FLOW EQUATIONS FOR N POINT FUNCTIONS AND BOUND STATES

Ulrich Ellwanger\thanks{Supported by a DFG Heisenberg fellowship, e-mail: I96 at VM.URZ.UNI-HEIDELBERG.DE}

Institut für Theoretische Physik
Universität Heidelberg
Philosophenweg 16, D-69120 Heidelberg, FRG

Abstract:
We discuss the exact renormalization group or flow equation for the effective action and its decomposition into one particle irreducible N point functions. With the help of a truncated flow equation for the four point function we study the bound state problem for scalar fields. A combination of analytic and numerical methods is proposed, which is applied to the Wick-Cutkosky model and a QCD-motivated interaction. We present results for the bound state masses and the Bethe-Salpeter wave function.
1 Introduction

For many considerations in quantum field theory it is a helpful idea to integrate out high frequency or short distance modes, and to study a scale-dependent effective theory for the remaining low frequency modes. This concept was first made concrete on the lattice within the framework of block spin transformations [1]. When applied to continuum quantum field theory, it can be cast into the form of exact renormalization group equations [2]-[10], which describe the cutoff dependence of the effective theory in a compact way. Polchinski and others [2]-[7] employed these ideas in order to study the dependence of physical Green functions on the UV cutoff with the aim to simplify perturbative proofs of renormalizability. The integration of the corresponding flow equations with respect to an infrared cutoff, on the other hand, provides a new exact method to compute effective low energy Lagrangians [8]-[10] or average actions [11]. Within perturbative or 1/N expansions this scheme has been applied successfully to the computation of effective potentials and “running” wave function normalizations and allows, e.g., to investigate critical phenomena in two and three dimensions [11], [12], high temperature phase transitions [13] and universality within the Higgs top system [10].

These results employed, to a large extent, an expansion of the effective action in powers of momenta (as it is appropriate for investigations of the effective potential). The first purpose of the present paper is the derivation of an expansion of the exact flow equations for the effective action in powers of fields, keeping the momentum dependence exact. This way one finds an infinite set of coupled differential equations for one-particle irreducible N point functions.

Then we concentrate on the four point function, whose singularities as a function of the c.m. energy squared s encode informations on possible bound states of theory. We study a truncation of the infinite set of flow equations, which corresponds to the ladder approximation. It leads to a simple flow equation including just the four point function itself. Already in this approximation we see, that the flow equation allows to find both the masses and the wave functions of bound states in a theory, and the equivalence to the Bethe-Salpeter equation can be shown.

Next we discuss, as an essential step towards numerical integrations of the flow equations, the Laplace transform of the four point function with respect to the Lorentz invariant products of momentum variables. Numerical methods can then be introduced after discretization in this “Laplace space”. Integration of the flow equation becomes a simple algorithm to update the corresponding “Laplace lattice” consecutively. The informations on bound states can be obtained from the numerical results in combination with analytic methods.

As a first application we study the Wick-Cutkosky model, which contains two massive complex scalars interacting through the exchange of a massless real scalar and ressembles, in the nonrelativistic limit, to the Positronium problem. We find good agreement between our results, both on the coupling constant dependence of the bound state mass and on the bound state wave function, and known formulas. Our method does not make use, however, of the O(4) symmetry of the Wick-Cutkosky model and can straightforwardly be applied to the bound state problem in
the case of arbitrary interactions. We present formulas, which allow to study interactions which correspond to potentials, in the non-relativistic limit, of the form \( r^\alpha \) with \( \alpha \) arbitrary. We end up with results for a QCD-motivated potential including a linearly rising confining part.

2 Flow Equations for N Point Functions

We will present the general features of the flow equations in the context of a single scalar field \( \varphi \) for simplicity, and we work in Euclidean space. The aim of the flow equations is the computation of Green functions within a theory, which is regularized in the UV in terms of a cutoff \( \Lambda \) in the propagator. In addition an infrared cutoff \( k \) is introduced below, and an important role is played by the corresponding propagator

\[
P_k^\Lambda(q^2) \equiv (R_k^\Lambda(q^2))^{-1} = \frac{h_\Lambda(q^2) - h_k(q^2)}{q^2 + m^2}
\]  

(2.1)

with

\[
\begin{align*}
    h_\Lambda(q^2) &\to 1 \quad \text{for} \quad q^2 \ll k^2 \\
    h_k(q^2) &\to 0 \quad \text{for} \quad q^2 \gg k^2.
\end{align*}
\]  

(2.2)

The starting point is the UV-regularized generating functional of connected Green functions \( G^\Lambda(J) \), which can be represented as

\[
e^{-G^\Lambda(J)} = \mathcal{N} \int \mathcal{D}\varphi e^{-\frac{1}{2}(\varphi,R_0^\Lambda\varphi)-S_{\text{int}}^\Lambda(\varphi)+(J,\varphi)}. \]  

(2.3)

Here \((J, \varphi)\) etc. is a short-hand notation for

\[
(J, \varphi) \equiv \int \frac{d^4q}{(2\pi)^4} J(q)\varphi(-q)
\]  

(2.4)

and we have represented the kinetic term in terms of the inverse UV-regularized propagator \( R_0^\Lambda(q^2) \). An alternative and useful representation of \( G^\Lambda(J) \) is given by

\[
e^{-G^\Lambda(J)} = e^{\frac{1}{2}(J,D_0^\Lambda J)} e^{D_0^\Lambda e^{-S_{\text{int}}^\Lambda(\varphi)}} \bigg|_{\varphi=P_0^\Lambda J}.
\]  

(2.5)

with

\[
D_0^\Lambda = \frac{1}{2}(P_0^\Lambda \frac{\delta}{\delta \varphi}, \frac{\delta}{\delta \varphi}).
\]  

(2.6)

Next we introduce an infrared cutoff \( k \) and define an effective interaction \( S_{\text{int}}(\varphi, k) \) by

\[
e^{-S_{\text{int}}(\varphi, k)} = e^{D_k^\Lambda e^{-S_{\text{int}}^\Lambda(\varphi)}},
\]  

