ON BOHR’S EQUIVALENCE THEOREM

MATTIA RIGHETTI

ABSTRACT. In this note we prove a converse of Bohr’s equivalence theorem for Dirichlet series under some natural assumptions.

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1. Introduction

Bohr’s interest in the Riemann zeta function led him to study the set of values taken by Dirichlet series in their half-plane of absolute convergence. For this problem Bohr developed a new method: associating to any Dirichlet series a power series with infinitely many variables (see [3]). He then introduced an equivalence relation among Dirichlet series and showed that equivalent Dirichlet series take the same set of values in certain open half-planes (see [4]). We give here a very brief account of this theory; for a complete treatment we refer to Bohr’s original work [4] and to Chapter 8 of Apostol [1].

We call general Dirichlet series any complex function $f(s)$, in the variable $s = \sigma + it$, that has a series representation of the form

$$f(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda(n)s},$$

where the coefficients $a(n)$ are complex and the sequence of exponents $\Lambda = \{\lambda(n)\}$ consists of real numbers such that $\lambda(1) < \lambda(2) < \cdots$ and $\lambda(n) \to \infty$ as $n \to \infty$.

Note that this class of general Dirichlet series includes both power series, when $\lambda(n) = n$, and ordinary Dirichlet series, when $\lambda(n) = \log(n)$.

Remark. The above definition of general Dirichlet series is the one that is given in the work of Bohr [4], and it is more restrictive than the usual definition without conditions on the sequence of exponents, which is already present in later works of Bohr (see e.g. [5]). Restricting to the above setting has the advantage that the region of absolute convergence of the series is a right half-plane, like for ordinary Dirichlet series (see e.g. [1, §8.2]). Since this is fundamental in the proof of the converse theorem, we have decided to work in the same setting of [4]. However, as we remarked in our Ph.D. thesis [8], Bohr’s equivalence theorem holds true mutatis mutandis even for the usual general Dirichlet series.

Following Bohr (see e.g. [1, §8.3]), given a sequence of exponents $\Lambda = \{\lambda(n)\}$, we say that a sequence of real numbers $B = \{\beta(n)\}$ is a basis for $\Lambda$ if it satisfies the following conditions:

(i) the elements of $B$ are linearly independent over the rationals;
(ii) for every $n$, $\lambda(n)$ is expressible as a finite linear combination over $\mathbb{Q}$ of elements of $B$;
(iii) for every $n$, $\beta(n)$ is expressible as a finite linear combination over $\mathbb{Q}$ of elements of $\Lambda$.

We may express the above conditions in matrix notation by considering $\Lambda$ and $B$ as infinite column vectors (see [1, §8.4]). In particular, if $B$ is a basis for $\Lambda$, we may write $\Lambda = RB$ and $B = TA$ for some Bohr matrices $R$ and $T$. 

1
We fix a sequence of exponents $\Lambda = \{\lambda(n)\}$ and a basis $B$ of $\Lambda$, so that we may write $\Lambda = RB$. Consider two general Dirichlet series with the same sequence of exponents $\Lambda$, say

$$f(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda(n)s} \quad \text{and} \quad g(s) = \sum_{n=1}^{\infty} b(n)e^{-\lambda(n)s}.$$ 

Then, we say that $f(s)$ and $g(s)$ are equivalent ($f \sim g$), with respect to $B$, if there exists a sequence of real numbers $Y = \{y(n)\}$ such that

$$b(n) = a(n)e^{i(RY)n} \quad \text{for every } n.$$ 

We may now state Bohr’s equivalence theorem (cf. Theorem 8.16 of Apostol [1]), which is, roughly speaking, a combination of Kronecker’s approximation theorem, Rouché’s theorem and the absolute convergence of the Dirichlet series.

**Theorem A (Bohr, [4, Satz 4]).** Let $f_1(s)$ and $f_2(s)$ be equivalent general Dirichlet series absolutely convergent for $\sigma > \alpha$. Then in any open half-plane $\sigma > \sigma_0 \geq \alpha$ the functions $f_1(s)$ and $f_2(s)$ take the same set of values.

**Remark.** Although Bohr’s equivalence theorem is usually stated for open half-planes, we have already remarked in [9] and [8] that Theorem A holds true also for open vertical strips.

