Horizontal normal map on the Heisenberg group

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Abstract

We investigate the notion of H–subdifferential and H–normal map of a function \( u \) on the Heisenberg group, based on the sub–Riemannian structure of \( H \). We show that some properties of the subdifferential in the Euclidean setting are inherited. In particular, a characterization of the convexity of a function is given via the nonemptiness of the H–subdifferential \( \partial_H u(g) \) at every point \( g \). Concerning the H–normal map, we prove a monotonicity result when suitable strictly convex radial functions are considered. Finally, we suggest a definition of the Monge–Ampèrè measure of a function \( u \) via its H–normal map, and we extend a well–known integration result by Rockafellar.

Key words: Heisenberg group, convex function, horizontal subdifferential, horizontal normal map, Monge–Ampèrè measure, Rockafellar function

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1 Introduction

The notion of subdifferential and normal map for a function defined on Euclidean spaces, or, more generally, in Banach spaces, is a classical concept (see, for instance, [22], [18]). The most interesting features concern the properties of the subdifferential arising from convex functions. Indeed, in this case, it enjoys some more interesting properties, among them its uniqueness at a point characterizes the differentiability of the function at the same point.

The notion of normal map has been exploited in order to define the weak solutions to the Monge–Ampère equation \( \det D^2 u = f \) (see, for instance, [12] and the references therein). In particular, a pointwise estimate for convex functions, the Alexandrov’s maximum principle, is of great importance in the theory of weak solutions for the Monge–Ampère equation; its proof relies on a monotonicity property of the normal map of convex functions, and it is based on geometric features of their graphs.

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A natural question rises whether similar properties and maximum principle results can be stated in the more general setting of Carnot groups or, more specifically, of the Heisenberg group.

Many papers have been devoted to the study of different types of convexity in Carnot groups. Seemingly, the most interesting and fruitful notion of convexity is the so–called weakly H–convexity (see [6], [17]). In [6], in order to deal with the horizontal version of the Monge–Ampère operator in a Carnot group, the authors worked out a notion of horizontal subgradient and of horizontal normal map of a function \( u \) that are strictly related to the definition of weak H–convexity; in the paper, we will refer to weak H–convexity simply as to convexity.

In [14], the authors investigated a Monge–Ampère type operator on the Heisenberg group, but they followed a different route with respect to the Euclidean framework (see [12]), expressing their doubts about the existence of a suitable definition of normal map in \( I_H \) useful to state maximum comparison results.

The aim of this paper is to shed some more light about properties and potential of the normal map of a function in the Heisenberg setting.

In Section 2 we provide the main definitions, while in Section 3 we define the H–subdifferential and the associated H–normal map for a function \( u : I_H \to \mathbb{R} \), that takes into account the sub–Riemannian structure of the Heisenberg group.

In Section 4 we state some results concerning the H–subdifferential of a convex function on \( I_H \); in particular, we show that, for convex functions, the H–subdifferential is nonempty at every point. Moreover, the uniqueness of the H–subdifferential at a point \( g_0 \) is equivalent to the existence of \( \mathcal{X} u(g_0) \).

The main purpose of Section 5 is the investigation of the normal map \( \partial_H u \); in particular, we are interested in those properties of this map that are inherited from the corresponding properties of the map in Euclidean spaces. We show that the image of compact subsets of \( I_H \) under the map \( \partial_H u \) are compact subsets of \( V_1 \). Furthermore, we prove a monotonicity result for the H–normal map of strictly convex, radial functions satisfying an additional assumption.

In Section 6, in order to show how the H–subdifferential of a function \( u \) carries much information about the function itself, we suggest a definition of the Monge–Ampère measure of \( u \) via its H–normal map, and we prove an extension of a well–known integration result due to Rockafellar.

2 Basic notions

The Heisenberg group \( I_H = I_H^1 \) is the Lie group given by the underlying manifold \( I^3 \) with the non commutative group law

\[
gg' = (x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')).
\]

The unit element is \( e = (0, 0, 0) \), and the inverse of \( g = (x, y, t) \) is \( g^{-1} = (-x, -y, -t) \). Left translations and anisotropic dilations are, in this setup, \( L_{g_0}(g) = g_0g \) and \( \delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t) \).
The differentiable structure on $\mathbb{H}$ is determined by the left invariant vector fields
\[ X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad \text{with} \quad [X,Y] = -4T. \]

The vector field $T$ commutes with the vector fields $X$ and $Y$; $X$ and $Y$ are called horizontal vector fields.

The Lie algebra of $\mathbb{H}$, $\mathfrak{h}$, is the stratified algebra $\mathfrak{h} = \mathbb{R}^3 = V_1 \oplus V_2$, where $V_1 = \text{span} \{X,Y\}$, $V_2 = \text{span} \{T\}$; $\langle \cdot , \cdot \rangle$ will denote the inner product. Via the exponential map $\exp$ we identify the vector $\alpha X + \beta Y + \gamma T$ in $\mathfrak{h}$ with the point $(\alpha,\beta,\gamma)$ in $\mathbb{H}$; the inverse $\xi : \mathbb{H} \to \mathfrak{h}$ of the exponential map has the unique decomposition $\xi = (\xi_1, \xi_2)$ with $\xi_i : \mathbb{H} \to V_i$.

The main issue in the analysis of the Heisenberg group is that the classical first and second order differential operators are considered only in terms of horizontal fields. For a given open subset $\Omega \subset \mathbb{H}$, the class $\Gamma^k(\Omega)$ represents the Folland–Stein space of functions having continuous derivatives up to order $k$ with respect to the vector fields $X$ and $Y$; we denote as usual by $C^k(\Omega)$ the class of functions having continuous derivatives up to order $k$ with respect to the differential structure of $\mathbb{R}^3$.

Let us recall that the horizontal gradient of a function $u \in \Gamma^1(\Omega)$ at $g \in \Omega$ is the 2–vector
\[ (\nabla_h u)(g) = ((Xu)(g), (Y u)(g)), \]
written with respect to the basis $\{X,Y\}$ of $V_1$; we denote by $\nabla u$ the element in $V_1$ defined as follows
\[ \nabla u = (Xu)X + (Y u)Y. \]

The horizontal Hessian of $u \in \Gamma^2(\Omega)$ at $g \in \Omega$ is the $2 \times 2$ matrix
\[ (\nabla^2_h u)(g) = \begin{pmatrix} (X(Xu))(g) & (X(Y u))(g) \\ (Y(Xu))(g) & (Y(Y u))(g) \end{pmatrix}, \]
while the symmetrized horizontal Hessian is the $2 \times 2$ symmetric matrix
\[ \left[ (\nabla^2_h u)(g) \right]^* = \frac{1}{2} \left\{ (\nabla^2_h u)(g) + \left[ (\nabla^2_h u)(g) \right]^T \right\}. \]

Given a point $g_0 \in \mathbb{H}$, the horizontal plane $H_{g_0}$ associated to $g_0$ is the plane in $\mathbb{H}$ defined by
\[ H_{g_0} = L_{g_0} (\exp(V_1)) = \{ g = (x,y,t) \in \mathbb{H} : t = t_0 + 2y_0 x - 2x_0 y \}. \]

Notice that, given $g \in \mathbb{H}_{g_0}$, the set $\{ g_0 \delta \lambda (g_0^{-1} g), \lambda \in [0,1] \}$ is the segment in $H_{g_0}$ joining $g_0$ with $g$ (i.e., the convex closure, in the Euclidean sense, of the set $\{ g_0, g \}$). We call it horizontal segment.

By contrast with Euclidean spaces, where the Euclidean distance is the most natural choice, in the Heisenberg group several distances have been introduced for different purposes. However, all of these distances are homogeneous, namely, they are left invariant and satisfy the relation $\rho(\delta g', \delta g) = r \rho(g', g)$ for every $g', g \in \mathbb{H}$, and $r > 0$. In particular, the Euclidean distance to the origin $| \cdot |$ on $\mathfrak{h}$ induces a
homogeneous pseudo–norm $|| \cdot ||$ on $\mathfrak{h}$ and, via the exponential map, on the group $I H$ in the following way: for $v \in \mathfrak{h}$, with $v = v_1 + v_2$, $v_i \in V_i$, we let

$$||v|| = (|v_1|^4 + v_2^2)^{1/4},$$

and then define the pseudo–norm on $I H$ by the equation

$$\rho(g) = ||v||,$$

if $g = \exp v$.

The distance between $g$ and $g'$ is given by $\rho(g^{-1}g')$.

### 2.1 Pansu differentiability

Let $\rho$ be any homogeneous distance on $I H$; to simplify the notation, we denote by $\rho(g)$ the distance $\rho(g, e)$.

Let $u : I H \to \mathbb{R}^k$. We say that $u$ is Pansu differentiable at $g \in I H$ if there exists a $G$–linear map $Du(g) : I H \to \mathbb{R}^k$, i.e., a group homomorphism that satisfies the relation $Du(g)(\delta_r h) = r Du(g)(h)$ for every $h \in I H$ and $r > 0$, and such that

$$\lim_{\rho(h) \to 0} \frac{|u(gh) - u(g) - Du(g)(h)|}{\rho(h)} = 0.$$  

We call the map $Du(g)$ the Pansu differential of $u$ in $g$.

