ON $p$-ADIC MULTIPLE BARNES-EULER ZETA FUNCTIONS AND THE CORRESPONDING LOG GAMMA FUNCTIONS

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Abstract. Suppose that $\omega_1, \ldots, \omega_N$ are positive real numbers and $x$ is a complex number with positive real part. The multiple Barnes-Euler zeta function $\zeta_{E,N}(s, x; \omega)$ with parameter vector $\omega = (\omega_1, \ldots, \omega_N)$ is defined as a deformation of the Barnes multiple zeta function as follows

$$\zeta_{E,N}(s, x; \omega) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_N=0}^{\infty} \frac{(-1)^{t_1+\cdots+t_N}}{(x + \omega_1 t_1 + \cdots + \omega_N t_N)^s}.$$

In this paper, based on the fermionic $p$-adic integral, we define the $p$-adic analogue of multiple Barnes-Euler zeta function $\zeta_{p,E,N}(s, x; \omega)$ which we denote by $\zeta_{D,E,N}(s, x; \omega)$. We prove several properties of $\zeta_{p,E,N}(s, x; \omega)$, including the convergent Laurent series expansion, the distribution formula, the difference equation, the reflection functional equation and the derivative formula. By computing the values of this kind of $p$-adic zeta function at nonpositive integers, we show that it interpolates the higher order Euler polynomials $E_{N,n}(x; \omega)$ $p$-adically.

Furthermore, we define the corresponding multiple $p$-adic Diamond-Euler Log Gamma function. We also show that the multiple $p$-adic Diamond-Euler Log Gamma function $\mathrm{Log} \Gamma_{D,E,N}(x; \omega)$ has an integral representation by the multiple fermionic $p$-adic integral, and it satisfies the distribution formula, the difference equation, the reflection functional equation, the derivative formula and also the Stirling’s series expansions.

1. Introduction

Throughout this paper, we use the following notations.

$\mathbb{C}$ — the field of complex numbers.
$\mathbb{R}^+$ — the set of positive real numbers.
$p$ — an odd prime number.
$\mathbb{Z}_p$ — the ring of $p$-adic integers.
$\mathbb{Q}_p$ — the field of fractions of $\mathbb{Z}_p$.
$\mathbb{C}_p$ — the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$.

$v_p$ — the $p$-adic valuation of $\mathbb{C}_p$ normalized so that $|p|_p = p^{-v_p(p)} = p^{-1}$

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The Hurwitz-type Euler zeta function is defined as follows

\[ \zeta_E(s, x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + x)^s} \]

for \( \text{Re}(s) > 0 \) (see [20] and [10, Eq. (1.6)]), which is a deformation of the Hurwitz zeta functions

\[ \zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s} \]

for \( \text{Re}(s) > 1 \). The Riemann zeta function is given by \( \zeta(s) = \zeta(s, 1) \), which is absolutely convergent for \( \text{Re}(s) > 1 \). When \( x = 1 \), (1.1) reduces to the Euler zeta functions (also called the Dirichlet eta functions)

\[ \zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \]

This function can be analytically continued to the complex plane without any pole. In fact, in [34, Theorem 2.5], it is shown that the Euler zeta functions \( \zeta_E(s) \) is summable (in the sense of Abel) to \( (1 - 2^{1-s})\zeta(s) \) for all values of \( s \).

During the 1960s, Kubota and Leopoldt gave \( p \)-adic analogues of the Riemann zeta functions and the corresponding \( L \)-series (Dirichlet \( L \)-functions). Their generalizations to the \( p \)-adic Hurwitz series, named the \( p \)-adic Hurwitz zeta functions, have been systematically studied by Cohen [3] and Tangedal and Young [32]. Recently, Young found that the \( p \)-adic Hurwitz zeta function has a connection with his definition of \( p \)-adic Arakawa-Kaneko zeta function by an identity (see [41, Corollary 1]).

The multiple Hurwitz zeta function in the complex field

\[ \zeta_N(s, x; \omega_1, \ldots, \omega_N) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_N=0}^{\infty} \frac{1}{(x + \omega_1 t_1 + \cdots + \omega_N t_N)^s}, \]

has a long history, it began with Barnes’s study on the theory of the multiple Gamma function \( \Gamma_k(a) \). (See [30, Sec. 2.1]). In 1977, Shintani [28] evaluated the special values of certain \( L \)-functions attached to real quadratic number fields in terms of Barnes’s multiple Hurwitz zeta functions, and the work of Barnes has also been applied in the study of the determinants of the Laplacians around 1980s and 1990s (see [35] and [30, Sec. 5.1, 5.2]). Recently, Tangedal and Young [32] considered the \( p \)-adic analogue of Barnes multiple zeta functions \( \zeta_{p,N}(s, x; \omega_1, \ldots, \omega_N) \) and the corresponding \( p \)-adic multiple log gamma functions \( G_{p,N}(x; \omega_1, \ldots, \omega_N) \).

In [33], Tangedal and Young showed that, for any real quadratic field with prime \( p \) splits completely in it, the \( p \)-adic multiple log gamma function \( G_{p,2}(x; \omega_1, \omega_2) \) presents a representation for the derivative at \( s = 0 \) of the \( p \)-adic partial zeta function associated with any element in certain narrow ray class groups.
The main aim of this paper is to define the $p$-adic analogue of multiple Barnes-Euler zeta functions
\[ \zeta_{E,N}(s, x; \bar{\omega}) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_N=0}^{\infty} \frac{(-1)^{t_1+\cdots+t_N}}{(x + \omega_1 t_1 + \cdots + \omega_N t_N)^s}, \]
and study their properties. Our approaches are mainly based on the theory of fermionic $p$-adic integrals. So we recall the definition and applications of these integrals in the following subsection.

1.1. The fermionic $p$-adic integral and its applications. Let $UD(\mathbb{Z}_p)$ be the space of all uniformly (or strictly) differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$ (see [3, §11.1.2]). The fermionic $p$-adic integral $I_{-1}(f)$ on $\mathbb{Z}_p$ of a function $f \in UD(\mathbb{Z}_p)$ is defined by
\[ I_{-1}(f) = \int_{\mathbb{Z}_p} f(a) d\mu_{-1}(a) = \lim_{N \to \infty} \sum_{a=0}^{p^N-1} f(a)(-1)^a. \]
In view of (1.4), for any $f \in UD(\mathbb{Z}_p)$, we have
\[ \int_{\mathbb{Z}_p} f(t+1) d\mu_{-1}(t) + \int_{\mathbb{Z}_p} f(t) d\mu_{-1}(t) = 2f(0) \]
(see [15, p. 782, (7)]).

The fermionic $p$-adic integral (1.4) were independently found as special cases by Katz [14, p. 486] (in Katz’s notation, the $\mu^{(2)}$-measure), Shiratani and Yamamoto [29], Osipov [23], Lang [19] (in Lang’s notation, the $E_{1,2}$-measure), T. Kim [15] from very different viewpoints. And they have many applications in number theory. First, they are nice tools for studying many special numbers and polynomials, in particular, the Euler numbers and polynomials.

The Euler polynomials $E_k(x), k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, are defined by the generating function
\[ \frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!}, \]
and the integers $E_k = 2^k E_k(1/2), k \in \mathbb{N}_0$, are called Euler numbers. For example, $E_0 = 1, E_2 = -1, E_4 = 5$, and $E_6 = -61$. The Euler numbers and polynomials (so called by Scherk in 1825) appear in Euler’s famous book, Institutiones Calculi Differentiales (1755, pp.487–491 and p.522). Notice that the Euler numbers with odd subscripts vanish, that is, $E_{2m+1} = 0$ for all $m \in \mathbb{N}_0$.

In 1875, Stern [26] gave a brief sketch of a proof of the following congruence of Euler numbers modulo powers of two:
\[ E_k \equiv E_l \pmod{2^n} \iff k \equiv l \pmod{2^n}, \]
for any $n \in \mathbb{Z}^+$, and $k, l$ be even. In 1910, Frobenius gave a detail for Stern’s sketch. In 1979, Ernvall [7] argued that he could not understand Frobenius’s proof and provided his own proof using umbral calculus. In 2002, Wagstaff
[39] presented an induction proof and in 2004, Sun [31] gave a combinational proof.

The $p$-adic approach to the above formula seems to be simple and general. It is based on the following Witt’s formula of Euler numbers, which gives a $p$-adic integral representation of Euler numbers:

$$E_n = \int_{\mathbb{Z}_p} (2a + 1)^n d\mu_{-1}(a).$$

In 2009, by using the above Witt’s formula, Kim [16, Theorem 2.11] gave a brief proof of the above Stern’s congruence. In 2010, also using the theory of $p$-adic integration, Maïga [21, Theorem 6] proved a Kummer type congruence for higher order Euler numbers:

$$E_{rp^k+s}^{(q)} \equiv E_{rp^k+s}^{(q)} \pmod{p^k\mathbb{Z}_p},$$

where $r, k$ and $s$ are integers such that $(r, s) = 1$, $k \geq 1$ and $s \geq 0$.

The fermionic $p$-adic integral may also apply to many other special polynomials and numbers, for example, the Genocchi numbers. In [2], by using the fermionic $p$-adic integral $I_{-1}(f)$, Cangul, Ozden and Simsek defined generating functions of higher order $w$-Genocchi numbers $G_{n,w}^{(N)}$:

$$\int_{\mathbb{Z}_p^N} w^{x_1+\cdots+x_N} e^{t(x_1+\cdots+x_N)} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_N) = \sum_{n=0}^{\infty} G_{n,w}^{(N)} \frac{t^n}{n!},$$

(1.8)

$$= 2^N \left( \frac{t}{we^t + 1} \right)^N$$

(see [2, (7)]), they also presented a Witt-type integral representations for these numbers:

$$\int_{\mathbb{Z}_p^N} (x_1 + \cdots + x_N)^n w^{x_1+\cdots+x_N} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_N) = \frac{G_{n,w}^{(N)}}{N! \binom{n+k}{N}}$$

(1.9)

(see [2, Theorem 3]).

In [17], using the fermionic $p$-adic integral [14], we also defined $\zeta_{p,E}(s, x)$, the $p$-adic analogue of Hurwitz-type Euler zeta functions [11], which interpolates [11] at nonpositive integers (see [17, Theorem 3.8(2)]), so called the $p$-adic Hurwitz-type Euler zeta functions.

We have also proved several properties of $\zeta_{p,E}(s, x)$, including the analyticity, the convergent Laurent series expansion, the distribution formula, the difference equation, the reflection functional equation, the derivative formula and the $p$-adic Raabe formula.

