Lattice Determination of Semileptonic Form Factors

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We report on the Lattice determination of the semileptonic form factors by lattice QCD. Comparison with the light-cone QCD sum rules are made for \( B \to \pi \nu \), \( B \to \rho \nu \) semileptonic decays.

1 Introduction

Computation of the semileptonic form factors for \( B \to \pi(\rho)\nu \), \( B \to D^{(*)}\nu \) and \( B \to K^{(*)}\ell^+\ell^- \) decays is a key to the determination of the CKM matrix elements \( |V_{ub}| \), \( |V_{cb}| \) and \( |V_{td}| \). In this report, I review the recent results by lattice QCD methods. I will mainly focus on the semileptonic decay \( B \to \pi \nu \), because lattice calculations in various approaches are already at hand \([1,2,3,4,5]\) and because most of the problems which exists in this decay are common to other processes.

Lattice QCD calculations of \( B \to \pi \nu \) form factors suffer from (1) the error from the large heavy quark mass \( m_Q \) in lattice unit, (2) the chiral extrapolations in the light quark mass region are made for \( m_Q \) using the simulation results with relatively large mass region \( m_Q > m_s/2 \), and (3) the limitations from the accessible kinematic range of \( q^2 \) from statistical and systematic errors.

In order to solve the first problem, two different approaches are made. One is to avoid the large discretization errors of \( O(a m_Q) \) by computing the form factors with the conventional relativistic quark action for charm quark mass region, and then extrapolate the results in \( 1/m_Q \). The other is to use HQET effective theory with \( 1/m_Q \) corrections. Since both approaches have their own advantages and disadvantages, it would be important to have both results and study whether they give consistent results within errors. I will present some of the major calculations from these two approaches and discuss the consistency of the results.

The second problem already gives a source of errors in the quenched calculations but it would become even more serious in unquenched QCD. Unfortunately, there are still no results in the unquenched QCD. Since the chiral limit of the form factors is ill-defined in the quenched QCD, all we can do with the present lattice data is to discuss the light quark mass dependence in the intermediate mass regime in the quenched QCD. However, one may be able to give an estimate of the low energy coefficients of the chiral perturbation theory based on the present lattice data. I will briefly review new studies of quenched chiral perturbation theory (QChPT) and partially quenched chiral perturbation theory (PQChPT) \([6]\) for \( B \to \pi \nu \) form factors, which give a phenomenological estimate on the quenching errors and chiral extrapolations. These studies will be even more useful once new calculations in unquenched QCD will be made.

The third problem arises because in semileptonic decays large energies are released to the final states so that the lepton pair invariant mass \( q^2 \) can range from 0 to \( (m_b - m_\pi)^2 \). However, due to the discretization errors of \( O(a E) \) as well as the statistical errors which grow as \( \sim \exp(\text{const} \times (E - m_\pi t)) \) where \( E \) is the energy of the pion, lattice QCD can cover only large \( q^2 \) region. The dispersion relation is a possible solution to give bounds for smaller \( q^2 \) region \([7]\). The light-cone QCD sum rule (LCSR) predicts form factors for small \( q^2 \), which is complementary to the lattice results. I will give a comparison of the recent LCSR results with the lattice results to see whether they will give consistent results.

I also review the form factors in other processes. Some of the recent work on the lattice QCD calculations of the \( B \to \rho \nu \) form factors in relativistic formalism are presented. Very precise calculations of semileptonic form factors for \( B \to D^{(*)}\nu \) at zero recoil and the calculations of the slope of the Isgur Wise function are presented.

2 \( B \to \pi \nu \)

The exclusive semileptonic decay \( B \to \pi \nu \) determines the CKM matrix element \( |V_{ub}| \) through the following formula,

\[
\frac{d\Gamma}{dq^2} = \frac{G_F^2}{24\pi^3}|k_\pi|^2|V_{ub}|^2 f^+(q^2),
\]

where the form factor \( f^+ \) is defined as

\[
\langle \pi(k_\pi)|\bar{q}q^\mu b|B(p_B)\rangle = f^+(q^2)\left[(p_B + k_\pi)^\mu - \frac{m_B^2 - m_\pi^2}{q^2} q^\mu\right] + f^0(q^2)\frac{m_B^2 - m_\pi^2}{q^2} q^\mu,
\]

with \( q = p_B - k_\pi \) and \( q^2 = m_B^2 + m_\pi^2 - 2m_Bv \cdot k_\pi \). The following parameterization proposed by Burdman et al. \([8]\) for \( f^+(q^2) \) gives a phenomenological estimate on the quenching errors and chiral extrapolations.
\[ \langle \pi(k_\pi) | \bar{q} \gamma^\mu b | B(v) \rangle = 2 \left[ f_1(v \cdot k_\pi) \nu^\mu + f_2(v \cdot k_\pi) \frac{\nu^2}{v \cdot k_\pi} \right], \quad (3) \]

is also useful for discussing the heavy quark symmetry and the chiral symmetry of the form factors in a transparent way.

