The tangent splash in PG(6, q)

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Abstract

Let \( B \) be a subplane of PG(2, \( q^3 \)) of order \( q \) that is tangent to \( \ell_\infty \). Then the tangent splash of \( B \) is defined to be the set of \( q^2 + 1 \) points of \( \ell_\infty \) that lie on a line of \( B \). In the Bruck-Bose representation of PG(2, \( q^3 \)) in PG(6, q), we investigate the interaction between the ruled surface corresponding to \( B \) and the planes corresponding to the tangent splash of \( B \). We then give a geometric construction of the unique order-\( q \)-subplane determined by a given tangent splash and a fixed order-\( q \)-subline.

1 Introduction

Let \( B \) be a subplane of PG(2, \( q^3 \)) of order \( q \) that is tangent to \( \ell_\infty \) in the point \( T \). The tangent splash \( S_T \) of \( B \) is defined to be the set of points of \( \ell_\infty \) that lie on a line of \( B \). So \( S_T \) has \( q^2 + 1 \) points, and contains the point \( T \), called the centre. Properties of the tangent splash were investigated in [4], and this investigation is continued here. We will work in the Bruck-Bose representation of PG(2, \( q^3 \)) in PG(6, q), and study the
tangent splash in this setting. Sections 2 and 3 comprise the relevant background material for this article.

The order-$q$-subplane $\mathcal{B}$ corresponds to a ruled surface in $\text{PG}(6, q)$, and the tangent splash $S_T$ corresponds to a set of $q^2+1$ planes in $\text{PG}(6, q)$. In Section 4, we investigate the geometric interaction between this ruled surface and plane set. Section 5 uses this interaction to describe the tangent subspace of a point $P \in \mathcal{B}$ (defined in [4]) in the Bruck-Bose setting.

In [4], the following result is proved.

**Theorem 1.1** In $\text{PG}(2, q^3)$, let $S_T$ be a tangent splash of $\ell_\infty$ and let $\ell$ be an order-$q$-subline disjoint from $\ell_\infty$ lying on a line which meets $S_T \setminus \{T\}$. Then there is a unique order-$q$-subplane tangent to $\ell_\infty$ that contains $\ell$ and has tangent splash $S_T$.

In Section 6 we start with a tangent splash $S_T$ and such an order-$q$-subline $\ell$, and show how to construct this unique order-$q$-subplane. This is a geometric construction in the Bruck-Bose representation in $\text{PG}(6, q)$.

## 2 The Bruck-Bose representation of $\text{PG}(2, q^3)$ in $\text{PG}(6, q)$

This section contains the necessary definitions, notation and results about the Bruck-Bose representation of $\text{PG}(2, q^3)$ in $\text{PG}(6, q)$.

### 2.1 The Bruck-Bose representation

We begin with a brief introduction to 2-spreads in $\text{PG}(5, q)$, see [11] for more details. A 2-*spread* of $\text{PG}(5, q)$ is a set of $q^3+1$ planes that partition $\text{PG}(5, q)$. A 2-*regulus* of $\text{PG}(5, q)$ is a set of $q+1$ mutually disjoint planes $\pi_1, \ldots, \pi_{q+1}$ with the property that if a line meets three of the planes, then it meets all $q+1$ of them. Three mutually disjoint planes in $\text{PG}(5, q)$ lie in a unique 2-regulus. A 2-spread $\mathcal{S}$ is *regular* if for any three planes in $\mathcal{S}$, the 2-regulus containing them is contained in $\mathcal{S}$.

The following construction of a regular 2-spread of $\text{PG}(5, q)$ will be needed. Embed $\text{PG}(5, q)$ in $\text{PG}(5, q^3)$ and let $g$ be a line of $\text{PG}(5, q^3)$ disjoint from $\text{PG}(5, q)$. Let $g^q$, 

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$g^q$ be the conjugate lines of $g$; both of these are disjoint from $\text{PG}(5, q)$. Let $P_i$ be a point on $g$; then the plane $\langle P_i, P_i^q, P_i^{q^2} \rangle$ meets $\text{PG}(5, q)$ in a plane. As $P_i$ ranges over all the points of $g$, we get $q^3 + 1$ planes of $\text{PG}(5, q)$ that partition $\text{PG}(5, q)$. These planes form a regular spread $\mathcal{S}$ of $\text{PG}(5, q)$. The lines $g$, $g^q$, $g^{q^2}$ are called the (conjugate skew) transversal lines of the spread $\mathcal{S}$. Conversely, given a regular 2-spread in $\text{PG}(5, q)$, there is a unique set of three (conjugate skew) transversal lines in $\text{PG}(5, q^3)$ that generate $\mathcal{S}$ in this way.

We now describe the Bruck-Bose representation of a finite translation plane $\mathcal{P}$ of order $q^3$ with kernel containing $GF(q)$, an idea which was developed independently by André [1] and Bruck and Bose [6, 7]. Let $\Sigma_\infty$ be a hyperplane of $\text{PG}(6, q)$ and let $\mathcal{S}$ be a 2-spread of $\Sigma_\infty$. We use the phrase a subspace of $\text{PG}(6, q) \setminus \Sigma_\infty$ to mean a subspace of $\text{PG}(6, q)$ that is not contained in $\Sigma_\infty$. Consider the following incidence structure: the points of $\mathcal{A}(\mathcal{S})$ are the points of $\text{PG}(6, q) \setminus \Sigma_\infty$; the lines of $\mathcal{A}(\mathcal{S})$ are the 3-spaces of $\text{PG}(6, q) \setminus \Sigma_\infty$ that contain an element of $\mathcal{S}$; and incidence in $\mathcal{A}(\mathcal{S})$ is induced by incidence in $\text{PG}(6, q)$. Then the incidence structure $\mathcal{A}(\mathcal{S})$ is an affine plane of order $q^3$. We can complete $\mathcal{A}(\mathcal{S})$ to a projective plane $\mathcal{P}(\mathcal{S})$; the points on the line at infinity $\ell_\infty$ have a natural correspondence to the elements of the 2-spread $\mathcal{S}$. The projective plane $\mathcal{P}(\mathcal{S})$ is the Desarguesian plane $\text{PG}(2, q^3)$ if and only if $\mathcal{S}$ is a regular 2-spread of $\Sigma_\infty \cong \text{PG}(5, q)$ (see [5]). For the remainder of this article we work in $\text{PG}(2, q^3)$, so $\mathcal{S}$ denotes a regular 2-spread of $\text{PG}(5, q)$.

We use the following notation: if $P$ is an affine point of $\text{PG}(3, q^3)$, we also use $P$ to refer to the corresponding affine point in $\text{PG}(6, q)$. If $T$ is a point of $\ell_\infty$ in $\text{PG}(2, q^3)$, we use $[T]$ to refer to the spread element of $\mathcal{S}$ in $\text{PG}(6, q)$ corresponding to $T$. More generally, if $X$ is a set of points in $\text{PG}(2, q^3)$, we use $[X]$ to denote the corresponding set in $\text{PG}(6, q)$.

As $\mathcal{S}$ is a regular spread, $\mathcal{P}(\mathcal{S}) \cong \text{PG}(2, q^3)$ and we can relate the coordinates of $\text{PG}(2, q^3)$ and $\text{PG}(6, q)$ as follows. Let $\tau$ be a primitive element in $GF(q^3)$ with primitive polynomial
\[ x^3 - t_2x^2 - t_1x - t_0, \]
where $t_0, t_1, t_2 \in GF(q)$. Then every element $\alpha \in GF(q^3)$ can be uniquely written as $\alpha = a_0 + a_1\tau + a_2\tau^2$ with $a_0, a_1, a_2 \in GF(q)$. Points in $\text{PG}(2, q^3)$ have homogeneous coordinates $(x, y, z)$ with $x, y, z \in GF(q^3)$. Let the line at infinity $\ell_\infty$ have equation $z = 0$; so the affine points of $\text{PG}(2, q^3)$ have coordinates $(x, y, 1)$. Points in $\text{PG}(6, q)$ have homogeneous coordinates $(x_0, x_1, x_2, y_0, y_1, y_2, z)$ with $x_0, x_1, x_2, y_0, y_1, y_2, z \in GF(q)$. Let $\Sigma_\infty$ have equation $z = 0$. Let $F = (\alpha, \beta, 1)$ be a point of $\text{PG}(2, q^3)$. We can write $\alpha = a_0 + a_1\tau + a_2\tau^2$ and $\beta = b_0 + b_1\tau + b_2\tau^2$ with $a_0, a_1, a_2, b_0, b_1, b_2 \in GF(q)$. 

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Then the map
\[
\sigma : \text{PG}(2, q^3) \setminus \ell_\infty \to \text{PG}(6, q) \setminus \Sigma_\infty \\
(\alpha, \beta, 1) \mapsto (a_0, a_1, a_2, b_0, b_1, b_2, 1)
\]
is the Bruck-Bose map. More generally, if \( z \in \text{GF}(q) \), then we can generalise the map to \( \sigma(\alpha, \beta, z) = (a_0, a_1, a_2, b_0, b_1, b_2, z) \). Note that if \( z = 0 \), then \( T = (\alpha, \beta, 0) \) is a point of \( \ell_\infty \), and \( \sigma(\alpha, \beta, 0) \) is a single point in the spread element \([T]\) corresponding to \( T \).

In [2], the coordinates of the transversals of the regular spread \( S \) are calculated.

**Lemma 2.1** The line \( g \) of \( \text{PG}(6, q^3) \) joining the points \((t_1 + t_2 \tau - \tau^2, t_2 - \tau, -1, 0, 0, 0, 0)\) and \((0, 0, 0, t_1 + t_2 \tau - \tau^2, t_2 - \tau, -1, 0)\) is one of the transversals of the regular spread.

### 2.2 Subplanes and sublines in the Bruck-Bose representation

An order-\( q \)-subplane of \( \text{PG}(2, q^3) \) is a subplane of \( \text{PG}(2, q^3) \) of order \( q \). An order-\( q \)-subline of \( \text{PG}(2, q^3) \) is a line of an order-\( q \)-subplane of \( \text{PG}(2, q^3) \). Note that an order-\( q \)-subplane is the image of the subplane \( \text{PG}(2, q) \) under the collineation group \( \text{PGL}(3, q^3) \), and an order-\( q \)-subline is the image of the subline \( \text{PG}(1, q) \) under \( \text{PGL}(3, q^3) \). In [2], the authors determine the representation of order-\( q \)-subplanes and order-\( q \)-sublines of \( \text{PG}(2, q^3) \) in the Bruck-Bose representation in \( \text{PG}(6, q) \), we quote the results we need here. We first introduce some notation to simplify the statements. A special conic \( \mathcal{C} \) is a conic in a spread element, such that when we extend \( \mathcal{C} \) to \( \text{PG}(6, q^3) \), it meets the transversals of the regular spread \( S \). Similarly, a special twisted cubic \( \mathcal{N} \) is a twisted cubic in a 3-space of \( \text{PG}(6, q) \setminus \Sigma_\infty \) about a spread element, such that when we extend \( \mathcal{N} \) to \( \text{PG}(6, q^3) \), it meets the transversals of \( S \). Note that a special twisted cubic has no points in \( \Sigma_\infty \).

**Theorem 2.2** [2] Let \( b \) be an order-\( q \)-subline of \( \text{PG}(2, q^3) \).

1. If \( b \subset \ell_\infty \), then in \( \text{PG}(6, q) \), \( b \) corresponds to a 2-regulus of \( S \). Conversely every 2-regulus of \( S \) corresponds to an order-\( q \)-subline of \( \ell_\infty \).

