Abstract. If $S$ is a scheme of characteristic $p$, we define an $F$-zip over $S$ to be a vector bundle with two filtrations plus a collection of semi-linear isomorphisms between the graded pieces of the filtrations. For every smooth proper morphism $X \to S$ satisfying certain conditions the de Rham bundles $H^m_{\text{dR}}(X/S)$ have a natural structure of an $F$-zip. We give a complete classification of $F$-zips over an algebraically closed field by studying a semi-linear variant of a variety that appears in recent work of Lusztig. For every $F$-zip over $S$ our methods give a scheme-theoretic stratification of $S$. If the $F$-zip is associated to an abelian scheme over $S$ the underlying topological stratification is the Ekedahl-Oort stratification. We conclude the paper with a discussion of several examples such as good reductions of Shimura varieties of PEL type and K3-surfaces.

Introduction

Let $f: X \to S$ be a smooth proper morphism of schemes in characteristic $p > 0$. We say that $f$ satisfies condition (D) if the sheaves $R^b f_* \Omega^i_{X/S}$ are locally free and if the Hodge-de Rham spectral sequence degenerates at $E_1$-level. The de Rham cohomology sheaves $M = H^m_{\text{dR}}(X/S)$ are then locally free $O_S$-modules that come naturally equipped with a descending filtration $C^\bullet$ (the Hodge filtration), an ascending filtration $D^\bullet$ (the conjugate filtration), and with $O_S$-linear isomorphisms $\varphi_i: (\text{gr}_C^i)^{(p)} \xrightarrow{\sim} \text{gr}_D^i$ given by the (inverse) Cartier operator. We call such a structure $M = (M, C^\bullet, D^\bullet, \varphi^\bullet)$ an $F$-zip over $S$.

It turns out that $F$-zips over $k = \overline{k}$ are essentially combinatorial objects. In order to state our result, let us first define the type of an $F$-zip $M$ over a connected basis $S$ as the function $\tau: \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ given by $\tau(i) = \text{rank}_{O_S}(\text{gr}_C^i)$. In the geometric example considered above, the type is given by the Hodge numbers $h^{i,m-i}$ of the fibres of $f$. Our first main result is the following (cf. section (4.4)).

**Theorem 1.** Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $\tau: \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support $i_1 < \cdots < i_r$. Let $n_j := \tau(i_j)$, write $J = (n_r, \ldots, n_1)$, and let $n := n_1 + \cdots + n_r$. Then there is a bijection

$$\left\{ \text{isomorphism classes of } F\text{-zips of type } \tau \text{ over } k \right\} \longleftrightarrow (S_{n_r} \times \cdots \times S_{n_1}) \backslash S_n =: J^W.$$

More precisely, to each $u \in J^W$ we associate a "standard $F$-zip" $\overline{M}_u$ over $\mathbb{F}_p$ such that any $F$-zip $\overline{M}$ over $k$ is isomorphic to $\overline{M}_u \otimes_{\mathbb{F}_p} k$ for some uniquely determined $u \in J^W$.

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In the case of an abelian variety \(X\) over a perfect field \(k\), the \(F\)-zip structure on \(H^1_{\text{dR}}(X/k)\) gives the Dieudonné module of the \(p\)-kernel group scheme \(X[p]\). In this special case, our classification theorem was proven (up to differences in terminology) by Kraft in [14]. It was realized by Ekedahl and Oort that this can be used to define a stratification of the moduli space \(\mathcal{A}_g\) of abelian varieties in characteristic \(p\). This Ekedahl-Oort stratification is a very useful tool in the study of \(\mathcal{A}_g\); see Oort, [22] and [23].

Our theory of \(F\)-zips enables us to extend these ideas to arbitrary families \(f: X \to S\) satisfying condition (D), and to de Rham cohomology in arbitrary degree. We define a generalized Ekedahl-Oort stratification of the base scheme \(S\). In fact, our theory gives a natural scheme-theoretic definition of these strata, which is new even in the case of abelian varieties. The result can be stated as follows (cf. section (4.9)).

**Theorem 2.** Let \(\tau\) and \(J W\) be as in Theorem 1. Let \(M = (M, C^*, D_\bullet, \varphi^\bullet)\) be an \(F\)-zip of type \(\tau\) over a scheme \(S\) of characteristic \(p\). For \(u \in J W\) we define a subfunctor \(S^u_M\) of \(S\) by the condition that a morphism \(g: T \to S\) factors through \(S^u_M\) if and only if \(g^* M\) is \(\text{fpf-locally isomorphic to} \ M^u \otimes_{F_p} O_T\). Then \(S^u_M \subset S\) is representable by a locally closed subscheme of \(S\), and the map

\[
\coprod_{u \in J W} S^u_M \longrightarrow S
\]

is a bijective monomorphism (i.e., a partition of \(S\)).

For the proof of Theorem 1 and Theorem 2 we study the \(\mathbb{F}_p\)-scheme \(X_\tau\) whose \(S\)-valued points are the triples \((C^*, D_\bullet, \varphi^\bullet)\) such that \((O_S^g, C^*, D_\bullet, \varphi^\bullet)\) is an \(F\)-zip of type \(\tau\) over \(S\). The algebraic group \(\text{GL}_{n,\mathbb{F}_p}\) naturally acts on \(X_\tau\). Theorem 1 amounts to a classification of the \(\text{GL}_n\)-orbits in \(X_\tau\).

We think of \(X_\tau\) as a “mod \(p\) analogue” of a compactified period domain. Indeed, if we let \(\# S \to S\) be the \(\text{GL}_n\)-torsor of trivialisations of the underlying vector bundle \(M\) then we get a natural “mod \(p\) period map” \(\# S \to X_\tau\), analogous to the period maps arising in Hodge theory. It turns out that there is a unique open \(\text{GL}_n\)-orbit \(X_\tau^\text{ord} \subset X_\tau\), the “ordinary locus” (cf. section (4.5)), which is to be thought of as the interior of the period domain. In this picture, the other strata correspond to degenerations of the data that constitute an \(F\)-zip.

In order to study the \(\text{GL}_n\)-orbits in \(X_\tau\), we express the latter in more group-theoretical terms. We introduce varieties \(Z_J\) that are semi-linear variants of the varieties studied by Lusztig in [15]. We consider these varieties in the general context of a (not necessarily connected) reductive group \(G\) over a finite field. Write \((W, I)\) for the Weyl group of \(G\) with its set of simple reflections. As further input for the definition of \(Z_J\) we need two subsets \(J, K \subset I\), and a Weyl group element \(x \in W\) satisfying certain assumptions (see section (3.2)). Write \(U_P\) for the unipotent radical of a parabolic \(P \subset G\). Then \(Z_J\) is the Zariski sheafification of the functor that classifies triples \((P, Q, [g])\) with \(P\) and \(Q\) parabolic subgroups of types \(J\) and \(K\), respectively, and with \([g]\) a double coset in \(U_Q \backslash G / F(U_P)\) such that \(Q\) and \(\# F(P)\) are in relative position \(x\). We prove that \(Z_J\) is a smooth variety of dimension equal to \(\dim(G)\). The group \(G\) naturally acts on \(Z_J\).

The connection with the theory of \(F\)-zips is as follows. Let \(\tau\) and \(J\) be as in Theorem 1, and take \(G = \text{GL}_{n,\mathbb{F}_p}\). We identify \(W = S_n\). The ordered partition \(J = (n_r, \ldots, n_1)\) corresponds to
a subset of the set $I$ of simple reflections. For $K \subseteq I$ we take the subset corresponding to the opposite partition $(n_1, \ldots, n_r)$, and for $x$ we take the element of minimal length in the double coset $W_K w_0 W_J$, where $W_J$ and $W_K \subset W$ are the subgroups generated by $J$ and $K$, respectively, and where $w_0 \in W$ is the longest element. We show that with these choices, there is a $GL_n$-equivariant isomorphism between $X_\tau$ and the variety $Z_J$. Theorem 1 is then a consequence of the following general result about the varieties $Z_J$ (cf. section (3.25)).

**Theorem 3.** There is a bijection between the set of $G$-orbits in $Z_J$ and the set $^JW \subset W$ of elements $w \in W$ that are of minimal length in their coset $W_Jw$. (So $^JW$ is in bijection with $W_J \setminus W$.)

The idea for the proof of this theorem is the following. Let $(P, Q, [g])$ be a point of $Z_J$. We define a new pair of parabolics $(P_1, Q_1)$ by

$$P_1 := (P \cap Q)U_P, \quad Q_1 := (Q \cap gF(P_1)g^{-1})U_Q.$$ 

In a sense that can be made precise, the pair $(P_1, Q_1)$ is a refinement of the pair $(P, Q)$. Repeating this process, we get a sequence of pairs $(P_n, Q_n)$ that stabilizes. Then the bijection in Theorem 3 is obtained by sending the point $(P, Q, [g])$ to the element of $W$ that measures the relative position of $P_n$ and $Q_n$ for $n \gg 0$.

The same ideas as sketched here can be applied to study $F$-zips with certain additional structures, such as a bilinear form or an action of a semi-simple algebra. We apply this to abelian varieties, K3-surfaces, and to good reductions of PEL-Shimura varieties. In this last case, we give a new proof of the dimension formula for Ekedahl-Oort strata that was obtained in [20] using the results of [27]. In fact, this is a consequence of the following general result on the dimensions of the $G$-orbits in $Z_J$ (cf. section (3.20)).

**Theorem 4.** For $u \in J$, let $O_u \subset Z_J$ be the corresponding $G$-orbit under the bijection of Theorem 3. Then

$$\text{codim}(O_u, Z_J) = \dim(\text{Par}_J) - \ell(u),$$

where $\ell(u)$ is the length of $u$ in the Coxeter group $W$, and where $\text{Par}_J$ is the variety of parabolics of type $J$.

We will now give an overview of the structure of the paper. In the first section we give the definition of $F$-zips over a base scheme of characteristic $p$, and we define standard $F$-zips. Section 2 contains some notations and lemmas on parabolic subgroups of reductive groups, their relative position, and their Levi subgroups.

Section 3 is the technical heart of the paper. Here we introduce and study the varieties $Z_J$ discussed above. The main goal of this section is the study of the fppf-quotient $G \setminus Z_J$ and, as an application, the proof of Theorems 3 and 4. Our method is a variation on ideas of Lusztig in [16].

In Section 4 we prove Theorems 1 and 2 announced above. The proof is an easy application of our classification of the $G$-orbits in $Z_J$.

In Section 5 we briefly discuss some examples of $F$-zips with additional structure. Finally, in Section 6 we discuss applications to geometry. We explain how a morphism $X \to S$ satisfying
condition (D) gives rise to an $F$-zip structure on the de Rham cohomology. There is also a version of this for log-schemes. We show that it is possible to detect ordinariness, in the sense of Illusie and Raynaud [10], from our partition of $S$. Next we consider good reductions of Shimura varieties of PEL-type. The partition obtained in this case is the generalized Ekedahl-Oort stratification studied earlier by the authors in [17], [18] and [27]. Finally, we study K3-surfaces $X \rightarrow S$, and we make the connection between the stratification of $S$ given by the height and the Artin invariant, and the generalized Ekedahl-Oort stratification obtained by our methods.

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1 Filtrations and Flags

(1.1) Throughout this section, $p$ is a prime number and $q$ is a fixed power of $p$. For a scheme $S$ of characteristic $p$ we denote by $F_S: S \rightarrow S$ the morphism which is the identity on the underlying topological space and the homomorphism $x \mapsto x^q$ on the sheaves of rings. For an $\mathcal{O}_S$-module $M$ we set $M^{(q)} = F^*_SM$.

(1.2) Let $S$ be a scheme, and let $M$ be a locally free $\mathcal{O}_S$-module of finite rank. By a descending filtration $C^\bullet$ of $M$ we mean a sequence $(C^i)_{i \in \mathbb{Z}}$ of $\mathcal{O}_S$-submodules $C^i \subset M$ such that $C^i$ is locally on $S$ a direct summand of $C^{i-1}$ and such that $C^i = M$ for $i \ll 0$ and $C^i = (0)$ for $i \gg 0$. We set $gr_C^{-i}(M) = gr_C^i = C^i/C^{i+1}$. We have an analogous definition of an ascending filtration $D^\bullet$, with associated graded modules $gr_D^i(M) = gr_D^{-i} = D_i/D_{i-1}$.

(1.3) Let $M$ be as above. A flag of $M$ is a set $\Delta$ of $\mathcal{O}_S$-submodules of $M$ which are locally direct summands, such that $\Delta$ contains $(0)$ and $M$ and is totally ordered by inclusion. Every (descending or ascending) filtration $C^\bullet$ defines a flag by forgetting the enumeration.

The set of flags of $M$ is partially ordered by inclusion. We say that $\Delta$ is a refinement of $\Delta'$ if $\Delta \supset \Delta'$.

(1.4) Let $S$ be a scheme and let $C^\bullet$ be a descending filtration of a locally free $\mathcal{O}_S$-module $M$ of finite type. For $s \in S$, consider the function $\tau^\bullet_C: \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ given by $m \mapsto dim_{\kappa(s)}(gr_{C^\otimes \kappa(s)}^m)$. As the $gr_{C^\otimes}$ are locally free, the function $\tau: s \mapsto \tau^\bullet_C$ is locally constant; it takes values in the set of maps $\mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ with finite support. We refer to $\tau$ as the type of the filtration. If $S$ is connected then $\tau$ is given by a single function $\mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ with finite support.

A similar definition applies to ascending filtrations.

(1.5) Definition: Let $S$ be an $\mathbb{F}_q$-scheme. An $F$-zip over $S$ is a tuple $\mathcal{M} = (M, C^\bullet, D^\bullet, \phi^\bullet)$ where
- $M$ is a locally free $\mathcal{O}_S$-module of finite rank,
- $C^\bullet$ is a descending filtration of $M$,
- $D_\bullet$ is an ascending filtration of $M$,
- $\varphi_\bullet$ is a family of $\mathcal{O}_S$-linear isomorphisms
  $$\varphi_n: (\text{gr}_C^n(q)) \sim \rightarrow \text{gr}_D^n$$
  for $n \in \mathbb{Z}$.

The rank of $M$ is called the **height** of $M$. The type of the filtration $C^\bullet$ is called the **type** of $M$.

We have the obvious notion of a morphism of $F$-zips (morphisms are not required to be strict for the filtrations) and hence get the category of $F$-zips over $S$, which is an $\mathbb{F}_q$-linear rigid tensor category. Note that this category is not abelian.

(1.6) For an $\mathbb{F}_q$-scheme $S$, let $\mathbf{F}$-zip$(S)$ be the category which has as objects the $F$-zips over $S$ and as morphisms the isomorphisms of $F$-zips over $S$. For a morphism of schemes $f: T \rightarrow S$ we have an obvious pullback functor $f^*: \mathbf{F}$-zip$(S) \rightarrow \mathbf{F}$-zip$(T)$. In this way we obtain a stack $\mathbf{F}$-zip, fibered over the category of $\mathbb{F}_q$-schemes endowed with the fpqc topology.

(1.7) **Proposition**: The stack $\mathbf{F}$-zip is a smooth Artin stack over $\mathbb{F}_q$. If $\tau: \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ is a function with finite support, the substack $\mathbf{F}$-zip$^\tau$ of $F$-zips of type $\tau$ is an open and closed substack of $\mathbf{F}$-zip and we obtain a decomposition

$$\mathbf{F}$-zip = \coprod_\tau \mathbf{F}$-zip$^\tau.$$

The Artin stacks $\mathbf{F}$-zip$^\tau$ are quasicompact.

The easy proof is omitted.

(1.8) **Example**: Assume $q = p$ and let $S = \text{Spec}(R)$ with $R$ a perfect ring of characteristic $p$.

Consider a BT$_1$-Dieudonné module over $S$, by which we mean a triple $(M, F, V)$ with $M$ a projective $R$-module of finite type, $F: M \rightarrow M$ an $\text{Frob}_R$-linear map, $V: M \rightarrow M$ an $\text{Frob}^{-1}_R$-linear map, such that $\text{Ker}(F) = \text{Im}(V)$ and $\text{Im}(F) = \text{Ker}(V)$ are locally direct summands of $M$.

The category of BT$_1$-Dieudonné modules can be identified with the category of $F$-zips whose type $\tau$ has support contained in $\{0, 1\}$: To $(M, F, V)$ we associate the $F$-zip $(M, C^\bullet, D_\bullet, \varphi_\bullet)$ with

- $C^0 = M \supset C^1 = \text{Ker}(F) \supset C_2 = (0)$
- $D_{-1} = (0) \subset D_0 = \text{Im}(F) \subset D_1 = M$,

with $\varphi_0: (M/\text{Ker}(F))^{(p)} \sim \rightarrow M$ the (linearization of the) isomorphism induced by $F$, and $\varphi_1: \text{Ker}(F)^{(p)} \sim \rightarrow M/\text{Im}(F)$ the inverse of the (linearized) isomorphism induced by $V$.

(1.9) **Standard $F$-zips**: We fix an integer $n \geq 1$ and a map $\tau: \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ with $\sum_{i \in \mathbb{Z}} \tau(i) = n$.

Let $i_1 < \cdots < i_r$ be the support of $\tau$ and $J = (n_r, \ldots, n_1)$ be the ordered partition of $n$ with $n_j = \tau(i_j)$. (Note the order of the $n_j$.) Let $W = S_n$ be the group of permutations of
\{1, \ldots, n\} and consider \( W_J = S_{n_1} \times \cdots \times S_{n_1} \) as a subgroup of \( S_n \) in the usual way. We set \( m_j = n_1 + \cdots + n_j \). Let \( x \in W \) be defined by
\[
x(i) = i + m_j + m_{j-1} - n, \quad \text{if } n - m_j < i \leq n - m_{j-1},
\]
i.e., \( x \) is the element of minimal length in \( w_0 W_J \), where \( w_0 \) is the longest element in \( W \). Finally let \( J W \) be the set of permutations \( u \in W \) with the property that
\[
u^{-1}(n - m_j + 1) < u^{-1}(n - m_j + 2) < \cdots < u^{-1}(n - m_{j-1})
\]
for all \( j = 1, \ldots, r \), i.e., \( J W \) consists of those \( u \in W \) which are of minimal length in their left coset \( W_J u \).

