Research Article

Hermite–Jensen–Mercer Type Inequalities for Caputo Fractional Derivatives

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In this article, certain Hermite–Jensen–Mercer type inequalities are proved via Caputo fractional derivatives. We established some new inequalities involving Caputo fractional derivatives, such as Hermite–Jensen–Mercer type inequalities, for differentiable mappings whose derivatives in the absolute values are convex.

1. Introduction

In recent years, inequality theory attracts many researchers due to its applications in our daily life and within the mathematics [1–9]. Let \(0 < x_1 \leq x_2 \leq \cdots \leq x_n\) and let \(\mu = (\mu_1, \mu_2, \ldots, \mu_n)\) be nonnegative weights such that \(\sum_{r=1}^{n} \mu_r = 1\). The Jensen inequality [10] states that \(h\) is a convex function on the interval \([u, v]\); then,

\[
h\left(\sum_{r=1}^{n} \mu_r x_r\right) = \left(\sum_{r=1}^{n} \mu_r h(x_r)\right),
\]

(1)

where \(\forall x_r \in [u, v]\) and all \(\mu_r \in [0, 1]\), \((r = 1, 2, \ldots, n)\).

The Hermite–Hadamard inequality asserts that if \(h: J \subseteq \mathbb{R} \rightarrow \mathbb{R}\) is a convex function on \(J\) with \(u, v \in J\), \(u < v\), then

\[
h\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_{u}^{v} h(\tau) \, d\tau \leq \frac{h(u) + h(v)}{2}.
\]

(2)

The reverse direction in the above inequality holds when \(h\) is concave.

**Theorem 1** (see [11]). If \(h\) is a convex function on \([u, v]\), then

\[
h\left(u + v - \sum_{r=1}^{n} \mu_r x_r\right) = h(u) + h(v) - \sum_{r=1}^{n} \mu_r h(x_r),
\]

(3)

\(\forall x_r \in [u, v]\) and all \(\mu_r \in [0, 1]\), \((r = 1, 2, \ldots, n)\).

Inequality (3) is known as the Jensen–Mercer inequality. Recently, inequality (3) has been generalized, see ([12–15]). For more recent and related results connected with Jensen–Mercer inequality, see ([11, 16–18]).

The previous era of fractional calculus is as old as the history of differential calculus. Several fractional operators are introduced that generalize ordinary integrals. However, the fractional derivatives have some basic properties than the corresponding classical ones. On the contrary, besides the smooth requirement, the Caputo derivative does not coincide with the classical derivative [19]. Caputo fractional derivatives are introduced by the Italian mathematician Caputo in 1967. Since then, a lot of research involves Caputo fractional derivatives [20–22].

The Caputo fractional derivatives are defined as in [23–26].

**Definition 1**. Suppose \(\alpha > 0\) and \(\alpha \notin \{1, 2, 3, \ldots\}\), \(n = [\alpha] + 1\), and \(h \in C^{\alpha}[u, v]\). The Caputo fractional derivatives of order \(\alpha\) are defined as follows:
\[
\left(\frac{d}{dt}\right)^n h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{\alpha-n+1} h^{(n)}(s) \, ds, \quad z > u,
\]

\[
\left(\frac{d}{dt}\right)^n h(t) = \frac{(-1)^n}{\Gamma(n-C)} \int_z^1 (r-t)^{\alpha-n+1} h^{(n)}(r) \, dr, \quad z < v.
\]

(5)

If \(\alpha = n \in \{1, 2, 3, \ldots\}\) and usual derivatives of \(h\) of order \(n\) exist, then the Caputo fractional derivatives \(\left(\frac{d}{dt}\right)^n h(t)\) coincide with \(h^{(n)}(t)\).

Specifically, we get

\[
\left(\frac{d}{dt}\right)^n h(t) = \left(\frac{d}{dt}\right)^n h(t) = h(t),
\]

where \(n = 1\) and \(\alpha = 0\).

In this article, by using the Jensen–Mercer inequality, we proved Hermite–Hadamard’s inequalities for fractional integrals and established some Hermite–Hadamard type inequalities for differentiable mappings whose derivatives in absolute value are convex.

2. Hermite–Hadamard–Mercer Type Inequalities for Caputo Fractional Derivatives

By using Jensen–Mercer inequalities, Hermite–Hadamard type inequalities can be expressed in Caputo fractional derivatives as follows.