(2.7)

where \( D_k^\Lambda \) is given as in (2.6) with \( P_0^\Lambda \) replace by \( P_k^\Lambda \). Clearly \( S_{\text{int}}(\varphi, k) \) is equal to the bare interaction \( S_{\text{int}}^\Lambda(\varphi) \) at \( k = \Lambda \), and from (2.5) we have

\[
S_{\text{int}}(\varphi, 0) \bigg|_{\varphi=P_0^\Lambda J} = G^\Lambda(J) + \frac{1}{2}(J, P_0^\Lambda J).
\]  

(2.8)
The flow equation for $S_{\text{int}}(\phi, k)$ follows easily after differentiation of the defining equation (2.7) with respect to $k$ (subsequently we use $\partial_k \equiv d/dk^2$)

\[
\partial_k S_{\text{int}}(\phi, k) = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \partial_k P_k^2(q^2) \cdot \left\{ \frac{\delta^2 S_{\text{int}}(\phi, k)}{\delta \phi(q) \delta \phi(-q)} - \frac{\delta S_{\text{int}}(\phi, k)}{\delta \phi(q)} \frac{\delta S_{\text{int}}(\phi, k)}{\delta \phi(-q)} \right\}. \tag{2.9}
\]

The generating functional of connected Green functions including the infrared cutoff $k$, $G_k^A(J)$, is related to $S_{\text{int}}(\phi, k)$ through

\[
S_{\text{int}}(\phi, k) \vert_{\phi=P_k^A} = G_k^A(J) + \frac{1}{2}(J, P_k^A J). \tag{2.10}
\]

Inserting (2.10) into (2.9) and taking into account that $J$ is implicitly $k$-dependent because of the $\phi = P_k^A \cdot J$ prescription in (2.10), one finds that $G_k^A(J)$ satisfies a similar flow equation:

\[
\partial_k G_k^A(J) = -\frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \partial_k R_k^A \cdot \left\{ \frac{\delta^2 G_k^A(J)}{\delta J(q) \delta J(-q)} - \frac{\delta G_k^A(J)}{\delta J(q)} \frac{\delta G_k^A(J)}{\delta J(-q)} \right\}. \tag{2.11}
\]

In (2.11) we have neglected a $J$-independent term in $G_k^A(J)$. For many considerations in quantum field theory, however, it is more convenient to deal with the generating functional of one-particle irreducible Green functions $\Gamma(\phi)$. Its constant part gives the effective potential, and also all other informations on a theory are encoded in $\Gamma(\phi)$ in a more compact way. In the presence of a UV cutoff $\Lambda$ and an infrared cutoff $k$ the effective action $\Gamma_k^A(\phi)$ is given by the Legendre transform of $G_k^A(J)$,

\[
\Gamma_k^A(\phi) = G_k^A(J) + (J, \phi). \tag{2.12}
\]

Inserting the Legendre transformation into (2.11) it is possible to derive the flow equation for $\Gamma_k^A(\phi)$. Thereby the implicit $k$-dependences of $J$ and $\phi$ have to be taken into account, but at the end they cancel and one is left with the simple result

\[
\partial_k \Gamma_k^A(\phi) = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \partial_k R_k^A(q^2) \left\{ \varphi(q) \varphi(-q) + \left( \frac{\delta^2 \Gamma_k^A(\phi)}{\delta \varphi(q) \delta \varphi(-q)} \right)^{-1} \right\}. \tag{2.13}
\]

Such an equation has also been found be Wetterich [1] and, in a different context (as a differential equation with respect to the UV cutoff $\Lambda$ instead of $k$) in [4]. If one splits off a bare kinetic part of $\Gamma_k^A(\phi)$,

\[
\Gamma_k^A(\phi) = \frac{1}{2}(\varphi, R_k^A \varphi) + \bar{\Gamma}_k(\phi), \tag{2.14}
\]

one obtains a flow equation for $\bar{\Gamma}_k(\phi)$ of the form

\[
\partial_k \bar{\Gamma}_k(\phi) = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \partial_k R_k^A(q^2) \left( R_k^A(q^2) + \frac{\delta^2 \bar{\Gamma}_k(\phi)}{\delta \varphi(q) \delta \varphi(-q)} \right)^{-1}. \tag{2.15}
\]

In general, however, $\bar{\Gamma}_k^A$ will still contain terms quadratic in $\varphi$. The boundary condition for $k \to \Lambda$ is most easily given for $\bar{\Gamma}_k(\phi)$: From the relation between
\[ \Gamma^A_k(\varphi) \] and \( S_{\text{int}}(\phi, k) \) via eqs. (2.12) and (2.10), and a careful consideration of the limit \( P^A_k \to 0 \) for \( k \to \Lambda \), one finds

\[ \tilde{\Gamma}^A_\Lambda(\varphi) = S^A_{\text{int}}(\phi) \bigg|_{\phi=\varphi} \quad (2.16) \]

where \( S^A_{\text{int}}(\phi) \) is the bare action of eqs. (2.3) or (2.5).

Let us now consider an expansion of \( \Gamma^A_k(\varphi) \) in powers of \( \varphi \), with one particle irreducible \( N \) point functions \( \Gamma^A_N \) as coefficients.

For simplicity we will assume a discrete symmetry \( \varphi \to -\varphi \) such that only even powers of \( \varphi \) appear. Instead of writing the convolutions over momenta, we switch to a tensor notation (as appropriate in the case of discretized momenta as, e.g., on a torus) according to

\[ \varphi(q) \to \varphi_i, \quad \Gamma^A(q_1...q_N) \to \Gamma^A_{i_1...i_N}, \quad \int \frac{d^4q}{(2\pi)^n} \to \sum_i, \text{etc.} \quad (2.17) \]

Thus we have

\[ \Gamma^A_k(\varphi) = \frac{1}{2} \varphi_i\varphi_j \Gamma^A_{ij} + \frac{1}{4!} \varphi_i\varphi_j\varphi_k\varphi_l \Gamma^A_{ijkl} + \frac{1}{6!} \varphi_i\varphi_j\varphi_k\varphi_l\varphi_m\varphi_n \Gamma^A_{ijklmn} + ... \quad (2.18) \]

