Computational methods for Bayesian semiparametric Item Response Theory models

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Abstract

Item response theory (IRT) models are widely used to obtain interpretable inference when analyzing data from questionnaires, scaling binary responses into continuous constructs. Typically, these models rely on a normality assumption for the latent trait characterizing individuals in the population under study. However, this assumption can be unrealistic and lead to biased results. We relax the normality assumption by considering a flexible Dirichlet Process mixture model as a nonparametric prior on the distribution of the individual latent traits. Although this approach has been considered in the literature before, there is a lack of comprehensive studies of such models or general software tools. To fill this gap, we show how the NIMBLE framework for hierarchical statistical modeling enables the use of flexible priors on the latent trait distribution, specifically illustrating the use of Dirichlet Process mixtures in two-parameter logistic (2PL) IRT models. We study how different sets of constraints can lead to model identifiability and give guidance on eliciting prior distributions. Using both simulated and real-world data, we conduct an in-depth study of Markov chain Monte Carlo posterior sampling efficiency for several sampling strategies. These strategies consider various combinations of model parameterizations, identifiability constraints and sampling algorithms. In the simulated scenarios, we find that some choices of parameterization and identifiability constraints can yield order-of-magnitude differences in sampling efficiency compared to others. We also find that there is little inferential penalty in using

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a semiparametric model when a parametric model would be correct, supporting the benefit of greater robustness to mis-specification. We illustrate the 2PL semiparametric IRT models with two real datasets related to education and medical assessments: the 2007 Trends in International Mathematics and Science Study (TIMSS) and the 1996 Health Survey for England. In the former case, semiparametric results are similar to the parametric 2PL model, while in the latter case the semiparametric model identifies distinct ability clusters missed by the parametric model. We conclude that having access to semiparametric models can be broadly useful, as it allows inference on the entire underlying ability distribution and its functionals, with NIMBLE being a flexible framework for estimation of such models.

*Keywords*: 2PL model, Dirichlet Process Mixture, MCMC strategies, NIMBLE.
1 Introduction

Traditional approaches in item response theory (IRT) modeling rely on the assumption that subject-specific latent traits follow a normal distribution. This assumption is often considered for computational convenience, but there are many situations in which it may be unrealistic (Samejima, 1997). For example, Micceri (1989) gives a comprehensive review of many psychometric datasets where the distribution of latent individual trait does not respect the normality assumption and presents instead asymmetries, heavy-tails or multimodality. In addition, estimation of IRT parameters in the presence of non-normal latent traits has been shown to produce biased estimates of the parameters (see, for example Finch & Edwards, 2016; Kirisci, chi Hsu, & Yu, 2001; Schmitt, Mehta, Aggen, Kubarych, & Neale, 2006; Seong, 1990).

Different proposals have been made in the general IRT literature for relaxing the normality assumption for the distribution of the latent individual trait, using either Markov chain Monte Carlo (MCMC) or Marginal Maximum Likelihood (MML) estimation methods. One option is to rely on more general parametric assumptions. For example, Azevedo, Bolfarine, and Andrade (2011) considered a skew-normal distribution (Azzalini, 1985), while others have suggested finite mixtures of normal distributions (Bambirra Gonçalves, da Costa Campos Dias, & Machado Soares, 2018; Bolt, Cohen, & Wollack, 2001). Alternatively, one can refrain from making distributional assumption on the latent abilities by using nonparametric maximum likelihood estimation (Laird, 1978; Mislevy, 1984), B-splines (Johnson, 2007; Woods & Thissen, 2006) or empirical histograms (Woods, 2007).

This paper considers the Bayesian nonparametric approach that uses a Dirichlet process mixture (Escobar & West, 1995; Ferguson, 1973; Lo, 1984) as a nonparametric prior on the distribution of the individual latent trait, in the context of logistic IRT models for binary responses. Bayesian nonparametric extensions of binary IRT models have been previously considered in the literature. Such models are semiparametric because they retain other, parametric, assumptions of binomial mixed models, such as the functional form of the link function. Within this approach, the semiparametric 1PL model has been the focus of more effort as well as software (Jara, Hanson, Quintana, Müller, & Rosner, 2011, DPpackage, no longer actively maintained). San Martín, Jara, Rolin, and Mouchart (2011) investigated a semiparametric generalization of the 1PL (Rasch-type) models from a theoretical perspective, while Finch and Edwards (2016) provided results from simulation studies. An example using the semiparametric 2PL model is given in Duncan and MacEachern (2008), but there is a lack of comprehensive studies of such models or general software tools. Other nonparametric IRT models relax assumptions about the functional form of the item response probabilities, considering a general monotonic function in place of the logistic/probit link function, sometimes referred as NIRT models. While we do not pursue this direction in this paper, focusing instead on nonparametric modeling of
the latent trait distribution, such an extension is relatively straightforward.

In this paper we fill three major gaps that hinder widespread application of semiparametric Bayesian item-response theory models. First, we implement semiparametric 1PL and 2PL models in NIMBLE (de Valpine et al., 2017) (R package nimble, de Valpine et al. (2020)), a flexible R-based system for hierarchical modeling. In particular, NIMBLE provides functionality for fitting hierarchical models that involve Dirichlet process priors either via a Chinese Restaurant Process (CRP) (Blackwell, MacQueen, et al., 1973) or a truncated stick-breaking (SB) (Sethuraman, 1994) representation of the prior. Hence, NIMBLE supports a much wider class of models than those that are implemented in standard software packages. Code is provided for all examples, along with guidelines on prior elicitation.

Second, we study the efficiency of several MCMC sampling strategies to estimate 2PL models in both simulated and real-data scenarios. In particular, we use results for parametric 2PL models to understand and make recommendations for effective MCMC strategies when using semiparametric 2PL models, which are inherently slower to mix than parametric 2PL models. This approach also allows us to compare various random walk Metropolis-Hastings MCMC strategies to the Hamiltonian Monte Carlo (HMC) strategy implemented in the widely used Stan package (Stan Development Team, 2018). This is important because HMC algorithms are not readily available for Dirichlet process prior models, since HMC cannot sample discrete parameters (the component indicators), which also cannot be integrated out in infinite mixture models.

Finally, we present a comparison of inferential results for item and person parameters under parametric and semiparametric assumptions. To make these comparisons fair, we carefully elicitate prior distributions for the models by matching the prior predictive distribution of the data to a common distribution. We also illustrate how to estimate the entire distribution of latent traits and its functionals under the two assumptions.

The remaining of the paper is organized as follow: in Section 2 we present the standard IRT model and the Bayesian semiparametric extension along with considerations for identifiability. We then discuss the goals of our numerical experiments to assess MCMC strategies and the datasets used in those experiments (Section 3). We present a variety of potential sampling strategies (Section 4), give guidance on selecting prior distributions (Section 5), and then show results for MCMC efficiency and statistical inference (Sections 6 and 7).

2 IRT models and background

IRT models are widely used in various social science disciplines to scale binary responses into continuous constructs. For conciseness, in this section we introduce model notation in the context of educational assessment, where typically data are answers to exam questions from a set of individuals. In particular, let $y_{ij}$ denote the answer of individual $j$ to item $i$ for $j = 1, \ldots, N$ and $i = 1, \ldots, I$, with $y_{ij} = 1$ when the answer is correct and 0 otherwise.
Responses from different individuals are assumed to be independent, while responses from the same individual are assumed independent conditional on the latent trait (this is sometimes called the *local independence assumption* in the psychometric literature).

