Dynamics of Vortex Pair in Radial Flow

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The problem of vortex pair motion in two-dimensional plane radial flow is solved. Under certain conditions for flow parameters, the vortex pair can reverse its motion within a bounded region. The vortex-pair translational velocity decreases or increases after passing through the source/sink region, depending on whether the flow is diverging or converging, respectively. The rotational motion of two corotating vortexes in a quiescent environment transforms into motion along a logarithmic spiral in the presence of radial flow. The problem may have applications in astrophysics and geophysics.

I. INTRODUCTION

In the number of astrophysical and geophysical problems the situation arises in which a system of vortices (in the simplest case, a vortex pair) interacts with diverging or converging radial flow. For example, in the dipolar vortex model of an active galactic nucleus (AGN) [1], the obscuring tori3 [2 – 5] are toroidal vortices in which self-gravitation is balanced by centrifugal forces [6]. (Possible geophysical applications are discussed at the end of this paper.)

The vortex motion inside an obscuring torus transform it from a pure geometrical object into a complex dynamical system. The interaction of the vortex pair with the radial flow can lead to phenomena such as vortex reversal, acceleration, or deceleration of translational motion, and modification of the rotational motion. These phenomena can have implications for the dynamics of vortices in various astrophysical and geophysical contexts.

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1 The idea of obscuring tori underlies the unified model of AGN [3, 4] in which the difference between types of active nuclei, such as radio galaxies and quasars (as well as between types of Seyfert galaxies), is attributed to the position of the obscuring tori relative to the line of sight. For example, in the case of quasars it is possible to observe the central AGN region adjacent to the black hole at the center of a galaxy, whereas the central region of a radio galaxy is obscured by the surface of a torus. In 2004, the idea was substantiated by direct observations of the obscuring torus with an optical interferometer combining 8.2 m telescopes at the Southern European Observatory [5].
FIG. 1: Dipolar vortex model of AGN obscuring tori (symmetry plane cross section) [1]: $M_c$ is the central mass and $R$ is the major radius of a torus.

to the dynamical one that allows to describe dynamical processes in AGNs. The vortex motion can be caused by interaction between the torus and the radial outflow of wind and radiation from the AGN central region, which is responsible also for the dipolar structure of the vortex motion in the torus (see Fig. 1).

The foregoing discussion motivates a study of the dynamics of interaction between vortices and radial flow. Problems of this kind seem to have never been analyzed. The simplest model is the planar interaction between radial flow and a dipole vortex pair. (The corresponding flow pattern can be envisaged from Fig. 1, but the present analysis is restricted to a plane two-dimensional problem.) In this paper, we present an exact solution for the vortex motion in an inviscid fluid. Possible exactly solvable generalization for rotating radial flow is considered at the end of the paper.

II. PLANAR VORTEICES IN DIVERGING RADIAL FLOW

The input time-dependent streamfunction equation in the plane case has the form [7]

$$\frac{\partial \Delta \psi}{\partial t} + J(\psi, \Delta \psi) = 0,$$

where the flow velocity $\mathbf{v}$ is related to the streamfunction $\psi$ as

$$v_x = -\frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \psi}{\partial x}, \quad \Delta \psi = \text{curl}_z \mathbf{v},$$

and $J(\alpha, \beta)$ is the Jacobian determinant. The equation (1) is the $z$ component of the vorticity conservation equation in two-dimensional case

$$\frac{d}{dt} \text{curl}_z \mathbf{v} = 0, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.$$
We represent the streamfunction as a decomposition into a regular part $\psi_r$ describing the flow and a singular part $\psi_s$ whose components describe point vortices:

$$\psi = \psi_r + \psi_s ,$$

where

$$\psi_s = \frac{1}{2\pi} \sum_m A_m \ln |r - r_m| ,$$

$A_m$ is the $m$th vortex strength, and $r_m$ is the corresponding position vector.\(^2\)

