QCD Corrections and the Endpoint of the Lepton Spectrum in Semileptonic $B$ Decays

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Abstract

Recently, Neubert has suggested that a certain class of nonperturbative corrections dominates the shape of the electron spectrum in the endpoint region of semileptonic $B$ decay. Perturbative QCD corrections are important in the endpoint region. We study the effects of these corrections on Neubert’s proposal. The connection between the endpoint of the electron spectrum in semileptonic $B$ decay and the photon spectrum in $b \rightarrow s \gamma$ is outlined.

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I. INTRODUCTION

The electron energy spectrum near its endpoint in semileptonic $B$ meson decay arises from $b \rightarrow u$ transitions and provides one method for the extraction of the Kobayashi-Maskawa mixing angle $V_{ub}$ from experiment. The spectrum must be known accurately within a few hundred MeV of its endpoint, since it is only in this region that the large background due to the dominant $b \rightarrow c$ weak transition is kinematically forbidden. Thus, the separation of the rare $b \rightarrow u$ decay from the inclusive spectrum relies upon a theoretical understanding of the shape of the spectrum in this small region. Unfortunately, it is precisely this region which is the least well understood theoretically.

The endpoint region of inclusive semileptonic $B$ decay has been studied extensively. The first approaches relied on QCD models. Grinstein et al. \cite{1} used a constituent quark model to sum over exclusive charmless final states in this region, assuming that the spectrum is dominated by a few low-lying resonances. Altarelli et al. \cite{2} computed the spectrum in the free $b$ quark decay model, augmented by the inclusion of a model of the Fermi motion of the $b$ quark in the $B$ meson. More recently, a QCD-based approach has been formulated in the context of the heavy quark effective theory (HQET). Using an operator product expansion (OPE) and the HQET, Chay, Georgi and Grinstein \cite{3} have shown that the free $b$ quark decay model describes inclusive semileptonic $B$ decay to leading and first subleading order in a systematic expansion in $1/m_b$, where $m_b$ is the $b$ quark mass of the HQET. The first non-vanishing corrections to the free quark decay result are of order $1/m_b^2$, and have now been computed \cite{4,6}. These corrections arise from higher order terms in the OPE whose matrix elements contain information about the state of the $b$ quark inside the hadron.

At leading order, the electron spectrum is governed by quark kinematics with an endpoint at $E_e = m_b/2$, rather than at the physical endpoint $M_B/2$ which is determined by the $B$ meson mass $M_B$. The higher order terms in the $1/m_b$ expansion produce corrections to the free quark decay spectrum, causing it to “leak” beyond the free quark endpoint. Understanding this process is crucial for extracting $V_{ub}$, since the difference $(M_B - m_b)/2$ is
expected to be several hundred MeV, and is comparable to the 330 MeV energy difference between the $b \to u$ and $b \to c$ endpoints. Recently, Neubert has shown that the most singular terms in the $1/m_b$ expansion can be used to define a “shape function” of the spectrum, which is determined by a certain set of nonperturbative matrix elements and is model-independent [7]. This shape function describes the electron energy spectrum beyond the kinematic endpoint of the free quark decay (neglecting QCD radiative corrections). In this paper we examine the influence of perturbative QCD corrections on the endpoint region. These corrections are particularly important due to the presence of a Sudakov double-logarithmic suppression of the free quark decay rate at the endpoint.

This paper is organized as follows. In Section 2, we review the operator product expansion analysis of the differential decay width for the endpoint region, neglecting perturbative QCD radiative corrections. The summation of the leading nonperturbative singularities to the shape of the endpoint spectrum is presented. We show that this summation can be obtained from the free $b$ quark decay result by suitably averaging the free quark decay result over the residual momentum of the $b$ quark inside the $B$ meson. Since the leading nonperturbative corrections can be generated by this procedure, radiative corrections can be included by computing radiative corrections to the free quark decay result and then averaging over the residual momentum of the $b$ quark. In Section 3, we consider the radiative corrections to free quark decay and show how they modify the shape of the endpoint of the electron spectrum. Numerical results and conclusions are presented in Section 4.