### 2.1 The two-parameter logistic model

Let $\pi_{ij}$ denote the probability that individual $j$ answers item $i$ correctly, given the model parameters $\eta_j, \lambda_i, \beta_i$, i.e. $\pi_{ij} = \Pr(y_{ij} = 1|\eta_j, \lambda_i, \beta_i)$ for $i = 1, \ldots, I$ and $j = 1, \ldots, N$. The parameter $\eta_j$ represents the latent ability of the $j$-th individual, while $\beta_i$ and $\lambda_i$ encode the item characteristics for the $i$-th item. In the two-parameter logistic (2PL) model, the probability $\pi_{ij}$ is determined using the logistic function as

$$\text{logit}(\pi_{ij}) = \lambda_i(\eta_j - \beta_i), \quad i = 1, \ldots, I, \quad j = 1, \ldots, N. \quad (1)$$

A further assumption for the latent abilities is that they are independently and identically distributed according to some distribution $G$,

$$\eta_j \sim_{\text{iid}} G, \quad j = 1, \ldots, N, \quad (2)$$

with $G$ traditionally a standard normal distribution. The parameter $\lambda_i > 0$ is often referred to as *discrimination*, since items with a large $\lambda_i$ are better at discriminating between subjects with similar abilities, while $\beta_i$ is called *difficulty* because for any fixed $\eta_j$ the probability of a correct response to item $i$ is decreasing in $\beta_i$. When $\lambda_i = 1$ for all $i = 1, \ldots, I$, the model in (1) reduces to the one-parameter logistic (1PL) model, also known as Rasch model (Rasch, 1990).

Often, the log-odds in (1) are reparameterized as

$$\lambda_i \eta_j + \gamma_i, \quad (3)$$

with $\gamma_i = -\lambda_i \beta_i$. These two parameterizations are sometimes referred to as *IRT parameterization* (1) and *slope-intercept parameterization* (3). While the slope-intercept parameterization is often considered for computational convenience, the IRT parameterization is the most traditional in terms of interpretation. In exploring different strategies for Bayesian estimation, we will consider both alternatives and investigate potential differences in terms of computational performance.

### 2.2 Semiparametric 2PL

The classical formulation of IRT models assumes that the latent abilities in (2) have a normal distribution. This assumption can be relaxed, modeling the distribution of ability as a mixture of normal distributions, where the number of mixture components does not need
to be specified in advance but rather is learned from the data. This can be achieved using a Dirichlet Process mixture (DPM) model for the distribution of ability. In particular, the latent distribution \( G \) in (2) can be constructed as a convolution involving a DP prior, i.e.

\[
G = \int K(\eta_j|\theta)F(d\theta), \quad F \sim \text{DP}(\alpha, G_0),
\]

where \( K(\cdot|\theta) \) is a suitable probability kernel (e.g., a normal distribution) indexed by the parameter \( \theta \), while \( \alpha \) and \( G_0 \) are, respectively, the concentration parameter and the base distribution of the Dirichlet Process. A random distribution \( F \) drawn from such a Dirichlet Process can be constructed as

\[
F(\cdot) = \sum_{k=1}^{\infty} w_k \delta_{\tilde{\theta}_k}
\]

where \( \delta_a \) is the Dirac probability measure concentrated at \( a \) and \( \tilde{\theta}_1, \tilde{\theta}_2, \ldots \) is a sequence of independent draws from \( G_0 \). The weights are calculated using a stick-breaking construction:

\[
w_k = v_k \prod_{l=1}^{k-1} (1 - v_l) \text{ with } v_1, v_2, \ldots \text{ a sequence of independent draws from a Beta}(1, \alpha) \text{ distribution (Sethuraman, 1994).}
\]

This construction makes it clear that, as long as the kernel \( K(\cdot|\theta) \) is continuous, the latent ability distribution \( G \) is also continuous, but \( F \) is almost surely discrete, naturally inducing clustering via ties in the parameter indexing the distribution of ability.

The DP is often characterized in terms of the predictive distribution associated with its draws (Blackwell et al., 1973). More specifically, let \( \theta_1, \ldots, \theta_N \) be an independent sample from \( F \). Integrating over \( F \) one can obtain the joint prior distribution on \( (\theta_1, \ldots, \theta_N) \), which can be written as the product of a sequence of conditional distributions, where

\[
(\theta_j|\theta_{j-1}, \ldots, \theta_1) \sim \frac{\alpha}{\alpha + j - 1} G_0 + \sum_{l=1}^{j-1} \frac{1}{\alpha + j - 1} \delta_{\theta_l},
\]

for \( j = 1, \ldots, N \). Note that this construction clearly indicates that there is a positive probability of ties among the \( \theta_j \)'s.

The distribution in (5) it is often interpreted in terms of the Chinese Restaurant Process (CRP) (see Aldous, 1985; Pitman, 1996). The name comes from the analogy used to describe the process. Consider a Chinese restaurant with an infinite number of tables, each table serving one dish shared by all customers sitting at that table. In this metaphor, each table represents a possible cluster, while each dish represents the parameter indexing the distribution associated with the cluster. Customers entering the restaurant can seat themselves at a previously occupied table and share the same dish (with probability proportional to the number of customers already sitting at the table), or go to a new table and order another dish (with probability proportional to \( \alpha \)).
Using \( z_j \) to denote the table chosen by the \( j \)th customer and \( \theta^*_k \) to denote the dish served in table \( k \), this can be formally stated as

\[
p(z_j = k | z_{j-1}, \ldots, z_2, z_1, \alpha) = \begin{cases} 
\frac{n_{j-1}^{k-1}}{\alpha + j - 1}, & k = 1, \ldots, K^{j-1}, \\
\frac{\alpha}{\alpha + j - 1}, & k = K^{j-1} + 1, 
\end{cases} \tag{6}
\]

and \( \theta^*_k \sim G_0 \), where \( K^{j-1} \) is the total number of occupied tables by the first \( j - 1 \) customers, and \( n_{j-1}^{k-1} \) is the number of customers at table \( k \) among the first \( j - 1 \). The concentration parameter \( \alpha \) controls the distribution of the number of tables (clusters), with larger values favoring more tables.

NIMBLE allows one to use either this CRP representation or a finite stick-breaking approximation for DP models. In this work we use the CRP. Using the indicators \( \mathbf{z} = \{z_j, j = 1, \ldots, N\} \), and denoting by \( \mathbf{z}|\alpha \sim CRP(\alpha) \) the joint distribution induced by (6), a Dirichlet process mixture (DPM) model for the distribution of ability can alternatively be written as

\[
\eta_j|\theta, z_j \overset{ind}{\sim} \mathcal{K}(\cdot|\theta_{z_j}), \quad j = 1, \ldots, N, \\
\mathbf{z}|\alpha \sim CRP(\alpha), \\
\theta^*_k \overset{iid}{\sim} G_0, \quad k = 1, 2, \ldots.
\]

In the context of 1PL and 2PL models, it seems natural to employ normal kernels, \( \mathcal{K}(\cdot|\theta) \), indexed by parameters \( \theta = \{\mu, \sigma^2\} \). This allows one to represent the abilities as a mixture of normal distributions, where the number of mixture components is unknown. Furthermore, under this choice, taking \( \alpha \to 0 \) leads to the original parametric model discussed in Section 2.1. Parameters characterizing each mixture component are drawn from the base distribution, \( G_0 \). For computational convenience the base distribution is typically the product of a normal distribution for \( \mu \) and an inverse-gamma distribution for \( \sigma^2 \).

### 2.3 Identifiability and constraints

Without additional constraints, the parameters of the 1PL and 2PL models are not identifiable (Bafumi, Gelman, Park, & Kaplan, 2005; Geweke & Singleton, 1981) (see Appendix A for details). For example, increasing all \( \eta_j \) and \( \beta_i \) values by the same amount yields the same probabilities in (1) for all \( i \) and \( j \). More generally, the ability parameters are known up to a linear transformation, and constraints are needed to identify them. To address this problem, traditional work on parametric IRT models assumes that latent abilities in (2) come from a standard normal distribution, i.e. \( G \equiv N(0, 1) \), and constrains the discrimination parameters \( \lambda_i \) for \( i = 1, \ldots, I \) to be positive in the 2PL model.

