Direct calculation of length contraction and clock retardation

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Abstract. For simple electromagnetic models of a rod and a clock, a change of the shape of the rod and of the rate of the clock when they are set in uniform motion is calculated exactly, employing the correct equation of motion of a charged particle in the electromagnetic field and the universal boostability assumption. Thus it is demonstrated that, for the simple system considered, length contraction and clock retardation can be interpreted as dynamical cause-and-effect phenomena, and not as kinematical effects as is usually construed in conventional presentations of Special Relativity. It is argued that the perspectival relativistic change of an object (corresponding to observations from two inertial frames), while certainly an acausal effect, has a dynamical content in the sense that it is tantamount to an actual dynamical change of the object in one frame.

1. Introduction

Recently, I attempted to clarify some basic concepts and results of Einstein’s special relativity theory [1], noting the paramount importance of what I called ‘the universal boostability assumption’ for construction of the theory. The fundamental assumption reads: ‘It is possible to set a measuring rod or clock in a uniform motion or bring it back to a permanent rest without changing the rest length of the rod or the rest rate of the clock, i.e. it is possible to boost them in such a way that they remain standards of length and time in their rest frame, regardless of the constitution of these objects.’ I pointed out that Einstein used a stronger assumption in his original foundation of Special Relativity [2], namely that the measuring capacity of a measuring rod or clock remains untouched under arbitrary boosts; I argued that the stronger assumption is unwarranted. Particularly, I analysed in detail the well-known relation for relativistic length contraction,

\[ l_v = l_0' \sqrt{1 - v^2/c^2}, \]  \hspace{1cm} (1)

relating lengths \( l_v \) and \( l_0' \) of one and the same rod as measured in two inertial frames \( S \) and \( S' \) in standard configuration (\( S' \) is uniformly moving at speed \( v \) along the common positive \( x, x' \)-axes, and the \( y \)- and \( z \)-axis of \( S \) are parallel to the \( y' \)- and \( z' \)-axis of \( S' \)), respectively; \( S' \) is the rest frame of the rod, and \( S \) is the lab frame, with respect to
which the rod is in uniform motion along its length at speed $v$. I recalled that Einstein in [2] stated that if the rod to be measured is at rest in $S$, then ‘in accordance with the principle of relativity’ its length as measured in $S$, $l_0$, must be equal to $l'_0$,

$$l_0 = l'_0,$$

(2)

employing the same measuring rod as in the earlier measurements. Equations (1) and (2) imply

$$l_v = l_0 \sqrt{1 - v^2/c^2}.$$  

(3)

Thus, according to Einstein, a rod initially at rest in an inertial frame, after a constant velocity $v$ is imparted to it in an arbitrary way so that the rod moves freely and uniformly along its length is contracted (its length is reduced), all with respect to that frame, as expressed by equation (3). However, I argued in [1], noting the relevance of rest properties–preserving accelerations, that all one may infer on the basis of Special Relativity is that, in general, equation (1) always applies, whereas equation (2) and thus equation (3) do not necessarily apply. While length contraction and clock retardation are generally regarded, starting from Einstein, as purely kinematical results of Special Relativity, obtained directly from the Lorentz transformations, I pointed out that equation (3) (which involves rest length–preserving accelerations), encapsulates the actual dynamical change of the rod in the $S$ frame due to the action of some forces on the rod in that frame. Moreover, even equation (1), which expresses the relativistic perspectival change of the rod (involving measurements from two different frames $S$ and $S'$ and, clearly, involving no forces acting on the rod by a mere transition to another inertial frame) has a natural dynamical content.

In a recent paper [3], I continued my attempts to clarify Special Relativity. To avoid possible terminological and conceptual muddle, I proposed to call the contents of equations (1) and (3), the relativistic length reduction and the relativistic FitzGerald–Lorentz contraction, respectively. I noted what I consider to be fallacies in the existing literature devoted to teaching of relativity, particularly the contention that in the perspectival change of an object in Special Relativity (corresponding to observations from two different inertial frames), there is no change in the object, it is only the reference frame that is changed from $S$ to $S'$. (More precisely, some authors explain the differences in observations between two inertial frames as a purely kinematical effect due to the relativity of simultaneity, ‘a consequence of our way of regarding things’ (cf, e.g., [4, 5]), while other authors (cf, e.g., [6, 7]) argue that the differences are basically of a dynamical origin, due to a dynamical change of standards of length and time when transferring the standards between the frames $S$ and $S'$. ) On the one hand, the relativistic perspectival change of an object is certainly an acausal phenomenon (there is no change in the object in the standard physicists’ sense of the word, referring to different properties of the object with time in one frame); on the other hand, as is pointed out in [1, 3], there is a dynamical content of the phenomenon which seems to be somewhat neglected in the literature.
The purpose of the present note is to illustrate deliberations presented in references [1] and [3] with simple examples, using elementary models of standards of length and time. Since measuring rods and clocks are physical devices and are subject to the laws of physics in accordance with which they are constructed, one must employ physical laws whose validity is well confirmed in an inertial frame (laboratory). The good candidates are Maxwell’s equations and the Lorentz force expression for the force acting on a charge $q^*$ in an electromagnetic field,

$$F_L = q^*E + q^*v \times B,$$

(4)

where $v$ is the instantaneous velocity of the charge, $E$ is the electric field and $B$ is the magnetic flux density. This has to be combined with the equation of motion of the charge $q^*$ in the electromagnetic field

$$\frac{d}{dt} \left( \frac{mv}{\sqrt{1 - v^2/c^2}} \right) = q^*E + q^*v \times B,$$