II. LEADING NONPERTURBATIVE SINGULARITIES

The inclusive differential decay distribution for $B \to X_{u,c} e \overline{\tau}$ is determined by the imaginary part of the time-ordered product of two weak currents,

$$T^{\mu\nu} = -i \int d^4x e^{-iq\cdot x} \langle B | T \{ J^{\mu\dagger}(x), J^{\nu}(0) \} | B \rangle,$$

where $J^\mu = \overline{\tau} \gamma^\mu (1 - \gamma^5) b$ and $q = u, c$. The time-ordered product may be expanded in inverse powers of the $b$ quark mass using an operator product expansion [3], and in powers
of $\alpha_s(m_b)$. In this section we will concentrate on the $1/m_b$ expansion. From the operator product expansion of the hadronic tensor, one obtains an expression for the inclusive electron energy spectrum, $d\Gamma/dy$, where $y$ is the rescaled electron energy, $y = 2E_e/m_b$. The leading term in the $1/m_b$ expansion produces the result of the free quark decay model, in which the inclusive semileptonic decay rate is given by the decay of a free, on-shell $b$ quark. The endpoint of the electron spectrum is at $y = 1$. The subleading terms represent corrections to free quark decay, in which certain features of the motion of the $b$ quark inside the $B$ meson are taken into account. The expansion is in powers of

$$\epsilon = \Lambda/m_b,$$ 

(2.2)

where $\Lambda$ is a scale typical of the strong interactions of QCD, perhaps 300 to 500 MeV.

Neglecting perturbative $\alpha_s(m_b)$ corrections, the electron energy spectrum for $B \to X_u e \bar{\nu}$ decay is given by

$$\frac{1}{\Gamma_0} \frac{d\Gamma}{dy} = \left\{ 2(3-2y)y^2 + 4(3-y)y^2 E_b - \frac{4y^2(9+2y)}{3} K_b - \frac{4y^2(15+2y)}{3} G_b \right\} \theta(1-y)$$

$$+ \left\{ 2E_b - \frac{4}{3} K_b + \frac{16}{3} G_b \right\} \delta(1-y) + \frac{2}{3} K_b \delta'(1-y),$$

(2.3)

up to corrections of order $\epsilon^3$, where $\Gamma_0$ is the free quark decay width

$$\Gamma_0 = |V_{ub}|^2 \frac{G_F^2 m_b^5}{192\pi^3},$$

(2.4)

and $\theta(x)$ is 1 if $x > 0$ and zero otherwise. $E_b$, $K_b$ and $G_b$ are hadronic matrix elements of order $\epsilon^2$, defined by

$$E_b = G_b + K_b,$$

$$K_b = \langle B(v) | \bar{b}_v \frac{D^2}{2m_b^2} b_v | B(v) \rangle,$$

$$G_b = \langle B(v) | \bar{b}_v g_{\alpha\beta} G^{\alpha\beta} b_v | B(v) \rangle,$$

(2.5)

*$\text{Eq. (2.3) holds for massless leptons. Lepton mass effects may be included, but they do not change the behavior of the endpoint spectrum in an important way.}$
where \( b \) is the \( b \) quark field in the HQET. The factor of \( \theta(1 - y) \) in the first term is required because the tree level decay distribution does not vanish at the boundary of the Dalitz plot. The \( \delta(1 - y) \) and \( \delta'(1 - y) \) singularities arise because some higher order terms in the \( 1/m_b \) expansion have the form of derivatives with respect to \( y \) of lower order terms. Since the free quark decay distribution does not vanish at the endpoint, this generates singular terms in the decay spectrum. These singularities imply that the \( 1/m_b \) expansion breaks down at \( y = 1 \).

Eq. (2.3) is the decay spectrum including all corrections of order \( 1/m_b^2 \). To all orders in \( 1/m_b \), the decay spectrum \( d\Gamma/dy \) obtained from the OPE at zeroth order in \( \alpha_s \) has the structure

\[
\frac{1}{\Gamma_0} \frac{d\Gamma}{dy} = \theta(1 - y) \left( \epsilon^0 + 0\epsilon + \epsilon^2 + \cdots \right) + \delta(1 - y) \left( 0\epsilon + \epsilon^2 + \cdots \right) + \delta'(1 - y) \left( \epsilon^2 + \epsilon^3 + \cdots \right) + \cdots + \delta^{(n)}(1 - y) \left( \epsilon^{n+1} + \epsilon^{n+2} + \cdots \right) + \cdots, \tag{2.6}
\]

where \( \epsilon^n \) denotes a term of that order, which may include a smooth function of \( y \). It is a nontrivial prediction of the heavy quark effective theory that the terms proportional to \( \epsilon \) in this expansion vanish [3], as is evident in eq. (2.3). Although the theoretical expression for \( d\Gamma/dy \) is singular at the endpoint \( y = 1 \), the total semileptonic width is not. The contribution to the total rate of a term \( \epsilon^m \delta^{(n)}(1 - y) \) is of order \( \epsilon^m \), so the semileptonic width has a well-behaved expansion in powers of \( 1/m_b \),

\[
\Gamma = \Gamma_0 \left( 1 + 0\epsilon + \epsilon^2 + \epsilon^3 + \cdots \right), \tag{2.7}
\]

where the term proportional to \( \epsilon \) vanishes.

