Exact recovery of Planted Cliques in Semi-random graphs

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Abstract

In this paper, we study the Planted Clique problem in a semi-random model. Our model is inspired from the Feige-Kilian model [FK01] which has been studied in many other works [Ste17, MMT20]. Our algorithm and analysis is on similar lines to the one studied for the Densest k-subgraph problem in the recent work of Khanna and Louis [KL20]. However since our algorithm fully recovers the planted clique w.h.p., we require some new ideas.

As a by-product of our main result, we give an alternate SDP based rounding algorithm (with matching guarantees) for solving the Planted Clique problem in a random graph. Also, we are able to solve special cases of the $DkSReg(n, k, d, \delta, \gamma)$ and $DkSExpReg(n, k, d, \delta, d', \lambda)$ models introduced in [KL20], when the planted subgraph $G[S]$ is a clique instead of an arbitrary $d$-regular graph.

Keywords: Planted cliques, Semi-random models, Approximation Algorithms, Semidefinite programming, Beyond worst-case analysis.

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1 Introduction

Given an undirected graph, the decision problem of checking whether it contains a $k$-clique, i.e., a subgraph of size $k$ which contains all the possible edges, is famously a NP-hard problem and appears in the list of twenty one NP-complete problems in the early work of Karp [Kar72]. This is a notoriously hard problem in the worst-case. The best known approximation algorithm by the work of Boppana and Halldórsson has an approximation factor of $O\left(\frac{n}{(\log n)^2}\right)$ [BH92]. The results by Hästad and Zuckerman shows that no polynomial time algorithm can approximate this to a factor better than $n^{1-\epsilon}$ for every $\epsilon > 0$, unless $P = NP$ [Hås97, Zuc06]. This was improved by Khot et al. [KP06], who showed that there is no algorithm which approximates the maximum clique problem (in the general case) to a factor better than $n/2(\log n)^{3/4+\epsilon}$ for any constant $\epsilon > 0$ assuming $NP \subseteq BPTIME\left(2(\log n)^{O(1)}\right)$.

These hardness results led to studying this problem in the average-case, i.e., we plant a clique of size $k$ in a Erdős-Rényi random graph ($G(n, p)$), and study the ranges of parameters of $k$ and $p$ for which this problem can be solved. We give a brief survey in Section 1.3.

Another direction is to consider the problem in a restricted family of graphs or “easier” instances. This allows us to design new and interesting algorithms with much better guarantees (as compared to the worst-case models) and might possibly help us get away from the adversarial examples which causes the problem to be hard in the first place. This way of studying hard problems falls under the area of “Beyond worst-case analysis”. We take this approach and in this work, we study the Planted Clique problem in a semi-random model. This is a model generated in multiple stages via a combination of adversarial and random steps. Such generative models have been studied in the early works of [FK01, SB02, CO07]. We refer the reader to [KL20] and the references therein for a survey of variety of graph problems which have been subjected to such a study.

1.1 Model

In this section, we describe our semi-random model.

**Definition 1.1.** An instance of our input graph $G = (V, E) \sim \text{Clique}(n, k, p, q, r, s, t, d, \gamma, \delta, \lambda)$ is generated as follows,

1. (Partition step) We partition the vertex set $V (|V| = n)$ into two sets, $S$ and $V \setminus S$ with $|S| = k$. We further partition $V \setminus S$ into sets $\Delta, \Lambda, \Pi$ such that
   - $\Delta$ is arbitrarily partitioned into sets $\Delta_1, \Delta_2, \ldots, \Delta_q$ such that for all $\ell \in [q]$, $|\Delta_\ell| \geq 0$.
   - $\Lambda$ is arbitrarily partitioned into sets $\Lambda_1, \Lambda_2, \ldots, \Lambda_r$ such that for all $\ell \in [r]$, $|\Lambda_\ell| = s$.
   - $\Pi$ is arbitrarily partitioned into sets $\Pi_1, \Pi_2, \ldots, \Pi_t$ such that for all $\ell \in [t]$, $|\Pi_\ell| \geq 0$.

2. (Adding random edges) We add edges between the following sets of pairs
   - $S \times (V \setminus S)$.
   - $\Delta_i \times \Delta_j$ for $i, j \in [q], i \neq j$.
   - $\Lambda_i \times \Lambda_j$ for $i, j \in [r], i \neq j$.
   - $\Pi_i \times \Pi_j$ for $i, j \in [t], i \neq j$.
   - $\Delta_i \times \Lambda_j$ for $i \in [q], j \in [r]$.
   - $\Lambda_i \times \Pi_j$ for $i \in [r], j \in [t]$.
   - $\Pi_i \times \Delta_j$ for $i \in [t], j \in [q]$. 

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independently with probability $p$ and of weight 1.

3. **(Adding a clique on $S$)** We add edges between pairs of vertices in $S$ such that the graph induced on $S$ is a clique. For the sake of brevity, we also add a self loop on each of the vertices of $V$, this will make the arithmetic cleaner (like the average degree of $S$ is now $k$ instead of $k - 1$) and has no severe consequences. Note that we assume the subgraph $G[S]$ is unweighted.

4. **(Adding edges in $\Delta_i$’s)** For each $i \in [q]$, we add edges of arbitrary non-negative weights between arbitrary pairs of vertices in $\Delta_i$, such that the graph induced on $\Delta_i$ is a bounded degree graph such that the degree of a vertex is at most $\delta k$.

5. **(Adding edges in $\Lambda_i$’s)** For each $i \in [r]$, we add edges of arbitrary non-negative weights between arbitrary pairs of vertices in $\Lambda_i$, such that the graph induced on $\Lambda_i$ is a $(s, d, \lambda)$-expander graph. See Definition 1.2 for more details.

6. **(Adding edges in $\Pi_i$’s)** For each $i \in [t]$, we add edges of arbitrary non-negative weights between arbitrary pairs of vertices in $\Pi_i$, such that the graph induced on $\Pi_i$ has the following property,

$$\max_{V' \subseteq \Pi_i} \left\{ \frac{\sum_{i,j \in V'} w(\{i, j\})}{2 |V'|} \right\} \leq \gamma k.$$ 

Or in other words, for each $i \in [t]$, the maximum average degree of the subgraph $G[V']$ is at most $\gamma k$ for some $\gamma \in (0, 1)$ and $V' \subseteq \Pi_i$. Here for an edge $e \in E$, $w(e)$ denotes the weight of edge $e$.

