Abstract. We propose domain decomposition preconditioners for the solution of the integral equation formulation of forward and inverse acoustic scattering problems with point scatterers. We study independently both problems and propose preconditioning techniques to accelerate the iterative solvers. In the forward scattering case, first, we extend to integral equations the domain decomposition based preconditioning techniques presented for partial differential equations in “A restricted additive Schwarz preconditioner for general sparse linear systems”, SIAM Journal on Scientific Computing, 21 (1999), pp. 792–797. Next, we propose a new preconditioner that is a low-rank correction of the domain decomposition based preconditioner. In the inverse scattering case, we use the low-rank corrected preconditioner proposed for the forward problem as the building block for constructing a preconditioner for the Gauss-Newton Hessian. Our numerical results show that both preconditioning strategies work well. In particular, for the inverse scattering problem, our preconditioner outperform low-rank approximations, which are the state of the art.

1. Introduction. There are diverse applications of inverse scattering in several fields of science such as medical imaging [4, 27, 30, 38, 40, 41, 43, 44, 45, 49], remote sensing [28, 50, 51], ocean acoustics [10, 14], nondestructive testing [5, 15, 17, 36, 35, 31], geophysics [2, 25, 46, 48, 56], and defense with radar and sonar [6, 13, 16]. Often, Newton-like methods are used for the solution of the inverse scattering problem. Those methods require several solutions of the forward scattering problem. More importantly, sometimes, due to the size of the problem being considered it is not possible to build the system of equations and solve it directly. Instead, we require the use of an iterative method to solve the system of linear equations, such as GMRES [42]. In this work, we present preconditioning strategies to speed-up the convergence of the iterative methods used in the solution of the forward and inverse scattering problems in their integral forms.

Problem statement: We consider the forward and inverse scattering problems in two dimensions, see Figure 1.1. Assume that \( q(x) \) represents a collection of point scatterers. We define the forward scattering operator \( F \) that maps \( q \) into the scattered field off of \( q \) by

\[
F(q; u^{inc}) = u^{scat},
\]

where \( u^{inc} \) is the incident field and \( u^{scat} \) is the scattered field off of \( q \). The operator \( F \) in (1.1) is well-defined since the forward scattering problem is well-posed [16]. To obtain the value of the scattered field, given the incident plane wave \( u^{inc} \), we have to solve the Lippmann-Schwinger Equation [16]. Considering that the domain is formed by \( N \) point scatterers, we must solve a dense linear system of equations, with an \( N \times N \) matrix on the left hand-side. Sometimes it is not feasible to construct the left-hand-side matrix to solve the system. In this case, we apply an iterative solver such as GMRES [42]. In case a fast method, such as the FMM [24], is not used to simulate the matrix vector multiplication of the forward problem operator, the computational complexity necessary for the convergence of the GMRES method to a set tolerance \( \epsilon_F \) is \( O(\kappa_F N^2) \), where \( \kappa_F \) is the number of iterations for the forward problem. One of the main interests of this paper is to study domain decomposition based preconditioning methods to accelerate the convergence of GMRES, decreasing the number of iterations \( \kappa_F \). (Henceforth, in the complexity analysis of the algorithms presented, we will not consider the use of fast methods.)

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Next, we solve the inverse scattering problem. In this problem, we find an approximation $\tilde{q}$ of the scatterer $q$ given measurements $d$ of the scattered field off of $q$ by solving

$$\tilde{q} = \arg \min_q \frac{1}{2} \left\| d - F(q) \right\|_{\partial B}^2,$$

where $d$ is the vector with components being the measurements of the scattered field $u_{\text{meas}}$ at receivers, $F(q)\big|_{\partial B}$ is the vector with values of the forward operator applied at $q$ evaluated at the receivers. This problem is nonlinear and ill-posed. We use the Gauss-Newton method for the nonlinearity and Tykhonov regularization for the ill-posedness. At each step of the Gauss-Newton iteration we solve the equation

$$H\delta q = J^* \left( d - F(q) \big|_{\partial B} \right), \quad (1.2)$$

where $H = J^*J + \beta I$ is the $N \times N$ Gauss-Newton Hessian matrix of the system, $J$ is the $N_dN_r \times N$ matrix representing the Fréchet derivative of the forward operator, $J^*$ is the adjoint of $J$, $N_q$ is the number of incident waves used to obtain measurements of the scattered field, $N_r$ is the number of receivers where the scattered field is measured and $\beta I$ is the regularization term. As with the forward scattering problem, constructing the matrix $H$ is expensive. To solve Equation (1.2) we use a matrix free iterative method. In each application of the Fréchet derivative, we need to solve $N_d$ forward scattering problems. Supposing that each of those problems takes approximately the same number of iterations $\kappa_F$ to converge, the total number of operations for the multiplication of $J$ by a vector is $O(\kappa_F N_d N_r N^2)$. The cost of solving (1.2) is going to be the cost of the application of the operator $H$ times the number of iterations $\kappa_I$ for the convergence of the iterative method, which is proportional to $\kappa_I \kappa_F N_d N^2$. The second main interest of this paper is to provide a preconditioner to speed-up the convergence to the solution of (1.2) using an iterative method. To construct the preconditioner for the inverse problem, we present a procedure that uses the preconditioners proposed in the first part of the paper for the forward problem.

**Notation:** We present the most common symbols used in this paper in Table 1.1.

**Methodology:** We present and compare preconditioning strategies to solve both the forward and inverse scattering problems.

1. For the forward problem, we apply and compare the domain decomposition based overlapping preconditioners presented in [8] to solve iteratively the Lippmann-Schwinger Equation for point scatterers. We use as preconditioners the Additive Schwarz (AS), the Restricted Additive Schwarz (RAS), the Additive Harmonic Schwarz (AHS) and the Symmetric Restricted Additive Schwarz (SRAS). We also propose a preconditioner that is obtained by applying a low rank correction to a domain decomposition preconditioner. We call this the RC preconditioner (rank-corrected preconditioner).

2. For the inverse problem, we use the RC preconditioner proposed for the forward problem to construct an approximation of the Gauss-Newton Hessian, which is then used to precondition Equation (1.2). We call this the HRC preconditioner.

A list of the numerical experiments with their respective descriptions and results is given in Table 1.2.

**Contributions:** We can summarize the contributions as follow:

- We extend and compare the preconditioners presented in article [8] for the integral equation formulation of the forward scattering problem for point scatterers;
Table 1.1: List of main symbols used in this article.

| Symbol | Description |
|--------|-------------|
| \(q\)  | Collection of point scatterers representing the medium |
| \(N\)  | Number of point scatterers |
| \(N_r\) | Number of receivers |
| \(N_{\lambda}\) | Number of singular values of the forward operator used for the RC preconditioner |
| \(N_s\) | Number of subdomains used in the partition of the domain |
| \(\theta\) | Incident direction of plane wave \(u^{inc}\) |
| \(k\) | Wavenumber (or frequency) of the incident plane wave |
| \(N_d\) | Number of incident waves |
| \(F\) | Forward scattering operator mapping \(q\) to the scattered field |
| \(G\) | Matrix from points in the domain to points in the domain |
| \(G_r\) | Matrix from points in the domain to receivers |
| \(I\) | Identity matrix |
| \(A\) | Matrix \(I + k^2 G Q\) of the system for the forward problem |
| \(J\) | Matrix of the Fréchet derivative of the forward operator \(F\) |
| \(H\) | Gauss-Newton Hessian for Equation (1.2) |
| \(\beta\) | Regularization parameter in Equation (1.2) |
| \(\partial B\) | Circle enclosing support of the scatterer where receivers are located |
| \(u^{inc}\) | Incident plane wave |
| \(u^{scat}\) | Scattered field off of \(q\) |
| \(u^{meas}\) | Scattered field off of \(q\) measured at the receivers |
| \(d\) | Vector with measurements of the scattered field on the receivers |
| \(\Omega\) | Full domain |
| \(\Omega_\delta\) | Subdomain with overlap \(\delta\) |
| \(R_i\) | Restriction operator for subdomain \(i\) with overlap \(\delta\) |
| \(A^{-1}\) | Preconditioner for the forward problem |
| \(H^{-1}\) | Preconditioner for the inverse problem |
| \(\kappa_F\) | Number of iterations for unpreconditioned GMRES for the forward problem |
| \(\kappa_I\) | Number of iterations for unpreconditioned GMRES for the inverse problem |
| \(e_{rel}\) | Relative error of the solution of the forward problem |

Table 1.2: List of numerical experiments with the respective Tables and Figures.

| Experiment | Description | Tables | Figures |
|------------|-------------|--------|---------|
| F.1        | GMRES performance using DD preconditioning | 3.3, B.2 | B.1, B.2 |
| F.2        | Comparison of the geometries for the partition | 3.4, B.3 | X       |
| F.3        | Multiple scattering effects | 3.5, B.4 | X       |
| F.4        | Scalability of the DD preconditioning | 3.6, B.5 | X       |
| F.5        | RC preconditioner performance | 3.7, B.6 | X       |
| I.1        | Influence of the size of the overlapping | 4.2 | 4.1, B.3 |
| I.2        | Influence of the number of subdomains | 4.2 | 4.2, B.4 |
| I.3        | Scalability of the inverse problem preconditioning | 4.3, B.7, B.8 | X |
| I.4        | Comparison with low-rank preconditioner | X | 4.3, B.5, B.6 |
| I.5        | Full inverse problem | 4.4 | 4.5, 4.6 |
(a) Forward scattering problem

Ω

supp(q)

(b) Inverse scattering problem

Figure 1.1: Scattering from a compactly supported inhomogeneity with scatterer \( q(x) \). In the forward scattering problem, \( q(x) \) is known and one seeks to compute the scattered field given the incident field, either within \( \Omega \) or on the boundary \( \partial \Omega \) of a disk, see Figure 1.1a. In the inverse scattering problem, \( q(x) \) is unknown and we seek to determine it from measurements of the scattered field at the receivers located on \( \partial \Omega \), see Figure 1.1b.

- We propose the RC preconditioner for the forward problem obtained by a low-rank update of the domain decomposition preconditioner for the forward problem;
- We introduce a robust novel preconditioner strategy for accelerating the solution of the inverse scattering problem using the RC preconditioner; and
- We use the preconditioners developed in the solution of a multifrequency full aperture inverse scattering problem.

Our key contribution is the preconditioner for the Gauss-Newton Hessian operator. The state of the art are low rank preconditioners. However, in our case, the Hessian’s rank can be high and our preconditioner scales much better.

Limitations: Our approach is subject to the following limitations:
- We do not have theory that connects the overlap of the partition of the domain to the final error of the iterative method;
- For the forward problem, it is not clear why the RAS and AHS behave better than the AS.
- Also, for the forward problem, the choice of the number of singular values used for the low-rank correction of the domain decomposition preconditioner to find the RC preconditioner is found computationally. The examples show that the number of singular values should increase with increasing frequency and decreasing overlapping, however in most cases we chose the parameter empirically.
- For the inverse problem, since the RC preconditioner is used, the number of singular values to obtain the preconditioner is chosen empirically.

Related work: The topic of domain decomposition for the solution of the partial differential equation version of the forward problem, the Helmholtz Equation, has been extensively studied, see [3, 8, 12, 18, 19, 20, 21, 22, 23, 29, 37, 47, 53, 54]. There is less work on the study of preconditioners in the solution of the integral equation form of the forward problem [1, 7, 39, 52, 55]. For the inverse scattering problem, some authors use spectral strategies to speed-up the solution of the Gauss-Newton iteration, see [6, 9, 26, 33, 34]. We are not aware of any work that uses the tradi-
tional domain decomposition preconditioners to generate a preconditioner. In our work, to make
the simulations simpler, we make some assumptions to simplify the problem, such as considering
the problem in two dimensions and that our domain of integration is composed of point charges
uniformly distributed without self-interactions. None of those assumptions, however, impose any
limitations on the applicability of this method in more complex settings, such as problems in three
dimensions or in problems with scatterers represented by continuous functions.

**Article Outline:** In Section 2, we describe the forward and inverse scattering problem for-
mulations and how to obtain their numerical solutions. In Section 3, we give a brief introduction
about domain decomposition techniques used in this article, we present and compare the domain
decomposition preconditioners used for the forward problem, and we present the low rank correction
procedure to obtain the RC preconditioner. In Section 4, we present our preconditioning strategy
for the inverse scattering problem. Concluding remarks are made in Section 5.

2. The forward and inverse scattering problem. In this section, we present the formula-
tion of both the forward and inverse scattering problems with the assumptions used to simplify those
problems. We follow by presenting the discrete system that we want to solve for both problems.

2.1. Forward scattering problem. The operator $F$ in (1.1) is well-defined since the forward
scattering problem is well-posed [16]. To find the value of the scattered field we solve the equation
\[
\Delta u^{\text{scat}} + k^2 (1 + q) u^{\text{scat}} = -k^2 q u^{\text{inc}}
\]  
where $u^{\text{scat}}$ satisfies the Sommerfeld radiation condition, $u^{\text{inc}}$ is an incoming incident plane wave,
and $k$ is the wavenumber.

Using Green’s identity and the Sommerfeld radiation condition, we obtain the integral form of
Equation (2.1) which is
\[
\begin{aligned}
  u^{\text{scat}}(x) + k^2 \int_{\mathbb{R}^2} G(x - y) q(y) u^{\text{scat}}(y) dy = & -k^2 \int_{\mathbb{R}^2} G(x - y) q(y) u^{\text{inc}}(y) dy,
\end{aligned}
\]
the Lippman-Schwinger Equation.

