On the Chow ring of Fano varieties on the Fatighenti–Mongardi list

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ABSTRACT
Conjecturally, Fano varieties of K3 type admit a multiplicative Chow–Künneth decomposition, in the sense of Shen–Vial. We prove this for many of the families of Fano varieties of K3 type constructed by Fatighenti–Mongardi. This has interesting consequences for the Chow ring of these varieties.

1. Introduction

Given a smooth projective variety \( Y \) over \( \mathbb{C} \), let \( A^i(Y) := CH^i(Y)_{\mathbb{Q}} \) denote the Chow groups of \( Y \) (i.e., the groups of codimension \( i \) algebraic cycles on \( Y \) with \( \mathbb{Q} \)-coefficients, modulo rational equivalence). The intersection product defines a ring structure on \( A^*(Y) = \bigoplus_i A^i(Y) \), the Chow ring of \( Y \) [11]. In the case of K3 surfaces, this ring structure has a remarkable property:

**Theorem 1.1 (Beauville–Voisin [2]).** Let \( S \) be a K3 surface. The \( \mathbb{Q} \)-subalgebra

\[
R^*(S) := \langle A^1(S), c_j(S) \rangle \subset A^*(S)
\]

injects into cohomology under the cycle class map.

Motivated by the cases of K3 surfaces and abelian varieties, Beauville [1] has conjectured that for certain special varieties, the Chow ring should admit a multiplicative splitting. To make concrete sense of Beauville’s elusive “splitting property conjecture”, Shen–Vial [34] have introduced the concept of multiplicative Chow–Künneth decomposition. It seems both interesting and difficult to better understand the class of special varieties admitting such a decomposition.

In [25], the following conjecture is raised:
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A multiplicative Chow

This program. The main result is as follows:

Then \( X \) has a multiplicative Chow–Kühneth decomposition.

**Conjecture 1.2.** Let \( X \) be a smooth projective Fano variety of K3 type (i.e., \( \dim X = 2d \) and the Hodge numbers \( h^{p,q}(X) = 0 \) for all \( p \neq q \) except for \( h^{d−1,d−1}(X) = h^{d+1,d−1}(X) = 1 \)). Then, \( X \) has a multiplicative Chow–Kühneth decomposition.

This conjecture is verified in some special cases [9, 22–25]. This article aims to contribute to this program. The main result is as follows:

**Theorem 1 (=Theorem 3.1).** Let \( X \) be a smooth Fano variety in one of the families of Table 1. Then \( X \) has a multiplicative Chow–Kühneth decomposition.

Table 1 lists Fano varieties \( X \) of K3 type that were constructed by Fatighenti–Mongardi [6] as hypersurfaces in products of Grassmannians. The K3 surfaces \( S \) in Table 1 are shown in [6] to be associated to \( X \) on the level of Hodge theory, and on the level of derived categories. In some cases, the geometric relation between \( X \) and \( S \) is straightforward (e.g., for \( B1 \) and \( B2 \) the Fano variety \( X \) is a blow-up with center the K3 surface \( S \)); in other cases the geometric relation is more indirect (e.g., for \( M1, M6, M7, M8, M9, M10 \) the Fano variety \( X \) is related to the K3 surface \( S \) via the so-called “Cayley’s trick,” cf. [6] and subsection 2.3 below).

To prove Theorem 1.1, we have devised a general criterion (Proposition 3.2), which we hope might apply to other Fano varieties of K3 type. To verify the criterion, one needs a motivic relation between the Fano variety \( X \) and the associated K3 surface \( S \), and one needs a certain instance of the Franchetta property.

As a consequence of our main result, the Chow ring of these Fano varieties behaves like the Chow ring of a K3 surface:

**Corollary (=Corollary 4.1).** Let \( X \subset U \) be the inclusion of a Fano variety \( X \) in its ambient space \( U \), where \( X, U \) are as in Table 1. Let \( \dim X = 2d \). Let \( R^i(X) \subset A^*(X) \) be the \( \mathbb{Q} \)-subalgebra

\[
R^i(X) := \langle A^i(X), A^2(X), ..., A^d(X), c_j(X), \text{Im}(A^*(U) \to A^*(X)) \rangle \subset A^*(X).
\]

Then, \( R^i(X) \) injects into cohomology under the cycle class map.

We end this introduction with a challenge. Fatighenti–Mongardi have constructed some more Fano varieties of K3 type for which it would be nice to settle Conjecture 1.2 (in particular the families labeled M13 and S1 in [6], for which I have not been able to check condition (c3) or (c3') of the general criterion Proposition 3.2).

Additionally, the following are some Fano varieties of K3 type in the literature for which Conjecture 1.2 is still open, and for which the methods of the present article are not sufficiently strong: Küchle fourfolds of type c5, Plücker hyperplane sections of \( \text{Gr}(3,10) \), intersections of

Table 1. Families of Fano varieties of K3 type. (As in [6], \( \text{Gr}(k,m) \) denotes the Grassmannian of \( k \)-dimensional subspaces of an \( m \)-dimensional vector space.

| Label in [6] | \( X \subset U \) | \( \dim X \) | \( p(X) \) | Genus of associated K3 | Also occurs in |
|--------------|------------------|-------------|-----------|-------------------------|---------------|
| B1           | \( X_{(2,1,1)} \subset \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \) | 4           | 3         | 7                       | [24]          |
| B2           | \( X_{(2,1)} \subset \text{Gr}(2,4) \times \mathbb{P}^3 \) | 4           | 3         | 5                       | [24]          |
| M1           | \( X_{(1,1,1)} \subset \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \) | 8           | 3         | 3                       | [14]          |
| M2           | \( X_{(1,1)} \subset \text{Gr}(2,5) \times \mathbb{P}^3 \) | 10          | 2         | 6                       |               |
| M3           | \( X_{(1,1)} \subset \text{Gr}(2,5) \times \mathbb{P}^3 \) | 10          | 2         | 6                       |               |
| M4           | \( X_{(1,1)} \subset \text{Gr}(2,5) \times \mathbb{P}^3 \) | 8           | 2         | 6                       |               |
| M6           | \( X_{(1,1)} \subset \mathbb{S}_2 \times \mathbb{P}^2 \) | 16          | 2         | 7                       |               |
| M7           | \( X_{(1,1)} \subset \text{Gr}(2,6) \times \mathbb{P}^5 \) | 12          | 2         | 8                       |               |
| M8           | \( X_{(1,1)} \subset \text{Gr}(2,6) \times \mathbb{P}^3 \) | 10          | 2         | 8                       |               |
| M9           | \( X_{(1,1)} \subset \mathbb{S}_2 \text{Gr}(2,6) \times \mathbb{P}^3 \) | 8           | 2         | 8                       |               |
| M10          | \( X_{(1,1)} \subset \text{Gr}(3,6) \times \mathbb{P}^3 \) | 8           | 2         | 9                       |               |
| S2           | \( X_{1} \subset O\text{Gr}(2,8) \) | 8           | 2         | 7                       | [25]          |

