Quantum solitons with emergent interactions in a model of cold atoms on the triangular lattice

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Cold atoms bring new opportunities to study quantum magnetism, and in particular, to simulate quantum magnets with symmetry greater than SU(2). Here we explore the topological excitations which arise in a model of cold atoms on the triangular lattice with SU(3) symmetry. Using a combination of homotopy analysis and analytic field–theory we identify a new family of solitonic wave functions characterised by integer charge \( Q = (Q_A, Q_B, Q_C) \), with \( Q_A + Q_B + Q_C = 0 \). We use a numerical approach, based on a variational wave function, to explore the stability of these solitons on a finite lattice. We find that solitons with charge \( Q = (1, 1, -2) \) spontaneously decay into a pair of solitons with elementary topological charge, and emergent interactions. This result suggests that it could be possible to realise a new class of interacting soliton, with no classical analogue, using cold atoms. It also suggests the possibility of a new form of quantum spin liquid, with gauge–group U(1)×U(1).

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While many aspects of quantum systems can be understood at a local level, it is their non–local, topological properties which offer the deepest and most surprising insights. Topology underlies our understanding of such highly correlated systems as \(^3\)He and the fractional quantum Hall effect [1], and has come to play an important role in the theory of metals [2, 3], superconductors [4], and even systems as seemingly conventional as band insulators [5]. The study of topological excitations in magnets has also enjoyed a recent renaissance, as it has become possible to study the interplay between topological excitations, such as skyrmions, and itinerant electrons [6].

At the same time, cold atoms have brought an opportunity to study quantum many-body physics in a new context [7, 8]. The phases realized include analogues of both magnetic metals and magnetic insulators [9, 10], with the exciting new possibility of extending spin symmetry from the familiar SU(2) to SU(N) [11, 12]. Spin models with enlarged symmetry bring with them the possibility to study new kinds of topological excitation, but to date, these remain relatively unexplored [13].

In this Communication, we consider the topological excitations which arise in an SU(3) antiferromagnet that could be realised quite naturally using cold atoms. Starting from the most general model for SU(3) spins on the triangular lattice, we use a combination of field–theory and homotopy analysis to categorise stable topological defects. We find a new kind of stable lump soliton with \(^2\)nd homotopy group \( \pi_2 = \mathbb{Z} \times \mathbb{Z} \), characterized by the integer charges \( (Q_A, Q_B, Q_C) \), with \( Q_A + Q_B + Q_C = 0 \), and obtain an analytic wave function for solitons with charge \( (-Q, Q, 0) \). We then study these solitons numerically, introducing a new quantum variational approach. The numerical analysis confirms the stability of the soliton with elementary topological charge \( (-1, 1, 0) \). We find that the soliton for higher topological charge with \( (1, 1, -2) \) decompose into solitons with elementary charge, with emergent repulsive interactions. An example of a soliton with elementary charge \( Q = (-1, 1, 0) \) is shown in Fig. 1.

The model we consider is the SU(3)–symmetric generalisation of the Heisenberg model on a triangular lattice

\[
H_{\text{exchange}}^{\text{SU}(3)} = J \sum_{(l,m)} s_{1m}^{\text{SU}(3)}
\] (1)
where $P_{1m}^{SU(3)}$ is a permutation operator exchanging the states of the atoms on sites $I$ and $m$. Magnetism of this type can be realised using the hyperfine multiplets of repulsively–interacting Fermi atoms $[12, 13, 18]$, an approach which has already been shown to work for SU(2) Mott Insulators $[9, 10]$, and was recently also demonstrated in experiment for an SU(6) Mott insulator $[10]$ — indeed SU(N) systems with $N > 2$ may have advantages for cooling $[16, 17]$.