(5)

where $m$ is the mass of the charge and $c$ is the speed of light in vacuo; the last equation fits the experimental facts if the additional independent assumption that $m$ is time-independent is introduced (cf, e.g. [8]). For simple models of a rod and a clock, operating on the basis of Maxwell’s equations and equation (5), it will be shown that when the rod and the clock are set in a uniform motion with respect to the laboratory frame, they exhibit the FitzGerald–Lorentz contraction and the Larmor clock retardation in the lab, assuming rest properties–preserving accelerations. Thus, for the rod and clock under consideration, a dynamical content of the effects is clearly revealed.

Dynamical analyses of length contraction and clock retardation in the spirit of the present one, based on electromagnetic laws, have been published occasionally [9, 10, 7]. Unfortunately, models proposed in [9, 10, 7] either cannot be solved analytically [9], or introduce for clocks, tacitly [7] or explicitly [10], a confusing assumption that the velocity of moving clock is much larger than the maximum velocity occurring in its clockwork. Thus they are not very convincing. Another point is that some authors [9, 7] attempt a constructive dynamical approach to Special Relativity, what seems to be an impossible mission, apart from the fact that Maxwell’s theory cannot account for the empirical stability of solid matter. Namely, if one starts from known and conjectured good laws of physics in any one inertial frame, one can learn that if a constant velocity is imparted to a rod and a clock, the moving rod is contracted and the moving clock runs slower. However, rod contraction and clock retardation are necessary but not sufficient conditions for the Lorentz transformations: to construct another inertial frame and to derive the Lorentz transformations, one has to introduce, at one place or another, Einstein’s two postulates of Special Relativity aided with the universal boostability assumption. (Rod contraction and clock retardation in the $S$ frame imply that one clock–two way speed of light is $c$ also in the $S'$ frame but this does not suffice to ‘spread time over space’ in $S'$.) Moreover, only on the basis of Einstein’s principle approach one knows that candidates for good physical laws in one inertial frame must be (or can be...
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made to be) Lorentz–covariant. (Note that despite repeated statements in the literature \[11, 12, 6\], Lorentz–covariance of Maxwell’s equations is not fulfilled automatically, as is pointed out, e.g. in \[13, 14\].) Thus, it appears that the laws of physics in any one reference frame cannot ‘account for all physical phenomena, including the observations of moving observers,’ contrary to Bell’s claim in \[9\].†

On the other hand, the standard ‘kinematical’ derivation of rod contraction and clock retardation \[2\], conceals the fundamental fact that rods and clocks in hand must be relativistically valid, that is, represent physical apparatuses or devices operating in accordance with the laws of physics which are (or can be made to be) Lorentz–covariant. In the standard approach, unexpected qualities of rods and clocks in motion appear as a dry consequence of the Lorentz transformations, which are achieved from logically entangled postulates, and which deal with rods and clocks in abstracto, regarded as primitive entities (cf \[1\], p 199). However, paraphrasing Møller \[15\], students of the theory of relativity would like to see—at least on a simple model—that rod contraction and clock retardation indeed follow from the structure of a physical system and the dynamical laws governing it, considered in one frame only (cf also \[16\]). Therefore I believe that a simple and exact model of a relativistically valid clock would enhance the students’ understanding of Special Relativity. Also, a dynamical content of the relativistic perspectival change in an object, and the universal boostability assumption, seem to be either neglected or misrepresented in the literature. Thus the present note, which aims to complement the standard principle approach to Special Relativity by providing simple illustrations of its dynamical contents, could perhaps be of some interest.

2. Direct calculation of length contraction

Firstly, as a relativistically valid standard of length I discuss an elementary model of solid body proposed by Sorensen \[17\].

2.1. Rod at rest

Consider four equal charges \( q \) of the same sign, at rest in the \( S \) frame (laboratory), placed at the vertices of a square \( ABCD \) of a side \( a \) (\( A \) is the bottom left hand vertex, and the vertices \( B, C \) and \( D \) run counterclockwise). Employing the Coulomb law, one finds that placing a charge of opposite sign, \( q_c = -q(1 + 2\sqrt{2})/4 \), at the centre of the square, the resultant of the forces acting on each charge is zero. Thus the system of

† Miller in \[7\] criticizes Bell’s anticipation of the equation of motion (5) as a limitation of Bell’s approach \[9\]. In his own constructive dynamical attempt to derive Special Relativity, Miller avoided the use of equation (5). Instead, he tacitly postulated that Maxwell’s equations apply not only in the original rest frame of a physical system, but also in its final rest frame, cf the argument leading to equation (6) in \[7\]. Thus Miller’s approach is at best a combination of a constructive dynamical, and the principle approaches.
the five charges is in the electrostatic equilibrium. From Earnshaw’s theorem, we know that the equilibrium is unstable (cf, e.g. \[18\]).