In particular, one gets immediately the following more practical version of the above theorem, in the sense that in the applications to the value distribution of $L$-functions one usually appeals to this statement. Similar results in particular cases may be found for example in Bohr [2, §2], Titchmarsh [10, §11.4], Bombieri and Mueller [7, Lemma 1] and Bombieri and Ghosh [6, p. 240].

**Theorem B (Bohr).** Let $f(s)$ be a general Dirichlet series absolutely convergent for $\sigma > \alpha$, and let $S_f(\sigma_1, \sigma_2)$ is the set of values taken by $f(s)$ in the strip $\sigma \leq \sigma_1 < \sigma < \sigma_2 \leq \infty$, then

$$S_f(\sigma_1, \sigma_2) = \{g(\sigma) \mid \sigma \in (\sigma_1, \sigma_2), g \sim f\}.$$ 

If moreover $\Lambda$ has an integral basis $B$, i.e. the Bohr matrix $R$ such that $\Lambda = RB$ has only integer entries, and $V_f(\sigma_0)$ is the set of values taken by $f(s)$ on the vertical line $\sigma = \sigma_0 > \alpha$, then

$$V_f(\sigma_0) = \{g(\sigma_0) \mid g \sim f\}.$$ 

The first statement follows immediately from Theorem A, while the second is Satz 3 in [4].

**Remark.** Note that (1) doesn’t hold in general. Indeed, consider the following example given by Bohr [4, pp. 151–153]: $\lambda(n) = 2n - 1 + \frac{1}{2^n}$. $f(s) = \sum_{n=1}^{\infty} e^{-\lambda(n)s}$ and $g(s) = -f(s)$. Then the only equivalent functions to $f(s)$ are its vertical shifts and, as proved by Bohr, $g(s)$ is not equivalent to $f(s)$. On the other hand if $\tau_m = 2\pi \prod_{n \leq m} (2n - 1)$ then $f(s + i\tau_m)$ converges uniformly to $g(s)$, and conversely $g(s + i\tau_m)$ converges uniformly to $f(s)$. Hence, using Rouché’s theorem one may show that $g(\sigma_0) \in V_f(\sigma_0)$ for any $\sigma_0 > \alpha$.

Actually the same argument shows that $f(s)$ and $g(s)$ have the same values in any open right half-plane, so in general there is no converse to Theorem A. This is because, as the example shows, the set of general Dirichlet series equivalent to a certain general Dirichlet series may not be closed with respect to uniform convergence on compact subsets. However this is a closed set if the sequence of exponents has an integral basis, e.g. for ordinary Dirichlet series.

This leads us to the following converse theorem.

**Theorem C.** Let $f_1(s)$ and $f_2(s)$ be general Dirichlet series with the same sequence of exponents $\Lambda = \{\lambda(n)\}$ and absolutely converging for $\sigma > \alpha$ for some $\alpha$. Suppose that $f_1(s)$ and $f_2(s)$ take the same set of values in any open half-plane $\sigma > \sigma_0 > \alpha$. Then $f_2(s)$ belongs to the closure, with respect to uniform convergence on compact subsets of $\sigma > \alpha$, of the set of general
Dirichlet series equivalent to \( f_1(s) \) and vice versa.
If furthermore \( \Lambda \) has an integral basis, then \( f_1(s) \) and \( f_2(s) \) are equivalent.

**Remark.** The requirement that \( f_1(s) \) and \( f_2(s) \) should have the same exponents is not really restrictive: one just needs to take the union of the sets of exponents. On the other hand *a fortiori* they would have the same exponents in the sense that \( a_1(n) = 0 \) if and only if \( a_2(n) = 0 \).

