On the positive solutions to some quasilinear elliptic partial differential equations

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Abstract. We establish that the elliptic equation $\Delta u + f(x, u) + g(|x|)x \cdot \nabla u = 0$, where $x \in \mathbb{R}^n$, $n \geq 3$, and $|x| > R > 0$, has a positive solution which decays to 0 as $|x| \to +\infty$ under mild restrictions on the functions $f, g$. The main theorem extends and complements the conclusions of the recent paper [M. Ehrnström, O.G. Mustafa, On positive solutions of a class of nonlinear elliptic equations, Nonlinear Anal. TMA 67 (2007), 1147–1154]. Its proof relies on a general result about the long-time behavior of the logarithmic derivatives of solutions for a class of nonlinear ordinary differential equations and on the comparison method.

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1 Introduction

This note, motivated by the recent papers [7, 9], is concerned with the existence of a positive solution to the boundary value problem

\[
\begin{align*}
    u'' + F(t, u) &= 0, \quad t \geq t_0 > 0, \\
    q_-(t) \cdot \frac{u(t)}{t} &\leq u'(t) \leq q_+(t) \cdot \frac{u(t)}{t}, \quad t \geq t_0, \\
    u(t) &= o(t) \quad \text{as } t \to +\infty,
\end{align*}
\]

for a certain class of continuous functions \( F : [t_0, +\infty) \times \mathbb{R} \rightarrow [0, +\infty) \).

Here, the functions \( q_\pm : [t_0, +\infty) \rightarrow [0, 1] \) are assumed continuous.

The interest in studying the \( q_\pm \) - problem (1) comes from an investigation of the existence and decay rates of the positive, vanishing at \(+\infty\), solutions to the quasilinear elliptic equation of second order

\[
\Delta u + f(x, u) + g(|x|)x \cdot \nabla u = 0, \quad x \in G_R,
\]

where \( G_R = \{x \in \mathbb{R}^n : |x| > R\} \) and \( n \geq 3 \). For an account of recent literature on this topic, we refer to the studies [2, 3, 5, 8, 12, 15, 16, 18].

Following [2, 13], we consider that the functions \( f : G_R \times \mathbb{R} \rightarrow \mathbb{R} \) and \( g : [R, +\infty) \rightarrow \mathbb{R} \) are locally Hölder continuous. Moreover,

\[
0 \leq f(x, U) \leq m(|x|, U), \quad x \in G_R, \quad U \in [0, \varsigma],
\]

for some \( \varsigma > 0 \) and the continuous application \( m : [R, +\infty) \times [0, \varsigma] \rightarrow [0, +\infty) \). The regularity assumptions upon \( f, g \) are sufficient for applying the comparison method [10] to the analysis of (2). In fact, given \( u(t) \) a positive solution of (1), the function

\[
U(x) = U(|x|) = \frac{u(t)}{t}, \quad \text{where } |x| = \theta(t) = \left( \frac{t}{n-2} \right)^{\frac{1}{n-2}}
\]

and \( t \geq t_0 = (n-2)R^{n-2} \), will be a super-solution to (2) satisfying the additional restriction

\[
x \cdot \nabla U(x) \leq 0, \quad x \in G_R.
\]

It has been noticed in [8] that, when \( g \) takes only nonnegative values, the additional requirement (5) for the solution \( U \) of the elliptic partial differential equation

\[
\Delta U + m(|x|, U) = 0, \quad x \in G_R,
\]

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allows for a complete removal of the integral conditions regarding $g$ from the hypotheses of various theorems in the recent literature. Further developments of this observation are given in [7, 9].

Condition (5), translated into the language of ordinary differential equations, reads as

$$u'(t) - \frac{u(t)}{t} \leq 0, \quad t \geq t_0. \quad (7)$$

It is obvious now that the $q_\pm$ - problem (1) constitutes an improvement of (7).

