NONCOMMUTATIVE HARMONIC ANALYSIS ON SEMIGROUPS

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Abstract. In this paper we obtain some noncommutative multiplier theorems and maximal inequalities on semigroups. As applications, we obtain the corresponding individual ergodic theorems. Our main results extend some classical results of Stein and Cowling on one hand, and simplify the main arguments of Junge-Le Merdy-Xu’s related work [15].

1. Introduction and Preliminaries

Let $L$ be a densely defined positive operator on $L_2(\Omega)$, where $\Omega$ is a $\sigma$-finite measure space. Suppose that $\{P_\lambda\}$ is the spectral resolution of $L$:

$$Lf = \int_0^\infty \lambda dP_\lambda f, \quad f \in \text{Dom}(L).$$

If $m$ is a bounded function on $[0, \infty)$, then by the spectral theorem, the multiplier operator $m(L)$ defined by

$$m(L)f = \int_0^\infty m(\lambda)dP_\lambda f, \quad f \in L_2(\Omega),$$

is bounded on $L_2(\Omega)$. Let $(T_t)_{t>0} = (e^{-tL})_{t>0}$ be the operator semigroup, which we always assume satisfies the contraction property:

$$\|T_t f\|_p \leq \|f\|_p, \quad f \in L_2(\Omega) \cap L_p(\Omega),$$

wherever $1 \leq p \leq \infty$. Stein [29] developed a Littlewood-Paley theory for such semigroups, with some additional hypotheses. By use of transference techniques, Coifman and Weiss [4], Cowling [5] presented an alternative and simpler approach to obtain some multiplier results and maximal inequalities. Indeed, Cowling [5] showed that $m(L)$, originally defined on $L_2(\Omega)$ via the spectral theorem, is exactly a bounded operator on $L_p(\Omega)$ for $1 < p < \infty$, whenever $m$ has a bounded analytic extension on some sector $\Sigma_\phi$ with $\phi > \pi|1/p - 1/2|$, where

$$\Sigma_\phi = \{z \in \mathbb{C} : |\text{arg } z| < \phi\}.$$

More precisely, the following estimate holds:

$$\|m(L)f\|_p \leq C_{\phi,p}\|m\|_\infty\|f\|_p, \quad f \in L_p(\Omega).$$

(1.1)

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In other words, the generators of the symmetric contraction semigroups have a $H^\infty$ functional calculus on each sector $\Sigma_\phi$ with $\phi > \pi|1/p - 1/2|$.

Recently, more attention was turned to diffusion semigroups on noncommutative space $L_p(M)$ associated to a von Neumann algebra, see for instance [15, 16, 17, 19, 20, 24]. In this paper, we consider the semigroup $(T_t)_{t>0}$ acting on noncommutative $L_p$-space associated with $(\mathcal{M}, \tau)$, where $\mathcal{M}$ is a von Neumann algebra with a normal finite faithful trace $\tau$. Under reasonable hypotheses, we obtain some noncommutative multiplier theorems, the noncommutative version of (1.1), which positively answers the question raised in [15, Remark 5.9]. Namely, the generators of the noncommutative diffusion semigroups also have a $H^\infty$ functional calculus on each sector $\Sigma_\phi$ with $\phi > \pi|1/p - 1/2|$. By means of the multiplier theorems, we establish some noncommutative maximal inequalities and individual ergodic theorems. It is worth pointing out that the key point of Cowling’s method in [5] is to combine the transference technique and Fendler’s dilation theorem. A noncommutative version of Fendler’s dilation theorem has been recently achieved by Junge-Ricard-Shlyakhtenko [18] (see also Dabrowski [8]). Armed with this result, we can extend Cowling’s method to the noncommutative setting. In this way, we recover the main results of [15] by a very simple method. This is a major advantage of our method over that of [15].

Now we introduce some preliminaries which will be used in the sequel. We shall work on a von Neumann algebra $\mathcal{M}$ equipped with a normal finite faithful trace $\tau$. For $1 \leq p < \infty$, let $L_p(\mathcal{M}, \tau)$ or simply $L_p(\mathcal{M})$ be the associated noncommutative $L_p$ space. Namely, $L_p(\mathcal{M})$ is the completion of $\mathcal{M}$ with the norm $\|x\|_p = (\tau(|x|^p))^{1/p}$, where $|x| = (x^*x)^{1/2}$ is the modulus of $x$. By convention we set $L_{\infty}(\mathcal{M}) = \mathcal{M}$, equipped with the operator norm. Like the commutative $L_p$-spaces, one has the duality: $L_p(\mathcal{M})^* = L_q(\mathcal{M})$ via $(x, y) \mapsto \tau(xy)$, for $x \in L_p(\mathcal{M}), y \in L_q(\mathcal{M})$ with $1 \leq p < \infty$ and $1/p + 1/q = 1$. It is also well known that $L_p(\mathcal{M})$ has UMD property for $1 < p < \infty$. We refer to [28] for more information and more historical references on noncommutative $L_p$-spaces.

We say an operator $T$ on $\mathcal{M}$ is completely positive if $T \otimes I_n$ is positive on $\mathcal{M} \otimes M_n$ for each $n$. Here, $M_n$ is the algebra of $n \times n$ matrices and $I_n$ is the identity operator on $M_n$. Now we introduce the standard noncommutative semigroup. That is, $(T_t)$ is a semigroup of completely positive maps on a finite von Neumann algebra $\mathcal{M}$ satisfying the following conditions:

1) Every $T_t$ is normal on $\mathcal{M}$ such that $T_t(1) = 1$;
2) Every $T_t$ is selfadjoint with respect to the trace $\tau$, i.e. $\tau(T_t(x)y) = \tau(xT_t(y))$;
3) The family $(T_t)$ is strongly continuous, i.e. $\lim_{t \to 0} T_t x = x$ with respect to the strong operator topology in $\mathcal{M}$ for any $x \in \mathcal{M}$.

Let us note that the first two conditions imply that $\tau(T_t x) = \tau(x)$ for all $x$, so $T_t$ is faithful and contractive on $L_1(\mathcal{M})$. By interpolation technique, $T_t$ can be extend to a contraction on $L_p(\mathcal{M})$ for $1 \leq p < \infty$ and satisfies $\lim_{t \to 0} T_t x = x$ in $L_p(\mathcal{M})$ for any $x \in L_p(\mathcal{M})$. Let us recall that such a semigroup admits an infinitesimal generator $L$, i.e., $T_t = e^{-tL}$. We refer to [19] for more details.