2. If \( b \) meets \( \ell_\infty \) in a point, then in \( \text{PG}(6, q) \), \( b \) corresponds to a line of \( \text{PG}(6, q) \setminus \Sigma_\infty \). Conversely every line of \( \text{PG}(6, q) \setminus \Sigma_\infty \) corresponds to an order-\( q \)-subline of \( \text{PG}(2, q^3) \) tangent to \( \ell_\infty \).
3. If \( b \) is disjoint from \( \ell_\infty \), then in \( \text{PG}(6, q) \), \( b \) corresponds to a special twisted cubic. Further, a twisted cubic \( \mathcal{N} \) of \( \text{PG}(6, q) \) corresponds to an order-\( q \)-subline of \( \text{PG}(2, q^3) \) if and only if \( \mathcal{N} \) is special.

**Theorem 2.3** [2] Let \( \mathcal{B} \) be an order-\( q \)-subplane of \( \text{PG}(2, q^3) \).

1. If \( \mathcal{B} \) is secant to \( \ell_\infty \), then in \( \text{PG}(6, q) \), \( \mathcal{B} \) corresponds to a plane of \( \text{PG}(6, q) \) that meets \( q + 1 \) spread elements. Conversely, any plane of \( \text{PG}(6, q) \) that meets \( q + 1 \) spread elements corresponds to an order-\( q \)-subplane of \( \text{PG}(2, q^3) \) secant to \( \ell_\infty \).

2. Suppose \( \mathcal{B} \) is tangent to \( \ell_\infty \) in the point \( T \). Then \( \mathcal{B} \) determines a set \([\mathcal{B}]\) of points in \( \text{PG}(6, q) \) (where the affine points of \( \mathcal{B} \) correspond to the affine points of \([\mathcal{B}]\)) such that:

   (a) \([\mathcal{B}]\) is a ruled surface with conic directrix \( \mathcal{C} \) contained in the plane \([T] \in \mathcal{S}\), and twisted cubic directrix \( \mathcal{N} \) contained in a 3-space \( \Sigma \) that meets \( \Sigma_\infty \) in a spread element (distinct from \([T]\)). The points of \([\mathcal{B}]\) lie on \( q + 1 \) pairwise disjoint generator lines joining \( \mathcal{C} \) to \( \mathcal{N} \).

   (b) The \( q + 1 \) generator lines of \([\mathcal{B}]\) joining \( \mathcal{C} \) to \( \mathcal{N} \) are determined by a projectivity from \( \mathcal{C} \) to \( \mathcal{N} \).

   (c) When we extend \([\mathcal{B}]\) to \( \text{PG}(6, q^3) \), it contains the conjugate transversal lines \( g, g^q, g^{q^2} \) of the regular spread \( \mathcal{S} \). So \( \mathcal{C} \) and \( \mathcal{N} \) are special.

   (d) \([\mathcal{B}]\) is the intersection of nine quadrics in \( \text{PG}(6, q) \).

The converse of this second correspondence is also true.

**Theorem 2.4** [3] Let \([\mathcal{B}]\) be a ruled surface of \( \text{PG}(6, q) \) defined by a projectivity from a conic directrix \( \mathcal{C} \) to a twisted cubic directrix \( \mathcal{N} \). Then \([\mathcal{B}]\) corresponds to an order-\( q \)-subplane of \( \text{PG}(2, q^3) \) if and only if \( \mathcal{C} \) is a special conic in a spread element \( \pi \), \( \mathcal{N} \) is a special twisted cubic in a 3-space about a spread element distinct from \( \pi \), and in the cubic extension \( \text{PG}(6, q^3) \) of \( \text{PG}(6, q) \), \([\mathcal{B}]\) contains the transversals of the regular spread \( \mathcal{S} \).

### 2.3 The collineation group in the Bruck-Bose representation

Consider a homography \( \alpha \in \text{PGL}(3, q^3) \) acting on \( \text{PG}(2, q^3) \) that fixes \( \ell_\infty \) as a set of points. There is a corresponding homography \([\alpha] \in \text{PGL}(7, q)\) acting on
the Bruck Bose representation of \( \text{PG}(2, q^3) \) as \( \text{PG}(6, q) \). Note that \([\alpha]\) fixes the hyperplane \( \Sigma_{\infty} \) at infinity of \( \text{PG}(6, q) \), and permutes the elements of the spread \( S \) in \( \Sigma_{\infty} \). Consequently subgroups of \( \text{PGL}(3, q^3) \) fixing \( \ell_{\infty} \) correspond to subgroups of \( \text{PGL}(7, q) \) fixing \( \Sigma_{\infty} \) and permuting the spread elements in \( \Sigma_{\infty} \). We will need some properties of these collineations groups, which we present here.

We first consider the simpler Bruck-Bose representation of \( \text{PG}(1, q^3) \) in \( \text{PG}(3, q) \). A homography \( \alpha \) of \( \text{PG}(1, q^3) \) is an element of \( \text{PGL}(2, q^3) \) and can be represented by a \( 2 \times 2 \) matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \text{GF}(q^3),
\]

where \( |A| \neq 0 \). If \( \alpha \) fixes the point \((1, 0)\), then we can set \( c = 0, d = 1 \). To convert this to a homography \([\alpha]\) in \( \text{PGL}(4, q) \) that acts on \( \text{PG}(3, q) \), write

\[
a = a_0 + a_1 \tau + a_2 \tau^2, \quad b = b_0 + b_1 \tau + b_2 \tau^2,
\]

for \( a_i, b_i \in \text{GF}(q) \), and let \( \sigma(a) = (a_0, a_1, a_2) \), \( \sigma(b) = (b_0, b_1, b_2) \). Let

\[
T = \begin{pmatrix} 0 & 0 & t_0 \\ 1 & 0 & t_1 \\ 0 & 1 & t_2 \end{pmatrix},
\]

then \([\alpha]\) has matrix

\[
[A] = \begin{pmatrix} \sigma(a)^t & T \sigma(a)^t & T^2 \sigma(a)^t & \sigma(b)^t \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Note that \([A]\) fixes the plane \( \pi_{\infty} \) of \( \text{PG}(3, q) \) with homogeneous coordinates \([0, 0, 0, 1]\). Further, the group \( \langle [T] \rangle \) is a Singer group acting on the points of \( \pi_{\infty} \) that is regular on the points, and on the lines of \( \pi_{\infty} \) (see [8, Chapter 4] for more information on Singer cycles).

As a homography of \( \text{PG}(1, q^3) \) that fixes the infinite point \((1, 0)\) gives rise to a homography of \( \text{PG}(3, q) \) that fixes the infinite plane \([0, 0, 0, 1]\), the following result is immediate.

**Theorem 2.5** Let \( E \) be the subgroup of \( \text{PGL}(2, q^3) \) fixing the infinite point \((1, 0)\) of \( \text{PG}(1, q^3) \). Consider the Bruck-Bose representation of \( \text{PG}(1, q^3) \) in \( \text{PG}(3, q) \) with plane at infinity \( \pi_{\infty} = [0, 0, 0, 1] \). Let \([E]\) be the subgroup of \( \text{PGL}(4, q) \) corresponding to \( E \), so \([E]\) acts on \( \text{PG}(3, q) \) and fixes the plane \( \pi_{\infty} \). Then

1. \( |[E]| = q^3(q^3 - 1) \).
2. $E$ is transitive on the affine points of $\text{PG}(1, q^3)$, and so $[E]$ is transitive on the affine points of $\text{PG}(3, q)$.

3. $E$ is transitive on the $q^3(q^3 - 1)$ order-$q$-sublines of $\text{PG}(1, q^3)$ not containing $(1, 0)$, and so $[E]$ is transitive on the special twisted cubics of $\text{PG}(3, q)$.

We now consider the Bruck-Bose representation of $\text{PG}(2, q^3)$ in $\text{PG}(6, q)$. Let $\alpha \in \text{PGL}(3, q^3)$ be a homography of $\text{PG}(2, q^3)$ fixing the line $\ell_\infty$ of $\text{PG}(2, q^3)$. Then $\alpha$ has matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c, d, e, f \in \text{GF}(q^3),$$

with $|A| \neq 0$. The corresponding homography $[\alpha] \in \text{PGL}(7, q)$ of $\text{PG}(6, q)$ has a $7 \times 7$ matrix

$$[A] = \begin{pmatrix} \sigma(a)^t & T\sigma(a)^t & T^2\sigma(a)^t \\ \sigma(d)^t & T\sigma(d)^t & T^2\sigma(d)^t \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma(b)^t & T\sigma(b)^t & T^2\sigma(b)^t \\ \sigma(e)^t & T\sigma(e)^t & T^2\sigma(e)^t \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma(c)^t \\ \sigma(f)^t \\ 1 \end{pmatrix}$$

with $T$ as in (1). It is straightforward to prove the following result that $T$ gives rise to a Singer cycle in each spread element in $\text{PG}(6, q)$.

**Theorem 2.6** Consider the homography $\Theta \in \text{PGL}(7, q)$ with $7 \times 7$ matrix

$$M = \begin{pmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where} \quad T = \begin{pmatrix} 0 & 0 & t_0 \\ 0 & 1 & t_1 \\ 0 & 1 & t_2 \end{pmatrix}.$$  

Then in $\text{PG}(6, q)$, $\Theta$ fixes each plane of the spread $\mathcal{S}$, and $\langle \Theta \rangle$ acts regularly on the set of points, and on the set of lines, of each spread element.

### 3 Background results on tangent splashes

In [4], a number of group theoretic properties of tangent splashes are proved, the most useful is the following.

**Theorem 3.1** The subgroup of $\text{PGL}(3, q^3)$ acting on $\text{PG}(2, q^3)$ and fixing the line $\ell_\infty$ is transitive on all the order-$q$-subplanes tangent to $\ell_\infty$, and transitive on the tangent splashes of $\ell_\infty$. 

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This means that if we want to prove a result about tangent order-$q$-subplanes or tangent splashes, then we can without loss of generality prove it for a particular tangent subplane or tangent splash. We use the following tangent order-$q$-subplane $\mathcal{B}$ that is coordinatised in full detail in [4, Section 6]. The labelling we use for the points and lines of $\mathcal{B}$ are illustrated in Figure 1, and their coordinates are given in Table 1. The tangent splash of $\mathcal{B}$ is

$$S_T = \{(a + b\tau, 1, 0) \mid a, b \in \text{GF}(q)\} \cup \{(1, 0, 0)\}.$$ 

If we look at the points of a tangent splash $S_T$ in the Bruck-Bose representation in $\text{PG}(6,q)$, we have a set $[S_T]$ of $q^2 + 1$ planes of the spread $\mathcal{S}$ in $\Sigma_\infty \cong \text{PG}(5,q)$. In [4], it is shown that there is an interesting set of cover planes in $\text{PG}(5,q)$ meeting every element of the tangent splash and contained entirely within the tangent splash. These cover planes will be useful in our construction later.

**Theorem 3.2** Let $S_T$ be a tangent splash of $\ell_\infty$ with centre $T$, and let $[S_T]$ be the corresponding set of planes in $\Sigma_\infty$ in the Bruck-Bose representation in $\text{PG}(6,q)$. There are exactly $q^2 + q + 1$ planes of $\Sigma_\infty \cong \text{PG}(5,q)$ that meet every plane of $[S_T]$, called cover planes. These cover planes each meet the centre $[T]$ in distinct lines, and meet every other plane of $[S_T]$ in distinct points, and hence are contained entirely within the splash.