### 2.1 Lattice results

Lattice calculation is possible only in limited situations. Spatial momenta must be much smaller than the cutoff, i.e. \(|\vec{p}_H|, |\vec{k}_d| < 1\) GeV. This means \(v \cdot k_\pi \equiv E_\pi < 1\) GeV or equivalently \(q^2 > 18\) GeV\(^2\). Another limitation is that due to the slowing down, simulations with very small light quark masses are difficult so that usual mass range for the light quark masses in practical simulations is \(m_\pi/3 < m_q \leq m_\pi\) or \(m_\pi = 0.4 \sim 0.8\) GeV. Therefore in order to obtain physical results chiral extrapolations in the light quark masses are necessary.

So far all the lattice calculations of the form factors are done only in quenched approximation. APE collaboration [11] and UKQCD collaboration computed \(B \to \pi \nu\) form factors for a fine lattice with the inverse lattice spacing \(a^{-1} \sim 2.7\) GeV. They used relativistic formalism for the heavy quark and extrapolated the results of heavy-light meson around charm quark masses to the bottom quark mass. Fermilab collaboration [3] used the Fermilab formalism for the heavy quark and computed the form factors on three lattices with \(a^{-1} = 1.2 \sim 2.6\) GeV. JLQCD collaboration [12] computed the form factors using NRQCD formalism for the heavy quark on a \(a^{-1} = 1.64\) GeV. NRQCD collaboration [5] also used NRQCD formalism for the heavy quark and an improved light quark action (D234 action) on an anisotropic lattice with \(a^{-1} = 1.2\) GeV (spatial), 3.3 GeV (temporal). In all of these calculations the light pseudoscalar meson masses are \(0.4 \sim 0.8\) GeV. Fig. 1 shows the result by different lattice groups, \(f^+(q^2)\) agrees within systematic errors while \(f^0(q^2)\) shows deviations among different methods.

The reason for the discrepancies in \(f^0\) can be attributed to the systematic error in the chiral extrapolation and heavy quark mass extrapolation (interpolation) error. In the following, we examine these errors in more detail. Light quark mass \(m_q\) dependence of form factors with fixed spatial momenta \(a p = \frac{2n}{3}(1, 0, 0)\) is shown in Fig. 2. In contrast to the JLQCD data, Fermilab data shows a significant increase towards the chiral limit. Large difference in Fermilab results and JLQCD results for \(f^0\) in the chiral limit arises from different \(m_q\) dependence, but the raw data for similar quark masses are not so different. Shigemitsu et al. studied the mass dependence of \(f_1 + f_2\) and find similar behavior as JLQCD. Further studies to clarify the light quark mass dependence are required.

![Figure 1. B \to \pi \nu\) form factors by different lattice groups.](image)

The error of the form factors in the present calculations is around 20%. Some of the major errors are the quenching error, chiral extrapolation error statistical error in all calculations. In addition, a large discretization error appears in JLQCD results and a large \(1/M\) extrapolation error is contained in APE and UKQCD results.

There are several proposals to improve the form factor determination. The quenching error can be resolved only by performing the unquenched calculations. Recently, JLQCD and UKQCD collaborations have accumulated \(n_f = 2\) unquenched lattice configurations with \(O(a)\)-improved Wilson fermions and \(n_f = 2 + 1\) unquenched configurations with improved staggered fermions have been produced by the MILC collaboration. These unquenched QCD data should be applied to form factor calculations.

In order to reduce the chiral extrapolation error, simulation with even smaller light quark masses are necessary. For Wilson type fermions, simulations with \(m_\pi < 0.4\) GeV will be very slow and also appearance of exceptional configuration may prevent the simulation for very light quark mass range. On the other hand, MILC collaboration is now carrying out simulations with \(m_\pi = 0.3 \sim 0.5\) GeV, which corresponds to \(m_q = 1/5m_\pi - 1/2m_\pi\) [9]. Since \(n_f = 2 + 1\) simulations are performed by taking the square root or quar-
tic root of the Dirac operator, one possible concern might be the flavor symmetry breaking effect which still exists in the improved stagger fermions. It would be important to understand how this flavor breaking affects distorts the chiral behavior. Although it is possible to simulate with very light quark mass without encountering any theoretical problems, the simulation cost at this stage is very high. A breakthrough in the algorithm is needed.

