(De)Constructing Dimensions and Non-commutative Geometry

Mohsen Alishahiha

Institute for Theoretical Physics, University of Amsterdam,
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

Abstract

In this paper the model considered by Arkani-Hamed, Cohen and Georgi in the context of (de)constructing dimensions has been studied by making use of non-commutative geometry. The non-commutative geometry provides a natural framework to study this model with or without gravity.
We are all agreed that your theory is crazy. The question which divides us is whether it is crazy enough.

- Niels Bohr

1 Introduction

Although it seems we live in a four-dimensional world, it has been suspected that at short distance, shorter than it has been probed yet, the best description of our world could be provided by a theory with more than four-dimensional spacetime. The simplest example could be that with four extended dimensions plus one compact dimension. In this case, at the distance much bigger than the size of the radius of the compact direction, the theory looks like a four-dimensional theory, while for the distance comparable to the size of the compact direction, the effects of the five-dimensional theory will be appeared. A generic feature of the theories with a small compact dimension is that, the higher dimensional theory appears at high energy (UV limit), while the lower dimension description emerges at large distance (IR limit).

In an reversal picture, the authors in [1] considered a theory in which the higher dimensional description is given in the IR limit. In fact, in UV limit, where theories with higher than four dimensions are going to be problematic, the theory is well described in terms of a four dimensional theory, and actually in the extreme UV, the theory is perfectly four dimensional! By making use of this strategy the authors in [1] suggested a way for a UV completions of the higher dimensional field theories.

In fact, this is a generic property of the field theory on a four extended dimensions plus some discrete extra dimensions. Here we shall only consider one extra discrete dimension. Suppose \( a \) be a length scale of the discrete dimension and \( l \) the scale of the four-dimensional theory. In general, the different phases of the theory are parameterized by a dimensionless parameter \( \zeta = a/l \). In the limit \( \zeta \gg 1 \) (UV) the theory is four dimensional and for \( \zeta \ll 1 \) (IR) the description of the theory is given by a five dimensional field theory.

If we think about this theory as a five-dimensional field theory latticized in one dimension, the parameter \( a \) plays the role of the lattice size and IR limit is the limit in which \( a \to 0 \) where we recover five continuous dimensional theory.

The model which has been studied in [1] is a quiver ( "moose" in their notation) model with \( SU(n)^N \) gauge theory coupled to \( N \) non-linear sigma model which can be obtained by starting with a quiver model with gauge group \( \prod_{i=1}^{N} SU_i(n) \times SU_i(m) \times SU_{i+1}(n) \) where \( i = 1 \) is periodically identified with \( i = N + 1 \). There are also fermions transforming bi-linearly under nearest-neighbor pairs of the gauge transformation. Suppose \( \Lambda_n \) and \( \Lambda_m \) be the energy scales of the gauge groups \( SU(n) \) and \( SU(m) \), respectively. For the limit where \( \Lambda_m \gg \Lambda_n \) and in the energy of order \( \Lambda_m \), the \( SU(m) \) groups become strong, causing the fermions to condense in pairs. The confining strong interactions also produce a spectrum of hadrons with masses
on the order of $\Lambda_m$. Therefore, below the scale $\Lambda_m$ the theory can be described as a gauge theory with gauge group $SU(n)^N$ coupled to a $N$ non-linear sigma model.\footnote{An infinite arrays of gauge theories has also been studied in\cite{2}, where an infinite number of gauge theories are linked by scalars to get an infinite tower of massiv e vector mesons ("hadrons") with a small coupling only for the single zero mass photon. It seems that this theory is in the same class as one considered in\cite{1}.}

One could think about this theory as a latticization of a five-dimensional gauge theory with lattice size $a \sim (g\Lambda_m)^{-1}$ where $g$ is the gauge coupling of the gauge group $SU(n)$. While the theory is a four-dimensional gauge theory in UV, at large distance it turns out to be a five-dimensional gauge theory compactified on a circle of circumference $R = Na$. In this sense, the extra dimension has been generated dynamically.\footnote{A possible connection between the non-commutative geometry and those theories with extra dimensions, like the Randall-Sundrum model\cite{4}, has also been observed in\cite{5}.}

In general, having a manifold with a discrete dimension would technically cause a problem, as the classical notion of the differential geometry fails for such a manifold. In particular, the notion of curvature and torsion, which we need if we wish to add gravity in the game, are not well-defined for such a manifold in terms of the classical differential geometry. Fortunately, there is a generalization of the classical geometry for such a manifold in the context of, so called, non-commutative geometry.\cite{3} Non-commutative geometry provides a strong tools to study a manifold with discrete dimension, and in fact, using non-commutative geometry one can study manifolds which could have no well-known geometrical picture. Indeed, a well-known example in the non-commutative geometry is what we are interested in, i.e. a four-dimensional manifold times a discrete set of $N$ points which altogether can be thought as a five-dimensional spacetime with four extended dimensions and one discrete dimension.

This is the aim of this note to reconsider the model studied in\cite{1} in the framework of the non-commutative geometry.\footnote{An infinite arrays of gauge theories has also been studied in\cite{2}, where an infinite number of gauge theories are linked by scalars to get an infinite tower of massive vector mesons ("hadrons") with a small coupling only for the single zero mass photon. It seems that this theory is in the same class as one considered in\cite{1}.} An advantage using this point of view is that, by making use of the non-commutative geometry one can easily add the gravity in the theory, much similar to what we would like to do for Yang-Mills sector. Indeed, one could have both Yang-Mills and gravity sectors in the same time. In particular, the dynamically generating dimension procedure\footnote{A possible connection between the non-commutative geometry and those theories with extra dimensions, like the Randall-Sundrum model\cite{4}, has also been observed in\cite{5}.} works for both gravity and Yang-Mills sectors in the same way.