7. **(Monotone adversary step)** Arbitrarily delete any of the edges added in step 2, step 4, step 5, and step 6.

8. Output the resulting graph.

Also see the below figure (Figure 1.1) for a pictorial representation of the model.

Figure 1: Clique($n, k, p, q, r, s, t, d, \gamma, \delta, \lambda$). A monotone adversary may remove any edges outside the subgraph $G[S]$. 

Also see the below figure (Figure 1.1) for a pictorial representation of the model.
**Definition 1.2** (Restatement of Definition 1.10 from [KL20]). A graph \( H = (V, E, w) \) is said to be a \((s, d, \lambda)\)-expander if \( |V| = s \), \( H \) is \( d \)-regular and \( |\lambda| \leq \lambda \), \( \forall i \in [s] \setminus \{1\} \), where \( \lambda_1 \geq \lambda_2 \ldots \geq \lambda_s \) are the eigenvalues of the weighted adjacency matrix of \( H \).

In this paper, the problem which we study is as follows: Given a graph generated from the above described model (Definition 1.1), the goal is to recover the planted clique \( G[S] \) with high probability. We show that for a “large” range of the input parameters, we can indeed solve this problem.

Our algorithm is based on rounding a standard semidefinite programming relaxation of Densest \( k \)-subgraph problem but we also add the following set of constraints to it,

\[
\langle X_i, X_j \rangle = 0 \quad \forall \ (i, j) \notin E
\]

For the sake of completeness, we rewrite the complete SDP in Appendix A. This is a key difference as compared to the Densest \( k \)-subgraph problem and we will use the above set of constraints crucially in our analysis, much of which is inspired from [KL20]. We will describe this in more detail in Section 2 and Section 3.

### 1.2 Main Result

In this section, we describe our main results and its interpretation.

**Theorem 1.3.** There exist universal constants \( \kappa, \xi \in \mathbb{R}^+ \) and a deterministic polynomial time algorithm, which takes an instance of \( \text{Clique}(n, k, p, q, r, s, t, d, \gamma, \delta, \lambda) \) where

\[
\nu = \frac{36\xi^2(np)(q + r + t + 1)}{k^2 \left(1 - 9p - \delta - 2\gamma - \frac{d'}{s} - \frac{\lambda}{k}\right)^2},
\]

satisfying \( \nu \in (0, 1) \), and \( p \in \left[\kappa \log n/n, 1\right) \), and recovers the planted clique \( S \) with high probability (over the randomness of the input).

It is important to note that our results do not depend on the size of the subgraphs \( G[\Delta_\ell]'s \) and \( G[\Pi_\ell]'s \) but only on their counts, i.e, parameters \( q \) and \( t \). Even our model is not parameterized by the sizes of \( \Delta_\ell's \) and \( \Pi_\ell's \). In other words, all the \( \Pi_\ell's \) can be of different sizes but as long as the average degree requirement of subgraphs \( G[\Pi_\ell]'s \) (the one stated in step 6) is met, our results hold. A similar statement holds for the subgraphs \( G[\Delta_\ell]'s \).

We see some interesting observations from the above theorem (Theorem 1.3). Firstly, there are a few conditions for the algorithm to work,

1) \( p = \Omega\left(\frac{\log n}{n}\right)^3 \), or to be verbose, \( p \) should be “large enough”.

2) The function \( \nu \) (which is dependent on the input parameters) should lie in the range \((0, 1)\), or stated in other words, \( \nu \) should be “small”.

A setting of input parameters when the value of \( \nu \) is “small” is as follows:

\[
k = \Omega\left(\sqrt{np(q + r + t + 1)}\right), \quad \delta, \gamma = \mathcal{O}(1), \quad s = \Omega(d'), \quad \text{and} \quad k = \Omega(\lambda)\).
\]

The above values of different input parameters suggest that the algorithm will work only when any subgraph of size \( k \) will be far from dense (or a clique) inside \( G[V \setminus S] \). Also since the subgraph \( G[S \times V \setminus S] \) is a

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1This condition is more of a technical requirement than an interesting setting.
when \( k \) Alon, Krivelevich, and Sudakov [AKS01] give a spectral algorithm to find the clique when which find a clique of size \( \log n \). This is also a nearly linear time algorithm which succeeds w.h.p. when \( k = \Omega(\sqrt{n}) \). Thus in this case, our problem reduces to recovering the planted clique in a random graph and we get a similar threshold value of \( k \) to the one already studied in literature [AKS01, BHK+16].

- Recall the model, DkSReg\((n, k, d, \delta, \gamma)\) introduced in the work of [KL20]. In this model, the subgraph \( G[S] \) is an arbitrary \( d \)-regular graph of size \( k \), \( G[S \times V \setminus S] \) is a random graph with parameter \( p \), and the subgraph \( G[V \setminus S] \), has the following property,

\[
\text{max}_{V' \subseteq V \setminus S} \left\{ \frac{\sum_{i,j \in V'} w((i,j))}{2|V'|} \right\} \leq \gamma d.
\]

Clearly, this is analogous to the case when we have only one such \( \Pi_1 \) comprising the whole of \( G[V \setminus S] \) such that the maximum average degree of any subgraph of \( G[\Pi_1] \) is at most \( \gamma k \). Now when \( q = r = 0 \) and \( t = 1 \), our model reduces to the case when DkSReg\((n, k, d, \delta, \gamma)\) has a clique on \( S \) (instead of a \( d \)-regular subgraph). Note that this case can be solved using our algorithm efficiently and we can recover the planted clique i.e. \( S \) w.h.p. This is a much stronger guarantee as compared to the one in [KL20] where they output a vertex set with a large intersection with the planted set (but not completely), with the same threshold on \( k \), i.e., \( k = \Omega(np) \).

- Similarly, in the model, DkSExpReg\((n, k, d, \delta, d', \lambda)\) introduced in the work of [KL20]. In this model, the subgraph \( G[S] \) is an arbitrary \( d \)-regular graph of size \( k \), \( G[S \times V \setminus S] \) is a random graph with parameter \( p \), and the subgraph \( G[V \setminus S] \), is a \( (n - k, d', \lambda) \)-expander graph. This is analogous to the case when we have only one such \( \Lambda_1 \) comprising the whole of \( G[V \setminus S] \). Now when \( q = t = 0 \) and \( r = 1 \), our model reduces to the case when DkSExpReg\((n, k, d, \delta, d', \lambda)\) has a clique on \( S \). And similar to the previous point, this case can also be solved using our algorithm efficiently and we can recover the planted clique i.e. \( S \) w.h.p.

