Horocycle dynamics: new invariants and eigenform loci in the stratum $\mathcal{H}(1,1)$

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Abstract

We study dynamics of the horocycle flow on strata of translation surfaces, introduce new invariants for ergodic measures, and analyze the interaction of the horocycle flow and real Rel surgeries. We use this analysis to complete and extend results of Calta and Wortman classifying horocycle-invariant measures in the eigenform loci. In addition we classify the horocycle orbit-closures and prove that every orbit is equidistributed in its orbit-closure. We also prove equidistribution results describing limits of sequences of measures. Our results have applications to the problem of counting closed trajectories on translation surfaces of genus 2.
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1. Introduction

Translation surfaces arise naturally in many different mathematical contexts, e.g. complex analysis, geometric group theory, geometry and polygonal billiard dynamics. See [MaTa, Zo, W3] for surveys of translation surface theory from a dynamical perspective. A stratum is a moduli space of translation surfaces of a given topological type (detailed definitions will be given in §2). There is a natural geometric action of $\text{SL}_2(\mathbb{R})$ on strata. The one-parameter subgroup $U$ of upper triangular unipotent matrices plays an especially interesting role. Understanding the dynamics of the action of $\text{SL}_2(\mathbb{R})$ and $U$ is very useful in understanding translation surface dynamics and geometry. We will give an example of this shortly.

The simplest stratum is the stratum $\mathcal{H}(0)$ of the torus with one marked point. This stratum has the structure of the homogeneous space $\text{SL}_2(\mathbb{R})/\text{SL}_2(\mathbb{Z})$. In this case we can identify the flow $U$ with the classical horocycle flow. The dynamical analysis of the horocycle flow in this setting is due to Hedlund [He] and Dani [D1, D2]. For general dynamical systems we have a description for typical orbits, but there are usually orbits with exceptional behavior, e.g. we may expect orbit closures to be fractal sets. In stark contrast with a typical dynamical system, for the horocycle flow, the results of Hedlund and Dani fully describe the distribution of every orbit: every orbit is either closed or dense, and in particular each orbit closure is a manifold and supports a natural measure. In addition every dense orbit is uniformly distributed with respect to the globally supported measure $\mu$. Limits of measures supported on closed horocycles converge to $\mu$, as do limits of expanded circle measures.

In the case of the torus, the $U$ flow on a stratum falls can be analyzed as a particular case of a unipotent flow on a homogeneous space. The analogy between homogeneous dynamics and moduli space dynamics has played a major role in the development of the theory. The simplest stratum which is not a homogeneous space is that of translation surfaces of genus two. We do not currently have a theory of the $U$-action on the genus two strata but we can refine the question. Since $U \subset \text{SL}_2(\mathbb{R})$ any $\text{SL}_2(\mathbb{R})$ orbit closure is $U$-invariant. We can ask about the dynamics of horocycle flows on $\text{SL}_2(\mathbb{R})$ orbit closures which we will call loci and denote by $L$. In fact we will be able to understand the horocycle dynamics on proper loci in genus two stratum components.

Some interesting loci in genus two were discovered by Veech. He found 3-dimensional loci corresponding to regular pentagon and decagon billiard tables. These loci correspond to closed $\text{SL}_2(\mathbb{R})$-orbits and have the form $\text{SL}_2(\mathbb{R})/\Gamma$ where $\Gamma \subset \text{SL}_2(\mathbb{R})$ is a non-uniform lattice. We will call such loci Veech loci. In particular they are examples of homogeneous spaces, and the horocycle flow on these loci behaves like the horocycle flow in the torus case. Five dimensional loci in genus two were discovered by Calta and McMullen and are called eigenform loci. We denote them as $E_D(1, 1)$ where $D \equiv 4 \text{ mod } 4$ 4. When $D$ is not a square such a $D$ is the discriminant of a unique order in a quadratic number field and these orders play a central role in the construction of $E_D(1, 1)$. When $D$ is a square $E_D(1, 1)$ consists of surfaces which cover tori. We refer to this setting as the arithmetic case. See Section 7 for more detailed information. The collection of all loci in genus two was classified by McMullen [McM3]. He showed that they are all Veech loci or eigenform loci.
The first description of ergodic invariant measures for eigenform loci is due to Calta and Wortman, see [CW]. In this paper we will show that every result for the horocycle flow in $\text{SL}_2(\mathbb{R})/\Gamma$ has an analog for eigenform loci. In Theorem 1.1 we prove a classification of ergodic invariant measures, extending [CW]. In Theorem 1.2 we completely describe orbit-closures and we describe the distribution of each orbit. In Theorem 1.3 we describe how families of closed horocycle orbits of increasing length behave. In Theorem 1.6 we prove that circles of increasing radii equidistribute relative to the smooth measure.

In various settings a good understanding of the horocycle action on a locus $\mathcal{L}$ has lead to a quantitative understanding of the rate of growth of closed orbits on translation surfaces in $\mathcal{L}$ and the rate of growth of closed billiard orbits for rational polygonal billiard tables corresponding to surfaces in $\mathcal{L}$. In particular proving the equidistribution of large circles leads to this result. See [Ve1, Ve2, EMS, EMaMo] and Theorem 1.7 for more information.

The question discussed in this paper concerning $U$-invariant ergodic measures and $U$ orbit-closures is finer than the similar questions for $\text{SL}_2(\mathbb{R})$-invariant measures and orbit-closures. For the latter question, recent breakthroughs were made by Eskin, Mirzakhani and Mohammadi [EMi, EMaMo], extending the earlier work of McMullen [McM3]. We refer to [W3] for a survey of these developments.

1.1. The horocycle flow on eigenform loci. Before stating our results we point out some ways in which horocycle dynamics in the eigenform loci differ from those in Veech loci. For any locus $\mathcal{L}$ we can distinguish two types of horocycle invariant orbits, those that are invariant under $\text{SL}_2(\mathbb{R})$ and those that are not. Horocycle measures which are not $\text{SL}_2(\mathbb{R})$-invariant come in continuous families which we call beds. For example in the Veech loci these are continuous families of closed horocycle orbits, and for each Veech locus there are finitely many such beds. These can be described in terms of surfaces with cylinder decompositions of a fixed combinatorial type. They also form orbits under the action of the upper triangular group, which normalizes the horocycle flow. Both of these descriptions admit an interesting generalization in the context of eigenform loci.

Surfaces with horizontal cylinder decompositions have horizontal saddle connections and these are preserved by the horocycle flow. In the setting of eigenform loci there are surfaces with horizontal saddle connections which do not have cylinder decompositions as well as those that do. A useful invariant to describe these beds is the ‘horizontal data diagram’ $\Xi(\mu)$ (see §5), recording topological and geometric structures which are invariant under the $U$-action. This includes the number and length of horizontal saddle connections, the topology of the complements, and the incidence of these saddle connections at singularities. Horizontal data diagrams play a central role in describing certain beds.

It is a general principle of homogeneous dynamics that if we are interested in a flow then the normalizer and centralizer of that flow take orbit closures to orbit closures. The normalizer of the horocycle flow includes the geodesic flow and for Veech loci, each bed is just a union of closed horocycles which are permuted by the geodesic flow. In eigenform loci we have additional centralizing elements. These involve the real Rel vector fields, which correspond to local motion in a direction in a stratum which moves one singularity horizontally with respect to another, while fixing absolute periods of the surface. As observed by Calta [C], these vector fields commute with the $U$-action and can be used to create new $U$-orbits out of old
U-orbits. Combining these vector fields with the geodesic flow vector field provides directions a family of vector fields which normalize the U-action, sending U-orbits to U-orbits, while inducing a time-change on the orbits.

There is a fundamental difference between homogeneous spaces and strata of translation surfaces, in connection with these vector fields. On homogeneous spaces, natural vector fields can be integrated to define group actions on the space, and the centralizer and the normalizer subgroups of the U-action play an important role in studying the dynamics. In the case of strata, motion in the real Rel directions is not globally defined; i.e. the solution curves for the differential equation defined by these vector fields may not be defined for all times. This implies that the centralizer and normalizer of the horocycle flow make sense locally but do not correspond to globally defined actions of Lie groups. In [EMaMo] issues related to locally defined flows are discussed using the terminology of ‘pseudo group-actions’. We will avoid this terminology here and instead use the more standard language of vector fields and the trajectories they generate.

In the first part of this paper (§2 – §5) we discuss invariants for the U-action which are defined in terms of the horizontal data diagram and the real Rel vector field. These are discussed for arbitrary strata and arbitrarily sub-loci. Then (from §7 onwards) we specialize to the eigenform loci in the stratum $\mathcal{H}(1,1)$, and use these invariants to fully describe the horocycle dynamics. The following result is the following measure classification result, which is the basis for all of our subsequent results on the eigenform loci.

**Theorem 1.1.** Let $D$ be as above and let $\mu$ be a U-invariant U-ergodic Borel probability measure on $\mathcal{E}_D(1,1)$. Then one of the following holds:

1. Every surface in $\text{supp}\, \mu$ has a horizontal cylinder decomposition and $\mu$ is the length measure on a periodic U-orbit.
2. Every surface in $\text{supp}\, \mu$ has a horizontal cylinder decomposition into three cylinders and $\mu$ is the area measure on a 2-dimensional minimal set for the U-action. In this case $\mu$ is invariant under the real Rel operation.
3. For every $M \in \text{supp}\, \mu$, the horizontal data diagram $\mathcal{X}(M)$ contains two saddle connections, joining distinct singularities, whose union divides $M$ into two isogenous tori glued along a slit. In this case $\mu$ is the image of the $\text{SL}_2(\mathbb{R})$-invariant measure on a quotient $\text{SL}_2(\mathbb{R})/\Gamma$ for some lattice $\Gamma$, via a Borel U-equivariant map.
4. For every $M \in \text{supp}\, \mu$, $\mathcal{X}(M)$ contains one saddle connection joining distinct singularities, and $\mu$ is the image of the $\hat{G}$-invariant measure on a quotient $\hat{G}/\Gamma$ for some lattice $\Gamma$ in the 3-fold connected cover $\hat{G}$ of $\text{SL}_2(\mathbb{R})$, via a Borel U-equivariant map.
5. The set of surfaces with horizontal saddle connections has $\mu$ measure zero and $\mu$ is the image of the $\text{SL}_2(\mathbb{R})$-invariant measure on a closed $\text{SL}_2(\mathbb{R})$-orbit in $\mathcal{H}(1,1)$ via a Borel U-equivariant map. In this case $D$ is either a square or is equal to 5.
6. For every $M \in \text{supp}\, \mu$, $\mathcal{X}(M)$ contains two saddle connections joining distinct singularities, whose complement in $M$ is a torus with two parallel slits of equal length, which are images of each other under a translation by an exactly $d$-torsion element of the torus. In this case $D = d^2$ is a square and $\mu$ is the image, via a Borel U-equivariant map, of the $\text{SL}_2(\mathbb{R})$-invariant measure on the space of tori.
(7) $\mu$ is the canonical flat measure on $\mathcal{E}_D(1,1)$ obtained from period coordinates.

We describe these measures in detail in §8. The construction of most of the measures in this list involves the real Rel operation on $\mathcal{H}(1,1)$ described earlier. Any $U$-invariant measure $\mu$, which is not preserved by real Rel, gives rise to a one-parameter family of $U$-invariant translates of $\mu$ by real Rel. This observation of Calta [C, CW] is crucial to our analysis, and the measures (3)–(6) all arise in this way as real Rel translates of $\text{SL}_2(\mathbb{R})$-invariant measures. The measures (5) are the real Rel translates of the natural measures on closed $\text{SL}_2(\mathbb{R})$-orbits in $\mathcal{H}(1,1)$. Loosely speaking, the measures (3),(4), and (6) are all pushforwards of measures on closed $\text{SL}_2(\mathbb{R})$-orbits in a suitable boundary component in a bordification of $\mathcal{H}(1,1)$, where the maps pushing the measure consist of the composition of real Rel with a map passing from the boundary component to $\mathcal{H}(1,1)$. In (4), the closed $\text{SL}_2(\mathbb{R})$-orbit belongs to the stratum $\mathcal{H}(2)$, while in (3) it belongs to $\mathcal{H}(0) \times \mathcal{H}(0)$, and in (6) it belongs to $\mathcal{H}(0,0)$ (where $\mathcal{H}(0)$ and $\mathcal{H}(0,0)$ denote respectively the moduli spaces of genus one translation surfaces with one or two marked points).

The equivariant maps in cases (3)–(6) arise from the real Rel operation and are not defined for every point in $\text{SL}_2(\mathbb{R})/\Gamma$. They define Borel isomorphisms between the supports of these measures and homogeneous spaces (that is, quotients $\text{SL}_2(\mathbb{R})/\Gamma$ for lattices $\Gamma \subset \text{SL}_2(\mathbb{R})$). We emphasize however that these maps are not everywhere defined, but rather on a dense open set of full measure, and thus their existence does not imply the existence of homeomorphisms of the supports of these measures with homogeneous spaces of $\text{SL}_2(\mathbb{R})$. In fact such homeomorphisms do not exist in general; in a forthcoming paper [SW3], we show that the supports of these measures are not homeomorphic to homogeneous spaces. In particular the supports of the measures appearing in case (5) above are manifolds with nonempty boundaries and infinitely generated fundamental groups.

Theorem 1.1 extends a theorem of Calta and Wortman [CW]. In [CW] it was assumed that $D$ is not a square, and thus case (6) did not arise. Also [CW] assumed that $\mu$ is not invariant under the real Rel operation, and case (2) did not arise. Theorem 1.1 is proved in a more general form in §9. The statement is inspired by Ratner’s measure classification theorem [KSS, Thm. 3.3.2], and its proof is inspired by [EMaMo, CW], which in turn employs arguments of Ratner. We provide some technical shortcuts and clarify some delicate steps. Our treatment relies on the general results worked out in §4.3, §4.4 and §6.

The measures in cases (1)–(6) of Theorem 1.1 are naturally organized in beds. In case (5) the continuous parameter involves a real Rel perturbation, and in case (2) there is a natural two parameter family of perturbations.

Applying our measure classification, we classify all orbit-closures for the $U$-action on $\mathcal{E}_D(1,1)$. In fact we prove a stronger statement. Recall that if $\mu$ is a $U$-invariant ergodic probability measure on a stratum $\mathcal{H}$, then $M \in \mathcal{L}$ is said to be generic for $\mu$ if for any continuous compactly supported function $f$ on $\mathcal{H}$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(u_s M) \, ds = \int f \, d\mu.$$  

The assertion that $M$ is generic for $\mu$ constitutes a quantitative strengthening of the assertion that $\text{supp} \mu = \text{supp} M$. The following statement is an analog of a genericity theorem proved by Ratner [KSS, Thm. 3.3.10]:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(u_s M) \, ds = \int f \, d\mu.$$
Theorem 1.2. For any $M \in \mathcal{E}_D(1,1)$, there is a measure $\mu$ described in Theorem 1.1 such that $M$ is generic for $\mu$, and belongs to the support of $\mu$.

In §11 we deduce Theorem 1.2 from a more explicit result Theorem 11.1 which explains, given $M$, how to detect the measure $\mu$ for which $M$ is generic. Continuing the analogy with related results for homogeneous flows, we prove several equidistribution results. We mention four results of this type here, and refer the reader to §12 for more results along these lines.

Theorem 1.3. Let $\mu$ be the length measure on a periodic $U$-orbit in $\mathcal{E}_D(1,1)$, let $\{g_t\}$ denote the geodesic flow, and suppose supp $\mu$ is not contained in a closed $\text{SL}_2(\mathbb{R})$-orbit. Then the measures $g_t \cdot \mu$ converge to the flat measure on $\mathcal{E}_D(1,1)$ as $t \to \infty$.

This could be viewed as a non-homogeneous counterpart of a theorem that Shah proves in a homogeneous setting [KSS, Thm. 3.7.6]. The following result also has an analog in the homogeneous setting (see [KSS, §3.7]):

Theorem 1.4. Let $\mu$ be the $\text{SL}_2(\mathbb{R})$-invariant measure on a closed $\text{SL}_2(\mathbb{R})$-orbit in $\mathcal{E}_D(1,1)$. Let $t \in \mathbb{R}$ and let $\mu_t$ be the measure obtained by applying the real Rel operation to surfaces in supp $\mu$, with rel parameter $t$. Then as $t \to \infty$ or $t \to -\infty$, $\mu_t$ converges to the flat measure on $\mathcal{E}_D(1,1)$.

In response to a question of Giovanni Forni, we prove:

Theorem 1.5. Let $\mu$ be any ergodic $U$-invariant measure on $\mathcal{E}_D(1,1)$, then as $t \to +\infty$, $g_t \cdot \mu$ converges to a $\text{SL}_2(\mathbb{R})$-invariant measure, and as $t \to -\infty$, either $g_t \cdot \mu$ converges to a $\text{SL}_2(\mathbb{R})$-invariant measure or $g_t \cdot \mu$ is divergent in the space of probability measures on $\mathcal{H}(1,1)$.

We also prove an equidistribution result for large circles:

Theorem 1.6. For any $M \in \mathcal{E}_D(1,1)$ which is not a lattice surface, let $\mu_t$ be the measure on $\mathcal{E}_D(1,1)$ defined by

$$
\int \varphi \, d\mu_t = \frac{1}{2\pi} \int_0^{2\pi} \varphi(g_t r_\theta M) \, d\theta, \quad \text{for all } \varphi \in C_c(\mathcal{E}_D(1,1)).
$$

Then $\mu_t$ converges to the flat measure on $\mathcal{E}_D(1,1)$ as $t \to \infty$.

Following a strategy of Eskin and Masur [EM], we use this result to solve a counting problem. We obtain:

Theorem 1.7. For any $M \in \mathcal{E}_D(1,1)$, the limit

$$
C_D = \lim_{T \to \infty} \frac{\# \{\text{saddle connections on } M \text{ of length } \leq T\}}{T^2}
$$

exists.

Theorem 1.7 was proved in [EMS] for the case that $D$ is a square, and in [B2] for the case that $D$ is not a square. In both of these papers precise formulæ for the constants $C_D$ were given. In fact, when $D$ is not square, $C_D = 4\pi$, which agrees with the value of this constant for a generic surface in $\mathcal{H}(1,1)$. The proof given in [B2] contained a gap in the case $D = 5$ which our results fill.
1.2. Guide to the paper. In section §2 we define strata of translation surfaces and establish some of their basic properties.

In §3 we give introduce ‘blowups of translation surfaces’ along with their corresponding moduli spaces and mapping class groups. These notions are useful throughout our discussion: for marking singularities, marking distinguished horizontal prongs, distinguishing horizontal data diagrams, resolving orbifold issues for both the stratum and for the structure of individual rel leaves, and for discussing surgeries involving a stratum and nearby boundary strata obtained from it as limits of rel operations. An interesting novelty of our approach is that certain finite covers \( \hat{G} \) of SL\(_2\)(\( \mathbb{R} \)) appear as the groups which act naturally on our covers (see §3.5). The finite covers we consider have arisen in topological contexts (see [Boi]), in the study of interval exchange transformations (see [Y]) and in computations of monodromy representations (see e.g. [MYZ]), as well as in our forthcoming [SW3]. We believe that the terminology we introduce will be useful in future work on translation surfaces.

In the Sections §4 we describe certain natural vector fields on strata arising from relative cohomology which we call Rel vector fields. In subsection §4.2 we describe certain subset of Rel fields which we call real Rel vector fields and show that in the appropriate sense these commute with the horocycle flow. In §4.3 we make a connection between real Rel and stabilizers of measures. We also describe centralizers and normalizers of the horocycle flow in the context of vector fields.

The domains of well-defined motion in the \( Z \) and \( N \) directions are invariants of a \( U \)-invariant ergodic measure \( \mu \), which we denote by \( Z(\mu) \) and \( N(\mu) \) (see §4.3). Moreover within these domains of well-defined motion, are Lie groups \( Z_\mu \) and \( N_\mu \) which do act on supp \( \mu \) and are the stabilizer subgroups of \( \mu \) within the centralizer and normalizer (see §4.3). The analysis of generic points plays a major role in the analysis of ergodic measures on homogeneous spaces. The Rel surgery depends on a parameter \( T \) and as remarked above, the Rel surgery is not globally defined. As a consequence, the set of surfaces for which it is defined for all \( T \) is not locally compact and it is not a priori clear that the centralizer of the \( U \)-action should map \( U \)-generic points to \( U \)-generic points. We clarify this in §4.4.

In §5 we introduce invariants of beds related to horizontal saddle connections. In §6 we explicitly identify the domain of definition of real Rel surgeries, continuing work done in [MW2, McM8, B2].

In §7–§12, we give a complete picture of the horocycle dynamics in the genus 2 eigenform loci.

Starting in §7 we describe the eigenform loci.

We describe ergodic invariant measures for the horocycle flow in detail in §8.

In §9 we revisit Ratner’s argument for transverse divergence of nearby horocycle orbits and apply it to eigenform loci to characterize invariant measures.

In §10 we develop a linearization technique for strata to analyze the behavior of \( U \)-orbits which are near beds. This is an analog of the ‘linearization technique’ developed by Dani and Margulis for homogeneous spaces.

In §11, we use ideas from the previous section to classify all orbit-closures for the \( U \)-action on \( E_2(1,1) \) and prove that every orbit is equidistributed in its closure.

In §12 we prove Theorem 1.6 and equidistribution results for several other naturally occurring sequences of measures.
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2. Strata

In this section we define our objects of study: translation surfaces, moduli spaces of translation surfaces and dynamics on moduli spaces. For background and alternate treatments we refer the reader to [EMZ, FM, MaTa, MaSm, W3, Zo]. Our treatment expands on previous work by discussing in detail blowups of translation surfaces and associated orbifold covering spaces of strata, which are needed in later sections. Our discussion also has interesting consequences of independent interest about fundamental groups of strata (see §3.6).

2.1. Translation surfaces. A translation surface, or a surface with translation structure can be defined in several equivalent ways. We will describe it by gluing polygons, in terms of an atlas using the language of \(pG,Xq\) structures, or as a holomorphic differential.

Let \(M\) be a surface obtained from a finite collection of polygons in \(\mathbb{R}^2\) which are glued together by isometries of the edges which are restrictions of translations. Each polygon edge receives an orientation when viewed as part of the boundary of a polygon and we assume that the edges of the polygons are partitioned into pairs of edges which are parallel, have the same length, and opposite orientations, and these are identified by the gluing isometries. This construction typically produces a finite set of singular points corresponding to the vertices of the polygons at which the cone angle is larger than \(2\pi\). The cone angle at a singularity is \(2\pi(n + 1)\) for some natural number \(n\). We call \(n\) the order of the singularity. In addition to the singular points that we have defined it is often useful to mark a finite set of points at which the cone angles are \(2\pi\). Let \(\Sigma\) be a finite subset of \(M\) which contains all of the singular points and possibly some additional ‘marked’ points.

A translation structure on a surface determines an atlas of charts for the surface \(M \setminus \Sigma\) taking values in \(\mathbb{R}^2\) where the transition maps are restrictions of translations. A translation structure can be determined by specifying this atlas. If we have a space \(X\) equipped with an action of a Lie group \(G\) a \((G,X)\)-structure is an atlas of charts with overlap functions in \(G\). Thus a translation surface produces a \((G,X) = (\mathbb{R}^2,\mathbb{R}^2)\) structure on \(M \setminus \Sigma\) where the first \(\mathbb{R}^2\) represents a Lie group acting on the second \(\mathbb{R}^2\) by translation. See [Th] for more information about \((G,X)\)-structures.

We can use the atlas of charts on \(M\) to define geometric structures on \(M\) which are naturally associated with the translation structure. Since the one-forms \(dx\) and \(dy\) are translation invariant on \(\mathbb{R}^2\) these charts allow us to build globally defined one-forms \(dx\) and \(dy\) on \(M\). Similarly we can use the planar charts to define a metric, an area form, and an orientation on \(M\). A saddle connection on \(M\) is a path with endpoints in \(\Sigma\), which is a straight line in each chart of this atlas, and does not contain singularities in its interior. The one-forms \(dx\) and \(dy\) are closed and represent cohomology classes in \(H^1(M,\Sigma;\mathbb{R})\). For an oriented path \(\gamma\) connecting points in \(\Sigma\) write \(\text{hol}(M,\gamma)\) for \(\left(\int_\gamma dx, \int_\gamma dy\right)\). We can think of \(\text{hol}(M,\gamma)\) as giving a homomorphism from \(\pi_1(M \setminus \Sigma)\) to \(\mathbb{R}^2\) or as determining an element of \(H^1(M,\Sigma;\mathbb{R}^2)\). This is the holonomy homomorphism. If \(\tilde{M}\) is the universal cover of \(M \setminus \Sigma\) then a map from \(\tilde{M}\) to \(\mathbb{R}^2\) with derivative equal to the identity is a developing...
map. Developing maps exist and are unique up to translation. The developing map is equivariant with respect to the holonomy homomorphism. This is an instance of the general principle that for any \((G, X)\)-structure on a manifold \(M\), there is a developing map \(\text{dev} : \tilde{M} \to X\) and a holonomy homomorphism \(\pi_1(M) \to G\).

The atlas for a translation surface gives a trivialization of the tangent bundle at non-singular points. Let us say that \(f : M \to N\) is a smooth map between translation surfaces if it preserves singular sets, and is smooth away from the singular set of \(M\). Using the trivialization of the tangent space, we can view the derivative of a smooth map \(f\) as a \(2 \times 2\) real matrix-valued function. We say that a smooth map \(f\) is a translation equivalence if it is a homeomorphism and its derivative is the identity matrix. The notion of translation equivalence gives a natural equivalence relation on translation surfaces. We say that \(M\) and \(N\) are affinely equivalent if there is an orientation-preserving smooth map \(f\) between them which is a homeomorphism and for which \(Df\) is constant but not necessarily equal to the identity. The affine equivalence classes are orbits of a \(\text{GL}_2^+(\mathbb{R})\)-action we will discuss shortly (where \(\text{GL}_2^+(\mathbb{R})\) is the group of orientation-preserving linear automorphisms of \(\mathbb{R}^2\)).

If we identify \(\mathbb{R}^2\) with \(\mathbb{C}\) then the coordinate charts of a translation structure induce a conformal structure on \(M \setminus \Sigma\). This conformal structure extends to \(M\) so that the points in \(\Sigma\) correspond to punctures. We can define the complex-valued one-form \(dz = dx + idy\). This is a holomorphic one-form or Abelian differential with zeros at the singular points where the order of the zero is the order of the singular point. If we are given a surface with a conformal structure \(X\) and we wish to define a compatible translation surface then this is determined by a nonzero holomorphic one-form \(\omega\). We use the notation \((X, \omega)\) to denote this pair. For the relations between these three points of view, see [Zo] and the references cited there.

### 2.2. Strata as sets

Strata are moduli spaces of translation surfaces. In this section we describe the set of surfaces which comprise a given stratum and the relevant notion of equivalence. In section §2.4 we will show that the set of these equivalence classes of surfaces can be given natural orbifold structures. Let \(k \in \mathbb{N}\) and let \(a_1, \ldots, a_k\) be a sequence of non-negative integers. Let \(M\) be a translation surface and let \(\Sigma\) be a finite subset of \(M\) which consists of \(k\) points labeled \(\xi_1, \ldots, \xi_k\) and contains all singular points of \(M\). (Note that our conventions imply that \(\Sigma\) has at least one point.) We say that \(M\) is a translation surface of type \((a_1, \ldots, a_k)\) if the cone angle at \(\xi_j\) is \(2\pi(a_j + 1)\). We refer to \(a_j\) as the order of \(\xi_j\). The points \(\xi_j\) with order zero are not singular and we will refer to them as marked points.

We want to construct a version of the stratum in which singular points have well-defined labels. To this end we say that two surfaces of type \((a_1, \ldots, a_k)\) are label preserving translation equivalent if they are translation equivalent by means of a translation equivalence preserving the labelling of the singular points. (When no confusion will result we will drop the expression ‘label preserving’.)

Let \(\mathcal{H} = \mathcal{H}(a_1, \ldots, a_k)\) denote the stratum of translation surfaces of type \((a_1, \ldots, a_k)\), considered up to label preserving translation equivalence. The surfaces in \(\mathcal{H}\) have genus \(g\) where \(2g - 2 = \sum a_j\). It is sometimes convenient to denote this stratum as \(\mathcal{H}_g(a_1, \ldots, a_k)\). In particular note that \(g \geq 1\) since the \(a_j\) are non-negative.

### 2.3. Strata of marked surfaces

Let \(S\) be an oriented surface of genus \(g\) and let \(\Sigma\) be a subset consisting of \(k\) points labeled \(\xi_1, \ldots, \xi_k\). The pair \((S, \Sigma)\) will serve
as a topological model for a translation surface. A marked translation surface is a translation surface $M$ of type $(a_1, \ldots, a_k)$, equipped with an orientation-preserving homeomorphism $f : S \to M$ where $\Sigma$ maps to the appropriate distinguished points of the translation structure, respecting the labels. We will consider $S$ as fixed and sometimes write maps $f : S \to M$ as pairs $(f, M)$. Two marked translation surfaces $(f_1, M_1)$ and $(f_2, M_2)$ are considered to be equivalent as marked translation surfaces if there is a label preserving translation equivalence $\varphi : M_1 \to M_2$ so that $\varphi \circ f_1$ and $f_2$ are isotopic via an isotopy that fixes the points in $\Sigma$.

Let $\mathcal{H}_m = \mathcal{H}_m(a_1, \ldots, a_k)$ denote the set of marked translation surfaces, up to the equivalence relation described above. As discussed above, a surface $M \in \mathcal{H}$ determines a cohomology class $\text{hol}(M, \cdot) \in H^1(M, \Sigma; \mathbb{R}^2)$. A marked translation surface $f : S \to M$ in $\mathcal{H}_m$ determines a pullback cohomology class $f^*(\text{hol}(M, \cdot)) \in H^1(S, \Sigma; \mathbb{R}^2)$ where $f^*(\text{hol}(M, \gamma)) = \text{hol}(M, f_*(\gamma))$. It is not hard to check that the pullback depends only on the equivalence class of $f$, i.e. if $(f_1, M_1)$ and $(f_2, M_2)$ are equivalent then the cohomology classes $f_i^*(\text{hol}(M_i, \cdot))$, $i = 1, 2$ are the same. We denote by

$$\text{dev} : \mathcal{H}_m \to H^1(S, \Sigma; \mathbb{R}^2)$$

the map which takes a marked translation surface to the corresponding element of $H^1(S, \Sigma; \mathbb{R}^2)$. This map is often called the period map.

### 2.4. Strata as spaces

We now define a topology on $\mathcal{H}_m$ using an atlas of charts which is defined via triangulations of translation surfaces. Let $\tau$ be a triangulation of $S$ so that the vertices are points in $\Sigma$. We do not require that edges have distinct endpoints in $S$ though we do require that they have distinct endpoints in the universal cover of $S$ or equivalently that no two edges are homotopic relative to their endpoints. This triangulation gives $S$ the structure of a $\Delta$-complex (for a definition see [H, p. 102]). Let $\mathcal{U}_\tau \subset \mathcal{H}_m$ be the set of marked translation surfaces containing a representative $f : S \to M$ which takes the edges of $\tau$ to saddle connections in $M$.

The function that maps an oriented edge to its holonomy vector is a 1-cochain in the cochain complex associated to the triangulation $\tau$. The condition that the sums of vectors on the boundary of a triangle is zero means that this 1-cochain is a cocycle. Using the fact that $\tau$ is a $\Delta$-complex we can identify the space of such cocycles with $H^1(S, \Sigma; \mathbb{R}^2)$ (see [H]).

Using the edges of the triangulation $\tau$ allows us to define comparison maps between two marked translation surfaces lying in a given chart $\mathcal{U}_\tau$. Namely, suppose $(f, M)$ and $(f', M')$ are two representatives of marked translation surfaces in the same $\mathcal{U}_\tau$. Then $f' \circ f^{-1} : M \to M'$ is a homeomorphism but it is only well defined up to isotopy. We define a map $F$ in this isotopy class by requiring that for each triangle $\Delta$ in $\tau$, $F$ is an affine map when restricted to $f(\Delta)$. This requirement and the triangulation $\tau$ determine $F = F(\tau, M, M')$, up to isotopy once isotopy classes of $f$ and $f'$ are fixed, and we call it the comparison map between $(f, M)$ and $(f', M')$.

**Proposition 2.1.** The map $\text{dev}|_{\mathcal{U}_\tau} : \mathcal{U}_\tau \to H^1(S, \Sigma; \mathbb{R}^2)$ is injective and its image is open.

**Proof.** Suppose $(f, M)$ and $(f', M')$ are in $\mathcal{U}_\tau$ and have the same image under $\text{dev}$. A priori the comparison map is affine on each triangle. In this case the
comparison map \( F(\tau, M, M') : M \to M' \) is a translation equivalence since the holonomy of corresponding edges are equal. We conclude that \((f, M)\) and \((f', M')\) are equivalent and the developing map is injective on \( U_r \).

A cohomology class \( \phi \) is in the image of \( \text{dev}|_{U_r} \), if the values of \( \phi \) on the sides of every triangle \( \Delta \) in \( \tau \) correspond to the coordinates of a non-degenerate triangle in \( \mathbb{R}^2 \) with the appropriate orientation — an open condition in \( H^1(S, \Sigma; \mathbb{R}^2) \). If this condition is satisfied then an appropriate translation surface can be built by gluing together triangles in \( \mathbb{R}^2 \) with edge coordinates given by \( \phi \). In particular the image \( \text{dev}(U_r) \) is open in \( H^1(S, \Sigma; \mathbb{R}^2) \). \( \square \)

It was shown in [MaSm] that every translation surface \( M \) with \( \Sigma \) nonempty admits a triangulation of this type. Thus the charts \( U_r \) cover \( \mathcal{H}_m \) as \( \tau \) ranges over triangulations of \( S \). The change of coordinate maps for this system of charts are linear. These charts on the sets \( U_r \) give \( \mathcal{H}_m \) an affine manifold structure. This affine manifold structure can also be discussed using the terminology of \((G, X)\)-structures. Specifically we can take \( X \) to be \( H^1(S, \Sigma; \mathbb{R}^2) \) and the structure group \( G \) can be taken to be the group of linear automorphisms of \( H^1(S, \Sigma; \mathbb{R}^2) \). Our choice of \( \text{dev} \) as notation for the map in (3) is motivated by the fact that this map is the developing map for the affine structure.

We have described triangulations of \((S, \Sigma)\) in terms of a homeomorphism with a \( \Delta \)-complex. We note that such a homeomorphism is determined up to isotopy by knowing the relative homotopy classes of the edges of the triangulation (see [FaMa, Lemma 2.9]). Furthermore distinct edges are not homotopic to each other relative to their endpoints. In the sequel we will use \( \tau \) to denote the homotopy classes of edges of a triangulation.

We now make use of the marked stratum to put a topology on the stratum. Consider the group of isotopy classes of orientation-preserving homeomorphisms of \( S \) fixing \( \Sigma \) pointwise. This is sometimes called the pure mapping class group (see [Th]). We will simply refer to it as the mapping class group and we will denote it by \( \text{Mod}(S, \Sigma) \). Up to the action of \( \text{Mod}(S, \Sigma) \) there are finitely many charts \( U_r \). It can be checked that the \( \text{Mod}(S, \Sigma) \)-action on \( \mathcal{H}_m \) is properly discontinuous. This equips the quotient \( \mathcal{H} = \mathcal{H}_m/\text{Mod}(S, \Sigma) \) with the structure of an affine orbifold, with respect to which the map \( \mathcal{H}_m \to \mathcal{H} \) is an orbifold covering map (see [Th] for the definition and basic properties of properly discontinuous actions and orbifolds).

2.5. The area form and the area one locus. Let \( \tilde{\mathcal{H}}^{(1)} \) and \( \mathcal{H}^{(1)} \) denote the subset of area-one surfaces in \( \tilde{\mathcal{H}} \) and \( \mathcal{H} \) respectively. With respect to the charts afforded by the map \( \text{dev} \), \( \tilde{\mathcal{H}}^{(1)} \) is a submanifold cut out by a quadratic equation.

We can give these quadratic equations explicitly. Identify the coefficients \( \mathbb{R}^2 \) with \( \mathbb{C} \) and define a Hermitian form on \( H^1(S, \Sigma; \mathbb{R}^2) \) by

\[
\langle \alpha, \beta \rangle = \frac{1}{2i} \int_M \alpha \wedge \bar{\beta}.
\]

The area of \( M \) is \( \langle \omega, \omega \rangle \), where \( \omega = \text{dev}(M) \).

We now explain how this Hermitian form can be obtained in more topological terms, from the cup product and a particular choice of coefficient pairing. First note that if we take \( z \) and \( w \) to represent two sides of a triangle, then the signed area of the triangle is equal to \( \Re(z \bar{w}/2i) \). Now recall that the cup product of two \( \mathbb{R} \)-valued simplicial cochains is defined on a simplex by taking the product of values of the cochains on the simplex, and is then extended by linearity to chains. If we
have coefficients which are not in \( \mathbb{R} \), we can replace the operation of multiplying values of cochains by a bilinear pairing of the values of cochains. Motivated by the above observation, we will use the coefficient pairing

\[
\mathbb{C} \times \mathbb{C} \ni (z, w) \mapsto zw/2i \in \mathbb{C}.
\]

If we interpret the 1-forms in (4) as complex valued cohomology classes, interpret integration as evaluation on the fundamental class of \( M \), and use evaluation on the fundamental class to identify \( H^2(S, \Sigma; \mathbb{C}) \) with \( \mathbb{C} \), then our Hermitian form is the cup product

\[
H^1(S, \Sigma; \mathbb{C}) \otimes H^1(S, \Sigma; \mathbb{C}) \to H^2(S, \Sigma; \mathbb{C}) \cong \mathbb{C},
\]

and the particular choice of coefficient pairing (5) is responsible for the connection with the area of translation surfaces.

