A GEVREY MICROLOCAL ANALYSIS OF
MULTI-ANISOTROPIC DIFFERENTIAL OPERATORS

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Abstract. We give a microlocal version of the theorem of iterates
in multi-anisotropic Gevrey classes for multi-anisotropic hypoellip-
tic differential operators.

1. Introduction

A fundamental result of Gevrey microlocal regularity due to Hörmander
is

(1.1) \( WF_s(u) \subset WF_s(P(x, D)u) \cup \text{Char}(P) \),

where \( P(x, D) \) denotes a differential operator with analytic coefficients
in \( \Omega \), \( \text{Char}(P) \) its set of characteristic points \((x, \xi) \in \Omega \times \mathbb{R}^n \)
and \( WF_s(u) \) is the Gevrey wave front of the distribution \( u \in \mathcal{D}'(\Omega) \).

Let \( WF_s(u, P(x, D)) \), see [2], be the Gevrey wave front of the distri-
bution \( u \in \mathcal{D}'(\Omega) \) with respect to the iterates of the operator \( P(x, D) \),
then the result (1.1) is made more precise by the following inclusion

(1.2) \( WF_s(u) \subset WF_s(u, P(x, D)) \cup \text{Char}(P) \),

since

(1.3) \( WF_s(u, P(x, D)) \subset WF_s(P(x, D)u) \).

Various extensions and generalizations of results (1.1) and (1.2) have
been obtained, according as one considers the classes of elliptic or hy-
opeelliptic differential operators or one considers different notions of
homogeneity associated to these classes of operators, see e. g. [2], [3],
[8], [15], [16] and [17].

In [14] a microlocal analysis of the so called inhomogeneous Gevrey
classes [15], see also [6], has been introduced. The \( \varphi \)–inhomogeneous
Gevrey wave front of a distribution \( u \in \mathcal{D}'(\Omega) \), denoted \( WF_\varphi(u) \), is
defined with respect to a weight function \( \varphi \).

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polyhedron; Multiquasielliptic differential operators; Gevrey spaces; Gevrey-
Hypoellipticity.
The method of Newton’s polyhedron, see [2] or [1], permits to approach differential operators with respect to their multi-quasihomogeneity. In this situation the $\varphi-$inhomogeneous Gevrey wave front $WF_{\varphi}(u)$ is characterized by a weight $\varphi$ equals to the function $|\xi|_P$ defined by the Newton’s polyhedron $P$ of the operator $P(x, D)$ and it is denoted by $WF_{s,P}(u)$.

An interpretation of the $\varphi-$inhomogeneous Gevrey microlocal analysis to the multi-anisotropic case is given in the paper [8], where a theorem in the spirit of the result (1.1) for a class of multi-quasihomogeneous hypoelliptic differential operators is obtained.

The aim of this paper is to obtain a result in the spirit of (1.2) for a class of multi-anisotropic hypoelliptic differential operators including the classes of operators studied in [2], [4], [7], [8], [10], [16] and [17]. The section 2 is an adapted modification of the $\varphi-$inhomogeneous Gevrey wave front of Liess-Rodino, see [14] and [8], to our multi-anisotropic case in the spirit of [13] and [16]. In section 3 we introduce and study the multi-anisotropic Gevrey wave front with respect to the iterates of an operator $P(x, D)$ and its Newton’s polyhedron $P$, denoted $WF_{s,P}(u, P(x, D))$, the following section 4 gives the microlocal result of type (1.2) for the studied class of differential operators. This class is microlocally characterized by the following definition.

**Definition 1.** Let $x_0 \in \Omega, \xi_0 \in \mathbb{R}^n \setminus \{0\}$ and $P(x, D)$ be a differential operator with coefficients in the anisotropic Gevrey class $G^{s,q}(\Omega)$, we say $(x_0, \xi_0) \notin \sum_{\rho, \delta, s}^{\mu, \mu'} P$ if there exists an open neighbourhood $U \subset \Omega$ of $x_0$, an open $q$-quasiconic neighbourhood $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ of $\xi_0$ and a constant $c > 0$ such that $\forall (x, \xi) \in U \times \Gamma$,

$$
\begin{align*}
|\xi|_{P}^\mu &\leq c |P(x, \xi)|, \\
|D_x^\alpha D_\xi^\beta P(x, \xi)| &\leq c^{\alpha+1} < \alpha, q >^{s\mu < \alpha, q>} |P(x, \xi)||\xi|^{|\xi|_{P}^\delta|\alpha| - \rho|\beta|},
\end{align*}
$$

where the numbers $\rho, \delta, \mu'$ and $\mu$ satisfy $0 \leq \delta < \rho \leq 1$ and $\delta \mu < \mu' \leq \mu$.

The principal result of this work is the following theorem.

