Dynamics of Cycles in Polyhedra I: 
The Isolation Lemma

Jan Kessler 
Institute of Mathematics 
University of Cologne, Germany

Jens M. Schmidt* 
Institute of Mathematics 
University of Cologne, Germany

Abstract

A cycle $C$ of a graph $G$ is isolating if every component of $G - V(C)$ is a single vertex. We show that isolating cycles in polyhedral graphs can be extended to larger ones: every isolating cycle $C$ of length $8 \leq |E(C)| < \frac{2}{3}(|V(G)| + 3)$ implies an isolating cycle $C'$ of larger length that contains $V(C)$. By “hopping” iteratively to such larger cycles, we obtain a powerful and very general inductive motor for proving and computing long cycles (we will give an algorithm with running time $O(n^2)$).

This provides a method to prove lower bounds on Tutte cycles, as $C'$ will be a Tutte cycle of $G$ if $C$ is. We also prove that $E(C') \leq E(C) + 3$ if $G$ does not contain faces of size five, which gives a new tool for proving results about cycle spectra and evidence that these face sizes obstruct long cycles. As a sample application, we test our motor on a conjecture on essentially 4-connected graphs.

A planar graph is essentially 4-connected if it is 3-connected and every of its 3-separators is the neighborhood of a single vertex. Essentially 4-connected graphs have been thoroughly investigated throughout literature as the subject of Hamiltonicity studies. Jackson and Wormald proved that every essentially 4-connected planar graph $G$ on $n$ vertices contains a cycle of length at least $\frac{2}{3}(n+2)$, and this result has recently been improved multiple times, culminating in the lower bound $\frac{5}{8}(n+2)$. However, the best known upper bound is given by an infinite family of such graphs in which every graph $G$ on $n$ vertices has no cycle longer than $\frac{5}{6}(n+4)$; this upper bound is still unmatched.

Using isolating cycles, we improve the lower bound to match the upper (up to a summand +1). This settles the long-standing open problem of determining the circumference of essentially 4-connected planar graphs.

1 Introduction

One of the unchallenged milestones in planar graph theory is the result by Tutte [11] that every 4-connected planar graph on $n$ vertices is Hamiltonian, i.e. has circumference $n$, where the circumference $\text{circ}(G)$ of a graph $G$ is the length of a longest cycle of $G$. However,
decreasing the connectivity assumption from 4 to 3 reveals infinitely many planar graphs that do not have long cycles: Moon and Moser [9] showed that there are infinitely many 3-connected planar (i.e., polyhedral) graphs that have circumference at most $9n^{\log_2 2}$ for $n := |V(G)|$, and this upper bound is best possible up to constant factors, as there is a constant $c > 0$ such that every polyhedral graph contains a cycle of length at least $cn^{\log_2 2}$.

One of the biggest remaining open problems in this area ever since is to characterize the connectivity properties between connectivity 3 and 4 that imply long cycles. Essential 4-connectivity is such a property and will be a focus of this paper.

Indeed, essentially 4-connected graphs have been thoroughly investigated throughout literature for this purpose. An upper bound for the circumference of essentially 4-connected planar graphs was given by an infinite family of such graphs on $n \geq 14$ vertices in which every graph $G$ satisfies $\text{circ}(G) = \frac{2}{3}(n + 4)$ [2]; the graphs in this family are in addition maximal planar. Regarding lower bounds, Jackson and Wormald [8] proved that $\text{circ}(G) \geq \frac{2}{3}(n + 2)$ for every essentially 4-connected planar graph $G$ on $n$ vertices. Fabrici, Harant and Jendröl [2] improved this lower bound to $\text{circ}(G) \geq \frac{5}{8}(n + 2)$; this result in turn was recently strengthened to $\text{circ}(G) \geq \frac{2}{3}(n + 2)$ [4], and then further to $\text{circ}(G) \geq \frac{5}{8}(n + 2)$ [5]. For the restricted case of maximal planar essentially 4-connected graphs, the matching lower bound $\text{circ}(G) \geq \frac{2}{3}(n + 4)$ was proven in [3]; however, the method used there is specific to maximal planar graphs. For the general polyhedral case, it is still an open conjecture that every essentially 4-connected planar graph $G$ on $n$ vertices satisfies $\text{circ}(G) \geq \frac{2}{3}(n + 4)$ and thus the upper bound; while this conjecture has been an active research topic at workshops for over a decade\footnote{personal communication with Jochen Harant}, it was only recently explicitly stated [3, Conjecture 2].

Here, we show that $\text{circ}(G) \geq \frac{2}{3}(n + 3)$ for every essentially 4-connected planar graph; this matches the upper bound up to the summand +1. Moreover, this is only an implication of a much more general result about polyhedral graphs, which we present here. While one part of the proof scheme follows the established approach of using Tutte cycles in combination with the discharging method, we contribute an intricate intersection argument on the weight distribution between groups of neighboring faces, which we call tunnels. This methods differs substantially from the result in [3] (which exploits the inherent structure of maximal planar graphs) and is, unlike the previous results, able to harness the dynamics of extending cycles in polyhedral graphs. In particular, we will discharge weights along an unbounded number of faces, which was an obstacle that was not needed to solve for the weaker bounds.

In fact, we will prove this tool in the following variant that provides also a method to prove many different cycle lengths. Let a cycle $C$ of a graph $G$ be extendable if $G$ contains a larger isolating cycle $C'$ such that $V(C) \subset V(C')$ and $|V(C')| \leq |V(C)| + \max\{3, 2 + n_5(G)\}$, where $n_5(G)$ is the number of faces of size five in $G$.

**Lemma 1** (Isolation Lemma). Every isolating cycle of length $c < \min\{\frac{2}{3}(n + 3), n\}$ in a polyhedral graph $G$ on $n$ vertices is extendable.

The assumption $c < n$ is redundant if and only if $c \geq 8$. The Isolation Lemma may be seen as a polyhedral relative of Woodall’s Hopping Lemma [13] that allows cycle extensions
through common neighbors of cycle vertex pairs even when none of these pairs have distance two in \( C \). Despite this correlation, the Isolation Lemma makes inherently use of planarity; in fact, it fails hard for non-planar graphs, as the graphs \( K_{c,n-c} \) for any \( c \geq 3 \) and any (isolating) cycle of length \( c \) in these show. We state some immediate corollaries.

**Corollary 2.** If a polyhedral graph \( G \) contains an isolating cycle of length \( c \geq 8 \), \( G \) contains cycles of at least \( \frac{2}{3}(n+3) - c \) different lengths in \( \{c, \ldots, \lceil \frac{2}{3}(n+3) \rceil \} \).

**Corollary 3.** If a bipartite polyhedral graph \( G \) contains an isolating cycle of length \( c \geq 8 \), \( G \) contains a cycle of length \( l \) for every even \( l \in \{c, \ldots, \lceil \frac{2}{3}(n+3) \rceil \} \).

The conclusion of Corollary 3 holds even when the polyhedral graph \( G \) does not have faces of size 5. In view of the sheer number of results in Hamiltonicity studies that use subgraphs involving faces of size five (see e.g. the Tutte Fragment or the one of Faulkner and Younger), this provides evidence that these faces are indeed key to a small circumference. Another corollary is that polyhedral graphs on \( n \) vertices, in which all cycles have length less than \( \min \{\frac{2}{3}(n+3), n\} \), do not contain any isolating cycle; for example, this holds for the sufficiently large Moon-Moser graphs [9] and the 18 graph classes of [7, Theorem 1] that have shortness exponent less than 1.

