SEARCH TECHNIQUES FOR ROOT-UNITARY POLYNOMIALS

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Abstract. We give an anecdotal discussion of the problem of searching for polynomials with all roots on the unit circle, whose coefficients are rational numbers subject to certain congruence conditions. We illustrate with an example from a calculation in $p$-adic cohomology made by Abbott, Kedlaya, and Roe, in which we recover the zeta function of a surface over a finite field.

Introduction

In this note, we give an anecdotal discussion of the problem of searching for polynomials with roots on a prescribed circle whose coefficients are rational numbers subject to certain congruence conditions. We were led to this problem by the use of $p$-adic cohomology to compute zeta functions of varieties over finite fields; in that context, one is looking for certain Weil polynomials (monic integer polynomials with complex roots all on a circle of radius $p^{i/2}$, for some prime number $p$ and some nonnegative integer $i$), and the cohomology calculation imposes congruence conditions on the coefficients. In fact, the main purpose of this note is to show that in a particular example from [1], the conditions obtained from the cohomology calculation indeed suffice to uniquely determine the zeta function being sought. We also illustrate with a larger example provided by Alan Lauder.

1. Definitions

A polynomial $P(z) = \sum_{i=0}^{n} a_i z^i \in \mathbb{C}[z]$ of degree $n$ is self-inversive if there exists $u \in \mathbb{C}$ with $|u| = 1$ such that

$$a_i = u \overline{a_{n-i}} \quad (i = 0, \ldots, n);$$

it is equivalent to require the roots of $P$ to be invariant, as a multiset, under inversion through the unit circle. It appears that the class of self-inversive polynomials first occurs in a theorem of Cohn [3, 5]; it occurs naturally in the study of the locations of roots of polynomials and their derivatives, as in the Schur-Cohn-Marden method [13, p. 150].

We will call a polynomial root-unitary if its roots all lie on the unit circle; this clearly implies self-inversivity. This class of polynomials has been widely studied, but does not seem to have a standard name: the term “unimodular polynomial” refers to a polynomial whose coefficients lie on the unit circle, while “unitary polynomial” is often read as a synonym for “monic polynomial” (particularly by speakers of French, in which a monic polynomial is standardly a “polynôme unitaire”).
Let \( P(z) \in \mathbb{R}[z] \) be a real root-unitary polynomial; then (1.1) must hold with either \( u = +1 \) or \( u = -1 \), in which case we say \( P \) is \textit{reciprocal} or \textit{antireciprocal}, respectively. (The terms \textit{palindromic} and \textit{antipalindromic} are also sometimes used.) If \( \deg(P) \) is odd, then \( P \) must be divisible by \( z + 1 \) or \( z - 1 \), depending on whether \( P \) is reciprocal or antireciprocal. If \( \deg(P) \) is even and \( P \) is antireciprocal, then \( P \) must be divisible by \((z + 1)(z - 1)\). This allows reduction of many questions about real root-unitary polynomials to the reciprocal case.

2. The basic problem

The basic problem is to identify rational polynomials with roots on a prescribed circle, given a few initial coefficients and a congruence condition on the remaining coefficients. One can renormalize in order to talk about root-unitary polynomials; as noted above, there is no real harm in only looking at reciprocal root-unitary polynomials. In any case, here is the precisely formulated question we will consider.

\textbf{Problem 2.1.} Fix positive integers \( n, k, q \) with \( n \geq k \). Also fix positive integers \( m_0, \ldots, m_{2n} \) such that \( m_j \) divides \( m_i \) for \( 0 \leq i \leq j \leq n \), and \( m_i = m_{2n-i} \) for \( 0 \leq i \leq 2n \). Given integers \( a_0, \ldots, a_{2n} \) with \( a_i = q^{n-i}a_{2n-i} \) for \( 0 \leq i \leq 2n \), and \( a_{2n} \neq 0 \), find all polynomials \( P(z) \) with all roots on the circle \(|z| = \sqrt{q}\) of the form

\[
P(z) = \sum_{i=0}^{2n} (a_i + c_i m_i) z^i,
\]

where the \( c_i \in \mathbb{Z} \) must satisfy \( c_i = q^{n-i}c_{2n-i} \) for \( 0 \leq i \leq 2n \), and \( c_i = 0 \) for \( i \geq 2n - k \).