Alternative constraints can also establish identifiability and could yield different computational performance for MCMC sampling. A common alternative for the parametric 1PL
and 2PL models considers sum-to-zero constraints for the item parameters (Fox, 2010)

\[ \sum_{i=1}^{I} \beta_i = 0, \quad \left( \text{or} \quad \sum_{i=1}^{I} \gamma_i = 0 \right), \quad \sum_{i=1}^{I} \log(\lambda_i) = 0. \]  

(7)

Centering the difficulty parameters addresses the invariance to translations, while centering the log-scale of the discrimination parameters to set their product to one, accounts for the invariance to rescalings of the latent space. Another potential set of constraints, popular in political science applications, involves fixing the value of the latent traits for two individuals (e.g., see Clinton, Jackman, & Rivers, 2004). Whatever the set of constraints however, it is worthwhile to note that they can be either directly incorporated in the model as part of the prior (and, therefore in the structure of the sampling algorithms), or they can be applied as a postprocessing step (after running an unconstrained MCMC). This last approach is typical of parameter-expanded algorithms, which embed targeted models in a larger specification. Parameter expansion has been proposed in the literature to accelerate EM (C. Liu, Rubin, & Wu, 1998) and Gibbs sampler (J. S. Liu & Wu, 1999) convergence, as well as a mechanism to induce new classes of priors (Gelman, 2004). Although targeting the same posterior, constrained priors and parameter expansion can lead to very different results in terms of convergence and mixing of the MCMC algorithms.

Similar arguments apply for the semiparametric 1PL and 2PL model extension using a Dirichlet Process mixture prior the distribution of ability. In that setting, one identifiability strategy may be to constrain the base distribution \( G_0 \), e.g., by letting \( G_0 \sim \mathcal{N}(0, 1) \) (for example, see Duncan & MacEachern, 2008). However, even if the prior expectation and variance of \( G_0 \) are zero and one, the corresponding posterior quantities can deviate substantially from these values, leading to biased inference (Yang & Dunson, 2010). More general centering approaches have been proposed in the literature when a DP distribution is used to model random effects or latent variables in a hierarchical model (Li, Müller, & Lin, 2011; Yang & Dunson, 2010; Yang, Dunson, & Baird, 2010). These approaches rely on parameter expansion by sampling from the unconstrained DP model and then applying a post-processing procedure to the posterior samples. This post-processing procedure requires the analytical evaluation of the posterior mean and variance of the DP random measure, with Li et al. (2011) providing results under the CRP representation and Yang et al. (2010) under the stick-breaking one. Although such strategies are useful for general hierarchical models to avoid identifiability issues, for the semiparametric 1PL and 2PL models it is simpler to use the sum-to-zero constraints on the item parameters in (7), and that is the approach we adopt in this work. As in the parametric case, we can either include these constraints in the prior or use the parameter expansion approach by sampling and then centering and rescaling the posterior samples as appropriate.
3 Aims and data sets

Our evaluations of computational and inferential performance rely on a series of numerical experiments that use both simulated and real datasets. The specific questions we consider are:

1. For parametric 2PL models, what are efficient Metropolis-Hastings-based MCMC sampling strategies? Do different strategies work better in different scenarios? We consider different parameterizations, constraints, and samplers.

2. For the parametric 2PL models, how does efficiency of random walk Metropolis-Hastings sampling compare to Hamiltonian Monte Carlo, as implemented in the popular Stan package? This question is of interest because HMC is not readily available for Dirichlet process prior models.

3. How does MCMC efficiency of a semiparametric 2PL model compare to that of a parametric 2PL model? Does this comparison differ when the parametric 2PL model is correct vs. incorrect?

4. To what degree does the use of a parametric model when its assumptions are violated yield bad inference? Does use of a semiparametric model change inference even when a parametric model would be valid?

5. How much do results differ between the semiparametric and parametric models for the real data examples?

3.1 Synthetic data

We specified two different simulation scenarios for the distribution assumed for the latent abilities: unimodal and bimodal distributions. For both scenarios, we simulate responses from $N = 2,000$ individuals on $I = 15$ binary items. Values for the discrimination parameters $\{\lambda_i\}_{i=1}^{15}$ are sampled from a Uniform$(0.5, 1.5)$ distribution, while values for difficulty parameters $\{\beta_i\}_{i=1}^{15}$ are taken to be equally spaced in $(-3, 3)$. We then centered the log of the discrimination parameters on zero (the difficulty parameters are already centered based on how they are generated). We considered two different underlying distributions for the latent abilities, $\eta_j$ for $j = 1, \ldots, 2,000$. In the unimodal scenario, latent abilities are generated from a normal distribution with mean 0 and variance $(1.25)^2$, while in the bimodal scenario, we used a equal-weights mixture of two normal distributions with means $\{-2, 2\}$ and common variance $(1.25)^2$. 
3.2 Real world data

The first example is a subset of data from the 1996 Health Survey for England (Joint Health Surveys Unit of Social and Community Planning Research and University College London, 2017), a survey conducted yearly to collect information concerning health and behavior of households in England. In particular, we have data for 10 items measuring Physical Functioning (PF-10), which is a sub-scale of the SF-36 Health Survey (Ware, 2003) administered to people aged 16 and above. In this case the latent trait quantifies the physical status of a given individual (Hays, Morales, & Reise, 2000; McHorney, Haley, & Ware Jr, 1997).

Participants in the survey were asked whether they perceived limitations in a variety of physical activities (e.g., running, walking, lifting heavy objects) and if so the degree of limitation. We list the original questions in the Appendix C. Answers to items comprised three possible responses (“yes, a lot”, “yes, limited a little”, “no, not limited at all”); however, in our analysis we consider the dichotomous indicator for not being limited at all. The left panel of Figure 1 shows the distribution of raw scores. For simplicity, we analyzed complete case data from 14,525 individuals out of 15,592 respondents, although the model can easily accommodate missing data.

The second example uses data from the TIMSS (Trends in International Mathematics and Science Study) survey, which is an international comparative educational survey dedicated to improving teaching and learning in mathematics and science for students around the world (http://timssandpirls.bc.edu/TIMSS2007/about.html). We used data from the 2007 eighth-grade mathematics assessment for the United States (N = 7,377), publicly available at https://timssandpirls.bc.edu/TIMSS2007/idb_ug.html. The dataset comprises 214 items, with 192 of them dichotomous, while the remaining 22 have three category responses (“incorrect”, ”partially correct”, ”correct”). We dichotomized these latter questions, considering partially correct answers as incorrect ones. Like other large-scale assessments, participants in TIMSS only received a subset of the items according to a booklet design, resulting in 28-32 item responses per student. Raw scores for the data are showed in the right panel of Figure 1. In addition, participants in the survey were sampled according to a complex two-stage clustered sampling design that we did not consider in our application. In other contexts the design is typically taken into account using sampling weights for model estimation, as discussed for example by Rutkowski, Gonzalez, Joncas, and von Davier (2010).
Figure 1: Distribution of the raw scores and relative percentages for the real data examples: health data (left panel) and TIMSS data (right panel).

## 4 Sampling strategies for the 2PL model

In this work we explore different sampling strategies for Bayesian estimation of the 2PL parametric and semiparametric models. We define a sampling strategy to include the combination of model parameterization, identifiability constraints and sampling algorithms, as summarized in Table 1.

| Model constraints          | Parametric | Semi-parametric |
|----------------------------|------------|-----------------|
|                            | Slope-intercept | IRT            | Slope-intercept | IRT            |
| Constrained abilities      | MH/conjugate  | MH/conjugate    | MH/conjugate    | MH/conjugate*  |
|                            | Centered     | HMC (Stan)      |                |                |
| Constrained item parameters| MH/conjugate  | MH/conjugate*   | MH/conjugate    | MH/conjugate*  |
| Unconstrained              | MH/conjugate  | MH/conjugate    | MH/conjugate    | MH/conjugate   |
|                            | Centered     | Centered        | Centered        |                |

Table 1: Summary of the 14 sampling strategies considered for the parameter and semiparametric 2PL model. Each of the 14 entries is a different strategy with “MH/conjugate”, “Centered”, and “HMC (Stan)” referring to three different sampling algorithms discussed below. The asterisk symbol denotes the sampling strategies that lead directly to posterior samples following model (8).