The $p$-adic Hurwitz-type Euler zeta function $\zeta_{p,E}(s, x)$ has been used by Hu, Kim and Kim to give a definition for the $p$-adic Arakawa-Kaneko-Hamahata zeta functions, which interpolate Hamahata’s poly-Euler polynomials at non-positive integers (see [11, Definition 1.4]). And from the properties of $\zeta_{p,E}(s, x)$, they also obtained many identities on the $p$-adic Arakawa-Kaneko-Hamahata zeta functions, including their derivative formula, difference equation and reflection formula (see [11, Theorems 2.1, 2.3 and 2.5]). Recently, the $p$-adic function $\zeta_{p,E}(s, x)$ becomes as a special case of Young’s
definition of a $p$-adic analogue for the generalized Barnes zeta functions of order zero associated to the function $f(t) = r \text{Li}_{k}((1 - e^{-t})/(1 - e^{-t})$, where $\text{Li}_{k}(z) = \sum_{k=1}^{\infty} \frac{z^{m}}{m^{k}}$ (see [13] (2.35)).

In [17] Sec.5], using these zeta functions as building blocks, we also gave a definition for the corresponding $L$-functions $L_{p,E}(\chi, s)$, so called $p$-adic Euler $L$-functions. In [10], we showed that a case of Gross’s refined Dedekind zeta functions (see [9] Sec.1], [10] Sec.2] and [1] Sec.7]) may be represented by a product of Euler $L$-functions (see [10] Propositions 3.2 and 3.3)), so the $p$-adic function $L_{p,E}(\chi, s)$ gives a growth formula for the $(S, \{2\})$-refined class numbers of the maximal real subfields of $p$-cyclotomic number fields (see [10] Theorem 1.2 and Sec.4]).

In [18], also using the fermionic $p$-adic integral $I_{-1}(f)$ (1.4), we defined the corresponding $p$-adic Log Gamma functions, named the $p$-adic Diamond-Euler Log Gamma functions. For example, in [18] Sec. 6], using the $p$-adic Hurwitz-type Euler zeta functions, we found that the derivative of the $p$-adic Hurwitz-type Euler zeta functions $\zeta_{p,E}(s, x)$ at $s = 0$ may be represented by the $p$-adic Diamond-Euler Log Gamma functions. This led us to connect the $p$-adic Hurwitz-type Euler zeta functions to the $(S, \{2\})$-version of the abelian rank one Stark conjecture (see [36]).

1.2. Main works. Suppose that $\omega_{1}, \ldots, \omega_{N}$ are positive real numbers and $x$ is a complex number with positive real part. The Barnes multiple zeta function $\zeta_{N}(s, x; \bar{\omega})$ with parameter vector $\bar{\omega} = (\omega_{1}, \ldots, \omega_{N})$ is defined for $\text{Re}(s) > N$ by

\begin{equation}
\zeta_{N}(s, x; \omega_{1}, \ldots, \omega_{N}) = \sum_{t_{1}=0}^{\infty} \cdots \sum_{t_{N}=0}^{\infty} \frac{1}{(x + \omega_{1}t_{1} + \cdots + \omega_{N}t_{N})^{s}},
\end{equation}

which may also be written more concisely as

\begin{equation}
\zeta_{N}(s, x; \bar{\omega}) = \sum_{t \in \mathbb{Z}_{0}^{N}} (x + \bar{\omega} \cdot t)^{-s}
\end{equation}

(see [32] (2.2)). When $N = 1$ and $\omega_{1} = 1$ this reduces to the Hurwitz zeta functions (1.2). As a function of $s$, $\zeta_{N}(s, x; \bar{\omega})$ is analytic for $\text{Re}(s) > N$ and it has a meromorphic continuation to all of $\mathbb{C}$ with simple poles at $s = 1, \ldots, N$. The Barnes multiple zeta function $\zeta_{N}(s, x; \bar{\omega})$ also satisfies the difference equation (see [32] (2.3)), the derivative formula [32] (2.5)), and it also interpolates the higher order Bernoulli polynomials $B_{N,n}(x; \bar{\omega})$ at nonpositive integers (see [32] (2.6)).

Suppose that $\omega_{1}, \ldots, \omega_{N} \in \mathbb{C}_{p}^{*}$, and let $\Lambda$ denote the $\mathbb{Z}_{p}$-linear span of $\{\omega_{1}, \ldots, \omega_{N}\}$. For $x \in \mathbb{C}_{p} \setminus \Lambda$, Tangedal and Young [32] (3.1) defined the $p$-adic analogue of the Barnes multiple zeta function $\zeta_{N}(s, x; \bar{\omega})$ with parameter vector $\bar{\omega} = (\omega_{1}, \ldots, \omega_{N})$ by the following equality

\begin{equation}
\zeta_{p,N}(s, x; \bar{\omega}) = \frac{1}{\omega_{1} \cdots \omega_{N}(s - 1) \cdots (s - N)}
\end{equation}

\begin{equation}
\times \int_{\mathbb{Z}_{p}} \cdots \int_{\mathbb{Z}_{p}} \frac{(x + \omega_{1}t_{1} + \cdots + \omega_{N}t_{N})^{N}}{(x + \omega_{1}t_{1} + \cdots + \omega_{N}t_{N})^{s}} dt_{1} \cdots dt_{N}.
\end{equation}
In which, the multiple Volkenborn integrals are all computable as iterated Volkenborn integrals. Here the Volkenborn integral for a function \( f \) which is strictly differentiable on \( \mathbb{Z}_p \) is defined by (comparing with (1.4) above of the definition of fermionic \( p \)-adic integrals)

\[
\int_{\mathbb{Z}_p} f(a) da = \lim_{N \to \infty} \frac{1}{p^N} \sum_{a=0}^{p^N-1} f(a)
\]

and this limit always exists when \( f \in UD(\mathbb{Z}_p) \) (see \[24, p. 264\] and \[27, \S 55\]). This integral was introduced by Volkenborn \[37\] and he also investigated many important properties of \( p \)-adic valued functions defined on the \( p \)-adic domain (see \[37, 38\]). At the \( k \)-th iteration, with \( 1 \leq k \leq N \), we integrate

\[
\int_{\mathbb{Z}_p} F_k(t_k, t_{k+1}, \ldots, t_N) dt_k,
\]

if \( F_k(t_k, t_{k+1}, \ldots, t_N) \) are strictly differentiable on \( \mathbb{Z}_p \) for each fixed vector \( (t_{k+1}, \ldots, t_N) \in \mathbb{Z}_p^{N-k} \).

Tangedal and Young \[32\] proved several properties of \( \zeta_{p,N}(s, x; \bar{\omega}) \), including the convergent Laurent series expansion (see \[32\] Theorem 4.1], the distribution formula, the difference equation, the reflection functional equation and the derivative formula (see \[32\] Theorem 3.2]). They also showed that \( \zeta_{p,N}(s, x; \bar{\omega}) \) nicely interpolates the higher order Bernoulli polynomials \( B_{N,n}(x; \bar{\omega}) \) \( p \)-adically at nonpositive integers (see \[32\] Theorem 3.2(v)). Furthermore, they \[32\] (3.11]) defined the \( p \)-adic multiple log gamma function \( G_{p,N}(x, \bar{\omega}) \) as the derivative of \( \zeta_{p,N}(s, x; \bar{\omega}) \) at \( s = 0 \). They also showed that the \( p \)-adic multiple log gamma function \( G_{p,N}(x; \bar{\omega}) \) has an integral representation by the multiple Volkenborn integral (see \[32\] (3.12]), and it satisfies the distribution formula, the difference equation, the reflection functional equation, the derivative formula (see \[32\] Theorem 3.4]) and the Stirling’s series expansions (see \[32\] Theorem 4.2]).

Recently, using the reflection formula for \( p \)-adic multiple zeta functions defined in \[32\], Young obtained strong congruences for sums of the form \( \sum_{n=0}^{N} B_n V_{n+1} \), where \( B_n \) denotes the Bernoulli number and \( V_n \) denotes a Lucas sequence of the second kind (see \[42\]).

Suppose that \( \omega_1, \ldots, \omega_N \) are positive real numbers and \( x \) is a complex number with positive real part. The multiple Barnes-Euler zeta function \( \zeta_{E,N}(s, x; \bar{\omega}) \) with parameter vector \( \bar{\omega} = (\omega_1, \ldots, \omega_N) \) is defined as a deformation of the Barnes multiple zeta function \( \zeta_N(s, x; \bar{\omega}) \) \[1.10\] as follows

\[
\zeta_{E,N}(s, x; \omega_1, \ldots, \omega_N) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_N=0}^{\infty} \frac{(-1)^{t_1+\cdots+t_N}}{(x + \omega_1 t_1 + \cdots + \omega_N t_N)^s}
\]

for \( \text{Re}(s) > 0 \), which may also be written more concisely as

\[
\zeta_{E,N}(s, x; \bar{\omega}) = \sum_{\bar{t} \in \mathbb{Z}_N^0} (-1)^{|\bar{t}|} (x + \bar{\omega} \cdot \bar{t})^{-s}.
\]
Here $|t| = t_1 + \cdots + t_N$. When $N = 1$ and $\omega_1 = 1$ this reduces to the Hurwitz-type Euler zeta functions (1.1). When $N = 0$ there is no parameter vector $\omega$ but we may still regard the above equation as defining the function $\zeta_{E,N}(s,x;\omega) = x^{-s}$. As a function of $s$, $\zeta_{E,N}(s,x;\omega)$ is analytic for $\text{Re}(s) > 0$ and continued analytically to $s \in \mathbb{C}$ without any pole (see (2.3)). The multiple Barnes-Euler zeta function $\zeta_{E,N}(s,x;\omega)$ also satisfies the difference equation (see Lemma 2.1(1)), the derivative formula (see Lemma 2.1(3)), and it also interpolates the higher order Euler polynomials $E_{N,n}(x;\omega)$ at nonpositive integers (see Lemma 2.3).

Suppose that $\omega_1,\ldots,\omega_N \in \mathbb{C}_p^\times$, and let $\Lambda$ denote the $\mathbb{Z}_p$-linear span of $\{\omega_1,\ldots,\omega_N\}$. In this paper, inspired by Tangedal and Young’s work [32], based on the fermionic $p$-adic integral (1.4), we define the $p$-adic analogue of multiple Barnes-Euler zeta function $\zeta_{E,N}(s,x;\omega)$, which we denote by $\zeta_{p,E,N}(s,x;\omega)$ (see (3.7)). It is analytic in certain areas for $s \in \mathbb{C}_p$ and $x \in \mathbb{C}_p \setminus \Lambda$ (see Theorem 3.3). We will prove several properties of $\zeta_{p,E,N}(s,x;\omega)$, including the convergent Laurent series expansion (see Theorem 3.9), the distribution formula, the difference equation, the reflection functional equation and the derivative formula (see Theorem 3.4). We will also show that $\zeta_{p,E,N}(s,x;\omega)$ interpolates the higher order Euler polynomials $E_{N,n}(x;\omega)$ $p$-adically at nonpositive integers (see Theorem 3.4(5)).