One can verify that for five point charges \(q, q, q, q\) and \(q_c\), the only static equilibrium shape is a square and not a rectangle or any other shape, as Sorensen \[17\] pointed out. Note that equilibrium conditions fix only the shape of the equilibrium configuration and not its size (the side of the square can be arbitrary). (Incidentally, the electrostatic potential energy of the static configuration is always zero, regardless of the value of \(a\)).

2.2. Rod in uniform motion

Assume now that the system considered has been accelerated, starting from rest until reaching a steady velocity \(v_0 = v_0 \hat{x}\), so that all five charges are uniformly moving in the plane of the initial square (the \(xy\) plane) parallel to the \(x\) axis; take that \(v_0\) is perpendicular to the sides \(AD\) and \(BC\) of the square. Assume also that that the acceleration was gentle, in the sense that, after all transient effects have died out, the system of five uniformly moving charges is again in a stationary (time-independent) configuration. The question arises, is there such a moving configuration at all.

Now we have to take into account that at the location of each charge, in addition to the electric field, there will be also a magnetic field, since the remaining charges are in motion. The \(\mathbf{E}\) and \(\mathbf{B}\) fields of a point charge \(q\) moving with constant velocity \(v_0\) were first obtained by Oliver Heaviside in 1888 and the \(\mathbf{B}\) field was rederived by J J Thomson in 1889 (\[19, 20, 21\], cf \[22\] and references therein), long before the advent of Special Relativity. The electric field is radial but not spherically symmetrical (contrary to the electrostatic field of \(q\)), and is given by

\[
\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{r}}{r^3} \left(1 - \frac{v_0^2}{c^2} \right)^{3/2},
\]

(6)

where \(\mathbf{r}\) is the position vector of a field point with respect to the instantaneous (at the same instant \(t\)) position of \(q\), \(\theta\) is the angle between \(\mathbf{r}\) and the velocity \(\mathbf{v}_0\), and \(c^2 \equiv 1/\varepsilon_0\mu_0\). (Recall that throughout the relativity paper \[2\], Einstein used the same symbol \(V\) for the speed of light \(\text{in vacuo}\) and the speed of electromagnetic waves \(\text{in vacuo}\) \(V \equiv 1/\sqrt{\varepsilon_0\mu_0}\), linking thus Special Relativity with Maxwell’s theory (cf \[1\], p 197).) The magnetic flux density is

\[
\mathbf{B}(\mathbf{r}, t) = \varepsilon_0\mu_0 v_0 \times \mathbf{E}(\mathbf{r}, t).
\]

(7)

Equation (7) and the Lorentz force expression (4) imply that the total electromagnetic force on each of the five charges of our uniformly moving system vanishes if and only if the \(\mathbf{E}\) field vanishes at the location of each charge. The symmetry suggests that the equilibrium configuration we are looking for has a rectangular shape with \(q_c\) at the centre. Therefore we assume that stationary configuration is a uniformly moving rectangle \(ABCD\), with sides \(AD\) and \(BC\) perpendicular to \(\mathbf{v}_0\). Denote lengths of the sides \(AB\) (parallel to \(\mathbf{v}_0\)) and \(AD\) (transverse to \(\mathbf{v}_0\)) by \(b\) and \(d\), respectively. Consider equilibrium conditions at the vertex \(A\). A little analysis reveals that the condition that
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the \( E \) field at \( A \) has no component in the direction perpendicular to the diagonal \( AC \), and along \( AC \), implies that

\[
\frac{d}{b^2} = \frac{b}{d^2(1 - v_0^2/c^2)^{3/2}},
\]

and

\[
\frac{1}{b} + \frac{1}{d(1 - v_0^2/c^2)^{3/2}} = \frac{2\sqrt{2}}{\sqrt{d^2 + b^2[1 - v_0^2d^2/c^2(d^2 + b^2)]^{3/2}}},
\]

respectively. Equation (8) gives

\[
b = d\sqrt{1 - v_0^2/c^2}.
\]

It is easy to check that, with this value of \( b \), equation (9) is satisfied identically. As can be seen, equation (10) is necessary and sufficient condition for the moving rectangular configuration \( ABCD \) to be the equilibrium one, i.e. the stress free state. (Incidentally, Sorensen assumed relation (10) from the outset. Thus he did not demonstrate that ‘to be in equilibrium [...] the five charges must have this rectangular shape, shortened in the dimension of the direction of motion by the Lorentz contraction as compared to the transverse direction,’ contrary to his claim in [17]. Instead, he proved only that equation (10) is sufficient condition for the moving rectangle to be in equilibrium.)