Finally, the Isolation Lemma implies also the following theorem.

**Theorem 4.** Every essentially 4-connected planar graph \( G \) on \( n \) vertices contains an isolating Tutte cycle of length at least \( \min \{\frac{2}{3}(n+3), n\} \) and such a cycle can be computed in time \( O(n^2) \).

**Proof.** We only consider existence and will give an algorithm at the end of this paper. It is well-known that every 3-connected plane graph on at most 10 vertices is Hamiltonian [1]. Since these graphs contain in particular the essentially 4-connected planar ones, this implies the theorem if \( n \leq 10 \); we therefore assume \( n \geq 11 \). For \( n \geq 11 \), it was shown in [2, Lemma 4(i)+(ii)] that \( G \) contains an isolating Tutte cycle \( C \) of length at least 8. If \( c \geq \frac{2}{3}(n+3) \), \( C \) is already long enough, since \( \frac{2}{3}(n+3) \leq n \); otherwise, applying iteratively the Isolation Lemma to \( C \) gives the claim and preserves a Tutte cycle, as no vertices of \( C \) are deleted.

Theorem 4 encompasses and strengthens most of the results known for the circumference of essentially 4-connected planar graphs, some of which can be found in [2, 6, 14].

### 2 Preliminaries

We use standard graph-theoretic terminology and consider only graphs that are finite, simple and undirected. For a vertex \( v \) of a graph \( G \), denote by \( \deg_G(v) \) the degree of \( v \) in \( G \). We omit subscripts if the graph \( G \) is clear from the context. Two edges \( e \) and \( f \) are *adjacent* if they share at least one end vertex. The *distance* of two edges in a connected graph is the length of a shortest path that contains both. We denote a path of \( G \) that visits the vertices \( v_1, v_2, \ldots, v_i \) in the given order by \( v_1v_2\ldots v_i \).

A *separator* \( S \) of a graph \( G \) is a subset of \( V(G) \) such that \( G - S \) is disconnected; we call \( S \) a \( k \)-*separator* if \( |S| = k \). Let a cycle \( C \) of a graph \( G \) be *isolating* if every component
of \( G - V(C) \) is a single vertex (see Figure 1i for an example). We do not require that these single vertices have degree three (this differs e.g. from [2, 4, 5]). A chord of a cycle \( C \) is an edge \( vw \notin E(C) \) for which \( v \) and \( w \) are in \( C \). According to Whitney [12], every 3-connected planar graph has a unique embedding into the plane (up to flipping and the choice of the outer face). Hence, we assume in the following that such graphs are equipped with a fixed planar embedding, i.e., are plane. Let \( F(G) \) be the set of faces of a plane graph \( G \).

## 3 Proof of the Isolation Lemma

Let \( G = (V, E) \) be a 3-connected plane graph on \( n \) vertices, and let \( C \) be an isolating cycle of \( G \) of length \( c < \min\{\frac{2}{3}(n + 3), n\} \). We assume to the contrary that \( C \) is not extendable.

Let \( V^- \) be the subset of \( V \) that is contained in the maximal path-connected open set (i.e. region) of \( \mathbb{R}^2 - C \) that is bounded (hence, strictly inside \( C \)), and \( V^+ := V - V(C) - V^- \). Without loss of generality, we assume \(|V^+| \leq |V^-| \). Since \( c < n \) and \(|V^+| \leq |V^-| \), we have \( V^- \neq \emptyset \). Let \( H \) be the plane graph obtained from \( G \) by deleting either all chords of \( C \) if \( V^+ = \emptyset \) or otherwise all chords of \( C \) whose interior point set is contained in the bounded region of \( \mathbb{R}^2 - C \) (see Figure 1i). Let \( H^+ := H - V^- \) and \( H^- := H - V^+ - (E(H^+) - E(C)) \).

(i) An isolating cycle \( C \) (fat edges) of an essentially 4-connected plane graph \( G \); vertices in \( V^+ \) are not drawn. Here, all vertices of \( V^- = \{a, b, c, d, e, g\} \) have degree three in \( G \), and \( H^- \) has no minor 1-face but would have one after contracting \( yz \). The dashed chord \( vw \) of \( C \) is in \( G \) but not in \( H \), so that \( f \) is a (major 1-) face of \( H \) but not a face of \( G \). The minor face \( f_8 \) has the two arches \( pgs \) and \( pr \) (\( ps \) is not an arch).

(ii) The subgraph \( H^- \) of \( G \) (solid edges) and a tree \( T^- \) constructed from \( H^- \) (dashed edges). There are \(|M^-| = 9 \geq |V^-| + 2 = 8 \) minor faces in \( H^- \) (depicted in grey), each of which is a 2-face that corresponds to a leaf of \( T \).

![Figure 1](image-url)

For a face \( f \) of \( H \) or of \( G \), the edges of \( C \) that are incident with \( f \) are called \( C \)-edges of \( f \) and their number is denoted by \( m_f \). A \( C \)-vertex of \( f \) is a vertex that is incident to a \( C \)-edge of \( f \); a \( C \)-vertex of \( f \) is extremal if it is incident to at most one \( C \)-edge of \( f \).
and non-extremal otherwise. A face $f$ is called $j$-face if $m_f = j$. If $f$ has an odd number of $C$-vertices or $C$-edges, we call their unique vertex or edge in the middle the middle $C$-vertex or $C$-edge of $f$.

Let two faces $f$ and $g$ of $H$ be opposite if $f$ and $g$ have a common $C$-edge. If $f$ has an $C$-edge $e$, let the $e$-opposite face of $f$ be the face of $H$ that is different from $f$ and incident to $e$. Let a face $f$ of $H$ be thin if $V^+ = \emptyset$ and $f$ is in $H^+$; otherwise, let $f$ be thick. A face $f$ of $H$ is called minor if it is either thick and incident to exactly one vertex of $V^- \cup V^+$ or thin and incident to exactly one edge that is not in $C$; otherwise, $f$ is called major. Let $M^-$ and $M^+$ be the sets of minor faces of $H^-$ and $H^+$, respectively. For a minor thick face $f$ of $H$, let $v_f$ be the unique vertex of $V^- \cup V^+$ incident to $f$. A 2-sandwiched vertex is the middle $C$-vertex of a thick minor 2-face of $H$.

Construct a graph $T^-$ (see Figure 1ii) obtained from $H^-$ with vertex set $M^- \cup V^-$ and the following edge-set. First, for every face $f \in M^-$, add the edge $fv_f$ to $T^-$. Second, for every major face $f$ of $H^-$ (in arbitrary order), fix any vertex $v \in V^-$ that is incident to $f$ and add the edge $vw$ to $T^-$ for every vertex $w \in V^- \setminus \{v\}$ that is incident to $f$. We first prove that $T^-$ is a tree.

