ON BLOW-UPS OF THE QUINTIC DEL PEZZO 3-FOLD AND VARIETIES OF POWER SUMS OF QUARTIC HYPERSURFACES

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ABSTRACT. We construct new subvarieties in the varieties of power sums for certain quartic hypersurfaces. This provides a generalization of Mukai’s description of smooth prime Fano threefolds of genus twelve as the varieties of power sums for plane quartics. In fact in [TZ08] we show that these quartics are exactly the Scorza quartics associated to general pairs of trigonal curves and ineffective theta characteristics and this enables us to prove there the main conjecture of [DK93].

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1. INTRODUCTION

1.1. Varieties of power sums.

The problem of representing a homogeneous form as a sum of powers of linear forms has been studied since the last decades of the 19th century. This is called the Waring problem for a homogeneous form. We are interested in the study of the global structure of a suitable compactification of the variety parameterizing all such representations of a homogeneous form. A precise definition of the claimed compactification is the following:

Definition 1.1.1. Let $V$ be a $(v+1)$-dimensional vector space and let $F \in S^m \hat{V}$ be a homogeneous forms of degree $m$ on $V$, where $\hat{V}$ is the dual vector space of $V$. Set

$$\text{VSP} (F, n)^o := \{ ([H_1], \ldots, [H_n]) \mid H_1^m + \cdots + H_n^m = F \} \subset \text{Hilb}^n (\mathbb{P}^* \hat{V}).$$

The closed subset $\text{VSP} (F, n) := \text{VSP} (F, n)^o$ is called the varieties of power sums of $F$.

Sometimes $\mathbb{P}^* \hat{V}$ will be denoted by $\hat{\mathbb{P}}^v$.

As far as we know, the first global descriptions of positive dimensional VSP’s were given by Mukai.

1.2. Mukai’s result.

Let $A_{22}$ be a smooth prime Fano threefold of genus twelve, namely, a smooth projective threefold such that $-K_{A_{22}}$ is ample, the class of $-K_{A_{22}}$ generates $\text{Pic} A_{22}$, and the genus $g(A_{22}) := (-K_{A_{22}})^3 + 1$ is equal to twelve. The linear system $| - K_{A_{22}}|$ embeds $A_{22}$ into $\mathbb{P}^{13}$.

Mukai discovered the following remarkable theorem ([Muk92], [Muk04]):

Theorem 1.2.1. Let $\{ F_4 = 0 \} \subset \mathbb{P}^2$ be a general plane quartic curve. Then

1. $\text{VSP} (F_4, 6) \subset \text{Hilb}^6 \hat{\mathbb{P}}^2$ is an $A_{22}$; and conversely,
2. every general $A_{22}$ is of this form.

His motivation to discover this result was a characterization of a general $A_{22}$. For this purpose, he noticed that the Hilbert scheme of lines on a general $A_{22} \subset \mathbb{P}^{13}$ is isomorphic to a smooth plane quartic curve $\mathcal{H}_1 \subset \mathbb{P}^2$ (the notation $\mathbb{P}^2$ will be compatible with $\hat{\mathbb{P}}^2$ in Theorem 1.2.1). He wanted to recover $A_{22}$ by $\mathcal{H}_1$; for this, one more data was necessary. In fact he proved that the correspondence on $\mathcal{H}_1 \times \mathcal{H}_1$ defined by intersections of lines on $A_{22}$ gives an ineffective theta characteristic $\theta$ on $\mathcal{H}_1$. More precisely, $\theta$ is constructed so that the following two sets in $\mathcal{H}_1 \times \mathcal{H}_1$ coincide:

$$\{ ([l], [m]) \mid l \cap m \neq \emptyset, l \neq m \} = \{ ([l], [m]) \mid h^0(\theta + [l] - [m]) > 0 \}.$$

Now a deep and beautiful result of G. Scorza asserts that, associated to the pair $(\mathcal{H}_1, \theta)$, there exists another plane quartic curve $\{ F_4 = 0 \}$ in the same ambient plane as $\mathcal{H}_1$. (By saluting Scorza, $\{ F_4 = 0 \}$ is called the Scorza quartic.) Then, finally, Mukai proved that $A_{22}$ is recovered as $\text{VSP} (F_4, 6)$. This is the result (2) of theorem 1.2.1. We recall also that since the number of the moduli of $A_{22}$ is equal to $\dim \mathcal{M}_4 = 6$, (1) follows from (2).
Moreover, Mukai observed that conics on $A_{22}$ are parameterized by the plane $\mathcal{H}_2$ and $\mathcal{H}_2$ is naturally considered as the plane $\mathbb{P}^2$ dual to $\mathbb{P}^2$ since, for a conic $q$ on $A_{22}$, the lines intersecting $q$ form a hyperplane section of $\mathcal{H}_1$.

Further, he showed that the six points $[H_1], \ldots, [H_6]$ such that $([H_1], \ldots, [H_6]) \in \text{VSP}^o(F_4,6)$ correspond to six conics through one point of $A_{22}$.

To sum up, even if it is not evident from the statement, the content of Mukai’s theorem is a new interpretation of the geometry of lines and conics on $A_{22}$.

1.3. Generalization.

We study the relation between the concept of varieties of power sums and the geometry of lines and conics of other classes of 3-folds.

To do that, consider the smooth quintic del Pezzo threefold $B$ namely, a smooth projective threefold such that $-K_B = 2H$, where $H$ is the ample generator of Pic $B$ and $H^3 = 5$. It is well known that the linear system $|H|$ embeds $B$ into $\mathbb{P}^6$.

Now, following Iskovskih we doubly project $A_{22}$ from a general line, that is we consider the following diagram:

$$
\begin{array}{cccc}
A' & \rightarrow & A & \rightarrow \\
\downarrow & & \downarrow & \\
A_{22} & & B, & \\
\end{array}
$$

where

- $f'$ is the blow-up along a general line $l$,
- $A' \rightarrow A$ is a flop,
- $f$ is the blow-up along a smooth rational curve of degree five, where the degree is measured by $H$. We consider $B \subset \mathbb{P}^6$ by $\Phi|_H$.

(See also the section 6 for more information).

It is known that a general line on $A_{22}$ is mapped to a general line on $B$ intersecting $C$, and a general conic on $A_{22}$ is mapped to a general conic on $B$ intersecting $C$ twice. These facts are easy to see since the exceptional divisor of $f$ is the strict transform of the unique hyperplane section vanishing along $l$ with multiplicity 3.

This situation is generalizable by considering a general smooth rational curve $C$ of degree $d$ on $B$, where $d$ is an arbitrary integer greater than or equal to 5 (mainly $d \geq 6$) and the sets of the secant lines of $C$ and of the multi-secant conics of $C$ respectively. This led to the following definition:

**Definition 1.3.1.** (1) A pair $(l, t)$ of a line $l$ on $B$ and a point $t \in C \cap l$ is called a marked line.

(2) A pair of a conic $q$ on $B$ and a zero-dimensional subscheme $\eta \subset C$ of length two contained in $q|_C$ is called a marked conic.

We can prove:

**Proposition 1.3.2.** Marked lines are parameterized by a smooth trigonal canonical curve $\mathcal{H}_1$ of genus $d - 2$.

See the subsection 4.1 for the proof. Here is a sketch of the proof. It is known that there are three lines (counted with multiplicities) through a point of $B$ (see the subsection 2.1). This
gives the triple cover $\mathcal{H}_1 \to C$ such that $(l, t) \mapsto t$. Moreover, points where ‘special lines’ pass through form a divisor $E \in |2H|$ and the intersection of this divisor and $C$ is nothing but the branch locus of this triple cover. We can show that all ramifications are simple. Thus it holds

$$2g(\mathcal{H}_1) - 2 = 3(-2) + 2d,$$

namely, $g(\mathcal{H}_1) = d - 2$.

As Mukai did, we can define an ineffective theta characteristic $\theta$ on $\mathcal{H}_1$ and construct the Scorza quartic hypersurface $\{F_4 = 0\}$ associated to this in the sense of [DK93, §9]. This quartic hypersurface lives in the projective space $\mathbb{P}^d$ rather indirect, hence we give a more direct construction of the Scorza constructed in this paper is actually Scorza in the forthcoming paper [TZ08].

For the construction of the quartic $\{F_4 = 0\}$, we make use of marked conics, which we study in the subsection 4.2 in detail. Among other things, we prove the following:

**Proposition 1.3.3.** If $d \geq 6$, then marked conics are parameterized by a so-called White surface $\mathcal{H}_2$ obtained by blowing up $S^2C \simeq \mathbb{P}^2$ at $\binom{d-2}{2}$ points. $\mathcal{H}_2$ is embedded by $|d-3)H - \sum_{i=1}^6 e_i|$ into $\mathbb{P}^{d-3}$, where $h$ is the pull-back of a line, $e_i$ are the exceptional curves of $\mathcal{H}_2 \to \mathbb{P}^2$ and $s := \binom{d-2}{2}$.

Here we use the notation $\mathbb{P}^{d-3}$ since the ambient projective spaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ are recdually dual as in the Mukai’s case. If $d = 6$, then $\mathcal{H}_2$ is a cubic surface. In general, Gimigliano [Gim99] shows that $\mathcal{H}_2$ is the intersection of cubics.

The proof of this proposition is more involved than that of Proposition 1.3.2. See Corollary 4.2.10 and Theorem 4.2.15 for the proof. Here is a sketch of the proof. The morphism $\mathcal{H}_2 \to \mathbb{P}^2$ is just a natural one $\mathcal{H}_2 \to S^2C \simeq \mathbb{P}^2$ mapping $(g, \eta) \mapsto \eta$. Let $\beta_i$ be a bi-secant line of $C$. It is shown that there exist $s := \binom{d-2}{2}$ bi-secant lines of $C$ (see Corollary 4.1.2). Then for the length two subscheme $\beta_{i_1}C$, there exist infinitely many marked conics $(\beta_i \cup \alpha, \beta_{i_1}C)$, where $\alpha$ are lines intersecting $\beta_i$, and it is known that such $\alpha$’s form one-dimensional family (see Proposition 2.1.3 (5)). This indicates why $\mathcal{H}_2 \to S^2C$ is the blow-up at $s$ points, which are $[\beta_{i_1}C] \in S^2C$. Moreover, birationality of $\mathcal{H}_2 \to \mathbb{P}^{d-2}$ follows from the fact that there exists a unique conic on $B$ through two points $t_1$ and $t_2$ if there is no line on $B$ through $t_1$ and $t_2$. This can be seen by the double projection from $t_1$ (see Corollary 3.2.3).

 Actually we consider the curves on $A$ called lines and conics on $A$ corresponding one to one to marked lines and conics respectively.

In [DK93, §9], the quartic $F_4$ is constructed for $(\mathcal{H}_1, \theta)$, which is a data of intersections of marked lines. Here to construct $F_4$ we need data of intersections of marked conics.

In fact assume that $d \geq 6$. Consider the locus $D_l \subset \mathcal{H}_2$ parameterizing marked conics which intersect a fixed marked line $l$. The locus $D_l$ turns out to be a divisor linearly equivalent to $(d - 3)H - \sum_{i=1}^6 e_i$ on $\mathcal{H}_2$. Moreover, $|D_l|$ is very ample and embeds $\mathcal{H}_2$ in $\mathbb{P}^{d-3}$ (see Theorem 4.2.15 (1)). Set $D_2 := \{([q_1], [q_2]) \in \mathcal{H}_2 \times \mathcal{H}_2 \mid q_1 \cap q_2 \neq \emptyset\}$ and denote by $D_q$ the fiber of $D_2 \to \mathcal{H}_2$ over a point $[q]$. It is easy to verify $D_q \sim 2D_l \simeq O_{\mathcal{H}_2}(2)$. By the seesaw theorem, it holds that $D_2 \sim p_1^*D_q + p_2^*D_q$. Since $\mathcal{H}_2$ is projectively Cohen-Macaulay and is not contained in a quadric (Theorem 4.2.15 (4)), it holds $H^0(\mathcal{H}_2 \times \mathcal{H}_2, D_2) \simeq H^0(\mathbb{P}^{d-3} \times \mathbb{P}^{d-3}, O_{(2, 2)})$. Thus $D_2$ is the restriction of a unique $(2, 2)$-divisor $D'_2$ on $\mathbb{P}^{d-3} \times \mathbb{P}^{d-3}$. Since $D'_2$ is symmetric, we may assume its equation $D_2$ is also symmetric. By restricting $D_2$ to the diagonal, we obtain a quartic hypersurface $\{F_4 = 0\}$ in $\mathbb{P}^{d-3}$. We can show that $F_4$ is non-degenerate in the sense of
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Dol04 (see the appendix). Then there exists a unique quartic hypersurface \( \{ F_4 = 0 \} \) in \( \mathbb{P}^{d-3} \) called the quartic form dual to \( \tilde{F}_4 \).

Now we can state our main result, which generalizes (2) of Theorem 1.2.1:

**Theorem 1.3.4.** Let \( f: A \to B \) be the blow-up along \( C \), and let \( \rho: \tilde{A} \to A \) be the blow-up of \( A \) along the strict transforms \( \beta_i^\prime \) of \( \binom{d-2}{2} \) bi-secant lines \( \beta_i \) of \( C \) on \( B \). Then there exists an injection from \( \tilde{A} \) to \( \text{VSP}(F_4, n) \), where \( n := \binom{d-1}{2} \). Moreover the image is uniquely determined by the incident variety \( D_2 \) and is an irreducible component of

\[
\text{VSP}(F_4, n; \mathcal{H}_2) := \{(\{[H_1], \ldots, [H_n]\} \mid [H_i] \in \mathcal{H}_2) \subset \text{VSP}(F_4, n)\}.
\]

See Theorem 5.4.1 and Proposition 5.4.3.

Actually, the number \( n \) is equal to the number of multi-secant conics of \( C \) through a general point of \( B \) (see Corollary 3.2.8). Moreover, rather importantly,

\begin{equation}
(1.1) \quad n \text{ is equal to the dimension of quadric forms on } \mathbb{P}^{d-3}.
\end{equation}

We give an outline of the proof of the main result. Let \( \mathcal{U}_2 \to \mathcal{H}_2 \) be the universal family of conics on \( A \), and consider the natural projection \( \psi: \mathcal{U}_2 \subset A \times \mathcal{H}_2 \to A \). The morphism \( \psi \) is not finite (see Proposition 1.2.12). Nevertheless the blow-up \( \tilde{\mathcal{U}}_2 \to \mathcal{U}_2 \) along \( \cup \beta_i^\prime \times \mathcal{H}_2 \) \( \cap \mathcal{U}_2 \) is Cohen-Macaulay and the natural projection \( \tilde{\psi}: \tilde{\mathcal{U}}_2 \to \tilde{A} \) is finite of degree \( n \) (Proposition 5.1.3). Therefore, since \( \tilde{\mathcal{U}}_2 \subset \tilde{A} \times \mathcal{H}_2 \), \( \tilde{\psi} \) is a flat family of 0-dimensional subschemes \( \subset \mathcal{H}_2 \) of length \( n \) parameterized by \( \tilde{A} \). Geometrically, the fiber over a general point \( \tilde{a} \in \tilde{A} \) corresponds to \( n \) conics through the image of \( \tilde{a} \) on \( A \). The morphism \( \tilde{\psi} \) defines \( \tilde{A} \to \text{Hilb}^n \mathbb{P}^{d-3} \) which is the one claimed in the main theorem. To understand its image, we need to understand the double polars of the special quartic \( F_4 \).

By the construction of \( \tilde{F}_4 \) and the theory of polarity (see the appendix), it holds that, for a conic \( q \) on \( A \) and the hyperplane section \( \{ H_q = 0 \} \subset \mathbb{P}^{d-3} \) corresponding to the point \( [q] \in \mathbb{P}^{d-3} \), the locus \( D_q \) is equal to \( \{ \tilde{D}_q := P_{H_q^2}(\tilde{F}_4) = 0 \} \cap \mathcal{H}_2 \). By definition of the dual quartic form \( F_4 \), it holds

\begin{equation}
(1.2) \quad P_{\tilde{D}_q}(F_4) = H_q^2.
\end{equation}

Moreover, by definition of \( D_q \), it holds that, for \( n \) conics \( q_1, \ldots, q_n \) on \( A \) corresponding to a general fiber of \( \tilde{\psi} \),

\begin{equation}
(1.3) \quad \tilde{D}_{q_i}([q_i]) \neq 0 \text{ and } \tilde{D}_{q_j}([q_j]) = 0 \ (i \neq j).
\end{equation}

Now the main theorem follows from a more or less formal argument of the theory of polarity from (1.1), (1.2), and (1.3).

We believe that, even by reading the proof of Theorem 5.4.1 after reading only this introduction and possibly the appendix, the readers can understand at least the reason why the variety of power sums appears.

1.4. **Structure of the paper.**

We add some explanations about the structure of the paper.

In the section 2, we construct smooth rational curves \( C_d \) of degree \( d \) on \( B \) and study in detail the relation of general \( C_d \) with lines and conics on \( B \).
In the section 3, we describe the projection of $B$ from a line or a conic, and the double projection of $B$ from a point. These operations are useful for counting the number of multi-secant conics of $C$ satisfying various pre-specified geometric conditions. For example, using double projection from a general point of $B$, we can show that the number of multi-secant conics of $C$ through a general point of $B$ is equal to $n$ (see Corllary 3.2.8).

Sections 2 and 3 are rather technical as far as the proofs it concerns but the results are really easy to be understood by a general reader and at least one of them, we mean Proposition 3.2.5 is of unexpected geometrical content; Proposition 3.2.5 or its restatement Corollary 3.2.6 shows that the number of multi-secant conics of $C$ through any point of $B$ outside $C$ is finite. This will be refined to finiteness results contained into Propositions 4.2.12 and 5.1.3.

In the section 4, we mostly study marked lines and conics, and lines and conics on the blow-up $A$ of $B$ along a smooth rational curve $C$ of degree $d$ as we mentioned in the subsection 1.3.

In the section 5, we show the main theorem.

In the section 6, we explain Mukai’s result from our viewpoint.

Finally we add an appendix which forms the section 7, where we explain some very basic facts on the theory of polarity for the readers’ convenience.

1.5. Forthcoming paper.

This work lays the foundations for the results of [TZ08].

As we mentioned in the abstract, there we show that the quartic $\{F_4 = 0\}$ coincides with the Scorza quartic associated to $(H_1, \theta)$ and the theta charasteristic $\theta$ is constructed explicitely.

Following [DK93], we also study other geometric objects associated to $(H_1, \theta)$. As an amazing application, we show the existence of the Scorza quartics for any general pairs of curves and ineffective theta characteristics. This is an affirmative answer to the conjecture stated by Dolgachev and Kanev in [DK93, §9].

Moreover, we can study the moduli spaces of spin curves, especially of trigonal spin curves relating this with the Hilbert schemes of smooth rational curves on $B$. In fact we prove that $H_1$ is a general trigonal curve if $C$ is general.

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2. Rational curves on the quintic del Pezzo threefold $B$

Let $V$ be a vector space with $\text{dim}_C V = 5$. The Grassmannian $G(2, V)$ embeds into $\mathbb{P}^9$ and we denote the image by $G \subset \mathbb{P}^9$. It is well-known that the quintic del Pezzo 3-fold, i.e., the Fano 3-fold $B$ of index 2 and of degree 5 can be realized as $B = G \cap \mathbb{P}^6$, where $\mathbb{P}^6 \subset \mathbb{P}^9$ is transversal to $G$ (see [Fuj81], [Isk77, Thm 4.2 (iii), the proof p.511-p.514]).
First we collect basic known facts on lines and conics on $B$ almost without proof. Let $\mathcal{H}_1^B$ and $\mathcal{H}_2^B$ be the Hilbert scheme, respectively, of lines and of conics on $B$.

2.1. Lines on $B$.

Let $\pi: \mathbb{P} \to \mathcal{H}_1^B$ be the universal family of lines on $B$ and $\varphi: \mathbb{P} \to B$ the natural projection. By [FN89a, Lemma 2.3 and Theorem 1], $\mathcal{H}_1^B$ is isomorphic to $\mathbb{P}^2$ and $\varphi$ is a finite morphism of degree three. In particular the number of lines passing through a point is three counted with multiplicities. We recall some basic facts about $\pi$ and $\varphi$ which we use in the sequel.

Before that, we fix some notation.

Notation 2.1.1. For an irreducible curve $C$ on $B$, denote by $M(C)$ the locus $\subset \mathbb{P}^2$ of lines intersecting $C$, namely, $M(C) := \pi(\varphi^{-1}(C))$ with reduced structure. Since $\varphi$ is flat, $\varphi^{-1}(C)$ is purely one-dimensional. If $\deg C \geq 2$, then $\varphi^{-1}(C)$ does not contain a fiber of $\pi$, thus $M(C)$ is a curve. See Proposition 2.1.3 for the description of $M(C)$ in case $C$ is a line.

Definition 2.1.2. A line $l$ on $B$ is called a special line if $N_{l/B} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

Remark. If $l$ is not a special line on $B$, then $N_{l/B} = \mathcal{O}_l \oplus \mathcal{O}_l$.

Proposition 2.1.3. It holds:

1. for the branched locus $B_\varphi$ of $\varphi: \mathbb{P} \to B$ we have:
   1. $B_\varphi \subset |-K_B|$, and
   2. $\varphi^*B_\varphi = R_1 + 2R_2$, where $R_1 \simeq R_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $\varphi: R_1 \to B_\varphi$ and $\varphi: R_2 \to B_\varphi$ are injective,

2. $R_2$ is contracted to a conic $Q_2$ by $\pi: \mathbb{P} \to \mathcal{H}_1^B$. Moreover $Q_2$ is the branched locus of the finite double cover $\pi|_{R_1}: R_1 \to \mathcal{H}_1^B$,

3. $Q_2$ parameterizes special lines,

4. if $l$ is a special line, then $M(l)$ is the tangent line to $Q_2$ at $[l]$. If $l$ is not a special line, then $\varphi^{-1}(l)$ is the disjoint union of the fiber of $\pi$ corresponding to $l$, and the smooth rational curve dominating a line on $\mathbb{P}^2$. In particular, $M(l)$ is the disjoint union of a line and the point $[l]$.

