Generalized Metrical Multi-Time Lagrange Model
for General Relativity and Electromagnetism

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Abstract

Section 1 contains some physical and geometrical aspects that motivates us to study the generalized metrical multi-time Lagrange space of General Relativity, denoted by $\text{GRGML}^n_p$, whose vertical fundamental metrical d-tensor is

$$G^{(\alpha)(\beta)}(t^\gamma, x^k, x^\gamma_k) = h^{\alpha\beta}(t^\gamma)e^{2\sigma(t^\gamma, x^k, x^\gamma_k)}\varphi_{ij}(x^k).$$

Section 2 develops the geometry of this space, in the sense of d-connections, d-torsions and d-curvatures. Section 3 constructs the Einstein equations of gravitational potentials of this generalized metrical multi-time Lagrange space. The conservation laws of the stress-energy d-tensor of $\text{GRGML}^n_p$ are also described. Section 4 describes the Maxwell equations which govern the electromagnetic field of this space.

Mathematics Subject Classification (2000): 53B40, 53C60, 53C80.

Key words: 1-jet fibre bundle, nonlinear connection, Cartan canonical connection, Einstein equations, Maxwell equations.

1 Geometrical and physical aspects

In this century, a lot of geometrical models for gravitational and electromagnetic theories was created. We refer especially to the well known Riemannian, Finslerian or, more general, Lagrangian theories.

The usual point of view that the underlying geometry of space-time is Riemannian is thought of to be further strengthened by the constructive-axiomatic formulation (EPS conditions) of General Relativity due to Ehlers, Pirani and Schild [4]. Within this scheme the geometry of space-time is regarded in terms of its main substructures: conformal and projective structures, which are in turn thought of to be fixed by light propagation and freely falling non-rotating neutral test particles, respectively.

In Finslerian context, R. Tavakol and Van den Berg [21] have showed that the geometrical framework within which theories of gravity are sought can be generalized without at the same time contradicting the EPS conditions.

More general, in Lagrangian terms, a natural geometrical-axiomatic approach of the EPS conditions is given by Miron and Anastasiei [5]. In order to discuss the EPS conditions, in Lagrangian terminology, we recall that a generalized Lagrange space $GL^n = (M, g_{ij}(x, y))$ is defined as a pair which consists of a real, $n$-dimensional manifold $M$ coordinated by $x = (x^i)_{i=1,n}$ and a fundamental metrical d-tensor $g_{ij}(x, y)$.
on $TM$, of rank $n$ and having a constant signature on $TM \setminus \{0\}$. We point out that $g_{ij}(x, y)$ is not necessarily 0-homogenous with respect to the direction $y = (y^1, \ldots, y^n)$. Let us assume that the generalized Lagrange space $GL^n = (M^n, g_{ij}(x, y))$ satisfies the following axioms:

**a.1** The fundamental tensor field $g_{ij}(x, y)$ is of the form

$$g_{ij}(x, y) = e^{2\sigma(x, y)} \varphi_{ij}(x).$$

**a.2** The space $GL^n$ is endowed with the non-linear connection

$$N^j_{ij}(x, y) = \gamma^j_{jk}(x)y^k,$$

where $\gamma^j_{jk}(x)$ are the Christoffel symbols for the semi-Riemannian metric $\varphi_{ij}(x)$.

The axiom **a.1** asserts that the metrical d-tensor $g_{ij}(x, y)$ is conformally to the semi-Riemannian metric $\varphi_{ij}(x)$. Therefore the spaces $GL^n$ and $R^n = (M, \varphi_{ij})$ have the same causal structure.

The axiom **a.2** ensures us that the autoparallel curves of the non-linear connection $N^j_{ij}(x, y)$ of the space $GL^n$ coincide to the geodesics of $R^n$.

Consequently, under above axiomatic assumptions, the generalized Lagrange space $GL^n$ becomes a convenient mathematical model for General Relativity, because it verifies the EPS conditions. The differential geometry of the generalized Lagrange space $GL^n = (M, e^{2\sigma(x, y)} \varphi_{ij}(x))$ is now completely developed in [1], [7], [9].

It is well known that the jet fibre bundle of order one $J^1(T, M)$ is a basic object in the study of classical and quantum field theories. In a general setting, in a previous paper [10], Neagu creates a natural geometry of physical fields induced by a Kronecker $h$-regular vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}(\gamma, x^k, x^\gamma_k)$ on the total space of the 1-jet vector bundle $J^1(T, M) \to T \times M$, where $(T, h)$ is a smooth, real, $p$-dimensional semi-Riemannian manifold coordinated by $t = (t^\alpha)_{\alpha=1, p}$, whose physical meaning is that of "multidimensional time". We point out that $J^1(T, M)$ is coordinated by $(t^\alpha, x^j, x^\gamma_k)$, where $x^\gamma_k$ have the physical meaning of partial directions.

In the multi-temporal context, the fundamental geometrical concept used in the geometrization of a vertical multi-time metric d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}(\gamma, x^k, x^\gamma_k)$ is that of *generalized metrical multi-time Lagrange space* [10]. This geometrical concept with physical meaning is represented by a pair $GML^p = (J^1(T, M), G^{(\alpha)(\beta)}_{(i)(j)})$ consisting of the 1-jet space and a *Kronecker $h$-regular* vertical multi-time metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)}$, that is

$$G^{(\alpha)(\beta)}_{(i)(j)}(\gamma, x^k, x^\gamma_k) = h^{\alpha\beta}(\gamma)g_{ij}(\gamma, x^k, x^\gamma_k),$$

where $g_{ij}(\gamma, x^k, x^\gamma_k)$ is a d-tensor on $J^1(T, M)$, symmetric, of rank $n$ and having a constant signature. The d-tensor $g_{ij}(\gamma, x^k, x^\gamma_k)$ is called the *spatial metrical d-tensor of $GML^p$*.

