Equivalence of Two Methods to Solve Static Electromagnetic Potentials

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In electromagnetic statics, the standard procedure to determine the electric scalar potential or magnetic vector potential in a bounded space is to solve Poisson’s equation subject to certain boundary conditions. On the other hand, as a direct generalization of Coulomb’s law or Biot-Savart law, the static electromagnetic potentials may also be obtained by directly integrating the electric charge or current distributions over the region (either volume or surface) where they are spread out. What is the relation between these two formalisms? In this article, we prove that they are in fact equivalent to each other in mathematics. Examples are also presented to explicitly show the validity of this equivalence.

I. INTRODUCTION

In the theory of electrostatics, Coulomb’s law indicates that the electric potential \( \phi(\mathbf{x}) \) in free space can be determined up to a constant provided the charge distribution \( \rho \)

\[
\phi(\mathbf{x}) = \frac{1}{4\pi \varepsilon} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV'
\]

(1)

where \( \mathbf{x} (\mathbf{x}') \) is the field (source) point, and \( \varepsilon \) is the dielectric constant. By applying the fact \( \nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} = -4\pi \delta(\mathbf{x} - \mathbf{x}') \), it can be shown that Eq. (1) is the solution to Poisson’s equation

\[
\nabla^2 \phi = -\frac{\rho}{\varepsilon}.
\]

(2)

Several preconditions are needed to justify the above result: There is only one kind of material (including the vacuum) of permittivity \( \varepsilon \) filling in the whole space, and the electric charge density \( \rho \) is well behaved. In the realistic world, there exist various materials which have various shapes of boundaries. The find the electric potential, the routine method is to solve Poisson’s equation (2) in confined spaces with respect to certain boundary conditions, which is discussed in any text book on electrodynamics.

If a boundary is electrically charged, the surface charge density, which can be thought of as a special type of volume charge density, can be determined up to a constant provided the charge distribution \( \sigma \)

\[
\phi(\mathbf{x}) = \frac{1}{4\pi \varepsilon} \int_{S'} \int dV' \frac{\sigma(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \delta(\mathbf{x}' - \mathbf{x})
\]

(3)

where \( \mathbf{x} \) is the normal vector of \( S' \), pointing from region 1 to region 2. To solve this boundary value problem, we expand \( \phi \) for each term in the linear expansion can be determined. In this way, the electric potential is fully solved.

II. PROOF OF EQUIVALENCE

A. Electrostatic Field

For convenience, we refer the method by solving Poisson’s equation as the routine method. The second method is to sum all contributions to the electric potential produced by both volume (Eq. (1)) and surface (Eq. (3)) charge densities, and we refer it as the integral method. Without loss of generality, we consider a simple space configuration where there exists only one closed boundary \( S' \) that separates the space into two parts, labelled by 1 (inside) and 2 (outside) respectively. Let the volume charge density be uniformly described by \( \rho \). By the routine method, the electric potential is obtained via solving Poisson’s equation

\[
\left\{ \begin{array}{ll}
\nabla^2 \phi_1 &= -\frac{\rho}{\varepsilon}, & \text{inside } S', \\
\nabla^2 \phi_2 &= -\frac{\rho}{\varepsilon}, & \text{outside } S',
\end{array} \right.
\]

(4)

under the boundary condition

\[
\phi_1|_{S'} = \phi_2|_{S'},
\]

(5)

\[
\varepsilon \hat{n} \cdot (\nabla \phi_1|_{S'} - \nabla \phi_2|_{S'}) = -\sigma|_{S'}.
\]

(6)

