A REDUCTION THEORY FOR OPERATORS IN TYPE I, VON NEUMANN ALGEBRAS

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Abstract. In this paper, we study the structure of operators in a type I, von Neumann algebra $\mathcal{A}$. Inspired by the Jordan canonical form theorem, our main motivation is to figure out the relation between the structure of an operator $A$ in $\mathcal{A}$ and the property that a bounded maximal abelian set of idempotents contained in the relative commutant $\{A\}' \cap \mathcal{A}$ is unique up to similarity. Furthermore, we classify this class of operators with the property by $K$-theory for Banach algebras. Some views and techniques are from von Neumann’s reduction theory.

1. Introduction

It is well-known that the Jordan canonical form theorem states that each operator $A$ in $M_n(\mathbb{C})$ is similar to a direct sum of Jordan matrices and the direct sum is unique up to similarity. An equivalent statement is that any two (bounded) maximal abelian sets of idempotents $\mathcal{P}$ and $\mathcal{Q}$ in $\{A\}' \cap M_n(\mathbb{C})$ are similar to each other in $\{A\}' \cap M_n(\mathbb{C})$.

There are two natural ways when we consider to generalize the Jordan canonical form theorem. One way is to consider the generalization in the type $I_\infty$ factor instead of the type $I_n$ factor $M_n(\mathbb{C})$. We started our study in [6] and carried on in [10, 7]. In this ‘infinite’ case, we found that not every normal operator $N$ possesses the property that any two bounded maximal abelian sets of idempotents $\mathcal{P}$ and $\mathcal{Q}$ in $\{N\}'$ are similar to each other in $\{N\}'$. The multiplicity function of $N$ plays an important role, and it is required to be bounded if we want $N$ with the property. Our results in [10, 7] are proved based on the ‘bounded multiplicity’ condition. On the other hand, another way is to consider the generalization in type $I_n$ von Neumann algebras. It is also a natural question to ask whether the above property holds for every operator in a type $I_n$ von Neumann algebra $\mathcal{A}$. In this paper, we investigate the structure of operators in a type $I_n$ von Neumann algebra $\mathcal{A}$ and operators with the property in $\mathcal{A}$, and the relation between them.

Throughout this article, we only discuss Hilbert spaces which are complex and separable. Denote by $L(\mathcal{H})$ the set of bounded linear operators on a Hilbert space $\mathcal{H}$. Unless there is a danger of confusion, we will assume from now on that $\mu$ is (the completion of) a finite positive regular Borel measure supported on a compact subset $\Lambda$ of $\mathbb{C}$. For the sake of simplicity, we consider elements in $L^\infty(\mu)$ as multiplication operators on $L^2(\mu)$ and matrices in $M_n(L^\infty(\mu))$ as bounded linear operators on $(L^2(\mu))^n$. In this sense every operator $A$ in $M_n(L^\infty(\mu))$ is in the

2000 Mathematics Subject Classification. Primary 47A15, 47A65; Secondary 47C15.

Key words and phrases. Strongly irreducible operator, similarity invariant, reduction theory of von Neumann algebras, $K$-theory, finite frame.
form

\[ A = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}_{n \times n} L^2(\mu) \]

(1.1)

where the multiplication operator \( M_{f_{ij}} \) is abbreviated as \( f_{ij} \) in \( L^\infty(\mu) \) and \( i, j = 1, \ldots, n \). An idempotent \( P \) on \( \mathcal{H} \) is an operator in \( \mathcal{L}(\mathcal{H}) \) such that \( P^2 = P \). A projection \( Q \) in \( \mathcal{L}(\mathcal{H}) \) is an idempotent such that \( Q = Q^* \). For an operator \( A \) in \( M_n(L^\infty(\mu)) \), the relative commutant of \( A \) with respect to \( M_n(L^\infty(\mu)) \) is denoted by \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) = \{ B \in M_n(L^\infty(\mu)) : AB = BA \} \).

For an operator \( A \) in a type \( I \) von Neumann algebra \( M_n(L^\infty(\mu)) \), we need to introduce the following definition.

**Definition 1.1.** Let \( A \) be an operator in \( M_n(L^\infty(\mu)) \). We say that the strongly irreducible decomposition of \( A \) is unique up to similarity with respect to the relative commutant \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \) if for every two bounded maximal abelian sets of idempotents \( \mathcal{P} \) and \( \mathcal{Q} \) in \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \), there exists an invertible element \( X \) in \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \) such that \( X \mathcal{P} X^{-1} = \mathcal{Q} \). We abbreviate such a relative commutant \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \) as a relative commutant with Property ‘UDSR’.

By Definition 1.1, for every matrix \( A \) in \( M_n(\mathbb{C}) \), the strongly irreducible decomposition of \( A \) is unique up to similarity with respect to \( \{ A \}^\prime \cap M_n(\mathbb{C}) \). (Our reason to focus on the relative commutant \( \{ A \}^\prime \cap M_n(\mathbb{C}) \) is that \( M_n(\mathbb{C}) \) can be embedded into many \( C^* \) algebras, and in this sense \( \{ A \}^\prime \) is not always equal to \( \{ A \}^\prime \cap M_n(\mathbb{C}) \).

In the present paper, to generalize the Jordan canonical form theorem for operators in \( M_n(L^\infty(\mu)) \), the first question we need to deal with is whether the relative commutant \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \) contains a bounded maximal abelian set of idempotents for every operator \( A \) in \( M_n(L^\infty(\mu)) \). We figure out the answer to this question is negative in Example 2.9. Thus we characterize several necessary and sufficient conditions for the relative commutant \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \) containing a bounded maximal abelian set of idempotents, and these conditions are also can be used to distinguish for which kinds of operators in \( M_n(L^\infty(\mu)) \), the strongly irreducible decompositions of these operators are unique up to similarity with respect to the relative commutants. Precisely, one of our main theorems is stated as follows.

**Theorem 1.2.** Let \( A \) be an operator in \( M_n(L^\infty(\mu)) \). Then the following statements are equivalent:

1. the relative commutant \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \) contains a finite frame \( \{ P_k \}_{k=1}^m \);
2. the relative commutant \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \) contains a bounded maximal abelian set of idempotents;
3. the local structures of \( A \) are stated as in Lemma 2.8;
4. there exists an invertible element \( X \) in \( M_n(L^\infty(\mu)) \) and a unitary operator \( U \) such that \( UXAX^{-1}U^* \) is a direct integral of strongly irreducible operators with respect to a diagonal algebra \( \mathcal{P} \) and \( U^* \mathcal{P} U \subseteq M_n(L^\infty(\mu)) \).

If one of the above condition holds for \( A \), then for every two bounded maximal abelian sets of idempotents \( \mathcal{P} \) and \( \mathcal{Q} \) in \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \), there exists an invertible element \( X \) in \( \{ A \}^\prime \cap M_n(L^\infty(\mu)) \) such that \( X \mathcal{P} X^{-1} = \mathcal{Q} \).

For a building block \( J_{n_k} \) in \( M_{n_k}(L^\infty(\mu)) \) stated in the ‘Local Structure Lemma’ (Lemma 2.8), we compute the \( K_0 \) group of \( J_{n_k}^\prime \cap M_{n_k}(L^\infty(\mu)) \) in Lemma 4.4. On
The other hand, by Lemma 2.8 we apply the $K_0$ groups of the relative commutants to classify operators $A$ in $M_n(L^\infty(\mu))$ with the property that the relative commutant $\{A\}' \cap M_n(L^\infty(\mu))$ contains a finite frame.

The present paper is organized as follows. In section 2, we prove several preliminary lemmas and introduce the concept for a frame to be finite in a relative commutant. In Proposition 2.5, we prove that the statements (1) and (2) in Theorem 1.2 are equivalent. In Lemma 2.8, we characterize the local structures of $A$ with respect to the center of $M_n(L^\infty(\mu))$, for an operator $A$ in $M_n(L^\infty(\mu))$ with $\{A\}' \cap M_n(L^\infty(\mu))$ containing a finite frame. Then we present an example in which an operator in $M_2(L^\infty(\mu))$ is constructed such that the relative commutant contains no finite frames. By this example and the proof of the claims in Lemma 2.8, we can construct many examples of this type. Section 3 is mainly devoted to the proof of Theorem 3.9. Theorem 1.2 is mainly a combination of Theorem 3.9, Proposition 2.5, Lemma 2.8 and Proposition 4.3. As a corollary, we prove that every normal operator in $M_n(L^\infty(\mu))$ possesses the properties mentioned in Theorem 1.2. In section 4, with the aid of Section 2 and Section 3, we show the connection between the direct integrals of strongly irreducible operators and the operators in $M_n(L^\infty(\mu))$ with the property mentioned in (2) of Theorem 1.2. Then we discuss the ‘local’ $K$-theory of the relative commutant of $A$ in $M_n(L^\infty(\mu))$ with respect to the center of $M_n(L^\infty(\mu))$ and Lemma 2.8. By virtue of the discussion, we prove that the ‘local’ $K$-theory of the relative commutant of $A$ in $M_n(L^\infty(\mu))$ can be used as a complete similarity invariant to classify operators with the properties mentioned in Theorem 1.2.

2. THE LOCAL STRUCTURES OF $A$ IN $M_n(L^\infty(\mu))$ FOR $\{A\}' \cap M_n(L^\infty(\mu))$ CONTAINING A FINITE FRAME

The following two lemmas are devoted to proving Lemma 2.3. In the proof of Lemma 2.3, we introduce an algorithm to find an invertible operator $X$ in $M_n(L^\infty(\mu))$ such that $XPX^{-1}$ is diagonal, for a given idempotent $P$ in $M_n(L^\infty(\mu))$. This algorithm is also used in the proof of the main theorem.

**Lemma 2.1.** Let $P$ be a non-zero idempotent in $M_n(\mathbb{C})$ of the form

$$P = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}
\end{pmatrix}_{n \times n},$$

where $\text{rank}(P) = r > 0$ and $\alpha_{11} \neq 0$. Let $X$ be a lower triangular matrix of the form

$$X = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
-\alpha_{21} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_{n1} & \alpha_{11} & \cdots & 1
\end{pmatrix}_{n \times n}.$$  

Then $X$ is invertible in $M_n(\mathbb{C})$ such that the $(1,1)$ entry of $XPX^{-1}$ is 1 and the $(i,1)$ entry of $XPX^{-1}$ is 0 for $i = 2, \ldots, n$. 

Proof: If the rank of \( P \) is \( n \), then we obtain \( P = I_n \), where we denote by \( I_n \) the unit of \( M_n(\mathbb{C}) \). For \( 0 < \operatorname{rank}(P) < n \), by a computation, we obtain that \( X^{-1} \) is of the form

\[
X^{-1} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
\alpha_{21} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1} & \alpha_{n2} & \cdots & 1
\end{pmatrix}_{n \times n}.
\tag{2.3}
\]

Notice that \( \operatorname{rank}(P) = \operatorname{rank}(XP) \) and the \((i,1)\) entry of \( XP \) is 0 for \( i = 2, \ldots, n \). Without loss of generality, we assume that \( XP \) is of the form

\[
XP = \begin{pmatrix}
\beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\
0 & \beta_{22} & \cdots & \beta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \beta_{n2} & \cdots & \beta_{nn}
\end{pmatrix}_{n \times n},
\tag{2.4}
\]

and the first \( r \) rows \( \{\beta_i\}_{i=1}^{r} \) of \( XP \) are linear independent, where we denote by \( \beta_i \) the \( i \)-th row of \( XP \). Note that \( \alpha_{ii} = \beta_{ii} \) for \( i = 1, \ldots, n \).

We assert that every element of \( \{\beta_i\}_{i=r+1}^{n} \) is a linear combination of \( \beta_2, \ldots, \beta_r \).

By the foregoing assumption, every element of \( \{\beta_i\}_{i=r+1}^{n} \) is a linear combination of \( \beta_1, \ldots, \beta_r \). Assume that

\[
\beta_{r+s} = \lambda_1 \beta_1 + \cdots + \lambda_r \beta_r
\tag{2.5}
\]

where \( \lambda_i \in \mathbb{C} \) for \( i = 1, \ldots, r, s = 1, \ldots, n - r \). If \( \lambda_1 \neq 0 \), then the \((r+s,1)\) entry of \( XP \) will not be 0. This contradicts (2.4). Thus we obtain the assertion.

By the preceding assertion, there exists a lower triangular invertible matrix \( Y \) of the form

\[
Y = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \lambda_{r+1,2} & \cdots & \lambda_{r+1,r} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \lambda_{n,2} & \cdots & \lambda_{n,r} & 0 & \cdots & 1
\end{pmatrix}_{n \times n},
\tag{2.6}
\]

such that the \((r+s)\)-th row of \( YXP \) is 0 for \( s = 1, \ldots, n - r \).

Note that \( YXPX^{-1}Y^{-1} \) is an idempotent of the form

\[
YXPX^{-1}Y^{-1} = \begin{pmatrix}
P_{11} & R \\
0 & 0
\end{pmatrix}_{\mathbb{C}^{(r)}(\mathbb{C}^{(n-r)})},
\tag{2.7}
\]

where \( P_{11} \) is in \( M_r(\mathbb{C}) \). Thus \( P_{11} \) is an idempotent. Note that the standard trace of an idempotent \( Q \) (denoted by \( \operatorname{Tr}(Q) \)) in \( M_n(\mathbb{C}) \) is equal to the rank of \( Q \). Therefore, we obtain that

\[
r = \operatorname{rank}(P) = \operatorname{rank}(YXPX^{-1}Y^{-1}) \\
= \operatorname{Tr}(YXPX^{-1}Y^{-1}) \\
= \operatorname{Tr}(P_{11}) \\
= \operatorname{rank}(P_{11}).
\tag{2.8}
\]
Thus $P_{11} = I_r$ is the unit of $M_r(\mathbb{C})$. By the construction of $Y$, the $(i,1)$ entries of $XPX^{-1}$, $YXPX^{-1}$ and $YXPX^{-1}Y^{-1}$ are the same for $i = 1, \ldots, r$. Since $P_{11} = I_r$, we obtain that the $(1,1)$ entry of $XPX^{-1}$ is 1 and the $(i,1)$ entry of $XPX^{-1}$ is 0 for $i = 2, \ldots, r$. The equality $(YXP)X^{-1} = Y(XPX^{-1})$ yields that the $(i,1)$ entry of $YXPX^{-1}$ is 0 for $i = r+1, \ldots, n$. By the construction of $Y$ and the fact that the $(i,1)$ entry of $XPX^{-1}$ is 0 for $i = 2, \ldots, r$, we obtain that the $(i,1)$ entry of $XPX^{-1}$ is 0 for $i = r+1, \ldots, n$. The proof is finished. \hfill $\square$

It is well-known that in $M_n(\mathbb{C})$ every idempotent is similar to a diagonal projection. The reason that we restate this in the following lemma is to develop an algorithm which leads to a solution of a related problem raised in $M_n(L^\infty(\mu))$.

