Kac–Moody algebras and controlled chaos

Daniel H Wesley

Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

E-mail: D.H.Wesley@damtp.cam.ac.uk

Received 7 November 2006, in final form 5 December 2006
Published 12 January 2007
Online at stacks.iop.org/CQG/24/F7

Abstract
Compactification can control chaotic Mixmaster behaviour in gravitational systems with \( p \)-form matter: we consider this in light of the connection between supergravity models and Kac–Moody algebras. We show that different compactifications define ‘mutations’ of the algebras associated with the noncompact theories. We list the algebras obtained in this way, and find novel examples of wall systems determined by Lorentzian (but not hyperbolic) algebras. Cosmological models with a smooth pre-big bang phase require that chaos is absent: we show that compactification alone cannot eliminate chaos in the simplest compactifications of the heterotic string on a Calabi–Yau, or M theory on a manifold of \( G_2 \) holonomy.

PACS numbers: 11.25.Mj, 98.80.Cq, 04.50.+h, 05.45.Gg

Introduction

It has long been known that gravitational systems typically approach a big crunch chaotically—a phenomenon sometimes called ‘Belinskii–Khalatnikov–Lifshitz (BKL) oscillations’ or ‘Mixmaster behaviour’ [1]. Spacetime becomes highly anisotropic, and the directions and rates of contraction (and expansion) oscillate chaotically until the big crunch is reached. This phenomenon deserves study for several reasons. For one, it is quite common among suitably general solutions of the Einstein equations near spacelike singularities. Also, the study of supergravity models in this regime has revealed algebraic structures that are related to a conjectured underlying U-duality symmetry group of M theory [2]. Lastly, avoiding chaotic behaviour is essential for any cosmological model with a pre-big bang phase; the universe must enter the expanding era in a nearly isotropic and homogeneous state, which seems unlikely after a chaotic epoch.

Here we characterize the chaotic properties of the low-energy supergravities obtained from simple compactifications of the heterotic string and M theory. We begin with the remarkable fact that the dynamics of these theories in the BKL limit is controlled by the \( BE_{10} \) and \( E_{10} \)
Kac–Moody algebras [3]. This structure is invariant under toroidal compactification [4], but non-toroidal compactification changes the standard picture [5] by deleting and adjoining roots of these algebras according to simple rules. Each compactification thereby produces a ‘mutation’ of the original algebra. We apply this fact and obtain novel examples where the dynamics is controlled by Lorentzian algebras that are not hyperbolic, previously seen only in Einstein gravity in spacetime dimension $\geq 11$ or when the noncompact spectrum is constrained as in [6]. We also list the compactifications for which chaos is suppressed: these provide a set of potential cosmological models whose pre-big bang phases are free of chaos.

In the present work, we consider only ‘simple’ Kaluza–Klein compactifications: quantum effects, fluxes, D- or M-branes, orbifolds, conifolds, etc, are not included, though it would be interesting to discover how these affect our conclusions. We also work in a free field approximation, but even if potentials are included, the only possibility relevant for chaos appears to be a matter component with $P > \rho$ [7], as required by the cyclic universe [8]. Most other interactions [9], as well as matter with $P < \rho$, are irrelevant near a big crunch.

The wall system

We first review some essential facts regarding the wall systems corresponding to gravitational theories, following [3]. We study cosmological spacetimes with string frame metric

$$\text{d} s^2 = -N^2 e^{-2\beta_0(t)} \text{d} t^2 + \sum_{i=1}^d e^{-2\beta_i(t)} [\omega_i]^2 ,$$

in which spatial curvature is included through the choice of $\omega_i = \omega_i(x) \text{d} x^i$, and $d = 9$ or 10, depending on whether we are studying a ten-dimensional superstring theory or eleven-dimensional supergravity. In the superstring case, we define $\beta_0(t) = \sum_{i=1}^9 \beta_i(t) + 2 \Phi(t)$, and fix $N = n \exp(-\beta_0)$, while for the M theory case $N = n \exp(-\sum_{j=1}^{10} \beta_j)$. In either case, we will refer to the variables $\beta_\mu$ with $\mu = 0, \ldots, 9$ or 1, $\ldots, 10$ as required. The spacetime action becomes

