In the present work the case of a chromium Bose–Einstein condensate is considered. The model includes not only the presence of the so–called contact interaction but also a long range and anisotropic dipole–dipole interaction has been included. Some thermodynamical properties are analyzed. For instance, the size of the condensate, chemical potential, speed of sound, number of particles, etc., are deduced. It will be shown that this dipole–dipole interaction implies the emergence of anisotropy, for example, in the speed of sound. The possible use of this anisotropy as a tool for the analyze of dissipative mechanisms, for instance, Landau’s criterion for superfluidity, will be also discussed.

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I. INTRODUCTION

Many of the properties of ultra–cold quantum gases are determined by the features of the interactions among the atoms of the corresponding system. For instance, symmetry characteristics, intensity, or range of the interaction play a relevant role in the observed phenomena. The changes that may appear, in this context, can be surprising. Indeed, for attractive short range interatomic interactions a Bose–Einstein condensate (BEC) is unstable with regard to local collapses. Nevertheless, the presence of a trapping potential allows, under certain conditions, the existence of metastable states. In other words, the system goes from unstable to metastable with the introduction of a trapping potential. This isotropic short range interaction, based upon an effective contact interaction, can be tuned resorting to Feshbach resonances and many interesting quantum features of these systems have been analyzed with the use of this tuning possibility.

The realization of a BEC resorting to 52 Cr atoms opens up an interesting window in the context of ultra–cold quantum gases. Indeed, the large magnetic moment that some atomic species possess, among them 52 Cr, offers the possibility of obtaining a dipolar degenerate quantum gas. One of the advantages of 52 Cr comprises the fact that they have a dipole moment of 6 Bohr magnetons and, in consequence, the intensity of the ensuing magnetic dipole–dipole interactions among the atoms is much larger than the corresponding for alkali atoms. The interest in dipolar degenerate quantum gases lies in the fact that novel phenomena are expected to emerge. For instance, its expansion, after releasing it from an anisotropic trap, entails that the anisotropy of the dipole–dipole interaction can be tracked down to a detectable anisotropic deformation of the expansion of the system. Clearly, additional effects shall emerge, as an example we may add that this dipole–dipole interaction, which has a long range, shall modify the shape of the condensate. It should be no surprise that this dipole–dipole interaction must modify the speed of sound in this kind of systems. Indeed, the absence of interatomic interactions means that sound cannot exist. In other words, interatomic interactions are a key ingredient in the definition of the speed of sound, therefore, the presence of an additional force, this magnetic dipole–dipole feature, should impinge upon the speed of sound. The analysis of the speed of sound allows us to include anisotropy as an additional characteristic in this physical parameter.

In the present work we consider a BEC in which the trapping potential is endowed with the mathematical form of an anisotropic harmonic oscillator and, in addition, we take into account the presence of a magnetic dipolar moment such that a dipole–dipole interaction emerges as a key element in the description of our system. Some parameters are deduced, for instance, speed of sound, radii of the BEC, chemical potential, and number of particles. An interesting point is related to the fact that the speed of sound is anisotropic.

II. DIPOLE–DIPOLE INTERACTION AND CONDENSATION

A. Dipole–Dipole Mean Field Theory Energy

The system under study is a gas of chromium atoms in which an appropriate rotating magnetic field entails the emergence of dipolar interactions among the atoms of the system. This kind of interactions are long–ranged and imply the presence of anisotropy. These two features
are in contrast with the properties of those pertaining to a short–range and isotropy interaction contained in the so–called scattering length. We introduce, from the very beginning, the Thomas–Fermi limit, and, within this context, we deduce the chemical potential, radii of the condensate, number of particles, and speed of sound.