We consider the fundamental representation of SU(3), for which

$$\mathcal{H}_{\text{exchange}}^{SU(3)} = J \sum_{(I,m)} T_I \cdot T_m ,$$

where $T$ is an 8–component vector, comprising the 8 independent generators of SU(3) $[19]$. These generators can be expressed terms of a quantum spin–1 $[1, 20–22, 24]$, with

$$T^1, T^2, T^3 = (S_x, S_y, S_z) ,$$

while the remaining five components of $T$ are given by the quadrupole moments

$$\begin{pmatrix}
T^4 \\
T^5 \\
T^6 \\
T^7 \\
T^8
\end{pmatrix} = \left(\begin{array}{c}
\frac{(S_x)^2 - (S_y)^2}{2S_xS_y + S_yS_z} \\
\frac{(S_y)^2 - (S_z)^2}{2S_yS_z + S_zS_x} \\
\frac{(S_z)^2 - (S_x)^2}{2S_zS_x + S_xS_y}
\end{array}\right) ,$$

familiar from the theory of liquid crystals $[25]$. Viewed this way, the SU(3) symmetry takes on a clear physical meaning — spin quadrupoles and dipoles enter into Eq. (1) on an equal footing. In addition, SU(3) rotations permit dipole moments to be transformed continuously into quadrupole moments, and vice–versa $[11, 21, 22, 24]$. This is a process which has no analogue in classical magnets or liquid crystals, and has vital implications for topological defects.

In addition to providing a natural description of an SU(3)–symmetric Mott insulator, Eq. (1) can also be thought of as a SU(3)–symmetric limit of the spin–1 biquadratic (BBQ) model $[20]$, a model which may also be realised using cold atoms $[26, 27]$. The spin–1 BBQ model has been widely studied on the triangular lattice, where it supports both conventional magnetic ground states, and quadrupolar phases analogous to liquid crystals $[11, 21, 22, 30]$. These, in turn, play host to a rich variety of topological excitations $[33, 37–43]$. In particular, unconventional solitons have been shown to arise in the SU(3)–symmetric Heisenberg ferromagnet, i.e. Eq. (1), for $J < 0$ $[2]$. As yet, however, little is known about the topological defects of Eq. (1) for anti-ferromagnetic interactions $J > 0$, the case which arises most naturally for cold atoms $[12, 13, 18]$.

The topological excitations of a given state follow from the structure of its ground–state manifold $[24, 35]$. The ground state of Eq. (1) on the triangular lattice, for $J > 0$, is known to break spin–rotation symmetry, and to have 3–sublattice order $[11, 18, 29]$. However, since the SU(3) symmetry permits rotations between quadrupole and dipole moments of spin, this ordered state does not correspond to any single 3–sublattice dipolar or quadrupolar state, but rather a continuously connected manifold $[24, 35]$. In what follows we construct a representation of this ground state manifold and use it to classify the topological excitations of Eq. (1). While these results are completely general, they can most easily be understood through a mean–field description of Eq. (1).

Following $[24, 29]$, we write

$$\mathcal{H}^{\text{MFT}}_{SU(3)} = 2J \sum_{(I,m)} |d_I \cdot d_m|^2 + \text{const} ,$$

where $d$ is a complex vector with unit norm, expressing the most general wave function for a quantum spin–1

$$|d\rangle = d^x|x\rangle + d^y|y\rangle + d^z|z\rangle ,$$

in terms of a basis of orthogonal spin quadrupoles

$$|x\rangle = i\frac{|1\rangle - |1\rangle}{\sqrt{2}} , |y\rangle = \frac{|1\rangle + |1\rangle}{\sqrt{2}} , |z\rangle = -i|0\rangle ,$$

and $|1\rangle$ is the state with $S^z = 1$, etc. For $J > 0$, Eq. (3) supports a manifold of 3–sublattice ground states satisfying the orthogonality condition

$$d_{A} \cdot d_{A'} = \delta_{AA'} ,$$

where $A, A' = \{ A, B, C \}$. The simplest wave function satisfying Eq. (6) is the 3–sublattice antiferroquadrupolar (AFQ) state

$$d_A = (1, 0, 0) , d_B = (0, 1, 0) , d_C = (0, 0, 1) ,$$

illustrated in the inset of Fig. 1. We take this a reference state, and denote it $|\gamma\rangle$.