Since the Hermitian form on \( \mathcal{H} \) was defined purely topologically, it is preserved by the \( \text{Mod}(S, \Sigma) \)-action. Thus \( \mathcal{H}^{(1)} \) and \( \mathcal{H}^{(1)} \) have \((G, X)\)-structures where \( X \) is a quadric in \( H^1(S, \Sigma; \mathbb{C}) \) and \( G \) can be taken to be the subgroup of the general linear group which preserves this quadric.

3. Blowups of translation surfaces

We now discuss a ‘blowup’ construction that replaces a singularity \( \xi \) by a boundary circle. This is a special case of a more general construction of a ‘real oriented blowup’ (see e.g. \([\text{HPV}]\)). The real oriented blowup of a point \( \xi \) in \( \mathbb{R}^2 \) is a new space \( \text{Blo}_{\xi}(\mathbb{R}^2) \) together with a collapsing map \( c : \text{Blo}_{\xi}(\mathbb{R}^2) \to \mathbb{R}^2 \) with the property that the inverse image of any point other than \( \xi \) is a single point while the inverse image of \( \xi \) is the circle of directions at \( \xi \) which we can identify with \( (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}^+ \) (or with the unit circle) and denote by \( S^1 \). The space \( \text{Blo}_{\xi}(\mathbb{R}^2) \) has the property that a smooth path in \( \mathbb{R}^2 \) landing at \( \xi \) and with non-zero derivative at \( \xi \) has a lift to the space \( \text{Blo}_{\xi}(\mathbb{R}^2) \) which takes the endpoint of the path to a point in the circle of directions.

If we blow up the vertex \( \psi \) of a polygon \( P \) in \( \mathbb{R}^2 \) then we obtain the space \( \text{Blo}_\psi(P) \) which is the result of replacing the vertex \( \psi \) in \( P \) by an interval. We describe this construction explicitly. Applying a translation, assume \( \psi \) is at the origin. Say that the two edges of the polygon incident to \( \psi \) are in directions \( \theta_1, \theta_2 \) with \( 0 < \theta_2 - \theta_1 < 2\pi \), and there is \( \varepsilon > 0 \) such that

\[
\{r(\cos \theta, \sin \theta) : \theta \in [\theta_1, \theta_2], \ r \in [0, \varepsilon]\}
\]

parametrizes a neighborhood of \( \psi \) in \( P \). The blowup of this neighborhood in \( \text{Blo}_\psi(P) \) corresponds to the rectangle

\[
\{(r, \theta) : \theta \in [\theta_1, \theta_2], \ r \in [0, \varepsilon]\}
\]

and the collapsing map \( c \) from \( \text{Blo}_\psi(P) \) to \( P \) takes \((r, \theta)\) to \( r(\cos \theta, \sin \theta) \). In particular the interval \( \{(0, \theta) : \theta \in [\theta_1, \theta_2]\} \) is collapsed to the point \( \psi \). Note that this interval has a natural ‘angular coordinate’ with values in the unit circle.

Given a translation surface \( M \) with singular points \( \Sigma \) we construct a surface with boundary \( \tilde{M} \) by blowing up all of the points of \( \Sigma \). We can do this as follows. Choose a triangulation of \( M \). Blow up each vertex of each triangle thereby creating a family of hexagons where each hexagon contains edges of two types: those corresponding to edges of triangles and those corresponding to vertices of triangles. Glue the (triangle edge) sides of the hexagons together according to the gluing pattern of the original triangles. The result is the surface \( \tilde{M} \) which is in fact independent
of the particular choice of triangulation. At a singular point $\xi_j$ of the surface the intervals mapping to $\xi_j$ glue together to form a circle which we denote $\partial_j M$. The angular coordinates glue together to give us a map $\varphi_j : \partial_j M \to S^1$. The total angular measure of $\partial_j M$ is $2\pi(a_j + 1)$ which is the cone angle at $\xi_j$. We can choose an identification of the circle $\partial_j M$ with the circle $\mathbb{R}/(2\pi(a_j + 1))\mathbb{Z}$ so that the angular coordinate of a point is equal to its circle coordinate modulo $2\pi$. This identification of $\partial_j M$ with the circle is well-defined up to translation by $2\pi$ while the map $\varphi_j$ is defined independently of any choices.

If we associate a point $\nu$ in the $j$-th boundary component $\partial_j M$ with a short ray heading away from $\xi_j$, then $\varphi_j(\nu)$ is the direction of the ray. We define the prongs to be points on boundary circles corresponding to horizontal rays, i.e. points whose angular parameter is $\pi k$ with $k \in \mathbb{Z}$. We will call a prong right- or left-pointing, if $k$ is even (resp. odd), that is, according to the orientation that the prong inherits from the plane. A particular choice of an identification of $\partial_j M$ with $\mathbb{R}/2\pi(a_j + 1)\mathbb{Z}$ is equivalent to the choice of a right-pointing prong. The boundary circles of $M$ inherit boundary orientations as boundary components of the oriented manifold $M$. With respect to the boundary orientation the maps $\varphi_j$ are covering maps of degree $-2(a_j + 1)$ where the negative sign reflects the fact that moving in the direction of the boundary orientation corresponds to decreasing the angular coordinate.

### 3.1. Strata of boundary marked surfaces

In this subsection we define a notion of marked surface appropriate to surfaces with boundary and a corresponding mapping class group. To this end we define a ‘model surface’ $\hat{S}$, which will capture some of the structure common to the surfaces $M$ for $M \in \mathcal{H}(a_1, \ldots, a_k)$. Let $\hat{S}$ be a surface with boundary which has genus $g$ where $2g - 2 = \sum a_j$ and $k$ boundary components labeled $\partial_1 \hat{S}, \ldots, \partial_k \hat{S}$. The circles inherit a boundary orientation from $S$. We equip each boundary circle $\partial_j \hat{S}$ with orientation reversing homeomorphisms $q_j : \partial_j \hat{S} \to \mathbb{R}/(2\pi(a_j + 1))\mathbb{Z}$ which give angular coordinates on the boundaries. A marked translation surface rel boundary is a surface $M$ which is a blowup of a translation surface $\hat{M}$ of type $(a_1, \ldots, a_k)$, equipped with an orientation preserving homeomorphism $\hat{f} : \hat{S} \to M$ respecting the labels, and such that on each boundary circle $\partial_j \hat{S}$ we have $\varphi_j \circ \hat{f} = q_j \mod 2\pi$. Note that a boundary marking of $M$ induces an explicit coordinate on $\partial M$ and an explicit choice of a prong, namely the image under $\hat{f}$ of the prong on $\partial_j \hat{S}$ corresponding to angular parameter zero.

We say that two boundary marked translation surfaces rel boundary $(\hat{f}_1, \hat{M}_1)$ and $(\hat{f}_2, \hat{M}_2)$ are equivalent if there is a translation equivalence $\gamma : M_1 \to M_2$ such that $\hat{f}_1 \circ \gamma$ and $\hat{f}_2$ are equal on $\partial \hat{S}$ and are isotopic via an isotopy that fixes the boundary. Let $\hat{\mathcal{H}} = \mathcal{H}(a_1, \ldots, a_k)$ denote the set of boundary marked translation surfaces, up to equivalence. We call $\hat{\mathcal{H}}$ a stratum of boundary marked translation surfaces.

There is a natural collapsing map $c : \hat{M} \to M$ which collapses each boundary component to a single point. A map $\hat{f} : \hat{S} \to \hat{M}$ induces a map $f : S \to M$, where $c \circ \hat{f} = f \circ c$. If $\hat{f}_1$ and $\hat{f}_2$ are equivalent (in the sense of §3.1), then so are $f_1$ and $f_2$ (in the sense of §2.2). We say that $f$ is obtained from $\hat{f}$ by projection. The forgetful map which takes $(f, M)$ to the pair $(f, M)$ where $f$ is obtained by projection gives a map $pr : \mathcal{H} \to \mathcal{H}_m$.

We now define a mapping class group rel boundary. We say a homeomorphism $\hat{f} : \hat{S} \to \hat{S}$ is admissible if $\hat{f}$ takes each boundary circle $\partial_j \hat{S}$ to itself and the
restriction of \( \tilde{f} \) to \( \partial_j \tilde{S} \) is a rotation by a multiple of \( 2\pi \) with respect to the circle coordinate given by \( q_j \). We say that two admissible homeomorphisms are isotopic rel boundary if they agree on \( \partial S \) and if they are isotopic by an isotopy which is the identity when restricted to \( \partial \tilde{S} \). We denote by \( \text{Mod}(\tilde{S}, a_1, \ldots, a_k) \) the group of isotopy classes rel boundary of admissible homeomorphisms of \( \tilde{S} \). When there is no chance of confusion we will abbreviate this to \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \). There is a natural right action of the mapping class group \( \text{Mod}(\tilde{S}, a_1, \ldots, a_k) \) on the stratum \( \hat{\mathcal{H}}(a_1, \ldots, a_k) \) by precomposition.

3.2. Relative homotopy classes of paths. A useful tool in the study of mapping class groups is the study of their action on curves. In order to analyze \( \hat{\mathcal{H}} \) and \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \) we will consider the action on homotopy classes of paths with endpoints in the boundary. We consider paths in \( \tilde{S} \) with endpoints in \( \partial \tilde{S} \). We say that two paths are homotopic if there is a homotopy between them that keeps the endpoints in \( \partial \tilde{S} \). We say that two homotopic paths are relatively homotopic if the homotopy can be chosen to fix the endpoints of the path. The collection of relative homotopy classes of paths is a natural set to consider but, unlike homotopy classes of paths in \((M, \Sigma)\), it is not discrete. We analyze its topology below. There is a well-defined action of \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \) on relative homotopy classes of paths. We say that a path \( \alpha \) is peripheral if both endpoints lie in the same boundary component and \( \alpha \) is homotopic to a path contained in that boundary component. Let \( \sigma : [0,1] \to \tilde{S} \) be a non-peripheral oriented path with \( \sigma(0) \in \partial \tilde{S}_j \) and \( \sigma(1) \in \partial \tilde{S}_k \). Let \( \mathcal{C}_\sigma(\tilde{S}) \) be the set of relative homotopy classes of oriented paths in \( \tilde{S} \) which are homotopic to \( \sigma \).

Definition 3.1. Let \( \hat{\sigma} \) and \( \hat{\sigma}' \) be elements of \( \mathcal{C}_\sigma(\tilde{S}) \) going from \( \partial_j \tilde{S} \) to \( \partial_k \tilde{S} \), and let \( \varepsilon > 0 \). We say that \( \hat{\sigma}' \) is \( \varepsilon \)-close to \( \hat{\sigma} \) if there are intervals \( I_1 \subset \partial_j \tilde{S} \) and \( I_2 \subset \partial_k \tilde{S} \) of length \( \varepsilon \), containing the endpoints of \( \hat{\sigma} \) and \( \hat{\sigma}' \), such that \( \hat{\sigma}' \) is homotopic to \( \hat{\sigma} \) through a family of paths each of which has one endpoint in \( I_1 \) and one endpoint in \( I_2 \).

Sets of \( \varepsilon \)-close homotopy classes of curves for given intervals \( I_1 \) and \( I_2 \) form the basis for a topology on \( \mathcal{C}_\sigma(\tilde{S}) \). We have an endpoint map \( \varepsilon : \mathcal{C}_\sigma(\tilde{S}) \to \partial_j \tilde{S} \times \partial_k \tilde{S} \) which takes a relative homotopy class of curves to its endpoints. With respect to the topology on \( \mathcal{C}_\sigma(\tilde{S}) \) the endpoint map is continuous.

Lemma 3.2. Say that \( S \) is a surface with boundary with negative Euler characteristic. The endpoint map \( \varepsilon : \mathcal{C}_\sigma(\tilde{S}) \to \partial_j \tilde{S} \times \partial_k \tilde{S} \) is a covering map and \( \mathcal{C}_\sigma(\tilde{S}) \) is the universal cover of \( \partial_j \tilde{S} \times \partial_k \tilde{S} \). It follows that the space of relative homotopy classes of paths in this homotopy class is homeomorphic to the product of the universal covers \( \tilde{\partial_j \tilde{S}} \) and \( \tilde{\partial_k \tilde{S}} \).

We will apply this result to the surfaces with boundary \( \tilde{S} \) arising as ‘model surfaces’ corresponding to some family of translation surfaces. According to our conventions these surfaces always have negative Euler characteristic. This result captures the idea that the distinction between homotopy and relative homotopy for paths is measured by the amount of twisting around each boundary component.

Proof. Since the Euler characteristic of \( \tilde{S} \) is negative we can give \( \tilde{S} \) a hyperbolic structure so that the boundaries are geodesics. The universal cover \( \tilde{\tilde{S}} \) is isometric to a convex subset of the hyperbolic plane with geodesic boundary.
Choose a boundary component $B_1$ of $\tilde{S}$ corresponding to $\partial_j \tilde{S}$. Identify $B_1$ with $\partial_j \tilde{S}$. Choose a lift of $\sigma$ to a path $\tilde{\sigma}$ in $\tilde{S}$ starting in $B_1$. The other endpoint of $\tilde{\sigma}$ lands in a component $B_2$ which maps to $\partial_k \tilde{S}$. Identify $B_2$ with $\partial_k \tilde{S}$. Since the path is non-peripheral, $B_1$ and $B_2$ are distinct. Any path homotopic to $\sigma$ has a lift to a path from $B_1$ to $B_2$ and this lift is unique since the subgroup of the deck group that stabilizes $B_1$ and $B_2$ is trivial. This follows from the fact that a hyperbolic isometry that fixes four points is the identity.

We get a map from the space of relative homotopy classes of paths homotopic to $\sigma$ to $B_1 \times B_2 = \partial_j \tilde{S} \times \partial_k \tilde{S}$ as follows. Given a path homotopic to $\sigma$ we lift it to a path from $B_1 \times B_2$ and we associate this path to its endpoints. Given a pair of points $(p_1, p_2) \in B_1 \times B_2$ we associate the projection to $\tilde{S}$ of the unique geodesic from $p_1$ to $p_2$. The fact that any path between $\partial_j \tilde{S}$ and $\partial_k \tilde{S}$ is relatively homotopic to a unique geodesic implies that these maps are inverses. This map is a continuous bijection with respect to the natural topology on $C_\ast(\tilde{S})$.

We now describe explicitly some elements of $\text{Mod}(\tilde{S}, \partial \tilde{S})$ which correspond to partial Dehn twists around boundary components. Let $A_j$ be an annular neighborhood of $\partial_j \tilde{S}$ where we choose coordinates $\{(t, \theta) : t \in [0, 1], \theta \in \mathbb{R}/2\pi(a_j + 1)\mathbb{Z}\}$. Here $t = 0$ corresponds to the boundary circle $\partial_j \tilde{S}$, and the $\theta$ coordinate of $A_j$ is compatible at $t = 0$ with the $\theta$ coordinate of the boundary circle. We will define a particular homeomorphism $\tau_j \in \text{Mod}(\tilde{S}, \partial \tilde{S})$ as follows. On $A_j$ we define $\tau_j(t, \theta) = (t, \theta + 2\pi(1 - t))$ so that $\tau_j$ rotates $\partial_j \tilde{S}$ by $2\pi$ and is the identity on the other boundary of $A_j$. We extend $\tau_j$ to a map $\tau_j : \tilde{S} \to \tilde{S}$ by setting it to be the identity outside of $A_j$. The map $\tau_j$ represents an element of $\text{Mod}(\tilde{S}, \partial \tilde{S})$ which we call a fractional Dehn twist by angle $2\pi$. In particular $\tau_j^{a_j+1}$ is a full Dehn twist around the boundary curve $\partial_j \tilde{S}$. Note that $\tau_j^{a_j+1}$ is a right Dehn twist and that it also makes sense to describe $\tau_j$ as a right fractional Dehn twist (see [FaMa]). The collapsing map $c : \tilde{S} \to S$ induces a map $c_* : \text{Mod}(\tilde{S}, \partial \tilde{S}) \to \text{Mod}(S, \Sigma)$. Denote by $FT$ the group generated by the fractional Dehn twists $\tau_1, \ldots, \tau_k$. Note that $FT$ depends on $(a_1, \ldots, a_k)$.

**Lemma 3.3.** We have a short exact sequence

$$1 \to FT \to \text{Mod}(\tilde{S}, \partial \tilde{S}) \xrightarrow{c_*} \text{Mod}(S, \Sigma) \to 1,$$

where the group $FT$ is the free Abelian group generated by the $\tau_j$ and is central in $\text{Mod}(\tilde{S}, \partial \tilde{S})$.

**Proof.** We can see from the definition of a fractional Dehn twist that an element of $FT$ is isotopic to the identity by an isotopy which moves points in the boundary of $S$. These isotopies descend to isotopies of $(S, \Sigma)$. It follows that $FT$ belongs to the kernel of $c_*$. The fact that the kernel of $c_*$ is exactly $FT$ follows from the arguments used in [FaMa, Prop. 3.20].

To see the surjectivity of $c_*$, let $h$ be a homeomorphism of $S$ fixing points of $\Sigma$. The homeomorphism $h$ is isotopic to a diffeomorphism (see [FaMa, Thm. 1.13]) which, using the properties of the real blowup, has a lift to a homeomorphism $h'$ from $\tilde{S}$ to itself. This homeomorphism induces a homeomorphism of $\partial_j \tilde{S}$ for each $j$. By applying an isotopy in the annular neighborhood $A_j$ of each $\partial_j \tilde{S}$ as above, we can replace $h'$ by a map $h''$, which is the identity on $A_j$. In particular $h$ and $h''$
represent the same element of Mod(S, Σ), and h" fixes each point of ∂S. This shows that the equivalence class of h contains a representative which is in the image of c*, proving surjectivity.

Since h" is the identity on each A_j and τ_j is the identity on the complement of A_j we see that h" and τ_j have disjoint support so they commute which implies that FT is central. In particular FT is abelian.

We now show that there are no additional relations between elements of FT. Consider a word in the collection of twists that represents a relation. Since FT is abelian we can write it as w = τ_1^{m_1} · · · τ_j^{m_j} · · · τ_k^{m_k}. Now since S has negative Euler characteristic we can find a non-peripheral path from a boundary component ∂jS to itself. The effect of w on this path is to shift both endpoints by 2πm_j. By Lemma 3.2, since w acts trivially, m_j = 0. Since j was arbitrary, w = 1. □

3.3. Strata of boundary marked surfaces as spaces. In this subsection we define a topology on Ŧ in a manner somewhat analogous to the method used for defining the topology on ℋ_m. We construct a cover of Ŧ by sets on which the developing map is an injection into H^1(S, Σ; R^2) and then we use these maps to endow each such set with the topology induced by the developing map. We show that with respect to this topology the map pr is a covering map, thus we conclude that these charts give not only a topology on Ŧ but a compatible affine structure.

To construct these charts fix a point (f, M) ∈ Ŧ and a geodesic triangulation τ of M. We caution the reader that in contrast to §2.4, here τ denotes a geodesic triangulation of M rather than a topological triangulation of S. We have canonical lifts of the edges σ of the triangulation τ to edges σ in M so that the endpoints of σ lie in ∂M. Let τ̂ be the collection of paths in Ŧ of the form f°(σ) for σ an edge of τ. These paths are embedded and we refer to them as arcs. These arcs of τ̂ decompose Ŧ into hexagons where edges of the hexagons consist alternately of arcs and intervals in boundary circles.

We now define what it means for two hexagon decompositions to be ε-close. Firstly we require that the arcs in the two decompositions are pairwise homotopic. Secondly we require that these pairs of homotopic arcs are ε-close in the sense of Definition 3.1. Note that any triangulation of a translation surface has the property that distinct edges lie in distinct homotopy classes, thus there is no ambiguity in comparing homotopy classes of arcs in the two decompositions.

Given a geodesic triangulation τ of M, let τ̂ be its pullback under a marking of blown up translation surfaces Ŧ → M, and let Ûε,τ consist of (f, M') ∈ Ŧ for which there is a geodesic triangulation σ of M' which lifts to a hexagon decomposition of Ŧ of M' so that the pullback of σ under f° is ε-close to τ̂.

Lemma 3.4. The developing map is injective on the set Ûε,π/2.

Remark 3.5. The developing map for Ŧ is most naturally defined to take values in H^1(S, ∂S; R^2) but the the collapsing map c : Ŧ → S induces a map c* : H^1(S, Σ; R^2) → H^1(S, ∂S; R^2) which is an isomorphism. In the sequel we will make use of this isomorphism to identify the two spaces.

Proof. Assume that (f_1, M_1) and (f_2, M_2) map to the same point in H^1(S, Σ; R^2) and both lie in Ûε,π/2. Let τ_1 and τ_2 be triangulations of M_1 and M_2 so that τ_1 = f_1°(τ̂_1) and τ_2 = f_2°(τ̂_2) are π/2-close to τ̂, and hence π-close to each other. Since M_1 and M_2 have geodesic triangulations such that corresponding edges have...
the same image in $\mathbb{R}^2$, the comparison map $F(\tau, M_1, M_2)$ is a translation equivalence, and we denote it by $g : M_1 \to M_2$. In order to show that $(f_1, \hat{M}_1)$ and $(\bar{f}_2, \hat{M}_2)$ represent the same element of $\tilde{\mathcal{H}}$ we need to show that $\tilde{g} \circ \bar{f}_1$ and $\tilde{g} \circ \bar{f}_2$ agree on $\partial^* \hat{S}$ and that they are isotopic via an isotopy which fixes $\partial \hat{S}$. Both $\tilde{\tau}_1$ and $\tilde{\tau}_2$ produce collections of arcs in $\hat{S}$. There is a unique correspondence between these collections of arcs so that corresponding arcs are homotopic. We want to show that corresponding arcs are not just homotopic but in fact relatively homotopic.

We begin by working with a single edge. Let $\sigma$ be an edge of $\tilde{\tau}$ and let $\sigma_1$ and $\sigma_2$ be corresponding oriented edges of $\tilde{\tau}_1$ and $\tilde{\tau}_2$. Our first objective is to show that $\sigma_1$ and $\sigma_2$ have the same endpoints and are relatively homotopic. The relative homotopy classes of $\sigma_1$ and $\sigma_2$ determine points $[\sigma_1]$ and $[\sigma_2]$ in $\tilde{C}_\sigma(\hat{S})$. It suffices to show that these two points are the same.

Say that $\sigma_1$ and $\sigma_2$ run from $\partial_i \hat{S}$ to $\partial_j \hat{S}$. We have the endpoint map $\epsilon : C_\sigma(\hat{S}) \to \partial_i \hat{S} \times \partial_j \hat{S}$. According to Lemma 3.2 this is a covering map. We also have projection maps $\pi_k : \partial_k \hat{S} \to \mathbb{R}/2\pi \mathbb{Z}$ which are covering maps (in either $S$ or $\hat{M}$). Consider the composition $\Pi = (p_i \times p_j) \circ \epsilon : C_\sigma(\hat{S}) \to \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R}/2\pi \mathbb{Z}$. This is a covering map. The oriented segments $\sigma_1$ and $\sigma_2$ have the same holonomy so they point in the same direction in $\mathbb{R}^2$ hence $\Pi([\sigma_1]) = \Pi([\sigma_2])$. Since both triangulations lie in $U_{\tau, \pi/2}$ there are intervals $I \subset \partial_i \hat{S}$ and $J \subset \partial_j \hat{S}$ of length $\pi$ and a homotopy from $\sigma_1$ to $\sigma_2$ for which the endpoints of the paths remain in $I$ and $J$. This homotopy gives a path $\rho$ from $[\sigma_1]$ to $[\sigma_2]$ in $C_\sigma(\hat{S})$ which projects to $I \times J \subset \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R}/2\pi \mathbb{Z}$ under the covering map $\Pi$. The image of $\rho$ is a loop, which is contractible since it is contained in the contractible set $I \times J$. Covering theory implies that $\rho$ itself must be a loop so $[\sigma_1] = [\sigma_2]$. It follows that $\tilde{g} \circ \bar{f}_1(\sigma_1)$ and $\tilde{g} \circ \bar{f}_2(\sigma_2)$ are relatively homotopic.

Proposition 2.1 shows that $g \circ \bar{f}_1$ and $\bar{f}_2$ induce isotopic maps on $(S, \Sigma)$. It remains to show that they agree as maps on $(\hat{S}, \partial \hat{S})$. According to Lemma 3.3 this means that there is no boundary twisting but boundary twisting is detected by the effect on relative homotopy classes of the paths $\sigma$. We conclude that $g \circ \bar{f}_1$ and $\bar{f}_2$ are isotopic relative to their boundaries.

We can use the injectivity of the developing maps on the sets $U_{\tau, \pi/2}$ to define a topology on $\tilde{\mathcal{H}}$. With respect to this topology the developing map becomes a local homeomorphism. The next proposition refers to this topology.

**Proposition 3.6.** The projection map $pr$ from $\tilde{\mathcal{H}}$ to $\mathcal{H}_m$ has the path lifting property.

Recall that the path lifting property for the map $pr$ means that if we are given a path $f : I \to \mathcal{H}_m$ and a lift $\hat{x}_0$ of the endpoint $f(0) = x_0$ then there is a unique path $\hat{f} : I \to \mathcal{X}$ with $\hat{f}(0) = \hat{x}_0$ which satisfies $pr \circ \hat{f} = f$ (see [H, p. 60] for more information).

**Proof.** It suffices to construct lifts of paths locally. Consider an open set $U_{\tau} \subset \mathcal{H}_m$ corresponding to a triangulation $\tau$ as in §24. Let $\phi_t = (f_t, \hat{M}_t)$ for $t \in [0, 1]$ be a path taking values in $U_{\tau}$ and let $(\hat{f}_0, \hat{M}_0)$ represent a point in $\tilde{\mathcal{H}}$ such that $pr(\hat{f}_0, \hat{M}_0) = (f_0, M_0)$. We define a path $\hat{\phi}_t = (\hat{f}_t, \hat{M}_t)$ for $t \in [0, 1]$, satisfying

$$\hat{\phi}_0 = (\hat{f}_0, \hat{M}_0) \quad \text{and} \quad \hat{\phi}_t(0) = \hat{f}_0$$

as follows.
Let $F(\tau, M_0, M_t) : M_0 \to M_t$ be the comparison maps defined in §2.4. These maps are piecewise linear and hence they have unique extensions to the blowups, that is there are homeomorphisms $\tilde{F}(\tau, M_0, M_t) : \tilde{M}_0 \to \tilde{M}_t$ of the blown up surfaces, satisfying
\[ c \circ \tilde{F}(\tau, M_0, M_t) = F(\tau, M_0, M_t) \circ c. \]
We define $\tilde{F}_t = \tilde{F}(\tau, M_0, M_t) \circ \tilde{f}_0 : \tilde{S} \to \tilde{M}_t$, and denote the restriction of $\tilde{F}_t$ to $\partial_j \tilde{S}$ by $\partial_j \tilde{F}_t$. We would like to set $\tilde{\phi}_t = (\tilde{F}_1, M_t)$ but there is a problem in that the maps $\tilde{F}_t$ need not preserve the boundary coordinates given by the maps $p_j : \partial_j M_t \to S^1$. In other words, it will not generally be the case that $p_j \circ \partial_j \tilde{F}_t = p_j$. We claim that there is a unique (up to isotopy) way to modify the maps $\tilde{F}_t$ by precomposition with a continuous family of maps of $\tilde{S}$, so that they do satisfy the boundary coordinate condition, and so that (7) holds. We will do this by precomposing the maps $\tilde{F}_t$ with homeomorphisms $H_t : \tilde{S} \to \tilde{S}$ supported in a neighborhood of the boundary and then set $\tilde{f}_t = \tilde{F}_t \circ H_t$.

In order to prove the existence of the required homeomorphisms $H_t$, we consider the condition that they will need to satisfy. For each boundary component $\partial_j \tilde{S}$ the restriction $\partial_j H_t$ should satisfy
\[ p_j \circ \partial_j \tilde{F}_t \circ \partial_j H_t = p_j \quad \text{and} \quad \partial_j H_0 = \text{Id}. \]
We rewrite this as
\[ p_j \circ \partial_j \tilde{F}_t = p_j \circ (\partial_j H_t)^{-1}. \]
Setting $\ell_{j,t} = (\partial_j H_t)^{-1}$, we see that $\ell_{j,t} : \partial_j \tilde{S} \to \partial_j \tilde{S}$ is a solution to the homotopy lifting problem
\[ p_j \circ \ell_{j,t} = p_j \circ \partial_j \tilde{F}_t \quad \text{and} \quad \ell_{j,0} = \text{Id}, \]
see Figure 1.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (S) {$S^1$};
  \node (tildeS) [above of=S] {$\tilde{S}$};
  \node (tildeSj) [left of=tildeS] {$\partial_j \tilde{S}$};
  \node (F) [above of=tildeSj] {$\tilde{F}_t$};
  \node (H) [above of=F] {$H_t$};
  \node (p) [above of=tildeSj] {$p_j$};

  \draw[->] (S) -- (tildeS);
  \draw[->] (S) -- (tildeSj);
  \draw[->] (tildeS) -- (F); 
  \draw[->] (tildeSj) -- (F);
  \draw[->] (tildeSj) -- (H);
  \draw[->] (H) -- (p);
  \draw[->, dashed] (tildeSj) -- (p);
  \draw[->] (p) -- (F);
  \draw[->, dashed] (p) -- (H);

  \node at (0,0) {$\ell_{j,t}$};
  \node at (-2,-2) {$\partial_j \tilde{S}$};
  \node at (2,-2) {$\partial_j \tilde{F}_t$};
  \node at (-1,-3) {$\partial_j H_t$};
  \node at (1,-3) {$\text{Id}$};
\end{tikzpicture}
\caption{The boundary map $\ell_{j,t} = (\partial_j H_t)^{-1}$ rectifies the discrepancies of $\partial_j \tilde{F}_t$ along boundary circles.}
\end{figure}

Since $p_j$ is a covering map the Homotopy Lifting Theorem [H, Prop 1.30] asserts that lifts $\ell_{j,t}$ exist and are unique. Thus $H_t = (\ell_{j,t})^{-1}$ is defined on the boundary of $\tilde{S}$. It remains to extend $H_t$ to annular neighborhoods of the boundaries. As in §3.1 we have a family of disjoint annuli $A_j$ in $\tilde{S}$ parametrized by $\{(r, \theta) : r \in [0, 1], \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ where in these coordinates, $\{r = 0\}$ is the $j$-th boundary component of $\tilde{S}$. Then we define $H_t$ to be the identity outside the union of annuli, to be equal to the prescribed map $\partial_j H_t$ on $\{r = 0\}$, and be given by the formula $H_t(r, \theta) = (r, \psi_{(1-r)}(\theta))$ for $0 \leq r < 1$.

It is clear that with these definitions, the path $\tilde{\phi}_t = (\tilde{f}_t, M_t)$ with $\tilde{f}_t = \tilde{F}_t \circ H_t$ satisfies (7).
surjectivity in (10) follows from the fact that the fractional twists \( f_1^s : \tilde{S} \to \tilde{M}_1 \), which gives us an isotopy from \( \tilde{f}_1^0 \) to \( \tilde{f}_1^1 \), fixing the boundary. That is, the lifts of homotopic paths have the same endpoint in \( \tilde{\mathcal{H}} \).

The uniqueness (up to isotopy) of the maps \( H_t \) (where \( H_0 \) is the identity and (8) holds), follows from the uniqueness of the lifts \( \ell_{j,t} \) and standard properties of annuli, and is left as an exercise. \( \square \)

**Corollary 3.7.** The map \( \text{pr} : \tilde{\mathcal{H}} \to \mathcal{H}_m \) is a covering map and \( \tilde{\mathcal{H}} \) has an affine structure given by period coordinates.

**Proof.** A local homeomorphism to a sufficiently nice space which has the path lifting property is a covering map. See [dC, p. 383] for more details and a proof. \( \square \)

The group \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \) acts on \( \tilde{\mathcal{H}} \) by precomposition. It acts continuously and properly discontinuously on \( \tilde{\mathcal{H}} \), the quotient is \( \mathcal{H} \) and according to Lemma 3.3 the subgroup \( FT \) acts simply transitively on each fiber of \( \text{pr} \). We warn the reader that the spaces \( \tilde{\mathcal{H}} \) and \( \mathcal{H}_m \) are not in general connected. Typically they have infinitely many components.

### 3.4. The space of framed surfaces.

Our next objective is to define and analyze the covering space of framed surfaces. Since \( \mathcal{H}_m \) is an affine manifold and \( \text{pr} : \tilde{\mathcal{H}} \to \mathcal{H}_m \) is a covering map, we have equipped \( \mathcal{H} \) with the structure of an affine manifold. Since the action of \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \) is properly discontinuous, for each subgroup \( \Gamma \) of \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \), we can form the quotient \( \tilde{\mathcal{H}}/\Gamma \). By Lemma 3.3 we have \( \mathcal{H} = \tilde{\mathcal{H}}/\text{Mod}(\tilde{S}, \partial \tilde{S}) \) and \( \mathcal{H}_m = \mathcal{H}/FT \). Moreover each \( \tilde{\mathcal{H}}/\Gamma \) is an orbifold cover of \( \mathcal{H} \). Note that this is not the Galois correspondence relating connected covers to subgroups of the fundamental group, since \( \tilde{\mathcal{H}} \) is not connected. Nevertheless we can define the space of framed surfaces via a group-theoretic approach.

We start with a discussion of subgroups of \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \). While an element of the group \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \) is only defined up to isotopy on the interior of \( \tilde{S} \), it is well-defined on the boundary circles and acts on each circle \( \partial_j \tilde{S} \) by rotations which are multiples of \( 2\pi \). Let \( \text{Mod}(\tilde{S}) \) be the subgroup of \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \) represented by homeomorphisms that fix the boundary pointwise. Let \( PR \) be the **prong rotation group**, consisting of homeomorphisms of the boundary \( \partial \tilde{S} \) which, on each boundary component \( \partial_j \tilde{S} \), are rotations by an integral multiple of \( 2\pi \). Since \( \partial_j \tilde{S} \) is parametrized by a circle of length \( 2\pi(a_j + 1) \), as a group \( PR \) is isomorphic to \( \prod_{j=1}^{k} \mathbb{Z}/(a_j + 1)\mathbb{Z} \). We have a short exact sequence

\[
1 \to \text{Mod}(\tilde{S}) \to \text{Mod}(\tilde{S}, \partial \tilde{S}) \to PR \to 1.
\]

Surjectivity in (10) follows from the fact that the fractional twists \( \tau_j \in \text{Mod}(\tilde{S}, \partial \tilde{S}) \) map to a collection of generators for \( PR \).

A **framed** translation surface is a translation surface \( M \) equipped with a right-pointing horizontal prong at each singular point. We will call this prong (considered as an element of \( \partial_j M \)) the **selected prong**. Equivalently a framed surface is equipped with a choice of boundary coordinate on \( C_j \) taking values in the circle \( \mathbb{R}/2\pi(a_j + 1)\mathbb{Z} \) so that the map \( p_j \) is reduction modulo \( 2\pi \) and the selected prong corresponds to the angle 0.
The space of framed translation surfaces is naturally a finite cover of \( \mathcal{H} \) and we will denote it by \( \tilde{\mathcal{H}} \). The connected components of \( \tilde{\mathcal{H}} \) were classified by Boissy [Boi]. We recover \( \mathcal{H} \) as the quotient of \( \tilde{\mathcal{H}} \) by the group \( \text{Mod}(\tilde{S}) \) in (10):

**Proposition 3.8.** We have \( \tilde{\mathcal{H}} / \text{Mod}(\tilde{S}) = \mathcal{H} \).

**Proof.** We define a map from \( \tilde{\mathcal{H}} \) to \( \mathcal{H} \) as follows. For each boundary component \( \partial_j \tilde{S} \), let \( 0_j \) denote the point in \( \partial_j \tilde{S} \) with angular coordinate 0. Given a marked blown-up surface \( (\tilde{f}, \tilde{M}) \), define a framed surface by letting the point \( \tilde{f}(0_j) \) be the selected prong. Say that two surfaces \( (\tilde{f}_1, \tilde{M}_1) \) and \( (\tilde{f}_2, \tilde{M}_2) \) map to the same surface in \( \mathcal{H} \). Thus there is a translation equivalence \( h : M_1 \to M_2 \) so that \( h \) takes selected prongs in each \( \partial_j \tilde{M}_1 \) to selected prongs in \( \partial_j \tilde{M}_2 \). This means that the restriction of the map \( (\tilde{f}_2)^{-1} \circ h \circ \tilde{f}_1 \) to \( \partial_j \tilde{S} \) fixes \( 0_j \). Since this map is a rotation of the circle with a fixed point it is the identity map. Thus \( (\tilde{f}_2)^{-1} \circ h \circ \tilde{f}_1 \in \text{Mod}(\tilde{S}) \) so \( (\tilde{f}_1, \tilde{M}_1) \) and \( (\tilde{f}_2, \tilde{M}_2) \) are \( \text{Mod}(\tilde{S}) \)-equivalent. \[ \Box \]

We summarize our constructions in Figure 2.

**Figure 2.** The vertical arrow corresponds to factoring by the action of \( \text{Mod}(\tilde{S}, \partial \tilde{S}) \), and the pair of arrows on the left and right correspond to factoring by the groups appearing in the sequences (6) and (10) respectively.