This paper is organized as following: In the section 2, we shall present a brief review of the non-commutative geometry. In section 3, we will consider a non-commutative space which is given by a four-dimensional flat manifold times a discrete set of $N$ points which can be thought as $N$ parallel four dimensional layers (sheets). Then we construct a gauge theory in this space which corresponds to a four-dimensional gauge theory with gauge group $SU(n)^N$ coupled to $N$ charged scalers which in general have a quartic potential. This model is very similar to that considered in\cite{1}. In section 4, we will study the gravity in this space. Finally, we shall give our conclusions in the section 5.
We note, however, that the formulas given in this paper have already been presented in the literature, mostly, for the case of $N = 2$ in the context of the non-commutative geometry applied to standard model of the particle physics. The thing is new in this paper would be the generalization of those results for $N$-points space, especially for the gravity sector, plus a new interpretation in the context of the dynamically generating dimension.

In this paper we use the following convention: the signature of the metric is $(-,+,+,\cdots,+)$ and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $[\gamma^\mu, \gamma^\nu] = 2\gamma^{\mu\nu}$, $(\gamma^5)^2 = 1$.

2 A Brief Review of Non-commutative Geometry

There is a well-known theorem due to Gelfand and Naimark that a smooth manifold, $\mathcal{M}$, can be studied by analyzing the commutative algebra $C^\infty(\mathcal{M})$ of smooth functions defined on $\mathcal{M}$. In other words, the smooth manifold $\mathcal{M}$ can be reconstructed from the structure of $C^\infty(\mathcal{M})$. The basic idea in the non-commutative geometry is how to define a compact, non-commutative space in terms of a unital, non-commutative $\ast$-algebra $\mathcal{A}$ [3].

Given a unital, non-commutative $\ast$-algebra $\mathcal{A}$ one can define the universal, differential algebra $\Omega(\mathcal{A})$ for the non-commutative space. For this purpose, assume $d$ to be an abstract differential operator which acts on elements of $\mathcal{A}$ and satisfies the Leibniz rule, with $d1 = 0, d^2a = 0$ where $1, a \in \mathcal{A}$. Therefore we have

$$\Omega(\mathcal{A}) = \bigoplus_{n=0}^{\infty} \Omega^n(\mathcal{A})$$

with $\Omega^0(\mathcal{A}) = \mathcal{A}$ and

$$\Omega^n(\mathcal{A}) = \left\{ \sum_i a_0^i da_1^i \cdots da_n^i \mid a_j^i \in \mathcal{A}, \forall i, j \right\}, \quad n = 1, 2, \cdots \quad (2)$$

In fact $\Omega^n(\mathcal{A})$ plays the role of space of $n$-form in the non-commutative geometry.

The next ingredient which plays an important role in the differential structure of the non-commutative geometry is the notion of Dirac $K$-cycle for $\mathcal{A}$. The Dirac $K$-cycle is defined by a doublet $(\mathcal{H}, D)$ where $\mathcal{H}$ is a Hilbert space and $D$ a selfadjoint operator on $\mathcal{H}$ (Dirac operator), together with an involutive representation, $\pi$, of $\mathcal{A}$ on $\mathcal{H}$

$$\pi : \mathcal{A} \rightarrow B(\mathcal{H}), \quad \pi(a^*) = \pi(a)^*, \quad \forall a \in \mathcal{A}$$

where $B(\mathcal{H})$ is the algebra of the bounded operator on $\mathcal{H}$.

Given a Dirac $K$-cycle for $\mathcal{A}$, one can define an involutive representation of $\Omega(\mathcal{A})$ on $\mathcal{H}$. This is provided by the map $\pi : \Omega(\mathcal{A}) \rightarrow B(\mathcal{A})$ in such a way that, for any element $\sum_i a_0^i da_1^i \cdots da_n^i \in \Omega^n(\mathcal{A})$, $n = 1, 2, \cdots$ we have

$$\pi \left( \sum_i a_0^i da_1^i \cdots da_n^i \right) = \sum_i \pi(a_0^i)[D, \pi(a_1^i)] \cdots [D, \pi(a_n^i)].$$

(4)
We note, however, that the representation $\pi$ is ambiguous \cite{6}. This can be seen as following. Suppose $\rho \in \Omega(A)$ be a one form. If $\pi(\rho)$ is set to zero, $\pi(d\rho)$ is not necessarily zero. This fact leads us to define a set of auxiliary fields which appear because of this ambiguity. By making use of the space of the auxiliary fields we can correct the definition of the space of forms such that the ambiguity will be removed.

The space of the auxiliary fields is defined by $\text{Aux} = \text{Ker} \pi + d \text{Ker} \pi$, where

$$
\text{Ker} \pi = \bigoplus_{n=0}^{\infty} \left\{ \sum_i a_i^0 da_i^1 \cdots da_i^n | \pi \left( \sum_i a_i^0 da_i^1 \cdots da_i^n \right) = 0 \right\},
$$

$$
d \text{Ker} \pi = \bigoplus_{n=0}^{\infty} \left\{ \sum_i da_i^0 da_i^1 \cdots da_i^n | \pi \left( \sum_i a_i^0 da_i^1 \cdots da_i^n \right) = 0 \right\}. \quad (5)
$$

The space of the auxiliary fields is a two-sided ideal in $\Omega(A)$ and this can be used to define the correct space of the forms as $\Omega_D(A) = \Omega(A)/\text{Aux}$. Therefore for a element $\sum_i a_i^0 da_i^1 \cdots da_i^n \in A$,

$$
\left\{ \sum_i \pi(a_i^0)[D, \pi(a_i^1)] \cdots [D, \pi(a_i^p)] + \pi(\alpha) | \alpha \in \text{Aux} \right\}
$$

represents an $n$-form (mod Aux) in $\Omega^n_D$ as an equivalence class of bounded operators on the Hilbert space $\mathcal{H}$.

