Asymptotic Stability of the Landau–Lifshitz Equation

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Abstract

The Landau–Lifshitz equation describes the behaviour of magnetic domains in ferromagnetic structures. Recently such structures have been found to be favourable for storing digital data. Stability of magnetic domains is important for this. Consequently, asymptotic stability of the equilibrium points in the Landau–Lifshitz equation are established. A suitable Lyapunov function is presented.

Key words: Asymptotic stability, Equilibrium Points, Hysteresis, Lyapunov function, Nonlinear control systems, Partial differential equations

1 Introduction

The Landau–Lifshitz equation is a coupled set of nonlinear partial differential equations. One of its first appearances is in a 1935 paper [17], in which this equation describes the behaviour of magnetic domains within a ferromagnetic structure. In recent applications, structures such as ferromagnetic nanowires have appeared in electronic devices used for storing digital information [22]. In particular, data is encoded as a specific pattern of magnetic domains within a ferromagnetic nanowire. Consequently the Landau–Lifshitz equation continues to be widely explored, and its stability is of growing interest [3,6,7,10,12,13,14,15,16,18,25]. Stability of equilibrium points is also related to hysteresis [9,20], and investigating stability lends insights into the hysteretic behaviour that appears in the Landau–Lifshitz equation [4,9,23,24].

Stability results are often based on linearization [6,12,16]; however, in the proceeding discussion, asymptotic stability of the Landau-Lifshitz equation is established using Lyapunov theory. This is preferred because linearization leads only to an approximation.

The difficulty with Lyapunov theory is often in the construction of an appropriate Lyapunov function. Working in infinite-dimensions makes this more difficult; however, Lyapunov functions have been found for the Landau–Lifshitz equation [10,13]. In both these works, a Lyapunov function establishes that the equilibrium points of the Landau-Lifshitz equation are stable. The work in [10] is extended here and asymptotic stability is shown. In particular, a nonlinear control is shown to steer the system to an asymptotically stable equilibrium point. Control of the Landau–Lifshitz equation is crucial as this means the behaviour of the domain walls, which contains the encoded data, can be fully determined [22].

The control objective is to steer the system dynamics to any arbitrary equilibrium point, which requires asymptotic stability. This is presented in Theorem 4, which is the main result and can be found in Section 3. A summary and future avenues are in the last section. To begin, a brief mathematical review of the Landau–Lifshitz equation is discussed next.

2 Landau-Lifshitz Equation

Consider a one dimensional ferromagnetic nanowire of length \( L > 0 \). Let \( \mathbf{m}(x,t) = (m_1(x,t), m_2(x,t), m_3(x,t)) \) be the magnetization of the ferromagnetic nanowire for some position \( x \in [0,L] \) and time \( t \geq 0 \) with initial magnetization \( \mathbf{m}(x,0) = \mathbf{m}_0(x) \). These dynamics are determined by

\[
\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{u}) - \nu \mathbf{m} \times (\mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{u})) \tag{1}
\]

\[
\mathbf{m}_x(0,t) = \mathbf{m}_x(L,t) = \mathbf{0}, \tag{2}
\]

\[
||\mathbf{m}(x,t)||_2 = 1. \tag{3}
\]

where \( \times \) denotes the cross product and \( || \cdot ||_2 \) is the Euclidean norm. Equation (1) is the one-dimensional controlled Landau–Lifshitz equation [2,9,10,11,13]. It satisfies the constraint in (3), which means the magnitude of
the magnetization is uniform at every point of the ferromagnet. The exchange energy is $m_{xx}$. Mathematically, $m_{xx}$ denotes magnetization differentiated with respect to $x$ twice. The parameter $\nu \geq 0$ is the damping parameter, which depends on the type of ferromagnet. The applied magnetic field, denoted $u(t)$, acts as the control, and hence, when $u(t) = 0$, equation (1) can be thought of as the uncontrolled Landau–Lifshitz equation. Neumann boundary conditions (2) are used here.

Existence and uniqueness results can be found in [1,5,8,10] and references therein. Solutions to (1) are defined on $L^2 = L_2([0, L]; \mathbb{R}^3)$ with the usual inner-product and norm with domain

$$D = \{ m \in L^2_0 : m_x \in L^2_0, \quad m_{xx} \in L^2_0, m_x(0) = m_x(L) = 0 \}.$$ 

The notation $\| \cdot \|_{L^2_0}$ is used for the norm.

3 Asymptotic Stability

For $u(t) = 0$, the set of equilibrium points is

$$E = \{ a = (a_1, a_2, a_3) : a_1, a_2, a_3 \text{ constants and } \|a\|_2 = 1 \},$$

which satisfies the boundary conditions in (2) [13, Theorem 6.1.1]. It is clear $E$ contains an infinite number of equilibria. A particular $a \in E$ is stable but not asymptotically stable [13, Proposition 6.2.1]; however, the set $E$ is asymptotically stable in the $L^2_0$-norm [8,10].

Let $r$ be an arbitrary equilibrium point of $E$ with $r_1 \neq 0$ to ensure $\|r\|_2 = 1$, that is, (3), is satisfied. Define the control in (1) to be

$$u = k(r - m)$$

where $k$ is a scalar constant. This is the same control used in [10]. For this control, $r$ is an equilibrium point of the controlled Landau-Lifshitz equation (1). It is shown in Theorem 4 that $r$ is locally asymptotically stable, which is the main result. Simulations demonstrating asymptotic stability of the Landau-Lifshitz equation given the control in (5) are shown in [10].

