A NOTE ON THE LAPLACE TRANSFORM AND THE VARIABLE-ORDER DIFFERENTIAL OPERATORS

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Abstract. In this short note, using the variable-order differential operator introduced by means of the inverse Laplace transform [1], we questioned the result obtained by Yang and Tenreiro Machado [2].

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1. Introduction

Over the time, fractional calculus has provided researchers with several important discoveries, in particular, applications in various areas of knowledge [3, 4, 5]. Some authors have shown themselves to be interested in proposing new operators of fractional differentiation and integration, until then with non-variable order. Recently, Almeida [6] introduced the \( \psi \)-Caputo fractional derivative of non-variable order of a function \( f \) with respect to another function \( \varphi \) and, Yang and Tenreiro Machado [2], presented a new operator of fractional integration of variable order and consequently made use of it, to introduce an operator, the \( \psi \)-Caputo operator of variable order with respect to another function.

Fractional differential operators in the Caputo or Riemann-Liouville sense, among others, to model certain natural phenomena, has been one of the most important tool to work with fractional calculus [3, 5]. The Laplace transform is widely used to solve such problems. In this note, we present a discussion about the Laplace transform of the Caputo fractional derivative of variable order with respect to another function, recently proposed by Yang and Tenreiro Machado [2]. By means of the variable order differential operator introduced as an inverse Laplace transform [1], we present some results, specifically, we evaluate the Laplace transform of the fractional derivative and make some comparisons.

2. Preliminaries

In this section we recover some results which are useful in the sequel of the note. First, we review the Laplace transform and the corresponding inverse Laplace transform; second we introduce the \( \varphi \)-Caputo fractional derivative of the function \( \psi (\sigma, t) \) of two-variable order
Let \( \psi(t) \) be a function of exponential order. The Laplace transform operator, denoted by \( \mathcal{L}(\cdot) \), acting on a function \( \psi(t) \), is defined by means of an improper integral with a non-singular kernel, as

\[
\mathcal{L}[\psi(t)] := \Psi(s) = \int_0^\infty e^{-st} \psi(t) \, dt, \quad \text{Re}(s) > 0
\]

with \( s \in \mathbb{C} \) is the parameter associated with the Laplace transform.

The corresponding inverse Laplace transform is obtained by means of the following expression

\[
\mathcal{L}^{-1}[\Psi(s)] := \psi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \Psi(s) \, ds
\]

with \( t > 0 \) and the contour, in the complex plane, contains all singularities on the left of the straightright line \( \text{Re}(s) = c > 0 \).

Suppose that \( T \) is an interval \( 0 \leq a < b < \infty \), \( \varphi \in C^1(T) \) with \( \varphi^{(1)}(t) \neq 0 \) and \( \psi \in C^1(T) \), for \( \forall t \in T \). The left and right fractional integral of the function \( \psi(\sigma, t) \) of two-variable order \( \xi(\sigma, t) \) \( (0 < \xi(\sigma, t) < 1) \), with respect to another function \( \varphi \) are given by

\[
I_{a+}^{\xi(\sigma,t)} \varphi \psi(\sigma, t) := \frac{1}{\Gamma(\xi(\sigma, t))} \int_a^t \varphi^{(1)}(\sigma, s) (\varphi(\sigma, t) - \varphi(\sigma, s))^{\xi(\sigma, t)-1} \psi(\sigma, s) \, ds
\]

and

\[
I_{b-}^{\xi(\sigma,t)} \varphi \psi(\sigma, t) := \frac{1}{\Gamma(\xi(\sigma, t))} \int_t^b \varphi^{(1)}(\sigma, s) (\varphi(\sigma, s) - \varphi(\sigma, t))^{\xi(\sigma, t)-1} \psi(\sigma, s) \, ds,
\]

respectively, where \( \Gamma(\cdot) \) is the gamma function.

The left and right \( \varphi \)-Caputo fractional derivatives of the function \( \psi(\sigma, t) \) with two-variable order \( \xi(\sigma, t) \) \( (0 < \xi(\sigma, t) < 1) \), with respect to another function \( \varphi \), are defined by

\[
C_{a+}^{\xi(\sigma,t)} \psi(\sigma, t) := \frac{1}{\Gamma(1 - \xi(\sigma, t))} \int_a^t (\varphi(\sigma, t) - \varphi(\sigma, s))^{-\xi(\sigma, t)} \psi^{(1)}(\sigma, s) \, ds
\]

and

\[
C_{b-}^{\xi(\sigma,t)} \psi(\sigma, t) := \frac{-1}{\Gamma(1 - \xi(\sigma, t))} \int_t^b (\varphi(\sigma, s) - \varphi(\sigma, t))^{-\xi(\sigma, t)} \psi^{(1)}(\sigma, s) \, ds,
\]

respectively, and \( \psi^{(1)}(\cdot, \cdot) \) denotes the first order derivative.

