ON THE ABSENCE OF MCSHANE-TYPE IDENTITIES FOR THE OUTER SPACE

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Abstract. A remarkable result of McShane states that for a punctured torus with a complete finite volume hyperbolic metric we have

\[ \sum_{\gamma} \frac{1}{e^{\ell(\gamma)} + 1} = \frac{1}{2} \]

where \( \gamma \) varies over the homotopy classes of essential simple closed curves and \( \ell(\gamma) \) is the length of the geodesic representative of \( \gamma \).

We prove that there is no reasonable analogue of McShane’s identity for the Culler-Vogtmann outer space of a free group.

1. Introduction

Let \( T \) be the one-punctured torus and let \( \rho \) be a complete finite-volume hyperbolic structure on \( T \). Let \( S \) be the set of all free homotopy classes of essential simple closed curves in \( T \) that are not homotopic to the puncture. Denote

\[ E(\rho) := \sum_{\gamma \in S} \frac{1}{e^{\ell_{\rho}(\gamma)} + 1}, \]

where \( \ell_{\rho}(\gamma) \) is the smallest \( \rho \)-length among all curves representing \( \gamma \). Thus \( E \) can be regarded as a function on the Teichmüller space of \( T \). A remarkable result of McShane [9] shows that this function is constant and that

\[(*) \quad E(\rho) = \frac{1}{2} \]

for every \( \rho \). We refer to (*) as McShane’s identity for \( T \). Since the thesis of McShane [9], other proofs of McShane’s identity for the punctured torus have been produced (particularly, see the work of Bowditch [3]) and McShane’s identity has been generalized to other hyperbolic surfaces and other contexts [4,10,11,12,13,14].

Note that if \( \psi \) is an element of the mapping class group of \( T \) then \( \psi \) permutes the elements of \( S \) and hence, clearly, \( E(\rho) = E(\psi\rho) \). Thus \( E \) obviously factors through to a function on the moduli space of \( T \) and (*) says that this function is identically equal to \( 1/2 \).

Let \( F_k = F(a_1, \ldots, a_k) \) be a free group of rank \( k \geq 2 \) with a free basis \( A = \{a_1, \ldots, a_k\} \). For \( F_k \) the best analogue of the Teichmüller space is the so-called Culler-Vogtmann outer space \( CV(F_k) \). Instead of actions on the hyperbolic plane the elements of the outer space are represented by minimal discrete isometric actions of \( F_k \) on \( \mathbb{R} \)-trees. Equivalently, one can think about a point of the outer space as being represented by a marked volume-one metric graph structure on \( F_k \), that is,

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an isomorphism $\phi : F_k \to \pi_1(\Gamma, p)$, where $\Gamma$ is a finite graph without degree-one
and degree-two vertices, equipped with a metric structure $L$ that assigns to each non-oriented
edge of $\Gamma$ a positive number called the length of this edge. The volume
of a metric structure on $\Gamma$ is the sum of the lengths of all non-oriented edges of
$\Gamma$. As we noted, the metric structures that appear in the description of the points
of the outer space, given above, are required to have volume equal to one. If
$(\phi : F_k \to \pi_1(\Gamma, p), L)$ represents a point of the outer space, the metric structure $L$
naturally lifts to the universal cover $\widehat{\Gamma}$, turning $\widehat{\Gamma}$ into an $\mathbb{R}$-tree $X$. The group $F_k$
acts on this $\mathbb{R}$-tree $X$ via $\phi$ by isometries minimally and discretely with the quotient
being equal to $\Gamma$. Similarly to the marked length spectrum in the Teichmüller space
context, a marked metric graph structure ($\phi : F_k \to \pi_1(\Gamma, p), L$) defines a hyperbolic
length function $\ell : C_k \to \mathbb{R}$ where $C_k$ is the set of all nontrivial conjugacy classes in
$F_k$. If $g \in F_k$, then $\ell([g])$ is the translation length of $g$ considered as the isometry of the
$\mathbb{R}$-tree $X$ described above. Alternatively, we can think about $\ell([g])$ as follows: $\ell([g])$ is the $L$-length of the shortest free homotopy representative of the curve $\phi(g)$ in $\Gamma$, that is, the $L$-length of the "cyclically reduced" form of $\phi(g)$ in $\Gamma$. Two volume-
one metric graph structures on $F_k$ represent the same point of $CV(F_k)$ if and only if their corresponding hyperbolic length functions are equal, or, equivalently, if the corresponding $\mathbb{R}$-trees are $F_k$-equivariantly isometric.