By abuse of notation, we denote by $M(l)$ the one-dimensional part of $M(l)$ for any line $l$. Vice-versa, any line in $\mathcal{H}_1^B$ is of the form $M(l)$ for some line $l$, and

5. the locus swept by lines intersecting $l$ is a hyperplane section $T_l$ of $B$ whose singular locus is $l$. For every point $b$ of $T_l \setminus l$, there exists exactly one line which belongs to $M(l)$ and passes through $b$. Moreover, if $l$ is not special, then the normalization of $T_l$ is $\mathbb{F}_1$ and the inverse image of the singular locus is the negative section of $\mathbb{F}_1$, or, if $l$ is special, then the normalization of $T_l$ is $\mathbb{F}_3$ and the inverse image of the singular locus is the union of the section and a fiber.

Proof. See [FN89a, §2] and [Hi94a, §1]. □

By the proof of [FN89a] we see that $B$ is stratified according to the ramification of $\varphi: \mathbb{P} \to B$ as follows:

$$B = (B \setminus B_\varphi) \cup (B_\varphi \setminus C_\varphi) \cup C_\varphi,$$

where $C_\varphi$ is a smooth rational normal sextic and if $b \in B \setminus B_\varphi$ exactly three distinct lines pass through it, if $b \in (B_\varphi \setminus C_\varphi)$ exactly two distinct lines pass through it, one of them is special, and finally $C_\varphi$ is the loci of $b \in B$ through which it passes only one line, which is special.
2.2. Conics on $B$.

**Proposition 2.2.1.** The Hilbert scheme $\mathcal{H}_2^B$ of conics on $B$ is isomorphic to $\mathbb{P}^4 = \mathbb{P}_s \mathbb{V}$. The support of a double line is a special line and the double lines are parameterized by a rational normal quartic curve $\Gamma \subset \mathbb{P}_s \mathbb{V}$ and the secant variety of $\Gamma$ is a singular cubic hypersurface which is the closure of the loci parameterizing reducible conics.

**Proof.** See [Ili94, Proposition 1.2.2].

The identification in the first statement is given by the map $sp: \mathcal{H}_2^B \to \mathbb{P}_s \mathbb{V}$ with $[c] \mapsto \langle Gr(c) \rangle = \mathbb{P}^3_\mathbb{c} \subset \mathbb{P}_s \mathbb{V}$, where for a general conic $c \subset B$ we set $Gr(c) := \bigcup \{ r \in \mathbb{P}_s \mathbb{V} \mid [r] \in c \} \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

2.3. Construction of rational curves $C_d$ of degree $d$ on $B$.

We construct smooth rational curves of degree $d$ on $B$ by smoothing the union of a smooth rational curve of degree $d-1$ and one of its uni-secant lines.

**Definition 2.3.1.** Let $C$ and $\gamma$ be smooth curves on $B$. We say that $\gamma$ is a secant curve of $C$ if $C \cap \gamma \neq \emptyset$. Moreover, we say that $\gamma$ is a $k$-secant curve (resp. a multi-secant curve) if $\gamma|_C$ is a 0-dimensional subscheme of length $k$ (resp. of length greater than or equal to 2). For $k = 1, 2, \ldots$, we say uni-secant, bi-secant, $\ldots$, instead.

**Proposition 2.3.2.** There exists a smooth rational curve $C_d$ of degree $d$ on $B$ such that

(a) a general line on $B$ intersecting $C_d$ is uni-secant,

(b) $C_d$ is obtained as a smoothing of the union of a smooth rational curve $C_{d-1}$ of degree $d-1$ on $B$ and a general uni-secant line of it on $B$, and

(c) $\mathcal{N}_{C_d/B} \simeq \mathcal{O}_{p_1}(d-1) \oplus \mathcal{O}_{p_1}(d-1)$. In particular $h^1(\mathcal{N}_{C_d/B}) = 0$ and $h^0(\mathcal{N}_{C_d/B}) = 2d$.

**Proof.** We argue by induction on $d$.

If $d = 1$, we have the assertion since $\mathcal{N}_{C_1/B} \simeq \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_1}$ for a general line $C_1$.

Now assume that $C_{d-1}$ is a smooth rational curve of degree $d-1$ on $B$ constructed inductively. By induction, a general secant line $l$ of $C_{d-1}$ on $B$ is uni-secant. Set $Z := C_{d-1} \cup l$ and $\mathcal{N}_{Z/B} := \mathcal{Hom}_{\mathcal{O}_B}(\mathcal{I}_Z, \mathcal{O}_B)$. By induction, the normal bundle of $C_{d-1}$ satisfies (c). Thus, by $\mathcal{N}_{l/B} \simeq \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_1}$ and [HHS5 Theorem 4.1 and its proof], it holds $h^1(\mathcal{N}_{Z/B}) = 0$, and moreover $Z := C_{d-1} \cup l$ is strongly smoothable, namely, we can find a smoothing $C_d$ of $Z$ with the smooth total space. By the upper semi-continuity theorem, $h^1(\mathcal{N}_{C_d/B}) = 0$ and, by the Riemann-Roch theorem, $h^0(\mathcal{N}_{C_d/B}) = 2d$.

We check the form of the normal bundle of $C_d$. Set $\mathcal{N}_{C_d/B} := \mathcal{O}_{p_1}(a_d) \oplus \mathcal{O}_{p_1}(b_d)$ ($a_d \geq b_d$) for the smoothing $C_d$ of $Z$. We show that $a_d = b_d = d-1$. It suffices to prove $h^0(\mathcal{N}_{Z/B}(-d)) = 0$. In fact, then, by the upper semi-continuous theorem, we have $h^0(\mathcal{N}_{C_d/B}(-d)) = 0$ and $a_d, b_d \leq d-1$. Thus, by $a_d + b_d = 2d-2$, it holds $a_d = b_d = d-1$. By noting $\mathcal{N}_{C_{d-1}/B} = \mathcal{O}_{p_1}(d-2) \oplus \mathcal{O}_{p_1}(d-2)$, the equality $h^0(\mathcal{N}_{Z/B}(-d)) = 0$ easily follows from the following three exact sequences, where $t := C_{d-1} \cap l$:

$$0 \to \mathcal{N}_{Z/B} \to \mathcal{N}_{Z/B|C_{d-1}} \oplus \mathcal{N}_{Z/B|l} \to \mathcal{N}_{Z/B} \otimes \mathcal{O}_t \to 0.$$ $$0 \to \mathcal{N}_{C_{d-1}/B} \to \mathcal{N}_{Z/B|C_{d-1}} \to T^1_t \to 0.$$
0 \to \mathcal{N}_{1/B} \to \mathcal{N}_{Z/B}\ell \to T^1_\ell \to 0.

We can inductively show that a general line \(m\) intersecting \(C_{d-1}\) does not intersect \(l\), thus \(m\) is a uni-secant line of \(C_{d-1} \cup l\). This implies (a) for \(C_d\) by a deformation theoretic argument. \(\square\)

**Corollary 2.3.3.** Let \(C_d\) be a smooth rational curve of degree \(d\) constructed as in Proposition 2.3.2. The Hilbert scheme of smooth rational curves on \(B\) of degree \(d\) is smooth at \([C_d]\) and is of dimension \(2d\).

**Proof.** The assertion follows from Proposition 2.3.2 (c). \(\square\)

### 2.4. Relations of \(C_d\) with lines and conics.

We study multi-secant lines and conics of \(C_d\).

**Proposition 2.4.1.** A general \(C_d\) as in Proposition 2.3.2 satisfies the following conditions:

1. there exist no \(k\)-secant lines of \(C_d\) on \(B\) with \(k \geq 3\),
2. there exist at most finitely many bi-secant lines of \(C_d\) on \(B\), and any of them intersects \(C_d\) simply,
3. bi-secant lines of \(C_d\) on \(B\) are mutually disjoint,
4. neither a bi-secant line nor a line through the intersection point between a bi-secant line and \(C_d\) is a special line, and
5. there exist at most finitely many points \(b\) outside \(C_d\) such that all the lines through \(b\) intersect \(C_d\), and such points exist outside bi-secant lines of \(C_d\).

**Proof.** We can prove the assertions by simple dimension counts based upon Proposition 2.3.2. We assume that \(d \geq 4\) since otherwise we can verify the assertion easily.

1. Let \(D\) be the closure of the set

   \[
   \{([C_d], [l]) \mid C_d \cap l \text{ consists of 3 points}\} \subset \mathcal{H}^B_d \times \mathcal{H}^B_1.
   \]

   Let \(\pi_d: D \to \mathcal{H}^B_d\) and \(\pi_1: D \to \mathcal{H}^B_1\) be the natural morphisms induced by the projections. The claim follows if we show that \(\dim C D \leq 2d - 1\) since \(\dim \mathcal{H}^B_1 = 2d\).

   Thus we estimate \(\dim C \text{Hom}^{2d}(\mathbb{P}^1, B; p_i \mapsto s_i, i = 1, 2, 3)\) at \([\pi]\), where \(p_i, i = 1, 2, 3\) are fixed points of \(\mathbb{P}^1\), \([\pi]\) is a general point and the degree is measured by \(-K_B\). By \(d \geq 4\) and Proposition 2.3.2 (c), it holds that \(h^0(\mathbb{P}^1, \pi^*T_B(-p_1 - p_2 - p_3)) = 2d - 6\) and \(h^1(\mathbb{P}^1, \pi^*T_B(-p_1 - p_2 - p_3)) = 0\). Then

   \[
   \dim C \text{Hom}^{2d}(\mathbb{P}^1, B, p_i \mapsto s_i, i = 1, 2, 3)[\pi] = h^0(\pi^*T_B(-p_1 - p_2 - p_3)) = 2d - 6.
   \]

   This implies that \(\dim C \pi^{-1}([l]) \leq 2d - 6 + 3 = 2d - 3\) since the three points can be chosen arbitrarily. Then \(\dim C D \leq 2d - 1\) since \(\dim C \mathcal{H}^B_1 = 2\).

2. Now let \(D\) be the closure of the set

   \[
   \{([C_d], [l]) \mid C_d \cap l \text{ consists of 2 points}\} \subset \mathcal{H}^B_d \times \mathcal{H}^B_1.
   \]

   As before, let \(\pi_d: D \to \mathcal{H}^B_d\) and \(\pi_1: D \to \mathcal{H}^B_1\) be the natural morphisms induced by the projections. By \(d \geq 4\) and Proposition 2.3.2 (c), it holds that \(h^0(\mathbb{P}^1, \pi^*T_B(-p_1 - p_2)) = 2d - 3\) and \(h^1(\mathbb{P}^1, \pi^*T_B(-p_1 - p_2)) = 0\). Then

   \[
   \dim C \text{Hom}^{2d}(\mathbb{P}^1, B, p_i \mapsto s_i, i = 1, 2)[\pi] = h^0(\pi^*T_B(-p_1 - p_2)) = 2d - 3.
   \]
Since \( \dim \text{Aut}(\mathbb{P}^1, p_1, p_2) = 1 \) it holds that \( \dim \pi_1^{-1}([l]) \leq 2d - 3 + 2 - 1 = 2d - 2 \). Hence \( \dim D = 2d \). Hence \( C_d \) has only a finite number of bi-secant lines.

We now show that the loci where \( C_d \) has a tangent bi-secant is a codimension one loci inside \( \mathcal{H}_B^d \). Let \( B_t \) be the blow-up of \( B \) in a point \( t \in C_d \) and let \( l \) be a bi-secant which is tangent to \( C_d \) at \( t \) (if it exists). Let \( E \) be the exceptional divisor, and \( C' \) and \( l' \) the strict transforms of \( C \) and \( l \) respectively. By hypothesis there exists a unique point \( s \in E \cap C' \cap l' \). We estimate \( \dim \text{Hom}_d^d(\mathbb{P}^1, B_t, p \mapsto s)_{[\pi]} \), where \( p \) is a fixed point of \( \mathbb{P}^1 \), \( [\pi] \) is a general point, and the degree is measured by \( -K_{B_t} \). In this case \( h^0(\pi^*T_{B_t}(-p)) = 2d - 2 \) hence \( \dim \pi_1^{-1}([l]) \leq 2d - 2 + 1 - 2 = 2d - 3 \). This implies the claim.

The cases (3), (4) and (5) are similar. Thus we only give few comments for (5). Set \( D \) be the closure of the set

\[
\{(\{\{C_d\}, [l_1], [l_2], [l_3]\}) \mid C_d \cap l_i \neq \emptyset \quad (i = 1, 2, 3), \\
l_1 \cap l_2 \cap l_3 \neq \emptyset, l_1 \cap l_2 \cap l_3 \notin C_d, l_i \text{ are distinct}\} 
\subset \mathcal{H}_B^1 \times \mathcal{H}_B^1 \times \mathcal{H}_B^1.
\]

For the former half of (5), we have only to prove that \( \dim D \leq 2d \). This can be carried out by a similar dimension count as above. For the latter half of (5), we use the inductive construction of \( C_d \) besides dimension count.

We can prove the following by a similar method hence we omit the proof.

**Proposition 2.4.2.** A general \( C_d \) as in Proposition 2.3.2 satisfies the following conditions:
1. there exist no \( k \)-secant conics of \( C_d \) with \( k \geq 5 \),
2. there exist at most finitely many quadri-secant conics of \( C_d \) on \( B \), and no quadri-secant conic is tangent to \( C_d \), and
3. \( q_{C_d} \) has no point of multiplicity greater than two for any multi-secant conic \( q \).

**Notation 2.4.3.** The bisecant lines of \( C_d \) are denoted by \( \beta_i \) where \( i = 1, \cdots, s \).

In the following proposition, we describe some more relations of \( C_d \) with lines on \( B \) which can be translated into the geometry of \( \mathcal{H}_B^1 \). More explicitly, we prove that \( M(C_d) \) is sufficiently general if \( C_d \) is general (recall the notation of the subsection 2.1).

**Proposition 2.4.4.** A general \( C_d \) as in Proposition 2.3.2 satisfies the following conditions:
1. \( C_d \) intersects \( B_\varphi \) simply,
2. \( M_d := M(C_d) \) intersects \( Q_2 \) simply,
3. \( M_d \) is an irreducible curve of degree \( d \) with only simple nodes (recall that in Proposition 2.1.3 (4), we abuse the notation by denoting the one-dimensional part of \( \pi(\varphi^{-1}(C_1)) \) by \( M(C_1) \)),
4. for a general line \( l \) intersecting \( C_d \), \( M_d \cup M(l) \) has only simple nodes as its singularities, and
5. \( M_d \cup M(\beta_i) \) has only simple nodes as its singularities.

**Proof.** We show the assertion inductively using the smoothing construction of \( C_d \) from the union of \( C_{d-1} \) and a general uni-secant line \( l \) of \( C_{d-1} \).

In case of \( d = 1 \), by letting \( C_1 \) be a general line, the assertion follows from Proposition 2.1.3.

By induction on \( d \) assume that we have a smooth \( C_{d-1} \) (\( d \geq 2 \)) satisfying (1)–(5). We verify
Proof. Let \( C_{d-1} \subseteq H \) intersect \( B \), simply by (1) for \( C_{d-1} \) and generality of \( l \).

(1) \( C_{d-1} \subseteq H \) intersects \( B \).

(2) \( M_{d-1} \subseteq M(l) \) intersects \( Q_2 \) simply by (2) for \( C_{d-1} \) and generality of \( l \).

(3) \( M_{d-1} \subseteq M(l) \) is not irreducible but is of degree \( d \) and has only simple nodes by (4) for \( C_{d-1} \).

(4) \( M_{d-1} \subseteq M(l) \cup M(m) \) has only simple nodes as its singularities for a general line \( m \) intersecting \( C_{d-1} \).

Indeed, since \( m \) is also general, \( M_{d-1} \subseteq M(m) \) has only simple nodes by (4) for \( C_{d-1} \). Thus we have only to prove that \( M_{d-1} \cap M(l) \cap M(m) = \emptyset \), namely, there is no secant line of \( C_{d-1} \) intersecting both \( l \) and \( m \). Fix a general \( l \) and move \( m \). If there are secant lines \( r_m \) of \( C_{d-1} \) intersecting both \( l \) and \( m \) for general \( m \)'s, then \( r_m \) moves whenever we have \( M(l) \subseteq M_{d-1} \), a contradiction.

(5) For a bi-secant line \( \beta \) of \( C_{d-1} \subseteq H \) except the lines through \( C_{d-1} \cap l \), the curve \( M_{d-1} \subseteq M(l) \cup M(\beta) \) has only simple nodes as its singularities.

Indeed, if \( \beta \) is a bi-secant line of \( C_{d-1} \), then the assertion follows from (5) for \( C_{d-1} \) by a similar way to the proof of (4). Suppose that \( \beta \) is a uni-secant line of \( C_{d-1} \) intersecting \( l \). We have only to prove that there is no secant line of \( C \) intersecting both \( l \) and \( \beta \). If there is such a line \( r \), then \( l, \beta \) and \( r \) pass through one point. This does not occur for general \( l \) and \( \beta \) by Proposition 2.4.1 (5).

Thus, by a deformation theoretic argument, we see that \( C_d \) satisfies (1)–(5). \qed

2.5. On irreducibility of families of rational curves on \( B \).

We discuss about irreducibility of the Hilbert scheme of smooth rational curves on \( B \) of a fixed degree though we do not need it fully.

For a smooth projective variety \( X \) in some projective space, let \( \mathcal{H}_d^0(X) \) be the Hilbert scheme of smooth rational curves on \( X \) of degree \( d \). By [Per02], \( \mathcal{H}_d^0(G(a,b)) \) is non-empty and irreducible, where \( G(a,b) \) is the Grassmannian parameterizing \( a \)-dimensional subvector spaces in a fixed \( b \)-dimensional vector space.

Let \( \mathcal{H}_d^0(X) \) be the open subset of \( \mathcal{H}_d^0(X) \) parameterizing smooth rational curves on \( X \) of degree \( d \) with linear hull of maximal dimension.

Let \( \mathcal{H}_d^B \) be the Hilbert scheme of general smooth rational curves on \( B \) of degree \( d \) obtained inductively as in Proposition 2.3.2.

We can show inductively that \( \mathcal{H}_d^B \subseteq \mathcal{H}_d^0(B) \), thus we can ask the following:

Question 2.5.1. \( \overline{\mathcal{H}_d^B} = \overline{\mathcal{H}_d^0(B)} \) ? (here we take the closure in the Hilbert scheme.) Are they irreducible ?

We have a partial answer to this question as follows:

Proposition 2.5.2. \( \mathcal{H}_d^B \) with any \( d \) and \( \mathcal{H}_d^0(B) \) with \( d \leq 6 \) are irreducible. \( \overline{\mathcal{H}_d^0(B)} = \mathcal{H}_d^B \) for \( d \leq 6 \).

Proof. The claim is true for \( d = 1 \) since \( \overline{\mathcal{H}_1^0(B)} = \overline{\mathcal{H}_1^B} \simeq \mathbb{P}^2 \).

First we prove \( \overline{\mathcal{H}_d^B} \) is irreducible for any \( d \). By induction let us assume that \( \overline{\mathcal{H}_{d-1}^B} \) is irreducible. Let \( [C_{d-1}] \in \overline{\mathcal{H}_{d-1}^B} \) be a generic element. The family of lines \( [l] \in \mathcal{H}_1^B \) which intersect a generic element of \( \overline{\mathcal{H}_{d-1}^B} \) is irreducible by Proposition 2.4.1 (3). This implies that the family \( \mathcal{H}_{d-1,1}^B \) of reducible curves \( C_{d-1}^0 = C_{d-1}^0 \cup l \) such that \( [C_{d-1}^0] \in \mathcal{H}_{d-1}^B \), \( [l] \in \mathcal{H}_1^B \) and length \( C_{d-1}^0 \cap l = 1 \) is
irreducible. As in the proof of Proposition 2.3.2, the Hilbert scheme is smooth at the point \([C_d^0]\). Thus \(\mathcal{H}_d^B\) is irreducible.

Second we prove \(\mathcal{H}_d^0(B)\) with \(d \leq 6\) is irreducible. Let \(\mathcal{B}\) be the irreducible family of del Pezzo 3-folds \(B = G(2,5) \cap \mathbb{P}^6\), where \(\mathbb{P}^6 \subset \mathbb{P}^9\) is transversal to \(G(2,5)\). Let

\[
J = \{([C_d^0], [B]) \in \mathcal{H}_d^0(G(2,5)) \times \mathcal{B} \mid C_d^0 \subset B\}.
\]

If \(d \leq 6\), then it is known that a general smooth rational curve of degree \(d\) on \(G(2,5)\) is a normal rational curve, and is contained in a smooth 3-dimensional linear section of \(G(2,5)\), namely, a smooth quintic del Pezzo 3-fold. Indeed, we can construct such a rational curve with \(d \leq 5\) explicitly on a smooth quintic del Pezzo surface, which is contained in a smooth quintic del Pezzo 3-fold. For \(d = 6\), \(C_6\) as in the subsection 2.1 is an example of such a rational curve \(C_6\) on a smooth quintic del Pezzo 3-fold.