We will also need the following result about cover planes.
Table 1: Coordinates of points in the order-$q$-subplane $\mathcal{B}$, $(e, d, f, h \in \text{GF}(q))$

| Notation | Coordinates | Description |
|----------|-------------|-------------|
| $T$      | $(1, 0, 0)$ | $\mathcal{B} \cap \ell_\infty$ |
| $P_\infty$ | $(1, 1, 1)$ | lines of $\mathcal{B}$ through $T$ |
| $m_e$    | $[0, e + \tau, -e]$ | points of $\mathcal{B}$ on $m_0$ |
| $m_\infty$ | $[0, 1, -1]$ | points of $\mathcal{B}$ on $m_\infty$ |
| $S_d$    | $(d, 0, 1)$ | $P_\infty S_d$ |
| $U_f$    | $(1 + f\tau, 1, 1)$ | $m_e \cap \ell_d$ |
| $\ell_d$ | $[1, d - 1, -d]$ | $P_{e,d} U_f$ |
| $P_{e,d}$ | $(e + d\tau, e, e + \tau)$ | $\ell_{e,d,f} \cap m_h$ |
| $\ell_{e,d,f}$ | $[-1, ef - d + 1 + f\tau, d - ef]$ | $\ell_{e,d,f} \cap m_\infty = U_f$ |
| $R_{e,d,f,h}$ | $(h + (fh - fe + d)\tau, h, h + \tau)$ |   |
| $R_{e,d,f,\infty}$ | $(1 + f\tau, 1, 1)$ |   |

**Theorem 3.3** Let $[S_T]$ be a tangent splash in the Bruck-Bose representation of $\text{PG}(2, q^3)$ in $\text{PG}(6, q)$. Let $\pi$ be a plane which meets the centre $[T]$ in a line, and meets three further elements $[U], [V], [W]$ of $[S_T]$, where $[T], [U], [V], [W]$ are not in a common 2-regulus. Then $\pi$ is a cover plane of $[S_T]$.

**4 Structure of the splash in $\text{PG}(6, q)$**

Let $\mathcal{B}$ be an order-$q$-subplane of $\text{PG}(2, q^3)$ tangent to $\ell_\infty$ at the point $T$ with tangent splash $S_T$. By Theorem 2.3, in $\text{PG}(6, q)$, $\mathcal{B}$ corresponds to a ruled surface $[\mathcal{B}]$ with special conic directrix $C$ in the spread element $[T]$. Further, $S_T$ corresponds to a set $[S_T]$ of $q^2 + 1$ planes of the regular spread $\mathcal{S}$. In this section we work in the Bruck-Bose representation in $\text{PG}(6, q)$ and investigate the interaction between the ruled surface $[\mathcal{B}]$ and the tangent splash $[S_T]$, and in particular with the cover planes of $[S_T]$. The interaction is complex, so we begin with a verbal description and diagram before stating the main result.

In $\text{PG}(2, q^3)$, let $\ell$ be an order-$q$-subline of $\mathcal{B}$ not through $T$, so $\ell$ is disjoint from $\ell_\infty$. Let $\ell$ be the extension of $\ell$ to $\text{PG}(2, q^3)$, and let $\ell \cap \ell_\infty = L$. By Theorem 2.2, in $\text{PG}(6, q)$, $\ell$ corresponds to a special twisted cubic $[\ell]$ in the 3-space $[\ell]$ about the spread element $[L]$. 
We will show that the following geometric relationship holds in \( \text{PG}(6, q) \). Through a point \( P \in [\ell] \), there is a unique tangent line to the twisted cubic \([\ell]\); it meets \( \Sigma_\infty \) in a point of \([L]\) which we denote \( I_{P,\ell} \). The set of \( q + 1 \) points \( D = \{ I_{P,\ell} \mid P \in [\ell] \} \) in \([L]\) is called the shadow of \([\ell]\). We will show that through each point of \( D \), there is a unique cover plane \( I_m \) of the tangent splash \([S_T]\). The cover plane \( I_m \) meets \([T]\) in a line, and we show that the resulting set of \( q + 1 \) lines in \([T]\) are the tangent lines of the conic directrix \( C \) of \([B]\). We will show that the cover plane \( I_m \) can be constructed from the points \( I_{P,\ell} \) as follows. Let \( m \) be a line of \([B]\) through \( T \) with points \( T, P_1, \ldots, P_q \), see Figure 2. Label the lines through \( P_1 \) by \( \ell_1, \ldots, \ell_q \). We show that the points \( \{ I_{P_1,\ell_1}, \ldots, I_{P_q,\ell_q} \} \) lie on a line \( I_{P_1} \) that meets \([T]\), and that the lines \( \{ I_{P_1}, \ldots, I_{P_q} \} \) lie in a plane \( I_m \) as required.

![Figure 2: Defining the points \( I_{P,\ell} \), lines \( I_P \) and planes \( I_m \)](image)

The main result is stated now.

**Theorem 4.1** Let \( B \) be an order-\( q \)-subplane of \( \text{PG}(2, q^3) \) tangent to \( \ell_\infty \) at the point \( T \) with tangent splash \( S_T \). In \( \text{PG}(6, q) \), \( B \) corresponds to a ruled surface \([B]\) with conic directrix \( C \) in the spread element \([T]\).

1. Let \( P \) be a point in \( B \setminus \{ T \} \). In \( \text{PG}(6, q) \), the \( q \) points \( \{ I_{P,\ell} \mid \ell \text{ is an order-}q\text{-subline of } B, P \in \ell, T \notin \ell \} \) lie on a line denoted \( I_P \) that meets \([T]\) in a point of \( C \).
2. Let $m$ be an order-$q$-subline of $\mathcal{B}$ through $T$. In $\text{PG}(6, q)$, the $q$ lines $\{I_P | P \in m, P \neq T\}$ lie in a cover plane $I_m$ of $[S_T]$ that meets $[T]$ in a tangent line of $\mathcal{C}$.

3. Let $L \in S_T \setminus \{T\}$, and let $\ell$ be the unique order-$q$-subline of $\mathcal{B}$ whose extension contains $L$. In $\text{PG}(6, q)$, the $q + 1$ planes $\{I_m | m \text{ is an order-$q$-subline of } \mathcal{B} \text{ through } T\}$ meet $[L]$ in the shadow $\mathcal{D} = \{I_{P, \ell} | P \in [\ell]\}$ of $[\ell]$, and meet $[T]$ in the $q + 1$ tangent lines of $\mathcal{C}$.

We prove this result using coordinates. By Theorem 3.1, we can without loss of generality prove this for the order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3. By Theorem 2.3, the order-$q$-subplane $\mathcal{B}$ corresponds to a ruled surface $[\mathcal{B}]$ in $\text{PG}(6, q)$ with a special conic directrix $\mathcal{C}$ in the spread element $[T]$. We will need the coordinates of this conic, and the next result calculates them. We use the following notation

$$\theta(e) = (e + \tau)^q = (e + \tau^q),$$

$$\theta(e) = (e + \tau)^q + 1 = (e + \tau)(e + \tau^q),$$

where $e \in \text{GF}(q)$. Note that since $\theta(e)^q = \theta(e)$, it follows that $\theta(e) \in \text{GF}(q)$. The next lemma uses the generalised Bruck-Bose map $\sigma$ defined in Section 2.1.

**Lemma 4.2** The conic directrix $\mathcal{C}$ in $[T]$ in $\text{PG}(6, q)$ of the order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3 has points $C_e = \sigma(\theta(e) \tau, 0, 0) = (t_0, e^2 + et_2, -e, 0, 0, 0, 0)$ for $e \in \text{GF}(q)$, and $C_{\infty} = \sigma(\tau, 0, 0) = (0, 1, 0, 0, 0, 0, 0)$. Further, the tangent line to $\mathcal{C}$ in $[T]$ at the point $C_e$ is given by the line $C_e T_e$ where $T_e = \sigma(\theta(e)^2 \tau, 0, 0)$ for $e \in \text{GF}(q)$, and $T_{\infty} = \sigma(\tau^2, 0, 0)$.

**Proof** We use the notation for points and lines of $\mathcal{B}$ given in Section 3. By Theorem 2.2, the order-$q$-subline $m_e$ of $\mathcal{B}$ corresponds in $\text{PG}(6, q)$ to an affine line $[m_e]$ contained in the ruled surface $[\mathcal{B}]$, $e \in \text{GF}(q) \cup \{\infty\}$. Hence the $q + 1$ generators of the ruled surface $[\mathcal{B}]$ are the lines $[m_e]$, $e \in \text{GF}(q) \cup \{\infty\}$. Thus the points of the conic directrix $\mathcal{C}$ in $[T]$ are $C_e = [m_e] \cap \Sigma_{\infty}$, $e \in \text{GF}(q) \cup \{\infty\}$.

If $e = \infty$, consider the two points $U_0 = (1, 1, 1)$ and $U_1 = (1 + \tau, 1, 1)$ of $\mathcal{B}$ which lie on the line $m_{\infty}$. So $[m_{\infty}] = \langle \sigma(1, 1, 1), \sigma(1 + \tau, 1, 1) \rangle$, and subtracting gives $C_{\infty} = [m_{\infty}] \cap \Sigma_{\infty} = \sigma(\tau, 0, 0) = (0, 1, 0, 0, 0, 0, 0)$. For $e \in \text{GF}(q)$, consider the two distinct points $P_{0,e} = (e, e + \tau)$, $P_{1,e} = (e + \tau, e, e + \tau)$ of $\mathcal{B}$ on the line $m_e$. Multiplying by $\theta(e)$ we have $P_{0,e} = (e \theta(e), e \theta(e), \theta(e))$ and $P_{1,e} = (\theta(e), e \theta(e), \theta(e))$. Mapping these to points in $\text{PG}(6, q)$, and subtracting gives $C_e = \sigma(\theta(e) \tau, 0, 0)$ as
required. Note that \( \theta(e)\tau = \tau(e+\tau^2)(e+\tau^2) = e^2\tau + e\tau(\tau^q + \tau^q) + \tau\tau^q\tau^q = e^2\tau + e\tau(t_2 - \tau) + t_0 = t_0 + (e^2 + et_2)\tau - e\tau^2 \), and so \( \sigma(\theta(e)\tau, 0, 0) = (t_0, e^2 + et_2, -e, 0, 0, 0, 0) \).

To calculate the tangents for \( C \) in \([T]\), let \( T_e = \sigma(\theta(e)^2\tau, 0, 0) \) for \( e \in \text{GF}(q) \), and consider the line \( t_e = C_e T_e = \{ rC_e + sT_e \mid r, s \in \text{GF}(q) \cup \{ \infty \} \} \) in \([T]\). We show that the point \( C_f \) of \( C \) is on the line \( t_e \) if and only if \( e = f \). Suppose \( C_f = rC_e + sT_e \) for some \( r, s \in \text{GF}(q) \cup \{ \infty \} \), then \( r\theta(e)\tau + s\theta(e)^2\tau = \theta(f)\tau \). Multiplying by \( (e + \tau^2)(e + \tau) / \tau \) yields \( r(f + \tau)(e + \tau)^2 \theta(e) + s(f + \tau)\theta(e)^2 = \theta(f)(e + \tau)^2 \), and so \( (r\theta(e) + s\theta(e)^2\tau^2) + (r\theta(e) + s\theta(e)^2\tau) + \theta(e)\tau^2 = \theta(f)e^2 + 2e\theta(f)\tau + \theta(f)\tau^2 \).

Equating the coefficients (in \( \text{GF}(q) \)) of \( \tau^2, \tau, 1 \) gives

\[
\begin{align*}
    r\theta(e) &= \theta(f) \quad (2) \\
    re\theta(e) + rf\theta(e) + s\theta(e)^2 &= 2e\theta(f) \quad (3) \\
    re\theta(e) + sf\theta(e)^2 &= \theta(f)e^2. \quad (4)
\end{align*}
\]

Note that as \( \theta(f) \neq 0 \), we have \( r \neq 0 \). Substituting (2) into (3) gives \( s\theta(e) = r(e-f) \). Substituting this and (2) into (4) gives \( r\theta(e)(e-f)^2 = 0 \), and so \( e = f \). Thus the line \( C_e T_e \) meets \( C \) in the point \( C_e \), and so is a tangent to \( C \) for \( e \in \text{GF}(q) \).