We say that a standard semigroup $(T_t)$ on a finite von Neumann algebra $\mathcal{M}$ admits a Markov dilation if there exists a larger finite von Neumann algebra $\mathcal{N}$, an increasing filtration
\((N_s)_{s \geq 0}\) with conditional expectation \(N_s = E_s(N)\) and trace preserving \(*\)-homomorphisms \(\pi_s : M \rightarrow N\) such that \(\pi_s(M) \subset N_s\) and
\[
E_s(\pi_t(x)) = \pi_s(T_{t-s}x), \quad 0 \leq s < t < \infty, \quad x \in M.
\]

In [18], the authors proved that every semigroup of completely positive unital selfadjoint maps on a finite von Neumann algebra admits a Markov dilation. Moreover, the authors in [18] extended the Markov dilation above to all of \(\mathbb{R}\) by using the ultraproduct argument. Namely, there exists a new finite von Neumann algebra \(N\), an increasing filtration \((N_s)_{-\infty < s < \infty}\) with conditional expectation \(N_s = E_s(N)\) and trace preserving \(*\)-homomorphisms \(\pi_s : M \rightarrow N\) such that \(\pi_s(M) \subset N_s\) and
\[
E_s(\pi_t(x)) = \pi_s(T_{t-s}x), \quad -\infty < s < t < \infty, \quad x \in M.
\]

Our paper is organized as follows. Section 2 is on noncommutative multiplier theorems. The noncommutative maximal inequalities are proved in Section 3. As applications, in Section 4, we give some individual ergodic theorems.

In the rest of the paper we use the same letter \(C\) to denote various positive constants which may change at each occurrence. Variables indicating the dependency of constants \(C\) will be often specified in the parenthesis. We use the notation \(X \lesssim Y\) or \(Y \gtrsim X\) for nonnegative quantities \(X\) and \(Y\) to mean \(X \leq CY\) for some inessential constant \(C > 0\). Similarly, we use the notation \(X \approx Y\) if both \(X \lesssim Y\) and \(Y \lesssim X\) hold.

2. Noncommutative multiplier theorems

In this section, we first give a noncommutative Fourier multiplier theorem by applying the following Junge-Ricard-Shlyakhtenko dilation theorem [18, Theorem 5 and Corollary 4.5], which plays a crucial role in our proof.

**Theorem 2.1.** Let \((T_t)\) be a semigroup of completely positive unital and selfadjoint maps on a finite von Neumann algebra \((M, \tau)\). Then \((T_t)\) admits a Markov dilation.

In the sequel, we suppose that the spectral projection \(P_0\) onto the kernel of \(L\) is trivial on \(L_p(M)\) and we hence do not need to consider the definition of \(m(0)\).

**Theorem 2.2.** Suppose that \(m\) is a bounded holomorphic function on the sector \(\Sigma_{\pi/2}\). Let \(\Phi\) be the distribution on \(\mathbb{R}\) whose Fourier transform is the bounded function defined (almost everywhere) by the formula
\[
\hat{\Phi}(v) = m(iv), \quad v \in \mathbb{R},
\]
where \(m(iv)\) is the non-tangential limit. If for some \(p \in [1, \infty]\) and all \(f\) in \(L_p(\mathbb{R}, L_p(M))\)
\[
\|\Phi \ast f\|_{L_p(\mathbb{R}, L_p(M))} \leq C\|f\|_{L_p(\mathbb{R}, L_p(M))},
\]
for some positive constant \(C\), then for all \(x \in M\),
\[
\|m(L)x\|_p \leq C\|x\|_p. \tag{2.1}
\]
Consequently, \(m(L)\) extends uniquely to a bounded operator on \(L_p(M)\), still denoted by \(m(L)\), of norm at most \(C\).
Proof. We proceed the proof by a standard transference argument. Since the distribution \( \Phi \) is defined as follows:

\[
\hat{\Phi}(v) = \int_{-\infty}^{+\infty} e^{-ivu} \Phi(u) du = m(iv), \quad v \in \mathbb{R}.
\]

Hence by the Paley-Wiener theorem, \( \Phi \) must be supported in \([0, \infty)\). A concrete computation (see, page 77 in [10]) shows that

\[
\int_{0}^{+\infty} e^{-\lambda u} \Phi(u) du = m(\lambda), \quad \lambda \in \mathbb{R}^+.
\]

Thus by the spectral theory

\[
\int_{0}^{+\infty} e^{-\lambda L} \Phi(u) du = \int_{0}^{+\infty} \left( \int_{0}^{+\infty} e^{-\lambda u} dP_\lambda \right) \Phi(u) du = \int_{0}^{+\infty} m(\lambda) dP_\lambda = m(L).
\]

By the noncommutative Markov dilation Theorem 2.1, \((T_t)\) admits a Markov dilation. We notice that the dilation property can be extended verbatim to all of \( \mathbb{R} \) by using the ultraproduct argument (see [18, Corollary 4.3, page 51 and Corollary 4.5, page 54] for more details. Namely, there exists a larger finite von Neumann algebra \( \mathcal{N} \), an increasing filtration \((\mathcal{N}_s)_{s \in \mathbb{R}}\) with conditional expectation \( E_s = E_s(\mathcal{N}) \) and trace preserving *-homomorphisms \( \pi_s : \mathcal{M} \rightarrow \mathcal{N} \) such that \( \pi_s(\mathcal{M}) \subset \mathcal{N}_s \) and

\[
E_s(\pi_t(x)) = \pi_s(T_{t-s} x), \quad -\infty < s \leq t < \infty, \quad x \in \mathcal{M}.
\]

Where \( \pi_s(x) := \beta(\pi_t(x)) \) for all \( x \in \mathcal{M}, \beta \) is the automorphism of \( \mathcal{N} \) appeared in Corollary 4.5 in [18]. Especially, for any \( t \in [0, +\infty) \) and \( s \in \mathbb{R} \)

\[
E_s(\pi_t x) = \pi_s T_{t-s} x, \quad \forall x \in \mathcal{M}.
\]