To \( \tau \) and \( u \in J W \) we associate a standard \( F \)-zip \( M^u_\tau = (M^u_\tau, (C^\bullet)_\tau, (D_\bullet)_\tau, (\varphi^\bullet)_\tau) \) over \( F_p \), where

- \( M^u_\tau = F_p^n \),
- \( (C^\bullet)_\tau \) is the unique filtration of type \( \tau \) such that the associated flag is given by
  \[
  F_p^n \supseteq F_p^{\{u(1), u(2), \ldots, u(m_{r-1})\}} \supseteq F_p^{\{u(1), u(2), \ldots, u(m_{r-2})\}} \supseteq \cdots \supseteq F_p^{\{u(1), u(2), \ldots, u(1)\}} \supseteq (0),
  \]
- \( (D_\bullet)_\tau \) is the unique filtration of type \( \tau \) such that the associated flag is given by
  \[
  (0) \subset F_p^{\{1, \ldots, n-m_{r-1}\}} \subset F_p^{\{1, \ldots, n-m_{r-2}\}} \subset \cdots \subset F_p^{\{1, \ldots, n-m_0\}} = F_p^n,
  \]
- \( (\varphi^\bullet)_\tau \) is zero for \( i \notin \{i_1, \ldots, i_r\} \) and for \( i = i_j \) it is the isomorphism
  \[
  (gr^D_C)^{(p)}(F_p^{\{u(m_{r-j}+1), \ldots, u(m_r-1)\}}) \rightarrow gr^D F_p^{\{n-m_{r-j+1}, \ldots, n-m_{r-j}\}}
  \]
  induced by the permutation matrix associated to \( x^{-1} u^{-1} \).

2 The relative position of parabolics over an arbitrary base

In this section we introduce some notations and collect some facts about parabolics of a reductive group \( G \) over an arbitrary base. At the end we explain all these notions for the case \( G = \text{GL}_n \).

(2.1) Let \( G \) be a group, \( X \subset G \) a subset and \( g \in G \). Then we set \( g X g^{-1} \).

If \( P \) is a parabolic subgroup of some reductive group scheme, we denote by \( U_P \) its unipotent radical.

(2.2) Let \( k \) be a field and let \( k^{\text{sep}} \) be a separable closure of \( k \). Recall that the functor \( X \mapsto X(k^{\text{sep}}) \) gives an equivalence of the category of (finite) étale \( k \)-schemes with the category of finite discrete sets endowed with a continuous action of \( \text{Gal}(k^{\text{sep}}/k) \).

(2.3) We fix the following notations: Let \( k \) be a field and let \( k^{\text{sep}} \) be a separable closure of \( k \). We denote by \( S \) an arbitrary \( k \)-scheme. If \( X \) is a \( k \)-scheme, write \( X_S := X \times_k S \).

Let \( G \) be a connected quasi-split reductive group over \( k \). We denote by \( (W, I) \) the Weyl group of the abstract based root datum of \( G \), together with its set of simple reflections. It is a finite Coxeter system carrying a continuous action of \( \text{Gal}(k^{\text{sep}}/k) \).
For subsets $J, K \subset I$ we denote by $W_J$ the subgroup of $W$ generated by $J$ and by $JW^K$ the set of elements $w \in W$ that are of minimal length in their double coset $W_JwW_K$. We write $JW = JW^\emptyset$ and $W^K = W^\emptyset K$.

**2.4** Let $\text{Par}$ be the smooth proper $k$-scheme that parametrizes the parabolic subgroups of $G$. It carries a $G$-action and the fppf quotient $G \backslash \text{Par}$ is representable by a finite étale $k$-scheme $\mathcal{D}$, see SGA3 ([7], Exp. XXVI, section 3, where $\mathcal{D}$ is called $\mathcal{P}(\text{Dyn}(G))$). We denote by $t: \text{Par} \to \mathcal{D}$ the canonical morphism. For $P \in \text{Par}(S)$ we call $t(P) \in \mathcal{D}(S)$ the *type* of $P$. If $J$ is a section of $\mathcal{D}$ over $S$ we denote by $\text{Par}_J = t^{-1}(J) \subset \text{Par}_S$ the scheme of parabolics of type $J$.

Under the equivalence of (2.2), the scheme $\mathcal{D}$ corresponds to the powerset of $I$ with its natural $\text{Gal}(k_{\text{sep}}/k)$-action. For $J \subset I$ we obtain the usual notion of parabolics of type $J$. The section of $\mathcal{D}$ corresponding to the empty subset of $I$ is defined over $k$ and $\mathcal{P}_\emptyset$ is the scheme of Borel subgroups of $G$.

We denote by $\mathcal{P} \to \text{Par}$ the universal parabolic subgroup and for $J \in \mathcal{D}(S)$ we write $\mathcal{P}_J$ for its pullback to $\text{Par}_J \to \text{Par}$. We denote by $U_J$ the unipotent radical of $P_J$.

**2.5** Let $S$ be a $k$-scheme. Let $P$ and $Q$ two parabolic subgroups of $G_S$. We say that $P$ and $Q$ are in *standard position* if the following conditions hold (which are mutually equivalent by [7], XXVI, 4.5):

1. The intersection $P \cap Q$ is smooth.
2. Locally for the Zariski topology on the basis, $P \cap Q$ contains a maximal torus of $G$.
3. Locally for the fpqc-topology on the basis, $P \cap Q$ contains a maximal torus of $G$.

Let $\mathcal{S}\mathcal{P}$ be the subfunctor of $\text{Par} \times \text{Par}$ of pairs $(P, Q)$ that are in standard position. By loc. cit., $\mathcal{S}\mathcal{P}$ is representable by a smooth quasi-projective scheme over $k$. If $S$ is the spectrum of a field, any two parabolics of $G_S$ are in standard position. Hence the monomorphism $\mathcal{S}\mathcal{P} \to \text{Par} \times \text{Par}$ is bijective. In fact, it can be shown that $\mathcal{S}\mathcal{P}$ is the disjoint union of the $G$-orbits in $\text{Par} \times \text{Par}$, in the scheme-theoretic sense. For sections $J, K \in \mathcal{D}(S)$ we denote by $\mathcal{S}\mathcal{P}_{J, K}$ the inverse image of $\text{Par}_J \times \text{Par}_K$ in $\mathcal{S}\mathcal{P}_S$.

The group $G$ acts on $\mathcal{S}\mathcal{P}$ by simultaneous conjugation. The fppf quotient $G \backslash \mathcal{S}\mathcal{P}$ is representable by a finite étale $k$-scheme $\mathcal{R}\mathcal{P}$. Let

$$r: \mathcal{S}\mathcal{P} \to \mathcal{R}\mathcal{P}$$

be the canonical morphism. There exists a unique surjective morphism of finite étale $k$-schemes $q: \mathcal{R}\mathcal{P} \to \mathcal{D} \times \mathcal{D}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{S}\mathcal{P} & \xrightarrow{r} & \mathcal{R}\mathcal{P} \\
\downarrow & & \downarrow q \\
\text{Par} \times \text{Par} & \xrightarrow{t \times t} & \mathcal{D} \times \mathcal{D}
\end{array}$$

is commutative.

On $k_{\text{sep}}$-valued points we have

$$\mathcal{R}\mathcal{P}(k_{\text{sep}}) \cong \prod_{J, K \subset I} JW^K$$
as sets with \( \text{Gal}(k^{\text{sep}}/k) \)-action, and \( J W^K = q^{-1}(J, K) \). Hence we obtain a morphism \( \iota : R P \to W \) whose restriction to \( q^{-1}(J, K) \) is the inclusion \( J W^K \hookrightarrow W \). We set

\[
\text{relopos} := \iota \circ r : \mathcal{S} P \to W.
\]

Whenever we write \( \text{relopos}(P, Q) \) it shall be understood that \( P \) and \( Q \) are in standard position. For \( x \in W \) we define \( \mathcal{S} P^x := \text{relopos}^{-1}(x) \).

\((2.6)\) Over \( k^{\text{sep}} \) (or any other separably closed extension of \( k \)) we can describe the morphism “relopos” as follows: We use the canonical isomorphism of the Weyl group of the abstract root datum of \( G \) with the set of \( G(k^{\text{sep}}) \)-orbits in \( \text{Par}_B(k^{\text{sep}}) \times \text{Par}_B(k^{\text{sep}}) \). For \( (B, B') \in \text{Par}_B(k^{\text{sep}}) \times \text{Par}_B(k^{\text{sep}}) \) we denote by \( \text{relopos}(B, B') \in W \) the corresponding \( G(k^{\text{sep}}) \)-orbit.

Now let \( J \) and \( K \) be arbitrary subsets of \( I \). For \( P \in \text{Par}_J(k^{\text{sep}}) \) and \( Q \in \text{Par}_K(k^{\text{sep}}) \) the relative position \( \text{relopos}(P, Q) \in J W^K \) is the unique minimal element (with respect to the Bruhat order) in the set

\[
\{ \text{relopos}(B, B') \mid B \subset P, B' \subset Q \}.
\]

The map \( (P, Q) \mapsto \text{relopos}(P, Q) \) gives a bijection between the set of \( G(k^{\text{sep}}) \)-orbits in \( \text{Par}_J(k^{\text{sep}}) \times \text{Par}_K(k^{\text{sep}}) \) and the set \( J W^K \).

Alternatively we can compute \( \text{relopos}(P, Q) \) as follows: Choose a maximal torus \( T \) which is contained in \( P \cap Q \). The choice of \( T \) provides an identification of \( W \) with \( N_G(T)/T \). There exists an \( n \in N_G(T) \) such that \( P \) and \( n(Q) \) contain a common Borel subgroup and the class of \( n \) in \( W_J \backslash W/W_K \) depends only on \( (P, Q) \). Its unique representative in \( J W^K \) is equal to \( \text{relopos}(P, Q) \).

\((2.7)\) For \( (P, Q) \in \mathcal{S} P(S) \) define the refinement of \( P \) with respect to \( Q \) to be

\[
\text{Ref}_Q(P) := (P \cap Q) U_P = U_P (P \cap Q).
\]

This is again a parabolic subgroup of \( G \) whose unipotent radical is \( U_P (P \cap U_Q) = (P \cap U_Q) U_P \). Indeed, it suffices to show this locally for the fpqc topology hence we can assume that \( P \cap Q \) contains a split maximal torus. Then the proof is the same as in [1], 4.4.

We refer to (2.13) for the description of \( \text{Ref}_Q(P) \) in the case that \( P \) and \( Q \) are parabolics of \( \text{GL}_N \).

Suppose \( P \in \mathcal{P}_J(S) \) and \( Q \in \mathcal{P}_K(S) \) are in standard position, with \( \text{relopos}(P, Q) = w \in J W^K \). Then \( \text{Ref}_Q(P) \) is of type \( J \cap w K \).

\((2.8)\) Let \( P \) and \( Q \) be two parabolic subgroups of \( G_S \). We say that \( P \) and \( Q \) are in good position if the following equivalent assertions hold:

\begin{enumerate}
  \item Zariski-locally on the basis, \( P \) and \( Q \) contain a common Levi subgroup.
  \item fpqc-locally on the basis, \( P \) and \( Q \) contain a common Levi subgroup.
  \item \( P \) and \( Q \) are in standard position and for every geometric point \( \bar{s} \) of \( S \) we have that \( P_{\bar{s}} \) and \( Q_{\bar{s}} \) contain a common Levi subgroup.
  \item \( P \) and \( Q \) are in standard position and for every geometric point \( \bar{s} \) of \( S \) we have \( J_{\bar{s}} = w_{\bar{s}}(K_{\bar{s}}) \), where \( J_{\bar{s}} \) and \( K_{\bar{s}} \) are the types of \( P_{\bar{s}} \) and \( Q_{\bar{s}} \), respectively, and where \( w_{\bar{s}} = \text{relopos}(P_{\bar{s}}, Q_{\bar{s}}) \).
\end{enumerate}

(This corresponds to what in [18], section 3, was called “in optimal position”.)
Lemma: Let $J, K \subset I$ be sets of simple roots and let $x \in K W J$ be such that $K = x J$. Let $Q$ be a parabolic subgroup of $G_S$ of type $K$ and let $M$ be a Levi subgroup of $Q$. Then there exists a unique parabolic subgroup $P$ of $G_S$ of type $J$ such that $M$ is a common Levi subgroup of $P$ and $Q$ and such that $\text{relops}(Q, P) = x$. (In particular, $P$ and $Q$ are then in good position).

Proof: Let $P$ and $P'$ be two parabolics of type $J$ such that $\text{relops}(Q, P) = \text{relops}(Q, P') = x$ and such that $M \subset P \cap P'$. Then it follows from [16], 8.4, that $\text{relops}(P, P') = 1$ and hence $P = P'$. This proves the unicity.

We omit the proof of the existence as we will not need this in the sequel.

Lemma: Let $P$ and $Q$ be two parabolics of $G_S$ which are in good position. Then we have $U_P \cap Q = U_P \cap U_Q$.

Proof: The question is local on $S$ for the fppf topology; hence we can assume that there exists a common Levi subgroup of $P$ and $Q$ whose connected center is a split torus. Now the proof is the same as in [16], 8.6.

Lemma: Let $P \in \text{Par}_J(S)$ and $Q \in \text{Par}_K(S)$ with $K = w_0 J$ where $w_0$ is the longest element of $W$. Let $x$ be the element of minimal length in the double coset $W_J w_0 W_K$. Then $P$ and $Q$ are in opposition (i.e., $P \cap Q$ is a common Levi subgroup of $P$ and $Q$), if and only if $\text{relops}(P, Q) = x$.

Lemma: Let $P$ and $Q$ be two parabolics of $G_S$ which are in good position. Then every parabolic subgroup $P'$ of $P$ is in standard position with $Q$ and we have $\text{relops}(P', Q) = \text{relops}(P, Q)$. The maps $P' \mapsto \text{Ref}_{P'}(Q)$ and $Q' \mapsto \text{Ref}_{Q'}(P)$ define mutually inverse bijections

$$\{\text{parabolic subgroups of } P\} \longleftrightarrow \{\text{parabolic subgroups of } Q\}.$$ Moreover, $P'$ and $\text{Ref}_{P'}(Q)$ are in good position and we have

$$\text{relops}(P', \text{Ref}_{P'}(Q)) = \text{relops}(P, Q).$$

In particular we see that $\text{Ref}_P(Q) = Q$.

Example: Let $G = \text{GL}_n$. Then $G$ is split over $k$. Associating to a flag in $\mathcal{O}_S^n$ its stabilizer defines an isomorphism between the scheme of flags and the scheme Par. We use this isomorphism to identify flags in $\mathcal{O}_S^n$ and parabolics of $G_S$.

The Weyl group $W$ can be identified with $S_n$ such that $I$ is the set of transpositions $\tau_\alpha = (\alpha \alpha + 1)$ for $\alpha = 1, \ldots, n - 1$. If $\Gamma = (\Gamma^i)$ is a flag such that all $\Gamma^i$ have constant rank, its type $J \subset I$ is determined by the rule that $\tau_\alpha \not\in J$ if and only if there exists an index $i$ with $\text{rk}_{\mathcal{O}_S}(\Gamma^i) = \alpha$.

Let $\Gamma = (\Gamma^i)$ and $\Delta = (\Delta^j)$ be two flags in $\mathcal{O}_S^n$. Then the following conditions are equivalent:

1. The parabolics associated to $\Gamma$ and $\Delta$ are in standard position.
2. For all $i$ and $j$, the submodule $\Gamma^i + \Delta^j \subset \mathcal{O}_S^n$ is locally a direct summand.
3. Zariski-locally on $S$ there exists a basis $\{e_1, \ldots, e_n\}$ of $\mathcal{O}_S^n$, such that for all $i$ and $j$ there exists a subset $I_{i,j}$ of $\{1, \ldots, n\}$ with $\Gamma^i + \Delta^j = \bigoplus_{\alpha \in I_{i,j}} \mathcal{O}_S \cdot e_\alpha$.

If these conditions are satisfied, the relative position of $\Gamma$ and $\Delta$ is completely determined by the function $(i, j) \mapsto \text{rk}_{\mathcal{O}_S}(\Gamma^i + \Delta^j)$.
As an example, for $J, K \subset I$, let $x$ be the element of minimal length in $W_J w_0 W_K$, where $w_0$ is the longest element in $W$. Let $\Gamma$ and $\Delta$ be flags of types $J$ and $K$, respectively, which are in standard position. Then we have

$$\relpos(\Gamma, \Delta) = 1 \iff \rk_{\mathcal{O}_S}(\Gamma^i + \Delta^j) = \max(\rk_{\mathcal{O}_S}(\Gamma^i), \rk_{\mathcal{O}_S}(\Delta^j)) \quad \text{for all } i, j,$$

and

$$\relpos(\Gamma, \Delta) = x \iff \rk_{\mathcal{O}_S}(\Gamma^i + \Delta^j) = \min(n, \rk_{\mathcal{O}_S}(\Gamma^i) + \rk_{\mathcal{O}_S}(\Delta^j)) \quad \text{for all } i, j.$$

If $\Gamma$ and $\Delta$ are flags in standard position with stabilizers $P$ and $Q$, respectively, the flag corresponding to $\Ref_Q(P)$ is given by the collection of submodules $(\Gamma^{i-1} \cap \Delta^j) + \Gamma^i$ for all $i$ and $j$. This is a refinement of the flag $\Gamma$.

If $\Gamma$ is a flag with associated parabolic $P$, the choice of a Levi subgroup of $P$ corresponds to the choice of a decomposition $\mathcal{O}_S^2 = \bigoplus_{j=1}^r M_j$ such that $\Gamma^i = \bigoplus_{j>i} M_{\pi(j)}$ for some permutation $\pi \in S_r$.

### 3 A semi-linear variation on a theme of Lusztig

In this section we consider a reductive group $G$ over $\mathbb{F}_q$. As in Lusztig’s paper [16], we define, for $J$ a set of simple reflections in the Weyl group, a variety $Z_J$ equipped with an action of $G$. This is a semi-linear variant of the variety examined by Lusztig. The main result of this section, Theorem (3.25), concerns a classification of the $G$-orbits in $Z_J$. This result will be used in the next section to prove our main classification theorem for $F$-zips.

