LOCAL WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR
THE DEGENERATE ZAKHAROV SYSTEM

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Abstract. The aim of this paper is to investigate well-posedness of the Cauchy
problem for the degenerate Zakharov system. Local well-posedness holds for
anisotropic Sobolev data by applying $U^2, V^2$ type spaces. We give the Schrödinger
initial data $H^{s_k, s'}$ and the wave data $H^{s_l, s'}$ where $s_k > (d - 1)/2, s_l > (d -
2)/2, s_k - s_l = 1/2$ and $s' > 1/2$.

1. Introduction

We consider the Cauchy problem for the degenerate Zakharov system:

$$
\begin{align*}
&i(\partial_t u + \partial_{x_1} u) + \Delta_\perp u = nu, \\
&\partial_t^2 n - \Delta_\perp n = \Delta_\perp |u|^2, \\
&(u, n, \partial_t n)|_{t=0} = (u_0, n_0, n_1),
\end{align*}
$$

where $d \geq 2$, $\Delta_\perp = \sum_{i=1}^{d-1} \partial_{x_i}^2$, $u$ is complex valued function and $n$ is real valued
function. For three spatial dimension, (1.1) describes the laser propagation when
the paraxial approximation is used and the effect of the group velocity is negligible
[17, 15, 11]. For the Cauchy problem for the Zakharov system

$$
\begin{align*}
&i\partial_t u + \Delta u = nu, \\
&\partial_t^2 n - \Delta n = \Delta |u|^2, \\
&(u, n, \partial_t n)|_{t=0} = (u_0, n_0, n_1),
\end{align*}
$$

where $\Delta = \sum_{i=1}^{d} \partial_{x_i}^2$, the well-posedness for (1.2) is well studied with low regularity
initial data $(u_0, n_0, |\nabla_x|^{-1} n_1) \in H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^l(\mathbb{R}^d)$, for instance Ginibre,
Tsutsumi and Velo [9] for all spatial dimensions, Bejenaru, Herr, Holmer and Tataru
[3] for $d = 2$, Bejenaru and Herr [4] for $d = 3$ and Bejenaru, Guo, Herr and Nakanishi
[2], Candy, Herr and Nakanishi [6] or the author and Tsugawa [12] for $d \geq 4$ case.

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On the other hand, the degenerate Zakharov system $\{1\}$ with low regularity initial data is much less considered than that for the Zakharov system. M. Colin and T. Colin [7] derived the Cauchy problem for $\{1\}$ in three spatial dimension. Later Linares, Ponce and Saut [15] proved local well-posedness in a suitable function space by deriving the local smoothing estimate and the maximal function estimate. More precisely, the lowest regularity initial data is $(u_0, n_0, n_1) \in \tilde{H}^5(\mathbb{R}^3) \times H^5(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$ and $\partial_{x_3} n_1 \in H^1(\mathbb{R}^3)$, where

$$\tilde{H}^s(\mathbb{R}^3) = \{ f \in H^s(\mathbb{R}^3) \mid \partial_{x_1}^{1/2} \partial^\alpha f, \partial_{x_2}^{1/2} \partial^\alpha f \in L^2(\mathbb{R}^3), \mid \alpha \mid \leq s, \alpha \in (\mathbb{Z}_{\geq 0})^3 \}. $$

Barros and Linares [1] obtained local well-posedness for initial data $(u_0, n_0, n_1) \in \tilde{H}^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and $\partial_{x_3} n_1 \in H^1(\mathbb{R}^3)$. The key is improving the regularity of the maximal function estimate $\{1.3\}$ and deriving the Strichartz estimate for the Schrödinger equation $\{1.4\}$ below. We may see lack of dispersion in $x_3$ variable in $\{1.4\}$.

$$\| S(t) f \|_{L^2_{t,x_3} L^\infty_{x_2}} \leq c(T, s) \| f \|_{\tilde{H}^s(\mathbb{R}^3)}, \quad s > 3/2, \tag{1.3}$$

$$\| S(t) g \|_{L^4_{t} L^4_{x_1} L^\infty_{x_2,x_3}} \leq c \| g \|_{L^2_{x_1,x_2}}. \tag{1.4}$$

where $2/q = 1 - 2/r$, $2 \leq r < \infty$, $c(T, s)$ and $c$ are constants, $S(t)$ is the free Schrödinger operator, see section 2 for precise definition.

The aim of this paper is to prove local well-posedness by $U^2, V^2$ type spaces in all spatial dimensions $(d \geq 2)$. The nonlinear estimate namely Proposition $3.1$ plays an important role in this paper. To prove Proposition $3.1$ we remark that (i) we use the anisotropic Sobolev data $H^{s,s'}(\mathbb{R}^d)$, (ii) we construct the solution for the Schrödinger equation in the intersection space between $V^2$ based space and the spaces associated to the local smoothing and the maximal function estimate, (iii) we consider $s_k - s_l = 1/2$, where $s_k, s_l$ denotes regularity of the initial data for the Schrödinger and the wave equation respectively. Concerning (i), the Schrödinger equation in $\{1.1\}$ does not have a dispersion in the $x_d$ variable, it seems hard to find smoothing property including the $x_d$ direction. Indeed, the local smoothing estimate Proposition $2.1$ $\{2.2\}$ does not have a smoothing with respect to the $x_d$ variable. On the other hand, from the following modulation estimate $\{1.5\}$ we can recover almost $1/2$ regularity with respect to $\Xi_1 = (\xi_1, \xi_1^\prime) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Let $\tau_3 = \tau_1 - \tau_2$, $\xi_3 = \xi_1 - \xi_2$, $|\xi_1| \gg |\xi_2|$ and $|\xi_1|^2 \gg \max\{|\xi_1^\prime|, |\xi_2^\prime|\}$. Then it holds that

$$\max\{|\tau_1 + |\xi_1|^2 + |\xi_1^\prime|, |\tau_2 + |\xi_2|^2 + |\xi_2^\prime|, |\tau_3 \pm |\xi_3|\| \gtrsim |\Xi_1| \geq |\Xi_1|. \tag{1.5}$$