\( \text{SGr}(k,m), \mathbb{S}_2 \text{Gr}(k,m) \) and \( O\text{Gr}(k,m) \) denote the symplectic resp. bisymplectic resp. orthogonal Grassmannian. \( \mathbb{S}_2 \) denotes a connected component of \( \text{OGr}(5,10) \), and \( Q_m \) is an \( m \)-dimensional smooth quadric.)
Gr(2, 8) with 4 Plücker hyperplanes, Gushel–Mukai fourfolds and sixfolds. It would be interesting to devise new methods to treat these families.

**Conventions.** In this article, the word variety will refer to a reduced irreducible scheme of finite type over \( \mathbb{C} \). A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

**All Chow groups will be with rational coefficients:** we denote by \( A_j(Y) \) the Chow group of \( j \)-dimensional cycles on \( Y \) with \( \mathbb{Q} \)-coefficients; for \( Y \) smooth of dimension \( n \) the notations \( A_j(Y) \) and \( A^{n-j}(Y) \) are used interchangeably. The notation \( A^i_{\text{hom}}(Y) \) will be used to indicate the subgroup of homologically trivial cycles.

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [30, 33]) will be denoted \( \mathcal{M}_{\text{rat}} \).

### 2. Preliminaries

#### 2.1. MCK decomposition

**Definition 2.1 (Murre [29]).** Let \( X \) be a smooth projective variety of dimension \( n \). We say that \( X \) has a CK decomposition if there exists a decomposition of the diagonal

\[
\Delta_X = \pi_X^0 + \pi_X^1 + \cdots + \pi_X^{2n} \quad \text{in} \quad A^n(X \times X),
\]

such that the \( \pi_X^i \) are mutually orthogonal idempotents and \( \langle \pi_X^i, H^j(X, \mathbb{Q}) \rangle = H^i(X, \mathbb{Q}) \).

(NB: “CK decomposition” is shorthand for “Chow–Künneth decomposition.”)

**Remark 2.2.** The existence of a CK decomposition for any smooth projective variety is part of Murre’s conjectures [15, 29].

**Definition 2.3 (Shen–Vial [34]).** Let \( X \) be a smooth projective variety of dimension \( n \). Let \( \Delta_X^{sm} \in A^{2n}(X \times X \times X) \) be the class of the small diagonal

\[
\Delta_X^{sm} := \{(x, x, x) \mid x \in X\} \subset X \times X \times X.
\]

An MCK decomposition is a CK decomposition \( \{\pi_X^i\} \) of \( X \) that is multiplicative, i.e., it satisfies

\[
\pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \times \pi_X^j) = 0 \quad \text{in} \quad A^{2n}(X \times X \times X) \quad \text{in} \quad i + j \neq k.
\]

(NB: “MCK decomposition” is shorthand for “multiplicative Chow–Künneth decomposition.”)

**Remark 2.4.** The small diagonal (seen as a correspondence from \( X \times X \) to \( X \)) induces the multiplication morphism

\[
\Delta_X^{sm} : h(X) \otimes h(X) \to h(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

Let us assume \( X \) has a CK decomposition

\[
h(X) = \bigoplus_{i=0}^{2n} h^i(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}.
\]

By definition, this decomposition is multiplicative if for any \( i, j \) the composition

\[
h^i(X) \otimes h^j(X) \to h(X) \otimes h(X) \xrightarrow{\Delta_X^{sm}} h(X) \quad \text{in} \quad \mathcal{M}_{\text{rat}}
\]

factors through \( h^{i+j}(X) \).

If \( X \) has an MCK decomposition, then, setting

\[
A^i_{(j)}(X) := (\pi_X^{2i-j})_* A^i(X),
\]
one obtains a bigraded ring structure on the Chow ring: that is, the intersection product sends
\[ A^i_j(X) \otimes A^r_s(Y) \to A^{i+r}_{j+s}(X). \]
It is expected that for any \( X \) with an MCK decomposition, one has
\[ A^i_j(X) \equiv 0 \text{ for } j < 0, A^i_0(X) \cap A^0_{\text{hom}}(X) \equiv 0; \]
this is related to Murre’s conjectures \( B \) and \( D \), that have been formulated for any CK decomposition [29].

The property of having an MCK decomposition is restrictive, and is closely related to Beauville’s “splitting property” conjecture [1]. To give an idea: hyperelliptic curves have an MCK decomposition \[34, \text{example 8.16}], but the very general curve of genus \( \geq 3 \) does not have an MCK decomposition \[9, \text{example 2.3}]. As for surfaces: a smooth quartic in \( \mathbb{P}^3 \) has an MCK decomposition, but a very general surface of degree \( \geq 7 \) in \( \mathbb{P}^3 \) should not have an MCK decomposition \[9, \text{proposition 3.4}]. For more detailed discussion, and examples of varieties with an MCK decomposition, we refer to \[34, \text{section 8}\], as well as \[9, 10, 22, 26, 28, 35, 36\].

## 2.2. The Franchetta property

**Definition 2.5.** Let \( X \to B \) be a smooth projective morphism, where \( X, B \) are smooth quasi-projective varieties. We say that \( X \to B \) has the Franchetta property in codimension \( j \) if the following holds: for every \( C \in A^j(X) \) such that the restriction \( C_{|X_b} \) is homologically trivial for the very general \( b \in B \), the restriction \( C_b \) is zero in \( A^j(X_b) \) for all \( b \in B \).

We say that \( X \to B \) has the Franchetta property if \( X \to B \) has the Franchetta property in codimension \( j \) for all \( j \).

This property is studied in \[3, 7, 8, 32\].

**Definition 2.6.** Given a family \( X \to B \) as above, with \( X := X_b \) a fiber, we write
\[ \text{GDA}_B(X) := \text{Im}(A^j(X) \to A^j(X)) \]
for the subgroup of generically defined cycles. In a context where it is clear to which family we are referring, the index \( B \) will often be suppressed from the notation.