The $k \times 3$ matrix representing the Pansu differential $Du$ of $u = (u^1, u^2, \ldots, u^k)$ can be written as follows

$$
\begin{pmatrix}
X u^1 & Y u^1 & 0 \\
X u^2 & Y u^2 & 0 \\
\vdots & \vdots & \vdots \\
X u^k & Y u^k & 0
\end{pmatrix}.
$$

(1)

The horizontal jacobian $J_H u(g)$ of $u$ at $g$ is defined by taking the standard jacobian of the matrix $u$; to this concern see [8]. In the particular case $k = 1$, an easy computation gives us that $u$ is Pansu differentiable at $g$ if

$$Du(g)(h) = \lim_{\lambda \to 0^+} \frac{u(g\delta_{\lambda}(h)) - u(g)}{\lambda}$$

exists for every $h \in I H$.

Moreover, if $u \in \Gamma^1(I H)$, the Pansu differential $Du(g)$ is given by the formula

$$Du(g)(h) = \langle Xu(g), \xi_1(h) \rangle,$$

for every $g$ and $h$ in $I H$ (see [6]).

Let us recall the following relevant definition concerning the degree of regularity of a function.
Definition 2.1 Let $\Omega \subset I^H$ be a bounded open subset, and $0 < \alpha \leq 1$. A bounded function $u : \Omega \to I^R$ is said to belong to the class $\Gamma^0,\alpha(\Omega)$ if there exists a positive constant $L_\alpha > 0$ such that
\[
|u(g) - u(g')| \leq L_\alpha \rho(g, g')^\alpha, \quad g, g' \in \Omega.
\]
A function $u \in \Gamma^0,1(\Omega)$ is said to belong to the class $\Gamma^1,\alpha(\Omega)$ if both $Xu$ and $Yu$ exist in $\Omega$ and $Xu, Yu \in \Gamma^0,\alpha(\Omega)$.

As usual we say that $u$ is Lipschitz continuous if $u \in \Gamma^0,1$; the symbol $\Gamma^0,1_{\text{loc}}(\Omega)$ denotes the class of locally Lipschitz continuous functions on $\Omega$.

In the fundamental paper [21], Pansu provides a Rademacher–Stefanov type result in the Carnot group setting; in particular, he proves that the Lipschitz continuous functions are differentiable almost everywhere in the horizontal directions. A further result, due to Danielli, Garofalo and Salsa ([7], Th. 2.7), will play a crucial role for some results in the sequel. We state it assuming that the Carnot group is the Heisenberg group.

Theorem 2.1 Let $\Omega$ be an open subset of $I^H$, and consider $u : \Omega \to I^R$, with $u \in \Gamma^0,1(\Omega)$. Then there exists a set $E \subset \Omega$ of Haar measure zero such that the Pansu differential $Du(g)$ and the horizontal gradient $Xu(g)$ exist for every $g \in \Omega \setminus E$, and
\[
Du(g)(h) = \langle Xu(g), \xi_1(h) \rangle, \quad \text{for every } h \in I^H.
\]
Furthermore, $Xu \in L^\infty(\Omega)$.

2.2 Convexity

In the Heisenberg group, and in Carnot groups in general, several definitions of convexity have been introduced and studied for both sets (see [20], [3]) and functions (see [6], [17]). As a matter of fact, the results obtained in literature suggest that, among them, the most suitable and satisfactory is the notion of weak H–convexity. In the sequel, for the sake of brevity, we will refer to weak H–convexity as to convexity; to avoid misunderstandings, the classical convexity will be called Euclidean convexity.

Definition 2.2 A subset $\Omega$ of $I^H$ is said to be convex if, for every $g \in \Omega$ and for every $g' \in H_g \cap \Omega$,
\[
g\delta_\lambda(g^{-1}g') \in \Omega, \quad \forall \lambda \in [0, 1].
\]

Definition 2.3 Let $\Omega$ be a convex subset of $I^H$. A function $u : \Omega \to I^R$ is called convex if
\[
u(g, \lambda) \leq u(g) + \lambda (u(g') - u(g))
\]
for all $g \in \Omega$, $g' \in H_g \cap \Omega$, and $\lambda \in [0, 1]$. This is equivalent to say that
\[
u(g \exp(\lambda v)) \leq u(g) + \lambda (u(g \exp(v)) - u(g))
\]
for every \( g \in \Omega \), \( v \in V_1 \) and \( \lambda \in [0, 1] \).

We say that \( u \) is strictly convex if \( u \) is convex and the equality in (4) holds, whenever \( g \neq g' \), if and only if \( \lambda = 0 \) or \( \lambda = 1 \).

Observe that \( u \) is a convex function on \( \Omega \) if and only if \( u \) is Euclidean convex on any horizontal segment.

In the sequel, without saying it explicitly, we will assume that the domain of a convex function is an open convex set.

Next theorem (see, for instance, [6]) provides a useful second order condition for convexity, based on the behaviour of the symmetrized Hessian of \( u \).

**Theorem 2.2** Let \( \Omega \) be an open convex subset of \( IH \) and let \( u \in \Gamma^2(\Omega) \). Then, \( u \) is convex on \( \Omega \) if and only if the symmetrized horizontal Hessian \( \left[(\nabla^2_h u)(g)\right]^* \) is positive semidefinite for every \( g \in \Omega \).

In [4] the authors provide a characterization of quasi–convex functions in \( C^2(IH) \) involving the symmetrized horizontal Hessian as well.

It is worthwhile to mention the following regularity result about convex functions on \( IH \) due to Balogh and Rickly:

**Theorem 2.3** (see [2], Theorem 1.2) Let \( u : IH \to IR \) be a convex function. Then \( u \in \Gamma^0_{loc} \).

### 3 H–subdifferential and H–normal map

Let \( \Omega' \subset IR^n \) be an open set. Let us recall (see [22]) that for every function \( f : \Omega' \to IR \), the subdifferential of \( f \) at a point \( x_0 \) is defined as follows:

\[
\partial f(x_0) = \{ p \in IR^n : f(x) \geq f(x_0) + \langle p, x - x_0 \rangle, \quad \forall x \in \Omega' \}.
\]

(5)

If \( \partial f(x_0) \) is not empty, we say that \( f \) is subdifferentiable at \( x_0 \).

The normal map of \( f \) is the set–valued function \( \partial f : \mathcal{P}(\Omega') \to \mathcal{P}(IR^n) \) defined by

\[
\partial f(E) = \bigcup_{x \in E} \partial f(x),
\]

for every \( E \subset \Omega' \).

In [6] a notion of horizontal subdifferential that takes into account the sub–Riemannian structure of \( IH \) is given.

**Definition 3.1** Let \( u : \Omega \to IR \), with \( \Omega \) open subset of \( IH \). The horizontal subdifferential (or H–subdifferential) of \( u \) at \( g_0 \in \Omega \) is the set

\[
\partial_H u(g_0) = \{ p \in V_1 : u(g) \geq u(g_0) + \langle p, \xi_1(g) - \xi_1(g_0) \rangle, \quad \forall g \in H_{g_0} \cap \Omega \}.
\]
As in the classical context, we say that \( u \) is \textit{horizontally subdifferentiable} (shortly, \( H \)-subdifferentiable) at \( g_0 \) if \( \partial u_H(g_0) \) is not empty. Moreover, if \( p \in \partial u_H(g_0) \), we say that \( p \) is a \( H \)-\textit{subgradient} of \( u \) at \( g_0 \).

In [6] the authors proved the following result:

**Proposition 3.1** (see [6], Proposition 10.6). Let \( u \) be a function in \( \Gamma^1(\Omega) \), and \( \Omega \subset \mathbb{H} \) open. If \( \partial_H u(g) \neq \emptyset \), then \( \partial_H u(g) = \{ \mathbb{X} u(g) \} \).

### 3.1 An equivalent notion of \( H \)-subdifferentiability

In the Euclidean context another notion of subdifferentiability can be done and we say (see, for instance, [10] and [15]) that \( f \) is subdifferentiable at \( x_0 \in \Omega' \subset \mathbb{R}^n \) if there exists \( p \in \mathbb{R}^n \) such that

\[
    f(x) \geq f(x_0) + \langle p, x - x_0 \rangle + o(|x - x_0|)
\]

as \( |x - x_0| \to 0 \). This notion is useful in the study of optimal mass transportation problems together with the notions of \( c \)-subdifferentiability, \( c \)-convexity and Legendre–Fenchel transform.

Recently, these concepts have been investigated in the framework of the Heisenberg group. For instance, in [1] the authors defined \( c \)-subdifferentiability and \( c \)-convexity for functions on \( \mathbb{H} \); in [5], taking into account the horizontal structure, the Fenchel transform was introduced for functions on \( \mathbb{H} \), and a characterization of convexity was provided.