The use of (1), in the particular case of $q_-(t) = 0$, $q_+(t) = (\ln t)^{-1}$ throughout $[t_0, +\infty)$, has been observed in [7]. To give it a self-contained presentation, let us recall first the essence of the reduction technique of [8]. The comparison ordinary differential equation in the study of super-solutions of (2) (that is, a rewriting of $\Delta U + m(|x|, U) + g(|x|)x \cdot \nabla U = 0$, $|x| > R$, which takes into account (3)) being displayed as

$$u'' + H(t, u) + h(t) \left( u' - \frac{u}{t} \right) = 0, \quad t \geq t_0, \quad (8)$$

where

$$H(t, u) = \frac{1}{n - 2} \theta(t)\theta'(t)m\left( \theta(t), \frac{u}{t} \right), \quad h(t) = \theta(t)\theta'(t)g(\theta(t)), \quad (9)$$

the method in [8] consists of removing the quantity ”$h(t) \left( u' - \frac{u}{t} \right)$” from (8) whenever $g$ is nonnegative-valued and the problem (1), with $q_-(t) \equiv 0$, $q_+(t) \equiv 1$, has a positive solution. The approach allows for total freedom of $g$, however, it keeps intact all the restrictions concerned with $m(|x|, U)$. Instead of this, we can use the next modification of (8), namely the quasilinear ordinary differential equation

$$u'' + \left| \frac{H(t, u)}{u} - [1 - q_+(t)] \cdot \frac{h(t)}{t} \right| u = 0, \quad t \geq t_0. \quad (10)$$

In the new setting (10), the functional quantity $\frac{H(t, u)}{u}$ is controlled only partially by the hypotheses of various comparison-type results.
We also notice that, since

$$\frac{H(t,u)}{u} = b(t,u) + [1 - q_+(t)] \cdot \frac{h(t)}{t}, \quad t \geq t_0,$$

for a continuous function $b : [t_0, +\infty) \times \mathbb{R} \to \mathbb{R}$ subjected to certain integral restrictions, see the cited literature, it is desirable to look for positive solutions of the $q_\pm$ - problem (1) with the functions $q_\pm$ described by

$$q_\pm(t) = o(1) \quad \text{when } t \to +\infty. \quad (11)$$

This will allow for a "free of restrictions" part of $\frac{H(t,u)}{u}$ as large as possible by such an approach.

In this note, using a special result about (1), a flexible criterion for the existence of positive solutions to (2) that decay to 0 as $|x| \to +\infty$ is established. It extends and complements the conclusions of [7, 9].

2 A boundary value problem for the logarithmic derivative: statement and application

Let us consider the problem

$$\begin{align*}
    u'' + F(t, u) &= 0, \quad t \geq t_0 > 0, \\
    \alpha(t) &\leq \frac{u'(t)}{u(t)} \leq \beta(t), \quad t \geq t_0, \\
    \int_{t_1}^{t_2} \frac{F(s, u(s))}{u(s)} ds &\leq \gamma(t_1, t_2), \quad (t_1, t_2) \in \Gamma, \\
    u(t_0) &= u_0 > 0,
\end{align*} \quad (12)$$

where the functions $F : [t_0, +\infty) \times \mathbb{R} \to [0, +\infty)$, $\alpha, \beta : [t_0, +\infty) \to [0, +\infty)$ and $\gamma : \Gamma = \{(t_1, t_2) : t_1 \geq t_0, t_2 \in [t_1, t_1 + p]\} \to [0, +\infty)$ are continuous. Here, $p > 0$ is fixed.

It is assumed that

$$\lim_{t \to +\infty} \alpha(t) = \lim_{t \to +\infty} \beta(t) = 0, \quad \alpha, \beta \in L^2((t_0, +\infty), \mathbb{R}). \quad (13)$$

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Also, for every $\varepsilon > 0$ there exists $\zeta(\varepsilon) \in (0, p)$ such that
\[
\gamma(t_1, t_2) < \varepsilon, \quad 0 \leq t_2 - t_1 \leq \zeta(\varepsilon), \quad (t_1, t_2) \in \Gamma.
\]

The particular case of interest herein is given by
\[
\alpha(t) = \frac{q_-(t)}{t} \quad \text{and} \quad \beta(t) = \frac{q_+(t)}{t},
\]
\[\text{together with}
F(t, u) = \left| \frac{H(t, u)}{u} - [1 - q_+(t)] \cdot \frac{h(t)}{t} \right| u
\]
for all $t \geq t_0$ and $u \in C$ (to be defined later in this section).