**Theorem 1.** Let $u \in \mathcal{D}'(\Omega)$, $P(x, D)$ a differential operator with coefficients in $G^{s,q}(\Omega)$ and $\rho, \delta, \mu', \mu$ such that $0 \leq \delta < \rho \leq 1$ and $\delta \mu < \mu' \leq \mu$, then

$$
WF_{s',P}(u) \subset WF_{s,P}(u, P) \cup \sum_{\rho, \delta, s}^{\mu, \mu', \rho} (P),
$$

where $s' = \max \left( \frac{s \mu}{\mu' - \delta \mu}, \frac{s}{\rho - \delta} \right)$. 

2. Multi-anisotropic Gevrey wave front

This section is an adaptation with a slight modification of the inhomogeneous Gevrey microlocal analysis introduced in [14], see also [15] and [8], to the multi-anisotropic case.

Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $P(x, D)$ be a linear partial differential operator with coefficients in $C^\infty(\Omega)$,  

$$P(x, D) = \sum_{\alpha \in \Lambda} a_\alpha(x) D^\alpha,$$

where $\Lambda$ is a finite subset of $\mathbb{Z}^n_+$. 

**Definition 2.** Let $x_0 \in \Omega$, the Newton's polyhedron of the operator $P(x, D)$ at the point $x_0$, denoted $P(x_0)$, is the convex hull of 

$$\{0\} \cup \{\alpha \in \mathbb{Z}^n_+, a_\alpha(x_0) \neq 0\}.$$

**Remark 1.** A Newton's polyhedron $P$ is always characterized by 

$$P = \cap_{a \in A} \{\alpha \in \mathbb{R}^n_+, <\alpha, a> \leq 1\},$$

where $A(P)$ is a finite subset of $\mathbb{R}^n$.

**Definition 3.** The Newton's polyhedron $P$ is said to be regular if for any $a = (a_1, ..., a_j, ..., a_n) \in A$ we have $a_j > 0, \forall j = 1, ..., n$.

**Definition 4.** The operator $P(x, D)$ is said regular if it satisfies the following conditions:

1. $P(x_0) = P, \forall x_0 \in \Omega.$
2. $P$ is a regular polyhedron.

**Remark 2.** In this paper we consider only regular operators.

Let $P$ be a regular polyhedron, we set 

$$V(P) = \{s^0 = 0, s^1, .., s^m\}$$

the set of the vertices of $P$. 

$$\mu_j = \max a_j^{-1}, a \in A.$$ 

$$\mu = \max \mu_j.$$ 

$$q = \left(\frac{\mu}{\mu_1}, ..., \frac{\mu}{\mu_n}\right).$$ 

$$k(\alpha) = \inf \{t > 0, t^{-1} \alpha \in P\} = \max_{a \in A} <\alpha, a>.$$ 

$$|\xi|_P = \left(\sum_{i=1}^{m} (\xi^2 s^i)^{1/\mu}\right)^{1/2}.$$ 

$$|\xi|_q = \left(\sum_{j=1}^{n} (\xi_j^2 q_j)^{1/2}\right)^{1/2}.$$
Definition 5. Let $s \geq 1$ and $\mathbb{P}$ be a regular polyhedron, we denote $G^{s,\mathbb{P}}(\Omega)$ the space of functions $u \in C^\infty(\Omega)$ such that $\forall K$ compact of $\Omega$, $\exists C > 0, \forall \alpha \in \mathbb{Z}^n_+$,

$$\sup_K |D^\alpha u| \leq C^{|\alpha|+1} k(\alpha)^{\mu k(\alpha)}.$$ (2.1)

Example 1. If the operator $P$ is $l$-quasi-elliptic of order $m$, with the weight $l = (l_1, ..., l_n)$, so its Newton’s polyhedron $\mathbb{P}$ is the simplex of vertices $\{0, m_je_j, j = 1, ..., n\}$, which is obviously regular. In this case the set $A$ coincides with the vector $\sum_{j=1}^n m_j e_j$, and we have $\mu_j = m_j$, $\mu = m$, $l = q = (\frac{m}{m_1}, ..., \frac{m}{m_n})$. If $\alpha \in \mathbb{Z}^n_+$, then $k(\alpha) = m^{-1} < \alpha, q >$ and we obtain $G^{s,\mathbb{P}}(\Omega) = G^{s,q}(\Omega)$ the anisotropic Gevrey space, i.e. the space of functions $u \in C^\infty(\Omega)$ such that $\forall K$ compact of $\Omega$, $\exists C > 0, \forall \alpha \in \mathbb{Z}^n_+$,

$$\sup_K |D^\alpha u| \leq C^{|\alpha|+1} \alpha_1^{q_1} ... \alpha_n^{q_n}.$$ 

The following lemma, obtained in [8], gives the existence of a truncation sequence, following the fundamental lemma 2.2 of [13] in the multi-anisotropic case. The quasihomogeneous case is a result of [16, lemma 1.2].

