Quantum adiabatic algorithm. The quantum adiabatic algorithm is a generic algorithm proposed for the solution of a variety of combinatorial optimization and decision problems involving binary variables from the NP-complete family. One considers a Hamiltonian involving \( N \) of a variety of combinatorial optimization and decision problems involving binary variables from the NP-complete family. An instance of random exact cover problem is a set of clauses \( C \), each clause \( C_k \) containing \( d_k \) bits in a state that is the symmetric superposition of all \( 2^N \) possible bit assignments. The system is initially prepared at the end of the algorithm. By adiabatic theorem, the system will remain in its ground state with high probability at the time of the algorithm (its complexity) is given by \( \lambda(T) = 1 \) for which the gap is minimal will be referred to as the bottleneck of the algorithm.

The Hamiltonian is implementable using only polynomially large number of gadgets. The system is initially prepared in a state that is the symmetric superposition of all \( 2^N \) possible bit assignments — an exact ground state of the second (“driver”) term in (1), which corresponds to the uniform magnetic field \( \lambda \) in the direction orthogonal to the quantization axis of computational basis \(|\hat{x}\rangle\). The parameter \( \lambda \) is changed in time from \( \lambda(0) \gg 1 \) initially to \( \lambda(T) = 0 \) at the end of the algorithm. By adiabatic theorem, the system will remain in its ground state with high probability provided that \( d\lambda/df \ll \Delta^2(\lambda) \), where \( \Delta(\lambda) = E_1(\lambda) - E_0(\lambda) \) is the energy gap between the ground state of \( \hat{H}(\lambda) \) and its first excited state. At time \( t = T \) the system will be in a superposition state of configurations with the optimal cost and one of the optimal solutions may be obtained by preforming a final measurement on the qubits. The running time of the algorithm (its complexity) is given by \( T \sim 1/\Delta_{\text{min}}^2 \), where \( \Delta_{\text{min}} \) is the minimum value of the gap as a function of \( \lambda \). The value \( \lambda = \lambda_\text{c} \) for which the gap is minimal will be referred to as the bottleneck of the algorithm.

It is known that the minimum gap can be exponentially small in \( N \) in the worst case. A really interesting question is how adiabatic algorithm performs on random instances of combinatorial optimization problems — the typical-case complexity. Historically, the benchmark problem for the quantum adiabatic algorithm has been the exact cover problem. An instance of random exact cover problem is a set of \( N \) bits and \( M \) clauses, each clause \( C \) containing three bits \((x_i, x_{iC}, x_{kC})\) chosen uniformly at random. One seeks an assignment such that bits in each clause add up to 1: 

\[
\sum_{j \neq k} x_{ij} + x_{ik} + x_{kj} = 1.
\]

A cost \( (x_{ij} + x_{ij} + x_{kj})^2 \geq 0 \) is assigned to each clause so that the total cost \( E(x) \) (given by the sum over individual clauses) is zero for satisfying assignments. In terms of Pauli operators [where \( \hat{\sigma}_i^z|x_i\rangle = (-1)^{x_i}|x_i\rangle, \hat{\sigma}_i^z|x_i\rangle = |\bar{x}_i\rangle \)], the quantum Hamiltonian is written as (cf. Eq. (2) of Ref. [1]):

\[
\hat{H}(\lambda) = M - \frac{1}{2} \sum_i B_i \hat{\sigma}_i^z + \frac{1}{2} \sum_{\langle ij \rangle} (\hat{\sigma}_i^x \hat{\sigma}_j^x + \hat{\sigma}_i^y \hat{\sigma}_j^y + \hat{\sigma}_i^z \hat{\sigma}_j^z) - \lambda \sum_i \hat{\sigma}_i^z,
\]

where \( B_i \) is the number of clauses in which bit \( i \) appears and the sum in the third term is over all clauses \( \langle ij \rangle \).
The first three terms describe the problem Hamiltonian that is diagonal in $\hat{\sigma}^z$-representation, while the fourth term describes a magnetic field in the transverse direction.

