$D = 5$ Simple Supergravity on $AdS_3 \times S^2$ and $N = 4$ Superconformal Field Theory

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Abstract

We study the Kaluza-Klein spectrum of $D = 5$ simple supergravity on $S^2$ with special interest in the relation to a two-dimensional $N = 4$ superconformal field theory. The spectrum is obtained around the maximally supersymmetric Freund-Rubin-like background $AdS_3 \times S^2$ by closely following the well-known techniques developed in $D = 11$ supergravity. All the vector excitations turn out to be “(anti-)self-dual”, having only one dynamical degree of freedom. The representation theory for the Lie superalgebra $SU(1,1|2)$ is developed by means of the oscillator method. We calculate the conformal weight of the boundary operator by estimating the asymptotic behavior of the wave function for each Kaluza-Klein mode. All the towers of particles are shown to fall into four infinite series of chiral primary representations of $SU(1,1|2) \times SL(2,\mathbb{R})$ (direct product), or $OSp(2,2; -1) \cong SU(1,1|2) \times SL(2,\mathbb{R})$ (semi-direct product).

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1 Introduction

For some time now, we have recognized that string/M theory on an Anti-de Sitter space times some compact space is intimately related to the conformal field theory on the boundary. The original conjecture on this relation [1] was clarified in the subsequent studies [2, 3] and further explored by a number of authors. See [4] for a recent review and an exhaustive reference.

In this paper we study the correspondence of $D = 5$ simple supergravity compactified on $S^2$ to a two-dimensional $N = 4$ superconformal field theory on the boundary of the Anti-de Sitter space $AdS_3$. $D = 5$ simple supergravity has a BPS solitonic string solution with a similar world-sheet structure to that of M5-brane [5, 6] in particular, $(4,0)$ unbroken world-sheet supersymmetry and near-horizon geometry $AdS_3 \times S^2$. Thus one naturally expects the correspondence between $D = 5$ simple supergravity on $AdS_3 \times S^2$ and a two-dimensional chiral $N = 4$ superconformal field theory at long distances. See [7] - [19] for other works on $AdS_3$.

Let us comment on the relation of the setting in this paper to M-theory compactifications. M5-brane wrapped around a holomorphic 4-cycle of a Calabi-Yau space represents at low energies a black string in $D = 5$, $N = 2$ (“$\mathcal{N} = 1$”) supergravity with worldsheet $(4,0)$ supersymmetry [20]. $D = 5$ black string with the $AdS_3 \times S^2$ near-horizon geometry also arises as the $T^6$ compactification of the orthogonal intersection of three M5-branes [21, 22] with leaving the common string and three overall transverse dimensions uncompactified. The Kaluza-Klein spectrum of D=11 supergravity in such a geometry was already obtained in refs. [16, 17]. Although there is some overlap between their and our results, we would like to stress a few new contributions made in this paper:

First, we present the complete detail of the derivation, including various gauge conditions and the harmonic expansions. This enables us to discuss the nature of the pure-gauge doubleton-like modes. Secondly, we clarify the existence of the central element in the oscillator representation of $SU(1,1|2)$, to which no analogue exists in $SU(1,1|M)$ for $M \geq 3$ [23]. We address the issue of the possibility that the symmetry group of the spectrum can be $OSp(2,2|2; \alpha \rightarrow -1)$. (See below in this section.) Finally, we emphasize the striking parallelism between $D = 5$ simple supergravity on $AdS_3 \times S^2$ and $D = 11$ supergravity on $AdS_7 \times S^4$ [24] - [26], which is buried in a number of moduli fields of the $D = 5$, $N = 8$ supergravity. Note that no Calabi-Yau
or orbifold compactification is known to yield \( D = 5 \) simple supergravity, although it may be realized as a consistent truncation. It seems hard to orbifold away the dilaton \(^{27}\), but we mention an attempt to construct some no-moduli theories \(^{28}\). In view of the similarity between \( D = 11 \) and \( D = 5 \) simple supergravity theories \(^{29, 30, 5, 6}\), it is tempting to suspect that the latter might also be a low-energy theory of something fundamental. This issue is beyond the scope of the present article.

Since the solitonic string may be regarded as an analogue of M5-brane, it is useful to gain insights from the \( AdS_7 \times S^4 \) compactification of \( D = 11 \) supergravity. It is well-known in the \( AdS_7 \times S^4 \) case that the Kaluza-Klein spectrum falls into the representation of the supergroup \( OSp(6, 2|4) \) \(^{26}\). Hence the first guess is that our supergroup might be \( OSp(2, 2|2) \), since it has the maximal subgroup \( O(2, 2) \times USp(2) \cong SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R \times SU(2) \), which is nothing but the isometry group of \( AdS_3 \times S^2 \). However, it turns out to be unsuited for our expectation; the generators of \( SL(2, \mathbb{R})_L \) do not decouple but are present in the anticommutators of right-moving supercharges. \( OSp(2, 2|2) \) has, on the other hand, infinitely many “cousins” who all contain \( SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R \times SU(2) \) bosonic subgroup, namely the one-parameter family of supergroups \( OSp(2, 2|2; \alpha) \). \( OSp(2, 2|2) \) itself can be written as \( OSp(2, 2|2; -\frac{1}{2}) \). A remarkable property of \( OSp(2, 2|2; \alpha) \) is that it has a decoupling limit \( \alpha \to -1 \), in which it factorizes into a semi-direct product \( SU(1, 1|2)_R \times SL(2, \mathbb{R})_L \). Since the Lie algebra of \( SU(1, 1|2) \) is the special subalgebra of the two-dimensional minimal \( N = 4 \) superconformal algebra, \( OSp(2, 2|2; \alpha \to -1) \) can be a candidate for our supergroup. Note that this is not a direct product, because the decoupled \( SL(2, \mathbb{R})_L \) does act as a rotation and a scale transformation on the supercharges.

To obtain the mass spectrum, we will closely follow the techniques which were already developed in 80’s in the studies of the \( 11 = 7 + 4 \) \(^{24} - ^{26}\) or \( 4 + 7 \) compactification \(^{31} - ^{35}\). (See \(^{36}\) for a review and further references therein.) In spite of the similarity, some of our results are new and even surprising. For example, we will show that all the massive vector excitations are “(anti-)self-dual” in the sense that the gauge potential is proportional to its three-dimensional curl. This property should be compared with the “(anti-)self-dual” three-form gauge field in \( D = 7 \) gauged supergravity \(^{37}\). Another unexpected observation is that the zero modes on the solitonic string do not fall into the “doubleton” representation, but to what

\(^{1}\)or, (an appropriate real form of) \( D(2, 1|\alpha) \).
may be called “quarteton” representation; although ultra-short doubleton can also be constructed in our case, it appears to play no role in the story. The quarteton is made up of two pairs of super-oscillators, and turns out to be a pure gauge mode just like singleton or doubleton [38] in M theory. The “massless” supergravity matter multiplet comes next, with four pairs of super-oscillators.

The remainder of this paper is organized as follows. In sect.2, we will briefly review some basic facts about $D=5$ simple supergravity and $AdS_3$. In sect.3, the Kaluza-Klein mass spectrum around the maximally supersymmetric vacuum $AdS_3 \times S^2$ is derived. The highest weight representations of $SU(1,1|2)$ are studied in sect.4 by using the oscillator method. In sect.5, we will calculate the conformal weight of the boundary operators. In sect.6, we will show how each of the Kaluza-Klein modes falls into an $SU(1,1|2) \times SL(2,\mathbb{R})$ chiral primary multiplet. The final section summarizes our results.

Before closing this section, we remark on our convention of the spacetime signature. The “mostly positive” $([-+\cdots+])$ metric is usually used in the modern literature of string theory, whereas the “mostly negative” $([+-\cdots-])$ one was commonly used in the Kaluza-Klein supergravity literature. To facilitate comparison with other literature, we adopt $[+-\cdots-]$ in sect.3, where the Kaluza-Klein spectrum is derived, while we use $[-+\cdots+]$ in all the other sections.

2 $D=5$ Simple Supergravity and $AdS_3$

2.1 Solitonic String in $D=5$ Simple Supergravity

It has been shown [3, 4] that the magnetic BPS string solution of $D=5$ simple supergravity has a very similar structure to that of M5-brane. The metric of this solution is given by

$$ds_5^2 = H^{-1}(-dt^2 + dy^2) + H^2(dr^2 + r^2d\Omega_2), \quad H \equiv 1 + \frac{Q}{r}$$

(2.1)

with the radial coordinate $r$ and the area element $d\Omega_2$ of the unit two-sphere. The $U(1)$ gauge field is

$$F^{ij} = -\sqrt{3}\epsilon^{ijk}H^{-4}\partial_kH$$

(2.2)

for the transverse space indices $i, j, k$, and $F^{MN} = 0$ otherwise.
This string solution approaches Minkowski space as $r \to \infty$, and $AdS_3 \times S^2$ as $r \to 0$. The former statement is obvious. To see the latter, let us consider the new radial coordinate $r'$ such that

$$r'^2 \equiv 4Q^2 r^{-1}.$$  \hspace{1cm} (2.3)

The horizon is located at $r' = \infty$. By using this $r'$, the metric can be written as

$$ds_5^2 \to \infty \frac{4Q^2}{r'^2} \left( 1 + \frac{4Q^2}{r'^2} \right)^{-1} (-dt^2 + dy^2) + \left( 1 + \frac{4Q^2}{r'^2} \right)^2 \left( \frac{4Q^2}{r'^2} dr'^2 + Q^2 d\Omega_2 \right)$$

$$r' \to \infty \frac{4Q^2}{r'^2} (-dt^2 + dy^2 + dr'^2) + Q^2 d\Omega_2.$$ \hspace{1cm} (2.4)

The first term of the right-hand side of (2.4) is nothing but the $AdS_3$ metric in the holospheric coordinate. We also see that the ratio of the radii of $AdS_3$ and $S^2$ is 2:1. As we shall see in sect.3, this is the maximally supersymmetric configuration. Thus the string solution interpolates between two maximally supersymmetric vacua, just as M5-brane does, thereby justifying the name of “solitonic string”.

