On partial uniqueness of complete non-compact Ricci flat metrics

Yuanqi Wang

Abstract

Using techniques for Caccioppoli inequality, on a fairly general class of complete non-compact Kähler manifolds with sub-quadratic volume growth, we show uniqueness of bounded $C^{1,1}$ solution to Monge-Ampere equation. This does not a priori require any decay of the solution.

1 Introduction

In an arbitrary Kähler class on a closed Kähler manifold with vanishing $c_1$, uniqueness of Ricci flat metric can be proved via integration by parts [2, 18]. On a non-compact complete Kähler manifold, we show that under sub-quadratic volume growth and other mild conditions, even without decay, there is still an integration by parts argument for uniqueness of bounded solution to Monge-Ampere equation. The technique follows Caccioppoli inequality on regularity for elliptic equations. For example, see [1, 7, 8, 14].

A sub-harmonic function on a compact manifold must be a constant. In contrast and in general, the same fails on Euclidean domains even if we prescribe constant boundary value. For example, the function $|X|^2 - 1$ is sub-harmonic and vanishes on the boundary of the unit ball centered at the origin. Via this viewpoint, a complete non-compact Riemannian manifold is in between: it is not compact, but it is complete as a metric space and has no boundary. Our result shows, under the volume growth condition, a sub-solution to Monge-Ampere equation with constant density 1 is still a constant.

Theorem 1.1. Let $(M, \omega)$ be a complete non-compact Kähler manifold of complex dimension $n \geq 2$ and sub-quadratic volume growth. Let $f$ be a continuous function bounded from above on $M$.

1. Suppose $\omega$ has strictly sub-quadratic volume growth. Then any bounded $C^{1,1}$ sub-solution, super-solution, or solution $\phi$ to

$$ (\omega + i\partial\bar{\partial}\phi)^n = \omega^n $$

is a constant.

*University of Kansas, Lawrence, KS, USA. yqwang@ku.edu.
2. Suppose $e^f - 1$ is integrable i.e. $\int_M |e^f - 1|\omega^n < +\infty$. Then any bounded $C^{1,1}$ solution $\phi$ to

$$
(\omega + i\partial \bar{\partial} \phi)^n = e^f\omega^n
$$

has bounded Dirichlet energy i.e. $\int_M |
\nabla \phi|^2 \omega^n \leq c_{DE} < +\infty$, where

$$
c_{DE} = 1000n^{4n+4} |\phi|_{C^0(M)} [(1 + K)^{2n} c_{vol} + |e^f - 1|_{L^1(M,\omega^n)}].
$$

3. Suppose additionally that $\omega$ satisfies weak Neumann Poincare inequality on annulus. Then any bounded $C^{1,1}$ solution $\phi$ to (1) is a constant.

The constant (3) might not be optimal, but is effective. The terms involved are defined below.

**Definition 1.2.** A bounded $C^{1,1}$ solution, sub-solution, or super-solution to (2) is a real-valued function $\phi$ with the following properties.

- $\phi$ is twice continuously differentiable under the holomorphic (smooth) manifold structure (not necessarily with norm bound).
- $\phi$ satisfies (2), $(\omega + i\partial \bar{\partial} \phi)^n \geq e^f\omega^n$, or $(\omega + i\partial \bar{\partial} \phi)^n \leq e^f\omega^n$ respectively.
- There is a positive number $K (< \infty)$ such that

$$
|\phi|_{C^0(M)} = \sup_M |\phi| < \infty \text{ and } 0 < \omega_\phi \leq K \omega \text{ on the whole } M.
$$

Fix a point $o \in M$ as distance origin and center of balls. We say a complete non-compact Kähler metric $\omega$ has sub-quadratic volume growth, if there is a positive sequence $\rho_i \to +\infty$ such that

$$
\limsup_{i \to \infty} \frac{Vol[B(2\rho_i) \setminus B(\rho_i)]}{\rho_i^2} < +\infty.
$$