We assume that $q(y)$ is a set of point scatterers distributed in a regular $\sqrt{N} \times \sqrt{N}$ grid of $x_i$
points in the square $[-0.5, 0.5]^2$, so that
\[
q(x) = \sum_{i=1}^{N} q_i \delta(x - x_i),
\]  
where $q_i$ is the charge of the point scatterer located at $x_i$.

Using the domain definition (2.2) and ignoring self iterations, we calculate the scattered field
on the points $x_i$ of the grid by solving
\[
(I + k^2 G Q) u^{\text{scat}} = -k^2 G Q u^{\text{inc}},
\]  
where $G$ is the $N \times N$ matrix with elements $(G)_{ij} = G(k \|x_i - x_j\|)$ when $i \neq j$ and $(G)_{ii} = 0$,
for $i, j = 1, \ldots, N$, $I$ is the $N \times N$ identity matrix, $Q$ is the $N \times N$ diagonal matrix with diagonal
elements $(Q)_{ii} = q_i$, $u^{\text{inc}}$ is a vector with coordinates $u^{\text{inc}}(x_i)$ and the solution vector
$u^{\text{scat}}$ is such that for each coordinate we have $(u^{\text{scat}})_i = u^{\text{scat}}(x_i)$. To simplify the notation, we
denote the matrix of the system for the forward problem $A = (I + k^2 G Q)$. 
Once the value of the scattered field is obtained at the scatterer points positions, the scattered field can be calculated at any point of the domain by solving

$$F(q)_{\partial\Omega} = u^{\text{meas}} = -k^2 G_r Q u^{\text{inc}} - k^2 G_r Q u^{\text{scat}}, \quad (2.4)$$

where $G_r$ is the $N_r \times N$ matrix with elements $(G_r)_{ij} = G(k\|y_i - x_j\|)$, where $y_i$ are the coordinates of the $N_r$ receivers, $u^{\text{meas}}$ is a vector with coordinates $(u^{\text{meas}})_i = u^{\text{meas}}(x_i)$, and $u^{\text{meas}}(x_i)$ is the measured scattered field at the receivers.

### 2.2. Inverse scattering problem.

In the inverse scattering problem, given scattered data off of an unknown scatterer $q$, we want to find an approximation $\tilde{q}$ of the scatterer such that

$$\tilde{q} = \arg \min_q \frac{1}{2} \left\| d - F(q)_{\partial\Omega} \right\|^2, \quad (2.5)$$

where $d$ is the vector with components being the measurements of the scattered field at the receivers.

To solve problem (2.5) we use the Gauss-Newton method. In this method, we start from an initial guess $q(0)$, and update the solution at each step making $q^{(i+1)} = q^{(i)} + \delta q$, for $i = 0, \ldots$, where $\delta q$ is obtained by solving

$$J \delta q = \left( d - F(q^{(i)})_{\partial\Omega} \right). \quad (2.6)$$

The matrix $J$ is the discrete version of the Fréchet derivative of $F$. Since, in most cases the number of measurements is higher than the number of scatterer points, the system (2.6) is overdetermined. From the application of perturbation analysis in Equation (2.4) we obtain the matrix for the Fréchet derivative

$$J = -k^2 G_r U^{\text{tot}} + k^4 G_r Q A^{-1} G U^{\text{tot}},$$

where $U^{\text{tot}}$ is the $N \times N$ diagonal matrix with diagonal elements $(U^{\text{tot}})_{i,i} = u^{\text{inc}}(x_i) + u^{\text{scat}}(x_i)$.

In each iteration, the Thykonov regularized Gauss-Newton step becomes

$$H \delta q = J^* \left( d - F(q^{(i)})_{\partial\Omega} \right),$$

where $H = (J^* J + \beta I)$ is the Hessian of the problem, $\beta$ is the regularization parameter and $J^*$ is the adjoint of $J$. The Gauss-Newton method is summarized in Algorithm 1.

### 3. Preconditioning of the forward problem.

In some cases, it is expensive to assemble the matrix $A$ and solve (2.3) using a direct method. In these cases, the solution of (2.3) is obtained by using an iterative method such as GMRES which takes $O(\kappa F N^2)$ operations to converge. To speed-up the GMRES convergence for the solution of the forward problem, we apply a domain decomposition based preconditioning strategy.

#### 3.1. Domain decomposition preliminaries.

Consider that $q$ is composed of $N$ point scatterers distributed in a $\sqrt{N} \times \sqrt{N}$ regular grid in the domain $\Omega = [-.5,.5]^2$, where $N$ is a perfect square integer. We partition $\Omega$ into $N_s$ nonoverlapping subdomains. Without loss of generality we
Algorithm 1 Gauss-Newton method for the inverse scattering problem.

1: **Input:** data \( d \), initial guess \( q_0 \), tolerances \( \epsilon_1, \epsilon_2 \), and maximum number of iterations \( N_{it} \).
2: Set \( q := q_0 \), \( \delta q := 0 \), and \( it := 0 \).
3: **while** \( \|d - F(q)\| \geq \epsilon_1 \) or \( it < N_{it} \) or \( \delta q \geq \epsilon_2 \) **do**
4: Solve \( Au_{\text{scat}} = -k^2 G Qu_{\text{inc}} \) using GMRES
5: Calculate \( F(q)_{|\partial B} = -k^2 G_r Qu_{\text{inc}} - k^2 G_r Qu_{\text{scat}} \)
6: Solve \( H \delta q = J^* \left( \frac{d - F(q)}{|\partial B} \right) \) using GMRES
7: Update \( q \leftarrow q + \delta q \)
8: Update \( it \leftarrow it + 1 \)
9: **end while**
10: The approximate solution is \( \tilde{q} := q \).

consider that the domain is partitioned in a perfect square number of same size squares as in Figure 3.1. We have

\[
\Omega = \bigcup_{i=1}^{N_s} \Omega_i^0.
\]

We define the overlapping partition of \( \Omega \), as follows. Let \( \Omega_i^\delta \) be the overlapping partition of \( \Omega_i^0 \), where \( \Omega_i^\delta \supset \Omega_i^0 \) is obtained by increasing the size of \( \Omega_i^0 \) by \( \delta \), where \( \delta \) is a measure of size of the overlap of the domains (it can be a percentage of the size of \( \Omega_i^0 \) or it can be the number of points in \( \Omega_i^\delta \) not in \( \Omega_i^0 \)). With this definition, we have:

\[
\Omega = \bigcup_{i=1}^{N_s} \Omega_i^\delta.
\]

Associated with each \( \Omega_i^0 \), we define a restriction operator \( R_i^0 \). In matrix terms, \( R_i^0 \) is a matrix whose diagonal elements are set to one if the corresponding point belongs to \( \Omega_i^0 \) and to zero otherwise. We define \( R_i^\delta \) in an analogous way. From the definition, we have

\[
\sum_{i=1}^{N_s} R_i^0(j,j) = 1
\]

for \( j = 1, \cdots, N \), where \( R_i^0(j,j) \) is the \( j \)th diagonal element of \( R_i^0 \), and

\[
\sum_{i=1}^{N_s} R_i^\delta(j,j) = m
\]

for \( j = 1, \cdots, N \), where \( R_i^\delta(j,j) \) is the \( j \)th diagonal element of \( R_i^\delta \) and \( m \) is the number of subdomains that contain the equivalent \( j \)th scatterer.

3.2. Domain decomposition preconditioning of the forward problem. To simplify the notation, we denote \( u = u^{\text{scat}} \) and \( b = -k^2 G Qu^{\text{inc}} \). We want to speed-up the convergence of GMRES to solve the system

\[
Au = b.
\]
Using a domain decomposition based preconditioner, the system to be solved becomes
\[ \tilde{A}^{-1}Au = \tilde{A}^{-1}b, \]
where \( \tilde{A}^{-1} \) is the proposed preconditioner.

We start by defining the matrix
\[ A_{ii} = R_i^\delta AR_i^\delta. \]
Table 3.1: List of preconditioners.

| Method name                                    | Abbreviation | Preconditioner                      |
|------------------------------------------------|--------------|-------------------------------------|
| Additive Schwarz                               | AS           | $\hat{A}_{AS}^{-1} = \sum_{i=1}^{N_s} (R_i^δ)^\top A_{ii}^{-1} R_i^δ$ |
| Restricted Additive Schwarz                    | RAS          | $\hat{A}_{RAS}^{-1} = \sum_{i=1}^{N_s} (R_0^i)^\top A_{ii}^{-1} R_0^i$ |
| Additive Harmonic Schwarz                      | AHS          | $\hat{A}_{AHS}^{-1} = \sum_{i=1}^{N_s} (R_δ^i)^\top A_{ii}^{-1} R_δ^i$ |
| Symmetrized Restricted Additive Schwarz        | SRAS         | $\hat{A}_{SRAS}^{-1} = \sum_{i=1}^{N_s} (R_0^i)^\top A_{ii}^{-1} R_0^i$ |

Note that although $A_{ii}$ is not invertible, we can invert its restriction to the subspace

$$A_{ii}^{-1} = \left( A_{ii} \bigg| L_i \right)^{-1},$$

where $L_i$ is the vector space spanned by the scatterers points in the domain $\Omega^δ_i$ in $\mathbb{R}^N$.

In [8], the authors proposed several variants of the Additive Schwarz preconditioners and compared these variants for systems of equations obtained from using the finite element method to solve the Helmholtz Equation. In this subsection, we apply all the non weighted methods listed in [8] to a system of equations obtained from the Lippman-Schwinger equation applied in point scatterers. All the methods proposed and compared in [8] are listed in Table (3.1) with their respective preconditioners.

**Remark 3.1.** It is also possible to construct weighted versions of the preconditioners in Table 3.1. In that case, we use the weighted restriction operator $R_{i, w}^δ$. This operator is defined such that

$$\sum_{i=1}^{N_s} R_{i, w}^δ (j, j) = 1$$

for $j = 1, \cdots, N$. The value of $R_{i, w}^δ$ at each point is going to be obtained by a linear interpolation, meaning that points closer to the boundary of the domain $\Omega^δ_i$ will have values closer to zero.

**Complexity analysis:** If $N$ is the total number of scatterers, the solution of the system (3.1) using a direct method like LU is $O(N^3)$. If we use an iterative method, such as GMRES, the complexity becomes $O(\kappa_F N^2)$, where $\kappa_F$ is the number of GMRES iterations. Applying the domain decomposition preconditioner with $N_s$ subdomains and considering that $\delta \ll N$, the complexity becomes

- Construction of the preconditioner: $O(N^3/N_s^2)$; and
- GMRES solve: $O(\hat{\kappa}_F (N^2/N_s + N^2))$,

where $\hat{\kappa}_F$ are the number of iterations for the preconditioned GMRES. Even though this numbers show a reasonable improvement when $\hat{\kappa}_F$ is much smaller than $\kappa_F$, we should take into consideration that domain decomposition preconditioners are extremely parallelizable, which means that the matrix in each subdomain can be inverted independently in a different core. In this case, the first term of the computational cost of the construction of the preconditioner becomes $O((N)^3/N_s^2)$. Note that several accelerators can be applied to speed-up the matrix-vector multiplication such as the FMM [24].
3.3. Rank correction of the preconditioner. With the intent to improve further the speed of convergence of the iterative method, we create a new preconditioner by adding to the domain decomposition based preconditioner a low-rank correction, calling it the RC preconditioner. In this subsection, we use $\tilde{A}^{-1}$ for the domain decomposition preconditioner and $\tilde{A}_{RC}^{-1}$ for the RC preconditioner.

To obtain the RC preconditioner, first, we calculate the singular value decomposition $USV^* = (F - \tilde{A})$. Next, we set the submatrices $U_{N_\lambda} = U(:,1:N_\lambda)$, $S_{N_\lambda} = S(1:N_\lambda,1:N_\lambda)$ and $V_{N_\lambda} = V(:,1:N_\lambda)$ and construct

$$\tilde{A}^{-1}_{RC} = (\tilde{A} + U_{N_\lambda}S_{N_\lambda}V_{N_\lambda}^*)^{-1}. \quad (3.2)$$

We can use the Woodbury formula to calculate $\tilde{A}^{-1}_{RC}$ and obtain

$$\tilde{A}^{-1}_{RC} = \tilde{A}^{-1} + \tilde{A}^{-1}U_{N_\lambda}(S_{N_\lambda}^{-1} - V_{N_\lambda}\tilde{A}^{-1}U_{N_\lambda})^{-1}V_{N_\lambda}\tilde{A}^{-1}. \quad (3.3)$$

The matrix $\tilde{A}^{-1}_{RC}$ can be used as a preconditioner for the solution of Equation (3.1) or even, depending on the number of singular values used in the rank correction, as an approximation to $A^{-1}$.

**Complexity analysis:** The complexity of this scheme is the same as the complexity of building the domain decomposition based preconditioner plus the complexity of obtaining the low rank correction. The complexity of obtaining the low-rank correction becomes

- Application of randomized SVD to obtain $N_\lambda$ largest singular values and associated singular vectors: $O(N^2 \log(N_\lambda))$ using randomized algorithms, or $O(N^2 N_\lambda)$ using classical algorithms; and
- Application of the Woodbury formula: $O(N_\lambda^3 + NN_\lambda + N^2)$.