Starting from (7), another notion of \( H \)-subdifferentiability can be given:

**Definition 3.2** Let \( u : \mathbb{H} \to \mathbb{R} \). We say that \( u \) is \textit{horizontally subdifferentiable} at \( g_0 \in \Omega \) if there exists \( p \in \mathbb{V}_1 \) such that

\[
    u(g) \geq u(g_0) + \langle p, \xi_1(g) - \xi_1(g_0) \rangle + o(||\xi_1(g) - \xi_1(g_0)||), \quad \text{for } g \in H_{g_0} \text{ and } g \to g_0.
\]

In next Proposition we show that, in the case of convex functions, the two notions of \( H \)-subdifferentiability given in Definitions 3.1 and 3.2 are equivalent.

**Proposition 3.2** Let \( u \) be a convex function. Then \( u \) is \( H \)-subdifferentiable at \( g_0 \) (in the sense of definition 3.2) if and only if \( \partial u_H(g_0) \) is not empty.

**Proof:** If \( \partial_H u(g_0) \neq \emptyset \), the \( u \) is trivially \( H \)-subdifferentiable at \( g_0 \) according to Definition 3.1. Assume that there exists \( p \in \mathbb{V}_1 \) such that definition 3.2 is fulfilled at \( g_0 \). Let us prove that \( p \in \partial_H u(g_0) \). By contradiction, let \( g' \in H_{g_0} \) such that

\[
    u(g') - u(g_0) - \langle p, \xi_1(g') - \xi_1(g_0) \rangle = \alpha < 0.
\]

Define the function \( U : [0, 1] \to \mathbb{R} \) as follows:

\[
    U(\lambda) = u(g_\lambda) - u(g_0) - \langle p, \xi_1(g_\lambda) - \xi_1(g_0) \rangle = u(g_\lambda) - u(g_0) - \lambda \langle p, \xi_1(g') - \xi_1(g_0) \rangle,
\]
where \( g_\lambda = g_0 \delta_\lambda (g_0^{-1} g') \) varies along the horizontal segment \([g_0, g']\) as \( \lambda \) varies in \([0, 1]\).

The function \( U \) is Euclidean convex, \( U(0) = 0, U(1) = \alpha < 0 \); in particular, for every \( \lambda \in [0, 1] \),

\[
U(\lambda) \leq (1 - \lambda)U(0) + \lambda U(1)
\]

\[
= \lambda \alpha. \quad (8)
\]

At the same time, by the assumption of H–differentiability of \( u \) at \( g_0 \),

\[
U(\lambda) \geq o(||\xi_1(g_\lambda - g_0)||) = ||\xi_1(g') - \xi_1(g_0)|| o(|\lambda|), \quad \lambda \to 0^+.
\]

Putting together (8) and (10), we get

\[
||\xi_1(g') - \xi_1(g_0)|| o(|\lambda|) \leq \lambda \alpha, \quad \lambda \to 0^+,
\]

or, dividing by \( \lambda \),

\[
||\xi_1(g') - \xi_1(g_0)|| o(1) \leq \alpha, \quad \lambda \to 0^+,
\]

a contradiction, since \( \alpha < 0 \).

In the sequel, we prefer to deal with the notion of H–subdifferentiability of Definition 3.1.

### 3.2 The Horizontal normal map

The notion of horizontal normal map associated to the horizontal subdifferential arises naturally:

**Definition 3.3** Let \( u : \Omega \to (-\infty, +\infty] \), \( \Omega \) open. The horizontal normal map of \( u \) (or H–normal map) is the set valued function \( \partial H u : \mathcal{P}(\Omega) \to \mathcal{P}(V_1) \) defined by

\[
\partial H u(E) = \bigcup_{g \in E} \partial H u(g),
\]

for every \( E \subset \Omega \).

One of the purposes of this paper is to establish whether the H–normal map can play a suitable role in dealing with the Monge–Ampère measure of a function (see [13]), or with some maximum or comparison principles for convex functions (see [14]).

Let us recall that, given a map \( \mathcal{F} : \mathcal{P}(\Omega) \to \mathcal{P}(Y) \), where \( \Omega \subset X \), the graph of \( \mathcal{F} \) is defined as the set

\[
\{(x, y) \in X \times Y : x \in \Omega, y \in \mathcal{F}(x)\}.
\]

A possible extension of the concept of continuity of a function to maps is provided by the notion of closed graph.

**Definition 3.4** Let \( X, Y \) be topological spaces. A map \( \mathcal{F} : X \to \mathcal{P}(Y) \) is said to have closed graph if for every \( \{x_n\} \subset X, x_n \to x \in X, \{y_n\} \subset Y, \) with \( y_n \in \mathcal{F}(x_n) \), then

\[
y_n \to y \implies y \in \mathcal{F}(x).
\]
4 H–subdifferentiability and convex functions

The aim of this section is to investigate some properties of the H–subdifferential and of the H–normal map of convex functions on $\mathcal{H}$. Our main result is Theorem 4.1 where a convex function is characterized via its H–subdifferentiability on the domain. As a consequence, the uniqueness of the H–subdifferential of a convex function $u$ at a point $g$ is equivalent to the existence of $Xu(g)$.

If $u : \Omega \to \mathbb{R}$ is convex, then, for every $g \in \Omega$, the limit
\[
\lim_{\lambda \to 0^+} \frac{u(g \exp(\lambda v)) - u(g)}{\lambda}
\]
exists in $\mathbb{R}$, for every $v \in V_1$. We set
\[
u'(g; v) = \lim_{\lambda \to 0^+} \frac{u(g \exp(\lambda v)) - u(g)}{\lambda}.
\]

Let us first state the following useful characterization of H–subdifferentials of a convex function.

**Proposition 4.1** Let $u : \Omega \to \mathbb{R}$ be a convex function, and $g \in \Omega$. Then $p \in \partial_H u(g)$ if and only if
\[ u'(g; v) \geq \langle p, v \rangle, \quad \text{for every } v \in V_1. \tag{11} \]
In particular, if $u$ is Pansu differentiable at $g$ and (3) holds, then
\[ Xu(g) \in \partial_H u(g). \]

**Proof:** Suppose that $p \in \partial_H u(g)$. Hence, for every $\lambda \in [0, 1]$ and $v \in V_1$, we have
\[ u(g \exp(\lambda v)) \geq u(g) + \lambda (p, v). \]

The previous inequality (12) holds if and only if
\[ \frac{u(g \exp(\lambda v)) - u(g)}{\lambda} \geq \langle p, v \rangle, \]
for every $\lambda \in (0, 1]$ and $v \in V_1$; hence (11) follows obviously.

Conversely, assume that (11) holds. Since $u$ is convex, we have
\[ u(g \exp(\lambda v)) \leq u(g) + \lambda (u(g \exp(v)) - u(g)), \tag{12} \]
for every $\lambda \in [0, 1]$ and $v \in V_1$. By (11) and (12) we get that
\[ \langle p, v \rangle \leq \lim_{\lambda \to 0^+} \frac{u(g \exp(\lambda v)) - u(g)}{\lambda} \leq u(g \exp(v)) - u(g), \]
for every $v \in V_1$; hence $p \in \partial_H u(g)$. 

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Let us suppose now that $u$ is Pansu differentiable at $g$ and (3) holds; then, by (2),

$$\langle Xu(g), \xi_1(h) \rangle = Du(g)(h) = \lim_{\lambda \to 0^+} \frac{u(g\delta_\lambda(h)) - u(g)}{\lambda},$$

for every $h \in \mathcal{H}$. The convexity of $u$ gives us that $\frac{u(g\delta_\lambda(h)) - u(g)}{\lambda}$ decreases when $\lambda \to 0^+$, for every fixed $g$ and $h \in H_e$. Hence

$$\langle Xu(g), \xi_1(h) \rangle \leq \frac{u(g\delta_\lambda(h)) - u(g)}{\lambda}, \quad \forall \lambda \in (0, 1],$$

so that $Xu(g)$ is a $\mathcal{H}$-subdifferential. \qed

In [6], Danielli, Garofalo and Nhieu proved the following

**Proposition 4.2** (see [6], Proposition 10.5). Let $u : \Omega \to \mathbb{R}$, where $\Omega$ is an open and convex subset of $\mathcal{H}$. If $\partial_H u(g) \neq \emptyset$ for every $g \in \Omega$, then $u$ is convex.

In order to show that the converse holds, we need next result. Indeed, the following Lemma will be crucial also in the proof of Theorem 5.1.

**Lemma 4.1** Let $\Omega$ be an open subset of $\mathcal{H}$, and consider a function $u \in \mathcal{C}(\Omega)$. Then the map $\partial_H u : \mathcal{P}(\Omega) \to \mathcal{P}(V_1)$ has closed graph.

