On $\mathcal{U}_h(sl(2))$, $\mathcal{U}_h(e(3))$ and their Representations

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Abstract

By solving a set of recursion relations for the matrix elements of the $\mathcal{U}_h(sl(2))$ generators, the finite dimensional highest weight representations of the algebra were obtained as factor representations. Taking a nonlinear combination of the generators of the two copies of the $\mathcal{U}_h(sl(2))$ algebra, we obtained $\mathcal{U}_h(so(4))$ algebra. The latter, on contraction, yields $\mathcal{U}_h(e(3))$ algebra. A nonlinear map of $\mathcal{U}_h(e(3))$ algebra on its classical analogue $e(3)$ was obtained. The inverse mapping was found to be singular. It signifies a physically interesting situation, where in the momentum basis, a restricted domain of the eigenvalues of the classical operators is mapped on the whole real domain of the eigenvalues of the deformed operators.

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1 Introduction

The enveloping Lie algebra $\mathcal{U}(sl(2))$ has two distinct quantizations: The first one is called the Drinfeld-Jimbo deformation (standard $q$-deformation) \cite{1, 2}, whereas the second one is called the Jordanian deformation (nonstandard $h$-deformation) \cite{3, 4} and may be obtained as a contraction of the Drinfeld-Jimbo one \cite{3}. Recently there is much interest in studies relating to various aspects of the $h$-deformed algebra $\mathcal{U}_h(sl(2))$. In particular, a two para- metric deformation of the dual algebra $\mathcal{F}_{un}_{h,h'}(GL(2))$ was obtained in \cite{6}. This author also constructed \cite{8} the differential calculus in the quantum plane. Quantum de Rham complexes associated with the $h$-deformed algebra $\mathcal{F}_{un}_h(sl(2))$ was given in \cite{7}. The universal $\mathcal{R}$-matrix of the algebra $\mathcal{U}_h(sl(2))$ was obtained \cite{8, 9}. Various non-semisimple $h$-deformed algebras were constructed at contraction limits \cite{10, 11, 9, 12}. The $h$-deformation was also extended to the case of superalgebras \cite{13}.

One of the studies made in the present article is as follows. Using the standard singular vector construction method, we study the finite dimensional highest weight representations of $\mathcal{U}_h(sl(2))$ algebra. Along similar lines, investigations were made before in \cite{14}. Compared to \cite{14}, a distinctive feature in our approach is that we develop a set of recursion relations, which may be easily solved to determine the matrix elements of the generators of the algebra. These matrix elements, in turn, specify the singular vectors leading to finite dimensional irreducible representations. This representations may also be obtained by exploiting a recently \cite{15} proposed nonlinear invertible map between the generators of $\mathcal{U}_h(sl(2))$ and the classical $sl(2)$ generators. We wish to stress, however, that the construction of the finite dimensional highest weight representations via the singular vector technique relies on first principles; and, therefore, may be useful for other nonlinear algebras, which exhibit no such maps on the corresponding linear algebras. In a continuation of our studies on the $\mathcal{U}_h(sl(2))$ algebra \cite{15}, we consider the contracted $\mathcal{U}_h(e(2))$ algebra \cite{11}. The latter may be also be mapped on the classical algebra $e(2)$.

We also consider the $\mathcal{U}_h(e(3))$ algebra obtained as a contraction of the $\mathcal{U}_h(so(4))$ algebra \cite{10, 9}, that may be realized from two copies of $\mathcal{U}_h(sl(2))$. It is of interest that, parallel to the scenario discussed for $\mathcal{U}_h(sl(2))$ algebra, $\mathcal{U}_h(e(3))$ algebra may also be realized in terms of the $e(3)$ generators. This enormously simplifies the problem of finding the irreducible representations of $\mathcal{U}_h(e(3))$ algebra. In contrast to a similar map obtained in \cite{16} for $\mathcal{U}_q(e(3))$ algebra and also the realization \cite{17} of the $\kappa$-Poincaré algebra \cite{18, 19} in terms of the classical Poincaré generators, the corresponding map for the present $\mathcal{U}_h(e(3))$ algebra on classical $e(3)$ exhibits a singularity. This may be of physical interest.

Let $h$ be an arbitrary complex parameter. The algebra $\mathcal{U}_h(sl(2))$ is then an associative algebra over $\mathbb{C}$ generated by $H$, $X$ and $Y$, satisfying the commutation relations \cite{3}

$$[H, X] = 2\frac{\sinh hX}{h},$$
$$[H, Y] = -Y(\cosh hX) - (\cosh hX)Y,$$
$$[X, Y] = H.$$ \hspace{1cm} (1.1)
The coalgebra structure of $\mathcal{U}_h(sl(2))$ reads \[4\]
\[\begin{align*}
\Delta(X) &= X \otimes 1 + 1 \otimes X, \\
\Delta(Y) &= Y \otimes e^{hX} + e^{-hX} \otimes Y, \\
\Delta(H) &= H \otimes e^{hX} + e^{-hX} \otimes H, \\
\varepsilon(X) &= \varepsilon(Y) = \varepsilon(H) = 0, \\
S(X) &= -X, \quad S(Y) = -e^{hX} Y e^{-hX}, \quad S(H) = -e^{hX} H e^{-hX}.
\end{align*}\] (1.2)

The Casimir element of $\mathcal{U}_h(sl(2))$ is given by \[10\]
\[C = \frac{1}{2h} \left( Y (\sinh hX) + (\sinh hX) Y \right) + \frac{1}{4} H^2 + \frac{1}{4} (\sinh hX)^2.\] (1.3)

2 Representations of $\mathcal{U}_h(sl(2))$ algebra

The finite dimensional highest weight representations of $\mathcal{U}_h(sl(2))$ as the factor-representations of the corresponding Verma modules were considered in \[14\]. The factorization scheme was carried out by the standard singular vector construction method. In \[14\], the operators $H, Y, \cosh(hX)$ and $\sinh(hX)$ were chosen as generators. We follow the same route here. However, a different choice of the generators of the algebra $\mathcal{U}_h(sl(2))$ allows us to express the matrix elements in terms of a simple set of recursion relations which may be easily solved. These matrix elements, in turn, completely determine the singular vectors at arbitrary levels. The appearance of a singular vector in a Verma module signals its reducibility. To obtain a finite dimensional irreducible representation, the submodule generated by treating the singular vector as a highest weight vector must be factored off. This scheme provides all irreducible representations at arbitrary dimensions. Alternately, these irreducible representations may be obtained via a recently developed \[13\] nonlinear invertible map between the $\mathcal{U}_h(sl(2))$ generators and the generators of the classical undeformed ($h = 0$) $sl(2)$ algebra. We will briefly elucidate this procedure later.

