Modelling the Evaporation of Non-singular Black Holes

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We present a model for studying the formation and evaporation of non-singular (quantum corrected) black holes. The model is based on a generalized form of the dimensionally reduced, spherically symmetric Einstein–Hilbert action and includes a suitably generalized Polyakov action to provide a mechanism for radiation back-reaction. The equations of motion describing self-gravitating scalar field collapse are derived in local form both in null co-ordinates and in Painleve–Gullstrand (flat slice) co-ordinates. They provide the starting point for numerical studies of complete space-times containing dynamical horizons that bound a compact trapped region. Such spacetimes have been proposed in the past as solutions to the information loss problem because they possess neither an event horizon nor a singularity. Since the equations of motion in our model are derived from a diffeomorphism invariant action they preserve the constraint algebra and the resulting energy momentum tensor is manifestly conserved.

I. INTRODUCTION AND BACKGROUND

General relativity predicts the existence of singularities at the center of all black holes, while Hawking’s famous quantum calculation implies that event horizons shrink by emitting thermal radiation. These two properties of black holes, combined with the fact that special relativity forbids information from escaping from an event horizon, lead to one of the deepest puzzles of modern theoretical physics: the so-called information loss paradox. In its simplest form, the question is: What is the endpoint of black hole formation and evaporation and what happens to the information about the state of the matter that formed the black hole?

A variety of end states have been suggested over the years. Among them:

1. The black hole evaporates completely via thermal radiation so that the information is lost to the outside world (asymptotic region of the black hole spacetime) forever. This scenario entails a breakdown of the unitarity of quantum mechanics.

2. At late stages the radiation is no longer thermal and the information emerges via quantum gravity related, presumably causality violating, corrections.

3. The black hole stops emitting radiation when it nears the Planck scale, leaving behind a microscopic remnant that hides forever an enormous amount of information.

4. Most recently it has been suggested that a firewall of as yet unknown origin exists just outside the event horizon that destroys the entanglement between infalling particles and outgoing radiation, effectively eliminating the information loss problem, albeit at a significant cost.

The purpose of the present paper is to set up a semi-classical model that explicitly realizes an alternative proposal to resolve the information loss problem. This proposal is based on the principle that there are no true singularities in nature. Following we start with the assumption that in the correct theory the classical singularity is will be replaced by a semi-classical non-singular region. Moreover, we assume that the dynamics of gravitational collapse and evaporation in this region can be described by effective semi-classical equations of motion. Our first goal is to construct a suitable “quantum corrected” lagrangian, potentially relevant to four dimensional black holes, from which to derive these semi-classical equations. Such an action must have three key attributes:

1. It must incorporate modifications to Einstein gravity at short distances that resolve the singularity.

2. It must have a radiation back reaction term.

3. At large distance scales it must accurately reproduce Einstein’s theory.

The expectation is that the resulting gravitational collapse and subsequent evaporation will produce neither

1 See also Varadarajan
2 These quantum corrections ultimately derive from the underlying, as yet unknown, quantum gravity theory. Later on we will discuss corrections to the action which mimic Hawking radiation. These corrections are derived from quantization of matter fields but, to avoid confusion, we will refer to them as radiation corrections or radiation terms.
an event horizon nor a singularity. Instead the result will be a completely non-singular spacetime that contains a closed dynamical horizon bounding a compact trapped region. At late times there will therefore be no impediment to information about the collapsed matter escaping to null infinity. Such a scenario was, to the best of our knowledge, first discussed by Ashtekar and Bojowald [3] and made explicit by Sean Hayward [5]. More recently, a compact, dynamical trapping horizon was realized via the numerical simulation of the spherical collapse of a massless scalar field in [6]. In this work the singularity resolution and energy loss were modelled by introducing explicit modifications to the gravitational potential in the equations of motion. Since the equations were not derived from a diffeomorphism invariant action, they did not preserve the constraint algebra and were non-conservative. A more systematic energy conserving model was later constructed [7] and used to show that non-singular black holes could indeed be formed via gravitational collapse [3] and provided that the effective quantum corrections to the gravitational potential were introduced within the framework of a (spherical symmetry preserving) diffeomorphism invariant action. The equations in this model were derived from a variational principle and hence are conservative, but they did not contain a mechanism for describing Hawking radiation.

In the following, we continue the above program by constructing a set of dynamical equations for four D (and higher) gravity that lead to non-singular collapse, are energy conserving and include radiation back-reaction. Modelling the effect of Hawking radiation is difficult and so far it is not well understood in more than two dimensions. In two dimensions, however, the conformal anomaly can be calculated at one loop order and integrated to derive the non-local Polyakov action [10],

\[
I_{Poly} \sim - \int d^2x \sqrt{-g} \mathcal{R} \frac{1}{D^2} \mathcal{R},
\]

where \( g \) is the determinant of the 2D metric, \( g_{\mu\nu} \), \( \mathcal{R} \) is the Ricci scalar calculated using the \( n \) dimensional gravitational constant \( \mathcal{G}(n) \). It has been studied extensively in the context of 2D models for black hole evaporation [11–14]. It has also been used in a toy model to simulate Hawking radiation in four dimensional, spherically symmetric Einstein gravity [15–16]. This is the approach we will adopt. The new, and crucial features of our analysis are two-fold: first we will formulate the theory in terms of a local, diffeomorphism invariant effective action. Our equations are therefore energy conserving and allow for the use of slicings that extend into the horizon. Secondly, the action which is our starting point is a generalization of the spherically symmetric Einstein action that allows us to use quantum motivated corrections that resolve the singularity in the vacuum solution. Since the class of Lagrangians we consider obey a Birkhoff theorem, this guarantees that gravitational collapse will yield an exterior spacetime that asymptotes to the corresponding non-singular solution. We therefore expect a complete absence of singularities in the dynamical evolution of the collapsing matter as found [7]. This will in turn permit us to examine via rigorous calculations, albeit in a simplistic model, the question posed above: How does singularity resolution affect the end point of gravitational collapse and Hawking radiation? One interesting question in this regard concerns the proof by Bergman and Roman [17] that the existence of trapped surfaces in a non-singular spacetime requires violations of the weak equivalence principle near the outer apparent horizon. Our system of equations modelling non-singular black hole formation will allow these potential violations to be examined quantitatively in a dynamical context.

In the present paper, we describe the model and derive the equations of motion both in conformal gauge and in flat slice or so-called Painleve-Gullstrand [18, 19] (P-G) type coordinates. The numerical simulations of these equations are relegated to a subsequent paper.

The organization is as follows: Section III discusses the classical action (without radiating terms). Section IIIA discusses the addition of the radiation corrections via the generalized Polyakov action in local form. Section IIIF derives the equations of motion in conformal gauge and discusses singularity resolution in the energy momentum tensor while Section IIIF presents the Hamiltonian analysis and the equations of motion in P-G type coordinates. Section IIIF closes with conclusions and prospects.

