Causal interpretation of quantum mechanics: The Schrödinger equation as a condition of stability

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Abstract. This work presents a further development of causal interpretation of quantum mechanics. Existence of ‘non-local hidden variables’ like a physical field is taken as an assumption. Under this assumption, on the basis of the causal approach it is proven that Schrödinger equation is a necessary condition for stability of the motion of a particle. For the case of a hydrogen atom this approach gives a mathematical base with which to suggest that (1) the electron’s spin in an atom is precessing; (2) the energy of the precessional motion on Bohr orbits satisfies the Rydberg formula, and (3) structures are formed in the physical vacuum which stabilize the motion of an electron on Bohr orbits.

1. Causal interpretation and ‘hidden variables’

1.1. ‘Nonlocal hidden variables’
The causal interpretation of quantum formalism centres about the existence of ‘hidden variables’. Historically, the issue of incompleteness in the description of physical reality by quantum mechanics was put forward for the first time by Einstein, Podolsky, and Rosen in 1935 (the EPR paradox). They proposed the existence of ‘hidden variables’, which uniquely characterize the given state of the system; thus, allowing a quantum system to be consistent with the deterministic theory. In 1964, however, John S. Bell advanced his famous inequalities. It followed from the violation thereof in quantum theory that any theory of hidden variables claiming to be able to describe experimental results, must be ‘nonlocal’ only. In other words, any theory that proposes the causal interpretation of quantum formalism must imply either the existence of a physical field which allows superluminal communication or the existence of long-distance forces, or the existence of an informational field that can transmit instantaneously a signal (information) of a disturbance that occurs at some point of space on large distances.

1.2. ‘A quantum potential’ and its nonlocality from D. Bohm’s point of view.

D. Bohm was a proponent of the causal interpretation of quantum theory and, therefore, existence of hidden variables. In 1952 he published two articles [1] in which he proposed an alternative to the Copenhagen interpretation, that was based on the notion of a particle that is acted upon by a new kind of field. This field was characterized by a potential which Bohm called ‘a quantum potential’. Bohm assumed that an elementary particle has a certain coordinate \( \vec{r}(x, y, z) \) and is moving along a definite trajectory with a definite velocity \( \vec{V}(\vec{r}, t) \). Bohm represented the Schrödinger wave function in the form

\[
\psi(\vec{r}) = A(\vec{r}) \exp\left(\frac{iS}{\hbar}\right)
\]

where \( S \) is the Hamilton's principal function and
\[ \vec{F} = \frac{1}{m} \nabla S \]  
Then he substituted \( \psi \) in this form into the Schrödinger equation for a single particle

\[ \left( \frac{\hbar^2}{2m} \Delta \psi (\vec{r}) + (\varepsilon - U) \psi (\vec{r}) = 0 \right. \tag{3} \]

(from this point forward, we will consider for simplicity the time-independent Schrödinger equation).

He separated real and imaginary parts and obtained two equations: 1) the modified Hamilton-Jacobi equation

\[ \frac{(\nabla S)^2}{2m} + U + U_Q = \varepsilon \]  
containing not only the classical potential \( U \), but also ‘the quantum potential’,

\[ U_Q = -\frac{\hbar^2}{2m} \nabla^2 A \tag{5} \]

and 2) a continuity equation for \( m |A|^2 \)

\[ 2(\nabla A) (\nabla S) + A \nabla^2 S = 0 \tag{6} \]

(Bohm understood \( |A|^2 \) as the probability distribution of particles in a statistical ensemble of similar systems). It follows from Eq. (1) and Eq. (5) that the quantum potential depends on the solution of the Schrödinger equation. For different solutions it is a different function of coordinates.

Bohm made the following statement about the quantum potential: \( U_Q \) has ‘the new feature of nonlocality’, which implied ‘an instantaneous connection between distant particles’, and its effects might not fall off with the distance. Bohm wrote: ‘The quantum potential is different in many ways from classical potentials. The first key difference is that multiplication of the wave function by a constant does not change the quantum potential. The quantum potential can therefore still be large even when the wave function is small. This means that its effects do not necessarily fall off with the distance... The quantum information potential has the new feature of nonlocality, implying an instantaneous connection between distant particles.’