Here \( \hat{n} \) is the normal vector of \( S' \), pointing from region 1 to region 2. To solve this boundary value problem, we expand \( \phi_{1,2} \) by a complete and orthogonal set of functions, usually the solutions of the associated Laplace’s equation \( \nabla^2 \phi_{1,2} = 0 \). Next, by applying boundary conditions, the coefficients of each term in the linear expansion can be determined. In this way, the electric potential is fully solved.
By the integral method, the electric potential in each region can be uniformly expressed as
\[
\phi(x) = \frac{1}{4\pi \varepsilon} \int dV \frac{\rho(x')}{|x - x'|} + \frac{1}{4\pi \varepsilon} \oint_{S'} dS' \frac{\sigma(x')}{|x - x'|} \tag{7}
\]
if all kinds of charge distributions are given. To show the equivalence of the two methods, we only need to verify that \(\phi(x')\) is inside the boundary (see Fig.1), a point can be simply represented by \(\phi(x') = \phi_1(x')\) for the special case where \(\sigma = 0\) and \(\phi_1 = 0\). The uniqueness theorem then guarantees that it is the only solution. We first prove this for generic situations, then give a more pedagogical discussion for the special case where \(S'\) is a spherical surface.

Figure 1: A thin Gaussian pillbox on the surface \(S'\).

**Proof:**

The Poisson’s equation (3) is satisfied by plugging in Eq.(7)
\[
\nabla^2 \phi(x) = -\frac{\rho(x)}{\varepsilon} - \frac{1}{\varepsilon} \int_{S'} dS' \sigma(x') \delta(x - x') = -\frac{\rho(x)}{\varepsilon}. \tag{8}
\]
Here the surface integral vanishes since \(x \neq x'\). The boundary \(S\) is illustrated in Fig.1. In spherical coordinates, a point on \(S\) is denoted by \(x' = (r', \theta', \phi')\), where \(r' = |x'|\) and \(\theta'\) and \(\phi'\) are the polar and azimuthal angles. The surface \(S\) can be simply represented by \(r'(\theta', \phi')\), i.e. \(r' = f(\theta', \phi')\). The first boundary condition, the continuity of \(\phi\) across \(S\), seems quite apparent since the electric potentials in region 1, 2 are both expressed by Eq.(3). However, since it includes a surface integral (obviously we don’t need to worry about the volume integral), it is safe to check whether this term breaks the continuity or not. Assuming the origin of the coordinate system is inside the boundary (see Fig.1), a point \(x\) with \(r = |x| = r' + \varepsilon\) (\(\varepsilon\) is a small positive number) is close to the outer/inner boundary. Expanding \(\frac{1}{|x - x'|}\) by Legendre polynomials
\[
\frac{1}{|x - x'|} = \sum_{l=0}^{\infty} \frac{(r' - \varepsilon)^l}{(r' + \varepsilon)^l+1} P_l(\cos \alpha), \quad \text{inside the boundary,}
\]
\[
\sum_{l=0}^{\infty} \frac{(r' + \varepsilon)^l}{(r' + \varepsilon)^l+1} P_l(\cos \alpha), \quad \text{outside the boundary,}
\]
where \(\alpha\) is the angle between \(x\) and \(x'\), we get
\[
\phi_1|_{S'} - \phi_2 |_{S'} \approx \oint_{S'} dS' \sigma(x') \sum_{l=0}^{\infty} \frac{1}{(r' + \varepsilon)^l+1} P_l(\cos \alpha). \tag{9}
\]
As \(\varepsilon \to 0\), the condition \(\phi_1|_{1'_{S'}} - \phi_2 |_{S'} = 0\) is recovered.

