Is the Free Locally Convex Space $L(X)$ Nuclear?

Arkady Leiderman and Vladimir Uspenskij

Dedicated to María Jesús Chasco on the occasion of her birthday.

Abstract. Given a class $\mathcal{P}$ of Banach spaces, a locally convex space (LCS) $E$ is called multi-$\mathcal{P}$ if $E$ can be isomorphically embedded into a product of spaces that belong to $\mathcal{P}$. We investigate the question whether the free locally convex space $L(X)$ is strongly nuclear, nuclear, Schwartz, multi-Hilbert or multi-reflexive. If $X$ is a Tychonoff space containing an infinite compact subset then, as it follows from the results of Außenhofer (Topol Appl 134:90–102, 2007), $L(X)$ is not nuclear. We prove that for such $X$ the free LCS $L(X)$ has the stronger property of not being multi-Hilbert. We deduce that if $X$ is a $k$-space, then the following properties are equivalent: (1) $L(X)$ is strongly nuclear; (2) $L(X)$ is nuclear; (3) $L(X)$ is multi-Hilbert; (4) $X$ is countable and discrete. On the other hand, we show that $L(X)$ is strongly nuclear for every projectively countable $\omega$-space (in particular, for every Lindelöf $\omega$-space) $X$. We observe that every Schwartz LCS is multi-reflexive. It is known that if $X$ is a $k_\omega$-space, then $L(X)$ is a Schwartz LCS (Außenhofer et al. in Stud Math 181(3):199–210, 2007), hence $L(X)$ is multi-reflexive. We show that for every first-countable paracompact (in particular, for every metrizable) space $X$ the converse is true, so $L(X)$ is multi-reflexive if and only if $X$ is a $k_\omega$-space, equivalently if $X$ is a locally compact and $\sigma$-compact space. Similarly, we show that for any first-countable paracompact space $X$ the free abelian topological group $A(X)$ is a Schwartz group if and only if $X$ is a locally compact space such that the set $X^{(1)}$ of all non-isolated points of $X$ is $\sigma$-compact.

Mathematics Subject Classification. Primary 46A03; Secondary 46B25, 54D30.

Keywords. Free locally convex space, nuclear locally convex space, Schwartz locally convex space, Hilbert space, reflexive Banach space

Published online: 12 October 2022
1. Introduction

All topological spaces are assumed to be Tychonoff and all topological vector spaces and groups are Hausdorff. All vector spaces are considered over the field of real numbers \( \mathbb{R} \). The abbreviation LCS means locally convex space.

Every Tychonoff space \( X \) can be considered as a subspace of its free locally convex space \( L(X) \) characterized by the following property: every continuous mapping \( f: X \to E \) to an LCS \( E \) uniquely extends to a continuous linear mapping \( \hat{f}: L(X) \to E \).

Similarly, one defines the free abelian group \( A(X) \) over a Tychonoff space \( X \). \( X \) is a subspace of the generators of \( A(X) \) and the topology of \( A(X) \) is characterized by the property that every continuous mapping \( f: X \to G \) to an abelian topological group \( G \) uniquely extends to a continuous homomorphism \( \hat{f}: A(X) \to G \). It is known that \( A(X) \) naturally embeds into \( L(X) \) \([33,35]\).

Free topological groups were introduced in 1941 by A. A. Markov in the short note \([21]\). The complete construction appeared 4 years later, in \([22]\), where certain basic properties of a free topological group \( F(X) \) and free abelian topological group \( A(X) \) over a Tychonoff space \( X \) were established. In particular, Markov answered in the negative Kolmogorov’s question as to whether every topological group is a normal space. Several proofs of the existence of free topological groups have been given, a proof, due to Graev \([15,16]\), is based on extending continuous pseudometrics from a Tychonoff space \( X \) to invariant continuous pseudometrics on \( F(X) \) (or \( A(X) \)). Free topological vector spaces were mentioned by Markov \([21]\), without any details. Later, in 1964, D. A. Raikov constructed the free locally convex space \( L(X) \) for a uniform space \( X \) \([26]\).

Free LCS constitute a very important subclass of locally convex spaces. It suffices to mention that every locally convex space is a linear quotient of a free LCS and that every Tychonoff space \( X \) embeds as a closed subspace into \( L(X) \). Analogous basic facts are valid for \( F(X) \) and \( A(X) \).

Given a class \( \mathcal{P} \) of Banach spaces, an LCS \( E \) is called multi-\( \mathcal{P} \) if \( E \) can be isomorphically embedded in a product of spaces in \( \mathcal{P} \). The following result was obtained in \([36]\): for every Tychonoff space \( X \) the free LCS \( L(X) \) can be isomorphically embedded in the product of Banach spaces of the form \( \ell^1(\Gamma) \), in other words, \( L(X) \) is multi-\( \mathcal{L}^1 \), where \( \mathcal{L}^1 \) is the class of all Banach spaces of the form \( \ell^1(\Gamma) \). It follows that, as a topological group, \( L(X) \) admits a
topologically faithful unitary representation, that is, $L(X)$ is isomorphic to a subgroup of the unitary group.

Vladimir Pestov raised the following

**Question 1.1.** When is $L(X)$ multi-Hilbert?

The question was motivated by the fact that $L(X)$ is nuclear, hence multi-Hilbert, if $X$ is a countable discrete space [31, Ch. 3, Theorem 7.4]; in that case $L(X)$ is the locally convex direct sum of countably many one-dimensional spaces, and will be denoted by $\varphi$.

In Sect. 2 we prove that $L(X)$ is not multi-Hilbert whenever $X$ contains an infinite compact subset. As a consequence we deduce that if $X$ is a $k$-space, then the following properties are equivalent: (1) $L(X)$ is strongly nuclear; (2) $L(X)$ is nuclear; (3) $L(X)$ is multi-Hilbert; (4) $X$ is countable and discrete.

On the other hand, we show that $L(X)$ is strongly nuclear (hence nuclear) for every projectively countable $P$-space (in particular, for every Lindelöf $P$-space) $X$.

Recall the definition of nuclear maps and nuclear spaces.

**Definition 1.2.** A linear map $u: E \to B$ from an LCS $E$ to a Banach space $B$ is called nuclear if it can be written in the form

$$u(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x)y_n,$$

where $\sum |\lambda_n| < +\infty$, $(f_n)$ is an equicontinuous sequence of linear functionals on $E$, and $(y_n)$ is a bounded sequence in $B$. An LCS $E$ is called nuclear if any continuous linear map from $E$ to a Banach space is nuclear.

The definition of nuclear LCS and the following important permanence results about nuclear LCS are due to Grothendieck [18] (see also [31, Ch. 3, §7]).

- Every subspace of a nuclear LCS is nuclear.
- Every Hausdorff quotient space of a nuclear LCS is nuclear.
- The product of an arbitrary family of nuclear LCS is nuclear.
- The locally convex direct sum of a countable family of nuclear LCS is a nuclear space.
- A closed bounded subset of a nuclear Fréchet space is compact, and then there are no infinite-dimensional Banach spaces that are nuclear.
- Every nuclear LCS is multi-Hilbert.

Note also

- The class of multi-Hilbert LCS is closed under Hausdorff quotients.

Some of the most important occurrences of the nuclear LCS in analysis are: (1) The Schwartz space of smooth functions on $\mathbb{R}^n$ for which the derivatives of all orders are rapidly decreasing; (2) The space of entire holomorphic functions on the complex plane endowed with the compact open topology.

Lemma 2.5 below also should be attributed to A. Grothendieck. Nuclear groups were introduced and investigated by Banaszczyk [5]. Note that the
class of nuclear groups contains all nuclear LCS. For the definition and a study of free abelian nuclear groups we refer the reader to [2].

Michael Megrelishvili asked whether $L(X)$ is multi-reflexive. It turns out that this problem is tightly connected with the properties of the Schwartz locally convex spaces. Schwartz LCS play an important role in analysis and its applications. Remark that in several aspects the properties of Schwartz LCS resemble the finite-dimensional Banach spaces, for instance, their bounded subsets are precompact. The latter property for Banach spaces is equivalent to finite dimensionality. The notion of Schwartz LCS was introduced by Grothendieck [17]. In this paper we will use the following equivalent versions:

**Definition 1.3.** An LCS $E$ is called a *Schwartz space* if one of the following equivalent conditions holds:

(a) Every continuous linear map from $E$ to a Banach space is compact, that is, sends a neighborhood of zero to a set with a compact closure;

(b) For every neighborhood $U$ of zero in $E$, there exists another neighborhood $V$ of zero such that for every $\lambda > 0$ the set $V$ can be covered by finitely many translates of $\lambda U$.