It is natural to ask if there is an analogue of McShane’s identity in the outer
space context. The (right) action of $\psi \in Out(F_k)$ on $CV(F_k)$ takes a hyperbolic
length function $\ell$ to $\ell \circ \psi$, that is, $\psi$ simply permutes the domain $C_k$ of $\ell$. Therefore
the real question, as in the Teichmüller space case, is if there is an analogue of
McShane’s identity for the moduli space $M_k = CV(F_k)/Out(F_k)$. The elements of
$M_k$ are represented by unmarked finite connected volume-one metric graphs $(\Gamma, L)$
without degree-one and degree-two vertices and with $\pi_1(\Gamma) \simeq F_k$.

To simplify the picture, and also since our results will be negative, we will consider a subset $\Delta_k$ of $CV(F_k)$ consisting of all volume-one metric structures on the
date W_k of $k$ circles wedged at a base-vertex $v_0$. We orient the circles and label
them by $a_1, \ldots, a_k$. This gives us an identification $\pi_1(W_k, v_0) = F(a_1, \ldots, a_k)$ of
$F_k = F(a_1, \ldots, a_k)$ with $\pi_1(W_k, v_0)$, so that indeed $\Delta_k \subseteq CV(F_k)$.

A volume-one metric structure $L$ on $W_k$ is a $k$-tuple $(L(a_1), \ldots, L(a_k))$ of positive
numbers with $\sum_{i=1}^k L(a_i) = 1$. Thus $\Delta_k$ has the natural structure of an open
$(k - 1)$-dimensional simplex in $\mathbb{R}^k$. As in the general outer space context, every
$L \in \Delta_k$ defines a hyperbolic length function $\ell_L : C_k \to \mathbb{R}$, where for $g \in F_k$ $\ell_L([g])$
is the $L$-length of the cyclically reduced form of $g$ in $F_k = F(a_1, \ldots, a_k)$. The open simplex $\Delta_k$ has a distinguished point $L_* := (\frac{1}{k}, \ldots, \frac{1}{k})$. Note that for every $[g] \in C_k$ we have $\ell_L([g]) = ||g||/k$, where $||g||$ is the cyclically reduced length of $g$
in $F_k = F(a_1, \ldots, a_k)$.

There is no perfect analogue for the notion of a simple closed curve in the free
group context. The closest such analogue is given by primitive elements, that is, elements belonging to some free basis of $F_k$. Let $P_k$ denote the set of conjugacy
classes of primitive elements of $F_k$. We will consider two versions of possible gen-
eralizations of McShane’s identity for free groups: the first involving all conjugacy
classes in $F_k$ and the second involving the conjugacy classes of primitive elements
of $F_k$. We will see that, under some reasonable assumptions, there are no analogues
of McShane’s identity in either context.
**Definition 1.1** (McShane-type functions on $\Delta_k$). Let $f : (0, \infty) \to (0, \infty)$ be a monotone non-increasing function and let $k \geq 2$. Define

$$C_f : \Delta_k \to (0, \infty], \quad C_f(\mathcal{L}) = \sum_{w \in \mathcal{C}_k} f(\ell_{\mathcal{L}}(w)) \quad \text{where } \mathcal{L} \in \Delta_k$$

and

$$P_f : \Delta_k \to (0, \infty], \quad P_f(\mathcal{L}) = \sum_{w \in \mathcal{P}_k} f(\ell_{\mathcal{L}}(w)) \quad \text{where } \mathcal{L} \in \Delta_k.$$

Obviously, $0 < P_f < C_f \leq \infty$ on $\Delta_k$.

Motivated by McShane’s identity, it is interesting to ask if there exist functions $f$ such that either $C_f$ or $P_f$ is constant on $\Delta_k$. To make the question meaningful we need to require $P_f$ (or, correspondingly, $C_f$) be finite at some point $\mathcal{L} \in \Delta_k$. Thus it is necessary to assume that $\lim_{x \to \infty} f(x) = 0$ and that this convergence to zero is sufficiently fast.

We establish the following negative results regarding the existence of analogues of McShane’s identity in the outer space context:

**Theorem A.** Let $k \geq 2$ be an integer and let $F = F(a_1, \ldots, a_k)$. Let $f : (0, \infty) \to (0, \infty)$ be a monotone non-increasing function such that:

1. $$\limsup_{x \to \infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.$$  
2. $$\liminf_{x \to \infty} f(x)^{1/x} > 0.$$  

Then:

(a) We have $P_f \leq C_f < \infty$ on some neighborhood $U$ on $\mathcal{L}_*$ in $\Delta_k$ (moreover, only the assumption (1) on $f$ above is required for this conclusion).

(b) We have $C_f \neq \text{const}$ on $\Delta_k$.

(c) If $k \geq 3$ then $P_f \neq \text{const}$ on $\Delta_k$.

