HebbNets: Dynamic network with Hebbian learning rule

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Abstract

It has been demonstrated that one of the most striking features of the nervous system, the so called 'plasticity' (i.e high adaptability at different structural levels) is primarily based on Hebbian learning which is a collection of slightly different mechanisms that modify the synaptic connections between neurons. The changes depend on neural activity and assign a special dynamic behavior to the neural networks. From a structural point of view, it is an open question what network structures may emerge in such dynamic structures under 'sustained' conditions when input to the system is only noise. In this paper we present and study the 'HebbNets', networks with random noise input, in which structural changes are exclusively governed by neurobiologically inspired Hebbian learning rules. We show that Hebbian learning is able to develop a broad range of network structures, including scale-free small-world networks.

Key words: small world, Hebbian learning, central nervous system, scale-free network
PACS: 89.75.Da

1 Introduction

In the last few years research on complex interactive systems (CISs) has become one of the most fascinating areas. One generally applied way to describe such systems is based on graphs with nodes (vertices) and (directed) edges, representing constituents of the system and their interactions. Classification
of CISs is grounded on their structural and dynamic network properties. Similar network structures may be found in many different fields spanning from social connection systems to biochemical processes [1,2,3,4,5,6,7,8,9,10]. Due to the recognition of many common characteristics in both natural and human (artificial) CISs, several general models have been designed to describe the emergence of such structures, e.g., by random restructuring of the links among a finite number of ‘nodes’ [1] or by ‘preferential attachment’ [3,4], or by optimizing the link structure of finite systems [9].

Probably the most complex network is inside us: the most exciting properties of our brain have a lot to do with the special connection system among its units. It is widely accepted that activity correlation between the computing units (i.e. different forms of the so called Hebbian learning mechanism) plays a fundamental role in forming the complex neural structures and maintaining its intrinsic plasticity. Its essence is that the connection strength between the communicating units is modified according to the simultaneous activity correlation of the signal sender and receiver.

It is worth noting that the concept of Hebbian learning has undergone revolutionary changes in the last few years. The original suggestion of Hebb [11] has been modified by recent findings [12,13,14]. For a review, see, e.g., [15]. A unifying description is called spike-time dependent synaptic plasticity (STDP) and it allows different time shift patterns between the units’ activities.

2 Description of HebbNet

In this letter we examine what network structures may emerge in a simplistic neural system by applying pure Hebbian dynamics without any special additional constraints. This neuronal network model will be referred as to HebbNet. We assume that the network is sustained by inputs with no spatio-temporal structure; the input is random noise. Our models consist of $N$ number of simplified integrate-and-fire like ‘neurons’ or nodes. The dynamics of the internal activity is written as

$$\frac{\Delta a_i}{\Delta t} = \sum_j w_{ij} a_j^s + x_{i}^{(ext)},$$

for $i = 1, 2, \ldots, N$. ($N$ was 200 in our simulations.) Variable $x_{i}^{(ext)} \in (0, 1)^N$ denotes the randomly generated input from the environment, $a_i$ is the internal activity of neuron $i$, $w_{ij}$ is $ij^{th}$ element of matrix $W$, i.e., the connection strength from neuron $j$ to neuron $i$. If $\Delta t = 1$ then we have a discrete-time network and each parameter has a time index, or if $\Delta t$ is infinitesimally small.
then Eq. 1 becomes a set of coupled differential equations. The neuron $j$ outputs a spike (neuron $j$ fires) when $a_j$ exceeds a certain level, the threshold parameter $\theta$. Spiking means that the output of the neuron $a_j^s$ (superscript $s$ stands for 'spiking') is set to 1. Otherwise, $a_j^s = 0$. Amount of excitation received by neuron $i$ from neuron $j$ is $w_{ij} a_j^s$ when neuron $j$ fires. After firing, $a_j$ is set to zero at the next time step. For continuous case $a_j$ is set to zero after a very small time interval. Equation 1 describes the simplest form of ‘integrate–and–fire’ network models which is still plausible from a neurobiological point of view. No temporal integration occurs for the discrete case provided that the left hand side of Eq. 1 is replaced by $a_i^+$ where superscript + denotes time shifting. In this limiting case, and if the threshold is high enough, ‘binary neurons’ emerge. This model resembles the original model of McCullough and Pitts [16].