**Lemma 5.** $T^-$ is a tree with inner vertex set $V^-$, leaf set $M^-$ and no vertex of degree two.

**Proof.** Consider two faces $f$ and $g$ of $H^-$ that are incident to a common edge $e$. Since $H^-$ does not contain any chord of $C$, $e$ is not a chord of $C$, so that $f$ and $g$ are incident to a common vertex $v \in V^-$. By construction of $T^-$, all inner vertices of $T^-$ that are incident to $f$ or $g$ (in particular, $v$) are connected in $T^-$. Hence, $T^-$ is connected. As $C$ is isolating, every two faces of $H^-$ are incident to at most one common vertex of $V^-$. Hence, the union of the acyclic graphs that are constructed for every major face of $H^-$, and thus $T^-$ itself, is acyclic. We conclude that $T^-$ is a tree with inner vertex set $V^-$ and leaf set $M^-$.

Note that $T^-$ may contain vertices of unbounded degree even when every vertex of $V^-$ has degree three in $G$ (for example, $\deg_{T^-}(c) = 4$ in Figure 1ii). If $V^+ \neq \emptyset$, Lemma 5 holds by symmetry also for the tree $T^+$ that is constructed from $H^+$ in the same way as $T^-$ from $H^-$. 

**Lemma 6.** We have $|M^-| \geq |V^-| + 2$. If $V^+ \neq \emptyset$, $|M^+| \geq |V^+| + 2$.

**Proof.** Since $V^- \neq \emptyset$, $T^-$ has at least four vertices by Lemma 5. It is well-known that every tree $T$ on at least two vertices has exactly $2 + \sum_{v \in V(T), \deg(v) \geq 3} (\deg(v) - 2)$ leaves, where $x$ is the number of vertices of degree at least 3 in $T$. Since $T^-$ has no vertex of degree two, this gives the first claim by Lemma 5. The second claim follows from $V^+ \neq \emptyset$ by the same argument applied to $T^+$.

In both equalities of Lemma 6, the last summand is non-negative but may be zero, as a longest cycle of the graph obtained from the octahedron by inserting a vertex of degree three in every face shows (see also the graph obtained from Figure 1ii by deleting $d$).

Consider a minor 1-face $f$ of $H$ with $C$-edge $vw$; since there are no thin minor 1-faces, $f$ is thick. Then the cycle obtained from $C$ by replacing $vw$ with the path $v v_f w$ shows
that $C$ is extendable, which contradicts our assumption. We conclude that $H$ has no minor 1-face. Since $C$ is isolating and has no chords in $H$, $H$ has no minor 0-face. To summarize our assumptions so far, we know that $C$ is not extendable, $c < \min\{\frac{2}{3}(n+3), n\}$, $|V^+| \leq |V^-| \geq 1$ and every minor face $f$ of $H$ satisfies $m_f \geq 2$.

For the final contradiction, we aim to prove

$$2c \geq 4(|M^-| + |M^+|) \text{ if } V^+ \neq \emptyset \text{ and }$$
$$2c \geq 4|M^-| + |M^+| \text{ if } V^+ = \emptyset$$

(1)

Assume $V^+ \neq \emptyset$. By Lemma 6, $|M^-| \geq |V^-| + 2$ and $|M^+| \geq |V^+| + 2$. Since $|V^-| + |V^+| = n - c$, Inequality 1 implies $c \geq 2(n - c + 4)$ and thus the claim $c \geq \frac{2}{3}(n + 4) \geq \frac{2}{3}(n + 3)$. In the special case $V^+ = \emptyset$, we have $|V^-| = n - c$, so that Inequality 2 implies $c \geq 2(n - c + 2 + \frac{1}{2})|M^+|$. Since every minor face of $H^+$ has a non-extremal $C$-vertex (since $H$ has no minor 1-faces), minimum degree 3 in $G$ implies that we have at least one chord of $C$ in $H^+$, so that $|M^+| \geq 2$. This gives $c \geq \frac{2}{3}(n + 3)$ (hence, we are only off by +1 in the case $V^+ = \emptyset$).

In order to prove Inequalities (1) and (2), we will charge every $j$-face of $H$ with weight $j$; hence, the total charge has weight $2c$. Then we discharge (i.e., move) these weights to minor faces such that no face has negative weight. We will prove that after the discharging every minor face of $H$ has sufficiently high weight to satisfy the inequalities.

### 3.1 Arches and Tunnels

For a face $f$ of $H$, a path $A$ of $G$ is an arch of $f$ if $f$ is minor and $A$ is either the maximal path in $H - E(C)$ all of whose edges are incident to $f$ (in this case we say that $A$ is proper; then $A$ has length one or two depending on whether $f$ is thin or thick) or a chord of $C$ whose inner point set is contained in $f$ and that does not join the two extremal $C$-vertices of $f$ (see Figure 2i). Hence, an arch $A$ is proper if and only if $A \subseteq H$, so that every minor face $f$ has exactly one proper arch. The face $f(A)$ of an arch $A$ is the minor face of $H$ that contains the inner point set of $A$.

Let the archway of an arch $A$ be the path in $C$ between the extremal vertices of $A$ whose edges are incident to $f(A)$ in $H$. Since $A$ and its archway close a face $f$ in the graph $A \cup C$, we define $m_A$, thickness, and the $C$-vertices and $C$-edges of $A$ just as the corresponding terms for the face $f$; in particular, $m_A$ denotes the number of edges of the archway of $A$, and $A$ is thick if $|A|$ is thick. Then every arch has exactly two extremal $C$-edges, as $H$ has no minor 1-face and, by the last condition of the definition of arches, two arches of $f$ have never the same pair of extremal $C$-edges (this prevents that two arches of the same face “overlap”). We call an arch $A$ a $j$-arch if $m_A = j$. If $f$ and $g$ are arches or faces of $H$, let $m_{fg}$ be the number of $C$-edges that $f$ and $g$ have in common; then $f$ and $g$ are opposite if $m_{fg} > 0$. An arch $A$ of an arch $B$ is an arch of $f(B)$ such that every $C$-edge of $A$ is a $C$-edge of $B$; we also say that $B$ has arch $A$.

Consider the 3-arches $T_1, \ldots, T_k$ in Figure 2 and assume that every $T_i$ is thick and proper, so that every $f(T_i)$ is a minor 3-face. Since every $f(T_i)$ receives only initial weight 3 and $k$ is unbounded, every local method of transferring weights to reach weight 4 per minor
face is bound to fail. Unfortunately, 3-faces are not the only example where non-local methods are needed: in fact, there is an unbounded number of faces in which weights must be transferred non-locally. We will therefore design the upcoming discharging rule in such a way that weight transfers are not dependent on minor face but instead on arches; this will reduce all structures that have to be handled non-locally to one common structure (called tunnel), which is the one in Figure 2.