The assumptions on $f(x)$ in Theorem A require $f(x)$ to decay both at least and at most exponentially fast; condition (1) assures that the value of $C_f$ is finite near $\mathcal{L}_*$. The idea of the proof of parts (b) and (c) of Theorem A uses the notion of volume entropy for a metric structure $\mathcal{L}$ on $W_k$ (see [6, 11, 8]). Roughly speaking, there are points $\mathcal{L}$ near the the boundary of $\Delta_k$ where the exponential growth rate, as $R \to \infty$, of the number of conjugacy classes with $\ell_\mathcal{L}$-length at most $R$ is bigger than the exponential rate of decay of the function $f$. This forces $C_f$ to be equal to $\infty$ at $\mathcal{L}$.

For $k = 2$ the set of conjugacy classes of primitive elements has quadratic rather than exponential growth. Therefore we modify the assumptions on $f(x)$ accordingly and obtain a somewhat stronger conclusion then in part (c) of Theorem A. For $k = 2$ the open 1-dimensional simplex $\Delta_2 \subseteq \mathbb{R}^2$ consists of all pairs $\mathcal{L}_t := (t, 1-t)$ where $t \in (0, 1)$. Therefore we may identify $\Delta_2$ with $(0, 1)$ and define $P_f(t) := P_f(\mathcal{L}_t)$. With this convention we prove:

**Theorem B.** Let $k = 2$ and $F = F(a, b)$. Let $f : (0, \infty) \to (0, \infty)$ be a monotone non-increasing function such that:

1. We have $f''(x) > 0$ for every $x > 0$.
2. There is some $\epsilon > 0$ such that $\lim_{x \to \infty} x^{3+\epsilon} f(x) = 0$. 


Then the following hold:

(a) We have \(0 < P_f(t) < \infty\) for every \(t \in (0, 1)\).

(b) The function \(P_f(t)\) is strictly convex on \((0, 1)\) and achieves a unique minimum at \(t_0 = 1/2\). In particular, \(P_f(t)\) is not a constant locally near \(t_0 = 1/2\) and thus \(P_f \neq \text{const}\) on \((0, 1)\).

The proof of Theorem B uses convexity considerations as well as some results about the explicit structure of primitive elements in \(F(a, b)\) \([5, 12]\).

Finally, we combine the volume entropy and the convexity ideas to obtain:

**Theorem C.** Let \(k \geq 2\) and let \(f : (0, \infty) \to (0, \infty)\) be a monotone decreasing function such that the following hold:

1. The function \(f(x)\) is strictly convex on \((0, \infty)\).
2. \(\limsup_{x \to \infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}\).

Then there exists a convex neighborhood \(U\) of \(L^*\) in \(\Delta_k\) such that \(0 < P_f < \infty\) on \(U\) and both \(C_f\) and \(P_f\) are strictly convex on \(U\). In particular, \(C_f \neq \text{const}\) on \(U\) and \(P_f \neq \text{const}\) on \(U\).

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## 2. Volume Entropy

In this section we will prove Theorem A, which is obtained as a combination of Theorem 2.4 and Theorem 2.5 below.

**Convention 2.1.** For the remainder of this section let \(k \geq 2\) be an integer and \(F_k = F(a_1, \ldots, a_k)\) be free of rank \(k\) with a free basis \(A = \{a_1, \ldots, a_k\}\). We identify \(F_k\) with \(\pi_1(W_k, v_0)\), as explained in the introduction. For \(g \in F_k\) we denote by \(|g|\) the freely reduced length of \(g\) with respect to \(A\) and we denote by \(||g||\) the cyclically reduced length of \(g\) with respect to \(A\).

We denote by \(CR_k\) the set of all cyclically reduced elements of \(F_k\) with respect to \(A\).

Let \(L\) be a metric graph structure on \(W_k\). For every \(g \in F_k\) there is a unique edge-path in \(W_k\) labelled by the freely reduced form of \(g\) with respect to \(A\). We denote the \(L\)-length of that path by \(L(g)\). As before, we denote by \(\ell_L : C_k \to \mathbb{R}\) the hyperbolic length function corresponding to \(L\). Thus if \(g \in F_k\) then \(\ell_L([g]) = L(u)\) where \(u\) is the cyclically reduced form of \(g\) with respect to \(A\).

**Definition 2.2 (Volume Entropy).** Let \(L\) be a metric structure on \(W_k\). The volume entropy \(h_L\) of \(L\) is defined as

\[
h_L = \lim_{R \to \infty} \frac{\log \# \{ g \in F_k : L(g) \leq R \}}{R}.
\]

It is well-known and easy to see that the limit in the above expression exists and is finite. We refer the reader to \([6, 11, 8]\) for a detailed discussion of volume entropy in the context of metric graphs.

