THE SPECTRUM OF HYPERSURFACE SINGULARITIES

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Abstract. This text is the write-up of a series of lectures on the asymptotic mixed Hodge theory of isolated hypersurface singularities, held at the Third Latin American school on Algebraic Geometry and its applications (ELGA 3) in Guanajuato, Mexico, in august 2017. Its focus is on the classical application of the semi-continuity of the spectrum due to Varchenko and Steenbrink to the problem of bounding the possible singularities on a projective hypersurface.

1. Lecture 1, Monday August 7, 2017

1.1. Motivation from classical algebraic geometry. A hypersurface $Z$ of degree $d$ in projective space $\mathbb{P}^n$ is given as solution set $\{F=0\}$ of a single homogeneous polynomial

$$F(x_0,x_1,\ldots,x_n) \in k[x_0,x_1,\ldots,x_n]^d$$

of degree $d$. For curves in $\mathbb{P}^2$ and surfaces in $\mathbb{P}^3$ and $k=\mathbb{R}$ one can make nice pictures of these varieties, and their beauty has inspired generations of mathematicians. In the words of A. Clebsch:

“Es ist die Freude an der Gestalt in einem höheren Sinne, die den Geometer ausmacht.”

Alfred Clebsch (1833-1872) and his cubic surface.

For a general choice of coefficients the hypersurface $Z$ is smooth, but for special choice of coefficients the $n+1$ equations

$$\partial_0 F = \partial_1 F = \ldots = \partial_n F = 0, \quad \partial_i F := \frac{\partial F}{\partial x_i}$$

\(^1\)“It is the joy of shape in a higher sense that makes the geometer”, written by A. Clebsch in his obituary for J. Plücker, Zum Gedächtnis an Julius Plücker, Dieterichschen Buchhandlung, Göttingen, (1872).
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may acquire one or more solutions and define the singular set $\Sigma$ of $Z$. The question what types of singularities may occur on a hypersurface of degree $d$ is known only in a very restricted number of cases. For cubic curves this goes back essentially to NEWTON, for cubic surfaces to SCHLÄFLI, who listed all possible types of singularities. The simplest type of singularity is the node, also called ordinary double point or $A_1$-singularity.

Such singularities appear as isolated points on the hypersurface and one may ask:

**Question:**

What is the maximal number $\mu_n(d)$ of $A_1$-singularities that can occur on a degree $d$ hypersurface in $\mathbb{P}^n$?

The value of $\mu_n(d)$ is known only for few values of $n$ and $d$.

Let us first look at the case $n = 2$, the case of curves of degree $d$ in $\mathbb{P}^2$. If $C \subset \mathbb{P}^2$ is an irreducible plane curve of degree $d$ and

$$n : \tilde{C} \to C$$

the normalisation map, then the genus of $\tilde{C}$ is given by the formula

$$g(\tilde{C}) = \frac{(d - 1)(d - 2)}{2} - \delta,$$

where

$$\delta = \sum_{p \in C} \delta(C, p).$$

Here $\delta(C, p)$ is the so-called $\delta$-invariant of the curve singularity $(C, p)$. Classically it is known under the name of virtual number of double points, as it can be shown to be equal to the number of $A_1$-singularities that emerge from $(C, p)$ if we perturb the normalisation map slightly. As $g(\tilde{C}) \geq 0$, we clearly have

$$\#A_1 \text{- singularities on } C \leq \frac{(d - 1)(d - 2)}{2}.$$

Equality happens precisely if the normalisation $\tilde{C}$ is rational. However, if we allow $C$ to be reducible, more double points can be created, and it is not hard to see that the maximal number is realised by curves $C$ that are union of $d$ lines in general position:

$$\mu_2(d) = \frac{d(d - 1)}{2}.$$

The case $n = 3$ of surfaces in three-space is rooted deeply in classical algebraic geometry and was a major research direction in Italian algebraic geometry. Here the problem is incomparably more difficult than for curves and the number $\mu_3(d)$ is currently known only for $d \leq 6$.

| $d$  | 1  | 2  | 3  | 4  | 5  | 6  | 7   |
|------|----|----|----|----|----|----|-----|
| $\mu_3(d)$ | 0  | 1  | 4  | 16 | 31 | 65 | $99 \leq \mu_3(d) \leq 104$ |
The is a rich corpus of algebraic geometry around the surfaces with many singularities.

Cayleys four nodal cubic and Kummers 16-nodal quartic.

In 1906 A.B. Basset gave the general upper bound
\[
\mu_3(d) \leq \frac{1}{2} (d(d-1)^2 - 5 - \sqrt{d(d-1)(3d-14) + 25})
\]
which was for a long time the best result known.

The bound of Miyaoka and Yau
\[
\mu_3(d) \leq \frac{4}{9} d(d-1)^2, \quad (d \geq 4)
\]
for the number of nodes on a surface of general type is the best currently known for degree \(d \geq 14\).

1.2. Spectral upper bound. In 1981 J. B. Bruce used results on the topology of hypersurface singularities to give a new upper bound for the number of nodes on a projective hypersurface. Probably inspired by this result, V. I. Arnol’d formulated in the same year a general upper bound for \(\mu_n(d)\) in terms of the counting of lattice points in a hypercube between two hyperplanes orthogonal to the long diagonal. This conjecture was proven by A. Varchenko.

**Theorem (A. Varchenko, 1983):**
\[
\mu_n(d) \leq A_n(d),
\]
where
\[
A_n(d) := \# \left\{ (k_1, k_2, \ldots, k_n) \in (0, d)^n \cap \mathbb{Z}^n \mid \frac{1}{2} nd - d + 1 < \sum_{i=1}^{n} k_i \leq \frac{1}{2} nd \right\}
\]
is the so-called Arnol’d number. It is elementary to give closed formulas for \(A_n(d)\) for fixed \(n\). For example
\[
A_3(d) = \begin{cases} 
\frac{23}{48} d^3 - \frac{9}{8} d^2 + \frac{5}{6} d & d \equiv 0 \mod 2 \\
\frac{23}{48} d^3 - \frac{23}{16} d^2 + \frac{73}{48} d - \frac{9}{16} & d \equiv 1 \mod 2,
\end{cases}
\]
so for large \(d\)
\[
A_3(d) \sim \frac{23}{48} d^3.
\]
Varchenko’s bound comes from properties of the \textit{mixed Hodge structure} on the vanishing cohomology of a hypersurface singularity, more specifically from the properties of the \textit{spectrum} of such singularities, a notion introduced by J. Steenbrink in 1976. The definition and proofs involve a mix of topology, algebra and analysis and will be the subject of these lectures.

1.3. **Hypersurface singularities.** We will consider the ring of germs of holomorphic functions

\[
\mathbb{C}\{x_0, x_1, \ldots, x_n\} = \mathcal{O}(\mathbb{C}^{n+1}, 0) =: \mathcal{O}_{n+1} =: \mathcal{O}.
\]

It is a local \(\mathbb{C}\)-algebra, with maximal ideal

\[
\mathcal{M} = (x_0, x_1, \ldots, x_n) \subset \mathcal{O}.
\]

The powers of \(\mathcal{M}\) form a descending filtration on \(\mathcal{O}\):

\[
\mathcal{O} \supset \mathcal{M} \supset \mathcal{M}^2 \supset \ldots \supset \mathcal{M}^k \supset \ldots \supset (0).
\]

The series \(f \in \mathcal{M}^k\) are precisely those with vanishing Taylor series up to order \(k - 1\). If \(f \in \mathcal{M}^2\), the series has no linear term and thus has a \textit{critical point} at the origin.

We will refer to all series \(f \in \mathcal{M}\) as a \textit{singularity}.

The automorphisms of the algebra \(\mathcal{O}\) can be identified with the group of coordinate transformations. One says that \(f, g \in \mathcal{O}\) are \textit{right equivalent}, notation \(f \sim g\), if \(f\) and \(g\) differ by a coordinate transformation. For example

\[
x^n + x^{n+1} = (x \sqrt{1 + x^n}) \sim x^n,
\]

where we use the coordinate transformation \(x \mapsto x \sqrt{1 + x}\). If \(f \in \mathcal{M}\), but \(f \notin \mathcal{M}^2\), then the implicit function theorem implies

\[
f \sim x_1.
\]

The \textit{Morse-lemma} states that if \(\phi \in \mathcal{M}^3\), then

\[
x_0^2 + x_1^2 + \ldots + x_n^2 + \phi \sim x_0^2 + x_1^2 + \ldots + x_n^2.
\]

The classification of functions up to right equivalence starts with the A-D-E list, and has been pushed to much further. But of course the complexity increases without bound, and there is no way in which such classification can ever be finished.

The \textit{Jacobian ideal} of \(f \in \mathcal{O}\) is the ideal

\[
J_f = (\partial_0 f, \partial_1 f, \ldots, \partial_n f) \subset \mathcal{O}, \quad \partial_i f := \frac{\partial f}{\partial x_i}.
\]

The \textit{Milnor algebra} of \(f\) is the factor ring

\[
\mathcal{O}/J_f
\]

and the \textit{Milnor number} \(\mu(f)\) is defined to be

\[
\mu(f) := \dim \mathcal{O}/J_f.
\]

If

\[
\mu(f) < \infty,
\]
then one says that \( f \) defines an isolated singularity. As an example, take \( f = x^3 + y^4 \). Its Jacobian ideal is \((x^2, y^3)\). So the Milnor algebra has a basis consisting of (the classes of) the monomials
\[
x^i y^j, \quad i = 0, 1, \quad j = 0, 1, 2
\]
so \( \mu(f) = 6 \). Note that \( \mu(f) \) is an invariant of a singularity: if \( f \sim g \), then \( \mu(f) = \mu(g) \).

A fundamental result in singularity theory is the

*Finite Determinacy Theorem*:
If \( \mu = \mu(f) < \infty \) and \( \phi \in M^{\mu+2} \), then
\[
f + \phi \sim f.
\]
In particular, any isolated singularity is right equivalent to a polynomial!

A *Brieskorn-Pham singularity* is a singularity of the form
\[
f = x_0^{d_0} + x_1^{d_1} + x_2^{d_2} + \ldots + x_n^{d_n},
\]
where \( d_i \geq 2 \). Its Milnor-algebra has a basis of monomials of the form
\[
x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n}, \quad 0 \leq k_i \leq d_i - 2,
\]
and hence
\[
\mu(f) = (a_0 - 1)(a_1 - 1)\ldots(a_n - 1).
\]
In particular, if all \( a_i \) are equal to \( d \), the Milnor algebra has a basis consisting of the monomials inside a \((n + 1)\)-dimensional cube.

### 1.4. Spectrum of a hypersurface singularity

We will give a precise definition of the spectrum later. Here we will give the main properties that allows one to compute the spectrum of a wide range of singularities.

1. The spectrum \( sp(f) \) is a (multi)-set of \( \mu := \mu(f) \) rational numbers called *spectral numbers*:
   \[
   sp(f) = \{ a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_\mu \}.
   \]
   Sometimes it is convenient to pack the information of the spectrum in a *spectral polynomial*:
   \[
   Sp(f) = \sum_{\alpha \in \mathbb{Q}} n_\alpha s^\alpha \in \mathbb{Z}[s^\alpha, \alpha \in \mathbb{Q}],
   \]
   where \( n_\alpha \) denotes the multiplicity with which \( \alpha \) appears in the spectrum of \( f \). Sometimes the spectral numbers are called the *exponents* of a singularity.

2. The spectrum is an invariant of a singularity: if \( f \sim g \), then \( sp(f) = sp(g) \). In fact, something stronger is true: if \( f \sim u.g \), where \( u \) is a unit of \( \mathcal{O} \), we say that \( f \) and \( g \) are contact equivalent and then \( sp(f) = sp(g) \).

3. Range: \( sp(f) \subset (0, n + 1) \).
(4) Symmetry: $\alpha_i + \alpha_{\mu-i} = n + 1$.

(5) Thom-Sebastiani: If $f \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$ and $g \in \mathbb{C}\{y_0, y_1, \ldots, y_m\}$ are two series in separate sets of variables, then

$$f \oplus g = f(x_0, x_1, \ldots, x_n) + g(y_0, y_1, \ldots, y_m) \in \mathbb{C}\{x_0, \ldots, y_m\}$$

is called the Thom-Sebastiani sum of $f$ and $g$. Then:

$$sp(f \oplus g) = \{\alpha + \beta \mid \alpha \in sp(f), \beta \in sp(g)\},$$

which can be expressed in term of the spectral polynomials as

$$Sp(f \oplus g) = Sp(f) \cdot Sp(g)$$

(6) $sp(x^m) = \{\frac{1}{m}, \frac{2}{m}, \frac{3}{m}, \ldots, \frac{m-1}{m}\}$

From these rules one can compute the spectrum for any Brieskorn-Pham singularity.