1.3. Critical notes on Bohm’s interpretation of the quantum potential.

1.3.1. Let us note that the property of ‘an instantaneous connection between distant particles’ is not new for potentials. Thus, action at a distance is a postulate of the Newtonian potential theory. Another example, is the pressure in an ideal incompressible fluid. The pressure is transmitted simultaneously over the entire volume. Instantaneity here is merely a part of the physical model. In real fluids, there exists a pressure wave, a ‘precursor’, which propagates with speeds much greater than that of the moving body.

The quantum potential, according to Bohm, however, is differ from other known potentials, because Bohm assumed its effects do not necessarily fall off with the distance. That was the reason Bohm assigned the meaning of information to potential \( U_Q \).

Below we will analyse the Bohm method and discuss where Bohm’s assumption about nonlocality of a quantum potential was incorrect. We will show that \( U_Q \) is determined by Eq. (5) only for the solutions that satisfy the Schrödinger equation, and is not defined on the entire space. For each solution of the Schrödinger equation, \( U_Q \) is a particular finite function of the coordinates.

1.3.2. Bohm assumed that in the limit \( \hbar \to 0 \), a quantum potential \( U_Q \to 0 \). This assumption, however, was incorrect which can be proven by calculating \( U_Q \) directly from the solutions of the Schrödinger equation for a hydrogen atom. Indeed, as it is known, these solutions can be written in the form
\[
\psi_{nlk} (r, \vartheta, \varphi) = A_{nlk} (r, \vartheta) \exp(i k \varphi)
\]
\[n = 1, 2, \ldots, \infty; \quad l \leq n - 1; \quad k = 0, \pm 1, \ldots, \pm l\]

So, for \(n = 1\) there is one solution

\[
\psi_{100} = A_{100} = \frac{1}{\sqrt{\pi}} \exp \left( -\frac{r}{r_0} \right)
\]

As follows from Eq. (5), the quantum potential for this solution has a form

\[
U_q = -\frac{\hbar^2}{2m_e} \frac{V^2 A_{100}}{A_{100}} A = \frac{\hbar^2}{2m_e r_0} \left( \frac{2}{r} - \frac{1}{r_0} \right)
\]

For \(n = 2, \quad k = \pm 1\), we have two solutions
\[
\psi_{2111} = \frac{1}{8\sqrt{\pi} r_0} \frac{r}{2r_0} \sin \vartheta \exp(\pm i \varphi)
\]

and the quantum potential takes the form

\[
U_q = -\frac{\hbar^2}{2m_e} \frac{\Delta A_{2111}}{A_{2111}} = \frac{\hbar^2}{2m_e} \left( \frac{1}{r^2 \sin^2 \vartheta} \frac{2}{r^0} + \frac{1}{(2r_0)^2} \right)
\]

where \(m_e\) is the mass of the electron and \(r_0\) is the Bohr radius given by

\[
r_0 = \frac{\hbar^2}{2m_e}
\]

After substituting expression for \(r_0\) into Eq. (8) or Eq. (9), we see that \(U_q\) tends to infinity as \(\hbar \to 0\). The fact that the quantum potential becomes infinitely large as \(\hbar \to 0\) agrees with the statement that the Schrödinger equation describes only the quantum phenomena.

2. 'Non-separability'

Let us now discuss the probabilistic interpretation of quantum theory which, currently, is the most common one. It holds that quantum mechanics deals with probabilities of observed quantities and measurements only. Followers of the Copenhagen school insist on point of view that physics is the science that rests solely with measurements and a question such as ‘where was a particle before it was detected by a device’ makes no sense. Joint probabilities of noncommitting operators in the Copenhagen interpretation cannot be used because direct measurement experiments cannot be conducted.

It should be noted, that the proponents of the Copenhagen interpretation are having difficulty explaining results of experiments with ‘the essentially quantum effects’ (e.g., teleportation of polarization of the photon). In these experiments the problem of interpretation of quantum formalism is further intensified: it is necessary to assume, that although, some properties of reality exist before measurements only potentially, however, there is a correlation between them. If one assumes that quantum objects have a priori properties corresponding to noncommuting operators, ‘hidden variables’ must be introduced in quantum physics.

To have some way for interpreting the experiments with ‘the essentially quantum effects’ while at the same time avoiding the introduction of long-distance forces or a physical field which allows the superluminal communication, the followers of the probabilistic interpretation of quantum theory began to talk about non-separability of quantum mechanics (in other words about existence of some type of an information ‘link’, between remote quantum object). That is, quantum mechanics is non-separable, local theory {separability means that spatially separated systems exist in independent states. Locality assumes that the state of the system may only be modified through effects propagating at sublight speeds}.