The highest weight vector $w_0$, where $\lambda$ is the highest weight, satisfies the relations
\[X.w_0 = 0, \quad H.w_0 = \lambda w_0.\] (2.1)
The Verma module $M$ is generated by the repeated actions of $Y$ on $w_0$ as
\[w_n = Y^n.w_0, \quad n \in \mathbb{N}.\] (2.2)
Using the commutations relations (1.1), it is evident that the actions of $X$ and $H$ on the vector space are described as
\[X.w_n = \sum_{k=0}^{[(n-1)/2]} X^{n-1-2k} w_{n-1-2k},\] (2.3)
\[H.w_n = \sum_{k=0}^{[n/2]} H^{n-2k} w_{n-2k},\] (2.4)
where $H_0^0 = \lambda$ and $[x]$ denotes the integer part in $x$. We note here that the actions of $X$ and $H$ on a state in the Verma module create the sequences of states differing in their indices by two. Our task is now to develop the recursion relations between the above matrix elements. To this end, we use (2.2) to obtain

$$X.w_n = ([X, Y] + Y X)w_{n-1}, \quad (2.5)$$
$$H.w_n = ([H, Y] + Y H)w_{n-1}. \quad (2.6)$$

The commutation relations (1.1) may now be exploited to obtain the following recursion relations

$$X^m_n = X^{m-1}_{n-1} + H^{m-1}_{n-1}, \quad (2.7)$$
$$H^n_n = H^{n-1}_{n-1} - 2, \quad (2.8)$$
$$H^{m+2n}_{m+2n} = H^{m-1}_{m+2n-1} - \sum_{k=1}^{n} \frac{h^{2k}}{(2k)!} \sum_{\delta=0}^{1} \sum_{\{\Delta_i\} = (1,2,\cdots,2k)} Z_{m,2n,\delta,\{\Delta_i\}}, \quad (2.9)$$

where the primed sum in the rhs is performed over the following all possible partitions of $2n$ among an even number of positive odd integers

$$\Delta_1 + \Delta_2 + \cdots + \Delta_{2k} = 2n, \quad \Delta_i \text{ mod } 2 = 1 \text{ for } i = (1,2,\cdots,2k) \quad (2.10)$$

and

$$Z_{m,2n,\delta,\{\Delta_i\}} = X^{m+2n-\delta-\Delta_1}_{m+2n-\delta} X^{m+2n-\delta-\Delta_1-\Delta_2}_{m+2n-\delta-\Delta_1} \cdots X^{m-\delta}_{m+2n-\delta-\Delta_1-\Delta_2-\cdots-\Delta_{2k-1}}. \quad (2.11)$$

The partition (2.10) reduces the problem of finding the matrix elements of $H$ and $X$ to a combinatorial exercise. This permits us to write down, starting from the known values at $(h = 0)$ limit, the matrix elements of these operators at arbitrary dimensions. We demonstrate this in the present section. To cite an example of the partition (2.10), we enlist all possible cases of partitioning $8$ (= $2n$)

$$\{(7, 1), (1, 7); (5, 3), (3, 5); (5, 1, 1, 1) \text{ and permutations}; (3, 1, 1, 1) \text{ and permutations}; (3, 1, 1, 1, 1, 1) \text{ and permutations}; (1, 1, 1, 1, 1, 1, 1, 1)\}.$$

Rewriting (2.7) differently, we obtain

$$X^m_{m+2n+1} = \sum_{k=0}^{m} H^k_{k+2n}; \quad n \geq 0. \quad (2.12)$$

The matrix elements $\sim O(1)$ surviving in the classical limit $(h = 0)$ can be immediately obtained from (2.8) and (2.12)

$$H^n_n = \lambda - 2n, \quad (2.13)$$
$$X^n_{n+1} = (n + 1)(\lambda - n). \quad (2.14)$$

From (2.9), (2.12), (2.13) and (2.14), it is evident that the matrix elements $X^m_{m+2n+1}$ and $H^m_{m+2n}$ are $\sim O(h^{2n})$. We now demonstrate that starting with the known matrix elements...
(2.13) and (2.14) at \( \sim O(1) \), we may explicitly evaluate the matrix elements appearing at an arbitrary power of \( h \) by solving the recursion relations (2.9) and (2.12). To determine the matrix elements \( H^n_{n+2} \sim O(h^2) \), we obtain from (2.9),

\[
H^n_{n+2} = -h^2 \sum_{k=0}^{n-1} X_{k+2}^k X_{k+1}^k - \frac{h^2}{2} X_{n+2}^{n+1} X_{n+1}^n.
\] (2.15)

Using the explicit values (2.14) of the elements \( X_{n+1}^n \), we obtain

\[
H^n_{n+2} = h^2 \rho_n^{(2)},
\] (2.16)

where

\[
\rho_n^{(2)} = -\sum_{k=0}^{n-1} (k+1)(k+2)(\lambda - k)(\lambda - k - 1) - \frac{1}{2} (n+1)(n+2)(\lambda - n)(\lambda - n - 1).
\] (2.17)

The elements \( X_{n+3}^n \sim O(h^2) \) now follow from (2.12) and (2.16) as

\[
X_{n+3}^n = h^2 \sigma_n^{(2)},
\] (2.18)

where

\[
\sigma_n^{(2)} = \sum_{k=0}^{n} \rho_k^{(2)}.
\] (2.19)

This completes our explicit evaluation of all matrix elements \( \sim O(h^2) \). Exploiting these explicitly known elements \( \sim O(1), O(h^2) \) in (2.13), (2.14), (2.16) and (2.18), we now determine the elements \( H^n_{n+4} \) and \( X^n_{n+5} \sim O(h^4) \). We enlist the result as follows

\[
H^n_{n+4} = h^4 \rho_n^{(4)},
\] (2.20)

where

\[
\rho_n^{(4)} = -\sum_{k=0}^{n-1} \left( (k+4)(\lambda - k - 3)\sigma_k^{(2)} + (k+1)(\lambda - k)\sigma_{k+1}^{(2)} + 2(4!) \frac{k+4}{4} \right) \left( \lambda - k \right) \left( \lambda - k - 4 \right)
\] 

\[
-\frac{1}{2} \left( (n+4)(\lambda - n - 3)\sigma_n^{(2)} + (n+1)(\lambda - n)\sigma_{n+1}^{(2)} + 2(4!) \frac{n+4}{4} \right) \left( \lambda - n \right) \left( \lambda - n - 4 \right)
\] (2.21)

and

\[
X^n_{n+5} = h^4 \sigma_n^{(4)},
\] (2.22)

where

\[
\sigma_n^{(4)} = \sum_{k=0}^{n} \rho_k^{(4)}.
\] (2.23)
The general scheme of determination of an arbitrary matrix element is evident now. Assuming that all the matrix elements appearing up to the order \( \sim O(h^{2(n-1)}) \) have been explicitly determined, we proceed to determine the matrix elements \( H_{m+2n}^m \) and \( X_{m+2n+1}^m \) (\( \sim O(h^{2n}) \)). The relation (2.9) may be reorganized as follows

\[
H_{m+2n}^m = -\sum_{k=1}^{n} \frac{h^{2k}}{(2k)!} \sum_{\{\Delta, i=1, \ldots, 2k\}} \left( 2 \sum_{l=0}^{m-1} Z_{l, 2n, 0, \{\Delta_i\}} + Z_{m, 2n, 0, \{\Delta_i\}} \right),
\]

(2.24)

where the primed sum in the rhs has been described in (2.10). The matrix elements appearing in the rhs of (2.24) are already known and may be utilized to explicitly evaluate \( H_{m+2n}^m \). Equations (2.12) and (2.24) now determine \( X_{m+2n+1}^m \). This completes our determination of all the matrix elements of the operator \( X \) and \( H \) acting on the Verma module.