Several relevant problems in the theory of Schwartz LCS were solved in [6, 7, 27, 28, 32] (our list of references does not attempt to be complete). Note that the class of all Schwartz LCS (as well as the classes of all strongly nuclear, all nuclear, all multi-Hilbert and all multi-reflexive LCS) constitutes a *variety*, i.e. is closed under the formation of subspaces (not necessarily closed), Hausdorff quotients, arbitrary products and isomorphic images (see [8]). Also, multi-reflexive LCS sometimes are called *infra-Schwartz* (see [19, Section 21.1]).

We observe that every Schwartz LCS is multi-reflexive. This follows, for example, from [19, 17.2.9 and 21.1.1(c)]; we give a short proof of this assertion in Theorem 3.3. Thus, the interplay between the four properties introduced above can be summarized in the following diagram (all implications below in general are not reversible). For every locally convex space $L$ the following holds:

\[
\begin{array}{c}
L \text{ is nuclear} \quad \iff \quad L \text{ is Schwartz} \\
\quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
L \text{ is multi-Hilbert} \quad \iff \quad L \text{ is multi-reflexive}
\end{array}
\]

It is known that $L(X)$ is a Schwartz space for every $k_\omega$-space $X$ [4, Theorem 5.2]. In Sect. 3 we provide an alternative proof of this fact (Theorem 3.5). Recall that a topological space $X$ satisfies the *first axiom of countability* if each point of $X$ has a countable base of neighborhoods. In Sect. 3 we give a complete answer to the question: when is $L(X)$ multi-reflexive?—for paracompact spaces $X$ satisfying the first axiom of countability; namely, if $X$ is such a space, then the following properties are equivalent: (1) $L(X)$ is a Schwartz space; (2) $L(X)$ is multi-reflexive; (3) $X$ is locally compact and $\sigma$-compact ($= X$ is the union of countably many compact sets). In particular, for a metrizable space $X$, $L(X)$ is multi-reflexive if and only if $X$ is locally
compact space and has a countable base. We also give a short direct proof of the fact that $L(J)$ is not multi-reflexive, where $J$ denotes the space of irrational numbers (Example 3.14).

A group version of the concept of a Schwartz space is introduced and studied in the paper [4] (see also [3]). In Sect. 4 we give a complete answer to the question: when is the free abelian group $A(X)$ a Schwartz group?—for paracompact spaces $X$ satisfying the first axiom of countability. In that case, $A(X)$ is a Schwartz group if and only if $X$ is a locally compact space such that the set $X^{(1)}$ of all non-isolated points of $X$ is $\sigma$-compact.

Our notations are standard, the reader is advised to consult with the monographs [10,11,31] for the notions which are not explicitly defined in the text. At the end of the article, we pose several open questions.

2. Nuclear and Multi-Hilbert $L(X)$

Let $X$ be a Tychonoff space containing an infinite compact subset $K$. Every continuous pseudometric defined on the compact space $K$ extends to a continuous pseudometric on $X$, therefore $L(K)$ can be identified by a linear topological isomorphism with a subspace of $L(X)$ [35]. Since the free abelian group $A(K)$ naturally embeds into $L(K)$, we observe that $A(K)$ is isomorphic to a topological subgroup of $L(X)$. Hence, if $L(X)$ is a nuclear LCS, then $A(K)$ is a nuclear group. However, the free abelian group $A(K)$ is nuclear if and only if a compact space $K$ is finite [2, Theorem 7.3]. This observation leads to

Proposition 2.1. If a Tychonoff space $X$ contains an infinite compact subset, then $L(X)$ is not nuclear.

Below we prove a significantly stronger result, the proof of which requires more elaborate arguments from functional analysis.

Theorem 2.2. If a Tychonoff space $X$ contains an infinite compact subset, then $L(X)$ is not multi-Hilbert.

To prove Theorem 2.2 we need some preparations. Let us say that a subset $C$ of a Banach space $E$ is an ellipsoid if $C$ is the image of a closed ball in a Hilbert space $H$ under a bounded linear mapping $H \to E$. Note that ellipsoids are weakly compact, hence closed in $E$.

Lemma 2.3. There exists a Banach space $E$ and a countable sequence $S = (x_n)$ converging to zero in $E$ such that $S$ is not contained in any ellipsoid in $E$.

Proof. Let $E$ be a Banach space without the approximation property. According to [12, Theorem 1.2], there is a compact subset $K$ in $E$ such that for any Banach space $F$ with a basis and any injective bounded operator $T:F \to E$ the image $T(F)$ does not contain $K$. Then $K$ is not contained in any ellipsoid. Indeed, if $H$ is a Hilbert space and $A:H \to E$ is a bounded linear operator, we can write $A$ as the composition $H \to \tilde{H} \to E$, where $\tilde{H}$
is the quotient of $H$ by the kernel of the operator $A$. Then $\tilde{H}$ is a Hilbert space and the operator $T: \tilde{H} \to E$ is injective, hence $K$ is not contained in the image $T(\tilde{H}) = A(H)$.

Now we find a countable sequence $S = (x_n)$ in $E$ converging to zero and such that $K$ is contained in the closed convex hull of $S$. If an ellipsoid contains $S$, then it also contains $K$, which is impossible. \qed

Remark 2.4. Note that a sequence with the properties stated in Lemma 2.3 can be found in any Banach space $E$ not isomorphic to a Hilbert space. For the proof, one can use, for example, the following: in every Banach space which is non-isomorphic to a Hilbert space there is a sequence $F_n$ of finite-dimensional subspaces whose Banach–Mazur distances from the Euclidean spaces of the same dimension goes to infinity [20].

Lemma 2.5. Let $L$ be a multi-Hilbert LCS. Then every continuous linear map $f: L \to E$ from $L$ to a Banach space $E$ can be represented as a composition $f = p \circ g$, where $g: L \to H$ and $p: H \to E$ are two continuous linear maps and $H$ is a Hilbert space.

Proof. We assume that $L$ is isomorphically embedded into the product $\prod_{i \in I} E_i$, where all $E_i$ are Hilbert spaces. Denote by $V$ the unit open ball of $E$.

Since $f$ is continuous, and by the definition of the product topology, there is a finite face $E_A = \prod_{i \in A} E_i$ of the product $\prod_{i \in I} E_i$, and a neighborhood $U$ of zero in $E_A$, such that $f(g^{-1}(U))$ is contained in $V$. Here the set of indexes $A \subset I$ is finite and $g = \pi |_L : L \to E_A$ is the restriction to $L$ of the corresponding canonical projection $\pi: \prod_{i \in I} E_i \to E_A$. Obviously, $E_A$ is isomorphic to a Hilbert space being a finite product of Hilbert spaces.

We claim that $\ker(g) \subset \ker(f)$. Indeed, if $x \in L$ and $g(x) = 0$, then $g(tx) = 0$ for every scalar $t$, so $tx \in g^{-1}(U))$. This means that $f(tx) = t(f(x))$ belongs to the same unit ball $V$ for every scalar $t$, which implies that $f(x) = 0$. So there is some map $p: g(L) \to E$ such that $f = p \circ g$. Clearly, $p$ is a linear map. It is also bounded, because $p(U \cap g(L)) = f(g^{-1}(U)) \subseteq V$. Further, the linear continuous map $p$ is defined on the vector subspace $g(L)$ of the Hilbert space $E_A$ and this map $p$ can be extended by continuity to a linear continuous map (for which we keep the same notation $p$) defined on the closure $H$ of $g(L)$ in $E_A$. A closed vector subspace $H$ of a Hilbert space is Hilbert itself, and we get the required factorization. \qed

Fact 2.6. Let $S$ be a non-trivial countable convergent sequence. Then $L(S)$ is not multi-Hilbert.