We are now in a position to determine the symmetry of the ground state manifold, and the topological excitations which follow from it. The universal covering group, G $[13]$, is given by a global SU(3) rotation acting on $|\gamma\rangle$. However the order parameters $\langle T_A \rangle, \langle T_B \rangle, \langle T_C \rangle$ are unchanged if the following matrices act on $|\gamma\rangle$:

$$f = \begin{pmatrix}
e^{i\theta_1} & 0 & 0 \\
0 & e^{-i\theta_1} & 0 \\
0 & 0 & 1
\end{pmatrix} \times \begin{pmatrix}
e^{i\theta_2} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-i\theta_2}
\end{pmatrix} ,$$

where $\theta_1$ and $\theta_2$ are two freely–chosen phases. It follows that the isotropy subgroup of our model is

$$H = U(1) \times U(1) ,$$

where each U(1) refers to a phase $0 \leq \theta < 2\pi$. Hence, the ground state manifold of Eq. (1), for $J > 0$, is given by $G/H = SU(3)/(U(1) \times U(1))$. The topological excitations
supported by Eq. \([1]\) follow directly from this result, and a standard application of homotopy theory \([15]\) gives
\[
\pi_1(\text{SU}(3)/(U(1) \times U(1))) = 0 ,
\]
\[
\pi_2(\text{SU}(3)/(U(1) \times U(1))) = \mathbb{Z} \times \mathbb{Z} ,
\]
where the trivial first homotopy group \(\pi_1\) implies the absence of point–like defects, while the non–zero second homotopy group \(\pi_2\) implies the existence of a stable soliton classified by two integers. This result should be contrasted with that obtained for ferromagnetic interactions, \(J < 0\), in which case \(G/H = \text{SU}(3)/(\text{SU}(2) \times U(1)) = \mathbb{CP}^2\), and stable solitons are classified by a single integer \(\mathbb{Z}\) \([2,4]\).

We can gain more understanding of this new class of solitons by constructing explicit wave function describing them. To this end, we consider the continuum limit of Eq. \((5)\) and, following Refs. \([24,35]\), write
\[
\mathcal{H}_{\text{SU}(3)}^{\text{eff}} = \frac{2J}{\sqrt{3}} \int dx \sum_{\mu=x,y} \left( |d_\mu^A \cdot \partial_\mu d_B|^2 + |d_\mu^B \cdot \partial_\mu d_C|^2 + |d_\mu^C \cdot \partial_\mu d_A|^2 \right) .
\]
Building on previous work on CP\(^2\) solitons \([3]\), we introduce a real scalar field
\[
A_{\mu}^\lambda = \frac{1}{2} \left[ d_\mu^\lambda \cdot \partial_\mu d_\lambda - (\partial_\mu d_\lambda^\ast \cdot d_\lambda) \right] ,
\]
and use identities of the form
\[
|d_\mu^A \cdot \partial_\mu d_A|^2 + |d_\mu^C \cdot \partial_\mu d_A|^2 = |(d_\mu^B \partial_\mu |d_A)|^2 + |(d_\mu^C \partial_\mu |d_A)|^2
\]
\[
= (\partial_\mu (d_\lambda^A)) \partial_\mu (d_\lambda^A) - |(d_\lambda^A \partial_\mu |d_A)|^2 ,
\]
where \(\sum_{\lambda=A,B,C} |d_\lambda^\lambda \rangle \langle d_\lambda| = 1\), to express Eq. \((14)\) as
\[
\mathcal{H}_{\text{SU}(3)}^{\text{eff}} = \sum_{\lambda=A,B,C} \frac{J}{\sqrt{3}} \int dx \sum_{\mu=x,y} |D_\mu^\lambda d_A|^2 ,
\]
\[
D_\mu^\lambda = \partial_\mu + iA_\mu^\lambda .
\]
Here \(A_\mu^\lambda\) transforms as \(A_\mu^\lambda \rightarrow A_\mu^\lambda - \partial_\mu \Lambda\) under the gauge transformation \(d^\prime(x) = e^{i\Lambda} d(x)\). We note that the orthogonality condition, Eq. \((8)\), implies that any two of the vectors \(d_\lambda\) uniquely determine the third, leaving only phase degrees of freedom.