With this notation, the Franchetta property amounts to saying that \( \text{GDA}_B^*(X) \) injects into cohomology, under the cycle class map.

## 2.3. Cayley’s trick and motives

**Theorem 2.7 (Jiang [16]).** Let \( E \to U \) be a vector bundle of rank \( r \geq 2 \) over a smooth projective variety \( U \), and let \( S := s^{-1}(0) \subset U \) be the zero locus of a regular section \( s \in H^0(U, E) \) such that \( S \) is smooth of dimension \( \dim U - \text{rank } E \). Let \( X := w^{-1}(0) \subset \mathbb{P}(E) \) be the zero locus of the regular section \( w \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) \) that corresponds to \( s \) under the natural isomorphism \( H^0(U, E) \cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) \), and assume \( X \) is smooth. There is an isomorphism of Chow motives
\[ h(X) \cong h(S)(1 - r) \oplus \bigoplus_{i=0}^{r-2} h(U)(-i) \text{ in } \mathcal{M}_{\text{rat}}. \]

**Proof.** This is [16, corollary 3.2], which more precisely gives an isomorphism of integral Chow motives. For later use, we now give some details about the isomorphism as constructed in loc. cit. Let
\[ \Gamma := X \times_U S \subset X \times S \]

\[ h(X) \cong h(S)(1 - r) \oplus \bigoplus_{i=0}^{r-2} h(U)(-i) \text{ in } \mathcal{M}_{\text{rat}}. \]
be correspondences inducing the maps \((\pi_i)_*\) of loc. cit., i.e.,
\[
(\Pi_i)_* = (\pi_i)_* := (q_{i+1})_* i_* : A^i(X) \to A^{i-i}(U),
\]
where \(i : X \to \mathbb{P}(E)\) is the inclusion morphism, and the \((q_{i+1})_* : A_*(\mathbb{P}(E)) \to A_*(U)\) are defined in loc. cit. in terms of the projective bundle formula for \(q : E \to U\). As indicated in [16, corollary 3.2] (cf. also [16, text preceding corollary 3.2]), there is an isomorphism
\[
(\Gamma, \Pi_0, \Pi_1, \ldots, \Pi_{r-2}) : h(X) \xrightarrow{\sim} h(S)(1 - r) \oplus \bigoplus_{i=0}^{r-2} h(U)(-i) \text{ in } \mathcal{M}_{rat}.
\]

**Remark 2.8.** In the setup of Theorem 2.7, a cohomological relation between \(X\) and \(S\) was established in [18, prop. 4.3] (cf. also [13, section 3.7], as well as [4, proposition 46] for a generalization). A relation on the level of derived categories was established in [31, theorem 2.10] (cf. also [13, section 3.7], as well as [4, proposition 46]).

We now make the natural observation that the isomorphism of Theorem 2.7 behaves well with respect to families, in the following sense:

**Notation 2.9.** Let \(X, S, U\) and \(E \to U\) be as in Theorem 2.7. Let \(B \subset \mathbb{P}H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))\) be the Zariski open such that both \(X := X_b \subset \mathbb{P}(E)\) and \(S := S_b \subset U\) are smooth of the expected dimension. Let
\[
\mathcal{X} \to B, \mathcal{S} \to B
\]
denote the universal families.

**Proposition 2.10.** Let \(X, S, U\) be as in Theorem 2.7. Assume \(U\) has trivial Chow groups. For any \(m \in \mathbb{N}\), there are injections
\[
GDA^i(X^m) \hookrightarrow GDA^{i+mmr}(S^m) \oplus \bigoplus GDA^*(S^{m-1}) \oplus \cdots \oplus \mathbb{Q}^i.
\]

**Proof.** (NB: we will not really need this proposition below, but we include it because it makes some arguments easier, cf. footnote 1 below.)

Let us first do the case \(m = 1\). The isomorphism of Theorem 2.7 is generically defined, i.e., there exist relative correspondences \(\Gamma_B, \Pi^1_B\) fitting into a commutative diagram
\[
\begin{align*}
A^i(\mathcal{X}) & \xrightarrow{((\Gamma_B)_*, (\Pi^0_B)_*, \ldots, (\Pi^{r-2}_B)_*))} A^{i+1-r}(\mathcal{S}) \oplus \bigoplus_{i=0}^{r-2} A^{i-i}(U \times B) \\
\downarrow & \downarrow \\
A^i(\mathcal{X}) & \xrightarrow{((\Gamma, (\Pi_B)_*, \ldots, (\Pi^{r-2}_B)_*))} A^{i+1-r}(\mathcal{S}) \oplus \bigoplus_{i=0}^{r-2} A^{i-i}(U),
\end{align*}
\]
where vertical arrows are restrictions to a fiber, and the lower horizontal arrow is the isomorphism of Theorem 2.7. Indeed, \(\Gamma_B\) can be defined as
\[
\Gamma_B := \mathcal{X} \times_{U \times B} \mathcal{S} \subset \mathcal{X} \times_B \mathcal{S}.
\]

The \(\Pi_i\) are also generically defined (just because the graph of the embedding \(i : X \to \mathbb{P}(E)\) is generically defined). This gives relative correspondences \(\Gamma_B, \Pi^1_B\) over \(B\) such that the restriction to a fiber over \(b \in B\) gives back the correspondences \(\Gamma, \Pi\) of Theorem 2.7. The fact that this makes diagram (1) commute is [30, lemma 8.1.6]. The commutative diagram (1) implies that
there is an injective map
\[ GDA^i(X) \hookrightarrow GDA^{i+1-t}(S) \oplus \bigoplus A^*(U) = GDA^{i+1-t}(S) \oplus \mathbb{Q}^t. \] (2)

The argument for \( m > 1 \) is similar: the isomorphism of motives of Theorem 2.7, combined with the fact that \( U \) has trivial Chow groups (and so \( h(U) \cong \bigoplus 1(\ast) \)) induces an isomorphism of Chow groups
\[ A^i(X^m) \cong A^{i+m-mr}(S^m) \oplus \bigoplus A^*(S^m-1) \oplus \cdots \oplus \mathbb{Q}^t. \] (3)

Here, the map from left to right is given by various combinations of the correspondences \( \Gamma \) and \( \Pi_i \). As we have seen these correspondences are generically defined, and so their products are also generically defined. It follows as above that the map (3) preserves generically defined cycles.

\[ \square \]