It has been observed that the bottleneck of simulated annealing (which can be thought of as a classical counterpart of quantum adiabatic algorithm) is the vicinity of temperature-driven phase transition. One might conjecture that the bottleneck of quantum adiabatic algorithm is the vicinity of transverse-field-driven quantum phase transition at finite $\lambda = \lambda_0 > 0$. This indeed had been confirmed in a few random NP-complete problems [3]. However, there is no reason to expect that this scenario is universal; even the existence of the phase transition cannot be guaranteed in some models [4]. Ref. [1] asserts that the bottleneck of the quantum adiabatic algorithm for random exact cover is unconnected to the quantum phase transition but is due to “accidental” avoided crossings of energy levels corresponding to localized states for infinitesimal transverse fields ($\lambda \to 0$ as $N \to \infty$). The associated gap is related to the overlap between localized states and is expected to be exponentially small. It is claimed that this mechanism is not peculiar to the random exact cover problem, but applies to a wide range of NP-complete problems defined on random hypergraphs.

The possibility of avoided crossings for $\lambda < \lambda_0$ had been raised before, for a model with a quasi-continuous (level spacings $\ll 1$) spectrum [5], but was thought not to occur for models with a discrete spectrum, such as exact cover, K-SAT, etc. Ref. [1] predicts avoided crossings close to the end of the algorithm whereas recent quantum Monte Carlo (QMC) simulations show the bottleneck in the middle of the algorithm [6]. However, QMC studies consider a different instance ensemble (extremely rare instances with a unique satisfying assignment are chosen) and the problem sizes considered ($N = 256$) may be too small (Ref. [1] estimates that the described mechanism may not kick in until $N \sim 10^5$). We will demonstrate that the exponential degeneracy of the ground state, which is a distinguishing feature of random NP-complete problems with discrete spectrum addressed in [7], dooms the proposed mechanism. Note that when the instance is not drawn from a uniformly random ensemble but is instead crafted to contain exactly one global and one local minimum separated by $N$ bit flips, the avoided crossing does take place for $\lambda \to 0$ [7].

**Overview of perturbation theory analysis.** Ref. [1] starts with the classical Hamiltonian for an instance with $N$ bits and $M$ clauses and develops a perturbation theory in a small parameter $\lambda \ll 1$. In this limit the perturbation theory is expected to be **locally convergent**. One may consider a global minimum $E(x_0) = 0$ and a local minimum at $E(x_1) = 1$. For small $\lambda > 0$, these energy levels acquire perturbative corrections

$$E^{(M)}_{x_0}(\lambda) \approx \delta E^{(M)}_{x_0}(\lambda), \quad E^{(M)}_{x_1}(\lambda) \approx 1 + \delta E^{(M)}_{x_1}(\lambda).$$

(3)

For some value $\lambda$ such that $\delta E^{(M)}_{x_0}(\lambda) - \delta E^{(M)}_{x_1}(\lambda) = 1$, the levels corresponding to states localized near $x = x_0$ and $x = x_1$ will be equal in energy. The minimum gap will be non-zero, but exponentially small, provided that $x_0$ and $x_1$ differ by $O(N)$ bit flips.

In a drastic simplification, Ref. [1] uses a clever trick to show that it suffices to examine only the global minima. Once a clause contradicting $x_1$ is removed, both $x_0$ and $x_1$ will have zero cost. Writing $E^{(M-1)}_x(\lambda)$ for the energy of eigenstate localized near $x$ for the instance with $M - 1$ clauses, and $\lambda_*$ denoting the solution to

$$E^{(M-1)}_{x_0}(\lambda_*) - E^{(M-1)}_{x_1}(\lambda_*) = 1,$$

(4)

it can be argued that an instance with $M$ clauses should have an avoided crossing for some $\lambda < \lambda_*$. This follows from inequality satisfied for small $\lambda$,

$$0 < E^{(M)}_x(\lambda) - E^{(M-1)}_x(\lambda) < 1,$$

(5)

obtained by treating the $M$-th clause as a perturbation. Eqs. (4) and (5) together imply $E^{(M)}_{x_0}(\lambda_*) > E^{(M)}_{x_1}(\lambda_*)$, but the opposite inequality holds for $\lambda = 0$. Therefore, the energies, being continuous functions of $\lambda$, must be equal for some $\lambda < \lambda_*$. This construction is visualized in Fig. [1](left).

