GRAM MATRIX IN INNER PRODUCT MODULES OVER 
$C^*$-ALGEBRAS

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Abstract. Let $(X, \langle \cdot, \cdot \rangle)$ be a semi-inner product module over a $C^*$-algebra $\mathcal{A}$. It is known that, for an arbitrary $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, the Gram matrix $[\langle x_i, x_j \rangle]$ is a positive element of the matrix algebra $M_n(\mathcal{A})$. We show, by defining a suitable new semi-inner product on $X$, that a stronger inequality, namely $[\langle x_i, x_j \rangle] \geq \frac{1}{\|z\|^2} [\langle x_i, z \rangle \langle z, x_j \rangle]$, holds true for all $z \in X$ such that $\langle z, z \rangle \neq 0$. As an application, we obtain an improvement of the Ostrowski inequality and a generalization of the covariance–variance inequality. By the same technique, we show that the inequality $[\langle x_i, x_j \rangle] \geq \frac{1}{\|z\|^2} [\langle x_i, z \rangle \langle z, x_j \rangle]$ can be refined by a sequence of nested inequalities. The paper ends with some operator-theoretical consequences, including an effective algorithm for inverting a positive invertible matrix.

1. Introduction and preliminaries

In this paper we study the Gram (or Gramian) matrix $[\langle x_i, x_j \rangle] \in M_n(\mathcal{A})$, where $x_1, \ldots, x_n$ are arbitrary elements in a semi-inner product module $(X, \langle \cdot, \cdot \rangle)$ over a $C^*$-algebra $\mathcal{A}$. If $X$ is an inner-product space (i.e. if $\mathcal{A} = \mathbb{C}$) this matrix becomes exactly the classical Gram matrix of $n$ vectors.

It is known that the matrix $[\langle x_i, x_j \rangle]$ is a positive element of the $C^*$-algebra $M_n(\mathcal{A})$ (see [8, Lemma 4.2]). Considering $X$ as a semi-inner product $C^*$-module with respect to another semi-inner product, we get a new inequality which generally improves the initial one.

Some applications are also considered. First, we improve the Ostrowski inequality for elements of an inner-product $C^*$-module which is recently obtained in [1]. Then, we define the covariance $\text{cov}_z(x, y)$ between $x$ and $y$ with respect to $z$ and variance $\text{var}_z(x)$ of $x$ with respect to $z$ and prove the generalized covariance–variance inequality. In the concluding section we show that the initial inequality

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([(x_i, x_j)] \geq 0) can be refined by a sequence of nested inequalities. This, in turn, leads to some interesting operator-theoretical consequences. In particular, we construct, for an invertible positive Hilbert space operator \( a \), a sequence that converges in norm to \( a^{-1} \). When applied to a positive invertible matrix, this gives us an effective algorithm for computing \( a^{-1} \) in which the number of steps does not exceed the number of elements of the spectrum of \( a \).

Before stating the results, we recall the definition of a semi-inner product \( C^* \)-module and introduce our notation.

Let \( \mathcal{A} \) be a \( C^* \)-algebra. A (right) semi-inner product \( \mathcal{A} \)-module is a linear space \( X \) which is a right \( \mathcal{A} \)-module with a compatible scalar multiplication \((\lambda xa) = x(\lambda a) = (\lambda x)a \) for all \( x \in X, a \in \mathcal{A}, \lambda \in \mathbb{C} \) endowed with an \( \mathcal{A} \)-semi-inner product \( \langle \cdot, \cdot \rangle : X \times X \to \mathcal{A} \) such that for all \( x, y, z \in X, \lambda \in \mathbb{C}, a \in \mathcal{A} \), it holds

(i) \( \langle x, x \rangle \geq 0 \);
(ii) \( \langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle \);
(iii) \( \langle x, ya \rangle = \langle x, y \rangle a \);
(iv) \( \langle x, y \rangle^* = \langle y, x \rangle \).

Obviously, every semi-inner product space is a semi-inner product \( C \)-module. We can define a semi-norm on \( X \) by \( \| x \| = \| \langle x, x \rangle \|^{\frac{1}{2}} \), where the latter norm denotes that in the \( C^* \)-algebra \( \mathcal{A} \). A pre-Hilbert \( \mathcal{A} \)-module (or an inner-product module) is a semi-inner product module over \( \mathcal{A} \) in which \( \| \cdot \| \) defined as above is a norm. If this norm is complete then \( X \) is called a Hilbert \( C^* \)-module. Each \( C^* \)-algebra \( \mathcal{A} \) can be regarded as a Hilbert \( \mathcal{A} \)-module via \( \langle a, b \rangle = a^*b \) \( (a, b \in \mathcal{A}) \). When \( X \) is a Hilbert \( C^* \)-module, we denote by \( \mathcal{B}(X) \) the algebra of all adjointable operators on \( X \).

An element \( a \) of a \( C^* \)-algebra \( \mathcal{A} \) is called positive (we write \( a \geq 0 \)) if \( a = x^*x \) for some \( x \in \mathcal{A} \). If \( a \in \mathcal{A} \) is positive, then there is a unique positive \( b \in \mathcal{A} \) such that \( a = b^2 \); such an element \( b \) is called the positive square root of \( a \) and denoted by \( a^{1/2} \). For every \( a \in \mathcal{A} \), the positive square root of \( a^*a \) is denoted by \( |a| \). For two self-adjoint elements \( a, b \), one can define a partial order \( \leq \) by \( a \leq b \iff b - a \geq 0 \). For \( n \in \mathbb{N} \), \( M_n(\mathcal{A}) \) denotes the matrix \( C^* \)-algebra of all \( n \times n \) matrices with entries from \( \mathcal{A} \). For more details on matrix algebras we refer the reader to [15, 19].
Throughout the paper, $\mathcal{A}$ stands for the minimal unitization of $\mathcal{A}$. If $\mathcal{A}$ is unital then $\mathcal{A} = A$ and if $\mathcal{A}$ is non-unital then $\mathcal{A} = A \oplus \mathbb{C}$ with $(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda \mu)$ and $(a, \lambda)^* = (a^*, \overline{\lambda})$. By $e$ we denote the unit in $\mathcal{A}$. If $X$ is an $\mathcal{A}$-module then it can be regarded as an $\mathcal{A}$-module via $xe = x$.

For every $x \in X$ the absolute value of $x$ is defined as the unique positive square root of the positive element $\langle x, x \rangle$ of $A$, that is, $|x| = \langle x, x \rangle^{1/2}$. Some standard references for $C^*$-algebras and $C^*$-modules are [8, 12, 16, 19].

2. The Gram matrix

We begin with a comparison of positivity of the Gram matrix for two elements of a semi-inner product $\mathcal{A}$-module $\mathcal{X}$ with the Cauchy–Schwarz inequality

$$\langle x, y \rangle \langle y, x \rangle \leq \|y\|^2 \langle x, x \rangle. \quad (2.1)$$

Let us write (2.1) in a matrix form. First recall that a matrix $\begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \in M_2(\mathcal{A})$ with invertible $c \in \mathcal{A}$ (resp. $a \in \mathcal{A}$) is positive if and only if $a \geq 0, c \geq 0$ and $bc^{-1}b^* \leq a$ (resp. $a \geq 0, c \geq 0$ and $b^*a^{-1}b \leq c$); see [3]. Therefore, we can write (2.1) as

$$\begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle x, y \rangle^* & \|\langle y, y \rangle\|e \end{bmatrix} \geq 0,$$  

(2.2)

where $e \in \mathcal{A}$ is the unit. Since

$$\begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle x, y \rangle^* & \|\langle y, y \rangle\|e \end{bmatrix} \geq \begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle x, y \rangle^* & \langle y, y \rangle \end{bmatrix} \geq 0,$$

it follows that positivity of the Gram matrix sharpens the Cauchy–Schwarz inequality.