The differential geometry of the generalized metrical multi-time Lagrange spaces together with its attached field theory are now considerably developed in [10]. Following the general physical and geometrical development from [10], the aim of this paper is to study the generalized metrical multi-time Lagrange space $GRGML^p$, whose spatial metrical d-tensor is of the form

$$g_{ij}(\gamma, x^k, x^\gamma_k) = e^{2\sigma(\gamma, x^k, x^\gamma_k)} \varphi_{ij}(x^k),$$

where $\sigma(\gamma, x^k, x^\gamma_k)$ have the physical meaning of the spatial scalar field on the space $GML^p$. Therefore we point out that $GML^p$ describe a natural geometry of physical fields induced by the spatial metrical d-tensor $g_{ij}(\gamma, x^k, x^\gamma_k)$ on the total space of the 1-jet vector bundle $J^1(T, M) \to T \times M$, where $(T, h)$ is a smooth, real, $p$-dimensional semi-Riemannian manifold coordinated by $t = (t^\alpha)_{\alpha=1, p}$, whose physical meaning is that of "multidimensional time". We point out that $J^1(T, M)$ is coordinated by $(t^\alpha, x^j, x^\gamma_k)$, where $x^\gamma_k$ have the physical meaning of partial directions.
where \( \varphi_{ij}(x^k) \) is a semi-Riemannian metric on the spatial manifold \( M \) and \( \sigma : J^1(T, M) \to R \) is a conformal smooth function, which gives the magnitude of directions \( x^\alpha_i \).

**Remark 1.1** From physical point of view, the interesting properties of this space are obtained considering the special conformal functions:

i) \( \sigma = U^{(i)}(t^\gamma, x^k)x^k_i \),

ii) \( \sigma = h^{\alpha\beta}(t^\gamma)A_i(x^k)A_j(x^k)x^i_\alpha x^j_\beta \),

iii) \( \sigma = \varphi_{ij}(x^k)X^\alpha(t^\gamma)X^\beta(t^\gamma)x^i_\alpha x^j_\beta \),

where \( U^{(i)}(t^\gamma, x^k) \) is a \( \delta \)-tensor on \( E \), \( A_i(x^k) \) is a covector field on \( M \), and \( X^\alpha(t^\gamma) \) is a vector field on \( T \). For more details, see \( \text{[7]} \).

In order to develop the geometry of this generalized metrical multi-time Lagrange space, we need a nonlinear connection \( \Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j}) \) on \( J^1(T, M) \). In this direction, we fix "a priori" the nonlinear connection \( \Gamma \) defined by the temporal components

\[
(1.5) \quad M^{(i)}_{(\alpha)\beta} = -H^\mu_{\alpha\beta} x^i_\mu 
\]

and the spatial components

\[
(1.6) \quad N^{(i)}_{(\alpha)j} = \gamma^i_{jm} x^m_\alpha,
\]

where \( H^\alpha_{\beta\gamma} \) (resp. \( \gamma^i_{jk} \)) are the Christoffel symbols of the semi-Riemannian metric \( h_{\alpha\beta} \) (resp. \( \varphi_{ij} \)).

**Remarks 1.2** i) The previous nonlinear connection \( \Gamma \) is dependent only the vertical fundamental metrical \( \delta \)-tensor \( G^{(\alpha)(\beta)}_{(i)(j)} \) of \( GRGML^n_\alpha \). This fact emphasize the metrical character of the geometry attached to this space, i.e., all geometrical objects are directly arised from \( G^{(\alpha)(\beta)}_{(i)(j)} \).

ii) The spatial components \( N^{(i)}_{(\alpha)j} \) of the fixed nonlinear connection \( \Gamma \) are without torsion \( [i] \).

Investigating the possibility of realising the multi-time EPS conditions for General Relativity, in a general setting, let us start with the generalized metrical multi-time Lagrange space

\[
(1.7) \quad GRGML^n_\alpha = (J^1(T, M), G^{(\alpha)(\beta)}_{(i)(j)} = h^{\alpha\beta}(t^\gamma)g_{ij}(t^\gamma, x^k, x^k)),
\]

which verifies the axioms A.1 and A.2 from below:

**A.1** The vertical fundamental tensor field \( G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k, x^k) \) is of the form

\[
(1.8) \quad G^{(i)(j)}_{(i)(j)}(t^\gamma, x^k, x^k) = h^{\alpha\beta}(t^\gamma)e^{2\sigma(t^\gamma, x^k, x^k)} \varphi_{ij}(x^k).
\]

**A.2** The generalized metrical multi-time Lagrange space \( GRGML^n_\alpha \) is endowed with the non-linear connection \( \Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j}) \), defined by the components

\[
(1.9) \quad M^{(i)}_{(\alpha)\beta} = -H^\mu_{\alpha\beta} x^i_\mu, \quad N^{(i)}_{(\alpha)j} = \gamma^i_{jm} x^m_\alpha,
\]
where $H^\alpha_{\beta\gamma}$ (resp. $\gamma^i_{jk}$) are the Christoffel symbols of the semi-Riemannian metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$).

Let us consider the Lagrangian function $L : J^1(T, M) \to R$, used in the Polyakov model of bosonic strings,

\begin{equation}
L(t^\gamma, x^k, x'^k) = h^{\alpha\beta}(t^\gamma)\varphi_{ij}(x^k)x^i_{\alpha}x^j_{\beta},
\end{equation}

and its vertical fundamental metrical d-tensor,

\begin{equation}
g^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \frac{\partial^2 L}{\partial x^i_{\alpha} \partial x^j_{\beta}} = h^{\alpha\beta}(t^\gamma)\varphi_{ij}(x^k).
\end{equation}

Remark 1.3 The extremals of the Lagrangian $\mathcal{L} = L\sqrt{|h|}$ are exactly the harmonic maps between the semi-Riemannian spaces $(T, h)$ and $(M, \varphi)$.