Solitonic solutions of Eq. \((17)\) are characterised by a finite topological charge. In order to parameterise this, we consider the based, second homotopy group
\[
\pi_2(\text{SU}(3)/(U(1) \times U(1))) , \ b)
\]
where the base–point \(b\) is given by
\[
b = f |r\rangle .
\]
Following Ref. \([15]\), this is determined by the first homotopy group of the isotropy subgroup \(\pi_1(H) = \mathbb{Z} \times \mathbb{Z}\) [cf. Eq. \((13)\)]. The two independent integers \(\mathbb{Z}\) distinguish the different solitons which are possible within this order–parameter space and, thereby, their topological charge.

We can evaluate the topological charge associated with a given soliton by considering how the vectors \(d_\lambda\) evolve on a closed path \(C(l)\) for \(0 \leq l \leq 1\) (with \(C(0) = C(1)\)), which encloses the soliton in the two–dimensional lattice space. For practical purposes, this closed path could be the boundary of a finite–size cluster. The simplest example is a soliton characterised by the integers \(\mathbb{Z} \times \mathbb{Z} = (1,0)\), in which case the state on the path \(C(l)\) is given by
\[
f_0(l) |r\rangle = \begin{pmatrix} e^{-i2\pi l} & 0 & 0 \\ 0 & e^{i2\pi l} & 0 \\ 0 & 0 & 1 \end{pmatrix} |r\rangle ,
\]
where \(0 \leq l \leq 1\) and \(C(0) = C(1)\). The contribution to the topological charge from each sublattice can be identified with the winding number associated with the diagonal elements of \(f_0\). In the case of the A–sublattice
\[
d_A(l) = f_0(l) d_A = (e^{-i2\pi l}, 0, 0) ,
\]
and it follows that the winding number on the path \(C(l)\) is unity, and the associated topological charge is given by \(Q_A = 1\). More formally, we can calculate this winding number as
\[
Q_A = \frac{i}{2\pi} \int_{C(l)} d\lambda \cdot (d_\lambda^A(x) \nabla d_A(x))
\]
\[
= \frac{i}{2\pi} \int dx \left( \partial_\lambda (d_\lambda^A(x)) (\partial_\lambda d_A(x)) - (\partial_\lambda d_\lambda^A(x)) (\partial_\lambda d_A(x)) \right)
\]
\[
= \frac{i}{2\pi} \int dx \epsilon_{\mu\nu} (D_\mu^A d_A)^\ast \cdot D_\nu^A d_A = 1 ,
\]
where the two–dimensional integral \(\int dx\) is carried out over the area enclosed by the path \(C(l)\). By inspection of Eq. \((20)\), the contribution of the B–sublattice is \(Q_B = -Q_A = -1\), while the contribution from the (topologically–trivial) C–sublattice vanishes. Therefore, for this example, \(Q = (Q_A, Q_B, Q_C) = (1, -1, 0)\).

This approach to evaluating the topological charge remains valid, regardless of the base point, and can be applied to any spin configuration. So quite generally, we can write
\[
Q_\lambda = \frac{i}{2\pi} \int dx \epsilon_{\mu\nu} (D_\mu^\lambda d_\lambda)^\ast \cdot D_\nu^\lambda d_\lambda ,
\]
subject to the constraint
\[
Q_A + Q_B + Q_C = 0 .
\]
condition on the three vectors $d_\lambda=A,B,C$, as defined in Eq. (8). We note that our definition of the topological charge, Eq. (23), differs by a sign convention from that definition used by in Ref. 2.

Viewed this way, the continuum theory, Eq. (17), comprises three copies of a CP⁰ nonlinear sigma model, linked by the orthogonality condition, Eq. (6). The Cauchy–Schwartz inequality [4] enables us to place a lower bound on the energy of a soliton, based on its charge

$$E_Q \geq \frac{2\pi J}{\sqrt{3}} (|Q_A| + |Q_B| + |Q_C|) .$$  \hspace{1cm} (25)

Equality in Eq. (25), also known as the Bogomol’nyi–Prasad–Sommerfield (BPS) bound [47], is achieved where

$$D_\lambda^A d_\lambda = \pm i \epsilon_{\mu\nu} D_\lambda^A d_\lambda \quad \forall \quad \lambda = A,B,C .$$ \hspace{1cm} (26)

