Modules over group rings of locally soluble groups with a certain condition of minimality

O. YU. Dashkova
MODULES OVER GROUP RINGS OF LOCALLY SOLUBLE GROUPS WITH A CERTAIN CONDITION OF MINIMALITY

O. YU. DASHKOVA

Received 10 May, 2014

Abstract. Let $A$ be a $RG$-module, where $R$ is an associative ring, $A/C_A(G)$ is an infinite $R$-module, $C_A(A) = 1$, $G$ is a locally soluble group. Let $L_{nf}(G)$ be the system of all subgroups $H \leq G$ such that quotient modules $A/C_A(H)$ are infinite $R$-modules. The author studies an $RG$-module $A$ such that $L_{nf}(G)$ satisfies the minimal condition as an ordered set. It is proved that a locally soluble group $G$ with these conditions is soluble. The structure of $G$ is described.

2010 Mathematics Subject Classification: 20F19; 20H25

Keywords: group ring, locally soluble group, module

1. INTRODUCTION

Let $A$ be a vector space over a field $F$. The subgroups of the group $GL(F, A)$ of all automorphisms of $A$ are called linear groups. If $A$ has a finite dimension over $F$ then $GL(F, A)$ can be considered as the group of non-singular $(n \times n)$-matrices, where $n = dim_F A$. Finite dimensional linear groups have played an important role in various fields of mathematics, physics and natural sciences, and have been studied many times. When $A$ is infinite dimensional over $F$, the situation is totally different. Infinite dimensional linear groups have been investigated little. The study of this class of groups requires additional restrictions. In [5] it was introduced the definition of the central dimension of an infinite dimensional linear group. Let $H$ be a subgroup of $GL(F, A)$. $H$ acts on the quotient space $A/C_A(H)$ in a natural way. The authors define $centdim_H$ to be $dim_F (A/C_A(H))$. The subgroup $H$ is said to have a finite central dimension if $centdim_H$ is finite and $H$ has infinite central dimension otherwise. Let $G \leq GL(F, A)$. In [5] it was considered the system $L_{id}(G)$ of all subgroups of $G$ of infinite central dimension. In order to investigate infinite dimensional linear groups that are close to finite dimensional, it is natural to consider the case where the system $L_{id}(G)$ is “very small”. The authors have studied locally soluble infinite dimensional linear groups such that $L_{id}(G)$ satisfies the minimal condition as an ordered set [5].

© 2014 Miskolc University Press
If $G \leq GL(F, A)$ then $A$ can be considered as an $FG$-module. The natural generalization of this case is the consideration of an $RG$-module $A$, where $R$ is a ring whose structure is near to a field. At this point the generalization of the notion of the central dimension of a subgroup of a linear group is the notion of the cocentralizer of a subgroup. This notion was introduced in [8]. Let $A$ be an $RG$-module, $R$ be an associative ring, $G$ be a group. If $H \leq G$ then the quotient module $A/C_A(H)$ considered as an $R$-module is called the cocentralizer of $H$ in the module $A$.

Modules over group rings of finite groups have been considered by many authors. Recently this class of modules was investigated in [6]. Study of modules over group rings of infinite groups requires some additional restrictions as in the case of infinite dimensional linear groups. In [2] it was investigated an $RG$-module $A$ such that $R$ is a dedekind domain and the cocentralizer of $G$ in the module $A$ is not an artinian $R$-module. It was considered the system $L_{nad}(G)$ of all subgroups of $G$ such that their cocentralizers in the module $A$ are not artinian $R$-modules which is ordered by the usual inclusion. It is investigated an $RG$-module $A$ such that the system $L_{nad}(G)$ satisfies the minimal condition as an ordered set, $G$ is a locally soluble group, $C_G(A) = 1$. The analogous problem for the ring of integers $R$ was investigated in [3].

In [1] we have studied an $RG$-module $A$ such that $R$ is the ring of integers, the cocentralizer of $G$ in the module $A$ is not a noetherian $R$-module and $C_G(A) = 1$. Let $L_{nd}(G)$ be the system of all subgroups of $G$ such that their cocentralizers in the module $A$ are noetherian $R$-modules. It was investigated an $RG$-module $A$ such that $L_{nd}(G)$ satisfies the minimal condition as an ordered set and $G$ is locally soluble.