Throughout this section, $q$ is a fixed power of a prime number $p$, and $S$ is a scheme over $\mathbb{F}_q$. Note that in [16], Lusztig writes $P^Q$ for what we call $\Ref_Q(P)$.

#### (3.1) Let $\tilde{G}$ be a possibly disconnected reductive group over $\mathbb{F}_q$ and denote by $G$ its identity component. We keep the notations of (2.3); note that $G$ is indeed quasi-split. Further we fix a connected component $G^1$ of $\tilde{G}$. Let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}_q$ and denote by $\sigma: x \mapsto x^q$ the arithmetic Frobenius in $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q)$. It acts on $(W, I)$.

There is a unique $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q)$-equivariant isomorphism $\delta: (W, I) \to (W, I)$ of Coxeter systems such that for all $g \in G^1(\overline{\mathbb{F}})$ and $P \in \mathcal{P}_J(\overline{\mathbb{F}})$ we have $^g P \in \mathcal{P}_{\delta(J)}(\overline{\mathbb{F}})$.

If there is no risk of confusion we simply write $F: \tilde{G} \to \tilde{G}$ for the morphism $F_{\tilde{G}}: \tilde{G} \to \tilde{G}^{(q)} = \tilde{G}$ that was defined in (1.1). It is an endomorphism of $\tilde{G}$.

#### (3.2) We fix the following data: Let $J$ and $K$ be subsets of $I$ and $x \in W$ such that $^x \delta(J) = K$ and $x \in K^1 W^{\delta(J)}$. We assume that $J$ and $x$ (hence also $K$) are defined over $\mathbb{F}_q$, i.e., $\sigma(J) = J$ and $\sigma(x) = x$.

#### (3.3) Let $\tilde{Z}_J$ be the $\mathbb{F}_q$-scheme given by the cartesian square

$$\begin{array}{ccc}
\tilde{Z}_J & \longrightarrow & \mathcal{S}\mathcal{P}^x \\
\downarrow & & \downarrow \\
\text{Par}_J \times \text{Par}_K \times G^1 & \xrightarrow{f} & \text{Par}_K \times \text{Par}_{\delta(J)}
\end{array}$$

where the morphism $f$ is given on points by $(P, Q, g) \mapsto (Q, {}^g F(P))$. If $S$ is an $\mathbb{F}_q$-scheme then the $S$-valued points of $\tilde{Z}_J$ are the triples $(P, Q, g)$ with $P$ and $Q$ parabolics of $G_S$ of types $J$.
and \( K \), respectively, and with \( g \in G^1(S) \) an element such that \( Q \) and \( ^gF(P) \) are in relative position \( x \). In particular, \( Q \) and \( ^gF(P) \) are then in good position; see (2.8). The forgetful morphism \((P,Q,g) \mapsto (P,Q)\) makes \( \tilde{Z}_J \) into a scheme over \( \text{Par}_J \times \text{Par}_K \).

We define an action of \( G \) on \( \tilde{Z}_J \) given on \( S \)-valued points by
\[
h \cdot (P,Q,g) := (hP, ^hQ, ^{hg}F(h)^{-1}).
\]

It is easily seen that this is well-defined.

(3.4) For \( u \in J^iW^K \), let \( \tilde{Z}_J^u \) be the subscheme of \( \tilde{Z}_J \) of triples \((P,Q,g)\) with \( \text{relpos}(P,Q) = u \). The natural morphism
\[
\prod_{u \in J^iW^K} \tilde{Z}_J^u \to \tilde{Z}_J
\]
is a bijective monomorphism.

Fix \( u \in J^iW^K \). Let \( L := \delta(J \cap ^u \delta(J)) = \delta(J \cap ^u K) \) and consider the morphism \( h: \tilde{Z}_J^u \to \text{Par}_K \times \text{Par}_L \) given on points by \((P,Q,g) \mapsto (Q, ^gF(\text{Ref}_Q(P)))\). Define a scheme \( \tilde{Y}_J^u \) by the fibre product diagram
\[
\begin{array}{ccc}
\tilde{Y}_J^u & \to & \text{SP}_{K,L} \\
\downarrow & & \downarrow \\
\tilde{Z}_J^u & \xrightarrow{h} & \text{Par}_K \times \text{Par}_L.
\end{array}
\]

On points this means that we are considering triples \((P,Q,g)\) in \( \tilde{Z}_J^u \) with the additional requirement that \( Q \) and \( ^gF(\text{Ref}_Q(P)) \) are in standard position.

Note that the \( G \)-action on \( \tilde{Z}_J \) preserves \( \tilde{Z}_J^u \) and \( \tilde{Y}_J^u \).

(3.5) Let \( J_1 := J \cap ^u \delta(J) = J \cap ^u K \) and \( K_1 := ^x \delta(J_1) \).

Define a morphism \( \tilde{\varphi}: \tilde{Y}_J^u \to \tilde{Z}_J \) by \( \tilde{\varphi}(P,Q,g) = (P_1,Q_1,g) \) with
\[
P_1 := \text{Ref}_Q(P) \quad \text{and} \quad Q_1 := \text{Ref}_{^x F(\text{Ref}_Q(P))}(Q) = \text{Ref}_{^x F(P_1)}(Q).
\]

To see that \( \tilde{\varphi} \) is well-defined, we need to check that \( P_1 \) and \( Q_1 \) are parabolics of types \( J_1 \) and \( K_1 \), respectively, and that \( \text{relpos}(Q_1, ^x F(P_1)) = x \). That \( P_1 \) has type \( J_1 \) is immediate from (2.7). Next remark that \( ^x F(P_1) \subset ^x F(P) \), so by (2.12) we have \( \text{relpos}(Q, ^x F(P_1)) = x \).

Again using (2.7) we then easily verify that \( Q_1 \) has type \( K_1 \), and by (2.12) we conclude that \( \text{relpos}(Q_1, ^x F(P_1)) = x \).

(3.6) Consider a sequence \( u = (u_0,u_1,\ldots) \) of elements of \( W \). Define a sequence of subsets \( J_n \subset I \) by setting \( J_0 := J \) and \( J_{n+1} := J_n \cap ^u \delta(J_n) \). Set \( K_n := ^x \delta(J_n) \). Let \( T(J) \) be the set of sequences \( u = (u_0,u_1,\ldots) \) such that for all \( n \geq 0 \) we have
\[
(3.6.1) \quad u_n \in J^nW^K \quad \text{and} \quad u_{n+1} \in W_{J_{n+1},u_nW_K}.
\]

(These conditions imply that in fact \( u_{n+1} \in u_nW_{K} \)). By construction, \( J_{n+1} \subseteq J_n \) and \( K_{n+1} \subseteq K_n \) for all \( n \). Hence there exists an index \( N \) such that \( J_{N+1} = J_N \) and \( K_{N+1} = K_N \) for all \( n \geq N \).

Writing \( J_\infty := J_N \) and \( K_\infty := K_N \) for \( n \gg 0 \), we find that \( J_\infty = u_nK_\infty \) for \( n \geq N \). If \( u \in T(J) \) and \( n \geq N \) then the two conditions in (3.6.1) readily imply that \( u_{n+1} = u_n \). Set \( u_\infty := u_n \) for any \( n \geq N \).
(3.7) Lemma: The map $T(J) \to W$ defined by $u \mapsto u_\infty$ gives a bijection $T(J) \leftrightarrow JW$.

Proof: Set $\tilde{J} = K_0$ and $\tilde{J}' = J$ and let $\varepsilon$ be the automorphism $w \mapsto \delta(w^{-1}wx)$ of $W$. Then the set $T(J)$ is nothing but Lusztig’s set $T(\tilde{J}, \varepsilon)$ as defined in [16], 2.2, and our claim follows from [16], 2.5.

(3.8) Let $u = (u_0, u_1, \ldots) \in T(J)$. Let $N(u)$ be the smallest non-negative integer such that $J_{n+1} = J_n$ for all $n \geq N(u)$; as we have seen this implies that also $K_{n+1} = K_n$ and $u_{n+1} = u_n$ for all $n \geq N(u)$.

For $r \geq 0$ we write $u_r := (u_r, u_{r+1}, \ldots)$, which is an element of $T(J_r)$. In particular, $u = u_0$. Note that $N(u_r) = \max\{0, N(u_0) - r\}$.

By induction on $N(u)$ we now define schemes $\tilde{Y}_J^u$ together with morphisms $\tilde{Y}_J^u \to \tilde{Y}_J^{u_0}$. If $N(u) = 0$ then we set $\tilde{Y}_J^u := \tilde{Y}_J^{u_0}$, mapping identically to itself. Next assume that $N(u) = N$ and that for all $L \subset I$ and $v \in T(L)$ with $N(v) < N$ the morphism of schemes $\tilde{Y}_J^v \to \tilde{Y}_J^{u_0}$ has been defined. Then we define $\tilde{Y}_J^u$ by the fibre product diagram

$$
\begin{array}{ccc}
\tilde{Y}_J^u & \longrightarrow & \tilde{Y}_J^{u_0} \\
\downarrow & & \downarrow \varphi \\
\tilde{Y}_{J_1}^u & \rightarrow & \tilde{Y}_{J_1}^{u_1} \\
\end{array}
$$

On points this means the following. If $N(u) = 0$ then $u = (u, u, \ldots)$ is a constant sequence, and we just consider the scheme $\tilde{Y}_J^u$. Next suppose $N(u) = 1$, which means that $u = (u_0, u_1, u_1, \ldots)$ for some $u_0 \neq u_1$. In this case, the points of $\tilde{Y}_J^u$ are the points $(P, Q, g)$ of $\tilde{Y}_J^{u_0}$ such that the associated triple $(P_1, Q_1, g) := \varphi(P, Q, g)$ lies in $\tilde{Y}_{J_1}^u \hookrightarrow \tilde{Z}_{J_1}$. In general we have a diagram

$$
\begin{array}{ccc}
\tilde{Y}_J^{u_0} & \longrightarrow & \tilde{Z}_J \\
\downarrow & & \downarrow \varphi \\
\tilde{Y}_{J_1}^{u_1} & \rightarrow & \tilde{Z}_{J_1} \\
\downarrow & & \downarrow \varphi \\
\tilde{Y}_{J_2}^{u_2} & \rightarrow & \tilde{Z}_{J_2} \\
& \vdots & \\
\end{array}
$$

and the points of $\tilde{Y}_J^u$ are those triples $(P, Q, g)$ in $\tilde{Y}_J^{u_0}$ that under each subsequent map $\varphi$ land inside $\tilde{Y}_{J_n}^{u_n} \hookrightarrow \tilde{Z}_{J_n}$.

Note that the map $\tilde{Y}_J^u \to \tilde{Y}_J^{u_0}$ is a monomorphism and that the $G$-action on $\tilde{Y}_J^{u_0}$ preserves $\tilde{Y}_J^u$.

(3.9) Let $u = (u_0, u_1, \ldots) \in T(J)$ and set $u = u_0$. The schemes

$$(3.9.1) \quad \tilde{Y}_J^u \hookrightarrow \tilde{Y}_J^u \hookrightarrow \tilde{Z}_J^u \hookrightarrow \tilde{Z}_J$$

are schemes over $\text{Par}_J \times \text{Par}_K$. Recall that we denote by $\mathcal{P}_J$ the universal parabolic group scheme over $\text{Par}_J$ and by $\mathcal{U}_J$ its unipotent radical. Then $F(\mathcal{P}_J)$ is again a parabolic subgroup scheme of $G \times \text{Par}_J$ over $\text{Par}_J$, which has $F(\mathcal{U}_J)$ as its unipotent radical. (In fact, as $J$ is defined over $\mathbb{F}_q$, so is $\text{Par}_J$, and $F(\mathcal{U}_J)$ is none other than the pull-back of $\mathcal{U}_J$ via the morphism $F_{\text{Par}_J}: \text{Par}_J \to \text{Par}_J$.)
Write $\mathcal{U}_{J,K}$ for the scheme $F(\mathcal{U}_J) \times \mathcal{U}_K$, but with a new group scheme structure given on points by $(v_1, v_2) \cdot (v'_1, v'_2) = (v'_1 v_1, v_2 v'_2)$. Then $\mathcal{U}_{J,K}$ acts from the left on all four schemes in (3.9.1) by

$$(v_1, v_2) \cdot (P, Q, g) = (P, Q, v_2 g v_1).$$

We define

$$Y^J_f \longleftarrow Y^J_f \longleftarrow Z^J_f \longleftarrow Z_J$$

to be the fppf quotient sheaves of the schemes in (3.9.1) by this action of $\mathcal{U}_{J,K}$. More informally we could write $Z_J = \mathcal{U}_K \backslash \mathcal{Z}_J/F(\mathcal{U}_J)$, and similarly for the other quotients. If $(P, Q, g) \in \mathcal{Z}_J(S)$ then we write $[P, Q, g]$ for its image in $Z_J(S)$.

It readily follows from the definitions that the $G$-action on $\mathcal{Z}_J$ induces a $G$-action on $Z_J$, and hence on all other quotients in (3.9.2). Further it follows from [7], Exp. XXVI, 2.2 that for an affine scheme $S$ the canonical morphism $\mathcal{Z}_J(S) \to Z_J(S)$ is surjective.

(3.10) Our next goal is to show that the sheaves in (3.9.2) are representable by schemes.

The quotient $\mathcal{P}_J/\mathcal{U}_J$ is representable by a reductive group scheme over $\text{Par}_J$. Let $H$ be defined by the cartesian diagram

$$
\begin{array}{ccc}
H & \longrightarrow & \mathcal{P}_J/\mathcal{U}_J \\
\downarrow & & \downarrow \\
\text{Par}_J \times \text{Par}_K & \longrightarrow & \text{Par}_J,
\end{array}
$$

with $\alpha$ given by $(P, Q) \mapsto F(P)$. For an affine scheme $S$, the $S$-valued points of $H$ are given by triples $(P, Q, y U_P F(P))$, where $P$ and $Q$ are parabolic subgroups of $G_S$ of types $J$ and $K$, respectively, and where $y \in F(P)(S)$.

Define a right action

$$Z_J \times_{(\text{Par}_J \times \text{Par}_K)} H \longrightarrow Z_J$$

as follows: For an affine scheme $S$, a point $z = [P, Q, g] \in Z_J(S)$, and $h = (P, Q, y U_P F(P)) \in H(S)$, we set

$$z \cdot h = [P, Q, g y] \in Z_J(S).$$

(3.11) Lemma: This action makes $Z_J$ into an $H$-torsor over $\text{Par}_J \times \text{Par}_K$ for the Zariski topology.

Proof: Let $P \in \text{Par}_J(S)$ and $Q \in \text{Par}_K(S)$ and suppose we have $g, \ g' \in G^1(S)$ such that $\text{relo}(Q, g F(P)) = \text{relo}(Q, g' F(P)) = x$. Locally on $S$ we can find $b \in Q$ such that $g' = b g F(P)$. Let $M$ be a common Levi subgroup of $g F(P)$ and $Q$ (which we can find Zariski-locally on $S$, as $g F(P)$ and $Q$ are in good position). Then we have $b = v m$ with $v \in U_Q$ and $m \in M$. As $M \subset g F(P)$, we have $g' \in v m g F(P) = v g F(P)$. This proves that the action is transitive.

Now assume that for $g \in G^1$ with $\text{relo}(Q, g F(P)) = x$ there exist elements $y, y' \in F(P)$ such that $g y' \in U_Q g y U_P F(P)$. Then possibly after multiplying $y$ from the right by an element of $U_{F(P)}$ we may assume that there is a $v \in U_Q$ with $g y' = v g y$. But then $v \in U_Q \cap g F(P) = U_Q \cap g U_{F(P)}$, where the last equality holds by (2.10). Hence there exists a $v' \in U_{F(P)}$ such that $v = g v' g^{-1}$. This gives that $y' \in U_{F(P)} \cdot y = y \cdot U_{F(P)}$, proving that the action is free.
(3.12) Corollary: The fppf sheaves $Y^u$, $Y^v$, $Z^u_j$ and $Z_j$ are representable by schemes.

(3.13) Lemma: The morphism $\tilde{\vartheta} : \tilde{Y}^u_j \to \tilde{Z}_{j_1}$ induces a morphism $\vartheta : Y^u_j \to Z_{j_1}$.

Proof: Let $(P, Q, g)$ be an $S$-valued point of $\tilde{Y}^u_j$, and let $(P_1, Q_1, g)$ be its image under $\tilde{\vartheta}$. Then $P_1 \subseteq P$ and $Q_1 \subseteq Q$, so $U_{F(P)} \subseteq U_{F(P_1)}$ and $U_Q \subseteq U_{Q_1}$. But then it is immediate from the definitions that the composed morphism $\tilde{Y}^u_j \to \tilde{Z}_{j_1} \to Z_{j_1}$ factors modulo the action of $U_{I, K}$.

(3.14) Let $u = (u_0, u_1, u_2, \ldots) \in \mathcal{T}(J)$. For $n \geq 0$ let $u_n = (u_n, u_{n+1}, \ldots) \in \mathcal{T}(J_n)$. As an immediate consequence of the definition of the schemes $\tilde{Y}^u_j$ and their quotients $Y^u_j$ we obtain $G$-equivariant morphisms

$$\tilde{\vartheta} : \tilde{Y}^u_j \to \tilde{Y}^u_{j_1}.$$ 

inducing $G$-equivariant morphisms

$$\vartheta : Y^u_j \to Y^u_{j_1}.$$ 

(3.15) The natural map $\coprod_{u \in \mathcal{T}(J)} \tilde{Y}^u_j \to \tilde{Z}_j$ is a bijective monomorphism. Passing to quotients modulo $U_{I, K}$ we readily find that $\coprod_{u \in \mathcal{T}(J)} Y^u_j \to Z_j$ is a bijective monomorphism, too. (Use that $\tilde{Z}_j \to Z_j$ is surjective on underlying topological spaces.) In particular, if $k$ is an algebraically closed field then we have a bijection

$$\coprod_{u \in \mathcal{T}(J)} Y^u_j(k) \xrightarrow{\sim} Z_j(k).$$

Our main goal for the rest of this section is to show that the $G$-action on the schemes $Y^u_j$ is transitive. Along the way we shall also compute the dimension of the schemes $Y^u_j$.