3.5. **Action of** \( G = \text{SL}_2(\mathbb{R}) \) **and its covers on** \( \tilde{\mathcal{H}} \). The affine equivalence classes of translation structures are orbits of a group actions which we now define. Recall that \( \text{GL}_2^+(\mathbb{R}) \) and \( \text{GL}_2^0(\mathbb{R}) \) denote respectively the group of orientation preserving invertible \( 2 \times 2 \) real matrices, and its universal cover group. Given \( g \in \text{GL}_2^0(\mathbb{R}) \) and a translation surface \( M \) we construct a new surface \( gM \) as follows. As discussed in §2.1, in the language of \( (G, X) \)-structures, a translation surface can be given by an atlas of charts on \( M \) with overlap functions taking values in the group of translations \( \mathbb{R}^2 \). The element \( g \) is a linear map \( \mathbb{R}^2 \to \mathbb{R}^2 \), and postcomposing each chart in an atlas, we obtain a new translation atlas. Let \( gM \) be this new translation surface and let \( \phi_g \) be the identity map from \( M \) to \( gM \) (the underlying surfaces are the same). The map \( \phi_g \) is an affine map with derivative \( g \).

The group \( g \in \text{GL}_2^0(\mathbb{R}) \) acts on \( \tilde{\mathcal{H}}_m \) as follows. If \( (f, M) \in \tilde{\mathcal{H}}_m \) then define \( g(f, M) \) to be \( (\phi_g \circ f, gM) \). The condition that \( g \in \text{GL}_2^0(\mathbb{R}) \) insures that \( \phi_g \circ f \) is orientation preserving. Since the action of \( \text{Mod}(S, \Sigma) \) on marked surfaces is by pre-composition, this action induces a well-defined action on \( \tilde{\mathcal{H}}_m \). If we let \( \text{GL}_2^0(\mathbb{R}) \)
act on $H^1(S, \Sigma; \mathbb{R}^2)$ by acting on the coefficients via the linear action on the plane, then the map $\text{dev} : \mathcal{H}_m \to H^1(S, \Sigma; \mathbb{R}^2)$ will be equivariant.

In the remainder of the paper we write $G$ for $\text{SL}_2(\mathbb{R})$.

It is a general principle that if a connected topological group acts on a topological space $X$, then its universal cover acts on any cover of $X$. Since $\text{GL}_2(\mathbb{R})$ and its subgroup $G$ act on $\mathcal{H}_m$, and $\tilde{\mathcal{H}} \to \mathcal{H}_m$ is a covering map, we conclude that their universal covering groups $\tilde{\text{GL}}_2(\mathbb{R})$ and $\tilde{G}$ act on $\tilde{\mathcal{H}}$. For related discussions in the case of strata of meromorphic quadratic differentials see [BS] and [HKK].

It will be useful for us to not only know that this action exists but to have an explicit description of the action. An element of $\tilde{g} \in \tilde{\text{GL}}_2(\mathbb{R})$ can be represented by a pair $(\rho, g)$ where $g \in \text{GL}_2(\mathbb{R})$ and $\rho : [0,1] \to \text{GL}_2(\mathbb{R})$ is a path with $\rho(0) = \text{Id}$ and $\rho(1) = g$. Two such representations are equivalent if the corresponding paths are homotopic relative to their endpoints. Given an element of $\tilde{H}$, $(\tilde{f}, \tilde{M})$ and an element $\tilde{g} = (\rho, g)$ of $\tilde{\text{GL}}_2(\mathbb{R})$ we have a path $\alpha(t) = \rho(t)(f, M)$ in $\mathcal{H}_m$ and a lift of the initial point of this path $(f, M)$ to $(\tilde{f}, \tilde{M}) \in \tilde{\mathcal{H}}$. According to Proposition 3.6 this path lifts uniquely to a path $\tilde{\alpha}(t) \in \tilde{\mathcal{H}}$ and we set $\tilde{g}(\tilde{f}, \tilde{M}) = \tilde{\alpha}(1)$. The resulting element is independent of the choice of path $\rho$.

Let

$$u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \quad \text{and} \quad r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$  

We will refer to the action of $u_s$ as the horocycle flow and $g_t$ as the geodesic flow and write

$$U = \{u_s : s \in \mathbb{R}\}, \quad A = \{g_t : t \in \mathbb{R}\}, \quad \text{SO}_2(\mathbb{R}) = \{r_\theta : \theta \in \mathbb{R}\}, \quad B = AU.$$  

Note that $B$ is the connected component of the identity in the group of upper-triangular matrices and it normalizes $U$. Since $U$ is simply connected, the connected component of the identity in the pre-image of $U$ in $\tilde{G}$ is a subgroup isomorphic to $U$. Thus we can identify $U$ with a subgroup of $\tilde{G}$, and the same is true for the groups $B$ and $A$. The groups $G$ and $\text{GL}_2(\mathbb{R})$ are both homotopy equivalent to $\text{SO}_2(\mathbb{R}) \cong S^1$, and $\tilde{G}$ and $\tilde{\text{GL}}_2(\mathbb{R})$ are both homotopy equivalent to $\text{SO}_2(\mathbb{R}) \cong \mathbb{R}$.

As we have seen the fundamental groups of both groups $\tilde{G}$ and $\tilde{\text{GL}}_2(\mathbb{R})$ are isomorphic to $\mathbb{Z}$. Thus for any positive integer $n$, there is a unique connected $n$-fold cover of $G$. Also, since $\text{SO}_2(\mathbb{R}) \subset G \subset \text{GL}_2(\mathbb{R})$ and the induced maps on fundamental groups are isomorphisms it will cause no confusion to identify the fundamental groups of these three groups. This group is infinite cyclic and we denote it by $C$. We have three short exact sequences:

$$1 \to C \to \tilde{\text{GL}}_2(\mathbb{R}) \to \text{GL}_2(\mathbb{R}) \to 1$$

$$1 \to C \to \tilde{G} \to G \to 1$$

$$1 \to C \to \tilde{\text{SO}}_2(\mathbb{R}) \to \text{SO}_2(\mathbb{R}) \to 1.$$  

We will write the element of $\tilde{\text{SO}}_2(\mathbb{R})$ corresponding to $\theta \in \mathbb{R}$ as $\tilde{r}_\theta$, so that $\tilde{r}_\theta \to r_\theta \mod 2\pi$ is the projection $\tilde{\text{SO}}_2(\mathbb{R}) \to \text{SO}_2(\mathbb{R})$. The group $C$ is central in $\text{GL}_2(\mathbb{R})$ and in these coordinates it is identified with $\{\tilde{r}_{2\pi n} : n \in \mathbb{Z}\}$. We will explicitly describe the action of $\tilde{\text{SO}}_2(\mathbb{R})$ on $\tilde{\mathcal{H}}$. 
Proposition 3.9. The left action of \( \hat{r}_{2\pi} \) on \( \tilde{\mathcal{H}} \) is equal to the right action of \( \tau^{-1} \) where \( \tau = \tau_1 \cdots \tau_k \in FT \). That is, for \((f, \hat{M}) \in \tilde{\mathcal{H}} \) we have \( \hat{r}_{2\pi}(\hat{f}, \hat{M}) = (f \circ \tau^{-1}, \hat{M}) \).

Proof. Given a marked translation surface rel boundary \( \hat{f} : \hat{S} \to \hat{M} \) we follow the definition of the action of \( \text{GL}_2(\mathbb{R}) \) on \( \tilde{\mathcal{H}} \) given above. Let \( D_\theta \) be the map supported in annuli around boundary components of \( \hat{S} \) that performs a Dehn twist by \( \theta \in \mathbb{R} \) on each boundary component. Let \( \theta_0 \in \mathbb{R} \), and assume for concreteness that \( \theta_0 > 0 \). We will describe the action of \( \hat{r}_{\theta_0} \) by explicitly lifting the action of \( \{ r_\theta : \theta \in [0, \theta_0] \} \) using the procedure described in Proposition 3.6.

Let \((f, \hat{M}) \in \mathcal{H}_m \) be a marked translation surface, and let \((\hat{f}, \hat{M}) \) be an element of \( \tilde{\mathcal{H}} \) projecting to \((f, \hat{M}) \). We want to lift the path \( \theta \mapsto r_\theta(M, f) = (f, r_\theta M) \) to \( \tilde{\mathcal{H}} \). As in \S 3.3, fix a triangulation \( \tau \) of \( S \). By a compactness argument it suffices to analyze the lift of the path \( \theta \mapsto r_\theta(M, f) \), for the subset \( \{ \theta \in [0, \theta_0] : r_\theta(M, f) \in \mathcal{U}_\tau \} \), and we can assume with no loss of generality that \((f, M) \in \mathcal{U}_\tau \).

Let \( F(\tau, \tau, r_\theta M) \) be the comparison map defined in \S 2.4. An important observation, which follows immediately from the definition of the comparison maps, is that the derivative of \( F(\tau, M, r_\theta M) \) is everywhere equal to the matrix \( r_\theta \), and hence the comparison map is in fact independent of the triangulation \( \tau \). Let \( F(\tau, M, r_\theta M) \) denote the extension of \( F(\tau, M, r_\theta M) \) to blown-up surfaces, and let \( \hat{F}_\theta = \hat{F}(\tau, M, r_\theta M) \circ \hat{f} \). Then as discussed in \S 3.3, \( \hat{F}_\theta \) does not preserve boundary coordinates, but the composition \( D_{-\theta} \circ \hat{F}_\theta \) does. Thus the path \( \theta \mapsto (\hat{F}_\theta \circ D_{-\theta}, r_\theta \hat{M}) \) satisfies (7), so by uniqueness, is the desired lift of the path \((f, r_\theta M) \) to \( \tilde{\mathcal{H}} \).

In particular, setting \( \theta_0 = 2\pi \) and using the fact that \( r_{2\pi} = \text{Id} \), we get \((\hat{f} \circ D_{-2\pi}, \hat{M}) \) and \( D_{-2\pi} = \tau^{-1} \) with \( \tau = \tau_1 \cdots \tau_k \).

3.6. Fundamental groups of covers of strata. We note some consequences of the above discussion.

Corollary 3.10. Every path component of \( \mathcal{H}_m \) has an infinite fundamental group. In particular it is never the case that the space of marked surfaces is the universal cover of the stratum.

Proof. Let \( C \) be a path component of \( \mathcal{H}_m \) and let \( M \in C \). Proposition 3.9 shows that lifting the closed path \( \{ r_\theta M : \theta \in [0, 2\pi] \} \) to \( \tilde{\mathcal{H}} \), we get a non-closed path whose endpoints differ by an application of the element \( \tau^{-1} \in FT \). This element has infinite order in \( FT \). Since the group \( FT \) acts freely on the fiber \( \text{pr}^{-1}(M) \), none of the lifted paths are closed. It follows that \( C \) has an infinite fundamental group. \( \square \)

An instructive example is the case \( \mathcal{H} = \mathcal{H}(0) \) (where the model surface \( S \) is the torus with one marked point and \( \hat{S} \) is the torus with an open disk removed). The covering space \( \mathcal{H}_m \) can be identified with \( \text{GL}_2(\mathbb{R}) \) (the connected component of the identity in \( \text{GL}_2(\mathbb{R}) \)), and is not simply connected, whereas \( \tilde{\mathcal{H}} \) is identified with its universal cover group \( \tilde{\text{GL}}_2(\mathbb{R}) \) and is simply connected. A generator of the fundamental group of \( \tilde{\text{GL}}_2(\mathbb{R}) \) acts by a boundary Dehn twist on \( S \).

We present another useful consequence of our lifting construction. Let \( \Gamma_0 \) be a subgroup of \( \text{Mod}(\hat{S}, \partial \hat{S}) \), let \( \mathcal{H}_{\Gamma_0} \) denote the quotient \( \mathcal{H}/\Gamma_0 \), let \( C \) be a path component of \( \mathcal{H}_{\Gamma_0} \) and let \( p \in C \) and \( q \in \tilde{\mathcal{H}} \) such that \( q \) projects to \( p \). We define a
homomorphism $\rho_q : \pi_1(C, p) \to \Gamma_0$ as follows. Let $\phi$ be a loop based at $p$. Let $[\phi]$ be the element of $\pi_1$ that it represents. We can lift $\phi$ to a path starting at $q$. The endpoint of this lifted path maps to $p$ so it has the form $r\gamma$ for some $\gamma \in \Gamma_0$ (where our notation reflects the fact that $\text{Mod}(\bar{S}, \partial\bar{S})$ acts by precomposition, so defines a right-action). Define $\rho_q([\phi])$ to be $\gamma$. The homotopy lifting property shows that $\rho_q([\phi])$ depends only on $q$ and on the homotopy class $[\phi]$, and not the particular loop $\phi$ chosen to represent it.

The construction of $\rho_q$ depends on the choice of the point $q$. If we were to choose a different point $q'$ mapping to $p$ then $q' = qa$ for some $\alpha \in \Gamma_0$. In this case the lift of $\phi$ starting at $qa$ is the path $\phi a$ and the other endpoint is $r\gamma\alpha = p\alpha(\alpha^{-1}\gamma\alpha)$. Thus $\rho_{q'}(\phi) = \alpha^{-1}\rho_q(\phi)$ so $\rho_{q'}$ differs from $\rho_q$ by an inner automorphism of $\Gamma_0$.

In other words we have constructed a preferred homomorphism $\rho : \pi_1(C) \to \Gamma_0$, well-defined up to a choice of an inner automorphism of $\Gamma_0$.

Let $n \in \mathbb{N}$ and let $C_n$ denote the subgroup of $C$ generated by $r_{2\pi n}$. Then $C_n$ is central in $G$ and $\hat{G}_n = G/C_n$ is the unique connected $n$-fold cover of $G$.

**Corollary 3.11.** Let $\Gamma_0$ be a subgroup of $\text{Mod}(\bar{S}, \partial\bar{S})$. Then $\hat{G}_n$ acts on $\hat{H}/\Gamma_0$ if and only if $\tau^n \in \Gamma_0$. In particular, suppose $n$ is the least common multiple of the numbers $\{a_i + 1 : i = 1, \ldots, k\}$. Then $G_n$ acts on $\mathcal{H}_n$, but $\hat{G}_m$ does not act when $m < n$.

**Proof.** Let $\mathcal{H}_{\Gamma_0} = \hat{H}/\Gamma_0$, and let $(\hat{f}, \hat{M}) \in \hat{H}$. If the action of $\hat{r}_{2\pi n}$ is well-defined on $\mathcal{H}_{\Gamma_0}$ then $\hat{r}_{2\pi n}(\hat{f}, \hat{M})$ and $(\hat{f}, \hat{M})$ are equivalent in $\mathcal{H}_{\Gamma_0}$, so by Proposition 3.9,

$$\hat{f} \circ \gamma = \hat{r}_{2\pi n}(\hat{f}, \hat{M}) = (\hat{f}, \hat{M})$$

for some $\gamma \in \Gamma_0$. That is, $\hat{f} \circ \gamma$ and $\hat{f} \circ \tau^n$ are isotopic via an isotopy fixing $\partial\bar{S}$.

In particular they represent the same elements in $\text{Mod}(\bar{S}, \partial\bar{S})$ and $\tau^n \in \Gamma_0$. Conversely if $\tau^n \in \Gamma_0$ then $(\hat{f}, \hat{M})$ and $\hat{r}_{2\pi n}(\hat{f}, \hat{M})$ represent the same surface in $\mathcal{H}_{\Gamma_0}$. \hfill $\square$

### 4. The Rel foliation and Rel vectorfields

It is a general principle that geometric structures on $H^1(S, \Sigma; \mathbb{R}^2)$ which are invariant under the action of the mapping class group induce geometric structures on the stratum. We will now see an example of this principle. Consider the following exact sequence:

$$0 \longrightarrow H^0(S) \longrightarrow H^0(\Sigma) \longrightarrow H^1(S, \Sigma) \xrightarrow{\text{Res}} H^1(S) \longrightarrow 0.$$  

Let us take the coefficients in cohomology groups to be $\mathbb{R}^2$.

Let $\mathcal{R}$ denote the image of $H^0(\Sigma; \mathbb{R}^2)$ in $H^1(S, \Sigma; \mathbb{R}^2)$. We call $\mathcal{R}$ the rel subspace. We can identify $\mathcal{R}$ with the kernel of the restriction map $\text{Res} : H^1(S, \Sigma; \mathbb{R}^2) \to H^1(S; \mathbb{R}^2)$. We can identify $H^0(\Sigma; \mathbb{R}^2)$ with the set of functions from $\Sigma$ to $\mathbb{R}^2$ and we can identify the image of $H^0(\Sigma; \mathbb{R}^2)$ in $H^0(S; \mathbb{R}^2)$ with the subspace of constant functions. Thus $\mathcal{R}$ can be seen as $\mathbb{R}^2$-valued functions on $\Sigma$ modulo constant functions. If $k$ is the cardinality of $\Sigma$ then the real dimension of $\mathcal{R}$ is $2(k-1)$. Beginning with §7 we will work in a stratum for which $\Sigma$ consists of two points, so that $\text{dim}_\mathbb{R}(\mathcal{R}) = 2$.

We will explicitly describe the action of $\mathcal{R}$ on the period coordinate space $H^1(S, \Sigma; \mathbb{R}^2)$. Pick $v \in \mathcal{R}$ and let $\gamma \in H^1(S, \Sigma; \mathbb{R}^2)$. We will define $\gamma + v \in$
$H^1(S, \Sigma; \mathbb{R}^2)$. Explicitly, the elements $\gamma$ and $\gamma + v$ are determined by their values on oriented paths in $S$ with endpoints in $\Sigma$. Let $\sigma$ be one such oriented path starting at $\xi_i$ and ending at $\xi_j$. Since $\mathcal{R} \cong H^0(\Sigma; \mathbb{R}^2) / H^0(S; \mathbb{R}^2)$, $v$ is an equivalence class of functions $\tilde{v} : \Sigma \to \mathbb{R}^2$, where functions are equivalent if they differ by a constant. We define $(\gamma + v)(\sigma) = \gamma(\sigma) + \tilde{v}(\xi_j) - \tilde{v}(\xi_i)$. Since representatives of $v$ differ by constants, the preceding formula does not depend on the choice of $\tilde{v}$. Also $v(\sigma)$ gives the same value for any $\sigma$ from $\xi_i$ to $\xi_j$.

The group $G$ acts equivariantly on the terms of the exact sequence (13). If we think of the terms as vector spaces of $\mathbb{R}^2$-valued functions then $G$ acts on these functions by acting on their values. In particular there is a natural action of $G$ on $\mathcal{R}$ since it is the quotient of the first two terms.

A subspace $W$ of a vector space $V$ defines a linear foliation of $V$ where the leaves are the translates of $W$. In this way the subspace $\mathcal{R}$ defines a foliation of $H^1(S, \Sigma)$. Since the mapping class group $\text{Mod}(S, \Sigma)$ preserves the short exact sequence it preserves this foliation and thus the foliation descends to a well-defined foliation on $\mathcal{H}$. We call this the Rel foliation. The names ‘kernel foliation’ and ‘absolute period foliation’ have also been used in the study of this foliation, see [Zo, Sch, McM7].

**Proposition 4.1.** Two surfaces in the same Rel leaf have the same area.

**Proof.** As we have seen in (4) the area of a surface $M$ can be written as $\langle \omega, \omega \rangle$ where $\omega = \text{dev}(M)$ and the bilinear form is the cup product with a certain coefficient pairing. So it suffices to show that the cup product of two classes in $H^1(S, \Sigma)$ depends only on the image of the classes in absolute cohomology $H^1(S)$; the latter statement follows from the fact that the cup product is natural with respect to the inclusion $(M, \emptyset) \to (M, \Sigma)$; i.e. diagram (14) below commutes. We refer to [H] for the definition of the cup product in the relative case, and to [H, Prop. 3.10] for a proof of naturality.

\[
\begin{align*}
H^1(S, \Sigma) \times H^1(S, \Sigma) & \xrightarrow{\cup} H^2(S, \Sigma) \\
H^1(S) \times H^1(S) & \xrightarrow{\cup} H^2(S)
\end{align*}
\]

We will define a notion of parallel translation on the leaves of the Rel foliation. We can identify the elements of $\mathcal{R}$ with constant vector fields on the vector space $H^1(S, \Sigma; \mathbb{R}^2)$. Recall that we have insisted on labeling the points in $\Sigma$, and that the mapping class group fixes $\Sigma$ pointwise. With these conventions, $\text{Mod}(S, \Sigma)$ acts trivially on $\mathcal{R}$. Thus the vector fields corresponding to $\mathcal{R}$ are invariant under $\text{Mod}(S, \Sigma)$ and induce well-defined vector fields on $\mathcal{H}_m$ and $\mathcal{H}$. The leaves of the Rel foliation have natural translation structures and these are the coordinate vector fields.

**4.1. Extending Rel paths.** The constant vector field associated with $v \in \mathcal{R}$ can be integrated on the vector space $H^1(S, \Sigma; \mathbb{R}^2)$ to give a one-parameter flow. Our next objective is to lift this flow, to the extent possible, to $\mathcal{H}_m$.

**Definition 4.2.** Let $M_0$ be point in $\mathcal{H}$ and let $v \in \mathcal{R}$. Let $\bar{v}$ denote the rel vector field on $\mathcal{H}$ corresponding to $v$. We say that $\text{Rel}_v(M_0)$ is defined and equal
to $M_1$ if there is a smooth path $\phi(t)$ in $\mathcal{H}$ with $\phi(0) = M_0$, $\frac{d}{dt}\phi(t) = \vec{v}(\phi(t))$ and $\phi(1) = M_1$.

The translation structures on the leaves of the Rel foliation are not complete in general and this means that the trajectories of the vector fields cannot always be defined for all time.

**Proposition 4.3.** Let $\Omega \subset \mathcal{H} \times \mathcal{R}$ be the set of pairs $(M, v)$ for which $\text{Rel}_v(M)$ is defined. Then $\Omega$ is open and the map $(M, v) \mapsto \text{Rel}_v(M)$ is continuous when viewed as a map from $\Omega$ to $\mathcal{H}$.

**Proof.** This follows from properties of solutions of first order ordinary differential equations. □

Our next result deals with the interaction between the natural actions of $G$ on $\mathcal{H}$ and $\mathcal{R}$, and the partially defined maps $\text{Rel}_v$.

**Proposition 4.4.** Let $M \in \mathcal{H}$, $v \in \mathcal{R}$ and $g \in G$. If $\text{Rel}_v(M)$ is defined then $\text{Rel}_{gv}(gM)$ is defined and $g(\text{Rel}_v(M)) = \text{Rel}_{gv}(gM)$.

**Proof.** Let $\vec{v}$ denote the vector field corresponding to $v \in \mathcal{R}$. To say that $\text{Rel}_v(M)$ is defined means that there is a smooth path $\phi(t)$ with $\phi(0) = M$, $\frac{d}{dt}\phi(t) = \vec{v}(\phi(t))$, and in this case $\phi(1) = \text{Rel}_v(M)$. Consider the path $t \mapsto g(\phi(t))$. It has the property that $g(\phi(0)) = gM$, $\frac{d}{dt}g(\phi(t)) = g(\frac{d}{dt}\phi(t)) = g\vec{v}(\phi(t)) = (gv)(\phi(t))$ and $g(\phi(1)) = g(\text{Rel}_v(M))$. The existence of this path shows that $\text{Rel}_{gv}(gM)$ is defined and $g(\text{Rel}_v(M)) = \text{Rel}_{gv}(gM)$. □

We can think of $\mathcal{R}$ as a Lie group acting on $H^1(S, \Sigma; \mathbb{R}^2)$. The fact that we can lift elements of the Lie group action on $H^1(S, \Sigma; \mathbb{R}^2)$ to $\mathcal{H}_m$ does not imply that the relations in the Lie group necessarily lift. For example the transformations $\text{Rel}_v$ and $\text{Rel}_w$ acting on $H^1(S, \Sigma; \mathbb{R}^2)$ commute but the corresponding lifted transformations of $\mathcal{H}_m$ need not commute where they are defined. The following result gives criteria for a composition law and for commutation.

**Proposition 4.5.** Let

$$\square = [0, 1]^2 \text{ and } \triangle = \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq 1\},$$

and let $v, w \in \mathcal{R}$. Then:

(i) If $\text{Rel}_{sv+tw}(M)$ is defined for all $(s, t) \in \triangle$ then

$$\text{Rel}_v \circ \text{Rel}_w(M) = \text{Rel}_{v+w}(M). \quad (15)$$

(ii) If $\text{Rel}_{sv+tw}(M)$ is defined for all $(s, t) \in \square$ then

$$\text{Rel}_v \circ \text{Rel}_w(M) = \text{Rel}_w \circ \text{Rel}_v(M) = \text{Rel}_{v+w}(M). \quad (16)$$

**Proof.** It suffices to prove the result in $\mathcal{H}_m$, since the maps $\text{Rel}_v$ are $\text{Mod}(S, \Sigma)$-equivariant. Note that (ii) follows immediately from (i), so we prove (i). Define

$$\sigma : \triangle \to H^1(S, \Sigma; \mathbb{R}^2) \text{ by } \sigma(s, t) = \text{dev}(M) + sv + tw.$$

Recall that we have a developing map $\text{dev} : \mathcal{H}_m \to H^1(S, \Sigma; \mathbb{R}^2)$. The developing map is a local homeomorphism. This does not imply that paths can be lifted but it does mean that when paths can be lifted the lifts are unique (see [E] for more information). The hypothesis that $\text{Rel}_{sv+tw}(M)$ is defined for all $(s, t) \in \triangle$ means that every path $r \mapsto \sigma(rs, rt)$ for $0 \leq r \leq 1$ lifts to $\mathcal{H}_m$. Let $\tilde{\sigma}(s, t) = \text{Rel}_{sv+tw}(M)$
be the lift of $\sigma$ to $\mathcal{H}_m$. Arguing as in Proposition 1.11 in [E] we see that $\tilde{\sigma}$ is a continuous map. By construction $\tilde{\sigma}(0, 1) = \text{Rel}_w(M)$ and $\tilde{\sigma}(1, 1) = \text{Rel}_v+\sigma(M)$. The path $\rho_0(r) = \tilde{\sigma}(r, 1)$ satisfies $\rho_0(0) = \text{Rel}_w(M)$ and $\rho'_0 = v$, so by the definition of Rel, $\rho_0(1) = \text{Rel}_v(\text{Rel}_w(M))$. Also $\rho_0(1) = \tilde{\sigma}(1, 1) = \text{Rel}_v+\sigma(M)$, and (15) follows.

In the application of Ratner’s techniques to strata in Proposition 9.4 we need to deal with the following situation. We have $M_j \to M^{(1)} \in \mathcal{H}$ and $\text{Rel}_{v_j}(g_j M_j) \to M^{(2)}$. We know that $g_j \to g_{\infty}$ and that $v_j \to v_{\infty}$ and we would like to conclude that $\text{Rel}_{v_{\infty}}(g_{\infty} M^{(1)}) = M^{(2)}$. This assertion does not hold in general but it does follow with one additional assumption.

**PROPOSITION 4.6.** Say that $M_j \to M^{(1)} \in \mathcal{H}$, $v_j \to v_{\infty}$ in $\mathfrak{R}$, $g_j \to g_{\infty}$ in $G$ and $\text{Rel}_{v_j}(g_j M_j) \to M^{(2)}$. If $\text{Rel}_{v_{\infty}}(g_{\infty} M^{(1)})$ exists then $\text{Rel}_{v_{\infty}}(g_{\infty} M^{(1)}) = M^{(2)}$.

**PROOF.** The continuity of the $G$ action as a map from $G \times \mathcal{H}$ to $\mathcal{H}$ implies that $g_j M_j \to g_{\infty} M^{(1)}$. If $\text{Rel}_{v_{\infty}}(g_{\infty} M^{(1)})$ exists then according to Proposition 4.3, there are neighborhoods of $U$ of $g_{\infty} M^{(1)}$ and $V$ of $v_{\infty}$ for which Rel is defined and continuous as a function on $U \times V$. It follows that $\text{Rel}_{v_j}(g_j M_j)$ converges to $\text{Rel}_{v_{\infty}}(g M^{(1)})$. Since $\text{Rel}_{v_j}(g_j M_j)$ converges to $M^{(2)}$ by hypothesis, we have $\text{Rel}_{v_{\infty}}(g_{\infty} M^{(1)}) = M^{(2)}$. □

4.2. **Real Rel.** Let us write $\mathbb{R}^2$ as $\mathbb{R}_x \oplus \mathbb{R}_y$. We then write

\[ H^1(S, \Sigma; \mathbb{R}^2) \cong H^1(S, \Sigma; \mathbb{R}_x) \oplus H^1(S, \Sigma; \mathbb{R}_y), \]

and we refer to $H^1(S, \Sigma; \mathbb{R}_x)$ as the horizontal space. Let $Z$ denote the intersection of $\mathfrak{R}$ and the horizontal space. We will refer to $Z$ as real Rel. Since the subgroup $B$ of $\text{SL}_2(\mathbb{R})$ preserves the horizontal directions $\mathbb{R}_x$, its action on $\mathfrak{R}$ leaves $Z$ invariant.

A special case which will concern us here are strata with two singularities. In this case $\mathfrak{R}$ can be identified with $\mathbb{R}^2$, and we make the identification explicit. Label the singularities of the model surface $S$ by $\xi_1$ and $\xi_2$, we will identify $\mathfrak{R}$ with $\mathbb{R}^2$ as follows: a cochain $v : H_1(S, \Sigma) \to \mathbb{R}^2$ which vanishes on cycles represented by closed curves is identified with the vector $v(\delta)$ for some (any) directed path $\delta$ from $\xi_1$ to $\xi_2$. In this case $Z$ is one dimensional, and we write $\text{Rel}_v(M)$ for $\text{Rel}_{v}(M)$, where $v = (t, 0) \in \mathbb{R}^2 \cong \mathfrak{R}$ via the identification above.

Figures 3 and 4 show the effect of flowing along the real Rel vector field on a decagon with opposite sides identified. In Figure 3 the flow has the effect of shortening the top saddle connection. The flow cannot be continued past the point at which the length of the top saddle connection shrinks to zero. In Figure 4 the lengths of saddle connections are preserved since they connect vertices of the same color, and hence represent a saddle connection from a singularity to itself. In this case the flow can be continued for all time.

**DEFINITION 4.7.** Let $v \in \mathfrak{R}$. We denote by $\mathcal{H}_v'$ the set of $M \in \mathcal{H}$ for which $\text{Rel}_v(M)$ is defined.

Proposition 4.5(i), with $w = -v$, implies $\text{Rel}_{-v} \circ \text{Rel}_v(M) = M$ for $M \in \mathcal{H}_w'$, and this yields a useful equivariance property:

\[ \text{Rel}_v(\mathcal{H}_w') = \mathcal{H}_{-v}. \]

In the case of two singularities keeping in mind our convention regarding the identification $\mathfrak{R} \cong \mathbb{R}^2$, we will continue to use the notation $\mathcal{H}_v'$ for $v \in \mathbb{R}^2$. Then
Proposition 4.4 implies that for \( v \in \mathbb{R}^2 \) and \( g \in G \) we have \( g(\mathcal{H}'_v) = \mathcal{H}'_{gv} \), where \( gv \) is the image of \( v \) under the linear action of \( g \) on \( \mathbb{R}^2 \). We now introduce some more notation for discussing directions belonging to \( Z \).

**Definition 4.8.** We define \( \mathcal{H}'_z \) to be \( \bigcap_{z \in Z} \mathcal{H}'_z \) (i.e. the set of \( M \in \mathcal{H} \) on which \( \text{Rel}_z \) is defined for all \( z \in Z \)).

When \( \mathcal{H} \) is a stratum with two singularities and \( t \) is a real number let \( \mathcal{H}'_v \) denote the set \( \mathcal{H}'_v \) with \( v = (t, 0) \) (i.e. the subset of \( M \in \mathcal{H} \) on which \( \text{Rel}_t \) is defined).

**Definition 4.9.** For a fixed \( M \in \mathcal{H} \), let
\[
\mathcal{Z}^{(M)} = \{ z \in Z : M \in \mathcal{H}'_z \}
\]
(i.e., the subset of \( Z \) corresponding to surgeries which are defined on \( M \)). Thus
\[
\mathcal{H}'_{\mathcal{Z}} = \{ M \in \mathcal{H} : \mathcal{Z}^{(M)} = Z \}.
\]

Recall that if \( V \) is a vector space and \( V_0 \subset V \), we will say that \( V_0 \) is a *star body* if
\[
v \in V_0, s \in [0, 1] \implies sv \in V_0.
\]
We denote the convex hull of a subset \( W \subset V \) by conv \( W \).

**Proposition 4.10.** The set \( \mathcal{Z}^{(M)} \) has the following properties:
(i) It is an open star body in $Z$.
(ii) If $b \in B$ then $Z(bM) = b(Z(M))$.
(iii) If $z \in Z(M)$ then $-z \in Z(\text{Rel}_z(M))$.
(iv) If $z, z' \in Z$ and $\text{conv}\{0, z, z'\} \subset Z(M)$, then $\text{Rel}_z(M)$ and $\text{Rel}_{z'}(\text{Rel}_z(M))$ are defined and $\text{Rel}_{z'}(\text{Rel}_z(M)) = \text{Rel}_{z + z'}(M)$.

**Proof.** The fact that $Z(M)$ is open follows from Proposition 4.3. The fact that it is a star body is immediate from Definition 4.2. This proves (i). Assertions (ii), (iii), (iv) follow respectively from Propositions 4.4, (18), and 4.5(i). □

We will need a significant strengthening of Proposition 4.10:

**Proposition 4.11.** For any $M$, the set $Z(M)$ is convex.

Whereas the proof of Proposition 4.10 relies only on general principles, Proposition 4.11 relies on additional information about Rel and will be proved in §6.

Recall that $G$ acts on $H^1(S, \Sigma; \mathbb{R})$ via its linear action on the coefficients $\mathbb{R}^2$. Since the linear action of $U$ on $\mathbb{R}^2$ preserves horizontal vectors, it fixes elements of $\mathbb{R}_x$. This implies that real Rel commutes with the horocycle flow. Namely, by Proposition 4.4, if $z \in Z$ and $u \in U$ then $uz = z$, and hence $u(\mathcal{H}'_z) = \mathcal{H}'_z$, and $u(\text{Rel}_z(M)) = \text{Rel}_z(uM) = \text{Rel}_z(uM)$, for $M \in \mathcal{H}'_z$.

**Definition 4.12.** Now define

$$N = \{(b, z) : b \in B, z \in Z\} \quad \text{and} \quad L = \{(g, v) : g \in G, v \in \mathcal{R}\}.$$

We equip $N$ and $L$ with the natural group structures as semi-direct products $N = B \ltimes Z$ and $L = G \ltimes \mathcal{R}$, which are compatible with their actions on period coordinates. For the action of the first factor on the second in this semi-direct product $L$, we take the natural action of $G$ on $\mathcal{R}$, and its restriction to the $B$-invariant subspace $Z$. In particular in the case of two singularities, $G$ acts on $\mathcal{R} \cong \mathbb{R}^2$ via its standard linear action, and $B$ acts on $Z \cong \mathbb{R}_x$ via the restriction of its linear action on $\mathbb{R}_x^2$, to the horizontal axis. We will write this semidirect product group law explicitly as

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1g_2, v_1 + g_1v_2) \quad \text{for} \quad g_i = (g_i, v_i) \in L, \quad i = 1, 2,$$

where in the expression $g_1v_2$ we mean the action of $G$ on $\mathcal{R}$ described above. We can associate a partially defined transformation of $\mathcal{H}$ to each element of $L$ as follows:

$$gM + v = \text{Rel}_v(gM) \quad \text{when} \quad (g, v) \in L \quad \text{and} \quad gM \in \mathcal{H}'_z.$$

We will use a different notation for the restriction to $N$. Namely we write

$$nM = \text{Rel}_z(bM) \quad \text{when} \quad n = (b, z) \in N \quad \text{and} \quad bM \in \mathcal{H}'_z.$$

Note that (19) and (20) give two different notations for the same transformations. The reason for this is a fundamental difference between the behavior of the same operation on $N$ and on $L$. When dealing with all of $L$, these operations need not obey a group action law. Indeed, it may happen that $(M + v) \ast w \neq (M + w) \ast v$. However, for $N$ we have the following weak form of a group action law:

**Proposition 4.13.** Let $n_1 = (b_1, z_1)$ and $n_2 = (b_2, z_2)$ be two elements of $N$. Suppose that $n_2M$ and $n_1(n_2M)$ are defined. Then $(n_1n_2)M$ is defined, and $n_1(n_2M) = (n_1n_2)M$. 


Using Proposition 4.10 we find that
\[(21) \quad \text{conv}\{0, -z_2, b_1^{-1}z_1\} \subset Z^{(n_2M)}.
\]

Then
\[(22) \quad n_1(n_2(M)) = \text{Rel}_{z_1}(b_1n_2(M))
\]
\[(23) \quad = \text{Rel}_{b_1^{-1}z_1}(n_2(M))
\]
\[(24) \quad = \text{Rel}_{b_1^{-1}z_1+z_2} \circ \text{Rel}_{-z_2}(n_2(M))
\]
\[(25) \quad = \text{Rel}_{b_1^{-1}z_1+z_2}(b_2(M))
\]
\[(26) \quad = \text{Rel}_{z_1+b_1z_2}(b_1b_2(M))
\]
\[= (n_1n_2)(M).
\]

Here Proposition 4.4 is used to derive (23) from (22) and to derive (26) from (25), and (24) is obtained from (23) by Proposition 4.10 and (21). □

The following is immediate from Propositions 4.3, 4.10 and 4.13 (and was proved previously in [MW2]):

**Corollary 4.14.** The set \(H_x'\) is \(N\)-invariant. The map \(N \times H_x' \to H_x'\) defined by \((ℓ, M) \mapsto ℓM\) defines a continuous action of \(N\) on \(H_x'\). The \(Z\)-orbits in this action are the real \(\text{Rel}\) leaves in \(H_x'\).