A number of arguments can be simplified if we use positivity of the Gram matrix. For example, it was proved in [7, Theorem 2.1] that for $x, y \in \mathcal{X}$ such that $|y|$ belongs to the center of $\mathcal{A}$, a stronger version of the Cauchy–Schwarz inequality holds, namely, $\langle x, y \rangle \langle y, x \rangle \leq \langle x, x \rangle \langle y, y \rangle$. From positivity of the Gram matrix it follows that for every $x, y \in \mathcal{X}$ and every $\epsilon > 0$ we have

$$\begin{bmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle x, y \rangle^* & \langle y, y \rangle + \epsilon e \end{bmatrix} \geq 0,$$
Proof. Let us first prove that \( \langle x, y \rangle (\langle y, y \rangle + \varepsilon e)^{-1} \langle y, x \rangle \leq \langle x, x \rangle \). If \( |y| \) belongs to the center of \( \mathcal{A} \), we get, by multiplying by \((\langle y, y \rangle + \varepsilon e)^{\frac{1}{2}}\) on both sides, \( \langle x, y \rangle \langle y, x \rangle \leq \langle x, x \rangle (\langle y, y \rangle + \varepsilon e) \). Since \( \varepsilon > 0 \) is arbitrary, we have \( \langle x, y \rangle \langle y, x \rangle \leq \langle x, x \rangle \langle y, y \rangle \).

An interesting fact about the inequality \( [\langle x_i, x_j \rangle] \geq 0 \) is that it is self-improving. We use this property in the proof of the following theorem.

**Theorem 2.1.** Let \( \mathcal{A} \) be a \( C^* \)-algebra and \((\mathcal{X}, \langle \cdot, \cdot \rangle)\) a semi-inner product \( \mathcal{A} \)-module. Let \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in \mathcal{X} \). Then for every \( z \in \mathcal{X} \) we have

\[
\|z\|^2 \left[ \langle x_i, x_j \rangle \right] \geq \left[ \langle x_i, z \rangle \langle z, x_j \rangle \right].
\]

(2.3)

**Proof.** Let us first prove that \([\langle x_i, x_j \rangle]\) is positive in \( M_n(\mathcal{A}) \) (the proof is included for the convenience of the reader, see [8, Lemma 4.2]). Since \( \langle \cdot, \cdot \rangle \) is a semi-inner product on \( \mathcal{X} \) it holds that

\[
\left\langle \sum_{i=1}^{n} x_i a_i, \sum_{i=1}^{n} x_i a_i \right\rangle \geq 0, \quad (a_1, \ldots, a_n \in \mathcal{A}).
\]

Then

\[
\sum_{i,j=1}^{n} a_i^* a_j \langle x_i, x_j \rangle a_j \geq 0, \quad (a_1, \ldots, a_n \in \mathcal{A}).
\]

(2.4)

By [18, Lemma IV.3.2] we know that a matrix \([c_{ij}] \in M_n(\mathcal{A})\) is positive if and only if \( \sum_{i,j=1}^{n} a_i^* c_{ij} a_j \geq 0 \) for all \( a_1, \ldots, a_n \in \mathcal{A} \). Therefore, (2.4) means that the matrix \([\langle x_i, x_j \rangle]\) is positive.

For an arbitrary \( z \in \mathcal{X} \) we define

\[
\langle \cdot, \cdot \rangle_z : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{A}, \quad \langle x, y \rangle_z := \|z\|^2 \langle x, y \rangle - \langle x, z \rangle \langle z, y \rangle.
\]

(2.5)

Observe that \( \langle x, y_1 + \lambda y_2 \rangle_z = \langle x, y_1 \rangle_z + \lambda \langle x, y_2 \rangle_z, \langle x, y a \rangle_z = \langle x, y \rangle_z a \) and \( \langle y, x \rangle_z^* = \langle y, x \rangle_z \), for all \( a \in \mathcal{A}, \lambda \in \mathbb{C}, x, y, y_1, y_2 \in \mathcal{X} \). Furthermore, by the Cauchy–Schwarz inequality

\[
\langle x, z \rangle \langle z, x \rangle \leq \|z\|^2 \langle x, x \rangle \quad (x \in \mathcal{X})
\]

(2.6)

we have \( \langle x, x \rangle_z \geq 0 \) for all \( x \in \mathcal{X} \), so \( \langle \cdot, \cdot \rangle_z \) is another semi-inner product on \( \mathcal{X} \). Therefore, \([\langle x_i, x_j \rangle_z]\) \geq 0, which is exactly (2.3). \( \square \)

We first state a few direct consequences of the preceding theorem.

**Corollary 2.2.** Let \( \mathcal{X} \) be a Hilbert \( \mathcal{A} \)-module and \( A \in \mathbb{B}(\mathcal{X}) \) a positive map. Then for all \( x_1, \ldots, x_n, z \in \mathcal{X} \)

\[
\|A^\frac{1}{2} z\|^2 \left[ \langle Ax_i, x_j \rangle \right] \geq \left[ \langle Ax_i, z \rangle \langle z, Ax_j \rangle \right] \geq 0.
\]
Proof. Immediate from (2.3) if we replace \(x_1, \cdots, x_n, z\) by \(A^{\frac{1}{2}}x_1, \cdots, A^{\frac{1}{2}}x_n, A^{\frac{1}{2}}z\).

Let \(\mathcal{B}\) be a \(C^*\)-subalgebra of \(\mathcal{A}\). A positive linear mapping \(\Phi : \mathcal{A} \to \mathcal{B}\) is called a left multiplier if \(\Phi(ab) = \Phi(a)b\) \((a \in \mathcal{A}, b \in \mathcal{B})\). Then any semi-inner product \(\mathcal{A}\)-module \(X\) becomes a semi-inner product \(\mathcal{B}\)-module with respect to
\[
[x, y]_\Phi = \Phi(\langle x, y \rangle), \quad (x, y \in \mathcal{X}).
\] (2.7)

By (2.3), it holds
\[
\|\Phi(\langle z, z \rangle)\|\|\Phi(\langle x_i, x_j \rangle)\| \geq \|\Phi(\langle x_i, z \rangle)\Phi(\langle z, x_j \rangle)\|.
\] (2.8)

Thus we get

**Corollary 2.3.** Let \((\mathcal{X}, \langle \cdot, \cdot \rangle)\) be a semi-inner product \(\mathcal{A}\)-module, \(\mathcal{B}\) a \(C^*\)-subalgebra of \(\mathcal{A}\) and \(\Phi : \mathcal{A} \to \mathcal{B}\) a positive left multiplier. Then
\[
\|\Phi(\langle z, z \rangle)\|\|\Phi(\langle x_i, x_j \rangle)\| \geq \|\Phi(\langle x_i, z \rangle)\Phi(\langle z, x_j \rangle)\|
\] (2.8)
for all \(x_1, \cdots, x_n, z \in \mathcal{X}\).

By choosing special \(\mathcal{X}\) we obtain some known results (see [2, 9, 13]). In our first corollary \(\mathcal{X}\) is a \(C^*\)-algebra regarded as a Hilbert \(C^*\)-module over itself. Since every conditional expectation \(\Phi : \mathcal{A} \to \mathcal{B}\) is a completely positive left multiplier (cf. [18, IV, §3]), the following corollary is an extension of [2, Theorem 1] for conditional expectations.

**Corollary 2.4.** Let \(\mathcal{B}\) be a \(C^*\)-subalgebra of a \(C^*\)-algebra \(\mathcal{A}\) and \(\Phi : \mathcal{A} \to \mathcal{B}\) a positive left multiplier. Then
\[
\|\Phi(c^*c)\|\|\Phi(a_i^*a_j)\| \geq \|\Phi(a_i^*c)\Phi(c^*a_j)\|
\] (2.9)
for all \(a_1, \cdots, a_n, c \in \mathcal{A}\).

In the next corollary \(\mathcal{X}\) is the space \(\mathbb{B}(\mathcal{H}, \mathcal{K})\) of all bounded linear operators between Hilbert spaces regarded as a Hilbert \(C^*\)-module over \(\mathcal{A} = \mathbb{B}(\mathcal{H})\) via \(\langle A, B \rangle = A^*B\).