![Figure 2: An acyclic tunnel track (T₁, T₂, B, T₄, T₅) with exit pair (g', e'). Here, (g', e') is on-track with itself, (f(T₁), v₀v₁), (h, v₂v₃), (g, e), (f, v₆v₇) and (f(T₅), v₈v₉), but not with (g, v₂v₃), which is on-track with (f, e). While (f(T₁), v₀v₁), (h, v₂v₃) and (g, e) are transfer pairs, (f, v₆v₇) and (f(T₅), v₈v₉) are not (the former, as neither v₆v₇ nor v₇v₈ is an extremal C-edge of f; the latter, as T₅ has only one opposite face).](image)

Let two 3-arches A and B be consecutive if $m_{A,B} = 1$. The reflexive and transitive closure of this symmetric relation partitions the set of 3-arches; we call the sets of this partition tunnels (see Figure 2). Since $G$ is plane, $G$ imposes a notion of clockwise and counterclockwise on $C$; in the following, both directions always refer to $C$. The counterclockwise track of a tunnel $T$ (which will transfer weights counterclockwise around $C$) is the sequence $(T₁, T₂, . . . , Tₖ)$ of all 3-arches of $T$ such that $Tᵢ₊₁$ is the clockwise consecutive successor of $Tᵢ$ for every $1 ≤ i < k$. We call a tunnel track $(T₁, T₂, . . . , Tₖ)$ and its tunnel $T$ cyclic if $k ≥ 3$ and $Tₖ$ and $T₁$ are consecutive, and acyclic otherwise.

The exit pair $(g, e)$ of a counterclockwise track consists of the counterclockwise extremal $C$-edge $e$ of $T₁$ and the $e$-opposite face $g$ of $f(T₁)$. Clockwise tracks and exit pairs are defined analogously. The exit pairs $(g, e)$ and $(g', e')$ of a tunnel $T$ are then the two exit pairs of the counterclockwise and clockwise tracks of $T$; we call $g$ and $g'$ exit faces of $T$).

Clearly, we have $e = e'$ if and only if $T$ is cyclic, and if so, $g'$ and $g$ are opposite faces, so that $g ≠ g'$. Hence, the exit pairs of every acyclic tunnel are different, while the exit faces may be identical.

In order to describe the weight transfers through tunnels, we define the following reflexive and symmetric relation for faces $g$ and $g'$ of $H$ and extremal $C$-edges $e$ and $e'$ of (not necessarily different) 3-arches of an acyclic tunnel $T$ such that $e$ and $e'$ are incident to $g$ and $g'$, respectively. Let $(g, e)$ be on-track with $(g', e')$ if the following statements are equivalent (see Figure 2).

- $g$ and $g'$ are contained in the same region of $R^2 - C$
- the distance between $e$ and $e'$ in the union of the $C$-edges of $T$ (measured by the
length of a path that does not exceed $T$) is a multiple of 4.

Clearly, this relation is an equivalence relation. Moreover, if $e$ is an extremal $C$-edge of a 3-arch $A$ of a tunnel $T$, $(f(A), e)$ is on-track with exactly one exit pair of $T$. Tunnels will serve as objects through which we can pull weight over long distances. We will later prove that tunnels transfer weights only one-way, i.e. use (the on-track pairs of) at most one of their tracks. Based on the structure of $G$, weight may not be transferred through the whole tunnel track; the following definition restricts the parts where weight transfers may occur.

Let $T$ be an acyclic tunnel track, $e$ an extremal $C$-edge of an arch $B$ of $T$ such that $(g, e) := (f(B), e)$ is on-track with the exit pair of $T$, $b$ the extremal $C$-edge of $B$ different from $e$, and $h$ the $b$-opposite face of $g$ (see Figure 2; informally, $(h, b)$ is the on-track pair in $T$ that precedes $(g, e)$). Recursively, we define that $(g, e)$ is a transfer pair of $T$ if $(h, b)$ is either the exit pair of $T$ or a transfer pair, and

- the $e$-opposite face $f$ of $g$ is minor and $m_f \geq 3$,
- $e$ is either an extremal $C$-edge of $g$ or adjacent to that edge, and in the latter case the middle $C$-edge of $B$ is incident to $f$ or a major face, and
- the middle $C$-edge of $B$ is not incident to $h$ (in particular, $h \neq f$).

For a tunnel track $T$, an arch $B$ of $T$ that has an extremal $C$-edge $e$ such that $(f(B), e)$ is a transfer pair of $T$ is called transfer arch of $T$. Note that a transfer arch of $T$ is not necessarily a transfer arch of the other tunnel track of $T$.

### 3.2 Discharging Rule

By saying that a face $g$ pulls weight $x$ over its $C$-edge $e$ for a positive weight $x$, we mean that $x$ is added to $g$ and subtracted from the $e$-opposite face of $g$; we sometimes omit $x$ if the precise value is not important, but positive.

**Definition 7** (Discharging Rule). For every minor face $g$ of $H$ and every $C$-edge $e$ of $g$ (both in arbitrary order), $g$ pulls weight 1 over $e$ from the $e$-opposite face $f$ of $g$ for every of the following conditions that is satisfied (see Figure 3).

**C1:** $f$ is major

**C2:** $f$ is minor, and $g$ is a thick 2-face

**C3:** $f$ is minor and $m_f \geq 3$, $e$ is the middle $C$-edge of a 3-arch $B$ of $g$ and not an extremal $C$-edge of a 3-arch of $f$, either $g$ is a 3-face or an extremal $C$-edge of $B$ is an extremal $C$-edge of $g$ and the other extremal $C$-edge $b$ of $B$ is incident to an extremal $C$-vertex of $f$ and not incident to a major face such that, if $b$ is incident to $f$, $g$ has a 4-arch that has $B$

**C4:** $f$ is minor, $e$ is a non-extremal $C$-edge of a 4-arch $B$ of $g$ such that the extremal $C$-edge of $B$ that is adjacent to $e$ is an extremal $C$-edge of $g$ and the other extremal $C$-edge $e$ of $B$ is incident to a thick minor 2-face $h$, $e$ is not the middle $C$-edge of a 3-arch of $g$, and $m_{f,B} \geq 2$
(i) Condition C2: The arrow depicts that $g$ pulls weight 1 over $e$; we do not indicate weights pulled over other edges here. The vertex $v_g$ of a minor face $g$ is drawn only if (as here) $g$ is known to be thick. The red dotted arches do not exist in $G$.

(ii) Condition C3: $e$ is not an extremal $C$-edge of a 3-arch of $f$, and either $g$ is a minor 3-face or $V_1V_2$ is an extremal $C$-edge of $g$ such that $b$ is incident to an extremal $C$-vertex of $f$ and not incident to a major face such that, if $b$ is incident to $f$, $g$ has a 4-arch that has $B$. Arches like $B$ that are not known to be proper (i.e., that are not known to be in $H$) are drawn dashed. Note that $g$ may have more than three $C$-edges.

(iii) Condition C4: $v_0v_3$ is not an arch of $g$.

(iv) Condition C5: $h \neq f$, $(h, b)$ is a transfer pair, no 2-arch has extremal $C$-vertex $v_{-1}$, $B$ has no 3-arch, and $e$ is not an extremal $C$-edge of a 3-arch of $f$. We color arches that are known to be transfer arches for some tunnel track grey.

(v) Condition C6: $h \neq f$, and $e$ is not the middle $C$-edge of a 3-arch of $g$.

(vi) Condition C7: $(g, e)$ is a transfer pair of the acyclic tunnel track $(T_1, T_2, T_3)$, and $(g', e')$ satisfies at least one of $C1$–$C6$.

