The Raney Numbers and \((s, s + 1)\)-Core Partitions

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Abstract

The Raney numbers \(R_{p, r}(k)\) are a two-parameter generalization of the Catalan numbers. In this paper, we give a combinatorial proof for a recurrence relation of the Raney numbers in terms of coral diagrams. Using this recurrence relation, we confirm a conjecture posed by Amdeberhan concerning the enumeration of \((s, s + 1)\)-core partitions \(\lambda\) with parts that are multiples of \(p\). As a corollary, we give a new combinatorial interpretation for the Raney numbers \(R_{p+1, r+1}(k)\) with \(0 \leq r < p\) in terms of \((kp + r, kp + r + 1)\)-core partitions \(\lambda\) with parts that are multiples of \(p\).

Keywords: Raney number, Catalan number, core partition, hook length, poset, order ideal, coral diagram

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1 Introduction

In this paper, we build a connection between the Raney numbers and \((s, s + 1)\)-core partitions with parts that are multiples of \(p\). We show that the number of \((kp + r, kp + r + 1)\)-core partitions with parts that are multiples of \(p\) equals the Raney number \(R_{p+1, r+1}(k)\), confirming a conjecture posed by Amdeberhan \([1]\).

The Raney numbers \(R_{p, r}(k)\) were introduced by Raney in his investigation of functional composition patterns \([13]\) and these numbers have also been used in probability theory \([11, 12]\). The Raney numbers \(R_{p, r}(k)\) are defined as follows:

\[
R_{p, r}(k) = \frac{r}{kp + r} \left( \begin{array}{c} kp + r \\ k \end{array} \right).
\]
The Raney numbers are a two-parameter generalization of the Catalan numbers. To be more specific, if \( r = 1 \), the Raney numbers specialize to the Fuss-Catalan numbers \( C_p(k) \) \[8, 9\], where \( C_p(k) \) are the numbers of \( p \)-ary trees with \( k \) internal vertices and

\[
C_p(k) = R_{p,1}(k) = \frac{1}{kp+1} \binom{kp+1}{k}.
\]

If we further set \( p = 2 \), we obtain the classical Catalan numbers \( C_k \), that is,

\[
R_{2,1}(k) = C_k = \frac{1}{k+1} \binom{2k}{k}.
\]

Let \( C_p(x) \) and \( R_{p,r}(x) \) denote the generating functions of the Fuss-Catalan numbers \( C_p(k) \) and the Raney numbers \( R_{p,r}(k) \), respectively, namely,

\[
C_p(x) = \sum_{k \geq 0} C_p(k) x^k = \sum_{k \geq 0} \frac{1}{kp+1} \binom{kp+1}{k} x^k,
\]

\[
R_{p,r}(x) = \sum_{k \geq 0} R_{p,r}(k) x^k = \sum_{k \geq 0} \frac{r}{kp+r} \binom{kp+r}{k} x^k.
\]

It is easily seen that \( C_p(x) = R_{p,1}(x) \). The following theorem gives more relations of the generating functions \( C_p(x) \) and \( R_{p,r}(x) \).

**Theorem 1.1** \([8, 9]\) Let \( p \) be a positive integer and let \( r, k \) be nonnegative integers. Then we have

\[
C_p(x) = 1 + xC_p(x)^p, \quad (1.2)
\]

\[
R_{p,r}(x) = C_p(x)^r. \quad (1.3)
\]

Notice that \( C_p(x) = R_{p,1}(x) \). The following theorem is followed directly by equating the coefficients of \( x^k \) in \((1.2)\) and \((1.3)\).

**Theorem 1.2** Let \( p \) be a positive integer and let \( r, k \) be nonnegative integers. Then the number \( R_{p,r}(k) \) satisfies the recurrence relations

\[
R_{p,1}(k) = \sum_{i=0}^{k-1} R_{p,1}(i) R_{p,p-1}(k - 1 - i), \quad (1.4)
\]

\[
R_{p,r}(k) = \sum_{i=0}^{k} R_{p,1}(i) R_{p,r-1}(k - i), \quad \text{for } r > 1, \quad (1.5)
\]

with the initial values \( R_{p,r}(0) = 1 \) if \( r \geq 0 \) and \( R_{p,0}(k) = 0 \) if \( k > 0 \).
Notice that $C_k = R_{2,1}(k)$. Substituting $p = 2$ into (1.4), we obtain the recurrence relation for the Catalan numbers $C_k = \sum_{i=0}^{k-1} C_i C_{k-1-i}$.