Lemma 1. Let $K$ be a compact set of $\mathbb{R}^n$ and let $s \geq 1$, then there exists a sequence $(\chi_N) \subset C^\infty_0(\mathbb{R}^n)$ such that $\chi_N = 1$ on $K$ and

$$|D^\alpha \chi_N| \leq C (CN^{\mu})^{<\alpha,a>} \text{ if } <\alpha,a> \leq N, \forall a \in A, N = 1, 2, ... .$$ (2.2)

A characterization of $G^{s,\mathbb{P}}(\Omega)$ using the Fourier transform is given by the following theorem.

Theorem 2. Let $x_0 \in \Omega$ and $u \in \mathcal{D}'(\Omega)$, then $u$ is $G^{s,\mathbb{P}}$ in a neighbourhood of $x_0$ if, and only if there exists a neighbourhood $U$ of $x_0$ and a sequence $(u_N)$ in $\mathcal{E}'(\Omega)$ such that

i) $u_N = u$ in $U$, $N = 1, 2, ... .$

ii) $u_N$ is bounded in $\mathcal{E}'(\Omega)$.

iii) $|\hat{u}_N(\xi)| \leq C \left(\frac{CN^\mu}{|\xi|^\mu}\right)^{\mu N}$, $N = 1, 2, ... .$

Proof. See [14] and [8].

We give now a microlocalization of the definition [5] It is an adapted modification, in the spirit of [13] and [16], of the $\varphi$-inhomogeneous Gevrey wave front of Liess-Rodino, see [14] and [8], to our multi-anisotropic case. It coincides exactly with the classical definition of the quasihomogeneous case.
Definition 6. Let \( x_0 \in \Omega, \xi_0 \in \mathbb{R}^n \setminus \{0\} \) and \( u \in \mathcal{D}^\prime(\Omega) \), we say that \( u \) is \( G_{s,P} \)-microregular at \((x_0, \xi_0)\), if there exists \( C > 0 \), a neighbourhood \( U \) of \( x_0 \) in \( \Omega \), a \( q \)-quasiconic neighbourhood \( \Gamma \) of \( \xi_0 \) in \( \mathbb{R}^n \setminus \{0\} \) and a sequence \((u_N) \subset E_{\Omega}^\prime\) such that

i) \( u_N = u \) in \( U, \ N = 1, 2, ... \).

ii) \( u_N \) is bounded in \( E_{\Omega}^\prime \).

iii) \( |\hat{u}_N(\xi)| \leq C \left( \frac{CN_s}{|\xi|_P} \right)^\mu_N, \ N = 1, 2, ..., \xi \in \Gamma \).

Remark 3. The definition \( \ref{def:GsP} \) coincides exactly with the quasihomogeneous case, see \([16]\), if the polyhedron \( \mathcal{P} \) is the simplex of vertices \( \{0, m_j e_j, j = 1, ..., n\} \).

Using the truncation sequence \((\chi_N)\) we obtain the following lemma, see \([8]\).

Lemma 2. Let \( u \in \mathcal{D}^\prime(\Omega) \) and \((x_0, \xi_0) \notin WF_{s,P}(u) \) and let \( U, \Gamma \) be as in definition \( \ref{def:GsP} \). If \( K \) is a compact neighbourhood of \( x_0 \) in \( U \), \( F \) is a \( q \)-quasiconic compact neighbourhood of \( \xi_0 \) in \( \mathbb{R}^n \setminus \{0\} \) equal to 1 on \( K \) satisfying \( \ref{eq:2.2} \), then there exists \( p_0 \in \mathbb{Z}_+, N_0 \in \mathbb{Z}_+ \) such that the sequence \((\chi_{p_0 N_0 + N_0} u)\) satisfies i)-iii) in \( K \) and \( F \).

We define the \( G_{s,P} \)-singsupp\((u)\) as the complementary of the biggest open subset of \( \Omega \) where \( u \) is \( G_{s,P} \). The relation between the multi-anisotropic Gevrey wave front and the multi-anisotropic Gevrey singular support is given by the following proposition.

Proposition 1. Let \( u \) be a distribution in \( \Omega \), then the projection of \( WF_{s,P}(u) \) on \( \Omega \) is the \( G_{s,P} \)-singsupp\((u)\).

Proof. It follows the similar proof of \([8]\). \( \square \)

The microlocal property of the differential operator \( P(x,D) \) with respect to the \( G_{s,P} \)-wave front \( WF_{s,P}(u) \) is given by the following theorem.

