Potential model calculations and predictions for $c\bar{s}$ quarkonia

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We investigate the spectroscopy and decays of the charm-strange quarkonium system in a potential model consisting of a relativistic kinetic energy term, a linear confining term including its scalar and vector relativistic corrections and the complete perturbative one-loop quantum chromodynamic short distance potential. The masses and wave functions of the various states are obtained using a variational technique, which are then used in a perturbative treatment of the potential to find the mass spectrum of the $c\bar{s}$ system and radiative decay widths. Our results compare well with the available data for the spectrum of $D_s$ states. We include a discussion of the effect of mixing and an investigation of the Lorentz nature of the confining potential.

1. INTRODUCTION

Recently we reported on a study of the charmonium and upsilon systems in a semi-relativistic model which includes all $v^2/c^2$ and one-loop QCD corrections for the interaction of a quark and antiquark of equal mass[1]. This semi-relativistic potential model successfully describes the spectra and leptonic and radiative decays of those systems. We have now extended this modelling approach to systems in which the quark and antiquark have different masses.

Interest in the modelling of light-heavy quarkonia is over 25 years old [2]. A variety of modelling approaches have been employed with varying success [3–8]. Renewed and continuing interest in the modelling of $c\bar{s}$ quarkonia is fueled by, in particular, the recent discovery of the $2^3S_1$ state [9] as well as ongoing efforts to determine the masses and decays of the $D_s$ mesons [10].

We have revised and extended the approach of our earlier papers in order to investigate the spectroscopy and decays of the $D_s$ system, as well as to discuss other questions of modelling interest. In addition, we investigate the the scalar/vector mixture of the phenomenological confining potential

In the next Section, we describe the potential model in some detail. This is followed, in Section 3, by an outline of our calculational approach. In Section 4, we present our results for the $D_s$ system, and then give some conclusions in Section 5. The conventions we use for our treatment of the mixing of the $J = 1$ $p$–states are given in the Appendix.

2. SEMI-RELATIVISTIC MODEL

In our analysis, we use a semi-relativistic Hamiltonian of the general form

$$H = \sqrt{\vec{p}^2 + m_1^2} + \sqrt{\vec{p}^2 + m_2^2} + Ar - \frac{4\alpha_S}{3r} \left[1 - \frac{3\alpha_S}{2\pi} + \frac{\alpha_S}{6\pi} (33 - 2n_f) \left(\ln(\mu r) + \gamma_E\right)\right] + V_L + V_S$$

$$= H_0 + V_L + V_S,$$

where $H_0$ is the non-interacting kinetic energy term and $V_L$ and $V_S$ are the linear confining and scalar/ vector potential terms, respectively. The parameter $Ar$ is the four-dimensional momentum transfer, and $\mu$ is a renormalization constant. The quantities $\alpha_S$ and $n_f$ are the strong coupling constant and the number of active quark flavors, respectively. The constants $\gamma_E$ and $\ln(\mu r)$ are the Euler-Mascheroni constant and the natural logarithm of the scale $\mu r$, respectively.

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where \( m_1 \) and \( m_2 \) are the quark masses, \( \mu \) is the renormalization scale, \( n_f \) is the effective number of light quark flavors and \( \gamma_E \) is Euler’s constant. \( V_L \) contains the \( v^2/c^2 \) corrections to the linear confining potential

\[
V_L = -(1 - f_V) \frac{A}{4r} \left[ \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \vec{L} \cdot \vec{S} + \left( \frac{1}{m_1} - \frac{1}{m_2^2} \right) \vec{L} \cdot (\vec{S}_1 - \vec{S}_2) \right] + f_V \frac{A}{4r} \left[ \left( \frac{1}{m_1} + \frac{1}{m_2} + \frac{16}{3m_1m_2} \right) \vec{S}_1 \cdot \vec{S}_2 + \left( \frac{1}{m_1} + \frac{1}{m_2} + \frac{4}{m_1m_2} \right) \vec{L} \cdot \vec{S} + \left( \frac{1}{m_1} - \frac{1}{m_2^2} \right) \vec{L} \cdot (\vec{S}_1 - \vec{S}_2) \right] + \frac{4}{3m_1m_2} (3\vec{S}_1 \cdot \vec{r} \vec{S}_2 - \vec{S}_1 \cdot \vec{S}_2) ,
\]