In [4] we have considered the similar problem where $R$ is the ring of integers and the noetherian condition is replaced by the minimax condition.

In this paper we investigate $RG$-module, where $R$ is an associative ring, $A/C_A(G)$ is an infinite $R$-module, $C_G(A) = 1$, $G$ is a locally soluble group. Let $L_{nf}(G)$ be the system of all subgroups $H \leq G$ such that $A/C_A(H)$ are infinite $R$-modules. We study an $RG$-module $A$ such that $L_{nf}(G)$ satisfies the minimal condition as an ordered set. It is proved that a locally soluble group $G$ with these conditions is soluble and the structure of $G$ is described.

The main results of this paper are Theorems 2 and 3.

2. PRELIMINARY RESULTS

We reduce some elementary facts about $RG$-modules.

Later on it is considered an $RG$-module $A$ such that $C_G(A) = 1$.

Let $A$ be an $RG$-module where $G$ is a group, $R$ is an associative ring. Recall that if $K \leq H \leq G$ and the cocentralizer of $H$ in the module $A$ is a finite $R$-module then the cocentralizer of $K$ in the module $A$ is a finite $R$-module also. If $U, V$ are
subgroups of $G$ such that their cocentralizers in the module $A$ are finite $\mathbb{R}$-modules, then $A/(C_A(U) \cap C_A(V))$ is a finite $\mathbb{R}$-module also.

Suppose that a group $G$ satisfies the condition $mn - nf$. If $H_1 > H_2 > H_3 > \cdots$ is an infinite strictly descending chain of subgroups of $G$ then there is the natural number $n$ such that the cocentralizer of $H_n$ in the module $A$ is a finite $\mathbb{R}$-module. Moreover, if $N$ is a normal subgroup of $G$ and the cocentralizer of $N$ in the module $A$ is an infinite $\mathbb{R}$-module then $G/N$ satisfies the minimal condition on subgroups.

**Lemma 1.** Let $A$ be an $\mathbb{R}G$-module, $G$ be a group, $\mathbb{R}$ be an associative ring. Suppose that $G$ satisfies the condition $mn - nf$, $X, H$ are subgroups of $G$ and $\Lambda$ is an index set such that

1. $X = Dr_{\lambda \in \Lambda}X_\lambda$, where $1 \neq X_\lambda$ is an $H$-invariant subgroup of $X$, for each $\lambda \in \Lambda$.
2. $H \cap X \leq Dr_{\lambda \in \Gamma}X_\lambda$ for some subset $\Gamma$ of $\Lambda$.

If the set $\Omega = \Lambda \setminus \Gamma$ is infinite, then the cocentralizer of $H$ in the module $A$ is a finite $\mathbb{R}$-module.

*Proof.* Suppose that the set $\Omega$ is infinite and let $\Omega_1 \supset \Omega_2 \supset \cdots$ be a strictly descending chain of subsets of the set $\Omega$. Since $H \cap Dr_{\lambda \in \Omega}X_\lambda = 1$, the chain of subgroups $\langle H, X_\lambda | \lambda \in \Omega_1 \rangle > \langle H, X_\lambda | \lambda \in \Omega_2 \rangle > \cdots$ is strictly descending. It follows that for some natural number $d$ the cocentralizer of the subgroup $\langle H, X_\lambda | \lambda \in \Omega_d \rangle$ in the module $A$ is a finite $\mathbb{R}$-module. Therefore the cocentralizer of $H$ in the module $A$ is a finite $\mathbb{R}$-module also. \qed

**Lemma 2.** Let $A$ be an $\mathbb{R}G$-module, $G$ be a group, $\mathbb{R}$ be an associative ring, $G$ satisfy the condition $mn - nf$, $H, K$ be subgroups of $G$ such that $K$ is a normal subgroup of $H$. Suppose that there exists an index set $\Lambda$ and subgroups $H_\lambda$ of $G$ such that $K \leq H_\lambda$ for all $\lambda \in \Lambda$, $H/K = Dr_{\lambda \in \Lambda}H_\lambda/K$, and the set $\Lambda$ is infinite. Then the cocentralizer of $H$ in the module $A$ is a finite $\mathbb{R}$-module.