We notice that, given $\lambda \in (0, 1)$, condition (11) yields the existence of $t_\lambda \geq t_0$ such that $\sup_{s \geq t_\lambda} q_+(s) \leq \lambda$. Consequently, we have
\[
\exp \left( \int_{t_0}^{t} \frac{q_+(s)}{s} ds \right) = O(t^\lambda) = o(t) \quad \text{when } t \to +\infty.
\]

In particular, the solution $u$ of (12) for $\alpha, \beta$ given by (15) will obey the long-time law
\[
\lim_{t \to +\infty} \frac{u(t)}{t} = 0,
\]
thus being a solution of (1).

The boundary value problem (12) is, bluntly speaking, about the existence of a positive solution to a nonlinear ordinary differential equation that has prescribed long-time behavior for its logarithmic derivative. In this way, the present problem is in the spirit of the investigation from [14].

The integral condition in the statement of (12) has a very simple particular case illuminating its presence:
\[
0 < \frac{F(t, u)}{u} \leq k < +\infty \quad \text{when } u \geq u_0 \exp \left( \int_{t_0}^{t} \alpha(s) ds \right), \quad t \geq t_0,
\]
where $k$ is a constant. Here, $\gamma(t_1, t_2) = k(t_2 - t_1)$ for all $(t_1, t_2) \in \Gamma$. Since the problem \([12]\) describes a non-oscillatory solution to the equation

$$u'' + F(t, u) = 0, \quad t \geq t_0,$$

while the comparison equation

$$u'' + ku = 0, \quad t \geq t_0,$$

is oscillatory, the analysis in \([17, \text{Theorem 5}]\) establishes that the presence in \([12]\) of the condition involving $\gamma$ does not diminish in any way the applicability of our technique to the problem presented in the introduction.

The main hypotheses regarding $F(t, u)$ are given following the lines of the Hale-Onuchic integration theory \([11]\). By introducing the set $B$ of functions $b \in C([t_0, +\infty), \mathbb{R})$ which obey for all $t \geq t_0$ the inequalities

$$\alpha(t) \leq b(t) \leq \beta(t), \quad (18)$$

we ask that $F$ be confined by

$$B_-(t) \leq \frac{1}{u_0} \int_{t}^{+\infty} F\left(s, u_0 \exp\left(\int_{t_0}^{s} b(\tau)d\tau\right)\right) \times \exp\left(-\int_{t_0}^{s} b(\tau)d\tau\right) ds \leq B_+(t), \quad t \geq t_0, \quad (19)$$

where

$$B_-(t) = \alpha(t) - \int_{t}^{+\infty} [\alpha(s)]^2 ds, \quad B_+(t) = \beta(t) - \int_{t}^{+\infty} [\beta(s)]^2 ds,$$

and also by

$$\frac{1}{u_0} \int_{t_1}^{t_2} F\left(s, u_0 \exp\left(\int_{t_0}^{s} b(\tau)d\tau\right)\right) \times \exp\left(-\int_{t_0}^{s} b(\tau)d\tau\right) ds \leq \gamma(t_1, t_2), \quad (t_1, t_2) \in \Gamma, \quad (20)$$
for all \( b \in \mathcal{B} \).

In the particular cases of \( F(t, u) = A(t)u \) and \( F(t, u) = A(t)u^\sigma \), where \( \sigma \in (0, 1) \), the restriction (19) reads as

\[
B_-(t) \leq \int_t^{+\infty} A(s)ds \leq B_+(t)
\]

and respectively as

\[
B_-(t) \leq u_0^{\sigma - 1} \int_t^{+\infty} A(s) \times \exp \left( - (1 - \sigma) \int_{t_0}^{s} b(\tau) d\tau \right) ds \leq B_+(t)
\]

throughout \([t_0, +\infty)\). An immediate simplification of (22), by means of (18), is given via the system of inequalities

\[
\begin{cases}
B_-(t) \leq u_0^{\sigma - 1} \int_t^{+\infty} A(s) \exp \left( - (1 - \sigma) \int_{t_0}^{s} \beta(\tau) d\tau \right) ds \\
B_+(t) \geq u_0^{\sigma - 1} \int_t^{+\infty} A(s) \exp \left( - (1 - \sigma) \int_{t_0}^{s} \alpha(\tau) d\tau \right) ds.
\end{cases}
\]

Here, in the first particular case of \( F \), we have

\[
\gamma(t_1, t_2) = \int_{t_1}^{t_2} A(s)ds,
\]

while in the second particular case

\[
\gamma(t_1, t_2) = u_0^{\sigma - 1} \int_{t_1}^{t_2} A(s) \exp \left( - (1 - \sigma) \int_{t_0}^{s} \alpha(\tau) d\tau \right) ds
\]

for all \((t_1, t_2) \in \Gamma\).