Figure 3: Conditions C2–C7
C5: $f$ is minor, $e$ is a non-extremal $C$-edge of a 4-arch $B$ of $g$ such that the extremal $C$-edge of $B$ that is adjacent to $e$ is an extremal $C$-edge of $g$ and the other extremal $C$-edge $b$ of $B$ is an extremal $C$-edge of a 3-arch $A_h$ of a face $h \neq f$, $(h,b)$ is a transfer pair, no 2-arch has extremal $C$-vertex $v_{-1}$, there is no 3-arch of $B$, $e$ is not an extremal $C$-edge of a 3-arch of $f$, and $m_{f,B} \geq 2$.

C6: $f$ is minor, $e$ is a non-extremal $C$-edge of a thick minor 4-face $g$ such that the extremal $C$-edge of $g$ that is not adjacent to $e$ is the middle $C$-edge of a 3-arch $A_h$ of a face $h \neq f$, and $e$ is not the middle $C$-edge of a 3-arch of $g$.

C7: $f$ is minor, $(g,e)$ is a transfer pair of an acyclic tunnel track $T$, and the exit pair $(g',e')$ of $T$ satisfies (in the notation $g$ and $e$) at least one of the conditions C1–C6.

Note that the weight transfers of this rule are solely dependent on $G$ and $C$ (and not on the current weight transfers). This holds in particular for the ones caused by C7, as these do not depend on C7 (but instead of C1–C6 on the exit pair). Since tunnels partition the set of 3-arches, it suffices to evaluate C7 once for the transfer pairs of each tunnel track.

After the discharging rule has been applied, C7 effectively routes weight 1 through an extremal part of a tunnel track towards its exit face if this exit face pulls weight from $T$ by any other condition. By definition of C1–C6, the only faces that do not have an arch of a tunnel $T$ (i.e. reside “outside” $T$) and pull over a $C$-edge of such an arch are the exit faces of $T$; in this sense, weight may leave $T$ only through an exit face of $T$.

### 3.3 Structure of Tunnels and Transfers

We give further insights into the structure of tunnels and the location of edges over which our discharging rule pulls (positive) weight.

**Lemma 8.** Every tunnel track $(T_1,\ldots,T_k)$ with $k \geq 3$ satisfies $m_{T_1,T_k} = 0$. In particular, every tunnel is acyclic.

We remark that it is possible, but slightly more involved, to prove Lemma 8 solely by using the discharging rule of Definition 7. Using Lemma 8, we assume from now on that every tunnel is acyclic. We now show that $G$ does not contain the dotted edges of Figure 3 for the respective conditions; this sheds first light on the implications that are triggered by the assumption that $C$ is not extendable.

**Lemma 9.** For any satisfied condition $X \in \{C2,\ldots,C6\}$, none of the red dotted arches in the respective Figure 3i–3v exist in the depicted face of $H$. If $X = C6$, $v_{-3v_{-1}} \in E(G)$.

**Proof.** We use the notation of Figure 3. Assume $X = C2$. If $v_0v_1$ (or, by symmetry, $v_{-1}v_0$) in Figure 3i is a $C$-edge of a 2-arch $A$ of $f$, $v_0$ is an extremal $C$-vertex of $A$, since $G$ is polyhedral. Then $C$ is extendable by the path replacement $v_{-1}v_0v_1v_0Av_2$, which adds one or two new vertices to $C$ (depending on whether $A$ is thick). If $v_0v_1$ (or, by symmetry, $v_{-1}v_0$) is the middle $C$-edge of a 3-arch, we have $v_0v_2 \in E(G)$, as $\deg_G(v_0) \geq 3$, since $G$ is polyhedral; hence, neither $v_0v_1$ nor $v_{-1}v_0$ is the middle $C$-edge of a 3-arch. Using the same argument, $f$ has no 4-arch with extremal $C$-edges $v_{-2}v_{-1}$ and $v_1v_2$. 

10
Assume $X = C3$. Then $v_0v_1$ is not the middle $C$-edge of a 3-arch of $f$, as $G$ is polyhedral. By definition of $C3$, $v_0v_1$ is not an extremal $C$-edge of a 3-arch of $f$.

Assume $X = C4$. Then the first argument for $X = C2$ implies $v_{−1}v_1 \notin E(G)$. In addition, $v_{−1}v_2 \notin E(G)$, as otherwise $C$ is extendable by the path replacement $v_{−2}v_hv_0v_1v_{−1}Bv_2$. Hence, $v_1v_2$ is not an extremal $C$-edge of a 3-arch of $g$. If $v_1v_2$ is an extremal $C$-edge of a 3-arch $A$ of $f$, $C$ is extendable by the path replacement $v_{−2}v_hv_0v_{−1}Bv_3v_2v_1Av_4$, as this adds at most three new vertices to $C$. Note that we have $v_A \neq v_h$ in this replacement if $A$ is proper ($A$ is thick because of $h$), as $H$ has no minor 1-face (here, with $C$-edge $v_0v_1$). In addition, $v_1v_2$ is not the middle $C$-edge of a 3-arch of $f$, as otherwise $\{v_0, v_3\}$ would be a 2-separator of $G$ by the previous claims.

Assume $X = C5$. By definition of $C5$, it only remains to prove that $v_1v_2$ is not the middle $C$-edge of a 3-arch $A$ of $f$. If it is, $\{v_0, v_3\}$ is a 2-separator of $G$, since $v_{−1}v_1$ and $v_{−1}v_2$ are not contained in $G$. This contradicts that $G$ is polyhedral.

Assume $X = C6$. Then $v_{−1}v_1 \notin E(G)$, as otherwise $C$ is extendable by the replacement $v_{−3}A_hv_0v_{−1}v_{−1}v_{−2}v_gv_2$. Since $G$ is polyhedral, $v_{−1}$ has degree at least three in $G$, which implies $v_{−3}v_{−1} \in E(G)$ as only remaining option. Then $v_{−2}v_1 \notin E(G)$, as otherwise $C$ is extendable by the replacement $v_{−3}v_{−1}v_0v_1v_{−2}v_gv_2$. Since $G$ is polyhedral, this implies that there is no 2-arch of $f$ with extremal $C$-vertices $v_0$ and $v_2$. Assume that $f$ has a 2-arch $A$ with extremal $C$-vertices $v_1$ and $v_3$. Then $A$ is thick, as $A_h$ has a 2-arch, but non-proper, as otherwise $f$ would be a minor 1-face of $H$. Hence, $v_1v_3 \in E(G)$, so that $C$ is extendable by the replacement $v_{−3}A_hv_0v_{−1}v_{−2}v_gv_2v_1v_3$. This implies in addition that $v_0v_1$ is not an extremal $C$-edge of a 3-arch of $f$, as $v_1$ would have degree two in $G$.\[\]

For a $C$-edge $e$ of a face $g$ of $H$ and a condition $X \in \{C1, C2, \ldots, C7\}$, let $g \xrightarrow{e} X$ denote that $X$ is satisfied for $g$ and $e$ in Definition 7. For notational convenience throughout this paper, whenever $g$ pulls weight from a face $f$, we denote by $v_0$ the extremal $C$-vertex of $f$ whose clockwise neighbor $v_1$ in $C$ is $C$-vertex of $f$, and denote by $v_i$ the $i$th vertex modulo $c$ in a clockwise traversal of $C$ starting at $v_1$.