**Proof:** We prove the lemma by contradiction. Assume that $g_0 \in \Omega$, and there exist sequences $\{g_n\} \subset \Omega$ and $\{p_n\}$, with $p_n \in \partial_H u(g_n)$, such that

$$g_n \to g_0, \quad p_n \to p, \quad p \notin \partial_H u(g_0).$$

Consequently, there exists $g' \in H_{g_0} \cap \Omega$ such that

$$u(g') - u(g_0) = \langle p, \xi_1(g') - \xi_1(g_0) \rangle - \alpha,$$

for a suitable $\alpha > 0$. From the assumptions, $u$ is continuous.

Denote by $B(e, r)$ the set of points $g \in \mathcal{H}$ such that $\rho(g) < r$. Let us consider the balls $B(e, r)$ and $B(e, r')$ in such a way that

i) $|u(g'h') - u(g')| < \alpha/10$, for every $h' \in B(e, r)$;

ii) $|u(g_0h) - u(g_0)| < \alpha/10$, for every $h \in B(e, r)$;

iii) $H_{g_0h} \cap L_{g'}(B(e, r')) \neq \emptyset$, for every $h \in B(e, r)$.

The reader can easily convince himself that such balls exist using suitable continuity arguments on $u$ and on the displacement of the horizontal planes of points moving close to others. For every $h \in B(e, r)$ and $h' \in B(e, r')$, i) and ii) imply the following inequality:

$$\langle p, \xi_1(g') - \xi_1(g_0) \rangle - \alpha = u(g') - u(g_0) \geq u(g'h') - u(g_0h) - \alpha/5. \quad (13)$$
Take $N$ such that $g_n \in L_{g_0}(B(e, r))$ for every $n \geq N$, and denote by $h_n$ the point in $B(e, r)$ such that $g_n = g_0h_n$. Notice that $h_n \to e$. Moreover, from the choice of the balls, there exists $h'_n \in B(e, r')$ such that

$$g'_n = g'h'_n \in Hg_0h_n, \quad g'_n \to g'.$$

Then, taking into account the assumptions and (13), we get

$$\langle p, \xi_1(g') - \xi_1(g_0) \rangle - \alpha \geq u(g'h'_n) - u(g_0h_n) - \alpha/5 \geq \langle p_n, \xi_1(g'h'_n) - \xi_1(g_0h_n) \rangle - \alpha/5 \geq \lim inf \langle p_n, \xi_1(g'h'_n) - \xi_1(g_0h_n) \rangle - \alpha/5 = \langle p, \xi_1(g') - \xi_1(g_0) \rangle - \alpha/5,$$

therefore obtaining

$$\langle p, \xi_1(g') - \xi_1(g_0) \rangle - \alpha \geq \langle p, \xi_1(g') - \xi_1(g_0) \rangle - \alpha/5,$$

a contradiction. \hfill \Box

We are now able to prove the following interesting characterization:

**Theorem 4.1** Let $u : \Omega \subset H \to \mathbb{R}$, where $\Omega$ is open and convex. Then $u$ is convex if and only if, for every $g \in \partial H u(g) \neq \emptyset$.

**Proof:** The “if” part is the result of Proposition 4.1.

From Theorem 2.3 any convex function $u$ belongs to $\Gamma^{0,1}_{loc}(\Omega)$; in particular, it is continuous. By contradiction, we assume that there exists $g_0 \in \Omega$ such that $\partial Hu(g_0) = \emptyset$. Let us consider a neighborhood $B(e, r)$ of the origin such that $u$ belongs to $\Gamma^{0,1}(L_{g_0}(B(e, r)))$. From Theorem 2.1 there exists a subset $E$ of $L_{g_0}(B(e, r))$ with null measure such that for every $g \in L_{g_0}(B(e, r)) \setminus E$ there exists the Pansu differential $Du(g)$ and (3) holds; Proposition 4.1 shows that, for such $g$, $\nabla H u(g) \in \partial H u(g)$. Since $\nabla H u \in L^{\infty}(L_{g_0}(B(e, r)))$ (see [6], Theorem 9.1), there exists $k$ such that $||\nabla H u(g)|| \leq k$, for a.e. $g \in L_{g_0}(B(e, r))$. Therefore there exists a sequence $\{g_n\}$ in $L_{g_0}(B(e, r)) \setminus E$ such that

$$g_n \to g_0, \quad \nabla H u(g_n) \to p, \quad \nabla H u(g_n) \in \partial H u(g_n),$$

for some $p \in V_1$. Then, since $\partial H u$ has closed graph (see Lemma 4.1), $p \in \partial H u(g_0)$, a contradiction. \hfill \Box

As a matter of fact, the uniqueness of the H–subdifferential for a convex function characterizes the H–differentiability. We are able to state the following:

**Theorem 4.2** Let $u$ be a convex function on $\Omega$. Then, $\nabla H u(g)$ exists for some $g \in \Omega$ if and only if $u$ has a unique $H$–subgradient at $g$. Moreover, in both cases, we have that $\partial H u(g) = \{\nabla H u(g)\}$. 

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Proof: Let \( u \) be convex. By Theorem 4.1, there exists \( p = p_1X + p_2Y \in \partial u(g) \). By Proposition 4.1, if we put \( h = (1, 0, 0) \in H_e \), we get

\[
p_1 = \langle p, \xi_1(h) \rangle \\
\leq \lim_{\lambda \to 0^+} \frac{u(g\delta(h)) - u(g)}{\lambda} \\
= \lim_{\lambda \to 0^+} \frac{u(g\exp(\lambda X)) - u(g)}{\lambda}.
\]

Now, taking \( h = (-1, 0, 0) \in H_e \), similar computations give

\[
-p_1 \leq - \lim_{\lambda \to 0^-} \frac{u(g\exp(\lambda X)) - u(g)}{\lambda}.
\]

Hence, we have

\[
p_1 = \lim_{\lambda \to 0} \frac{u(g\exp(\lambda X)) - u(g)}{\lambda} = Xu(g).
\]

Similar arguments show that \( p_2 = Yu(g) \).

Conversely, assume that \( \partial u(g_0) = \{p\} \). Let us suppose \( \Omega = \mathcal{H} \), for the sake of simplicity. Fix \( v \in V_1 \), and consider the linear space \( \mathcal{L}(v) = \{av, a \in \mathcal{R}\} \). Define on \( \mathcal{L}(v) \) the linear functional

\[
L_v(w) = au'(g_0; v),
\]

for \( w = av \). Notice that \( L_v(w) = u'(g_0; w) \) whenever \( w = av \), with \( a > 0 \). Indeed,

\[
u'(g_0; w) = \lim_{\lambda \to 0^+} \frac{u(g_0\exp(\lambda w)) - u(g_0)}{\lambda} = \lim_{\lambda \to 0^+} \frac{u(g_0\exp(\lambda v)) - u(g_0)}{\lambda} = L_v(w).
\]

By the convexity of \( u \), the function \( t \to u(g_0\exp(tv)) \) is Euclidean convex; in particular, the following inequality holds

\[
u'(g_0; -v) \geq -u'(g_0; v).
\]

Assume that \( w = av \), with \( a < 0 \). Then

\[
u'(g_0; w) = u'(g_0; av) = (-a)u'(g_0; -v) \geq au'(g_0; v) = L_v(w).
\]

Since the linear functional \( L_v \) satisfies on \( \mathcal{L}(v) \) the inequality

\[
L_v(w) \leq u'(g_0; w),
\]

by the Hahn–Banach theorem there exists \( p_v \in V_1 \) such that

\[
\langle p_v, w \rangle \leq u'(g_0; w), \quad \forall w \in V_1.
\]

From Proposition 4.1, \( p_v \in \partial u(g_0) \), and, by the assumptions, \( p_v = p \). In particular, \( p_v \) is independent on \( v \). Since \( \langle p, v \rangle = u'(g_0; v) \), and \( v \) is any vector in \( V_1 \), we can conclude that

\[
\langle p, v \rangle = u'(g_0; v), \quad \forall v \in V_1;
\]

thus \( u'(g_0; \cdot) \) is linear on \( V_1 \). Hence, if \( w = X \) then

\[
u'(g_0; w) = -u'(g_0; -w) = \lim_{\lambda \to 0^-} \frac{u(g_0\exp(\lambda X)) - u(g_0)}{\lambda}.
\]

This implies the existence of \( Xu(g_0) \), that should be equal to \( u'(g_0; X) \). Similar arguments prove the existence of \( Yu(g_0) \); in particular, \( p = Xu(g_0)x + Yu(g_0)y \). \(\square\)
5 The H–normal map

The aim of this section is to investigate some properties of those subsets of $V_1$ that are images, via the H–normal map, of subsets of $\Omega$. We start by studying the properties of the H–normal map of a single point, i.e. the H–subdifferential.

**Proposition 5.1** Let $u : \Omega \to \mathbb{R}$, with $\Omega \subset \mathcal{H}$ open; let $g \in \Omega$. Then $\partial u_H(g)$ is a convex set. Moreover, if $u$ is locally bounded, then $\partial u_H(g)$ is compact.