Inserting this expansion into the flow equation (2.13) and ordering the result according to powers of \( \varphi \), one obtains the following infinite system of equations:

\[ \partial_k \Gamma^A_{ij} = \partial_k R^A_{kl} \left[ \delta^A_k \delta^A_{ij} - \frac{1}{2} \Gamma^{-1}_{2km} \Gamma^A_{mnij} \Gamma^{-1}_{2nh} \right] \]

\[ \partial_k \Gamma^A_{ijkl} = \partial_k R^A_{nm} \left[ 3 \Gamma^{-1}_{2no} \Gamma^A_{npij} \Gamma^{-1}_{2pq} \Gamma^A_{qrkl} \Gamma^{-1}_{2rm} - \frac{1}{2} \Gamma^{-1}_{2no} \Gamma^A_{npijkl} \Gamma^{-1}_{2pm} \right] \]

\[ \text{etc.} \quad (2.19) \]

The r.h.s. of the flow equations for the individual \( N \) point functions \( \Gamma_N \) have the following properties: In contrast to standard \( \beta \) functions each of them is exact, there are no higher order terms beyond the ones shown explicitly. They have a simple diagrammatic interpretation shown in fig. 1. The flow equation for \( \Gamma_N \) includes always a term involving \( \Gamma_{N+2} \) on its r.h.s., thus a truncation of the series leaves no exact result. On the other hand, the iterative solution of these differential equations generates the loop expansion of standard perturbation theory, because the r.h.s. of each equation involves effectively an overall factor of \( \hbar \).

### 3 Flow Equation for the Four Point Function

In this section we will consider a truncation of the flow equation of the four point function, which corresponds to the ladder approximation in the Bethe-Salpeter
framework. First, we will neglect the contributions to the flow of the full inverse propagator $\Gamma_2$ induced by the interactions and identify $\Gamma_2$ with the bare (regularized) inverse propagator, $\Gamma_2 = R^A_k$ (or $\Gamma_2^{-1} = P^A_k$). Second, we neglect the contribution of $\Gamma_6$ to the flow of $\Gamma_4$, which amount to taking only bubble-type diagrams into account.

We will slightly change the field content, namely include two distinct complex scalar fields $\varphi_a, \varphi_b$ with identical mass $m$ for simplicity. The expansion of the effective action reads

$$\Gamma_k^{\Lambda}(\varphi_a, \varphi_b) = \varphi_a^\dagger R^A_k j \varphi_a + \varphi_b^\dagger R^A_k j \varphi_b + \varphi_a^\dagger \varphi_a \varphi_b^\dagger \varphi_b \Gamma_{4ji}^{ih} + \ldots$$  \hspace{1cm} (3.1)

and the flow equation for $\Gamma_4$ becomes in the present approximation

$$\partial_k \Gamma_{4ji}^{ih} = -2\partial_k P^{Am}_k \Gamma_{4hi}^{mh} P^{Ao}_k \Gamma_{4jm}^{ip}.$$  \hspace{1cm} (3.2)

With momenta written explicitly this equation reads

$$\partial_k \Gamma_4(p_1, p_2, p_3, p_4) = -2 \int \frac{d^4q}{(2\pi)^4} \Gamma_4(p_1, p_2, q, -q - p_1 - p_2) \cdot \Gamma_4(-q, q + p_1 + p_2, p_3, p_4) \cdot \partial_k P^A_k(q^2) \cdot P^A_k((q + p_1 + p_2)^2).$$  \hspace{1cm} (3.3)

The equivalence to the Bethe-Salpeter equation can be established as follows: Let us introduce a two-particle propagator $P^{(2)}$ through

$$P^{(2)}_{ij\ mn} \equiv P^A_k m P^A_k n.$$  \hspace{1cm} (3.4)

Then eq. (3.2) can formally be written as

$$\partial_k \Gamma_4 = -\Gamma_4 \otimes \partial_k P^{(2)} \otimes \Gamma_4$$  \hspace{1cm} (3.5)

with the formal solution

$$\Gamma_4 = \frac{\Gamma_4^A}{1 + P^{(2)} \otimes \Gamma_4^A}.$$  \hspace{1cm} (3.6)

where $\Gamma_4^A$ is the boundary value or bare four point function. In our convention, the non-amputated four point function $\tilde{\Gamma}_4$, which also includes a disconnected piece, is related to $\Gamma_4$ through

$$\tilde{\Gamma}_4 = -P^{(2)} \otimes \Gamma_4 \otimes P^{(2)} + P^{(2)}$$  \hspace{1cm} (3.7)

and satisfies the Bethe-Salpeter equation [14]

$$\tilde{\Gamma}_4 = P^{(2)} + P^{(2)} \otimes K \otimes \tilde{\Gamma}_4$$  \hspace{1cm} (3.8)

where $K$ denotes the interaction kernel. The formal solution of (3.8) reads

$$\tilde{\Gamma}_4 = \frac{P^{(2)}}{1 - K \otimes P^{(2)}} = P^{(2)} + \frac{P^{(2)} \otimes K \otimes P^{(2)}}{1 - K \otimes P^{(2)}}.$$  \hspace{1cm} (3.9)

Using eq. (3.7), one finds that this solution of the B.-S. equation coincides with the solution (3.6) of the flow equation after identifying $K$ with $-\Gamma_4^A$. Thus, although
these equations are quite different, they have the same physical content. We expect
that a general equivalence between the full set of flow equations (2.19), beyond
the ladder approximation, and the full set of Schwinger-Dyson equations can be
established.

This discussion showed already an equivalence between the boundary value of
the running four point function $\Gamma_4$ at $k = \Lambda$, denoted by $\Gamma^\Lambda_4$ in eq. (3.6), and a
general interaction kernel $K$. In fact, $\Gamma^\Lambda_4$ does not necessarily have to be a bare
coupling of a theory with only $\varphi_a, \varphi_b$ fields, but could be the result of interactions
with different fields which have already been completely integrated out. An example
is given by the Wick-Cutkosky model [14], which starts out with an additional real
massless scalar field $\phi$ beyond the fields $\varphi_a, \varphi_b$. The original interaction involves no
bare four point coupling as in eq. (3.1), but just trilinear interactions of the form

$$\Gamma_{int} = g\phi(\varphi^\dagger_a \varphi_a + \varphi^\dagger_b \varphi_b).$$

(3.10)

It is straightforward to integrate out the field $\phi$, which gives rise to an effective
(nonlocal) four point function for the field $\varphi_a, \varphi_b$ as in (3.1) with

$$\Gamma_4^\Lambda(p_1, p_2, p_3, p_4) = \frac{g^2}{(p_1 - p_3)^2} = \frac{g^2}{(p_2 - p_4)^2}.$$  (3.11)

The dynamics of the remaining fields $\varphi_a, \varphi_b$ in the presence of the interaction (3.11)
can now either be obtained by means of the B.S. equation or, as we propose here, by
integrating the flow equations for $\Gamma_4$ as given by eq. (3.3) with (3.11) as boundary
condition. It is easy to imagine more general boundary conditions obtained, e.g.,
after integrating out photons, gluons, or the like.