(3)

where \( A \) is the linear coupling strength. The first line in Eq.(3) is the contribution from scalar exchange while the second and third lines give the contribution from vector exchange, with \( f_V \) representing the fraction of vector exchange in the interaction. The short distance potential is [2]

\[
V_S = V_{HF} + V_{LS} + V_T + V_{SI} + V_{MIX} ,
\]

(4)

with

\[
V_{HF} = \frac{32\alpha_s \vec{S}_1 \cdot \vec{S}_2}{9m_1m_2} \left\{ (1 - \frac{19\alpha_s}{6\pi}) \delta(\vec{r}) - \frac{\alpha_s}{8\pi} (\frac{m_1 - m_2}{m_1 + m_2} + \frac{m_1 + m_2}{m_1 - m_2}) \ln \frac{m_2}{m_1} \right\} - \frac{\alpha_s}{24\pi^2} (33 - 2\alpha_f) \vec{S} \cdot \vec{S} \left[ \ln \frac{\mu r + \gamma_E}{r} \right] + \frac{21\alpha_s}{16\pi^2} \ln (\frac{m_1m_2}{r}) \left[ \ln (\frac{m_1m_2}{r}) \right] \]

(5a)

\[
V_{LS} = \frac{\alpha_s \vec{L} \cdot \vec{S}}{3m_1^2m_2r^3} \left\{ (m_1 + m_2)^2 + 2m_1m_2 \left[ \frac{8}{3} - 6 (\ln (m_1m_2) \frac{\vec{S}}{r} + \gamma_E - 1) \right] - \frac{3\alpha_s}{2\pi} (m_1^2 - m_2^2) \ln \frac{m_2}{m_1} \right\}
\]

(5b)

\[
V_T = \frac{4\alpha_s (3\vec{S}_1 \cdot \vec{r} \vec{S}_2 - \vec{S}_1 \cdot \vec{S}_2)}{3m_1m_2r^3} \left\{ 1 + \frac{4\alpha_s}{3\pi} + \frac{\alpha_s}{6\pi} \left[ (33 - 2n_f) (\ln \frac{\mu r + \gamma_E - 4}{3}) \right] \right\}
\]

(5c)

\[
V_{SI} = \frac{2\pi\alpha_s}{3} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \left\{ (1 - \frac{3\alpha_s}{2\pi}) \delta(\vec{r}) - \frac{\alpha_s}{6\pi} (33 - 2n_f) \vec{S} \cdot \vec{S} \right\}
\]

(5d)

\[
V_{MIX} = \frac{\alpha_s \vec{L} \cdot (\vec{S}_1 - \vec{S}_2)}{3m_1^2m_2r^3} \left\{ (m_1^2 - m_2^2) \left[ \frac{8}{3} - 6 (\ln (m_1m_2) \frac{\vec{S}}{r} + \gamma_E - 1) \right] - \frac{3\alpha_s}{2\pi} (m_1^2 + m_2^2) \ln \frac{m_2}{m_1} \right\}
\]

(5e)

We have chosen \( H_0 \) such that it contains the relativistic kinetic energy and the leading order spin-independent portions of the long-range confining potential and the one-loop QCD short-range potential. It is important to recall that the potential given by Eq.(4) does not reduce to the potential in Ref.1 due to the presence of annihilation terms in the equal-mass quark-antiquark potential. It should also be noted that in calculating the matrix elements of the \( \delta(\vec{r}) \) terms in Eqs. (5a) and (5d), we ‘soften’ their singularity by adopting the quasistatic approximation of Ref.[11], which leads to the replacement

\[
\delta(\vec{r}) \rightarrow \frac{m^2}{\pi r} e^{-2mr}
\]

(6)

where \( m \) is the quark mass. This softening helps the stability of the eigenvalue calculation.
The $c\bar{s}$ mass spectrum and corresponding wave functions are obtained using the variational approach described in Ref. 1. In this approach, we expand the wave functions as

$$\psi_{jls}(\vec{r}) = \sum_{k=0}^{n} C_k \left( \frac{r}{R} \right)^{k+\ell} e^{-r/R} Y_{jls}^{m}(\Omega),$$