In general, one considers a random instance with $M - 1$ clauses chosen uniformly at random and some solution $x_1$. A new uniformly random instance with $M$ clauses is formed by adding a new random clause. With **finite probability**, the new clause is violated by $x_1$ so that levels corresponding to $x_1$ and some other solution $x_0$ satisfied by the new clause cross for $\lambda$ such that $\Delta E_{10} = E^{(M-1)}_{x_0}(\lambda) - E^{(M-1)}_{x_1}(\lambda) \sim 1$. Failing that, random instances with $M + 1$, $M + 2$, etc. clauses may be generated by adding more random clauses, which ensures that avoided crossing takes place with probability tending to one.

The necessary condition for this mechanism is the convergence of the perturbation theory. Ref. [1] justifies its use by showing that $\lambda_* \to 0$ as $N \to \infty$. Within ordinary (non-degenerate) perturbation theory up to the 4th order, the energy of the state corresponding to a solution with zero cost is (cf. Eq. (33) in Ref. [1])

$$E_x(\lambda) \approx \text{common term} + \lambda^4 \sum_{ijk} \left( \frac{4/(B_i B_k)}{1 - 4/(B_j + B_k)^2} x_i + \frac{4/(B_i B_k)}{1 - 4/(B_i + B_k)^2} x_j + \frac{4/(B_i B_j)}{1 - 4/(B_i + B_j)^2} x_k \right).$$

(6)
Therefore, Ref. [1] claims that avoided crossings take place for \( \lambda \) such that avoided crossings takes place for \( \lambda = \lambda_* \), the levels of the original instance will cross for somewhat smaller \( \lambda \). Right: The effect of exponential degeneracy is depicted here. Level splitting (bottom figure) is large only for randomly chosen levels (blue lines) so they will intersect (top figure) with high probability. Level with the smallest perturbation correction \( (\mathbf{x}_0, \mathbf{x}_1) \) may not intersect until \( \lambda \) is large as the level splitting does not scale with \( N \).

The common term is the same for all configurations but the second term is configuration-dependent. For uniformly random ensemble, \( B_i \) [defined in the text surrounding Eq. (2)] are random Poisson-distributed variables with mean \( 3M/N = O(1) \), each term in the sum over \( M = O(N) \) clauses is a random \( O(1) \) variable. By central limit theorem, the sum is approximately a Gaussian of width \( O(\sqrt{N}) \) so that for two different bit configurations \( \Delta E_{10} \sim \sqrt{N}\lambda^4 \). Therefore, Ref. [1] claims that avoided crossings take place for \( \lambda \sim 1/N^{1/8} \ll 1 \), well within the region of applicability of perturbation theory.

**Effects of exponential degeneracy.** The argument at the end of the previous section overlooks the fact that the values of \( \mathbf{x}_0 \) and \( \mathbf{x}_1 \) are correlated with realizations of random instances. While in many circumstances neglecting correlations may not lead to qualitative changes, an important factor in this case is the large number of solutions with zero cost. Even if perturbative corrections to all solutions are assumed independent random variables, it can only be established that *randomly chosen* energy levels corresponding to \( E = 0 \) and \( E = 1 \) may intersect for \( \lambda \sim 1/N^{1/8} \), as depicted in Fig. 1 (right). Since we are interested in the intersections with the ground state, we require that \( \mathbf{x}_1 \) and \( \mathbf{x}_0 \) correspond to the ground states of Hamiltonian with \( M-1 \) and \( M \) clauses respectively, i.e. that \( E^{(M-1)}_{\mathbf{x}_1}(\lambda) \) and \( E^{(M)}_{\mathbf{x}_0}(\lambda) \) be smallest. But with this restriction, we will see that \( E_{\mathbf{x}_0}(\lambda) - E_{\mathbf{x}_1}(\lambda) \sim \lambda^4 \), so that avoided crossings are unlikely until \( \lambda \sim 1 \), which may be outside the radius of convergence of perturbation theory.