Thus a general fiber \(J \to \mathcal{B}\) is equal to \(\mathcal{H}_d^0(B)\) and is non-empty. Moreover, any fiber of \(J \to \mathcal{H}_d^0(G(2,5))\) is isomorphic to \(G(\mathbb{P}^d, \mathbb{P}^6)\). Since \(\mathcal{H}_d^0(G(2,5))\) is irreducible and \(\mathcal{H}_d^0(G(2,5))\) is an open subset of \(\mathcal{H}_d^0(G(2,5))\), it holds \(J\) is irreducible. By the argument of [MT01, Proof of Theorem 3.1 p.17], we have only to show that there is one particular component of a general fiber \(J \to \mathcal{B}\) invariant under monodromy. Actually, this is nothing but \(\mathcal{H}_d^B\).

Corollary 2.5.3. Let \(C_d\) be a general smooth rational curve constructed as in Proposition 2.3.2.

(1) If \(d = 5\), then \(C_5\) is a normal rational curve and is contained in a unique hyperplane section \(S\), which is smooth. If \(d \geq 6\), then \(C_d\) is not contained in a hyperplane section.

3. VARIOUS PROJECTIONS OF \(B\)

3.1. Projection of \(B\) from a line or a conic.

Proposition 3.1.1. (1) Let \(l\) be a line on \(B\). Then the projection of \(B\) from \(l\) is decomposed as follows:

\[
\begin{array}{c}
B_l \\
\downarrow \pi_{1l} \\
B \\
\downarrow \\
\pi_{2l} \\
Q,
\end{array}
\]

where \(\pi_{1l}\) is the blow-up along \(l\) and \(B \longrightarrow Q\) is the projection from \(l\) and \(\pi_{2l}\) contracts onto a rational normal curve of degree 3 the strict transform of the loci swept by the lines of \(B\) touching \(l\). Moreover

\[
-K_{B_l} = H + H_Q,
\]

where \(H\) and \(H_Q\) are the pull backs of general hyperplane sections of \(B\) and \(Q\) respectively. We denote by \(E_l\) the \(\pi_{1l}\)-exceptional divisor.

(2) Let \(q\) be a smooth conic on \(B\). Then the projection of \(B\) from \(q\) behaves as follows:

\[
\begin{array}{c}
B_q \\
\downarrow \pi_{1q} \\
B \\
\downarrow \\
\pi_{2q} \\
\mathbb{P}^3,
\end{array}
\]
where \( \pi_{1q} \) is the blow-up of \( B \) along \( q \) and \( \pi_{2q} \): \( B_q \to \mathbb{P}^3 \) is the divisorial contraction of the strict transform \( T_q \) of the loci swept by the lines touching \( q \). Moreover

\[
(3.4) \\
-K_{B_q} = H + H_{\mathbb{P}},
\]

where \( H \) and \( H_{\mathbb{P}} \) are the pull backs of general hyperplane sections of \( B \) and \( \mathbb{P}^3 \) respectively.

**Proof.** These results are more or less well-known. For (1), refer [Fu81], and for (2) and (1), refer [MM81], No. 22 for (2) (No. 26 for (1)). See also [MM85], p.533 (7.7) for a discussion. \( \square \)

We give several applications of the projection of \( B \) from a line or a conic.

Let \( C := C_d \) be a general rational curve of degree \( d \) constructed as in Proposition 2.3.2 and \( l_1 \) and \( l_2 \) two general secant lines of \( C \) such that \( l_1 \cap l_2 = \emptyset \). We need to count the number of multi-secant conics of \( C \) intersecting \( l_1 \) and \( l_2 \) in the proof of Theorem 4.2.15.

**Lemma 3.1.2.** Assume that \( d \geq 3 \). Let \( B \to Q \to \mathbb{P}^2 \) be the successive linear projections from \( l_1 \) and then the strict transform of \( l_2 \) on \( Q \). Let \( l \) be another general secant line of \( C \), and \( C'' \) and \( l' \subset \mathbb{P}^2 \) be the images of \( C \) and \( l \) respectively. Then \( C \cup l \to C'' \cup l' \) is generically one to one and \( \deg C'' \cup l' = d - 1 \). Moreover, \( C'' \cup l' \) has only simple nodes as its singularities. In particular (since \( \deg C'' = d - 2 \) and \( C'' \) is rational) \( C'' \) has \( \frac{(d-3)(d-4)}{2} \) simple nodes, equivalently, there exist \( \frac{(d-3)(d-4)}{2} \) bi-secant conics of \( C \) intersecting both \( l_1 \) and \( l_2 \).

**Remark.** The line \( l \) is needed for the inductive proof as below.

**Proof.** We show the assertion using the inductive construction of \( C = C_d \). The assertion follows for \( d = 3 \) directly. Consider a smoothing from \( C_{d-1} \cup m \) to \( C_d \). Let \( m_1 \) and \( m_2 \) two general secant lines of \( C_{d-1} \) such that \( m_1 \cap m_2 = \emptyset \). Let \( B \to Q \to \mathbb{P}^2 \) be the successive linear projections from \( m_1 \) and then from the strict transform of \( m_2 \) on \( Q \). Let \( r \) be another general secant line of \( C_{d-1} \), and \( C'_d \), \( m' \) and \( r' \subset \mathbb{P}^2 \) be the images of \( C_{d-1} \), \( m \) and \( r \) respectively. Then we have only to show that \( C_{d-1} \cup m \cup r \to C'_d \cup m' \cup r' \) is generically one to one, \( \deg C'_d \cup m' \cup r' = d - 1 \) and \( C'_d \cup m' \cup r' \) has only simple nodes as its singularities assuming \( C_{d-1} \cup r \to C'_d \cup r' \) is generically one to one, \( \deg C'_d \cup r' = d - 2 \) and \( C'_d \cup r' \) has only simple nodes as its singularities.

Since \( m \) is also general, \( C_{d-1} \cup m \to C'_d \cup m' \) is generically one to one, \( \deg C'_d \cup m' = d - 2 \) and \( C'_d \cup m' \) has only simple nodes as its singularities. Thus \( C_{d-1} \cup m \cup r \to C'_d \cup m' \cup r' \) is generically one to one and \( \deg C'_d \cup m' \cup r' = d - 1 \). To show \( C'_d \cup m' \cup r' \) has only simple nodes as its singularities, it suffices to prove that there are no secant conics of \( C_{d-1} \) intersecting all the \( m_1 \), \( m_2 \), \( m \) and \( r \). This follows from the fact that a secant conic \( q \) of \( C_{d-1} \) intersects finitely many secant lines of \( C_{d-1} \) by \( M(q) \not\subseteq M(C_{d-1}) \).

The last statement follows from that, by generality of \( l_1 \) and \( l_2 \), any multi-secant conic of \( C \) intersecting \( l_1 \) and \( l_2 \) is bi-secant. \( \square \)

The following is a variant of Lemma 3.1.2 which is also need in the proof of Theorem 4.2.15.

**Lemma 3.1.3.** Assume that \( d \geq 4 \). Let \( l_0 \) be a general uni-secant line of \( C \). Let \( B \to Q \to \mathbb{P}^2 \) be the successive linear projections from \( l_0 \) and then the strict transform of a bi-secant line \( \beta_i \) on \( Q \). Let \( l \) be another general uni-secant line of \( C \), and \( C' \) and \( l' \subset \mathbb{P}^2 \) be the images of \( C \) and \( l \) respectively. Then \( C \cup l \to C' \cup l' \) is generically one to one, \( \deg C' \cup l' = d - 2 \), and \( C' \cup l' \) has only simple nodes as its singularities. In particular (since \( \deg C' = d - 3 \) and \( C' \) is
rational) $C'$ has $\frac{(d-4)(d-5)}{2}$ simple nodes, equivalently, there exist $\frac{(d-4)(d-5)}{2}$ bi-secant conics of $C$ intersecting $\beta_i$ and $l_0$ except conics containing $\beta_i$.

Proof. Similarly to the previous lemma, we show the assertion using the inductive construction of $C = C_d$. The assertion follows for $d = 4$ directly. Consider a smoothing from $C_{d-1} \cup m$ to $C_d$. Let $m_0$ be a general uni-secant line of $C_{d-1}$, and $\beta$ a bi-secant line of $C_{d-1} \cup m$ different from any of the remaining two lines through $C_{d-1} \cap m$. Let $B \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ be the successive linear projections from $m_0$ and then the strict transform of $\beta$ on $Q$. Let $r$ be another general uni-secant line of $C_{d-1}$, and $C'_{d-1}$, $m'$ and $r' \subset \mathbb{P}^2$ be the images of $C_{d-1}$, $m$ and $r$ respectively.

First we suppose that $\beta$ is a bi-secant line of $C_{d-1}$. Then we have only to show that $C_{d-1} \cup m \cup r \dashrightarrow C'_{d-1} \cup m' \cup r'$ is generically one to one, deg $C'_{d-1} \cup m' \cup r' = d - 2$, and $C'_{d-1} \cup m' \cup r'$ has only simple nodes as its singularities assuming $C_{d-1} \cup r \dashrightarrow C'_{d-1} \cup r'$ is birational and $C'_{d-1} \cup r'$ has only simple nodes as its singularities. The proof is the same as that of Lemma 3.1.2 so we omit it.

Next suppose that $\beta$ is a uni-secant line of $C_{d-1}$ intersecting $m$ outside $C_{d-1} \cap m$. Note that, by the projection $B \dashrightarrow \mathbb{P}^2$, $m$ is contracted to a point. Moreover, $\beta$ is a general uni-secant line since so is $m$. Thus, by Lemma 3.1.2, $C_{d-1} \cup m \cup r \dashrightarrow C'_{d-1} \cup m' \cup r'$ is generically one to one, deg $C'_{d-1} \cup m' \cup r' = d - 2$, and $C'_{d-1} \cup m' \cup r'$ has only simple nodes as its singularities.

Let $f: A \to B$ be the blow-up of $B$ along a general smooth rational curve $C_d$. The following lemma can be regarded as the assertion of generality of $C_d$. We need this in the subsection 5.1.

Lemma 3.1.4. Let $\beta'_i \subset A$ be the strict transform of a bi-secant line $\beta_i$ of $C_d$. It holds:

$$N_{\beta'_i/A} = O_{\beta'_i}(-1) \oplus O_{\beta'_i}(-1).$$

Proof. We prove the assertion by using the inductive construction of $C_d$. The assertion is clear for $d = 1$ since $C_1$ has no bi-secant line.

Suppose the assertion holds for $C_{d-1}$. Choose a general uni-secant line $l \subset B$ of $C_{d-1}$. Let $m_1, \ldots, m_{d-2}$ be the lines on $B$ intersecting both $C_{d-1}$ and $l$ outside $C_{d-1} \cap l$. By generality of $C_{d-1}$ we can assume that $m_1, \ldots, m_{d-2}$ are uniseant of $C_{d-1}$.

Let $A' \to B$ be the blow-up along $C_{d-1} \cup l$. Note that the smoothing $C_{d-1} \cup l$ to $C_d$ induces that of $A'$ to $A$. Let $\tilde{m}_i$ be the strict transform of $m_i$ on $A'$. By the smoothing construction of $C_d$ from $C_{d-1} \cup l$ and the assumption on induction, we have only to prove $N_{\tilde{m}_i/A'} = O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1)$. Let $A'_1 \to B$ be the blow-up along $l$ and $A'_2 \to A'_1$ the blow-up along the strict transform of $C_{d-1}$. Denote by $m'_i$ and $m''_i$ the strict transform of $m_i$ on $A'_1$ and $A'_2$ respectively. Then $N_{\tilde{m}_i/A'} = N_{m''_i/A'_2}$. We consider the projection of $B$ from the line $l$ as in Proposition 3.1.1 (2). Since $m'_i$ is a fiber of $A'_1 \to Q$, we have $N_{m'_i/A'_1} = O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1)$. Let $F$ be the exceptional divisor of $A'_1 \to Q$ and $F'$ the strict transform of $F$ on $A'_2$. We may suppose $F$ and $C'_{d-1}$ intersect transversely, thus $F' \to F$ is the blow-up at $d - 2$ points $m''_i \cap C''_{d-1}$ ($i = 1, \ldots, d - 2$). Thus $F' \cdot m''_i = -1$ and $N_{m''_i/F'} = O_{\mathbb{P}^1}(-1)$, and this implies the assertion.

3.2. Double projection of $B$ from a point.

Definition 3.2.1. Let $b$ be a point of $B$. We call the rational map from $B$ defined by the linear system of hyperplane sections singular at $b$ the double projection from $b$.

Proposition 3.2.2. Let $b$ be a point of $B$. Then
(1) the target of the double projection from b is $\mathbb{P}^2$, and the double projection from b and the projection $B \rightarrow \overline{B}_b$ from b fit into the following diagram:

\[
\begin{array}{ccc}
\pi_{1b} & \rightarrow & B_b \\
\phantom{B} & \searrow & B \\
\phantom{B} & \swarrow & B_b \\
\phantom{B} & \nearrow & \overline{B}_b \\
\pi_{2b} & \rightarrow & \mathbb{P}^2,
\end{array}
\]

where $\pi_{1b}$ is the blow-up of B at b, $B_b \rightarrow B'_b$ is the flop of the strict transforms of lines through b, and $\pi_{2b}: B'_b \rightarrow \mathbb{P}^2$ is a (unique) $\mathbb{P}^1$-bundle structure. Moreover, $\overline{B}_b \rightarrow \mathbb{P}^2$ is the projection from the plane which is the image of $\pi_{1b}$-exceptional divisor.

We denote by $E_b$ the $\pi_{1b}$-exceptional divisor and by $E'_b$ the strict transform of $E_b$ on $B'_b$.

\[
L = H - 2E'_b \quad \text{and} \quad -K_{B'_b} = H + L,
\]

where $H$ is the strict transform of a general hyperplane section of B, and L is the pull back of a line on $\mathbb{P}^2$.

(3) Case (a)

If $b \not\in B_\varphi$, then the strict transforms $l'_i$ of three lines $l_i$ through b on $B_b$ have the normal bundle $O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1)$. The flop $B_b \rightarrow B'_b$ is the Atiyah flop. In particular, $E'_b \rightarrow E_b$ is the blow-up at the three points $E_b \cap l'_1$.

Case (b)

If $b \in B_\varphi \setminus C_\varphi$, then $E_b \rightarrow E'_b$ can be described as follows: let l and m be two lines through b, where l is special, and m is not special. Let $l'$ and $m'$ be the strict transforms of l and m on $B_b$. First blow up $E_b$ at two points $t_1 := E_b \cap l'$ and $t_2 := E_b \cap m'$ and then blow up at a point $t_3$ on the exceptional curve $e$ over $t_1$. Finally, contract the strict transform of $e$ to a point. Then we obtain $E'_b$ (this is a degeneration of the case (a)).

Case (c)

See [FN89] in case of $b \in C_\varphi$, and

(4) a fiber of $\pi_{2b}$ not contained in $E'_b$ is the strict transform of a conic through b, or the strict transform of a line $\not\in b$ intersecting a line through b.

The description of the fibers of $\pi_{2b}$ contained in $E'_b$ is as follows:

Case (a)

If $b \not\in B_\varphi$, then $\pi_{2b}: E'_b \rightarrow \mathbb{P}^2$ is the blow-down of the strict transforms of three lines connecting two of $E_b \cap l'_i$, namely, $E_b \rightarrow \mathbb{P}^2$ is the Cremona transformation.

Case (b)

Assume that $b \in B_\varphi \setminus C_\varphi$. Then $\pi_{2b}: E'_b \rightarrow \mathbb{P}^2$ is the blow-down of the strict transforms of two lines, one is the line connecting $t_1$ and $t_2$, the other is the line whose strict transform passes through $t_3$. $E_b \rightarrow \mathbb{P}^2$ is a degenerate Cremona transformation.

Case (c)

See [FN89] in case of $b \in C_\varphi$.

Proof. This is a standard result in the birational geometry of Fano 3-folds but is less known than Proposition 3.1.1. We have only found the paper [FN89], in which they deal with the most difficult case (c). Here we sketch the construction of the flop in the middle case (b) to intend the reader to get a feeling of birational maps from B.
Let \( b \) be a point of \( B_2 \setminus C_\gamma \). We use the notation of the statement of (3). The flop of \( m' \) is the Atiyah flop. We describe the flop of \( l' \). By \( \mathcal{N}_{l/B} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \), it holds that \( \mathcal{N}_{l/B_0} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \). Hence the flop of \( l' \) is a special case of Reid’s one [Rei83, Part II]. We show that the width is two in Reid’s sense. Let \( T_1 \) be the normalization of \( T_1 \). By Proposition 2.1.3 (5), \( T_1 \simeq \mathbb{F}_3 \) and the inverse image of the singular locus of \( T_1 \) is the union of the negative section \( C_0 \) and a fiber \( r \). Let \( \mu: \tilde{B}_b \rightarrow B_b \) be the blow-up along \( l' \) and \( F \) the exceptional divisor. Let \( T_2 \) be the strict transform of \( T_1 \) on \( \tilde{B}_b \). Then \( T_2 \) is the blow-up of \( T_1 \) at two points \( s_1 \in C_0 \) and \( s_2 \in r \). Denote by \( C'_0 \) and \( r' \) the strict transforms of \( C_0 \) and \( r \). We prove that \( \mathcal{N}_{r'/\tilde{B}_b} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \). Note that \( F \cap T_2 = C'_0 \cup r' \). The curves \( C'_0 \) and \( r' \) are two sections on \( F \). Let \( T'_1 \) be the image of \( T_2 \) on \( B_b \). By \( \mathcal{N}_{r'/B_b} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \) and \( T_2 = \mu^*T'_1 - 2F \), it holds \( F \simeq \mathbb{F}_2 \), and \( T_2|_{F} \sim 2G_0 + 3\gamma \), where \( G_0 \) is the negative section of \( F \) and \( \gamma \) is a fiber of \( F \rightarrow l' \). Note that \( F \cdot C'_0 = (F|_{T'_2} \cdot C'_0)|_{T_2} = -3 \) and \( F \cdot r' = (F|_{T_2} \cdot r')|_{T_2} = 0 \), and \( F \cdot G_0 = 0 \) and \( F \cdot (G_0 + 3\gamma) = -3 \). Thus we have \( C'_0 \sim G_0 + 3\gamma \) and \( r' = G_0 \) on \( F \). Now we see that \( -K_{\tilde{B}_b} \cdot r' = (\mu^*(-K_{B_b}) - F) \cdot r' = 0 \). Therefore, by \( (r')^2 = -1 \) on \( T_2 \), it holds that \( \mathcal{N}_{r'/\tilde{B}_b} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \).

It is easy to see that we can flop \( r' \). Let \( \tilde{B}_b \rightarrow \tilde{B}'_b \) be the flop of \( r' \) (now we consider locally around \( r' \)). Let \( F'' \) be the strict transform of \( F \) on \( \tilde{B}'_b \). By [Rei83], \( F'' \simeq F \) and there is a blow-down \( \tilde{B}_b \rightarrow \tilde{B}''_b \) of \( F'' \) such that \( \tilde{B}''_b \) is smooth. \( \tilde{B}_b \rightarrow \tilde{B}''_b \) is the flop of \( l' \).

By this description of the flop, we can easily obtain (3).

As a first application of the above operations, we have the following result, which we often use:

**Corollary 3.2.3.** Let \( b_1 \) and \( b_2 \) be two (possibly infinitely near) points on \( B \) such that there exists no line on \( B \) through them. Then there exists a unique conic on \( B \) through \( b_1 \) and \( b_2 \).

**Proof.** We project \( B \) from \( b_1 \) as in (3.5). Then the assertion follows by the description of fibers of \( \pi_{2b} \), as in Proposition 3.2.2 (4). \( \square \)

**Notation 3.2.4.** Consider the double projection from \( b \), see proposition 3.2.2. Throughout the paper, we denote by \( C'_b, C''_b \) and \( C_b \) the strict transforms of \( C := C_d \) on \( B_b, B'_b \) and \( \mathbb{P}^2 \) respectively.

The following result is one of the key results for the proof of the main result. Its importance and difficulty lies in the actual fact that it holds not only for a general \( b \in B \) but also for every \( b \in B \).

**Proposition 3.2.5.** Let \( C_d \) be a general smooth rational curve of degree \( d \) on \( B \) constructed as in Proposition 2.3.2. Assume that \( d \geq 5 \). Then, for any point \( b \in B \), the restriction of \( \pi_b \) to \( C_d \) is birational.

**Proof.** We prove the assertion by induction based on the construction of \( C_d \) from \( C_{d-1} \cup l \), where \( l \) is a general uni-secant line of \( C_{d-1} \) on \( B \).

