Generating functions for descents over words which avoid a consecutive pattern

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Abstract

In this paper, we extend the reciprocity method introduced by Jones and Remmel to study the distributions of descents over words which have no $u$-matches for words $u$ that have at most one descent.

1 Introduction

Let $\mathbb{P} = \{1, 2, \ldots \}$ denote the set of positive integers and for any $k \in \mathbb{P}$, let $[k] = \{1, \ldots, k\}$. We let $\mathbb{P}^* ([k]^*)$ denote the set of all words over the alphabet $\mathbb{P} ([k])$. We let $\epsilon$ denote the empty word and we say $\epsilon$ has length 0. We let $\mathbb{P}^+ = \mathbb{P}^* - \{\epsilon\}$ and $[k]^+ = [k]^* - \{\epsilon\}$.

If $u = u_1 \ldots u_j$ and $v = v_1 \ldots v_i$ are words in $\mathbb{P}^*$, we let $uv = u_1 \ldots u_jv_1 \ldots v_i$ denote the concatenation of $u$ and $v$. Suppose that we fix $j \geq 1$. Then for any word $w = w_1 \ldots w_n$, we say that a word $u = u_1 \ldots u_j$ is a prefix of a word $w$ if there is a word $v$ such that $uv = w$, is a suffix of $w$ if there is a word $v$ such that $vu = w$, and is a factor of $w$ if there are words $f$ and $v$ such that $fuv = w$.

Now suppose that $n \geq 1$ and $w = w_1 \ldots w_n \in \mathbb{P}^n$. Then we let $|w| = n$ denote the length of $w$. We let

\begin{align*}
Des(w) &= \{i : w_i > w_{i+1}\} \\
Rise(w) &= \{i : w_i < w_{i+1}\} \\
des(w) &= |Des(w)| \\
\text{rise}(w) &= |Rise(w)| \\
Lev(w) &= \{i : w_i = w_{i+1}\} \\
\text{lev}(w) &= |Lev(w)|.
\end{align*}

We shall refer to elements of $Des(w)$, $WDes(w)$, $Rise(w)$, $WRise(w)$, and $Lev(w)$ as descents, weak descents, rises, weak rises, and levels of $w$, respectively. We let $\pi^w$ denote $z_{w_1} \ldots z_{w_n}$. We let $\text{red}(w)$ denote the word that results from $w$ by replacing all occurrences of the $i$th smallest letter in $w$ by $i$. For example, if $w = 44537792$, then $\text{red}(w) = 33425561$.

Let $u = u_1 \ldots u_j \in \mathbb{P}^j$ and $w = w_1 \ldots w_n \in \mathbb{P}^n$. Then if $\text{red}(u) = u$, a $u$-match in $w$ is a factor $v$ of $w$ such that $\text{red}(v) = u$. An exact $u$-match in $w$ is a factor $v$ of $w$ such that $v = u$. 


We let \( \text{umch}(w) \) denote the number of \( u \)-matches in \( w \) if \( \text{red}(u) = u \) and \( e\text{umch}(w) \) denote the number of exact \( u \)-matches in \( w \). For example, if \( w = 31442521337792 \) and \( u = 213 \), then \( w \) has three \( u \)-matches, namely 314, 425, and 213, but only one exact \( u \)-match. Thus \( \text{umch}(w) = 3 \) and \( e\text{umch}(w) = 1 \). For any word \( w \in \mathbb{P}^* \) and \( i, j \in \mathbb{P} \), we let \( i\]j)(w) denote the number of exact matches of \( ij \) in \( w \).

For any word \( u = u_1 \ldots u_j \in [k]^j \) such that \( \text{red}(u) = u \), let \( St(P)(u) (St(k)(u)) \) equal the set of \( 1 < s \leq j \) such that there exists a word \( w = w_1 \ldots w_{s+j-1} \) in \( \mathbb{P}^* ([k]^*) \) such that \( \text{red}(w_1 \ldots w_j) = u \) and \( \text{red}(w_s \ldots w_{s+j-1}) = u \). That is, \( St(P)(u) (St(k)(u)) \) is the set of positions \( 1 < s \leq j \) such that there is a word \( w \) in \( \mathbb{P}^* ([k]^*) \) in which there is a pair of overlapping \( u \)-matches such that the first \( u \)-match starts at position 1 and the second \( u \)-match starts at position \( s \). We say that \( u \) is \( \mathbb{P} \)-minimal overlapping (\( [k] \)-minimal overlapping) if \( St(P)(u) = \{j\} \) \( (St(k)(u) = \{j\}) \). Thus \( u \) is \( \mathbb{P} \)-minimal overlapping if any two consecutive \( u \)-matches can share at most one letter which must be the last letter of the first \( u \)-match and the first letter of the second \( u \)-match. We say that \( u \) has the \( \mathbb{P} \)-weakly decreasing (\( \mathbb{P} \)-weakly increasing, \( \mathbb{P} \)-level) overlapping property if \( s \in St(P)(u) \) implies that \( u_1 \geq u_s \) \( (u_1 \leq u_s) \), \( u_1 = u_s \). We say that \( u \) has the \( [k] \)-weakly decreasing \( ([k] \)-weakly increasing, \( [k] \)-level) overlapping property if \( s \in St(k)(u) \) implies that \( u_1 \geq u_s \) \( (u_1 \leq u_s) \), \( u_1 = u_s \). If \( k \geq 2 \), then we say that \( u \) is \([k] \)-non-overlapping if \( St(k)(u) = \emptyset \). For example, suppose that \( u = 123234 \). Then

1. \( w(1) = 123234345 \) witnesses that \( 4 \in St(P)(u) \),
2. \( w(2) = 1232345456 \) witnesses that \( 5 \in St(P)(u) \), and
3. \( w(3) = 12323456567 \) witnesses that \( 6 \in St(P)(u) \).

It is easy to see that in each case, \( w(i) \) uses the smallest alphabet possible. Clearly 2 and 3 are not in \( St(P)(u) \) or \( St(k)(u) \) for any \([k] \). It thus follows that \( St(P)(u) = St(k)(u) = \{4, 5, 6\} \) for any \( k \geq 7 \). However, \( St(4)(u) = \emptyset \) so that \( u \) is \([4] \)-non-overlapping, \( St(5)(u) = \{4\} \), and \( St(6)(u) = \{4, 5\} \). Note that \( u \) has the \( \mathbb{P} \)-weakly increasing overlapping property and the \([k] \)-weakly increasing overlapping property for any \( k \geq 5 \). Next suppose that \( v = 345123 \). Then

1. \( w(4) = 567345123 \) witnesses that \( 4 \in St(P)(v) \),
2. \( w(5) = 4561345123 \) witnesses that \( 5 \in St(P)(v) \), and
3. \( w(6) = 34512345123 \) witnesses that \( 6 \in St(P)(v) \).

Again it is easy to see that in each case, \( w(i) \) uses the smallest alphabet possible and 2 and 3 are not in \( St(P)(u) \) or \( St(k)(v) \) for any \([k] \). It follows that \( St(P)(v) = St(k)(v) = \{4, 5, 6\} \) for any \( k \geq 7 \). However, \( St(5)(u) = \{6\} \) so that \( u \) is \([5] \)-minimal overlapping and \( St(6)(u) = \{5, 6\} \). Note that \( u \) has the \( \mathbb{P} \)-weakly decreasing overlapping property and the \([k] \)-weakly decreasing overlapping property for any \( k \geq 5 \) but that it also has the \([5] \)-weakly increasing overlapping property and the \([5] \)-level overlapping property.

We can also make similar definitions for exact matchings. That is, for \( u = u_1 \ldots u_j \in [k]^j \), let \( ESV(P)(u) (ESV(k)(u)) \) equal the set of \( 1 < s \leq j \) such that there exists a word \( w = w_1 \ldots w_{s+j-1} \) in \( \mathbb{P}^* ([k]^*) \) such that \( w_1 \ldots w_j = u \) and \( w_s \ldots w_{s+j-1} = u \). That is, \( ESV(P)(u) (ESV(k)(u)) \) is the set of positions \( 1 < s \leq j \) such that there is a word \( w \) in \( \mathbb{P}^* ([k]^*) \) in which there is a pair of overlapping exact \( u \)-matches such that the first exact \( u \)-match starts at position 1 and the second exact \( u \)-match starts at position \( s \). We say that \( u \) is \( \text{exact} \mathbb{P} \)-minimal overlapping (\( \text{exact} \mathbb{P} \)-level overlapping)
In this way, we can show that the functions $N$ contribute to $u$ if any two consecutive exact $u$-matches can share at most one letter which must be the last letter of the first exact $u$-match and the first letter of the second exact $u$-match. For example $u = 131$ is a word that has the exact $\mathbb{P}$-minimal overlapping property. We say that $u$ is exact $\mathbb{P}$-non-overlapping (exact $[k]$-non-overlapping if $ES\mathcal{t}(\mathbb{P})(u) = \emptyset$) if $ES\mathcal{t}([k])(u) = \emptyset$. For example $u = 132$ is a word that has the exact $\mathbb{P}$-non-overlapping property.

Let $z_k = z_1, \ldots, z_k$ and $z_\infty = z_1, z_2, \ldots$. Then for any $u \in [k]^j$, we let

$$EN_{n,u}^{(k)}(x, z_k) = \sum_{w \in [k]^n, \text{umch}(w) = 0} x^{\text{des}(w)+1} w$$

and

$$EN_{n,u}^{(\mathbb{P})}(x, z_\infty) = \sum_{w \in \mathbb{P}^n, \text{umch}(w) = 0} x^{\text{des}(w)+1} w.$$

Similarly for $u \in [k]^j$ such that $\text{red}(u) = u$, we let

$$N_{n,u}^{(k)}(x, z_k) = \sum_{w \in [k]^n, \text{umch}(w) = 0} x^{\text{des}(w)+1} w$$

and

$$N_{n,u}^{(\mathbb{P})}(x, z_\infty) = \sum_{w \in \mathbb{P}^n, \text{umch}(w) = 0} x^{\text{des}(w)+1} w.$$

The main goal of this paper is to study the generating functions

$$\mathcal{E}\mathcal{N}_{u}^{(k)}(x, z_k, t) = 1 + \sum_{n \geq 1} EN_{n,u}^{(k)}(x, z) t^n$$

and

$$\mathcal{E}\mathcal{N}_{u}^{(\mathbb{P})}(x, z_\infty, t) = 1 + \sum_{n \geq 1} EN_{n,u}^{(\mathbb{P})}(x, z) t^n,$$

in the case where $u$ is a word with $\text{des}(u) \leq 1$ and the generating functions

$$\mathcal{N}_{u}^{(k)}(x, z_k, t) = 1 + \sum_{n \geq 1} N_{n,u}^{(k)}(x, z) t^n$$

and

$$\mathcal{N}_{u}^{(\mathbb{P})}(x, z_\infty, t) = 1 + \sum_{n \geq 1} N_{n,u}^{(\mathbb{P})}(x, z) t^n,$$

in the case where $\text{red}(u) = u$ and $\text{des}(u) \leq 1$.

When $k$ and $|u|$ are small, there are well-known recursive methods to compute $N_{n,u}^{(k)}(x, z_k)$ or $EN_{n,u}^{(k)}(x, z_k)$. That is, suppose that $|u| = r$. For any word $v \in [k]^*$, we let $B_v^{(k)} = \{w \in [k]^* : v \text{ is a prefix of } w\}$ and

$$\mathcal{N}_{u,v}^{(k)}(x, z_k, t) = 1 + \sum_{n \geq 1} t^n \sum_{w \in B_v^{(k)} \cap [k]^n, \text{umch}(w) = 0} x^{\text{des}(w)+1} w.$$

For example, if $k = 3$, $u = 123$, and $v = 12$, then the words in $B_{12}^{(3)}$ are of the form 12 or 1 concatenated with either a word in $B_{21}^{(3)}$, $B_{22}^{(3)}$, or $B_{23}^{(3)}$. Words of the form $1 \ast B_{23}^{(3)}$ cannot contribute to $\mathcal{N}_{u,v}^{(k)}(x, z_k, t)$ since they all start with a 123-match. It follows that

$$\mathcal{N}_{u,12}^{(3)}(x, z_3, t) = x z_1 z_2 t^2 + z_1 t \mathcal{N}_{u,21}^{(3)}(x, z_3, t) + z_1 t \mathcal{N}_{u,22}^{(3)}(x, z_3, t).$$

In this way, we can show that the functions $\mathcal{N}_{u,v}^{(3)}(x, z_3, t)$ where $|v| = |u| - 1$ satisfy simple recursions. Bringing the terms that do not involve the generating functions to one side, one can
rewrite these equations in the form

\[
\vec{v} = M \begin{pmatrix}
N_{u,11}^{(3)}(x, z, t) \\
N_{u,12}^{(3)}(x, z, t) \\
N_{u,13}^{(3)}(x, z, t) \\
N_{u,21}^{(3)}(x, z, t) \\
N_{u,22}^{(3)}(x, z, t) \\
N_{u,23}^{(3)}(x, z, t) \\
N_{u,31}^{(3)}(x, z, t) \\
N_{u,32}^{(3)}(x, z, t) \\
N_{u,33}^{(3)}(x, z, t)
\end{pmatrix}.
\]

Then if one can invert the matrix \(M\), one can solve for the generating functions \(N_{u,ij}^{(3)}(x, z, t)\) from which one can easily recover the desired generating function \(N_u^{(3)}(x, z, t)\). More details on the method can be found in [12]. The problem with this method is that it requires us to invert a \(|u|-1\) matrix with multivariable entries which is impractical to compute as \(k\) and \(|u|\) get large.

The method that we will employ is what Jones and Remmel [15, 16] call the reciprocal method. The basic idea is the following. We assume that we can write the generating function \(N_u^{(3)}(x, z, t)\) as

\[
N_u^{(3)}(x, z, t) = \frac{1}{U_{1}^{(p)}(x, z, t)}
\]

where \(U_{1}^{(p)}(x, z, t) = 1 + \sum_{n \geq 1} U_{1,n}^{(p)}(x, z, t)t^n\). (1)

Thus

\[
U_{1}^{(p)}(x, z, t) = \frac{1}{1 + \sum_{n \geq 1} N_{u,n}^{(p)}(x, z, t)t^n}.
\]

(2)

One can then use the homomorphism method to give a combinatorial interpretation to the right-hand side of (2) which can be used to find a combinatorial interpretation for \(U_{1}^{(p)}(x, z, t)\). The homomorphism method derives generating functions for various statistics on permutations and words by applying a ring homomorphism defined on the ring of symmetric functions \(\Lambda\) in infinitely many variables \(x_1, x_2, \ldots\) to simple symmetric function identities such as

\[
H(t) = 1/E(-t)
\]

where \(H(t)\) and \(E(t)\) are the generating functions for the homogeneous and elementary symmetric functions given by

\[
H(t) = \sum_{n \geq 0} h_n t^n = \prod_{i \geq 1} \frac{1}{1 - x_it} \quad \text{and} \quad E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i \geq 1} 1 + x_it.
\]

(4)

See, for example, [4, 21, 22, 23, 24, 25] or the recent book by Mendes and Remmel [26]. In our case, we define a homomorphism \(\Theta_u\) on \(\Lambda\) by setting

\[
\Theta_u(e_n) = (-1)^n N_{u,n}^{(p)}(x, z, t).
\]

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Hence
\[ U^{(P)}_u(x, z_\infty, t) = \frac{1}{\Theta_u(E(-t))} = \Theta_u(H(t)) \]
which implies that
\[ \Theta_u(h_n) = U^{(P)}_{u,n}(x, z_\infty). \] (5)

Thus if we can compute \( \Theta_u(h_n) \) for all \( n \geq 1 \), then we can compute the polynomials \( U^{(P)}_{u,n}(x, z_\infty) \) and the generating function \( U^{(P)}_u(x, z_\infty, t) \) which in turn allows us to compute the generating function \( N^{(P)}_u(x, z_\infty, t) \). The same method can be applied to find combinatorial interpretations for \( U^{(k)}_u(x, z_k, t) \), \( EU^{(P)}_u(x, z_\infty, t) \), and \( EU^{(k)}_u(x, z_k, t) \) where

\[
N^{(k)}_u(x, z_k, t) = \frac{1}{U^{(k)}_u(x, z_k, t)},
\]
\[
\mathcal{E}N^{(P)}_u(x, z_\infty, t) = \frac{1}{EU^{(P)}_u(x, z_\infty, t)}, \quad \text{and}
\]
\[
\mathcal{E}N^{(k)}_u(x, z_k, t) = \frac{1}{EU^{(k)}_u(x, z_k, t)}.
\]

The final steps of the reciprocity method that we employ will be different from the ones used by Jones and Remmel [15, 16] for permutations. For the generating function for permutations that they studied, Jones and Remmel used the combinatorial interpretation that arose from the analogue of \( \Theta_u(h_n) \) to obtain simple recursions satisfied by their analogue of \( U^{(P)}_{u,n}(x, z_\infty) \). In our case, we shall use the combinatorial interpretation of \( \Theta_u(h_n) \) that comes out of the homomorphism method plus a map which we call the “collapse map” to show that we can obtain a closed expression for the generating functions \( U^{(P)}_u(x, z_\infty, t) \) or \( U^{(k)}_u(x, z_k, t) \) by an appropriate substitution in certain other generating functions for words.

The generating functions that we will substitute in will depend on the relative order of \( u_1 \) and \( u_j \) where \( u = u_1 \ldots u_j \). In each case our generating function will be over the variables \( x_{ij} \) where \( i, j \in \mathbb{P} \), the variables \( z_i \) where \( i \in \mathbb{P} \), and \( t \) which we denote as \((x_\infty, z_\infty, t)\). In the case where \( u_1 > u_j \), our final expression for our desired generating functions \( U^{(P)}_u(x, z_\infty, t) \) or \( U^{(k)}_u(x, z_k, t) \) will be a substitution into the generating function

\[
D^{(P)}(x_\infty, z_\infty, t) = \sum_{w \in \mathbb{P}^*} t^{\vert w \vert} z^w \prod_{i < j} x_{ji}^{[\bar{w}]}.
\]

In the case where \( u_1 < u_j \) and \( u \) has the \( \mathbb{P} \)-weakly increasing overlapping property ([k]-weakly increasing overlapping property), our final expression for our desired generating function \( U^{(P)}_u(x, z_\infty, t) \) (\( U^{(k)}_u(x, z_k, t) \)) will be a substitution into the generating function

\[
R^{(P)}(x_\infty, z_\infty, t) = \sum_{w \in \mathbb{P}^*} t^{\vert w \vert} z^w \prod_{i < j} x_{ji}^{[\bar{w}]}.
\]
In the case were \( u_1 = u_j \) and \( u \) has the \( \mathbb{P} \)-level overlapping property (\([k]-level overlapping property\)), our final expression for our desired generating function \( U^{(\mathbb{P})}_u(x, z_\infty, t) \) will be a substitution into the generating function

\[
\mathcal{L}^\mathbb{P}(x_\infty, z_\infty, t) = \sum_{u = u_1 \leq u_2 \leq \cdots \leq u_n \in \mathbb{P}^*} t^{\left\lfloor w \right\rfloor u} \prod_{i} x_i^{u_i(t)}.
\]

If \( u \) does not have the \( \mathbb{P} \)-level overlapping property (\([k]-level overlapping property\)), it will still be the case that \( u \) has the \( \mathbb{P} \)-weakly decreasing overlapping property (\([k]-weakly decreasing overlapping property\)). In such a case, our final expression for our desired generating function \( U^{(\mathbb{P})}_u(x, z_\infty, t) \) will be a substitution into the generating function

\[
\mathcal{W}^\mathbb{P}(x_\infty, z_\infty, t) = \sum_{u \in \mathbb{P}^*} t^{\left\lfloor w \right\rfloor u} \prod_{j \leq i} x_i^{u_j(t)}.
\]

We will prove the following theorems for the generating functions \( \mathcal{D}^\mathbb{P}(x_\infty, z_\infty, t), \mathcal{L}^\mathbb{P}(x_\infty, z_\infty, t), \) and \( \mathcal{R}^\mathbb{P}(x_\infty, z_\infty, t) \). Given a set \( S \subseteq \mathbb{P} \), we let

\[
DXZ(S) = \begin{cases} 
    z_j & \text{if } S = \{j\}, \\
    z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_i+1} - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \text{ where } k \geq 2.
\end{cases}
\]

and

\[
RXZ(S) = \begin{cases} 
    \frac{z_j}{1-z_j t} & \text{if } S = \{j\}, \\
    \left(\prod_{i=1}^{k} \frac{z_i}{1-z_i t}\right) \prod_{j=1}^{k-1} x_{j,j+1} & \text{if } S = \{j_1 < \cdots < j_k\} \text{ where } k \geq 2.
\end{cases}
\]

Let \( \mathcal{W}^\mathbb{P}_* \) (\( \mathcal{W}[k]^* \)) denote the set of all weakly decreasing words in \( \mathbb{P}^* \) (\([k]^* \)). Given a nonempty word \( v \) in \( \mathcal{W}^\mathbb{P}_* \), we let

\[
WDXZ(v) = \begin{cases} 
    z_j & \text{if } v = j, \\
    z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_i,j_i+1} - 1) & \text{if } v = j_1 \geq \cdots \geq j_k \text{ where } k \geq 2.
\end{cases}
\]

**Theorem 1.**

\[
\mathcal{D}^\mathbb{P}(x_\infty, z_\infty, t) = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXZ(S)}.
\]

**Theorem 2.**

\[
\mathcal{W}^\mathbb{P}(x_\infty, z_\infty, t) = \frac{1}{1 - \sum_{n \geq 1} t^n \sum_{v \in \mathcal{W}^\mathbb{P}_*, |v| = n} WDXZ(v)}.
\]

**Theorem 3.**

\[
\mathcal{R}^\mathbb{P}(x_\infty, z_\infty, t) = 1 + \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} RXZ(S).
\]

**Theorem 4.**

\[
\mathcal{L}^\mathbb{P}(x_\infty, z_\infty, t) = \prod_{i \geq 1} \left(1 + \frac{z_i t}{1-z_i z_i t}\right)
\]
The main advantage of our approach is that we obtain a uniform way to find expressions for the generating functions \( U_u(x, z_{\infty}, t) \), \( U_u^{(k)}(x, z_k, t) \), \( EU_u(x, z_{\infty}, t) \), and \( EU_u^{(k)}(x, z_k, t) \) which are independent of the length of \( u \) as long as \( \text{des}(u) = 1 \) and \( u \) satisfies the appropriate overlapping conditions. In fact our general methods can be applied even in cases where \( \text{des}(u) > 1 \). However in such cases the combinatorial interpretation of \( \Theta_u(h_u) \) that comes out the homomorphism method is significantly more complicated so that we will not pursue such results in this paper.