**Proposition 2.3.** Let \(k \geq 2\) and \(L\) be as in definition 2.2.

Then the limits

\[
h'_L = \lim_{R \to \infty} \frac{\log \# \{ g \in CR_k : L(g) \leq R \}}{R},
\]

exist and are finite. We refer the reader to \([6, 11, 8]\) for a detailed discussion of volume entropy in the context of metric graphs.
and

\[
h''_L = \lim_{R \to \infty} \frac{\log \# \{ w \in \mathcal{C}_k : \ell_L(w) \leq R \}}{R}.
\]

exist and

\[h_L = h'_L = h''_L.\]

\[\text{Proof.}\] Let \(M := \max\{|a_i|_\mathcal{L} : i = 1, \ldots, k\}\) and \(m := \min\{|a_i|_\mathcal{L} : i = 1, \ldots, k\}\).

For each \(g \in F\) there exists a cyclically reduced word \(v_g\) such that \(|g| = |v_g|\) and such that \(g\) and \(v_g\) agree except possibly in the last letter. Then \(|\mathcal{L}(g) - \mathcal{L}(v_g)| \leq M\). Moreover, the function \(F_k \to CR_k, g \mapsto v_g\) is at most \(2k\)-to-one. Therefore for every integer \(R > 0\)

\[\# \{ g \in CR_k : \mathcal{L}(g) \leq R \} \leq \# \{ g \in F_k : \mathcal{L}(g) \leq R \} \leq 2k \# \{ g \in CR_k : \mathcal{L}(g) \leq R + M \}\]

and

\[\# \{ w \in \mathcal{C}_k : \ell_L(w) \leq R \} \leq \# \{ g \in CR_k : \mathcal{L}(g) \leq R \} \leq \frac{R}{m} \# \{ w \in \mathcal{C}_k : \ell_L(w) \leq R \}.\]

This implies the statement of the proposition. \(\square\)

**Theorem 2.4.** Let \(k \geq 2\) be an integer and let \(F = F(a_1, \ldots, a_k)\). Let \(f : (0, \infty) \to (0, \infty)\) be a monotone non-increasing function such that:

1. \[\limsup_{x \to \infty} f(x)^{1/x} < \frac{1}{(2k-1)^k}.\]
2. \[\liminf_{x \to \infty} f(x)^{1/x} > 0.\]

Then:

(a) We have \(0 < C_f < \infty\) on some neighborhood \(U\) on \(\mathcal{L}_*\) in \(\Delta_k\) (moreover, only the assumption (1) on \(f\) above is required for this conclusion).

(b) We have \(C_f \neq \text{const on } \Delta_k\).

\[\text{Proof.}\] The assumptions on \(f(x)\) imply that there exist \(N > 0\) and \(0 < \sigma_1 < \sigma_2 < \frac{1}{(2k-1)^k}\) such that for every \(x \geq N\)

\[\sigma_1 x \leq f(x) \leq \sigma_2 x.\]

Let \(\mathcal{L}_* = \left(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_k}\right) \in \Delta_k\). For any \(g \in F_k\) we have \(\mathcal{L}_*(g) = |g|/k\). Then an easy direct computation shows that \(h_{\mathcal{L}_*} = k \log(2k-1)\), so that \(e^{h_{\mathcal{L}_*}} = (2k-1)^k < \frac{1}{\sigma_2}\).

Since the volume entropy \(h\) is a continuous function on \(\Delta_k\) (see, for example, [6]), there exist a neighborhood \(U\) of \(\mathcal{L}_*\) in \(\Delta_k\) and \(0 < c < \frac{1}{\sigma_2}\) such that \(e^{h_{\mathcal{L}}} < c\) for every \(\mathcal{L} \in U\).

Observe now that \(C_f < \infty\) on \(U\). Let \(\mathcal{L} \in U\) be arbitrary. There there exist \(M > 0\) and \(c_1\) with \(c < c_1 < \frac{1}{\sigma_2}\) such that for every integer \(R > 0\) we have

\[\# \{ w \in \mathcal{C}_k : \ell_\mathcal{L}(w) \leq R \} \leq M \sigma_1 R.\]
Therefore

\[ C_f(L_t) = \sum_{w \in C_k} f(\ell_L(w)) = \sum_{i=0}^{\infty} \sum_{w \in C_k, i < \ell_L(w) \leq i+1} f(\ell_L(w)) \leq \sum_{i=0}^{\infty} \sum_{w \in C_k, i < \ell_L(w) \leq i+1} f(i) \leq \sum_{i=0}^{\infty} M c_i^{i+1} f(i) < \infty \]

where the last inequality holds since \( c_1 < \frac{1}{\sigma_2} \) and \( f(x) \leq \sigma_2^2 \) for all \( x \geq N \). Thus indeed \( C_f(L) < \infty \), so that \( C_f < \infty \) on \( U \).

For \( 0 < t < \frac{1}{(k-2)t} \) let \( L_t = (t, t, \ldots, t, 1 - (k - 1)t) \in \Delta_k \). Then, as follows from the proof of Theorem B of [6] (specifically the proof of Theorem 9.4 on page 25 of [6]),

\[ \lim_{t \to 0} h_{L_t} = \infty. \]

Indeed, let \( \Gamma \) be the subgraph of \( W_k \) consisting of the loops labelled by \( a_1 \) and \( a_k \). The restriction \( L_t' \) of \( L_t \) to \( \Gamma \) is a metric structure on \( \Gamma \) of volume \( 1 - (k - 2)t \).