Now consider the case of \( C_\infty = \sigma(\tau, 0, 0), T_\infty = \sigma(\tau^2, 0, 0) \). In a similar manner to the above, if \( r\tau + s\tau^2 = \theta(e)\tau \) we multiply by \( (e + \tau^2) / \tau \) to get \( r(e + \tau) + s(e + \tau)\tau = \theta(e) \). Equating coefficients of \( \tau^2 \) and then \( \tau \) gives \( s = 0 \) and \( r = 0 \). Hence \( C_\infty T_\infty \) is a tangent as required. Note that if we put \( e = \infty \) in the expression for \( T_e = \sigma(\theta(e)^2\tau, 0, 0) \), we get \( \sigma(\tau, 0, 0) \), which is \( C_\infty \). So we need to define \( T_\infty \) differently to ensure that \( C_\infty T_\infty \) is a line.

Finally, we use these coordinates to verify that \( C \) is a special conic. If we regard \([T]\) as \( \text{PG}(2, q) \) with points \( (x, y, z) \), then \( C \) has equation \( t_0 z^2 - xy - t_2xz = 0 \). This is easy to verify as the point \((t_0, e^2 + et_2, x - e)\) corresponding to \( C_e \) satisfies the equation, and the point \((0, 1, 0)\) corresponding to \( C_\infty \) also satisfies the equation.

To prove that \( C \) meets the transversals of \( S \), we recall from Lemma 2.1 that one transversal point is \( R = (t_1 + t_2\tau - \tau^2, t_2 - \tau, -1) \). Substituting \( R \) into the equation of \( C \) gives \( t_0(-1)^2 - (t_2 - \tau)(t_1 + t_2\tau - \tau^2) - t_2(-1)(t_1 + t_2\tau - \tau^2) = t_0 + t_1\tau + t_2\tau^2 - \tau^3 = 0 \), hence \( R \in C \). Since the equation of \( C \) is over \( \text{GF}(q) \), the points \( R^{\ell}, R^{\ell^2} \) are also on \( C \).

Theorem 4.1 is proved in the next lemma using the following three steps. Part 1 calculates the coordinates of the points \( I_{P,\ell} \) for all pairs \((P, \ell)\), where \( P \) is a point of \( \mathcal{B} \) incident with an order-\( q \)-subline \( \ell \) of \( \mathcal{B} \) \((P \neq T, \ell \) not through \( T) \). Part 2 shows that by varying the order-\( q \)-subline \( \ell \) through \( P \) we construct a set of \( q \) collinear
points \( \{I_P \mid \ell \text{ a line of } \mathcal{B} \text{ through } P, T \notin \ell \} \). These points lie on a line denoted \( I_P \) that meets \( [T] \) in a point of the conic directrix \( \mathcal{C} \). In part 3 of the lemma we show that the set of \( q \) lines \( \{I_P \mid P \) is on a line \( m \) of \( \mathcal{B} \) through \( T, P \neq T \} \) lie on a cover plane \( I_m \) that meets \( [T] \) in a tangent line of \( \mathcal{C} \).

Lemma 4.3 Let \( \mathcal{B} \) be the order-\( q \)-subplane coordinatised in Section 3. Consider the order-\( q \)-sublines \( \ell_{e,d,f} \) of \( \mathcal{B} \) with points \( P_{e,d} \) and \( U_f, e, d, f \in GF(q) \). Then

1. \( I_{P_{e,d},\ell_{e,d,f}} = \sigma((1 - d + ef)\tau + f\tau^2)\theta(e)^2, \tau\theta(e)^2, 0), (e, d, f \in GF(q)) \),
   \( I_{U_f,\ell_{e,d,f}} = \sigma((1 - d + ef)\tau + f\tau^2, \tau, 0), (e, d, f \in GF(q)) \).

2. For fixed \( e, d \in GF(q) \), the set \( I_{P_{e,d}} = \{I_{P_{e,d},\ell_{e,d,f}} \mid f \in GF(q)\} \cup \{C_e\} \) is a line,
   For fixed \( f \in GF(q) \), the set \( I_{U_f} = \{I_{U_f,\ell_{e,d,f}} \mid e, d \in GF(q)\} \cup \{C_{\infty}\} \) is a line.

3. For fixed \( e \in GF(q) \), the set \( I_{m_e} = \{I_{P_{e,d}} \mid d \in GF(q)\} \cup \{C_eT_e\} \) is a cover plane of \( [S_T] \),
   The set \( I_{m_{\infty}} = \{I_{U_f} \mid f \in GF(q)\} \cup \{C_{\infty}T_{\infty}\} \) is a cover plane of \( [S_T] \).

Proof We use the coordinates for points and lines of \( \mathcal{B} \) given in Table 1. Note that every line of \( \mathcal{B} \) not through \( T \) is \( \ell_{e,d,f} \) for some \( e, d, f \in GF(q) \), and every point \( P \) of \( \mathcal{B} \) not on the line \( m_{\infty} \) can be written as \( P_{e,d} \) for some \( e, d \in GF(q) \). Further, the point \( P_{e,d} \) lies on the line \( \ell_{e,d,f} \) for \( f \in GF(q) \). Hence the pairs \( (P_{e,d}, \ell_{e,d,f}) \) and \( (U_f, \ell_{e,d,f}) \) cover all the pairs \( (P, \ell) \) such that \( P \) is a point of \( \mathcal{B} \) distinct from \( T \), and \( \ell \) is an order-\( q \)-subline of \( \mathcal{B} \) not through \( T \). Hence it suffices to calculate \( I_{P_{e,d}} \) for these pairs.

Consider a line \( \ell_{e,d,f} \) for fixed \( e, d, f \in GF(q) \). The points of \( \ell_{e,d,f} \) are \( R_{e,d,f,h} \) for \( h \in GF(q) \cup \{\infty\} \). Note that \( R_{e,d,f,e} = P_{e,d} \) and \( R_{e,d,f,\infty} = U_f \). To calculate the coordinates of the point \( I_{P_{e,d},\ell_{e,d,f}} \) in PG(6, \( q \)), we need to look at the tangent line to the twisted cubic \( [\ell_{e,d,f}] \) at the point \( P_{e,d} \). We consider the secant line \( P_{e,d}R_{e,d,f,h} \) of \( [\ell_{e,d,f}] \) and calculate where it meets \( \Sigma_{\infty} \). Then letting \( h = e \) will give us the coordinates of \( I_{P_{e,d},\ell_{e,d,f}} \). To find where this secant line meets \( \Sigma_{\infty} \), we take the coordinates of \( P_{e,d} \) and \( R_{e,d,f,h} \) in PG(2, \( q^3 \)), and write them with last coordinate in GF(\( q \)). This allows us to use the generalised Bruck-Bose map \( \sigma \) defined in Section 2.1 to convert them to coordinates in PG(6, \( q \)).

Let \( X = \theta(e)\theta(h)P_{e,d} \) and \( Y = \theta(e)\theta(h)R_{e,d,f,h} \) so

\[
X = \sigma((e + d\tau)(h + \tau)\theta(e)\theta(h), e(h + \tau)\theta(e)\theta(h), \theta(e)\theta(h))
\]

\[
Y = \sigma((h + (fh - fe + d)\tau)(e + \tau)\theta(e)\theta(h), h(e + \tau)\theta(e)\theta(h), \theta(e)\theta(h))
\]

13
For part 2, we fix the point $P$. This completes the proof of part 1. Taking the limit of this point as $h \to \infty$ gives $I_{e,d,f} = \sigma((1 - d + ef)\tau + f\tau^2, \tau\theta(e)^2, 0)$ as required.

Now consider the point $U_f = R_{e,d,f,\infty}$ on $\ell_{e,d,f}$. Similar to the above let $X = \theta(h)U_f$ and $Y = \theta(h)R_{e,d,f,h}$. Then

$$
X = \sigma((1 + f \tau)(h + \tau)\theta(h), (h + \tau)\theta(h), \theta(h))
$$

and

$$
X - Y = \sigma(((1 - d + ef)\tau + f\tau^2)\theta(h), \tau\theta(h), 0),
$$

so

$$
\frac{1}{h^2}(X - Y) = \sigma\left((1 - d + ef)\tau + f\tau^2\right)\left(1 + \frac{\tau q}{h}\right)\left(1 + \frac{\tau q^2}{h}\right), \tau\left(1 + \frac{\tau q}{h}\right)\left(1 + \frac{\tau q^2}{h}\right), 0.
$$

Taking the limit of this point as $h \to \infty$ gives $I_{e,d,f,\infty} = \sigma((1 - d + ef)\tau + f\tau^2, \tau, 0)$. This completes the proof of part 1.

For part 2, we fix the point $P_{e,d}$ (so fix $e, d \in \text{GF}(q)$) and look at the lines of $B$ through $P_{e,d}$, but not through $T$, namely the $q$ lines $\ell_{e,d,f}$ for $f \in \text{GF}(q)$. We show that the $q$ points $I_{e,d,f}$ for $f \in \text{GF}(q)$ lie on a line that contains the point $C_e = \sigma(\tau\theta(e), 0, 0)$ of the conic directrix $C$. First note that $I_{e,d,f,0} = \sigma((1 - d)\tau\theta(e)^2, \tau\theta(e)^2, 0)$. Now

$$
I_{e,d,f,\infty} = \sigma(((1 - d + ef)\tau + f\tau^2)\theta(e)^2, \tau\theta(e)^2, 0)
$$

as $f \theta(e) \in \text{GF}(q)$. Hence for fixed $e, d \in \text{GF}(q)$, the points $I_{e,d,f}$ for $f \in \text{GF}(q)$ all lie on the line joining $C_e$ and $I_{e,d,0}$, as required.

Now we consider the remaining points $U_f$ of $B$. For fixed $f \in \text{GF}(q)$, the lines $\ell_{e,d,f}$, $e, d \in \text{GF}(q)$ consist of the $q$ lines of $B$ through $U_f$, not through $T$. We show that
the \( q \) points \( I_{U_f, \ell_{e,d}, f}, e,d \in GF(q) \), lie on a line through \( C_\infty = \sigma(\tau,0,0) \). First note that \( I_{U_f, \ell_{0,0}, f} = \sigma(\tau + f\tau^2, \tau,0) \). Now

\[
I_{U_f, \ell_{e,d}, f} = \sigma((1 + ef - d)\tau + f\tau^2, \tau,0) = \sigma((ef - d)\tau + (\tau + f\tau^2), \tau,0) = (ef - d)\sigma(\tau,0,0) + \sigma(\tau + f\tau^2, \tau,0) = (ef - d)C_\infty + I_{U_f, \ell_{0,0}, f},
\]

as \( ef - d \in GF(q) \). Hence for fixed \( f \in GF(q) \), the points \( I_{U_f, \ell_{e,d}, f} \) for \( e,d \in GF(q) \) all lie on the line joining the points \( C_\infty \) and \( I_{U_f, \ell_{0,0}, f} \), as required.

For part 3, note that for fixed \( e \in GF(q) \), the lines \( I_{P_e,d} \) for \( d \in GF(q) \) and the tangent line \( C_eT_e \) all share the point \( C_e \). Also note that the \( I_{P_e,d} \) for \( d \in GF(q) \) are distinct lines, since they meet different splash elements. Thus to show that they lie in a plane, it is sufficient to find a line not through \( C_e \) that meets \( C_eT_e \) and meets each \( I_{P_e,d} \) for \( d \in GF(q) \). The line \( n = \langle \sigma(\tau\theta(e)^2, \tau\theta(e)^2,0), T_e \rangle \) does this as the line \( I_{P_e,d} \) contains the point \( I_{P_e,d, \ell_{e,d,0}} \) (for any \( d \in GF(q) \)) and

\[
I_{P_e,d, \ell_{e,d,0}} = \sigma((1 - d)\tau\theta(e)^2, \tau\theta(e)^2,0) = \sigma(\tau\theta(e)^2, \tau\theta(e)^2,0) - dT_e \in n.
\]

Hence \( I_{m_e} \) is a plane. It is a cover plane of \( [S_T] \) as it is completely contained in \( [S_T] \).