II. EFFECTIVE GRAVITATIONAL ACTION

We start from Einstein gravity in \( n \) space-time dimensions minimally coupled to a massless scalar field, whose action is:

\[
I_{(n)} = \frac{1}{16\pi \mathcal{G}(n)} \int d^n x \sqrt{-\bar{g}} \left( \mathcal{R}(\bar{g}) - 8\pi \mathcal{G}(n)(\mathcal{D}\psi)^2 \right),
\]

where \( \mathcal{G}(n) \) is the higher dimensional gravitational constant, \( \mathcal{R}(\bar{g}) \) is the Ricci scalar calculated using the \( n \) dimensional metric, \( \bar{g}, \mathcal{D} \) is the derivative compatible with \( \bar{g} \) and \( \psi \) is a scalar field. After imposing spherical symmetry, integrating out the angular variables and absorbing a factor of \( \sqrt{8\pi \mathcal{G}(n)} \) into the scalar field to make

\[^4\text{We are grateful to T. Roman for bringing this reference to our attention.}\]
it dimensionless, this action takes the form \[20, 21\]:

\[
I_{(2)} = \frac{1}{l^{n-2}} \int d^2x \sqrt{-g} \left[ R^{n-2} \mathcal{R} + (n-2)(n-3)R^{(n-4)}(DR)^2 + (n-2)(n-3)R^{(n-4)} - R^{n-2}(D\psi)^2 \right] \tag{3}
\]

where \( R \) is the areal radius, \( \mathcal{R} \) is the Ricci scalar of \( g \), which is the \((t, x)\) part of the higher dimensional metric:

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu + R^2d\Omega^{(n-2)} \tag{4}
\]

and we have defined the length parameter:

\[
l^{(n-2)} := \frac{16\pi G^{(n)}}{A_{(n-2)}} \tag{5}
\]

with \( A_{(n-2)} \) being the invariant volume of a unit \( n-2 \)-sphere. In terms of these parameters the well known Schwarzschild-Tangherlini solution is:

\[
ds^2 = -\left(1 - \frac{l^{(n-2)}M}{(n-2)R^{n-3}}\right)dt^2 + \left(1 - \frac{l^{(n-2)}M}{(n-2)R^{n-3}} \right)^{-1}dR^2 + R^2d\Omega^{(n-2)} \tag{6}
\]

where \( M \) is the ADM mass in the vacuum case.

We now generalize the above allowing the coefficients of each of the three terms in \((8)\) to be arbitrary functions of the areal radius \( R(x) \), as,

\[
l_{\text{general}} = \frac{1}{l^{n-2}} \int d^2x \sqrt{-g} \left\{ \phi(R)\mathcal{R} + h(R)(DR)^2 + V(R) + B(R)(D\psi)^2 \right\} \tag{7}
\]

The above is the most general action that contains at most two derivatives of the metric and areal radius and yields a Hamiltonian that is quadratic in their conjugate momenta. It can in principle be further generalized to take the general form of a dimensionally reduced higher curvature Lovelock gravity, whose equations of motion are second order, but higher order in momenta. See \[22–24\] for a Hamiltonian analysis of spherically symmetric Lovelock gravity. We relegate the investigation of such theories to future work.

For \((7)\) to reduce to the GR case, two conditions must be satisfied,

\[
\phi = -B = R^{n-2} \tag{8}
\]

\[
h = V = \phi'' \tag{9}
\]

where a prime means differentiation with respect to the areal radius. We will not assume either of these to be true for most of this work but much of the interesting physics occurs when \((9)\) is obeyed. This will be discussed throughout this work when relevant and particularly in Appendices \[B\] and \[C\].

The action \((7)\) belongs to the class of theories called generic 2-D dilaton gravity (see \[25\] and \[26\] for reviews). These theories obey a Birkhoff theorem \[27\]. The most general vacuum solution can be found by defining a new metric \( \tilde{g}_{\mu\nu} := \omega^2 g_{\mu\nu} \) which puts the action in the form

\[
I = l^{-(n-2)} \int d^2x \sqrt{-\tilde{g}} \left( \phi R(\tilde{g}) + V/\omega^2 \right) \tag{10}
\]

whose equations of motion can readily be solved \[27\]. The corresponding metric in the current parameterization is:

\[
ds^2 = \frac{j}{\omega^2} \left[ -\left(1 - \frac{l^{(n-2)}M}{j} \right)dt^2 + \left(1 - \frac{l^{(n-2)}M}{j} \right)^{-1}dR^2 + R^2d\Omega^2 \right] \tag{11}
\]

where

\[
\ln(\omega^2) := \int \frac{h}{\phi'}dR \tag{12}
\]

\[
j := \int (\phi' V/\omega^2)dR. \tag{13}
\]

The solution contains a single parameter, \( M \), and has at least one Killing vector \((\partial/\partial t)\). There are horizons whenever \( j = l^{(n-2)}M \), and the Killing vector is timelike in the asymptotic region, \( j > l^{(n-2)}M \). Most importantly, the arbitrary functions of the areal radius \( \phi(R) \), \( h(R) \) and \( V(R) \) can, as we shall see, be chosen so that the resulting vacuum solutions are non-singular.

As an example of a non-singular metric that can be obtained as a solution to the equations derived from an action of the form \((7)\) consider the following metric that was originally proposed by Poisson and Israel \[28\]:

\[
ds^2 = -\left(1 - \frac{l^2MR^2}{2(R^3 + \nu^3)} \right)dt^2 + \left(1 - \frac{l^2MR^2}{2(R^3 + \nu^3)} \right)^{-1}dR^2 + R^2d\Omega^2. \tag{14}
\]

This is of the same form as \((11)\) with the following iden-
tifications:
\[ \phi' = j = 2 \frac{R^3 + \nu^3}{R^2} \]
\[ \phi = 2 R^3 - 2 \nu^3 R \]
\[ \omega^2 = j = 2 \frac{R^3 + \nu^3}{R^2} \]
\[ h = \ln(\omega^2) \to j' = 2 \frac{R^3 - 2 \nu^3}{R^3} \]
\[ V = \omega^2 \]

In the above \( \nu \) is the parameter that determines the scale of the quantum corrections. The above metric approaches a deSitter metric as \( R \to 0 \) and hence is manifestly non-singular. The resulting spacetime contains two horizons and the effective stress energy tensor violates the weak energy condition at short length scales (of the order of \( \nu \)). The properties of this metric are discussed more fully in [7]. They can easily be generalized to higher dimensions with:

\[ j(R) = (n - 2) \frac{R^{n-1} + \nu^{n-1}}{R^2} \]

A qualitatively different example of a non-singular static spacetime was derived from polymer quantum gravity in [23] and [31]. This spacetime contains a single bifurcative horizon that surrounds an infinite Kasner type universe on the interior. It describes, in effect, a wormhole whose minimum throat radius shrinks to the polymerization scale before re-expanding to infinity. Details of how it fits into the current formalism are given in Appendix A.

III. RADIATION TERMS: THE GENERALIZED POLYAKOV ACTION

The radiation term [14] assumes that the matter under consideration is minimally coupled to gravity, ie \( B \) is a constant in (7). We will use a more general version that allows for more general matter couplings [31, 34],

\[ I_{Poly} \sim - \int d^2 x \sqrt{-g} \left[ R \frac{1}{D^2} R - b(R) \left( \frac{1}{D^2} R - \ln \mu^2 \right) (D R)^2 + c(R) R \right], \]

where \( \mu \) is a constant related to the renormalization procedure used to obtain the Polyakov action. The forms of \( b \) and \( c \) have been suggested by [34] to be \( b = -3(B')^2/B^2 \) and \( c = -6 \ln B \). We will leave them general in this paper.