### 1.3 Related Work

**Random models for the clique problem.** For the Erdős-Rényi random graph: \( G(n, 1/2) \), it is known that the largest clique has a size approximately \( 2 \log_2 n \) [Mat76]. There are several poly-time algorithms which find a clique of size \( \log_2 n \), i.e., with an approximation factor roughly \( 1/2 \) [GM75]. It is a long standing open problem to give an algorithm which finds a clique of size \( (1 + \varepsilon) \log_2 n \) for any fixed \( \varepsilon > 0 \). This conjecture has a few interesting cryptographic consequences as well [JP00].

**Planted models for the clique problem.** In the planted clique problem, we plant a clique of size \( k \) in \( G(n, 1/2) \) and study the ranges of \( k \) for which this problem can be solved. The work by Kucera [Kuc95] shows that if \( k = \Omega(\sqrt{n \log n}) \), then the planted clique essentially comprises of the vertices of the largest degree. Alon, Krivelevich, and Sudakov [AKS01] give a spectral algorithm to find the clique when \( k = \Omega(\sqrt{n}) \). There is also a nearly linear time algorithm which succeeds w.h.p. when \( k \geq (1 + \varepsilon) \sqrt{n / \varepsilon} \) for any \( \varepsilon > 0 \) [DM15]. When \( k = o(\sqrt{n}) \), the work by Barak et al. rules out the possibility for a sum of squares algorithm to work [BHK+16].
Semi-random models for related problems. The semi-random model studied in this paper is inspired from a combination of two works. First is the Feige-Kilian model [FK01], this is a very generic model. In this model, we plant an independent set on \( S (|S| = k) \), the subgraph \( G[S \times V \setminus S] \) is a random graph with parameter \( p \), while the subgraph \( G[V \setminus S] \) can be an arbitrary graph. Then an adversary is allowed to add edges anywhere without disturbing the planted independent set. McKenzie, Mehta, and Trevisan [MMT20] show that for \( k = \Omega \left( \frac{n^{2/3}}{p^{1/3}} \right) \), their algorithm finds a “large” independent set. And for the range \( k = \Omega \left( \frac{n^{2/3}}{p^{1/3}} \right) \), their algorithm outputs a list of independent sets, one of which is \( S \) with high probability. Restrictions of this model has also been studied in the works of [Ste17, CSV17].

It is important to note that the above model is a pretty generic model and also solves the semi-random model which we study in our paper, however there are some key differences. Firstly, ours is an exact deterministic algorithm based on the SDP relaxation of the \( k \)-clique while they use a “crude” SDP (this idea was introduced in [KMM11]) which is not a relaxation of the independent set (or the complementary clique problem). But both the SDPs “clusters” the vectors corresponding to the planted set. Secondly the algorithmic guarantee of the work by [MMT20] is of a different nature where they output a list of independent sets one of which is the planted set, as described above.

The second relevant model is studied by Khanna and Louis [KL20] for the Densest \( k \)-subgraph problem. They plant an arbitrary dense subgraph on \( G[S] \), the subgraph \( G[S \times V \setminus S] \) is a random subgraph, and the subgraph \( G[V \setminus S] \) has a property (step 3 of model construction) like the one of \( \Lambda_1 \)'s or \( \Pi_1 \)'s of this paper. A monotone adversary can delete edges outside \( G[S] \). Our algorithm, model, and the analysis is inspired from their work. We study the problem in the case when \( G[S] \) is a clique on \( k \) vertices instead of an arbitrary \( d \)-regular graph. We get a full recovery of the clique in this paper instead of a “large” recovery of the planted set, for a “wide” range of input parameters. A key result by Charikar [Cha00] is used to prove our bounds.

The idea of using SDP based algorithms for solving semi-random models of instances has been explored in multiple works for a variety of graph problems, some of which are [BCC+10, KMM11, MMV12, MMV14, MNS16, HWX16, LV18, LV19, MMT20, KL20].

1.4 Proof Idea

Our algorithm is based on rounding a SDP solution. The basic idea is to show that the vectors corresponding to the planted set \( S \) are “clustered” together. This is shown by bounding the contribution of the vectors towards the SDP mass from the rest of the graph (i.e. everything except \( G[S] \)). This allows us to exploit the geometry of vectors to recover a part of the planted clique. This is possible only because the subgraph \( G[V \setminus S] \), which is a combination of expanders, low-degree graphs etc. and thus is a sparse graph by construction (step 2, step 4, step 5, step 6) and the random bipartite subgraph \( G[S \times V \setminus S] \) (step 2) will not have any dense sets either. Thus qualitatively the SDP should put most of the mass on the edges of \( S \). We show for a “large” range of input parameters, this indeed happens.

The rest of the vertices can be recovered using a greedy algorithm. Note that for this to work, we crucially use the orthogonality constraints added for each non-edge pair (equation (1)) and this additional recovery step works only because the planted set is a clique and not an arbitrary dense subgraph.

Remark 1.4. Our algorithm (Algorithm 1) is based on rounding a SDP (SDP A.1) and is robust against a monotone adversary (step 7 of the model construction). This is an important point because many of the algorithms based on spectral or combinatorial methods are not always robust and do not work with the presence of such adversaries.
2 Analysis

In this section, we bound the SDP mass corresponding to different subgraphs. The idea is to show that the SDP (SDP A.1) puts a large fraction of its total mass on \( G[S] \). Most of the bounds closely follows that from [KL20]. A monotonicity argument (similar to [KL20]) can be used to ignore the action of the adversary. We restate it for the sake of completeness in Remark 2.1.

Remark 2.1 (Restating from [KL20]). In the analysis below, without loss of generality we can ignore the adversarial action (step 7 of the model construction) to have taken place. Let us assume the monotone adversary removes edges arbitrarily from the subgraphs \( G[V \setminus S] \) & \( G[S', V \setminus S] \) and the new resulting adjacency matrix is \( A' \). Then for any feasible solution \( \{Y_{ij}\}_{ij=1}^r \) of the SDP, we have \( \sum_{i \in P, j \in Q} A'_{ij} \langle Y_i, Y_j \rangle \leq \sum_{i \in P, j \in Q} A_{ij} \langle Y_i, Y_j \rangle \) for \( \forall P, Q \subseteq V \). This holds because of the non-negativity constraint (10). Thus the upper bounds on SDP contribution by vectors in \( G[S, V \setminus S] \) and \( G[V \setminus S] \) as claimed by the different claims below are intact and the rest of the proof follows exactly. Hence, without loss of generality, we can ignore this step in the analysis of our algorithm.