Let $EDML^n_p = (J^1(T, M), L)$ be the metrical multi-time Lagrange space of electrodynamics corresponding to $L$. In this context, we have the following important result

**Theorem 1.1** i) The generalized metrical multi-time Lagrange space $GRGML^n_p$ has the same conformal structure as the autonomous metrical multi-time Lagrange space of electrodynamics $EDML^n_p = (J^1(T, M), L)$.

ii) The harmonic maps of the nonlinear connection $\Gamma$ of $GRGML^n_p$ coincide with those of the canonical nonlinear connection of $EDML^n_p = (J^1(T, M), L)$. Moreover, these are exactly the harmonic maps between the semi-Riemannian spaces $(T, h)$ and $(M, \varphi)$.

**Proof.** The axiom $A.1$ ensures us that both spaces $GRGML^n_p$ and $EDML^n_p$ have the same causal structure.

Following the paper, we deduce that the components of the canonical nonlinear connection of $EDML^n_p$ are described exactly by formulas, that is, those from the axiom $A.2$. Moreover, the relation between sprays and the components of a nonlinear connection, described in [14], and the definition of harmonic maps attached to a given multi-time dependent spray on $J^1(T, M)$, imply what we were looking for.

In conclusion, we can assert that the generalized metrical multi-time Lagrange space $GRGML^n_p$, which verifies the assumptions $A.1$ and $A.2$, represents a convenient relativistic model, in the multi-temporal context, since it has the same conformal and projective properties as the autonomous metrical multi-time Lagrange space from Polyakov model of bosonic strings.

### 2 Cartan canonical connection

In this section, we will apply the general geometrical development of the generalized metrical multi-time Lagrange spaces, to the particular space

\begin{equation}
GRGML^n_p = (J^1(T, M), \mathcal{G}^{(\alpha)(\beta)}_{(i)(j)} = h^{\alpha\beta}(t^\gamma)e^{2\sigma(t^\gamma, x^k, x'^k)}\varphi_{ij}(x^k)),
\end{equation}
endowed with the nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$, where

\begin{equation}
(2.2) \quad M^{(i)}_{(\alpha)\beta} = -H^{i}_{\alpha\beta}x^{i}_{\mu}, \quad N^{(i)}_{(\alpha)j} = \gamma^{i}_{jm}x^{m}_{\alpha}.
\end{equation}

Let \( \{ \frac{\delta}{\delta t^{\alpha}}, \frac{\delta}{\delta x^{i}}, \frac{\partial}{\partial x^{j}} \} \) be the adapted bases of the nonlinear connection $\Gamma$, where

\begin{equation}
(2.3) \quad \begin{cases}
\frac{\delta}{\delta t^{\alpha}} = \frac{\partial}{\partial t^{\alpha}} - M^{(i)}_{(\beta)\alpha} \frac{\partial}{\partial x^{i}} \\
\frac{\delta}{\delta x^{i}} = \frac{\partial}{\partial x^{i}} - N^{(i)}_{(\beta)j} \frac{\partial}{\partial x^{j}} \\
\delta x^{j} = dx^{j} + M^{(i)}_{(\alpha)j} dt^{\beta} + N^{(i)}_{(\alpha)j} dx^{j}.
\end{cases}
\end{equation}

Following the paper [10], by a direct calculation, we can determine the Cartan canonical connection of $GRGML^{n}_{p}$, together with its torsion and curvature local d-tensors.

**Theorem 2.1** The Cartan canonical connection $CT = (H^{\gamma}_{\alpha\beta}, C^{k}_{\gamma\gamma}, L^{i}_{ij}, C^{(i)}_{j(k)})$ of $GRGML^{n}_{p}$ has the adapted coefficients

\begin{equation}
(2.4) \quad H^{\gamma}_{\alpha\beta} = H^{\gamma}_{\alpha\beta}, \quad C^{k}_{\gamma\gamma} = \sigma^{k}_{\gamma}\delta^{k}_{\gamma}, \quad L^{k}_{ij} = \gamma_{ij}^{k} + \Lambda^{k}_{ij}, \quad C^{(i)}_{j(k)} = \sigma^{(i)}_{(k)}\delta^{i}_{(k)} + \sigma^{(i)}_{(j)}\delta^{k}_{(j)} - \varphi_{ij}\sigma^{(i)}\delta^{k}_{(j)},
\end{equation}

where

\begin{equation}
(2.5) \quad \sigma^{k}_{\gamma} = \frac{\delta\sigma}{\delta t^{\gamma}}, \quad \sigma^{k}_{i} = \frac{\delta\sigma}{\delta x^{i}}, \quad \sigma^{(i)}_{(k)} = \frac{\delta\sigma}{\delta x^{k}}, \quad \sigma^{k}_{ij} = \varphi^{km}_{ij}\sigma^{(m)}_{(i)} + \Lambda^{k}_{ij} = \sigma^{k}_{\gamma}\delta^{k}_{\gamma} + \sigma^{k}_{\gamma}\delta^{k}_{i} - \varphi_{ij}\sigma^{k}_{\gamma}.
\end{equation}