**Corollary 2.5.** Let \(\mathcal{B}\) be a \(C^*\)-subalgebra of \(\mathbb{B}(\mathcal{H})\) and \(\Phi : \mathbb{B}(\mathcal{H}) \to \mathcal{B}\) a positive left multiplier. Then
\[
\|\Phi(C^*C)\|\|\Phi(A_i^*A_j)\| \geq \|\Phi(A_i^*C)\Phi(C^*A_j)\|
\] is valid for all \(A_1, \cdots, A_n, C \in \mathbb{B}(\mathcal{H}, \mathcal{K})\).
In the case of a unital $C^*$-algebra $\mathcal{A}$ with the unit $e$, another example of a positive left multiplier is a positive linear functional, in particular a vector state $\varphi(a) = \langle au, u \rangle$ corresponding to a unit vector $u \in \mathcal{H}$ on which $\mathcal{A}$ acts.

Let us now discuss some applications of Theorem 2.1.

2.1. An Ostrowski type inequality. Here we show that some inequalities of Ostrowski type can be viewed as the Cauchy–Schwarz inequality with respect to a new semi-inner product.

It was proved in [11] that for any three elements in a real inner product space $(H, (\cdot | \cdot))$ it holds

$$
\left| \left| z \right|^2 (x|y) - (x|z)(y|z) \right|^2 \leq \left( \left| z \right|^2 \left| x \right|^2 - (x|z)^2 \right) \left( \left| z \right|^2 \left| y \right|^2 - (y|z)^2 \right).
$$

Since here $H$ is a real vector space, this may be written as

$$
\left| (x|y) \right|^2 \leq (x|x)_z (y|y)_z
$$

and this is exactly the Cauchy–Schwarz inequality for $(\cdot | \cdot)_z$. Therefore, the Cauchy–Schwarz inequality for a semi-inner product $(\cdot, \cdot)_z$ on a semi-inner product module $\mathcal{X}$, i.e.

$$
(\left| z \right|^2 \langle x, x \rangle - \langle y, z \rangle \langle z, x \rangle)(\left| z \right|^2 \langle x, y \rangle - \langle x, z \rangle \langle z, y \rangle) \\
\leq \left| z \right|^2 \langle x, x \rangle - \langle y, z \rangle \langle z, x \rangle \left( \left| z \right|^2 \langle y, y \rangle - \langle y, z \rangle \langle z, y \rangle \right) \tag{2.10}
$$

generalizes the result from [11]. In the special case when $\langle x, z \rangle = 0$ we get

$$
\left| \langle z, y \rangle \right|^2 \leq \frac{\left| z \right|^2}{\left| x \right|^2} (\left| x \right|^2 \left| y \right|^2 - \left| \langle x, y \rangle \right|^2), \tag{2.11}
$$

which is the Ostrowski inequality in a semi-inner product $C^*$-module (see [1]). Since (2.10) is the Cauchy–Schwarz inequality and (2.11) is its special case, Theorem 2.1 improves both of them. Namely,

$$
\begin{bmatrix}
    \langle x, x \rangle_z & \langle x, y \rangle_z \\
    \langle x, y \rangle_z^* & \langle y, y \rangle_z
\end{bmatrix} \geq 0,
$$

(which is exactly (2.3) for $n = 2$) improves (2.10), and it improves (2.11) in the case $\langle x, z \rangle = 0$. 
2.2. A Covariance–variance inequality. The next application of Theorem 2.1 is the covariance–variance inequality in semi-inner product $C^*$-modules. The interested reader is referred to [2, 5, 10, 13] for some generalizations of covariance–variance inequality. Let us begin with a definition and some known examples.

**Definition 2.6.** Let $\mathcal{A}$ be a $C^*$-algebra, $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be a semi-inner product $\mathcal{A}$-module and $x, y, z \in \mathcal{X}$. The covariance $\text{cov}_z(x, y)$ between $x$ and $y$ with respect to $z$ is defined to be the element $\langle x, y \rangle_z$ of $\mathcal{A}$. The element $\text{cov}_z(x, x)$ is said to be the variance of $x$ with respect to $z$ and denoted by $\text{var}_z(x)$.

**Example 2.7.** Given a Hilbert space $\mathcal{H}$, vectors $x, y \in \mathcal{H}$ and operators $S, T \in \mathcal{B}(\mathcal{H})$, covariance and variance of operators was defined in [10] as

$$\text{cov}_{x,y}(S, T) = \|y\|^2(Sx|Tx) - (Sx|y)(y|Tx).$$

Observe that $\text{cov}_{x,y}(S, T) = (Sx|Tx)y$. In the case where $\|x\| = 1$ and $y = x$ we get the notion of covariance of two operators $T$ and $S$ introduced in [5] as

$$\text{cov}_x(S, T) = (Sx|Tx) - (Sx|x)(x|Tx).$$

A notion of covariance and variance of Hilbert space operators was investigated in [5, 17]. In addition, Enomoto [4] showed a close relation of the operator covariance–variance inequality with the Heisenberg uncertainty principle and pointed out that it is exactly the generalized Schrödinger inequality. It is notable that several mathematicians who have been toying with such ideas ever since von Neumann provided the setting of Hilbert space operators for quantum theory and self-adjoint operators got to be considered as non-commutative random variables (or observables), see [14, Section 5] for more information.

Another remarkable fact is that for a unit vector $x \in \mathcal{H}$, the determinant of the positive semidefinite Gram matrix

$$\begin{pmatrix}
(Sx|Sx) & (Sx|Tx) & (Sx|x) \\
(Tx|Sx) & (Tx|Tx) & (Tx|x) \\
(x|Sx) & (x|Tx) & (x|x)
\end{pmatrix}
$$

is the difference $\text{var}_x(S)\text{var}_x(T) - |\text{cov}_x(S, T)|^2$ and is nonnegative; see [6].

**Example 2.8.** Recall that if $(\Omega, \mu)$ is a probability measure space, then $Ef = \int_{\Omega} f d\mu$ is the expectation of the random variable $f \in L^2(\Omega, \mu)$. Then the covariance between $f$ and $g$ is defined to be $\text{cov}(f, g) = E(fg) - EFf Eg$ and variance of
$f$ is $\text{cov}(f, f)$. We can obtain this by considering $L^2(\Omega, \mu)$ as a Hilbert $\mathbb{C}$-module via the usual inner product $\langle f, g \rangle = \int_\Omega \overline{f}g$.

Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{X}$ be a semi-inner product $\mathcal{A}$-module. Cauchy–Schwarz inequality for $\text{cov}(\cdot, \cdot)$ is known as the covariance–variance inequality. Therefore, Theorem 2.1 can be also stated in the following form.

**Theorem 2.9** (Generalized Covariance-Variance Inequality). Let $\mathcal{A}$ be a $C^*$-algebra and $\mathcal{X}$ be a semi-inner product $\mathcal{A}$-module. Let $x_1, \ldots, x_n, z \in \mathcal{X}$. Then the matrix $[\text{cov}(x_i, x_j)] \in M_n(\mathcal{A})$ is positive.

Assume that $\mathcal{A}$ is a $C^*$-algebra acting on a Hilbert space, $\mathcal{B}$ is one of its $C^*$-subalgebras and $\mathcal{X}$ is a semi-inner product $\mathcal{A}$-module. If we fix a unit vector $x \in \mathcal{X}$ and a positive left multiplier mapping $\Phi$ and take operators $A$ and $B$ in $\mathcal{B}(\mathcal{X})$, then we could define the covariance of $A, B$ and variance of $A$ by

$$\text{cov}(A, B) = \Phi(\langle Ax, Bx \rangle) - \Phi(\langle Ax, x \rangle)\Phi(\langle x, Bx \rangle)$$

and $\text{var}(A) = \text{cov}(A, A)$, respectively; see [4]. Observe that, if we regard $X$ as a semi-inner product $\mathcal{A}$-module with respect to $[\cdot, \cdot]_{\Phi}$ defined by (2.7), then we have $\text{cov}(A, B) = \text{cov}_x(Ax, Bx)$. Therefore,

$$\begin{bmatrix}
\text{var}(A) & \text{cov}(A, B) \\
\text{cov}(A, B)^* & \text{var}(B)
\end{bmatrix} = \begin{bmatrix}
\text{var}_x(Ax) & \text{cov}_x(Ax, Bx) \\
\text{cov}_x(Ax, Bx)^* & \text{var}_x(Bx)
\end{bmatrix} \geq 0.$$