Theorem 3. Let \( u \in \mathcal{D}^\prime(\Omega) \) and \( P(x,D) \) be a differential operator with coefficients in \( G_{s,q}(\Omega) \), then

\[
(2.3) \quad WF_{s,P}(Pu) \subset WF_{s,P}(u).
\]

Proof. See \([14]\) and \([15]\). \( \square \)
Lemma 3. The space of Gevrey vectors of distributions $u \in \mathcal{D}'(\Omega)$ such that $\forall \chi \in \mathcal{E}'(\Omega)$, and $f \in G^{s,p}(\Omega)$, then $gf \in G^{s,p}(\Omega)$, see [11]. This justifies the optimal choice of the regularity of the coefficients of the operator $P(x,D)$.

3. Multi-anisotropic Gevrey wave front with respect to the iterates of a differential operator

The Gevrey microlocal analysis with respect to the iterates of a differential operator has been introduced for the first time by P. Bolley and J. Camus in [2] in the homogeneous case. L. Zanghirati in [16] has adapted it to the quasihomogeneous case. The aim of this section is to extend this analysis to the multi-quasihomogeneous case.

Definition 7. Let $r \in \mathbb{R}$ and $s \geq 1$, we denote $G^{s,p}_r(\Omega, P)$ the space of distributions $u \in \mathcal{D}'(\Omega)$ such that $\forall K$ compact of $\Omega$, $\exists C > 0$, $\forall N \in \mathbb{Z}_+$,
\[
\left\| P^N u \right\|_{H^r(K)} \leq C (CN^s)^{\mu N}.
\]
The space of Gevrey vectors of the operator $P$ is by definition
\[
G^{s,p}(\Omega, P) = \bigcup_{r \in \mathbb{R}} G^{s,p}_r(\Omega, P).
\]

The space of Gevrey vectors $G^{s,p}(\Omega, P)$ of the operator $P$ is described with the help of the Fourier transform in the following lemma.

Lemma 3. Let $x_0 \in \Omega$ and $u \in \mathcal{D}'(\Omega)$, then $u \in G^{s,p}(V, P)$ for a neighbourhood $V$ of $x_0$ if, and only if, there exists a neighbourhood $U$ of $x_0$, $U \subset V$, $C > 0$, $M \in \mathbb{R}$ and a sequence $(f_N)$ in $\mathcal{E}'(V)$ such that
\begin{enumerate}[(i)]
\item $f_N = P^N u$ in $U$, $N = 0, 1, \ldots$
\item $\left| \hat{f}_N(\xi) \right| \leq C (CN^s)^{\mu N} (1 + |\xi|^M)$, $\xi \in \mathbb{R}^n, N = 0, 1, \ldots$
\end{enumerate}

Proof. It follows the proof of proposition 1.4 of [2].

Remark 4. The product of two functions of the space $G^{s,p}(\Omega)$ does not belong in general to $G^{s,p}(\Omega)$, but if $g \in G^{s,q}(\Omega)$ and $f \in G^{s,p}(\Omega)$, then $gf \in G^{s,p}(\Omega)$, see [11].
Proof. It is sufficient to see at first that $G^{s,q}(\Omega) \subset G^{s,\mu u}(\Omega)$, $\forall a \in A$ and for any $a, b \in Z$ in $C$ and apply after lemma 2.3 of [16] or adapt lemma 5.3 of [13]. □

\[ (\mu < \alpha, a >)^{(\mu < a, a >)} \leq (\mu^2 < \alpha, b >)^{(\mu^2 < a, b >)}, \alpha \in Z^+ \]

and apply after lemma 2.3 of [16] or adapt lemma 5.3 of [13]. □

Thanks to the truncation sequence $(\chi_N)$, if $u \in \mathcal{D}'(\Omega)$, the sequence $u_N = \chi_N u$ is bounded in $\mathcal{E}'(\Omega)$ and then $\exists C > 0$, $|\widehat{u_N}(\xi)| \leq C (1 + |\xi|)^M$, $\xi \in \mathbb{R}^n, N \in \mathbb{Z}_+$. In the problem of iterates this property is precised by the following result.

Lemma 5. Let $K$ be a compact subset of $\Omega$ and let $(\chi_N)$ be a sequence in $C^\infty_0(K)$ satisfying \(2\), then $\forall u \in \mathcal{D}'(\Omega)$, $\exists p_0 > 0, \forall p > p_0, \forall r \in \mathbb{Z}_+$, the sequence $f_N = \chi_{pN+r}P^N$ satisfies
\[
\widehat{f_N}(\xi) \leq C (C (N^{s\mu} + |\xi|_p))^\mu N^M, \xi \in \mathbb{R}^n, N \in \mathbb{Z}_+.
\]

Proof. It does not differ substantially from its quasihomogeneous similar lemma 2.4 of [16]. □

The belonging to the space $G^{s,F}(\Omega,P)$ is microlocally characterized by the following definition.