**Theorem 2.2** The torsion $T$ of the Cartan canonical connection of $GRGML^{n}_{p}$ is determined by seven effective local d-tensors, namely,

\begin{equation}
(2.6) \quad T^{m}_{\alpha} = -\sigma^{m}_{\alpha}\delta^{m}_{\gamma}, \quad P^{m}_{(\beta)\alpha}(j) = C^{m}_{i(j)}(\beta), \quad P^{m}_{(\mu)\alpha}(j) = -\sigma^{m}_{\alpha}\delta^{m}_{\mu}\delta^{m}_{\gamma}, \quad P^{(m)}_{(\mu)\alpha}(j) = -L^{m}_{i(j)}\delta^{m}_{\mu}, \quad S^{(m)}(\alpha)(\beta) = \delta^{m}_{\mu}C^{m}_{i(j)}(\beta) - \delta^{m}_{\mu}C^{m}_{i(j)}(\alpha), \quad R^{(m)}_{i(j)} = -\delta^{m}_{\alpha}x^{m}_{\alpha}, \quad R^{m}_{(\mu)\alpha} = 0, \quad R^{(m)}_{(\mu)ij} = \sigma^{m}_{ijk}\delta^{m}_{\alpha}.
\end{equation}

where $H^{\gamma}_{\mu\alpha\beta}$ (resp. $\varphi_{ij}^{m}$) are the local curvature tensors of the semi-Riemannian metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$).

In order to describe the local curvature d-tensors of Cartan canonical connection of $GRGML^{n}_{p}$, let us consider $BG = (H^{\gamma}_{\alpha\beta}, 0, \gamma_{ij}^{k}, 0)$, the Berwald h-normal G-linear connection attached to the semi-Riemannian metrics $h_{\alpha\beta}$ and $\varphi_{ij}$. We denote by $"/\alpha\"$, $"\|\"$, and $"\|^{(\alpha)}\"$, the local covariant derivatives induced by $BG$. Now, taking into account the expressions of these local covariant derivatives [13], by a direct calculation, we deduce
Proposition 2.3 The Berwald connection $BT$ of $GRGL^m_p$ has the following metrical properties:

\[
\begin{align*}
& h_{\alpha\beta}/\gamma = 0, \quad h_{\alpha\beta}\|k = 0, \quad h_{\alpha\beta}\|^{(\gamma)}_{(k)} = 0, \\
& \varphi_{ij}/\gamma = 0, \quad \varphi_{ij}\|k = 0, \quad \varphi_{ij}\|^{(\gamma)}_{(k)} = 0, \\
& g_{ij}/\gamma = 2\sigma_{ij}g_{ij}, \quad g_{ij}\|k = 2\sigma_{kj}g_{ij}, \quad g_{ij}\|^{(\gamma)}_{(k)} = 2\sigma^{(\gamma)}_{(k)}g_{ij},
\end{align*}
\]

where $g_{ij}(t^\gamma, x^k, x^{k^\gamma}) = e^{2\sigma(t^\gamma, x^k, x^{k^\gamma})}\varphi_{ij}(x^k)$.

In these conditions, using the general expressions of the local curvature d-tensors attached to the Cartan canonical connection of a generalized metrical multi-time Lagrange space $GML^n_p$, by computations, we obtain

Theorem 2.4 The curvature $R$ of the Cartan canonical connection of $GRGL^m_p$ is determined by seven effective local d-tensors, expressed by,

\[
\begin{align*}
& H^\alpha_{\eta\beta\gamma} = \frac{\partial H^\alpha_{\eta\beta}}{\partial \gamma} - \frac{\partial H^\alpha_{\eta\beta}}{\partial \beta} + H^\mu_{\eta\gamma}H^\alpha_{\mu\beta} - H^\alpha_{\gamma\mu}H^\mu_{\eta\beta}, \\
& R^l_{i\beta\gamma} = \left[ C^{l(\mu)}_{i(\mu)} - \delta^l_i\delta^{m(\mu)}_{\beta} \right] R^{(m)}_{\gamma}, \\
& R^l_{i\beta k} = \left[ \varphi_{ik}\varphi^m_{l\beta} - \delta^l_i\delta^m_{\beta} \right] \sigma_{mk}/\beta, \\
& R^l_{ijk} = r^l_{ijk} + \Lambda^l_{ij}\|k - \Lambda^l_{ik}\|j + \Lambda^l_{ik}\Lambda^m_{mk} - \Lambda^l_{ik}\Lambda^l_{mj} + C^{l(\mu)}_{i(\mu)} R^{(m)}_{\gamma}, \\
& P^l_{i\beta(k)} = \frac{\partial C^l_{i\beta}}{\partial \gamma} - \frac{\partial C^l_{i\beta}}{\partial \beta} - C^l_{i\beta}\gamma, \\
& P^l_{i\beta(k)} = \Lambda^l_{ij}\|^{(\gamma)}_{(k)} - C^l_{i(\gamma)} C^m_{i(k)} + \Lambda^m_{ij} C^l_{m(k)} + C^l_{i(\gamma)} C^m_{i(k)} C^l_{m(k)} + C^m_{i(\gamma)} C^l_{m(k)} + C^l_{i(\gamma)} C^m_{i(k)} C^l_{m(k)}, \\
& S^l_{i\beta(k)} = C^l_{i\beta}\|^{(\gamma)}_{(k)} - C^l_{i(\gamma)} C^m_{i(k)} + C^m_{i(\gamma)} C^l_{m(k)} + C^m_{i(\gamma)} C^l_{m(k)} + C^l_{i(\gamma)} C^m_{i(k)} C^l_{m(k)} + C^m_{i(\gamma)} C^l_{m(k)} + C^l_{i(\gamma)} C^m_{i(k)} C^l_{m(k)},
\end{align*}
\]

where $H^\alpha_{\eta\beta\gamma}$ and $r^l_{ijk}$ are the curvature tensors of the semi-Riemannian metrics $h_{\alpha\beta}$ and $\varphi_{ij}$.