The value of the existing limit superior is denoted by $c_{vol}$. We say such an $\omega$ has strictly sub-quadratic volume growth if $c_{vol} = 0$ i.e.

$$
\lim_{i \to \infty} \frac{Vol[B(2\rho_i) \setminus B(\rho_i)]}{\rho_i^2} = 0.
$$

On the other hand, we say it satisfies weak Neumann Poincare inequality on annulus if there is $\mu_i \to \infty$ and $\mu_i$, such that $\mu_i \leq \rho_i$ when $i$ is large, and

$$
\int_{B(2\rho_i) \setminus B(\rho_i)} |\phi - \mu_i|^2 \leq c_P \rho_i^2 \int_{\{r \geq \rho_i\}} |\nabla \phi|^2,
$$

where $c_P$ is independent of $i$ or $\phi$, as long as $\phi$ is twice continuously differentiable. Our argument [19] below is independent of $\mu_i$. 


Suppose we have two solutions $\phi_1$ and $\phi_2$ to the general volume form equation (2). For uniqueness, as long as the conditions hold, we can apply Theorem 1.1 or 1.3 with reference metric being $\omega + i\partial\bar{\partial}\phi_1$ or $\omega + i\partial\bar{\partial}\phi_2$, and $\phi$ being $\pm(\phi_2 - \phi_1)$ respectively.

Theorem 1.1 partially addresses the uniqueness of Tian-Yau solutions [17, Theorem 1.1]. This particular result is under sub-quadratic volume growth, and their solution is bounded $C^{1,1}$. Let “unique...(up to constant)” abbreviates “unique...up to addition by a real constant”.

**Corollary 1.3.** Let $2 > \alpha \geq 1$. If the $(K, 2, \beta)$—polynomial growth condition is strengthened to $(K, \alpha, \beta)$, the Tian-Yau solution $\varphi$ in [17, Theorem 1.1] is the unique bounded $C^{1,1}$ solution (up to constant) to the Monge-Ampere equation [17, (1.1)]. The solution $u$ in Hein’s version [10, Prop 4.1], under $SOB(\beta)$—condition, $\beta \leq 2$, is the unique bounded $C^{1,1}$ solution (up to constant) to the Monge-Ampere equation therein. Consequently, the following holds.

- The Tian-Yau Ricci-flat space [17, Theorem 4.1] of volume growth $O(r^{2n+1})$ is the unique solution (up to constant) to the Monge-Ampere equation [17, (1.1)] with reference metric $\omega_N$ [17, (4.4)].

- All the gravitational instantons in Hein’s construction [11, Theorem 1.5] are unique solutions (up to constant) to corresponding Monge-Ampere equations in [10, Prop 4.1].

Our sub-quadratic volume growth is a sequential condition, and is apparently implied by that ball of radius $R$ (centered at the base point) has volume $\leq CR^2$, for large $R$ cf. [17, Definition 1.1]. Suppose the weak Poincare inequality (7) on annulus is implied by $(K, 2, \beta)$—polynomial growth and other conditions in [17, Theorem 1.1]. Then uniqueness of Tian-Yau solution [17, Theorem 1.1] holds in full generality. Nevertheless, for Hein’s version [10, Prop 4.1], we do have Poincare inequality [10, Prop 3.4]. Therefore uniqueness also holds for rigorous quadratic volume growth i.e. $SOB(2)$—case.

Under faster than quadratic volume growth, uniqueness of Monge-Ampere solutions is implied by certain decay on the Kähler potential $\phi$. See [13, 8.5 Theorem A4] for example. Geometric uniqueness of Ricci flat metrics, as in [5, 6, 9, 10, 11, 13, 15], usually involves both Monge-Ampere uniqueness and $i\partial\bar{\partial}$—lemma under decay conditions. For recent work on Liouville theorem of Monge-Ampere equations on product manifolds, see Hein [12] for example. For recent work on Liouville theorem of metric Laplacians on certain non-compact complete manifolds, see Sun-Zhang [15] and Carron [3] for examples. For earlier work on rigidity of harmonic functions on non-compact complete Riemannian manifolds, see Cheng-Yau [4] for example.
2 Proof

Convention: Unless otherwise specified, the metric for gradient is $\omega$. The integrals and volumes are with respect to the top degree form $\omega^n$.