Definition 8. Let $u \in \mathcal{D}'(\Omega)$, $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ and $P(x,D)$ be a differential operator with coefficients in $G^{s,q}(\Omega)$. We say that $u$ is $G^{s,F}$-microregular with respect to the iterates of $P(x,D)$ at $(x_0, \xi_0)$, we denote $(x_0, \xi_0) \notin WF_{s,F}(u,P)$, if there exists $C > 0, M \in \mathbb{R}$, a neighbourhood $U$ of $x_0$ in $\Omega$, a $q$-quasiconic neighbourhood $\Gamma$ of $\xi_0$ in $\mathbb{R}^n \setminus \{0\}$ and a sequence $(f_N) \subset \mathcal{E}'(\Omega)$ such that
\[
\begin{align*}
\text{i)} f_N = P^N u & \quad \text{in } U, N \in \mathbb{Z}_+ . \\
\text{ii)} |\widehat{f_N}(\xi)| & \leq C (C (N^{s\mu} + |\xi|_p))^\mu N^M, \xi \in \mathbb{R}^n, N \in \mathbb{Z}_+ . \\
\text{iii)} |\widehat{f_N}(\xi)| & \leq C (C N^s)^\mu N (1 + |\xi|)^M, \xi \in \Gamma, N \in \mathbb{Z}_+ .
\end{align*}
\]

The following proposition gives the link between the $G^{s,F}$-singularities of a distribution $u \in \mathcal{D}'(\Omega)$ with respect to the iterates of $P(x,D)$ and the wave front $WF_{s,F}(u,P)$.

Proposition 2. Let $u \in \mathcal{D}'(\Omega)$ and $P(x,D)$ be a differential operator with coefficients in $G^{s,q}(\Omega)$, then the projection of $WF_{s,F}(u,P)$ on $\Omega$ is the complement of the biggest open subset $\Omega'$ of $\Omega$ where $u \in G^{s,F}(\Omega',P)$.

Proof. It follows the steps of the proofs of the classical theorems in the homogeneous case, see [2], and the quasihomogeneous case, see [16], and makes use essentially of the following lemma.
Lemma 6. Let \( u \in {\mathcal{D}}' (\Omega) \) and \( (x_0, \xi_0) \notin WF_{s,p} (u, P) \), \( U \) and \( \Gamma \) be as in the definition \( K \) a compact neighbourhood of \( x_0 \) in \( U \), \( F \) be a \( q \)-quasiconic compact neighbourhood of \( \xi_0 \) in \( \Gamma \) and \( (\chi_N) \subset C_0^\infty (U) \) be a sequence equals to 1 on \( K \) satisfying \( (2.4) \), then there exists \( p_0 \in \mathbb{Z}_+ \), \( N_0 \in \mathbb{Z}_+ \) such that the sequence \( (\chi_{p_0N+N_0}P^Nu) \) satisfies \( jjj \) in \( F \).

The microlocal property of the operator \( P(x, D) \) with respect to the wave front \( WF_{s,p} (u, P) \) is the following result.

Theorem 4. Let \( u \in {\mathcal{D}}' (\Omega) \) and \( P(x, D) \) be a differential operator with coefficients in \( G^{s,q} (\Omega) \), then

\[
WF_{s,p} (u, P) \subset WF_{s,p} (Pu) \subset WF_{s,p} (u)
\]

Proof. Suppose that \( (x_0, \xi_0) \notin WF_{s,p} (u) \), then there exists a neighbourhood \( U \) of \( x_0 \), a \( q \)-quasiconic neighbourhood \( \Gamma \) of \( \xi_0 \) and a bounded sequence \( (u_N) \in \mathcal{E}' (\Omega) \) such that \( u_N = u \) in \( U \) and \( |\hat{u}_N (\xi)| \leq C \left( \frac{CN_N}{|\xi|^\mu} \right) \), \( N = 1, 2, \ldots, \xi \in \Gamma \). Let \( K \) be a compact neighbourhood of \( x_0 \) in \( U \), \( F \) be a \( q \)-quasiconic compact neighbourhood of \( \xi_0 \) in \( \Gamma \) and let \( (\chi_N) \subset C_0^\infty (U) \) equal to 1 on \( K \) satisfying \( (2.2) \). Choose \( p \geq p_0 + N_0 \) and set \( f_N = \chi_{pN}P^Nu \), we will show that this sequence satisfies \( jjj \) since \( j \) is true and \( jj \) is fulfilled according to lemma \( 5 \).