It would be mistaken, however, to think that the idea of the quantum non-separability does not contradict the theory of relativity. There is an implicit postulate of separability in the theory of relativity (the state of any composite system is wholly determined by the separate states of its subsystems). Even
Bohm noted that to measure the speed of light it is necessary to have a receiver and a transmitter that are not only separated in space, but are also autonomous in their behaviour.

2.1 Model of superfluid vacuum.

Many of quantum mechanics paradoxes could be explained, if we introduce a hypothesis that the ‘nonlocal hidden variables’ are a physical field that reflects processes in the physical vacuum. If we, in addition, assume that the physical vacuum is like a superfluid medium (for example like superfluid He-3) then such model will provide an explanation of the natural frequencies of an atom. Indeed, homogeneous precessing domains are observed in superfluid He-3 where all spins of the fluid particles precess with the same frequency and phase. In the superfluid vacuum the electron’s motion should create structures (quasi-particles) similar to such domains. Below we will prove that in the case of a hydrogen atom, the angular frequency of the precession of the quasi-particle’s spin is determined by the Rydberg formula.

It is interesting, that directed interactions were observed between two domains that were separated in space and were precessing with the same frequencies but different phases in the superfluid He-3. The possibilities of such interactions are explained by existence of spin currents in the superfluid medium. These interactions occurred without dissipation due to the superfluid properties of the medium.

3. The Schrödinger equation as a condition of stability

As early as 1929 N.G. Chetaev, a well-known expert in the theory of stability, worked in the University of Göttingen where he must have become familiar with Schrödinger’s work [2]. In 1931, when Chetaev returned to the USSR, he published an article [3] in which he attempted to derive the time-independent Schrödinger equation from the viewpoint of the theory of stability. Chetaev assumed that Schrödinger equation extracted from solutions of the Hamilton-Jacobi equation only those that satisfy the condition of stability. However, Chetaev was unable to obtain the Schrödinger equation exactly because he did not introduce ‘hidden variables’ like a physical field into the Hamilton-Jacobi equation (as mentioned above, the causal interpretation assumes the existence of ‘hidden variables’). Besides that, he, like Bohm, erroneously believed that \( U \) represents the potential energy of small disturbing forces.

Here we present the derivation of the Schrödinger equation using Chetaev’s approach but correcting his mistakes. Unlike Chetaev, we introduce an unknown potential \( \Phi \) in the Hamilton-Jacobi equation, which is an operator dependent on a trajectory. On trajectories that are obtained from the classical approach to the Schrödinger equation, operator \( \Phi \) equals the known function \( U \) defined by Eq. (5).

This mysterious potential \( \Phi \) is a mathematical representation of the physical properties of ‘hidden variables.’ We can only make various hypothesis of the physical nature of the ‘hidden variables.’ It is important to note, that potential \( \Phi \) does not only act on the centre of mass of a particle, but can also act on the rotational degrees of freedom, for example on the particle’s spin. In this case the equation of motion of the centre of mass of the particle moving along a trajectory and the equation of motion about the centre of mass may not be independent, and because of that they cannot be analysed separately. The Hamilton's principal function, in this cases, as well as generalized motion integral, depends on variables corresponding to both rotational and translational motion. In this work we restrict ourselves the rotational motion of a fast gyroscope (a spin). If the gyroscope axis is almost perpendicular to the plain of the orbit the generalized motion integral does not contain the kinetic energy of the gyroscope precessional motion, it has only the energy of the field that acts on the gyroscope [see Appendix].

The Hamilton-Jacobi equation with given potential energy \( U \) and additional term \( \Phi \) has a form

\[
\frac{(\nabla S)^2}{2m} + U + \Phi = \epsilon
\]  

Consider now the motion of a particle that it would have if small disturbing forces with potential energy \( W \) are present. In this case Eq. (11) takes the form:
\[ \frac{(\nabla S)^2}{2m} + U + \Phi + W = \varepsilon \tag{12} \]

Of all possible motions of the particle, we will consider only those for which the arbitrary constants of the complete integral of the Hamilton-Jacobi equation are assigned fixed values, and we will call the collection of these motions a \textit{packet}. Following the Chetaev’s method, we assume that influence of the perturbing forces on a packet at an arbitrary point is proportional to the density of trajectories \[ A^2 \] at that point. For the packet consisting of stable trajectories this influence must be minimal. That is, the action of perturbing forces is relatively less for the packets for which

\[ \int W A^2 d\tau = \int W \psi^* \psi d\tau = \min \tag{13} \]

Here \( \psi \) is determined by substitution (1). The integrals are taken over the entire volume of configuration space.