The semileptonic decay width for \( b \to u \) is difficult to measure because of background contamination from the dominant \( b \to c \) semileptonic decays. It is therefore important to be able to compute the semileptonic decay rate for \( b \to u \) transitions near the endpoint \( y = 1 \), since the kinematic endpoint of the \( b \to c \) spectrum is below the \( b \to u \) endpoint. One way to calculate the endpoint spectrum is to weight the differential distribution \( d\Gamma/dy \) by a normalized function of width \( \sigma \) around \( y = 1 \). We will refer to this procedure as “smearing.”
Most of the details of the smearing procedure are unimportant; the only quantity of relevance is the width $\sigma$ of the smearing region. A physically meaningful result can be obtained by smearing over a large enough region in $y$ such that the singular corrections to $d\Gamma/dy$ are small. In ref. [4], it was shown that the singular corrections are small if the smearing width is chosen so that $\sigma \gg \epsilon$. We will now show that by summing the leading singularities, one can choose $\sigma$ of order $\epsilon$.

The singular distribution $\epsilon^m\delta^{(n)}(1 - y)$ (where $m > n$) smeared over a region of width $\sigma$ gives a contribution of order $\epsilon^m/\sigma^{n+1}$ to $d\Gamma/dy$. If the width $\sigma$ of the smearing region is of order $\epsilon^p$, the generic term $\epsilon^m\delta^{(n)}(1 - y)$ yields a contribution of order $\epsilon^{m-(n+1)p}$. Since $m > n$, this shows that the $1/m_b$ expansion for the spectrum breaks down unless $p \leq 1$, i.e. the smearing region cannot be made narrower than of order $\epsilon$. If $p > 1$, the $1/m_b$ expansion breaks down because it is dominated by an infinite number of terms at large values of $n$. This divergence is not associated with the failure of the OPE due to the presence of resonances with masses of order the QCD scale [5]. The region in which such resonances dominate the final state is of width $\epsilon^2$, while the expansion breaks down upon smearing over any region of size $\epsilon^{1+\delta}$, where $\delta > 0$.

If the smearing region is chosen to be of order $\epsilon$, the form of the expansion (2.6) shows that the leading terms of the form $\theta(1 - y)$ and $\epsilon^{n+1}\delta^{(n)}(1 - y)$ all contribute at order unity to $d\Gamma/dy$, all terms of the form $\epsilon^{n+2}\delta^{(n)}(1 - y)$ contribute at order $\epsilon$, etc. Thus one can, in principle, obtain the decay spectrum smeared over a width of order $\epsilon$ if one can sum the leading singularities in eq. (2.6). The sum of the leading singularities produces a distribution $d\Gamma/dy$ of width $\epsilon$, and with a height of the same magnitude as the free quark decay distribution for $d\Gamma/dy$, i.e. with a height of order one. The subleading singularities produce a distribution which is also of width $\epsilon$, but has a height of order $\epsilon$ times the distribution obtained by summing the leading singularities. The decay distribution $d\Gamma/dy$ cannot be obtained with a resolution finer than $\epsilon$ without summing all the subleading singularities.

Neubert has shown that the series of leading singularities
\[
\frac{1}{\Gamma_0} \frac{d\Gamma}{dy} = A_0 \theta(1 - y) + 0 \epsilon \delta(1 - y) + A_2 \epsilon^2 \delta'(1 - y) + \cdots
\]  

(2.8)

may be resummed into a “shape function”, which describes the behavior of the theoretical spectrum in the region beyond the free quark decay endpoint at \( y = 1 \) [7]. These terms arise in a particularly simple way in the OPE, because they come only from the expansion of the quark propagator which connects the two currents. The shape function has a width of order \( \epsilon \) and height of order one.

The series of leading singularities (2.8) can be obtained by averaging the free quark decay result over the residual momentum of the \( b \) quark in the \( B \) meson [4]. This simple procedure is important since it will also enable us to obtain the leading nonperturbative singularities for the radiative corrections by only calculating radiative corrections to free quark decay.

The differential decay distribution is obtained from the tensor \( T^{\mu\nu} \) defined in eq. (2.1). This tensor is a function of the momentum transfer to the leptons, \( q \), and the velocity of the \( B \) meson, \( v \). The differential decay distribution is proportional to the hadronic tensor contracted with the lepton tensor \( L_{\mu\nu} \), which depends on the electron and neutrino momenta, \( k_e \) and \( k_\nu \):

\[
\frac{d\Gamma}{dx \ dy \ dq^2} \propto W^{\mu\nu} L_{\mu\nu} ,
\]