We have

\[
\hat{f}_N (\xi) = \int e^{-i<x,\xi>} \chi_{pN}P^Nu dx = \int u \, ^tP^N (e^{-i<x,\xi>}) \chi_{pN} \, dx.
\]

Set \( ^tP (x, D) = \sum_{\alpha \in \mathbb{Z}_+^n \cap \mathbb{P}} a'_\alpha (x) D^\alpha \) and let \( 0 = k_0 < k_1 < \ldots < k_r = 1 \), be the elements of the set \( \{ k = k (\alpha), \alpha \in \mathbb{Z}_+^n \cap \mathbb{P} \} \). Then

\[
^tP (e^{-i<x,\xi>}) \chi_{pN+r} = e^{-i<x,\xi>} |\xi|^\mu P \chi_{pN+r},
\]

where \( R (x, \xi, D) = R_0 + \ldots + R_r \) and

\[
R_l (x, \xi, D) = \sum_{\alpha \in \mathbb{Z}_+^n \cap \mathbb{P}} \sum_{\beta \leq \alpha \atop k(\beta) = k_l} (-1)^{|\beta|} a'_\alpha (x) \frac{\xi^\beta}{|\xi|^\mu} D^{n-\beta}.
\]

By iteration we find

\[
^tP^N (e^{-i<x,\xi>}) \chi_{pN} = e^{-i<x,\xi>} |\xi|^\mu \sum_{0 \leq l_1 \leq \ldots \leq r \leq N} R_{l_1} \ldots R_{l_N} \chi_{pN}.
\]
Since the coefficients of $R_l$ are in $G^{s,a} (\Omega)$, $\forall \xi \in \mathbb{R}^n$, then from lemma 2, we obtain for $< \alpha, a > \leq N, a \in \mathcal{A},$

$$|D^a R_l ... R_l \chi_{pN+r}| \leq C_1^{N+1} N \left( \sum_{1 \leq i \leq N} \sum_{\mu} k_i \right) \left( \sum_{1 \leq i \leq N} k_i \right),$$

since $|\xi|^\beta \leq |\xi|^\mu$, $\forall \beta \in \mathbb{Z}_+$. Then for $|\xi|^\rho \geq N^{s\mu}, < \alpha, a > \leq N, a \in \mathcal{A}$, we get

$$(3.5) \quad |D^a \left( R^N \chi_{pN} \right) | \leq C_2^{N+1} N^{s\mu}. $$

From (5.3), (5.4), (5.5) and lemma 2, we obtain

$$\left| \hat{f}_N (\xi) \right| = \left| \left( (R^N \chi_{pN}) u \right)^{\rho} (\xi) \right| \leq C \left( C N^s \right)^{\rho}, \xi \in F, |\xi|^\rho \geq N^{s\mu},$$

so $(x_0, \xi_0) \notin WF_{s,q} (u, P)$, hence $WF_{s,p} (u, P) \subset WF_{s,p} (u)$.

Since

$$WF_{s,p} (u, P) = WF_{s,p} (Pu, P) \subset WF_{s,p} (Pu)$$

and $WF_{s,p} (Pu) \subset WF_{s,p} (u)$, according to theorem 3, so the proof of theorem 4 is complete.

4. THE MULTI-ANISOTROPIC GEVREY MICROLOCAL REGULARITY

We obtain in this section a result of Gevrey microlocal regularity for a class of multi-anisotropic hypoelliptic differential operators characterized by the following definition.

**Definition 9.** Let $x_0 \in \Omega, \xi_0 \in \mathbb{R}^n \setminus \{0\}$ and $P (x, D)$ be a differential operator with coefficients in $G^{s,a} (\Omega)$, we denote $(x_0, \xi_0) \notin \sum_{\rho, \delta, s} \mu (P)$ if there exists an open neighbourhood $U \subset \Omega$ of $x_0$, an open $q$-quasiconic neighbourhood $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ of $\xi_0$ and a constant $c > 0$ such that $\forall (x, \xi) \in U \times \Gamma$,

$$(4.1) \quad \left\{ \begin{array}{ll}
|\xi|^{\mu} / c \leq c |P (x, \xi)|, \\
D^a D^\beta P (x, \xi) / c^{a+1} \leq c^{q < \alpha, \rho > |P (x, \xi)| |\xi|^{\mu} - |\rho|^{\delta} |, \\
\end{array} \right.$$

where the numbers $\rho, \delta, \mu$ and $\mu'$ satisfy $0 \leq \delta < \rho \leq 1$ and $\delta \mu < \mu' \leq \mu$.

We need the following lemma which is a modification of the similar result of 2 lemm 3.8.

**Lemma 7.** Under the notations of definition 2, if $\chi_N \in C_0^\infty (U)$ satisfies (2.2), so there exists $C > 0$ such that for $(x, \xi) \in U \times \Gamma, h_1, ..., h_j \in$
$\mathbb{Z}_+, a \in \mathcal{A}, \alpha < \alpha_1 + .. + \alpha_j, a > \leq N, \beta_1, .., \beta_{j-1} \in \mathbb{Z}_+^n$:

$$\left| D^{\alpha_1} P_{h_1} P^{(\beta_1)} ... D^{\alpha_{j-1}} P_{h_{j-1}} P^{(\beta_{j-1})} D^{\alpha_j} P_{h_j} \chi_N \right| \leq C^{N+1+|h_1|+..+|h_j|} < \alpha, a >^{s_{\alpha,a}} \left| P(x, \xi) \right|^{h_1+..+h_j+j-1} |\xi|_{P}^{\beta_a} - \rho|\beta|,$$

where $\alpha = \alpha_1 + .. + \alpha_j, \beta = \beta_1 + .. + \beta_{j-1}$.