Examples:

(1) $sp(x^2) = \{\frac{1}{2}\}$, so

$$sp(x_0^2 + x_1^2 + \ldots + x_n^2) = \left\{\frac{n+1}{2}\right\}.$$

The spectrum consists of a single point at the center of the range $(0, n+1)$.

(2) $sp(x^2 + y^3) = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{2} + \frac{2}{3}\right\} = \left\{\frac{5}{6}, \frac{7}{6}\right\}$.

(3) $sp(x^4 + y^4) = \left\{\frac{2}{4}, \frac{3}{4}, \frac{3}{4}, \frac{4}{4}, \frac{4}{4}, \frac{4}{4}, \frac{5}{4}, \frac{5}{4}, 6\right\}$.

Here the spectral numbers appear with a multiplicity, which one often writes above the spectral number:

$$\begin{array}{cccccc}
\frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{2}{4} & \frac{1}{2} \\
\frac{2}{4} & \frac{3}{4} & \frac{5}{4} & \frac{6}{4} & \frac{6}{4}
\end{array}$$

(4) In general, for a Brieskorn-Pham singularity

$$f = x_0^{a_0} + x_1^{a_1} + \ldots + x_n^{a_n},$$

we can assign to each monomial

$$x_0^{k_0} x_1^{k_1} \ldots x_n^{k_n}, \quad 0 \leq k_i \leq a_i - 2,$$

of the Milnor-algebra $O/J_f$ the number

$$\sum_{i=0}^{n} \frac{k_i + 1}{a_i}.$$

The (multi)-set of numbers so obtained is precisely $sp(f)$. 
In fact, as we will see, it is more natural to interpret the number \( \sum_{i=0}^{n} \frac{k_i + 1}{a_i} \) as the quasihomogeneous weight of the corresponding differential \((n + 1)\)-form

\[x_0^{k_0}x_1^{k_1} \ldots x_n^{k_n}dx_0 \wedge dx_1 \wedge \ldots \wedge dx_n,\]

where we put

\[w(x_i) = \frac{1}{a_i},\]

so that \( w(f) = 1. \)

Recall that more generally a polynomial \( f \) is called quasi-homogeneous with weights \( w_0, w_1, w_2, \ldots, w_n, w_i \in \mathbb{Q}_{>0} \), if all monomials \( x_0^{v_0}x_1^{v_1} \ldots x_n^{v_n} \) occurring in \( f \) are on the hyperplane with the equation

\[w_0v_0 + w_1v_1 + \ldots + w_nv_n = 1.\]

We call

\[w(x_0^{k_0}x_1^{k_1} \ldots x_n^{k_n}dx_0 \wedge dx_1 \wedge \ldots \wedge dx_n) := \sum_{i=0}^{n} w_i(k_i + 1)\]

the weight of the monomial differential form. We define the Milnor module as

\[\Omega_f := \Omega^{n+1}/df \wedge \Omega^n = (\mathcal{O}/f) \ dx_0 \wedge dx_1 \wedge \ldots \wedge dx_n.\]

Then the spectral polynomial of \( f \) is just the weighted Poincaré-series of the \(\mathbb{Q}\)-graded Milnor module

\[\Omega_f = \bigoplus_{\alpha \in \mathbb{Q}} \Omega_{f,\alpha},\]

that is:

\[Sp(f) = \sum_{\alpha \in \mathbb{Q}} \dim(\Omega_{f,\alpha})s^\alpha.\]

1.5. Varchenko’s theorem. The aim of these lectures is to describe a proof of the following powerful theorem:

**Theorem (Varchenko, 1983):**

Let \( Z \subset \mathbb{P}^n \) be a hypersurface of degree \( d \) with isolated singular points \( p_1, p_2, \ldots, p_N \). Let the singularity \((\overline{Z}, p_i)\) be described by \( f_i \in \mathbb{C}\{x_1, x_2, \ldots, x_n\} \). Then for each \( \alpha \in \mathbb{R} \) one has an inequality

\[\#(\alpha, \alpha + 1) \cap sp(x_0^d + x_1^d + \ldots + x_n^d) \geq \sum_{i=1}^{N} \#(\alpha, \alpha + 1) \cap sp(f_i).\]

The combined number of spectral numbers for all singularities on a degree \( d \) hypersurface in any open interval of length one is bounded above by the number of spectral numbers in the corresponding interval for the singularity \( x_0^d + x_1^d + \ldots + x_n^d \).

The spectral numbers of \( f = x_0^d + x_1^d + \ldots + x_n^d \) are the weights

\[\sum_{i=0}^{n} \frac{k_i}{d}\]
of the monomials
\[ x_0^{k_0-1} x_1^{k_1-1} \cdots x_n^{k_n-1} dx_0 \wedge dx_1 \wedge \cdots \wedge dx_n, \]
where
\[ 0 < k_i < d. \]
The open interval \((\frac{n-1}{2} + \epsilon, \frac{n+1}{2} + \epsilon)\) contains the spectral number \(\frac{n+1}{2}\) of the \(A_1\)-singularity, so the number of \(A_1\) singularities is bounded by the number of spectral numbers of \(f\) in the corresponding interval, which is exactly the Arnold-number \(A_n(d)\!\).

**Examples:**

(1) The Milnor number of \(f = x^d + y^d\) is \((d-1)^2\). There are precisely \(d - 1\) spectral numbers equal to 1. So the intervals \((0, 1)\) and \((1, 2)\) both contain \[((d - 1)^2 - (d - 1))/2 = (d - 1)(d - 2)/2\) spectral numbers. An interval of length 1 containing 1 thus contains at least \((d - 1) + (d - 1)(d - 2)/2 = d(d - 1)/2\) spectral numbers. We find that there can be at most \(d(d - 1)/2\) nodes on a degree \(d\) curve.

(2) From now we write the multiplicities above the corresponding spectral number:

\[ sp(x^3 + y^3 + z^3) = \begin{array}{cccc}
1 & 3 & 3 & 1 \\
3 & 4 & 3 & 6 \\
4 & 5 & 3 & 6 \\
3 & 6 & 4 & 6 \\
1 & 3 & 6 & 7
\end{array} \]

The interval \((\frac{2}{3}, \frac{5}{3})\) contains 1 + 3 = 4 spectral numbers, so the maximum number of nodes on a cubic surface is at most four. The Cayley cubic realises this maximum.

(3)

\[ sp(x^4 + y^4 + z^4) = \begin{array}{cccccccc}
1 & 3 & 6 & 7 & 6 & 3 & 1 \\
3 & 4 & 5 & 6 & 4 & 7 & 8 & 9 & 4
\end{array} \]

The interval \((\frac{3}{4}, \frac{7}{4})\) contains 3 + 6 + 7 = 16 spectral numbers, so the maximum number of nodes on a quartic surface is at most 16. Any Kummer surface realises this maximum.

(4)

\[ sp(x^5 + y^5 + z^5) = \begin{array}{cccccccc}
1 & 3 & 6 & 10 & 12 & 10 & \cdots \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 5 & \cdots
\end{array} \]

The interval \((\frac{3}{5}, \frac{8}{5})\) contains 3 + 6 + 10 + 12 = 31 spectral numbers, so the maximum number of nodes on a quintic surface is at most 31. Any Togliatti-surface realises this maximum.

**Exercises:**

1) Work out the spectral bound for the number of \(A_2\)-singularities on a surface of degree \(\leq 5\).
2) Look up the equations for the A-D-E surface singularities and commit them to memory. Determine their spectra.

3) Schläfli determined all possible combinations of singularities that can occur on a cubic surface. Check that these combinations are exactly those that are allowed by the spectral bound. Verify for example the spectrum does not allow an $A_6$-singularity, but $A_5$ plus an additional $A_1$ is not excluded.

3) (i) Show that

$$x + y + z + u = 0, \quad x^3 + y^3 + z^3 + \frac{1}{4}u^3 = 0$$

defines a cubic surface with four nodes (Cayley’s cubic). Show that

$$x + y + z + u + v = 0, \quad x^3 + y^3 + z^3 + u^3 + v^3 = 0$$

defines a cubic in $\mathbb{P}^4$ with 10 nodes (Segres cubic).

(ii) Determine the number of nodes of the following cubic in $\mathbb{P}^n$:

$$\sum_{i=0}^{n+1} x_i = 0, \quad \sum_{i=0}^{n+1} x_i^3 = 0, \quad (n \text{ even})$$

$$\sum_{i=0}^{n} x_i = 0, \quad \sum_{i=0}^{n} x_i^3 + \frac{1}{4}x_{n+1}^3 = 0, \quad (n \text{ odd})$$

(iii) Show that $A_3(3) = 4$, $A_4(3) = 10$, $A_5(3) = 15$ and in general

$$A_n(3) = \binom{n + 1}{\left\lfloor \frac{n}{2} \right\rfloor}$$

and conclude the following result contained in the thesis of T. Kalker:

$$\mu_3(n) = A_3(n).$$

4) A general line through a general point $P$ of the Segre cubic $S \subset \mathbb{P}^4$ intersects $S$ in two further points and defines a birational map $\phi_P : S \to \mathbb{P}^3$ of degree 2, that ramifies along a Kummer surfaces, i.e. a quartic surface with 16 ordinary double points. Verify these facts.

2. Lecture 2, Tuesday August 8, 2017

The spectrum has a lot of deeper properties that we will encounter later. One of these is the following: if $\alpha \in \text{sp}(f)$, then

$$\lambda := \exp(2\pi i \alpha)$$

is an eigenvalue of the cohomological monodromy transformation. In other words, the spectral numbers $\alpha_1, \alpha_2, \ldots, \alpha_\mu$ are specific logarithms of the monodromy eigenvalues. To appreciate this, we first have to look deeper into the topology of an isolated hypersurface singularity.

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2 The spectrum shares this property with the roots of the b-function.
2.1. **Good representative of a germ.** A power series \( f \in M^2 \subset \mathcal{O} \) with \( \mu(f) < \infty \) defines a holomorphic function on a neighbourhood \( U \) and by shrinking we may suppose that \( 0 \) is the only critical point of the function \( f : U \to \mathbb{C} \), which is a singularity of the level set \( f^{-1}(0) \). For \( \epsilon \) small enough, the \( \epsilon \)-ball around \( 0 \)

\[
B_\epsilon := \{ x \in \mathbb{C}^{n+1} \mid |x| \leq \epsilon \}
\]

is contained in \( U \). It is a fundamental non-trivial fact about the analytic set \( f^{-1}(0) \) that one can find \( \epsilon > 0 \) with the property that the boundary \( \partial B_\epsilon \) is transverse to \( f^{-1}(0) \) for all \( 0 < \epsilon' \leq \epsilon \). We now take \( 0 < \eta \) such that for all \( t \in S_\eta := \{ t \in \mathbb{C} \mid |t| \leq \eta \} \) the level set \( f^{-1}(t) \) is transverse to \( \partial B_\epsilon \). One now puts

\[
X := X_{\epsilon, \eta} := B_\epsilon \cap f^{-1}(S_\eta),
\]

\[
S := S_\eta.
\]

The function \( f : U \to \mathbb{C} \) restricts to a function, again called \( f \),

\[
f : X \longrightarrow S
\]

which is called a **good representative of the germ** \( f \in \mathcal{O} \). One puts for \( t \in S \):

\[
X_t := f^{-1}(t) \subset X, \quad X^\ast = X \setminus X_0, S^\ast := S \setminus \{0\}
\]

\( f \) restricts to a map

\[
f^* : X^\ast \longrightarrow S^\ast
\]

that is a locally trivial \( C^\infty \)-fibre bundle by the **Ehresmann fibration theorem**. It is called the **Milnor-fibration** associated to \( f \). All fibres \( X_t, t \in S^\ast \) are diffeomorphic complex \( n \)-dimensional manifolds with boundary the real \((2n-1)\)-dimensional manifold

\[
\partial X_t = X_t \cap \partial B_\epsilon.
\]

The manifolds \( X_t \) are called the **Milnor fibres** of our our germ \( f \in \mathcal{O} \). The fibration is trivial near the boundary and consequently all \( \partial X_t \) can be identified with \( L := \partial X_0 \), the **link** of the singularity. The space \( X_0 \) is homeomorphic to the topological cone over this link, so in particular it is contractible.
2.2. **Two fundamental theorems.** For the basic understanding of the topology of hypersurface singularities the following theorems are of fundamental importance:

**The bouquet theorem** (Milnor 1968): Let \( \mu := \mu(f) < \infty \).
Then the fibres \( X_t \) of the fibration \( X^* \longrightarrow S^* \) have the homotopy type of a bouquet of \( \mu \) \( n \)-spheres:
\[
X_t \equiv \vee_{i=1}^{\mu} S^n.
\]

**Corollary:** The homology of the Milnor fibre is non-vanishing only in degree 0 and \( n \). In fact, for the (reduced) homology in degree \( n \) one has:
\[
\tilde{H}_n(X_t) = \mathbb{Z}^\mu.
\]
There is more topological information in the fibre bundle \( X^* \longrightarrow S^* \). As \( S^* \) is homotopy equivalent to a circle, the space \( X^* \) can constructed from \( X_t \times [0,1] \) by identifying \( X_t \times \{0\} \) with \( X_t \times \{1\} \) using a glueing map
\[
\tau : X_t \longrightarrow X_t.
\]
The map \( \tau \) can be chosen to be the identity near \( \partial X_t \) and is called the **geometric monodromy**. It induces a homological monodromy transformation
\[
T = \tau_* : H_n(X_t, \mathbb{Z}) \longrightarrow H_n(X_t, \mathbb{Z})
\]
which, after a choice of basis, is represented by a \( \mu \times \mu \) integral matrix.