For \( \lambda = 2j \) (\( j = 0, \frac{1}{2}, 1, \ldots \)), singular vectors \( \{w_s^{(j)} | (2j + 1) \in \mathbb{N}\} \) appear in the Verma module, which is, thus, rendered reducible. An irreducible representation of \( (2j + 1) \) dimension is obtained by taking the quotient module \( L^j = M/\{U_h(sl(2)).w_s^{(j)}\} \). The singular vectors \( \{w_s^{(j)}\} \) are annihilated by \( X \) and are eigenvectors of \( H \)

\[
X.w_s^{(j)} = 0,
\]

(2.25)

\[
H.w_s^{(j)} = \lambda^{(j)}w_s^{(j)}.
\]

(2.26)

We note that (2.25), together with the commutation relation (1.1), demands (2.26) to be satisfied. The zero mode condition (2.25) gives rise to a set of linear equations determining the singular vectors \( w_s^{(j)} \) completely. We describe this bellow. The matrix elements obtained before suggest an ansatz for \( \{w_s^{(j)}\} \)

\[
w_s^{(j)} = w_{2j+1} + \sum_{p=1}^{[j]} C_p^{(j)} w_{2j-2p+1}.
\]

(2.27)

Substituting (2.27) in (2.25), we get

\[
(2j + 1)(\lambda - 2j)w_{2j} + \sum_{r=1}^{[j]} X_{2j+1}^{2j-2r} w_{2j-2r} + \sum_{p=1}^{[j]} C_p^{(j)} \sum_{r=0}^{[j]-p} X_{2j+1-2p}^{2j-2p-2r} w_{2j-2r-2p} = 0.
\]

(2.28)

For \( \lambda = 2j \), the first term in (2.28) vanishes. Making a change in the index in the second summation, we obtain

\[
\sum_{r=1}^{[j]} X_{2j+1}^{2j-2r} w_{2j-2r} + \sum_{p=1}^{[j]} C_p^{(j)} \sum_{r=p}^{[j]-p} X_{2j+1-2p}^{2j-2r} w_{2j-2r} = 0.
\]

(2.29)

In (2.29), it is understood that the matrix elements of \( X \) are evaluated at \( \lambda = 2j \). Reversing the order of summation in the second term (2.29), we get

\[
\sum_{r=1}^{[j]} \left( X_{2j+1}^{2j-2r} + \sum_{p=1}^{r} C_p^{(j)} X_{2j+1-2p}^{2j-2r} \right) w_{2j-2r} = 0.
\]

(2.30)
This enforces the coefficients $C_p^{(j)} (p = 1, \cdots, [j])$ to satisfy $[j]$ linear (in fact, triangular) equations, where the matrix elements of $X$ are evaluated at $\lambda = 2j$:

$$X^{2j-2r}_{2j+1} (\lambda = 2j) + \sum_{p=1}^{r} C_p^{(j)} X^{2j-2r}_{2j+1-2p} (\lambda = 2j) = 0, \quad r = (1, \cdots, [j]).$$  \tag{2.31}

The equations (2.31) may be easily solved. We complete our discussion of the singular vectors corresponding to arbitrary $(2j + 1)$ dimensional representations by enlisting the first few singular vectors for different choices of $\lambda$

| Value of $\lambda$ (= $2j$) | Singular vector |
|-----------------------------|-----------------|
| 0                           | $w_1$           |
| 1                           | $w_2$           |
| 2                           | $w_3 + h^2w_1$  |
| 3                           | $w_4 + 6h^2w_2$ |
| 4                           | $w_5 + 21h^2w_3 + 36h^4w_1$ |
| 5                           | $w_6 + 56h^2w_4 + 460h^4w_2$ |
| 6                           | $w_7 + 126h^2w_5 + 3105h^4w_3 + 8100h^6w_1$ |
| 7                           | $w_8 + 252h^2w_6 + 14796h^4w_4 + 166320h^6w_2$ |

The analytical expression derived in [14] allows one to find the singular vectors for $\lambda \leq 3$. Using REDUCE, this author also derived expressions necessary for finding singular vectors up to $\lambda \leq 6$. In our recursive scheme, the problem of determination of singular vectors is much simplified. Determination of the matrix elements of $X$ by our recursive method also yields automatically the singular vectors at arbitrary levels. The matrix element of the operator $Y$ may, now, be readily determined. The recipe is as follows. In order to extract the irreducible representations of dimension $(2j + 1)$, the singular vector $w_s^{(j)}$ existing for $\lambda = 2j$ is identified with the null vector

$$w_s^{(j)} \approx 0.$$  \tag{2.33}

This, in conjunction with (2.2) and (2.27), now leads to

$$Y.w_{2j} = - \sum_{p=1}^{[j]} C_p^{(j)} w_{2j-2p+1}.$$  \tag{2.34}

Our construction of states (2.2) and the coefficients $\{C_p^{(j)}\}$, already determined by the system of linear equations (2.31), now provide all the matrix elements of the operator $Y$.

We supplement our preceding description with an example, where we construct the irreducible representations for $j = \frac{7}{2}$ case. The factor module now consists of the vector space

$$\{w_i|i = 0, \cdots, 7\},$$

which may be identified as follows

$$(w_i)_j = \delta_{ij}, \quad (i, j) = (0, 1, \cdots, 7).$$  \tag{2.35}
Using our construction of the matrix elements (2.12), (2.13), (2.14) and (2.24) we obtain the representations of the operators $X$ and $H$. The relations (2.2), (2.31) and (2.34) provide the representations of the operator $Y$. The $j = \frac{7}{2}$ representation reads:

$$X = \begin{pmatrix}
0 & 7 & 0 & -42h^2 & 0 & 252h^4 & 0 & 58968h^6 \\
0 & 0 & 12 & 0 & -216h^2 & 0 & 4176h^4 & 0 \\
0 & 0 & 0 & 15 & 0 & -600h^2 & 0 & 22500h^4 \\
0 & 0 & 0 & 0 & 16 & 0 & -1224h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 15 & 0 & -2058h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$Y = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},$$

$$H = \begin{pmatrix}
7 & 0 & -42h^2 & 0 & 252h^4 & 0 & 58968h^6 & 0 \\
0 & 5 & 0 & -174h^2 & 0 & 3924h^4 & 0 & 88776h^6 \\
0 & 0 & 3 & 0 & -384h^2 & 0 & 18324h^4 & 0 \\
0 & 0 & 0 & 1 & 0 & -624h^2 & 0 & 49212h^4 \\
0 & 0 & 0 & 0 & -1 & 0 & -834h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & -966h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -7
\end{pmatrix}. \ (2.36)$$