The outline of this paper is as follows. In Section 2, we shall review the basic background on symmetric functions that one will need for the paper. In Section 3, we shall describe the use of the reciprocal method to obtain combinatorial interpretations for \( U_{u,n}(x, z_{\infty}, t) \), \( U_{u,n}(x, z_k, t) \), \( EU_{u,n}(x, z_{\infty}, t) \), and \( EU_{u,n}^{(k)}(x, z_k, t) \). In section 4, we shall show how to use Theorem 1 to find expressions for \( U_u^{(P)}(x, z_{\infty}, t) \), \( U_u^{(k)}(x, z_k, t) \), \( EU_u^{(P)}(x, z_{\infty}, t) \), and \( EU_u^{(k)}(x, z_k, t) \) in the case where \( u = u_1 \ldots u_j \), \( u_1 > u_j \), and \( \text{des}(u) = 1 \). In section 5, we shall show how to use Theorem 3 to find expressions for \( U_u^{(P)}(x, z_{\infty}, t) \), \( U_u^{(k)}(x, z_k, t) \), \( EU_u^{(P)}(x, z_{\infty}, t) \), and \( EU_u^{(k)}(x, z_k, t) \) in the case where \( u = u_1 \ldots u_j \), \( u_1 < u_j \), \( \text{des}(u) = 1 \), and \( u \) has the \( P \)-weakly increasing overlapping property or \( [k] \)-weakly increasing overlapping property. In section 6, we shall show how to use Theorems 2 and 3 to find expressions for \( U_u^{(P)}(x, z_{\infty}, t) \), \( U_u^{(k)}(x, z_k, t) \), \( EU_u^{(P)}(x, z_{\infty}, t) \), and \( EU_u^{(k)}(x, z_k, t) \) in the case where \( u = u_1 \ldots u_j \), \( u_1 = u_j \), \( \text{des}(u) \leq 1 \), and \( u \) has the \( P \)-level overlapping property or \( [k] \)-level overlapping property. In section 6, we shall prove Theorems 1, 2, 3, and 4. Finally, in Section 7, we shall discuss some further extensions of our methods. For example, we will discuss how we can replace the statistic \( \text{des}(w) \) in our formulas by \( w_{\text{des}}(w) \) or \( \text{lev}(w) \) and we will discuss how we can extend our methods to handle cases where \( u \) has more than one descent.

## 2 Symmetric Functions

In this section we give the necessary background on symmetric functions needed for our proofs. We shall consider the ring of symmetric functions, \( \Lambda \), over infinitely many variables \( x_1, x_2, \ldots \).

The homogeneous symmetric functions, \( h_n \in \Lambda \), and elementary symmetric functions, \( e_n \in \Lambda \), are defined by the generating functions

\[
H(t) = \sum_{n \geq 0} h_n t^n = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} \quad \text{and} \quad E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i=1}^{\infty} (1 + x_i t).
\]

The \( n \)-th power symmetric function, \( p_n \in \Lambda \), is defined as \( p_n = \sum_{i=1}^{\infty} x_i^n \).

Let \( \lambda = (\lambda_1, \ldots, \lambda_{\ell}) \) be an integer partition; that is, \( \lambda \) is a finite sequence of weakly increasing non-negative integers. Let \( \ell(\lambda) \) denote the number of nonzero integers in \( \lambda \). If the sum of these integers is \( n \), we say that \( \lambda \) is a partition of \( n \) and write \( \lambda \vdash n \). For any partition \( \lambda = (\lambda_1, \ldots, \lambda_{\ell}) \), define \( h_\lambda = h_{\lambda_1} \cdots h_{\lambda_{\ell}} \), \( e_\lambda = e_{\lambda_1} \cdots e_{\lambda_{\ell}} \), and \( p_\lambda = p_{\lambda_1} \cdots p_{\lambda_{\ell}} \). The well-known fundamental theorem of symmetric functions, see [10], says that \( \{e_\lambda : \lambda \vdash n\} \) is a basis for \( \Lambda_n \), the space of symmetric functions which are homogeneous of degree \( n \). Equivalently, the fundamental theorem of symmetric functions states that \( \{e_0, e_1, \ldots\} \) is an algebraically independent set of generators for the ring \( \Lambda \). It follows that one can completely specify a ring homomorphism \( \Gamma : \Lambda \to R \) from \( \Lambda \) into a ring \( R \) by giving the values of \( \Gamma(e_n) \) for \( n \geq 0 \).
Next we give combinatorial interpretations to the expansion of $h_{\mu}$ in terms of the elementary symmetric functions. Given partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ and $\mu \vdash n$, a $\lambda$-brick tabloid of shape $\mu$ is a filling of the Ferrers diagram of shape $\mu$ with bricks of size $\lambda_1, \ldots, \lambda_\ell$ such that each brick lies in single row and no two bricks overlap. For example, Figure 1 shows all the $\lambda$-brick tabloids of shape $\mu$ where $\lambda = (1,1,2,2)$ and $\mu = (2,4)$.

![Figure 1: The four (1,1,2,2)-brick tabloids of shape (2,4).](image)

If $T$ is a brick tabloid of shape $(n)$ such that the lengths of the bricks, reading from left to right, are $b_1, \ldots, b_\ell$, then we shall write $T = (b_1, \ldots, b_\ell)$. For example, the brick tabloid $T = (2,3,1,4,2)$ is pictured in Figure 2.

![Figure 2: The brick tabloid $T = (2,3,1,4,2)$.](image)

Let $B_{\lambda,\mu}$ denote the set of all $\lambda$-brick tabloids of shape $\mu$ and let $B_{\lambda,\mu} = |B_{\lambda,\mu}|$. Eğecioğlu and Remmel proved in [10] that

$$h_{\mu} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,\mu} e_\lambda. \quad (13)$$

### 3 The reciprocal method

In this section, we shall apply the reciprocal method to give combinatorial interpretations to $U_{u}(P)(x, z_\infty, t)$, $U_{u}(k)(x, z_k, t)$, $EU_{u}(P)(x, z_\infty, t)$, and $EU_{u}(k)(x, z_k, t)$.

Fix a word $u$ such that $\text{des}(u) \leq 1$. We will start out by considering $U_{u}(P)(x, z_\infty, t)$ as the other cases are similar. Recall that

$$U_{u}(P)(x, z_\infty, t) = \frac{1}{1 + \sum_{n \geq 1} N_{u,n}(x, z_\infty)t^n}. \quad (14)$$

Thus if we let $\Theta_u(e_n) = (-1)^n N_{u,n}(x, z_\infty)$ for $n \geq 1$ and $\Theta_u(e_0) = 1$, we see that

$$\Theta_u(H(t)) = 1 + \sum_{n \geq 1} \Theta_u(h_n)$$

$$= \Theta_u \left( \frac{1}{1 - E(-t)} \right) = \frac{1}{1 + \sum_{n \geq 1} (-1)^n \Theta_u(e_n)}$$

$$= \frac{1}{1 + \sum_{n \geq 1} N_{u,n}(x, z_\infty)t^n} = U_{u}(P)(x, z_\infty, t).$$

Thus it follows that $\Theta_u(h_n) = U_{u,n}(x, z_\infty)$.
By (13), we have that
\[
\Theta_u(h_n) = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \Theta_u(e_\lambda)
\]
\[
= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \sum_{(b_1, \ldots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} (-1)^{b_i} N_{u,b_i}^{(P)}(x, z_\infty)
\]
\[
= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{(b_1, \ldots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} N_{u,b_i}^{(P)}(x, z_\infty).
\] (15)

Our next goal is to give a combinatorial interpretation to the right-hand side of (15). Fix a partition \(\lambda\) of \(n\) and a \(\lambda\)-brick tabloid \(B = (b_1, \ldots, b_{\ell(\lambda)})\). We will interpret \(\prod_{i=1}^{\ell(\lambda)} N_{u,b_i}^{(P)}(x, z_\infty)\) as the number of ways of picking words \((w^{(1)}, \ldots, w^{(\ell(\lambda))})\) such that for each \(i\), \(w^{(i)} \in \mathbb{P}_{b_i}\) is a word such that \(w^{(i)}\) is obtained by reading the elements in the cells of \(\lambda\) in \(\ell(\lambda)\)-tuple to be \(\prod_{i=1}^{\ell(\lambda)} x^{\text{des}(w^{(i)})+1} \prod_{i=1}^{\ell(\lambda)} z_{w^{(i)}}\).

We can then use the pair \((B, (w^{(1)}, \ldots, w^{(\ell(\lambda))}))\) to construct a filled-labeled-brick tabloid \(O_{(B,(w^{(1)},\ldots,w^{(\ell(\lambda))})}\) as follows. First for each brick \(b_i\), we place the word \(w^{(i)}\) in the cells of the brick, reading from left to right. Then we label each cell of \(b_i\) that starts a descent of \(w^{(i)}\) with a \(x\) and we also label the last cell of \(b_i\) with \(x\). This accounts for the factor \(x^{\text{des}(w^{(i)})+1}\). Finally, we use the factor \((-1)^{\ell(\lambda)}\) to change the label of the last cell of each brick from \(x\) to \(-x\). For example, suppose \(n = 17\), \(u = 312\), \(B = (3,7,4,3)\) \(w^{(1)} = 1\ 1\ 7\), \(w^{(2)} = 3\ 6\ 6\ 5\ 2\ 5\ 1\), \(w^{(3)} = 3\ 4\ 7\ 6\), and \(w^{(4)} = 2\ 5\ 2\). Then we have pictured the filled-labeled-brick tabloid \(O_{(B,(w^{(1)},\ldots,w^{(4)}))}\) from the pair \((B, (w^{(1)}, \ldots, w^{(4)}))\) in Figure 3.

![Figure 3: The construction of a filled-labeled-brick tabloid.](image)

Clearly, we can recover the pair \((B, (w^{(1)}, \ldots, w^{(\ell(\lambda))}))\) and the labels on the cells from \(B\) and the word \(w\) which is obtained by reading the elements in the cells of \(O_{(B,(w^{(1)},\ldots,w^{(\ell(\lambda))})}\) from left to right. Thus we shall specify the filled-labeled-brick tabloid \(O_{(B,(w^{(1)},\ldots,w^{(\ell(\lambda))})}\) by \((B, w)\).

We let \(O_{u,n}^{(P)}\) denote the set of all filled-labeled-brick tabloids constructed in this way. That is, \(O_{u,n}^{(P)}\) consists of all pairs \(O = (B, w)\) where

1. \(B = (b_1, \ldots, b_{\ell(\lambda)})\) is brick tabloid of shape \((n)\),

2. \(w = w_1 \ldots w_n \in \mathbb{P}_n\) such that there is no \(u\)-match of \(\sigma\) which is entirely contained in a single brick of \(B\), and

3. if there is a cell \(c\) such that a brick \(b_i\) contains both cells \(c\) and \(c+1\) and \(w_c > w_{c+1}\), then cell \(c\) is labeled with a \(x\) and the last cell of any brick is labeled with \(-x\).

The sign of \(O\), \(\text{sgn}(O)\), is \((-1)^{\ell(\lambda)}\) and the weight of \(O\), \(\text{wt}(O)\), is \(x^{\ell(\lambda)+\text{intdes}(\sigma)}\) where \(\text{intdes}(w)\) denotes the number of \(i\) such that \(w_i > w_{i+1}\) and \(w_i\) and \(w_{i+1}\) lie in the same brick. We shall refer to such \(i\) as an internal descent of \(O\). Note that the labels on \(O\) are completely determined.
by the underlying brick tabloid $B = (b_1, \ldots, b_{\ell(\lambda)})$ and the underlying word $w$. Thus the filled-labeled-brick tabloid $O$ pictured in Figure 3 equals $((3, 7, 4, 3), 1 1 7 3 6 6 5 2 5 1 3 4 7 6 2 5 2)$.

It follows that

$$\Theta_u(h_n) = \sum_{O \in \mathcal{O}_{u,n}^{(p)}} sgn(O) wt(O). \quad (16)$$

Next we define a weight-preserving, sign-reversing involution $I_u$ on $\mathcal{O}_{u,n}^{(p)}$. Given an element $O = (B, w) \in \mathcal{O}_{u,n}^{(p)}$ where $B = (b_1, \ldots, b_k)$ and $w = w_1 \ldots w_n$, scan the cells of $O$ from left to right looking for the first cell $c$ such that either

(i) $c$ is labeled with a $x$ or

(ii) $c$ is a cell at the end of a brick $b_i$, $w_c > w_{c+1}$, and there is no $u$-match of $w$ that lies entirely in the cells of bricks $b_i$ and $b_{i+1}$.

In case (i), if $c$ is a cell in brick $b_j$, then we split $b_j$ into two bricks $b'_j$ and $b''_j$ where $b'_j$ contains all the cells of $b_j$ up to and including cell $c$ and $b''_j$ consists of the remaining cells of $b_j$ and we change the label on cell $c$ from $x$ to $-x$. In case (ii), we combine the two bricks $b_i$ and $b_{i+1}$ into a single brick $b$ and change the label on cell $c$ from $-x$ to $x$. If neither case (i) nor case (ii) applies, then we define $I_u(O) = O$. For example, consider the element $O \in \mathcal{O}_{312,17}^{(p)}$ pictured in Figure 3. Note that even though the number in the last cell of brick 1 is greater than the number in the first cell of brick 2, we can not combine these two bricks because $7 3 6$ would be a 312-match. Thus the first place that we can apply the involution is on cell 6 which is labeled with an $x$ so that $I_u(O)$ is the object pictured in Figure 4.

```
1 1 7 3 6 6 5 2 5 1 3 4 7 6 2 5 2
```

Figure 4: $I_u(O)$ for $O$ in Figure 3

We claim that whenever $u$ is a word such that $\text{red}(u) = u$ and $\text{des}(u) \leq 1$, $I_u$ is an involution, i.e. $I_u^2$ is the identity. First we consider the case where $\text{des}(u) = 1$. Now suppose that we are in case (i) where we split a brick $b_j$ at cell $c$ which is labeled with a $x$. In that case, we let $a$ be the number in cell $c$ and $a'$ be the number in cell $c+1$ which must also be in brick $b_j$. It must be the case that there is no cell labeled $x$ before cell $c$ since otherwise we would not use cell $c$ to define the involution. However, we have to consider the possibility that when we split $b_j$ into $b'_j$ and $b''_j$, we might then be able to combine the brick $b_{j-1}$ with $b'_j$ because the number in that last cell of $b_{j-1}$ is greater than the number in the first cell of $b'_j$ and there is no $u$-match in the cells of $b_{j-1}$ and $b'_j$. Since we always take an action on the left most cell possible when defining $I_u(O)$, we know that we cannot combine $b_{j-1}$ and $b_j$ so that there must be a $u$-match in the cells of $b_{j-1}$ and $b_j$. Clearly, that $u$-match must have involved the number $a'$ and the number in cell $d$ which is the last cell in brick $b_{j-1}$. But that is impossible because then there would be two descents among the numbers between cell $d$ and cell $c + 1$ which would violate our assumption that $u$ has only one descent. Thus whenever we apply case (i) to define $I_u(O)$, the first action that we can take is to combine bricks $b'_j$ and $b''_j$ so that $I_u^2(O) = O$.

If we are in case (ii), then again we can assume that there are no cells labeled $x$ that occur before cell $c$. When we combine bricks $b_i$ and $b_{i+1}$, then we will label cell $c$ with a $x$. It is clear
that combining the cells of \(b_i\) and \(b_{i+1}\) cannot help us combine the resulting brick \(b\) with an earlier brick since it will be harder to have no \(w\)-matches with the larger brick \(b\). Thus the first place cell \(c\) where we can apply the involution to \(I_u(O)\) will again be cell \(c\) which is now labeled with a \(x\) so that \(I_u^2(O) = O\) if we are in case (ii).

The case where \(\text{des}(u) = 0\) is even easier. Suppose that \(a\) is number in the the last cell of \(b_j\) and \(a'\) is the number in the first cell of \(b_{j+1}\) and \(a > a'\). Then there can be no \(w\)-match of \(w\) that is contained in the cells of \(b_j\) and \(b_{j+1}\) because by our definitions there is no \(w\)-match in the cells of \(b_j\) and there is no \(w\)-match in the cells of \(b_{j+1}\) so that the only possible \(w\)-match in the cells of \(b_j\) and \(b_{j+1}\) would have to involve \(a\) and \(a'\) which is impossible if \(\text{des}(u) = 0\). It easily follows that we will apply the involution to the first possible cell \(c\) which is labeled with either \(x\) or \(\overline{x}\) and what ever action we take at cell \(c\) to create \(I_u(O)\), we will come back to cell \(c\) to undo that action to define \(I^2(O)\).

Our definitions ensure that if \(I_u(O) \neq O\), then \(\text{sgn}(O)wt(O) = -\text{sgn}(I_u(O))wt(I_u(O))\). Hence, if we let \(\mathcal{I}O_{u,n}^{(P)}\) denote set set all \(O = (B, w) \in \mathcal{O}_{u,n}^{(P)}\) such that \(I_u(O) = O\), then

\[
\Theta_u(h_n) = \sum_{O \in \mathcal{I}O_{u,n}^{(P)}} \text{sgn}(O)wt(O) = \sum_{O \in \mathcal{I}O_{u,n}^{(P)}} \text{sgn}(O)wt(O).
\]

Thus we must examine the fixed points of \(I_u\). So assume that \((B, w)\) is a fixed point of \(I_u\).

There are two cases to consider.

**Case 1.** \(\text{des}(u) = 0\).

Suppose that \((B, w) \in \mathcal{I}O_{u,n}\) where \(B = (b_1, \ldots, b_k)\) and \(w = w_1 \ldots w_n\). There can be no cell \(c\) which is labeled with \(x\) in \((B, w)\) since we could use such a cell to define \(I_u\) which would violate our assumption that \((B, w)\) is a fixed point of \(I_u\). Similarly there can be no cell \(c\) which is at the end of a brick \(b_j\) such that \(w_c > w_{c+1}\) since again we could use such a cell to define \(I_u(O)\). This means that \(w\) must be weakly increasing within any brick and if \(c\) is a cell at the end of brick \(b_j\) which is followed by another brick \(b_{j+1}\), then \(w_c \leq w_{c+1}\). Thus \((B, w)\) is a fixed point if and only if \(w\) is a weakly increasing word such that \(w\) has no \(w\)-match that lies entirely within one of the brick of \(B\). If \(B\) has \(k\) bricks, then then weight of \((B, w)\) is just \((-x)^{k-w}\). We let \(\mathcal{W}I\mathcal{O}_{u,n} = \{(B, w) \in \mathcal{I}O_{u,n}^{(P)} : w_1 \leq w_2 \leq \cdots \leq w_n\}\) denote the set of elements of \(\mathcal{I}O_{u,n}^{(P)}\) where \(w\) is weakly increasing. Then we have the following lemma. Let \(Q(x, z_\infty)\) be the set of rational functions in the variables \(x\) and \(z_\infty\) over the rationals \(Q\).

**Lemma 5.** Suppose that \(u\) is a word in \(\mathcal{P}^+\) such that \(\text{red}(u) = u\) and \(\text{des}(u) = 0\). Let \(\Theta_u : \Lambda \to Q(x, z_\infty)\) be the ring homomorphism defined by setting \(\Theta_u(e_0) = 1\) and \(\Theta_u(e_n) = (-1)^nN_{u,n}^{(P)}(x, z_\infty)\) for \(n \geq 1\). Then

\[
U_{u,n}^{(P)}(x, z_\infty) = \Theta_u(h_n) = \sum_{((b_1, \ldots, b_k), w) \in \mathcal{W}I\mathcal{O}_{u,n}} (-x)^{k-w}.
\]

**Case 2.** \(\text{des}(u) = 1\).

First it is easy to see that there can be no cells which are labeled with \(x\) so that numbers in each brick of \(O\) must be weakly increasing. Second we cannot combine two consecutive bricks \(b_i\)
and \(b_{i+1}\) in \(O\) which means that either there is an weak increase between the bricks \(b_i\) and \(b_{i+1}\) or there is a decrease between the bricks \(b_i\) and \(b_{i+1}\), but there is a \(u\)-match in the cells of the bricks \(b_i\) and \(b_{i+1}\). Thus we have proved the following.