The fact that Problem 2.1 is a finite problem follows easily from the estimates

\[
|a_i + c_i m_i| \leq \binom{2n}{i} q^{i/2} |a_{2n}| \quad (i = 0, \ldots, 2n);
\]

when \( n \) is small, these estimates carry most of the information from the condition that \( P(z\sqrt{q}) \) must be root-unitary. However, for \( n \) large, this is quite far from true. Indeed, by [7, Proposition 2.2.1] (see also [16] for a generalization in the context of Mahler measures), the space of monic root-unitary reciprocal polynomials of degree \( 2n \) has volume

\[
\frac{2^n}{n!} \prod_{j=1}^{n} \left( \frac{2j}{2j - 1} \right)^{n+1-j} \leq \frac{2^n}{n!} \prod_{j=1}^{n} 2^{n+1-j} = \frac{2^{(n^2+3n)/2}}{n!}
\]

whereas the space of monic reciprocal polynomials of degree \( 2n \) whose coefficient of \( z^i \) has norm \( \leq \binom{2n}{i} \) for \( i = 0, \ldots, 2n - 1 \) has volume

\[
\prod_{j=1}^{n} \left( \frac{2^n}{j} \right) = \prod_{j=1}^{n} 2 \prod_{i=0}^{j-1} \frac{2n - i}{j - i} \geq \prod_{j=1}^{n} 2 \prod_{i=0}^{j-1} 2 = 2^{(n^2+3n)/2}.
\]

For \( n \) large, these are wildly discrepant, so one expects the restriction of root-unitarity to carry much more information than the simple bound on the size of coefficients.
3. Exhaustion over a tree

We now describe our basic approach to Problem 3.1 starting with a change of variable also used in [7]. Define a polynomial \( Q(z) \in \mathbb{Z}[z] \) of degree \( n \) by the formula

\[
P(z) = z^n Q(z + q/z).
\]

Then for \( i = 0, \ldots, n \), the coefficients of \( z^{n-i}, \ldots, z^n \) of \( Q \) are obtained from \( a_{2n-i}, \ldots, a_{2n} \) by an invertible linear transformation over \( \mathbb{Z} \).

**Problem 3.1.** Fix positive integers \( n, k, q \) with \( n \geq k \). Also fix positive integers \( m_0, \ldots, m_n \) such that \( m_j \) divides \( m_i \) for \( 0 \leq i \leq j \leq n \). Given integers \( b_0, \ldots, b_n \) with \( b_n \neq 0 \), find all polynomials \( Q(z) \in \mathbb{Z}[z] \) with all roots real and lying in the interval \([-2\sqrt{q}, 2\sqrt{q}]\), such that

\[
Q(z) = \sum_{i=0}^{n} (b_i + d_i m_i) z^i
\]

for some \( d_i \in \mathbb{Z} \) with \( d_i = 0 \) for \( i \geq n - k \).

Our approach to Problem 3.1 is via enumeration of a certain rooted tree.

**Proposition 3.2.** Fix notation as in Problem 3.1. Then there exist sets \( S_j \subseteq \mathbb{Z}^j \) for \( j = 0, \ldots, n - k \) satisfying the following conditions.

(a) The set \( S_0 \) consists of the empty \( 0 \)-tuple.

(b) For \( 0 < j \leq n - k \), if \((d_{n-k-1}, \ldots, d_{n-k-j}) \in S_j\), then \((d_{n-k-1}, \ldots, d_{n-k-j+1}) \in S_{j+1}\).

(c) For \( 0 \leq j \leq n - k \), if \((d_{n-k-1}, \ldots, d_{n-k-j}) \in S_j\), then \(Q_0(z) = \sum_{i=0}^{n} b_i z^i + \sum_{i=n-k-j}^{n-1} d_i m_i z^i\) has the property that \(Q_0^{n-k-j} \) has all roots in \([-2\sqrt{q}, 2\sqrt{q}]\).

(d) Every tuple \((d_{n-k-1}, \ldots, d_0) \in \mathbb{Z}^{n-k}\) such that \(\sum_{i=0}^{n} b_i z^i + \sum_{i=0}^{n-k-1} d_i m_i z^i\) has all roots in \([-2\sqrt{q}, 2\sqrt{q}]\) belongs to \(S_{n-k}\).