The integral of a form $\beta \in \Omega(A)$ over a non-commutative space $A$ is defined by

$$
\int \beta = \text{Tr}_\omega \left( \pi(\beta)D^{-d} \right), \quad (7)
$$

where $\text{Tr}_\omega$ is the Dixmier trace and $d$ is the dimension of the space represented by $A$. The Dixmier trace is defined by

$$
\text{Tr}_\omega(|T|) = \lim_{\omega \to 0} \frac{1}{\log N} \sum_i \mu_i(T), \quad (8)
$$

where $T$ is a compact operator, and $\mu_i$ are the eigenvalues of $|T|$.

One can also define a vector bundle over a non-commutative space $A$, which is a free, projective $A$-module. In fact a vector bundle, $E$, is defined by the vector space $E$ of its section which is going to be a free, projective, left $A$-module. Here we are interested in the case $E = A$.

By making use of the structure of the non-commutative geometry, we will be able to formalize a gauge theory on a non-commutative space. The procedure to define the Yang-Mills action is as following. As in the commutative case, we would like to have a gauge connection and curvature which are one- and two-form, respectively. Suppose $A \in \Omega^1(A)$ be a gauge connection. It can be expressed as

$$
A = \sum_\alpha g_\alpha \, df_\alpha \, . \quad (9)
$$
with the condition $\sum \alpha g_\alpha f_\alpha = 1$. We need to impose this condition in order to get correct gauge transformation under the unitary gauge group $U(A) = \{g \in A \mid g^* g = 1\}$ (see for example [7]). Of course this is no loss in generality, as the field $\sum \alpha g_\alpha f_\alpha$ is independent. As the usual case, the curvature is defined by $F = dA + A^2$. Finally the Yang-Mills action is given by

$$S_{YM} = \frac{1}{8} \text{Tr}_\omega \left( \pi^2(F) D^{-4} \right),$$

(10)

here we assumed that the manifold represented by $\mathcal{M}$ is a four-dimensional manifold. In the case we are interested in, the action (10) reads (for example see [7])

$$S_{YM} = \frac{1}{8} \int d^4x \sqrt{\text{det}(g)} \text{Tr} \left( \pi^2(F) \right),$$

(11)

where $g$ is the metric and the trace, Tr, is taken over both the Clifford algebra and the matrix structure.

### 3 N layers Model

In this section we shall consider a non-commutative space which is taken to be a product of a continuous four-dimensional manifold times a discrete set of $N$ points. Here, we assume that the four-dimensional space is a flat space, and therefore this system could be thought as $N$ parallel four dimensional layers. The proper algebra for this model is (we will only consider the case with $N \geq 3$)

$$\mathcal{A} = C^\infty(\mathcal{M}_4) \otimes ( \oplus_{i=1}^N M_n(\mathbb{C}) ).$$

(12)

The Dirac operator can be chosen as follow

$$D = \sum_{i=1}^N \left[ \gamma^\mu \partial_\mu e_{i,i} + \gamma^5 \frac{K}{\sqrt{2}} (e_{i,i+1} + e_{i,i-1}) \right],$$

(13)

where $e_{i,j}$ is an $N \times N$ matrix with $(e_{i,j})_{ab} = \delta_{ia} \delta_{jb}$. $K$ is an $n \times n$ matrix which in our case, it is chosen to be diagonal $K = M \mathbb{1}$. Here, we used the notation in which $e_{1,0} \equiv e_{N,1}$ and $e_{N,N+1} \equiv e_{1,N}$, that means the $(N+1)$-th layer is identified with the first one. In other words, we are dealing with a compact discrete direction which could be considered as a circle with circumference $R = Na$ with $a = M^{-1}$.

A representation of any elements $f \in \mathcal{A}$ in $\mathcal{H}$ is

$$\pi(f) = \sum_{i=1}^N f^i(x) e_{i,i},$$

(14)

---

3 Using the notion of distance in the non-commutative geometry, one can see that the distance between the $(i+1)$-th and $i$-th layers is $a = M^{-1}$. For recent discussion on the notion of distance in non-commutative geometry see [8].
where $f^i(x) := f(x, y + ia)$ is a function on the manifold $\mathcal{M}_4$ defined at $i$-th layer and $y$ is the coordinate of the discrete direction. Now, we would like to study a gauge theory on this space. This model corresponds to a four dimensional gauge theory with the gauge group $SU(n)^N$ coupled to $N$ charged scalars which in general have quartic potential.

