Incomplete Localization for Disordered Chiral Strips

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Abstract
We prove that a disordered analog of the Su-Schrieffer-Heeger model exhibits dynamical localization (i.e. the fractional moments condition) at all energies except possibly zero energy, which is singled out by chiral symmetry. Localization occurs at arbitrarily weak disorder, provided it is sufficiently random. If furthermore the hopping probability measures are properly tuned so that the zero energy Lyapunov spectrum does not contain zero, then the system exhibits localization also at that energy, which is of relevance for topological insulators [18]. The method also applies to the usual Anderson model on the strip.

1 Introduction
The Anderson model in one-dimension was long known [25] to exhibit complete localization, that is, localization regardless of the strength of the disorder or the energy at which the system is probed. A major advancement in this direction was made in [24], which handles the strip with singular distributions of the onsite disorder. However, as it turns out, additional constraints on the randomness can make complete localization fail at some special energy values or ranges, as was demonstrated already e.g. for the random polymer model [22]. Such special energy values may be linked to the existence of rich topological phases. Indeed, the question of localization for the present model emerged from the study [18] of chiral topological insulators (class AIII in the Altland-Zirnbauer classification [3]) in 1D, where a link was realized between the failure of localization at zero energy and topological phase shifts. Heuristically, this is precisely what makes the Anderson model in 1D topologically trivial [20, Table 1]–it cannot have any phase transitions, being completely localized. Thus the impetus of the present paper was to show that indeed the topological study of strongly disordered systems in [18] did not involve an empty set of models since its assumptions are fulfilled with a probability of either zero or one, as shall be demonstrated.

The chiral model dealt with in the present study—a disordered analog of the SSH model [35]—is characterized by having no on-site potential and possibly alternating distributions of the nearest-neighbor hopping. We further generalize it by working with matrix-valued hopping terms, i.e., we work on a strip. The model exhibits dynamical localization at all non-zero energies, in the sense of the fractional-moments (FM henceforth) condition, as long as the hopping matrix distributions are continuous w.r.t. the Lebesgue measure on $\text{Mat}_N(\mathbb{C})$ and have finite moments. If the zero-energy Lyapunov spectrum does not contain zero (which depends on how the model is tuned), then localization holds also at zero energy, the possibility of this failing being in stark contrast to the Anderson model. As noted above if the zero energy Lyapunov spectrum does contain zero, then the system could exhibit a topological phase shift, so that it makes sense that localization should fail then (as the topological indices are only defined in when the system is localized at zero energy).

There is also an independent interest in using the FM method for localization proofs rather than the multi-scale analysis, since its consequences for dynamical localization are somewhat easier to establish and more readily apply to topological insulators (compare [1] with [16]).
Hence we note that the Anderson model on the strip can also be handled via our methods, but as always using the FM method, we need regularity of the probability distributions (cf. the treatment of [24, 10]). The two key differences for the localization proof between the ordinary Anderson model (which has onsite disorder) and the chiral model (which has disorder only in the kinetic energy) are the a-priori bounds on the Green’s function (Section 5) and the proof of irreducibility necessary for Furstenberg’s theorem (Section 4.1).

Chapman and Stolz [14] deal with a similar case of disordered hopping, but stay within the multi-scale framework of [24]. Studies of the Lyapunov spectrum for various symmetry classes have also been conducted in [29, 26, 4] among others.

This paper is organized as follows. After presenting the model and our assumptions about it, we connect the Green’s function to a product of transfer matrices in Section 3. In Section 4 we use Furstenberg’s theorem in order to show that the Lyapunov spectrum at non-zero real energies must be simple. In Section 5 we employ rank-2 perturbation theory to get an a-priori bound on the $x, x - 1$ entries of the Green’s function, and finally in Section 6 we can tie everything together to obtain localization at non-zero energies. In this section we also show that any decay of the Greens function implies exponential decay, which is typical for 1D models. The study at zero energy requires separate treatment in Section 7. In Section 8 we briefly comment on so called chiral random band matrices at zero energy and the $\sqrt{N}$ conjecture, i.e., what happens to the Lyapunov spectrum when the size of the strip is taken to infinity. Along the way some technical results of independent interest in random operators are presented, such as Proposition 6.4 (Combes-Thomas estimate without assuming compact support for the hopping terms), Proposition 6.5 (upgrading the decay of the Greens function off the real axis, uniformly in the height) and Lemma 6.6 (any decay of the Greens function implies exponential decay, in 1D).

2 The model and the results

We study the Schrödinger equation on $\psi \in \mathcal{H} := \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$ at energy $E$ given by

\[(H\psi)_n := T^*_n \psi_{n+1} + T_n \psi_{n-1} = E\psi_n, \quad (n \in \mathbb{Z}). \tag{2.1}\]

where $(T_n)_{n \in \mathbb{Z}}$ is a random sequence of invertible $N \times N$ complex matrices. Here $N \in \mathbb{N}_{\geq 1}$ is some fixed (once and for all) number of internal degrees of freedom at each site of the chain.

We take $(T_n)_{n \in \mathbb{Z}}$ to be an alternatingly-distributed, independent random sequence. To describe its law, let $\alpha_0, \alpha_1$ be two given probability distributions on $\text{GL}_N(\mathbb{C})$. We assume for both $i = 0, 1$ the following:

Assumption 2.1. (Fatness) $\text{supp} \alpha_i$ contains some subset $U_i \in \text{Open}(\text{GL}_N(\mathbb{C}))$.

Assumption 2.2. (Uniform $\tau$-Hölder continuity) Fix $i = 0, 1$ and $k, l \in \{1, \ldots, N\}$. Let $\mu_{R,I}$ be the conditional probability distributions on $\mathbb{R}$ obtained by ”wiggling” only the real, respectively imaginary part of the random hopping matrix $M_{k,l}$, that is,

\[\mu_{R,I} = \alpha_i(\cdot | (M_{k',l'})_{k',l' \neq k,l}, (M_{kl})_{I,R}).\]

Then we assume that for some $\alpha \in (0, 1]$, $\mu_{R,I}$ is a uniformly $\tau$-Hölder continuous measure (see [1]): there is some constant $C > 0$ such that for all intervals $J \subseteq \mathbb{R}$ with $|J| \leq 1$ one has

\[\mu_{R,I}(J) \leq C|J|^\tau.\]

Assumption 2.3. (Regularity) For $i = 0, 1$, $\alpha_i$ has finite moments in the following sense:

\[\int_{M \in \text{GL}_N(\mathbb{C})} \|M^{\pm 1}\|^2 d\alpha_i(M) < \infty. \tag{2.2}\]
Remark 2.4. These three assumptions are not optimal for the proof of localization that shall follow (cf. [24]), and were chosen as a good middle way optimizing both simplicity of proofs and strong results. Note that we do not require that $\alpha_i$ has a density with respect to the Haar measure.

We then draw $(T_n)_n$ such that all its even members follow the law of $\alpha_0$ and all its odd members follow the law of $\alpha_1$. This makes $H$ into a random ergodic Hamiltonian.

The following definition’s phrasing makes sense in light of the theorems we shall prove at zero energy:

**Definition 2.5.** We say that the system exhibits localization at zero energy if $\alpha_0$ and $\alpha_1$ are such that the Lyapunov spectrum $\{\gamma_j(0)\}_{j=1}^{2N}$ (see (4.2) for the definition) of the zero-energy Schrödinger equation $H\psi = 0$ does not contain zero:

$$0 \notin \{\gamma_j(0)\}_{j=1}^{2N}. \quad (2.3)$$

In what follows, let $(\delta_n)_n \in \mathbb{Z}$ be the canonical (position) basis of $\ell^2(\mathbb{Z})$ and the map $G(n, m; z) = \langle \delta_n, (H - z)^{-1}\delta_m \rangle : \mathbb{C}^N \to \mathbb{C}^N$ acts between the internal spaces of $n$ and $m$; $\| \cdot \|$ is the trace norm of such maps.

The main result of this paper is the following theorem:

**Theorem 2.6.** (I) Under Assumptions 2.1 to 2.3, the FM condition holds at all non-zero energies: $\forall \lambda \in \mathbb{R} \setminus \{0\}, \exists s \in (0, 1) : \exists 0 < C, \mu < \infty$:

$$\sup_{\eta \neq 0} \mathbb{E}[||G(n, m; \lambda + i\eta)||^s] \leq Ce^{-\nu|n-m|}, \quad \forall n, m \in \mathbb{Z}. \quad (2.4)$$

(II) If moreover (2.3) holds, then (2.4) extends to $\lambda = 0$ as well.

With $P := \chi(-\infty, 0)(H)$ the Fermi projection, the theorem implies via [1] the following

**Corollary 2.7.** Under the above assumptions, including (2.3), [18, Assumption 1] is true almost surely: With probability one, for some deterministic $\mu, \nu > 0$ and random $C > 0$ we have

$$\sum_{n, n' \in \mathbb{Z}} \|P(n, n')\|(1 + |n|)^{-\nu}e^{\mu|n-n'|} \leq C < +\infty.$$  

We also have [18, Assumption 2] fulfilled almost-surely due to the FMC being satisfied at zero energy:

**Corollary 2.8.** Under the above assumptions, including (2.3), zero is almost-surely not an eigenvalue of $H$ (that is, [18, Assumption 2]).

**Proof.** The inequality (2.4) at zero energy implies that almost-surely,

$$\lim_{\eta \to 0^+} \sup |\eta||G(n, m; i\eta)|| = 0.$$  

In particular for $n = m$ we find that

$$\lim_{\eta \to 0^+} \eta\|\text{Im} G(n, n; i\eta)\| = 0,$$

for any $n$. However, zero is an eigenvalue of $H(\omega)$ iff

$$\lim_{\eta \to 0^+} \eta\|\text{Im} G(\omega, n, n; i\eta)\| > 0$$

for some $n$ (see [5, Jakšić: Topics in spectral theory]).

In conclusion, taking the (full-measure) intersection of the two above sets, we get that with probability one [18, Assumptions 1 and 2] hold under the condition (2.3).
Relation to earlier model. Any realization of the random Hamiltonian (2.1) is of the same (deterministic) type as considered in [18, eq. (2.1)], at least up to a unitary map implementing notational changes. Let us first recall that Hamiltonian $H'$ and then set up the unitary map. The Hilbert space there was given by

$$\mathcal{H}' := \mathcal{H} \otimes \mathbb{C}^2 \ni \psi = \begin{pmatrix} \psi^+_{m} \\ \psi^-_{m} \end{pmatrix}_{m \in \mathbb{Z}},$$

(2.5)

the Hamiltonian by $H' = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}$ with $S$ acting on $\mathcal{H}$ as $(S\psi^+)_{m} := A_{m}\psi^+_{m-1} + B_{m}\psi^+_{m}$; Here $A, B : \mathbb{Z} \to \text{GL}_N(\mathbb{C})$ are two given deterministic sequences.

We define the unitary map $U : \mathcal{H}' \rightarrow \mathcal{H}$ via

$$U(\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix})_{n} := \begin{pmatrix} \psi^+_{n} \\ \psi^-_{n} \end{pmatrix}, \quad (m = 2n + 1)$$

$$\begin{pmatrix} \psi^+_{n} \\ \psi^-_{n} \end{pmatrix}, \quad (m = 2n),$$

so that $U H' U^* = H$ upon setting

$$T_n := \begin{pmatrix} B^*_{n} \\ A_{n} \end{pmatrix}, \quad (m = 2n + 1)$$

$$\begin{pmatrix} B^*_{n} \\ A_{n} \end{pmatrix}, \quad (m = 2n).$$

Thus our model is a random version of the model studied in [18]. The reason we use $H$ instead of $H'$ as before is that to estimate Green’s functions (our ultimate goal in localization) there is no need to distinguish between two types of sites, and hence no reason to group them into dimers. In the present context, the chirality symmetry operator is $\Pi := (-1)^X$ with $X$ the position operator, and of course we still have the chiral symmetry constraint $\{ H, \Pi \} = 0$.

The almost-sure spectrum. Here we will give conditions for the almost-sure spectrum to contain a dense pure point interval about zero of localized states, i.e., that there will not be a spectral gap about zero, but merely a mobility-gap [17]. Our goal here is to show that this situation is in fact generic.

The idea is that the almost-sure spectrum of an ergodic family is determined by the union of all periodic configurations within the support of the probability measure defining the model [21, 23].

The following two claims are adaptations of [23, Prop. 3.8, 3.9] from the Anderson model to our chiral model. The proof of the first is precisely the same as that of [23, Prop. 3.8] and is hence omitted.

Only until the end of this section, for convenience of notation, we consider our Hilbert space where every pair of sites is grouped together as in (2.5). Hence we define $\mu := \alpha_0 \otimes \alpha_1, \Omega := (\text{GL}_N(\mathbb{C}))^2 \mathbb{Z}$ and $P := \bigotimes_{\mathbb{Z}} \mu$ is a probability measure on $\Omega$. We label a pair $(A_n, B_n) \in \text{GL}_N(\mathbb{C})^2$ of hopping matrices by $S_n$ so that $(S_n)_{n \in \mathbb{Z}} \in \Omega$.

**Proposition 2.9.** There is a subset $\Omega_0 \subseteq \Omega$ with $P(\Omega_0) = 1$ such that: for any $(S_n)_{n} \in \Omega_0$, any $\Lambda \subseteq \mathbb{Z}$ finite, any finite sequence $(q_n)_{n \in \Lambda}$ such that $q_n \in \text{supp}(\mu)$ for all $n \in \Lambda$, and any $\varepsilon > 0$, there is a sequence $(j_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}$ with $|j_n| \rightarrow \infty$ such that

$$\sup_{l \in \Lambda} \| q_l - S_{l-j_n} \| < \varepsilon \quad (n \in \mathbb{N}).$$

**Proposition 2.10.** For any $\bar{S} = (\bar{A}, \bar{B}) \in \text{supp}(\mu)$, the almost-sure spectrum of the random model defined by 2.1 contains the spectrum of the 2-periodic deterministic Hamiltonian $H$ associated with $\bar{S}$.
Proof. Let \( \lambda \in \sigma(\hat{H}) \) be given. By Weyl’s criterion for the spectrum, there is a sequence of normalized vectors \((\psi_n)_{n \in \mathbb{N}} \subseteq \mathcal{H}\) such that \(\| (\hat{H} - \lambda 1) \psi_n \| \to 0\) as \(n \to \infty\). WLOG we may also assume that for each \(n\), \(\psi_n\) has compact support.

By the preceding proposition, we can build a sequence \((j_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}\) such that with probability 1,

\[
\sup_{l \in \text{supp}(\psi_n)} \| \bar{S} - S_{l+j_n} \| < \frac{1}{n} \quad (n \in \mathbb{N}).
\]

Let us define \(\Psi_n := \psi_n(\cdot - j_n)\) for any \(n \in \mathbb{N}\), and estimate

\[
\|(\hat{H} - \lambda 1) \Psi_n\| \leq \|(\hat{H} - \lambda 1) \psi_n\| + \|(\hat{H} - \bar{H}) \Psi_n\|.
\]

Now the first term converges to zero by construction. The second one, after using Holmgren’s bound, is estimated by (2.6) and hence also converges to zero. We conclude that \((\Psi_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}\) is a Weyl sequence for \((\hat{H}, \lambda)\) with probability one and hence \(\lambda \in \sigma(\mathcal{H})\) with the same probability.