In order to deal with eq. (3.3) we next observe that, since the framework is
completely Lorentz-covariant, $\Gamma_4$ should be a function of Lorentz-invariant products
of momenta only. Off-shell six independent invariants can be formed, which we
denote by

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \quad t = (p_2 - p_3)^2 = (p_2 - p_4)^2,$$

$$v_1 = p_1^2 + p_2^2, \quad v_2 = p_3^2 + p_4^2, \quad w_1 = p_1^2 - p_2^2, \quad w_2 = p_3^2 - p_4^2.$$  (3.12)

Although $\Gamma_4$ will in general be an arbitrary function of those six variables, we will
first study the case where $\Gamma_4$ can be written as a product as follows:

$$\Gamma_4 = f(v_1, w_1) \cdot D_k(s) \cdot f(v_2, w_2).$$  (3.13)

If one inserts this ansatz into the flow equation (3.3), one finds that it is stable
and thus corresponds to a “fix point”. Furthermore variables can be separated, and
one finds that the function $f$ can depend on $k$ only via an overall factor, which can
be included in $D_k(s)$ by definition. This latter function satisfies a flow equation of
the form

$$\partial_k D_k(s) = -2D_k^2(s) \cdot F_k(s, f)$$  (3.14)
with
\[ F_k(s, f) = \int \frac{d^4q}{(2\pi)^4} f(q, -q - p_1 - p_2) f(-q, q + p_1 + p_2) \partial_k P^A_k(q^2) P^A_k((q + p_1 + p_2)^2) \]  
(3.15)

The general solution of the flow equation (3.14) is easily written down in terms of an arbitrary boundary condition \( D_{k'}(s) \) at \( k = k' \):
\[ D_k(s) = \frac{D_{k'}(s)}{1 - 2D_{k'}(s) \int_k^{k_2} d\hat{k}^2 F_k(s, f)} \]  
(3.16)

On the one hand, this shows that any problem where \( \Gamma^4 \) starts out to be of the form (3.13) is analytically solvable. On the other hand, if the theory contains a bound state in the channel \((\varphi_a, \varphi_b) \rightarrow (\varphi_a, \varphi_b)\), which manifests itself as a pole in \( s \) of \( \Gamma_4 \), we actually expect that \( \Gamma_4 \) approaches the form (3.13) for \( k \rightarrow 0 \) and \( s \) in the vicinity of the pole. Then the function \( f \) corresponds to the amplitude of the propagator and contains the information on the position of the pole. In fact we will see that the factorized form (3.13) of \( \Gamma_4 \) is an infrared-attractive “fix point” of the flow equations, which will be of great help in extracting informations on possible bound states analytically.

In order to proceed towards concrete calculations we have to make a choice on the form of the regularization of the propagator \( P^A_k \) or on the function \( h_k(q^2) \) of eqs. (2.1) and (2.2). The following choice turns out to be particularly useful:
\[ h_k(q^2) = e^{-\frac{q^2 + m^2}{k^2}} \]  
(3.17)

or
\[ P^A_k(q^2) = e^{-\frac{q^2 + m^2}{\Lambda^2}} - e^{-\frac{q^2 + m^2}{k^2}} = \int_{1/\Lambda^2}^{1/k^2} d\alpha e^{-\alpha(q^2 + m^2)}. \]  
(3.18)

This regularized propagator has the following properties: For general nonvanishing \( k^2 \), there is no pole at \( q^2 = -m^2 \), instead it behaves like \( 1/k^2 \), at \( q^2 = -m^2 \), for \( k^2 \rightarrow 0 \). Beyond the “pole”, for \( q^2 < -m^2 \), it increases exponentially for \( k^2 \rightarrow 0 \). The absence of singularities at finite \( k^2 \) is also clear from the boundedness of the Feynman parameter integral \( d\alpha \).

Similarly we have to make a choice on the parametrization of the dependence of \( \Gamma_4 \) on the six Lorentz invariants shown in (3.12). First we make a technical simplification, namely we neglect the dependence of \( \Gamma_4 \) on the variables \( w_1, w_2 \) of (3.12). In the Wick-Cutkosky model \( \Gamma_4 \) does not depend on \( w_1, w_2 \) neither at its starting point at \( k \rightarrow \Lambda \), given by eq. (3.11) in the form \( g^2/t \), nor for \( k \rightarrow 0 \) in the weak coupling limit. In this limit the wave function \( f \) is known to depend on \( t \) only.

Now it turns out to be very convenient to switch from \( \Gamma_4(s, t, v_1, v_2) \) to its Laplace transform with respect to the three variables \( t, v_1 \) and \( v_2 \):
\[ \Gamma_4(s, t, v_1, v_2) = \int_0^\infty dl_0 dl_1 dl_2 C_k(s, t, l_0, l_1, l_2) e^{-l_0 t - l_1 v_1 - l_2 v_2} \]  
(3.19)
After inserting this Laplace transformation together with the parametrization (3.18) of the propagator into the flow equation (3.3), one finds that the $d^4 q$ integration can easily be performed, and one is left with the following flow equation for $C_k$:

$$\partial_k C_k(s, l_0, l_1, l_2) = \frac{1}{8\pi^2 k^2} \int_{k^2/\Lambda^2}^1 \frac{d\beta}{\Lambda} \int_0^\infty dn_0 dn_1 dn_2 dm_0 dm_1 dm_2$$

$$\cdot \frac{1}{B^2} \cdot e^{-\frac{k}{B^2} \left(m^2(1+\beta) + \frac{4m^2}{B^2}\right)} \delta(l_0 - \frac{k^2n_0m_0}{B}) \delta(l_1 - \frac{n_0(b+b')}{2B} - n_1)$$

$$\cdot \delta(l_2 - \frac{m_0(b+b')}{2B} - m_2) \cdot C_k(s, n_0, n_1, n_2) \cdot C_k(s, m_0, m_1, m_2)$$