Using the heavy quark symmetry is another way for improvements. Since CLEO-c experiment can measure form factors for \( D \to \pi \ell v \) to a few percent accuracy, their results will be a good approximation for the \( B \to \pi \ell v \) form factors. Then the task for lattice QCD is to predict the \( 1/m_Q \) dependence of the form factors. The B/D ratio

\[
\frac{d\Gamma(B \to \pi \ell v)}{d\Gamma(D \to \pi \ell v)}
\]

with the same recoil energy \( v \cdot k_\pi \) would be a nice quantity to measure on the lattice, since a large part of the statistical error, the perturbative error and the chiral extrapolation errors are expected to cancel in this ratio.

### 2.2 QChPT and (PQ)ChPT

Becirevic et al. \[2\] made estimates of the quenching effect and the chiral extrapolation error from low energy effective meson theory, namely QChPT and (PQ)ChPT for quenched QCD and full QCD respectively. In their analysis the non perturbative low energy coupling constant are (1) \( a, g \) and \( f \) for \( f_B \) \( B' \pi \) coupling and \( f_\pi \), (2) \( L_4 \) and \( L_5 \) for the two \( O(p^3) \) terms, and (3) \( m_0 \) and \( g' \) for the two parameters of quenching effect. The parameters are estimated by collecting all the knowledge from the quenched lattice QCD simulation, the full QCD simulation, the experimental values and large \( N \) order estimation.

They found that the quenching errors \( Q_2 \equiv (f_2^{\text{full}} - f_2^{\text{quench}})/f_2^{\text{full}} \) and \( Q_{12} \equiv ((f_1 + f_2)^{\text{full}} - (f_1 + f_2)^{\text{quench}})/((f_1 + f_2)^{\text{full}}) \) are as large as 25%-50% depending on \( v \cdot k_\pi \) and \( g' \).

They also estimated the chiral extrapolation error by comparing the result with linear extrapolation and linear plus chiral log corrections for unquenched theory. They found that the linear extrapolation of \( m_{q^2} > 0.4 \) GeV data gives an overestimate of \( f_{\perp} \approx f_2 \) by 2-7% for \( v \cdot k_\pi = 0.19 \) GeV and 5-15% for \( v \cdot k_\pi = 0.54 \) GeV.

They proposed an extrapolation strategy in the partially quenched theory for \( m_{q^2} > 0.4 \) GeV simulation data in which one first takes a linear extrapolation of \( m_{valence} \) and then make an extrapolation in \( m_{valence} \) with a linear + log form. In this method they estimate that the chiral extrapolation error in unquenched theory in under 5% level.

### 2.3 \( q^2 \) dependence from the dispersive bound and LCSR

Model independent bounds for the whole \( q^2 \) range can be obtained with the dispersion relation, perturbative QCD, and the lattice QCD data \[4\]. Fig. 4 shows the pioneering result by Lellouch \[7\]. Reducing the lattice errors or having other inputs would significantly improve the results. More elaborate studies along this line would be important.

The light-cone QCD sum rule (LCSR) \[10\] \[11\] \[12\] can also give form factors for small \( q^2 \) region. The basic idea is to compute the following matrix element \( CF_v \)

\[
CF_v \equiv i \int d^4 \eta e^{i \eta \cdot (\pi(p))} \langle \pi(p) | T[\bar{q}(p) \gamma \mu b(p)](y) j_B(0)|0 \rangle,
\]
in two different methods and equate the results. The two computational methods are: (1) light cone expansion which is expressed by the pion light-cone wavefunction \( \phi_\pi(u) \) and (2) the dispersion relation which takes the following sum of the physical poles

\[
CF_\nu \sim \frac{m_B^2 f_B}{m_b} f^+(q^2) \left( \frac{1}{m_B^2 - p_B^2} \right) + \text{higher poles},
\]

where higher poles are suppressed by Borel transformation with \( M \) and approximated by the light-cone expansion results above a threshold \( s_0^2 \).

The theoretical input parameters are the parameters \( a_i \)'s of light-cone wavefunctions in the Gegenbauer polynomial expansion,

\[
\phi_i = 6u(1-u)[1 + a_2 C_2^{1/2}(2u - 1) + \cdots],
\]

the \( B \) meson decay constant \( f_B \), the b quark mass \( m_b \), the threshold \( s_0^2 \), and the parameter \( M \) for the Borel transformation.

We here give the new results of Ref. \[12\] as an example. It is found that the radiative correction is about 10% and the correction from higher twists (twist 3) is \( \sim 30\% \). The results for \( q^2 < 14 \text{ GeV}^2 \) are well fitted by

\[
f^+(q^2) = \frac{F(0)}{1 - a q^2/m_B^2 + b(q^2/m_B^2)^2}.
\]

Light-cone QCD sum rule results for \( q^2 < 14 \text{ GeV}^2 \) can also be fitted by the pole dominance ansatz.

\[
f^+(q^2) = \frac{c}{1 - q^2/m_B^2},
\]

where \( c = f_B g_{BB\pi}/(2m_B) = 0.414^{+0.016}_{-0.018} \) plus systematic errors.