For later comparison purpose, we also express the $j = \frac{7}{2}$ representation of the $U_h(sl(2))$ algebra, where the operator $H$ has been diagonalized. It reads

$$X = \begin{pmatrix}
0 & 7 & 0 & 105h^2 & 0 & 3780h^4 & 0 & 56700h^6 \\
0 & 0 & 12 & 0 & 240h^2 & 0 & 6480h^4 & 0 \\
0 & 0 & 0 & 15 & 0 & 300h^2 & 0 & 3780h^4 \\
0 & 0 & 0 & 0 & 16 & 0 & 240h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 15 & 0 & 105h^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$
\[ Y = \begin{pmatrix} 0 & \frac{-21h^2}{2} & 0 & \frac{315h^4}{4} & 0 & 4725h^6 & 0 & 99225h^8 \\ 1 & 0 & -33h^2 & 0 & 180h^4 & 0 & 8100h^6 & 0 \\ 0 & 1 & 0 & -\frac{105h^2}{2} & 0 & 225h^4 & 0 & 4725h^6 \\ 0 & 0 & 1 & 0 & -60h^2 & 0 & 180h^4 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{105h^2}{2} & 0 & 315h^4 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{4}{21h^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{2}{0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ H = \text{diag}(7, 5, 3, 1, -1, -3, -5, -7). \]  

(2.37)

This completes our recipe for constructing an arbitrary \((2j + 1)\) dimensional irreducible representation of the algebra \(\mathcal{U}_h(sl(2))\) by the process of factorization of the Verma module \(M\). The above method, developed from the first principles, may now be compared with an alternative way of obtaining the irreducible representations by exploiting a recently developed invertible map between the generators of the \(\mathcal{U}_h(sl(2))\) algebra and the undeformed \(sl(2)\) generators. The map reads

\[ X = \frac{2}{h} \text{arctanh}(\frac{h J_+}{2}), \]
\[ Y = \sqrt{1 - \frac{h^2 J_+^2}{4}} J_- \sqrt{1 - \frac{h^2 J_-^2}{4}}, \]
\[ H = J_0, \]  

(2.38)

where \((J_\pm, J_0)\) satisfy the \(sl(2)\) algebra

\[ [J_0, J_\pm] = \pm 2J_\pm, \quad [J_+, J_-] = J_0 \]  

(2.39)

and the following cocommutative coproduct relations

\[ \triangle(J_i) = J_i \otimes 1 + 1 \otimes J_i, \quad i = (\pm, 0). \]  

(2.40)

The action of these generators on the basis states \(\{w^j_m \mid j = (0, \frac{1}{2}, 1, \cdots), -j \leq m \leq j\}\) may be taken as

\[ J_+ w^j_m = (j - m)(j + m + 1)w^j_{m+1}, \]
\[ J_- w^j_m = w^j_{m-1} \quad \text{for} \quad m \geq -j + 1, \quad J_- w^j_{-j} = 0, \]
\[ J_0 w^j_m = 2m w^j_m. \]  

(2.41)

For \(j = \frac{7}{2}\), the representation (2.37) may be immediately constructed by using the map (2.38).
It is interesting to note that, following the map (2.38) an induced cocommutative co-product structure \( \tilde{\Delta} \) may be ascribed to the generators \((X, Y, H)\) as follow

\[
\tilde{\Delta}(X) = \frac{2}{\hbar} \arctanh\left( \frac{\hbar}{2} \frac{\Delta(J_+)}{2} \right),
\]

\[
\tilde{\Delta}(Y) = \sqrt{1 - \frac{\hbar^2 \Delta(J_+)^2}{4}} \Delta(J_-) \sqrt{1 - \frac{\hbar^2 \Delta(J_+)^2}{4}},
\]

\[
\tilde{\Delta}(H) = \Delta(J_0).
\] (2.42)

It is to be emphasized, however, that the coproduct structure \( \tilde{\Delta} \) treats the generators \((X, Y, H)\) as elements of undeformed \( U(sl(2)) \) algebra and is unrelated to the coproduct structure of the Hopf algebra \( U_h(sl(2)) \). Here we repeat the comment about usefulness of our systematic of development of the recursive scheme of finding the irreducible representations of the \( U_h(sl(2)) \). This procedure is developed from the first principles and may be useful for other nonlinear algebras, where the maps, similar to (2.38), to the corresponding linear algebras may not be available.

In the rest of the present section, we consider the \( U_h(e(2)) \) algebra, which may be obtained \([11]\) as a contraction of the \( U_h(sl(2)) \) algebra. Starting with an undeformed \( e(2) \) generated by \((J, \mathcal{P}_\pm)\)

\[
[J, \mathcal{P}_\pm] = \pm \mathcal{P}_\pm, \quad [\mathcal{P}_+, \mathcal{P}_-] = 0,
\] (2.43)

we, parallel to our previous prescription in (2.38), define

\[
\chi = \frac{2}{\hbar} \arctanh\left( \frac{\hbar}{2} \frac{\mathcal{P}_+}{2} \right),
\]

\[
\eta = \sqrt{1 - \left( \frac{\hbar}{2} \mathcal{P}_+ \right)^2} \mathcal{P}_- \sqrt{1 - \left( \frac{\hbar}{2} \mathcal{P}_+ \right)^2} = \left( 1 - \left( \frac{\hbar}{2} \mathcal{P}_+ \right)^2 \right) \mathcal{P}_-, \]

\[
\zeta = 2J.
\] (2.44)

Then it follows

\[
[\zeta, \chi] = \frac{2}{\hbar} \sinh(\hbar \chi), \quad [\zeta, \eta] = -2 \eta \cosh(\hbar \chi), \quad [\chi, \eta] = 0.
\] (2.45)

The Casimir is now

\[
C = \mathcal{P}_+ \mathcal{P}_- = \frac{\eta}{\hbar} \sinh(\hbar \chi).
\] (2.46)

This corresponds to the algebra considered in \([11]\), except that we have not implicitly changed the signature of the metric from Euclidean to Lorentzian. The representations of \( e(2) \) again, as before, provide explicitly those for the \( h \)-deformed case \( U_h(e(2)) \). The standard unitary representations of \( e(2) \) are, of course, infinite dimensional.
3  On $\mathcal{U}_h(e(3))$ and its Nonlinear Map

One of the techniques used in the studies of the nonsemisimple quantized algebras is contraction. A singular transformation is performed in the vector space of the universal enveloping algebra of a suitable semisimple quantum algebra. These singular transformations were applied to study various $q$-deformed inhomogeneous algebras. Another important application of this technique was made in the study of the $\kappa$-deformed Poincaré algebra. Contractions can be implemented in different ways implying different consequences. There are two distinct classes. In the first, $q$ goes to unity at the contraction limit, but, nonetheless, the deformation persists, as the ratio of $(\ln q)$ with the contraction parameter remains finite. The above works are in this category. In an alternate scheme, contraction is performed, by retaining the value of $q$ (generic or root of unity) invariant. When $q$ is a root of unity, the second procedure is useful to construct, for example, the periodic representations for the contracted algebra. In the present work, we limit our consideration to the first type.