However applicable case is limited hence it is natural to use anisotropic Sobolev data $H^{s,s'}(\mathbb{R}^d)$ for $\{1.1\}$. In Theorem $1.1$ we take $1/2 + \varepsilon'$ regularity with respect
to the $x_d$ variable for any $\varepsilon' > 0$. This regularity comes from the natural Sobolev embedding $H^{1/2 + \varepsilon'}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. For (ii), we cannot replace $V^2$ by $L_t^\infty L_x^2$ since one of the nonlinear estimates for the Schrödinger equation, for $J_{4,3,1}$ in the proof of Proposition 3.1, we have to apply $U_3^\infty \hookrightarrow L_t^\infty L_x^2$ since the frequency $N_1$ of the wave equation is low. Moreover it seems difficult to control in the framework of $X^{s,b}$ defined by the Fourier restriction norm method. For the estimate of $J_{1,1}$ in the proof of Proposition 3.1, we need to gain 1/2 regularity by the local smoothing estimate, however we run short of $\varepsilon$ regularity with time variable by applying the $X^{s,b}$ space. To solve the problem, we use the $U^2, V^2$ type spaces which are enhanced $X^{s,b}$ spaces. Now we mention (iii), namely $s_k - s_l = 1/2$ balances the two nonlinear estimates for the Schrödinger and the wave equations. Once we take $s_k, s_l$ satisfying $s_k - s_l > 1/2$, we may find that the nonlinear estimate for the Schrödinger equation, for instance $J_{1,1}$ is hard to estimate since we need to gain more than 1/2 regularity.

By applying the local smoothing estimate, we only gain 1/2 regularity hence it is not sufficient to derive the estimate for $J_{1,1}$. We note that $J_{1,1}$ contains resonant region and cannot apply the modulation estimate (1.5) there. Also, the bilinear Strichartz estimate Proposition 4.1 does not gain enough regularity to overcome the difficulty.

The case $s_k - s_l < 1/2$ contains similar problem, hence $s_k - s_l = 1/2$ is important in our analysis.

If we transform $n_\pm := n \pm i \omega^{-1} \partial_t n, \omega := (-\Delta_\perp)^{1/2}$, then (1.1) is equivalent to the following:

\[
\begin{cases}
    i(\partial_t + \partial_{x_d})u + \Delta_\perp u = (1/2)(n_+ + n_-)u, & (t, x) \in [0, T] \times \mathbb{R}^d, \\
    (i\partial_t \mp \omega)n_\pm = \pm \omega |u|^2, & (t, x) \in [0, T] \times \mathbb{R}^d, \\
    (u, n_\pm)|_{t=0} = (u_0, n_\pm)
\end{cases}
\]

(1.6)

The main result is as follows.

**Theorem 1.1.** Let $d \geq 2, s > (d - 1)/2, s' > 1/2$ and assume the initial data $(u_0, n_{\pm 0}) \in H^{s, s'}(\mathbb{R}^d) \times H^{s-1/2, s'}(\mathbb{R}^d)$. Then, (1.6) is locally well-posed in $H^{s, s'}(\mathbb{R}^d) \times H^{s-1/2, s'}(\mathbb{R}^d)$.

Now we mention the critical exponent for (1.6). The scaling between the first and the second equation of (1.6) is different, hence there is no critical index for (1.6) in the strict sense. However, if we neglect $\mp \omega n_\pm$ in (1.6), then under the scaling

\[
u_\lambda(t, \tilde{x}, x_d) := \lambda^{-3/2} u(\lambda^{-2} t, \lambda^{-1} \tilde{x}, \lambda^{-2} x_d), \quad \lambda > 0,
\]

\[
n_{\pm \lambda}(t, \tilde{x}, x_d) := \lambda^{-2} n_\pm(\lambda^{-2} t, \lambda^{-1} \tilde{x}, \lambda^{-2} x_d), \quad \lambda > 0,
\]

\[
\tilde{x} = (x_1, ..., x_{d-1}) \in \mathbb{R}^{d-1}, \quad \lambda > 0.
\]
the critical exponent is \( k + 2k' = (d - 2)/2 \) and \( l + 2l' = (d - 3)/2 \), where \((u_0, n_{\pm 0}) \in H^{k, k'}(\mathbb{R}^d) \times H^{l, l'}(\mathbb{R}^d)\). Hence, we expect that Theorem 1.1 is not optimal. However our result improves almost 1 derivative than [1] for \( d = 3 \).

Finally, we mention the conservation laws for (1.1): \( \|u(t)\|_{L_x^2} \) and

\[
\|(-\Delta)^{1/2}u\|_{L_x^2}^2 + \text{Im} \int \bar{u} \partial_x u \, dx + \int |u|^2 \, dx + \frac{1}{2}\|n\|_{L_x^2}^2 + \frac{1}{2}\|(-\Delta)^{-1/2}\partial_t n\|_{L_x^2}^2
\]

are conserved for (1.1). There is no global well-posedness result for (1.1), hence by using these quantities we expect obtaining global well-posedness in future.

In section 2, we introduce notations, lemmas and solution spaces \( X_S, Z_{W_{\pm}} \). In section 3, we show the crucial nonlinear estimate to obtain Theorem 1.1. Finally in section 4, we derive the bilinear Strichartz estimate for future reference.

2. Notations and Preliminary Lemmas

In this section, we prepare some lemmas, propositions and notations to prove the main theorem. \( A \lesssim B \) means that there exists \( C > 0 \) such that \( A \leq CB \). Also, \( A \sim B \) means \( A \lesssim B \) and \( B \lesssim A \). Let \( a \pm := a \pm \varepsilon \) for sufficiently small \( \varepsilon > 0 \) and \( \tilde{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1} \). Let \( u = u(t, x) \). \( \mathcal{F}_t u, \mathcal{F}_x u \) denote the Fourier transform of \( u \) in time, space, respectively. We denote \( \|f\|_2 = \|f\|_{L_t^2 L_x^2} \) and we define \( H^{s, s'}(\mathbb{R}^d) \) with the norm

\[
\|g\|_{H^{s, s'}} := \left( \int \langle \xi \rangle^{2s} \langle \xi' \rangle^{2s'} |\mathcal{F}_x g(\xi, \xi')|^2 d\xi d\xi' \right)^{1/2}
\]

for \( \xi \in \mathbb{R}^{d-1}, \xi' \in \mathbb{R}, \langle \cdot \rangle := (1 + |\cdot|^2)^{1/2} \).