Introducing \( \xi(\sigma, t) = \xi \) and \( \varphi(\sigma, t) = t \) in Eq. \((2.3)\) and Eq. \((2.4)\), we have the left and right Caputo fractional derivatives

\[
C_{a+}^{\xi} \psi(t) := \frac{1}{\Gamma(1 - \xi)} \int_a^t (t - s)^{-\xi} \psi^{(1)}(s) \, ds
\]
and

\begin{equation}
C^\xi D_{b-\psi} e^\xi (t) := \frac{-1}{\Gamma (1 - \xi)} \int_t^b (s - t)^{-\xi} e^{(1)}(s) \, ds,
\end{equation}

respectively, with $0 < \xi < 1$ and $e^{(1)}(\cdot)$ denotes the first order derivative.

First, we consider $a = 0$ in Eq. (2.5) and taking the respective Laplace transform, we get

\begin{equation}
L[ C^\xi D_0^\xi \psi(t) ] = s^\xi \Psi(s) - \sum_{k=0}^{n-1} s^{n-1-k} \psi^{(k)}(0).
\end{equation}

Considering $n = 1$ in Eq. (2.7) we have $L[ C^\xi D_0^\xi (f(t)) ] = s^\xi \Psi(s) - \psi(0)$.

For continuity of work it is appropriate to consider the following function: $\psi(t) = \frac{t^{-1-\alpha}}{\Gamma(-\alpha)}$, where $t$ is the independent variable and $\alpha$, with $0 < \alpha < 1$, is a real constant. Taking its Laplace transform, we have

\begin{equation}
L[\psi(t)] = L \left[ \frac{t^{-1-\alpha}}{\Gamma(-\alpha)} \right] = \frac{1}{\Gamma(-\alpha)} L[t^{-1-\alpha}].
\end{equation}

3. Comments on the paper by Yang & Tenreiro Machado

In a recent paper [1] the author introduces a differential operator of variable order by means of the inverse Laplace transform of the function $s^q(t')$, defined by [2]

\begin{equation}
L^{-1}[s^q(t')] = \frac{t^{-1-q(t')}}{\Gamma(-q(t'))}.
\end{equation}

It is important to note that: the variable associated with the inverse Laplace transform is $s$, with $s \neq t'$ and $q(t')$ is the order of the derivative of variable-order. Then, taking the Laplace transform on both sides of Eq. (3.1) we get

\begin{equation}
s^q(t') = L \left[ \frac{t^{-1-q(t')}}{\Gamma(-q(t'))} \right] = \frac{1}{\Gamma(-q(t'))} L[t^{-1-q(t')}] \end{equation}

because $t$ is considered the variable of integration in the Laplace transform.

Choosing $q(t') = q$, a constant in Eq. (3.2), we have

\begin{equation}
s^q = \frac{1}{\Gamma(-\alpha)} L[t^{-1-q}],
\end{equation}

which is the same as in Eq. (2.8) for $q = \alpha$. 

On the other hand, as defined in [1] we conclude
\[
\mathcal{L} \left[ \frac{t^{-1-q(t')}}{\Gamma(-q(t'))} \right] = \frac{1}{\Gamma(-q(t'))} \mathcal{L}[t^{-1-q(t')} \neq \mathcal{L} \left[ \frac{t^{-1-q(t)}}{\Gamma(-q(t))} \right]
\]
because \( t \neq t' \).

Also, in this sense, in a recent paper [2], the authors write its Eq.(15) in the following form
\[
\mathcal{L} \left[ \frac{t^{-1-\xi(t)}}{\Gamma(-\xi(t))} \right] = s^{\xi(t)}.
\]
But, as we have just shown it should be written as follows
\[
\mathcal{L} \left[ \frac{t^{-1-\xi(t')}}{\Gamma(-\xi(t'))} \right] = s^{\xi(t')}
\]
or in the following form \( \frac{1}{\Gamma(-\xi(t'))} \mathcal{L}[t^{-1-\xi(t')} \neq s^{\xi(t')} \).