*Proof.* Let $\Gamma$ and $\Omega$ be infinite disjoint subsets of the set $\Lambda$ such that $\Lambda = \Gamma \cup \Omega$. Let $U/K = Dr_{\lambda \in \Gamma}H_\lambda/K$, let $V/K = Dr_{\lambda \in \Omega}H_\lambda/K$, and let $\Gamma_1 \supset \Gamma_2 \supset \cdots$ be a strictly descending chain of subsets of the set $\Gamma$. Then we construct an infinite strictly descending chain of subgroups

$$\langle V, H_\lambda | \lambda \in \Gamma_1 \rangle > \langle V, H_\lambda | \lambda \in \Gamma_2 \rangle > \cdots .$$

It follows from the condition $mn - nf$ that the cocentralizer of $V$ in the module $A$ is a finite $\mathbb{R}$-module. Likewise, we obtain that the cocentralizer of $U$ in the module $A$ is a finite $\mathbb{R}$-module. Since $H = UV$, it follows that the cocentralizer of $H$ in the module $A$ is a finite $\mathbb{R}$-module. \qed

**Lemma 3.** Let $A$ be an $\mathbb{R}G$-module, $G$ be a group, $\mathbb{R}$ be an associative ring, $G$ satisfy the condition $mn - nf$. If an element $g \in G$ has infinite order then the cocentralizer of $\langle g \rangle$ in the module $A$ is a finite $\mathbb{R}$-module.
Proof. Let \( p, q \) are distinct primes greater than 3 and let \( u = g^p, v = g^q \). Then there is an infinite descending chain of subgroups \( \langle u \rangle > \langle u^2 \rangle > \langle u^4 \rangle > \cdots \). It follows from the condition \( \min - nf \) that there exists the natural number \( k \) such that the cocentralizer of the subgroup \( \langle u^{2k} \rangle \) in the module \( A \) is a finite \( R \)-module. Similarly, there exists a natural number \( l \) such that the cocentralizer of the subgroup \( \langle v^{3l} \rangle \) in the module \( A \) is a finite \( R \)-module. Therefore the cocentralizer of the subgroup \( \langle g \rangle = \langle u^{2k} \rangle \langle v^{3l} \rangle \) in the module \( A \) is a finite \( R \)-module.

\[ \square \]

The following result gives an important information about the derived quotient group under the condition \( \min - nf \).

**Lemma 4.** Let \( A \) be an \( RG \)-module, \( G \) be a group, \( R \) be an associative ring. Suppose that the cocentralizer of \( G \) in the module \( A \) is an infinite \( R \)-module, and \( G \) satisfies the condition \( \min - nf \). Then the quotient group \( G' / G'' \) is Chernikov.

**Proof.** Suppose that the quotient group \( G / G' \) is not Chernikov group. Let \( \mathfrak{S} \) be the family of all subgroups \( H \leq G \) such that \( H / H' \) is not Chernikov and the cocentralizer of \( H \) in the module \( A \) is an infinite \( R \)-module. Since \( G \in \mathfrak{S} \) then \( \mathfrak{S} \neq \emptyset \). Since the set \( \mathfrak{S} \) satisfies the minimal condition, then it has a minimal element. Let \( D \) be this minimal element. If \( U, V \) are proper subgroups of the group \( D \) such that \( D = UV \) and \( U \cap V = D' \), then at least one of these subgroups, \( U \) say, such that its cocentralizer in the module \( A \) is an infinite \( R \)-module. The choice of \( D \) implies that \( U / U' \) is Chernikov. It follows with regard to the isomorphism \( U / D' \simeq (U / U') / (D' / U') \) that \( U / D' \) is also Chernikov. Since the cocentralizer of \( U \) in the module \( A \) is an infinite \( R \)-module it follows that the abelian quotient group \( D / U \) is also Chernikov. Hence the quotient group \( D / D' \) is Chernikov. Contrary to the choice of \( D \). Therefore \( D / D' \) is indecomposable. Hence \( D / D' \) is isomorphic to a subgroup of quasi-cyclic group \( C_{q^\infty} \), for some prime \( q \). Contradiction.

\[ \square \]

Let \( A \) be an \( RG \)-module, \( G \) be a group, \( R \) be an associative ring. Let \( FFD(G) \) be the set of all elements \( x \in G \) such that the cocentralizer of \( \langle x \rangle \) in the module \( A \) is a finite \( R \)-module. Since \( C_A(x^g) = C_A(x)g \) for all \( x, g \in G \), it follows that \( FFD(G) \) is a normal subgroup of \( G \).