To verify the condition (6), we choose a thin Gaussian pillbox covering a tiny patch of \(S'\), as shown in Fig.1. We assume that the center of the pillbox is located at \(x = (r, \theta, \phi)\), and its thickness is \(2\varepsilon\). Thus, the top/bottom of the pillbox is represented by \(r \equiv r \pm \varepsilon\), of which the area is \(\Delta S = r^2 \sin \theta d\theta d\phi\). Multiply the left-hand-side of Eq.(6) by \(\Delta S\) and use Gauss’s theorem, we get
\[
\left[ -\varepsilon \hat{n} \cdot \nabla \phi_2 |_{S'}, + \varepsilon \hat{n} \cdot \nabla \phi_1|_{S'} \right] \Delta S
\]
\[
= \oint_{\partial \text{pillbox}} \mathbf{D} \cdot dS = \int_{\text{pillbox}} \nabla \cdot dV = -\int_{S'} dV \nabla \cdot \phi(x) dV = -\int_{r'}^{-r_0} d\varepsilon \nabla^2 \phi(x) \Delta S
\]
\[
= -\int_{r'}^{-r_0} d\varepsilon \nabla^2 \phi(x) r^2 \sin \theta d\theta d\phi + \int_{r'}^{-r_0} d\varepsilon \nabla^2 \phi(x) r^2 \sin \theta d\theta d\phi
\]
\[
= \int_{r'}^{-r_0} d\varepsilon \int_{0}^{2\pi} d\phi' \int_{0}^{\pi} d\theta' \nabla^2 \phi(x) r^2 \sin \theta' d\theta' d\phi' \delta(r - r') \sin \theta' d\theta' d\phi' \delta(\phi - \phi') \Delta S. \tag{11}
\]
Here \(dV\) is replaced by \(dr \sin \theta d\theta d\phi\) since it measures the change of \(r\), and \(r_0\) is replaced by \(r' \pm \varepsilon\) since \(x\) is now at \(S'\). Next, substitute the spherical-coordinate-expression of \(\delta(x - x')\)
\[
\delta(x - x') = \frac{1}{r^2 \sin \theta} \delta(r - r') \delta(\theta - \theta') \delta(\phi - \phi') \tag{13}
\]
we get
\[
\left[ -\varepsilon \hat{n} \cdot \nabla \phi_2 |_{S'}, + \varepsilon \hat{n} \cdot \nabla \phi_1|_{S'} \right] \Delta S
\]
\[
= \int_{r'}^{r_0} dr \int_{0}^{2\pi} d\phi' \int_{0}^{\pi} d\theta' \nabla^2 \phi(x) r^2 \sin \theta' d\theta' d\phi' \delta(r - r') \delta(\theta - \theta') \delta(\phi - \phi') \Delta S
\]
\[ = \sigma(r(\theta, \phi), \theta, \phi) \Delta S, \]  
(14)

which finally leads to
\[ \epsilon \hat{n} \cdot (\nabla \phi_1|_{S'} - \nabla \phi_2|_{S'}) = -\sigma|_{S'}. \]  
(15)

Hence, the two methods are indeed equivalent.

If \( S' \) is a geometrically regular boundary, the proof may be more perceivable. For simplicity, we only care about the condition (6). Assuming \( S' \) is a spherical surface of radius \( R_0 \), the expansion (9) is now expressed as
\[ \frac{1}{|x - x'|} = \begin{cases} \sum_{l=0}^{\infty} \frac{R_0^l}{R_0^{l+1}} P_l(\cos \alpha), & r < R_0, \\ \sum_{l=0}^{\infty} \frac{R_0^l}{R_0^{l+1}} P_l(\cos \alpha), & r > R_0. \end{cases} \]  
(16)

Substituting this into Eq.(7), and applying the addition formula of the spherical harmonics
\[ P_l(\cos \alpha) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi), \]  
(17)

the left-hand-side of Eq.(6) becomes
\[ \epsilon \left( \frac{\partial \phi_2}{\partial r} \right)_{r=R_0} - \frac{\partial \phi_1}{\partial r} \right)_{r=R_0} \int_{S'} \sigma(x') \sum_{l=0}^{\infty} \left[ (-1)^{l+1} - \frac{R_0^l}{R_0^{l+1}} \right] P_l(\cos \alpha) \]  
\[ = -\sigma_{lm} Y_{lm}(\theta, \phi) \]  
(18)

where \( \sigma_{lm} = \int_{S'} \sin \theta' \int_{0}^{2\pi} d\phi' \sigma(\theta', \phi') Y_{lm}^*(\theta', \phi') \) is the \( m \)-th coefficient of the series when \( \sigma(\theta, \phi) \) is expanded by spherical harmonics.