The non-local character of the actions [11] and (21) is often dealt with by working in double-null coordinates,

\[ ds^2 = e^{2f} du dv, \]

where \( f \) is a scalar function and the Ricci scalar takes the form

\[ R = -8 e^{-2f} f_{, uv} = 2 D^2 f. \]

It is sometimes advantageous, however, to work in other coordinate systems, particularly when performing numerical simulations that go past horizon formation (see numerics section of [35]). In this case auxiliary fields, \( z_1 \) and \( z_2 \) can be used to write the action as [34, 33, 37]

\[ I_{Poly} \sim - \int \sqrt{-g} R (z_1 + z_2) + D_A z_2 D^A z_1 + b(D R)^2 (z_1 - \ln \mu^2) + c R \]

where \( A = 0,1 \). The equations of motion for \( z_1 \) and \( z_2 \) are given by

\[ D^2 z_1 - R = 0, \]

\[ D^2 z_2 - R - b(D R)^2 = 0. \]

Inserting (24) and (26) into (21) gives (21).

Inspired by [4] and [24] we define the action that we will consider for the rest of this work as

\[ I = \int \frac{1}{l^{n-2}} d^2 x \sqrt{-g} \left[ \phi(R) R + h(R) (D R)^2 + V(R) \right] + W \left[ R (z_1 + z_2) + D_A z_2 D^A z_1 \right] \]

\[ + b(R) (D R)^2 (z_1 - \ln \mu^2) + c(R) R \] + \( B(R) (D \psi)^2 \}

where \( W \) is a coupling constant.

It is worth noting here that Ayal and Piran [15] (see also [16]) considered a model similar to this one. In fact they considered the equations of motion, in null gauge, that would come from varying (27) with the fields given as those for the 4D GR case with \( B = c = 0 \) and \( z_1 = z_2 \). They eventually modified those equations of motion to remove a singularity at \( R = 0 \) in the effective stress energy tensor. The subsequent equations did not obey the Bianchi identities, however, since they altered the equations of motion directly and not their corresponding action.

IV. DOUBLE NULL COORDINATES

In this section we derive the covariant equations of motion, constraints and effective energy momentum tensor
from the action (27) in null gauge.

A. Action

We start with the metric

$$g_{\mu\nu} = \begin{bmatrix} 0 & e^{2f}/2 \\ e^{2f}/2 & 0 \end{bmatrix}$$

(28)

from which the Ricci scalar is given by (23). Using (28)

where we have absorbed the factor of $l^{-2}$ into the action.

B. Equations of Motion

Varying (24) gives the equations of motion

$$\frac{\delta I}{\delta R} = 0 = -4\phi f_{,uv} - 2h'R_{,uv} - 4hR_{,uv} + (e^{2f}/2)V'$$

$$+ 2b(R)R_{,uv}(z_1 - \ln \mu^2) - 4c(R)f_{,uv}$$

$$+ 2B(R)^2_{,uv}$$

(29)

where we have absorbed the factor of $l^{-2}$ into the action.

Equations (33) and (34) can be used to write (30) and (31) without any dependence on $z_1$ or $z_2$ as

$$-4\phi f_{,uv} - 2h'R_{,uv} - 4hR_{,uv} + (e^{2f}/2)V' + 2B'\psi_{,uv}$$

$$+ 2b(R)R_{,uv}(z_1 - \ln \mu^2) - 4c(R)f_{,uv}$$

$$+ 4b(R)_{,uv} + 4\phi'f_{,uv} - 4R_{,uv} - 4R_{,uv}f' - 4R_{,uv}f''$$

$$= 0$$

(35)

and

$$-4R_{,uv}\phi' - 4R_{,uv}\phi'' + e^{2f}V$$

$$+ W\left[16f_{,uv} - 4bR_{,uv} - 4R_{,uv}e' - 4R_{,uv}e''\right] = 0$$

(36)

Using the GR values for $\phi$, $h$, $V$ and $B$ as well as $W = -\alpha/8\pi$, $b = c = 0$, $n = 4$ and $u \to -u$ (due to different sign conventions) we find that (35) and (36) reduce to (11a) and (11b) of [15] (where $\alpha$ is the coupling constant used in that work).

C. Constraint Equations

To find the constraint equations we use the most general metric

$$g_{\mu\nu} = \begin{bmatrix} A & e^{2f}/2 \\ e^{2f}/2 & C \end{bmatrix}$$

(37)

which corresponds to the Ricci scalar

$$R = (-g)^{-2}$$

$$\begin{cases} A \left[ (C,_{,u})^2 / 2 + C,_{,uv}A,_{,u}/2 - e^{-2f}C,_{,uf,uv} \right] \\
+ C \left[ (A,_{,v})^2 / 2 + C,_{,uv}A,_{,u}/2 - e^{-2f}C,_{,uf,uv} \right] \\
+ (e^{2f}/2) \left[ -e^{2f}C,_{,uf,uv} - e^{2f}C,_{,uf,uv} \right] \\
+ A,_{,uv}C,_{,u}/2 - C,_{,uv}A,_{,u} + 8ACf,_{,uv}f,_{uv} \end{cases}$$

$$+ (-g)^{-1} \begin{bmatrix} A,_{,uv} - 2e^{2f}f,_{uv} + C,_{,uv} \end{bmatrix}$$

(38)

From (38) we can see the following useful relationships (up to boundary terms)
There is also a second constraint equation which can be
re-written using (34) to give

\[
(2\phi''(R_v)^2 + 2\phi'R_{vv} - 4f_v\phi'R_v) e^{-2f} (39)
\]

This gives the constraint

\[
\left( \frac{\delta}{\delta A} \int d^2x \sqrt{-g} \delta R \right)_{A=C=0} = 0 = \nonumber \\
2e^{-2f}[\phi'^2(R_v)^2 + \phi'R_{vv} - 2\phi'f_v R_v] - 2e^{-2f}h(R_v)^2 \nonumber \\
- 2e^{-2f}B(\psi,v)^2 \nonumber \\
+ W\{2e^{-2f}[z_{1,vv} + z_{2,v} - 2f_vz_{1,v} - 2f_vz_{2,v}] \nonumber \\
- 2e^{-2f}z_{1,vv}z_{2,v} - 2e^{-2f}b(z_1 - \ln \mu^2)(R_v)^2 \nonumber \\
+ 2e^{-2f}c'(R_v)^2 + c'R_{vv} - 2\phi f_v R_v\} (41)
\]

Note from (33) that \(z_1(u,v) = -2f(u,v) + p_u(u) + p_v(v)\), where \(p_u\) and \(p_v\) are some functions. If we assume that \(p_u\) and \(p_v\) are constants and re-arrange (11) and use (33) and (34) we get

\[
-R_{vv} + 2f_vR_v = \frac{\phi'' + 2Wb(f + \ln |\mu|) - h}{\phi'} (R_v)^2 \nonumber \\
- \frac{B}{\phi'}(\psi,v)^2 - \frac{4W}{\phi'} \{f_{vv} - (f_v)^2 - \tilde{b}/4\} (42)
\]

where we define

\[
\tilde{\phi} := \phi + Wc \quad (43)
\]

\[
\tilde{b}(v) := \int du (bR_{uu}R_v) \quad (44)
\]

and we have used (34) to write

\[
z_{2,vv} = -2f_v + \tilde{b}_v. (45)
\]