**Proof.** Create \( S_{n-k} \) by taking all solutions of Problem 3.1 then let \( S_j \) be the set of initial segments of length \( j \) occurring among elements of \( S_{n-k} \). Property (c) holds by Rolle’s theorem.

We may identify a system of sets as in Proposition 3.2 with a rooted tree, where the children of a \( j \)-tuple in \( S_j \) are its extensions to a \((j+1)\)-tuple in \( S_{j+1}\). To solve Problem 3.1 in practice, we perform a depth-first enumeration of such a tree, and read off the solutions of Problem 3.1 as the elements of \( S_{n-k} \). To describe such a tree and its enumeration, it suffices to describe how to compute the list of children of a given node. (One could also perform a breadth-first exhaustion, but in practice this seems to be inferior because of increased overhead.)

Note that if one wishes to decide as soon as possible whether the number of solutions is 0, 1, or more than 1, it may be advantageous to visit the children of a given node in “inside-out order” rather than in ascending or descending order. For instance, if a given tuple can be extended by 5, 6, 7, 8, 9, we would visit these extensions in the order 7, 6, 8, 5, 9.

4. First approach: root-finding

We now describe our first algorithmic approach to Problem 3.1 and its implementation [12] in the case where \( q = 1 \). (See Section 8 for comments on the remaining cases.)
One can interpret this more geometrically by drawing the graph of \( R \) and \( s \). Conditions occur for some \( i \) values of \( c = -2 \) and treat \( x \) and \( x \) as local maximum or minimum depending on whether \( q \) is even or odd.

Let \( x_1 \leq \cdots \leq x_{d-1} \) be the roots of \( R' \) counted with multiplicity, and put \( x_0 = -2\sqrt{q} \) and \( x_d = 2\sqrt{q} \). For \( i = 0, \ldots, d \), put \( y_i = R(x_i) \). Then the values of \( c \) we want are those for which

\[
\begin{align*}
y_{d-2i} + c &\geq 0 \quad (i = 0, \ldots, [d/2]) \\
y_{d-1-2i} + c &\leq 0 \quad (i = 0, \ldots, [(d-1)/2]).
\end{align*}
\]

One can interpret this more geometrically by drawing the graph of \( R \) over \([-2\sqrt{q}, 2\sqrt{q}]\). The values of \( c \) are the negatives of the integral \( y \)-values between the highest local minimum and the lowest local maximum of \( R \) (inclusive), provided that we treat \( 2\sqrt{q} \) as a local maximum, and treat \(-2\sqrt{q}\) as a local maximum or minimum depending on whether \( d \) is even or odd.

Our principal method for treating Problem 4.1 is to compute numerical approximations to the \( x_i \) and \( y_i \). We throw an exception if these approximations are not sufficiently accurate, unless \( x_i = x_{i+1} \) for some \( i \); we can both detect and resolve this case using exact arithmetic.

**Algorithm 4.2.** Consider inputs as in Problem 4.1 together with a positive integer \( p \). Using \( \text{GSL} \), compute numerical approximations \( \tilde{x}_1 \leq \cdots \leq \tilde{x}_{d-1} \) to the roots of \( R' \), presumed (but not guaranteed) correct to within \( 2^{-p} \). For \( i = 1, \ldots, d-1 \), put \( r_i = [\tilde{x}_i 2^{p-1} - 1] 2^{-p+1} \) and \( s_i = r_i + 2^{-p+3} \); also put \( r_0 = s_0 = -2\sqrt{q} \) and \( r_d = s_d = 2\sqrt{q} \). If any of the following conditions occur for some \( i \in \{1, \ldots, d-1\} \):

\begin{itemize}
  \item \( s_i \geq r_{i+1} \);
  \item \( (-1)^{d-i} R''(r_i) > 0 \);
  \item \( R'(r_i) \) and \( R'(s_i) \) have the same sign;
\end{itemize}