**Lemma 5.** Let \( A \) be an \( RG \)-module, \( G \) be a group, \( R \) be an associative ring. Suppose that the cocentralizer of \( G \) in the module \( A \) is an infinite \( R \)-module, and \( G \) satisfies the condition \( \min - nf \). Then \( G \) is either periodic or \( G \neq FFD(G) \).

**Proof.** We suppose to the contrary that \( G \) is neither periodic nor \( G \neq FFD(G) \). Let \( \mathfrak{S} \) be the family of all subgroups \( H \leq G \) such that \( H \) is not periodic and \( H \neq FFD(H) \). \( \mathfrak{S} \) is non-empty. If \( H \neq FFD(H) \) then there is an element \( h \in H \) such that the quotient module \( A / C_A(h) \) is an infinite \( R \)-module. Hence \( \mathfrak{S} \subseteq L_{nf}(G) \), and
therefore $\mathcal{G}$ satisfies the minimal condition. Let $D$ be the minimal element of $\mathcal{G}$, let $L = FFD(D)$. Note that $L \neq 1$, since $D$ is not a periodic group. If $L \leq S \leq D$ and $S \neq D$, then $S = FFD(S)$ so $S \leq L$. Hence $D/L$ has order $q$ for some prime $q$. Let $x \in D \setminus L$. If an element $a$ has infinite order, then the choice of $D$ implies that $(x,a) = D$. It follows that $L$ is finitely generated and since $L = FFD(L)$, the quotient module $A/C_A(L)$ is a finite $R$-module. Since the subgroup $L$ is normal in $D$, then $C = C_A(L)$ is an $RD$-submodule of $A$. It follows that $A$ has the finite series of $RD$-submodules

$$\langle 0 \rangle \leq C \leq A,$$

such that $A/C$ is a finite $R$-module. Since $A/C$ is a finite $R$-module then $D/C_B(A/C)$ is finite. As $C = C_A(L)$ then $L \leq C_B(C)$. It follows that $D/C_B(C)$ is finite too.

Let $W = C_B(C) \cap C_B(A/C)$. By Remak theorem

$$D/W \leq D/C_B(C) \times D/C_B(A/C).$$

It follows that the quotient group $D/W$ is finite. $W$ acts trivially on each factor of the series $\langle 0 \rangle \leq C \leq A$. Therefore $W$ is abelian.

Let $U$ be a normal subgroup of finite index of $D$. The subgroup $U$ is not periodic and so $\langle U,x \rangle$ is neither periodic nor $\langle U,x \rangle \neq FFD(\langle U,x \rangle)$. The choice of $D$ implies that $D = \langle U,x \rangle$ and hence the quotient group $D/U$ is abelian. If $E$ is the finite residual of $D$, it follows that the quotient group $D/E$ is abelian. Since $E \leq W$ then $D/W$ is also abelian. It follows that $D/(W \cap L)$ is abelian. Since $W \cap L \leq W$, then the subgroup $W \cap L$ is abelian, and so $D$ is a finitely generated metabelian subgroup. By theorem of P.Hall (Theorem 9.51 [9]) $D$ is residually finite. As above, $D$ is therefore abelian. Since $D = U(x)$ for every subgroup $U$ of finite index, it follows that the group $D$ is infinite cyclic. By Lemma 3 $D = FFD(D)$. We have the contradiction with the choice of $D$.

\[ \square \]

3. Locally soluble groups with the condition $\text{min} - n f$

**Lemma 6.** Let $A$ be an $RG$-module, $G$ be a periodic locally soluble group, $R$ be an associative ring. Suppose that the cocentralizer of $G$ in the module $A$ is an infinite $R$-module and $G$ satisfies the condition $\text{min} - n f$. Then $G$ either satisfies the minimal condition on subgroups or $G = FFD(G)$.