(3.16) Lemma: The morphism $\tilde{\vartheta} : \tilde{Y}^u_{j_n} \to \tilde{Y}^u_{j_{n+1}}$ is an isomorphism.

Proof: Without loss of generality we can assume $n = 0$. Suppose $\tilde{\vartheta}(P, Q, g) = \tilde{\vartheta}(P', Q', g') =: (P_1, Q_1, g_1)$. Clearly, $g = g_1 = g'$. Further, $P$ and $P'$ are parabolics of the same type, and they both contain $P_1$; hence $P = P'$. The same argument shows that $Q = Q'$. Hence $\tilde{\vartheta}$ is a monomorphism.

Now let $(P_1, Q_1, g)$ be an $S$-valued point of $\tilde{Y}^u_{j_1}$. Let $P$ be the unique parabolic of type $J$ that contains $P_1$, and let $Q$ be the unique parabolic of type $K$ containing $Q_1$. These exist by [7], Exp. XXVI, 3.8. Then $(P_1, Q_1)$ is in standard position, $P \supset P_1$, and $Q \supset Q_1$; hence $(P, Q)$ is in standard position, too. In a similar way we see that $(\vartheta(F(\text{Ref}_Q(P)), Q)$ is in standard position.

By definition of $\mathcal{T}(J)$ we have $\text{relops}(P_1, Q_1) \in uW_K$ with $u = u_0 \in jW_K$. It follows that $\text{relops}(P, Q) = u$. Similarly, as $x \in K^{W_{\mathcal{O}(J)}}$ and $\text{relops}(Q, \vartheta(F(P_1))) = x$, we also have $\text{relops}(Q, \vartheta(F(P))) = x$. Hence it remains to see that $\text{Ref}_Q(P) = P_1$ and $\text{Ref}_Q(F(P_1))(Q) = Q_1$. For this we may work fppf-locally on $S$.

We write $\text{relops}(P_1, Q_1) = uw$ with $w \in W_K$. Working fppf-locally we can assume that $G_S$ is split and hence we can find Borel subgroups $B \subset P_1$ and $C \subset Q_1$ such that $\text{relops}(B, C) = uw$. As we have $\ell(uw) = \ell(u) + \ell(w)$, there exists a Borel subgroup $D$ of $G_S$ such that $\text{relops}(B, D) = u$ and $\text{relops}(D, C) = w$. As $C \subset Q_1 \subset Q$ and $w \in W_K$, we see that $D \subset Q$. As $B \subset P$ and $D \subset Q$ and $\text{relops}(B, D) = \text{relops}(P, Q)$, we have $B \subset \text{Ref}_Q(P)$. But then $P_1$ and $\text{Ref}_Q(P)$ have the same type and have a Borel subgroup in common; hence they are equal.
It remains to be shown that \( Q_1 = \text{Ref}_{sF(P_1)}(Q) \). Set \( Q'_1 := \text{Ref}_{sF(P_1)}(Q) \). By (2.12) we have relpos\((Q_1, gF(P_1)) = x = \text{relpos}(Q'_1, gF(P_1)) \), so there exists an \( h \in gF(P_1) \) with \( Q_1 = hQ'_1 \). Moreover we have \( hQ \supset hQ'_1 = Q_1 \) and \( Q \supset Q_1 \) and hence \( hQ = Q \). Therefore,

\[
Q_1 = hQ'_1 = h\text{Ref}_{sF(P_1)}(Q) = \text{Ref}_{sF(P_1)}(Q),
\]

and the proof is complete.

(3.17) Lemma: The morphism \( \vartheta: Y_{J_n}^m \to Y_{J_{n+1}}^m \) induces an isomorphism of fppf quotient sheaves

\[
\tilde{\vartheta}: G\backslash Y_{J_n}^m \to G\backslash Y_{J_{n+1}}^m.
\]

Proof: Without loss of generality we can assume \( n = 0 \). We consider \( S \)-valued points, where \( S \) is an affine \( \mathbb{F}_q \)-scheme. Note that the quotient map \( \tilde{Y}_n^m(S) \to Y_n^m(S) \) is surjective by [7], Exp. XXVI, 2.2. Further let us recall that for \( (P, Q, g) \in \tilde{Y}_n^m(S) \) we denote by \( [P, Q, g] \) its image in \( Y_n^m \).

By (3.16) we only have to show that \( \tilde{\vartheta} \) is a monomorphism. Let \( [P, Q, g] \) and \( [P', Q', g'] \) be two \( S \)-valued points of \( Y_n^m \) such that \( \vartheta([P, Q, g]) = \vartheta([P', Q', g']) \) are in the same \( G(S) \)-orbit. We want to show that \( [P, Q, g] \) and \( [P', Q', g'] \) are fppf-locally in the same \( G(S) \)-orbit. As \( \vartheta \) is \( G \)-equivariant, we can assume that \( \vartheta([P, Q, g]) = \vartheta([P', Q', g']) =: [P_1, Q_1, g_1] \).

As \( P \) and \( P' \) have the same type and both contain \( P_1 \), we get \( P = P' \). Similarly, \( Q = Q' \).

By definition, \( \text{relpos}(Q, gF(P)) = \text{relpos}(Q', g'F(P)) = x \); hence \( gF(P) \) and \( g'F(P) \) are both in good position to \( Q \). Let \( L \) (resp. \( L' \)) be a common Levi subgroup of \( gF(P) \) and \( Q \) (resp. of \( g'F(P) \) and \( Q \)). There exists a unique \( \xi \in U_Q \) with \( \xi L = L' \) ([7], Exp. XXVI, 1.8). By (2.9) this implies that \( \xi gF(P) = g'F(P) \), and therefore \( \xi g_1 \in g'F(P) \). We can replace \( g \) by \( \xi g \) and write \( g' = gy \) with \( y \in F(P) \). In particular, we now have \( gF(P) = g'F(P) \).

We have

\[
\text{Ref}_{sF(\text{Ref}_Q(P))}(Q) = Q_1 = \text{Ref}_{sF(\text{Ref}_Q(P))}(Q)
\]

and this is a parabolic subgroup of \( Q \). As \( Q \) and \( gF(P) = g'F(P) \) are in good position, (2.12) implies that \( gF(\text{Ref}_Q(P)) = g'F(\text{Ref}_Q(P)) \); in other words

\[
gF(P_1) = g'F(P_1).
\]

By hypothesis, \( g' \in U_{Q_1}gU_{F(P_1)} \), at least fppf-locally. Hence can write \( g' = vgu \) with \( v \in U_{Q_1} = (U_{sF(P_1)} \cap Q)U_Q \) and \( u \in U_{F(P_1)} \). Changing \( g \) on the left by an element of \( U_Q \), we may assume that \( v \in U_{sF(P_1)} \cap Q \). Write \( v = gu'g^{-1} \) with \( u' \in U_{F(P_1)} \) and replace \( u \) by \( u'u \). Then we have \( g' = gu \) with \( u \in U_{F(P_1)} = U_{F(P)}(F(P) \cap U_{F(Q)}) \); see (2.7).

Write \( u = u_1u_2 \) with \( u_1 \in U_{F(P)} \) and \( u_2 \in F(P) \cap U_{F(Q)} \). Replacing \( g \) by \( gu_1 \) and \( u \) by \( u_2 \) we can further assume that

\[
(3.17.1) \quad g' = gu, \quad \text{with } u \in F(P) \cap U_{F(Q)}.
\]

Note that we did not use the \( G \)-action so far.

To finish the proof, all we now have to remark is that fppf-locally on \( S \) we can write \( u = F(v) \) with \( v \in P \cap U_Q \) (as \( F: P \cap U_Q \to F(P) \cap U_{F(Q)} \) is an epimorphism of fppf sheaves), and then

\[
v^{-1} \cdot [P, Q, g] = [v^{-1}P, v^{-1}Q, v^{-1}gF(v)] = [P, Q, v^{-1}g'] = [P, Q, g'].
\]
Hence $\bar{\vartheta}$ is indeed injective.

**Lemma (3.18):** Let $S$ be an affine scheme. For $[P_1, Q_1, g_1] \in Y_{J_1}^n(S)$, choose $[P, Q, g] \in Y_J^n(S)$ with $\vartheta([P, Q, g]) = [P_1, Q_1, g_1]$. (This is possible by (3.16) and the surjectivity of the map $\tilde{Y}_{J_1}^n(S) \to Y_{J_1}^n(S).$) Then we have a well-defined morphism

$$\kappa : F(P) \cap U_{F(Q)} \to \vartheta^{-1}([P_1, Q_1, g_1])$$

given on points by $v \mapsto [P, Q, gv]$, and this induces an isomorphism

$$\left(F(P) \cap U_{F(Q)}\right)/\left(U_{F(P)} \cap U_{F(Q)}\right) \cong \vartheta^{-1}([P_1, Q_1, g_1]).$$

**Proof:** It is easy to check that $\kappa$ is well-defined. The arguments of (3.17), resulting in the relation (3.17.1), show that $\kappa$ is an epimorphism of sheaves.

It is clear that if $v = v'y$ for some $y \in U_{F(P)}$ then $\kappa(v) = \kappa(v')$. Conversely, assume that $\kappa(v) = \kappa(v')$. Then we have

$$gv' \in U_QgvU_{F(P)} = U_QgU_{F(P)}v,$$

so we may write $gv' = wguv$ with $w \in U_Q$ and $u \in U_{F(P)}$. But then $w = gv'u^{-1}g^{-1} \in U_Q \cap gF(P)$, so

$$wgu^{-1} = (U_Q \cap gF(P))gU_{F(P)} = U_{\text{Ref}_{Q(gF(P))}} = U_{F(P)};$$

where the last equality holds because $Q$ and $gF(P)$ are in good position. It follows that $wgu = gy$ for some $y \in U_{F(P)}$; hence $v' = yv \in U_{F(P)} \cdot v = v \cdot U_{F(P)}$.

**Lemma (3.19):** Let $\mathcal{P}_J$ and $\mathcal{P}_K$ be the universal parabolic subgroups over $\text{Par}_J$ and $\text{Par}_K$, respectively. Define an action of $F(\mathcal{P}_J) \cap U_{F(\mathcal{P}_K)}$ on $\tilde{Z}_J$ over $\text{Par}_J \times \text{Par}_K$ by

$$v \cdot (P, Q, g) = (P, Q, gv).$$

For $u \in J^nW^K$ this action preserves $\tilde{Y}_J^n \cong \tilde{Z}_J$. Moreover, $\tilde{\vartheta}(v \cdot (P, Q, g)) = \tilde{\vartheta}(P, Q, g)$ so $F(\mathcal{P}_J) \cap U_{F(\mathcal{P}_K)}$ acts on the fibres of $\tilde{\vartheta}$. Obviously, this action descends to an action on $Z_J$ and $Y_J^n$. Hence for a scheme $S$ over $\text{Par}_J \times \text{Par}_K$ and a section $y \in Y_J^n(S)$, we have that $(F(\mathcal{P}_J) \cap U_{F(\mathcal{P}_K)})_S$ acts on the fibre $\vartheta^{-1}(\vartheta(y))$. Now (3.18) shows that $\vartheta^{-1}(\vartheta(y))$ is a torsor under the affine group scheme $(F(\mathcal{P}_J) \cap U_{F(\mathcal{P}_K)})/U_{F(\mathcal{P}_J)}$.

Hence the dimension of the fibres of $\vartheta$ equals

$$\dim(F(U_{J_1})) - \dim(F(U_J)) = \dim(U_{J_1}) - \dim(U_J) = \dim(\text{Par}_{J_1}) - \dim(\text{Par}_J).$$

Repeating this argument, we obtain a chain of morphisms

$$Y_J^n \xrightarrow{\vartheta_1} Y_{J_1}^{n_1} \xrightarrow{\vartheta_2} \ldots \xrightarrow{\vartheta_n} Y_{J_\infty}^{n_\infty}$$

where each of the morphisms $\vartheta_n$ is a torsor under a unipotent group of dimension $\dim(\text{Par}_{J_n}) - \dim(\text{Par}_{J_{n-1}}).$
By (3.11) the forgetful morphism \( \pi: Z_{J_\infty} \to \text{Par}_{J_\infty} \times \text{Par}_{K_\infty} \) is smooth and surjective of relative dimension \( \dim(G) - 2 \dim(\text{Par}_{J_\infty}) \). The inverse image of \( \mathcal{SP}^u \subset \text{Par}_{J_\infty} \times \text{Par}_{K_\infty} \) under \( \pi \) is nothing but \( Y_{J_\infty}^u = Y_{J_\infty}^{u_\infty} \) as all pairs \((P, Q) \in \mathcal{SP}^u \) are in good position.

In particular, we see that \( Y_{J_\infty}^u \) and hence \( Y_J^u \) are nonempty.

(3.20) Proposition: For \( u \in T(J) \) let \( u_\infty \in JW \) be the corresponding element as in (3.7). Then

\[
\text{codim}(Y_J^u, Z_J) = \dim(\text{Par}_J) - \ell(u_\infty).
\]

Proof: By (3.19) we have

\[
\dim(Y_J^u) = \dim(\text{Par}_{J_\infty}) - \dim(\text{Par}_J) + \dim(Y_{J_\infty}^{u_\infty})
\]

\[
= \dim(\text{Par}_{J_\infty}) - \dim(\text{Par}_J) + \dim(G) - 2 \dim(\text{Par}_{J_\infty}) + \dim(\mathcal{SP}^{u_\infty})
\]

\[
= \dim(G) - \dim(\text{Par}_J) - \dim(\text{Par}_{J_\infty}) + \ell(u_\infty) + \dim(\text{Par}_{J_\infty})
\]

\[
= \dim(G) - \dim(\text{Par}_J) + \ell(u_\infty).
\]

On the other hand, (3.11) implies that \( \dim(Z_J) = \dim(G) \) which proves our claim.

(3.21) Suppose given an element \( u \in JW^K \), rational over \( \mathbb{F}_q \), with the property that \( J = {}^u K = {}^{ux} \delta(J) \). The case we have in mind is when \( J = J_\infty, K = K_\infty \) and \( u = u_\infty \) for some element \( u \in T(J_0) \) as in (3.6).

We fix a triple \((P_0, Q_0, L_0)\) consisting of a parabolic subgroup \( P_0 \subset G \) of type \( J \), a parabolic subgroup \( Q_0 \subset G \) of type \( K \), and a subgroup \( L_0 \subset G \) such that \( \text{relpos}(P_0, Q_0) = u \) and such that \( L_0 \) is a common Levi subgroup of \( P_0 \) and \( Q_0 \). Such triples exists (rationally over \( \mathbb{F}_q \)) because \( G \) is quasi-split and \( J, K \) and \( u \) are all defined over \( \mathbb{F}_q \). Note in particular that \( F(P_0) = P_0 \) and \( F(Q_0) = Q_0 \).

Let \( X_J^u \) be the \( \mathbb{F}_q \)-scheme whose \( S \)-valued points are the elements \( g \in G^1(S) \) such that

1. \( \text{relpos}(Q_0, {}^g P_0) = x \);
2. \( {}^g L_0 = L_0 \);
3. \( L_0 \) is a Levi subgroup of \( {}^g P_0 \).

We have an action of \( L_0 \) on \( X_J^u \) by left multiplication. We claim that this makes \( X_J^u \) an \( L_0 \)-pseudo-torsor in the étale topology. (As we will see below, \( X_J^u \) is nonempty, so it is in fact a true \( L_0 \)-torsor.) To see this, suppose we have \( g, h \in X_J^u(S) \). Then \( \text{relpos}(Q_0, {}^g P_0) = x = \text{relpos}(Q_0, {}^h P_0) \) and \( L_0 \) is a common Levi subgroup of \( Q_0, {}^g P_0 \) and \( {}^h P_0 \). By (2.9) this implies that \( {}^g P_0 = {}^h P_0 \), hence the element \( y := g^{-1}h \) lies in \( P_0(S) \). But we also know that \( y \) normalizes \( L_0 \), so \( y \) lies in the normalizer of \( L_0 \) inside \( P_0 \), which is \( L_0 \) itself.

(3.22) We have a second action of \( L_0 \) on \( X_J^u \), given on points by \( y \cdot g = ygF(y^{-1}) \). (Note that \( ygF(y^{-1}) \) is again in \( X_J^u \), as \( F(y^{-1}) \) is in \( F(L_0) = L_0 \subset P_0 \).) We denote this action by \( \rho: L_0 \times X_J^u \to X_J^u \).

We have chosen \( P_0 \) and \( Q_0 \) such that they are in good position; in particular, \( \text{Ref}_{Q_0}(P_0) = P_0 \). Hence if \( g \in X_J^u \) then \( Q_0 \) and \( gF(\text{Ref}_{Q_0}(P_0)) = gF(P_0) = {}^g P_0 \) are in standard position and we obtain a well-defined morphism

\[
f: X_J^u \to Y_J^u, \quad g \mapsto [P_0, Q_0, g].
\]

Clearly \( f \) is equivariant with respect to \( L_0 \)-actions, where we take the \( \rho \)-action on \( X_J^u \).
(3.23) Lemma: Notation and assumption as in (3.21). The morphism $G \times X^g_j \to Y^g_j$ given on points by

$$(h,g) \mapsto (hP_0, hQ_0, hgF(h^{-1}))$$

is an epimorphism of fppf sheaves. In particular, if $k$ is an algebraically closed extension field of $\mathbb{F}_q$ then every $G(k)$-orbit in $Y^g_j(k)$ meets the image of $X^g_j(k)$ under the morphism $f$.

Proof: The last assertion follows from the first because every fppf covering of $\text{Spec}(k)$ has a section. To prove the first assertion, let $S$ be an $\mathbb{F}_q$-scheme and let $y \in Y^g_j(S)$. After fppf-localization on $S$ we may represent $y$ by a triple $(P, Q, g)$ in $Y^g_j$. Possibly after a further localization we can find an element $\gamma$ in $G$ with $\gamma Q = Q_0$. Replacing $(P, Q, g)$ by $(\gamma P, \gamma Q, \gamma gF(\gamma^{-1}))$ we may from now on assume that $Q = Q_0$.