Consider two equal atoms, one located at \( \vec{r} \) and the second one at \( \vec{R} \), then a potential energy related to a dipole–dipole interaction between them appears, here \( \gamma \) denotes the gyromagnetic ratio \( [8] \):

\[
V_d(\vec{r} - \vec{R}) = \frac{\mu_0 \gamma^2}{4\pi ||\vec{r} - \vec{R}||^3} \left[ \vec{S}_1 \cdot \vec{S}_2 - \frac{3}{2} \left( \vec{S}_1 \cdot (\vec{r} - \vec{R}) \right) \frac{||\vec{r} - \vec{R}||}{||\vec{r} - \vec{R}||^2} \right].
\]

(1)

An external magnetic field polarizes all our chromium atoms along the \( z \)–axis, such that

\[
\vec{S}_1 = \vec{S}_2 = 3\hbar \hat{z}.
\]

(2)

Under these circumstances the dipolar potential reads

\[
V_d(\vec{r} - \vec{R}) = \frac{\mu_0 (6\mu_B)^2}{4\pi} \frac{1}{||\vec{r} - \vec{R}||^3} \left[ 1 - 3 \frac{(z - Z)^2}{||\vec{r} - \vec{R}||^2} \right].
\]

(3)

This energy will be considered in the context of the mean field theory, namely,

\[
E_d = \frac{1}{2} \int d\vec{r} d\vec{R} V_d(\vec{r} - \vec{R}) n(\vec{r}) n(\vec{R}).
\]

(4)

The time–independent Gross–Pitaevskii equation reads

\[
\mu \psi(\vec{r}) = \frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + V_t(\vec{r}) \psi(\vec{r}) + U_0 |\psi(\vec{r})|^2 \psi(\vec{r}) + \int d\vec{R} V_d(\vec{r} - \vec{R}) n(\vec{R}) \psi(\vec{r}).
\]

(5)

Here \( U_0 = 4\pi a \hbar^2 / m \) \( [3] \) (\( a \) is the scattering length) and \( V_t \) denotes the trapping potential and for our case we have that it depicts an anisotropic harmonic oscillator

\[
V_t(\vec{r}) = \frac{m}{2} \left[ \omega_0^2 x^2 + \omega_0^2 y^2 + \omega_z^2 \right].
\]

(6)

B. Chemical Potential and Geometry of the Condensate

At this point we introduce two simplifications, namely; (i) the Thomas–Fermi approximation, i.e., we neglect the kinetic term; (ii) in the integral term of (5) we consider

\[
n(\vec{r}) \Rightarrow n(0)(\vec{r}) = (\mu - V_t(\vec{r}))/U_0, \text{ in other words, for the calculation of this integral term we introduce for the density the corresponding function appearing in the case of vanishing dipole–dipole interaction} \ [10].
\]

In order to simplify the calculations let us point out the following identity

\[
\frac{1}{||\vec{r} - \vec{R}||^3} - 3 \frac{(z - Z)^2}{||\vec{r} - \vec{R}||^5} = - \frac{\partial^2}{\partial z^2} \left[ \frac{1}{||\vec{r} - \vec{R}||} \right] - \frac{4\pi}{3} \delta(\vec{r} - \vec{R}).
\]

(7)

This last expression entails

\[
\int d\vec{R} V_d(\vec{r} - \vec{R}) n(0)(\vec{R}) = -\mu_0 (6\mu_B)^2 \frac{1}{3} n(0)(\vec{r}) + \frac{\partial^2}{\partial z^2} (\phi(\vec{r}))
\]

where

\[
\phi(\vec{r}) = \frac{1}{4\pi} \int \frac{n(0)(\vec{R})}{||\vec{r} - \vec{R}||} d^3 R.
\]

(8)

We must now provide the volume of integration. In the absence of dipole–dipole interaction and with an isotropic trap, i.e., \( \omega_0 = \omega_z \), the volume is a sphere whose radius \( R \) can be obtained, within the context of the Thomas–Fermi approximation, imposing the condition

\[
n(r = R) = 0 \Rightarrow \mu = V_t(r = R).
\]

(10)

The value of the chemical potential equals the value of the trap evaluated at the boundary of the condensate. In the case of an anisotropic trap we do not expect to have a sphere \([10]\). For a situation like the one considered here, in connection with the trap, the geometry of the condensate should be derived, as in the case of an isotropic trap, from our model. Clearly, we expect to have, due to the symmetry of the trap, to equal sizes \( R_0 \) and a second one different \( R_z \). These parameters do not determine, completely, the geometry of the system. The presence of a dipole–dipole interaction along a certain direction complicates even more the question of the corresponding geometry. Indeed, this kind of interactions may change, drastically, the geometry of the condensate \([11]\).