(3.20)

with

$$b = 1 + k^2(n_2 + m_1), \quad b' = \beta + k^2(n_2 + m_1), \quad B = b + b' + k^2(n_0 + m_0).$$

(3.21)

Of course the previously discussed simplification in the case of factorization remain valid: If $\Gamma_4$ factorized as in eq. (3.13), $C_k$ factorizes as

$$C_k(s, l_0, l_1, l_2) = \delta(l_0) \tilde{f}(l_1) D_k(s) \tilde{f}(l_2)$$

(3.22)

where $\tilde{f}(l)$ is the Laplace transform of the wave function $f(v)$. The solution (3.16) for $D_k(s)$ remains the same with $F_k(s, f)$ now given by

$$F_k(s, f) = \frac{1}{16\pi^2 k^2} \int_{k^2/\Lambda^2}^1 \frac{d\beta}{\Lambda} \int_0^\infty dl_1 dl_2 \tilde{f}(l_1) \tilde{f}(l_2) \frac{B^2}{B^2} e^{-\frac{k}{B^2} \left(m^2(1+\beta) + \frac{4m^2}{B^2}\right)}$$

(3.23)

and $b = 1 + k^2(l_1 + l_2), \quad b' = \beta + k^2(l_1 + l_2), \quad B = b + b'$. The right-hand side of eq. (3.20) shows already an expected analytic property of the four point function: If we continue the variable $s$ into the Minkowskian regime ($s < 0$), we expect a cut at $s = -4m^2$. Indeed, in the limit $k^2 \to 0$ and for $\beta \to 1$ the exponent on the r.h.s. of eq. (3.20) becomes $-(2m^2 + s/2)/k^2$ and thus explodes for $s < -4m^2$.

In the case of general interactions, i.e. general boundary conditions for $\Gamma_4$, or $C_k$ at $k = \Lambda$, the flow equations either in the form (3.3) or in the form (3.20) cannot be solved analytically. Now we propose to solve eq. (3.20) numerically after discretization of the variables $l_0, l_1$ and $l_2$. $C_k(s, l_0, l_1, l_2)$ becomes a function $C_k(s)_{i_0i_1i_2}$, which lives on a three-dimensional lattice, where the integers $i_0, i_1, i_2$ denote the lattice points. After discretization of the variable $k^2$ eq. (3.20) provides an algorithm on how to update the lattice in each step from $C_k(s)$ to $C_{k-\Delta k}(s)$.

It is sensible to start this algorithm with values of $k$ large compared with the scale $m$ of the problem; choosing the starting point $\Lambda$ to be $\Lambda^2 \sim 10m^2$ turns out to be sufficient for our purposes. At the beginning one has to fix $C_{\Lambda}$, the Laplace transform of $\Gamma_4$, and the variable $s$.

In the presence of a bound state with mass $M$ we expect the following behaviour of $C_k$ in analogy to the behavior of the regularized propagator $P_k^\Lambda$ of eq. (3.18): For $s > -M^2$ $C_k$ remains finite for $k \to 0$, whereas for $s \lesssim -M^2$ $C_k$ approaches a factorized form as in eq. (3.22), and the factor $D_k(s)$ diverges for $s < -M^2$ and $k \to 0$. A divergence of $D_k$ for $k \to 0$ thus indicates, that the variable $s$ satisfies $s < -M^2$. This behaviour is indeed the result of our numerical investigations.
Since the factorized form (3.22) turns out to be infrared stable, eq. (3.16) can help us to decide whether, at a given value of $s$, $C_{k\to0}$ diverges or not: If we observe that for a certain value of $k = k'$, $C_{k'}$ assumes the form (3.22), we can read off the function $\tilde{f}(l)$ and the value of $D_{k'}(s)$. Then we can compute the function $F_k(s, f)$ of eq. (3.23), insert it into eq. (3.16), and we see immediately that $D_{k\to0}(s)$ diverges if

$$2D_{k'}(s) \int_0^{k^2} d\hat{k}^2 F_k(s, f) > 1.$$  

(3.24)

There is thus no need to integrate numerically down to $k = 0$, but the following procedure turns out to be sufficient: We integrate numerically the flow equation for $C_k$, at a given value of $s$, down to $k^2 = k'^2 \simeq m^2/10$. Then we check whether $C_k$ has assumed the factorized form (3.22). If not, this remains so even for smaller values of $k^2$ (as we have tested numerically) and we conclude, that the corresponding value of $s$ satisfies $s > -M^2$, where $M$ is the mass of the lowest lying bound state. Thus we decrease $s$ and start again, until we get to a value of $s$ where $C_{k'}(s)$ factorizes. Then we read off $D_{k'}(s)$ and $\tilde{f}(l)$ and check with the help of eq. (3.24) whether $D_{k\to0}(s)$ diverges, i.e. whether $s$ satisfies $s < M^2$. By repeating this procedure for different values of $s$, we can pin down $M^2$ as accurately as we like, and of course we have also obtained the Laplace-transformed wave function $\tilde{f}(l)$.

The general picture of the use of flow equations is the following: For $k^2 \gg m^2$, the r.h.s. of the flow equations is small except for Green functions which correspond to marginal or relevant operators, which are thus the only ones to vary significantly (but which are not present in the present approximation). In this regime, at least for small couplings, the result of the integration of the flow equations could also be obtained by standard perturbative methods, eventually improved by the use of any mass-independent standard renormalization group. For $k^2 \sim m^2$, however, dynamical effects like the formation of bound states take place, which generally require the use of numerical methods. For $k^2 \ll m^2$, the r.h.s. of the flow equations are again small (damped exponentially) except for external momenta, where Green functions develop singularities like poles or cuts. This is a feature both of the bare propagator (3.18) and of the four point function discussed here. The development of singularities, however, has simple universal properties like the factorized behaviour (3.13) of the four point function, which allow it to be treated analytically again.

