Concrete Categories in Homotopy Type Theory

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November 11, 2013

Abstract

We introduce some classes of genuine higher categories in homotopy type theory, defined as well-behaved subcategories of the category of types. We give several examples, and some techniques for showing other things are not examples. While only a small part of what is needed, it is a natural construction, and may be instructive for people seeking to provide a fully general construction.

1 Introduction

1.1 Categories in homotopy type theory

Homotopy type theory is a recently-developed foundational approach to mathematics. The key idea is that a certain flavour of type theory can be given a homotopical interpretation, in which a type is viewed as a space, and a function is viewed as a continuous map. This interpretation provides a rich semantics, and many internal definitions can be made which harmoniously represent classical results in homotopy theory as structural results about type theory. The basic reference is the book [6].

Inevitably, any foundational approach to mathematics will be judged in some small part on its ability to comfortably represent category theory, which has become an essential tool in organising modern mathematics [3,12]. Thus far the author knows of one attempt, by Ahrens, Kapulkin and Shulman [2,6 Chapter 9], to undertake category theory in homotopy type theory. That attempts deals with 1-categories, rather than (∞,1)-categories. In other words, the type of homomorphisms between any two objects is homotopically discrete. (Just as ordinary homotopy theory contains a theory of sets, which can be represented as discrete spaces, homotopy type theory contains a more classical type theory within, made up of those types satisfying a similar kind of discreteness condition).

However, homotopy type theory studies types which are more general than sets. Usually maps between structures built from such types cannot be expected to be discrete.

Indeed, since homotopy type theory uses (∞,0)-categories to model types; it is natural to feel that (∞,1)-categories are the most appropriate concept of category in this setting, just as Joyal [9] and Lurie [11] have provided ample evidence that they are often tractable and useful in topology.

The purpose of this paper is to produce some genuine (∞,1)-categories in homotopy type theory. Our approach is certainly not fully general, but our
examples encompass a range of ($\infty, 1$)-categories that one might wish to work with. We also attempt to sketch some perceptions of the limitations of our approach.

Also available is a library of code\textsuperscript{8} written in the dependently-typed programming language Agda\textsuperscript{1}, demonstrating these concepts in practice; this is based on a homotopy type theory library provided by Brunerie and coworkers\textsuperscript{4}. At appropriate points in what follows, we reference this library.

1.2 Conventions

This paper is written in an informal form of type theory, roughly as used in the book \textsuperscript{6}. We do not emphasise universes; the reader who cares can identify appropriate universe levels for themselves.

We write $x \equiv_A y$ for the path type between two elements $x, y : A$ (we avoid using the phrase identity type, saving the word identity for use in its categorical sense) and write simply $x \equiv y$ if $A$ is obvious from context.

We use the dot $\cdot$ to denote composition of paths.

We call a $(−1)$-truncated type a proposition; the book \textsuperscript{6} calls these mere propositions, but we have no use for any other meaning of the word and do not wish to sound demeaning about them. Similarly, a $(−2)$-truncated type is called a set.

We use the phrase subcategory in a very vague sense: we mean the domain of a certain sort of functor. The functor in question is always required to satisfy some kind of faithfulness condition (which we will make clear as required), but never any kind of injectivity, or essential injectivity, on objects. We feel that this is less uncommon than it sounds: most practical uses of subcategories in mathematics are similar.

2 Inbuilt coherence: the category of types

There are grave problems associated with naive attempts to define categories, or higher categories, in homotopy type theory: there is an infinite amount of data required, of a type which increases progressively in complexity.

We may start with a set of objects $\text{obj} : \text{Type}$, and a dependent type of morphisms $\text{hom} : \text{obj} \to \text{obj} \to \text{Type}$. This lacks the basic structure of identities and composition, so we require elements as follows:

\[
\text{id} : (x : \text{obj}) \to \text{hom}(x, x)
\]

and

\[
\text{cmp} : (x, y, z : \text{obj}) \to \text{hom}(y, z) \to \text{hom}(x, y) \to \text{hom}(x, z).
\]

This lacks the associative and unit laws of composition, so we must add something (for example) whose content is that

\[
\text{cmp}(\text{cmp}(f, g), h) = \text{cmp}(f, \text{cmp}(g, h))
\]

for all composable strings of morphisms.

However, in homotopy type theory, this does not assert that those two are indistinguishable, merely that they are homotopic. As it happens, we can form
two different chains of composites of such homotopies showing that
\[
\text{cmp}(\text{cmp}(\text{cmp}(f, g), h), k) = \text{cmp}(f, \text{cmp}(g, \text{cmp}(h, k))).
\]
We must assert that these are equal, but this does not make contractible the space of composites of five maps. Things continue becoming more complex in this way. While appropriate structures in traditional foundations are well-known\[10\], the problem of specifying the resulting data in homotopy type theory is unsolved at the time of writing.

The starting point of our work is the observation that, while nobody has written down a general definition of \((\infty, 1)\)-categories in homotopy type theory, there is one fully coherent example built in. That category is the category Type of types, and functions between them.

In Agda, for example, we define the composition of two functions by the usual formula
\[
(g \circ f)(x) = g(f(x)).
\]
But the result is that Agda normalises both \((h \circ g) \circ f\) and \(h \circ (g \circ f)\) to
\[
\lambda x. h(g(f(x))),
\]
and, as a result, the associativity of composition \((h \circ g) \circ f \equiv h \circ (g \circ f)\) is a triviality.

The same goes for higher associativity laws. For example, consider the “associativity pentagon”:

If we wish to verify that this commutes: that the two chains of associativity identities connecting \(((f \circ g) \circ h) \circ k\) and \(f \circ (g \circ (h \circ k))\) agree, then Agda can verify immediately that both are simply reflexivity on \(\lambda x. k(h(g(f(x))))\), and so are equal, by reflexivity of identity. This pattern continues: all the structure of Type as a higher category is supplied in a straightforward fashion by the underlying type theory.