then abort or return according as Algorithm 4.4 aborts or returns. If none of the conditions occur, put \( l = -\infty \) and \( u = +\infty \). For \( i = d, d-1, \ldots, 0 \) in turn:
• if \( r_i = s_i \), put \( t = R(r_i) \);
• if \( r_i < s_i \) and \( d - i \) is even, let \( t \) be the value computed by applying Algorithm 4.3 with \([r, s] = [r_i, s_i]\) and \( t_0 = -l \), then replace \( l \) by \( \max\{-t, l\} \);
• if \( r_i < s_i \) and \( d - i \) is odd, let \( t \) be the value computed by applying Algorithm 4.3 with \([r, s] = [r_i, s_i]\) and \( t_0 = u \) after replacing \( R \) by \(-R\), then replace \( u \) by \( \min\{t, u\} \);
• if now \( l > u \), return the empty set.

Return the range \( \mathbb{Z} \cap [l, u] \); this solves Problem 4.1 if not aborted.

\textbf{Proof.} The only thing that needs to be noted here is that failure to invoke Algorithm 4.4 ensures that the intervals \([r, s] = [r_i, s_i]\) are disjoint and contain one root of \( R' \) apiece, so the input to Algorithm 4.3 is valid. \( \square \)

In order to determine the roundings of the \( y_i \), we use exact arithmetic as follows.

\textbf{Algorithm 4.3.} Let \( R(z) \in \mathbb{Q}[z] \) be a polynomial such that \( R' \) has all roots real and distinct. Let \( r, s \in \mathbb{Q} \) be such that \( R'(r) \leq 0 \), and the interval \([r, s]\) contains a local maximum of \( R \) and no other roots of \( R' \). Let \( t_0 \in \mathbb{Z} \cup \{+\infty\} \). Compute

\[
t = \lfloor R(r) \rfloor \quad u = \lceil R(r) + (s - r)R'(r) \rceil.
\]

If \( t \geq t_0 \), then return \( t_0 \) (this can be checked before computing \( u \)). Otherwise, while \( t \neq u \), repeat the following: for \( v = \lceil \frac{t+u}{2} \rceil \), if \( R - v \) has any roots in \([r, s]\) as determined by \textsc{polsturm}, then replace \( t \) by \( v \), otherwise replace \( u \) by \( v - 1 \). Return \( t \); then for \( x \) the unique root of \( R' \) contained in \([r, s]\), either \( t \geq t_0 \) or \( t = \lfloor R(x) \rfloor \).

\textbf{Proof.} Since \( R' \) has all roots real and distinct, \( x \) must be an isolated root of \( R' \). Since \( x \) is a local maximum for \( R \), \( R' \) must undergo a sign crossing at \( x \) from positive to negative. Since \( R' \) has no other roots in \([r, s]\), \( R' \) must be positive in \([r, x]\) and negative in \((x, s]\).

The roots of \( R'' \) interlace those of \( R' \) by Rolle’s theorem, so in \((r, x]\) we have either zero or one root of \( R'' \). The root occurs if and only if there is a sign crossing; since \( R''(x) < 0 \) and \( R''(r) \leq 0 \), we deduce that there is no root, and \( R''(z) < 0 \) for all \( z \in (r, x] \).

This implies that \( R'(r) \geq R'(z) \) for \( z \in [r, x] \); since \( R'(r) > 0 \),

\[
R(x) = R(r) + \int_r^x R'(z) \, dz \leq R(r) + (x - r)R'(r) \leq R(r) + (s - r)R'(r).
\]

This yields the claim. \( \square \)

Note that to a certain extent, taking \( p \) small in Algorithm 4.2 is beneficial to Algorithm 4.3 because it keeps the heights of the rationals \( r_i, s_i \), small. However, it may happen that if \( p \) is too small, then the gap between the initial values of \( t \) and \( u \) in Algorithm 4.3 may be quite large, and a great deal of time may be wasted narrowing the gap.

Recall that Algorithm 4.2 does not treat cases of Problem 4.1 in which \( R' \) has repeated roots, or \( R'(-2,\sqrt{q})R'(2,\sqrt{q}) = 0 \); here is a simple treatment. In practice, these cases seem to be exceedingly rare; for instance, they do not occur at all in the example of Section 6.