### 2.2 Energy and Spin in $AdS_3$

The Anti-de Sitter space $AdS_3$ is the homogeneous space $SO(2, 2)/SL(2, R)$. It can be thought of as the hyperboloid

$$-u^2 - v^2 + x^2 + y^2 = -l^2$$ \hspace{1cm} (2.5)

embedded in the flat space $\mathbb{R}^4$ with the metric

$$ds^2 = l^2 (-du^2 - dv^2 + dx^2 + dy^2).$$ \hspace{1cm} (2.6)

$l(>0)$ is the size of $AdS_3$. Several convenient parameterizations are known. Among them we take

$$u = l \cosh \rho \cos \tau, \quad v = l \cosh \rho \sin \tau,$$  \hspace{1cm} (2.7)

$$x = l \sinh \rho \cos \phi, \quad y = l \sinh \rho \sin \phi.$$ \hspace{1cm} (2.8)

Then the metric (2.6) becomes

$$ds^2 = l^2 (d\rho^2 - \cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2).$$ \hspace{1cm} (2.9)
It has the topology $S^1 \times \mathbb{R}^2$ covered by $-\frac{\pi}{2} \leq \tau \leq \frac{3\pi}{2}$ and $\rho \geq 0$, $0 \leq \phi \leq 2\pi$, respectively. One usually considers its universal covering space to unwrap the closed timelike curve along $\tau$. In this case $\tau$ runs over $-\infty < \tau < \infty$, and the topology becomes $\mathbb{R}^3$. Null and space-like infinity is represented by $\rho \to \infty$; it has a topology of a cylinder.

It is well-known that $SO(2, 2)$ is not a simple group, but a direct product $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. Let $M_{ij}$ ($i, j = 0, 1, 2, 3$) be the generators of $SO(2, 2)$ satisfying

$$[M_{ij}, M_{kl}] = \eta_{jk} M_{il} + \eta_{il} M_{jk} - \eta_{jl} M_{ik} - \eta_{ik} M_{jl} \tag{2.10}$$

with $\eta_{ij} = \text{diag}(-1, +1, +1, -1)$. The indices $i, j, \ldots$ are raised by $\eta^{ij}$. Defining

$$M_{i}^{\pm} \equiv \frac{1}{2} \epsilon^{ijk} M_{\pm ij}, \quad M_{i}^{ij} \equiv M^{ij} \pm \frac{1}{2} \epsilon^{ijkl} M_{kl} \quad (i, j, \ldots = 0, 1, 2), \tag{2.11}$$

each set of $M_{i}^{\pm}$ ($i = 0, 1, 2$) satisfies the commutation relations of $SO(2, 1) \cong SL(2, \mathbb{R})$ Lie algebra.

In general, the energy and the spin of a particle in a $(d + 1)$-dimensional Anti-de Sitter space are defined as the eigenvalue of $M_{0}^{d+1}$ and the representation of the $SO(d)$ subgroup, respectively. Thus the energy $E$ and the spin $S$ in $AdS_3$ are labeled by two $SO(2) \cong U(1)$-charges $M_0^0$ and $M_1^1$. Therefore, one may identify

$$L_0 = -\frac{i}{2} M_0^0, \quad L_{\pm 1} = \frac{i}{2} (M_+^2 \mp i M_+^1),$$

$$\overline{L}_0 = -\frac{i}{2} M_0^0, \quad \overline{L}_{\pm 1} = \frac{i}{2} (M_+^2 \mp i M_+^1) \tag{2.12}$$

with

$$E = L_0 + \overline{L}_0, \quad S = L_0 - \overline{L}_0. \tag{2.13}$$

They induce the special conformal transformations on the boundary.\footnote{Note that $L_{\pm 1}$ and $\overline{L}_{\pm 1}$ are complex linear combinations of $M_{i}^{\pm}$; they cannot be expressed with pure imaginary (nor real) coefficients alone. This means that each of two $SL(2, \mathbb{R})$ factors of $SO(2, 2)$ is not the special conformal subgroup itself. Rather, one needs to go to its complexification $SL(2, \mathbb{C})$, and then consider another real form. These two real forms are related by an inner automorphism in this $SL(2, \mathbb{C})$. Of course, the origin of this “twist” is the fact that the boundary metric has a Lorentzian signature, whereas the Virasoro algebra is the symmetry of Euclidean conformal field theories.}
3 Kaluza-Klein Mass Spectrum

Let us now consider the compactification of $D = 5$ simple supergravity on $S^2$. We will first derive a Freund-Rubin-like solution [40].

The Lagrangian of $D=5$ simple supergravity is given by

$$\mathcal{L} = e_5 \left[ -\frac{1}{4} R - \frac{1}{4} F_{MN} F^{MN} - \frac{i}{2} \bar{\psi}_M \tilde{\Gamma}^{MNP} D_N \left( \frac{3\omega - \bar{\omega}}{2} \right) \psi_P + \bar{\psi}_P D_N \left( \frac{3\omega - \bar{\omega}}{2} \right) \tilde{\Gamma}^{MNP} \psi_M \right]$$

$$- \frac{1}{6\sqrt{3}} e^{-1} \epsilon^{MNPQR} F_{MN} F_{PQ} A_R$$

$$- \frac{\sqrt{3}i}{8} \psi_M (\tilde{\Gamma}^{MNPQ} + 2g^{[P} g^{Q]}N) \psi_N (F_{PQ} + \tilde{F}_{PQ}) \right], \quad (3.1)$$

where we adopt the notation used in [30]. $\tilde{\Gamma}^M$ are five-dimensional gamma matrices. In the background $\psi_M = 0$, Einstein’s and Maxwell’s equations read, respectively,

$$R_{MN} - \frac{1}{2} g_{MN} R = -(2F_{MP} F_N^P - \frac{1}{2} g_{MN} F^2), \quad (3.2)$$

$$F_{MN} \tilde{\Gamma}^P = \frac{1}{2\sqrt{3}} e^{-1} \epsilon^{NPQRS} F_{PQ} F_{RS}, \quad (3.3)$$

where $;$ denotes the covariant derivative. We write $\mu, \nu, \ldots = 0, 1, 2$ as three-dimensional spacetime indices, and $m, n, \ldots = 3, 4$ as $S^2$ indices. The basic assumptions are

$$g_{\mu m} = F_{\mu m} = 0, \quad F_{mn} = f\epsilon_{mn} \quad (3.4)$$

for some non-zero real constant $f$. We also require that the Killing spinor equation

$$\delta \psi_M = 0 = \epsilon_{;M} + \frac{1}{4\sqrt{3}} (\tilde{\Gamma}_M^{PQ} - 4\delta_P^M \tilde{\Gamma}^Q) F_{PQ} \epsilon \quad (3.5)$$

be satisfied by maximally many supersymmetry parameters factorized in the form $\epsilon(x^\mu, y^m) = \epsilon(x^\mu) \eta(y^m)$ with three- and two-dimensional Dirac spinors $\epsilon(x^\mu)$ and $\eta(y^m)$, respectively. Taking the representation

$$\tilde{\Gamma}^\mu = \gamma^\mu \otimes \Gamma^5, \quad \tilde{\Gamma}^m = 1 \otimes \Gamma^m, \quad \Gamma^5 = i\Gamma^3 \Gamma^4 \quad (3.6)$$

in terms of three- and two-dimensional gamma matrices $\gamma^\mu$ and $\Gamma^m$, we obtain

$$\eta_{;m} - \frac{i}{\sqrt{3}} f\Gamma^5 \Gamma_m \eta = 0,$$

$$\epsilon_{;\mu} - \frac{i}{2\sqrt{3}} f\gamma_\mu \epsilon = 0, \quad (3.7)$$

$$F_{\mu\nu} = 0.$$
in the background (3.4). Then it follows from the the integrability conditions for (3.7) that the metric must satisfy:

\[ R_{mnpq} = \frac{-4}{3} f^2 (g_{mp}g_{nq} - g_{mq}g_{np}), \]
\[ R_{\mu\nu\rho\sigma} = \frac{1}{3} f^2 (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \] (3.8)

Therefore, the background metric is that of \( AdS_3 \times S^2 \). We henceforth set \( f = \frac{\sqrt{3}}{2} \) so that the radius of \( S^2 \) is normalized to be 1.