The general procedure of combined numerical and analytic methods for the search for bound states as described above will be applied to some concrete models in the next section.

4 Results for Bound States

In this chapter we will study the bound state problem in the case of two complex scalar fields using the methods derived in section 3. We integrate the flow equations for the four point function $\Gamma_4$ in an approximation, where the scalar propagator remains the free one, and where the contribution of the six point function to the
r.h.s. of the flow equation for $\Gamma_4$ is neglected. After restricting the momentum dependence of $\Gamma_4$ to the four invariants $s, t, v_1$ and $v_2$ of (3.12) and after the Laplace transformation with respect to $t, v_1$ and $v_2$, the flow equation has the form (3.20).

The information on the underlying model, i.e. the bare interaction $\Gamma_4^\Lambda$ or the kernel of the corresponding Bethe-Salpeter equation, is entirely and only specified by the boundary condition at $k = \Lambda$. In the case of the Wick-Cutkosky model, which we study first, this boundary condition is given by $\Gamma_4^\Lambda = g^2/t$ (see eq. (3.11) or, in terms of the Laplace transform $C_k$ of (3.19), by

$$C_\Lambda(s, l_0, l_1, l_2) = g^2 \delta(l_1)\delta(l_2).$$

As described in the previous chapter, we discretize the variables $l_0, l_1$ and $l_2$. We used lattice sizes up to 20 in all three directions. We integrate the flow equation (3.20) numerically from $k^2 = \Lambda^2 = 10m^2$ down to $k^2 = k_0^2 = m^2/10$ at fixed c.m. energy $s$ with $-4m^2 < s < 0$. At $k^2 = k_0^2$ we check, whether $C_k$ has at least approximately assumed the factorized form (3.22). If this is the case, we extract the function $\tilde{f}(l)$ and the value of $D_k(s)$. Then we compute the l.h.s. side of (3.23), still at fixed $s$, in order to check whether $s$ satisfies $s > -M^2$ or $s < -M^2$, where $M$ is the mass of the lowest-lying bound state.

Indeed this method works very well in the present model. If $s$ is chosen close to or below $-M^2$, the numerical deviations of $C_k$, from the factorized form (3.22), are less than one percent. For such values of $s$, $C_k$ is seen to increase rapidly towards small values of $k$. For larger values of $s$, on the other hand, $C_k$ is seen to remain finite for $k \to 0$. If we would take $C_k(s, 0, 0, 0) > 10^5$ as a criterium for $s < -M^2$, this would actually match our criterium for $s < -M^2$ based on eq. (3.24) already within a few percent of the value of $s$.

Within the Wick-Cutkosky model it is convenient to study the dependence of the mass $M$ of the lowest lying bound state on a coupling $\lambda$ defined by

$$\lambda = g^2/16\pi m^2.$$  

Analytic results are known in the weak coupling limit $\lambda \to 0$, where

$$M^2 \simeq (4 - \lambda^2)m^2,$$

and in the case of a vanishing bound state mass [14]:

$$M = 0 \quad \text{for} \quad \lambda = 2\pi.$$  

A result of a numerical solution of the Bethe-Salpeter equation can be found in [15]:

$$M = (2 - .082)m \quad \text{for} \quad \lambda = 1.$$  

In fig. 2 we plot our results for $M^2$ in units of $m^2$ versus $\lambda$. We indicate, for $\lambda < .3$, eq. (4.3) as a short line, and the two results (4.4) and (4.5) as crosses. We show our results as error bars, which were obtained on a $20^3$ lattice. The error bars are due to varying the prescriptions on how to discretize the variables $l_0, l_1$ and $l_2$, and
due to changing the physical size of the lattice: The ranges of $l_0$, $l_1$ and $l_2$ are given by $0 \leq l_0 \leq l_{0\text{max}}$ and $0 \leq l_1, l_2 \leq l_{\text{max}}$, and we varied $l_{0\text{max}}$ between 10 and 40, and $l_{\text{max}}$ between 6 and 10 (in units of $m^{-2}$). (Larger values of $l_{\text{max}}$ are irrelevant, because $C_k(s, l_0, l_1, l_2)$ decreases exponentially with $l_1$ and $l_2$, see below.)

In the case of large values of the coupling $\lambda$ it should be remembered that we neglected the dependence of $\Gamma_4$ on the momentum variables $w_1, w_2$ of eq. (3.12). This is known to be a good approximation, for $\Gamma_4$ with $k \ll \Lambda$, only in the nonrelativistic or weak coupling limit. Therefore the deviations of our results from the known behaviour (4.4) for $\lambda \to 2\pi$ (strong coupling) are understandable.

In the extreme weak coupling limit $\lambda \to 0$, on the other hand, the physical size of a bound state, in ordinary space, is known to increase. In terms of our “Laplace space”, spanned by the variables $l_i$, large distance also correspond to large values of $l_i$. Since this “Laplace space” necessarily has a finite size, it is again understandable that this method cannot describe phenomena accurately, where very large distances play an important role, as in the case of very weakly coupled bound states.

For moderate values of $\lambda$ we find a large region, however, where our results seem to be free of those systematic errors and match the known ones suprising well. The dependence of our results on the lattice size might be of some interest. Therefore we plot, for $\lambda = 1$, our results for $M^2$ versus the lattice size in fig. 3. (The error bars are of the same origin as discussed above.) We see the steady convergence, with increasing lattice size, towards the known result, which makes us believe that the method has no systematic limitations beyond the ones mentioned before.

The procedure required already to extract the function $\tilde{f}(l)$ out of $C_{k'}$, which satisfies (3.22) for $s$ near $-M^2$. Up to a normalization this function is the Laplace transform of the amputated Bethe-Salpeter wave function. In the weak coupling limit of the Wick-Cutkosky model this amputated wave function is known to be

\[ f(p_1^2, p_2^2) = 1/(p_1^2 + p_2^2 + 2m^2) = 1/(v_1 + 2m^2) \] (4.6)

with $v_1$ as in (3.12). Accordingly $\tilde{f}(l)$ should read

\[ \tilde{f}(l) = \exp(-2m^2l). \] (4.7)

In fig. 4 we plot our result for $\log[\tilde{f}(l)]$, at $\lambda = 1$, versus $l$ in units of $m^{-2}$ for a lattice size of 10. We normalized our result with respect to the known one at $l = 1$, and we see very nicely the agreement with the known exponential decrease, until finite lattice size effects start to play a role.