**Theorem 2.** Suppose that a positive function \(h: [u, v] \rightarrow R\) with \(0 \leq u < v\) and \(h \in C^n[u, v]\). If \(h^{(n)}\) is a convex function on \([u, v]\), then the following inequalities for Caputo fractional derivatives hold:

\[
h^{(n)}\left(u + v - \frac{x + y}{2}\right) \leq h^{(n)}(u) + h^{(n)}(v)
\]

\[
- \frac{\Gamma(n-\alpha+1)}{2(y-x)\Gamma(\alpha)} \left\{ \left(\frac{d}{dt}\right)^\alpha h(y) + (-1)^n \left(\frac{d}{dt}\right)^\alpha h(x) \right\}
\]

\[
\leq h^{(n)}(u) + h^{(n)}(v) - h^{(n)}\left(\frac{x + y}{2}\right),
\]

\(\forall x, y \in [u, v], \alpha > 0,\) and \(\Gamma(\cdot)\) is the gamma function.

Proof. Using the Jensen–Mercer inequality, we get

\[
h^{(n)}\left(u + v - \frac{u_1 + v_1}{2}\right) \leq h^{(n)}(u) + h^{(n)}(v)
\]

\[
- \frac{h^{(n)}(u_1) + h^{(n)}(v_1)}{2}, \quad \forall u_1, v_1 \in [a, b].
\]

(8)

Now, by changing of variables \(u_1 = \tau x + (1 - \tau)y\) and \(v_1 = (1 - \tau)x + \tau y\), \(\forall x, y \in [u, v],\) and \(\tau \in [0, 1]\) in (8), we get

\[
h^{(n)}\left(u + v - \frac{x + y}{2}\right) \leq h^{(n)}(u) + h^{(n)}(v)
\]

\[
- \frac{h^{(n)}(\tau x + (1 - \tau)y) + h^{(n)}((1 - \tau)x + \tau y)}{2}.
\]

(9)

Multiplying both sides by \(\tau^{n-\alpha-1}\) and then integrating with respect to \(\tau\) over \([0, 1]\), we get

\[
\frac{1}{\Gamma(n-\alpha)} h^{(n)}\left(u + v - \frac{x + y}{2}\right) \leq \frac{1}{\Gamma(n-\alpha)} \left\{ h^{(n)}(u) + h^{(n)}(v) \right\}
\]

\[
- \frac{1}{2} \left\{ \int_0^1 \tau^{n-\alpha-1} \left[h^{(n)}(\tau x + (1 - \tau)y) + h^{(n)}((1 - \tau)x + \tau y)\right] d\tau \right\}.
\]

(10)

After simplification, we get

\[
h^{(n)}\left(u + v - \frac{x + y}{2}\right)
\]

\[
\leq h^{(n)}(u) + h^{(n)}(v) - \frac{\Gamma(n-\alpha+1)}{2(y-x)\Gamma(\alpha)} \left\{ \left(\frac{d}{dt}\right)^\alpha h(y) + (-1)^n \left(\frac{d}{dt}\right)^\alpha h(x) \right\},
\]

and so the first inequality of (2.1) is proved.

Now, for the proof of the second inequality of (2.1), we first note that if \(h^{(n)}\) is a convex function, then for \(\tau \in [0, 1]\), it gives

\[
h^{(n)}\left(\frac{x + y}{2}\right) = h^{(n)}\left(\frac{\tau x + (1 - \tau)y + (1 - \tau)x + \tau y}{2}\right)
\]

\[
\leq \frac{h^{(n)}(\tau x + (1 - \tau)y) + h^{(n)}((1 - \tau)x + \tau y)}{2}.
\]

(12)

Multiplying both sides by \(\tau^{n-\alpha-1}\) and then integrating the resulting inequality with respect to \(\tau\) over \([0, 1]\), we get

\[
\frac{1}{\Gamma(n-\alpha)} h^{(n)}\left(\frac{x + y}{2}\right) \leq \frac{1}{\Gamma(n-\alpha)} \left\{ \int_0^1 \tau^{n-\alpha-1} \left|h^{(n)}(\tau x + (1 - \tau)y) + h^{(n)}((1 - \tau)x + \tau y)\right| d\tau \right\}
\]

\[
+ h^{(n)}((1 - \tau)x + \tau y))\right|\}
\]

h^{(n)}\left(\frac{x + y}{2}\right) \leq \frac{\Gamma(n-\alpha+1)}{2(y-x)\Gamma(\alpha)} \left\{ \left(\frac{d}{dt}\right)^\alpha h(y) + (-1)^n \left(\frac{d}{dt}\right)^\alpha h(x) \right\}.
\]

(13)

Multiplying by \((-1)^n\) and adding \(h^{(n)}(u) + h^{(n)}(v)\) both sides in (13), we get the second inequality of (2.1), which completes the proof.