In some situations, the integral expression (17) provides a more straightforward way to calculate the electrostatic field if the charge distribution is known. We will make a direct comparison by an explicit example in our later discussions.

**B. Magnetostatic Field**

A parallel theorem can be deduced in magnetostatics. In the standard routine, the magnetostatic potential is determined by solving Poisson’s equation
\[ \nabla^2 A = -\mu_0 \mathbf{J} \]  
(19)

with respect to the boundary conditions
\[ \left\{ \begin{align*} (A_1 - A_2)|_{S'} &= 0, \\ \left( \frac{1}{\mu} \nabla \times A_1 - \frac{1}{\mu} \nabla \times A_2 \right)|_{S'} &= \hat{n} \times \vec{a}, \end{align*} \]  
(20)

where \( \vec{a} \) is the surface current density. Similarly, the magnetic vector potential can also be obtained from a generalization to Biot-Savart law
\[ A(x) = \frac{\mu}{4\pi} \int_{S'} dS' \frac{\vec{a}(x') \delta(x - x')}{|x - x'|}, \]  
(21)

where the second integral is implied by applying \( \mathbf{J}(x) = \vec{a}(x') \delta(x - x') \), just as the evaluation in Eq.(3). The uniqueness theorem requires that this is a solution to the boundary value problem specified by Eqs.(19) and (20). Once again we emphasize that this theorem only applies to the situation that there exists only one type of material.

**Figure 2:** A small rectangular loop on the surface \( S' \).

**Proof:**
Similar to Eq.(8), Eq.(21) is the solution to Poisson’s equation (19). For boundary conditions, we consider an Amperian loop as shown in Fig. 2 of which the length and width are \( \Delta l \) and \( 2\epsilon \) such that \( \Delta l \gg \epsilon \) (For end points inside/outside the boundary, the distance between them and the origin are denoted by \( r'_1/r'_2 \) with \( r'_1 - r'_2 = 2\epsilon \). Let \( \hat{t} \) be the tangent direction along the length size, and \( \hat{n} \) the normal direction of boundary. Define \( \hat{\beta} = \hat{n} \times \hat{t} \), indicating the normal direction of the Amperian rectangle. Note \( \mathbf{J} \) is stationery, then \( \mathbf{A} \) satisfies the Coulomb gauge condition \( \nabla \cdot \mathbf{A} = 0 \). Applying Stokes’s theorem to evaluate the contour integral of \( \mathbf{H} \) along the Amperian rectangle, we have
\[ \int_{C} \mathbf{H} \cdot d\mathbf{l} = \int_{C} \frac{1}{\mu} \nabla \times \mathbf{A} \cdot d\mathbf{l} \]  
\[ = \oint_{C} \frac{1}{\mu} \nabla \times (\nabla \times \mathbf{A}) \cdot d\mathbf{S} \]  
\[ = -\oint_{C} \frac{1}{\mu} \nabla^2 \mathbf{A} \cdot d\mathbf{S} \]  
\[ = -\oint_{C} \frac{1}{\mu} \nabla^2 \mathbf{A} \cdot d\mathbf{S} \]  
\[ = -\oint_{C} \frac{1}{\mu} \nabla^2 \mathbf{A} \cdot d\mathbf{S} \]  
\[ = \int_{S} dS' \frac{\vec{a}(x') \cdot \hat{\beta} \delta(x - x') dS'}. \]  
(22)
where the expression (21) has been plugged in. Here $\Delta S$ is the area surrounded by the Amperian loop

$$\Delta S \approx 2\epsilon \Delta l = \iint_{\Delta S} \mathrm{d}S \approx \Delta l \int_{r_c}^{r_c'} \mathrm{d}r,$$