### 3. Improving the Inequality $[\langle x_i, x_j \rangle] \geq 0$

Let $\mathcal{A}$ be a $C^*$-algebra and $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ a semi-inner product $\mathcal{A}$-module. We have proved in Proposition 2.1 that

$$\|z\|^2[\langle x_i, x_j \rangle] \geq [\langle x_i, z \rangle \langle z, x_j \rangle],$$

for any $n \in \mathbb{N}$ and $z, x_1, \ldots, x_n \in \mathcal{X}$. As we already noticed, this inequality emerged as a self-improving property of positivity of the Gram matrix. In this section we show that this self-improving property extends even further. More precisely, we will obtain a sequence of inequalities nested between $\|z\|^2[\langle x_i, x_j \rangle]$ and $[\langle x_i, z \rangle \langle z, x_j \rangle]$. Let us first fix some notation.
Let $\mathcal{A}$ be a $C^*$-algebra with the unit $e$. For a positive element $a \in \mathcal{A}$, $a \not= 0$, define

$$f_0(a) = a, \quad g_0(a) = \|f_0(a)\|e - f_0(a),$$

$$f_1(a) = f_0(a)g_0(a), \quad g_1(a) = \|f_1(a)\|e - f_1(a),$$

$$\ldots$$

$$f_m(a) = f_{m-1}(a)g_{m-1}(a), \quad g_m(a) = \|f_m(a)\|e - f_m(a),$$

$$\ldots$$

(3.1)

Observe that all $f_m(a)$ and $g_m(a)$ are polynomials in $a$. An easy inductive argument shows that all $f_m(a)$ and $g_m(a)$ are positive elements as well and for all $m \geq 0$ it holds

$$f_{m+1}(a) = f_m(f_1(a)), \quad g_{m+1}(a) = g_m(f_1(a)).$$

(3.2)

It may happen that $f_m(a) = 0$ for some $m \in \mathbb{N}$ (see Proposition 3.3 below); then, obviously, $f_k(a) = 0$ for all $k \geq m$. On the other hand, if $f_m(a) \not= 0$ for some $m$, then, by definition, $f_j(a) \not= 0$, $\forall j \leq m$. Thus, for each $m$ such that $f_m(a) \not= 0$ we can define

$$p_0(a) = \frac{e}{\|f_0(a)\|},$$

$$p_1(a) = \frac{e}{\|f_0(a)\|} + \frac{g_0(a)^2}{\|f_0(a)\|\|f_1(a)\|},$$

$$p_2(a) = \frac{e}{\|f_0(a)\|} + \frac{g_0(a)^2}{\|f_0(a)\|\|f_1(a)\|} + \frac{g_0(a)^2g_1(a)^2}{\|f_0(a)\|\|f_1(a)\|\|f_2(a)\|},$$

$$\ldots$$

$$p_m(a) = \frac{e}{\|f_0(a)\|} + \sum_{i=1}^m \left( \frac{1}{\prod_{k=0}^{i-1} \|f_k(a)\|} \prod_{k=0}^{i-1} g_k(a)^2 \right).$$

(3.3)

It is convenient here to make the following convention: if $m$ is the last index such that $f_m(a) \not= 0$ then we define

$$p_j(a) = p_m(a), \quad (j > m).$$

(3.4)

Thus, we can treat $(p_m(a))$ as an infinite sequence of positive elements in $\mathcal{A}$ even in the case when there is $m \geq 0$ such that $f_m(a) = 0$.

It is obvious that $0 \leq p_0(a) \leq p_1(a) \leq \ldots \leq p_m(a) \leq \ldots$. Further, observe also that $p_m(a)$’s which are defined by (3.3) are all different. Indeed, suppose $p_{m-1}(a)$ and $p_m(a)$ are defined by (3.3) and $p_{m-1}(a) = p_m(a)$. Then

$$\frac{1}{\prod_{k=0}^{m-1} \|f_k(a)\|} \prod_{k=0}^{m-1} g_k(a)^2 = 0$$
which implies $g_0(a)g_1(a) \cdots g_{m-1}(a) = 0$ and therefore
\[
\begin{align*}
  f_m(a) &= f_{m-1}(a)g_{m-1}(a) = f_{m-2}(a)g_{m-2}(a)g_{m-1}(a) \\
  &= \ldots = f_0(a)g_0(a) \cdots g_{m-1}(a) = 0.
\end{align*}
\]
This is the contradiction, since $p_m(a)$ is defined by (3.3).

**Theorem 3.1.** Let $X$ be a module over a $C^*$-algebra $A$, and let $\langle \cdot, \cdot \rangle$ be any $A$-valued semi-inner product on $X$. Let $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$. For each $z \in X$ such that $\langle z, z \rangle \neq 0$ it holds
\[
\langle x_i, x_j \rangle \geq \ldots \geq \langle x_i, z \rangle p_m(\langle z, z \rangle) \langle z, x_j \rangle \geq \ldots \geq \langle x_i, z \rangle p_0(\langle z, z \rangle) \langle z, x_j \rangle = \frac{1}{\|z\|^2} \langle x_i, z \rangle \langle z, x_j \rangle \geq 0.
\]

**Proof.** We will prove by induction that
\[
\begin{align*}
  \langle x_i, x_j \rangle &\geq \langle x_i, z \rangle p_m(\langle z, z \rangle) \langle z, x_j \rangle \quad (3.5)
\end{align*}
\]
holds true for all $m \geq 0$, for each $z \in X$ and for every $A$-valued semi-inner product $\langle \cdot, \cdot \rangle$ on $X$ such that $\langle z, z \rangle \neq 0$.

For $m = 0$ this is precisely the statement of Theorem 2.1.

Suppose that (3.5) holds for some $m$ and for all $z$ and $\langle \cdot, \cdot \rangle$ such that $\langle z, z \rangle \neq 0$. Choose an arbitrary semi-inner product $\langle \cdot, \cdot \rangle$ on $X$ such that $\langle z, z \rangle \neq 0$. If $f_{m+1}(\langle z, z \rangle) = 0$, there is nothing to prove since then, by our convention, $p_{m+1}(\langle z, z \rangle) = p_m(\langle z, z \rangle)$.

Suppose now that $f_{m+1}(\langle z, z \rangle) \neq 0$. Then, by (3.2), $f_{m}(f_1(\langle z, z \rangle)) = f_{m}(\langle z, z \rangle) \neq 0$. By the inductive assumption (for the semi-inner product $\langle \cdot, \cdot \rangle$) it holds
\[
\begin{align*}
  \langle x_i, x_j \rangle &\geq \langle x_i, z \rangle p_m(\langle z, z \rangle) \langle z, x_j \rangle,
\end{align*}
\]
that is,
\[
\begin{align*}
  \|z\|^2 \langle x_i, x_j \rangle &\geq \langle x_i, z \rangle \langle z, x_j \rangle + \langle x_i, z \rangle p_m(\langle z, z \rangle) \langle z, x_j \rangle \quad (3.6)
\end{align*}
\]
Observe that $\langle z, z \rangle = f_1(\langle z, z \rangle)$, so $\|z\|^2 = \langle z, z \rangle$ and $p_m(\langle z, z \rangle)$ commute. Therefore
\[
\begin{align*}
  \langle x_i, z \rangle p_m(\langle z, z \rangle) \langle z, x_j \rangle &= (\|z\|^2 \langle x_i, z \rangle - \langle x_i, z \rangle \langle z, z \rangle) p_m(\langle z, z \rangle) \\
  &= (\|z\|^2 \langle z, x_j \rangle - \langle z, z \rangle \langle z, x_j \rangle) \\
  &= \langle x_i, z \rangle (\|z\|^2 - \langle z, z \rangle) p_m(\langle z, z \rangle) \langle z, x_j \rangle \\
  &= \langle x_i, z \rangle (g_0(\langle z, z \rangle)^2 p_m(f_1(\langle z, z \rangle)) \langle z, x_j \rangle.
\end{align*}
\]
Since \( f_{m+1}(\langle z, z \rangle) \neq 0 \) and \( f_m(\langle z, z \rangle) \neq 0 \), the elements \( p_{m+1}(\langle z, z \rangle) \) and \( p_m(\langle z, z \rangle) \) are defined by (3.3) (not by (3.4)). It is easy to verify, using (3.1) and (3.2), that

\[
e + g_0(\langle z, z \rangle)^2p_m(f_1(\langle z, z \rangle)) = \|z\|^2p_{m+1}(\langle z, z \rangle),
\]

which, together with (3.6), gives \([\langle x_i, x_j \rangle] \geq [\langle x_i, z \rangle p_{m+1}(\langle z, z \rangle) \langle z, x_j \rangle]\).