We decompose the SDP objective into multiple parts (corresponding to different subgraphs) and bound each of them separately (Section 2.1) and then combine these bounds in the end (Section 2.2). We also highlight the lemma where we bound the corresponding sum.

\[
\sum_{i,j \in V} A_{ij} \langle X_i, X_j \rangle = \sum_{i,j \in S} A_{ij} \langle X_i, X_j \rangle + 2 \sum_{i \in S, j \notin V \setminus S} A_{ij} \langle X_i, X_j \rangle + \sum_{\ell=1}^q \sum_{i \in P, j \in Q} A_{ij} \langle Y_i, Y_j \rangle + \sum_{\ell=1}^q \sum_{i \in S, j \notin V \setminus S} A_{ij} \langle X_i, X_j \rangle + \sum_{\ell=1}^q \sum_{i \in S, j \notin V \setminus S} A_{ij} \langle X_i, X_j \rangle \\
\text{Lemma 2.4} + \sum_{\ell=1}^r \sum_{i \in \Lambda_2} A_{ij} \langle X_i, X_j \rangle + \sum_{\ell=1}^r \sum_{i \in \Lambda_2} A_{ij} \langle X_i, X_j \rangle + \sum_{\ell=1}^r \sum_{i \in \Pi, j \notin (V \setminus S) \setminus P} A_{ij} \langle X_i, X_j \rangle \\
\text{Lemma 2.11} + \sum_{\ell=1}^r \sum_{i \in \Pi, j \notin (V \setminus S) \setminus P} A_{ij} \langle X_i, X_j \rangle \\
\text{Lemma 2.13} + \sum_{\ell=1}^r \sum_{i \in \Pi, j \notin (V \setminus S) \setminus P} A_{ij} \langle X_i, X_j \rangle \\
\text{Lemma 2.15}
\]

Note that the highlighted terms in equation (2) corresponds to the subgraphs \( G[S] \), \( G[\Lambda_1] \forall i_1 \in [q], \) \( G[\Lambda_2] \forall i_2 \in [r], \) and \( G[\Pi_i] \forall i_3 \in [i] \) respectively while the unhighlighted terms (i.e. the contribution from the random subgraph) can be further split as follows. This is also called the “centering” trick.

\[
2 \sum_{i \in S, j \notin V \setminus S} A_{ij} \langle X_i, X_j \rangle = 2p \sum_{i \in S, j \notin V \setminus S} \langle X_i, X_j \rangle + 2 \sum_{i \in S, j \notin V \setminus S} (A_{ij} - p) \langle X_i, X_j \rangle \\
\text{Lemma 2.5}
\]

\[
\sum_{\ell=1}^q \sum_{i \in \Lambda_2} A_{ij} \langle X_i, X_j \rangle = p \sum_{\ell=1}^q \sum_{i \in \Lambda_2} \langle X_i, X_j \rangle + \sum_{\ell=1}^q \sum_{i \in \Lambda_2} (A_{ij} - p) \langle X_i, X_j \rangle \\
\text{Lemma 2.6 Part(a)}
\]

\[
\sum_{\ell=1}^r \sum_{i \in \Pi, j \notin (V \setminus S) \setminus P} A_{ij} \langle X_i, X_j \rangle = p \sum_{\ell=1}^r \sum_{i \in \Pi, j \notin (V \setminus S) \setminus P} \langle X_i, X_j \rangle + \sum_{\ell=1}^r \sum_{i \in \Pi, j \notin (V \setminus S) \setminus P} (A_{ij} - p) \langle X_i, X_j \rangle \\
\text{Lemma 2.6 Part(b)}
\]
\[ \sum_{i} \sum_{j \in (\mathcal{V} \setminus \mathcal{S}) \setminus \Pi_{\ell}} A_{ij} \left< X_{i}, X_{j} \right> = p \sum_{i} \sum_{j \in (\mathcal{V} \setminus \mathcal{S}) \setminus \Pi_{\ell}} \left< X_{i}, X_{j} \right> + \sum_{i} \sum_{j \in (\mathcal{V} \setminus \mathcal{S}) \setminus \Pi_{\ell}} (A_{ij} - p) \left< X_{i}, X_{j} \right> \] (6)

Lemma 2.6 Part (c)

Note that there are two kinds of terms in equations (3), (4), (5), and (6), one, which only depends on the SDP constraints, and the second, which uses the adjacency matrix of \( \mathcal{G} \). Before proceeding, we introduce a new matrix for convenience.

Definition 2.2. Let \( B \) be a \( n \times n \) sized centered matrix (i.e. \( \mathbb{E}[B] = 0 \)) defined as follows. Here \( A \) denotes the adjacency matrix of the input graph (before the action of monotone adversary).

\[
B_{ij} \overset{\text{def}}{=} \begin{cases} 0 & \text{if } i, j \in \mathcal{S} \text{ or } i, j \in \Delta_{1} \text{ or } \ldots \text{ or } i, j \in \Delta_{q} \text{ or } i, j \in \Lambda_{1} \text{ or } \ldots \text{ or } i, j \in \Lambda_{r} \text{ or } i, j \in \Pi_{1} \text{ or } \ldots \text{ or } i, j \in \Pi_{t}. \\ A_{ij} - p & \text{otherwise} \end{cases}
\]

This definition (Definition 2.2) allows us to rewrite the centered terms as follows.

\[
2 \sum_{i \in \mathcal{S}, j \in \mathcal{V} \setminus \mathcal{S}} (A_{ij} - p) \left< X_{i}, X_{j} \right> + \sum_{\ell=1}^{q} \sum_{i \in \Delta_{\ell}, j \in (\mathcal{V} \setminus \mathcal{S}) \setminus \Delta_{\ell}} (A_{ij} - p) \left< X_{i}, X_{j} \right> + \sum_{\ell=1}^{r} \sum_{i \in \Lambda_{\ell}, j \in (\mathcal{V} \setminus \mathcal{S}) \setminus \Lambda_{\ell}} (A_{ij} - p) \left< X_{i}, X_{j} \right>
\]

\[
+ \sum_{i} \sum_{j \in (\mathcal{V} \setminus \mathcal{S}) \setminus \Pi_{\ell}} (A_{ij} - p) \left< X_{i}, X_{j} \right> = \sum_{i, j \in \mathcal{V} \setminus \mathcal{S}} B_{ij} \left< X_{i}, X_{j} \right> .
\] (7)

Lemma 2.9

2.1 Bounding the SDP terms

In this section, we show an upper bound on the various terms of the SDP objective (discussed as above). First, we introduce some notation.