The singular component satisfies the Poisson Equation [9, 10]

$$\Delta \psi_s = \sum_m A_m \delta(x - x_m)\delta(y - y_m).$$

Substituting (3) into (1) we obtain the streamfunction equation

$$\frac{\partial \Delta \psi_r}{\partial t} + J(\psi_s + \psi_r, \Delta \psi_r) = 0$$

and expressions for the velocity components of the $m$th vortex

$$\dot{x}_m = -\frac{\partial(\psi_r + \psi_s^m)}{\partial y} \bigg|_{r=r_m} ,$$

$$\dot{y}_m = \frac{\partial(\psi_r + \psi_s^m)}{\partial x} \bigg|_{r=r_m} ,$$

where $\psi_s^m$ is the streamfunction $\psi_s$ minus the contribution of the $m$th vortex.

If the regular component $\psi_r$ corresponds to a diverging radial flow with source at the origin,\(^3\)

$$\psi_r = -Q\varphi, \quad Q = \text{const} > 0$$

then vorticity vanishes outside the source:

$$\Delta \psi_r = \frac{\partial^2 \psi_r}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi_r}{\partial \varphi^2} = 0 .$$

Now, Eq.(5) becomes an identity, and expressions (6) combined with (7) yield

$$\dot{x}_m = -\frac{\partial \psi_s^m}{\partial y} \bigg|_{r=r_m} + Q\frac{x_m}{r_m^2} ,$$

$$\dot{y}_m = \frac{\partial \psi_s^m}{\partial x} \bigg|_{r=r_m} + Q\frac{y_m}{r_m^2} .$$

\(^2\) We use the approach developed in [8].

\(^3\) Note that the assumption of incompressible flow is inapplicable within a small neighborhood of the radial flow source depending on sound velocity.
In the case of two vortices (Fig. 2), Eqs. (9) reduce to

\[
\begin{align*}
\dot{x}_1 &= -\frac{A_2 y_{12}}{2\pi r_{12}^2} + Q\frac{x_1}{r_1^2}, \\
\dot{y}_1 &= \frac{A_2 x_{12}}{2\pi r_{12}^2} + Q\frac{y_1}{r_1^2}, \\
\dot{x}_2 &= \frac{A_1 y_{12}}{2\pi r_{12}^2} + Q\frac{x_2}{r_2^2}, \\
\dot{y}_2 &= -\frac{A_1 x_{12}}{2\pi r_{12}^2} + Q\frac{y_2}{r_2^2},
\end{align*}
\]

(10)

where \((x_{12}, y_{12}) = (x_1 - x_2, y_1 - y_2)\) and \(r_{12} = |r_1 - r_2|\). Consider a dipole vortex pair \((A_2 = -A_1)\) symmetric about the x axis (Fig.3) in the initial moment of time. Since \(x_{12} = 0\) and \(y_2 = -y_1\), it will suffice to analyze the equations of motion of the one vortex located at \(y = y_1 > 0\):

\[
\begin{align*}
\dot{x} &= -\frac{A}{4\pi y} + Q\frac{x}{r^2}, \\
\dot{y} &= Q\frac{y}{r^2},
\end{align*}
\]

(11a)

(11b)

where \(x = x_1 = x_2, r = \sqrt{x^2 + y^2}\), and \(A = A_2 > 0\). In a quiescent environment, the pair moves with velocity \(A/4\pi y\) antiparallel to the x axis. It follows from Eq.(11b) that \(\dot{y} > 0\) i.e., the separation between the vortices increases with time elapsed, while the interaction between them (the first term in Eq.(11a)) decreases in a diverging flow.
III. HAMILTONIAN FORMULATION AND INTEGRATION OF GOVERNING EQUATIONS

An arbitrary system of vortices satisfying Eqs. (9) can be described in terms of the Hamiltonian

\[ \hat{H} = \frac{1}{4\pi} \sum_{m \neq n} A_m A_n \ln r_{mn} - Q \sum_m A_m \arccot \frac{x_m}{y_m}. \]