and $\vec{a} \cdot \mathrm{d}S = \hat{\beta} \cdot \mathrm{d}S$ since $\hat{\beta}$ is the normal vector of the infinitesimal patch $\mathrm{d}S$. Similar to the electric part, the boundary $S'$ is parameterized by $r'(\theta', \phi')$, then $\mathrm{d}S' = r'^2 \sin \theta' \, d\theta' \, d\phi'$. Applying Eqs. (23) and (13), Eq. (22) further yields

$$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} \approx \int_{r_c}^{r_c'} \int_{0}^{2\pi} \mathrm{d}\theta' \int_{0}^{\pi} \mathrm{d}\phi' \cdot \hat{\beta} \hat{n} \, \Delta l = \hat{\alpha}(r(\theta, \phi), \theta, \phi) \cdot \hat{\beta} \Delta l. \quad (24)$$

Here the coordinate set $(r, \theta, \phi)$ denotes a point in $\Delta S$, and the delta functions also force it to lie at the boundary $S'$. In other words, it is at the intersecting line between $\Delta S$ and $S'$. Next, applying $\hat{\beta} = \hat{n} \times \hat{r}$, we get

$$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \hat{\alpha} \cdot \hat{n} \Delta l = \hat{\alpha} \cdot (\hat{n} \times \hat{r}) \Delta l = \hat{\alpha} \cdot (\hat{n} \times r) \Delta l. \quad (25)$$

Note the contour integral of $H$ can also be evaluated as

$$\oint \mathbf{H} \cdot \mathrm{d}\mathbf{l} \approx (\mathbf{H}_2 - \mathbf{H}_1) \bigg|_{S'} \cdot \hat{n} \Delta l = \frac{1}{\mu} (\nabla \times \mathbf{A}_2 - \nabla \times \mathbf{A}_1) \bigg|_{S'} \cdot \hat{n} \Delta l, \quad (26)$$

the arbitrariness of $\hat{n}$ leads to

$$\left( \frac{1}{\mu} \nabla \times \mathbf{A}_2 - \frac{1}{\mu} \nabla \times \mathbf{A}_1 \right) \bigg|_{S'} = \hat{\alpha} \times \hat{n}. \quad (27)$$

Therefore, the solution (21) indeed satisfies the boundary condition. The continuity of $\mathbf{A}$ across the surface can be simply verified in the same way as that of $\phi$, and we omit the details here.

### III. EXAMPLES

As a comparison between the two methods, we use two examples to show their equivalence.

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**Example 1.** Assume the surface charge density on a thin dielectric sphere of radius $R_0$ is maintained as $\sigma = \sigma_0 \cos \theta$, find the electrical potentials inside and outside the sphere.

**Solution:**

**Routine Method:**

The electric potentials inside and outside the boundary can be respectively expressed as

$$\phi_1(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta),$$

$$\phi_2(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{R_0^l} P_l(\cos \theta). \quad (28)$$

Applying the boundary conditions, we get

$$A_l R_0^l = \frac{B_{l+1}}{R_0^l} \sum_{m=0}^{\infty} \left( |A_l| R_0^{l-1} + (l + 1) \frac{B_l}{R_0^{l+2}} \right) P_l(\cos \theta) = \frac{\sigma}{\varepsilon_0}. \quad (29)$$

Note $\sigma = \sigma_0 P_1(\cos \theta)$, we have

$$A_1 + \frac{2B_1}{R_0} = \frac{\sigma_0}{\varepsilon_0}, \quad lA_l R_0^{l-1} + (l + 1) \frac{B_l}{R_0^{l+2}} = 0, \quad \text{for} \ l \geq 3. \quad (30)$$