Let \( \{\mathcal{F}_y^{-1}[\varphi_n](y)\}_{n \in \mathbb{Z}} \subset \mathcal{S}(\mathbb{R}) \) be the Littlewood-Paley decomposition, that is to say

\[
\begin{align*}
\varphi(\eta) &\geq 0, \\
\text{supp } \varphi(\eta) &\subset \{\eta \mid 2^{-1} \leq |\eta| \leq 2\},
\end{align*}
\]

\[
\varphi_n(\eta) := \varphi(2^{-n}\eta), \quad \sum_{n=-\infty}^{\infty} \varphi_n(\eta) = 1 \quad (\eta \neq 0), \quad \psi(\eta) := 1 - \sum_{n=0}^{\infty} \varphi_n(\eta).
\]

Let \( N, N' \in 2^\mathbb{Z} \) be dyadic numbers. \( P_{N, N'} \) and \( P_0 = P_{0, 0} \) denote

\[
\mathcal{F}_x[P_{N, N'}f](\xi, \xi') := \varphi(|\xi|/N)\varphi(|\xi'|/N')\mathcal{F}_x[f](\xi, \xi') = \varphi_n(|\xi|)\varphi_{n'}(|\xi'|)\mathcal{F}_x[f](\xi, \xi'),
\]

\[
\mathcal{F}_x[P_{0}f](\xi, \xi') := \psi(|\xi|)\psi(|\xi'|)\mathcal{F}_x[f](\xi, \xi').
\]
Let $S(t) = \exp\{t(i\Delta - \partial_{x_k})\} : L^2_x \to L^2_x$ be the linear operator associated to the Schrödinger equation:

$$i(\partial_t + \partial_{x_k})u + \Delta u = 0.$$ 

Namely, $\mathcal{F}_x[S(t)f](\xi, \xi') := e^{-it(|\xi|^2 + \xi'^2)}\mathcal{F}_x f(\xi, \xi')$ for $\xi \in \mathbb{R}^{d-1}, \xi' \in \mathbb{R}$. Similarly, we define the wave unitary operator $W_\pm(t) := \exp\{\mp it \omega\} : L^2_x \to L^2_x$ such that $\mathcal{F}_x[W_\pm(t)g](\xi, \xi') := e^{\pm i|\xi|} \mathcal{F}_x g(\xi, \xi')$ for $\xi \in \mathbb{R}^{d-1}, \xi' \in \mathbb{R}$.

Let $Z$ be the set of finite partitions $-\infty = t_0 < t_1 < \cdots < t_K = \infty$.

**Definition 1.** For $\{t_k\}_{k=0}^K \in Z$ and $\{\phi_k\}_{k=0}^{K-1} \subset L^2_x$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{L^2_x}^2 = 1$ and $\phi_0 = 0$, we call the function $a : \mathbb{R} \to L^2_x$ given by

$$a = \sum_{k=1}^K \mathbf{1}_{(t_{k-1}, t_k)} \phi_{k-1}$$

a $U^2$-atom. Furthermore, we define the atomic space

$$U^2 := \left\{ u = \sum_{j=1}^\infty \lambda_j a_j \mid a_j : U^2\text{-atom}, \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^\infty |\lambda_j| < \infty \right\},$$

with norm

$$\|u\|_{U^2} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| \mid u = \sum_{j=1}^\infty \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j : U^2\text{-atom} \right\}.$$

**Definition 2.** We define $V^2$ as the normed space of all functions $v : \mathbb{R} \to L^2_x$ such that $\lim_{t \to \pm \infty} v(t)$ exist and for which the norm

$$\|v\|_{V^2} := \sup_{\{t_k\}_{k=0}^K \in Z} \left( \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2_x}^2 \right)^{1/2}$$

is finite, where we use the convention that $v(-\infty) := \lim_{t \to -\infty} v(t)$ and $v(\infty) := 0$. Note that $v(\infty)$ does not necessarily coincide with the limit at $\infty$. Likewise, let $V^-_x$ denote the closed subspace of all $v \in V^2$ with $\lim_{t \to -\infty} v(t) = 0$.

**Definition 3.** For $A = S$ or $W_\pm$, we define

(i) $U^2_A = A(\cdot)U^2$ with norm $\|u\|_{U^2_A} = \|A(\cdot)u\|_{U^2}$,

(ii) $V^2_A = A(\cdot)V^2$ with norm $\|u\|_{V^2_A} = \|A(\cdot)u\|_{V^2}$.

See [10], [11] for more detail.

**Definition 4.** For a Hilbert space $H$ and a Banach space $E \subset C(\mathbb{R}; H)$, we define $B_r(H) := \{ f \in H \mid \|f\|_H \leq r \}$ and

$$E([0, T]) := \{ u \in C([0, T]; H) \mid \exists \tilde{u} \in E, \tilde{u}(t) = u(t), t \in [0, T] \},$$
endowed with the norm \( \|u\|_{E([0,T])} = \inf \{ \|\tilde{u}\|_E | \tilde{u}(t) = u(t), t \in [0,T] \} \).

Hereafter, we denote \( E \) instead of \( E([0,T]) \) for brevity. We introduce the solution spaces \( X_S \) and \( Z_{W^\pm} \) below.

