A LARGE GAP IN A DILATE OF A SET

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Abstract. Let $A \subset \mathbb{F}_p$ with $|A| > 1$. We show there is a $d \in \mathbb{F}_p^\times$ such that $d \cdot A$ contains a gap of size at least $2p/|A| - 2$.

1. Introduction

Let $p$ be a prime and $A \subset \mathbb{F}_p$. We let $g(A)$ be the largest gap in $A$, that is the largest integer $g(A)$ such that there is a $t \in \mathbb{F}_p$ satisfying

$$(\{1, \ldots, g(A)\} + t) \cap A = \emptyset.$$ 

By the pigeon-hole principle,

$$g(A) \geq p/|A| - 1 \quad \ldots \quad (1)$$

For $d \in \mathbb{F}_p^\times$ we define

$$d \cdot A := \{da : a \in A\}.$$ 

We seek lower bounds for

$$L(A) := \frac{|A|}{p} \sup_{d \in \mathbb{F}_p^\times} g(d \cdot A).$$

Note $L(A)$ is translation and dilation invariant. By (1), we have $L(A) \geq 1 - |A|/p$. Our goal is to double this bound.

Theorem 1 Let $p$ be a prime and $A \subset \mathbb{F}_p$ with $|A| > 1$. Then

$$L(A) \geq 2(1 - \frac{|A|}{p}),$$

or equivalently

$$\sup_{d \in \mathbb{F}_p^\times} g(d \cdot A) \geq 2\left(\frac{p}{|A|} - 1\right).$$

We prove Theorem 1 using the polynomial method, or more precisely Redei’s method [Re73]. Our work bears some similarity to the recent work [BSW20]. As asked by Ben Green [Gr20+], it would be of interest to better understand $L(A)$, especially in the special case $|A| \sim \sqrt{p}$. In this case, is $L(A) \geq C$ for any fixed $C$? We remark that for $|A| \leq 1/100 \log p$ or $|A| \sim cp$, with $0 < c < 1$, one may improve upon Theorem 1 by applying Dirichlet’s box principle and Szemerédi’s theorem [Sz75], respectively.

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2. Proof of Theorem 1

Proof of Theorem 1. We apply the polynomial method. We may suppose $|A| < p$ and set

$$m = \sup_{d \in \mathbb{F}_p^*} g(d \cdot A) + 1.$$  

Since $|A| > 1$, we have $m < p$. Let $B = \{1, \ldots, m\}$ and

$$P = A \times B \subset \mathbb{F}_p^2.$$  

Thus the line $y = dx + t$ intersects $P$ for every $d \in \mathbb{F}_p^\times$ and $t \in \mathbb{F}_p$. Let

$$k := |A||B| - p + 1,$$

and

$$w(d, t) = d \prod_{a \in A, b \in B} (b + da + t).$$

It follows that $w$ vanishes on $\mathbb{F}_p^2$ and so by [A199, Theorem 1], we have

$$w(d, t) = (t^p - t)u(d, t) + (d^p - d)v(d, t),$$

for some $u, v$ of degree at most $k$. Taking the homogeneous part of degree $|B||A| + 1$ and setting $d = 1$, we find a $g$ and $h$ of degree at most $k$ such that

$$f(t) := \prod_{a \in A} (t + a)^{|B|} = t^p g(t) + h(t).$$

Then for every $a \in A$,

$$(t + a)^{|B|-1} f'(t), \quad f'(t) = t^p g'(t) + h'(t),$$

and so $(t + a)^{|B|-1}$ must also divide

$$(t^p g'(t) + h'(t))g(t) - (t^p g(t) + h(t))g'(t) = h'(t)g(t) - h(t)g'(t).$$

In other words,

$$\prod_{a \in A} (t + a)^{|B|-1},$$

divides a polynomial of degree at most $2k - 1$. We conclude $(|B| - 1)|A| \leq 2k - 1$, which implies

$$p \leq k + |A|, \quad (2)$$

or

$$h'(t)g(t) = h(t)g'(t). \quad (3)$$

We claim if $(3)$ holds with $\deg(g), \deg(h) < p$, then $g(t)$ and $h(t)$ have the same roots (with multiplicity) in an algebraic closure of $\mathbb{F}_p$. Indeed, suppose that $(t + \alpha)^d \mid g(t)$ but $(t + \alpha)^{d+1} \nmid g(t)$ for some $\alpha \in \overline{\mathbb{F}}_p$. Thus there is a $g_2 \in \overline{\mathbb{F}}_p[t] \setminus (t + \alpha)\overline{\mathbb{F}}_p[t]$ such that

$$g(t) = g_2(t)(t + \alpha)^d, \quad g'(t) = (t + \alpha)^{d-1}(dg_2(t) + (t + \alpha)g_2'(t)).$$

We have $(t + \alpha)^{d-1} \mid g'(t)$ and since $d < p$, we also have $(t + \alpha)^d \mid g'(t)$. Thus by $(3)$ we find $(t + \alpha)|h(t)$. Then we may let $g(t) = (t + \alpha)g_1(t)$ and $h(t) = (t + \alpha)h_1(t)$ for some $g_1, h_1 \in \overline{\mathbb{F}}_p[t]$. Substituting these into $(3)$ and simplifying reveals

$$h_1'(t)g_1(t)(t + \alpha)^2 = h_1(t)g_1'(t)(t + \alpha)^2.$$
Thus (3) is satisfied for $g_1(t)$ and $h_1(t)$. The claim follows by induction on $\deg(g)$.

Thus if $k < p$ and (3) holds then

$$h(t) = cg(t), \quad c \in \mathbb{F}_p,$$

and so

$$f(t) = (t^p + c)g(t) = (t + c)^p g(t).$$

Since $|B| = m < p$, this is impossible in light of

$$f(t) = \prod_{a \in A} (t + a)^{|B|}.$$  

Thus (2) holds or $k \geq p$ and in either case we find

$$|A||B| \geq 2p - |A| - 1$$

and so

$$\sup_{d \in \mathbb{F}_p^\times} g(d \cdot A) \geq 2p/|A| - 2 - 1/|A|.$$  

We now remove the $1/|A|$, using that the left hand side is an integer. If $|A| = 2$, then one easily checks that Theorem 1 holds. Otherwise $2 < |A| < p$ and so $|A| \nmid 2p$. Thus $2p/|A| - 2$ is not an integer. Since the fractional part of $2p/|A| - 2$ is at least $1/|A|$, we have that

$$\lceil 2p/|A| - 2 - 1/|A| \rceil \geq 2p/|A| - 2,$$

and so

$$\sup_{d \in \mathbb{F}_p^\times} g(d \cdot A) \geq 2p/|A| - 2.$$  

We remark the above proof fails for general point sets that are not necessarily cartesian products. Indeed, take a blocking set construction [Mo07, Page 107] in $\mathbb{P}^2(\mathbb{F}_p^2)$ of size $\sim 3/2p$. After projective transformation, make the line at infinity contain precisely one point, say $(1, 0, 0)$. Deleting this point creates a subset, $P$, of affine space such that every non-horizontal line intersects $P$. The proof does remain valid if we restrict to point sets $P$ such that $\pi(P)$ is small, where $\pi$ is a projection onto a coordinate axis.

References

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