For Theorem 1.1.1

Write $\omega_\phi$ for $\omega + i \overline{\partial} \phi$. We have the difference

$$\omega^n_\phi - \omega^n = i \overline{\partial} \phi \wedge Q.$$  \hspace{1cm} (8)

where $Q = \omega^{n-1}_\phi + \omega^{n-2}_\phi \wedge \omega + ... + \omega^{n-1}_\phi$. By positivity and $C^{1,1}$-condition (4), we verify

$$\omega^{n-1} \leq Q \leq n^{2n} (1 + K)^n \omega^{n-1}.$$  

Actually, any constant depending only on the data in Theorem 1.1 and Definition 1.2 suffices, but we want explicit constant, though not necessarily optimal. Moreover,

$$\begin{cases}
i \overline{\partial} \phi \wedge Q \geq 0 & \text{for subsolution, and} \\
i \overline{\partial} \phi \wedge Q \leq 0 & \text{for super-solution.} \end{cases}$$  \hspace{1cm} (9)

Because $\phi$ is bounded,

- if it is a sub-solution, by adding a constant if necessary, we assume $\min \phi \geq 1$;
- if it is a super-solution, by adding a constant if necessary, we assume $\max \phi \leq -1$.

Let $\chi$ be compactly supported Lipschitz function. We multiply both hand sides in (9) by $\chi^2 \phi$. In either case, because of the definite sign of $\chi^2 \phi$, we find

$$\chi^2 \phi \cdot i \overline{\partial} \phi \wedge Q \geq 0.$$  \hspace{1cm} (10)

We integrate (10) by parts:

$$\int_M \chi^2 i \overline{\partial} \phi \wedge \overline{\partial} \phi \wedge Q \leq -2 \int_M \phi \chi \cdot i \overline{\partial} \phi \wedge \overline{\partial} \phi \wedge Q.$$  \hspace{1cm} (11)

The left side of (11) is bounded from below by

$$\int_M \chi^2 i \overline{\partial} \phi \wedge \overline{\partial} \phi \wedge Q \geq \int_M \chi^2 i \overline{\partial} \phi \wedge \overline{\partial} \phi \wedge \omega^{n-1} \geq \frac{1}{2n} \int_M \chi^2 |\nabla \phi|^2.$$

On the other hand, Cauchy-Schwartz yields an upper bound on the right side of (11):

$$\begin{align*}
| -2 \int_M \phi \chi i \overline{\partial} \phi \wedge \overline{\partial} \phi \wedge Q | & \leq 2n^{2n} (1 + K)^n \int_M |\phi| |\chi| |\nabla \phi| |\nabla \phi| \\
& \leq 2n^{2n} (1 + K)^n (\int_M \chi^2 |\nabla \phi|^2)^{\frac{1}{2}} (\int_M \phi^2 |\nabla \chi|^2)^{\frac{1}{2}}.
\end{align*}$$
The above two inequalities and (11) imply
\[ \int_M \chi^2 |\nabla \phi|^2 \leq 4n^{2n+1}(1 + K)^n \left( \int_M \chi^2 |\nabla \phi|^2 \right)^{1/2} \left( \int_M \phi^2 |\nabla \chi|^2 \right)^{1/2}. \] (12)

Either \( \int_M \chi^2 |\nabla \phi|^2 \) is 0 or not, we find
\[ \int_M \chi^2 |\nabla \phi|^2 \leq 16n^{4n+2}(1 + K)^{2n} \int_M \phi^2 |\nabla \chi|^2. \] (13)