### 2.4. A Franchetta-type result

**Proposition 2.11.** Let \( Y \) be a smooth projective variety with trivial Chow groups (i.e., \( A^i_{\text{hom}}(Y) = 0 \)). Let \( L_1, \ldots, L_r \rightarrow Y \) be very ample line bundles, and let \( \mathcal{X} \rightarrow B \) be the universal family of smooth complete intersections of type \( X = Y \cap H_1 \cap \cdots \cap H_n \), where \( H_j \in |L_j| \). Assume the fibers \( X \) have \( H^i_{\text{tr}}(X, \mathbb{Q}) \neq 0 \). There is inclusion
\[ \ker(GDA^i_B(X \times X) \rightarrow H^{2i}(X \times X, \mathbb{Q})) \subset \langle (p_1)^*GDA^*_B(X), (p_2)^*GDA^*_B(X) \rangle. \]

**Proof.** This is essentially equivalent to Voisin’s “spread” result [39, proposition 1.6] (cf. also [27, proposition 5.1] for a reformulation). For completeness, we include a quick proof. Let \( \tilde{B} := \mathbb{P}H^0(Y, L_1 \oplus \cdots \oplus L_r) \) (so that \( B \subseteq \tilde{B} \) is a Zariski open), and let us consider the projection
\[ \pi : \mathcal{X} \times_B \mathcal{X} \rightarrow Y \times Y. \]

Using the very ampleness assumption, one finds that \( \pi \) is a \( \mathbb{P}^q \)-bundle over \( (Y \times Y) \setminus \Delta_Y \), and a \( \mathbb{P}^q \)-bundle over \( \Delta_Y \). That is, \( \pi \) is what is termed a *stratified projective bundle* in [7]. As such, [7, proposition 5.2] implies the equality
\[ GDA^i_B(X \times X) = \text{Im}(A^*(Y \times Y) \rightarrow A^*(X \times X)) + \Delta, GDA^i_B(X), \] (4)
where \( \Delta : X \rightarrow X \times X \) is the inclusion along the diagonal. Since \( Y \) has trivial Chow groups, one has \( A^*(Y \times Y) \cong A^*(Y) \otimes A^*(Y) \). Base-point freeness of the \( L_i \) implies \( \mathcal{X} \rightarrow Y \) has the structure of a projective bundle; it is then readily seen (by a direct argument or by simply applying once more [7, proposition 5.2]) that
\[ GDA^i_B(X) = \text{Im}(A^*(Y) \rightarrow A^*(X)). \]

The equality (4) thus reduces to
\[ GDA^i_B(X \times X) = \langle (p_1)^*GDA^*_B(X), (p_2)^*GDA^*_B(X), \Delta_X \rangle \]
(where \( p_1, p_2 \) denote the projection from \( X \times X \) to first resp. second factor). The assumption that \( X \) has non-zero transcendental cohomology implies that the class of \( \Delta_X \) is not decomposable in cohomology. It follows that
\[ \text{Im}(GDA^i_B(X \times X) \rightarrow H^{2i}(X \times X, \mathbb{Q})) = \text{Im}(\text{Dec}^i(X \times X) \rightarrow H^{2i}(X \times X, \mathbb{Q})) \oplus \mathbb{Q}[\Delta_X], \]
where we use the shorthand
\[ \text{Dec}^i(X \times X) := \langle (p_1)^*GDA^*_B(X), (p_2)^*GDA^*_B(X) \rangle \cap A^i(X \times X) \]
for the decomposable cycles. We now see that if $\Gamma \in GDA^{\dim X}(X \times X)$ is homologically trivial, then $\Gamma$ does not involve the diagonal and so $\Gamma \in \text{Dec}^{\dim X}(X \times X)$. This proves the proposition. \qed

**Remark 2.12.** Proposition 2.11 has the following consequence: if the family $\mathcal{X} \to B$ has the Franchetta property, then $\mathcal{X} \times_B \mathcal{X} \to B$ has the Franchetta property in codimension $\dim X$.

### 2.5. HPD and motives

**Theorem 2.13.** Let $Y_1, Y_2 \subset \mathbb{P}(V)$ be smooth projective varieties with trivial Chow groups (i.e., $A^i_{\text{hom}}(Y_j) = 0$), and let $Y^\vee_2 \subset \mathbb{P}(V^\vee)$ be the HPD dual of $Y_2$. Let $H \subset \mathbb{P}(V) \times \mathbb{P}(V)$ be a $(1,1)$-divisor, and let $f_H : \mathbb{P}(V) \to \mathbb{P}(V^\vee)$ be the morphism defined by $H$. Assume that the varieties

\[
X := (Y_1 \times Y_2) \cap H,
\]

\[
S := Y_1 \cap (f_H)^{-1}(Y^\vee_2)
\]

are smooth and dimensionally transverse. Assume moreover that the Hodge conjecture holds for $S$, that $H^j(S, \mathbb{Q})$ is algebraic for $j \neq \dim S$ and that $H^{\dim S}(S, \mathbb{Q})$ is not completely algebraic. Then, there is a split injection of Chow motives

\[
h(X) \leftarrow h(S)(-m) \oplus 1(*)_{\text{in} \mathcal{M}_{\text{rat}}},
\]

where $m := \frac{1}{2}(\dim X - \dim S)$. (In particular, one has vanishing

\[
A^i_{\text{hom}}(X) = 0 \quad \forall \ j > \frac{1}{2}(\dim X + \dim S).
\]

**Proof.** Using the HPD formalism, it is proven in [6, proposition 2.4] that there exists a semi-orthogonal decomposition

\[
D^b(X) = \langle D^b(S), A_1, \ldots, A_s \rangle,
\]

where the $A_j$ are some exceptional objects. Using Hochschild homology and the Kostant–Rosenberg isomorphism (cf. for instance [19, sections 1.7 and 2.5]), this implies that there exist correspondences $\Phi'$ and $\Xi'$ such that

\[
H^+_tr(X, \mathbb{Q}) \xrightarrow{(\Phi')} H^+_tr(S, \mathbb{Q}) \xrightarrow{(\Xi')} H^+_tr(X, \mathbb{Q})
\]

is the identity. (Here, $H^+_tr(S, \mathbb{Q})$ denotes the orthogonal complement of the algebraic part of cohomology.) By assumption $H^+_tr(S, \mathbb{Q}) = H^{\dim S}_tr(S, \mathbb{Q})$, and by weak Lefschetz $H^+_tr(X, \mathbb{Q}) = H^{\dim X}_tr(X, \mathbb{Q})$, and so we actually have that