Therefore \( \frac{1}{1 - (k - 2)t} L_t' \) is a volume-one metric structure on \( \Gamma \) with respect to which the length of \( a_1 \) goes to 0 as \( t \to 0 \). Therefore, as established in the proof of Theorem 9.4 of [6],

\[ \lim_{t \to 0} h_{L_t'} = \infty. \]

However,

\[ h_{L_t'} = \frac{1}{1 - (k - 2)t} h_{\frac{1}{1 - (k - 2)t} L_t} \]

and therefore

\[ \lim_{t \to 0} h_{L_t'} = \infty. \]

It is obvious from the definition of volume entropy that \( h_{L_t'} \leq h_{L_t} \) and hence

\[ \lim_{t \to 0} h_{L_t} = \infty, \]

as claimed.

Hence there exists \( t_0 \in (0, \frac{1}{(k-2)t}) \) such that for every \( t \in (0, t_0) \) we have \( e^{h_{L_t'}} > \frac{1}{\sigma_1} + 2 \). Let \( t \in (0, t_0) \) be arbitrary. We claim that \( C_f(L_t) = \infty \).

Since \( e^{h_{L_t'}} > \frac{1}{\sigma_1} + 2 \), by Proposition 2.3 there is \( R_0 > N > 0 \) such that for every \( R \geq R_0 \) we have

\[ \# \{ w \in C_k : L_t(w) \leq R \} \geq (\frac{1}{\sigma_1} + 1)^R. \]

For every \( R \geq R_0 \)

\[ C_f(L_t) = \sum_{w \in C_k} f(L_t(w)) \geq \sum_{w \in C_k, L_t(w) \leq R} f(L_t(w)) \geq \sum_{w \in C_k, L_t(w) \leq R} f(R) \geq (\frac{1}{\sigma_1} + 1)^R f(R) \geq (\frac{1}{\sigma_1} + 1)^R \sigma_1^R = (1 + \sigma_1)^R. \]

Since this is true for every \( R \geq R_0 \), it follows that \( C_f(L_t) = \infty \).

Thus \( C_f(L) < \infty \) while \( C_f(L_t) = \infty \) for all sufficiently small \( t > 0 \). Therefore \( C_f \neq const \) on \( \Delta_k \). \( \square \)
Theorem 2.5. Let \( k \geq 3 \) be an integer and let \( F = F(a_1, \ldots, a_k) \). Let \( f : (0, \infty) \to (0, \infty) \) be as in Theorem 2.4.

Then:

(a) We have \( P_f \leq C_f < \infty \) on some neighborhood \( U \) on \( L_* \) in \( \Delta_k \).

(b) We have \( P_f \neq \text{const} \) on \( \Delta_k \).

Proof. Again, by assumptions on \( f(x) \), there exist there exist \( N > 0 \) and \( 0 < \sigma_1 < \sigma_2 < \frac{1}{(2k-1)^{k+1}} \) such that for every \( x \geq N \)

\[
\sigma_1^x \leq f(x) \leq \sigma_2^x.
\]

By Definition \( 0 \leq P_f \leq C_f \). By Theorem 2.4 we have \( C_f < \infty \) on some neighborhood \( U \) on \( L_* \) in \( \Delta_k \) and hence \( P_f \leq C_f < \infty \) on \( U \).

Put \( F_{k-1} := F(a_1, \ldots, a_{k-1}) \) so that \( F_k = F_{k-1} * (a_k) \). For \( 0 < t < \frac{1}{k-2} \) let

\[
L_t := \left( \frac{t}{2}, \frac{t}{2}, \ldots, \frac{t}{2}, \frac{1}{2} - (k-2) \frac{t}{2} \right) \in \Delta_k
\]

and

\[
\hat{L}_t := \left( \frac{t}{2}, \frac{t}{2}, \ldots, \frac{t}{2}, \frac{1}{2} - (k-2) \frac{t}{2} \right) \in \frac{1}{2} \Delta_{k-1}.
\]

Thus \( 2\hat{L}_t \in \Delta_{k-1} \) is a volume-one metric structure on \( W_{k-1} \). Since \( k \geq 3 \), we have \( k-1 \geq 2 \) and hence, exactly as in the proof of Theorem 2.4 \( \lim_{t \to 0} h_{2\hat{L}_t} = \infty \).