Let us give an overview of notation and terminology on partitions. A partition $\lambda$ of a positive integer $n$ is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$. We write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \vdash n$ and we say that $n$ is the size of $\lambda$ and $m$ is the length of $\lambda$. The Young diagram of $\lambda$ is defined to be an up- and left-justified array of $n$ boxes with $\lambda_i$ boxes in the $i$-th row. Each box $B$ in $\lambda$ determines a hook consisting of the box $B$ itself and boxes directly to the right and directly below $B$. The hook length of $B$, denoted $h(B)$, is the number of boxes in the hook of $B$.

For a partition $\lambda$, the $\beta$-set of $\lambda$, denoted $\beta(\lambda)$, is defined to be the set of hook lengths of the boxes in the first column of $\lambda$. For example, Figure 1 illustrates the Young diagram and the hook lengths of a partition $\lambda = (5, 3, 2, 2, 1)$. The $\beta$-set of $\lambda$ is $\beta(\lambda) = \{9, 6, 4, 3, 1\}$. Notice that a partition $\lambda$ is uniquely determined by its $\beta$-set.

Given a decreasing sequence of positive integers $(h_1, h_2, \ldots, h_m)$, it is easily seen that the unique partition $\lambda$ with $\beta(\lambda) = \{h_1, h_2, \ldots, h_m\}$ is $\lambda = (h_1 - (m - 1), h_2 - (m - 2), \ldots, h_{m-1} - 1, h_m)$.

![Figure 1: The Young diagram of $\lambda = (5, 3, 2, 2, 1)$](image)

For a positive integer $t$, a partition $\lambda$ is a $t$-core partition, or simply a $t$-core, if it contains no box whose hook length equal to $t$ (or equivalently, equal to a multiple of $t$). Let $s$ be a positive integer not equal to $t$, we say that $\lambda$ is an $(s, t)$-core if it is simultaneously an $s$-core and a $t$-core. For example, the partition $\lambda = (5, 3, 2, 2, 1)$ in Figure 1 is a $(5, 8)$-core.

Let $s$ and $t$ be two coprime positive integers. Anderson [3] showed that the number of $(s, t)$-core partitions equals $(s + t) / (s + t)$. It specializes to the Catalan number $C_s = \frac{1}{s+1} \binom{2s}{s}$ if $t = s + 1$. Ford, Mai and Sze [7] proved that the number of self-conjugate $(s, t)$-core partitions equals $\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}$. Furthermore, Olsson and Stanton [14] proved that there exists a unique $(s, t)$-core partition with the maximum size $(s^2 - 1)(t^2 - 1)/24$. A simpler proof was provided by Tripathi [17]. More results on $(s, t)$-core partitions can be found in [2, 4, 6, 10, 16, 18, 19].

In [1], Amdeberhan posed the following conjecture.
Conjecture 1.1 Let $s$ and $p$ be positive integers. The number of $(s, s+1)$-core partitions $\lambda$ with parts that are multiples of $p$ equals

$$s + 1 - p\left\lfloor \frac{s}{p} \right\rfloor \left( s + \left\lfloor \frac{s}{p} \right\rfloor \right).$$

(1.6)

We observe that the expression (1.6) appearing in Conjecture 1.1 equals the Raney number $R_{p+1,r+1}(k)$ if we write $s = kp + r$, where $0 \leq r < p$. In the next section, we present a combinatorial proof of the recurrence relations (1.4) and (1.5) of the Raney numbers by using coral diagrams. In Section 3, we give a characterization of the $\beta$-set of the conjugate of an $(s, s+1)$-core partition with parts that are multiples of $p$. Based on this characterization, we show that the number of $(kp+r, kp+r+1)$-core partitions with parts that are multiples of $p$ has the same recurrence relation with the Raney number $R_{p+1,r+1}(k)$. This proves Conjecture 1.1.

2 A combinatorial proof of Theorem 1.2

In this section, we investigate the Raney numbers and give a combinatorial proof of Theorem 1.2 by using coral diagrams. Let us begin with an introduction of some graph theoretic terminology.