**The monodromy theorem (Brieskorn 1970):** The homological monodromy transformation
\[
T : H_n(X_t, \mathbb{Z}) \longrightarrow H_n(X_t, \mathbb{Z})
\]
is quasi-unipotent. More precisely, there exists a \( q \in \mathbb{N} \) such that
\[
(T^q - I\text{d})^{n+1} = 0
\]
In other words, the eigenvalues of \( T \) are roots of unity and the Jordan-blocks have size \( \leq n + 1 \).

The monodromy theorem is a general result in algebraic geometry and various proofs are known. Especially for the case of singularities a beautiful proof was given by Brieskorn in 1970.

**Examples:**

1) The \( A_1 \)-singularity.

If \( n = 0 \) we are dealing with the map \( x \mapsto x^2 \). The Milnor fibre consists of two points, which are interchanged under the monodromy. On the reduced homology group \( \tilde{H}_0(X, \mathbb{Z}) \) the monodromy transformation \( T \) is multiplication by \(-1\).

If \( n = 1 \) the Milnor fibre \( X_t \) defined by \( x^2 + y^2 = t \) has the topology of a cylinder; the homology group is generated by the real circle
\[
\delta(t) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = t\}.
\]
Projection of the Milnor fibre onto the $x$-line realises it as a two-fold cover, ramified at the two points $x = \pm \sqrt{t}$. If we run with $t$ once around the origin, these two ramification points interchange, but the vanishing cycle ‘stays the same’:

The monodromy acts as identity on $H_1(X_t, \mathbb{Z})$.

Similarly in higher dimensions: for $n$ even, the monodromy is multiplication with $-1$, whereas for $n$ odd, it is the identity.

However, it is important to realise that even for $n = 1$ the monodromy transformation is geometrically a highly non-trivial map.

From this sequence of pictures one sees that the monodromy map $\tau$ is what in topology is called a Dehn-twist. Its non-triviality can be detected by looking at its effect on the relative cycle $\gamma \in H_1(X_t, \partial X_t)$.

2) $A_2$-singularity $f = y^2 + x^3$. The Milnor fibre

$$X_t = \{ y^2 + x^3 = t \}$$

---

3Homologically there is a well-defined map

$$\text{Var} : H_1(X_t, \partial X_t, \mathbb{Z}) \to H_1(X_t, \mathbb{Z}), \ [\gamma] \mapsto [T \gamma - \gamma]$$

called the variation mapping.
has the topology of an elliptic curve, with a disc removed, so indeed has the homotopy type of the wedge of two circles. To ‘see’ the monodromy, we perturb the function $f$ and consider

$$f_\lambda = y^2 + x^3 - \lambda x$$

where $\lambda$ is a small positive real number. For a fixed value of $\lambda$, the function $f_\lambda$ has two critical points. These points appear when the cubic polynomial

$$x^3 - \lambda x - t = 0$$

has a repeated root. These points are of form $(x_+,0)$ and $(x_-,0)$ with critical values

$$t_+ = \sqrt{\frac{4\lambda^3}{27}}, \quad t_- = -\sqrt{\frac{4\lambda^3}{27}}.$$

We choose a basis of the homology $\delta_+ , \delta_-$ such that if we move towards $t_\pm$, the cycle $\delta_\pm$ is vanishing.

We orient the vanishing cycles such that the intersection product is given by

$$< \delta_+, \delta_- > = 1,$$

as indicated in the picture. We consider in the disc $S_\lambda \subset \mathbb{C}$ the straight line paths $\gamma_\pm$ from a base point $\ast$ to $t_\pm$. If we start from $\ast$, follow the path $\gamma_\pm$ to a point close to $t_\pm$ and encircle it in the positive direction, and then return to the base point $\ast$ we obtain two local monodromy transformations $T_\pm$.

From the analysis of the $A_1$-case we can see that

$$T_+(\delta_+) = \delta_+, \quad T_+(\delta_-) = \delta_- + \delta_+, \quad T_-(\delta_+) = \delta_+ - \delta_-, \quad T_-(\delta_-) = \delta_-$$
Hence, with respect to the basis $\delta_+, \delta_-$ these transformations $T_+$ and $T_-$ are represented by the following matrices:

$$T_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T_- = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$ 

The monodromy transformation $T$ of the $A_2$-singularity is obtained by encircling both critical points, so is given by the matrix product

$$T = T_- T_+ = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

It is easy to check that

$$T^6 = I$$

and the eigenvalues of $T$ are the primitive sixth roots of unity

$$e^{2\pi i/6} \text{ and } e^{-2\pi i/6}.$$ 

The above idea of using a small perturbation to analyse the topology of a singularity is useful in many other situations as well. Using these ideas that can be traced back to 

Lefschetz, Brieskorn gave an alternative proof of the bouquet theorem, that we will sketch now.

(1) We consider a perturbation of our function $f$ of the form

$$f_\lambda = f + \sum_{i=0}^n \lambda_i x_i.$$ 

If we take the $\lambda_i$ small enough, the function $f_\lambda$ will have a finite number of critical points $p_1, p_2, \ldots, p_N$ in the set $X_\lambda := f_\lambda^{-1}(S_\eta)$. As the critical space, defined by the vanishing of the partial derivatives

$$\partial_0 f_\lambda = \partial_1 f_\lambda = \ldots = \partial_n f_\lambda$$

is a complete intersection, it is Cohen-Macaulay, hence the sum of the Milnor numbers at the points $p_i, i = 1, \ldots, N$ add up to the value at $\mu = \mu(f)$ for $\lambda = 0$.

(2) By choosing the $\lambda_i$ not only small, but also generic, the critical points of $f_\lambda$ will all be non-degenerate and have different critical values under the map $f_\lambda$. In particular, there will be exactly $\mu$ critical points and $\mu$ critical values in the disc $S_\eta$. We obtain what is called a morsification of $f$.

(3) Now choose paths $\gamma_i, i = 1, 2, \ldots, \mu$ connecting a base point $* \in \partial S_\eta$ with the critical values $t_1, t_2, \ldots, t_\mu$. 


(4) The preimage of the disc $S_\eta$ is $X_\lambda$, which is contractible. Furthermore, the disc $S_\eta$ contracts to the union of the paths $\gamma_i$, and their preimage under $f_\lambda$ contracts to the fibre $X_*$ over the base point, to which we glue the Lefschetz thimbles $\Gamma_i, i = 1, 2, \ldots, \mu$, lying over the paths $\gamma_i, i = 1, 2, \ldots, \mu$. So we see that by glueing $\mu$-thimbles to the fibre $X_*$ we obtain a contractible space.

(5) The thimbles are topological discs of real dimension $n$. So by glueing $\mu$ $n$-discs to $X_*$ we obtain a contractible space. It follows from CW-topology that $X_*$ can only have the homotopy type of a bouquet of $n$-spheres!

It is far more difficult to understand the nature of the monodromy transformation from the above morsification picture.

**Theorem** (F. Pham, 1965):

For the singularity $f = x_0^{a_0} + x_1^{a_1} + \ldots x_n^{a_n}$ with Milnor number

$$\mu = (a_0 - 1)(a_1 - 1)\ldots(a_n - 1),$$

the monodromy

$$T : H_n(X_t) \longrightarrow H_n(X_t)$$
is of finite order

\[ e := \text{lcm}(a_0, a_1, \ldots, a_n). \]

The eigenvalues of \( T \) are the numbers

\[ \omega_0 \omega_1 \ldots \omega_n, \]

where the \( \omega_i \) are (non-trivial) \( a_i \)-th roots of unity.

Because \( f \) is obtained as Thom-Sebastiani sum from functions depending on a single variable, it is easy to prove the result, using the fact for a Thom-Sebastiani sum the homology and monodromy transformation can be identified with the tensor product of the factors:

\[
\tilde{H}_n(X(f \oplus g)) = \tilde{H}_n(X(f)) \otimes \tilde{H}_n(X(g)), \quad T(f \oplus g) = T(f) \otimes T(g)
\]

Furthermore, Lê had shown that the monodromy to be of finite order for any irreducible curve singularity and the question arose if the monodromy of any isolated singularity was always of finite order.

2.3. Examples of A’Campo and Malgrange. The answer is: NO! The first examples were given by N. A’Campo in 1973. Consider the function

\[ f = (x^2 + y^3)(x^3 + y^2) = x^2y^2 + x^5 + y^5 + x^3y^3 \sim x^2y^2 + x^5 + y^5 \]

Its zero-level consist of two transverse cuspidal branches.

Let us consider the embedded resolution of this function. Blowing up the origin introduces a component, along which the pull-back of \( f \) vanishes with order \( 4 = \text{mult}_0(f) \). The strict transform consists of smooth branches, which are tangent to this divisor. Blowing up at these two points introduces two further exceptional curves, along which the function (always pull-back) vanishes with order \( 4 + 1 = 5 \). The strict transforms now are transverse to the first divisor and the newly introduced divisors. Blowing once more at these two points introduces two further curves, along which the pull-back of \( f \) vanish with order \( 1 + 5 + 4 = 10 \). The configuration of exceptional curves now looks as follows.
The strict transforms of the two branches are indicated by an arrow. This configuration now leads to a very precise model of the Milnor-fibre, obtained from cyclic coverings of these curves determined by the multiplicities of \( f \). As explained in detail in the book of Brieskorn and Knörrer, the result looks as follows:

In this way A’Campo showed that the curve \( \gamma \) in the picture satisfies \((T^{10} - I)\gamma \neq 0\), so one has

\[
(T^{10} - I) \neq 0, \quad (T^{10} - I)^2 = 0
\]

In his paper, A’Campo asked if there were examples of isolated singularities in \( n + 1 \) variables for which the monodromy has a Jordan block of size \( n + 1 \). It is hard to analyse higher dimensional examples along these geometrical lines and the question was answered by Malgrange, using a totally different kind of argument.

**Theorem (B. Malgrange, 1973):**

The monodromy transformation \( T \) of the singularity

\[
f = (x_0x_1 \ldots x_n)^2 + x_0^{2n+4} + x_1^{2n+4} + \ldots + x_n^{2n+4}
\]

has a Jordan block of size \( n + 1 \).

The argument given by Malgrange runs as follows. For \( t \) real and positive, the set

\[
\delta(t) := \{ x \in \mathbb{R}^{n+1} \mid f(x) = t \} \subset X_t
\]

has the topology of an \( n \)-dimensional sphere. It is the boundary of the set

\[
E(t) := \{ x \in \mathbb{R}^{n+1} \mid f(x) \leq t \}.
\]
Clearly, $E(t)$ is topologically an $(n+1)$-ball; it can be seen as the Lefschetz thimble of the vanishing cycle $\delta(t)$. Now consider

$$I(t) := \int_{\delta(t)} x_0 dx_1 dx_2 \ldots dx_n = \int_{E(t)} dx_0 dx_1 \ldots dx_n = \text{Vol}(E(t)).$$

MALGRANGE claims that for $t \to 0$, the integral $I(t)$ behaves like

$$I(t) \sim Ct^{1/2}(\log t)^n,$$

where $C$ is a constant. If we analytically continue $I(t)$ along a path going once around the origin in the positive direction, $I(t)$ changes its value to

$$(-t^{1/2})(\log t + 2\pi i)^n,$$

and will not return to its original value by going around any number of times. On the other hand, this is equal to the integral of $\omega = x_0 dx_1 dx_2 \ldots dx_n$ over the transformed cycle $T\delta(t)$. Clearly, $T$ can not be of finite order, and a little further thinking shows that the $(\log t)^n$-term creates a Jordan block of size $n+1$.