Using the non-commutative formalizem of gauge theory introduced in the previous section, we find the following expression for the gauge connection

$$\pi(A) = \sum_\alpha \pi(g) [D, \pi(f)]$$

$$= \sum_{i=1}^N \left[ \gamma^\mu A^i_\mu e_{i,i} + \gamma^5 \frac{M}{\sqrt{2}} (\phi^{i,i+1} e_{i,i+1} + \phi^{i,i-1} e_{i,i-1}) \right], \quad (15)$$

where

$$A^i_\mu = \sum_\alpha g^i_\alpha \partial_\mu f^i_\alpha,$$

$$\phi^{i,i+1} = \sum_\alpha g^i_\alpha (f^{i+1}_\alpha - f^i_\alpha),$$

$$\phi^{i,i-1} = \sum_\alpha g^i_\alpha (f^{i-1}_\alpha - f^i_\alpha). \quad (16)$$

Similarly, one can also write down the representation of the curvature, $\pi(F) = \pi(dA) + \pi(A)^2$. Plugging the result into the equation (10), we can find the Yang-Mills action. Setting $U^{i,j} = \phi^{i,j} + 1$, we get

$$S_{YM} = \int d^4 x \sum_{i=1}^N \text{tr} \left[ -\frac{1}{4g^2} F^i_{\mu\nu} F^i_{\mu\nu} - \frac{1}{2} \frac{f_s^2}{D_\mu U^{i,i+1} D_\mu U^{i,i+1} + \ldots} \right], \quad (17)$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + [A^i_\mu, A^i_\nu],$$

$$D_\mu U^{i,i+1} = \partial_\mu U^{i,i+1} + A^i_\mu U^{i,i+1} - U^{i,i+1} A^i_\mu,$$

$$\overline{D_\mu U^{i,i+1}} = \partial_\mu U^{i+1,i} + A^{i+1,i}_\mu U^{i+1,i} - U^{i+1,i} A^i_\mu, \quad (18)$$

and $f_s^2 = M^2/g^2$ with $g^2$ being the gauge coupling. The dots represent a combination of the potential for the scalars $U^{i,j}$ and the auxiliary fields as well, which their forms are not important for our purpose. Actually, as we already mentioned in the previous section, the auxiliary fields can be quotiented out. Alternatively, they can eliminated by their equations of motion, as they are not a dynamical field. Doing so, we will get a quartic potential for the scalars. These scalars can get vacuum expectation values, and therefore this model would be equivalent to one considered

\[^4\text{For precise form of the auxiliary fields and their role in the non-commutative geometry, the reader is referred to, for example, [9].}\]
as a theory which dynamically generates fifth-dimension. Indeed, in the non-commutative geometry framework, this fifth-dimension is nothing but the discrete dimension.

It is worth to note that as the distance between layers gets smaller and smaller, we will recover a five-dimensional gauge theory with the gauge group $SU(n)$. Physically, what we mean by $a \to 0$ is that, we are approaching the IR limit where the energy scale of the theory is much smaller than the scale of the discrete dimension $g f_s$. To see this, we note that in the non-commutative geometry the scalars $\phi^{i,j}$ play the role of the gauge field in the discrete direction. To make this statement clear, we rewrite the scalars as follow

$$M\phi^{i,i+1} = \sum_\alpha g^i_\alpha \partial_5 f^i_\alpha := A^i_5,$$

$$M\phi^{i,i-1} = -\sum_\alpha g^i_\alpha \bar{\partial}_5 f^i_\alpha := -\bar{A}^i_5,$$

where $\partial_5 (\bar{\partial}_5)$ is left (right) discrete derivative

$$\partial_5 f(y) = \frac{f(y+a) - f(y)}{a}, \quad \bar{\partial}_5 f(y) = \frac{f(y) - f(y-a)}{a}.$$  

As $a \to 0$, these two derivatives become equal and therefore we get $A^i_5 = \bar{A}^i_5$. Using this definition, the gauge connection (15) reads

$$\pi(A) = \sum_{i=1}^N \left[ \gamma^\mu A^i_\mu e_{i,i} + \frac{\gamma^5}{\sqrt{2}} (A^i_5 e_{i,i+1} - \bar{A}^i_5 e_{i,i-1}) \right].$$

Therefore, we get the following Yang-Mills action

$$S_{YM} = \int d^4x \sum_{i=1}^N \frac{1}{4g^2} F^i_{\mu\nu} F^i_{\mu\nu} - \frac{1}{2g^2} \bar{F}^i_{\mu5} F^i_{\mu5} + \cdots,$$

here the dots represent the auxiliary field which can be integrated out. In fact, if we had started with the corrected space of form, we would not have seen the dots in the expression (22). Moreover

$$F^i_{\mu5} = \partial_\mu A^i_5 - \partial_5 A^i_\mu + A^i_\mu A^i_5 - A^i_5 A^i_{\mu+1},$$

$$\bar{F}^i_{\mu5} = \partial_\mu \bar{A}^i_5 - \bar{\partial}_5 A^i_\mu + \bar{A}^i_\mu \bar{A}^i_5 - \bar{A}^i_5 \bar{A}^i_{\mu-1}.$$  

Here we have applied the definition of the right and left discrete derivative to $A^i_\mu$.

As $a \to 0$, we have $\bar{F}^i_{\mu5} = \bar{F}^i_{\mu5}$, and moreover the summation can be replaced by an integral, more precisely we have $\sum_{i=1}^N \to \frac{1}{a} \int_0^N dy$, therefore the action (22) reads

$$S_{YM} = \int d^4x \int dy \text{ tr} \left[ -\frac{1}{4g^2} F_{pq} F_{pq} \right],$$

here the dots represent the auxiliary field which can be integrated out.
where
\[ F_{pq} = \partial_p A_q - \partial_q A_p + [A_p, A_q], \quad p, q = 1, \cdots, 5 \tag{25} \]
is the five-dimensional curvature and \( g_5^2 = Rg_4^2 \) with \( g_4 = g/N \), which is the gauge coupling of the diagonal subgroup of the original gauge group.