Proof. On the contrary, suppose that $L(S)$ is multi-Hilbert. By Lemma 2.3 we define a 1-to-1 continuous map $f: S \to E$ such that $f(S)$ is not contained in an ellipsoid. Extend $f$ to a linear continuous map $\hat{f}: L(S) \to E$. Now, by Lemma 2.5, we can represent $\hat{f}$ as a composition $\hat{f} = p \circ \hat{g}$, where $\hat{g}: L(S) \to H$ and $p: H \to E$ are two continuous linear maps, $\hat{g}$ extends the map $g: S \to H$ and $H$ is a Hilbert space. The following diagram illustrates our construction.
The compact set $g(S) \subset H$ is contained in a ball, hence $f(S) = p(g(S))$ is contained in an ellipsoid. The obtained contradiction finishes the proof. □

**Proof of Theorem 2.2.** Let $K$ be an infinite compact subset of $X$ and assume that $L(X)$ is multi-Hilbert. As we have noted before $L(K)$ can be identified with a linear subspace of $L(X)$, hence $L(K)$ is multi-Hilbert. $K$ is an infinite compact space, therefore we can choose a continuous map $\pi: K \to [0, 1]$ with an infinite image. Let $M = \pi(K)$. We have that $\pi: K \to M$ is a closed (hence, quotient) continuous surjective mapping. In this situation $\pi$ lifts to a linear continuous quotient mapping $\hat{\pi}$ from $L(K)$ onto $L(M)$. The latter fact can be proved analogously to [24, Proposition 1.8]. Since the class of multi-Hilbert LCS is closed under Hausdorff quotients we conclude that $L(M)$ is also multi-Hilbert. However, $M$ is an infinite compact subset of the segment $[0, 1]$, therefore $M$ contains a copy of a convergent sequence $S$. This would mean that $L(S)$ is multi-Hilbert contradicting Fact 2.6. The obtained contradiction finishes the proof of Theorem 2.2. □

Recall that a topological space $X$ is called a $k$-space whenever $F \subset X$ is closed iff the intersection $F \cap K$ is closed in $K$ for every compact $K \subset X$. If $X$ can be covered by countably many compact subsets $K_n$ such that $F \subset X$ is closed iff all intersections $F \cap K_n$ are closed, then $X$ is said to be a $k_\omega$-space [13].

An LCS $E$ is called strongly nuclear [27, 1.4] if for every continuous seminorm $q$ on $E$ there exist a rapidly decreasing sequence $(\lambda_n)$ (this means that $\sum n^k|\lambda_n| < \infty$ for every $k$) and an equicontinuous sequence $(a_n)$ of linear functionals on $E$ such that $q(x) \leq \sum |\lambda_n|\langle x, a_n \rangle$ for all $x \in E$. It is easy to verify that $L(\mathbb{N}) = \varphi$ is strongly nuclear, where $\mathbb{N}$ denotes the discrete space of natural numbers.

Following [8] denote by $\mathcal{V}(\varphi)$ the variety of LCS generated by $\varphi$. Note that $\mathcal{V}(\varphi)$ is the (unique) second smallest variety, in the sense that every variety properly containing $\mathcal{V}(\mathbb{R})$ contains $\mathcal{V}(\varphi)$ [8].

**Corollary 2.7.** Let $X$ be a $k$-space. The following conditions are equivalent:

(i) $L(X)$ is strongly nuclear;
(ii) $L(X)$ is nuclear;
(iii) $L(X)$ is multi-Hilbert;
(iv) $X$ is a countable discrete space.

**Proof.** Only the implication (iii) $\Rightarrow$ (iv) is new and needs to be proved. By Theorem 2.2, every compact subset of $X$ is finite, therefore the $k$-space $X$ is discrete. Every multi-Hilbert space is multi-reflexive, and we will show later that $L(X)$ is not multi-reflexive if $X$ is an uncountable discrete space (Fact 3.9). □
It follows from Theorem 2.8 below that there exist non-separable spaces \( X \) such that \( L(X) \) are strongly nuclear, hence nuclear. We say that a Tychonoff space \( X \) is projectively countable if every metrizable image of \( X \) under a continuous map is countable. Recall that a \( P \)-space is a topological space in which the intersection of countably many open sets is open. To see that Lindelöf \( P \)-spaces are projectively countable, observe that if \( X \) is a \( P \)-space, all points of \( Y \) are \( G_{\delta} \), and \( f : X \to Y \) is a continuous map, then the inverse images of points from \( Y \) form a disjoint open cover of \( X \). It is known that \( \omega \)-modification of a topology of any scattered Lindelöf space produces a Lindelöf \( P \)-space. A concrete example of an uncountable Lindelöf \( P \)-space is the so-called one-point Lindelöfication \( X = \Gamma \cup \{ p \} \) of an uncountable discrete space \( \Gamma \). All open neighborhoods of the point \( p \) in \( X \) are of the form \( A \cup \{ p \} \), where \( A \subset \Gamma \) and \( |\Gamma \setminus A| \leq \aleph_0 \).

**Theorem 2.8.** If \( X \) is a projectively countable \( P \)-space (for instance, a Lindelöf \( P \)-space), then \( L(X) \) is strongly nuclear.

**Proof.** Let \( M \) be a metrizable space and \( f : X \to M \) be a continuous map. Then \( f(X) \) is countable. Every point \( m \in M \) is a \( G_{\delta} \)-set in \( M \), therefore every fiber \( f^{-1}(m) \) is open in the \( P \)-space \( X \). We see that \( X \) is the disjoint countable union of open fibers \( f^{-1}(m) \), \( m \in f(X) \). It follows that every continuous map \( f : X \to M \) admits a factorization \( X \to \mathbb{N} \to M \), where \( \mathbb{N} \) is the discrete space of natural numbers, hence any linear continuous map \( L(X) \to B \) to a Banach space \( B \) admits a factorization \( L(X) \to L(\mathbb{N}) \to B \). Using [27, Remark 3.4] we can conclude that \( L(X) \) can be isomorphically embedded into a power of the space \( \varphi \). This means that \( L(X) \) belongs to \( \mathcal{V}(\varphi) \) which is a sub-variety of the variety of all strongly nuclear spaces [8], since \( L(\mathbb{N}) = \varphi \) is strongly nuclear. \( \square \)

It turns out that there are projectively countable \( P \)-spaces which are not Lindelöf \( P \)-spaces. A Tychonoff space \( X \) is called cellular-Lindelöf if for every disjoint family \( \mathcal{U} \) of open nonempty subsets of \( X \) there is a Lindelöf subspace \( L \) of \( X \) such that \( L \) meets every member of the family \( U \). It is easy to see that every cellular-Lindelöf \( P \)-space is projectively countable. Hence, Theorem 2.8 implies

**Proposition 2.9.** If \( X \) is a cellular-Lindelöf \( P \)-space, then \( L(X) \) is a strongly nuclear LCS.

Another quick consequence provides sufficient conditions for the free abelian group \( A(X) \) to be nuclear:

**Corollary 2.10.** If \( X \) is a cellular-Lindelöf \( P \)-space, then \( A(X) \) is a nuclear topological group.

See [34, Theorem 3.16] for an example of a cellular-Lindelöf \( P \)-space which is not even weakly Lindelöf.

Besides the requirement that \( X \) should contain no infinite compact subspaces, there are other necessary conditions for \( L(X) \) to be multi-Hilbert. First, if \( L(X) \) is multi-Hilbert, then \( L(X) \) is multi-reflexive, and then every metrizable image of \( X \) under a continuous map must be separable by Corollary 3.19. However, the following question remains open.
Problem 2.11. Let $L(X)$ be nuclear (multi-Hilbert). Does it imply that $X$ must be projectively countable?

We finish Sect. 2 by providing necessary conditions on $X$ towards the solution of Problem 2.11.

Theorem 2.12. If $L(X)$ is multi-Hilbert, then $X$ cannot be continuously mapped onto the closed segment $I = [0, 1]$.

Proof. Recall that any metric space $M$ can be isometrically embedded into the Lipschitz-free Banach space over $M$, which is usually denoted by $\mathcal{F}(M)$ (see [14]). It is easy to see that $\mathcal{F}(M)$ is not Hilbert. By Remark 2.4 we know that in every Banach space which is not isomorphic to a Hilbert space, there exists a convergent sequence that does not lie in any ellipsoid.