How to find the asymptotic expansion of the integral $I(t)$? For this Malgrange suggests the following method. Let

$$F(t) := \{ x \in \mathbb{R}^{n+1} \mid (x_0 x_1 \ldots x_n)^2 \leq t, x_i^{2n+4} \leq t \}.$$

Then it is easy to see that

$$F \left( \frac{t}{n+1} \right) \subset E(f) \subset F(t),$$

so that it is sufficient to study the asymptotic expansion of $\text{Vol}(F(t))$, which is elementary. To illustrate the point, let us look at the case $n = 2$, so we look at the curve $\delta(t) : (xy)^2 + x^6 + y^6 = t$ for various values of $t$

![Diagram](image)

**The vanishing cycle $\delta(t)$ for smaller and smaller values of $t$.**

The volume of the region $F(t)$ is, up to a factor four, equal to the area of the subset of the quadrant $x \geq 0, y \geq 0$ given by the inequalities

$$xy \leq t^{1/2}, \quad x \leq t^{1/6}, \quad y \leq t^{1/6}.$$
Doing the integral, we find

\[ \text{Area}(F(t)/4) = t^{1/3}t^{1/6} + \int t^{1/6}t^{1/2}dx/x \]

\[ = t^{1/2} - \frac{1}{6}t^{1/2}\log t. \]

So just by looking at the shape of \( \delta(t) \) for small \( t \) is is ‘clear’ that a logarithm has to appear in the expansion, and hence that the monodromy is of infinite order!

**Exercise:** 1) Show that the Milnor fibre of the \( A_1 \)-singularity in dimension \( n \) can be identified with the disc bundle inside the cotangent bundle to the \( n \)-sphere.

2) Give an analysis of the monodromy of the \( A_3 \)-singularity using a morsification. Determine the three local monodromies and compute their product. What is the order of the resulting monodromy transformation?

3) Give an analysis of the Milnor-fibre of the \( A_2 \)-singularity using an embedded resolution. Show that one obtains the elliptic curve with an automorphism of order 6.

4) Work out the details for the expansion of the integrals considered by Malgrange. Can you find the sub-leading order?

**3. Lecture 3, Wednesday, August 9**

We saw yesterday in an example that it was useful to consider integrals of differential forms over vanishing cycles. Such integrals turn out to expand in a series of the form

\[ I(t) = \int_{\delta(t)} \omega = Ct^\alpha \log t^k + \ldots \]

Such an integral represents a multi-valued function on the punctured disc \( S^* \). If we fix a determination of this multi-valued function near the point \( t_0 \in \partial S \) and continue along the path \( t_0e^{i\phi} \), where the argument \( \phi \) runs from 0 to \( 2\pi \), the function \( I(t) \) changes to

\[ T(I(t)) = I(t \exp(2\pi i)). \]

Note that we will have the relation

\[ T(I(t)) = \int_{T\delta(t)} \omega. \]

It is easy to analyse what happens with the terms in the expansion: upon analytic continuation, the function \( t^\alpha \) changes to \( t^{\alpha + 2\pi i\alpha} \), so

\[ T(t^\alpha) = e^{2\pi i\alpha}t^\alpha. \]

In words, it gets multiplied by the number \( e^{2\pi i\alpha} \), which is a root of unity precisely when \( \alpha \in \mathbb{Q} \).

The logarithm \( \log t \) changes to \( \log t + 2\pi i \):

\[ T(\log t) = \log t + 2\pi i. \]
So we see that the pair of functions $\log t, 1$ create a Jordan-block of size 2:
\[
T \begin{pmatrix} \log t \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \log t \\ 1 \end{pmatrix}
\]
and
\[
(T - 1)(\log t) \neq 0, \quad (T - 1)^2(\log t) = 0.
\]
Similarly,
\[
T \begin{pmatrix} (\log t)^2 \\ \log t \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 4\pi i & (2\pi i)^2 \\ 0 & 1 & 4\pi i \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (\log t)^2 \\ \log t \\ 1 \end{pmatrix}
\]
so,
\[
(T - 1)^2(\log t)^2 \neq 0, \quad (T - 1)^3(\log t)^2 = 0.
\]

In general, if $\alpha = k/q$, then one has
\[
(T^q - I)^{n+1} \alpha (\log t)^n = 0.
\]

3.1. **What integrals to consider?** First, let us recall some aspects of the Poincaré residue. If $Z$ is a complex manifold and $V$ a smooth divisor defined by a holomorphic function $f$, we say that a holomorphic differential form $\eta$ is residue of a meromorphic form $\omega$ on $Z$, if we can write
\[
\omega = \eta \wedge df + \omega',
\]
where $\omega'$ is regular on $Z$. One writes
\[
\text{Res}_V(\omega) = \eta|_V.
\]
The residue theorem from the elementary theory of holomorphic functions has the following generalisation:
\[
\int_\delta \text{Res}_V(\omega) = \frac{1}{2\pi i} \int_\Delta \omega
\]
Here $\Delta \subset Z \setminus V$ is obtained from $\delta$ by applying the tube operation: we replace each point of the cycle $\delta$ by a small circle encircling the divisor $V$ in the positive direction.

We will apply this in particular to the situation that arises in the geometry of a good representative of an isolated hypersurface singularity. So, we consider a holomorphic $(n + 1)$-form on $X$:
\[
\omega \in \Gamma(X, \Omega_X^{n+1}).
\]
Let us fix $t \in S^*$ and consider the following differential form on the Milnor fibre $X_t$:
\[
\eta_t := \text{Res} \left( \frac{\omega}{f - t} \right) \in \Gamma(X_t, \Omega_{{X_t}}^n).
\]
The period integrals we want to consider are of the form
\[
I(t) := \int_{\delta(t)} \text{Res} \left( \frac{\omega}{f - t} \right) = \frac{1}{2\pi i} \int_{\Delta(t)} \frac{\omega}{f - t}.
\]
where $\delta(t) \in H_n(X_t, \mathbb{Z})$. If we fix a sector $W$ in the punctured disc $S^*$ and use parallel transport of a cycle $\delta(t_0) \in H_n(X_{t_0}, \mathbb{Z})$ to obtain cycles $\delta(t) \in H_n(X_t, \mathbb{Z})$ for $t \in W$, we obtain a well-defined holomorphic function $I(t)$ in $W$.

**Example:** Let us consider a quasi-homogeneous singularity $f$ with weights $w_i = \omega(x_i)$. Put

$$\omega = x_0^{\nu_0} x_1^{\nu_1} \ldots x_n^{\nu_n} dx_0 dx_1 \ldots dx_n.$$

For any choice of cycle $\delta(t)$, the integral is seen to have a scaling property: if we replace $x_i$ be $\lambda^{w_i} x_i$ in the integral, we find

$$I(t) = \frac{1}{2\pi i} \int_{\Delta(t)} \frac{\lambda^{w(\omega)} \omega}{\lambda f(x) - t} = \lambda^{w(\omega)} - 1 I(t/\lambda).$$

So putting $t = \lambda$ we get the scaling relation:

$$I(t) = t^{w(\omega)} - 1 I(1)$$

These period integrals are rather trivial. They have finite monodromy, but the weight of the differential form is reflected in the speed of vanishing of the period integral.

**The regularity theorem:**

All period integrals of the form

$$I(t) := \int_{\delta(t)} \text{Res} \left( \frac{\omega}{\lambda f(x) - t} \right)$$

attached to an isolated hypersurface singularity have an expansion (convergent on any sector $W \subset S^*$) of the form

$$I(t) = \sum_{\alpha, k} C_{\alpha, k} t^{\alpha} (\log t)^k$$

Furthermore:

- $e^{2\pi i \alpha}$ is an eigenvalue of the monodromy $T : H_n(X_t, \mathbb{Z}) \rightarrow H_n(X_t, \mathbb{Z})$.
- $0 \leq k \leq n + 1$.
- $\alpha > -1$.

The last inequality is of particular importance, as was first recognised by Malgrange. It can be deduced from the following

**Positivity Lemma (Malgrange (1974)):**

If $\eta \in \Gamma(X, \Omega^n_X)$ is a global $n$-form on $X$ and if we put

$$\eta_t := \eta|_{X_t} \in \Gamma(X_t, \Omega^n_{X_t})$$

then, if $t \rightarrow 0$:

$$\int_{\delta(t)} \eta_t \rightarrow 0.$$

The idea is that the integral of a holomorphic differential form over cycles that shrinks to 0 has to go to zero, just because we integrating over smaller and smaller sets.
3.2. **Connection on the cohomology bundle.** We have seen that the restriction of \( f : X \longrightarrow S \) to \( X^\ast \) defines a is a locally trivial fibre bundle over \( S^\ast \). The direct image
\[
R^n f_* \mathbb{Z}_{X^\ast}
\]
is a *local system* with fibre over \( t \in S^\ast \) equal to \( H^n(X_t, \mathbb{Z}) \). The monodromy transformation \( T \) describes the non-trivial twisting in this lattice bundle. We will denote its complexification by \( H_S^* \):
\[
H^*_S = \mathbb{C} \otimes_{\mathbb{Z}} R^n f_* \mathbb{Z}_{X^\ast} = R^n f_* \mathbb{C}_{X^\ast}
\]
One may associate to this local system of \( \mu \)-dimensional vector spaces a vector bundle of rank \( \mu \) by tensoring this with the holomorphic functions on \( S^\ast \):
\[
\mathcal{H}^*_S := \mathcal{O}_S \otimes_{\mathbb{C}} H^*_S
\]
which we will call the *cohomology bundle*. Its sections describe cohomology classes that depend holomorphically on \( t \in S^\ast \). On the vector bundle \( \mathcal{H}^*_S \) there is a tautological connection
\[
\nabla : \mathcal{H}^*_S \longrightarrow \Omega^1_S \otimes \mathcal{H}^*_S,
\]
defined by
\[
\nabla (g \otimes h) = dg \otimes h.
\]
If we denote by \( t \) the local coordinate on \( S \), then we have the standard vector field \( \frac{\partial}{\partial t} \) dual to \( dt \) and we get a map
\[
\partial_t := \nabla_{\partial/\partial t} : \mathcal{H}^*_S \longrightarrow \mathcal{H}^*_S, \quad g \otimes h \mapsto \frac{dg}{dt} \otimes h.
\]
The local system of *horizontal sections* of the cohomological bundle is identified with the local system we started with:
\[
H^*_S = \text{Ker}(\partial_t) \subset \mathcal{H}^*_S.
\]
One would like to extend \( \mathcal{H}^*_S \) to a vector bundle \( \mathcal{H}_S \) on \( S \supset S^\ast \). This is easy and can be done in many possible ways. As \( S^\ast \) is a Stein space, the vector bundle \( \mathcal{H}^*_S \) is in fact a *trivial* vector bundle. We can pick *any* basis of sections \( e_1, e_2, \ldots, e_\mu \in \Gamma(S^\ast, \mathcal{H}^*_S) \) and consider the vector bundle
\[
\mathcal{H}_S := \bigoplus_{i=1}^\mu \mathcal{O}_S e_i \subset j_*\mathcal{H}^*_S,
\]
where \( j : S^\ast \longrightarrow S \) is the inclusion map.

What happens to the connection? Well, we can write out the effect of \( \partial_t \) on the basis sections \( e_i \):
\[
\partial_t e_i = \sum_{i=1}^\mu A_{ij}(t)e_j,
\]
and obtain a matrix of holomorphic function on \( S^\ast \)
\[
A(t) = (A_{ij}(t)) \in \text{Mat}(\mu \times \mu, \mathcal{O}_S^*),
\]
called the connection matrix. A section
\[
v = \sum_{i=1}^\mu v_i(t)e_i
\]
will be horizontal if and only if the coefficients $v_i(t)$ solve the system of differential equations

$$\frac{d}{dt} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_\mu \end{pmatrix} = -A(t) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_\mu \end{pmatrix}.$$ 

It is of importance to realise that the matrix $A(t)$ will not extend holomorphically over $0$. If that were to happen, the coefficients of the horizontal sections would be holomorphic at $0$ too, and as a consequence, the monodromy would be trivial. So the best one can hope for is a matrix $A(t)$ with a pole at $0$.

We will describe a nice tautological way to do this with a pole of order one.

There is no such thing as the Milnor-fibre, as for each $t \in S^*$ we obtain a different manifold $X_t$. It is somewhat surprising that there is an object that might be called the canonical Milnor fibre: one simply takes the pull-back $X_\infty$ of the fibration $X^* \rightarrow S^*$ over the universal covering $\exp : \tilde{S}^* \rightarrow S^*$, i.e. we form the pull-back diagram

$$
\begin{array}{ccc}
X_\infty & \longrightarrow & X^* \\
\downarrow & & \downarrow \\
\tilde{S}^* & \xrightarrow{\exp} & S^* \\
\end{array}
$$

and we form the groups

$$H_Z := H^n(X_\infty, \mathbb{Z}), \quad H_C := \mathbb{C} \otimes \mathbb{Z} H_Z = H^n(X_\infty, \mathbb{C}).$$

The monodromy will induce an automorphism of $H_Z$ and $H_C$ that we will denote by $T$.