In order to find the Kaluza-Klein spectrum of the compactified five-dimensional theory, we need the equation of motion of a massless scalar. In the model we are considering, it is given by
\[ \text{Tr}[ D, [D, \pi(\psi)]] = 0 \tag{26} \]

where the trace is taken over both the Clifford algebra and the matrix structure. Setting \( \psi_j = \varphi(x) \exp(ik(y + ja)) \), we find
\[ g^\mu\nu \partial_\mu \partial_\nu \varphi(x) + \left( \frac{2}{\alpha} \right)^2 \sin^2\left( \frac{ka}{2} \right) \varphi(x) = 0. \tag{27} \]

Note that since the discrete direction is compact we have \( k = 2\pi l/Na \) for \( l = 0, 1, \cdots, N \). Therefore, in the limit of \( l \ll N \), we recover precisely the correct Kaluza-Klein spectrum\(^5\)
\[ M_{KK} = \frac{2\pi l}{R}. \tag{28} \]

It can be also seen that in this limit, the five-dimensional Lorentz invariant is automatically restored.

### 4 Gravity Sector

In this section we are going to introduce the gravity in our model. In fact one advantage of looking at the model considered in \( \square \) from the non-commutative geometry point of view is that, in the framework of the non-commutative geometry we will be able to formalize the theory of gravity on the non-commutative space much similar to what we have for the gauge theory. Although we shall only study the gravity sector, it is possible to have the gravity coupled to the gauge sector in the same time. We note that the gravity in the non-commutative geometry has been studied in \( \square \) as the gravity sector of Standard Model. Actually the content of this section is generalization of that in \( \square \) to the \( N \)-point space, though, our point of view is a little different.

Consider a space which is taken to be a product of a continuous four dimensional manifold times a discrete set of \( N \) point. It is very similar to what we had in the previous section, though, here we will drop the assumption of the flatness of the four dimensional spacetime. Moreover the distance between the layers is not taken to be constant. Nevertheless, the algebra \( \mathcal{A} \) has the same structure
\[ \mathcal{A} = C^\infty(\mathcal{M}_4) \otimes \left( \bigoplus_{i=1}^N M_n(\mathcal{C}) \right). \tag{29} \]
We would also like to introduce a local orthonormal basis for the cotangent bundle, $\Omega^1_D(A)$. Here we use the following convention for indices: the capital letters $A, B, \cdots$ run from 1 to 5 and the indices $a, b, \cdots$ run from 1 to 4. The basis of the cotangent bundle, $\{e^a\}$, is

$$\pi(e^a) = \sum_{i=1}^N \gamma^a e_{i,i}, \quad \pi(e^5) = \sum_{i=1}^N \frac{\gamma^5}{\sqrt{2}} (e_{i,i+1} - e_{i,i-1}),$$

(30)

with $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $(\gamma^5)^2 = 1$. The hermitian structure on $\Omega^1_D(A)$ with the proper normalized trace, $Tr$, is given by

$$\langle e^A, e^B \rangle = Tr(e^A (e^B)^*) = \delta^{AB},$$

(31)

which is essentially defined in terms of the Dixmier trace.

The Dirac operator can be chosen as follow

$$D = \sum_{i=1}^N \left[ \gamma^a e^\mu_a \partial_\mu e_{i,i} + \frac{\gamma^5}{\sqrt{2}} K_i (e_{i,i+1} + e_{i,i-1}) \right]$$

(32)

where $K_i = \lambda \phi_i(x) 1$. This means that the distance between the four-dimensional spaces is different. Nevertheless, we assume that the expectation value of $\phi_i$ is constant of order one. Essentially, $\lambda$ plays the same role as $M$ in the previous section, in particular, the length of the compact discrete direction is $R = N\lambda^{-1}$, which we shall assume to be fixed. Note that, $\{e^\mu_a\}$ in (32) is a vierbein, i.e. an orthonormal basis of the section of the tangent bundle, so that

$$e^\mu_a g_{\mu\nu} e^\nu_b = \eta_{ab}, \quad e^\mu_a \eta^{ab} e^\nu_b = g^{\mu\nu}.$$  

(33)

Suppose $\rho = \sum_\alpha g_\alpha df_\alpha$ be a one form, i.e. $\rho \in \Omega^1_D(A)$, using the Dirac operator (32), we get

$$\pi(\rho) = \sum_{i=1}^N \left[ \gamma^a e^\mu_a \rho_\mu e_{i,i} + \frac{\gamma^5}{\sqrt{2}} \lambda \phi_i(x) (\rho_{5i} e_{i,i+1} - \bar{\rho}_{5i} e_{i,i-1}) \right],$$

(34)

where $\rho_\mu, \rho_{5i}, \bar{\rho}_{5i}$ are defined the same as those in (16), of course with a different sign for $\bar{\rho}_{5i}$. Using the spacetime gamma matrices, $\gamma^\mu = \gamma^a e^\mu_a$, $\gamma^5 = \gamma^5 e^5_5$, with $e^5_5 = \phi_i(x)$, the expression of the one-form (34) can be recast as

$$\pi(\rho) = \sum_{i=1}^N \left[ \gamma^\mu \rho_\mu e_{i,i} + \frac{\gamma^5}{\sqrt{2}} \lambda (\rho_{5i} e_{i,i+1} - \bar{\rho}_{5i} e_{i,i-1}) \right],$$

(35)

We use indices with dot for cotangent or tangent space in order not to be confused with spacetime indices.
which is essentially analogous to (21) for non-flat space. It can also be shown that
the expression of
\[ \pi(\partial \rho) \]
modulo the auxiliary fields is
\[ \pi(\partial \rho) = \sum_{i=1}^{N} \left[ \gamma^{\mu \nu} \partial_{\mu} \rho_{\nu i} e_{i,i} + \frac{\gamma^{\mu \nu} i}{\sqrt{2}} \left( \partial_{\mu} \rho_{5i} + \rho_{\mu i} - \rho_{\mu i+1} \right) e_{i,i+1} - \frac{\gamma^{\mu \nu} i}{\sqrt{2}} \left( \partial_{\mu} \rho_{5i} + \rho_{\mu i-1} - \rho_{\mu i} \right) e_{i,i-1} \right] . \] (36)