So far, a tunnel might transfer weights through both of its tracks simultaneously. The next lemma shows that this never happens.

**Lemma 10.** Let $(g, e)$ and $(g′, e′)$ be the exit pairs of a tunnel $T$ such that $g$ pulls weight over $e$. Then

(i) $g$ is minor, $g \xrightarrow{e} C2$ and no other condition is satisfied for $(g, e)$,

(ii) every 2-arch $A$ of an arch of $T$ has a $C$-edge $b$ such that $(f(A), b)$ is on-track with $(g, e)$,

(iii) $g \neq g′$ and there is no 2-arch of $g′$ that has $C$-edge $e′$,

(iv) $g′$ does not pull any weight over $e′$.

(v) for every 4-arch $A$ that has an arch $T_i$ of $T$, the common extremal $C$-edge $b$ of $A$ and $T_i$ satisfies that $(f(A), b)$ is on-track with $(g, e)$,

(vi) every arch $T_i$ of $T$ that is consecutive to two transfer arches of $T$ satisfies $m_{f(T_i)} \leq 4$.
By Lemma 10(iv), all weight transfers that are caused within a tunnel by Condition C7 go one-way, i.e., use only one track of T. As an immediate implication, the following lemma shows that all weight transfers strictly within a tunnel are solely dependent on the weight transfers on its exit pairs.

**Lemma 11.** Let \((g,e)\) be a transfer pair of a tunnel track \(T\) with exit pair \((g',e')\) such that the \(e\)-opposite face \(f\) of \(g\) is minor. Then \(g\) pulls weight over \(e\) if and only if \(g'\) pulls weight over \(e'\) (and if so, \(g \xleftarrow{e} C7\) and \(g' \xleftarrow{e'} C2\)).

**Proof.** Assume that \(g'\) pulls weight over \(e'\). By Lemma 8, \(T\) is acyclic. By Lemma 10(i), \(g' \xleftarrow{e} C2\). Hence, \(C7\) is satisfied for \((g,e)\), so that \(g\) pulls weight over \(e\).

Assume to the contrary that \(g \not\xleftarrow{e} X\) for some \(X \in \{C1, \ldots, C7\}\) and \(g'\) does not pull any weight over \(e'\). The latter implies \(X \not\in C7\). Since \(f\) is minor, \(X \not\in C1\). Since \(e\) is a \(C\)-edge of a 3-arch \(B\) of \(T\) with \(f(B) = g, X \not\in C2\). By planarity, \(X \not\in C3\). By Lemma 9, \(X \not\in \{C4, C5, C6\}\), which is a contradiction. \(\square\)

An immediate implication of the discharging rule in Definition 7 is that every face pulls an non-negative integer weight over every edge, as every satisfied condition adds 1 to that weight. We next prove that no two of the conditions \(C1\)–\(C7\) are satisfied simultaneously for the same face \(g\) and edge \(e\); hence, \(g\) pulls either weight 0 or 1 over \(e\). This is crucial for keeping the amount of upcoming arguments on a maintainable level; in fact, our conditions were designed that way.

**Lemma 12.** The total weight pulled by a face of \(H\) over its \(C\)-edge \(e\) is either 0 or 1. If it is 1, the \(e\)-opposite face does not pull any weight over \(e\).

**Proof.** Assume to the contrary that \(e\) is incident to two faces \(f\) and \(g\) of \(H\) such that \(f \xleftarrow{e} X\) and \(l \xleftarrow{e} Y\) for conditions \(X\) and \(Y\) and \(l \in \{f, g\}\); without loss of generality, we assume that \(Y\) is not stated before \(X\) in Definition 7. In general, \(X = Y\) implies \(l = g\). If \(X = C1\), \(g\) is major, which implies \(Y = C1\) and thus \(l = g\); then \(f\) is major, which contradicts \(f \xleftarrow{e} X\). Hence, \(X \not\in C1\), so that both \(f\) and \(g\) are minor. If \(Y = C7\), Lemmas 10(iv) and 11 imply that \(X \not\in C7\) and \(e\) is an extremal \(C\)-edge of two consecutive 3-arches; then \(X \not\in \{C3, \ldots, C6\}\) by Lemma 9 and \(X \not\in C2\) by planarity, which is a contradiction. Hence \(Y \not\in C7\), which implies \(X \not\in C7\).

We distinguish the remaining options for \(X\) and \(Y\) in \(\{C2, \ldots, C6\}\).

Assume \(X = C2\); by our notational convention, \(v_1\) is the 2-sandwiched \(C\)-vertex of \(f\). Then \(m_f = 2\), which implies \(l = g\), as the remaining options \(Y \in \{C3, C4, C5, C6\}\) for \(l = f\) require \(m_f \geq 3\). Since \(f\) is thick, \(v_f\) exists. By Lemma 9 (for \(C2\)), \(Y \not\in \{C2, C3\}\). Assume \(Y = C4\). If \(f\) is not incident to any extremal \(C\)-edge of \(B\) (see Figure 3iii), \(G\) contains neither \(v_1v_1\) nor \(v_1v_3\) by Lemma 9 (for \(C2\)), which contradicts \(\deg_G(v_1) \geq 3\). Otherwise, consider Figure 4i. Then \(\deg_G(v_1) \geq 3\) implies \(v_1v_1 \in E(G)\), which contradicts Lemma 9 (for \(C4\)). If \(Y = C5\), \(m_{f,g} = 2\) contradicts the assumption \(m_{f,g} = 3\) of that case. If \(Y = C6\), Lemma 9 implies that neither \(v_1v_1\) nor \(v_1v_1\) is contained in \(G\), which contradicts \(\deg_G(v_1) \geq 3\).

Assume \(X = C3\). Then \(e\) is the middle \(C\)-edge of a 3-arch \(A_f\) of \(f\), which does not satisfy \(l = f\) and \(Y \in \{C4, C5, C6\}\) by definition of these conditions. We conclude \(l = g\). Then \(Y \in \{C3, C4, C5\}\) contradicts Lemma 9, and \(Y = C6\) contradicts that \(G\) is plane.
Assume $X = C4$. If $l = f$, $Y \notin \{C5, C6\}$, as 2-faces do not have 3-arches, so let $l = g$. Assume $Y \in \{C4, C5\}$ (see Figure 4ii for $Y = C4$). By Lemma 9 (applied for $A_f$ and $B$), neither $v_{-1}$ nor $v_4$ is adjacent to a vertex of $\{v_1, v_2\}$ in $G$, so that $\{v_0, v_3\}$ is a 2-separator of $G$. If $Y = C6$, we get a contradiction to planarity.

Assume $X = C5$. If $l = f$, $Y \neq C6$ by planarity, so let $l = g$. Assume $Y = C5$. By Lemma 9 (applied for $A_f$ and $B$), neither $v_{-1}$ nor $v_4$ is adjacent to a vertex of $\{v_1, v_2\}$ in $G$, so that $\{v_0, v_3\}$ is a 2-separator of $G$. If $Y = C6$, we get a contradiction to planarity.

Assume $X = C6$. Then $Y = C6$ and thus $l = g$, which contradicts that $G$ is plane. □

By Lemma 12, we know that whenever weight 1 is pulled over some edge $e$ by some condition $C1$–$C7$, no other condition is satisfied on $e$ and 1 is the final amount of weight transferred over $e$.