**Lemma 6.** Suppose that \(u \in \mathbb{P}^+\), \(\text{red}(u) = u\), and \(\text{des}(u) = 1\). Let \(\Theta_u : \Lambda \rightarrow \mathbb{Q}(x, z_\infty)\) be the ring homomorphism defined by setting \(\Theta_u(e_0) = 1\) and \(\Theta_u(e^n) = (-1)^n N_{u,n}^{(P)}(x, z_\infty)\) for \(n \geq 1\). Then

\[
U_{u,n}^{(P)}(x, z_\infty) = \Theta_u(h_n) = \sum_{O \in \mathcal{O}_{u,n}^{(P)}, I_u(O) = O} \text{sgn}(O) \text{wt}(O)
\]

where \(\mathcal{O}_{u,n}^{(P)}\) is the set of objects and \(I_u\) is the involution defined above. Moreover \(O = (B, w)\), where \(B = (b_1, \ldots, b_k)\) and \(w = w_1 \ldots w_n\), is a fixed point of \(I_u\) if and only if it has the following two properties:

1. there are no cells labeled with \(x\) in \(O\), i.e., the elements of \(w\) in each brick of \(O\) are weakly increasing and
2. if \(b_i\) and \(b_{i+1}\) are two consecutive bricks in \(O\), then either (a) there is a weak increase between \(b_i\) and \(b_{i+1}\), i.e., \(w_{\sum_{j=1}^{i} |b_j|} \leq w_{1+\sum_{j=1}^{i} |b_j|}\) or (b) there is a decrease between \(b_i\) and \(b_{i+1}\), i.e., \(w_{\sum_{j=1}^{i} |b_j|} > w_{1+\sum_{j=1}^{i} |b_j|}\), but there is a \(u\)-match contained in the elements of the cells of \(b_i\) and \(b_{i+1}\) which must necessarily involve \(w_{\sum_{j=1}^{i} |b_j|}\) and \(w_{1+\sum_{j=1}^{i} |b_j|}\).

Clearly, if we restrict to the alphabet \([k]\) instead of \(\mathbb{P}\), we will get the same two lemmas except that the words all have to be in \([k]^*\) rather than in \(\mathbb{P}^*\).

Next we want to consider what happens when we replace \(u\)-matches by exact \(u\)-matches. We can follow the same steps to interpret \(EU_{u,n}^{(P)}(x, z_\infty, t)\). That is,

\[
EU_{u}^{(P)}(x, z_\infty, t) = \frac{1}{1 + \sum_{n \geq 1} EN_{u,n}^{(P)}(x, z_\infty) t^n}.
\]

Thus if we let \(\Gamma_u(e_n) = (-1)^n EN_{u,n}^{(P)}(x, z_\infty)\) for \(n \geq 1\) and \(\Gamma_u(e_0) = 1\), we see that

\[
\Gamma_u(H(t)) = 1 + \sum_{n \geq 1} \Gamma_u(h_n) = 1 + \sum_{n \geq 1} \frac{1}{1 + \sum_{n \geq 1} (-1)^n \Gamma_u(e_n)} = \frac{1}{1 + \sum_{n \geq 1} EN_{u,n}^{(P)}(x, z_\infty) t^n} = EU_{u}^{(P)}(x, z_\infty, t).
\]

Thus it follows that \(\Gamma_u(h_n) = EU_{u,n}^{(P)}(x, z_\infty)\).

By (13), we have that

\[
\Gamma_u(h_n) = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \Gamma_u(e_\lambda) = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \sum_{(b_1, \ldots, b(\lambda)) \in B_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} (-1)^{b_i} EN_{u,b_i}^{(P)}(x, z_\infty) = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{(b_1, \ldots, b(\lambda)) \in B_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} EN_{u,b_i}^{(P)}(x, z_\infty)
\]

(21)
Again we can give a combinatorial interpretation to the right-hand side of (21). Fix a partition \( \lambda \) of \( n \) and a \( \lambda \)-brick tableau \( B = (b_1, \ldots, b_{\ell(\lambda)}) \). We will interpret \( \prod_{i=1}^{\ell(\lambda)} \mathcal{EN}^{(P)}_{u,b_i}(x,z) \) as the number of ways of picking words \( (w^{(1)}, \ldots, w^{(\ell(\lambda))}) \) such that for each \( i \), \( w^{(i)} \in \mathbb{P}^{b_i} \) is a word such that \( \text{eumch}(w) = 0 \) and assigning a weight to this \( \ell(\lambda) \)-tuple to be \( \prod_{i=1}^{\ell(\lambda)} x^{\text{des}(w^{(i)})+1} w^{(i)} \).

Following the same steps that we did to interpret \( \Theta_u(h_n) \), we let \( \mathcal{EO}_{u,n}^{(P)} \) denote the set of all filled-labeled-brick tabloids constructed in this way. That is, \( \mathcal{EO}_{u,n}^{(P)} \) consists of all pairs \( O = (B, w) \) where

1. \( B = (b_1, \ldots, b_{\ell(\lambda)}) \) is brick tableau of shape \( (n) \),

2. \( w = w_1 \ldots w_n \in \mathbb{P}^n \) such that there is no exact \( u \)-match of \( \sigma \) which is entirely contained in a brick of \( B \), and

3. if there is a cell \( c \) such that a brick \( b_i \) contains both cells \( c \) and \( c+1 \) and \( w_c > w_{c+1} \), then cell \( c \) is labeled with a \( x \) and the last cell of any brick is labeled with \( -x \).

The sign of \( O, \text{sgn}(O) \), is \((-1)^{\ell(\lambda)}\) and the weight of \( O, \text{wt}(O) \), is \( x^{\ell(\lambda)+\text{intdes}(\sigma)} z^w \). Then as before we can conclude

\[
\Gamma_u(h_n) = \sum_{O \in \mathcal{EO}_{u,n}^{(P)}} \text{sgn}(O)\text{wt}(O). \tag{22}
\]

At this point, we can define an involution \( J_u \) exactly as we did for \( I_u \) except replace \( u \)-match by exact \( u \)-matches in the definitions. This will allow us to prove the following two lemmas.

**Lemma 7.** Suppose that \( u \) is a word in \( \mathbb{P}^+ \) such that \( \text{des}(u) = 0 \). Let \( \Gamma_u : \Lambda \rightarrow \mathbb{Q}(x) \) be the ring homomorphism defined by setting \( \Gamma_u(e_0) = 1 \) and \( \Gamma_u(e_n) = (-1)^n \mathcal{EN}^{(P)}_{u,n}(x,z) \) for \( n \geq 1 \). Then

\[
\mathcal{EU}_{u,n}^{(P)}(x,z) = \theta_u(h_n) = \sum_{(b_1, \ldots, b_k, w) \in \mathcal{WO}_{u,n}} (-x)^{b_k} z^w \tag{23}
\]

where \( \mathcal{WO}_{u,n} \) is the set all \( (B, w) \in \mathcal{EO}_{u,n}^{(P)} \) such that \( J_u(B, w) = (B, w) \) and \( w \) is weakly increasing.

**Lemma 8.** Suppose that \( u \in \mathbb{P}^+ \) and \( \text{des}(u) = 1 \). Let \( \Gamma_u : \Lambda \rightarrow \mathbb{Q}(y) \) be the ring homomorphism defined by setting \( \Gamma_u(e_0) = 1 \) and \( \Gamma_u(e_n) = (-1)^n \mathcal{EN}^{(P)}_{u,n}(x,z) \) for \( n \geq 1 \). Then

\[
\mathcal{EU}_{u,n}^{(P)}(x,z) = \Gamma_u(h_n) = \sum_{O \in \mathcal{EO}_{u,n}^{(P), J_u(O) = O}} \text{sgn}(O)\text{wt}(O) \tag{24}
\]

where \( \mathcal{EO}_{u,n}^{(P)} \) is the set of objects and \( J_u \) is the involution defined above. Moreover \( O = (B, w) \), where \( B = (b_1, \ldots, b_k) \) and \( w = w_1 \ldots w_n \), is a fixed point of \( J_u \) if and only if it has the following two properties:

1. there are no cells labeled with \( x \) in \( O \), i.e., the elements of \( w \) in each brick of \( O \) are weakly increasing and

2. if \( b_i \) and \( b_{i+1} \) are two consecutive bricks in \( O \), then either (a) there is a weak increase between \( b_i \) and \( b_{i+1} \), i.e., \( w_{\sum_{j=1}^{i-1} |b_j|} \leq w_{1+\sum_{j=1}^{i} |b_j|} \), or (b) there is a decrease between \( b_i \) and \( b_{i+1} \), i.e., \( w_{\sum_{j=1}^{i} |b_j|} > w_{1+\sum_{j=1}^{i} |b_j|} \), but there is an exact \( u \)-match contained in the elements of the cells of \( b_i \) and \( b_{i+1} \) which must necessarily involve \( w_{\sum_{j=1}^{i-1} |b_j|} \) and \( w_{1+\sum_{j=1}^{i} |b_j|} \).
4 The case where \( u = u_1 \ldots u_j, \) \( \text{des}(u) = 1, \) and \( u_1 > u_j \)

In this section, we shall consider the problem of computing the generating functions
\( N_{\text{des}}^{(P)}(x, z_{\text{\infty}}, t), \) \( N_{\text{des}}^{(k)}(x, z_k, t), \) \( E N_{\text{des}}^{(P)}(x, z_{\text{\infty}}, t), \) and \( E N_{\text{des}}^{(k)}(x, z_k, t) \) for \( u = u_1 \ldots u_j \) such that \( \text{des}(u) = 1, \) and \( u_1 > u_j. \)

Now suppose that \( u = u_1 \ldots u_j, \) \( \text{red}(u) = u, \) \( u_1 > u_j, \) and \( \text{des}(u) = 1. \) Let \( 1 \leq s < j \) be the position such that \( u_s > u_{s+1} \) so that \( u_1 \leq \cdots \leq u_s > u_{s+1} \leq \cdots \leq u_j. \) Then \( S_{\text{red}}^{(P)}(u) \subseteq \{ s + 1, \ldots, j \} \) since if we try to start a match at one of the positions \( 2, \ldots, s, \) the descent in the second match would not be in the right place. It follows that \( u \) automatically has the \( \mathbb{P}-\)weakly decreasing overlapping property and the \( [k] \)-weakly decreasing overlapping property for any \( k \geq 2. \)

We start by considering a special class of words \( u = u_1 \ldots u_j \) which have the \( \mathbb{P} \)-minimal overlapping property \( ([k] \)-minimal overlapping property). This means that any two consecutive \( u \)-matches can share at most one letter. For example \( u = 2341 \) has the \( \mathbb{P} \)-minimal overlapping property while \( u = 3412 \) does not have the \( \mathbb{P} \)-minimal overlapping property since in the word \( w = 563412, \) the \( u \)-matches 5634 and 3412 share two letters.

Thus assume that \( u = u_1 \ldots u_j, \) \( \text{red}(u) = u, \) \( \text{des}(u) = 1, \) \( u_1 > u_j, \) and \( u \) has the \( \mathbb{P} \)-minimal overlapping property. First we introduce what we shall call the collapse map which maps fixed points of \( I_u \) or \( J_u \) to a certain subset of words in \( \mathbb{P}^*. \) This is best explained through an example. Suppose that \( u = 2341 \) and we want to compute \( U_{2341}^{(7)}(x, z_7, t). \) By (25), we know that

\[
U_{2341}^{(7)}(x, z_7) = \sum_{O \in O_{\text{\infty}}, J_u(O) = O} \text{sgn}(O) \text{wt}(O). \tag{25}
\]

Now suppose that we are given a fixed point \( (B, w) \) of \( I_u \) where \( B = (b_1, \ldots, b_k) \) and \( w = w_1 \ldots w_n \) such as the one pictured in Figure 5. We know that to be a fixed point of \( I_u, \) \( w \) must be weakly increasing within bricks of \( B \) and that for any \( i < k, \) if \( c \) is last cell in brick \( b_i \) and \( w_c > w_{c+1}, \) then there must be a \( u \)-match in \( w \) which is contained in the cells of \( b_i \) and \( b_{i+1}. \) In our particular example, since \( u = 2341 \) has a single descent, this match must involve the last three cells of \( b_i \) and the first cell of \( b_{i+1}. \) In Figure 5, we have indicated the two such 2341-matches in our example by placing stars below the cells in the 2341-matches. In this case the collapse map just maps \( (B, w) \) to the word \( v = C(B, w, u) \) which is the result of starting with \( w \) and removing the letters in all such matches that do not correspond to the end points of the match. This process is pictured in Figure 5 where again we have starred the elements in \( C(B, w, u) \) that remain from the original 2341-matches in \( w. \) What makes the case where \( u \) has the minimal overlapping property easier is that, since any two consecutive \( u \)-matches can share at most letter, there is no possibility that an end point of \( u \)-match in \( w \) occurs in the middle of another \( u \)-match in \( w \) so that the letters that we remove from \( w \) for any pair of \( u \)-matches are disjoint from each other.

The next question that we want to consider is how can we construct all the fixed points of \( (B, w) \) of \( I_u \) such that \( C(B, w, v) \) is equal to a given word \( v = v_1 \ldots v_n. \) First it easy to see that the only descents that appear in a word \( C(B, w, u) \) must come from 2341-matches that straddled two bricks in \( B. \) Thus if \( v_s > v_{s+1}, \) then \( v_s \) must have played the role of 2 in the original 2341-match and \( v_{s+1} \) must have played the role of 1 in the original 2341-match. Such a requirement rules out certain words from being in the range of the collapse map \( C. \) For example, suppose that the underlying alphabet is [7]. Then if \( v_s = 6 \) and \( v_{s+1} = 1 \) where \( i < 6, \) then \( v \) could not have come from the collapse of 2341-match because we can not add two letters which could play
Figure 5: A fixed point of $I_{2341}$.

the role of 3 and 4 in the 2341-match. If we consider the first descent 32 in the $C(B, w, u)$ of Figure 5, then we see there are many ways that we could add the two middle letters. That is, the original 2341-match could have been any $3cd2$ where $c < d$ and $c, d \in \{4, 5, 6, 7\}$. It follows that the extra weight from these possibilities that is not included in $C(B, w, u)$ in this case would be $-xt^2 \sum_{4 \leq c < d \leq 7} z_c z_d$. Here the $-x$ comes from the fact that we know that the original match straddled two bricks and there is a weight of $-x$ associated with the end point of the first of those two bricks. On the other hand, if $v_s \leq v_{s+1}$, then we have only two choices. That is, either cell $s$ was the end of a brick or cell $s$ was an internal cell of a brick. This implies that each weak rise in $v$ contributes a factor of $(1 - x)$ since if $s$ is at the end of a brick, there is a weight of $-x$ associated with the last cell of a brick. In this way, we can associate a weight with each weak rise or descent of $v$ which will allow us to compute

$$\sum_{(B, w) \text{ is a fixed point of } I_u \atop C(B, w, u) = v} sgn(B, w)wt(B, w).$$

In our case where $u = 2341$ and $k = 7$, the weights associated with the descents are given in the table below.

| Descents   | $wt_{2341,7}(ji)$ |
|------------|-------------------|
| $i \ (i < 7)$ | 0                 |
| $6i \ (i < 6)$ | 0                 |
| $5i \ (i < 5)$ | $-xz_6z_7t^2$    |
| $4i \ (i < 4)$ | $-x(z_5z_6 + z_5z_7 + z_6z_7)t^2$ |
| $3i \ (i < 3)$ | $-x(z_4z_5 + z_4z_6 + z_5z_6 + z_5z_7 + z_6z_7)t^2$ |
| $21$       | $-x(z_3z_4 + z_3z_5 + z_3z_6 + z_3z_7 + z_4z_5 + z_4z_6 + z_4z_7 + z_5z_6 + z_5z_7 + z_6z_7)t^2$ |

The weights $wt_{2341,7}(ji)$.

However, if $u = 2341$ and we want to compute $U_{u,k}^{(P)}(x, z_{\infty})$, the weights for any descent $ji$ would be $-xt^2 \sum_{j < c < d} z_c z_d$ which is an infinite sum.

Going back to our example where $u = 2341$ and $k = 7$, it follows that for any $v \in [7]^+$,

$$\sum_{(B, w) \text{ is a fixed point of } I_u \atop C(B, w, u) = v} sgn(B, w)wt(B, w) = -x^{\text{rise}(v)}(1 - x)^{\text{rise}(v)} \prod_{s \in \text{Des}(v)} wt_{2341,7}(v_s v_{s+1}). \quad (26)$$

15
Here the initial $-x$ comes from the fact that the last cell of $(B, w)$ always contributes a $-x$ since the last cell is at the end of a brick. It follows that

$$U^{(7)}_{2341}(x, z_7, t) = 1 + \sum_{n \geq 1} U^{(7)}_{2341,n}(x, z_7) t^n = 1 + \sum_{v \in [7]^+} -x(1 - x)^{\text{rise}(v)} z^{|v|} t^{|v|} \prod_{s \in \text{Des}(v)} \text{wt}_{2341,7}(v_s v_{s+1}).$$  \hspace{1cm} (27)

Hence we could compute $N^{(7)}_{2341,n}(x, z_7, t) = 1 / U^{(7)}_{2341,n}(x, z_7, t)$ if we can compute the right-hand side of (27).

The case of exact matches is even simpler. In that case, we want to compute

$$\sum_{(B, w) \text{ is a fixed point of } J_u} \text{sgn}(B, w) \text{wt}(B, w).$$

Going back to our example of $u = 2341$ over the alphabet $[7]$, we see the only descents that appear in a word $v = C(B, w, u)$ must come from exact 2341-matches that straddled two bricks in $B$. Thus if $v_s > v_{s+1}$, then it must be the case that $v_s = 2$, $v_{s+1} = 1$ and we must have eliminated a 3 and 4 from $w$. Thus we want to compute $EU^{(P)}_{2341,n}(x, z_\infty)$ or $EU^{(k)}_{2341,n}(x, z_k)$ for $k \geq 4$, the weights would be the following.

| Descents | weight $\text{ewt}_{2341,P}(ji)$ |
|----------|----------------------------------|
| $ji$ where either $j \neq 2$ or $i \neq 1$ | 0 |
| 21       | $-xz_3z_4t^2$ |

The weights $\text{ewt}_{2341}(ji)$.

It follows that for any $v \in P^+$,

$$\sum_{(B, w) \text{ is a fixed point of } J_{2341}} \text{sgn}(B, w) \text{wt}(B, w) = -x z^{|v|} t^{|v|}(1 - x)^{\text{rise}(v)} \prod_{s \in \text{Des}(v)} \text{ewt}_{2341,P}(v_s v_{s+1}).$$  \hspace{1cm} (28)

and

$$EU^{(P)}_{2341,n}(x, z_\infty, t) = 1 + \sum_{n \geq 1} EU^{(P)}_{2341,n}(x, z_\infty) t^n = 1 + \sum_{v \in P^+} -x z^{|v|} t^{|v|}(1 - x)^{\text{rise}(v)} \prod_{s \in \text{Des}(v)} \text{ewt}_{2341,P}(v_s v_{s+1}).$$  \hspace{1cm} (29)

In our case, $\prod_{s \in \text{Des}(v)} \text{ewt}_{2341,P}(v_s v_{s+1}) = 0$ unless the only descents in $v$ are of the from 21. It follows that the only nonempty words $v$ that can contribute to (29) are words $v$ of the form $w$ or of the form $1a_1 2b_1 211a_2 2b_2 21 \ldots 1a_r 2b_r 21w$ for some $r \geq 1$ where $w$ is a weakly increasing word. Let

$$W(x, z_\infty, t) := \prod_{i=1}^\infty \frac{1}{(1 - (1 - x) z_i t)}.$$
The generating function of \(-x^{|v|}(1-x)^{\text{rise}(v)} \prod_{s \in \text{Des}(v)} \text{ewt}_{2341,7}(v_s v_{s+1})\) over all nonempty weakly increasing words is just
\[
\frac{-x}{1-x} \left( -1 + W(x, z_\infty, t) \right). \tag{30}
\]

The generating function of \(-x^{|v|}(1-x)^{\text{rise}(v)} \prod_{s \in \text{Des}(v)} \text{ewt}_{2341,7}(v_s v_{s+1})\) over all words \(v\) of the form \(1^a 2^b 2\) is
\[
\frac{z_2 t}{(1-(1-x)z_1 t)(1-(1-x)z_2 t)}. \tag{31}
\]

The generating function of \(-x^{|v|}(1-x)^{\text{rise}(v)} \prod_{s \in \text{Des}(v)} \text{ewt}_{2341,7}(v_s v_{s+1})\) over all words \(v\) of the form \(1^a 2^b 211^a 2^b 21 \ldots 1^a 2^b 21 w\) where \(w\) is weakly increasing is
\[
-xW(x, z_\infty, t) \left( \frac{z_2 t}{(1-(1-x)z_1 t)(1-(1-x)z_2 t)} \right)^r (-x z_1 z_3 z_4 t^3)^r (1-x)^{r-1}. \tag{32}
\]

Here the term \((-x z_1 z_3 z_4 t^3)^r\) comes from the weights \(\text{ewt}_{2341,7}(21)\) that arise from the descents 21 and \((1-x)^{r-1}\) comes from the weights of the rises coming from the first \(r-1\) 1s which are the second elements of the descents 21. It follows that the generating function of \(-x^{|v|}(1-x)^{\text{rise}(v)} \prod_{s \in \text{Des}(v)} \text{ewt}_{2341,7}(v_s v_{s+1})\) over all \(v\) such that \(v\) is of the form \(1^a 2^b 211^a 2^b 21 \ldots 1^a 2^b 21 w\) for some \(r \geq 1\) where \(w\) is a weakly increasing word is equal to
\[
-xW(x, z_\infty, t) \sum_{r \geq 1} \left( \frac{-x z_1 z_2 z_3 z_4 t^4}{(1-(1-x)z_1 t)(1-(1-x)z_2 t)} \right)^r (1-x)^{r-1} =
\]
\[
-xW(x, z_\infty, t) \left( \frac{1}{1-x} - \frac{1}{1 - x z_1 z_2 z_3 z_4 t^4 (1-x)} (1-(1-x)z_1 t)(1-(1-x)z_2 t) \right). \tag{31}
\]