Indeed, multiplying Eqs. (9) by $A_m$ and verifying that

\[ \dot{x}_m \frac{\partial \hat{H}}{\partial x_m} + \dot{y}_m \frac{\partial \hat{H}}{\partial y_m} = 0, \quad m = 1, 2, ..., \]

we can write

\[ A_m \dot{x}_m = -\frac{\partial \hat{H}}{\partial y_m}, \quad A_m \dot{y}_m = \frac{\partial \hat{H}}{\partial x_m}. \]

In the case of a symmetrically moving vortex pair, Eqs. (11) can be rewritten in Cartesian coordinates in the canonical form

\[ \dot{x} = -\frac{\partial H}{\partial y}, \quad \dot{y} = \frac{\partial H}{\partial x}. \] (12)

with the time-independent Hamiltonian

\[ H(x, y) = \frac{A}{4\pi} \ln y - Q \arccot \frac{x}{y}. \] (13)

The trajectory of the vortex pair is parameterized by the "energy integral" of the system (11):

\[ E = \frac{A}{4\pi} \ln y - Q \arccot \frac{x}{y}. \] (14)

Hence, the "phase" of movement along the trajectory $\Phi \equiv \arccot(x/y)$ can be expressed as

\[ \Phi = \frac{A}{4\pi Q} \ln y - \frac{E}{Q}. \] (15)

Depending on vortex and flow parameters relation the case of reverse vortex motion may be possible. The vortex pair reverses its motion between $x_-$ and $x_+$, where

\[ x_{\pm} = y \frac{2\pi Q}{A} \left( 1 \pm \sqrt{1 - \left( \frac{A}{2\pi Q} \right)^2} \right), \] (16)

are the turning points where $\dot{x}$ changes sign. For existence of the turning points it has to be fulfilled the inequality

\[ (2\pi Q/A)^2 \geq 1. \] (17)
If $A > 0$, then the pair moves antiparallel to the $x$ axis and infinity corresponds the phase values

$$
\Phi = +0, \quad x = +\infty,
$$

$$
\Phi = \pi - 0, \quad x = -\infty.
$$

(18)

If $A < 0$ the pair moves parallel to the $x$ axis. The cotangent argument lies in this case in the interval $(-\pi, 0)$ and the phases corresponding to infinity are

$$
\Phi = -0, \quad x = -\infty,
$$

$$
\Phi = -\pi + 0, \quad x = +\infty.
$$

(19)

It convenient to regard the vortex movement in polar coordinates, $x = r \cos \varphi$, $y = r \sin \varphi$, $\varphi = \arccot(x/y)$. In this case the Eqs.(11) take the form

$$
\dot{r} = -\frac{A}{4\pi r} \cot \varphi + \frac{Q}{r},
$$

(20a)

$$
\dot{\varphi} = \frac{A}{4\pi r^2}.
$$

(20b)

These equations can also be represented in Hamiltonian form:

$$
\dot{\xi} = -\frac{\partial \tilde{H}}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial \tilde{H}}{\partial \xi},
$$

(21)

$$
\tilde{H} = A \frac{\ln \xi}{4\pi} + A \frac{\ln \sin^2 \varphi}{4\pi} - 2Q \varphi,
$$

where $\xi = r^2$ and Hamiltonian $\tilde{H}(\xi, \varphi)$ is twice as large than Hamiltonian $H(x, y)$ in Cartesian coordinates $\tilde{H}(\xi, \varphi) = 2H(x, y)$ for the same system. Thus, the Hamiltonian equations of motion written in these coordinates are

$$
\dot{\xi} = 2Q - \frac{A}{2\pi} \cot \varphi,
$$

$$
\dot{\varphi} = \frac{A}{4\pi \xi}.
$$

(22)

The distance from the center of source to the vortex reaches a minimum $r_*$ when $\varphi_* = \arccot(4\pi Q/A)$. The equation of trajectory $r(\varphi)$ parameterized by the energy integral is solved directly relative to the distance from the center:

$$
\xi(\varphi) \equiv r^2(\varphi) = \exp \left(\frac{8\pi}{A} E + \frac{8\pi Q}{A} \varphi - \ln \sin^2 \varphi\right)
$$