**Definition 5.** Let \( s_k > (d - 1)/2, s_l > (d - 2)/2, s > 1/2 \) and \( \varepsilon > 0 \) is sufficiently small such that \( s_k - (d - 1)/2 - \varepsilon > 0 \). We define the function spaces \( X_S, Y_{s_k, s}', Y_{s_l, s}' \), \( Z_{W^\pm} \) as follows:

\[
X_S := \{ u \in C([0, T]; H^{s_k, s'}(\mathbb{R}^d)) | \|u\|_{X_S} < \infty \}, \\
Y_{s_k, s}' := \{ u \in C([0, T]; H^{s_k, s'}(\mathbb{R}^d)) | \|u\|_{Y_{s_k, s}'} < \infty \}, \\
Z_{W^\pm} := \{ n \in C([0, T]; H^{s_l, s'}(\mathbb{R}^d)) | \|n\|_{Z_{W^\pm}} < \infty \},
\]

where

\[
\|u\|_{X_S} = J + K + M, \quad \|u\|_{Y_{s_k, s}'} = J, \quad \|n\|_{Z_{W^\pm}} = R,
\]

\[
J := \|P_0 u\|_{V^2_s} + \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \|P_{N, N'} u\|_{V^2_s}^2 \right)^{1/2},
\]

\[
K := \max_{1 \leq i \leq d-1} \left( \|P_0 u\|_{L^2_s L^{\infty}_{x_1} \ldots L^\infty_{x_{i-1} x_{i+1} \ldots x_d t}} + \left( \sum_{N, N' \geq 1} N^{2(s_k + 1/2)} N'^{2s'} \|P_{N, N'} u\|_{L^2_s L^{\infty}_{x_1} \ldots L^\infty_{x_{i-1} x_{i+1} \ldots x_d t}}^2 \right)^{1/2} \right),
\]

\[
M := \max_{1 \leq i \leq d-1} \left( \|P_0 u\|_{L^2_s L^\infty_{x_1} \ldots L^\infty_{x_{i-1} x_{i+1} \ldots x_d t}}^2 + \left( \sum_{N, N' \geq 1} N^{2(s_k - (d-1)/2 - \varepsilon)} N'^{2s'} \|P_{N, N'} u\|_{L^2_s L^{\infty}_{x_1} \ldots L^\infty_{x_{i-1} x_{i+1} \ldots x_d t}}^2 \right)^{1/2} \right),
\]

\[
R := \|P_0 n\|_{V^2_{W^\pm}} + \left( \sum_{N, N' \geq 1} N^{2s_l} N'^{2s'} \|P_{N, N'} n\|_{V^2_{W^\pm}}^2 \right)^{1/2}.
\]

Similarly we define \( Y_{W^\pm} \) as follows.

\[
Y_{W^\pm} := \{ n \in C([0, T]; H^{s_l, s'}(\mathbb{R}^d)) | \|n\|_{Y_{W^\pm}} < \infty \},
\]

\[
\|n\|_{Y_{W^\pm}} := \|P_0 n\|_{V^2_{W^\pm}} + \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \|P_{N, N'} n\|_{V^2_{W^\pm}}^2 \right)^{1/2}.
\]
The integral equations for (1.6) are as follows.

\[ I_T, S(n_+, n_-, u) = S(t)u_0 - \frac{i}{2} \int_0^t S(t - t')(n_+(t') + n_-(t'))u(t') \, dt', \]

\[ I_T, W_\pm(u, v) = W_\pm(t)n_{\pm0} \pm \int_0^t W_\pm(t - t')\omega(u(t')(\overline{v(t')})) \, dt'. \]

The proof of the following proposition for \( d = 3 \) can be found in [15]. Analogously, we can show (2.2).

**Proposition 2.1.** For \( d \geq 2, f \in L^2 \), it holds that

\[ \| S(t)P_0f \|_{L^\infty_{x_1}L^2_{x_2 \ldots x_{d-1}, t}} \lesssim \| P_0f \|_2, \]  \tag{2.1}

\[ \| D^{1/2}_{x_1}S(t)f \|_{L^\infty_{x_1}L^2_{x_2 \ldots x_{d-1}, t}} \lesssim \| f \|_2, \]  \tag{2.2}

where \( F_{x_1}[D^{1/2}_{x_1}u] = |\xi_1|^{1/2}F_{x_1}[u] \). (2.1), (2.2) hold exchanging \( x_1 \) for \( x_i, i \in \{2, \ldots, d - 1\} \).

**Remark 2.1.** From (2.2), we have \( \| D^{1/2}_{x_1}f \|_{L^\infty_{x_1}L^2_{x_2 \ldots x_{d-1}, t}} \lesssim \| f \|_{V^2_x} \).

**Proposition 2.2.** For \( d \geq 2, s > (d - 1)/2 \) and \( g \in L^2, h \in H^s_xL^2_{x_d} \), it holds that

\[ \| S(t)P_0g \|_{L^2_{x_1}L^\infty_{x_2 \ldots x_{d-1}, t}L^2_{x_d}} \lesssim \| P_0g \|_2, \]  \tag{2.3}

\[ \| S(t)h \|_{L^2_{x_1}L^\infty_{x_2 \ldots x_{d-1}, t}L^2_{x_d}} \lesssim \| h \|_{H^s_xL^2_{x_d}}, \]  \tag{2.4}

The above estimates hold exchanging \( x_1 \) for \( x_i, i \in \{2, \ldots, d - 1\} \).

**Proposition 2.3.** For \( d \geq 2, f \in S(\mathbb{R}^{d+1}) \), it holds that

\[ \left\| \int_0^t S(t - t')P_0f(t') \, dt' \right\|_{L^\infty_{t}L^2_{x}} \lesssim \| P_0f \|_{L^1_{x_1}L^2_{x_2 \ldots x_{d-1}, t}}, \]  \tag{2.5}

\[ \left\| D^{1/2}_{x_1} \int_0^t S(t - t')f(t') \, dt' \right\|_{L^\infty_{x_1}L^2_{x}} \lesssim \| f \|_{L^1_{x_1}L^2_{x_2 \ldots x_{d-1}, t}}. \]  \tag{2.6}

The above estimates hold exchanging \( x_1 \) for \( x_i, i \in \{2, \ldots, d - 1\} \).