### 3.4 The Proof

Throughout this section, let $w$ denote the weight function on the set of faces of $H$ after our discharging rule has been applied. Clearly, $\sum_{f \in F(H)} w(f) = 2c$ still holds. For $S \subseteq E(C)$ and the set $F$ of $C$-edges of a face $f$ of $H$, let the (weight) contribution of $S$ to $f$ be $|S \cap F|$ (i.e. the initial weight these edges give to $w(f)$) plus the sum of weights pulled by $f$ over edges in $S \cap F$ minus the sum of weights pulled by opposite faces of $f$ over edges in $S \cap F$. The contribution of an arch $A$ to $f$ is the contribution of the $C$-edges of $A$ to $f$; in particular, every proper arch $A$ contributes weight $w(f(A))$ to $f(A)$. Note that we have $w(f) \geq x$ if a set $S$ contributes weight $x$ to $f$, as $f$ may lose weight at most 1 for every of its $C$-edges that is not in $S$ by Lemma 12, which cancels the initial weight 1 given by this $C$-edge.

**Lemma 13.** For two $C$-edges $a$ and $b$ of a minor face $f$, let $h \leftarrow X$ and $g \leftarrow Y$ such that $f \notin \{g, h\}$ and $X$ and $Y$ are not contained in $\{C2, C7\}$. Then

(i) $a$ (and thus $b$) is either an extremal $C$-edge of $f$ or adjacent to that, and
(ii) $a$ and $b$ have distance at least three in $C$.

Proof. For Claim (i), assume to the contrary that $a$ is neither an extremal $C$-edge of $f$ nor adjacent to that. Then $m_f \geq 5$ and $X \notin \{C2, C7\}$. Since $f$ is minor, $X \neq C1$. Since the definition of $C3$ requires an edge that is not incident to $f$, $X \neq C3$. For $X \in \{C4, C5, C6\}$, the edge $v_{-1}v_0$ in Figures 3ii–3v is not incident to $f$, which contradicts the choice of $a$.

For Claim (ii), assume to the contrary that $a$ and $b$ have distance at most two in $C$. Since $f$ is minor, $C1 \notin \{X, Y\}$. Let $X = C3$ and let $B$ be the 3-arch of $h$ that has $C$-edge $e$. Then the existence of $B$ and planarity imply $Y \neq C3$, and Lemma 9 implies $Y \notin \{C4, C5, C6\}$. Hence, $X \in \{C4, C5, C6\}$. Then $Y \notin \{C3, C4, C5, C6\}$ by planarity and the fact that, in the notation of the conditions $C4$, $C5$ and $C6$, we have $h \neq f$ and no 3-arch with $C$-edge $e$. This is a contradiction. □

Lemma 14. Let $g^{\overleftarrow{e}}-X$, $f$ be the $e$-opposite face of $g$, $B$ be the arch of $g$ shown in Figure 3 (for $X \in \{C2, C6\}$, let $B$ be the proper arch of $g$), and $S$ be the set of common $C$-edges of $f$ and $B$.

- If $X = C3$ and $|S| = 3$, $S$ contributes weight at least 2 to $f$.
- If $X \in \{C4, C5\}$, $S$ contributes weight at least $|S| - 1$ to $f$.
- If $X = C6$, $S$ contributes weight at least 1 to $f$.

Lemma 15. For a minor face $f$, let $A$ be an arch of $f$ with minimal $m_A$ such that a face $h \neq f$ pulls weight over a $C$-edge $a$ of $A$ by Condition $Y \in \{C2, C7\}$. Then $w(f) \geq 4$.

In particular, we may choose $A$ in Lemma 15 as the proper arch of $f$. This implies the following helpful corollary.

Corollary 16. Every minor face $f$ that has a $C$-edge over which an opposite face of $f$ pulls weight by $C2$ or $C7$ satisfies $w(f) \geq 4$.

We now show that Inequalities 1 and 2 hold, which proves the Isolation Lemma.

Lemma 17. Let $f$ be a face of $H$. Then $w(f) \geq 0$, $w(f) \geq 4$ if $f$ is thick and minor, $w(f) \geq 1$ if $f$ is a thin minor 2-face, and $w(f) \geq 3$ if $f$ is thin and minor such that $m_f \geq 3$.

Proof. By Lemma 12, an opposite face of $f$ pulls weight at most weight one over any $C$-edge of $f$. Since the initial weight of such an edge for $f$ is one, we have $w(f) \geq 0$. In the remaining proof, let $f$ be minor. Assume that $f$ has a $C$-edge $e'$ such that the $e'$-opposite face of $f$ pulls weight over $e'$. Then $w(f) \geq 4$ by Lemma 15. We therefore assume that no opposite face of $f$ pulls weight over a $C$-edge of $f$ by Condition $C2$ or $C7$.

Let $m_f = 2$. If $f$ is thick, Condition $C2$ and Lemma 12 imply $w(f) = 4$. If $f$ is thin, assume to the contrary that $w(f) = 0$. Then $f$ has a $C$-edge $e$ such that $g_{e}^{\overleftarrow{e}}-X$ for the $e$-opposite face $g$ of $f$. Since $X \in \{C3, C4, C5, C6\}$, Lemma 13 implies that the $C$-edge of $f$ different from $e$ contributes weight one to $f$. This contradicts $w(f) = 0$.

Let $m_f = 3$. Assume to the contrary that $f$ has an $C$-edge $e$ such that $g_{e}^{\overleftarrow{e}}-X$ for the $e$-opposite face $g$ of $f$. Then, for every $X \in \{C3, C4, C5, C6\}$, Lemma 9 contradicts that
e is a C-edge of f. Hence, $w(f) \geq 3$. If f is thin, this gives the claim, so let f be thick. Let b be the middle C-edge of f. Then b is not an extremal C-edge of a 3-arch, as then C is extendable by the replacement consisting of the two 3-arches and their two common C-edges. Hence, $f \not\prec C3$, which gives the claim $w(f) \geq 4$.

Let $m_f = 4$. Assume that f has an C-edge e such that $g \preceq X$ for the e-opposite face g of f, as otherwise $w(f) \geq 4$. Let $e = v_0v_1$. If $v_1v_3 \in E(G)$ (this is not possible for $X = C6$ by Lemma 9), C is extendable by Figure 5i for $X = C3$ and by analogous replacements for $X \in \{C4, C5\}$. If $v_{-1}v_1 \in E(G)$ (this is not possible for $X = C6$), no opposite face of f pulls weight over $v_3v_4$, as then C is extendable by Figure 5ii (or analogues for $C4, C5$ and $C6$). By Lemma 13(ii), no opposite face of f pulls weight over a non-extremal C-edge of f. Hence $w(f) \geq 3$ if f is thin and $w(f) \geq 4$ if f is thick, as then f pulls weight over $v_2v_3$ by Condition C4.