The following will be useful. For \(u \in U\) and \(n = (b_n, z_n) \in N\), write \(u^{(b_n)} = b_nb^{-1}\), so that \(nu = u^{(b_n)}n\) as elements of \(N\). Then we have:

**Corollary 4.15.** For any \(u \in U\), \(n \in N\) and \(M \in H\), if \(n(uM)\) is defined then \(nM\) is defined and \(n(uM) = u^{(b_n)}(nM)\).

**Proof.** We apply Proposition 4.13 with \(b_2 = u\) and \(z_2 = 0\). □

### 4.3. The stabilizer group of a measure

Proposition 4.10 implies the invariance property \(Z^{(uM)} = Z^{(M)}\). We now extend this to semi-direct products and use it to define the stabilizer of a measure, within a collection of partially defined transformations.

For fixed \(M \in H\) we write
\[N^{(M)} = \left\{(b, z) \in N : z \in Z^{(bM)}\right\}.
\]

(i.e. the set of \(n \in N\) for which \(nM\) is defined). Then it follows from Proposition 4.3 that \(N^{(M)}\) is open for each \(M\), and Corollary 4.15 implies \(N^{(uM)} = N^{(M)}\).

**Proposition 4.16.** Suppose \(M\) is in the \(U\) orbit-closure of \(M'\). Then \(N^{(M)} \subset N^{(M')}\).

**Proof.** If \(n \in N^{(M)}\) then by Proposition 4.3 there is a neighborhood \(W\) of \(M\) in \(H\) such that \(n \in N^{(M_1)}\) for any \(M_1 \in W\). Let \(u \in U\) such that \(uM' \in W\). Then \(z \in N^{(uM')} \subset N^{(M')}\). □

We also need the following:
Proposition 4.17. Given an ergodic $U$-invariant measure $\mu$ there is a subset $\Omega \subset H$ such that $\mu(\Omega) = 1$ and for any $M_1, M_2 \in \Omega$, $Z^{(M_1)} = Z^{(M_2)}$ and $N^{(M_1)} = N^{(M_2)}$.

Proof. To explain the idea, we first prove the assertion for $Z^{(M)}$ in case $\dim Z = 1$. In this case, $Z^{(M)}$ is an open interval for every $M \in H$, which we can write as

$$Z^{(M)} = (a_M, b_M), \text{ for some } -\infty \leq a_M < b_M \leq \infty.$$  

The maps $M \mapsto a_M, M \mapsto b_M$ are measurable maps with values in the extended real line so by ergodicity, are constant $\mu$-a.e., and the statement follows.

In the general case the proofs for $Z^{(M)}$ and $N^{(M)}$ are identical, so we discuss the case of $Z^{(M)}$. The map $M \mapsto Z^{(M)}$ is a map from $H$ to the collection of open subsets of $Z$. Rather than worry about measurability issues, we proceed as follows. Let $Z_0$ be a dense countable subset of $Z$. For each $z \in Z_0$, the set

$$\Omega_z = \{M \in H : z \in Z^{(M)}\}$$  

is a measurable invariant set, so has measure 0 or 1 by ergodicity. We define

$$\Omega = \bigcap_{\mu(\Omega_z) = 1} \Omega_z \setminus \bigcup_{\mu(\Omega_z') = 0} \Omega_z',$$

where $z, z'$ range over the countable set $Z_0$. Then $\mu(\Omega) = 1$, and for any $M \in \Omega$,

$$Z^{(M)} \cap Z_0 = \{z \in Z_0 : \mu(\Omega_z) = 1\},$$  

does not depend on $M$. Since an open set is the interior of the closure of its intersection with a dense set, we see that $Z^{(M)} = Z^{(M')}$ for any $M_1, M_2 \in \Omega$.  

We will denote by $Z^{(\mu)}$ and $N^{(\mu)}$ the sets $Z^{(M)}, N^{(M)}$ which appear in Proposition 4.17 for $M \in \Omega$. If $z \in Z^{(\mu)}$ we define a pushforward

$$Rel_z(\mu)(X) = \mu(\text{Rel}_{-z}(X \cap H'_z)), \text{ for all measurable } X \subset H.$$  

Note that $\mu(H'_z) = 1$, and now it follows from (18) that $Rel_z(\mu)$ is a probability measure. Moreover, the partially defined map $Rel_z$ is a measurable conjugacy between $(H, \mu)$ and $(H, Rel_z(\mu))$, thought of as dynamical systems for the $U$-action.

Thus $Rel_z(\mu)$ is again an ergodic $U$-invariant measure. The same observations are valid for $n \in N^{(\mu)}$. Namely, if we denote

$$H'_n = \{M \in H : n(M) \text{ is defined}\} = \{M \in H : z \in Z^{(bM)}\}$$

(where $n = (b, z)$), then we have an equivariance property

$$n(H'_n) = H'_{n-1}$$

and we can define an ergodic $U$-invariant measure

$$n_*(\mu)(X) = \mu(n^{-1}(X \cap H'_{n-1})), \text{ for all measurable } X \subset H.$$  

Corollary 4.15 now implies that $n_*\mu$ is a $U$-invariant measure, and the partially defined map $M \mapsto n(M)$ is equivariant for the action of $U$ on $(H, \mu)$ and the “time-changed” action of $U$ on $(H, n_*\mu)$ via

$$u \cdot M = u^{(n)}M, \text{ for } n_*\mu \text{ a.e. } M$$

(where $u^{(n)} = nun^{-1}$).
The collection of Borel probability measures on a locally compact space $X$ can be given the weak-$*$ topology by embedding it in the dual space of the space $C_c(X)$ of continuous functions with compact support.

**Proposition 4.18.** Let $\mu$ be an ergodic $U$-invariant probability measure. The map which takes $n \in N(\mu)$ to $n_\# \mu$ is continuous with respect to the weak-$*$ topology.

**Proof.** Let $n_j$ be a sequence of elements of $N(\mu)$ converging to $n_x \in N(\mu)$. In order to show that $n_j_\# \mu$ converges to $n_x_\# \mu$ we need to show that for any continuous compactly supported function $\varphi$ on $\mathcal{H}$, we have

$$
\lim_{j \to \infty} \int_{\mathcal{H}} \varphi(n_j_\# \mu) = \int_{\mathcal{H}} \varphi(n_x_\# \mu).
$$

For each $j$, the set $\mathcal{H}'_{n_j}$ has full $\mu$-measure, and hence so does $\mathcal{H}'_0 = \bigcap_{1 \leq j \leq \infty} \mathcal{H}'_{n_j}$.

Now we have:

$$
\lim_{j \to \infty} \int_{\mathcal{H}} \varphi(n_j_\# \mu) = \lim_{j \to \infty} \int_{\mathcal{H}_{n_j}(\mathcal{H}_0')} \varphi(n_j_\# \mu)
$$

$$
= \lim_{j \to \infty} \int_{\mathcal{H}_0'} \varphi \circ n_j \, d\mu
$$

$$
= \int_{\mathcal{H}_0'} \varphi \circ n_x \, d\mu
$$

$$
= \int_{\mathcal{H}_0'} \varphi \, d(n_x_\# \mu)
$$

$$
= \int_{\mathcal{H}} \varphi \, d(n_x_\# \mu).
$$

The equalities (29) and (34) follow from the fact that each $n_j_\# \mu$ assigns full measure to $n_j(\mathcal{H}_0')$ respectively. The lines (30) and (33) follow from the definition of the pushforward of a measure. Line (31) follows from Lebesgue’s Dominated Convergence Theorem using the fact that, since $f$ has compact support, it is bounded and hence the family of functions $f \circ n_j$ is uniformly bounded and also using the fact that constant functions are in $L^1(\mu)$ since $\mu$ is a probability measure. Line (32) follows from Proposition 4.3 and the continuity of $\varphi$. □

**Definition 4.19.** For any ergodic $U$-invariant measure $\mu$ we define

$$
N_\mu = \left\{ n \in N(\mu) : n_\# \mu = \mu \right\}.
$$

**Corollary 4.20.** $N_\mu$ is a closed subgroup of $N$.

**Proof.** The fact that $N_\mu$ is closed follows from Proposition 4.18, and in order to prove that $N_\mu$ is closed under compositions, it suffices to show that for $\mu$-almost every $M$, $n_1(n_2M) = (n_1n_2)M$. This follows from Proposition 4.13. □

**Proposition 4.21.** If $N_\mu$ contains a non-unipotent element (i.e. an element in $N \setminus UZ$) and $\dim Z = 1$ then $N_\mu \cap Z$ is connected. In particular, if $N_\mu$ contains
a non-unipotent element of \( N \) and a nontrivial element of \( Z \) then it contains all of \( Z \).

**Proof.** Write \( Z_1 = UZ \cong \mathbb{R}^2 \) and \( N_1 = N_\mu \cap Z_1 \), and let \( a \in N_\mu \setminus N_1 \). By Corollary 4.20, \( N_\mu \) is a closed subgroup of \( N \), and hence \( N_1 \) is a closed subgroup of \( Z_1 \) containing \( U \). If it is not connected then there is a minimal positive distance between two distinct cosets for \( N_1 \) in \( Z_1 \). However \( N_1 \) is invariant under conjugation by the elements \( a \) and \( a^{-1} \), and one of these acts on \( Z_1 \) by contractions. This contradiction proves the claim. \( \square \)

### 4.4. Generic points and real Rel.

Recall that if \( \mu \) is a \( U \)-invariant ergodic probability measure on a closed subset \( L \) of a stratum \( \mathcal{H} \), then \( M \in \mathcal{L} \) is said to be **generic for \( \mu \)** if for any continuous compactly supported function \( f \) on \( L \), we have

\[
\lim_{T \to \pm \infty} \frac{1}{T} \int_0^T f(u_s M) \, ds = \int f \, d\mu.
\]

**Definition 4.22.** Let \( \mu \) be an invariant \( U \)-ergodic measure on \( \mathcal{H} \). Then \( \Omega_\mu \) denotes the set of generic points for \( \mu \).

We collect some well-known facts about generic points.

**Proposition 4.23.** If \( \mu \) is ergodic then \( \Omega_\mu \) has full \( \mu \) measure. If \( \mu \neq \nu \) then \( \Omega_\mu \) and \( \Omega_\nu \) are disjoint. If \( M \in \Omega_\mu \) then the support of \( \mu \) is contained in the orbit closure of \( M \).

In light of Proposition 4.16, this implies:

**Corollary 4.24.** If \( M \in \Omega_\mu \) then \( N^{(\mu)}(M) \subset N^{(M)}(\mu) \).

It will prove useful to reformulate the notion of genericity in terms of convergence of measures. It is important to keep in mind that we are dealing with spaces which are locally compact but not compact. The weak-* topology on the space of Borel measures is defined in terms of integrals of continuous functions of compact support. In this topology the total mass of a measure need not be a continuous function on the space of measures since the constant function 1 need not have compact support.

**Definition 4.25.** Let \( \nu(M,T) \) be the probability measure defined by

\[
\int \varphi \, d\nu(M,T) = \frac{1}{T} \int_0^T \varphi(u_s M) \, ds, \quad \text{for all } \varphi \in C_c(\mathcal{H}).
\]

Then \( M \) is generic for \( \mu \) if

\[
\lim_{T \to \pm \infty} \nu(M,T) = \mu
\]

where the limit is taken with respect to the weak-* topology on measures on \( \mathcal{H} \).

In preparation for Proposition 4.27 below, we collect some results about integrals of continuous bounded functions along orbits. The space of probability measures on a locally compact space can be given another topology by embedding it in the dual space of \( C_b(X) \), the space of continuous bounded functions on \( X \). This topology is called the strict topology (see [Bo] for more information). Clearly convergence in the strict topology implies convergence in the weak-* topology. The following result gives a simple criterion for showing that weak-* convergence implies strict convergence.
Proposition 4.26 ([Bo] Prop. 9, p. 61). Suppose X is a locally compact space, \( \mu_j \) is a sequence of probability measures on X, and \( \mu \) is also a measure on X. If \( \mu_j \to \mu \) with respect to the weak-* topology, and if \( \mu \) is a probability measure, then \( \mu_j \to \mu \) with respect to the strict topology.

If the limit measure \( \mu \) is not a probability measure then it has total mass less than one. This phenomenon is referred to as ‘loss of mass’. In §10 we will give conditions that establish that loss of mass does not occur for measures \( \nu(M,T) \).

We will repeatedly use the following:

Proposition 4.27. Let \( \mu \) be an ergodic \( U \)-invariant measure. If \( n \in N^{(\mu)} \) then \( n(\Omega_{\mu}) \) is the set of generic points for \( n_\mu \).

Proof. Let \( M \in \Omega_{\mu} \). According to Corollary 4.24, \( nM \) is defined. We use the reformulation of genericity in terms of weak-* convergence of measures given above. Thus we are given that \( \lim_{T \to \infty} \nu(M,T) = \mu \) with respect to the weak-* topology on measures on \( \mathcal{H} \) and we want to show that \( \lim_{T \to \infty} \nu(nM,T) = n_\mu \) with respect to the same topology.

Since \( N \) normalizes \( U \), there is \( c > 0 \) such that \( n_{cs} = u_{cs} n \). We have \( n_{s} \nu(M,T) = \nu(nM,cT) \). Thus it suffices to show:

\[
\lim_{T \to \infty} n_{s} \nu(M,T) = n_{s} \mu.
\]

Let \( \mathcal{H}'_1 = \mathcal{H}'_n \) as in (27) and let \( \mathcal{H}'_2 = n(\mathcal{H}'_1) = \mathcal{H}'_{n-1} \). Let \( \mathfrak{M} \) denote the set of probability measures on \( \mathcal{H} \) which assign mass 1 to \( \mathcal{H}'_1 \), and let \( \mathfrak{N} \) denote the set of probability measures on \( \mathcal{H} \) which assign mass 1 to \( \mathcal{H}'_2 \). By Proposition 4.3, \( \mathcal{H}'_1 \) and \( \mathcal{H}'_2 \) are open subsets of \( \mathcal{H} \), and in particular are locally compact. Moreover any continuous compactly supported function on \( \mathcal{H}'_1 \) extends to a continuous function on \( \mathcal{H} \) by setting it equal to zero on \( \mathcal{H} \setminus \mathcal{H}'_1 \). It follows that \( \lim_{T \to \infty} \nu(M,T) = \mu \) with respect to the weak-* topology on \( \mathfrak{M} \).

Since \( n \) is a homeomorphism from \( \mathcal{H}'_1 \) to \( \mathcal{H}'_2 \), the map \( n_\mu : \mathfrak{M} \to \mathfrak{N} \) is continuous and \( \lim_{T \to \infty} n_{s} \nu(M,T) = n_{s} \mu \) with respect to the weak-* topology on \( \mathfrak{N} \). Since \( n_{s} \mu \) is a probability measure which assigns full measure to \( \mathcal{H}'_2 \), and since \( \mathcal{H}'_2 \) is locally compact, we can use Proposition 4.26 to conclude that \( \lim_{T \to \infty} n_{s} \nu(M,T) = n_{s} \mu \) in the strict topology on measures on \( \mathcal{H}'_2 \).

We need to show that convergence also holds with respect to the weak-* topology on measures on \( \mathcal{H} \). Given a function \( \varphi \) on \( \mathcal{H} \) which is continuous and compactly supported, its restriction to \( \mathcal{H}'_2 \) is a bounded continuous function. Therefore

\[
\lim_{T \to \infty} \int_{\mathcal{H}'_2} \varphi \, dn_{s} \nu(M,T) = \int_{\mathcal{H}'_2} \varphi \, dn_{s} \mu
\]

by strict convergence which gives

\[
\lim_{T \to \infty} \int_{\mathcal{H}} \varphi \, dn_{s} \nu(M,T) = \int_{\mathcal{H}} \varphi \, dn_{s} \mu,
\]

and establishes convergence with respect to the weak-* topology on measures on \( \mathcal{H} \).

Corollary 4.28. If \( \mu \) is an ergodic \( U \)-invariant probability measure and if \( M_1 \) and \( M_2 \) are in \( \Omega_{\mu} \) and there is an element \( n \in N^{(\mu)} \) of \( N \) such that \( M_2 = nM_1 \), then \( n \in N_{\mu} \).
PROOF. Let $M_1$ and $M_2$ be generic for $\mu$ and $M_2 = nM_1$ where $n \in N$. According to Proposition 4.27, $\mu$ and $n\ast\mu$ share a generic point and hence, by Proposition 4.23, they must coincide. \qed

5. Horizontal equivalence of surfaces

Given $M \in H$ with singularity set $\Sigma$, we denote by $\Xi(M)$ the set of horizontal saddle connections for $M$. We would like to use $\Xi(M)$ to define two equivalence relations on surfaces: topological horizontal equivalence and geometrical horizontal equivalence. These will serve as invariants of ergodic $U$-invariant measures in the same way that the sets $Z(\mu)$ and $N(\mu)$ appearing in Proposition 4.17 do. Let $M$ be the blowup of $M$ and let $(\hat{f}, \hat{M}) \in \hat{H}$ be a marking of $\hat{M}$ rel boundary. Then $\hat{f}^{-1}(\Xi(M))$ is a subset of $\hat{S}$, which we denote by $\hat{f}^*(\Xi)$. We take $\Xi(M)$ to include all of the points of $\Sigma$ and hence $\hat{f}^*(\Xi)$ contains the boundary $\partial\hat{S}$. In addition, for each edge of $\Xi(M)$, $\hat{f}^* (\Xi)$ contains an edge in the interior of $S$, which intersects the boundary $\partial S$ at points with angular parameters which are multiples of $\pi$, even or odd according as the point is the initial or terminal point of the edge.

**Definition 5.1.** We will say that $M_1$ and $M_2$ are topologically horizontally equivalent if there are markings rel boundary $(\hat{f}_i, \hat{M}_i)$ such that $\hat{f}_i^*(\Xi)$ and $\hat{f}_i^*(\Xi)$ can be obtained from each other by an isotopy of $\hat{S}$ that does not move points of $\partial\hat{S}$. We will say that $M_1$ and $M_2$ are geometrical horizontally equivalent if they are topologically horizontally equivalent, and if for any edge $\delta$ in $\hat{f}_i^*(\Xi(M_i))$, $\text{hol}(M_1, \hat{f}_1(\delta)) = \text{hol}(M_2, \hat{f}_2(\delta))$, where $\hat{f}_1$ and $\hat{f}_2$ are as in the definition of topological equivalence.

**Proposition 5.2.** If $M \in H$, $b \in B$ and $u \in U$ then $M$ and $bM$ are topologically horizontally equivalent, and $M$ and $uM$ are geometrically horizontally equivalent.

**Proof.** Since $B$ acts by postcomposition on the charts, and preserves the horizontal direction, it preserves the set $\Xi(M)$ (possibly changing the lengths of edges), and hence $M$ and $bM$ are topologically equivalent. Moreover if $b = u \in U$ then the lengths of edges are unaffected and so $M$ and $uM$ are geometrically horizontally equivalent. \qed

**Corollary 5.3.** If $\mu$ is an ergodic $U$-invariant measure on $H$, then there is a subset $X \subset H$ of full $\mu$-measure such that for any $M_1, M_2 \in X$, $M_1$ and $M_2$ are geometrically horizontally equivalent.

We will define a combinatorial invariant of topological horizontal equivalence – the horizontal data diagram. This diagram is an analog of the separatrix diagram introduced by Kontsevich and Zorich [KoZo] and captures some of the properties of $\Xi(M)$ which depend only on the class of $M$. Note that a graph embedded on a translation surface has a ribbon graph structure, namely at each vertex it inherits from the surface a cyclic order of edges incident to the vertex. Its vertices are labeled by the labels $\xi_1, \ldots, \xi_k$ of the corresponding singularities. If the graph consists of horizontal saddle connection, each edge inherits an orientation from the translation structure and we take these orientations as part of the horizontal data diagram structure.

For any $(\hat{f}, \hat{M}) \in \hat{H}$, $\hat{f}^*(\Xi)$ is a graph embedded in $\hat{S}$. By projecting it to $S$ we thus obtain a ribbon graph with labeled singularities and oriented edges. The graphs $\hat{f}^*(\Xi)$ carry additional information, namely the angular distance separating
consecutive edges incident at a vertex. This angle is an integer multiple of $\pi$, and the orientations of consecutive edges agree if and only if this distance is an even multiple of $\pi$. It is clear that all of this information is an invariant of geometric horizontal equivalence.

The horizontal data diagram of $M$ records the graph structure induced by $\tilde{f}^*(\Xi)$, as well as the orientation of edges, labeling of vertices, and cyclic structure at each singularity. In order to record the information of angular separation of edges at vertices, we indicate as dashed lines additional left and right pointing horizontals. Thus at the singularity $\xi_i$ there are $2(a_i + 1)$ prongs, some of which correspond to edges of the graph. Figures 5, 6 and 7 depict graphs which can occur as $\Xi(M)$ for some $M$ in $\mathcal{H}(1,1)$.

![Figure 5. A horizontal data diagram on a blown up surface.](image)

In Figure 6 the angle between any two adjacent ends is exactly $\pi$, but this is not the case in Figure 7. We capture this angle information in the diagrams in Figure 7 by inserting additional dotted lines at each vertex indicating the ends of horizontal separatrices which are not saddle connections. We can determine the angle between two prongs by counting the number of ends of separatrices between them. Note that it makes sense to give orientations to separatrices so that the orientations of ends at a given vertex alternate with respect to the cyclic ordering.

It follows from Corollary 5.3 that for any ergodic $U$-invariant measure $\mu$, there is a subset $X$ of $\mathcal{H}$ such that $\mu(X) = 1$ and every $M \in X$ has the same horizontal data diagram. We call it the horizontal data diagram of $\mu$ and denote it by $\Xi(\mu)$.

If a horizontal data diagram is maximal, i.e. at each vertex, all prongs are initial or terminal prongs of edges, then the horizontal data diagram coincides with the separatix diagram of $[\text{KoZo}]$.

We will sometimes need an extension of $\Xi(M)$. Recall that at each $\xi_i \in \Sigma$ there are $2(a_i + 1)$ prongs. Some of these are part of saddle connections in $\Xi(M)$, and we call the others the unoccupied prongs of $\Xi(M)$. Let $\Xi_-(M)$ denote the graph which contains $\Xi(M)$ and has one additional prong edge for every unoccupied prong, with one vertex at a singular point and one vertex at a point of $M \setminus \Sigma$, where the edge is realized by a path in $M$ which lies on the horizontal separatrix issuing at the corresponding prong.
Figure 6. List of all maximal horizontal data diagrams in $\mathcal{H}(1,1)$ up to switching labels and orientations. See Figure 4 for a polygonal presentation of a cylinder decomposition of type A.

Figure 7. These figures represent some non-maximal horizontal data diagrams in $\mathcal{H}(1,1)$.

We will sometimes need to be specific about the length of edges. In that case, given $L > 0$, we will let $\Xi_\cdot(M,L)$ denote the graph $\Xi_\cdot(M)$ described above, where the prong edges all have length $L$. Note that since all of the prong edges are part of an infinite separatrix in the horizontal direction, the graph $\Xi_\cdot(M,L)$ can be embedded in $M$ for any $L$.

6. An explicit surgery for real $\text{Rel}$

In this section, for fixed $M \in \mathcal{H}$ and $z \in Z^{(M)}$, we will present an explicit presentation of $\text{Rel}_z M$ in terms of glued polygons. Our explicit surgery generalizes the special cases treated in [McM8] and [B2, §3]. See also the general result in [MW2].

As a by-product, it will enable us to determine $Z^{(M)}$ explicitly from the geometry of $M$. This analysis makes it possible to analyze limits $\lim_{j \to \infty} \text{Rel}_{z_j} M$, for $z_j \in Z^{(M)}$ with $\lim_{j \to \infty} z_j \in \partial Z^{(M)}$. Such limits do not exist as elements of $\mathcal{H}$; loosely speaking they belong to a bordification of $\mathcal{H}$ obtained by adjoining boundary strata. We will not construct this bordification in this paper but hope to return to it in future work. A particularly simple case of this bordification arises when one takes limits of surfaces in $\mathcal{H}(1,1)$ for which a segment joining the two singularities collapses. Even in this relatively simple case, which will arise in Theorem 6.5, the construction we use differs from earlier related work (see [KoZo, EMZ]) and leads naturally to the use of framed surfaces.
We will establish some terminology and make a construction which will be used in stating Theorem 6.1. Let $L$ and $\varepsilon$ be positive numbers. Given $M$, we let $\Xi(M, L)$ and $\Xi(M, L, L)$ be as above. We say that $N = N(L, \varepsilon) \subset M$ is the $(L, \varepsilon)$-rectangle thickening of $\Xi(M, L)$ if $N$ is a union of rectangles $R^+_e$ and $R^-_e$ in $M$, with sides parallel to the coordinate axes, where $e$ ranges over the edges of $\Xi(M, L)$, $R^+_e$ has vertical sides of length $\varepsilon$, and the edge $e$ runs along the bottom of $R^+_e$ and the top of $R^-_e$. Here the words ‘bottom’ and ‘top’ refer to the orientation provided by the translation surface structure and need not correspond to the directions shown in our figures. The edge identification maps of the rectangles are inherited from $\Xi(M, L)$, namely they are as follows. Each $R^-_e$ is attached to $R^+_e$ along $e$, and the bottom of $R^-_e$ (resp. top of $R^+_e$) is unattached. If the right end of $e$ is not a singularity (i.e. $e$ is right-pointing prong edge) then the right hand boundaries of $R^-_e$ are unattached. If the right end of $e$ is the singularity $\xi$ then the right hand boundary of $R^-_e$ (resp. $R^+_e$) is unattached at $\xi$. For the gluing rule for the left edges, replace left with right and clockwise with counterclockwise in the above description. See Figure 8.

Note that for any $L$ there is $\varepsilon_0 = \varepsilon_0(M, L)$ such that for all $\varepsilon < \varepsilon_0$, the $(L, \varepsilon)$-rectangle thickening of $\Xi(M)$ exists and is embedded in $M$. Moreover the constant $\varepsilon_0(M, L)$ can be chosen to be independent of $M$ and $L$, for $L$ in a bounded set of numbers and $M$ in a compact set of topologically horizontally equivalent surfaces.

![Figure 8. A rectangle thickening of $\Xi(M)$ in $\mathcal{H}(2)$ with 10 rectangles.](image)

Given translation surfaces $M, M'$, and a subset $M_0 \subset M$ we say that $M'$ can be obtained from $M$ by modifying $M_0$ if there are polygon representations of $M$ and $M'$, and a subset $M'_0 \subset M'$, such that $M_0, M'_0$ are unions of polygons, and there is a homeomorphism $f : (M \setminus M_0) \to (M' \setminus M'_0)$, which is a translation in each coordinate chart of the translation surface structures on $M$ and $M'$.

**Theorem 6.1.** Let $M \in \mathcal{H}$. For each $\delta \in \Xi(M)$, let $Z(M, \delta)$ denote the connected component of 0 in $\{z \in Z : \text{hol}(M, \delta) + z(\delta) \neq 0\}$. Then

$$Z(M) = \bigcap_{\delta \in \Xi(M)} Z(M, \delta).$$

(36)
For any $z \in Z^M$ there is $L = L_z > 0$, such that for any $\varepsilon < \varepsilon_0(M, L)$, $\text{Rel}_z M$ can be obtained from $M$ by modifying $\mathcal{N}(L, \varepsilon)$. Moreover, the function $z \mapsto L_z$ can be taken to be bounded when $z$ varies in a bounded subset of $Z^M$.

**Proof.** For $\delta \in \Xi(M)$, we have $\text{hol}(M, \delta) = (x, 0)$ for some nonzero $x \in \mathbb{R}$, and in this proof we change our notation slightly and denote this number $x$ by $\text{hol}(M, \delta)$.

First note that $Z^M(M) = \bigcap_{\delta \in \Xi(M)} Z^{(M, \delta)}$. Indeed, if $z \in Z^M(M)$ then the straight line path $\{\text{Rel}_z M : t \in [0, 1]\}$ is embedded in $\mathcal{H}$. In particular, for any saddle connection $\delta \in \Xi(M)$ and any $t \in [0, 1],$

\[
\text{hol}(M, \delta) + tz(\delta) = \text{hol}(\text{Rel}_z M, \delta) \neq 0.
\]

So the path $\{tz : t \in [0, 1]\}$ is contained in $Z^{(M, \delta)}$ and hence $z \in Z^{(M, \delta)}$.

Now let $z \in \bigcap_{\delta \in \Xi(M)} Z^{(M, \delta)}$. Recall that we may think of an element $z$ of $Z \cong H^0(\Sigma; \mathbb{R})/H^0(S; \mathbb{R})$ as being represented by a function $\tilde{z} : \Sigma \to \mathbb{R}$. All such representatives $\tilde{z}$ differ by constants, and

\[
|z_{ij}| = \tilde{z}(\xi_j) - \tilde{z}(\xi_i) = z(\delta_{ij}), \tag{37}
\]

for any path $\delta_{ij} : [0, 1] \to S$ with $\delta_{ij}(0) = \xi_i$, $\delta_{ij}(1) = \xi_j$. We fix one representative $\tilde{z}$ of $z$, and let

\[
L > \max_{i=1, \ldots, k} |\tilde{z}(\xi_i)|. \tag{38}
\]

Our assumption about $z$ implies that

\[
\text{if } \delta \text{ is from } \xi_i \text{ to } \xi_j \text{ and } \text{hol}(M, \delta) > 0, \text{ then } \text{hol}(M, \delta) + z_{ij} > 0. \tag{39}
\]

Let $\varepsilon < \varepsilon_0(M, L)$ and let $\mathcal{N}_0 = \mathcal{N}(L, \varepsilon)$. Choose a marking $(f, M) \in \mathcal{H}_m$ of $M$. For each $t \in [0, 1]$, we will define a translation surface $\mathcal{M}_t$ and a marking $f_t : S \to \mathcal{M}_t$, as follows. To each rectangle $R = R^\pm_\varepsilon$ in $\mathcal{N}_0$ we define a trapezoid $R_t = R^\pm_{\varepsilon, t}$, by choosing a plane development $\bar{R}$ of $R$ and moving the vertices of $\bar{R}$ horizontally, where points of $\bar{R}$ which come from $\partial \mathcal{N}_0$ are not moved, and points which correspond to the singularity $\xi_i$ are moved by adding $t \tilde{z}(\xi_i)$ to their horizontal component. See Figures 9 and 10.

Note that for any $t \in [0, 1]$, the lengths of edges of $R_t$ do not vanish. Indeed, $R^\pm_{\varepsilon, t}$ has either one or two vertices which are in $\Sigma$, depending on whether $\varepsilon$ is a prong edge or an edge of $\Xi(M)$. In case it is a prong edge, the length of horizontal sides of $R$ is $L$ and (38) implies that the sidelength of $R_t$ is positive, and in case it is an edge of $\Xi(M)$, (39) implies that the sidelength of $R_t$ is positive. We glue the different trapezoids $\{R_t\}$ to each other along their edges, using the same gluing that defines $\mathcal{N}_0$, to obtain a complex $\mathcal{N}_t$. Since the sides of $R$ corresponding to $\partial \mathcal{N}_0$ have the same length in $R_t$, their boundaries $\partial \mathcal{N}_t$ and $\partial \mathcal{N}_0$ can be identified by a translation, and so we can glue $\mathcal{N}_t$ to $M \setminus \mathcal{N}_0$ along $\partial \mathcal{N}_0$. We denote the resulting translation surface by $M_t$; clearly it is obtained from $M$ by modifying $\mathcal{N}_0$.

On each rectangle $R$ we choose a homeomorphism $f_t : R \to R_t$ which sends vertices of $R$ to the corresponding vertices of $R_t$, and acts affinely on each boundary edge of $\partial R_t$. This choice ensures that $f_t$ can be extended consistently from rectangles to their union, defining a map $f_t : \mathcal{N}_0 \to \mathcal{N}_t$, and then extended to a homeomorphism $f_t : M \to M_t$. We set $f_t = f_t \circ f$. With this choice $(f_t, M_t)$ is a path in $\mathcal{H}_m$, and the maps $f_t \circ f^{-1}$ are isotopic to the identity via an isotopy fixing $\Sigma$. 

We claim that for each $t$, the pullback $\text{dev}(f_t, M_t) = f_t^*\text{hol}(M_t, \cdot)$ is the cohomology class $\text{dev}(f, M) + tz$. This will imply that $M_t = \text{Rel}_{\xi_t}(M)$ for all $t \in [0, 1]$, which implies all the required conclusions of the Theorem. We verify this formula on each path $\gamma : [0, 1] \rightarrow M$ between singularities $\xi_i$ and $\xi_j$. The path $\gamma$ is homotopic to a concatenation of linear segments $\delta_1, \ldots, \delta_t$ which begin and end at singular points and are completely contained in $N_0$, and linear segments $\delta'_1, \ldots, \delta'_m$ which begin and end at singular points and are not completely contained in $N_0$. Each of the segments $\delta = \delta_i$ is a saddle connection in $\Xi(M)$, thus it is one of the edges $e$ of the rectangles $R_e$, and by construction $\text{hol}(f_t^* M_t, \delta) = \text{hol}(f^* M, \delta) + tz(\delta)$. If $\delta' = \delta'_i$ is a path not contained in $N_0$ we can subdivide it into a concatenation of paths $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, where $\sigma_1$ goes from $\xi_1$ to $\partial N_0$, $\sigma_2$ goes from $\partial N_0$ to $\xi_j$, $\sigma_3$ (resp. $\sigma_4$) is a union of segments with interior completely inside (resp.
outside) \( \mathcal{N} \) from \( \partial \mathcal{N}_0 \) to \( \partial \mathcal{N}_0 \). Moreover by applying a homotopy we can assume that on the initial surface \( M \), each of the segments in \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) proceeds along a vertical line in the rectangles \( R^+_t \). Now we compute the difference \( \text{hol}(f^*_t M_1, \sigma) - \text{hol}(f^* M, \sigma) \) in each case. We have \( \text{hol}(f^*_t M_1, \sigma_1) - \text{hol}(f^* M, \sigma_1) = -z(\xi_1) \) and \( \text{hol}(f^*_t M_1, \sigma_2) - \text{hol}(f^* M, \sigma_2) = z(\xi_2) \) by definition of the deformed flat structure on \( R^+_t \). We also have \( \text{hol}(f^*_t M_1, \sigma_4) - \text{hol}(f^* M, \sigma_4) = 0 \), since \( \sigma_4 \) is in the complement of \( \mathcal{N}_0 \) where the two flat structures are the same, and we have \( \text{hol}(f^*_t M_1, \sigma_3) - \text{hol}(f^* M, \sigma_3) = 0 \) since each of the segments of \( \sigma_3 \) passes through both \( R^+_t \) and \( R^-_t \) for some \( e \), leaving and exiting at symmetric points, and the change to the holonomy in these two rectangles cancel each other. All together we have \( \text{hol}(f^* M, \delta') - \text{hol}(f^*_t M_1, \delta') = z(\xi_3) - z(\xi_4) \), as required. 

The following result says that the only obstruction to defining the Rel flow is the one illustrated in Figure 3. Let \( z_{ij} \) be defined as in equation (37).

**Corollary 6.2.** Suppose \( Z \) is the real Rel subspace for a stratum \( \mathcal{H} \) as above and let \( z \in Z \), \( M \in \mathcal{H} \). Then \( M \in \mathcal{H}'_{\delta} \) exactly when there is no horizontal saddle connection \( \delta \) on \( M \) from singularity \( \xi_i \) to singularity \( \xi_j \), and \( t \in [0, 1] \) such that \( \text{hol}(M, \delta) + t z_{ij} = 0 \). In particular \( \mathcal{H}'_{\delta} \) is the set of surfaces which have no horizontal saddle connections joining distinct singularities.

Furthermore, if \( \mathcal{H} \) has two singularities, and \( v \in \mathcal{R} \cong \mathbb{R}^2 \), then \( M \notin \mathcal{H}'_{\delta} \) if and only if there is a saddle connection \( \delta \) on \( M \) from \( \xi_2 \) to singularity \( \xi_1 \), with \( \text{hol}(M, \delta) = tv \) for some \( t \in [0, 1] \).

**Proof.** The first assertion is a restatement of (36), and the second assertion follows immediately from the first. For the third assertion, let \( r_\theta \) be the rotation matrix for which \( r_\theta v \) is horizontal. We obtain the assertion by applying the first assertion to the surface \( r_\theta M \) and using Proposition 4.4 with \( g = r_\theta \).

Corollary 6.2 was proved in [MW2, Thm. 11.2]. See also [McM8].

Figures 3 and 4 illustrate the meaning of Corollary 6.2. In Figure 3 the saddle connection at the top and bottom of the decagon violates the first condition, when \( v = (T, 0) \) for \( T \) which is at least as large as the length of this segment. In Figure 4 there are no horizontal saddle connections joining distinct singularities, and as a consequence of Corollary 6.2, \( \text{Rel}_{(T, 0)} M \) is defined for all \( T \).

We now derive some consequences. As a first consequence we have:

**Proof of Proposition 4.11.** Each \( Z^{(M, \delta)} \) is a half-space, and in particular is convex. Thus Proposition 4.11 follows immediately from (36). 

An immediate consequence of the explicit surgery we have presented in the proof of 6.1 is the following useful statement:

**Corollary 6.3.** For any \( M \) and any \( t \) for which \( \text{Rel}_t M \) is defined, there is a natural bijection between horizontal saddle connections on \( M \) and on \( \text{Rel}_t M \), and for each saddle connection \( \delta \) directed from \( \xi_2 \) to \( \xi_1 \), \( \text{hol}(\text{Rel}_t M, \delta) = \text{hol}(M, \delta) - (t, 0) \). In particular \( M \) and \( \text{Rel}_t M \) are topologically horizontally equivalent.

The following will be crucial for analyzing \( U \)-invariant measures in §8.

**Definition 6.4.** Suppose \( \mathcal{H} = \mathcal{H}(1, 1) \), and let \( T \in \mathbb{R} \setminus \{0\} \). Let \( \mathcal{H}'_{\delta} \) denote the set of surfaces \( M \in \mathcal{H} \) for which...
Let there be a map which is a lift of \( \Phi \).  