We examined the effect of local activity threshold and global activity constraint (selection of a given percent of nodes with the highest activity). The former one is more realistic biologically, while the latter one is more convenient: in this way the ratio of active units is always known and fixed. For these two cases, computer simulations showed negligible differences. Synaptic strengths were modified as follows:

$$\frac{\Delta w_{ij}}{\Delta t} = \sum_{(t_i,t_j)} K(t_j - t_i) a_i^{s_i} a_j^{s_j},$$

(2)

where $K$ is a kernel function which defines the influence of the temporal activity correlation on synaptic efficacy and $\Delta w_{ij}/\Delta t$ may be taken over discrete or over infinitesimally small time intervals. Possible examples are depicted in Fig. 1. The kernel is a function of the time differences. When the input is made of noise, as in our studies, only the ratio of the positive (strengthening) and the negative (weakening) parts of the kernel function should count. This is the result of the lack of temporal correlations in the input. Temporal grouping and reshaping of the kernel would not modify our results as long as the said ratio is kept constant. In turn, our results concern both types of kernels depicted in Fig. 1.

In the first place, we have been interested in the emerging local and global connectivity structure of $W$. Instead of using global structural property ($L$, characteristic path length which is the average number of edges on the shortest paths) and the clustering coefficient ($C$) proposed by Watts and Strogatz [1] we applied the so called connectivity length measure based on the concept of network efficiency [7]. This single measure is more appropriate for weighted networks [6], equally well applicable for describing global and local properties and offers a unified theoretical background to characterize our system. According to the definition [6,8], local efficiency between nodes $i$ and $j$ in a weighted network with connectivity matrix $W$ is $\epsilon_{ij} = 1/d_{ij}$, where $d_{ij} =$
Figure 1. **Kernel functions**

Two temporal kernels as a function of time difference between spiking time of neuron \(i\) and \(j\) \((t_i - t_j)\). Relevant parameter of the shape for noise-sustained systems is the ratio \((r_{A^+/A^-})\) of the areas/sums of positive and negative parts/components of the kernel, \(A^+\) and \(A^-\), respectively \((r_{A^+/A^-} = A^+/A^-)\).

\[
\min_{n,k_1,...k_n} \left( \frac{1}{w_{ij}} + \frac{1}{w_{ik_1}} + ... + \frac{1}{w_{k_n-1,k_n}} + \frac{1}{w_{k_n,j}} \right) (k_m \in (1, 2, ... N)) \text{ for every } 1 \leq m < N - 1 \text{ and } 1 < n \leq N.
\]

For graphs with connection strengths of values 0 or 1, \(d_{ij}\) corresponds to the shortest distance between nodes \(i\) and \(j\). The average of these values \((E[d_{ij}] = \frac{1}{N(N-1)} \sum_{i\neq j} \epsilon_{ij})\) characterizes the efficiency of the whole network. The local harmonic mean distance for node \(i\) is defined as

\[
D_h(i) = \frac{n^{(i)}}{\sum_{j:w_{ij}>0} \epsilon_{ij}},
\]

where \(n^{(i)}\) is the number of neurons around neuron \(i\) with \(w_{ij} > 0\). In terms of efficiency, this inverse of this value describes how good the local communication is amongst the first neighbors of node \(i\) with node \(i\) removed. It is a measure of the fault tolerance of the system. The mean global distance in the network is defined by the following quantity:

\[
D_h = \frac{N(N-1)}{\sum_{i,j} \epsilon_{ij}}.
\]

Global distance provides a measure for the size (or the diameter) of the network, which influences the average time of information transfer. According to [6,8] local harmonic mean distance measure behaves like \(1/C\) (inverse of the clustering coefficient), whereas the global value corresponds to \(L\). It can be shown that \(L\) is a good approximation of \(D_h\) (or \(1/L\) for the global efficiency) under certain conditions [8].
3 Results and Discussion

These connectivity length measures allowed us to study the emerging network structures as the function of the following parameters: (i) the magnitude of the external excitation (defined by the average percentage of neurons receiving excitation from the environment and (ii) the strengthening–weakening area ratio of the kernel, $K$. The binary neuron model was also investigated. Figures 2 and 3 summarize our findings in different parameter regions. The figure displays the appearance of scale free nets as a function of the excitation level and $r_{A_+} / A_-$. The length of the scale-free regions was determined by first plotting the distribution of the sum of the weights of outgoing connections (averaged over 10000 samples taken from 20 networks) for every parameter set studied. Results were depicted on loglog plot. Supposing a power-law distribution ($P(k^*) \approx k^{*-\gamma}e^{-k^*/\xi}$, where $k^*$ denotes the discretized values of the connection strength), a linear fitting was made to approximate $\gamma$. The width of the scale-free region was estimated by the length of the region with power-law distribution relative to the full length covered on the log scale. Maximum error of the linear fit was set to $10^{-3}$ STD. That is, for 100 discretization points, the width of a region spreading an order of magnitude on the loglog plot is equal to 0.5.