**Proof:** Take any $p_1, p_2 \in \partial u_H(g)$. For every $\lambda \in [0,1]$ we have that

$$u(g') = (1 - \lambda)u(g') + \lambda u(g') \geq (1 - \lambda)(u(g) + \langle p_1, \xi_1(g') - \xi_1(g) \rangle) + \lambda(u(g) + \langle p_2, \xi_1(g') - \xi_1(g) \rangle) = u(g) + \langle (1 - \lambda)p_1 + \lambda p_2, \xi_1(g') - \xi_1(g) \rangle,$$

for every $g' \in H_g$, therefore showing that $(1 - \lambda)p_1 + \lambda p_2$ is a H–subgradient of $u$.

Let $\{p_k\}$ be a sequence in $\partial u_H(g)$. For every $k$ and for every $w \in V_1$, with $||w|| = 1$, we have

$$u(g\delta_\lambda(\exp(w))) \geq u(g) + \langle p_k, \xi_1(g\delta_\lambda(\exp(w))) - \xi_1(g) \rangle = u(g) + \lambda \langle p_k, w \rangle.$$

In particular for every $k$ with $p_k \neq 0$, if we put $w = p_k/||p_k||$, then we obtain

$$\sup_{||w|| = 1} u(g\delta_\lambda(\exp(w))) \geq u(g) + \frac{\lambda}{||p_k||} \langle p_k, p_k \rangle = u(g) + \lambda ||p_k||. \quad (14)$$

Take any $\lambda$ sufficiently small such that $g\delta_\lambda(\exp(w)) \in \Omega$ for every $w$ with $||w|| = 1$: since $u$ is locally bounded, (14) gives us that $\{p_k\}$ in a bounded subset of $V_1$. Hence there exists a convergent subsequence $\{p_{k_n}\}$ such that $p_{k_n} \to p$. Clearly

$$u(g') \geq u(g) + \langle p_{k_n}, \xi_1(g') - \xi_1(g) \rangle,$$

for every $n$ and for every $g' \in H_g \cap \Omega$. Letting $n \to \infty$, we obtain that $p \in \partial u_H(g)$. □

The investigation of the images of the H–normal map is actually more awkward. Indeed, if we shift from a point $g$ to another point $g'$, and consider $p \in \partial u_H(g)$ and $p' \in \partial u_H(g')$, then the H–subdifferentials $p$ and $p'$ support the function on the different planes $H_g$ and $H_{g'}$.

First of all, it easy to see that, as in the Euclidean case (see [12]), if $K$ is Euclidean convex, then $\partial u_H(K)$ is not necessary Euclidean convex in $V_1$.

To obtain information about $\partial u_H(\Omega)$, we need some regularity assumptions on $u$.

**Proposition 5.2** Let $u : \Omega \to \mathbb{R}$ and $u \in \Gamma^{0,1}(\Omega)$, with Lipschitz constant $L$. Then $||p|| \leq L$, for every $p \in \partial u_H(\Omega)$; in particular, $\partial u_H(\Omega)$ is a bounded set.
Proof: Let $p \in \partial u_H(\Omega)$. Then there exists $g \in \Omega$ such that, for every $h \in H$ and $\lambda > 0$, we have
\[
\frac{u(g\delta_\lambda(h)) - u(g)}{\lambda} \geq \langle p, \xi_1(h) \rangle.
\]
Since $u$ is Lipschitz continuous, there exists $L > 0$ such that $|u(g) - u(g')| \leq L\rho(g, g')$, for every $g' \in \Omega$. Hence
\[
\frac{u(g\delta_\lambda(h)) - u(g)}{\lambda} \leq L\rho(h).
\]
The previous two inequalities give us that
\[
\langle p, \xi_1(h) \rangle \leq L\rho(h),
\]
for every $h \in H$. If we put $h = \xi_1^{-1}(p)$, we obtain $||p||^2 \leq L||p||$. □

Theorem 5.1 Let $u : \Omega \to \mathbb{R}$ be a function in $\Gamma^{0,1}_{\text{loc}}(\Omega)$. Then, for every compact set $K \subset \Omega$, the set $\partial u_H(K)$ is compact.

Proof: By the assumption, for every $g \in \Omega$ there exists a neighborhood $B_g$ such that $u \in \Gamma^{0,1}(B_g)$, i.e., there is a constant $L_g$ such that
\[
|u(g'') - u(g')| \leq L_g\rho(g'', g'),
\]
for every $g'', g' \in B_g$.

Let $K$ be any compact subset of $\Omega$; then $K \subset \cup_{i=1}^N B_{g_i}$ for a suitable finite set of points $\{g_1, g_2, \ldots, g_N\}$.

Take any $p \in \partial u_H(K)$, and denote by $g$ a point in $K$ such that $p \in \partial u_H(g) \subset \partial_H u(B_{g_i})$, for some $i$. Hence, by Proposition 5.2, since $u \in \Gamma^{0,1}(B_{g_i})$ for every $i = 1, 2, \ldots, N$, we have that
\[
||p|| \leq L_{g_i} \leq \max\{L_{g_i} : 1 \leq i \leq N\}.
\]
Hence $\partial u_H(K)$ is bounded.

Let us now consider a sequence $\{p_n\} \subset \partial u_H(K)$, and assume that $p_n \to p$, and denote by $g_n$ a point in $K$ such that $p_n \in \partial_H u(g_n)$. Since $K$ in compact, there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to g$ for some $g \in K$. Since the normal map $\partial_H u$ has closed graph (see lemma 4.1), $p \in \partial_H u(g)$. □

Next result follows trivially from Theorems 5.1 and 2.3.

Remark 5.1 Let $u$ be a convex function on $\Omega$. Then, for every compact set $K \subset \Omega$, the set $\partial u_H(K)$ is compact.
5.1 Monotonicity property of the H–normal map

The purpose of this subsection is to investigate whether, as in the Euclidean context, a monotonicity property of this type holds:

**Problem 5.1** Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and denote by $u$ and $v$ two convex functions such that

$$u(g) = v(g), \quad \forall g \in \partial \Omega$$

and

$$u(g) \leq v(g), \quad \forall g \in \Omega.$$ 

Then

$$\partial_H v(\Omega) \subset \partial_H u(\Omega).$$

We are not able to prove the result in the general case; at the moment, several difficulties rise in the proof. In the sequel we will consider a particular case of the situation described in Problem (5.1); we assume to work with functions $u : \mathbb{R}^n \to \mathbb{R}$ with the following property:

$$u(x, y, t) = U(r, t),$$

for every $(x, y, t)$ where

$$r = \sqrt{x^2 + y^2}.$$ 

By abuse of language, we will call these functions “radial”.

We think that the ideas contained in the proof of this special case could be of some interest to deal with the general one.

For every $t \in \mathbb{R}$ and $R \geq 0$, we will denote by $C(t, R)$ the set

$$C(t, R) = \{(x, y, t) \in \mathbb{R}^n : x^2 + y^2 = R^2\},$$

and by $D(t, R)$ the “open” disc in $H_{(0,0,t)}$ defined by

$$D(t, R) = \{(x, y, t) \in \mathbb{R}^n : x^2 + y^2 < R^2\}.$$ 

In order to prove our monotonicity result, we need to state the following technical propositions, that explains some geometric features of the normal map of radial functions.

**Proposition 5.3** Let $u : \mathbb{R}^n \to \mathbb{R}$ be a radial, convex function in $C^1(\mathbb{R}^n)$. Then, for every $t \in \mathbb{R}$ and $R \geq 0$,

$$\partial_H u(C(t, R)) = \{p \in V_1 : ||p|| = R'\},$$

for a suitable $R' \geq 0$.

**Proof:** Under the assumptions, for every $g \in \mathbb{R}^n$ the map $\partial_H u : \mathbb{R}^n \to V_1$ is single–valued, and $\partial_H u(g) = \{Xu(g)\}$ (see Theorem [4.2]).
For every $g = (r \cos \theta, r \sin \theta, t) \in \mathcal{H}$, we have

$$X u(g) = U_r(r, t) \cos \theta + r U_t(r, t) \sin \theta, \quad Y u(g) = U_r(r, t) \sin \theta - r U_t(r, t) \cos \theta,$$

and

$$||X u(g)|| = \sqrt{U_r^2(r, t) + 4r^2 U_t^2(r, t)}.$$

If $\theta$ varies in $[0, 2\pi)$ we get the thesis. \hfill $\square$

**Proposition 5.4** Let $u : \mathcal{H} \to \mathbb{R}$ be a radial, convex function in $C^1(\mathcal{H})$. Denote by $\Omega$ the (nonempty) sublevel set

$$\Omega = \{ g \in \mathcal{H} : u(g) < 0 \}.$$

Then, $\partial H u(\Omega \cap H(0,0,t))$ is a disc (open or closed) in $V_1$, centered at the origin. Consequently, the set $\partial H u(\Omega)$ is a disc (open or closed) in $V_1$, centered at the origin.