Finally we need to understand the spectrum of a deterministic periodic realization:

Lemma 2.11. Let \(A, B \in \text{GL}_N(\mathbb{C})\). If \(H\) is the 2-periodic Hamiltonian (2.1) such that all even, odd hopping terms are equal to \(A, B\) respectively, then

\[
\inf \sigma(H^2) \leq \|A\| \|B\| \text{dist}(\sigma(|A^{-1}B|), 1) \text{dist}(\sigma(|AB^{-1}|), 1).
\]

Proof. After Bloch decomposition we see that

\[
\sigma(H^2) = \cup_{k \in \mathbb{T}^1} \sigma(\|S(k)\|^2)
\]

where \(S(k) = A \exp(-ik) + B\) for all \(k \in \mathbb{T}^1\). Hence

\[
\inf \sigma(|H|) = \inf_{k \in \mathbb{T}^1} \inf \sigma(|A \exp(-ik) + B|)
\]

\[
= \inf_{k \in \mathbb{T}^1} \|(A \exp(-ik) + B)^{-1}\|^{-1}
\]

\[
\leq \inf_{k \in \mathbb{T}^1} \|A\| \|\exp(-ik)1_N + A^{-1}B\|^{-1}
\]

\[
\leq \|A\| \text{dist}(\sigma(|A^{-1}B|), 1).
\]

To finish the argument, we use the fact we could’ve factored \(B\) instead of \(A\), and the basic estimate \(\min\{a, b\} \leq \sqrt{ab}\).

We obtain

Theorem 2.12. If there are sequences \((A_n)_{n}, (B_n)_{n} \subseteq \text{supp}(\alpha_0), \text{supp}(\alpha_1)\) such that the RHS of (2.7) vanishes as \(n \to \infty\) but is strictly positive for any finite \(n\), then almost-surely, \(H\) has no spectral gap.

3 Transfer matrices

In this section we will estimate the Green’s function in terms of transfer matrices, which arise by looking at the Schrödinger equation (with \(H\) acting to the right or to the left),

\[
(H - z)\psi = 0, \quad \varphi(H - z) = 0, \quad (z \in \mathbb{C})
\]

(3.1)
as a second order difference equation, where \( \psi = (\psi_n)_{n \in \mathbb{Z}}, \varphi = (\varphi_n)_{n \in \mathbb{Z}} \) are (possibly unbounded) sequences with \( \psi_n, \varphi_n \in \mathbb{C}^N \) viewed as column, respectively row vectors. Evaluated at \( n \in \mathbb{Z} \), eqs. (3.1) read

\[
T_n \psi_{n-1} + T_n^* \psi_{n+1} = z \psi_n, \\
\varphi_{n-1} T_n^* + \varphi_{n+1} T_n = z \varphi_n.
\]  

(3.2)

For any two sequences \( \psi, \varphi \) of columns and rows respectively, let

\[
C_n(\varphi, \psi) := \varphi_n T_{n+1}^* \psi_{n+1} - \varphi_{n+1} T_{n+1} \psi_n
\]

be their Wronskian (or Casoratian); if they solve (3.1), then their Wronskian is independent of \( n \), as seen from the identity

\[
C_n(\varphi, \psi) - C_{n-1}(\varphi, \psi) = \varphi_n (T_{n+1}^* \psi_{n+1} + T_n \psi_{n-1}) - (\varphi_{n+1} T_{n+1} + \varphi_{n-1} T_n^*) \psi_n.
\]

Moreover, the Wronskian may also be expressed as

\[
C_n(\varphi, \psi) = (\varphi_{n+1} T_{n+1} - \varphi_n) \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} \begin{pmatrix} T_{n+1}^* \psi_{n+1} \\ \psi_n \end{pmatrix},
\]

which prompts us to associate to any sequence \( \psi : \mathbb{Z} \rightarrow \mathbb{C}^N \) the sequence \( \Psi : \mathbb{Z} \rightarrow \mathbb{C}^{2N} \) by

\[
\Psi_n := \begin{pmatrix} T_{n+1}^* \psi_{n+1} \\ \psi_n \end{pmatrix}.
\]

(3.3)

The transfer matrix is the map

\[
A_n(z) : \mathbb{C}^{2N} \rightarrow \mathbb{C}^{2N}, \quad \Psi_{n-1} \mapsto \Psi_n
\]

defined by the first equation (3.2):

\[
\Psi_n = A_n(z) \Psi_{n-1},
\]

(3.4)

holds for any \( n \in \mathbb{Z} \) such that that equation holds. The transfer matrix is thus given by the square matrix of order \( 2N \)

\[
A_n(z) = \begin{pmatrix} z T_0^* & -T_n \\ T_n^* & 0 \end{pmatrix}
\]

(3.5)

with the abbreviation \( M^* := (M^*)^{-1} = (M^{-1})^* \). Since the second equation (3.1) is equivalent to \((H - \tau) \varphi^* = 0\), the constancy of the Wronskian implies

\[
A_n(z)^* J A_n(z) = J, \quad J := \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix},
\]

which can of course also be verified directly from (3.5). Another property that can be so verified is that \( \det A_n(z) \) is independent of \( z \) and moreover

\[
| \det A_n(z) | = 1 \quad (z \in \mathbb{C}, n \in \mathbb{Z}).
\]

(3.6)

The matrix \( J \) defines the symplectic structure of \( \mathbb{C}^{2N} \). In particular for \( z = \lambda \in \mathbb{R} \) the transfer matrix is Hermitian symplectic, \( A_n(\lambda) \in Sp_{2N}^*(\mathbb{C}) \), where

\[
Sp_{2N}^*(\mathbb{C}) \equiv \{ A \in \text{Mat}_{2N}^*(\mathbb{C}) | A^* J A = J \}
\]

(3.7)

is the Hermitian symplectic group, not to be confused with the complex symplectic group \( Sp_{2N}(\mathbb{C}) \equiv \{ A \in \text{Mat}_{2N}(\mathbb{C}) | A^T J A = J \} \) (and see Section 9.2). But we keep \( z \in \mathbb{C} \) for now so that \( A_n(z) \notin Sp_{2N}^*(\mathbb{C}) \) as a rule.

Let \( I \subseteq \mathbb{Z} \) be an interval (in the sense of \( \mathbb{Z} \)) and assume (3.4) for all \( n \in I \). Then clearly

\[
\Psi_n = B_{n,m}(z) \Psi_{m-1}
\]

(3.8)

for \( n, m \in I, n \geq m \) with the matrix of order \( 2N \)

\[
B_{n,m}(z) := A_n(z) \cdots A_m(z).
\]
Remark 3.1. The relations between eqs. (3.1–3.4) trivially extend to matrix solutions \( \psi \) and \( \Psi \) respectively which are obtained by placing \( l = 1, \ldots, 2N \) solutions next to one another in guise of columns. Seen that way they become linear maps \( \psi : \mathbb{C}^l \to \mathbb{C}^N \), \( \Psi : \mathbb{C}^l \to \mathbb{C}^{2N} \).

Lemma 3.2. For \( I, n, m \) and \( \Psi \) as just stated we have

\[
|\Psi_{m-1}|^2 \leq \frac{\text{tr}((\wedge^{l-1}(BP))^2)}{\text{tr}((\wedge^l (BP))^2)}|\Psi_n|^2, \tag{3.9}
\]

where we use \( |M|^2 = M^*M \), set \( B = B_{n,m}(z) \), and where \( P : \mathbb{C}^{2N} \to \mathbb{C}^{2N} \) is an orthogonal projection of rank \( l \) such that \( P\Psi_{m−1} = \Psi_{m−1} \).

The proof rests on the following lemma, to be proven below.

Lemma 3.3. Let \( W \subseteq V \) be linear spaces; let \( V \) be equipped with an inner product and let \( P : V \to V \) be the orthogonal projection onto \( W \). Let \( B : V \to V \) be a linear map and \( B' := B|_W : W \to V \) its restriction to \( W \). Then

\[
\text{tr}(|B'|^2) = \frac{\text{tr}((\wedge^{l-1}(BP))^2)}{\text{tr}((\wedge^l (BP))^2)}\tag{3.10}
\]

with \( l := \dim W \).

Proof of Lemma 3.2. By (3.8) we have

\[
|\Psi_n|^2 = |B\Psi_{m-1}|^2 = |B'\Psi_{m-1}|^2
\]

with \( B' := B|_W, W := \text{im} P \); and thus

\[
|\Psi_{m-1}|^2 \leq \text{tr}(|B'|^2)|\Psi_n|^2
\]

in view of

\[
Q \geq \|Q^{-1}\|^{-1}, \quad \|Q^{-1}\| \leq \text{tr}(Q^{-1})
\]

for any \( Q > 0 \), applied to \( Q := |B'|^2 \). We conclude by (3.10).

Proof of Lemma 3.3. We recall [14, Lemma C.12]: Let \( L \geq l \geq 1 \) be integers and let \( v_1, \ldots, v_l \in \mathbb{C}^L \) be linearly independent. Define \( w := v_1 \wedge \cdots \wedge v_l \) and \( w_k := v_1 \wedge \cdots \wedge \hat{v}_k \wedge \cdots \wedge v_l \) \( (k = 1, \ldots, l) \) with \( \hat{\cdot} \) denoting omission and (for \( l = 1 \)) the empty product being \( w_1 = 1 \in \wedge^0 \mathbb{C}^L = \mathbb{C} \). Let \( G \) be the Gramian matrix (of order \( l \)) for \( v_1, \ldots, v_l \), i.e.

\[
G_{jk} = (v_j, v_k), \quad (j, k = 1, \ldots, l).
\]

Then

\[
\text{tr}(G^{-1}) = \frac{\sum_{k=1}^l \|w_k\|^2}{\|w\|^2}. \tag{3.11}
\]

This having been done, let \( (e_1, \ldots, e_l) \) be an orthonormal basis of \( W \), whence

\[
\varepsilon = e_1 \wedge \cdots \wedge e_l
\]

\[
\varepsilon_k = e_1 \wedge \cdots \wedge \hat{e}_k \wedge \cdots \wedge e_l, \quad (k = 1, \ldots, l)
\]

are orthonormal bases of \( \wedge^l W \) and \( \wedge^{l-1} W \), respectively. We then apply (3.11) with \( L = 2N \) to \( v_i = B'e_i \), and thus to \( w = (\wedge^l B')\varepsilon \), \( w_k = (\wedge^{l-1} B')\varepsilon_k \) and to \( G_{jk} = (|B'|^2)_{jk} \), the result being

\[
\text{tr}(|B'|^2) = \frac{\text{tr}(\wedge^{l-1} B'|^2)}{\text{tr}(\wedge^l B'|^2)}\tag{3.10}
\]

Finally, just as we have \( \text{tr}_W(|B'|^2) = \text{tr}(|BP|^2) \), so we do

\[
\text{tr}_W(|\wedge^k B'|^2) = \text{tr}(|\wedge^k (BP)|^2),
\]

because \( (\wedge^k B)(\wedge^k P) = \wedge^k (BP) \).
The entries $G_{nk} = G_{nk}(z)$ of the Green’s function are matrices of order $N$ which may be looked at in their dependence on $n$ at fixed $k$. So viewed $G_k = (G_{nk})_{n \in \mathbb{Z}}$ satisfies

$$(H - z)G_k = \delta_k$$

with $\delta_k = (\delta_{nk}1)_{n \in \mathbb{Z}}$. Thus $\psi = G_k$ satisfies (3.1) at sites $n \neq k$ and in the matrix sense of Remark 3.1 (with $\ell = N$). Likewise,

$$G_k = (G_{nk})_{n \in \mathbb{Z}}, \quad G_{nk} = \begin{pmatrix} T_{n+1}^*G_{n+1,k} & G_{nk} \end{pmatrix}$$

satisfies (3.4) with $\Psi = G_k$ by (3.3). In particular, it does for $n \in I = (-\infty, k - 1]$ whence Lemma 3.2 applies to $m \leq n = k - 1$. The result is as follows:

**Lemma 3.4.** We have

$$|G_{m-1,k}(z)|^2 \leq \frac{\operatorname{tr}(|\Lambda^{-1}(BP)|^2)}{\operatorname{tr}(|\Lambda^1(BP)|^2)} |G_{k-1,k}(z)|^2,$$

where $B = B_{k-1,m}(z)$ and $P$ is a projection onto the range of $G_{m-1,k}$.

**Proof.** Given the preliminaries it suffices to observe that $|G_{m-1,k}|^2 \leq |G_{m-1,k}|^2$.

In particular, we can pick $G^+$ as the right half-line Green’s function on $[m - 1, \infty)$ and $P$ a projection onto the first $N$ dimensions of $\mathbb{C}^{2N}$. As we’ll see below, this will then relate $G_{m-1,k}(z)$ to the $N$th Lyapunov exponent of the transfer matrices, times a ”constant” (in $|m - k|$) factor $G_{k-1,k}(z)$ which will be controlled via the a-priori bounds Section 5.

## 4 The Lyapunov spectrum and localization

In this section we define the Lyapunov spectrum associated to the random sequence of matrices $(A_n(\lambda))_n$. The main result here will be Corollary 4.10 which will show that the smallest positive exponent is strictly positive for all $\lambda \in \mathbb{R} \setminus \{0\}$. Since it encodes in it the localization length, at least morally this already implies localization, and we shall show this rigorously using the FM method.

For brevity we define the maps $a, S$ into $\text{Sp}_{2N}(\mathbb{C})$ by

$$\text{GL}_N(\mathbb{C}) \ni X \mapsto \begin{pmatrix} X^0 & 0 \\ 0 & X \end{pmatrix}$$

$$\mathbb{R} \ni \lambda \mapsto \begin{pmatrix} \lambda & -I_N \\ I_N & 0 \end{pmatrix}.$$ 

Then we may factorize the transfer matrix (3.5) as $A_n(z) = S(z)a(T_n)$, the first factor being deterministic. We note that the matrices $A_n(z)$ are independent, but not identically distributed since they are distributed differently for $n$ even and odd. Hence $(A_{2n+1}A_{2n})_n$ is an i.i.d. random sequence.

For $\lambda \in \mathbb{R}$, let $\mu_\lambda$ be the push forward measure induced by

$$\text{GL}_N(\mathbb{C})^2 \ni (X, Y) \mapsto S(\lambda)a(X)S(\lambda)a(Y) \in \text{Sp}_{2N}(\mathbb{C})$$

where $X, Y$ are distributed with $\alpha_1, \alpha_0$ respectively. Below we sometimes leave $\lambda$ implicit in the notation.