Now let \( \chi \) be the following piece-wise linear function in the distance \( r \) from the base point \( o \), and \( \rho > 0 \).
\[ \chi(r) = \begin{cases} 1 & \text{when } r \leq \rho, \\ 2 - \frac{r}{\rho} & \text{when } \rho \leq r \leq 2\rho, \\ 0 & \text{when } r \geq 2\rho. \end{cases} \] (14)

Apparently, \( \chi \) is Lipschitz. We find
\[ \int_{B(\rho)} |\nabla \phi|^2 \leq \frac{16n^{4n+2}(1 + K)^{2n}}{\rho^2} \int_{B(2\rho) \setminus B(\rho)} |\phi|^2 \leq 16n^{4n+2}(1 + K)^{2n} |\phi|_{C^0(M)} \cdot \frac{\text{Vol}[B(2\rho) \setminus B(\rho)]}{\rho^2}. \] (15)

Let \( \rho = \rho_i \) and \( i \to \infty \). Monotone convergence theorem implies \( |\nabla \phi|^2 \) is integrable on \( M \) and
\[ \int_M |\nabla \phi|^2 \leq 16n^{4n+2}(1 + K)^{2n} |\phi|_{C^0(M)} \cdot \limsup_{i \to \infty} \frac{\text{Vol}[B(2\rho_i) \setminus B(\rho_i)]}{\rho_i^2} = 0. \] (16)
This means \( \phi \) is a constant.

For Theorem 1.1.2

In this case we do not add any constant to \( \phi \). It is a solution by assumption. The equality still holds if we multiply \( i\partial s \phi \wedge Q = (e^f - 1)\omega^n \) by \( \chi^2 \phi \). The same argument (8)—(12) with \( e^f \omega^n \) instead of \( \omega^n \) yields
\[ \int_M \chi^2 |\nabla \phi|^2 \leq 4n^{2n+1}(1 + K)^n \left( \int_M \chi^2 |\nabla \phi|^2 \right)^{1/2} \left( \int_M \phi^2 |\nabla \chi|^2 \right)^{1/2} + \int_M \chi^2 |\phi(e^f - 1)| \leq 4n^{2n+1}(1 + K)^n \left( \int_M \chi^2 |\nabla \phi|^2 \right)^{1/2} \left( \int_M \phi^2 |\nabla \chi|^2 \right)^{1/2} + |\phi|_{C^0(M)} \cdot |e^f - 1|_{L^1(M)}. \]

The following elementary claim is proved by completing square.
Claim 2.1. Let \( a, b, c, d \) be non-negative numbers such that
\[
a \leq 2d\sqrt{a} \cdot \sqrt{b} + c.
\]
Then \( a \leq 2d^2b + c + 2\sqrt{d^2b(d^2b + c)} \leq 4d^2b + 3c \).

We then find
\[
\int_M \chi^2|\nabla \phi|^2 \leq 64n^{4n+2}(1 + K)^{2n} \int_M \phi^2|\nabla \chi|^2 + 3|\phi|_{C^0(M)} \cdot |e^f - 1|_{L^1(M)} \tag{17}
\]
Still let \( \rho = \rho_i \) and \( i \to \infty \). Same argument as \( \text{(15)} \) and \( \text{(16)} \) yields
\[
\int_M |\nabla \phi|^2 \leq 64n^{4n+2}(1 + K)^{2n}c_\text{pol}|\phi|_{C^0(M)} + 3|\phi|_{C^0(M)} \cdot |e^f - 1|_{L^1(M)}. \tag{18}
\]
We enlarge the constant to \( 3 \).