### 3.1 Boson Masses

The next step is the linearization of the field equations. Let us separate the fluctuations around the background as

\[ g^{MN} = \hat{g}^{MN} + h^{MN}, \]
\[ F_{MN} = \hat{F}_{MN} + 2a_{[N;M]}, \] (3.9)

where the fields with \( \circ \) stand for the background (3.4)(3.8), and the notations \( \hat{\eta}_{mn} = \hat{\varepsilon}_{2m} \) and \( \hat{\eta}_{\mu\rho} = \hat{\varepsilon}_{3\mu\rho} \) will be used later. By writing out three- and two-dimensional indices, we find from Einstein’s equation:

\[ -\frac{1}{2} (h^M_{\mu;\nu;M} + h^M_{\nu;\mu;M} - h^M_{M;\mu;\nu} - h^M_{\mu;\nu;M}) \]
\[ = -\frac{1}{2} h_{\mu\nu} + \frac{2}{\sqrt{3}} \hat{h}_{\mu\nu} \hat{\eta}^{pq} a_{qp}, \] (E1)
\[ -\frac{1}{2} (h^M_{\mu;m;M} + h^M_{m;\mu;M} - h^M_{M;\mu;m} - h^M_{\mu;m;M}) \]
\[ = -2\sqrt{3} \hat{h}_{m}^p a_{[p;\mu]} - \frac{1}{2} h_{mn}, \] (E2)
\[ -\frac{1}{2} (h^M_{m;n;M} + h^M_{n;m;M} - h^M_{M;m;n} - h^M_{m;n;M}) \]
\[ = -2\sqrt{3} (\hat{h}_{n}^p a_{[p;m]} + \hat{h}_{m}^p a_{[p;n]}) + \frac{2}{\sqrt{3}} \hat{h}_{mn} \hat{\eta}^{pq} a_{qp} + h_{mn}, \] (E3)

and from Maxwell’s equations:

\[ a^{[\mu;M]}_M - \frac{\sqrt{3}}{4} h^{\mu\nu;n} \hat{h}_{mn} - \hat{\eta}^{\mu\rho} a_{\rho\nu} = 0, \] (M1)
\[ a^{[m;M]}_M + \frac{\sqrt{3}}{4} (h^{mpq} \hat{h}_{pq} + h_{pM}^M \hat{h}_{pq}^p - \frac{1}{2} h_{M;p} h_{p}^{pm}) = 0. \] (M2)
We fix the five-dimensional $U(1)$ gauge- and diffeomorphism degrees of freedom by imposing
\[ a_m^m = 0, \quad h^{m\mu} = h^{\mu m} = 0. \]  
(3.10)

It is known, however, that the latter five conditions do not fix all the diffeomorphisms. Indeed, the coordinate transformation on the $S^2$:
\[ \delta y^m \equiv \xi^m = u \cdot Y^m \]  
(3.11)

for an arbitrary function $u(x^\mu)$ on $AdS_3$ and any particular spherical harmonics $Y(y^m)$ preserves the gauge conditions (3.10) invariant if it is accompanied by the coordinate transformation on the $AdS_3$:
\[ \delta x^\mu \equiv \xi^\mu = -u^\mu \cdot Y. \]  
(3.12)

Since this $\xi^M$ causes the change of the metric
\[ \delta h^{mn} = 2u \cdot Y^{;mn}, \]  
(3.13)

one may additionally impose
\[ h^m_m = 0 \]  
(3.14)

using this degree of freedom.

To obtain the mass spectrum, one expands the fields in terms of spherical harmonics. Since the rank of $SU(2)$ is one, the only independent harmonics on $S^2$ are the scalar harmonics; all the vector or tensor harmonics can be expressed by covariant derivatives of the scalar harmonics [41]. Thus the fields are expanded as
\[ h_{\mu\nu} = \sum_k H^{(k)}_{\mu\nu} Y(k), \]
\[ h_{\mu m} = \sum_k (B^{(k)}_{1\mu} Y(k);m + B^{(k)}_{2\mu} \hat{\eta}_{mn} Y(k)^{;n}), \]
\[ h_{mn} = \sum_k (\phi^{(k)}_1 Y(k);m;n + \phi^{(k)}_2 Y(k);m;l + \phi^{(k)}_3 \hat{\eta}_{mn} Y(k)), \]
\[ a_\mu = \sum_k a^{(k)}_\mu Y(k), \]
\[ a_m = \sum_k (b^{(k)}_1 Y(k);m + b^{(k)}_2 \hat{\eta}_{mn} Y(k)^{;n}), \]  
(3.15)
where \( Y_{(k)} \) are the spherical harmonic functions on \( S^2 \) satisfying

\[
\Delta Y_{(k)} = k(k+1)Y_{(k)}
\]

for the Laplacian \( \Delta \) on \( S^2 \). By substituting the gauge conditions (3.10)(3.14), we obtain

\[
\begin{align*}
    h_{\mu m} :_m &= 0 \Rightarrow B^{(k)}_{1\mu} = 0, \\
    h_{\mu m} :_\mu &= 0 \Rightarrow B^{(k);\mu}_{2\mu} = 0, \\
    h^m_m &= 0 \Rightarrow k(k+1)\phi^{(k)}_1 + 2\phi^{(k)}_3 = 0, \\
    a^m_m &= 0 \Rightarrow b^{(k)}_1 = 0.
\end{align*}
\]

(3.17)

Since eq. (3.8) for \( R_{mn} \) does not allow \( \phi^{(k)}_2 \) but a constant, one may ignore it. Thus every bosonic linearized field (3.15) is expressed by a single mode function. In the calculations below, \( h_{\mu\nu} \), \( h_{\mu m} \), ... will be regarded as modes of some particular harmonics.

**Scalars**

Let us define

\[
\begin{align*}
    h &\equiv h^\mu_{\mu}, \quad \varphi \equiv h_{\mu\nu} :^{\mu;\nu}, \quad \psi \equiv h^m_{mn}, \quad da \equiv \eta^{pq}a_{pq}.
\end{align*}
\]

(3.18)

The trace of the equation (E1) yields

\[
\Box + \frac{1}{2} \Delta + \frac{1}{2} h - \varphi - 2\sqrt{3}da = 0.
\]

(3.19)

\( \Box \) is the three-dimensional d’Alembertian. Taking covariant derivatives \( E_2^{\mu;m} \), we obtain

\[
\Box \psi - \Box \Delta h + \Delta \varphi - 2\sqrt{3} \Box da = 0.
\]

(3.20)

The two-dimensional curl of \( E_2 \) gives

\[
(\Box + \Delta)da - \frac{\sqrt{3}}{4} \Delta h = 0.
\]

(3.21)

Only two of four fields \( h, \varphi, \psi \) and \( da \) are shown to be independent. Indeed, the trace of (E3) gives the constraint

\[
\frac{1}{2} \Delta h - \psi + \frac{8}{\sqrt{3}} da = 0.
\]

(3.22)
Also, using (3.22), one can eliminate $\psi$ in (3.20) to find
\[
\Box \left( -\frac{1}{2} \Delta h + \frac{2\sqrt{3}}{3} da \right) + \Delta \varphi = 0.
\] (3.23)

Eqs. (3.19), (3.21) and (3.23) yield another constraint
\[
- \left( \frac{1}{2} \Delta + \frac{3}{2} \right) h - \varphi + \frac{10\sqrt{3}}{3} da = 0.
\] (3.24)

Using this constraint (3.24), we can eliminate $\varphi$ in (3.19):
\[
(\Box + \Delta + 2) h - \frac{16\sqrt{3}}{3} da = 0.
\] (3.25)

One may read off the mass matrix $M_{\text{scalar}}$ from (3.21) and (3.25):
\[
M_{\text{scalar}}^2 = \begin{bmatrix}
\Delta + 2 & -\frac{16\sqrt{3}}{3} \\
-\frac{\sqrt{3}}{3} \Delta & \Delta \\
\end{bmatrix}.
\] (3.26)

Diagonalizing $M_{\text{scalar}}^2$, we obtain the two towers of eigenvalues
\[
- m_{\text{scalar}}^2 = \Box = -(k^2 - k), \quad -(k^2 + 3k + 2).
\] (3.27)

**Vectors**

We find from the two-dimensional curl of (E2) that
\[
(\Box + \Delta + \frac{1}{2}) H_\mu + 2\sqrt{3} \Delta a_\mu = 0
\] (3.28)

with
\[
H_\mu \equiv \hat{\eta}^{pq} h_{\mu q;p},
\] (3.29)

while from (M1) we get
\[
(\Box + \Delta - \frac{1}{2}) a_\mu - 2 \text{rot} a_\mu + \frac{\sqrt{3}}{2} H_\mu = 0,
\] (3.30)

where
\[
\text{rot} a_\mu \equiv \hat{\eta}^\mu_{\rho\nu} a_{\rho\nu}.
\] (3.31)
With the help of the identity for vector fields:

\[(\text{rot})^2 = -\Box + \frac{1}{2}, \quad (3.32)\]

we find from the equations (3.28)-(3.30) that the operator rot has four different eigenvalues \(\omega\) for each \(SU(2)\)-charge (labeled by \(k\)):

\[\omega = -k - 2, \quad k - 1, \quad -k, \quad k + 1. \quad (3.33)\]

This means that all the massive vectors are “self-dual” in the sense that they satisfy the first-order differential equation:

\[\text{rot} a^\mu = \omega a^\mu \quad (3.34)\]

for some constant \(\omega\). Therefore, while ordinary vector fields have two massive states in three-dimensions, they have only one dynamical degree of freedom. Such “self-dual” bosons in odd dimensions were first recognized in the \(S^4\) compactification of \(D = 11\) supergravity in [37], and played an important role in the construction of \(D = 7\) gauged supergravity.