Furthermore, we will define the multiple $p$-adic Diamond-Euler Log Gamma function $\log \Gamma_{D,E,N}(x,\omega)$ as the derivative of $x\zeta_{p,E,N}(s,x;\omega)/(s-1)$ at $s = 0$ (see (3.13)). As a function of $x$, it is locally analytic on $\mathbb{C}_p \setminus \Lambda$ (see Theorem 3.5). We will also show that the multiple $p$-adic Diamond-Euler Log Gamma function $\log \Gamma_{D,E,N}(x,\omega)$ has an integral representation by the multiple fermionic $p$-adic integral (see Proposition 3.6), so it extends the definition of the $p$-adic Diamond-Euler Log Gamma function given in [18, (2.1)]. And it satisfies the distribution formula, the difference equation, the reflection functional equation, the derivative formula (see Theorem 3.8) and also the Stirling’s series expansions (see Theorem 3.10).

## 2. Multiple Barnes-Euler zeta functions and Euler polynomials of higher order

In this section, we give a brief review of the properties of the multiple Barnes-Euler zeta functions (the reader may refer to [4] for an excellent modern treatment). The Euler polynomials of higher order arise naturally in the study of these functions and in the first part of section 3 we give a $p$-adic realization of these polynomials in terms of multiple fermionic $p$-adic integrals.

Let $\mathbb{N}_0$ denote the set of nonnegative integers, that is $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}^+$ the set of positive real numbers. Suppose that $\omega_1,\ldots,\omega_N$ are positive real numbers and $z$ is a complex number with positive real part. The Euler polynomial $E_{N,n}(x;\omega)$ of order $N$ and degree $n$ with parameter vector $\omega =$
$(\omega_1, \ldots, \omega_N)$ is defined by

$$\frac{2^N e^{xt}}{\prod_{j=1}^{N} (e^{\omega_j t} + 1)} = \sum_{n=0}^{\infty} E_{N,n}(x; \bar{\omega}) \frac{t^n}{n!}. \quad (2.1)$$

For $N = 1$ and $\omega_1 = 1$, they coincide with usual Euler polynomials $E_n(x)$, that is, $E_{1,n}(x; 1) = E_n(x)$ for all $n \in \mathbb{N}_0$. When $N = 0$ there is no parameter vector $\bar{\omega}$ but we still use the generating function to define

$$E_{0,n}(x; -) = x^n. \quad (2.2)$$

If all $\omega_i = 1$ then $E^{(N)}_n(x) = E_{N,n}(x; 1, \ldots, 1)$ are the polynomials studied in [4], [22] and [40].

As the classical Riemann zeta functions (1.3), the multiple Barnes-Euler zeta function $\zeta_{E,N}(s, x; \bar{\omega})$ defined in (1.14) may also be represented by the Mellin transform as follows

$$\zeta_{E,N}(s, x; \bar{\omega}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \prod_{j=1}^{N} (1 + e^{-\omega_j t}) dt \prod_{j=1}^{N} \left(1 + e^{-\omega_j t}\right)^{-1} \frac{dt}{t}, \quad (2.3)$$

where $\text{Re}(s) > 0$, since if substituting the following power series into (2.3)

$$\prod_{j=1}^{N} (1 + e^{-\omega_j t})^{-1} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} (-1)^{m_1+\cdots+m_N} e^{-t(m_1\omega_1+\cdots+m_N\omega_N)} \quad (2.4)$$

and noticing that

$$\int_0^\infty t^{s-1} e^{-\omega t} \frac{dt}{t} = \omega^{-s} \Gamma(s), \quad (2.5)$$

then we have (1.14), which is used as a starting point of our work. By (2.3), $\zeta_{E,N}(s, x; \bar{\omega})$ can be continued analytically to $s \in \mathbb{C}$ without any pole.

**Lemma 2.1.** The following identities hold:

1. $\zeta_{E,N}(s, x + \omega_N; \bar{\omega}) + \zeta_{E,N}(s, x; \bar{\omega}) = \zeta_{E,N-1}(s, x; \omega_1, \ldots, \omega_{N-1})$.
2. $\zeta_{E,N}(s, cx; c\bar{\omega}) = c^{-s} \zeta_{E,N}(s, x; \bar{\omega})$ for all $c \in \mathbb{R}^+$.
3. $\frac{\partial^m}{\partial x^m} \zeta_{E,N}(s, x; \bar{\omega}) = (-1)^m (s)_m \zeta_{E,N}(s + m, x; \bar{\omega})$, where $(s)_m$ is the Pochhammer symbol defined by $(s)_m = s(s+1) \cdots (s+m-1)$ if $m \in \mathbb{N}$ and 1 if $m = 0$. 


Following Ruijsenaars (see [25, (3.8)]), in (2.5), replace $s$ by $s + k - 1$, it clear that $\zeta_{E,N}(s, x; \bar{\omega})$ satisfies the equation (2.6)

$$
\zeta_{E,N}(s, x; \bar{\omega}) = \frac{1}{\Gamma(s)} \sum_{k=0}^{M} \frac{(-1)^k}{k!} E_{N,k}(0; \bar{\omega}) x^{1-s-k} \Gamma(s + k - 1) 
\bigg[ \frac{\Gamma(s - 1)}{\Gamma(s)} - \frac{\Gamma(s - k)}{\Gamma(s)} \bigg] 
\sum_{k=0}^{M} \frac{(-1)^k}{k!} E_{N,k}(0; \bar{\omega}) t^{k-1} 
\bigg] \bigg[ \frac{\Gamma(s - k)}{\Gamma(s)} \bigg] 
\prod_{l=0}^{s-1} (s + l) 
+ \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^s e^{-xt} \bigg[ \frac{1}{\prod_{j=1}^{N} (1 + e^{-\omega_j t})} - \sum_{k=0}^{M} \frac{(-1)^k}{k!} E_{N,k}(0; \bar{\omega}) t^{k-1} \bigg] \frac{dt}{t}.
$$

We set (2.7)

$$
\log \Gamma_{E,N}(x; \bar{\omega}) = \frac{\partial}{\partial s} \zeta_{E,N}(s, x; \bar{\omega}) \bigg|_{s=0}.
$$

Then using $\frac{1}{\Gamma(s)} = s + O(s^2)$ for $s \to 0$ and using $E_{N,0}(0; \bar{\omega}) = 1$, Barnes’s asymptotic formula for the derivative of $\zeta_{E,N}(s, x; \bar{\omega})$ at $s = 0$ can be written as:

**Theorem 2.2.** We have

$$
\log \Gamma_{E,N}(x; \bar{\omega}) = x(\log x - 1) + E_{N,1}(0; \bar{\omega}) \log x 
+ \sum_{k=2}^{M} \frac{(-1)^k}{k!} E_{N,k}(0; \bar{\omega}) x^{-k+1}(k - 2)! + R_{N,M}(x),
$$

where

$$
R_{N,M}(x) = \int_{0}^{\infty} e^{-xt} \bigg[ \frac{1}{\prod_{j=1}^{N} (1 + e^{-\omega_j t})} - \sum_{k=0}^{M} \frac{(-1)^k}{k!} E_{N,k}(0; \bar{\omega}) t^{k-1} \bigg] \frac{dt}{t}.
$$

Using different methods as in [4], we obtain the following result which connections the special values of $\zeta_{E,N}(s, x; \bar{\omega})$ to the Euler polynomials of order $N$.

**Lemma 2.3.** The special value of $\zeta_{E,N}(s, x; \bar{\omega})$ at a rational integer $k$ ($k \in \mathbb{N}_0$) is given by equating the coefficients of powers of $t$ in the identity

$$
\sum_{k=0}^{\infty} \frac{\zeta_{E,N}(-k, x; \bar{\omega}) t^k}{k!} = \frac{e^{xt}}{\prod_{j=1}^{N} (e^{\omega_j t} + 1)}.
$$

In particular, we have

$$
\zeta_{E,N}(-k, x; \bar{\omega}) = \frac{1}{2^N} E_{N,k}(x; \bar{\omega}).
$$
Proof. Applying the procedure used to get uniformly convergent in the wider sense of the multiple series, we obtain the following generating function of $E_{N,n}(x; \bar{\omega})$:

\begin{equation}
2^N e^{xt} \prod_{j=1}^{N} (e^{\omega_j t} + 1) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} (-1)^{m_1 + \cdots + m_N} e^{(x + \omega_1 m_1 + \cdots + \omega_N m_N)t}.
\end{equation}

Letting $s = -k \ (k \in \mathbb{N}_0)$ in (1.14), we get

\begin{equation}
\zeta_{E,N}(-k, x; \bar{\omega}) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_N=0}^{\infty} (-1)^{t_1 + \cdots + t_N} (x + \omega_1 t_1 + \cdots + \omega_N t_N)^k.
\end{equation}

Since \( \left( \frac{d}{dt} \right)^k e^{(x + m_1 \omega_1 + \cdots + m_N \omega_N)t} \bigg|_{t=0} = (x + \omega_1 m_1 + \cdots + \omega_N m_N)^k \), it is clear from (2.1) and (2.9) that

\begin{equation}
\zeta_{E,N}(-k, x; \bar{\omega}) = \left( \frac{d}{dt} \right)^k \left( \sum_{k=0}^{\infty} \zeta_{E,N}(-k, x; \bar{\omega}) \frac{t^k}{k!} \right)
\end{equation}

\begin{align*}
&= \left( \frac{d}{dt} \right)^k \sum_{m_1=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} (-1)^{m_1 + \cdots + m_N} e^{(x + \omega_1 m_1 + \cdots + \omega_N m_N)t} \bigg|_{t=0} \\
&= \frac{1}{2^N} \left( \frac{d}{dt} \right)^k \prod_{j=1}^{N} (e^{\omega_j t} + 1) \bigg|_{t=0} \\
&= \frac{1}{2^N} \left( \frac{d}{dt} \right)^k \sum_{n=0}^{\infty} E_{N,n}(x; \bar{\omega}) \frac{t^n}{n!} \bigg|_{t=0},
\end{align*}

so that

\begin{equation}
\zeta_{E,N}(-k, x; \bar{\omega}) = \frac{1}{2^N} E_{N,k}(x; \bar{\omega}).
\end{equation}