It is important to realize that the number of solutions of random exact cover is exponential in \( N \), even near the satisfiability threshold \( \alpha_s = M/N \approx 0.626 \) where the random instance is satisfiable with probability \( 1/2 \). Just prior to adding a random clause which makes an instance unsatisfiable, some bits are frozen (have the same values in all solutions) while others are not. The latter, “soft” bits, contribute to the exponential degeneracy. In the numerical simulations of Ref. [1] all bits that do not appear in any clause as well as clauses with two or more bits that do not belong to any other clause are removed. This ensures that flipping two bits does not lead to another solution \( E = 0 \) as that would make the expression formally infinite, indicating that a degenerate perturbation theory should be used instead. Such hypergraph trimming does not affect the satisfiability of the instance, can be done in polynomial time prior to running quantum adiabatic algorithm, and removes “trivial degeneracies”. However, it does not remove all degeneracies: there will remain soft bits; moreover whether a given a given bit is soft depends on the assignment...
of “hard”, or frozen bits. In Fig. 2 (left) we plot the number of solutions of trimmed hypergraph as a function of $N$ at satisfiability threshold. It is seen that the number of solutions grows exponentially as $N \approx \exp(0.021N)$. The smallness of exponent is the reason this effect only starts to manifest itself for $N \gtrsim 100$.

Once the presence of exponentially many solutions [$N \sim \exp(cN)$] is taken into account, the density of states $E(x)(\lambda)$ for all solutions is written as

$$\rho(E) = N \frac{1}{\sqrt{2\pi N \sigma}} \exp \left[ -\frac{(E - \bar{E})^2}{2\sigma^2 N} \right] \sim \frac{1}{\sqrt{N}} \exp \left[ cN - \frac{(E - \bar{E})^2}{2\sigma^2 N} \right],$$

where the energy levels are assumed to have Gaussian distribution with mean $\bar{E}$ and variance $\sigma \sqrt{N}$, and where $\sigma = O(\lambda^4)$. Then the energy of the ground state $E_0(\lambda)$ (corresponding to configuration with the smallest perturbation theory correction) can be estimated by solving $p(E_0(\lambda)) \approx 1$. This implies

$$E_0 \approx \bar{E} - N \sigma \sqrt{c} + O \left( \frac{\log N}{N} \right).$$

Notice that this correction is proportional to $N$ rather than $\sqrt{N}$. The fluctuations of $E_0$ are $O(\sigma)$ and have a Gumbel distribution $\tilde{\mathcal{G}}$. This linear scaling is verified numerically in Fig. 2 (right) where we plot the difference of the largest perturbative correction and the smallest perturbative correction (so that the common term cancels out) as a function of $N$.

Next, we show that the level spacing is only $O(\sigma) \equiv O(\lambda^4)$ and does not scale with $N$. Let us compute the probability that the gap to the first excited state is at least $\Delta$. First, pick a reference energy $E_{\text{ref}}$ and write down the probability that $\exp(cN) - 1$ levels have higher energy $E_0 = E_{\text{ref}} - \Delta$:

$$\exp(cN) \left[ 1 - \int_{-\infty}^{E_{\text{ref}}} e^{-(E - \bar{E})^2/(2\sigma^2 N)} \frac{dE}{\sigma \sqrt{2\pi N}} \right]^{\exp(cN) - 1} \frac{1}{\sigma \sqrt{2\pi N}} e^{-(E_{\text{ref}} - E - \Delta)^2/(2\sigma^2 N)} dE_{\text{ref}}.$$

This expression is non-negligible only if $E_{\text{ref}} - \bar{E} \approx -N\sigma \sqrt{2c}$. The desired expression is an integral over $E_{\text{ref}}$,

$$p(E_1 - E_0 > \Delta) = \int \exp \left( -\frac{E - E_{\text{ref}}}{\sigma^2 N} - \frac{\Delta^2}{2\sigma^2 N} \right) A(E_{\text{ref}}) dE_{\text{ref}},$$

where $A(E_{\text{ref}})$ is a complicated expression independent of $\Delta$. Replacing $E_{\text{ref}}$ with its approximate value in the exponential and neglecting the term quadratic in $\Delta$, we obtain

$$p(E_1 - E_0 \geq \Delta) = \exp \left( -\frac{\sqrt{2c} \Delta}{\sigma} \right),$$

FIG. 2: Left: Number of solutions (for satisfiable instances) as a function of $N$. The data fit exponential dependence $N \sim \exp(cN)$ with $c \approx 0.0209 \pm 0.0002$. Right: The difference between the largest and the smallest 4th order perturbative correction to the ground state, as a function of $N$. A linear fit with the coefficient $0.0535 \pm 0.0004$ is obtained. A leaf removal algorithm has been applied to randomly generated instances to insure that no clause contains more than one bit not appearing in other clauses. Errorbars correspond to one standard deviation (68% confidence interval). Linear fits are for the interval $100 \leq N \leq 1000$. 
where we also used the fact that the probability is 1 when $\Delta = 0$. The same result is obtained for $E_2 - E_1$, $E_3 - E_2$, etc. At the edge of the spectrum, the spacings are exponentially distributed with mean $\sigma/\sqrt{2c}$, i.e. levels have Poisson statistics. These results are not new: they are well-known in extreme value statistics and appear in a solution of Derrida’s random energy model.