Consequently, keeping the support of \( \Phi \) in \([0, \infty)\) in mind, for any \( x \in \mathcal{M} \),

\[
\|m(L)x\|_{L_p(\mathcal{M})} = \|\pi_s \int_{0}^{+\infty} \Phi(t) T_{t-s} x dt\|_{L_p(\mathcal{N})} = \| \int_{0}^{+\infty} \Phi(t) E_{-s} \pi_t x dt\|_{L_p(\mathcal{N})}
\]

\[
= \| \int_{-\infty}^{+\infty} \Phi(t) E_{-s} \pi_{t-s} x dt\|_{L_p(\mathcal{N})} = \| E_{-s} \int_{-\infty}^{+\infty} \Phi(t) \pi_{t-s} x dt\|_{L_p(\mathcal{N})}
\]

\[
\leq \| \int_{-\infty}^{+\infty} \Phi(t) \pi_{t-s} x dt\|_{L_p(\mathcal{N})}, \quad \forall s \in \mathbb{R},
\]

which implies that for any \( N \in \mathbb{N} \)

\[
\tau \left( \|m(L)x\|^p \right) \leq \frac{1}{2N} \int_{-N}^{N} \tau \left( \| \int_{-\infty}^{+\infty} \Phi(t) \pi_{t-s} x dt\|^p \right) ds.
\]
For any $M > 0$, let $\chi(-M - N, M + N)(w)$ be the characteristic function of $(-M - N, M + N)$ and $f_x(-w) = \pi_w x \chi(-M - N, M + N)(w)$, then $f_x(\cdot) \in L_p(\mathbb{R}, L_p(M))$. By our assumption, we obtain
\[
\left\| \Phi * f_x \right\|_{L_p(\mathbb{R}, L_p(M))} \leq C \left\| f_x \right\|_{L_p(\mathbb{R}, L_p(M))}.
\]

Enlarging the integral domain from $[-N, N]$ to $\mathbb{R}$ we see that the expression on the right in (2.3) is smaller than
\[
\frac{1}{2N} \int_{-\infty}^{\infty} \left( \tau \left| \int_{-\infty}^{\infty} \Phi(t) \pi_{t-s} x \chi_{(-M-N, M+N)}(t-s) dt \right|^p \right) ds
\]
\[
= \frac{1}{2N} \left\| \Phi * f_x \right\|_{L_p(\mathbb{R}, L_p(M))}^p \leq \frac{C_p}{2N} \left\| f_x \right\|_{L_p(\mathbb{R}, L_p(M))}^p
\]
\[
= \frac{C_p}{2N} \int_{-\infty}^{\infty} \left\| \pi_{-w} x \chi_{(-M-N, M+N)}(w) \right\|_{L_p(M)}^p dw
\]
\[
= \frac{C_p(2N + 2M)}{2N} \left\| x \right\|_{L_p(M)}^p, \quad x \in M.
\]

Let $N \rightarrow \infty$, we get
\[
(2.4) \quad \left\| m(L)x \right\|_p \leq C \left\| x \right\|_p, \quad x \in M.
\]
Since $M$ is dense in $L_p(M)$, $m(L)$ then extends uniquely to a bounded operator on $L_p(M)$. The proof is complete. □

For later use, we record the following corollary on the imaginary powers of $L$.

**Corollary 2.3.** Suppose that $1 < p < \infty$, and that $u \in \mathbb{R}$. Then the operator $L^{iu}$ is bounded on $L_p(M)$: for any $x$ in $L_p(M) \cap L_2(M)$,
\[
(2.5) \quad \left\| L^{iu} x \right\|_p \leq C \left( \frac{p^2}{p - 1} \right) \left( 1 + |u| \right) \exp\left( \frac{\pi}{2} |u| \right) \left\| x \right\|_p,
\]
where $C$ is an absolute constant.

**Proof** Let $\Phi$ be the distribution with Fourier transform
\[
\hat{\Phi}(v) = (iv)^{iu}, \quad v \in \mathbb{R}.
\]
In this case, $\Phi(t) = \Gamma(-iu)^{-1} t^{-iu-1}$ if $t > 0$, otherwise $\Phi(t) = 0$, where and in the sequel $\Gamma(\cdot)$ is the Gamma function. $\hat{\Phi}$ is the boundary value of the holomorphic function
\[
m(z) = z^{iu} = \exp\left( iu \log |z| - u \arg z \right).
\]
We note that $m$ satisfies the Hörmander-Mihlin condition:
\[
|m(iv)| \leq \exp\left( \frac{\pi}{2} |u| \right), \quad \left| \frac{\partial}{\partial v} m(iv) \right| = \left| \frac{i u}{v} m(iv) \right| \leq \frac{|u|}{|v|} \exp\left( \frac{\pi}{2} |u| \right), \quad v \in \mathbb{R}.
\]
By the Hörmander multiplier theorem (see [1] or [32]) and the UMD property of noncommutative $L_p(\mathcal{M})$ space, we deduce that for any $f \in L_p(\mathbb{R}, L_p(\mathcal{M}))$

$$\|\Phi \ast f\|_{L_p(\mathbb{R}, L_p(\mathcal{M}))} \leq C(p)(1 + |u|)\exp\left(\frac{\pi}{2}|u|\right)\|f\|_{L_p(\mathbb{R}, L_p(\mathcal{M}))}.$$  

By a result of Parcet in [26], we can give an explicit estimate of the constant $C(p) = C[p + \frac{1}{p}]$. In fact, let $T$ be the Calderón-Zygmund operator associated to the kernel $\Phi(t)$. Since

$$|\Phi(t)| \leq \exp\left(\frac{\pi}{2}|u|\right) \lesssim \frac{1}{|t|},$$

and

$$|\Phi(s) - \Phi(s')| \lesssim (1 + |u|)\exp\left(\frac{\pi}{2}|u|\right) \lesssim \frac{|s - s'|}{|s|^2} \quad \text{if} \quad |s - s'| \leq \frac{1}{2}|s|.$$

Namely, the kernel $\Phi(t)$ satisfies the size and smoothness conditions with the Lipschitz smoothness parameter $\gamma = 1$. It follows from Theorem A in [26] that for any $f \in L_p(\mathbb{R}, L_p(\mathcal{M}))$

$$\|\Phi \ast f\|_{L_p(\mathbb{R}, L_p(\mathcal{M}))} = \|Tf\|_{L_p(\mathbb{R}, L_p(\mathcal{M}))} \leq C(p)\frac{p^2}{p - 1}(1 + |u|)\exp\left(\frac{\pi}{2}|u|\right)\|f\|_{L_p(\mathbb{R}, L_p(\mathcal{M}))},$$

where $C$ is a constant independently on $p$. Hence we deduce from Theorem 2.2 that

$$(2.6) \quad \|L^u x\|_p \leq C(p)\frac{p^2}{p - 1}(1 + |u|)\exp\left(\frac{\pi}{2}|u|\right)\|x\|_p, \quad x \in L_p(\mathcal{M}) \cap L_2(\mathcal{M}).$$

\[\square\]

Remark 2.4. Note that the operator norm of $L^u$ on $L_2(\mathcal{M})$ is equal to 1 by the spectral theory. Hence, it is possible to improve the constant in (2.5) by the Riesz-Thorin interpolation theorem. Using verbatim the standard method stated in [10, Corollary 6.3.1, page 78], we can improve (2.5) as follows which is needed in Section 3

$$\|L^u x\|_p \leq C(p)\frac{p^2}{p - 1}(1 + |u|)\exp\left(\frac{\pi}{2}|u|\right)\|x\|_p,$$

where $C$ is an absolute constant. Moreover, by using Cowling’s argument [5, Corollary 1, page 270], we can further improve the power index on $|u|$, we leave it to the reader.