An existence result for problem (12) now reads as follows.

**Theorem 1** Suppose that the functions \( F, \alpha, \beta, \gamma \) satisfy the conditions (13), (14), (19) and (20). Then, the problem (12) has a solution \( u \). In particular, its logarithmic derivative lies in \( \mathcal{B} \).
Its proof is presented in section 3.

To apply the conclusions of Theorem 1 to the analysis of equation (2), let us introduce the functions $q_-(t) \equiv 0$ and $q_+$ subjected to (11) and (17). Here, $t_0 = (n-2)R^{n-2}$, $R > 0$. Define also the functions $H$, $h$ on the basis of (9) and assume, for simplicity, that

$$
\frac{H(t, u(t))}{u(t)} \geq [1 - q_+(t)] \cdot \frac{h(t)}{t}, \quad u \in \mathcal{C}, \quad (23)
$$

throughout $[t_0, +\infty)$. The set $\mathcal{C}$ consists of all the functions $u \in C^1([t_0, +\infty), (0, +\infty))$ with $u(t_0) = u_0$ and $\frac{u'}{u} \in \mathcal{B}$. We fix $u_0 > 0$ such that $\frac{u_0}{t_0} \leq \varsigma$.

The condition (23) says, practically, that $m$ from (3) is in a certain sense larger than $g$. We recall that [9, Theorem 1, Remark 1] dealt with the complementary case, namely $a(r) \leq lr^{n-2}g(r)$ for all $r \geq R$, where $m(r, U) = a(r)w(U)$ and $l > 0$ is a constant.

Our main contribution here is the next result.

**Theorem 2** Assume that $g(r) \geq 0$ for all $r \geq R$. Suppose further that, when $r \geq R$ and $R \leq r_1 \leq r_2 \leq \left(\frac{p}{n-2}\right)\frac{1}{n-2}$, one has

$$
\int_{r}^{+\infty} M(\tau, u((n-2)\tau^{n-2}))d\tau \leq B_+((n-2)r^{n-2})
$$

and

$$
\int_{r_1}^{r_2} M(\tau, u((n-2)\tau^{n-2}))d\tau \leq \gamma((n-2)r_1^{n-2}, (n-2)r_2^{n-2})
$$

for all $u \in \mathcal{C}$, where

$$
M(\tau, u) = \frac{\tau}{n-2} \left\{ m\left(\tau, \frac{u}{(n-2)\tau^{n-2}}\right) - [1 - q_+((n-2)\tau^{n-2})]\frac{g(\tau)}{\tau^{n-2}} \right\}.
$$

Then, the equation (2) has a positive solution $u(x)$, defined in $G_R$, such that $\lim_{|x| \to +\infty} u(x) = 0$.

Its proof is presented in the next section.
3 Proofs

Proof of Theorem 1. The existence of a solution to the problem (12) will be demonstrated by employing the Schauder fixed point theorem.

Let $A(t_0)$ be the real linear space of continuous functions $b : [t_0, +\infty) \to \mathbb{R}$ which satisfy

$$\lim_{t \to +\infty} b(t) = l_b \in \mathbb{R}.$$ 

If endowed with the standard sup-norm $\| \cdot \|$, $A(t_0)$ becomes a Banach space. Avramescu’s criterion [1] for relative compactness of subsets in this space asks from $S \subset A(t_0)$ to be norm-bounded, equicontinuous (meaning that all the functions from $S$ are uniformly continuous in the same way) and equiconvergent (that is, all the functions $b \in S$ approach their limits $l_b$ in an uniform way) in order to be relatively compact.

We start by noticing that $B \subset A(t_0)$ with $l_b = 0$ for all $b \in B$. Since $\|b\| \leq \|\beta\|$ throughout $B$, the set is bounded. It is also easy to conclude that it is convex and closed in the norm topology of $A(t_0)$.