To complete the proof it only remains to recall that \( p_m(\langle z, z \rangle) \geq p_m(\langle z, z \rangle) \geq \ldots \geq p_0(\langle z, z \rangle) \geq 0 \), for every \( z \in \mathcal{B} \) such that \( f_m(\langle z, z \rangle) \neq 0 \).

**Remark 3.2.** It was tacitly assumed in the preceding proof that the underlying \( C^* \)-algebra has the unit element \( e \). If \( \mathcal{A} \) is a non-unital algebra, the same proof applies by working in the minimal unitization \( \widetilde{\mathcal{A}} \). However, although the unit element \( e \in \widetilde{\mathcal{A}} \) appears in the expressions \( p_m(a) \), inequality (3.5) involves only elements from the original \( C^* \)-algebra \( \mathcal{A} \).

Observe that, if \( b \in \mathcal{A} \) is positive and such that \( \|zb^\frac{1}{2}\| \leq \|z\| \), then, by Theorem 2.1, we have

\[
\|z\|^2[\langle x_i, x_j \rangle] \geq \|zb^\frac{1}{2}\|^2[\langle x_i, x_j \rangle] \geq [\langle x_i, zb^\frac{1}{2}\rangle \langle zb^\frac{1}{2}, x_j \rangle] = [\langle x_i, z \rangle b \langle z, x_j \rangle].
\]

Thus, the inequalities from Theorem 3.1 can alternatively be derived from the inequality \( \|zp_m(\langle z, z \rangle)^\frac{1}{2}\| \leq \|z\| \). Instead of proving this inequality directly, we opted for the inductive approach from the proof of Theorem 3.1 since it leads naturally to the sequence \( (f_m) \) and gives us more insight into the sequence \( p_m(a) \) (which, as we will see below, has many interesting properties).

If \( \langle z, z \rangle \) is not a scalar multiple of the unit, then \( f_1(\langle z, z \rangle) \neq 0 \) and the preceding theorem strictly refines the inequality from Theorem 2.1. Moreover, if \( f_m(\langle z, z \rangle) \neq 0 \) for all \( m \in \mathbb{N} \), Theorem 3.1 provides an infinite sequence of inequalities. On the other hand, if \( f_m(\langle z, z \rangle) = 0 \) for some \( m \geq 0 \), then, by (3.4), only finitely many inequalities are obtained. The following proposition characterizes all such elements \( z \in \mathcal{B} \). It turns out that the sequence of inequalities obtained in Theorem 3.1 is finite precisely when \( \langle z, z \rangle \) has a finite spectrum.

**Proposition 3.3.** Let \( a \) be a positive element of a \( C^* \)-algebra \( \mathcal{A} \subseteq \mathcal{B}(\mathcal{H}) \). Then there exists \( m \in \mathbb{N} \) such that \( f_m(a) = 0 \) if and only if \( a \) has a finite spectrum.

**Proof.** Suppose that there is \( m \in \mathbb{N} \) such that \( f_m(a) = 0 \). Let \( \lambda \in \sigma(a) \). Then \( f_m(\lambda) \in f_m(\sigma(a)) = \sigma(f_m(a)) = \{0\} \). This shows that \( \sigma(a) \) is contained in a finite set, namely in the set of all zeros of the polynomial \( f_m \).
To prove the converse, suppose that \( \sigma(a) \) is a finite set. Let \( \sigma(a) \setminus \{0\} = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) where \( \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0 \). By the spectral theory there exist orthogonal projections \( P_1, P_2, \ldots, P_k \in \mathbb{B}(\mathcal{H}) \) which are mutually orthogonal and such that \( a = \sum_{i=1}^{k} \lambda_i P_i \). Let \( Q = I - \sum_{i=1}^{k} P_i \). Observe that \( Q = 0 \) if and only if \( 0 \not\in \sigma(a) \). Let us now compute \( f_m(a), m \in \mathbb{N} \).

\[
\begin{align*}
  f_0(a) &= a = \sum_{i=1}^{k} \lambda_i P_i, & \|f_0(a)\| &= \lambda_1, \\
  g_0(a) &= \lambda_1 I - f_0(a) = \lambda_1 (Q + \sum_{i=1}^{k} P_i) - \sum_{i=1}^{k} \lambda_i P_i = \lambda_1 Q + \sum_{i=2}^{k} (\lambda_1 - \lambda_i) P_i, \\
  f_1(a) &= f_0(a) g_0(a) = \sum_{i=2}^{k} \lambda_i (\lambda_1 - \lambda_i) P_i,
\end{align*}
\]

Observe that the number of non-zero elements of the spectrum of \( f_1(a) \) is \( k - 1 < k \). Suppose now, without loss of generality, that \( \lambda_2 (\lambda_1 - \lambda_2) \geq \lambda_i (\lambda_1 - \lambda_i) \) for all \( i = 2, \ldots, k \). Then \( \|f_1(a)\| = \lambda_2 (\lambda_1 - \lambda_2) \) and

\[
\begin{align*}
  g_1(a) &= \lambda_2 (\lambda_1 - \lambda_2) I - f_1(a) \\
  &= \lambda_2 (\lambda_1 - \lambda_2) (Q + \sum_{i=1}^{k} P_i) - \sum_{i=2}^{k} \lambda_i (\lambda_1 - \lambda_i) P_i \\
  &= \lambda_2 (\lambda_1 - \lambda_2) Q + \sum_{i=3}^{k} \lambda_i (\lambda_1 - \lambda_i) P_i \\
  &= \lambda_2 (\lambda_1 - \lambda_2) Q + \sum_{i=3}^{k} (\lambda_1 - \lambda_2 - \lambda_i) (\lambda_2 - \lambda_i) P_i,
\end{align*}
\]

\[
\begin{align*}
  f_2(a) &= f_1(a) g_1(a) = \sum_{i=3}^{k} (\lambda_1 - \lambda_i) (\lambda_2 - \lambda_i) (\lambda_1 - \lambda_2 - \lambda_i) \lambda_i P_i.
\end{align*}
\]

This shows that \( f_2(a) \) has at most \( k - 2 \) non-zero elements in its spectrum. We now proceed inductively and conclude that there is \( m \) such that \( f_m(a) = 0 \).

**Remark 3.4.** Suppose that \( \mathcal{A} = \mathbb{C} \), i.e. that \( \mathcal{D} \) is a semi-inner product space. Then for each \( z \in \mathcal{D} \) the spectrum \( \sigma(\langle z, z \rangle) \) is a singleton, so \( f_1(\langle z, z \rangle) = 0 \). Hence, in this situation, the sequence of inequalities from Theorem 3.1 terminates already at the first step. In other words, Theorem 3.1 reduces then to Theorem 2.1. Therefore, Theorem 3.1 gives us a new (possibly finite) sequence of inequalities only if the underlying \( C^* \)-algebra is different from the field of complex numbers.

In the rest of the paper we shall show that the sequence \( (p_m(a)) \) that emerged from the proof of Theorem 3.1 has some interesting properties. Let us first consider the case from the preceding proposition, when the sequence of the inequalities is finite. The following result is interesting in its own. If \( a \in \mathcal{A} \) is such that \( f_M(a) \neq 0 \) and \( f_{M+1}(a) = 0 \) for some \( M \in \mathbb{N} \), we show that, roughly speaking, \( p_M(a) \) is the inverse of \( a \).

**Proposition 3.5.** Let \( a \neq 0 \) be a positive element in a \( C^* \)-algebra \( \mathcal{A} \subseteq \mathbb{B}(\mathcal{H}) \) with a finite spectrum. Let \( M \) be the number with the property \( f_M(a) \neq 0 \),
$f_{M+1}(a) = 0$. Then $ap_M(a)$ is the projection to the image of $a$. In particular, if $a$ is an invertible operator, then $p_M(a) = a^{-1}$.