Finally, we consider the set \( I_{m_\infty} = \{ I_{U_f} \mid f \in GF(q) \} \cup \{ C_\infty T_\infty \} \). In a similar manner to the previous case, we have

\[
I_{U_f, \ell_{0,0}, f} = \sigma(\tau + f\tau^2, \tau,0) = fT_\infty + \sigma(\tau,\tau,0).
\]

So \( \langle T_\infty, \sigma(\tau,\tau,0) \rangle \) is a line not through \( C_\infty \) that meets \( I_{U_f}, f \in GF(q) \) and \( C_\infty T_\infty \), hence \( I_{m_\infty} \) is a plane. \( \square \)

Note that this lemma completes the proof of Theorem 4.1. We will later need the coordinates of the shadow for the line \( \ell_1 = \ell_{e,1,0} \) calculated in this lemma, so we state this as a corollary.

**Corollary 4.4** The order-\( q \)-subline \( \ell_1 = \{ P_{e,1} \mid e \in GF(q) \} \cup \{ P_\infty = U_0 \} \) has shadow \( \mathcal{D} = \{ D_e = I_{P_{e,1}, \ell_{e,1,0}} = \sigma(0,\theta(e)^2\tau,0) \mid e \in GF(q) \} \cup \{ D_\infty = \sigma(0,\tau,0) \} \).

5 The tangent subspace of a point

Let \( \mathcal{B} \) be an order-\( q \)-subplane in PG(2, \( q^3 \)) tangent to \( \ell_\infty \) at the point \( T \). For each affine point \( P \) of \( \mathcal{B} \), we can construct an order-\( q \)-subplane \( P^\perp \) that contains \( P \) and
is secant to \( \ell_{\infty} \) as follows. Let \( \ell_1, \ldots, \ell_{q+1} \) be the \( q + 1 \) lines of \( \mathcal{B} \) through \( P \). Then \( m = \{ \ell_i \cap \ell_{\infty} \mid i = 1, \ldots, q + 1 \} \) is an order-\( q \)-subline of \( \ell_{\infty} \) through \( T \). Now \( m \) and \( PT \cap \mathcal{B} \) are two order-\( q \)-sublines through \( T \), and so lie in a unique order-\( q \)-subplane which we denote by \( P^\perp \), and call the tangent subspace. Recall that in \( \text{PG}(6, q) \), \( \mathcal{B} \) corresponds to a ruled surface \([\mathcal{B}]\) and \( P^\perp \) corresponds to a plane \([P^\perp]\). In \([4]\) it was shown that in \( \text{PG}(6, q) \), the plane \([P^\perp]\) is the tangent space to the ruled surface \([\mathcal{B}]\) at the point \( P \). We can use Theorem 4.1 to investigate the structure of \([P^\perp]\) in \( \text{PG}(6, q) \) in more detail.

Label the order-\( q \)-sublines of \( \mathcal{B} \) through \( P \) by \( TP, \ell_1, \ldots, \ell_q \). In \( \text{PG}(6, q) \), the order-\( q \)-subline \( \ell_i \) corresponds to a twisted cubic \([\ell_i]\), \( i = 1, \ldots, q \). We show that the \( q + 1 \) lines through \( P \) in the plane \([P^\perp]\) consist of the generator line of \([\mathcal{B}]\) through \( P \) and the tangent line to the twisted cubic \([\ell_i]\) at the point \( P \), for \( i = 1, \ldots, q \).

**Lemma 5.1** The plane \([P^\perp]\) contains a generator line of \([\mathcal{B}]\).

**Proof** Using Theorem 3.1 and \([4, \text{Lemma 4.4}]\), we can without loss of generality prove this for the order-\( q \)-subplane \( \mathcal{B} \) coordinatised in Section 3 and the point \( P_{e,d} \) of \( \mathcal{B} \). In \([4, \text{Corollary 10.4}]\), it is shown that the point \( C_e = \sigma(\tau \theta(e), 0, 0) \) lies in the plane \([P_{e,d}^\perp]\). By Lemma 4.2, \( C_e \) is a point of the conic directrix of \([\mathcal{B}]\), so the plane \([P_{e,d}^\perp]\) meets the conic directrix. The line \( P_{e,d}C_e \) is in the plane \([P_{e,d}^\perp]\), so in \( \text{PG}(2, q^3) \) it corresponds to the order-\( q \)-subline \( m_e \) of \( \mathcal{B} \). Thus in \( \text{PG}(6, q) \), \([P_{e,d}^\perp]\) contains the generator \( P_{e,d}C_e \) of the ruled surface \([\mathcal{B}]\). \( \square \)

**Theorem 5.2** For each affine point \( P \) of \( \mathcal{B} \), we have \([P^\perp] = \langle P, I_P \rangle\).

**Proof** Let \( \mathcal{B} \) be the order-\( q \)-subplane coordinatised in Section 3 and let \( P_{e,d} \) be a point of \( \mathcal{B} \). By \([4, \text{Corollary 10.4}]\), \([P_{e,d}^\perp]\) contains the conic directrix point \( C_e \) and the point \( J_{e,d} = \sigma(((1 - d)\tau \theta(e)^2, \tau \theta(e)^2, 0) \). Using Lemma 4.3(1), we have \( J_{e,d} = I_{P_{e,d},e,d,0} \). By Lemma 4.3(2), \( I_{P_{e,d},e,d} \) contains \( C_e \) and \( J_{e,d} \). As \( P_{e,d} \notin I_{P_{e,d},e,d} \), it follows that \([P_{e,d}^\perp] = \langle P_{e,d}, C_e J_{e,d} \rangle = \langle P_{e,d}, I_{P_{e,d},e,d} \rangle\). \( \square \)

**Corollary 5.3** In \( \text{PG}(2, q^3) \), let \( \mathcal{B} \) be an order-\( q \)-subplane tangent to \( \ell_{\infty} \) at \( T \), and let \( P \) be an affine point of \( \mathcal{B} \). The lines of \( \mathcal{B} \) through \( P \) correspond in \( \text{PG}(6, q) \) to \( q \) twisted cubics \([\ell_i]\), \( i = 1, \ldots, q \), and one generator line of \([\mathcal{B}]\) (the ruled surface corresponding to \( \mathcal{B} \)). The tangent to the twisted cubic \([\ell_i]\) at \( P \) lies in the tangent subspace \([P^\perp]\) for \( i = 1, \ldots, q \).
6 Constructing an order-\(q\)-subplane from a splash and a subline

In PG(2, \(q^3\)), let \(S_T\) be a tangent splash of \(\ell_\infty\) with centre \(T\) and let \(\ell\) be an order-\(q\)-subline disjoint from \(\ell_\infty\) whose extension \(\tilde{\ell}\) to PG(2, \(q^3\)) meets \(\ell_\infty\) in a point \(L = \tilde{\ell} \cap \ell_\infty\) of \(S_T \setminus \{T\}\). By Theorem 1.1, there is a unique order-\(q\)-subplane tangent to \(\ell_\infty\) that contains \(\ell\) and has tangent splash \(S_T\). In this section we give a geometric construction of this subplane.

We work in the Bruck-Bose representation of PG(2, \(q^3\)) in PG(6, \(q^3\)), and also use the cubic extension PG(6, \(q^3\)) of PG(6, \(q\)). We use the following notation for the cubic extension PG(6, \(q^3\)). If \(K\) is a subspace or a curve of PG(6, \(q\)) then denote by \(K^*\) the natural extension of \(K\) to a subspace or curve of PG(6, \(q^3\)).

In PG(6, \(q\)), the order-\(q\)-subline \(\ell\) corresponds to a special twisted cubic \([\ell]\) in a 3-space \([\tilde{\ell}]\) about a spread element \([L]\) (by Theorem 2.2). In order to construct the required order-\(q\)-subplane of PG(2, \(q^3\)), we use Theorem 2.4 and construct a ruled surface \(V\) in PG(6, \(q\)) with twisted cubic directrix \([\ell]\) and a conic directrix \(C\) in the centre \([T]\), such that in the cubic extension PG(6, \(q^3\)), \(V^*\) contains the transversals of the regular spread \(S\).

Recall from Theorem 3.2 that the tangent splash \([S_T]\) in PG(6, \(q\)) has an associated set of \(q^2 + q + 1\) cover planes that each meet the centre \([T]\) in distinct lines, and meet every other plane of \([S_T]\) in distinct points. Our construction will exploit the rich geometrical nature of the cover planes of \([S_T]\) and their interaction with the shadow points of the twisted cubic \([\ell]\) (this interaction is described in Section 4). The construction is now stated, then is proved using a series of lemmas.

**Construction 1** In PG(2, \(q^3\)), let \(S_T\) be a tangent splash of \(\ell_\infty\) with centre \(T\) and let \(\ell\) be an order-\(q\)-subline disjoint from \(\ell_\infty\) whose extension to PG(2, \(q^3\)) meets \(\ell_\infty\) in a point of \(S_T \setminus \{T\}\). Then in PG(6, \(q\)):

1. For each point \(N_i, i = 1, \ldots, q+1,\) of the twisted cubic \([\ell]\), let \(D_i\) be the intersection of the tangent to \([\ell]\) at \(N_i\) with \([L]\). The set of points \(D = \{D_1, \ldots, D_{q+1}\}\) in \([L]\) is called the shadow of \([\ell]\).

2. Map the shadow \(D\) in \([L]\) to a special conic \(C\) in \([T]\) by

   (a) Through each point \(D_i \in D\) there is a unique cover plane of \(S_T\), this cover plane meets \([T]\) in a line \(t_i\).
(b) The lines $t_1, \ldots, t_{q+1}$ are tangent lines to a unique special conic $C$ of $[T]$.

3. Let $t_i \cap C = C_i$, then the lines $N_iC_i$, $i = 1, \ldots, q + 1$, form a ruled surface that corresponds in $\text{PG}(2, q^3)$ to the unique order-$q$-subplane tangent to $\ell_\infty$ containing $\ell$ and with tangent splash $S_T$.

We begin with three lemmas that investigate the shadow of an order-$q$-subline. We first determine the structure of a shadow in a spread element $[L]$, and show that it is different for $q$ even and $q$ odd.

**Lemma 6.1** Let $\ell$ be an order-$q$-subline of $\text{PG}(2, q^3)$ disjoint from $\ell_\infty$, and let $D$ be the shadow of $\ell$ in $\text{PG}(6, q)$.

1. For $q$ even, the shadow $D$ is a special conic of $[L]$.

2. For $q$ odd, the shadow $D$ has the property that no 3 points of $D$ lie in a special conic of $[L]$.

**Proof** Let $\ell$ be an order-$q$-subline of $\text{PG}(2, q^3)$ disjoint from $\ell_\infty$, such that its extension $\bar{\ell}$ meets $\ell_\infty$ in a point $L$ of $S_T \setminus \{T\}$. By Theorem 2.2, in $\text{PG}(6, q)$, $\mathcal{N} = [\ell]$ is a special twisted cubic in a 3-space $[\bar{\ell}]$ that meets $\Sigma_\infty$ in the spread element $[L]$. Let the point $N_i \in \mathcal{N}$, $i = 1, \ldots, q + 1$, have tangent $n_i$ to $\mathcal{N}$ and let $D_i = n_i \cap [L]$. So the shadow of $[\ell]$ is $D = \{D_1, \ldots, D_{q+1}\} \subset [L]$. In the cubic extension $\text{PG}(6, q^3)$, let $[L]^*$ meet the transversal lines $g, g^q, g^{q^2}$ of the regular spread $S$ in the points $P, P^q, P^{q^2}$ respectively. As $\mathcal{N}$ is special, $P, P^q, P^{q^2}$ lie in $\mathcal{N}^*$, the extension of $\mathcal{N}$ to $\text{PG}(6, q^3)$.