Gravitons

The \(R_{\mu\nu}\) equation (E1) yields

\[(\Box + \Delta - \frac{1}{2}) h_{(\mu\nu)} = 0 \quad (3.35)\]

for the traceless transverse part of \(h_{\mu\nu}\). Thus we obtain a single tower for massive gravitons:

\[-m_{\text{graviton}}^2 = \Box = -(k^2 + k - \frac{1}{2}). \quad (3.36)\]

3.2 Fermion Masses

We next turn to the fermions. Up to quadratic terms, the field equation for \(\psi_M\) is

\[-i\Gamma^{MNP}\psi_{P,N} - \frac{\sqrt{3}}{4}i(\Gamma^{MNPQ} + 2g^{P}[g^Q]^N)\psi_N F_{PQ} = 0. \quad (3.37)\]
In the background (3.4), this yields the following system of equations
\begin{align}
\gamma^{\mu\nu\psi_{\rho,\nu}} + \gamma^{\mu\nu} \Gamma^m \Gamma^m (\psi_{m,\nu} - \psi_{\nu,m}) + \gamma^\mu \Gamma^{mn} \psi_{n,m} - \frac{3}{4} i \epsilon^{\mu\nu} \psi_\nu &= 0, \\
\gamma^{\mu\nu} \Gamma^m \psi_{\nu,\mu} - \Gamma^{mn} \Gamma^5 \gamma^\nu (\psi_{n,\nu} - \psi_{\nu,n}) + \frac{3}{4} e_2^{-1} \epsilon^{mn} \psi_n &= 0,
\end{align}
in which $\psi_\mu$ and $\psi_m$ are mixed. To decouple one from the other in the field equations, we take a convenient gauge-fixing condition for local supersymmetry [33, 34]
$$\tilde{\Gamma}^M \psi_M = 0. \quad (3.40)$$
Using the field equation (3.37), one finds
$$\psi^M;_{M} = -\frac{1}{4} i\gamma^\mu \psi_\mu = \frac{1}{4} i \Gamma^5 \Gamma^m \psi_m. \quad (3.41)$$
This enables us to rewrite the system of field equations as
\begin{align}
\gamma^\nu \psi^\mu;_{\nu} - \frac{i}{2} \gamma^{\mu\nu} \psi_\nu + \frac{i}{4} \psi^\mu + \Gamma^5 \Gamma^m \psi^m;_{m} &= 0, \\
\gamma^\mu \psi^{m;}_m + \Gamma^5 \Gamma^m \psi^m;_n + i(\frac{1}{4} \psi^m + \Gamma^{mn} \psi_n) &= 0, \quad (3.43)
\end{align}
where $\psi_\mu$ and $\psi_m$ appear in the separate equations.

In fact, the condition (3.40) again does not fix all the freedom of the supersymmetry. Using this residual degree of freedom, one may additionally set [33, 34]:
$$\hat{D}_m \psi^m = \psi^m;_m - \frac{i}{2} \Gamma^5 \Gamma_m \psi^m = 0. \quad (3.44)$$

We will use this condition shortly.

Gravitini

The equation (3.42) determines the mass spectrum for the spin-$\frac{3}{2}$ fields. Let us consider the eigenvalue problem for the Dirac operator on the two-sphere:
$$\Gamma^5 \Gamma^m D_m \phi = i\zeta \phi, \quad (3.45)$$
where $D_m$ is the covariant derivative for a two-component complex spinor $\phi$ on the two-sphere. We assume the form of $\phi$ as
\begin{align}
\phi(y) &= a Y \eta(y) + b \partial_m Y \Gamma^m \eta(y) \\
&\quad + c Y \Gamma^5 \eta(y) + d \partial_m Y \Gamma^m \Gamma^5 \eta(y), \quad (3.46)
\end{align}
where \( \eta(y) \) is the two-dimensional part of the Killing spinor (3.7). Plugging (3.46) into (3.45), one obtains the characteristic polynomial equation

\[
\det \begin{bmatrix}
-\imath \zeta - \imath & 0 & 0 & k(k + 1) \\
0 & -\imath \zeta & -1 & 0 \\
0 & k(k + 1) & -\imath \zeta + \imath & 0 \\
-1 & 0 & 0 & -\imath \zeta
\end{bmatrix} = 0.
\] (3.47)

The solutions are

\[
\zeta = \pm k, \quad \pm (k + 1).
\] (3.48)

Each eigenvalue corresponds to a gravitino with mass \(|\zeta + \frac{1}{4}|\):

\[
\gamma^\nu \psi^\mu_{\cdot \nu} - \frac{i}{2} \gamma^{\mu\nu} \psi_{\nu} + i(\zeta + \frac{1}{4}) \psi^\mu = 0.
\] (3.49)

**Spinors**

The masses of spin-\(\frac{1}{2}\) particles are determined by the eigenvalues of the mass operator in (3.43):

\[
\Gamma^5 \Gamma^n D_n \psi^m + i \Gamma^{mn} \psi_n = i \kappa \psi^m.
\] (3.50)

In general, one needs eight spherical harmonics to expand \( \psi^m \):

\[
\psi^m = a_1 Y^{m \eta} + a_2 Y^{m \Gamma^5 \eta} + a_3 (\bar{\psi} Y^{m \eta}) + a_4 (\bar{\psi} Y^{m \eta}) \Gamma^5 \eta + a_5 \bar{\eta}^{mn} Y_n \eta + a_6 \bar{\eta}^{mn} Y_n \Gamma^5 \eta + a_7 \bar{\eta}^{mn} (\bar{\psi} Y_n) \eta + a_8 \bar{\eta}^{mn} (\bar{\psi} Y_n) \Gamma^5 \eta.
\] (3.51)

This expansion includes spurious modes; one can remove them by using the condition (3.44). After a little calculation, it turns out that the following relations among the expansion coefficients hold:

\[
a_1 = a_3 = 0, \quad a_2 = a_8, \quad [k(k + 1) - 1]a_4 - a_6 - i a_7 = 0.
\] (3.52)

Thus the expansion of the physical wave function can be written as

\[
\psi_{\text{phys}}^m = a_2 (Y^{m \Gamma^5 \eta} + \bar{\eta}^{mn} (\bar{\psi} Y_n) \Gamma^5 \eta)
+ a_4 ((\bar{\psi} Y^{m}) \Gamma^5 \eta + (k(k + 1) - 1) \bar{\eta}^{mn} Y_n \Gamma^5 \eta)
+ a_5 \bar{\eta}^{mn} Y_n \eta
+ a_7 (\bar{\eta}^{mn} (\bar{\psi} Y_n) \eta - i \bar{\eta}^{mn} Y_n \Gamma^5 \eta).
\] (3.53)
Thus the problem is reduced to the following $4 \times 4$ matrix eigenvalue equation

$$\det \begin{bmatrix}
i\kappa & 0 & -k(k+1) & 0 \\
0 & i\kappa & 0 & k(k+1) \\
1 & 0 & i\kappa + i & 0 \\
0 & -1 & 0 & i\kappa - i
\end{bmatrix} = 0. \tag{3.54}$$

The solutions are

$$\kappa = \pm k, \pm (k+1). \tag{3.55}$$

With these solutions our Dirac equation is given by

$$\gamma^\mu \psi^m_{;\mu} + i(\kappa + \frac{1}{4})\psi^m = 0. \tag{3.56}$$

The Kaluza-Klein mass spectrum obtained in this section is summarized in Table 1.

4 $SU(1,1|2)$ Lie Superalgebra

In the previous section we obtained the Kaluza-Klein spectrum of $D = 5$ simple supergravity on $AdS_3 \times S^2$. We now study the representation theory of the Lie superalgebra $SU(1,1|2)$ into which each Kaluza-Klein mode is to fit.

The Lie superalgebra $SU(1,1|2)$ is defined by the following super-commutation relations denoted by $[,]$ among the fourteen generators $X_\mu$ ($\mu = 1, 2, \cdots, 14$) [2, 3].

$$[X_\mu, X_\nu] = X_\mu X_\nu - (-1)^{p(\mu)p(\nu)} X_\nu X_\mu
= i f_{\mu\nu\rho} X_\rho \tag{4.1}$$

for some structure constants $f_{\mu\nu\rho}$ (see Appendix A). The fermion number $p(\mu)$ is 0 if $\mu \in \{1, 2, \cdots, 6\}$, or 1 if $\mu \in \{7, 8, \cdots, 14\}$. Several remarks are in order. First, the naively defined $SL(2|2)$ (the complexification of $SU(1,1|2)$) as the algebra of the supertraceless $4 \times 4$ matrices necessarily contains the obvious central element $1_4 = \text{diag}(1,1,1,1)$. Therefore, one considers the residue class

$$\text{Supertraceless } 4 \times 4 \text{ matrices}/\{1_4\}. \tag{4.2}$$

By $SU(1,1|2)$ we mean this quotient algebra in this paper. Secondly, the difference between $SU(2|2)$ and $SU(1,1|2)$ should be explained. Both are the real forms of $SL(2|2)$ and contain the three-dimensional bosonic subalgebra generated by $X_\mu$ ($\mu = 1, 2, 3$). For the former algebra, the structure constants $f_{\mu\nu\rho}$ are the same as those of $SU(2)$, i.e. $f_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}$ (Levi-Civita tensor) while they are those of $SU(1,1) \cong SL(2,\mathbb{R})$, $f_{123} = -1, f_{231} = +1, f_{312} = +1$ for the latter.
4.1 \( N = 4 \) Minimal Superconformal Algebra

The Lie superalgebra \( SU(1,1|2) \) is the finite subalgebra \( \{L_0, L_{\pm 1}, T_{0}^{i}, G_{\pm 1/2}^{i}, \overline{G}_{\pm 1/2}^{i}\} \) of the minimal \( N = 4 \) superconformal algebra \[44\] in the Neveu-Schwarz sector. The explicit super-commutation relations are:

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{1}{2}km(m^2 - 1)\delta_{m+n,0}, \\
\{G^a_r, G^b_s\} &= \{\overline{G}^a_r, \overline{G}^b_s\} = 0, \\
\{G^a_r, \overline{G}^b_s\} &= 2\delta^{ab}L_{r+s} - 2(r - s)\sigma^i_{ab}T^i_{r+s} + \frac{1}{2}k(4r^2 - 1)\delta_{r+s,0}, \\
[T^i_m, T^j_n] &= i\epsilon^{ijk}T^k_{m+n} + \frac{1}{2}km\delta_{m+n,0}\delta^{ij}, \\
[T^i_m, G^a_r] &= -\frac{1}{2}\sigma^i_{ab}G^b_m + r, \\
[T^i_m, \overline{G}^a_r] &= -\frac{1}{2}\sigma^i_{ab}\overline{G}^b_m - r, \\
[L_m, G^a_r] &= (\frac{1}{2}m - r)G^a_{m+r}, \\
[L_m, \overline{G}^a_r] &= (\frac{1}{2}m - r)\overline{G}^a_{m+r}, \\
[L_m, T^i_n] &= -nT^i_{m+n},
\end{align*}
\]

where \( \sigma^i \) is the Pauli spin matrix, \( m \) and \( n \) run over integers, \( r \) and \( s \) are half odd integers, \( a, b = 1 \) or \( 2 \), and \( i \) is the \( SU(2) \) index taking the value 1, 2 or 3.