Further, we define the integral operator $T : B \to A(t_0)$ through the formula

$$T(b)(t) = \int_{t_0}^{+\infty} [b(s)]^2 ds + \frac{1}{u_0} \int_{t_0}^{+\infty} F \left( s, u_0 \exp \left( \int_{t_0}^{s} b(\tau)d\tau \right) \right) \times \exp \left( -\int_{t_0}^{s} b(\tau)d\tau \right) ds, \quad t \geq t_0.$$ 

The assumptions (18), (19) imply that $T(B) \subseteq B$. (24)

We shall establish that $T : B \to B$ is continuous.

Set $\varepsilon > 0$. There exists $T_{\varepsilon} \geq t_0$ such that $\beta(T_{\varepsilon}) \leq \frac{\varepsilon}{6}$. Introduce also the functions

$$g(t, w) = (u_0 e^w)^{-1} F(t, u_0 e^w) \quad \text{and} \quad x(t; b) = \int_{t_0}^{t} b(\tau)d\tau$$

for all $w \geq 0$ and $b \in B$. 

Since \( g : [t_0, T_\varepsilon] \times [0, w_\varepsilon] \to \mathbb{R} \) is uniformly continuous, where \( w_\varepsilon = u_0 \exp\left( \int_{t_0}^{T_\varepsilon} \beta(\tau)d\tau \right) \), there exists \( \delta(\varepsilon) > 0 \) such that
\[
|g(s, w_1) - g(s, w_2)| \leq \frac{\varepsilon}{3T_\varepsilon}, \quad s \in [t_0, T_\varepsilon],
\]
for all \( w_{1,2} \in [0, w_\varepsilon] \) with \( |w_1 - w_2| \leq \delta(\varepsilon) \). Notice that
\[
|x(t; b_1) - x(t; b_2)| \leq \int_{t_0}^{t} |b_1(\tau) - b_2(\tau)|d\tau \leq T_\varepsilon \|b_1 - b_2\|.
\]

Now, given \( b_{1,2} \in B \) with \( \|b_1 - b_2\| \leq \eta(\varepsilon) = \min\left\{ \frac{\varepsilon}{6T_\varepsilon \|\beta\|}, \frac{\delta(\varepsilon)}{T_\varepsilon} \right\} \), we have the estimates
\[
|T(b_1)(t) - T(b_2)(t)|
\leq \int_{t_0}^{T_\varepsilon} |[b_1(s)]^2 - [b_2(s)]^2|ds + 2 \int_{T_\varepsilon}^{+\infty} [\beta(s)]^2 ds
+ \int_{t_0}^{T_\varepsilon} |g(s, x(s; b_1)) - g(s, x(s; b_2))|ds + 2B_+(T_\varepsilon)
\leq \int_{t_0}^{T_\varepsilon} |b_1(s) + b_2(s)|ds \cdot \|b_1 - b_2\| + 2 \int_{T_\varepsilon}^{+\infty} [\beta(s)]^2 ds
+ \frac{\varepsilon}{3} + 2B_+(T_\varepsilon)
\leq 2\|\beta\| \cdot T_\varepsilon \|b_1 - b_2\| + 2\beta(T_\varepsilon) + \frac{\varepsilon}{3} \leq \varepsilon,
\]
that is, \( \|T(b_1) - T(b_2)\| \leq \varepsilon \).

In order to apply Schauder’s fixed point theorem to operator \( T \), it remains to prove that the set \( T(B) \) verifies the hypotheses of Avramescu’s criterion.

The first one, namely the boundedness of \( T(B) \), follows from (24).

The equicontinuity of \( T(B) \) is a consequence of the estimate
\[
0 \leq T(b)(t_1) - T(b)(t_2) \leq \|\beta\|^2(t_2 - t_1) + \gamma(t_1, t_2), \quad (t_1, t_2) \in \Gamma,
\]
together with (14).
The equiconvergence of $T(B)$ is implied by
\[ 0 \leq T(b)(t) = |T(b)(t) - l_{T(b)}| \leq \beta(t) = o(1) \quad \text{as } t \to +\infty \]
for all $b \in B$.

The solution of problem (12) has the formula
\[ u(t) = u_0 \exp \left( \int_{t_0}^{t} b_0(s)ds \right), \quad t \geq t_0, \]
where $b_0 \in B$ is the fixed point of operator $T$. □

The following lemma is of use for Theorem 2.

Lemma 1 (see [13]) If there exist a nonnegative subsolution $w$ and a positive supersolution $v$ to (2) in $G_R$, such that $w(x) \leq v(x)$ for $x \in \overline{G}_R$, then (2) has a solution $u$ in $G_R$ such that $w \leq u \leq v$ throughout $\overline{G}_R$. In particular, $u = v$ on $|x| = R$.