We know that $\text{relpos}(F, Q_0) = u = \text{relpos}(P_0, Q_0)$. Hence fppf-locally on $S$ we can find $\eta \in Q_0$ with $\eta P = P_0$. Replacing $(P, Q, g)$ by $(\eta P, \eta Q, \eta gF(\eta^{-1}))$ we arrive at the situation where $P = P_0$ and $Q = Q_0$.

The assumption that $\text{relpos}(Q, gF(P)) = x$ implies that $Q$ and $gF(P)$ are in good position, so fppf-locally on $S$ there is a common Levi $M$ of $Q$ and $gF(P) = gP$. There is a unique $v \in U_Q$ such that $\eta M = L_0$. Replacing $g$ by $vg$ we get that $L_0$ is a common Levi of $P$, $Q$ and $gP$. But then there is a unique $w \in U_P = U_{F(P)}$ with $gL_0 = wL_0$. Replacing $g$ by $gw^{-1}$ we finally arrive at a triple $(P, Q, g)$ that is in the image of $X^g_j$ under $f$.

(3.24) Lemma: Notation and assumption as in (3.21). The morphism $\Psi: L_0 \times X^g_j \to X^g_j \times X^g_j$ given on points by $(y, g) \mapsto (ygF(y^{-1}), g)$ is finite étale and surjective. In particular, if $k$ is an separably closed field then $L_0(k)$ acts transitively on $X^g_j(k)$.

Proof: It follows from (3.23) and (3.20) that $X^g_j$ is nonempty. Choose a finite field extension $\mathbb{F}_q \subset k$ such that $X^g_j(k) \neq \emptyset$. It suffices to show that $\Psi$ is finite étale surjective after base change to $k$. If $g \in X^g_j(k)$ then we get an isomorphism $L_{0,k} \cong X^g_j_{0,k}$ by $z \mapsto zg$, and $\Psi$ becomes the morphism $L_{0,k} \times L_{0,k} \to L_{0,k} \times L_{0,k}$ given by $(y, z) \mapsto (yzgF(y^{-1})g^{-1}, z)$.

Consider the morphism $h: L_{0,k} \to L_{0,k}$ given by $y \mapsto yF(y^{-1})g^{-1}$. We claim that $h$ is finite étale and surjective. In fact, it suffices to show this for the morphism $h_1: L_{0,k} \to L_{0,k}$ given by $y \mapsto yF(y^{-1})$. By Lang’s theorem, $h_1$ is surjective. The fibres of $h_1$ are principal homogeneous under (right multiplication by) $L_0(\mathbb{F}_q)$, and using [17], Thm. 23.1 we find that $h_1$, hence also $h$, is finite faithfully flat. Looking at tangent spaces we see that it is even étale.

We view $L_{0,k} \times L_{0,k}$ as a scheme over $L_{0,k}$ via the second projection. Note that $\Psi$ is a morphism over $L_{0,k}$. After base change over the morphism $h$ we obtain

$$h^*\Psi: L_{0,k} \times L_{0,k} \to L_{0,k} \times L_{0,k}$$

given on points by $(c, d) \mapsto (cdF((cd)^{-1})g^{-1}, d)$. Writing $\mu: L_0 \times L_0 \to L_0$ for the group law, we have an isomorphism $(\mu, pr_2): L_0 \times L_0 \cong L_0 \times L_0$, and we find that $h^*\Psi = (h \times \text{id}) \circ (\mu, pr_2)$. Hence $h^*\Psi$ is finite étale surjective, and since these properties are local for the fppf topology, the lemma follows.

(3.25) Theorem: Let $u \in T(J)$ and let $u_\infty \in J^W$ be the corresponding element as in (3.7). The $G$-scheme $Y^u_j$ is equi-dimensional of codimension $\dim(\text{Par}_J) - \ell(u_\infty)$ in $Z_J$. The group $G$
acts transitively on $Y^n J$, in the sense that the morphism

$$G \times Y^n J \to Y^n J \times Y^n J,$$

given by $(g, y) \mapsto (y, g \cdot y)$

is an epimorphism of fppf sheaves.

In particular, for any algebraically closed extension $k$ of $\mathbb{F}_q$ there is a natural bijection between the $G(k)$-orbits in $Z_J(k)$ and the set $J^W$.

**Proof:** The dimension formula was proven in (3.20). It follows from (3.17) that the $G$-action is transitive on $Y^n J$ if and only if it is transitive on $Y^n_{\text{ord}} = Y^n_{\text{ord}}$. But this is the case because of (3.23) and (3.24). The last assertion now follows from (3.7) and (3.15).

(3.26) Note that in $Z_J$ there is a unique orbit $Z_J^{\text{ord}}$ of maximal dimension. Therefore this orbit is open and dense in $Z_J$. We call $Z_J^{\text{ord}}$ the ordinary orbit. There is also a unique orbit of minimal dimension, which is therefore a closed orbit; it has dimension $\dim(G) - \dim(\text{Par}_J)$.

In our applications to $F$-zips we shall have that $\delta = \text{id}$ and $K = w_0 J$, where $w_0 \in W$ is the longest element. For $x \in K^W J$ we take the minimal representative of $W_K w_0 W_J$; this element is in fact the unique element of maximal length in $K^W J$. With these choices, if $R$ is an $\mathbb{F}_q$-algebra, $Z_J^{\text{ord}}(R)$ consists of those points $[P, Q, g]$ in $Z_J$ such that $\text{relpos}(P, Q)$ is the element of maximal length $u_{\text{max}} = x^{-1} \in J^W K$. Indeed, if $\text{relpos}(P, Q) = u_{\text{max}}$ then $P$ and $Q$ are in good position (2.8). Therefore $(P, Q) = (P_1, Q_1, \cdots)$, and we see that $u_{\infty} = u_0 = u_{\text{max}}$. Conversely, under the bijection of Lemma (3.7), the element $u_{\text{max}} \in J^W K \subset J^W$ corresponds to the constant sequence $u = (u_{\text{max}}, u_{\text{max}}, \ldots)$.

4 Applications to F-zips

(4.1) Fix an integer $n \geq 0$. Let $V$ be an $\mathbb{F}_p$-vector space of dimension $n$. We shall apply the theory of Section 3 with $\hat{G} = G := \text{GL}(V)$. (So $q = p$.) Let $(W, I)$ be the Weyl group with its set of simple reflections; see Example (2.13) for an explicit description.

If $S$ is an $\mathbb{F}_p$-scheme, write $V_S := V \otimes \mathcal{O}_S$ and $V_S^{(p)} := F^p_S V_S = V_S \otimes_{\mathcal{O}_S, F_p} \mathcal{O}_S$. We have a canonical $\mathcal{O}_S$-linear isomorphism $\xi_S: V_S^{(p)} \sim \to V_S$ by $(v \otimes x) \otimes y \mapsto v \otimes x^p y$ for $v \in V$ and $x, y$ local sections of $\mathcal{O}_S$.

An $S$-valued point of $G$ is given by an $\mathcal{O}_S$-linear automorphism $g$ of $V_S$. The Frobenius endomorphism $F: G \to G$ is given by $F(g) = \xi_S \circ g^{(p)} \circ \xi_S^{-1}$, where $g^{(p)} = F^p_S(g)$ is the automorphism $g \otimes \text{id}$ of $V_S^{(p)}$.

We fix a function $\tau: \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with $\sum_{i \in \mathbb{Z}} \tau(i) = n$. Let $C^\bullet$ be any filtration of type $\tau$ on $V$. The stabilizer $\text{Stab}(C^\bullet) \subset G$ is a parabolic subgroup; its type $J \subset I$ is independent of the choice of $C^\bullet$. We refer to $J$ as the parabolic type associated to $\tau$.

Let $w_0 \in W$ be the longest element, and set $K := w_0 J$. Let $x \in K^W J$ be the minimal representative of the double coset $W_K w_0 W_J$. It is easily verified that $K = x J$, so we are in the situation of (3.2). (Note that $\delta$ is the identity.)

(4.2) Let $X_{\tau}$ be the scheme over $\mathbb{F}_p$ whose $S$-valued points are the triples $(C^\bullet, D^\bullet, \varphi^\bullet)$ such that $(V_S, C^\bullet, D^\bullet, \varphi^\bullet)$ is an $F$-zip of type $\tau$. The group $G$ acts on $X_{\tau}$; on points:

$$g \cdot (C^\bullet, D^\bullet, \varphi^\bullet) = (g(C^\bullet), g(D^\bullet), \varphi^\bullet),$$

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where $\psi_i$ is the composition
\[
(g(C^i)/g(C^{i+1}))^{(p)} \xrightarrow{\sim} g^{(p)}((C^i/C^{i+1})^{(p)}) \xrightarrow{g^{(p)},-1} (C^i/C^{i+1})^{(p)} \xrightarrow{\psi_i} D_{i}/D_{i-1} \xrightarrow{u} g(D_i)/g(D_{i-1}).
\]

\textbf{(4.3) Lemma:} With notation as above, there is a $G$-equivariant isomorphism of $\mathbb{F}_p$-schemes $X_\tau \xrightarrow{\sim} Z_J$.

\textbf{Proof:} Consider the $\mathbb{F}_p$-scheme $\tilde{X}_\tau$ whose $S$-valued points are the tuples
\[
(C^\bullet, \{A^i\}_{i \in \mathbb{Z}}, D_\bullet, \{B_i\}_{i \in \mathbb{Z}}, \varphi_\bullet)
\]
with $(C^\bullet, D_\bullet, \varphi_\bullet)$ in $X_\tau$, with $\{A^i\}$ a splitting of $(C^\bullet)^{(p)}$ and $\{B_i\}$ a splitting of $D_\bullet$. (By this we mean that $\{A^i\}_{i \in \mathbb{Z}}$ is a collection of subspaces of $V_S^{(p)}$ such that $(C^j)^{(p)} = \oplus_{i \geq j} A^i$ for all $j$; similarly for $\{B_i\}$.) We have a forgetful morphism $\tilde{X}_\tau \to X_\tau$.

We may view $X_\tau$, hence also $\tilde{X}_\tau$, as schemes over $\text{Par}_J \times \text{Par}_K$ by associating to $(C^\bullet, D_\bullet, \varphi_\bullet)$ the pair $(P, Q)$ with $P = \text{Stab}(C^\bullet)$ and $Q = \text{Stab}(D_\bullet)$. Let $U_{J,K}$, with underlying scheme $F(U_J) \times U_K$, be the group scheme over $\text{Par}_J \times \text{Par}_K$ as in (3.9). It acts from the left on $\tilde{X}_\tau$ over $X_\tau$ by
\[
(u_1, u_2) \cdot (C^\bullet, \{A^i\}_{i \in \mathbb{Z}}, D_\bullet, \{B_i\}_{i \in \mathbb{Z}}, \varphi_\bullet) = (C^\bullet, \{\xi_S^{-1} u_1^{-1} \xi_S(A^i)\}_{i \in \mathbb{Z}}, D_\bullet, \{u_2(B_i)\}_{i \in \mathbb{Z}}, \varphi_\bullet).
\]

The set of splittings of a filtration $\Gamma^\bullet$ (descending or ascending) is principal homogeneous under the unipotent radical of the associated parabolic $\text{Stab}(\Gamma^\bullet)$. Using this fact it readily follows that $X_\tau$ is the fppf quotient of $\tilde{X}_\tau$ modulo $U_{J,K}$.

It remains to be shown that we have an isomorphism $\tilde{X}_\tau \xrightarrow{\sim} \tilde{Z}_J$, equivariant with respect to both the $G$-actions and the $U_{J,K}$-actions. Define $\alpha: \tilde{X}_\tau \to \tilde{Z}_J$ by associating to an $S$-valued point $(C^\bullet, \{A^i\}_{i \in \mathbb{Z}}, D_\bullet, \{B_i\}_{i \in \mathbb{Z}}, \varphi_\bullet)$ the triple $(P, Q, g)$ with $P = \text{Stab}(C^\bullet)$ and $Q = \text{Stab}(D_\bullet)$, and with $g \in G(S)$ the composition
\[
V_S \xrightarrow{\xi_S^{-1}} V_S^{(p)} = \bigoplus_{i \in \mathbb{Z}} A^i \cong \bigoplus_{i \in \mathbb{Z}} (\text{gr}_{\xi_S}^{(p)}(C^i)) \xrightarrow{\cdot g^{(p)}} \bigoplus_{i \in \mathbb{Z}} \text{gr}_{\xi_S}^{(p)} D_i \cong \bigoplus_{i \in \mathbb{Z}} B_i = V_S.
\]
By construction, $g(\xi_S(C^\bullet)^{(p)})$ is in opposition with $D_\bullet$; hence $\text{relos}(g, g^F(P)) = x$ and $\alpha$ is well-defined. It is straightforward to check that $\alpha$ is equivariant with respect to the actions of $G$ and $U_{J,K}$.

Next we define a morphism $\beta: \tilde{Z}_J \to \tilde{X}_\tau$. Start with an $S$-valued point $(P, Q, g) \in \tilde{Z}_J$. Then $Q$ and $g^F(P)$ are two parabolics in opposition, which means that $M := Q \cap g^F(P)$ is a common Levi subgroup. Hence $L := g^{-1} M$ is a Levi subgroup of $F(P)$. Now use the correspondences between parabolics and flags, and between Levi subgroups and splittings of a flag. More concretely, let $C^\bullet$ be the unique filtration of $V_S$ of type $\tau$ such that $P = \text{Stab}(C^\bullet)$, let $\{A^i\}$ be the splitting of $(C^\bullet)^{(p)}$ corresponding to the Levi subgroup $L \subset F(P)$, let $D_\bullet$ be the filtration of $V_S$ of type $\tau$ corresponding to $Q$, and let $\{B_i\}$ be the splitting of $D_\bullet$ corresponding to the Levi subgroup $M \subset Q$. Because $gL = M$, there is a permutation $\pi$ of $\mathbb{Z}$ such that $g(\xi_S(A^i)) = B_{\pi(i)}$ for all $i \in \mathbb{Z}$. The assumption that $Q$ and $g^F(P)$ are in opposition then implies that we in fact have $\xi_S(\xi_S^{-1} \xi_S(A^i)) = B_i$ for all $i$. Hence we can define $\varphi_i$ to be the composition
\[
(\text{gr}_{\xi_S}^{(p)}(C^i)) \cong A^i \xrightarrow{\xi_S}\ B_i \cong D_i.
\]
Then $(C^*, \{A^i\}_{i \in \mathbb{Z}}, D^*, \{B^i\}_{i \in \mathbb{Z}}, \varphi^*)$ is a well-defined element of $X_{\tau}(S)$. As it is clear from the construction that $\alpha$ and $\beta$ are inverse to each other, the lemma is proven.

**Theorem (4.4)** Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $n \geq 0$ be an integer, let $G = \text{GL}_n$, and let $(W, I)$ be the Weyl group with its subset of simple reflections. Let $\tau : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with $\sum_{i \in \mathbb{Z}} \tau(i) = n$, and let $J \subset I$ be the associated parabolic type. Then there is a bijection

$$\{(\text{isomorphism classes of } F\text{-zips of type } \tau \text{ over } k) \sim J W \cong W_J \backslash W.\}

In particular, every $F$-zip of type $\tau$ is isomorphic to a standard $F$-zip $M^u_\tau \otimes \mathbb{F}_p$ as in (1.9), for a unique $u \in \mathcal{J} W$.

**Proof:** The first statement is the conjunction of Thm. (3.25) and the above lemma. For the second assertion one verifies that $M^u_\tau \otimes \mathbb{F}_p$ corresponds, under (4.4.1), precisely with the element $u \in \mathcal{J} W$.

**Theorem (4.5)** We call the $\text{GL}_n$-orbit of $X_\tau$ corresponding via the bijection (4.3) to the open dense orbit $Z_J^\text{red}$ in $Z_J$ (3.26) the ordinary orbit. It parametrizes $F$-zips $(M, C^*, D^*, \varphi^*)$ of type $\tau$ such that the filtration $C^*$ and $D^*$ are in opposition, i.e., the rank of $C^i \cap D_j$ is as small as possible for all $i, j \in \mathbb{Z}$ (2.13).

**Theorem (4.6)** The standard $F$-zips $M^u_\tau$ defined in (1.9) correspond, under the isomorphism of (4.3), to certain standard triples $[P, Q, g]$ in $Z_J$. As we shall discuss now, these can be defined independent of the language of $F$-zips, for an arbitrary reductive group $\tilde{G}$ as in (3.1).

Let $L$ be the splitting field of $G$. We choose an $\mathbb{F}_p$-rational Borel pair $(T, B)$ of $G$ such that $T$ is split over $L$. Via this choice we identify the Weyl group $W$ with $N_G(T)/T$. Moreover, we choose a set-theoretic section $s : W(L) \to N_G(T)(L)$. For $u \in \mathcal{J} W$ let $u = (u_0, u_1, \ldots) \in \mathcal{T}(J)$ be the corresponding family under (3.7). We apply the definitions and the notation of (3.6); in particular, $u = u_\infty$.

We denote by $(P^u_\infty, Q^u_\infty, g^u) \in \tilde{Z}_J(L)$ the triple satisfying:

(a) $P^u_\infty = u_\infty \cdot P'$, where $P'$ is the parabolic subgroup of type $J_\infty$ containing $B$;

(b) $Q^u_\infty$ is the parabolic subgroup of $G_L$ of type $K_\infty = u_\infty^{-1} J$ containing $B$;

(c) $g^u = s((u_\infty x)^{-1}) \in N_G(T)(L)$.

Note that $K_\infty = u_\infty^{-1} J$ is already defined over $\mathbb{F}_q$; hence the same is true for $Q^u_\infty$ and $F(Q^u_\infty) = Q^u_\infty$. By definition we have

$$\text{relpos}(P^u_\infty, F(Q^u_\infty)) = u_\infty,$$

$$\text{relpos}(Q^u_\infty, g^u P^u_\infty) = x.$$

Therefore, $(P^u_\infty, Q^u_\infty, g^u) \in \tilde{Y} J^u = \tilde{Y} J^u_\infty$.