3.4. Numerical Experiments. We present experiments that verify the effectiveness of the domain decomposition preconditioners in different scenarios for the forward scattering problem. Initially, we compare GMRES without preconditioner with the preconditioned GMRES using the domain decomposition strategies. We intend to show the effect of the number of subdomains and overlapping on the preconditioners as well as, the use of a different geometry for the partitioning of the domain, and the effect of multiple scattering on the methods. Next, we present an experiment that shows the scalability of the domain decomposition based preconditioners as the number of points scatterers increases. Finally, after we choose the domain decomposition preconditioner with the best performance, we use this preconditioner as the basis for the RC preconditioner. We finish this section by presenting an experiment showing the performance of the RC preconditioner for different values of $N_\lambda$. A list of the experiments for the forward problem with their descriptions and results is provided in Table 4.1.

For this section we define the relative error of the iterative solution with respect to the direct method solution $e_{rel} := ||u_{GMRES} - u_{LU}||/||u_{LU}||$, where $u_{GMRES}$ is the solution obtained by GMRES and $u_{LU}$ is the solution obtained by the LU direct solver. The LU solution is obtained by solving (3.1) using the backslash in MATLAB.

In this section, we concentrate in showing the number of iterations used for convergence of the methods. We present supplemental results for the problems in Appendix B.

In all the experiments we used the following parameters for the MATLAB GMRES iterative solver...
Table 3.2: List of forward problem experiments.

| Experiment | Description | Tables | Figures |
|------------|-------------|--------|---------|
| F.1        | GMRES performance using DD preconditioning | 3.3, B.2 | B.1,B.2 |
| F.2        | Comparison of the geometries for the partition | 3.4, B.3 | X       |
| F.3        | Multiple scattering effects | 3.5, B.4 | X       |
| F.4        | Scalability of the DD preconditioning | 3.6, B.5 | X       |
| F.5        | Comparison of DD and RC preconditioners | 3.7, B.6 | X       |

- GMRES configuration with preconditioners: tolerance of $10^{-13}$, no restart, and maximum number of iterations $N - 1$.  
- GMRES configuration with no preconditioners: tolerance of $10^{-11}$, no restart, and maximum number of iterations $N - 1$.

**Experiment F.1 – GMRES performance using DD preconditioning:** This example aims to compare the performance of the preconditioners presented in Table 3.1 to solve Equation (3.1). We analyze the effects of the number of subdomains and size of the overlap at different wavenumbers. The following parameters are used:

- **Incoming wave:** the incoming wave is given by $u^{inc}(x, y) = \exp(ikx)$ with $k/(2\pi) = 5$ and 20;
- **Scatterer:** we use a regular grid of $64^2$ point scatterers and their magnitudes are given by the equation

$$q_4(x = (x, y)) := \begin{cases} 0.1, & \text{if } \cos^2(2\pi x) + \cos^2(2\pi y) > 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

A plot of $q_4$ can be seen in Figure 3.2.

- **Domain decomposition:** the number of subdomains is $N_s = 4, 16$ and 64, and the overlap parameter is $\delta = 1$ and 6;

As a matter of illustration, we present in Figure 3.3 the plot of the real part of the scattered field off of $q_4$ when the incident plane wave has incidence direction $\theta = (1, 0)$ and wavenumber $k/(2\pi) = 5, 10, 20$ and 40.

In Table 3.3 we present the number of iterations necessary for the convergence of GMRES with no preconditioners and with the preconditioners in Table 3.1. In Table B.2 of Appendix B, we present the relative error $e_{rel}$ of the solution using GMRES with no preconditioners and with the domain decomposition based preconditioners.

In Figure (B.1) of Appendix B, we plot the eigenvalues of (a) $A$, (b) $\tilde{A}_{AS}^{-1}A$, (c) $\tilde{A}_{RAS}^{-1}A$ and (d) $\tilde{A}_{SRAS}^{-1}A$ when $N_s = 16$ and $\delta = 6$. The eigenvalues for $\tilde{A}_{RAS}^{-1}A$ are more clustered than the eigenvalues for $A$, $\tilde{A}_{AS}^{-1}A$ and $\tilde{A}_{SRAS}^{-1}A$, justifying its best performance. In Figure (B.2) in Appendix B, we present the eigenvalues of $\tilde{A}_{RAS}^{-1}A$ with (a) $N_s = 4$ and $\delta = 1$, (b) $N_s = 4$ and $\delta = 6$, (c) $N_s = 64$ and $\delta = 1$ and (d) $N_s = 64$ and $\delta = 6$. Note that the eigenvalues are more clustered for smaller $N_s$ and larger $\delta$.

**Summary:** the preconditioning schemes applied to the integral equation work similarly to when applied to the partial differential equation. As with the PDEs, the RAS and AHS have very similar performance and both work better than the AS and SRAS. Regarding the size of the overlap parameter, as expected the smaller $\delta$ translates into worse performance. Regarding the number of
subdomains, even though the increase of subdomains decreases the computational time for the construction of the preconditioners, it also precludes an increase in the number of iterations for the convergence of the method.

**Experiment F.2 – Comparison of the geometries for the partition:** In this example we compare the choice of the geometry of the partition. We use two different geometries. In the first choice, called $G_1$, the domain is subdivided into square domains of same size. In the second choice, called $G_2$, the domain is divided in vertical bands. The following parameters are used:

- Incoming wave: the incoming wave is given by $u^{inc}(x = (x, y)) = \exp(ikx)$ with $k/(2\pi) = 10$ and 40;
- Scatterer: we use a regular grid of 64$^2$ point scatterers. The scatterers magnitudes are given by $q_4$;
- Domain decomposition: the number of subdomains is $N_s = 4, 9$ and 16; and the overlap parameter is $\delta = 1$ and 3;

In Table 3.4 we present the number of iterations necessary for the convergence of GMRES with no preconditioner and using the AS, RAS and AHS preconditioners when using the two geometries for the partition of the domain. In Table B.3 of Appendix B, we present the relative error $e_{rel}$.

**Summary:** for this case in particular, we were not able to experience any difference in performance between the two geometries used when both domains have almost the same amount of points. The methods behave similarly as in the previous example, with the RAS and AHS being faster and presenting similar results.

**Experiment F.3 – Multiple scattering effects:** In this example we compare the results for two different scatterers $q_4$ and $q_{16}$. The scatterer $q_{16}$ (Figure 3.4) is given by the formula

$$q_{16}(x = (x, y)) := \begin{cases} 0.1, & \text{if } \cos^2(4\pi x) + \cos^2(4\pi y) > 0.5; \\
0, & \text{otherwise.} \end{cases}$$

The following parameters are used:
Figure 3.3: The real part of the scattered field off of $q_4$ when the incident plane wave has incidence direction $\theta = (1,0)$ and wavenumber: (a) $k/(2\pi) = 5$, (b) $k/(2\pi) = 10$, (c) $k/(2\pi) = 20$ and (d) $k/(2\pi) = 40$.

- Incoming wave: the incoming wave is given by $u^{inc}(x=(x,y)) = \exp(ikx)$ with $k/(2\pi) = 10$ and 40;
- Scatterers: we use a regular grid of $128^2$ scatterer points. The scatterers magnitudes are given by $q_4$ and $q_{16}$;
- Domain decomposition: the number of subdomains is $N_s = 4$ and 16, and the overlap parameter is $\delta = 1$ and 16.

In Figure 3.5, we show the plot of the real part of the scattered field off of $q_{16}$ when the incident plane wave has incidence direction $\theta = (1,0)$ and wavenumber $k/(2\pi) = 10$ and 40.

In Table 3.5, we present the number of iterations necessary for the convergence of GMRES with no preconditioner and using the AS, RAS and AHS preconditioners when the scatterer is $q_4$ and $q_{16}$. In Table B.4, we present the relative error of the solution $e_{rel}$.

**Summary:** as expected, due to the increasing of the effect of multiple scattering, the number of iterations necessary for GMRES to converge is higher for the domain $q_{16}$. The RAS and AHS
Table 3.3: (Experiment F.1) We present the number of iterations necessary for the convergence of GMRES with and without using domain decomposition preconditioning strategies. The incoming plane wave has horizontal direction of propagation and wavenumbers \( k/(2\pi) = 5 \) and 20. We use a regular grid of \( N = 64^2 \) point scatterers with magnitude given by the function \( q_4 \). The number of subdomains used is \( N_s = 4, 16, \) and 64 and the overlap parameter is \( \delta = 1 \) and 6.

| \( k/2\pi \) | GMRES | \( N_s \) | overlap | AS | RAS | AHS | SRAS |
|-----------|-------|---------|---------|----|-----|-----|------|
| 5         | 97    | 4       | 1       | 18 | 16  | 15  | 16   |
|           |       | 6       | 17      | 15 | 15  | 15  | 15   |
|           |       | 16      | 54      | 44 | 44  | 69  | 69   |
|           |       | 6       | 30      | 17 | 17  | 74  | 74   |
|           |       | 64      | 71      | 44 | 44  | 68  | 68   |
|           |       | 6       | 46      | 18 | 18  | 74  | 74   |
| 20        | 293   | 4       | 43      | 42 | 42  | 42  | 42   |
|           |       | 6       | 43      | 41 | 41  | 41  | 41   |
|           |       | 16      | 104     | 74 | 74  | 130 | 130  |
|           |       | 6       | 62      | 42 | 42  | 143 | 143  |
|           |       | 64      | 138     | 76 | 76  | 129 | 129  |
|           |       | 6       | 97      | 41 | 42  | 143 | 143  |

Figure 3.4: The (a) isometric view and (b) top view of the domain \( q_{16} \) used in the Experiment F.3.

still converge faster than GMRES without preconditioners and with the AS preconditioner.

Experiment F.4 – Scalability of the DD preconditioning: This example aims to check the scalability of the methods presented in Table 3.1. We fix the number of subdomains and increase the number of points in the grid. The following parameters are used:

- Incoming wave: the incoming wave is given by \( u^{inc}(x = (x,y)) = \exp(ikx) \) with \( k/(2\pi) = \)
Table 3.4: (Experiment F.2) We present the number of iterations necessary for the convergence of GMRES with no preconditioner and with the AS, RAS and AHS preconditioning strategies. The incoming plane waves have wavenumbers $k/(2\pi) = 10$ and $40$. We use a regular grid of $N = 64^2$ point scatterers with magnitude given by the function $q_4$. $G_1$ represents the partition composed of equal sized squares and $G_2$ is the partition composed of vertical bands. The number of subdomains used is $N_s = 4, 9, 16$ and the overlap parameter is $\delta = 1$ and $3$.

Table 3.5: (Experiment F.3) We present the number of iterations necessary for the convergence of GMRES with no preconditioner and with the AS, RAS and AHS preconditioners. The incoming plane waves have wavenumbers $k/(2\pi) = 10$ and $40$. We use $N = 64^2$ scatterers points with magnitude given by the functions $q_4$ and $q_{16}$. The number of subdomains used is $N_s = 4$ and $16$, and the overlap parameter is $\delta = 1$ and $16$. 

| $k/2\pi$ | GMRES | $N_s$ | $\delta$ | AS | RAS | AHS |
|---|---|---|---|---|---|---|
| 10 | 164 | 4 | 1 | 26 48 25 40 25 41 | 3 | 26 34 24 26 25 26 | 9 | 1 | 42 59 35 40 35 40 | 3 | 36 60 24 27 24 27 | 16 | 1 | 72 99 57 63 58 63 | 3 | 60 60 32 36 32 36 |
| 40 | 398 | 4 | 1 | 71 152 67 136 67 138 | 3 | 71 85 67 68 67 68 | 9 | 1 | 123 176 125 135 124 139 | 3 | 92 155 77 79 77 79 | 16 | 1 | 274 269 240 225 239 226 | 3 | 147 146 99 124 100 124 |

| $k/2\pi$ | GMRES | $N_s$ | $\delta$ | AS | RAS | AHS |
|---|---|---|---|---|---|---|
| 10 | 211 | 4 | 1 | 23 45 21 42 21 42 | 16 | 23 34 21 22 21 22 | 1 | 69 81 60 77 60 77 | 16 | 23 85 21 41 21 41 | 16 | 23 85 21 41 21 41 | 16 | 134 279 100 273 100 271 | 16 | 65 207 61 152 61 152 |
Figure 3.5: The real part of the scattered field off of $q_{16}$ when the incident plane wave has incidence direction $\theta = (1, 0)$ and wavenumber: (a) $k/(2\pi) = 10$ and (b) $k/(2\pi) = 40$.

10, 20 and 40:
- Scatterer: we use regular grids of $N = 64^2$, $128^2$ and $256^2$ scatterer points. The scatterer points magnitudes are given by $q_4$;
- Domain decomposition: the number of subdomains is $N_s = 16$, we use the overlap parameter $\delta = 3$ for the grid with $64^2$ scatterer points, $\delta = 6$ for $128^2$ scatterer points and $\delta = 12$ for $256^2$ scatterer points;

In Table 3.6, we present the number of iterations for the convergence of GMRES with no preconditioner and with the preconditioners AS, RAS and AHS. We present the relative error of the solution $e_{rel}$ in Table B.5 on Appendix B.

**Summary:** the results show that to have the same accuracy, when keeping the number of subdomains constant, we need to increase the overlap parameter as the number of scatterers in the grid increases. We can conclude that the methods are fully scalable for increasing number of scatterers in the domain.