The authors go beyond, in the same paper [2], they consider \( t^{-1-\xi(\sigma, t')} \Gamma(-\xi(\sigma, t')) \) with \( \sigma \) a real constant, and conclude that (its Eq.(16))
\[
(3.3) \quad \frac{1}{\Gamma(-\xi(\sigma, t'))} \mathcal{L}[t^{-1-\xi(\sigma, t')} \neq s^{\xi(\sigma, t')}
\]
with \( t \neq \sigma \neq t' \).

So, if we choose \( \sigma = 0 \) in the last equation, we get its Eq.(15), with the identification \( \xi(0, t') = \xi(t') \) and, also, if \( \sigma = t' = 0 \) we recover Eq.(2.8) with the identification \( \xi(0, 0) = \alpha \).

Now, in the same paper [2], the authors consider a function \( \varphi(t) \), defined by
\[
(\varphi(t))^{-1-\xi(\sigma, t')} \Gamma(-\xi(\sigma, t'))
\]
whose Laplace transform was given as
\[
\mathcal{L} \left[ (\varphi(t))^{-1-\xi(\sigma, t')} \right] = [\varphi^{(1)}(t)] s^{\xi(\sigma, t')}
\]
which can be written in the following form
\[
(3.4) \quad \frac{1}{\Gamma(-\xi(\sigma, t'))} \mathcal{L}[\varphi(t) \neq [\varphi^{(1)}(t)] s^{\xi(\sigma, t')}
\]
For \( \varphi(t) = t \), we have \( \varphi^{(1)}(t) = 1 \). So that Eq.(3.4) becomes Eq.(3.3), that is
\[
(3.5) \quad [\varphi^{(1)}(t)] s^{\xi(\sigma, t')} = [1] s^{\xi(\sigma, t')} = s^{\xi(\sigma, t')}.
In this case, we have
\begin{equation}
\mathcal{L} \left[ \psi^{(1)}(\sigma, t) \right] = s \left\{ \psi(\sigma, s) - \left[ \varphi^{(1)}(t) \right] s^{-1} \psi(\sigma, 0) \right\}.
\end{equation}

As an example, will be check Eq.(22) of the paper [2]. Taking the Laplace transform on both sides of Eq.(2.3), we have
\begin{equation}
\mathcal{L} \left[ C_{a_+} D_{a_+}^{\xi(\sigma,t);\varphi} \psi(\sigma, t) \right] = \mathcal{L} \left[ \frac{s}{\Gamma(1-\xi(\sigma,t))} \int_{a}^{t} \left( \varphi(\sigma,t) - \varphi(\sigma,s) \right)^{-\xi(\sigma,t)} \psi^{(1)}(\sigma,s) ds \right] \\
= \mathcal{L} \left[ \frac{\varphi(\sigma, t)^{1-\xi(\sigma,t)}}{\Gamma(-\xi(\sigma,t))} \times \psi^{(1)}(\sigma, s) \right] \\
= \mathcal{L} \left[ \frac{\varphi(\sigma, t)^{1-\xi(\sigma,t)}}{\Gamma(-\xi(\sigma,t))} \right] \times \mathcal{L} \left[ \psi^{(1)}(\sigma, s) \right],
\end{equation}

where * denotes the convolution.

Note that, we can use Eq.(3.6), in part (II) of the product between (I) and (II). But, we can’t use Eq.(15) as the authors report in the paper [2], because as seen above, the definition of differential operator of variable order by means of the inverse Laplace transform, was used in a wrong way.

In this sense, we conclude that
\begin{equation}
\mathcal{L} \left[ C_{a_+} D_{a_+}^{\xi(\sigma,t);\varphi} \psi(\sigma, t) \right] \neq \left[ \varphi^{(1)}(t) \right] s^{\xi(\sigma,t)} \psi(s) - \left[ \varphi^{(1)}(t) \right] s^{\xi(\sigma,t)-1} \psi(0).
\end{equation}

4. Concluding remarks

Using the variable-order differential operator introduced by means of the inverse Laplace transform, we present a proof that the Laplace transform of \( \varphi\)-Caputo fractional derivative of the function \( \psi(\sigma,t) \) of two-variable order \( \xi(\sigma,t) \) \( (0 < \xi(\sigma,t) < 1) \) with respect to another function \( \varphi \), is not the identity Eq.(22) of the mentioned paper [2].

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