It follows that there exists such a metric on the convergent sequence $S$ that the Lipschitz-free Banach space $\mathcal{F}(S)$ has the property: $S$ is not contained in any ellipsoid lying in $\mathcal{F}(S)$. Call such a metric on $S$ wild. Extend such a wild metric to $I$ to get a wild metric on $I$. If we replace a wild metric by any larger metric, we still get a wild metric. Summing an appropriate series of metrics on $I$, we can construct such a metric on $I$ that it is wild on any segment $[a, b] \subset I$. Now assume $L(X)$ is multi-Hilbert and there exists an onto map $\pi: X \to I$. Equip $I$ with a hereditarily wild metric, as above, and consider the extension $\widetilde{\pi}: L(X) \to \mathcal{F}(I)$. The mapping $\widetilde{\pi}$ factorizes through a Hilbert space: $L(X) \to H \to \mathcal{F}(I)$, and the image of $H \to \mathcal{F}(I)$ contains $I$. Since $H$ is the union of a sequence of balls, $I$ is covered by a sequence of ellipsoids. Ellipsoids are weakly compact sets, hence they are closed in $\mathcal{F}(I)$. The Baire Category Theorem implies that a certain segment $[a, b]$ is covered by an ellipsoid, contradicting the fact that our metric was wild on every segment. \hfill \Box

A continuous image of the space of irrationals $J$ is called an analytic space.

Corollary 2.13. If $L(X)$ is multi-Hilbert, and $Y$ is a continuous image of $X$, then every analytic subspace of $Y$ is countable.

Proof. On the contrary, assume that $Y$ admits an uncountable analytic subspace. Every uncountable analytic space contains a homeomorphic copy of the Cantor set (e.g. see [29, Corollary 3.5.2]). Cantor set can be mapped continuously onto $[0, 1]$, and this mapping extends to a continuous mapping defined on $Y$. We conclude that $X$ can be continuously mapped onto $[0, 1]$, which is impossible by Theorem 2.12. The obtained contradiction implies that $Y$ cannot have uncountable analytic subspaces. \hfill \Box

Remark 2.14. Analyzing the proof of Theorem 2.12 we see that it may be generalized as follows: if $L(X)$ is multi-Hilbert, and $Y$ is a metrizable continuous image of $X$, then $Y$ can be represented as the union of the at most countable set of isolated points and a meager set $Y^{(1)}$ of non-isolated points (i.e. $Y^{(1)}$ is a countable union of nowhere-dense subsets). Evidently, every countable metrizable $Y$ is such a space. The converse is not true: there exists an uncountable subspace of $[0, 1]$ without isolated points which is meager in itself.
3. Schwartz and Multi-reflexive $L(X)$

Recall that an LCS is multi-reflexive if it is isomorphic to a subspace of a product of reflexive Banach spaces. For instance, every LCS endowed with its weak topology is multi-Hilbert (hence multi-reflexive), since it embeds into a product of the real lines. However, a reflexive LCS need not be multi-reflexive [23, Remark 4.14 (a)].

Our proof of Theorem 2.2 is based on the fact that certain compact subsets of Banach spaces are not contained in ellipsoids. However, every weakly compact subset of a Banach space is contained in the image of the closed unit ball of a reflexive space under a bounded linear mapping (see [11, Theorem 13.22]). This motivates the following question (suggested to us by Michael Megrelishvili):

**Question 3.1.** When is $L(X)$ multi-reflexive?

We have seen that $L(X)$ is not multi-Hilbert if $X$ is an infinite compact space (Theorem 2.2). Nevertheless, $L(X)$ is a Schwartz LCS for every compact space $X$ [4]. We provide an alternative proof of this statement (Theorem 3.5) and due to the fact that every Schwartz LCS is multi-reflexive (Theorem 3.3), we conclude that $L(X)$ is multi-reflexive for every compact space $X$ (Theorem 3.6).

In general, $L(X)$ need not be Schwartz or multi-reflexive. We provide a complete characterization of first-countable paracompact spaces $X$ such that $L(X)$ is Schwartz or multi-reflexive (Theorem 3.12). In particular, if $X$ is metrizable, then $L(X)$ is multi-reflexive iff $L(X)$ is a Schwartz LCS iff $X$ is locally compact and has a countable base (Corollary 3.13).

**Lemma 3.2.** If $L$ is a Schwartz LCS, then every continuous linear map $L \to B$ to a Banach space $B$ admits a factorization $L \to B_1 \to B$, where $B_1$ is a Banach space, $L \to B_1$ is a continuous linear map and $B_1 \to B$ is a compact operator.

**Proof.** Let $U$ be a balanced convex neighborhood of zero in $L$ such that the image of $U$ in $B$ has a compact closure. Let $p$ be the gauge functional corresponding to $U$. We can take for $B_1$ the completion $\tilde{L}_U$ of the normed space $L_U$ associated with the seminorm $p$, as in [31, Ch. 3, §7].

**Theorem 3.3.** Every Schwartz LCS is multi-reflexive.

**Proof.** Let $L$ be a Schwartz LCS. We want to prove that continuous linear maps from $L$ to reflexive Banach spaces generate the topology of $L$. Since maps of $L$ to all Banach spaces generate the topology of $L$, it suffices to prove that every continuous linear map $L \to B$ to a Banach space admits a factorization $L \to B_1 \to B$, where $B_1$ is a reflexive Banach space. According to Lemma 3.2, we can find a factorization $L \to B_2 \to B$, where $B_2 \to B$ is a compact operator between Banach spaces. Compact operators are weakly compact, and in virtue of the Davis–Figiel–Johnson–Pelszynski theorem [11, Theorem 13.33] weakly compact operators between Banach spaces admit a factorization through reflexive spaces. Thus we have a required factorization $L \to B_2 \to B_1 \to B$ for some reflexive Banach space $B_1$. 

$\square$
Below we collect several basic facts about the topology of $L(X)$ and associated Banach spaces. For a space $X$ we denote by $L_0(X)$ the hyperplane

$$L_0(X) = \left\{ \sum c_i x_i : \sum c_i = 0, \ c_i \in \mathbb{R}, \ x_i \in X \right\}$$

of $L(X)$. The topology of $L_0(X)$ is generated by the seminorms $\bar{d}$ of the following form. For a continuous pseudometric $d$ on $X$, the seminorm $\bar{d}$ on $L_0(X)$ is defined by

$$\bar{d}(v) = \inf \left\{ \sum |c_i| d(x_i, y_i) : v = \sum c_i (x_i - y_i), \ c_i \in \mathbb{R}, \ x_i, y_i \in X \right\}. \ (3.1)$$

Denote by $B_d$ the Banach space associated with the seminormed space $(L_0(X), \bar{d})$, that is, the completion of the quotient by the kernel of $\bar{d}$. The dual Banach space $B_d^*$ is naturally isomorphic to the quotient $C_d/\mathbb{R}$ of the space of $d$-Lipschitz functions on $X$ by the one-dimensional subspace of constant functions. The seminorm on $C_d$ assigns to each $d$-Lipschitz function $f$ its Lipschitz constant

$$\|f\| = \inf \left\{ k \geq 0 : |f(x) - f(y)| \leq k \cdot d(x, y) \text{ for all } x, y \in X \right\}. \ (3.2)$$

**Lemma 3.4.** If $X$ is a compact space and $d$ is a continuous pseudometric on $X$, then the natural operator $B_{\sqrt{d}} \to B_d$ is compact.