First we prove the assertion for \( d = 5 \). Assume by contradiction that \( \pi_{b|C_b} \) is not birational for a point \( b \). Then, since \( C \rightarrow C_b \) is a composite of linear projections, \( C_b \) is a line or conic in \( \mathbb{P}^2 \). Let \( S \) be the pull-back of \( C_b \) by \( \pi_{2b} \). If \( C_b \) is a line, then \( C_5 \) is contained in a singular hyperplane section, which is the strict transform of \( S \) on \( B \) (recall that \( B \rightarrow \mathbb{P}^2 \) is the double projection
from $b$). This contradicts Corollary 2.5.3. Assume that $C_b$ is a conic. The only possibility is that $L \cdot C_b'' = 4$ and $C_b'' \to C_b$ is a double cover since $L \cdot C_b'' = 1$ and $\deg(C_b'' \to C_b) \leq 5$. Since the flop does not change the intersection numbers between the canonical divisor and curves, we have $-K_{B_b'} \cdot C_b'' = -K_{B_b} \cdot C_b'$. If $b \in C$, then we have $-K_{B_b'} \cdot C_b'' = 8$. Thus, by Proposition 3.2.2 (2) and $L \cdot C_b'' = 4$, it holds $H \cdot C_b'' = 4$. By $L = H - 2E_b''$, this shows that $E_b' \cdot C_b'' = 0$. This is, however, a contradiction since $E_b' \cap C_b'' \neq \emptyset$. Thus $b \not\in C$, and, by Proposition 3.2.2 (2), it holds $H \cdot C_b'' = 6$. By $L = H - 2E_b''$, we have $E_b' \cdot C_b'' = 1$. We compute $E_b'^2S$. Note that $-K_{B_b'} = 2H - 2E_b'' = 2(L + 2E_b'') - 2E_b'' = 2(L + E_b'')$. We have

$$E_b'^2L = \frac{1}{4}(-K_{B_b'} - 2L)^2L = \frac{1}{4}(-K_L - L|_L)^2 = 1.$$ 

Thus we have $E_b'^2S = 2E_b'^2L = 2$. The surface $S$ is a Segre-del Pezzo scroll. Let $C_b$ be the negative section of $S$ and $l$ is a fiber of $S \to C_b$ and set $e := -C_b^2$. We can write $E_{b|S} = C_0 + pl$ and $C_b'' \sim 2C_0 + ql$ ($p, q \geq 0$). By $E_b' \cdot C_b'' = 1$ and $E_b'^2S = 2$, we have $q + 2p - 2e = 1$ and $2p - e = 2$. Thus $e = 2p - 2$ and $q = 2p - 3$. Since $C_b''$ is irreducible, $q \geq 2e$, whence $2p - 3 \geq 2(2p - 2)$, i.e., $p = 0$ and $q = -3$, a contradiction.

Assume that $d \geq 6$. Let $C \to \Delta$ be the one-parameter smoothing of $C_{d-1} \cup l$ such that $C$ is smooth (as we saw in the proof of Proposition 2.3.2 this is possible). We consider the trivial family of the double projections $B \times \Delta \to \mathbb{P}^2 \times \Delta$ from $b \times \Delta$. Denote by $C_b', C_b''$ and $C_b$ the strict transforms of $C$ on $B_b \times \Delta, B_b' \times \Delta$ and $\mathbb{P}^2 \times \Delta$ respectively. We also denote by $C_{d-1, b}, C_{d-1, b}'$, and $C_{d-1, b}$ the strict transforms of $C_{d-1}$ on $B_b, B_b'$ and $\mathbb{P}^2$ respectively. To prove the proposition, it suffices to show that, for any $b$, there exists at least one point on $C_{d-1, b}$ over which $C \to C_b$ is isomorphic. First, admitting this claim, we finish the proof of the proposition. Indeed, set

$$N := \{(b, t) \in B \times \Delta \mid C \to C_b \text{ is not isomorphic over any point of } C_{b, t}\}$$

and let $\Delta' \subset \Delta$ be the image of $N$ by the projection to $\Delta$. $N$ is a closed subset, and so is $\Delta'$ since $B \times \Delta \to \Delta$ is proper. Thus $\Delta'$ consists of finitely many points since the origin is not contained in $\Delta'$ by admitting the above claim. Therefore, for a point $t \in \Delta$ sufficiently near the origin, $C_t \to C_{t, b}$ is birational for any $b$, which implies the proposition.

Now we show the above claim. By induction, we may assume that $C_{d-1} \to C_{d-1, b}$ is birational for any $b$. Note that $C_{d-1, b}$ is not a line since otherwise $C_{d-1}$ is contained in a singular hyperplane section as we see above in the case of $C_5$, a contradiction. We investigate the image of $l$ on $\mathbb{P}^2$. Recall the description of the fibers of $\pi_{2b}$ (Proposition 3.2.2 (4)). If $b \not\in l$, then the image of $l$ is a line or a point on $\mathbb{P}^2$. If $b \in l$, then the strict transform of $l$ on $B_b$ is a flopping curve. Thus $C_b$ contains the image of the flopped curve, which is a line. We investigate the other possible irreducible components of the central fiber $C_{b, 0}$ of $C_b \to \Delta$. If $b \not\in C_{d-1} \cup l$, then the only possibility is that $C_{b, 0}$ contains the image of a flopped curve, which is a line on $\mathbb{P}^2$. Suppose $b \in C_{d-1} \cup l$. Let $m_b'$ be the exceptional curve for $C_b' \to C$. Since $C$ is a smooth surface, $m_b'$ is a line on $E_b$. The curve $C_{b, 0}$ contains the strict transform $m_b$ of $m_b'$. This is the only possibility of the other components of $C_{b, 0}$. Let $l_b'$ be the strict transform of $l$ on $B_b$. If $b \in l$, then by the description of $E_b \to \mathbb{P}^2$, $m_b$ is a line since $l_b'$ is a flopping curve. Suppose that $b \in C_{d-1} \setminus l$. If $m_b'$ intersects a flopping curve, $m_b$ is a line or a point. In the other case, $m_b$ is a conic. If $b \not\in l \cup \beta_1$, then deg $C_{d-1, b} = d - 3$ by Proposition 3.2.2 (2). By $d \geq 6$, $C_{d-1, b}$ is not a conic. Thus $C_{d-1, b} = m_b$. Assume $b \in \beta_i$. Then deg $C_{d-1, b} = d - 4$. Thus, if $d \geq 7$,
then $C_{d-1,b} \neq m_b$. We show that even if $d = 6$, it holds $C_{d-1,b} \neq m_b$. By Proposition 2.4.1 (4), the flop $B_b \to B'_b$ is of type (a) in Proposition 3.2.2 (3). The strict transform $m''_b$ of $m'_b$ on $B''_b$ intersects the three fibers of $\pi_b$ contained in $E''_b$, which are the strict transforms of three lines on $E_b$. On the other hand, by $E'_b \cdot C''_{d-1,b} = 2$, the curve $C''_{d-1,b}$ intersects at most two fibers of $\pi$ contained in $E'_b$. Thus it holds $C_{d-1,b} \neq m_b$.

The above investigation shows that $\mathcal{C} \to \mathcal{C}_b$ is isomorphic over a point of $C_{d-1,b}$. □

We restate the proposition in terms of the relation between $C_d$ and multi-secant conics of $C_d$ on $B$ as follows:

**Corollary 3.2.6.** Let $b$ be a point of $B$ not in any bi-secant line of $C_d$ on $B$. If $d \geq 5$, then there exist finitely many $k$-secant conics of $C_d$ on $B$ through $b$ with $k \geq 2$ if $b \notin C_d$ (resp. with $k \geq 3$ if $b \in C_d$).

**Proof.** For a point $b \in B$ outside bi-secant lines of $C_d$ on $B$, there exist a finite number of singular multi-secant conics of $C_d$ through $b$ since the number of lines through $b$ is finite, and the number of lines intersecting both a line through $b$ and $C_d$ is also finite by Proposition 2.4.4 (3). Therefore we have only to consider smooth multi-secant conics $q$ of $C_d$ through $b$. By Proposition 3.2.2 (4), the strict transform $q'$ of such a conic $q$ on $B'_b$ is a fiber of $\pi_{gb}$. If $b \notin C_d$, then $q'$ intersects $C'_b$ twice or more counted with multiplicities, thus by Proposition 3.2.5 the finiteness of such a $q$ follows. We can prove the assertion in case of $b \in C_d$ similarly, thus we omit the proof. □

**Remark.** We refine this statement in Lemmas 4.2.12 and 5.1.3.

**Lemma 3.2.7.** Let $l$ be a general uni-secant line of $C$ and $l_b \subset \mathbb{P}^2$ the image of $l$ by the double projection from a point $b$. For a general point $b \notin C$, $\deg C_b = d$ and $C_b \cup l_b$ has only simple nodes. Assume that $d \geq 3$. For a general point $b$ of $C$, $\deg C_b = d - 2$ and $C_b \cup l_b$ has only simple nodes.

**Proof.** The claim for $\deg C_b$ follows from Propositions 3.2.2 (2) and 3.2.3. As for the singularity of $C_b \cup l_b$, the claim follows from simple dimension count. For simplicity, we only prove that for a general point $b \notin C$, the curve $C_b$ has only simple nodes. By Proposition 2.4.2, we may assume that any multi-secant conic through $b$ is smooth, bi-secant and intersects $C$ simply. Let $q$ be a smooth bi-secant conic through $b$. We may assume that $\mathcal{N}_{q/B} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Let $q'$ be the strict transform of $q$ on $B'_b$. Let $\tilde{B}' \to B'_b$ be the blow-up along $q'$, $E_{q'}$ the exceptional divisor and $\tilde{C}''$ the strict transform of $C''_b$. Note that $E_{q'} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ since $\mathcal{N}_{q/B'_b} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Then $C_b$ has simple nodes at the image of $q'$ if and only if the two points in $E_{q'} \cap \tilde{C}''$ does not belong to the same ruling with the opposite direction to a fiber of $E_{q'} \to q'$. Let $\tilde{B}_q \to B$ be the blow-up along $q$, $E_q$ the exceptional divisor and $\tilde{C}$ the strict transform of $C$. It is easy to see that a ruling of $E_q$ with the opposite direction to a fiber of $E_q \to q$ corresponds to that of $E_{q'}$ with the opposite direction to a fiber of $E_{q'} \to q'$. Thus $C_b$ has simple nodes at the image of $q'$ if and only if the two points in $E_q \cap \tilde{C}$ does not belong to the same ruling with the opposite direction to a fiber of $E_q \to q$. We can show that this is the case for a general $b$ by simple dimension count. □

**Corollary 3.2.8.** (1) The number of multi-secant conics of $C$ through a general point of $B$ is $n := \frac{(d-1)(d-2)}{2}$. 


(2) The number of $k$-secant conics of $C$ with $k \geq 3$ through a general point of $C$ is $\frac{(d-3)(d-4)}{2}$.

(3) Let $l$ be a general uni-secant line of $C$. Then the number of multi-secant conics of $C$ intersecting $l$ and passing through a general point of $C$ is $d - 3$.

Proof. We only prove (1) since the other statement can be proved similarly.

Let $b \not\in C$ be a general point of $B$. Recall that, by Corollary 3.2.6, there exist only finitely many multi-secant conics of $C$ through $b$. Moreover, since $C_b$ is a nodal rational curve of degree $d$ by Lemma 3.2.7, the number of its nodes is exactly $\frac{(d-3)(d-4)}{2}$.

As we saw in Corollary 3.2.8 (1), a general point of $B$ gives $n$ multi-secant conics of $C$ through it. Conversely, we ask whether mutually intersecting $n$ multi-secant conics of $C$ actually pass through one point or not. The next lemma partially answer this question and it is sufficient for our purpose in the proof of Theorem 4.4.1. We remark that the case $d = 5$ is treated in Dol04 4.3.

Lemma 3.2.9. Let $q_1, \ldots, q_n$ be mutually intersecting $n$ distinct multi-secant conics of $C$ such that

1. all $q_i$ are smooth,
2. no two of $q_i$ intersect at a point of $C \cup \cup_i \beta_i$, and
3. if three of $q_i$ pass through a point $b$, then any other $q_i$ does not intersect a line through $b$ outside $b$.

Then all $q_i$ pass through one point.

Remark. The set of $n$ conics through a general point satisfies the conditions of the lemma.

Proof.

Step 1. Let $b \in B$ be a point such that five of $q_i$, say, $q_1, \ldots, q_5$ pass through $b$. Then all the $q_i$ pass through $b$.

By the double projection from $b$, $q_1, \ldots, q_5$ are mapped to points $p_1, \ldots, p_5$ on $\mathbb{P}^2$. Suppose by contradiction that a smooth conic $q_j$ does not pass through $b$. Let $q'_j$, $q''_j$ and $\tilde{q}_j$ be the strict transforms of $q_j$ on $B_b$, $B'_b$ and $\mathbb{P}^2$, and set $S := \pi_{2b}^{-1} \tilde{q}_j$. By the assumption (3), $q_j$ does not intersect a line through $b$. Thus $\tilde{q}_j$ is a smooth conic through $p_1, \ldots, p_5$. The conic $\tilde{q}_j$ is unique since a conic through five points is unique. It holds that $-K_{B_b} \cdot q'_j = 4$ and $S \cdot q''_j = 4$, thus $S \simeq \mathbb{F}_2$ and $q''_j$ is the negative section. This implies that $q_j$ is also unique. By reordering, we may assume that $j = n$. We have the configuration such that all the conics pass through $b$ except $q_n$. Denote by $p_i$ the image of $q_i$ ($i \neq n$). Then $\tilde{q}_n$ and $C_b$ intersect at $p_i$. By $d \geq 6$, it holds deg $C_b \geq 3$, thus $\tilde{q}_n \not\in C_b$. By the assumption (2), $b \not\in C$. Therefore $\tilde{q}_n$ and $C_b$ intersect at $n - 1$ singular points of $C_b$. Since deg $C_b \leq d$, it holds $2(n - 1) \leq 2d$, a contradiction.

Step 2. If four conics $q_1, \ldots, q_4$ pass through one point $b$, then all the conics pass through $b$.

By contradiction and Step 1, we may assume that all the conics except $q_1, \ldots, q_4$ do not pass through $b$. Pick up two any conics, say, $q_5$ and $q_6$, not passing through $b$. Considering the double projection from $b$ as in Step 1. Denote by $\tilde{q}_j$ ($j \geq 5$) the image of $q_j$ on $\mathbb{P}^2$. By the assumption (3), $q_5$ and $q_6$ do not intersect a line through $b$, thus $\tilde{q}_5$ and $\tilde{q}_6$ are conics on $\mathbb{P}^2$. Therefore $q_5 \cap q_6$ lies on one of $q_1, \ldots, q_4$ since otherwise $\tilde{q}_5$ and $\tilde{q}_6$ would intersect at five points and this is a contradiction as in Step 1. Thus any two conics intersect on $q_1, \ldots, q_4$. Let $p_i$ be the intersection $q_i \cap q_5$ for $i = 1, \ldots, 4$. Then $q_j$ ($j \geq 5$) pass through one of $p_i$. Thus one of
We start the study of the geometry of lines. A contradiction to Corollary 2.5.3. A of curves on $H$. 4.1.1 and Theorem 4.2.15 for a quick view of their properties.

Curve 4.1. Construction of $H$. Then by Steps 1 and 2, points are different. Thus

$$q$$

Correction of $H$.

$m$ points or more by the assumption (2). But 2

Construction 4.1.1. We complete the proof.

Assume by contradiction that $q_1, \ldots, q_n$ do not pass through one point on $B$. If $d \geq 7$, then, by Steps 1 and 2,

$$(3.6) \quad \text{at most three of } q_i \text{'s pass through any intersection point.}$$

Let $m$ be the number of conics in a maximal tree $T$ of $q_i$'s such that two conics in $T$ pass through any intersection point. Note that $T$ is connected since $q_i$'s mutually intersect. The number of the intersection points of $q_i$'s contained in $T$ is $\frac{m(m-1)}{2}$.

By the maximality of $T$, a conic not belonging to $T$ passes through one of the intersection points of conics in $T$. By (3.6), no two conics not belonging to $T$ pass through one of the intersection point of conics in $T$. Hence it holds $\frac{m(m-1)}{2} + m \geq n$. This implies that $m \geq d - 2$ by $n = \frac{(d-1)(d-2)}{2}$. By reordering, we assume that $q_1, \ldots, q_m$ belong to $T$. If $d = 6$, then we take $q_1, \ldots, q_4$ as in the last part of Step 2. Consider the projection $B \rightarrow \mathbb{P}^3$ from the conic $q_1$. Then $q_2, \ldots, q_m$ are mapped to lines $l_2, \ldots, l_m$ intersecting mutually on $\mathbb{P}^3$ and the intersection points are different. Thus $l_2, \ldots, l_m$ span a plane, which in turn shows that $q_1, \ldots, q_m$ span a hyperplane section $H$ on $B$. Since $C$ intersects $q_i$ at two point or more, $C$ intersects $H$ at $2m$ points or more by the assumption (2). But $2m \geq 2(d-2) > d$, $C$ must be contained in $H$, a contradiction to Corollary 2.5.3.

4. Lines and Conics on $A$

We fix a general $C := C_d$ as in the subsection 2.3. Let $f: A \rightarrow B$ be the blow-up along $C$. We start the study of the geometry of $A$. In the subsections 4.1 and 4.2, we study the families of curves on $A$ of degree one or two with respect to the anti-canonical sheaf of $A$ (we call them lines and conics on $A$ respectively). The curve $H_1$ parameterizing lines on $A$ and the surface $H_2$ parameterizing conics on $A$ are two of the main characters in this paper. See Corollary 4.1.1 and Theorem 4.2.15 for a quick view of their properties.

4.1. Curve $H_1$ parameterizing marked lines.

4.1.1. Construction of $H_1$ and marked lines.

Set $H_1 := \varphi^{-1}C \subset \mathbb{P}$ and $M := M_d$. We begin with a few corollaries of Proposition 2.4.4.

**Corollary 4.1.1.** If $d \geq 2$, then $H_1$ is a smooth curve of genus $d-2$ with the triple cover $H_1 \rightarrow C$. In particular, if $d \geq 5$, then $H_1$ is a smooth non-hyperelliptic trigonal curve of genus $d-2$.

**Proof.** By Propositions 2.1.3 (1) and 2.4.4 (1), it holds that $H_1$ is smooth and the ramification for $H_1 \rightarrow C$ is simple. Since $B_\varphi \in |-K_B|$ and $d = \deg C$, we can compute $g(H_1)$ by the Hurwitz formula:

$$2g(H_1) - 2 = 3 \times (-2) + d \times 2,$$

equivalently, $g(H_1) = d - 2$. 

\[\square\]
Corollary 4.1.2. The number of nodes of \( M \) is \( s := \frac{(d-2)(d-3)}{2} \), whence \( C \) has \( \frac{(d-2)(d-3)}{2} \) bi-
secant lines on \( B \).

Proof. By Proposition 2.4.4 (3), we see that \( \pi|_{\mathcal{H}_1} : \mathcal{H}_1 \to M \) is birational and \( p_0(M) = \frac{(d-1)(d-2)}{2} \).
Then by \( g(\mathcal{H}_1) = d - 2 \), the number of nodes of \( M \) is \( \frac{(d-1)(d-2)}{2} - (d - 2) = \frac{(d-2)(d-3)}{2} \).
The latter half follows since a bi-secant line of \( C \) corresponds to a node of \( M \). \( \square \)

Now we select some lines on \( B \) which we use in the sequel. Note that
\[
\mathcal{H}_1 = \{([l], t) | [l] \in M, t \in C \cap l \} \subset M \times C,
\]
and the elements of \( \mathcal{H}_1 \) deserve a name:

Definition 4.1.3. A pair of a secant line \( l \) of \( C \) on \( B \) and a point \( t \in C \cap l \) is called a marked line.

Let \( (l, t) \) be a marked line. If \( C \cap l \) is one point, then \( \{t\} = C \cap l \) is uniquely determined.
For a bi-secant line \( \beta_i \) of \( C \), there are two choices of \( t \). Thus \( \mathcal{H}_1 \) parameterizes marked lines.

4.1.2. Lines on the blow-up \( A \) of \( B \) along \( C \).

We prove that each marked line corresponds to a curve of anticanonical degree 1 on the blow-up \( A \) of \( B \) along \( C \). This gives us a suitable notion of line on \( A \).

Notation 4.1.4.
(1) Let \( f : A \to B \) be the blowing up along \( C \) and \( E_C \) the \( f \)-exceptional divisor,
(2) \( \{p_{11}, p_{12}\} = C \cap \beta_i \subset B \),
(3) \( \zeta_{ij} = f^{-1}(p_{ij}) \subset E_C \subset A \), and
(4) \( \beta_i \cap \zeta_{ij} = p'_{ij} \subset E_C \subset A \),
where \( i = 1, \ldots, s \) and \( j = 1, 2 \).

Definition 4.1.5. We say that a connected curve \( l \subset A \) is a line on \( A \) if \( -K_A \cdot l = 1 \) and \( E_C \cdot l = 1 \).

We point out that since \( -K_A = f^*(-K_B) - E_C \) and \( E_C \cdot l = 1 \) then \( f(l) \) is a line on \( B \) intersecting \( C \). More precisely:

Proposition 4.1.6. A line \( l \) on \( A \) is one of the following curves on \( A \):

(i) the strict transform of a uni-secant line of \( C \) on \( B \), or
(ii) the union \( l_{ij} = \beta_i' \cup \zeta_{ij} \), where \( i = 1, \ldots, s \) and \( j = 1, 2 \).

In particular \( l \) is reduced and \( p_0(l) = 0 \).

Notation 4.1.7. For a line \( l \) on \( A \), we usually denote by \( \overline{l} \) its image on \( B \).

Corollary 4.1.8. The curve \( \mathcal{H}_1 \subset \mathbb{P} \) is the Hilbert scheme of the lines of \( A \).

Proof. Let \( \mathcal{H}'_1 \) be the Hilbert scheme of lines on \( A \), which is a locally closed subset of the Hilbert scheme of \( A \). By the obstruction calculation of the normal bundles of the components of lines on \( A \), it is easy to see that \( \mathcal{H}'_1 \) is a smooth curve. Denote by \( \mathcal{U}_1 \to \mathcal{H}'_1 \) the universal family of the lines on \( A \) and let \( \overline{\mathcal{U}}_1 \) be the image of \( \mathcal{U}_1 \) on \( B \times \mathcal{H}'_1 \) (with induced reduced structure).