Remark 2.3 (Restatement of Notation from [KL20]). We define probability distributions \( \mu \) over finite sets \( \Omega \). For a random variable (r.v.) \( X : \Omega \rightarrow \mathbb{R} \), its expectation is denoted by \( \mathbb{E}_{\mu}[X] \). In particular, we define the distribution which we use below. For a vertex set \( \mathcal{V}' \subset \mathcal{V} \), we define a probability (uniform) distribution \( (f_{\mathcal{V}'}) \) on the vertex set \( \mathcal{V}' \) as follows. For a vertex \( i \in \mathcal{V}' \), \( f_{\mathcal{V}'}(i) = \frac{1}{|\mathcal{V}'|} \). We use \( i \sim \mathcal{V}' \) to denote \( i \sim f_{\mathcal{V}'} \) for clarity.

Lemma 2.4 (Restatement of Lemma 3.2 from [KL20] with the value \( d = k \)).

\[
\sum_{i, j \in \mathcal{S}} A_{ij} \left< X_{i}, X_{j} \right> \leq k^{2} \left( \mathbb{E}_{i \sim \mathcal{S}} \|X_{i}\|^{2} \right).
\]

Lemma 2.5 (Restatement of Lemma 2.5 from [KL20]).

\[
\sum_{i \in \mathcal{S}, j \in \mathcal{V} \setminus \mathcal{S}} \left< X_{i}, X_{j} \right> \leq 3k^{2} \left( 1 - \mathbb{E}_{i \sim \mathcal{S}} \|X_{i}\|^{2} \right).
\]

Lemma 2.6. (a)

\[
\sum_{\ell=1}^{q} \sum_{i \in \Delta_{\ell}, j \in (\mathcal{V} \setminus \mathcal{S}) \setminus \Delta_{\ell}} \left< X_{i}, X_{j} \right> \leq k^{2} \left( 1 - \mathbb{E}_{i \sim \mathcal{S}} \|X_{i}\|^{2} \right).
\]

(b)

\[
\sum_{\ell=1}^{r} \sum_{i \in \Lambda_{\ell}, j \in (\mathcal{V} \setminus \mathcal{S}) \setminus \Lambda_{\ell}} \left< X_{i}, X_{j} \right> \leq k^{2} \left( 1 - \mathbb{E}_{i \sim \mathcal{S}} \|X_{i}\|^{2} \right).
\]
Proof of Part (a). Note that for all $\ell \in [q]$,

$$\sum_{i \in \Lambda_{t}, j \in (V \setminus S) \setminus \Lambda_{t}} \langle X_{i}, X_{j} \rangle \leq k \sum_{i \in \Lambda_{t}} \langle X_{i}, X_{i} \rangle \leq k \sum_{i \in \Lambda_{t}} \langle X_{i}, X_{i} \rangle \leq k^2 \left( 1 - \mathbb{E}_{i \sim S} \|X_i\|^2 \right)$$

The first inequality just follows from the SDP constraint (12) (non-negativity) and the second one follows from the constraint (10). Summing up for all $\ell \in [q]$,

$$\sum_{\ell = 1}^{q} \sum_{i \in \Lambda_{t}, j \in (V \setminus S) \setminus \Lambda_{t}} \langle X_{i}, X_{j} \rangle \leq k \sum_{\ell = 1}^{q} \sum_{i \in \Lambda_{t}} \langle X_{i}, X_{i} \rangle \leq k \sum_{\ell = 1}^{q} \sum_{i \in V \setminus S} \langle X_{i}, X_{i} \rangle \leq k^2 \left( 1 - \mathbb{E}_{i \sim S} \|X_i\|^2 \right) \quad \text{(By SDP constraint (9))}$$

The proofs of Parts (b) and (c) follow similarly.

Lemma 2.7 (Restatement of Lemma 2.6 from [KL20]).

$$\sum_{i \in S, j \in (V \setminus S) \setminus \Delta_{1}} B_{ij} \langle X_{i}, X_{j} \rangle \leq \|B\| \sqrt{\sum_{t \in S} \|X_t\|^2} \sqrt{\sum_{t \in (V \setminus S) \setminus \Delta_{1}} \|X_t\|^2}.$$  

Lemma 2.8 (Restatement of Corollary 2.8 from [KL20]). There exists universal constants $\kappa, \xi \in \mathbb{R}^+$ such that if $p \in \left[ \frac{\kappa \log n}{n}, 1 \right)$, then

$$\|B\| \leq \xi \sqrt{n p}$$

with high probability (over the randomness of the input).\(^2\)

Lemma 2.9. With high probability (over the randomness of the input),

$$\sum_{i, j \in V} B_{ij} \langle X_{i}, X_{j} \rangle \leq 2\xi k \sqrt{np} \sqrt{(q + r + t + 1)} \left( 1 - \mathbb{E}_{i \sim S} \|X_i\|^2 \right)$$

if $p \in \left[ \frac{\kappa \log n}{n}, 1 \right)$, where $\kappa, \xi \in \mathbb{R}^+$ are universal constants.