□

**Theorem 3.** Suppose that a positive function \(h: [u, v] \rightarrow R\) with \(0 \leq u < v\) and \(h \in C^n[u, v]\). If \(h^{(n)}\) is a convex function on
[u, v], then the following inequalities for Caputo fractional derivatives hold:
\[
h^{(n)}(u + v - \frac{x + y}{2}) \leq \frac{2^{n-\alpha-1} \Gamma(n-\alpha+1)}{(y-x)^{\alpha-n}} \times \left(D^{\alpha}_{(u+v-(x+y)/2)}h\right)(u + v - x) + (-1)^{\alpha} \left(D^{\alpha}_{(u+v-(x+y)/2)}h\right)(u + v - y)
\]
\[
+ \frac{1}{2} \left(h^{(n)}(x) + h^{(n)}(y)\right).
\]
(14)

where \(\forall x, y \in [u, v]\), \(\alpha > 0\), and \(\Gamma(\cdot)\) is the gamma function.

Proof. To prove the first part of inequality, we use convexity of \(h^{(n)}\) to get
\[
h^{(n)}(u + v - \frac{u_1 + v_1}{2}) = h^{(n)}\left(\frac{u + v - u_1 + u + v - v_1}{2}\right)
\]
\[
\leq \frac{1}{2} h^{(n)}(u + v - u_1) + h^{(n)}(u + v - v_1),
\]
(15)
for all \(u_1, v_1 \in [u, v]\). Now by writing variables \(u_1 = (\tau/2)x + (2 - \tau/2)y\) and \(v_1 = (2 - \tau/2)x + (\tau/2)y\) for \(x, y \in [u, v]\) and \(\tau \in [0, 1]\), we get
\[
2h^{(n)}\left(u + v - \frac{x + y}{2}\right) \leq h^{(n)}\left(u + v - \left(\frac{\tau}{2}x + \frac{2 - \tau}{2}y\right)\right)
\]
\[
+ h^{(n)}\left(u + v - \left(\frac{2 - \tau}{2}x + \frac{\tau}{2}y\right)\right).
\]
(16)

Multiplying both sides of (16) by \(\tau^{n-\alpha-1}\) and then integrating with respect to \(\tau\) over \([0, 1]\), we get
\[
2h^{(n)}\left(u + v - \frac{x + y}{2}\right) \int_0^1 \tau^{n-\alpha-1} \, d\tau \leq \int_0^1 \tau^{n-\alpha-1} \left(h^{(n)}\left(u + v - \left(\frac{\tau}{2}x + \frac{2 - \tau}{2}y\right)\right) + h^{(n)}\left(u + v - \left(\frac{2 - \tau}{2}x + \frac{\tau}{2}y\right)\right)\right) \, d\tau.
\]
(17)

Further simplifying gives
\[
h^{(n)}\left(u + v - \frac{x + y}{2}\right) \leq \frac{2^{n-\alpha-1} \Gamma(n-\alpha+1)}{(y-x)^{\alpha-n}} \times \left(D^{\alpha}_{(u+v-(x+y)/2)}h\right)(u + v - x) + (-1)^{\alpha} \left(D^{\alpha}_{(u+v-(x+y)/2)}h\right)(u + v - y)
\]
\[
\]
\[
+ \frac{1}{2} \left(h^{(n)}(x) + h^{(n)}(y)\right).
\]
(18)

and so the first inequality of (14) is proved.