Claim 4.1.9. \( \overline{\mathcal{U}} \to \mathcal{H}'_1 \) is a \( \mathbb{P}^1 \)-bundle.
Proof of the claim. Let $\mathcal{L}$ be the pull-back of the ample generator of $\text{Pic} \ B$ by

$$U_1 \hookrightarrow A \times \mathcal{H}'_1 \to B \times \mathcal{H}'_1 \to B.$$ 

Since $\varphi : U_1 \to \mathcal{H}'_1$ is flat and $h^0(l, \mathcal{L}|l) = 2$ for a line $l$ on $B$, $\mathcal{E} := \varphi_* \mathcal{L}$ is a locally free sheaf of rank two. $\mathbb{P}(\mathcal{E})$ is nothing but the $\mathbb{P}^1$-bundle contained in $B \times \mathcal{H}'_1$ whose fiber is the image of a line on $A$. This implies that $\mathbb{P}(\mathcal{E}) = \overline{U}$ as schemes and $\overline{U}$ is a $\mathbb{P}^1$-bundle. □

By the claim, we have a natural morphism $\mathcal{H}'_1 \to \mathbb{P}^2$, whose image is $M$. By Proposition 4.1.6 $\mathcal{H}'_1 \to M$ is birational and surjective. Since $\mathcal{H}'_1$ and $\mathcal{H}_1$ are smooth, they are both normalizations of $M$, hence $\mathcal{H}'_1 \simeq \mathcal{H}_1$. □

Remark. For a bi-secant line $\beta_i$, we have two choices of marking, $p_{i1}$ or $p_{i2}$. We describe which line on $A$ corresponds to $(\beta_i, p_{ij})$. Denote by $U_1 \to \mathcal{H}_1$ the universal family of the lines on $A$ and consider the following diagram:

$$
\begin{array}{ccc}
U_1 & \subset & A \times \mathcal{H}_1 \\
\downarrow & & \downarrow \\
\overline{U}_1 & \subset & B \times \mathcal{H}_1.
\end{array}
$$

Then $U_1 \to \overline{U}_1$ is the blow-up along $(C \times \mathcal{H}_1) \cap \overline{U}_1$, which is the union of a section of $\overline{U}_1 \to \mathcal{H}_1$ consisting markings and finite set of points $(p_{i,3-j}, [\beta_i, p_{ij}])$. Thus the marked line $(\beta_i, p_{ij})$ corresponds to the line $l_{i,3-j}$.

4.2. Surface $\mathcal{H}_2$ parameterizing marked conics.

Now we define a notion of conic on $A$. We proceed as in the case of lines, first defining the notion of marked conic.

4.2.1. Construction of $\mathcal{H}_2$ and marked conics.

Definition 4.2.1. A pair of a multi-secant conic $q$ on $B$ and a zero-dimensional subscheme $\eta \subset C$ of length two contained in $q_\mathcal{C}$ is called a marked conic.

From now on, we assume that $d \geq 3$.

Marked conics are parameterized by

$$\mathcal{H}_2' := \{(\beta, \eta) \mid \beta \in \overline{\mathcal{H}_2}, \eta \subset q_\mathcal{C}\} \subset \overline{\mathcal{H}_2} \times S^2 C$$

with reduced structure, where $\overline{\mathcal{H}_2} \subset \mathbb{P}^4$ is the locus of multi-secant conics of $C$ on $B$.

By Corollary 3.2.3 and $d \neq 1$, the natural projection of $\mathcal{H}_2' \to S^2 C$ is one to one outside $[\beta_{i|C}]$ and the diagonal of $S^2 C$.

We denote by $\mathcal{E}_i'$ the fiber of $\mathcal{H}_2' \to S^2 C$ over a $[\beta_{i|C}]$. Since $B$ is the intersection of quadrics, any conic cannot intersect a line twice properly. Thus any conic $\supset [\beta_{i|C}]$ contains $\beta_i$. This implies that $\mathcal{E}_i' \simeq \mathbb{P}^1$, and $\mathcal{E}_i'$ parameterizes marked conics of the form

$$\{(\beta_i \cup \alpha, [\beta_{i|C}]) \mid \alpha \text{ is a line such that } \alpha \cap \beta_i \neq \emptyset\}.$$ 

Over the diagonal of $S^2 C$, $\mathcal{H}_2' \to S^2 C$ is finite since for $t \in C$, there exist a finite number of reducible conics with $t$ as a singular point or conics tangent to $C$ at $t$.

Hence $\mathcal{H}_2'$ is the union of the unique two-dimensional component, which dominates $S^2 C$, and possibly lower dimensional components mapped into the diagonal of $S^2 C$ or $\mathcal{E}_i'$. Note that
$\mathcal{H}'_2 \rightarrow \overline{\mathcal{H}}_2$ is finite since choices of markings of a multi-secant conic of $C$ is finitely many by $d \geq 3$.

**Claim 4.2.2.** $e'_i$ is contained in the unique two-dimensional component of $\mathcal{H}'_2$.

**Proof.** We have only to prove that $\overline{\mathcal{H}}_2$ is two-dimensional near the generic point of the image of $e'_i$ since $\mathcal{H}'_2 \rightarrow \overline{\mathcal{H}}_2$ is one to one near the generic point of the image of $e'_i$. Let $\mathcal{V}_2 \rightarrow \mathcal{H}'_2 \cong \mathbb{P}^4$ be the universal family of conics on $B$ and $\overline{\mathcal{H}}'_2$ the inverse image of $C \times C$ by $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2 \rightarrow B \times B$. Since the morphism $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2 \rightarrow \mathcal{V}_2 \rightarrow \mathbb{P}^4$ is flat, $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2$ is purely six-dimensional. Thus any component of $\overline{\mathcal{H}}'_2$ has dimension greater than or equal to two. Though the inverse image of the diagonal of $C \times C$ is three-dimensional, any other component of $\overline{\mathcal{H}}'_2$ is at most two-dimensional by a similar investigation to $\mathcal{H}'_2$. Thus $\overline{\mathcal{H}}_2$ is two-dimensional near the generic point of the image of $e'_i$ since $\overline{\mathcal{H}}_2$ is the image of the two-dimensional part of $\overline{\mathcal{H}}'_2$ by $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2 \rightarrow \mathbb{P}^4$ near the generic point of the image of $e'_i$. \(\square\)

**Notation 4.2.3.** Let $\mathcal{H}_2$ be the normalization of the unique two-dimensional component of $\mathcal{H}'_2$ and $\overline{\mathcal{H}}_2 \subset \overline{\mathcal{H}}_2$ the image of $\mathcal{H}_2$. Denote by $\eta$ the natural morphism $\mathcal{H}_2 \rightarrow S^2C$. Set

$$c_i := [\beta_i|C] \in S^2C \cong \mathbb{P}^2,$$

and

$$e_i := \eta^{-1}(c_i),$$

where $i = 1, \ldots, s$.

By the above consideration, $\eta: \mathcal{H}_2 \rightarrow S^2C$ is isomorphic outside $[\beta_i|C]$ by the Zariski main theorem, and $\mathcal{H}_2 \rightarrow \overline{\mathcal{H}}_2$ is the normalization. Thus we see that $\mathcal{H}_2$ parameterizes marked conics in one to one way outside the inverse image of $c_i$. We need to understand the inverse image by $\eta$ of the diagonal.

**Claim 4.2.4.** Assume that $([q], [2b]) \in \mathcal{H}_2$ for $b \in C$ and a conic $q$. Then

(1) $q$ is reduced,

(2) if $q$ is smooth at $b$, then $q$ is tangent to $C$ at $b$, and

(3) if $q$ is singular at $b$, then the strict transform of $q$ is connected on $A$. Moreover, $b \not\in \beta_i$ nor $B_\varphi$.

**Proof.** We use the double projection from $b$. By Proposition [3.2.2] (4) and a degeneration argument, $q$ corresponds to the fiber of $\pi_{2b}$ through the point $t'$ in $C''_b \cap E'_b$ coming from $t := C'_b \cap E_b$.

(1) Assume by contradiction that $q$ is non-reduced. By Proposition [2.2.1] $q$ is a multiple of a special line $l$. By Proposition [2.4.1] (4), $l$ is a uni-secant line of $C$. Let $m$ be the other line through $b$ (by generality of $C$, we have $l \neq m$). Let $l'$ and $m'$ be the strict transforms of $l$ and $m$ on $B_b$ respectively. By Proposition [3.2.2] (4), the fiber of $\pi_{2b}$ through $t'$ is the strict transform of the line in $E_b$ joining $l' \cap E_b$ and $m' \cap E_b$. Hence by the assumption, $l' \cap E_b$, $m' \cap E_b$ and $C'_b \cap E_b$ are collinear. By dimension count similar to the proof of Proposition [2.4.1] we can prove that a general $C$ does not satisfy this condition.

(2) This follows from the previous discussion.

(3) Set $q = l_1 \cup l_2$, where $l_1$ and $l_2$ are the irreducible components of $q$, and let $l'_i$ be the strict transform of $l_i$ on $B_b$. By (1), it holds $l_1 \neq l_2$. Then the fiber of $\pi_{2b}$ corresponding to $q$ is the
strict transform of the line on \( E_b \) through \( E_b \cap l'_1 \) and \( E_b \cap l'_2 \). Note that \( A \) is obtained from \( B_b \) by blow-up \( B_b \) along \( C'_b \) and then contracting the strict transform of \( E_b \). Thus the former half of the assertion follows. The latter half follows again by simple dimension count. \( \square \)

4.2.2. Conics on \( A \).

**Definition 4.2.5.** We say that a connected and reduced curve \( q \subset A \) is a conic on \( A \) if \(-K_A \cdot q = 2\) and \( E_C \cdot q = 2 \).

Using this definition, we can classify conics on \( A \) similarly to Proposition 4.1.6.

**Proposition 4.2.6.** Let \( q \) be a conic on \( A \). Then \( \overline{q} := f(q) \subset B \) is a multi-secant conic of \( C \). Moreover one of the following holds:

(a) \( \overline{q} \) is smooth at \( \overline{q} \cap C \), \( q \) is the union of the strict transform \( q' \) of \( \overline{q} \) and \( k - 2 \) distinct fibers \( \zeta_1, \ldots, \zeta_{k-2} \) of \( E_C \) such that \( \zeta_i \cap q' \neq \emptyset \),

(b) \( \overline{q} \) is the union of the two uni-secant lines \( \overline{l} \) and \( \overline{m} \) such that \( C \cap \overline{l} \cap \overline{m} \neq \emptyset \). \( q \) is the union of the strict transforms \( l \) and \( m \) of \( \overline{l} \) and \( \overline{m} \) respectively (we assume that \( l \cap m \neq \emptyset \)), or

(c) \( \overline{q} \) is the union of \( \beta_i \) and a line \( \overline{r} \) through a \( p_{ij} \). \( q \) is the union of the fiber \( \zeta_{ij} \) over \( p_{ij} \) and the strict transforms \( \beta'_i \) and \( r' \) of \( \beta_i \) and \( \overline{r} \) respectively.

**Notation 4.2.7.** We usually denote by \( \overline{q} \subset B \) the image of a conic \( q \) on \( A \).

Let \( \mathcal{H}_2^A \) be the normalization of the two-dimensional part of the Hilbert scheme of conics on \( A \), which is a locally closed subset of the Hilbert scheme of \( A \). Let \( \mu : U_2 \to \mathcal{H}_2^A \) be the pull-back of the universal family of conics on \( A \).

**Lemma 4.2.8.** Let \( \overline{U}_2 \) be the image of \( U_2 \) on \( B \times \mathcal{H}_2^A \) (with induced reduced structure) then \( \overline{U}_2 \to \mathcal{H}_2^A \) is a conic bundle.

**Proof.** The proof is similar to that of Claim 4.1.9.

Let \( \mathcal{L} \) be the pull-back of the ample generator of Pic \( B \) by

\[
U_2 \hookrightarrow A \times \mathcal{H}_2^A \to B \times \mathcal{H}_2^A \to B.
\]

Since \( \mu : U_2 \to \mathcal{H}_2^A \) is flat and \( h^0(q, \mathcal{L}|_q) = 3 \) for a conic \( q \) on \( A \) (recall that \( q \) is reduced), then \( \mathcal{E} := \mu_* \mathcal{L} \) is a locally free sheaf of rank 3. Letting \( \mathbb{P}^6 = \langle B \rangle \), \( \mathbb{P}(\mathcal{E}) \) is the \( \mathbb{P}^2 \)-bundle contained in \( \mathbb{P}^6 \times \mathcal{H}_2^A \) whose fiber is the plane spanned by the image of a conic on \( A \). Let \( Q := (B \times \mathcal{H}_2^A) \cap \mathbb{P}(\mathcal{E}) \), where the intersection is taken in \( \mathbb{P}^6 \times \mathcal{H}_2^A \). A scheme theoretic fiber of \( Q \to \mathcal{H}_2^A \) is the image of a conic of \( A \) since \( B \) is the intersection of quadrics. Then \( Q = \overline{U}_2 \) as schemes and \( \overline{U}_2 \) is a conic bundle. \( \square \)

**Proposition 4.2.9.** There exists a natural bijection between the set of marked conics belonging to \( \mathcal{H}_2 \) and the set of conics on \( A \). Moreover, the two surfaces \( \mathcal{H}_2^A \) and \( \mathcal{H}_2 \) are isomorphic.

**Proof.** The first assertion follows from Claim 4.2.4 (1) and (3), and Proposition 4.2.6.

By Lemma 4.2.8, there exists a natural morphism \( \overline{\nu} : \mathcal{H}_2^A \to \overline{\mathcal{H}}_2 \). By Proposition 4.2.6, \( \overline{\nu} \) is finite and birational, hence \( \overline{\nu} \) lifts to the morphism \( \nu : \mathcal{H}_2^A \to \mathcal{H}_2 \) since \( \mathcal{H}_2 \to \overline{\mathcal{H}}_2 \) is the normalization. By the Zariski main theorem, \( \nu \) is an inclusion. By Claim 4.2.4 (1) and (3), and Proposition 4.2.6, \( \nu \) is also surjective. \( \square \)
By Proposition 4.2.9 we can pass freely from conics on $A$, that is elements of $\mathcal{H}_2^A$ to marked conics and vice-versa according to the kind of argument we will need. In particular we can speak of the universal family $\mu: \mathcal{U}_2 \to \mathcal{H}_2$ of marked conics meaning $\mathcal{U}_2 := \mathcal{U}_2^A$ and $\mathcal{H}_2^A$ identified to $\mathcal{H}_2$ via $\iota$.

**Corollary 4.2.10.** The Hilbert scheme of conics on $A$ is an irreducible surface (and $\mathcal{H}_2$ is the normalization). The normalization is injective, namely, $\mathcal{H}_2$ parameterizes conics on $A$ in one to one way.

**Proof.** By Proposition 4.2.6, the image of $\mathcal{H}_2$ in the Hilbert scheme of $A$ parameterizes all the conics on $A$, thus the first part follows.

For the second part, we have already seen that $\mathcal{H}_2$ parameterizes marked conics belonging to $\mathcal{H}_2$ in one to one way outside $\cup_i e_i$. Thus, by Proposition 4.2.9 $\mathcal{H}_2$ parameterizes conics on $A$ in one to one way outside $\cup_i e_i$. Let $\alpha$ be a general line intersecting $\beta_i$, and $\alpha'$ the strict transform of $\alpha$ on $A$. By easy obstruction calculation, we see that the Hilbert scheme of conics on $A$ is smooth at $[\beta_i \cup \alpha']$. Thus general points of $e_i$ also parameterizes conics on $A$ in one to one way. Then, however, since $e'_i \simeq \mathbb{P}^1$, where $e'_i$ is the inverse image of $[\beta_i|C]$ by $\mathcal{H}_2' \to S^2C$, it holds that $e_i \simeq e'_i \simeq \mathbb{P}^1$ ($\mathcal{H}_2 \to S^2C$ has only connected fibers). This implies the assertion. \(\Box\)

In subsection 4.2.5 we complete the description of $\mathcal{H}_2$. In 4.2.3 and 4.2.4, we give some preliminary results for this purpose.

### 4.2.3. Quasi-finiteness of $\psi: \mathcal{U}_2 \to A$.

**Notation 4.2.11.** For a point $b \in C$, set

$$L_b := \{[q] \in \mathcal{H}_2 | \exists b' \neq b, f(q) \cap C = \{b, b'\}\}.$$

By Corollary 3.2.3, $\eta(L_b)$ is a line in $S^2C \simeq \mathbb{P}^2$.

Let $\psi: \mathcal{U}_2 \to A$ be the morphism obtained via the universal family $\mu: \mathcal{U}_2 \to \mathcal{H}_2$. The following result refines Proposition 3.2.5. Here we need this result technically for the discussion in 4.2.4 but this is important for the proof of the main result and is refined again in 5.1 (Proposition 5.1.3).

From now on in this paper, we assume that $d \geq 5$.

**Proposition 4.2.12.** The morphism $\psi$ is finite of degree $n = \frac{(d-1)(d-2)}{2}$ and flat outside $\cup_{i=1}^s \beta'_i$.

**Proof.** Let $a \in A \setminus \cup_{i=1}^s \beta'_i$ and set $b := f(a)$. If $b \not\in C$, then the finiteness of $\psi$ over $a$ follows from Corollary 3.2.6. Moreover, by Corollary 3.2.8, the number of conics through a general $a$ is $n$. Thus $\deg \psi = n$. We will prove that $\psi$ is finite over $a \in E_C \setminus \cup_{i=1}^s \beta'_i$. Once we prove this, the assertion follows. Indeed, $\mathcal{U}_2$ is Cohen-Macaulay since $\mathcal{H}_2$ is smooth and any fiber of $\mathcal{U}_2 \to \mathcal{H}_2$ is reduced, thus $\psi$ is flat.

Let $a \in E_C \setminus \cup_{i=1}^s \beta'_i$. The assertion is equivalent to that only finitely many conics belonging to $L_b$ pass through $a$. If $b \not\in \cup_{i=1}^s \beta_i$, then $L_b$ is irreducible. If $b \in \cup_{i=1}^s \beta_i$, then $L_b = L'_b \cup e_i$, where $L'_b$ is the strict transform of $\eta(L_b)$ whence is irreducible. Note that almost all the conics belonging to $e_i$ does not pass through a $b \not\in \cup_{i=1}^s \beta'_i$. Let $S_b \subset A$ be the locus swept by the conics of the family $L_b$ if $b \not\in \cup_{i=1}^s \beta_i$, or the locus swept by the conics of the family $L'_b$ if $b \in \cup_{i=1}^s \beta_i$. Then $S_b$ is irreducible. Let $\overline{S}_b := f(S_b)$, $\overline{S}'_b$ and $\overline{S}''_b$ the strict transforms of $\overline{S}_b$ on $B_b$ and $B'_b$.
respective. Then \( S'_b = \pi_{2b'} S_b \). Let \( d_b := \deg C_b \). By Proposition \[3.2.2\] (2), \( d_b = d - 2 \) if \( b \not\in \cup^n {i=1} \beta_i \), or \( d = 3 \) if \( b \in \cup^n {i=1} \beta_i \). Since \( S'_b \sim d_b L \) and \( L = H - 2E_b \), we have \( S'_b|E_b \) is a curve of degree \( 2d_b \) in \( E_b \simeq P^2 \).

Since \( A \) is obtained from \( B_b \) by blowing up \( C'_b \) and then contracting the strict transform of \( E_b \), a point \( a \) over \( b \) corresponds to a line \( l_a \) in \( E_b \) through \( t := E_b \cap C'_b \). The image on \( B_b \) of the strict transform of a conic on \( A \) through a intersects \( E_b \) at a point of \( l_a \cap S'_b \). If \( C''_b \) does not intersect fibers of \( \pi_{2b} \) contained in \( E'_b \), then \( S''_b|E'_b \) is irreducible. Thus no \( l_a \) is contained in \( S''_b|E'_b \) and we are done. Assume that \( C''_b \) intersects a fiber \( l' \) of \( \pi_{2b} \) contained in \( E'_b \). This is a situation as in Claim \[4.2.4\] (3), hence \( b \not\in B_\varphi \) nor \( b \not\in \cup^n {i=1} \beta_i \) for a general \( C \). Since \( L_b \) is irreducible by \( b \not\in \cup^n {i=1} \beta_i \), it suffices to prove the finiteness and nonemptyness of the set of conics through a general point \( a \) over \( b \). Equivalently, we have only to show that a general \( l_a \) intersects \( S''_b|E'_b \) outside \( t \). Since \( l' \) intersects \( C''_b \) simply at one point, \( C_b \) is smooth at the image \( t' \) of \( l' \) on \( \mathbb{P}^2 \). Thus \( S''_b|E'_b = C''_b + l \), where \( C''_b \) and \( l \) are the strict transforms of \( C_b \) and \( l' \). Note that \( C''_b \) is smooth at \( t \) and \( \deg C''_b = 2d_b - 1 = 2d - 5 \geq 5 \) by \( d \geq 5 \). Thus a general \( l_a \) intersect \( C''_b \) outside \( t \).

Remark. Though we do not need it later, we describe the fiber of \( \psi \) over a general point \( a \in C \setminus \cup^n {i=1} \beta_i \) for reader’s convenience.

Set \( b := f(a) \). As in the proof of Proposition \[4.2.12\] a point \( a \) over \( b \) corresponds to a line \( l_a \) in \( E_b \) passing through \( E_b \cap C'_b \). By Lemma \[3.2.7\] it holds that \( \deg C_b = d - 2 \) and \( C_b \) has \( \frac{(d - 3)(d - 4)}{2} \) simple nodes for a general \( b \in C \). This means that \( \frac{(d - 3)(d - 4)}{2} \) tri-secant conics pass through \( b \). By Proposition \[4.2.6\] corresponding to a tri-secant conic \( \mathfrak{q} \), there is a unique conic \( q \) on \( A \) containing the fiber of \( E_C \) over \( b \) and such a conic on \( A \) contains \( a \). Thus we obtain \( \frac{(d - 3)(d - 4)}{2} \) conics through \( a \). By definition of \( L_b \), these conics do not belong to \( L_b \).