(2.9)

where \( W^{\mu\nu} \) is the discontinuity of \( T^{\mu\nu} \) across the physical cut, \( W^{\mu\nu} = -\text{Im} \ T^{\mu\nu}/\pi \). The constant of proportionality in eq. (2.9) involves \( G_F^2 \) and the mixing angle \( |V_{ub}|^2 \). The dimensionless variables \( x \), \( y \) and \( q^2 \) are defined by

\[
x = \frac{2k_\nu \cdot v}{m_b}, \quad y = \frac{2k_e \cdot v}{m_b}, \quad q^2 = \frac{q^2}{m_b^2} .
\]

(2.10)

The lowest order (in \( 1/m_b \)) decay distribution \( d\Gamma_{\text{tree}}/dx \ dy \ dq^2 \) is the decay distribution for a free on-shell \( b \) quark with mass \( m_b \) and the same velocity \( v \) as the \( B \) meson. However, the \( b \) quark in the \( B \) meson is off-shell with a distribution of residual momentum \( k \). The off-shell \( b \) quark, with momentum \( m_b v + k \), may be viewed as an on-shell quark with mass \( m'_b \) and velocity \( v' \), where \( m'_b v' = m_b v + k \). The decay rate for such a quark is obtained by
evaluating the lowest order expression for \( d\Gamma_{\text{free}}/dy \) in the rest frame of the moving quark, and then boosting back to the rest frame of the \( B \) meson,

\[
d\Gamma = \frac{1}{v \cdot v'} d\Gamma_{\text{free}}(x', y', \hat{q}^2, m'_b).
\] (2.11)

Note that all scaled quantities depend implicitly on \( m_b \), and hence must be primed. We now replace \( m'_b v' \rightarrow m_b v + k \) and average over the residual momentum \( k^\mu \). Expanding in \( k^\mu/m_b \), we obtain a series of the form [4]

\[
\langle d\Gamma \rangle = \langle 1 + 2v \cdot k/m_b + k^2/m_b^2 \rangle^{1/2} \left[ 1 + k^\mu_1 \frac{\partial}{\partial m_b v^\mu_1} + \frac{1}{2} k^\mu_1 k^\mu_2 \frac{\partial}{\partial m_b v^\mu_1} \frac{\partial}{\partial m_b v^\mu_2} + \ldots \right] d\Gamma_{\text{free}}^{\text{free}}(x', y', \hat{q}^2, m'_b),
\] (2.12)

where \( \langle \cdot \rangle \) denotes an average with respect to the distribution of the momentum \( k \) of the \( b \) quark in the \( B \) meson. The derivatives with respect to \( m_b v^\mu \) can be rewritten as derivatives with respect to \( x, y \) and \( \hat{q}^2 \) using the chain rule. Terms with \( n \) derivatives with respect to \( m_b v^\mu \) in eq. (2.12) turn into terms with \( n_x, n_y \) and \( n_q \) derivatives with respect to \( x, y \) and \( \hat{q}^2 \) respectively, where \( n_x + n_y + n_q \leq n \). The expansion of \( d\Gamma/dy \) is then obtained by integrating the expansion of \( d\Gamma/dx \, dy \, d\hat{q}^2 \) with respect to \( x \) and \( \hat{q}^2 \). The explicit computations to order \( 1/m_b^2 \) are given in ref. [4].

In this paper, we are interested in summing the most singular terms in \( d\Gamma/dy \) near \( y = 1 \) to all orders in \( 1/m_b \). These terms are found by retaining the terms in eq. (2.12) with the maximum number of \( y \)-derivatives at each order in \( 1/m_b \). This corresponds to only retaining the \( \partial^n/\partial y^n \) term in \( \partial^n/\partial m_b v^\mu_1 \ldots \partial m_b v^\mu_n \) in eq. (2.12) and ignoring the prefactor \( [1 + 2v \cdot k/m_b + k^2/m_b^2]^{1/2} / [1 + v \cdot k/m_b] \). Terms with derivatives with respect to \( x \) or \( \hat{q}^2 \) do not generate derivatives with respect to \( y \) on integration over \( x \) and \( \hat{q}^2 \), and are less singular than the terms we have retained. The most singular terms are thus obtained using

\[
\left( \frac{\partial}{\partial m_b v^\mu} \right)^n \rightarrow \left( \frac{\partial y}{\partial m_b v^\mu} \frac{\partial}{\partial y} \right)^n \rightarrow \left( \frac{2}{m_b} \left( \hat{k}_v - y v^\mu \right) \frac{\partial}{\partial y} \right)^n \xrightarrow{y=1} \left( \frac{2}{m_b} \left( \hat{k}_v - v \right) \mu \frac{\partial}{\partial y} \right)^n,
\] (2.13)