\textbf{Algorithm 4.4.} Consider inputs as in Problem 4.1. Put \( T = \gcd(R', (z^2 - 4q)R'') \); if \( T \) is constant, then abort. Otherwise, let \( S_1, \ldots, S_k \) denote the distinct irreducible factors of \( \gcd(R', (z^2 - 4q)R'') \). Determine whether the quotients upon dividing \( R \) by each \( S_i \) are all equal to a single integer \(-c\). If so, use \textsc{polsturm} to check whether \( R(z) + c \) has all roots real.
and in $[-2\sqrt{q}, 2\sqrt{q}]$; if so, return the singleton set $\{c\}$. In all other cases, return the empty set. This solves Problem 4.1 if not aborted.

Proof. Suppose that $T$ is nonconstant and $R(z) + c$ has all roots real and in $[-2\sqrt{q}, 2\sqrt{q}]$. Let $r$ be a root of $T$. If $r = -2\sqrt{q}$, then by Rolle’s theorem, $R(z) + c$ has a root less than or equal to $-2\sqrt{q}$, hence $R(-2\sqrt{q}) + c = 0$. Similarly, if $r = 2\sqrt{q}$, then $R(2\sqrt{q}) + c = 0$. If $-2\sqrt{q} < r < 2\sqrt{q}$, then $r$ is a root of $R''$ and so must be a multiple root of $R'$; by Rolle’s theorem, $r$ must be a root of $R(z) + c$. This proves the claim. 

5. Second approach: power sums

Inspection of the enumeration of the maximal tree in some examples suggests that it is rather bushy, in the sense of having many vertices with many children but few deep descendants. This in turn suggests that a more refined tree construction might be able to achieve substantial runtime improvements. Our second approach, implemented in [11] using SAGE and components as in the previous section (but again restricted to the case $q = 1$), does this; it is based on estimations of power sums, as in the work of Boyd [4] and subsequent authors (most notably [8]) on searching for polynomials with small Mahler measure.

Given a polynomial $R(z) = \sum_{i=0}^{n} c_i z^i$ with $c_n \neq 0$, with roots $r_1, \ldots, r_n$, the power sums of $R$ are defined as

$$s_j = r_1^j + \cdots + r_n^j \quad (j = 0, 1, \ldots).$$

They are related to the coefficients of $R$ via the Newton identities:

$$j c_{n-j} + \sum_{i=0}^{j-1} c_{n-i} s_{j-i} = 0 \quad (j = 1, \ldots, n).$$

In particular, given $c_n$, one can recover $c_{n-1}, \ldots, c_{n-j}$ from $s_1, \ldots, s_j$ via an invertible linear transformation over $\mathbb{Q}$. Moreover, the $j$-th power sum of $R(z) + \sum_{i=0}^{n-j} c_i z^i$ equals $s_j - j c_{n-j}/c_n$. Note that PARI provides a routine polsym to generate the power sums of a polynomial.

In this tree enumeration, we will generate some nodes which do not actually belong to the tree, because they do not satisfy (c); hence our first step when considering a proposed node will be to check (c) using polsturm. (Profiling data in some examples suggests that this step is a bottleneck in the computation; some improvement may be derived by instead using Sturm-Habicht sequences, as described in [2], or perhaps even using real root isolation techniques. We plan to investigate this further.) If (c) is satisfied, and the node is not at maximum depth, we enumerate its children by generating and solving an instance of the following problem.

Problem 5.1. Given a polynomial $R(z) = \sum_{i=0}^{n} c_i z^i$ with $c_n \neq 0$, and an integer $1 \leq j \leq n$, find $l, u \in \mathbb{Z}$ such that for any real numbers $d_{n-j}, \ldots, d_0$ with $d_{n-j} \in \mathbb{Z}$ and $R(z) + \sum_{i=0}^{n-j} d_i z^i$ having roots in $[-2\sqrt{q}, 2\sqrt{q}]$, we have $d_{n-j} \in [l, u]$.