If $q$ is even, then the tangents to a twisted cubic lie in a regulus [9, Theorem 21.1.2]. This regulus lies in a unique hyperbolic quadric $\mathcal{H}$ of the 3-space $[\bar{\ell}]$. As the twisted cubic $\mathcal{N}$ is disjoint from $[L]$, $\mathcal{H}$ meets $[L]$ in a non-degenerate conic, so the shadow $D$ is a non-degenerate conic. In the cubic extension $\text{PG}(6, q^3)$, $\mathcal{H}^*$ contains the twisted cubic $\mathcal{N}^*$, and so $\mathcal{H}^*$ contains the transversal points $P, P^q, P^{q^2}$. So the conic $D^*$ contains $P, P^q, P^{q^2}$, and hence $D$ is a special conic of $[L]$.

Now suppose $q$ is odd. Consider the points $N_1, N_2 \in \mathcal{N}$ with tangents $n_1, n_2$. By [9, Lemma 21.1.6(Cor 2)], $\mathcal{N} \setminus n_1, n_2$ lie in a hyperbolic quadric $\mathcal{H}$. By [3, Lemma 2.6], the points $D_1, D_2$ lie in a unique special conic $C$ of $[L]$. As $\mathcal{H}^*$ and $C^*$ both contain the five points $D_1, D_2, P, P^q, P^{q^2}$, $\mathcal{H}^*$ contains $C^*$ and so $\mathcal{H}$ contains $C$. 

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We now show that $C$ contains no further point of the shadow $D$. Suppose to the contrary that $C$ contains the point $D_3$ (so $D_3$ lies on the tangent $n_3$). By [9, Lemma 21.1.7], the tangents $n_1, n_2$ lie in one regulus $R$ of $H$, and $R$ consists of chords of $N$. As $D_3 \in C \subset H$, it follow that $D_3$ is on line $n$ of the regulus $R$. Note that $n$ is a chord of $N$ as $R$ consists of chords. By [9, Lemma 21.1.8], $R$ contains at most two tangents of $N$, hence as $R$ contains $n_1, n_2$, $R$ does not contain $n_3$. So $n$ is not the tangent $n_3$. Hence through the point $D_3$, there are two chords of $N$, namely $n$ and the tangent $n_3$. This contradicts [9, Theorem 21.1.9]. Hence a special conic of $[L]$ contains at most two points of a shadow. \hfill \Box

Next we look at the action in $\text{PG}(6, q)$ of a certain Singer cycle acting on the shadow of an order-$q$-subline, and also acting on the conic directrix of an order-$q$-subplane.

**Lemma 6.2** Let $B$ be an order-$q$-subplane of $\text{PG}(2, q^3)$ tangent to $e_\infty$ and let $\ell$ be an order-$q$-subline of $B$ disjoint from $e_\infty$. In the Bruck-Bose representation in $\text{PG}(6, q)$, consider the collineation $\Theta$ of $\text{PG}(6, q)$ defined in Theorem 2.6.

1. The conic directrix $C$ of $[B]$ has an orbit of size $q^2 + q + 1$ under $\langle \Theta \rangle$.

2. The shadow $D$ of $[\ell]$ has an orbit of size $q^2 + q + 1$ under $\langle \Theta \rangle$.

**Proof** By Theorem 3.1, we can without loss of generality prove this for the order-$q$-subplane coordinatised in Section 3. The coordinates of the conic directrix $C$ of $[B]$ are calculated in Lemma 4.2. So $C = \{\sigma(\tau, 0, 0) \cup \{\sigma(\vartheta(e)^i, 0, 0) \mid e \in \text{GF}(q)\}$, and $\Theta^i(C) = \{\sigma(\varpi^{i+1}, 0, 0) \cup \{\sigma(\vartheta(e)^i, 0, 0) \mid e \in \text{GF}(q)\}$. Suppose $C = \Theta^i(C)$ for some $i$ with $1 \leq i < q^2 + q + 1$. Suppose firstly that $q \geq 3$, so $C$ contains at least four points. Hence there exists $a, b, e, f \in \text{GF}(q)$ with $a \neq b$ and $e \neq f$, such that $\sigma(\vartheta(a)^i, 0, 0) \equiv \sigma(\vartheta(e)^i, 0, 0)$ and $\sigma(\vartheta(b)^i, 0, 0) \equiv \sigma(\vartheta(f)^i, 0, 0)$, thus there exists nonzero $c, d \in \text{GF}(q)$ such that

$$\vartheta(a)^i = c\vartheta(e)^i, \quad \vartheta(b)^i = d\vartheta(f)^i.$$  

Hence $\vartheta(a)/\vartheta(b) = (c/d)\vartheta(e)/\vartheta(f)$ and so $d\vartheta(a)\vartheta(f) = c\vartheta(b)\vartheta(e)$. Multiplying both sides by $(a + \tau)(a + \varpi)(e + \varpi)(f + \varpi)$ yields

$$d\vartheta(a)\vartheta(f)(b + (b + e)\varpi + \tau^2) = c\vartheta(b)\vartheta(e)(af + (a + f)\varpi + \tau^2).$$  

Equating the coefficient of $\varpi^2$ gives $d\vartheta(a)\vartheta(f) = c\vartheta(b)\vartheta(e)$, and substituting back into (6) gives $b + (b + e)\varpi = a + (a + f)\varpi$. Equating the coefficient of $\varpi$ gives $b = a + f - e$ and using the constant term gives $(a + f - e)e = af$, hence $(f - e)(e - a) = 0$. As
\[ e \neq f \text{ we have } e = a. \] So by (5) we have \[ \theta(a)\tau = c\theta(a)\tau^{i+1} \] and so \[ 1/c = \tau^i. \] Now \[ \tau^i \in \text{GF}(q) \implies q^2 + q + 1 \mid i, \] contradicting \[ 1 \leq i < q^2 + q + 1. \] Hence the orbit of \( C \) under \( \langle \Theta \rangle \) is of size at least \( q^2 + q + 1 \). As \( \tau^{q^2+q+1} \in \text{GF}(q) \), \( \Theta \tau^{q^2+q+1}(C) = C \). Hence the size of the orbit of \( C \) is exactly \( q^2 + q + 1 \).

Now consider the case \( q = 2 \). Let \( \mathcal{C} = \{C_{1,0} = \sigma(\tau,0,0), C_{2,0} = \sigma(\theta(0)\tau,0,0), C_{3,0} = \sigma(\theta(1)\tau,0,0)\} \) and so \( \Theta^1(\mathcal{C}) = \{C_{1,i} = \sigma(\tau^{i+1},0,0), C_{2,i} = \sigma(\theta(0)\tau^{i+1},0,0), C_{3,i} = \sigma(\theta(1)\tau^{i+1},0,0)\} \). If \( \Theta^i(C_{j,0}) = C_{j,i} \) for some \( j \), then \( \tau^i \in \text{GF}(q) \) and so \( q^2 + q + 1 \mid i \), contradicting \( 1 \leq i < q^2 + q + 1 \). If \( \mathcal{C} = \Theta^i(\mathcal{C}) \) and \( \Theta^i(C_{j,0}) \neq C_{j,i} \) for \( j = 1, 2, 3 \), then we have either

\[
\tau = b\theta(0)\tau^{i+1}, \quad \theta(0)\tau = c\theta(1)\tau^{i+1}, \quad \theta(1)\tau = d\tau^{i+1},
\]
or
\[
\tau = r\theta(1)\tau^{i+1}, \quad \theta(1)\tau = s\theta(0)\tau^{i+1}, \quad \theta(0)\tau = t\tau^{i+1},
\]

for some nonzero \( b, c, d, r, s, t \in \text{GF}(q) \). Multiplying the three equations together in each set gives \( 1 = bcd\tau^{3i} \) or \( 1 = rst\tau^{3i} \). As \( q^2 + q + 1 = 7 \) and \( (3, 7) = 1 \), this cannot happen for \( i < q^2 + q + 1 = 7 \).

We now prove part 2. By [4, Lemma 4.4(6)], we can without loss of generality prove this for the shadow whose coordinates are calculated in Corollary 4.4, namely \( \mathcal{D} = \{(0, \tau, 0) \cup \{(0, \theta(e)^2\tau, 0) \mid e \in \text{GF}(q)\} \). Using a similar argument to part 1, if \( q \geq 3 \), there exists \( a, b, e, f \in \text{GF}(q) \) with \( a \neq b \) and \( e \neq f \), and so there exists nonzero \( c, d \in \text{GF}(q) \) such that

\[ \theta(a)^2\tau = c\theta(e)^2\tau^{i+1}, \quad \theta(b)^2\tau = d\theta(f)^2\tau^{i+1}. \]

Hence \( \theta(a)^2/\theta(b)^2 = (c/d)\theta(e)^2/\theta(f)^2 \). Thus \( c/d \) is a square in \( \text{GF}(q^3) \), and hence is a square in \( \text{GF}(q) \). Writing \( c/d = x^2 \) with \( x \in \text{GF}(q) \) yields \( \theta(a)\theta(f) = \pm x\theta(b)\theta(e) \). The result now follows using a similar argument to part 1. The case \( q = 2 \) is similar.

Consider a 3-space \( \Sigma \) of \( \text{PG}(6, q) \setminus \Sigma_\infty \) about a splash element \( [L] \). There are many special twisted cubics in \( \Sigma \), each gives rise to a shadow in \( [L] \). We now show that there are only \( q^2 + q + 1 \) distinct shadows in \( [L] \).

**Lemma 6.3** There are \( q^2 + q + 1 \) distinct shadows in a non-centre splash element.

**Proof** Let \( \overline{\ell} \) be a line of \( \text{PG}(2, q^3) \) that meets \( \ell_\infty \) in the point \( L \). We look at certain collineations from \( \text{PGL}(2, q^3) \) acting on the line \( \overline{\ell} \). In particular, let \( E \) be the subgroup of \( \text{PGL}(2, q^3) \) fixing the infinite point \( L \) of \( \overline{\ell} \) (\( E \) is defined in Theorem 2.5).
In $\PG(6,q)$, $\ell$ corresponds to a 3-space $[\ell]$ that meets $\Sigma_{\infty}$ in the spread element $[L]$, and $[E]$ is the subgroup of $\PGL(4,q)$ acting on $[\ell] \cong \PG(3,q)$ that fixes $\pi_{\infty} = [L]$.

It is straightforward to show that there are $q^3(q^3-1)$ order-$q$-sublines of $[\ell]$ disjoint from $L$. By Theorem 2.5, $E$ is transitive on these order-$q$-sublines. Hence using Theorem 2.2, in $\PG(6,q)$, $[E]$ is transitive on the special twisted cubics of $[\ell]$. However, $|E| = q^3(q^3-1)$, hence $[E]$ acts regularly on the special twisted cubics in $[\ell]$.

To find the number of distinct shadows in $[L]$, we consider the subgroup $[D]$ of $[E]$, where $[D]$ contains only the collineations of $\PG(3,q)$ which fix $\pi_{\infty} = [L]$ pointwise. So $D$ contains the elations and homologies of $\PG(3,q)$ with axis $\pi_{\infty}$. As the product of two collineations with axis $\pi_{\infty}$ is also a collineation with axis $\pi_{\infty}$, $[D]$ contains exactly the elations and homologies with axis $\pi_{\infty}$. The group of elations with axis $\pi_{\infty}$ and centre a point of $\pi_{\infty}$ is of size $q$, so the number of non-identity elations of $[D]$ is $(q^2 + q + 1) \times (q - 1)$. The group of homologies with axis $\pi_{\infty}$ and centre a point not on $\pi_{\infty}$ has size $q - 1$, hence the number of non-identity homologies of $[D]$ is $q^3 \times (q - 2)$. As the product of two collineations with axis $\pi_{\infty}$ is again a collineation with axis $\pi_{\infty}$, $|[D]| = (q^2 + q + 1) \times (q - 1) + q^3 \times (q - 2) + 1 = q^3(q - 1)$. Note that all the special twisted cubics in an orbit of $[D]$ have the same shadow, since $[D]$ fixes $\pi_{\infty}$ pointwise, and hence fixes the shadow pointwise. Hence there are at most $|[E]|/[|D|] = q^2 + q + 1$ distinct shadows. However, by Lemma 6.2, there are at least $q^2 + q + 1$ distinct shadows and so the result follows. □

We now begin the proof of step 2 of Construction 1. We want to construct a map from the shadow of a non-centre splash element $[U]$ to a special conic of the centre $[T]$. We begin by investigating a map arising from the cover planes of $\ST$.