There is also a second constraint equation which can be found by swapping \(u\) and \(v\) in (42).

\[\text{D. Effective Stress Energy Tensor}\]

We now use the equations of motion and Einstein’s equations to calculate the energy momentum tensor. Using (2.18) of [21] we find that the non-angular components of the Einstein tensor in \(n\) dimensions are given by

\[
G_{uu}^{(n)} = \frac{(n-2)}{R} (-R_{uu} + 2f_u R_u) \quad (46)
\]

\[
G_{vv}^{(n)} = \frac{(n-2)}{R} (-R_{vv} + 2f_v R_v) \quad (47)
\]

\[
G_{uv}^{(n)} = \frac{(n-2)}{R} \left[ R_{uv} - \frac{(n-3)}{R} \left( \frac{e^{2f}}{4} - R_{u}R_v \right) \right] \quad (48)
\]

Concentrating on the \(u-v\) component, using the equation of motion (36) to solve for \(R_{uv}\) and the expression for the mass (13)

\[
(DR)^2 = \frac{j\omega^2}{\phi'^2} \left( 1 - \frac{M}{j} \right) \quad (49)
\]

\[
= 4e^{-2f}R_u R_v \quad (50)
\]

(\(j = \frac{1}{2}\)) gives the following expression for the effective stress tensor

\[
G_{uv}^{(n)} = \frac{(n-2)}{\phi'R} \left\{ \frac{e^{2f}}{4} \frac{M}{j} \left( \tilde{\phi}'' - \frac{(n-3)\tilde{\phi'}}{R} \right) \right. \nonumber \\
+ W \left[ 4f_{uv} - \frac{e^{2f}}{4} \left( \tilde{c}'' + b \left( 1 - \frac{M}{j} \right) \right) \right] \right\}. \quad (51)
\]

In the above we have assumed only that (17) is satisfied, which also implies that \(j = \phi' = \omega^2\). This is necessary in order to eliminate terms in the stress energy tensor that are singular at \(R = 0\). These conditions are satisfied for the two horizon quantum corrected black hole described by (15)– (19). Assuming reasonable regularity conditions on the mass function and \(c\) and \(b\) at the origin, the singularity observed by Piran and Ayal in the effective stress tensor is resolved as long as the singularity in \(1/j\) is resolved. Note that for the effective stress tensor of the vacuum metric (11) \((M \neq 0)\) to be non-singular \(1/j\) must vanish at least as fast as \(R^2\) as \(R \to 0\). Similar statements apply to the diagonal components of the energy momentum tensor as we can see by comparing (47) to (42).
V. HAMILTON’S EQUATIONS IN NON–NULL COORDINATES

To find the equations of motion in non-null coordinates. We start with the general, ADM metric \[38\],
\[ds^2 = -N^2 dt^2 + \Lambda^2 (N_r dt + dx)^2.\] (52)
For later use, we compute the following quantities:
\[(DR)^2 = -R_y^2 + \Lambda^{-2} R_x^2,\] (53)
\[\sqrt{-g(z)} D^2 R = -\partial_t (\Lambda R_y) + \partial_x (\Lambda N_r R_y + \Lambda^{-1} N R_x),\] (54)
where we define the operator, \(\beta, y\), acting on some field, \(\beta\) by
\[\beta, y := N^{-1} (\beta, t - N_r \beta, x).\] (55)
Using the metric \[52\] the Ricci scalar can be written as
\[\sqrt{-g} R = 2(N_r x N_y \Lambda^{-1} - 2 N_r x \Lambda, y - \Lambda(N_r, y) + N^{-1} \Lambda(N_r, y)^2 + N \Lambda, yy - (N_x \Lambda^{-1}, x))\] (56)
To write the action in a form that lends itself to Hamiltonian analysis we define
\[\Phi(R, t, x, t, x) := \phi(R) + W(z_1 + z_2) + Wc(R)\]
\[= \phi(R) + W(z_1 + z_2)\] (57)
and
\[h_z(R, t, x, z_1(t, x)) := h(R) + Wb(R)(z_1 - \ln \mu^2)\] (58)
With these definitions the action looks like
\[I = \frac{1}{l^{n-2}} \int d^2 x \sqrt{-g} \times \]
\[\left\{ \Phi R + h_z(DR)^2 + V + WD_A z_2 D^A z_1 + B(D\psi)^2 \right\}\] (59)
We first work with the term containing the Ricci scalar. From \[66\] we can calculate this term up to total derivatives (t.d.) to be
\[\sqrt{-g} R \Phi = 2 \Phi, y (N_r, x \Lambda - \Lambda A, y) - 2 N (\Phi, x \Lambda^{-1}, x) + t.d.\] (60)
Using this the action \[59\] becomes
\[I = \frac{1}{l^{n-2}} \int d^2 x \times \]
\[\left\{ 2N^{-1} (\Phi, t - N_r \Phi, x) (N_r, x \Lambda - (\Lambda, t - N_r \Lambda, x))\right\} \]
\[- 2 N (\Phi, t \Lambda^{-1}, x) + \Lambda \Lambda V\]
\[+ h_z [-N^{-1} \Lambda R_y^2 + 2 N_r N^{-1} \Lambda R_r R_x + (\Lambda \Lambda^{-1} - N^2 N^{-1} \Lambda) R_x^2]\]
\[+ W[-N^{-1} \Lambda z_1, t z_2, t + N_r N^{-1} \Lambda (z_1, x z_2, t + z_1, t z_2, x)]\]
\[+ (\Lambda \Lambda^{-1} - N^2 N^{-1} \Lambda) z_1, x z_2, x]\]
\[+ B[-N^{-1} \Lambda \psi^2 + 2 N_r N^{-1} \Lambda \psi, t \psi, x]\]
\[+ (\Lambda \Lambda^{-1} - N^2 N^{-1} \Lambda) \psi^2] \right\}\] (61)
We can see from \[61\] that the conjugate momenta corresponding to \(N\) and \(N_r\) are zero, \(P_N = P_{N_r} = 0\). The remaining conjugate momenta, corresponding to \(\Lambda, R, z_1, z_2\) and \(\psi\) are given by
\[P_{\Lambda} = \frac{-l^{-(n-2)} 2 N^{-1} \Lambda \phi'(R) - N R_x)}{62}\]
\[= \frac{-l^{-(n-2)} 2 \phi' R_y + W(z_1 + z_2), y)\] (62)
\[P_{R} = \frac{l^{-(n-2)} 2 N^{-1} \phi' (N_r, x \Lambda - (\Lambda, t - N_r \Lambda, x))}{63}\]
\[= \frac{-l^{-(n-2)} 2 \phi' R_y + W(z_1 + z_2), y)\]
\[= \frac{-l^{-(n-2)} 2 N^{-1} \phi' (N_r, x \Lambda - 2 \phi' \Lambda, y - 2 \Lambda h_z R_y)}{63}\]
\[= \frac{-l^{-(n-2)} 2 N^{-1} \phi' (N_r, x \Lambda - 2 \phi' \Lambda, y - 2 \Lambda h_z R_y)}{63}\]
\[P_{z_1} = \frac{l^{-(n-2)} 2 N^{-1} W (N_r, x \Lambda - (\Lambda, t - N_r \Lambda, x))}{64}\]
\[= \frac{-l^{-(n-2)} 2 N^{-1} W (N_r, x \Lambda - 2 \Lambda, y - \Lambda z_2, y)\] (64)
\[P_{z_2} = \frac{l^{-(n-2)} 2 N^{-1} W (N_r, x \Lambda - (\Lambda, t - N_r \Lambda, x))}{65}\]
\[= \frac{-l^{-(n-2)} 2 N^{-1} W (N_r, x \Lambda - 2 \Lambda, y - \Lambda z_2, y)\] (65)
and
\[P_{\psi} = \frac{l^{-(n-2)} B[-2 N^{-1} \Lambda \psi, t + 2 N_r N^{-1} \Lambda \psi, t]}{66}\]
\[= \frac{-2 l^{-(n-2)} \Lambda \psi, t + 2 N R, N^{-1} \Lambda \psi, x}{66}\]
respectively. By combining \[62, 63, 64\] and \[65\] we get
where, for ease of notation we have defined