**Lemma 2.2.** If $P$ is an idempotent in $M_n(\mathbb{C})$, then there exists an invertible matrix $X$ in $M_n(\mathbb{C})$ such that $XPX^{-1}$ is diagonal in $M_n(\mathbb{C})$, where $X$ is a composition of finitely many invertible matrices as in (2.2) and row-switching unitary matrices and an invertible block matrix of the form

$$
\begin{pmatrix}
I_r & R \\
0 & I_{n-r}
\end{pmatrix} \in \mathbb{C}^{(r)} \times \mathbb{C}^{(n-r)}.
$$

**Proof.** If the idempotent $P$ is trivial in $M_n(\mathbb{C})$, then the invertible matrix $X$ can be chosen to be the unit. If $P$ is nontrivial, then we assume $0 < \text{rank}(P) = r < n$. This yields $0 < \text{Tr}(P) = \text{rank}(P) = r$. Thus there exists a nonzero entry denoted by $(i,i)$ in the main diagonal of $P$. Let $U_1$ be the elementary matrix switching all matrix elements on row 1 with their counterparts on row $i$. Then the $(1,1)$ entry of $U_1PU_1^*$ equals the $(i,i)$ entry of $P$. With respect to $U_1^*PU_1^*$, we construct an invertible operator $X_1$ as in (2.2). Then $P_1 = X_1U_1P(X_1U_1)^{-1}$ can be expressed in the form

$$
P_1 = \begin{pmatrix}
1 & \alpha_{12} & \cdots & \alpha_{1n} \\
0 & \alpha_{22} & \cdots & \alpha_{2n} \\
& \vdots & \ddots & \vdots \\
0 & \alpha_{n2} & \cdots & \alpha_{nn}
\end{pmatrix}_{n \times n}.
$$

Let $P_{22}$ be the matrix of the form

$$
P_{22} = \begin{pmatrix}
\alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \ddots & \vdots \\
\alpha_{n2} & \cdots & \alpha_{nn}
\end{pmatrix}_{(n-1) \times (n-1)}.
$$

Then $P_{22}$ is an idempotent in $M_{n-1}(\mathbb{C})$ and $\text{Tr}(P_{22}) = r - 1$. If $r = 1$, then $P_{22} = 0$. If $r > 1$, then there exists a nonzero entry $\alpha_{jj}$ in the main diagonal of $P_{22}$. Let $U_2$ be the elementary matrix in $M_n(\mathbb{C})$ switching all matrix elements on row 2 with their counterparts on row $j$. Then the (2,2) entry of $U_2X_1U_1P(U_2X_1U_1)^{-1}$ equals the $(j,j)$ entry of $X_1U_1P(X_1U_1)^{-1}$. With respect to $P_{22}$, we construct an invertible operator $X_2$ as in (2.2). Then $X_2P_{22}X_2^{-1}$ can be expressed in the form

$$
\begin{pmatrix}
1 & \beta_{23} & \cdots & \beta_{2n} \\
0 & \beta_{33} & \cdots & \beta_{3n} \\
& \vdots & \ddots & \vdots \\
0 & \beta_{n3} & \cdots & \beta_{nn}
\end{pmatrix}_{(n-1) \times (n-1)}.
$$
Let $X_2$ be of the form
\[
X_2 = \begin{pmatrix} 1 & 0 \\ 0 & \hat{X}_2 \end{pmatrix} \mathbb{C}^{(n-1)}.
\] (2.13)

Then $P_2(= X_2 U_2 X_1 U_1 P (X_2 U_2 X_1 U_1)^{-1})$ can be expressed in the form
\[
P_2 = \begin{pmatrix} 1 & 0 & \beta_{13} & \cdots & \beta_{1n} \\ 0 & 1 & \beta_{23} & \cdots & \beta_{2n} \\ 0 & 0 & \beta_{33} & \cdots & \beta_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \beta_{n3} & \cdots & \beta_{nn} \end{pmatrix}_{(n \times n)}.
\] (2.14)

After $r$ steps, we obtain $P_r$ of the form
\[
P_r = \begin{pmatrix} I_r & R \\ 0 & 0_{n-r} \end{pmatrix} \mathbb{C}^{(r)}_{\mathbb{C}^{(n-r)}}.
\] (2.15)

Let $X_{r+1}$ be of the form
\[
X_{r+1} = \begin{pmatrix} I_r & R \\ 0 & I_{n-r} \end{pmatrix} \mathbb{C}^{(r)}_{\mathbb{C}^{(n-r)}}.
\] (2.16)

Then $X_{r+1}$ is invertible and $X_{r+1}^{-1}$ is of the form
\[
X_{r+1}^{-1} = \begin{pmatrix} I_r & -R \\ 0 & I_{n-r} \end{pmatrix} \mathbb{C}^{(r)}_{\mathbb{C}^{(n-r)}}.
\] (2.17)

Thus $X_{r+1} P_r X_{r+1}^{-1}$ is diagonal in $M_n(\mathbb{C})$. And we obtain the invertible matrix $X = X_{r+1} X_r U_r \cdots X_1 U_1$ such that $XPX^{-1}$ is diagonal in $M_n(\mathbb{C})$. \hfill \Box

With the algorithm in the foregoing two lemmas, we prove the following lemma.

**Lemma 2.3.** If $P$ is an idempotent in $M_n(L^\infty(\mu))$, then there exists an invertible operator $X$ in $M_n(L^\infty(\mu))$ such that $XPX^{-1}$ is diagonal.

**Proof.** For the sake of simplicity, we write $P$ of the form
\[
P = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}_{n \times n} L^2(\mu),
\] (2.18)

where every $f_{ij}$ is in $L^\infty(\mu)$ for $i, j = 1, \ldots, n$. We use the relaxed convention of treating Borel representatives $f_{ij}$ as elements in $L^\infty(\mu)$ for $i, j = 1, \ldots, n$ such that $P(\lambda)$ is an idempotent in $M_n(\mathbb{C})$ for every $\lambda$ in $\Lambda$. Thus we obtain that
\[
f = \sum_{i=1}^{n} f_{ii}
\] (2.19)
is an integer-valued Borel simple function. With respect to the spectrum of $f$, we obtain a Borel partition $\{\Lambda_k\}_{k=0}^{n}$ of $\Lambda$ such that the standard trace of $P(\lambda)$ is $k$ for every $\lambda$ in $\Lambda_k$ and $k = 0, \ldots, n$. Notice that $\Lambda_k$ may be of $\mu$-measure zero for some $k$s. Without loss of generality, we assume $\Lambda = \Lambda_r$ for $1 \leq r \leq n-1$.

Before we apply Lemma 2.1 and Lemma 2.2 to construct the invertible operator, we need to modify the form in (2.18).
We observe that there exists a Borel function \( f_{i_1 i_1} \) in the main diagonal of \( P \) such that the Borel subset \( \Lambda_{r_1} \) of \( \Lambda_r \) of the form
\[
\Lambda_{r_1} \triangleq \{ \lambda \in \Lambda_r : |f_{i_1 i_1}(\lambda)| \geq \frac{r}{n} \}
\]
is not of \( \mu \)-measure zero. In a similar way, there exists a Borel function \( f_{i_2 i_2} \) in the main diagonal of \( P \) such that the Borel subset \( \Lambda_{r_2} \) of \( \Lambda_r \setminus \Lambda_{r_1} \) of the form
\[
\Lambda_{r_2} \triangleq \{ \lambda \in \Lambda_r \setminus \Lambda_{r_1} : |f_{i_2 i_2}(\lambda)| \geq \frac{r}{n} \}
\]
is not of \( \mu \)-measure zero. In this way, we obtain a Borel partition \( \{ \Lambda_{r_j} \}_{j=1}^k \) of \( \Lambda_r \) with \( k \leq n \). There exists a unitary operator \( U_1 \) in \( M_n(L^\infty(\mu)) \) such that \( U_1(\lambda) \) is an elementary matrix in \( M_n(\mathbb{C}) \) switching all matrix entries in row \( i_j \) with their counterparts in row 1 for every \( \lambda \) in \( \Lambda_{r_j} \) and \( j = 1, \ldots, k \). Thus the absolute value of the \((1, 1)\) entry of \( U_1PU_1^*(\lambda) \) for every \( \lambda \) in \( \Lambda_r \) is not less than \( rn^{-1} \). We write the operator \( U_1PU_1^* \) of the form
\[
U_1PU_1^* = \begin{pmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{pmatrix}_{n \times n} L^2(\mu)
\]
Thus we obtain \( |h_{11}(\lambda)| \geq \frac{r}{n} \) for every \( \lambda \) in \( \Lambda_r \). Let \( X_1 \) be constructed as in (2.2)
\[
X_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -h_{21}/h_{11} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -h_{n1}/h_{11} & 0 & \cdots & 1 \end{pmatrix}_{n \times n} L^2(\mu)
\]
Write \( P_1 = X_1U_1PU_1^{-1}X_1^{-1} \). By Lemma 2.1, \( P_1 \) can be expressed in the form
\[
P_1 = \begin{pmatrix} 1 & \varphi_{12} & \cdots & \varphi_{1n} \\ 0 & \varphi_{22} & \cdots & \varphi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \varphi_{n2} & \cdots & \varphi_{nn} \end{pmatrix}_{n \times n} L^2(\mu)
\]
Combining the preceding construction of \( U_1 \) and the algorithm developed in the proof of Lemma 2.2, we can construct an invertible operator \( X \) in \( M_n(L^\infty(\mu)) \) dependent on \( P \) such that \( XPX^{-1} \) is diagonal.

By applying the preceding lemma, we define a \( \mu \)-measurable function \( \text{Tr}(P) \) of the form
\[
\text{Tr}(P)(\lambda) \triangleq \text{Tr}(P(\lambda)) = \text{rank}(P(\lambda))
\]
for every idempotent \( P \) in \( M_n(L^\infty(\mu)) \) and almost every \( \lambda \) in \( \Lambda \), where \( \text{Tr} \) is the standard trace on \( P(\lambda) \). Denote by \( \mathcal{E}_n \) the set of central projections in \( M_n(L^\infty(\mu)) \). For every central projection \( E \) in \( \mathcal{E}_n \) there exists a Borel subset \( \Lambda_E \) of \( \Lambda \) such that
\[
\text{Tr}(E)(\lambda) = \begin{cases} 
1, & \text{if } \lambda \in \Lambda_E, \\
0, & \text{if } \lambda \in \Lambda \setminus \Lambda_E.
\end{cases}
\]

On generalizing the Jordan canonical form theorem, we need to introduce the concept ‘finite frame’ in the relative commutant of an operator \( A \) in \( M_n(L^\infty(\mu)) \).
Definition 2.4. Let $A$ be an operator in $M_n(L^\infty(\mu))$. A finite subset $\{P_k\}_{k=1}^m$ of idempotents in $\{A\}' \cap M_n(L^\infty(\mu))$ is said to be a finite frame of $\{A\}' \cap M_n(L^\infty(\mu))$ if the following conditions are satisfied:

1. $P_i P_j = P_j P_i = 0$ for $i \neq j$ and $i, j = 1, \ldots, k$;
2. the idempotent $\sum_{k=1}^m P_k$ equals to the identity of $M_n(L^\infty(\mu))$;
3. $P_k$ is minimal in the set $\{P \in \mathcal{P} : P(\lambda) \neq 0 \ a.e. \ [\mu] \text{ on } \Lambda_k\}$, where we write $\mathcal{P}_k = \{P \in \{A\}' \cap M_n(L^\infty(\mu)) : P^2 = P, PP_k = P_k P\}$ and $\Lambda_k = \{\lambda \in \Lambda : P_k(\lambda) \neq 0\}$ is the support of $\text{Tr}(P_k)$;
4. the first three items also hold for every $\{P_k E\}_{k=1}^m$ restricted on $\text{ran}E$, where $E$ is a central projection and $\text{Tr}(E)$ is supported on a Borel subset $\Lambda_E$ of $\Lambda$.

Furthermore, a finite frame $\{P_k\}_{k=1}^m$ in $\{A\}' \cap M_n(L^\infty(\mu))$ is said to be self-adjoint if $P_k$ is self-adjoint for $k = 1, \ldots, m$.

This concept is inspired by the ‘cross section’ in fibre bundles. The difference is that every element in a finite frame here is Borel measurable. In the rest of this section, we establish a relation between the local structures of $A$ in $M_n(L^\infty(\mu))$ and a finite frame in $\{A\}' \cap M_n(L^\infty(\mu))$. First, we give a necessary and sufficient condition for $\{A\}' \cap M_n(L^\infty(\mu))$ containing a finite frame.

Proposition 2.5. Let $A$ be an operator in $M_n(L^\infty(\mu))$. Then $\{A\}' \cap M_n(L^\infty(\mu))$ contains a finite frame if and only if $\{A\}' \cap M_n(L^\infty(\mu))$ contains a bounded maximal abelian set of idempotents $\mathcal{P}$.

Proof. Note that if $\{P_k\}_{k=1}^m$ is a finite frame of $\{A\}' \cap M_n(L^\infty(\mu))$, then $\{P_k\}_{k=1}^m$ and $\mathcal{E}$ generate a bounded maximal abelian set of idempotents $\mathcal{P}$ in $\{A\}' \cap M_n(L^\infty(\mu))$. Actually, every idempotent $P$ in $\mathcal{P}$ is a finite combination of idempotents $\{P_k\}_{k=1}^m$ cut by some central projections.

On the other hand, if $\mathcal{P}$ is a bounded maximal abelian set of idempotents in $\{A\}' \cap M_n(L^\infty(\mu))$, then we write $\mathcal{P}_0 = \{P \in \mathcal{P} : \text{Tr}(P) \text{ is supported on } \Lambda\}$. The set $\mathcal{P}_0$ is not empty, since $\mathcal{P}_0$ contains the identity of $M_n(L^\infty(\mu))$. Note that every $\text{Tr}(P)$ is a $\mu$-measurable integer-valued simple function for $P$ in $\mathcal{P}$. Thus there exists a maximal totally-ordered subset of $\mathcal{P}_0$. Since $\mathcal{P}$ is bounded and closed in the weak-operator topology, there exists a minimal idempotent $P_1$ in $\mathcal{P}_0$, which means there is no proper sub-idempotents of $P_1$ in $\mathcal{P}_0$. If $P_1 = I$, then $\{P_1\}$ is a finite frame of $\mathcal{P}$. Otherwise, denote by $\Lambda_1$ the support of $I - P_1$ and write $\mathcal{P}_1 = \{P \in (I - P_1)\mathcal{P} : P \text{ is supported on } \Lambda_1\}$. Thus there exists a bounded maximal totally-ordered subset of $\mathcal{P}_1$. By reduction, there exists a minimal idempotent $P_2$ in $\mathcal{P}_1$. If $P_2 \neq I - P_1$, then we iterate the preceding procedure. After finite steps the procedure stops and we obtain a finite subset $\{P_k\}_{k=1}^m$ in $\mathcal{P}$. It’s obvious that the set $\{P_k E\}_{k=1}^m$ satisfies the first three items restricted on $\text{ran}E$ in Definition 2.4 for every central projection $E$ in $M_n(L^\infty(\mu))$. Therefore, $\{P_k\}_{k=1}^m$ is a required finite frame in $\mathcal{P}$. □

In the following lemma, for an operator $A$ in $M_n(L^\infty(\mu))$ such that the relative commutant $\{A\}' \cap M_n(L^\infty(\mu))$ contains a finite frame $\{P_k\}_{k=1}^m$, we modify the finite frame to be self-adjoint.

Lemma 2.6. For an operator $A$ in $M_n(L^\infty(\mu))$ such that $\{A\}' \cap M_n(L^\infty(\mu))$ contains a finite frame $\{P_k\}_{k=1}^m$, there exists an invertible operator $X$ in $M_n(L^\infty(\mu))$ such that every $X P_k X^{-1}$ is diagonal for $k = 1, \ldots, m$. 
Proof. Assume that \( A \) is an operator in \( M_n(L^\infty(\mu)) \) such that \( \{A\}' \cap M_n(L^\infty(\mu)) \) contains a finite frame \( \{P_k\}_{k=1}^m \). There exists a finite Borel partition \( \{\Lambda_j\}_{j=1}^r \) of \( \Lambda \) such that for every central projection \( E_k \) as in (2.26) and \( k = 1, \ldots, m, \) the function \( \text{Tr}(P_kE_k) \) takes a constant a.e. \([\mu]\) on \( \Lambda \). Therefore without loss of generality, we assume that \( \text{Tr}(P_k(\lambda)) = r_k \) (a constant function) a.e. \([\mu]\) on \( \Lambda \) for \( k = 1, \ldots, m. \) By Lemma 2.3, there exists an invertible operator \( X_1 \) in \( M_n(L^\infty(\mu)) \) such that \( X_1P_1X_1^{-1} \) is diagonal and \( X_1P_2X_1^{-1} \) is of the form

\[
X_1P_2X_1^{-1} = \begin{pmatrix}
0 & 0 \\
0 & Q_k
\end{pmatrix}
\begin{pmatrix}
(L^2(\mu))^{(r_1)} \\
(L^2(\mu))^{(n-r_1)}
\end{pmatrix},
\]

where \( Q_k \) is an idempotent in \( M_{n-r_1}(L^\infty(\mu)) \) for \( k = 2, \ldots, m. \) Again by Lemma 2.3, there exists an invertible operator \( Y_2 \) in \( M_{n-r_1}(L^\infty(\mu)) \) such that \( Y_2Q_2Y_2^{-1} \) is diagonal and \( Y_2Q_2Y_2^{-1} \) is of the form

\[
Y_2Q_2Y_2^{-1} = \begin{pmatrix}
0 & 0 \\
0 & R_k
\end{pmatrix}
\begin{pmatrix}
(L^2(\mu))^{(r_2)} \\
(L^2(\mu))^{(n-r_2)}
\end{pmatrix},
\]

where \( R_k \) is an idempotent in \( M_{n-r_1-r_2}(L^\infty(\mu)) \) for \( k = 3, \ldots, m. \) Let \( X_2 \) be of the form

\[
X_2 = \begin{pmatrix}
I & 0 \\
0 & Y_2
\end{pmatrix}
\begin{pmatrix}
(L^2(\mu))^{(r_1)} \\
(L^2(\mu))^{(n-r_1)}
\end{pmatrix}.
\]

Then \( X_2 \) is invertible in \( M_n(L^\infty(\mu)) \). Iterating the preceding procedure, we obtain \( m - 1 \) invertible operators \( \{X_k\}_{k=1}^{m-1} \) in \( M_n(L^\infty(\mu)) \). The invertible operator \( X = X_{m-1} \cdots X_1 \) is as required in this lemma. \( \square \)

By Lemma 2.6, we investigate the local structures of \( A \) in \( M_n(L^\infty(\mu)) \) if the relative commutant \( \{A\}' \cap M_n(L^\infty(\mu)) \) contains a finite frame. For this purpose, we need to introduce the following definition.