**Proof.** The dual estimate of Proposition 2.1 (2.1) is

\[ \left\| \int_{-\infty}^\infty S(-t')P_0f(t')dt' \right\|_{L^2_{x}} \lesssim \| P_0f \|_{L^1_{x_1}L^2_{x}}. \]

Since \( S(t) \) is the unitary operator on \( L^2_{x_1} \),

\[ \left\| \int_{-\infty}^\infty S(t - t')P_0f(t')dt' \right\|_{L^2_{x}} \lesssim \| P_0f \|_{L^1_{x_1}L^2_{x}}. \]
Replacing $f(t')$ by $1_{[0, t]}(t')f(t')$, then take $L^\infty_t$ norm and (2.5) follows. (2.6) follows from [15] Proposition 2.1 for $d = 3$. The other cases are treated similarly, hence we omit the proof. \hfill \Box

**Proposition 2.4.** For $d \geq 2$, $f \in S(\mathbb{R}^{d+1})$, the following estimates hold:

\[
\left\| \int_0^t S(t - t') P_0 f(t') \, dt' \right\|_{L^\infty_x L^2_t} \lesssim \left\| P_0 f \right\|_{L^1_x L^2_t} \lesssim \left\| P_0 f \right\|_{L^1_x L^2_{x, t}^{d-1, d}}, \tag{2.7}
\]

\[
\left\| \partial_{x_1} \int_0^t S(t - t') f(t') \, dt' \right\|_{L^\infty_x L^2_t} \lesssim \left\| f \right\|_{L^1_x L^2_{x, t}^{d-1, d}}. \tag{2.8}
\]

The above estimates hold exchanging $x_1$ for $x_i$, $i \in \{2, \ldots, d-1\}$.

**Proof.** From the Sobolev inequality and (2.5), we have

\[
\left\| \int_0^t S(t - t') P_0 f(t') \, dt' \right\|_{L^\infty_x L^2_t} \lesssim T^{1/2} \left\| \nabla_{x_1} \right\|^{1/2} \left\| \int_0^t S(t - t') P_0 f(t') \, dt' \right\|_{L^\infty_t L^2_x} \lesssim \left\| \int_0^t S(t - t') P_0 f(t') \, dt' \right\|_{L^\infty_t L^2_x} \lesssim \left\| P_0 f \right\|_{L^1_x L^2_t} \lesssim \left\| P_0 f \right\|_{L^1_x L^2_{x, t}^{d-1, d}}.
\]

Thus (2.7) follows. The proof of (2.8) for $d = 3$ can be found in [15] Proposition 2.1. Analogously we have (2.8). \hfill \Box

**Proposition 2.5.** Let $d \geq 2$, $s > (d - 1)/2$, $s' > 1/2$, $f \in S(\mathbb{R}^{d+1})$. It holds that

\[
\left\| \int_0^t S(t - t') P_0 f(t') \, dt' \right\|_{L^\infty_x L^{s, s'}_{x, t}^{d-1, d}} \lesssim \left\| P_0 f \right\|_{L^1_x L^{2, \infty}_{x, t}^{d-1, d}}, \tag{2.9}
\]

\[
\left\| D_{x_1}^{1/2} \int_0^t S(t - t') P_N, N' f(t') \, dt' \right\|_{L^\infty_x L^{s, s'}_{x, t}^{d-1, d}} \lesssim N^s \left\| P_N, N' f \right\|_{L^1_x L^{2, \infty}_{x, t}^{d-1, d}}. \tag{2.10}
\]

The above estimates hold exchanging $x_1$ for $x_i$, $i \in \{2, \ldots, d-1\}$.

**Proof.** We only check (2.10). We assume $|\xi_1| = \max_{1 \leq i \leq d-1} |\xi_i|$. From (2.4) and the dual estimate of (2.2)

\[
\left\| \int_{\mathbb{R}} S(-t') D_{x_1}^{1/2} f(t') \, dt' \right\|_{L^2_x} \lesssim \left\| f \right\|_{L^1_x L^{2, \infty}_{x, t}^{d-1, d}},
\]

we obtain for $s > (d - 1)/2$

\[
\left\| D_{x_1}^{1/2} \int_0^T S(t - t') P_N, N' f(t') \, dt' \right\|_{L^\infty_x L^{s, s'}_{x, t}^{d-1, d}} \lesssim N^s \left\| P_N, N' f \right\|_{L^1_x L^{2, \infty}_{x, t}^{d-1, d}}.
\]

Then we replace $f(t')$ by $1_{[0, t]}(t')f(t')$, we obtain the desired result. \hfill \Box

We introduce the Strichartz estimate which was proved by [1] when $d = 3$. 
Proposition 2.6. Let \( d \geq 2, u_0 \in L^2 \). For the admissible pair \((q, r)\), namely \( 2/q = (d-1)(1/2 - 1/r) \), \((q, r, d) \neq (2, \infty, 3)\), it holds that
\[
\|S(t)u_0\|_{L_t^qL_x^rL_{x,d}^2} \lesssim \|u_0\|_2.
\]

Remark 2.2. For \( d \geq 3 \), \((q, r) = (4, 2(d-1)/(d-2))\) is an admissible pair, from the property of \( U^p, V^p \), it holds that \( V_2^2 \hookrightarrow U_S^4 \hookrightarrow L^2_tL_x^{2(d-1)/(d-2)}L_{x,d}^2 \). When \( d = 2 \), \((q, r) = (4, \infty)\) is an admissible pair, then \( V_2^2 \hookrightarrow U_S^4 \hookrightarrow L^4_tL_x^2L_{x,2}^2 \) holds. See \([10]\). The following trilinear estimates will be applied in the proof of Proposition 3.1.