3 Inheriting coherence: \(n\)-concreteness

As we have seen, the category Type has excellent properties within homotopy type theory. Even so, it’s only a single example of an \((\infty, 1)\)-category, not a general approach to \((\infty, 1)\)-category theory.
However, we can use it as a starting point for building more: for any \( n : \mathbb{N}_{\geq 2} \), we can define a notion of \((\infty, 1)\)-categories with a functor to Type, which is a \( n \)-truncated map on each homtype. We can call these \((n + 2)\)-concrete \((\infty, 1)\)-categories.

The coherence of the category Type automatically supplies the desired coherence of our category in high degrees. However we must explicitly supply the categorical structure in low degrees; this structure is that of a fully weak \((n + 1, 1)\)-category, so general definitions require machinery and work which we are unwilling to undertake here (see Leinster’s book[10] for a survey of approaches).

The situation is perhaps best made clear by example; we talk through the cases \( n = 0, 1, 2, 3 \) below: there are uncontroversial definitions of categories and bicategories.

### 3.1 0-concrete \((\infty, 1)\)-categories

The notion of 0-concrete \((\infty, 1)\)-category is the notion of a full subcategory of Type.

We could specify such a thing simply by giving a type of objects \( \text{obj} \) and a realisation map \( \text{obj}^+ : \text{obj} \to \text{Type} \), and define

\[
\text{hom}(x, y) = (\text{obj}^+(x) \to \text{obj}^+(y)).
\]

More elaborately, one could give a type of objects and a realisation map as above, and choose a map

\[
\text{hom} : \text{obj} \to \text{obj} \to \text{Type}
\]

together with, for each \( x \) and \( y \), an equivalence

\[
\text{conf}(x, y) : \text{hom}(x, y) \cong \text{hom}'(x, y),
\]

the “conformity map” where \( \text{hom}'(x, y) = (\text{obj}^+(x) \to \text{obj}^+(y)) \) as used before. Clearly these are the equivalent concepts, and it is a matter of convenience which we choose to use; we shall use the latter in what follows, since it is more similar to our other definitions.

While we will give some useful examples below, our interest in the notion of 0-concrete \((\infty, 1)\)-categories is largely due to their status as the simplest of the family of \( n \)-concrete \((\infty, 1)\)-categories.

Here is a result that gives some idea of the limitation of this concept:

**Theorem 1.** Consider the disjoint union \(* \sqcup *\) of two copies of the terminal category. This is not a 0-concrete \((\infty, 1)\)-category.

**Proof.** In fact, it’s not a full subcategory of the category of spaces under ordinary foundations. Suppose the two spaces representing the two objects are \( X \) and \( Y \). Now, if \( X \) has a point \( x \), then there is a constant map from \( Y \) to \( X \) with image \( x \). However, if \( X \) is empty, then there is an inclusion map from \( X \) to \( Y \). Either way, there is some map between them.

This argument does not work as stated in homotopy type theory, since it uses the law of excluded middle to argue that \( X \) must either be empty or have
a point. However, we can use double negation to recover something similar: for types $X$ and $Y$, if $\neg(X \to Y)$, we can show that $\neg(\neg X)$ and $\neg Y$.

We get

$$\neg(X \to Y) \to \neg Y$$

which of course means

$$( (X \to Y) \to \bot ) \to (Y \to \bot )$$

by composing with the constant map $Y \to (X \to Y)$.

And we get

$$\neg(X \to Y) \to \neg(\neg X)$$

from the inclusion $\bot \to Y$.

Hence, if we have two objects in a 0-concrete $(\infty, 1)$-category, then we cannot simultaneously have $\neg(\text{obj}^+(x) \to \text{obj}^+(y))$ and $\neg(\text{obj}^+(y) \to \text{obj}^+(x))$: the former implies $\neg\neg\text{obj}^+(x)$ and the latter implies $\neg\text{obj}^+(x)$, a contradiction. \(\square\)

## 3.2 1-concrete $(\infty, 1)$-categories

A 1-concrete $(\infty, 1)$-category is a subcategory of Type where, as morphisms, we choose some connected components of the homtypes in Type. We cannot do this freely: we must choose the connected components of the identity morphisms, and given any two morphisms we have chosen, we must also choose the connected component of their composite.

More formally, it consists of:

- A type $\text{obj}$ of objects;
- An object realisation map $\text{obj}^+ : \text{obj} \to \text{Type}$;
- For every pair $x, y : \text{obj}$, a type of homomorphisms $\text{hom}(x, y)$;
- For every pair $x, y : \text{obj}$, a homomorphism realisation map $\text{hom}^+ : \text{hom}(x, y) \to \text{hom}'(x, y)$, where $\text{hom}'(x, y) = (\text{obj}^+(x) \to \text{obj}^+(y))$;
- For every pair $x, y : \text{obj}$, an element $\text{conf}(x, y)$ of the proposition that $\text{hom}^+ : \text{hom}(x, y) \to \text{hom}'(x, y)$ is 1-truncated (the "conformity");
- For every $x : \text{obj}$, an element $\text{ident}'(x)$ of the homotopy fibre of $\text{hom}^+$ at the point $\text{id}_{\text{obj}^+(x)}$.
- For every $x, y, z : \text{obj}$, and $g : \text{hom}(y, z)$ and $f : \text{hom}(x, y)$, an element $\text{cmp}'(g, f)$ of the homotopy fibre of $\text{hom}^+$ at the point $\text{hom}^+(g) \circ \text{hom}^+(f)$.

This notion is already quite powerful, and using it we can comfortably express many categories that we might choose to care about, as will be seen in the next section.
3.3 2-concrete $(\infty, 1)$-categories

The next stage up, the 2-concrete $(\infty, 1)$-category, is a subcategory of Type where we are allowed to choose a set of copies of each morphism.