In order to give a more natural and beautiful form to the local curvature d-tensors, we need the following notations:

\[
\begin{align*}
& \varphi_{kl} = \varphi^{ij}\varphi_{kl} - \delta^l_i\delta^j_k, \\
& \sigma_{jk} = \sigma_{j\|k} - \sigma_j\sigma_k + \frac{1}{2}\varphi_{jk} < \sigma, \sigma >, \\
& \sigma^{(\gamma)}_{j(k)} = \sigma_{j\|^{(\gamma)}_{(k)}} - \sigma_j\sigma^{(\gamma)}_{(j)} + \frac{1}{2}\varphi_{jk} < \sigma, \sigma >^\gamma, \\
& \sigma^{(\beta)}_{(j)k} = \sigma^{(\beta)}_{(j)\|k} - \sigma_j\sigma^{(\beta)}_{(j)} + \frac{1}{2}\varphi_{jk} < \sigma, \sigma >^\beta, \\
& \sigma^{(\beta)}_{(j)(k)} = \sigma^{(\beta)}_{(j)\|^{(\gamma)}_{(k)}} - \sigma_{(j)}\sigma^{(\gamma)}_{(k)} + \frac{1}{2}\varphi_{jk} < \sigma, \sigma >^{\beta\gamma},
\end{align*}
\]
Proposition 2.5

The following tensorial identities are true:

\[ \begin{align*}
\sigma_{jk} - \sigma_{kj} &= -r_{ij}^{m} \sigma_{(m)}{^{(\mu)}} x_{\mu}, \\
\sigma_{ij}^{(\beta)} - \sigma_{ij}^{(\beta)} &= 0, \\
\sigma_{(i)(j)} - \sigma_{(j)(i)} &= 0.
\end{align*} \]

Consequently, by simple computations, we obtain

\[ \begin{align*}
H_{\eta\beta\gamma} &= H_{\eta\beta\gamma}^\alpha, \\
R_{i\beta\gamma}^{l} &= -\varphi_{jl}^{m} \sigma_{(m)}{^{(\mu)}} x_{\mu}, \\
R_{i\beta\gamma}^{l} &= \varphi_{ik}^{m} \sigma_{(m)}{^{(\mu)}} x_{\mu}, \\
P_{i\beta(k)}^{(\gamma)} &= \sigma_{l}^{(\mu)} \sigma_{(l)}{^{(\gamma)}}^{(\mu)} + \delta_{\beta}^{(\gamma)} \delta_{l}^{(\gamma)}, \\
P_{i\beta(k)}^{(\gamma)} &= \sigma_{l}^{(\mu)} \sigma_{(l)}{^{(\gamma)}}^{(\mu)} + \delta_{\beta}^{(\gamma)} \delta_{l}^{(\gamma)}, \\
S_{i\beta(k)}^{(\gamma)} &= \sigma_{(l)}{^{(\gamma)}}^{(\mu)} - \sigma_{(l)}{^{(\gamma)}}^{(\mu)} + \varphi_{ijs} \sigma_{(s)}{^{(\gamma)}}^{(\mu)} - \varphi_{ik} \sigma_{(k)}{^{(\gamma)}}^{(\mu)} + \varphi_{jk} \sigma_{(j)}{^{(\gamma)}}^{(\mu)}.
\end{align*} \]

where \( \sigma^l = \varphi^m \sigma_m \) and \( \sigma^{(\mu)} = \varphi^m \sigma_{(m)}{^{(\mu)}}. \)

3 Einstein equations of gravitational field

In order to develop the gravitational theory on \( GRGL_{\eta}^n \), we point out that the vertical metrical d-tensor \( \varphi^l \) and its fixed nonlinear connection \( \sigma_{(i)}{^{(\mu)}} \) induce a natural gravitational \( h \)-potential on the 1-jet space \( J^1(T, M) \) (i. e. a Sasaki-like metric), which is expressed by \( h \)

\[ G = h_{\alpha\beta} dt^{\alpha} \otimes dt^{\beta} + e^{2\varphi} \varphi_{ij} dx^i \otimes dx^j + h^{\alpha\beta} e^{2\varphi} \varphi_{ij} \delta x^i_{\alpha} \otimes \delta x^j_{\beta}. \]
Let $CT = (H^\alpha_{\alpha \beta}, G^\gamma_{\gamma \gamma}, L^i_{jk}, G_{j(k)})$ be the Cartan canonical connection of $GRGML^n_C$.

We postulate that the gravitational equations which govern the gravitational $h$-potential $G$ of $GRGML^n_C$ are the Einstein equations attached to the Cartan canonical connection and the adapted metric $G$ on $J^1(T, M)$, that is,

$$Ric(CT) - \frac{Sc(CT)}{2}G = \mathcal{K}T,$$

where $Ric(CT)$ represents the Ricci d-tensor of the Cartan connection, $Sc(CT)$ is its scalar curvature, $\mathcal{K}$ is the Einstein constant and $T$ is an intrinsic distinguished tensor of matter which is called the stress-energy d-tensor.