We need more \( n - \frac{(d - 3)(d - 4)}{2} = 2d - 5 \) conics through \( a \). We show that there exist \( 2(d - 2) - 1 \) conics through \( a \) on \( A \) coming from the family parameterized by \( L_b \). We use the notation of the proof of Proposition \[4.2.12\]. For a general \( b \in C \), \( C'_b \) does not intersect fibers of \( \pi_{2b} \) contained in \( E'_b \). Thus \( S'_b|E'_b \) is an irreducible curve of degree \( 2(d - 2) \) on \( E_b \). Thus there are \( 2(d - 2) \) intersection points of \( S'_b|E'_b \) and \( l_a \). Among those, the intersection point \( C'_b \cap E_b \) does not correspond to a conic on \( A \) through \( a \) since it comes from the tangent of \( C \). Thus we have \( 2(d - 2) - 1 \) conics as desired.

4.2.4. Intersection of lines and conics on \( A \).

To understand better \( \eta: \mathcal{H}_2 \rightarrow \mathbb{P}^2 \) we need to find special loci inside \( \mathcal{H}_2 \). A natural step is to study the locus of conics which intersect a fixed line. This locus turn out to be a good divisor of \( \mathcal{H}_2 \).

Let \( U_1' \subset U_2 \times \mathcal{H}_1 \) be the pull-back of \( U_1 \) via the following diagram:

\[ (4.1) \]

\[
\begin{array}{ccc}
U_1' & \subset & U_2 \times \mathcal{H}_1 \\
\downarrow & & \downarrow \\
\mathcal{D}_1 & \subset & \mathcal{H}_2 \times \mathcal{H}_1 \\
& \longrightarrow & \mathcal{H}_1,
\end{array}
\]

where \( \mathcal{D}_1 \) is the image of \( U_1' \) on \( \mathcal{H}_2 \times \mathcal{H}_1 \).
By definition
\[ \hat{D}_1 = \{ ([q], [l]) \mid q \cap l \neq \emptyset \} \subset H_2 \times H_1. \]

First we need to know which component of \( \hat{D}_1 \) is divisorial or dominates \( H_1 \). For this purpose, we study mutual intersection of a conic and a line in special cases. Let \( F \subset H_2 \times H_1 \) be the image in \( H_2 \times H_1 \) of the inverse image of \( ((\cup \beta'_i) \times H_1) \cap U_1 \); that is
\[ F := \{ ([q], [l]) \mid q \cap \beta'_i \cap l \neq \emptyset \}. \]

A point \( ([q], [l]) \in F \) iff i) \( l = l_{ij} := \beta'_i \cup \zeta_{ij} \) and \( q \cap \beta'_i \neq \emptyset \) or ii) \( l \neq l_{ij} \), and \( q \cap \beta'_i \cap l \neq \emptyset \). For every \( i = 1, \ldots, s, j = 1, 2 \) the family of those \( ([q], [l]) \) which satisfies i) or ii) has dimension one and clearly does not dominate \( H_1 \).

**Corollary 4.2.13.** Any component of \( \hat{D}_1 \) which is not contained in \( F \) dominates \( H_1 \). Moreover, any non-divisorial component of \( \hat{D}_1 \) outside \( F \) (if it exists) is a one-dimensional component whose generic point parameterizes reducible conics, namely, a one-dimensional component of
\[ \{ ([q], [l]) \mid l \subset q \}. \]

**Remark.** Here we leave the possibility that a one-dimensional component whose generic point parameterizes reducible conics is contained in a divisorial component of \( \hat{D}_1 \). We, however, prove that this is not the case in Corollary 4.2.17. Hence, finally, the fiber of \( \hat{D}_1 \to H_1 \) over a general \( [l] \in H_1 \) parameterizes conics which properly intersect \( l \).

**Proof.** By Proposition 4.2.12, \( U_2 \to A \) is finite and flat outside \( \cup \beta'_i \). Then \( U_2 \times H_1 \to A \times H_1 \) is flat outside \( (\cup \beta'_i) \times H_1 \). By base change, \( U'_1 \to U_1 \) is flat and finite outside \( (\cup \beta'_i) \times H_1 \cap U_1 \). Every irreducible component of \( U'_1 \) which is not mapped to \( (\cup \beta'_i) \times H_1 \cap U_1 \) is two-dimensional, and dominates \( U_1 \), hence dominates \( H_1 \). Therefore any component of \( \hat{D}_1 \) which is not contained in \( F \) dominates \( H_1 \).

We find a possible non-divisorial component of \( \hat{D}_1 \) outside \( F \). Let \( \gamma \subset U'_1 \) be a curve mapped to a point, say, \( ([q], [l]) \) on \( H_2 \times H_1 \). The image of \( \gamma \) on \( A \) is an irreducible component of \( q \), say, \( q_1 \). The image of \( \gamma \) on \( U_1 \) is \( q_1 \times [l] \), thus \( q_1 \) is also an irreducible component of \( l \). We have the following three possibilities:

1. \( l \) is irreducible, hence \( q_1 = l \) and \( q = l \cup m \), where \( m \) is another line. Such \( ([q], [l]) \) form the one-dimensional family of reducible conics,
2. \( l = l_{ij} \) and \( \beta'_i \subset q \). Namely \( [q] \in e_i \), or \( q \supset \beta'_i \cup \alpha \cup \zeta_{ik} \), where \( \alpha \) is the strict transform of a line on \( B \) intersecting \( \beta_i \) and \( C \) outside \( \beta_i \cap C \), or
3. \( l = l_{ij} \) and \( \zeta_{ij} \subset q \) and \( f(q) \) is a tri- or quadri-secant conic of \( C \) such that \( p_{ij} \in f(q) \).

Thus we have the second assertion. \( \square \)

**Notation 4.2.14.** Let \( D_1 \subset H_2 \times H_1 \) be the divisorial part of \( \hat{D}_1 \). Since \( H_1 \) is a smooth curve \( D_1 \to H_1 \) is flat. Let \( D_1 \) be the fiber of \( D_1 \to H_1 \) over \([l] \in H_1 \). Clearly we can write \( D_1 \hookrightarrow H_2 \).

**4.2.5. Description of \( H_2 \).**

Now we reach the precise description of \( H_2 \).

**Theorem 4.2.15.** (1) The morphism \( \eta: H_2 \to \mathbb{P}^2 \) is the blow-up at \( c_1, \ldots, c_s \) and \( e_i \) are \( \eta \)-exceptional curves. It holds:
\[ D_1 \sim (d - 3)h - \sum_{i=1}^{s} e_i, \]
where $h$ is the strict transform of a general line on $\mathbb{P}^2$.

\begin{equation}
(2) \quad h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}((d - 4)h - \sum_{i=1}^{s} e_i)) = 0.
\end{equation}

(3) $|D_l|$ is base point free. In case of $d = 5$, the image of $\Phi|_{D_l}$ is $\mathbb{P}^2$. In case of $d \geq 6$, $D_l$ is very ample and $|D_l|$ embeds $\mathcal{H}_2$ into $\mathbb{P}^{d-3}$.

Here we use the dual notation $\mathbb{P}^{d-3}$ for later convenience.

(4) If $d \geq 6$, then $\mathcal{H}_2 \subset \mathbb{P}^{d-3}$ is projectively Cohen-Macaulay, equivalently,

\begin{equation}
\begin{aligned}
(3) \quad h^i(\mathbb{P}^{d-3}, \mathcal{I}_{\mathcal{H}_2}(j)) = 0 & \quad \text{for } i = 1, 2 \text{ and } j \in \mathbb{Z}, \\
\end{aligned}
\end{equation}

where $\mathcal{I}_{\mathcal{H}_2}$ is the ideal sheaf of $\mathcal{H}_2$ in $\mathbb{P}^{d-3}$. Moreover, $\mathcal{H}_2$ is the intersection of cubics.

Remark. If $d \geq 6$, then $\mathcal{H}_2 \subset \mathbb{P}^{d-3}$ is so called the White surface (see [Whi21] and [Gim89]). In [Man01], the White surface attains the maximal degree among projectively Cohen-Macaulay rational surfaces in a fixed projective space.

Proof. (1) First we compute the intersection number $D_l \cdot L_b$ for general $l$ and $b$ (this intersection number will be well-defined since the intersection points of $D_l$ and $L_b$ are contained in the smooth locus of $\mathcal{H}_2$). We prove that $D_l$ and $L_b$ intersect simply. Indeed, let $\pi_C : C \times C \to S^2C$ be the natural projection and $L_b$ a ruling of $C \times C \to C$ in one fixed direction such that $\pi_C(L_b) = \eta(L_b)$. By applying the Bertini theorem to $|L_b|$, we see that $\pi_C^*\eta(D_l)$ and $L_b$ intersect simply for a general $b \in C$ whence $\eta(D_l)$ intersects $\eta(L_b)$ simply since $\pi_C$ is etale at $\pi_C^*\eta(D_l) \cap L_b$. Then $D_l$ intersects $L_b$ simply since $\eta$ is isomorphic at $D_l \cap L_b$. Thus we have only to count the number of points in $D_l \cap L_b$, which is $d - 3$ by Corollary 3.2.8 (3). Now we see $D_l \cdot L_b = d - 3$ whence $\eta(D_l)$ is a curve of degree $d - 3$.

Second, we compute the intersection number $D_{l_1} \cdot D_{l_2}$ for two general lines $l_1$ and $l_2$ on $A$. The images $\overline{l}_1 := f(l_1)$ and $\overline{l}_2 := f(l_2)$ be two general secant lines of $C$ such that $\overline{l}_1 \cap \overline{l}_2 = \emptyset$. By Lemma 3.1.2, $\#(D_{l_1} \cap D_{l_2}) = \frac{(d - 3)(d - 4)}{2}$. This immediately gives for the intersection product $D_{l_1} \cdot D_{l_2} \geq \frac{(d - 3)(d - 4)}{2}$. Unfortunately, we cannot show the intersection is simple apriori so we need some argument. On the other hand, $D_l \cap e_i \neq \emptyset$ for a general $l$ since $D_l \cap e_i$ contains the point corresponding to a marked conic $(\beta_i \cup \alpha, \beta_{\iota(C)})$, where $\alpha$ is the unique line intersecting $\beta_i$ and $l$. Moreover, for two general $l_1$ and $l_2$, $D_{l_1} \cap D_{l_2} \cap e_i = \emptyset$, and $D_{l_1} \cap e_i$ and $D_{l_2} \cap e_i$ are contained in the smooth locus of $\mathcal{H}_2$. Thus, by taking the minimal resolution of $\mathcal{H}_2$ near $e_i$ if necessary, we can see that $D_{l_1} \cdot D_{l_2} \leq (d - 3)^2 - s = \frac{(d - 3)(d - 4)}{2}$. Therefore $D_{l_1} \cdot D_{l_2} = \frac{(d - 3)(d - 4)}{2}$. Moreover $e_i^2 = -1$ and since $e_i \cap e_j = \emptyset$ we obtain that $\eta : \mathcal{H}_2 \to \mathbb{P}^2$ is the blow-up at $e_1, \ldots, e_s$. Consequently, $D_l \sim (d - 3)h - \sum e_i$ for a general $[l] \in \mathcal{H}_1$, and, by the flatness of $\mathcal{D}_1 \to \mathcal{H}_1$, that holds for any $[l] \in \mathcal{H}_1$.

(2) Let $L'_{p_{i,j}} = L_{p_{i,j}} - e_i$ (note that $e_i \subset L_{p_{i,j}}$). We see that $L'_{p_{i,j}} \subset D_{t_{i,j}}$ and $D_{t_{i,i}} - L'_{p_{i,1}} = D_{t_{i,2}} - L'_{p_{i,2}}$, which we denote by $D_{t_{i}i}$. Note that

\begin{equation}
D_{t_{i}i} \sim (d - 4)h - \sum_{k \neq i} e_k.
\end{equation}
It is easy to see that $D_{\beta_i}$ have the following properties:

\begin{align}
(4.2) && D_{\beta_i} \cap e_i = \emptyset. \\
(4.3) && D_{\beta_i} \cap D_{\beta_j} \cap D_{\beta_k} = \emptyset.
\end{align}

We only prove (4.2). Since $D_{\beta_i} \cap e_i \neq \emptyset$ would imply that $e_i$ is a component of $D_{\beta_i}$, it suffices to prove that, for a general $l$, $D_{\beta_i} \cap D_l$ does not contain a point of $e_i$. By Lemma 3.1.3, $D_{\beta_i} \cap D_l$ contains $\frac{(d-4) (d-5)}{2}$ points corresponding to bi-secant conics intersecting $\beta_i$ and $l$ except conics containing $\beta_i$. On the other hand, we have $D_l \cdot D_{\beta_i} = \frac{(d-4) (d-5)}{2}$, thus the conics we count in Lemma 3.1.3 correspond to all the intersection of $D_{\beta_i} \cap D_l$. Consequently, $D_{\beta_i} \cap D_l$ does not contain a point of $e_i$.

By (4.2) and the trivial equality

$$
(d-4) h - \sum_{i \geq k+1} e_i = D_{\beta_k} + e_1 + \cdots + e_{k-1},
$$

we obtain $e_k \not\in \text{Bs}|(d-4) h - \sum_{i \geq k+1} e_i|$. By Lemma 3.1.3, $D_{\beta_i} \cap D_l$ contains $\frac{(d-4) (d-5)}{2}$ points corresponding to bi-secant conics intersecting $\beta_i$ and $l$ except conics containing $\beta_i$. On the other hand, we have $D_l \cdot D_{\beta_i} = \frac{(d-4) (d-5)}{2}$, thus the conics we count in Lemma 3.1.3 correspond to all the intersection of $D_{\beta_i} \cap D_l$. Consequently, $D_{\beta_i} \cap D_l$ does not contain a point of $e_i$.

$$
H^0(\mathcal{O}_2, (d-4) h - \sum_{i \geq k+1} e_i) \to H^0(\mathcal{O}_2, e_k)
$$

is surjective. Hence by the exact sequence

$$
0 \to \mathcal{O}_2((d-4) h - \sum_{i \geq k} e_i) \to \mathcal{O}_2((d-4) h - \sum_{i \geq k+1} e_i) \to \mathcal{O}_e \to 0,
$$

we have $H^1(\mathcal{O}_2, ((d-4) h - \sum_{i=1} e_i)) \simeq H^1(\mathcal{O}_2, \mathcal{O}_2((d-4) h))$. Since it is easy to see that $h^1(\mathcal{O}_2, \mathcal{O}_2((d-4) h)) = 0$, we have (2).

(3) Since no conic on $A$ intersects all the line on $A$, $|D_l|$ has no base point. In case $d = 5$, the image of $\Phi|D_l|$ is $\mathbb{P}^2$ by $(D_l)^2 = 1$.

Assuming $d \geq 6$, we prove that $D_l$ is very ample. By (2) and [DGSS, Theorem 3.1], it suffices to prove that

$$
h^0(\mathcal{O}_2, (h - \sum_{j=1}^{d-3} e_{i_j})) = 0
$$

for any set of $d-3$ exceptional curves $e_{i_1}, \ldots, e_{i_{d-3}}$. Assume by contradiction that there exists an effective divisor $L \in |h - \sum_{j=1}^{d-3} e_{i_j}|$ for a set of $d-3$ exceptional curves $e_{i_1}, \ldots, e_{i_{d-3}}$.

By $\frac{(d-2)(d-3)}{2} - (d-3) \geq 3$, we find at least three $e_i$ such that $i \not\in \{j_1, \ldots, j_{d-3}\}$. For an $i \not\in \{j_1, \ldots, j_{d-3}\}$, noting $D_l \sim D_{\beta_i} + h - e_i$, $D_l \cdot L = 0$, and $L \cdot (h - e_i) > 0$, we have $L \subset D_{\beta_i}$. This contradicts (4.3) since the number of $i$ such that $i \not\in \{j_1, \ldots, j_{d-3}\}$ is at least 3.

We show that $h^0(\mathcal{O}_2(D_l)) = d - 2$. By the Riemann-Roch theorem, $\chi(\mathcal{O}_2(D_l)) = d - 2$. Since $h^2(\mathcal{O}_2, \mathcal{O}_2(D_l)) = h^0(\mathcal{O}_2, \mathcal{O}_2(-D_l + K_{\mathbb{P}^2})) = 0$, we see that $h^0(\mathcal{O}_2, \mathcal{O}_2(D_l)) = d - 2$ is equivalent to $h^1(\mathcal{O}_2, \mathcal{O}_2(D_l)) = 0$. Since $|D_l|$ has no base point, so is $|(d-3) h - \sum_{i \geq k+1} e_i|$. Thus the proof that $h^1(\mathcal{O}_2, \mathcal{O}_2(D_l)) = 0$ is almost the same as the above one showing (2) and we omit it.

(4) follows from [Gim89, Proposition 1.1].
Corollary 4.2.16. \(H^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(i)) \simeq H^0(\mathbb{P}^{d-3}, \mathcal{O}_{\mathbb{P}^{d-3}}(i))\) for \(i = 1, 2\).

Proof. The assertion follows from Theorem 4.2.15 (4).

Corollary 4.2.17. For a general \([l] \in \mathcal{H}_1\), \(D_l\) does not contain any point corresponding to the line pairs \(l \cup m\) with \([m] \in \mathcal{H}_1\), and hence \(D_l\) parameterizes all conics which properly intersect \(l\).

Proof. Fix \([m] \in \mathcal{H}_1\) such that \(l \cup m\) is a line pair. If \((\overline{m}, b)\) is the marked line given by \(m\), then we have \(d - 2\) line pairs \(l \cup m, l_1 \cup m, \ldots, l_{d-3} \cup m\). Since \(L_0 \sim h\) then \(h \cdot D_l = d - 3\) and definitely \([l_1 \cup m], \ldots, [l_{d-3} \cup m] \in D_l\). Thus \([l \cup m] \notin D_l\).

5. VARIETIES OF POWER SUMS FOR SPECIAL QUARTICS \(F_4\)

In Proposition 4.2.12 we have seen that \(\psi: \mathcal{U}_2 \to A\) is finite and flat outside \(\cup_{i=1}^s \beta_i\). We can modify the morphism \(\psi: \mathcal{U}_2 \to A\) to obtain a finite one. See Proposition 5.1.3 which is the goal of the subsection 5.1.

5.1. Finiteness of \(\tilde{\psi}: \tilde{\mathcal{U}}_2 \to \tilde{A}\)

Let \(\rho: \tilde{A} \to A\) be the blow-up along \(\cup_{i=1}^s \beta_i\). We add the following piece of notation:

Notation 5.1.1. (1) \(E_i := \rho^{-1}(\beta_i)\). By Lemma 3.1.4 \(E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1\), (2) \(\tilde{E}_C := \) the strict transform of \(E_C\), and (3) \(\tilde{\eta}_{ij}\) := the strict transform of the fiber \(\eta_{ij}\) of \(E_C\) over \(p_{ij} \in C \cap \beta_i\), where \(i = 1, \ldots, s\) and \(j = 1, 2\).

The domain of the finite morphism is \(\tilde{\mathcal{U}}_2 := \mathcal{U}_2 \times_A \tilde{A}\); in other words, \(\tilde{\mathcal{U}}_2\) is the blow-up of \(\mathcal{U}_2\) along \(\Gamma := \mathcal{U}_2 \cap (\cup_{i=1}^s \beta_i \times \mathcal{H}_2)\). We obtain that the natural morphism \(\tilde{\mathcal{U}}_2 \to \tilde{A}\) is finite after a local analysis of the morphism \(\mathcal{U}_2 \to A\) in the neighborhood of \(\Gamma\).

It is easy to describe \(\Gamma\) set-theoretically. Note that, by Proposition 2.4.4 (5), there are \(d - 4\) lines \(\alpha_{i1}, \ldots, \alpha_{id-4}\) distinct from \(\beta_i\) and intersecting both \(C\) and \(\beta_i\) outside \(C \cap \beta_i\). Set \(t_{ik} := \alpha_{ik} \cap C\). Corresponding to \(\alpha_{ik}\), there are two marked conics \((\alpha_{ik} \cup \beta_i; p_{ik}, t_{ik})\) and \((\alpha_{ik} \cup \beta_i; p_{ik}, t_{ik})\). We denote by \(\xi_{ijk}\) the conics on \(A\) corresponding to \((\alpha_{ik} \cup \beta_i; p_{ij}, t_{ik})\), where \(i = 1, \ldots, s, j = 1, 2, k = 1, \ldots, d - 4\). Let \(D_{\beta_i}\) be as in the proof of Theorem 4.2.15. Now we can state that \(\Gamma\) is set-theoretically the union of \(\beta_i \times e_i\),

\[\Gamma_i := \{(x, [q]) \mid [q] \in D_{\beta_i}, x = q \cap \beta_i\},\]

which is a section of \(\mu\) over \(D_{\beta_i}\), and

\[\Gamma_{ijk} := \beta_i \times [\xi_{ijk}] (i = 1, \ldots, s, j = 1, 2, k = 1, \ldots, d - 4).\]

The conic \(\xi_{ijk}\) does not belong to \(e_i\) by the choice of marking. Moreover, we have the following:

Lemma 5.1.2. The conic \(\xi_{ijk}\) does not belong to \(D_{\beta_i}\).
Proof. We consider the projection of $B$ from a bi-secant line $\beta_i$ (see Proposition 3.1.1 (1)). Let $C' \subset Q$ be the image of $C$ by this projection and $p'_{ij}$ the point of $C'$ corresponding to $p_{ij}$, where $p_{ij}$ is one of the two points of $C \cap \beta_i$. By this projection, the line $\alpha_{ik}$ maps to a point, which we denote by $s_{ik}$. Let $F$ be the exceptional divisor of the blowing up along $\beta_i$, and $F'$ the image of $F$ on $Q$. We say a ruling of $F' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is horizontal if it does not come from a fiber of $F \to \beta_i$. Note that the image $q' \subset Q$ of a general conic $q$ belonging to $D_{\beta_i}$ is a bi-secant line of $C'$. Thus, if $[\xi_{ijk}] \in D_{\beta_i}$, then $\xi_{ijk}$ would also correspond to a bi-secant line of $C'$, which must be the horizontal ruling of $F'$ through $p'_{ij}$ and $s_{ik}$. By inductive construction of $C$, however, we can prove that $p'_{ij}$ and $s_{ik}$ do not lie on a horizontal ruling (cf. the proof of Lemma 3.1.4). Thus we have the claim. \qed

We can conclude that all of $\beta_i' \times e_i$, $\Gamma_i$ and $\Gamma_{ijk}$ are disjoint ($i = 1, \ldots, s$, $j = 1, 2$, $k = 1, \ldots, d - 4$).

