Abstract. Given certain $n \times n$ invertible matrices $A_1, \ldots, A_m$ and $0 \leq \alpha < n$, we obtain the $H^{p(\cdot)}(\mathbb{R}^n) \rightarrow L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of the integral operator with kernel $k(x, y) = |x - A_1 y|^{-\alpha_1} \cdots |x - A_m y|^{-\alpha_m}$, where $\alpha_1 + \cdots + \alpha_m = n - \alpha$ and $p(\cdot), q(\cdot)$ are exponent functions satisfying log-Hölder continuity conditions locally and at infinity related by $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$. We also obtain the $H^{p(\cdot)}(\mathbb{R}^n) \rightarrow H^{q(\cdot)}(\mathbb{R}^n)$ boundedness of the Riesz potential operator.

1. Introduction

Given a measurable function $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ such that $0 < \inf_{x \in \mathbb{R}^n} p(x) \leq \sup_{x \in \mathbb{R}^n} p(x) < \infty$, let $L^{p(\cdot)}(\mathbb{R}^n)$ denote the space of measurable functions such that for some $\lambda > 0$,

$$
\int \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty.
$$

Let

$$
\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.
$$

We see that $(L^{p(\cdot)}(\mathbb{R}^n), \|f\|_{p(\cdot)})$ is a quasi normed space. As usual we will denote $p_+ = \sup_{x \in \mathbb{R}^n} p(x)$ and $p_- = \inf_{x \in \mathbb{R}^n} p(x)$.
These spaces are referred to as the variable $L^p$ spaces. In the last years many authors have extended the machinery of classical harmonic analysis to these spaces. See, for example [1], [2], [4], [5], [7].

In the famous paper [6], C. Fefferman and E. Stein defined the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < \infty$, with the norm given by

$$\|f\|_{H^p} = \left\| \sup_{t > 0} \sup_{\varphi \in \mathcal{F}_N} \left| t^{-n} \varphi(t^{-1} \cdot) \ast f \right| \right\|_p,$$

for a suitable family $\mathcal{F}_N$. In [9], E. Nakai and Y. Sawano defined the Hardy spaces with variable exponents, replacing $L^p$ by $L^{p(.)}$ in the above norm and they investigated their properties.

Let $0 \leq \alpha < n$, let $p(.) : \mathbb{R}^n \to (0, \infty)$ be a measurable function, and let $q(.)$ be defined by $\frac{1}{q(.)} = \frac{1}{p(.)} - \frac{\alpha}{n}$. Given certain invertible matrices $A_1, \ldots, A_m$, $m \geq 1$, we study

$$T_\alpha f(x) = \int |x - A_1 y|^{-\alpha_1} \ldots |x - A_m y|^{-\alpha_m} f(y) \, dy,$$

where $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Observe that in the case $\alpha > 0$, $m = 1$ and $A_1 = I$, $T$ is the classical fractional integral operator $I_\alpha$ (also known as the Riesz potential).

With respect to classical Lebesgue or Hardy spaces, in the case $m > 1$, in [10], we obtained the $H^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ boundedness of these operators and we showed that we cannot expect the $H^p(\mathbb{R}^n) - H^q(\mathbb{R}^n)$ boundedness of them. This is an important difference with respect to the case $m = 1$. Indeed, in [12], M. Taibleson and G. Weiss, using the molecular characterization of the real Hardy spaces, obtained the boundedness of $I_\alpha$ from $H^p(\mathbb{R}^n)$ into $H^q(\mathbb{R}^n)$, $0 < p \leq 1$. In this paper we will extend both results to the setting of variable exponents. Here and below we shall postulate the following conditions on $p(.)$:

$$\left| p(x) - p(y) \right| \leq \frac{c}{-\log |x - y|}, \quad |x - y| < \frac{1}{2},$$

and

$$\left| p(x) - p(y) \right| \leq \frac{c}{\log (e + |x|)}, \quad |y| \geq |x|.$$

We note that the condition (3) is equivalent to the existence of constants $C\infty$ and $p\infty$ such that

$$\left| p(x) - p\infty \right| \leq \frac{C\infty}{\log (e + |x|)}, \quad x \in \mathbb{R}^n.$$