Theorem 6.5. There is a map 

\[
\Phi : \mathcal{H}_T^p \to \mathcal{H}(2)
\]

which is affine in charts (hence continuous) and \( U \)-equivariant. For each \( M \in \mathcal{H}_T^p \), \( \Phi(M) \) is obtained by modifying \( N(\varepsilon, L) \) for some \( \varepsilon > 0, L > T \) depending on \( M \).  

There is a map 

\[
\Phi_t = \Phi^{(T)}_t : \mathcal{H}_T^p \to \mathcal{H}(2),
\]

which is a lift of \( \Phi \) (i.e. \( \Phi = P \circ \Phi_t \) where \( P : \mathcal{H}(2) \to \mathcal{H}(2) \) is the projection), and \( \Phi_t \) is a homeomorphism onto its image.

Suppose \( T > 0 \). Note that assumption (ii) implies that \( [0, T) \subset Z^{(M)} \), and assumption (i) implies that \( T \notin Z^{(M)} \); that is \( T \notin \partial Z^{(M)} \). A topology on \( \mathcal{H}(1,1) \cup \mathcal{H}(2) \) can be constructed in which the map \( \Phi \) can be recovered as \( \Phi(M) = \lim_{s \to T^-} \text{Rel}_s(M) \). 

We will not construct this topology here.

Proof. We will assume throughout the proof that \( T > 0 \). The case in which \( T < 0 \) can be dealt with by repeating the arguments below, switching the labels of the two singularities. Let \( \delta' \) be as in (i) in the definition of \( \mathcal{H}_T^p \) and let \( e' \) be the corresponding edge of \( \Xi(M) \). Let \( L > |T| \) and define \( N(L, \varepsilon) \) as in the discussion prior to the statement of Theorem 6.1, where \( \varepsilon > 0 \) is small enough so that the rectangles \( R^\pm \) are all embedded in \( M \). Define the polygons \( \mathcal{R}^\pm \) as in the proof of Theorem 6.1. For all \( e \neq e' \) condition (ii) ensures that \( \mathcal{R}^\pm \) is a nondegenerate trapezoid. For \( e' \), the length of the edge corresponding to \( e' \) is zero and so it is a triangle. See Figure 11. We glue the two triangles \( R^\pm \) to the trapezoids \( R^\pm \), \( e \neq e' \) to obtain the complex \( \mathcal{N}_T \) which we glue to \( M \setminus \mathcal{N}_0 \) as before, to obtain \( M_T \). We note that \( \mathcal{N}_T \) is a thickening of a graph obtained from \( \Xi(M, L) \) by collapsing the edge \( e' \). One can compute explicitly that there is one singular point in \( \mathcal{N}_T \) and that it has cone angle 6\( \pi \); that is \( M_T \in \mathcal{H}(2) \). We set \( \Phi(M) = M_T \).

![Figure 11. The developing image of degenerate rectangles. The trapezoid \( R^+ \) has degenerated to a triangle, the edge \( e' = e_2 \) has disappeared, and the two singular points have coalesced.](image)

We now show that \( \Phi \) is affine in charts. We first explain what this means. Let \( \mathcal{R}^\pm \) be the pre-image of \( \mathcal{H}_T^p \) in \( \mathcal{H}(1,1) \), and let \( \text{dev}_{(1,1)} : \mathcal{H}(1,1) \to H^1(S, \{\xi_1, \xi_2\}; \mathbb{R}^2) \) be the developing map as in (3). Condition (i) in Definition 6.4 can be expressed as a linear condition on the image of \( \text{dev}_{(1,1)} \) and condition (ii) is an open condition on the image of \( \text{dev}_{(1,1)} \), and thus \( \mathcal{R}^\pm \) is an affine submanifold of \( \mathcal{H}(1,1) \). Also let \( \text{dev}_{(2)} : \mathcal{H}(2) \to H^1(S', \{\xi'\}; \mathbb{R}^2) \), for some model...
Figure 12. The complex $N_0$, shown topologically, with $\delta'$ connecting the two singularities.

Figure 13. The developing image of rectangles. The double lines represent two different edges on the surface.

Figure 14. The developing image of rectangles when $\delta'$ shrinks to a point and the two singularities coalesce.

Let $(f, M) \in \mathcal{H}_{m,T}$ be a marked surface projecting to $M \in \mathcal{H}_T$, and define the map $f_T : M \to M_T$ as in the proof of Theorem 6.1. Let $f_T = f \circ f_T : S \to M_T$. The map $f_T$ is injective on the complement of $e'$ and maps all points in $e'$, including its endpoints $\xi_1$ and $\xi_2$, to the unique singularity of $M_T$, which we denote by $\xi$. Note that $f^{-1}(e')$ is a simple path connecting the two points $\xi_1$ and $\xi_2$. Let $S'$ be the surface obtained from $S$ by collapsing $f^{-1}(e')$ to a point $\xi'$, and let $p : S \to S'$ be the quotient map. Since $f^{-1}(e')$ is contractible, $S'$ is also a genus 2 surface, and since $f_T$ is constant on $f^{-1}(e')$, it descends to a homeomorphism $S' \to M_T$, which we continue to denote by $f_T$. We see that

$$\Phi_m : \mathcal{H}_{m,T} \to \mathcal{H}_m(2), \quad \Phi_m(f, M) = (f_T, M_T)$$
is a lift of $\Phi$. Since $f^{-1}(\epsilon')$ is contractible, the pullback $p^* : H^1(S'; \mathbb{R}^2) \to H^1(S; \mathbb{R}^2)$ is an isomorphism. Let $\text{Res}_{(1,1)}$ and $\text{Res}_{(2)}$ denote the restriction maps in (13), in the two cases corresponding respectively to $H(1,1)$ and $H(2)$. Since $\xi_1, \xi_2$ are contained in $f^{-1}(\epsilon')$, and the holonomies of absolute periods are on the same on $(f, M)$ and $(f_T, M_T)$, we find that $\text{Res}_{(2)} \circ \text{dev}_{(2)} \circ \Phi \circ \text{Res}_{(1,1)} \circ \text{dev}_{(1,1)}$. Since $\text{Res}_{(2)}$ is an isomorphism, this yields (40) with $L = \text{Res}_{(2)} \circ p^* \circ \text{Res}_{(1,1)}$. This computation, and the fact that the action of $U$ does not move the points of $\epsilon'$, also shows that $\Phi$ is $U$-equivariant.

We now show that $\Phi$ lifts to a continuous map $\Phi_f : H^2(T) \to H_f^2(\mathcal{T})$. In view of the discussion in §3.4, in order to lift $\Phi$ to a map to $H_f(\mathcal{T})$ we need to equip $M_T$ with a right-pointing horizontal prong at the singular point $\xi$. Let $\delta' = \delta'(M)$ be as in the definition of $H^2_f$ and let $q(M)$ be the prong which is obtained by moving an angle $\pi$ in the counterclockwise direction from the terminal prong of $\delta'$, at the singularity $\xi_2$. Then $q(M)$ is in the complement of $\delta'$ and so is mapped by $f_T$ to a horizontal prong on $M_T$. See Figures 11 and 15, where $q(M)$ is marked as $q_1$.

We need to show that with this choice of selected prong, $\Phi_f$ is continuous. In light of Proposition 3.8, it is enough to show two things: (i) that the choice of prong $M \mapsto q(M)$ is continuous with respect to the coordinates given by the developing map, for any fixed triangulation; and (ii), that $M \mapsto q(M)$ is $\text{Mod}(S)$-invariant. It is clear from our description of $q(M)$ that (i) and (ii) are satisfied.

Finally, since $\Phi_f$ is a lift of a locally affine map in charts, in order to show that it is a homeomorphism onto its image we need only verify that it is injective, and for this we explicitly construct its inverse. We first pick one basepoint $M_0 = \Phi_f(M_0)$ in each connected component of the image of $\Phi$, choose a horizontal prong at $M_0$ using the direction of $\delta$ in $M_0$ as in the preceding paragraph, and extend this choice continuously to all surfaces in the image of $\Phi$. Now for any $M'$ in the image of $\Phi$, let

---

**Figure 15.** The corresponding topologically correct picture. The chosen prong edge is labeled $q_1$.  

\( \mathcal{N}' \) be the \((L, \varepsilon)\)-rectangle thickening of \( \Xi \) \((M') \) for \( L > |T| \) and small enough \( \varepsilon \). We consider \( \mathcal{N}' \) with its decomposition into rectangles as in the preceding discussion, and we now modify this decomposition. Let \( q_1 \) be the chosen right-pointing prong on \( M' \) and let \( q_2 \) be the left-pointing prong which is clockwise from \( q_1 \). Similarly, let \( q_3, q_4 \) be the left- and right-pointing prongs which are at angular distance \( 3\pi \) from \( q_1, q_2 \) respectively. Let \( \sigma_1 \) (resp. \( \sigma_2 \)) be the two vertical segment of lengths \( \varepsilon \) between \( q_1, q_2 \) (resp., between \( q_3 \) and \( q_4 \)) connecting \( \xi \) to the boundary of \( \partial \mathcal{N}' \). For \( i = 1, 2 \), the segments \( \sigma_1, \sigma_2 \) are boundary segments of two rectangles \( R_i^\pm \) of the complex \( \mathcal{N}' \), one on each side. Let \( \Delta_i \) be a triangle which is embedded in the union of \( R_i^+ \cup R_i^- \), has an apex at \( \xi_i \), and has \( \sigma_i \) as an altitude contained in its interior. We now replace each \( R_i^\pm \) with \( R_i^\pm \setminus \Delta_i \), and add the \( \Delta_i \) to the polygonal decomposition of \( \mathcal{N}' \). Thus we have a decomposition of \( \mathcal{N}' \) into rectangles, trapezoids, and two triangles. To each of them we apply the map described in the proof of Theorem 6.1, with \(-T\) instead of \( T \). That is, we do not move points on \( \partial \mathcal{N}' \) and the non-boundary edges. The two triangles are thought of as degenerate trapezoids. The choice of the prongs at \( \xi_i \), and the fact that \( M' \) is in the image of \( \Phi \), ensure that these operations are well-defined, that is for all \( t \) strictly between 0 and \( T \), the deformed shapes are nondegenerate trapezoids. Gluing them to each other using the gluing map of \( \mathcal{N}' \) and gluing the resulting complex to \( M \setminus \mathcal{N}' \) completes the definition of the inverse of \( \Phi_t \).

The image of \( \Phi \) in Theorem 6.5 can be described explicitly in terms of the choice of horizontal prong at the singularity. Namely suppose again that \( T > 0 \) and that \( q_1 \) is the chosen prong. Let \( q_2, \ldots, q_6 \) be the additional prongs at \( \xi \) in counterclockwise order (note that this differs from the labeling in Figure 15). Then the image of \( \Phi \) is the set of \( \mathcal{M} \in \mathcal{H}_I(2) \) which have no horizontal saddle connections of length at most \( T \) from any one of \( q_1 \) or \( \bar{q}_3 \), to one of \( q_4 \) or \( \bar{q}_6 \).

The inverse of \( \Phi \) appearing in Theorem 6.5 is the operation of ‘splitting open a singularity’ which was discussed in [EMZ].

7. The eigenform locus

In this section we will define the eigenform locus \( \mathcal{E}_D \). We describe its intersection with \( \mathcal{H}(1, 1) \) and \( \mathcal{H}(2) \) and describe how it meets some boundary strata. We summarize some properties of surfaces in the eigenform locus.

7.1. Definition of the eigenform loci. The eigenform loci were defined by Calta [C] and McMullen [McM2]. Calta made use of the \( J \) invariant and McMullen used properties of real multiplication on Jacobians. Here we follow the approach of McMullen.

For every positive integer \( D \equiv 0, 1 \pmod{4} \) with \( D \geq 4 \) there is a closed, connected, \( G \)-invariant locus \( \mathcal{E}_D \subset \mathcal{H}(2) \cup \mathcal{H}(1, 1) \), called the eigenform locus, which we now describe.

An order in a number field \( F \) is a subring \( \mathcal{O} \) of the ring of integers \( \mathcal{O}_F \) which is finite index as an abelian group. Orders in quadratic fields are particularly simple as they can be classified by a single integer \( D \), the discriminant. More precisely, for every positive integer \( D \equiv 0, 1 \pmod{4} \), we consider the real quadratic order

\[
\mathcal{O}_D = \mathbb{Z}[T]/(T^2 + bT + c),
\]
where \(b, c \in \mathbb{Z}\) are such that \(b^2 - 4c = D\). If \(D\) is not square, it is a subring of the real quadratic field, \(F_D = \mathbb{Q}[T]/(T^2 + bT + c)\). We also allow \(D\) to be square, in which case \(F_D\) is isomorphic to \(\mathbb{Q} \oplus \mathbb{Q}\) as a \(\mathbb{Q}\)-algebra. In either case the isomorphism classes of \(\mathcal{O}_D\) and \(F_D\) depend only on \(D\). We fix a choice of a ring homomorphism \(\iota: F_D \to \mathbb{R}\). When \(D\) is not square, \(\iota\) is a field embedding. If \(D\) is square, \(\iota\) is obtained from the projection of \(\mathbb{Q} \oplus \mathbb{Q}\) onto its first factor. A more detailed discussion of orders in number fields appears in [BoSh].

Consider a genus two Riemann surface \(X\) with Jacobian variety \(\text{Jac}(X) = \Omega(X)^* / H_1(X; \mathbb{Z})\), where \(\Omega(X)\) is the space of holomorphic one-forms on \(X\). Real multiplication by \(\mathcal{O}_D\) on \(\text{Jac}(X)\) is a ring homomorphism \(\rho: \mathcal{O}_D \to \text{End}^0(\text{Jac}(X))\), where \(\text{End}^0(\text{Jac}(X))\) is the ring of endomorphisms of \(\text{Jac}(X)\) which are self-adjoint with respect to the intersection form on \(H_1(X; \mathbb{Z})\). We also require \(\rho\) to be proper, in the sense that it does not extend to \(\mathcal{O}_E \supseteq \mathcal{O}_D\) for some \(E \mid D\).

Real multiplication by \(\mathcal{O}_D\) induces a representation of \(\mathcal{O}_D\) on \(\Omega(X)\), and by self-adjointness, a decomposition of \(\Omega(X)\) into complementary eigenspaces. A nonzero holomorphic one-form on \(X\) is an eigenform if it lies in one of these eigenspaces. We say that a pair \((X, \omega)\) is an eigenform for real multiplication if \(\text{Jac}(X)\) has real multiplication with \(\omega\) an eigenform.

Real multiplication for curves in genus two is very special, as it can be detected from knowledge of the absolute periods of a single one-form on the curve. That is to say, real multiplication in genus two has a ‘purely flat’ description. More precisely:

**Proposition 7.1 ([B2])**. A genus two translation surface \(M\) is an eigenform for real multiplication by \(\mathcal{O}_D\) if and only if there is a proper monomorphism \(\rho_0: \mathcal{O}_D \to \text{End}^0(H_1(M; \mathbb{Z}))\) such that

\[
\text{hol}(M, \rho_0(\lambda) \cdot \gamma) = \iota(\lambda)\text{hol}(M, \gamma)
\]

for each \(\lambda \in \mathcal{O}_D\) and \(\gamma \in H_1(M; \mathbb{Z})\).

The \(\rho_0\) in this Proposition is simply the action on homology induced by the real multiplication \(\rho: \mathcal{O}_D \to \text{End}^0(\text{Jac}(M))\). See also [McM2, Lemma 7.4], and see [CaSm] for an alternative approach.

We define the eigenform locus \(\mathcal{E}_D \subset \mathcal{H}(2) \cup \mathcal{H}(1, 1)\) to be the locus of eigenforms for real multiplication by \(\mathcal{O}_D\), and we define \(\mathcal{E}_D(2)\) and \(\mathcal{E}_D(1, 1)\) to be the intersections of \(\mathcal{E}_D\) with the respective strata. Similarly we denote the corresponding subsets of area-one surfaces by \(\mathcal{E}_D^{(1)}(2)\) and \(\mathcal{E}_D^{(1)}(1, 1)\).

The locus \(\mathcal{E}_D(1, 1)\) is \(\text{GL}_2(\mathbb{R})\)-invariant, as it can be easily seen that the condition of Proposition 7.1 is \(G\)-invariant (this was first proved in [McM2] and [C]). It is also \(\text{Rel}\) invariant since this condition only involves absolute periods. Moreover \(\mathcal{E}_D(1, 1)\) is a six dimensional linear submanifold of \(\mathcal{H}(1, 1)\) with respect to the period coordinates from \(\S 2.2\). To see this explicitly, choose two generators \(\gamma_1, \gamma_2\) of \(H_1(M; \mathbb{Z})\) (as an \(\mathcal{O}_D\)-module) and complete to a set of four generators as a \(\mathbb{Z}\)-module, e.g. by adjoining a multiple of each \(\gamma_i\) by a generator of \(\mathcal{O}_D\) over \(\mathbb{Z}\). Equation (41) now gives linear equations which the vectors \(\text{hol}(M, \gamma)\) must satisfy, and these equations define \(\mathcal{E}_D(1, 1)\) locally. As a consequence \(\mathcal{E}_D^{(1)}(1, 1)\) is a five-dimensional manifold locally defined in period coordinates by linear equations and one quadratic equation.

This dimension count easily implies that \(\text{GL}_2(\mathbb{R})\)-orbits and \(\text{Rel}\) leaves locally fill \(\mathcal{E}_D(1, 1)\).
Recall the semi-direct product $L = G \ltimes \mathbb{R}^2$ introduced in §4.2. The group $L$ is embedded in $\text{GL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ which is an open set in a 6-dimensional vector space.

**Proposition 7.2.** For any $M \in \mathcal{E}^{(1)}_D(1,1)$ there is a neighborhood $\mathcal{U}$ of the identity in $L$ and a neighborhood $\mathcal{U}'$ of $M$ in $\mathcal{E}^{(1)}_D(1,1)$ such that the map $p : \mathcal{U} \to \mathcal{U}'$ defined by

$$p(g, v) = gM + v$$

is the restriction of an affine homeomorphism to $\mathcal{U}$.

**Proof.** Consider a precompact neighborhood $\mathcal{W}$ of the identity in $\text{GL}_2(\mathbb{R})$. For some $\varepsilon > 0$, no surface in $\mathcal{W} \cdot M$ has saddle connections of length less than $\varepsilon$. By Corollary 6.2, $p$ is well-defined on $\mathcal{V} = \mathcal{W} \times B(0) \subset \text{GL}_2(\mathbb{R}) \times \mathbb{R}^2$. Possibly decreasing $\varepsilon$ so that the image of $p$ is contained in an affine coordinate chart as defined above, $p$ is a homeomorphism onto its image. Since the action of Rel preserves the area of surfaces, $p$ sends $L$ into the locus of area-one surfaces. Intersecting $\mathcal{V}$ with $L$ and the image of $p$ with the locus of area-one surfaces, we obtain $\mathcal{U}$ and $\mathcal{U}'$ with the required properties. $\square$

Recall that the Rel operations were defined in §4.2 in terms of a vector field $\mathfrak{R}$ and in $\mathcal{H}(1,1)$ we have $\mathfrak{R} \cong \mathbb{R}^2$. Also note that the $G$-action on a stratum makes it possible to associate to each element of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $G$, a vector field on $\mathcal{H}$. From Proposition 7.2 and the Frobenious integrability criterion (see e.g. [Wa]) we obtain:

**Corollary 7.3.** For any $D$, the vector fields on $\mathcal{E}^{(1)}_D(1,1)$ obtained as the infinitesimal generators of the $G$-action and the Rel operations, taken with the natural Poisson bracket, have the structure of the Lie algebra $\mathfrak{l} = \mathfrak{sl}_2(\mathbb{R}) \ltimes \mathbb{R}^2$. For any Lie subalgebra $\mathfrak{l}_0 \subset \mathfrak{l}$, the plane field corresponding to the vector fields in $\mathfrak{l}_0$ are integrable and define a foliation of $\mathcal{E}^{(1)}_D(1,1)$.

The eigenform locus has a more elementary description in the case where $D$ is a square. A translation surface $X$ is a torus cover if there is a branched cover $p : X \to T$ which is a local translation for some flat torus $T$ (note that the branch points of $p$ are not required to lie over a single point of $T$). We say $p$ is primitive if it does not factor through a torus cover of smaller degree, equivalently if the map on homology $p_* : H_1(X; \mathbb{Z}) \to H_1(T; \mathbb{Z})$ is onto. McMullen established in [McM3] that $\mathcal{E}_{d_2}$ is the locus of primitive degree $d$ torus covers.

**7.2. G-invariant measures in genus two.** We now discuss the $G$-invariant measures in genus two. Of course each of these is also a horocycle invariant measure. These were classified by McMullen in [McM3]. Measures supported on the full strata were constructed by Masur [Ma1] and Veech [Ve1] using period coordinates on these strata. In [McM3] McMullen constructed measures on the eigenform loci in an analogous way using period coordinates.

We may use Proposition 7.2 to define a measure on $\mathcal{E}_D(1,1)$ by locally pushing forward Haar measure on $L$. More precisely, recall that $L$ is unimodular, that is its Haar measure is invariant under both right and left multiplications. Given $E$ in the image $\mathcal{U}'$ of $p$, we assign to $E$ the Haar measure of $p^{-1}(E) \subset L$. Right-invariance of Haar measure implies that the measure of $E$ doesn’t depend on the choice of basepoint, and left-invariance implies that the measure is $G$-invariant.
McMullen [McM3] proved that this measure is finite. We call this the flat measure on \( E_D(1,1) \).

Here is an alternative description of the flat measure which can be generalized to define a measure on any linear submanifold of a stratum. Suppose \( f : S \to M \) is a marked translation surface, write \( M = (X, \omega) \), and suppose that \( \text{Jac}(X) \) admits real multiplication by \( O_D \). The real multiplication on \( \text{Jac}(X) \) gives \( H_1(S; \mathbb{Z}) \) the structure of an \( O_D \)-module. A choice of embedding \( O_D \to \mathbb{R} \) makes \( \mathbb{R}^2 \) an \( O \)-module as well. We define \( H^2_{O_D}(S, \Sigma; \mathbb{R}^2) \subset H^1(S, \Sigma; \mathbb{R}^2) \) to be the subspace of cocycles for which the induced period map \( H_1(S; \mathbb{Z}) \to \mathbb{R}^2 \) is \( O_D \)-linear. This is in other words the space of cocycles satisfying (41). By Proposition 7.1, the linear subspace \( H^1_{O_D}(S, \Sigma; \mathbb{R}^2) \) parameterizes the eigenform locus in \( \mathcal{H}_m \) near \( M \).

We have the commutative diagram of cohomology groups (all coefficients in \( \mathbb{R}^2 \)):

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0(\Sigma)/H^0(S) & \longrightarrow & H^1_{O_D}(S, \Sigma) & \longrightarrow & H^1_{O_D}(S) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\Sigma)/H^0(S) & \longrightarrow & H^1(S, \Sigma) & \longrightarrow & H^1(S) & \longrightarrow & 0
\end{array}
\]

We give \( E_D(1,1) \) a measure by defining a measure on the linear space \( H^1_{O_D}(S, \Sigma) \) on which it is modeled which is invariant under the monodromy action. To define such a measure, we give a measure on the other two terms of the short exact sequence. The left term is canonically \( \mathbb{R}^2 \), and the monodromy action is trivial. We give it the usual Euclidean area form. The \( H^1(S) \) term has a symplectic form arising from the intersection form, which is preserved by the action of the mapping class group. This descends to a symplectic form on \( H^1_{O_D}(S) \) which is preserved by monodromy. This form is non-degenerate; this could be checked by direct computation in this case, and was proved in complete generality by Avila, Eskin and Möller [AEM]. Therefore the symplectic form defines a volume form on \( H^1_{O_D}(S) \) which is also monodromy invariant. The product of these volume forms induces one on \( H^1_{O_D}(S, \Sigma) \) which defines a measure on \( E_D(1,1) \). Finally, we apply the standard cone construction (meaning we push forward the restriction of the measure to surfaces in \( E_D(1,1) \) of area at most 1, by the canonical projection onto the locus of surfaces of area one; see [Zo] for details) to obtain a \( G \)-invariant measure on \( E_D^{(1)}(1,1) \).

The eigenform locus \( E_D(1,1) \) is nonempty for each \( D \geq 4 \) and \( D \equiv 0 \) or 1 (mod 4). In each case it is connected by [McM2]. The \( D = 1 \) case can be regarded as the locus of degenerate eigenforms where one separating curve has been pinched. This locus lies in the boundary of \( E_D(1,1) \) and is called the “product locus” in [B1].

In the stratum \( \mathcal{H}(2) \), the locus of eigenforms \( E_D(2) \), is called the Weierstrass curve in McMullen’s papers. By [McM5], \( E_D(2) \) consists of a single \( G \)-orbit if \( D \equiv 1 \) (mod 8) and \( D \geq 5 \) (note \( E_5(2) \) is empty), or if \( D = 9 \). Otherwise \( E_D(2) \) consists of two orbits. It is equipped with a finite measure coming from Haar measure on \( G \).

The square-discriminant eigenform locus \( E_D \) also contains a countable, dense collection of closed \( G \)-orbits. A translation surface \( X \) is called a square-tiled surface if it is a branched cover of the standard square torus with all of the branching lying over a single point. Every square-tiled surface has a closed \( G \)-orbit, and the square-tiled surfaces are dense in each \( E_D \).
Closed $G$-orbits inherit a measure from the Haar measure on $G$, and this measure is finite by a result of Smillie (see [SW1]). We will refer to this measure as the Haar measure on the closed $G$-orbit. The decagon surface is the surface obtained by identifying opposite sides of the regular 10-gon. It was shown by Veech [Ve1] that it has a closed $G$-orbit. It belongs to $\mathcal{H}(1,1)$ and in fact to the eigenform locus $\mathcal{E}_5(1,1)$. We write $L_{\text{dec}} \subset \mathcal{E}_5(1,1)$ for its $G$-orbit. Closed $G$-orbits in genus two were constructed by Calta [C] and McMullen [McM2] and then classified by McMullen [McM3].

**Theorem 7.4.** Each connected component of $\mathcal{E}_D(2)$ is a closed $G$-orbit, and every closed $G$-orbit in $\mathcal{H}(2)$ is of this form.

Every closed $G$-orbit in $\mathcal{H}(1,1)$ is either the $G$-orbit of a square-tiled surface or is $L_{\text{dec}}$.

In [McM3], McMullen showed that the measures defined above are the full list of ergodic $G$-invariant measures in genus two:

**Theorem 7.5.** Every ergodic $G$-invariant measure on $\mathcal{H}(2)$ is either the flat measure on the full stratum or Haar measure on a closed $G$-orbit.

Every ergodic $G$-invariant measure on $\mathcal{H}(1,1)$ is either the flat measure on the full stratum, the flat measure on some $\mathcal{E}_D(1,1)$, or the Haar measure on a closed $G$-orbit.

### 7.3. Degenerate eigenform surfaces

We will also be interested in eigenforms which are not genus two surfaces but can be thought of as surfaces lying in a bordification of $\mathcal{H}(1,1)$. We will consider two cases, where the role of “boundary strata” will be played respectively by $\mathcal{H}(0) \times \mathcal{H}(0)$ and $\mathcal{H}(0,0)$.

Given a pair of genus one translation surfaces $E_1$ and $E_2$, we may consider the one-point connected sum $X = E_1 \# E_2$. These degenerations arise from families of genus two surfaces where a separating curve has been pinched. As we have the direct sum decompositions $H_1(X) = H_1(E_1) \oplus H_1(E_2)$ and $\Omega(X) = \Omega(E_1) \oplus \Omega(E_2)$, the Jacobian of $X$ is simply the product of $E_1$ and $E_2$:

$$\text{Jac}(X) = \Omega^*(X)/H_1(X; \mathbb{Z})$$

$$\cong \Omega^*(E_1)/H_1(E_1; \mathbb{Z}) \oplus \Omega^*(E_2)/H_1(E_2; \mathbb{Z})$$

$$\cong E_1 \times E_2$$

Just as for the smooth case, we say that $X$ is an eigenform for real multiplication if $\text{Jac}(X)$ has real multiplication, with the one-form defining the translation structure belonging to one of the eigenspaces in $\Omega(X)$. McMullen gave a more explicit description of real multiplication for these degenerate surfaces in terms of isogenies of the $E_i$.

Recall that $E_1$ and $E_2$ are isogenous if there is a holomorphic covering map $p: E_1 \to E_2$. The isogeny $p$ is primitive if it cannot be written as a composition of an isogeny of lower degree with a self-covering of $E_2$. Existence of $p$ yields a dual isogeny $\tilde{p}: E_2 \to E_1$, so isogeny is an equivalence relation. In translation coordinates, an isogeny $p$ is of the form $p(z) = \lambda z + c$ for some complex number $\lambda$ which we call the scaling factor of $p$. In our setting, $\lambda$ will always be real, in which case $p$ preserves the horizontal direction but scales the metric by a factor of $\lambda$.

**Proposition 7.6 ([McM5]).** The surface $E_1 \# E_2$ is an eigenform for real multiplication by $\mathcal{O}_D$ if and only if there exists a primitive degree $m$ isogeny $p: E_1 \to E_2$...
E_2, together with an integral solution \((e, \ell)\) to the equation \(e^2 + 4m\ell^2 = D\) with \(\ell > 0\) and \(\gcd(e, \ell) = 1\), such that the scaling factor \(\lambda\) is the unique real positive root to the equation \(\lambda^2 - e\lambda - \ell^2m = 0\).

We define \(\mathcal{P}_D \subset \mathcal{H}(0) \times \mathcal{H}(0)\) to be the locus of pairs \((E_1, E_2)\) such that \(E_1 \neq E_2\) is an eigenform for real multiplication by \(\mathcal{O}_D\). With the diagonal \(G\)-action on \(\mathcal{H}(0) \times \mathcal{H}(0)\), the locus \(\mathcal{P}_D\) is \(G\)-invariant by Proposition 7.6.

By [McM5], the locus \(\mathcal{P}_D\) consists of finitely many closed \(G\)-orbits. We recall McMullen’s classification of these \(G\)-orbits. A prototype for real multiplication by \(\mathcal{O}_D\) is a triple of integers \((e, \ell, m)\) such that \(D = e^2 + 4\ell^2m\), with \(\ell, m > 0\) and \(\gcd(\ell, m) = 1\). A prototype \((e, \ell, m)\) determines a prototypical form in \(\mathcal{P}_D\) as follows. Let \(\lambda\) be the unique positive solution of \(\lambda^2 = e\lambda + \ell^2m\). We define a pair of lattices in \(\mathbb{C}\):

\[
\Lambda_1 = \mathbb{Z}(\lambda, 0) \oplus \mathbb{Z}(0, \lambda) \quad \Lambda_2 = \mathbb{Z}(\ell m, 0) \oplus \mathbb{Z}(0, \ell),
\]

and associated genus one translation surfaces \(E_i = (\mathbb{C}/\Lambda_i, dz)\). Multiplication by \(\lambda\) defines an isogeny \(p: E_1 \to E_2\) of degree \(\ell m\).

**Proposition 7.7 ([McM5]).** Each \(G\)-orbit of \(\mathcal{P}_D\) contains a unique prototypical form.

Given a prototype \((e, \ell, m)\), we define \(\mathcal{P}_D(e, \ell, m)\) to be the closed \(G\)-orbit containing the prototypical form associated to \((e, \ell, m)\). By the above proposition, this gives a bijection between prototypes and components of \(\mathcal{P}_D\). We say that two pairs of genus-one forms have the same combinatorial type if they lie on the same component of \(\mathcal{P}_D\).

By [McM5, Theorem 2.1], the \(G\)-orbit corresponding to the prototype \((e, \ell, m)\) is isomorphic to the modular curve \(G/\Gamma_0(m)\), where \(\Gamma_0(m) \subset \text{SL}(2, \mathbb{Z})\) is the group of matrices which are upper-triangular \(\text{mod}\ m\).

Finally, in the case when \(D = d^2\) with \(d > 1\), we consider the moduli space \(\mathcal{H}(0, 0)\) of genus one translation surfaces \(E\) with two marked points \(p\) and \(q\). Again, there is a natural \(G\)-action on this space. We define \(\mathcal{S}_\mathcal{P} \subset \mathcal{H}(0, 0)\) to be the locus of \((E, p, q)\) such that \(p - q\) is exactly \(d\)-torsion in the group law on \(E\) (that is \(d(p - q) = 0\) and \(d'(p - q) \neq 0\) for any \(d' < d\), in particular this implies \(p \neq q\)). By [B1], \(\mathcal{S}_\mathcal{P}\) is a closed \(G\)-orbit isomorphic to \(G/\Gamma_1(d)\), where

\[
\Gamma_1(d) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) : a \equiv e \equiv 1 \text{ (mod } d\text{)} \text{ and } c \equiv 0 \text{ (mod } d\text{)} \right\}.
\]

We can regard \((E, p, q)\) as a degenerate degree \(d\) torus cover obtained by pinching a nonseparating curve. Since \(p - q\) is a point of order \(d\) in the group law on \(E\), we have a degree \(d\) map \(\pi: E \to F\), where \(F\) is the quotient of \(E\) by the order \(d\) subgroup generated by \(p - q\). This \(\pi\) is the torus cover of minimal degree sending \(p\) and \(q\) to the same point.

The precise sense in which surfaces in the loci \(\mathcal{P}_D, \mathcal{S}_\mathcal{P},\) and \(\mathcal{E}_D(2)\) can all be regarded as lying in the boundary of \(\mathcal{E}_D(1, 1)\), is explained in [B1]. We will not explicitly use this point of view.

**7.4. Properties of eigenform surfaces.** We will require the following properties of surfaces in \(\mathcal{E}_D(1, 1)\). Recall that the modulus of a flat cylinder is defined to be its height divided by its circumference.
Proposition 7.8. In any cylinder decomposition of a surface in $E_D(1,1)$ with more than one cylinder, there is at least one rational relation among the moduli of horizontal cylinders.

Proof. Given a horizontally periodic surface $M$ with $n$ cylinders, the orbit-closure $\overline{UM}$ is a torus of dimension $n - r$, where $r$ is the number of independent rational relations among the moduli of the cylinders (see [SW1, Proposition 4]). We must then show that in any case the dimension of $\overline{UM}$ is smaller than $n$.

First, suppose $n = \dim \overline{UM} = 3$. We may parameterize a neighborhood of $M$ in $H(1,1)$ by the three holonomy vectors $(x_i, y_i)$ of a saddle connection joining cone points on the two boundary components of the $i$-th cylinder $C_i$. Since the $y_i$ are constant on $\overline{UM}$ and $\dim \overline{UM} = 3$, the $x_i$ are arbitrary on $\overline{UM}$. However, since $\overline{UM}$ is contained in the eigenform locus, the $x_i$ satisfy the nontrivial real-linear equation (41) which defines the eigenform locus (see also equation (6.5) of [B2]), where this equation is given explicitly), a contradiction.

Now suppose $n = \dim \overline{UM} = 2$. There must then be a horizontal saddle connection joining distinct zeros. Applying the map $\Phi$ of Theorem 6.5, i.e., the real-rel flow so that the length of this saddle connection tends to 0, yields a surface $M'$ in $H(2)$ with two horizontal cylinders with the same moduli. A neighborhood of $M'$ in $H(2)$ is parameterized by two holonomy vectors $(x_i, y_i)$. Again, if $\dim \overline{UM} = 2$, the $x_i$ would be arbitrary on $\overline{UM}$, which contradicts the real-linear equation (41) which defines the eigenform locus. Alternatively, since $GM'$ is closed, $UM'$ must also be closed, and so is $UM$, a translate of $UM$ by real-rel. (Note in the two-cylinder case, the claim also follows directly from [McM3, Theorem 9.1].) □

Proposition 7.9. The $U$-action on each $E_D(1,1)$, with respect to the flat measure, is ergodic.

Proof. In a $G$-action, ergodicity of the geodesic flow implies ergodicity of the $U$-action by the “Mautner phenomenon” (see e.g. [EW]). So it suffices to prove ergodicity of the $G$-action. This can be proved by applying the Hopf argument to the geodesic flow as in [Ma1]. □

Recall that a periodic direction of a translation surface is a direction in which the surface is a union of parallel cylinders and saddle connections. A translation surface is completely periodic if for any direction which contains a cylinder, the surface is a union of parallel cylinders and saddle connections.

Theorem 7.10 ([McM4, C]). Every genus two surface $M$ which is an eigenform is completely periodic. Moreover for $M \in E_D(1,1)$, if $\Xi(M)$ contains a saddle connection joining a singularity to itself, then the horizontal direction of $M$ is periodic.

Given two genus one translation surfaces $E_1$ and $E_2$, consider a horizontal segment $I \subset \mathbb{R}^2$ which embeds into each $E_i$. We may form the connected sum $X = E_1 \#_I E_2$ by removing the image of $I$ from each $E_i$, and then gluing the resulting boundary components. The result is a surface $X \in H(1,1)$ with two horizontal saddle connections whose union is a loop separating $X$ into two tori (see [McM3] for details).

Theorem 7.11. Suppose that $D$ is not square. Let $M \in E_D$, and suppose $M$ contains a loop which is a union of horizontal saddle connections. Then either
• The horizontal direction of $M$ is periodic, or
• $M$ is obtained by gluing two genus one translation surfaces with uniquely
  ergodic horizontal directions along a horizontal slit.

In particular, in the second case $\Xi(M)$ consists of two horizontal saddle connections
joining distinct singularities which are interchanged by the hyperelliptic involution
(for the corresponding horizontal data diagram, see diagram 5 in Figure 7).