We finally get

$$\left\{ \begin{array}{l} \phi_1(r, \theta) = \frac{\sigma_0}{\varepsilon_0} r \cos \theta, \quad r < R_0, \\
\phi_2(r, \theta) = \frac{\sigma_0 R_0}{3\varepsilon_0} \cos \theta, \quad r > R_0. \end{array} \right. \quad (31)$$

**Integral Method:**

Applying Eq. (7), we get

$$\phi(x) \equiv \phi(r, \theta) = \frac{1}{4\pi\varepsilon_0} \iint \mathrm{d}S \frac{\sigma(x')}{|x - x'|}, \quad (32)$$

where $x' = (R_0, \theta', \phi')$ denotes a point on the sphere, and $\sigma(x') = \sigma(\theta') = \sigma_0 \cos \theta'$. Let the angle between $x$ and $x'$ be $\alpha$. Using the expression (17), the electric potential inside the sphere is

\[
\phi_1(r, \theta) = \frac{1}{4\pi\varepsilon_0} \iint \mathrm{d}S' \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r' R_0^2}{4\pi\varepsilon_0 R_0^{l+2}} \int_{0}^{2\pi} \mathrm{d}\phi' \int_{0}^{\pi} \sin \theta' \mathrm{d}\theta' \frac{4\pi}{2l+1} Y_{lm}(\theta', \phi') Y_{lm}(\theta, \phi). \quad (33)
\]

and evaluate the integral over $\phi'$, only the $m = 0$ term is non-
zero. Hence we have
\[ \phi_1(r, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^\infty \left[ \int_0^\pi \sin \theta' \sigma(\theta') P_l(\cos \theta') \frac{r_l P_l(\cos \theta)}{R_l^{|0|^2}} \right] P_l(\cos \theta) \]
\[ = \frac{\sigma_0}{2\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} r_l \int_0^\pi \sin \theta' P_l(\cos \theta') P_l(\cos \theta) \]
\[ = \frac{\sigma_0}{3\varepsilon_0} r^2 \cos \theta, \]  
(34)
where we have applied the orthogonality relation
\[ \int_0^1 dx P_l(x) P_m(x) = \frac{2}{2l+1} \delta_{lm}. \]  
(35)
In exactly the same way, the electric potential outside the sphere \(|x| = r > R_0\) is given by
\[ \phi_2(r, \theta) = \frac{R_0^2 \sigma_0}{3r^2 \varepsilon_0} \cos \theta. \]  
(36)
The result agrees with that obtained by the routine method.

To explicit show the equivalence theorem in magnetostatics, we choose an example in Griffiths's book \([1]\).

*Example 2.* A spherical thin shell, of radius \(R_0\), carrying a uniform surface charge \(\sigma\), is set spinning at the angular velocity \(\hat{\omega} = \omega \hat{e}_z\). Find the magnetic field \(H\) inside and outside the sphere.