A state \( |\phi\rangle \) is said chiral primary if

\[
\begin{align*}
G^a_{-1/2}|\phi\rangle &= \overline{G}^a_{-1/2}|\phi\rangle = 0, \quad (4.4) \\
G^a_{n+1/2}|\phi\rangle &= \overline{G}^a_{n+1/2}|\phi\rangle = 0, \quad \text{for } n \geq 0, \quad a = 1, 2. \quad (4.5)
\end{align*}
\]

A chiral primary state \( |\phi\rangle \) satisfies

\[
L_0|\phi\rangle = T^3_0|\phi\rangle. \quad (4.6)
\]

4.2 Oscillator Realizations

To construct the representations of the \( SU(1,1|2) \) algebra, we will consider the explicit realizations of the algebra in terms of super-oscillators \[23, 45\].

The doubleton representation

The smallest representation, the doubleton representation, is given in terms of the two boson-fermion pairs \( (a, \alpha) \) and \( (b, \beta) \) as

\[
\begin{align*}
L_0 &= \frac{1}{2}(a^\dagger a + bb^\dagger), \quad L_1 = ab, \quad L_{-1} = a^\dagger b^\dagger, \\
T^3_0 &= \frac{1}{2}(\alpha^\dagger \beta + \beta^\dagger \alpha), \quad T^2_0 = \frac{1}{2}(\alpha^\dagger \beta - \beta^\dagger \alpha), \quad T^3_0 = \frac{1}{2}(\alpha^\dagger \alpha - \beta^\dagger \beta). \quad (4.7)
\end{align*}
\]
\[
G^1_+ = \sqrt{2}b\alpha, \quad G^2_+ = \sqrt{2}b\beta, \quad G^1_- = \sqrt{2}a^\dagger\alpha, \quad G^2_- = \sqrt{2}a^\dagger\beta,
\]
\[
G^1_1 = \sqrt{2}a\alpha^\dagger, \quad G^2_1 = \sqrt{2}a\beta^\dagger, \quad G^1_{-1} = \sqrt{2}b\alpha^\dagger, \quad G^2_{-1} = \sqrt{2}b\beta^\dagger,
\]
provided that
\[
\nu_1 \equiv a^\dagger a - b^\dagger b + \alpha^\dagger\alpha + \beta^\dagger\beta = 1.
\] (4.8)

Note that \(\nu_1\) commutes with all these generators. One may achieve the quotient algebra with respect to the center \(\nu_1\) by imposing the condition (4.8) on the space of states. The restricted Fock space consists of the states with \(\nu_1\)-charge +1. \(a^\dagger, \alpha^\dagger\) and \(\beta^\dagger\) carry charge +1, while \(b^\dagger\) does −1. Introducing the Fock vacuum \(|0\rangle\) with the property
\[
a|0\rangle = b|0\rangle = \alpha|0\rangle = \beta|0\rangle = 0,
\] (4.9)
one can obtain a four-dimensional basis (Fig.1(a))
\[
|\frac{1}{2}, \frac{1}{2}\rangle = \alpha^\dagger|0\rangle, \quad |\frac{1}{2}, -\frac{1}{2}\rangle = \beta^\dagger|0\rangle, \quad |1, 0\rangle^{(1)} = a^\dagger|0\rangle, \quad |1, 0\rangle^{(2)} = b^\dagger\beta^\dagger\alpha^\dagger|0\rangle.
\] (4.10)
The states with the lower \(L_0\) are fermionic in the doubleton representation. One of the two lower \(L_0\) states is a chiral primary state \((h = q = \frac{1}{2})\), and the other is an anti-chiral primary state \((h = -q = \frac{1}{2})\), which are mapped onto each other by \(T^i_0\).

The quartetion, massless and massive representations

The above realization can be easily extended to a higher-dimensional one. For this, we replace a single boson-fermion pair by two or more pairs as
\[
a \rightarrow (a_1, a_2, \ldots, a_r), \quad b \rightarrow (b_1, b_2, \ldots, b_r),
\]
\[
\alpha \rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_r), \quad \beta \rightarrow (\beta_1, \beta_2, \ldots, \beta_r),
\] (4.11)
where \(r\) is an integer \((r = 2, 3, \ldots)\). The products of the generators are correspondingly replaced as
\[
ab \rightarrow a \cdot b \equiv \sum_{i=1}^{r} a_i b_i, \quad \text{etc.}
\] (4.12)
and the constraint \(\nu_1 = 1\) by
\[
\nu_r = a^\dagger \cdot a - b^\dagger \cdot b + \alpha^\dagger \cdot \alpha + \beta^\dagger \cdot \beta \equiv r.
\] (4.13)
The basis of the representation can be constructed in a similar way. For instance, let us consider the case $r = 2$, which we call quartet representation (Fig.1(b)). The three lowest-$L_0$ states are

\begin{align}
|1, 1\rangle &= \alpha_1^\dagger \alpha_2^\dagger |0\rangle, \\
|1, 0\rangle &= (\alpha_1^\dagger \beta_2^\dagger + \beta_1^\dagger \alpha_2^\dagger) |0\rangle, \\
|1, -1\rangle &= \beta_1^\dagger \beta_2^\dagger |0\rangle,
\end{align}

which are bosonic. The level-one descendants consist of two states, each of which is doubly degenerated. They are

\begin{align}
|\frac{3}{2}, \frac{1}{2}\rangle^{(1)} &= (a_1^\dagger \alpha_2^\dagger - a_2^\dagger \alpha_1^\dagger) |0\rangle \\
|\frac{3}{2}, \frac{1}{2}\rangle^{(2)} &= (b_1^\dagger \alpha_2^\dagger \beta_1^\dagger + b_2^\dagger \alpha_1^\dagger \beta_2^\dagger) |0\rangle, \\
|\frac{3}{2}, -\frac{1}{2}\rangle^{(1)} &= (b_1^\dagger \beta_2^\dagger - b_2^\dagger \beta_1^\dagger) |0\rangle \\
|\frac{3}{2}, -\frac{1}{2}\rangle^{(2)} &= (a_1^\dagger \alpha_2^\dagger \beta_1^\dagger + a_2^\dagger \alpha_1^\dagger \beta_2^\dagger) |0\rangle.
\end{align}

The unique level-two descendant is

\begin{align}
|2, 0\rangle &= (a_1^\dagger \alpha_2^\dagger - a_2^\dagger \alpha_1^\dagger) (b_1^\dagger \beta_1^\dagger + b_2^\dagger \beta_2^\dagger) |0\rangle.
\end{align}

One may easily see that all the states above are antisymmetric with respect to the exchange of the indices 1 and 2.

The states in the massless ($r = 4$) and general massive ($r = 6, 8, \ldots$) representations are shown in Fig.1(c). They contain the massless and massive representations of $SU(1, 1)$ constructed in [23], respectively. One may also construct similar representations for odd $r$, but (like the doubleton) they do not appear in the spectrum, either.

5 Conformal Weights of the Boundary Operators

We will now calculate the conformal dimension of a boundary field which couples to each Kaluza-Klein mode in the bulk. The metric of $AdS_3$ is given by (2.9) with $l = 2$:

\begin{equation}
\begin{aligned}
ds^2 &= 4(- \cosh^2 \rho \, d\tau^2 + \sinh^2 \rho \, d\phi^2 + d\rho^2).
\end{aligned}
\end{equation}

We consider a $p$-form $\mathcal{C}$ on $AdS_3$ with the spin $|S|$ and a boundary field $\mathcal{O}$ interacting with $\mathcal{C}$ at the boundary $\partial AdS_3$. The interaction between them is given by

\begin{equation}
\int_{\partial AdS_3} \mathcal{C} \wedge \mathcal{O}.
\end{equation}

\textsuperscript{3}We give a special name to this representation because, as we will show in the next section, it is this representation that plays a similar role to that of the doubleton in the $7+4$ compactification; the doubleton constructed above does not appear in the spectrum of our $AdS_3 \times S^2$ compactification.
Suppose that the wave function behaves as $C \sim e^{\lambda \rho}$ near the boundary $\rho = \infty$. Then the sum of the left and right conformal dimensions is given by

$$h_L + h_R = \lambda + 2 - p, \quad |h_L - h_R| = |S|,$$  \hspace{1cm} (5.3)

where $h_R$ (resp. $h_L$) is the eigenvalue of $L_0$ (resp. $\overline{L}_0$) (2.12).

**Scalars**

We will first study the asymptotic behavior of the Klein-Gordon field $\varphi$ on $AdS_3$ obeying

$$(\Box - m^2_{\text{scalar}})\varphi = 0.$$  \hspace{1cm} (5.4)

Using the explicit metric (5.1), one can rewrite this as

$$\left[\frac{1}{4} \left( \frac{\partial^2}{\partial \rho^2} + \frac{2}{\cosh 2\rho} \frac{\partial}{\partial \rho} - \frac{1}{\cosh^2 \rho} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sinh^2 \rho} \frac{\partial^2}{\partial \phi^2} \right) - m^2_{\text{scalar}}\right] \varphi = 0.$$  \hspace{1cm} (5.5)

If the asymptotic form at the boundary is $\varphi \sim e^{\lambda \rho}$, $\lambda$ must satisfy

$$\lambda^2 + 2\lambda - 4m^2_{\text{scalar}} = 0.$$  \hspace{1cm} (5.6)

Since a scalar has equal left and right conformal dimensions $h_L = h_R$ ($p = |S| = 0$ in eq.(5.3)), the equation we will solve is

$$2h_L = \lambda + 2.$$  \hspace{1cm} (5.7)

From (3.27), there are two distinct masses $m^2_{\text{scalar}} = k^2 - k$ and $k^2 + 3k + 2$ for a given integer $k$. They have the degeneracy $2k + 1$ coming from the spherical harmonics $Y_{(k)}$. We mean by “$SU(2)$-charge $q$” this degeneracy, i.e. $q = k$ in these cases.