**Proposition 4.1.** We have $\int \log^+ (\|g\|) d\mu(g) < \infty$ where $\log^+$ is the positive part of $\log$. 

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Proof. By definition we have
\[ \int \log^+(\|g\|)d\mu(g) = \int \log^+(\|Sa(X)Sa(Y)\|)d\alpha_1(X)d\alpha_0(Y). \]
Since \( \log^+ \) is monotone increasing,
\[ \log^+(\|Sa(X)Sa(Y)\|) \leq 2\log^+(\|S\|) + \log^+(\|a(X)\|) + \log^+(\|a(Y)\|). \]
Hence it is sufficient to show that
\[ \int_{\text{GL}_N(\mathbb{C})} \log^+(\|a(X)\|)d\alpha_i(X) < \infty. \] (4.1)
This follows from Assumption 2.3 by \( \|a(X)\| = \max(\|X^\circ\|, \|X\|), \|X^\circ\| = \|X^{-1}\|, \) and \( \log^+(t) \leq t^2 \) for all \( t > 0. \)

Corollary 4.2. Using [6, pp. 6] we have that the \( 2N \) Lyapunov exponents (henceforth LE)
\[ \gamma_j(z) \equiv \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(\sigma_j(B_n(z)))], \] (4.2)
where \( \sigma_j \) is the \( j \)-th singular value of a matrix (ordered such that \( \sigma_1 \) is the largest), are well-defined and take values in \( [-\infty, \infty) \). We use the abbreviation \( B_n(z) = B_{n,1}(z). \)

Remark 4.3. While (3.6) alone does not allow us to conclude a symmetry property for the exponents, if we restrict to \( z \in \mathbb{R} \), then the Hermitian symplectic condition implies that the exponents are symmetric about zero, that is, \( \gamma_j(z) = -\gamma_{2N-(j-1)}(z) \) for all \( j \in \{1, \ldots, N\} \).

Proof. Since we are taking the logarithm, the symmetry of the singular values of the Hermitian symplectic matrix \( B_n(z) \) about one (as shown in Proposition 9.8) implies a symmetry of the exponents about zero.

4.1 Irreducibility associated with the transfer matrices

In order to prove the irreducibility of the semigroup generated by \( \text{supp}(\mu_\lambda) \), we study the map that takes a hopping matrix to the transfer matrix and its relation to \( \text{Sp}^*_2(\mathbb{C}) \).

Proposition 4.4. Let
\[ M = \begin{pmatrix} \lambda & 0 \\ \lambda T^\circ & -T \end{pmatrix} \] (4.3)
with \( \lambda \in \mathbb{R} \setminus \{0\} \). Then the map
\[ \text{GL}_N(\mathbb{C})^3 \ni (T_1, T_2, T_3) \mapsto M_1M_2M_3 \in \text{Sp}^*_2(\mathbb{C}) \]
is a submersion, i.e. its tangent map has maximal rank.

Lemma 4.5. Let
\[ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_{2N}(\mathbb{C}) \]
with \( D \in \text{GL}_N(\mathbb{C}) \). Then \( M \in \text{Sp}^*_2(\mathbb{C}) \) iff
\[ B = RD, \quad C = DS \] (4.4)
with \( R = R^*, \) \( S = S^* \) and
\[ A = D^*(I + B^*C). \] (4.5)
The proof may be found in Section 9.2.

Let $\mathcal{S}_0 \subseteq \text{Sp}_{2N}(\mathbb{C})$ be the open subset (and hence submanifold) given by $\mathcal{S}_0 = \{M \in \text{Sp}_{2N}(\mathbb{C})| \det D \neq 0\}$. By Lemma 4.5, the map $\mathcal{S}_0 \to \text{GL}_N(\mathbb{C}) \times \text{Herm}_N(\mathbb{C}) \times \text{Herm}_N(\mathbb{C})$, $M \mapsto (D, R, S)$ is a coordinate chart of $\mathcal{S}_0$; for short $M \equiv (D, R, S)$. In particular

$$\dim \text{Sp}_{2N}(\mathbb{C}) = \dim \mathcal{S}_0 = 2N^2 + N^2 + N^2 = 4N^2.$$ 

In the following $\lambda \in \mathbb{R}$, $\lambda \neq 0$ is fixed.

Let $\mathcal{A}_1 \subseteq \mathcal{S}_0$ be the submanifold which in terms of the chart consists of matrices $M \equiv (D, R = \lambda \mathbf{1}, S) =: (D, S)$. In particular $\dim \mathcal{A}_1 = 3N^2$. Moreover, we consider matrices of the form (4.3) with $T \in \text{GL}_N(\mathbb{C})$. Then $M \in \text{Sp}_{2N}(\mathbb{C})$, as remarked before (3.7), though they are not of the form discussed in Lemma 4.5 because of $D = 0$. However, their products are, $M_1M_2 \in \mathcal{A}_1$, with

$$M_1M_2 \cong (D, S) = (-T_1^0T_2, -\lambda(T_2^0T_2)^{-1})$$

(4.6)

in terms of the chart. In fact

$$M_1M_2 = \begin{pmatrix} * & -\lambda T_1^0T_2 \\ \lambda T_1^0T_2 & -T_1^0T_2 \end{pmatrix}$$

from which the claims about $D$ and $R$, i.e. $D \in \text{GL}_N(\mathbb{C})$ and $R = \lambda \mathbf{1}$, are evident by comparison with (4.4); the one about $S$ follows from $T_2(T_2^0T_2)^{-1} = T_2^2$.

Let $M$ be as in (4.3) and $M' \equiv (D, S) \in \mathcal{A}_1$. Then $MM' \in \mathcal{S}_0$ with $MM' \equiv (\tilde{D}, \tilde{R}, \tilde{S})$, where

$$\tilde{D} = \lambda T^0D, \quad \tilde{R} = \lambda - \lambda^{-1}|T|^2, \quad \tilde{S} = S + \lambda^{-1}|D|^{-2}.$$  

(4.7)

In fact,

$$\begin{pmatrix} \lambda T^0 & -T \\ T^0 & 0 \end{pmatrix} \begin{pmatrix} A & \lambda D \\ DS & D \end{pmatrix} = \begin{pmatrix} * & (\lambda^2T^0 - T)D \\ T^0A & \lambda T^0D \end{pmatrix},$$

where $A = D^0 + \lambda DS$ by (4.5). Then the claim (4.7) is obvious for $\tilde{D}$ and by (4.4) amounts for the rest to

$$T^0A = \tilde{D} \tilde{S}, \quad (\lambda^2T^0 - T)D = \tilde{R} \tilde{D}.$$  

The two conditions simplify to $D^{-1}D^0 + \lambda S = \lambda \tilde{S}$ and to $\lambda^2 - T(T^0)^{-1} = \lambda \tilde{R}$, which by $D^{-1}D^0 = |D|^{-2}$ are satisfied.

The next lemma concludes the proof of Proposition 4.4.

**Lemma 4.6.** The above product maps

$$\begin{align*}
\text{GL}_N(\mathbb{C}) \times \text{GL}_N(\mathbb{C}) & \to \mathcal{A}_1, \\
(T_1, T_2) & \mapsto M_1M_2 = M', \\
\text{GL}_N(\mathbb{C}) \times \mathcal{A}_1 & \to \mathcal{S}_0, \\
(T, M') & \mapsto MM'
\end{align*}$$

are submersions.

**Proof.** We represent domain and codomain of both maps in their charts:

$$\begin{align*}
(T_1, T_2) & \mapsto (D, S), \\
(T, D, S) & \mapsto (\tilde{D}, \tilde{R}, \tilde{S})
\end{align*}$$

(4.8)

with right hand sides given by (4.6, 4.7). As a preparation we observe that the following maps are submersions:

- $\text{GL}_N(\mathbb{C}) \ni T \mapsto T^{-1}, T^*, TM$ or $MT$ for some fixed $M \in \text{GL}_N(\mathbb{C})$, etc.
GL_N(C) \cap \text{Herm}_N(C), S \mapsto S^{-1}

GL_N(C) \to \text{Herm}_N(C), T \mapsto |T|^2 or |T^*|^2,

\text{Herm}_N(C) \supset : S \mapsto S + M for some fixed M \in \text{Herm}_N(C).

Only the map \( T \mapsto |T|^2 \) may deserve comment. It has a differentiable right inverse \( S \mapsto S^{1/2} \) on the open subset \( \{S > 0\} \subseteq \text{Herm}_N(C) \), whence the claim.

The first map (4.8) is then written as a concatenation

\[
(T_1, T_2) \mapsto (D, T_2), \quad T_2 \mapsto S
\]

of maps that are seen to be submersions. Likewise for the second map:

\[
(T, D, S) \mapsto (T, D, \tilde{S}), \quad (T, D) \mapsto (T, \tilde{D}), \quad T \mapsto \tilde{T}.
\]

**Proposition 4.7.** Let \( \lambda \in \mathbb{R} \setminus \{0\} \). Then the semigroup \( T_{\mu\lambda} \) generated by \( \text{supp} \mu_\lambda \) contains an open subset of \( \text{Sp}_N^q(C) \).

Proof. By definition, the measure \( \mu_\lambda \) is the one induced by \( \alpha_0, \alpha_1 \) through (4.3) and the map \( (M_1, M_2) \mapsto B := M_2 M_1 \). Now \( T_{\mu\lambda} \) contains for sure all matrices \( B'B = M_2' M_1' M_2 M_1 \). By Assumption 2.1, Proposition 4.4 and the submersion theorem [9, Theorem 7.1], the matrices \( M_2' M_2 M_1 \) cover an open subset of \( \mathcal{N}_0 \) and hence of \( \text{Sp}_N^q(C) \), and so does \( B'B \). \( \square \)

**Remark 4.8.** For the usual Anderson model on a strip, as in [24, 34] for example, the transfer matrix is rather

\[
A_x(z) = \begin{pmatrix}
    z - V_x & -\mathbb{1}_N \\
    \mathbb{1}_N & 0_N
\end{pmatrix}
\]

with \((V_x)_{x \in \mathbb{Z}}\) the independent, identically distributed sequence of onsite potentials (which in general should take values in \( \text{Herm}_N(C) \)). Then known results, say, in [24] (and references therein) show that the semigroup generated by the support of this transfer matrix contains an open subset of the Hermitian symplectic group (for all \( z \in \mathbb{C} \)). Taking this result (which holds also for \( z = 0 \)) to replace our Proposition 4.7 and proving an a-priori bound, one could extend the analysis here to the Anderson model on the strip as well.

**Corollary 4.9.** If \( \lambda \in \mathbb{R} \setminus \{0\} \), then the Lyapunov spectrum is simple: \( \gamma_j(\lambda) \neq \gamma_{j'}(\lambda) \) for all \( j \neq j' \) in \( \{1, \ldots, 2N\} \).

Proof. We apply [6, Proposition IV.3.5], which goes through even though it is applied on \( \text{Sp}_N^q(\mathbb{R}) \) whereas here we apply it on \( \text{Sp}_N^q(C) \). \( \square \)

**Corollary 4.10.** If \( \lambda \in \mathbb{R} \setminus \{0\} \), then \( \gamma_N(\lambda) > 0 \).

Proof. We always have \( \gamma_N \geq \gamma_{N+1} \) because this is how we choose the ordering of the labels. By Remark 4.3 we have that \( \gamma_N(\lambda) = -\gamma_{N+1}(\lambda) \) and by Corollary 4.9 \( \gamma_N(\lambda) \neq \gamma_{N+1}(\lambda) \). \( \square \)

**Remark 4.11.** \( \gamma_N(0) = 0 \) is possible though generically false. See also Section 8.

Proof. When \( z = 0 \), via (3.4), for even sites:

\[
\Psi_{2x} \equiv \begin{pmatrix}
    T_{2x+1}^* \psi_{2x+1} \\
    \psi_{2x}
\end{pmatrix} = A_{2x}(0) \Psi_{2x-1}
\]

\[
= \begin{pmatrix}
    0 & -T_{2x} \\
    T_{2x} & 0_N
\end{pmatrix} \begin{pmatrix}
    T_{2x}^* \psi_{2x} \\
    \psi_{2x-1}
\end{pmatrix} = \begin{pmatrix}
    -T_{2x} \psi_{2x-1} \\
    \psi_{2x}
\end{pmatrix}.
\]
That is,
\[ \psi_{2x+1} = -T_{2x+1}^0 T_{2x} \psi_{2x-1}, \] (4.9)
and similarly for the odd sites,
\[ \psi_{2x+2} = -T_{2x+2}^0 T_{2x+1} \psi_{2x}. \]

As a result, within its zero eigenspace, \( H \) commutes with the operator \((-1)^X\) where \( X \) is the position operator (this operator gives the parity of the site) and the problem splits into two independent first-order difference equations, with transfer matrices \( -T_{2x+1}^0 T_{2x} \) and \( -T_{2x+2}^0 T_{2x+1} \) respectively. These transfer matrices are not symplectic, nor is the absolute value of their determinant equal to one—they are merely elements in \( \text{GL}_N(\mathbb{C}) \), and only their direct sum has these properties.

Hence the theorem insuring the simplicity of the Lyapunov spectrum, [6, pp. 78, Theorem IV.1.2] does not help in this case, since the simplicity of the Lyapunov spectrum of these transfer matrices will not imply that none of the exponents are zero. Instead we are reduced to the more direct question of whether any of the exponents are zero or not.

Here is a (trivial) example where there is a zero exponent: when \( N = 1 \), there is only one exponent for each (separate) chirality sector, and the hopping matrices are merely complex numbers. Then for the (even) positive chirality e.g. we have
\[
\gamma^+_1(0) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(|(-T_{2n+1}^0 T_{2n}) (-T_{2n-1}^0 T_{2n-2}) \ldots (-T_{2}^0 T_{2})|)]
\]
\[
= \lim_{n \to \infty} \frac{1}{n} \left( - \sum_{l=3, l \text{ odd}}^{2n+1} \mathbb{E}[\log(|T_l|)] + \sum_{l=2, l \text{ even}}^{2n} \mathbb{E}[\log(|T_l|)] \right)
\]

(independence property)
\[
= \mathbb{E}_{\alpha_1}[\log(| \cdot |)] - \mathbb{E}_{\alpha_0}[\log(| \cdot |)].
\]

We immediately see that when \( \alpha_1 = \alpha_0 \), the exponent is zero, and when \( \alpha_1 \neq \alpha_0 \) and both have non-zero log-expectation value (not the same value), then the exponent is non-zero.

In general, as can be seen from the formula for the transfer matrices, if \( \text{supp} \alpha_1 \subseteq \text{supp} \alpha_0 \) then none of the exponents will be zero at zero energy.

In the rest of this section we establish continuity properties of the exponents and finally connect \( \gamma_N(\lambda) \) to the decay rate (in \( n \)) of a product of \( n \) transfer matrices. This is a simple extension of the analysis in [24, Section 2] to the complex-valued case with off-site randomness. Thus we frequently use the notation and conventions of [24] below sometimes without explicit reference.