**Proof.** Let us first observe that, since the spectrum of $a$ is finite, $\text{Im} a$ is a closed subspace of $\mathcal{H}$. For every $\lambda \in \mathbb{R}$ and $l \in \mathbb{N}$ it holds

$$
\lambda \prod_{k=0}^{l-1} g_k(\lambda) = f_0(\lambda)g_0(\lambda)g_1(\lambda) \cdots g_{l-1}(\lambda) = f_1(\lambda)g_1(\lambda) \cdots g_{l-1}(\lambda) = \ldots = f_l(\lambda).
$$

Since $f_M(a) \neq 0$ and $f_{M+1}(a) = 0$ we conclude that $p_0(a), \ldots, p_M(a)$ are defined by (3.3), while, by (3.4), $p_j(a) = p_M(a)$ for $j \geq M + 1$. Therefore, for $\lambda \neq 0$ and $m = 1, \ldots, M$ it holds

$$
p_m(\lambda) = \frac{1}{\|f_0(a)\|} + \frac{1}{\lambda^2} \sum_{l=1}^{m} \frac{f_l(\lambda)^2}{\prod_{k=0}^{l-1} \|f_k(a)\|}. \tag{3.8}
$$

Let $m \in \{1, \ldots, M\}$. Since $f_m(a) \geq 0$ (and $f_m(a) \neq 0$), $\|f_m(a)\|$ is the maximum of the set $\sigma(f_m(a)) = f_m(\sigma(a))$. Let $\lambda_m \in \sigma(a)$ be such that $f_m(\lambda_m) = \|f_m(a)\|$. Then $g_m(\lambda_m) = \|f_m(a)\| - f_m(\lambda_m) = 0$ and therefore $f_j(\lambda_m) = 0$ for all $j \geq m + 1$. Since obviously $\lambda_m \neq 0$, (3.8) gives $p_j(\lambda_m) = p_m(\lambda_m)$ for all $j \in \{m, \ldots, M\}$. Therefore, for $j \in \{m, \ldots, M\}$ we have

$$
p_j(\lambda_m) = p_m(\lambda_m) = \frac{1}{\|f_0(a)\|} + \frac{1}{\lambda_m^2} \left( \sum_{l=1}^{m-1} \frac{f_l(\lambda_m)^2}{\prod_{k=0}^{l-1} \|f_k(a)\|} + \frac{f_m(\lambda_m)^2}{\prod_{k=0}^{m-1} \|f_k(a)\|} \right).
$$

Using $f_m(\lambda_m) = \|f_m(a)\|$, for all $j \in \{m, \ldots, M\}$ we get

$$
p_j(\lambda_m) = \frac{1}{\|f_0(a)\|} + \frac{1}{\lambda_m^2} \left( \sum_{l=1}^{m-2} \frac{f_l(\lambda_m)^2}{\prod_{k=0}^{l-1} \|f_k(a)\|} + \frac{f_{m-1}(\lambda_m)^2}{\prod_{k=0}^{m-1} \|f_k(a)\|} + \frac{f_m(\lambda_m)^2}{\prod_{k=0}^{m-1} \|f_k(a)\|} \right) \tag{3.9}
$$

Observe that for every $k \in \mathbb{N}$ and every $\lambda \in \mathbb{R}$ it holds: $f_{k-1}(\lambda)^2 + f_k(\lambda) = f_{k-1}(\lambda)^2 + f_{k-1}(\lambda)g_{k-1}(\lambda) = f_{k-1}(\lambda)(f_{k-1}(\lambda) + g_{k-1}(\lambda)) = f_{k-1}(\lambda)\|f_{k-1}(a)\|$. 
Therefore, for \( j \in \{m, \ldots, M\} \),
\[
p_j(\lambda_m) = \frac{1}{\|f_0(a)\|} + \frac{1}{\lambda_m^2} \left( \sum_{l=1}^{m-2} \frac{f_l(\lambda_m)^2}{\prod_{k=0}^{l-1} \|f_k(a)\|} + \frac{f_{m-1}(\lambda_m)\|f_{m-1}(a)\|}{\prod_{k=0}^{m-2} \|f_k(a)\|} \right)
\]
\[
= \frac{1}{\|f_0(a)\|} + \frac{1}{\lambda_m^2} \left( \sum_{l=1}^{m-2} \frac{f_l(\lambda_m)^2}{\prod_{k=0}^{l-1} \|f_k(a)\|} + \frac{f_{m-1}(\lambda_m)\|f_{m-1}(a)\|}{\prod_{k=0}^{m-2} \|f_k(a)\|} \right).
\]
(3.10)

We now proceed recursively in the same way as (3.10) is obtained from (3.9) to get
\[
p_j(\lambda_m) = \frac{1}{\|f_0(a)\|} + \frac{1}{\lambda_m^2} \left( \sum_{l=1}^{m-2} \frac{f_l(\lambda_m)^2}{\prod_{k=0}^{l-1} \|f_k(a)\|} + \frac{f_{m-1}(\lambda_m)\|f_{m-1}(a)\|}{\prod_{k=0}^{m-2} \|f_k(a)\|} \right)
\]
\[
= \frac{1}{\|f_0(a)\|} + \frac{1}{\lambda_m^2} \left( \frac{f_1(\lambda_m)^2}{\|f_0(a)\|\|f_1(a)\|} + \frac{f_{m-1}(\lambda_m)\|f_{m-1}(a)\|}{\|f_0(a)\|\|f_1(a)\|} \right)
\]
\[
= \frac{1}{\|f_0(a)\|} + \frac{1}{\lambda_m^2} \left( \frac{f_1(\lambda_m)\|f_1(a)\|}{\|f_0(a)\|\|f_1(a)\|} \right)
\]
\[
= \frac{1}{\|f_0(a)\|} + \frac{1}{\lambda_m^2} \frac{\lambda_m g_0(\lambda_m)}{\|f_0(a)\|}
\]
\[
= \frac{\lambda_m + g_0(\lambda_m)}{\lambda_m\|f_0(a)\|} = \frac{\|f_0(a)\|}{\lambda_m\|f_0(a)\|} = \frac{1}{\lambda_m}
\]
for \( j \in \{m, \ldots, M\} \).

After all, we have proved: if \( \lambda_m \in \sigma(a) \) is such that \( \|f_m(a)\| = f_m(\lambda_m) \) for
some \( m \in \{0, \ldots, M\} \) then \( p_j(\lambda_m) = \frac{1}{\lambda_m} \) for all \( j \in \{m, \ldots, M\} \).

Let us take particular \( \lambda \in \sigma(a), \lambda \neq 0 \). From \( f_{M+1}(a) = 0 \) it follows that
\( f_{M+1}(\lambda) = 0 \). Then there exists \( m \leq M \) such that \( f_m(\lambda) \neq 0 \) and \( f_{m+1}(\lambda) = 0 \).
Then from \( f_{m+1}(\lambda) = f_m(\lambda)g_m(\lambda) \) we get \( g_m(\lambda) = 0 \), i.e., \( f_m(\lambda) = \|f_m(a)\| \).
This means that for every \( \lambda \in \sigma(a) \) there is \( m \leq M \) such that \( p_j(\lambda) = \frac{1}{\lambda} \) for all \( j \in \{m, \ldots, M\} \). Since \( \sigma(a) \) is finite, there is \( m \leq M \) such that
\[
p_m(\lambda) = \frac{1}{\lambda}, \quad \forall \lambda \in \sigma(a) \setminus \{0\}
\]
and therefore
\[
p_M(\lambda) = \frac{1}{\lambda}, \quad \forall \lambda \in \sigma(a) \setminus \{0\}.
\]
Then
\[
\lambda p_M(\lambda) = \begin{cases} 
1, & \lambda \in \sigma(a) \setminus \{0\}, \\
0, & \lambda \in \sigma(a) \cap \{0\}.
\end{cases}
\]
This is precisely what we need to conclude that \( a p_M(a) \) is the orthogonal projection to \( \text{Im} \, a \). In the case when \( a \) is invertible, then \( \lambda p_M(\lambda) = 1 \) for all \( \lambda \in \sigma(a) \),
so \( a p_M(a) = e \).
Remark 3.6. We note that the preceding proposition provides us with an effective algorithm for inverting any positive invertible matrix.