Substituting the expression for \( W \) from Eq. (12) into Eq. (13), we obtain the following variational problem

\[ \delta \left[ \frac{(\nabla S)^2}{2m} + U + \Phi - \varepsilon \right] \psi^* \psi d\tau = 0 \tag{14} \]

Using Eq. (1) we have

\[ \frac{i}{\hbar} \frac{\partial S}{\partial x} = \frac{1}{\psi} \frac{\partial \psi}{\partial x} - \frac{1}{A} \frac{\partial A}{\partial x} - \frac{1}{\psi^*} \frac{\partial \psi^*}{\partial x} + \frac{1}{A} \frac{\partial A}{\partial x}, \ldots \tag{15} \]

In view of (15) variational problem (14) takes the form

\[ \delta \left[ \frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial y^2} \right) - \left( \frac{\partial A}{\partial x} \right)^2 - \ldots \right] \psi^* \psi d\tau = 0 \]

The Euler–Lagrange equation for this variational problem has the following form

\[ \int \delta \psi^* \left[ \frac{\hbar^2}{2m} \left( -\nabla^2 \psi + \frac{\nabla^2 A}{A} \psi \right) + (U + \Phi - \varepsilon) \psi^* \psi \right] d\tau = 0 \tag{16} \]

If we set

\[ \Phi_0 = U_0 = -\frac{\hbar^2}{2m} \frac{\nabla^2 A}{A} \tag{17} \]

and

\[ \frac{\partial \Phi}{\partial A} = 0 \tag{18} \]

then the Euler–Lagrange equation for this variational problem is the Schrödinger equation (3).

This way, on the stable trajectories that are determined by the Schrödinger equation, the generalized motion integral (19) takes the form

\[ (1/2)m\psi^2 + U + U_0 = \varepsilon \tag{19} \]

Note that potential \( \Phi \) equals the quantum potential only on stable orbits, not on the entire space as Bohm suggested.

In general, there are two approaches to the study of stability: one is to study the stability of a non-perturbed motion with the addition of small perturbation forces, while the other is to study the stability of motion under variation of the initial coordinates and momenta with the same primary forces. The first approach was used in the above derivation. The second, which is the most widely used one, is based on the Poincaré variational equations. It can be shown that the second approach leads to the same result as the first, that the Schrödinger equation is a necessary condition for stability.
4. Electron in the field of an atomic nucleus

Now, using the causal approach, let us analyse the Schrödinger equation for the hydrogen atom

\[ \nabla^2 \psi + \frac{2m_e}{\hbar^2}(\varepsilon + \frac{e^2}{r}) \psi = 0 \]  

(20)

The solutions of this equation can be written in the form (7). [Note here, that these solutions can be obtained strictly mathematically without introducing any operators from quantum mechanics].

Now, obtain the velocity and the trajectory of the electron’s motion using Eq. (2). The phase of the wave function (1) can be written as \( S / \hbar = k \varphi \). As known, \( \nabla S \) in spherical coordinates has the form:

\[ \nabla S = \frac{\partial S}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial S}{\partial \vartheta} \hat{\vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial S}{\partial \varphi} \hat{\varphi} \]  

(21)

The component of the velocity along the radius \( \hat{r} \) is zero, because \( \frac{\partial S}{\partial r} = 0 \); the component of the velocity along \( \hat{\vartheta} \) is also equal to zero because \( \frac{\partial S}{\partial \vartheta} = 0 \). The only nonzero component of the velocity is the \( \hat{\varphi} \) component. Thus, for the velocity of the electron on the trajectories obtained from the Schrödinger equation we have

\[ \vec{V} = \frac{1}{m_e} \nabla S = \frac{\hbar k}{m_e r \sin \vartheta} \hat{\varphi} \]  

(22)

It follows from Eq. (22) that only two situations are possible: either the centre of mass of the electron is at rest (\( k = 0 \)) or it is moving in a circular orbit lying in a plane parallel to the \( xy \) plane with the centre of the orbit located on the \( z \) axis (the above reasoning can be applied to any coordinate axis). Note, that these circular trajectories satisfy only the necessary condition of stability, therefore, among these trajectories there can be trajectories that are not stable.