For \( R \geq 1 \)

\[
b_{R,t} := \# \{ g \in F_{k-1} : \hat{L}_t(g) \leq R \}.
\]

Then

\[
s_t := \lim_{R \to \infty} \frac{\log b_{R,t}}{R} = h_{\hat{L}_t} = 2h_{2\hat{L}_t}.
\]

and therefore

\[
\lim_{t \to 0} s_t = \infty.
\]

Hence there exists \( 0 < t_0 < \frac{1}{k-2} \) such that for every \( t \in (0, t_0) \) we have \( e^{s_t} > \frac{1}{\sigma_1} + 2 \).

Fix an arbitrary \( t \in (0, t_0) \). Since \( e^{s_t} > \frac{1}{\sigma_1} + 2 \), by there is \( R_0 > N > 0 \) such that for every \( R \geq R_0 \) we have

\[
b_{R,t} = \# \{ g \in F_{k-1} : \hat{L}_t(g) \leq R \} \geq \left( \frac{1}{\sigma_1} + 1 \right)^R.
\]

Note that for every \( g \in F_{k-1} \) the element \( ga_k \in F_k \) is primitive in \( F \). Moreover, if \( g_1 \neq g_2 \) are distinct elements of \( F_{k-1} \) then \( g_1a_k \) and \( g_2a_k \) are not conjugate in \( F_k \).

Recall that by definition of \( L_t \) we have \( L_t(a_k) = \frac{1}{2} \). For \( R \geq 1 \) denote

\[
p_{R,t} := \# \{ w \in P_k : \ell_{L_t}(w) \leq R \}.
\]

Then for every \( R \geq R_0 + \frac{1}{2} \) we have

\[
p_{R,t} \geq b_{R-\frac{1}{2},t} \geq \left( \frac{1}{\sigma_1} + 1 \right)^{R-\frac{1}{2}}.
\]
Theorem 3.5. Let \( f \) be monotone decreasing on \((0, \infty)\). Hence for every \( R \geq R_0 + \frac{1}{2} \)
\[
P_f(L_t) = \sum_{w \in \mathcal{P}_k} f(L_t(w)) \geq \sum_{w \in \mathcal{P}_k, L_t(w) \leq R} f(L_t(w)) \geq \sum_{w \in \mathcal{P}_k, L_t(w) \leq R} f(R) \geq \left( \frac{1}{\sigma_1} + 1 \right)^{R-\frac{1}{2}} f(R) \geq \left( \frac{1}{\sigma_1} + 1 \right)^{R-\frac{1}{2}} \sigma_1^R = (1 + \sigma_1)^R \left( \frac{1}{\sigma_1} + 1 \right)^{-\frac{1}{2}}.
\]

Since this is true for every \( R \geq R_0 + \frac{1}{2} \), it follows that \( P_f(L_t) = \infty \).

Thus \( P_f(L_\infty) < \infty \) while \( P_f(L_t) = \infty \) for all sufficiently small \( t > 0 \). Therefore \( P_f \neq \text{const} \) on \( \Delta_k \).

\[\square\]

3. Primitive elements in \( F(a,b) \)

In this section we will prove Theorem \( \mathbb{E} \).

Convention 3.1. Throughout this section let \( F_2 = F(a,b) \) be a free group of rank two.

Let \( \alpha : F(a,b) \to \mathbb{Z}^2 \) be the abelianization homomorphism, that is, \( \alpha(a) = (1,0) \) and \( \alpha(b) = (0,1) \). Then \( \alpha \) is constant on every conjugacy class and therefore \( \alpha \) defines a map \( \beta : \mathbb{C}_2 \to \mathbb{Z}^2 \).

Definition 3.2 (Visible points). A point \((p,q) \in \mathbb{Z}^2 \) is called visible if \( \gcd(p,q) = 1 \). We denote the set of all visible points in \( \mathbb{Z}^2 \) by \( V \).

We will need the following known facts about primitive elements in \( F(a,b) \) (see, for example, \( [5, 12] \)):

Proposition 3.3. The following hold:

1. The restriction of \( \beta \) to \( \mathcal{P}_2 \) is a bijection between \( \mathcal{P}_2 \) and the set of visible elements \( V \subseteq \mathbb{Z}^2 \).
2. Let \( w \in F(a,b) \) be a cyclically reduced primitive element and let \( \alpha(w) = (p,q) \in \mathbb{Z}^2 \).

Then every occurrence of \( a \) in \( w \) has the same sign (either \(-1,0 \) or \( 1 \)) as \( p \) and every occurrence of \( b \) in \( w \) has the same sign (again either \(-1,0 \) or \( 1 \)) as \( q \). Thus the total number of occurrences of \( a^{\pm 1} \) in \( w \) is equal to \( |p| \) and the total number of occurrences of \( b^{\pm 1} \) in \( w \) is equal to \( |q| \).