**Proof.** We suppose to the contrary that $G$ is neither satisfies the minimal condition on subgroups nor $G \neq FFD(G)$. Let $\mathcal{G}$ be the family of all subgroups $H \leq G$ such that $H$ does not satisfy the minimal condition on subgroups and $H \neq FFD(H)$. Then $\mathcal{G} \neq \emptyset$. If $H \neq FFD(H)$ then the cocentralizer of $H$ in the module $A$ is an infinite $R$-module and hence $\mathcal{G} \subseteq L_{nf}(G)$. Therefore $\mathcal{G}$ satisfies the minimal condition. Let $D$ be the minimal element and let $L = FFD(D)$. There exists an infinite strictly descending chain of subgroups of $D$:
Since $D$ satisfies the condition $\min -nf$ then there exists the natural number $k$ such that the cocentralizer of $H_k$ in the module $A$ is a finite $R$-module. Therefore $H_k \leq L$, and hence $L$ does not satisfy the minimal condition. If $x \in D \setminus L$ then it follows from the choice of the subgroup $D$ that $\langle x, L \rangle = D$. Hence the quotient group $D/L$ has the order $q$ for prime $q$. If it is necessary we replace $x$ by the suitable power and obtain that $x$ has the order $q^r$ for some natural number $r$. Since the group $D$ is not Chernikov then by D.I.Zaicev’s theorem [10], $D$ contains $\langle x \rangle$-invariant abelian subgroup $B = Dr_{n\in\mathbb{N}}(b_n)$ and we may assume that the elements $b_n$ have prime orders for all $n \in \mathbb{N}$. Let $1 \neq c_1 \in B$ and $C_1 = \langle c_1 \rangle^{\langle x \rangle}$. Then $C_1$ is finite and there is the subgroup $E_1$ such that $B = C_1 \times E_1$. Let $U_1 = core_{\langle x \rangle}E_1$. Therefore $U_1$ has finite index in $B$. If $1 \neq c_2 \in U_1$ and $C_2 = \langle c_2 \rangle^{\langle x \rangle}$ then $C_2$ is a finite $\langle x \rangle$-invariant subgroup and $\langle C_1, C_2 \rangle = C_1 \times C_2$. Continuing in this manner, we can construct a family of subgroups $\{C_n | n \in \mathbb{N}\}$ for which $\langle C_n | n \in \mathbb{N} \rangle = Dr_{n\in\mathbb{N}}C_n$. By Lemma 1 $x \in L$. Contradiction.

From Lemmas 5 and 6 it follows the theorem.

**Theorem 1.** Let $A$ be an $RG$-module, $G$ be a locally soluble group, $R$ be an associative ring. Suppose that the cocentralizer of $G$ in the module $A$ is an infinite $R$-module, and $G$ satisfies the condition $\min -nf$. Then $G$ either satisfies the minimal condition on subgroups or $G = FFD(G)$.

**Lemma 7.** Let $A$ be an $RG$-module, $G$ be a locally soluble group. Suppose that the cocentralizer of $G$ in the module $A$ is a finite $R$-module. Then $G$ is almost abelian.

**Proof.** Let $C = C_A(G)$. Then $A$ has the series of $RG$-submodules $\langle 0 \rangle \leq C \leq A$, where $A/C$ is a finite $R$-module. Since $G \leq C_G(C)$ then $G/C_G(C)$ is trivial. As $A/C$ is a finite $R$-module then $G/C_G(A/C)$ is finite.

Let $H = C_G(C) \cap C_G(A/C)$. Each element of $H$ acts trivially on every factor of the series $\langle 0 \rangle \leq C \leq A/C$. By Kaluzhnin Theorem (p. 144 [7]) $H$ is abelian. By Remak’s Theorem

$$G/H \leq G/C_G(C) \times G/C_G(A/C).$$

It follows that $G/H$ is finite. Then $G$ is an almost abelian group.

**Lemma 8.** Let $A$ be an $RG$-module, $G$ be a locally soluble group, $R$ be an associative ring, and if the cocentralizer of $G$ in the module $A$ is an infinite $R$-module then $G$ satisfies the condition $\min -nf$. Then either $G$ is soluble or $G$ has an ascending series of normal subgroups $1 = W_0 \leq W_1 \leq \cdots \leq W_n \leq \cdots \leq W_0 = \cup_{n\in\mathbb{N}}W_n \leq G$. 

$$H_1 > H_2 > H_3 > \cdots$$
such that the cocentralizer of each subgroup \(W_n\) in the module \(A\) is a finite \(R\)-module, the factors \(W_{n+1}/W_n\) are abelian for \(n = 1, 2, \cdots\), and \(G/W_\omega\) is a Chernikov group.