Theorem 2.5. Suppose that $m$ is a bounded holomorphic function on the sector $\Sigma_\phi$ with $\pi/2 < \phi \leq \pi$. Then for $1 < p < \infty$,

$$\|m(L)x\|_p \leq C(p, \phi, m)\|x\|_p, \quad \forall x \in L_p(\mathcal{M}),$$

$C(p, \phi, m)$ is a constant depending only on $p, \phi$ and $m$.

Proof. Let $\Phi$ be the distribution on $\mathbb{R}$ satisfying

$$\hat{\Phi}(v) = m(iv), \quad v \in \mathbb{R}.$$  

Since $m$ extends analytically to $\Sigma_\phi$, $\Phi$ must be supported in $\mathbb{R}^+ \cup \{0\}$. If $v \in \mathbb{R} \setminus \{0\}$, then the disc $D$ with center $iv$ and radius $r$ is contained in $\Sigma_\phi$ provided that

$$r < \sin(\phi - \pi/2)|v|.$$
By the Cauchy formula, we have

\[
\frac{dm}{dz}(iv) = \frac{1}{2\pi i} \int_{\partial D} \frac{m(\xi)}{(\xi - iv)^2} d\xi.
\]

This implies that

\[
|v \frac{\partial}{\partial v} m(iv)| \leq \|m\|_{\infty} \csc(\phi - \frac{\pi}{2}).
\]

Consequently, \(\hat{\Phi}(v)\) satisfies the Hörmander-Mihlin condition. Since the noncommutative \(L_p(\mathcal{M})\) space has UMD property, we claim that for \(1 < p < \infty\),

\[
\|\Phi \ast f\|_{L_p(\mathbb{R}, L_p(\mathcal{M}))} \leq C(p, \phi, m) \|f\|_{L_p(\mathbb{R}, L_p(\mathcal{M}))}, \quad f \in L_p(\mathbb{R}, L_p(\mathcal{M})�).
\]

The desired result immediately follows from Theorem 2.2. \(\square\)

The theorem above shows that \(L\) admits a bounded \(H^\infty(\Sigma_\phi)\) functional calculus with \(\pi/2 < \phi \leq \pi\). By a standard angle reduction principle for noncommutative semigroup, see [15, Proposition 5.8], \(L\) actually admits a bounded \(H^\infty(\Sigma_\phi)\) functional calculus for any \(\phi > |\frac{1}{p} - \frac{1}{2}|\pi\), which positively answers the question raised in [15, Remark 5.9]. Then we summarize the main result of this section as follows.

**Theorem 2.6.** Suppose that \(m\) is a bounded holomorphic function on the sector \(\Sigma_\psi\). If \(|\frac{1}{p} - \frac{1}{2}|\pi < \psi\), then for \(1 < p < \infty\),

\[
\|m(L)x\|_p \leq C(p, \psi, m) \|x\|_p, \quad x \in L_p(\mathcal{M}),
\]

for some constant \(C(p, \psi, m)\).

**Remark 2.7.** By tensoring \(\mathcal{M}\) with \(M_n\), the algebra of \(n \times n\) matrices, for any \(n\), the theorem above implies that \(L\) has a completely bounded \(H^\infty\) functional calculus in \(\Sigma_\psi\) with \(\phi > |\frac{1}{p} - \frac{1}{2}|\pi\). We refer the interested reader to [15] for more information on \(H^\infty\) functional calculus.

Since every bounded \(H^\infty\) functional calculus implies square function estimates, we have the following corollary from Theorem 2.6. We refer to [15, Theorem 7.6 or Corollaries 7.7 and 7.10] for more details on square functions.

**Corollary 2.8.** Suppose that \(m\) is a bounded holomorphic function on the sector \(\Sigma_\psi\) with \(\psi > |\frac{1}{p} - \frac{1}{2}|\pi\).

Then for any \(x \in L_p(\mathcal{M})\) the following hold:
(1) For $1 < p < 2$, 
\[
\|x\|_{L_p(M)} \approx \inf \left\{ \left\| \left( \int_0^\infty t \left| \frac{\partial}{\partial t}(T_t(x_1)) \right|^2 dt \right)^{1/2} \right\|_{L_p(M)} + \right. \\
\left. \left\| \left( \int_0^\infty t \left| \frac{\partial}{\partial t}(T_t(x_2)) \right|^2 dt \right)^{1/2} \right\|_{L_p(M)} \right\},
\]
where the infimum runs over all $x_1, x_2 \in L_p(M)$ such that $x = x_1 + x_2$.

(2) For $2 \leq p < \infty$, 
\[
\|x\|_{L_p(M)} \approx \max \left\{ \left\| \left( \int_0^\infty t \left| \frac{\partial}{\partial t}(T_t(x)) \right|^2 dt \right)^{1/2} \right\|_{L_p(M)}, \right. \\
\left. \left\| \left( \int_0^\infty t \left| \frac{\partial}{\partial t}(T_t(x))^* \right|^2 dt \right)^{1/2} \right\|_{L_p(M)} \right\}.
\]

\textbf{Remark 2.9.} It is worth noticing that a similar result is obtained in [15, pp. 68] with a different method. We should emphasize here that the ours is much simpler and works without the hypothesis of $H^\infty$-functional calculus for $L$.

3. Noncommutative maximal inequalities

We now turn to maximal inequalities. We first recall the definition of noncommutative maximal functions introduced by Pisier (see [27]) and Junge (see [14]). Let $1 \leq p \leq \infty$, we define $L_p(M, \ell_\infty)$ to be the space of all sequences $x = (x_n)_{n \geq 1}$ in $L_p(M)$, which admit a factorization of the following form: there exist $a, b \in L_{2p}(M)$ and a bounded sequence $y = (y_n) \subset L_\infty(M)$ such that 
\[
x_n = ay_nb, \quad n \geq 1.
\]
The norm of $x$ in $L_p(M, \ell_\infty)$ is given by 
\[
\|x\|_{L_p(M, \ell_\infty)} = \inf \left\{ \|a\|_{2p} \sup_n \|y_n\|_\infty \|b\|_{2p}, \right. \\
\left. \text{where the infimum is taken over all factorizations of } x \text{ as above. It is easy to see that } \right.
\]
$L_p(M, \ell_\infty)$ is a Banach space with the norm $\|\cdot\|_{L_p(M, \ell_\infty)}$, and a positive sequence $x = (x_n)$ belongs to $L_p(M, \ell_\infty)$ if and only if there is $a \in L_p^+(M)$ such that $x_n \leq a$ for all $n$. Moreover, in this case, 
\[
\|x\|_{L_p(M, \ell_\infty)} = \inf \left\{ \|a\|_p : a \in L_p^+(M) \text{ such that } x_n \leq a, \forall n \geq 1 \right\}.
\]
The norm of $x$ in $L_p(\mathcal{M}, \ell_\infty)$ is conventionally denoted by $\|\sup_{n \geq 1}^+ x_n\|_p$. Please note that $\|\sup_{n \geq 1}^+ x_n\|_p$ is just a notation since $\sup_{n \geq 1} x_n$ does not make any sense in the noncommutative setting. We use this notation only for convenience.