We have been able to construct an exact wave function satisfying the Eq. (26), for the special case $Q = (Q, Q, 0)$. This corresponds to Q pairs of orthogonal CP⁰ solitons [2, 4], found on two of the three sublattices, while the third remains topologically trivial. Specifically, we find

$$d_A(z) = \frac{\xi u_1 + \left[ \Pi_{k=1}^Q (z - z_k) \right] v_1}{\sqrt{\xi^2 + \Pi_{k=1}^Q (z - z_k)^2}} ,$$

$$d_B(z) = \frac{-\xi v_1 + \left[ \Pi_{k=1}^Q (z^* - z_k^*) \right] u_1}{\sqrt{\xi^2 + \Pi_{k=1}^Q (z^* - z_k^*)^2}} ,$$

$$d_C(z) = y_1 ,$$ \hspace{1cm} (27)

where $z = x + iy$ is a complex coordinate, $u_1$, $v_1$ and $y_1$ are complex orthonormal vectors, taken to be

$$v_1 = (1, 0, 0) , \quad u_1 = (0, 1, 0) , \quad y_1 = (0, 0, 1) .$$ \hspace{1cm} (28)

The coordinate $z_k$ specifies the positions of each soliton. Since the energy of this family of solitons is completely determined by its charge [cf. Eq. (25)], it is independent of their size, which is set by the real parameter $\xi$. The wave function, Eq. (27), for a soliton with elementary charge $Q = (-1, 1, 0)$, is illustrated in Fig. 1.

In order to gain more insight into solitons with general charge, for which no closed–form solution exists, we now switch to numerical analysis. We adopt a variational approach, based on a general product wave function, and minimise the energy of this wave function using simulated annealing [5], as described in the supplemental materials. Simulations were carried out for hexagonal clusters, and seeded with a trial wave function with definite topological charge $Q$.

In the simplest case, a soliton of elementary charge $Q = (-1, 1, 0)$, we can use the BPS bound Eq. (27) as a trial wave function, and simulations converge on a state like that shown in Fig. 1, confirming the stability of the analytic solution on a finite lattice [19]. We next consider

![Fig. 2: (Color online). Illustration of how repulsive interactions cause a single soliton with charge $Q = (1, 1, -2)$ to break into two separate pieces. (a) Quadrupole moment $Q_C x^2 - y^2$ on the C sublattice associated with the trial wave function Eq. (29). (b) Quadrupole moment after numerical minimisation of a variational wave function.](image-url)
FIG. 3: (Color online). Details of the variational wave function describing a state with topological charge $Q = (1, 1, -2)$, resolved onto A, B and C sublattices. (a)–(c) wave function near to the maximum in Fig. 2(b), showing a soliton with elementary charge $Q = (1, 0, -1)$. (d)–(f) wave function near to the minimum in Fig. 2(b), showing a soliton with elementary charge $Q = (0, 1, -1)$. The probability–surface for each spin-1 (defined in the supplemental materials), is rendered in blue, while the color underlay shows the dipole moments $S^y$ [(a)–(c)], and $S^x$ [(d)–(f)], induced by the soliton.

bound, comprising orthogonal CP$^2$ solitons on two of the three sublattices A, B, C [cf. Fig. 1]. Numerical simulations were used to confirm the stability of these solutions, and to explore the structure of solitons with more general topological charge. We find that solitons with charge $Q = (1, 1, -2)$ spontaneously decay into solitons with “elementary” charge $Q = (0, 1, -1)$ and $Q = (1, 0, -1)$ [cf. Fig. 3]. We infer that solitons with different elementary charge interact repulsively.

To the best of our knowledge, these results represent the first example of quantum solitons characterised by two integers, with emergent repulsive interactions. The model solved has direct application to experiments on cold atoms in an optical lattice, where quantum magnets with SU(3) symmetry arise quite naturally [12, 13, 18]. It is also interesting to speculate that these new solitons might survive as dynamical excitations in a spin–1 magnet where SU(3) symmetry was broken, as argued for the CP$^2$ solitons found in the SU(3)–symmetric Heisenberg ferromagnet [2]. And, since topological defects determine the type of quantum spin liquid which follows when classical order melts [33, 41, 50], these results suggest the possibility of a new class of quantum spin liquid with an underlying U(1)$\times$U(1) gauge structure. We hope that this may help to shed light on the quantum spin–liquids found in two–dimensional quantum magnets which might otherwise support 3–sublattice order [51–55].