Proof. We regard $M$ as a Riemann surface $X$ equipped with a holomorphic
one-form $\omega$. By [McM1], if $\text{Re}\, \omega$ has zero “Galois flux” and $\Xi(M)$ contains a loop,
then we are in one of these two cases. Zero flux follows from equation $4.2$ of
[McM1] and Theorem 5.1 of [McM3].

The final statement is proved in the following lemma.

Lemma 7.12. Let $M \in \mathcal{H}(1,1)$ be a surface with two horizontal saddle connections $I$ and $I'$ joining distinct singularities. The loop $\gamma = I \cup I'$ disconnects $M$ if
and only if the hyperelliptic involution interchanges $I$ and $I'$.

Proof. Let $\eta: M \to M$ denote the hyperelliptic involution and $\omega$ the holomorphic one-form on $M$ induced by the translation structure.

Suppose that we are in the case where $\eta$ interchanges $I$ and $I'$. Since $\eta^*\omega = -\omega$,
the map $\eta$ sends $\gamma$ to itself preserving the orientation. Thus $[\eta_\# \gamma] = [\gamma] \in H_1(M; \mathbb{Z})$.

But the hyperelliptic involution acts on $H_1(M; \mathbb{Z})$ as $-\text{Id}$. It follows that $\gamma$ is
homologous to zero, so it separates $M$.

Suppose now that $\eta$ fixes $I$ and $I'$. Since $\eta$ reverses the orientation of each of
these saddle connections, each must contain a single fixed point of $\eta$ (i.e. one of the
six Weierstrass points of $M$). If $f: M \to S^2$ is the quotient map induced by
the hyperelliptic involution, then in this case the union $f(I) \cup f(I')$ is a smooth
embedded path joining two branch points of $f$. Since an embedded path does not
disconnect the sphere, and $f$ has branch points disjoint from $f(I) \cup f(I')$, the
complement $M \setminus (I \cup I')$ is also connected, a contradiction.

When $D = d^2$, there is one more possible configuration of horizontal saddle
connections. Suppose $(E, p, q)$ is a genus one translation surface with two marked
points $p, q$ whose difference $p - q$ is exactly $d$-torsion in the group law of $E$. There
is then a genus one surface $F$ and degree $d$ cover $\pi: E \to F$ with $\pi(p) = \pi(q)$.
Let $I \subset \mathbb{R}^2$ be a segment which may be embedded in $E$ by a translation to yield
disjoint parallel segments $I'$ and $I''$ beginning at $p$ and $q$ respectively. We may
then form the self-connected-sum $M = (E, p, q)\# I$ by cutting along $I'$ and $I''$ and
then regluing to obtain a genus two surface. Since the gluings are compatible with
the covering $\pi: E \to F$, the surface $M$ is a primitive degree $d$ branched cover of
$F$, so $M \in \mathcal{E}_d^2(1,1)$. If the slope of $I$ is not a periodic direction on $E$, the surface
$M$ has exactly two saddle connections of this slope of the same length, and the
complement of this pair of saddle connections is a genus one surface. As in the
second case in Theorem 7.11, the horizontal diagram is once again diagram 5 of
Figure 7, but the complement of the slit has a different topology than in the case
that $D$ is not a square.

The following theorem says when $D$ is square, this is the only additional con-
figuration of saddle connections.
Theorem 7.13. Suppose that $D = d^2$. Let $M \in \mathcal{E}_D$, and suppose $M$ contains a loop which is a union of horizontal saddle connections. Then one of the following holds.

- The horizontal direction of $M$ is periodic.
- $M$ is obtained by gluing two genus one translation surfaces with uniquely ergodic horizontal directions along a horizontal slit.
- $M$ is a self-connected sum of $(E, p, q)$ along two horizontal slits of the same length based at $p$ and $q$ respectively, where $E$ is a genus one translation surface with uniquely ergodic horizontal direction, and $p - q$ has order exactly $d$ in the group law of $E$.

In particular, in the second case $\Xi(M)$ consists of two horizontal saddle connections joining distinct singularities which are interchanged by the hyperelliptic involution. In the third case, $\Xi(M)$ consists of two horizontal saddle connections of the same length fixed by the hyperelliptic involution.

Proof. Suppose that $M$ is not periodic. If $M$ has a saddle connection joining a singularity to itself, this saddle connection is the boundary of a cylinder, which implies that the horizontal direction of $M$ is periodic by Theorem 7.10, a contradiction. Therefore $\Xi(M)$ consists of two horizontal saddle connections $I$ and $I'$ joining singularity $\xi_1$ to singularity $\xi_2$.

Let $\pi: M \to F$ be a primitive degree $d$ cover of a genus one translation surface. The horizontal direction of $F$ is not periodic, otherwise $M$ would be periodic as well. The singularities $\xi_i$ are also the branch points of $\pi$, and the images $\pi(I)$ and $\pi(I')$ are horizontal segments on $F$ joining $\pi(\xi_1)$ to $\pi(\xi_2)$. Since the horizontal direction of $F$ is not periodic, the images $\pi(\xi_1)$ and $\pi(\xi_2)$ must be distinct. There is then a unique horizontal segment on $F$ joining these points which $I$ and $I'$ are both mapped to injectively by $\pi$. It follows that $I$ and $I'$ have the same length.

By Lemma 7.12, the loop $I \cup I'$ disconnects $M$ if and only if $I$ and $I'$ are interchanged by the hyperelliptic involution.

Suppose we are in the case where $I$ and $I'$ are fixed by the hyperelliptic involution. The complement $M \setminus (I \cup I')$ is then a genus one translation surface with two boundary components, each a union of two horizontal segments of the same length. The holonomy around each boundary component is trivial, so we have exhibited $M$ as a self-connected-sum $(E, p, q)\# I$. The branch covering $\pi$ must arise from a primitive covering $E \to F$ which identifies $p$ and $q$. This implies that $p - q$ is exactly $d$-torsion in the group law of $E$. \qed

8. Construction of $U$-invariant ergodic measures in $\mathcal{E}_D$

In the previous section we discussed $G$-invariant measures on $\mathcal{E}_D$. In this section we will discuss $U$-invariant measures which are not $G$-invariant.

8.1. Minimal sets. A minimal set for a flow is a minimal (with respect to inclusion) nonempty closed invariant subset. An example is a closed orbit, but more complicated examples may arise. The minimal sets for the horocycle flow in any stratum were classified in [SW1], where the following was shown (see also [HW, §6.1]).

Proposition 8.1. For $M \in \mathcal{H}$, the following are equivalent:

- $\overline{UM}$ is minimal.
\section{Construction of $U$-invariant ergodic measures in $E_D$}

- $UM$ is compact.
- $M$ has a horizontal cylinder decomposition, with $\Xi(M)$ consisting of the boundaries of these cylinders.

In case these hold, $UM$ is topologically conjugate to the quotient of a torus by a finite group, via a map which is affine in charts. The dimension of this torus is the dimension over $\mathbb{Q}$ of the span of the moduli of the cylinders in the horizontal direction of $M$, and under this conjugacy, the $U$-action on the torus is a straight-line linear flow. The closure of any $U$-orbit in $H$ contains some $M$ satisfying the above conditions.

\textbf{Corollary 8.2.} Each minimal set supports a unique $U$-invariant measure (which in particular is ergodic). The measure is linear with respect to the linear structure on the torus which is the minimal set.

\textbf{Proof.} These follow from well-known properties of straight line flows on tori, and from the isomorphism above. \hfill $\Box$

\textbf{Corollary 8.3.} Let $M \in E_D$ be a surface with a horizontal cylinder decomposition as in Proposition 8.1. For cylinder decompositions of type (A) of Figure 6, the corresponding orbit-closure is a minimal set which is one-dimensional or two-dimensional. In the former case it is a closed horocycle, and in the latter, the corresponding measure is invariant under $UZ$, and this group acts transitively on the support of the measure. For cylinder decompositions of types (B), (C) and (D), $U$-orbits are periodic.

\textbf{Proof.} By Proposition 8.1 the dimension of the $U$-orbit closure is the dimension of the $\mathbb{Q}$-span of the moduli, which by Proposition 7.8 is either 1 or 2. Suppose that it is 2. Again by Proposition 7.8, there must be three cylinders, so we are in case (A). The description in [SW1] shows that the additional dimension of the orbit closure is given by changing only the twists around cylinders, and it is straightforward to check using the explicit coordinates used in [SW1], that the $Z$-action also affects only the twists, changing them linearly in a way which is distinct from the $U$ action. Thus the $UZ$-orbits are 2 dimensional, and hence $UZ$ is transitive on the orbit-closure and the corresponding measure is $UZ$-invariant.

In cases (B) and (C) there are only two cylinders which, by Proposition 7.8, have rationally related moduli. In case (D) there is only one cylinder. So in all of these cases the minimal set is 1-dimensional. \hfill $\Box$

\section{Parameter spaces of minimal sets}

For every discriminant $D$, there are examples in $E_D$ of surfaces belonging to minimal sets of type (A), for which the $U$-orbit closure is two-dimensional, i.e. is an orbit of $UZ$. These minimal spaces are naturally grouped in continuous families, which we refer to as beds. Their structure was studied in detail in [B2, §6] when $D$ is not square. We summarize this structure here. This analysis will not be required in the sequel. Contrary to the rest of the paper, this discussion will be in the context of surfaces with unlabeled singularities as the combinatorial structure is simpler.

Each $E_D(1, 1)$ contains finitely many four (real) dimensional beds which parameterize unit area surfaces with periodic horizontal directions having three cylinders, where the ratios of circumferences of these cylinders are fixed, while twist parameters and heights are allowed to vary, subject to the linear condition which defines the eigenform locus. These beds are invariant under the horocycle flow, geodesic
flow and real and imaginary rel. Each bed is foliated by a family of 2-tori defined by fixing the heights as well as the circumferences of the cylinders, and varying the twist parameters. Each torus is a closed $UZ$-orbit and is either a $U$-minimal set or a union of closed $U$-orbits, depending on whether or not moduli of the cylinders have rational ratios. We will see in Theorem 12.3 that applying the geodesic flow to one of these tori gives a family of measures which equidistribute in $E_D(1,1)$.

As we vary the heights of the cylinders in one of these families of three-cylinder surfaces (see Figure 16), the height of a cylinder may tend to 0, yielding a family of two-cylinder surfaces in the boundary of the family of three-cylinder surfaces (see Figure 17). For a given three cylinder surface there are two cylinders which are candidates for degeneration and for each candidate cylinder there is a path in the family of three cylinder surfaces leading to a surface in which this cylinder is degenerate. Thus there are two distinct families of two-cylinder surfaces lying in the boundary of each three-cylinder family. These two-cylinder families are three-dimensional and composed of surfaces having horizontal data diagram of type (B) or (C). By Corollary 8.3, each two-cylinder family is a union of closed $U$-orbits.

Each family of two-cylinder surfaces is in turn in the boundary of exactly two families of three-cylinder surfaces. Starting from a given family of three-cylinder surfaces, we may continue this process of moving in the family so that the cylinders degenerate to one of the boundary two-cylinder families and then by undoing a different degeneration we get to a different three-cylinder family. After passing through a finite sequence of three-cylinder families, we eventually return to the
original one. These families of minimal sets then comprise finitely many disjoint cycles.

If $O_D$ is a maximal order, the set of cycles is naturally in bijection with the ideal class group of $O_D$. The bijection is given by associating to a periodic surface in one of these families the fractional ideal in $F_D$ obtained by taking the $\mathbb{Z}$-span of the circumferences of its cylinders. These circumferences are constant (up to scale) in each two- or three-cylinder family of minimal sets, and when a three-cylinder family degenerates to a two-cylinder family, one cylinder is lost but the fractional ideal generated by their circumferences is unchanged. If $O_D$ is not maximal the analysis becomes more involved. In the case where $D$ is square, the structure of the collection of minimal sets is more complicated due to the presence of minimal sets consisting of one-cylinder surfaces, in addition to two and three cylinder decompositions.

### 8.3. Framing and splitting.

Let $\mathcal{H}_t(2) \rightarrow \mathcal{H}(2)$ be the threefold cover corresponding to choosing a right-pointing horizontal prong at the singularity, as in §3.4. This cover is connected, since as we have seen in Proposition 3.9, the $SO(3)$-orbits of a point contain all of its pre-images under the map $\mathcal{H}_t(2) \rightarrow \mathcal{H}(2)$. We denote by $\tilde{E}_D(2)$ the pre-image of $E_D(2)$ in $\mathcal{H}_t(2)$. By the same reasoning as above, each connected component of $E_D(2)$ has connected inverse image. Recall that since $U$ is simply connected it lifts to $\tilde{G}$ as a subgroup, so it acts on $\tilde{E}_D(2)$.

**Proposition 8.4.** *The action of $U$ is ergodic on any connected component of $\tilde{E}_D(2)$.***

**Proof.** Since any connected component of $E_D(2)$ is a $G$-orbit (see Theorem 7.4), $\tilde{G}$ acts transitively on each connected component of $\tilde{E}_D(2)$, and in particular, ergodically. The inclusion $U \subset \tilde{G}$ has the Mautner property (see e.g. [EW]) and hence any ergodic $\tilde{G}$-action is also ergodic for $U$. □

### 8.4. Other $U$-invariant measures.

We now discuss the remaining ergodic $U$-invariant measures. Each of these arise from applying the rel flow to a closed $G$-orbit, possibly lying in a boundary stratum of $E_D$. To avoid working with compactifications, we will not always take this point of view explicitly, in favor of more elementary constructions.

We classify the remaining measures in terms of their horizontal data diagram $\Xi(\mu)$, defined in §5. We have the following possibilities.

**Proposition 8.5.** *Let $\mu$ be an ergodic $U$-invariant measure, which is not supported on a minimal set. There are four possibilities for $\Xi(\mu)$:

(i) $\Xi(\mu) = \emptyset$.

(ii) $\Xi(\mu)$ consists of one saddle connection joining distinct singularities. For $\mu$-a.e. surface, this saddle connection is fixed by the hyperelliptic involution.

(iii) $\Xi(\mu)$ consists of two saddle connections joining distinct singularities which are interchanged by the hyperelliptic involution. For $\mu$-a.e. surface $M$, the union of these saddle connections disconnects $M$ into a pair of tori with slits removed.

(iv) $\Xi(\mu)$ consists of two saddle connections joining distinct singularities which are fixed by the hyperelliptic involution. For $\mu$-a.e. surface $M$, the complement of these saddle connections is a torus with two slits removed, and $D$ is a square.*
In the case where $M$ belongs to a minimal set, $\Xi(M)$ is one of configurations pictured in Figure 6.

Our goal in this section will be to provide examples of $U$-ergodic $U$-invariant measures in each of these cases.

For fixed discriminant $D$, let $C$ denote one of the following:

(i) The $G$-orbit of the regular decagon $L_{\text{dec}} \subset \mathcal{E}_5(1,1)$, or if $D$ is square, the $G$-orbit of a square-tiled surface in $\mathcal{E}_D(1,1)$.

(ii) A component of $\tilde{E}_D(2)$ in $\mathcal{H}_d(2)$.

(iii) A component of $P_D \subset \mathcal{H}(0) \times \mathcal{H}(0)$.

(iv) If $D = d^2$, the $d$-torsion locus $S_D \subset \mathcal{H}(0,0)$.

Note that in this case and subsequently we use the same symbol to denote subsets of different spaces; that is, more formally we should replace $C$ with one of $C_{\text{i}} - C_{\text{iv}}$ depending on the cases above. We will continue with this slight inaccuracy with the set $\mathcal{B}$ and map $\Psi$ below, and this should cause no confusion.

In each case, $C$ has a natural $G$ or $G/P$ action, and $C$ is isomorphic to $G/\Gamma$ or $G/P$ for some lattice $\Gamma$. Let $\mathcal{B} \subset C$ denote the subset consisting of surfaces without horizontal saddle connections. In case (iii) or (iv), we interpret a saddle connection to be either a closed geodesic or a segment joining two marked points.

Proposition 8.6. The set $\mathcal{B} \subset C$ is the complement of the set of closed horocycles.

Proof. In cases (i) and (ii) this follows from the Veech alternative [Ve2].

In case (iii), recall that every point of $C$ represents a pair of isogenous genus one translation surfaces, so the horizontal direction is periodic for one if and only if it is periodic for the other. A torus with periodic horizontal direction is stabilized by an infinite cyclic subgroup of $U$, and for two isogenous tori, these subgroups are commensurable. Thus a point of $C$ representing a pair of tori with periodic horizontal direction lies on a closed horocycle.

In case (iv), every point of $C$ represents a torus with two marked points $(E, p, q)$ where $p - q$ is $d$-torsion. It has a horizontal saddle connection exactly when the horizontal direction of $E$ is periodic, in which case $E$ is stabilized by an infinite cyclic subgroup of $U$. Given $E$, there are finitely many choices of $p$ and $q$ which differ by $d$-torsion, up to isomorphism of $E$. Thus $(E, p, q)$ is also stabilized by an infinite cyclic subgroup of $U$, so lies on a closed horocycle.

Note that $\mathcal{B}$ is a dense $G_\delta$ subset of a locally compact metrizable space, and hence the Borel $\sigma$-algebra structure on $\mathcal{B}$ is a standard Borel space (see [K]).

Corollary 8.7. In each case the action of $U$ on $\mathcal{B}$ is uniquely ergodic, i.e. there is a unique $U$-invariant regular Borel measure on $\mathcal{B}$.

Proof. By Proposition 8.6, the action of $U$ on $\mathcal{B}$ is measurably conjugate to the horocycle flow on the complement of the set of horocycles in $G/\Gamma$ or $G/P$ for some lattice $\Gamma$. Dani [D1] classified the $U$-invariant measures on $G/\Gamma$ and $G/P$, showing that they are either supported on closed horocycles or are the global measure induced by Haar measure. Thus Dani’s theorem implies that the $U$-action on each $\mathcal{B}$ is uniquely ergodic.
In each of the four cases introduced above, we define a map \( \Psi : \mathcal{B} \times (0, \infty) \to \mathcal{H}(1,1) \) as follows:

(i) \( \Psi(M,T) = \text{Rel}_T M \). This is well-defined by Corollary 6.2.

(ii) \( \Psi(M,T) \) is the inverse of the map \( \Phi^{(T)} \) of Theorem 6.5. Informally, \( \Psi(M,T) \) is the surface obtained by splitting the singularity of angle \( 6\pi \) to two singularities of angle \( 4\pi \) each, and moving them apart using \( \text{Rel}_T \). The way in which the two singularities are to be split apart is made explicit by the framing.

(iii) Given a pair of genus one translation surfaces \( (E_1, E_2) \in \mathcal{B} \subset \mathcal{P}_D \), we define \( \Psi((E_1, E_2), T) \) to be the surface obtained by removing a horizontal slit of length \( T \) from each \( E_i \) and then gluing the resulting surfaces along their boundaries to obtain a genus two surface in \( \mathcal{H}(1,1) \). The resulting surface is an eigenform, since real multiplication is preserved by deformations which leave absolute periods constant by Proposition 7.1.

(iv) Given a genus one surface \( E \) with aperiodic horizontal foliation and with marked points \( p, q \) which differ by \( d \)-torsion, we define \( \Psi((E, p, q), T) \) to be the surface obtained by cutting \( E \) along two horizontal slits of length \( T \) with left endpoints \( p \) and \( q \) and then gluing the resulting two boundary components to obtain a genus two surface in \( \mathcal{H}(1,1) \). We saw above that there is a primitive degree \( d \) torus covering \( \pi : E \to F \) sending \( p \) and \( q \) to the same point. Our gluing identifies only pairs of points which have the same image, so the resulting surface is also a primitive degree \( d \) torus cover belonging to \( \mathcal{E}_D(1,1) \).

In each case, we label the singularities of the resulting surface so that singularity \( \xi_1 \) is on the left hand side of the horizontal saddle connection of length \( T \), and singularity \( \xi_2 \) is on the right hand side.

We extend the definition of \( \Psi(M,T) \) to \( T < 0 \) as follows. In case (i), we simply apply \( \text{Rel}_T \). In the remaining cases, we repeat the same construction using \( |T| \) in place of \( T \), but choose the opposite labeling for the singularities.

For fixed \( T \neq 0 \), let \( \Psi_T : \mathcal{B} \to \mathcal{E}_D(1,1) \) be defined by \( \Psi_T(M) = \Psi(M,T) \). In cases (ii), (iii), and (iv), we denote by \( \mathcal{B}_T \) the image of \( \Psi_T \).

We summarize this discussion in the following:

**Proposition 8.8.** For each \( T \), \( \Psi_T \) is continuous on \( \mathcal{B} \). In case (i), the image is a set of surfaces with no horizontal saddle connections. In the remaining cases, the image \( \mathcal{B}_T \) is the set of surfaces in \( \mathcal{E}_D(1,1) \) with

- in case (ii) there is exactly one horizontal saddle connection, with length \( |T| \), which joins distinct singularities.
- in case (iii) there are exactly two horizontal saddle connections of length \( |T| \) joining distinct singularities which are interchanged by the hyperelliptic involution.
- in case (iv) there are exactly two horizontal saddle connections of length \( |T| \) joining distinct singularities which have the same length and are both fixed by the hyperelliptic involution.

These saddle connections are all positively oriented in the case \( T > 0 \) and negatively oriented in the case \( T < 0 \).
In each case, the pushforward under $\Psi_T$ of the $G$-invariant (or $\hat{G}$-invariant) measure on $B$ is the unique $U$-invariant and $U$-ergodic measure $\mu$ on $E_D(1,1)$, such that $\Xi(\mu)$ is as in cases (i)-(iv) above.

**Proof.** Most of the statements in the Proposition were established in the preceding discussion. To obtain continuity of $\Psi_T$, in case (i), continuity follows from Proposition 4.3, and in case (ii), from Theorem 6.5. In the other cases, it follows immediately from the explicit definition of the surgeries described above that continuity holds in a neighborhood of $M$, for all $T$ sufficiently small (depending on this neighborhood). Continuity for arbitrary $T$ follows by combining this with Proposition 4.3.

We now show that for each of the cases (i)–(iv), each surface $M$ with $\Xi(M) = \Xi(\mu)$ as described in the list, is in the image of the map $\Psi_T$. To show that $M$ is in the image of $\Psi_T$, we define the inverse of $\Psi_T$ by inverting the surgery construction used to define it. Thus for case (ii), we use Theorem 6.5. In case (iii), $\Psi_T$ associates to a pair of isogenous tori $(E,F)$, the connected sum $E \# I F$. Given a surface $M$ with two horizontal saddle connections interchanged by the hyperelliptic involution, we cut $M$ along these saddle connections to obtain a pair of isogenous tori $(E,F)$, and so $M = \Psi_T(E,F)$. The other cases are similar.

The uniqueness of the measures $\mu$ for which $\Xi(\mu)$ is as in each of the cases above, now follows from the unique ergodicity of the $U$-action on the corresponding set $B$. □

9. Classification of ergodic measures

In this section we show that the only $U$-invariant $U$-ergodic measures in $E_D(1,1)$ are those described in Section 8. The following is our full measure classification result, which is an expanded version of Theorem 1.1.

**Theorem 9.1.** Let $\mu$ be a $U$-invariant $U$-ergodic Borel probability measure on $E_D$. Then exactly one of the following cases holds:

1. $\mu$ is length measure on a periodic $U$-orbit, $\Xi(\mu)$ is a complete separatrix diagram of type (A), (B), (C) or (D) (see Figure 6), and all the cylinders in the corresponding cylinder decomposition have commensurable moduli.
2. $\mu$ is the flat measure on a 2-dimensional torus which is a $U$-minimal set, $\Xi(\mu)$ is a complete separatrix diagram of type (A), and the cylinders in the corresponding cylinder decomposition do not have commensurable moduli.
3. $\Xi(\mu)$ consists of two saddle connections joining distinct singularities, which disconnect $M$. The complement consists of two isogenous tori glued along a slit as in case (iii) of §8.4, and there is a $T \in \mathbb{R} \setminus \{0\}$ such that $\mu$ is the pushforward of Haar measure on a connected component of $\mathcal{P}_D$, via the map $\Psi_T$.
4. $\Xi(\mu)$ consists of one saddle connection joining distinct singularities, as in case (ii) of §8.4, and there is a $T \in \mathbb{R} \setminus \{0\}$ such that $\mu$ is the image under $\Psi_T$ of Haar measure on a finite volume $\hat{G}$-orbit in $\mathcal{H}_4(2)$, where $\hat{G}$ and $\mathcal{H}_4(2)$ are the threefold covers of $G$ and $\mathcal{H}(2)$ described in §3.5.
5. $\Xi(\mu) = \emptyset$ and there is $T \in \mathbb{R}$ such that $\mu$ is the image under Rel$_T$, of Haar measure on a closed $G$-orbit in $E_D(1,1)$. In this case $D$ is either a square or is equal to 5.
(6) \( \Xi(\mu) \) contains two saddle connections joining distinct singularities, whose complement in \( M \) is a torus with two parallel slits of equal length, as in case (v) of §8.4, and there is a \( T \in \mathbb{R} \setminus \{0\} \) such that \( \mu \) is the image under \( \Psi_T \) of a \( G \)-invariant measure on the space of tori with two marked points.

In this case \( D \) is a square.

(7) \( \mu \) is the flat measure on \( \mathcal{E}_D(1,1) \).

**Definition 9.2.** The numbering in Theorem 9.1 corresponds to that in Theorem 1.1. If \( \mu \) is a measure satisfying the conditions of item (j) in Theorem 9.1 (where \( j \in \{1, \ldots, 7\} \)) then we will refer to \( j \) as the type of the measure \( \mu \).

Recall from Proposition 7.2 and Corollary 7.3 that \( \mathcal{E}_D(1,1) \) is an affine manifold in period coordinates, that the tangent space to \( \mathcal{E}_D(1,1) \) at a point \( M \) is naturally identified with the Lie algebra \( \mathfrak{t} \cong \mathfrak{sl}_2(\mathbb{R}) \times \mathbb{R}^2 \), and every Lie subalgebra of \( \mathfrak{t} \) defines a foliation of \( \mathcal{E}_D(1,1) \). Theorem 9.1 can be reformulated in Lie algebra terms, as follows. For any ergodic \( U \)-invariant measure \( \mu \) on \( \mathcal{E}_D(1,1) \), there is a Lie subalgebra \( \mathfrak{l}_\mu \) such that one leaf of the foliation corresponding to \( \mathfrak{l}_\mu \) is of full \( \mu \)-measure, and \( \mu \) is invariant under the vector fields corresponding to the elements of \( \mathfrak{l}_\mu \). The following list identifies \( \mathfrak{l}_\mu \) in each of the cases of Theorem 9.1:

1. \( \mathfrak{l}_\mu = \mathfrak{u} = \text{Lie}(U) \).
2. \( \mathfrak{l}_\mu = \mathfrak{u} \oplus \mathfrak{z} \) where \( \mathfrak{z} \) is the Lie algebra of \( Z \).
3. \( \mathfrak{l}_\mu \) is the Lie algebra of the subgroup of \( L \) fixing the holonomy of the two saddle connections in \( \Xi(\mu) \) (note that these vectors have the same holonomy).
4. \( \mathfrak{l}_\mu \) is the Lie algebra of the subgroup of \( L \) fixing the holonomy vector of either of the two saddle connections in \( \Xi(\mu) \) (note that these vectors have the same holonomy).
5. \( \mathfrak{l}_\mu \) is the Lie algebra of the subgroup of \( L \) fixing the vector \( (T,0) \in \mathbb{R}^2 \).
6. \( \mathfrak{l}_\mu \) is the Lie algebra of the subgroup of \( L \) fixing the holonomy of either of the two vectors in \( \Xi(\mu) \).
7. \( \mathfrak{l}_\mu = \mathfrak{t} \).

The proof of Theorem 9.1 occupies the rest of this section. It is modeled on arguments of Ratner dealing with horocycle orbits in homogeneous spaces (see [R] for a survey), which were first applied to spaces of translation surfaces in [EMaMo] and applied to the eigenform locus by Calta and Wortman in [CW]. We follow the outline of [CW] but make several improvements. These enable us to bypass some entropy arguments used in [CW] and give a clearer justification of Proposition 9.4 below.

### 9.1. The structure of Ratner’s proof

Ratner’s argument hinges on the analysis of transverse divergence of nearby trajectories for the \( U \)-action which she calls the \( R \)-property (see [R, p. 22]). Suppose we want to compare two orbits \( u_tM \) and \( u_tM' \) where \( M \) and \( M' \) are close. We can write \( M' = gM + v \). The case when \( (g,v) \in N \) is somewhat special and we consider it first. In this case \( g \) normalizes \( U \) and satisfies \( u_tg = gu_{\lambda t} \) for some \( \lambda > 0 \) independent of \( t \), and therefore

\[
 u_tM' = u_t(g,v)M = (u_tg,v)M = (gu_{\lambda t},v)M = (g,v)u_{\lambda t}M. 
\]

The divergence is caused by two factors. Since \( (g,v) \) is independent of \( t \) and small we think of \( u_tM' \) and \( u_{\lambda t}M \) as being close. In particular if \( \lambda \) is not 1 then the primary divergence of these two points is in the horocycle direction. Rescaling
time along the second orbit by replacing $t$ with $\lambda t$ has the effect of removing this divergence. If we refer to the remaining orbit divergence as transverse divergence, then in case $(g, v) \in N$ we see that the transverse divergence is constant.

Now let us consider the general case where $M' = gM + v$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, and $v$ arbitrary. In order to pick out the divergence in the directions transverse to the horocycle direction we will rescale the time factor along the second orbit. Specifically let $u_t M$ be the first orbit and write the second orbit as $u_s M'$ with $s = s(t)$. Unlike the previous case, the time parameter $s(t)$ for the second orbit will depend on the time parameter $t$ in the first orbit in a nonlinear way and the rescaling will only work in a finite interval of time. We will show that the time change is not too far from being linear. In order to do this we choose a family of transversals to the horocycle orbits and choose the function $s(t)$ so that $u_t M$ and $u_s M'$ stay in the same transversal. Recall from Proposition 7.2 that locally we can model $E_D(1, 1)$ on $L$. We choose as our family of transversals the elements of $L$ corresponding to elements $(g, v)$ where $g$ is lower triangular and $v$ is in $\mathbb{R}^2$, and both $g, v$ are small. This defines a submanifold transverse to $U$ of complementary dimension.

Since $M' = gM + v$ is defined, Proposition 4.4 tells us that $u_s gM + u_t v$ is also defined and is equal to $u_s (gM + v) = u_s M'$. Since $u_s gM = (u_s gu_{s^{-1}}) u_t M$, the pair $(u_s gu_{s^{-1}}, u_t v)$, considered as an element of $L$, gives a well-defined map which moves $u_t M$ to $u_s M'$. We calculate:

$$u_s gu_{s^{-1}} = \begin{pmatrix} a + sc & b - at + s(d - ct) \\ c & d - ct \end{pmatrix}.$$

In order for $u_t M$ and $u_s M'$ to lie on the same transversal, we require that the matrix $u_s gu_{s^{-1}}$ be close to the identity and be lower triangular.

**Definition 9.3.** For $t \in \mathbb{R}$ we set

$$s = s(t, g) = \frac{at - b}{d - ct}.$$

With this choice we have:

$$u_s(t) gu_{s^{-1}} = \begin{pmatrix} \frac{1}{d - ct} & 0 \\ c & d - ct \end{pmatrix}$$

(here $s(t) = s(t, g)$), so that $u_t M$ and $u_s(t) M'$ lie in the same transverse leaf as long as $t$ is chosen such that $d - tc$ is close to 1.

Note that while the initial displacement may consist of displacement in the Rel direction as well as the $G$-direction, the time change only depends on the initial displacement in the $G$ direction.

Let $\mu$ be a $U$-invariant ergodic measure on $E_D(1, 1)$ with $\Xi(\mu) = \emptyset$. Since $N$ acts on the set of surfaces with $\Xi(\mu) = \emptyset$ we can consider the subgroup of $N$ that preserves $\mu$. Let $N^\circ_\mu$ to be the connected component of the identity in $N_\mu$, where $N_\mu$ is the stabilizer of $\mu$ as in Definition 4.19.

**Proposition 9.4.** Let $\mu$ be a $U$-invariant ergodic measure on $E_D(1, 1)$ with $\Xi(\mu) = \emptyset$. For any $\varepsilon > 0$ there is a compact subset $K$ of $E_D(1, 1)$, consisting of generic points for $\mu$, with $\mu(K) \geq 1 - \varepsilon$ such that if $M \in K$ is a limit of $M_k = h_k M + v_k \in K$, with $h_k \in G$ and $v_k = (x_k, y_k) \in \mathbb{R}^2$, and $(h_k, v_k) \to (1_G, 0)$, $(h_k, v_k) \notin U$ for infinitely many $k$, then $\dim N^\circ_\mu \geq 2$. 
PROOF. Let $\Omega_\mu$ be the set of $\mu$-generic points for the $U$-action. The set $\Omega_\mu$ has full $\mu$ measure. Given $\varepsilon$, let $\Omega_1$ be a compact subset of $\Omega_\mu$ such that $\mu(\Omega_1) \geq 1 - \varepsilon$ and every surface in $\Omega_1$ satisfies $\Xi(\mu) = \emptyset$. By the Birkhoff ergodic theorem there is a compact subset $K$ of $\Omega_\mu \cap \mathcal{E}_D(1,1)$ of measure $\mu(K) \geq 1 - \varepsilon$, and a $T_0 > 0$, such that for all $M \in K$ and all $T > T_0$,

$$\frac{1}{T} \left| \left\{ t \in [0, T] : u_t M \in \Omega_1 \right\} \right| \geq 1 - 2\varepsilon. \tag{44}$$

We will show that this choice of $K$ satisfies the conclusion of the Proposition. So let $M_k, M, h_k$ and $v_k = (x_k, y_k)$ be as in the statement of the Proposition. Write $c_k = c(h_k)$ and $d_k = d(h_k)$.

By Corollary 4.20, $N_\mu$ is a closed subgroup of $N$ and hence a Lie group. Thus $N_\mu$ is a connected Lie group that contains $U$. In order to show that $\dim N_\mu \geq 2$ it suffices to rule out the possibility $U = N_\mu$, that is, in any neighborhood of the identity in $N$ we need to find an element in $N \smallsetminus U$ preserving $\mu$. If the sequence $(h_k, v_k)$ contains infinitely many elements of $N \smallsetminus U$ then it follows from Corollary 4.28 that we have such a sequence of elements. We may thus assume that $(h_k, v_k) \notin N$ for infinitely many $k$.

Write elements of $Z$ as $\text{Rel}_s$ (where defined). Assume first that there is a subsequence of indices so that

$$y_k = o(c_k). \tag{45}$$

In this case we will show that for each sufficiently small $\delta > 0$, there is an element

$$\ell = \ell(\delta) = g_{r, \text{Rel}_s} \tag{46}$$

preserving $\mu$, such that

$$\frac{\varepsilon \delta}{2} \leq \tau \leq 2\delta \quad \text{and} \quad \sigma = O(\delta). \tag{47}$$

The fact that $\ell$ can be written as $g_{r, \text{Rel}_s}$ means that it lies in $N$. The fact that $\tau \neq 0$ means that $\ell$ does not lie in $U$.

In light of Corollary 4.28, in order to show that $\ell \in N$ belongs to $N_\mu$, it suffices to prove that there are surfaces $M^{(1)}$ and $M^{(2)}$ in $\Omega_\mu$ such that $\ell M^{(1)} = M^{(2)}$. We will find $M^{(1)}$ and $M^{(2)}$ as limits of convergent subsequences $M_j \to M^{(1)}$ with $M_j \in \Omega_1$ and $g_j M_j + v_j \to M^{(2)}$ with $g_j M_j + v_j \in \Omega_1$ using the compactness of $\Omega_1$. We will show that $g_j$ converges to an element of $g_{r, s} \in B$ and $v_j$ converges to an element of $v \in Z$. Let $\ell = (g_{r, s}, v)$. Since elements of $K$ have no horizontal saddle connections and $g_{r, s} \in B$ it follows that $g_{r, s} M^{(1)}$ has no horizontal saddle connections. Since $v \in Z$, Theorem 6.1 then tells us that $\text{Rel}_{r}(g_{r, s} M^{(1)})$ exists so we can apply Proposition 4.6 to conclude that $\ell(M^{(1)}) = M^{(2)}$.

Recall that in order for $u_t M$ and $u_{s(t)} M'$ to be close it is necessary that $t$ be chosen so that $d - tc$ is close to 1.

**Definition 9.5.** For $g \in G$ set

$$I_g = \{ t \in \mathbb{R} : |\ln(d - tc)| \leq \delta \}, \quad \text{where} \quad d = d(g), c = c(g).$$

An elementary computation shows that there is a neighborhood $W$ of 1$_G$ such that if $g \in W$ then $0 \in I_g$, and for any $t_1, t_2$ in the connected component of $I_g$ containing 0, we have

$$\frac{|t_1 - t_2|}{2} \leq |s(t_1, g) - s(t_2, g)| \leq 2|t_1 - t_2|. \tag{48}$$
Here the $s(t, g)$ are defined by equation (42) and $\delta$ is assumed to be sufficiently small.