$$S = \int \left( \eta_\mu^\nu \frac{d\beta^\mu}{dt} \frac{d\beta^\nu}{dt} - n V(\beta) \right) \text{d} t,$$

where $\eta_\mu^\nu$ is a flat metric of signature $(- + \cdots +)$. The effects of $p$-form energy densities, spatial gradients, and curvature in the physical spacetime are thereby encoded in the Toda-type potential

$$V(\beta) = \sum_A \{c_A \exp(-2w_A \beta^\mu)$$

for the motion of the point $\beta^\mu(t)$ in an auxiliary spacetime. The $\{w_A\}$ are ‘wall forms,’ indexed by $A$ and with components $w_A^\mu$. They are determined by the $p$-form menu, and each corresponds to a Kasner stability condition [1, 3, 5]. The coefficients $c_A$ depend on the initial energy densities in $p$-form fields, curvature perturbations, spatial gradients, and other contributions.

It is conventional to simplify matters by restricting our attention to a smaller set $\{r_A\} \subset \{w_A\}$ of ‘dominant walls,’ which are not hidden behind other walls. In the case of superstrings and M theory, the Cartan matrix $A_{AB} = 2(r_A \cdot r_B)/(r_A \cdot r_A)$, computed using the natural metric $\eta_\mu^\nu$ on the $\beta$-space, is precisely that of the $E_{10}$ and $BE_{10}$ Kac–Moody algebras [3]. The dominant walls play the role of simple roots of the algebra. For the $E_{10}$ and $BE_{10}$ cases, the point $\beta^\mu(t)$ is trapped by the dominant walls, and so the corresponding string and M theory models have only chaotic solutions, undergoing an infinite number of BKL oscillations as they approach the big crunch.
Compactification and mutation

Compactification changes the wall system associated with a given theory. To see why, consider that when spacetime is noncompact, each \( p \)-form possesses a spatially homogeneous mode that can grow rapidly near the big crunch. These modes correspond to dominant walls and are responsible for chaos. The wall system is unchanged by toroidal compactification \([3, 4]\), but it was shown in \([5]\) that compactification on more general manifolds can avoid chaos by forbidding the spatially homogeneous modes of some \( p \)-form fields. The energy density in the remaining modes scales like dust or radiation, which are irrelevant during a contracting phase.

The influence of compactification on the wall system is expressed by a ‘selection rule’ \([5]\): if a Betti number \( b_j \) of the compact manifold vanishes, then remove from \( \{w_A\} \) all \( p \)-form walls arising from the electric modes of a \( j \)-form or the magnetic modes of a \((j-1)\)-form. (We take the ‘\( p \)’ in ‘\( p \)-form’ to be the number of indices on its gauge potential.) There is a corresponding rule for gravitational walls, which we will not employ here. The selection rules are subject to a genericity assumption that the compact manifold \( M \) does not factor, both topologically and metrically, as \( M = M_1 \times M_2 \).

Since compactification deletes walls, it modifies the dominant wall set. If the Cartan matrix for the new dominant wall set obeys the generalized Cartan conditions \([10]\), it defines a new algebra, the ‘mutation’ of the original algebra. There is no \textit{a priori} reason that special properties of the dominant wall set should survive compactification, and in the next section we will give examples where the new wall systems do not define an algebra.

A natural question arises: can compactification remove enough walls to allow non-chaotic solutions to a previously chaotic theory, such as string or M theory? This was addressed in \([5]\), where some non-chaotic solutions were found. Here we will give a more complete answer, using the ‘coweights’ \( \Lambda^\Lambda_A \) introduced in \([3]\). If the \( \{r_A\} \) are linearly independent and complete, the coweights are defined by the condition \( r_A \mu^\Lambda \Lambda^\Lambda_B = \delta_A^\Lambda \). The region far from the walls is then given by the cone \( W^+ \) of linear combinations of coweights with nonnegative coefficients. Non-chaotic solutions of \((2)\) are null rays whose velocities lie in \( W^+ \). These solutions exist only when there are both spacelike and timelike coweights; thus proving the existence of non-chaotic solutions reduces to computing the norms of the coweights.