Proof. A similar calculation to the one done in Lemma 2.7, we can easily show that, $\forall i_1 \in [q]$,

$$\sum_{i \in \Lambda_{i_1}, j \in (V \setminus S) \setminus \Lambda_{i_1}} B_{ij} \langle X_{i}, X_{j} \rangle \leq \|B\| \sqrt{\sum_{i \in \Lambda_{i_1}} \|X_t\|^2} \sqrt{\sum_{t \in (V \setminus S) \setminus \Lambda_{i_1}} \|X_t\|^2} \leq \|B\| \sum_{i \in \Lambda_{i_1}} \|X_t\|^2 \sqrt{\sum_{t \in (V \setminus S) \setminus \Lambda_{i_1}} \|X_t\|^2}.$$  

$\forall i_2 \in [r]$,

$$\sum_{i \in \Lambda_{i_2}, j \in (V \setminus S) \setminus \Lambda_{i_2}} B_{ij} \langle X_{i}, X_{j} \rangle \leq \|B\| \sqrt{\sum_{i \in \Lambda_{i_2}} \|X_t\|^2} \sqrt{\sum_{t \in (V \setminus S) \setminus \Lambda_{i_2}} \|X_t\|^2} \leq \|B\| \sum_{i \in \Lambda_{i_2}} \|X_t\|^2 \sqrt{\sum_{t \in (V \setminus S) \setminus \Lambda_{i_2}} \|X_t\|^2}.$$

\(^2\)Note that the matrix $B$ is defined differently in the two papers however this is not a critical issue and the spectral norm bound still holds.
∀i_3 ∈ [r],
\[ \sum_{i \in \Pi_{i_3}, j \in (V \setminus S) \setminus \Pi_{i_3}} B_{ij} \langle X_i, X_j \rangle \leq \|B\| \left( \sum_{i \in S} \|X_i\|^2 \right)^{1/2} \left( \sum_{i = 1}^q \sum_{i \in A_1} \|X_i\|^2 \right)^{1/2} \left( \sum_{i = 1}^r \sum_{i \in A_2} \|X_i\|^2 \right)^{1/2} \left( \sum_{i = 1}^t \sum_{i \in A_3} \|X_i\|^2 \right)^{1/2} \left( \sum_{i \in V \setminus S} \|X_i\|^2 \right)^{1/2}. \]

Summing up for \( S, \Delta_{i_3}', \Lambda_{i_3}', \), and \( \Pi_{i_3}' \), we get,
\[ \sum_{i,j \in V} B_{ij} \langle X_i, X_j \rangle \leq 2 \|B\| \left( \sum_{i = 1}^q \sum_{i \in \Delta_1} \|X_i\|^2 \right)^{1/2} \left( \sum_{i = 1}^r \sum_{i \in \Delta_2} \|X_i\|^2 \right)^{1/2} \left( \sum_{i = 1}^t \sum_{i \in \Pi_{i_3}} \|X_i\|^2 \right)^{1/2} \left( \sum_{i \in V \setminus S} \|X_i\|^2 \right)^{1/2} \leq 2 \|B\| \left( q + r + t + 1 \right) \left( \sum_{i = 1}^q \sum_{i \in \Delta_1} \|X_i\|^2 \right)^{1/2} \left( \sum_{i = 1}^r \sum_{i \in \Delta_2} \|X_i\|^2 \right)^{1/2} \left( \sum_{i = 1}^t \sum_{i \in \Pi_{i_3}} \|X_i\|^2 \right)^{1/2} \left( \sum_{i \in V \setminus S} \|X_i\|^2 \right)^{1/2} \leq 2 \|B\| \left( q + r + t + 1 \right) k \sqrt{k \left( 1 - \mathbb{E}_{i \notin S} \|X_i\|^2 \right)} \text{ (by SDP constraint (9))} \]

Lemma 2.10 (Restatement of Proposition 3.2 from [KL20] with the value \( d \leq \delta k \)). For all \( \ell \in [q] \),
\[ \sum_{i,j \in \Delta_{i_\ell}} A_{ij} \langle X_i, X_j \rangle \leq \delta k \sum_{i \in \Delta_{i_\ell}} \|X_i\|^2. \]

Lemma 2.11.
\[ \sum_{\ell = 1}^q \sum_{i \in \Delta_{i_\ell}} A_{ij} \langle X_i, X_j \rangle \leq \delta k \left( 1 - \mathbb{E}_{i \notin S} \|X_i\|^2 \right). \]

Proof. Summing up for all \( \ell \in [q] \) and using Lemma 2.10 for each sum, we get,
\[ \sum_{\ell = 1}^q \sum_{i \in \Delta_{i_\ell}} A_{ij} \langle X_i, X_j \rangle \leq \delta k \sum_{\ell = 1}^q \sum_{i \in \Delta_{i_\ell}} \|X_i\|^2 \leq \delta k \sum_{i \in V \setminus S} \|X_i\|^2 = \delta k \left( 1 - \mathbb{E}_{i \notin S} \|X_i\|^2 \right) \text{ (by SDP constraint (9))}. \]

Lemma 2.12 (Restatement of Proposition 2.13 from [KL20] with the value \( n = s \)). For a \((s, d', \lambda)\)-expander graph, the value of the SDP (SDP A.1) is at most \( k^2 \frac{d'}{s} + k \lambda \).

Lemma 2.13.
\[ \sum_{\ell = 1}^r \sum_{i \in \Delta_{i_\ell}} A_{ij} \langle X_i, X_j \rangle \leq k^2 \left( \frac{d'}{s} + \frac{\lambda}{k} \right) \left( 1 - \mathbb{E}_{i \notin S} \|X_i\|^2 \right). \]

Proof. Summing up for all \( \ell \in [r] \) and using Lemma 2.12 for each sum and a scaling factor of \( \sum_{i \in \Delta_{i_\ell}} \|X_i\|^2 / k \), we get,
\[ \sum_{\ell = 1}^r \sum_{i \in \Delta_{i_\ell}} A_{ij} \langle X_i, X_j \rangle \leq k^2 \left( \frac{d'}{s} + \frac{\lambda}{k} \right) \left( \sum_{i \in \Delta_{i_\ell}} \|X_i\|^2 \right) \leq k^2 \left( \frac{d'}{s} + \frac{\lambda}{k} \right) \left( 1 - \mathbb{E}_{i \notin S} \|X_i\|^2 \right) \]
where we used SDP constraint (9) in the second inequality.
Lemma 2.14 (Restatement of Proposition 3.12 from [KL20] with the value \(d = k\)). For all \(\ell \in [t]\),

\[
\sum_{i,j \in \Pi_\ell} A_{ij} \langle X_i, X_j \rangle \leq 2\gamma k \sum_{i \in \Pi_\ell} \|X_i\|^2.
\]

Lemma 2.15.

\[
\sum_{\ell=1}^{t} \sum_{i,j \in \Pi_\ell} A_{ij} \langle X_i, X_j \rangle \leq 2\gamma k^2 \left(1 - \mathbb{E}_{i \sim S} \|X_i\|^2\right).
\]

Proof. Summing up for all \(\ell \in [t]\) and using Lemma 2.14 for each sum, we get,

\[
\sum_{\ell=1}^{t} \sum_{i,j \in \Pi_\ell} A_{ij} \langle X_i, X_j \rangle \leq 2\gamma k \sum_{i \in \mathcal{V} \setminus S} \|X_i\|^2 \leq 2\gamma k^2 \left(1 - \mathbb{E}_{i \sim S} \|X_i\|^2\right) \quad \text{(by SDP constraint (9))}.
\]

\[\square\]

2.2 Putting things together

In this section we combine the above bounds.