This requires still more data and axioms to be given by hand. We need a choice of preimage of the identity maps, and of each composition, much as before. But we now need to impose category axioms on this structure: we need to impose the left and right unit axioms, and the associativity axioms, to ensure that those choices of connected components give genuine categories.

More formally, the structure consists of all the structure of a 1-concrete $(\infty, 1)$-category, except that the conformity element conf asserts that the maps $\text{hom}^+$ are 0-truncated, and elements of the following types (for all $x, y, z, w : \text{obj}$, $f : \text{hom}(x, y)$, $g : \text{hom}(y, z)$ and $h : \text{hom}(z, w)$ as appropriate):

\[
\begin{align*}
\text{unit}^l &: \text{cmp}(\text{ident}(y), f) \equiv f \\
\text{unit}^r &: \text{cmp}(f, \text{ident}(x)) \equiv f \\
\text{assoc} &: \text{cmp}(\text{cmp}(h, g), f) \equiv \text{cmp}(h, \text{cmp}(g, f)).
\end{align*}
\]

Here we define $\text{ident}$ and $\text{cmp}$ to be the first component of $\text{ident}'$ and $\text{cmp}'$ respectively, so that they have types

\[
\begin{align*}
\text{ident} &: (x : \text{obj}) \to \text{hom}(x, x) \\
\text{cmp} &: \text{hom}(y, z) \to \text{hom}(x, y) \to \text{hom}(x, z).
\end{align*}
\]

3.4 3-concrete $(\infty, 1)$-categories and beyond

By now, hopefully the pattern is becoming clear. A 3-concrete $(\infty, 1)$-category will have a conformity type that is weaker still: it only asserts that the maps $\text{hom}^+$ are 1-truncated.

This means that more structure should be supplied by hand: the pentagon and triangle identities, familiar from the definition of a bicategory (or a monoidal category) as in \[3\], need to be imposed to ensure coherence of the unit and associativity laws.

In general, each time we increase the concreteness level, we need to add more axioms simulating a weak $n$-category.

4 Examples

4.1 $(\infty, 1)$-categories of types, sets, $n$-groupoids, etc

The obvious examples of 0-concrete $(\infty, 1)$-categories simply consist of full subcategories of the category of types on special sorts of types.

The trivial case is, of course, the 0-concrete $(\infty, 1)$-category of types itself.

We could take as objects, instead, the $n$-truncated types for any $n : \mathbb{N}_{\geq 2}$. For $n = 0, 1, \ldots$ we get the $(\infty, 1)$-category of sets, or of 1-groupoids, and so on.

Another family of examples is what we get from using a singleton as set of objects: this is a coherent version of the endomorphism monoid of a type $X$, regarded as a 1-object category.
We can produce the category of finite sets: there is a standard model for nonempty finite ordered sets: we define $\text{Fin}(n)$ for $n : \mathbb{N}$ by the constructors:

$$
0 : \text{Fin}(n + 1) \\
S : \text{Fin}(n) \to \text{Fin}(n + 1).
$$

This gives us a 0-concrete $(\infty, 1)$-category with obj $= \mathbb{N}$ and obj$^+$ $= \text{Fin}$.

### 4.2 The simplicial category $\Delta$

The simplex category $\Delta$, the category of nonempty finite ordered sets and order-preserving maps, fits into this scheme.

We also provide a convenient model $\text{Ord}(0, 0)$ for the ordered maps from $\text{Fin}(m)$ to $\text{Fin}(n)$, with three constructors:

$$
0 : \text{Ord}(0, 0) \\
S^l : \text{Ord}(m, n + 1) \to \text{Ord}(m + 1, n + 1) \\
S^r : \text{Ord}(m, n) \to \text{Ord}(m, n + 1).
$$

The semantics of 0 are obvious; those of $S^l$ are defined by

$$
S^l(f)(0) = 0, \\
S^l(f)(i + 1) = f(i);
$$

and those of $S^r$ are defined by

$$
S^r(f)(i) = f(i) + 1;
$$

This recursively defines a map

$$
\text{Ord}^+(m, n) : \text{Ord}(m, n) \to (\text{Fin}(m) \to \text{Fin}(n))
$$

for every $m$ and $n$.

It is straightforward to recursively define identities and compositions for the type family $\text{Ord}$, and also to show that these coincide with the genuine identities and compositions under $\text{Ord}^+$.

Moreover, standard methods permit one to show that $\text{Fin}(n)$ and $\text{Ord}(m, n)$ are both sets.

**Theorem 2.** The category $\Delta$ is a 1-concrete $(\infty, 1)$-category.

**Proof.** We use object set obj $= \mathbb{N}$, and the object realisation map $\text{Fin} \circ S$ (the suspension is so that we get only nonempty finite ordered sets).

Then we use $\text{Ord}$ to define $\text{hom}$, and then $\text{hom}^+$ is the recursively-defined map $\text{Ord}^+$ defined above.

Identities and composition have already been discussed. All that remains is the conformity. We find it helpful to prove the following:

**Claim 2.1.** An injection into a set has propositions as homotopy fibres.

**Proof of Claim.** It is easy to show that any two elements of the homotopy fibre are equal. 

We put this claim to work on the map $\text{Ord}^+(m, n)$. The codomain is the type of functions $\text{Fin}(m) \to \text{Fin}(n)$, which is a set since $\text{Fin}(n)$ is one. It is not hard to prove that $0 \neq S(i)$ for all $i : \text{Fin}(n)$, and thence to show recursively that the map $\text{Ord}^+$ is injective.
4.3 Ahrens-Kapulkin-Shulman 1-categories

The work of Ahrens, Kapulkin and Shulman provides examples of our theory. We refer to the notion of category they consider as AKS-categories.