Putting \((30)\) and \((31)\) together we see that
\[
EU_{2341,n}(x, z_\infty, t) = 1 + \frac{-x}{1-x} \left( -1 + W(x, z_\infty, t) \right) +
\]
\[
-xW(x, z_\infty, t) \left( \frac{1}{1-x} - \frac{1}{1 - x z_1 z_2 z_3 z_4 t^4 (1-x)} (1-(1-x)z_1 t)(1-(1-x)z_2 t) \right) =
\]
\[
1 + \frac{x}{1-x} - \frac{xW(x, z_\infty, t)}{1-x} \left( \frac{1}{1-x} + \frac{x z_1 z_2 z_3 z_4 t^4 (1-x)}{(1-(1-x)z_1 t)(1-(1-x)z_2 t)} \right). \tag{32}
\]

Thus
\[
\mathcal{E}N_{2341}(x, z_\infty, t) = \frac{1}{1 + \frac{x}{1-x} - \frac{xW(x, z_\infty, t)}{1-x}} - \frac{1}{1 + \frac{x z_1 z_2 z_3 z_4 t^4 (1-x)}{(1-(1-x)z_1 t)(1-(1-x)z_2 t)}}. \tag{33}
\]

It should be clear from our arguments that the only role that the 2 and 3 played in the final form \(\mathcal{E}N_{2341}(x, z_\infty, t)\) was to contribute a factor of \(z_3 z_4 t^2\) to the expression \(\frac{x z_1 z_2 z_3 z_4 t^4 (1-x)}{(1-(1-x)z_1 t)(1-(1-x)z_2 t)}\) on the right hand side of \((33)\). Thus our arguments show that if \(v = 2\alpha 1\) where \(\alpha\) is non-empty weakly increasing word in \(\{2, 3, \ldots\}^*\), then we have the following theorem.
Theorem 9. Let \( u = 2\alpha_1 \) where \( \alpha \) is non-empty weakly increasing word in \( \{2, 3, \ldots\}^* \). Then
\[
\mathcal{E}N_{2\alpha_1}^{(p)}(x, z, t) = \frac{1}{1 + \frac{x}{1-x}} - \frac{xW(x, z, t)}{1-x} \frac{1}{\frac{xz_{1z_{2z_{\alpha}}}}{1-(1-x)z_{1z_{2z_{\alpha}}}}}. 
\] (34)

Other examples where the weights \( wt_{u, P}(ij) \) are easy to compute are words of the form \( u = 2^r1 \) or \( u = 21^r \) where \( r \geq 2 \). It is easy to see that both \( 2^r1 \) and \( 21^r \) have the minimal overlapping property. In this case, the only \( u \)-matches are of the form \( b^r a \) where \( b > a \geq 1 \) if \( u = 2^r1 \) or \( ba^r \) where \( b > a \geq 1 \) if \( u = 21^r \). For example, suppose that \( u = 231 \). Then in Figure 6 we have pictured a fixed point of \( I_u \) where we have indicated the two \( 231 \)-matches in our example by placing stars below the cells in the \( 231 \)-matches. In the case the collapse map just maps \( (B, w) \) to the word \( v = C(B, w, u) \) which is the result of starting with \( w \) and removing the letters in all such matches that do not correspond to the end points of the match. This process is pictured in Figure 6b where again we have starred the elements in \( C(B, w, u) \) that remain from the original \( 231 \)-matches in \( w \).

\[
\begin{align*}
(B, w) &= \begin{bmatrix}
\begin{array}{cccc}
3 & 3 & 3 & 3 \\
1 & 3 & 4 & 5 \\
2 & 4 & 4 & 4 \\
4 & 5 & 5 \\
\end{array}
\end{bmatrix} \\
C(B, w, u) &= \begin{bmatrix}
\begin{array}{cccc}
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 \\
4 & 5 & 5 \\
\end{array}
\end{bmatrix}
\end{align*}
\]

Figure 6: A fixed point of \( I_{2221} \).

In this case, if we have a descent \( ji \), then \( wt_{231, k}(ji) = wt_{2221, P}(ji) = -xz_2^2 t^2 \) since we will always add back two \( j \)s for each descent of the form \( ji \). Thus if \( u = 231 \) it follows that for any \( v \in P^+ \),
\[
\sum_{(B, w) \text{ is a fixed point of } I_{2221} \atop C(B, w, 2221) = v} \text{sgn}(B, w)wt(B, w) = -x(1-x)^{\text{rise}(v)} \prod_{s \in \text{Des}(v)} -xz_2^2 t^2. \tag{35}
\]
and
\[
U_{2221, n}^{(P)}(x, z, t) = 1 + \sum_{n \geq 1} U_{2221, n}^{(P)}(x, z, t) t^n \\
= 1 + \sum_{v \in P^+} -x(1-x)^{\text{rise}(v)} \prod_{s \in \text{Des}(v)} -xz_2^2 t^2. \tag{36}
\]

Hence we could compute \( \mathcal{A}_{2221, n}^{(P)}(x, z, t) = \frac{1}{U_{2221, n}^{(P)}(x, z, t)} \) if we can compute the right-hand side of (36).

When \( u \) does not have the minimal overlapping property, we can obtain similar results but the collapse maps and the weights \( wt_u(ji) \) are more complicated. Again this is best explained through an example. Suppose that \( u = 3412 \) and \( k = 8 \).
When $u$ does not have the minimal overlapping property, then we can have a situation such as the one pictured in Figure 7. If we look at the descents between bricks 1 and 2 which correspond to the $u$-match 7846, we see that we would like to eliminate the 8 and 4. However, this $u$-match overlaps the $u$-match associated with the descent between bricks 2 and 3 which is 4623. Thus we would also like to eliminate the 6 and 2. We will say that two such matches are linked if one of the end points of first match is one of middle elements of the second match. Depending on the pattern we could have a series of $u$-matches in a fixed point of $(B, w)$ which are linked. The fact that we are assuming that $u = u_1 \ldots u_j$ where $u_1 > u_j$ and $\text{des}(u) = 1$ implies that $u$ has the $\mathbb{P}$-weakly decreasing overlapping property or $[k]$-weakly decreasing overlapping property. It is then easy to see that if $w = w_1 \ldots w_n$ is a word such that there is a $u$-match starting position 1 and a $u$-match ending at position $n$ and any two consecutive $u$-matches in $w$ are linked, then $w_1 > w_n$. In such a situation, the collapse map will eliminate all the symbols except for the first element of the first match and last element of the last match in a maximal sequence of linked $u$-matches which will result in a descent. This is illustrated in Figure 7 where we have two maximal blocks of linked 3412-matches. Thus in the linked 3412-matches in cells 2 through 7, we keep only the 7 and the 3 and in the linked matches in cells 9 through 14, we keep only the 5 and the 2. Because we are assuming that $u_1 > u_j$, we know that maximal blocks of linked $u$-matches must be finite since the end point of such matches must strictly decrease. When we see a descent $ji$ in a word $C(B, w, u)$, the weight associated with such a decent is now more complicated. For example, in our case where $u = 3412$ and $k = 8$, a decent of the form 72 can correspond to a single 3412-match which would have to be of the form 7812, it could correspond to a maximum block with 2 linked 3412-matches in which case it must be of the form 78cd12 where $3 \leq c < d \leq 6$, or it could correspond to a maximum block with 3 linked 3412-matches in which case it must be 78563412. Thus

$$wt_{3412}(72) = -xz_1z_8t^2 + x^2t^4z_1z_8 \left( \sum_{3 \leq c < d \leq 6} z_c z_d \right) - x^3t^6z_1z_3z_4z_5z_6z_8$$

On the other hand a descent of the form $ji$ where $j - i \leq 2$ can only correspond to a single 3412-match so that $wt_{3412}(ji) = -xt^2(\sum_{j < s \leq 8} z_s)(\sum_{1 \leq t < i} z_t)$.

We give the weights associated with the descents for $u = 3412$ and $k = 8$ in the following table.
Hence of Proposition 1, we know that

\[ \sum_{j \geq 1} \sum_{i < j} z_{ij} = 0 \]

Next suppose that we replace \( t \) by \( yt \) and \( x_{ij} \) by \( \frac{x_{ij}}{y} \). Under this substitution the left-hand side of (40) becomes

\[ \sum_{w = w_1 \ldots w_n \in P^+} y^{|w|} \frac{\text{rise}(w)}{y} \prod_{i \in \text{Des}(w)} x_{w_i w_{i+1}}. \]
Note that for \( S = \{ j_1 < \cdots < j_k \} \) where \( k \geq 2 \), our substitution replaces \( t^k DXZ(S) \) by

\[
y^k t^k z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} \left( \frac{x_{j_{i+1} j_i}}{y} - 1 \right) = yt^k z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_{i+1} j_i} - y).
\]

Thus if we let

\[
DXYZ(S) = \begin{cases} 
z_j & \text{if } S = \{ j \}, \\
z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_{i+1} j_i} - y) & \text{if } S = \{ j_1 < \cdots < j_k \} \text{ where } k \geq 2,
\end{cases}
\]

then we see that right-hand side of (40) becomes

\[
y \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXYZ(S)
\]

\[
1 - y \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXYZ(S).
\]

It follows that

\[
-x \sum_{w = w_1 \cdots w_n \in \mathbb{P}^+} t^{|w|} y^{\text{wris}(w)} \prod_{i \in \text{Des}(w)} x_{w_i w_{i+1}} = \frac{-x \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXYZ(S)}{1 - y \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXYZ(S)}.
\]

Thus

\[
1 - x \sum_{w = w_1 \cdots w_n \in \mathbb{P}^+} t^{|w|} y^{\text{wris}(w)} \prod_{i \in \text{Des}(w)} x_{w_i w_{i+1}} = \frac{1 - (x + y) \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXYZ(S)}{1 - y \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXYZ(S)}.
\]

By setting \( z_i = 0 \) for \( i > k \), we also obtain that

\[
1 - x \sum_{w = w_1 \cdots w_n \in \mathbb{P}^+} t^{|w|} y^{\text{wris}(w)} \prod_{i \in \text{Des}(w)} x_{w_i w_{i+1}} = \frac{1 - (x + y) \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXYZ(S)}{1 - y \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXYZ(S)}.
\]

Note that if we replace \( y \) by \( (1 - x) \) and \( x_{j_i} \) by \( wt_u(j_i) \), the left-hand side of (42) becomes \( U_u^{(P)}(x, z_\infty, t) \) and the left-hand side of (43) becomes \( U_u^{(k)}(x, z_k, t) \). Similarly, if we replace \( y \) by \( (1 - x) \) and \( x_{j_i} \) by \( ewt_u(j_i) \), the left-hand side of (42) becomes \( EU_u^{(P)}(x, z_\infty, t) \) and the left-hand side of (43) becomes \( EU_u^{(k)}(x, z_k, t) \). Then using the fact that \( N_u^{(P)}(x, z_\infty, t) = 1/U_u^{(P)}(x, z_\infty, t) \) and that \( E N_u^{(P)}(x, z_\infty, t) = 1/EU_u^{(P)}(x, z_\infty, t) \), we have the following theorem.

**Theorem 10.** Suppose that \( u = u_1 \ldots u_j \in \mathbb{P}^* \), \( \text{red}(u) = u \), \( \text{des}(u) = 1 \), \( u_1 > u_j \). Then

\[
N_u^{(P)}(x, z_\infty, t) = \frac{1 - (1 - x) \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXTZ_u(S)}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXTZ_u(S)}
\]

and

\[
EN_u^{(P)}(x, z_\infty, t) = \frac{1 - (1 - x) \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} EDTXZ_u(S)}{1 - \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} EDTXZ_u(S)}
\]

where

\[
DXTZ_u(S) = \begin{cases} 
z_j & \text{if } S = \{ j \}, \\
z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} (wt_u(j_{i+1} j_i) + x - 1) & \text{if } S = \{ j_1 < \cdots < j_k \} \text{ where } k \geq 2
\end{cases}
\]
and

\[ \text{EDXTZ}_u(S) = \begin{cases} z_j & \text{if } S = \{j\}, \\
 z_1 \cdots z_k \prod_{i=1}^{k-1} (ewt_u(j_{i+1}j_i) + x - 1) & \text{if } S = \{j_1 < \cdots < j_k\} \text{ where } k \geq 2. \end{cases} \]  

(47)

If we set \( z_i = 0 \) for all \( i > k \), then we obtain the following theorem.

**Theorem 11.** Now suppose that \( u = u_1 \ldots u_j \in [k]^* \), \( \text{red}(u) = u \), \( \text{des}(u) = 1 \), \( u_1 > u_j \). Then

\[ \mathcal{N}_u^{(k)}(x, z_k, t) = \frac{1 - (1 - x) \sum_{n=1}^k t^n \sum_{S \subseteq [k], |S| = n} \text{EDXTZ}_u(S)}{1 - \sum_{n=1}^k t^n \sum_{|S| = n} \text{EDXTZ}_u(S)} \]  

(48)

and

\[ \mathcal{E}\mathcal{N}_u^{(k)}(x, z_k, t) = \frac{1 - (1 - x) \sum_{n=1}^k t^n \sum_{S \subseteq [k], |S| = n} \text{EDXTZ}_u(S)}{1 - \sum_{n=1}^k t^n \sum_{S \subseteq [k], |S| = n} \text{EDXTZ}_u(S)} \]  

(49)

It follows from Theorem 11 that to compute the generating function \( \mathcal{N}_u^{(k)}(x, z_k, t) \), we need only compute sums of the form

\[ P_{n,u}(x, t) = \sum_{S \subseteq [k], |S| = n} \text{EDXTZ}_u(S) \]

for \( 1 \leq n \leq k \). As an example, suppose that \( k = 5 \) and we want to compute \( \mathcal{N}_u^{(5)}(x, z_5, t) \) where we set \( z_i = 1 \) for all \( i \). Then with this specialization, it is easy to see that

1. \( \text{wt}_{2341}(21) = -3xt^2 \),
2. \( \text{wt}_{2341}(3i) = -xt^2 \) for all \( i < 3 \),
3. \( \text{wt}_{2341}(4i) = 0 \) for all \( i < 4 \), and
4. \( \text{wt}_{2341}(5i) = 0 \) for all \( i < 5 \).

It follows that that

1. \( \text{DXTZ}_{2341}([1, 2])|_{z_i=1} = -3xt^2 + x - 1 \),
2. \( \text{DXTZ}_{2341}([i, 3])|_{z_i=1} = -xt^2 + x - 1 \) for all \( i < 3 \),
3. \( \text{DXTZ}_{2341}([i, 4])|_{z_i=1} = x - 1 \) for all \( i < 4 \), and
4. \( \text{DXTZ}_{2341}([i, 5])|_{z_i=1} = x - 1 \) for all \( i < 5 \).

One can then compute that

1. \( P_{1,2341}(x, t) = 5 \),
2. \( P_{2,2341}(x, t) = -10 + 10x - 5xt^2 \),
3. \( P_{3,2341}(x, t) = 10 - 20x + 14t^2x + 10x^2 - 14t^2x^2 + 3t^4x^2 \),
4. \( P_{4,2341}(x, t) = -5 + 15x - 13t^2x - 15x^2 + 26t^2x^2 - 6t^4x^2 + 5x^3 - 13t^2x^3 + 6t^4x^3 \), and
5. \( P_{5,2341}(x, t) = (-3x^2 + x - 1)(-xt^2 + x - 1)(x - 1)^2. \)

Thus
\[
\sum_{w \in [5]^*, 2341-\mch(w) = 0} x^{\text{des}(w) + 1} = \frac{1 - (1 - x)(\sum_{n=1}^{5} P_{n,2341}(x, t)t^n)}{1 - \sum_{n=1}^{5} P_{n,2341}(x, t)t^n}.
\] (50)

We have computed that the initial terms of this series are
\[
1 + 5x + 5(3x + 2x^2)t^2 + 5(7x + 16x^2 + 2x^3)t^3 + 5(14x + 72x^2 + 37x^3 + x^4)t^4 +
\]
\[
(126x + 1210x^2 + 1492x^3 + 246x^4 + x^5)t^5 + (210x + 3387x^2 + 7921x^3 + 3522x^4 + 210x^5)t^6 +
\]
\[
(330x + 8344x^2 + 32461x^3 + 28902x^4 + 5471x^5 + 120x^6)t^7 + \cdots .
\]

One can obtain several interesting generating functions from \( N_{2341}^{(5)}(x, z_5, t) \). For example, setting \( x = 0 \) in \( \frac{1}{2} \frac{\partial^2}{\partial x^2} N_{2341}^{(5)}(x, z_5, t) \), one finds that the generating function for the number of words \( w \) in \([5]^*\) such that \( \text{des}(w) = 1 \) and \( 2341-\mch(w) = 0 \) is
\[
\frac{t^2(10 - 20t + 10t^2 + 13t^4 + 4t^5)}{(1 - t)^{10}}.
\]

Similarly setting \( x = 0 \) in \( \frac{1}{2} \frac{\partial^2}{\partial x^2} N_{2341}^{(5)}(x, z_5, t) \), one finds that the generating function for the number of words \( w \) in \([5]^*\) such that \( \text{des}(w) = 2 \) and \( 2341-\mch(w) = 0 \) is
\[
\frac{t^3 Q(t)}{(1 - t)^{15}}
\]
where
\[Q(t) = 10 + 35t - 233t^2 + 416t^3 - 219t^4 - 266t^5 + 458t^6 - 167t^7 - 161t^8 + 198t^9 - 83t^{10} + 13t^{11}.
\]

If \( u = 2^\mathbf{1} \) where \( s \geq 2 \) and we set \( z_i = 1 \) for all \( i \), then it is easy to see that \( \text{wt}_{2^\mathbf{1}}(ji) = -xt^{s-1} \) for all \( j > i \) and that \( DXTZ(S)|_{z_i=1} = (-xt^{s-1} + 1 - x)|_{S=1} \) for all \( S \subseteq [k] \) where \( |S| \geq 1 \). It then easily follows from Theorem [11]
\[
\sum_{w \in [k]^*, 2^\mathbf{1}-\mch(w) = 0} x^{\text{des}(w) + 1} = \frac{1 - (1 - x)(\sum_{n=1}^{k} \binom{k}{n}(-xt^{s-1} + 1 - x)^{n-1}t^n)}{1 - \sum_{n=1}^{k} \binom{k}{n}(-xt^{s-1} + 1 - x)^{n-1}t^n}.
\] (51)

As an example,
\[
\sum_{w \in [5]^*, 2^\mathbf{1}-\mch(w) = 0} x^{\text{des}(w) + 1} = \frac{1 - (1 - x)(\sum_{n=1}^{5} \binom{5}{n}(-xt^2 + x - 1)^{n-1}t^n)}{1 - \sum_{n=1}^{5} \binom{5}{n}(-xt^2 + x - 1)^{n-1}t^n}.
\] (52)

We have computed that the initial terms of this series are
\[
1 + 5xt + 5(3x + 2x^2)t^2 + 5(7x + 16x^2 + 2x^3)t^3 + 5(14x + 71x^2 + 37x^3 + x^4)t^4 +
\]
\[
(126x + 1166x^2 + 1486x^3 + 246x^4 + x^5)t^5 + 5(42x + 634x^2 + 1553x^3 + 704x^4 + 42x^5)t^6 +
\]
\[
(330x + 7554x^2 + 30998x^3 + 28662x^4 + 5471x^5 + 120x^6)t^7 + O[t]^8.
\]
5 The case \(u = u_1 \ldots u_j\), \(\text{des}(u) = 1\), and \(u_1 < u_j\)

In this section, we shall consider the problem of computing the generating functions
\(N^{(P)}_u(x, z, t)\), \(N^{(k)}_u(x, z, t)\), \(E N^{(P)}_u(x, z, t)\), and \(E N^{(k)}_u(x, z, t)\) for \(u = u_1 \ldots u_j\) such that \(\text{des}(u) = 1\), \(u_1 < u_j\), and \(u\) has the \(P\)-weakly increasing overlapping property (\([k]\)-weakly increasing overlapping property).

Again the simplest case is when \(u\) has the \(P\)-minimal overlapping property in which case \(u\) automatically has the \(P\)-weakly increasing overlapping property. For example, suppose that \(u = 12433\). Now suppose that we are given a fixed point \((B, w)\) of \(I_u\), where \(B = (b_1, \ldots, b_k)\) and \(w = w_1 \ldots w_n\), such as the one pictured in Figure 8. We know that to be a fixed point of \(I_u\), \(w\) must be weakly increasing within bricks of \(B\) and that for any \(i < k\), if \(c\) is last cell in brick \(b_i\) and \(w_c > w_{c+1}\), then there must be a \(u\)-match in \(w\) which is contained in the cells of \(b_i\) and \(b_{i+1}\). In our particular example, since \(u = 12433\) has a single descent, this match must involve the last three cells of \(b_i\) and the first two cells of \(b_{i+1}\). In Figure 8 we have indicated the three such matches in our example by placing stars below the cells in the 12433-matches. In this case, the collapse map just maps \((B, w)\) to the word \(v = C(B, w, u)\) which is the result of starting with \(w\) and removing the letters in all such matches that do not correspond to the end points of the match. This process is pictured in Figure 8 where again we have starred the elements in \(C(B, w, u)\) that remain from the original 12433-matches in \(w\). In this case, the resulting word \(C(B, w, u)\) must be weakly increasing.

\[
(B, w) = \begin{array}{cccccccccc}
1 & 2 & 5 & 3 & 3 & 3 & 3 & 3 & 7 & 6 & 6 & 7 & 9 & 8 & 8 & x \\
* & * & * & * & x & * & * & * & * & * & * & * & * & x
\end{array}
\]

\[C(B, w, u) = 1 \underline{3} \underline{3} \underline{3} \underline{3} \underline{6} \underline{8} \underline{8}
\]

\[** \ * * *
\]

Figure 8: A fixed point of \(I_{12433}\).