\[
H^{\dim X}_tr(X, \mathbb{Q}) \xrightarrow{(\Phi')} H^{\dim S}_tr(S, \mathbb{Q}) \xrightarrow{(\Xi')} H^{\dim X}_tr(X, \mathbb{Q})
\]

is the identity. Again using Hochschild homology and the Kostant–Rosenberg isomorphism, we see that the Hodge conjecture for $S$, plus the decomposition (5), implies the Hodge conjecture for $X$. This means that we can find correspondences $\Phi$ and $\Xi$ such that

\[
H^*(X, \mathbb{Q}) \xrightarrow{\Phi} H^{\dim S}(S, \mathbb{Q}) \oplus \bigoplus \mathbb{Q}(\dim X - j) \xrightarrow{\Xi} H^*(X, \mathbb{Q})
\]

is the identity, i.e., the cycle

\[
\Delta_X = \Xi \circ \Phi \in A^{\dim X}(X \times X)
\]

is homologically trivial.

We now consider things family-wise, i.e., we construct universal families $\mathcal{X} \to B$ and $S \to B$, where
parametrizes all divisors $H$ such that both $X := X_H$ and $S := S_H$ are smooth and dimensionally transverse.

Applying Voisin’s Hilbert schemes argument [37, proposition 3.7] (cf. also [21, proposition 2.11]) to this setup, we may assume that the correspondences $U$ and $N$ are generically defined (with respect to $B$), and so in particular

$$\Delta_X - \Xi \circ \Phi \in GDA^{\dim X}(X \times X).$$

We observe that $H_{tr}^{\dim X}(X, \mathbb{Q}) \cong H_{tr}^{\dim S}(S, \mathbb{Q})$ (this follows from the decomposition (5)), and so $H_{tr}^{\dim X}(X, \mathbb{Q}) \neq 0$; all conditions of Proposition 2.11 are fulfilled. Applying Proposition 2.11 to the cycle $\Delta_X - \Xi \circ \Phi$, we find that a modification of this cycle vanishes:

$$\Delta_X - \Xi \circ \Phi - \gamma = 0 \text{ in } A^{\dim X}(X \times X),$$

where

$$\gamma \in \langle (p_1)^*GDA^*(X), (p_2)^*GDA^*(X) \rangle$$

is a decomposable cycle. This translates into the fact that (up to adding some trivial motives $1(\ast)$ and modifying the correspondences $\Phi$ and $\Xi$) the composition

$$h(X) - h(S)(-m) \oplus 1(\ast)\Xi h(X) \text{ in } \mathcal{M}_{rat}$$

is the identity, which proves the proposition.

(Finally, the statement in parentheses is a straightforward consequence of the injection of motives: taking Chow groups, one obtains an injection

$$A_{hom}^j(X) \hookrightarrow A_{hom}^{j-m}(S).$$

But the group on the right vanishes for $j - m > \dim S$, which means $j > \frac{1}{2}(\dim X + \dim S)$.)

**Example 2.14.** Here is a sample application of Theorem 2.13. Let $Y_1 = Y_2 = \text{Gr}(2, 5) \subset \mathbb{P}^9$. Then, $Y_2' = \text{Gr}(2, 5) \subset (\mathbb{P}^9)^{\vee}$ and $S := Y_1 \cap (f_H)^{-1}(Y_2')$ is 3-dimensional (for $H$ sufficiently general). We consider the 11-dimensional variety

$$X := (\text{Gr}(2, 5) \times \text{Gr}(2, 5)) \cap H \subset \mathbb{P}^9 \times \mathbb{P}^9,$$

where $H$ is a general $(1, 1)$-divisor. This $X$ is a Fano variety of Calabi–Yau type, considered in [13, section 3.3]. Theorem 2.13 implies that one has

$$A_{hom}^j(X) = 0 \forall j > 7,$$

i.e., $X$ has Niveau$(A^*(X)) \leq 3$ in the sense of [20].

### 3. Main result

This section contains the proof of our main result, which is as follows:

**Theorem 3.1.** Let $X \subset U$ be the inclusion of a Fano variety $X$ in its ambient space $U$, where $X, U$ are as in Table 1. Then $X$ has an MCK decomposition. The Chern classes $c_j(X)$, and the image $\text{Im}(A^*(U) \rightarrow A^*(X))$, lie in $A^{\ast}_{\{0\}}(X)$. 


3.1. A criterion

To prove Theorem 3.1, we will use the following general criterion:

**Proposition 3.2.** Let $\mathcal{X} \to B$ be a family of smooth projective varieties. Assume the following conditions:

1. (c0) each fiber $X$ has dimension $2d \geq 8$;
2. (c1) each fiber $X$ has a self-dual CK decomposition $\{\pi_X^i\}$ which is generically defined (with respect to $B$), and $h^j(X) \cong \oplus 1(*)$ for $j \neq 2d$;
3. (c2) there exists a family of surfaces $S \to B$ where $B$ is a countable intersection of non-empty Zariski opens, and for each $b \in B$ there is a split injection of motives
   $$h(X_b) \hookrightarrow h(S_b)(1-d) \oplus \bigoplus 1(*)$$
   in $M_{\text{rat}}$.
4. (c3) the family $S \times_B S \to B^o$ has the Franchetta property.

Then for each fiber $X$, $\{\pi_X^i\}$ is an MCK decomposition, and $\text{GDA}^*(X) \subset A^*_0(X)$.

Moreover, condition (c3) may be replaced by the following:

1. (c3') $S \to B^o$ is a family of $K3$ surfaces, which is the universal family of smooth sections of a direct sum of very ample line bundles on some smooth projective ambient space $V$ with trivial Chow groups (i.e., $A^*_\text{hom}(V) = 0$), and $S \to B^o$ has the Franchetta property.

**Proof.** Using Voisin’s Hilbert schemes argument [37, proposition 3.7] (cf. also [21, proposition 2.11]), one may assume that the split injection of (c2) is generically defined (with respect to $B^o$).