Generalising the preceding example somewhat, examples of their theory give examples of our theory:

**Theorem 3.** Any AKS-category yields a 2-concrete \((\infty, 1)\)-category.

**Proof.** Suppose we have such a category, with object type \(\text{obj}\) and morphism types \(\text{hom}(x, y)\) for \(x, y: \text{obj}\).

We inherit the type of objects as is. The object realisation map \(\text{obj}^+\) takes an object \(x\) to the type of pairs consisting of an object \(y\) and an element \(f: \text{hom}(y, x)\).

Note that \(\text{obj}^+(x)\) is a 1-truncated type. That is because it is a \(\Sigma\)-type; \(\text{obj}\) is 1-truncated (this is Lemma 3.8]) and homsets in an AKS-category are genuine sets: they’re 0-truncated and hence 1-truncated. Hence, also, for all \(x\) and \(y\) the type of maps from \(\text{obj}^+(x)\) to \(\text{obj}^+(y)\) is 1-truncated.

We define the realisation \(\text{hom}^+\) as follows:

\[
\text{hom}^+(f)(z, g) = (z, f \circ g).
\]

It is straightforward to define units and composition using this definition; the maps require the left unit and associativity axioms respectively.

The problem that remains is conformity. Given a map \(f\) from \(\text{obj}^+(x)\) to \(\text{obj}^+(y)\), we must show that the homotopy fibre of \(f\) under \(\text{hom}^+\) is a set. A \(\Sigma\)-type is \(n\)-truncated if the base and all fibres are \(n\)-truncated. In this case the base is a set because one axiom of an AKS-category is that homomorphisms form sets, and the fibre is a set because it’s a path type of the 1-truncated type \(\text{obj}^+(x) \to \text{obj}^+(y)\).

4.4 Types as \((\infty, 0)\)-categories

Given a type \(X\), it is natural to wish to regard \(X\) as an \(\infty\)-groupoid, which is an \((\infty, 0)\)-category: a degenerate case of an \((\infty, 1)\)-category where all morphisms (given by path types) are equivalences.

We can do this:

**Theorem 4.** A type \(X\) can be given the structure of a 1-concrete \((\infty, 1)\)-category.

**Proof.** We naturally take \(\text{obj} = X\), and we choose \(\text{obj}^+\) to be the type of paths to \(x\). As promised, we also choose \(\text{hom}(x, y) = (x \equiv X y)\). The proper definition of \(\text{hom}^+\) is very much like that used in the subsection above:

\[
\text{hom}^+(e)(z, \epsilon) = (z, \epsilon \cdot e).
\]

The identity and composites are quick checks, and conformity is also rapidly proved by path induction.

8
4.5 Automorphism groups and categories of $n$-truncated maps

Given a map $f : X \to Y$ between two types, there is a proposition expressing that $f$ is $n$-truncated (for any $n : \mathbb{N} \geq 2$). It is also true that identity maps are $n$-truncated (for all $n$), and composites of $n$-truncated maps are $n$-truncated.

That gives that there are 1-concrete $(\infty, 1)$-categories of types and $n$-truncated maps, or of any given type of types and $n$-truncated maps between them.

One special case is when we take the object type to be a singleton and $n = -2$: the resulting category is the one-object category of self-equivalences of some given type: this $(\infty, 1)$-category can be regarded as the automorphism group of that type.

4.6 Free categories

One might reasonably wish to discuss the free category (on a specified type of objects and specified types of morphisms between them).

So suppose given a type $\text{obj} : \text{Type}$ and a family $\text{arr} : \text{obj} \to \text{obj} \to \text{Type}$.

We can define the homomorphisms inductively, as linked lists of composable arrows:

\[
\begin{align*}
\text{nil} : & \text{hom}(x, x) \\
\text{cons} : & \text{hom}(y, z) \to \text{arr}(x, y) \to \text{hom}(x, z).
\end{align*}
\]

As is normal for linked lists, there is a unital and associative composition operation, which we denote by $\circ$.

This structure fits into our framework:

**Theorem 5.** A free category is a 1-concrete $(\infty, 1)$-category.

**Proof.** We have $\text{obj}$ and $\text{hom}$ already. We define $\text{obj}^+(x)$ to be the type of “homs to $x$”, in other words the $\Sigma$-type of pairs consisting of an element $y : \text{obj}$ and an element $f : \text{hom}(y, x)$.

Then $\text{hom}^+$ is defined by composition:

\[
\text{hom}^+(f)(x, g) = (x, g \circ f).
\]

Using this definition, $\text{ident}'$ and $\text{cmp}'$ are clear from the algebraic properties of the composition.

What’s left is the conformity. Standard methods, as in \[6\] Section 2.12, will prove that for any $x, y : \text{obj}$ and for any $f, g : \text{hom}(x, y)$, the type $\text{hom}^+(f) \equiv \text{hom}^+(g)$ is equivalent to the type $f \equiv g$. This enables us to show that $\text{hom}^+$ is 1-truncated, via the following lemma:

**Lemma 5.1.** Suppose $k$ is any function such that, for any $x, y$, the map

\[
\text{ap}(k) : (x \equiv y) \to (k(x) \equiv k(y))
\]

is an equivalence. Then the homotopy fibre of $k$ at any point is a proposition.