**Proof:** Notice that

$$\partial H u(\Omega) = \bigcup_{t \in \mathbb{R}} \partial H u(\Omega \cap H(0,0,t)),$$

and $\Omega \cap H(0,0,t) = D(t, R(t))$. By Proposition 5.3 and by the continuity of $X u$ we easily get the thesis. \hfill $\square$

We would like to emphasize that, in general, $||X u(r \cos \theta, r \sin \theta, t)||$ is not an increasing function of $r$; this explains why, without further conditions on $u$, one cannot infer that $\partial H u(\partial D(t, R)) = \partial(\partial H u(D(t, R)))$. The radius $\bar{R}$ of $\partial H u(D(t, R))$ is given by the expression

$$\bar{R}(t, R) = \sup\{||X u(r \cos \theta, r \sin \theta, t)||, 0 \leq r < R, \theta \in [0, 2\pi)\}.$$

**Theorem 5.2** Let $u \in C^1(\mathcal{H})$, $v \in C^2(\mathcal{H})$ be radial, strictly convex functions such that $u \leq v$. Denote by $\Omega$ the (nonempty) level set

$$\Omega = \{ g \in \mathcal{H} : u(g) < 0 \} = \{ g \in \mathcal{H} : v(g) < 0 \};$$

assume that $\Omega$ is bounded and

$$\partial \Omega = \{ g \in \mathcal{H} : u(g) = 0 \} = \{ g \in \mathcal{H} : v(g) = 0 \}.$$

If $\bar{g} \in \partial \Omega$, then there exists $\bar{s} = \bar{s}(\bar{g}) \in (0, 1]$ such that

$$X v(\bar{g}) = \bar{s} X u(\bar{g}).$$

In addition, suppose that the function $V$ defined as $V(r, t) = v(x, y, t)$ satisfies in $\overline{\Omega}$ the assumption

$$r^3 V_r^2 - V_r V_{rr} < 0. \quad (15)$$

Then,

$$\partial H v(\Omega) \subset \partial H u(\Omega). \quad (16)$$
**Proof:** Take any point \( \bar{y} = (\bar{x}, \bar{y}, \bar{t}) \), such that \( \bar{y} \in \partial \Omega \cap H_{(0,0,\bar{t})} \). Notice that \( \mathbb{X}u(\bar{y}) \neq 0 \), since \( u \) is strictly convex and \( \mathbb{X}u(0,0,\bar{t}) = 0 \).

Let us consider the function \( F^u_{\bar{y}} : \mathbb{R}^2 \to \mathbb{R} \) obtained by restricting \( u \) to \( H_{\bar{y}} \) and defined as follows:

\[
F^u_{\bar{y}}(x,y) = u(x,y,\bar{t} + 2\bar{y}x - 2\bar{y}y).
\]

Denote by \( \Omega^u_{\bar{y}} \) the sublevel set

\[
\Omega^u_{\bar{y}} = \{(x,y) \in \mathbb{R}^2 : F^u_{\bar{y}}(x,y) < 0\},
\]

and consider, in particular, its boundary

\[
\partial \Omega^u_{\bar{y}} = \{(x,y) \in \mathbb{R}^2 : F^u_{\bar{y}}(x,y) = 0\}.
\]

This set is not empty, since \( F^u_{\bar{y}}(\bar{x}, \bar{y}) = 0 \); moreover, from the inequality

\[
\mathbb{X}u(\bar{y}) = \frac{\partial F^u_{\bar{y}}}{\partial x}((\bar{x}, \bar{y}))X + \frac{\partial F^u_{\bar{y}}}{\partial y}((\bar{x}, \bar{y}))Y \neq 0,
\]

the implicit function theorem assures that, at least locally, there exists a unique curve \( \gamma_{\bar{y}} : I \to \mathbb{R}^2 \), \( \gamma_{\bar{y}}(s) = (x(s), y(s)) \), with \( (x(s), y(s)) \in C^1(I) \) and \( F^u_{\bar{y}}(x(s), y(s)) = 0 \); moreover, \( \nabla F^u_{\bar{y}}(x, y) \) is orthogonal to \( \dot{\gamma}_{\bar{y}}(0) \). Since \( \nabla F^u_{\bar{y}}(x, y) \) represents the increasing direction of \( F^u_{\bar{y}} \) at the point \((x, y)\), the vector \( -\nabla F^u_{\bar{y}}(x, y) \) points towards \( \Omega^u_{\bar{y}} \); this implies that \((x, y) - z\nabla F^u_{\bar{y}}(x, y)) \) belongs to \( \Omega^u_{\bar{y}} \), at least for small values of \( z > 0 \).

Let us start from the function \( v \) instead of \( u \). By the same arguments applied to \( F^u_{\bar{y}} = v(x,y,\bar{t} + 2\bar{y}x - 2\bar{y}y) \), taking into account that, from the assumptions on \( u \) and \( v \), we have \( \Omega^u_{\bar{y}} = \Omega^v_{\bar{y}} \), we find out that \( \nabla F^u_{\bar{y}}(x, y) \) and \( \nabla F^v_{\bar{y}}(x, y) \) should satisfy the equality

\[
\nabla F^u_{\bar{y}}(x, y) = \overline{s} \nabla F^v_{\bar{y}}(x, y),
\]

for some positive \( \overline{s} \); in other words,

\[
\mathbb{X}v(\bar{y}) = \overline{s} \mathbb{X}u(\bar{y}). \tag{17}
\]

Let us prove that \( \overline{s} \leq 1 \). Consider the functions \( f^u, f^v \) defined in \( [0, \epsilon) \) for a suitable small \( \epsilon \) as follows

\[
f^u(z) = F^u_{\bar{y}}((x, y) - z\nabla F^u_{\bar{y}}(x, y)), \quad f^v(z) = F^v_{\bar{y}}((x, y) - z\nabla F^v_{\bar{y}}(x, y)).
\]

Since \( f^u(0) = f^v(0) = 0 \), and \( f^u(z) \leq f^v(z) \leq 0 \) if \( 0 < z < \epsilon \), standard arguments of real analysis entail that \((f^u)'(0) \leq (f^v)'(0) \leq 0 \). From \((f^u)'(0) = -||\mathbb{X}u(\bar{y})||^2 \) and \((f^v)'(0) = -||\mathbb{X}v(\bar{y})||^2 = -\overline{s}^2 ||\mathbb{X}u(\bar{y})||^2 \), we obtain that \( \overline{s} \leq 1 \).

Let us consider the function

\[
r \mapsto ||\mathbb{X}v(g)||^2 = V_r^2(r,t) + 4r^2 V_t^2(r,t),
\]

where \( g = (r \cos \theta, r \sin \theta, t) \). By the assumption (15), standard computations imply that this is an increasing function; in particular, from Propositions 5.3 and 5.4

\[
\partial_H v(\partial (\Omega \cap H_{(0,0,t)})) = \partial_H v(\Omega \cap H_{(0,0,t)})) = \partial_H v(\Omega \cap H_{(0,0,t)})).
\]
Moreover, taking into account that (17) holds with \( s \leq 1 \), we get
\[
\partial_H v(\Omega \cap H_{(0,0,t)}) \subset \partial_H u(\Omega \cap H_{(0,0,t)}).
\]
Suppose now that \( p \) is in \( \partial_H v(\Omega) \), i.e., \( p = Xv(g') \) for some \( g' = (r' \cos \theta, r' \sin \theta, t') \in \Omega \). Then
\[
p \in \partial_H v(C(t',r')) \subset \partial_H v(\Omega \cap H_{(0,0,t')}) \subset \partial_H u(\Omega \cap H_{(0,0,t')}),
\]
thereby proving (19). \( \square \)

Following the idea in [2], we consider radial functions of the type
\[
v(x, y, t) = ((x^2 + y^2)^2 + z(t))^{1/4},
\]
where \( z : \mathbb{R} \to \mathbb{R} \) is assumed to be twice continuously differentiable and positive. Theorem 2.2 and easy computations (see [1]) show that \( u \) is convex on \( \mathbb{H} \) if and only if
\[
4z(1 + z'') \geq 3(z')^2, \quad \text{on } \mathbb{R}.
\]
Condition (15) is equivalent to the inequality
\[
16(z(t))^2 + r^4(16z(t) - (z'(t))^2) \geq 0.
\]

6 Applications

The aim of this section is to show that, like in the Euclidean framework, the H–subdifferential of a function on the Heisenberg group carries a lot of information about the function itself.

6.1 The Monge–Ampère measure and H–normal map

In the Heisenberg group, the Monge–Ampère type operator \( S_{ma} \) (see [1] and [14]) is a fully nonlinear operator on \( u \) defined by
\[
S_{ma}(u) = \det[\nabla_H^2 u]^* + \frac{3}{4}([X, Y] u)^2 = \det[\nabla_H^2 u]^* + 12(Tu)^2
\]
In [14] the authors proved the following result:

**Theorem 6.1** Given a convex function \( u \in C(\Omega) \), there exists a unique Borel measure \( \mu_u \) such that, when \( u \in C^2(\Omega) \),
\[
\mu_u(E) = \int_E [S_{ma}(u)](g) dg,
\]
for any Borel set \( E \subset \Omega \).
We call $\mu_u$ the Monge–Ampère measure of $u$.