When a new random clause is added, a fraction of the solutions will disappear. If we neglect any correlations as before, we can assume that each solution will satisfy the new clause with finite probability $p < 1$. Conditioned on the fact that the “old” ground state contradicts the new clause, the probability that old $k$-th excited state satisfies it, but 1st, 2nd, $(k-1)$-st excited states contradict it,

$$p_k = p(1 - p)^{k-1}. \tag{11}$$

The gap between old ground and $k$-th excited state $E_k - E_0$ is distributed with probability density

$$\rho_k(x) = \left(\frac{\sqrt{2c}}{\sigma}\right)^k \frac{x^{k-1}}{(k-1)!} e^{-\sqrt{2c}x}. \tag{12}$$

Therefore, the distribution of spacing between the old ground and lowest-lying excited state satisfying the new clause is

$$\rho(x) = \sum_{k=1}^{\infty} p_k \rho_k(x) = \frac{p\sqrt{2c}}{\sigma} e^{-\sqrt{2c}x}, \tag{13}$$

an exponential distribution with mean $\frac{\sigma}{p\sqrt{2c}}$. Strictly speaking, this is not the same as the distribution of the correct quantity $\Delta E_{10}(\lambda) = E_{x_0}^{(M-1)}(\lambda) - E_{x_1}^{(M-1)}(\lambda)$, where $x_1$ and $x_0$ correspond to the ground state of instance with $M - 1$ and $M$ clauses respectively. The addition of new clause introduces a configuration-dependent correction $O(\lambda^4)$ which is comparable to $O(\lambda^4)$ level spacing. This means that the levels are somewhat “reshuffled”, i.e. old $(k+1)$-st excited state may become smaller in energy than old $k$-th excited state. We therefore expect that the distribution of energy differences will deviate from true exponential, but the characteristic scale should still be $O(\lambda^4)$ with no $N$-dependence. Fig. [3] illustrates the distribution of $\Delta E_{10}$ for a particular random instance; for large $N$ the distribution still has an exponential tail, but the middle of the distribution slightly deviates from true exponential.

We should mention that approximating the distribution of configuration-dependent 4th order corrections can be approximated by a Gaussian only for $E - \bar{E} \sim \sqrt{N}\lambda^4$, but $E_0$ corresponds to the tail of the distribution where this

![FIG. 3: The values of normalized level splittings $\Delta E_{10}/\lambda^4$ for 4000 random instances with $N = 200$ and $N = 1000$, sorted in a decreasing order. Each dot’s y-coordinate is the value of the splitting and the x-coordinate is its index $k$ in the decreasing sequence. A straight line on semilogarithmic plot would correspond to exponential distribution. Deviation from true exponential is noticeable for $N = 1000$. A clause-to-variable ratio is fixed to $M/N = 0.62$.](attachment:fig3.png)
approximation is not valid. The probability density of the sum of $O(N)$ random variables, each having variance $O(\lambda^4)$ is expected to be exponentially small when we are $O(N\lambda^4)$ away from the mean; the exact dependence can be computed by considering optimal fluctuations. Hence, we still expect that $\bar{E} - E_0 \sim N\lambda^4$. Similarly, level spacing $E_1 - E_0 \sim \lambda^4$, although there is no guarantee that it is exponential-distributed. Therefore, our conclusions are independent of this approximation.

Since $\Delta E_{10} \sim \lambda^4$, avoided crossings should not take place until $\lambda \sim 1$. But for these values of $\lambda$, higher orders of perturbation theory may not be discarded and the perturbation theory itself may become divergent, as it should near the quantum phase transition.