**Remark 3.1.** The definition of $L_p(\mathcal{M}, \ell_\infty)$ can be extended to an arbitrary index set $I$. Then $L_p(\mathcal{M}, \ell_\infty(I))$ can be defined similarly as before. More precisely, $L_p(\mathcal{M}, \ell_\infty(I))$ consists of all families $(x_i)_{i \in I}$ in $L_p(\mathcal{M})$ which can be factorized as $x_i = ay_i b$ with $a, b \in L_{2p}(\mathcal{M})$ and a bounded family $(y_i)_{i \in I} \subset L_\infty(\mathcal{M})$. The norm of $(x_i)_{i \in I}$ in $L_p(\mathcal{M}, \ell_\infty(I))$ is defined as

$$\inf \left\{ \|a\|_{2p} \sup_i \|y_i\|_{\infty} \|b\|_{2p} \right\},$$

the infimum running over all factorizations as above. As before, this norm is also denoted by $\|\sup_{i \in I}^+ x_i\|_p$.

**Remark 3.2.** One can easily check that for any index set $I$ and $1 \leq p \leq \infty$, a family $(x_i)_{i \in I}$ in $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M}, \ell_\infty(I))$ if and only if

$$\sup_{J \subset I, J \text{ is a finite set}} \|\sup_{i \in J}^+ x_i\|_p < \infty.$$

If this is the case, then

$$\|\sup_{i \in I}^+ x_i\|_p = \sup_{J \subset I, J \text{ is a finite set}} \|\sup_{i \in J}^+ x_i\|_p.$$

The main result of this section is relevant to the noncommutative maximal function $\|\sup_{x \in \Sigma_\psi}^+ T_x x\|$, where $\Sigma_\psi$ is a sector in $C$, which generalizes the Theorem 5.1 and Corollary 5.11 in [19], and Corollary 5.7 in [17]. Moreover, we will see in the next section that our maximal inequality implies that $(T_x x)_c$ converges bilaterally almost uniformly for any $x \in L_p(\mathcal{M})$. In addition, for $p > 2$ the bilateral almost uniform convergence can be improved to the almost uniform convergence. For formulating this result we need further notation from [9]. Let $2 \leq p \leq \infty$ and $I$ be an index set. We define the space $L_p(\mathcal{M}, \ell_\infty^c(I))$ as the family of all $(x_i)_{i \in I} \subset L_p(\mathcal{M})$ for which there are an $a \in L_p(\mathcal{M})$ and $(y_i)_{i \in I} \subset L_\infty(\mathcal{M})$ such that

$$x_i = y_ia \quad \text{and} \quad \sup_{i \in I} \|y_i\|_\infty < \infty.$$ 

$$\|\langle x_i \rangle\|_{L_p(\mathcal{M}, \ell_\infty^c(I))}$$

is then defined to be the infimum $\{\sup_{i \in I} \|y_i\|_\infty \|a\|_p\}$ over all factorizations of $(x_i)$ as above. It is easy to check that $\|\cdot\|_{L_p(\mathcal{M}, \ell_\infty^c(I))}$ is a norm, which makes $L_p(\mathcal{M}, \ell_\infty^c(I))$ a Banach space. Note that $(x_i) \in L_p(\mathcal{M}, \ell_\infty^c(I))$ if and only if $(x_i^* x_i) \in L_p(\mathcal{M}, \ell_\infty(I))$. If $I = \mathbb{N}$, $L_p(\mathcal{M}, \ell_\infty^c(I))$ is simply denoted by $L_p(\mathcal{M}, \ell_\infty^c)$. To state our maximal inequalities we also need the following lemma.

**Lemma 3.3.** [5] Let

$$m_\theta(\lambda) = \exp(-e^{i\theta} \lambda) - \int_0^1 \exp(-t\lambda) dt,$$

where $|\theta| \leq \pi/2$, and $n_\theta = m_\theta \circ \exp$. Then for any $u \in \mathbb{R}$

$$\hat{n}_\theta(u) = \left(e^{-\theta u} - (1 + iu)^{-1}\right)\Gamma(-iu) \quad \text{and} \quad |\hat{n}_\theta(u)| \leq \exp \left(\left|\theta - \frac{\pi}{2}\right||u|\right).$$
Theorem 3.4. Suppose that \( 1 < p < \infty \), and

\[
0 \leq \psi / \pi < 1/2 - |1/p - 1/2|.
\]

Let \( \Sigma_\psi \) be the sector \( \{ z \in \mathbb{C} : |\arg z| < \psi \} \). Then there exists a constant \( C \) depending only on \( p \) and \( \psi \) such that

\[
\| \sup_{z \in \Sigma_\psi}^{+} T_z x \|_p \leq C \| x \|_p, \quad x \in L_p(M).
\] (3.2)

Moreover, if \( p > 2 \), then

\[
\| (T_z x)_{z \in \Sigma_\psi} \|_{L_p(M, \ell_\infty(\Sigma_\psi))} \leq C \| x \|_p, \quad \forall x \in L_p(M).
\] (3.3)

Proof Let \( n_\theta(t) = m_\theta(\exp(t)) \), \( t \in \mathbb{R} \) and

\[
m_\theta(\lambda) = \exp(-e^{i\theta} \lambda) - \int_0^1 \exp(-t\lambda) dt, \quad \lambda \in \mathbb{R}^+,
\] (3.4)

where \( |\theta| < \psi \). It follows from Fourier transform that

\[
m_\theta(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{n}_\theta(u) \lambda^{iu} du, \quad \lambda \in \mathbb{R}^+.
\]

By functional calculus we have,

\[
m_\theta(tL)x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{n}_\theta(u) t^{iu} L^{iu} x du, \quad t \in \mathbb{R}^+.
\] (3.5)