Let $P(t)$ be any linear function, that is $P$ is of the form $P(t) = At + B$ for some $A$ and $B$, and for $a < b$ let $\|P\| = \max_{x \in [a, b]} |P(x)|$. Then it is easy to check that

$$\frac{|\{t \in [a, b] : |P(t)| < \varepsilon \|P\|\}|}{b - a} < 2\varepsilon.$$  \hspace{1cm} (49)

The definition of $I_g$ and equation (43) ensure that when $t$ is at the an endpoint of $I_g$, the $g_t$-component of the displacing element $u_s gu_{-t}$ is $2\delta$. Namely, given $c \neq 0$ and $d$, the formula $T = T(c, d) = \frac{d - c^2 + k}{d}$ gives the two solutions to the equation $|\ln(d - cT)| = \delta$. Since $d_k \to 1$ and $c_k \to 0$, for all sufficiently large $k$ there is a unique such $T_k = T(c_k, d_k)$ with $T_k > 0$, and moreover for all sufficiently large $k$, we will have $T_k \geq T_0$. Here we take $\delta$ sufficiently small and $k$ sufficiently large so that

$$\frac{1}{2} \leq \frac{|d_k - c_k t - 1|}{\ln(d_k - c_k t)} \leq 2.$$ \hspace{1cm} (50)

We apply the above estimates for $T = T_k$ and $P(t) = d_k - c_k t - 1$ and obtain the existence of a subset of $t \in [0, T]$, of measure at least $1 - 8\varepsilon$, for which all of the following hold:

- $u_t M \in \Omega_1$ (by (44)).
- $u_s M_k \in \Omega_1$, where $s = s(h_k, t)$ (by (48), (44) and since $M_k \in K$).
- $|d_k - c_k t - 1| \geq \frac{\varepsilon}{T}$, and hence $|\ln(d_k - c_k t)| \geq \frac{\varepsilon}{T}$ (by (49) and (50)).

In particular, if we choose $\varepsilon < 1/8$ (which we may with no loss of generality), then the set of $t$ satisfying these three properties is nonempty. Fixing such a $t$, it follows from (43) and Proposition 4.4 that

$$u_s M_k = u_s (h_k M \ast v_k) = u_s h_k u_{-t} u_t M \ast u_s v_k$$

$$= \begin{pmatrix} \frac{1}{d_k - c_k t} & 0 \\ c_k & d_k - c_k t \end{pmatrix} u_t M \ast (x_k + s y_k, y_k).$$ \hspace{1cm} (51)

For each $k$ we let $t_k$ and $s_k = s(h_k, t_k)$ satisfy (51). Using the compactness of $\Omega_1$ we can take a subsequence along which we have convergence

$$u_{t_k} M \to M^{(1)}, \ u_{s_k} M_k \to M^{(2)}.$$  \hspace{1cm} (52)

By (51), we have $u_{s_k} M_k = \ell_k u_{t_k} M$ where

$$\ell_k = \begin{pmatrix} \frac{1}{d_k - c_k t_k} & 0 \\ c_k & d_k - c_k \end{pmatrix}, (x_k + s(t_k, h_k) y_k, y_k).$$

We claim that $M^{(2)} = \ell M^{(1)}$, for $\ell \in N$ satisfying the bounds (46), (47). It is enough to show that after passing to a further subsequence, the sequence $\ell_k$ converges to an element $\ell$ satisfying these bounds. Since $|\ln(d_k - c_k t)| \geq \frac{\varepsilon}{T}$, since $c_k \to 0$, and by definition of the interval $I_{h_k}$, the matrices $\begin{pmatrix} \frac{1}{d_k - c_k t_k} & 0 \\ c_k & d_k - c_k \end{pmatrix}$ converge to $g_\tau$ with $\tau$ as in (47). Since $x_k, y_k \to 0$, in order to obtain the required bound on $\sigma$ we
need to show that \( s_k y_k = O(\delta) \). But
\[
\begin{align*}
    s_k y_k &= y_k \frac{a_k t_k - b_k}{d_k - c_k t_k} = \frac{y_k}{c_k} \frac{a_k t_k - b_k}{d_k/c_k - t_k} \\
    \quad \overset{\text{(45)}}{=} o(1) \frac{a_k t_k - b_k}{c_k/d_k - t_k} = o(1) = O(\delta),
\end{align*}
\]
as required.

In case the condition of equation (45) does not hold, we have
\[ c_k = O(y_k). \]
In this case we can employ a similar argument, and find an element \( \ell(\delta) = g_\tau \text{Rel}_\sigma \) as in (46) which satisfies (instead of (47)) the estimates
\[ \sigma \geq c \delta \] for some \( c > 0 \), and \( \tau = O(\delta) \), \( \sigma = O(\delta) \).

We leave the details of this computation to the reader. In particular (52) implies that \( \ell(\delta) \not\in U \), as required. \( \square \)

**Remark 9.6.** The proof of [CW, Lemma 2], which is the statement analogous to Proposition 9.4, is incomplete. The formula preceding the final paragraph of the proof in [CW] would hold for an honest group action, but in this case requires justification.

**Proof of Theorem 9.1.** To each ergodic horocycle invariant measure \( \mu \) we associate a horizontal data diagram via Corollary 5.3. The possibilities for \( \Xi(\mu) \) given in the statement of Theorem 9.1 cover all cases by Proposition 8.5. In §8, for each possibility for \( \Xi(\mu) \), a measure was constructed. If it is a complete separatrix diagram, then by Proposition 8.1 and Corollary 8.2, we must be in cases (1) or (2). In all other cases for which \( \Xi(\mu) \neq \emptyset \), it was shown in Proposition 8.8 that there are no additional measures. This shows that if \( \Xi(\mu) \neq \emptyset \), then \( \mu \) must have type (1), (2), (3), (4) or (6).

We now treat the case \( \Xi(\mu) = \emptyset \). Recall from Definition 4.8 and Corollary 6.2 that \( \mathcal{H}'_\infty \) is the set of surfaces in \( \mathcal{H}(1,1) \) with no horizontal saddle connections joining distinct singular points. Then \( \mu(\mathcal{H}'_\infty) = 1 \) and \( N = B \times Z \) acts on \( \mathcal{H}'_\infty \). Recall that \( N^\circ_\mu \) is the connected component of the stabiliser of \( \mu \) so that \( U \subset N^\circ_\mu \).

We first claim that there is no \( U \)-orbit with positive measure. Indeed, if this were the case then by ergodicity, \( \mu \) is supported on one orbit which, by Poincaré recurrence, must be a periodic orbit. A periodic orbit is a special case of a minimal set so by Proposition 8.1, \( \Xi(\mu) \) consists of a full separatrix diagram. This contradicts our assumption that \( \Xi(\mu) = \emptyset \).

We now claim that we cannot have \( N^\circ_\mu = U \), i.e. that \( \dim N^\circ_\mu \geq 2 \). To see this, let \( \Omega_\mu \subset \mathcal{H}'_\infty \) be the set of generic points as in Proposition 4.27, and let \( K \) be as in Proposition 9.4. We have \( K \subset \Omega_\mu \). Since \( \mu \) assigns zero measure to individual \( U \)-orbits and \( K \) is a compact set of positive measure, \( K \) contains a sequence \( \{M_k\} \) such that \( M_k \to M \in K \) and none of the \( M_k \) are on the \( U \)-orbit of \( M \). According to Proposition 9.4, \( \dim N^\circ_\mu \geq 2 \).

Now suppose \( Z \subset N^\circ_\mu \). We will show that in this case \( \mu \) is the flat measure on \( \mathcal{L}^D(1,1) \). Let \( \mathcal{L} \) be an affine \( G \)-invariant subspace of \( \mathcal{H}(1,1) \), and let \( \mathcal{L}^{(1)} \) be its subset of area-one surfaces. We define the horospherical foliation for \( \{g_t\} \) to be the foliation into leaves locally modelled on the intersection of the tangent space of \( \mathcal{L}^{(1)} \) with the horizontal space \( H^1(S, \Sigma; \mathbb{R}_x) \) (the first summand in the splitting
This terminology is motivated by the fact that the \( \{ g_t \} \)-flow is non-uniformly hyperbolic and the above leaves are generically its strong unstable leaves. For the case \( L = E_D(1, 1) \), the leaves of the horospherical foliation are just the orbits of \( UZ \), so in the case we are now considering, the measure \( \mu \) is invariant under the horospherical foliation, and must be flat measure in light of the following:

**Claim 1.** Suppose that:

- \( L \) is \( GL_2(\mathbb{R}) \)-invariant and affine in period coordinates, and is defined by real-linear equations in \( H^1(S, \Sigma; \mathbb{R}^2) \cong H^1(S, \Sigma; \mathbb{C}) \).
- The flat measure on \( L^{(1)} \) obtained from applying the cone construction to \( L \) is \( G \)-invariant and ergodic.

Then any strong-stable invariant measure \( \mu \) on \( L^{(1)} \), with \( \Xi(\mu) = \emptyset \) coincides with the flat measure.

This result was proved by Lindenstrauss and Mirzakhani in [LM] under the additional hypothesis that \( L \) is the principal stratum, i.e. \( L = \mathcal{H}(1, \ldots, 1) \). In [SW4] we adapt the argument of Lindenstrauss and Mirzakhani to prove the more general statement given above. The special case we require here, namely \( L = E_D(1, 1) \), was explained by Calta and Wortman, see [CW, §6].

Now suppose \( \dim N^\circ_\mu \geq 2 \) but \( N^\circ_\mu \) does not contain \( Z \). Since \( Z \) is a normal subgroup of \( N \), \( N^\circ_\mu \) does not contain a conjugate of \( Z \). Let \( A \) be the diagonal group in \( N \). Up to conjugacy, the only three one-parameter subgroups of \( N \) are \( U, Z \) and \( A \), so we find that \( N^\circ_\mu \) contains a conjugate of \( A \). We can write this conjugate as \( A' = nAn^{-1} \) where \( n \in N \), and since \( N = B \times Z \), we can write \( n = \text{Rel}_Tua \) where \( a \in A, u \in U \) and \( T \in \mathbb{R} \). Since \( N^\circ_\mu \) contains \( U \) and \( A = aAa^{-1} \) we find that \( N^\circ_\mu \) contains \( \text{Rel}_T \)A(\text{Rel}_-T). Therefore the measure \( \mu' = (\text{Rel}_-T)\ast \mu \) is \( B \)-invariant. By a result of Eskin, Mirzakhani and Mohammadi [EMiMo], \( \mu' \) is \( G \)-invariant, and thus by McMullen’s classification of \( G \) invariant measures [McM3, McM4], the following are the only possibilities:

- \( \mu' \) is the \( G \)-invariant measure on a closed \( G \)-orbit, and \( D \) is either a square or \( D = 5 \). This implies that \( \mu = \text{Rel}_T \ast \mu' \) is as described in case (5).
- \( \mu' \) is the flat measure on \( E_D(1, 1) \). Since the flat measure is \( Z \)-invariant, \( \mu = \mu' \) and we are in case (7).

\[ \Box \]

### 10. Injectivity and nondivergence

In this section we will prove some results which will be used in the proof of Theorem 11.1. The strategy of proof involves showing that typical horocycle orbits do not spend too much time close to a closed \( G \)-orbit \( L \) or translations of closed \( G \)-orbits by the \( \text{Rel} \) flow. The argument depends on the fact that a point in an eigenform locus which is close to the \( \text{Rel}_1 \) orbit of \( L \) but not in the \( \text{Rel}_1 \) orbit of \( L \) drifts slowly in the direction of the \( \text{Rel}_1 \) vector field. In order to exploit this phenomenon it is useful to have coordinates close to the \( \text{Rel}_1 \) orbit of \( L \). This argument is similar in spirit to the linearization method of Dani-Margulis, see [KSS, §3.4].

The following is an analog of [KSS, Cor. 3.4.8]. For a stratum \( \mathcal{H} \) and \( r > 0 \), let \( \mathcal{H}_r \) be the set of surfaces \( M \) in \( \mathcal{H} \) for which there are no horizontal saddle
Corollary 6.2. We prove injectivity. Suppose that $M$, $M'$ on $L_x$ are such that $\text{Rel} \, r$ then the map $L_x$ is well-defined and injective on $L_x \times [−T, T] \times \{0\}$, where $L_x = \mathcal{L} \cap H_r$. In particular, if we set $L_x = \mathcal{L} \cap H_x$ then the map (53) is well-defined and injective on $L_x \times \mathbb{R} \times \{0\}$.

Proof. We begin by proving that the map (53) is well-defined and injective on $L_x \times \mathbb{R} \times \{0\}$. The assertion that the map is well-defined on $L_x$ follows from Corollary 6.2. We prove injectivity. Suppose that $M_1, M_2 \in L_x$, and $x_1, x_2 \in \mathbb{R}$ are such that $\text{Rel}_{x_1} M_1 = M_1 + (x_1, 0) = M_2 + (x_2, 0) = \text{Rel}_{x_2} M_2$. If $x_1 = x_2$ then by applying $\text{Rel}_{−x_1}$ we find that $M_1 = M_2$, so we can assume $x_1 \neq x_2$. Let $x_3 = x_1 − x_2$. Then by Proposition 4.5

(55) \[ \text{Rel}_{x_1}(\text{Rel}_{x_2} M_1) = \text{Rel}_{x_3}(\text{Rel}_{x_2} M_2) = \text{Rel}_{x_1} M_2, \]

and $\text{Rel}_{x_3}$ is not the identity element of $Z$. Our next objective is to exploit this relationship.

Let $\nu$ denote the $G$-invariant measure on $\mathcal{L}$ coming from the Haar measure on $G$, and let $\nu_1 = \text{Rel}_{x_1, *} \nu$. By [DS], the set of $U$-generic points for $\nu$ in $\mathcal{L}$ is the set of surfaces which are not on periodic $U$-orbits, and by [Ve2], this is the set of surfaces without horizontal saddle connections. By Proposition 4.27, the same conclusion holds for $\nu_1$, that is the set of $U$-generic points for $\nu_1$ in $\text{Rel}_{x_1} \mathcal{L}$ are the surfaces without horizontal saddle connections. By (55), $\text{Rel}_{x_3}$ maps $\text{Rel}_{x_1} M_1$, which is a generic point for $\nu_1$, to $\text{Rel}_{x_1} M_2$, which is another generic point. In light of Corollary 4.28, $\text{Rel}_{x_1}$ preserves $\nu_1$. The stabilizer of $\nu$ in $N$ contains the diagonal subgroup $A$ and hence the stabilizer of $\nu_1$ in $N$ contains a non-unipotent element (take for example a nontrivial element of the conjugate $\text{Rel}_{x_1, A \text{Rel}_{−x_1}}$ in $B \times Z$). Proposition 4.21 implies that the stabilizer of $\nu_1$ properly contains the group $ZU$. According to Theorem 9.1, the only $U$-invariant measure on $\mathcal{E}_D(1, 1)$ whose stabilizer properly contains $ZU$ is the flat measure, so $\nu_1$ is the flat measure. This implies that $\nu = \text{Rel}_{−x_1, g} \nu_1$, is also the flat measure contradicting the assumption that $\nu$ is supported on a closed $G$ orbit. This proves the second assertion of the Lemma.

We now prove the first assertion of the Lemma. According to Corollary 6.2, the map (53) is well-defined on $L_x \times [−T, T] \times \{0\}$ if $r > T$. It remains to show that this map is injective when $r$ is sufficiently large. Set $M' = \text{Rel}_{x_1} M_1$. Suppose that $M' = \text{Rel}_{x_2} M_2$ for $M_1, M_2 \in L_x$ and $x_1, x_2 \in \mathbb{R}$. According to the first part of the proof, at least one of the surfaces $M_i$ has horizontal saddle connections. Since the $Z$-action preserves the property of having horizontal saddle connections (by Corollary 6.3), both surfaces have horizontal saddle connections, and since
\(\mathcal{L}\) is a closed \(G\)-orbit, they lie on closed \(U\)-orbits and have a horizontal cylinder decomposition. Since circumferences and heights of cylinders are the same for \(M_i\) and \(M'\), they are the same for \(M_1\) and \(M_2\) and this ensures that \(M_1\) and \(M_2\) lie on the union of one of finitely many closed horocycles \(Ux_1, \ldots, Ux_t\) of equal length. Suppose first that this length is 1, and denote the union of the horocycles of length 1 by \(A_1\). Let \(r_1 > 0\) be the length of the shortest horizontal saddle connection on any surface in \(A_1\), and choose \(T_1 \in (0, r_1)\) for which the map (53) is injective on \(A_1 \times [-T_1, T_1] \times \{0\}\). Such a \(T_1\) exists because the Rel vector field is transverse to \(\mathcal{L}\), and by a compactness argument. Now suppose the length of the closed horocycles is \(L\). Then for \(r_0 \equiv \log L\), the union of the closed horocycles is \(A_L \equiv g_{r_0} A_1\), the length of the shortest saddle connections on these horocycles is \(r_L \equiv e^{r_0/2} r_1\) and the map (53) is injective on \([-T_L, T_L]\) where \(T_L \equiv e^{r_0/2} T_1\). That is, the ratio \(r_L/T_L\) is independent of \(L\). In particular, for any \(T > 0\), we can choose \(L\) so that \(T = T_L\) and choose \(r > r_L\). Since

\[\mathcal{L}_r \subset \mathcal{L} \setminus \bigcup_{r_L \leq r} A_L,\]

for these choices, the map (53) will be injective on \(\mathcal{L}_r \times [-T, T] \times \{0\}\). \(\Box\)

**Remark 10.3.** The condition that \(M \in \mathcal{L}_r\) is necessary for the validity of Lemma 10.2. When \(D\) is a square and \(\mathcal{L}\) is the closed \(G\)-orbit of a square-tiled surface, there are surfaces \(M \in \mathcal{L}\), with a horizontal cylinder decomposition of type (A), whose real Rel orbit is compact. When \(D = 5\) the ZU-orbit of a surface in \(\mathcal{L}\) with a cylinder decomposition of type (A) is also not injectively embedded. That is, in both cases, injectivity will fail if \(M\) has short horizontal saddle connections.

We will also need the following quantitative nondivergence results for the horocycle flow. The following results are proved in [MW1]:

**Theorem 10.4.** For any stratum \(\mathcal{H}\) there are constants \(C\) and \(\alpha\) such that for any \(M \in \mathcal{H}\), any \(\rho \in (0, 1]\) and any \(T > 0\), if for any saddle connection \(s\) for \(M\), \(\max_{s \in [0, T]} \|\text{hol}(u_s M, \delta)\| \geq \rho\) then

\[|\{s \in [0, T] : u_s M\} has saddle connections of length < \varepsilon\}| < CT \left(\frac{\varepsilon}{\rho}\right)^{\alpha}.\]

In particular:

(I) For any \(\varepsilon > 0\) and any compact \(K' \subset \mathcal{H}\) there is a compact \(K \subset \mathcal{H}\) such that for any \(T > 0\) and any \(M \in K'\),

\[(56)\]

\[\frac{1}{T} |\{s \in [0, T] : u_s M \notin K\}| < \varepsilon.\]

(II) For any \(\varepsilon > 0\) there is a compact \(K \subset \mathcal{H}\) such that for any \(M \in \mathcal{H}_K\), there is \(T_0 > 0\) such that for all \(T > T_0\), (56) holds.

We will need a refinement of statement (I).

**Proposition 10.5.** For any positive constants \(\varepsilon\) and \(r\), and any compact set \(L \subset \mathcal{H}_K\), there is an open neighborhood \(W\) of \(L\) and a compact \(\Omega \subset \mathcal{H}\) containing \(W\), such that surfaces in \(\Omega\) have no horizontal saddle connections shorter than \(r\), and such that for any surface \(M\) and any interval \(I \subset \mathbb{R}\) for which \(u_s M \in W\) for
some \( s \in I \), we have

\[
\frac{|\{ s \in I : u_s M \neq \Omega \}|}{|I|} < \varepsilon.
\]

**Proof.** If \( s_0 \in I \) is such that \( u_{s_0} M \in W \) then by making the change of variables \( s \mapsto s - s_0 \) we can assume \( M \in W \) and \( I = [a, b] \) with \( a \leq 0 \leq b \). By considering separately the subintervals \([a, 0]\) and \([0, b]\), we can assume that \( I = [0, T] \) for some \( T > 0 \). Given a surface \( M \) we let \( \text{hol}(M) \) be the subset of \( \mathbb{R}^2 \) consisting of the holonomies of the saddle connections of \( M \), and set

\[
K_{r_1, r_2} = \{ M \in \mathcal{H} : \text{hol}(M) \cap ((-r_1, r_1) \times (-r_2, r_2)) = \emptyset \}.
\]

Then \( K_{r_1, r_2} \) is compact for any \( r_1, r_2 \).

For any surface \( M \) and saddle connection \( \delta \) we define \( x(M, \delta), y(M, \delta) \) via the formula

\[
\text{hol}(M, \delta) = (x(M, \delta), y(M, \delta)).
\]

Given \( \varepsilon, r \) and \( L \) as in the statement of the proposition, let \( \eta \) satisfy \( C\eta^\alpha < \varepsilon \), where \( C \) and \( \alpha \) are as in Theorem 10.4, and let \( t_0 = 2 \log \left( \frac{\eta}{\sqrt{2}r} \right) \) and \( \sigma' = \frac{\eta^2}{2r} \).

These choices guarantee

\[
e^{t_0/2} = \frac{\eta}{\sqrt{2}} \quad \text{and} \quad e^{-t_0/2}\sigma' = \frac{\eta}{\sqrt{2}}.
\]

Let \( L'' = g_{t_0}(L) \). Then \( L'' \) is a compact subset of \( \mathcal{H}_\infty \). Compactness implies that there is a constant \( \theta > 0 \) with the property that for any \( M \in L'' \) with a saddle connection \( \delta \) of length less than 1, the vertical component \( y = y(M, \delta) \) satisfies \( |y| \geq \theta \). Moreover by making \( \theta \) smaller if necessary, we can ensure that the same property holds for all surfaces in a neighborhood \( W'' \) of \( L'' \). Let \( T_1 = \frac{\theta}{r} \). Then for any \( M \in W'' \) and any saddle connection \( \delta \) for \( M \), either \( \|\text{hol}(M, \delta)\| \geq 1 \) or

\[
\|\text{hol}(u_{T_1} M, \delta)\| \geq |x(u_{T_1} M, \delta)| = |x(M, \delta) + T_1 y(M, \delta)| \geq 2 \frac{\theta}{\varepsilon} |y(M, \delta)| - 1 \geq 1.
\]

That is, \( \max_{s \in [0, T_1]} \|\text{hol}(u_s M, \delta)\| \geq 1 \), so we can apply Theorem 10.4 with \( \rho = 1 \). We obtain that for all \( T \geq T_1 \), and all \( M \in W'' \),

\[
\frac{1}{T} |\{ s \in [0, T] : u_s M \text{ has saddle connections of length } < \eta \}| < \varepsilon.
\]

We now claim that if we set \( \Omega = K_{r, \sigma} \) for any \( \sigma \leq \sigma' \) then (57) holds when \( M \in W' \) and \( I = [0, T] \), \( T > e^{-t_0} T_1 \). Indeed suppose \( M \in W' \), \( \delta \) is a saddle connection for \( M \), and \( s \in [0, T] \) such that \( \text{hol}(u_s M, \delta) \in (-r, r) \times (-\sigma, \sigma) \).

Let \( M'' = g_{t_0} M \in W'' \), \( s' \) be \( e^{-t_0} s \) so that \( g_{t_0} u_s M = u_{s'} M'' \). By (59),

\[
\text{hol}(u_{s'} M'', \delta) = g_{t_0} \text{hol}(u_s M, \delta) \in \left( -\frac{\eta}{\sqrt{2}}, \frac{\eta}{\sqrt{2}} \right) \times \left( -\frac{\eta}{\sqrt{2}}, \frac{\eta}{\sqrt{2}} \right)
\]

and in particular \( \|\text{hol}(u_{s'} M'', \delta)\| < \eta \). Since the change of variables \( s \mapsto s' \) is linear, (57) follows from (60).

This completes the proof for the interval \( I = [0, T] \) when \( T > e^{-t_0} T_1 \). Finally, since \( L \subset \mathcal{H}_\infty \) is compact, by making \( \sigma \) small enough, we see that \( \Omega \) contains a neighborhood of \( L \), so that taking a small enough neighborhood \( W \) of \( L \) contained
in \(W\), we can ensure that \(W \subset \Omega\) and also that for any \(s \in [0,e^{-t_0} T_1]\) and any \(M \in W\), \(\text{hol}(u_s M) \cap (-r, r) \times (-\sigma, \sigma) = \emptyset\).

In light of Proposition 4.3, the following is an immediate consequence of Lemma 10.2 and Proposition 10.5:

**Corollary 10.6.** Given positive numbers \(T\) and \(\varepsilon\), and a compact subset \(L \subset \mathcal{L}_\mathcal{X}\), there is \(\delta > 0\), a neighborhood \(W\) of \(L\) and a compact set \(\Omega \subset \mathcal{L}_\mathcal{X}\), containing \(W\), such that the map (53) is well-defined, continuous and injective on \(\Omega \setminus [-T, T] \times [-\delta, \delta]\), and for any interval \(I\) and any \(M \in \mathcal{H}\), if there is \(s_0 \in I\) such that \(u_{s_0} M \in W\) then equation (57) holds.

\[ \square \]

We take this opportunity to record a consequence of Proposition 10.5, which will not be used in this paper but is of independent interest.

**Corollary 10.7.** For any \(\varepsilon > 0\) and any compact \(L \subset \mathcal{H}_\mathcal{X}\), there is a compact \(\Omega \subset \mathcal{H}_\mathcal{X}\) containing \(L\) such that for any \(M \in L\) and any interval \(I \subset \mathbb{R}\) containing \(0\), (57) holds.

**Proof.** For any \(\varepsilon > 0\) and \(j = 1, 2, \ldots\), let \(\varepsilon_j = \frac{\varepsilon}{2^j}\). By Proposition 10.5, we can find \(\Omega_j\) containing \(L\) such that all surfaces in \(\Omega_j\) have no horizontal saddle connections of length less than \(j\), and (57) holds with \(\varepsilon_j\) in place of \(\varepsilon\) for any interval \(I\) containing \(0\) and any \(M \in L\). Then \(\Omega = \bigcap_j \Omega_j\) has the required properties. \(\square\)

11. All horocycle orbits are generic

The goal of this section is to prove the following more detailed version of Theorem 1.2.

**Theorem 11.1.** Let \(M \in \mathcal{E}_D(1,1)\). Then \(M\) is generic for a \(U\)-invariant ergodic measure \(\mu_M\), and the type of \(\mu_M\) (as described in Definition 9.2) is as follows:

- \(\mu_M\) has type (1) when \(\Xi(M)\) is a complete separatrix diagram of types (B), (C) or (D), or a complete separatrix diagram of type (A) with commensurable moduli.
- \(\mu_M\) has type (2) when \(\Xi(M)\) is a complete separatrix diagram of type (A) with two cylinders of incommensurable moduli.
- \(\mu_M\) has type (3) when \(\Xi(M)\) consists of two saddle connections joining distinct singularities, which disconnect \(M\) into two isogenous tori glued along a slit, as in case (iii) of §8.4.
- \(\mu_M\) has type (4) when \(\Xi(M)\) consists of one saddle connection joining distinct singularities, as in case (ii) of §8.4.
- \(\mu_M\) has type (5) when \(\Xi(M) = \emptyset\) and there is \(s \in \mathbb{R}\) so that \(\text{Rel}_s M\) is a lattice surface.
- \(\mu_M\) has type (6) when \(\Xi(M)\) is a pair of saddle connections which do not disconnect \(M\), as in case (iv) of §8.4.
- \(\mu_M\) has type (7) if it does not correspond to one of the previous cases i.e. \(\Xi(M) = \emptyset\) and \(M\) is not the result of applying the \(\text{Rel}\) flow to a lattice surface.

**Remark 11.2.** It would be interesting to characterize case (5) explicitly. That is, give a geometric characterization of those surfaces \(M \in \mathcal{E}_D(1,1)\) with \(\Xi(M) = \emptyset\), for which there is \(s \in \mathbb{R}\) such that \(\text{Rel}_s M\) is a lattice surface.
The proof of Theorem 11.1 relies on an analog of the ‘linearization method’, see [KSS, §3.4]. We need the following notion.

**Definition 11.3.** The support of a $U$-invariant $U$-ergodic measure on $\mathcal{E}_D(1,1)$ is called a sheet. The type of a sheet of the sheet is the type of the corresponding measure.

Sheets of the same type typically appear in families. It will be convenient to partition the sheets into families, called beds. In the sequel, a bed will be a measurable set which is a union of sheets in $\mathcal{E}_D(1,1)$. We further require the sheets to be of a fixed type $(j)$, for $j \in \{3, 4, 5, 6, 7\}$, or to be of one of the two types $(1)$ or $(2)$. A bed corresponding to sheets of type $(1)$ or $(2)$ will be called a bed of minimal sets and a bed corresponding to sheets of type $j \in \{3, \ldots, 7\}$ will be called a bed of type $(j)$. If $\mathcal{B}$ is a bed corresponding to measures $\mu$ of a certain type, we say that $\mu$ belongs to $\mathcal{B}$, and we define $\Xi(\mathcal{B})$ to be $\Xi(\mu)$ for $\mu$ belonging to $\mathcal{B}$.

The reason for combining sheets of type $(1)$ and $(2)$ into one bed, is that an arbitrarily small perturbation of a sheet of type $(1)$ can be a sheet of type $(2)$ and conversely (the condition that the moduli of the cylinders are rationally related is not stable under small perturbations).

In our definition of beds we only required them to be measurable but in fact they can be chosen so that they have a nicer structure. Continuing with the discussion following Definition 9.2, one can partition the sheets into beds which are almost everywhere locally modeled on affine subspaces of the Lie algebra $\mathfrak{t}$. However the corresponding affine subspaces of $\mathfrak{t}$ need not correspond to Lie subalgebras. Additionally beds may have complicated topology (e.g. boundary). This will be discussed further in [SW3].

For our analysis, the following property of a bed will be helpful:

**Definition 11.4.** We will say that a sequence of sets $K_1, K_2, \ldots$ exhausts $\mathcal{B}$ if the $K_i$ are compact, and $\mu(\bigcup_i K_i) = 1$ for any measure $\mu$ which belongs to $\mathcal{B}$.

**Proposition 11.5.** For each type $j \leq 6$, all sheets in $\mathcal{E}_D(1,1)$ of type $(j)$ are contained in a finite or countable union of beds, and for each bed $\mathcal{B}$ there is a countable sequence of compact sets which exhausts $\mathcal{B}$. Explicitly these properties are satisfied by the following choices:

- For types $j \in \{3, 4, 6, 7\}$, we let $\mathcal{B}$ be the union of all sheets of type $(j)$, and we let $K_1$ be the closure of the set of $M \in \mathcal{E}_D(1,1)$ which have no saddle connections shorter than $1/i$, and such that $\{\delta \in \Xi(M) : ||\text{hol}(M, \delta)|| \leq i\}$ and $\Xi(\mathcal{B})$ are the same (as horizontal data diagrams).
- For types $j \in \{1, 2\}$ we let $\mathcal{B}$ be the union of all sheets of type $1$ or $2$, and we define $K_i$ as before.
- If $j = 5$, for each closed $G$-orbit $\mathcal{L}$ we take $\mathcal{B} = \bigcup_{t \in \mathbb{R}} \text{Rel}_t(\mathcal{L})$. Let $\nu_0$ be the Haar measure on $\mathcal{L}$, let $\mathcal{L}_x$ be as in (54), let $L_1, L_2, \ldots$ be a nested sequence of compact subsets of $\mathcal{L}_x$ with $\nu_0(\bigcup_i L_i) = 1$, and let
  \[ K_i = \bigcup_{||t|| \leq i} \text{Rel}_t(L_i). \]

**Proof.** It is clear that the beds listed above contain all sheets of type $j$. In all cases except $j = 5$ there is just one bed. In the case $j = 5$ the number of beds is at most countable, since each $\mathcal{E}_D(1,1)$ contains at most finitely many closed $G$-orbits.
We now show that in all cases, the sets $K_i$ listed above exhaust the bed. Suppose first that $Ξ(B) \neq \emptyset$. It is clear that each $K_i$ as in the statement of the proposition is compact. The set $\bigcup K_i$ contains all $M \in B$ for which $Ξ(M) = Ξ(B)$, so Corollary 5.3 implies that $µ(∪_i K_i) = 1$ for any measure $µ$ which belongs to $B$.

Now suppose $B$ is a bed of type (5). Let $L, ν_0, L_i, K_i$ be as above. Each $K_i$ is compact by Proposition 4.3. Also, for any measure $µ$ so Corollary 5.3 implies that $ν_0$ arbitrary we must have $µ$ such that $ν_0 \in L_0$. Now suppose $ν_0(∪_i L_i) = 1$. For each $t$, since $∪_i K_i$ contains $∪_i Rel(L_i)$ for each $t$, we have $ν_t(∪_i K_i) = 1$ for each $t$.

The following result summarizes a strategy for proving equidistribution results which we will use repeatedly.

**Proposition 11.6.** Let $\{µ_t\}$ be a collection of measures where $t$ ranges over either the positive integers or non-negative real numbers. Suppose the following hold:

(a) The sequence $(µ_t)$ has no escape of mass as $t \to \infty$: i.e. for any $ε > 0$ there is a compact $K \subseteq E_D(1,1)$ and $t_0$ such that for all $t \geq t_0$, $µ_t(K) \geq 1 - ε$.
(b) Any convergent subsequence $(µ_{t_k})$ of $(µ_t)$ with $t_k \to \infty$ converges to a measure which is invariant under a conjugate of $U$ by an element of $G$.
(c) For any bed $B \subseteq E_D(1,1)$ there is a sequence $K_1, K_2, \ldots$ of sets which exhaust $B$, and for any i and any $ε > 0$, there is $t_0$ and an open set $U$ containing $K_i$ such that for all $t \geq t_0$, $µ_t(U) < ε$.

Then the sequence $µ_t$ converges to the flat measure on $E_D(1,1)$ as $t \to \infty$.

**Proof.** It suffices to show that any subsequence $(µ_{t_k})$ of $(µ_t)$, with $t_k \to \infty$ contains a further subsequence converging to the flat measure. So re-indexing, suppose $(µ_t)$ is already a subsequence. Since there is no escape of mass, the set $(µ_t)$ is precompact with respect to the weak-* topology. So passing to a subsequence we can assume that $µ_k$ converges weak-* to a probability measure $ν$ and we need to show that $ν$ is the flat measure. By assumption (b), $ν$ is invariant under a conjugate $U' = g^{-1} Ug$ of $U$, and hence $g_\ast ν$ is $U$-invariant. We need to show that $g_\ast ν$ is the flat measure, as this will imply that $ν$ is the flat measure as well. To simplify notation we therefore replace $g_\ast ν$ with $ν$ to assume that $ν$ is $U$-invariant. Using the ergodic decomposition of $ν$ and the fact that there are countably many beds, we can write $ν = \sum_j ν_j$ where each $ν_j$ is supported on the beds of type $j$, and (after normalizing $ν_j$ to be a probability measure) is a convex combination of the measures belonging to these beds. We must show that $ν_1 = \cdots = ν_6 = 0$. We derive this from assumption (c), as follows.

Fix a type $j \leq 6$ and let $B \subseteq E_D(1,1)$ be a bed of type $j$. Let $K_1, K_2, \ldots$ be compact sets as described in assumption (c). Since $µ(∪ K_i) = 1$ for every $U$-invariant ergodic measure $µ$ belonging to $B$, in order to show $ν_j = 0$ it suffices to show that $ν_j(∪ K_i) = 0$. Suppose by contradiction that $a = ν_j(E_D(1,1))$ is strictly positive. For any $i$, and any $ε > 0$, let $t_0$ and $U$ be as in (c). There is a continuous compactly supported function $ϕ : E_D(1,1) \to [0,1]$ which is identically 1 on $K_i$ and vanishes outside $U$. The definition of the weak-* topology and condition (c) now ensure that $ν_j(K_i) \leq \int_{E_D(1,1)} ϕ \, dν_j \leq \frac{1}{a} \lim_k \int_{E_D(1,1)} ϕ \, dµ_k \leq \frac{ε}{a}$. Since $ε > 0$ was arbitrary we must have $ν_j(K_i) = 0$ for each $i$, and hence $ν_j(∪ K_i) = 0$. □
11. ALL HOROCYCLE ORBITS ARE GENERIC

Proof of Theorem 11.1. In this proof we will say that $M$ is of type (j) (where $j \in \{1, \ldots, 7\}$) if $\Xi(M)$ is topologically equivalent to $\Xi(\mu)$ and $\mu$ is of type $j$.

Step 1: $M$ is of type (1) through (6). If $\Xi(M)$ decomposes $M$ into horizontal cylinders then the $U$-action on the orbit-closure of $M$ is conjugate to an irrational straightline flow on a torus and so every orbit is equidistributed. This is what happens when $\Xi(M)$ is a complete separatrix diagram, i.e. in cases (1) or (2). When $\Xi(M) \neq \emptyset$ but $\Xi(M)$ is not a complete separatrix diagram, the $U$-action on $UM$ is obtained from the $U$-action on a simpler space via a $U$-equivariant map $\Psi_T$. Namely, by Proposition 8.8, in each of the cases (3), (4) and (6) this simpler space is a finite volume homogeneous space $G/\Gamma$, and $M = \Psi_T(M_0)$ where $M_0 \in G/\Gamma$ does not lie on a periodic $U$-orbit, since $\Xi(M)$ is not a complete separatrix diagram. The equidistribution of $UM_0$ in $G/\Gamma$ follows from [DS], and the equidistribution of $M$ follows from the fact that $\Psi_T$ is $U$-equivariant and continuous. The same argument applies when $\Xi(M) = \emptyset$ and $M$ belongs to $Rel_iL$, for a closed $G$-orbit $L$ and $s \in \mathbb{R}$.