**Proof.** If the quotient module \(A/C_A(G)\) is a finite \(R\)-module then \(G\) is soluble by Lemma 7. Therefore it seemed reasonable to study locally soluble groups \(G\) such that \(A/C_A(G)\) is an infinite \(R\)-module.

Later we consider the case when the cocentralizer of \(G\) in the module \(A\) is an infinite \(R\)-module. At first we prove that \(G\) is hyperabelian. To accomplish this we show that every non-trivial image of \(G\) contains a non-trivial normal abelian subgroup.

Let \(H\) be a proper normal subgroup of \(G\). Suppose that the cocentralizer of \(H\) in the module \(A\) is an infinite \(R\)-module. Then \(G/H\) satisfies the minimal condition on subgroups. Therefore \(G/H\) is a Chernikov group, and contains a non-trivial normal abelian subgroup. Now we suppose that the the cocentralizer of \(H\) in the module \(A\) is a finite \(R\)-module. Let \(\Sigma = \{M_\sigma/H | \sigma \in \Sigma\}\) be the family of all non-trivial normal abelian subgroups of the quotient group \(G/H\). At first we consider the case when for all \(\sigma \in \Sigma\) the cocentralizer of \(M_\sigma\) in the module \(A\) is an infinite \(R\)-module. We shall prove that the quotient group \(G/H\) satisfies the minimal condition on normal subgroups. Let \(\{M_\delta/H\}\) be a non-empty subset of \(\Sigma\). The cocentralizer of a subgroup \(M_\delta\) in the module \(A\) is an infinite \(R\)-module for all \(\delta\). By the condition \(min(n - \eta)\) the set \(\{M_\delta\}\) has the minimal element \(M\). Then \(M/H\) is the minimal element of subset \(\{M_\delta/H\}\). Therefore \(G/H\) satisfies the minimal condition on normal subgroups. It follows that the quotient group \(G/H\) is hyperabelian and contains a non-trivial normal abelian subgroup. In the case when for some \(\gamma \in \Sigma\) the cocentralizer of \(M_\gamma\) in the module \(A\) is a finite \(R\)-module, the subgroup \(M_\gamma\) is soluble. Then \(M_\gamma/H\) is a non-trivial normal soluble subgroup of \(G/H\). Therefore the quotient group \(G/H\) contains a non-trivial normal abelian subgroup and so \(G\) is hyperabelian.

Let \(1 = H_0 \leq H_1 \leq \cdots \leq H_\alpha \leq \cdots \leq G\) be a normal ascending series with abelian factors and let \(\alpha\) be the least ordinal such that the cocentralizer of \(H_\alpha\) in the module \(A\) is an infinite \(R\)-module. Then, as above, the subgroup \(H_\beta\) is soluble for all \(\beta < \alpha\). Moreover, the quotient group \(G/H_\alpha\) satisfies the minimal condition on subgroups, and so is a soluble Chernikov group.

At first we suppose that \(\alpha\) is not a limit ordinal. Then the subgroup \(H_\alpha\) is soluble and it follows that \(G\) is soluble also. Now we consider the case when \(\alpha\) is a limit ordinal, and \(G\) is not soluble. For all natural numbers \(k\) there exists an ordinal \(\beta_k\) such that \(\beta_k < \alpha\), \(H_{\beta_k}\) has derived length at least \(k\). Moreover, we may assume that \(\beta_i < \beta_{i+1}\) for all natural numbers \(i\). Let \(T_i = H_{\beta_i}\) for all natural numbers \(i\). It follows that \(G\) has an ascending series of normal soluble subgroups \(1 = T_0 \leq T_1 \leq \cdots \leq T_\omega\). Then the subgroup \(T_\omega = \cup_{n \in N} T_n\) is not soluble and so \(T_\omega = H_\alpha\). A series \(1 = W_0 \leq W_1 \leq \cdots \leq W_n \leq \cdots \leq W_\omega = \cup_{n \in N} W_n \leq G\) with the properties referred in the theorem can be obtained from the series \(1 = T_0 \leq T_1 \leq \cdots \leq T_\omega \leq G\). \(\square\)
Lemma 9. Let $A$ be an $RG$-module, $G$ be a group, $R$ be an associative ring. Suppose that the cocentralizer of $G$ in the module $A$ is an infinite $R$-module, $G$ satisfies the condition $\min -nf$ and $G = FFD(G)$. Then the quotient group $G/G^3$ is finite.