Heterotic string and M theory

We now possess the tools required to describe the chaotic properties of a theory, given its \( p \)-form menu and the Betti numbers of its compactification manifold, which we now apply to the supergravities obtained from the heterotic string and M theory. For each theory, we begin with the full set of billiard walls as given in \([3, 4]\). We then remove walls in accordance with the selection rules described in the previous section and in \([5]\), and find the new relevant walls. From these we calculate the new coweights and new Cartan matrices, which determine the chaotic properties of the compactified theories as per our discussion above.

The results for the heterotic string are summarized in figure 1 and table 1. We assume compactification on a six-manifold, which has three relevant Betti numbers \( b_1, b_2 \) and \( b_3 \), and therefore eight possible vanishing Betti number combinations. Our results indicate that controlled chaos requires either \( b_3 \) or both \( b_1 \) and \( b_2 \) to vanish. This agrees with numerical searches for Kasner solutions with controlled chaos carried out by the author. The formulation reported herein is superior to that of \([5]\) and the numerical search since we can say definitively that these are the \textit{only} solutions with controlled chaos. Significantly, Calabi–Yau compactifications, which have \( b_1 = 0 \) but \( b_2, b_3 > 0 \), appear incompatible with controlled chaos.
Figure 1. The root systems arising from the compactification of the heterotic string on a six-manifold. There are two unnamed rank ten diagrams, denoted $X_{10}$ and $X'_{10}$. ‘$G$’ denotes a gravitational wall.

Table 1. Summary of the compactifications of the heterotic theory on $M_6$. Vanishing Betti numbers are denoted by a ‘0’, non-vanishing ones with a ‘+’. ‘$G$’ denotes a gravitational wall form that, when deleted, leaves a remaining root system of Lie type. Two previously unnamed root systems appear, denoted $X_{10}$ and $X'_{10}$. The $b_1 = b_3 = 0$ examples possess special properties as described in the text.

| Betti numbers | Number of $\Lambda^+$ that are: |
|---------------|----------------------------------|
| $b_1$ $b_2$ $b_3$ Chaotic? | s-like | null | t-like | ‘Root system’ |
| + + + Yes | 0 | 2 | 8 | $BE_{10}$ |
| 0 + + Yes | 0 | 3 | 7 | $DE_{10}$ |
| + 0 + Yes | 0 | 2 | 8 | $BE_{10}$ |
| 0 0 + No | 1 | 8 | 1 | $G + E_9^{(1)}$ |
| + + 0 No | 5 | 2 | 3 | $X_{10}$ |
| 0 + 0 No | - | - | - | $G + X'_{10}$ |
| + 0 0 No | 5 | 2 | 3 | $G + B_9$ |
| 0 0 0 No | 1 | 1 | 8 | $AE_9$ |
A wide variety of root systems arise from compactifications of the heterotic string. While the original wall system is described by a Dynkin diagram, one sees from figure 1 that this property is not shared by all of the wall systems after compactification. However, a valid Dynkin diagram is always obtained by omitting a single gravitational wall. The diagrams so obtained include algebras of Lorentzian \( (X_{10}, X'_{10}) \), hyperbolic \( (BE_{10}, DE_{10}, AE_{9}) \), affine \( (E(1)_{9}) \), and finite \( (B_9) \) type.

Compactification provides novel examples of systems where the dynamics is controlled by algebras that are Lorentzian but not hyperbolic—i.e. the simple root Gram matrix has signature \((-++\cdots+)\), but deleting any single node does not yield a finite or affine algebra. The relevant new algebras are \( X_{10} \) and \( X'_{10} \) in figure 1. They are examples of the ‘very extended’ Lie algebras of the type studied in [11], with various possible central node assignments. The appearance of these algebras is notable, since previous examples of algebras arising in the BKL limit have all been hyperbolic, except for the \( AE_d \) series with \( d \geq 10 \) for pure gravity in \((d + 1)\) dimensions, and some Lorentzian algebras obtained by geometric constraints on M theory fields [6].