The deduction of some of the properties of the condensate requires the knowledge of the geometry of the system, for instance, we must calculate the integral appearing in \([3]\), a fact that needs the integration volume. For our case we will assume that the system has an ellipsoidal geometry, with two equal axes \( R_0 \), along the \( x \) and \( y \) axes, and the third one different \( R_z \).

Our coordinate system will be chosen to be prolate spheroidal \([12]\):

\[
x = R_z \sinh u \cos v \cos \phi,
\]

\[
y = R_z \sinh u \cos v \sin \phi,
\]

\[
z = R_z \cosh u \sin v.
\]

(11)
In these last expressions we have that $0 \leq v \leq \pi$; $0 \leq \phi \leq 2\pi$; $0 \leq u \leq \tilde{u}$. The mean field version of the condensate is given by

$$n(\vec{r}) = \frac{1}{U_0} \left[ \mu - V_t(\vec{r}) - \int V_0(\vec{r} - \vec{R}) n(\vec{r}) n(\vec{R}) d\vec{R} \right].$$

In other words, the condition $n(\vec{r} = R_0 \vec{e}_z) = 0$ implies

$$\mu = \left( 1 + \frac{\gamma}{3} (1 + \cosh(\tilde{u}) - \tilde{R}_0) \right)^{-1} \left\{ \frac{m \omega_0^2 R_0^2}{2} + \gamma \left[ \frac{m \omega_0^2 R_0}{6} + \frac{m \omega_0^2 R_0^2}{2} \left[ \frac{\tilde{R}_0}{R_0} \right] \right] + 4 \frac{m \omega_0^2 R_0^2}{105} \left( \frac{2 \tilde{R}_0 R_0^2}{R_0^3} - \tilde{R}_0^3 \right) \right\}. \quad (18)$$

We may find a relation between $R_0$ and $R_z$ recalling that $n(\vec{r} = R_z \vec{e}_z) = 0$ provides also the chemical potential.

$$\mu = \left( 1 + \frac{\gamma}{3} (\cosh(\tilde{u}) - 3\pi) \right)^{-1} \left\{ \frac{m \omega_0^2 R_0^2}{2} + \gamma \left[ \frac{m \omega_0^2 R_0^2}{6} + \frac{m \omega_0^2 R_0^2}{2} \left( \cosh(\tilde{u}) - 1 \right) + \frac{\pi}{16} \right] + 2 \frac{m \omega_0^2 R_0^2}{105} \right\}. \quad (19)$$

The comparison between these two last expressions renders a transcendental equation which determines the ratio $R_0/R_z$ as a function of $\omega_0$, $\omega_z$, and $\gamma$.

$$\frac{\omega_0 R_0}{\omega_z R_z} = \frac{1 + \frac{\gamma}{3} (1 + \cosh(\tilde{R}_0/R_z) - \tilde{R}_0/R_z)}{1 + \frac{\gamma}{3} (\cosh(\tilde{R}_0/R_z) - 3\pi)} \times \left\{ 1 + \frac{\gamma}{3} [1 + (\omega_0/\omega_z)^2] \left[ \frac{\cosh(\tilde{R}_0/R_z) - 2 + \pi/2) \right] \times \left\{ 1 + \frac{\gamma}{3} \left[ 1 + \frac{3 R_0^2}{5 \tilde{R}_0} \left( \cosh(\tilde{R}_0/R_z) - 2 \tilde{R}_0/R_z \right) + \frac{8 R_0 \omega_0^2}{7 \tilde{R}_0 \omega_0^2} (1 - R_0^2/R_z^3) \right] \right\}^{-1}. \quad (20)$$

This last expression will be satisfied only for certain values of $R_0/R_z$ (assuming $\omega_0$, $\omega_z$, and $\gamma$ are known) but it does not provide the value of $R_0$ (or of $R_z$). In order to obtain these parameters we require an additional equation.