For Theorem 1.1.3

By monotone convergence theorem, the established Dirichlet energy bound implies
\[
\lim_{\rho \to \infty} \int_{\{r \geq \rho\}} |\nabla \phi|^2 = 0.
\]
We apply the argument last section to \( \phi - \mu_i \), which is also a solution. Multiplying \( i\partial \bar{\partial} \phi \wedge Q = 0 \) by \( \chi^2(\phi - \mu_i) \), we still have equality. The assumed Poincaré inequality \( \text{(7)} \) and derivation of \( \text{(15)} \) shows
\[
\int_{B(\rho_i)} |\nabla \phi|^2 = \int_{B(\rho_i)} |\nabla (\phi - \mu_i)|^2 \leq \frac{16n^{4n+2}(1 + K)^{2n}}{\rho_i^2} \int_{B(2\rho_i) \setminus B(\rho_i)} |\phi - \mu_i|^2
\]
\[
\leq 16n^{4n+2}(1 + K)^{2n}c_P \int_{\{r \geq \rho\}} |\nabla \phi|^2 \text{ which } \to 0 \text{ as } i \to \infty. \tag{19}
\]
This again says \( \int_M |\nabla \phi|^2 = 0 \) and \( \phi \) is a constant.

For Corollary 1.3

It is obvious from the general uniqueness \( \text{[11]} \) and the volume growth condition. Because the density data \( f \) in \( \text{[17]} \) Theorem 1.1] and \( \text{[10]} \) Proposition 3.4] are bounded on the whole \( M \) i.e. there exists a positive (finite) number \( c_0 \) such that \( |f| \leq c_0 \), any bounded \( C^{1,1} \)-solution \( v \) yields \( \omega_0 + i\partial \bar{\partial} v \) quasi-isometric to the reference metric \( \omega_0 \) i.e. there is a \( c_1 > 0 \) possibly depending on \( v \) and \( f \) such that
\[
\frac{\omega_0}{c_1} \leq \omega_0 + i\partial \bar{\partial} v \leq c_1\omega_0.
\]
In conjunction with the remark above Corollary \( \text{[13]} \) if there are two solutions \( \phi_1 \) and \( \phi_2 \), use \( \omega_{\phi_1} \) as the new reference metric and denote it by \( \omega \), and denote \( \phi_2 - \phi_1 \) by \( \phi \). The volume form of \( \omega \) and \( \omega_0 \) coincide. We apply Theorem \( \text{[11]} \)
When $\alpha < 2$ in the $(K, \alpha, \beta)$-condition [17, Definition 1.1], or $\beta < 2$ in Hein’s $SOB(\beta)$-condition, by quasi-isometry, $\omega$ has strict sub-quadratic volume growth. Then Theorem [11,1] yields the result.

We elaborate more for $SOB(2)$. We do not know whether the solution $\omega$ is $SOB(2)$, though $\omega_0$ is by assumption. Nevertheless, the interior Neumann Poincare inequality [10, Proposition 3.4] for $SOB(2)$—reference metric $\omega_0$ and the quasi-isometry

$$\frac{\omega}{c_2} \leq \omega_0 \leq c_2 \omega$$

still implies the weak Neumann Poincare inequality (7) for $\omega$. Namely, fix a single base point $o$ for both $\omega_0$ and $\omega$. For large enough $c_3$ independent of $i$ such that the ball $B(4\rho_i)$ with respect to $\omega$ is contained in the ball $B(c_3\rho_i)$ with respect to $\omega_0$, and the ball $B(\rho_i)$ with respect to $\omega$ contains the ball $B(\frac{c_3}{c_3})$ with respect to $\omega_0$, we assign the data

$$r_1 = \frac{\rho_i}{c_3}, \quad s = \frac{\rho_i}{1000c_3}, \quad r_2 = c_3\rho_i, \quad \kappa = 0$$

on radius and other to the ball and annuli in [10, Proposition 3.4] for the $SOB(2)$—reference metric $\omega_0$. When $i$ is large, [10 (3.4)] implies

$$\int_{B(2\rho_i) \setminus B(\rho_i)} |h - h_A(\frac{\rho_i}{c_3}, c_3\rho_i)|^2 \omega^n$$

$$\leq c_3' \int_{B(c_3\rho_i) \setminus B(\frac{\rho_i}{c_3})} |h - h_A(\frac{\rho_i}{c_3}, c_3\rho_i)|^2 \omega_0^n \leq c_4\rho_i^2 \int_{\{r \geq \frac{\rho_i}{1000c_3}\}} |\nabla \omega_0f|^2 \omega_0^n$$