Let $p$ be a positive integer. Then a $p$-star is a rooted tree with $p$ terminal edges lying above a single base vertex. A coral diagram of type $(p, r, k)$ is a rooted tree which is constructed from an $r$-star at its base via the repeated placement of $k$ $p$-stars atop terminal edges. Let $D(p, r, k)$ denote the set of coral diagrams of type $(p, r, k)$. We can construct a coral diagram $D \in D(p, r, k)$ by attaching $p$-stars one “tier” at a time. We begin with the base tree and work upward. Figure 2 illustrates a coral diagram of type $(2, 3, 3)$.

![Figure 2: Construction of a coral diagram of type (2,3,3)](image)

We note that the definition of coral diagrams differs slightly from the one defined by Beagley and Drube [5]. In [5], a coral diagram of type $(p, r, k)$ is a rooted tree which is constructed from an $(r+1)$-star via the repeated placement of $k$ $p$-stars atop terminal edges that are not the leftmost edge adjacent to the root. Not attaching $p$-stars to the
leftmost edge adjacent to the root gives them a consistent way of selecting a base vertex for planar embedding. Anyhow, the coral diagrams under the two definitions have the same enumeration.

We need the following theorem due to Beagley and Drube \[5\].

**Theorem 2.1** Let \( p \) be a positive integer and let \( r, k \) be nonnegative integers. Then the number of coral diagrams of type \((p, r, k)\) equals the Raney number \( R_{p,r}(k) \), that is,

\[
|D(p, r, k)| = R_{p,r}(k).
\]

Using Theorem 2.1, we present a combinatorial proof of Theorem 1.2.

**Proof of Theorem 1.2.** The initial values for \( R_{p,r}(k) \) follow directly from Theorem (2.1). Now we assume that \( k > 0 \) and \( r > 0 \). Let \( D \) be a coral diagram in \( D(p, r, k) \). There are two cases.

If \( r > 1 \), we divide the coral diagram \( D \) into two coral diagrams \( D_1 \) and \( D_2 \) by splitting the root of \( D \) into two vertices such that the root of \( D_1 \) is only adjacent to the leftmost edge and the root of \( D_2 \) is adjacent to the remaining \( r - 1 \) edges. See Figure 3 as an example. Assume that the coral diagram \( D_1 \) has \( i \) \( p \)-stars, then the coral diagram \( D_2 \) has \( k - i \) \( p \)-stars. It follows that \( D_1 \in D(p, 1, i) \) and \( D_2 \in D(p, r - 1, k - i) \). Since the coral diagram \( D \) is uniquely determined by \( D_1 \) and \( D_2 \), we obtain the relation (1.5), that is,

\[
R_{p,r}(k) = \sum_{i=0}^{k} R_{p,1}(i)R_{p,r-1}(k-i).
\]

If \( r = 1 \), then \( D \) is constructed from a 1-star. Contracting the edge of this 1-star into a new vertex, we obtain a new coral diagram \( D' \). It is easily seen that \( D' \) is a coral diagram in \( D(p, p, k - 1) \). Hence we have \( R_{p,1}(k) = R_{p,p}(k-1) \). If further \( p = 1 \), since \( R_{1,0}(k) = 0 \) when \( k > 0 \), relation (1.4) is equivalent to \( R_{1,1}(k) = R_{1,1}(k-1) \). By the expression (1.1) for the Raney numbers \( R_{p,r}(k) \), it can be easily checked that \( R_{1,1}(k) = R_{1,1}(k-1) = 1 \). Hence relation (1.4) holds if \( r = p = 1 \). Now assume that \( p > 1 \). Combining relation (1.5) and \( R_{p,1}(k) = R_{p,p}(k-1) \), we obtain (1.4). This completes the proof.

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**3 Proof of Conjecture 1.1**

In this section, we give a characterization of the \( \beta \)-set of the conjugate of an \((s, s + 1)\)-core partition with parts that are multiples of \( p \). Using this characterization, we prove Conjecture 1.1 posed by Amdeberhan. As a corollary, we give a new combinatorial interpretation for the Raney numbers \( R_{p+1,r+1}(k) \) with \( 0 \leq r < p \) in terms of \((kp + r, kp + r + 1)\)-core partitions \( \lambda \) with parts that are multiples of \( p \).
We observe that the expression (1.6) appearing in Conjecture 1.1 equals the Raney number $R_{p+1,r+1}(k)$ if we write $s = kp + r$, where $0 \leq r < p$. That is,

$$\frac{s + 1 - p\left\lfloor \frac{s}{p} \right\rfloor}{s + 1} \left( s + \left\lfloor \frac{s}{p} \right\rfloor \right) = \frac{kp + r + 1 - kp}{kp + r + 1} \left( \frac{kp + r + k}{kp + r} \right)$$

$$= \frac{r + 1}{k(p + 1) + r + 1} \left( \frac{k(p + 1) + r + 1}{k} \right) = R_{p+1,r+1}(k).$$

Hence Conjecture 1.1 can be restated as the following equivalent theorem.