As anticipated in Section 2, we explore both parameterizations of the 2PL model men-
tioned in Section 2.1: the IRT and the slope-intercept parameterization. To compare estimates obtained from different parameterizations on a common scale, we post-process posterior samples (using transformations described in Appendix A) to respect the following base parameterization

\[
\text{logit}(\pi_{ij}) = \lambda_i(\eta_j - \beta_i), \quad i = 1, \ldots, I,
\]

\[
\sum_{i=1}^{I} \log(\lambda_i) = 0 \quad \sum_{i=1}^{I} \beta_i = 0,
\]

\[\eta_j \sim G, \quad j = 1, \ldots, N, \quad (8)\]

where \(G\) denotes a general distribution for the latent abilities, either parametric or nonparametric. The model in (8) follows the IRT parameterization with sum-to-zero identifiability constraints, which is typically the targeted one for inference for interpretability reasons.

4.1 Identifiability constraints

Our targeted inferential model in (8) can be estimated directly, accounting for identifiability constraints. This can be achieved by introducing in the model formulation a set of auxiliary item parameters, \(\{\lambda^*_i, \beta^*_i\}\) for each \(i = 1, \ldots, I\) and defining \(\{\lambda_i, \beta_i\}\) as

\[
\log(\lambda_i) = \log(\lambda^*_i) - \frac{1}{I} \sum_{i=1}^{I} \log(\lambda^*_i) \quad \beta_i = \beta^*_i - \frac{1}{I} \sum_{i=1}^{I} \beta^*_i, \quad i = 1, \ldots, I. \quad (9)
\]

In Table 1 we label this model as the constrained item parameters model. Unconstrained priors are then placed on the auxiliary parameters \(\{\lambda^*_i, \beta^*_i\}\). The same formulation applies under the slope-intercept parameterization, where the sum-to-zero constraints are placed on the pairs \(\{\log(\lambda_i), \gamma_i\}\) for \(i = 1, \ldots, I\).

Alternatively, we can consider unconstrained versions of the 2PL model, treating the unconstrained model as a parameter-expanded version of the targeted inferential model in (8), where the redundant parameters are the means of the difficulty and log discrimination parameters. Hence we conduct MCMC sampling with known model unidentifiability, and before using the results for inference, we transform samples to follow the model in (8), as described in Appendix A.

Finally, for the parametric case only, we consider the traditional version of the 2PL model that assumes abilities parameters \(\eta_j\) for \(j = 1, \ldots, N\) following a standard normal distribution (constrained abilities model).

4.2 Sampling algorithms

All the versions of the 2PL model are estimated in the Bayesian framework via MCMC methods. The NIMBLE system provides a suite of different sampling algorithms along with
the possibility to code user-defined samplers. For all models we consider NIMBLE’s default sampling configuration (MH/conjugate algorithm). In addition, under the slope-intercept parameterization we propose a custom algorithm to jointly sample item parameters (centered algorithm, discussed below). In the parametric setting only, we also investigate performance of Hamiltonian Monte Carlo as implemented in Stan (Carpenter et al., 2017) (HMC (Stan) algorithm).

NIMBLE’s MCMC uses an overall Gibbs sampling strategy, cycling over individual parameters, or parameter blocks for parameters with a multivariate prior, in each iteration. By default, specific sampler types are assigned to the parameters or parameter blocks. NIMBLE’s default MCMC configuration assigns a conjugate (sometimes called “Gibbs”) sampler where possible, meaning that parameters are directly sampled from the corresponding full conditional posterior distribution. For non-conjugate continuous-valued parameters, NIMBLE assigns a conjugate sampler for binary-valued or categorical parameters. For the parametric versions of the 2PL model, the default NIMBLE assignment corresponds to these conjugate and adaptive random walk Metropolis-Hastings samplers, with the latter also used for most parametric components of the semiparametric 2PL. Specialized samplers are assigned when Bayesian nonparametric priors are considered in the semiparametric 2PL.

NIMBLE offers high flexibility for customizing the algorithms, allowing inclusion of user-programmed custom samplers. In the case of the slope-intercept parameterization, we take advantage of this flexibility to propose a custom sampling strategy (centered sampler). This strategy relies on the use of an adaptive random walk Metropolis-Hastings sampler with a joint proposal for each pair of item parameters \(\{\lambda_i, \gamma_i\}\) for \(i = 1, \ldots, I\). The proposal is made under a reparameterization of the model that centers the abilities to have mean zero. Implementation details are provided in Appendix B.

Finally, for the parametric 2PL model we also consider a Hamiltonian Monte-Carlo (HMC) algorithm, as implemented in the Stan software. Stan implements an adaptive HMC sampler (Betancourt, Byrne, Livingstone, & Girolami, 2014) based on the No-U-Turn sampler (NUTS) by Hoffman and Gelman (2014). HMC algorithms are known to produce samples that are much less autocorrelated than those of other samplers but at more computational cost given the need to calculate the gradient of the log-posterior. In this work, we limit the comparison to the IRT parameterization with constraints on the abilities distribution, as that is the model provided in the edstan R package (Furr, 2017).

5 Choice of prior distributions

Past research on Bayesian IRT models has warned about the use of either vague priors or highly informative priors when there is little information about the parameters (Natesan,
In particular Natesan et al. (2016) investigated the use of different prior choices in 1PL and 2PL models using MCMC and variational Bayes algorithms and found that the use of vague priors tends to produce biased inference or convergence issues. Similarly, it is well known that highly informative prior distributions on parameters can strongly affect model comparison procedures. In our experiments we elicit the parameters of our priors by matching the prior predictive distributions across all the models we compare. This “predictive matching approach” has been widely used to guide prior elicitation in model comparison settings (Bedrick, Christensen, & Johnson, 1996; Berger & Pericchi, 1996; Ibrahim, 1997).

In the context of 1PL and 2PL IRT models, we aim to match the prior marginal predictive distribution of a response \( y_{ij} \), which in turn can be achieved by matching the induced prior distribution on the marginal prior probability of a correct response, \( \pi_{ij} = \expit\{\lambda_i(\eta_j - \beta_i)\} \). Note that all the priors discussed in this paper are separately exchangeable, which means that this prior marginal will be the same for any values of \( i \) and \( j \). In particular, we attempt to match a Beta(0.5, 0.5) distribution, which is both the reference and the Jeffreys prior for the Bernoulli likelihood in the fully exchangeable case (Berger, Bernardo, & Sun, 2009; Bernardo, 1979). A similar approach to prior elicitation in the context of latent space models for networks can be found in Guhaniyogi and Rodriguez (2020) and Sosa and Rodriguez (2021).

Because there are no analytical expressions available for the prior distribution of \( \pi_{ij} \), we use simulations to estimate the shape of the prior distribution and obtain an approximate match. This is facilitated by our implementation in NIMBLE. Indeed, one of the advantages of the NIMBLE system is that it provides a seamless way to simulate from the model of interest. Histograms of samples from the resulting induced priors can be seen in Figure 2 for a set of parametric and semiparametric models. Further details are presented in the following subsections.
Figure 2: Histogram of samples from the induced prior on $\pi_{ij}$ under each of the considered models. Dashed line indicates the density function of a Beta(0.5, 0.5) distribution. Samples for the semiparametric models use a prior distribution Gamma(2, 4) for the DP concentration parameter $\alpha$, but similar results are obtained under the other settings presented Section 5.