Let us fix an endomorphism $A \in \text{End}(H_C)$ with the property that

$$e^{2\pi i A} T = \text{Id}.$$

Clearly, if $T$ is a Jordan matrix with eigenvalue $\lambda$, then the choices for $A$ are determined by choices of exponents $\alpha$ such that

$$e^{-2\pi i \alpha} = \lambda.$$

From the vector space $H_C$ and $A$ we now can construct a specific extension of the cohomology bundle. Let $h \mapsto h(t)$ be the isomorphism between $H_C$ and $H^n(X_t, \mathbb{C})$ induced from the maps

$$X_t \hookrightarrow X_\infty, \quad t \in \tilde{S}^*$$

Then

$$s[h] := t^A h(t)$$

can be considered as a single valued section of $\mathcal{H}_{S^*}$: if we replace $t$ by $e^{2\pi i \phi} t$ and let $\phi$ increase from $0$ to $1$, we obtain

$$t^A t^{2\pi i A}(T(h))(t) = t^A h(t).$$

Now let

$$\mathcal{L} \subset j_*(\mathcal{H}_S)$$
be the $\mathcal{O}_S$-module generated by these $s[h]$, $h \in H_C$. It is called the canonical extension of $H_{S^*}$ associated to $A$. Note that
\[
t \frac{d}{dt} t^A = A t^A
\]
so that on $\mathcal{L}$ the connection has tautologically a first order pole:
\[
t \partial_t : \mathcal{L} \to \mathcal{L}.
\]
Note that the map
\[
H_C \to \mathcal{L}; \ h \to t^A h(t)
\]
induces a natural isomorphism
\[
H_C \xrightarrow{\sim} \mathcal{L} / t^\mathcal{L}
\]
under which the map $A \in \text{End}(H_C)$ is identified with $t \partial_t$, the so-called residue of the connection.

3.3. The Brieskorn module. We denote by
\[
\Omega^p := \Omega^p_{C^n,0}
\]
the $\mathcal{O}$-module of germs at the origin of holomorphic differential $p$-forms on $C^{n+1}$. The factor space
\[
\mathcal{H}^{(0)} := \Omega^{n+1} / df \wedge d\Omega^{n-1}
\]
is called the Brieskorn module. Note that what is divided out here is not an $\mathcal{O}$-submodule, so the Brieskorn module is not an $\mathcal{O}$-module. As we will see, this innocent looking algebraic object contains a wealth of information. We will see later that $\mathcal{H}^{(0)}$ can be seen as stalk at 0 of a specific extension of the cohomology bundle $H_{S^*}$.

It is readily verified that the Brieskorn module $\mathcal{H}^{(0)}$ carries two operations
\[
t : \mathcal{H}^{(0)} \longrightarrow \mathcal{H}^{(0)}, [\omega] \mapsto [f \omega]
\]
\[
b : \mathcal{H}^{(0)} \longrightarrow \mathcal{H}^{(0)}, [\omega = d\eta] \mapsto [df \wedge \eta]
\]
which satisfy the relation
\[
t b - b t = b^2.
\]
We now explain how the Brieskorn module $\mathcal{H}^{(0)}$ is naturally associated to the period integrals considered above. First note that the period integrals
\[
I(t) = \int_{\delta(t)} \text{Res}_{X_i} \left( \frac{\omega}{f-t} \right)
\]
only depend on the class $[\omega]$ of $\omega \in \mathcal{H}^{(0)}$: for $\omega = df \wedge d\lambda$ we find
\[
I(t) = \int_{\delta(t)} \text{Res}_{X_i} \left( \frac{d(f-t) \wedge d\lambda}{f-t} \right) = \int_{\delta(t)} d\lambda = 0
\]
as the integral of an exact form on a cycle vanishes by Stokes theorem. Furthermore,
\[
0 = \int_{\Delta(t)} \frac{(f-t)\omega}{f-t},
\]
so that indeed
\[
\int_{\delta(t)} \operatorname{Res} \left(\frac{f \omega}{f - t}\right) = t \int_{\delta(t)} \operatorname{Res} \left(\frac{\omega}{f - t}\right).
\]

Of particular importance is the derivative of a period integral. By differentiating under the integral sign and using the fact that one can freely 'move' the cycle \(\Delta(t)\) without changing the integral, we see that
\[
\frac{d}{dt} \int_{\Delta(t)} \frac{\omega}{f - t} = \int_{\Delta(t)} \left(\frac{\omega}{f - t}\right)^2.
\]

But this is no longer an integral of the type we were considering, as now we encounter a pole of order two along \(X_t\). However, one has the following key fact:

**Proposition:**
\[
\frac{d}{dt} \int_{\Delta(t)} b \omega = \int_{\Delta(t)} \frac{\omega}{f - t}
\]

**proof:** We write \(\omega = d\eta\) and have by definition \(df \wedge \eta = b\omega\). Then:
\[
\frac{d}{dt} \int_{\Delta(t)} \frac{df \wedge \eta}{f - t} = \int_{\Delta(t)} \frac{df \wedge \eta}{(f - t)^2} = \int_{\Delta(t)} \frac{\omega}{f - t}
\]
where we used the easy-to-check pole order reduction formula:
\[
\frac{df \wedge \eta}{(f - t)^2} = \frac{d\eta}{f - t} - d\left(\frac{\eta}{f - t}\right).
\]

The integral over the second term over \(\Delta\) clearly gives 0. \(\diamondsuit\)

This proposition shows what the mysterious operation \(b\) on the Brieskorn module really is: it is to be identified with the inverse of differentiation with respect to \(t\). One often writes:
\[
\partial_t^{-1} := b
\]

The following result of fundamental importance of was conjectured by Brieskorn (1970):

**Theorem (Sebastiani (1970):)** If \(f\) is an isolated singularity with Milnor-number \(\mu\), then \(\mathcal{H}^{(0)}\) is a free \(\mathbb{C}\{t\}\)-module of rank \(\mu\).

We will see later that the freeness is a consequence of the positivity lemma.

As a direct consequence, the factor space
\[
\mathcal{H}^{(0)}/t\mathcal{H}^{(0)} = \Omega^{n+1}/(df \wedge d\Omega^{n-1} + f\Omega^{n+1})
\]
is a \(\mathbb{C}\)-vector space of dimension \(\mu\). Note that the space
\[
\mathcal{H}^{(0)}/\partial_t^{-1}\mathcal{H}^{(0)} = \Omega^{n+1}/df \wedge \Omega^n =: \Omega_f
\]
is also a \(\mathbb{C}\)-vector space of dimension \(\mu\), but there is no simple canonical isomorphism between the two, unless \(f\) is weighted homogeneous, in which case we learn from the Euler relation
\[
\sum w_ix_i\partial_if = f
\]
that the two subspaces that are divided out are equal.

If we choose a basis

\[ \omega_1, \omega_2, \ldots, \omega_\mu \]

for \( \mathcal{H}^{(0)} \) as \( \mathbb{C}\{t\}\)-module, one can in principle write out the action of \( b = \partial_t^{-1} \) on this basis and obtain a \( \mu \times \mu \)-matrix \( B(t) \) of homomorphic function germs in \( t \) such that

\[
\partial_t^{-1} \left( \begin{array}{c} \omega_1 \\ \omega_2 \\ \cdots \\ \omega_\mu \end{array} \right) = B(t) \left( \begin{array}{c} \omega_1 \\ \omega_2 \\ \cdots \\ \omega_\mu \end{array} \right)
\]

For the corresponding period integrals

\[ I_i := \int_{\delta(t)} \operatorname{Res} \left( \frac{\omega_i}{f - t} \right) \]

over some vanishing cycle \( \delta(t) \) we thus find

\[
\begin{pmatrix} I_1 \\ I_2 \\ \cdots \\ I_\mu \end{pmatrix} = \frac{d}{dt} (B(t)) \begin{pmatrix} I_1 \\ I_2 \\ \cdots \\ I_\mu \end{pmatrix}
\]

which amounts to a linear system of differential equations for the period integrals

\[
\frac{d}{dt} \left( \begin{array}{c} I_1 \\ I_2 \\ \cdots \\ I_\mu \end{array} \right) = A(t) \left( \begin{array}{c} I_1 \\ I_2 \\ \cdots \\ I_\mu \end{array} \right),
\]

where

\[
A(t) = B(t)^{-1} \left( I - \frac{d}{dt} B(t) \right).
\]

As we know from the positivity lemma that all period integrals have moderate growth, it follows from the classical theory of systems of linear differential equations that the above linear system is in fact regular singular: after an appropriate change of basis we can achieve that \( A(t) \) has at most a pole of order one:

\[
A(t) = \frac{1}{t} A_{-1} + A_0 + A_1 t + \ldots
\]

and all solutions admit an expansion of the form

\[
\sum_{a,k} C_{a,k} t^a (\log t)^k.
\]

Exercises:
(1) Verify that
\[(t \partial_t - \alpha)^k t^\alpha (\log t)^k \neq 0, \quad [(t \partial_t - \alpha)^{k+1} t^\alpha (\log t)^k = 0, \]

(2) Solve the differential equation \(t^2 \partial_t f = 1\) on a punctured disc \(S^*\).

(3) Give an explicit description of the Brieskorn lattice for \(f = y^2 + x^3\). Use a monomial basis \(x^n y^m dx dy\) to describe the elements of \(\Omega^{n+1}\). Indicate the relations between these monomials coming from the equivalences induced by the relations
\[df \wedge dx^a y^b\]
Find a basis for \(\mathcal{H}^{(0)}\). What is the action of \(t, \partial_t^{-1}\) on the basis elements.

(4) Verify that \(tb - bt = b^2\).

(5) Verify that from \(\partial_t t - t \partial_t = 1\) one obtains formally
\[t \partial_t^{-1} \partial_t^{-1} t = \partial_t^{-2}\].

(6) Use the Euler relation
\[\sum_{i=0}^{n} w_i x_i \partial_i f = f\]
to show that the two sub-spaces
\[df \wedge \Omega^n, \quad \text{and} \quad df \wedge d\Omega^{n-1} + f\Omega^{n+1}\]
of \(\Omega^{n+1}\) are equal if \(f\) is weighted homogeneous.

4. Lecture 4, Thursday, August 10, 2017

In the last lecture we have seen how the Brieskorn module \(\mathcal{H}^{(0)}\) leads to an algebraic approach to determine the (complexified) monodromy transformation. However, the construction was rather ad hoc and today we will see how the Brieskorn module fits into a standard approach using the relative deRham complex of a good representative \(f : X \rightarrow S\) of our singularity. The most natural interpretation however is in terms of the Gauß-Manin system, which corresponds to the direct image in the category of \(\mathcal{D}\)-modules.

4.1. The deRham and relative deRham complexes. The Poincaré lemma states that if \(Z\) is a smooth complex manifold of dimension \(n\), then the constant sheaf \(\mathcal{C}_Z\) is resolved by the de Rham-complex on \(Z\):
\[\mathcal{C}_Z = [\mathcal{O}_Z \xrightarrow{d} \Omega^1_Z \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n_Z]\]
It follows that the cohomology of \(Z\) is equal to the hypercohomology of the de Rham complex:
\[H^i(Z, \mathcal{C}) = H^i(\mathcal{C}_Z) = H^i(\Omega^n_Z)\).
In the special case that \(Z\) is a Stein space, it follows from Cartan theorem B that the higher cohomology \(H^i(\Omega^n_Z)\) vanishes for \(i > 0\), so in that case the cohomology of \(Z\) can be described by the complex of global holomorphic differential forms on \(Z\):
\[H^i(Z, \mathcal{C}) = H^i(\Gamma(Z, \Omega^n_Z))\).
This applies in particular to the total space $X$ of a good representative and also to the Milnor fibres $X_t$.

If $f: Z \to T$ is a smooth submersion between complex manifolds of fibre dimension $n$, one can use the relative de Rham complex $(\Omega^*_Z/T, d)$ with terms

$$\Omega^p_{Z/T} := \Omega^p_Z / df \wedge \Omega^{p-1}_Z,$$

and differential induced by $d$ to describe the cohomology of the fibres of the map $f$. The relative Poincaré lemma states that the relative de Rham complex resolves the sheaf $f^{-1}\mathcal{O}_T$ of holomorphic functions that are constant along the fibres:

$$f^{-1}(\mathcal{O}_T) = [\mathcal{O}_Z \to \Omega^1_{Z/T} \to \cdots \to \Omega^n_{Z/T}].$$

Taking the $i$-th direct image we obtain a description of the cohomology bundle as the hyperdirect image of relative de Rham:

$$\mathcal{O}_T \otimes R^i f_* \mathcal{C}_Z = R^i f_*(\Omega^*_Z/T).$$

In case $Z \to T$ is a Stein-mapping, the higher direct image sheaves of $\Omega^p_{Z/T}$ vanish, and we are left with

$$\mathcal{O}_T \otimes R^i f_* \mathcal{C}_Z = \mathcal{H}^i(f_*(\Omega^*_Z/T)),$$

i.e. we can describe the cohomology bundle using the cohomology of the global relative de Rham complex.

Let us see what remains from this picture if we apply it to the map $f: X \to S$, which is non-submersive at 0.