**Theorem 3.1** Let $s$ and $p$ be positive integers. Suppose that $s = kp + r$, where $0 \leq r < p$. Then the number of $(s, s+1)$-core partitions $\lambda$ with parts that are multiples of $p$ equals the Raney number $R_{p+1,r+1}(k)$.

To prove the above theorem, we give a characterization of the $\beta$-set $\beta(\lambda^c)$, where $\lambda^c$ is the conjugate of an $(s, s+1)$-core partition $\lambda$ with parts that are multiples of $p$. Let us recall some notation and terminology on posets.

Let $P$ be a poset. For two elements $x$ and $y$ in $P$, we say $y$ covers $x$ if $x < y$ and there exists no element $z \in P$ satisfying $x < z < y$. The Hasse diagram of a finite poset $P$ is a graph whose vertices are the elements of $P$, whose edges are the cover relations, and such that if $y$ covers $x$ then there is an edge connecting $x$ and $y$ and $y$ is placed above $x$. An order ideal of $P$ is a subset $I$ such that if any $y \in I$ and $x \leq y$ in $P$, then $x \in I$. Let $J(P)$ denote the set of order ideals of $P$. For more details on posets, see Stanley [15].

Anderson [3] gave a characterization of the $\beta$-set of a $t$-core partition. Based on this characterization, Stanley [16] obtained the following lemma which gives a correspondence between core partitions and order ideals of a certain poset by mapping a partition to its $\beta$-set.
Lemma 3.2 Let $s, t$ be two coprime positive integers, and let $\lambda$ be a partition of $n$. Then $\lambda$ is an $(s, t)$-core partition if and only if $\beta(\lambda)$ is an order ideal of $P_{(s,t)}$, where

$$P_{(s,t)} = \mathbb{N}^+ \setminus \{n \in \mathbb{N}^+ | n = k_1s + k_2t \text{ for some } k_1, k_2 \in \mathbb{N} \}$$

and $y$ covers $x$ in $P_{(s,t)}$ if $y - x \in \{s, t\}$.

For example, let $s = 3$ and $t = 4$. We can construct all $(3, 4)$-core partitions by finding order ideals of $P_{(3,4)}$. It is easily checked that $P_{(3,4)} = \{1, 2, 5\}$ with the partial order $5 > 2$ and $5 > 1$. Hence the order ideals of $P_{(3,4)}$ are $\emptyset$, $\{1\}$, $\{2\}$, $\{2, 1\}$ and $\{5, 2, 1\}$. The corresponding $(3, 4)$-core partitions are $\emptyset$, $(1)$, $(2)$, $(1, 1)$ and $(3, 1, 1)$, respectively. Let $T_s = P_{(s,s+1)}$. From Lemma 3.2, the Hasse diagram of $T_s$ can be easily constructed. For example, Figure 4 illustrates the Hasse diagram of the poset $T_6$.

![Hasse diagram](image.png)

Figure 4: The Hasse diagram of $T_6 = P_{(6,7)}$

The following lemma gives a characterization of the $\beta$-set of the conjugate $\lambda^c$ of a partition $\lambda$ with parts equaling multiples of $p$.

Lemma 3.3 Let $\lambda$ be a partition and let $\lambda^c$ be the conjugate of $\lambda$. Then $\lambda$ is a partition with parts that are multiples of $p$ if and only if the length of each maximal sequence of consecutive integers in $\beta(\lambda^c)$ is divisible by $p$.

Proof. Suppose that $\lambda^c = (a_1^{m_1}, a_2^{m_2}, \ldots, a_n^{m_n})$, where $a_1 > a_2 > \cdots > a_n$ and $a_i^{m_i}$ means $m_i$ occurrences of $a_i$. It is easily seen that $\lambda$ is a partition with parts that are multiples of $p$ if and only if $m_i(1 \leq i \leq n)$ is a multiple of $p$. Then the lemma follows directly from the relation of a partition and its $\beta$-set.