which gives the leading singularities,
\[
\frac{d\Gamma}{dy} = \frac{d\Gamma_{\text{free}}}{dy} + \langle k^{\mu_1} \rangle \left( \frac{2}{m_b} \right) (\hat{k}_e - v)_{\mu_1} \frac{\partial}{\partial y} \left( \frac{d\Gamma_{\text{free}}}{dy} \right) + \cdots + \frac{1}{n!} \langle k^{\mu_1} \cdots k^{\mu_n} \rangle \left( \frac{2}{m_b} \right)^n (\hat{k}_e - v)_{\mu_1} \cdots (\hat{k}_e - v)_{\mu_n} \frac{\partial^n}{\partial y^n} \left( \frac{d\Gamma_{\text{free}}}{dy} \right) + \cdots
\]

\[
= \sum_{n=0}^{\infty} \frac{2^n}{m_b^n n!} (\hat{k}_e - v)_{\mu_1} \cdots (\hat{k}_e - v)_{\mu_n} \langle k^{\mu_1} \cdots k^{\mu_n} \rangle \frac{\partial^n}{\partial y^n} \left( \frac{d\Gamma_{\text{free}}}{dy} \right),
\]

where \( \hat{k}_e = k_e/m_b \). Eq. (2.14) sums the leading nonperturbative corrections in the endpoint region, provided one interprets the residual momentum \( k \) in eq. (2.9) as the operator \( iD \) and the average as the expectation value of the resulting operator in the \( B \)-meson state. There is no operator ordering ambiguity for the leading singularity in this identification, because \( D^{\mu_1} \cdots D^{\mu_n} \) is contracted with the completely symmetric tensor \( (\hat{k}_e - v)^{\mu_1} \cdots (\hat{k}_e - v)^{\mu_n} \), and so the commutator \([D^\mu, D^\nu]\) does not contribute. Only the part of the matrix element \( \langle B(v)|iD^{\mu_1} \cdots iD^{\mu_n}|B(v)\rangle \) proportional to the tensor structure \( v^{\mu_1} \cdots v^{\mu_n} \) contributes to the most singular terms, since \( (\hat{k}_e - v)^2 \) vanishes at \( y = 1 \). Neglecting perturbative \( \alpha_s(m_b) \) radiative corrections, the most singular terms in eq. (2.14) are \( \delta \)-functions and their derivatives, which arise from differentiating the factor of \( \theta(1 - y) \) in \( d\Gamma_{\text{free}}/dy \). Dropping the \( n = 0 \) term in eq. (2.14) and allowing the derivatives to act only on the \( \theta \)-function gives Neubert’s shape function

\[
S(y) = \sum_{n=1}^{\infty} \frac{2^n}{m_b^n n!} (\hat{k}_e - v)_{\mu_1} \cdots (\hat{k}_e - v)_{\mu_n} \langle B(v)|iD^{\mu_1} \cdots iD^{\mu_n}|B(v)\rangle \frac{\partial^n}{\partial y^n} \theta(1 - y).
\]

This procedure for averaging over residual momentum produces the same result for the leading singularities as the operator product expansion. As discussed in ref. [4], one can use reparameterization invariance [9] to show that averaging over residual momentum gives the same answer as the OPE, provided one neglects the commutator \([D^\mu, D^\nu]\) and higher dimension operators involving light quark fields. The commutator and higher dimension operators do not contribute to the most singular terms, and so averaging over residual momentum will be adequate for this discussion.

It is simple to understand how this averaging procedure generates a shape function which extends beyond the free quark decay endpoint. If the energy of the \( b \) quark is allowed to
fluctuate from its on-shell value, occasionally it will have an energy larger than its free value \( m_b \). This fluctuation corresponds to a situation in which the quark has temporarily absorbed some energy from the light degrees of freedom in the \( B \) meson; if it decays weakly at this moment, then an energy \( E_e > m_b/2 \) may be given to the electron.

### III. RADIATIVE CORRECTIONS

The advantage of the averaging procedure for obtaining the leading nonperturbative singularities as \( y \to 1 \) is that it generalizes straightforwardly to the case when radiative corrections are included. The averaging procedure applied to the free quark decay distribution including radiative corrections yields the leading nonperturbative singularities including radiative corrections.