4.1. Bohr orbits

Let us consider circular orbits with the centres at the origin. In this case

\[ \vec{V} = \frac{\nabla S}{m_e} = \frac{\hbar k}{m_e} \hat{\varphi} \]  

(23)

According to Newton’s second law

\[ \frac{m_e V^2}{r} = \frac{\hbar^2 k^2}{m_e r^2} = F \]  

(24)

Therefore, besides the known Coulomb force acting on the centre of mass of the electron there is an additional force

\[ F_a = F - \frac{e^2}{r^2} = \frac{\varepsilon^2}{r^2} \left[ \frac{\hbar^2 k^2}{m_e e^2} - r \right] \]  

(25)

It follows from Eq. (25) that \( F_a = 0 \) on Bohr orbits with radius given by

\[ r_{B_k} = \frac{\hbar^2 k^2}{m_e e^2} \]  

(26)

Therefore, under the causal approach, Bohr orbits are the solutions of the Schrodinger equation.

Note, that the second condition of stability, Eq. (18), is also valid on the Bohr orbits. Indeed, in the case of the hydrogen atom, on the trajectories \( r = \text{const} \) we have

\[ \frac{\partial \Phi}{\partial \varphi} = \frac{\partial \Phi}{\partial r} = \frac{\partial U}{\partial r} = F_a = 0 \]

Hence, Bohr orbits (and only these orbits) satisfy not only condition of stability (16) but also condition of stability (18).
4.2. The precession of the electron’s spin in an atom

It follows from Eq. (5) and Eq. (7) that the generalized motion integral (11) for the Bohr orbit is

$$\frac{m_{e}V_{k}^{2}}{2} - \frac{e^{2}}{r_{n_{k}}} - \frac{\hbar^{2}}{2m_{e}A_{nk}} = \varepsilon_{n}$$

(27)

where $\varepsilon_{n}$ is the Bohr’s energy, given by Eq. (15).

As can be seen from Eq. (24) and Eq. (25) the only force that acts on the centre of mass on the Bohr orbit is the Coulomb force. Thus, for the centre of mass of the electron the motion integral is as follows

$$\frac{m_{e}V_{k}^{2}}{2} - \frac{e^{2}}{r_{n_{k}}} = \varepsilon_{k}$$

(28)

Comparing the expressions (27) and (28) yields for the Bohr orbit

$$U_{Q} = \frac{\hbar^{2}}{2m_{e}A_{nk}} = \varepsilon_{n} - \varepsilon_{k}$$

(29)

Suppose that the motion about the centre of mass is the precessional motion of the electron’s spin. We can describe the spin precession as precession of a classical gyroscope because there is a preferable direction in this problem (the normal to a plain of an orbit). It follows from the fast-gyroscopic theory [see Appendix] that for the precessional motion of a gyroscope which is limited to small precessional angles, the energy of the motion $E$ is numerically equals the product of the intrinsic angular momentum and the angular frequency of precession $\omega_{e}$

$$E = \hbar \omega_{e} / 2$$

(30)

In our case

$$U_{Q} = \varepsilon_{n} - \varepsilon_{k}$$

(31)

Thus, on Bohr orbit potential $U_{Q}$ (‘a quantum potential’) equals the energy of the precessional motion of the electron’s spin. It can be seen from Eq. (39) that the formula for the energy of the precessional motion is, in fact, the Rydberg formula.

4.3. De Broglie’s ‘law of phase harmony’

As is known, Louis de Broglie was a supporter of the causal interpretation of quantum mechanics. De Broglie assumed that every elementary particle possesses an internal oscillatory process, although his theory does not address the nature of this process. De Broglie introduced two objects: a particle and an accompanying stationary wave. Considering these objects separately he came to his famous ‘law of phase harmony’: the phase of the $\psi$ function associated with the moving particle must always be in accord with the phase of the stationary wave that is accompanying the particle. The ‘stationary wave’, however, requires a medium, presumably the ‘physical vacuum’, in which it propagates. But the vacuum having non-uniform energy density and pressure does not agree with the theory of relativity.

To bring the existence of ‘the accompanying stationary wave’ in agreement with the requirements of the theory of relativity Jean-Pierre Vigier, who was a De Broglie’ follower, proposed the stochastic interpretation of quantum mechanics. He represented the physical vacuum as a medium that consists of particle-antiparticle pairs. However, Vigier did not finish his work. The attempts of others of de Broglie’ followers to combine the pilot-wave theory with the special relative did not succeed.