In the adapted basis $(X_A) = \left(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^\alpha}\right)$ attached to $\Gamma$, the curvature d-tensor $R$ of the Cartan connection is expressed locally by $R(X_C, X_B)X_A = R^D_{ABC}X_D$. Hence, it follows that we have $R_{AB} = Ric(CT)(X_A, X_B) = R^D_{ABD}$ and $Sc(CT) = G^{AB}R_{AB}$, where

$$G^{AB} = \begin{cases} h_{\alpha \beta}, & \text{for } A = \alpha, B = \beta \\ e^{-2\sigma} \varphi^{ij}, & \text{for } A = i, B = j \\ h_{\alpha \beta}e^{-2\sigma} \varphi^{ij}, & \text{for } A = (i)_{(\alpha)}, B = (j)_{(\beta)} \\ 0, & \text{otherwise.} \end{cases}$$

Taking into account the expressions of the local curvature d-tensors of the Cartan connection of $GRGML^n_C$, we obtain without difficulties

**Theorem 3.1** The Ricci d-tensor $Ric(CT)$ of $GRGML^n_C$ is determined by seven effective local d-tensors, expressed, in adapted basis, by:

$$R_{\alpha \beta} = H_{\alpha \beta} = H^\mu_{\alpha \beta \mu},$$

$$R_{ij} = (1-n)\sigma_{ij/\beta},$$

$$R_{ij} = r_{ij} + (2-n)\sigma_{ij} - \varphi_{ij} < \sigma > + m_{ij}\sigma_{(i)}^{(m)} x^m - \varphi_{ir} m_{ijp} \sigma_{pir} x^m,$$

$$P_{(i)\beta}^{(a)} = (1-n)\sigma_{(i)/\beta}^{(a)} + \sigma_{(i)}^{(a)} \sigma_{\beta},$$

$$P_{(i)j}^{(a)} = \sigma_{ij}^{(a)} + (1-n)\sigma_{(i)}^{(a)} - \varphi_{ij} < \sigma > + \sigma_{(i)}^{(a)} \sigma_{(j)},$$

$$S_{(i)(j)}^{(\beta)} = \sigma_{(i)(j)}^{(\beta)} + \sigma_{(i)(j)}^{(\beta)} - \varphi_{jk} < \sigma > + \sigma_{(j)}^{(\beta)} \sigma_{(k)}^{(\gamma)} - \sigma_{(k)}^{(\beta)} \sigma_{(j)}^{(\gamma)},$$

where $n = \dim M, H_{\alpha \beta}$ (resp. $r_{ij}$) are the local Ricci tensors of the metric $h_{\alpha \beta}$ (resp. $\varphi_{ij}$), $< \sigma > = \varphi^{rs} \sigma_{rs}, < \sigma > = \varphi^{rs} \varphi_{rs}^{(a)} r_{(s)}$ and $< \sigma > = \varphi^{rs} \varphi_{rs}^{(a)} r_{(s)}^{(a)}$.

Let us denote $H = h^{\alpha \beta} H_{\alpha \beta}, R = e^{2\sigma} \varphi^{ij} R_{ij}$ and $S = h_{\alpha \beta} e^{2\sigma} \varphi^{ij} S_{(i)(j)}^{(\alpha)(\beta)}$. In this context, by a simple calculation, it follows
Theorem 3.2  The scalar curvature of the Cartan connection $CT$ of $GRGML^n_p$ is given by
\begin{equation}
Sc(CT) = H + R + S,
\end{equation}
where
\begin{equation}
H = h^{\alpha\beta}h_{\alpha\beta},
\end{equation}
\begin{equation}
R = e^{-2\sigma} \left[ r + 2(1-n) <\sigma > + 2r_{ms}^n \sigma^m x^m_\mu \right],
\end{equation}
\begin{equation}
S = 2(1-n)e^{-2\sigma}<<\sigma>>,
\end{equation}
where $n = \dim M$, $H$ (resp $r$) is the scalar curvature of the semi-Riemannian metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$) and $<<\sigma>> = h^{\alpha\beta}h_{\alpha\beta}$. 

Following the gravitational field theoretical exposition on a generalized metrical multi-time Lagrange space $GML^n_p$ from the paper [10], by local computations, we can give

Theorem 3.3  If $p > 2$ and $n > 2$, the Einstein equations which govern the gravitational $h$-potential $G$ of $GRGML^n_p$ are
\begin{equation}
\begin{cases}
H_{\alpha\beta} - \frac{H}{2}h_{\alpha\beta} = \mathcal{K}\tilde{T}_{\alpha\beta}
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
r_{ij} - \frac{r}{2}\varphi_{ij} + \rho_{ij} = \mathcal{K}\tilde{T}_{ij}
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
S^{(\alpha)(\beta)}_{(i)(j)} + (n-1) <<\sigma>> h^{\alpha\beta}\varphi_{ij} = \mathcal{K}\tilde{T}^{(\alpha)(\beta)}_{(i)(j)},
\end{cases}
\end{equation}

where $\tilde{T}_{\alpha\beta}$, $\tilde{T}_{ij}$ and $\tilde{T}^{(\alpha)(\beta)}_{(i)(j)}$ represent the components of the new stress-energy d-tensor $\tilde{T}$, expressed by
\begin{equation}
\begin{cases}
\tilde{T}_{\alpha\beta} = T_{\alpha\beta} + \frac{R + S}{2\mathcal{K}}h_{\alpha\beta}
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\tilde{T}_{ij} = T_{ij} + \frac{H + S}{2\mathcal{K}}e^{2\sigma}\varphi_{ij}
\end{cases}
\end{equation}

\begin{equation}
\begin{cases}
\tilde{T}^{(\alpha)(\beta)}_{(i)(j)} = T^{(\alpha)(\beta)}_{(i)(j)} + \frac{H + R}{2\mathcal{K}}h^{\alpha\beta}e^{2\sigma}\varphi_{ij},
\end{cases}
\end{equation}

and
\begin{equation}
\rho_{ij} = (2-n)\sigma_{ij} - (3-2n) <\sigma > \varphi_{ij} - [\varphi_{is} r_{mp}^s + 2r_{mp}\varphi_{ij} - r_{mj}\varphi_{ip}]\sigma_{\mu\nu}x^m_\mu.
\end{equation}