5.1 Priors for the item parameters

In Bayesian IRT modeling, normal distributions are typically chosen as priors for the item parameters. This is true under both parameterizations. In addition, the discrimination parameters, $\{\lambda_i\}_{i=1}^I$, are typically assumed positive, so we considered a normal distribution on the log-scale. To summarize, priors on the item parameters are:

$$\log \lambda_i \sim \mathcal{N}(\mu_{\lambda}, \sigma_{\lambda}^2), \quad \beta_i \sim \mathcal{N}(0, \sigma_{\beta}^2), \quad \gamma_i \sim \mathcal{N}(0, \sigma_{\gamma}^2) \quad i = 1, \ldots, I.$$ 

By default, we center on the difficulty parameters $\beta_i$ (or the reparameterized version $\gamma_i$) on 0 for $i = 1, \ldots, I$, while we set $\sigma_{\beta}^2 = \sigma_{\gamma}^2 = 3$. For the discrimination parameters, we set $\mu_{\lambda} = \sigma_{\lambda}^2 = 0.5$ such that the prior probability mass on the original scale is mostly in the range (0.5, 2.5).

5.2 Priors for the distribution of ability

In choosing priors for the abilities, we distinguish between the parametric and semiparametric cases. In the parametric case, excluding the strategies in which the distribution is a standard normal, we assume $G \equiv \mathcal{N}(\mu_\eta, \sigma_\eta^2)$. We specify hyperpriors for the unknown mean and
variance, using a normal distribution for the mean $\mu_\eta \sim \mathcal{N}(0, 3)$, and an inverse-gamma distribution for the variance, $\sigma^2_\eta \sim \text{InvGamma}(2.01, 1.01)$ as in Paulon, De Iorio, Guglielmi, and Ieva (2018), with hyperparameter values implying an a priori marginal expected value of 1 and a priori variance equal to 100.

In the semiparametric case, we need to specify the base distribution $G_0$ of the DP mixture prior along with the hyperparameters. We choose $G_0 \equiv \mathcal{N}(0, \sigma^2_0) \times \text{InvGamma}(\nu_1, \nu_2)$ where InvGamma($\nu_1, \nu_2$) denotes an inverse-gamma distribution with shape parameter $\nu_1$ and mean $\nu_2/(\nu_1 - 1)$. In choosing values for the hyperparameters $\{\sigma^2_0, \nu_1, \nu_2\}$, we first considered the concentration parameter $\alpha$ as fixed and evaluated the induced prior distribution on $\pi$ for values of $\alpha \in \{0.01, 0.05, 0.5, 1, 1.5, 2\}$. Recall that $\alpha$ controls the prior expectation and variance of the number of clusters induced by the DP, which are both of the order $\alpha \log(N)$.

We discuss prior choice for the $\alpha$ in Section 5.3. As in the parametric case, we center the normal distribution for the mixture component means on 0 with $\sigma^2_0 = 3$ and set $\nu_1 = 2.01$ and $\nu_2 = 1.01$ for the inverse-gamma distribution. Given these settings, we found that choosing $\alpha \in \{0.01, 0.05, 0.5, 1, 1.5, 2\}$ does not have much effect on the marginal prior distribution of the $\pi_{ij}$s.

5.3 Prior on the DP concentration parameter

One may be interested in placing a prior distribution on the concentration parameter $\alpha$ of the Dirichlet Process. A typical choice for the DP concentration parameter is a Gamma($a, b$), with shape $a > 0$ and scale $b > 0$, due to its computational convenience (Escobar & West, 1995). As previously stated, the concentration parameter controls the prior distribution of the number of clusters (Escobar & West, 1995; J. S. Liu, 1996). In choosing values $a$ and $b$, we considered the implied prior mean and variance of the number of clusters.

Let $K_N$ denote the number of clusters for a sample of size $N$. Results from Antoniak (1974) and J. S. Liu (1996) show that the expected value and variance of $K_N$ given $\alpha$ is

$$E(K_N|\alpha) = \sum_{i=1}^{\alpha} \frac{\alpha}{\alpha + N - i}, \quad \text{Var}(K_N|\alpha) = \sum_{i=1}^{\alpha} \frac{\alpha(i - 1)}{\alpha + N - i}^2. \quad (10)$$

We exploit these results to choose values $a$ and $b$ that lead to reasonable a priori values for the moments of the number of clusters for each of our applications. For a given $N$ and for different values of $a$ and $b$, we evaluated the marginal expectation and variance of the quantities in (10) via Monte Carlo approximation. We sample $\alpha_r$ for $r = 1, \ldots, R$ from its prior and compute

$$\hat{E}(K_N) = \frac{1}{R} \sum_{r=1}^{R} E[K_N|\alpha_r], \quad \hat{\text{Var}}(K_N) = \frac{1}{R} \sum_{r=1}^{R} \text{Var}(K_N|\alpha_r) + \hat{\text{Var}}(E[K_N|\alpha]),$$

where $\hat{\text{Var}}(E[K_N|\alpha]) = R^{-1} \sum_{r=1}^{R} \left[ E[K_N|\alpha_r] - \hat{E}[K_N] \right]^2$. 
Table 2: Approximate expectation and variance of the a priori number of clusters, $K_N$, under different choices of the concentration parameter distribution, for $N = \{2,000, 14,525, 7,377\}$, as in our data.

We explored a few prior choices and tabulate approximated moments in Table 5.3, for the values of $N$ in our datasets. We consider the popular choice of $a = 2$, $b = 4$ for the hyperparameters as in Escobar and West (1995) along with values favoring a small number of clusters ($a = 1, b = 3$) and values leading to a more vague prior ($a = 1, b = 1$). For our applications we decided to favor a relatively small number of clusters, choosing $a = 2, b = 4$ as hyperparameters for the simulated data, and $a = 1, b = 3$ for the real-world data.

6 Comparing MCMC efficiency

To compare sampling strategies, we measure their performance in terms of MCMC efficiency, which we define for each parameter as the effective sample size (ESS) divided by computation time. The effective sample size gives the equivalent number of independent samples that would contain the same statistical information as the actual non-independent samples. Computation time is measured for the actual MCMC run, not accounting for steps to prepare for a run, thereby focusing on the algorithms of interest rather than the software. Comparison between HMC and MCMC algorithms raises the question of how to fairly account for computation times, given that these two classes of algorithms use different tuning phases. Hence, we decided to consider different timings when using the two algorithms: (i) sampling time, which accounts only for the time to draw the posterior samples, hence discarding the time needed for the burn-in and warm-up phases of the two algorithms and (ii) total time comprising also the burn-in and warm-up phases. When computing efficiency based on the sampling time, we can assess pure efficiency of sampling from the posterior. Using total time accounts for potentially different times needed for warm-up/burn-in by the different algorithms but introduces the difficulty of determining the optimal burn-in/warm-up time, which we avoided here in favor of use of basic defaults.

We would like to use a single metric of MCMC performance to compare different sampling strategies, in particular the minimum MCMC efficiency across all model parameters, which corresponds to the worst mixing parameter (Nguyen et al., 2020). However our sampling models and sampling strategies have a variety of specifications for the ability distributions, which makes it difficult to compare performance with respect to the abilities. Therefore, we compare MCMC performance using the minimum MCMC efficiency across the item
parameters, after mapping the posterior samples to follow our target inferential model (8). There are several ways to estimate ESS, but we use `effectiveSize` function in the R `coda` package (Plummer, Best, Cowles, & Vines, 2006) since this function provides stable estimates of ESS.

For all MCMCs using NIMBLE, we used a total of 50,000 iterations, with a 10% burnin of 5,000 for all examples. We inspected traceplots to assess convergence of the chains. When running the HMC algorithm via Stan, we used a total of 8,000 iterations, with the first 4,000 iterations as warm-up steps (setting half the iterations for warm-up is the Stan default). Under these settings, the NIMBLE- and Stan-based MCMCs take approximately the same total time for a given dataset. Note that we ran the MCMC chains for many iterations in these experiments so that we could obtain reliable estimates of ESS. To ensure the chains were long enough, we used multiple runs for a portion of the experiments (not shown).