Definition 3.4 (Admissible function). We say that a function \( f : (0, \infty) \to [0, \infty) \) is admissible if it satisfies the following conditions:

1. We have \( f''(x) > 0 \) for every \( x > 0 \).
2. There is some \( \epsilon > 0 \) such that \( \lim_{x \to \infty} x^{3+\epsilon} f(x) = 0 \).

The second condition means that \( f(x) \) converges to zero asymptotically faster than \( \frac{1}{x^\epsilon} \) as \( x \to \infty \). Note that an admissible function must be strictly positive and monotone decreasing on \((0, \infty)\).

Theorem 3.5. Let \( f \) be any admissible function. Then the following hold:

1. We have \( 0 < P_f(t) < \infty \) for every \( t \in (0,1) \).
2. The function \( P_f(t) \) is strictly convex on \((0,1)\) and achieves a unique minimum at \( t = 1/2 \). In particular, \( P_f(t) \) is not a constant locally near \( t = 1/2 \).
Moreover, conclude that because of condition (2) in the definition of admissibility of \( f \)
for every \( \ell \), we have
\[
\ell_{V_r}(w) = t|p| + (1 - t)|q|.
\]

Let \( V' := \{(p, q) \in V : |p| > |q|\} \). Then we have
\[
\sum_{(p, q) \in V'} f(t|p| + (1 - t)|q|) = \sum_{(p, q) \in V'} g_{p,q}(t).
\]

Fix some \( t \in (0, 1) \). We can also represent \( P_f(t) \) as
\[
P_f(t) = \sum_{N=1}^{\infty} \sum_{(p, q) \in V, \max\{|p|, |q|\} = N} f(t|p| + (1 - t)|q|).
\]

Since \( f(x) \) is a monotone non-increasing function, if \( (p, q) \in V, \max\{|p|, |q|\} = N \), we have
\[
f(t|p| + (1 - t)|q|) \leq \min\{f(tN), f((1 - t)N)\} = f(cN)
\]
where \( c = \max\{t, 1 - t\} \). For every integer \( N \geq 1 \) the number of points \( (p, q) \in \mathbb{Z}^2 \)
with \( |p| \leq N, |q| \leq N \) is \((2N + 1)^2\).

Therefore
\[
P_f(t) \leq \sum_{N=1}^{\infty} \sum_{(p, q) \in V, \max\{|p|, |q|\} = N} f(cN) \leq \sum_{N=1}^{\infty} (2N + 1)^2 f(cN) < \infty.
\]
because of condition (2) in the definition of admissibility of \( f(x) \). Thus \( 0 < P_f(t) < \infty \) for every \( t \in (0, 1) \).

Note that for each \( (p, q) \in V' \)
\[
g_{p,q}'(t) = f'(t|p| + (1 - t)|q|)(|p| - |q|) + f'(t|q| + (1 - t)|p|)(|q| - |p|)
\]
\[
g_{p,q}''(t) = f''(t|p| + (1 - t)|q|)(|p| - |q|)^2 + f''(t|q| + (1 - t)|p|)(|q| - |p|)^2.
\]

Since \( |p| > |q| \) and, by definition of admissibility, \( f''(x) > 0 \) for every \( x \in \mathbb{R} \), we conclude that \( g_{p,q}''(t) > 0 \) for every \( t \in (0, 1) \). Hence \( g_{p,q} \) is strictly convex on \((0, 1)\).

Moreover,
\[
g_{p,q}'(t) = f'\left(\frac{|p|}{2} + \frac{|q|}{2}\right)(|p| - |q|) + f'\left(\frac{|q|}{2} + \frac{|p|}{2}\right)(|q| - |p|) = 0.
\]

Since \( g_{p,q}' > 0 \) on \((0, 1)\), it follows that \( g_{p,q} \) is strictly convex on \((0, 1)\) and achieves
a unique minimum on \((0, 1)\) at \( t = \frac{1}{2} \).
Since 0 < \( P_f < \infty \) on (0, 1) and \( P_f = \sum_{(p,q) \in V'} g_{p,q} \), it also follows that \( P_f \) is strictly convex on (0, 1) and achieves a unique minimum on (0, 1) at \( t = \frac{1}{2} \). \( \square \)

4. Exploiting convexity

In this section we combine the ideas of the previous two sections and establish Theorem C from the introduction.

**Theorem 4.1.** Let \( k \geq 2 \) and let \( f : (0, \infty) \to (0, \infty) \) be monotone decreasing function such that the following hold:

(1) The function \( f(x) \) is strictly convex on (0, \( \infty \)).

(2) \( \limsup_{x \to \infty} f(x)^{1/x} < \frac{1}{(2k-1)^k} \).

Then there exists a convex neighborhood \( U \) of \( L \) in \( \Delta_k \) such that 0 < \( P_f < C_f < \infty \) on \( U \) and both \( C_f \) and \( P_f \) are strictly convex on \( U \). In particular, \( C_f \neq \text{const} \) on \( U \) and \( P_f \neq \text{const} \) on \( U \).