Note that the character of forces governing the equilibrium is such that equilibrium conditions determine the shape of the configuration and not its size (\( d \) is arbitrary), analogously to the electrostatic case. Thus, accelerating square of side \( a \) until reaching the steady velocity \( v_0 \), one can arrive at a moving stationary rectangle with sides \( a\sqrt{1 - v_0^2/c^2} \) and \( a \) in the direction of motion and transverse to it, respectively, but also with sides \( d\sqrt{1 - v_0^2/c^2} \) and \( d \), where \( d \neq a \). Clearly, only in the first case acceleration was rest length–preserving. Namely, according to Special Relativity, observing the moving rectangles from their rest frame \( S' \), they will be squares of sides \( a \) and \( d \neq a \), respectively, since Maxwell’s equations can be made to be Lorentz–covariant, and the Lorentz force expression should apply in every frame. Thus equation (1) always applies, whereas equations (2) and (3) do not necessarily apply, as is pointed out in [1].

As a historical aside, recall that Lorentz argued long ago that if to a system \( \Sigma' \) of particles in the equilibrium configuration at rest relative to the ether ‘the velocity \( v = v\hat{x} \) is imparted, it will of itself change into the system \( \Sigma' \) which is got from \( \Sigma' \) by the deformation \( (\sqrt{1 - v^2/c^2}, 1, 1) \) ([24], pp 5–7, 21–23, 27–28, cf also [3], pp 60–1). However, Lorentz was wrong here; the change \( \Sigma' \rightarrow \Sigma \) can also be effectuated by the transformation \( (l\sqrt{1 - v^2/c^2}, l) \), where \( l \neq 1 \), as the present Section 2 reveals. (From \( vdl/dv = 0 \), Lorentz deduced that \( dl/dv = 0 \), \( l = const \), and concluded: ‘The value of the constant must be unity, because we know already that, for \( v = 0 \), \( l = 1 \’ ([24], p 27). But, all one can deduce from \( vdl/dv = 0 \) is that \( dl/dv = 0 \) for \( v \neq 0 \) ! Thus \( l \) may have arbitrary (constant) value for \( v \neq 0 \).)
3. Direct calculation of clock retardation

The same equilibrium system of five charges, providing a standard of length in the preceding section, will be employed as an exact and yet simple model of a relativistically valid clock.

3.1. Clock at rest

Let four identical charges \( q \) be now fixed at the vertices of the square \( ABCD \) of side \( a \) at rest in the lab frame \( S \). Denote the axis perpendicular to the plane of the square which passes through its centre as the \( z \) axis; choose the origin at the centre and the \( x \) and \( y \) axes parallel to the sides \( AB \) and \( AD \) of the square, respectively. Remove the charge \( q_c \) from its central equilibrium position to the point on the positive \( z \) axis with \( z = A \) and release it with zero initial velocity to move under the action of the electrostatic field of the remaining four charges.

The exact equation of motion of the charge \( q_c \) in the electrostatic field is

\[
m \frac{d}{dt} \left( \frac{v}{\sqrt{1 - v^2/c^2}} \right) = q_c E,
\]

where the mass \( m \) of \( q_c \) is assumed to be time-independent, as is pointed out in the Introduction. Equation (11) and identity

\[
v \cdot \frac{d}{dt} \left( \frac{v}{\sqrt{1 - v^2/c^2}} \right) \equiv c^2 \frac{d}{dt} \left( \frac{1}{\sqrt{1 - v^2/c^2}} \right),
\]

imply that

\[
mc^2 \frac{d}{dt} \left( \frac{1}{\sqrt{1 - v^2/c^2}} \right) = q_c E \cdot v.
\]

Specifying to our problem, \( E \) is the electrostatic field given by

\[
E(0, 0, z) = \frac{\kappa 4q z \hat{z}}{(z^2 + a^2/2)^{3/2}},
\]

where \( \kappa = 1/4\pi\varepsilon_0 \), and \( v = v_z \hat{z} \), since the motion is along the \( z \) axis. Using equations (11), (13) and (14) one obtains

\[
\frac{dv_z}{dt} = \frac{q_c}{m} \left(1 - \frac{v_z^2}{c^2}\right)^{3/2} \frac{\kappa 4q z}{(z^2 + a^2/2)^{3/2}}.
\]