Now, for the proof of the second inequality of (14), we first note that if \(h^{(n)}\) is a convex function, then by employing Jensen–Mercer inequality (3) for \(\tau \in [0, 1]\) gives
\[
h^{(n)}\left(u + v - \left(\frac{\tau}{2}x + \frac{2 - \tau}{2}y\right)\right) \leq \frac{\tau}{2} h^{(n)}(x) + \frac{2 - \tau}{2} h^{(n)}(y).
\]
(19)

By adding the inequalities of (19) and (20), we get
\[
h^{(n)}\left(u + v - \left(\frac{\tau}{2}x + \frac{2 - \tau}{2}y\right)\right) + h^{(n)}\left(u + v - \left(\frac{2 - \tau}{2}x + \frac{\tau}{2}y\right)\right) \leq 2\left(h^{(n)}(u) + h^{(n)}(v)\right) - \left(h^{(n)}(x) + h^{(n)}(y)\right).
\]
(21)

Multiplying both sides by \(\tau^{n-\alpha-1}\) and then integrating over \(\tau \in [0, 1]\), we get
\[
\int_0^1 \tau^{n-\alpha-1} \left(h^{(n)}\left(u + v - \left(\frac{\tau}{2}x + \frac{2 - \tau}{2}y\right)\right) + h^{(n)}\left(u + v - \left(\frac{2 - \tau}{2}x + \frac{\tau}{2}y\right)\right)\right) \, d\tau \leq \left\{2\left(h^{(n)}(u) + h^{(n)}(v)\right) - \left(h^{(n)}(x) + h^{(n)}(y)\right)\right\} \int_0^1 \tau^{n-\alpha-1} \, d\tau.
\]
(22)

After simplification, we get
\[
\frac{2^{n-\alpha-1} \Gamma(n-\alpha+1)}{(y-x)^{\alpha-n}} \times \left(D^{\alpha}_{(u+v-(x+y)/2)}h\right)(u + v - x) + (-1)^{\alpha} \left(D^{\alpha}_{(u+v-(x+y)/2)}h\right)(u + v - y)
\]
\[
\leq \left(h^{(n)}(u) + h^{(n)}(v)\right) - \frac{1}{2} \left(h^{(n)}(x) + h^{(n)}(y)\right).
\]
(23)

Now, concatenating (18) and (23), we get (14).

Now, we introduce some new lemmas involving Caputo fractional derivatives.

Lemma 1. Suppose that \(h: [u, v] \rightarrow R\) is a differentiable mapping on \((u, v)\) with \(u < v\) and \(h \in C^{n+1} [u, v]\), then the following equality for Caputo fractional derivatives holds:
where $\forall x, y \in [u, v]$, $\alpha > 0$, $\tau \in [0, 1]$, and $\Gamma(\cdot)$ is the gamma function.

Proof. It suffices to note that
\[
I = \frac{y-x}{2} [I_1 - I_2],
\] (25)
where

\[
I_1 = \int_0^1 \left( y^{-\alpha} \right) h^{n+1}(u + v - (\tau x + (1 - \tau) y)) d\tau
\]
and replacing the values of $I_1$ and $I_2$ in (25), we get (24). \(\square\)

Remark 1. If we take $x = u$ and $y = v$ in Lemma 1, then it reduces to Remark 2.5 in [25].

Lemma 2. Suppose that $h: [u, v] \rightarrow \mathbb{R}$ is a differentiable mapping on $(u, v)$ with $u < v$ and $h \in C^{n+1}([u, v])$, then the following equality for Caputo fractional derivatives holds:

\[
\begin{align*}
  h^{(n)}(u + v - x + y) &- \frac{2^{n-1} \Gamma(n - \alpha + 1)}{(y - x)^n} (\xi) h^{n+1}(u + v - (\tau x + (1 - \tau) y)) d\tau \\
  &= \frac{y-x}{4} \int_0^1 \left( y^{-\alpha} h^{n+1} \left( u + v - \left( \frac{\tau x + 2 - \tau}{2} y \right) \right) d\tau - \int_0^1 \left( y^{-\alpha} h^{n+1} \left( u + v - \left( \frac{2 - \tau x + 2 \tau}{2} y \right) \right) d\tau \right) \right].
\end{align*}
\] (27)