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2. Proof of Theorem C

Let \( B = \{ \beta(j) \} \) be a basis for \( \Lambda \) and let \( R = (r_{ij}) \) be the Bohr matrix such that \( \Lambda = RB \). We work by induction: we want to show that for any \( n \) there exist \( Y_n \) such that

\[
a_2(k) = a_1(k)e^{i(Ry_n)k}, \quad k = 1, \ldots, n.
\]

For \( n = 1 \), we show that there exist sequences \( \{ \sigma_m \} \), with \( \sigma_m \to \infty \), \( \{ t_m \} \) and \( \{ t'_m \} \) such that

\[
e^{\lambda(1)\sigma_m}[f_1(\sigma_m + it_m) - f_2(\sigma_m + it'_m)] \to 0, \quad m \to \infty.
\]

We construct \( \{ \sigma_m \} \) in the following way. It is well known that for every \( v \in \mathbb{C} \) the Dirichlet series has a zero-free right half-plane which is maximal, say \( \sigma > \sigma^*(v) \). Hence we take \( \sigma_m = \sigma^*(f_2(m)) \), \( m = 1, 2, \ldots \). Note that by definition and by hypothesis for any \( \varepsilon > 0 \) we have

\[
f_2(m) \in S_f_j(\sigma_m - \varepsilon, \infty) \setminus S_f_j(\sigma_m, \infty), \quad j = 1, 2.
\]

Therefore, for any \( m \), there exist \( \sigma_{m,1} \) and \( \sigma_{m,2} \) such that \( 0 < \sigma_m - \sigma_{m,j} < \varepsilon \), \( j = 1, 2 \), and \( t_m \) and \( t'_m \) such that

\[
f_1(\sigma_{m,1} + it_m) = f_2(\sigma_{m,2} + it'_m) = f_2(m).
\]

Hence, taking \( \varepsilon = e^{-|\lambda(1)|\sigma_m/|\lambda(1)| + 1} \) we get

\[
|f_1(\sigma_m + it_m) - f_2(\sigma_m + it'_m)| \leq |f_1(\sigma_m + it_m) - f_1(\sigma_{m,1} + it_m)|
+ |f_2(\sigma_{m,2} + it'_m) - f_2(\sigma_m + it'_m)|
\leq \sum_{n=1}^{\infty} |a_1(n)|e^{-\lambda(n}\sigma_m}|e^{\lambda(n)(\sigma_m - \sigma_{m,1})} - 1|
+ \sum_{n=1}^{\infty} |a_2(n)|e^{-\lambda(n)\sigma_m}|e^{\lambda(n)(\sigma_m - \sigma_{m,2})} - 1|
\ll (|a_1(1)| + |a_2(1)|)e^{-|\lambda(1)|\sigma_m}
+ \sum_{n=2}^{\infty} |a_1(n)|e^{-\lambda(n)\sigma_{m,1}}
+ \sum_{n=2}^{\infty} |a_2(n)|e^{-\lambda(n)\sigma_{m,2}}.
\]

From this we have that (3) immediately follow.

On the other hand, we recall that \( R \) has rational entries, so we may write \( r_{ij} = b_{ij}/q_{ij} \), and for any \( i \) we have \( r_{ij} \neq 0 \) only for a finite number \( j \)s. Hence we define

\[
d_h = \text{l.c.m.} \{ q_{ij} : i = 1, \ldots, h, \ j \text{ s.t. } r_{ij} \neq 0 \}, \quad h = 1, 2, \ldots.
\]

If we apply Helly’s selection principle (see e.g. Lemma 1, §8.12 of [1]) to the uniformly bounded double sequences \( \{-t_m \beta(j) \bmod 2\pi d_1\} \) and \( \{-t'_m \beta(j) \bmod 2\pi d_1\} \), then there exist sequences
\[ \Theta = \{ \theta(j) \} \text{ and } \Theta' = \{ \theta'(j) \} \text{ and a subsequence } \{ m_k \} \text{ such that, when } k \to \infty, \]
\[ e^{\lambda(1) \sigma_{m_k}} [f_1(\sigma_{m_k} + it_{m_k}) - f_2(\sigma_{m_k} + it_{m_k}')] \to a_1(1)e^{i(\Theta')_1} - a_2(1)e^{i(\Theta')_1}. \]

By (3) we have that (2) holds for \( n = 1 \) by taking \( Y_1 = \Theta - \Theta' \).