As in the previous section, we want to construct the set of fixed points of \((B, w)\) of \(I_u\) such that \(C(B, w, v)\) is equal to a given word \(v = v_1 \ldots v_n\) where \(v_1 \leq \cdots \leq v_n\).

If \(v_s < v_{s+1}\), then we have three possibilities: (i) \(v_s v_{s+1}\) could lie in the same brick \(b_i\) of \(B\), (ii) \(v_s\) could end a brick \(b_i\) and \(v_{s+1}\) could start the brick \(b_{i+1}\) in \(B\), or (iii) \(v_s v_{s+1}\) arose from a collapse across two bricks \(b_i\) and \(b_{i+1}\) where there was a decrease between bricks \(b_i\) and \(b_{i+1}\) and \(v_s\) played the role of 1 in the \(u\)-match and \(v_{s+1}\) plays the role of second 3 in the \(u\)-match that must cross the bricks \(b_i\) and \(b_{i+1}\). For example, suppose that the underlying alphabet is \([9]\). If \(v_s = 8\) and \(v_{s+1} = 9\), then \(v\) could not have come from the collapse of 12433-match because we can not add a letter which could play the role of 4 in the 12433-match. Hence, the weight associated to a rise 89 is just \(-x\). If we consider the first rise 13 in the \(C(B, w, u)\) of Figure 8 then we see there are many ways that we could add the three letters middle letters. That is, the original 12433-match could have been any \(12c33\) where \(c \in \{4, 5, 6, 7, 8, 9\}\). It follows that the extra weight from these possibilities that is not included in \(C(B, w, u)\) is
\[-x t^3 z_2 z_3 \sum_{1 \leq c \leq 9} z_c\]. Here the \(-x\) comes from the fact that we know that the original match
straddled two bricks and there is a weight of $-x$ associated with the end point of the first of those two bricks. Thus the weight associated with the rise 13 is $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$. If we consider the second rise 36 in the $C(B, w, u)$ of Figure 8, then we see there are many ways that we could add the three letters middle letters. That is, the original 12433-match must have been 67988. It follows that the extra weight in this case that is not included in $\sum_{x, t} \text{sgn}(B, w) wt(B, w)$.

In our case where $u = 12433$ and $k = 9$, the weights associated with the rises are given in the table below.

| Rises | $wt_{12433,9}(ij)$ | $1 - x$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ |
|-------|-------------------|----------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| i9 ($i \leq 7$) or $i(i + 1)$ | $1 - x$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ |
| i8 ($i \leq 6$) | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ |
| i7 ($i \leq 5$) | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ |
| i6 ($i \leq 4$) | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ |
| i5 ($i \leq 3$) | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ |
| i4 ($i \leq 2$) | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ |
| 13 | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ | $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ |

The weights $wt_{12433,9}(ij)$.

However, if $u = 12433$ and we want to compute $U_{u, n}^{(9)}(x, z_\infty)$, the weights for any rise $ij$ where $i + 1 < j$ would be $1 - x - xt^3z_2z_3 \sum_{4 \leq c \leq 9} z_c$ which is an infinite sum.

Going back to our example where $u = 12433$ and $k = 9$, it follows that for any $v \in [9]^+$,

$$
\sum_{(B, w) \text{ is a fixed point of } I_u} \text{sgn}(B, w) wt(B, w) = -xz^{|v|}(1 - x)^{lev(v)} \prod_{s \in \text{Rise}(v)} wt_{12433,9}(v_s v_{s+1}).
$$

As in the previous section, then initial $-x$ comes from the fact that the last cell of $(B, w)$ always contributes a $-x$ since the last cell is at the end of a brick. But then we know that

$$
U_{12433}^{(9)}(x, z_9, t) = 1 + \sum_{n \geq 1} U_{12433, n}^{(9)}(x, z_9) t^n = 1 + \sum_{v \in [9]^+, \text{des}(v) = 0} -x(1 - x)^{lev(v)} z^{|v|} \prod_{s \in \text{Rise}(v)} wt_{12433,9}(v_s v_{s+1}).
$$
Hence we could compute $N_{12433}^{(9)}(x, z_0, t) = \frac{1}{U_{12433}^{(9)}(x, z_0, t)}$ if we can compute the right-hand side of (54).

As in the previous section, the case of exact matches is much simpler. In that case, we want to compute

$$\sum_{(B, w) \text{ is a fixed point of } J_u} \text{sgn}(B, w) \text{wt}(B, w).$$

Going back to our example of $u = 12433$ over the alphabet $[9]$, we see that the weight associated to a rise $v < v_{s+1}$ is $1 - x$ unless $v = 1, v_{s+1} = 3$. If $v = 1, v_{s+1} = 3$, then we must have eliminated a 243 from $w$. Thus if we want to compute $EU_{12433, n}^{(9)}(x, z, \infty)$ or $EU_{12433, n}^{(k)}(x, z_k)$ for $k \geq 4$, the weights associated to rises are given in the following table.

| Rise          | weight $\text{ewt}_{12433,9}(ij)$ |
|---------------|-----------------------------------|
| $ij$ where either $i \neq 1$ or $j \neq 3$ | $1 - x$ |
| 13            | $1 - x - xz_2z_3z_4t^3$           |

The weights $\text{ewt}_{12433}(ij)$.

It follows that for any $v \in [9]^+$,

$$\sum_{(B, w) \text{ is a fixed point of } J_{12433}} \text{sgn}(B, w) \text{wt}(B, w) = -x^{e^v}t^{|v|}(1 - x)^{\text{lev}(v)} \prod_{s \in \text{Rise}(v)} \text{ewt}_{12433,9}(v, v_{s+1})$$

and

$$EU_{12433}^{(9)}(x, z_0, t) = 1 + \sum_{n \geq 1} EU_{12433, n}^{(9)}(x, z_0)t^n$$

$$= 1 + \sum_{v \in [9]^+, \text{des}(v) = 0} -x^{e^v}t^{|v|}(1 - x)^{\text{lev}(v)} \prod_{s \in \text{Rise}(v)} \text{ewt}_{12433,9}(v, v_{s+1}).$$

(55)

When $u$ does not have the minimal overlapping property, we can obtain similar results if $u$ has the $P$-weakly increasing overlapping property or the $[k]$-weakly increasing overlapping property. For example suppose that $u = u_1, \ldots, u_j$, $\text{des}(u) = 1$, $u_1 < u_j$, and $u$ has the $P$-weakly increasing overlapping property. Now suppose that $w = w_1 \ldots w_n$ is a maximal sequence of linked $u$-matches. That is, we assume $w$ starts and ends with a $u$-match and any two consecutive $u$-matches share at least two letters. Then if the $u$-matches in $w$ start at positions $1 = i_1 < i_2 < \cdots < i_k$, then the $P$-weakly increasing overlapping property in $w$ ensures that $w_1 = w_{i_1} \leq \cdots \leq w_{i_k} < w_{n}$. Thus in a collapse map, if we eliminate $w_2 \ldots w_{n-1}$ we will be left with a rise $w_1 w_n$. This may not happen if $u$ does not have the $P$-weakly increasing overlapping property. For example, suppose $u = 2413$, then the words $w^{(1)} = 472613, w^{(2)} = 472614$, and $w^{(3)} = 472615$ have $u$-matches starting at positions 1 and 3. Thus in such a case, we have no control over the relationship between first and last letter of a maximal sequence of linked $u$-matches.

Thus assume that $u = u_1 \ldots u_j$, $\text{des}(u) = 1$, $u_1 < u_j$ and $u$ has the $P$-weakly increasing overlapping property. Then we shall see that the collapse map still works but the weight function $\text{wt}_u(ij)$ is more complicated. As we saw in the previous section, we must pay attention to overlapping $u$-matches that share more than one letter. We will consider the example where $u = 11124333$ and $k = 7$. Clearly $u$ has the weakly increasing overlapping property. In this
case, \( u \)-matches can overlap in either one, two, or three letters. As in the previous section, the collapse map will keep only the first and last letters of a consecutive sequence of \( u \)-matches such that each consecutive pair share at least two letters. For example, at the top of Figure 9 we have given an example where two consecutive \( u \)-matches share 3 letters and at the bottom of Figure 9 we have given an example where two consecutive \( u \)-matches share 2 letters.

\[
(B, w) = \begin{pmatrix}
1 & 1 & 2 & 7 & 3 & 3 & 4 & 6 & 5 & 5 & 6 & 6 & 6 & 7 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & -
\end{pmatrix}
\]

\[
C(B, w, u) = 1 5 6 6 6 7
\]

\[
(B, w) = \begin{pmatrix}
1 & 2 & 2 & 3 & 6 & 4 & 4 & 4 & 5 & 7 & 6 & 6 & 6 & 6 & 7 \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & -
\end{pmatrix}
\]

\[
C(B, w, u) = 1 2 6 6 6 7
\]

**Figure 9:** A fixed point of \( I_{11124333} \).

As before, if we are given a weakly increasing word \( v = v_1 \ldots v_n \in [7]^+ \), we want to find the sum of the weights of all fixed points \((B, w)\) of \( I_u \) such that \( C(B, w, u) = v \). Now if \( v_s = v_{s+1} \), then either \( v_s v_{s+1} \) lie in the same brick which contributes a factor of 1 or \( v_s v_{s+1} \) lie in different bricks which contributes a factor of \(-x\) for the brick that ends at \( v_s \). Thus we obtain a factor of \( 1 - x \) for each level of \( v \). For the rises of \( v \), we should observe that the start and the end of any two consecutive \( u \)-matches which share more than one letter must differ by at least 4. Similarly, the start and the end of any three consecutive \( u \)-matches in which each two consecutive \( u \)-matches share more than one letter must differ by at least 6. Hence, for \( k = 7 \), we can not have three consecutive \( u \)-matches in which each two consecutive \( u \)-matches share more that one letter because the smallest starting point is 1 the smallest ending point is 7 which leaves no room for a letter which is larger than the last three letters in such a sequence. For each pair, \( v_s < v_{s+1} \) which occurs in \( v \), we get a factor of \( 1 - x \) as we did for levels. However in this case, we must also consider the possible collapses that could give rise to \( v_s v_{s+1} \). These are as follows.

1. Rises of the form \( i(i + 1) \) or \( i7 \) where \( 1 \leq i \leq 5 \) can not arise from the collapse map in our case so that \( wt_{11124333, 7}(v_s v_{s+1}) = 1 - x \) in these cases.

2. \( v_s v_{s+1} = 13 \). In this case, a \( u \)-match that could give rise to 13 under the collapse map must be of the form 1112a333 where \( a \in \{4, 5, 6, 7\} \). Thus

\[
wt_{11124333, 7}(v_s v_{s+1}) = 1 - x - xt^6 z_1 z_2^2 (z_4 + z_5 + z_6 + z_7) z_3^2.
\]

3. \( v_s v_{s+1} = 14 \). In this case, a \( u \)-match that could give rise to 14 under the collapse map
must be of the form 111ab444 where \(a \in \{2, 3\}\) and \(b \in \{5, 6, 7\}\). Thus

\[
wt_{11124333,7}(v_s v_{s+1}) = 1 - x - x^6 z_1^2(z_2 + z_3)(z_5 + z_6 + z_7)z_4^2.
\]

4. \(v_s v_{s+1} = 15\). In this case, a single \(u\)-match that could give rise to 15 under the collapse map must be of the form 111ab555 where \(a \in \{2, 3, 4\}\) and \(b \in \{6, 7\}\). There are also two possibilities for linked \(u\)-matches that could give rise to 15 under the collapse map, namely, (i) 1112a333ab555 or (ii) 1112a33334b555 where \(a \in \{4, 5, 6, 7\}\) and \(b \in \{6, 7\}\). Thus

\[
wt_{11124333,7}(v_s v_{s+1}) = 1 - x - x^6 z_1^2(z_2 + z_3 + z_4)(z_5 + z_6 + z_7)z_5^2 - \\
x t^{11} z_1^2 z_2(z_4 + z_5 + z_6 + z_7)z_3^3 z_4(z_5 + z_6 + z_7)z_6^2 - \\
x t^{12} z_1^2 z_2(z_4 + z_5 + z_6 + z_7)z_3^3 z_4(z_5 + z_6 + z_7)z_5^2.
\]

5. \(v_s v_{s+1} = 16\). In this case, a single \(u\)-match that could give rise to 16 under the collapse map must be of the form 111a7666 where \(a \in \{2, 3, 4, 5\}\). There are also four possibilities for linked \(u\)-matches that could give rise to 16 under the collapse map, namely, (i) 1112a333b7666 \(a \in \{4, 5, 6, 7\}\) and \(b \in \{4, 5\}\), (ii) 1112a3333b7666 where \(a \in \{4, 5, 6, 7\}\) and \(b \in \{4, 5\}\), (iii) 11ab44457666 \(a \in \{2, 3\}\) and \(b \in \{5, 6, 7\}\), or (iv) 11ab444457666 where \(a \in \{2, 3\}\) and \(b \in \{5, 6, 7\}\). Thus

\[
wt_{11124333,7}(v_s v_{s+1}) = 1 - x - x^6 z_1^2(z_2 + z_3 + z_4 + z_5)z_7 z_6^2 - \\
x t^{11} z_1^2 z_2(z_4 + z_5 + z_6 + z_7)z_3^3(z_4 + z_5)z_7 z_6^2 - \\
x t^{12} z_1^2 z_2(z_4 + z_5 + z_6 + z_7)z_3^3(z_4 + z_5)z_7 z_6^2 - \\
x t^{11} z_1^2 z_2(z_5 + z_6 + z_7)z_3^3 z_5 z_7 z_6^2 - \\
x t^{12} z_1^2(z_2 + z_3)(z_5 + z_6 + z_7)z_4^4 z_5 z_7 z_6^2.
\]

6. \(v_s v_{s+1} = 24\). In this case, a \(u\)-match that could give rise to 24 under the collapse map must be of the form 2223a444 where \(a \in \{5, 6, 7\}\). Thus

\[
wt_{11124333,7}(v_s v_{s+1}) = 1 - x - x^6 z_2^2 z_3(z_5 + z_6 + z_7)z_4^2.
\]

7. \(v_s v_{s+1} = 25\). In this case, a \(u\)-match that could give rise to 25 under the collapse map must be of the form 222ab555 where \(a \in \{3, 4\}\) and \(b \in \{6, 7\}\). Thus

\[
wt_{11124333,7}(v_s v_{s+1}) = 1 - x - x^6 z_2^2(z_3 + z_4)(z_6 + z_7)z_5^2.
\]

8. \(v_s v_{s+1} = 26\). In this case, a single \(u\)-match that could give rise to 26 under the collapse map must be of the form 222a7666 where \(a \in \{3, 4, 5\}\). There are also two possibilities for linked \(u\)-matches that could give rise to 26 under the collapse map, namely, (i) 2223a44457666 or (ii) 2223a444457666 where \(a \in \{5, 6, 7\}\). Thus

\[
wt_{11124333,7}(v_s v_{s+1}) = 1 - x - x^6 z_2^2(z_3 + z_4 + z_5)z_7 z_6^2 - \\
x t^{11} z_2^2 z_3(z_5 + z_6 + z_7)z_3^3 z_5 z_7 z_6^2 - \\
x t^{12} z_2^2 z_3(z_5 + z_6 + z_7)z_3^3 z_5 z_7 z_6^2.
\]
9. \( v_s v_{s+1} = 35 \). In this case, a \( u \)-match that could give rise to 35 under the collapse map must be of the form \( 333a555 \) where \( a \in \{6, 7\} \). Thus

\[
wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_5^2 z_4 (z_6 + z_7) z_5^2.
\]

10. \( v_s v_{s+1} = 36 \). In this case, a \( u \)-match that could give rise to 36 under the collapse map must be of the form \( 333a7666 \) where \( a \in \{4, 5\} \). Thus

\[
wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_5^2 (z_4 + z_5) z_7 z_6^2.
\]

11. \( v_s v_{s+1} = 46 \). In this case, a \( u \)-match that could give rise to 46 under the collapse map must be of the form \( 44457666 \). Thus

\[
wt_{11124333,7}(v_s v_{s+1}) = 1 - x - xt^6 z_4 z_5 z_7 z_6 z_2.
\]

It follows that for any \( v \in [7]^+ \) such that \( v \) is weakly increasing,

\[
\sum_{(B, w) \text{ is a fixed point of } I_{11124333} \atop C(B, w, 11124333) = v} \sgn(B, w) wt_{11124333,7}(B, w) = - \frac{x^{|v|}}{t^{|v|}} (1 - x)^{\lev(v)} \prod_{s \in \text{Rise}(v)} wt_{11124333,7}(v_s v_{s+1}).
\] (57)

and

\[
U_{11124333}^{(7)}(x, z_7, t) = 1 + \sum_{n \geq 1} U_{11124333, n}^{(7)}(x, z_7) t^n = 1 + \sum_{v \in [7]^+, \des(v) = 0} \frac{-x^{|v|}}{t^{|v|}} (1 - x)^{\lev(v)} \prod_{s \in \text{Rise}(v)} wt_{11124333,7}(v_s v_{s+1}).
\] (58)

What we need to be able to compute the right-hand sides of either (54), (56), or (58), is the generating function over all weakly increasing words \( v \in \mathbb{P}^* \) where we not only keep track of the rises of \( P \) but also the type of rises.

By Theorem 33 we know that

\[
\sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} RXZ(S) = \sum_{w = w_1 \leq \cdots \leq w_n \in \mathbb{P}^+} t^{|w|} z^{|w|} \prod_{i \in \text{Rise}(w)} x_{w_1 w_{i+1}}.
\] (59)

If we first replace \( t \) by \( yt \) and \( x_{ij} \) by \( x_{ij}/y \) in (59) and then divide by \( y \), the right-hand side becomes

\[
\sum_{w = w_1 \leq \cdots \leq w_n \in \mathbb{P}^+} t^{|w|} z^{|w|} y^{\lev(w)} \prod_{i \in \text{Rise}(w)} x_{w_1 w_{i+1}}
\]

and the left-hand side becomes

\[
\sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{P}, |S| = n} RXY Z(S)
\]

29
where
\[
RXYZ(S) = \begin{cases} 
\frac{z_i}{1-z_i y t} & \text{if } S = \{j\}, \text{ and} \\
\left( \prod_{i=1}^k \frac{z_{i j} t}{1-z_{i j} y t} \right) \prod_{i=1}^{k-1} x_{i j, i+1} & \text{if } S = \{j_1 < \cdots < j_k\} \text{ where } k \geq 2.
\end{cases}
\]

Hence
\[
1 - x \sum_{w=w_1 \leq \cdots \leq w_n \in \mathbb{F}^+} t^{|w|} y^{\text{lev}(w)} z^w \prod_{i \in \text{Rise}(w)} x_{w_i, w_{i+1}} = 1 - x \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{F}, |S| = n} RXYZ(S). \tag{61}
\]

If we set \( z_i = 0 \) for \( i > k \), then we obtain that
\[
1 - x \sum_{w=w_1 \leq \cdots \leq w_n \in [k]^+} t^{|w|} y^{\text{lev}(w)} z^w \prod_{i \in \text{Rise}(w)} x_{w_i, w_{i+1}} = 1 - x \sum_{n=1}^k t^n \sum_{S \subseteq [k], |S| = n} RXYZ(S). \tag{62}
\]

Note that if we replace \( y \) by \((1-x)\) and \( x_{ij} \) by \( w t_u(ij) \), the left-hand side of (61) becomes \( U_u^{(P)}(x, z_\infty, t) \) and the left-hand side of (62) becomes \( U_u^{(k)}(x, z_k, t) \). Similarly, if we replace \( y \) by \((1-x)\) and \( x_{ij} \) by \( e w t_u(ij) \), the left-hand side of (61) becomes \( E U_u^{(P)}(x, z_\infty, t) \) and the left-hand side of (62) becomes \( E U_u^{(k)}(x, z_k, t) \). Then using the fact that \( N_u^{(P)}(x, z_\infty, t) = 1/U_u^{(P)}(x, z_\infty, t) \) and that \( E N_u^{(P)}(x, z_\infty, t) = 1/E U_u^{(P)}(x, z_\infty, t) \), we have the following theorem.

**Theorem 12.** Suppose that \( u = u_1 \ldots u_j \in \mathbb{P}^* \), \( \text{red}(u) = u \), \( \text{des}(u) = 1 \), \( u_1 < u_j \), and \( u \) has the \( \mathbb{P} \)-weakly increasing overlapping property. Then
\[
N_u^{(P)}(x, z_\infty, t) = \frac{1}{1 - x \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{F}, |S| = n} RXTZ(S)} \tag{63}
\]
and
\[
E N_u^{(P)}(x, z_\infty, t) = \frac{1}{1 - x \sum_{n \geq 1} t^n \sum_{S \subseteq \mathbb{F}, |S| = n} ERXTZ(S)} \tag{64}
\]
where
\[
RXTZ_u(S) = \begin{cases} 
\frac{z_j}{1-z_j t} & \text{if } S = \{j\}, \text{ and} \\
\left( \prod_{i=1}^k \frac{z_{i j} t}{1-z_{i j} t} \right) \prod_{i=1}^{k-1} (w t_u(j_i j_{i+1})) & \text{if } S = \{j_1 < \cdots < j_k\} \text{ where } k \geq 2.
\end{cases}
\]

and
\[
ERXTZ_u(S) = \begin{cases} 
\frac{1}{1 - (1-x) z_j t} & \text{if } S = \{j\}, \text{ and} \\
\left( \prod_{i=1}^k \frac{z_{i j} t}{1-(1-x) z_{i j} t} \right) \prod_{i=1}^{k-1} e w t_u(j_i j_{i+1}) & \text{if } S = \{j_1 < \cdots < j_k\} \text{ where } k \geq 2.
\end{cases}
\]
If we specialize the variables so that \( z_i = 0 \) for all \( i > k \), then we have the following theorem.