**Experiment F.5 – RC preconditioner performance:** This example aims to show the performance of the RC preconditioner to solve Equation (3.1) when the correction is applied in the RAS preconditioner. We analyze the effects of the number of subdomains, size of the overlap and number of singular values used for the correction at different wavenumbers. The following parameters are used:
- Incoming wave: the incoming wave is given by $u^{inc}(x = (x,y)) = \exp(i k x)$ with $k/(2\pi) = 5$ and 20;
- Scatterer: we use a regular grid of $64^2$ scatterer points. The scatterer points magnitudes are given by $q_4$;
- Domain decomposition: number of subdomains is $N_s = 4$ and 16, and the overlap parameter is $\delta = 1$ and 8;
- RC preconditioner: we use the RAS preconditioner $\hat{A}^{-1}_{\text{RAS}}$ to construct the preconditioner $\hat{A}^{-1}_{\text{RC}}$. We choose $N_\lambda = 20$, 40, 60 and 80 for wavenumber $k/(2\pi) = 5$ and $N_\lambda = 40$, 80, 120 and 160 for wavenumber $k/(2\pi) = 20$. 


(a) Iterations

| k/2\pi | \sqrt{N} | GMRES | \delta | AS | RAS | AHS | SRAS |
|--------|----------|-------|--------|----|-----|-----|------|
| 10     | 64       | 164   | 3      | 60 | 32  | 32  | 105  |
|        | 128      | 211   | 6      | 55 | 24  | 24  | 116  |
|        | 256      | 277   | 12     | 52 | 21  | 21  | 121  |
| 20     | 64       | 293   | 3      | 78 | 45  | 45  | 143  |
|        | 128      | 413   | 6      | 71 | 31  | 31  | 152  |
|        | 256      | 526   | 12     | 65 | 27  | 27  | 169  |
| 40     | 64       | 398   | 3      | 147| 99  | 100 | 286  |
|        | 128      | 674   | 6      | 112| 62  | 61  | 200  |
|        | 256      | 957   | 12     | 101| 45  | 45  | 230  |

Table 3.6: (Experiment F.4) We present the number of iterations for the convergence of the GMRES with no preconditioner and with the domain decomposition preconditioners in Table 3.1. The incoming plane wave has horizontal direction of propagation and frequencies \( k/(2\pi) = 10, 20 \) and 40. We use regular grids of \( N = 64^2, 128^2 \) and \( 256^2 \) point scatterers with magnitude given by the function \( q_4 \). The number of subdomains used is \( N_s = 16 \) and the overlapping parameter is \( \delta = 3, 6 \) and 12 for \( N = 64^2, N = 128^2 \) and \( N = 256^2 \) respectively.

In Table 3.7 we present the number of iterations for the convergence of GMRES with no preconditioner, using the RAS preconditioner and using the RC preconditioner with different numbers of singular values for the correction. In Table B.6 in Appendix B, we present the relative error of the solution \( e_{rel} \).

**Summary:** the number of singular values used for the construction of the RC preconditioner is dependent on the number of subdomains and overlap used for the RAS preconditioner and it also dependent on the wavenumber of the incoming wave. From the results, we have that as the wavenumber of the incoming wave increases, more singular values are necessary to obtain better precision and less iterations using the RC preconditioner.

3.5. **Conclusion preconditioners for the forward scattering problem.** The preconditioning methods tested in this section provide improvements on the speed-up of convergence of the standard GMRES approach decreasing its computational cost. The RAS and AHS outperform the AS and SRAS in all of our examples. As expected, the number of iterations decreases when the partition overlap is larger, and increases if the number of subdomains increases too much. The methods presented are easily translated to higher dimensions and they are scalable with the increase of the number of scatterers.

4. **Preconditioning of the inverse problem.** To obtain the update of the domain at each step of the Gauss-Newton method, we must solve Equation (1.2) using an iterative method such as GMRES. Each time we multiply the matrix \( \mathbf{J} \) by a vector, we need to use GMRES to solve a system with left-hand-side matrix \( \mathbf{A} \) which costs \( \mathcal{O}(\kappa_F N^2) \) operations. The total cost of of multiplying a vector by \( \mathbf{J} \) is \( C_{\mathbf{J}} = \mathcal{O}(N_d N_r N + N_d \kappa_F N^2) \). Since \( \mathbf{H} = \mathbf{J}^\ast \mathbf{J} + \beta \mathbf{I} \), the cost of applying \( \mathbf{H} \) is \( C_{\mathbf{H}} = C_{\mathbf{J}} + C_{\mathbf{J}^\ast} \), where the cost of applying \( C_{\mathbf{J}^\ast} \) is the cost of applying the adjoint of \( \mathbf{J} \), which is very similar to the cost \( C_{\mathbf{J}} \). Finally, the total number of operations to solve the system obtained by
the Gauss-Newton method is $O(\kappa_I C_H)$, where $\kappa_I$ is the number of iterations necessary for GMRES to converge to a prescribed tolerance. With the intent of decreasing the number of iterations $\kappa_I$ for the solution of (1.2), we propose a strategy to construct a preconditioner using the forward problem RC preconditioner $\tilde{A}^{-1}_{RC}$.

### 4.1. Preconditioning using an approximation of the inverse forward operator.
At each iteration, the system that needs to be solved is:

$$
\mathbf{H}\delta \mathbf{q} = \mathbf{J}^\ast \left(\mathbf{d} - \mathbf{F}(q)\right)|_{\partial\Omega},
$$

(4.1)

where $\mathbf{H} = \mathbf{J}^\ast \mathbf{J} + \beta \mathbf{I}$ and $\beta$ is the regularization parameter.

The matrix for the Fréchet derivative of the forward problem with an incoming plane with direction $\theta$ is given by

$$
\mathbf{J}_\theta = -k^2 \mathbf{G}_r \mathbf{U}^{tot}_{\theta} + k^4 \mathbf{G}_r \mathbf{Q}(I + k^2 \mathbf{GQ})^{-1} \mathbf{GU}^{tot}_{\theta}.
$$

(4.2)

In the case that we have $N_d$ incoming waves with incoming direction $\theta_j$, the Fréchet derivative becomes

$$
\mathbf{J} = [\mathbf{J}_{\theta_1}; \mathbf{J}_{\theta_2}; \cdots; \mathbf{J}_{\theta_n}].
$$

(4.3)
Note that as consequence of the use of multiple incoming plane waves the matrix (4.3) is better conditioned than the operator (4.1).

To speed-up the iterative method for solving Equation 4.1, we intend to use a preconditioner that approximates $H^{-1}$. Our first attempt was to use the preconditioner $\tilde{H} = J^*J + \beta I$, where

$$\tilde{J}_θ = -k^2 G_r U^{tot}_θ + k^4 G_r Q \tilde{A}^{-1} GU^{tot}_θ,$$

$\tilde{J} = [\tilde{J}_θ; \tilde{J}_θ; \cdots; \tilde{J}_θ]$ and $\tilde{A}^{-1}$ is a domain decomposition preconditioner in Table 3.1. Unfortunately, the domain decomposition based preconditioners are not a good approximation of the inverse of the forward operator, henceforth, the preconditioner $\tilde{H}$ does not speed-up the solution of our system.

Next, we use the RC preconditioner $\tilde{A}_{RC}^{-1}$ as an approximation of $F^{-1}$. We construct the approximation of the matrix $J_θ$ by doing

$$\tilde{J}_θ = -k^2 G_r U^{tot}_θ + k^4 G_r Q \tilde{A}_{RC}^{-1} GU^{tot}_θ.$$

Using all directions, we have $\tilde{J}_{RC} = [\tilde{J}_θ; \tilde{J}_θ; \cdots; \tilde{J}_θ]$. To ease the notation, henceforth, we will drop the $θ$ index, unless otherwise stated. The quality of the approximation of $J$ by $\tilde{J}$ depends on the number of eigenvalues $N_λ$ chosen for the rank correction, as seen in Theorem 4.1.

**Theorem 4.1.** Consider the singular value decomposition $USV^* = (A - \tilde{A})$ and the submatrices $U_{N_λ} = U(:,1:N_λ)$, $S_{N_λ} = S(1:N_λ,1:N_λ)$ and $V_{N_λ} = V(:,1:N_λ)$. Let $ε > 0$, if we choose $N_λ$ large enough such that

$$\| \tilde{A} - A - U_{N_λ}S_{N_λ}V_{N_λ}^* \| ≤ ε$$

then we have that

$$\| J - \tilde{J}_{RC} \| ≤ C(k, q, N_λ)ε,$$

where $C(k, q, N_λ)$ is a constant that depends on $k$, $\|Q\|_{∞}$, and $N_λ$.

**Proof.** According to [32], setting a tolerance $ε > 0$, we can obtain $N_λ$ such that

$$\| \tilde{A} - A - U_{N_λ}S_{N_λ}V_{N_λ}^* \| ≤ ε.$$

Since $\tilde{A}_{RC}^{-1} = (\tilde{A} - U_{N_λ}S_{N_λ}V_{N_λ}^*)^{-1}$, we have

$$\tilde{A} - \tilde{A}_{RC} = U_{N_λ}S_{N_λ}V_{N_λ}^*$$

From (4.5) and (4.6), we obtain

$$\| \tilde{A}_{RC} - \tilde{A} \| ≤ ε$$

The norm of the difference between $J$ and its approximation $\tilde{J}_{RC}$ is given by

$$\| J - \tilde{J}_{RC} \| ≤ k^4 \| G_r Q (A^{-1} - \tilde{A}_RC^{-1}) GU^{tot} \|$$

$$≤ k^4 \| G_r \| \| Q \| \| A^{-1} - \tilde{A}_RC^{-1} \| \| G \| \| U^{tot} \|.$$

We have that $\|G\|$, $\|G_r\|$ and $\|U^{tot}\|$ are bounded by a constant depending on $k$ and $\|Q\| ≤ \|Q\|_{∞}$. 

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Since \( (\mathbf{A}^{-1} - \mathbf{A}_R^{-1}) = \mathbf{A}_R^{-1} (\mathbf{A}_R - \mathbf{A}) \mathbf{A}^{-1} \), applying properties of matrices norms, we get

\[
\|\mathbf{A}^{-1} - \mathbf{A}_R^{-1}\| \leq \|\mathbf{A}_R^{-1}\| \|\mathbf{A} - \mathbf{A}_R\| \|\mathbf{A}^{-1}\| \leq \tilde{C}(k, q, N_\lambda) \|\mathbf{A} - \mathbf{A}_R\|
\]

where \( \tilde{C}(k, q, N_\lambda) \) is a constant that depends on \( k \), \( \|q\|_\infty \) and \( N_\lambda \).

Using the bounds of the norm of the matrices and (4.9), we obtain

\[
\|\mathbf{J} - \tilde{\mathbf{J}}_{RC}\| \leq C(k, q, N_\lambda) \|\mathbf{A} - \mathbf{A}_R\|
\]

where we merged the constants and reuse \( C(k, q, N_\lambda) \) to denote the final constant. \( \Box \)

Next, we construct the preconditioner \( \mathbf{H}_{RC} = \tilde{\mathbf{J}}_{RC}^* \mathbf{J}_{RC} + \beta \mathbf{I} \). We refer to this preconditioner, as the **Hessian rank corrected preconditioner** (HRC preconditioner). The procedure to build the HRC preconditioner for the inverse problem is summarized in Algorithm 2.

**Algorithm 2 Algorithm for the HRC preconditioner**

1. **Input:** Given \( \mathbf{A}_R^{-1} \) and \( N_\lambda \).
2. Use SVD to calculate \([\mathbf{U}, \mathbf{S}, \mathbf{V}] = \text{svd}(\mathbf{A} - \mathbf{A}_R)\).
3. Set \( \mathbf{U}_{N_\lambda} = \mathbf{U}(:,1:N_\lambda) \), \( \mathbf{S}_{N_\lambda} = \mathbf{S}(1:N_\lambda,1:N_\lambda) \) and \( \mathbf{V}_{N_\lambda} = \mathbf{V}(1:,1:N_\lambda) \).
4. Use the Woodbury Formula to calculate \( \mathbf{A}_{RC}^{-1} = (\mathbf{A} + \mathbf{U}_{N_\lambda} \mathbf{S}_{N_\lambda} \mathbf{V}_{N_\lambda}^T)^{-1} \).
5. Calculate for each direction \( \theta_j \) the approximation of the Fréchet derivative

\[
\tilde{\mathbf{J}}_{\theta_j} = -k^2 \mathbf{G}_r \mathbf{U}_{\theta_j}^{tot} + k^4 \mathbf{G}_r \mathbf{Q} \tilde{\mathbf{A}}_{RC}^{-1} \mathbf{G} \mathbf{U}_{\theta_j}^{tot}.
\]

6. Set \( \tilde{\mathbf{J}}_{RC} = [\tilde{\mathbf{J}}_{\theta_1}; \tilde{\mathbf{J}}_{\theta_2}; \cdots; \tilde{\mathbf{J}}_{\theta_{N_\lambda}}] \).
7. Set the preconditioner \( \mathbf{H}_{RC} = \tilde{\mathbf{J}}_{RC}^* \tilde{\mathbf{J}}_{RC} + \beta \mathbf{I} \).

**Computational complexity:** When Equation (4.1) is solved using GMRES, \( \mathbf{H} \) is applied a total of \( \kappa_I \) times. Our objective is to decrease the value of \( \kappa_I \) using the proposed HRC preconditioner. To construct the preconditioner, we can apply the randomized SVD in \( \mathbf{A} - \mathbf{A}_R \), which costs \( \mathcal{O}(N_\lambda N^2) \), and consequently apply the Woodbury formula to create \( \mathbf{A}_R^{-1} \).