Recall from Theorem 3.2 that a line $m$ of the centre $[T]$ lies in a unique cover plane which meets any non-centre splash element $[U]$ in a unique point $M$. Conversely, any point $M$ of a non-centre splash element $[U]$ lies in a unique cover plane that meets $[T]$ in a line $m$. We call this bijection from points of $[U]$ to lines of $[T]$, and lines of $[T]$ to points of $[U]$ the cover plane map.

**Theorem 6.4** In $\PG(6,q)$, let $\ST$ be a tangent splash with centre $[T]$ and let $[U]$ be any non-centre element of $\ST$. Then the cover plane map takes the $q + 1$ lines of $[T]$ through a fixed point of $\ST$ to a special conic of $[U]$. Further, the points of a special conic in $[U]$ are mapped under the cover plane map to lines of $[T]$ through a common point.

**Proof** By Theorem 3.1, we can without loss of generality prove the result for the
tangent splash with coordinates \( S_T = \{(a + b\tau, 1, 0) | a, b \in \text{GF}(q)\} \cup \{T = (1, 0, 0)\} \) (given in Section 3). Note that by [4, Theorem 5.5], this tangent splash is uniquely determined by the centre \( T = (1, 0, 0) \) and the three points \( U = (0, 1, 0), V = (1, 1, 0) \) and \( W = (\tau, 1, 0) \). We now work in the Bruck-Bose representation in PG(6, \( q \)) and make use of the generalised Bruck-Bose map for our coordinates. Recall that these coordinates use \( \tau^3 = t_0 + t_1\tau + t_2\tau^2 \).

We will use the function \( f : \text{GF}(q) \to \text{GF}(q^3) \) defined by \( f(a) = 1 + at_2 - a^2t_1 + a\tau + a^2\tau^2 \). Consider the \( q + 1 \) lines \( m_a, a \in \text{GF}(q) \cup \{\infty\} \), of \( T \) through the point \( P_i = \sigma(1, 0, 0), \) where \( m_\infty = \langle \sigma(1, 0, 0), \sigma(\tau^2, 0, 0) \rangle \) and \( m_a = \langle \sigma(1, 0, 0), \sigma(\tau + a\tau^2, 0, 0) \rangle \). Also consider the \( q + 1 \) points of \( [U] \) with coordinates \( U_\infty = \sigma(0, -t_1 + \tau^2, 0), U_a = \sigma(0, f(a), 0), a \in \text{GF}(q) \). We construct \( q + 1 \) planes \( \pi_a = \langle m_a, U_a \rangle, a \in \text{GF}(q) \cup \{\infty\} \) and show that these planes are cover planes of \( S_T \). As \( T, U, V, W \) are not on a common order-\( q \)-subline of \( PG(2, q^3) \), by Theorem 2.2, in \( PG(6, q) \) \([T], [U], [V], [W]\) are not contained in a common 2-regulus. Hence by Lemma 3.3, it suffices to show the planes \( \pi_a \) meet \([T]\) in a line, and \([U], [V], [W]\) in points.

Clearly the planes \( \pi_a \) each meet \([T]\) in a line and \([U]\) in a point. Consider the point \( V_a = \sigma(f(a), f(a), 0) \in [V], a \in \text{GF}(q). \) We have \( V_a = (1 + at_2 - a^2t_1)\sigma(1, 0, 0) + a\sigma(\tau + a\tau^2, 0, 0) + U_a, \) so \( V_a \in \pi_a. \) Similarly, \( V_\infty = \sigma(-t_1 + \tau^2, -t_1 + \tau^2, 0) \) is a point in \([V]\) that also lies in \( \pi_\infty. \) Consider the point \( W_a = \sigma(f(a)\tau, f(a), 0) \in [W], a \in \text{GF}(q) \). Now \( f(a) = (1 + at_2 - a^2t_1 + a\tau + a^2\tau^2)\tau = a^2t_0 + (1 + at_2)(\tau + a\tau^2), \) so \( W_a = a^2t_0\sigma(1, 0, 0) + (1 + at_2)\sigma(\tau + a\tau^2, 0, 0) + U_a, \) so \( W_a \in \pi_a. \) Similarly \( W_\infty = \sigma(t_0 + t_2\tau^2, -t_1 + \tau^2, 0) \) is a point in \([W]\) that is also in \( \pi_\infty. \) Hence the planes \( \pi_a, a \in \text{GF}(q) \cup \{\infty\}, \) are cover planes of \( S_T. \)

We now show that the cover planes \( \pi_a, a \in \text{GF}(q) \cup \{\infty\} \) meet \([U]\) in \( q + 1 \) points that lie on a special conic. The points of \([U]\) have coordinates \((0, 0, 0, y_0, y_1, y_2, 0), \) so we can consider them as points \((y_0, y_1, y_2) \in PG(2, q). \) In this setting, \( U_a = (1 + at_2 - a^2t_1, a, a^2) \) and \( U_\infty = (-t_1, 0, 1). \) It is straightforward to show that \( U_a, U_\infty \) lie on the conic \( C_1 \) of equation \( y_1^2 - t_1y_2^2 + t_2y_1y_2 - y_0y_2 = 0. \) To prove that \( C_1 \) is special, we use Lemma 2.1 where the transversal point \([U] \cap g \) is calculated to be \( Q = (t_1 + t_2\tau - \tau^2, t_2 - \tau, -1). \) It is straightforward to show that \( Q \) satisfies the equation of \( C_1, \) so \( Q \in C_1^*. \) Hence \( Q^q, Q^{q^2} \in C_1^* \) and so \( C_1 \) is special.

Thus the pencil of lines of \([T]\) through a point \( P_i \) of \([T]\) maps to a special conic \( C_i \) of \([U]\) via the cover plane map. If \( i \neq j, \) then \( C_i \neq C_j \) as the cover plane map maps distinct points of \([U]\) to distinct lines of \([T]. \) There are \( q^2 + q + 1 \) distinct pencils in \([T]\) which map to \( q^2 + q + 1 \) distinct special conics of \([U]. \) By [3, Lemma 2.6] there are exactly \( q^2 + q + 1 \) special conics in \([U]. \) Hence the cover plane map acts as a
bijection from pencils of $[T]$ to special conics of $[U]$. \hfill \square

We can now prove step 2 of Construction 1.

**Theorem 6.5** There is a bijection from the $q^2 + q + 1$ shadows $D$ in a non-centre splash element $[U]$ to the $q^2 + q + 1$ special conics $C$ in the centre $[T]$ as follows. Firstly, map the point $D_i \in D$ under the cover plane map to a line $t_i \in [T]$. Then

1. if $q$ is even, the lines $t_i$ are concurrent in a point $N$; and there is a unique special conic of $[T]$ with nucleus $N$.
2. if $q$ is odd, the set of lines $t_i$ form a conic envelope, and the tangential points form a special conic of $[T]$.

**Proof** Suppose first that $q$ is even. By Lemma 6.1, the shadow is a special conic in $[U]$. Hence by Theorem 6.4, the cover plane map maps this to a set of $q + 1$ lines through a point $N$ in $[T]$. In PG($6,q^3$), there is a unique conic $C^*$ of $[T]^*$ that contains the three transversal points $Q = g \cap [T]^*$, $Q^0$, $Q^{q^2}$ and has nucleus $N$. As $N \in [T]$, $C$ is a conic of $[T]$. Further, $C$ is non-degenerate as all special conics are non-degenerate by [3, Lemma 2.6]. So we have a map from shadows of $[U]$ to special conics of $[T]$ with distinct shadows mapping to distinct special conics. As there are $q^2 + q + 1$ shadows in $[U]$ and $q^2 + q + 1$ special conics in $[T]$ (by Lemma 6.3 and [3, Lemma 2.6]), this map is a bijection.

Now suppose $q$ is odd. By Lemma 6.1, the shadow is a set of $q + 1$ points in $[U]$, no three in a special conic of $[U]$. Hence by Theorem 6.4, the cover plane map maps this to a set of $q + 1$ lines in $[T]$, no three concurrent. By Segre [12] this is a dual conic, and so the tangential points form a conic of $[T]$. To show that this conic is special, we need to use coordinates.

As in the proof of Theorem 6.4, without loss of generality we let $s_T$ be the tangent splash with centre $T = (1,0,0)$ and containing the three points $U = (0,1,0)$, $V = (1,1,0)$ and $W = (\tau,1,0)$. We now work in PG($6,q$) using coordinates. By [4, Lemma 4.4(6)], we can without loss of generality prove the result for the shadow whose coordinates are calculated in Corollary 4.4, namely $D = \{D_e = \sigma(0,e^2\tau,0) \mid e \in GF(q) \cup \{\infty\}\}$. We will show that the map described in the statement of the theorem maps this shadow to the tangent lines of the special conic $C = \{C_e = \sigma(e\theta(e)^2\tau,0) \mid e \in GF(q) \cup \{\infty\}\}$ in $[T]$; and so maps it to the conic $C$. By Lemma 4.2, the tangent to $C$ at the point $C_e$ is the line $C_eT_e$, with $T_e = \sigma(\tau\theta(e)^2,0,0), e \in GF(q)$ and $T_{\infty} = \sigma(\tau^2,0,0)$. 

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For \( e \in GF(q) \cup \{ \infty \} \), consider the plane \( \pi_e = \langle D_e, C_e, T_e \rangle \). It contains a line \( C_e T_e \) of \([T]\) and a point \( D_e \) of \([U]\). By Lemma 3.3, to show that \( \pi_e \) is a cover plane of the tangent splash \([S_T]\), we just need to show that \( \pi_e \) meets \([V]\) and \([W]\). Suppose first that \( e \in GF(q) \). Consider the point \( V_e = \sigma(\tau \theta(e)^2, \tau \theta(e)^2, 0) \) of \([V]\), \( e \in GF(q) \), as \( V_e = T_e + D_e, V_e \in \pi_e \), so \( \pi_e \) meets \([V]\) in a point. Consider the point \( W_e = \theta(e)C_e - eT_e + D_e \in \pi_e \), so \( \pi_e \) meets \([W]\) in a point. Hence \( \pi_e \) is a cover plane of the splash \([S_T]\), \( e \in GF(q) \).

Now let \( V_\infty = \sigma(\tau, \tau, 0) \in [V] \), and \( W_\infty = \sigma(\tau^2, \tau, 0) \in [W] \). As \( V_\infty = C_\infty + D_\infty \) and \( W_\infty = T_\infty + D_\infty \), we have \( V_\infty, W_\infty \in \pi_\infty \). Hence \( \pi_\infty \) is a cover plane of \([S_T]\).

Thus the cover plane map maps the point \( D_e \) of \( \mathcal{D} \) to the tangent \( C_e T_e \) of \( \mathcal{C} \), \( e \in GF(q) \cup \{ \infty \} \). Hence we have a map from the shadow \( \mathcal{D} \) of \([U]\) to the special conic \( \mathcal{C} \) of \([T]\). By Lemma 6.3, there are \( q^2 + q + 1 \) shadows in \([U]\), and by [3, Lemma 2.6] there are \( q^2 + q + 1 \) special conics in \([T]\). As the cover plane map maps distinct points of \([U]\) to distinct lines of \([T]\), it follows that distinct shadows are mapped to distinct special conics. Hence our map from \( \mathcal{D} \) to \( \mathcal{C} \) is a bijection from shadows of \([U]\) to special conics of \([T]\). □

This completes the proof of step 2 of Construction 1.