In the context the Jordanian deformations, a construction $\mathcal{U}_h(so(4)) = \mathcal{U}_h(sl(2)) \oplus \mathcal{U}_{-h}(sl(2))$ was used to obtain the quantized algebra $\mathcal{U}_h(so(4))$. The $h$-deformed 3-dimensional Euclidean algebra $\mathcal{U}_h(e(3))$, among others, was realized as a contraction of $\mathcal{U}_h(so(4))$ algebra. In the connection between the classical Euclidean algebra $e(3)$ and the non-linear $\sigma$-model was pointed out. In that respect, the study of the various deformations of $e(3)$ assumes importance. In searching for an alternate $h$-deformation of the $e(3)$ algebra, here we follow a closely parallel approach. Starting with a different choice of the generators of the $\mathcal{U}_h(so(4))$ algebra, we obtain a new $\mathcal{U}_h(e(3))$ algebra at a contraction limit. The present $\mathcal{U}_h(e(3))$ has several interesting properties. In particular, the rotation algebra is preserved after the contraction. More importantly, reminiscent of the scenario discussed for the $\mathcal{U}_h(sl(2))$ algebra, the generators of this $\mathcal{U}_h(e(3))$ algebra may be realized, via a nonlinear map, in terms of the undeformed $e(3)$ generators. This tremendously simplifies the study of the irreducible representations of the proposed $\mathcal{U}_h(e(3))$ algebra. Along similar lines, the corresponding $q$-deformed algebra $\mathcal{U}_q(e(3))$ was studied before. The comparison between the two cases turns out to be of interest.

The $h$-deformed algebra $\mathcal{U}_h(so(4))$ is considered as a direct sum $\mathcal{U}_h(so(4)) = \mathcal{U}_h(sl(2)) \oplus \mathcal{U}_{-h}(sl(2))$. The choice of the oppositely signed deformation parameters $(\pm h)$ is necessary to avoid singularities in the coproducts after the contraction is achieved. Let $(X_1, Y_1, H_1)$ and $(X_2, Y_2, H_2)$ be two mutually commuting sets of generators, where each triplet satisfies the commutation relations. Their coalgebraic structure reads

\[
\begin{align*}
\triangle(X_i) &= X_i \otimes 1 + 1 \otimes X_i, \\
\triangle(Y_i) &= Y_i \otimes e^{h\theta_i}X_i + e^{-h\theta_i}X_i \otimes Y_i, \\
\triangle(H_i) &= H_i \otimes e^{h\theta_i}X_i + e^{-h\theta_i}X_i \otimes H_i, \\
\varepsilon(X_i) &= \varepsilon(Y_i) = \varepsilon(H_i) = 0, \\
S(X_i) &= -X_i,
\end{align*}
\]
of the generators (3.1). The advantage of the algebra (3.3) in that the representation of the subalgebra consisting of the complete structure (3.3). The coalgebraic structure is as follows

\[ S(Y_i) = -e^{\theta_i X_i} Y_i e^{-\theta_i X_i}, \]
\[ S(H_i) = -e^{\theta_i X_i} H_i e^{-\theta_i X_i}, \]

where \( i = (1, 2) \) and \( \theta_1(\theta_2) = 1(-1) \). In contrast to the usual practice [10, 9], we make the following choice of the \( U_h(so(4)) \) generators

\[
\begin{align*}
J_+ &= X_1 + X_2, & K_+ &= X_1 - X_2, \\
J_- &= Y_1 e^{h X_2} + e^{-h X_1} Y_2, & K_- &= Y_1 e^{h X_2} - e^{-h X_1} Y_2, \\
J_0 &= H_1 e^{h X_2} + e^{-h X_1} H_2, & K_0 &= H_1 e^{h X_2} - e^{-h X_1} H_2. 
\end{align*}
\]

This choice is motivated by our intention to preserve a subalgebra \( (J_\pm, J_0) \) satisfying the commutation relations of the \( U_h(sl(2)) \) generators. The generators (3.2) may be expressed in terms of the generators used in [10, 9]. The algebraic structure of \( U_h(so(4)) \) now reads

\[
\begin{align*}
[J_0, J_+] &= [K_0, K_+] = \frac{2}{h} \sinh(h J_+), \\
[J_0, J_-] &= -J_- \cosh(h J_+) - \cosh(h J_+) J_, \\
[J_+, J_-] &= [K_+, K_-] = J_0, \\
[K_0, K_-] &= -J_- e^{-h K_+} - e^{-h K_+} J_- - K_- \sinh(h J_+) - \sinh(h J_+), \\
[J_0, K_+] &= [K_0, J_+] = \frac{2}{h} (\cosh(h J_+) - e^{-h K_+}), \\
[J_0, K_-] &= -K_- \cosh(h J_+) - \cosh(h J_+) K_- - \frac{h}{8} (J_0 + K_0) e^{-h J_+} \\
&\quad + e^{-h J_+} (J_0 + K_0) \left( (J_0 - K_0) e^{h J_+} + e^{h J_+} (J_0 - K_0) \right), \\
[K_0, J_-] &= -K_- e^{-h K_+} - e^{-h K_+} K_- - \sinh(h J_+) J_- - J_- \sinh(h J_+) + \\
&\quad \frac{h}{8} (J_0 + K_0) e^{-h J_+} + e^{h J_+} (J_0 + K_0) \left( (J_0 - K_0) e^{h J_+} + e^{h J_+} (J_0 - K_0) \right), \\
[J_+, K_-] &= [K_+, J_-] = K_0, \\
[J_+, K_+] &= 0, \\
[J_-, K_-] &= -\frac{h}{4} (J_+ + K_+) \left( e^{-h J_+} (J_0 - K_0) e^{h J_+} + (J_0 - K_0) \right) \\
&\quad - \frac{h}{4} \left( (J_0 + K_0) e^{-h J_+} + e^{-h J_+} (J_0 + K_0) \right) (J_- - K_-) e^{h J_+} \\
[J_0, K_0] &= 2 J_0 \sinh(h J_+) + 2 K_0 (e^{-h K_+} - \cosh(h J_+)). \tag{3.3}
\end{align*}
\]