The conformal dimensions $h_L$, $h_R$ and the $SU(2)$-charges for a given $k$ are summarized in Table 1(a). The $SU(2)$ multiplet with $m^2_{\text{scalar}} = k^2 - k$ proves to contain a chiral primary state $h_R = q$, while the other does not.

**(Anti-)Self-dual Vectors**

Next we consider the problem for the (anti-)self-dual vector fields $\varphi_\mu$ satisfying

$$(\text{rot} \: \varphi)^\mu \equiv \tilde{\eta}^{\mu \rho \sigma} \varphi_{\rho, \sigma} = \omega \varphi^\mu.$$  \hspace{1cm} (5.8)
for some constant $\omega$. Plugging the $AdS_3$ metric (5.1) into this equation, we find

\begin{align}
(\text{rot } \varphi)^\rho &= \frac{1}{2 \sinh 2\rho} \partial_\tau \varphi_\rho = \omega \varphi^\rho = \frac{1}{4} \omega \varphi_\rho, \\
(\text{rot } \varphi)^\tau &= \frac{1}{2 \sinh 2\rho} \partial_\rho \varphi_\tau = \omega \varphi^\tau = -\frac{\omega}{4 \cosh^2 \rho} \varphi_\tau, \\
(\text{rot } \varphi)^\phi &= \frac{1}{2 \sinh 2\rho} \partial_\phi \varphi_\rho = \omega \varphi^\phi = \frac{\omega}{4 \sinh^2 \rho} \varphi_\phi
\end{align}

Taking the limit $\rho \to \infty$ in (5.9), we obtain the following reduced equation for $\varphi_\mu$ at the boundary:

\begin{align}
\varphi_\rho &= 0, \quad \partial_\rho \varphi_\phi = 2 \omega \varphi_\tau, \quad \partial_\rho \varphi_\tau = 2 \omega \varphi_\phi.
\end{align}

Thus the asymptotics are

\begin{align}
\varphi_+ &\equiv \varphi_\tau + \varphi_\phi \sim e^{2\omega \rho} \quad \text{if} \quad \omega > 0, \\
\varphi_- &\equiv \varphi_\tau - \varphi_\phi \sim e^{-2\omega \rho} \quad \text{if} \quad \omega < 0.
\end{align}

Substituting $\lambda = 2|\omega|$ and $p = |S| = 1$ in (5.3), we obtain

\begin{align}
h_L + h_R = 2|\omega| + 1, \quad |h_L - h_R| = 1.
\end{align}

In (3.33), we have found four towers of excitations $\omega = k + 1, -k, k - 1, -k - 2$ for a given integer $k$. They have $SU(2)$-charge $q = k$, and $h_R > h_L$ ($h_R < h_L$) if $\omega > 0$ ($\omega < 0$). Only in the cases for $\omega = -k$ and $k - 1$, the (anti-)self-dual vector fields are chiral primary (Table 1(b)).

**Gravitons**

The equation of motion for the graviton $\varphi_{\mu\nu}$ is

\begin{align}
\square - m_{\text{graviton}}^2 \varphi_{\mu\nu} = 0.
\end{align}

Plugging the metric (5.1) again, we see that

\begin{align}
\varphi_{\rho\rho}, \varphi_{\rho\tau}, \varphi_{\rho\phi} \ll \varphi_{\tau\tau}, \varphi_{\phi\phi}, \varphi_{\tau\phi}.
\end{align}

Thus we solve

\begin{align}
\left[ \frac{1}{4} \left( \partial_\rho^2 - 2 \partial_\rho - 2 \right) - m_{\text{graviton}}^2 \right] \varphi_{\mu\nu} = 0 \quad (\mu, \nu) = (\tau, \tau), (\phi, \phi), (\tau, \phi).
\end{align}
The conformal dimensions are shown, due to \( p = |S| = 2 \), to be
\[
h_L + h_R = \lambda = 1 + (4m^2_{\text{graviton}} + 3)^{1/2}, \quad |h_L - h_R| = 2. \tag{5.16}
\]
Using (5.30), we obtain the conformal dimensions shown in Table 1(c). Again, the states in the bottom row contain a chiral primary state, while the states in the first row do not.

**Spinors**

Consider the Dirac equation
\[
(\gamma^\mu \nabla_\mu + m_{1/2})\psi \equiv \left[ \gamma^\mu \left( \partial_\mu + \frac{1}{4} \omega^{ij}_\mu \gamma_{ij} \right) + m_{1/2} \right] \psi = 0. \tag{5.17}
\]
The non-zero components of the spin connection are
\[
\omega^{12}_\phi = \cosh \rho, \quad \omega^{02}_\tau = \sinh \rho, \tag{5.18}
\]
where the indices \((0, 1, 2)\) are those of the local Lorentzian coordinate with the signature \((-+, +, +)\). We adopt the convention for the gamma matrices:
\[
\gamma_0 = i\sigma_2, \quad \gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_3, \tag{5.19}
\]
where \(\sigma_{1,2,3}\) are Pauli’s spin matrices. In this representation one can rewrite the Dirac equation (5.17) as
\[
0 = \left[ \frac{1}{2 \cosh \rho} \gamma^0 (\partial_\tau + \frac{1}{2} \sinh \rho \gamma_{02}) + \frac{1}{2 \sinh \rho} \gamma^1 (\partial_\phi + \frac{1}{2} \cosh \rho \gamma_{12}) + \frac{1}{2} \gamma^2 \partial_\rho + m_{1/2} \right] \psi \\
\rightarrow \left[ \frac{1}{2} (\partial_\rho + 1) \sigma_3 + m_{1/2} \right] \psi, \tag{5.20}
\]
which implies
\[
\lambda = 2|m_{1/2}| - 1. \tag{5.21}
\]
In the previous section we have found (eq.(3.53))
\[
m_{1/2} = \kappa - \frac{1}{4}, \quad SU(2)\text{-charge} = |\kappa| - \frac{1}{2}, \quad \kappa = \pm k, \pm (k + 1) \tag{5.22}
\]
for each integer \(k\). Using these values with \(p = 0, |S| = 1/2\) in (5.3), the conformal dimensions are calculated as shown in Table 1(e), where the rule that \(h_R > h_L\)
(h_R < h_L) if κ is positive (negative) is used, similarly to the case of the (anti-)self-dual vectors.

Gravitini

Finally, we consider the conformal dimensions of the gravitini. We consider the Dirac equation with fully covariantized derivative

\[(\gamma^\nu D_\nu + m)\psi^\mu = 0,\]

\[D_\nu \psi^\mu \equiv (\partial_\nu + \omega_\nu^{ij}\gamma_{ij})\psi^\mu + \Gamma^\mu_\nu_\chi \psi^\chi. \tag{5.23}\]

Substituting the metric (5.1) and taking the limit \(\rho \to \infty\), one can show that \(\psi^\tau, \psi^\phi\) and \(v \equiv e^{-\rho}\psi^\rho\) have the same scaling behavior. The Dirac equation is reduced to

\[
\left[\frac{1}{2}\sigma_3(\partial_\rho + 2) + \left( m + \frac{1}{4} \right) \right] \psi^\tau - \frac{1}{2}\sigma_3 \psi^\phi + \sigma_- v = 0, \\
\left[\frac{1}{2}\sigma_3(\partial_\rho + 2) + \left( m + \frac{1}{4} \right) \right] \psi^\phi - \frac{1}{2}\sigma_3 \psi^\tau + \sigma_- v = 0, \\
\left[\frac{1}{2}\sigma_3(\partial_\rho + 2) + \left( m + \frac{1}{4} \right) \right] v + \frac{1}{2}\sigma_- \psi^\tau - \frac{1}{2}\sigma_- \psi^\phi = 0, \tag{5.24}\]

where \(\sigma_\pm = \sigma_1 \pm i\sigma_2\). After a lengthy calculation, the asymptotic form is shown to be

\[\psi^\tau, \psi^\phi, v \sim e^{(\lambda-1)\rho}, \quad \lambda - 1 = |2\zeta + \frac{3}{2}| - 1. \tag{5.25}\]

Using (3.48) with \(p = 1\), \(|S| = 3/2\), we obtain Table 1(d).

6 AdS/CFT Correspondence

6.1 SU(1, 1|2)_R \times SL(2, \mathbb{R})_L Structure of the Spectrum

We will now show how these infinite towers of particles fit into representations of \(SU(1, 1|2)_R \times SL(2, \mathbb{R})_L\), the finite dimensional subalgebras of the right \(N = 4\) superconformal and left Virasoro algebras. Since the right supersymmetry does not change the left conformal dimension \(h_L\), an irreducible supermultiplet consists of fields with a common value of \(h_L\). Thus we first assemble the following fields having equal \(h_L = k\):

- A scalar with \(h_L = h_R = q = k\) (\(S_I\)),
- A pair of spinors with \(h_L = k\), \(h_R = k + \frac{1}{2}\), and \(q = k - \frac{1}{2}\) (\(Sp_I, Sp_{II}\)),

21
• A self-dual vector with $h_L = k$, $h_R = k + 1$, and $q = k - 1$ ($V_I$).