This means that there exists a relative correspondence $\Phi$ fitting into a commutative diagram

$$
\begin{array}{ccc}
A^i(\mathcal{X}) & \xrightarrow{\Phi} & A^{i+1-d}(S) \oplus \bigoplus A^*(B^o) \\
\downarrow & & \downarrow \\
A^i(X) & \xrightarrow{(\Phi|_X)} & A^{i+1-d}(S) \oplus \mathbb{Q}^i,
\end{array}
$$

where vertical arrows are restrictions to a fiber, and the lower horizontal arrow is induced by the injection of (c2). The same then applies to $X \times X$, i.e., there is a commutative diagram

$$
\begin{array}{ccc}
A^i(\mathcal{X} \times_B \mathcal{X}) & \to & A^{i+2-2d}(S \times_B S) \oplus \bigoplus A^*(S) \oplus \bigoplus A^*(B^o) \\
\downarrow & & \downarrow \\
A^i(X \times X) & \leftarrow & A^{i+2-2d}(S \times S) \oplus \bigoplus A^*(S) \oplus \bigoplus \mathbb{Q}^i,
\end{array}
$$

where the lower horizontal arrow is split injective thanks to (c2). That is, there is an injection

$$\text{GDA}^i_{B^o}(X \times X) \hookrightarrow \text{GDA}^{i+2-2d}_{B^o}(S \times S) \oplus \bigoplus \text{GDA}^i_{B^o}(S) \oplus \mathbb{Q}^i.$$

It then follows from (c3) that $\mathcal{X} \times_B \mathcal{X} \to B^o$ has the Franchetta property.

Let us now ascertain that the CK decomposition $\{\pi_X^i\}$ is multiplicative. What we need to check is that for each $X = X_b$ one has

$$\pi_X^i \circ \Delta_X^{sm} \circ (\pi_X^j \times \pi_X^k) = 0 \text{ in } A^{4d}(X \times X \times X) \text{ in } i + j \neq k. \quad (6)$$
A standard spread lemma (cf. [38, lemma 3.2]) shows that it suffices to prove this for all $b \in B^c$, so we will henceforth assume that $X = X_b$ with $b \in B^c$. We note that the cycle in (6) is generically defined, and homologically trivial.

Let us assume that among the three integers $(i,j,k)$, at least one is different from $2d$. Using the hypothesis $h^t(X) = \bigoplus 1(*)$ for $j \neq 2d$, we find there is a (generically defined) split injection

$$(\pi_X^{4d-i} \times \pi_X^{4d-j} \times \pi_X^k)_* A^d(X \times X \times X) \rightarrow A^s(X \times X).$$

Since

$$\pi_X^k \circ \Delta_X^{sm} \circ (\pi_X^i \times \pi_X^j) = (\pi_X^i \times \pi_X^i \times \pi_X^k)_*(\Delta_X^{sm}) = (\pi_X^{4d-i} \times \pi_X^{4d-j} \times \pi_X^k)_*(\Delta_X^{sm})$$

(where the first equality is an instance of Lieberman’s lemma), the required vanishing (6) now follows from the Franchetta property for $X \times_B X \rightarrow B$.

It remains to treat the case $i = j = k = 2d$. Using the split injection of motives (2) and taking the tensor product, we find there is a split injection of Chow groups

$$A^i(X \times X \times X) \rightarrow A^{i+3-3i}(S^3) \oplus \bigoplus A^i(S^2) \oplus \bigoplus A^i(S) \oplus \mathbb{Q}^i.$$  

Moreover (just as we have seen above for $X^2$), this injection respects generically defined cycles, i.e., there is an injection

$$GDA^i(X \times X \times X) \rightarrow GDA^{i+3-3i}(S^3) \oplus \bigoplus GDA^i(S^2) \oplus \bigoplus GDA^i(S) \oplus \mathbb{Q}^i.$$  

In particular, taking $j = 4d$ we find an injection

$$GDA^{4d}(X \times X \times X) \rightarrow GDA^{d+3}(S^3) \oplus \bigoplus GDA^d(S^2) \oplus \bigoplus GDA^d(S) \oplus \mathbb{Q}^d.$$  

By assumption, $d \geq 4$ and so the summand $GDA^{d+3}(S^3)$ vanishes for dimension reasons. The required vanishing (6) then follows from the Franchetta property for $S \times_B S$. This proves that \{\pi_X^k\} is MCK.

To see that $GDA^i(X) \subset A^i_{(0)}(X)$, it suffices to note that

$$(\pi_X^k)_*, GDA^i(X)(k \neq 2j)$$

is generically defined, and homologically trivial. The Franchetta property for $X \rightarrow B$ (which is implied by the Franchetta property for $X \times_B X$) then implies the vanishing

$$(\pi_X^k)_*, GDA^i(X) = 0(k \neq 2j),$$

and so $GDA^i(X) \subset (\pi_X^k)^* A^i(X) =: A^i_{(0)}(X)$.

Let us now proceed to show that condition (c3′) implies condition (c3). The hypotheses of (c3′) imply that $B^c$ is a Zariski open in some $\bar{B} := \mathbb{P} H^0(V, \oplus_{j=1}^s L_j)$ which is isomorphic to $\mathbb{P}^r$. The very ampleness assumption implies that

$$\pi : S \times_B S \rightarrow V \times V$$

is a $\mathbb{P}^{r-2s}$-bundle over $(V \times V) \setminus \Delta_V$ and a $\mathbb{P}^{r-2}$-bundle over $\Delta_V$. That is, $\pi$ is a stratified projective bundle in the sense of [7]. As such, [7, proposition 5.2] implies the equality

$$GDA^*(S \times S) = \text{Im}(A^*(V \times V) \rightarrow A^*(S \times S)) + \Delta_S GDA^*(S),$$

where $\Delta : S \rightarrow S \times S$ is the inclusion along the diagonal. Since $V$ has trivial Chow groups, one has $A^*(V \times V) \cong A^*(V) \otimes A^*(V)$. Moreover, $S \rightarrow V$ is a projective bundle and so [7, proposition 5.2] gives $GDA^*(S) = \text{Im}(A^*(V) \rightarrow A^*(S))$. It follows that the above equality reduces to

$$GDA^*(S \times S) = \langle (p_1)^* GDA^*(S), (p_2)^* GDA^*(S), \Delta_S \rangle$$  \hspace{1cm} (7)

(where $p_1$, $p_2$ denote the projection from $S \times S$ to first resp. second factor). By assumption, $S$ is a K3 surface and the Franchetta property holds for $S \rightarrow B^c$, which means that
Table 2. Fano varieties $X$ and their associated K3 surface $S$.