Remarks 3.1  i) It is remarkable that, in the particular case $\sigma \equiv 0$, the Einstein equations $(E'_1)$ of $GRGML^n_p$ reduce to the classical ones.

ii) Note that, in order to have the compatibility of the Einstein equations, it is necessary that the certain adapted local components of the stress-energy d-tensor vanish "a priori".
From physical point of view, the stress-energy d-tensor $T$ must verify the local conservation laws $T^B_A|B = 0$, $\forall A \in \{\alpha, i \}^{(\alpha)}$, where $T^B_A = G^{BD}\mathcal{T}_{DA}$, “"$A$”, represent one of the local covariant derivatives "$\gamma\beta","\gamma^n$" or "$\delta_{(j)}"$, associated to the Cartan canonical connection $C$ \[13\].

In this context, let us denote

$$
\tilde{T}_T = h^{\alpha\beta}\tilde{T}_{\alpha\beta}, \quad \tilde{T}_M = e^{-2\sigma}\varphi^{ij}\tilde{T}_{ij}, \quad \tilde{T}_e = h^{\mu\nu}e^{-2\sigma}\varphi^{mr}\tilde{T}_{(\mu)(r)},
$$

(3.9)

$$
\tilde{T}^\mu = h^{\alpha\mu}\tilde{T}_{\alpha\mu}, \quad \tilde{T}^m = e^{-2\sigma}\varphi^{im}\tilde{T}_{mj}, \quad \tilde{T}^{(i)(\beta)} = h_{\alpha\mu}e^{-2\sigma}\varphi^{im}\tilde{T}_{(\mu)(i)}.
$$

Following again the development of gravitational generalized metrical multi-time theory from \[10\], we find

**Theorem 3.4** If $p > 2$, $n > 2$, the new stress-energy d-tensors $\tilde{T}_{\alpha\beta}, \tilde{T}_{ij}$ and $\tilde{T}^{(i)(\beta)}_{(i)(\beta)}$ must verify the following conservation laws:

$$
\begin{cases}
\tilde{T}^\mu_{(\beta)} + \frac{1}{2-n}\tilde{T}_{M/\beta} + \frac{1}{2-pn}\tilde{T}_v = -R^m_{\beta|m} - P^{(m)}_{(\beta)/m} \\
\frac{1}{2-p}\tilde{T}_{T|j} + \tilde{T}^m_{|j} + \frac{1}{2-pn}\tilde{T}_v = -P^{(m)}_{(\beta)|j} \\
\frac{1}{2-p}\tilde{T}^{(i)}_{(\alpha)} + \frac{1}{2-n}\tilde{T}^m_{(\alpha)} + \tilde{T}^{(m)}_{(\alpha)} + P^{(m)}_{(\beta)} = -P^{(m)}_{(\beta)|m}.
\end{cases}
$$

(3.10)

where

$$
P_{\beta} = e^{-2\sigma}\varphi^{im}R_{m\beta}, \quad P_{(i)\beta} = e^{-2\sigma}\varphi^{im}h_{\alpha\mu}P^{(\mu)}_{(m)\beta},
$$

(3.11)

$$
P^{(i)\beta}_{(j)} = e^{-2\sigma}\varphi^{im}P^{(i)\beta}_{m(j)}, \quad P^{i\beta}_{(j)} = e^{-2\sigma}\varphi^{im}h_{\alpha\mu}P^{(\mu)}_{m(j)}.
$$

**Remark 3.2** If the conformal function $\sigma$ is independent of partial directions $x^i_\alpha$, in other words, all functions $\sigma^{(i)}_{(\alpha)}$ vanish, then the conservation laws take the following simple form:

$$
\tilde{T}^\mu_{(\beta)/\mu} = 0, \quad \tilde{T}^m_{i|m} = 0, \quad \tilde{T}^{(m)}_{(\alpha)(\beta)/m} = 0.
$$

(3.12)

\section{4 Maxwell equations of electromagnetic field}

In order to develop the electromagnetic theory on the generalized metrical multi-time Lagrange space $GRGML^p_{\mu}$, let us consider the canonical Liouville d-tensor $D = x^i_\alpha \frac{\partial}{\partial x^i_\alpha}$ on $J^1(T, M)$. Using the Cartan canonical connection $CT$ of $GRGML^p_{\mu}$, we construct the metrical deflection d-tensors \[10\]

$$
\tilde{D}^{(i)}_{(i)\beta} = \left[ G^{(i)(\mu)}_{(i)(m)} x^m_{\beta} \right],
$$

(4.1)

$$
\tilde{D}^{(i)}_{(i)j} = \left[ G^{(i)(\mu)}_{(i)(m)} x^m_{\mu} \right]_{j},
$$

$$
\tilde{D}^{(i)(\beta)}_{(i)(j)} = \left[ G^{(i)(\mu)}_{(i)(m)} x^m_{\beta} \right]_{(j)},
$$

10
where \( G^{(\alpha)(\beta)}_{(i)(j)} = h^{\alpha\beta} e^{2\sigma} \varphi_{ij} \) is the vertical fundamental metrical d-tensor of \( GRGML^n_p \) and \( "_{(j)}^\beta \) are the local covariant derivatives of \( CT \).