Proposition 2.16. With high probability (over the randomness of the input),

\[
\mathbb{E}_{i \sim S} \|X_i\|^2 \geq 1 - \frac{4\xi^2 (np)(q + r + t + 1)}{k^2 \left(1 - 9p - \delta - 2\gamma - \frac{d'}{s} - \frac{\lambda}{k}\right)^2}
\]

if \(p \in \left[\frac{\kappa \log n}{n}, 1\right)\), where \(\kappa, \xi \in \mathbb{R}^+\) are a universal constants.

Proof. Since our SDP is a maximization relaxation (See Appendix A), we have that,

\[
k^2 \leq \sum_{i,j \in \mathcal{V}} A_{ij} \langle X_i, X_j \rangle
\]

\[
k^2 \leq k^2 \left(\mathbb{E}_{i \sim S} \|X_i\|^2\right) + 9pk^2 \left(1 - \mathbb{E}_{i \sim S} \|X_i\|^2\right) + 2\xi k \sqrt{np} \sqrt{(q + r + t + 1) \left(1 - \mathbb{E}_{i \sim S} \|X_i\|^2\right)} + 6k^2 \left(1 - \mathbb{E}_{i \sim S} \|X_i\|^2\right)
\]

\[
+ \left(\frac{d'}{s} + \frac{\lambda}{k}\right) \left(1 - \mathbb{E}_{i \sim S} \|X_i\|^2\right) + 2\gamma k^2 \left(1 - \mathbb{E}_{i \sim S} \|X_i\|^2\right).
\]

where we used decomposition of the sum \(\sum_{i,j \in \mathcal{V}} A_{ij} \langle X_i, X_j \rangle\), and the results from the Section 2.1. Rearranging, cancelling terms, and using the fact that the function \(x \mapsto x^2\ \forall x \in \mathbb{R}^+\) is increasing, we get,

\[
\mathbb{E}_{i \sim S} \|X_i\|^2 \geq 1 - \frac{4\xi^2 (np)(q + r + t + 1)}{k^2 \left(1 - 9p - \delta - 2\gamma - \frac{d'}{s} - \frac{\lambda}{k}\right)^2}.
\]

\[\square\]

Definition 2.17. Let \(\psi\) be a function over the input parameters defined as, \(\psi \defeq \frac{4\xi^2 (np)(q + r + t + 1)}{k^2 \left(1 - 9p - \delta - 2\gamma - \frac{d'}{s} - \frac{\lambda}{k}\right)^2}\) for the sake of brevity.
3 Recovering the planted clique

In the previous section (Section 2), we showed that under some mild conditions over the input parameters (namely when, $p$ is “large” and $\psi$ is “small”) and with high probability (over the randomness of the input), we have,

$$\mathbb{E} ||X||^2 \geq 1 - \psi.$$ 

We define a vertex set

$$T \overset{\text{def}}{=} \{ i \in V : ||X_i||^2 \geq 1 - \alpha \psi \}$$

where $1 < \alpha < 1/\psi$ is a parameter to be chosen later.

We will next show that for a cleverly chosen value of $\alpha$, we can show that $T$ is also a clique, and using the facts that $|T \cap S| > 0$ and that the boundary of the subgraph $G[S]$ is random, we further show that $T \subseteq S$. Once we have established this, it is easy to recover the rest of the vertices of $S \setminus T$ using a simple greedy heuristic. Before that, we recall two important technical results from [KL20].

**Lemma 3.1.** Let $\{Y_i\}_{i=1}^n$ be any feasible solution of the SDP and $V' \subseteq V$ such that for all $i \in V'$, $||Y_i||^2 \geq 1 - \varepsilon$ where $0 \leq \varepsilon \leq 1$, then for all $i, j \in V'$, $\langle Y_i, Y_j \rangle \geq 1 - 3\varepsilon$.

We defer the proof of Lemma 3.1 to Appendix B since it is standard in the literature.

**Lemma 3.2** (Restatement of Lemma 3.5 from [KL20]). With high probability (over the randomness of the input),

$$|T \cap S| \geq \left(1 - \frac{1}{\alpha}\right)k.$$  

The next lemma (Lemma 3.3) is perhaps the most important technical result of this paper.

**Lemma 3.3.** For $\alpha = 1/(3 \sqrt[3]{\psi})$ and $\psi \in (0, 1/9)$. With high probability (over the randomness of the input), the subgraph $G[T]$ is a clique and moreover, $T \subseteq S$.

**Proof.** By applying Lemma 3.1 to the set $T$, we get for all $i, j \in T$: $\langle X_i, X_j \rangle \geq 1 - 3\alpha \psi$. We set $\alpha$ such that $1 - 3\alpha \psi > 0 \iff \alpha < 1/(3\psi)$. Thus we can set $\alpha = 1/(3 \sqrt[3]{\psi})$. It does satisfy the bounds on $\alpha$, namely $\alpha \in (1, 1/\psi)$ when $\psi \in (0, 1/9)$. By the SDP constraints, $\langle X_i, X_j \rangle = 0 \forall (i, j) \notin E$ (the extra added constraint, or, equation (1)), we have that the subgraph induced on $T$ is a clique. This is easy to see. Consider any two vertices $u, v \in T$ such that there is no edge between $u$ and $v$, then by the above SDP constraint, $\langle X_u, X_v \rangle = 0$, however by the definition of set $T$, $\langle X_u, X_v \rangle > 0$. This is a contradiction and thus $T$ is indeed a clique.