Example 3.7. Let \( a = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \) = \( \text{diag}(5, 4, 2, 1) \). Here we have

\[
\begin{align*}
 f_0(a) &= \text{diag}(5, 4, 2, 1), \quad \|f_0(a)\| = 5, \quad g_0(a) = \text{diag}(5, 4, 2, 1), \\
 f_1(a) &= \text{diag}(0, 4, 6, 4), \quad \|f_1(a)\| = 6, \quad g_1(a) = \text{diag}(6, 2, 0, 2), \\
 f_2(a) &= \text{diag}(0, 8, 0, 8), \quad \|f_2(a)\| = 8, \quad g_2(a) = 0, \\
 f_3(a) &= 0.
\end{align*}
\]

By Proposition 3.5, \( p_2(a) = \frac{e}{\|f_0(a)\|} + \frac{g_0(a)^2}{\|f_0(a)\| \cdot \|f_1(a)\|} + \frac{g_1(a)^2 \cdot g_2(a)^2}{\|f_0(a)\| \cdot \|f_1(a)\| \cdot \|f_2(a)\|} \) is the inverse of \( a \). Indeed,

\[
p_2(a) = \text{diag}(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) + \text{diag}(0, \frac{1}{30}, \frac{9}{30}, \frac{16}{30}) + \text{diag}(0, \frac{4}{240}, 0, \frac{64}{240}) = \text{diag}(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 1).
\]

Let us now consider the sequence \( (p_m(a)) \) in full generality. Again, we assume that \( a \) is a positive operator on some Hilbert space \( \mathcal{H} \) (i.e. \( \mathcal{A} \) is represented faithfully on \( \mathcal{H} \)). Denote by \( p \) the orthogonal projection to \( \text{Im} a \). If \( a \) has a finite spectrum we have seen in the preceding proposition that \( ap_M(a) \) is equal to \( p \), where \( M \) is less than or equal to the number of elements of \( \sigma(a) \). In particular, if \( a \) is an invertible operator, then \( p_M(a) = a^{-1} \).

If \( \sigma(a) \) is an infinite set we know that \( f_m(a) \) is never equal to 0; thus, \( (p_m(a)) \) is in this case an increasing sequence of positive elements of \( \mathcal{B}(\mathcal{H}) \). It would be natural to expect that in this situation the sequence \( (ap_m(a)) \) converges to \( p \) in norm. However, this is not true in general, as the following example shows.

Example 3.8. Suppose that \( a \) is a positive compact operator with an infinite spectrum and that the sequence \( (ap_m(a)) \) converges to \( p \) in norm. Observe that \( p_m(a) \in C^*(a) \), for all \( m \geq 0 \), where \( C^*(a) \) denotes the \( C^* \)-algebra generated by \( a \). Since \( C^*(a) \) is closed, that would imply \( p \in C^*(a) \). But this is impossible: since \( \sigma(a) \) is an infinite set, \( \text{Im} a \) is an infinite dimensional subspace and hence \( p \) (as a non-compact operator) cannot belong to \( C^*(a) \).

In this light, the following proposition is the best possible extension of Proposition 3.5.
Proposition 3.9. Let $a$ be a positive element in a $C^*$-algebra $\mathfrak{A} \subseteq B(\mathcal{H})$ with an infinite spectrum. Then $\lim_{m \to \infty} ap_m(a) = a$. In particular, if $a$ is an invertible operator, $\lim_{m \to \infty} p_m(a) = a^{-1}$.

Proof. Since $\sigma(a)$ is not finite, $f_m(a) \neq 0$ for all $m \in \mathbb{N}$, so every $p_m(\lambda)$ is defined by (3.3). Take an arbitrary $m \in \mathbb{N}$. For every $\lambda \in \sigma(a)$ we have

$$1 - \lambda p_m(\lambda) = \left(1 - \frac{\lambda}{\|a\|}\right) - \sum_{l=1}^{m} \frac{\lambda \prod_{k=0}^{l-1} g_k(\lambda)}{\prod_{k=0}^{l} ||f_k(a)||}$$

$$= \frac{g_0(\lambda)}{\|a\|} - \sum_{l=1}^{m} \frac{f_1(\lambda) \prod_{k=0}^{l-1} g_k(\lambda)}{\prod_{k=0}^{l} ||f_k(a)||}$$

$$= \left(\frac{g_0(\lambda)}{\|a\|} - \frac{f_1(\lambda) g_0(\lambda)}{\|a\||f_1(a)||}\right) - \sum_{l=2}^{m} \frac{f_1(\lambda) \prod_{k=0}^{l-1} g_k(\lambda)}{\prod_{k=0}^{l} ||f_k(a)||}$$

$$= \frac{g_0(\lambda) g_1(\lambda)}{\|a\||f_1(a)||} - \sum_{l=2}^{m} \frac{f_1(\lambda) \prod_{k=0}^{l-1} g_k(\lambda)}{\prod_{k=0}^{l} ||f_k(a)||}$$

$$= \ldots$$

$$= \frac{\prod_{k=0}^{m-1} g_k(\lambda)}{\prod_{k=0}^{m} ||f_k(a)||} - \frac{f_m(\lambda) \prod_{k=0}^{m-1} g_k(\lambda)}{\prod_{k=0}^{m} ||f_k(a)||}$$

$$= \frac{\prod_{k=0}^{m-1} g_k(\lambda)}{\prod_{k=0}^{m} ||f_k(a)||}.$$

Then

$$\lambda - \lambda^2 p_m(\lambda) = \frac{\lambda \prod_{k=0}^{m-1} g_k(\lambda) \prod_{k=0}^{m} ||f_k(a)||}{\prod_{k=0}^{m} ||f_k(a)||} = \frac{f_{m+1}(\lambda) \prod_{k=0}^{m} ||f_k(a)||}{\prod_{k=0}^{m} ||f_k(a)||}$$

and therefore

$$\|a - ap_m(a)a\| = \sup_{\lambda \in \sigma(a)} \{|\lambda - \lambda^2 p_m(\lambda)|\}$$

$$= \sup_{\lambda \in \sigma(a)} \{|f_{m+1}(\lambda)| \prod_{k=0}^{m} ||f_k(a)||\} \leq \frac{\|f_{m+1}(\lambda)\|}{\prod_{k=0}^{m} ||f_k(a)||}.$$

From $\sigma(f_k(a)) \subseteq [0, \|f_k(a)||]$ and $f_{k+1}(\lambda) = \|f_k(a)||f_k(\lambda) - f_k(\lambda)^2$ it follows that $\sigma(f_{k+1}(a)) \subseteq [0, \frac{1}{4}\|f_k(a)||^2]$, so $\|f_{k+1}(a)\| \leq \frac{1}{4}\|f_k(a)||^2$ for all $k$. Then

$$\|a - ap_m(a)a\| \leq \frac{\|f_{m+1}(\lambda)\|}{\prod_{k=0}^{m} ||f_k(a)||} \leq \frac{1}{4} \frac{\|f_m(\lambda)\|^2}{\prod_{k=0}^{m} ||f_k(a)||} \leq \frac{1}{4} \frac{\|f_m(\lambda)\|^2}{\prod_{k=0}^{m} ||f_k(a)||} \leq \frac{1}{4^2} \frac{\|f_{m-1}(\lambda)\|^2}{\prod_{k=0}^{m-1} ||f_k(a)||} \leq \frac{1}{4^2} \frac{\|f_{m-1}(\lambda)\|^2}{\prod_{k=0}^{m-1} ||f_k(a)||} \leq \ldots$$

$$\leq \frac{1}{4^m} \frac{\|f_1(\lambda)\|}{\prod_{k=0}^{m} ||a||} \leq \frac{1}{4^{m+1}} \frac{\|a\|^2}{\prod_{k=0}^{m} ||a||} \leq \frac{1}{4^{m+1}} \|a\|.$$ 

Since $m$ is arbitrary, we conclude that $\lim_{m \to \infty} ap_m(a) = a$. \qed
Remark 3.10. (a) From \( \lim_{m \to \infty} ap_m(a)a = a \) one easily concludes \( \lim_{m \to \infty} ap_m(a) = p \) in the strong operator topology.