It should be noted that, in order to obtain all the results of figs. 2 to 4 from the numerical integration of the flow equation (3.20), a computer time on a work station of only a few minutes was needed. Furthermore no use was made of the $O(4)$ symmetry of the Wick-Cutkosky model [14], which was exploited in the numerical solution of the Bethe-Salpeter equation in [15]. Thus the method can straightforwardly be applied to the relativistic bound state problem with arbitrary interactions.

Of particular interest are interaction kernels, which are motivated by QCD and which correspond to a confining potential in the nonrelativistic limit. Let us first discuss potentials $V(r)$, which behave like $r^{\alpha}$ as a function of the distance $r$ in
ordinary three-dimensional space. The corresponding Lorentz-invariant momentum-dependent interaction kernel is actually not unique; it becomes unique, however, if one assumes that it depends only on the momentum transfer \( q^2 = t \). We adopt the notation to our procedure, where the interaction kernel coincides with the boundary value \( \Gamma^A_4 \) of the four point function; then the relation between \( V(r) \) and \( \Gamma^A_4(t) \) reads

\[
\Gamma^A_4(t) = \gamma 2^{3+\alpha} \pi^{3/2} t^{-3-\alpha} \frac{\Gamma\left(\frac{3+\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)} \quad \text{for} \quad V(r) = \gamma r^\alpha. \tag{4.8}
\]

In our formalism we actually need the Laplace transform \( C_\Lambda \) in analogy to (4.1); its corresponding form is given by

\[
C_\Lambda(l_0) = \gamma \frac{2^{3+\alpha} \pi^{3/2}}{\Gamma(-\frac{\alpha}{2})} l_0^{\frac{\alpha}{2}}. \tag{4.9}
\]

The possible singularities in \( \alpha \) in eqs. (4.8), (4.9) originate from analytic continuation in \( \alpha \), which is required if the functions \( V(r) \) or \( C_\Lambda(l_0) \) are not well behaved at large distances \( r \rightarrow \infty \) or \( l_0 \rightarrow \infty \). Within our procedure, however, \( C_\Lambda(l_0) \) will only be nonvanishing for a finite range of \( l_0 \) due to the finite size of the lattice. From the general relation between \( V(r) \) and \( C_\Lambda(l_0) \),

\[
V(r) = \frac{1}{4 \pi^{3/2} r} \int_0^\infty \frac{dx}{\sqrt{x}} C_\Lambda(l_0) e^{-x} \quad \text{with} \quad l_0 = \frac{r^2}{4x}, \tag{4.10}
\]

one finds the following: If \( C_\Lambda(l_0) \) vanishes for \( l_0 \geq a \), \( V(r) \) decays exponentially for \( r^2 \gg a \). On the other hand, for \( r^2 \ll a \), \( V(r) \) and \( C_\Lambda(l_0) \) are related as given by eqs. (4.8), (4.9) up to a constant in \( V(r) \) depending on \( a \). Such a constant is actually quite welcome; mesonic spectra in QCD, in the nonrelativistic limit, are typically derived from potentials of the form

\[
V(r) = \lambda r - \frac{4}{3} \alpha_{QCD} r^{-1} - c \tag{4.11}
\]

with a constant \( c \) of \( O(1 \text{ GeV}) \).

We applied our method to relativistic interactions, which are motivated by the potential (4.11). Of course we are dealing with scalars instead of fermionic quarks, but scalars are known to reproduce the spectrum of fermionic bound states in the nonrelativistic limit (where spin effects become negligible), provided the Coulomb interaction induced by vector boson exchange is multiplied by a factor 4.

Thus we integrated the flow equations (3.2) with a boundary condition \( C_\Lambda(l_0) \) of the form

\[
C_\Lambda(l_0) = -\theta\frac{\pi c^2}{4\lambda^2} - l_0)(8\pi \lambda l_0 + 16\pi \cdot \frac{4}{3} \alpha_{QCD}), \tag{4.12}
\]

which reproduces (4.11) for \( r \ll c/\lambda \) according to (4.10). For the string tension \( \lambda \) we choose .25 GeV/\( \sqrt{\alpha} \), .3 for \( \alpha_{QCD} \) and 1 GeV for the constant \( c \). In table 1 we give our results for the bound state mass \( M \) for different values of the scalar mass \( m \) in the range 1.5 GeV to 5.5 GeV. Again the results agree with general expectations:
Small, but increasing binding energy towards larger scalar masses, and vanishing binding energy for \( m \lesssim 1\text{GeV} \) [16]. (Our present method does not allow us to find bound states above threshold or with \( M > 2m \), since for \( s < -4m^2 \) the behaviour of the four point function is dominated by the two-particle cut, which reveals itself as a singularity for \( k \to 0 \).) The increasing errors towards larger scalar masses and binding energies likely indicate that here the neglect of the dependence of \( \Gamma_4 \) on the momentum variables \( w_1, w_2 \) is not a good approximation. Nevertheless the results show certainly the feasibility of the method in the case of general interactions.

5 Conclusions

In this paper we presented a derivation of the exact flow equation (2.13), which allows to compute effective low energy actions in terms of arbitrary high energy actions. It corresponds to an infinite set of flow equations for one particle irreducible Green functions, where each equation is again exact. Solutions to a finite subset of these equations are already nonperturbative in \( \bar{\hbar} \), and correspond to summations of certain sets of Feynman diagrams. They contain much more information, however, than the finite number of running coupling constants in standard renormalization theory.

Whereas the practical use of flow equations, often in different formulations, has been demonstrated before in context of expansions of effective actions in powers of momenta [10]-[13], we concentrated here on the bound state problem. Our tool was a truncated flow equation for the four point function, corresponding to the summation of “ladder type” diagrams, with the full dependence on the momenta left intact.