(7)

where $Y_{jls}^{m}(\Omega)$ denotes the orbital-spin wave function for a specific total angular momentum $j$, orbital angular momentum $\ell$, and total spin $s$. The $C_k$'s are determined by minimizing

$$E = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle},$$

(8)

with respect to variations in these coefficients. This procedure results in a linear eigenvalue equation for the $C_k$'s and the energies, and is equivalent to solving the Schrödinger equation. The wave functions corresponding to different eigenvalues are orthogonal and the $k^{th}$ eigenvalue $\lambda_k$ is an upper bound on the exact energy $E_k$. For $n = 14$, the lowest four eigenvalues for any $\ell$ are stable to one part in $10^6$. We performed a perturbative calculation, using $H_0$ as the unperturbed Hamiltonian and all other terms treated as first-order perturbations.

An optimal set of the parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ were found by minimizing the $\chi^2$ function

$$\chi^2 = \sum_{i=1}^{N} \frac{(O_{\exp i} - O_{\th(i)}(\alpha))^2}{\sigma_i^2},$$

(9)

where the $O_i$ denote the experimental and theoretical values of some quarkonium observable and the $\sigma_i$ are the associated errors. In this work, the $O_{\exp i}$ consist of a subset of the measured $D_s$ masses. For the masses, the $\sigma_i$ are taken to be the actual experimental error and a common intrinsic theoretical error added in quadrature. The latter error reflects the experimental uncertainty associated with omitting corrections beyond one-loop and is estimated by requiring the $\chi^2$/degree of freedom to be approximately unity. Typically, this error is a few MeV. The minimization of $\chi^2$ with respect to variations of the parameters $\alpha$ is accomplished using the search program STEPIT [12]. The choices of $\alpha_S$ and $m_c$ were kept consistent with the results of running these parameters from the charmonium scale of Ref.[1] by introducing a Gaussian prior in the $\chi^2$ function for each one. For additional discussion of calculational details, see Appendix A of Ref.1.

4. RESULTS

We summarize our results in the following tables. The parameters resulting from our fit are given in Table I.

| Parameter | Value   |
|-----------|---------|
| $A$ (GeV$^2$) | 0.115   |
| $\alpha_S$ | 0.391   |
| $m_c$ (GeV) | 1.66    |
| $m_S$ (GeV) | 0.346   |
| $\mu$ (GeV) | 1.19    |
| $f_V$ | 0.00    |

**TABLE I:** Fitted Parameters for the $c\bar{s}$ system

The results for our determination of the $D_s$ levels are shown in Table II [13]. Overall our fit to the spectrum is quite good.

As is usual in potential model treatments [5, 7, 14–16], the radiative widths were calculated in the dipole approximation. We obtained the $E_1$ matrix elements by using the variational radial wave functions to construct
The widths in this case are
\[ \Gamma_f = \frac{4\alpha}{9} \omega^3 \left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2 \frac{E_f}{M_i} \]
for the $D_s^* \rightarrow D_s + \gamma$. In the case of the $p$–state magnetic transitions $D_{s1} \rightarrow D_{s0} + \gamma$ and $D'_{s1} \rightarrow D_{s0} + \gamma$, both the singlet and triplet states of the mixtures in Eq. (11) contribute to the widths. If we use the perturbative wave functions, then the relative phase of the triplet contribution with respect to the singlet contribution is $\pi/2$.