**4.2. Numerical Experiments.** We present experiments that verify the effectiveness of the HRC preconditioner for the inverse scattering problem. The aim of the first two numerical experiments is to understand the influence of the overlap parameter and the number of subdomains in the choice of the number of singular values used for the construction of HRC preconditioner. In the third experiment, we check the scalability of the HRC preconditioner as the number of scatterers points increases. In the fourth experiment, we compare the HRC preconditioner with a state-of-the-art **low-rank preconditioner**, which we call henceforth as the LR preconditioner. Finally, in the last experiment, we use full aperture data from several incoming directions at multiple frequencies to obtain a full reconstruction of the scatterer. We apply the recursive linearization algorithm (RLA) [11] using GMRES without preconditioner, with the LR preconditioner and with the HRC preconditioner, and we compare the total number of iterations necessary to obtain the solutions. A list of the experiments in this section with their description and results is provided in Table 4.1. A full description of the RLA is out of the scope of this paper. For a short summary, please see Appendix A, and for a more detailed exposition see [11].
Table 4.1: List of inverse problem experiments.

| Experiment | Description                                      | Figures | Tables |
|------------|--------------------------------------------------|---------|--------|
| I.1        | Influence of the size of the overlapping         | 4.1, B.3 | 4.2    |
| I.2        | Influence of the number of subdomains            | 4.2, B.4 | 4.2    |
| I.3        | Scalability of the inverse problem preconditioning| X       | 4.3, B.7, B.8 |
| I.4        | Comparison with low-rank preconditioner          | 4.3, B.5, B.6 | X |
| I.5        | Full inverse problem                            | 4.5, 4.6 | 4.4    |

To simulate Equation 4.1, first we obtain the right-hand-side \( d - F(\bar{q}) \big|_{\partial B} \). The scattered data \( d \) is measured at \( N_r \) receivers at \( R(\cos(2m\pi/N_r),\sin(2m\pi/N_r)) \), with \( m = 0, 1, \ldots, N_r - 1 \). The data is generated by \( N_d \) incoming incident waves with direction of propagation \( \theta_j = (\cos(2j\pi/N_d),\sin(2j\pi/N_d)) \), for \( j = 0, 1, \ldots, N_d - 1 \). To decrease the ill-posedness of the system and guarantee that the step will be at the local set of convexity of the method at wavenumber \( k \), the initial guess \( \bar{q} \) is chosen to be a tiny perturbation of \( q \) given by

\[
\bar{q}(x) = q(x) + 10^{-2}\|q\|_N^{(0,1)} kN^2
\]

where \( N(0,1) \) is a uniformly distributed random number in \([0,1]\). With \( \bar{q} \) we are able to calculate \( F(\bar{q})\big|_{\partial B} \).

To treat the ill-posedness of the operator \( H \), we start by normalizing the equation by diving it by the largest singular value of \( H \). Next, we add the penalty term \( \beta I \) with regularization parameter \( \beta \). For Examples I.1-4, we use \( \beta = 10^{-6} \). For Example I.5, we use a different parameter that depends on the wavenumber, more details follow in the specific example.

The GMRES iterative solver is used to solve Equation (4.1). We tried other iterative solvers such as CG and LSQR, however the number of iterations to obtain the same accuracy on the solution was much higher than GMRES. We do not report the results of this experiment here.

In all experiments, we use the RAS domain decomposition preconditioner \( \tilde{A}^{-1}_{\text{RAS}} \) as the basis to construct the RC preconditioner \( \tilde{A}^{-1}_{\text{RC}} \). The RAS preconditioner was chosen because it presented the best results among the domain decomposition preconditioners in the forward problem experiments.

For this section we define the relative error of the iterative solution with respect to the direct method solution \( e_{rel} := \|\delta q_{\text{GMRES}} - \delta q_{\text{LU}}\|/\|u_{\text{LU}}\| \), where \( \delta q_{\text{GMRES}} \) is the solution obtained by GMRES and \( \delta q_{\text{LU}} \) is the solution obtained by the LU direct solver. The LU solution is obtained by solving (4.1) using the backslash in MATLAB.

**Experiment I.1 – Influence of the size of the overlapping:** We analyze the performance of the HRC preconditioner when we change the size of the overlapping when keeping the number of subdomains constant. The following parameters are used:

- Incoming waves: the incoming waves are given by the wave \( u^{inc}(x = (x,y)) = \exp(ikx \cdot \theta_j) \) with \( k/(2\pi) = 5, 10 \) and 20, and \( \theta_j \) prescribed as in the beginning of the subsection with \( j = 0, \ldots, 7 \);
- Receivers: the receivers are located at \( x_r = (x_r,y_r) = 0.8(\cos(2\pi r/N_r),\sin(2\pi r/N_r)) \), with \( r = 1, \ldots, N_r \) and \( N_r = 2,000 \);
- Scatterer: the scatterers points are distributed in an uniform grid of points with \( N = 64^2 \) scatterers and their magnitude is given by the function \( q_4 \);
| $k/(2\pi)$ | Iterations | Error     |
|-----------|------------|-----------|
| 5         | 255        | 2.4e-4    |
| 10        | 118        | 2.9e-4    |
| 20        | 237        | 8.3e-4    |

Table 4.2: (Experiment I.1-2) Number of iterations and error using GMRES without preconditioner to solve Equation 4.1 at frequencies $k = 5(2\pi), 10(2\pi)$ and $20(2\pi)$.

- Domain decomposition: the number of subdomains is fixed at $N_s = 16$ and the overlap parameter is $\delta = 3, 6$ and $9$.
- HRC preconditioner: we use for the low-rank correction $N_{\lambda} = 10 + 10m$, $m = 0, \ldots, M_{\lambda}$, where $M_{\lambda}$ differs for each problem and it is the maximum number needed to obtain the prescribed error.

In Figure 4.1, we present the number of iterations necessary for the HRC preconditioned GMRES to obtain $e_{rel} \approx O(10^{-4})$ for different $N_{\lambda}$ at: (a) $k/(2\pi) = 5$, (b) $k/(2\pi) = 10$ and (c) $k/(2\pi) = 20$. In each figure, we have three lines and each line represents the number of iterations necessary for convergence when using the HRC preconditioner obtained with different overlap parameter $\delta = 3, 6$ and $9$. In Table 4.2, we present the number of iterations with its respective error in the solution using GMRES without preconditioner at wavenumbers $k/(2\pi) = 5, 10$ and $20$.

In Figure B.3 of Appendix B, we present respectively $\|\tilde{A}_{RC}^{-1}A - A^{-1}\|/\|A^{-1}\|$ and $\|\tilde{A}_{RC} - A\|/\|A\|$ for different $N_{\lambda}$ at $k/(2\pi) = 5$ ((a),(b)), $10$ ((c),(d)) and $20$ ((e),(f)). Each line represents the error of the approximation for $\delta = 3, 6$ and $9$.

Summary: The value of the parameter $N_{\lambda}$ needs to be larger at higher frequencies to obtain the same accuracy in the approximation of the inverse of the forward operator. This is expected due to the fact that the singular values of $A$ decay faster at lower frequencies than at higher frequencies. We also notice that, in accordance with intuition, if we use a larger overlap parameter to create $A_{\text{RAS}}^{-1}$ than we need smaller values for the parameter $N_{\lambda}$ to obtain a prescribed fixed accuracy.

Experiment I.2 – Influence of the number of subdomains: We analyze the performance of the HRC preconditioner when we change the number of subdomains when we have a constant overlap parameter. The following parameters are used:

- Incoming waves: the incoming waves are given by the wave $u^{inc}(x = (x, y)) = \exp(ikx \cdot \theta_j)$ with $k/(2\pi) = 5, 10$ and $20$, and $\theta_j$ prescribed as in the beginning of the subsection with $j = 0, \ldots, 7$;
- Receivers: the receivers are located at $x_r = (x_r, y_r) = 0.8(\cos(2\pi r/N_r), \sin(2\pi r/N_r))$, with $r = 1, \ldots, N_r$, and $N_r = 2,000$;
- Scatterer: the scatterers points are distributed in an uniform grid of points with $N = 64^2$ scatterers and their magnitude is given by the function $q_4$;
- Domain decomposition: the number of subdomains is $N_s = 4, 16, 25, 36$ and $64$, and the overlap parameter is $\delta = 8$;
- HRC preconditioner: we use for the low-rank correction $N_{\lambda} = 10 + 10m$, $m = 0, \ldots, M_{\lambda}$, where $M_{\lambda}$ differs for each problem and it is the maximum number needed to obtain the prescribed error.

In Figure 4.2, we present the number of iterations necessary for the HRC preconditioned GM-
Figure 4.1: (Experiment I.1) Plot of the number of iterations necessary to obtain $e_{rel} \approx \mathcal{O}(10^{-4})$ using the HRC preconditioner for the GMRES with different $N_\lambda$ at wavenumbers: (a) $k/(2\pi) = 5$, (b) $k/(2\pi) = 10$ and (c) $k/(2\pi) = 20$. Each line represents the number of iterations obtained when the preconditioner is obtained using $\delta = 3, 6$ and 9. The number of subdomains is $N_s = 16$.

RES to obtain $e_{rel} \approx \mathcal{O}(10^{-4})$ for different $N_\lambda$ at different wavenumbers: (a) $k/(2\pi) = 5$, (b) $k/(2\pi) = 10$ and (c) $k/(2\pi) = 20$. In each figure, we have five lines and each line represents the number of iterations necessary for convergence when using the HRC preconditioner obtained with different number of subdomains.

In Figure B.4 of Appendix B, we present respectively $\|\tilde{A}^{-1}_{RC} - A^{-1}\|$/$\|A^{-1}\|$ and $\|\tilde{A} - A\|$/$\|A\|$ for different $N_\lambda$ at $k/(2\pi) = 5$ ((a),(b)), 10 ((c),(d)) and 20 ((e),(f)). Each line represents the error of the approximation for $N_s = 4, 16, 25, 36$ and 64.

**Summary:** We note that as we increase the number of subdomains used, we require a larger $N_\lambda$ parameter to obtain better results and eventually it deteriorates when we have a very large number of subdomains.

**Experiment I.3 – Scalability of the inverse problem preconditioning:** This example aims to check the scalability of the HRC preconditioner. We fix the number of subdomains and
increase the number of points in the grid and the overlap parameter. The following parameters are used:

- **Incoming waves**: the incoming waves are given by the wave \( u^{\text{inc}}(x = (x, y)) = \exp(ikx \cdot \theta_j) \) with \( k/(2\pi) = 5 \) and 20, and \( \theta_j \) prescribed as in the beginning of the subsection with \( j = 0, \ldots, 7 \);
- **Receivers**: the receivers are located at \( x_r = (x_r, y_r) = 0.8(\cos(2\pi r/N_r), \sin(2\pi r/N_r)) \), with \( r = 1, \ldots, N_r \), and \( N_r = 10,000 \);
- **Scatterer**: we use a regular grid with \( N = 64^2 \), \( 128^2 \) and \( 256^2 \) scatterers and their magnitude is given by the function \( q_4 \);
- **Domain decomposition**: the number of subdomains is \( N_s = 16 \) constant, we use the overlap parameter \( \delta = 3 \) for the grid with \( 64^2 \) scatterer points, \( \delta = 6 \) for \( 128^2 \) scatterer points and \( \delta = 8 \).
\[ \delta = 12 \] for 256\(^2\) scatterer points;

- HRC preconditioner: we choose \( N_{\lambda} = 20, 40, 60 \) and 80 for wavenumber \( k/(2\pi) = 5 \) and \( N_{\lambda} = 40, 90 \) and 120 for wavenumber \( k/(2\pi) = 20 \).

We present in Table 4.3 the number of GMRES iterations necessary for the method to converge to the relative error \( e_{rel} \) with order of magnitude \( O(\Phi) \), with \( \Phi = 10^{-2}, 10^{-3}, 10^{-4} \) and \( 10^{-5} \). We present the respective \( e_{rel} \) in Table B.8 of Appendix B.

**Summary:** The results show that the method is fully scalable with the increase of the number of points in the domain, requiring approximately the same number of iterations to obtain the same accuracy for increasing domain size.

**Experiment I.4 – Comparison with a low-rank preconditioner:** we compare the HRC preconditioner with a low-rank preconditioner (called here LR preconditioner) obtained by inverting the regularized low-rank approximation of the operator \( \mathbf{H} \). A very similar version of this preconditioner was previously presented in [26].