The next proposition contains the final finiteness result we need.

Proposition 5.1.3. $\widetilde{U}_2$ is Cohen-Macaulay and the natural morphism $\tilde{\psi}: \widetilde{U}_2 \to \tilde{A}$ is finite (of degree $n := \frac{(d-1)(d-2)}{2}$). In particular, $\tilde{\psi}$ is flat.

Lemma 5.1.4. $\Gamma$ is a reduced scheme and $U_2$ is smooth along $\Gamma$.

First we finish the proof of Proposition 5.1.3 by admitting this lemma:

Proof of Proposition 5.1.3. By Lemma 5.1.4, the morphism $\bar{U}_2 \to U_2$ is the blow-up along the reduced subscheme $\Gamma$ contained in the smooth locus of $U_2$. The subscheme $\beta_i' \times e_i$ is a Cartier divisor of $U_2$, thus $\bar{U}_2 \to U_2$ is isomorphic over $\beta_i' \times e_i$. The curve $\Gamma_{ijk}$ is smooth and the curve $\Gamma_i$ has only planar singularities since so is $D_{\beta_i}$. Thus $\bar{U}_2$ is Cohen-Macaulay since so is $U_2$.

We have only to prove that $\tilde{\psi}$ is finite. By Proposition 4.1.12 $\tilde{\psi}$ is finite outside $\cup E_i$. Note that $\tilde{\psi}^{-1}(E_i)$ is nothing but the inverse images of $\beta_i' \times e_i$, $\Gamma_i$ and $\Gamma_{ijk}$ by $\bar{U}_2 \to U_2$, all of which are $\mathbb{P}^1$-bundles over curves and are mapped to $E_i$ finitely. Hence we are done. \qed

Proof of Lemma 5.1.4. We study $U_2$ locally along $\Gamma$.

Let $q$ be a conic on $A$ belonging to $D_{\beta_i}$. Then, by Proposition 4.1.1 (4), Lemma 5.1.3 and the fact that $D_{\beta_i} \cap e_i = \emptyset$ (see the proof of Theorem 4.2.15 (4.2)), we see that $q$ is smooth near $\beta_i'$ and intersects $\beta_i'$ transversely. This implies that $U_2$ is smooth along $\Gamma_i$. Note that, near $\Gamma_i$, the morphism $\psi: U_2 \to A$ is finite, hence flat. Since $\Gamma$ is the pull-back of $\beta_i'$ near $\Gamma_i$ and $\Gamma_i$ is not contained in the ramification locus of $\psi$, it holds that $\Gamma$ is reduced along $\Gamma_i$.

Let $q$ be the fiber of $U_2 \to H_2$ given by $[\xi_{ijk}]$ or a point of $e_i$. Note that $q$ is a conic on $A$ and has only nodes as its singularities. We show that $h^1(N_{q/A}) = 0$ and the natural map $H^0(N_{q/A}) \to H^0(T^1_p) \simeq C$ is surjective, where $p$ is any node of $q$ and $T^1_p$ is the local deformation space of $p$. As in the proof of [HH85] Proposition 1.1], this implies that $H_2$ coindices with the Hilbert scheme of conics on $A$ at $[\xi_{ijk}]$ or a point of $e_i$, and $U_2$ is smooth near $q$.

First we treat the case where $q = \xi_{ijk} = \alpha'_{ik} \cup \beta'_i \cup \zeta_{i,3-j}$. Note that $\mathcal{N}_{\alpha'_{ik}/A} \simeq \mathcal{O}_{P_1} \oplus \mathcal{O}_{P_2}(-1)$, $\mathcal{N}_{\beta'_i/A} \simeq \mathcal{O}_{P_1}(-1)^{\oplus 2}$, and $\mathcal{N}_{\zeta_{i,3-j}/A} \simeq \mathcal{O}_{P_1} \oplus \mathcal{O}_{P_2}(-1)$. We apply [HH85] Theorem 4.1] by setting $X = \xi_{ijk}$, $C = \beta'_i$ and $D = \alpha'_{ik} \cup \zeta_{i,3-j}$. We check the conditions a) and b) of [ibid.]. The condition a) clearly holds. The condition b) follows from the following two facts:

1. Let $F$ be the exceptional divisor of the blow up of $B$ along $\alpha_{ik}$. Note that $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

We call a fiber of $F \to \mathbb{P}^1$ in the other direction to $F \to \alpha_{ik}$ a horizontal fiber. Then the
intersection points of the strict transform of $C$ and $F$, and the strict transform of $\beta_i$ and $F$ do not lie on a common horizontal fiber.

This can be proved by the inductive construction of $C = C_d$ in a similar fashion to the proof of Lemma [3.1.4] or by a straightforward dimensional computation as the one of Proposition [2.4.4] (2), and

(2) let $G$ be the exceptional divisor of the blow up of $A$ along $\zeta_{i,3-j}$. Note that $G \cong \mathbb{P}_1$. Then the intersection points of the strict transform of $\beta'_i$ and $G$ does not lie on the negative section of $G$.

Indeed, since $E_C \cdot \zeta_{i,3-j} = -1$, the intersection of $G$ and the strict transform of $E_C$ is the negative section of $G$. On the other hand, the strict transforms of $E_C$ and $\beta'_i$ are disjoint. Thus, by [HH85, Theorem 4.1], $\xi_{ijk}$ satisfies the desired properties.

Secondly, we treat the case $q$ is a fiber over a point of $e_i$. Note that $\bar{q} = \beta_i \cup \alpha$, where $\alpha$ is a line intersecting $\beta_i$. Denote by $\alpha'$ the strict transform of $\alpha$. We make the following case division:

(a) $\alpha \cap C = \emptyset$ and $N_{\alpha/B} = \mathcal{O}_{\mathbb{P}_1}^{\oplus 2}$.
(b) $\alpha \cap C = \emptyset$ and $N_{\alpha/B} = \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(1)$.
(c) $\alpha = \alpha_{ik}$ for some $k$.
(d) $\alpha$ passes through a point of $\beta_i \cap C$.

In the case (a) or (b), it is easy to see the proof of [HH85, Theorem 4.1] works as above by setting $X = q, C = \beta'_i$ and $D = \alpha'$. In the case (c) or (d), we need to modify the proof of [ibid.]. We only treat the case (c) since we can treat the case (d) similarly. Note that $q = \beta'_i \cup \alpha_{ik}' \cup \gamma_{ik}$, where $\gamma_{ik}$ is the fiber of $E_C$ over $t_{ik}$. Note that $C$ is smooth. By [HH85, Corollary 3.2] and simple dimension count, we can describe the restrictions of the normal bundle $N_{q/A}$ to the components of $q$ as follows:

$$N_{q/A|\beta'_i} = \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1), N_{q/A|\alpha_{ik}'} = \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1), \text{ and } N_{q/A|\gamma_{ik}} = \mathcal{O}_{\mathbb{P}_1}^{\oplus 2}.$$ 

Set $C = \beta'_i \cup \gamma_{ik}$ and $D = \alpha_{ik}'$. As in [HH85, Theorem 4.1], set $S := C \cap D$. By the description of $N_{q/A|\beta'_i}, \ N_{q/A|\alpha_{ik}'}$, and $N_{q/A|\gamma_{ik}}$, it holds that $H^1(N_{q/A|C}) = H^1(N_{q/A|D}) = \{0\}$. Moreover, considering the tautological linear systems of $\mathbb{P}(N_{q/A|\beta'_i}), \mathbb{P}(N_{q/A|\alpha_{ik}'}), \mathbb{P}(N_{q/A|\gamma_{ik}})$, and $\mathbb{P}(N_{q/A})$, we see that $H^0(N_{q/A|C}) \oplus H^0(N_{q/A|D}) \to H^0(N_{q/A|S})$ is surjective. Thus $h^1(N_{q/A}) = 0$ holds. By [HH85, Corollary 3.2] again, we have the following exact sequences (cf. [HH85, (3) in the proof of Theorem 4.1]):

$$0 \to \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-2) \to N_{q/A|\beta'_i} \to N_{q/A|S} \to 0,$$

$$0 \to \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-2) \to N_{q/A|\alpha_{ik}'} \to N_{q/A|S} \to 0,$$

$$0 \to \mathcal{O}_{\mathbb{P}_1}^{\oplus 2}(-1) \to N_{q/A|\gamma_{ik}} \to N_{q/A|S} \to 0.$$ 

Thus we can consider that $H^0(N_{q/A|C})$ and $H^0(N_{q/A|D})$ is a subspace of $H^0(N_{q/A|S})$. By [HH85, (2) in the proof of Theorem 4.1], we see that $H^0(N_{q/A|C}) \to H^0(T_p)$ and $H^0(N_{q/A|D}) \to H^0(T_p)$ are surjective. Moreover, considering the tautological linear systems of $\mathbb{P}(N_{q/A|\beta'_i}), \mathbb{P}(N_{q/A|\alpha_{ik}'}), \mathbb{P}(N_{q/A|\gamma_{ik}})$, and $\mathbb{P}(N_{q/A})$, we see that the kernels of $H^0(N_{q/A|C}) \to H^0(T_p)$ and $H^0(N_{q/A|D}) \to H^0(T_p)$ does not coincide for any $p \in S$. Thus any non-zero element of $H^0(T_p) \cong \mathbb{C}$ comes from that of $H^0(N_{q/A|C}) \cap H^0(N_{q/A|D})$ as in the end of the proof of [HH85, Theorem 4.1]. This implies that the natural map $H^0(N_{q/A}) \to H^0(T_p)$ is surjective for any $p \in S$. 

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Note that, near $e_i$, the family $\mathcal{U}_2 \to \mathcal{H}_2$ is locally a deformation of a node with smooth discriminant locus $e_i$. Thus a local computation shows that $\Gamma$ is reduced along $\beta'_i \times e_i$.

Now we prove that $\Gamma$ is reduced along $\Gamma_{ijk}$. We have only to prove that $\mathcal{U}_2 \to A$ is unramified along $\Gamma_{ijk}$ since then $\Gamma$ is the étale pull-back of $\beta'_i$ near $\Gamma_{ijk}$, hence is reduced.

Recall that we set $S = (\alpha'_k \cap \beta'_l) \cup (\zeta_{i,j} \cap \beta'_l)$. By simple dimension count and the following exact sequence:

$$0 \to N_{\beta'_i/A} \to N_{\xi_{ijk}/A}|_{\beta'_i} \to T_S^1 \to 0,$$

we can prove that $N_{\xi_{ijk}/A}|_{\beta'_i} \simeq \mathcal{O}_{\mathbb{P}^1}$. Thus $H^0(N_{\xi_{ijk}/A}) \otimes \mathcal{O}_{\xi_{ijk}} \to N_{\xi_{ijk}/A}$ is surjective at a point of $\Gamma_{ijk}$ since it factor through the surjection $H^0(N_{\xi_{ijk}/A}|_{\beta'_i}) \otimes \mathcal{O}_{\beta'_i} \to N_{\xi_{ijk}/A}|_{\beta'_i}$. Thus $\mathcal{U}_2 \to A$ is unramified along $\Gamma_{ijk}$.

From now on we assume that $d \geq 6$ and we consider $\mathcal{H}_2 \subset \bar{\mathbb{P}}^{d-3}$.

Consider the following diagram:

(5.1)

\[
\begin{array}{ccc}
\mathcal{H}_2 & \xrightarrow{\tilde{\mu}} & \mathcal{U}_2 \\
\downarrow & & \downarrow \tilde{\psi} \\
\tilde{\mathcal{A}} & & \\
\end{array}
\]

**Definition 5.1.5.** Let $\tilde{a}$ be a point of $\tilde{A}$. We say that $[\tilde{\psi}^{-1}(\tilde{a})] \in \text{Hilb}^n[\bar{\mathbb{P}}^{d-3}]$ is the cluster of conics attached to $\tilde{a}$ and denote it by $[Z_{\tilde{a}}]$. A conic $q$ such that $[q] \in \text{Supp} Z_{\tilde{a}}$ is called a conic attached to $\tilde{a}$.

**Remark.** Though we do not need it later, we describe the fiber of $\tilde{\psi}$ over a general point $\tilde{a} \in E_i$ for some $i$ for reader’s convenience. In other words, we exhibit $n$ conics attached to $\tilde{a}$.

Set $a := \rho(\tilde{a}) \in A$ and $b := f(a) \in \beta_i$. We use notations of Proposition 4.2.12 Since $\deg C_b = d - 2$, the number of bi-secant conics through $b$ not belonging to the family $e_i$ is given by the number of double points of $C_b$, which is $\frac{(d-3)(d-4)}{2}$. Moreover $2(d-4)$ conics $\xi_{ijk}$ through $a$.

The number of remaining conics is $3 = n - \frac{(d-3)(d-4)}{2} - 2(d-4)$. Such conics will belong to $e_i$. We look for three such conics. Let $S_i$ be the strict transform on $\tilde{A}$ of the locus of lines intersecting $\beta_i$. Then it is easy to see that $S_i|_{E_i}$ does not contain any fiber $\gamma_i$ of the second projection $\sigma_i: E_i \to \bar{\mathbb{P}}^1$. Moreover $S_i|_{E_i} \sim 2\gamma_i + 3f_i$, where $f_i$ is a fiber of $E_i \to \beta'_i$. Let $\gamma'_i$ be the fiber of $\sigma_i$ through $\tilde{a}$. Then $\gamma'_i$ intersect $S_i$ at three points. Corresponding to these three points, there are three lines on $B$ intersecting $\beta_i$. Denote by $l_1, l_2$ and $l_3 \subset A$ the strict transforms of these three lines. Then $\beta'_i \cup l_j$ ($j = 1, 2, 3$) are the conics on $A$ what we want.

By Proposition 5.1.3 and the universal property of Hilbert schemes, we obtain a naturally defined map $\Psi: \tilde{A} \to \text{Hilb}^n[\bar{\mathbb{P}}^{d-3}]$. This is clearly injective because $n$ conics attached to a point $\tilde{a} \in \tilde{A}$ uniquely determines $\tilde{a}$.

To understand the image of $\Psi$, we construct the special quartic hypersurface which live in the dual projective space to the ambient of $\mathcal{H}_2$.

### 5.2. Intersection of conics and conics on $A$.

To construct the special quartic hypersurface, we need the incidence variety defined by the intersections of conics.
Similarly to (4.1), we consider the following diagram:

\[ \begin{array}{ccc}
\mathcal{U}'_2 \subset \mathcal{U} \times \mathcal{H}_2 & \xrightarrow{(\psi, \text{id})} & A \times \mathcal{H}_2 \supset \mathcal{U}_2 \\
\downarrow & & \downarrow \\
\hat{\mathcal{D}}_2 \subset \mathcal{H} \times \mathcal{H}_2 & \xrightarrow{} & \mathcal{H}_2,
\end{array} \]

where \( \mathcal{U}'_2 \subset \mathcal{U} \times \mathcal{H}_2 \) is the base change of \( \mathcal{U}_2 \) and \( \hat{\mathcal{D}}_2 \) the image of \( \mathcal{U}'_2 \) on \( \mathcal{H} \times \mathcal{H}_2 \). Similarly to the investigation of the diagram (4.1), we see that the image \( F' \) in \( \mathcal{H}_2 \times \mathcal{H}_2 \) of the inverse image of \( \cap_{i=1}^n \mathcal{D}'_i \times \mathcal{H}_2 \) is not divisorial nor does not dominate \( \mathcal{H}_2 \). Moreover, any component of \( \hat{\mathcal{D}}_2 \) outside \( F' \) dominates \( \mathcal{H}_2 \), and is divisorial or possibly the diagonal of \( \mathcal{H}_2 \times \mathcal{H}_2 \). Note that dislike the diagram (4.1), there is no other non-divisorial component in this case. Compare the proof of Corollary 4.2.13. Here we leave the possibility that the diagonal of \( \mathcal{H}_2 \times \mathcal{H}_2 \) is contained in the divisorial component of \( \hat{\mathcal{D}}_2 \). We, however, prove this is not the case in Lemma 5.3.2.

Let \( D_2 \subset \mathcal{H}_2 \times \mathcal{H}_2 \) be the union of the divisorial components of \( \hat{\mathcal{D}}_2 \) with reduced structure. \( D_2 \) is Cartier since \( \mathcal{H}_2 \times \mathcal{H}_2 \) is smooth. \( D_2 \to \mathcal{H}_2 \) is flat since \( D_2 \) is Cohen-Macaulay, \( \mathcal{H}_2 \) is smooth and \( D_2 \to \mathcal{H}_2 \) is equi-dimensional. Let \( D_q \) be the fiber of \( D_2 \to \mathcal{H}_2 \) over \([q] \in \mathcal{H}_2 \) via the projection to the second factor.

5.3. Construction of the special quartics.

**Lemma 5.3.1.** \( D_q \sim 2(d-3)h - 2 \sum_{i=1}^e e_i \) for a conic \( q \). \( D_q \) is a quadric section of \( \mathcal{H}_2 \subset \mathbb{P}^{d-3} \).

**Proof.** The proof of the former half is almost identical to the one of Theorem 4.2.13 (1). The latter half follows from Corollary 4.2.16. \( \square \)

Now we proceed to construct the quartic hypersurface.

From now on, we write \( \mathbb{P}^{d-3} = \mathbb{P}_s V \), where \( V \) is the \( d - 2 \)-dimensional vector space. The crucial point in the following considerations is the equality:

\[ n = \dim S^2 V. \]

By the seesaw theorem, it holds that \( D_2 \sim p_1^* D_q + p_2^* D_q \). Consider the morphism \( \mathcal{H}_2 \times \mathcal{H}_2 \) into \( \mathbb{P}^{d-2} \times \mathbb{P}^{d-3} \) defined by \( |p_1^* D_1 + p_2^* D_1| \), which is an embedding since \( d \geq 6 \). By Corollary 4.2.16 it holds

\[ H^0(\mathcal{H}_2 \times \mathcal{H}_2, D_2) \simeq H^0(\mathbb{P}^{d-3} \times \mathbb{P}^{d-3}, O(2, 2)). \]

Therefore \( D_2 \) is the restriction of a unique \((2, 2)\)-divisor on \( \mathbb{P}^{d-3} \times \mathbb{P}^{d-3} \), which we denote by \( \{ \tilde{D}_2 = 0 \} \). Since \( \{ \tilde{D}_2 = 0 \} \) is symmetric, we may assume the equation \( \tilde{D}_2 \) is also symmetric. Actually, the desired quartic is obtained by restricting \( \tilde{D}_2 \) to the diagonal and taking the dual in the sense of Dolgachev (see the appendix), but we need more argument for the proof of the main theorem.

For \([q] \in \mathcal{H}_2 \), we denote by \( \tilde{D}_q \) the restriction of \( \tilde{D}_2 \) to the fiber over \([q] \). Note that \( \tilde{D}_2 \in S^2 V \otimes S^2 V \), so \( \tilde{D}_2 \) defines a linear map \( \lambda : S^2 V \otimes S^2 V \to S^2 V \). Let \( H_q \) be a linear form on \( V \) corresponding to \( q \). It holds that \( \lambda(H_q^2) = \tilde{D}_q \) up to scalar, so we may choose \( H_q \) such that \( \lambda(H_q^2) = \tilde{D}_q \) holds. We prove that \( \lambda \) is an isomorphism.
Lemma 5.3.2. \( D_2 \) does not contain the diagonal of \( \mathcal{H}_2 \times \mathcal{H}_2 \). In particular we have the following:

Let \( \tilde{a} \) be a general point of \( \tilde{A} \) and \( q_1, q_2, \ldots, q_n \in \mathcal{H}_2 \) the conics attached to \( \tilde{a} \). Then

\[
D_{q_i}([q_i]) \neq 0
\]

for \( 1 \leq i \leq n \).

Proof. Here we assume \( d \geq 3 \). It suffices to prove that \( D_q([q]) \neq 0 \) for a general \([q] \in \mathcal{H}_2 \). This is equivalent to that the image \( D_q^\flat \) on \( \mathcal{H}_2 \) of \( D_q \) does not contain \([\mathcal{H}]\). Note that \( D_q^\flat \) is the closure of the locus of multi-secant conics of \( C \) intersecting properly \([\mathcal{F}]\). Now the assertion follows from the inductive construction of \( C_d \) from \( C_{d-1} \cup \mathcal{J} \). From now on, we denote \( D_q^\flat \) for \( C_d \) by \( D_{q,d}^\flat \). If \( d = 3 \), then \( D_q \sim 0 \), thus the assertion trivially true. If \( D_{q,d-1}^\flat([\mathcal{F}]) \neq 0 \) for a general multi-secant conic \( \mathcal{F} \) of \( C_{d-1} \), then \( D_{q,d}^\flat([q]) \neq 0 \) for a general multi-secant conic \( \mathcal{F} \) of \( C_d \). \( \square \)

Let \( \tilde{a} \) be a general point of \( \tilde{A} \) and \( q_1, \ldots, q_n \) are the conics attached to \( \tilde{a} \). By the definition of \( \tilde{D}_{q_i} \) and generality of \( \tilde{a} \), we have the following:

\[
\text{(5.4)} \quad \tilde{D}_{q_j}([q_i]) = 0 \quad (j \neq i) \quad \text{and} \quad \tilde{D}_{q_i}([q_i]) \neq 0.
\]

(5.4) implies \( \tilde{D}_{q_1}, \ldots, \tilde{D}_{q_n} \) are linearly independent, and, by (5.3), they span the vector space \( S^2\mathcal{V} \). Thus \( \lambda \) is an isomorphism.