\( \dagger \) The force \( q_c E \) is always parallel to the instantaneous velocity \( v \) of the charge \( q_c \) so that one can derive equation (15) using the concept of ‘longitudinal’ mass, taking into account that the Lorentz force expression is a pure force (cf [11, 8]). I preferred not to employ here the obsolete concepts of ‘transverse’ and ‘longitudinal’ mass, while they appear occasionally in the literature [11, 10].
Obviously, the charge $q_c$ does not perform a simple harmonic motion. However, noting that $q_c < 0$, and also that at $t = 0$, $z = A$, $v_z = 0$, a little analysis reveals that the solution of equation (15) satisfying these initial conditions is a periodic function of $t$,

$$z_R = f(t);$$

(16)

subscript $R$ serves as a reminder that the square is at rest. Denote the period of that function by $T_0$; the period comprises continuous changes of position of the charge $q_c$ from $z = A$ to $z = -A$ and vice versa. Clearly, the square and the charge may be considered as a simple model of a clock, and can be used for measuring time in terms of the number of periods $T_0$. Namely, as Jefimenko pointed out [10], ‘as a physical entity, time is defined in terms of specific measurement procedures, which for the purpose of the present discussion may be described simply as “observing the rate of the clocks.”’ However, for observing the rate of a clock in uniform motion with respect to the lab frame $S$, one has first to ‘spread time over space,’ i.e. to synchronise distant clocks at rest in $S$. From Maxwell’s equations one knows that a natural means for that purpose is the propagation of electromagnetic waves in vacuo.

3.2. Clock in uniform motion

Assume now that the same clock is set in uniform motion with constant velocity $v_0 = v_0 \hat{x}$ along the positive $x$ axis, so as to be relativistically valid, i.e. to serve as an identical standard of time also for a co-moving inertial observer. From the preceding discussion it follows that now four identical charges $q$ have to be fixed in their rest length–preserving equilibrium positions, that is at the vertices of the moving rectangle $ABCD$ with sides $AB = a\sqrt{1 - v_0^2/c^2}$ and $AD = a$. Remove the charge $q_c$ co-moving with the rectangle from its central equilibrium position, to the co-moving point on the axis of the rectangle with $z = A$, and release it with initial velocity $v_0$ to move under the action of the electromagnetic field of the remaining four charges.

The exact equation of motion of the charge $q_c$ in the field, obtained from equation (5) assuming that $m$ is constant, reads

$$m \frac{d}{dt} \left( \frac{v}{\sqrt{1 - v^2/c^2}} \right) = q_c E + q_c v \times B;$$

(17)

obviously, equation (13) applies in this case too. Using equation (6), after a somewhat cumbersome but in every step simple calculation, one finds that the electric field on the co-moving axis (which is perpendicular to the plane of the moving rectangle $ABCD$ and passes through its centre) is

$$E = \kappa q z \frac{1}{(z^2 + a^2/2)^{3/2}} \sqrt{1 - v_0^2/c^2} = E_z \hat{z},$$

(18)

For the magnetic flux density at the same point on the co-moving axis, using equation (7), one finds

$$B = \frac{-v_0}{c^2} E_z \hat{y} = \frac{-v_0}{c^2} \frac{\kappa q z}{(z^2 + a^2/2)^{3/2}} \frac{1}{\sqrt{1 - v_0^2/c^2}} = B_y \hat{y}.$$

(19)
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Using equations (17), (13), (18) and (19), and taking into account initial conditions (at the time \( t = 0 \), the charge \( q_c \) is at the point \( z = \mathcal{A} \) on the co-moving axis, and components of its velocity are \( v_x = v_0, \ v_y = v_z = 0 \), a little analysis reveals that the charge \( q_c \) will forever move along the co-moving axis, i.e. so that \( v_x = v_0, \ v_y = 0 \).

Passing details, we give the final equation of motion of the charge along the co-moving axis:

\[
\frac{d v_z}{d t} = \frac{q_c}{m} \left( 1 - \frac{v^2}{c^2} \right)^{3/2} \frac{\kappa 4 q z}{(z^2 + a^2/2)^{3/2} \sqrt{1 - v_0^2/c^2}}. \tag{20}
\]

Now since \( v^2 = v_0^2 + v_z^2 \) one has

\[
\left( 1 - \frac{v^2}{c^2} \right)^{3/2} = \left( 1 - \frac{v_0^2}{c^2} \right)^{3/2} \left[ 1 - \frac{v_z^2}{c^2(1 - v_0^2/c^2)} \right]^{3/2}, \tag{21}
\]

and introducing

\[
v^*_z = \frac{v_z}{\sqrt{1 - v_0^2/c^2}} = \frac{d z}{d t^*}, \tag{22}
\]

where

\[
t^* \equiv t \sqrt{1 - v_0^2/c^2}. \tag{23}
\]

equation (20) can be recast as

\[
\frac{d v^*_z}{d t^*} = \frac{q_c}{m} \left( 1 - \frac{v^*_z^2}{c^2} \right)^{3/2} \frac{\kappa 4 q z}{(z^2 + a^2/2)^{3/2}}. \tag{24}
\]