**Definition 2.7.** For an operator \( A = (A_{ij})_{1 \leq i,j \leq n} \) in \( M_n(L^\infty(\mu)) \), the \( k \)-diagonal entries are those \( A_{ij} \) with \( j = i + k. \)

If \( \{A\}' \cap M_n(L^\infty(\mu)) \) contains a finite frame \( \{P_k\}_{k=1}^m \), then we observe that every \( \text{Tr}(P_k) \) is a \( \mu \)-measurable simple function on \( \Lambda \) for \( k = 1, \ldots, m. \) Thus there exists a finite Borel partition \( \{\Lambda_j\}_{j=1}^r \) of \( \Lambda \) with respect to \( \{\text{Tr}(P_k)\}_{k=1}^m \) such that every \( \text{Tr}(P_k) \) takes a constant a.e. \([\mu]\) on every \( \Lambda_j \) for \( j = 1, \ldots, r. \) Therefore, we can focus on the local structure of \( A \) restricted on each \( (L^2(\Lambda_j, \mu))^{(n)} \) respectively, and then combine them together. In this sense, we assume that every \( \text{Tr}(P_k) \) takes a constant a.e. \([\mu]\) on \( \Lambda \) for \( k = 1, 2, \ldots, m, \) and we obtain the following proposition.

**Proposition 2.8.** Let \( A \) be an operator in \( M_n(L^\infty(\mu)) \) and \( \{A\}' \cap M_n(L^\infty(\mu)) \) contains a finite frame \( \{P_k\}_{k=1}^m \) such that every \( \text{Tr}(P_k) \) takes a constant a.e. \([\mu]\) on \( \Lambda. \) Then there exists an invertible operator \( X \) in \( M_n(L^\infty(\mu)) \) such that \( XAX^{-1} \) is of the block diagonal form

\[
XAX^{-1} = \begin{pmatrix}
A_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_m
\end{pmatrix}_{m \times m},
\]
and $A_k$ in $M_{n_k}(L^\infty(\mu))$ is of the upper triangular form

$$A_k = \begin{pmatrix} A_{11}^k & A_{12}^k & \cdots & A_{1n_k}^k \\ 0 & A_{22}^k & \cdots & A_{2n_k}^k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n_kn_k}^k \end{pmatrix} \in L^2(\mu) \quad \text{for } k = 1, \ldots, m.$$ (2.31)

such that:

1. the equality $\sum_{k=1}^m n_k = n$ holds;
2. the equality $A_{jj}^k = A_{11}^k$ holds for $j = 2, \ldots, n_k$ and $k = 1, \ldots, m$;
3. the support of $A_{jj}^k + 1$ equals $\Lambda$ for $j = 1, \ldots, n_k - 1$ and $k = 1, \ldots, m$.

Proof. By Lemma 2.6, there exists an invertible operator $Y$ in $M_n(L^\infty(\mu))$ such that every $YP_kY^{-1}$ is a diagonal projection in $M_n(L^\infty(\mu))$ for $k = 1, \ldots, n$. Note that $\{YP_kY^{-1}\}_{k=1}^m$ is a finite frame in $\{YAY^{-1}\}' \cap M_n(L^\infty(\mu))$.

If $\{YP_kY^{-1}\}_{k=1}^m$ contains only one element, the identity of $M_n(L^\infty(\mu))$, then this frame and $E_n$ the set of central projections in $M_n(L^\infty(\mu))$, generate a bounded maximal abelian set of idempotents in $\{YAY^{-1}\}' \cap M_n(L^\infty(\mu))$ which is also $E_n$.

We prove (2) and (3) mentioned for $A_k$ in the proposition by two claims.

By (23) and for the sake of simplicity, there exists a unitary operator $U$ in $M_n(L^\infty(\mu))$ such that $UYAY^{-1}U^* = A_0$ is of the upper triangular form

$$A_0 = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ 0 & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{nn} \end{pmatrix} \in L^2(\mu). \quad \text{(2.32)}$$

(i) Based on this $n$-by-$n$ upper triangular operator-valued matrix, we claim that the equality $f_{11} = f_{ii}$ holds for $i = 1, 2, \ldots, n$.

If it is not true, then there exist an $\epsilon > 0$ and a Borel subset $\Lambda_\epsilon$ of $\Lambda$ with $\mu(\Lambda_\epsilon) > 0$ such that either the inequality $|f_{ii}(\lambda) - f_{11}(\lambda)| \geq \epsilon$ holds for every $\lambda$ in $\Lambda_\epsilon$ or the equality $f_{ii}(\lambda) = f_{11}(\lambda)$ holds for every $\lambda$ in $\Lambda_\epsilon$ for $i = 1, 2, \ldots, n$, where we can choose Borel representatives to fulfill the relations for every $\lambda$ in $\Lambda_\epsilon$.

We construct a nontrivial idempotent in $\{A_0\}' \cap M_n(L^\infty(\mu))$ to draw a contradiction. For this purpose, we switch entries in the main diagonal by similar transformations to simplify the computation. In the discussion that follows, we restrict $A_0$ in $M_n(L^\infty(\Lambda_\epsilon, \mu))$.

If $f_{k+1,k+1} = f_{11}$ on $\Lambda_\epsilon$ and $|f_{kk}(\lambda) - f_{11}(\lambda)| \geq \epsilon$ holds for every $\lambda$ in $\Lambda_\epsilon$, then we obtain the equality for $\phi = f_{k,k+1}/(f_{kk} - f_{k+1,k+1})$:

$$\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_{kk} & f_{k,k+1} \\ 0 & f_{k+1,k+1} \end{pmatrix} = \begin{pmatrix} f_{kk} & 0 \\ 0 & f_{k+1,k+1} \end{pmatrix} \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}. \quad \text{(2.33)}$$

Then we can switch the two main diagonal entries by a row-switching transformation:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{kk} & 0 \\ f_{k+1,k+1} & f_{kk} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{k+1,k+1} & 0 \\ 0 & f_{kk} \end{pmatrix}. \quad \text{(2.34)}$$

We construct an invertible operator $T = I_{M_{n-k}(L^\infty(\Lambda_\epsilon, \mu))} \oplus S_2 \oplus I_{M_{n-k}(L^\infty(\Lambda, \mu))}$ in $M_n(L^\infty(\Lambda_\epsilon, \mu))$, where $I_{M_{n-k}(L^\infty(\Lambda, \mu))}$ is the identity of $M_{k}(L^\infty(\Lambda_\epsilon, \mu))$ and $S_2$ in...
must be null. In the case of triangles

The expression of

Thus we obtain

By the construction of \( T \), we obtain that \( TA_0 T^{-1} \) is of the upper triangular form and the \((k, k)\) and \((k + 1, k + 1)\) entries of \( A_0 \) are switched to the \((k + 1, k + 1)\) and \((k, k)\) entries of \( TA_0 T^{-1} \) respectively. Iterating this construction and making the related similar transformation to \( A_0 \) one by one for finite steps, we obtain an upper triangular matrix \( A_1 \) in \( M_n(L^\infty(A_\epsilon, \mu)) \) of the form

\[
A_1 = \begin{pmatrix}
h_{11} & h_{12} & \cdots & h_{1n} \\
0 & h_{22} & \cdots & h_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & h_{nn}
\end{pmatrix} L^2(A_\epsilon, \mu)
\]

such that the first \( r \) main diagonal entries are equal to \( f_{11} \) on \( A_\epsilon \) and the equality \( |h_{kk}(\lambda) - h_{11}(\lambda)| \geq \epsilon \) holds for every \( \lambda \) in \( A_\epsilon \) for \( k = r + 1, r + 2, \ldots, n \).

With the preceding preparation, we construct a nontrivial idempotent \( P \) in \( \{A_1\}' \cap M_n(L^\infty(A_\epsilon, \mu)) \) of the form

\[
P = \begin{pmatrix} I_{h_{11}(L^\infty(A_\epsilon, \mu))} & R \\ 0 & 0 \end{pmatrix}
\]

It is sufficient to ensure the existence of \( R \) in \( M_{r \times (n-r)}(L^\infty(A_\epsilon, \mu)) \). Let \( R \) be of the form

\[
R = \begin{pmatrix}
\phi_{1, r+1} & \phi_{1, r+2} & \cdots & \phi_{1n} \\
\phi_{2, r+1} & \phi_{2, r+2} & \cdots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{r, r+1} & \phi_{r, r+2} & \cdots & \phi_{rn}
\end{pmatrix} L^2(A_\epsilon, \mu)
\]

If the equality \( A_1 P = PA_1 \) holds for the \((r, r + 1)\) entry of \( A_1 P \), we obtain the equality

\[
h_{rr} \phi_{r, r+1} = h_{r, r+1} + \phi_{r, r+1} h_{r+1, r+1}.
\]

Thus we obtain

\[
\phi_{r, r+1} = h_{r, r+1}/(h_{rr} - h_{r+1, r+1}).
\]

For the \((r - 1, r + 1)\) entry of \( A_1 P \), we obtain the equality

\[
h_{r-1, r+1} \phi_{r-1, r+1} = h_{r-1, r+1} + \phi_{r-1, r+1} h_{r+1, r+1}.
\]

Thus we obtain

\[
\phi_{r-1, r+1} = (h_{r-1, r+1} - h_{r-1, r+1} \phi_{r+1})/(h_{r-1, r+1} - h_{r+1, r+1}).
\]

In this way, we obtain \( \phi_{k, r+1} \) one by one for \( k = r, r - 1, \ldots, 1 \). For the \((r, r + 2)\) entry of \( A_1 P \), we obtain the equality

\[
h_{rr} \phi_{r, r+2} = h_{r, r+2} + \phi_{r, r+1} h_{r+1, r+2} + \phi_{r+2, r+1} h_{r+2, r+2}.
\]

Thus we obtain

\[
\phi_{r, r+2} = (h_{r, r+2} + \phi_{r, r+1} h_{r+1, r+2})/(h_{rr} - h_{r+2, r+2}).
\]

In this way and by entries in column \((r + 1)\) of \( P \), we obtain \( \phi_{k, r+2} \) one by one for \( k = r, r - 1, \ldots, 1 \). Similarly, we obtain the left columns of \( P \) one after another. And this \( P \) does exist in \( \{A_1\}' \cap M_n(L^\infty(A_\epsilon, \mu)) \). Notice that \( n > \text{Tr}(P)(\lambda) = r > 0 \) (a
constant) on \( \Lambda \) and \( A \) is similar to \( A \) in \( M_n(L^\infty(\Lambda, \mu)) \). Therefore there exists an idempotent \( Q \) in \( \{A\}' \cap M_n(L^\infty(\Lambda, \mu)) \) such that \( n > \text{Tr}(Q)(\lambda) = r > 0 \) (a constant) on \( \Lambda \). Let \( \text{Tr}(Q)(\lambda) = n \) on \( \Lambda \backslash \Lambda_r \). This is a contradiction between the existence of \( Q \) and the assumption that the frame only contains the identity of \( M_n(L^\infty(\mu)) \). Hence the equality \( f_{11} = f_{ii} \) holds for \( i = 1, 2, \ldots, n \). The claim is true.

(ii) Based on the \( n \)-by-\( n \) upper triangular operator-valued form (2.32), we claim that the support of \( f_{i,i+1} \) is \( \Lambda \) for \( i = 1, 2, \ldots, n-1 \), if a finite frame in the relative commutant contains only the identity of \( M_n(L^\infty(\mu)) \).

If the claim is not true, then there exist a positive integer \( r \) in \( \{2, \ldots, n\} \) and a Borel subset \( \Lambda_0 \) of \( \Lambda \) with \( \mu(\Lambda_0) > 0 \) such that:

1. \( f_{r-1,r} = 0 \) on \( \Lambda_0 \);
2. \( f_{i,i+1} \) is supported on \( \Lambda_0 \) for \( i = 1, \ldots, r-2 \) if \( r \geq 3 \).

By the proof of the preceding claim, without of loss of generality, we assume that:

1. \( f_{r-1,r} = 0 \) on \( \Lambda \);
2. there exists an \( \epsilon > 0 \) such that the inequality

\[
|f_{i,i+1}(\lambda)| \geq \epsilon
\]  

holds for every \( \lambda \) in \( \Lambda \) and \( i = 1, \ldots, r-2 \) if \( r \geq 3 \).

By the preceding claim and the assumption in (2.45), we construct an invertible operator \( S_r \) in \( M_n(L^\infty(\mu)) \) such that the equality

\[
\begin{pmatrix}
  f_{11} & \cdots & f_{1,r-1} & f_{1r} \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & f_{r-1,r-1} & 0 \\
  0 & \cdots & 0 & f_{rr}
\end{pmatrix}
\begin{pmatrix}
  f_{11} & \cdots & f_{1,r-1} & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & f_{r-1,r-1} & 0 \\
  0 & \cdots & 0 & f_{rr}
\end{pmatrix}
\]

holds. Let \( S'_r \) be of the form

\[
S'_r = \begin{pmatrix}
  1 & 0 & \cdots & 0 & \phi_{1r} \\
  0 & 1 & \cdots & 0 & \phi_{2r} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 1 & \phi_{r-1,r} \\
  0 & 0 & \cdots & 0 & 1
\end{pmatrix}_{r \times r}
\]

With respect to the \((r-2, r)\) entries of the products on both sides of the equality in (2.46), the equality

\[
f_{r-2,r-2}\phi_{r-2,r} + f_{r-2,r-1}\phi_{r-1,r} + f_{r-2,r} = \phi_{r-2,r}f_{rr}
\]

yields that \( \phi_{r-1,r} = -f_{r-2,r}/f_{r-2,r-1} \). Thus with respect to the \((r-3, r)\) entries of the products on both sides of the equality in (2.46), the equality

\[
f_{r-3,r-3}\phi_{r-3,r} + f_{r-3,r-2}\phi_{r-2,r} + f_{r-3,r-1}\phi_{r-1,r} + f_{r-3,r} = \phi_{r-3,r}f_{rr}
\]

yields that \( \phi_{r-2,r} = -(f_{r-3,r-1}\phi_{r-1,r} + f_{r-3,r})/f_{r-3,r-2} \). In this way, we obtain \( \phi_{ir} \) for \( i = r-1, r-2, \ldots, 2 \). We can choose 0 to be \( \phi_{1r} \). The existence of \( S'_r \) is proved.

Note that \( S_r = S'_r \oplus I_{M_n(L^\infty(\mu))} \) is invertible in \( M_n(L^\infty(\mu)) \). The first \( r \) rows and \( r \) columns of \( S_rA_0S_r^{-1} \) form an \( r \)-by-\( r \) square matrix which is block diagonal.
Denote by \( A'_{r+1} \) the \((r+1)\)-by-\((r+1)\) square matrix formed by the first \((r+1)\) rows and \((r+1)\) columns of \( S_rA_0S_r^{-1} \). Thus \( A'_{r+1} \) is of the block form

\[
A'_{r+1} = \begin{pmatrix}
J_1 & 0 & R_1 \\
0 & J_2 & R_2 \\
0 & 0 & f_{r+1,r+1}
\end{pmatrix}
\]  

(2.50)

where \( J_1 \) is the \((r-1)\)-by-\((r-1)\) square matrix formed by the first \((r-1)\) rows and \((r-1)\) columns of \( S_rA_0S_r^{-1} \) and \( J_2 = f_{rr} \). We show that there exists a Borel subset \( \Lambda_1 \) of \( \Lambda \) with \( \mu(\Lambda_1) > 0 \) such that \( A'_{r+1} \) is similar to a block diagonal form restricted in \( M_{r+1}(L^\infty(\Lambda_1, \mu)) \).