(23)

where $\tilde{E} = 2E$ and $E$ is given by (14). Using (23), we integrate Eq.(20b) in quadratures:

$$
t - t_0 = \frac{4\pi}{A} \int_{\varphi_0}^{\varphi} d\varphi_1 \exp \left(2(\mu + \lambda \varphi_1) - \ln \sin^2 \varphi_1\right),
$$

(24)
where
\[ \mu = \frac{4\pi E}{A}, \quad \lambda = \frac{4\pi Q}{A}. \]

The function \( t(\varphi) \) may be expressed in terms of elementary functions. The inversion of this function and calculation of \( r(t) \) can be performed numerically.

Changing back to the Cartesian coordinates, we obtain the trajectory equation
\[ x = y \cot \left( \frac{A}{4\pi Q} \ln y - \frac{E}{Q} \right). \tag{25} \]

Hence, the vortex locations at infinite distance are found by using (19):
\[ y_{+\infty} = \exp \left( \frac{4\pi E}{A} \right), \]
\[ y_{-\infty} = \exp \left[ \frac{4\pi(E + \pi Q)}{A} \right]. \tag{26} \]

Thus, the vortex separation increases by a factor of \( \exp(4\pi^2 Q/A) \) after the pair has passed through the flow region.

### IV. BLOW-OFF OF THE VORTEX PAIR COMPONENTS

Suppose that the energy corresponds to the desired trajectory. For example, condition (17) must be satisfied for a trajectory with reverse motion. Using the expression for the trajectory, we represent (16) as
\[ x_{\pm} = y_{\pm} \frac{2\pi Q}{A} \left[ 1 \pm \sqrt{1 - \left( \frac{A}{4\pi Q} \right)^2} \right], \tag{27} \]
where the ordinates
\[ y_{\pm} = \exp \left[ \frac{4\pi}{A}(E + Q \cdot \arccot \Psi_{\pm}) \right] \tag{28} \]
are expressed in terms of the phases
\[ \Psi_{\pm} = \frac{2\pi Q}{A} \left[ 1 \pm \sqrt{1 - \left( \frac{A}{4\pi Q} \right)^2} \right]. \tag{29} \]

We take a value \( y_0 \) in the interval defined by (26) that lies sufficiently close to a turning point. Using the expression for the trajectory, we determine the corresponding value \( x_0 \), numerically calculate the function \( x(y) \), and invert it to find the required \( y(x) \).

Figure 4 demonstrates that a vortex pair components moves apart as it approaches the source from infinity and can indeed reverse its motion along the \( x \) axis. Having traveled to
a distance sufficiently far away from the source, where the radial flow is weaker, the pair components resume their motion in the direction determined by the sense of vortex rotation. The separation at negative infinity given by (26) is reached asymptotically.

V. ACCELERATION OF A VORTEX PAIR BY CONVERGING RADIAL FLOW

Consider a vortex pair in a converging flow, as shown in Fig. 6. Since $Q < 0$, we change to a new parameter according to $Q = -P = const < 0$ and rewrite the expression for the regular streamfunction component as

$$\psi_r = -Q\varphi = P\varphi .$$  (30)

It is clear from Eq. (11b) that $\dot{y} < 0$; i.e., the vortex separation decreases with time elapsed. According to (18), the separation at infinite distance is

$$y_{+\infty} = \exp \left( \frac{4\pi E}{A} \right), \quad y_{-\infty} = \exp \left[ \frac{4\pi (E - \pi P)}{A} \right].$$  (31)
It follows from (31) that the vortex separation decreases by a factor of \( \exp(4\pi^2P/A) \) after the pair has passed through the sink region.