**Proof.** We may assume that $d$ is a metric. Pick a point $x_0 \in X$, and identify the quotient $C_d/\mathbb{R}$ with the Banach space $\hat{C}_d = \{ f \in C_d ; f(x_0) = 0 \}$. A bounded linear operator between Banach spaces is compact if and only if its dual is compact, by the Shauder theorem [11, Theorem 15.3], so it suffices to prove that the embedding $\hat{C}_d \to \hat{C}_{\sqrt{d}}$ is compact. Take any sequence $(f_n)$ in the unit ball of $\hat{C}_d$, that is, a sequence of $d$-non-expanding functions such that $f_n(x_0) = 0$. By the Arzelà–Ascoli theorem, the sequence $(f_n)$ has a uniformly convergent subsequence. To simplify the notation, assume that the sequence $(f_n)$ itself uniformly converges to a limit $f$. We claim that $(f_n)$ converges to $f$ in $\hat{C}_{\sqrt{d}}$ as well. Let $\epsilon > 0$ be given. Put $g_n = f_n - f$. We must prove that for $n$ large enough we have

$$|g_n(x) - g_n(y)| \leq \sqrt{d(x, y)} \ (3.3)$$

for all $x, y \in X$. Note that each $g_n$ is 2-Lipschitz with respect to $d$. If $x$ and $y$ are close to each other, namely, if $d(x, y) \leq \epsilon^2/4$, then (3.3) holds:

$$|g_n(x) - g_n(y)| \leq 2d(x, y) = 2\sqrt{d(x, y)} \sqrt{d(x, y)} \leq \epsilon \sqrt{d(x, y)}.$$

On the other hand, for pairs $x, y$ such that $d(x, y) > \epsilon^2/4$ the inequality 3.3 holds for $n$ large enough because the sequence $(g_n)$ uniformly converges to zero. \qed

**Theorem 3.5.** If $X$ is a compact space, then $L(X)$ is a Schwartz LCS.

**Proof.** Let $T : L(X) \to B$ be a continuous linear operator, where $B$ is a Banach space. We want to prove that $T$ is compact. Let $d$ be the pseudometric on $X$ induced by $T$ from the metric on $B$. We have a factorization $L(X) \to B_d \to B$ of the operator $T$. By Lemma 3.4, the middle arrow in the factorization $L(X) \to B_{\sqrt{d}} \to B_d \to B$ is a compact operator. It follows that $T$ is a compact operator as well. \qed
Combining Theorems 3.3 and 3.5, we get the result we were aiming at.

**Theorem 3.6.** If $X$ is a compact space, then $L(X)$ is multi-reflexive.

Theorems 3.5 and 3.6 can be readily generalized to the case of $k_\omega$-spaces, similarly to [4]. Indeed, let $X = \bigcup_{n \in \mathbb{N}} K_n$ be a $k_\omega$-space, where $(K_n)$ is a sequence of compact subsets of $X$ witnessing the $k_\omega$-property. Then $X$ is a quotient of the free sum $\bigoplus_{n \in \mathbb{N}} K_n$ [13], and $L(X)$ is a quotient of $L\left(\bigoplus_{n \in \mathbb{N}} K_n\right)$ which is isomorphic to the countable locally convex sum of $L(K_n)$. Since Schwartz spaces are preserved by quotients and by locally convex countable sums [19, Proposition 21.1.7], we conclude that $L(X)$ is a Schwartz LCS.

In particular, we have established the following (which is a somewhat special case of [4]):

**Theorem 3.7.** If $X$ is a locally compact $\sigma$-compact space, then $L(X)$ is a Schwartz LCS and hence (Theorem 3.3) multi-reflexive.

Our next aim is to show that for a wide and natural class of Tychonoff spaces $X$ the converse of Theorem 3.7 is true: $L(X)$ is a Schwartz LCS if and only if $X$ is a locally compact and $\sigma$-compact space.

**Lemma 3.8.** If $L(X)$ is multi-reflexive, then for every Banach space $E$ and every continuous map $f: X \to E$ the image $f(X)$ can be covered by countably many weakly compact subspaces of $E$.

**Proof.** If $L(X)$ is multi-reflexive, by literally the same arguments as we have used in the proof of Lemma 2.5, we can represent $f$ as a composition $f = p \circ g$, where $g: X \to F$ is a continuous map to a reflexive Banach space $F$ and $p: F \to E$ is a bounded linear map. Since the reflexive Banach space $F$ is the union of countably many weakly compact sets, the same is true for the set $p(F)$ which is a subset of $E$ containing $f(X)$. □

**Fact 3.9.** If $X$ is an uncountable discrete space, then $L(X)$ is not multi-reflexive.

**Proof.** Consider the Banach space $E = \ell^1(\Gamma)$, where $\Gamma$ is a set of indexes with the same cardinality as $X$. Let $Y = \{e_\gamma: \gamma \in \Gamma\}$ be the canonical basis of $E$. Let $f$ be any bijection from $X$ onto $Y$, then $f: X \to E$ is a continuous map, because $X$ is discrete. We claim that $Y = f(X)$ cannot be covered by countably many weakly compact subspaces of $E$. This follows from the easily verified fact that $Y$ is a closed discrete subset of $E$ endowed with the weak topology (use the natural pairing $(a_\gamma, b_\gamma) = \sum a_\gamma b_\gamma$ between $E = \ell^1(\Gamma)$ and $E' = \ell^\infty(\Gamma)$, and consider $0$–$1$ sequences in $E'$). Alternatively, note that $\ell^1(\Gamma)$ has the Schur property, that is, every weakly converging sequence in $\ell^1(\Gamma)$ converges in the norm topology, and then every weakly compact subset of $\ell^1(\Gamma)$ is compact in the norm topology (see [11, Exercise 5.47]). Hence, no infinite subset of the closed discrete set $Y$ can be covered by a weakly compact subset of $E$, and the claim follows. □

The following known statement plays a crucial role in the sequel.
Theorem 3.10. Let $X$ be a paracompact (for instance, a metrizable) space. Then for every closed subspace $Y \subset X$, the natural map $L(Y) \to L(X)$ is a topological embedding of locally convex spaces. Analogously, the natural map $A(Y) \to A(X)$ is a topological embedding of topological groups.

The metric fan $M$ is the metrizable space defined as follows. As a set, $M$ is the set $\mathbb{N} \times \mathbb{N}$ of pairs of natural numbers plus a point $p$ “at infinity”; $p$ is the only non-isolated point of $M$, and a basic neighborhood $U_n$ of $p$ consists of $p$ and all pairs $(a, b)$ such that $b > n$. Thus, $M$ is the union of countably many sequences converging to $p$. It is easy to verify that the space $M$ is not locally compact.

Fact 3.11. Let $M$ be the metric fan defined above. Then the free LCS $L(M)$ is not multi-reflexive.

Proof. Consider the separable non-reflexive Banach space $E = \ell^1$, and let $(e_n)$ be the canonical basis. The collection of all vectors of the form $e_n/k, k = 1, 2, \ldots$, together with the zero vector, is a subspace $A \subset E$ homeomorphic to $M$. Let $f : M \to A$ be a homeomorphism, $\hat{f} : L(M) \to E$ its linear extension. We want to prove that there is no factorization $L(M) \to F \to E$ of $\hat{f}$, where $F$ is a reflexive Banach space. Suppose there is such a factorization. Let $\tilde{A}$ be the image of $M$ in $F$, and let $p$ be the non-isolated point of $\tilde{A}$. Denote the arrow $F \to E$ by $g$. We have continuous bijections $M \to \tilde{A} \to A$ such that their composition is a homeomorphism. It follows that the restriction $g |_{\tilde{A}} : \tilde{A} \to A$ is a homeomorphism. Let $U$ be the closed unit ball in $F$. Then $(p + U) \cap \tilde{A}$ is a neighborhood of $p$ in $\tilde{A}$. Since $g |_{\tilde{A}} : \tilde{A} \to A$ is a homeomorphism, $g((p + U) \cap \tilde{A})$ is a neighborhood of $g(p) = 0$ in $A$, and so is the bigger set $g(U) \cap A$. Thus for $k$ large enough all the vectors $e_n/k$ lie in the weakly compact set $g(U)$. Multiplying by $k$, we conclude that all the vectors $e_n$ lie in a weakly compact subset of $E$. That is a contradiction: we noted in the proof of Fact 3.9 that the set $\{e_n\}$ is closed and discrete with respect to the weak topology of $E$.

We now are ready to prove one of the main results of our paper.

Theorem 3.12. Let $X$ be a first-countable paracompact space. The following conditions are equivalent:

(i) $L(X)$ is a multi-reflexive LCS;
(ii) $L(X)$ is a Schwartz LCS;
(iii) $X$ is locally compact and $\sigma$-compact.

Proof. (iii) $\Rightarrow$ (ii): this is Theorem 3.7.
(ii) $\Rightarrow$ (i): this is Theorem 3.3.
(i) $\Rightarrow$ (iii): by Theorem 3.10 and Fact 3.11, $X$ does not contain a closed copy of the metric fan $M$. A first-countable paracompact space is locally compact if and only if it does not contain a closed subspace homeomorphic to $M$ (by [9, Lemma 8.3]). Hence $X$ is locally compact. Being locally compact and paracompact, $X$ can be represented as a disjoint union of clopen $\sigma$-compact subspaces (by [10, Theorem 5.1.27]). Since $X$ does not contain an
uncountable closed discrete subspace (by Fact 3.9 and Theorem 3.10), the number of components in the representation of $X$ above is at most countable and the proof is complete.