The principal result of this work is the following theorem.

**Theorem 5.** Let $\Omega$ be an open subset of $\mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$ and $P(x, D)$ be a differential operator with coefficients in $G^{s,q}(\Omega)$ and let $\rho, \delta, \mu'$ and $\mu$ be real numbers satisfying $0 \leq \delta < \rho \leq 1$ and $\delta \mu < \mu' \leq \mu$, then (4.2)

$$WF_{s',P}(u) \subset WF_{s,P}(u, P) \cup \sum_{P,\delta,s}^{\mu,\mu',P}(P),$$

where $s' = \max \left( \frac{s_{\mu}}{\mu'-\delta \mu}, \frac{s}{\rho-\delta} \right)$.

**Proof.** Let $(x_0, \xi_0) \notin WF_{s,P}(u, P) \cup \sum_{P,\delta,s}^{\mu,\mu',P}(P)$, then there exists $C > 0, M \in \mathbb{R}$, a neighbourhood $U$ of $x_0$ in $\Omega$, a $q$-quasiconic neighbourhood $\Gamma$ of $\xi_0$ in $\mathbb{R}^n \setminus \{0\}$ and a sequence $(f_n) \subset \mathcal{E}'(\Omega)$ such that the conditions j), j)), and j)) of definition 8 are fulfilled. Let $K$ be a compact neighbourhood of $x_0$ in $U, \Gamma$ a $q-$quasiconic compact neighbourhood of $\xi_0$ in $\Gamma$ such that (4.1) is hold and let $\chi_N \in C_0^\infty(U), \chi_N = 1$ on $K$ satisfying (2.2) and $p$ a large enough integer. Set $u_N = \chi_{pN} u$ and let’s prove that this sequence satisfies iii) since i) and ii) are fulfilled. We write

$$tP \left( e^{-i<x,\xi>} w \right) = e^{-i<x,\xi>} \left( tP(x, -\xi) (I - R) \right) w,$$

where

$$-R(x, \xi, D) = \sum_{\beta \neq 0} \frac{1}{\beta!} tP^{(\beta)}(x, -\xi) D^\beta.$$

By iteration we get

$$tP^N \left( e^{-i<x,\xi>} w \right) = e^{-i<x,\xi>} \left( tP(x, -\xi) (I - R) \right)^N w.$$

The fact that we can divide by $tP(x, -\xi)$ is due to the following lemma which can easily be proved.

**Lemma 8.** If $(x_0, \xi_0) \notin \sum_{P,\delta,s}^{\mu,\mu',P}(P)$, then $(x_0, -\xi_0) \notin \sum_{P,\delta,s}^{\mu,\mu',P}(tP)$.

Set

$$w_N = \sum_{h_1+..+h_N \leq \frac{\mu'}{\mu-\delta} N} R^{h_1} \left( tP \right)^{-1} ... R^{h_N} \left( tP \right)^{-1} \chi_{pN},$$
where \( tP = tP(x, -\xi) \). Then this function satisfies
\[
(tP (I - R))^N w_N = \chi_{pN} - e_N,
\]
where
\[
e_N = \sum_{j=1}^{N} (tP (I - R))^N \sum_{h_j+h_{N}=k} tP \rho_{j,h_{N}}^{-1} \cher_{N}^{-1} \chi_{pN}.
\]

Hence
\[
(4.3) \quad \hat{u}_N(\xi) = w_N \hat{f}_N(\xi) + \hat{e}_N u(\xi), \quad \xi \in F.
\]