C. Speed of Sound, Number of Particles, and energy per particle

The speed of sound ($c_s$) is given by

$$c_s^2 = m \left( \frac{\partial \mu}{\partial n} \right). \quad (13)$$

For our particular case we have ($\delta = R_0/R_z$)

$$c_s^2 = \frac{m \omega_0^2}{\omega_z^2} \mu \left[ \frac{1}{U_0} \left[ \mu - V_t(\vec{r}) - \int V_0(\vec{r} - \vec{R}) n(\vec{r}) n(\vec{R}) d\vec{R} \right] \right]. \quad (17)$$
\[
\epsilon_s^2 = \frac{\omega_s^2 R_s^2}{2} \left\{ 1 + \frac{\gamma}{3} \left[ \cosh(\delta) - 3\pi \right] \right\}^{-1} \left\{ 1 + \frac{\gamma}{3} \left[ 1 + (\omega_0/\omega_s)^2(2\cosh(\delta) - 2 + \pi/2) \right] \right\}
\]

This last expression provides us with the possibility of deducing the geometrical parameters of the condensate through the value of the speed of sound at the center of the condensate, i.e., \(\epsilon_s^2(\tau = 0) = \epsilon_s^2(0)\). Indeed,

\[
R_z^2 - 2 \frac{\omega_z^2 \epsilon_s^2(0)}{\omega_z^2} \left\{ 1 + \frac{\gamma}{3} \left[ \cosh(\delta) - 3\pi \right] \right\}^{-1} \left\{ 1 + \frac{\gamma}{3} \left[ 1 + (\omega_0/\omega_z)^2(2\cosh(\delta) - 2 + \pi/2) \right] \right\}
\]

\[
\left\{ 1 + \frac{\gamma}{3} \left[ 1 + \cosh(\delta) \right] \right\} \frac{\omega_z^2 \epsilon_s^2(0)}{2} \cosh(\delta).
\] (22)

This expression provides the value of \(R_z\) as a function of measurable parameters.

The number of particles can be obtained integrating \(\frac{dN}{dT}\) and we have, approximately

\[
N = \frac{4\pi \mu}{3U_0} \left\{ 1 + \frac{\gamma}{3} \right\} R_0^2 R_z - \left\{ 1 + \frac{\gamma}{3} \right\} \frac{2m\pi \omega_z^2 R_s^2}{15U_0} \left\{ 2 + (\omega_0/\omega_z)^2 \right\} \cosh^5(\delta) - \frac{2}{3} \left[ \frac{4(\omega_0/\omega_z)^2 - 1}{\omega_z^2} \right] \cosh^3(\delta) \left\{ 3(\omega_0/\omega_z)^2 \cosh(\delta) - \frac{4}{3} \left( \frac{\omega_0}{\omega_z} \right)^2 + 1 \right\} \right\}. \] (23)

We know that \(\mu = \frac{\partial E}{\partial N}(T,V)\) [13], therefore we may find the internal energy of the condensate. Indeed, \(E = \int \mu dN = \int \mu \frac{dN}{d\mu} d\mu\), and therefore, after a lengthy calculation we find that

\[
E = \frac{3}{2} \left\{ \left( \frac{a_1 + a_2 a_3}{a_2} \right) \left( \frac{2}{5} \right)^{7/2} - \frac{4}{5} b_2^{5/2} + \frac{2}{3} b_2^{3/2} \right\} + \left( \frac{a_1 b_2^{3/2}}{a_2} \right) \left( \frac{2}{5} \right) \left( \frac{2}{5} \right) - \frac{4}{3} b_2^{5/2} \right\} \right\}.
\] (24)

In these expressions we have the following parameters:

\[
z = a_2 \mu + b, \quad b = -m \epsilon_s^2(0), \quad a_2 = 1 + \frac{\gamma}{3} \left[ 1 + \cosh(\delta) \right],
\]

\[
a_1 = 4\pi \delta^2 \frac{3U_0}{1 + \frac{\gamma}{3}} \left( \frac{10}{m \omega_0^2 \cosh(\delta)} \right)^{3/2},
\] (25)

\[
a_3 = \left( 1 + \frac{\gamma}{3} \right) \left( \frac{10}{m \omega_0^2 \cosh(\delta)} \right)^{-5/2} \left\{ \left( \frac{\omega_0}{\omega_z} \right)^2 - \frac{2}{3} \left( \frac{\omega_0}{\omega_z} \right)^4 \right\} \cosh^5(\delta) - \frac{2}{3} \left( \frac{\omega_0}{\omega_z} \right)^2 \cosh^3(\delta) + 3 \left( \frac{\omega_0}{\omega_z} \right)^2 \cosh(\delta) - \frac{4}{3} \left( \frac{\omega_0}{\omega_z} \right)^2 \right\}. \] (26)

Clearly, this last result allows us to find the average energy, per particle, \(\epsilon = E/N\).