Up to now we did not introduce any hypothesis besides the hypothesis of existence of “hidden variables” like a physical field. As was said above, if we make an assumption that the physical field reflects motions of the physical vacuum which is like a superfluid medium in its properties then we can provide an explanation of the natural frequencies of the atom. Indeed, in superfluid He-3 structures (quasi-particles) like homogeneous precessing domains are observed, where all spins of the fluid particles precess with the same frequency and phase. These quasi-particles have spin proportional to $\hbar$. In such superfluid vacuum the electron’s motion must create structures similar to such domains.
The wave function of an electron with the non-stationary component is
\[ \Psi = \exp(-i \frac{E}{\hbar} t) \psi(\mathbf{r})\psi(\mathbf{r}). \]

If we accept the De Broglie’s ‘law of phase harmony’ the following equality is valid
\[ E = \omega_p \hbar / 2 = \omega'_p \hbar \]  \hspace{1cm} (32)
where \( \omega_p \) is the angular frequency of coherent precession of the spin of vacuum structure. It follows from Eqs. (30), (31) and (32) that the angular frequency of the precession of the vacuum structure's spin \( \omega_p \) is calculated from Rydberg formula \( \omega_p \hbar = \epsilon_n - \epsilon_s \), which describes the natural frequencies of the hydrogen atom. This approach is consistent with the gyroscopic models of emission – absorption of a resonance photon.

**References**

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**Appendix**

Consider the rotation about the centre of mass of a gyroscope. The gyroscope's angular speed can be expressed as
\[ \dot{\omega} = \omega_1 \dot{\hat{e}}_1 + \omega_2 \dot{\hat{e}}_2 + \omega_3 \dot{\hat{e}}_3 = \theta \dot{\hat{e}}_z + \phi' \dot{\hat{e}}_y + \psi' \dot{\hat{e}}_x, \]
where \( \dot{\hat{e}}_1, \dot{\hat{e}}_2, \dot{\hat{e}}_3 \) are the orthogonal unit vectors in a moving coordinate system associated with the gyroscope, \( \dot{\hat{e}}_3 \) is directed along the gyroscope's axis of symmetry, \( \phi' \) is the angular speed of the gyroscope precession along \( Z \) axis, \( \theta \) is an angular speed of nutation. If we take \( \theta \) to be an angle between \( Z \) axis and vector \( \hat{e}_3 \), the following equations is valid: \( \omega_1 = \theta', \omega_2 = \phi' \sin \theta, \omega_3 = \psi' + \phi' \cos \theta \). If in the gyroscope \( I_1 = I_2 \) (where \( I_1 \) and \( I_2 \) are the moments of inertia with respect to axis in the direction of \( \hat{e}_1 \) and \( \hat{e}_2 \) correspondingly) then the kinetic energy of an gyroscope is given by
\[ T = I_1 \left( \theta'^2 + \phi'^2 \sin^2 \theta \right) / 2 + I_1 \left( \psi' + \phi' \cos \theta \right)^2 / 2 \] \hspace{1cm} (A.1)
where \( I_1 \) is the gyroscope’s moment of inertia about its axis of symmetry. In the gyroscopic theory it is proven that for a fast gyroscope the amplitude of its nutation oscillations is small. The nutation oscillations almost disappear \( (\theta' \rightarrow 0) \) when the precession angle \( \theta \) is very small. If besides that we assume the intrinsic angular momentum is conserved: \( I_1 \dot{\omega}_1 = I_1 \left( \psi' + \phi' \cos \theta \right) = G = \text{const} \) then formula (29) can be written as \( T = G \left( \psi' + \phi' \cos \theta \right)^2 / 2 \). It can be seen that the both terms linearly depend on the angular velocity and, therefore, they will not appear in the law of conservation of energy. Hence, the conservation of energy of the motion of the centre of mass plus the precessional motion of the fast gyroscope about the centre of mass has the form
\[ mV^2 / 2 + U - E = \epsilon \] \hspace{1cm} (A.2)
where \( U \) is the potential energy, corresponding to the motion of the centre of mass, and \( E \) is the potential energy corresponding to the precessional motion. In the gyroscopic theory it is proven that \( E \approx G \phi' \) for small \( \theta \).

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