If \( R_2 = 0 \) on \( \Lambda \), then \( A'_{r+1} \) is unitarily equivalent to the block form

\[
A'_{r+1} \cong \begin{pmatrix}
J_2 & 0 & 0 \\
0 & J_1 & R_1 \\
0 & 0 & f_{r+1,r+1}
\end{pmatrix}
\]  

(2.51)

Thus if the \((r-1, r+1)\) entry \( f'_{r-1,r+1} \) of \( S_rA_0S_r^{-1} \) is 0, then a similar computation as for \( S'_r \) in (2.46) and (2.47) yields that the operator \( A'_{r+1} \) is similar to a block diagonal matrix of the form

\[
A'_{r+1} \sim \begin{pmatrix}
J_2 & 0 & 0 \\
0 & J_1 & 0 \\
0 & 0 & f_{r+1,r+1}
\end{pmatrix}
\]  

(2.52)

If \( R_2 \neq 0 \), then there exists a Borel subset \( \Lambda_1 \) of \( \Lambda \) with \( \mu(\Lambda_1) > 0 \) such that the inequality \( |f'_{r-1,r+1}(\lambda)| \geq \epsilon_1 \) holds for some \( \epsilon_1 > 0 \) and every \( \lambda \) in \( \Lambda_1 \). Thus let \( J'_1 \) be the operator restricted in \( M_{r}(L^\infty(\Lambda_1, \mu)) \) of the form

\[
J'_1 = \begin{pmatrix}
J_1 & R_1 \\
0 & f_{r+1,r+1}
\end{pmatrix}
\]  

(2.53)

Therefore \( J'_1 \) and \( J_2 \) restricted in \( M_r(L^\infty(\Lambda_1, \mu)) \) and \( L^\infty(\Lambda_1, \mu) \) respectively satisfy conditions (1’) and (2’) mentioned in (2.45), where the lower bound with respect to the 1-diagonal entries in \( J'_1 \) and \( J_2 \) is \( \min\{\epsilon, \epsilon_1\} \).

On the other hand, if \( R_2(\lambda) \neq 0 \) a.e. \( [\mu] \) on \( \Lambda \), then there exists a Borel subset \( \Lambda_1 \) of \( \Lambda \) with \( \mu(\Lambda_1) > 0 \) such that the inequality \( |R_{2}(\lambda)| \geq \epsilon_1 \) holds for some \( \epsilon_1 > 0 \) and every \( \lambda \) in \( \Lambda_1 \) in (2.50). We need to consider two subcases. In one subcase, the \((r-1, r+1)\) entry \( f'_{r-1,r+1} \) of \( S_rA_0S_r^{-1} \) is 0 on \( \Lambda_1 \), then a computation as for \( S'_r \) in (2.46) and (2.47) yields that \( A'_{r+1} \) is similar to a block diagonal form restricted in \( M_{r+1}(L^\infty(\Lambda_1, \mu)) \)

\[
A'_{r+1} \sim \begin{pmatrix}
J_1 & 0 & 0 \\
0 & J_2 & R_2 \\
0 & 0 & f_{r+1,r+1}
\end{pmatrix}
\]  

(2.54)

Let \( J'_2 \) be the operator restricted in \( M_2(L^\infty(\Lambda_1, \mu)) \) of the form

\[
J'_2 = \begin{pmatrix}
J_2 & R_2 \\
0 & f_{r+1,r+1}
\end{pmatrix}
\]  

(2.55)

Therefore \( J'_1 \) and \( J'_2 \) restricted in \( M_{r+1}(L^\infty(\Lambda_1, \mu)) \) respectively satisfy conditions (1’) and (2’) mentioned in (2.45), where the lower bound with respect to the 1-diagonal entries in \( J'_1 \) and \( J_2 \) is \( \min\{\epsilon, \epsilon_1\} \).

In the other subcase, without loss of generality, we assume that the inequality \( |f'_{r-1,r+1}(\lambda)| \geq \epsilon_2 \) holds a.e. \( [\mu] \) on \( \Lambda_1 \) for the \((r-1, r+1)\) entry \( f'_{r-1,r+1} \) of \( S_rA_0S_r^{-1} \).
If \( r = 2 \), then \( A_{r+1}' \) is similar to a block diagonal form as in (2.54). If \( r > 2 \), then \( A_{r+1}' \) is similar to a block diagonal form as in (2.51).

Let \( S_{r+1}' \) be the invertible operator in \( M_{r+1}(L^\infty(\Lambda_1, \mu)) \) such that \( S_{r+1}'A_{r+1}'S_{r+1}^{-1} \) is of the block diagonal form as discussed in the preceding paragraphs. The operator \( S_{r+1}' = S_{r+1}' \oplus I_{M_{n-r+1}(L^\infty(\Lambda_1, \mu))} \) is invertible in \( M_n(L^\infty(\Lambda_1, \mu)) \) and we write \( A_{r+1}' = S_{r+1}'S_A A_0 S^{-1} S^{-1}_{r+1} \). The first \((r+1)\) rows and \((r+1)\) columns of \( A_{r+1}' \) form an \((r+1)\)-by-\((r+1)\) square matrix which is block diagonal. Denote by \( A_{r+2}' \) the \((r+2)\)-by-\((r+2)\) square matrix formed by the first \((r+2)\) rows and \((r+2)\) columns of \( A_{r+1}' \). As the preceding discussion, there exists a Borel subset \( \Lambda_2 \) of \( \Lambda_1 \) with \( \mu(\Lambda_2) > 0 \) and an invertible operator \( S_{r+2}' \) in \( M_n(L^\infty(\Lambda_2, \mu)) \) such that \((r+2)\)-by-\((r+2)\) square matrix formed by the first \((r+2)\) rows and \((r+2)\) columns of \( S_{r+2}'A_{r+1}'S_{r+2}^{-1} \) is of the block diagonal form. Iterating this construction for finite steps, we obtain a Borel subset \( \Lambda_m \) of \( \Lambda \) with \( \mu(\Lambda_m) > 0 \) and an invertible operator \( S_m \) in \( M_n(L^\infty(\Lambda_m, \mu)) \) such that \( S_m A_0 S^{-1}_m \) is of the block diagonal form

\[
S_m A_0 S^{-1}_m = \begin{pmatrix}
J_{r_1} & 0 & \cdots & 0 \\
0 & J_{r_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & J_{r_k}
\end{pmatrix},
\]

(2.56)

where \( J_{r_i} \) in \( M_{r_i}(L^\infty(\Lambda_m, \mu)) \) for \( i = 1, 2, \ldots, k \) satisfies conditions (1') and (2') mentioned in (2.45) for some \( \epsilon > 0 \) and the \( \Lambda_m \) instead of \( \Lambda \). By the construction of \( S_m A_0 S^{-1}_m \), we obtain that \( 1 \leq r_1 < n \). Therefore there exists a nontrivial idempotent \( P \) in the relative commutant \( \{A_0\}' \cap M_n(L^\infty(\mu)) \) such that \( 0 < \text{Tr}(P)(\lambda) < n \) for every \( \lambda \) in \( \Lambda_m \) and \( \text{Tr}(P)(\lambda) = n \) for every \( \lambda \) in \( \Lambda \setminus \Lambda_m \). This is a contradiction between the existence of \( P \) and the assumption that a frame in the relative commutant \( \{A_0\}' \cap M_n(L^\infty(\mu)) \) only contains the identity of \( M_n(L^\infty(\mu)) \). Hence we prove the second claim mentioned between (2.44) and (2.45).

If \( \{YP_kY^{-1}\}_{k=1}^m \) contains more than one element, then \( YAY^{-1} \) can be expressed in the diagonal form

\[
YAY^{-1} = \begin{pmatrix}
B_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & B_m
\end{pmatrix}_{m \times m},
\]

(2.57)

where every \( B_k \) is the restriction of \( YAY^{-1} \) on the range of \( YP_kY^{-1} \) for \( k = 1, \ldots, m \). Write \( \text{Tr}(P_k)(\lambda) = n_k \) a.e. \([\mu]\) on \( \Lambda \). Then the restriction of \( YP_kY^{-1} \) on its range becomes the identity of \( M_{n_k}(L^\infty(\mu)) \) and \( B_k \) is in \( M_{n_k}(L^\infty(\mu)) \). In particular, the identity of \( M_{n_k}(L^\infty(\mu)) \) forms a finite frame in \( \{B_k\}' \cap M_{n_k}(L^\infty(\mu)) \). Therefore by (2, Theorem 2), there exists a unitary operator \( U_k \) in \( M_{n_k}(L^\infty(\mu)) \) such that \( U_k B_k U_k^* \) is of the upper triangular form

\[
U_k B_k U_k^* = \begin{pmatrix}
A_{11}^k & A_{12}^k & \cdots & A_{1n_k}^k \\
0 & A_{22}^k & \cdots & A_{2n_k}^k \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{n_kn_k}^k
\end{pmatrix}_{n_k \times n_k},
\]

(2.58)

Write \( A_k = U_k B_k U_k^* \). Then by the foregoing proof, we obtain that \( A_k \) is with the required properties for \( k = 1, \ldots, m \).

\[ \square \]
Note that the reverse assertion is obvious by (10, Lemma 3.1). Therefore for an operator in \( M_2(L^\infty(\mu)) \), we obtain a necessary and sufficient condition to connect the structures of \( A \) and a finite frame in the relative commutant \( \{A\}' \cap M_2(L^\infty(\mu)) \).

We finish this section with an example to show that there exists an operator in \( M_2(L^\infty(\mu)) \) such that the relative commutant contains no finite frames.

**Example 2.9.** Let \( A \in M_2(L^\infty(\mu)) \) be of the form

\[
A = \begin{pmatrix} f & 1 \\ 0 & -2f \end{pmatrix} L^2(\mu)'
\]

(2.59)

where \( f \) is injective and the essential range of \( f \) (as a function in \( L^\infty(\mu) \)) is the interval \([0, 1]\). If \( P \) is an idempotent in the relative commutant \( \{A\}' \cap M_2(L^\infty(\mu)) \), then we can express \( P \) in the form

\[
P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} L^2(\mu)'
\]

(2.60)

where \( p_{ij} \) is in \( L^\infty(\mu) \) for \( 1 \leq i, j \leq 2 \). The equality \( AP = PA \) yields that

\[
\begin{cases}
-2fp_{22} = p_{21} - 2p_{22}f; \\
fp_{12} + p_{22} = p_{11} - 2p_{12}f.
\end{cases}
\]

(2.61)

Therefore, we obtain

\[
\begin{aligned}
p_{21} &= 0; \\
3fp_{12} &= p_{11} - p_{22}.
\end{aligned}
\]

(2.62)

Since \( P \) is an idempotent, we obtain that \( p_{ii} \) is a characteristic function in \( L^\infty(\mu) \) for \( i = 1, 2 \). A further computation shows that the idempotents in the relative commutant \( \{A\}' \cap M_2(L^\infty(\mu)) \) form an abelian set. Note that this abelian set of idempotents is unbounded. Thus there does not exist an idempotent \( Q \) in \( \{A\}' \cap M_2(L^\infty(\mu)) \) such that \( \text{Tr}(Q)(\lambda) = 1 \) a.e. \([\mu]\) on \( \Lambda \). If \( \{A\}' \cap M_2(L^\infty(\mu)) \) contains a finite frame \( \{P_{k_i}\}_{k_i=1}^m \), then there exists a \( P_{k_0} \) in the finite frame such that \( f(\text{supp}(\text{Tr}(P_{k_0}))) \) contains an open neighborhood \( 0 \). Thus \( \text{Tr}(P_{k_0}) \) can not take a constant on its support and we can construct a proper sub-idempotent of \( P_{k_0} \) in \( \{A\}' \cap M_2(L^\infty(\mu)) \) supported on \( \text{supp} \text{Tr}(P_{k_0}) \). This is a contradiction.

3. Bounded maximal abelian sets of idempotents in \( \{A\}' \cap M_2(L^\infty(\mu)) \)

For a matrix \( A \in M_n(\mathbb{C}) \), any two (bounded) maximal abelian sets of idempotents \( \mathcal{S} \) and \( \mathcal{D} \) in \( \{A\}' \cap M_n(\mathbb{C}) \) are similar to each other in \( \{A\}' \cap M_n(\mathbb{C}) \). In this section, for \( A \) in \( M_n(L^\infty(\mu)) \) such that \( \{A\}' \cap M_n(L^\infty(\mu)) \) contains a finite frame, we prove that every two bounded maximal abelian sets of idempotents \( \mathcal{S} \) and \( \mathcal{D} \) in \( \{A\}' \cap M_n(L^\infty(\mu)) \) are similar to each other in \( \{A\}' \cap M_n(L^\infty(\mu)) \).

We need to mention that in (10, Theorem 3.3) we prove a special case for the motivation mentioned in the preceding paragraph. That is if an operator \( A \) is as in the form of (2.31)

\[
A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{pmatrix}_{n \times n} L^2(\mu)
\]

(3.1)
such that:

1. $A_{11}$ corresponds to an injective Borel function $\phi$ in $L^\infty(\mu)$;
2. the equality $A_{jj} = A_{11}$ holds for $j = 2, \ldots, n$;
3. the support of $A_{j,j+1}$ equals $\Lambda$ for $j = 1, \ldots, n - 1$,

then every two bounded maximal abelian sets of idempotents $\mathcal{P}$ and $\mathcal{Q}$ in the relative commutant $\{A^{(k)}\}' \cap M_{nk}(L^\infty(\mu))$ are similar to each other in $\{A^{(k)}\}' \cap M_{nk}(L^\infty(\mu))$, where $k$ is a positive integer. We generalize the models applied in ([10], Theorem 3.3) to be the forms characterized in (2.30) and (2.31) to continue our study in this section.

Part of our main theorem is abstracted from the following example.

**Example 3.1.** Let $A$ in $M_4(L^\infty(\mu))$ be of the form

$$
A = \begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}, \quad \text{and} \quad A_1 = \begin{pmatrix} f_1 & f_2 \\ 0 & f_1 \end{pmatrix} L^2(\mu), \quad A_2 = \begin{pmatrix} f_2 & f_2 \\ 0 & f_1 \end{pmatrix} L^2(\mu),
$$

(3.2)

where $f_i$ in $L^\infty(\mu)$ is supported on $\Lambda$ for $i = 1, 2$. Then we prove that for every two bounded maximal abelian sets of idempotents $\mathcal{P}$ and $\mathcal{Q}$ in $\{A\}' \cap M_4(L^\infty(\mu))$, there exists an invertible operator $X$ in $\{A\}' \cap M_4(L^\infty(\mu))$ such that $X \mathcal{P} X^{-1} = \mathcal{Q}$. We divide the proof into two parts.

First, we assert that for every idempotent $P$ in $\{A\}' \cap M_4(L^\infty(\mu))$, there exists an invertible operator $X$ in $\{A\}' \cap M_4(L^\infty(\mu))$ such that $X P X^{-1}$ is diagonal.