\[
\begin{align*}
\dot{z}_1 &= \omega_p \omega_r^2 R_{y} - P_{21} - P_{22} \\
&= -S_3 - \frac{P_{21}}{W_A}, \\
\dot{z}_2 &= \omega_p \omega_r^2 R_{y} - P_{21} + P_{22} \\
&= -S_3 - \frac{P_{21}}{W_A},
\end{align*}
\]

and

\[
\begin{align*}
\dot{z}_1 &= \omega_p \omega_r^2 R_{y} - P_{21} - P_{22} \\
&= -S_3 - \frac{P_{21}}{W_A}, \\
\dot{z}_2 &= \omega_p \omega_r^2 R_{y} - P_{21} + P_{22} \\
&= -S_3 - \frac{P_{21}}{W_A},
\end{align*}
\]

and

\[
\begin{align*}
\dot{R}_{y} &= \frac{\omega_p \omega_r^2 (\omega_p \omega_r^2 R_{y} + 2 W P_{R})}{\Lambda (\omega_p \omega_r^2 + 4 W h_z)} \\
&= \frac{\omega_p \omega_r^2 (S_1 - 2 W P_{R})}{S_2},
\end{align*}
\]

where, for ease of notation we have defined

\[
\begin{align*}
S_1 &= \omega_p \omega_r^2 R_{y} - P_{21} - P_{22}, \\
S_2 &= \Lambda (\omega_p \omega_r^2 + 4 W h_z), \\
S_3 &= \frac{2 h_z S_1 - \omega_p \omega_r^2 P_{R}}{S_2}
\end{align*}
\]

and we have absorbed a factor of \( l^{n-2} \) into all of the conjugate momenta. Plugging (67), (68), (69) as well as (70) and (71) into the action (61) it can be shown that the Hamiltonian density, \( \mathcal{H} \), can be written as a sum of two constraints,

\[
\mathcal{H} = NH + N_r H_r + t.d.,
\]

where the constraints are given by

\[
\begin{align*}
H &= 2 \omega_p \omega_r^2 R_{y} - \frac{h_z R_{y}^2}{\Lambda} - NV \\
&+ \frac{W}{\Lambda} \left[ -z_{1,x}^2 - 2 \frac{z_{1,x} + z_{2,x}}{\Lambda} \right] \\
&+ \frac{h_z (S_1^2 - \omega_p \omega_r^2 P_{R} S_1 - W P_{R}^2)}{S_2} - \frac{P_{21} P_{22}}{W_A} \\
&- \frac{P_{21}^2}{4\Lambda B} - \frac{B \psi_{,x}^2}{\Lambda},
\end{align*}
\]

and

\[
H_r = -P_{\Lambda,x} + P_{R} R_{,x} + P_{21} z_{1,x} + P_{22} z_{2,x} + P_{\psi} \psi_{,x}.
\]

Note that we absorbed a factor of \( l^{n-2} \) into \( H^M \), \( H^{(M)} \), \( H^{(G)} \) and \( H_r^{(G)} \) (in addition to all of the conjugate momenta).

From (73) we can write down the equations of motion as

\[
R_{,t} = NR_{,y} + N_r R_{,x}
\]

and

\[
\begin{align*}
P_{R,t} &= -\frac{2 N_x}{\Lambda} \left( \frac{\omega_p \omega_r^2}{\Lambda},x \right) - 2 \frac{N_x \omega_p \omega_r^2 R_{y}}{\Lambda^2} + 2 NV \\
&- 2 W \left[ N \frac{z_{1,x} + z_{2,x}}{\Lambda},x \right] + N_r x \left( \frac{z_{1,x} + z_{2,x}}{\Lambda^2} \right)
\end{align*}
\]

and

\[
\begin{align*}
P_{\Lambda,t} &= -\frac{2 N}{\Lambda} \left( \frac{\omega_p \omega_r^2}{\Lambda},x \right) - 2 \frac{N_x \omega_p \omega_r^2 R_{y}}{\Lambda^2} + 2 NV \\
&- 2 W \left[ N \frac{z_{1,x} + z_{2,x}}{\Lambda},x \right] + N_r x \left( \frac{z_{1,x} + z_{2,x}}{\Lambda^2} \right)
\end{align*}
\]

and

\[
\begin{align*}
P_{\psi,t} &= -\frac{N P_{\psi}}{2\Lambda B} + N_r \psi_{,x}
\end{align*}
\]

and

\[
\begin{align*}
P_{\psi,t} &= -\frac{N P_{\psi}}{2\Lambda B} + N_r \psi_{,x}
\end{align*}
\]

and

\[
\begin{align*}
P_{\psi,t} &= -\frac{N P_{\psi}}{2\Lambda B} + N_r \psi_{,x}
\end{align*}
\]
\[
\begin{align*}
P_{z_1,t} &= \left[ -\frac{W(Nz_{2,x} + 2N_x)}{\Lambda} + N_rP_{z_1} \right].x \\
&\quad - NAWb \left[ R_y^2 - R_x^2 \frac{2}{\Lambda^2} \right] \\
\end{align*}
\]
where we used the following useful identity,
\[
\frac{S_1^2}{4W} - \frac{S_2 R_y^2}{2} = \frac{h_x S_1^2 - \tilde{\phi}'P_R S_1 - WP_R^2}{S_2}.
\]
(83)

Since our Hamiltonian is the sum of two first class constraints we must now pick two gauge choices. We will pick
\[
\chi := R - x = 0.
\]
(87)
which makes the spatial coordinate the areal radius.