Lemma 2.7. We denote \( P_{N_1,N_1'} = n_{N,N'}, u_{N_2,N_2'}, v_{N_2,N_2'} \), \( P_{N,N'}v = u_{N,N'} \) for dyadic numbers \( N_1, N_2, N, N_1', N_2', N' \). Then the following estimates hold: (i)
\[
\left| \int_{\mathbb{R}^{d+1}} 1_{[0,T]}n_{N_1,N_1'}u_{N_2,N_2'}v_{N,N'} \, dxdt \right| \lesssim T^{1/2}N^{-1/2}N_{\min}^{1/2} \left\| n_{N_1,N_1'} \right\|_{L_t^\infty L_x^2} \left\| u_{N_2,N_2'} \right\|_{L_x^2 L_{x,d}^{\infty}} \left\| v_{N,N'} \right\|_{L_x^2}, \quad (2.11)
\]
(ii) If \( d \geq 3 \), we have
\[
\left| \int_{\mathbb{R}^{d+1}} 1_{[0,T]}n_{N_1,N_1'}u_{N_2,N_2'}v_{N,N'} \, dxdt \right| \lesssim T^{1/2}N_1^{(d-3)/2}N_{\min}^{1/2} \left\| n_{N_1,N_1'} \right\|_{L_t^\infty L_x^2} \left\| u_{N_2,N_2'} \right\|_{V_S^2} \left\| v_{N,N'} \right\|_{V_S^2}, \quad (2.12)
\]
(iii) If \( d = 2 \), we have
\[
\left| \int_{\mathbb{R}^3} 1_{[0,T]}n_{N_1,N_1'}u_{N_2,N_2'}v_{N,N'} \, dxdt \right| \lesssim T^{3/4}N_{\min}^{1/2} \left\| n_{N_1,N_1'} \right\|_{L_t^\infty L_x^2} \left\| u_{N_2,N_2'} \right\|_{V_S^2} \left\| v_{N,N'} \right\|_{V_S^2}, \quad (2.13)
\]
where \( N_{\min} = \min\{N', N_1', N_2'\}\). \((2.11)\) holds exchanging \( x_1 \) for \( x_i \), \( i \in \{2, \ldots, d-1\} \).

Proof. We only show \( N_2' = \min\{N', N_1', N_2'\} \) since the other cases are treated similarly. (i) Without loss of generality, we assume \( |\xi_i| = \max_{1 \leq i \leq d-1} |\xi_i| \) where \( \xi_i \) denotes the \( i \)-th component of \( \xi \). By the Hölder inequality and the Sobolev inequality, we have
\[
(L.H.S. of (2.11)) \lesssim T^{1/2}N_{2'}^{1/2} \left\| n_{N_1,N_1'} \right\|_{L_t^\infty L_x^2} \left\| u_{N_2,N_2'} \right\|_{L_x^2 L_{x,d}^{\infty}} \left\| v_{N,N'} \right\|_{V_S^2}, \quad (2.14)
\]
Then Remark 2.1 leads the desired result. (ii) follows from the Hölder inequality, the Sobolev inequality and Remark 2.2. Indeed,
\[
(L.H.S. of (2.12)) \lesssim \left\| n_{N_1,N_1'} \right\|_{L_t^\infty L_x^2} \left\| u_{N_2,N_2'} \right\|_{L_x^2 L_{x,d}^{(d-1)/(d-2)}} \left\| v_{N,N'} \right\|_{L_x^2 L_{x,d}^{2(d-1)/(d-2)}} \lesssim T^{1/2}N_1^{(d-3)/2}N_{2'}^{1/2} \left\| n_{N_1,N_1'} \right\|_{L_t^\infty L_x^2} \left\| u_{N_2,N_2'} \right\|_{V_S^2} \left\| v_{N,N'} \right\|_{V_S^2}.
\]
By the Hölder inequality, the Sobolev inequality and Remark 2.2, we have
\[
(L.H.S. \text{ of } (2.13)) \lesssim T^{3/4} \|n_{N_1, N_1'} \|_{L_t^\infty L_x^2} \|u_{N_2, N_2'} \|_{L_t^\infty L_x^2} \|v_{N, N'} \|_{L_t^1 L_x^2}.
\]
\[
\lesssim T^{3/4} N_2^{\alpha/2+} \|n_{N_1, N_1'} \|_{L_t^\infty L_x^2} \|u_{N_2, N_2'} \|_{L_t^\infty L_x^2} \|v_{N, N'} \|_{V_3^2}.
\]
Then from \( V_3^2 \hookrightarrow L_t^\infty L_x^2 \), (iii) follows. \( \square \)

3. NONLINEAR ESTIMATES

In this section, we derive the nonlinear estimate for the Duhamel term. For the linear part, we see from Proposition 2.1 and Proposition 2.2 that
\[
\| P_0 S(t) u_0 \|_{V_3^2} \lesssim \| P_0 u_0 \|_2,
\]
\[
\left( \sum_{N, N' \geq 1} N^{2s_k} N^{2s'} \| P_{N, N'} S(t) u_0 \|_{L_t^\infty L_x^2}^2 \right)^{1/2} \lesssim \| u_0 \|_{H^{s_k, s'}}.
\]
\[
\left( \sum_{N, N' \geq 1} N^{2(s_k+1/2)} N^{2s'} \| P_{N, N'} S(t) u_0 \|_{L_t^\infty L_x^2}^2 \right)^{1/2} \lesssim \| u_0 \|_{H^{s_k, s'}}.
\]
for \( i = 1, \ldots, d-1 \). Moreover,
\[
\| P_0 W_\pm(t) n_{\pm 0} \|_{U_{W, \pm}} \lesssim \left( \sum_{N_1, N_1' \geq 1} N^{2s_k} N^{2s'} \| P_{N_1, N_1'} W_{\pm}(t) n_{\pm 0} \|_{U_{W, \pm}}^2 \right)^{1/2} \lesssim \| n_{\pm 0} \|_{H^{s_k, s'}}.
\]

Now, we state and prove the nonlinear estimate.