6.1 Efficiency results for simulated data

For the two simulation scenarios (unimodal and bimodal) we estimated the 2PL parametric model using the different sampling strategies summarized in Table 1. Figure 3 compares efficiency for all these strategies using the minimum ESS per second, computed with respect to the total and sampling time. Using different time baselines when computing efficiency does not affect the ranking of the algorithms, but it highlights the trade-off for the HMC algorithm between sampling efficiency and computational cost. While the HMC is highly efficient in producing samples with low correlation, warm-up steps are computationally expensive, taking 2/3 and 4/5 of the total time in our two simulation scenarios, respectively.
Figure 3: Minimum ESS per second for various sampling strategies used to estimate the 2PL parametric model for the unimodal (left column) and bimodal (right column) scenarios. Results are computed using total time (top row) and sampling time (bottom row). Note that scales are different for the different rows.

Amongst the non-HMC strategies, using the slope-intercept parameterization is better for the unimodal scenario and worse for the bimodal scenario. Unconstrained scenarios generally mix well, as do scenarios with constraints on the abilities. However, imposing constraints on the item parameters directly in the sampling performs poorly because obtaining each sample is time-consuming. This is because the constraints in (9) require that all the likelihood terms be calculated for each parameter update, whereas for other strategies only the likelihood terms for individuals’ responses on the item under consideration need to be calculated. While incorporating constraints on the item parameters is time-consuming, such a strategy
could be useful in more complicated hierarchical models, in particular when it is unclear how to rescale posterior samples. The centered strategy for the slope-intercept parameterization has little impact in the unimodal scenario but appears to help in the bimodal scenario.

Moving to the semiparametric 2PL, recall we did not consider identifiability constraints on the abilities. We either included identifiability constraints on the item parameters in the sampling or sampled from the unconstrained model and rescaled the posterior samples. As expected when using a more complicated model, we observed some reduction in efficiency in the semiparametric model compared to the parametric model, but not a drastic one (Figure 4). Results for the semiparametric case were similar in relative terms, but not in absolute magnitudes, when comparing the different parameterizations and constraints.

6.2 Efficiency results for real-world data

We can carry out similar comparisons using the real-data examples (Figure 5), noting that we excluded the constrained items scenario given its poor performance on the simulated
datasets. Here we see that HMC is best for the TIMSS data but not for the health data, with other strategies being comparable. Which parameterization and constraint scenarios are best depends on the dataset.

Figure 5: Minimum ESS per second (computed using the total time) for semiparametric 2PL models (bottom row) and their parametric versions (top row), in the health data (left column) and TIMSS data (right column) applications. Note the scales are different for the different rows.

The efficiency is lower than for the simulated datasets because the real data have many more individuals or items, and therefore more parameters. Related to this, when considering the posteriors for the abilities in the semiparametric model for the health data, we saw evidence for multimodality and some difficulty moving between modes for individuals with high raw scores. The multimodality is likely related to it being difficult for the semiparametric model to identify the exact magnitude of the ability for such individuals. The use of more informative priors, with careful elicitation of the prior distribution, may be important in such cases.
7 Comparing results in terms of statistical inference

In this section we compare results for the parametric and semiparametric models in terms of statistical inference, regardless of the sampling strategy used to obtain posterior samples. All posterior samples follow the parameterization in (8), and we use samples from the most efficient sampling strategy for each dataset. We use posterior means as point estimates for the item and ability parameters. For the simulated datasets, we measure how well the models recover the (known) true value of the parameters using absolute error, e.g., \(|\hat{\beta}_i - \beta_i|\), and squared error, e.g., \((\hat{\beta}_i - \beta_i)^2\).

A crucial point of this paper is to make inference on the distribution of latent abilities. An estimate of this distribution is sometimes based on the posterior means of the abilities (Bambirra Gonçalves et al., 2018; Duncan & MacEachern, 2008), and histograms or kernel density plots are reported. Such an estimate ignores uncertainty in the estimates of individual abilities. Instead, one should directly obtain the point estimate of the posterior distribution of the latent abilities \(p(\eta|Y)\) (for any value of \(\eta\)) using the posterior samples. In the parametric case, this reduces to:

\[
\hat{p}(\eta|Y) = \frac{1}{T} \sum_{t=1}^{T} N(\eta; \mu^{(t)}, \sigma^2(t)),
\]

(11)

with \(N(\cdot; \mu, \sigma^2)\) indicating the probability density function of a normal distribution with mean \(\mu\) and variance \(\sigma^2\). In the semiparametric case, an estimate of \(p(\eta|Y)\) is the posterior mean of the mixing measure \(G\) of the Dirichlet Process. This can be obtained using posterior samples, averaging over the DP conditional distribution in (5) computed for each iteration \(t = 1, \ldots, T\),

\[
\hat{p}(\eta|Y) = \frac{1}{T} \sum_{t=1}^{T} \left\{ \frac{n_k^{(t)}}{\alpha^{(t)}} + \frac{\alpha^{(t)}}{N} N(\eta; \mu_k^{(t)}, \sigma_k^{2(t)}) \right\} + \frac{\alpha^{(t)}}{\alpha^{(t)} + N} N(\eta; \mu_{K^{(t)}+1}, \sigma_{K^{(t)}+1}^2),
\]

(12)

with \(n_k^{(t)}\) the number of observations in cluster \(k\) at iteration \(t\), \(K^{(t)}\) the total number of clusters at iteration \(t\), and \(\mu_{K^{(t)}+1}\) and \(\sigma_{K^{(t)}+1}^2\) sampled from \(G_0\) (conditional on the data). We graphically compare estimates for the distribution of ability resulting from (11)–(12) with the estimates obtained using the posterior means.

In the semiparametric setting, it is possible to make full inference on \(p(\eta|Y)\); this requires sampling from the posterior of the mixing distribution \(F\). A computational approach to obtain the entire posterior distribution has been presented in Gelfand and Kottas (2002), a version of whose algorithm is implemented in NIMBLE in the function \texttt{getSamplesDPMeasure}. This function provides samples of a truncated version of the infinite mixture to a level \(L\). The value of \(L\) varies at each iteration of the MCMC’s output when \(\alpha\) is random, while it is the same at each iteration when \(\alpha\) is fixed. In our case, for every MCMC iteration,
we can obtain samples of the vector of mixture weights $\{w_1^{(t)}, \ldots, w_L^{(t)}\}$ and parameters of the mixture components. We can use these samples to make inference on functionals of the distribution, such as the percentile for an individual, $100 \times p_j$, where $p_j = \int_{-\infty}^{\eta_j} p(\eta|Y) d\eta$, typically paired with test scores when giving results for educational assessments. For an individual $j$ for $j = 1,\ldots,N$ we estimate $p_j$ at each MCMC iteration as

$$p_j^{(t)} = \sum_{l=1}^{L(t)} w_l^{(t)} F_N(\eta_j^{(t)}; \mu^{(t)}, \sigma^2(t)),$$

(13)

where $F_N$ denotes the distribution function of the normal distribution. For comparison, we define the parametric counterpart as $p_j^{(t)} = F_N(\eta_j^{(t)}; \mu^{(t)}, \sigma^2(t))$.