(\begin{itemize}
\item[(i)] $e = v_0v_1$ and $v_1v_3 \in E(G)$ for $X = C3$.
\item[(ii)] $e = v_0v_1$ and $v_{-1}v_1 \in E(G)$ for $X = C3$.
\end{itemize})

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure5.png}
\caption{$m_f = 4$.}
\end{figure}

If $v_1v_3 \not\in E(G)$ and $v_{-1}v_1 \not\in E(G)$, $\deg_G(v_1) \geq 3$ implies that f has a 3-arch B with extremal C-vertex $v_1$. Since $v_2v_3$ is not an extremal C-edge of another 3-arch by the previous replacements, and for the same reason no opposite face of f pulls weight over $v_3v_4$. By C3, f pulls then weight over $v_2v_3$, which gives $w(f) \geq 4$.

By Lemma 13(i) and symmetry, the only remaining case is $e = v_1v_2$. Then $X \neq C6$. Since $\{v_0, v_3\}$ is not a 2-separator of G, Lemma 9 implies that f has a 2- or 3-arch and is therefore thick. If $X = C3$, the corresponding 3-arch has only f as opposite face, so that there is a 4-arch A with $m_{f,A} = 3$ by definition of C3. For $X \in \{C4, C5\}$, such a 4-arch A with $m_{f,A} = 3$ exists by definition. Then C is extendable by the replacement that consists of A, the proper arch of f and the three common C-edge of A and f.

Let $m_f \geq 5$. By Lemma 13(i) and (ii), there are at most two C-edges of f over which opposite faces of f pull weight. Hence, $w(f) \geq m_f - 2 \geq 3$. This gives the claim if f is thin or if f is thick and $m_f \geq 6$. Hence, let f be thick and $m_f = 5$ and assume to the contrary that $w(f) \leq 3$. Then there are C-edges e and b such that $g \preceq X$ and $h \preceq Y$ for opposite faces g and h of f; by Lemma 13(ii), e and b have distance at least three in C, so that at least one of them, say e, is an extremal C-edge of f. By symmetry, we assume without loss of generality $e = v_0v_1$.

If $v_1v_4 \in E(G)$, C is extendable by Figure 6i for $X = C4$ and by analogous replacements
(i) \( v_1v_3 \in E(G) \) for \( X = C_4 \).

(ii) \( b = v_3v_4 \) and \( v_{-1}v_1 \in E(G) \) for \( X = C_4 \) and \( Y = C_3 \).

(iii) \( b = v_4v_5 \) and \( v_{-1}v_1 \in E(G) \) for \( X = C_4 \) and \( Y = C_3 \).

(iv) \( v_1v_3 \in E(G) \) and \( v_0v_4 \) for \( X = C_4 \).

Figure 6: \( m_f = 5 \).
for $X \in \{C3, C5, C6\}$, so assume otherwise. Let $b = v_3v_4$; then $Y = C6$ is not possible. Since $\{v_2, v_5\}$ is not a 2-separator of $G$, a vertex of $\{v_0, v_1\}$ is adjacent to a vertex of $\{v_3, v_4\}$ in $G$; hence, $v_1v_5 \notin E(G)$. Since $\deg_G(v_1) \geq 3$, $G$ contains $v_1v_3$ or $v_1v_5$. Then $C$ is extendable by Figure 6ii or the replacement $v_0v_1v_5v_4v_3v_1v_2v_0$. Hence, $b = v_4v_5$. Since both $e$ and $b$ are now extremal $C$-edges, assume by symmetry that if at least one of $\{v_0, v_5\}$ is the middle $C$-vertex of a 2-arch, $v_0$ is.

Let $v_1v_3 \in E(G)$; this implies $X \neq C6$. Then $C$ is extendable by Figure 6iii (the analogue for $X = C5$ is the same as the one given for $C4$; all may be used separately on both the left and right hand side). Hence, assume $v_1v_3 \notin E(G)$ and by our previous assumption thus $v_4v_6 \notin E(G)$. By $\deg_G(v_1) \geq 3$ and Lemma 9, $G$ contains $v_1v_3$ or $v_1v_5$ for every $X \in \{C3, C4, C5, C6\}$. By symmetry, the same holds for $v_4$, so that we have either $v_1v_3 \in E(G)$ and $v_0v_4$ or $v_2v_4 \in E(G)$ and $v_1v_5$, say the first one by symmetry. Then $C$ is extendable by Figure 6iv. This proves the claim.

\section{Algorithm}

We sketch how a cycle of length at least $\frac{3}{5}(n+2)$ in any essentially 4-connected planar graph $G$ on $n$ vertices can be computed. For $n \leq 10$, all such graphs are known to be Hamiltonian and we may compute even a longest cycle in constant time, so assume $n \geq 11$. Now we may use the quadratic-time algorithm from [10] to find a starting isolating (Tutte-)cycle $C$ (this requires to choose the start and end-edges of $C$ carefully, as described in [4]).

If the cycle has a thick minor 1-face, we extend the cycle by its arch. Otherwise, we can detect in in linear time which case of the case distinction of this paper we are. Now every extension may be carried out in linear time as well (we always maintain the current cycle $C$ and the types of the faces that are incident to $C$). Iterating this implies a total running time of $O(n^2)$.

\textbf{Acknowledgments.} The second author wants to thank Jochen Harant for introducing him to this topic and Igor Fabrici, Jochen Harant and Samuel Mohr for many discussions on that topic.

\section*{References}

[1] M. B. Dillencourt. Polyhedra of small order and their Hamiltonian properties. \textit{Journal of Combinatorial Theory, Series B}, 66(1):87–122, 1996.

[2] I. Fabrici, J. Harant, and S. Jendrol. On longest cycles in essentially 4-connected planar graphs. \textit{Discussiones Mathematicae Graph Theory}, 36:565–575, 2016.

[3] I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt. Circumference of essentially 4-connected planar triangulations. submitted.

[4] I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt. Longer cycles in essentially 4-connected planar graphs. \textit{Discussiones Mathematicae Graph Theory}, 40:269–277, 2020.
[5] I. Fabrici, J. Harant, S. Mohr, and J. M. Schmidt. On the circumference of essentially 4-connected planar graphs. *Journal of Graph Algorithms and Applications*, 24(1):21–46, 2020.

[6] B. Grünbaum and J. Malkevitch. Pairs of edge-disjoint Hamilton circuits. *Aequationes Mathematicae*, 14:191–196, 1976.

[7] B. Grünbaum and H. Walther. Shortness exponents of families of graphs. *Journal of Combinatorial Theory, Series A*, 14(3):364–385, 1973.

[8] B. Jackson and N. C. Wormald. Longest cycles in 3-connected planar graphs. *Journal of Combinatorial Theory, Series B*, 54:291–321, 1992.

[9] J. W. Moon and L. Moser. Simple paths on polyhedra. *Pacific J. Math.*, 13(2):629–631, 1963.

[10] A. Schmid and J. M. Schmidt. Computing 2-walks in polynomial time. *ACM Transactions on Algorithms*, 14(2):22.1–22.18, 2018.

[11] W. T. Tutte. A theorem on planar graphs. *Transactions of the American Mathematical Society*, 82:99–116, 1956.

[12] H. Whitney. Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54(1):150–168, 1932.

[13] D. R. Woodall. The binding number of a graph and its Anderson number. *Journal of Combinatorial Theory, Series B*, 15(3):225–255, 1973.

[14] C.-Q. Zhang. Longest cycles and their chords. *Journal of Graph Theory*, 11:341–345, 1987.