Special care is required when all three Betti numbers vanish, for the relevant wall set is linearly independent but incomplete. We can recover the coweight description by introducing a (spacelike) covector \( \hat{\Lambda}^\vee \) such that \( r_{\alpha\mu} \hat{\Lambda}^\vee_{\mu} = 0 \) for all \( A \). The \( \Lambda^\vee_{\alpha} \) are then determined up to \( \Lambda^\vee_{\alpha} = \Lambda^\vee_{\alpha} + k^\alpha \hat{\Lambda}^\vee \) for constants \( k^\alpha \), which we fix by requiring \( \eta_{\mu\nu} \Lambda^\vee_{\alpha} \hat{\Lambda}^\vee_{\nu} = 0 \), which amounts to minimizing the norm of the coweights. Now by analogy to the usual case we can treat \( \hat{\Lambda}^\vee \) as a coweight and write points in \( W^\ast \) as \( \hat{\lambda} \hat{\Lambda}^\vee + \sum A \lambda_A \Lambda^\vee_A \), with \( \lambda_A \geq 0 \). Since \( \hat{\lambda} \) may take any value, \( W^\ast \) is no longer a cone. Nonetheless, \( \hat{\Lambda}^\vee \) is spacelike and therefore non-chaotic solutions exist. When only \( b_2 > 0 \), the dominant walls are not linearly independent, and so a coweight description is impossible; however, it is clear that this compactification has non-chaotic solutions since the \( b_3 = 0 \) compactification does.

We next consider eleven-dimensional supergravity. The results are shown in figure 2. The chaotic properties are entirely controlled by two Betti numbers, \( b_1 \) and \( b_3 \), and in compactifications to four dimensions we have \( b_1 = b_3 \) by Poincaré duality. Therefore the only relevant algebras for compactification to four dimensions are \( E_{10} \) (when \( b_4 > 0 \)) and \( AE_{10} \) (when \( b_3 = b_4 = 0 \)). \( E_{10}^+ \) was found to control the dynamics of an M theory truncation in [6], though via a different construction and without the additional gravitational wall present here.

Our results imply that chaos can be controlled if \( b_3 = b_4 = 0 \). This yields \( AE_{10} \), which is Lorentzian (but not hyperbolic) and possesses spacelike coweights. This is expected: with this compactification the massless bosonic sector is identical to vacuum Einstein gravity, whose billiard system is described by \( AE_{10} \) and is not chaotic [1].
Conclusions

We have studied how compactification influences the algebraic structures controlling the dynamics of string and M theory models near a big crunch. Different compactifications of the heterotic string and M theory, defined by their vanishing Betti numbers, lead to ‘mutations’ of the $BE_{10}$ and $E_{10}$ algebras that control the BKL dynamics of the noncompact theories. We have listed the set of algebras obtained from simple compactifications, and have found new examples where the BKL dynamics is controlled by Lorentzian (but not hyperbolic) algebras, denoted here by $X_{10}$, $X'_{10}$ and $E^{+++}_{7}$. Previously, algebras in this class have only been observed controlling the BKL dynamics of pure Einstein gravity in $(d + 1) \geq 11$ dimensions, where the $AE_{d}$ series appears, or in [6].

Our results rule out controlling chaos in the sense of [5] within the simplest string models of four-dimensional physics. For the heterotic string compactified on a Calabi–Yau manifold [12], the first Betti number $b_1 = 0$, but both $b_2$ and $b_3$ are nonzero. As table 1 indicates, compactification with only $b_1$ vanishing is insufficient to control chaos. A similar problem arises with M theory, compactified on a seven manifold of $G_2$ holonomy [13]. In this case $b_1$ and $b_2$ are nonzero, and we have shown here that chaos is inevitable for a compactification with these Betti numbers.