Our prediction is in apparent disagreement with the results of numerical simulations of Ref. [1] that seem to support the claim that $\Delta E_{10} \sim \sqrt{N}\lambda^4$. Ref. [1] correctly examined the edge of the spectrum: all solutions were enumerated and the 4th order perturbation theory corrections both before and after adding the new clause were computed for $x_1$ and zeros in $x_0$, suggesting a finite limit as $E_1 - E_0 \sim \lambda^4$. However, as we mentioned earlier, for $N \sim 100$ the effects of exponential degeneracy are not yet prominent. Had the simulation been extended to larger values of $N$, the flattening of the curves would have been observed suggesting a finite limit as $N \to \infty$.

**Numerical results.** We have extended the numerical study of Ref. [1] to much larger values of $N$. A complete enumeration of all solutions becomes prohibitively time-consuming as the number of solutions explodes. However, we are really interested in a solution with the smallest 4th order perturbation theory correction. From Eq. (6) it is seen that this correction is linear in binary variables. Finding a solution corresponding to the ground state is equivalent to solving integer linear programming (ILP) problem, for which we utilize standard software packages [9]. ILP algorithms are more efficient than approaches based on a complete enumeration as entire branches corresponding to suboptimal solutions are pruned using, e.g, LP relaxations as a lower bound.

![Figure 4: Left: Average as well as 75th, 50th (median), and 25th percentiles of distribution of $(\Delta E_{10}/\lambda^4)^2$ for different values of $N$ (cf. Fig. 2 from Ref. [1]). Dashed lines are linear fits for $50 \leq N \leq 200$. Right top: Just the median of the distribution as a function of $N$. Right bottom: Average and percentiles of $p(\Delta E_{10}/\lambda^4)$ (not squared!) on a log-log plot. Dashed lines correspond to power-law fits in the interval $100 \leq N \leq 1000$. The exponents obtained were: $0.33 \pm 0.01$ (average), $0.23 \pm 0.01$ (75%), and $0.13 \pm 0.01$ (median and 25%); here error estimates refer only to the goodness of fit. The estimates of the exponents are unreliable since only one decade of value of $N$ was included in the fit. Errorbars correspond to one standard deviation (68%). A clause-to-variable ratio is $M/N = 0.62$. The results are about 20% larger than in Ref. [1]. The discrepancy might be due to minor difference in the numerical procedure: we chose the added clause at random among those that contradict $x_1$, while Ref. [1] restarted from scratch if random clause did not contradict $x_1$. The difference is not essential since the probability that a random clause contradicts a particular assignment is finite, but since this probability depends on the number of ones and zeros in $x_1$, the distributions are not identical.](image-url)
In Fig. 4 one can see that the curves are leveling off for larger values of \( N \), in agreement with our argument that \( \Delta E_{10} \) should not scale with \( N \). The fact that the average square of the gap and the 75th percentile are so close to each other for \( N \leq 200 \) (also seen in Ref. [1]) is not coincidental. It is an indirect evidence that the distribution of \( \Delta E \) is close to exponential since \( 1 - e^{-\sqrt{7}} \approx 0.757 \). For larger \( N \), the distribution is not exponential, possibly due to above-mentioned reshuffling of energy levels as their density is increased.

The flattening of the curve corresponding to the median is quite pronounced. Since average is more sensitive to the tails of the probability distribution, even larger values \( N \) may be needed to show its approach to the limiting value at \( N \to \infty \).

Of course, the present numerical study cannot completely rule out the possibility that \( \Delta E_{10} \) still increases with \( N \) with a power-law exponent smaller than \( 1/2 \). Indeed, the median (which is more statistically robust measure of scale than the average) seems to grow as \( N^{0.13} \) in the interval \( 100 \leq N \leq 1000 \). Tails of the distribution might be responsible for larger exponents observed for the 75th percentile and the average. If the corrections were to grow indefinitely, for sufficiently large \( N \) they would be large enough to cause avoided crossings. With the assumption that corrections increase as \( N^{1/2} \), Ref. [1] claims that the mechanism may only set in for very large \( N > N_c \), where the threshold had been estimated as either \( N_c \approx 5400 \) or \( N_c \approx 86000 \) depending on assumptions made. If the corrections were to rise only as \( N^{0.13} \) rather than \( N^{0.5} \), the value of \( N_c \) would be pushed even higher. We expect that an observed power-law fit with a finite value of the exponent is an artifact of using too short an interval (between 100 and 1000).