One advantage of the algebra (3.3) in that the representation of the subalgebra consisting of the generators \( (J_\pm, J_3) \) is completely known. This may help in finding the representation of the complete structure (3.3). The co-algebraic structure is as follows

\[ \triangle(J_+) = J_+ \otimes 1 + 1 \otimes J_+, \]
\[ \Delta(J_-) = J_- \otimes \cosh(hJ_+) + e^{-hK_+} \otimes J_- + K_- \otimes \sinh(hJ_+), \]
\[ \Delta(J_0) = J_0 \otimes \cosh(hJ_+) + e^{-hK_+} \otimes J_0 + K_0 \otimes \sinh(hJ_+), \]
\[ \Delta(K_+) = K_+ \otimes 1 + 1 \otimes K_+, \]
\[ \Delta(K_-) = K_- \otimes \cosh(hJ_+) + e^{-hK_+} \otimes K_- + J_- \otimes \sinh(hJ_+), \]
\[ \Delta(K_0) = K_0 \otimes \cosh(hJ_+) + e^{-hK_+} \otimes K_0 + J_0 \otimes \sinh(hJ_+), \]
\[ \varepsilon(\xi) = 0 \quad \text{for} \quad \xi = (J_+, J_-, J_0, K_+, K_-, K_0), \]
\[ S(J_+) = -J_+, \]
\[ S(J_-) = -e^{hK_+} \left( J_- \cosh(hJ_+) - K_- \sinh(hJ_+) \right), \]
\[ S(J_0) = -e^{hK_+} \left( J_0 \cosh(hJ_+) - K_0 \sinh(hJ_+) \right), \]
\[ S(K_+) = -K_+, \]
\[ S(K_-) = -e^{hK_+} \left( K_- \cosh(hJ_+) - J_- \sinh(hJ_+) \right), \]
\[ S(K_0) = -e^{hK_+} \left( K_0 \cosh(hJ_+) - J_0 \sinh(hJ_+) \right). \] (3.4)

The universal \( \mathcal{R} \)-matrix for the algebra \( \mathcal{U}_h(sl(2)) \) was obtained \cite{8, 9} recently. For the algebra \( \mathcal{U}_h(so(4)) \), the corresponding universal \( \mathcal{R} \)-matrix was constructed \cite{9} by considering two copies of the \( \mathcal{R} \)-matrix of the algebra \( \mathcal{U}_h(sl(2)) \). For our choice of the generators (3.2), the universal \( \mathcal{R} \)-matrix for the \( \mathcal{U}_h(so(4)) \) algebra assumes the form

\[ \mathcal{R} = \exp \left[ \frac{h}{8} (\Delta - \Delta') \right] \left[ \frac{(J_0 + K_0) (1 - e^{-h(J_+ + K_+)})(J_+ + K_+) + (J_0 - K_0) (e^{h(J_+ + K_+)} - 1)(J_+ - K_+)}{\cosh(hJ_+) - \cosh(hK_+)} \right], \] (3.5)

where \( \Delta' \) is the flipped coproduct map.

In order to construct the \( \mathcal{U}_h(e(3)) \) algebra by using the contraction procedure, we define a complex parameter \( \omega \) and the generators \( (P_\pm, P_0) \) as follows

\[ \omega = \frac{h}{\epsilon}, \quad P_\pm = \epsilon K_\pm, \quad P_0 = \epsilon K_0, \] (3.6)

while the generators \( (J_\pm, J_0) \) are not transformed. In the limit \( \epsilon \to 0 \), the following algebraic structure of \( \mathcal{U}_h(e(3)) \) is obtained from (3.3):

\[ [J_0, J_\pm] = \pm 2J_\pm, \quad [J_+, J_-] = J_0, \]
\[ [P_0, P_\pm] = 0, \quad [P_+, P_-] = 0, \]
\[ [J_0, P_+] = [P_0, J_+] = \frac{2}{\omega} (1 - e^{-\omega P_+}), \]
\[ [J_0, P_-] = -2P_+ + \frac{\omega}{2} P_0^2, \]
\[ [P_0, J_-] = -2e^{-\omega P_+} P_- - \frac{\omega}{2} P_0^2, \]
\[ [J_+, P_-] = [P_+, J_-] = P_0, \]
\[ [J_+, P_+] = 0, \quad [J_-, P_-] = \omega P_0 P_-, \quad [J_-, P_+] = \omega P_0 P_+. \quad (3.7) \]

A distinctive feature of this algebra is that, unlike the results obtained in [9] for the corresponding case, here the rotation group is preserved. The coalgebra maps for \( U_h(e(3)) \) are read from (3.4) at the contraction limit

\[ \Delta(J_+) = J_+ \otimes 1 + 1 \otimes J_+, \]
\[ \Delta(J_-) = J_- \otimes 1 + e^{-\omega P_+} \otimes J_- + \omega P_- \otimes J_+, \]
\[ \Delta(J_0) = J_0 \otimes 1 + e^{-\omega P_+} \otimes J_0 + \omega P_0 \otimes J_+, \]
\[ \Delta(P_+) = P_+ \otimes 1 + 1 \otimes P_+, \]
\[ \Delta(P_-) = P_- \otimes 1 + e^{-\omega P_+} \otimes P_-, \]
\[ \Delta(P_0) = P_0 \otimes 1 + e^{-\omega P_+} \otimes P_0, \]
\[ \varepsilon(\xi) = 0 \quad \text{for} \quad \xi = (J_+, J_-, J_0, P_+, P_-, P_0), \]
\[ S(J_+) = -J_+, \quad S(J_-) = -e^{\omega P_+}(J_- - \omega P_- J_+), \quad S(J_0) = -e^{\omega P_+}(J_0 - \omega P_0 J_+), \]
\[ S(P_+) = -P_+, \quad S(P_-) = -e^{\omega P_+} P_-, \quad S(P_0) = -e^{\omega P_+} P_0. \quad (3.8) \]

The universal \( \mathcal{R} \)-matrix for the \( U_h(e(3)) \) algebra may be obtained from (3.5) at the \( \epsilon \to 0 \) limit as the singular terms cancel:

\[ \mathcal{R} = \exp \left[ \frac{\omega}{4}(\Delta - \Delta^\dagger) \right] \frac{(P_0 J_+ + J_0 P_+)(1 - e^{\omega P_+}) + \omega P_0 P_+ J_+ e^{\omega P_+}}{1 - \cosh(\omega P_+)} \]. \quad (3.9) \]

Closely paralleling our earlier description of the \( U_h(sl(2)) \) algebra, we demonstrate here that the algebra (3.7) may be mapped on the classical \( e(3) \) algebra. To this end, we define the generators

\[ \Pi_+ = \frac{1}{\omega}(e^{\omega P_+} - 1), \quad \Pi_- = P_- - \frac{\omega}{4} P_0^2 e^{\omega P_+}, \quad \Pi_0 = P_0 e^{\omega P_+}. \quad (3.10) \]