**Proof of lemma.** We show that any two elements of the homotopy fibre are equal. This can be simplified by using path induction to simplify one (but not both) of the second components of the homotopy fibre to refl.
So we aim to show that an element \((a, u)\) is equal to an element \((b, \text{refl})\). Writing \(e\) for the hypothesis that \(\text{ap}(k)\) is an equivalence, we can show that \(a \equiv b\) immediately from \(e\) (as \(\pi_1(\pi_1(e(u)))\)); we are left with the check

\[
\text{transport}(\lambda x \to f(x) \equiv f(b), \pi_1(\pi_1(e(u))))
\]

\[
= \text{transport}(\lambda x \to x \equiv f(b), \text{ap}(f)(\pi_1(\pi_1(e(u))))), u)
\]

\[
= \text{ap}(f)(\pi_1(\pi_1(e(u)))) \cdot u
\]

\[
= \text{refl}.
\]

This completes the proof.

\[
\checkmark
\]

5 Pointed types: a cautionary tale

Recall that the type of pointed types is defined by

\[
\text{Type}^* = \Sigma(\text{Type}, \text{id}),
\]

so that a pointed type \((X, x)\) consists of a type \(X\) and an element \(x : X\) (the basepoint).

Given two pointed types \((X, x)\) and \((Y, y)\), the type of pointed maps between them is defined by

\[
(X, x) \rightarrow^* (Y, y) = \Sigma(X \to Y, \lambda f \to f(x) \equiv y).
\]

This concept is ubiquitous in algebraic topology, and so one would naturally want to form the category of pointed types and pointed maps accordingly.

Unfortunately, the obvious approach doesn’t work. This would be to model it as a concrete \((\infty, 1)\)-category by forgetting the basepoint, so taking \(\text{obj}^+(X, x) = X\) and \(\text{hom}^+(f, p) = f\). But then, consider what happens when we take \(X\) to be the singleton type \(1\), and \(*\) to be its unique element, then the type of pointed maps

\[
(1, *) \rightarrow^* (Y, y)
\]

is contractible (since it is equivalent to the type of paths to \(y\) in \(Y\)), but the type \(\text{hom}^+((1, *), (Y, y))\) is the type \(1 \to Y\), which is equivalent to \(Y\). The map \(\text{hom}^+\) is, under these equivalences, the inclusion of \(y\) into \(Y\).

The problem is that this inclusion has homotopy fibre \(y \equiv_\forall y\), the loop space of \(Y\) at \(y\), and this can only be expected to be \(n\)-truncated if \(Y\) is \((n + 1)\)-truncated (more precisely, if the basepoint component of \(Y\) is). Thus we cannot form an \(n\)-concrete category of all pointed types by this process for any \(n\).

One could wonder whether this was just an unfortunate choice of \(\text{obj}^+\) and \(\text{hom}^+\), but one cannot do any better:

**Theorem 6.** The category of pointed types is not an \(n\)-concrete category for any \(n\).

**Proof.** Suppose that the category of pointed types can be described as an \(n\)-concrete category.

It is not possible for \(\text{obj}^+(X, x)\) to be \(k\)-truncated for every pointed type \((X, x)\). Indeed, if it were, then \(\text{hom}^+((X, x), (Y, y))\) would be \(k\)-truncated for
every pair of pointed types, and then, as $\text{hom}^+((X, x), (Y, y))$ is an $n$-truncated map, $\text{hom}((X, x), (Y, y))$ would be $(n + k)$-truncated, which is certainly not true for all pairs of pointed spaces!

But then, the argument of the special case above goes through: the map $\text{hom}^+((1, *), (Y, Y))$ goes from a contractible type to a type which is not in general $k$-truncated for any $k$, and hence cannot be an $n$-truncated map in general. \hfill \square

Plainly enough, the same difficulties may be expected to apply to most other categories of structured types.

There are certain compromises that can be made. For example, the category of pointed $n$-truncated types will certainly form an $(n + 1)$-concrete category. That is certainly less than the homotopy theorist would wish for, but may be of considerable utility to the algebraist.

Another trick is to truncate the defining equation of a pointed map, defining instead

$$\text{hom}((X, x), (Y, y)) = \Sigma((X \to Y), \lambda f \to \tau_i(f(x) \equiv y)),$$

where $\tau_i$ is the $i$-truncation operator. In the case $i = -1$, this may be interpreted as describing the category of pointed types and maps which preserve only the connected component of the basepoint.

It is hard to escape the conclusion that this highlights an essential deficiency of type theory in dealing with pointed types: it is hard to see any way of dealing with them without explicitly having to handle coherence at all levels.

Remark 7. This analysis shows that we also can’t hope to form the arrow category of Type as a concrete category: the category whose morphisms are pairs of types $X, Y$ equipped with a map $X \to Y$. Indeed, were this to be possible, we could obtain a category of pointed types by imposing the restriction that $X$ be contractible (which is a proposition).

6 Spans of types: an open problem

The category of spans, and its variants, can be expected to be of some importance; some of their uses in homotopy theory are described in the author’s PhD thesis[7].

We aim to describe a category with obj = Type, and where $\text{hom}(X, Y)$ is the $\Sigma$-type of spans: pairs consisting of a type $U$ and morphisms $f : U \to X$ and $g : U \to Y$. We write such things as $(f; U; g)$; and draw them where possible as roof-shaped diagrams:

```
U
\downarrow \downarrow \downarrow
X \quad Y.
```

The identity span on a type $X$ consists entirely of identities on $X$: it is $(\text{id}_X; X; \text{id}_X)$. Composition is defined by pullbacks: the composite of spans $X \leftarrow U \to Y$ and
In our symbolic notation, the composite of a span \((f; U; g)\) from \(X\) to \(Y\) with a span \((h; V; k)\) from \(Y\) to \(Z\) is
\[(h; V; k) \circ (f; U; g) = (f \pi_1; U \times V; k \pi_2).\]

This definition inspires an appropriate choice of \(\text{obj}^+\): we might take \(\text{obj}^+ X\) to be the type \(\text{Type}_{/X}\) of types over \(X\): that is, types equipped with a map to \(X\). We could then define \(\text{hom}^+\) by a “pull-push” construction:
\[
\text{hom}^+(f; U; g)(A) = g^* f^*(A).
\]

Here \(f^* : \text{Type}_{/X} \to \text{Type}_{/U}\) denotes the pullback along \(f\): it replaces a set \(\alpha : A \to X\) over \(X\) with the set \(f^* \alpha : Z \times_X U \to U\). Also, \(g^* : \text{Type}_{/U} \to \text{Type}_{/Y}\) denotes the pushforward along \(g\): it replaces a set \(\beta : B \to U\) with the set \(g \beta : B \to Y\).