**Theorem 13.** Suppose that \( u = u_1 \ldots u_j \in [k]^* \), \( \text{red}(u) = u \), \( \text{des}(u) = 1 \), \( u_1 < u_j \), and \( u \) has the \([k]\)-weakly increasing overlapping property. Then

\[
N_u^{(k)}(x, z_k, t) = \frac{1}{1 - x \sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S| = n} \text{ERXTZ}(S)}
\]  

(67)

and

\[
\mathcal{E}N_u^{(k)}(x, z_k, t) = \frac{1}{1 - x \sum_{n=1}^{k} t^n \sum_{S \subseteq [k], |S| = n} \text{ERXTZ}(S)}
\]  

(68)

It follows from Theorem 13 that to compute the generating function we need to \( N_u^{(k)}(x, z_k, t) \), we need only compute sums of the form

\[
P_{n,u}(x, t) = \sum_{S \subseteq [k], |S| = n} \text{ERXTZ}_u(S)
\]

for \( 1 \leq n \leq k \) and that to compute the generating function we need to \( \mathcal{E}N_u^{(k)}(x, z_k, t) \), we need only compute sums of the form

\[
P_{n,u}(x, t) = \sum_{S \subseteq [k], |S| = n} \text{ERXTZ}_u(S)
\]

for \( 1 \leq n \leq k \).

For example, suppose that we want to compute \( \mathcal{E}N_u^{(9)}(x, z_9, t) \) where \( u = 12433 \) and we set \( z_i = 1 \) for \( i = 1, \ldots, 9 \). For each set singleton \( S = \{j\} \), \( \text{ERXTZ}_u(S) = \frac{1}{(1 - (1 - x)t)^j} \). For sets \( S \) of cardinality greater than 2, there are two types of sets \( S = \{j_1 < j_2 < \ldots j_n\} \) to consider, namely, those where \( j_1 = 1 \) and \( j_2 = 3 \) and those sets where it is not the case that \( j_1 = 1 \) and \( j_2 = 3 \). If \( S = \{j_1 < j_2 < \ldots j_n\} \) where it is not the case that \( j_1 = 1 \) and \( j_2 = 3 \), then we know that \( \text{ERXTZ}_u(S) = \frac{(1 - x)^{k-1}}{(1 - (1 - x)t)^j} \). If \( S \) is of the form \( \{1, 3\} \cup T \) where \( T \subseteq \{4, 5, 6, 7, 8, 9\} \), then

\[
\text{ERXTZ}_u(S) = (1 - x - xt^3)(1 - x)^{|T|} \frac{1}{(1 - (1 - x)t)^{|T|+2}}.
\]

If follows that

\[
\sum_{n=1}^{9} t^n \sum_{S \subseteq [9], |S| = n} \text{ERXTZ}(S) = \sum_{n=1}^{p} \binom{9}{k} \frac{k^k (1 - x)^{k-1}}{(1 - (1 - x)t)^k} - \sum_{j=0}^{6} \binom{6}{j} \frac{t^{j+2}(1 - x)^{j+1}}{(1 - (1 - x)t)^{j+2}} + \sum_{j=0}^{6} \binom{6}{j} \frac{t^{j+2}(1 - x - xt^3)(1 - x)^j}{(1 - (1 - x)t)^{j+2}}
\]

\[
= \sum_{n=1}^{p} \binom{9}{k} \frac{k^k (1 - x)^{k-1}}{(1 - (1 - x)t)^k} - \sum_{j=0}^{6} \binom{6}{j} \frac{t^{j+2}(1 - x)^j}{(1 - (1 - x)t)^{j+2}}.
\]
Thus if we let
\[ A_{12433,9}(x, t) = 1 - x \left( \sum_{k=1}^{9} \binom{9}{k} \frac{t^k (1 - x)^k - 1}{(1 - (1 - x)t)^k} - \sum_{j=0}^{6} \binom{6}{j} \frac{t^{j+2} (x^3)(1 - x)^j}{(1 - (1 - x)t)^{j+2}} \right), \]
then
\[ \mathcal{E} N_{12433}(x, 9, t) \big|_{z_i = 1} = \frac{1}{A_{12433,9}(x, t)}. \]  

We have used (69) to compute the first few terms in the series of \( \mathcal{E} N_{12433}(x, 9, t) \big|_{z_i = 1}. \)

\[
\begin{align*}
\mathcal{E} N_{12433}(x, 9, t) \big|_{z_i = 1} &= 1 + 9tx + t^2 (45x + 36x^2) + t^3 (165x + 480x^2 + 84x^3) + \\
&+ t^4 (495x + 3510x^2 + 2430x^3 + 126x^4) + \\
&+ t^5 (1287x + 18612x^2 + 31212x^3 + 7812x^4 + 126x^5) + \\
&+ t^6 (3003x + 79925x^2 + 262626x^3 + 167826x^4 + 17976x^5 + 84x^6) + \\
&+ t^7 (6435x + 294616x^2 + 1683836x^3 + 2132496x^4 + 634446x^5 + 31536x^6 + 36x^7) + \\
&+ t^8 (12870x + 965709x^2 + 885187x^3 + 19458252x^4 + 11854197x^5 + \\
&+ 1826577x^6 + 43677x^7 + 9x^8) + \\
&+ t^9 (24310x + 2881330x^2 + 40454572x^3 + 140542120x^4 + 149803150x^5 + \\
&+ 49462810x^6 + 4200670x^7 + 48610^8 + x^9) + \cdots.
\end{align*}
\]

We end this section with a remark about the case where \( u = u_1 \ldots u_j, \) \( \text{des}(u) = 1, \) \( u_1 < u_j, \)
and \( u \) does not have the weakly increasing overlapping property. There are two problems in this case. First, as we saw earlier, it is possible that the end points of collapse \( u \)-match in a fixed \((B, w)\) point of \( I_u \) can lead to a rise, a level, or a descent in \( C(B, w, u). \) This means that the weights \( w_{u, P}(ij) \) or \( w_{u, [k]}(ij) \) are much more complicated. The second problem is to find \( U_u^{(P)}(x, z_\infty, t), \) we would need to substitute in a generating function of the form
\[
1 + \sum_{n \geq 1} t^n \sum_{w = w_1 \ldots w_n \in P} \prod_{i=1}^{n-1} x^{w_i w_{i+1}} \]
and we do not know of any way to find a compact form for such a generating function.

6 The case \( u = u_1 \ldots u_j, \) \( \text{des}(u) = 1, \) and \( u_1 = u_j \)

In this section, we shall consider the problem of computing the generating functions
\( \mathcal{N}^{(P)}_u(x, z_\infty, t), \mathcal{N}^{(k)}_u(x, z_k, t), \mathcal{E} N^{(P)}_u(x, z_\infty, t), \) and \( \mathcal{E} N^{(k)}_u(x, z_k, t) \) for \( u = u_1 \ldots u_j \) such that \( \text{des}(u) = 1 \) and \( u_1 = u_j. \)

As in the previous sections, we need to compute \( U_u^{(P)}(x, z_\infty, t), U_u^{(k)}(x, z_k, t), EU_u^{(P)}(x, z_\infty, t), \) and \( EU_u^{(k)}(x, z_k, t). \) To compute these generating functions, we use Theorem 2 or 4 plus the collapse map.

First assume that \( u = u_1 \ldots u_j, \) \( \text{red}(u) = u, \) \( \text{des}(u) = 1, \) \( u_1 = u_j, \) and \( u \) has the \( P- \) minimal overlapping property. We can define the collapse map to fixed points of \( I_u \) or \( J_u \) exactly as in the
previous sections. For example, suppose that \( u = 12311 \) and we want to compute \( U_{12311}^{(7)}(x, z_7, t) \). By (19), we know that

\[
U_{12311}^{(7)}(x, z_7) = \sum_{O \in \mathcal{O}_{12311,n}^{(k)}} \text{sgn}(O) wt(O). \tag{71}
\]

As before, we know that if \((B, w)\) is a fixed point of \( I_{12311} \), then elements in the bricks are weakly increasing and if there is a decrease between two brick \( b_i \) and \( b_{i+1} \), there must be a 12311-match that involves the last 3 cells of \( b_i \) and the first three cells of \( b_{i+1} \). We have pictured such a fixed point in Figure 10.

![Figure 10](image_url)

The difference between this case and the previous case where \( u_i > u_j \) is that a 12311-match of the form \( ijkii \) will just be replaced by \( ii \) so that only factors of the form \( ii \) could have come from a 12311-match in the collapse of a fixed point of \( I_{12311} \). The fact that 12311 has the \( \mathbb{P} \)-minimal overlapping property ensures that any two such 12311-matches can only intersect at the right-hand endpoint of the first match and left-hand endpoint of the second match. It follows that \( C(B, w, u) \) will always be a weakly increasing word. We claim that in this case a factor of the form \( ii \) must have weight \( 1 - x - xt^3 z_i \sum_{i < c < d \leq k} z_c z_d \) if we are computing \( U_{12311,n}^{(k)}(x, z_k) \) and \( 1 - x - xt^3 z_i \sum_{i < c < d} z_c z_d \) if we are computing \( U_{12311,n}^{(7)}(x, z_7) \). That is, the 1 corresponds to the case where \( ii \) are in the same brick, the \( -x \) corresponds to the case where the first \( i \) is in last cell of some brick \( b_j \) and the second \( i \) is in the first cell of the next brick, and the third term corresponds to the cases where we have a decrease between two consecutive bricks and we deleted the second, third, and fourth elements of the 12311-match between the two bricks. In our example, the weight of the levels for computing \( U_{12311,n}^{(7)}(x, z_7) \) would be as follows.

| Levels | \( wt_{12311,7}(ii) \) |
|--------|---------------------|
| 77     | \( 1 - x \)         |
| 66     | \( 1 - x \)         |
| 55     | \( 1 - x - xt^3 z_5 z_6 z_7 \) |
| 44     | \( 1 - x - xt^3 z_4 (\sum_{4 < c < d \leq 7} z_c z_d) \) |
| 33     | \( 1 - x - xt^3 z_3 (\sum_{3 < c < d \leq 7} z_c z_d) \) |
| 22     | \( 1 - x - xt^3 z_2 (\sum_{2 < c < d \leq 7} z_c z_d) \) |
| 11     | \( 1 - x - xt^4 z_1 (\sum_{1 < c < d \leq 7} z_c z_d) \) |

The weights \( wt_{12311,7}(ii) \).
In this case, rises in \( C(B, w, 12311) \) of the form \( ij \) where \( i < j \) correspond to a factor of \( 1 - x \) where the 1 comes from the case where \( ij \) are in the same brick and the \(-x\) corresponds to the case where \( i \) and \( j \) are in different bricks.

It follows that for any \( v \in [7]^+ \) which is weakly increasing,

\[
\sum_{(B, w) \text{ is a fixed point of } I_{12311}} \text{sgn}(B, w) wt_{12311}(B, w) = -x^{\varepsilon(v)}(1-x)^{\text{rise}(v)} \prod_{s \in \text{Lev}(v)} wt_{12311, 7}(v_s v_{s+1}).
\]

and

\[
U_{12311}^{(7)}(x, z_7, t) = 1 + \sum_{n \geq 1} U_{12311, n}^{(7)}(x, z_7) t^n
= 1 + \sum_{v \in [7]^+, \text{des}(v) = 0} -x^{\varepsilon(v)}(1-x)^{\text{rise}(v)} \prod_{s \in \text{Lev}(v)} wt_{12311, 7}(v_s v_{s+1}).
\]  

(72)

Next suppose that \( u = u_1 \ldots u_j, \text{red}(u) = u, \text{des}(u) = 1, u_1 = u_j, \) and \( u \) has the \( \mathbb{P} \)-level overlapping property or the \([k]\)-level overlapping property, but \( u \) does not have the \( \mathbb{P} \)-minimal overlapping property. The fact that \( u \) has the \( \mathbb{P} \)-level overlapping property (\([k]\)-level overlapping property) ensures that if \( w = w_1 \ldots w_n \) is word which starts and ends with a \( u \)-match and any two consecutive \( u \)-matches in \( w \) share at least two letters, then it must be the case that \( w_1 = w_n \). Thus under the collapse map, any collapse will end up with a level of the form \( ii \).

The main difference in this case is that it is possible to have the weights \( wt_{u, k}(ii) \) or \( wt_{u, P}(ii) \) correspond to infinite families of words of different lengths even in the case where the alphabet is finite. For example, suppose that \( u = 11211 \). Then it is possible that in a fixed point \((B, w)\) of \( I_{11211}, w \) has a factor where consecutive occurrences of the pattern 11211 are linked of the form \( iyi_1 iyi_2 iyi_3 iyi \ldots iyi_n ii \) where \( y_1, \ldots, y_n > i \) like those that occur in the first 14 cells of the fixed point pictured in Figure [III]. For each given maximal sequence of this type, the collapse map would eliminate all the symbols between the first and the last \( i \). In such a case, the weight corresponding to the symbols that are eliminated for such a string in the collapse map would be \((1-x)^n z_i^{2n} z_{y_1} \ldots z_{y_n} t^{3n} \). It would follow that if we are working in \( \mathbb{P}^* \), then

\[
wt_{11211, P}(ii) = 1 - x + \frac{-xz_i^2 (\sum_{s > i} z_s) t^3}{1 + xz_i^2 (\sum_{s > i} z_s) t^3}
\]

while if we are working in \([k]^* \), then for \( 1 \leq i < k \),

\[
wt_{11211, k}(ii) = 1 - x + \frac{-xz_i^2 (\sum_{s = i+1}^k z_s) t^3}{1 + xz_i^2 (\sum_{s = i+1}^k z_s) t^3}
\]

and

\[
wt_{11211, k}(kk) = 1 - x.
\]

That is, in each of these expressions the 1 corresponds to the case where both \( is \) are part of the same brick, the \(-x\) corresponds to the case where the two \( is \) are the last and first elements of two consecutive bricks, and the series \( \frac{-xz_i^2 (\sum_{s > i} z_s) t^3}{1 + xz_i^2 (\sum_{s > i} z_s) t^3} \) corresponds the fact that we could have eliminated sequences of the form \( iyi_1 iyi_2 iyi_3 iyi \ldots iyi_n i \) for any \( n \geq 1 \) between the two \( is \).
Figure 11: A fixed point of $I_{11211}$.

Nevertheless, we can still apply the same reasoning as above to prove that for any $v \in [7]^+$ which is weakly increasing,

$$
\sum_{(B,w) \text{ a fixed point of } I_{11211}} sgn(B,w)wt_{11211}(B,w) = -x^{\lceil \frac{v}{|v|} \rfloor} (1 - x)^{\text{rise}(v)} \prod_{s \in \text{Lev}(v)} wt_{11211,7}(v_s v_{s+1}).
$$

and

$$
U^{(7)}_{11211}(x, z_7, t) = 1 + \sum_{n \geq 1} U^{(7)}_{11211,n}(x, z_7) t^n
$$

$$
= 1 + \sum_{v \in [7]^+, \text{des}(v) = 0} -x^{\lceil \frac{v}{|v|} \rfloor} (1 - x)^{\text{rise}(v)} \prod_{s \in \text{Lev}(v)} wt_{11211,7}(v_s v_{s+1}).
$$

(75)

We should note that as patterns get more complicated, it becomes increasingly difficult to compute $wt_{u, P(ii)}$ or $wt_{k(ii)}$. For example, suppose $u = 3^5451235$. Then linked patterns can overlap at either 1, 2, 3, 4, or 5 symbols.

It follows from Theorem 4 that

$$
\sum_{v \in \mathbb{P}^+, \text{des}(v) = 0} x_{v_i v_i} = -1 + \prod_{i \geq 1} \left( 1 - \frac{z_i t}{1 - x_i z_i t} \right).
$$

(76)

Replacing $t$ by $yt$ and $x_{ij}$ by $x_{ii}/y$, we see that

$$
\sum_{v_1, \ldots, v_n \in \mathbb{P}^+, v_1 \leq v_2 \leq \cdots \leq v_n} x_{v_i v_i} = -1 + \prod_{i \geq 1} \left( 1 + \frac{yz_i t}{1 - x_i z_i t} \right).
$$

(77)

Thus

$$
1 + \sum_{v_1, \ldots, v_n \in \mathbb{P}^+, v_1 \leq v_2 \leq \cdots \leq v_n} -x^{\lceil \frac{v}{|v|} \rfloor} y^{\text{rise}(v)} \prod_{i \in \text{Lev}(v)} x_{v_i v_i} = 1 + \frac{x}{y} \left( -1 + \prod_{i \geq 1} \left( 1 + \frac{yz_i t}{1 - x_i z_i t} \right) \right).
$$

(78)

and

$$
1 + \sum_{v_1, \ldots, v_n \in [k]^+, v_1 \leq v_2 \leq \cdots \leq v_n} -x^{\lceil \frac{v}{|v|} \rfloor} y^{\text{rise}(v)} \prod_{i \in \text{Lev}(v)} x_{v_i v_i} = 1 + \frac{x}{y} \left( -1 + \prod_{i=1}^k \left( 1 + \frac{yz_i t}{1 - x_i z_i t} \right) \right).
$$

(79)
But then it follows that if \( u = u_1 \ldots u_j \), \( \text{red}(u) = u \), \( \text{des}(u) = 1 \), \( u_1 = u_j \), and \( u \) has the \( \mathbb{P} \)-level overlapping property, then
\[
U_{u}^{(\mathbb{P})}(x, z_{\infty}, t) = 1 + \sum_{v \in \mathbb{P}^+, \text{des}(v) = 0} -x^{|v|}(1 - x)^{\text{rise}(v)} \prod_{i \in \text{lev}(v)} \text{wt}_{u, \mathbb{P}}(v_i v_i)
\]
\[
= 1 + \frac{-x}{1 - x} \left( -1 + \prod_{i \geq 1} \left( 1 + \frac{(1 - x)z_i t}{1 - \text{wt}_{u, \mathbb{P}}(ii) z_i t} \right) \right)
\]
and, for all \( k \geq 1 \), if \( u = u_1 \ldots u_j \), \( \text{red}(u) = u \), \( \text{des}(u) = 1 \), \( u_1 = u_j \), and \( u \) has the \([k]\)-level overlapping property, then
\[
U_{u}^{(k)}(x, z_{k}, t) = 1 + \sum_{v \in [k]^+, \text{des}(v) = 0} -x^{|v|}(1 - x)^{\text{rise}(v)} \prod_{i \in \text{lev}(v)} \text{wt}_{u, k}(v_i v_i)
\]
\[
= 1 + \frac{-x}{1 - x} \left( -1 + \sum_{i=1}^{k} \left( 1 + \frac{(1 - x)z_i t}{1 - \text{wt}_{u, k}(ii) z_i t} \right) \right)
\]

Thus we have the following theorem.

**Theorem 14.** If \( u = u_1 \ldots u_j \in \mathbb{P}^+ \) is such that \( \text{red}(u) = u \), \( \text{des}(u) = 1 \), \( u_1 = u_j \), and \( u \) has the \( \mathbb{P} \)-level overlapping property, then
\[
N_{u}^{(\mathbb{P})}(x, z_{\infty}, t) = \frac{1}{1 - \frac{x}{1 - x} \left( -1 + \prod_{i \geq 1} \left( 1 + \frac{(1 - x)z_i t}{1 - \text{wt}_{u, \mathbb{P}}(ii) z_i t} \right) \right)}.
\]  
(80)

If \( u = u_1 \ldots u_j \in [k]^+ \) is such that \( \text{red}(u) = u \), \( \text{des}(u) = 1 \), \( u_1 = u_j \), and \( u \) has the \([k]\)-level overlapping property, then
\[
N_{u}^{(k)}(x, z_{k}, t) = \frac{1}{1 - \frac{x}{1 - x} \left( -1 + \sum_{i=1}^{k} \left( 1 + \frac{(1 - x)z_i t}{1 - \text{wt}_{u, k}(ii) z_i t} \right) \right)}
\]  
(81)

Note that if \( u = u_1 \ldots u_j \), \( \text{des}(u) = 1 \), \( u_1 = u_j \), then \( u \) automatically has the exact \( \mathbb{P} \)-level overlapping property (exact \([k]\)-level overlapping property).

**Theorem 15.** If \( u = u_1 \ldots u_j \in \mathbb{P}^+ \) is such that \( \text{des}(u) = 1 \) and \( u_1 = u_j \), then
\[
\mathcal{E}N_{u}^{(\mathbb{P})}(x, z_{\infty}, t) = \frac{1}{1 - \frac{x}{1 - x} \left( -1 + \prod_{i \geq 1} \left( 1 + \frac{(1 - x)z_i t}{1 - \text{wt}_{u, \mathbb{P}}(ii) z_i t} \right) \right)}.
\]  
(82)

and if \( u = u_1 \ldots u_j \in [k]^+ \) is such that \( \text{des}(u) = 1 \) and \( u_1 = u_j \), then
\[
\mathcal{E}N_{u}^{(k)}(x, z_{k}, t) = \frac{1}{1 - \frac{x}{1 - x} \left( -1 + \sum_{i=1}^{k} \left( 1 + \frac{(1 - x)z_i t}{1 - \text{wt}_{u, k}(ii) z_i t} \right) \right)}
\]  
(83)

For example, suppose we want to compute \( N_{123411}^{(7)}(x, z_7, t) \) where we set \( z_i = 1 \) for all \( i \). It follows from (81) that
\[
N_{123411}^{(7)}(x, z_7, t) = \frac{1}{1 - \frac{x}{(1 - x)} \left( -1 + \prod_{i=1}^{7} Q_i(x, t) \right)}
\]  
(84)

where
1. \( Q_1(x, t) = 1 + \frac{(1-x)t}{1-(1-x-15xt^3)t} \),

2. \( Q_2(x, t) = 1 + \frac{(1-x)t}{1-(1-x-10xt^2)t} \),

3. \( Q_3(x, t) = 1 + \frac{(1-x)t}{1-(1-x-6xt)t} \),

4. \( Q_4(x, t) = 1 + \frac{(1-x)t}{1-(1-x-3xt^3)t} \),

5. \( Q_5(x, t) = 1 + \frac{(1-x)t}{1-(1-x-2xt^2)t} \),

6. \( Q_6(x, t) = 1 + \frac{(1-x)t}{1-(1-x)t} \), and

7. \( Q_7(x, t) = 1 + \frac{(1-x)t}{1-(1-x)t} \).