For this purpose, we characterize the relative commutant of $A$ in $M_4(L^\infty(\mu))$. For an element $B$ in $\{A\}' \cap M_4(L^\infty(\mu))$, we can write $B$ in the block form

$$
B = \begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}
$$

(3.3)

such that $B_{ij}$ satisfies the equality $A_i B_{ij} = B_{ij} A_j$ for $i, j = 1, 2$. By a similar computation as in (2.60), we obtain that $B_{ij}$ can be expressed in the form

$$
B_{ij} = \begin{pmatrix}
\varphi_{ij}^{11} & \varphi_{ij}^{12} \\
\varphi_{ij}^{21} & \varphi_{ij}^{22}
\end{pmatrix} L^2(\mu),
$$

(3.4)

where $\varphi_{kl}^{ij}$ is in $L^\infty(\mu)$ for $k, l = 1, 2$. The equality $A_i B_{ii} = B_{ii} A_i$ yields that $f_i \varphi_{ii}^{jj} = \varphi_{ij}^{11} f_j$ for $i = 1, 2$. By $f_j(\lambda) \neq 0$ for almost every $\lambda$ in $\Lambda$ and $i = 1, 2$, we obtain that $\varphi_{ij}^{ii} = \varphi_{ij}^{ii}$ a.e. $[\mu]$ on $\Lambda$ and $i = 1, 2$. Thus we abbreviate $\varphi_{ij}^{ii}$ as $\varphi^{ii}$ for $j = 1, 2$. The equality $A_i B_{ij} = B_{ij} A_j$ yields that $f_i \varphi_{jj}^{ij} = \varphi_{ij}^{11} f_j$ for $i = 1, 2$ and $i \neq j$. Thus $B$ can be expressed in the form

$$
B = \begin{pmatrix}
\varphi^{11} & \varphi_{12}^{11} & \varphi_{12}^{12} & L^2(\mu) \\
0 & \varphi^{11} & \varphi_{12}^{12} & L^2(\mu) \\
\varphi_{11}^{21} & \varphi_{12}^{21} & \varphi_{22}^{22} & L^2(\mu) \\
0 & \varphi^{22} & \varphi_{22}^{22} & L^2(\mu)
\end{pmatrix}
$$

(3.5)

Let $U \in M_4(L^\infty(\mu))$ be of the form

$$
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} L^2(\mu)
$$

(3.6)
Then $UBU^* \in \{ UAU^* \} \cap M_4(L^\infty(\mu))$ is of the form

$$UBU^* = \begin{pmatrix} \varphi_{11}^{11} & \varphi_{11}^{12} & \varphi_{12}^{11} & \varphi_{12}^{12} \\ \varphi_{21}^{11} & \varphi_{22}^{11} & \varphi_{21}^{12} & \varphi_{22}^{12} \\ 0 & 0 & \varphi_{11}^{21} & \varphi_{22}^{21} \\ 0 & 0 & \varphi_{21}^{22} & \varphi_{22}^{22} \end{pmatrix} L^2(\mu).$$

(3.7)

Note that $\varphi_{11}^{ij}$ and $\varphi_{22}^{ij}$ satisfy the equality $\int_{ij} \varphi_{22}^{ij} = \varphi_{11}^{ij} f_j$ for $i \neq j$ and $i, j = 1, 2$.

Let $P$ be an idempotent in $\{ UAU^* \} \cap M_4(L^\infty(\mu))$. Then by (3.7) the idempotent $P$ can be expressed in the form

$$P = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}, \quad P_{11} = \begin{pmatrix} \varphi_{11}^{11} & \varphi_{11}^{12} \\ \varphi_{21}^{11} & \varphi_{22}^{11} \end{pmatrix}, \quad P_{22} = \begin{pmatrix} \varphi_{11}^{12} & \varphi_{21}^{12} \\ \varphi_{22}^{12} & \varphi_{22}^{22} \end{pmatrix},$$

where we obtain that $P_{ii}$ is an idempotent in $M_4(L^\infty(\mu))$ for $i = 1, 2$. Let $R$ be of the form

$$R = \begin{pmatrix} 0 & P_{12} \\ 0 & 0 \end{pmatrix}.$$

(3.9)

Then $R$ is in $\{ UAU^* \} \cap M_4(L^\infty(\mu))$. Denote by $\sigma_{\{ UAU^* \} \cap M_4(L^\infty(\mu))}(R)$ the spectrum of $R$ in the unital Banach algebra $\{ UAU^* \} \cap M_4(L^\infty(\mu))$. Thus the equality

$$\sigma_{\{ UAU^* \} \cap M_4(L^\infty(\mu))}(SR) = \sigma_{\{ UAU^* \} \cap M_4(L^\infty(\mu))}(RS) = \{ 0 \}$$

(3.10)

holds for every operator $S$ in $\{ UAU^* \} \cap M_4(L^\infty(\mu))$. Thus the equality

$$(2P - I)(2P - I - R) = I - (2P - 1)R$$

(3.11)

yields that $2P - I - R$ is invertible in $\{ UAU^* \} \cap M_4(L^\infty(\mu))$. (Note that $2P - I$ is invertible in $\{ UAU^* \} \cap M_4(L^\infty(\mu))$. Since the equality $(P - R)^2 = P + R^2 - RP - PR = P - R$ yields that $R^2 - RP - PR + R = 0$, we obtain

$$(P - R)(2P - I - R) = P - PR - 2RP + R + R^2 = P - RP = (2P - I - R)P,$$

(3.12)

which means $P$ is similar to $P - R$ in $\{ UAU^* \} \cap M_4(L^\infty(\mu))$. The idempotent $P - R$ is of the form

$$P - R = \begin{pmatrix} \varphi_{11}^{11} & \varphi_{11}^{12} & 0 & 0 \\ \varphi_{11}^{21} & \varphi_{11}^{22} & 0 & 0 \\ 0 & 0 & \varphi_{11}^{11} & \varphi_{12}^{21} \\ 0 & 0 & \varphi_{21}^{22} & \varphi_{22}^{22} \end{pmatrix} L^2(\mu).$$

(3.13)

Next, we need to construct an invertible operator in $\{ UAU^* \} \cap M_4(L^\infty(\mu))$ such that $P - R$ is similar to a diagonal projection.

Without loss of generality, we assume that $\text{Tr}(P_{11})(\lambda) = 1$ for every $\lambda$ in $\Lambda$. By the proof of Lemma 2.3, there exists a unitary operator $V$ in $M_4(L^\infty(\mu))$ such that $V(P - R)V^*$ is of the form

$$V(P - R)V^* = \begin{pmatrix} \psi_{11}^{11} & \psi_{11}^{12} & 0 & 0 \\ \psi_{11}^{21} & \psi_{11}^{22} & 0 & 0 \\ 0 & 0 & \psi_{11}^{11} & \psi_{12}^{12} \\ 0 & 0 & \psi_{21}^{22} & \psi_{22}^{22} \end{pmatrix} L^2(\mu).$$

(3.14)
where $|\psi^{11}(\lambda)| \geq 1/2$ for every $\lambda$ in $\Lambda$. Let $Y$ be in the form

$$
Y = \begin{pmatrix}
1 & 0 & 0 & 0 & L^2(\mu) \\
-\psi_{11}/\psi^{11} & 1 & 0 & 0 & L^2(\mu) \\
0 & 0 & 1 & 0 & L^2(\mu) \\
0 & 0 & -\psi_{22}/\psi^{11} & 1 & L^2(\mu)
\end{pmatrix}
$$

(3.15)

Then by Lemma 2.3, we obtain that $YV(P - R)V^*Y^{-1}$ is diagonal in $M_4(L^\infty(\mu))$. By the constructions of $V$ and $Y$, the operator $V^*YV$ is invertible in $\{UAU^*\}' \cap M_4(L^\infty(\mu))$. Thus $V^*YV(2P - I - R)$ is an invertible operator in $\{UAU^*\}' \cap M_4(L^\infty(\mu))$. Therefore $U^*V^*YV(2P - I - R)U$ is an invertible operator in $\{A\}' \cap M_4(L^\infty(\mu))$ as required and we achieve the assertion.

Let $\mathcal{E}$ be the set of diagonal projections in $\{A\}' \cap M_4(L^\infty(\mu))$. Then by a similar computation as in (2.61) and (2.62), we can verify that $\mathcal{E}$ is a bounded maximal abelian set of idempotents in $\{A\}' \cap M_4(L^\infty(\mu))$. Assume that $\mathcal{P}$ is a bounded maximal abelian set of idempotents in $\{A\}' \cap M_4(L^\infty(\mu))$. Then by Proposition 2.5, $\{A\}' \cap M_4(L^\infty(\mu))$ contains a finite frame $\mathcal{P}_0$. By the preceding assertion there exists an invertible operator $X$ in $\{A\}' \cap M_4(L^\infty(\mu))$ such that every element in $X\mathcal{P}X^{-1}$ is diagonal. Thus $X\mathcal{P}X^{-1}$ is included in $\mathcal{E}$. The maximality of $X\mathcal{P}X^{-1}$ in $\{A\}' \cap M_4(L^\infty(\mu))$ ensures that $X\mathcal{P}X^{-1} = \mathcal{E}$. Therefore, every bounded maximal abelian set of idempotents is similar to $\mathcal{E}$ in $\{A\}' \cap M_4(L^\infty(\mu))$. This example is finished. 

To generalize Example (3.1), we need the following lemmas.

**Lemma 3.2.** Let $A$ in $M_n(L^\infty(\mu))$ be of the upper triangular form

$$
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
0 & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{nn}
\end{pmatrix}_{n \times n}
$$

(3.16)

such that

1. $A_{ii} = A_{ij}$ for $i, j = 1, 2, \ldots, n$;
2. $A_{i,i+1}$ is supported on $\Lambda$ for $i = 1, 2, \ldots, n - 1$.

If $B$ is an operator in $\{A\}' \cap M_n(L^\infty(\mu))$, then $B$ is of the form

$$
B = \begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1n} \\
0 & B_{22} & \cdots & B_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_{nn}
\end{pmatrix}_{n \times n}
$$

(3.17)

such that $B_{ii} = B_{jj}$ for $i, j = 1, 2, \ldots, n$.

**Proof.** For the sake of simplicity, we use the relaxed convention of treating Borel representatives as elements in $L^\infty(\mu)$ and consider $A_{ij}$ and $B_{ij}$ as functions in
$L^\infty(\mu)$ for $i, j = 1, \ldots, n$. Let $B$ in $\{A\}' \cap M_n(L^\infty(\mu))$ be of the form

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix} \in L^2(\mu)$$

(3.18)

For the $(n, 2)$ entry of $AB$, by $AB = BA$ we obtain the equality

$$A_{nn}B_{n2} = B_{n1}A_{12} + B_{n2}A_{22}. \quad (3.19)$$

By $A_{22} = A_{nn}$ the equality (3.19) yields that $B_{n1}A_{12} = 0$. Since $A_{12}$ is supported on $\Lambda$, we obtain that $B_{n1} = 0$ a.e. $[\mu]$ on $\Lambda$. Thus for the $(n, 3)$ entry of $AB$, we obtain the equality

$$A_{nn}B_{n3} = B_{n2}A_{23} + B_{n3}A_{33}. \quad (3.20)$$

By $A_{33} = A_{nn}$, the equality (3.20) yields that $B_{n2}A_{23} = 0$. Since $A_{23}$ is supported on $\Lambda$, we obtain that $B_{n2} = 0$ a.e. $[\mu]$ on $\Lambda$. In this way and by the equalities with respect to the entries in the $n$-th row of $AB$, we obtain that $B_{ni} = 0$ a.e. $[\mu]$ on $\Lambda$ for $i = 1, \ldots, n - 1$. Based on this, by the equalities with respect to the entries in the $(n - 1)$-th row of $AB$ and the fact that $A_{i,i+1}$ is supported on $\Lambda$ for $i = 1, \ldots, n - 1$, we obtain that $B_{n-1,i} = 0$ a.e. $[\mu]$ on $\Lambda$ for $i = 1, \ldots, n - 2$. Therefore, we finally obtain that $B_{ij} = 0$ for $i > j$.

For the $(i, i+1)$ entry of $AB$, by $AB = BA$ we obtain the equality

$$A_{ii}B_{i,i+1} + A_{i,i+1}B_{i+1,i+1} = B_{ii}A_{i,i+1} + B_{i+1,i+1}A_{i+1,i+1}. \quad (3.21)$$

By $A_{ii} = A_{i+1,i+1}$ the equality (3.21) yields that $A_{i,i+1}B_{i+1,i+1} = B_{ii}A_{i,i+1}$. Since $A_{i,i+1}$ is supported on $\Lambda$, we obtain that $B_{ii} = B_{i+1,i+1}$ a.e. $[\mu]$ on $\Lambda$.

By Lemma 3.2, we obtain that every idempotent in $\{A\}' \cap M_n(L^\infty(\mu))$ is diagonal. In the following lemma, we deal with a general case, but there is some loss in the result compared with the result in the preceding lemma.

**Lemma 3.3.** Let $A$ in $M_m(L^\infty(\mu))$ and $B$ in $M_n(L^\infty(\mu))$ be of the upper triangular forms

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{pmatrix} \in L^2(\mu)$$

(3.22)

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ 0 & B_{22} & \cdots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{nn} \end{pmatrix} \in L^2(\mu)$$

(3.23)

such that

1. $A_{ii} = B_{jj}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$;
2. $A_{i,i+1}$ and $B_{j,j+1}$ are supported on $\Lambda$ for $i = 1, 2, \ldots, m - 1$ and $j = 1, 2, \ldots, n - 1$. 

If \( m \geq n \), and the equalities \( AC = CB \) and \( BD = DA \) hold for operator-valued matrices \( C = (C_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \) in \( M_{m \times n}(L^\infty(\mu)) \) and \( D = (D_{ij})_{1 \leq i \leq m, 1 \leq j \leq m} \) in \( M_{n \times m}(L^\infty(\mu)) \), then \( C_{ij} = 0 \) for \( i > j \) and \( D_{ij} = 0 \) for \( i > j - (m - n) \).

**Proof.** By \( AC = CB \), we consider the \((m-1,1)\) entry of \( AC \) and obtain that

\[
A_{m-1,m-1}C_{m-1,1} + A_{m-1,m}C_{m,1} = C_{m-1,1}B_{11}. \tag{3.24}
\]

By \( A_{m-1,m-1} = B_{11} \), the equality (3.24) yields that \( A_{m-1,m}C_{m,1} = 0 \). Since \( A_{m-1,m} \) is supported on \( \Lambda \), we obtain that \( C_{m,1} = 0 \) a.e. \([\mu]\) on \( \Lambda \). Thus we consider the \((m-2,1)\) entry of \( AC \) and obtain the equality

\[
A_{m-2,m-2}C_{m-2,1} + A_{m-2,m-1}C_{m-1,1} = C_{m-2,1}B_{11}. \tag{3.25}
\]

By \( A_{m-2,m-2} = B_{11} \) and the fact \( A_{m-2,m-1} \) is supported on \( \Lambda \), the equality (3.25) yields that \( C_{m-1,1} = 0 \) a.e. \([\mu]\) on \( \Lambda \). By the equalities with respect to the entries in the first column of \( AC \), we obtain that \( C_{i1} = 0 \) for \( i = m, \ldots, 2 \). Based on this, the equalities with respect to the entries in the second column of \( AC \) yield that \( C_{2i} = 0 \) for \( i = m, \ldots, 3 \). In this way, we finally obtain that \( C_{ij} = 0 \) for \( i > j \).

By \( BD = DA \), we consider the \((n,2)\) entry of \( BD \) and obtain that

\[
B_{m}D_{n2} = D_{n1}A_{12} + D_{n2}A_{22}. \tag{3.26}
\]

By \( A_{22} = B_{nn} \), the equality (3.26) yields that \( D_{n1}A_{12} = 0 \). Since \( A_{12} \) is supported on \( \Lambda \), we obtain that \( D_{n1} = 0 \) a.e. \([\mu]\) on \( \Lambda \). Thus we consider the \((n,3)\) entry of \( BD \) and obtain the equality

\[
B_{mn}D_{n3} = D_{n2}A_{23} + D_{n3}A_{33}. \tag{3.27}
\]

By \( A_{33} = B_{nn} \), the equality (3.27) yields that \( D_{n2}A_{23} = 0 \). Since \( A_{23} \) is supported on \( \Lambda \), we obtain that \( D_{n2} = 0 \) a.e. \([\mu]\) on \( \Lambda \). By the equalities with respect to the entries in the \( n \)-th row of \( BD \), we obtain that \( D_{ni} = 0 \) for \( i = 1, \ldots, m-1 \). Based on this, the equalities with respect to the entries in the \((n-1)\)-th row of \( BD \) yield that \( D_{n-1,i} = 0 \) for \( i = 1, \ldots, m-2 \). In this way, we finally obtain that \( D_{ij} = 0 \) for \( i + (m-n) > j \). \( \square \)

From Lemma 3.4 to Lemma 3.8, we make preparations for proving Theorem 3.9. For each operator \( A \) in \( M_n(L^\infty(\mu)) \) and the relative commutant \( \{A^\prime \cap M_n(L^\infty(\mu))\} \) containing a finite frame, we can decompose \( A \) into two fundamental cases by Proposition 2.8 and the discussion preceding it. In the following paragraphs, Lemma 3.4 and Proposition 3.5 deal with one case while Lemma 3.6 and Proposition 3.7 deal with another case. Lemma 3.8 shows the reason why we make the decomposition.

**Lemma 3.4.** For \( k = 1, \ldots, m \), let \( A_k = (A_{ij}^k)_{1 \leq i,j \leq n} \) be an upper triangular operator-valued matrix in \( M_n(L^\infty(\mu)) \) such that

1. \( A_{i1}^k = A_{lj}^l \) for \( k, l = 1, 2, \ldots, m \) and \( i, j = 1, 2, \ldots, n \);
2. \( A_{i,i+1}^k \) is supported on \( \Lambda \) for \( k = 1, 2, \ldots, m \) and \( i = 1, 2, \ldots, n-1 \).