The trajectory of the pair is found by following an analogy with the analysis above. It is clear that the vortex components separation decreases, and the pair can reverse its motion within a certain portion of its trajectory near the sink if the parameters satisfy certain conditions, as illustrated by Fig.7. Having approached the sink to a sufficiently close distance, the pair accelerates as the vortex separation decreases and continues to move with a higher velocity determined by the sense of vortex rotation after it has passed the second turning point. At negative infinity, the pair reaches an asymptotic steady state with a smaller vortex separation and a higher translational velocity. In the example illustrated by Fig.7, the velocity of the pair “thrown” out of the sink region is higher by three orders of magnitude.

VI. LAGRANGIAN DESCRIPTION

Hamiltonian (21) in polar coordinates can be represented as

\[
\tilde{H} = \tilde{K}(\xi) + \tilde{U}(\varphi).
\]

Here, \( \varphi \) and \( \xi \) are interpreted as generalized coordinate and momentum, respectively (cf. (22));
\( \tilde{K}(\xi) = (A/4\pi) \ln \xi \) is the kinetic energy and potential energy is expressed as

\[
\tilde{U}(\varphi) = \frac{A}{4\pi} \ln \sin^2 \varphi - 2Q\varphi .
\]

The corresponding Lagrangian is

\[
\tilde{L}(\varphi, \dot{\varphi}) = \frac{A}{4\pi} \ln \dot{\varphi} - \frac{A}{4\pi} \ln \sin^2 \varphi + 2Q\varphi + \text{const} ,
\]

where the generalized velocity \( \dot{\varphi} = A/4\pi \xi \) is inversely proportional to \( \xi \).\(^4\) Given \( \tilde{L}(\dot{\varphi}, \varphi) \), we integrate the Euler-Lagrange equation

\[
\frac{\ddot{\varphi}}{\varphi^2} = \frac{4\pi}{A} \frac{\partial U(\varphi)}{\partial \varphi},
\]

to find the solution

\[
t - t_0 = C \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sin^2 \varphi} \exp \left( \frac{8\pi Q\varphi}{A} \right),
\]

which is equivalent to solution (24) obtained by using the Hamiltonian approach. In this case

\[
C = \frac{4\pi}{A} \exp \left( \frac{8\pi E}{A} \right).
\]

**VII. COROTATING VORTEICES IN RADIAL FLOW**

We consider two corotating vortices with equal circulations symmetric with respect to the source of the radial flow (Fig. 8).

It is well known [9] that the separation \( 2r_0 \) between the vortices in a quiescent environment remains constant and they rotate about the center of vorticity with an angular velocity of \( A/2\pi r_0^2 \). In radial flow, equations of motion (11) for a vortex are modified as follows:

\[
\dot{x} = -\frac{A}{4\pi r^2} y + \frac{Qx}{r^2},
\]

\[
\dot{y} = \frac{A}{4\pi r^2} x + \frac{Qy}{r^2}.
\]

where \( y = y_1 = -y_2 \) and \( A_1 = A_2 = A \).

System (35) yields \( r^2 = 2Qt + r_0^2 \); i.e., the squared separation between vortices in a diverging flow is a linear increasing function of time (Fig.9a). In a converging flow, the vortex

\(^4\) The inverse proportion between generalized velocity \( \dot{\varphi} \) and momentum \( \xi \) implies that with accuracy to unessential constant summand \( \tilde{L} = \xi \dot{\varphi} - \tilde{H} = -\tilde{H} \).
FIG. 8: Initial state of two corotating vortices with $A_1 = A_2 = A$. In a quiescent environment, the vortices circumrotate.

FIG. 9: Trajectory of a component of the pair of corotating vortices of equal strength $A = 4\pi$:
(a) diverging radial flow with $Q = 0.1$; (b) converging flow with $P = -Q = 0.1$.

separation vanishes in the finite time $t = r_0^2/2P$ (Fig.9b), while the angular velocity of vortex rotation indefinitely increases (see Footnote 3).