Equation (24) for the clock in motion has exactly the same form as equation (15) for the same clock at rest, the only difference being that variable \( t \) in the latter is replaced by \( t^* \equiv t \sqrt{1 - v_0^2/c^2} \) in the former. Since the solution of equation (15) is a periodic function (16) with period \( T_0 \), it is clear that the solution of equation (24) satisfying identical initial conditions \( (z = \mathcal{A} \text{ and } v_z = v^*_z = 0, \text{ at } t = t^* = 0) \) is the same function of \( t^* \),

\[
z_M = f(t^*) = f(t \sqrt{1 - v_0^2/c^2}), \tag{25}
\]

where subscript \( M \) serves as a reminder that the clock is in uniform motion; period of the clock in motion is obviously

\[
T_M = \frac{T_0}{\sqrt{1 - v_0^2/c^2}}. \tag{26}
\]

Thus, the above model of a clock, however fragile, provides a simple and yet exact illustration of the Larmor clock retardation. For this clock, the retardation appears to be a dynamical, cause–and–effect phenomenon, as was the case with length contraction.

*Note that when the clock (the square + the charge \( q_c \)) moves along the axis of the square (the \( z \) axis), so that \( v_0 = v_0 \hat{z} \) (in the same way as Jefimenko’s clock \# 1 [10]), an exact one frame derivation of equation (26) appears rather challenging.*
Rewrite equation (26) in the form
\[ T_v = \frac{T_0}{\sqrt{1 - v^2/c^2}}, \]
where now \( v \) is the speed of the rectangle in uniform motion, so as to be analogous to equation (3). Since rest rate–preserving acceleration is assumed in the above derivation of equation (27), one has that
\[ T_0 = T'_0, \]
where \( T'_0 \) is period of the moving clock as observed by an inertial co-moving observer. Equations (27) and (28) imply
\[ T_v = \frac{T'_0}{\sqrt{1 - v^2/c^2}}, \]
which is analogous to equation (1). As can be seen, \( \textit{mutatis mutandis} \), remarks analogous to those for the uniformly moving rod, presented in the last two paragraphs of Section 2, apply to the case of the uniformly moving clock. Particularly, equation (29) always applies, whereas equations (28) and (27) need not apply.

Note that the above dynamical derivation of equation (26) applies to the simple clock considered. On the other hand, in the framework of relativistic \textit{kinematics}, Møller argued: ‘In view of the fact that an arbitrary physical system can be used as a clock, we see that any physical system which is moving relative to a system of inertia must have a slower course of development than the same system at rest’ [12]. Here one has to take into account that, according to Special Relativity, any physical system must conform to some Lorentz–covariant \textit{dynamical} laws, however complex the system is. Since the exact form of the laws is generally unknown, ‘an all-inclusive dynamic (causal) interpretation of time dilation is hardly possible,’ as Jefimenko pointed out [10]. Fortunately, the principle approach to Special Relativity predicts equation (26) \textit{indirectly}, via the Lorentz transformations, without the need to enter into details of the phenomenon that serves as a clock. Namely, one need not know the exact laws governing the operation of a clock; it suffices to know that the laws have to be Lorentz–covariant. However, one must admit that any clock retardation hides a complex dynamical process and also involves the universal boostability assumption. Finally, note that the above simple clock model illustrating a dynamical content of equation (26) represents an \textit{ideal} clock. Namely, a real clock necessarily involves damping, which is in our case due to

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What is the origin of labelling rod contraction and clock retardation as kinematical effects (i.e., that they can be dealt with without involving actions, forces, masses)? This appears to be relics of Newton’s absolute space and time concepts, where it is tacitly assumed that ‘a moving rigid body at the epoch \( t \) may in geometrical respects be perfectly represented by the same body at rest in a definite position’ [2], and analogously for a moving clock. While ‘kinematical’ is fitting in the context of the Galilean transformation, it masquerades the dynamical contents of the Lorentz transformation.
the radiation reaction force. Construction of Special Relativity requires of course ideal standard clocks so our simple model may perhaps be to the point.

3.3. Clock retardation in detail

Equation (15) can obviously be recast into

$$\frac{dv_z}{dz} v_z = -\frac{|q_c q| \kappa^4}{m} \left( 1 - \frac{v_z^2}{c^2} \right)^{3/2} \frac{z}{(z^2 + a^2)^{3/2}}.$$  \hspace{1cm} (30)

Separating variables and integrating, setting $v_z = 0$ when $z = \pm A$ and solving for $v_z$ yields

$$v_z = \frac{dz}{dt} = \mp c \{\ldots\}^{1/2},$$  \hspace{1cm} (31)

where

$$\{\ldots\} \equiv \{1 - [1 + (|q_c q| \kappa^4/\hbar c^2)(1/\sqrt{z^2 + a^2} - 1/\sqrt{A^2 + a^2})]^{-2}\},$$  \hspace{1cm} (32)

and $-$ and $+$ sign corresponds to the motion of $q_c$ in the direction of decreasing $z$ and increasing $z$, respectively. Equation (31) implies that, for the oscillator at rest, passage of $q_c$ from $z$ to $z + dz$ lasts time interval

$$dt = \mp (1/c) dz \{\ldots\}^{-1/2}.$$  \hspace{1cm} (33)