This is our assertion for the value of $\zeta_{E,N}(s, x; \bar{\omega})$ at non-positive integers. \( \square \)

**Lemma 2.4.** The following identities hold:

1. For an even integer $M$, we have

\[ \sum_{j=0}^{M-1} (-1)^j E_{K,n+1}(x + j \omega_{K+1}; \omega_1, \ldots, \omega_K) = \frac{1}{2} (E_{K+1,n+1}(x; \omega_1, \ldots, \omega_{K+1}) - E_{K+1,n+1}(x + M \omega_{K+1}; \omega_1, \ldots, \omega_{K+1})). \]

2. For an odd integer $M$, we have

\[ \sum_{j=0}^{M-1} (-1)^j E_{K,n+1}(x + j \omega_{K+1}; \omega_1, \ldots, \omega_K) = \frac{1}{2} (E_{K+1,n+1}(x; \omega_1, \ldots, \omega_{K+1}) + E_{K+1,n+1}(x + M \omega_{K+1}; \omega_1, \ldots, \omega_{K+1})). \]
Proof. Suppose that $M$ is an odd integer. To see Part (2), from (2.9) and Lemma 2.3 note that

\begin{equation}
E_{K+1,n+1}(x + M \omega_{K+1}; \omega_1, \ldots, \omega_{K+1})
\end{equation}

\begin{equation}
= 2^{K+1} \sum_{t_1=0}^{\infty} \cdots \sum_{t_{K+1}=0}^{\infty} (-1)^{t_1+\cdots+t_{K+1}} (x + \omega_1 t_1 + \cdots + \omega_{K+1} (t_{K+1} + M))^{n+1}
\end{equation}

\begin{equation}
= -2^{K+1} \sum_{t_1=0}^{\infty} \cdots \sum_{t_{K+1}=M}^{\infty} (-1)^{t_1+\cdots+t_{K+1}} (x + \omega_1 t_1 + \cdots + \omega_{K+1} t_{K+1})^{n+1}
\end{equation}

and

\begin{equation}
E_{K+1,n+1}(x; \omega_1, \ldots, \omega_{K+1})
\end{equation}

\begin{equation}
= 2^{K+1} \sum_{t_1=0}^{\infty} \cdots \sum_{t_{K+1}=0}^{\infty} (-1)^{t_1+\cdots+t_{K+1}} (x + \omega_1 t_1 + \cdots + \omega_{K+1} t_{K+1})^{n+1}
\end{equation}

\begin{equation}
= 2^{K+1} \sum_{t_1=0}^{\infty} \cdots \sum_{t_{K+1}=M}^{\infty} (-1)^{t_1+\cdots+t_{K+1}} (x + \omega_1 t_1 + \cdots + \omega_{K+1} t_{K+1})^{n+1}
\end{equation}

\begin{equation}
+ 2^{K+1} \sum_{t_1=0}^{\infty} \cdots \sum_{t_{K+1}=0}^{M-1} (-1)^{t_1+\cdots+t_{K+1}} (x + \omega_1 t_1 + \cdots + \omega_{K+1} t_{K+1})^{n+1}.
\end{equation}

On the other hand, we have

\begin{equation}
2^{K+1} \sum_{t_1=0}^{\infty} \cdots \sum_{t_{K+1}=0}^{M-1} (-1)^{t_1+\cdots+t_{K+1}} (x + \omega_1 t_1 + \cdots + \omega_{K+1} t_{K+1})^{n+1}
\end{equation}

\begin{equation}
= 2 \sum_{j=0}^{M-1} (-1)^j E_{K,n+1}(x + j \omega_{K+1}; \omega_1, \ldots, \omega_{K}).
\end{equation}

Thus, Part (2) follows from (2.12), (2.13) and (2.14). The calculation of Part (1) is similar. \square

3. \textbf{p-Adic Multiple Barnes-Euler Zeta and Multiple p-adic Diamond-Euler Log Gamma Functions}

In this section, we define the \textit{p-}adic analogue of the multiple Barnes-Euler zeta function $\zeta_{E,N}(s, x; \bar{\omega})$ and the corresponding \textit{p-}adic Log Gamma functions, and we also study their properties.

For a vector $\bar{\omega} = (\omega_1, \ldots, \omega_N) \in \mathbb{C}_p^N$ we define its norm $||\bar{\omega}||_p$ by

\begin{equation}
||\bar{\omega}||_p = \max \{|\omega_1|_p, \ldots, |\omega_N|_p\}.
\end{equation}

The multiple fermionic \textit{p-}adic integrals considered here are defined as iterated integrals. At the $k$-th iteration, with $1 \leq k \leq N$, we integrate

\begin{equation}
\int_{\mathbb{Z}_p} F_k(t_k, t_{k+1}, \ldots, t_N) d\mu_{-1}(t_k),
\end{equation}
give that \( F_k(t_k, t_{k+1}, \ldots, t_N) \) are continuous functions on \( \mathbb{Z}_p \) for each fixed vector \((t_{k+1}, \ldots, t_N) \in \mathbb{Z}_{p}^{N-k} \). Under these conditions, we use the notation

\[
\int_{\mathbb{Z}_p^N} f(\vec{t}) \, d\mu_{-1}(\vec{t}), \quad \text{where } \vec{t} = (t_1, \ldots, t_N),
\]

to denote the \( N \)-fold iterated fermionic \( p \)-adic integral

\[
\int_{\mathbb{Z}_p^N} f(\vec{t}) \, d\mu_{-1}(\vec{t}) = \lim_{l \to \infty} \lim_{l \to \infty} \sum_{t_1=0}^{p^l-1} \cdots \sum_{t_N=0}^{p^l-1} f(t_1, \ldots, t_N)(-1)^{t_1 + \cdots + t_N}.
\]

**Lemma 3.1.** For any \( x \in \mathbb{C}_p \), \( \omega_i \in \mathbb{C}_p^\times \) (\( i = 1, \ldots, N \)) and \( 1 \leq k \leq N \), we have

\[
\int_{\mathbb{Z}_p} E_{k-1,n}(x + \omega kt_k + \cdots + \omega_N t_N; \omega_1, \ldots, \omega_{k-1}) \, d\mu_{-1}(t_k)
= E_{k,n}(x + \omega_{k+1} t_{k+1} + \cdots + \omega_N t_N; \omega_1, \ldots, \omega_k),
\]

where \( E_{0,n}(x; -) = x^n \) and \( \omega_{k+1} t_{k+1} + \cdots + \omega_N t_N = 0 \) when \( k = N \).

**Proof.** By (1.4) and Lemma 2.4(2), we have

\[
\int_{\mathbb{Z}_p} E_{k-1,n}(x + \omega kt_k + \cdots + \omega_N t_N; \omega_1, \ldots, \omega_{k-1}) \, d\mu_{-1}(t_k)
= \lim_{l \to \infty} \sum_{t_k=0}^{p^{l-1}} (-1)^{t_k} E_{k-1,n}(x + \omega_{k+1} t_{k+1} + \cdots + \omega_N t_N + \omega_k t_k; \omega_1, \ldots, \omega_{k-1})
= \lim_{l \to \infty} \frac{1}{2}(E_{k,n}(x + \omega_{k+1} t_{k+1} + \cdots + \omega_N t_N; \omega_1, \ldots, \omega_k)
+ E_{k,n}(x + \omega_{k+1} t_{k+1} + \cdots + \omega_N t_N + p^l \omega_k; \omega_1, \ldots, \omega_k))
= E_{k,n}(x + \omega_{k+1} t_{k+1} + \cdots + \omega_N t_N; \omega_1, \ldots, \omega_k)
\]
as desired. \( \square \)

**Corollary 3.2.** For any \( x \in \mathbb{C}_p \) and \( \omega_i \in \mathbb{C}_p^\times \) (\( i = 1, \ldots, N \)), we have

\[
\int_{\mathbb{Z}_p^N} (x + \vec{\omega} \cdot \vec{t})^n \, d\mu_{-1}(\vec{t}) = E_{N,n}(x; \vec{\omega}),
\]

where \( n \in \mathbb{N}_0 \).

**Proof.** If we set \( k = 1 \) in Lemma 3.1, we have

\[
\int_{\mathbb{Z}_p} (x + \omega_1 t_1 + \cdots + \omega_N t_N)^n \, d\mu_{-1}(t_1)
= \int_{\mathbb{Z}_p} E_{0,n}(x + \omega_1 t_1 + \cdots + \omega_N t_N; -) \, d\mu_{-1}(t_1)
= E_{1,n}(x + \omega_2 t_2 + \cdots + \omega_N t_N; \omega_1)
\]
since $E_{0,n}(x; -) = x^n$. Using (3.3), if we apply Lemma 3.1 inductively for $k = 2, \ldots, N$, we obtain
\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (x + \omega_1 t_1 + \cdots + \omega_N t_N)^n d\mu_{-1}(t_1)\mu_{-1}(t_2) = \int_{\mathbb{Z}_p} E_{1,n}(x + \omega_2 t_2 + \cdots + \omega_N t_N; \omega_1) d\mu_{-1}(t_2) = E_{2,n}(x + \omega_3 t_3 + \cdots + \omega_N t_N; \omega_1, \omega_2).
\]
Hence the assertion is clear. \hfill \Box

The projection function $\langle x \rangle$ for all $x \in \mathbb{C}_p^\times$ defined by Kashio [13] and Tangedal-Young [32] will play a key role in our definitions. We recall the definition of $\langle x \rangle$ in the following paragraph.

Fixing an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$, $p^\mathbb{Q}$ denotes the image in $\mathbb{C}_p^\times$ of the set of positive real rational powers of $p$ under this embedding, $\mu$ denotes the group of roots of unity in $\mathbb{C}_p^\times$ of order not divisible by $p$. For $x \in \mathbb{C}_p$, $|x|_p = 1$, there exists a unique elements $\hat{x} \in \mu$ such that $|x - \hat{x}|_p < 1$ (called the Teichmüller representative of $x$); it may be defined by $\hat{x} = \lim_{n \to \infty} x^{p^n}$. We extend this definition to $x \in \mathbb{C}_p^\times$ by
\[
\hat{x} := \left( x/p^v_p(x) \right),
\]
that is, we define $\hat{x} = \hat{u}$ if $x = p^r u$ with $p^r \in p^\mathbb{Q}$ and $|u|_p = 1$, then we define the function $\langle \cdot \rangle$ on $\mathbb{C}_p^\times$ by
\[
\langle x \rangle = p^{-v_p(x)} x / \hat{x}.
\]
For $x \in \mathbb{C}_p$ with $|x|_p < 1$ the Iwasawa logarithm function is defined by the usual power series
\[
\log_p x = -\sum_{n=1}^{\infty} \frac{(1 - x)^n}{n}
\]
and is extended to a continuous function on $\mathbb{C}_p^\times$ by defining $\log_p x = \log_p \langle x \rangle$.