Proof. We suppose for a contradiction that the quotient group $G/G^3$ is infinite. Then $G$ has an infinite strictly descending series of normal subgroups $G > N_1 > N_2 > \cdots$, such that the quotient groups $G/N_i$ are finite for each $i$. Therefore there exists $k$ for which the quotient group $G/N_k$ is finite and the cocentralizer of $N_k$ in the module $A$ is a finite $R$-module. Since $G = FFD(G)$, there is the subgroup $H$ such that its cocentralizer in the module $A$ is a finite $R$-module and $G = H \tilde{N}_k$. Hence the cocentralizer of $G$ in the module $A$ is a finite $R$-module. Contradiction.

Lemma 10. Let $A$ be an $RG$-module, $G$ be a locally soluble group, $R$ be an associative ring. Suppose that the cocentralizer of $G$ in the module $A$ is an infinite $R$-module and $G$ satisfies the condition $\min -nf$. If $G$ has an ascending series of normal subgroups $1 = W_0 \leq W_1 \leq \cdots \leq W_n \leq \cdots \leq \cup_{n \geq 1} W_n = G$, in which the cocentralizer of each subgroup $W_n$ in the module $A$ is a finite $R$-module, and each factor $W_{n+1}/W_n$ is abelian, then $G$ is soluble.

Proof. Since the quotient module $A/C_A(W_k)$ is a finite $R$-module for each $k \in \mathbb{N}$ then there is the series of $RG$-submodules $A = A_0 \geq A_1 \geq A_2 \geq \cdots \geq A_k \geq \cdots \geq A_\omega = C_A(G)$, such that $A_k = C_A(W_k)$ and each factor $A_k/A_{k+1}$ is a finite $RG$-module. Let $H = \bigcap_{j \geq 0} C_G(A_j/A_{j+1})$. Then $G/C_G(A_j/A_{j+1})$ is finite for each $j \in \mathbb{N}$. Since $G/H$ embeds in the Cartesian product of the quotient groups $G/C_G(A_j/A_{j+1})$, it follows that $G/H$ is residually finite. Moreover, $G$ is a union of subgroups such that their cocentralizers in the module $A$ are finite $R$-modules. Hence $G = FFD(G)$. By Lemma 9 the quotient group $G/H$ is finite.

Since $G = FFD(G)$ then the cocentralizer of $H$ in the module $A$ is an infinite $R$-module. We shall prove that $H$ is soluble. Let $L_j = C_H(A/A_j)$, $j = 1, 2, \cdots$. Let $H \neq L_j$ for some $j$. The quotient group $H/L_j$ is finite for each $j = 1, 2, \cdots$. We suppose that there is the number $j$ such that the cocentralizer of $L_j$ in the module $A$ is a finite $R$-module. Let $j$ be the least number with this property. It follows that the cocentralizer of $L_{j-1}$ in the module $A$ is an infinite $R$-module. On the other hand since the quotient group $L_{j-1}/L_j$ is finite and $G = FFD(G)$, then the cocentralizer of $L_{j-1}$ in the module $A$ is a finite $R$-module. We have contradiction. Therefore the cocentralizer of each subgroup $L_j$ in the module $A$ is an infinite $R$-module. Since $H$ satisfies the condition $\min -nf$ then there exists the number $m$ such that $L_j = L_m$ for all $j \geq m$. From this fact and from the choice of subgroup $L_j$ it follows that the subgroup $L_m$ is soluble. Since the quotient group $H/L_m$ is finite then $H$ is also soluble. Then $G$ is soluble.
From the obtained results it follows Theorem 2.

**Theorem 2.** Let $A$ be an $\mathbf{RG}$-module, $G$ be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that if the cocentralizer of $G$ in the module $A$ is an infinite $\mathbf{R}$-module, $G$ satisfies the condition $\min nf$. Then $G$ is soluble.

**Theorem 3.** Let $A$ be an $\mathbf{RG}$-module, $G$ be a locally soluble group, $\mathbf{R}$ be an associative ring. Suppose that the cocentralizer of $G$ in the module $A$ is an infinite $\mathbf{R}$-module and $G$ satisfies the condition $\min nf$. Then $G$ has the normal abelian subgroup $H$ such that $G/H$ is Chernikov.