Changing to the complex coordinate $w = x + iy$, we have $|w|^2 = r^2$. Accordingly, Eqs. (35) are equivalent to

$$\frac{\dot{w}}{w} = \left( Q + \frac{A}{4\pi} i \right) \frac{1}{|w|^2} = \left( Q + \frac{A}{4\pi} i \right) \frac{1}{2Qt + r_0^2}.$$  \hspace{1cm} (36)

Integrating Eq. (36), we obtain

$$w = w_0 \left( 1 + \frac{2Qt}{r_0^2} \right)^{\frac{1}{2} \left( 1 + \frac{A}{4\pi} i \right)}.$$  \hspace{1cm} (37)
where \( w_0 \) is the initial value of the complex coordinate. In the limit of \( Q \to 0 \), the solution describes uniform rotation. Indeed, changing to the new variable \( \eta = 2Qt/r_0^2 \), we have
\[
 w^2 = w_0^2 \cdot [\frac{(1 + \eta)^{1/\eta}}{\eta}]^{\frac{iAt}{2\pi r_0^2}} .
\] (38)

Since \( \lim_{\eta \to 0} (1 + \eta)^{1/\eta} = e \), we obtain
\[
 w \to w_0 \exp \left( \frac{iAt}{2\pi r_0^2} \right) .
\]

In the polar coordinates, the motion is particularly simple. It is obvious that the instantaneous angular velocity is expressed as follows \((\ddot{H} = A \ln \xi / 4\pi - 2Q\varphi)\):
\[
 \dot{\varphi}(t) = \frac{A}{4\pi(2Qt + r_0^2)} .
\]

Hence,
\[
 \varphi(t) = \frac{A}{8\pi Q} \ln \left( 1 + \frac{2Q}{r_0^2}t \right) + \varphi(0) ;
\]
i.e., the trajectory is a logarithmic spiral. The limit as \( Q \to 0 \) is easily taken.

VIII. VORTICES IN ROTATING RADIAL FLOW

Flow rotation can be taken into account by adding a corresponding term to the expression for the regular streamfunction component,
\[
 \psi_r = -Q\varphi + \psi^\Omega_r ,
\]
where \( \psi^\Omega_r = \Omega r^2/2 \) represents rigid-body rotation with angular velocity \( \Omega \). Then, Eqs. (10) are rewritten in terms of \( w = x + iy \) as
\[
 \frac{\partial w_1}{\partial t} - i\Omega w_1 = \frac{iA_2}{2\pi r_{12}^2} (w_1 - w_2) + \frac{Q}{r_1^2} w_1 ,
\]
\[
 \frac{\partial w_2}{\partial t} - i\Omega w_2 = -\frac{iA_1}{2\pi r_{12}^2} (w_1 - w_2) + \frac{Q}{r_2^2} w_2 .
\] (39)

In the rotating reference frame we substitute \( w = w' \exp(i\Omega t) \) and use the relations \( r^2 = |w|^2 = |w'|^2 \) and \( r_{12}^2 = |w_1 - w_2|^2 = |w'_1 - w'_2|^2 \) to derive the equations of motion for vortices in irrotational radial flow, which makes it possible to use the results obtained above.
IX. CONCLUSIONS

We have shown that the problem of vortex pair motion in two-dimensional radial flow has an exact solution. Under certain conditions for flow parameters, the vortex pair can reverse its motion. In a diverging flow, the separation between the vortices increases, but the pair does not break apart. In a converging flow, the vortices move inwards, and the translational velocity of the pair increases. In both cases, the vortex separation in the pair changes by a finite amount as it passes through the source region. The squared separation between corotating vortices of equal strength is a linear function of time. In a diverging flow, they move apart indefinitely. In a converging flow, the vortex separation vanishes in a finite time. Note that the problem of vortex motion in rotating radial flow may have qualitative applications in planetary geophysics. For example, the recently discovered double-lobed vortex swirling around Venus south pole [12] may be described by a similar model if there is meridional flow in the polar region. The present model cannot describe vortices spun up by the wind, since circulation is conserved. Varying circulation can be treated by allowing for dissipation. It would be interesting to analyze the asymmetrical motion of a vortex pair in a radial flow. Finally, also of interest is the motion of vortex rings in radial flow.

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