**III. DISCUSSION**

We have considered a bosonic system comprised by chromium atoms in which the corresponding trap has two equal frequencies. In addition, a magnetic field polarizes all the chromium atoms and, in consequence, dipole–dipole interactions emerge as an important element in the physics of the system. Concerning the chemical potential in this scheme we have that it is given by [13]. It is readily seen that the chemical potential is a nonlinear function of the parameters of the trap and the scattering length, a fact which is no surprise, since previous results have shown this kind of dependence [13]. At this point we must underline that in these aforementioned examples it has been assumed that the strength of the contact interaction is larger than that stemming from the dipole–dipole term. In the present situation this assumption has not been imposed and, in this sense, it provides a more general case. Of course, we may recover the results of these previous works. Indeed, consider, for instance, the case of vanishing dipole–dipole interaction and an isotropic harmonic oscillator. Under these conditions the Thomas–Fermi approximation defines the following relation among the radius of the condensate \(R_i\), frequency \((\omega_i)\), and chemical potential \((\mu)\), namely, \(\mu = \omega_i R_i^2\), where \(i = 0, z\). Expressions (18) and (19) imply this value of the chemical potential if we set \(\gamma = 0\). Of course, in the present situation the dependence of the chemical potential upon the frequencies, scattering length, and dipole–dipole interaction is more complicated. Since the value of the chemical potential in the region of temperatures between \(T = 0\) and \(T = T_c\) is a constant and it coincides with the energy of the ground state then we know that \(\epsilon_0 = \mu\).

Clearly, (20) cannot provide us with the value of \(R_0\) or \(R_z\), but only with the ratio \(R_0/R_z\). This fact is no surprise at all and it is also present in the simplest case in which the dipole–dipole interaction and the contact interaction are switched off. Indeed, for this last case we know that there is no diffusion of particles within the system when the chemical potential has the same value at all points, hence \(\omega_0^2 R_0^2 = \omega_z^2 R_z^2 \Rightarrow \omega_0^2 / \omega_z^2 = R_z^2 / R_0^2\). In other words, thermodynamical arguments do not determine \(R_0\) or \(R_z\), only its ratio. For our situation, given the frequencies and \(\gamma\), (20) provides us the ratio...
\( R_0/R_z \) which is unique, namely, there is only one case in which \( R_0 \) and \( R_z \) are both positive and fulfill this foresaid expression. In order to determine the value of \( R_0 \) or \( R_z \) we may resort to expression (22) where we may deduce \( R_z \) as a function of the speed of sound at the center of the condensate, of \( \delta \), frequencies, etc.

Concerning the speed of sound in this system (21), it is readily seen that it is not only position–dependent but also anisotropic. To fathom better this last statement notice that (21) implies that the motion along the \( z \)–axis happens at a different speed that along the \( x \)–axis. Indeed, if in (21) we impose the condition \( y = z = 0 \) and calculate the corresponding speed of sound, and afterwards we do the same, but with the condition \( x = y = 0 \), the results differ. The reason for this lies not only on the fact that the frequencies along \( x \) and \( z \) are different (see the term \( \omega_0^2 x^2 + \omega_0^2 y^2 + \omega_z^2 z^2 \) in (21)) but also on the fact that there is dipole–dipole interaction only along the \( z \)-axis. For instance, the speed of sound includes the term \(-\frac{3\pi}{2m} R_z\) but it does not include terms of the form \(-\frac{3\pi}{2m} R_0 \) or \(-\frac{\gamma_3}{3m} R_0 \). This absence can be comprehended as a consequence of a dipole–dipole interaction only along the \( z \)-axis.

The possible relevance of this anisotropic behavior of the speed of sound is related to fact that it could provide us with a tool to investigate some dissipative mechanisms, in particular the case of the Landau criterion for superfluidity [14] seems to be a feasible case [7]. Notice that our calculation confirms the conjecture mentioned in [7] about the loss of isotropy in the speed of sound for a chromium BEC, and, in consequence, makes sounder the possibility of studying dissipative mechanisms with systems in which dipole–dipole interactions play a relevant role in the definition of its physical properties.

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