Figure 2. Scale-free region with negligible interaction
Left: exponent of the power law, right: relative percentage of the power-law domain as a function of $r_{A_+} / A_-$ and $r_{ex}$ (the ratio of excited neurons). Contribution of other neurons to the neuronal inputs is negligibly small. Difference between binary and integrate-and-fire neurons disappears in this limiting case. Results are averaged over 20 runs, all sampled 50 times, $\theta = 0.5$. Stripes denote unstable region: components of matrix $W$ may vanish. Log-log plots corresponding to points (a)–(d) are shown in Fig. 3. Power-law with negative (positive) exponent: cases (a) and (d) (case (c)). Positive exponents are thresholded to zero on the figure. For visualization purposes, the data have been interpolated between the calculated grid points.

Fig. 4 displays the emerging connections of a HebbNet for two different parameter sets. We compared the resulting HebbNet structures with a random net, in which the same weights of the dynamic network have been randomly
Figure 3. Log-log plots for different parameters

The four diagrams display typical distributions \(P(k^*)\) for parameters shown in Fig. 2 by (a), (b), (c) and (d). Cases (a) and (d) are arbitrary examples from the power law region.

assigned to different node pairs. The two inlets show the HebbNet connection matrices. While inlet (c) belonging to case (c) in Fig. 2 resembles a random connection matrix, inlet (d) belonging to case (d) in Fig. 2 represents a sparse structure. (Note that most elements are not zero, but very small.)

Fig. 4 highlights clearly the emerging small-world properties, i.e., small local connectivity values (high clustering coefficients) for case (d). Although the global connectivity length was almost the same for all HebbNets and their corresponding random nets, local distances are much smaller in case (d). That is, connectivity structure is sparse but information flow is still fault tolerant and efficient.

The robustness of the network to the external excitation is illustrated on the next figure. By increasing the excitation level, the average local connectivity length of the random net is drastically increasing, whereas the efficiency of the small-world network does not change too much in the same region. For the network with parameters \(r_{A^+/A^-} = 0.1\) (Fig. 5(A)), there is a sharp cut-off around excitation level 0.55, where local distances suddenly drop, due to the high ratio of excitation. Qualitatively similar behavior can be seen for \(r_{A^+/A^-} = 0.6\) (Fig. 5(B)), but the cut-off is around \(r_{ex} = 0.9\).

For networks with significant interaction we have experienced a convergence of the exponent of the power-law distribution to -1. The width of the scale-free region was relatively broad (see, Fig. 6).
Figure 4. Harmonic mean distances
Local harmonic mean distances in ascending order are shown. For better visualization not all data points are marked and the points are connected with a solid line. Lines with upward triangle markers: STDP learning. Lines with circles: same but randomly redistributed weights. Line with empty (solid) markers: HebbNet of case (c) (case (d)). Global harmonic mean distances for the original and for the randomized networks in case (c) of Fig. 2 (case (d) of Fig. 2) are about the same $D_h \approx D_r \approx 5.5$ ($D_h \approx D_r \approx 10$). The two inlets show the resulting connection matrices.

Figure 5. Average local distance vs. excitation ratio
A: $r_{A^+/A^-} = 0.1$, B: $r_{A^+/A^-} = 0.6$. Diamonds: average local distances for the evolving network. Circles: average local distances for the corresponding random net.

4 Summary

In summary, we have demonstrated that small-world architectures with scale-free domains may emerge in sustained networks under STDP Hebbian learning rule without any other specific constraints on the evolution of the net. Although one always has to remember that results from simplified models may not carry over to biophysically realistic networks, we feel that some intriguing conjectures can be made based on our findings. The role of noise in the central nervous system [17,18] is unclear. The existence of such ‘HebbNets’ may support the speculative view of Kandel et al. [19] that structural development
and learning plasticity in CNS may have a common basis. According to our results, evolution and plasticity of the networks may be maintained by noise randomly generated within the CNS. We conjecture that the sustained nature of noise and the competition imposed by small $r_{A^+/A^-}$ values are the two relevant components of plasticity and learning. It might be equally important that exponents of HebbNets with significant interaction amongst neurons are similar in a broad range of parameters.

As far as other evolving networks are considered, the profound implication of our result is that local (Hebbian) learning rules may be sufficient to form and maintain an efficient network in terms of information flow. This feature differs from existing models, such as the model on preferential attachment [4], the global optimization scheme [9], and also from the original Watts and Strogatz model [1].

5 Acknowledgements

This work was partially supported by the Hungarian National Science Foundation, under Grant No. OTKA 32487.
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