Note that this problem is somewhat open-ended: if $l_i, u_i$ is a solution of Problem 5.1 for $i = 1, \ldots, k$, then so is $l, u$ for $l = \max_i \{l_i\}, u = \min_i \{u_i\}$. It thus suffices to exhibit a list of inequalities satisfied by the coefficients of a polynomial $R(z) = \sum_{i=0}^{n} c_i z^i$ with all roots in $[-2\sqrt{q}, 2\sqrt{q}]$; equivalently, we may exhibit inequalities satisfied by the power sums $s_i$ of $R$. Here are some convenient ones; adding additional inequalities should provide even better results, although at some point adding a new inequality will eliminate so few cases that it
(1) For \( i \) even, 
\[
    s_i - 4qs_{i-2} \leq 0 \quad (i \geq 2).
\]

(2) Let \( T_i(z) = \sum_{k=0}^{i} t_{i,k} z^k \) be the polynomial of degree \( i \) for which \( T_i(2\sqrt{q}\cos \theta) = 2\sqrt{q} \cos i\theta \) (a rescaled Chebyshev polynomial of the first kind); then
\[
    \left| \sum_{k=0}^{i} t_{i,k} s_k \right| \leq 2n\sqrt{q} \quad (i \geq 0)
\]
\[
    \left| \sum_{k=0}^{i-2} t_{i-2,k} (s_{k+2} - 2qs_k) \right| \leq 4nq\sqrt{q} \quad (i \geq 2).
\]

(3) Put \( s'_i = \sum_{k=0}^{i} \binom{i}{k} (2\sqrt{q})^{i-k} s_k \); then
\[
    s'_i s'_{i-2} - (s'_{i-1})^2 \geq 0 \quad (i \geq 2),
\]
\[
    s'_i - 4\sqrt{q} s'_{i-1} \leq 0 \quad (i \geq 1).
\]

(4) Put \( s''_i = \sum_{k=0}^{i} \binom{i}{k} (2\sqrt{q})^{i-k} (-1)^k s_k \); then
\[
    s''_i s''_{i-2} - (s''_{i-1})^2 \geq 0 \quad (i \geq 2),
\]
\[
    s''_i - 4\sqrt{q} s''_{i-1} \leq 0 \quad (i \geq 1).
\]

6. An example

Here is an example of the basic problem, excerpted from [1, §4.2], and some results obtained using the algorithms and implementations described above.

Consider the smooth quartic surface \( X \) in the projective space over the finite field \( \mathbb{F}_3 \) defined by the homogeneous polynomial
\[
    x^4 - xy^3 + xy^2w + xzw + y^3w^2 - xzw^2 + y^4 + y^3w - y^2zw + z^4 + w^4.
\]
(As described in [1], this polynomial was chosen essentially at random except for a skew towards sparseness.) Since \( X \) is a K3 surface, the Hodge diamond of \( X \) is
\[
\begin{array}{ccc}
1 & & \\
0 & 0 & \\
1 & 20 & 1 \\
0 & 0 & \\
1 & & \\
\end{array}
\]
and the Hodge polygon of primitive middle cohomology has vertices \((0, 0), (1, 0), (20, 19), (21, 21)\). Consequently, the zeta function of \( X \) has the form
\[
    \zeta_X(T) = \exp \left( \sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbb{F}_{3^n}) \right) = \frac{1}{(1 - T)(1 - 3T)(1 - 9T)R(T)},
\]
where \( R(T) \in \mathbb{Z}[T] \) is a polynomial of degree 21 such that \( R(0) = 1 \), the complex roots of \( R \) lie on the circle \(|T| = 3^{-1}\), and (by an inequality of Mazur) the Newton polygon of \( R \) lies above the Hodge polygon. In particular, the polynomial \( S(T) = 3R(T/3) \) is root-unitary and has integral coefficients.
Define
\[ S_0(T) = 3T^{21} + 5T^{20} + 6T^{19} + 7T^{18} + 5T^{17} + 4T^{16} + 2T^{15} - T^{14} - 3T^{13} - 5T^{12} \\
- 5T^{11} - 5T^{10} - 5T^9 - 3T^8 - T^7 + 2T^6 + 4T^5 + 5T^4 + 7T^3 + 6T^2 + 5T + 3; \]
one easily checks that \( S_0 \) is root-unitary. By explicitly enumerating \( X(\mathbb{F}_{q^n}) \) for \( n \leq 5 \), one finds
\[ S(T) \equiv S_0(T) \pmod{T^6}; \]
by a 3-adic cohomology computation described in [1], one finds
\[ S(T) \equiv S_0(T) \pmod{3^5}. \]
Having performed these computations, one wants to verify whether these restrictions suffice to ensure \( S(T) = S_0(T) \). Moreover, one also wants to know to what extent they can be weakened while still forcing \( S(T) = S_0(T) \), as the enumeration and cohomology calculations become significantly more cumbersome as the strength of their results is forced to increase.