**Theorem (Brieskorn (1970))**: For a good representative $f: X \to S$ of an isolated singularity we set

$$\mathcal{H}^p(X/S) := R^p f_*(\Omega^*_X/S) = \mathcal{H}^p(f_*(\Omega^*_X/S)).$$

Then

1. $\mathcal{H}^p(X/S)$ is $\mathcal{O}_S$-coherent.
2. $\mathcal{H}^p(X/S)|_{S^*} = \mathcal{H}^p_{S^*} = \mathcal{O}_{S^*} \otimes R^p f_*(\mathcal{C}_X)$
3. there is an exact sequence of sheaves on $S$:

$$0 \to R^p f_* \mathcal{C}_X \otimes \mathcal{O}_S \to \mathcal{H}^p(X/S) \to f_*(\mathcal{H}^p(\mathcal{O}^*_f)) \to 0$$

where the complex $\Omega^*_f$ has terms

$$\Omega^p_f := \Omega^p / df \wedge \Omega^{p-1}$$

and differential induced by $d$.

The deepest fact here is the appearance of coherence of the cohomology in this non-proper situation. It is an application of an important functional analytic principle that goes back to H. Cartan and L. Schwarz: on the one hand, shrinking of $X$ does not change the cohomology, but on the other hand such a shrinking is a compact operator, which implies the coherence (Theorem of Forster-Knorr). The second fact is obvious from the previous discussion, and shows that $\mathcal{H}^p(X/S)$ can be seen as a
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specific extension of the cohomology bundle. The third point exhibits the stalk at 0 of the sheaf in the middle as the cohomology of a very concrete complex of \(O\)-modules.

Given this theorem of Brieskorn, Malgrange has given a purely algebraic proof that the cohomology \(H^p(\Omega_f^\bullet)\) is non-zero only for \(p = 0\) and \(p = n\). This gives an algebraic proof that the groups \(H^p(X_t, \mathbb{C})\) are only non-vanishing for \(p = 0\) and \(p = n\).

The argument runs as follows: take \(\omega \in \Omega^p\), representing a class \([\omega] \in H^p := H^p(\Omega_f^\bullet)\). So we have

\[d\omega = df \wedge \eta\]

for some \(\eta \in \Omega^p\). So

\[df \wedge d\eta = -d(df \wedge \eta) = -d(d\omega) = 0.\]

For an isolated singularity \(f\), the partial derivatives form a regular sequence and as a consequence, the corresponding Koszul-complex resolves the Milnor-algebra \(O / J_f\). This means that the complex \((\Omega^\bullet, df \wedge)\) is exact in degrees \(< n + 1\), we may conclude from this that \(d\eta = df \wedge \lambda\), for some \(\lambda \in \Omega^p\) (This step fails if \(p = n\)). In this way we obtain a well defined operation

\[\partial_t : H^p \longrightarrow H^p, \quad [\omega] \mapsto [\eta].\]

If we let

\[t : H^p \longrightarrow H^p, \quad [\omega] \mapsto [f\omega],\]

we obtain the structure of a module over the ring

\[\mathcal{D} := \mathbb{C}\{t\}[\partial_t].\]

But the operation \(\partial_t\) is invertible on \(H^p\): given \([\eta] \in H^p\), the form \(df \wedge \eta\) is closed:

\[d(df \wedge \eta) = -df \wedge d\eta = -df \wedge df \wedge ... = 0.\]

If \(p \neq 0\) the Poincaré-lemma implies that \(df \wedge \eta = d\omega\) for some \(\omega\). Then \([\omega]\) maps to \([\eta]\) under \(\partial_t\). But this can happen only if \(H^p = 0\).

The only interesting group is the \(n\)-cohomology \(H := H^n(\Omega_f^\bullet)\). It maps into

\[H' := \Omega^n / df \wedge \Omega^{n-1} + d\Omega^{n-1},\]

which in turn maps to the Brieskorn module

\[H'' := \Omega^{n+1} / df \wedge d\Omega^{n+1} = \mathcal{H}^{(0)}\]

via the map

\[\eta \mapsto df \wedge \eta.\]

4.2. The Gauss-Manin system. The modern perspective on these matters is via the theory of \(\mathcal{D}\)-modules, that was developed by Z. Mebkhout, M. Kashiwara and others. We now give a very rough sketch of this important theory.

On any complex manifold \(Z\) there is a sheaf \(\mathcal{D}_Z\) of differential operators on \(Z\). This is a sheaf of non-commutative rings, locally generated by \(O_Z\) and the sheaf of vector fields \(\Theta_Z\). A \(\mathcal{D}_Z\)-module is a a sheaf on which \(\mathcal{D}_Z\) acts and this notion
generalises the notion of vector bundle with connection on \( Z \). There is a so-called de Rham-functor \( dR \), which maps a \( \mathcal{D}_Z \)-module to its de Rham complex
\[
 dR(\mathcal{M}) = [\mathcal{M} \to \mathcal{M} \otimes \Omega^1_Z \to \mathcal{M} \otimes \Omega^2_Z \to \ldots]
\]
It induces a functor from
\[
 D^b_{\text{hol}}(Z) \xrightarrow{dR} D^b_c(Z)
\]
from the derived category of \( \mathcal{D}_Z \)-modules to the derived category of constructible sheaves on \( Z \). For example, the \( \mathcal{D}_Z \)-module \( \mathcal{O}_Z \) is mapped to the ordinary de Rham complex \( (\Omega^\bullet, d) \) of \( Z \), which by the Poincaré-lemma is quasi-isomorphic to \( C_Z \), the constant sheaf on \( Z \). What is more, there is the formalism of the six operations, which commute with \( dR \). In particular, for a map \( f : Z \to T \) one can form the direct image \( f^\bullet \mathcal{O}_Z \) which is (a complex of) \( \mathcal{D}_T \)-module(s), which via the de Rham-functor should correspond to the direct image \( R^\bullet f_* C_Z \) of the constant sheaf.

Without going into any further detail, the lesson is that there should exist a natural \( \mathcal{D}_S \)-module associated to a good representative \( f : X \to S \) of an isolated singularity. It is called the Gauss-Manin system of \( f \) and has a very simple concrete description that we will give now.

**Definition:** The Gauss-Manin-system of \( f \) is defined
\[
 \mathcal{G} := \mathcal{H}^{n+1}(\Omega^\bullet, d).
\]
Let us spell this out in detail. We denote, as before, by \( \Omega^\bullet \) the \( \mathcal{O} \)-module of germs of holomorphic differential forms on \((\mathbb{C}^{n+1}, 0)\). We consider an abstract symbol \( D \) and form \( \Omega^\bullet[D] \), whose elements are polynomials
\[
 \sum \omega_k D^k
\]
in the symbol \( D \), with coefficients from \( \Omega^\bullet \). On \( \Omega^\bullet[D] \) there is defined a differential \( d \), which is defined on a basis element \( \omega D^k \) as follows:
\[
 d(\omega D^k) = d\omega D^k + df \wedge \omega D^{k+1}.
\]
Then one has \( dd = 0 \), which follows from the anti-commutation of the two operations
\[
 d : \Omega^p \to \Omega^{p+1}, \quad df \wedge : \Omega^p \to \Omega^{p+1}.
\]
One can depict \( \Omega^\bullet[D] \) as a double complex of a particular shape, that is indicated in the following picture.

\[
\begin{array}{cccccccccccc}
& & & & & & & & & & & \\
& & & & & & & & & & & \\
0 & \to & \mathcal{O} & \xrightarrow{d} & \Omega^1 & \to & \ldots & \to & \Omega^n & \xrightarrow{d} & \Omega^{n+1} & \to & 0 & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & 0 & \to & \mathcal{O} & \xrightarrow{d} & \Omega^1 & \to & \ldots & \to & \Omega^n & \xrightarrow{d} & \Omega^{n+1} & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & 0 & \to & 0 & \to & \mathcal{O} & \xrightarrow{d} & \Omega^1 & \to & \ldots & \to & \Omega^n & \xrightarrow{d} & \Omega^{n+1} \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\end{array}
\]
The horizontal arrows are induced by the exterior derivative $d$, the vertical maps are induced by $df\wedge$. The $k$-th row comes with a power $D^k$, which is suppressed from the diagram.

On the complex $(\Omega^*[D], d)$ one can define two further operations

$$\partial_t(d^k) = d^{k+1},$$

$$t \cdot (d^k) = f\omega^k - k\omega^{k-1}.$$  

It is a pleasant exercise to check that these operations commute with $d$ and satisfy

$$\partial_t t - t\partial_t = 1,$$

so that the cohomology groups of $(\Omega^*[D], d)$ are $D := \mathbb{C}\{t\}[\partial_t]$-modules.

This is the result of taking the direct image in the category of $D$-modules and it all looks all rather mysterious. But in fact there is a clear interpretation of the terms in the complex, if we keep in mind the following correspondence:

$$\omega^k \leftrightarrow \text{Res} \left( \frac{k!\omega}{(f-t)^{k+1}} \right)$$

so that we are dealing with an abstract version of general period integrals

$$\frac{1}{2\pi i} \int_{\Delta(t)} \frac{k!\omega}{(f-t)^{k+1}}$$

From this one can reverse engineer the operations $d$, $t$ and $\partial_t$ on $\Omega^*[D]$.

**Proposition:** The cohomology groups $\mathcal{H}^p(\Omega^*[D], d)$ are non-zero only for $p = 1$ and $p = n + 1$.

The interesting cohomology group is the one in degree $n + 1$.

4.3. **The Brieskorn module and the Hodge-Filtration.** Let us look at the term $\Omega^{n+1}$ sitting at the lower right corner of the complex $(\Omega^*[D], d)$ and let us try to understand the kernel of the map

$$\Omega^{n+1} \rightarrow G.$$

**Proposition:** The kernel of the above map is exactly

$$df \wedge d\Omega^{n-1}$$

**proof:** If the element

$$\eta := \sum_{k=0}^{N} \eta_k D^k \in \Omega^n[D]$$

maps under to $\omega$ under $d$, we get the equations

$$\omega = d\eta_0, \quad d\eta_1 + df \wedge \eta_0 = 0, d\eta_2 + df \wedge \eta_1 = 0, \ldots, df \wedge \eta_N = 0.$$ 

From the last equation and the exactness of $df\wedge$ we get

$$\eta_N = df \wedge \lambda_N$$
When we substitute this in the penultimate equation we get

\[ d(df \wedge \lambda_N) + df \wedge \eta_{N-1} = 0, \]

which amounts to

\[ df \wedge (-d\lambda_N + \eta_{N-1}) = 0 \]

or

\[ -d\lambda_N + \eta_{N-1} = df \wedge \lambda_{N-1}. \]

Continuing in this way we see that there exist \( \lambda_k \) such that

\[ \eta_k = d\lambda_{k+1} + df \wedge \lambda_k. \]

In particular, it follows

\[ \omega = d\eta_0 \in df \wedge d\Omega^{n-1} \]

**Corollary:** the image of \( \Omega^{n+1} \) in \( \mathcal{G} \) is isomorphic to the Brieskorn module \( \mathcal{H}^{(0)} \):

\[ \mathcal{H}^{(0)} \subset \mathcal{G} \]

If we apply \( \partial_t \) to \( \mathcal{H}^{(0)} \) we end up in a larger space

\[ \mathcal{H}^{(1)} := \partial_t \mathcal{H}^{(0)} \subset \mathcal{G}, \]

which is exactly the image of the sub-complex

\[
\begin{array}{ccc}
\Omega^{n+1} & \longrightarrow & 0 \\
\uparrow df \wedge & \uparrow & \\
\Omega^n & \xrightarrow{d} & \Omega
\end{array}
\]

in \( \mathcal{G} \). In general, the image of the sub-complex

\[ \bigoplus \bigoplus_{q=0}^{k-n+1} \Omega^{n+1-k+q} D^q \]

obtained from vertical truncation of the above double complex is exactly

\[ \mathcal{H}^{(k)} := \partial_t^k \mathcal{H}^{(0)} \]

Recall that there was a natural operation of \( \partial_t^{-1} \) on \( \mathcal{H}^{(0)} \). We can extend this to \( k \in \mathbb{Z} \) and define

\[ \mathcal{H}^{(-k)} := \partial_t^{-k} \mathcal{H}^{(0)} \]

This is the real significance of the Brieskorn-module: it is a free \( \mathbb{C}\{t\} \)-submodule of \( \mathcal{G} \) that defines a filtration

\[ \ldots \subset \mathcal{H}^{(-2)} \subset \mathcal{H}^{(-1)} \subset \mathcal{H}^{(0)} \subset \mathcal{H}^{(1)} \subset \mathcal{H}^{(2)} \subset \ldots \subset \mathcal{G} \]

on it. It is commonly called, after a shift

\[ \mathcal{F}^{n-k} = \mathcal{H}^{(k)} \]

the Hodge filtration on \( \mathcal{G} \). Note that the quotient

\[ \mathcal{H}^{(0)}/\mathcal{H}^{(-1)} = \Omega^{n+1}/df \wedge \Omega^n =: \Omega_f \]

is our Milnor-module of dimension \( \mu \) and using the isomorphism induced by \( \partial_t \) one sees that all the successive quotients

\[ \mathcal{H}^{(k)}/\mathcal{H}^{(k-1)} \]
are isomorphic to $\Omega_f$.