Suppose that $\omega_1, \ldots, \omega_N \in \mathbb{C}_p^\times$, and let $\Lambda$ denote the $\mathbb{Z}_p$-linear span of $\{\omega_1, \ldots, \omega_N\}$. For all $x \in \mathbb{C}_p \setminus \Lambda$, we define the $p$-adic multiple Barnes-Euler zeta function $\zeta_{p,E,N}(s, x; \bar{\omega})$ by
\[
\zeta_{p,E,N}(s, x; \bar{\omega}) = \int_{\mathbb{Z}_p^N} \langle x + \bar{\omega} \cdot \bar{t} \rangle^{1-s} d\mu_{-1}(\bar{t}).
\]
Here, the function with $N = 0$ is understood to be ‘there is no parameter vector $\bar{\omega}$.’ This suggests that the integral
\[
\zeta_{p,E,0}(s, x; -) = \int_{\mathbb{Z}_p^N} \langle x \rangle^{1-s} d\mu_{-1}(t) = \langle x \rangle^{1-s}
\]
is derived easily from the relation $\int_{\mathbb{Z}_p^N} d\mu_{-1}(t) = 1$ (see [17, Proposition 2.1]).

By [32, Proposition 2.1] and (3.7), we have the following result on the analytic of $\zeta_{p,E,N}(s, x; \bar{\omega})$ in certain areas for $s \in \mathbb{C}_p$ and $x \in \mathbb{C}_p \setminus \Lambda$. 

Proposition 3.3. For any choices of \( \omega_1, \ldots, \omega_N, x \in \mathbb{C}_p^\times \) with \( x \not\in \Lambda \) the function \( \zeta_{p,E,N}(s, x; \bar{\omega}) \) is a \( C^\infty \) function of \( s \) on \( \mathbb{Z}_p \), and is an analytic function of \( s \) on a disc of positive radius about \( s = 0 \); on this disc it is locally analytic as a function of \( x \) and independent of the choice made to define the \( \langle \cdot \rangle \) function. If \( \omega_1, \ldots, \omega_N, x \) are so chosen to lie in a finite extension \( K \) of \( \mathbb{Q}_p \) whose ramification index over \( \mathbb{Q}_p \) is less than \( p - 1 \) then \( \zeta_{p,E,N}(s, x; \bar{\omega}) \) is analytic for \( s \in \mathbb{C}_p \) such that \( |s|_p < |\pi|_p^{-1/p-1} \), where \( (\pi) \) is the maximal ideal of the ring of integers \( O_K \) of \( K \). If \( s \in \mathbb{Z}_p \), the function \( \zeta_{p,N}(s, x; \bar{\omega}) \) is locally analytic as a function of \( x \) on \( \mathbb{C}_p \setminus \Lambda \).

Theorem 3.4. The function \( \zeta_{p,E,N}(s, x; \bar{\omega}) \) has the following properties:

1. For all \( x \in \mathbb{C}_p \setminus \Lambda \) the function \( \zeta_{p,E,N}(s, x; \bar{\omega}) \) satisfies the difference equation

\[
\zeta_{p,E,N}(s, x + \omega; \bar{\omega}) + \zeta_{p,E,N}(s, x; \bar{\omega}) = 2\zeta_{p,E,N-1}(s, x; \omega_1, \ldots, \omega_{N-1}),
\]

with the convention \( \zeta_{p,E,0}(s, x; -) = \langle x \rangle^{1-s} \) (see (3.3)) and \( \bar{\omega} = (\omega_1, \ldots, \omega_N) \).

2. For all \( c \in \mathbb{C}_p^\times \) and all \( x \in \mathbb{C}_p \setminus \Lambda \) we have

\[
\zeta_{p,E,N}(s, cx; c\bar{\omega}) = \langle c \rangle^{1-s} \zeta_{p,E,N}(s, x; \bar{\omega}).
\]

3. For all \( x \in \mathbb{C}_p \setminus \Lambda \) we have the reflection function equation

\[
\zeta_{p,E,N}(s, |\bar{\omega}| - x; \bar{\omega}) = \zeta_{p,E,N}(s, x; \bar{\omega}).
\]

Here we will write \( |\bar{\omega}| = \omega_1 + \cdots + \omega_N \).

4. For all \( x \in \mathbb{C}_p \setminus \Lambda \) and all positive odd integers \( m \) we have the multiplication formula (distribution relation)

\[
\zeta_{p,E,N}(s, x; \bar{\omega}) = \langle m \rangle^{1-s} \sum_{0 \leq j_i < m} (-1)^{|j|} \zeta_{p,E,N} \left( s, \frac{x + \bar{j} \cdot \bar{\omega}}{m} ; \bar{\omega} \right),
\]

where the sum is over all vectors \( \bar{j} = (j_1, \ldots, j_N) \) with \( 0 \leq j_i < m \). In particular for any \( k \in \mathbb{N} \) we have

\[
\zeta_{p,E,N}(s, x; \bar{\omega}) = \sum_{0 \leq j_i < p_k} (-1)^{|j|} \zeta_{p,E,N} \left( s, \frac{x + \bar{j} \cdot \bar{\omega}}{p^k} ; \bar{\omega} \right).
\]

5. Suppose that \( \omega_1, \ldots, \omega_N \in \mathbb{Q} \) are positive real numbers and \( x \in \mathbb{Q} \) is a complex number with positive real part. Under our fixed embedding of \( \mathbb{Q} \) into \( \mathbb{C}_p \), suppose that \( |x|_p > ||\bar{\omega}||_p \). Then for all \( k \in \mathbb{N} \), we have

\[
\zeta_{p,E,N}(1 - k, x; \bar{\omega}) = 2^N \left( \frac{|x|}{x} \right)^k \zeta_{E,N}(-k, x; \bar{\omega}).
\]

6. Suppose \( \omega_1, \ldots, \omega_N, x \in \mathbb{C}_p^\times \) and \( |x|_p > ||\bar{\omega}||_p \). Then for any \( m \in \mathbb{N}_0 \), the identity

\[
\frac{\partial^m}{\partial x^m} \zeta_{p,E,N}(s, x; \bar{\omega}) = (-1)^m \left( \frac{|x|}{x} \right)^m (s - 1)_m \zeta_{p,E,N}(s + m, x; \bar{\omega}).
\]
holds if $s \in \mathbb{Z}_p$; this identity also holds for $|s|_p < |\pi|^{-1}p^{-1/(p-1)}$ if $x$ and all $\omega_i$ lie in a finite extension $K$ of $\mathbb{Q}_p$ whose ring of integers has maximal ideal $(\pi)$ with ramification index over $\mathbb{Q}_p$ less than $p - 1$.

**Proof.** (1) Following Tangedal and Young (see [32, Theorem 3.2]), we define a function $f(t_N)$ on $\mathbb{Z}_p$ by the $(N-1)$-fold iterated fermionic $p$-adic integral as follows

$$f(t_N) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \frac{1}{\langle x + \omega_1 t_1 + \cdots + \omega_N t_N \rangle^{s-1}} d\mu_{-1}(t_1) \cdots d\mu_{-1}(t_{N-1}).$$

From (1.5), we have

$$\int_{\mathbb{Z}_p} f(t_N + 1) d\mu_{-1}(t_N) + \int_{\mathbb{Z}_p} f(t_N) d\mu_{-1}(t_N) = 2f(0).$$

If we put $t_N = 0$ in (3.9), use (3.7) and (3.10), we get

$$\zeta_{p,E,N}(s, x + \omega_N; \bar{\omega}) + \zeta_{p,E,N}(s, x; \bar{\omega})$$

$$= \int_{\mathbb{Z}_p} f(t_N + 1) d\mu_{-1}(t_N) + \int_{\mathbb{Z}_p} f(t_N) d\mu_{-1}(t_N)$$

$$= 2f(0)$$

$$= 2 \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \frac{1}{\langle x + \omega_1 t_1 + \cdots + \omega_N t_N \rangle^{s-1}} d\mu_{-1}(t_1) \cdots d\mu_{-1}(t_{N-1})$$

$$= 2\zeta_{p,E,N-1}(s, x; \omega_1, \ldots, \omega_{N-1}),$$

which prove (1).

(2) From the fact that $\langle \cdot \rangle$ is a multiplicative homomorphism, we have

$$\zeta_{p,E,N}(s, cx; c\bar{\omega}) = \int_{\mathbb{Z}_p} \frac{1}{\langle c(x + \bar{\omega} \cdot \tilde{t}) \rangle^{s-1}} d\mu_{-1}(\tilde{t})$$

$$= \frac{1}{\langle c \rangle^{s-1}} \zeta_{p,E,N}(s, x; \bar{\omega}).$$

Then the assertion is clear.

(3) We define

$$f(x, \tilde{t}) = \frac{1}{\langle x + \bar{\omega} \cdot \tilde{t} \rangle^{s-1}}.$$

Since

$$\int_{\mathbb{Z}_p} f(t + 1) d\mu_{-1}(t) = \int_{\mathbb{Z}_p} f(-t) d\mu_{-1}(t)$$

\( \zeta_{p,E,N}(s, |\bar{\omega}| - x; \bar{\omega}) \)
\[
= \int_{Z_p^N} f(\omega_1 + \cdots + \omega_N - x, \bar{t}) d\mu_{-1}(\bar{t}) \\
= \int_{Z_p^N} \frac{1}{\langle \omega_1 + \cdots + \omega_N - x + \omega_1 t_1 + \cdots + \omega_N t_N \rangle^{s-1}} d\mu_{-1}(\bar{t}) \\
= \int_{Z_p^N} \frac{1}{\langle \omega_1 + \cdots + \omega_{N-1} - x + \omega_1 t_1 + \cdots + \omega_N(t_N + 1) \rangle^{s-1}} d\mu_{-1}(\bar{t}) \\
= \int_{Z_p^N} f(\omega_1 + \cdots + \omega_{N-1} - x, t_1, \ldots, t_N, t_N + 1) d\mu_{-1}(\bar{t}) \\
= \int_{Z_p^N} f(\omega_1 + \cdots + \omega_{N-1} - x, t_1, \ldots, t_{N-1} - t_N) d\mu_{-1}(\bar{t}) \\
= \ldots \\
= \int_{Z_p^N} f(-x, -t_1, \ldots, -t_N) d\mu_{-1}(\bar{t}) \\
= \int_{Z_p^N} f(x, \bar{t}) d\mu_{-1}(\bar{t}) = \zeta_{p,E,N}(s, x; \bar{\omega}).
\]

The validity of the penultimate equality depends on the fact that \( \langle -z \rangle^s = \langle z \rangle^s \); when \( p \neq 2 \) this is valid since \( \langle -z \rangle = \langle z \rangle \). For \( p = 2 \) we have \( \langle -z \rangle = -\langle z \rangle \) so the equation is not valid for general \( s \), but there is a disc about \( s = 0 \) on which \( \langle -z \rangle^s = \exp_p(s \log_p(-\langle z \rangle)) = \exp_p(s \log(z)) = \langle z \rangle^s \), so on this disc the equation is valid (see [32, p. 1250]).