Our work connects with other results regarding modifications of the algebras that control BKL dynamics. Lorentzian subalgebras of $E_{10}$ were uncovered in [6] through various truncations of the spectrum of M theory. In [4] chains of theories controlled by the same (hyperbolic) algebra are obtained by dimensional oxidation and reduction. The results presented here indicate that it is possible to jump between chains through oxidation or reduction on manifolds with vanishing Betti numbers.

It would be interesting to understand how the selection rules are modified in more general string and M theory compactifications (including flux, D- or M-branes, orbifolds, etc—see [14]). These wider classes of string models hold the promise of providing a realistic low energy particle spectrum, and we expect that the techniques employed here would be useful for determining whether controlled chaos is possible. It would be significant for cosmology if a satisfactory compactification could be found, as it might form the basis of a new string cosmological model with a smooth pre-big bang phase.

Acknowledgments

We are grateful to Marc Henneaux and Daniel Persson for many useful comments, assistance with terminology, identifying the $E^{+++}_{7}$ algebra, and for pointing out a number of interesting references. We enjoyed some informative conversations with Malcolm Perry and Neil Turok regarding Kac–Moody algebras and their significance in physics during the later stages of this work. We thank Katie Mack for a careful reading of this communication, and Latham Boyle and Andrew Tolley for their comments on an earlier version.

References

[1] Lifshitz E M and Khalatnikov I M 1963 Adv. Phys. 12 185
Misner C W 1969 Phys. Rev. Lett. 22 1071
Belinskii V A and Khalatnikov I M 1973 Sov. Phys.—JETP 36 591
Demaret J, Henneaux M and Spindel P 1985 Phys. Lett. B 164 27
[2] Julia B 1985 Lectures in Applied Mathematics vol 21 (Providence, RI: American Mathematical Society/SIAM) p 335
Julia B 1980 Report No LPTENS 80/16 Invited paper presented at Nuffield Gravity Workshop (Cambridge, 22 June–12 July 1980) (Cambridge: Cambridge University Press)

[3] Damour T, Henneaux M and Nicolai H 2002 Phys. Rev. Lett. 89 221601

Damour T and Henneaux M 2000 Phys. Rev. Lett. 85 920

Damour T and Henneaux M 2001 Phys. Rev. Lett. 86 4749

Damour T, Henneaux M, Julia B and Nicolai H 2001 Phys. Lett. B 509 323

Henneaux M and Julia B 2003 J. High Energy Phys. JHEP05(2003)047

[4] Damour T, de Buyl S, Henneaux M and Schomblond C 2002 J. High Energy Phys. JHEP08(2002)030

de Buyl S and Schomblond C 2004 J. Math. Phys. 45 4464

[5] Wesley D H, Steinhardt P J and Turok N 2005 Phys. Rev. D 72 063513

[6] Henneaux M, Leston M, Persson D and Spindel P 2006 J. High Energy Phys. JHEP10(2006)021

[7] Erickson J K, Wesley D H, Steinhardt P J and Turok N 2004 Phys. Rev. D 69 063514

[8] Khoury J, Ovrut B A, Steinhardt P J and Turok N 2001 Phys. Rev. D 64 123522

Steinhardt P J and Turok N 2002 Phys. Rev. D 65 126003

[9] Damour T, Henneaux M, Rendall A D and Weaver M 2002 Ann. H. Poincare 3 1049

[10] Kac V 1990 Infinite Dimensional Lie Algebras (Cambridge: Cambridge University Press)

[11] Gaberdiel M R, Olive D I and West P C 2002 Nucl. Phys. B 645 403

[12] Candelas P, Horowitz G T, Strominger A and Witten E 1985 Nucl. Phys. B 258 46

[13] Duff M J, Nilsson B E W and Pope C N 1986 Phys. Rep. 130 1

Joyce D D 2000 Compact Manifolds with Special Holonomy (Oxford: Oxford University Press)

[14] Brown J, Ganor O J and Helfgott C 2004 J. High Energy Phys. JHEP08(2004)063

Brown J, Ganguli S, Ganor O J and Helfgott C 2005 J. High Energy Phys. JHEP06(2005)057

Bagnoud M and Carlevaro L 2006 J. High Energy Phys. JHEP11(2006)003 (Preprint hep-th/0607136)