In the Euclidean context (see [12]), the Monge–Ampère measure $M_f$ associated to a function $f$ is defined via the notion of normal map $\partial f$ of $f$ (see (5) and (6)). In particular, if $f \in C(\Omega')$, with $\Omega'$ open in $\mathbb{R}^n$, the Monge–Ampère measure is the set function $M_f : E' \rightarrow [0, \infty]$ defined by

$$M_f(E') = |\partial f(E')| = \int_{\partial f(E')} 1 dp, \quad \forall E' \in E',$$

(18)

where $E' = \{E' \subset \Omega' : \partial f(E') \text{ is Lebesgue measurable}\}$ and $|A|$ denotes the Lebesgue measure of $A$. If $f$ is an Euclidean convex function in $C^2(\Omega')$, we have that

$$M_f(E') = \int_{E'} \det[D^2 f(x)] dx,$$

(19)

for every Borel set $E' \subset \Omega'$. The proof of (19) (see [12]) exploits the property that if $f$ is Euclidean convex and $C^2(\Omega')$, we can identify $\partial f$ with $\nabla f$ and $\nabla f$ is one–to–one on the set $\{x \in \Omega' : D^2 f(x) > 0\}$. Hence every point $p \in \partial f(E')$ is the image of a single point $x \in E'$: this is the reason to put the integrand function in (18) equal to 1, for every $p \in \partial f(\Omega')$.

Our purpose is to suggest a definition (see Theorem 6.3) of the Monge–Ampère measure of $u$ in the Heisenberg context, on the analogy of the Euclidean framework, using the H–normal map $\partial_H u$ of $u$.

We know that if $u$ is a convex function in $\Gamma^1(\Omega)$, then $\partial_H u(g) = \{\nabla_H u(g)\}$; however, it is unreasonable to require that $\nabla_H u : \Omega \rightarrow V_1$ is a one–to–one map, since $\Omega \subset \mathbb{H}$ and $V_1$ is essentially $\mathbb{R}^2$. In other words, every point $v \in \partial_H u(E)$ is the image of a set of points $\Sigma^E_u \subset E$. Therefore we need to replace the weight “1” in integral (18) with a convenient weight. For every $p \in V_1$, the weight of $p$ will be the 2–dimensional spherical Hausdorff measure of $\Sigma^E_u$.

In order to do this, we recall the following coarea formula proved by Magnani in [19]. We refer to [9] for all the relevant definitions about spherical Hausdorff measures.

**Theorem 6.2** Let $F : \Omega \rightarrow \mathbb{R}^2$ be a Lipschitz map, where $\Omega \subset \mathbb{H}$ is a measurable set. Then, for every measurable function $z : \Omega \rightarrow [0, \infty]$, the following formula holds

$$\int_{\Omega} z(g) J_H F(g) dg = \int_{\mathbb{R}^2} \left( \int_{F^{-1}(v) \cap \Omega} z(w) dS^2_{H}(w) \right) dv,$$

(20)

where $dS^2_{H}$ denotes the 2–dimensional spherical Hausdorff measure.

Let $u$ be a convex function in $\Gamma^2(\Omega)$, and consider the function $F : \Omega \rightarrow \mathbb{R}^2$ defined by $F(g) = (X u(g), Y u(g))$. Clearly,

$$DF(g) = \begin{pmatrix} XX u(g) & Y X u(g) & 0 \\ XY u(g) & YY u(g) & 0 \end{pmatrix}.$$
Standard computations give us that
\[ J_H F(g) = \det[\nabla^2 H u](g) = \det[\nabla^2 H u(g)]^* + 4(T u(g))^2. \]

If we consider \( z = 1 \) and \( E \subset \Omega \) measurable, by the formula (20) we obtain

\[
\int_E (\det[(\nabla^2 H u)(g)]^* + 4((T u)(g))^2) \, dg = \int_E J_H F(g) \, dg = \int_{\mathbb{R}^2} \left( \int_{F^{-1}(v) \cap E} dS^2_{\nabla H u}(w) \right) \, dv = \int_{\mathbb{R}^2} (S^2_{\nabla H u}((\nabla H u)^{-1}(v) \cap E)) \, dv = \int_{\partial H u(E)} S^2_{\nabla H u}(\Sigma^E_v) \, dv,
\]

where, for every \( v \in \mathbb{R}^2 \), the set \( \Sigma^E_v \subset \mathbb{H} \) is defined by

\[
\Sigma^E_v = E \cap [(\nabla H u)^{-1}(v)].
\] (21)

Taking into account the arguments above, we state the following theorem where a possible definition for a Monge–Ampère measure associated to \( u \) is provided.

**Theorem 6.3** Let \( u \in \Gamma^{0,1}(\Omega) \), with \( \Omega \) open in \( \mathbb{H} \). Let us consider the function \( \nu_u : \mathcal{E} \to [0, \infty] \) defined by

\[
\nu_u(E) = \int_{\partial H u(E)} S^2_{\nabla H u}(\Sigma^E_v) \, dv, \quad \forall E \in \mathcal{E},
\]

where \( \Sigma^E_v \) is given in (21) and \( \mathcal{E} = \{ E \subset \Omega : E \text{ and } \partial H u(E) \text{ are Lebesgue measurable} \} \).

Then

i. \( \nu_u \) is non negative and \( \sigma \)-additive;

ii. if \( u \) is convex and \( u \in \Gamma^2(\Omega) \), then

\[
\nu_u(E) = \int_E (\det[(\nabla^2 H u)(g)]^* + 4((T u)(g))^2) \, dg
\]

for every \( E \in \mathcal{E} \).

We call \( \nu \), with an abuse of language, the Monge–Ampère measure associated to \( u \). Up to now, we are not able to prove that \( \mathcal{E} \) is a \( \sigma \)-algebra. Indeed, while it is quite trivial that the numerable union of sets in \( \mathcal{E} \) is still a set in \( \mathcal{E} \), it is not clear what happens about the complement of a set in \( \mathcal{E} \). Notice that, for every \( E \in \mathcal{E} \), we have

\[
\partial H u(E^c) = (\partial H u(\Omega) \setminus \partial H u(E)) \cup (\partial H u(\Omega \setminus E) \cap \partial H u(E)).
\]

---

1 we identify \( \partial H u(E) \) with a subset of \( \mathbb{R}^2 \), as we did with \( \nabla H u(g) \).
The main problem is to show that \( \partial_H u(\Omega \setminus E) \cap \partial_H u(E) \) is Lebesgue–measurable; notice that, in the Euclidean framework, this set has null measure.

**Proof:** First of all, let us notice that for every \( E \in \mathcal{E} \) and for every \( v \in V_1 \), the set \( \Sigma_v^E \) is a Borel set; from the Borel regularity of \( S_H^2 \), the set \( \Sigma_v^E \) is \( S_H^2 \)-measurable.

Let us consider a sequence \( \{E_i\}_{i=1}^\infty \) of disjoint sets in \( \mathcal{E} \). It is straightforward that \( \Sigma_v^{\cup_i E_i} = \cup_i \Sigma_v^{E_i} \), and \( \{\Sigma_v^{E_i}\}_i \) is a family of disjoint subsets of \( \Omega \). We get

\[
\nu_u(\cup_i E_i) = \int_{\mathbb{R}^2} S_H^2(\Sigma_v^{\cup_i E_i}) \, dv
= \int_{\mathbb{R}^2} S_H^2(\cup_i \Sigma_v^{E_i}) \, dv
= \int_{\mathbb{R}^2} \sum_i S_H^2(\Sigma_v^{E_i}) \, dv
= \sum_i \int_{\mathbb{R}^2} S_H^2(\Sigma_v^{E_i}) \, dv
= \sum_i \nu_u(E_i).
\]

Hence, \( \mu_u \) is \( \sigma \)-additive. Clearly, ii. is obvious for previous computations. \( \square \)

### 6.2 The Rockafellar function in \( I \mathbb{H} \)

In the Euclidean framework, as well as in the more general Banach setting, a convex function can be detected using its subdifferential at every point via the Rockafellar function (for a new and recent proof, see [10]). We are going to prove that a similar integrability property is inherited by convex functions on the Heisenberg group, where the \( H \)-subdifferential plays nearly the same role. Indeed, the following result holds:

**Theorem 6.4** Let \( u : I \mathbb{H} \to \mathbb{R} \) be a convex function. Then,

\[
u_u(\cup_i E_i) = \int_{\mathbb{R}^2} S_H^2(\cup_i \Sigma_v^{E_i}) \, dv
= \int_{\mathbb{R}^2} S_H^2(\cup_i \Sigma_v^{E_i}) \, dv
= \sum_i \int_{\mathbb{R}^2} S_H^2(\Sigma_v^{E_i}) \, dv
= \sum_i \nu_u(E_i).
\]

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Indeed, let us consider a particular sequence

\[
\{g_i\}
\]