From the algebra (3.7), it follows that \( (J_\pm, J_0, \Pi_\pm, \Pi_0) \) obey classical algebra \( e(3) \) algebra, where \( (\Pi_\pm, \Pi_0) \) play the role of generators of translations. We obtain

\[ [J_0, J_\pm] = \pm 2J_\pm, \quad [J_+, J_-] = J_0, \quad [\Pi_0, \Pi_\pm] = 0, \quad [\Pi_+, \Pi_-] = 0, \]
\[ [J_+, \Pi_+] = 0, \quad [J_+, \Pi_-] = P_0, \quad [J_+, \Pi_0] = -2\Pi_+, \]
\[ [J_-, \Pi_+] = -\Pi_0, \quad [J_-, \Pi_-] = 0, \quad [J_-, \Pi_0] = 2\Pi_-, \]
\[ [J_0, \Pi_\pm] = \pm 2\Pi_\pm, \quad [J_0, \Pi_0] = 0. \quad (3.11) \]
The Casimir operators are
\begin{align}
C_1 &= \Pi_+ \Pi_- + \frac{1}{4} \Pi_0^2, \\
&= \frac{1}{\omega} (\omega P_+ - 1)(P_+ - \omega P_0^2 e^{\omega P_+}) + \frac{1}{4} P_0^2 e^{2\omega P_+}, \\
C_2 &= J_+ \Pi_- + J_- \Pi_+ + \frac{1}{2} J_0 \Pi_0 \\
&= J_+(P_- - \frac{\omega}{4} P_0^2 e^{\omega P_+}) + \frac{1}{\omega} J_-(\omega P_+ - 1) + \frac{1}{2} J_0 P_0 e^{\omega P_+}.
\end{align}
(3.12)

(3.13)

The representation theory of the classical \(\mathfrak{e}(3)\) algebra may now be readily used to obtain the representations of the algebra (3.7). For the classical \(\mathfrak{e}(3)\) generated by \((J_\pm, J_0, \Pi_\pm, \Pi_0)\) we may introduce the standard representations in the momentum or the angular momentum basis. Concerning the action of the generators \((P_\pm, P_0)\) on momentum bases, we note the following. The inverse map of (3.10) reads
\begin{align}
P_+ &= \frac{1}{\omega} \ln(1 + \omega \Pi_+), \\
P_- &= \Pi_- + \frac{\omega}{4} \frac{\Pi_0^2}{(1 + \omega \Pi_+)}, \\
P_0 &= \frac{\Pi_0}{(1 + \omega \Pi_+)}
\end{align}
(3.14)

On the momentum basis (where the eigenvalues of \(\Pi_\pm\) and \(\Pi_0\) are taken to be \(\tilde{\Pi}_\pm\) and \(\tilde{\Pi}_0\) respectively), it is particularly evident from (3.14) that the eigenvalues of the operators \((P_\pm, P_0)\) diverge for \((1 + \omega \tilde{\Pi}_+) = 0\). Moreover the eigenvalue of \(P_+\) develops an imaginary part for \((1 + \omega \tilde{\Pi}_+) < 0\). In fact the whole real domain for the eigenvalues of operators \((P_\pm, P_0)\) correspond to the restricted domain of the classical eigenvalues given by \((1 + \omega \Pi_+) > 0\).

The singularities in the inverse map (3.14) contrast very sharply with a similar construction obtained by one of us [16] in the context of \(q\)-deformed algebra \(U_q(\mathfrak{e}(3))\). We will briefly review the mapping of the nonlinear algebraic structure of \(U_q(\mathfrak{e}(3))\) on the classical \(\mathfrak{e}(3)\) algebra in the Appendix. From (A.4), it is evident that, there the nonlinearity enters in the map through the positive definite ‘invariant mass’ \(C_1\), and, consequently, the map is invertible without any singularity. It is known [17] that the \(\kappa\)-deformed Poincaré algebra may be realized in terms of the generators of the classical Poincaré algebra. We note the analogies and contrasts between the map [17] for the \(\kappa\)-Poincaré algebra and the corresponding maps for the algebras \(U_h(\mathfrak{e}(3))\) and \(U_q(\mathfrak{e}(3))\). As noted earlier, the nonlinearity enters the map (3.14) for the \(U_h(\mathfrak{e}(3))\) algebra through the classical momentum operator, which is not positive definite, and, therefore, the map shows singularity. As for the \(U_q(\mathfrak{e}(3))\) example discussed in the Appendix, there is no singularity in the map. For the \(\kappa\) deformed algebra, the nonlinearity enters [17] through the classical energy operator and, for the positive energy solutions, the map shows no singularity.

4 Conclusion

We obtained the finite dimensional highest weight representations of \(U_h(\mathfrak{sl}(2))\) algebra as factor representations by using the standard singular vector treatment. This was done by
solving a set of recursion relations valid for the matrix elements of the $\mathcal{U}_h(sl(2))$ generators. These irreducible representations may also be determined by mapping the $\mathcal{U}_h(sl(2))$ algebra on the classical $sl(2)$ algebra. The corresponding map of the $\mathcal{U}_h(e(2))$ algebra was obtained by using a contraction method. Taking a nonlinear combination of generators of two copies of a $h$-deformed $sl(2)$ algebra, the $\mathcal{U}_h(so(4))$ algebra was constructed. An $\mathcal{U}_h(e(3))$ algebra was constructed by contracting $\mathcal{U}_h(so(4))$. The $\mathcal{U}_h(e(3))$ may be mapped on the classical $e(3)$ algebra. A physically interesting feature is that this map, unlike the previously known cases of $\mathcal{U}_q(e(3))$ algebra and the $\kappa$-Poincaré algebra, exhibits a singular behavior. In the momentum basis, a restricted domain of the eigenvalues of the classical operators is mapped on the whole real domain of the eigenvalues of the deformed operators.

**Acknowledgments:**

One of us (RC) wants to thank A. Chakrabarti for a kind invitation. He is also grateful to the members of the CPTH group for their kind hospitality.
Appendix

Here we deviate from the main body of the paper and give a summary of the construction of the $q$-deformed algebra $\mathcal{U}_q(e(3))$, where the algebraic structure may also be mapped [10] on the undeformed $e(3)$ algebra. Unlike the $h$-deformed case discussed in (3.10) and (3.14), the map obtained in [10] is invertible and non-singular.