We adopt the following viewpoint which is customary in the study of product of random matrices. Since we are analyzing matrices in \( \text{Sp}^*_2(\mathbb{C}) \), we’ll be interested in isotropic subspaces (the subspaces of \( \mathbb{C}^{2N} \) on which \( J \) restricts to the zero bilinear form) which are left invariant by \( \text{Sp}^*_2(\mathbb{C}) \). Correspondingly we study the isotropic Grassmannian manifold, \( \bar{L}_k \), the set of isotropic subspaces of dimension \( k \) within \( \mathbb{C}^{2N} \) with \( k \in \{1, \ldots, N\} \). A convenient way to parametrize such isotropic subspaces is via exterior powers: a simple vector \( u_1 \wedge \cdots \wedge u_k \in \Lambda^k \mathbb{C}^{2N} \) with \( \{u_i\}_i \) linearly independent and \( \langle u_i, Ju_j \rangle = 0 \) for all \( i, j \in \{1, \ldots, k\} \) defines a point in \( \bar{L}_k \), and one can talk about the action of \( \text{Sp}^*_2(\mathbb{C}) \) on such a point by lifting \( g \in \text{Sp}^*_2(\mathbb{C}) \) to \( \Lambda^k g \) on \( \Lambda^k \mathbb{C}^{2N} \).

**Definition 4.12.** For any \( [v] \in \bar{L}_k \) and \( z \in \mathbb{C} \) let
\[
\Phi_z([v]) := \mathbb{E}[\log(\frac{\|\Lambda^k A_1(z)v\|}{\|v\|})].
\]
Proposition 4.13. \([v] \mapsto \Phi([v])\) is continuous.

Proof. This is [24, Prop. 2.4] or [11, V.4.7 (i)] for our model. Via Proposition 9.12 and Proposition 4.1 we find that the “sequence” \([v] \mapsto \log(\frac{\|A^k[v]\|}{\|v\|})\) is bounded by the integrable function \(k(2N-1)\log(\|A_1||)\). So using the Lebesgue dominated convergence theorem and the fact that \([v] \mapsto \log(\frac{\|A^k[v]\|}{\|v\|})\) is continuous we find our result.

Proposition 4.14. \(z \mapsto \Phi_z([v])\) is Lipschitz continuous, uniformly in \([v]\), as long as \(z\) ranges in a compact subset of \(\mathbb{C}\).

Proof. This is [24, Prop. 2.4] or [11, V.4.7 (ii)] for our model. First note that since \(\|Mu\| = \|ML^{-1}Lu\| \leq \|ML^{-1}\|\|Lu\|\) we have for all \([v]\),

\[
\Phi_z([v]) - \Phi_w([v]) \leq E[\log(\|A_1(z)(A_1(w))^{-1}\|)].
\]

By symmetry,

\[
\Phi_w([v]) - \Phi_z([v]) \leq E[\log(\|A_1(w)(A_1(z))^{-1}\|)]
\]

\[
\text{(det}(A_1(w)(A_1(z))^{-1}) = 1) = (2N-1)E[\log(\|A_1(z)(A_1(w))^{-1}\|)],
\]

so that

\[
|\Phi_z([v]) - \Phi_w([v])| \leq (2N-1)E[\log(\|A_1(z)(A_1(w))^{-1}\|)].
\]

Next we have

\[
S(z)a(X)S(z)a(Y)[S(w)a(X)S(w)a(Y)]^{-1} = S(z)a(X)S(z)S(w)^{-1}a(X)^{-1}S(w)^{-1}.
\]

We remark that

\[
S(z)S(w)^{-1} = \begin{pmatrix} z & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \begin{pmatrix} 0_N & \mathbb{1}_N \\ -\mathbb{1}_N & w \end{pmatrix} = \begin{pmatrix} \mathbb{1}_N & (z-w)\mathbb{1}_N \\ 0 & \mathbb{1}_N \end{pmatrix} = \mathbb{1}_{2N} + (z-w)\begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix},
\]

so that

\[
\|S(z)a(X)S(z)S(w)^{-1}a(X)^{-1}S(w)^{-1}\| = \left\|S(z)a(X)[\mathbb{1}_{2N} + (z-w)\begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix}]a(X)^{-1}S(w)^{-1}\right\|
\]

\[
\leq 1 + |z-w|\left\|\begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix}\right\| + |z-w|\|S(z)\|\|a(X)\|\times
\]

\[
\times \left\|\begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix}\right\|\|a(X)^{-1}\|\|S(w)^{-1}\|.
\]

Now \(\left\|\begin{pmatrix} 0 & \mathbb{1}_N \\ 0 & 0 \end{pmatrix}\right\| = 1, \|a(X)^{-1}\| = \|a(X)\|\) and \(\log(1 + x) \leq x\) for all \(x \in \mathbb{R}\) so that we find

\[
\log(\|A_1(z)(A_1^w)^{-1}\|) \leq |z-w|(1 + \|S(z)\|\|S(w)^{-1}\|\|a(X)\|^2).
\]

Since \(z \mapsto \|S(z)\|\) is continuous, \(z\) ranges in a compact set, and using Assumption 2.3, we find

\[
|\Phi_z([v]) - \Phi_w([v])| \leq |z-w|(1 + \sup_z\|S_z\|^2\|a(T_1)\|^2),
\]

obtaining the claim for \(k = 1\); the other cases being an easy generalization.

\(\square\)
We state without proof the following corollary and proposition which may be found in [24, Prop. 2.5], [24, Prop. 2.6] respectively. Their proof in the present setting relies on Propositions 4.13 and 4.14.

**Corollary 4.15.** The map \( z \mapsto \gamma_j(z) \) is continuous for all \( j \) as \( z \) ranges in a compact subspace of \( \mathbb{C} \).

**Proposition 4.16.** For each \( j \in \{1, \ldots, N\} \) and \( [x] \in \bar{L}_j \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(\frac{\|\wedge^j B_n(\lambda)x\|}{\|x\|})] = \sum_{l=1}^j \gamma_l(\lambda),
\]

the limit being uniform as \( \lambda \) ranges in a compact subspace of \( \mathbb{R} \) and also uniform in \([x]\).

The following result finally connects \( \gamma_N(\lambda) > 0 \) with the exponential decay of a product of transfer matrices. Its proof is included in the appendix for the reader’s convenience, as it appeared merely as an outline in [24, Prop. 2.7].

**Proposition 4.17.** For any compact \( K \subseteq \mathbb{R} \), \( j \in \{1, \ldots, N\} \) such that \( \gamma_j(\lambda) > 0 \) for all \( \lambda \in K \), there exist some \( s \in (0, 1) \), \( N \in \mathbb{N} \) and \( C > 0 \) such that

\[
\mathbb{E}[\frac{\|\wedge^{j-1} B_n(\lambda)y\|}{\|y\|} \frac{\|x\|}{\|\wedge^j B_n(\lambda)x\|}] \leq \exp(-Cn)
\]

for all \( n \in \mathbb{N}_{\geq N} \), for all \( [x] \in \bar{L}_j \) and for all \( [y] \in \bar{L}_{j-1} \).

## 5 An a-priori bound

In this short section we establish the a-priori boundedness of one-step Green’s functions, which is a staple of the fractional-moments method. The fact that for our model one uses the one-step Green’s function rather than the diagonal one is due to the fact we do not have on-site randomness, which forces the usage of rank-2 perturbation theory.

**Proposition 5.1.** For any \( s \in (0, 1) \) we have some strictly positive constant \( C_s \) such that

\[
\mathbb{E}[\|G(x, x-1; z)\|^s] \leq C_s < \infty,
\]

for all \( x \in \mathbb{Z} \) and for all \( z \in \mathbb{C} \). So \( C_s \) depends on \( s, \alpha_1 \) and \( \alpha_0 \).

**Proof.** Using finite-rank perturbation theory, we can find the explicit dependence of the complex number \( G(x, x-1; z)_{i,j} \) (for some \((i, j) \in \{1, \ldots, N\}^2\)) on the random hopping \((T_x)_{i,j} \). Indeed, \((T_x)_{i,j} = (\delta_x \otimes e_i, H(T), \delta_{x-1} \otimes e_j) \). We find that \( G(x, x-1; z)_{i,j} \) is equal to the bottom-left matrix element of the \( 2 \times 2 \) matrix

\[
(A + \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix})^{-1}, \tag{5.1}
\]

where \( A \) is the inverse of the \( 2 \times 2 \) matrix

\[
\begin{pmatrix} \tilde{G}(x-1, x-1; z)_{i,j} & \tilde{G}(x-1, x; z)_{i,i} \\ \tilde{G}(x, x-1; z)_{i,j} & \tilde{G}(x, x; z)_{i,i} \end{pmatrix},
\]

with \( \tilde{H} \) being \( H \) with \((T_x)_{i,j} \) “turned off” (i.e. set to zero), and \( \lambda := (T_x)_{i,j} \) for convenience. Thus our goal is to bound the fractional moments with respect to \( \lambda \in \mathbb{C} \) of the off-diagonal entry of the matrix in (5.1) where \( A \) is some given \( 2 \times 2 \) matrix with complex entries. The only thing we know about \( A \) is that \( \text{Im}\{A\} > 0 \) though we won’t actually use this (cf. [19, Lemma 5]).
First note that \( \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} = \Re \{ \lambda \} \sigma_1 + \Im \{ \lambda \} \sigma_2 \) and also that for any \( 2 \times 2 \) matrix \( M \) we have \( \frac{1}{2} \text{tr}(M\sigma_1) = \frac{1}{2}(M_{12} + M_{21}) \) whereas \( \frac{1}{2} \text{tr}(M\sigma_2) = \frac{1}{2}(M_{12} - M_{21}) \). Thus by the triangle inequality,
\[
\mathbb{E}[|M_{12}|^s] = \mathbb{E}[\frac{1}{2} \text{tr}(M\sigma_1)^s] + \mathbb{E}[\frac{1}{2} \text{tr}(M\sigma_2)^s].
\]
So, if we get control on each summand separately we could bound \( \mathbb{E}[|M_{12}|^s] \).
We write
\[
\tilde{A} + \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} = \tilde{A} + \Re \{ \lambda \} \sigma_1
\]
and we expand \( \tilde{A} = a_0 + a_i \sigma_i \) for some \( \begin{pmatrix} a_0 \\ a \end{pmatrix} \in \mathbb{C}^4 \), so that if \( M := (a_0 + a_i \sigma_i + \Re \{ \lambda \} \sigma_1)^{-1} \)
we find
\[
\frac{1}{2} \text{tr}(M\sigma_1) = \frac{a_1 - \Re \{ \lambda \}}{a_0^2 - a_2^2 - a_3^2 - (a_1 + \Re \{ \lambda \})^2}.
\]
We now apply Proposition 5.2 just below with \( z = -a_1 - \Re \{ \lambda \} \), \( c = a_0^2 - a_2^2 - a_3^2 \) to get the result (using the Layer-Cake representation), which relies on Assumption 2.2. The term \( \mathbb{E}[\frac{1}{2} \text{tr}(M\sigma_2)^s] \) is dealt with in precisely the same way. \( \square \)

**Proposition 5.2.** Consider the subset of the complex plane
\[
D_{c,t} := \{ z \in \mathbb{C} | |z - \frac{c}{2}| > t \}
\]
for \( c \in \mathbb{C} \) and \( t > 0 \) and the line \( \mathbb{R} \ni \lambda \mapsto z := \alpha \lambda + \beta \) (\( \alpha, \beta \in \mathbb{C} \), \(|\alpha| = 1\)) parametrized by arclength. Then its intersection with \( D_{c,t} \) is bounded in Lebesgue measure as
\[
|\{ \lambda \in \mathbb{R} | z \in D_{c,t} \}| \leq \frac{4}{t}.
\]
**Proof.** Since the statement is invariant w.r.t. rotations of \( c, \alpha, \beta \) about the origin, we may assume \( c \geq 0 \). We then estimate the measure when \( D_{c,t} \) is replaced by its intersection with the right half-plane \( \{ z \in \mathbb{C} | \Re \{ z \} \geq 0 \} \) (and likewise for the left one). Then
\[
|\frac{z}{z^2 - c^2}| = \frac{1}{|z - c|^2} \frac{|z|}{|z + c|} \leq \frac{1}{|z - c|}
\]
because \( \Re \{ z \} + c \geq \Re \{ z \} \) there, which implies \( D_{c,t} \cap \{ z | \Re \{ z \} \geq 0 \} \subseteq \{ z | |z - c| < t^{-1} \} \). The intersection of that disk with any line is of length \( 2t^{-1} \) at most. \( \square \)

**Corollary 5.3.** For any \( s \in (0, 1) \) we have some strictly positive constant \( C_s \) (not the same as the one from above) such that
\[
\mathbb{E}[\|G(x, x; z)\|^s] < \frac{1}{|z|^s} C_s,
\]
for all \( x \in \mathbb{Z} \) and for all \( z \in \mathbb{C} \setminus \{0\} \). \( C_s \) depends only on \( s \in (0, 1) \), \( \alpha_1 \) and \( \alpha_0 \).

**Proof.** From the relation \( (H - z)R(z) = I \) we find
\[
-zG(x, x) + T_{x+1}G(x + 1, x; z) + T_xG(x - 1, x; z) = I_N,
\]
so that
\[
G(x, x; z) = z^{-1}[I_N - T_{x+1}G(x + 1, x; z) + T_xG(x - 1, x; z)].
\]
Hence by the triangle inequality, Hölder’s inequality, Assumption 2.3 and Proposition 5.1 we find the result. \( \square \)

**Remark 5.4.** The last two statements hold equally well if we replace \( G(x, x) \) or \( G(x - 1, x) \) with the half-line Green’s function or even a finite-volume Green’s function.
6 Localization at non-zero energies

In this section we establish localization for all non-zero energies. We do this in two steps: first at real energies (and hence finite volume) due to the fact that the Furstenberg analysis requires the transfer matrices to be $\text{Sp}_{2N}(\mathbb{C})$-valued which needs the energy to be real. We then extend this exponential decay off the real axis using the harmonic properties of the Green’s function to get polynomial decay at complex energies, which in turn implies exponential decay via a decoupling-type lemma. Once localization of finite volume at complex energies is established, the infinite volume result is implied via the strong-resolvent convergence of the finite volume Hamiltonian to the infinite volume one.

6.1 Localization at finite volume and real energies

**Theorem 6.1.** For any compact $K \subset \mathbb{R} \setminus \{0\}$, there is some $s \in (0, 1)$ and $\mu > 0$ such that

$$\sup_{\lambda \in K} \mathbb{E}[|G_{[x,y]}(x, y; \lambda)|^s] < |\lambda|^{-s} e^{-\mu|x-y|},$$

for all $(x, y) \in \mathbb{Z}^2$ with $|x - y|$ sufficiently large. Here $G_{[x,y]}$ is the Green’s function of the finite-volume restriction of $H$ to $[x, y] \subseteq \mathbb{Z}$.