Naturally one could restrict various parts of the structure: for example, restricting the objects only to certain families of types; restricting the central objects in the spans, or restricting the class of morphisms which are permitted in the spans. We might call any such structure an \emph{category of spans}, but in the discussion below we will continue to assume there are no such restrictions for simplicity.

It is of course reasonable to ask whether this structure, as described, does indeed produce any \(n\)-concrete \((\infty, 1)\)-categories of spans (for any \(n\)).

We can give a sense of the nature of this question by looking between two simple examples of pairs of objects.

Firstly, we consider morphisms from \(\emptyset\) to \(1\). In this case the type of spans is contractible: given a diagram \(\emptyset \from U \to 1\), the \(U\) must be empty (and there is a contractible type of empty types) and the maps are then chosen from a contractible type of possibilities.

The type \(\text{Type}_{/\emptyset}\) is contractible, and the type \(\text{Type}_{/1}\) is equivalent to \(\text{Type}\). Thus the type \(\text{hom}^+(\emptyset, 1)\) is \((1 \to \text{Type}) \simeq \text{Type}\).

The map \(\text{hom}^+\) is, under these equivalences, the map \(1 \to \text{Type}\) picking out the empty type. This map can certainly be seen to be \((-1)\)-truncated: emptiness is a proposition.

Secondly, however, we consider morphisms from \(1\) to \(1\). In this case the type of spans is equivalent to \(\text{Type}\): all we do is freely choose the intervening object \(U\) in a diagram \(1 \from U \to 1\), and then we have a contractible type of choices for the maps.

The map \(\text{hom}^+\) is then the map
\[
\text{Type} \to (\text{Type} \to \text{Type})
\]
sending $U$ to the map $(U \times -)$.

Now, suppose we investigate what happens if we attempt to prove that this map is $(-1)$-truncated. Suppose we have a map $F : \text{Type} \to \text{Type}$; is its homotopy fibre $n$-truncated?

That is, given two pairs $(U, \alpha)$ and $(V, \beta)$, where $U, V : \text{Type}$, and $\alpha : F \equiv (U \times -)$ and $\beta : F \equiv (V \times -)$, what can we say about the type $(U, \alpha) \equiv_{hfibre(F)} (V, \beta)$?

To start with, we have

$$U \equiv U \times 1 \equiv F(1) \equiv V \times 1 \equiv V,$$

using $\alpha$ and $\beta$ respectively.

So, using path induction, we may as well suppose that $U = V$ and simply ask about $\alpha \equiv_{F \equiv (U \times -)} \beta$; or, better yet, discuss the type $(U \times -) \equiv (U \times -)$, which contains the element $\alpha^{-1} \cdot \beta$.

This is not going to be $n$-truncated in general for any $n$: if $U$ has interesting self-equivalences, they will extend to $(U \times -)$. For example, if $U$ is the boolean type $1 \sqcup 1$, then $\alpha^{-1} \cdot \beta$ may well exchange the summands.

However, even if $U$ is contractible, it is not clear what we can say: while the author does not believe it is possible to write down any element of the type

$$(V : \text{Type}) \to V \equiv_{\text{Type}} V$$

except $\lambda V \to \text{refl}$, he has been unable to show that this type is contractible, and hence it is unclear, at least with the standard axioms of homotopy type theory, how to show that any category of spans is $n$-concrete.

7 Constructions

In this section we list a few general constructions on concrete $(\infty, 1)$-categories.

7.1 Increasing the concreteness level

As one might expect, an $n$-concrete $(\infty, 1)$-category can be viewed as an $(n + 1)$-concrete $(\infty, 1)$-category. In general, the conformity axiom for an $n$-concrete $(\infty, 1)$-category trivially implies the conformity axiom for an $(n + 1)$-concrete $(\infty, 1)$-category, and it also provides the extra structure in degree $n$.

7.2 Disjoint unions

For any $n \geq 1$, the disjoint union $\mathcal{C} \sqcup \mathcal{D}$ of two $n$-concrete $(\infty, 1)$-categories $\mathcal{C}$ and $\mathcal{D}$ has the structure of an $n$-concrete $(\infty, 1)$-category.

Indeed, we take $\text{obj} = \text{obj}_\mathcal{C} \sqcup \text{obj}_\mathcal{D}$, and we take $\text{obj}^+$ to be defined as $\text{obj}^+_\mathcal{C}$ on $\text{obj}_\mathcal{C}$ and as $\text{obj}^+_{\mathcal{D}}$ on $\text{obj}_\mathcal{D}$.

All the other structure is defined as it is in $\mathcal{C}$ or $\mathcal{D}$ as appropriate (there is nothing to define whenever objects from both $\mathcal{C}$ and $\mathcal{D}$ are involved).

As a result, the category $\ast \sqcup \ast$ is a $1$-concrete $(\infty, 1)$-category.
7.3 Products

Products of \( n \)-concrete \((\infty, 1)\)-categories are \( n \)-concrete \((\infty, 1)\)-categories, for \( n \geq 1 \). We demonstrate this explicitly for \( n = 1 \):

**Theorem 8.** Let \( C \) and \( D \) be 1-concrete \((\infty, 1)\)-categories. Then \( C \times D \) is also a 1-concrete \((\infty, 1)\)-category.

**Proof.** We take \( \text{obj} = \text{obj}_C \times \text{obj}_D \), and \( \text{obj}^+(x, y) = \text{obj}_C^+(x) \sqcup \text{obj}_D^+(y) \).