Proof of Theorem 2. Consider the positive, twice continuously differentiable functions given by
\[ U(x) = y(r) = \frac{u_1(t)}{t}, \quad t \geq t_0, \]
where $r = |x| = \theta(t)$. Here, $u_1$ is the solution of problem (12) obtained at Theorem 11. Since the range of $q_+$ is a subset of $[0, 1]$, we have
\[ \frac{u(t)}{t} \leq \frac{u_0}{t_0} \exp \left( \int_{t_0}^{t} \frac{q_+(s)}{s}ds \right) \leq \frac{u_0}{t_0} \leq \varsigma, \quad t \geq t_0, \]
for all $u \in \mathcal{C}$. This estimate allows us to use the comparison inequality (3) in the following.

By a straightforward computation we get that \[ t\theta'(t) = \frac{1}{n-2} \theta(t) \quad (25) \]
and
\[ \begin{cases} \frac{d\theta}{dt} = y + t\theta'(t) \frac{dy}{dt} \\ \frac{d^2\theta}{dt^2} = \frac{n-1}{n-2} \theta'(t) \frac{dy}{dt} + \frac{\theta(t)\theta'(t)}{n-2} \frac{d^2y}{dt^2} \end{cases} \quad (26) \]
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Further, taking into account (25) and (26), we have

$$r^{n-1}(\Delta U + f(x, U) + g(|x|)x \cdot \nabla U)$$

$$= \frac{d}{dr} \left( r^{n-1} \frac{dy}{dr} \right) + r^{n-1} f(x, U) + r^n g(r) \frac{dy}{dr}$$

$$= \frac{n-2}{\theta(t) \theta'(t)} [\theta(t)]^{n-1} \left[ u''(t) + \frac{1}{n-2} \theta(t) \theta'(t) f(x, U) + \theta(t) \theta'(t) g(\theta(t)) \right] \left( u'_1(t) - \frac{u_1(t)}{t} \right),$$

for any $t \geq t_0$.

We have obtained that

$$|x|^{n-1}(\Delta U + f(x, U) + g(|x|)x \cdot \nabla U)$$

$$\leq \frac{n-2}{\theta(t) \theta'(t)} [\theta(t)]^{n-1} [u''(t) + F(t, u_1(t))] = 0,$$

where the function $F$ is given by (16).

Now, $U$ is a positive super-solution of (2). Also, the trivial solution of (2) is its (nonnegative) sub-solution. According to Lemma 1 there exists a nonnegative solution $u$ to (2), defined in $\overline{G}_R$. Since

$$(\Delta + g(|x|)x \cdot \nabla)(-u) = f(x, u) \geq 0,$$

the strong maximum principle (6) can be applied to $-u$. This means that the function $-u$ cannot attain a nonnegative maximum at a point of $G_R$ unless it is constant. Since $-u$ is negative on $\{x : |x| = R\}$ and $-u(x) \leq 0$ throughout $\overline{G}_R$ as $u$ is confined between 0 and a positive super-solution $U$, it follows that $-u$ cannot have zeros.

We conclude that $u$ is a positive solution of (2) that decays to 0 when $|x| \to +\infty$. Furthermore, via (17), we can compute the decay rate of $u$:

$$0 < u(x) \leq \frac{u_1((n-2)|x|^{n-2})}{(n-2)|x|^{n-2}} = O \left( |x|^{(\lambda-1)(n-2)} \right) \quad \text{when } |x| \to +\infty.$$

The proof is complete. □
4 Conclusion

To emphasize the significance of Theorem \[2\], let us consider a particular case of the comparison function \( m \) from \([3]\), namely

\[
m(|x|, U) = a(|x|)U, \quad x \in G_R, \quad U \in [0, \varsigma],
\]

(27)

for a continuous function \( a : [R, +\infty) \to [0, +\infty) \).