**Corollary 3.13.** Let $X$ be a metrizable space. The following conditions are equivalent:

(i) $L(X)$ is a multi-reflexive LCS;

(ii) $L(X)$ is a Schwartz LCS;

(iii) $X$ is locally compact and has a countable base.

**Example 3.14.** The space of irrational numbers $J$ is not locally compact, hence as a straightforward consequence of Corollary 3.13 we have that $L(J)$ is not multi-reflexive. We note that an alternative proof of this fact readily follows from Lemma 3.8.

Indeed, consider any separable non-reflexive Banach space $E$. For example, we can take the separable Banach space $E = \ell^1$. Then weakly compact subsets of $E$ have empty interior, and it follows from the Baire Category Theorem that $E$ is not the union of countably many weakly compact sets. On the other hand, $E$, like any Polish space, is the image of $J$ under a continuous map. Now apply Lemma 3.8.

The last part of this section is devoted to the following goals: (1) Find sufficient conditions on $X$, more general than those in Theorem 3.7, which implies that $L(X)$ is a Schwartz LCS (and $A(X)$ is a Schwartz group); (2) Find the most general necessary conditions for $L(X)$ to be multi-reflexive.

Recall that a Tychonoff space is pseudocompact if every continuous real-valued function defined on $X$ is bounded. Let us say that a Tychonoff space $X$ is $QS$-pseudocompact, if there is a countable sequence of pseudocompact spaces $(X_n)$ such that $X$ can be represented as a quotient of the free sum $\bigoplus_{n \in \mathbb{N}} X_n$.

**Theorem 3.15.** Let $X$ be a $QS$-pseudocompact space. Then $L(X)$ is a Schwartz LCS.

**Proof.** By the argument used in the paragraph after Theorem 3.6, the general case can be reduced to the case when $X$ is pseudocompact. In that case $L(X)$ is isomorphic to a linear subspace of $L(\beta X)$, where $\beta X$ is the Čech–Stone compactification of $X$ (see [33,35]). Since $L(\beta X)$ is Schwartz, by Theorem 3.5, we conclude that $L(X)$ also is Schwartz.

**Corollary 3.16.** Let $X$ be a free sum of a locally compact second-countable metrizable space and a $QS$-pseudocompact space. Then $L(X)$ is a Schwartz space.

**Example 3.17.** Let $X$ be the ordered topological space $[0, \omega_1)$ consisting of all countable ordinals. Then $X$ is a countably compact (hence pseudocompact) first-countable locally compact space. Further, $X$ is not $\sigma$-compact. Nevertheless, by Theorem 3.15, $L(X)$ is a Schwartz LCS. There is no contradiction with Theorem 3.12 because $X$ is not paracompact.
Now we provide several necessary conditions for $L(X)$ to be multireflexive.

**Proposition 3.18.** For any Tychonoff space $X$, if $L(X)$ is multi-reflexive, then every discrete collection of nonempty open subsets in $X$ is countable.

**Proof.** On the contrary, assume that \{${U_{\alpha}}; \alpha \in A$\} is an uncountable discrete collection of nonempty open subsets in $X$. Pick a point $y_{\alpha} \in U_{\alpha}$ for each $\alpha \in A$ and denote the set of all points $y_{\alpha}$ by $Y$. We prove that $L(Y)$ is topologically embedded into $L(X)$. It suffices to show that every continuous map $\varphi: Y \to E$, where $E$ is an arbitrary LCS, can be extended to a continuous linear operator from $L(X)$ to $E$. With this aim, let us fix continuous functions $f_{\alpha}: X \to [0, 1]$ such that supp ($f_{\alpha}$) $\subset U_{\alpha}$ and $f_{\alpha}(y_{\alpha}) = 1$, for each $\alpha \in A$. Now first we define an extension of $\varphi$ to $\hat{\varphi}: X \to E$ by the following formula: $\hat{\varphi}(x) = \sum_{\alpha \in A} f_{\alpha}(x) \varphi(y_{\alpha})$. Then a continuous map $\hat{\varphi}$ by definition can be extended to a linear continuous operator from $L(X)$ to $E$. Since $L(X)$ is assumed to be multi-reflexive, its vector subspace $L(Y)$ is multi-reflexive as well, which is not true by Fact 3.9. The obtained contradiction finishes the proof. □

**Corollary 3.19.** Let $L(X)$ be multi-reflexive. Then every metrizable image of $X$ under a continuous map must be separable.

The next assertion should be compared to Corollary 2.13.

**Proposition 3.20.** Let $L(X)$ be multi-reflexive. Assume that $X$ is mapped continuously onto a normal space $Y$. Then every closed subset $F$ of $Y$ which is analytic must be $\sigma$-compact.

**Proof.** Assume on the contrary that $F$ is analytic but not $\sigma$-compact. By [29, Theorem 3.5.4] $F$ contains a closed subset $K$ which admits a continuous (even perfect) map onto the space of irrationals $J$. Therefore, as in Example 3.14, $K$ admits a continuous map $f$ onto the separable non-reflexive Banach space $E = \ell^1$. Separable Banach spaces have the absolute extension property with respect to the class of normal spaces \cite{1} (see also \cite{25}). Since $K$ is a closed subspace of a normal space $Y$, we can extend $f$ to a continuous map from $Y$ onto $E$. Finally, we obtain a continuous map from a multi-reflexive LCS $L(X)$ onto a separable non-reflexive Banach space $E$, which is impossible by Example 3.14. The obtained contradiction finishes the proof. □

**Corollary 3.21.** Let $L(X)$ be multi-reflexive. Then every analytic image of $X$ under a continuous map must be $\sigma$-compact.

4. **Schwartz $A(X)$**

A concept of a Schwartz topological group which turned out to be coherent with the concept of a Schwartz locally convex space has been introduced in \cite{4}. Similarly to Schwartz LCS, the class of Schwartz groups enjoys analogous permanence properties: it is closed with respect to subgroups, Hausdorff
Let Theorem 4.3. conditions are equivalent: quasi-convex Schwartz groups are precompact [4].

A natural question arises: when is the free abelian topological group $A(X)$ Schwartz? Note that the authors of [4] found sufficient conditions on $X$ implying that $A(X)$ is a Schwartz group, which are identical to those conditions on $X$ implying that $L(X)$ is a Schwartz LCS. In this section, we give a complete characterization of first-countable paracompact spaces $X$ such that the free abelian topological group $A(X)$ is a Schwartz group. Obviously, obtained characterization applies also to metrizable spaces $X$.

For a subset $U$ of an abelian group $G$ such that $0 \in U$, and a natural number $n$, we denote $U_{(n)} = \{ x \in G: kx \in U: k = 1, 2, \ldots, n \}$ [4]. The next formulation is adjusted to Definition 1.3(b) of a Schwartz LCS.

**Definition 4.1.** [4] Let $G$ be an abelian topological group. We say that $G$ is a Schwartz group if for every neighborhood $U$ of zero in $G$ there exists another neighborhood $V$ of zero in $G$ and a sequence $(F_n)$ of finite subsets of $G$ such that $V \subseteq F_n + U_{(n)}$ for every $n \in \mathbb{N}$.

Below we collect several basic facts about the topology of $A(X)$ and its relations with the topology of $L(X)$ (for the details see [35]).