**Proof.** Let $\lambda \in K$ be given and $s \in (0, 1)$. By Lemma 3.4 we know that

$$\|G_{[x,y]}(x, y; \lambda)\| \leq |\lambda|^{-s} C(\varepsilon) \left( \sum_{j=1}^{N} \left\| \frac{\Lambda_{N}^{-1} B_{y-1,j-1}(\lambda) u_j}{\Lambda_{N}^{-1} B_{y-1,j-1}(\lambda) u_j} \right\|^2 \right)^{\frac{s}{2}}.$$

Here $u = e_1 \wedge \cdots \wedge e_N$ and $u_j = e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \cdots \wedge e_N$. Now using Hölder’s inequality we get

$$\mathbb{E}[|G_{[x,y]}(x, y; \lambda)|^s] \leq |\lambda|^{-s} \mathbb{E}[C(\varepsilon)^{\frac{s}{2}} \sum_{j=1}^{N} \mathbb{E}[\left| \frac{\Lambda_{N}^{-1} B_{y-1,j-1}(\lambda) u_j}{\Lambda_{N}^{-1} B_{y-1,j-1}(\lambda) u_j} \right|^s]].$$

$\mathbb{E}[C(\varepsilon)^{s/2}]$ is bounded uniformly in $\lambda$ using Proposition 5.1, Corollary 5.3 and Assumption 2.3. Now $u_j \in \bar{L}_{N-1}$ for any $j$ and $u \in \bar{L}_N$; $K \neq 0$, $\gamma_N(\lambda) > 0$ for all $\lambda \in K$ so that we may apply Proposition 4.17 to get that for $s_K$, for appropriate $\infty > C > 0$, for $|x - y| > n_K$

$$\mathbb{E}[|G_{[x,y]}(x, y; \lambda)|^{s_K}] < |\lambda|^{-s_K} C N \exp(-\frac{1}{2} C_J K|x-y|),$$

which implies the bound in the claim. Note that we have used stationarity of $\mathbb{P}$ to go from $0, x - y$ to $y, x$. \qed

6.2 Infinite volume complex energy

Obtaining polynomial decay off the real axis from exponential decay at the real axis was already accomplished in [2, Theorem 4.2] using properties of the Poisson kernel. Here we provide another proof of this fact and go on to show that any decay implies exponential decay for our one dimensional models.

Let $Q(a) \subseteq \mathbb{C}$ be an open square of side $a > 0$; its lower side is placed on the real axis $\mathbb{R} \subseteq \mathbb{C}$ with endpoints denoted $x_{\pm}$, $(x_{+} - x_{-} = a)$. By $\bar{Q}(a) \subseteq Q(a)$ we mean the symmetrically placed subsquare $\bar{Q}(a) = Q(a/2)$, see Figure 1.
In this section $f = f(z)$ is a subharmonic function defined on $Q$. By its boundary values we simply mean
\[ f(z) = \limsup_{z' \to z} f(z') \quad (\leq +\infty, z \in \partial Q). \]
We will use the maximum principle in the form
\[ f(z) \leq \sup_{w \in \partial Q} f(w), \quad (z \in Q). \tag{6.1} \]
where, as usual, $0 < s < 1$.

We present two lemmas. The first one says that if $f$ is bounded everywhere on $\partial Q$, except for some controlled divergence when the real axis is approached, then $f$ is bounded on $\tilde{Q}$ with explicit bounds, i.e. not just by compactness. The second lemma says that if $f$ is small everywhere on $\partial Q$, except very near the real axis, where it is just bounded, then $f$ is small on $\tilde{Q}$. The two lemmas may be used in concatenation.

**Lemma 6.2.** Let $M \geq 0$ and suppose
\[ f(z) \leq M, \quad (z \in (\partial Q)_h), \]
\[ f(z) \leq M \left( \frac{a}{\text{Im}\{z\}} \right)^s, \quad (z \in (\partial Q)_v), \tag{6.2} \]
where $(\partial Q)_h/v$ are the horizontal / vertical parts of $\partial Q$. Then
\[ f(z) \leq M(1 + s^{1-s}), \quad (z \in \tilde{Q}). \]

**Lemma 6.3.** Let $m, M \geq 0$ and let $I \subseteq (\partial Q)_v$ be the union of the two vertical intervals of length $b \leq a$ next to $x_{\pm}$. Suppose
\[ f(z) \leq m, \quad (z \in \partial Q \setminus I), \]
\[ f(z) \leq M, \quad (z \in I). \tag{6.3} \]
Then
\[ f(z) \leq 2^{1-s} M \left( \frac{b}{a} \right)^s + m. \]

As a preliminary to the proofs, we consider the function
\[ v(z) := -\text{Im}\{z^{-s}\} \]
defined on the first quadrant $\{z \in \mathbb{C} | \Re\{z\} > 0, \Im\{z\} > 0\}$. It is harmonic and the chosen branch is made clear using polar coordinates $z = re^{i\theta}, \quad (r > 0, 0 < \theta < \pi/2)$ as
\[ v(z) = r^{-s} \sin(s\theta). \]
In particular, \(0 < v(z) < r^{-s} \sin(\frac{\pi}{2} z)\). We also note its boundary values

\[ v(x) = 0, \quad v(iy) = \sin(\frac{\pi}{2} s)y^{-s}, \quad (x, y > 0). \]

The analogous function on the second quadrant is \(v(-\bar{z})\).

We will also make use of the harmonic function

\[ h(z) := \frac{C}{\sin(\frac{\pi}{2} z)}(v(z - x_{-}) + v(x_{+} - \bar{z})), \quad (z \in Q), \]

for some \(C > 0\) and boundary values

\[ h(z) \geq C(\text{Im} z)^{-s}, \quad (z \in (\partial Q)_{v}). \]

Moreover we have

\[ |z - x_{\pm}| \geq \frac{a}{2}, \quad (z \in \tilde{Q}), \]

which yields the upper bound

\[ h(z) \leq 2^{1-s} Ca^{-s}, \quad (z \in \tilde{Q}). \quad (6.4) \]

**Proof of Lemma 6.2.** We have

\[ h(z) \geq M(\frac{a}{\text{Im} z})^{-s}, \quad (z \in (\partial Q)_{v}) \]

for \(C := Ma^{s}\). Since \(h(z), M \geq 0\) anyway we have by (6.2)

\[ f(z) \leq h(z) + M, \quad (z \in \partial Q). \]

Since the difference of the two sides is still subharmonic, the inequality applies to \(z \in Q\) by the maximum principle (6.1). In particular, for \(z \in \tilde{Q}\) we have \(f(z) \leq 2^{1-s}M + M\) by (6.4) as claimed.

**Proof of Lemma 6.3.** We have

\[ h(z) \geq M, \quad (z \in I) \]

for \(Cb^{-s} := M\). So

\[ f(z) \leq h(z) + m, \quad (z \in \partial Q) \]

by (6.3) and, as before,

\[ f(z) \leq 2^{1-s}M(\frac{b}{a})^{-s} + m, \quad (z \in \tilde{Q}). \]

We now turn to a slight generalization of the well-known Combes-Thomas estimate \([15]\). The generalization is that we do not assume that \(\alpha_{i}\) have compact support such that \(\sup_{x} \|T_{x}\| \leq K \exists K > 0\).

**Proposition 6.4.** We have some \(C > 0\) such that

\[ \mathbb{E}[\|G(x, y; E + i\eta)\|^{4}] \leq \frac{2}{\eta} e^{-C\eta|x-y|} \quad (6.5) \]

for all \(x, y \in \mathbb{Z}\), for all \(\eta > 0\) and \(E \in \mathbb{R}\), regardless of the fate of localization at \(E\).
Proof. The first step is to note that because of finite-rank perturbation theory, we have
\[ G(x, y; z) = (\mathbb{1} - G(x, x - 1; z)T_x^*G(x, y; z)(\mathbb{1} - T_y G(x, \infty)(y + 1, y; z)) \]
so that with the a-priori bound Proposition 5.1 on \( E[G(x, x - 1; z)]^\alpha \), we only need to consider
the finite-volume Green’s function \( G(x, y; z) \). Throughout we use \( z = E + i \eta \).

The second step is to divide the expectation value, for arbitrary \( M > 0 \), using the trivial bound \( \| G(x, y; z) \|^8 < \eta^{-s} \), and the original Combes-Thomas bound (see below) on bounded hopping terms:
\[
\mathbb{E}\| G(x, y; z) \|^8 = \int_{\omega \in \Omega} \sup_{x', y \in [x, y]} \mathbb{E}\| G(x, y; z) \|^8 d\mathbb{P}(\omega) + 
\eta^{-s} \left( 1 - \mathbb{P}\left( \left\{ \omega \in \Omega \mid \sup_{x' \in [x, y]} \| T_{x'}(\omega) \| < M \right\} \right) \right) \]
\leq \frac{2^8}{\eta^s} \exp(-s - \frac{\eta}{8M} |x - y|) + \eta^{-s} \left( 1 - \mathbb{P}\left( \left\{ \omega \in \Omega \mid \sup_{x' \in [x, y]} \| T_{x'}(\omega) \| < M \right\} \right) \right).
\]
Now we have using the i.i.d. property (we ignore the fact that with the a-priori bound \( \| G(x, y; z) \|^8 < \eta^{-s} \),
and the original Combes-Thomas bound (see below) on bounded hopping terms:
\[
\mathbb{P}\left( \left\{ \omega \in \Omega \mid \sup_{x' \in [x, y]} \| T_{x'}(\omega) \| < M \right\} \right) = \mathbb{P}(\| T_1(\omega) \| < M) \] \]
\[ = (1 - \mathbb{P}(\| T_1(\omega) \| \geq M)) |x - y| \]
Now the usual Markov inequality is
\[
\mathbb{P}(\| T_1(\omega) \| \geq M) \leq \frac{\mathbb{E}[\| T_1(\omega) \|^8]}{M^8}.
\]
However, it is actually true that for any \( \alpha \geq 0 \),
\[
\mathbb{P}(\| T_1(\omega) \| \geq M) \leq \frac{\mathbb{E}[\| T_1(\omega) \|^\alpha]}{M^\alpha}.
\]
Hence we get, since \( q \mapsto 1 - (1 - q)^n \) is increasing for all \( n \) and \( q \ll 1 \) (\( M \) big),
\[
1 - \mathbb{P}(\| T_1(\omega) \| \geq M) \leq \left( 1 - \frac{\mathbb{E}[\| T_1(\omega) \|^\alpha]}{M^\alpha} \right)^{|x - y|} \]
\[ \leq |x - y| \frac{\mathbb{E}[\| T_1(\omega) \|^\alpha]}{M^\alpha}.
\]
If we now pick \( \alpha := 2 \) and \( M := \mathbb{E}[\| T_1(\omega) \|^2]^{-\frac{1}{2}} |x - y|^{-\frac{1}{2}} \) we find that
\[
\mathbb{E}[\| G(x, y; z) \|^8]^{\frac{|x - y|^{-\frac{1}{2}}}{2}} \mathbb{E}[\| G(x, y; z) \|^8]^{\frac{|x - y|^{-\frac{1}{2}}}{2}} = 0.
\]
Now since we know that any decay of \( \mathbb{E}[\| G(x, y; z) \|^8] \) implies exponential decay for our model, Lemma 6.6, we find the desired result.

For completeness we give now the proof of the usual Combes-Thomas estimate with bounded hopping (bounded by some \( K < \infty \)):

Without loss of generality let \( E = 0 \) and pick some \( \mu > 0 \) (to be specified later). Also pick some \( f \) bounded and Lipschitz such that \( |f(x) - f(y)| \leq \mu |x - y| \). We use the notation \( R(x) \equiv (H - z \mathbb{1})^{-1} \) for the resolvent and also define \( H_f := e^{f(X)}He^{-f(X)} \) with \( X \) the position operator and \( B := H_f - H \).
Then we have $(H_f \psi)(x) = e^{f(x)-f(x+1)} T_{x+1}^* \psi(x+1) + e^{f(x)-f(x-1)} T_x \psi(x-1)$ so that the matrix elements of $B$ are given by

$$B_{x,y} = (e^{f(x)-f(x+1)} - 1) T_{x+1}^* \delta_{y,x+1} + (e^{f(x)-f(x-1)} - 1) T_x \delta_{y,x-1},$$

whence it follows by Holmgren that

$$\|B\| \leq 2(e^{\mu} - 1) K =: b(\mu),$$

If we now choose $\mu := \log(1 + \frac{n}{2})$ then evidently $b(\mu) = \frac{1}{2} n$ so that $\|(H_f - z\mathbb{1}) \psi\| \geq (\eta - \|B\|)\|\psi\|$ and we get $\|(H_f - z\mathbb{1})^{-1}\| \leq \frac{2}{\eta}$. But $\|G(x,y;E+i\eta)\| = |e^{-f(x)+f(y)}||\langle \delta_x, (H_f - z\mathbb{1})^{-1} \delta_y \rangle| \leq |e^{-f(x)+f(y)}| \frac{1}{\eta}$ and we obtain the result by the freedom in choice between $f$ and $-f$. \hfill \qed

We proceed to obtain the decay of the Green’s function uniformly in $\eta \equiv \text{Im} z$.

**Proposition 6.5.** The finite-volume Green’s function $\mathbb{E}[||G(x,y)(x,y;z)||^s]$ decays in $|x-y|$ at any value of $\text{Im} z$ for all $\mathbb{R}$ $z \neq 0$.

**Proof.** Let $\lambda \in \mathbb{R} \setminus \{0\}$ be given and define $z := \lambda + i\eta$. Define $f(z) := \mathbb{E}[||G(x,y)(x,y;z)||^s]$ for fixed $x,y$ and $0 < s < 1$ sufficiently small such that the hypothesis for Theorem 6.1 holds.

Since the Green’s function is holomorphic, it follows that $\|G(x,y)(x,y;z)\|^s$ is subharmonic and hence so is $f$.

Let $0 < a < 4K$ be such that the interval about $\lambda$ does not include zero: $0 \notin (\lambda - a, \lambda + a)$. Then due to Theorem 6.1 and the basic fact that $\|G(x,y;z)\| \leq \text{Im} z^{-1}$, we know that there is some $M \geq 0$ such that the assumptions of Lemma 6.2 are fulfilled ($M$ depends on $a$ and the constants provided by Theorem 6.1). Hence we may conclude that $f(z) \leq M(1+2^{1-s})$ as $z$ ranges in $\hat{Q}(a)$.

Now pick any $b < a/2$. With $I := (\hat{Q}(a))/\cap \{z|\text{Im} z \leq b\}$, Proposition 6.4 and Theorem 6.1 imply that on $f(z) \leq m$ for all $z \in \partial Q(a) \setminus I$, with $m = Cb^s e^{-bd}$ with $d := |x-y|$ large enough, for some constant $C > 0$ (where we have used that log$(1+\beta) \geq \alpha \beta$ for all $\beta \leq \alpha^{-1}$ and $\alpha < 1$). We also have still from Lemma 6.2 that $f(z) \leq M(1+2^{1-s})$ for all $z \in I \subseteq \hat{Q}(a)$. Hence Lemma 6.3 applies to give us that $f(z) \leq 2^{1+2s} M(1+2^{1-s})(\frac{b}{a/2})^s + m$ for all $z \in \hat{Q}(a/2)$.

Put succinctly, we find $f(z) \leq C(b^s + b^{-s} e^{-bd})$ for all $z \in \hat{Q}(a/2)$ for some constant $C > 0$ (different constant than before), $0 < s < 1$ and $d \gg 1$, and we are free to choose $b \leq a/2$. Our goal is to get decay in $d$. If we pick $b := d^{-s}$ for example we get the desired decay in $d$. \hfill \qed

Finally we are ready to get the exponential decay of the infinite volume Green’s function, which concludes the proof for the first part of Theorem 2.6.