| Label in [6] | $X$ | $\dim X$ | $S \subseteq V$ |
|--------------|-----|----------|-----------------|
| M1           | $X_{(1,1,1)} \subset \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$ | 8 | $S^1 \subset \mathbb{P}^3 \times \mathbb{P}^3$ |
| M6           | $X_{(1,1)} \subset S_5 \times \mathbb{P}^2$ | 16 | $S^1 \subset S_5$ |
| M7           | $X_{(1,1)} \subset \text{Gr}(2,6) \times \mathbb{P}^5$ | 12 | $S^1 \subset \text{Gr}(2,6)$ |
| M8           | $X_{(1,1)} \subset S\text{Gr}(2,6) \times \mathbb{P}^4$ | 10 | $S^1 \subset S\text{Gr}(2,6)$ |
| M9           | $X_{(1,1)} \subset S_5\text{Gr}(2,6) \times \mathbb{P}^3$ | 8 | $S^1 \subset S_5\text{Gr}(2,6)$ |
| M10          | $X_{(1,1)} \subset S\text{Gr}(3,6) \times \mathbb{P}^3$ | 8 | $S^1 \subset S\text{Gr}(3,6)$ |

For the case $M_k$, let

$$GDA^+(S) = \mathbb{Q} \oplus GDA^1(S) \oplus \mathbb{Q}[o],$$

where $o \in A^2(S)$ is the Beauville–Voisin class [2]. Given a divisor $D \in A^1(S)$, it is known that

$$\Delta_S \cdot (p_j)^*(D) = \Delta_S(D) = D \times o + o \times D \text{ in } A^2(S \times S) \tag{7}$$

[2, proposition 2.6(a)]. Also, it is known that

$$\Delta_S \cdot (p_j)^*(o) = \Delta_S(o) = o \times o \text{ in } A^4(S \times S) \tag{8}$$

[2, proposition 2.6(b)]. It follows that the right-hand side of (7) is decomposable in codimension $> 2$, i.e.,

$$\langle (p_1)^*GDA^+(S), (p_2)^*GDA^+(S), \Delta_S \rangle \cap A^j(S \times S) = \langle (p_1)^*GDA^+(S), (p_2)^*GDA^+(S) \rangle \cap A^j(S \times S) \forall j \neq 2.$$

Since we know that $GDA^+(S)$ injects into cohomology (this is the Franchetta property for $S \rightarrow B^o$), equality (7) (plus the Künneth decomposition in cohomology) now implies that

$$GDA^j(S \times S) \rightarrow H^{2j}(S \times S, \mathbb{Q})$$

is injective for $j \neq 2$.

For the case $j=2$, it suffices to remark that $\Delta_S$ is linearly independent from the decomposable part in cohomology (for otherwise $H^{2,0}(S)$ would be zero, which is absurd). The injectivity of

$$GDA^2(S \times S) \rightarrow H^4(S \times S, \mathbb{Q})$$

then follows from (7) plus the injectivity of $GDA^+(S) \rightarrow H^*(S, \mathbb{Q})$. This shows that condition (c3') implies condition (c3); the proposition is proven. \hfill $\Box$

### 3.2. Verifying the criterion: part 1

**Proposition 3.3.** The following families verify the conditions of Proposition 3.2: the universal families $X \rightarrow B$ of Fano varieties of type $M1$, $M6$, $M7$, $M8$, $M9$, $M10$.

**Proof.** The existence of a generically defined CK decomposition is an easy consequence of the fact that the Fano varieties $X$ under consideration are complete intersections in an ambient space $U$ with trivial Chow groups, cf. for instance [37, lemma 3.6]. This takes care of conditions (c0) and (c1) of Proposition 3.2.

To verify condition (c2), we use Cayley’s trick (Theorem 2.7). The K3 surface $S$ associated to the Fano variety $X$ is a complete intersection in an ambient space $V$ as indicated in Table 2. Let us write $2d := \dim X$. The ambient spaces $V$ that occur all have trivial Chow groups, and so Theorem 2.7 gives the split injection of motives

$$h(X) \hookrightarrow h(S)(1-d) \oplus 1(*) \text{ in } \mathcal{M}_{\text{rat}},$$

i.e., condition (c2) is verified.
We observe that all ambient spaces $V$ in Table 2 have trivial Chow groups. To verify condition (c3'), it only remains to check the Franchetta property for the families $S \to B^\circ$. In all these cases, $S \to V$ is a projective bundle, and so (using the projective bundle formula, or lazily applying [7, proposition 5.2]) we find equality
\[
GDA^{l}_p(S) = \text{Im}(A^l(V) \to A^l(S)).
\]

Let us check that the right-hand side injects into cohomology. This is non-trivial only in codimension $j = 2$. For the family $M1$, it suffices to observe that $A^2(P^3 \times P^3)$ is generated by intersections of divisors, and so
\[
\text{Im}(A^2(P^3 \times P^3) \to A^2(S)) = \mathbb{Q}[\sigma]
\]
injects into cohomology. For the family $M7$, it suffices to check that the restriction of $c_2(Q) \in A^2(\text{Gr}(2,6))$ (where $Q$ denotes the universal quotient bundle) to $S$ is proportional to $\sigma$; this is done in [32, proposition 2.1]. For the family $M6$, we may as well verify that
\[
\text{Im}(A^2(\text{OGr}(5,10)) \to A^2(S)) = \mathbb{Q}[\sigma]
\]
(recall that $S_5$ is a connected component of $\text{OGr}(5,10)$ in its spinor embedding), this is taken care of in [32, proposition 2.1]. For the families $M8$ and $M9$, since $S\text{Gr}(2,6)$ and $S_2\text{Gr}(2,6)$ are complete intersections (of dimension 7 resp. 6) inside $\text{Gr}(2,6)$, there is an isomorphism
\[
A^2(S_2\text{Gr}(2,6)) \cong A^2(S\text{Gr}(2,6)) \cong A^2(\text{Gr}(2,6)).
\]

The case $M7$ then guarantees that $\text{Im}(A^2(V) \to A^2(S))$ is spanned by $\sigma$. Finally, for the case $M10$ one observes that $A^2(S\text{Gr}(3,6)) \cong \mathbb{Q}$ (this follows from [12, proposition 2.1], where $S\text{Gr}(3,6)$ is denoted $Y_3$), and so
\[
\text{Im}(A^2(S\text{Gr}(3,6)) \to A^2(S)) = \mathbb{Q}[h^2] = \mathbb{Q}[\sigma].
\]

**Remark 3.4.** It seems likely that the families $M7$, $M8$, $M9$, $M10$ can be related to one another via (a higher-codimension version of) the game of projections and jumps of [4, sections 3.3 and 3.4]. This might simplify the above argument.