For a space $X$ we denote by $A_0(X)$ the open subgroup

$$A_0(X) = \left\{ \sum c_i x_i: \sum c_i = 0, \ c_i \in \mathbb{Z}, \ x_i \in X \right\}$$

of $A(X)$. If $d$ is a continuous pseudometric on $X$, $d$ has a canonical translation-invariant extension over $A_0(X)$ defined by

$$\bar{d}(v, 0) = \inf \left\{ \sum |c_i|d(x_i, y_i): v = \sum c_i(x_i - y_i), \ c_i \in \mathbb{Z}, \ x_i, y_i \in X \right\}$$

and $\bar{d}(u, v) = \bar{d}(u - v, 0)$. We denote the seminorm $v \mapsto \bar{d}(v, 0)$ on $A_0(X)$ by the same symbol $\bar{d}$. This seminorm is the restriction of the seminorm on $L_0(X)$ denoted by the same symbol in the equality (3.1). In other words, in the definition above we may replace the condition $c_i \in \mathbb{Z}$ by $c_i \in \mathbb{R}$ without affecting the result [35]. The fact that in the equality (4.1) the minimum over $c_i \in \mathbb{Z}$ is the same as the minimum over $c_i \in \mathbb{R}$ follows from the integrality property for the transportation problem. Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ be non-negative integers. Consider the compact convex set of all $(m \times n)$-matrices $(c_{ij})$ with real entries $\geq 0$ such that the sum of all entries in the $i$th row is $a_i$ and the sum in the $j$th column is $b_j$ ($1 \leq i \leq m$, $1 \leq j \leq n$). Then all extreme points of this convex set are matrices with integral entries.

Since the seminorm $\bar{d}$ on $A_0(X)$ is the restriction of a seminorm on the LCS $L_0(X)$, we immediately deduce the following assertion.

**Lemma 4.2.** Let $d$ be a pseudometric on a set $X$, $\bar{d}$ the seminorm on $A_0(X)$ given by the equality (4.1). For every integer $n$ and $t \in A_0(X)$ we have $\bar{d}(nt) = |n|\bar{d}(t)$.

Denote by $X^{(1)}$ the set of all non-isolated points of $X$.

**Theorem 4.3.** Let $X$ be a first-countable paracompact space. The following conditions are equivalent:
(i) $A(X)$ is a Schwartz group;

(ii) $X$ is a locally compact space such that the set $X^{(1)}$ is $\sigma$-compact.

Proof. (ii) $\Rightarrow$ (i): $X$ is locally compact and paracompact, hence, as we have observed earlier in the proof of Theorem 3.12, $X$ can be represented as a disjoint union of clopen $\sigma$-compact subspaces $X_\alpha$ (by [10, Theorem 5.1.27]). The set of non-isolated points $X^{(1)}$ is a Lindelöf space, therefore we can pick countably many $X_\alpha$-s which cover $X^{(1)}$. Denote by $D$ the complement of the union of all those $X_\alpha$-s in $X$. Evidently, $D$ is closed in $X$. On the other hand, $D$ consists of points isolated in $X$. So we can represent $X$ as a disjoint sum $D \oplus Y$ of two clopen subsets $D$ and $Y$, where $D$ consists of isolated points of $X$ and $Y$ is locally compact and $\sigma$-compact. Further, the group $A(D)$ is discrete and the group $A(Y)$ is Schwartz, since $L(Y)$ is Schwartz, by Theorem 3.7.

We conclude that $A(X)$ is a Schwartz group, in view of the known fact that $A(X)$ is isomorphic to the product $A(D) \times A(Y)$.

(i) $\Rightarrow$ (ii): On the contrary, assume that $X$ is not locally compact. Then, as we have observed earlier in the proof of Theorem 3.12, $X$ contains a closed copy of the metric fan $M$. By Theorem 3.10, this means that the free abelian topological group $A(X)$ contains an isomorphic copy of the free abelian topological group $A(M)$. It remains to prove the following

Fact 4.4. Let $M$ be the metric fan. Then $A(M)$ is not a Schwartz group.

Proof. The fan $M = (\mathbb{N} \times \mathbb{N}) \cup \{p\}$ can be represented as a countable union of disjoint closed discrete layers $\{M_k: k \in \mathbb{N}\}$, $M_k = \mathbb{N} \times \{k\}$, and a single non-isolated point $p$ such that for every choice $x_k \in M_k$ the sequence $(x_k)$ converges to $p$ in $M$. Consider the natural metric $d$ on $M$ defined as follows: the distance $d(x, p) = 1/k$ for every $x \in M_k$; the distance $d(x, y) = 2/k$ between any two distinct points $x, y \in M_k$; and $d(x, y) = 1/k + 1/n$ if $x \in M_k, y \in M_n, k \neq n$. Let $U$ be the unit ball in $A_0(M)$ with respect to the metric $d$ on $A_0(M)$. If $V$ is any neighborhood of zero, we can find a natural $k$ such that all points in the $k$th layer $M_k$ of the fan are in $p+V$. Since $d(x, y) = 2/k$ for any two distinct points $x, y$ of $M_k$, we conclude that $d(nx, ny) = 2n/k$ (by Lemma 4.2). It follows that for $n > k$ the difference $x - y$ does not belong to $U(n) - U(n)$, so the balls $x + U(n)$, as $x$ runs over $M_k$, are pairwise disjoint. Therefore, $M_k$ cannot be covered by a set of the form $F_n + U(n)$, where $F_n$ is finite. The same is true for the translate $M_k - p$ of $M_k$ and moreover for the larger set $V$.

So, we obtained a contradiction assuming that $X$ is not locally compact. Therefore, $X^{(1)}$ as a closed subset of $X$ is itself locally compact (and paracompact). Then, as we have observed earlier in the proof of Theorem 3.12, being locally compact and paracompact, $X^{(1)}$ can be represented as a disjoint union of clopen (in $X^{(1)}$) $\sigma$-compact subspaces. Striving for a contradiction, assume that the number of clopen components in this representation is uncountable. Then we can find an uncountable closed discrete subset $W \subset X^{(1)} \subset X$. Paracompact spaces are collection wise normal [10, Theorem 5.1.18], hence there exists a discrete family $\{U_w: w \in W\}$ of open sets in $X$ such that $w \in U_w$ for every $w \in W$. Every $w \in W$ is the limit of a convergent sequence
$S_w \subset U_w$. Put $Y = \bigcup_{w \in W} S_w$. Then $Y$ is closed in $X$ (as the union of a discrete family of closed sets), and $Y$ is homeomorphic to a disjoint sum of uncountably many convergent sequences. In order to get a contradiction it now suffices to prove the following

**Fact 4.5.** Let $Y$ be the disjoint sum of uncountably many convergent sequences. Then $A(Y)$ is not a Schwartz group.

*Proof.* Let $S = \{1, 1/2, 1/3, \ldots\} \cup \{0\}$ be the standard convergent sequence together with its limit, $W$ be an uncountable index set (regarded as a discrete space), and $Y = S \times W$. Equip $Y$ with the natural metric $d$ such that the distance $d$ between $(1/k, w')$ and $(1/l, w'')$ is $|1/k - 1/l|$ if $w' = w''$ and 1 otherwise. Take $U = \{t \in A_0(Y) : \overline{d}(t) < 1\}$. It suffices to prove that there does not exist a neighborhood $V$ of zero in $A_0(Y)$ and finite sets $F_n$ such that $V \subset F_n + U(n)$ for all $n = 1, 2, \ldots$.

Assume the contrary: such $V$ and $F_n$’s exist. If $w \in W$ and $\epsilon > 0$, let us say that $V$ is $\epsilon$-fat in the direction $w$ if $x - y \in V$ for all points $x = (1/k, w)$ and $y = (1/k', w)$ such that $1/k \leq \epsilon$ and $1/k' \leq \epsilon$. For every $w \in W$ there exists an integer $k \in \mathbb{N}$ such that $V$ is $1/k$-fat in the direction $w$, so we can choose an infinite set $J \subset W$ and $k \in \mathbb{N}$ such that $V$ is $1/k$-fat in the direction $w$ for every $w \in J$.

For $w \in J$ consider the points $x_w = (1/k, w)$ and $y_w = (1/(k + 1), w)$. Note that the distance from $x_w$ to any other point in $Y$ is $\geq 1/(k + 1)$.

From the equality $d = f$ we conclude that $\overline{d}(t) \geq 1/(k + 1)$ for every $t \in A_0(Y)$ such that the support of $t$ contains $x_w$. (The support supp $t$ of $t = \sum_{y \in Y} n_y y$ is the set of all $y \in Y$ such that $n_y \neq 0$.) If $n \geq k(k+1)$, we have $\overline{d}(nt) = n \overline{d}(t) \geq 1$ (by Lemma 4.2), so $t \notin U(n)$. Since $x_w - y_w \in V \subset F_n + U(n)$ and the support of any $t \in U(n)$ does not contain $x_w$, it follows that there is $t_w \in F_n$ such that $x_w$ belongs to the support of $t_w$. Since this is true for every $w \in J$, we get a contradiction: the infinite set $\{x_w : w \in J\}$ is covered by finitely many finite sets supp $t$, $t \in F_n$. \(\square\)

We now can finish the proof of Theorem 4.3: $X^{(1)}$ can be represented as at most countable union of clopen $\sigma$-compact subspaces, and the proof is complete. \(\square\)

**Corollary 4.6.** Let $X$ be a metrizable space. The following conditions are equivalent:

(i) $A(X)$ is a Schwartz group;

(ii) $X$ can be represented as a disjoint union of two clopen subsets $D$ and $Y$, where $D$ consists of isolated points of $X$ and $Y$ is locally compact and has a countable base.