**Lemma 6.6.** For fixed $z$, uniformly in $\text{Im} \{z\}$: assume that $\mathbb{E}[||G(1,n)(1,n;z)||^s] \rightarrow 0$ as $|n| \rightarrow \infty$ for some $s \in (0,1)$. Then $\mathbb{E}[||G(1,n;z)||^{s'}] \leq Ce^{-\mu |n|}$ for all $n \in \mathbb{Z}$ sufficiently large, for some $s' \in (0,1)$ sufficiently small, $\mu > 0$.

**Proof.** We have $H_{[x,y]} = \sum_{x'=x+1}^y T_{x'}$ with $(T_j \psi)(x) := \delta_{j,x} T_j \psi_{j-1} + \delta_{j-1,x} T_j^* \psi_j$ as the finite volume Dirichlet restriction of $H$ onto $[x,y] \cap \mathbb{Z}$, $x < y$. Then the resolvent equation yields

$$R = R_{(-\infty,y)} - \sum_{x'=y+1}^\infty R_{(-\infty,y)} T_{x'} R.$$

Taking the $(x,y)$ matrix element yields (suppressing the $z$ variable for the moment)

$$G(x,y) = G_{(-\infty,y)}(x,y) - \sum_{x'=y+1}^\infty G_{(-\infty,y)}(x,x') T_{x'} G(x'-1,y) + G_{(-\infty,y)}(x,x'-1) T_{x'}^* G(x',y)$$

$$= G_{(-\infty,y)}(x,y) (1 - T_{y+1}^* G(y+1,y)),\]
where the second line follows because the matrix elements of $R_{(-\infty, y]}$ outside of $(-\infty, y]$ are zero. Next we have again by the resolvent equation

$$ R_{(-\infty, y]} = R_{[x, y]} - \sum_{x' = -\infty}^{x} R_{(-\infty, y]} T_{x'} R_{[x, y]}, $$

so that taking the $(x, y)$ matrix element we get

$$ G_{(-\infty, y]}(x, y) = G_{[x, y]}(x, y) - \sum_{x' = -\infty}^{x} G_{(-\infty, y]}(x, x') T_{x'} G_{[x, y]}(x', y) + G_{(-\infty, y]}(x, x' - 1) T_{x'}^* G_{[x, y]}(x', y) = (\mathbf{1} - G_{(-\infty, y]}(x, x - 1) T_{x'}^*) G_{[x, y]}(x, y), $$

where the second line follows again because the matrix elements of $R_{[x, y]}$ outside of $[x, y]$ are zero. So we find

$$ \mathbb{E}[\|G(x, y)\|^s] \leq C \mathbb{E}[\|G_{[x, y]}(x, y)\|^s'], $$

where $s' = 4s$ for example. To get $C$ one has to invoke the Hölder inequality twice as well as the a-priori bound which is known for $\mathbb{E}[\|G(x, x + 1; z)\|^s]$ uniformly in $z$.

The upshot is that we may concentrate on exponential decay of $g(n) := \mathbb{E}[\|G_{[1, n]}(1, n)\|^s]$ in $n$ (by stationarity it does not matter to shift the object by $x - 1$ and call $y - x + 1 =: n$).

Our next procedure is to get a one step bound between $g(n + m)$ and $g(n)g(m)$ for any $n, m$:

We use again the resolvent identity to get

$$ G_{[1, n+m]}(1, n + m) = -G_{[1, n+m]}(1, n) T_{n+1}^* G_{[n+1, n+m]}(n + 1, n + m), $$

(note $G_{[n+1, n+m]}(1, n + m) = 0$) and

$$ G_{[1, n+m]}(1, n) = G_{[1, n]}(1, n) - G_{[1, n]}(1, n) T_{n+1}^* G_{[1, n+m]}(n + 1, n), $$

so that

$$ G_{[1, n+m]}(1, n + m) = -G_{[1, n]}(1, n) T_{n+1}^* G_{[n+1, n+m]}(n + 1, n + m) + G_{[1, n]}(1, n) T_{n+1}^* G_{[1, n+m]}(n + 1, n) T_{n+1}^* G_{[n+1, n+m]}(n + 1, n + m). $$

Taking the fractional moments expectation value, using the triangle inequality as well as the submultiplicativity of the norm, we find

$$ \mathbb{E}[\|G_{[1, n+m]}(1, n + m)\|^s] \leq \mathbb{E}[\|G_{[1, n]}(1, n)\|^s] + \mathbb{E}[\|G_{[1, n]}(1, n)\|^s] T_{n+1}^* \mathbb{E}[\|G_{[n+1, n+m]}(n + 1, n + m)\|^s] + \mathbb{E}[\|G_{[n+1, n+m]}(n + 1, n)\|^s] \times \mathbb{E}[\|G_{[1, n+m]}(n + 1, n + m)\|^s]. $$

Note that in the first line, the first and last factors in the expectation are actually independent of each other and both independent of $T_{n+1}$. Hence that expectation factorizes. In the second line, again the first and last factors do not depend on $T_{n+1}$ (yet the middle one does) so that we can perform the integration over $T_{n+1}$ first, which would involve integration only over $\|T_{n+1}^*\|^{2s} \|G_{[n+1, n+m]}(n + 1, n)\|^s$. We use Hölder once and the a-priori bound on $G(n + 1, n)$, which requires only integration over $T_{n+1}$. After the bound on that integral the remaining integral factorizes as the two remaining factors are independent of each other. We find

$$ \mathbb{E}[\|G_{[1, n+m]}(1, n + m)\|^s] \leq C \mathbb{E}[\|G_{[1, n]}(1, n)\|^s] \mathbb{E}[\|G_{[n+1, n+m]}(n + 1, n + m)\|^s] $$

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with \( C := \mathbb{E}[\|T_{n+1}^*\|^s] + \mathbb{E}[\|T_{n+1}^*\|^4]^{1/2} \mathbb{E}[\|G_{[1,n+m]}(n+1,n)^s\|^{2s}]^{1/2} \). If \( s < 1 \) is sufficiently small and we assume that there are moments for \( \|T_{n+1}^*\|^s \) for such \( s \), then we find \( 0 < C < \infty \).

The crucial point now is that due to stationarity,

\[
\mathbb{E}[\|G_{[1,m]}(1,m)^s\|] = \mathbb{E}[\|G_{[n+1,n+m]}(n+1,n+m)^s\|].
\]

The final result is that

\[
g(n + m) \leq C g(n) g(m) \quad \forall n \in \mathbb{N}.
\]

Now we use the assumption that \( g \) is decaying, which means we could find some \( n_0 \in \mathbb{N} \) sufficiently large so that \( \beta := C g(n_0) < 1 \). Then for any \( n \in \mathbb{N} \), write \( n = pm_0 + q \) (for some \( p \in \mathbb{N}, q \in \mathbb{N} \) with \( 0 \leq q < n_0 \)). We find by iteration, defining \( \mu := -\log(\beta) > 0 \):

\[
g(n) = g(pm_0 + q) \leq C g(pm_0) g(q)
\leq (C g(n_0))^p g(q)
(g \text{ is decaying and } q < n_0)
\leq C' \beta^p = C' \exp(-\mu \left\lfloor \frac{n - q}{n_0} \right\rfloor) \leq C'' \exp(-\mu' n).
\]

7 Localization at zero energy

The foregoing discussion only worked at non-zero energies. There were two reasons for that:

1. We could not guarantee that the zero energy Lyapunov spectrum has a gap. This goes back to Proposition 4.7.

2. We could not get an a-priori bound on the diagonal matrix element of the Green’s function \( G(x, x; z) \) which is uniform as \( z \to 0 \). This goes back to Corollary 5.3.

In order to deal with that special situation, we have to consider the Schrödinger equation at zero energy and then conclude about slightly non-zero values of the energy. We note that \( H \) is not invertible, so an expression for the resolvent like \( R(0) \equiv H^{-1} \) does not make sense. Hence we use the finite-volume regularization in this section.

7.1 Finite volume localization

Thus we are considering the operator \( H_{[1,L]} \) for some \( L \in \mathbb{N} \), which is just \( H \) with Dirichlet boundary conditions. It is the finite \( L \times L \) “band” matrix of \( N \times N \) blocks given as

\[
H_{[1,L]} = \begin{pmatrix}
0 & T_2^* & & \\
T_2 & 0 & T_3^* & \\
& T_3 & \ddots & \\
& & \ddots & T_L^*
\end{pmatrix}.
\]

Proposition 7.1. \( H_{[1,2n+1]} \) is not invertible and \( H_{[1,2n]} \) is invertible, for all \( n \in \mathbb{N} \).

Proof. Using the left boundary condition we have \( \psi_0 = 0 \). Then the Schrödinger equation at zero energy implies that the wave function at all even sites is zero, by iteration:

\[
T_2^* \psi_2 + T_1 \psi_0 = 0
\]
and so on. Thus the even sites are all zero by the left boundary condition.

If we consider $H^{[1,2n]}$, then the right boundary condition is $\psi_{2n+1} = 0$, and then again by using the Schrödinger equation the wave function at all odd sites must be zero. Hence $H^{[1,2n]} \psi = 0$ implies $\psi = 0$, that is, $H^{[1,2n]}$ is invertible.

If on the other hand we have $H^{[1,2n+1]}$, then the right boundary condition is $\psi_{2n+2} = 0$, which doesn’t give any new information: it is merely compatible with having the wave function at all even sites zero. Hence, the wave function at odd sites is unconstrained. Once $\psi_{1} \in \mathbb{C}^{N}$ is chosen, we use the equation to obtain the wave function’s value at odd sites along the entire chain:

$$T_{3}^{*} \psi_{3} + T_{2} \psi_{1} = 0$$

and so on. Hence $\ker(H^{[1,2n+1]}) \cong \mathbb{C}^{N}$ and $\ker(H^{[1,2n]}) = \{0\}$. \hfill \Box

Consequently, it would not make sense to consider the resolvent for odd chain-lengths, and we shall restrict our attention to $H^{[1,2n]}$.

It turns out that it is easy to calculate the matrix elements of $R^{[1,2n]}(0) \equiv (H^{[1,2n]})^{-1}$. We only need to describe the elements of $R^{[1,2n]}(0)$ on the diagonal and above it due to the self-adjointness of $H^{[1,2n]}$.

**Proposition 7.2.** The only non-zero matrix elements $R^{[1,2n]}(0)$ on or above the diagonal are given by

$$G^{[1,2n]}(2k,2l+1; 0) = (-T_{2k}^{0} T_{2k-1}^{0}) \cdots (-T_{2l+4}^{0} T_{2l+3}^{0}) T_{2l+2}^{0}$$

for all $(k,l) \in \mathbb{N}^{2}$ such that $(2k,2l+1) \in [1,2n]^{2}$ and such that $k > l$.

**Proof.** Let $l \in \mathbb{N}$ be given such that $2l+1 \in [1,2n]$. We start from the left boundary condition, which is that $G^{[1,2n]}(0,2l+1; z) = 0$. We then evaluate the Schrödinger equation at zero energy in the left most position to find

$$T_{2}^{*} G^{[1,2n]}(2,2l+1; 0) = \delta_{2,2l+1}.$$ 

If $l = 0$ then we find $G^{[1,2n]}(2,1; 0) = T_{2}^{0}$. Otherwise $G^{[1,2n]}(2,2l+1; 0) = 0$. We continue in this fashion to find that $G^{[1,2n]}(2k,2l+1; 0) = 0$ as long as $k \leq l$ and the equation above once $k = l + 1$, and then iterate for $k > l + 1$.

We cannot proceed in the same way for $G^{[1,2n]}(1,2l+1; 0)$ because the boundary condition on the left doesn’t say anything about it. Instead we must use the boundary condition on the right, which says $G^{[1,2n]}(2n+1,2l+1; z) = 0$. In the same way we use the Schrödinger equation to conclude about $G^{[1,2n]}(2n-1,2l+1; 0) = 0$:

$$T_{2n+1}^{*} G^{[1,2n]}(2n+1,2l+1; 0) + T_{2n} G^{[1,2n]}(2n-1,2l+1; 0) = \delta_{2n,2l+1}.$$ 

Since the right hand side will always be zero (due to the difference in parity), we find that

$$G^{[1,2n]}(2k+1,2l+1; 0) = 0$$

for all $k$ such that $2k+1 \in [1,2n]$.

In a similar way we also find that

$$G^{[1,2n]}(2k,2l; 0) = 0$$

for all $2k$ and $2l$ within the chain, using the boundary condition on the left and then evolving to the right. \hfill \Box
Now that we know that all diagonal matrix elements of $R_{[1,2n]}(0)$ are zero, we proceed to get an expression at non-zero energy, but still finite volume:

**Proposition 7.3.** If the Lyapunov exponents are all non-zero for $z = 0$ (so we assume more than what Corollary 4.9 automatically gives), then we have

$$E[\|G_{[1,2n]}(x,x; z)\|^s] < C$$

for some constant uniformly in $z$ (as $z \to 0$), uniformly in $n$, and independent of $x$.

**Proof.** We use the resolvent identity to get

$$G_{[1,2n]}(x,x; z) = G_{[1,2n]}(x,x; z) - G_{[1,2n]}(x,x; 0) = \langle \delta_x, [R_{[1,2n]}(z) - R_{[1,2n]}(0)]\delta_x \rangle$$

(Resolvent identity)

$$= \langle \delta_x, zR_{[1,2n]}(z)R_{[1,2n]}(0)\delta_x \rangle$$

$$= \sum_{y=1}^{2n} zG_{[1,2n]}(x,y; z)G_{[1,2n]}(y,x; 0).$$

So

$$E[\|G_{[1,2n]}(x,x; z)\|^s] \leq \sum_{y=1}^{2n} |z|^s (E[\|G_{[1,2n]}(x,y; z)\|^{2s})^{\frac{1}{2}} (E[\|G_{[1,2n]}(y,x; 0)\|^{2s})^{\frac{1}{2}}.$$

We now use Lemma 6.6 (namely that the finite volume complex energy Green’s function is exponentially decaying) to conclude:

$$E[\|G_{[1,2n]}(x,x; z)\|^s] \leq \sum_{y=1}^{2n} |z|^s (|z|^{-s} e^{-\mu_s|x-y|}) (e^{-\mu_s'|x-y|})$$

$$< C.$$
with \((T_j \psi)(x) := \delta_{j,x} T_j \psi_{j-1} + \delta_{j-1,x} T_j^* \psi_j\) so that the resolvent identity gives

\[
R_{[-n+1,n]} = R + R(H - H_{[-n+1,n]})R_{[-n+1,n]} \\
= R + \sum_{j \in \mathbb{Z} \setminus \{-n+2, \ldots, n\}} RT_j R_{[-n+1,n]}.
\]

Now

\[
\| \sum_{j \in \mathbb{Z} \setminus \{-n+2, \ldots, n\}} T_j \psi \|^2 = \sum_{l \in \mathbb{Z}} \| \sum_{j \in \mathbb{Z} \setminus \{-n+2, \ldots, n\}} \langle \delta_l, T_j \psi \rangle \|^2
\]

\[
= \sum_{j \in \mathbb{Z}} \| \sum_{l \in \mathbb{Z} \setminus \{-n+2, \ldots, n\}} (\delta_j T_l \psi_{j-1} + \delta_{j,l-1} T_l^* \psi_l) \|^2
\]

\[
\leq \sum_{j \in \mathbb{Z} \setminus \{-n+2, \ldots, n\}} \| T_j \psi_{j-1} \|^2 + \sum_{j \in \mathbb{Z} \setminus \{-n+1, \ldots, n-1\}} \| T_{j+1}^* \psi_{j+1} \|^2.
\]

Next we observe that

\[
(R_{[-n+1,n]}(z) \psi)_j = -z^{-1} \psi_j \quad \forall j \in \mathbb{Z} \setminus \{-n+2, \ldots, n\}
\]

so that

\[
\| R(z) \sum_{j=-n+2}^{n} T_j R_{[-n+1,n]}(z) \psi \|^2 \leq \| R(z) \|^2 |z|^{-2} \sum_{j \in \mathbb{Z} \setminus \{-n+2, \ldots, n\}} \| T_j \psi_{j-1} \|^2 + \| R(z) \|^2 |z|^{-2} \sum_{j \in \mathbb{Z} \setminus \{-n+1, \ldots, n-1\}} \| T_{j+1}^* \psi_{j+1} \|^2.
\]

But \(\psi \in l^2\) and \(\| R(z) \| \leq |\text{Im}\{z\}|^{-1}\), so that the right hand side converges to zero as \(n \to \infty\), all at fixed \(z \neq 0\).