(4) We note that [17, Theorem 2.2(3)]

\[
\int_{Z_p} f(t) d\mu_{-1}(t) = \sum_{j=0}^{m-1} (-1)^j \int_{Z_p} f(j + mt) d\mu_{-1}(t),
\]
where \( m \) is an odd integer. We may use (3.11) and (3.12) to compute
\[
\zeta_{p,E,N}(s,x;\bar{\omega}) = \int_{\mathbb{Z}_p^N} f(x,\bar{t})d\mu_1(\bar{t})
\]
\[
= \int_{\mathbb{Z}_p^N} \left( \int_{\mathbb{Z}_p} f(x,t_1,\ldots,t_N)d\mu_1(t_1) \right) \cdots d\mu_1(t_1)
\]
\[
= \sum_{j_1=0}^{m-1} (-1)^{j_1} \int_{\mathbb{Z}_p^N} f(x,t_1,\ldots,j_1+mt_N)d\mu_1(t_1) \cdots d\mu_1(t_1)
\]
\[
= \cdots
\]
\[
= \sum_{0 \leq j_i < m} (-1)^{\bar{j}_i} \int_{\mathbb{Z}_p^N} f(x,\bar{j} + m\bar{t})d\mu_1(\bar{t})
\]
\[
= \sum_{0 \leq j_i < m} (-1)^{\bar{j}_i} \int_{\mathbb{Z}_p^N} f(x + \bar{j} \cdot \bar{\omega},m\bar{t})d\mu_1(\bar{t})
\]
\[
= \langle m \rangle^{1-s} \sum_{0 \leq j_i < m} (-1)^{\bar{j}_i} \int_{\mathbb{Z}_p^N} f \left( \frac{x + \bar{j} \cdot \bar{\omega}}{m},\bar{t} \right) d\mu_1(\bar{t})
\]
\[
= \langle m \rangle^{1-s} \sum_{0 \leq j_i < m} (-1)^{\bar{j}_i} \zeta_{p,E,N} \left( s, \frac{x + \bar{j} \cdot \bar{\omega}}{m}; \bar{\omega} \right)
\]
proving (4).

(5) Note that (32, p. 1251) if \( |x|_p > \|\bar{\omega}\|_p \), then
\[
(3.13) \quad \langle x + \bar{\omega} \cdot \bar{t} \rangle = \frac{\langle x \rangle_x}{x} (x + \bar{\omega} \cdot \bar{t}) \quad \text{for all} \ \bar{t} \in \mathbb{Z}_p^N.
\]
For given \( x \) and \( \bar{\omega} \) satisfying these hypotheses, we have
\[
(3.14) \quad \frac{\partial}{\partial x} (x + \bar{\omega} \cdot \bar{t})^s = s(x + \bar{\omega} \cdot \bar{t})^s(x + \bar{\omega} \cdot \bar{t})^{-1} \quad \text{uniformly for} \ \bar{t} \in \mathbb{Z}_p^N.
\]
Now by (3.7), (3.13), Corollary 3.2 and Lemma 2.3, we have
\[
\zeta_{p,E,N}(1-k,x;\bar{\omega}) = \int_{\mathbb{Z}_p^N} \langle x + \bar{\omega} \cdot \bar{t} \rangle^k d\mu_1(\bar{t})
\]
\[
= \left( \frac{\langle x \rangle_x}{x} \right)^k \int_{\mathbb{Z}_p^N} (x + \bar{\omega} \cdot \bar{t})^k d\mu_1(\bar{t})
\]
\[
= \left( \frac{\langle x \rangle_x}{x} \right)^k \int_{\mathbb{Z}_p^N} E_{0,k}(x + \bar{\omega} \cdot \bar{t}; -)d\mu_1(\bar{t})
\]
\[
= \left( \frac{\langle x \rangle_x}{x} \right)^k E_{N,k}(x; \bar{\omega})
\]
\[
= 2^N \left( \frac{\langle x \rangle_x}{x} \right)^k \zeta_{E,N}(-k,x;\bar{\omega}),
\]
which proves (5).
(6) From (3.13) and (3.14) we have
\[
\frac{\partial}{\partial x} \zeta_{p,E,N}(s, x; \bar{\omega}) = \int_{\Z_p^N} \frac{\partial}{\partial x} (x + \bar{\omega} \cdot \bar{t})^{1-s} d\mu_{-1}(\bar{t}) \\
= (1-s) \int_{\Z_p^N} (x + \bar{\omega} \cdot \bar{t})^{1-s} \frac{1}{x + \bar{\omega} \cdot \bar{t}} d\mu_{-1}(\bar{t}) \\
= (1-s) \langle x \rangle \int_{\Z_p^N} (x + \bar{\omega} \cdot \bar{t})^{-s} d\mu_{-1}(\bar{t}) \\
= (1-s) \langle x \rangle \zeta_{p,E,N}(s+1, x; \bar{\omega}),
\]
which proves (6) for \(m = 1\). The general statement of (6) follows by induction on \(m\). \(\Box\)

Given \(\omega_1, \ldots, \omega_N \in \C_p^\times\) and \(x \in \C_p \setminus \Lambda\), we define the multiple \(p\)-adic Diamond-Euler Log Gamma function \(\log \Gamma_{D,E,N}(x; \bar{\omega})\) by
\[(3.15) \quad \log \Gamma_{D,E,N}(x; \bar{\omega}) = \frac{x}{\langle x \rangle} \frac{\partial}{\partial s} \left( \frac{1}{s-1} \zeta_{p,E,N}(s, x; \bar{\omega}) \right) \bigg|_{s=0}.\]

When \(N = 1\) and \(\omega_1 = 1\) this is the \(p\)-adic Diamond-Euler Log Gamma function
\[\log \Gamma_{D,E,1}(x; 1) = \log \Gamma_{D,E}(x), \quad x \in \C_p \setminus \Z_p\]
(see [18, p. 4235, (2.1)]).

By Proposition 3.3 and (3.15), we have the following result on the analytic of \(\log \Gamma_{D,E,N}(x; \bar{\omega})\) as a function of \(x\).

**Theorem 3.5.** For any \(\omega_1, \ldots, \omega_N \in \C_p^\times\) the function \(\log \Gamma_{D,E,N}(x; \bar{\omega})\) is independent of the choice made to define the \(\langle \cdot \rangle\) function, and is locally analytic as a function of \(x \in \C_p \setminus \Lambda\).

In what follows, we shall give some properties of multiple \(p\)-adic Diamond-Euler Log Gamma function \(\log \Gamma_{D,E,N}(x; \bar{\omega})\).

Using the Volkenborn integral, Diamond [5] [6] gave a definition for the \(p\)-adic Log Gamma functions. Following Diamond, using the multiple fermionic \(p\)-adic integral on \(\Z_p\), we obtain an integral representation of \(\log \Gamma_{D,E,N}(x; \bar{\omega})\), which shows that it is indeed a generalization of the \(p\)-adic Diamond-Euler Log Gamma function defined in [18 (2.1)], as follows:

**Proposition 3.6.** For \(\omega_1, \ldots, \omega_N \in \C_p^\times\) and \(x \in \C_p \setminus \Lambda\) we have
\[\log \Gamma_{D,E,N}(x; \bar{\omega}) = \int_{\Z_p^N} ((x + \bar{\omega} \cdot \bar{t}) \log_p (x + \bar{\omega} \cdot \bar{t}) - (x + \bar{\omega} \cdot \bar{t})) d\mu_{-1}(\bar{t}),\]
where \(\log_p\) is the Iwasawa \(p\)-adic logarithm.

**Proof.** Note that \(\frac{\partial}{\partial x} (x + \bar{\omega} \cdot \bar{t})^{1-s} = -(x + \bar{\omega} \cdot \bar{t})^{1-s} \log_p (x + \bar{\omega} \cdot \bar{t})\). From (3.13) and (3.15) the assertion is clear, since \(\log_p x = \log_p \langle x \rangle\) on \(\C_p^\times\) (see [32, p. 1244, (2.18)]). \(\Box\)
Remark 3.7. Imai [12, Proposition 1] also defined a $p$-adic log multiple $\Gamma$ function and formulated a relationship with a special value of a $p$-adic analogue of the multiple $\zeta$-function, not with the derivatives.

We put

$$D = \frac{\partial}{\partial x}.$$  

Furthermore, if $D^k$ denotes the $n$th derivative, we define

$$\psi_{p,E,N}^{(k)}(x; \bar{\omega}) = D^k \log \Gamma_{D,E,N}(x; \bar{\omega})$$

for $x \in \mathbb{C}_p \setminus \Lambda$. We also denote $\psi_{p,E,N}^{(1)}(x; \bar{\omega})$ by $\psi_{p,E,N}(x; \bar{\omega})$.

**Theorem 3.8.** The function $\log \Gamma_{D,E,N}(x; \bar{\omega})$ has the following properties:

1. For all $x \in \mathbb{C}_p \setminus \Lambda$ the function $\log \Gamma_{D,E,N}(x; \bar{\omega})$ satisfies the difference equation

   $$\log \Gamma_{D,E,N}(x + \omega N; \bar{\omega}) + \log \Gamma_{D,E,N}(x; \bar{\omega}) = 2 \log \Gamma_{D,E,N-1}(x; \omega_1, \ldots, \omega_{N-1}),$$

   with the convention $\log \Gamma_{D,E,0}(x; -) = x(\log_p x - 1)$.

2. For all $c \in \mathbb{C}_p^\times$ and all $x \in \mathbb{C}_p \setminus \Lambda$ we have

   $$\log \Gamma_{D,E,N}(cx; c\bar{\omega}) = c \left( \log \Gamma_{D,E,N}(x; \bar{\omega}) + \frac{x}{\langle x \rangle} \zeta_{p,E,N}(0, x; \bar{\omega}) \log_p c \right).$$

3. For all $x \in \mathbb{C}_p \setminus \Lambda$ and $|x|_p > \|\bar{\omega}\|_p$ we have the reflection function equation

   $$\log \Gamma_{D,E,N}(|\bar{\omega}| - x; \bar{\omega}) + \log \Gamma_{D,E,N}(x; \bar{\omega}) = 0,$$

   where $|\bar{\omega}| = \omega_1 + \cdots + \omega_N$.