**Proposition 3.1.** Let \( d \geq 2, s_k > (d-1)/2, s_l > (d-2)/2, s_k - s_l = 1/2, s' > 1/2, u_0 \in H^{s_k, s'}, n_{\pm 0} \in H^{s_l, s'}, \omega = (-\Delta)_{1/2} \), and \( 0 < T < 1 \). Then, it holds that
\[
\left\| \int_0^t S(t-t') n(t') u(t') \, dt' \right\|_{Z_{W, \pm}^{s_k, s'}} \lesssim T^{1/2} \| n \|_{Z_{W, \pm}^{s_k, s'}} \| u \|_{X_S}, \quad (3.1)
\]
\[
\left\| \int_0^t W_\pm(t-t') (\hat{\omega} \hat{v})(t') \, dt' \right\|_{Z_{W, \pm}^{s_k, s'}} \lesssim T^{1/2} \| u \|_{X_S} \| v \|_{X_S}. \quad (3.2)
\]

**Proof.** To prove (3.1) we estimate the high and the low frequency part of the Duhamel term respectively. We firstly estimate the high frequency part of \( J \), namely
\[
\left( \sum_{N, N' \geq 1} N^{2s_k} N^{2s'} \| P_{N, N'} \int_0^t S(t-t') n(t') u(t') \, dt' \|_{V_3^2}^2 \right)^{1/2}.
\]
For brevity, we denote $P_{N, N'} f = f_{N, N'}$. We set $J_i, J_{4, i}, i = 1, 2, 3$ where

$$J_1 := \left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \sup_{\|v\|_{L^2} = 1} \left| \sum_{N_2 \leq N} \sum_{N_1 \sim N} \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1} u_{N_2} v_{N, N'} \, dxdt \right|^2 \right)^{1/2},$$

$$J_2 := \left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \sup_{\|v\|_{L^2} = 1} \left| \sum_{N_2 \gg N} \sum_{N_1 \sim N} \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1} u_{N_2} v_{N, N'} \, dxdt \right|^2 \right)^{1/2},$$

$$J_3 := \left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \sup_{\|v\|_{L^2} = 1} \left| \sum_{N_2 \sim N} \sum_{N_1 \sim N} \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1} u_{N_2} v_{N, N'} \, dxdt \right|^2 \right)^{1/2},$$

$$J_{4, 1} := \left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \left\| P_{N, N'} \int_0^t S(t - t') n(t') u(t') \, dt' \right\|_{V^2}^2 \right)^{1/2},$$

$$J_{4, 2} := \left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \gg N} \sum_{N_1 \sim N} n_{N_1} u_{N_2} \, dt' \right\|_{V^2}^2 \right)^{1/2},$$

$$J_{4, 3} := \left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \sim N} \sum_{N_1 \sim N} n_{N_1} u_{N_2} \, dt' \right\|_{V^2}^2 \right)^{1/2}.$$

Then from [12] Corollary 2.10

$$\left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \left\| P_{N, N'} \int_0^t S(t - t') n(t') u(t') \, dt' \right\|_{V^2}^2 \right)^{1/2} \lesssim \sum_{i=1}^3 (J_i + J_{4, i}).$$

Let us separate $J_1$ into the following three parts.

$$J_{1, 1} := \left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \sup_{\|v\|_{L^2} = 1} \left| \sum_{N_2 \leq N} \sum_{N_1 \sim N} \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1, N_2} u_{N_2, N'} \, dxdt \right|^2 \right)^{1/2},$$

$$J_{1, 2} := \left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \sup_{\|v\|_{L^2} = 1} \left| \sum_{N_2 \gg N} \sum_{N_1 \sim N} \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1, N_2} u_{N_2, N'} \, dxdt \right|^2 \right)^{1/2},$$

$$J_{1, 3} := \left( \sum_{N, N' \geq 1} N^{2s} N' N^{2s'} \sup_{\|v\|_{L^2} = 1} \left| \sum_{N_2 \sim N} \sum_{N_1 \sim N} \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1, N_2} u_{N_2, N'} \, dxdt \right|^2 \right)^{1/2}.$$

Similarly, we have $J_i \lesssim \sum_{i=1}^3 J_{1, i}, J_{4, k} \lesssim \sum_{i=1}^3 J_{4, k, i}, (i = 2, 3, k = 1, 2, 3)$. For brevity, we only consider the case $N'_2 \lesssim N' \sim N_1$, namely $J_{1, 1}$ since the other cases are treated similarly from $L^{\infty}(\mathbb{R}) \leftrightarrow H^{s'}(\mathbb{R})$ with $s' > 1/2$. Hereafter we consider $|\xi| = \max_{1 \leq i \leq d-1} |\xi_i|$ for brevity. Now we estimate $J_{1, 1}$. From Lemma 2.4.
\(s_k - s_l = 1/2, s_k - (d - 1)/2 - \varepsilon > 0\), the Cauchy-Schwarz inequality and \(\|n\|_{L_t^\infty H_x^{s'_2}} \lesssim R\), we have

\[
J_{1,1} \lesssim T^{1/2} \left( \sum_{N, N' \geq 1} N^{2s_k - 1} N^{2s'_2} \left( \sum_{N_1 \sim N, N'_1 \sim N'} \|n_{N_1}, N'_1\|_{L_t^\infty L_x^2} \left( \|P_0 u\|_{L_x^2 L_t^\infty s_{d-1} L_x^2} \right)^2 \right) \right)^{1/2}
\]

\[
+ \sum_{1 \leq N_2 \leq N, 1 \leq N'_2 \leq N'} N_2^{1/2} \|u_{N_2}, N'_2\|_{L_x^\infty s_{d-1} L_x^2}^2 \left( \sum_{N_1 \sim N, N'_1 \sim N'} \|n_{N_1}, N'_1\|_{L_t^\infty L_x^2} \left( \|P_0 u\|_{L_x^2 L_t^\infty s_{d-1} L_x^2} \right)^2 \right) \right)^{1/2}
\]

\[
\lesssim T^{1/2} \left( \sum_{N_1, N'_1 \geq 1} N_1^{2s_k} N'_1^{2s'_2} \|n_{N_1}, N'_1\|_{L_t^\infty L_x^2}^2 \left( \|P_0 u\|_{L_x^2 L_t^\infty s_{d-1} L_x^2}^2 \right) \right)^{1/2}
\]