Suppose now that (2) holds, then reasoning as above it is easy to get that (2) holds also for \( n + 1 \). Indeed, let \( \{ \sigma_m \} \) be the same sequence as above and let \( g_n(s) \) be the general Dirichlet series equivalent to \( f_1(s) \) defined by
\[ g_n(s) = \sum_k a_1(k)e^{i(RY_n)s}e^{-\lambda (k)s}. \]

By hypothesis and Theorem A we have that for any \( \varepsilon > 0 \)
\[ f_2(m) \in S_{g_n}(\sigma_m - \varepsilon, \infty) \setminus S_{g_n}(\sigma_m, \infty). \]

Therefore, for any \( m \), there exist \( \sigma_{m,1} \) and \( \sigma_{m,2} \) such that \( 0 < \sigma_{m} - \sigma_{m,j} < \varepsilon, j = 1, 2, \) and \( t_m \) and \( t_m' \) such that
\[ g_n(\sigma_{m,1} + it_m) = f_2(\sigma_{m,2} + it_m') = f_2(m). \]

Hence, taking \( \varepsilon = e^{-|\lambda(n+1)+|\lambda(n+1)|+1} \), since (2) holds, analogously as before we get
\[ |g_n(\sigma_{m} + it_m) - f_2(\sigma_{m} + it_m')| \ll (|a_1(n+1)| + |a_2(n+1)|)e^{-|\lambda(n+1)+|\lambda(n+1)|}\sigma_{m} \]
\[ + \sum_{k=n+2}^{\infty} |a_1(k)|e^{-\lambda(k)\sigma_{m,1}} + \sum_{k=n+2}^{\infty} |a_2(k)|e^{-\lambda(k)\sigma_{m,2}} \]

From this we deduce that
\[ e^{\lambda(n+1)\sigma_{m}} [g_n(\sigma_{m} + it_m) - f_2(\sigma_{m} + it_m') \to 0, \quad m \to \infty. \]

On the other hand, as before, if we apply Helly’s selection principle to the uniformly bounded double sequences \( \{-t_m \beta(j) \mod 2\pi d_{n+1}\} \) and \( \{-t_m' \beta(j) \mod 2\pi d_{n+1}\} \), then there exist \( \Theta = \{ \theta(j) \} \) and \( \Theta' = \{ \theta'(j) \} \) and a subsequence \( \{ m_k \} \) such that, when \( k \to \infty, \)
\[ e^{\lambda(n+1)\sigma_{m_k}} [g_n(\sigma_{m_k} + it_{m_k}) - f_2(\sigma_{m_k} + it_{m_k}')] \to a_1(n+1)e^{i(RY_{n+\Theta})_{n+1}} - a_2(n+1)e^{i(\Theta')_{n+1}}. \]

By the uniqueness of the limit we have that (2) holds for \( n + 1 \) by taking \( Y_{n+1} = Y_n + \Theta - \Theta' \).

Finally we note that by induction we have actually constructed a sequence \( \{g_n(s)\} \) of general Dirichlet series equivalent to \( f_1(s) \) and by (2) we have that \( g_n(s) \) converges uniformly on every compact subset of \( \sigma > \alpha \) to \( f_2(s) \). Since we may change \( f_1(s) \) with \( f_2(s) \), this proves the first statement.

If \( B \) is an integral basis for \( \Lambda \), then we may apply Helly’s selection principle to the uniformly bounded double sequence \( \{ y_n(j) \mod 2\pi \} \), where \( Y_n = \{ y_n(j) \} \) is the sequence obtained at each step of the induction process. Hence there exist a sequence \( Y \) and a subsequence \( \{ n_k \} \) such that for \( h \to \infty \) the functions \( g_{n_k}(s) \) converge uniformly on every compact subset of \( \sigma > \alpha \) to
\[ g(s) = \sum_k a_1(k)e^{i(RY)s}e^{-\lambda (k)s}, \]
which is equivalent to \( f_1(s) \). By the uniqueness of the limit and of the analytic continuation we have that \( g(s) = f_2(s) \) in the half-plane of absolute convergence, and the result follows.
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Dipartimento di Matematica, Università di Genova, via Dodecaneso 35, 16146, Genova, Italy
Current address: Centre de Recherches Mathématiques, Université de Montréal, P.O. box 6128, Centre-Ville Station, Montréal, Quebec H3C 3J7, Canada.
E-mail address: righetti@dima.unige.it