Proof. It suffices to note that

\[
I = \frac{y-x}{4} \{I_1 - I_2\},
\] (28)

\[
I_1 = \int_0^1 \left( y^{-\alpha} h^{n+1} \left( u + v - \left( \frac{\tau x + 2 - \tau}{2} y \right) \right) d\tau
\]

\[
= \frac{2}{y-x} h^{(n)} \left( u + v - \left( \frac{x + y}{2} \right) \right) - \frac{2(n-\alpha)}{y-x} \int_0^1 h^{(n)} \left( u + v - \left( \frac{\tau x + 2 - \tau}{2} y \right) \right) d\tau
\]

\[
= \frac{2}{y-x} h^{(n)} \left( u + v - \left( \frac{x + y}{2} \right) \right) - (-1)^n \frac{2^{n-1} \Gamma(n - \alpha + 1)}{y-x} (\xi) h^{n+1} \left( u + v - \left( \frac{\tau x + 2 - \tau}{2} y \right) \right) d\tau,
\] (29)
\[
I_2 = \int_0^1 t^{n-\alpha-1} h(t) \left( u + v - \frac{2}{2} x + \frac{\tau}{2} y \right) \, dt \\
= -\frac{2}{y-x} h^{(n)} \left( u + v - x + y \right) + \frac{2(n-\alpha)}{y-x} \int_0^1 t^{n-\alpha-1} h^{(n)} \left( u + v - \frac{2}{2} x + \frac{\tau}{2} y \right) \, dt \\
= -\frac{2}{y-x} h^{(n)} \left( u + v - x + y \right) + \frac{2^{n-\alpha+1} \Gamma(n-\alpha+1)}{(y-x)^{\alpha+1}} \left( c D_a^{\alpha-\alpha} \right) (u + v - x) 
\] (30)

and combining (29) and (30) with (28), we get (27).

**Remark 2.** If we take \( x = u \) and \( y = v \) in Lemma 2, then it reduces to Lemma 2 in [24].

\[
\left| \frac{h^{(n)}(u + v - x) + h^{(n)}(u + v - y)}{2} - \frac{\Gamma(n-\alpha+1)}{2(y-x)^{\alpha+1}} \left( c D_a^{\alpha-\alpha} \right) (u + v - x) \right| \\
\leq \frac{y-x}{n-\alpha+1} \left( |h^{n+1}(u)| + |h^{n+1}(v)| - \left( |h^{n+1}(x)| + |h^{n+1}(y)| \right) \right) 
\] (31)

where \( \forall \, x, y \in [u, v], \alpha > 0, \tau \in [0, 1] \), and \( \Gamma(\cdot) \) is the gamma function.

**Proof.** By using Lemma 1 and the Jensen–Mercer inequality, we get

\[
\left| \frac{h^{(n)}(u + v - x) + h^{(n)}(u + v - y)}{2} - \frac{\Gamma(n-\alpha+1)}{2(y-x)^{\alpha+1}} \left( c D_a^{\alpha-\alpha} \right) (u + v - x) \right| \\
\leq \frac{y-x}{2} \int_0^1 t^{n-\alpha-1} - (1-t)^{n-\alpha} |h^{n+1}(u + v - (\tau x + (1-\tau) y))| \, d\tau \\
\leq \frac{y-x}{2} \left[ I_1 + I_2 \right] 
\] (32)

where

\[
I_1 = \int_0^1 ((1-t)^{n-\alpha-1} - t^{n-\alpha}) \left( |h^{n+1}(u)| + |h^{n+1}(v)| - \left( (1-t)^{n+1}(x) + (1-t)^{n+1}(y) \right) \right) \, dt \\
= \left( |h^{n+1}(u)| + |h^{n+1}(v)| \right) \left( \frac{1}{n-\alpha+1} - \frac{2^{n-\alpha}}{n-\alpha+1} \right) \\
- \left( |h^{n+1}(x)| + |h^{n+1}(y)| \right) \left( \frac{1}{n-\alpha+2(n-\alpha+2)} + \frac{2^{n-\alpha+1}}{n-\alpha+2} \right) 
\]
\[ I_2 = \int_0^1 \left( x^{\alpha-1} - (1-x)^{\alpha-1}\right) \left[ |h^{n+1}(u)| + |h^{n+1}(v)| - \left( \frac{\tau|h^{n+1}(x)| + (1-\tau)|h^{n+1}(y)|}{2} \right) \right] \]

and putting the values of \( I_1 \) and \( I_2 \) in (32), we get (31). □

**Remark 3.** If we take \( x = u \) and \( y = v \) in Theorem 4, then it reduces to Corollary 2.7 in [25].

\[
\left| h^{(\alpha)}(u + v - \frac{x+y}{2}) - \frac{2^n - 1}{(y-x)^{\alpha-1}} \left[ (\mathcal{D}_{[u+v-\tau(x+y)/2]}^\alpha h)(u+v-x) + (1-\tau)(\mathcal{D}_{[u+v-\tau(x+y)/2]}^\alpha h)(u+v-y) \right] \right| \leq \frac{y-x}{2} \left\{ |h^{n+1}(u)| + |h^{n+1}(v)| - \left( \frac{\tau|h^{n+1}(x)| + (1-\tau)|h^{n+1}(y)|}{2} \right) \right\},
\]