Let $S$ be an integer set and let $p$ be a positive integer. We say that the set $S$ has property $\mathcal{H}_p$ if the length of each maximal sequence of consecutive integers in $S$ is divisible by $p$. For example, the integer set $\{1, 2, 3, 6, 7, 8, 9, 10, 11\}$ has property $\mathcal{H}_3$. The following lemma gives a complete characterization of the $\beta$-set of the conjugate $\lambda^c$ of an $(s, s+1)$-core partition $\lambda$ with parts that are multiples of $p$. 

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Lemma 3.4 Let \( s \) and \( p \) be two positive integers. Let \( \lambda \) be a partition and let \( \lambda^c \) be the conjugate of \( \lambda \). Then \( \lambda \) is an \((s, s + 1)\)-core partition with parts that are multiples of \( p \) if and only if \( \beta(\lambda^c) \) is an order ideal in \( T_s \) with property \( \mathcal{H}_p \).

Proof. From Lemma 3.3 we see that \( \lambda \) is a partition with parts that are multiples of \( p \) if and only if \( \beta(\lambda^c) \) has property \( \mathcal{H}_p \). It is easily verified that \( \lambda \) is an \((s, s + 1)\)-core partition if and only if \( \lambda^c \) is an \((s, s + 1)\)-core partition. Then from Lemma 3.2 we obtain that \( \lambda \) is an \((s, s + 1)\)-core partition if and only if \( \beta(\lambda^c) \) is an order ideal in \( T_s \). In conclusion, \( \lambda \) is an \((s, s + 1)\)-core partition with parts that are multiples of \( p \) if and only if \( \beta(\lambda^c) \) is an order ideal in \( T_s \) with property \( \mathcal{H}_p \). \( \blacksquare \)

Let \( s \) and \( p \) be two positive integers. Suppose that \( s = kp + r \), where \( 0 \leq r < p \). Let \( O_{p,r}(k) \) denote the number of order ideals in \( T_s \) with property \( \mathcal{H}_p \). We proceed to study the number \( O_{p,r}(k) \). To this end, we partition \( J(T_s) \) according to the smallest missing element of rank 0 in an order ideal. Note that the elements of rank 0 in \( T_s \) are the minimal elements. For \( 1 \leq i \leq s - 1 \), let \( J_i(T_s) \) denote the set of order ideals of \( T_s \) such that \( i \) is the smallest missing element of rank 0. Let \( J_s(T_s) \) denote the set of order ideals which contain all minimal elements in \( T_s \). Then we can write \( J(T_s) \) as

\[
J(T_s) = \bigcup_{i=1}^{s} J_i(T_s).
\]

Figure 5 gives an illustration of the elements contained in an order ideal in \( J_5(T_{12}) \). We see that an order ideal \( I \in J_5(T_{12}) \) must contain the elements labeled by squares, but does not contain any elements represented by open circles. The elements represented by solid circles may or may not appear in \( I \). That is, \( I \) can be decomposed into three parts, one is \( \{1, 2, 3, 4\} \), one is isomorphic to an order ideal of \( T_4 \) and one is isomorphic to an order ideal of \( T_7 \).

In general, for \( 1 \leq i \leq s \) and an order ideal \( I \in J_i(T_s) \), we can decompose it into three parts: one is \( \{1, 2, \ldots, i - 1\} \), one is isomorphic to an order ideal of \( T_{i-1} \) and one is isomorphic to an order ideal of \( T_{s-i} \) (some parts may be empty). We shall use this decomposition to prove the following recurrence relations for the number \( O_{p,r}(k) \).

Theorem 3.5 Let \( p \) be a positive integer and let \( r, k \) be nonnegative integers with \( 0 \leq r < p \). Then the number \( O_{p,r}(k) \) satisfies the recurrence relations

\[
O_{p,0}(k) = \sum_{i=0}^{k-1} O_{p,0}(i)O_{p,p-1}(k-1-i), \quad (3.1)
\]

\[
O_{p,r}(k) = \sum_{i=0}^{k} O_{p,0}(i)O_{p,r-1}(k-i), \quad \text{for } r > 0, \quad (3.2)
\]

with the initial value \( O_{p,r}(0) = 1 \) if \( r \geq 0 \).
Figure 5: The elements of an order ideal $I \in J_5(T_{12})$

Proof. If $k = 0$, that is, $s = r < p$. Then the unique order ideal in $T_r$ with property $P_p$ is $\emptyset$. By the definition of $O_{p,r}(k)$, we have that $O_{p,r}(0) = 1$.