### 7.1 Inferential results for the simulated datasets

We show results for the most efficient sampling strategies under both the parametric and semiparametric model. For the unimodal simulation, samples come from the unconstrained sampling strategy using the centered sampler (SI parameterization), while for the bimodal case we used samples from the unconstrained sampling strategy (IRT parameterization). Using the absolute error and the squared error for each parameter, we report in Table 3 the mean absolute error (MAE) and the mean squared error (MSE) across item and ability parameters.

|                 | Unimodal Simulation | Bimodal simulation |
|----------------|--------------------|--------------------|
|                | Parametric | Semi-parametric | Parametric | Semi-parametric |
| Difficulty parameters | 0.0996 0.0185 | 0.0988 0.0183 | 0.0895 0.0137 | 0.0673 0.0083 |
| Discrimination parameters | 0.0734 0.0069 | 0.0731 0.0070 | 0.0832 0.0105 | 0.0397 0.0020 |
| Ability parameters | 0.4836 0.3719 | 0.4836 0.3720 | 0.5944 0.5571 | 0.5501 0.4775 |

Table 3: MAE and MSE for the item and ability parameters estimates, under the unimodal and bimodal simulation, using samples from most efficient sampling strategies.
In the unimodal scenario, we observe similar performance for the parametric and semiparametric 2PL in estimating item parameters (Figure 6). As expected, we observe some differences when considering the bimodal scenario (Figure 7), with the semiparametric model outperforming the parametric one. This is especially evident when comparing estimates of the discrimination parameters, in particular for larger values.
Similar conclusions to those from Table 3 can be made about the ability parameters when estimating abilities using the posterior means of the individual abilities (Figure 8). However, results are very different when one looks at estimates of the distribution of ability (Figure 9). The normality assumption of the parametric model leads to unimodal density estimates, inconsistent with the true distribution, whereas the semiparametric model can recover it. The posterior means of individual abilities in Figure 8 are a compromise between the inferred distribution of ability and the information in the data, so with sufficient observations, one can obtain estimates of the distribution that are reasonable even with severe model mis-specification. In other words, for mis-specified parametric models, the in-sample predictions for observed individuals can be reasonable, while the out-of-sample predictions based on (11) for new individuals are poor. Note that inspection of the posterior means of individual abilities for the parametric model, relative to the assumed parametric distribution, can be used to assess model mis-specification, clearly indicating mis-specification in the bimodal simulation here.

Figure 8: Histogram and density estimate of individual posterior mean abilities, under unimodal scenario (left panel) and bimodal scenario (right panel), compared with the true density (dotted line).
Figure 9: Distribution of ability estimated under the unimodal scenario (left panel) and bimodal scenario (right panel), compared with the true density (dotted line). Dashed lines indicate 95% credible intervals for the estimated distributions.

Mis-specification of the distribution of ability has limited effect when estimating individual percentiles. In Figure 7.1 we compare the posterior mean estimates of individual percentiles with the true percentile of the simulation distribution, for a subset of 50 individuals. In the unimodal simulation the semiparametric and parametric model perform very similarly in ordering individuals, while there are modest differences under the bimodal one. MSE values for both models are very similar in both simulations.
Figure 10: Estimates of individual percentiles (with 95% credible interval) for a subset of 50 individuals with varying (true) ability levels under the unimodal scenario (left panel) and the bimodal scenario (right panel). Black dots correspond to true percentiles.

7.2 Inferential results for real-world data

For the real data examples we graphically inspect results from the parametric and semiparametric models, using samples from the most efficient strategy in Section 6. For the health data we used samples from the IRT unconstrained model, while for the TIMSS data we used samples from the SI unconstrained model using the centered sampler.

In Figure 11 we compare item parameter estimates from the two models for the health data application, while Figure 12 shows estimates for the distribution of abilities. Recall that, in this case, we interpret the latent ability as physical ability, with high values characterizing healthy individuals. As with the bimodal simulation, estimates from the parametric model of the distribution of physical ability are quite different than the distribution of individual posterior mean abilities. It is clear that the parametric model is badly mis-specified and would produce bad out-of-sample predictions. In contrast, the semiparametric model seems to nicely characterize multi-modality in the latent distribution. We observe in Figure 12 large credible intervals for high values of this distribution that can be explained by the presence of many individuals with high raw scores, (i.e., 9 or 10 out of 10, Figure 1) for whom the model can clearly determine that their physical abilities are high, but with the exact magnitudes being difficult to identify.

The two modeling assumptions yield different estimates of the item parameters (Figure 11), with this difference being higher for extreme values. However, the relative ranking of the items is roughly the same in both cases, with for example item 1 (Vigorous activities) being the most difficult item and the one with lowest value of the discrimination parame-
ter. According to the parametric model, discrimination parameters for item 3 (Lift/carry) and item 10 (Bathing/dressing) should have similar values, while the semiparametric model separates them.

Figure 11: Health data. Comparison of item parameter estimates from the parametric and semiparametric models. In each panel items are ordered by increasing values of the parameter estimate under the semiparametric model.

Figure 12: Health Data. Histogram and density estimate of the posterior means of the latent abilities (left panel) and estimate of the posterior distribution for the latent abilities (right panel). Dashed lines indicate 95% credible intervals for the estimated distributions.

For the TIMSS data we only compare point estimates of the item parameters (Figure 13), due to the large number of parameters. Estimates for the difficulty parameters are very similar between the two models, while estimates for the discrimination parameters are more
different, especially for larger values. Figure 14 shows estimates of the distribution of ability. In this case the semiparametric model estimate shows a moderate departure from the normal parametric assumption, with the estimated distribution being right-skewed.

Figure 13: TIMSS data. Comparison of posterior estimates of the item parameters between the parametric and semiparametric model both using the SI unconstrained centered sampling strategy.

Figure 14: TIMSS Data. Histogram and density estimate of the posterior means of the latent abilities (left panel), and estimate of the posterior distribution for the latent abilities (right panel). Dashed lines indicate 95% credible intervals for the estimated distributions.
The effect of departures from normality in the distribution of ability is evident when estimating individual percentiles. Figure 15 compares these estimates for both the health and TIMSS data for a sample of 50 individuals sorted according to estimated abilities from the semiparametric model. In particular, the parametric assumption in the case of the health data multi-modality yields some differences in estimating percentile values and individual ordering. This estimates are associated with larger intervals than in the semiparametric case. In the case of the skewness with the TIMSS data, the interval lengths are similar when comparing the parametric and semiparametric models.

8 Discussion

In this paper, we consider a semiparametric extension for 1PL and 2PL models, using Dirichlet process mixtures as a nonparametric prior to flexibly characterize the distribution of ability. We provide an overview of these models and study how different sets of constraints can address identifiability issue and lead to different MCMC estimation strategies.

Focusing on the 2PL model, we compare efficiency and inferential results under different sampling strategies based on model parametrization, constraints and sampling algorithms. We find that MCMC performance across strategies can vary in relation to underlying shape of the latent distribution. Hamiltonian Monte Carlo is often more efficient for the parametric 2PL model, particularly when the underlying distribution of ability is reasonably consistent with the normality assumption. However non-HMC samplers are generally competitive in terms of performance and are needed for semiparametric inference. When moving to semiparametric modeling, the computational cost can be high for large datasets, given that
sampling from the Dirichlet Process requires iteration through all individuals. However we find computational costs to be reasonable in our applications in light of the better inferential results.

In particular under model mis-specification, inference for item parameters worsens noticeably in the parametric model compared to the semiparametric model. With sufficient data, inference for the abilities of observed individuals can be decent even under mis-specification of the distribution of ability, but inference for the unknown latent distribution (i.e., the predictive distribution for new individuals) as a whole can be quite bad. Having access to semiparametric models can be broadly useful, as it allows inference on the entire underlying distribution of ability and its functionals, such as percentiles.

In this work we extensively use the NIMBLE software for hierarchical modeling, with code reproducing results in the paper available at https://github.com/salleuska/IRT_nimble_code, along with a small tutorial using a simulated example. Although there are other software solutions enabling Bayesian nonparametric modeling, these are often limited in the type of algorithms or in the class of models available. NIMBLE offers a high degree of flexibility in that the models considered in this paper could be easily embedded in more complicated ones. Sampler assignment can be highly customized by the user, including user-defined sampling algorithms. This customizability makes NIMBLE a powerful platform for comparing different sampling strategies. At the same time, NIMBLE allows easy sharing of the most successful strategies as block-box implementations for end users.
A. Identifiability

The 2PL model is not identifiable based on the likelihood. Here we demonstrate the non-identifiability for the two parameterizations we consider, showing how different linear transformations lead to the same probabilities. Note that these transformations are defined for every parameter associated with each item $i = 1, \ldots, I$ and individual $j = 1, \ldots, N$.