They have precisely the correct quantum numbers to fit into the three-floor trapezoid diagram in Fig.2(a)! One state in the diagram is mapped to another state on the same horizontal line by the action of the $SU(2)$ subalgebra $T_0^i$ in $SU(1,1|2)$. Borrowing the result on the asymptotic Virasoro algebra in $AdS_3$ [46], one concludes that the above set of three fields corresponds to a single chiral primary multiplet of $N = 4$ superconformal algebra. In this case the chiral primary state corresponds to the scalar with the highest $U(1)$ charge in the $SU(2)$ multiplet. Similarly, one may group

• An anti-self-dual vector with $h_L = k + 1$, $h_R = k$, and $q = k$ ($V_{II}$),

• A pair of spinors with $h_L = k + 1$, $h_R = k + \frac{1}{2}$, and $q = k - \frac{1}{2}$ ($\text{Sp}_{III}, \text{Sp}_{IV}$),

• A scalar with $h_L = h_R = k + 1$, and $q = k - 1$ ($S_{II}$).

These three fit to a trapezoid diagram as shown in Fig.2(b) and correspond to another chiral primary superconformal multiplet. The (highest component of the) self-dual vector corresponds to the chiral primary field in this case.

We will now turn to the graviton multiplets. A massive graviton in three dimensions has two dynamical degrees of freedom, i.e. the two “helicity” states $S = h_R - h_L = \pm 2$. Each of them is mapped to a different conformal field on the boundary. The trio of

• A self-dual vector field with $h_L = k - 1$, $h_R = k$, and $q = k$ ($V_{III}$),

• A pair of gravitini with $h_L = k - 1$, $h_R = k + \frac{1}{2}$, and $q = k - \frac{1}{2}$ ($Go_{I}, Go_{II}$),

• A graviton with $h_L = k - 1$, $h_R = k + 1$, and $q = k - 1$ ($G_1$)

contains the $S = +2$ graviton and fits to the trapezoid in Fig.2(c), whereas the other trio of

• A graviton with $h_L = k + 2$, $h_R = k$, and $q = k$ ($G_{II}$),

• A pair of gravitini with $h_L = k + 2$, $h_R = k + \frac{1}{2}$, and $q = k - \frac{1}{2}$ ($Go_{III}, Go_{IV}$),

• An anti-self-dual vector with $h_L = k + 2$, $h_R = k + 1$, and $q = k - 1$ ($V_{IV}$)
has the $S = -2$ graviton and corresponds to another diagram shown in Fig. 2(d). The chiral primary state in Fig. 2(c) is a self-dual vector field, while the one in Fig. 2(d) comes from a graviton.

We have thus shown that all the infinite towers of the Kaluza-Klein spectrum fall into four infinite series of chiral primary (short) multiplets of $SU(1, 1|2)_R \times SL(2, R)_L$.

### 6.2 Classification of the Mass Spectrum

Our results for the mass spectrum obtained in the previous sections are summarized in Table 2. Several remarks are in order:

i) All the multiplets appearing in the spectrum are short, i.e. chiral primary, multiplets. This phenomenon of multiplet-shortening is an essential property of the compactifications of $D = 11$ supergravity; otherwise the highest-spin state in a long multiplet would exceed the limit for the allowed spins. In contrast, even the long multiplet of $SU(1, 1|2)$ is “short enough” to be fit within the range of the allowed spins. Nevertheless, our multiplets are all short. These results, in turn, imply that the infinite series of graviton multiplets (the top row of the Table 2) do not exhaust all the multiplets in the spectrum; each column for a fixed “excitation level” is decomposed into four irreducible representations of $SU(1, 1|2)_R \times SL(2, R)_L$.

ii) The zero-modes on the solitonic string appear as a quartet in the first column ($n = 0$). There appears no doubleton in the table. It should be compared with the case of the $S^4$ compactification of $D = 11$ supergravity, where the zero-modes on M5-brane correspond to a doubleton. In fact, the quartet is a pure gauge mode just like singleton or doubleton in the higher-dimensional cases; for $k = 1$, $Y_{(1); mn}$ is proportional to $g_{mn}Y_{(1)}$ so that the scalar mode $\phi^{(1)}_3$ can be gauged away.

iii) The second column ($n = 1$) has the same matter field content as that of the $T^2$ compactification of $D = 5$ simple supergravity. After the experience in the $T^7/S^7$ or $T^4/S^4$ compactification, one may naturally expect the existence of an $SO(3)$ gauged supergravity in $D = 3$. Indeed, the self-dual vector transforms as 3, the correct representation to be the $SO(3) \cong SU(2)$ gauge field.

iv) Finally, we would like to mention the possibility that the symmetry of the mass spectrum might not be the direct product $SU(1, 1|2) \times SL(2, R)$ but $OSp(2, 2|2; -1)$. As we discuss in Appendix B, the bosonic generators of $OSp(2, 2|2; \alpha)$ form the direct
product $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SU(2)$. If we take the limit $\alpha \rightarrow -1$, one of the two $SL(2, \mathbb{R})$ groups decouples from the rest, and the $OSp(2, 2|2; -1)$ decomposes into the semi-direct $SU(1, 1|2) \times SL(2, \mathbb{R})$. It would be more natural if the decoupled $SL(2, \mathbb{R})$ will be “supplied” as $SL(2, \mathbb{R})_L$ from the larger (though semi-direct product) supergroup $OSp(2, 2|2; -1)$. More specifically, in the notations in Appendix B, one identifies

$$T_0 = -Q^1_z, \quad T_{\pm 1} = Q^1_{\pm} \pm iQ^1_y.$$ (6.1)

The analogy from the 7+4 compactification supports this picture, since the symmetry group is $OSp(6, 2|4)$ in that case. The minimal (i.e. $SU(2)$) $N = 4$ superconformal algebra does not contain the generator which distinguishes $G^1_{\pm \frac{1}{2}}$ and $\overline{G}^2_{\pm \frac{1}{2}}$ ($G^2_{\pm \frac{1}{2}}$ and $\overline{G}^2_{\pm \frac{1}{2}}$). Therefore, although $Q^1_z$ act on the supercharges as a rotation among these supercharges, all our result on the multiplet structure goes without any modification. Such a nontrivial $SL(2, \mathbb{R})_L$ action on the supercharges in the right sector leads to a two-dimensional conformal field theory of an unconventional type. It would be interesting to see the physical implication of the free parameter $\alpha$ in $OSp(2, 2|2; \alpha)$. We leave the study on this point to future research, but we mention that an interesting realization of the parameter $\alpha$ in $D = 5$ black holes has been discussed in [48].

7 Summary

We have studied the Kaluza-Klein spectrum of $D = 5$ simple supergravity on $S^2$ with special interest in the relation to a two-dimensional $N = 4$ superconformal field theory. A maximally supersymmetric Freund-Rubin-like background $AdS_3 \times S^2$ was found, and turned out to be the geometry near the horizon of the solitonic string in $D = 5$ simple supergravity.

The Kaluza-Klein spectrum was obtained by closely following the well-known techniques developed in the $S^4/S^7$ compactification of $D = 11$ supergravity. We found a single tower of particles for gravitons, 4 for vectors, 2 for scalars and 4 for each of spin-3/2 and spin-1/2 fields. All the vector excitations are “(anti-)self-dual” with having only half of what the ordinary massive vector has as its dynamical degrees of freedom (namely 1).

We next developed the representation theory for the Lie superalgebra $SU(1, 1|2)$. The oscillator method was used. A special care was taken for the central element which arises in the naive definition of $SU(1, 1|2)$. The quotient algebra $SU(1, 1|2)$ was
realized in a restricted Fock space. We constructed doubleton, quartet, massless and massive representations by using one, two, four and \( r (= 6, 8, \ldots) \) pair(s) of super-oscillators. They are all short (chiral primary) representations.

We then calculated the conformal weight of the boundary operator by estimating the asymptotic behavior of the wave function for each Kaluza-Klein field. We finally showed that all the towers of particles were classified into four infinite series of chiral primary representations of \( SU(1, 1|2) \times SL(2, \mathbb{R}) \) (direct product), or \( OSp(2, 2; -1) \cong SU(1, 1|2) \times SL(2, \mathbb{R}) \) (semi-direct product).

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Appendix

A \( SU(1, 1|2) \) and the Finite Subalgebra of \( N = 4 \) Superconformal Algebra

The Lie superalgebra \( SL(2|2) \) can be defined using \( 4 \times 4 \) supertraceless matrices

\[
X = \begin{bmatrix}
x_{11} & x_{12} & x_{13} & x_{14} \\
x_{21} & x_{22} & x_{23} & x_{24} \\
x_{31} & x_{32} & x_{33} & x_{34} \\
x_{41} & x_{42} & x_{43} & x_{44}
\end{bmatrix}
\]

(A.1)

with

\[
\text{str } X \equiv x_{11} + x_{22} - x_{33} - x_{44} = 0.
\]

(A.2)

\( x_{ij} (i, j = 1, \ldots, 4) \) are complex numbers. By definition the identity matrix

\[
\mathbf{1}_4 = \text{diag}(1, 1, 1, 1)
\]

(A.3)

is supertraceless. Thus the algebra of \( 4 \times 4 \) supertraceless matrices contains a center generated by \( \mathbf{1}_4 \). \( SL(2|2) \) is defined as the quotient algebra divided by this central element.
Now we consider the real form $SU(1,1|2)$ of $SL(2|2)$. We choose the following fourteen matrices as a basis:

\[
L_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_{-1} = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
T_0^3 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad T_1^1 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad T_{-1}^2 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix},
\]

\[
G_{\frac{1}{2}}^{\frac{1}{2}} = \sqrt{2} \begin{bmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_2^{\frac{1}{2}} = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
G_{\frac{1}{2}}^{\frac{1}{2}} = \sqrt{2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_2^{\frac{1}{2}} = \sqrt{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
\overline{G}_{\frac{1}{2}}^{\frac{1}{2}} = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{G}_2^{\frac{1}{2}} = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
\]

\[
\overline{G}_{\frac{1}{2}}^{\frac{1}{2}} = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{G}_2^{\frac{1}{2}} = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{bmatrix},
\]