**Corollary 7.5.** We have

\[
\lim_{n \to \infty} \| G_{[-n+1,n]}(x, x; z) \| = \| G(x, x; z) \|
\]

for all \(z \in \mathbb{C} \setminus \mathbb{R}\), so that using Fatou’s lemma,

\[
E[\| G(x, x; z) \|^s] \leq \lim_{n \to \infty} E[\| G_{[-n+1,n]}(x, x; z) \|^s].
\]

But since the bound Proposition 7.3 is uniform in \(n\), we find that \(E[\| G(x, x; z) \|^s]\) is bounded uniformly in \(|\text{Im}\{z\}|\) under the same assumptions on the Lyapunov spectrum as in Proposition 7.3.

As a result, we may now go back to the previous section and apply all the proofs there, extending them so that it holds uniformly including in the limit \(z \to 0\) as long as the Lyapunov spectrum doesn’t include zero, at all real energies, including zero energy. This concludes the proof of the second part of Theorem 2.6.

## 8 Chiral random band matrices and the \(\sqrt{L}\) conjecture

In this section we remark briefly about random band matrices from the point of view of our chiral model. To avoid confusion, now the size of our random matrices is \(W\) instead of \(N\) as in the rest of the paper. Random band matrices are random Hermitian matrices \(A\) of size\(^1\) \(L \times L\), for some \(L \in \mathbb{N}\), such that there is a parameter \(W \in \mathbb{N} < L/2\) with \(A_{ij} = 0\) for all \(|i - j| > W\).

\(^1\)To be compatible with the rest of the paper we denote a size of a matrix here by \(L\) instead of the usual \(N\) in the literature, the usual name of the conjecture is “the \(\sqrt{N}\) conjecture.”
I.e., they are of finite band width $W$. The random distribution that is usually considered is that where each entry within the non-zero band is independent and identically distributed (except the Hermitian constraint), say, as a Gaussian centered at zero (though this detail is largely unimportant for the relevant questions to follow). These models are interesting due to the fact that they exhibit a phase transition, from localization for $W \ll \sqrt{L}$ to delocalization $W \gg \sqrt{L}$. I.e., a transition is conjectured for $W \sim \sqrt{L}$ [13, 12]. Indeed, the case where $W$ is a constant (w.r.t. $L \to \infty$) is simply the Anderson model on a strip (of width $W$), which is completely localized, and the case $W = L/2$ is a random matrix from the Gaussian unitary ensemble, known to be delocalized. Both phases have been established mathematically (see [30, 28] for the localized regime for $W \sim L^{1/7}$ and e.g. [7, 8] and references therein for the delocalized regime for $W \sim L^{3/4}$) but the precise transition at $W \sim \sqrt{L}$ has yet to have been established (but see e.g. [33, 32] for some recent promising progress).

Using finite rank perturbation theory and a-priori bounds, the question of localization for random band matrices is equivalent to the question of the localization length for the Anderson model on the strip, or the following generalization of it:

$$(H\psi)_n = T_{n+1}^n\psi_{n+1} + T_n\psi_{n-1} + V_n\psi_n \quad (\psi \in l^2(\mathbb{Z}) \otimes \mathbb{C}^W; n \in \mathbb{Z})$$

(8.1)

where $\{V_n\}$ is a random i.i.d. sequence chosen from GUE($W$) and $\{T_n\}$ is a random i.i.d. sequence chosen from Ginibre($W$). Strictly speaking this model is called the Wegner-$W$ orbital model [28], and to get the random band model one should take $T_n$ to be distributed as random upper triangular. The distinction, however, should not matter for our purposes. Then the $\sqrt{L}$ conjecture from the localization side is equivalent to the model (8.1) exhibiting localization length of order $W$ (measured in distance between slices of width $W$); one way to formulate that is to establish the FM condition for (8.1) with decay rate $W$:

$$\sup_{\eta \neq 0} \mathbb{E}[\|G(x, y; E + i\eta)\|^s] \leq C \exp(-\mu|x-y|/W) \quad (x, y \in \mathbb{Z})$$

(8.2)

for some $s \in (0, 1)$ and $0 < C, \mu < \infty$ where $\mu \sim O(1)$ in $W \to \infty$. This should hold for any $E$ in the almost-sure spectrum, i.e., there is no expectation for a phase transition in $E$. Thus, the preceding sections which connect localization length with the smallest Lyapunov exponent lead us to re-formulate the $\sqrt{L}$ conjecture (at energy $\lambda$) from the localization side as:

$$\inf_{j \in \{1, \ldots, 2W\}} |\gamma_j(\lambda)| \gtrsim \frac{1}{W}.$$  

(8.3)

It turns out–and this is the point of this section–that for (2.1) (i.e. $V_n = 0$ in (8.1)) at zero energy, the Lyapunov spectrum can be calculated exactly for special choices of $\alpha_0, \alpha_1$.

**Theorem 8.1.** Consider the model (2.1) with distributions $\alpha_0, \alpha_1$ both Ginibre with variances $\sigma_0, \sigma_1$ respectively. If $\sigma_0 = \sigma_1$ then all zero energy Lyapunov exponents are zero. If, on the other hand, one chooses e.g. $\sigma_0 = \exp(-1/W)$ and $\sigma_1 = \exp(-2/W)$, the $\sqrt{W}$ conjecture holds at zero energy, i.e.,

$$\inf_{j \in \{1, \ldots, 2W\}} |\gamma_j(0)| \sim \frac{1}{W}.$$  

(8.4)

**Proof.** At zero energy, the Lyapunov spectrum breaks apart as in (4.9), so it is enough to study the smallest Lyapunov exponent of the sequence of $W \times W$ matrices $\{A_nB_n^\alpha\}$ where $\{A_n\}$ are distributed according to $\alpha_0$ and $\{B_n\}$ according to $\alpha_1$. We recall that the Ginibre distribution is given by

$$d\text{Ginibre}_\sigma(A) = (\pi\sigma^2)^{-W^2} \exp\left(-\frac{1}{\sigma^2} \text{tr}(|A|^2)\right) dA$$

26
where $dA$ denotes the Lebesgue measure on $\text{Mat}_W(\mathbb{C}) \cong \mathbb{C}^{W^2}$. Hence, the distribution $P_{\text{CBZE}}$ of each element of the sequence $\{A_n B_n^o\}_n$ is given by

$$E_{\text{CBZE}}[f] = \int_{A,B \in \text{Mat}_W(\mathbb{C})} (\pi^2 \sigma_0^2 \sigma_1^2)^{-W^2} \exp \left( -\frac{1}{\sigma_0^2} \text{tr}(|A|^2) - \frac{1}{\sigma_1^2} \text{tr}(|B|^2) \right) f(AB^o) dA dB$$

for any measurable $f : \text{Mat}_W(\mathbb{C}) \to \mathbb{C}$. Here the acronym CBZE stands for the distribution of “Chiral Band matrices at Zero Energy”.

In particular, we see that: (1) $P_{\text{CBZE}}[\text{GL}_W(\mathbb{C})^c] = 0$, and (2) for any fixed unitary $W \times W$ matrix $U$, $|AB^o|^2$ has the same distribution as $|AB^o U|^2$. Indeed, this is clear from the cyclicity of the trace, $U$ being unitary so its Jacobian is the identity, and

$$U^*(AB^o)^* AB^o U = (U^* BU)^{-1}(U^* A^* AU)(U^* BU) \longrightarrow (AB^o)^* AB^o.$$ 

These two conditions thus allow us to apply the same arguments as in [27] to conclude that the Lyapunov exponents $\{\xi_j\}_{j=1}^W$ of the sequence $\{A_n B_n^o\}_n$ are given by

$$\xi_1 + \cdots + \xi_k = E_{\text{CBZE}}[\log (\|AB^o e_1 \wedge \cdots \wedge e_k\|)].$$

But following the arguments in [27], even more is true. Since the choice of the basis $\{e_j\}_j$ does not matter for the calculation of the exponents, we could just as well use $\{B^* e_j\}_j$ instead so we also have

$$\xi_1 + \cdots + \xi_k = E_{\text{CBZE}}[\log (\|AB^o B^* e_1 \wedge \cdots \wedge e_k\|)]$$

so we see that the task reduces to calculation of the exponents $\{\xi_j^A\}_{j=1}^W$ of $A$ and $\{\xi_j^B\}_{j=1}^W$ of $B$ separately, and that $\xi_j = \xi_j^A - \xi_j^B$ (the distribution of $B$ is invariant under adjoints). Since both $A$ and $B$ are Ginibre, the very same calculation in [27] applies and we obtain

$$\xi_j = \log(\sigma_0/\sigma_1) \quad (j = 1, \ldots, W)$$

so all exponents are degenerate. The two statements in the theorem readily follow.

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## 9 Appendix

### 9.1 Deterministic bounds on the Fermi projection

Our goal in this subsection is to prove Corollary 2.7. Using [1, Theorem 2], one obtains that if $P := \chi_{(-\infty,E)}(H)$ where $E \in \mathbb{R}$ is an energy for which (2.4) holds (either automatically for all $E \neq 0$ by the first part of Theorem 2.6 or with further assumptions by its second part) then

$$E[\|P(x,y)\|] \leq C e^{-\mu|x-y|}$$

(9.1)

for some (deterministic) constants $C > 0$ and $\mu > 0$. The following claim proves Corollary 2.7 then.
Claim 9.1. Let $A : \Omega \to B(\mathcal{H})$ be an ergodic random operator (with the usual assumptions on the disorder probability space $(\Omega, A, \mathbb{P}, T)$ with $T$ being the ergodic action of $\mathbb{Z}$). If 
$$
\mathbb{E}[\|A(x, y)\|] \leq C e^{-\mu|x-y|}
$$
for some (deterministic) constants $C, \mu > 0$, then almost surely for some (random) constants $C', \nu > 0$ and deterministic constants $\mu', \nu > 0$,
$$
\sum_{x, y \in \mathbb{Z}} \|A(x, y)\|(1 + |x|)^{-\nu} e^{\mu'|x-y|} \leq C' < \infty.
$$

The proof of this claim has appeared already in many places, e.g., [31, Prop. A.1].

9.2 The Hermitian symplectic group

Since the Hermitian symplectic group is not a very canonical object in the mathematics literature (though it does appear, e.g., in [29]), we spell out its basic properties below for convenience.

Definition 9.2. The (Hermitian) symplectic group $\text{Sp}_{2N}^*(\mathbb{C})$ is defined as
$$
\text{Sp}_{2N}^*(\mathbb{C}) \equiv \{ M \in \text{Mat}_{2N \times 2N}(\mathbb{C}) | M^*JM = J \}
$$
with $J \equiv \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix}$ the standard symplectic form. Here we abbreviate $G := \text{Sp}_{2N}^*(\mathbb{C})$. In this section we write $M$ in $N$-block form as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C,$ and $D$ are unrelated to their meanings outside of this section.

Proposition 9.3. $G$ is a group under matrix multiplication.

Proof. Taking the determinant of 
$$
M^*JM = J
$$
shows that $|\det(M)| = 1$ so that $M$ is invertible if it is in $G$. From the defining relation we have 
$$
M = (M^*J)^{-1}J = M^{-1}A^{-1}J
$$
so that using the commutativity of adjoint and inverse, we get:
$$
M^{-1} = J^{-1}M^*J = J^{-1}((M^{-1})^*)^*J = J^{-1}((M^{-1})^*)^{-1}J = ((M^{-1})^*J)^{-1}J,
$$
so that
$$
(M^{-1})^*JM^{-1} = J \tag{9.2}
$$
and we find that $M^{-1} \in G$ as well. Hence the inverse map restricted to $G$ lands in $G$. If 
$$
M_i^*JM_i = J
$$
for $i \in \{1, 2\}$ then
$$
M_2^*M_1^*JM_1M_2 = M_2^*JM_2 = J.
$$
Hence matrix multiplication map restricted to $G^2$ lands in $G$.

Associativity is inherited by the associativity of general matrix multiplication. Finally, the identity matrix is of course conjugate symplectic. \qed
Remark 9.4. Note that $G$ is unitarily equivalent to $U(N, N)$, the indefinite unitary group.

Proposition 9.5. $G$ is closed under adjoint.

Proof. Let $M \in G$ be given. Then $M^*JM = J$. We want to show that $MJM^* = J$. Taking the inverse of (9.2) we find

$$((M^{-1})^*JM^{-1})^{-1} = J^{-1}.$$  

Use the fact that $J^{-1} = -J$ and again the commutativity of adjoint and inverse maps to find:

$$MJM^* = J,$$

so that $M^* \in G$ as desired.

Proposition 9.6. $G$ may be described as

$$G = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Mat}_{2N}(\mathbb{C}) \left| (A, B, C, D) \in \text{Mat}_N(\mathbb{C})^4 : \begin{array}{l} A^*C \text{ and } B^*D \text{ are S.A.} \\ \text{and } A^*D - C^*B = \mathbbm{1}_N \end{array} \right. \right\}. \quad (9.3)$$

Proof. Starting from $M^*JM \doteq J$ we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} 0 & -\mathbbm{1}_N \\ \mathbbm{1}_N & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} 0 & -\mathbbm{1}_N \\ \mathbbm{1}_N & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$= \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} -C & -D \\ A & B \end{pmatrix}$$

$$= \begin{pmatrix} -A^*C + C^*A & -A^*D + C^*B \\ -B^*C + D^*A & -B^*D + D^*B \end{pmatrix}$$

$$\doteq \begin{pmatrix} 0 & -\mathbbm{1}_N \\ \mathbbm{1}_N & 0 \end{pmatrix}.$$

Remark 9.7. By the last equation in (9.3), if $D \in \text{GL}_N(\mathbb{C})$, then $M$ is specified by only three block entries $B$, $C$ and $D$ and $A$ may be solved for to satisfy the symplectic constraint.