Let \( z = te^{i\theta} \) with \( |\theta| < \psi \) and \( t > 0 \), then it follows from (3.4), by functional calculus, that

\[
T_z x = e^{-zL} x = m_\theta(tL)x + \int_0^1 \exp(-stL) x ds
\]

\[
= m_\theta(tL)x + \frac{1}{t} \int_0^t T_s x ds.
\] (3.6)

Consequently,

\[
\| \sup_{z \in \Sigma_\psi}^{+} T_z x \|_p \leq \| \sup_{t>0,|\theta|<\psi}^{+} m_\theta(tL)x \|_p + \| \sup_{t>0}^{+} \frac{1}{t} \int_0^t T_s x ds \|_p.
\] (3.7)
Since
\[ \left\| \sup_{t>0,|\theta|<\psi} m_\theta(tL)x \right\|_p = \left\| \sup_{t>0,|\theta|<\psi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{n}_\theta(u) t^{iu} L^{iu} x du \right\|_p \]
\[ \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\| \sup_{t>0,|\theta|<\psi} \hat{n}_\theta(u) t^{iu} L^{iu} x \right\|_p du \]
\[ \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \sup_{t>0,|\theta|<\psi} \left| \hat{n}_\theta(u) t^{iu} \right| \exp\left(\frac{1}{\pi^2} |u| \right) \exp\left(\frac{1}{\pi^2} |u| \right) |L^{iu} x|_p du \]
\[ \lesssim \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(\frac{1}{\pi^2} |u| \right) \| L^{iu} \|_p du \right) \| x \|_p \]
\[ = C(p, \psi) \| x \|_p, \]
where \( \| L^{iu} \|_p \) is the operator norm of \( L^{iu} \) on \( L_p(M) \) and
\[ C(p, \psi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(\frac{1}{\pi^2} |u| \right) \| L^{iu} \|_p du. \]

It follows from Corollary 2.3 and Remark 2.4, we have
\[ C(p, \psi) \lesssim \frac{p^2}{p^2 - 12\pi} \int_{-\infty}^{\infty} (1 + |u|^2)^{\frac{p}{2} - 1} \exp\left(\pi \left| \frac{1}{p} - \frac{1}{2} \right| |u| \right) \exp\left(\frac{1}{\pi^2} |u| \right) du < \infty, \]
where the finiteness of last integral can be found in [10, page 81]. Thus we deduce that
\[ \left\| \sup_{t>0,|\theta|<\psi} m_\theta(tL)x \right\|_p \lesssim C(p, \psi) \| x \|_p, \quad \forall x \in L_p(M). \tag{3.9} \]

Similarly, we also get that
\[ \left\| (m_\theta(tL)x)_{t>0,|\theta|<\psi} \right\|_{L_p(M, L^\infty(\Sigma_\psi))} \lesssim C(p, \psi) \| x \|_p, \quad \forall x \in L_p(M). \tag{3.10} \]

On the other hand, Theorem 4.5 in [19] implies that
\[ \left\| \sup_{t>0} \frac{1}{t} \int_0^t T_s x ds \right\|_p \leq C_p \| x \|_p, \quad \forall x \in L_p(M) \quad \text{for} \quad 1 < p < \infty, \tag{3.11} \]
and
\[ \left\| \left( \frac{1}{t} \int_0^t T_s x ds \right)_{t>0} \right\|_{L_p(M, L^\infty(\mathbb{R}^+))} \leq C_p \| x \|_p, \quad \forall x \in L_p(M) \quad \text{for} \quad 2 < p < \infty. \tag{3.12} \]

Combining (3.7), (3.9) and (3.11) implies the desired estimate (3.2). And the desired estimate (3.3) follows from (3.7), (3.10) and (3.12). \( \square \)
4. Individual Ergodic Theorems

In this section, motivated by Proposition 7 in [31], we apply the maximal inequalities proved in the previous section to the pointwise ergodic convergence. To this end we need an appropriate analogue for the noncommutative setting of the usual almost everywhere convergence. This is the almost uniform convergence introduced by Lance in [22].

**Definition 4.1.** Let $\mathcal{M}$ be a von Neumann algebra equipped with a finite normal faithful trace $\tau$. Let $x_n, x \in L^0(\mathcal{M})$.

1. $(x_n)$ is said to converge bilaterally almost uniformly (b.a.u. in short) to $x$ if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that
   \[ \tau(e^\perp) < \varepsilon \quad \text{and} \quad \lim_{n \to \infty} \|e(x_n - x)e\|_\infty = 0. \]

2. $(x_n)$ is said to converge almost uniformly (a.u. in short) to $x$ if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that
   \[ \tau(e^\perp) < \varepsilon \quad \text{and} \quad \lim_{n \to \infty} \|(x_n - x)e\|_\infty = 0. \]

In the commutative case, both convergences in the definition above are equivalent to the usual almost everywhere convergence by virtue of Egorov’s theorem. However they are different in the noncommutative case. Similarly, we can introduce these notions of convergence for functions with values in $L^0(\mathcal{M})$ and for nets in $L_0(\mathcal{M})$.

**Theorem 4.2.** Suppose that $1 < p < \infty$, and that
   \[ 0 \leq \psi/\pi < 1/2 - |1/p - 1/2|. \]

Then for any $x \in L_p(\mathcal{M})$ the following hold:

1. The operators $T_{z}x$ converge bilaterally almost uniformly to $x$ for $1 < p \leq 2$ and almost uniformly to $x$ for $2 < p < \infty$ as $z$ tends to $0$ in $\Sigma_\psi$.
2. The operators $T_{z}x$ converge bilaterally almost uniformly to $P(x)$ for $1 < p \leq 2$ and almost uniformly to $P(x)$ for $2 < p < \infty$ as $z$ tends to $\infty$ in $\Sigma_\psi$, where $P$ denotes the projection from $L_p(\mathcal{M})$ onto the fixed point subspace of the semigroup $(T_t)_{t > 0}$.