The one-loop QCD contribution to the free quark decay process, including both virtual gluons and real gluon emission, has been computed [10]. The corrected electron spectrum takes the form

\[
\frac{d\Gamma_{\text{free}}}{dy} = \frac{d\Gamma_0}{dy} \left[ 1 - \frac{2\alpha_s}{3\pi} G(y, \hat{m}_q) + O(\alpha_s^2) \right],
\]

where \( \Gamma_0 \) is the tree level free quark decay rate. Perturbative QCD corrections do not extend the electron spectrum beyond the free quark decay endpoint \( y = 1 \). This can only occur because of the nonperturbative \( 1/m_b \) corrections discussed in the preceding section. In the interesting case \( \hat{m}_q = 0 \) relevant to the transition \( B \to X_u e \bar{\nu} \), \( G(y, 0) \) is given by [10,11]

\[
G(y, 0) = G(y) = \ln^2(1 - y) + \frac{31}{6} \ln(1 - y) + \pi^2 + \frac{5}{4} + (\text{vanishing as } y \to 1). \tag{3.2}
\]

The leading singularity at each order in perturbation theory is proportional to \( \alpha_s^n \ln^{2n}(1 - y) \). These singularities lead to a breakdown of the perturbative QCD expansion near the endpoint \( y = 1 \), unless they can be summed. The double logarithms have been shown to exponentiate [12], yielding an expression which formally has the structure

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\[
\frac{d\Gamma_{\text{free}}}{dy} = R(y) \frac{d\Gamma_0}{dy},
\]  

(3.3)

where

\[
R(y) = \exp \left\{ -\frac{2\alpha_s}{3\pi} \ln^2(1-y) \right\}.
\]  

(3.4)

This is the form for the decay spectrum used by Altarelli et al. [2]. The Sudakov form factor \( R(y) \) causes the electron spectrum to vanish at the free quark endpoint \( y = 1 \).

The contribution to the endpoint shape of the electron energy spectrum coming from the exponentiated double-logarithm in \( R(y) \) is a calculable effect. One might hope that once this leading radiative correction has been accounted for, it would be consistent to include the leading higher dimension operators using eq. (2.14) and neglect all subleading radiative corrections. However, we find that for very large \( m_b \) this is not the case; the perturbative expansion is so poorly behaved at large orders in \( \alpha_s(m_b) \) that it is necessary to sum an infinite number of infinite series before including nonperturbative effects with eq. (2.14). Nevertheless, for the case of interest \( m_b \approx 4.5 \text{ GeV} \), neglecting the subleading radiative corrections may provide a reasonable approximation for the endpoint of the electron energy spectrum.

Before analyzing the general structure of the radiative corrections, it is instructive to consider a simple example which illustrates the importance of subleading radiative corrections. Consider the order \( \alpha_s \) correction given in eq. (3.1). This correction has \( \ln^2(1-y) \) and \( \ln(1-y) \) singularities as \( y \to 1 \). The \( \ln^2(1-y) \) singularity is summed into the Sudakov form factor \( R(y) \), leaving the subleading \( \ln(1-y) \) singularity. This subleading logarithmic singularity must also be understood in order to determine the effect of radiative corrections on the endpoint energy spectrum [13]. To see this, note that it is possible to write two different expressions for the decay spectrum which contain the same Sudakov leading singularity, but which have very different behaviors as \( y \to 1 \). The first expression is the conventional definition

\[
\frac{d\Gamma_{\text{free}}}{dy} = R(y) \frac{d\Gamma_0}{dy} \left[ 1 - \frac{2\alpha_s}{3\pi} \tilde{G}(y) \right],
\]

(3.5)
where

\[ \bar{G}(y) = G(y) - \ln^2(1 - y). \]  

(3.6)

However, one can also rewrite the decay spectrum as

\[ \frac{d\Gamma_{\text{free}}}{dy} = \frac{d\Gamma_0}{dy} \left[ R(y) - \frac{2\alpha_s}{3\pi} \bar{G}(y) \right], \]  

(3.7)

which is equally valid to order \( \alpha_s \). The two expressions (3.5) and (3.6) have the same \( \ln^2(1 - y) \) singularity as \( y \to 1 \), but differ in the subleading terms. The first expression (3.5) vanishes as \( y \to 1 \), whereas eq. (3.6) diverges as \( y \to 1 \). Thus, the exact form of the subleading singularity is required in order to determine the shape of the spectrum very near the endpoint.

We will now demonstrate that in the limit \( m_b \to \infty \), summing the most singular \( 1/m_b \) corrections with eq. (2.14) cannot be used to improve the behavior of the electron spectrum near \( y = 1 \) without first summing an infinite number of subleading perturbative QCD singularities. In a schematic notation in which we include only the powers of \( \alpha_s \) and \( \ln(1 - y) \), the radiative corrections near \( y = 1 \) have the structure

\[ 
1 + \alpha_s \ln^2(1 - y) + \alpha_s \ln(1 - y) + \alpha_s \\
+ \alpha_s^2 \ln^4(1 - y) + \alpha_s^2 \ln^3(1 - y) + \alpha_s^2 \ln^2(1 - y) + \alpha_s^2 \ln(1 - y) + \alpha_s^2 \\
+ \alpha_s^3 \ln^6(1 - y) + \alpha_s^3 \ln^5(1 - y) + \alpha_s^3 \ln^4(1 - y) + \alpha_s^3 \ln^3(1 - y) + \cdots \\
+ \cdots.
\]  