**Proof.** By Theorem 2.4 there exists a convex neighborhood \( U \) of \( L \) in \( \Delta_k \), such that 0 < \( P_f < C_f < \infty \) on \( U \). We will prove that \( P_f \) and \( C_f \) are strictly convex on \( U \).

Let \( D \) be the set of all \( k \)-tuples of integers \( m = (m_1, \ldots, m_k) \) such that \( m_i \geq 0 \) for \( i = 1, \ldots, k \) and \( m_1 + \cdots + m_k > 0 \). For each \( m = (m_1, \ldots, m_k) \in D \) let \( Q_m \) be the set of all \( w \in \mathcal{C}_k \) such that \( w \) involves exactly \( m_i \) occurrences of \( a_i^{\pm 1} \) for \( i = 1, \ldots, k \) and let \( q_m := \#(Q_m) \). Note that for every \( w \in Q_m \), if \( L = (x_1, \ldots, x_k) \in \Delta_k \), then we have

\[ \ell_L(w) = m_1 x_1 + \cdots + m_k x_k. \]

Denote by \( f_m : \Delta_k \to \mathbb{R} \) the function defined as

\[ f_m(x_1, \ldots, x_k) := f(m_1 x_1 + \cdots + m_k x_k), \quad (x_1, \ldots, x_k) \in \Delta_k. \]

The function \( f(x) \) is convex on (0, \( \infty \)) and the function \( (x_1, \ldots, x_k) \mapsto m_1 x_1 + \cdots + m_k x_k \) is linear on \( \Delta_k \). Therefore \( f_m \) is convex on \( \Delta_k \).

Then for any \( L = (x_1, \ldots, x_k) \in \Delta_k \) we have

\[ C_f(L) = \sum_{m \in D} q_m f_m(\ell_L). \]

Since each \( f_m \) is convex on \( \Delta_k \), it follows that \( C_f \) is convex on \( \Delta \). We claim that \( C_f \) is strictly convex on \( U \). Let \( D_1 \) be the subset of \( D \) consisting of all the \( k \)-tuples having a single nonzero entry equal to 1, that is, \( D_1 \) is the union of the \( k \) standard unit vectors in \( \mathbb{Z}^k \). Let \( m_i = (0, \ldots, 1, \ldots, 0) \in D_1 \) where 1 occurs in the \( i \)-th position. Then \( Q_{m_i} = \{ [m_i], [a_i^{-1}] \} \) and \( q_{m_i} = 2 \). Also, \( f_m(x_1, \ldots, x_k) = f(x_i) \) for every \( (x_1, \ldots, x_k) \in \Delta_k \).

Put \( g := f_{m_1} + \cdots + f_{m_k} : \Delta_k \to \mathbb{R} \), so that

\[ g(x_1, \ldots, x_k) = f(x_1) + \cdots + f(x_k), \quad (x_1, \ldots, x_k) \in \Delta_k. \]

It is easy to see that \( g \) is strictly convex on \( \Delta_k \) since \( f \) is strictly convex on (0, \( \infty \)). We have:

\[ C_f = \sum_{m \in D} q_m f_m = 2g + \sum_{m \in D - D_1} q_m f_m. \]
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Since \( C_f < \infty \) on a convex set \( U \) and since \( g \) is strictly convex on \( U \) and \( \sum_{m \in D - D_1} q_m f_m \) is convex on \( U \), it follows that \( C_f \) is strictly convex on \( U \) as claimed.

The proof that \( P_f \) is strictly convex on \( U \) is exactly the same as for \( C_f \) above. The only change that needs to be made is to re-define \( Q_m \) for each \( m = (m_1, \ldots, m_k) \in D \) as the set of all \( w \in P_k \) such that \( w \) involves exactly \( m_i \) occurrences of \( a_i^{\pm 1} \) for \( i = 1, \ldots, k \).

□

Remark 4.2. Let \( Z_k \) be the set of all root-free conjugacy classes \( w \in C_k \), that is, conjugacy classes of nontrivial elements of \( F_k \) that are not proper powers. It is not hard to show, similar to Proposition 2.3, that if \( L \) is a metric structure on \( W_k \) then

\[
 h_L = \tilde{h}_L
\]

where

\[
 \tilde{h}_L := \lim_{R \to \infty} \frac{\log \# \{ w \in Z_k : \ell_L(w) \leq R \} \cdot R}{R}.
\]

If one now re-defines the McShane function \( C_f \) as \( S_f \):

\[
 S_f : \Delta_k \to (0, \infty], \quad S_f(L) = \sum_{w \in Z_k} f(\ell_L(w)) \quad \text{where } L \in \Delta_k,
\]

then the proofs of the parts of Theorem \( \Box \) and Theorem \( \Box \) dealing with \( C_f \) go through verbatim for \( S_f \).

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