The inverse \( \lambda^{-1}: S^2\mathcal{V} \rightarrow S^2\tilde{\mathcal{V}} \) defines an element \( \tilde{D}_2 \in S^2\tilde{\mathcal{V}} \otimes S^2\tilde{\mathcal{V}} \). We consider the polarization map \( \text{pl}_2: S^2\mathcal{V} \rightarrow \text{Sym}_2\mathcal{V} \) (see the appendix). We show that \( \tilde{U} := \text{pl}_2 \otimes \text{pl}_2(\tilde{D}_2) \in \text{Sym}_2\mathcal{V} \otimes \text{Sym}_2\mathcal{V} \subset \mathcal{V}^{\otimes 4} \) is actually contained in \( \text{Sym}_4\mathcal{V} \). This will implies that \( \text{pl}_2 \otimes \text{pl}_2(\tilde{D}_2) \) is the image of a quartic form \( \in S^4\mathcal{V} \) by \( \text{pl}_4 \).

The following argument is almost identical with the proof of [DK93, Theorem 9.3.1] (The identification will be clearer by constructing the theta characteristic on \( \mathcal{H}_2 \) in the forthcoming paper). Let \( \ell \) be a general line on \( A \) and \( l_1, \ldots, l_{d-2} \) the lines intersecting \( \ell \). Note that \( l_1, \ldots, l_{d-2} \) correspond to lines on \( B \) intersecting both \( C \) and the image \( \mathcal{J} \) of \( \ell \) on \( B \) except those through \( C \cap \mathcal{J} \). Thus the number of such lines is \( d - 2 \). Since \( \ell \) is general, so are \( l_1, \ldots, l_{d-2} \). We have \( d - 2 \) reducible conics \( r_1 := \ell \cup l_1, \ldots, r_{d-2} := \ell \cup l_{d-2} \). It holds that \( D_{r_i} = D_\ell + D_{l_i} \). By Corollary 4.2.16 \( \tilde{D}_\ell \) and \( \tilde{D}_{l_i} \) are defined by linear forms \( L \) and \( L_i \). We may assume that \( \lambda(H^2_{r_i}) = D_{r_i} = L_i L \). By Corollary 4.2.17 \( L_i([r_i]) \neq 0 \) and \( L_i([r_j]) = 0 \) for \( i \neq j \). In other words, it holds \( \langle L_i, H_i \rangle \neq 0 \) and \( \langle L_i, H_j \rangle = 0 \) for \( i \neq j \), where \( \langle, \rangle \) is the natural dual pairing. Thus \( L_1, \ldots, L_{d-2} \) and \( H_1, \ldots, H_{d-2} \) span \( \tilde{\mathcal{V}} \) and \( \mathcal{V} \), respectively since dim \( \mathcal{V} = d - 2 \). Moreover, \( \{H_i\} \) and \( \{L_i\} \) are dual to each other. Choose coordinates of \( \mathcal{V} \) and \( \tilde{\mathcal{V}} \) such that \( H_i \) and \( L_i \) are coordinate hyperplanes \( \{x_i = 0\} \) and \( \{u_i = 0\} \) respectively. Set \( L = \sum a_iu_i \). For any \( y = (y_1, \ldots, y_{d-2}) \in \mathcal{V} \), we have \( \lambda(\sum y_ix_i^2) = (\sum a_iu_i)(\sum y_iu_i) \) by \( \lambda(H^2_{r_i}) = L_i L \). Considering \( \tilde{U} \in \mathcal{V}^{\otimes 4} \), this implies that \( \tilde{U}(L, y, x, x) = \sum y_ix_i^2 = P_y(\sum x_i^2) \), where \( x = (x_1, x_2, \ldots, x_{d-2}) \) and \( P_y \) is the polar with respect to \( y \) (see the appendix). Thus we have \( \tilde{U}(L, y, x, z) = \sum y_ix_iz_i \) for \( z = (z_1, z_2, \ldots, z_{d-2}) \), hence \( \tilde{U}(L, y, x, z) \) is symmetric for \( y, x \) and \( z \). Since \( \tilde{U} \in \text{Sym}_2\mathcal{V} \otimes \text{Sym}_2\mathcal{V} \) and \( \tilde{D}_2 \) is symmetric, we have shown that \( \tilde{U} \in \text{Sym}_4\mathcal{V} \).
Let \( F_4 \) be the quartic form associated to \( \tilde{U} \), namely, \( F_4 := \tilde{U}(x, x, x, x) \). By the construction, it holds
\[
P_{D_q}(F_4) = H_q^2.
\]
By the theory of polarity (see the appendix), we can interpret what we have done as follows: \( \lambda^{-1} = \text{ap}_{F_4} \). Since \( \lambda^{-1} \) is an isomorphism, \( F_4 \) is non-degenerate.

5.4. Proof of the main theorem.

**Theorem 5.4.1.** \( \text{Im } \Phi \) is an irreducible component of
\[
\text{VSP}(F_4, n; \mathcal{H}_2) := \{(H_1, \ldots, H_n) \mid [H_i] \in \mathcal{H}_2 \} \subset \text{VSP}(F_4, n).
\]

**Proof.** Set
\[
Z := \{(H_1, \ldots, H_n) \in \text{Hilb}^{n}\mathbb{P}^{d-3} \mid H_1^4 + \ldots + H_n^4 = F_4, [H_i] \in \mathcal{H}_2 \}.
\]

For a general point \( \tilde{a} \) and conics \( q_1, \ldots, q_n \) attached to \( \tilde{a} \), we have (5.4). Conversely, \( n \) conics \( q_i \) satisfying (5.4) and the assumptions (1)–(3) of Lemma 3.2.9 determine a point of \( \tilde{A} \). Note that the assumptions (1)–(3) of Lemma 3.2.9 are open conditions. Thus we have only to prove that (5.4) is equivalent to
\[
\alpha_1 H_{q_1}^4 + \ldots + \alpha_n H_{q_n}^4 = F_4 \quad \text{with some nonzero } \alpha_i \in \mathbb{C}.
\]

We see that (5.6) is equivalent to
\[
(5.7) \quad \text{If } \{G = 0\} \subset \mathbb{P}^{d-3} \text{ is any quartic through } [q_1], \ldots, [q_n], \text{ then } P_{F_4}(G) = 0.
\]

Indeed, by the apolarity pairing, \( \langle G, H_q^4 \rangle = 0 \Leftrightarrow G([q_i]) = 0 \), thus the assumption on \( G \) is equivalent to \( G \in \langle H_{q_1}^4, \ldots, H_{q_n}^4 \rangle^\perp \). Therefore (5.6) is equivalent to \( \langle H_{q_1}^4, \ldots, H_{q_n}^4 \rangle^\perp \subset \langle F_4 \rangle^\perp \). Since \( F_4 \) is non-degenerate, this is equivalent to (5.6).

We show (5.4) implies (5.7). If (5.4) holds, then \( \tilde{D}_{q_i} \) (\( i \neq 1 \)) generate the space of quadric forms passing through \([q_1]\), we may write \( G = Q_2 \tilde{D}_{q_2} + \ldots + Q_n \tilde{D}_{q_n} \), where \( Q_i \) are quadratic forms on \( \mathbb{P}^{d-3} \). By \( G([q_i]) = 0 \) for \( i \neq 1 \), we have \( Q_i([q_i]) \tilde{D}_{q_i}([q_i]) = 0 \). \( \tilde{D}_{q_i}([q_i]) \neq 0 \) implies that \( Q_i([q_i]) = 0 \). Thus \( Q_i \) is a linear combination of \( \tilde{D}_{q_i} \) (\( j \neq i \)). Consequently, \( G \) is a linear combination of \( \tilde{D}_{q_i} \tilde{D}_{q_j} \) (\( 1 \leq i < j \leq n \)). Thus \( P_{F_4}(G) = 0 \) follows from that
\[
P_{F_4}(\tilde{D}_{q_i} \tilde{D}_{q_j}) = P_{H_{q_i}}(\tilde{D}_{q_j}) = \tilde{D}_{q_j}([q_i]) = 0.
\]

Finally we show (5.6) implies (5.4). By (5.6), it holds
\[
H_{q_i}^2 = P_{D_{q_i}}(F_4) = \sum \alpha_j(\tilde{D}_{q_i}, H_{q_j}^4) H_{q_j}^2.
\]

Since \( \tilde{D}_{q_i} \) are linearly independent, so are \( H_{q_j}^2 \). Thus (5.4) holds.

**Definition 5.4.2.** We say \( \text{Im } \Phi \) is the main component of \( \text{VSP}(n, F_4; \mathcal{H}_2) \).

The following proposition characterizes the main component of \( \text{VSP}(n, F_4; \mathcal{H}_2) \):
Proposition 5.4.3. Let $(\mathcal{H}_2^k)^o$ and $(\text{Hilb}^k\mathbb{P}^{d-3})^o$ ($k \in \mathbb{N}$) be the complements of all the small diagonals of $\mathcal{H}_2^k$ ($k$ times product of $\mathcal{H}_2$) and $\text{Hilb}^k\mathbb{P}^{d-3}$ respectively. Set

$$\text{VSP}^o(F_4, n; \mathcal{H}_2) := \{([H_1], \ldots, [H_n]) \mid [H_i] \in \mathcal{H}_2, H_1^m + \cdots + H_n^m = F_1\}.$$  

Let $V^o$ be the inverse image of $\text{VSP}^o(F_4, n; \mathcal{H}_2)$ by the natural projection $(\mathcal{H}_2^o)^o \to (\text{Hilb}^o\mathbb{P}^{d-3})^o$. Let $(\mathcal{H}_2^2)^o \to (\mathcal{H}_2^2)^o$ be the projection to any of two factors. Then a component of $V^o$ dominating $\mathcal{D}_2$ dominates the main component of $\text{VSP}(F_4, n; \mathcal{H}_2)$. In particular, the main component is uniquely identified from $\mathcal{D}_2$.

Proof. Let $([q_1], [q_2]) \in \mathcal{D}_2 \cap (\mathcal{H}_2^2)^o$ be a general point and $\{q_i\}$ ($i = 1, \ldots, n$) any set of mutually conjugate $n$ conics including $q_1$ and $q_2$. Since $q_1$ and $q_2$ are general, we may assume that all the $q_i$ are general. By Lemma 3.2.9 and Theorem 5.4.1, it suffices to prove that $q_1, \ldots, q_n$ satisfies the conditions (1)–(3) of Lemma 3.2.9

(1). Let $\tau_1$ and $\tau_2$ are mutually intersecting smooth conics on $B$ and $\tau_3$ a line pair on $B$ intersecting both $\tau_1$ and $\tau_2$. Since the Hilbert scheme of conics on $B$ is 4-dimensional, the pair of $\tau_1$ and $\tau_2$ depends on 7 parameters. If we fix $\tau_1$ and $\tau_2$, then $\tau_3$ depends on 1 parameter. Thus the configuration $\tau_1, \tau_2, \tau_3$ depends on 8 parameters. Fix $\tau_1, \tau_2$ and $\tau_3$. We count the number of parameters of $C_4$ such that $C_4$ intersects each of $\tau_i$ ($i = 1, 2, 3$) twice. The number of parameters is $h^0((\mathcal{O}_{\mathbb{P}^2}(d-1) \oplus \mathcal{O}_{\mathbb{P}^2}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-6)) + 6 = 2d - 12 + 6 = 2d - 6$, where $+6$ means the sum of the numbers of parameters of two points on $\tau_i$ ($i = 1, 2, 3$). By $2d - 6 + 8 = 2d + 2$, a general $C_d$ has 2-dimensional pairs of mutually intersecting bi-secant conics which intersect at least one bi-secant line pair of $C_d$. Thus general pairs of mutually intersecting bi-secant conics of $C_d$, which form a 3-dimensional family, do not intersect a bi-secant line pair of $C_d$.

(2). Assume by contradiction that $\tau_i, \tau_j$ and $\tau_k$ pass through a point $b$, and $\tau_i$ does not pass through $b$ but intersects a line through $b$. Then by the double projection from $b$, $\tau_i$ is mapped to a line through the three singular points of the image of $C_b$ corresponding to $\tau_i, \tau_j$ and $\tau_k$. Thus we have only to prove that for a general point of $b$ on $B$, three double points of the image of $C_b$ do not lie on a line.

Fix a general point $b \in B$. Let $\tau_1, \tau_2, \tau_3$ be three conics on $B$ through $b$ such that by the double projection from $b$, they are mapped to three colinear points on $\mathbb{P}^2$. The number of parameters of $C_d$’s intersecting each of $\tau_i$ twice is $h^0((\mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-6)) = 2d - 12$ since $h^1((\mathcal{O}_{\mathbb{P}^2}(d-1) \oplus \mathcal{O}_{\mathbb{P}^2}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-6)) = 0$. Note that the number of parameters of $\tau_1, \tau_2, \tau_3$ is 5 since that of lines in $\mathbb{P}^2$ is 2, and that of three points on a line is 3. Thus the number of parameters of $C_d$’s such that its image of the double projection from $b$ has three colinear double points is at most $2d - 1$. Hence a general $C_d$ does not satisfy this property.

(3). Let $r_1$ and $r_2$ be a general pair of mutually conjugate conics on $A$ such that $\tau_1$ and $\tau_2$ are smooth, and $\tau_1$ and $\tau_2$ intersect at a point on $C \cup \cup_i \beta_i$. Such general pairs of conics $r_1$ and $r_2$ form a two-dimensional family since $\dim C \cup \cup_i \beta_i = 1$ and if one point $t$ of $C \cup \cup_i \beta_i$ is fixed, then such pairs of conics such that $t \in \tau_1 \cap \tau_2$ form a one-dimensional family. For a general pair of $r_1$ and $r_2$, the number of the sets of $n$ mutually conjugate conics including $r_1$ and $r_2$ is finite since $D_{r_1}$ and $D_{r_2}$ has no common component. Thus $\{q_i\}$ does not contain such a pair by generality whence $\{q_i\}$ satisfies (3). \qed
6. Relation with Mukai’s result

Here we sketch how the argument goes on if $d = 5$ and explain a relation of our result with Theorem 1.2.1.

Assume that $d = 5$. Associated to the birational morphism $\mathcal{H}_2 \to \mathbb{P}^2$, there exists a nonfinite birational morphism

$$\Phi: \tilde{A} \to A_{22} := \operatorname{VSP}(F_4, 6) \subset \operatorname{Hilb}^6 \mathbb{P}^2,$$

which fits into the following diagram:

\[ \begin{array}{ccc}
\tilde{A} & \xrightarrow{\Phi} & A_{22} \\
\downarrow{\rho} & & \downarrow{f'} \\
A & \xrightarrow{f} & A' \\
\downarrow{\rho'} & & \downarrow{f'} \\
B & & A_{22},
\end{array} \]

where

- $A_{22}$ is a smooth prime Fano threefold of genus twelve,
- $\rho'$ is the blow-down of the three $\rho$-exceptional divisors $E_i$ ($i = 1, 2, 3$) over the strict transforms $\beta'_i$ in the other direction. In other words, $A \dashrightarrow A'$ is the flops of $\beta'_1$, $\beta'_2$ and $\beta'_3$ (cf. Lemma 3.1.4), and
- the morphism $f'$ contracts the strict transform of the unique hyperplane section $S$ containing $C$ (see Corollary 2.5.3) to a general line on $A_{22}$.

The rational map $A_{22} \dashrightarrow B$ is the famous double projection of $A_{22}$ from a general line $m$ first discovered by Iskovskih (see [Isk78]).

We explain how $f'$ and $\rho'$ are interpreted in our context. As we remarked after the proof of Theorem 4.2.15, the morphism $\mathcal{H}_2 \to \mathbb{P}^2$ defined by $|D_i|$ contracts three curves $D_{e_i}$ which parameterize conics intersecting $\beta'_i$. By noting $S$ is covered by the images of such conics, this corresponds to that the morphism $f'$ contracts the strict transform of $S$.

We can see that any conic on $A$ except one belonging to $D_{e_i}$ corresponds to that on $A_{22}$ in the usual sense, and the component of Hilbert scheme of $A_{22}$ parameterizing conics is naturally isomorphic to $\mathbb{P}^2$. The three conics on $A_{22}$ corresponding to the images of $D_{e_i}$ are $\beta''_i \cup m$, where $\beta''_i$ are the images of the flopped curve corresponding to $\beta'_i$.

Let $a \in E_i$. Then six conics on $A$ attached to $a$ are $\xi_{ij1}$ ($j = 1, 2$), a conic $q_a$ from $D_{e_i}$ and three conics from $e_i$ (see the remark at the end of 5.1). Moreover, if $a$ moves in a fiber $\gamma$ of the other projection $E_i \to \mathbb{P}^1$, then only the conic $q_a$ from $D_{e_i}$ varies. By the contraction $\mathcal{H}_2 \to \mathbb{P}^2$, there is no difference among points on $\gamma$. This is the meaning of the contraction $\rho'$ of $E_i$ in the other direction.

Finally we remark that $\mathcal{H}_1$ is also naturally isomorphic to the component of Hilbert scheme of $A_{22}$ parameterizing lines.

7. Appendix

We give a quick review of basic facts on the theory of polarity. The main references are [DK93, §1 and §2] and [Dol04, §2].
• Denote by $\text{Sym}_m V$ the image of the linear map

\[
\tilde{V}^\otimes m \rightarrow \tilde{V}^\otimes m \quad t \mapsto \sum_{\sigma \in S_m} \sigma(t).
\]

The map $\tilde{V}^\otimes m \rightarrow \text{Sym}_m V$ is decomposed as $\tilde{V}^\otimes m \xrightarrow{s_m} \text{Sym}_m V \xrightarrow{p_m} \text{Sym}_m V$, where $s_m$ is the natural quotient map. Denote by $p_m: \text{Sym}_m V \rightarrow \tilde{V}^\otimes m$ the map obtained from $p_m$ by dividing $m!$. This is called the polarization map. Let $r_m: \text{Sym}_m V \hookrightarrow \tilde{V}^\otimes m \xrightarrow{s_m} \text{Sym}_m V$ be the natural map. Then it holds that $p_m \circ r_m = r_m \circ pl_m = \text{id}$.

• For $F \in \text{Sym}_m \tilde{V}$, set $\tilde{F} := pl_m(F)$. Then $F(x) = \tilde{F}(x,x,\ldots,x)$ for $x \in V$.

• For $F \in \text{Sym}_m \tilde{V}$ and $a \in V$, set $P_a(F)(x) := \tilde{F}(a,x,\ldots,x)$. It is easy to verify

\[
P_a(F) = \frac{1}{m} \sum_i a_i \frac{\partial F}{\partial x_i},
\]

where $a_i$ and $x_i$ are coordinates of $a_i$, and on $V$ respectively. Similarly, by setting $P_{a,b,\ldots,c}(F) := \tilde{F}(a,b,\ldots,c,x,\ldots,x)$ (the number of $a,b,\ldots,c$ is $k$), it holds

\[
P_{a,b,\ldots,c,x,\ldots,x}(F) = \frac{(m-k)!}{m!} \sum_{i_1,\ldots,i_k} a_{i_1} b_{i_2} \cdots c_{i_k} \frac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}}.
\]

This is called the mixed polar of $F$ with respect to $a,b,\ldots,c$.

It is possible to regard this as the pairing between $F \in \text{Sym}_m \tilde{V}$ and $ab\cdots c \in \text{Sym}_k V$. By extending this pairing, we have

\[
\text{Sym}_k V \times \text{Sym}_m \tilde{V} \rightarrow \text{Sym}^{m-k} \tilde{V}
\]

\[
(G,F) \mapsto P_G(F).
\]

Further, by fixing $F$, we can write

\[
ap_F: \text{Sym}_k V \rightarrow \text{Sym}^{m-k} \tilde{V}
\]

\[
G \mapsto P_G(F).
\]

This is called the apolarity map.

When $m = k$, this pairing is sometimes denoted by $\langle G, F \rangle$ and is called the apolarity pairing.

• The following is a basic property of the apolarity pairing:

\[
\langle F, ab\cdots c \rangle = \tilde{F}(a,b,\ldots,c),
\]

where the number of $a,b,\ldots,c$ is $m$. In particular,

\[
\langle F, a^m \rangle = \tilde{F}(a,a,\ldots,a) = F(a).
\]

• If $m = 2k$, then $F$ is said to be non-degenerate if

\[
ap_F^k: \text{Sym}_k V \rightarrow \text{Sym}_k \tilde{V}
\]

is an isomorphism. In this case, there is $\bar{F} \in \text{Sym}_k V$ such that

\[
ap_F^{k-1} = \bar{F}.
\]

$\bar{F}$ is called the form dual to $F$. 
Usually, we consider the apolarity maps in the projective setting. Namely, we consider $a \in \mathbb{P}^* \mathcal{V}$ rather than $a \in \mathcal{V}$, etc. In this situation, we denote by $H_a \in \mathcal{V}$ an element corresponding to $a \in \mathbb{P}^* \mathcal{V}$, which is unique up to scalar. By abuse of notation, we sometimes continue to write $P_a(F)$ rather than $P_{H_a}(F)$.

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