If \( P \) is an idempotent in \( \{ \sum_{k=1}^m \oplus A_{ij}^k \cap M_{mn}(L^\infty(\mu)) \} \), then there exists an invertible operator \( X \) in \( \{ \sum_{k=1}^m \oplus A_{ij}^k \cap M_{mn}(L^\infty(\mu)) \} \) such that \( XPX^{-1} \) is diagonal.
Proof. Let \( P \) be an idempotent in \( \{ \sum_{k=1}^{m} \oplus A_k \}' \cap M_{mn}(L^\infty(\mu)) \) of the form

\[
P = \begin{pmatrix}
P_{11} & P_{12} & \cdots & P_{1m} \\
P_{21} & P_{22} & \cdots & P_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
P_{m1} & P_{m2} & \cdots & P_{mm}
\end{pmatrix},
\]

(3.28)

where \( P_{kl} \in M_n(L^\infty(\mu)) \) and \( A_k P_{kl} = P_{kl} A_l \) for \( k, l = 1, \ldots, m \). By Lemma 3.2 and Lemma 3.3, we obtain that \( P_{kl} \) is of the upper triangular form

\[
P_{kl} = \begin{pmatrix}
P_{11}^{kl} & P_{12}^{kl} & \cdots & P_{1n}^{kl} \\
0 & P_{22}^{kl} & \cdots & P_{2n}^{kl} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & P_{nn}^{kl}
\end{pmatrix}_{n \times n},
\]

(3.29)

where we treat \( P_{ij}^{kl} \) as an element in \( L^\infty(\mu) \) for \( k, l = 1, \ldots, m \) and \( i, j = 1, \ldots, n \).

The following facts are important:

1. the equality \( P_{ii}^{kl} = P_{11}^{kl} \) holds for \( k, l = 1, \ldots, m \) and \( i = 1, \ldots, n \);  
2. the equality \( A_{k,i}^{k} P_{i+1,i+1}^{kl} = P_{ii}^{kl} A_{i,i+1}^{l} \) holds for \( k, l = 1, \ldots, m \) and \( i = 1, \ldots, n-1 \). Thus by the assumption that \( A_{k,i}^{k} \) are all supported on \( \Lambda \) for \( k = 1, \ldots, m \) and \( i = 1, \ldots, n-1 \), the equality (3.30) yields that if \( P_{ii}^{kl} = 0 \) for some \( i \) then each main diagonal entry of \( P_{kl} \) is 0.

There exists a unitary operator \( U \) in \( M_{mn}(L^\infty(\mu)) \) which is a composition of finitely many row-switching transformations such that \( UPU^*=Q \) as an element in \( \{ U(\sum_{k=1}^{m} \oplus A_k)U^*' \}' \cap M_{mn}(L^\infty(\mu)) \) is of the upper triangular form

\[
Q = \begin{pmatrix}
Q_{11} & Q_{12} & \cdots & Q_{1n} \\
0 & Q_{22} & \cdots & Q_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & Q_{nn}
\end{pmatrix},
\]

(3.31)

where for \( i, j = 1, \ldots, n \), every \( Q_{ij} \) is of the form

\[
Q_{ij} = \begin{pmatrix}
P_{ij}^{11} & P_{ij}^{12} & \cdots & P_{ij}^{1m} \\
P_{ij}^{21} & P_{ij}^{22} & \cdots & P_{ij}^{2m} \\
\vdots & \vdots & \ddots & \vdots \\
P_{ij}^{m1} & P_{ij}^{m2} & \cdots & P_{ij}^{mm}
\end{pmatrix}_{m \times m},
\]

(3.32)

Note that \( Q_{ii} \) is an idempotent in \( M_m(L^\infty(\mu)) \) for \( i = 1, \ldots, n \).

We treat Borel representatives as elements in \( L^\infty(\mu) \). Without loss of generality, we assume that \( \text{Tr}(Q_{11}) \) defined as in (2.25) takes a constant \( r > 0 \) on \( \Lambda \). By the proof of Lemma 2.3, we may further assume that \( |P_{11}^{11}(\lambda)| \geq rn^{-1} \) for every \( \lambda \) in \( \Lambda \). Therefore we construct an invertible element \( X_{ii} \) in \( M_m(L^\infty(\mu)) \) for \( i = 1, \ldots, n \) of
the form
\[ X_{ii} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{P^{11}}{P_{ii}^{1}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{P^{m1}}{P_{ii}^{1}} & \cdots & \cdots & 1 \end{pmatrix}_{m \times m} L^2(\mu) \]  
(3.33)

Therefore by the proof of Lemma 2, the (1, 1) entry of \( X_{ii}Q_{ii}X_{ii}^{-1} \) is 1 and the
(k, 1) entries of \( X_{ii}Q_{ii}X_{ii}^{-1} \) are 0s for \( k = 2, \ldots, m \).

Based on the preceding discussion, let \( Y \) in \( M_{mn}(L^\infty(\mu)) \) be of the form
\[ Y = \begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1m} \\ Y_{21} & Y_{22} & \cdots & Y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{m1} & Y_{m2} & \cdots & Y_{mm} \end{pmatrix}, \]  
(3.34)

where \( Y_{kl} \in M_n(L^\infty(\mu)) \) and \( A_kY_{kl} = Y_{kl}A_j \) for \( k, l = 1, \ldots, m \) such that:

1. \( Y_{kk} \) equals the identity of \( M_n(L^\infty(\mu)) \) for \( k = 1, \ldots, m \);
2. \( Y_{kj}^{k1} = P_{kj}^{kl} / P_{ii}^{11} \) for every entry \( Y_{kj}^{k1} \) of \( Y_{kl} \) and \( k = 2, \ldots, m \);
3. other \( Y_{kj}^{k1} \) not mentioned in the first two items are 0s for \( k, l = 1, \ldots, m \).

Note that every \( Y_{kl} \in M_n(L^\infty(\mu)) \) is of the upper triangular form for \( k, l = 1, \ldots, m \). Then \( Y \) is an invertible element in \( \{ \sum_{k=1}^{m} \oplus A_k \} \cap M_{mn}(L^\infty(\mu)) \). We write \( X_1 = UYU^* \) for \( U \) in the discussion preceding (3.31). Then \( X_1 \) is of the upper triangular form
\[ X_1 = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ 0 & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{nn} \end{pmatrix}, \]  
(3.35)

where \( X_{ii} \) is in the form of (3.33) for \( i = 1, \ldots, n \). Therefore, \( X_i \) is invertible in
\( \{ U(\sum_{k=1}^{m} \oplus A_k )U^* \} \cap M_{mn}(L^\infty(\mu)) \). In \( X_iQX_1^{-1} \), the \((i, i)\) block entry \( X_{ii}Q_{ii}X_{ii}^{-1} \) possesses the property mentioned in the discussion preceding (3.34). By a similar proof of Lemma 2.3 and iterating the preceding discussion, we obtain an invertible element \( X \) in \( \{ U(\sum_{k=1}^{m} \oplus A_k )U^* \} \cap M_{mn}(L^\infty(\mu)) \) such that \( XQX^{-1} \) is of the upper triangular form
\[ XQX^{-1} = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ 0 & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{nn} \end{pmatrix}, \]  
(3.36)

such that \( R_{ii} \) is diagonal for \( i = 1, \ldots, n \). Thus \( R_{ii} = R_{jj} \) for \( i, j = 1, \ldots, n \). Note
that the diagonal matrix
\[ R_1 = \begin{pmatrix} R_{11} & 0 & \cdots & 0 \\ 0 & R_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{nn} \end{pmatrix}, \]  
(3.37)

is in \( \{ U(\sum_{k=1}^{m} \oplus A_k )U^* \} \cap M_{mn}(L^\infty(\mu)) \). Therefore \( XQX^{-1} - R_1 = R \) is in
\( \{ U(\sum_{k=1}^{m} \oplus A_k )U^* \} \cap M_{mn}(L^\infty(\mu)) \). For every element \( S \) in \( \mathcal{A} \), we obtain that
\[ \sigma_3(SR) = \sigma_3(RS) = \{0\}, \text{ where } R = \{U(\sum_{k=1}^{m} A_k)U^*\}' \cap M_{mn}(L^\infty(\mu)). \] Therefore the equality \((I - 2RXQX^{-1})(I - 2RXQX^{-1} + R) = I + (I - 2RXQX^{-1})R \) yields that \( I - 2RXQX^{-1} + R \) is invertible in \( \{U(\sum_{k=1}^{m} A_k)U^*\}' \cap M_{mn}(L^\infty(\mu)) \), since \((I - 2RXQX^{-1})^2 = I \). Hence the equality

\[
(XQX^{-1} - R)^2 = XQX^{-1} + R^2 - RXQX^{-1} - XQX^{-1}R
= XQX^{-1} - R
\]

yields that \( R^2 + R - RXQX^{-1} - XQX^{-1}R = 0 \), and we obtain

\[
(XQX^{-1} - R)(I - 2RXQX^{-1} + R)
= 2RXQX^{-1} - XQX^{-1} + XQX^{-1}R - R - R^2
= RXQX^{-1} - XQX^{-1}
= (R + I - 2RXQX^{-1})XQX^{-1}.
\]

This means that \( XQX^{-1} \) is similar to \( XQX^{-1} - R(= R_1) \) in the relative commutant \( \{U(\sum_{k=1}^{m} A_k)U^*\}' \cap M_{mn}(L^\infty(\mu)) \). Thus \( Z = U^*(R + I - 2RXQX^{-1})XU \) is the required invertible element in \( \{\sum_{k=1}^{m} A_k\}' \cap M_{mn}(L^\infty(\mu)) \) such that \( ZPZ^{-1} \) is diagonal. \hfill \( \square \)

By the preceding lemma and the proof of Lemma 2.6 we obtain the following proposition.

**Proposition 3.5.** For \( k = 1, \ldots, m \), let \( A_k = (A_{ij}^k)_{1 \leq i,j \leq n} \) be an upper triangular operator-valued matrix in \( M_n(L^\infty(\mu)) \) such that

1. \( A_{ii}^k = A_{ij}^k \) for \( k, l = 1, \ldots, m \) and \( i, j = 1, \ldots, n \);
2. \( A_{i,i+1}^k \) is supported on \( \Lambda \) for \( k = 1, \ldots, m \) and \( i = 1, \ldots, n - 1 \).

If \( \mathcal{P} \) and \( \mathcal{Q} \) are two bounded maximal abelian sets of idempotents in the relative commutant \( \{\sum_{k=1}^{m} A_k\}' \cap M_{mn}(L^\infty(\mu)) \), then there exists an invertible operator \( X \) in \( \{\sum_{k=1}^{m} A_k\}' \cap M_{mn}(L^\infty(\mu)) \) such that \( X\mathcal{P}X^{-1} = \mathcal{Q} \).

**Proof.** Denote by \( \mathcal{P} \) the set of diagonal projections in \( \{\sum_{k=1}^{m} A_k\}' \cap M_{mn}(L^\infty(\mu)) \). We can verify that \( \mathcal{P} \) is a bounded maximal abelian set of idempotents in the relative commutant \( \{\sum_{k=1}^{m} A_k\}' \cap M_{mn}(L^\infty(\mu)) \). By Proposition 2.5, there exists a finite frame \( \mathcal{P}_0 \) in \( \mathcal{P} \). Thus combining Lemma 3.4 and the proof of Lemma 2.6, there exists an invertible operator \( X \) in \( \{\sum_{k=1}^{m} A_k\}' \cap M_{mn}(L^\infty(\mu)) \) such that every element in \( X\mathcal{P}_0X^{-1} \) is diagonal. Furthermore, every element in \( X\mathcal{P}X^{-1} \) is diagonal. Therefore \( X\mathcal{P}X^{-1} \subseteq \mathcal{P} \). The maximality of \( \mathcal{P} \) yields the equality \( X\mathcal{P}X^{-1} = \mathcal{P} \). By the same way, there exists an invertible operator \( Y \) in \( \{\sum_{k=1}^{m} A_k\}' \cap M_{mn}(L^\infty(\mu)) \) such that \( Y\mathcal{Q}Y^{-1} = \mathcal{Q} \). Therefore \( \mathcal{P} \) and \( \mathcal{Q} \) are similar to each other in \( \{\sum_{k=1}^{m} A_k\}' \cap M_{mn}(L^\infty(\mu)) \). \hfill \( \square \)

In the following lemma, we pay attention to another case different from the one mentioned in Lemma 3.4.

**Lemma 3.6.** For \( k = 1, \ldots, m \), let \( A_k = (A_{ij}^k)_{1 \leq i,j \leq n_k} \) be an upper triangular operator-valued matrix in \( M_{n_k}(L^\infty(\mu)) \) such that

1. \( A_{ii}^k = A_{ij}^k \) for \( k, l = 1, \ldots, m, i = 1, \ldots, n_k \) and \( j = 1, \ldots, n_k \);
2. \( A_{i,i+1}^k \) is supported on \( \Lambda \) for \( k = 1, \ldots, m \) and \( i = 1, \ldots, n_k - 1 \);
3. \( n_1 > \cdots > n_m \) and \( \sum_{k=1}^{m} n_k = r \).
If \( P \) is an idempotent in \( \{ \sum_{k=1}^m \oplus A_k \}' \cap M_r(\mathbb{L}^\infty(\mu)) \), then there exists an invertible operator \( X \) in \( \{ \sum_{k=1}^m \oplus A_k \}' \cap M_r(\mathbb{L}^\infty(\mu)) \) such that \( XPX^{-1} \) is diagonal.

Proof. For the sake of simplicity, we only prove this lemma for \( m = 2 \) and \( A_1, A_2 \) are of the forms

\[
A_1 = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
0 & A_{22} & A_{33} \\
0 & 0 & A_{33}
\end{pmatrix} L^2(\mu), \quad A_2 = \begin{pmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{pmatrix} L^2(\mu)
\]

If \( P \) is an idempotent in the relative commutant \( \{ A_1 \oplus A_2 \}' \cap M_5(\mathbb{L}^\infty(\mu)) \) then by Lemma 3.2 and Lemma 3.3, \( P \) is of the form

\[
P = \begin{pmatrix}
P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\
0 & P_{22} & P_{23} & P_{24} & P_{25} \\
0 & 0 & P_{33} & P_{34} & P_{35} \\
0 & 0 & 0 & P_{44} & P_{45} \\
0 & 0 & 0 & 0 & P_{55}
\end{pmatrix} L^2(\mu)
\]

There exists a unitary element \( U \) in \( M_5(\mathbb{L}^\infty(\mu)) \) which is a composition of finitely many row-switching transformations such that \( UPU^* = Q \) as an element in the relative commutant \( \mathfrak{A} = \{ U(A_1 \oplus A_2)U^* \} ' \cap M_5(\mathbb{L}^\infty(\mu)) \) is of the form

\[
Q = \begin{pmatrix}
P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\
0 & P_{22} & P_{23} & P_{24} & P_{25} \\
0 & 0 & P_{33} & P_{34} & P_{35} \\
0 & 0 & 0 & P_{44} & P_{45} \\
0 & 0 & 0 & 0 & P_{55}
\end{pmatrix} L^2(\mu)
\]

We can verify that \( Q_1 = \text{diag}(P_{11}, P_{12}, P_{13}, P_{14}, P_{15}) \) is a projection in the relative commutant \( \mathfrak{A} \). Thus \( R = Q - Q_1 \) is in the relative commutant \( \mathfrak{A} \). By computation, we obtain that the equality \( \sigma_{\mathfrak{A}}(SR) = \sigma_{\mathfrak{A}}(RS) = \{0\} \) holds for every element \( S \) in \( \mathfrak{A} \). Therefore by an analogous proof following (3.37) in Lemma 3.4, there exists an invertible element \( X \) in the relative commutant \( \{ A_1 \oplus A_2 \}' \cap M_5(\mathbb{L}^\infty(\mu)) \) such that \( XPX^{-1} \) is diagonal. By iterating the preceding proof we achieve a generalized case mentioned in the lemma. \( \square \)

By combining Proposition 2.5, Lemma 3.6 and the method applied in the proof of Lemma 2.6, we obtain the following proposition.