### 3.3. Verifying the criterion: part 2

**Proposition 3.5.** The following families verify the conditions of Proposition 3.2: the universal families $X \to B$ of Fano varieties of type M3 and M4.

**Proof.** The existence of a generically defined CK decomposition follows as above. The difference with the above is that the families M3 and M4 are *not* in the form of Cayley’s trick; hence, to check condition (c2) we now apply Theorem 2.13 rather than Theorem 2.7.

For the case M3, Theorem 2.13 applies with $Y_1 = \text{Gr}(2,5)$ and $Y_2 = Q_5$ a 5-dimensional quadric embedded in $P^9$. Let $B^\circ \subset B$ be the open parametrizing Fano varieties $X$ of type M3 for which, in the notation of Theorem 2.13, $S$ is a smooth surface. For each $X = X_b$ with $b \in B^\circ$, Theorem 2.13 gives an injection of motives
\[
h(X) \hookrightarrow h(S)(-4) \oplus \bigoplus 1(\ast) \text{ in } M_{\text{rat}}.
\]

Since quadrics are projectively self-dual, this $S$ is the intersection of $\text{Gr}(2,5)$ with a quadric and 3 hyperplanes in $P^9$; this is Mukai’s model for the general K3 surface of genus 6. That the family $S \to B^\circ$ has the Franchetta property is proven in [32]. This takes care of conditions (c2) and (c3) of Proposition 3.2.
For the family M4, Theorem 2.13 applies again, with \( Y_1 = \text{SGr}(2, 5) \) and \( Y_2 = Q_4 \) a 4-dimensional quadric embedded in \( \mathbb{P}^9 \). Note that \( Y_1 \) is a hyperplane section of \( \text{Gr}(2, 5) \) under its Plücker embedding. Again, let \( B^0 \subset B \) denote the open where both \( X \) and \( S \) are smooth dimensionally transverse. Theorem 2.13 now gives an injection of motives

\[
 h(X) \hookrightarrow h(S)(-3) \oplus 1(*) \text{ in } \mathcal{M}_{\text{rat}},
\]

where \( S \) is again the intersection of \( \text{Gr}(2, 5) \) with a quadric and 3 hyperplanes in \( \mathbb{P}^9 \). The family \( S \rightarrow B^0 \) is now the family of all smooth 2-dimensional complete intersections of \( \text{SGr}(2, 5) \) with a quadric and 2 hyperplanes. One has that \( S \rightarrow \text{SGr}(2, 5) \) is a projective bundle, and so (as before)

\[
 \text{GDA}^2(S) = \text{Im}(\text{GDA}^2(\text{SGr}(2, 5)) \rightarrow \text{A}^2(S)).
\]

But \( \text{GDA}^2(\text{Gr}(2, 5)) \rightarrow \text{A}^2(\text{SGr}(2, 5)) \) is an isomorphism (weak Lefschetz), and so \( \text{GDA}^2(S) = \mathbb{Q}[a] \) as for the family M3. All conditions of Proposition 3.2 are verified.

\[\Box\]

### 3.4. Proof of theorem

**Proof.** (of Theorem 3.1) For the families B1 and B2 the result was proven in [24]. The family S2 was treated in [25]. For the remaining families, we have checked (Propositions 3.3 and 3.5) that Proposition 3.2 applies, which gives a generically defined MCK decomposition. The Chern classes \( c_j(X) \), as well as the image \( \text{Im}(A^*(U) \rightarrow A^*(X)) \), are clearly generically defined, and so they are in \( A^j_{\text{rat}}(X) \) thanks to Proposition 3.2.

\[\Box\]

### 4. A consequence

**Corollary 4.1.** Let \( X \subset U \) be the inclusion of a Fano variety \( X \) in its ambient space \( U \), where \( X, U \) are as in Table 1. Let \( \dim X = 2d \). Let \( R^*(X) \subset A^*(X) \) be the \( \mathbb{Q} \)-subalgebra

\[
 R^*(X) := \langle A^1(X), A^2(X), \ldots, A^d(X), c_j(X), \text{Im}(A^*(U) \rightarrow A^*(X)) \rangle \subset A^*(X).
\]

Then, \( R^*(X) \) injects into cohomology under the cycle class map.

**Proof.** This is a formal consequence of the MCK paradigm. We know (Theorem 3.1) that \( X \) has an MCK decomposition, and \( c_j(X) \) and \( \text{Im}(A^*(U) \rightarrow A^*(X)) \) are in \( A^j_{\text{rat}}(X) \). Moreover, we know that

\[
 A^j_{\text{hom}}(X) = 0 \forall j \neq d + 1 \tag{8}
\]

(Indeed, the injection of motives of Proposition 3.2(c2) induces an injection \( A^j_{\text{hom}}(X), \rightarrow A^{j-1-d}(S) \) where \( S \) is a K3 surface). This means that

\[
 A^j(X) = A^j_{\text{rat}}(X) \forall j \neq d + 1,
\]

and so

\[
 R^*(X) \subset A^j_{\text{rat}}(X).
\]

It only remains to check that \( A^j_{\text{rat}}(X) \) injects into cohomology under the cycle class map. In view of (8), this reduces to checking that the cycle class map induces an injection

\[
 A^{d+1}_{j}(X) \hookrightarrow H^{2d+2}(X, \mathbb{Q}).
\]

By construction, the correspondence \( \pi_X^{2d+2} \) is supported on a subvariety \( V \times W \subset X \times X \), where \( V, W \subset X \) are (possibly reducible) subvarieties of dimension \( \dim V = d + 1 \) and \( \dim W = d \).
As in [5], the action of $\pi_X^{2d+2}$ on $A^{d+1}(X)$ factors over $A^0(\tilde{W})$, where $\tilde{W} \to W$ is a resolution of singularities. In particular, the action of $\pi_X^{2d+2}$ on $A^{d+1}_{hom}(X)$ factors over $A^0_{hom}(\tilde{W}) = 0$ and so is zero. But the action of $\pi_X^{2d+2}$ on $A^{d+1}(X)$ is the identity, and so

$$A^{d+1}_{(0)}(X) \cap A^{d+1}_{hom}(X) = 0,$$

as requested.

\[\square\]

Note

1. (NB: in practice, one can often avoid recourse to the Hilbert scheme argument in this step. For instance, in the setting of Proposition 3.3 below, the split injection of (c2) is generically defined by construction, and one can apply Proposition 2.10 to conclude that $X \times_B X \to B$ has the Franchetta property.)

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