(3.8)

The first column, containing terms of the form \( \alpha_s^n \ln^{2n}(1 - y) \), exponentiates into the Sudakov factor \( R(y) \), after which the most singular terms remaining are of order \( \alpha_s^n \ln^{2n-1}(1 - y) \). We may write the \( m \)th column of the expansion (3.8) as an infinite series of the form

\[ C_m(y) = \sum_{n=[m/2]}^{\infty} b_{mn} \alpha_s^n \ln^{2n-m+1}(1 - y). \]  

(3.9)
The series of leading singularities corresponds to \( m = 1 \); for this case, and only this case, the coefficients

\[
b_{1n} = \frac{1}{n!} \left( -\frac{2}{3\pi} \right)^n
\]  

(3.10)

have been computed for all \( n \), and the sum \( C_1(y) \) is \( R(y) \). The series \( C_m(y) \) for \( m > 1 \) represent an infinite set of infinite series, for which the behavior of the coefficients \( b_{mn} \) for large \( n \) is not known.

The unknown subleading series \( C_m(y), m > 1 \), limit the accuracy with which one can determine the electron energy spectrum. For perturbation theory to be valid, one has to remain in a region in which all the subleading terms are small, since their structure is not known, i.e. all the terms beyond the first column of eq. (3.8) must be small. This condition requires that \( \alpha_s^n \ln^{2n-m}(1 - y) \ll 1 \) for all \( n \) and all \( m > 1 \), or that \( \alpha_s \ll 1 \) and

\[
\alpha_s \ln^2(1 - y) < 1,
\]

(3.11)

which is the condition required for \( n \to \infty \) with \( m \) fixed. If eq. (3.11) is satisfied, the first column sums to \( R(y) \), the second column is of order \( \sqrt{\alpha_s} \) times the first column, the third column is of order \( \sqrt{\alpha_s} \) times the second column, and so on. The condition (3.11) has converted the QCD perturbation series in eq. (3.8) into an expansion in \( \sqrt{\alpha_s} \). Summing all the leading singularities \( \alpha_s^n \ln^{2n}(1 - y) \), or summing any finite number of columns of eq. (3.8), does not increase the region of validity of the perturbation expansion, since the condition that the next column be small is still eq. (3.11). To increase the region of validity of the perturbative expansion, one must sum all the terms of the form \( \alpha_s^n \ln^{2n-m}(1 - y) \) for \( 0 \leq m \leq \lambda n \) and \( \lambda > 0 \), in which case one needs only the restriction \( \alpha_s \ln^{2-\lambda}(1 - y) < 1 \). That is, one must sum all the terms in (3.8) below a line which makes an angle \( \tan^{-1} \lambda \) with the vertical, which implies that one must sum a large number of subleading logarithms at high orders in perturbation theory.

At present, only the sum \( C_1(y) \) of the first column is known, so the condition for the subleading QCD radiative corrections to be small is that given by eq. (3.11). To determine the restriction on \( y \), we use eq. (3.11) in the form
\[ \frac{\alpha_s(m_b)}{\pi} \ln^2(1 - y) < 1, \] (3.12)

where we have noted that the perturbation series is really in \( \alpha_s/\pi \) rather than \( \alpha_s \). The condition for reliability of the QCD radiative corrections is

\[ 1 - y > e^{-\sqrt{\pi/\alpha_s}}. \] (3.13)

For very heavy quarks, this corresponds to a region that is much larger than the smearing width \( \epsilon \) of the \( 1/m_b \) corrections. To see this, take the limit \( m_b \to \infty \) with \( \alpha_s \) at high energies held fixed, i.e. \( \alpha_s(m_b) = 6\pi/[(33 - 2n_f) \ln m_b/\Lambda_{\text{QCD}}] \) \( (n_f = 5) \). Define the parameter \( t \) by

\[ \ln m_b/\Lambda_{\text{QCD}} = t^2. \]

Then eq. (3.13) becomes

\[ 1 - y > e^{-t\sqrt{23/6}}, \] (3.14)

whereas \( \epsilon \sim e^{-t^2} \). For large quark masses (large \( t \)), \( \epsilon \) is much smaller than the restriction (3.14) on \( 1 - y \).