We will estimate both terms of the second member of (4.3). Let \( a \in A \) and \( 0 = k_0 < k_1 < \ldots < k_r = 1 \), be the elements of the set \( \{k = < \alpha, a >, \alpha \in \mathbb{Z}^n_+ \cap \mathbb{P} \} \). We write \( R = R_1 + \ldots + R_r \) where
\[
-R_i(x, \xi, D) = \sum_{\beta, \alpha = k_i} \frac{tP(\beta) (x, -\xi)}{\beta!} (tP (x, -\xi))^{P (\beta)} D^\beta,
\]
then we have
\[
w_N = \sum_{h_1+\ldots+h_N=k} \sum_{h_1} \ldots \sum_{h_N} (R_{1_1} \ldots R_{1_{h_1}}) \ldots (R_{N_1} \ldots R_{N_{h_N}}),
\]
Since \( < \alpha, a > = |\alpha| \leq \mu < \alpha, a > \), so from lemma \( 4 \) we have for \( < \alpha, a > \leq N, a \in A \),
\[
|D^\alpha (R_{1_1} \ldots R_{1_{h_1}}) \ldots (R_{N_1} \ldots R_{N_{h_N}}) | \leq C^{N+1} N^{\mu < \alpha, a >} \ldots (tP (x, \xi))_{N}^{-|\alpha| - |\beta|} |\xi|^{|(\mu - \mu')^N|},
\]
where \( \sum^* \) means the sum over \( 1 \leq l \leq N, 1 \leq i \leq h_i, 1 \leq i \leq r \). Since the number of terms in the sum \( w_N \) is bounded from above by \( C_0 \), so \( \exists C > 0 \) such that, for \( < \alpha, a > \leq N, \xi \in \mathbb{F} \), \( \xi |^{|(\mu - \mu')^N|} \geq N^\mu \), we have
\[
|D^\alpha w_N| \leq C^{N+1} N^{\mu < \alpha, a > \ldots (tP (x, \xi))_{N}^{-|\alpha| - |\beta|} |\xi|^{|(\mu - \mu')^N|},
\]
from lemma \( 4 \) we obtain for \( \xi |^{|(\mu - \mu')^N|} \geq N^\mu \),
\[
\left| \hat{w}_N \hat{f}_N(\xi) \right| \leq C_1 (C_1 N^{\mu} |\xi|^M) \xi |^{(\mu - \mu')^N} \chi_{pN},
\]

\[
(4.4) \quad \leq C_2 \left( \frac{C_2 N^{\mu} |\xi|^M}{|\xi|_{P}} \right)^{\mu N} \chi_{pN}.
\]

\[
|\chi_{pN},
\]
By the same procedure in the estimate of $w_N$, we get for $e_N$,\[ |D^\alpha e_N| \leq C_3^{N+1} N^{s\mu} \langle \frac{N^{\rho-\delta}}{|\xi|^P} \rangle^{(\rho-\delta)\mu N}, \quad \langle \alpha, a \rangle \leq N, |\xi|^P \geq N^{s\mu}. \]

Let $M_1$ be the order of the distribution $u$ in $K$, so
\[
|\widehat{ue_N}(\xi)| \leq C_4^{N+1} |\xi|M_1 \left( \frac{N^{\rho-\delta}}{|\xi|^P} \right)^{(\rho-\delta)\mu N}.
\]

From (4.3), (4.4) and (4.5) we easily obtain that
\[
(x_0, \xi_0) \notin WF_{s', P}(u).
\]

\[\square\]

5. CONSEQUENCES

This section gives some corollaries of the obtained result.

**Corollary 1.** If $P(x, D)$ is a differential operator with analytic coefficients, satisfying (4.1) with $|\xi|^P = |\xi|_q$, then theorem [2] coincides with the principal theorem 5.1 of Bolley-Camus [2], i.e. $\forall s \geq 1$,
\[
WF_s(u) \subset WF_s(u, P) \cup \sum_{\rho, \delta, s} \mu, \mu' (P).
\]

**Remark 5.** The results of [7] and [10] can be included in this corollary.

**Corollary 2.** If the differential operator $P(x, D)$ is $q$-quasihomogeneous with coefficients in $G^{n,q}(\Omega)$, then $|\xi|^P = |\xi|_q$ and
\[
\sum_{\mu, \mu, P}(P) = \{(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\} : P_q(x, \xi) = 0\},
\]
where $P_q(x, \xi)$ is the principal $q$-quasihomogeneous part of $P(x, \xi)$. Consequently theorem [2] coincides with the principal theorem of [16], i.e. $\forall s \geq 1$
\[
WF_{s,q}(u) \subset WF_{s,q}(u, P) \cup \{(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\} : P_q(x, \xi) = 0\}.
\]

**Definition 10.** The operator $P(x, D)$ is said multi-quasielliptic in $\Omega$, if it is regular and $\forall x_0 \in \Omega$,
\[
\exists C > 0, \exists R \geq 0, (|\xi|^P)^{\mu(P)} \leq C |P(x_0, \xi)|, \forall \xi \in \mathbb{R}^n, |\xi| \geq R.
\]

The multi-anisotropic Gevrey regularity of the solutions of multi-quasielliptic differential equations, see [17] and [4], is obtained easily from the following microlocal result.
Corollary 3. Let $u \in \mathcal{D}'(\Omega)$ and $P(x, D)$ be a multi-quasielliptic differential operator with coefficients in $G^{s,q}(\Omega)$, then $\forall s \geq 1$, we have

$$WF_{s,P}(u) = WF_{s,P}(u, P) = WF_{s,P}(Pu).$$

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