An observation that the exponent is close to \( 1/\ln 1000 \) (corresponding to the largest size considered) suggests a possibility that corrections increase as a logarithm of \( N \). A logarithmic rise would violate the condition \( \lambda_* \leq 1/\log N \) given in Ref. [1]: indeed, a central point of its argument is the claim that corrections increase as a finite power of \( N \), or much faster than a logarithm. The less stringent condition conjectured there would be satisfied, but the corresponding value of \( N_c \) might be astronomically large.

Numerical results clearly contradict the square-root-of-\( N \) scaling, but cannot reliably distinguish an approach to a finite limit from an extremely slow increase with \( N \) (e.g., as a logarithm). Based on numerical study alone, this scenario cannot be ruled out, but the theoretical analysis of the previous section, although imprecise, suggests that the corrections approach a finite limit as \( N \to \infty \). But we can think of no reason that might cause a plausible logarithmic rise.

**Concluding remarks.** We want to highlight one important limitation of the perturbation theory approach. Even for the “trimmed” ensemble considered in Ref. [1], strictly speaking the largest configuration-dependent correction is not \( O(\lambda^4) \) but rather \( O(\lambda^3) \), the latter coming from degenerate perturbation theory. Indeed, consider two clauses connected to the remainder of the graph as depicted in Fig. 5 (left). If both \( x_1 = x_2 = 0 \) then \((x_3, x_4, x_5)\) can be assigned either \((0,1,0)\) or \((1,0,1)\). Since the two configurations with the same energy differ by 3 bit flips, the splitting caused by the degenerate perturbation theory causes \( O(\lambda^3) \) correction to the energy. It can be argued that such clauses can be removed: since they can be satisfied for any value of \( x_3 \) and \( x_2 \) they only contribute to trivial degeneracies. However, in a similar example involving three clauses [see Fig. 5 (right)], they cannot be removed and yet they contribute \( O(\lambda^4) \) due to the degenerate perturbation theory correction — the same order as the correction due to ordinary perturbation theory. In other problems the effect of degenerate perturbation theory can be stronger: for \( K\text{-SAT} \) it enters as \( O(\lambda) \) correction. The difficulty of dealing with contributions from the degenerate perturbation theory is a need to diagonalize matrix involving many solutions. Although ordinary perturbation theory is inadequate, we believe that our main contention, that \( \Delta E \) does not scale with \( N \), is still correct.

The crucial factor in our analysis is the existence of exponentially many solutions. This phenomenon is common to all combinatorial optimization problems defined on random hypergraphs. One might ask if in some models hypergraph “trimming” may lift this degeneracy. One such example is \( K\text{-XOR-SAT} \) problem, where exponential degeneracy can be removed right at the satisfiability threshold by such trimming. However, perturbative corrections are independent of bit assignments to all orders of perturbation theory, and the mechanism described in Ref. [1] is not applicable there. This is probably not coincidental: unless local energy landscapes are identical in the vicinity of all solutions, the exponential degeneracy may not be removed by only geometric transformations of the random hypergraph.

While we refute the claim that exponentially small gaps appear with high probability for \( \lambda \to 0 \), the general possibility of exponentially small gaps for finite \( \lambda < \lambda_c \) cannot be ruled out. But estimating the probability of their occurrence might require using non-perturbative approaches.

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FIG. 5: Left: An example of $O(\lambda^3)$ contribution from the degenerate perturbation theory. If $x_1 = x_2 = 0$, two allowed assignments of variables $(x_3, x_4, x_5)$: $(0, 1, 0)$ and $(1, 0, 1)$ differ by three spin flips. Right: An example of $O(\lambda^4)$ contribution from the degenerate perturbation theory. $(x_4, x_5, x_6, x_7)$ can be either $(0, 1, 0, 1)$ or $(1, 0, 1, 0)$ if $x_1 = x_2 = x_3 = 0$. The clauses cannot be removed without affecting the satisfiability of the instance: they prohibit an assignment $x_1 = x_3 = 1$, $x_2 = 0$. In each figure solid dots represent binary variables and triangles represent clauses in an instance of exact cover problem. Binary variables below the dashed lines are involved in other clauses as indicated by zigzag lines.

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