Thus, the period $T_0$ of the oscillator at rest is given by

$$T_0 = \frac{2}{c} \int_A^{-A} \{\ldots\}^{-1/2} dz.$$  \hspace{1cm} (34)

Incidentally, in the case of small oscillations, i.e. when $A \ll a$, from equation (34) one obtains that

$$T_0 \approx \frac{2}{c} \int_A^{-A} \{1 - [1 + (\mathcal{K}/2mc^2)(A^2 - z^2)]^{-2}\}^{-1/2} dz.$$  \hspace{1cm} (35)

where $\mathcal{K} \equiv |q_c q| \kappa^4/(a^2/2)^{3/2}$. As can be seen, this result coincides with the corresponding Møller’s result for the period of his “relativistic” oscillator’, equation (60) in [15], as it should. Finally, note that when $\mathcal{K} A^2 \ll mc^2$, from equation (35) one obtains the familiar expression for the period of the simple harmonic oscillator, $T_0 = 2\pi \sqrt{m/\mathcal{K}}$.

Analogously, using equations (22)-(24), for the oscillator in uniform motion one finds that passage of $q_c$ from $z$ to $z + dz$ lasts time interval

$$dt_M = \frac{1}{\sqrt{1 - v_0^2/c^2}} (\mp) (1/c) dz \{\ldots\}^{-1/2},$$  \hspace{1cm} (36)

which is $1/\sqrt{1 - v_0^2/c^2}$ times longer than the corresponding time interval (33) for the oscillator at rest. This embodies clock retardation. The period of the oscillator in motion is obviously given by

$$T_M = \frac{1}{\sqrt{1 - v_0^2/c^2}} 2 \int_A^{-A} \{\ldots\}^{-1/2} dz,$$  \hspace{1cm} (37)

which is equation (26).
4. Where do the perspectival changes come from?

To avoid confusion, begin with a few terminological comments.

By ‘the perspectival relativistic change’ of an object I mean that, according to Special Relativity, one and the same object (in the sense consisting of the same ‘atoms’) has distinct properties (say, the length of a rod or the period of a clock), depending on whether the properties are being measured in the rest frame of the object or in the laboratory frame (with respect to which the object is in a uniform translatory motion). Note that here we deal with two different states of motion of an object with respect to two inertial frames (‘observers’), and two respective ‘configurational states’ of the object. By ‘the actual physical change’ of an object in the standard physicists’ sense of the phrase, I mean that properties of the object become distinctly different under the action of certain forces (both external and internal), all with respect to one and the same inertial frame.

Now comes what is perhaps the key question of Special Relativity. When a single physical object is observed by two different inertial observers (or when a single observer changes frames), where do the differences between their observations come from? Particularly, when an object at rest in \( S' \) and thus in uniform motion with respect to the lab frame \( S \) is observed from the two frames, why the results of observations differ? Note that the perspectival change itself has nothing to do with previous history of the object either in \( S' \) or in \( S \), the history may be unknown to us; moreover, the object need not be free nor connected. Note also that there seems to be an overall consensus that we did nothing to the object by merely observing it from two different frames (or by accelerating an observer to another frame). Thus, there is no cause of the perspectival relativistic change, it is an acausal effect.

Miller argued in [7] that the differences among observations of different inertial observers ‘are due to the differences in their respective measuring instruments [...] these perspectival effects ultimately have a dynamical origin because the properties of measuring instruments are determined by the forces that keep them in equilibrium in their respective frames.’ The author explained, following Feinberg [6], that ‘when the measuring rods and clocks are moved between inertial observers, they suffer dynamical changes. When the observers use their dynamically altered rods and clocks to make measurements, it is not surprising that their results differ and that they differ by the same factors that are involved in the dynamical changes.’

While Feinberg and Miller advocate a force interpretation of the so-called kinematical effects of Special Relativity, a common thread in their discussions is that ‘there are no dynamical effects in the physical object being observed’; the differences in measuring instruments used by by different inertial observers are all that matters. Now, it is certainly true that nothing at all has happened to the object being observed by a mere transition to another inertial frame. (‘The body received no impact, pull or boosts, but is viewed from a system moving relative to it; [...] there has been no actual change in the body itself.’) However, I think that Feinberg and Miller’s interpretation
Direct calculation of length contraction and clock retardation falsifies the spirit of Special Relativity. While the perspectival relativistic change is an acausal effect, I will argue that there is a dynamical explanation of the effect in the sense that the perspectival change is tantamount to an actual physical change. This seems to be the gist of Special Relativity.

Firstly, each inertial observer possesses his or her own set of measuring instruments which are identical to one another. A measuring rod at rest in the lab frame $S$ is in all respects identical to a measuring rod of the same construction at rest in the ‘moving’ frame $S'$ under identical physical conditions; the rods embody the same length in their respective rest frames. That the rods can indeed be of the same construction is secured by the universal boostability assumption, as was illustrated in Section 2.