*Solution:*

**Routine Method:**

We can first solve Eq. (19) for \(A\) subject to the boundary conditions (20), and then the magnetic field is obtained via \(H = \frac{1}{\mu_0} \nabla \times A\). However, this will be very tedious since at least six sets of coefficients are needed to specify \(A\) inside and outside the spherical shell due to the fact that \(A\) is a vector field. A simpler and equivalent method is to introduce the magnetic scalar potential \(\phi_m\) \([11, 12]\), satisfying the Laplace's equation \(\nabla^2 \phi_m = 0\) inside and outside the sphere since the electric current is only distributed on the spherical shell. The boundary condition is given by
\[ B_{1r}|_{r=R_0} = B_{2r}|_{r=R_0}, \quad \hat{e}_r \times (H_2 - H_1)|_{r=R_0} = \hat{a}, \]  
(37)
where \(B_{1,2r}\) is the radial component of \(B_{1,2}\), and \(\hat{a}(x) = \sigma \hat{e}_x\) is the surface electric current. Note \(H = -\nabla \phi_m\), the boundary conditions for the magnetic scalar potential are then given by
\[ \left[ \frac{\partial \phi_m}{\partial r} \right]_{r=R_0} = \sigma \omega R_0 \sin \theta, \]
\[ \left[ \frac{1}{R_0} \frac{\partial \phi_m}{\partial \theta} \right]_{r=R_0} = \sigma \omega R_0 \sin \theta. \]  
(38)
The general solution can be expressed as
\[ \phi_m(r, \theta) = \sum_{l=1} A_l r^l P_l(\cos \theta), \quad r < R_0, \]
\[ \phi_m(r, \theta) = \sum_{l=1} B_l r^l P_l(\cos \theta), \quad r > R_0. \]  
(39)
Substituting them into the boundary conditions, we get
\[ A_l = \frac{(l+1)B_l}{2R_0^2 l(l+1)}, \quad B_l - A_1 R_0^1 = \sigma \omega R_0^2, \]
\[ B_l = A_l R_0^l \text{ if } l \neq 1. \]  
(40)
Hence the magnetic scalar potential is given by
\[ \begin{cases} \phi_{m1} = -\frac{2R_0 \sigma}{3} \hat{\omega} \cdot x, & r < R_0, \\ \phi_{m2} = \frac{R_0^2 \sigma}{3r^2} \hat{\omega} \cdot x, & r > R_0, \end{cases} \]  
(41)
and the magnetic field is
\[ H = -\nabla \phi_m = \begin{cases} \frac{2R_0 \sigma}{3} \hat{\omega} \cdot x - \frac{\hat{\omega}}{r^3} \cdot x, & r < R_0, \\ \frac{R_0^2 \sigma}{3r^2} \hat{\omega} \cdot x - \frac{\hat{\omega}}{r^3} \cdot x, & r > R_0. \end{cases} \]  
(42)

**Integral Method:**

Let the coordinate of a field point be \(x = (r, \theta, \varphi)\), and the coordinate of a point at the spherical shell be \(x' = (R_0, \theta, \varphi')\). The magnetic vector potential is then given by
\[ A(x) = \frac{\mu_0}{4\pi} \oint \frac{\sigma \hat{\omega} \times x'}{|x - x'|} dS', \]
\[ = \frac{\mu_0 R_0 \sigma \hat{\omega}}{4\pi} \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} d\varphi' \frac{\sin^2 \theta'}{|x - x'|} \hat{e}_\varphi'. \]  
(43)
Note the \(\hat{e}_\varphi'\) also depends on the position \(x'\)
\[ \hat{e}_\varphi' = -\sin \varphi' \hat{e}_x + \cos \varphi' \hat{e}_y, \]
\[ = \frac{1}{|x - x'|} \left[ (i \hat{e}_x + \hat{e}_y) e^{i \varphi'} - (i \hat{e}_x - \hat{e}_y) e^{-i \varphi'} \right]. \]  
(44)
This contributes to the integration too. Expanding \(1/|x - x'|\) by Legendre polynomials, we have
\[ \mathbf{A}(\mathbf{x}) = \begin{cases} \frac{\mu_0 R_0^3 \sigma \omega}{4\pi} \sum_{l=0}^{\infty} \frac{R_0^l}{r^{l+1}} \int_0^\infty \frac{d\theta'}{2\pi} \int_0^{2\pi} \sin \theta' P_l(\cos \alpha) \hat{e}_\phi, & \text{if } r < R_0, \\ \frac{\mu_0 R_0^3 \sigma \omega}{8\pi} \sum_{l=0}^{\infty} \frac{R_0^l}{r^l} \int_0^\infty \frac{d\theta'}{2\pi} \int_0^{2\pi} \sin \theta' P_l(\cos \alpha) \hat{e}_\phi, & \text{if } r > R_0, \end{cases} \] (45)

where \( \alpha \) is the angle between \( \mathbf{x} \) and \( \mathbf{x}' \). Next, by expanding \( P_l(\cos \alpha) \) by spherical harmonics and using Eq. (44), the integral in Eq. (45) is evaluated as