Naturally, we define \( \text{hom}((x_1, y_1), (x_2, y_2)) = \text{hom}_C(x_1, x_2) \times \text{hom}_D(y_1, y_2) \).

There is then an obvious candidate for the map \( \text{hom}^+ \), which has type \( \text{hom}^+(x_1, x_2) \times \text{hom}_D(y_1, y_2) \rightarrow \text{obj}_C^+(x_1) \sqcup \text{obj}_D^+(y_1) \rightarrow \text{obj}_C^+(x_2) \sqcup \text{obj}_D^+(y_2) \), namely to define \( \text{hom}^+(f, g) = \text{hom}^+_C(f) \sqcup \text{hom}^+_D(g) \).

In this setup, the existence of suitable \( \text{cmp}' \) and \( \text{ident}' \) is easy; the big problem is the conformity. This follows from the fact that \( \text{hom}^+(f, g) \) is the composite of two maps; firstly a map which could reasonably be called \( \sqcup \), from

\[
(\text{obj}_C^+(x_1) \rightarrow \text{obj}_C^+(x_2)) \times (\text{obj}_D^+(y_1) \rightarrow \text{obj}_D^+(y_2))
\]

to

\[
((\text{obj}_C^+(x_1) \sqcup \text{obj}_D^+(y_1)) \rightarrow (\text{obj}_C^+(x_2) \sqcup \text{obj}_D^+(y_2)))
\]

and \( \text{hom}^+_C(f) \times \text{hom}^+_D(g) \). It is not a difficult exercise to show that both these maps are \((-1)\)-truncated.

8 Equivalences and univalence

It is a normal demand of category theory to be able to define equivalences. It is particularly important in this setting: Ahrens, Kapulkin and Shulman\[2\] discuss the utility of imposing a univalence axiom, which states that, between any two objects \( x \) and \( y \), the natural map from the type of paths between \( x \) and \( y \) to the type of equivalences between them is an equivalence.

We proceed to define equivalences in \( n \)-concrete \((\infty, 1)\)-categories in the manner that one might expect: we use the machinery of equivalences of types, together with some extra data to check that the given structure in degrees up to \( n \) agrees with that machinery.

Accordingly, we assume given a type \( \text{is-equiv}(f) \) dependent upon types \( X \) and \( Y \) and a function \( f : X \rightarrow Y \), which expresses that \( f \) is an equivalence and which is a proposition for all \( f \). Several models are described in \[6\] Theorems 4.2.13, 4.3.2, 4.4.4. Using this we will define types \( \text{is-equiv}_n(f) \) for \( f \) a morphism \( f : \text{hom}(x, y) \) in an \( n \)-concrete \((\infty, 1)\)-category, for \( n = 0, 1, 2 \).

8.1 The 0-concrete case

The type \( \text{is-equiv}_0(f) \) is defined simply to be \( \text{is-equiv}(\text{hom}^+(f)) \). This, of course, is a proposition.
8.2 The 1-concrete case

We define the type is-equiv₁(f) to be the type of pairs consisting of:

- An element of is-equiv(hom⁺(f)); and
- An element of the homotopy fibre of hom⁺ over the inverse hom⁺(f)⁻¹ thus described.

In other words, a map f in a subcategory C of Type is invertible if its invertible in Type, and its inverse is also contained in C.

This, again, is a proposition: it’s a Σ-type whose base and fibre are both propositions.

If we wish to choose the model for is-equiv consisting of bi-invertible morphisms, we can simplify this description: it consists of morphisms g, g’ : hom(y, x) such that cmp(g, f) ≡ id(x) and cmp(f, g’) ≡ id(y).

8.3 The 2-concrete case

We define the type is-equiv₂(f) to be the type whose elements consist of:

- An element of is-equiv(hom⁺(f));
- An element (g, e) of the homotopy fibre of hom⁺ over the inverse hom⁺(f)⁻¹ thus described;
- Paths cmp(f, g) ≡ id and cmp(g, f) ≡ id.

Again, this is a proposition: it’s fibred over the proposition is-equiv(hom⁺(f)), and the standard proof that any two inverses are equal proves that any appropriate elements of the homotopy fibre of hom⁺ are equal.

As before, this admits a simplification if we use bi-invertibility as our definition of equivalence: again we just need g, g’ : hom(y, x) with cmp(g, f) ≡ id(x) and cmp(f, g’) ≡ id(y).

9 Functors

If one is serious about doing category theory with n-concrete (∞, 1)-categories, then one must certainly wish to define functors between them. A direct definition, sending objects to objects, and morphisms to morphisms, and so on, appears to have all the deficiencies that a direct definition of categories would have: the need for an infinite sequence of coherence data.

However, there is a standard trick for representing functors using only a well-developed theory of categories, using the notion of a cocartesian fibration. This approach is developed in the Joyal-Lurie theory of (∞, 1)-categories in \[11, Section 2.4 and thereafter\].

Suppose, therefore, we have an n-concrete (∞, 1)-category on object set A ⊔ B, and no homomorphisms from anything in B to anything in A:

(a : A)(b : B) → ¬hom(mir(b), inl(a)).

Given that, this category can be regarded as being over the category with two objects and one non-identity arrow. We call it an arrowlike category.
A morphism $f : \text{hom}(\text{inl}(a), \text{inr}(b))$ is \textit{cocartesian} if, for all $b' : B$, the map $\lambda g \to \text{cmp}(g, f)$ induces an equivalence

$$\text{hom}(\text{inr}(b), \text{inr}(b')) \to \text{hom}(\text{inl}(a), \text{inr}(b')).$$

We say that an arrowlike category as described above is a \textit{cocartesian fibration} if every object in $A$ has a cocartesian morphism out of it.