First, we describe how to construct the LR preconditioner \( \mathbf{H}_{LR}^{-1} \). Start by calculating the singular value decomposition of \( \mathbf{J}^* \mathbf{J} = \mathbf{U} \mathbf{S} \mathbf{U}^* \). Next, approximate \( \mathbf{J}^* \mathbf{J} \approx \mathbf{U}_{N_{\lambda}} \mathbf{S}_{N_{\lambda}} \mathbf{U}^*_{N_{\lambda}} \), where \( \mathbf{U}_{N_{\lambda}} = \mathbf{U}(\::,1:N_{\lambda}) \), \( \mathbf{S}_{N_{\lambda}} = \mathbf{S}(1:N_{\lambda},1:N_{\lambda}) \) and \( \mathbf{V}_{N_{\lambda}} = \mathbf{V}(\::,1:N_{\lambda}) \). Finally, set the LR preconditioner as

\[
\mathbf{H}_{LR}^{-1} = \mathbf{U}_{N_{\lambda}} (\mathbf{S}_{N_{\lambda}} + \beta \mathbf{I})^{-1} \mathbf{U}^*_{N_{\lambda}} + \beta^{-1} (\mathbf{I} - \mathbf{U}_{N_{\lambda}} \mathbf{U}^*_{N_{\lambda}}).
\]

The following parameters are used in this experiment:

- Incoming waves: the incoming waves are given by \( u^{inc}(x = (x,y)) = \exp(ikx \cdot \theta_j) \) with \( k/(2\pi) = 5, 10 \) and 20, and \( \theta_j \) prescribed as in the beginning of the subsection with \( j = 0, \ldots, 7 \);
- Receivers: the receivers are located at \( \mathbf{x}_r = (x_r,y_r) = 0.8(\cos(2\pi r/N_r),\sin(2\pi r/N_r)) \), with \( r = 1,\ldots,N_r \), and \( N_r = 2000 \);
- Scatterer: we use a regular grid with \( N = 64^2 \) scatterers and their magnitude is given by the function \( q_4 \);
- Domain decomposition: the number of subdomains is \( N_s = 16 \) and the overlap parameter is \( \delta = 4 \) and 8;
- HRC and LR preconditioners: we use for the low-rank correction \( N_{\lambda} = 10 + 10m \), \( m = 0,\ldots,M_{\lambda} \), where \( M_{\lambda} \) differs for each problem and it is the maximum number needed to obtain the prescribed error.

In Figure 4.3, we present the number of iterations necessary for the preconditioned GMRES to obtain relative error \( e_{rel} = O(10^{-4}) \) for different \( N_{\lambda} \) at different wavenumbers: (a) \( k/(2\pi) = 5 \), (b) \( k/(2\pi) = 10 \) and (c) \( k/(2\pi) = 20 \). In each figure, the blue line represents the number of iterations necessary using the LR preconditioner, the red line represents the number of iterations necessary using the HRC preconditioner with \( \delta = 4 \) (in the legend of the figure as HRC-4) and the brown line represents the number of iterations necessary using the HRC preconditioner with \( \delta = 8 \) (in the legend of the figure as HRC-8).

**Summary:** when using small \( N_{\lambda} \) the number of iterations for convergence of GMRES is larger when using the HRC preconditioner than when using the LR preconditioner. This behavior is reversed when using larger \( N_{\lambda} \) and the HRC preconditioner performance is much better than of the LR preconditioner.

**Experiment I.5 – Full inverse problem:** In this example, we compare the full reconstruction of a scatterer using GMRES with no preconditioner, GMRES with the LR preconditioner, and
Figure 4.3: (Experiment I.4) Plot of the number of iterations necessary to obtain \( e_{\text{rel}} \approx \mathcal{O}(10^{-4}) \) using LR preconditioned and HRC preconditioned GMRES with different \( N_{\lambda} \) at wavenumbers: (a) \( k/(2\pi) = 5 \), (b) \( k/(2\pi) = 10 \) and (c) \( k/(2\pi) = 20 \). In each picture, the blue line represents the number of iterations using the LR preconditioner, the red and brown lines represents the number of iterations using the HRC preconditioner with \( \delta = 4 \) and 8, respectively. The number of subdomains used for the HRC preconditioner is fixed, \( N_s = 16 \).

GMRES with the HRC preconditioner. We use the recursive linearization algorithm (RLA) to reconstruct the scatterer, given scattered data generated by incoming waves with multiple frequencies. The scatterer points magnitudes are given by the equation

\[
q_b(x, y) = 0.01 \exp \left( -\left( (x - 0.1)^2 + (y - 0.2)^2 \right)/0.03 \right),
\]

and can be seen in Figure 4.4. We have chosen this function because it is very easy to simulate and obtain a very accurate reconstruction of it using a relatively low frequency amount of data, in comparison to \( q_4 \).

The following parameters are used to simulate this experiment:

- Incoming waves: the incoming waves are given by \( u^{inc}(x, y) = \exp(ik_0x \cdot \theta_j) \) with
\( k_\ell = 1 + 0.25\ell \), for \( \ell = 1, \ldots, 37 \), and \( \theta_j \) prescribed as in the beginning of the subsection with \( j = 0, \ldots, 7 \);

- Receivers: the receivers are located at \( x_r = (x_r, y_r) = 0.8(\cos(2\pi r/N_r), \sin(2\pi r/N_r)) \), with \( r = 1, \ldots, N_r \), and \( N_r = 2000 \);
- Scatterer: we use a regular grid with \( N = 32^2 \) scatterers and their magnitude is given by the function \( q_b \);
- Domain decomposition: the number of subdomains is fixed set to \( N_s = 16 \) and the overlap parameter is \( \delta = 4 \);
- HRC and LR preconditioners: to keep the number of iterations low, the choice of the parameter \( N_\lambda \) must depend on the wavenumber \( k \). We decide to use the function \( N_\lambda(k) = \lceil 40k/9 + 140/9 \rceil \). With this function, \( N_\lambda(1) = 20 \) and \( N_\lambda(10) = 60 \);
- GMRES: the tolerance of the residual is \( 10^{-7} \) and the maximum number of iterations is 1,000, with no restarts being used;
- Gauss-Newton: the stopping criteria parameters for the Gauss-Newton method are the maximum number of iterations equal to 50, the norm of the update \( \delta q \) must be less than \( 10^{-3}/k \) and the norm of the objective functional must be less \( 10^{-4}/k \);
- Regularization: we choose \( \beta = 10^{-0.9k-3.7} \), so that at \( \beta(1) \approx 2.5 \times 10^{-5} \) and \( \beta(10) = 2 \times 10^{-13} \); and
- Initial guess: the initial guess is the regular grid scatterer points with the magnitude given by the function identically zero in the domain.

The reconstructions obtained using GMRES without preconditioner, the LR preconditioned GMRES and the HRC preconditioned GMRES can be seen in Figures 4.5a, 4.5b and 4.5c respectively. In Figure 4.6, we present at each wavenumber: (a) the relative error between the reconstruction and \( q_b \), (b) the number of iterations of the Gauss-Newton method and (c) the total number of GMRES iterations used.

In Table 4.4, we present at the wavenumbers \( k = 1, 2.5, 5, 7.5 \) and 10 the number of GMRES iterations with and without the preconditioners in the columns labeled “Step”. The total number of
iterations from wavenumber 1 up to the wavenumber $k$ is presented in the column labeled “Total”.

**Summary:** The total number of GMRES iterations when using the HRC preconditioner is ten times smaller than the total number of GMRES iterations without the preconditioner. The preconditioner is very effective to be used with the RLA and even though we can experience a small increase in the number of iterations with the increase of the wavenumber, this can be remedy by using a more aggressive choice of $N_\lambda$.

4.3. **Conclusion on the preconditioning for the inverse problem.** With the right choice of the number of singular values, we can construct the RC preconditioner to approximate the inverse of the forward operator and consequently use this approximation to construct the HRC preconditioner to speed-up the solution of the system (4.1). The HRC preconditioner does not only provide a drastic decrease in the number of iterations necessary for the convergence of GMRES, it is also scalable for domains of increasing size.

5. **Conclusions.** We have presented preconditioning strategies for the integral forms of both the forward and inverse acoustic scattering problems in two dimensions.

For the forward problem, initially, we extended to the integral equation case the domain decomposition based preconditioning strategies for PDEs: Additive Schwarz, Restricted Additive Schwarz, Additive Harmonic Schwarz and Symmetric Restricted Additive Schwarz. We presented examples comparing the methods using different number of subdomains and size of overlap parameter at different frequencies. The main conclusion for this part is that the convergence of GMRES using the RAS and AHS preconditioners is faster than with the other preconditioners. Regarding the partition of the domain, the convergence is faster when using larger overlap, and as the number of subdomains in the partition increases the convergence speed-up deteriorates. A great feature of the methods is their scalability. We finish the section by presenting the RC preconditioner, which is obtained by applying a rank correction procedure to a domain decomposition based preconditioner. The convergence of the iterative method using the RC preconditioner is even faster than using the domain decomposition preconditioners.

For the inverse problem, we used the forward problem RC preconditioner to construct the
Regarding the full reconstruction of the scatterer $q_b$ using RLA, we present at each wavenumber $k$: (a) the relative error of the reconstruction with respect to $q_b$, (b) the number of iterations necessary for the convergence of the Gauss-Newton method, and (c) the total number of GMRES iterations used. In each figure, the curve with □ marks has the values for the solution using GMRES with no preconditioner, the curve with the ○ marks has the values for the solution using LR preconditioned GMRES and the curve with the × marks has the values for the solution using the HRC preconditioned GMRES.

HRC preconditioner. Examples are presented to show the behavior of the HRC preconditioner using different number of subdomains and different size of overlap parameter for the partition of the domain. As we noted with the forward problem preconditioners, the convergence is faster when using larger overlap in the partition of the domain and as the number of subdomains in the partition increases the convergence speed-up worsens significantly. An example showing the scalability of the method for domains with increasing number of points at different frequencies is also presented. Finally, in the last example of the section, the reconstruction of a set of scatterers
points using the recursive linearization algorithm is presented. The convergence of the method using the HRC preconditioner is far superior to the GMRES solver without preconditioner and using the LR preconditioner, which is considered the state-of-the-art for this problem.

The preconditioning strategies presented are a viable alternative to speed-up the solution of the forward and inverse scattering problems specially when the size of the domain is extremely large, due to their scalability. They also can be easily adapted to three dimensions and to other problems such as electromagnetics.

In the future, we intend to expand the techniques in this article to the continuous case and use them to solve the multifrequency inverse scattering problem for penetrable media for large scale problems in two and three dimensions.

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Appendix A. Recursive Linearization Algorithm. At low frequencies, the inverse scattering problem is uniquely solvable; however, it presents poor stability, meaning that it is difficult to obtain high resolution of the contrast function. On the other hand, at higher frequencies, the objective function presents multiple minima but is very stable. This trade-off between frequency and stability of the problem forms the basis of the Recursive Linearization Algorithm. The RLA uses standard frequency continuation to solve a sequence of inverse single-frequency scattering problems at increasing frequencies, using the solution of each problem as the initial guess for the subsequent problem. The summarized description is in Algorithm 3.

Algorithm 3 Recursive Linearization Algorithm with Gauss-Newton method (RLA).

1: Input: data $d(k_j)$ for $j = 1, \ldots, Q$ with $k_1 < \cdots < k_Q$, initial guess $q_0$, tolerances $\epsilon_1(k), \epsilon_2(k)$ and maximum number of iterations $N_{it}$.
2: for $j = 1, \ldots, Q$ do
3:  Set $q := q_{j-1}$, $\delta q := 0$ and $it := 0$.
4:  while $\|d(k) - F(q)\|_{\partial B} \geq \epsilon_1(k)$ and $it < N_{it}$ and $\delta q \geq \epsilon_2(k)$ do
5:     Solve $H \delta q = J^* \left( d(k) - F(q) \right)_{|\partial B}$
6:     Update $q \leftarrow q + \delta q$
7:     Update $it \leftarrow it + 1$
8:  end while
9:  Set $q_j := q$.
10: end for

Appendix B. Supplemental numerical results. We present supplemental results regarding the experiments in the forward and inverse scattering problems. A list of the experiments, their related results and the description of the results is in Table B.1.
Table B.1: List of supplemental results for the numerical experiments.

| Experiment | Results | Description of Results |
|------------|---------|------------------------|
| F.1        | Table B.2 | Relative error $e_{rel}$ for the simulations in Experiment F.1. |
|            | Figure B.1 | Eigenvalues in the complex plane of $A$, $A_{AS}^{-1}A$, $A_{RAS}^{-1}A$ and $A_{S_{RAS}}^{-1}A$ when $N_s = 16$, $\delta = 6$ and $k/(2\pi) = 20$. |
|            | Figure B.2 | Eigenvalues in the complex plane of $A_{RAS}^{-1}A$ with $N_s = 4$ and 16, $\delta = 1$ and 6, and $k/(2\pi) = 20$. |
| F.2        | Table B.3 | Relative error $e_{rel}$ for the simulations in Experiments F.2. |
| F.3        | Table B.4 | Relative error $e_{rel}$ for the simulations in Experiments F.3. |
| F.4        | Table B.5 | Relative error $e_{rel}$ for the simulations in Experiments F.4. |
| F.5        | Table B.6 | Relative error $e_{rel}$ for the simulations in Experiments F.5. |
| I.1        | Figure B.3 | Plots of $\|A_{RC}^{-1} - A^{-1}\|/\|A^{-1}\|$ and $\|A_{RC}^{-1} - A\|/\|A\|$ at wavenumbers $k/(2\pi) = 5$, 10 and 20, with $N_s = 16$ and $\delta = 3$, 6 and 9. |
| I.2        | Figure B.4 | Plots of $\|A_{RC}^{-1} - A^{-1}\|/\|A^{-1}\|$ and $\|A_{RC}^{-1} - A\|/\|A\|$ at wavenumbers $k/(2\pi) = 5$, 10, and 20, with $N_s = 4$, 16, 25, 36 and 64, and $\delta = 8$. |
| I.3        | Table B.7 | Plots of $\|A_{RC}^{-1} - A^{-1}\|/\|A^{-1}\|$ and $\|A_{RC}^{-1} - A\|/\|A\|$ for $N = 32^2$, $64^2$ and $128^2$ scatterers when $k/(2\pi) = 5$ and 20, with different $N_{\lambda}$. |
| I.4        | Table B.8 | Relative error $e_{rel}$ for the simulations in Experiments I.3. |
| I.5        | Figure B.5 | Singular values of $H$, $H_{RC}^{-\lambda}H$ and $H_{RC}^{-\lambda}H$ using $N_{\lambda} = 140$, and $H_{RC}^{-\lambda}H$ using $N_{\lambda} = 40$ when $k/(2\pi) = 5$. |
| I.6        | Figure B.6 | Singular values of $H$, $H_{RC}^{-\lambda}H$ and $H_{RC}^{-\lambda}H$ using $N_{\lambda} = 240$, and $H_{RC}^{-\lambda}H$ using $N_{\lambda} = 100$ when $k/(2\pi) = 20$. |

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Figure B.1: (Experiment F.1 – Supplemental results) Plot in the complex plane of the eigenvalues of (a) $A$, (b) $\tilde{A}^{-1}A$, (c) $A^{-1}_{RAS}$ and (d) $A^{-1}_{SRAS}$, at $k/(2\pi) = 20$, using the domain decomposition parameters $N_s = 16$ and $\delta = 6$.