4. For all $x \in \mathbb{C}_p \setminus \Lambda$ and all positive odd integers $m$ we have the multiplication formula (distribution relation)

   $$\log \Gamma_{D,E,N}(x; \bar{\omega}) = m \sum_{0 \leq j_i < m} (-1)^{\lfloor \bar{j} \rfloor} \log \Gamma_{D,E,N} \left( \frac{x + \bar{j} \cdot \bar{\omega}}{m} ; \bar{\omega} \right)$$

   $$+ E_{N,1}(x; \bar{\omega}) \log_p m,$$

   where the sum is over all vectors $\bar{j} = (j_1, \ldots, j_N)$ with $0 \leq j_i < m$.

In particular for any $k \in \mathbb{N}$ we have

$$\log \Gamma_{D,E,N}(x; \bar{\omega}) = p^k \sum_{0 \leq j_i < p^k} (-1)^{\lfloor \bar{j} \rfloor} \log \Gamma_{D,E,N} \left( \frac{x + \bar{j} \cdot \bar{\omega}}{p^k} ; \bar{\omega} \right).$$

5. For all $x \in \mathbb{C}_p \setminus \Lambda$ and for $|x|_p > \|\bar{\omega}\|_p$ we have

   $$\psi_{p,E,N}^{(k+1)}(x; \bar{\omega}) = (-1)^{k+1} \int_{\mathbb{Z}_p^N} \frac{1}{(x + \bar{\omega} \cdot \bar{t})^k} d\mu_{N-1}(\bar{t})$$

   and we may write this as

   $$\psi_{p,E,N}^{(k+1)}(x; \bar{\omega}) = (-1)^{k+1} \left( \frac{\langle x \rangle}{x} \right)^k \zeta_{p,E,N}(k + 1, x; \bar{\omega}),$$
where \( k \in \mathbb{N} \). In particular \( \psi_{p,E,N}(x; \bar{\omega}) = \int_{\mathbb{Z}^N_p} \log_p(x + \bar{\omega} \cdot \bar{t}) d\mu_{-1}(\bar{t}) \).

Proof. (1) Suppose that \( N = 0 \). By (3.8) and (3.15), we get

\[
\log \Gamma_{D,E,0}(x; -) = x \langle x \rangle \left( \frac{1}{s-1} \langle x \rangle^{1-s} \right) \bigg|_{s=0} = x \log_p x - x.
\]

Therefore Part (1) is obtained from Theorem 3.4(1) by using (3.15).

(2) From Theorem 3.4(2) we have

\[
(3.18) \quad \log \Gamma_{D,E,N}(cx; c\bar{\omega}) = \frac{cx}{\langle cx \rangle} \frac{\partial}{\partial s} \left( \frac{1}{s-1} \langle cx \rangle^{1-s} \right) \zeta_{p,E,N}(s, cx; c\bar{\omega}) \bigg|_{s=0}.
\]

Therefore by (3.15) and (3.18) we obtain

\[
\log \Gamma_{D,E,N}(cx; c\bar{\omega}) = \frac{cx}{\langle cx \rangle} \frac{\partial}{\partial s} \left( \frac{1}{s-1} \zeta_{p,E,N}(s, x; \bar{\omega}) \right) \bigg|_{s=0} = \frac{cx}{\langle cx \rangle} \zeta_{p,E,N}(0, x; \bar{\omega}) \log_p c + c \log \Gamma_{D,E,N}(x; \bar{\omega}),
\]

which proves (2). In this calculation we have used the fact that \( \langle \cdot \rangle \) is a multiplicative homomorphism.

(3) We will write \( |\bar{\omega}| = \omega_1 + \cdots + \omega_N \). From Theorem 3.4(3) we have

\[
(3.19) \quad \log \Gamma_{D,E,N}(|\bar{\omega}| - x; \bar{\omega}) = \langle \bar{\omega} \rangle - x \frac{\partial}{\partial s} \left( \frac{1}{s-1} \zeta_{p,E,N}(s, |\bar{\omega}| - x; \bar{\omega}) \right) \bigg|_{s=0} = - \frac{1}{x} \langle x \rangle.
\]

Since \( \langle -z \rangle = \langle z \rangle \), we see easily that for \( |x|_p > ||\bar{\omega}||_p \)

\[
\frac{|\bar{\omega}| - x}{\langle |\bar{\omega}| - x \rangle} = \frac{|\bar{\omega}| - x}{\langle \bar{\omega} \rangle} \frac{\langle x \rangle}{\langle x \rangle} = -1.
\]

Hence we obtain, by (3.15) and (3.19), that

\[
\log \Gamma_{D,E,N}(|\bar{\omega}| - x; \bar{\omega}) = - \log \Gamma_{D,E,N}(x; \bar{\omega}),
\]

which proves (3).

(4) Since \( \langle \cdot \rangle \) is a multiplicative homomorphism, we can obtain the following identity

\[
(3.20) \quad \frac{x + \frac{j}{m} \bar{\omega}}{\langle x + \frac{j}{m} \bar{\omega} \rangle} = \frac{\langle m \rangle}{m} \frac{x}{\langle x \rangle}.
\]
Let $m$ be an odd integer. Then by Lemma 2.3, Theorem 3.4(4), Theorem 3.4(5) and (3.20) we have

\begin{equation}
E_{N,k}(x; \omega) = m^k \sum_{0 \leq j_i < m} (-1)^{\bar{j}} E_{N,k} \left( \frac{x + \bar{j} \cdot \omega}{m} ; \omega \right),
\end{equation}

where the sum is over all vectors $\bar{j} = (j_1, \ldots, j_N)$ with $0 \leq j_i < m$. From Theorem 3.4(4) it is easy to see that

\begin{equation}
\frac{\partial}{\partial s} \left( \frac{1}{s - 1} \zeta_{p,E,N}(s, x; \omega) \right) \bigg|_{s=0}
= \frac{\partial}{\partial s} \left( \frac{\langle m \rangle}{s - 1} \sum_{0 \leq j_i < m} (-1)^{\bar{j}} \zeta_{p,E,N} \left( s, \frac{x + \bar{j} \cdot \omega}{m} ; \omega \right) \right) \bigg|_{s=0}.
\end{equation}

In view of (3.15) and (3.20), the left side of (3.22) becomes

\begin{equation}
\left\langle \frac{\partial}{\partial s} \left( \frac{1}{s - 1} \zeta_{p,E,N}(s, x; \omega) \right) \bigg|_{s=0} \right\rangle = \frac{\langle m \rangle}{m} \log \Gamma_{D,E,N}(x; \omega).
\end{equation}

Since $\zeta_{p,E,N}(0, x; \omega) = \langle x \rangle E_{N,1}(x; \omega)$, by (3.15), (3.20), (3.21) with $k = 1$, (3.22) and (3.23), we get

\begin{align*}
\log \Gamma_{D,E,N}(x; \omega) &= \frac{\partial}{\partial s} \left( \frac{1}{s - 1} \zeta_{p,E,N}(s, x; \omega) \right) \bigg|_{s=0} \\
&= \log \Gamma_{D,E,N} \left( \frac{x + \bar{j} \cdot \omega}{m} ; \omega \right) \\
&+ \langle m \rangle \sum_{0 \leq j_i < m} (-1)^{\bar{j}} \log \Gamma_{D,E,N} \left( \frac{x + \bar{j} \cdot \omega}{m} ; \omega \right) \\
&= \log \Gamma_{D,E,N} \left( \frac{x + \bar{j} \cdot \omega}{m} ; \omega \right) \\
&+ \sum_{0 \leq j_i < m} (-1)^{\bar{j}} E_{N,1} \left( \frac{x + \bar{j} \cdot \omega}{m} ; \omega \right) \\
&+ \sum_{0 \leq j_i < m} (-1)^{\bar{j}} \log \Gamma_{D,E,N} \left( \frac{x + \bar{j} \cdot \omega}{m} ; \omega \right),
\end{align*}

which proves (4).

(5) Observe that for given $x$ and $\omega$, we have

\begin{equation}
\frac{\partial}{\partial x} \left[ (x + \omega \cdot \bar{t}) \log_p (x + \omega \cdot \bar{t} - 1) \right] = \log_p (x + \omega \cdot \bar{t}).
\end{equation}

Thus, by Proposition 3.6, we have $\psi_{p,E,N}(x; \omega) = \int_{\mathbb{Z}_p^N} \log_p (x + \omega \cdot \bar{t}) d\mu_{-1}(\bar{t})$ and the first equality of (5) follows by induction. Moreover for $|x|^p > \|\omega\|_p$ we have

\begin{equation}
\langle x + \omega \cdot \bar{t} \rangle = \frac{\langle x \rangle}{x} (x + \omega \cdot \bar{t}) \quad \text{for all } \bar{t} \in \mathbb{Z}_p^N.
\end{equation}
and from the definition of $\zeta_{p,E,N}(s, x; \bar{\omega})$ (3.7) and the first part of (5), we have

$$\psi_{p,E,N}^{(k+1)}(x; \bar{\omega}) = (-1)^{k+1} \int_{\mathbb{Z}_p^N} \frac{1}{(x + \bar{\omega} \cdot \bar{t})^k} d\mu_{-1}(\bar{t})$$

$$= (-1)^{k+1} \left( \frac{x}{x} \right)^k \zeta_{p,E,N}(k+1, x; \bar{\omega}),$$

which is the second equality of (5). □

Now we give computationally efficient formulas for our functions in the case that the argument $x$ has $p$-adic absolute value larger than the norm of $\bar{\omega}$.

**Theorem 3.9.** Suppose $\omega_1, \ldots, \omega_N, x \in \mathbb{C}_p^\times$ and $|x|_p > ||\bar{\omega}||_p$. Then there is an identity of analytic functions

$$\zeta_{p,E,N}(s, x; \bar{\omega}) = \langle x \rangle^{1-s} \sum_{j=0}^{\infty} \left( 1 - \frac{s}{j} \right) E_{N,j}(0; \bar{\omega}) x^{-j}$$

on a disc of positive radius about $s = 0$. If in addition $\omega_1, \ldots, \omega_N, x$ are so chosen to lie in a finite extension $K$ of $\mathbb{Q}_p$ whose ramification index over $\mathbb{Q}_p$ is less than $p - 1$, then this formula is valid for $s \in \mathbb{C}_p$ such that $|s|_p < |\pi|_p^{-1} p^{-1/(p-1)}$, where $(\pi)$ is the maximal ideal of the ring of integers $O_K$ of $K$.