The following lemmas are needed in Theorem 4. Lemmas 1 and 2 appear in [10], which can be obtained from the product rule and integration by parts, respectively.

**Lemma 1.** For $m \in L^2_0$, the derivative of $g = m \times m_x$ is $g_x = m \times m_{xx}$.

**Lemma 2.** For $m \in L^2_0$ satisfying (2),

$$\int_0^L (m - r)^T (m \times m_{xx}) dx = 0.$$
Substituting (8) and (9) into equation (7) leads to
\[ f(k)h^T m - m^T x T m = f(k)h^T (m \times m_{xx}) - \nu f(k) (m \times m_{xx})^T (h \times m) \]
\[ = - \nu f(k)||m \times h||^2_2 + k m^T x (m \times h) - \nu m \times m_{xx}||^2_2 \]
\[ = - \nu m^T x (m \times h) - \nu m^T x (m \times m_{xx}) \]
\[ = - k m^T x (m \times h) - \nu m^T x (m \times m_{xx}) \]
\[ = - k m^T x (m \times h) + \nu m \times m_{xx}||^2_2 \]
\[ + \nu (m \times h)^T (m \times m_{xx}) \].

(9)

Substituting (8) and (9) into equation (6) leads to
\[ \begin{align*}
\frac{dV}{dt} &= f(k)k \int_0^L ||m \times m_{xx}||^2_2 dx
\end{align*} \]
\[ - \nu f(k)k \int_0^L ||m \times h||^2_2 dx
\]
\[ - \nu f(k)k \int_0^L ||m \times m_{xx}||^2_2 dx
\]
\[ \frac{dV}{dt} \leq \nu f(k)k \int_0^L ||m \times m_{xx}||^2_2 dx
\]
\[ - \nu f(k)k \int_0^L ||m \times h||^2_2 dx
\]
\[ - \nu f(k)k \int_0^L ||m \times m_{xx}||^2_2 dx
\]
\[ \frac{dV}{dt} \leq \nu f(k)k ||m \times m_{xx}||^2_2 - \nu f(k)k ||m \times h||^2_2.
\]

Substituting this expression into equation (6) leads to
\[ \frac{dV}{dt} \leq \nu f(k)k ||m \times m_{xx}||^2_2 - \nu f(k)k ||m \times h||^2_2.
\]

Since \( ||m \times h||^2_2 \leq 1 \) from Lemma 3, then
\[ \frac{dV}{dt} \leq \nu f(k)k ||m \times m_{xx}||^2_2 - \nu f(k)k ||m \times h||^2_2.
\]

Since \( f(k)k \leq 1 \), then
\[ \frac{dV}{dt} \leq \nu ||m \times m_{xx}||^2_2.
\]

which is clearly less than or equal to zero, and the value of \( k \) can be any number in the interval (0, 1/2]. For instance, picking \( k = 1/4 \), the Lyapunov function is
\[ V(m) = \frac{1}{8} ||m \times m_{xx}||^2_2.
\]

Revisiting equation (10), if \( m = r \), then \( h = 0 \) and hence \( dV/dt = 0 \). On the other hand, \( dV/dt = 0 \) implies \( m \times h = 0 \), and hence \( r \times m = 0 \). This is a system of algebraic equations,
\[ \begin{align*}
r_2 m_3 - r_3 m_2 &= 0
r_3 m_1 - r_1 m_3 &= 0
r_1 m_2 - r_2 m_1 &= 0.
\end{align*} \]

The solution is \( m_2 = r_2 m_1 \) and \( m_3 = r_3 m_1 \) for any \( m_1 \) and \( r_1 \). Given \( ||m||_2 = 1 \) and \( ||r||_2 = 1 \), this leads to \( m_1^2 = r_1^2 \) and hence \( m_1 = \pm r_1 \), which leads to \( m_2 = \pm r_2 \) and \( m_3 = \pm r_3 \); that is, \( m = \pm r \). For \( V(m) \) on \( B(r,p) \), this implies \( m = r \) if \( dV/dt = 0 \). Local asymptotic stability follows from Lyapunov’s Theorem [19, Theorem 6.2.1].
4 Discussions

Asymptotic stability of an arbitrary equilibrium point of the Landau–Lifshitz equation with Neumann boundary conditions is shown in Theorem 4. This is established using Lyapunov theory. The result in Theorem 4 is an extension of the work presented in [10].

The control given in (5) can be used to control the hysteresis that often appears in magnetization dynamics including those described by the Landau–Lifshitz equation [8,9]. Figure 1 depicts the input–output map of the Landau–Lifshitz equation in (1). The output is the magnetization, \( \mathbf{m}(x, t) = (m_1(x, t), m_2(x, t), m_3(x, t)) \), and the input is a periodic function denoted \( \hat{u}(t) \) where \( \omega \) is the frequency of this input. For each \( m_i \) with \( i = 1, 2, 3 \), the input is the periodic function \( 0.01 \cos(\omega t) \). For this periodic input, equation (1) becomes

\[
\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{u}) - \nu \mathbf{m} \times (\mathbf{m} \times (\mathbf{m}_{xx} + \mathbf{u})) + \hat{u}(t).
\]

As the frequency of the (periodic) input approaches zero, loops appear in the input–output map for \( m_1(x, t) \), which indicates the presence of hysteresis [21].

Because hysteresis is characterized by multiple stable equilibrium points [9,20], the ability to control the stability of equilibrium points implies the ability to control hysteresis. Such a control for the Landau–Lifshitz equation is given in (5). This is a possible avenue for future exploration.

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