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Figure B.2: (Experiment F.1 – Supplemental results) Plot in the complex plane of the eigenvalues of $\tilde{A}_R^{-1}A$ at $k/(2\pi) = 20$, using the domain decomposition parameters (a) $N_s = 4$ and $\delta = 1$, (b) $N_s = 4$ and $\delta = 6$, (c) $N_s = 64$ and $\delta = 1$, and (d) $N_s = 64$ and $\delta = 6$.

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Figure B.3: (Experiment I.1 – Supplemental results) Plots of $\|\tilde{A}_{RC}^{-1} - A^{-1}\|/\|A^{-1}\|$ and $\|\tilde{A}_{RC} - A\|/\|A\|$ for different $N_\lambda$ at wavenumbers: (a) and (b) $k/(2\pi) = 5$, (c) and (d) $k/(2\pi) = 10$, and (e) and (f) $k/(2\pi) = 20$. In each plot, each line represents the error when using overlap parameter $\delta = 3$, 6 and 9 points. The number of subdomains is $N_s = 16$. 
Figure B.4: (Experiment I.2 – Supplemental results) Plots of $\|A^{-1}_{RC} - A^{-1}\|/\|A^{-1}\|$ and $\|A^{-1}_{RC} - A\|/\|A\|$ for different $N_\lambda$ at wavenumbers: (a) and (b) $k/(2\pi) = 5$, (c) and (d) $k/(2\pi) = 10$, and (e) and (f) $k/(2\pi) = 20$. Each line represents the number of iterations obtained when the preconditioner is obtained using $N_s = 4, 16, 25, 36$ and 64. The overlap parameter is $\delta = 8$. 
Figure B.5: (Experiment I.4 – Supplemental results) Plot of the singular values of (a) $H$ at $k/(2\pi) = 5$, (b) $H_{LR}^{-1}H$ at $k = 5/(2\pi)$ and $N_\lambda = 140$, (c) $H_{HRC}^{-1}H$ with $N_\lambda = 140$, $N_s = 16$ and $\delta = 4$, and (d) $H_{HRC}^{-1}H$ with $N_\lambda = 40$, $N_s = 16$ and $\delta = 8$ when the incoming incident waves have wavenumber $k/(2\pi) = 5$.

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Figure B.6: (Experiment I.4 – Supplemental results) Plot of the singular values of (a) $H$, (b) $H^{-1}\_LR$, (c) $H^{-1}\_HRC$ with $N_\lambda = 240$, (d) $H^{-1}\_HRC$ with $N_\lambda = 240$, $N_s = 16$ and $\delta = 4$, and (d) $H^{-1}\_HRC$ with $N_\lambda = 240$, $N_s = 16$ and $\delta = 8$ when the incoming incident waves have wavenumber $k/(2\pi) = 20$.
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Table 4.3: (Experiment I.3) Number of iterations necessary for the convergence of GMRES with the prescribed order of magnitude to the solution of the Equation (4.1) using the HRC preconditioner at (a) $k/(2\pi) = 5$ and (b) $k/(2\pi) = 20$. The prescribed order of magnitude of the error is $O(\Phi)$, with $\Phi = 10^{-2}, 10^{-3}, 10^{-4}$ and $10^{-5}$. The total number of scatterers in the domain are $N = 32^2$, $64^2$ and $128^2$. The number of subdomains is $N_s = 16$ and the overlap parameter $\delta = 3, 6$ and 12 respectively for the domains with $N = 32^2$, $64^2$ and $128^2$ scatterers. The low-rank approximation parameter is $N_\lambda = 20, 40, 60$ and 80 at $k = 5(2\pi)$ and $N_\lambda = 40, 90$ and 120 at $k = 20(2\pi)$. 

(a) Iterations for $k/(2\pi) = 5$

| $N$ | $O(\text{Error})$ | GMRES | 20 | 40 | 60 | 80 |
|-----|-------------------|--------|----|----|----|----|
| $32^2$ | $10^{-2}$ | 221 | 205 | 67 | 14 | 3 |
|     | $10^{-3}$ | 248 | 231 | 81 | 15 | 4 |
|     | $10^{-4}$ | 263 | 261 | 91 | 17 | 4 |
|     | $10^{-5}$ | 274 | 286 | 102 | 20 | 5 |
| $64^2$ | $10^{-2}$ | 198 | 159 | 37 | 11 | 8 |
|     | $10^{-3}$ | 238 | 183 | 45 | 12 | 9 |
|     | $10^{-4}$ | 255 | 198 | 51 | 13 | 9 |
|     | $10^{-5}$ | 266 | 217 | 58 | 15 | 11 |
| $128^2$ | $10^{-2}$ | 195 | 110 | 21 | 9 | 6 |
|     | $10^{-3}$ | 237 | 134 | 25 | 11 | 7 |
|     | $10^{-4}$ | 256 | 152 | 26 | 13 | 8 |
|     | $10^{-5}$ | 268 | 169 | 29 | 14 | 9 |

(b) Iterations for $k/(2\pi) = 20$

| $N$ | $O(\text{Error})$ | GMRES | 40 | 90 | 120 |
|-----|-------------------|--------|----|----|-----|
| $32^2$ | $10^{-2}$ | 112 | 112 | 33 | 2 |
|     | $10^{-3}$ | 158 | 131 | 40 | 3 |
|     | $10^{-4}$ | 208 | 150 | 44 | 4 |
|     | $10^{-5}$ | 250 | 166 | 46 | 4 |
| $64^2$ | $10^{-2}$ | 122 | 657 | 15 | 11 |
|     | $10^{-3}$ | 182 | 754 | 19 | 14 |
|     | $10^{-4}$ | 236 | 786 | 22 | 16 |
|     | $10^{-5}$ | 285 | 805 | 24 | 17 |
| $128^2$ | $10^{-2}$ | 123 | 544 | 10 | 9 |
|     | $10^{-3}$ | 191 | 630 | 12 | 12 |
|     | $10^{-4}$ | 266 | 694 | 14 | 13 |
|     | $10^{-5}$ | 330 | 732 | 16 | 13 |
The table below shows the total number of iterations of GMRES at each frequency with and without preconditioner. In the column labeled “Step”, we present the sum of the number of iterations of GMRES for the solution of the Gauss-Newton method at the respective wavenumber in the column $k$. In the column labeled “Total”, we present the number of iterations of GMRES used for the RLA from the wavenumber 1 up to the respective wavenumber in the column $k$.

$$\begin{array}{cccccc}
\hline
k & \text{Step} & \text{Total} & N_\lambda & \text{Step} & \text{Total} \\
\hline
1 & 151 & 151 & 20 & 39 & 39 & 15 & 15 \\
2.5 & 454 & 2019 & 27 & 192 & 699 & 36 & 190 \\
5 & 85 & 3886 & 38 & 56 & 1670 & 4 & 305 \\
7.5 & 145 & 5313 & 49 & 120 & 2773 & 10 & 385 \\
10 & 175 & 6934 & 60 & 205 & 4442 & 31 & 577 \\
\hline
\end{array}$$

Table 4.4: (Experiment I.5) Total number of iterations of GMRES at each frequency with and without preconditioner. In the column labeled “Step”, we present the sum of the number of iterations of GMRES for the solution of the Gauss-Newton method at the respective wavenumber in the column $k$. In the column labeled “Total”, we present the number of iterations of GMRES used for the RLA from the wavenumber 1 up to the respective wavenumber in the column $k$.

(a) Relative error $e_{rel}$ of the scattered field

$$\begin{array}{cccccc}
\hline
k/2\pi & \text{GMRES} & N_s & \text{overlap} & AS & RAS & AHS & SRAS \\
\hline
5 & 1.6e-08 & 4 & 1 & 4.3e-10 & 2.2e-10 & 3.3e-09 & 2.2e-10 \\
 & & & 6 & 2.7e-09 & 2.3e-09 & 2.1e-09 & 2.3e-09 \\
 & & 16 & 1 & 2.4e-09 & 5.9e-09 & 3.4e-09 & 1.1e-08 \\
 & & & 6 & 9.1e-10 & 3.4e-09 & 3.8e-09 & 1.3e-08 \\
 & & 64 & 1 & 1.4e-08 & 1.6e-08 & 4.2e-09 & 2.3e-08 \\
 & & & 6 & 4.4e-10 & 4.0e-10 & 2.4e-09 & 8.0e-09 \\
20 & 1.9e-10 & 4 & 1 & 1.8e-08 & 4.9e-09 & 4.2e-09 & 4.9e-09 \\
 & & & 6 & 7.9e-08 & 2.0e-08 & 8.3e-09 & 2.0e-08 \\
 & & 16 & 1 & 2.1e-07 & 1.8e-07 & 3.3e-08 & 2.0e-07 \\
 & & & 6 & 2.7e-08 & 3.0e-08 & 8.4e-09 & 3.6e-07 \\
 & & 64 & 1 & 2.3e-06 & 2.1e-07 & 1.5e-07 & 3.9e-07 \\
 & & & 6 & 3.7e-08 & 8.6e-09 & 1.5e-08 & 3.2e-07 \\
\hline
\end{array}$$

Table B.2: (Experiment F.1 – Supplemental results) We present the relative error $e_{rel}$ of the GMRES solution with the domain decomposition preconditioners and without using preconditioners. The incoming plane wave has horizontal direction of propagation and frequencies $k/(2\pi) = 5$ and 20. We use a regular grid of $N = 64^2$ point scatterers with magnitude given by the function $q_4$. The number of subdomains is $N_s = 4, 16,$ and 64 and the overlap parameter is $\delta = 1$ and 6.
(a) Relative error $e_{\text{rel}}$ of the scattered field

| $k/2\pi$ | GMRES | $N_s$ | $\delta$ | $\mathcal{G}_1$ | $\mathcal{G}_2$ | $\mathcal{G}_1$ | $\mathcal{G}_2$ | $\mathcal{G}_1$ | $\mathcal{G}_2$ |
|----------|--------|-------|---------|----------------|----------------|----------------|----------------|----------------|----------------|
| 10       | 6.1e-10| 4     | 1       | 4.5e-8        | 1.4e-8         | 1.5e-9         | 1.3e-7         | 1.7e-8         | 1.1e-7         |
|          |        | 3     | 2.7e-8  | 2.7e-8        | 3.8e-8         | 2.2e-8         | 5.1e-9         | 3.9e-8         |                 |
|          |        | 9     | 7.1e-10 | 7.2e-8        | 2.0e-8         | 7.4e-8         | 2.7e-8         | 9.6e-8         |                 |
|          |        | 16    | 7.6e-9  | 2.7e-8        | 1.7e-8         | 1.5e-8         | 2.0e-8         | 1.1e-8         |                 |
| 40       | 7.3e-11| 4     | 1       | 2.2e-8        | 1.1e-7         | 1.3e-7         | 6.7e-8         | 4.4e-8         | 7.4e-8         |
|          |        | 3     | 3.2e-8  | 4.9e-8        | 2.3e-8         | 4.2e-8         | 1.5e-8         | 1.3e-8         |                 |
|          |        | 16    | 8.3e-7  | 1.7e-7        | 2.6e-7         | 1.1e-6         | 3.2e-7         | 6.5e-8         |                 |

Table B.3: (Experiment F.2 – Supplemental results) We present the relative error $e_{\text{rel}}$ of the solution obtained by the iterative method without using preconditioners and using the preconditioners AS, RAS and AHS. The incoming plane waves have wavenumbers $k/(2\pi) = 10$ and 40. We use a regular grid of $N = 64^2$ point scatterers with magnitude given by the function $q_4$. $\mathcal{G}_1$ represents the partition composed of equal sized squares and $\mathcal{G}_2$ is the partition composed of vertical bands.