Under the IRT parameterization:

1. $\eta'_j = \eta_j / s$ and $\lambda'_i = s \lambda_i$

   $$\lambda'_i (\eta'_j - \beta_i) = s \lambda_i (\eta_j / s - \beta_i) = \lambda_i \eta_j - \lambda_i \beta_i = \lambda_i (\eta_j - \beta_i),$$

2. $\eta'_j = \eta_j + c$ and $\beta'_i = \beta_i + c$,

   $$\lambda_i (\eta'_j - \beta'_i) = \lambda_i (\eta_j + c - (\beta_i + c)) = \lambda_i (\eta_j - \beta_i).$$

Under the slope-intercept parameterization:

1. $\eta'_j = \eta_j / s$ and $\lambda'_i = s \lambda_i$,

   $$\lambda'_i \eta'_j + \gamma_i = s \lambda_i \eta_j / s + \gamma_i = \lambda_i \eta_j + \gamma_i,$$

2. $(\lambda_i \eta_j)' = \lambda_i \eta_j + c$ and $\gamma_i' = \gamma_i - c$, or $\eta'_j = \eta_j + c$ and $\gamma'_i = \gamma_i - \lambda_i c$

   $$\lambda_i \eta'_j + \gamma'_i = \lambda_i (\eta_j + c) + \gamma_i - \lambda_i c = \lambda_i \eta_j + \gamma_i.$$

Post-processing to satisfy identifiability constraints

This section reports the transformations we apply to item and ability parameters in order to satisfy the identifiability constraints in our base parameterization (8). These transformations are applied to each posterior sample.

Under the IRT parameterization, the set of transformations for each posterior sample of $\{\lambda_i, \beta_i, \eta_j\}$ for $i = 1, \ldots, I, j = 1, \ldots, N$ takes these forms:

$$\lambda^*_i = s \lambda_i, \quad \beta^*_i = \frac{\beta_i - b}{s}, \quad \eta^*_j = \frac{\eta_j - b}{s},$$

subject to $\prod_{i=1}^I \lambda^*_i = 1$, $\sum_{i=1}^I \beta^*_i = 0$. By solving the system of equations given by the transformations and the set of identifiability constraints, we obtain

$$s = \exp \left\{ \sum_{i=1}^I \log(\lambda_i) / I \right\}, \quad b = \frac{\sum_{i=1}^I \beta_i}{I}. \quad (A1)$$
Under the slope-intercept parameterization, the set of transformations for each posterior sample of \( \{ \lambda_i, \gamma_i, \eta_j \} \) for \( i = 1, \ldots, I, j = 1, \ldots, N \) takes these forms:

\[
\tilde{\lambda}_i = s \lambda_i, \quad \tilde{\gamma}_i = \gamma_i - \lambda_i c, \quad \tilde{\eta}_j = \eta_j + c / s,
\]

subject to \( \prod_{i=1}^I \tilde{\lambda}_i = 1 \), \( \sum_{i=1}^I \tilde{\gamma}_i = 0 \). Similarly, by solving the system of equations given by the transformations and the set of identifiability constraints, we obtain

\[
s = \exp \left\{ \sum_{i=1}^I \frac{\log(\lambda_i)}{I} \right\}, \quad c = \frac{\sum_{i=1}^I \gamma_i}{\sum_{i=1}^I \lambda_i}, \quad (A2)
\]

Finally, to get from the slope-intercept parameterization to the IRT parameterization, we define \( \tilde{\beta}_i := -\tilde{\gamma}_i / \tilde{\lambda}_i \) and then calculate \( \beta_i^* = \tilde{\beta}_i - \sum_i \tilde{\beta}_i / I \).

**Rescaling the DP density**

We can obtained posterior samples from the mixing distribution \( F \) via NIMBLE’s `getSamplesDPmeasure` function, allowing us to estimate the density for the latent ability distribution. However, when comparing these estimated densities between models, for some of the sampling strategies, we need to transform the estimated density to account for the transformations of the abilities from the scale on which sampling is done to the scale in (8).

As an example, consider the IRT parameterization without constraints. From the MCMC output we can obtain \( p(\tilde{\eta}) \) evaluated for different values of \( \tilde{\eta} \), but we want \( p(\tilde{\eta}^*) \) with \( \tilde{\eta}^* = (\tilde{\eta} - b) / s \). To do so we need the Jacobian of the transformation, which is simply \( s \). Then, we obtain \( p(\tilde{\eta}^*) \)

\[
p(\tilde{\eta}^*) = p_{\tilde{\eta}}(s\tilde{\eta}^* + b) \left| \frac{\partial(s\tilde{\eta}^* + b)}{\partial \tilde{\eta}^*} \right| = p_{\tilde{\eta}}(s\tilde{\eta}^* + b)s. \quad (A3)
\]

**B. Centered sampler**

We consider a custom sampler for the 2PL model under the slope-intercept parameterization. Intuition for this sampling strategy comes from the resemblance to a linear model. In order to sample \( \{ \lambda_i, \gamma_i \} \) efficiently, we propose centering the implied covariate, \( \eta_j \), to have mean zero. This is analogous to centering covariates in a linear model, but in this case the ”covariate” values are not fixed, so the centering needs to be done in each iteration. For a given item \( i \) for \( i = 1, \ldots, I \) we can rewrite:

\[
\lambda_i \eta_j + \gamma_i = \lambda_i (\eta_j - \bar{\eta}) + \lambda_i \bar{\eta} + \gamma_i, \\
= \lambda_i \eta_j^c + \gamma_i^c,
\]
such that the quantity $\eta^c_j = \eta_j - \bar{\eta}$ is centered. The idea is to propose a new value $\lambda^*_i$ in this new parameterization at each MCMC iteration, using a random walk on the log scale. Translating to the original parameterization, we have:

$$
\lambda^*_i \eta^c_j + \gamma^c_i = \lambda^*_i (\eta_j - \bar{\eta}) + \lambda_i \bar{\eta} + \gamma_i,
$$

$$
= \lambda^*_i \eta_j - \lambda^*_i \bar{\eta} + \lambda_i \bar{\eta} + \gamma_i.
$$

This means that we are proposing $\gamma^*_i = \gamma_i + \bar{\eta} (\lambda_i - \lambda^*_i)$. Thus we have a joint proposal $(\lambda^*_i, \gamma^*_i)$ that accounts for the usual correlation in a regression between intercept and slope. Apart from accounting for sampling $\lambda_i$ on the log scale, the proposal is symmetric, so no Hastings correction is needed. The original sampler for $\gamma_i$ can stay the same. This is because in the reparameterization with $\gamma^c_i$ above, shifting $\gamma_i$ by a certain amount is equivalent to shifting $\gamma^c_i$.

C. Health data questions

The following items are about activities you might do during a typical day. Does your health now limit you in these activities? If so, how much?

1. Vigorous activities: Vigorous activities, such as running, lifting heavy objects, participating in strenuous sports.
2. Moderate activities: Moderate activities, such as moving a table, pushing a vacuum cleaner, bowling or playing golf.
3. Lift/Carry: Lifting or carrying groceries.
4. Several stairs: Climbing several flights of stairs.
5. One flight stairs: Climbing one flight of stairs.
6. Bend/Kneel/Stoop: Bending, kneeling, or stooping.
7. Walk more mile: Walking more than a mile.
8. Walk several blocks: Walking several blocks.
9. Walk one block: Walking one block.
10. Bathing/Dressing: Bathing or dressing yourself.
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