**Corollary 6.6** These results give us a map \( \phi \) from the special twisted cubic \( \mathcal{N} = [\ell] \) to a special conic \( \mathcal{C} \) in \([T]\) by mapping: a point \( N_i \in \mathcal{N} \) to a point \( D_i \) in the shadow \( \mathcal{D} \) in \([L]\), then to a line \( t_i \) in \([T]\), then to a point \( C_i = t_i \cap \mathcal{C} \).

The next theorem shows that this map \( \phi \) from \( N_i \) to \( C_i \), \( i = 1, \ldots, q + 1 \), is a projectivity. Note that we can easily define a projectivity \( \eta \) from \( \mathcal{N} \) to \( \mathcal{C} \) as follows. Let \( \mathcal{N}^* \) have transversal points \( P = g \cap [L]^* \), \( P^a, P^a^2 \), and let \( \mathcal{C}^* \) have transversal points \( Q = g \cap [T]^* \), \( Q^a, Q^a^2 \). As a projectivity is uniquely determined by three points, there is a projectivity \( \eta \) from \( \mathcal{N}^* \) to \( \mathcal{C}^* \) that maps \( P \mapsto Q \), \( P^a \mapsto Q^a \), \( P^a^2 \mapsto Q^a^2 \). The next result shows that this projectivity \( \eta \) also maps \( N_i \) to \( C_i \), \( i = 1, \ldots, q + 1 \), and so \( \eta = \phi \). Note that the projectivity \( \eta \) determines a unique ruled surface \( \mathcal{V} \), so \( \mathcal{V} \) is the unique ruled surface with directrices \( \mathcal{C}, \mathcal{N} \) such that in the cubic extension \( PG(6, q^3) \), \( \mathcal{V}^* \) contains the transversals of the regular spread \( \mathcal{S} \).

**Theorem 6.7** Let \( \mathcal{N} \) be a special twisted cubic in a 3-space of \( PG(6, q) \setminus \Sigma_\infty \) about a non-centre splash element. Let \( \mathcal{C} \) be the special conic in \([T]\) constructed from \( \mathcal{N} \) by the map \( \phi \), where \( \phi \) maps a point \( N_i \in \mathcal{N} \) to a point \( C_i \in \mathcal{C}, i = 1, \ldots, q + 1 \) as described in Corollary 6.6. Let \( \mathcal{V} \) be the unique ruled surface with directrices \( \mathcal{C} \) and \( \mathcal{N} \) such that in the cubic extension \( PG(6, q^3) \), \( \mathcal{V}^* \) contains the transversals of
the regular spread $\mathcal{S}$. Let $\eta$ be the projectivity of $\mathcal{V}$ that maps $\mathcal{N}$ to $\mathcal{C}$. Then $\eta = \phi$, so $N_iC_i$, $i = 1, \ldots, q + 1$, are the generator lines of $\mathcal{V}$.

**Proof**  As in the proof of Theorem 6.4, without loss of generality we may take the splash $S_T$ to have centre $T = (1,0,0)$ and three points $U = (0,1,0)$, $V = (1,1,0)$ and $W = (\tau,1,0)$. Further, by [4, Lemma 4.4(5,6)] we can without loss of generality take the twisted cubic $\mathcal{N} = [\ell]$ to correspond to the order-$q$-subline $\ell = \{(e + \tau, e, e + \tau) \mid e \in \text{GF}(q)\} \cup \{(1,1,1)\}$ of $\text{PG}(2,q^3)$.

Using the notation from Table 1, we have $\ell = \ell_1 = \ell_{e,1,0}$. The line $\ell$ has points $P_{e,1} = (e + \tau, e, e + \tau) \equiv (\theta(e), \theta'(e), \theta(e))$, $e \in \text{GF}(q)$ and $P_\infty = U_0 = (1,1,1)$. In $\text{PG}(6,q)$ label the points of the twisted cubic $\mathcal{N} = [\ell]$ as $N_e = P_{e,1}$, $e \in \text{GF}(q)$, and $N_\infty = P_\infty$. By Corollary 4.4, the shadow of $N_e$ is the point $D_e = \sigma(0, \theta'(e)^2,0)$, $e \in \text{GF}(q)$, and the shadow of $N_\infty$ is $D_\infty = \sigma(0,\tau,0)$.

As $\tau$ satisfies the polynomial $x^3-t_2x^2-t_1x-t_0$ we have $\tau\tau^2\tau^2 = t_0$, $\tau + \tau^2 + \tau^3 = t_2$ and $t_0/\tau = -t_1 - t_2 \tau + \tau^2$. Hence for $e \in \text{GF}(q)$, $N_e = \sigma(\theta(e),\theta'(e),\theta(e)) = (\theta(e),0,0,e^3 + t_2e^2 - t_1e, -e^2 - t_2e, e, \theta(e))$. Also note that $\theta(e) = e^3 + t_2e^2 - t_1e + t_0$.

We can extend $\mathcal{N}$ to a twisted cubic $\mathcal{N}^*$ of $\text{PG}(6,q^3)$ where $\mathcal{N}^* = \{N_e \mid e \in \text{GF}(q^3) \cup \{\infty\}\}$. We can calculate $N_{-\tau} \equiv (0,0,0, t_2\tau + t_1 - \tau^2, t_2 - \tau, -1, 0)$. Hence by Lemma 2.1, $N_{-\tau} = g \cap [U]$. Similarly $N_{-\tau^2} = g^2 \cap [U]$ and $N_{-\tau^3} = g^3 \cap [U]$.

The proof of Theorem 6.5 shows that the bijection $\phi$ maps $\mathcal{N}$ to the conic $\mathcal{C} = \{C_e = \sigma(\theta(e)\tau,0,0) \mid e \in \text{GF}(q) \cup \{\infty\}\}$ in $[T]$ with $\phi(N_e) = C_e$, $e \in \text{GF}(q) \cup \{\infty\}$. Note that the proof of Lemma 4.2 shows that $\mathcal{C}$ is a special conic, and that when we expand $\sigma$, we have $C_e = (t_0, c^2 + t_2e, -e, 0,0,0,0)$, $e \in \text{GF}(q) \cup \{\infty\}$. We can extend $\mathcal{C}$ to a conic of $\text{PG}(6,q^3)$, by taking $e \in \text{GF}(q^3) \cup \{\infty\}$. Now $C_{-\tau} = (t_0, \tau^2 - t_2\tau, 0,0,0,0) \equiv (t_2\tau + t_1 - \tau^2, t_2 - \tau, -1, 0,0,0)$ which is the transversal point $g \cap [T]$. Similarly, $C_{-\tau^2} = g^2 \cap [T]$ and $C_{-\tau^3} = g^3 \cap [T]$.

We now show that the mapping $\phi: N_e \mapsto C_e$, $e \in \text{GF}(q) \cup \{\infty\}$, is a projectivity. Firstly, note that the map $(1,e,e^2,e^3) \mapsto (1,e,e^3)$ for $e \in \text{GF}(q) \cup \{\infty\}$ is a projectivity. Now the point $C_e$ is equivalent to $M_C(1,e,e^2)^t$ and the point $N_e$ is equivalent to $M_N(1,e,e^2,e^3)^t$ where

$$
M_C = \begin{pmatrix} t_0 & 0 & 0 \\ 0 & t_2 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad M_N = \begin{pmatrix} 0 & -t_1 & t_2 & 1 \\ 0 & -t_2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ t_0 & -t_1 & t_2 & 1 \end{pmatrix}.
$$
Hence the map $\phi$ is a projectivity. Extend $\phi$ in the natural way to a projectivity of $\operatorname{PG}(6, q^3)$. It is straightforward to verify that this projectivity maps $N_{-\tau}, N_{-\tau q}, N_{-\tau q^2}$ to $C_{-\tau}, C_{-\tau q}, C_{-\tau q^2}$ respectively. Thus the projectivity $\phi$ corresponds to the unique ruled surface with directrices $C$ and $N$ containing the transversals of $S$, that is $\phi = \eta$. □

We now complete the proof of Construction 1.

**Proof of Construction 1** In $\operatorname{PG}(2, q^3)$, let $S_T$ be a tangent splash of $\ell_\infty$ and let $\ell$ be a fixed order-$q$-subline such that $\ell$ meets $\ell_\infty$ in a point of $S_T \setminus \{T\}$. By Theorem 6.7, Steps 1, 2, 3 of Construction 1 give a ruled surface $V$ of $\operatorname{PG}(6, q)$ that corresponds to an order-$q$-subplane $\pi$ of $\operatorname{PG}(2, q^3)$. As $V$ contains the twisted cubic $N = [\ell]$, $\pi$ contains the order-$q$-subline $\ell$. It remains to show that $\pi$ has tangent splash $S_T$.

By Theorem 1.1, in $\operatorname{PG}(2, q^3)$, there is a unique order-$q$-subplane that has centre $T$, tangent splash $S_T$ and contains $\ell$, denote this unique subplane by $\mathcal{B}$. We will use coordinates to show that $\pi = \mathcal{B}$, by showing that in $\operatorname{PG}(6, q)$, $V = [\mathcal{B}]$.

As before, without loss of generality let $S_T = \{(a + b\tau, 1, 0) \mid a, b \in \operatorname{GF}(q)\} \cup \{T = (1, 0, 0)\}$ and $\ell = \ell_1$. So the unique order-$q$-subplane containing $\ell$ with splash $S_T$ is the order-$q$-subplane $\mathcal{B}$ coordinatised in Section 3. In Lemma 4.2, the coordinates of the conic directrix of $[\mathcal{B}]$ in $\operatorname{PG}(6, q)$ are calculated. The proof of Theorem 6.7 shows that $V$ has the same conic directrix. So $V$ and $[\mathcal{B}]$ are ruled surfaces with the same conic directrix and same twisted cubic directrix. By Theorem 2.3, in the cubic extension $\operatorname{PG}(6, q^3)$, $[\mathcal{B}]^*$ contains the three transversals of the regular spread $S$. By Theorem 6.7, $V^*$ also contains the three transversals of $S$. Since a projectivity is uniquely determined by the image of three points, both $V$ and $[\mathcal{B}]$ are determined by the same projectivity, and so $V = [\mathcal{B}]$. Hence in $\operatorname{PG}(2, q^3)$, $\mathcal{B}$ and $\pi$ are the same order-$q$-subplane, so the order-$q$-subplane $\pi$ constructed in Construction 1 is the unique order-$q$-subplane that has tangent splash $S_T$ and contains the order-$q$-subline $\ell$.

This completes the proof of Construction 1.

**7 Conclusion**

This paper concludes the study of tangent order-$q$-subplanes in $\operatorname{PG}(2, q^3)$ in the Bruck-Bose representation in $\operatorname{PG}(6, q)$. In [2], we investigated the order-$q$-sublines
and order-$q$-subplanes of $\text{PG}(2, q^3)$, and in particular showed that the tangent order-$q$-subplanes correspond to certain ruled surfaces in $\text{PG}(6, q)$. In [3], we characterised which ruled surfaces of $\text{PG}(6, q)$ correspond to tangent order-$q$-subplanes. The main results of [4] involved finding properties of the tangent splash of a tangent order-$q$-subplane. Further, we investigated the tangent space of a point of the ruled surface in $\text{PG}(6, q)$ representing a tangent order-$q$-subplane. In this paper, building on earlier results, we investigated properties of the tangent splash in the $\text{PG}(6, q)$ setting. Further, we gave a geometrical construction of a tangent order-$q$-subplane in $\text{PG}(6, q)$, beginning with a tangent splash and one appropriate order-$q$-subline.

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