A connection, \( \nabla \), on \( \Omega_{D}^{1}(A) \) is defined by
\[ \nabla e^{A} = -\omega^{AB} \otimes e^{B} \] with \( \omega^{AB} \in \Omega_{D}^{1}(A) \). Using equation (35), it can be seen that \( \pi(\nabla) \) in the basis \( \{ e^{A} \} \) has following general form
\[ \pi(\omega^{AB}) = \sum_{i=1}^{N} \left[ \gamma^{\mu \nu} \omega^{AB}_{\mu i} e_{i,i} + \frac{\gamma^{5 \nu} i}{\sqrt{2}} \left( \chi^{AB}_{i} e_{i,i+1} - \bar{\chi}^{AB}_{i} e_{i,i-1} \right) \right] , \] (37)

From the hermiticity property of \( \nabla \) we have
\[ \omega^{AB}_{\mu i} = -\omega^{BA}_{\mu i} , \quad \chi^{AB}_{i} = -\bar{\chi}^{BA}_{i} . \] (38)

The components of the torsion and Riemann curvature defined by
\[ T^{A} = T(\nabla) e^{A} \]
and \( \mathcal{R}(\nabla)e^{A} = \mathcal{R}^{AB} \otimes e^{B} \) respectively, are given by \[ T^{A} = \pi(de^{A}) + \pi(\omega^{AB}) \pi(e^{B}) , \]
\[ \mathcal{R}^{AB} = \pi(dw^{AB}) + \pi(\omega^{AC}) \pi(\omega^{CB}) . \] (39)

Using the most general expression of the one- and two-form, (35) and (36), the components of the torsion and curvature can be written as follow
\[ T^{a} = \sum_{i=1}^{N} \left[ \gamma^{\mu \nu} \left( \partial_{\mu} e^{a}_{\nu} + \omega^{ab}_{\mu i} e^{b}_{\nu} \right) e_{i,i} + \frac{\gamma^{5 \nu} i}{\sqrt{2}} \left( \omega^{5 \nu}_{\mu i} e^{5}_{5i} - \lambda \chi^{ab}_{5i} e^{b}_{\mu} \right) e_{i,i+1} - \frac{\gamma^{5 \nu} i}{\sqrt{2}} \left( \omega^{5 \nu}_{\mu i} e^{5}_{5i} - \lambda \bar{\chi}^{ab}_{5i} e^{b}_{\mu} \right) e_{i,i-1} \right] , \]
\[ T^{5} = \sum_{i=1}^{N} \left[ \gamma^{\mu \nu} \omega^{5 \nu}_{\mu i} e_{i,i} + \frac{\gamma^{5 \nu} i}{\sqrt{2}} \left( \partial_{\mu} e^{5}_{5i} - \lambda \chi^{5 \nu}_{5i} e^{b}_{\mu} \right) e_{i,i+1} - \frac{\gamma^{5 \nu} i}{\sqrt{2}} \left( \partial_{\mu} e^{5}_{5i} - \lambda \bar{\chi}^{5 \nu}_{5i} e^{b}_{\mu} \right) e_{i,i-1} \right] \] (40)

for the torsion, and for the curvature we find
\[ \mathcal{R}^{AB} = \sum_{i=1}^{N} \left[ \frac{1}{2} \gamma^{\mu \nu} \mathcal{R}^{AB}_{\mu i} e_{i,i} + \frac{\gamma^{5 \nu} i}{\sqrt{2}} \left( Q^{AB}_{\mu i} e_{i,i+1} - Q^{AB}_{\mu i} e_{i,i-1} \right) \right] , \] (41)
where
\[ \mathcal{R}^{AB}_{\mu i} = \partial_{\mu} \omega^{AB}_{\nu i} - \partial_{\nu} \omega^{AB}_{\mu i} + \omega^{AC}_{\mu i} \omega^{CB}_{\nu i} - \omega^{AC}_{\nu i} \omega^{CB}_{\mu i} , \]
\[ Q_{\mu i}^{AB} = \partial_\mu \chi_i^{AB} + \omega_\mu^{AB} - \omega_{\mu i+1}^{AB} + \omega_\mu^{AC} \chi_i^{CB} - \chi_i^{AC} \omega_{\mu i+1}^{CB}, \]

\[ \bar{Q}_{\mu i}^{AB} = \partial_\mu \bar{\chi}_i^{AB} + \omega_\mu^{AB} - \omega_{\mu i-1}^{AB} + \omega_\mu^{AC} \bar{\chi}_i^{CB} - \bar{\chi}_i^{AC} \omega_{\mu i-1}^{CB}. \]  

(42)

Finally the Einstein-Hilbert action is

\[ S_{EH} = \kappa^{-2} \langle \mathcal{R}^{AB} e^B, e^A \rangle \]

\[ = \kappa^{-2} \int d^4 x \text{Tr} (\mathcal{R}^{AB} e^B (e^A)^*) . \]  

(43)

From the equation (30), (31), and (41), the Einstein-Hilbert action, (43), reads

\[ S_{EH} = \int \sqrt{\text{det}(g)} d^4 x \sum_{i=1}^{N} \left[ \frac{1}{k^2} e^\mu e^\nu R_{\mu \nu i}^{ab} + \frac{\lambda}{2k^2} \delta_i^5 e^\mu \left( Q_{\mu i}^{ab} + \bar{Q}_{\mu i}^{ab} - Q_{\mu i}^{5a} - \bar{Q}_{\mu i}^{5a} \right) \right] . \]  