where \( \forall x, y \in [u,v], \alpha > 0, \tau \in [0,1], \) and \( \Gamma(\cdot) \) is the gamma function. **Proof.** By using Lemma 2 and the Jensen–Mercer inequality, we get

\[
\left| h^{(\alpha)}(u + v - \frac{x+y}{2}) - \frac{2^n - 1}{(y-x)^{\alpha-1}} \left[ (\mathcal{D}_{[u+v-\tau(x+y)/2]}^\alpha h)(u+v-x) + (1-\tau)(\mathcal{D}_{[u+v-\tau(x+y)/2]}^\alpha h)(u+v-y) \right] \right| \leq \frac{y-x}{4} \left\{ \int_0^1 \tau^{\alpha-1} \left[ h^{n+1}(u + v - \frac{\tau x + 2-\tau}{2} y) \right] \right\} \, d\tau 
\]

\[
\leq \frac{y-x}{4} \left\{ \int_0^1 \tau^{\alpha-1} \left[ h^{n+1}(u) + h^{n+1}(v) \right] - \left( \frac{\tau}{2} h^{n+1}(x) + \frac{(2-\tau)}{2} h^{n+1}(y) \right) \right\} \, d\tau 
\]

\[
\leq \frac{y-x}{2(n-\alpha+1)} \left\{ h^{n+1}(u) + h^{n+1}(v) \right\},
\]

**Theorem 5.** Suppose that \( h: [u,v] \rightarrow R \) is a differentiable mapping on \( (u,v) \) with \( u < v \) and \( h \in C^{n+1}([u,v]). \) If \( |h^{n+1}| \) is a convex function on \( [u,v], \) then the following inequality for Caputo fractional derivatives holds:
which completes the proof. \hfill □

**Theorem 6.** Suppose that \( h: \mathbb{[u,v]} \rightarrow \mathbb{R} \) is a differentiable mapping on \( (u,v) \) with \( u < v \) and \( h \in C^{n+1} [u,v] \). If \( |h^{n+1}|^q \) is a convex function on \([u,v]\) and \( q > 1 \), then the following inequality for Caputo fractional derivatives holds:

\[
\left| h^{(n)}(u + v - \frac{x + y}{2}) - \frac{2^{n-a-1}(n - \alpha + 1)}{\Gamma(n - \alpha + 1)} \left[ (\int_{(u-v)(x+y)/2)}^{x} h(t) dt \right] (u + v - x) + (-1)^q \left( \int_{(u-v)(x+y)/2)}^{x} h(t) dt \right) (u + v - y) \right| \leq \frac{y - x}{16} \left( \frac{4}{np - \alpha p + 1} \right) \left[ 4^{(1/q)} \cdot 2 \left( |h^{n+1}(u)| + |h^{n+1}(v)| \right) - (3^{(1/q)} + 1) \left( |h^{n+1}(x)| + |h^{n+1}(y)| \right) \right],
\]

where \( \forall x, y \in [u,v], \alpha > 0, \tau \in [0,1], \) and \( I(\cdot) \) is the gamma function.

**Proof.** By using Lemma 2 and applying the famous Hölder integral inequality, we get

\[
\left| h^{(n)}(u + v - \frac{x + y}{2}) - \frac{2^{n-a-1}(n - \alpha + 1)}{\Gamma(n - \alpha + 1)} \left[ (\int_{(u-v)(x+y)/2)}^{x} h(t) dt \right] (u + v - x) + (-1)^q \left( \int_{(u-v)(x+y)/2)}^{x} h(t) dt \right) (u + v - y) \right| \leq \frac{y - x}{4} \left( \frac{1}{np - \alpha p + 1} \right) \left[ (\int_{0}^{1} t^{(n-a)p} dt) \cdot \left( \int_{0}^{1} |h^{n+1}(u + v - (\frac{\tau}{2} x + \frac{2 - \tau}{2} y))|^q d\tau \right) \right]
\]