**Proposition 3.7.** For \( k = 1, \ldots, m \), let \( A_k = (A_{ij}^k)_{1 \leq i,j \leq n_k} \) be an upper triangular operator-valued matrix in \( M_{n_k}(\mathbb{L}^\infty(\mu)) \) such that

1. \( A_{ii}^k = A_{lj}^k \) for \( k \), \( l \) \( = 1, \ldots, m \), \( i = 1, \ldots, n_k \), and \( j = 1, \ldots, n_l \); 
2. \( A_{ki}^k \) is supported on \( \Lambda \) for \( k = 1, \ldots, m \) and \( i = 1, \ldots, n_k - 1 \); 
3. \( n_1 > \cdots > n_m \) and \( \sum_{k=1}^m n_k = r \).

If \( \mathcal{P} \) and \( \mathcal{Q} \) are two bounded maximal abelian sets of idempotents in the relative commutant \( \{ \sum_{k=1}^m \oplus A_k \}' \cap M_5(\mathbb{L}^\infty(\mu)) \), then there exists an invertible operator \( X \) in \( \{ \sum_{k=1}^m \oplus A_k \}' \cap M_5(\mathbb{L}^\infty(\mu)) \) such that \( X \mathcal{P} X^{-1} = \mathcal{Q} \).

By a similar computation as in Lemma 3.3, we obtain the following lemma. This lemma implies the reason for our consideration about the preceding two cases.
Lemma 3.8. For $k = 1, 2$, let $A_k = (A^k_{ij})_{1 \leq i, j \leq n_k}$ be an upper triangular operator-valued matrix in $M_{n_k}(L^\infty(\mu))$ such that

1. $A^k_{ii} = A^k_{jj}$ for $k = 1, 2$, $i, j = 1, \ldots, n_k$;
2. $A^1_{11}(\lambda) \neq A^2_{11}(\lambda)$ a.e. $[\mu]$ on $\Lambda$.

If $B$ is an element in $M_{n_1 \times n_2}(L^\infty(\mu))$ such that $A_1B = BA_2$, then $B = 0$.

By an analogous discussion preceding Proposition 2.8, for $A$ in $M_n(L^\infty(\mu))$ and the relative commutant $\{A\}' \cap M_n(L^\infty(\mu))$ containing a finite frame, there exists a finite $\mu$-measurable partition $\{A_i\}_{i=1}^m$ of $A$ such that for $k, l = 1, \ldots, m$ and based on the notations in Proposition 2.8, either $A^k_{i1}(\lambda) = A^l_{i1}(\lambda)$ a.e. $[\mu]$ on $\Lambda_t$ or $A^k_{i1}(\lambda) \neq A^l_{i1}(\lambda)$ a.e. $[\mu]$ on $\Lambda_t$ for $t = 1, \ldots, r$. Denote by $E_t$ the central projection in $M_n(L^\infty(\mu))$ such that $\Tr(E_t)$ is supported on $\Lambda_t$ for $t = 1, \ldots, r$. Then combining Proposition 3.5, Proposition 3.7 and Lemma 3.8, we obtain that for every two bounded maximal abelian sets of idempotents $\mathcal{P}$ and $\mathcal{Q}$ in the relative commutant $\{E_t\}' \cap M_n(L^\infty(\Lambda_t, \mu))$, there exists an invertible element $X$ in $\{E_t\}' \cap M_n(L^\infty(\Lambda_t, \mu))$ such that $X\mathcal{P}X^{-1} = \mathcal{Q}$. Thus by combining the ‘local results’, we obtain the following theorem.

Theorem 3.9. Let $A$ be an operator in $M_n(L^\infty(\mu))$. If the relative commutant $\{A\}' \cap M_n(L^\infty(\mu))$ contains a finite frame then for every two bounded maximal abelian sets of idempotents $\mathcal{P}$ and $\mathcal{Q}$ in the relative commutant $\{A\}' \cap M_n(L^\infty(\mu))$, there exists an invertible element $X$ in $\{A\}' \cap M_n(L^\infty(\mu))$ such that $X\mathcal{P}X^{-1} = \mathcal{Q}$.

By virtue of Theorem 3.9, we prove the following corollary.

Corollary 3.10. For every normal operator $A$ in $M_n(L^\infty(\mu))$, the strongly irreducible decomposition of $N$ is unique up to similarity with respect to the relative commutant $\{A\}' \cap M_n(L^\infty(\mu))$.

Proof. Let $A$ be a normal operator in $M_n(L^\infty(\mu))$ and $\mathcal{P}$ be a bounded maximal abelian set of projections in $\{A\}' \cap M_n(L^\infty(\mu))$.

First we assert that $\mathcal{P}$ is a bounded maximal abelian set of idempotents in $\{A\}' \cap M_n(L^\infty(\mu))$. Assume that $Q$ is an idempotent in $\{A\}' \cap M_n(L^\infty(\mu))$ such that the equality $QP = PQ$ holds for every projection in $\mathcal{P}$. By the polar decomposition of $Q$ we can express $Q$ in the form $Q = V|Q|$, where $V$ in $M_n(L^\infty(\mu))$ is a partial isometry with initial space $(\ker Q)\perp$ and final space ran $Q$. The equalities $PQ = QP$ and $PQ^* = Q^*P$ yield that $P|Q| = |Q|P$. Therefore the definition of $V$ and the equality

$$PV|Q| = PQ = QP = V|Q|P = VP|Q|$$

yield that the equality $PV = VP$ holds for every projection $P$ in $\mathcal{P}$. Denote by $\mathfrak{A}$ the abelian von Neumann algebra generated by $\mathcal{P}$. Since $\mathfrak{A}$ is a maximal abelian von Neumann algebra in $\{A\}' \cap M_n(L^\infty(\mu))$, we obtain that $A$ is contained in $\mathfrak{A}$. The facts that $PV = VP$ holds for every projection $P$ in $\mathcal{P}$ and $A$ is contained in $\mathfrak{A}$ yield that $AV = VA$ and $AV^* = V^*A$. Therefore $V$ and $V^*$ are contained in $\{A\}' \cap M_n(L^\infty(\mu))$. Thus $VV^*$ and $V^*V$ are contained in $\mathfrak{A}$. Since the equality $QP = PQ$ holds for every projection in $\mathcal{P}$, we obtain that $VV^*Q = QVV^*$. By the definition of $V$, $VV^*$ is the final projection of $V$ with ran $VV^* = \ker Q$. Hence the equality $Q = VV^*Q = QVV^* = VV^*$ holds. The assertion is achieved. By Proposition 2.5 and Theorem 3.9, we can finish the proof. \[\square\]
Corollary 3.11. Let \( A \) be a normal operator in \( M_\infty(L^\infty(\mu)) \). Then every bounded maximal abelian set of idempotents in \( \{A\}' \cap M_\infty(L^\infty(\mu)) \) is a bounded maximal abelian set of idempotents in \( \{A\}' \).

Proof. Note that every bounded maximal abelian set of idempotents in \( \{A\}' \cap M_\infty(L^\infty(\mu)) \) contains the set of central projections in \( M_\infty(L^\infty(\mu)) \). Let \( \mathcal{P} \) be a bounded maximal abelian set of idempotents in \( \{A\}' \cap M_\infty(L^\infty(\mu)) \) and \( Q \) be an idempotent in \( \{A\}' \cap \mathcal{P}' \). Since \( Q \) commutes with every central projection in \( M_\infty(L^\infty(\mu)) \), we obtain that \( Q \) is contained in \( M_\infty(L^\infty(\mu)) \). Thus \( Q \) is contained in \( \mathcal{P} \). Therefore \( \mathcal{P} \) is a bounded maximal abelian set of idempotents in \( \{A\}' \). \( \square \)

4. Direct integral forms and the ‘local’ \( K \)-theory for the relative commutants of operators in \( M_\infty(L^\infty(\mu)) \)

By virtue of the reduction theory of von Neumann algebras, for every operator \( A \) in \( M_\infty(L^\infty(\mu)) \), we can express \( A \) in a direct integral form with respect to an abelian von Neumann algebra in the relative commutant \( \{A\}' \cap M_\infty(L^\infty(\mu)) \). A natural question is what the direct integral form of \( A \) looks like with respect to a finite frame in \( \{A\]' \cap M_\infty(L^\infty(\mu)) \). For this purpose we need to introduce some concepts and notations that will be used in this section. First, we need two concepts from operator theory.

Definition 4.1. An operator \( A \) in \( \mathcal{L}(\mathcal{H}) \) is said to be irreducible if its commutant \( \{A\}' \equiv \{B \in \mathcal{L}(\mathcal{H}) : AB = BA\} \) contains no projections other than 0 and the identity operator \( I \) on \( \mathcal{H} \), introduced by P. Halmos in [4]. (The separability assumption of \( \mathcal{H} \) is necessary because on a nonseparable Hilbert space every operator is reducible.) An operator \( A \) in \( \mathcal{L}(\mathcal{H}) \) is said to be strongly irreducible if \( XAX^{-1} \) is irreducible for every invertible operator \( X \) in \( \mathcal{L}(\mathcal{H}) \), introduced by F. Gilfeather in [3]. This shows that the commutant of a strongly irreducible operator contains no idempotents other than 0 and \( I \). For more literature around strongly irreducible operators, the reader is referred to [5].

We also need some concepts from the reduction theory of von Neumann algebras.

Definition 4.2. For the most part, we follow [1][9]. Once and for all, let \( \mathcal{H}_1 \subset \mathcal{H}_2 \subset \cdots \subset \mathcal{H}_n \subset \mathcal{H}_\infty \) be a sequence of Hilbert spaces with \( \mathcal{H}_n \) having dimension \( n \) and \( \mathcal{H}_\infty \) spanned by the remaining \( \mathcal{H}_n \)'s. Let \( \mu \) be (the completion of) a finite positive regular Borel measure supported on a compact subset \( \Lambda \) of \( \mathbb{R} \). (This assumption makes sense by virtue of ([8], Theorem 7.12).) And let \( \{\Lambda_\infty\} \cup \{\Lambda_n\}_{n=1}^\infty \) be a Borel partition of \( \Lambda \). Then we form the associated direct integral Hilbert space

\[
\mathcal{H} = \int_\Lambda \mathcal{H}(\lambda)d\mu(\lambda),
\]

which consists of all (equivalence classes of) measurable functions \( f \) and \( g \) from \( \Lambda \) into \( \mathcal{H}_\infty \) such that:

1. \( f(\lambda) \in \mathcal{H}(\lambda) \equiv \mathcal{H}_n \) for \( \lambda \in \Lambda_n \);
2. \( \|f\|^2 \equiv \int_\Lambda \|f(\lambda)\|^2d\mu(\lambda) < \infty \);
3. \( (f,g) \equiv \int_\Lambda (f(\lambda),g(\lambda))d\mu(\lambda) \).

The element in \( \mathcal{H} \) represented by the measurable function \( \lambda \rightarrow f(\lambda) \) is denoted by \( \int_\Lambda f(\lambda)d\mu(\lambda) \). An operator \( A \) in \( \mathcal{L}(\mathcal{H}) \) is said to be decomposable if there exists a
such that:

$$\{ \text{Correspondingly, } D \text{ generating } M \text{ diagonal for } X \}$$

there exists an invertible element $E$ central projections of $M$ algebra.

For the sake of simplicity and without loss of generality, we assume that

For a direct integral of strongly irreducible operators, we obtain the following proposition.

**Proposition 4.3.** Let $A$ be in $M_n(L^\infty(\mu))$. Then $\{ A \} \cap M_n(L^\infty(\mu))$ contains a finite frame if and only if there exists an invertible element $X$ in $M_n(L^\infty(\mu))$ and a unitary operator $U$ such that $UXAX^{-1}U^*$ is a direct integral of strongly irreducible operators with respect to a diagonal algebra $\mathcal{D}$ and $U^*\mathcal{D}U \subseteq M_n(L^\infty(\mu))$.

**Proof.** If $\{ A \} \cap M_n(L^\infty(\mu))$ contains a finite frame $\{ P_k \}_{k=1}^m$, then by Lemma 2.6 there exists an invertible element $X$ in $M_n(L^\infty(\mu))$ such that every $XP_kX^{-1}$ is diagonal for $k = 1, 2, \ldots, m$. The set $\{ XP_kX^{-1} \}_{k=1}^m$ is a finite frame in the relative commutant $\{ XAX^{-1} \} \cap M_n(L^\infty(\mu))$. Therefore $\{ XP_kX^{-1} \}_{k=1}^m$ and $\mathcal{E}_n$, the set of central projections of $M_n(L^\infty(\mu))$, generate a maximal abelian von Neumann subalgebra $\mathcal{D}_0$ in the relative commutant $\{ XAX^{-1} \} \cap M_n(L^\infty(\mu))$. Since $\mathcal{D}_0$ contains $\mathcal{E}_n$, we can verify that $\mathcal{D}_0$ is also a maximal abelian von Neumann subalgebra in $\{ XAX^{-1} \}$. Thus by [8, Theorem 7.12], there exists a self-adjoint element $D_0$ generating $\mathcal{D}_0$.

For the sake of simplicity and without loss of generality, we assume that $A$ in $M_3(L^\infty(\mu))$ is of the form

$$A = \begin{pmatrix} f & f_{12} & f_{13} \\ 0 & f & f_{23} \\ 0 & 0 & f \end{pmatrix} L^2(\mu)$$

such that:

1. $A = \Lambda_1 \cup \Lambda_2$ and $\mu(\Lambda_i) > 0$ for $i = 1, 2$;
2. the support of $f_{12}$ is $\Lambda$;
3. the support of $f_{23}$ is $\Lambda_1$;
4. $f_{13}$ and $f_{23}$ vanish on $\Lambda_2$.

Correspondingly, $\{ P_1, P_2 \}$ is a finite self-adjoint frame contained in the relative commutant $\{ A \} \cap M_3(L^\infty(\mu))$ such that

1. $P_1$ is of the form

$$P_1(\lambda) = \begin{cases} \text{diag}(1, 1, 1) \in M_3(\mathbb{C}), & \forall \lambda \in \Lambda_1; \\ \text{diag}(1, 1, 0) \in M_3(\mathbb{C}), & \forall \lambda \in \Lambda_2; \end{cases}$$

(4.3)
(2) $P_2$ is of the form

$$P_2(\lambda) = \begin{cases} 0 \in M_3(\mathbb{C}), & \forall \lambda \in \Lambda_1; \\ \text{diag}(0, 0, 1) \in M_3(\mathbb{C}), & \forall \lambda \in \Lambda_2. \end{cases} \quad (4.4)$$

Thus the abelian von Neumann algebra $\mathcal{D}_0$ generated by $\{P_1, P_2\} \cup \mathcal{E}_3$ is maximal in the relative commutant $\{A\}' \cap M_3(L^\infty(\mu))$. By (S, Theorem 7.12), there exist self-adjoint injective elements $D_i$ in $L^\infty(\Lambda_i, \mu)$ for $i = 1, 2, 3$ such that:

1. $D_i$ is in $L^\infty(\Lambda_i, \mu)$ with $\{D_i\}'' = L^\infty(\Lambda_i, \mu)$ for $i = 1, 2$, and $D_3$ is in $L^\infty(\Lambda_2, \mu)$ with $\{D_3\}'' = L^\infty(\Lambda_2, \mu)$;
2. the spectra of $D_i$s denoted by $\Gamma_i$ for $i = 1, 2, 3$ are pairwise disjoint.