Since \(g_{i+1} \in H_{g_i}\) if and only if \(g_i \in H_{g_{i+1}}\), from \(p_{i+1} \in \partial_H u(g_{i+1})\), we get that

\[
u(g_i) \geq u(g_{i+1}) + \langle p_{i+1}, \xi_1(g_i) - \xi_1(g_{i+1}) \rangle.
\]

Then,

\[
u(g_{i+1}) - u(g_i) - \langle p_i, \xi_1(g_{i+1}) - \xi_1(g_i) \rangle \leq \langle p_{i+1}, \xi_1(g_{i+1}) - \xi_1(g_i) \rangle + \langle p_i, \xi_1(g_{i+1}) - \xi_1(g_i) \rangle
\]

\[
= \langle p_{i+1} - p_i, \xi_1(g_{i+1}) - \xi_1(g_i) \rangle,
\]

for every \(i\), \(0 \leq i \leq n - 1\). Hence, taking into account (23), we obtain:

\[
0 \leq u(g) - u(g_0) - \sum_{i=0}^{n-1} \langle p_i, \xi_1(g_{i+1}) - \xi_1(g_i) \rangle \leq \sum_{i=0}^{n-1} \langle p_{i+1} - p_i, \xi_1(g_{i+1}) - \xi_1(g_i) \rangle, \tag{24}
\]

In order to prove (22), we will show that, for every \(\epsilon > 0\), there exists a finite sequence in \(\mathcal{P}\) such that

\[
\sum_{i=0}^{n-1} \langle p_{i+1} - p_i, \xi_1(g_{i+1}) - \xi_1(g_i) \rangle \leq \epsilon. \tag{25}
\]

Let us consider, first, the case \(\xi_1(g_0) = \xi_1(g)\); this implies that \(g \not\in H_{g_0}\). Choose \(g' \in H_{g_0}\) and \(g'' \in H_{g'} \cap H_g\); from \(g_0 \neq g' \in H_{g_0}\), we have that \(\xi_1(g') \neq \xi_1(g)\) and hence \(H_{g'} \cap H_g \neq \emptyset\). Take \(p_0 \in \partial u_H(g_0)\), \(p' \in \partial u_H(g')\), \(p'' \in \partial u_H(g'')\) and \(p \in \partial u_H(g)\). For every \(\epsilon > 0\), denote by \(N\) a positive integer such that

\[
\epsilon N \geq \langle p' - p_0, \xi_1(g') - \xi_1(g_0) \rangle + \langle p'' - p', \xi_1(g'') - \xi_1(g') \rangle + \langle p - p'', \xi_1(g) - \xi_1(g'') \rangle. \tag{26}
\]

We will single out a set of points \(\{p_i\}_{i=0}^{3N}\) on the broken line \([g_0, g'] \cup [g', g''] \cup [g'', g]\). Indeed, let us consider a particular sequence \(\{(g_i, p_i)\}_{i=0}^{3N} \in \mathcal{P}\) defined as follows:

i) for \(i = 1, \ldots, N - 1\), we pick out \(g_i \in [g_0, g'] \subset H_{g_0}\) such that \(\xi_1(g_i) = (\xi_1(g') - \xi_1(g_0))i/N + \xi_1(g_0)\);

ii) we set \(g_N = g'\) and \(p_N = p'\);

iii) for \(i = N + 1, \ldots, 2N - 1\), we pick out \(g_i \in [g', g''] \subset H_{g'}\) such that \(\xi_1(g_i) = (\xi_1(g'') - \xi_1(g'))(i - N)/N + \xi_1(g')\);

iv) we set \(g_{2N} = g''\) and \(p_{2N} = p''\);

v) for \(i = 2N + 1, \ldots, 3N - 1\), we pick out \(g_i \in [g'', g] \subset H_g\) such that \(\xi_1(g_i) = (\xi_1(g) - \xi_1(g''))(i - 2N)/N + \xi_1(g'')\);

vi) we set \(g_{3N} = g\) and \(p_{3N} = p\);

vii) for every \(i\), with \(1 \leq i \leq 3N - 1\) and \(i\) different from \(N\) and \(2N\), we choose \(p_i \in \partial u_H(g_i)\).
Notice that \( g_{i+1} \in H_{g_i} \), for every \( i \), \( 0 \leq i \leq 3N - 1 \). From i)–vii) and (26), we obtain

\[
\sum_{i=0}^{3N-1} \langle p_{i+1} - p_i, \xi_1(g_{i+1}) - \xi_1(g_i) \rangle = \\
= \left( \sum_{i=0}^{N-1} + \sum_{i=N}^{2N-1} + \sum_{i=2N}^{3N-1} \right) \langle p_{i+1} - p_i, \xi_1(g_{i+1}) - \xi_1(g_i) \rangle \\
= \sum_{i=0}^{N-1} \langle p_{i+1} - p_i, \xi_1(g') - \xi_1(g_0) \rangle/N + \sum_{i=N}^{2N-1} \langle p_{i+1} - p_i, \xi_1(g'') - \xi_1(g') \rangle/N + \sum_{i=2N}^{3N-1} \langle p_{i+1} - p_i, \xi_1(g) - \xi_1(g'') \rangle/N \\
= \frac{\langle p_N - p_0, \xi_1(g') - \xi_1(g_0) \rangle}{N} + \frac{\langle p_{2N} - p_N, \xi_1(g'') - \xi_1(g') \rangle}{N} + \frac{\langle p_{3N} - p_{2N}, \xi_1(g) - \xi_1(g'') \rangle}{N} \\
\leq \epsilon.
\]

Hence (25) holds.

If \( g \notin H_{g_0} \) and \( \xi_1(g) \neq \xi_1(g_0) \), we set \( g' = g_0 \) and choose \( g'' \in H_{g_0} \cap H_g \). The proof is similar to the previous case.

Finally, if \( g \in H_{g_0} \), we set \( g' = g_0 \) and \( g'' = g \) : again, the proof follows the line of the previous case. □

References

[1] L. Ambrosio and S. Rigot. Optimal mass transportation in the Heisenberg group. *Journal of Functional Analysis*, 208, 2004.

[2] Z.M. Balogh and M. Rickly. Regularity of convex functions on Heisenberg groups. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 2:847–868, 2003.

[3] A. Calogero, G. Carcano, and R. Pini. Twisted convex hulls in the Heisenberg group. *J. Convex Anal.*, 14:607–619, 2007.

[4] A. Calogero, G. Carcano, and R. Pini. On weakly H–quasiconvex functions on the Heisenberg group. *J. Convex Anal.*, 15, 2008.

[5] A. Calogero and R. Pini. Note on the Fenchel transform in the Heisenberg group. *in preparation*.

[6] D. Danielli, N. Garofalo, and D.M. Nhieu. Notions of convexity in Carnot groups. *Comm. Anal. Geom.*, 11:263–341, 2003.
[7] D. Danielli, N. Garofalo, and S. Salsa. Variational inequalities with lack of ellipticity. I. Optimal interior regularity and non–degeneracy of the free boundary. Indiana Univ. Math. J., 52:361–398, 2003.

[8] L. C. Evans and R. F. Gariepy. Measure Theory and Fine Properties of Functions. CRC Press, 1991.

[9] H. Federer. Geometric Measure Theory. Springer, 1969.

[10] W. Gangbo and R.J. McCann. The geometry of optimal transportation. Acta Math., 177, 1996.

[11] N. Garofalo and F. Tournier. New properties of convex functions in the Heisenberg group. Trans. Amer. Math. Soc., 5(358):2011–2055, 2005.

[12] C. E. Gutiérrez. The Monge-Ampère Equation. Birkhäuser, Boston, MA, 2001.

[13] C. E. Gutiérrez and A. Montanari. On the second order derivatives of convex functions on the Heisenberg group. Ann. Sc. Norm. Super. Pisa Cl. Sci. III, 5:349–366, 2004.

[14] C. E. Gutiérrez and A. Montanari. Maximum and comparison principles on the Heisenberg group. Communications in Partial Differential Equations, 29:1305–1334, 2005.

[15] C. E. Gutiérrez and T. van Nguyen. On Monge–Ampère type equations arising in optimal transportation problems. Calculus of Variations, 28:275–316, 2007.

[16] M. Ivanov and N. Zlateva. A new proof of the integrability of the subdifferential of a convex function on a Banach space. Proc. Amer. Math. Soc., 136, 2008.

[17] G. Lu, J.J. Manfredi, and B. Stroffolini. Convex functions on the Heisenberg group. Calculus of Variations, 19:1–22, 2004.

[18] R. Lucchetti. Convex and Well–Posed Problems. Springer Verlag, 2006.

[19] V. Magnani. Blow–up of regular submanifolds in Heisenberg groups and applications. Central European Journal of Mathematics, 4 (1):82–109, 2006.

[20] R. Monti and M. Rickly. Geodetically convex sets in the Heisenberg group. J. Convex Analysis, 12:187–196, 2005.

[21] P. Pansu. Métriques de Carnot–Carathéodory et quasi–isométries des espaces symétriques de rang un. Ann. of Math, 129(2):1–60, 1989.

[22] R.T. Rockafellar. Convex Analysis. Princeton University Press, 1969.