Let $P^u$ (resp. $Q^u$) be the unique parabolic of type $J$ (resp. of type $K = x_\infty J$) containing $P^u_\infty$ (resp. $Q^u_\infty$). Now (3.16) implies that $(P^u, Q^u, g^u) \in \tilde{Y} J^u_\infty$. We call the image $[P^u, Q^u, g^u] \in Y_J^u$ the standard triple of type $u$ associated to $(T, B, s)$. Another choice of $(T, B, s)$ gives a point of $Y_J^u$ in the same $G$-orbit.

In the case $G = \text{GL}_n, \mathbb{F}_p$, we have $L = \mathbb{F}_p$. Take $T$ to be the diagonal torus, $B$ the Borel subgroup of upper triangular matrices, and $s : W = S_n \to N_G(T)$ the map that associates to
a permutation the corresponding permutation matrix. Further, fix a type \( \tau \) with \( \sum \tau(i) = n \). The triple \((P^u, Q^u, g^u) \in \tilde{Y}_J\) then corresponds, under the isomorphism as in the proof of (4.3), to a point of the scheme \( \tilde{X}_J \). It can be checked that this point is none other than the standard \( F \)-zip \( \underline{M}^u \) together with the obvious splitting of the filtrations \( C^\bullet \) and \( D_\bullet \), given by the basis \( \{e_1, \ldots, e_n\} \) of the underlying vector space.

\[
(4.7) \quad \text{Let } \tau: \mathbb{Z} \to \mathbb{Z}_{\geq 0} \text{ be a function with finite support. Let } n := \sum_{i \in \mathbb{Z}} \tau(i). \text{ As in (4.1) we fix an } \mathbb{F}_p \text{-vector space } V \text{ of dimension } n, \text{ we set } G := \text{GL}(V), \text{ and we let } (W, I) \text{ be the Weyl group of } G \text{ with its set of simple reflections. Let } J \subset I \text{ be the parabolic type associated to } \tau.
\]

Consider the scheme \( Z_J \). For \( \mathbf{u} \in \mathcal{T}(J) \) we have a locally closed subscheme \( Y^\mathbf{u} \rightsquigarrow Z_J \), stable under the action of \( G \), and the morphism

\[
(4.7.1) \quad \prod_{\mathbf{u} \in \mathcal{T}(J)} Y^\mathbf{u} \to Z_J
\]

is a bijective monomorphism.

The underlying reduced schemes \( (Y^\mathbf{u})_{\text{red}} \) are irreducible and non-singular, as \( G \) acts transitively on them.

We claim that (4.7.1) is a stratification. More precisely, for \( \mathbf{u}, \mathbf{v} \in \mathcal{T}(J) \), let us write \( \mathbf{v} \preceq \mathbf{u} \) if \( Y^\mathbf{v} \) meets the Zariski closure of \( Y^\mathbf{u} \). Then it follows from the general properties of orbits under a group action (see e.g. [27], 4.2) that \( \preceq \) is a partial ordering on \( \mathcal{T}(J) \) and that

\[
(4.7.2) \quad \overline{Y^\mathbf{v}} = \bigsqcup_{\mathbf{v} \preceq \mathbf{u}} Y^\mathbf{u}.
\]

This last identity has to be interpreted set-theoretically, or on points with values in an algebraically closed field. As a slight refinement, we shall prove in (4.11) below that if \( \mathbf{v} \preceq \mathbf{u} \) then \( Y^\mathbf{v} \) is in fact contained in the Zariski closure of \( Y^\mathbf{u} \) as a subscheme.

\[
(4.8) \quad \text{The Ekedahl-Oort stratification associated to an } F \text{-zip. We retain the notation of (4.7). Let } \underline{M} \text{ be an } F \text{-zip of type } \tau \text{ over a connected base scheme } S. \text{ Let } M \text{ be the underlying locally free } O_S \text{-module, which is of rank } n = \dim(V). \text{ Let } ^\# S \to S \text{ be the } G_S \text{-torsor of trivialisations of } M; \text{ so if } T \text{ is a scheme over } S \text{ then the } T \text{-valued points of } ^\# S \text{ are the isomorphisms } M_T \isom V_T := (V \otimes_{\mathbb{F}_p} O_T).
\]

We have a canonical \( G \)-equivariant morphism

\[
\nu: ^\# S \to X_\tau \cong Z_J.
\]

As explained in the introduction, we think of this map as a “mod p period map”.

For \( u \in \mathcal{J}W \) corresponding to \( \mathbf{u} \in \mathcal{T}(J) \), define \( ^\# S^u := \nu^{-1}(Y^\mathbf{u}) \), which is a locally closed subscheme of \( ^\# S \), preserved by the action of \( G \). Now define \( S^u \to S \) to be the quotient of \( ^\# S^u \) by the action of \( G \). Note that \( ^\# S \to S \) is locally trivial for the Zariski topology, and if \( U \subset S \) is an open subset over which we have a trivialisation \( \alpha: M_{\mathcal{I}U} \isom V_U \), then \( S^u \cap U \) is just the pull-back of \( ^\# S^u \subset ^\# S \) under the section \( U \to ^\# S \) corresponding to \( \alpha \). It is clear from the construction that

\[
(4.8.1) \quad \prod_{u \in \mathcal{J}W} S^u \to S
\]

is a bijective monomorphism.
Let $\mathcal{M}$ be the universal $F$-zip over $X_x \cong Z_J$. The locally closed subscheme $Y^n_J$ is the locus where $\mathcal{M}$ is fppf-locally isomorphic to the standard $F$-zip $\mathcal{M}_x^n$ using the notations (1.9) with $q = p$. Hence it follows that $S^u$ represents the subfunctor $S^u$ of $S$ which is defined by the property that a morphism $g: T \to S$ factors through $S^u$ if and only if $g^*\mathcal{M}$ is fppf-locally isomorphic to $\mathcal{M}_x^n \otimes_{\mathcal{O}_T} \mathcal{O}_T$.

We refer to the subschemes $S^u \to S$ as the Ekedahl-Oort loci in $S$ associated to the $F$-zip $\mathcal{M}$ and to (4.8.1) as the Ekedahl-Oort partition of $S$ associated to $\mathcal{M}$. (We use the terms “loci” and “partition” because (4.8.1) is not, in general, a stratification of $S$. In fact, the closure of an irreducible component of $S^u$ need not be a union of components of EO-loci.)

(4.9) **Definition:** In the above situation we say that the $F$-zip $\mathcal{M}$ is *isotrivial of type* $u$ if $S = S^u$. We say that $\mathcal{M}$ is a *constant $F$-zip of type* $u$ if it is isomorphic to $\mathcal{M}_x^n \otimes_{\mathcal{O}_S} S$.

As a corollary of our method of proof we obtain the following result.

(4.10) **Corollary:** Let $\mathcal{M}$ be an $F$-zip of type $\tau$ over $S$. Then the following assertions are equivalent:

1. The $F$-zip $\mathcal{M}$ is isotrivial of type $u$.
2. There exists a faithfully flat morphism $S' \to S$, locally of finite presentation, such that $\mathcal{M} \otimes_S S'$ is a constant $F$-zip of type $u$.

If $S$ is quasi-separated, these conditions are also equivalent to:

3. Zariski-locally on $S$ there exists a faithfully flat quasi-finite morphism $S' \to S$ of finite presentation such that $\mathcal{M} \otimes_S S'$ is a constant $F$-zip of type $u$.

**Proof:** Our results in Section 3 show that if (1) holds then $\mathcal{M}$ is fppf-locally constant; whence (2). The equivalence of (2) and (3) in the quasi-separated case follows from [10], IV, 17.16.2.

(4.11) **Remark:** Isotrivial $F$-zips are not, in general, étale-locally constant. E.g., if the base scheme is the spectrum of a field then in general we need a non-separable field extension to trivialize the $F$-zip. As a concrete example, let $k$ be a field of characteristic $p$, let $\gamma \in k$, and consider the $F$-zip $\mathcal{M}_k$, with underlying module $M = k^5 = k \cdot e_1 + \cdots + k \cdot e_5$, with

$$C^0 = M \supset C^1 = \text{Span}(e_1, e_3) \supset C^2 = (0)$$

and

$$D_{-1} = (0) \subset D_0 = \text{Span}(e_1, e_2, e_3) \subset D_1 = M;$$

with

$$\varphi_0: (M/C^1)^{(p)} \xrightarrow{\sim} D_0 \quad \text{given by} \quad e_2^{(p)} \mapsto e_1, \quad e_4^{(p)} \mapsto e_2, \quad e_5^{(p)} \mapsto \gamma e_2 + e_3,$$

and with

$$\varphi_1: (C^1)^{(p)} \xrightarrow{\sim} M/D_0 \quad \text{given by} \quad e_1^{(p)} \mapsto e_4, \quad e_3^{(p)} \mapsto e_5.$$

Then $\mathcal{M}_k \cong \mathcal{M}_0$ over $\bar{k}$, but to realize this isomorphism one has to extract a $p$th root of $\gamma$.

(4.12) **Lemma:** Let $u, v \in T(J)$ be elements with $v \preceq u$. Let $Y^n_J$ denote the scheme-theoretic Zariski closure of $Y^n_J$ (i.e., the scheme-theoretic image of $Y^n_J \to Z_J$). Then $Y^n_J$ is contained in $\overline{Y^n_J}$ as subschemes of $Z_J$.  

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Proof: It suffices to show that if \( R := k[\![ t ]\!/ (t^n) \) with \( k \) an algebraically closed field, then any point \( \mu \colon \text{Spec}(R) \to Y^{\gamma} \) factors through \( Y := \overline{\text{Y}}^{\gamma} \). But by (4.10), given an isotrivial \( F \)-zip of type \( v \) over \( R \), then there is a faithfully flat extension \( R \subset R' \) over which the \( F \)-zip is isomorphic to \( M^\gamma \otimes_{\mathbb{F}_p} R' \). In other words, if \( m \colon \text{Spec}(R') \to \text{Spec}(\mathbb{F}_p) \to Y^{\gamma} \) is the morphism corresponding to the constant \( F \)-zip \( M^\gamma \otimes R' \) then there is an element \( g \in G(R') \) such that \( \mu = g \cdot m \) in \( Z_J(R') \). Moreover, by (4.7.2) the point \( \text{Spec}(\mathbb{F}_p) \to Y^{\gamma} \) corresponding to \( M^\gamma \) factors through \( Y \), hence so does the point \( m \). But \( Y \), as a closed subscheme of \( Z_J \), is stable under the action of \( G \); hence \( \mu \in \overline{\text{Y}}(R') \).

5 \( F \)-zips with additional structure

The purpose of this section is to discuss how the main result of the previous section can be extended to \( F \)-zips with additional structure. Ultimately one might wish to have a theory of \( F \)-zips with \( G \)-structure, where \( G \) is an arbitrary reductive group. However, it is not clear to us how to define such a notion in full generality. Therefore we restrict the discussion to two simple examples.

(5.1) Let \( S \) be a scheme. Consider a pair \((M, \psi)\) consisting of a locally free \( \mathcal{O}_S \)-module of finite rank, together with a perfect pairing \( \psi : M \otimes_{\mathcal{O}_S} M \to \mathcal{O}_S \). Let \( b_\psi : M \iso M^\gamma \) be the isomorphism given on local sections by \( m \mapsto \psi(- \otimes m) \). For a locally direct summand \( N \subset M \), we define \( N^\perp \subset M \) to be the kernel of the composite map \( M \xrightarrow{\sim} M^\gamma \to N^\gamma \), where the first map is \( b_\psi \). We call \( N \) isotropic if \( N \subset N^\perp \); in that case \( \psi \) induces a perfect pairing on \( N^\perp / N \). Note that \( N^\perp \perp = N \).

Now assume that either \( \psi \) is symplectic, meaning that \( \psi(m, m) = 0 \) for all local sections \( m \), or symmetric; we shall consider the latter case only in characteristic \( \neq 2 \). A flag \( \Delta \) in \( M \) is called a symplectic (resp. orthogonal) flag if for every \( N \in \Delta \) we also have \( N^\perp \in \Delta \). As \( \Delta \) is totally ordered, either \( N \) or \( N^\perp \) is then isotropic. We call a filtration symplectic (resp. orthogonal) if the associated flag is.

Let \( S \) be a scheme of characteristic \( p \). Consider a tuple \( M := (M, \psi, C^\bullet, D^\bullet, \varphi^\bullet) \) such that \( M^\gamma := (M, C^\bullet, D^\bullet, \varphi^\bullet) \) is an \( F \)-zip over \( S \), with \( \psi \) a (perfect) symplectic or symmetric bilinear form on \( M \), and such that the flags \( C^\bullet \) and \( D^\bullet \) are symplectic, resp. orthogonal. Let \( \tau \) be the type of \( C^\bullet \). Let \( i \in \mathbb{Z} \) be an index such that \( \tau(i) \neq 0 \). There is a unique index \( j \in \mathbb{Z} \) such that

\[
(C^i)^\perp = C^{i+1} \quad \text{and} \quad (C^{i+1})^\perp = C^j,
\]

and \( b_\psi \) induces an isomorphism

\[
\alpha : \text{gr}_C^i \iso (\text{gr}_C^i)^\gamma.
\]

By an easy dimension count we then find that, for these same indices \( i \) and \( j \), we have

\[
D^i_{i-1} = D_j \quad \text{and} \quad D^i_i = D_{j-1},
\]

and we get an isomorphism

\[
\beta : \text{gr}_D^j \iso (\text{gr}_D^j)^\gamma.
\]

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(5.2) Definition: Let $S$ be a scheme of characteristic $p$, with $p > 2$ in the orthogonal case. By a symplectic $F$-zip over $S$, resp. an orthogonal $F$-zip over $S$, we mean a tuple $M = (M, \psi, C^\bullet, D^\bullet, \varphi_\bullet)$ as above, with $\psi$ symplectic, resp. symmetric, such that for all indices $i$ and $j$, the diagram

$$
\begin{array}{ccc}
(gr_C^i)^{(p)} & \xrightarrow{\iota^j} & gr_D^j \\
\alpha^i(p) \downarrow & & \downarrow \beta \\
(gr_C^i)^{(p), \vee} & \sim & (gr_D^j)^{\vee}
\end{array}
$$

is commutative.

(5.3) Let $(V, \psi)$ be a finite dimensional $\mathbb{F}_p$-vector space equipped with a perfect bilinear pairing $\psi$, assumed to be either symplectic or symmetric. If $\psi$ is symmetric we assume that $p > 2$ and also that $\dim(V)$ is odd.

In the symplectic case, set $G := \text{Sp}(V, \psi)$; in the symmetric case, $G := \text{SO}(V, \psi)$. As usual, let $(W, I)$ be the Weyl group with its set of simple reflections. We say that a type $\tau : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with support $i_1 < \cdots < i_r$ is admissible if $\tau(i_n) = \tau(i_{r+1-n})$ for all $n$. This is equivalent to the condition that for some field $k$ of characteristic $p$, there exists a symplectic (resp. orthogonal) filtration $C^\bullet$ of $V_k$ of type $\tau$. The stabilizer $\text{Stab}_{G}(C^\bullet)$ is then a parabolic subgroup of $G_k$; its type $J \subset I$ only depends on $\tau$. We call $J$ the parabolic type associated to $\tau$.

Define $X_\tau$ to be the $\mathbb{F}_p$-scheme whose $S$-valued points are the triples $(C^\bullet, D^\bullet, \varphi_\bullet)$ such that $(V_S, \psi_S, C^\bullet, D^\bullet, \varphi_\bullet)$ is a symplectic (resp. orthogonal) $F$-zip over $S$. We let $G$ act on $X_\tau$ by the same rule as in (4.2).

On the other hand, let $w_0$ be the element of maximal length in the Weyl group $W$ of $G$ and let $x \in W_{w_0(j)}w_0W_J$ be the element of minimal length. Consider the scheme $Z_J$ defined in (3.9) associated to $G$, $J$ and $x$. We claim that we again have a $G$-equivariant isomorphism of $\mathbb{F}_p$-schemes $X_\tau \xrightarrow{\sim} Z_J$. The proof of this is essentially the same as that of (4.3), provided we consider symplectic (resp. orthogonal) splittings of the filtrations $(C^\bullet)^{(p)}$ and $D^\bullet$. We leave the details to the reader. Note, however, that it is essential to have a bijective correspondence between symplectic (resp. orthogonal) flags and parabolic subgroups of $G$, as well as a correspondence between the symplectic (resp. orthogonal) splittings of a flag and the Levi subgroups of the corresponding parabolic. Such a correspondence fails for orthogonal groups in an even number of variables, which is why we assume that in the orthogonal case, $\dim(V)$ is odd.

(5.4) Corollary: Let $k$ be an algebraically closed field of characteristic $p$.

(i) Let $G = \text{Sp}(V, \psi)$ and $(W, I)$ be as above; symplectic case. Let $\tau$ be an admissible type with $\sum_{i \in \mathbb{Z}} \tau(i) = \dim(V)$ and with associated parabolic type $J \subset I$. Then there is a bijection

$$
\{\text{isomorphism classes of symplectic } F\text{-zips of type } \tau \text{ over } k\} \xrightarrow{\sim} J W \cong W_J \setminus W.
$$

(ii) Let $G = \text{SO}(V, \psi)$ and $(W, I)$ be as above; orthogonal case, with $\dim(V)$ odd. Let $\tau$ be an admissible type with $\sum_{i \in \mathbb{Z}} \tau(i) = \dim(V)$ and with associated parabolic type $J \subset I$. Then there is a bijection

$$
\{\text{isomorphism classes of orthogonal } F\text{-zips of type } \tau \text{ over } k\} \xrightarrow{\sim} J W \cong W_J \setminus W.
$$

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Note that in this result the $\mathbb{F}_p$-structure on $G$ plays no role, as $(W, I)$ only depends on $G_k$, which in turn only depends on $\dim(V)$.