We have computed that

\[
\mathcal{N}_{12311}^{(7)}(x, z; t) = \\
1 + 7xt + 7(4x + 3x^2)t^2 + 7(12x + 32x^2 + 5x^3)t^3 + \\
7(30x + 190x^2 + 118x^3 + 5x^4)t^4 + 7(66x + 823x^2 + 1236x^3 + 268x^4 + 3x^5)t^5 + \\
7(132x + 2912x^2 + 8500x^3 + 4770x^4 + 422x^5 + x^6)t^6 + \\
(1716x + 62532x^2 + 312558x^3 + 349315x^4 + 88852x^5 + 3424x^6 + x^7)t^7 + \\
7(429x + 24609x^2 + 194029x^3 + 374249x^4 + 197729x^5 + 25209x^6 + 429x^7)t^8 + \cdots .
\]

Finally we shall consider the case where \( u = u_1 \ldots u_j \), \( \text{red}(u) = u \), \( \text{des}(u) = 1 \), \( u_1 = u_j \) and \( u \) does not have the \( P \)-level overlapping property ([\( k \])-level overlapping property). Given such a \( u \), let \( s \) be the position such that \( u_s > u_{s+1} \). Then we must have that \( u_{s+1} \leq \cdots \leq u_j = u_1 \) and \( St^{(P)}(u) \subset \{ s+1, \ldots , j \} \) (\( St^{([k])}(u) \subset \{ s+1, \ldots , j \} \)). This means that \( u \) automatically has the \( P \)-weakly decreasing overlapping property ([\( k \]-weakly decreasing overlapping property) and \( u \) is not \( P \)-minimal overlapping ([\( k \]-minimal overlapping). Now suppose that \( w = w_1 \ldots w_n \) is a maximal sequence of linked \( u \)-matches. That is, we assume \( w \) starts and ends with a \( u \)-match and any two consecutive \( u \)-matches share at least two letters. Then if the \( u \)-matches in \( w \) start at positions \( 1 = i_1 < i_2 < \cdots < i_k \), then the \( P \)-weakly decreasing overlapping property ensures that \( w_1 = w_{i_1} \geq \cdots \geq w_{i_k} = w_n \). Thus in a collapse map, if we eliminate \( w_2 \ldots w_{n-1} \), then we will be left with a weak descent \( w_1 w_n \). Thus we must figure out the weights \( wt_u(ji) \) for \( j \geq i \).

To illustrate the process, we will consider the example where \( u = 2312 \) and the alphabet is [4]. If \( w = w_1 w_2 w_3 w_4 \in [4]^* \) and \( \text{red}(w) = 2312 \), then clearly \( w \) must start with either 2 or 3 since those are the only letters \( a \) which have at least one letter in [4] bigger than \( a \) and one letter in [4] which is less than \( a \). It follows that \( wt_{2312,4}(44) = wt_{2312,4}(11) = 1 - x \). Also \( wt_{2312,4}(4i) = 0 \) for \( i = 1, 2, 3 \) and \( wt_{2312,4}(j1) = 0 \) for \( j = 2, 3, 4 \).

Next consider \( wt_{2312,4}(22) \). There are only two possible words in [4]^4 that reduce to \( u \), namely, \( w = 2312 \) and \( v = 2412 \). Since there is no \( u \)-match that can start with 1, there cannot be a pair of linked \( u \)-matches that start with either \( w \) or \( v \). Thus there can be no maximal sequences of linked \( u \)-matches that start and end with 2. This means that when we collapsed to 22, either we started with 2312 and eliminated 31 or we started with 2412 and we eliminated 41. It follows that \( wt_{2312,4}(22) = 1 - x - x z_1(z_3 + z_4)t^2 \).
Next consider \( wt_{2312,4}(33) \). There are only two possible words in \([4]^4\) that reduce to \( u \), namely, \( w = 3413 \) and \( v = 3423 \). Since there is no \( u \)-match that can start with 1, there cannot be a pair of linked \( u \)-matches that start with \( w \). There is a pair of linked \( u \)-matches that start with \( v \), namely, 342312. However this pair can not be extended. Thus there can be no maximal sequences of linked \( u \)-matches that start and end with 3. This means that when we collapsed to 33 either we started with 3413 and eliminated 41 or we started with 3423 and eliminated 42. It follows that \( wt_{2312,4}(33) = 1 - x - xz_4(z_1 + z_2)t^2 \).

Finally we consider \( wt_{2312,4}(32) \). In this case, the only possible way to have a maximal sequence \( w \) of linked \( u \)-matches starting with a \( u \)-match whose first letter is 3 and ending with a \( u \)-match whose last letter is 2 is \( w = 342312 \). Since in the fixed points of \( I_{2312} \), the sequences in the bricks are weakly increasing, the only way that 32 occurs in the collapse of fixed point \((B, w)\) of \( I_{2312} \) is if we started with 342312, which means that a brick ended after 4 and a brick ended after the second 3, and eliminated 4231. Hence \( wt_{2312,4}(32) = x^2z_1z_2z_3z_4t^4 \).

Thus we have the following table for \( wt_{2312,4}(32) \):

| Weak Descents | \( wt_{2312,4}(ji) \) |
|---------------|----------------------|
| 44            | \( 1 - x \)          |
| 4i (i < 4)    | 0                    |
| 33            | \( 1 - x - xz_4(z_1 + z_2)t^2 \) |
| 32            | \( x^2z_1z_2z_3z_4t^4 \) |
| 31            | 0                    |
| 22            | \( 1 - x - xz_1(z_3 + z_4)t^2 \) |
| 21            | 0                    |
| 11            | \( 1 - x \)          |

The weights \( wt_{2312,4}(ji) \).

It follows that for any \( v \in [4]^+ \),

\[
\sum_{(B, w) \text{ is a fixed point of } I_{2312} \atop C(B, w; 2312) = v} sgn(B, w) wt(B, w) = -x^{\overline{v}}t^{\overline{v}}(1 - x)^{\text{rise}(v)} \prod_{s \in WDes(v)} wt_{2312,4}(v_sv_{s+1}).
\]  

(85)

Here the initial \(-x\) comes from the fact that the last cell of \((B, w)\) always contributes a \(-x\) since the last cell is at the end of a brick. It follows that

\[
\begin{align*}
U_{2312}^{(4)}(x, z_4, t) &= 1 + \sum_{n \geq 1} U_{2312,n}^{(4)}(x, z_4)t^n \\
&= 1 + \sum_{v \in [4]^+} -x(1 - x)^{\text{rise}(v)} x^{\overline{v}}t^{\overline{v}} \prod_{s \in WDes(v)} wt_{2312,4}(v_sv_{s+1}).
\end{align*}
\]

(86)

Hence we could compute \( \Lambda_{2312}^{(4)}(x, z_4, t) = \frac{1}{U_{2312}^{(4)}(x, z_4, t)} \) if we can compute the right-hand side of (86).

What we need to be able to compute the right-hand side of (86) is the generating function over all words \( v \in \mathbb{P}^* \) where we not only keep track of the weak descents of \( P \) but also of type of weak descents of \( P \).

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By Theorem 2, we know that
\[
1 - \sum_{n \geq 1} t^n \sum_{v \in WD^*} t^n \frac{1}{WDXZ(v)} = 1 + \sum_{w = w_1 \ldots w_n \in P^+} \frac{1}{WDXZ(v)} \prod_{i \in WD^*(w)} x_{w_i w_{i+1}}. \tag{87}
\]
Hence
\[
\sum_{w = w_1 \ldots w_n} t^n \frac{1}{z^n} \prod_{i \in WD^*(w)} x_{w_i w_{i+1}} = \left(1 - \sum_{n \geq 1} t^n \sum_{v \in WD^*} \frac{1}{WDXZ(v)} \right) - 1.
\]
Next suppose that we replace \( t \) by \( yt \) and \( x_{ij} \) by \( \frac{x_{ij}}{y} \). Under this substitution the left-hand side in (88) becomes
\[
\sum_{w = w_1 \ldots w_n} t^n \frac{1}{z^n} \prod_{i \in WD^*(w)} x_{w_i w_{i+1}}.
\]
Note that for \( v = j_1 \geq \cdots \geq j_k \) where \( k \geq 2 \), our substitution replaces \( t^k WDXZ(v) \) by
\[
y^k t^k z_j \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_{i+1} j_i} - y) = yt^{k} z_j \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_{i+1} j_i} - y).
\]
Thus if we let
\[
WDXZ(v) = \begin{cases} 
\frac{z_j}{y} & \text{if } v = j, \\
\frac{z_j \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_{i+1} j_i} - y)}{y} & \text{if } v = j_1 \geq \cdots \geq j_k \text{ where } k \geq 2,
\end{cases} \tag{89}
\]
then we see that the right-hand side of (88) becomes
\[
\frac{y \sum_{n \geq 1} t^n \sum_{v \in WD^*} \frac{1}{WDXZ(v)} = 1 - y \sum_{n \geq 1} t^n \sum_{v \in WD^*} \frac{1}{WDXZ(v)} WDXZ(v)}{1 - y \sum_{n \geq 1} t^n \sum_{v \in WD^*} \frac{1}{WDXZ(v)} WDXZ(v)}.
\]
It follows that
\[
-x \sum_{w = w_1 \ldots w_n} t^n \frac{1}{z^n} \prod_{i \in WD^*(w)} x_{w_i w_{i+1}} = \frac{-x \sum_{n \geq 1} t^n \sum_{v \in WD^*} \frac{1}{WDXZ(v)} WDXZ(v)}{1 - y \sum_{n \geq 1} t^n \sum_{v \in WD^*} \frac{1}{WDXZ(v)} WDXZ(v)}.
\]
Thus
\[
1 - x \sum_{w = w_1 \ldots w_n} t^n \frac{1}{z^n} \prod_{i \in WD^*(w)} x_{w_i w_{i+1}} = \frac{1 - (x + y) \sum_{n \geq 1} t^n \sum_{v \in WD^*} \frac{1}{WDXZ(v)} WDXZ(v)}{1 - y \sum_{n \geq 1} t^n \sum_{v \in WD^*} \frac{1}{WDXZ(v)} WDXZ(v)}. \tag{90}
\]
By setting $z_i = 0$ for $i > k$, we also obtain that

$$1 - x \sum_{w=w_1 \ldots w_n \in [k]^+} t^{\mid u \mid} y^{\text{rise}(u)} x_{w_1 w_{i+1}} \prod_{i \in W \text{Des}(u)} x_{w_i} = \frac{1 - (x + y) \sum_{n=1}^{k} t^{n} \sum_{v \in WD[k]^*, |v| = n} WDXYZ(v)}{1 - y \sum_{n=1}^{k} t^{n} \sum_{v \in WD[k]^*, |v| = n} WDXYZ(v)}.$$ \hspace{1cm} (91)

Note that if we replace $y$ by $(1 - x)$ and $x$ by $wt_u(ji)$, the left-hand side of (90) becomes $U_u^{(P)}(x, z, t)$ and the left-hand side of (91) becomes $U_u^{(k)}(x, z, t)$. Then using the fact that $N_u^{(P)}(x, z, t) = 1/U_u^{(P)}(x, z, t)$, we have the following theorem.

**Theorem 16.** Suppose that $u = u_1 \ldots u_j \in \mathbb{P}^*$, $\text{red}(u) = u$, $\text{des}(u) = 1$, $u_1 = u_j$, and $u$ does not have the $\mathbb{P}$-level overlapping property (so it automatically has the $\mathbb{P}$-weakly decreasing property). Then

$$N_u^{(P)}(x, z, t) = \frac{1 - (1 - x) \sum_{n=1}^{k} t^{n} \sum_{v \in WD^{P*}, |v| = n} WDXZ_u(v)}{1 - \sum_{n=1}^{k} t^{n} \sum_{v \in WD^{P*}, |v| = n} WDXZ_u(v)}$$ \hspace{1cm} (92)

where

$$WDXZ_u(v) = \begin{cases} z_j & \text{if } v = j, \text{ and} \\ z_1 \cdots z_k \prod_{i=1}^{k-1} (wt_u(j_i + 1 j_i) + x - 1) & \text{if } v = j_1 \geq \cdots \geq j_k \text{ where } k \geq 2. \end{cases} \hspace{1cm} (93)$$

If set $z_i = 0$ for all $i > k$, then we obtain the following theorem.

**Theorem 17.** Now suppose that $u = u_1 \ldots u_j \in [k]^*$, $\text{red}(u) = u$, $\text{des}(u) = 1$, $u_1 = u_j$, and $u$ does not have the $[k]$-level overlapping property (so it automatically has the $[k]$-weakly decreasing property). Then

$$N_u^{(k)}(x, z, t) = \frac{1 - (1 - x) \sum_{n=1}^{k} t^{n} \sum_{v \in WD[k]^*, |v| = n} WDXZ_u(v)}{1 - \sum_{n=1}^{k} t^{n} \sum_{v \in WD[k]^*, |v| = n} WDXZ_u(v)}.$$ \hspace{1cm} (94)

The key to be able to compute $N_u^{(k)}(x, z, t)$ in the case of Theorem 17 is to be able to compute $\sum_{n=1}^{k} t^{n} \sum_{v \in WD[k]^*, |v| = n} WDXZ_u(v)$, which is often complicated because of the large number of weakly decreasing words in $WD[k]^*$, but for certain patterns we can compute it. For example, consider the case where $u = 2312$, $k = 4$, and we set $z_i = 1$ for $i = 1, \ldots, 4$. With this substitution, the table of $WDXZ_{2312}(ij)$ becomes

| Weak Descents | $WDXZ_{2312}(ij)$ |
|---------------|------------------|
| 44            | 0                |
| 4i (i < 4)    | $x - 1$          |
| 33            | $-2xt^2$         |
| 32            | $x - 1 + x^2t^4$ |
| 31            | $x - 1$          |
| 22            | $-2xt^2$         |
| 21            | $x - 1$          |
| 11            | 0                |

The weights $WDXZ_{2312}(ij)$ in the case $z_i = 0$ for $i = 1, \ldots, 4$. \hspace{1cm} 40
Because $W D X T Z_{2312}(44) = W D X T Z_{2312}(11) = 0$, it follows that the only words that we have to consider are 1, 4, 41 and words in $(\epsilon + 4)(\{2\}^+ + \{3\}^+ + \{3\}^*32\{2\}^*)(\epsilon + 1)$. It is easy to see that

$$\sum_{n \geq 1} t^n \sum_{v \in \{3\}^+, |v| = n} W D X T Z_n(v) = \frac{t}{1 + 2xt^3}.$$ 

That is, the first 3 gives a factor of $t$ and each additional 3 gives a factor of $-2xt^3$. Similarly,

$$\sum_{n \geq 1} t^n \sum_{v \in \{2\}^+, |v| = n} W D X T Z_n(v) = \frac{t}{1 + 2xt^3}.$$ 

When considering words in $\{3\}^*32\{2\}^*$, the 32 gives a factor of $(x-1)t^2 + x^2t^6$ and each additional 3 to the left gives a factor of $-2xt^3$ and each additional 2 to the right gives a factor of $-2xt^3$. Thus

$$\sum_{n \geq 1} t^n \sum_{v \in \{3\}^*32\{2\}^*, |v| = n} W D X T Z_n(v) = \frac{(x-1)t^2 + x^2t^6}{(1 + 2xt^3)^2}.$$ 

Thus

$$\sum_{n \geq 1} t^n \sum_{v \in \{3\}^+ + \{2\}^+ + \{3\}^*32\{2\}^*, |v| = n} W D X T Z_n(v) = \frac{2t + 4xt^4 + (x-1)t^2 + x^2t^6}{(1 + 2xt^3)^2}.$$ 

Hence if $E = (\epsilon + 4)(\{3\}^+ + \{2\}^+ + \{3\}^*32\{2\}^*)(\epsilon + 1)$, it follows that

$$\sum_{n \geq 1} t^n \sum_{v \in E, |v| = n} W D X T Z_{2312}(v) = \frac{(1 + (x-1)t)^2(2t + 4xt^4 + (x-1)t^2 + x^2t^6)}{(1 + 2xt^3)^2}$$

since adding a 4 to the left of a word $w \in \{3\}^+ + \{2\}^+ + \{3\}^*32\{2\}^*$ gives rise to a factor of $(x-1)t$ and adding a 1 to the right of a word $w \in \{3\}^+ + \{2\}^+ + \{3\}^*32\{2\}^*$ gives rise to a factor of $(x-1)t$. It follows that

$$\sum_{n \geq 1} t^n \sum_{v \in W D|d|^*, |v| = n} W D X T Z_{2312}(v) = 2t + (x-1)t + \frac{(1 + (x-1)t)^2(2t + 4xt^4 + (x-1)t^2 + x^2t^6)}{(1 + 2xt^3)^2} = \frac{P(x,t)}{(1 + 2xt^3)^2}.$$ 

where

$$P(x,t) = 4t + (-6 + 6x)t^2 + (4 - 8x + 4x^2) t^3 + (-1 + 15x - 3x^2 + x^3) t^4 + (-12x + 12x^2) t^5 + (4x - 7x^2 + 4x^3) t^6 + (6x^2 + 2x^3) t^7 + (-3x^2 + 2x^3 + x^4) t^8.$$ 

Thus

$$N_{2312}^{(4)}(x, 1, 1, 1, 1, t) = \frac{1 - (x-1)\frac{P(x,t)}{(1 + 2xt^3)^2}}{1 - \frac{P(x,t)}{(1 + 2xt^3)^2}}. \quad (95)$$
We have used Mathematica to compute the first few terms in this series:

\[ 1 + 4xt + 2 \left(5x + 3x^2\right) t^2 + 4 \left(5x + 10x^2 + x^3\right) t^3 + \left(35x + 151x^2 + 65x^3 + x^4\right) t^4 + \]
\[ 4 \left(14x + 109x^2 + 111x^3 + 14x^4\right) t^5 + \left(84x + 1068x^2 + 2009x^3 + 716x^4 + 28x^5\right) t^6 + \]
\[ 2 \left(60x + 1166x^2 + 3561x^3 + 2535x^4 + 362x^5 + 4x^6\right) t^7 + \]
\[ (165x + 4670x^2 + 21400x^3 + 25650x^4 + 8172x^5 + 486x^6 + x^7) t^8 + \cdots. \]

7 The proofs of Theorems 1, 2, 3, and 4

In this section, we shall prove Theorems 1, 2, 3, and 4.

We start with the proof of Theorem 1.

Next we want to give a combinatorial interpretation to (99). First we pick a brick tabloid
\[ B, w, L \]
that contains all triples \( (a_1, \ldots, a_{\ell(\mu)}) \), such that \( S_j \) has size \( b_j \) and placing the elements of \( S_j \) in the cells of \( b_j \) in decreasing order for \( j = 1, \ldots, \ell(\mu) \). If \( S_j = \{a_1 > \cdots > a_{b_j}\} \), then we interpret the factor \( DXZ(S_j) = z_{a_1} \cdots z_{a_{b_j}} \prod_{i=1}^{b_j-1} \left( x_{a_i, a_{i+1}} - 1 \right) \) as the ways of labeling the cells of \( b_j \) that contain \( a_i \) where \( i < b_j \) with either \( z_{a_i}x_{a_i, a_{i+1}} \) or with \(-z_{a_i}\) and labeling the last of cell \( b_j \) with \( z_{a_{b_j}} \). We shall call all such objects created in this way filled labeled brick tabloids and let \( H_n \) denote the set of all filled labeled brick tabloids that arise in this way. Thus \( H_n \) consists of all triples \((B, w, L)\) such that
1. $B = (b_1, \ldots, b_k)$ is a brick tabloid of length $n$,
2. $w = w_1 \ldots w_n$ is a word in $\mathbb{P}^n$ such that $w$ is strictly decreasing in each brick, and
3. $L$ is a labeling of the cells of $B$ such that $L(i)$ is equal to $z_a$ if $i$ is the last cell of some brick $b_j$ which contains $a$ and $L(i) = -z_a$ or $L(i) = x_{ab}z_a$ if $i$ is not the last cell of a brick, cell $i$ contains $a$ and cell $i + 1$ contains $b$.

We then define the weight of $(B, w, L)$, $wt(B, w, L)$, to be the product of all the $x_{ab}$ and $z_a$ labels in $L$ and the sign of $(B, w, L)$, $\text{sgn}(B, w, L)$, to be the product of all the $-1$ factors in the labels in $L$. This process is illustrated in Figure 12 to construct an element $(B, w, L)$ of $H_{12}$ such that

\[
wt(B, w, L) = \frac{2}{2} \frac{2}{1} \frac{2}{1} \frac{2}{1} \frac{2}{1} \frac{2}{1} \frac{2}{1} \frac{2}{1} \frac{2}{1} \frac{2}{1} \frac{2}{1} \frac{2}{1}
\]

and $\text{sgn}(B, w, L) = -1$.

Thus

\[
\Gamma(h_n) = \sum_{(B, w, L) \in H_n} \text{sgn}(B, w, L)wt(B, w, L).
\] (100)

| $S_1 = \{1, 4, 5\}$ | $S_2 = \{2, 3, 4, 6\}$ | $S_3 = \{1, 2\}$ | $S_4 = \{1, 3, 6\}$ |
|---|---|---|---|
| $x_1 x_2 x_3 x_4 x_5$ | $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$ | $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9$ | $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10}$ |

Figure 12: A element $(B, w, L) \in H_{12}$.