$$\leq c_5\rho_i^2 \int_{\{r \geq \frac{\rho_i}{c_3}\}} |\nabla f|^2 \omega^n,$$  \hspace{1cm} (20)

where the large enough positive constants $c_3'$, $c_4$, $c_5$ are independent of $\rho_i$ or the arbitrary twice differentiable function $h$, and $h_A(\frac{\rho_i}{c_3}, c_3\rho_i)$ is the $\omega_0^n$—average of $h$ on the closed annulus therein. Note $h_A(\frac{\rho_i}{c_3}, c_3\rho_i)$ is also applied in the first line of (20) in the integration against $\omega^n$. Related to $c_4$, the volume ratio $N$ in [10 (3.4)] is bounded (from above) by the volume constants in $SOB(2)$-condition [10, Definition 3.1] (including the Ricci lower bound). This fulfills requirement (7) because $\frac{c_4}{c_3}$ still approaches $\infty$ as $i \to \infty$. Then apply Theorem [11,1].

References

[1] A. Björn, N. Marola. *Moser iteration for (quasi) minimizers on metric spaces*. Manuscripta Math. 121 (2006), no. 3, 339–366.

[2] E. Calabi. *On Kähler manifolds with vanishing canonical class*, *Algebraic Geometry and Topology*. Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 78–89. Princeton University Press, Princeton, N.J., 1957.
G. Carron. *Harmonic functions on manifolds whose large spheres are small*. Ann. Math. Blaise Pascal 23 (2016), no. 2, 249-261.

S-Y. Cheng, S-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*. Communications on Pure and Applied Mathematics 28 (1975), no. 3, 333-354.

R. Conlon, H.J. Hein. *Asymptotically conical Calabi-Yau manifolds* I. Duke Math. J. 162 (2013), no. 15, 2855-2902.

R. Conlon, H.J. Hein. *Asymptotically conical Calabi-Yau metrics on quasi-projective varieties*. Geom. Funct. Anal. 25 (2015), no. 2, 517–552.

E. DiBenedetto, N.S. Trudinger. *Harnack inequalities for quasiminima of variational integrals*. Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), no. 4, 295–308.

M. Giaquinta, L. Martinazzi. (2012) *L²–regularity: The Caccioppoli inequality*. In: *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*. Publications of the Scuola Normale Superiore. Edizioni della Normale, Pisa.

M. Haskins, H.-J. Hein, J. Nordström. *Asymptotically cylindrical Calabi–Yau manifolds*. J. Differential Geom. 101 (2015), no. 2, 213-265.

H.-J. Hein. *On gravitational instantons*. Thesis (Ph.D.)-Princeton University. 2010. 129 pp. ISBN: 978-1124-34891-9, ProQuest LLC

H.-J. Hein. *Gravitational instantons from rational elliptic surfaces*. J. Amer. Math. Soc. 25 (2012), no. 2, 355-393.

H.-J. Hein. *A Liouville theorem for the complex Monge-Ampère equation on product manifolds*. Comm. Pure Appl. Math. 72 (2019), 122-135.

D. Joyce. *Compact manifolds with special holonomy*. Oxford Mathematical Monographs.

J. Kinnunen, N. Shanmugalingam. *Regularity of quasi-minimizers on metric spaces*. Manuscripta Math. 105 (2001), no. 3, 401–423.

S. Sun, R.B. Zhang. *A Liouville theorem on asymptotically Calabi spaces*. Calculus of Variations and Partial Differential Equations 60 (3), 1-43.

Gábor Székelyhidi. *Uniqueness of some Calabi-Yau metrics on ℂ^n*. Geom. Funct. Anal. 30 (2020), no. 4, 1152–1182.

G. Tian, S.T. Yau. *Complete Kähler manifolds with zero Ricci curvature*. I. J. Amer. Math. Soc. 3 (1990), no. 3, 579-609.

S.T. Yau. *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation*. I. Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.