The resulting radiative widths are shown in Table III.
\begin{table}[h]
\centering
\begin{tabular}{l|c|c}
\hline
 & \(\Gamma_J\) (keV) & Model/Expt \\
\hline
\(D_s^* \rightarrow D_s\) & 1.91 & < 1.9 \times 10^3 \\
\(D_{s0}(2317) \rightarrow D_s^*\) & 4.92 & \\
\(D_{s1}(2460) \rightarrow D_s\) & 12.8 & BR = 0.18 \pm 0.04 \\
\(D_{s1}(2460) \rightarrow D_s^*\) & 15.5 & BR < 0.08 \\
\(D_{s1}(2460) \rightarrow D_{s0}(2317)\) & 5.74 & \\
\(D_{s1}^*(2536) \rightarrow D_s\) & 54.5 & \\
\(D_{s1}^*(2536) \rightarrow D_{s0}(2317)\) & 8.90 & possibly seen \\
\(D_{s2}(2575) \rightarrow D_s^*\) & 2.36 & \\
\(D_{s1}^*(2637) \rightarrow D_{s0}(2317)\) & 6.76 & \\
\(D_{s1}^*(2637) \rightarrow D_{s1}(2460)\) & 2.8 & \\
\(D_{s1}^*(2637) \rightarrow D_{s1}^*(2536)\) & 0.24 & \\
\(D_{s1}^*(2637) \rightarrow D_{s2}(2573)\) & 0.35 & \\
\(D_s(2485) \rightarrow D_{s1}(2460)\) & 0.01 & \\
\hline
\end{tabular}
\caption{The radiative decays of the \(D_s\) mesons are shown. These widths are computed using the mass values obtained directly from our calculation. This includes the \(n = 2\) pseudoscalar and vector states, the latter of which has recently been observed with a higher mass [13]. The widths are from [13].}
\end{table}

5. CONCLUSIONS

We have shown that a potential model consisting of the relativistic kinetic energy, a linear long-range confining potential together with its \(v^2/c^2\) relativistic corrections, and the full \(v^2/c^2\) plus one-loop QCD corrected short distance potential is capable of providing extremely good fits to the spectra of the \(D_s\) states by treating them as states of the \(c\bar{s}\) system. We find that in this perturbative treatment the long-range potential must be entirely due to scalar exchange.

The single photon widths can be obtained from the variational wave functions, but, apart from some branching ratio measurements, there are relatively little data available. Our theoretical results are comparable to those given in Refs.[5] and [7] allowing for the fact that both of these references use a substantially higher strange quark mass (419 MeV and 480 MeV, respectively). In every case, efforts to model these states will be greatly improved by the availability of additional data.

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APPENDIX A: DETAILS OF MIXING

The mixing of the \(^3P_1\) and \(^1P_1\) states is obtained by diagonalizing the 2 × 2 matrix

\[
\begin{pmatrix}
E_3 & V_{31} \\
V_{31} & E_1
\end{pmatrix},
\]

where \(E_3\) is the \(^3P_1\) energy, \(E_1\) is the \(^1P_1\) energy and \(V_{31}\) is the mixing matrix element. In perturbation theory, this is relatively simple since all of these matrix elements can be calculated using the unperturbed wave functions.
that are all the same. The energy eigenvalues are

$$E_{\pm} = \frac{1}{2} (E_3 + E_1) \pm \frac{1}{2} \sqrt{(E_3 - E_1)^2 + 4V_{31}^2},$$  

(A2)

and we fit the $D_{s1}(2536)$ and $D_{s1}(2460)$ to $E_+$ and $E_-$. Note that as $V_{31} \to 0$, $E_+ \to E_3$ and $E_- \to E_1$. To define the mixing angles in terms of known parameters, we assume that the eigenvector $\psi_+$ corresponding to $E_+$ behaves as

$$\psi_+ \overset{V_{31} \to 0}{\to} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$  

(A3)

With this assumption, $\psi_+$ is

$$\psi_+ = \frac{1}{\sqrt{(E_+ - E_1)^2 + V_{31}^2}} \begin{pmatrix} E_+ - E_1 \\ V_{31} \end{pmatrix}.$$  

(A4)

By writing $\psi_+$ as

$$\psi_+ = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix},$$  

(A5)

with

$$\tan(\theta) = -\frac{V_{31}}{E_+ - E_1},$$  

(A6)

we arrive at the decomposition Eq. (11). It is possible to obtain an estimate of the mixing angle by using the branching ratios of the $D_{s1}(2460) \to D_s\gamma$ and $D_{s1}(2460) \to D^*_s\gamma$, whose ratio gives

$$\frac{\Gamma(D_{s1}(2460) \to D^*_s\gamma)}{\Gamma(D_{s1}(2460) \to D_s\gamma)} \omega^3 = \tan^2(\theta),$$  

(A7)

where $\omega$ is the momentum of the photon in the $D_s$ transition and $\omega^*$ is the corresponding photon momentum in the $D^*_s$ transition. Using the published branching ratio information, with favorable assumptions, the data are consistent with $\theta = \pm 50^\circ$. Our calculation gives $\theta = 59.7^\circ$. 

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