**Proof.** Under the stated hypotheses, for all $\bar{t} \in \mathbb{Z}_p^N$, by (3.24), we have

$$\langle x + \bar{\omega} \cdot \bar{t} \rangle^{1-s} = \left( \frac{x}{x} \right)^{1-s} (x + \bar{\omega} \cdot \bar{t})^{1-s}$$

$$= \langle x \rangle^{1-s} \sum_{j=0}^{\infty} \left( 1 - \frac{s}{j} \right) (\bar{\omega} \cdot \bar{t})^j x^{-j}$$

as an identity of analytic functions. From (2.2), denote by

$$(\bar{\omega} \cdot \bar{t})^j = E_{0,j}(\bar{\omega} \cdot \bar{t}; -).$$

If integrating the above equality with respect to $t_1, \ldots, t_N$, then by (3.7) and Corollary 3.2, we have

$$\zeta_{p,E,N}(s, x; \bar{\omega}) = \int_{\mathbb{Z}_p^N} \langle x + \bar{\omega} \cdot \bar{t} \rangle^{1-s} d\mu_{-1}(\bar{t})$$

$$= \langle x \rangle^{1-s} \sum_{j=0}^{\infty} \left( 1 - \frac{s}{j} \right) x^{-j} \int_{\mathbb{Z}_p^N} (\bar{\omega} \cdot \bar{t})^j d\mu_{-1}(\bar{t})$$

$$= \langle x \rangle^{1-s} \sum_{j=0}^{\infty} \left( 1 - \frac{s}{j} \right) E_{N,j}(0; \bar{\omega}) x^{-j}.$$
Theorem 3.10 (Stirling’s series). Suppose $\omega_1, \ldots, \omega_N, x \in \mathbb{C}_p^\times$ and $|x|_p > \|\bar{\omega}\|_p$. Then
\[
\log \Gamma_{D,E,N}(x; \bar{\omega}) = x(\log_p x - 1) + E_{N,1}(0; \bar{\omega}) \log_p x + \sum_{j=2}^{\infty} \frac{(-1)^j}{j(j-1)} E_{N,j}(0; \bar{\omega}) x^{-j+1}.
\]

Proof. We have the following expansions (see [8, p. 38]):
\[
\langle x \rangle_1^{1-s} = \langle x \rangle (1 - s \log \langle x \rangle + \cdots)
\]
and
\[
\binom{1 - s}{j} = \frac{(-1)^{j+1}}{j(j-1)} s + \cdots,
\]
provided $j \geq 2$. Rewrite the expansion of Theorem 3.9 on a disc of positive radius about $s = 0$ as
\[
\langle x \rangle_1^{1-s} \sum_{j=0}^{\infty} \binom{1 - s}{j} E_{N,j}(0; \bar{\omega}) x^{-j} = \langle x \rangle^{1-s} \left(1 + (1-s)x^{-1}E_{N,1}(0; \bar{\omega}) + \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j(j-1)} s + \cdots \right) x^{-j} E_{N,j}(0; \bar{\omega})\right).
\]
By the definition (3.15), (3.25) and Leibniz’s rule, we obtain
\[
\log \Gamma_{D,E,N}(x; \bar{\omega}) = \frac{x}{\langle x \rangle} \frac\partial{\partial s} \left(\frac{1}{s-1} \zeta_{p,E,N}(s, x; \bar{\omega})\right) \bigg|_{s=0}
\]
\[
= \frac{x}{\langle x \rangle} \frac\partial{\partial s} \left[\frac{\langle x \rangle^{1-s}}{s-1} \left(1 + (1-s)x^{-1}E_{N,1}(0; \bar{\omega}) + \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j(j-1)} s + \cdots \right) x^{-j} E_{N,j}(0; \bar{\omega})\right]\bigg|_{s=0}
\]
\[
= (x \log_p x - x)(1 + x^{-1}E_{N,1}(0; \bar{\omega})) + E_{N,1}(0; \bar{\omega}) + \sum_{j=2}^{\infty} \frac{(-1)^j}{j(j-1)} x^{-j+1} E_{N,j}(0; \bar{\omega}).
\]
This completes the proof. \[\square\]

Remark 3.11. 1. Since $\log \Gamma_{D,E,1}(x; 1) = \log \Gamma_{D,E}(x)$ for $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$ and $E_{1,j}(0;1) = E_j(0)$ for $n \in \mathbb{N}_0$, when $N = 1$ and $\omega_1 = 1$ Theorem 3.10 becomes the known relation [18, Theorem 2.3]
\[
\log \Gamma_{D,E}(x) = x(\log_p x - 1) + E_1(0) \log_p x + \sum_{j=2}^{\infty} \frac{(-1)^j}{j(j-1)} E_j(0) x^{-j+1}.
\]
If $j$ is even, then we have $E_j(0) = 0$, and we may replace the symbol $(-1)^j$ by $-1$ in the above formula.

2. From Theorem 3.10 Stirling’s series expansion for $\log \Gamma_{D,E,N}(x; \bar{\omega})$ agrees exactly with the asymptotic expansion for $\log \Gamma_{E,N}(x; \bar{\omega})$ given by
Theorem 2.2 where the infinite series converges $p$-adically for large $x$ and the error term $R_{N,M}$ vanishes. This phenomenon was first observed by Diamond [5, Theorem 6] for $N = 1$, and by Tangedal and Young for general positive integers $N$ [32, Theorem 4.2].

By differentiating both sides of Theorem 3.10 with respect to $x$, we have the following Laurent series expansion of $\psi_{p,E,N}(x; \bar{\omega})$.

**Corollary 3.12.** Let $\omega_1, \ldots, \omega_N, x \in \mathbb{C}_p^\times$ and $|x|_p > \|\bar{\omega}\|_p$. Then

$$\psi_{p,E,N}(x; \bar{\omega}) = \log_p x + E_{N,1}(0; \bar{\omega}) \frac{1}{x} + \sum_{j=1}^{\infty} \frac{(-1)^j}{j+1} \frac{1}{x^{j+1}} E_{N,j+1}(0; \bar{\omega})$$

and

$$\psi^{(k+1)}_{p,E,N}(x; \bar{\omega}) = (k-1)! \sum_{j=0}^{\infty} (-1)^j \left( -\frac{j-1}{k} \right) \frac{1}{x^{k+j}} E_{N,j}(0; \bar{\omega}),$$

where $k \in \mathbb{N}$.

Although for $x \in \Lambda$ our functions $\zeta_{p,E,N}(s, x; \bar{\omega})$ and $\log \Gamma_{D,E,N}(x; \bar{\omega})$ are undefined, we may extend their definitions to $x \in \Lambda$ and $\|\bar{\omega}\|_p < 1$ as zero values, and denote them by $\zeta^*_p,E,N(s, x; \bar{\omega})$ and $L\log \Gamma_{p,E,N}(x; \bar{\omega})$ (see Kashio [13], Tangedal and Young [32]). The remaining case, when $\|\bar{\omega}\|_p \geq 1$ and $x \in \Lambda$, can be expressed in terms of the above definitions, as we show in what follows.

We consider the function

$$\zeta^*_p,E,N(s, x; \bar{\omega}) = \int_{\mathbb{Z}_p^N} f^*(x, \bar{t}) d\mu_{-1}(\bar{t}),$$

where

$$f^*(x, \bar{t}) = \begin{cases} f(x, \bar{t}) & \text{if } |x + \bar{\omega} \cdot \bar{t}|_p = 1, \\ 0 & \text{if } |x + \bar{\omega} \cdot \bar{t}|_p < 1 \end{cases}$$

and $f(x, \bar{t})$ is as defined in (3.11). Then we define

$$L\log \Gamma_{p,E,N}(x; \bar{\omega}) = \frac{x}{\langle x \rangle} \frac{\partial}{\partial s} \left( \frac{1}{s-1} \zeta^*_p,E,N(s, x; \bar{\omega}) \right) \bigg|_{s=0}. $$

**Theorem 3.13.** Suppose that $\|\bar{\omega}\|_p \geq 1$ and $x \in \Lambda$. Then we have

$$\zeta^*_p,E,N(s, x; \bar{\omega}) = \sum_{0 \leq j_i < p \atop |x + \bar{j} \cdot \bar{\omega}|_p = \|\bar{\omega}\|_p} (-1)^{\sum_j} \zeta_{p,E,N} \left( s, \frac{x + \bar{j} \cdot \bar{\omega}}{p} ; \bar{\omega} \right)$$

and

$$L\log \Gamma_{p,E,N}(x; \bar{\omega}) = \sum_{0 \leq j_i < p \atop |x + \bar{j} \cdot \bar{\omega}|_p = \|\bar{\omega}\|_p} (-1)^{\sum_j} \log \Gamma_{D,E,N} \left( \frac{x + \bar{j} \cdot \bar{\omega}}{p} ; \bar{\omega} \right),$$

where the sums are all over all vectors $\bar{j} = (j_1, \ldots, j_N)$ with $0 \leq j_i < p$ ($i = 1, \ldots, N$) and $|x + \bar{j} \cdot \bar{\omega}|_p = \|\bar{\omega}\|_p$.  

Proof. Let us assume that $||\tilde{\omega}||_p = 1$ and $x \in \Lambda$. From (3.12) and (3.26), we obtain

$$
\zeta_{p,E,N}^*(s, x; \tilde{\omega}) = \int_{\mathbb{Z}_p^n} f^*(x, \tilde{t}) d\mu_{-1}(\tilde{t})
$$

$$
= \sum_{0 \leq j, < p} (-1)^{[j]} \int_{\mathbb{Z}_p^n} f(x, \tilde{\omega} + p\tilde{t}) d\mu_{-1}(\tilde{t})
$$

$$
= \sum_{0 \leq j, < p} (-1)^{[j]} \int_{\mathbb{Z}_p^n} f(x + \tilde{\omega} \cdot t, p\tilde{t}) d\mu_{-1}(\tilde{t})
$$

(3.28)

$$
= \sum_{0 \leq j, < p} (-1)^{[j]} \int_{\mathbb{Z}_p^n} f(x + \tilde{\omega} \cdot \tilde{t}, p\tilde{t}) d\mu_{-1}(\tilde{t})
$$

which proves the first part in the case $||\tilde{\omega}||_p = 1$. The case of general $||\tilde{\omega}||_p \geq 1$ may be deduced from the dilation relation (Theorem 3.4(2) above and [13, Eq. (5.7)]) of $\zeta_{p,E,N}$ and $\zeta_{p,E,N}^*$. The second part follows from (3.27) and the first part. \hfill \Box

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