**Corollary 4.7.** Let $X$ be a free sum of a discrete space, a locally compact second-countable metrizable space and a QS-pseudocompact space. Then $A(X)$ is a Schwartz group.
5. Open Problems

Problem 5.1. Characterize all Tychonoff spaces $X$ such that $L(X)$ are Schwartz / multi-reflexive.

Problem 5.2. Does there exist a Tychonoff space $X$ such that $L(X)$ is multi-Hilbert but not nuclear?

Problem 5.3. Does there exist a Tychonoff space $X$ such that $L(X)$ is multi-reflexive but not Schwartz?

A more specific question is the following.

Problem 5.4. Let $p$ be a point from the remainder of the Stone-Čech compactification $\beta\mathbb{N} \setminus \mathbb{N}$. Denote by $X = \mathbb{N} \cup \{p\}$. Is $L(X)$ nuclear or multi-Hilbert?

Note that every Schwartz space can be isomorphically embedded in some sufficiently high power of the Banach space $c_0$ [28]. There are examples of Schwartz spaces which cannot be isomorphically embedded in any power of the Banach space $\ell^p (1 < p < \infty)$ [6]. On the other hand, every nuclear space can be isomorphically embedded in some sufficiently high power of every Banach space [30]. These facts provide motivation for the following questions.

Problem 5.5. Characterize all Tychonoff spaces $X$ such that $L(X)$ can be isomorphically embedded in some sufficiently high power of the Banach space $c_0$.

Problem 5.6. Characterize all Tychonoff spaces $X$ such that $L(X)$ can be isomorphically embedded in some sufficiently high power of the Banach space $\ell^p (1 < p < \infty)$.

Acknowledgements

We would like to express our gratitude to Vladimir Pestov and Michael Megrelishvili for the most stimulating questions and discussions on the topic. We thank Mikhail Ostrovskii for guiding us towards the paper [20]. The authors are grateful to the referee for careful reading of the paper and valuable suggestions and comments.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.
References

[1] Arens, R.: Extension of coverings, of pseudometrics, and of linear-space-valued mappings. Canad. J. Math. 5, 211–215 (1953)

[2] Außenhofer, L.: Free nuclear groups. Topol. Appl. 134, 90–102 (2007)

[3] Außenhofer, L.: On the Hausdorff variety generated by all locally $k_\omega$ groups and on Schwartz groups. Topol. Appl. 159, 2248–2257 (2012)

[4] Außenhofer, L., Chasco, M.J., Domínguez, X., Tarieladze, V.: On Schwartz groups. Stud. Math. 181(3), 199–210 (2007)

[5] Banaszczyk, W.: Additive Subgroups of Topological Vector Spaces, Lecture Notes in Math, vol. 1466. Springer, Berlin (1991)

[6] Bellenot, S.F.: Factorable bounded operators and Schwartz spaces. Proc. Am. Math. Soc. 42, 551–554 (1974)

[7] Bellenot, S.F.: Factorization of compact operators and finite representation of Banach spaces with applications to Schwartz spaces. Stud. Math. 62, 273–286 (1978)

[8] Diestel, J., Morris, S.A., Saxon, S.A.: Varieties of linear topological spaces. Trans. Am. Math. Soc. 172, 207–230 (1972)

[9] van Douwen, E.: The integers and topology. In: Kunen, K., Vaughan, J.E. (eds.) Handbook of Set-Theoretic Topology, pp. 111–167. North-Holland, Amsterdam (1984)

[10] Engelking, R.: General Topology, Revised and competed edition. Heldermann, Berlin (1989)

[11] Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V.: Banach Space Theory. The Basis for Linear and Nonlinear Analysis. CMS Books in Mathematics, Springer (2011)

[12] Fonf, V.P., Johnson, W.B., Plichko, A.M., Shevchyk, V.V.: Covering a compact set in a Banach space by an operator range of a Banach space with basis. Trans. Am. Math. Soc. 358(4), 1421–1434 (2006)

[13] Franklin, S.P., Smith Thomas, B.V.: A survey on $k_\omega$ spaces. Topol. Proc. 2, 111–124 (1977)

[14] Godefroy, G.: A survey on Lipschitz-free Banach spaces. Comment. Math. 55(2), 89–118 (2015)

[15] Graev, M. I.: Free topological groups, Amer. Math. Soc. Transl. (1) 8 (1962), 305–364, Russian original in: Izv. Akad. Nauk SSSR 12 (1948), 279–324

[16] Graev, M.I.: The theory of topological groups, I. Uspekhi Mat. Nauk 5, 3–56 (1950). (in Russian)

[17] Grothendieck, A.: Sur les espaces ($F$) et ($DF$). Summa Brasil. Math. 3, 57–121 (1954)

[18] Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires. Mem. Am. Math. Soc. 16 (1955)

[19] Jarchow, H.: Locally Convex Spaces. B. G. Teubner, Stuttgart (1981)

[20] Joichi, J.T.: Normed linear spaces equivalent to inner product spaces. Proc. Am. Math. Soc. 17, 423–426 (1966)

[21] Markov, A.A.: On free topological groups. Dokl. Akad. Nauk SSSR 31, 299–301 (1941)
[22] Markov, A. A.: On free topological groups, Amer. Math. Soc. Transl. (1) 8 (1962), 195–272, Russian original in: Izv. Akad Nauk SSSR 9 (1945), 3–64
[23] Megrelishvili, M.G.: Fragmentability and continuity of semigroup actions. Semigroup Forum 57, 101–126 (1998)
[24] Okunev, O.G.: A method for constructing examples of $M$-equivalent spaces. Topology Appl. 36, 157–171 (1990)
[25] Przymusiński, T.: Collectionwise normality and absolute retracts. Fund. Math. 98, 61–73 (1978)
[26] Raikov, D.A.: Free locally convex space for uniform spaces. Mat. Sb. 63, 582–590 (1964)
[27] Randtke, D.: Characterization of precompact maps, Schwartz spaces and nuclear spaces. Trans. Am. Math. Soc. 165, 87–101 (1972)
[28] Randtke, D.: A structure theorem for Schwartz spaces. Math. Ann. 201, 171–176 (1973)
[29] Rogers, C.A., Jayne, J.E.: $K$-analytic sets. In: Analytic Sets, pp. 1–181. Academic Press (1980)
[30] Saxon, S.A.: Embedding nuclear spaces in products of an arbitrary Banach space. Proc. Am. Math. Soc. 34, 138–140 (1972)
[31] Schaefer, H.H.: Topological Vector Spaces. Macmillan Company, New York (1966)
[32] Terzioğlu, T.: On Schwartz spaces. Math. Ann. 182, 236–242 (1969)
[33] Tkachenko, M.G.: On completeness of free abelian topological groups. Soviet Math. Dokl. 27, 341–345 (1983)
[34] Tkachuk, V.V.: Weakly linearly Lindelöf spaces revisited. Topol. Appl. 256, 128–135 (2019)
[35] Uspenskij, V.V.: Free topological groups of metrizable spaces. Math. USSR-Izv. 37, 657–680 (1991)
[36] Uspenskij, V. V.: Unitary representability of free abelian topological groups. Appl. Gener. Topol. 9(2), 197–204 (2008). arXiv: math.RT/0604253

Arkady Leiderman
Department of Mathematics
Ben-Gurion University of the Negev
P.O.B. 653
Beer Sheva
Israel
e-mail: arkady@math.bgu.ac.il

Vladimir Uspenskij
Department of Mathematics
Ohio University
321 Morton Hall
Athens OH45701
USA
e-mail: uspenski@ohio.edu

Received: October 23, 2021.
Revised: March 21, 2022.
Accepted: September 29, 2022.