4.4. The $V_\ast$-filtration. The $\mathcal{D}$-module $\mathcal{G}$ has three important properties. First, it is $\mathcal{D}$-coherent, second it regular singular and third, the operator

$$\partial_t : \mathcal{G} \longrightarrow \mathcal{G}$$

is an isomorphism. These properties all follow more or less directly from the results for the Brieskorn-module. The last property is sometimes referred to as the micro-local nature of the Gauss-Manin system.

The generalised eigenspaces of the operator

$$t\partial_t - \alpha : \mathcal{G} \longrightarrow \mathcal{G}$$

are denoted by

$$C_\alpha := \{ m \in \mathcal{G} \mid \exists N \ (t\partial_t - \alpha)^N m = 0 \}.$$ 

Furthermore, we denote by

$$V_\alpha \subset \mathcal{G},$$

the $\mathbb{C}\{t\}$-module generated by all $C_\beta$ with $\beta \geq \alpha$ and similarly

$$V_{>\alpha} \subset V_\alpha$$

generated by $C_\beta$ with $\beta > \alpha$. So

$$\text{Gr}^V_\alpha = V_\alpha / V_{>\alpha} \approx C_\alpha.$$ 

The positivity lemma of Malgrange implies

$$\mathcal{H}^{(0)} \subset V_1$$

and it is a very non-trivial fact that can be proved using the self-duality of the Gauss-Manin system that one also has

$$V_\mu \subset \mathcal{H}^{(0)}$$

The spaces $F_\mu = \mathcal{H}^{(n-p)}$ induce a filtration on the spaces $C_\alpha$ by putting

$$F_\mu C_\alpha := (F_\mu \cap V_\alpha + V_{>\alpha}) / V_\alpha.$$

4.5. The mixed Hodge Structure. It was shown by Steenbrink that there is a natural mixed Hodge structure on the cohomology of the canonical Milnor fibre $X_\infty$ of an isolated hypersurface singularity. Recall that $X_\infty$ is simply the pull-back $X_\infty$ of the fibration $X^* \longrightarrow S^*$ over the universal covering $\tilde{S}^* \longrightarrow S^*$, i.e. we form the pull-back diagram

$$\begin{array}{ccc}
X_\infty & \longrightarrow & X^* \\
\downarrow & & \downarrow \\
\tilde{S}^* & \longrightarrow & S^*
\end{array}$$

As before, we put $H_\mathbb{Z} := H^n(X_\infty, \mathbb{Z})$ and on it we have the monodromy transformation $T : H_\mathbb{Z} \longrightarrow H_\mathbb{Z}$. We write $T = T_sT_u = T_uT_s$, where $T_s$ is the semi-simple and $T_u$ the unipotent part of the monodromy. The monodromy logarithm is

$$N := \log T_u : H_\mathbb{Q} \longrightarrow H_\mathbb{Q}, \quad H_\mathbb{Q} = \mathbb{Q} \otimes_{\mathbb{Z}} H_\mathbb{Z},$$

and determines a unique monodromy weight filtration

$$M_0 \subset M_1 \subset \ldots M_{2k-1} \subset M_{2k}$$
characterised by the property that \( N(M_k) \subset M_{k-2} \) and

\[
N^k : Gr^M_{n+k} \rightarrow Gr^M_{n-k}.
\]

Furthermore, if we denote the generalised eigenspace for eigenvalue \( \lambda = \exp(2\pi i \alpha) \), then

\[
H^n(X_\infty, \mathbb{C})_\lambda = C_\alpha,
\]

so we can identify

\[
H^n(X_\infty, \mathbb{C}) = \bigoplus_{-1 < \alpha \leq 0} C_\alpha,
\]

and thus define

\[
F^p H^n(X_\infty, \mathbb{C}) = \bigoplus_{-1 < \alpha \leq 0} F^p C_\alpha.
\]

**Theorem (Steenbrink, Scherk, 1985):** With \( W_\bullet := M_\bullet \), the triple

\[(H_Z, W_\bullet, F^\bullet)\]

is a mixed Hodge structure.

This statement means that the filtration \( F^\bullet \) induces on all \( Gr^W_k H_Q \) a pure Hodge structure of weight \( k \); a \( \mathbb{Q} \)-Hodge structure of weight \( k \) is a \( \mathbb{Q} \)-vector space \( H_Q \) with a filtration \( F^\bullet \) on \( H_C = \mathbb{C} \otimes H_Q \), such that

\[
F^p \oplus F^{k-p+1} = H_C.
\]

Equivalently, we have a **Hodge decomposition**

\[
H_C = \bigoplus_{p+q=k} H^{p,q} \quad H^{p,q} := F^p \cap \overline{F^q} = \overline{H^{q,p}},
\]

from which one can reconstruct the Hodge filtration as

\[
F^p = \bigoplus_{p' \geq p} H^{p',k-p'}.
\]

The concept of mixed Hodge structure was introduced by Deligne, who showed that each cohomology group \( H^k(Z) \) of a possibly singular quasi-projective variety \( Z \) carries a natural structure of a mixed Hodge structure. Each algebraic map between quasi-projective varieties induces a map of mixed Hodge structures. The power of the theory lies in the fact that such maps are *strictly compatible* with respect to weight and Hodge filtration: taking \( Gr^W_k \) or \( Gr^F_p \) are exact functors.

The mixed Hodge structure on \( H^n(X_\infty, Z) \) defined above is of quite a different nature as the mixed Hodge structures defined by Deligne; they are called *limit or asymptotic* mixed Hodge structures.

Note that the above definition of the components of the mixed Hodge structure attached to a singularity is completely local, but all proofs of the Hodge properties use globalisation.
4.6. **The spectrum.** We now can give a general definition of the spectrum of a singularity:

**Definition:**

\[
Sp(f) = s \sum_{-1<\alpha\leq 0} \sum_p \dim_{\mathbb{C}} Gr_F^{-p} C_\alpha s^{\alpha + p}.
\]

So the eigenvectors which belong to the smallest Hodge space \(F^n\) are lifted to exponent in \((0,1]\), those in the next Hodge space \(F^{n-1}/F^n\) lift to exponents in \((1,2]\), etc.

There is an alternative definition using the \(V_\bullet\)-filtration on the Milnor module

\[
\Omega_f = \Omega^{n+1}/df \wedge \Omega^n = \mathcal{H}^{(0)}/\mathcal{H}^{(-1)}
\]

We set

\[
V_\alpha \Omega_f := (V_\alpha \mathcal{H}^{(0)}) + \mathcal{H}^{(-1)}/\mathcal{H}^{(-1)}
\]

It is an exercise to show that there is an isomorphism

\[
Gr_I^{-p} C_\alpha \xrightarrow{\partial_t^{-p}} Gr_{\alpha+p}^V \Omega_f
\]

so that one can alternatively define

\[
Sp(f) = s \sum_\alpha \dim_{\mathbb{C}} Gr_{\alpha}^V (\Omega_f)s^\alpha.
\]

The spectrum of a singularity probably hides many deep secrets. A particular nice one is the following, that was inspired by an analysis of the genus one partition function of the Frobenius manifold defined by the singularity \(f\):

**Variance Conjecture (Hertling):**

\[
\frac{1}{\mu} \sum_{i=1}^{\mu} \left( \alpha_i - \frac{n+1}{2} \right)^2 \leq \frac{\alpha_\mu - \alpha_1}{12}
\]

For details we refer to the monograph of Hertling.

**Exercises:**

1. Check the details of the argument of Malgrange for the vanishing of \(H^p(\Omega_f^*)\).
   In particular, show that a \(D\)-module \(H\) on which \(\partial_t\) is invertible, and which is finitely generated as on \(\mathbb{C}\{t\}\)-module has to be zero.

2. Using the correspondence

\[
\omega D^k \leftrightarrow Res \left( \frac{k! \omega}{(f-t)^{k+1}} \right)
\]

check that the action of multiplication by \(t\) and \(\partial_t\) on \(G\) correspond to the multiplication by \(t\) and differentiation with respect to \(t\).
(3) Check that the operations $t, \partial_t$ on the complex $(\Omega^*[D], \partial)$ are well defined and commute with $\partial$.

(4) Show that if an element $m \in G$ is $C\{t\}$-torsion (i.e. $t \cdot m = 0$), then it is contained in $V_{-1}G$. Conclude that the Brieskorn lattice is a free $C\{t\}$-module.

(5) Analyse the Gauss-Manin system for the $A_1$-singularity. Show that if the number of variables is even, $G$ contains $t$-torsion elements.

5. Lecture 5

We have sketched how from an isolated singularity $f \in \mathcal{O}$ one can obtain a mixed Hodge structure $(H_Z, W_*, F^*)$. Here $H_Z = H^n(X_\infty, \mathbb{Z})$ is the cohomology of the canonical Milnor fibre, $W_* = M_*$ the monodromy weight-filtration and $F^*$ the Hodge-filtration, that can be obtained from the Brieskorn-lattice inside the Gauss-Manin system of $f$. We defined the spectrum $sp(f)$, which lifts monodromy eigenvalues to exponents whose integer part records the Hodge space they are contained in. In this way, the Hodge spaces correspond to integer intervals $(k, k + 1]$ in the spectrum.

5.1. Some further results. For example for a curve singularity the spectrum is contained in $(0, 2)$. One has

$$\#(0, 1] \cap sp(f) = \delta(f) = \dim F^1, \quad \#(1, 2] \cap sp(f) = \delta(f) - r + 1 = \dim F^0 / F^1$$

where $r$ denotes the number of local branches of the curve $f = 0$.

For a surface singularity $V := \{f(x, y, z) = 0\}$, the spectrum is contained in $(0, 3)$ and

$$\#(0, 1] \cap sp(f) = p_g(V) = \dim F^2$$

Here $p_g(V)$ is the so-called geometrical genus, which is defined as the number of independent differential forms of top degree that do not extend on a resolution. To be precise, if

$$\pi : \tilde{V} \longrightarrow V$$

is a resolution of an $n$-dimensional isolated singularity $V$, and

$$E = \pi^{-1}(0)$$

the exceptional divisor, then

$$p_g(V) := \dim H^0(\tilde{V} \setminus E, \Omega^n_\tilde{V}) / H^0(\tilde{V}, \Omega^n_\tilde{V})$$

The singularities with $p_g(V) = 0$ are precisely the rational singularities, so in the surface case these are the $A - D - E$-singularities. The $A - D - E$-surface singularities can thus be characterised by the property that

$$sp(f) \subset (1, 2).$$

The spectral numbers in the first interval $(0, 1]$ are easy to compute using an embedded resolution. So if $f : X \longrightarrow S$ is a good representative of our singularity, we consider
\[ \rho : Y \longrightarrow X, \] 
such that \((f \circ \rho)^{-1}(0)\) is a divisor with normal crossings. Let \(E_i\) be the irreducible components of \(\rho^{-1}(0)\). If \(\omega \in \Omega^{n+1}\) is a differential form, we put

\[ w(\omega) := \min_i \text{mult}_{E_i}(\rho^*(\omega) + 1) / \text{mult}_{E_i}(f \circ \rho) \]

where \(\text{mult}_{E_i}\) denotes the vanishing order along \(E_i\).

**Theorem (Varchenko, 1982):** The spectral numbers in \((0, 1] \cap \text{sp}(f)\) are exactly the numbers \(w(\omega)\) which are \(\leq 1\).

These numbers can be defined in much greater generality and also appear under the name of *jumping coefficients* in the theory of multiplier ideals. The smallest spectral number is called classically *the complex singularity exponent*, but is now often called the *log canonical treshold*.

The Newton-diagram of a function \(f\) defined a canonical filtration on \(\Omega^{n+1}\) and thus determines an induced filtration on \(N_\bullet \Omega_f\). One has the following very useful theorem due to M. Saito:

**Theorem (M. Saito, 1988):** If \(f\) is non-degenerate with respect to its Newton-diagram, the Newton filtration \(N_\bullet\) on \(\Omega_f\) coincides with the \(V_\bullet\)-filtration, shifted by one:

\[ V_\alpha \Omega_f = N_{\alpha+1} \Omega_f. \]

### 5.2. Adjacencies

It is of great beauty and interest to study how complicated critical points can decompose into simpler ones. One considers holomorphic function germs \(f_\lambda(x) = F(x, \lambda) \in \mathbb{C}\{x_0, x_1, \ldots, x_n, \lambda\}\) that depend on an additional deformation parameter \(\lambda\). If for \(\lambda \neq 0\), small enough, the function \(f_\lambda\) has critical points \(p_1, p_2, \ldots, p_n\) with critical value 0 and \((f_\lambda, p_i) \sim (g_i, 0)\), we say that \(F\) defines an *adjacency* between \(f\) and \(g_1, g_2, \ldots, g_N\) and write \(f \sim g_1, g_2, \ldots, g_N\).