Taking into account the expressions of the local covariant derivatives of the Cartan canonical connection \( CT \), we find

**Proposition 4.1** The metrical deflection d-tensors of the space \( GRGML^n_p \) are given by the following formulas,

\[
\begin{align*}
\bar{D}^{(\alpha)}_{(ij)} & = e^{2\sigma} h^{\alpha\mu} \varphi_{im} \sigma_{\beta} x^m_{\mu}, \\
D^{(\alpha)}_{(ij)} & = -e^{2\sigma} h^{\alpha\mu} [\sigma_{j} \varphi_{im} - \sigma_{i} \varphi_{jm} + \sigma_{m} \varphi_{ij}] x^m_{\mu}, \\
\bar{d}^{(\alpha)(\beta)}_{(ij)} & = e^{2\sigma} \left[ h^{\alpha\beta} \varphi_{ij} + h^{\alpha\mu} \left( \sigma_{(j)}^{(\beta)} \varphi_{im} - \sigma_{(i)}^{(\beta)} \varphi_{jm} + \sigma_{(m)}^{(\beta)} \varphi_{ij} \right) x^m_{\mu} \right].
\end{align*}
\]

\[
(4.2)
\]

**Definition 4.1** The distinguished 2-form on \( J^1(T, M) \),

\[
F = F^{(\alpha)}_{(i)} \delta x^i_{\alpha} \wedge dx^j + \bar{f}^{(\alpha)(\beta)}_{(i)(j)} \delta x^i_{\alpha} \wedge \delta x^j_{\beta},
\]

where \( F^{(\alpha)}_{(i)} = \frac{1}{2} \left[ D^{(\alpha)}_{(i)} - D^{(\alpha)}_{(j)} \right] \) and \( \bar{f}^{(\alpha)(\beta)}_{(i)(j)} = \frac{1}{2} \left[ \bar{d}^{(\alpha)(\beta)}_{(i)(j)} - \bar{d}^{(\alpha)(\beta)}_{(j)(i)} \right] \), is called the distinguished electromagnetic 2-form of the generalized metrical multi-time Lagrange space \( GRGML^n_p \).

**Proposition 4.2** The local electromagnetic d-tensors of \( GRGML^n_p \) have the expressions,

\[
\begin{align*}
F^{(\alpha)}_{(i)} & = e^{2\sigma} h^{\alpha\mu} [\varphi_{jm} \sigma_{i} - \varphi_{im} \sigma_{j}] x^m_{\mu}, \\
\bar{f}^{(\alpha)(\beta)}_{(i)(j)} & = e^{2\sigma} h^{\alpha\mu} [\varphi_{im} \sigma_{(j)}^{(\beta)} - \varphi_{jm} \sigma_{(i)}^{(\beta)}] x^m_{\mu}.
\end{align*}
\]

**Theorem 4.3** The electromagnetic components \( F^{(\alpha)}_{(i)} \) and \( \bar{f}^{(\alpha)(\beta)}_{(i)(j)} \) of the generalized metrical multi-time Lagrange space \( GRGML^n_p \) are governed by the Maxwell equations:

\[
\begin{align*}
F^{(\alpha)}_{(i)(k)\beta} & = F^{(\alpha)}_{(i)(k)\beta} + x^{(\alpha)}_{(i)(k)} \sigma_{\beta} - x^{(\alpha)}_{(k)(i)} \sigma_{\beta}, \\
\bar{f}^{(\alpha)(\gamma)}_{(i)(k)\beta} & = 2 \bar{f}^{(\alpha)(\gamma)}_{(i)(k)\beta} + x^{(\alpha)}_{(i)(k)\beta} - x^{(\alpha)}_{(k)(i)\beta} \sigma^{(\gamma)}_{(i)\beta}, \\
\sum_{i,j,k} F^{(\alpha)}_{(i)(j)k} & = - \sum_{i,j,k} x^{(\alpha)}_{(i)(j)k} \sigma^{(\gamma)}_{mjk} x^m_{\mu}, \\
\sum_{i,j,k} \left\{ F^{(\alpha)(\gamma)}_{(i)(j)k} + \bar{f}^{(\alpha)(\gamma)}_{(i)(j)k} \right\} & = 0, \\
\sum_{i,j,k} f^{(\alpha)(\beta)(\gamma)}_{(i)(j)k} & = 0,
\end{align*}
\]

\[
(4.5)
\]

where \( x^{(\alpha)}_{(i)} = e^{2\sigma} h^{\alpha\mu} \varphi_{im} x^m_{\mu} \).
Proof. By a direct calculation, the following tensorial identities,

\[ D^{(\alpha)}_{(i)\beta} - x^{(\alpha)}_{(m)} T^m_{\beta i} = 2 x^{(\alpha)}_{(i)} \sigma_{\beta}, \]

\[ d^{(\alpha)(\beta)}_{(i)(j)} + x^{(\alpha)}_{(m)} C^m_{i(j)} = e^{2\sigma} \mathcal{H}_{\alpha\beta} \varphi_{ij} + 2 x^{(\alpha)}_{(i)} \sigma^{(\beta)}_{(j)}, \]

hold good. The identities (4.6) together with the general expressions of Maxwell equations for a generalized metrical multi-time Lagrange space, imply what we were looking for. ■

Remark 4.1 If the conformal function \( \sigma(t^\gamma, x^k, x^k) \) is of the form \( \sigma(x^k) \), then \( f^{(\alpha)(\beta)}_{(i)(j)} \) vanish, and the Maxwell equations of GRGML reduce to

\[
\begin{align*}
F^{(\alpha)}_{(i)k/\beta} &= F^{(\alpha)}_{(i)k} \sigma_{\beta} + x^{(\alpha)}_{(i)} \sigma_{\beta} |_{k} - x^{(\alpha)}_{(k)} \sigma_{\beta} |_{i} \\
\sum_{(i,j,k)} F^{(\alpha)}_{(i)j|k} &= 0 \\
\sum_{(i,j,k)} F^{(\alpha)}_{(i)j}(\gamma) &= 0.
\end{align*}
\]

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