In principle, the effective low energy action and hence the low energy four point function contains all informations available by summing perturbation theory. We presented some technical tools, which allow to make practical use of this principle. These tools involved the parametrization of the momentum dependence of Green function in terms of its Laplace transform with respect to the Lorentz-invariant products of momenta (except for the c.m. energy \( s \)). This procedure is obviously manifest Lorentz-invariant, and simplifies the r.h.s. of the flow equations such that they become accessible to a numerical treatment.

One has not to rely completely on numerical methods, however: In the presence of nontrivial analytic structures, such as bound state poles, the form of the Green functions simplifies at small scales such that the flow equations can be solved analytically in this regime. The convergence of the Green functions towards these simple structures (as the factorized form (3.13)) in the infrared represents a new form of universality.

We demonstrated the practical feasibility of the method in the framework of the Wick-Cutkosky model, which represents a non-trivial bound-state problem with some known results derived via the Bethe-Salpeter equation. We checked the possibility to continue the c.m. energy \( s \) towards negative values (into the Minkowski regime), and the possibility to obtain both the mass of the lowest-lying bound state as well as the corresponding wave function with the help of the proposed combination
of numerical and analytic methods.

The general applicability of these methods allowed us to investigate very different interactions such as QCD-motivated confining kernels. The relation between the Laplace transform of such kernels and confining nonrelativistic potentials including a constant term is actually an interesting subject for its own, we just briefly presented some general formulas on this relation. Again our results look quite promising, but this field deserves further studies.

Clearly our approach can be improved and extended in a straightforward manner: The dependence of the four point function on the momentum variables $w_1, w_2$ can be restored, and fermions can be included [17]. Furthermore the flow equation for the two point function or the exact propagator can be taken into account, and it is possible to go beyond the ladder approximation by including the six or even higher point functions. Also the consideration of gauge interactions will impose no intractable problems [18]: Of course the momentum cutoff has to be covariantized, which is straightforward since the UV cutoff $\Lambda$ remains fixed throughout the whole procedure. It will turn out, however, that an additional one-loop counterterm is required [18]; this field is the subject of present investigations.

In conclusion, besides a derivation and discussion of the flow equations we have shown that the bound-state problem is another field of practical applicability of this approach. It seems to be general and flexible enough to allow for studies of a wide range of phenomena in quantum field theory.

**Acknowledgement**

It is a pleasure to thank D. Gromes and C. Wetterich for stimulating discussions.

**Figure Captions**

Fig. 1: Diagrammatic representation of the flow equations for the two and four point functions as in eq. (2.19). The lines denote the full propagators $\Gamma^{-1}_2$, encircled numbers the corresponding amputated $N$ point function, and the cross in a box an insertion of $\partial_k R_k^\Lambda$.

Fig. 2: Plot of $M^2$ of the lowest-lying bound state in the Wick-Cutkosky model versus the coupling $\lambda$. The short line denotes the known analytic result in the weak coupling limit (see eq. (4.3)), and the crosses known results for $\lambda = 1$ (eq. (4.4)) and $\lambda = 2\pi$ (eq. (4.5)). Our results are shown as error bars.
Fig. 3: Plot of $M^2$, in the same model and for $\lambda = 1$, versus the lattice size. The known result (eq. (4.5)) is indicated as a cross.

Fig. 4: Plot of the logarithm of the Laplace transform of the wave function $\tilde{f}(l)$ of the Wick-Cutkosky model in the weak coupling limit. The known result (eq. (4.7)) is indicated as a straight line, and our results are normalized to the known one at $l = 1$. 
Result for the bound state mass $M$ for an interaction corresponding to a nonrelativistic potential (4.11) and parameters as indicated below eq. (4.12). $m$ denotes the different masses of the scalar constituents.

| $m$ [GeV] | $M$ [GeV]       |
|-----------|-----------------|
| 1.5       | 2.99–3          |
| 2.5       | 4.92–4.99       |
| 3.5       | 6.81–6.96       |
| 4.5       | 8.63–8.94       |
| 5.5       | 10.11–10.83     |
References

[1] K. G. Wilson and I. Kogut, Phys. Rep. 12 (1974) 75; F. Wegner, in: Phase Transitions and Critical Phenomena, vol. 6, eds. C. Domb and M. Green (Academic Press, New York 1975).

[2] J. Polchinski, Nucl. Phys. B231 (1984) 269.

[3] B. Warr, Annals of Physics 183 (1988) 1 and 59.

[4] T. Hurd, Commun. Math. Phys. 124 (1989) 153.

[5] G. Keller, C. Kopper, and M. Salmhofer, Helv. Phys. Acta 65 (1992) 32.

[6] G. Keller and C. Kopper, Phys. Lett. B273 (1991) 323, Commun. Math. Phys. 148 (1992) 445, Commun. Math. Phys. 153 (1993) 245, preprint UNIGOE-THPHY-4-93.

[7] M. Bonini, M. D’Attanasio, and G. Marchesini, preprint UPRF 92-360.

[8] U. Ellwanger, Z. Phys. C58 (1993) 619.

[9] C. Wetterich, preprint HD-THEP-92-64, to appear in Z. Phys. C, Phys. Lett. B301 (1993) 90, preprint HD-THEP-93-17.

[10] U. Ellwanger and L. Vergara, Nucl. Phys. B398 (1993) 52.

[11] C. Wetterich, Nucl. Phys. B352 (1991) 529.

[12] C. Wetterich, Z. Phys. C57 (1993) 451

[13] N. Tetradis and C. Wetterich, Nucl. Phys. B398 (1993) 659.

[14] C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, New York 1980), chapt. 10.

[15] B. Silvestre-Brac, A. Bilal, C. Gignoux, and P. Schuck, Phys. Rev. D29 (1984) 2275.

[16] D. Gromes, Z. Phys. C11 (1981) 147.

[17] C. Wetterich, Z. Phys. C48 (1990) 693; S. Bornholdt and C. Wetterich, Z. Phys. C58 (1993) 585.

[18] M. Reuter and C. Wetterich, Nucl. Phys. B391 (1993), preprint HD-THEP-92-62.
\[ \dot{\theta}_k - 2 = -\frac{1}{2} \]

\[ \dot{\theta}_k - 4 = 3 - \frac{1}{2} \]

Fig. 1
Fig. 2
Fig. 4