Before choosing the second gauge notice that the mass function (see Appendix [C]) is given by
\[
M = j \left( 1 - \frac{(\phi')^2}{j\omega^2} (DR)^2 \right)
\]
\[
= j \left( 1 - \frac{(\phi')^2}{j\omega^2} \left[ \frac{R_x^2}{\Lambda} - \left( -\tilde{\phi}' S_1 - 2WP_R \frac{S_2}{S_2} \right)^2 \right] \right)
\]
(88)
where we used (69). This inspires the choice of gauge
\[
\Lambda - \sqrt{\frac{(\phi'R_x)^2}{j\omega^2}} = 0.
\]
(89)
We are interested in the case where (9) is satisfied (which implies \((\phi'R_x)^2j^{-1}\omega^{-2} = 1\)) For this reason we choose
\[
\xi := \Lambda - 1 = 0.
\]
(90)

For both the GR case and the case of (15) - (19) the mass function is well defined and is the boundary term which must be added to the Hamiltonian to make the variational principle well defined (see Appendix [C]). For our gauge choices (and assuming (9)) the mass function is given by
\[
M = j \left( -\tilde{\phi}' S_1 - 2WP_R \frac{S_2}{S_2} \right)^2.
\]
(91)
By comparing (87) and (90) to (52) we can also see that this gauge choice is regular and spatially flat at horizon formation, i.e. when
\[
(\text{DR})^2 = 0.
\]
(92)
The consistency conditions on these two gauge choices can be obtained by setting \(\dot{R} = 0\) and \(\dot{\Lambda} = 0\) in (76) and (78). Note that we now use a dot to represent differentiation with respect to our time, \(T\), in this choice of gauge and that a prime still represents differentiation with respect to the areal radius, \(R\), which is now our spatial coordinate. The consistency conditions are given by
\[
\frac{N_r}{N} = -R_y
\]
(93)
and
\[
(-NR_y)' + \frac{S_3}{2} = 0,
\]
(94)
from which we can write \(N_r\) as
\[
N_r = -\exp \left[ \int dR \frac{S_3}{2R_y} \right]
\]
(95)
where we used (87) and (90) to redefine
\[
S_1 := P_\Lambda/2 - P_{z_1} - P_{z_2},
\]
(96)
\[
S_2 := \tilde{\phi}'^2 + 4W h_z,
\]
(97)
\[
S_3 := 2h_x S_1 - \tilde{\phi}' P_R \frac{S_2}{S_2}
\]
(98)
and
\[
R_y = \frac{-\tilde{\phi}' S_1 - 2WP_R}{S_2}.
\]
(99)

\(R\) and \(\Lambda\) are no longer phase space variables and neither are their conjugate momenta. \(P_R\) and \(P_\Lambda\) must be written in terms of the remaining phase space variables by setting the Hamiltonian and momentum constraints to zero. I.e, we must solve
\[ H = 0 = 2h'' - h_z - V + W \left[ -z_1' z_2' + 2(z_1'' + z_2'') \right] + \frac{h_z S_1^2 - \psi P_R S_1 - W P_R^2}{S_2} - \frac{P_{z_2}^2}{4B} - B \psi^2 \]

and

\[ H_r = 0 = -P^r_A + P_R + P_{z_1} z_1' + P_{z_2} z_2' + P_\psi \psi' \]  

for \( P_R \) and \( P_\Lambda \).

At this point we can consider singularity resolution at the origin in both the metric, as specified by \([12], [93], [95]\), as well as in the higher dimensional generalization of the energy momentum tensor, \([51]\). Consider first the metric: \( N \) goes to a constant at the origin so \([93]\) implies that as long as \( R_{yy} = \sqrt{l^{n-2}M/j} \) is not singular at \( R = 0 \) the metric is regular there. Thus, the singularities in both the energy momentum tensor and in the metric (in the case where \([9] \) is satisfied) can be resolved by choosing the functions \( V(R), \phi(R) \) and \( h(R) \) in the action so that the factors containing \( j (= \phi') \) are not singular at the origin. Note that this does not result in violations of the Bianchi identities since we derived our equations of motion from a variational principal.

With the gauge choices, \([54]\) and \([60]\), the equations of motion for the remaining phase space variables can now be written as

\[ \dot{\psi} = -N \frac{P_\psi}{2B} + N_r \psi', \]  

\[ \dot{P}_\psi = (-2NB \psi' + N_r P_\psi)', \]  

\[ \dot{z}_1 = -NS_3 - \frac{NP_{z_2}}{W} + N_r z_1', \]  

\[ \dot{P}_{z_1} = -(W(N z_2' + 2N') + N_r P_{z_1})' \]  

\[ - NWb(R_{yy}^2 - 1), \]  

\[ \dot{z}_2 = -NS_3 - \frac{NP_{z_2}}{W} + N_r z_2' \]  

and

\[ \dot{P}_{z_2} = (-W(N z_2' + 2N') + N_r P_{z_2})'. \]

These equations of motion, \([102], [103], [104], [105], [106], [107]\) along with the consistency conditions, \([83]\) and \([95]\), the constraints, \([100], [101]\) and the definitions, \([93], [97], [98], [99]\) can be used to evolve appropriate initial conditions forward in time to show the formation of a black hole with a radiating term.

VI. CONCLUSIONS

In this work we defined an action that models spherically symmetric, dimensionally reduced gravity with a radiating term. Although the radiating term was borrowed from the two dimensional conformal anomaly we also incorporated terms which account for the non-minimal coupling of the matter field characteristic of higher dimensional spherically symmetric gravity. We derived Lagrange’s equations in null gauge and found (in agreement with \([12]\)) that the energy momentum tensor is singular at the origin. We then showed that it is possible to remove the singularity in the effective stress tensor as well as the singularity in the vacuum solution by an appropriate choice of coefficients in the action. This is a key result of our paper since it leads to conservative equations that provide the starting point for numerical studies of the evaporation of non-singular black holes up to and including the end state of the evaporation process. In order to prepare for numerical studies, we then performed a detailed Hamiltonian analysis, including a prescription of the boundary conditions and corresponding boundary terms needed to make the variational principle well defined. We then chose a gauge such that the metric is well defined at apparent horizons, which in principle enables the study of the dynamics of the complete spacetime. In an upcoming paper we will present the results of numerical simulations using these equations.

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Appendix A: Loop Quantum Gravity Black Hole

An example of a non-singular, single horizon black hole in four dimensions that can be derived as a solution to
\[ ds^2 = -\left(1 - \frac{k^2}{R^2} - \frac{l^2 M}{2R}\right) dt^2 + \left(1 - \frac{k^2}{R^2} - \frac{l^2 M}{2R}\right)^{-1} \frac{dR^2}{1 - \frac{k^2}{R^2}} + R^2 d\Omega^2 \]

where \( k \) is the polymerization (quantum gravity) length scale. This is of the same form as (11) with the following identifications:

\[ j(R) = 2R\sqrt{1 - \frac{k^2}{R^2}} \quad (A2) \]
\[ \phi'(R) = \frac{j(R)}{1 - \frac{k^2}{R^2}} = \frac{2R}{\sqrt{1 - \frac{k^2}{R^2}}} \quad (A3) \]
\[ \phi(R) = R\sqrt{R^2 - k^2} + k^2 \ln(R + \sqrt{R^2 - k^2}) \quad (A4) \]
\[ \omega^2 = \frac{j(R)}{R\sqrt{1 - \frac{k^2}{R^2}}} = \frac{2R}{\sqrt{1 - \frac{k^2}{R^2}}} \quad (A5) \]
\[ h(R) = \ln(\omega^2) \phi'(R) = \frac{\phi'(R)}{R} = \frac{1}{\sqrt{1 - \frac{k^2}{R^2}}} \quad (A6) \]
\[ V(R) = \omega^2 \frac{j'(R)}{\phi'} = 2 \quad (A7) \]

As shown in (50), the above metric can be analytically continued to describe a complete non-singular spacetime containing a single bifurcate horizon. The area radius is bounded below by \( k \) and re-expands to infinity in the interior of the horizon. Note that for this black hole the conditions (14) are not satisfied, so that there are in principle terms in the effective stress tensor that are singular at \( R = 0 \). In the present case this is not an issue because, as previously mentioned, \( R = 0 \) is excluded from the complete, regular spacetime.