(b) It is obvious from Proposition 3.5 that in the case of a finite spectrum the sequence \((ap_m(a)a)\) converges to \(a\) in norm (since it becomes a constant sequence after the \(M\)-th term). One may ask whether the proof of Proposition 3.9 works in the case when \(\sigma(a)\) is finite. Namely, since then there is \(M\) such that \(f_M(a) \neq 0\) and \(f_{M+1}(a) = 0\), only \(p_0, \ldots, p_M\) are defined by (3.3) and then \(p_m = p_M\) for \(m \geq M\). Therefore, we can get only that \(\|a - ap_m(a)a\| \leq \frac{1}{\sqrt{\|a\|^r}} \|a\|\) for \(m \geq M\) and one cannot conclude from here that \(\lim_{m \to \infty} \|a - ap_m(a)a\| = 0\).

By the first part of the preceding remark, \(p\) is the only possible norm-limit of the sequence \((ap_m(a))\). In the following proposition we characterize those positive operators \(a\) for which the sequence \((ap_m(a))\) converges to \(p\) in norm. First we need a lemma. Keeping the notation from the preceding paragraphs, let us also fix the following notational conventions: for a positive operator \(a \in B(H)\) denote \(H_1 = \overline{\text{Im} a}\) and \(H_2 = \text{Ker} a\). According to the decomposition \(H = H_1 \oplus H_2\) we can write \(a = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}\). For the operators \(a\) and \(a_1\) we denote by \((f_m(a))\) and \((f_1^{(1)}(a_1))\) the sequences defined by (3.1) and by \((p_m(a))\) and \((p_1^{(1)}(a_1))\) those defined by (3.3) and (3.4).

Lemma 3.11. \(f_m(a) = \begin{bmatrix} f_1^{(1)}(a_1) & 0 \\ 0 & 0 \end{bmatrix}\) and \(p_m(a) = \begin{bmatrix} p_1^{(1)}(a_1) & 0 \\ 0 & 0 \end{bmatrix}\) for all \(m \geq 0\).

Proof. The first assertion is trivial for \(m = 0\). Let \(I_j\) denotes the identity operator on \(H_j\) for \(j = 1, 2\). Observe that \(\|a\| = \|a_1\|\) which means \(\|f_0(a)\| = \|f_0^{(1)}(a_1)\|\). This implies \(f_1(a) = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}\). A general inductive argument is obtained exactly in the same way.

The second assertion now follows from the first one combined with (3.8). \(\square\)

Proposition 3.12. Let \(a \in B(H)\) be a positive operator and \(p \in B(H)\) the orthogonal projection to \(\text{Im} a\). Then \((ap_m(a))_m\) converges to \(p\) in norm if and only if \(\text{Im} a\) is a closed subspace of \(H\).
Proof. Suppose first that $a$ has a closed range, i.e. $\text{Im} \ a = \overline{\text{Im} \ a}$. Then $a_1 = a|_{\text{Im} \ a} : \text{Im} \ a \to \text{Im} \ a$ is a bijection. Since $\text{Im} \ a$ is a Hilbert space, $a_1$ is an invertible operator. By Proposition 3.9, the sequence $(a_1 p_m^{(1)}(a_1))$ converges in norm and $\lim_{m \to \infty} a_1 p_m^{(1)}(a_1) = I_1$. By the preceding lemma $a p_m(a) = [a_1 p_m^{(1)}(a_1) \ 0 
 0 \ 0]$ converges in norm to $[I_1 \ 0 
 0 \ 0]$ which is the orthogonal projection to $\text{Im} \ a = \overline{\text{Im} \ a}$.

Conversely, suppose that $(a p_m(a))$ converges in norm. As we already noted, the limit is then necessarily $p$. By the second assertion of the preceding lemma, $I_1$ is then the norm-limit of the sequence $(a_1 p_m^{(1)}(a_1))$. Since the group of invertible operators is open, it follows that $a_1 p_m^{(1)}(a_1)$ is an invertible operator, for $m$ large enough. In particular, $a_1 p_m^{(1)}(a_1)$ is a surjection and hence $a_1$ is a surjection as well. Thus, $\text{Im} \ a_1 = \overline{\text{Im} \ a}$. Since, obviously, $\text{Im} \ a = \text{Im} \ a_1$, this shows that $a$ has a closed range. □

Notice that each positive operator with a finite spectrum has a closed range. Thus, the preceding proposition extends Proposition 3.5. At the same time, it provides another explanation of Example 3.8 since a compact positive operator with an infinite spectrum cannot have a closed range.

At the end, let us turn back to the sequence of inequalities from Theorem 3.1. If $z \in \mathcal{X}$ has the property that $\sigma(\langle z, z \rangle)$ is finite then, by Propositions 3.3 and 3.5, there exists $M \in \mathbb{N}$ such that $f_M(\langle z, z \rangle) = 0$, $f_{M+1}(\langle z, z \rangle) \neq 0$ and $\langle z, z \rangle p_M(\langle z, z \rangle)$ is the projection to $\overline{\text{Im} \ (z, z)}$. In this case, the sequence of inequalities from Theorem 3.1 is finite and the last term between $\frac{1}{\|z\|^2}[\langle x_i, z \rangle \langle z, x_j \rangle]$ and $[\langle x_i, x_j \rangle]$ is $[\langle x_i, z \rangle p_M(\langle z, z \rangle) \langle z, x_j \rangle]$ (for all $x_1, \ldots, x_n$). The following proposition explains the reason: the sequence terminates at that place just because $[\langle x_i, z \rangle p_M(\langle z, z \rangle) \langle z, x_j \rangle]$ is the maximal element of the set of positive matrices under consideration.

**Proposition 3.13.** Let $\mathcal{X}$ be a semi-inner product module over a $C^*$-algebra $\mathcal{A} \subseteq \mathbb{B}(\mathcal{H})$. For $z \in \mathcal{X}$ and $a = \langle z, z \rangle \in \mathcal{A}$, let $p \in \mathbb{B}(\mathcal{H})$ denotes the orthogonal projection to $\overline{\text{Im} \ a}$. Suppose that there exists a positive operator $h \in \mathbb{B}(\mathcal{H})$ such that for all $x_1, \ldots, x_n \in \mathcal{X}$ and every $m \geq 0$ it holds

$$[\langle x_i, x_j \rangle] \geq [\langle x_i, z \rangle h \langle z, x_j \rangle] \geq [\langle x_i, z \rangle p_m(\langle z, z \rangle) \langle z, x_j \rangle]. \quad (3.11)$$

Then $aha = a$ and $ah = p$. 
Proof. It follows from (3.11) that
\[ \langle z, z \rangle \geq (z, z)h(z, z) \geq (z, z)p_m((z, z))(z, z), \quad \forall m \geq 0, \]
that is, \( a \geq aha \geq ap_m(a)a \) for all \( m \geq 0 \). By Proposition 3.9 (or Remark 3.10(b)), it follows that \( aha = a \). This implies \( ah = p \).

Suppose now, as in the discussion preceding the proposition, that there exists \( M \in \mathbb{N} \) such that \( f_M(a) \neq 0 \) and \( f_{M+1}(a) = 0 \). Then \( a \) has a finite spectrum, \( \text{Im} \, a \) is a closed subspace, and \( ap_M(a) = p \). So, if \( h \) is as in Proposition 3.13, then \( ah = p \) and therefore \( ap_M(a) = ah \). By taking adjoints we get \( ha = p_M(a)a \) and this shows that \( h \) and \( p_M(a) \) coincide on \( \text{Im} \, a \).

If \( \sigma(a) \) is infinite, there is no \( M \) as above, but still the sequence \( (ap_m(a)a) \) converges in norm to \( a \). From the proof of Proposition 3.13 it follows that for any \( h \in \mathbb{B}(\mathcal{H}) \) which satisfies left hand side inequality of (3.11) it holds \( aha \leq a \). Therefore, we have a kind of best result even in this case, since \( h \) which appears in (3.11) is such that \( aha = \lim_{m \to \infty} ap_m(a)a = a \).

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