Let $D_0$ be a self-adjoint element in $\{A\}' \cap M_3(L^\infty(\mu))$ of the form

$$D_0(\lambda) = \begin{cases} \text{diag}(D_1(\lambda), D_1(\lambda), D_1(\lambda)) \in M_3(\mathbb{C}), & \forall \lambda \in \Lambda_1; \\ \text{diag}(D_2(\lambda), D_2(\lambda), D_3(\lambda)) \in M_3(\mathbb{C}), & \forall \lambda \in \Lambda_2. \end{cases} \quad (4.5)$$

Therefore $\sigma(D_0) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\{D_0\}'' = \mathcal{D}_0$. Denote by $E_{D_0}(\cdot)$ the spectral measure for $D_0$. Treating $\sigma(D_0)$ as the index set, we form the direct integral Hilbert space $\mathcal{H}$ as follows

$$\mathcal{H} = \int_{\sigma(D_0)}^\oplus \mathcal{H}(\gamma) d\nu(\gamma), \quad (4.6)$$

where

$$\mathcal{H}(\gamma) = \begin{cases} \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}, & \gamma \in \Gamma_1; \\ \mathbb{C} \oplus \mathbb{C} \oplus 0, & \gamma \in \Gamma_2; \\ 0 \oplus 0 \oplus \mathbb{C}, & \gamma \in \Gamma_3; \end{cases} \quad (4.7)$$

and $D_i^{-1}(\Delta)$ denotes the preimage of each Borel subset $\Delta$ with respect to (the Borel function) $D_i$. Define two $\nu$-measurable operator-valued functions $D(\cdot)$ and $\tilde{A}(\cdot)$ on $\sigma(D_0)$ of the form

$$D(\gamma) = \begin{cases} \text{diag}(\gamma, \gamma, \gamma) \in M_3(\mathbb{C}), & \gamma \in \Gamma_1; \\ \text{diag}(\gamma, 0, \gamma) \in M_3(\mathbb{C}), & \gamma \in \Gamma_2; \\ \text{diag}(0, 0, \gamma) \in M_3(\mathbb{C}), & \gamma \in \Gamma_3; \end{cases} \quad (4.9)$$

and $\tilde{A}(\gamma)$ to be the restriction of $A \circ D_i^{-1}(\gamma)$ on $\text{ran}(D(\gamma)) = \mathcal{H}(\gamma)$ for $\gamma \in \Gamma_i$, $i = 1, 2, 3$. It is easy to verify that there exists a unitary operator $U$ such that $UD_0U^* = D|_{\mathcal{H}}$, $UAU^* = A$ and $AD|_{\mathcal{H}} = D|_{\mathcal{H}}A$. By (H, Lemma 3.1), we obtain that $\tilde{A}(\gamma)$ is strongly irreducible a.e. $[\nu]$ on $\sigma(D_0)$. By a similar argument, we can prove the ‘if only’ part for the general case.

On the other hand, if $A \in M_\infty(L^\infty(\mu))$ and there exists a unitary operator $U$ such that $UAU^*$ is a direct integral of strongly irreducible operators with respect to a diagonal algebra $\mathcal{D}$ and $U^*\mathcal{D}U \subseteq M_\infty(L^\infty(\mu))$, then it is sufficient to show that the set of projections $\mathcal{D}$ in $U^*\mathcal{D}U$ is a bounded maximal abelian set of idempotents in the relative commutant $\{A\}' \cap M_\infty(L^\infty(\mu))$. If there exists a nontrivial idempotent $P$ in $\{A\}' \cap M_\infty(L^\infty(\mu)) \cap \mathcal{D}'$ and $P$ not in $\mathcal{D}$, then $UPU^*(\gamma)$ is not trivial for almost
every \( \gamma \) on \( \Gamma \) with respect to \( \nu \), where \( \Gamma \) is the index set for \( \mathcal{D} \) in the direct integral form and \( \nu \) is a finite positive regular Borel measure applied in the direct integral form. Let \( \Gamma_p \) be a subset of \( \Gamma \) with \( \nu(\Gamma_p) > 0 \) such that \( UPU^{*}(\gamma) \) is nontrivial for every \( \gamma \in \Gamma_p \). Since the equality \( UAU^{*}(\gamma)UPU^{*}(\gamma) = UPU^{*}(\gamma)UAU^{*}(\gamma) \) holds for almost every \( \gamma \) in \( \Gamma \) with respect to \( \nu \). Then \( UAU^{*}(\gamma) \) is not strongly irreducible for almost every \( \gamma \) in \( \Gamma_p \) with respect to \( \nu \). This is a contradiction. Therefore \( \mathcal{D} \) is a bounded maximal abelian set of idempotents in the relative commutant \( \{A\}' \cap M_n(L^\infty(\mu)) \). By Proposition 2.5 and the preceding argument, we prove the ‘if’ part.

Based on Proposition 2.8, we investigate the local K-theory for the relative commutant \( \{A\}' \cap M_n(L^\infty(\mu)) \), where \( A \in M_n(L^\infty(\mu)) \) and \( \{A\}' \cap M_n(L^\infty(\mu)) \) contains a finite frame. The following lemma shows the \( K_0 \) group of the relative commutant of a building block in (2.30).

**Lemma 4.4.** Let \( A \) be an upper triangular operator-valued matrix in \( M_n(L^\infty(\mu)) \) of the form

\[
A = \begin{pmatrix}
    f & f_{12} & \cdots & f_{1n} \\
    0 & f & \cdots & f_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & f
\end{pmatrix}
\]

(4.10)
such that \( f_{i,i+1} \) is supported on \( \Lambda \) for \( i = 1, 2, \ldots, n-1 \). Then the \( K_0 \) group of the relative commutant \( \{A\}' \cap M_n(L^\infty(\mu)) \) is of the form

\[
K_0(\{A\}' \cap M_n(L^\infty(\mu))) \cong \{ \phi : \Lambda \rightarrow \mathbb{Z} | \phi \text{ is bounded Borel} \}.
\]

**Proof.** By Lemma 3.2, we obtain that for every \( B \in \{A\}' \cap M_n(L^\infty(\mu)) \), \( B \) is of the upper triangular form

\[
B = \begin{pmatrix}
    \phi & \phi_{12} & \cdots & \phi_{1n} \\
    0 & \phi & \cdots & \phi_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & \phi
\end{pmatrix}
\]

(4.12)

Therefore, every \( B \in \{A^{(m)}\}' \cap M_{mn}(L^\infty(\mu)) \) is of the form

\[
B = \begin{pmatrix}
    B_{11} & B_{12} & \cdots & B_{1m} \\
    B_{21} & B_{22} & \cdots & B_{2m} \\
    \vdots & \vdots & \ddots & \vdots \\
    B_{m1} & B_{m2} & \cdots & B_{mm}
\end{pmatrix}
\]

(4.13)

where \( B_{kl} \in M_n(L^\infty(\mu)) \) and \( AB_{kl} = B_{kl}A \) for every \( k, l \) in \( \{1, \ldots, m\} \). Thus \( B_{kl} \) is of the upper triangular form as in (4.12)

\[
B_{kl} = \begin{pmatrix}
    B_{11}^{kl} & B_{12}^{kl} & \cdots & B_{1n}^{kl} \\
    0 & B_{22}^{kl} & \cdots & B_{2n}^{kl} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & B_{nn}^{kl}
\end{pmatrix}
\]

(4.14)

where we treat \( B_{ij}^{kl} \) as an element in \( L^\infty(\mu) \) and \( B_{ii}^{kl} = B_{jj}^{kl} \) for \( k, l = 1, \ldots, m \) and \( i, j = 1, \ldots, n \).
There exists a unitary operator $U$ in $M_{mn}(L^\infty(\mu))$ which is a composition of finitely many row-switching transformations such that $UBU^* = C$ as an element in $\{U(A^{(m)})U^*\}' \cap M_{mn}(L^\infty(\mu))$ is of the upper triangular form

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ 0 & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{nn} \end{pmatrix}, \quad (4.15)$$

where for $i, j = 1, \ldots, n$, every $C_{ij}$ is of the form

$$C_{ij} = \begin{pmatrix} B_{ij}^{11} & B_{ij}^{12} & \cdots & B_{ij}^{1m} \\ B_{ij}^{21} & B_{ij}^{22} & \cdots & B_{ij}^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{ij}^{m1} & B_{ij}^{m2} & \cdots & B_{ij}^{mm} \end{pmatrix}_{m \times m} \in L^2(\mu). \quad (4.16)$$

The equality $C_{ii} = C_{jj}$ holds for $i, j = 1, \ldots, n$ and $C_{11}$ can be every element in $M_{mn}(L^\infty(\mu))$. This fact is different from the counterpart mentioned from (3.30) to (3.32).

If $B$ is an idempotent, then so is $C$. By a similar proof from (3.30) to (3.39), we obtain an invertible element $X$ in $\{U(A^{(m)})U^*\}' \cap M_{mn}(L^\infty(\mu))$ such that $XCX^{-1}$ is of the diagonal form

$$XCX^{-1} = \begin{pmatrix} C_{11} & 0 & \cdots & 0 \\ 0 & C_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{nn} \end{pmatrix} \quad (4.15)$$

where $C_{ii}$ are the same idempotent for $i = 1, \ldots, n$. Furthermore, we assume that

$$\text{Tr}(C_{ii})(\lambda) = r \quad (4.16)$$

for an integer $1 \leq r \leq m$ a.e. $[\mu]$ on $\Lambda$. Then by Lemma (2.3), there exists an invertible element $Y$ in $M_{mn}(L^\infty(\mu))$ such that

$$YC_{ii}Y^{-1} = I_{M_{r}(L^\infty(\mu))} \oplus 0. \quad (4.17)$$

Note that $Y^{(n)}$ is an invertible element in $\{U(A^{(m)})U^*\}' \cap M_{mn}(L^\infty(\mu))$. In this sense, we form an abelian semigroup by equivalence classes of idempotents in $\bigcup_{m=1}^{\infty}\{A^{(m)}\}' \cap M_{mn}(L^\infty(\mu))$ up to similarity. Thus this semigroup is isomorphic to the abelian semigroup

$$\{\phi: \Lambda \rightarrow \mathbb{N}| \phi \text{ is bounded Borel} \} \quad (4.18)$$

by an isomorphism induced by the function $\text{Tr}()$ defined in (2.25). Then by a routine computation, we obtain the $K_0$ group of $\{A\}' \cap M_{\infty}(L^\infty(\mu))$ is of the form formulated as in (4.11). \qed

For a general case that $A$ in $M_{2}(L^\infty(\mu))$ and $\{A\}' \cap M_{\infty}(L^\infty(\mu))$ contains a finite frame, the $K_0$ group of $\{A\}' \cap M_{\infty}(L^\infty(\mu))$ may not be of a neat form as in (4.11). We see this from the following example.
Example 4.5. Let $A$ in $M_4(L^\infty(\mu))$ be of the form
\[
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad \text{and } A_1 = \begin{pmatrix} f_1 & f_1 \\ 0 & f_1 \end{pmatrix} L^2(\mu), \quad A_2 = \begin{pmatrix} f_2 & f_2 \\ 0 & f_2 \end{pmatrix} L^2(\mu), \tag{4.19}
\]
where $f_1(\lambda) = \lambda$ and $f_2(\lambda) = 1$ for every $\lambda$ in $\Lambda$ and we assume that $\Lambda = [0,1]$. Note that $0$ is in $\sigma(f_1)$. For every $\varepsilon > 0$, there exists an open ball $\{\lambda : |\lambda - 0| < \varepsilon\}$ with $\mu\{\lambda : |\lambda - 0| < \varepsilon\} > 0$ such that $f(\lambda) = |\lambda - 0| < \varepsilon$. Then the point $0$ in the unit interval $[0,1]$ makes the $K_0$ group of $\{A\} \cap M_4(L^\infty(\mu))$ different from the form as in (4.11). We define an equivalent relation $\sim$ in the group
\[
\mathcal{G} = \{ \phi : \Lambda \to \mathbb{Z}(2) | \phi \text{ is bounded Borel} \}. \tag{4.20}
\]
That is for $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ in $\mathcal{G}$, $\phi$ is said to be equivalent to $\psi$ (denoted by $\phi \sim \psi$) if there exists an $\varepsilon > 0$ such that:
1. $\phi(\lambda) = \psi(\lambda)$ for $|\lambda| < \varepsilon$;
2. $\phi(\lambda) = \psi(\lambda)$ for $|\lambda| \geq \varepsilon$.

Thus by a routine computation, we obtain that
\[
K_0(\{A\} \cap M_4(L^\infty(\mu))) \cong \mathcal{G} / \sim. \tag{4.21}
\]
For a general case, we assume that the support of $f_i$ is $\Lambda$ and $0 \in \sigma(f_i)$ for $i = 1,2$. Let $\{p_k^i\}_{i=1,2, k \in \mathbb{N}}$ be the set of points in $\Lambda$ such that:
1. $f_i(p_k^i) = 0$ for $i = 1,2$ and every $k \in \mathbb{N}$;
2. For every $p_k^i$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f_i(p) - \varepsilon$ for $|p - p_k^i| < \delta$.

Essentially, the equivalent relation mentioned preceding (4.21) is determined by the set of $\{p_k^i\}_{k \in \mathbb{N}}$ that $\Lambda$ chosen with respect to the 1-diagonal entries of $A_1$ and $A_2$.

Lemma 4.6. Let $A = A_1 \oplus A_2$ and $A_i$ in $M_n(L^\infty(\mu))$ is of the form
\[
A_i = \begin{pmatrix} A_{11}^i & A_{12}^i & \cdots & A_{1n_i}^i \\ 0 & A_{22}^i & \cdots & A_{2n_i}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n_i,n_i}^i \end{pmatrix}_{n_i \times n_i} L^2(\mu), \tag{4.22}
\]
such that
1. $n_1 > n_2$ and $n = n_1 + n_2$;
2. $A_{kk}^i = A_{ll}^i$ for $i = 1,2$ and $k,l = 1,\ldots,n_i$;
3. the support of $A_{k,k+1}^i$ is $\Lambda$ for $i = 1,2$ and $k = 1,\ldots,n_i - 1$.

Then the $K_0$ group of the relative commutant $\{A\} \cap M_n(L^\infty(\mu))$ is of the form
\[
K_0(\{A\} \cap M_n(L^\infty(\mu))) \cong \{ \phi : \Lambda \to \mathbb{Z}(2) | \phi \text{ is bounded Borel} \}. \tag{4.23}
\]

The proof of this lemma is a routine computation by applying Lemma 3.2, Lemma 3.3, Lemma 3.4, Lemma 3.6 and Lemma 3.8. Therefore, by Lemma 2.8, to tell when two operators in $M_n(L^\infty(\mu))$ are similar to each other in $M_n(L^\infty(\mu))$, it is sufficient to tell when two building blocks as in (2.31) are similar to each other. For two building blocks in $M_n(L^\infty(\mu))$, we present a method to distinguish them by the $K$-theory of the relative commutant of the direct sum of this two building blocks.
Proposition 4.7. Let $A = A_1 \oplus A_2$ and $A_i$ in $M_n(L^\infty(\mu))$ is of the form

$$A_i = \begin{pmatrix} A_{i1} & A_{i2} & \cdots & A_{in} \\ 0 & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn} \end{pmatrix}_{n \times n} L^2(\mu),$$

such that

1. $A_{ik} = A_{li}$ for $i, k = 1, 2$ and $l = 1, \ldots, n$;
2. the support of $A_{ik}, k+1$ is $\Lambda$ for $i = 1, 2$ and $k = 1, \ldots, n - 1$.

If the $K_0$ group of the relative commutant $\{A\}' \cap M_{2n}(L^\infty(\mu))$ is of the form

$$K_0(\{A\}' \cap M_{2n}(L^\infty(\mu))) \cong \{ \phi : \Lambda \to \mathbb{Z} | \phi \text{ is bounded Borel} \},$$

then $A_1$ and $A_2$ are similar to each other in $M_n(L^\infty(\mu))$.

By the methods we applied in Example 4.5 and other lemmas in this section, we obtain this proposition.

References

1. E. Azoff, C. Fong and F. Gilfeather, ‘A reduction theory for non-self-adjoint operator algebras.’ Trans. Amer. Math. Soc. 224 (1976), 351–366. MR0448109
2. Don Deckard and Carl Pearcy ‘On matrices over the ring of continuous valued functions on a Stonian space.’ Proc. Amer. Math. Soc. 14 (1963) 322–328. MR0147926
3. F. Gilfeather, ‘Strong reducibility of operators.’ Indiana Univ. Math. J. 22 (4) (1972), 393–397. MR0303322
4. P. Halmos, ‘Irreducible operators.’ Michigan Math. J. 15 (1968), 215–223. MR0231233
5. C. Jiang and Z. Wang, Structure of Hilbert space operators. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006. MR2221863
6. C. Jiang and R. Shi, ‘Direct Integrals of Strongly Irreducible Operators.’ J. Ramanujan Math. Soc. 26 (2) (2011), 165–180. MR2816786
7. C. Jiang and R. Shi, ‘A similarity invariant of a class of $n$-normal operators in terms of $K$-theory’ arXiv:1211.6243 [math.FA].
8. H. Radjavi and P. Rosenthal, Invariant subspaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77. Springer-Verlag, New York-Heidelberg, 1973. MR0367682
9. J. T. Schwartz, $W^*$-algebras. Gordon and Breach, New York, 1967. MR0232221
10. R. Shi, ‘On a generalization of the Jordan canonical form theorem on separable Hilbert spaces.’ Proc. Amer. Math. Soc. 140 (5) (2012), 1593–1604. MR2869143

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