The result obtained here is that already the conditions that \( S(T) \in \mathbb{Z}[T] \), \( S(T) \) is root-unitary, and
\[ S(T) \equiv S_0(T) \pmod{3^2T^1} \]
force \( S = S_0 \). Note that already the congruence \( S(T) \equiv S_0(T) \pmod{3} \) implies that \( S \) must be reciprocal rather than antireciprocal, so we may as well put
\[ P(T) = S(T)/(T + 1), \quad P_0(T) = S_0(T)/(T + 1). \]
Then the conditions we are interested in are that \( P(T) \in \mathbb{Z}[T] \), \( P(T) \) is root-unitary and reciprocal, and
\[ P(T) \equiv P_0(T) \pmod{3^iT^j} \]
for various \( i, j \). The asserted result is that these conditions for \( i = 2, j = 1 \) force \( P = P_0 \). (It turns out that \( i = 1 \) does not suffice even with \( j = 10 \).)

We checked the sufficiency of the conditions for \( i = 2, 3, 4, 5 \) and \( j = 1, 2, 3, 4, 5 \) by running the implementation [12] on one Opteron 246 CPU (64-bit, 2 GHz) of the computer dwork.mit.edu. The machine has 2GB of RAM.

The timings and sizes of the computations for various initial constraints are summarized in Table 1 (using root-finding, as in Section 4, with rounding precision \( p = 32 \)) and Table 2 (using power sums, as in Section 5). Each entry consists of the number of CPU seconds for the calculation, rounded up to the nearest tenth of a second, followed by the number of leaves (terminal nodes) in the tree over which we exhausted. Note the significant savings achieved by the second approach. (We did some additional experiments combining the two approaches, but the power sum method by itself seemed to outperform hybrid methods.)

### Table 1. Timings for recovery of \( P \) given (6.1), using root-finding.

|       | \( 3^2 \)   | \( 3^3 \)   | \( 3^4 \)   | \( 3^5 \)   |
|-------|-------------|-------------|-------------|-------------|
| \( T^1 \) | 564.2/1011788 | 2.2/3858 | 0.1/38 | 0.1/2 |
| \( T^2 \) | 267.9/501620 | 2.2/3784 | 0.1/38 | 0.1/2 |
| \( T^3 \) | 4.3/4714 | 0.1/63 | 0.1/6 | 0.1/1 |
| \( T^4 \) | 1.8/1838 | 0.1/51 | 0.1/6 | 0.1/1 |
| \( T^5 \) | 0.7/612 | 0.1/32 | 0.1/5 | 0.1/1 |
Table 2. Timings for recovery of $P$ given (6.1), using power sums.

|   | $3^2$     | $3^3$     | $3^4$     | $3^5$     |
|---|-----------|-----------|-----------|-----------|
| $T^1$ | 1.9/1157  | 0.1/6     | 0.1/1     | 0.1/1     |
| $T^2$ | 0.6/347   | 0.1/2     | 0.1/1     | 0.1/1     |
| $T^3$ | 0.3/117   | 0.1/1     | 0.1/1     | 0.1/1     |
| $T^4$ | 0.2/53    | 0.1/1     | 0.1/1     | 0.1/1     |
| $T^5$ | 0.1/23    | 0.1/1     | 0.1/1     | 0.1/1     |

7. Another example

The previous example shows the superiority of the power sum method over the root-finding method. This suggests trying a larger example to test the limits of the power sum method; here is an example provided by Alan Lauder.