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Supplemental material for: Quantum solitons with emergent interactions in a model of cold atoms on the triangular lattice

### A. Pictorial representation of wave functions

It is convenient to represent the wave function of the solitons studied in this Rapid Communication pictorially. The state of quantum spin–1 can conveniently be represented as a probability surface

\[
P(\theta, \phi) = |\langle \mathbf{d} | \Omega \rangle|^2
\]  

(30)

through its projection onto a spin coherent state

\[
|\Omega\rangle = R(\theta, \phi) |1\rangle
\]  

(31)

where \(|1\rangle\) is the eigenstate with \(S^z = +1\), and \(R(\theta, \phi)\) is an SU(2) rotation matrix. A pictorial representation of a soliton with elementary charge \(Q = (-1,1,0)\) is shown in Fig. 4 below. The results are taken from numerical simulation, as described below, and in the main text.

In many cases it is also interesting to examine the quadrupole, or dipole, moments of spin which are induced by the soliton, relative to the reference state \(|\text{quadrupole}, \text{or dipole, moments of spin which are in-}
\rangle\langle \text{turn, following the standard Metropolis algo-
}\text{rimth. Temperatures are reduced following a geometrical}
\text{progression}
\text{temperature of the}
\text{finite–size cluster, and |d}_l\rangle \text{is defined through Eq. (6) of the main text. Since d has unit norm, and the physical properties of |d}_l\rangle \text{are invariant under a change of phase,
\begin{align}
|d}_l\rangle \rightarrow e^{i\theta} |d}_l\rangle,
\end{align}

(35)
a wave function of this form has \(4^N\) variational parameters. These parameters are determined by simulated annealing [3], starting from an initial “guess” at the soliton wave function, with definite topological charge \(Q\).

In this approach, simulations are carried out at a temperature \(T\), which is decreased gradually, and at each temperature the wave function is updated using Markov–chain Monte Carlo sampling. Monte Carlo updates involve the random re-orientation of the vector \(\mathbf{d}\) for each spin in turn, following the standard Metropolis algorithm. Temperatures are reduced following a geometrical progression

\[
T_{k+1} = \alpha T_k \quad , \quad 0 < \alpha < 1
\]  

(36)

where \(T_k\) is the temperature of the \(k^{th}\) of \(N_{\text{annealing}}\) annealing steps. Typical values are

\[
\begin{align}
N_{\text{annealing}} &= 300 \\
\alpha &= 0.95 \\
T_{\text{initial}} &= 0.1 \, J \\
T_{\text{final}} &\approx 2.1 \times 10^{-8} \, J
\end{align}
\]  

(37)

We consider hexagonal clusters with the full symmetry of the triangular lattice, and impose a boundary condition at the edges of the cluster consistent with the reference state \(|\psi\rangle\) [Eq. (9) of main text]. In addition, to ensure that simulations are carried out at fixed topological charge, we impose a “smoothness condition”

\[
|\theta_\mu |_{d_\lambda}|^2 \leq 1 \quad , \quad \mu = x, y \quad , \quad \lambda = A, B, C.
\]  

(38)
FIG. 4: (Color online). Example of a soliton in an SU(3)–symmetric model of cold atoms on the triangular lattice. The soliton shown has topological charge \( Q = (-1, 1, 0) \), and is represented through the probability surface for a quantum spin-1 at each site [cf. Eq. (30)], rendered in blue. Red, green and blue bars show the orientation of the quadrupole moment on each of the three sublattices A, B, and C. The color underlay shows the variation of the dipole moment, \( S_z \) across the soliton. Results are taken from variational calculations for the SU(3)–symmetric Heisenberg model, \( \mathcal{H}_{\text{exchange}}^{\text{SU(3)}} \), and should be compared with Fig. 1 of the main text.

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