It is straightforward to check that they satisfy (4.3). For example,

\[
\{G_{\frac{1}{2}}, \overline{G}_{\frac{1}{2}}\} = \begin{bmatrix} 0 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2(L_0 - T_0^3).
\]

The last line is an equality in the quotient algebra.
The complex Lie superalgebra \( \text{OSp}(2m|2n) \) is defined as the superalgebra of \((2m + 2n) \times (2m + 2n)\) matrices \(X\) in the form
\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]
where the \(2m \times 2m\) and \(2n \times 2n\) matrices \(A\) and \(D\) satisfy
\[
^tDG + GD = 0, \quad ^tA = -A, \quad B = -^tCG,
\]
\[
G = \begin{bmatrix} O & 1_n \\ -1_n & O \end{bmatrix}.
\]

\(1_n\) is the \(n \times n\) identity matrix. If and only if \(m = 2\) and \(n = 1\), the above defined \(\text{OSp}(2m|2n)\) can be extended to a one-parameter family of Lie superalgebras \(\text{OSp}(4|2; \alpha)\) with a real parameter \(\alpha\). According to the notations in [43], we give the definition of Lie superalgebra \(\text{OSp}(4|2; \alpha)\). That is, among the seventeen generators \(Q^m_j (j = x, y, z; m = 1, 2, 3)\) and \(R_{\mu\nu\rho}\) (\(\mu, \nu, \rho = 1, 2\)), the super-commutation relations are:
\[
\begin{align*}
\{Q^m_j, Q^n_k\} &= i\delta^m_n\epsilon_{jkl}Q^l_k, \\
\{Q^1_j, R_{\nu\rho}\} &= \frac{1}{2}\sigma^{\nu\rho}_{\mu\nu}R_{\mu\nu\rho}, \quad [Q^2_j, R_{\nu\rho}] = \frac{1}{2}\sigma^{\nu\rho}_{\nu\rho}R_{\mu\nu\rho}, \quad [Q^3_j, R_{\nu\rho}] = \frac{1}{2}\sigma^{\nu\rho}_{\nu\rho}R_{\mu\nu\rho}, \\
\{R_{\mu\nu\rho}, R_{\mu'n'\rho'}\} &= \alpha_1 C_{\rho\rho'}C_{\nu\nu'}(\sigma^j)_{\mu\mu'}Q^1_j \\
&\quad + \alpha_2 C_{\rho\rho'}(\sigma^j)_{\nu\nu'}C_{\mu\mu'}Q^2_j + \alpha_3 (\sigma^j)_{\rho\rho'}C_{\nu\nu'}C_{\mu\mu'}Q^3_j
\end{align*}
\]
with
\[
\alpha_1 + \alpha_2 + \alpha_3 = 0
\]
and
\[
C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
The nine bosonic elements \(Q^m_j\) consist of three mutually commuting \(SL(2, \mathbb{C})\) algebras \(SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C})\), and the eight fermionic elements \(R_{\mu\nu\rho}\) form a fundamental representation. Since the algebra depends on \(\alpha_i\) only through the ratio
\[
\alpha_2 = 1.
\]
Thus there is only one free parameter $\alpha = \alpha_3 = -\alpha_1 - 1$ \cite{42}. When $\alpha = -1/2$, the algebra is reduced to the ordinary $OSp(4|2)$ algebra \cite{12,43}.

The most important case for this paper is the $\alpha \rightarrow -1$ limit. Let us consider the real form $OSp(2, 2|2; -1)$. In this case $Q^m_j$ form $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SU(2)$. At $\alpha = -1$, $\alpha_i$ can be fixed as

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = -1. \quad (B.7)$$

Let

$$T_0^1 = Q_x^2, \quad T_0^2 = Q_y^2, \quad T_0^3 = Q_z^2, \quad L_0 = -Q_z^3, \quad L_{\pm 1} = Q_x^3 \pm i Q_y^3, \quad G^1_{\pm 1/2} = R_{121}, \quad G^2_{\pm 1/2} = -R_{111}, \quad G^1_{-1/2} = R_{122}, \quad G^2_{-1/2} = -R_{112}, \quad (B.8)$$

then it can be shown that the $SL(2, \mathbb{R})$ generated by $Q^i_1$ decouples and that the other generators $\{L_{0, \pm 1}, T^i_0, G^a_{\pm 1/2}, \overline{G^a_{\pm 1/2}}\}$ form the closed algebra $SU(1, 1|2)$ \cite{13,49}. Note that the $SL(2, \mathbb{R})$ generated by $Q^i_1$ acts on $G, \overline{G}$ nontrivially, and hence the whole $OSp(2, 2|2; -1)$ is a semi-direct product.

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Figure Captions

Fig.1 Representations of $SU(1,1|2)$. (a) The doubleton representation. (b) The “quarteton” representation ($r = 2$). (c) The quartet ($r = 2$), massless ($r = 4$) and massive ($r = 6, 8, \ldots$) representations.

Fig.2 The boundary fields grouped into $SU(1,1|2) \times SL(2, \mathbb{R})$ (or $OSp(2,2|2; \alpha \to -1)$) multiplets. As $N = 4$ superconformal fields, the chiral primary fields corresponds to (a) a scalar (b) an anti-self-dual vector (c) a self-dual vector (d) a graviton.
Table 1: Conformal weights of the boundary fields corresponding to the Kaluza-Klein modes.

(a) Scalars.

| Series | $m_{\text{scalar}}^2$ | $h_L$ | $h_R$ | $q$ |
|--------|----------------------|-------|-------|-----|
| $S_I$  | $k^2 - k$            | $k$   | $k$   | $k$ |
| $S_{II}$ | $k^2 + 3k + 2$      | $k + 2$ | $k + 2$ | $k$ |

(b) Vectors.

| Series | $\omega$ | $h_L$ | $h_R$ | $q$ |
|--------|----------|-------|-------|-----|
| $V_I$  | $k + 1$  | $k + 1$ | $k + 2$ | $k$ |
| $V_{II}$ | $-k$     | $k + 1$ | $k$   | $k$ |
| $V_{III}$ | $k - 1$  | $k - 1$ | $k$   | $k$ |
| $V_{IV}$ | $-k - 2$ | $k + 3$ | $k + 2$ | $k$ |

(c) Gravitons.

| Series | $m_{\text{graviton}}^2$ | $h_L$ | $h_R$ | $q$ |
|--------|--------------------------|-------|-------|-----|
| $G_I$  | $k^2 + k - \frac{1}{7}$  | $k$   | $k + 2$ | $k$ |
| $G_{II}$ | $k^2 + k - \frac{1}{2}$ | $k + 2$ | $k$   | $k$ |

(d) Gravitini.

| Series | $\zeta$ | $h_L$ | $h_R$ | $q$ |
|--------|---------|-------|-------|-----|
| $G_{OI}$ | $k$     | $k - 1$ | $k + \frac{2}{7}$ | $k - \frac{1}{7}$ |
| $G_{OII}$ | $k + 1$ | $k$   | $k + \frac{2}{7}$ | $k + \frac{1}{7}$ |
| $G_{OIII}$ | $-k$   | $k + 2$ | $k + \frac{2}{7}$ | $k - \frac{1}{7}$ |
| $G_{OIV}$ | $-(k + 1)$ | $k + 3$ | $k + \frac{2}{7}$ | $k + \frac{1}{7}$ |

(e) Spinors.

| Series | $\kappa$ | $h_L$ | $h_R$ | $q$ |
|--------|----------|-------|-------|-----|
| $Sp_{I}$ | $k$      | $k$   | $k + \frac{1}{2}$ | $k - \frac{1}{2}$ |
| $Sp_{II}$ | $k + 1$  | $k + 1$ | $k + \frac{3}{2}$ | $k + \frac{1}{2}$ |
| $Sp_{III}$ | $-k$    | $k + 1$ | $k + \frac{3}{2}$ | $k - \frac{1}{2}$ |
| $Sp_{IV}$ | $-(k + 1)$ | $k + 2$ | $k + \frac{3}{2}$ | $k + \frac{1}{2}$ |
Table 2: Kaluza-Klein spectrum of $AdS_3 \times S^2$ supergravity.

| Field                        | Massive states ($n \geq 2$) | $AdS_3$ spin $S = h_R - h_L$ | Multiplicities |
|------------------------------|-----------------------------|-------------------------------|----------------|
| Graviton ($G_1$)             | 1                           | $+2$                          | 1 3 5 \ldots |
| Gravitino ($G_{01, II}$)     | 2                           | $+\frac{3}{2}$               | 2 4 6 \ldots |
| Self-dual vector ($V_{III}$) | 1                           | $+1$                          | 3 5 7 \ldots |
| Graviton ($G_{II}$)          | 1                           | $-2$                          | 3 5 \ldots   |
| Gravitino ($G_{0II, IV}$)    | 2                           | $-\frac{3}{2}$               | 2 4 \ldots   |
| Anti-self-dual vector ($V_{IV}$) | 1                        | $-1$                          | 1 3 \ldots   |
| Self-dual vector ($V_{I}$)   | 1                           | $+1$                          | 1 3 5 7 \ldots|
| Spinor ($Sp_{I, II}$)        | 2                           | $+\frac{1}{2}$                | 2 4 6 8 \ldots|
| Scalar ($S_I$)               | 1                           | 0                             | 3 5 7 9 \ldots|
| Anti-self-dual vector ($V_{II}$) | 1                        | $-1$                          | 3 5 \ldots   |
| Spinor ($Sp_{III, IV}$)      | 2                           | $-\frac{1}{2}$                | 2 4 \ldots   |
| Scalar ($S_{II}$)            | 1                           | 0                             | 1 3 \ldots   |
Fig. 1(c)
Fig. 2(a)

Fig. 2(b)
Fig. 2(c) $h_L = k - 1$

Fig. 2(d) $h_L = k + 2$