As we have already noted, the residual momentum averaging procedure discussed in Sect. 3 can be applied to the QCD corrected free quark decay spectrum. This procedure yields the leading \( 1/m_b \) singularities to all orders in \( \alpha_s \). When radiative corrections are neglected, we have shown that the leading \( 1/m_b \) singularities smear the decay spectrum by a width of order \( \epsilon \). Thus, a reliable determination of the decay spectrum near the endpoint requires knowing the lowest order (in \( 1/m_b \)) spectrum at least within a distance \( \epsilon \) of the endpoint. Eq. (3.13) implies that in the \( m_b \to \infty \) limit, the lowest order spectrum with perturbative QCD corrections is not known in a region near the endpoint which is much larger than \( \epsilon \).

\[ \alpha_s(m_b) \]

‡The leading singularities arise when the derivatives in eq. (2.14) act on the Sudakov suppression factor \( R(y) \), not on the \( \theta \)-functions as in eq. (2.15).
IV. NUMERICAL RESULTS AND CONCLUSIONS

The radiative corrections become large in a region given by eq. (3.13) which is much larger than \( \epsilon \) in the limit \( m_b \to \infty \). However, for a large but finite quark mass it is possible that one is in a regime where \( e^{-\sqrt{\pi/\alpha_s}} \) is not much larger than \( \epsilon \). For example, for \( m_b = 4.5 \) GeV, we find \( \alpha_s(m_b) \sim 0.2 \) and \( e^{-\sqrt{\pi/\alpha_s}} \sim 0.02 \). This value of \( 1 - y \) corresponds to a smearing width of approximately 50 MeV, which is smaller than \( \epsilon \). Whether this crude estimate is valid depends critically on the size of the coefficients of the subleading terms in the second and higher columns of eq. (3.8). For example, the subleading \( \ln(1 - y) \) and constant terms in \( G(y) \) (see eq. (3.2)) give

\[
\left(\frac{2\alpha_s}{3\pi}\right) \left[\left(\frac{31}{6}\right) \ln(1 - y) + \pi^2 + \frac{5}{4}\right] \approx -0.1, \tag{4.1}
\]

when the electron energy is 200 MeV away from the free quark endpoint \( y = 1 \).

Eq. (4.1) indicates that it may be a good approximation to include the effects of perturbative QCD corrections on the shape of the endpoint region using for \( d\Gamma_{\text{free}}/dy \) the free \( b \)-quark decay rate including the leading QCD double logarithms with eq. (3.3). The smallness of eq. (4.1) arises from a cancellation between the \( \ln(1 - y) \) and constant terms in \( G(y) \). Each of these separately is not particularly small. Thus we are not completely confident that higher order perturbative corrections are negligible. It may be possible to sum the second column of eq. (3.8) using the methods developed in [14]. Such a summation would provide useful information on the importance of higher order QCD corrections.

The shape of the endpoint region of the electron spectrum depends on the matrix elements \( \langle B(v) | iD^{\mu_1} \ldots iD^{\mu_n} | B(v) \rangle \). Neubert estimates these matrix elements using a quark model for the \( B \) meson [7]. Eventually, these matrix elements can be determined directly from experiment. For example, the same matrix elements occur in the \( 1/m_b \) corrections to semileptonic \( b \to c \) decay and in the decay \( b \to s\gamma \) [15]. Thus a precise measurement of the electron spectrum in \( b \to c \) semileptonic decay and the photon energy spectrum in \( b \to s\gamma \). The
order $\alpha_s$ radiative corrections have also been computed for $b \rightarrow s\gamma$ \cite{10}. Let $x_\gamma = 2E_\gamma/m_b$, and define

$$F(y) = \int_y^1 \frac{d\Gamma}{dx_\gamma} dx_\gamma,$$

(4.2)

where $d\Gamma/dx_\gamma$ is the inclusive photon energy spectrum in $b \rightarrow s\gamma$, neglecting the strange quark mass. $F(y)$ for $b \rightarrow s\gamma$ and $d\Gamma/dy$ for $b \rightarrow u$ semileptonic decays have the same $\ln^2(1 - y)$ but different $\ln(1 - y)$ singularities as $y \rightarrow 1$. However, $F(y)$ for $b \rightarrow s\gamma$ and $d\Gamma/dy$ for $c \rightarrow d$ semileptonic decays do have the same $\ln^2(1 - y)$ and $\ln(1 - y)$ singularities as $y \rightarrow 1$ in the order $\alpha_s$ radiative corrections.

The methods in this paper and ref. \cite{7} for describing the endpoint region of the electron spectrum apply when the endpoint is dominated by many states with masses of order $\sqrt{m_b \Lambda_{QCD}}$. However, in the non-relativistic constituent quark model estimate of ref. \cite{1}, the region beyond the $B \rightarrow X_c e\bar{\nu}_e$ endpoint is dominated by the single decay mode $B \rightarrow \rho e\bar{\nu}_e$. If $\rho$ dominance is found to hold experimentally, then the sum of the leading singularities is not a valid description of the endpoint in a region which is as small as the difference between the $B \rightarrow X_c e\bar{\nu}_e$ and $B \rightarrow X_u e\bar{\nu}_e$ endpoints.

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