### Appendix B: Misner-Sharp Mass Function

The following analysis describes the mass function used in this work.

1. **General Relativity**

In general relativity with no radiation terms the Misner-Sharp mass function is defined by (24):

\[ l^{n-2}M = (n-2)R^{n-3} \left(1 - (DR)^2\right) = (n-2)R^{n-3} \left(1 - \left[\Lambda^{-2}R^2 - R^2\right]\right) \quad (B1) \]

where, from now on we absorb a factor of \( l^{n-2} \) into the mass function. In the case of 4D GR, ie, \( \phi = \phi = R^2 \), \( h = \hat{h} = 2 = V \), the mass function is given by

\[ M = 2R \left(1 - \left[\frac{R_x}{\Lambda} - \left(\frac{P}{4\Lambda}\right)^2\right]\right) \quad (B2) \]

It is then easy to verify that:

\[ -\tilde{H} := -\frac{R_x}{\Lambda}H - \frac{P}{4\Lambda}H_r = M_x \quad (B3) \]

It therefore makes sense to write the Hamiltonian in terms of the new Hamiltonian constraint:

\[ \mathcal{H} = \tilde{N}\tilde{H} + \tilde{N}_rH_r = -\tilde{N}M_x + \tilde{N}_rH_r \quad (B4) \]

with suitably redefined Lagrange multipliers:

\[ \tilde{N} := \frac{\Lambda}{R_x}N \quad (B5) \]
\[ \tilde{N}_r := N_r - \frac{P}{4\Lambda}N \quad (B6) \]

Note that with asymptotically flat boundary conditions the only boundary terms that arise in the variation are of the form \( \tilde{N}(\delta M)_x \). The boundary term that needs to be added to the Hamiltonian in order to make the variational principle well defined is therefore:

\[ H_B = \tilde{N}_\infty M_\infty \quad (B7) \]

assuming that the mass function vanishes on the inner boundary: \( M_{R=0} = 0 \). This is discussed for the radiating case in Appendix C.

2. **General \( \phi, h \) and \( V \) with No Radiation**

The mass function for general \( \phi \), \( h \) and \( V \) with no radiating terms is given by

\[ M = j \left(1 - \frac{(\phi')^2}{j\omega^2}(DR)^2\right) = j \left(1 - \frac{(\phi')^2}{j\omega^2} \left[\left(\frac{R_x}{\Lambda}\right)^2 - \left(\frac{P}{2\phi}\right)^2\right]\right) \quad (B8) \]
This is a mass function in the sense that it can be written as a combination of constraints,
\[ -\dot{H} := -\frac{\phi' R_x}{\omega^2 \Lambda} H - \frac{P_\Lambda}{2\omega^2 \Lambda} H_r = M_{xx}. \]  
(B9)
and it reduces to the Misner-Sharp mass function for the appropriate values of \( \phi, h \) and \( V \). As in the GR case we define
\[ \mathcal{H} = \bar{\mathcal{H}} + \bar{N}_r H_r = -\bar{N} M_{xx} + \bar{N}_r H_r \]  
(B10)
with suitably redefined lagrangian multipliers:
\[ \tilde{N} := \frac{\omega^2 \Lambda}{\phi' R_x} N \]  
(B11)
\[ \tilde{N}_r := N_r - \frac{P_\Lambda}{2\phi' R_x} N \]  
(B12)

### 3. General \( \phi, h \) and \( V \) with Radiation

In the most general radiating case that we consider, the derivative of the mass function [88] cannot be written as a linear combination of the constraints. In the GR case where \( \phi'' = h = V \) (and therefore \( \phi'^2/j\omega^2 = 1 \)) it is shown in appendix C that the boundary term which must be added to the Hamiltonian is of the form [87] with \( M \) given by (B3). In terms of phase space variables, (69) is given by
\[ M = (n-2)R^{n-3} \times \] 
\[ \left( 1 - \left[ \left( \frac{R_x}{\Lambda} \right)^2 - \frac{(-\tilde{\phi}' S_1 - 2WP_R)}{S_2} \right]^2 \right) \]  
(B13)
The derivative of this mass function also cannot in general be written as a combination of the constraints in the full radiating case. Nonetheless, the analysis of appendix C applies to the non-GR case as long as the conditions \( \phi'' = h = V \) are satisfied in the asymptotic region. In this case, as \( R \to \infty \),
\[ M \to j \left( 1 - \left[ \left( \frac{R_x}{\Lambda} \right)^2 - \frac{(-\tilde{\phi}' S_1 - 2WP_R)}{S_2} \right]^2 \right) \]  
(B14)
and this provides the boundary term needed to make the variational principle well defined, as well as the corresponding conserved energy.

### Appendix C: Boundary Conditions and Boundary Term

Here we derive the boundary conditions at \( x \to \infty \) which give a finite Hamiltonian. For this to happen the Hamiltonian density [73] must go to zero faster than \( x^{-1} \). For the non-radiation fields we adopt the boundary conditions used in [23] and [24] for slices which approach flat slice coordinates as \( x \to \infty \),
\[ N \sim N_\infty(t) + O(x^{-\epsilon_N}), \]  
(C1)
\[ N_r \sim N_r^\infty(t)x^{-(n-3)/2}, \]  
(C2)
\[ \Lambda \approx 1, \]  
(C3)
\[ R \approx x + R_1(t)x^{-\epsilon_R}, \]  
(C4)
where \( \epsilon_N > 0, \epsilon_R > 1 \) for \( n = 4 \) and \( \epsilon_R > n-4 \) for \( n > 4 \). In this derivation we ignore the matter terms which are treated in [39]. For the radiation fields we assume the form
\[ z_1 \sim z_{10}^\infty(t) + z_{11}^\infty(t)x^{-\epsilon_{z_1}} \]  
(C5)
\[ z_2 \sim z_{20}^\infty(t) + z_{21}^\infty(t)x^{-\epsilon_{z_2}} \]  
(C6)
We will use the fact that the Hamiltonian should be finite to find the conditions on \( z_{10}, z_{20}, \epsilon_{z_1} \) and \( \epsilon_{z_2} \).

In this appendix we assume that the fields \( \phi, h, V \) and \( B \) are those of GR case and that \( b \) and \( c \) go to zero at least as fast as (grow at least as slowly as)
\[ b = -3(B'/B)^2 \]  
(C7)
\[ c = -6 \ln B \]  
(C8)
although we will discuss the applicability of this analysis to the singularity resolved case of [10], [13] and [19].