(44)

One can now impose the torsionless condition which leads to the following conditions: \( \omega_{\mu i}^{ab} = \omega_{\mu}^{ab} \) for all \( i \), and \( \chi_i^{AB} = \bar{\chi}_i^{AB} \). Moreover, we get

\[ \partial_\mu \delta_i^5 = \lambda \bar{\chi}_i^b \epsilon_\mu^b. \]  

(45)

Plugging these conditions into the (44), one finds the following action for the gravity

\[ S_{EH} = \int \sqrt{\text{det}(g)} d^4 x \left[ \frac{1}{k^2} e^\mu e^\nu R_{\mu \nu i}^{ab} - \frac{1}{2} \sum_{i=1}^{N} \partial_\mu \sigma_i \partial^\mu \sigma_i \right], \]  

(46)

where \( \phi_i(x) = e^{-\kappa \sigma_i(x)/2} \) and \( k^2 = \kappa^2/N \). As a conclusion, the Einstein-Hilbert action for the non-commutative space given by (29), turns out to be the gravity action plus \( N \) scalars. In order to understand the role of these scalars one has to consider the gravity coupled to the Yang-Mills sector. For this purpose, one can write the Yang-Mills action in the same way as we did in the previous section, but with the Dirac operator (32). Of course, this in not what we are going to do now, we would rather to consider the case where \( \lambda \to \infty \). Physically, this corresponds to the limit where the good description would be in the terms of a five-dimensional gravity, much similar to what we had in the Yang-Mills sector when \( M \to \infty \).

Using the notation of the left and right discrete derivative (20), and setting \( \lambda \chi_i^{AB} = \omega_{5i}^{AB} \) and \( \lambda \bar{\chi}_i^{AB} = \bar{\omega}_{5i}^{AB} \), we find

\[ \pi(\omega^{AB}) = \sum_{i=1}^{N} \left[ \gamma^\mu_{\omega,\mu i} e_{i,i} + \gamma_5^i \sqrt{2} \left( \omega_{5i}^{AB} e_{i,i+1} - \bar{\omega}_{5i}^{AB} e_{i,i-1} \right) \right] \]

\[ \mathcal{R}^{AB} = \sum_{i=1}^{N} \left[ \frac{1}{2} \gamma^\mu_{\mathcal{R},\mu i} e_{i,i} + \gamma_5^i \sqrt{2} \left( (\mathcal{R}_{\mu 5i}^{AB} - \mathcal{L}) e_{i,i+1} - (\bar{\mathcal{R}}_{\mu 5i}^{AB} - \bar{\mathcal{L}}) e_{i,i-1} \right) \right] \]  

(47)

where \( \mathcal{L} = \lambda^{-1} \omega_{5i}^{AC} \partial_5 \omega_{\mu i}^{CB} \), \( \bar{\mathcal{L}} = \lambda^{-1} \omega_{5i}^{AC} \bar{\partial}_5 \omega_{\mu i}^{CB} \) and

\[ \mathcal{R}_{\mu 5i}^{AB} = \partial_\mu \omega_{5i}^{AB} - \partial_5 \omega_{\mu i}^{AB} + \omega_{5i}^{AC} \omega_{\mu i}^{CB} - \omega_{\mu i}^{AC} \omega_{5i}^{CB} , \]
\[ \mathcal{R}^{AB}_{\mu\bar{s}i} = \partial_\mu \bar{\omega}^{AB}_{\bar{s}i} - \bar{\partial}_\bar{s} \omega^{AB}_{\mu i} + \omega^{AC}_{\mu i} \bar{\omega}^{CB}_{\bar{s}i} - \bar{\omega}^{AC}_{\bar{s}i} \omega^{CB}_{\mu i}. \] (48)

Keeping in mind that in the limit of \( \lambda \rightarrow \infty \) we have \( \omega^{AB}_{\mu i} = \bar{\omega}^{AB}_{\mu i} \), the action (43) reads

\[ S_{EH} = \int \sqrt{\det(g)} \, d^4x \sum_{i=1}^{N} \left[ \frac{1}{\kappa^2} e_\mu^a e_\nu^b \mathcal{R}^{ab}_{\mu\nu i} + \frac{1}{\kappa^2} e_5^a e_6^\mu \left( \mathcal{R}^{a5}_{\mu\bar{s}i} + \bar{\mathcal{R}}^{a5}_{\bar{s}i\mu} \right) \right]. \] (49)

Here we have dropped those terms which are proportional to \( \lambda^{-1} \). In the limit where \( \lambda \rightarrow \infty \), one has also to replace the summation with an integral. Doing so, we find the five-dimensional gravity action as following

\[ S_{EH} = \kappa_5^{-2} \int \sqrt{\det(G)} \, d^4x \, dy \sum_{pq} \left[ e_A^p e_B^q \mathcal{R}^{AB}_{pq} \right], \] (50)

where \( \mathcal{R}^{AB}_{pq} \), \( \det(G) = \phi^2(x, y) \det(g) \) and \( \kappa_5^2 = R \kappa_4^2 \) are five-dimensional curvature, metric and Newton constant, respectively.

Note that, in the geometry we are considering for our spacetime, by dimensional reduction from five dimensions to four dimensions we will not get a gravity coupled to gauge field, as we used to get in the ordinary Kaluza-Klein reduction, where the \( G_{\mu5} \) plays the role of the gauge field in the four-dimensional theory. In fact, in our case the four-dimensional Yang-Mills and gravity sectors are coming from the five-dimensional Yang-Mills and gravity sectors, respectively. As we shall discuss in the next section, the effects of the five-dimensional gravity in the four-dimensional Yang-Mills sector will be appeared in the potential for the scalars in the Yang-Mills sector.