(5.5) **Remark**: As remarked at the beginning of this section, it is not clear to us how to define the notion of an $F$-zip with $G$-structure, for $G$ an arbitrary reductive group. It is possible, though, to obtain rather complete results for $F$-zips equipped with an action of a semi-simple algebra and a hermitian form. For Dieudonné modules this was carried out in [18].

(5.6) Slightly changing notation, let $G_1 := \text{Sp}(V, \psi)$, resp. $G_1 := \text{SO}(V, \psi)$ be the reductive group over $\mathbb{F}_p$ considered in (5.3). Let $G_2 := \text{GL}(V)$. Let $(W_i, I_i)$ be the Weyl group of $G_i$.

If $\text{Par}(G_1)$ is the scheme of parabolic subgroups of $G_1$ then we have a canonical morphism $\text{Par}(G_1) \hookrightarrow \text{Par}(G_2)$; in terms of symplectic (resp. orthogonal) flags $\Delta$ in $V$ it sends $\text{Stab}_{G_1}(\Delta)$ to $\text{Stab}_{G_2}(\Delta)$. As $W_i$ can be identified with the set of $G_i$-orbits in $\text{Par}(G_i)^2$ (over any separably closed field), we obtain a natural homomorphism $\iota : W_1 \to W_2$, which is in fact injective.

Let $k = \bar{k}$. Let $M$ be a symplectic (resp. orthogonal) $F$-zip over $k$ with $\dim(M) = \dim(V)$. Write $M'$ for the underlying $F$-zip, obtained by forgetting the form $\psi$. Let $J_1 \subseteq I_1$ be the parabolic type associated to the type $\tau$ in the group $G_1$. Then $\iota$ maps $J_1 W_1$ into $J_2 W_2$. If $u_1 \in J_1 W_1$ is the element corresponding to $M$ under (5.4), and $u_2 \in J_2 W_2$ is the element corresponding to $M'$ under (4.4.1) then we have the relation $\iota(u_1) = u_2$.

For a more precise statement, consider the schemes $Z_{J_1}^{(1)} \sim X_\tau^{(1)}$ formed with respect to the group $G_1$ and the subset $J_1 \subseteq I_1$ and the schemes $Z_{J_2}^{(2)} \sim X_\tau^{(2)}$ formed with respect to $G_2$ and $J_2 \subseteq I_2$. Then the forgetful morphism $M \to M'$ defines a closed immersion $\alpha : X_\tau^{(1)} \to X_\tau^{(2)}$. If $u_1 \in J_1 W_1$ corresponds to the sequence $u_1 \in T(J_1)$ and $u_2 := \iota(u_1)$ corresponds to $u_2 \in T(J_2)$ then it can be shown that $\alpha$ induces an isomorphism between the subscheme $Y_{J_1}^{u_1} \sim Z_{J_1}^{(1)} \cong X_\tau^{(1)}$ and the subscheme $Y_{J_2}^{u_2} \cap Z_{J_2}^{(1)} \sim Z_{J_2}^{(2)} \cong X_\tau^{(2)}$.

6 **F-zips coming from geometry**

(6.1) Let $f : X \to S$ be a morphism of schemes in characteristic $p$. We denote by $\text{Frob}_S : S \to S$ the absolute Frobenius. By definition of the relative Frobenius $F_{X/S}$ we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{F_{X/S}} & X^{(p)} \\
\downarrow f & & \downarrow f^{(p)} \\
S & \xrightarrow{\text{Frob}_S} & S 
\end{array}
$$

where the square is cartesian.

Now assume that $f$ is smooth. Recall that we have two spectral sequences converging to the de Rham cohomology $H^*_\text{dR}(X/S) = R^*_f(\Omega^*_X/S)$, namely the Hodge-de Rham spectral sequence

$$
\begin{aligned}
H^0 F_{a+b} &= R^b f_* (\Omega^a_X/S) \Rightarrow H^{a+b}_{\text{dR}}(X/S)
\end{aligned}
$$

and the conjugate spectral sequence

$$
\begin{aligned}
\text{conj} F^{a+b} &= R^a f_* (\mathcal{H}^b(\Omega^*_X/S)) \Rightarrow H^{a+b}_{\text{dR}}(X/S).
\end{aligned}
$$
Moreover, there is a unique isomorphism of graded \( \mathcal{O}_{X(p)} \)-modules
\[
C^{-1}; \bigoplus_{i \geq 0} \Omega^i_{X(p)/S} \overset{\sim}{\longrightarrow} \bigoplus_{i \geq 0} \mathcal{H}^i(F_* (\Omega^i_{X/S})) ,
\]
the (inverse) Cartier isomorphism, which satisfies
\[
C^{-1}(1) = 1 \quad C^{-1}(d\sigma^{-1}(x)) = \text{class of } x^{p-1} dx \\
C^{-1}(\omega \wedge \omega') = C^{-1}(\omega) \wedge C^{-1}(\omega') .
\]

(6.2) Let \( f : X \rightarrow S \) be a smooth and proper morphism. We say that \( f \) satisfies condition (D) if the following two conditions hold:

(a) The \( \mathcal{O}_S \)-modules \( R^b f_* (\Omega^a_X/S) \) are locally free of finite rank for all \( a, b \geq 0 \).

(b) The Hodge-de Rham spectral sequence degenerates at \( E_1 \).

If \( f \) is satisfies (D), the formation of the Hodge-de Rham spectral sequences commutes with base change \( S' \rightarrow S \).

(6.3) Let \( f : X \rightarrow S \) be a smooth morphism of schemes of characteristic \( p \). For \( a, b \in \mathbb{Z}_{\geq 0} \) the (inverse) Cartier isomorphism \( C^{-1} \) of (6.1.1) defines an isomorphism
\[
R^a f_*^p (\Omega^b_{X(p)/S}) \overset{\text{conj}}{\sim} E_{2ab}^a = R^a f_* (H^a(\Omega^b_{X/S})) .
\]
If further the \( \mathcal{O}_S \)-modules \( R^p f_* (\Omega^q_X/S) \) are flat (e.g. if \( f \) satisfies condition (D)), we get an isomorphism
\[
\varphi^{ab} : \text{Frob}_S^p R^a f_* (\Omega^b_X/S) = \text{Frob}_S^p (H^a(E^{b,a})) \overset{\text{conj}}{\sim} E_{2ab}^a = R^a f_* (H^a(\Omega^b_{X/S})) .
\]
Using this, one can show (e.g. [13], 2.3.2), that if \( f \) is satisfies condition (D), the conjugate spectral sequence degenerates at \( E_2 \) and that its formation commutes with arbitrary base change.

(6.4) We list some examples of morphisms that satisfy condition (D). As usual, \( S \) is a scheme of characteristic \( p \).

1. Any abelian scheme \( f : A \rightarrow S \) is satisfies (D). (Degeneracy of the Hodge-de Rham spectral sequence at \( E_1 \) can be proven as in [21], Prop. 5.1.)

2. Any smooth proper curve \( f : C \rightarrow S \) satisfies (D). (Use the previous example.)

3. Any K3-surface \( X \rightarrow S \) satisfies (D). (This follows from [5], Prop. 2.2.)

4. Every smooth complete intersection in the projective space \( \mathbb{P}^n_S \) satisfies (D) as a scheme over \( S \). (See [3], Thm. 1.5.)

5. Let \( f : X \rightarrow S \) be a smooth proper morphism such that \( (F_{X/S})_* (\Omega^*_{X/S}) \) is decomposable (i.e., isomorphic in the derived category to a complex with zero differential). Then \( f \) satisfies (D) by results of Deligne and Illusie, see [6], Cor. 4.1.5. Moreover, this condition is satisfied if \( \dim(X/S) < p \) and \( f \) admits a smooth lifting \( \tilde{f} : \tilde{X} \rightarrow \tilde{S} \) with \( \tilde{S} \) a flat \( \mathbb{Z}/p^2 \mathbb{Z} \)-scheme (loc. cit., 3.7).

(6.5) Let \( f : X \rightarrow S \) be a morphism satisfying (D). Fix an integer \( n \) with \( 0 \leq n \leq 2 \dim(X/S) \). We associate to \( f \) an \( F \)-zip \( (M, C^*, D^*, \varphi^*) \) over \( S \) as follows: Set \( M = H^n_{\text{dR}}(X/S) \). Let \( C^* \) be
the Hodge filtration on \( M \), and define the filtration \( D_\bullet \) by \( D_1 = \text{conj} F^{n-i} H^a_{dR}(X/S) \). Finally, let
\[
\varphi_i := \varphi^{n-i,i} \cdot (\text{gr}^i_C)^{(p)} = \text{Frob}_S^* R^{n-i} f_*(\Omega^i_{X/S}) \rightarrow \text{gr}^i_D = R^{n-i} f_*(\Omega^i_{X/S})
\]
where \( \varphi^{n-i,i} \) is the isomorphism defined in (6.3).

Note that \( C^\bullet \) and \( D_\bullet \) are filtrations in the sense of (1.2). This follows from the fact that both the Hodge-de Rham spectral sequence and the conjugate spectral sequence are compatible with base change. (Use the fact that a homomorphism \( \iota \) satisfies (D)). A nontrivial example for a morphism of log-schemes satisfying condition (D) is the following case: Let \( S \) be the spectrum of a discrete valuation ring and let \( X \) be a complete intersection in a projective space over \( S \). Assume that \( X \) is a regular, flat over \( S \), and that its special fibre is a divisor with normal crossings. Then the structure morphism \( f: X \rightarrow S \) satisfies condition (D) if we endow \( X \) and \( S \) with their natural log-structures.

(6.6) In the situation of (6.5), if \( S \) is smooth over some scheme \( T \) then we have an additional structure on the F-zip \( H^a_{dR}(X/S) \), viz. a Gauss-Manin connection \( \nabla \) relative to \( T \). The filtration \( D_\bullet \) is horizontal with respect to \( \nabla \), the filtration \( C^\bullet \) satisfies Griffiths transversality, and the maps \( \varphi_i \) are horizontal with respect to the canonical connection on \( (\text{gr}^i_C)^{(p)} \) and the connection induced by \( \nabla \) on \( \text{gr}^i_D \). In this paper we shall make no attempt to further exploit this structure.

(6.7) There is also a logarithmic variant. For this we use the language of logarithmic schemes, as for instance in Kato’s paper [12]. Let \( f: (X,\mathcal{M}) \rightarrow (S,\mathcal{N}) \) be a morphism of schemes with fine log-structures in characteristic \( p \) and denote by \( \omega^\bullet_{X/S} \) the logarithmic de Rham complex. As in the non-logarithmic case, there are two spectral sequences converging to the logarithmic de Rham cohomology \( H^\bullet_{dR,\log}(X/S) = R^n f_*(\omega^\bullet_{X/S}) \):
\[
H^a E^{ab}_1 = R^b f_*(\omega^a_{X/S}) \Rightarrow H^{a+b}_{dR,\log}(X/S),
\]
\[
\text{conj} E^{ab}_2 = R^a f_*(H^b(\omega^\bullet_{X/S})) \Rightarrow H^{a+b}_{dR,\log}(X/S).
\]

If \( f \) is log-smooth and of Cartier type, there exists also a logarithmic variant \( C^{-1} \) of the Cartier isomorphism.

Similarly as above, we say that \( f \) satisfies condition (D) if the following conditions are satisfied:

(a) The log-structures \( \mathcal{M} \) and \( \mathcal{N} \) are fine and the morphism \( f \) is log-smooth and of Cartier type. Its underlying scheme morphism is proper.

(b) The logarithmic Hodge-de Rham spectral sequence degenerates at level \( E_1 \).

(c) The \( \mathcal{O}_S \)-modules \( R^b f_* \omega^a_{X/S} \) are locally free.

If the log-structures \( \mathcal{M} \) and \( \mathcal{N} \) are trivial (or more general if \( f \) is a strict morphism of log-schemes) then \( f \) satisfies condition (D) if and only if the underlying scheme morphism is satisfies (D). A nontrivial example for a morphism of log-schemes satisfying condition (D) is the following case: Let \( S \) be the spectrum of a discrete valuation ring and let \( X \) be a complete intersection in a projective space over \( S \). Assume that \( X \) is a regular, flat over \( S \), and that its special fibre is a divisor with normal crossings. Then the structure morphism \( f: X \rightarrow S \) satisfies condition (D) if we endow \( X \) and \( S \) with their natural log-structures.

Again one can show that condition (b) and the existence of the Cartier isomorphism imply that the conjugate spectral sequence degenerates at \( E_2 \). Moreover, condition (3) and the
existence of the Cartier isomorphism then imply that the formation of the logarithmic Hodge-de Rham spectral sequence and of the logarithmic conjugate spectral sequence commute with arbitrary base change.

For a log-smooth morphism \( f: (X, \mathcal{M}) \to (S, \mathcal{N}) \) of fine log-schemes, the sheaf of logarithmic differentials \( \omega^n_{X/S} \) is locally free of finite type. If its rank is constant we call this rank the relative dimension of \((X, \mathcal{M})\) over \((S, \mathcal{N})\) and denote it by \( \dim(X/S) \). (Note that in general the underlying scheme morphism of \( f \) need not even be flat.) If \( f \) now satisfies condition (D) then we obtain, as in the non-logarithmic case, an \( F \)-zip structure on \( \overline{H}^n_{dR, \log}(X/S) \) for every integer \( n \) with \( 0 \leq n \leq 2 \dim(X/S) \).

(6.8) Let \( f: X \to S \) be a morphism satisfying (D). Fix \( 0 \leq n \leq 2 \dim(X/S) \), and denote by \( FZ(n)(f) = (M, C^\bullet, D^\bullet, \varphi^\bullet) \) the corresponding \( F \)-zip with \( M = \overline{H}^n_{dR}(X/S) \). We assume that \( N(n) = \text{rk}_{O_S}(M) \) is constant on \( S \). Let \( J(n) \) be the parabolic type associated to \( C^\bullet \). Let \( W(n) = S_{N(n)} \) be the Weyl group of \( \text{GL}_N(n) \) and let \( w(n)_{\text{max}} \) be the unique maximal element in \( J(n)W(n) \) with respect to the Bruhat order.

The Ekedahl-Oort locus \( S^{w(n)_{\text{max}}} \) corresponding to \( w(n)_{\text{max}} \) and the choice of \( n \) is an open subscheme of \( S \). We set

\[
S_{\text{ord}} = \bigcap_n S^{w(n)_{\text{max}}}. 
\]

(6.9) Proposition: In the situation of (6.8) we have \( S_{\text{ord}} = S \) if and only if for every geometric point \( \overline{s} \) of \( S \) the \( \kappa(\overline{s}) \)-scheme \( X_{\overline{s}} \) is ordinary in the sense of [11], 4.12.

Proof: As \( S_{\text{ord}} \subseteq S \) is open, we can assume that \( S = \text{Spec}(k) \) for an algebraically closed field \( k \). By [11], 4.13, \( X \) is ordinary if and only if Hodge filtration and conjugate filtration in \( \overline{H}^n_{dR}(X/k) \) are in opposition, i.e., if and only if their relative position is equal to the maximal element in \( J_nW^n \) where \( K_n = w_0(J_n) \) and where \( w_0 \) is the maximal element in \( W_n \). By (4.5) this is the case if and only if the isomorphism type of \( FZ_n(f) \) corresponds, via (4.4.1), to the element \( w_{n, \text{max}} \).

(6.10) Let \( X \) be a (log-)smooth projective variety over an algebraically closed field \( k \) such that \( X \to \text{Spec}(k) \) satisfies condition (D). As suggested by the title of this paper, we may think of the \( F \)-zip structure on the de Rham cohomology as a discrete invariant of \( X \). As such, this contains certain discrete invariants previously studied by other authors, such as the \( a \)-number defined by van der Geer and Katsura in [9]. More precisely, if \( u_\infty \in JW \) is the element classifying the \( F \)-zip \( H^n_{dR}(X/k) \), and if \( u = (u_0, u_1, \ldots) \) is the sequence corresponding to \( u_\infty \) via (3.7), then the \( a \)-number only depends on \( u_0 \), which is the relative position of the Hodge and the conjugate filtration.

(6.11) \( F \)-zips and Shimura varieties of PEL-type. Let \( D = (B, *, V, (\cdot, \cdot), O_B, \Lambda, h) \) denote a Shimura-PEL-datum, integral and unramified at a prime \( p \), let \( G \) its associated reductive group over \( \mathbb{Q} \), and \([\mu]\) denotes the associated conjugacy class of cocharacters of \( G \). By this we mean that

- \( B \) is a finite-dimensional semi-simple \( \mathbb{Q} \)-algebra, such that \( B_{\mathbb{Q}_p} \) is isomorphic to a product of matrix algebras over unramified extensions of \( \mathbb{Q}_p \);
• * is a $\mathbb{Q}$-linear positive involution on $B$;
• $V \neq 0$ is a finitely generated left $B$-module;
• $\langle , \rangle$ is a nondegenerate alternating $\mathbb{Q}$-valued form on $V$ such that $\langle bv, w \rangle = \langle v, b^* w \rangle$ for all $v, w \in V$ and $b \in B$;
• $O_B$ is a $^*$-invariant $\mathbb{Z}_p(\rho)$-order of $B$ such that $O_B \otimes \mathbb{Z}_p$ is a maximal order of $B \otimes \mathbb{Q}_p$;
• $\Lambda$ is an $O_B$-invariant $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$, such that $\langle , \rangle|_{\Lambda \times \Lambda}$ is a perfect pairing of $\mathbb{Z}_p$-modules;
• $G$ is the $\mathbb{Q}$-group of $B$-linear symplectic similitudes of $V$, i.e., for any $\mathbb{Q}$-algebra $R$ we have

$$G(R) = \{ g \in GL_B(V \otimes R) \mid \langle gv, gw \rangle = c(g) \cdot \langle v, w \rangle \text{ for some } c(g) \in R^* \};$$

• $h: \text{Res}_{C/B}(G_{m,C}) \rightarrow G_{\mathbb{R}}$ is a homomorphism defining a complex structure on $V_{\mathbb{R}}$ which is compatible with $\langle , \rangle$;
• $[\mu]$ is the $G(\mathbb{C})$-conjugacy class of the cocharacter $\mu_h$ associated to $h$ (cf. [4], 1.1.1). Then $V_{\mathbb{C}}$ has only weights 0 and 1 with respect to any $\mu \in [\mu]$.