By applying Minkowski’s inequality, we get

\[
\left| h^{(n)}(u + v - \frac{x + y}{2}) - \frac{2^{n-a-1}(n - \alpha + 1)}{\Gamma(n - \alpha + 1)} \left[ (\int_{(u-v)(x+y)/2)}^{x} h(t) dt \right] (u + v - x) + (-1)^q \left( \int_{(u-v)(x+y)/2)}^{x} h(t) dt \right) (u + v - y) \right| \leq \frac{y - x}{16} \left( \frac{4}{np - \alpha p + 1} \right) \left[ (4^{(1/q)} \cdot 2 \left( |h^{n+1}(u)| + |h^{n+1}(v)| \right) - (3^{(1/q)} + 1) \left( |h^{n+1}(x)| + |h^{n+1}(y)| \right) \right]
\]
which completes the proof.

3. New Hölder’s and Improved Iscan Inequalities

Theorem 7. Suppose that $h: [u, v] \rightarrow R$ is a differentiable mapping on $(u, v)$ with $u < v$ and $h \in C^{n+1} [u, v]$. If $|h^{n+1}|^q$ is a convex function on $[u, v]$ and $q > 1$, then the following inequality for Caputo fractional derivatives holds:

\[
\left| h^{(n)}\left(u + v - \frac{x + y}{2}\right) - \frac{2^{n-a-1} \Gamma (n - \alpha + 1)}{(y - x)^{n-a}} \left[ (c D^a_{(u+v-(x+y)/2)}) h \right](u + v - x) + (-1)^n (c D^a_{(u+v-(x+y)/2)}) h \right)(u + v - y) \right|
\]

\[
\leq \frac{y - x}{4} \left\{ \left( \left( \frac{1}{((n - \alpha) p + 1)((n - \alpha) p + 2)} \right)^{(1/p)} \right) \left( \frac{1}{2} \left( |h^{n+1}(u)|^q + |h^{n+1}(v)|^q \right) \right)^{(1/q)} \right. 
\]

\[
- \left( \frac{1}{12} |h^{n+1}(x)|^q + \frac{5}{12} |h^{n+1}(y)|^q \right)^{(1/q)} + \left( \frac{1}{((n - \alpha) p + 1)((n - \alpha) p + 2)} \right)^{(1/p)} \left( \frac{1}{2} \left( |h^{n+1}(u)|^q + |h^{n+1}(v)|^q \right) \right)^{(1/q)} 
\]

\[
- \left( \frac{1}{6} |h^{n+1}(x)|^q + \frac{1}{3} |h^{n+1}(y)|^q \right)^{(1/q)} \right\} + \left\{ \left( \frac{1}{((n - \alpha) p + 1)(n - \alpha) p + 2) \right)^{(1/p)} \left( \frac{1}{2} |h^{n+1}(u)|^q \right)^{(1/q)} 
\]

\[
+ |h^{n+1}(v)|^q - \left( \frac{5}{12} |h^{n+1}(x)|^q + \frac{1}{12} |h^{n+1}(y)|^q \right)^{(1/q)} 
\]

\[
\left( \frac{1}{((n - \alpha) p + 2)} \right)^{(1/p)} \right\} \left( \frac{1}{2} \left( |h^{n+1}(u)|^q + |h^{n+1}(v)|^q \right) \right)^{(1/q)} \right\}, 
\]

where $\forall x, y \in [u, v], \alpha > 0$, $\tau \in [0, 1]$, and $\Gamma (\cdot)$ is the gamma function.

Proof. By using Lemma 2 with Jensen–Mercer inequality and applying the Hölder–Iscan integral inequality (Theorem 1.4 [27]), we get

\[
\left| h^{(n)}\left(u + v - \frac{x + y}{2}\right) - \frac{2^{n-a-1} \Gamma (n - \alpha + 1)}{(y - x)^{n-a}} \left[ (c D^a_{(u+v-(x+y)/2)}) h \right](u + v - x) + (-1)^n (c D^a_{(u+v-(x+y)/2)}) h \right)(u + v - y) \right|
\]

\[
\leq \frac{y - x}{4} \left\{ \left( \left( \int_0^1 (1 - \tau) t^{np-\alpha p} d\tau \right)^{(1/p)} \right) \left( \int_0^1 (1 - \tau) |h^{n+1}(u + v - \left( \frac{\tau x + \frac{2 - \tau}{2} y}{2} \right))|^q d\tau \right)^{(1/q)} 
\]