(a) Relative error $e_{\text{rel}}$ of the scattered field

| $k/2\pi$ | GMRES | $N_s$ | $\delta$ | $\mathcal{G}_4$ | $\mathcal{G}_{16}$ | $\mathcal{G}_4$ | $\mathcal{G}_{16}$ | $\mathcal{G}_4$ | $\mathcal{G}_{16}$ |
|----------|--------|-------|---------|----------------|----------------|----------------|----------------|----------------|----------------|
| 10       | 2.4e-9 | 7.4e-10| 4       | 1              | 7.4e-8        | 3.3e-9         | 1.7e-8         | 2.7e-8         | 3.2e-8         |
|          |        |       | 10      | 5.4e-7        | 9.7e-9        | 1.7e-8         | 1.1e-8         | 1.1e-8         | 1.5e-8         |
|          |        |       | 16      | 2.2e-8        | 4.3e-8        | 2.7e-8         | 2.0e-7         | 7.5e-8         | 3.1e-7         |
| 40       | 3.3e-10| 3.9e-10| 4       | 1              | 3.6e-6        | 4.3e-7         | 8.0e-7         | 6.4e-7         | 9.1e-7         |
|          |        |       | 16      | 2.6e-7        | 6.3e-7        | 9.2e-7         | 6.6e-7         | 1.2e-6         | 1.5e-6         |
|          |        |       | 16      | 4.6e-6        | 3.9e-7        | 1.4e-5         | 1.6e-6         | 3.4e-6         | 1.1e-6         |

Table B.4: (Experiment F.3 – Supplemental results) We present the relative error of the solution $e_{\text{rel}}$ obtained by the iterative method without using preconditioners and using the preconditioners AS, RAS and AHS. The incoming plane waves have wavenumbers $k/(2\pi) = 10$ and 40. We use a regular grid of $N = 64^2$ point scatterers with magnitude given by the functions $q_4$ and $q_{16}$. The number of subdomains is $N_s = 4$ and 16, and the overlap parameter is $\delta = 1$ and 8.
(a) Relative error $e_{rel}$ of the scattered field

| $k/2\pi$ | $\sqrt{N}$ | GMRES | $\delta$ | AS | RAS | AHS | SRAS |
|----------|-------------|--------|----------|----|-----|-----|------|
| 10       | 64          | 6.1e-10| 3        | 3.2e-08 | 2.3e-08 | 1.5e-08 | 7.1e-08 |
|          | 128         | 2.4e-09| 6        | 9.5e-08 | 1.0e-07 | 7.3e-08 | 2.3e-07 |
|          | 256         | 1.1e-08| 12       | 2.6e-06 | 6.9e-06 | 1.0e-06 | 8.8e-07 |
| 20       | 64          | 1.9e-10| 3        | 4.9e-08 | 3.5e-08 | 6.5e-09 | 1.2e-06 |
|          | 128         | 1.7e-09| 6        | 2.9e-07 | 1.2e-06 | 5.8e-07 | 6.1e-06 |
|          | 256         | 6.0e-09| 12       | 2.0e-06 | 1.6e-05 | 1.7e-05 | 2.7e-05 |
| 40       | 64          | 7.3e-11| 3        | 6.7e-07 | 2.0e-07 | 6.8e-07 | 1.1e-06 |
|          | 128         | 3.6e-10| 6        | 2.7e-06 | 4.5e-07 | 4.2e-07 | 7.3e-06 |
|          | 256         | 1.5e-09| 12       | 1.5e-04 | 1.2e-04 | 1.3e-04 | 1.6e-04 |

Table B.5: (Experiment F.4 – Supplemental results) We present the relative error $e_{rel}$ obtained by the iterative method without using preconditioners and using the preconditioners AS, RAS and AHS. The incoming plane wave has horizontal direction of propagation and frequencies $k/(2\pi) = 10$, 20 and 40. We use regular grids of $N = 64^2$, $128^2$ and $256^2$ point scatterers with magnitude given by the function $q_4$. The number of subdomains is $N_s = 16$ and the overlap parameter is $\delta = 3$, 6 and 12 for $N = 64^2$, $N = 128^2$ and $N = 256^2$ respectively.

(b) Relative error $e_{rel}$ of the scattered field at $k/(2\pi) = 5$

| GMRES | $N_s$ | $\delta$ | RAS   | $RC - N_{\lambda}$ |
|-------|------|----------|-------|---------------------|
| 4.0e-10 | 4    | 1        | 1.7e-09 | 5.1e-12              |
|       | 1    | 8        | 2.0e-09 | 4.5e-12              |
|       | 16   | 1        | 6.4e-09 | 5.6e-10              |
|       | 16   | 8        | 2.4e-09 | 5.1e-12              |

(b) Relative error $e_{rel}$ of the scattered field at $k/(2\pi) = 20$

| GMRES | $N_s$ | $\delta$ | RAS   | $RC - N_{\lambda}$ |
|-------|------|----------|-------|---------------------|
| 2.5e-10 | 4    | 1        | 4.9e-09 | 4.2e-09              |
|       | 1    | 8        | 1.1e-08 | 4.2e-09              |
|       | 16   | 1        | 1.8e-07 | 2.6e-07              |
|       | 16   | 8        | 1.7e-08 | 4.2e-09              |

Table B.6: (Experiment F.5 – Supplemental results) We present the relative error $e_{rel}$ for GMRES, RAS and the RC preconditioner using $N_{\lambda} = 20$, 40, 60 and 80 at $k = 5$ and $N_{\lambda} = 40$, 80, 120 and 160 at $k = 20$. The incoming plane waves have wavenumbers $k/(2\pi) = 5$ and 20 with incidence direction $(1, 0)$. We use a regular grid of $N = 64^2$ point scatterers with magnitude given by the function $q_4$. The number of subdomains is $N_s = 4$ and 16, and the overlap parameter is $\delta = 1$ and 8.
(a) Relative error of the matrix approximation at $k/(2\pi) = 5$

\[
\frac{\|\tilde{A}_{RC}^{-1} - A^{-1}\|}{\|A^{-1}\|} \quad \frac{\|\tilde{A}_{RC} - A\|}{\|A\|}
\]

| $N \lambda$ | 20 | 40 | 60 | 80 | 20 | 40 | 60 | 80 |
|--------------|----|----|----|----|----|----|----|----|
| 32$^2$       | 1.8e-1 | 5.8e-2 | 1.5e-2 | 2.3e-3 | 7.6e-2 | 1.8e-2 | 4.3e-3 | 7.7e-4 |
| 64$^2$       | 1.4e-1 | 4.6e-2 | 1.5e-2 | 8.6e-3 | 5.7e-2 | 8.3e-3 | 3.0e-3 | 1.4e-3 |
| 128$^2$      | 2.9e-1 | 1.3e-1 | 6.2e-2 | 3.5e-2 | 8.5e-2 | 2.6e-2 | 1.1e-2 | 5.8e-3 |

(b) Relative error of the matrix approximation at $k/(2\pi) = 20$

\[
\frac{\|\tilde{A}_{RC}^{-1} C - A^{-1}\|}{\|A^{-1}\|} \quad \frac{\|\tilde{A}_{RC} - A\|}{\|A\|}
\]

| $N \lambda$ | 40 | 90 | 120 | 40 | 90 | 120 |
|--------------|----|----|-----|----|----|-----|
| 32$^2$       | 5.8e-2 | 6.3e-4 | 1.2e-4 | 1.8e-2 | 2.0e-4 | 6.6e-7 |
| 64$^2$       | 2.0e0 | 2.3e-2 | 1.3e-2 | 2.6e-1 | 6.8e-3 | 3.7e-3 |
| 128$^2$      | 5.6e-1 | 6.3e-2 | 4.6e-2 | 2.5e-1 | 2.0e-2 | 1.1e-2 |

Table B.7: (Experiment I.3 – Supplemental results) Relative error of the approximation $\|\tilde{A}_{RC}^{-1} - A^{-1}\|/\|A^{-1}\|$ and $\|\tilde{A}_{RC} - A\|/\|A\|$ for domains composed of $N = 32^2$, $64^2$ and $128^2$ scatterers at wavenumbers (a) $k/(2\pi) = 5$ and (b) $k/(2\pi) = 20$. At $k/(2\pi) = 5$, the relative error of the matrix approximation is presented for $N \lambda = 40$, 90, and 120, while at $k = 20(2\pi)$ the relative error of the matrix approximation is presented for $N \lambda = 20$, 40, 60, and 80.
Table B.8: (Experiment I.3 – Supplemental results) Relative error of the iterative solution of Equation (4.1) using the HRC preconditioner with GMRES at (a) $k/(2\pi) = 5$ and (b) $k/(2\pi) = 20$. This table shows the relative error obtained for the number of iterations in Table 4.3. The relative error have order of magnitude $O(\Phi)$, with $\Phi = 10^{-2}$, $10^{-3}$, $10^{-4}$ and $10^{-5}$. The total number of scatterer points in the domain are $N = 32^2$, $64^2$ and $128^2$. The number of subdomains is fixed $N_s = 16$ and the overlap parameter is $\delta = 3$, 6 and 12 respectively for the domain with $N = 32^2$, $64^2$ and $128^2$. The number of singular values used is $N_\lambda = 20$, 40, 60 and 80 at $k/(2\pi) = 5$ and $N_\lambda = 40$, 90 and 120 at $k/(2\pi) = 20$. 

|       | $N$  | $O(\text{Error})$ | GMRES | $N_\lambda$ | 20  | 40  | 60  | 80  |
|-------|------|-------------------|-------|-------------|-----|-----|-----|-----|
| (a)   |      |                   |       |             |     |     |     |     |
|       |      |                   |       | $k/(2\pi) = 5$ |     |     |     |     |
|       |      |                   |       | 32$^2$      |     |     |     |     |
|       |      |                   |       | $10^{-2}$   | 1.8e-1 | 5.8e-2 | 5.0e-2 | 1.3e-2 | 6.4e-2 |
|       |      |                   |       | $10^{-3}$   | 2.4e-3 | 4.7e-3 | 4.2e-3 | 2.6e-3 | 3.7e-4 |
|       |      |                   |       | $10^{-4}$   | 2.2e-4 | 5.9e-4 | 6.2e-4 | 3.3e-4 | 3.7e-4 |
|       |      |                   |       | $10^{-5}$   | 1.9e-5 | 7.4e-5 | 6.6e-5 | 3.5e-5 | 6.5e-5 |
|       |      |                   |       | $64^2$      |     |     |     |     |
|       |      |                   |       | $10^{-2}$   | 1.9e-1 | 3.0e-2 | 3.6e-2 | 1.3e-2 | 1.3e-2 |
|       |      |                   |       | $10^{-3}$   | 2.8e-3 | 4.9e-3 | 5.4e-3 | 3.4e-3 | 3.3e-4 |
|       |      |                   |       | $10^{-4}$   | 2.4e-4 | 1.8e-4 | 4.4e-4 | 4.9e-4 | 3.3e-4 |
|       |      |                   |       | $10^{-5}$   | 3.0e-5 | 2.1e-5 | 8.8e-5 | 2.9e-5 | 1.5e-5 |
|       |      |                   |       | $128^2$     |     |     |     |     |
|       |      |                   |       | $10^{-2}$   | 1.8e-1 | 6.9e-2 | 1.8e-2 | 3.7e-2 | 2.4e-2 |
|       |      |                   |       | $10^{-3}$   | 3.8e-3 | 3.8e-3 | 1.5e-3 | 1.8e-3 | 2.0e-3 |
|       |      |                   |       | $10^{-4}$   | 2.6e-4 | 4.4e-4 | 6.5e-4 | 1.1e-4 | 2.9e-4 |
|       |      |                   |       | $10^{-5}$   | 3.2e-5 | 5.1e-5 | 8.2e-5 | 1.1e-5 | 2.3e-5 |
| (b)   |      |                   |       | $k/(2\pi) = 20$ |     |     |     |     |
|       |      |                   |       | 32$^2$      |     |     |     |     |
|       |      |                   |       | $10^{-2}$   | 4.0e-2 | 3.2e-2 | 1.2e-2 | 1.2e-2 | 2.2e-2 |
|       |      |                   |       | $10^{-3}$   | 7.6e-3 | 3.1e-3 | 1.9e-3 | 1.7e-3 |
|       |      |                   |       | $10^{-4}$   | 8.9e-4 | 3.0e-4 | 1.4e-4 | 1.5e-5 |
|       |      |                   |       | $10^{-5}$   | 8.9e-5 | 1.8e-5 | 2.2e-5 | 1.5e-5 |
|       |      |                   |       | $64^2$      |     |     |     |     |
|       |      |                   |       | $10^{-2}$   | 4.1e-2 | 1.6e-2 | 3.8e-2 | 3.7e-2 |
|       |      |                   |       | $10^{-3}$   | 5.8e-3 | 1.2e-3 | 3.0e-3 | 3.4e-3 |
|       |      |                   |       | $10^{-4}$   | 8.1e-4 | 1.3e-4 | 2.3e-4 | 3.6e-4 |
|       |      |                   |       | $10^{-5}$   | 8.5e-5 | 4.6e-5 | 4.2e-5 | 9.3e-5 |
|       |      |                   |       | $128^2$     |     |     |     |     |
|       |      |                   |       | $10^{-2}$   | 2.7e-2 | 3.8e-2 | 1.8e-2 | 1.6e-2 |
|       |      |                   |       | $10^{-3}$   | 5.8e-3 | 2.2e-3 | 1.4e-3 | 1.1e-3 |
|       |      |                   |       | $10^{-4}$   | 9.0e-4 | 2.3e-4 | 1.2e-4 | 4.4e-5 |
|       |      |                   |       | $10^{-5}$   | 8.2e-5 | 2.1e-5 | 1.1e-5 | 4.4e-5 |