We can write down the asymptotic form of the conjugate momenta, up to dominant terms, using [02], [63], [04] and [65].
\[ P_\Lambda \simeq 2(n - 2) N^{-1}_\infty (-R_{11} x^{n-3-\epsilon R} + N^\infty x^{(n-3)/2}) \]
\[ - 2W N^{-1}_\infty(z_{10,t} + z_{10,20,t} + z_{12,t} x^{-\epsilon_1} + z_{22,t} x^{-\epsilon_2}) \]
\[ + 6W(n - 2)^2 N^{-1}_\infty(z_{10} + z_{12,t} x^{-\epsilon_1} - \ln \mu) \times (R_{11} x^{-\epsilon R} - N^\infty x^{-(n+1)/2}), \]
\[ (C9) \]

\[ P_{\Lambda,x} \simeq 2(n - 2) N^{-1}_\infty((-n - 3 - \epsilon R) R_{11} x^{n-4-\epsilon R}) \]
\[ + N^\infty((n - 3)/2) x^{(n-5)/2} \]
\[ + 2W N^{-1}_\infty(\epsilon_{12} z_{12,t} x^{-\epsilon_1-1} + \epsilon_{12} z_{12,t} x^{-\epsilon_2-1}) \]
\[ (C10) \]

\[ P_R \simeq 2(n - 2) N^{-1}_\infty((-n - 3) R_{11} x^{n-4-\epsilon R}) \]
\[ + N^\infty((n - 3)/2) x^{(n-5)/2} \]
\[ + 6W(n - 2)^2 N^{-1}_\infty(z_{10} + z_{12,t} x^{-\epsilon_1} - \ln \mu) \times (R_{11} x^{-\epsilon R} - N^\infty x^{-(n+1)/2}), \]
\[ (C11) \]

\[ P_{z1} \simeq - W N^{-1}_\infty((n - 3) N^\infty x^{-(n-1)/2} + z_{10,t} + z_{12,t} x^{-\epsilon_1}) \]
\[ (C12) \]

\[ P_{z2} \simeq - W N^{-1}_\infty((n - 3) N^\infty x^{-(n-1)/2} + z_{10,t} + z_{22,t} x^{-\epsilon_2}) \]
\[ (C13) \]

At this point it is advantageous to consider the \( N P_{z1} P_{z2}/W \Lambda \) term in (79). The relevant, radiation terms in this term go as

\[ z_{10,t} z_{20,t} + z_{12,t} z_{22,t} x^{-\epsilon_1} \]
\[ + z_{12,t} z_{22,t} x^{-\epsilon_1} + z_{12,t} z_{22,t} x^{-(\epsilon_1 + \epsilon_2)}, \]
\[ (C14) \]

which are not cancelled by any of the other terms in (73). The form of (C14) means that \( \epsilon_1 + \epsilon_2 > 1 \) and \( z_{10,t} z_{20,t} = 0 \). In the case where \( b = c = 0 \rightarrow z_1 = z_2 := z/2, P_{z1} = P_{z2} := P_z \) we can show that \( z \sim x^{z/2} \). In the spirit of this limit we take

\[ \epsilon_1 > 1/2, \quad \epsilon_2 > 1/2, \quad z_{10} = 0, \quad z_{20} = 0. \]
\[ (C15) \]

Using these conditions and keeping only the biggest terms we then find

\[ P_\Lambda \simeq 2(n - 2) N^{-1}_\infty(-R_{11} x^{n-3-\epsilon R} + N^\infty x^{(n-3)/2}) \]
\[ (C16) \]

\[ P_{\Lambda,x} \simeq 2(n - 2) N^{-1}_\infty((-n - 3 - \epsilon R) R_{11} x^{n-4-\epsilon R}) \]
\[ + N^\infty((n - 3)/2) x^{(n-5)/2} \]
\[ (C17) \]

\[ P_R \simeq 2(n - 2) N^{-1}_\infty((-n - 3) R_{11} x^{n-4-\epsilon R}) \]
\[ + N^\infty((n - 3)/2) x^{(n-5)/2} \]
\[ (C18) \]

\[ P_{z1} \simeq - W N^{-1}_\infty(z_{12,t} x^{-\epsilon_1}) \]
\[ (C19) \]

\[ P_{z2} \simeq - W N^{-1}_\infty(z_{12,t} x^{-\epsilon_2}). \]
\[ (C20) \]

Plugging these and (C15) into (74) and (75) and dropping all non-dominant terms we can see that the Hamiltonian density reduces to the non-radiating case and the rest of the proof that the Hamiltonian is finite can be found in Appendix A.2 of [24].

It is important to note that there are some cancellations in the rest of the derivation which require \( 2\phi'' - h_z - V \rightarrow 0 \) as \( x \rightarrow \infty \). This occurs in the GR case as well as for (13), (18) and (19) assuming that \( b \) and \( c \) go to zero fast enough. These must be satisfied in order to use the boundary conditions given by (C1), (C2), (C3) and (C4).

We now take the variation of the action to find which boundary terms do not approach zero as \( x \rightarrow \infty \). The variation of the action is given by

\[ \delta I = \int dt \int dx \left\{ \partial_t (\delta \Lambda P_\Lambda) - \delta \Lambda \dot{P}_\Lambda + \hat{\Lambda} \delta P_\Lambda \right\} \]
\[ + \partial_t (\delta R P_R) - \delta R \dot{P}_R + \hat{R} \delta P_R \]
\[ + \partial_t (\delta z_1 P_{z1}) - \delta z_1 \dot{P}_{z1} + z_1 \delta P_{z1} \]
\[ + \partial_t (\delta z_2 P_{z2}) - \delta z_2 \dot{P}_{z2} + z_2 \delta P_{z2} \]
\[ - \delta N H - \delta N \dot{H} - \delta N_{\tau} H_r - \delta N_{\tau} \dot{H}_r \right\}, \]
\[ (C21) \]

Assuming that all variations vanish at the time end points the only contributions to the boundary term come from the last line of (C21). Using the boundary conditions (C1), (C2), (C3), (C4), (C5), (C6) and (C15) we calculate the boundary term. This calculation is tedious but straightforward and can be found in Appendix A.2 of [24] for the non-radiating case. The only boundary term which does not approach zero as \( x \rightarrow \infty \) is

\[ N_{\tau} \Lambda \delta P_\Lambda \simeq N^\infty x^{-(n-3)/2} \left( 2(n - 2) \delta (N^\infty/N_\infty) x^{(n-3)/2} \right) \]
\[ 
\[ = (n - 2) N_\infty \delta (N^{R=2}/N^\infty). \]
\[ (C22) \]

The Misner-Sharp mass in this case (see (1.35) of [24]), with \( l = 1 \), is given by

\[ M = (n - 2) R^{n-3}(N_{\tau}/N)^2 \]
\[ (C23) \]

which goes to \( (n - 2)(N^\infty/N_\infty)^2 \) as \( x \rightarrow \infty \) and so the the variation of the action is

\[ \delta I = \int dt dx (\text{dynamical terms}) + \int dt \left[ N \delta M \right]_{x=+\infty}^{x=-\infty}. \]
\[ (C24) \]

This requires the addition of \( NM|_{R=\infty} \) to the Hamiltonian.
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