\[
+ \left( \int_0^1 (1 - \tau) t^{np-\alpha p} d\tau \right)^{(1/p)} \left( \int_0^1 (1 - \tau) |h^{n+1}(u + v - \left( \frac{\tau x + \frac{2 - \tau}{2} y}{2} \right))|^q d\tau \right)^{(1/q)} 
\]

\[
+ \left( \int_0^1 (1 - \tau) t^{np-\alpha p} d\tau \right)^{(1/p)} \left( \int_0^1 (1 - \tau) f^{n+1}(u + v - \left( \frac{2 - \tau}{2} x + \frac{\tau}{2} y \right))|^q d\tau \right)^{(1/q)} 
\]

\[
+ \left( \int_0^1 (1 - \tau) t^{np-\alpha p} d\tau \right)^{(1/p)} \left( \int_0^1 (1 - \tau) f^{n+1}(u + v - \left( \frac{2 - \tau}{2} x + \frac{\tau}{2} y \right))|^q d\tau \right)^{(1/q)} 
\].

By the convexity of $|h^{n+1}|^q$,
\[
|h^{n+1}(u + v - \left(\frac{r}{2} x + \frac{2 - r}{2} y\right))|^q \leq |h^{n+1}(u)|^q + |h^{n+1}(v)|^q - \left(\frac{1}{2} |h^{n+1}(x)|^q + \frac{2 - r}{2} |h^{n+1}(y)|^q\right).
\]
(41)

\[
\begin{align*}
&\left|h^{(n)}(u + v - \left(\frac{r}{2} x + \frac{2 - r}{2} y\right)) - \frac{2^{n-1}(n-\alpha+1)}{(y-x)^{n-\alpha}} \left[\langle cD^a_{(u+v-(x+y)/2)} h \rangle (u + v - x) + (-1)^q \langle cD^a_{(u+v-(x+y)/2)} h \rangle (u + v - y)\right]\right|
\end{align*}
\]
\[
\leq \frac{y - x}{4} \left\{ \left(\frac{1}{(n-\alpha+1)(n-\alpha+2)}\right)^{1/(1-q)} \times \left(\frac{|h^{n+1}(u)|^q + |h^{n+1}(v)|^q}{(n-\alpha+1)(n-\alpha+2)} - \left(\frac{|h^{n+1}(x)|^q}{2(n-\alpha+2)(n-\alpha+3)} + \frac{|h^{n+1}(y)|^q}{2(n-\alpha+2)(n-\alpha+3)}\right)\right) \right. \\
&\quad + \left. \left(\frac{1}{(n-\alpha+2)}\right)^{1/(1-q)} \times \left(\frac{|h^{n+1}(a)|^q + |h^{n+1}(v)|^q}{(n-\alpha+1)(n-\alpha+2)} - \left(\frac{|h^{n+1}(x)|^q}{2(n-\alpha+2)(n-\alpha+3)} + \frac{|h^{n+1}(y)|^q}{2(n-\alpha+2)(n-\alpha+3)}\right)\right)\right\}.
\]
(42)

where $\forall x, y \in [u, v]$, $\alpha > 0$, $r \in [0, 1]$, and $\Gamma(\cdot)$ is the gamma function.

Proof. By using Lemma 2 with Jensen–Mercer inequality and applying the improved power-mean integral inequality (Theorem 1.5 [27]), we get

By using calculus tools, one can have required result. \qed

Theorem 8. Suppose that $h: [u, v] \rightarrow R$ is a differentiable mapping on $(u, v)$ with $u < v$ and $h \in C^{n+1} [u, v]$. If $|h^{n+1}|^q$ is a convex function on $[u, v]$, $p > 1$, and $q = (p/p - 1)$, then the following inequality for Caputo fractional derivatives holds:
After some simplifications, we get required result. □

4. Conclusion

In this paper, we presented Hermite–Hadamard–Mercer inequalities for convex functions via Caputo fractional derivatives. We also developed some new bounds using Hölder–Iscan and improved power-mean integral inequalities. Our results will attract attentions of many researchers working in the field of inequalities and enable them to think further for other generalized convex functions. One may think to extend these results for higher order convex functions. The works above can also build up for a convex function of two variables. For further directions, we refer to [28–33].

Data Availability

All data are included within this paper.

Conflicts of Interest

The authors of this paper declare that they have no conflicts of interest.

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