Step 2: $M$ is of type (7) and $\Xi(B) \neq \emptyset$: When $M$ is of type (7) we verify the hypotheses of Proposition 11.6 for the collection of measures $\{\mu_t : t > 0\}$ defined by averaging along the orbit of $M$; that is, $\mu_t = \nu(M, t)$, where $\nu(M, t)$ is as in (4.25). Hypothesis (a) of Proposition 11.6 follows from Theorem 10.4 (I) and hypothesis (b) is immediate from the definition of $\mu_t$. To verify (c) we first discuss beds $B$ for which $\Xi(B) \neq \emptyset$, adapting an argument of [EMaMo]. Let $K_1, K_2, \ldots$ be the exhaustion of the bed $B$, as in Proposition 11.5. Fix $i$ and let $\varepsilon > 0$. Then any surface in $K_i$ contains a horizontal saddle connection of length at most $i$. Let $C(\varepsilon)$ be the open set of surfaces whose shortest saddle connection is shorter than $\varepsilon$. By Theorem 10.4 (II) there is $\delta > 0$ (depending only on the stratum $H(1, 1)$) such that every surface $M'$ without horizontal saddle connections satisfies

\[ \limsup_{t \to \infty} \mu_{M', t}(C(\delta)) < \varepsilon \tag{61} \]

(where $\mu_{M', t} = \nu(M', t)$ is as in Definition 4.25). Let

\[ t_0 < 2(\log \delta - \log i), \tag{62} \]

so that if $M_1$ has a horizontal saddle connection of length less than $i$, then $g_{t_0}M_1$ has a saddle connection of length less than $\delta$; in other words, $K_1 \subset \mathcal{U} \text{s.t. } g_{t_0}(C(\delta))$. Since $g_{t_0}u_xg_{-t_0} = u_{e^{x}t}$, setting $M' = g_{t_0}M$, we have $g_{t_0}\mu_{M, t} = \mu_{M', e^{x}t}$ and (61) implies that

\[ \limsup_{t \to \infty} \mu_{M', t}(U) = \limsup_{t \to \infty} \mu_{M', t}(C(\delta)) < \varepsilon, \]

so (c) is satisfied.

Step 3: $M$ is of type (7) and $\Xi(B) = \emptyset$. That is $B$ is a bed of type (5). The measures belonging to $B$ are of the form $Rel_s\mu$, where $s \in \mathbb{R}$ and $\mu$ is the $G$-invariant measure on a closed $G$-orbit $L$. Let $L_{\infty}$ be as in (54). Set

\[ F(M, x, y) = M + (x, y), \tag{63} \]

postponing for the moment questions of domain of definition of $F$. Note that $F(M, x, 0) = Rel_xM$, and when there is no risk of confusion, we will write $F(M, x, 0)$ simply as $F(M, x)$. Let $L_i$ and $K_i$ be as in Proposition 11.5, so that the $K_i$ exhaust $B$. Note that $L_i \subset L_{\infty}$ for each $i$ and

\[ K_i = F(L_i \times [-i, i] \times \{0\}). \]
We will verify (c) for the sets $K_i$. Given $i$ and $\epsilon > 0$, choose
\[ j > \frac{(8 + \epsilon)(i + 1)}{\epsilon}. \]

Corollary 10.6 gives us a constant $\delta > 0$, an open set $W \subset \mathcal{L}$ containing $L_i$, and a compact set $\Omega$ containing $W$ such that: (1) $F$ is well-defined, continuous and injective on $\Omega \times [-j, j] \times [-\delta, \delta]$, and (2) whenever $M' \in \mathcal{L}$ and $I \subset \mathbb{R}$ is an interval such that $u_s M' \in W$ for some $s \in I$, we have
\[ \frac{1}{|I|} \left| \left\{ s \in I : u_s M' \notin \Omega \right\} \right| < \frac{\epsilon}{4}. \]

Now set
\[ U = F(W \times (-i + 1, i + 1) \times (-\delta, \delta)). \]

Then $U$ is a neighborhood of $K_i$, and we need to show that $\mu_{M,t}(U) < \epsilon$ for all sufficiently large $t$. That is, we need to find $t_0$ so that for $t > t_0$,
\[ \frac{|\hat{I} \cap [0, t]|}{t} < \epsilon, \quad \text{where we set } \hat{I} = \{ s \in \mathbb{R} : u_s M \in U \}. \]

Whenever $s_0 \in \hat{I}$, there are $M' = M'(s_0) \in W$ and $(x, y) = (x(s_0), y(s_0)) \in (-i + 1, i + 1) \times (-\delta, \delta)$ such that $u_{s_0} M = F(M', x, y)$. Since $M$ is not of type (5) we have $y(s_0) \neq 0$. We define the following intervals:
\[ \mathcal{I} = \mathcal{I}(s_0) \overset{\text{def}}{=} \{ s : |x + sy| \leq i + 1 \}, \]
\[ \mathcal{J} = \mathcal{J}(s_0) \overset{\text{def}}{=} \{ s : |x + sy| \leq j \}. \]

These are nested bounded intervals with common midpoint $s = -\frac{x(s_0)}{y(s_0)}$. If an interval $I$ contains an endpoint of $\mathcal{I}$ and intersects $\mathcal{I}$ then it contains one of the two connected components of $\mathcal{J} \cap \mathcal{I}$. These two components have equal lengths since the two intervals share the same midpoint and by (64),
\[ \frac{|\mathcal{I} \cap \mathcal{I}|}{|\mathcal{J} \cap \mathcal{I}|} \leq \frac{2(i + 1)|y_1|}{(j - i - 1)|y_1|} < \frac{\epsilon}{4}. \]

We claim that if $s \in \mathcal{J} \cap \mathcal{I}$ then either $u_s M' \notin \Omega$ or $u_{s_0 + s} M \notin U$. Indeed, suppose $s \in \mathcal{J} \cap \mathcal{I}$ and $u_s M' \in \Omega$. According to Proposition 4.4,
\begin{align*}
u_{s_0 + s} M &= u_s F(M', x, y) = u_s (M' + (x, y)) \\
&= u_s M' + (x + sy, y) = F(u_s M', x + sy, y).
\end{align*}

The injectivity of $F$ on $\Omega \times [-j, j] \times [-\delta, \delta]$ and $|x + sy| > i + 1$ imply that $u_{s_0 + s} M \notin U$. This proves the claim.

For each $s_0 \in \mathcal{I}$, denote the translates by
\[ \mathcal{J}'(s_0) = \mathcal{J}(s_0) + s_0, \quad \mathcal{I}'(s_0) = \mathcal{I}(s_0) + s_0. \]

The claim, combined with (65) and (68), imply that if $[0, t]$ contains an endpoint of $\mathcal{J}' = \mathcal{J}'(s_0)$, then
\[ \frac{|[0, t] \cap \mathcal{J}' \cap \hat{I}|}{|[0, t] \cap \mathcal{J}'|} \leq \frac{|[0, t] \cap \mathcal{I}'|}{|[0, t] \cap \mathcal{J}'|} + \frac{|\{ s \in [0, t] : u_s M' \notin \Omega \}|}{|[0, t] \cap \mathcal{J}'|} < \frac{\epsilon}{2}. \]
Since \( \hat{\mathcal{I}} \) is covered by the intervals \( \{ \mathcal{J}'(s_0) : s_0 \in \hat{\mathcal{I}} \} \), a standard covering argument shows that we can take a countable subcover \( \mathcal{J}'_1, \mathcal{J}'_2, \ldots \) satisfying
\[
s \in \hat{\mathcal{I}} \quad \Rightarrow \quad 1 \leq \# \{ \ell : s \in \mathcal{J}'_\ell \} \leq 2.
\]
In particular 0 is contained in at most two of the intervals \( \mathcal{J}'(s_\ell) \) and if we take \( t_0 \) to be larger than the right endpoint of these two intervals, and \( t > t_0 \), \([0, t]\) will contain at least one of the endpoints of any \( \mathcal{J}'(s_\ell) \) which intersects \([0, t]\). Then for any \( t > t_0 \) we will have:
\[
\left| [0, t] \cap \hat{\mathcal{I}} \right| \leq \sum_\ell \left| [0, t] \cap \mathcal{J}'_\ell \cap \hat{\mathcal{I}} \right| \leq \frac{\varepsilon}{2} \sum_\ell \left| [0, t] \cap \mathcal{J}'_\ell \right| \leq \frac{\varepsilon}{2} 2^L [0, t]
\]
and this proves (67).

\[ \square \]

12. Equidistribution results for sequences of measures

In this section we show that several natural sequences of measures equidistribute with respect to the flat measure on \( \mathcal{E}_D(1, 1) \). We prove equidistribution for sequences of periodic horocycles of increasing period in Theorem 1.3, for translates of \( G \)-invariant measures by the Rel flow in Theorem 1.4, for circle orbits of increasing radius in Theorem 1.7, and for pushforwards of invariant measures of minimal sets by the geodesic flow in Theorem 12.3.

**Proof of Theorem 1.4.** Let \( \mathcal{L} \) be a closed \( G \)-orbit in \( \mathcal{E}_D(1, 1) \), let \( \mu \) be the Haar measure on \( \mathcal{L} \), and let \( \mu_t = \text{Rel}_u \mu \). We will prove that as \( t \to +\infty \) or \( t \to -\infty \), the measure \( \mu_t \) converges to the flat measure on \( \mathcal{E}_D(1, 1) \). For definiteness we discuss the case \( t \to +\infty \), the second case being similar. It is enough to show that if \( t_n \to \infty \) is any sequence of real numbers for which \( \mu_{t_n} \) converges to some \( \nu \) (where \( \nu \) is not necessarily a probability measure), then \( \nu \) is the flat measure (and in particular is a probability measure).

First we show that \( \nu \) is a probability measure, i.e. that there is no escape of mass. This follows from statement (II) of Theorem 10.4 as follows. We need to show that for any \( \varepsilon > 0 \) there is a compact subset \( K_0 \) of \( \mathcal{E}_D(1, 1) \) such that for all \( t \), \( \mu_t(K_0) \geq 1 - \varepsilon \). Given \( \varepsilon \) let \( K \) be the intersection of \( \mathcal{E}_D(1, 1) \) with the compact set in statement (II) of Theorem 10.4. Let \( \varphi \) be a continuous compactly supported function on \( \mathcal{E}_D(1, 1) \), with values in \([0, 1]\), which is identically equal to 1 on \( K \), and let \( K_0 = \text{supp} \varphi \). Let \( M \) be a generic point for \( \mu_t \). According to Proposition 4.27, \( M = \text{Rel}_t(M') \) for \( M' \in \mathcal{L} \), such that \( M' \) is generic for \( \mu \). In particular \( M' \) has no horizontal saddle connections, and hence neither does \( M \). According to (II),
\[
\liminf_{T \to \infty} \frac{1}{T} \left| \{ s \in [0, T] : u_s M \in K \} \right| \geq 1 - \varepsilon,
\]
and therefore
\[
\mu_t(K_0) \geq \int \varphi \, d\mu_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(u_s M) \, ds
\]
\[
\geq \liminf_{T \to \infty} \frac{1}{T} \int_0^T 1_{K}(u_s M) \, ds \geq 1 - \varepsilon.
\]

Now we claim that \( \nu(\mathcal{H}_\varepsilon) = 1 \), that is \( \nu \) gives no mass to the set of surfaces with horizontal saddle connections. This follows by again expressing \( \mu_t \) as the limit of an integral along a generic horocycle orbit, and repeating the argument of Step 2 of the proof of Theorem 11.1. Thus, in view of Claim 1 (see the proof of Theorem...
it suffices to prove that $\nu$ is invariant under the ‘horospherical foliation’ $UZ$. It is clear that $\nu$ is $U$-invariant, and it remains to show that it is invariant under $\text{Rel}_s$ for any $s \in \mathbb{R}$.

Let $\varphi \in C_c(\mathcal{E}_D(1,1))$. Since $\varphi$ is uniformly continuous, for any $\varepsilon > 0$ there is a neighborhood $U$ of the identity in $G$ so that

$$|\varphi(M) - \varphi(gM)| < \varepsilon \quad \text{for any } M \in \mathcal{E}_D(1,1), \ g \in U.$$

Now define

$$\tau(t, s) = 2 \log \left(1 + \frac{s}{t}\right) \quad \text{and} \quad \tau_n = \tau(t_n, s).$$

By a matrix multiplication in $N$, and using Corollary 4.14 we see that these choices ensure that for any surface $M$ with no horizontal saddle connections,

$$g_{\tau_n} \text{Rel}_{t_n} M = \text{Rel}_{t_n + s \tau_n} M.$$

Moreover $g_{\tau_n} \to \text{Id}$ as $n \to \infty$. Then for $n$ large enough, by (69),

$$\left| \int \varphi \ d\text{Rel}_{t_n} \ast \mu - \int \varphi \circ g_{\tau_n} \ d\text{Rel}_{t_n} \ast \mu \right| < \varepsilon.$$

On the other hand, using (70) and the fact that $\mu$ is supported on $\mathcal{L}_x$, we obtain

$$\int \varphi \circ g_{\tau_n} \ d\text{Rel}_{t_n} \ast \mu = \int_{\mathcal{L}_x} \varphi(g_{\tau_n} \text{Rel}_{t_n}(M)) \ d\mu(M)$$

$$= \int_{\mathcal{L}_x} \varphi(\text{Rel}_{t_n + s \tau_n} M) \ d\mu(M) = \int_{\mathcal{L}_x} \varphi(\text{Rel}_{t_n + s} M) \ d\mu(M).$$

In the last line we used the fact that $\mu$ is invariant under $g_{\tau}$ for all $\tau$. Putting these together we find that for sufficiently large $n$,

$$\left| \int \varphi \ d\text{Rel}_{t_n} \ast \mu - \int \varphi \ d(\text{Rel}_{t_n + s}) \ast \mu \right| < \varepsilon,$$

and since $\varepsilon$ was arbitrary,

$$\nu = \lim_{n \to \infty} \text{Rel}_{t_n} \ast \mu = \lim_{n \to \infty} (\text{Rel}_{s + t_n}) \ast \mu = \text{Rel}_s \nu.$$

Generalizing Theorem 1.4 we have:

**Theorem 12.1.** In each of the cases (ii), (iii), (iv) of §8.4, let $T \neq 0$ and let $\Psi_T$ be the map described in §8.4. Let $\mu_T$ be the pushforward of Haar measure under $\Psi_T$, as in Proposition 8.8. Then as $|T| \to \infty$, $\mu_T$ tends to the flat measure on $\mathcal{E}_D(1,1)$.

The proof of Theorem 1.4 goes through almost verbatim. We leave the details to the reader.

**Proof of Theorem 1.3.** We apply Proposition 11.6 to

$$\mu_t \overset{\text{def}}{=} g_{t \ast} \mu,$$

where $\int f \ d\mu = \frac{1}{p} \sum_{s=0}^{p-1} f(u_s M) ds$ and $\{u_s M : s \in [0, p]\}$ is a closed horocycle of period $p$.

To verify that there is no escape of mass we use Theorem 10.4. Let $\rho_0$ be the length of the shortest horizontal saddle connection on $M$, let $t_0 = -2 \log \rho_0$ and let $t \geq t_0$. The choice of $t_0$ ensures that the shortest horizontal saddle connection
on $g_t M$ has length at least 1. Since the orbit $U g_t M$ is periodic, the measure $\mu_t$ is identical to the measure obtained by averaging along any integer multiple of the period $e^t p$ of this orbit. The set of saddle connections for $g_t M$ whose length is shorter than 1 is finite and none of these is horizontal. Thus for each such saddle connection $\delta$, $\operatorname{hol}(u_s g_t M, \delta)$ diverges as $s \to \infty$. Therefore if we take a sufficiently large multiple of the period $s_0 = k e^t p$, $k \in \mathbb{N}$, any saddle connection on $M$ will have length greater than 1 either for the surface $g_t M$ or for the surface $u_{s_0} g_t M$. As a consequence, the hypothesis of Theorem 10.4 is satisfied with $\rho = 1$, and $I = [0, s_0]$ and there is no escape of mass for a multiple of the period. By periodicity, and using the notation of Definition 4.25, we have $\mu_t = \nu(g_t M, s_0)$, and thus there is no escape of mass for the measures $\mu_t$ either. This verifies hypothesis (a) of Proposition 11.6, and hypothesis (b) is obvious.

To verify hypothesis (c) we adapt the argument given in the proof of Theorem 11.1, retaining the same notation. Namely, in step 2 we verify (c) for beds with Proposition 11.6, and hypothesis (b) is obvious.

Before proving (73), we claim that for all sufficiently large $t$, $\mu_t(U) < \varepsilon$, as required.

Continuing with Step 3, it remains to verify (c) for beds with $\Xi(B) = \emptyset$. We fix $i$ and $\varepsilon > 0$, and note that the argument in the preceding paragraph, using Theorem 10.4, implies that there is $\delta > 0$ such that if $t$ is large enough so that $g_t M$ has no horizontal saddle of length shorter than 1, then

$$\mu_t(C(\delta)) < \varepsilon. \tag{72}$$

Then we take $t_0$ small enough so that all surfaces in $g_{t_0}(K_i)$ have horizontal saddle connections shorter than $\delta$, and set $U = g_{-t_0}(C(\delta))$. Then (72) and $\mu_{t+t_0} = g_{t_0} \mu_t$ imply that for all sufficiently large $t$, $\mu_t(U) < \varepsilon$, as required.

Moreover, suppose otherwise, that is there are $t_n \to \infty$, $M_n' \in W$, $|x_n| = i + 1$ and $s_n \in [0, e^{t_n} p]$ such that

$$u_{s_n} g_{t_n} M = \operatorname{Rel}_{x_n}(M_n'). \tag{73}$$

Thus $M_n'$ lies in the image of $g_{t_n} M$, which implies

$$u_{s_n} M = \operatorname{Rel}_{x_n} g_{-t_n} M_n', \quad x_n = e^{-t_n} x_n \to 0.$$

Our hypothesis that $M$ does not belong to a closed $G$-orbit implies that $x_n \neq 0$. Since $U$ commutes with $\operatorname{Rel}_{x_n}$, it follows that

$$UM = \operatorname{Rel}_{x_n} U g_{-t_n} M_n', \tag{74}$$

and hence $U g_{-t_n} M_n'$ is a closed horocycle of period $p$ on $L$. There are only finitely many such closed horocycles on $L$ and their union is a compact set disjoint from the closed orbit $UM$. Since $x_n \to 0$, this contradicts (74), and proves the claim.

We now note that the argument given in the proof of Theorem 11.1 for proving (67) goes through, with the same notations, and proves (73). Indeed the only information we needed in the proof of (67) was that the number $y = y(s_0)$ considered in the proof was nonzero, which is exactly what we have shown in the preceding paragraph. \[\square\]
Remark 12.2. When $D$ is a square, Theorem 1.3 can also be proved by exploiting the connection between $\mathcal{E}_D(1,1)$ and a homogeneous space $\text{SL}_2(\mathbb{R}) \times \mathbb{R}^2/\Gamma$ (see [EMS]) and using a theorem of Shah [KSS, Thm. 3.7.6].

Generalizing Theorem 1.3 we have:

Theorem 12.3. Let $\mathcal{O}$ be a minimal set for the $U$-action, and let $\mu$ be the $U$-invariant measure on $\mathcal{O}$. Suppose that $\mathcal{O}$ is not contained in a closed $G$-orbit. Then $g_{t\ast}\mu$ tends to the flat measure on $\mathcal{E}_D(1,1)$.

Proof. If $\mathcal{O}$ is a closed $U$-orbit this follows from Theorem 1.3. According to Corollary 8.3, in the remaining case $\dim \mathcal{O} = 2$, the cylinder decomposition is of type (A), and the measure $\mu$ is invariant under $UZ$. Let $\nu$ be any limit point of $g_{t_n\ast}\mu$, for $t_n \to \infty$. By repeating the arguments given in the proof of Theorem 1.3 we find that $\nu$ is a probability measure, and gives zero mass to surfaces with horizontal saddle connections. Also clearly $\nu$ is $UZ$-invariant. Therefore, by Claim 1, $\nu$ is the flat measure on $\mathcal{E}_D(1,1)$. \hfill $\Box$

Proof of Theorem 1.5. We show the existence of $\nu = \lim_{t \to \pm\infty} g_{t\ast}\mu$ and describe this limit explicitly, for each of the measures $\mu$ in Theorem 9.1. We treat the cases $t \to +\infty$ and $t \to -\infty$ separately, beginning with the case $t \to +\infty$. If $\mu$ is of type 1, then the limit measure $\nu$ is either the flat measure on $E_D(1,1)$ or a $G$-invariant measure on a closed $G$-orbit, by Theorem 1.3. If $\mu$ is of type 2, then Theorem 12.3 implies that $\nu$ is flat measure. If $\mu$ is of type 3, 4, or 6, then there is some $T \neq 0$ such that $\mu = \mu_T$ is the pushforward of a $G$-invariant measure under $\Psi_T$, for the map $\Psi_T$ described in §8.4. Note that this map satisfies the following equivariance rule: for $M \in \mathcal{B}$, $g_T\Psi_T(M) = \Psi_T(g_TM)$. This can be seen by examining the definition of $\Psi_T$ in each case and using Proposition 4.4. Therefore we have $g_{t\ast}\mu = \mu_{e^T}$, and by Theorem 12.1, the limit $\nu$ is the flat measure. If $\mu$ is of type 5 then $\mu = \text{Rel}_{T\ast}\mu_0$, where $\mu_0$ is a $G$-invariant measure on a closed $G$-orbit. If $T = 0$ then $\mu = \mu_0$ is $G$-invariant and in particular $\nu = \mu_0$. If $T \neq 0$ then the relation $g_T\text{Rel}_T = \text{Rel}_{e^Tg_T}$ (see Proposition 4.4) implies that $g_{t\ast}\mu_0 = \text{Rel}_{e^Tg_T}\mu_0$ and by Theorem 1.4, the limit measure $\nu$ is the flat measure. Finally in case 7 the measure $\mu$ is $G$-invariant and there is nothing to prove.

When $t \to -\infty$, for each of the measures of type 1, 2, 3, 4 or 6, almost every surface in the support of $\mu$ has at least one horizontal saddle connection of some fixed length. As $t \to -\infty$, the length of this saddle connection tends to zero and hence $g_{t\ast}\mu$ diverges in the space of measures on $\mathcal{H}(1,1)$. If $\mu$ is of type 5 then $\mu = \text{Rel}_{T\ast}\mu_0$, where $\mu_0$ is a $G$-invariant measure on a closed $G$-orbit. The commutation relation $g_T\text{Rel}_T = \text{Rel}_{e^Tg_T}$ implies that $g_{t\ast}\mu = \text{Rel}_{e^Tg_T}\mu_0$ and since $t \to -\infty$, the measure tends to the $G$-invariant measure $\mu_0$. Finally in case 7 the measure $\mu$ is $G$-invariant and there is nothing to prove. \hfill $\Box$

We now collect some results which will be used in the proof of Theorem 1.7. Let $B(a, b) = [-a, a] \times [-b, b]$, and for any $t > 0$ and $v \in \mathbb{R}^2$, set

$$E_{t, v} = \{g_{t\theta}rv : \theta \in [0, 2\pi]\}.$$ 

The following is an elementary fact about ellipses whose proof we omit.

Proposition 12.4. Given $\varepsilon > 0$, suppose $r_1, r_2, \delta_1, \delta_2$ satisfy the inequalities

$$r_1 < \varepsilon r_2, \quad \delta_1 < \varepsilon \delta_2. \quad (75)$$
Then for any $t > 0$, either $E_{t,v} \subset B(r_2, \delta_2)$, or
\begin{equation}
(76) \quad \| \{ \theta \in [0, 2\pi] : g_{t \theta} \theta v \in B(r_1, \delta_1) \} \| \leq \varepsilon \| \{ \theta \in [0, 2\pi] : g_{t \theta} \theta v \in B(r_2, \delta_2) \} \|
\end{equation}

We will also need the following analog of Corollary 10.6.

**Proposition 12.5.** Given positive $\varepsilon, T, \eta,$ and a compact subset $L \subset \mathcal{L}_x$, there are positive $t_0, \delta, \varepsilon, r$, a neighborhood $W$ of $L$ and a compact set $\Omega \subset L$ containing $W$, such that the map (53) is well-defined, continuous and injective on $\Omega \times [-T, T] \times [-\delta, \delta]$, and for any $t \geq t_0$, for any interval $I \subset J \subset \left[ \frac{\pi}{2} - \eta, \frac{\pi}{2} + \eta \right] \cup \left[ \frac{3\pi}{2} - \eta, \frac{3\pi}{2} + \eta \right]$, and any $M \in \mathcal{H}$, if there is $\theta_0 \in I$ such that $g_{t \theta_0} g_{-t} M \in W$ then
\begin{equation}
(77) \quad \frac{| \{ \theta \in I : g_{t \theta} g_{-t} M \notin \Omega \} |}{|I|} < \varepsilon.
\end{equation}

**Proof.** Using Proposition 4.3, we see that it suffices to prove an analog of Proposition 10.5; namely, that for any positive $\eta, \varepsilon, r$ and any compact $L \subset \mathcal{L}_x$, there is a neighborhood $W$ of $L$, a compact $\Omega \subset \mathcal{H}$ containing $W$, and $t_0$, such that surfaces in $\Omega$ have no horizontal saddle connections shorter than $r$, and for any $t \geq t_0$, any interval $I \subset J$ which contains $\theta_0$ with $g_{t \theta_0} g_{-t} M \in W$, the estimate (77) holds. This statement can be obtained from Proposition 10.5 as follows.

A matrix computation shows that
\begin{equation}
(78) \quad u_{e_t \tan \theta} = a(t, \theta) g_{t \theta} g_{-t}, \quad \text{where } a(t, \theta) = \begin{pmatrix} (\cos \theta)^{-1} & 0 \\
-\varepsilon^{-t} \sin \theta & \cos \theta \end{pmatrix}.
\end{equation}

We write $a(t, \theta) = a_2(\theta) a_1(t, \theta)$, where
\begin{align*}
& a_2(\theta) = \begin{pmatrix} (\cos \theta)^{-1} & 0 \\
0 & \cos \theta \end{pmatrix}, \quad a_1(t, \theta) = \begin{pmatrix} 1 & 0 \\
-\varepsilon^t \tan \theta & 1 \end{pmatrix}.
\end{align*}

Thus $a_1(t, \theta) \to \text{Id}$ as $t \to \infty$, uniformly for $\theta \in J$, and $a_2(\theta)$ preserves horizontal saddle connections, changing their length by a factor of at most $C_1 \equiv \max_{\theta \in J} (\cos \theta)^{-1}$. Also let $C_2$ be an upper bound on the derivative of the map $\theta \mapsto \tan \theta$ on the interval $J$. Let $L' = \bigcup_{\theta \in J} a_2(\theta)L$, which is also a compact subset of $\mathcal{L}_x$. We apply Proposition 10.5 with $C_1(r+1), \varepsilon/C_2, L'$ in place of $r, \varepsilon, L$, to obtain a neighborhood $W'$ of $L'$ and a compact set $\Omega'$ containing $W'$, such that surfaces in $\Omega'$ have no horizontal saddle connections shorter than $C_1(r+1)$, and (57) holds. We define:
\begin{equation}
\Omega'' = \bigcup_{\theta \in J} a_2(\theta)^{-1} \Omega'.
\end{equation}

Then $\Omega''$ is a compact set, containing no surfaces with horizontal saddle connections of length less than $r+1$. Therefore for $t_0$ sufficiently large, surfaces in
\begin{equation}
\Omega \equiv \bigcup_{t \geq t_0} a_1(t, \theta)^{-1} \Omega'' = \bigcup_{t \geq t_0} a(t, \theta)^{-1} \Omega'
\end{equation}

have no horizontal saddle connections of length less than $r$. We will make $t_0$ larger below. We also note that $a(t, \theta)^{-1} \Omega' \subset \Omega$ for every $t \geq t_0, \theta \in J$. Now we set
\begin{equation}
W = \Omega \cap \bigcap_{t \geq t_0} a(t, \theta)^{-1} W',
\end{equation}

so that if $g_{t \theta_0} g_{-t} M \in W$ for some $\theta_0 \in J$ and $t \geq t_0$, then by (78), $u_{e_t \tan \theta_0} M \in W'$. Since (57) holds for $\Omega'$ (with $\varepsilon/C_2$ instead of $\varepsilon$), we obtain that (77) holds for $\Omega.$
Finally we note that if $t_0$ is chosen sufficiently large, then $\mathcal{W}$ contains an open set containing $L$.

**Proposition 12.6.** Suppose $\mathcal{L}$ is a closed $G$-orbit in $\mathcal{H}(1,1)$ and $M \notin \mathcal{L}$. Then there are positive $\delta, \bar{r}$ so that for all $t \geq \bar{r}$, if there is $\theta \in [0, 2\pi]$ such that $g_{t}\gamma_{\theta}M = M' + (x, y)$ with $M' \in \mathcal{L}$, then $e^{-t}x^2 + e^{t}y^2 \geq \delta$.

**Proof.** Assume by contradiction that there are $M_n \in \mathcal{L}$, $t_n \to \infty$, $\theta_n \in [0, 2\pi]$ and $x_n, y_n$ such that

\begin{equation}
\tag{79}
e^{-t_n}x_n^2 + e^{t_n}y_n^2 \to 0
\end{equation}

and $g_{t_n} \gamma_{\theta_n}M = M_n + (x_n, y_n)$. By Corollary 6.2, there are neighborhoods $\mathcal{U} \subset \mathcal{H}(1,1)$ containing $M$ and $\mathcal{V} \subset \mathbb{R}^2$ containing 0, such that the map $(M', \nu) \to M' + \nu$ is well-defined on $\mathcal{U} \times \mathcal{V}$. Set $v_n = r_{-\theta_n}g_{t_n}(x_n, y_n)$. Then (79) implies that $v_n \to 0$ and hence for all sufficiently large $n$, the maps $M' \to M' + \pm v_n$ are well-defined and continuous on $\mathcal{U}$. Set $\tilde{M}_n = M + -v_n$, so that $\tilde{M}_n \to M$. We have

\[
M = r_{-\theta_n}g_{-t_n}(M_n + (x_n, y_n)) = r_{-\theta_n}g_{-t_n}M_n + (r_{-\theta_n}g_{-t_n}(x_n, y_n))
\]

and this implies that $\tilde{M}_n = r_{-\theta_n}g_{-t_n}M_n$. We have found a sequence in $\mathcal{L}$ converging to $M$, contrary to the assumption that $M \notin \mathcal{L}$. \hfill $\Box$

Now we prove Theorem 1.6.

**Proof.** We will use Proposition 11.6. Hypothesis (a) was verified (for any translation surface $M$) in [EM]. To prove hypothesis (b), we note that $\mu_t$ is invariant under the conjugated group $\{g_{t}r_{\theta}g_{-t} : \theta \in [0, 2\pi]\}$. Conjugating we see that

\[
g_{t}r_{\theta}g_{-t} = \begin{pmatrix}
\cos \theta & -e^{t} \sin \theta \\
e^{-t} \sin \theta & \cos \theta
\end{pmatrix}.
\]

For every fixed $s$ and $t$ we set

\[
\theta(s, t) = \arcsin (-se^{-t}) , \text{ so that } -e^{t} \sin \theta(s, t) = s.
\]

Then $g_{t}r_{\theta(s, t)}g_{-t} \to u_s$ as $t \to \infty$. Therefore the limit measure $\nu$ is invariant under each $u_s$.

We now verify (c) for beds $\mathcal{B}$ with $\Xi(\mathcal{B}) \neq \emptyset$. Let $K_1, K_2, \ldots$ be the sequence filling out the bed $\mathcal{B}$, as in Proposition 11.5. Fix $i$ and let $\varepsilon > 0$. Then any surface in $K_i$ contains a horizontal saddle connection of length at most $\delta$. By [EM], there are positive $t_1, \delta$ such that for all $t \geq t_1$, $\mu_t(C(\delta)) < \varepsilon$. Let $t_0$ satisfy (62), so that $K_i \subset \mathcal{U}^{\perp} \equiv g_{-t_0}(C(\delta))$. Since $\mu_{t+t_0} = g_{t_0} \# \mu_t$, for all $t > t_0 + t_1$ we have

\[
\mu_t(\mathcal{U}) = \mu_{t_0} \# \mu_{t-t_0}(\mathcal{U}) = \mu_{t-t_0}(C(\delta)) < \varepsilon,
\]

so (c) is satisfied.

It remains to verify (c) for beds with $\Xi(\mathcal{B}) = \emptyset$. Let $F, L_i, K_i = F(L_i \times [-i, i])$ be as in the proof of Theorem 11.1. Given $i$ and $\varepsilon > 0$, choose

\begin{equation}
\tag{80}
\hat{j} > \frac{8(i+1)}{\varepsilon}.
\end{equation}
Choose \( \eta > 0 \) small enough so that

\[
|J| = 4\eta < \pi \varepsilon, \quad \text{where } J = \left[ \frac{\pi}{2} - \eta, \frac{\pi}{2} + \eta \right] \cup \left[ \frac{3\pi}{2} - \eta, \frac{3\pi}{2} + \eta \right].
\]

Let \( \mathcal{W}, \Omega, t_0, \delta \) be as in Proposition 12.5, where we apply the proposition with \( \eta \) and with \( j, L, \varepsilon/8 \) instead of \( T, L, \varepsilon \). Let \( \delta, \tilde{t} \) be as in Proposition 12.6. Making \( \delta \) smaller and \( t_0 \) larger, we can assume that \( t_0 \geq \tilde{t} \) and

\[
j \delta < \tilde{\delta}^2.
\]

Now let

\[
\delta' \in \left( 0, \frac{\varepsilon \delta}{8} \right)
\]

and set

\[
\mathcal{U} = F(\mathcal{W} \times (- (i + 1), i + 1) \times (- \delta', \delta')).
\]

Then \( \mathcal{U} \) is a neighborhood of \( K_i \), and we need to show that \( \mu_t(\mathcal{U}) < \varepsilon \) for all \( t \geq t_0 \). That is, we need to verify

\[
|\hat{\mathcal{I}}| < 2\pi \varepsilon, \quad \text{where } \hat{\mathcal{I}} = \{ \theta \in [0, 2\pi] : g_t r_{\theta} M \in \mathcal{U} \}.
\]

In light of (81) it suffices to show that \( |\hat{\mathcal{I}} \cap J| < \pi \varepsilon \). Whenever \( \theta_0 \in \hat{\mathcal{I}} \cap J \) there are \( M' = M'(\theta_0) \in \mathcal{W} \) and \( (x, y) = (x(\theta_0), y(\theta_0)) \in (- (i + 1), i + 1) \times (- \delta', \delta') \) such that \( g_t r_{\theta_0} M = F(M', x, y) \). We define the following intervals:

\[
\mathcal{I} = \mathcal{I}(\theta_0) \overset{\text{def}}{=} \{ \theta \in [0, 2\pi] : g_t r_{\theta-\theta_0} g_{-t}(x, y) \in B(i + 1, \delta') \}
\]
\[
\mathcal{J} = \mathcal{J}(\theta_0) \overset{\text{def}}{=} \{ \theta \in [0, 2\pi] : g_t r_{\theta-\theta_0} g_{-t}(x, y) \in B(j, \delta') \}.
\]

We think of \([0, 2\pi]\) as a circle by identifying 0 and \( 2\pi \), and think of these subsets as arcs on the circle. In view of Proposition 12.6 and the choice of \( t_0 \), we have \( e^{-t} x^2 + e^t y^2 \geq \tilde{\delta} \). Using (82) and considering the two choices of \( \theta \) which make \( r_{\theta-\theta_0} g_{-t}(x, y) \) horizontal and vertical, we see that \( \mathcal{J} \) does not coincide with the entire circle. Then (80), (83) and Proposition 12.4 imply that

\[
\frac{|\mathcal{I}|}{|\mathcal{J}|} < \frac{\varepsilon}{8}.
\]

We claim that if \( \theta \in \mathcal{J} \setminus \mathcal{I} \) then either \( g_t r_{\theta-\theta_0} g_{-t} M' \not\in \Omega \) or \( g_t r_{\theta} M \not\in \mathcal{U} \). Indeed, suppose \( \theta \in \mathcal{J} \setminus \mathcal{I} \) and \( g_t r_{\theta-\theta_0} g_{-t} M' \in \Omega \). According to Proposition 4.4,

\[
g_t r_{\theta} M = g_t r_{\theta-\theta_0} g_{-t} g_t r_{\theta_0} M = g_t r_{\theta-\theta_0} g_{-t} (M' \oplus (x, y))
\]
\[
= g_t r_{\theta-\theta_0} g_{-t} M' + g_t r_{\theta-\theta_0} g_{-t}(x, y).
\]

The injectivity of \( F \) on \( \Omega \times [-j, j] \times [-\delta, \delta] \) implies that \( g_t r_{\theta} M \not\in \mathcal{U} \). This proves the claim.

The claim, combined with (76) and (77), implies that

\[
\frac{|\hat{\mathcal{I}} \cap \mathcal{J}|}{|\mathcal{J}|} \leq \frac{|\mathcal{I}|}{|\mathcal{J}|} + \frac{|\{ \theta \in \mathcal{J} : g_t r_{\theta-\theta_0} g_{-t} M' \not\in \Omega \}|}{|\mathcal{J}|} < \frac{\varepsilon}{4}.
\]
We have covered $\hat{\mathcal{I}} \cap J$ by the intervals $\left\{ \mathcal{J}(\theta_0) + \theta_0 : \theta_0 \in \hat{\mathcal{I}} \right\}$, and we pass to a subcover such that

$$\theta \in \hat{\mathcal{I}} \cap J \implies \exists \# \{ \ell : \theta \in \mathcal{J}_\ell' \} \leq 2,$$

and obtain:

$$|\hat{\mathcal{I}} \cap J| \leq \sum_\ell |\hat{\mathcal{I}} \cap \mathcal{J}_\ell'| < \frac{\varepsilon}{4} \sum_\ell |\mathcal{J}_\ell'| < \pi \varepsilon.$$

\[ \blacksquare \]

**Proof of Theorem 1.7.** This follows immediately from Theorem 1.6 by an argument developed by Eskin and Masur [EM]. See [EMS, EMaMo, B2] for more details. \[ \blacksquare \]
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