\[
\left[ \frac{\mu_0 R_0^3 \sigma \omega}{4\pi} \sum_{l=0}^{\infty} \frac{R_0^l}{r^{l+1}} \int_0^\infty \frac{d\theta'}{2\pi} \int_0^{2\pi} \sin \theta' P_l(\cos \alpha) \hat{e}_\phi \right] = \pi \int_0^\pi d\theta' \sin^2 \theta' \left[ \frac{1}{(l+1)} P_l(\cos \theta') P_l(\cos \theta') e^{\imath \theta} \right.
\left. - l(l+1)(\hat{e}_x - \hat{e}_y) P_l(\cos \theta') P_l(\cos \theta') e^{\imath \theta} \right] = 2\pi \int_0^\pi d\theta' \sin^2 \theta' \frac{1}{l(l+1)} P_l(\cos \theta') P_l(\cos \theta') e^{\imath \theta},
\] (46)

where we have applied the relation \( \int_0^{2\pi} d\theta' \sin \theta' e^{\imath \theta'} = 2\pi \delta_{l1} \) in the second line, and the relation \( P_{l+1}^{-1}(\cos \theta) = \frac{1}{(l+1)} P_l(\cos \theta) \) in the last line. Furthermore, by using the identity \( P_l(\cos \theta') = -\sin \theta' P_l'(\cos \theta') \) we get

\[
\left[ \frac{\mu_0 R_0^3 \sigma \omega}{4\pi} \sum_{l=0}^{\infty} \frac{R_0^l}{r^{l+1}} \int_0^\infty \frac{d\theta'}{2\pi} \int_0^{2\pi} \sin \theta' P_l(\cos \theta') \hat{e}_\phi \right] = 2\pi \int_0^\pi d\theta' \sin^2 \theta' \frac{1}{l(l+1)} P_l(\cos \theta') P_l(\cos \theta') e^{\imath \theta}.
\] (47)

Substitute this into Eq. (46), we have

\[
\left[ \frac{\mu_0 R_0^3 \sigma \omega}{4\pi} \sum_{l=0}^{\infty} \frac{R_0^l}{r^{l+1}} \int_0^\infty \frac{d\theta'}{2\pi} \int_0^{2\pi} \sin \theta' P_l(\cos \theta') \hat{e}_\phi \right] = \frac{8\pi}{3} \delta_{l1} \sin \theta \hat{e}_\phi.
\] (48)

Finally we get

\[
\mathbf{A}(\mathbf{x}) = \begin{cases} \frac{\mu_0 R_0^3 \sigma \omega}{3} \sin \theta \hat{e}_\phi = \frac{\mu_0 R_0^3 \sigma \omega}{3} \frac{\hat{\omega}}{r} \times \mathbf{x}, & \text{if } r < R_0, \\ \frac{\mu_0 R_0^4 \sigma \omega}{3} \sin \theta \hat{e}_\phi = \frac{\mu_0 R_0^4 \sigma \omega}{3} \frac{\hat{\omega}}{r^3} \times \mathbf{x}, & \text{if } r > R_0. \end{cases}
\] (49)

The corresponding magnetic field is

\[
\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A} = \begin{cases} \frac{2R_0 \sigma \hat{\omega}}{3}, & \text{if } r < R_0, \\ \frac{3(\hat{\omega} \cdot \mathbf{x}) \mathbf{x} - \hat{\omega}}{r^3}, & \text{if } r > R_0, \end{cases}
\] (50)

which exactly agrees with Eq. (42).

**IV. CONCLUSION**

We have discussed the integral expressions of the static electromagnetic potentials as a generalization to Coulomb’s law/Biot-Savart law when there exist surface charge/current distributions in some situation. We proved that it is equivalent to solve Poisson’s equations subject to certain boundary conditions. Explicit examples are also provided to show the applications of such integral expressions.

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