Starting with a contraction $\mathcal{U}_q(so(4)) = \mathcal{U}_q(sl(2)) \oplus \mathcal{U}_q(sl(2))$ where $(\hat{j}^{(i)}_\pm, \hat{j}^0_\pm)$ are the generators of the two deformed $sl(2)$ algebras, we define

$$\hat{J}_0 = \hat{j}^{(1)}_0 + \hat{j}^{(2)}_0, \quad \hat{J}_\pm = \hat{j}^{(1)}_\pm q^{-\hat{j}^0_\pm} + \hat{j}^{(2)}_\pm q^{\hat{j}^0_\pm},$$

$$\hat{K}_0 = \hat{j}^{(1)}_0 - \hat{j}^{(2)}_0, \quad \hat{K}_\pm = \hat{j}^{(1)}_\pm q^{-\hat{j}^0_\pm} - \hat{j}^{(2)}_\pm q^{\hat{j}^0_\pm}. \quad \text{(Appendix.1)}$$

The modifications with respect to [21], in the definitions of $\hat{J}_\pm$, $\hat{K}_\pm$ are suitable for our purpose. We now perform the following contraction

$$\hat{K}_\pm = \frac{\hat{P}_\pm}{\epsilon}, \quad \hat{K}_0 = \frac{\hat{P}_0}{\epsilon}, \quad q = e^{\epsilon \Omega} \quad \text{(Appendix.2)}$$

and take the limit $\epsilon \to 0$. The $\mathcal{U}_q(so(4))$ algebra constructed with the generators (A.1) preserves the rotation subalgebra, both before and after the contraction. It also permits to exhibit a nonlinear mapping to the classical algebra in a direct fashion. After contraction, the $\mathcal{U}_q(e(3))$ algebra reads

$$[\hat{J}_0, \hat{J}_\pm] = \pm 2\hat{J}_\pm, \quad [\hat{J}_\pm, \hat{J}_-] = \hat{J}_0, \quad [\hat{P}_0, \hat{P}_\pm] = 0, \quad [\hat{P}_+, \hat{P}_-] = 0,$$

$$[\hat{J}_0, \hat{P}_\pm] = [\hat{P}_0, \hat{J}_\pm] = \pm 2\hat{P}_\pm, \quad [\hat{J}_0, \hat{P}_0] = 0,$$

$$[\hat{P}_\pm, \hat{J}_\pm] = \pm \Omega \hat{P}_\pm, \quad [\hat{J}_\pm, \hat{P}_\pm] = \pm \frac{1}{\Omega} (e^{\Omega \hat{P}_0} - 1). \quad \text{(Appendix.3)}$$

The algebra (A.3) may be mapped on the classical $e(3)$ algebra. Let $(\hat{J}_\pm, \hat{J}_0, \hat{\Pi}_\pm, \hat{\Pi}_0)$ be the generators of the classical $e(3)$ algebra. Then the map reads

$$e^{-\frac{\Omega}{2} \hat{P}_0} = (1 + \frac{1}{2} C_1 \Omega^2) - \frac{\Omega}{2} (1 + \frac{1}{4} C_1 \Omega^2)^{1/2} \hat{\Pi}_0,$$

$$\hat{P}_\pm e^{-\frac{\Omega}{2} \hat{P}_0} = (1 + \frac{1}{4} C_1 \Omega^2)^{1/2} \hat{\Pi}_\pm. \quad \text{(Appendix.4)}$$

where $C_1$ is the positive definite ‘invariant mass’ of the undeformed $e(3)$ algebra. In terms of the generators of $\mathcal{U}_q(e(3))$, the Casimir operator $C_1 = \hat{\Pi}_+ \hat{\Pi}_- + \frac{1}{4} \hat{\Pi}_0^2$ has the form

$$C_1 = \hat{P}_+ \hat{P}_- e^{-\frac{\Omega}{2} \hat{P}_0} + \frac{1}{\Omega^2} (e^{\frac{\Omega}{2} \hat{P}_0} + e^{-\frac{\Omega}{2} \hat{P}_0} - 2). \quad \text{(Appendix.5)}$$

Therefore the mapping (A.3) is invertible without any singularity. This clearly contrasts the map described in Section 3 for the $h$-deformed algebra $\mathcal{U}_h(e(3))$. 

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References

[1] V. G. Drinfeld, Quantum Groups, Proc. Int. Congress of Mathematicians, Berkeley, California, Vol. 1, Academic Press, New York (1986), 798.

[2] M. Jimbo, Lett. Math. Phys. 10 (1985) 63.

[3] E. E. Demidov, Yu. I. Manin, E. E. Mukhin and D. Z. Zhdanovich, Prog. Theor. Phys. Suppl. 102 (1990) 203.

[4] Ch. Ohn, Lett. Math. Phys. 25 (1992) 85.

[5] A. Aghamohammadi, M. Khorrami and A. Shariati, J. Phys.A: Math. Gen. 28 (1995) L225.

[6] A. Aghamohammadi, Mod. Phys. Lett. A8 (1993) 2607.

[7] V. Karimipour, Lett. Math. Phys. 35 (1995) 303.

[8] A. A. Vladimirov, Mod. Phys. Lett. A 8 (1993) 2573.

[9] A. Shariati, A. Aghamohammadi and M. Khorrami, Mod. Phys. Lett. A 11 (1996) 187.

[10] A. Ballesteros, F. J. Herranz, M. A. del Olmo and M Santander, J.Phys.A: Math. Gen. 28 (1995) 941.

[11] A. Ballesteros, F. J. Herranz, M. A. del Olmo, C. M. Perena and M. Santander, J.Phys.A: Math. Gen. 28 (1995) 7113.

[12] P. Parashar, Nonstandard Poincare and Heisenberg Algebras, SISSA:85/96/FM, q-alg/9606003.

[13] L. Dabrowski and P. Parashar, h-deformation of GL(1|1) SISSA-134-95-FM.

[14] V. K. Dobrev, Representations of the Jordanian Quantum Algebra $U_h(sl(2))$, IC/96/14.

[15] B. Abdesselam, A. Chakrabarti and R. Chakrabarti, Irreducible Representations of Jordanian Quantum Algebra $U_h(sl(2))$ Via a Nonlinear map, Ecole Polytechnique Preprint CPTH-S455.0696 (1996).

[16] A. Chakrabarti, Generators of q-deformed E(3) as nonlinear functions of the classical generators, Ecole polytechnique preprint, A/97-0992 (1992) (unpublished).

[17] P. Kosinski, J. Lukierski, J. Sobczyk and P. Maslanka, Modern Physics Letters A 34 (1995) 2599.

[18] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B 264 (1991) 331.
[19] J. Lukierski, A. Nowicki and H. Ruegg, Phys. Lett. B 293 (1992) 344.

[20] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, J. Math. Phys. 32 (1991) 1155.

[21] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, J. Math. Phys. 32 (1991) 1159.

[22] A. Chakrabarti, $IU(n)_q$: $q$-deformation of inhomogeneous algebras, (Goslar conference proceedings, 1991, P. 203, Quantum symmetries; Edited by H. D. Doebner and V.K. Dobrev; Pub. World Scientific, Singapore.

[23] A. Chakrabarti, J. Math. Phys. 35 (1994) 4247.

[24] A. Chakrabarti, $SO(5)_q$ and contraction (Karpacz conf- on $q$-groupes. 1994) quantum groups, P. 343, Ed. J. Lukierski, Z. Popwicz, J. Sobczyk (Polish Scientific Publishers PWN).

[25] L.S. Faddeev, Integrable Models in (1+1)-dimensional quantum field Theory, Les Houches Lectures 1982 (Elsevier, Amsterdam, 19984).