**Proof.** It should be noted that by Theorem 2 the group $G$ is soluble. To accomplish this proof we consider the case when $G$ is not Chernikov.

Let $G = D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots \supseteq D_n = 1$ be the derived series of $G$. There exists the number $m$ such that the cocentralizer of $D_m$ in the module $A$ is an infinite $\mathbf{R}$-module but the cocentralizer of $D_{m+1}$ in the module $A$ is a finite $\mathbf{R}$-module. By Lemma 4 the quotient groups $D_i/D_{i+1}$, $i = 0, 1, \ldots, m$, are Chernikov. Let $U = D_{m+1}$. Then the quotient group $G/U$ is Chernikov. Let $C = C_A(U)$. $C$ is an $\mathbf{RG}$-submodule of $A$. Therefore there exists the series of $-submodules

$$\{0\} \subseteq C \subseteq A,$$

such that $A/C$ is a finite $\mathbf{R}$-module. Then $G/C_G(A/C)$ is finite.

Let $H = C_G(C) \cap C_G(A/C)$. The subgroup $H$ acts trivially on each factor of the series $\{0\} \subseteq C \subseteq A$. Therefore $H$ is abelian. Since the quotient group $G/U$ is Chernikov and $U \subseteq C_G(C)$ then the quotient group $G/C_G(C)$ is also Chernikov. By Remak theorem $G/H \leq G/C_G(C) \times G/C_G(A/C)$. It follows that $G/H$ is Chernikov. Therefore $G$ contains the normal abelian subgroup $H$ such that $G/H$ is Chernikov. \(\square\)

In the paper the author have used the methods of the proofs of [5].

**REFERENCES**

[1] O. Dashkova, “Modules over integer group rings of locally solvable groups with the restrictions on some systems of subgroups. (Russian),” *Dopov. Nats. Akad. Nauk Ukr.*, no. 2, pp. 14–19, 2009.

[2] O. Dashkova, “On a class of modules over group rings of locally soluble groups. (Russian),” *Trudy Inst. Mat. i Mech. Ural Otd. Ros. Akad. Nauk*, vol. 15, no. 2, pp. 94–98, 2009. [Online]. Available: http://dx.doi.org/10.1134/S0081543809070062

[3] O. Dashkova, “On one class of modules over integer group rings of locally soluble groups. (Russian),” *Ukrainian Math. J.*, vol. 61, no. 1, pp. 44–51, 2009. [Online]. Available: http://dx.doi.org/10.1007/s11253-009-0197-x

[4] O. Dashkova, “Modules over integer group rings of locally soluble groups with minimax restriction. (Russian),” *Fundam. Prikl. Mat.*, vol. 17, no. 3, pp. 25–37, 2011/2012. [Online]. Available: http://dx.doi.org/10.1007/s10958-012-1055-1

[5] M. Dixon, M. Evans, and L. Kurdachenko, “Linear groups with the minimal condition on subgroups of infinite central dimension,” *J. Algebra*, vol. 277, no. 1, pp. 172–186, 2004. [Online]. Available: http://dx.doi.org/10.1016/j.jalgebra.2004.02.029
[6] P. M. Gudivok, V. P. Rud’ko, and V. A. Bovdi, *Crystallographic groups* (Ukrainian). Uzhgorod: Uzhgorods’ki˘ı Natsional’ni˘ı Universitet, 2006.

[7] M. Kargapolov and Y. Merzlyakov, *Fundamentals of group theory* (Russian). Moscow: Izdat. “Nauka”, 1972.

[8] L. Kurdachenko, “On groups with minimax classes of conjugate elements,” *Infinite groups and adjoining algebraic structures*, pp. 160–177, 1993.

[9] D. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, ser. Ergebnisse der Mathematik und ihrer Grenzgebiete. Berlin, Heidelberg, New York: Springer-Verlag, 1972, vol. 1,2.

[10] D. Zaicev, “Solvable subgroups of locally solvable groups,” *Dokl. Akad. Nauk SSSR*, vol. 214, no. 6, pp. 1250–1253, 1974.

Author’s address

**O. Yu. Dashkova**

The Branch of Moscow state university in Sevastopol, Department of Mathematics, 7 Sevastopol heroes street, 99001 Sevastopol, Russia

*E-mail address: odashkova@yandex.ru*