### 5 Conclusions

In this note we considered the Yang-Mills theory as well as the gravity in a non-commutative space given by a four-dimensional manifold times a set of discrete \( N \)-points. The Yang-Mills theory which we studied in this paper is a gauge theory with gauge group \( SU(n)^N \) coupled to \( N \)-charged scalars. In the high energy limit the theory is a four dimensional field theory, while in the IR limit, the theory behaves like a five-dimensional gauge theory with the gauge group \( SU(n) \). The same as that in [1], one can think about this procedure as a dynamically generating dimension.

This model can also be thought as a latticization of a five-dimensional gauge theory. From the non-commutative geometry point of view, this can be seen by noting that the non-commutative five-dimensional spacetime we have used so far, can be considered as a five-dimensional space, \( (x_i, y) \), \( i = 1, \cdots 4 \), with following non-commutative relation

\[ [y, dy] = ady, \] (51)
where $a$ is a constant. All other coordinates commute. Moreover, we have
\[
df(y) = dy (\partial_y f)(y) = (\bar{\partial}_y)(y) dy,
\]
where the left and right derivatives are defined as ($20$). One can show that a gauge
theory on this space will be a five-dimensional gauge theory latticized in one dimen-
sion, much similar to ($17$) (see for example ($13$)).

An advantage working in the framework of the non-commutative is that the grav-
ity can be added to the game in the same way as the Yang-Mills sector. Although,
in this paper we have only considered the gravity and Yang-Mills sectors separately,
one could consider both of them at the same time. In particular, in this case we will
get the following potential for the scalars in the Yang-Mills sector (see also ($3$))
\[
V_i \sim (|U_{i,i+1}|^2 - e^{-\kappa\sigma})^2.
\]
Here we set $\phi_i(x) = \phi(x) = e^{-\kappa\sigma/2}$. Furthermore, the Kaluza-Klein spectrum for
this case is
\[
M_{KK}^2 = e^{-\kappa\sigma} \left( \frac{2}{a} \right)^2 \sin^2 \left( \frac{k a}{2} \right).
\]
Note that, as we can see from ($53$), the effect of the five-dimensional gravity in four-
dimensional theory is given by a potential in the from of that in Randall-Sundrum
model ($\textit{3}$). In fact, one could think about the theory we are consider-
ing here, as $N$-copies of the Randall-Sundrum model. One could also add the fermions to the
theory.

We would like to note that, as far as the five-dimensional theory concerns, there
is no difference between a theory with parameters $(a, N)$ and $(a', N')$, provided of
course $aN = a'N'$. Nevertheless, as we are approaching the UV limit we will end up
with two different four-dimensional theories. For example, If we started with a five-
dimensional gauge theory with gauge group $SU(n)$, the four-dimensional theories
would be either $SU(n)^N$ gauge theory with parameters $g$ and $f_s$ or $SU(n)^{N'}$ gauge
theory with parameters $g'$ and $f'_s$. Moreover, we have the following relations between
the parameters of these two description
\[
g' = \frac{N'}{N} g, \quad f_s = f'_s.
\]
Therefore, it seems that starting from a five-dimensional gauge theory we will have
several UV completions. One might suspect that in the framework of the non-
commutative geometry, these issue would be related to the notion of the “Morita
equivalence”. It would be nice to see if we can make this relation more precise. We
hope to come back to this point in the future.

We also hope that the interpretation we have made here, could be used for the
further study of the (de)constructing dimensions story.

**Acknowledgments**: I would like to thank N. Arkani-Hamed for comments and
correspondence.
References

[1] N. Arkani-Hamed, A. G. Cohen and H. Georgi, “(De)Constructing Dimensions”, hep-th/0104005.

[2] M.B.Halpern and W.Siegel, “Electromagnetism as a Strong Interaction” Phys.Rev. D11 (1975) 2967.

[3] A. Connes, “Non-commutative Geometry”, Academic Press (1994).

[4] L. Randall and R. Sundrum, “An Alternative to Compactification”, Phys.Rev.Lett. 83 (1999) 4690; hep-th/9906064.

[5] F. Lizzi, G. Mangano and G. Miele, “Another Alternative to Compactification: Non-commutative Geometry and Randall-Sundrum Models”, Mod.Phys.Lett. A16 (2001) 1; hep-th/0009180.

[6] A. Connes and J. Lott, “Particle Models and Non-commutative Geometry”, Nucl.Phys.Proc.Suppl. 18B (1991) 29.

[7] A. H. Chamseddine, G. Felder and J. Fröhlich, “Grand Unification in Non-commutative Geometry”, Nucl.Phys. B395 (1993) 672; hep-ph/9209224.

[8] J. Dai, X-C. Song, “Connes’ Distance of One-Dimensional Lattices: General Cases”, math-ph/0104002.

[9] A. H. Chamseddine, “The Scalar Potential in Non-commutative Geometry”, Phys.Lett. B373 (1996) 61; hep-th/9510215.

[10] C. T. Hill, S. Pokorski and J. Wang, “Gauge Invariant Effective Lagrangian for Kaluza-Klein Modes”, hep-th/0104035.

[11] A. H. Chamseddine, “Gravity in Non-commutative Geometry”, Commun.Math.Phys. 155 (1993) 205; hep-th/9209043.

[12] A. Dimakis, F. Muller-Hoissen and T. Striker, “Non-commutative Differential Calculus and Lattice Gauge Theory”, J.Phys. A26 (1993) 1927.

A. Dimakis and F. Muller-Hoissen, “Some Aspects of Non-commutative Geometry and Physics”, physics/9712004.