Next we prove that w.h.p. $T \subseteq S$. By Lemma 3.2, $|T \cap S| \geq \left(1 - \frac{1}{\alpha}\right)k = \left(1 - \frac{1}{\sqrt[3]{\psi}}\right)k > 0$ when $\psi \in (0, 1/9)$. Then,

$$\mathbb{P}[T \subseteq S] \leq \mathbb{P}[\exists v \in V \setminus S \text{ which has an edge with all the vertices of } T \cap S] \leq np^{T \cap S} \leq np^{(1 - 3 \sqrt[3]{\psi})k} = o(1).$$

where we used the union bound in step 2 and the lower bound on $|T \cap S|$ in step 3.

We now have all the ingredients to prove our main result.

**Proof of Theorem 1.3.** By Lemma 3.3 we showed that $T \subseteq S$, now we can use a greedy strategy to recover the rest of $S$. We iterate over all vertices in $V \setminus T$ and add them to our set if it has edges to all of $T$. A calculation similar to the one shown above can be used to ensure that no vertex of $V \setminus S$ enters in this greedy step. Also note that

$$\alpha \psi = \frac{\psi}{3 \sqrt[3]{\psi}} = \frac{\sqrt[3]{\psi}}{3}.$$ We define $\nu \overset{\text{def}}{=} 9\psi$.
Here $\nu$ is nothing but a normalization of $\psi$ for a cleaner representation. We summarize this in the algorithm below (Algorithm 1). It is easy to see that the output of this algorithm, the set $Q$ is essentially the planted clique $S$ itself.

\begin{algorithm}
\begin{enumerate}
  \item Solve SDP A.1 to get the vectors $\{X_i\}_{i=1}^n$.
  \item Let $T = \{ i \in V : \|X_i\|^2 \geq 1 - (\sqrt{\nu}/9) \}$.
  \item Initialize $Q = T$.
  \item Iterate over all vertices $v \in V \setminus T$, such that if $v$ shares an edge with all the vertices in $Q$, then update $Q = Q \cup \{v\}$.
  \item Return $Q$.
\end{enumerate}
\caption{Recovering the planted set $S$.}
\end{algorithm}

4 Conclusions

In this paper, we looked at a semi-random model for the PLANTED CLIQUE problem. We showed that the natural SDP relaxation of the $k$-clique puts together the vectors corresponding to the planted clique “closely”. This allows us to recover a part of the planted solution. The rest of the solution can be recovered by using a greedy algorithm. Our model is inspired from the seminal work of Feige and Killian [FK01]. Our algorithm and the analysis closely follows from the work on DENSEST $k$-SUBGRAPH problem by Khanna and Louis [KL20].

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We use the following SDP relaxation to solve this problem.

SDP A.1.

\[
\begin{align*}
\text{maximize} & \quad \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} \langle X_i, X_j \rangle \\
\text{subject to} & \quad \sum_{i=1}^{n} \langle X_i, X_i \rangle = k \\
& \quad \sum_{j=1}^{n} \langle X_i, X_j \rangle \leq k \langle X_i, X_i \rangle \quad \forall i \in [n] \\
& \quad \langle X_i, X_j \rangle = 0 \quad \forall (i, j) \notin E \\
& \quad 0 \leq \langle X_i, X_j \rangle \leq \langle X_i, X_i \rangle \quad \forall i, j \in [n], \ i \neq j \\
& \quad \langle X_i, X_i \rangle \leq 1 \quad \forall i \in [n] \\
& \quad \langle X_i, I \rangle = \langle X_i, X_i \rangle \quad \forall i \in [n] \\
& \quad \langle I, I \rangle = 1
\end{align*}
\]
It is easy to see that when $G[S]$ is a clique, then the integral solution corresponding to $S$ does satisfy the above constraints. We state it now,

$$X_i = \begin{cases} \hat{v} & i \in S \\ 0 & i \in V \setminus S \end{cases} \quad \text{and} \quad I = \hat{v}$$

where $\hat{v}$ is any unit vector and this feasible solution gives an objective value of $k^2/2$.

**B Proof of Lemma 3.1**

The proof of this lemma follows along the lines of Lemma 2.3 of [KL20] however we restate it for completeness.

**Proof.** We first introduce vectors $Z_i \in \mathbb{R}^{n+1}$ and scalars $\alpha_i \in \mathbb{R}$ (for all $i \in V'$) such that $Y_i = \alpha_i I + Z_i$ and $\langle I, Z_i \rangle = 0$. Using SDP constraints (14) and (15) we get,

$$||Y||^2 = \langle Y_i, I \rangle = \langle \alpha_i I + Z_i, I \rangle = \alpha_i \langle I, I \rangle + \langle I, Z_i \rangle = \alpha_i .$$

Also note that, $||Y||^2 = \alpha_i^2 ||I||^2 + ||Z||^2 = ||Y||^4 + ||Z||^2 \implies ||Z|| = \sqrt{||Y||^2 - ||Y||^4} . \quad (16)$

For $i, j \in V'$,

$$\langle Y_i, Y_j \rangle = \left( ||Y||^2 I + Z_i, ||Y||^2 I + Z_j \right)$$

$$= ||Y||^2 \langle Y, I \rangle + ||Y||^2 \langle I, Z \rangle + ||Y||^2 \langle I, Z \rangle + \langle Z, Z \rangle$$

$$= ||Y||^2 \langle Y, I \rangle + \langle Z, Z \rangle \quad (\because \langle I, Z \rangle = 0)$$

$$\geq ||Y||^2 \langle Y, I \rangle - ||Z|| \langle Z, Z \rangle \quad \text{(since the maximum angle between them can be } \pi)$$

$$= ||Y||^2 \langle Y, I \rangle - \left( \sqrt{||Y||^2 - ||Y||^4} \left( \sqrt{||Y||^2 - ||Y||^4} \right) \right). \quad \text{(by equation (16))}$$

Since $||Y||^2 \geq 1 - \epsilon$ using this in above equation we get

$$\langle Y_i, Y_j \rangle = ||Y||^2 \langle Y, I \rangle - \left( \sqrt{||Y||^2 (1 - ||Y||^2)} \right) \left( \sqrt{||Y||^2 (1 - ||Y||^2)} \right)$$

$$\geq (1 - \epsilon)^2 - \sqrt{\epsilon} \sqrt{\epsilon} \quad (\because ||Y||^2 \leq 1 \text{ and } 1 - ||Y||^2 \leq \epsilon)$$

$$\geq (1 - \epsilon)^2 - \epsilon = 1 + \epsilon^2 - 2 \epsilon - \epsilon \geq 1 - 3 \epsilon .$$

$\square$