**Proof** (1) Let $x \in L_2(\mathcal{M}) \cap L_p(\mathcal{M})$ and $s > 0$. Let $D$ be the disc of center $s$ and radius $r$ with $r < s \sin \psi$. For any $z \in D$, by the vector-valued Cauchy formula, we have
   \[ T_z(x) = \frac{1}{2\pi i} \int_{\partial D} \frac{T_\zeta(x)d\zeta}{\zeta - z}. \]

Thus
   \[ T_z(x) - T_s(x) = \frac{z - s}{2\pi i} \int_{\partial D} \frac{T_\zeta(x)d\zeta}{(\zeta - z)(\zeta - s)}. \]

By the convexity of the operator valued function: $x \mapsto |x|^2$, $|T_z(x) - T_s(x)|^2 \leq \frac{|z - s|^2}{4\pi^2} \int_{\partial D} \frac{|d\zeta|}{|(\zeta - z)(\zeta - s)|^2} \int_{\partial D} |T_\zeta(x)|^2 |d\zeta| \leq C|z - s|^2 \alpha$
for \(|z-s| < \frac{\varepsilon}{2}\), where \(C\) denotes a positive constant independent of \(z\) and \(a = \int_{\partial D} |T_\zeta(x)|^2 d\zeta\). Note that \(a \in L_1(\mathcal{M})\). It follows that there exists a contraction \(u \in L_\infty(\mathcal{M})\) (depend on \(z\) and \(s\)) such that

\[
T_z(x) - T_s(x) = C|z - s|au^{1/2}.
\]

For any \(\varepsilon > 0\), let \(e = e_{(0,\|a^{1/2}\|_2/\varepsilon^{1/2})}(a^{1/2})\) be the spectral projection of \(a^{1/2}\) on the interval \((0, \|a^{1/2}\|_2/\varepsilon^{1/2})\). Then

\[
\tau(e^{1}) < \varepsilon \quad \text{and} \quad a^{1/2}e \in L_\infty(\mathcal{M}).
\]

Therefore

\[
\left\| \left( T_z(x) - T_s(x) \right) e \right\|_\infty = C|z - s|\|au^{1/2}e\|_\infty \leq C|z - s|\|a^{1/2}e\|_\infty.
\]

Consequently,

\[
\lim_{z \to s} \left\| \left( T_z(x) - T_s(x) \right) e \right\|_\infty = 0.
\]

Namely, \(\lim_{z \to s} T_z(x) = T_s(x)\) almost uniformly. It then follows that \(\lim_{z \to 0} T_z(T_s(x)) = T_s(x)\) almost uniformly for all \(x \in L_2(\mathcal{M}) \cap L_p(\mathcal{M})\). Since the linear span of \(\{ T_s(x) : x \in L_2(\mathcal{M}) \cap L_p(\mathcal{M}), s > 0 \}\) is dense in \(L_p(\mathcal{M})\), our desired results then follows from Theorem 3.4.

Indeed, for \(x \in L_p(\mathcal{M})\) and \(\varepsilon > 0\), take a sequence \((x_n)\) in the span of \(\{ T_s(x) : x \in L_2(\mathcal{M}) \cap L_p(\mathcal{M}), s > 0 \}\) such that

\[
\|x_n - x\|_p < \frac{1}{2^n} \left( \frac{\varepsilon}{2^n} \right)^{\frac{1}{p}}.
\]

Let \(e_{1,n} = e_{(0,1/2^n)}(|x - x_n|)\) be the spectral projection of \(|x - x_n|\) on the interval \((0,1/2^n)\), then

\[
\tau(1 - e_{1,n}) \leq \frac{\|x - x_n\|_p^p}{(1/2^n)^p} = \frac{\varepsilon}{2^n} \quad \text{and} \quad \|x - x_n|e_{1,n}\|_\infty < \frac{1}{2^n}.
\]

Set \(e_1 = \bigwedge_n e_{1,n}\). We have

\[
(4.2) \quad \tau(1 - e_1) < \varepsilon \quad \text{and} \quad \|x - x_n|e_1\|_\infty < \frac{1}{2^n}.
\]

From inequality (3.2) in Theorem 3.4, we know that

\[
\| \sup_{z \in \Sigma_\psi} T_z(x - x_n) \|_p \leq C(p, \psi) \|x - x_n\|_p.
\]

That is, there are \(a, b \in L_{2p}(\mathcal{M})\) and \((c_z) \subset L_\infty(\mathcal{M})\) such that

\[
\sup_{z \in \Sigma_\psi} \|c_z\|_\infty \leq 1, \quad \|a\|_{2p} = \|b\|_{2p} \lesssim \|x - x_n\|_p^{\frac{1}{p}},
\]

and

\[
T_z(x - x_n) = ac_zb.
\]
Let $e_{2,n} = e_{(0,1/2)\mathbb{R}}(|a|) \wedge e_{(0,1/2)\mathbb{R}}(|b|)$, then
\[
\tau(1 - e_{2,n}) \leq \frac{\|a\|_{2p}^{2p}}{(1/2^n)^p} \lesssim \frac{\varepsilon}{2^n} \quad \text{and} \quad \|e_{2,n}T_z(x - x_n)e_{2,n}\|_{\infty} \leq \frac{1}{2^n}.
\]

Set $e_2 = \bigwedge_{n} e_{2,n}$. We have
\[
\tau(1 - e_2) \lesssim \varepsilon \quad \text{and} \quad \|e_2T_z(x - x_n)e_2\|_{\infty} \leq \frac{1}{2^n}.
\]

Since $\lim_{z \to 0} T_z(x_n) = x_n$ almost uniformly for all $n$, there is a projection $e_{3,n}$ such that
\[
\tau(1 - e_{3,n}) < \frac{\varepsilon}{2^n} \quad \text{and} \quad \lim_{z \to 0} \|(T_zx_n - x_n)e_{3,n}\|_{\infty} = 0.
\]

Let $e_3 = \bigwedge_{n} e_{3,n}$, then
\[
\tau(1 - e_3) < \varepsilon \quad \text{and} \quad \lim_{z \in \Sigma_\psi} \|(T_zx_n - x_n)e_3\|_{\infty} = 0.
\]

Take $e = e_1 \wedge e_2 \wedge e_3$, then
\[
\tau(e^\perp) \lesssim \varepsilon,
\]

and
\[
\|e(T_zx - x)e\|_{\infty} \leq \|e(T_zx - T_zx_n)e\|_{\infty} + \|e(T_zx_n - x_n)e\|_{\infty} + \|e(x_n - x)e\|_{\infty}.
\]

Thus it follows from formulas (4.2)-(4.4) that
\[
\lim_{z \to 0} \|e(T_zx - x)e\|_{\infty} = 0.
\]

Thus the first part of (1) is proved. The case for $p > 2$ can be similarly proved by using inequality (3.3). This completes the proof of (1).