In some of the recent literature \([2, 3, 4, 8, 18]\) regarding \((2)\), the leading hypothesis was

\[
\int_R^{+\infty} r [a(r) + |g(r)|] dr < +\infty.
\]

(28)

The conclusion of these papers reads, practically, as follows: since the elliptic equation

\[
\Delta u + f(x, u) = 0, \quad |x| > R,
\]

(29)

has a positive solution decaying to 0 as \(|x| \to +\infty\) under the hypotheses \((3), (27)\) and \(\int_{R}^{+\infty} ra(r) dr < +\infty\), its ”small” perturbation by the term ”\(g(|x|) x \cdot \nabla u\)”, namely \((2)\), where the degree of ”smallness” is given by \((28)\), will preserve this feature.

The hypotheses of Theorem \([2]\) if verified, reveal that the behavior of certain solutions to \((2)\) for large \(|x|\)’s instead of being controlled by this summing action of functions \(a, g\) is actually governed by their coupling. In our case, that is, it might happen that for certain functions \(a, g\), where

\[
0 \leq g(r) \leq \frac{a(r)}{n - 2}, \quad r \geq R,
\]

and

\[
\int_{r}^{+\infty} n(\tau) d\tau \leq B_+ ((n - 2)r^{n-2}), \quad r \geq R,
\]

and

\[
\int_{r_1}^{r_2} n(\tau) d\tau \leq \gamma ((n - 2)r_1^{n-2}, (n - 2)r_2^{n-2}),
\]
where $R \leq r_1 \leq r_2 \leq \left( r_1^{n-2} + \frac{p}{n-2} \right)^{\frac{1}{n-2}}$ and

$$n(r) = \frac{r^{3-n}}{n-2} \left\{ \frac{a(r)}{n-2} - \left[ 1 - q_+ (n-2) r^{n-2} \right] g(r) \right\}, \quad r \geq R,$$

the unperturbed equation (29) does not have any vanishing at $+\infty$ solution besides the trivial solution.

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References

[1] C. Avramescu, Sur l’existence des solutions convergentes de systèmes d’équations différentielles non linéaires, Ann. Mat. Pura Appl. 81 (1969), 147–168

[2] A. Constantin, Existence of positive solutions of quasilinear elliptic equations, Bull. Austral. Math. Soc. 54 (1996), 147–154

[3] A. Constantin, Positive solutions of quasilinear elliptic equations, J. Math. Anal. Appl. 213 (1997), 334–339

[4] A. Constantin, On the existence of positive solutions of second order differential equations, Ann. Mat. Pura Appl. 184 (2005) 131–138

[5] J. Deng, Bounded positive solutions of semilinear elliptic equations, J. Math. Anal. Appl. 332 (2007), 475–486

[6] L.E. Fraenkel, Introduction to maximum principles and symmetry in elliptic problems, Cambridge Univ. Press, Cambridge, 2000

[7] S. Djebali, T. Moussaoui, O.G. Mustafa, Positive evanescent solutions of nonlinear elliptic equations, J. Math. Anal. Appl. 333 (2007), 863–870

[8] M. Ehrnström, Positive solutions for second-order nonlinear differential equations, Nonlinear Anal. TMA 64 (2006), 1608–1620
[9] M. Ehrnström, O.G. Mustafa, On positive solutions of a class of nonlinear elliptic equations, Nonlinear Anal. TMA 67 (2007), 1147–1154

[10] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin, 2001

[11] J.K. Hale, N. Onuchic, On the asymptotic behavior of solutions of a class of differential equations, Contributions Differential Equations 2 (1963), 61–75

[12] M. Hesaaraki, A. Moradifam, On the existence of bounded positive solutions of Schrödinger equations in two-dimensional exterior domains, Appl. Math. Lett. 20 (2007), 1227–1231

[13] E.S. Noussair, C.A. Swanson, Positive solutions of quasilinear elliptic equations in exterior domains, J. Math. Anal. Appl. 75 (1980), 121–133

[14] O.G. Mustafa, Positive solutions of nonlinear differential equations with prescribed decay of the first derivative, Nonlinear Anal. TMA 60 (2005), 179–185

[15] A. Orpel, On the existence of positive radial solutions for a certain class of elliptic BVPs, J. Math. Anal. Appl. 299 (2004), 690–702

[16] E. Wahlén, Positive solutions of second-order differential equations, Nonlinear Anal. TMA 58 (2004), 359–366

[17] J.S.W. Wong, On second order nonlinear oscillation, Funkc. Ekvac. 11 (1968), 207–234

[18] Z. Yin, Monotone positive solutions of second-order nonlinear differential equations, Nonlinear Anal. TMA 54 (2003), 391–403