Now suppose that $k \geq 1$. Let $I$ be an order ideal in $J(s)$ with property $H_p$. Then $I \in J_{ip+1}(T_s)$, where $0 \leq i \leq \left\lfloor \frac{s-1}{p} \right\rfloor$ since $I$ has property $H_p$. It is easily seen that $I$ can be decomposed into three parts: one is $\{1, 2, \ldots, ip\}$, one is isomorphic to an order ideal $I_1$ of $T_{ip}$ and one is isomorphic to an order ideal $I_2$ of $T_{s-1-ip}$. Since the absolute difference of any two numbers in different parts are larger than 1, we have that all of the three parts $\{1, 2, \ldots, ip\}$, $I_1$ and $I_2$ have property $H_p$. Conversely, given a pair $(I_1, I_2)$ of order ideals with property $H_p$, $I_1 \in T_{ip}$ and $I_2 \in T_{s-1-ip}$, we can recover an order ideal in $J_{ip+1}(T_s)$ by reversing the decomposition procedure. It is apparent the resulting order ideal has property $H_p$. Thus, the order ideals in $J_{ip+1}(T_s)$ with property $H_p$ are in one-to-one correspondence with pairs of order ideals with property $H_p$, one in $T_{ip}$ and one in $T_{s-1-ip}$.

It is easily seen that the number of order ideals in $T_{ip}$ with property $H_p$ is counted by $O_{p,0}(i)$. To enumerate the number of order ideals in $T_{s-1-ip}$ with property $H_p$, we consider two cases. If $r = 0$, namely $s = kp$, then $s - 1 - ip = (k - i - 1)p + (p - 1)$. It follows that the number of order ideals in $T_{s-1-ip}$ with property $H_p$ is counted by $O_{p,ip-1}(k - i - 1)$. If $r > 0$, then $s - 1 - ip = (k - i)p + r - 1$. So in this case, the number of order ideals in $T_{s-1-ip}$ with property $H_p$ is counted by $O_{p,r-1}(k - i)$, and the proof follows.

Proof of Theorem 3.1. From Lemma 3.4 we see that to prove Theorem 3.1 it suffices to
show that

\[ O_{p,r}(k) = R_{p+1,r+1}(k), \]

for all \( p > 0, 0 \leq r < p \) and \( k \geq 0 \). We proceed to show the assertion by induction on \( k \).

Recall that \( O_{p,r}(0) = 1 = R_{p+1,r+1}(0) \) for all \( r \geq 0 \). Assume that \( O_{p,r}(s) = R_{p+1,r+1}(s) \) for all \( p > 0, 0 \leq r < p \) and \( s < k \). Now we shall show that \( O_{p,r}(k) = R_{p+1,r+1}(k) \). In views of relations (1.4) and (3.1), we have \( O_{p,0}(k) = R_{p+1,1}(k) \) by induction hypothesis.

From relations (1.5) and (3.2), we have

\[ R_{p+1,2}(k) = \sum_{i=0}^{k} R_{p,1}(i) R_{p,1}(k-i) \]

and

\[ O_{p,1}(k) = \sum_{i=0}^{k} O_{p,0}(i) O_{p,0}(k-i). \]

Combining the above two relations with \( O_{p,0}(s) = R_{p+1,1}(s) \) for all \( 0 \leq s \leq k \), we can deduce that \( O_{p,1}(k) = R_{p+1,2}(k) \). By similar arguments, one can easily verify that \( O_{p,r}(k) = R_{p+1,r+1}(k) \) for all \( r > 1 \). Thus, we have \( O_{p,r}(k) = R_{p+1,r+1}(k) \) for all \( r \geq 0 \), which completes the proof. \( \blacksquare \)

Let \( p, r, k \) be nonnegative integers and \( r < p \). From Theorem 3.1, we see that the Raney numbers \( R_{p+1,r+1}(k) \) equal the numbers of \((kp+r, kp+r+1)\)-core partitions with parts that are multiples of \( p \). This gives a new combinatorial interpretation for these Raney numbers.

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