(2) Now we turn to the second part of the theorem. First $L$ induces a canonical splitting on $L_p(M)$ for $1 < p < \infty : L_p(M) = N(L) \oplus R(L)$. Moreover, $N(L)$ is the fixed point subspace of the semigroup $(T_t)_{t>0}$. Thus it suffices to prove that for any $x \in R(L)$, $T_zx$ converge bilaterally almost uniformly to $P(x)$ for $1 < p \leq 2$ when $z \to \infty$. Using (3.2) in Theorem 3.4 as in the previous part of the proof, we need only to do this for $x$ in a dense subset of $R(L)$. It is well known that $\{T_{t+s}(y) - T_s(y) : s > 0, t > 0, y \in L_p(M)\}$ is such a subset. Thus we are reduced to prove the above convergence for $x = T_{t+s}(y) - T_s(y)$. Let $\tilde{\Gamma}$ be the boundary of $\Sigma_\psi$. Noting that $\|\lambda(\lambda - L)^{-1}\|$ is bounded on $\tilde{\Gamma}$, we have the integral representation of $T_z$
\[
T_z = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} e^{-z\lambda}(\lambda - L)^{-1}d\lambda, \quad z \in \Sigma_\psi.
\]

Then for $x = T_{t+s}(y) - T_s(y)$,
\[
T_z(x) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \left( e^{-(z+t+s)\lambda} - e^{-(z+s)\lambda} \right)(\lambda - L)^{-1}(y)d\lambda.
\]
Again by the convexity of the operator valued function: \( x \mapsto |x|^2 \),

\[
|T_z(x)|^2 \leq \frac{i^2}{4\pi^2} \int_{\Gamma} |e^{-2xz\lambda}||d\lambda| \int_{\Gamma} |e^{-2(t+s)\lambda}||(\lambda - L)^{-1}(y)|^2|d\lambda| 
= C \frac{1}{|z|} a,
\]

where \( C \) denotes a positive constant independent of \( z \) and \( a \) is a positive operator in \( L_1(\mathcal{M}) \).

The remaining part of the proof is similar to that of (1) and we omit it here. \( \square \)

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**References**

[1] Burkholder D., Martingales and singular integral in Banach spaces, Handbook of Geometry of Banach spaces, Vol.1: 233-269, North-Holland, 2001.

[2] Chilin V., Litvinov S., Skalski A., A few remarks in non-commutative ergodic theory, J. Operator Theory, 2005, 53: 331-350.

[3] Coifman R., Rochberg R., Weiss G., Applications of transference: the \( L_p \)-version von Neuman’s inequality and the Littlewood-Paly-Stein theory, Linear Spaces and Approximation, 53–67, Basel, 1978.

[4] Coifman R., Weiss G., Transference methods in analysis, CBMS regional conference series in mathematics, No.31, A.M.S., Providence, R.I. 1976.

[5] Cowling M., Harmonic analysis on semigroups, Ann. Math., 1983, 117: 267–283.

[6] Cowling M., Leinert M., Pointwise convergence and semigroups acting on vector-valued functions, Bull. Aust. Math. Soc., 2011, 84: 44-48.

[7] Dabrowski Y., A non-commutative path space approach to stationary free stochastic differential equations, arXiv:1006.4351.

[8] Dabrowski Y., A free stochastic partial differential equation, Ann. Inst. H. Poincar Probab. Statist., 2014, 50: 1404-1455.

[9] Defant A., Junge M., Maximal theorems of Menchoff-Rademacher type in noncommutative \( L_q \) spaces, J. Funct. Anal., 2004, 206: 322-355.

[10] Fendler G., On dilation and transference for continuos one-parameter semegroup of positive contraction on \( L^p \)-spaces, Ann. Univ. Sarav. Ser. Math., 1998, 9(1), iv+97 pp.

[11] Goldstein M., Litvinov S., Banach principle in the space of \( \tau \)-measurable operators, Studia Math., 2000, 143: 33-41.

[12] Hansen F., An operator inequality, Math. Ann., 1979/80, 246: 249-250.

[13] Hytönen T., Littlewood-Paley-Stein theory for semigroups in UMD spaces, Rev. Mat. Iberoam., 2007, 23: 973-1009.

[14] Junge M., Doob’s inequality for noncommutative martingale, J.Reine Angew. Math., 2002, 549: 149-190.

[15] Junge, M., Le Merdy, C., Xu, Q. \( H_\infty \) functional calculus and square functions on noncommutative \( L_p \)-spaces, Astisrique 305, 2006.

[16] Junge M., Mei T., Noncommutative Riesz transforms-a probabilistic approach, Amer. J. Math., 2010, 132: 611-681.
[17] Junge M., Mei T., BMO spaces associated with semigroups of operators, Math. Ann., 2012, 352: 691-743.

[18] Junge M., Ricard E., Shlyakhtenko D., Noncommutative diffusion semigroups and free probability, to appear.

[19] Junge M., Xu Q., Noncommutative maximal ergodic theorems, J. Amer. Math. Soc., 2007, 20(2): 385-439.

[20] Kriegler C. Analyticity angle for non-commutative diffusion semigroups, J. London Math. Soc., 2011, 83: 168-186.

[21] Kunstmann P., Štrkalj, Ž: $H^\infty$-calculus for submarkovian generators, Proc. Amer. Math. Soc., 2003, 131(7): 2081-2088.

[22] Lance E., Ergodic theorems for convex sets and operator algbras, Invent.Math., 1976, 37(3): 201-214.

[23] Martineza T., Torrea J., Xu Q., Vector-valued Littlewood-Paley-Stein theory for semigroups, Adv. Math., 2006, 203: 430-475.

[24] Mei T., Tent spaces associated with semigroup of operators, J. Funct. Anal., 2008, 255: 3356-3406.

[25] Mei T., Parcet J., Pseudo-localization of singular integral and noncommutative Littlewood-Paley inequalities, Int. Math. Res. Not., 2009, 8: 1433-1487.

[26] Parcet J., Pseudo-localization of singular integral and noncommutative Calderón-Zygmund theory, J. Funct. Anal., 2009, 256: 509-593.

[27] Pisier, G., Noncommutative vector-valued $L^p$-spaces and completely $p$-summing maps, C. R. Acad. Sci. Paris Sr. I Math., 1993, 316(10): 1055-1060.

[28] Pisier,G., Xu Q., Noncommutative $L^p$ spaces, Handbook of Geometry of Banach spaces, 2003, 2: 1459-1517.

[29] Stein, E., Topics in Harmonic analysis related to the Littlewood-Paley Theory, Annals of Mathematics Studies, No. 63, Princeton University Press, 1970.

[30] Taggart R., Pointwise convergence for semigroups in vector-valued $L_p$ spaces, Math. Z., 2009, 261: 933-949.

[31] Xu Q., $H^\infty$ functional calculus and maximal inequalities for semigroups of contractions on vector-valued $L_{p'}$-spaces, Int. Math. Res. Not., 2014, rnu104, 18 pages, doi:10.1093/imrn/rnu104.

[32] Zimmermann F., On vector-valued Fourier multiplier theorems, Studia Math., 1989, 93: 201-222.

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