**Examples:**

![Examples of adjacencies](image-url)
It is an instructive exercise to write down the formulas that describe the above adjacencies.

5.3. **Semi-continuous invariants.** An invariant $I$ of a singularity is *semi-continuous* if for each adjacency $f \rightsquigarrow g_1 + g_2 + \ldots + g_N$ one has

$$I(f) \geq \sum_{i=1}^{N} I(g_i)$$

*Examples:*

1. $\mu(f) \geq \sum_{i=1}^{N} \mu(g_i)$
2. $\delta(f) \geq \sum_{i=1}^{N} \delta(g_i)$
3. $p_8(f) \geq \sum_{i=1}^{N} p_8(g_i)$

Arnol’d conjectured that the quantity

$$\#(-\infty, a] \cap sp(f)$$

is semi-continuous.

**Definition:** A subset $S \subset \mathbb{R}$ is called a semi-continuity set if

$$\#S \cap sp(f)$$

is semi-continuous.

**Theorem:**

(A. Varchenko, 1983): if $f$ is quasi-homogeneous then each open interval $(a, a + 1)$ is a semi-continuity set.
(J. Steenbrink, 1985): for general \( f \) each interval \((a, a+1)\) is semi-continuity set.

**Corollary:** In a \( \mu \)-constant family the spectrum is constant.

**Proof:** By semi-continuity, the number of exponents (spectral numbers) in each interval of length 1 can only increase under specialisation. As the total number of exponents is equal to \( \mu \) and thus stays constant by assumption, in each interval of length 1 the number of exponents is constant, which can only happen if all exponents stay constant!

### 5.4. The Bruce deformation.

Consider a projective hypersurface

\[ Z = \{F(x_0, x_1, \ldots, x_n) = 0\} \subset \mathbb{P}^n \]

where \( F \) is a homogeneous polynomial in \( x_0, x_1, \ldots, x_n \). We can write

\[ F = f_d + x_0 f_{d-1} + x_0^2 f_{d-2} + \cdots + x_0^d \]

where \( f_k \) is homogeneous of degree \( k \) in \( x_1, x_2, \ldots, x_n \). By putting \( x_0 = \lambda \in \mathbb{C} \), we obtain the equation for an affine hypersurface

\[ X_\lambda := \{F_\lambda = 0\} \subset \mathbb{C}^n, \]

where

\[ F_\lambda := f_d + \lambda f_{d-1} + \lambda^2 f_{d-2} + \cdots + \lambda^d \in \mathbb{C}[x_1, x_2, \ldots, x_n] \]

Note however that

\[ F_\lambda(\lambda x_1, \ldots, \lambda x_n) = \lambda^d F_1(x_1, \ldots, x_n), \]

which implies that for all \( \lambda \neq 0 \) these affine hypersurfaces \( X_\lambda \) are isomorphic to each other. As

\[ X_0 = \{f_d = 0\} \subset \mathbb{C}^n \]

we see that \( X_0 \) is the affine cone on the projective hypersurface \( Z \cap H \), where \( H \) is the hyperplane defined by \( x_0 = 0 \).

If \( Z \) has isolated singularities with singular set \( \Sigma := \{p_1, p_2, \ldots, p_N\} \) and coordinates are chosen such that \( H \cap \Sigma = \emptyset \), then the homogeneous affine hypersurface \( X_0 \) defined by \( f_d = 0 \) defines an isolated singularity, namely the affine cone over the smooth projective variety \( Z \cap H \), whereas for \( \lambda \neq 0 \) the affine hypersurface \( X_\lambda \) has isolated singularities, defined by \( g_1, g_2, \ldots, g_N \). Hence, we obtain an special adjacency

\[ f_d \rightsquigarrow g_1 + g_2 + \cdots + g_N, \]

that was already used by J. W. Bruce.

**Corollary:** The spectral bound for projective hypersurfaces holds.

The Varchenko semi-continuity theorem implies

\[ #(a, a+1) \cap sp(f_d) \geq \sum #(a, a+1) \cap sp(g_i) \]
But $f_d$ and $x_1^d + x_2^d + \ldots + x_n^d$ are connected via a $\mu$-constant deformation and thus have the same spectrum, so that we have
\[ #(a, a + 1) \cap sp(x_1^d + x_2^d + \ldots + x_n^d) \geq \sum #(a, a + 1) \cap sp(g_t) \]
As the spectrum at the left hand side is determined by the lattice point in a cube, we obtain the spectral bound as formulated in lecture 1.

5.5. Globalisation and variations of (mixed) Hodge structures. It follows from the fact that any isolated singularity is right equivalent to a polynomial that the Milnor fibration $f : X \rightarrow S$ can be globalised to a flat, projective family
\[ F : Y \rightarrow S \]
and we may assume that the fibre $Y_t = f^{-1}(t)$ over $t$ is smooth for $t \neq 0$ and $Y_0$ has a single isolated singular point, and after restriction we obtain our good representative $f : X \rightarrow S$.

We would like to express the idea that the difference between $Y_0$ and $Y_t$ is concentrated at the singularity. This can be done in great generality with the diagram that can be found in SGA 7.

\[
\begin{array}{cccccc}
Y_0 & \xrightarrow{i} & Y & \xleftarrow{j} & Y^* & \xleftarrow{\rho} & Y_\infty \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{0\} & \rightarrow & S & \leftarrow & S^* & \leftarrow & \tilde{S^*} \\
\end{array}
\]
The cohomology group $H^n(Y_t)$ of the smooth projective varieties $Y_t$, $t \in S^*$ each carry a pure Hodge structure and form what is called a variation of Hodge structures. This means that on the cohomology bundle $\mathcal{H}$ on $S^*$ we are given a filtration by vector bundles on $S^*$
\[ \mathcal{F}^n \subset \mathcal{F}^{n-1} \subset \ldots \subset \mathcal{F}^0 = \mathcal{H} \]
satisfying Griffiths transversality:
\[ \partial_t(\mathcal{F}^k) \subset \mathcal{F}^{k-1} \]
The construction of W. Schmid extends these vector bundles to $\mathcal{F}^p_e$ on all of $S$. Together with the monodromy filtration $M_*$ coming from $N = \log T_\nu$, one obtains a limiting mixed Hodge structure on the cohomology groups $H^k(Y_\infty)$, which were described geometrically using an semi-stable model for the degeneration by J. Steenbrink. One obtains an exact sequence of mixed Hodge structures
\[ \ldots H^n(Y_0) \rightarrow H^n_{\text{lim}}(Y_\infty) \rightarrow H^n(X_\infty) \rightarrow H^{n+1}(Y_0) \ldots \]
Here the MHS at the left group is Deligne’s MHS on the singular space $Y_0$; in the middle we have the MHS of Schmid and Steenbrink. The group at the right is attached to the singularity, and the MHS of Scherk and Steenbrink makes it into an exact sequence of MHS.

For our application we have to dig still deeper. If the general fibre $Y_t$ is singular itself, we end up with a variation of MHS with fibres $H^n(Y_t)$ and one has to consider the problem of how to extend this structure over the origin. It turns out there is no direct analogue of the theorem of W. Schmid, rather there is an additional condition.
leading to the notion of an admissible variation of mixed Hodge structures introduced by J. Steenbrink and S. Zucker, which is satisfied in the geometrical case. The admissibility amounts to existence of a relative weight filtration $M_\bullet$ attached to the monodromy logarithm $N$ and weight filtration $W_\bullet$ on $H^n(Y_\infty)$, characterised by

- $NM_k \subset M_{k-2}$
- $M_\bullet$ induced on $Gr_k^W$ the monodromy weight filtration of the nilpotent transformation $Gr_k^W : Gr_k^W \to Gr_k^W$.

So for each $k$ one obtains well defined limit mixed Hodge structures $Gr_k^W H^n(Y_\infty)$, whose weight filtration comes from the induced monodromy logarithm.

5.6. The argument. We consider a flat projective family of varieties with $n$-dimensional fibres over a base $T$. After Deligne, the group $H^i(Y_t)$ carries for each $t \in T$ a natural mixed Hodge structure. For each integer $p$ we consider the Euler-characteristic

$$\chi_p(t) := (-1)^n \sum (-1)^i Gr_F^p H^i(Y_t)$$

It follows from general arguments that this function is constructible, i.e. there exists a stratification for which $\chi_p$ is constant on the strata. If $h : S \to T$ maps all points of the punctured disc $S^*$ into one stratum, and $0$ into an adjacent stratum, then we put

$$\chi_p(h) := (-1)^n \sum \dim Gr_F^p R^i \Phi_h.$$ 

Proposition: One has the following jump formula:

$$\chi_p(h) = \chi_p(t) - \chi_p(s)$$

where $t = h(0)$, $s = h(t')$, $t' \neq 0$.

proof: From the triangle of nearby and vanishing cycles

$$\ldots \to C_{Y_t} \to R\psi_h \to R\phi_h \to \ldots$$

one obtains a long exact sequence

$$\ldots \to H^k(Y_t) \to H^k_{lim}(Y_\infty) \to R^k \phi_h \to H^{k+1}(Y_t) \ldots$$

Here the middle terms is the limit of the variation of mixed Hodge structures of the family over $h$. Now we use that

$$\dim Gr_F^p H_{lim}(Y_\infty) = \dim Gr_F^p H^p(Y_s)$$

Using this, and taking $\chi_p$ of the above exact sequence then give the jump formula $\diamond$

Definition: The map $f : Y \to T$ is called spherical if for all maps $h : S \to T$ as above one has:

$$Gr_F^p R^i \phi_h = 0 \text{ for } i \neq n.$$ 

Examples are families with at most isolated hypersurface singularities, or more generally, isolated complete intersection singularities. From the jump formula one gets immediately:

Corollary: If $f : Y \to T$ is spherical, then the function

$$t \mapsto \chi_p(t)$$
is upper semi-continuous.

Now let $s$ be a generic point of $T$; the difference between $\chi_p(t) - \chi_p(s)$ is also upper semi-continuous and furthermore can be identified with local contributions coming from the singularities.

If we consider an adjacency $f \rightsquigarrow g_1 + g_2 + \ldots + g_N$, we obtain

$$\text{Gr}^p_F H^n(X_\infty(f)) \geq \sum_{i=1}^N \text{Gr}^p_F H^n(X_\infty(g_i))$$

which means

$$\#(n-p, n-p+1] \cap \text{sp}(f) \geq \sum_{i=1}^N \#(n-p, n-p+1] \cap \text{sp}(g_i)$$

So all half open intervals $(k, k+1]$ are semi-continuity sets.

5.7. Varchenko’s trick. To extend the above argument to arbitrary intervals $(\alpha, \alpha + 1]$ one has somehow to shift the intervals. This can be achieved with the following trick: The spectrum of

$$F(x_0, x_1, \ldots, x_n, z) = f(x_0, x_1, \ldots, x_n) + z^m$$

consists of $m$-shifted copies of the spectrum of $f$:

$$\text{Sp}(F) = \text{sp}(f) + \frac{1}{m} \cup \text{sp}(f) + \frac{2}{m} \cup \ldots \cup \text{sp}(f) + \frac{m-1}{m} \text{sp}(f)$$

Furthermore, there is an action of $\mu_m$ on the vanishing cohomology of $F$, induced by

$$z \mapsto \zeta z$$

where $\zeta \in \mu_m$ is an $m$-th root of unity. Using now the equivariant version of the previous set-up gives that the intervals $(k + \frac{i}{m}, k + 1 + \frac{i}{m}]$ are also semi-continuity sets. As the spectral numbers are rational, this clearly implies that all intervals $(\alpha, \alpha + 1]$ are semi-continuity sets.

Exercises:

1) If $f \in C\{x_0, x_1, \ldots, x_n\}$ is an arbitrary (isolated) singularity, then $F := f + u^2 + v^2$ defines a rational singularity. So 3-dimensional rational singularities are at least as complex as arbitrary curve singularities.

2) Compute the spectrum of A’Campo’s singularity $x^2y^2 + x^5 + y^5$.

   (i) using the theorem M. Saito’s theorem.

   (ii) using the embedded resolution and Varchenko’s theorem.

3) Work out the details of Varchenko’s trick.

We have come to the end of these lectures, which really only pretend to offer a rough sketch of the ideas involved. I hope the reader will be sufficiently intrigued.
to make a more serious study of the original literature. I consider myself lucky for having been a witness of a part of the above described developments that define a unique golden era of singularity theory.

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6. Literature

The best way to learn a subject is to study the original papers. Only from reading these one can understand the motivations and see how the main ideas developed. It is a gratifying aspect of the internet era that it has become rather simple to get access to most of these original papers.

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