Next we define a weight-preserving sign-reversing involution $I : H_n \to H_n$. To define $I(C)$, we scan the cells of $C = (B, w, L)$ from left to right looking for the leftmost cell, $t$, such that either (i) $t$ is labeled with $-z_{wt}$ or (ii) $t$ is at the end of a brick, $b_j$, there is a brick $b_{j+1}$ immediately following $b_j$, and $w_t > w_{t+1}$. In case (i), $I(C) = (B, w', L')$ where $B'$ is the result of replacing the brick $b$ in $B$ containing $t$ by two bricks $b^*$ and $b^{**}$, where $b^* \text{ contains all the cells of } b \text{ weakly to the left of cell } t \text{ and } b^{**} \text{ contains all the cells of } b \text{ strictly to the right of cell } t$, $w' = w$, and $L'$ is the labeling that results from $L$ by changing the label of cell $t$ from $-z_{wt}$ to $z_{wt}$. In case (ii), $I(C) = (B', w', L')$ where $B$ is the result of replacing the bricks $b_j$ and $b_{j+1}$ in $B$ by a single brick $b$, $w' = w$, and $L'$ is the labeling that results from $L$ by changing the label of cell $t$ from $z_{wt}$ to $-z_{wt}$. If neither case (i) or case (ii) applies, then we let $I(C) = C$. For example, if $C$ is the element of $H_{12}$ pictured in Figure 12, then $I(C)$ is pictured in Figure 13.

| $S_1 = \{1, 4, 5\}$ | $S_2 = \{2, 3, 4, 6\}$ | $S_3 = \{1, 2\}$ | $S_4 = \{1, 3, 6\}$ |
|---|---|---|---|
| $x_1 x_2 x_3 x_4 x_5$ | $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8$ | $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9$ | $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10}$ |

Figure 13: $I(C)$ for $C$ in Figure 12.

It is easy to see that $I^2(C) = C$ for all $C \in H_n$ and that if $I(C) \neq C$, then $\text{sgn}(C)w(C) = -\text{sgn}(I(C))w(I(C))$. Hence $I$ is a weight-preserving and sign-reversing involution that shows

\[
\Gamma(h_n) = \sum_{C \in H_n, I(C) = C} \text{sgn}(C)w(C).
\] (101)

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Thus, we must examine the fixed points, \( C = (B, w, L) \), of \( I \). First, there can be no \( -z_a \) labels in \( L \) so that \( \text{sgn}(C) = 1 \). Moreover, if \( b_j \) and \( b_{j+1} \) are two consecutive bricks in \( B \) and \( t \) is the last cell of \( b_j \), then it cannot be the case that \( w_t > w_{t+1} \) since otherwise we could combine \( b_j \) and \( b_{j+1} \). Thus for each cell \( t \) such that \( w_t > w_{t+1} \), it must be the case that cells \( t \) and \( t+1 \) lie in the same brick and, hence, cell \( t \) is labeled with \( z_{w_t}x_{w_t,t+1} \).

It follows that \( \text{sgn}(C)w(C) = z^w \prod_{1 \leq i < j} x^{ji(w)} \). For example, Figure 14 shows a fixed point of \( I \) in \( H_{12} \).

Vice versa, if \( w \in \mathbb{P}^n \), then we can create a fixed point, \( C = (B, w, L) \), by having the bricks of \( B \) end at cells \( t \) such that either \( w_t \leq w_{t+1} \) or \( t = n \), labeling each cell \( t \) such that \( w_t > w_{t+1} \) with \( z_{w_t}x_{w_t,w_{t+1}} \) and labeling the remaining cells \( t \) with \( w_t \). Thus we have shown that

\[
\Gamma(h_n) = \sum_{w \in \mathbb{P}^n} z^w \prod_{i<j} x^{ji(w)}.
\]

as desired.

Applying \( \Gamma \) to the identity \( H(t) = \frac{1}{1-E(-t)} \), we get

\[
\sum_{n \geq 0} \Gamma(h_n)t^n = 1 + \sum_{n \geq 1} t^n \sum_{w \in \mathbb{P}^n} z^w \prod_{i<j} x^{ji(w)} = 1 + \sum_{n \geq 1} (-t)^n \Gamma(e_n) = \frac{1}{1 + \sum_{n \geq 1} (-1)^n t^n \sum_{S \subseteq \mathbb{P}, |S| = n} DXZ(S)}
\]

which proves (9).

**Proof of Theorem 2**

One can easily modify the proof of Theorem 1 to prove Theorem 2. Recalling that given a weakly decreasing word \( w \) from \( \mathbb{P}^* \), we let

\[
WDXZ(w) = \begin{cases} 
  z_j & \text{if } w = j, \\
  z_{j_1} \cdots z_{j_k} \prod_{i=1}^{k-1} (x_{j_i,j_i+1} - 1) & \text{if } w = j_1 \geq \cdots \geq j_k \text{ where } k \geq 2.
\end{cases}
\]

(102)

Define a ring homomorphism \( \Gamma_w : \Lambda \to \mathbb{Q}[x, z] \) by defining \( \Gamma(e_0) = 1 \) and, for \( n \geq 1, \)

\[
\Gamma_w(e_n) = (-1)^{n-1} \sum_{w \in \mathbb{P}^*, |w| = n} WDXZ(w).
\]

(103)
Then we claim that
\[
\Gamma_w(h_n) = \sum_{w \in \mathbb{P}^n} \varpi^n \prod_{i \leq j} x_{j_i}^{\mathbb{H}(w)}.
\]
That is,
\[
\Gamma_w(h_n) = \sum_{\mu \vdash n} (-1)^{-\ell(\mu)} B_{\mu,(n)} \Gamma_w(e_\mu)
\]

\[
= \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,(n)}} \prod_{j=1}^{\ell(\mu)} (-1)^{b_j-1} \sum_{w_j \subseteq \mathbb{D} \cap \mathbb{P}^*, |w_j|=b_j} \mathbb{DZ}(w_j)
\]

\[
= \sum_{\mu \vdash n} \sum_{(b_1, \ldots, b_{\ell(\mu)}) \in B_{\mu,(n)}} \prod_{j=1}^{\ell(\mu)} \sum_{w_j \subseteq \mathbb{D} \cap \mathbb{P}^*, |w_j|=b_j} \mathbb{DZ}(w_j)
\]

Next we want to give a combinatorial interpretation to (105). First we pick a brick tabloid $B = (b_1, \ldots, b_k)$ of length $n$. Then we interpret $\prod_{j=1}^{\ell(\mu)} \sum_{w_j \subseteq \mathbb{D} \cap \mathbb{P}^*, |w_j|=b_j} \mathbb{DZ}(w_j)$ as picking a sequent of words in $\mathbb{D} \cap \mathbb{P}^*$, $(w_1, \ldots, w_{\ell(\mu)})$, such that $w_j$ has length $b_j$ and placing the elements of $w_j$ in the cells of $b_j$ in $j = 1, \ldots, \ell(\mu)$. If $w_j = a_1 \geq \cdots \geq a_{b_j}$, then we interpret the factor $\mathbb{DZ}(w_j) = z_{a_1} \cdots z_{a_{b_j}} \prod_{i=1}^{b_j-1} (x_{a_i} - x_{a_{i+1}} - 1)$ as the ways of labeling the cells of $b_j$ that contain $a_i$ where $i < b_j$ with either $z_{a_i} x_{a_{i+1}}$ or with $-z_{a_i}$ and labeling the last cell $b_j$ with $z_{a_{b_j}}$. We shall call all such objects created in this way filled labeled brick tabloids and let $\mathbb{D}H_n$ denote the set of all filled labeled brick tabloids that arise in this way. Thus $\mathbb{D}H_n$ consists of all triples $(B, w, L)$ such that

1. $B = (b_1, \ldots, b_k)$ is a brick tabloid of length $n$,
2. $w = w_1 \cdots w_n$ is a word in $\mathbb{P}^n$ such that $w$ is weakly decreasing in each brick, and
3. $L$ is a labeling of the cells of $B$ such that $L(i)$ is equal to $z_a$ if $i$ is the last cell of some brick $b_j$ which contains a and $L(i) = -z_a$ or $L(i) = x_{ab} z_a$ if $i$ is not the last cell of a brick, cell $i$ contains a and cell $i+1$ contains $b$.

We then define the weight of $(B, w, L)$, $wt(B, w, L)$, to be the product of all the $x_{ab}$ and $z_a$ labels in $L$ and the sign of $(B, w, L)$, $sgn(B, w, L)$, to be the product of all the $-1$ factors in the labels in $L$. This process is illustrated in Figure 15 to construct an element $(B, w, L)$ of $H_{12}$ such that

\[
wt(B, w, L) = z_1 z_2^3 z_3^2 z_4^3 z_5^2 z_{64}^3 x_{64} x_{66} x_{32}
\]

and $sgn(B, w, L) = -1$.

![Figure 15: A element $(B, w, L) \in \mathbb{D}H_{12}$.](image)
At this point, the only difference in the proof is that we are dealing with filled brick tabloids which have weakly decreasing sequences in the bricks rather than strictly decreasing sequences in the bricks. This means that we can modify the involution \( I \) of Theorem 1 by splitting bricks at cells labeled with \(-z_i\) or combining two bricks such that the elements in the two bricks form a weakly decreasing sequence. Then essentially the same proof will show that (104) holds.

**Proof of Theorem 3.**

Given any weakly increasing word \( w = w_1 \ldots w_n \), we let \( S(w) \) denote the set of letters that appear in \( W \). For example, if \( w = 1123555 \), then \( S(w) = \{1, 2, 3, 5\} \). We claim that for any non-empty set \( S = \{j_1 < \cdots < j_k\} \) contained in \( P \),

\[
\sum_{w \in \mathcal{P}^+, S(w) = S} t^{\|w\|} \prod_{i<j} x_{ij}^{w} = t^{|S|} R_{XZ}(S).
\]

That is, if \( S = \{j\} \), then \( w \) must be of the for \( j^k \) for some \( k \geq 0 \) so that in this case

\[
\sum_{w \in \mathcal{P}^+, S(w) = S} t^{\|w\|} \prod_{i<j} x_{ij}^{w} = \frac{z_j^k}{1 - z_j t} = t^{|S|} \frac{z_j}{1 - z_j t} = t^{|S|} R_{XZ}(S).
\]

If \( S(w) = \{j_1 < \cdots < j_k\} \) where \( k \geq 2 \), then \( w \) must be of the form \( w = j_1^{a_1} j_2^{a_2} \cdots j_k^{a_k} \) where \( a_i \geq 1 \) for \( i = 1, \ldots, k \). For any such word, it is easy to see that

\[
\prod_{i<j} x_{ij}^{w} = \prod_{i=1}^{k-1} x_{j_i, j_{i+1}}.
\]

Hence,

\[
\sum_{w \in \mathcal{P}^+, S(w) = S} t^{\|w\|} \prod_{i<j} x_{ij}^{w} = \left( \prod_{i=1}^{k} \frac{z_j t}{1 - z_j t} \right) \prod_{i=1}^{k-1} x_{j_i, j_{i+1}}
= t^{|S|} \left( \prod_{i=1}^{k} \frac{z_j}{1 - z_j t} \right) \prod_{i=1}^{k-1} x_{j_i, j_{i+1}}
= t^{|S|} R_{XZ}(S).
\]

Thus

\[
\mathcal{R}(x_\infty, z_\infty, t) = 1 + \sum_{n \geq 1} t^n \sum_{S \subseteq \mathcal{P}, |S| = n} R_{XZ}(S).
\]

**Proof of Theorem 4.**

Consider a factor \( \left( 1 + \frac{z_i t}{1 - x_{ii} z_i t} \right) \). One can think of the choice of 1 in that factor as not choosing \( i \) to occur in the word where as the factor \( \frac{z_i t}{1 - x_{ii} z_i t} \) corresponds to choosing one of \( i, ii, iii, iiii, \ldots \) in word. Equation (12) easily follows.
8 Possible extensions

The methods that we have used in this paper can be modified to find generating functions of the form

$$\sum_{w \in \mathbb{P}^*, \text{umch}(w) = 0} t^{|w|} x^{\text{lev}(w)} + 1 \frac{1}{z},$$

$$\sum_{w \in [k]^*, \text{umch}(w) = 0} t^{|w|} x^{\text{lev}(w)} + 1 \frac{1}{z},$$

and

$$\sum_{w \in \mathbb{P}^*, \text{emch}(w) = 0} t^{|w|} x^{\text{des}(w)} + 1 \frac{1}{z},$$

$$\sum_{w \in [k]^*, \text{emch}(w) = 0} t^{|w|} x^{\text{des}(w)} + 1 \frac{1}{z},$$

in the case where \(\text{lev}(u) = 1\) and generating functions of the form

$$\sum_{w \in \mathbb{P}^*, \text{umch}(w) = 0} t^{|w|} x^{\text{des}(w)} + 1 \frac{1}{z},$$

$$\sum_{w \in [k]^*, \text{umch}(w) = 0} t^{|w|} x^{\text{des}(w)} + 1 \frac{1}{z},$$

and

$$\sum_{w \in \mathbb{P}^*, \text{emch}(w) = 0} t^{|w|} x^{\text{des}(w)} + 1 \frac{1}{z},$$

$$\sum_{w \in [k]^*, \text{emch}(w) = 0} t^{|w|} x^{\text{des}(w)} + 1 \frac{1}{z},$$

in the case where \(\text{des}(u) = 1\). The idea is that one can modify the reciprocal method presented in Section 3 to replace the statistic \(\text{des}(w) + 1\) by \(\text{lev}(w) + 1\) or \(\text{des}(w) + 1\). Then one can modify the collapse map appropriately. Finally, one needs appropriate modifications of Theorems 1, 2, 3, and 4 to produce generating functions which keep track of labeled rises, levels, or descents that can be specialized to compute the generating functions of interest. For example, in the case where we study the distribution of \(\text{lev}(w) + 1\) over words with no \(u\)-matches, the collapse map only produces words which have no consecutive repeated letters so that we need generating functions which keep track of labeled descents and rises over words which have no 11-match. These modifications will appear in the thesis of the second author.

By the isomorphism which sends a word \(w = w_1 \ldots w_n\) to its reverse, \(w^r = w_n \ldots w_1\), one can automatically produce similar generating functions where the statistics \(\text{des}(w) + 1\) and \(\text{des}(w) + 1\) are replaced by \(\text{rise}(w) + 1\) and \(\text{rise}(w) + 1\), respectively.

One can also easily modify the methods to keep track of restricted sets of descents. For example, given a word \(w = w_1 \ldots w_n \in \mathbb{P}^*\), let \(\text{des}(w) = |\{i : w_i > w_{i+1} \text{ and } w_i \text{ is even}\}|\). Then the techniques of this paper can be easily modified to find closed expressions for

$$\sum_{w \in \mathbb{P}^*, \text{umch}(w) = 0} t^{|w|} x^{\text{edes}(w)} + 1 \frac{1}{z},$$

$$\sum_{w \in [k]^*, \text{umch}(w) = 0} t^{|w|} x^{\text{edes}(w)} + 1 \frac{1}{z},$$

and

$$\sum_{w \in \mathbb{P}^*, \text{emch}(w) = 0} t^{|w|} x^{\text{edes}(w)} + 1 \frac{1}{z},$$

$$\sum_{w \in [k]^*, \text{emch}(w) = 0} t^{|w|} x^{\text{edes}(w)} + 1 \frac{1}{z},$$

in the case where \(\text{edes}(u) = 1\).

Finally, one can extend the reciprocal methods in this paper to give a combinatorial interpretation of \(U_{u,n}^{(P)}(x, z_{\infty}, t), U_{u,n}^{(k)}(x, z_k, t), EU_{u,n}^{(P)}(x, z_{\infty}, t), \) and \(EU_{u,n}^{(k)}(x, z_k, t)\) in the case where \(\text{des}(u) > 1\). Basically one has to modify the involution \(I_u\) presented in Section 3 appropriately. This has been done in the case of permutations by Quang Bach and the first author in the case of permutations \([2, 3]\). However, in the case where \(\text{des}(u) > 1\), the corresponding set of fixed points are much more complicated. For example, it will no longer be the case that in fixed points of the modified version of \(I_u\) that the underlying word will be weakly increasing in bricks. These more
complicated fixed points then require a more complicated version of the collapse map. Nevertheless, one can still come up with closed formulas for the generating functions $U_{u}^{(p)}(x, z_{\infty}, t)$, $U_{u}^{(k)}(x, z_{k}, t)$, $EU_{u}^{(p)}(x, z_{\infty}, t)$, and $EU_{u}^{(k)}(x, z_{k}, t)$. This work will appear in a subsequent paper.

References

[1] R.E.L. Aldred, M. Atkinson, and D.J. McCaughan, Avoiding consecutive patterns in permutations, Advances in Applied Mathematics, 45 (2010), 449-461.

[2] Q. Bach and J.B. Remmel, Generating functions for permutations which avoid consecutive patterns with multiple descents, Australian Journal of Combinatorics, 64 (2016), 194-231.

[3] Q. Bach and J.B. Remmel, Decent c-Wilf equivalence, [arXiv:1520.07190].

[4] D. Beck and J. Remmel, Permutation enumeration of the symmetric group and the combinatorics of symmetric functions, J. Combin. Theory Ser. A, 72 (1995), no. 1, 1–49.

[5] F. Brenti, Unimodal polynomials arising from symmetric functions, Proc. Amer. Math. Soc., 108 (1990), no. 4, 1133–1141.

[6] F. Brenti, Permutation enumeration symmetric functions, and unimodality, Pacific J. Math., 157 (1993), no. 1, 1–28.

[7] A. Duane and J. Remmel, Minimal overlapping patterns in colored permutations, Electronic J. Combinatorics, 18(2) (2011), P25, 34 pgs.

[8] S. Elizalde and M. Noy, Consecutive patterns in permutations, Adv. in Appl. Math., 30 (2003), no. 1-2, 110–125, Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001).

[9] S. Elizalde, The most and least avoided consecutive pattern, Proc. Lond. Math. Soc., 106 (2013), 957-979.

[10] O. Eğecioğlu and J. B. Remmel, Brick tabloids and the connection matrices between bases of symmetric functions, Discrete Appl. Math., 34 (1991), no. 1-3, 107–120, Combinatorics and theoretical computer science (Washington, DC, 1989).

[11] I.P. Goulden and D.M. Jackson, Combinatorial Enumeration, John Wiley & Sons 1983.

[12] S. Heubach and T. Mansour, Combinatorics of Compositions and Words, Discrete Mathematics and Its Applications, Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, London, New York, (2009).

[13] M. Jones and J.B. Remmel, Pattern Matching in the Cycle Structures of Permutations, Pure Math. and Applications, 22 (2011), 173-208.

[14] M. Jones and J.B. Remmel, A reciprocity approach to computing generating functions for permutations with no pattern matches, Discrete Mathematics and Theoretical Computer Science, DMTCS Proceedings, 23 International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), 119 (2011), 551-562.
[15] M. Jones and J.B. Remmel, A reciprocity method for computing generating functions over the set of permutations with no consecutive occurrence of $\tau$, Discrete Mathematics, 313 Issue 23 (2013), 2712-2729.

[16] M. Jones and J. Remmel, Generating functions for the number of permutations with no consecutive occurrences of $1p23\ldots (p-1)$ or $13\ldots (p-1)2p$, to appear in Pure Mathematics and Applications.

[17] S. Kitaev, Patterns in permutations and words, Springer-Verlag, 2011.

[18] A. Khoroshkin and B. Shapiro, Using homological duality in consecutive pattern avoidance, Electronic J. Combinatorics, 18 (2011), # P9

[19] I. G. Macdonald, Symmetric Functions and Hall Polynomials. 2nd ed. Oxford University Press, 1995.

[20] J. Liese and J.B. Remmel, Generating functions for permutations avoiding a consecutive pattern, Annals of Combinatorics, 14 (2010), 103-121.

[21] T. Langley and J.B. Remmel, Enumeration of $m$-tuples of permutations and a new class of power bases for the space of symmetric functions, Adv. Appl. Math., 36 (2006), 30-66.

[22] A. Mendes and J.B. Remmel, Generating functions for statistics on $C_k \wr S_n$, Séminaire Lotharingien de Combinatoire, B54At, (2006), 40 pp.

[23] A. Mendes and J.B. Remmel, Permutations and words counted by consecutive patterns, Adv. Appl. Math., 37 4, (2006) 443-480.

[24] A. Mendes and J.B. Remmel, Descents, major indices, and inversions in permutation groups, Discrete Mathematics, Vol. 308, Issue 12, (2008), 2509-2524.

[25] A. Mendes, J.B. Remmel, and A. Riehl, Permutations with $k$-regular descent patterns, Permutation Patterns (S. Linton, N. Ruskuc, and V. Vatter, eds.), London Math. Soc. Lecture Notes 376, 259-286, (2010).

[26] A. Mendes and J. Remmel, Counting with symmetric functions, Developments in Mathematics, vol. 43, Springer (Cham, Heidelberg, New York, Dordrecht, London), ISBN 978-3-309-23619-6 Electronic.

[27] D. Rawlings and M. Tiefenbruck, Consecutive Patterns: From Permutations to Column-Convex Polyominoes and Back, Electronic Journal of Combinatorics, 17 (1) (2010), #R62

[28] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, (1999).

[29] E. Steingrímsson: Generalized permutation patterns – a short survey, Permutation Patterns, St Andrews 2007, S.A. Linton, N. Ruskuc, V. Vatter (eds.), LMS Lecture Note Series 376, Cambridge University Press, (2010), 137-152.