We assume that $p > 2$ if $G$ is not connected.

Let $E$ be the associated reflex field, i.e., the field of definition of $[\mu]$. It is a finite extension of $\mathbb{Q}$. Fix an embedding of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$ into some algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Via this embedding we can consider $[\mu]$ as a $G(\overline{\mathbb{Q}}_p)$-conjugacy class of cocharacters. Denote by $v|p$ the place of $E$ given by the chosen embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and write $E_v$ for the $v$-adic completion of $E$. Let $\kappa = \kappa(v)$ be its residue class field.

Further fix an open compact subgroup $K^p \subset G(\mathbb{A}^p)$ and denote by $\mathfrak{A}_{D,K^p}$ the associated moduli space, defined by Kottwitz in [14]. We assume that $K^p$ is sufficiently small such that $\mathfrak{A}_{D,K^p}$ is representable. It is then a smooth equi-dimensional quasi-projective scheme over the localization of $O_B$ in $p$. It classifies tuples $(A, \lambda, \iota, \bar{\eta})$ where

• $A$ is an abelian scheme up to prime-to-$p$-isogeny;
• $\lambda$ is a $\mathbb{Q}$-homogeneous polarization of $A$ containing a polarization $\lambda \in \bar{\lambda}$ of degree prime to $p$;
• $\iota: O_B \rightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}(\rho)$ is an involution preserving $\mathbb{Z}(\rho)$-algebra homomorphism where the involution is $^*$ on $O_B$ and the Rosati involution given by $\bar{\lambda}$ on $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}(\rho)$;
• $\bar{\eta}$ is a $K^p$-level structure.

Further $(A, \lambda, \iota, \bar{\eta})$ should satisfy a determinant condition; see [14], §5 or [25], 3.23 a) for a precise formulation. We denote by $\mathfrak{A}_0$ the reduction $\mathfrak{A}_{D,K^p} \otimes \kappa$ at $v$.

(6.12) We denote by $\tilde{G}$ the reductive $\mathbb{F}_p$-group of $O_B/pO_B$-linear symplectic similitudes of $\Lambda_0 := \Lambda/p\Lambda$ and let $G$ be its identity component. Via the canonical bijection of $G(\overline{\mathbb{Q}}_p)$-conjugacy classes of cocharacters and $G(\mathbb{F}_p)$-conjugacy classes of cocharacters we consider $[\mu]$ as a $G(\mathbb{F}_p)$-conjugacy class of cocharacters. Its field of definition is $\kappa$. Let $(W, I)$ be the Weyl group of $G$ together with its set of simple reflections, and let $J \subset I$ be the subset of simple reflections corresponding to $[\mu]$. Then $J$ is defined over $\kappa$.

(6.13) Let $S$ be a $\kappa$-scheme and let $(A, \lambda, \iota, \bar{\eta})$ be an $S$-valued point of $\mathfrak{A}_0$. Every abelian scheme satisfies condition (D). We set $M = H^1_{\text{rig}}(A/S)$. By (6.5) we obtain the structure of an $F$-zip on $M$. The filtration $C^\bullet$ (resp. $D^\bullet$) is of the form $M = C^0 \supseteq C^1 \supseteq C^2 = (0)$ (resp. $(0) = D_{-1} \subset D_0 \subset D_1 = M$), where $C^1$ and $D_0$ are locally direct summands of rank equal to
dim(A/S). Moreover, the submodules $D_0$ and $C^1$ are $O_B/pO_B$-invariant and totally isotropic with respect to the perfect alternating form induced by any $\lambda \in \hat{\lambda}$ which is of order prime to $p$.

(6.14) Lemma: Locally for the étale topology the two skew Hermitian modules with $O_B/pO_B$-action $M$ and $\Lambda_{0,S} = \Lambda_0 \otimes \mathbb{F}_p \mathcal{O}_S$ are isomorphic.

Proof: This is a special case of [25], 3.16.

(6.15) We define two smooth coverings $\#A_0$ and $\tilde{A}_0$ of $A_0$ as follows: For every $\kappa$-scheme $S$ the $S$-valued points of $\#A_0$ are given by tuples $(A, \lambda, \iota, \eta, \alpha)$ where $(A, \lambda, \iota, \eta) \in A_0(S)$ and where $\alpha$ is an $O_B/pO_B$-linear symplectic similitude $H^1_{dR}(A/S) \sim \sim \Lambda_{0,S}$.

The $S$-valued points of $\tilde{A}_0$ are given by tuples $(A, \lambda, \iota, \eta, \alpha, C', D')$ with $(A, \lambda, \iota, \eta, \alpha) \in \#A_0$ and where $C'$ and $D'$ are $O_B/pO_B$-invariant totally isotropic complements of $C^1$ and $D_0$, respectively.

It follows from (6.14) that $\#A_0$ is a torsor for the étale topology over $A_0$ under the smooth group scheme $G$. Furthermore, because Zariski-locally on $S$ we can always find complements $C'$ and $D'$ as above, $\tilde{A}_0$ is a torsor over $\#A_0$ under the smooth unipotent group scheme $U_{J,K}$ defined in (3.9), where $J$ and $K$ are the parabolic types of the filtrations $C^\bullet$ and $D_\bullet$, respectively.

We relate this to $\tilde{Z}_J$ as defined in (3.3). The scheme $\tilde{Z}_J$ depends on some automorphism $\delta$ of the Weyl group of $G$ which takes into account that our group $\hat{G}$ might be disconnected. Hence we will write $\tilde{Z}_{J,\delta}$. Moreover we let $\tilde{Z}'_J$ be the disjoint union of the schemes $\tilde{Z}_{J,\delta}$ for the various possible $\delta$. We can do this also for the schemes $Z_J = Z_{J,\delta}$ and obtain a scheme $Z'_J$.

For every $S$-valued point $(A, \ldots)$ of $\tilde{A}_0$ we obtain an $F$-zip with underlying $O_S$-module $H^1_{dR}(A/S)$ with additional structures and with splittings for their filtrations. As in the proof of (4.3) we can associate to this $F$-zip an $S$-valued point of $\tilde{Z}'_J$. By passing to the quotients, we obtain a morphism

$$\pi: \tilde{A}_0 \longrightarrow [G\backslash Z'_J]$$

where on the right hand side we have the quotient stack.

Note that it follows from [19], 4.1, that we can decompose $A_0$ into the special fibres of individual Shimura varieties such that $\pi$ factors through one of the $[G\backslash Z_{J,\delta}]$. We omit the details.

For each connected component $Z_{J,\delta}$ of $Z'_J$ and for $\mathfrak{u} \in \mathcal{T}(J)$ we have defined a $G$-invariant subscheme $Y^\mathfrak{u}$ which gives by passage to the quotient a locally closed substack $[G\backslash Y^\mathfrak{u}]$ of $[G\backslash Z_{J,\delta}]$. The inverse images of these substacks $\tilde{A}_0^\mathfrak{u}$ in $\tilde{A}_0$ are by definition the Ekedahl-Oort strata in $A_0$. Note that these strata now carry a canonical scheme structure.

(6.16) It is shown for $p > 2$ in [27] that $\pi$ is the composition of a smooth morphism and a homeomorphism. In particular we obtain that the codimension of the Ekedahl-Oort stratum $A_0^\mathfrak{u}$ is the same as the codimension of $Y^\mathfrak{u}$ in $Z_{J,\delta}$ if it is nonempty. Hence we get

$$\text{codim}(A_0^\mathfrak{u}, A_0) = \dim(\text{Par}_J) - \ell(\mu_\infty)$$

by (3.20). This gives a new proof of the main result of [20].

By [19] 3.2.7 the inverse image of the union of the open strata in $[G\backslash Z'_J]$ is just the $\mu$-ordinary locus of $A_0$ in the sense of [26] and we obtain a new proof of the main result of [26], as was already pointed out in [19].
(6.17) Example: F-zips associated to strongly divisible lattices. Let $F$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $O_F$ and finite residue field $\kappa$ of cardinality $q$. Let $k$ be a perfect extension of $\kappa$ and let $L = W(k) \otimes_{W(\kappa)} F$. Let $\sigma_0$ be the automorphism of $W(k)$ over $W(\kappa)$ induced by the $q$th power Frobenius on $k$, and define $\sigma = \sigma_0 \otimes \text{id}_F$. We fix a uniformizing element $\pi$ of $O_F$, and we set $O_L := W(k) \otimes_{W(\kappa)} O_F$.

Let $(H, \Phi, \text{Fil}^*)$ be a filtered isocrystal over $L$. By this we mean that $H$ is a finite dimensional $L$-vector space, equipped with a $\sigma$-linear bijective operator $\Phi: H \to H$, and with a descending filtration $\text{Fil}^*$. Suppose that $\mathcal{M} \subset H$ is a strongly divisible lattice, i.e., an $O_L$-lattice such that

$$\mathcal{M} = \sum_{i \in \mathbb{Z}} \pi^{-i} \Phi(\mathcal{M} \cap \text{Fil}^i).$$

We claim that $M := \mathcal{M}/\pi \mathcal{M}$ naturally inherits the structure of an $F$-zip over $k$ with respect to the prime power $q$. The definition is as follows.

We let $C^\bullet$ be the descending filtration on $M$ induced by $\text{Fil}^*$, so

$$C^i = \{ m \in M \mid \exists y \in \mathcal{M} \cap \text{Fil}^i \text{ with } y \text{ mod } \pi \mathcal{M} = m \}. $$

Next define the ascending filtration $D_\bullet$ by

$$D_i = \{ m \in M \mid \exists y \in \mathcal{M} \text{ with } \pi^{-i} \Phi(y) \in \mathcal{M} \text{ and } \pi^{-i} \Phi(y) \text{ mod } \pi \mathcal{M} = m \}. $$

Define a $k$-linear map $\tilde{\varphi}_i: (C^i)^{(q)} \to D_i$ by $\tilde{\varphi}_i(m \otimes 1) = \pi^{-i} \Phi(y) \text{ mod } \pi \mathcal{M}$, where $y \in \mathcal{M} \cap \text{Fil}^i$ is any element with $y \text{ mod } \pi \mathcal{M} = m$. Using (6.17.1) we see that this is well-defined. It is easily seen that $\tilde{\varphi}_i$ vanishes on $(C^{i+1})^{(q)}$, so we may define $\varphi_i: (\text{gr}_C^i)^{(q)} \to \text{gr}^D_i$ to be the map induced by $\tilde{\varphi}_i$.

It remains to be seen that $\varphi_i$ is an isomorphism, and by a dimension count it suffices to show that each $\varphi_i$ is surjective. For this, consider an element $m \in D_i$. By assumption there is an element $y \in \mathcal{M}$ with $\pi^{-i} \Phi(y) \in \mathcal{M}$ and $\pi^{-i} \Phi(y) \text{ mod } \pi \mathcal{M} = m$. By (6.17.1) there are elements $z_j \in \mathcal{M} \cap \text{Fil}^i$ such that $\pi^{-i} \Phi(y) = \sum \pi^{-j} \Phi(z_j)$. Because $\Phi$ is injective, $\pi^{-i} y = \sum \pi^{-j} z_j$. Let $y' := y - \sum_{j < i} \pi^{-j} z_j$. Then $\Phi(y') \in \pi^i \mathcal{M}$, and if we write $m'$ for the class of $\pi^{-i} \Phi(y')$ modulo $\pi \mathcal{M}$ then $m'$ and $m$ represent the same class in $\text{gr}_C^D$. On the other hand, $y' = \sum_{j \geq i} \pi^{-j} z_j$ lies in $\text{Fil}^i \cap \mathcal{M}$. Hence it follows that $\tilde{m} \in \text{gr}_C^D$ is in the image of $\varphi_i$.

If we look more closely, we see that we not only get the structure of an $F$-zip on $M$, but that it comes equipped with a natural splitting of the $D_\bullet$-filtration. Namely, define $\tilde{M} \subset H$ by $\tilde{M} := \sum_{i \in \mathbb{Z}} \pi^{-i} (\mathcal{M} \cap \text{Fil}^i)$. Then $\Phi$ gives an isomorphism $\sigma^* \tilde{M} \sim \to \mathcal{M}$. Moreover, we have a natural isomorphism $\tilde{M}/\pi \tilde{M} \sim \to \oplus_{i \in \mathbb{Z}} \text{gr}_C^i(M)$. Hence $\Phi$ induces an isomorphism $\oplus (\text{gr}_C^i)^{(q)} \sim \to M$; it maps $(\text{gr}_C^i)^{(q)}$ into $D_i$, and the composition $(\text{gr}_C^i)^{(q)} \to D_i \to \text{gr}^D_i$ is of course $\varphi_i$. The image of $(\text{gr}_C^i)^{(q)}$ inside $D_i$ is the subspace $E_i \subset D_i$ given by $E_i = \{ m \in M \mid \exists y \in \mathcal{M} \cap \text{Fil}^i \text{ with } \pi^{-i} \Phi(y) \text{ mod } \pi \mathcal{M} = m \}$, and this is a complement for $D_{i-1}$ inside $D_i$.

(6.18) Example: F-zips associated to K3 surfaces. Fix a natural number $d$ and a prime number $p$ with $p \nmid 2d$. Let $S$ be a scheme of characteristic $p$, and let $(Y, L)$ be a K3 surface with polarization of degree $2d$ over $S$. This gives rise to a sequence

$$S = S_1 \supset S_2 \supset \cdots \supset S_{10} \supset S_\infty$$

(6.18.1) $\vdash S_{\infty, 10} \supset S_{\infty, 9} \supset \cdots \supset S_{\infty, 1}$. 

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Here $S_h \subset S$, for $h \in \{1, 2, \ldots, 10, \infty\}$, is the closed subscheme of $S$ given, loosely speaking, by the condition that the formal group of $X$ has height $\geq h$. For details we refer to Ogus’s paper [22]. On the supersingular locus $S_\infty$ we have a further set-theoretic stratification, letting $S_{\infty, i}$ be the locus of points $s \in S$ where $\sigma_0(Y_s) \leq i$; here $\sigma_0$ denotes the Artin invariant. For details we again refer to [22].

We now want to connect the stratification in (6.18.1) with our theory of $F$-zips. Let $f: Y \to S$ be the structural morphism, and consider the second de Rham cohomology $H := R^2f_*\Omega^*_Y/S$. It is a locally free $O_S$-module of rank 22, which comes equipped with a non-degenerate symmetric bilinear form $Q: H \times H \to O_S$. If $c_1(L) \in H(S)$ is the first Chern class of $L$ then the primitive cohomology

$$M := \langle c_1(L) \rangle^+ \subset H$$

is locally free of rank 21, and $Q$ restricts to a non-degenerate form on $M$, which we again call $Q$.

As in (6.5), let $C^\bullet$ be the Hodge filtration on $M$ and let $D_\bullet$ be the conjugate filtration (up to a renumbering). The type $\tau$ is given by $\tau(0) = \tau(2) = 1$ and $\tau(1) = 19$, and $C^\bullet$ and $D_\bullet$ are orthogonal filtrations. The inverse Cartier isomorphism gives isomorphisms

$$\varphi_i: (\text{gr}_{C^i}(p) \iso \text{gr}_{D^i}$$

such that $M = (M, Q, C^\bullet, D_\bullet, \varphi_s)$ is an orthogonal $F$-zip.

Let $(V, \psi)$ be an orthogonal space over $\mathbb{F}_p$, with $\dim(V) = 21$. Set $G := \text{SO}(V, \psi)$, which has root system of type $B_{10}$. We take a basis of simple roots as $\{\alpha_1, \ldots, \alpha_{10}\}$ as in [2], Planche II; thus, $\alpha_{10}$ is the short root. Let $I = \{s_1, \ldots, s_{10}\}$ be the corresponding set of simple reflections. We have

$$W \cong \{\rho \in S_21 \mid \rho(j) + \rho(22 - j) = 22 \text{ for all } j\},$$

with $s_i$ corresponding to the element $(i, i + 1) \cdot (21 - i, 22 - i)$ for $1 \leq i \leq 9$ and $s_{10}$ corresponding to the transposition $(10, 12)$. (Note that $\rho(11) = 11$ for all $\rho \in W$.)

Let $J := I \setminus \{s_1\}$. We have $W_J = \{\rho \in W \mid \rho(1) = 1\}$, so $W_J/W$ is a set of 20 elements. The set $JW$ of minimal representatives consists of elements $x_1, \ldots, x_{20}$, which we number in such a way that $\ell(x_j) = 20 - j$. In the Bruhat ordering we have $x_1 > x_2 > \cdots > x_{20}$.

Consider the covering $^\# S \to S$, such that the $T$-valued points of $^\# S$, for $T$ an $S$-scheme, are the isometries $M_T \iso V_T$. Then $^\# S$ is an torsor over $S$ in the étale topology, under the group $O(V, \psi)$. If $X_T$ is the scheme of orthogonal $F$-zip structures on $(V, \psi)$, as in (5.3), then we have an $O(V, \psi)$-equivariant morphism $\rho: ^\# S \to X_T$. If $X_T(j) \subset X_T$ is the stratum corresponding to the element $x_j$, let $S^{(j)} \subset S$ be the subscheme obtained as the quotient of $\rho^{-1}(X_T(j))$ under $O(V, \psi)$.

The connection between the EO-loci $S^{(j)}$ thus obtained and the strata in (6.18.1) is given by the following result.

**Proposition:** With notation as in (6.18.1) we have

$$S^{(j)} = S_j \setminus S_{j+1} \quad \text{for } 1 \leq j \leq 11,$$

where we let $S_{11} := S_\infty = S_{\infty, 10}$ and $S_{12} := S_{\infty, 9}$. Further, on $k$-valued points we have

$$S_{\infty, i}(k) = \prod_{i \geq 21 - l} S^{(i)}(k) \quad \text{for } 1 \leq l \leq 10.$$
This is essentially what Ogus proves in [22]. Thus, we see that our theory of $F$-zips gives a uniform and scheme-theoretic approach to the whole chain in (6.18.1).

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