Proof of Lemma 4.5. (4.5) is equivalent to the last condition of (9.3) when $D$ is invertible. So by Proposition 9.6 we need only verify that $A^*C$ and $B^*D$ are self-adjoint. But $B^*D = D^*RD$ which is self-adjoint by hypothesis on $R$ and $A^*C = (I + C^*B)D^{-1}DS = S + D^*RDS$, the last expression being a sum of two terms which are self-adjoint using the assumptions on $R$ and $S$.

Proposition 9.8. If $M \in G$, then its eigenvalues are symmetric about $S^1$ so that its singular values are symmetric about one.

Proof. We have $\det(J) = 1$ so that $|\det(M)| = 1$. So define $\theta \in S^1$ via $e^{i\theta} := \det(M^*)^{-1}$. We have $M = -JM^*J$ from the definition so that if $p_M(\lambda) \equiv \det(M - \lambda I_{2N})$ is the characteristic polynomial of $M$, we find

$$p_M(\lambda) = \det(-JM^*J - \lambda I_{2N})$$

$$= \det(M^*J - \lambda J)$$

$$= e^{-i\theta} \det(J - \lambda M^*J) \quad (\det(J) = 1)$$

$$= e^{-i\theta} \det(-J + \lambda M^*) \quad (e^{-i\theta} \det(M^*) = 1)$$

$$= e^{-i\theta} \overline{\lambda}^N \det(-\overline{\lambda}^{-1} + M) = e^{-i\theta} \lambda^2 \text{p}_M(\bar{\lambda}^{-1}).$$
Since $|M|^2$ is also symplectic, this relation holds true also for $p|M|^2$, hence for the singular values.

**Corollary 9.9.** For any $M \in G$, $\|M\| = \|M^{-1}\|$.

**Proof.** $\|M\|$ is the largest singular value, $\|M^{-1}\|^{-1}$ is the smallest singular value, but the singular values are symmetric about one.

### 9.3 Some technical proofs necessary for Proposition 4.17

**Proof of Proposition 4.17.** This is essentially [24, Prop. 2.7]. We spell out the proof here explicitly for the reader’s convenience because in the reference it is merely outlined.

We define the additive co-cycle $\zeta$ on $\bar{L}_{j-1} \times \bar{L}_j$, which is an $Sp^*_N(\mathbb{C})$-space, via the formula

$$
\zeta(g, [y], [x]) := \log(\frac{\|g y\|}{\|g\|} \frac{\|\zeta\|}{\|\zeta\|}) \quad \forall (g, [y], [x]) \in Sp^*_N(\mathbb{C}) \times \bar{L}_{j-1} \times \bar{L}_j.
$$

Indeed, we have

$$
\zeta(gh, [y], [x]) = \zeta(g, h[y], h[x]) + \zeta(h, [y], [x]).
$$

Now, we have

$$
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\zeta(B_n(z), [y], [x])] = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(\frac{\|\zeta\|}{\|\zeta\|})] = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(\frac{\|\zeta\|}{\|\zeta\|})] - \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log(\frac{\|\zeta\|}{\|\zeta\|})],
$$

and then by Proposition 4.16, $\frac{1}{n} \mathbb{E}[\zeta(B_n(z), [y], [x])] \xrightarrow{n \to \infty} -\gamma_j(z)$ uniformly in $z \in K$, $([y], [x]) \in \bar{L}_j \times \bar{L}_{j-1}$. As a result, for any $\varepsilon > 0$ there is some $n_K(\varepsilon)$ such that $\forall n \in \mathbb{N}_{\geq n_K(\varepsilon)}$ we have

$$
\frac{1}{n} \mathbb{E}[\zeta(B_n(z), [y], [x])] + \gamma_j(z) < \varepsilon
$$

and then

$$
\mathbb{E}[\zeta(B_n(z), [y], [x])] < n(-\gamma_j(z) + \varepsilon)
$$

so if we pick $n \geq n_K(1/\Gamma_j(K))$, we find $\mathbb{E}[\zeta(B_n(z), [y], [x])] < -\frac{1}{2}n\Gamma_j(K)$, with $\Gamma_j(K) := \inf_{z \in K} \gamma_j(z) > 0$ by hypothesis.

Before proceeding with the proof of Proposition 4.17, we give the following intermediate

**Proposition 9.10.** There is some $n_0(K)$ and some $s_K \in (0, 1)$ such that

$$
\mathbb{E}[\exp(s_K\zeta(B_n(z), [y'], [x']))] < 1 - \varepsilon_K \tag{9.4}
$$

for all $y'$, $x'$ and $z \in K$, for some $\varepsilon_K \in (0, 1)$.

**Proof.** This is the strip-analog of [10, Lemma 5.1]. First note that for all $\alpha \in \mathbb{R}$ we have $e^\alpha \leq 1 + \alpha + \alpha^2 e^{[\alpha]}$ so that

$$
\exp(s\zeta(B_n(z), [y'], [x'])) \leq 1 + s\zeta(B_n(z), [y'], [x']) + s^2\zeta(B_n(z), [y'], [x'])^2 e^{s\zeta(B_n(z), [y'], [x'])}.
$$

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Next,
\[
|\zeta(B_n(z), [y'], [x'])| \equiv | \log( \frac{\|\wedge^j B_n(z) y\|}{\|y\|} \frac{\|x'\|}{\|\wedge^j B_n(z) x\|}) | \leq | \log( \frac{\|\wedge^j B_n(z) y\|}{\|y\|}) | + \log( \frac{\|\wedge^j B_n(z) x\|}{\|x'\|}) ,
\]
and via Proposition 9.12 we have
\[
|\zeta(B_n(z), [y'], [x'])| \leq (j - 1)(2N - 1) \log(||B_n(z)||) + j(2N - 1) \log(||B_n(z)||) \\
\leq 4jN \log(||B_n(z)||) \\
\leq 4jN \sum_{k=1}^{n} \log(||A_k(z)||) .
\]

Hence by Hölder’s inequality and Proposition 9.11 right below we have:
\[
\mathbb{E}[\exp(s\zeta(B_n(z), [y'], [x']))] \leq 1 + s\mathbb{E}[\zeta(B_n(z), [y'], [x'])) + s^2(\mathbb{E}[\zeta(B_n(z), [y'], [x']))^4)]^{\frac{1}{2}} \times \\
\times (\mathbb{E}[e^{2s\zeta(B_n(z), [y'], [x'))}])^{\frac{1}{2}} \times (\mathbb{E}[\log(||A_1(z)||)^4])^{\frac{1}{2}} \times \\
\times (\mathbb{E}[\log(||A_1(z)||)])^{\frac{1}{2}}^2 .
\]

We note that $||A_1(z)||^{8sjN}$ is integrable by the proof of Proposition 4.1 and Assumption 2.3 if we pick $s \in (0, 1)$ sufficiently small.

Now with some choice of constants $C_1(K) := 16j^2N^2 \sup_{z \in K} \sqrt{\mathbb{E}[\log(||A_1(z)||^4)]} < \infty$ and $C_2(K) := \sup_{z \in K} \sqrt{\mathbb{E}[||A_1(z)||^{8sjN}]}$ we find
\[
\mathbb{E}[\exp(s\zeta(B_n(z), [y'], [x']))] \leq 1 + s\mathbb{E}[\zeta(B_n(z), [y'], [x'])) + s^2n^2C_1(K)C_2(K)^n .
\]

From the above we know that
\[
\mathbb{E}[\exp(s\zeta(B_n(z), [y'], [x']))] \leq 1 - s\frac{1}{2}nK(\frac{1}{2}\Gamma_j(K))\Gamma_j(K) + \\
+ s^2nK(\frac{1}{2}\Gamma_j(K))^2C_1(K)C_2(K)^nK(\frac{1}{2}\Gamma_j(K)) .
\]

Hence there is some $s \in (0, 1)$ (which depends on $K$) so that
\[
\varepsilon_K := \frac{1}{2}nK(\frac{1}{2}\Gamma_j(K))\Gamma_j(K) - snK(\frac{1}{2}\Gamma_j(K))^2C_1(K)C_2(K)^nK(\frac{1}{2}\Gamma_j(K)) .
\]
is positive.  

Continuing now with the proof of Proposition 4.17, we note that the object whose expectation we’re actually trying to bound is

\[
\exp(s\zeta(B_n(z), [y], [x])) = \exp(s \log(\frac{\|\wedge^j gy\|}{\|y\|} \frac{\|x\|}{\|\wedge^j gx\|})),
\]

\[
= \left(\frac{\|\wedge^j gy\|}{\|y\|} \frac{\|x\|}{\|\wedge^j gx\|}\right)^s,
\]

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and that
\[ \exp(s\zeta(B_{n+m}(z), [y], [x])) = \exp(s\zeta(A_{n+m}(z) \ldots A_{1+m}(z)B_m(z), [y], [x])) \]
(cocycle property)
\[ = \exp(s\zeta(A_{n+m}(z) \ldots A_{1+m}(z), B_m([y], [x]) + \zeta(B_m(z), [y], [x]))) \]
\[ = \exp(s\zeta(A_{n+m}(z) \ldots A_{1+m}(z), B_m([y], B_m([x]))) \times \exp(s\zeta(B_m(z), [y], [x])). \]

Hence due to the fact that \( \{A_n(z)\}_{n \in \mathbb{Z}} \) are independent, we can integrate first only over \( \{A_{n+m}(z), \ldots, A_{1+m}(z)\} \) so that in that integration \( B_m(z)y \) and \( B_m(z)x \) are fixed. Then via (9.4) we find
\[ \mathbb{E}[\exp(sK\zeta(B_{n_0(K)+m}(z), [y], [x]))] \leq (1 - \varepsilon_K)\mathbb{E}[\exp(sK\zeta(B_m(z), [y], [x]))]. \]

So if \( n \in \mathbb{N}_{\geq n_0(K)} \) is given, we write it as \( n = q_n n_0(K) + r_n \) with \( r_n \in \{0, \ldots, n_0(K) - 1\} \). We then have using Hölder’s inequality
\[ \mathbb{E}[\exp(\frac{1}{2}sK\zeta(B_n(z), [y], [x]))] \]
\[ \leq \mathbb{E}[\exp(\frac{1}{2}sK\zeta(A_{q_n n_0(K)+r_n}(z) \ldots A_{q_n n_0(K)+1}(z), B_{q_n n_0(K)}([y], B_{q_n n_0(K)}([x])) \times \exp(\frac{1}{2}sK\zeta(B_{q_n n_0(K)}([y], [x]))) \]
\[ \leq \sqrt{\mathbb{E}[\exp(sK\zeta(A_{q_n n_0(K)+r_n}(z) \ldots A_{q_n n_0(K)+1}(z), B_{q_n n_0(K)}([y], B_{q_n n_0(K)}([x])) \times \right. \]
\[ \left. \sqrt{\mathbb{E}[\exp(sK\zeta(B_{q_n n_0(K)}([y], [x]))) \]
\[ \leq \sqrt{\mathbb{E}[\norm{(\wedge)^{j-1}A_{q_n n_0(K)+r_n}(z) \ldots A_{q_n n_0(K)+1}(z)}^{\norm{(\wedge)^{j-1}A_{q_n n_0(K)+r_n}(z) \ldots A_{q_n n_0(K)+1}(z)}^{-1}} sK] \times \right. \]
\[ \left. \times (1 - \varepsilon_K)^{\frac{q_n}{q_n}} \right), \]

and using the proof of Proposition 9.12 and the independence condition (assuming again that \( s_K \) has to be redefined so that \( \norm{A_1}^{(j-1)s_K+(2N-1)js_K} \) is also integrable (via Assumption 2.3)) we find
\[ \mathbb{E}[\exp(\frac{1}{2}sK\zeta(B_n(z), [y], [x]))] \leq \mathbb{E}[\norm{A_1}^{(j-1)s_K+(2N-1)js_K}]^{\frac{q_n}{q_n}} (1 - \varepsilon_K)^{\frac{q_n}{q_n}} \]
\[ \leq (\sup_{z \in K} \mathbb{E}[\norm{A_1}^{(j-1)s_K+(2N-1)js_K}]^{n_0(K)}) (1 - \varepsilon_K)^{\frac{q_n}{q_n}}, \]
which implies the bound in the claim.

In the proof above we have used the following two propositions. The first one is a basic consequence of the multinomial theorem, Jensen’s inequality, and independence and hence its proof is omitted.

**Proposition 9.11.** We have
\[ \mathbb{E}[\left( \sum_{j=1}^n \log(\norm{A_j}) \right)^4] \leq n^4 \mathbb{E}[\log(\norm{A_1})^4] \]
where \( \{A_j\}_j \) are independent variables.
Proposition 9.12. ([6] Lemma III.5.4) If $M \in \text{Mat}_{L \times L}(\mathbb{C})$ has $|\det(M)| = 1$ then for any $j \in \{1, \ldots, L\}$ and $v \in \Lambda^j \mathbb{C}^L$ we have

$$|\log(\|\Lambda^j M\|)| \leq j(L - 1) \log(\|M\|)$$

and

$$|\log(\|\Lambda^j Mv\|/\|v\|)| \leq j(L - 1) \log(\|M\|).$$

Proof. First note that $\|\Lambda^j M\|$ is the product of the first $j$ singular values:

$$\|\Lambda^j M\| = \sigma_1(M) \cdots \sigma_j(M)$$

so that

$$\|\Lambda^j M\| \leq \|M\|^j.$$ 

Conversely,

$$\|\Lambda^j M\|^{-1} \leq \|\Lambda^j (M)^{-1}\| = \|\Lambda^j M^{-1}\| \leq \|M^{-1}\|^j.$$ 

For any invertible matrix we have

$$\|M^{-1}\| \leq \|M\|^L - 1/|\det(M)|,$$

so that in our case

$$\|\Lambda^j M\|^{-1} \leq \|M\|^{j(L-1)}.$$ 

We find using the fact that log is monotone increasing,

$$\log(\|\Lambda^j M\|) \leq j \log(\|M\|) \leq j(L - 1) \log(\|M\|)$$

and

$$-\log(\|\Lambda^j M\|) = \log(\|\Lambda^j M\|^{-1}) \leq j(L - 1) \log(\|M\|).$$

Next, we have

$$\frac{\|\Lambda^j Mv\|}{\|v\|} \leq \|\Lambda^j M\|,$$

whereas $\|v\| = \|(\Lambda^j M)^{-1}(\Lambda^j M)v\| \leq \|(\Lambda^j M)^{-1}\| \|\Lambda^j Mv\|$ so that

$$\left(\frac{\|\Lambda^j Mv\|}{\|v\|}\right)^{-1} \leq \|(\Lambda^j M)^{-1}\|$$

which gives the second inequality of the prop.

Finally, note that these inequalities indeed make sense: $1 \leq \|M\|^{-1} \leq \|M\|\|M\|^{-L} = \|M\|^L$ so that $\log(\|M\|) \geq 0$ always when $|\det(M)| = 1.$
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