Therefore, it is somewhat perplexing to account for the differences between the observations of the $S$– and $S'$–observer in terms of the differences in their respective measuring instruments, as Feinberg [6] and Miller [7] do. A natural explanation appears to be at hand.

As was noted above, in the perspectival relativistic change we deal with two different states of motion of an object with respect to two inertial observers, and two respective ‘configurational states’ of the object. Specifying to the simple system discussed in Sections 2 and 3, taking into account that the theory employed (Maxwell’s equations plus the Lorentz force equation (5)) is made to be Lorentz–covariant, it follows that different observations of the system considered in the lab frame $S$ and in the rest frame $S'$ are due to different states of motion of the system in the two frames, and thus to its different respective configurations. The differences in configurations are due to a different electromagnetic field produced by the moving field–producing charges, and hence to a different force acting on the moving field–experiencing charges.

On the other hand, if we start from the object at rest in the lab frame $S$, which is in the same configurational state as it was the object’s rest state in the ‘moving’ frame $S'$, and accelerate it until reaching the steady velocity $v = \hat{v}\hat{x}$ in a persistent state (thus being at rest with respect to the ‘moving’ frame) and if the acceleration was rest properties–preserving, we reach the same configurational state of the moving object as measured in the lab, as was found earlier as a result of the perspectival change. Now we deal with two configurational states of the same object, which are identical to the ones discussed above in the context of the perspectival change, corresponding to two different states of motion but now with respect to one inertial frame. In the ‘one frame scenario’, one has two stationary configurational states with distinct properties of the object due to different character of forces providing equilibrium in the two states of

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¶ According to conventional presentations of Special Relativity, measuring instruments need not be transferred between frames; instead, they can be constructed in each frame ‘from scratch’, following the same recipe. But it appears that the universal boostability assumption, or its equivalent, must be introduced at some step of the procedure. At first sight, the universal boostability assumption has basically the same contents as Born’s ‘principle of the physical identity of the units of measure’ [5] (cf also [1], footnote 12). However, Born seems to to imply that measuring capacity of a rod or clock remains untouched under arbitrary boosts, which is incorrect, cf Section 2. In the same way, Feinberg’s ‘universality with respect to the acceleration regime’ [23] does not generally hold.
motion; this actual physical change has a clear dynamical origin. Since the ‘two frames situation’ is perfectly equivalent to the corresponding ‘one frame situation’ under the assumptions stated, one must admit that not only the actual physical change but also the perspectival relativistic change of an object has a dynamical content. (Clearly, in the system discussed in Section 2, the change belongs to statics (equilibrium of forces and momenta).)

Basically, all that matters is the state of uniform motion of a physical system with respect to one (arbitrary but fixed) inertial observer, under the proviso that the (persistent) rest configuration of the system is conserved. It is irrelevant whether two different states of motion of the system are observed from two different inertial frames, respectively, or from one frame only, if in the latter case the two states of motion are related by a rest properties–preserving acceleration. This supremacy of any one inertial observer appears to be the gist of the principle of special relativity.

5. Summary

The calculations of rod contraction and clock retardation presented in this paper provide a dynamical cause–and–effect type interpretation of those so-called kinematical effects of Special Relativity. A dynamical content of the effects is clearly revealed at least in the case of the simple electromagnetic model employed, in terms of various character of forces governing the equilibrium in the state of motion and in the state of rest of the system under consideration. By means of the same model, the importance of the universal boostability assumption is illustrated. A dynamical content of the perspectival relativistic change is also discussed. It is argued that when a connected physical object in a persistent state is observed by observers in different inertial frames, the differences among their observations are due to changes in character of forces which determine the structure of the object with a change of its velocity, provided that the velocity change was performed in a rest properties–preserving way. The different inertial observers have a dynamical explanation of the differences among their observations in terms of an equivalent dynamical change in the object with respect to one inertial frame.

Incidentally, Feinberg [6] pointed out that, instead of transferring a physical object from its initial rest frame $S$ to its final rest frame $S'$ through a rest properties–preserving acceleration, the same final state of the object in motion relative to $S$ can be reached in a different way. Namely, instead of accelerating the object, one can accelerate a reference frame–copy of the $S'$ frame, initially at rest relative to $S'$, until the copy frame eventually becomes the $S$ frame (cf [2] and also a somewhat obscure attempt by Swann [25]); of course, the object is now assumed to be always at rest in $S'$. Feinberg asked why does the action on the measuring system of rods and clocks cause a contraction of the measured rod. After an explanation that I found obscure, he noted that ‘one may naturally still wonder why a symmetric result is obtained when there is such an enormous asymmetry in the transition to the final state of motion with the same relative velocity.’ It seems, however, that in his explanation Feinberg failed to take into account properly that the perspectival change is an acausal phenomenon, and also that measuring capacity of a rod or clock remains untouched under rest properties–preserving boosts.
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