To start with, the concept of a cocartesian morphism is well-behaved:

\textbf{Theorem 9.} The type of proofs that a morphism is cocartesian is a proposition.
\textit{Proof.} It’s a dependent function type, valued in types of equivalences, all of which are propositions. \hfill \square

In fact, more than this is true, providing we use a univalence axiom (as discussed in Section 8 above):

\textbf{Theorem 10.} In a univalent $n$-concrete $(\infty,1)$-category, the type of cocartesian morphisms out of any object is a proposition.
\textit{Proof.} In fact (following the pattern so far) we prove this in detail only for $n = 0, 1, 2$.

First we show that, given any two cocartesian morphisms $f : \text{hom}(x, y)$ and $g : \text{hom}(x, z)$, there is an equivalence between $y$ and $z$.

The cocartesian nature of $f$ gives an element $i : \text{hom}(y, z)$ such that $\text{cmp}(i, f) \equiv g$. Similarly, there is an element $j : \text{hom}(z, y)$ such that $\text{cmp}(j, g) \equiv f$.

Now,

$$\text{cmp}(\text{cmp}(i, j), g) \equiv \text{cmp}(i, \text{cmp}(j, g)) \equiv \text{cmp}(i, f) \equiv g,$$

but since $\text{cmp}(\_ , g)$ is an equivalence this means that $\text{cmp}(i, j) \equiv \text{id}(z)$.

Similarly $\text{cmp}(\text{cmp}(j, i), f) \equiv f$ and hence $\text{cmp}(j, i) \equiv \text{id}(y)$.

By the discussion in Section 8 this gives us our equivalence. In general, for an $n$-concrete $(\infty, 1)$-category for $n > 2$, we should have to work more.

Now the appropriate univalence axiom gives that $y \equiv z$ and $\text{hom}^+(i)$ maps to $\text{id}(\text{obj}^+(y))$, and hence that $f \equiv g$. \hfill \square

As a corollary, we get that the notion of cocartesian fibration is well-behaved:

\textbf{Theorem 11.} The type of proofs that an arrowlike category is a cocartesian fibration is a proposition.
\textit{Proof.} This type is an dependent function type, and by the previous theorem it is valued in propositions, and hence a proposition itself. \hfill \square

\textbf{Remark 12.} While we can define functors, we have no chance of forming concrete functor categories. Indeed, we can’t even form the arrow category $\text{Fun}(\Delta^1, \text{Type})$, as mentioned above in Remark 7.

\section{Prospects for further work}

\subsection{Further constructions}

Clearly the methods described above do not constitute a full development of category theory. One may reasonably ask about $n$-concrete versions of other popular constructions: with what truncation hypotheses can they be defined?
10.2 Uniform definitions

The definitions above depend on an understanding of notions of \((n,1)\)-category; we have restricted detailed discussion to cases of small \(n\) where appropriate definitions are well-known.

Nevertheless, families of general definitions exist\[10\], and it would perhaps be worthwhile to see if any of them can painlessly be implemented in homotopy type theory.

The aim would be a well-defined family of definitions of \(n\)-concrete \((\infty,1)\)-category, valid for all \(n \in \mathbb{N}\).

10.3 Concrete categories in exotic homotopy type theories

At present only one homotopy type theory has received extensive study: the one modelled by the homotopy theory of spaces.

It was once the case that the only homotopy theory that was studied was the homotopy theory of spaces. However, with the help of the language of model categories\[5\], it was progressively realised that this is just one in a vast family of homotopy theories, many of them helpful even in furthering understanding of spaces themselves. A complete list of examples would be longer than this paper; one very modest example is the theory of pointed spaces, where preservation of the basepoint is forced.

The author suspects that exotic homotopy type theories, corresponding to other homotopy theories, will soon receive heavy attention. This will naturally augment the collection of concrete categories: given a type theory containing universes of pointed types, the difficulties of subsection \([5]\) would vanish altogether.

References

[1] Agda, available at [http://wiki.portal.chalmers.se/agda/pmwiki.php](http://wiki.portal.chalmers.se/agda/pmwiki.php).
[2] Benedikt Ahrens, Chris Kapulkin, and Michael Shulman, Univalent categories and the Rezk completion (2013), 27 pp., available at [http://arxiv.org/abs/1303.0584](http://arxiv.org/abs/1303.0584).
[3] Francis Borceux, Handbook of categorical algebra. I, Encyclopedia of Mathematics and its Applications, vol. 50, Cambridge University Press, Cambridge, 1994. Basic category theory. MR1291599 (96g:18001a).
[4] Guillaume Brunerie, Development of homotopy type theory in Agda, available at [https://github.com/HoTT/HoTT-Agda](https://github.com/HoTT/HoTT-Agda).
[5] W. G. Dwyer and J. Spaliński, Homotopy theories and model categories, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126, DOI 10.1016/B978-044481779-2/50063-1, (to appear in print). MR1361887 (96h:55014).
[6] Homotopy Type Theory: Univalent Foundations of Mathematics, available at [http://homotopytypetheory.org/book/](http://homotopytypetheory.org/book/).
[7] James Cranch, Algebraic theories and \((\infty,1)\)-categories, PhD thesis, University of Sheffield, 2010, [arXiv:1011.3243v1[math.AT]](https://arxiv.org/abs/1011.3243v1).
[8] James Cranch, Homotopy type theory in Agda, available at [https://github.com/jcranch/HoTT-Agda](https://github.com/jcranch/HoTT-Agda).
[9] André Joyal, Quasi-categories and Kan complexes, J. Pure Appl. Algebra 175 (2002), no. 1-3, 207–222, DOI 10.1016/S0022-4049(02)00135-4. Special volume celebrating the 70th birthday of Professor Max Kelly. MR1935979 (2003h:55026).
[10] Tom Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004. MR2094071 (2005h:18030)

[11] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659 (2010j:18001)

[12] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR1712872 (2001j:18001)