The reciprocal polynomial

$$P(T) = 24017^{56} - 343T^{55} - 5439T^{54} - 1050T^{53} + 7156T^{52} + 5043T^{51} - 5829T^{50} - 7990T^{49}$$
$$+ 1437T^{48} + 6348T^{47} + 2115T^{46} - 332T^{45} - 1756T^{44} - 4639T^{43} - 1802T^{42} + 3938T^{41}$$
$$+ 4762T^{40} + 16T^{39} - 3366T^{38} - 2658T^{37} - 2051T^{36} + 1572T^{35} + 5810T^{34} + 2097T^{33}$$
$$- 5558T^{32} - 3955T^{31} + 2598T^{30} + 1931T^{29} - 831T^{28} + 1931T^{27} + \cdots$$

is root-unitary; it arises from a 7-adic cohomology calculation of the primitive middle cohomology of an elliptic surface over $\mathbb{F}_7$ with Hodge diamond

$$
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
4 & 49 & 4. \\
0 & 0 & & \\
1 & & & \\
\end{array}
$$

As in the previous example, we ask whether a reciprocal root-unitary polynomial $P_0(T)$ satisfying

$$(7.1) \quad P(T) \equiv P_0(T) \pmod{7^i T^j}$$

necessarily equals $P(T)$. In the following, each expression $(A/B)$ indicates that the indicated computation required $A$ CPU seconds and encountered $B$ terminal nodes.

- For $i = 2$, $P_0 = P$ is not forced for $j = 28$ (0.7/15).
- For $i = 3$, $P_0 = P$ is forced for $j = 25$ (336.2/355435) but not for $j = 24$ (711.7/755544).
- For $i = 4$, $P_0 = P$ is forced for $j = 16$ (331304.0/196405710). We were unable to find any value of $j$ for which $P_0 = P$ is not forced. (For comparison, the complexities for $j = 17, 18, 19, 20$ were 61787.7/36665858, 12464.5/7334642, 2275.5/1349860, 392.6/232783.)
- For $i = 5$, $P_0 = P$ is forced for $j = 1$ (93.7/13513).

One can explain this behavior heuristically by imposing only the condition that (in the notation of Section 5) $| \sum_{k=0}^l t_{i,k} s_k | \leq 2n$ for $i \geq 0$. This restriction constrains the coefficient
of \( T^j \) to a range of size \((4 \cdot 28)/(7^4 \cdot j)\). Once \( i \) is big enough that this range typically takes only a few elements, we can expect to be able to force \( P_0 = P \).

8. The case of nonsquare \( q \)

As noted earlier, our implementations so far have only covered the case \( q = 1 \). It is easy to reduce to this case from any case in which \( q \) is a square. For \( q \) not a square, there are several ways to proceed; we do not know which of these is best.

- One may repeat the methods as written above, but using exact arithmetic in the quadratic field \( \mathbb{Q}(\sqrt{q}) \).
- One may replace \( \sqrt{q} \) by an upper approximation by a rational number \( s \) and look for polynomials with roots in \([-2s, 2s]\), then screen out those which do not have roots in \([-2\sqrt{q}, 2\sqrt{q}]\).
- One may consider the polynomial \( S \) defined by \( S(z^2) = R(z)R(-z) \).

9. Further comments

Note that \textit{SAGE} runs primarily in the interpreted language Python, although many of its components either are compiled libraries, or have been migrated to C using \textit{Cython}. (Indeed, the latter progress has been ongoing, and this can be detected in the runtimes of our algorithms under different versions of \textit{SAGE}.) It is thus fair to ask whether some additional optimization could be achieved by porting everything to a compiled environment. We have already built in some savings by performing most polynomial manipulation in \textit{PARI} with limited conversions to/from \textit{SAGE}, and by porting some key subroutines into \textit{Cython}; it is not clear how much more room there is for improvement on this front.

Our depth-first search is implemented using a queue rather than recursion; this has the advantage of making it easily amenable to parallelization. Although quite sophisticated strategies have been devised for scheduling in the context of tree traversal (e.g., \[17\]), even implementing some simple scheduling mechanisms, such as work-stealing, would be helpful in a multiprocessor environment. Starting with version 2.0, \textit{SAGE} includes a subsystem called \textit{DSAGE} (Distributed \textit{SAGE}), which may facilitate this sort of simple parallelization.

One can use similar search techniques for polynomials with roots bounded in a convex subset of the complex plane, since the Gauss-Lucas theorem asserts that this property is also preserved by taking derivatives. We have not experimented with this in any detail.

Acknowledgments

Thanks to Alan Lauder and Chris Davis for feedback on early versions of this paper, and to Josh Kantor, William Stein, and Carl Witty for implementation advice.

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