Fig. 4 shows the result by the light-cone QCD sum rule. It is remarkable that the light-cone QCD sum rule give consistent results with lattice QCD.

### 3 \( B \to \rho \nu \)

Recently, UKQCD collaboration \[13\] and SPQcdR collaboration \[14\] started studies of \( B \to \rho \nu \) form factors. Both collaborations use \( O(a) \)-improved Wilson action for the heavy quark and extrapolate the numerical results of \( m_Q \to m_c \) towards the physical b quark mass. The lattice spacings are \( a^{-1} = 2.0 \) and \( 2.7 \) GeV for UKQCD and \( a^{-1} = 2.7 \) and \( 3.7 \) GeV for SPQcdR.

UKQCD fits the lattice data for \( q^2 > 14 \text{ GeV}^2 \) to the following form

\[
\frac{1}{\vert V_{cb} \vert^2} \frac{d\Gamma}{dq^2} = \frac{G_F^2 q^2 [\lambda(q^2)]^{1/2}}{192\pi^3 m_B^3} (a + b(q^2 - q_{\text{max}}^2)),
\]

The fit coefficients are \( a = 38^{+8}_{-5} \pm 5 \text{ GeV}^2 \) and \( b = 0 \pm 2 \pm 1 \), where the first error is statistical and the second is the extrapolation error for both \( a \) and \( b \).

SPQcdR collaboration obtains form factors for \( q^2 > 10 \text{ GeV}^2 \). They find the results which is consistent with the light-cone QCD sum rule results.

### 4 \( B \to D^{(*)} \nu \)

One can extract \( \vert V_{cb} \vert \) from the \( B \to D^{(*)} \nu \) semileptonic decay near zero recoil as

\[
\frac{d\Gamma}{d\omega}(B \to D^{(*)}) \propto \vert V_{cb} \vert^2 \vert \mathcal{F}_{B \to D^{(*)} \nu} \vert^2.
\]

where \( \omega \equiv v \cdot v' \) and \( \mathcal{F}_{B \to D^{(*)} \nu} \) are the linear combinations of form factors \( h_a, h_{A_{1,2,3}} \). One important outcome from the heavy quark symmetry is that the form factor \( \mathcal{F}_{B \to D^{(*)} \nu} \) is equal to unity at zero recoil up to perturbatively calculable
The deviation from unity in the double ratio only start from $1/m^{2}$ including systematic errors. The $B \rightarrow D^{*}$ result is

$$\mathcal{F}_{B \rightarrow D^{*}}(1) = 0.913^{+0.024}_{-0.017} \pm 0.016^{+0.003}_{-0.014} \pm 0.0006 \pm 0.016 - 0.014 \pm 0.014.$$  

where the first is statistical and the second is the error from $m_{b,c}$ and the third is the perturbative error.

UKQCD collaboration [16] evaluated the $\omega$ dependence of the form factors $B \rightarrow D^{(*)}l\nu$ from the 3 point functions in quenched QCD. Parameterizing the form factor as

$$h_f(\omega) \propto \xi(\omega) = 1 - \rho^2(\omega - 1) + \mathcal{O}(\omega - 1)^2$$

(14)

They obtain the slope

$$\rho^2 = 0.83^{+0.15+0.24}_{-0.11-0.01},$$

(15)

where the first and the second errors are statistical and systematic, respectively.

UKQCD collaboration [17] also computed the form factors $A_{b} \rightarrow A_{c}l\nu$. Gottlieb and Tamhankar [18] also made an exploratory study on he same form factors and obtained $\xi(\omega)$ for finite heavy quark mass. Both groups found that the heavy quark mass dependence is small.

5 Summary

$B \rightarrow \pi l\nu$ form factors are computed with 20% error for $q^2 > 18$ GeV$^2$. Different approaches show good agreements in the form factors for the same mass range of the heavy and the light quarks, although the extrapolation in $m_{q}$ and $1/M_{Q}$ gives deviations in $\rho^2$. Therefore, a better understanding of the quark mass dependence is required. Studies in QChPT and PQChPT suggests that the unquenching error may be quite large so that lattice results in unquenched QCD are required. Once unquenched lattice results will be available, studies in PQChPT suggest that chiral extrapolation errors can be controlled below 5% level. The form factor for smaller $q^2$ range can be obtained either by the dispersive bounds or Light-Cone QCD sum rule with the help of nonperturbative inputs. $B \rightarrow pl\nu$ form factors are in progress. It seems that the form factors in the range $q^2 > 14$ GeV$^2$ are feasible. $B \rightarrow D^{(*)}l\nu$ results are established in quenched QCD, whose extension to the unquenched calculation should be straightforward.

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