\[
+ \sum_{1 \leq N_2 \leq N_1, 1 \leq N'_2 \leq N'_1} N_2^{2(s_k - (d - 1)/2 - \varepsilon)} N'_2^{2s'_2} \|u_{N_2}, N'_2\|_{L_x^\infty s_{d-1} L_x^2}^2 \left( \sum_{N_1 \sim N, N'_1 \sim N'} \|n_{N_1}, N'_1\|_{L_t^\infty L_x^2} \left( \|P_0 u\|_{L_x^2 L_t^\infty s_{d-1} L_x^2} \right)^2 \right) \right)^{1/2}
\]

\[
\lesssim T^{1/2} M \|n\|_{L_t^\infty H_x^{s'_2}} \lesssim T^{1/2} MR.
\]

Let us estimate \(J_{2,1}\). From Lemma 2.7 (2.11), \(s_k - s_l = 1/2\), the Cauchy-Schwarz inequality, \(l^1 l^2 \rightarrow l^2 l^1\) and \(s_k - (d - 1)/2 - \varepsilon > 0\), we have

\[
J_{2,1} = \left( \sum_{N, N' \geq 1} N^{2s_k} N^{2s'_2} \sup_{\|\|_{L_x^2} = 1} \left( \sum_{N_2 \gg N, N'_2 \lesssim N} \| \sum_{N_1 \sim N_2, N'_1 \sim N'} \sum_{N_2 \gg N, N'_2 \lesssim N} \|n_{N_1}, N'_1\|_{L_t^\infty L_x^2} \left( \|P_0 u\|_{L_x^2 L_t^\infty s_{d-1} L_x^2} \right)^2 \right) \right)^{1/2}
\]

\[
\lesssim T^{1/2} \left( \sum_{N, N' \geq 1} N^{2s_k - 1} N^{2s'_2} \left( \sum_{N_2 \gg N, N'_2 \lesssim N} \|n_{N_1}, N'_1\|_{L_t^\infty L_x^2} \left( \|P_0 u\|_{L_x^2 L_t^\infty s_{d-1} L_x^2} \right)^2 \right) \right)^{1/2}
\]

\[
\lesssim T^{1/2} \left( \sum_{N_2 \gg 1} \sum_{N_1 \sim N_2} \left( N_1^{2s_k} \|n_{N_1}\|_{L_t^\infty L_x^2 H_x^{s'_2}}^2 \|u_{N_2}\|_{L_x^2 L_t^\infty s_{d-1} L_x^2} \right)^{1/2} \|n\|_{L_t^\infty H_x^{s'_2}} \right) \lesssim T^{1/2} MR.
\]
Let us estimate $J_{3,1}$. When $d \geq 3$, from Lemma 2.7 (2.12), $U_S^2 \mapsto V_S^2$, $s_t > (d - 2)/2$ and the Cauchy-Schwarz inequality we have

$$J_{3,1} = \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \sup_{\|v\|_{L^2}} \left| \sum_{N_2 \sim N, N'_2 \lesssim N'} \sum_{N_1 \sim N_2} N_1^{d-1/2} + N_2'^{1/2+} \right| \right)^{1/2} \lesssim T^{1/2} \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left( \sum_{N_2 \sim N, N'_2 \lesssim N'} \sum_{N_1 \sim N_2} N_1^{d-1/2} + N_2'^{1/2+} \right) \right)^{1/2} \lesssim T^{1/2} \|u\|_{Y_{s_k, s'}} \|n\|_{L^2_t H^1_x} \lesssim T^{1/2} J R.$$ 

For $d = 2$, by Lemma 2.7 (2.13) we have

$$J_{3,1} \lesssim T^{3/4} \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left( \sum_{N_2 \sim N, N'_2 \lesssim N'} \sum_{N_1 \sim N_2} N_1^{d-1/2} + N_2'^{1/2+} \right) \right)^{1/2} \lesssim T^{3/4} \|u\|_{Y_{s_k, s'}} \|n\|_{L^2_t H^1_x} \lesssim T^{3/4} J R.$$ 

We estimate $J_{4,1,1}$ below. From $|\xi_1| = \max_{1 \leq i \leq d-1} |\xi_i|$, Proposition 2.3 (2.6), $s_k - s_t = 1/2$, the Hölder inequality, $s_k - (d-1)/2 - \varepsilon > 0$ and the Cauchy-Schwarz inequality

$$J_{4,1,1} = \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left| \int_0^t S(t - t') \sum_{N_2 \lesssim N, N'_2 \lesssim N'} \sum_{N_1 \sim N_2} \left| n_{N_1, N'_1} u_{N_2, N'_2} \right| dt \right|^2 \right)^{1/2} \lesssim \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left| \sum_{N_2 \lesssim N, N'_2 \lesssim N'} \sum_{N_1 \sim N_2} \left| n_{N_1, N'_1} u_{N_2, N'_2} \right| \right|^2 \right)^{1/2} \lesssim (3.3) \lesssim T^{1/2} M \|n\|_{L^\infty_t H^1_x} \lesssim T^{1/2} M R.$$
We estimate $J_{4,2,1}$. From $|\xi_1| = \max_{1 \leq i \leq d-1} |\xi_i|$, Proposition 2.3 (2.6), $s_k - s_l = 1/2$, the Hölder inequality and the Sobolev inequality

$$J_{4,2,1} = \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \geq N, N'_2 \leq N' N_1 - N_2, N'_1 - N'} n_{N_1, N'_1} u_{N_2, N'_2} dt' \right\|_{L_t^2 L_x^2}^{2} \right)^{1/2}$$

$$\lesssim \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left\| \sum_{N_2 \geq N, N'_2 \leq N' N_1 - N_2, N'_1 - N'} n_{N_1, N'_1} u_{N_2, N'_2} \right\|_{L_t^2 L_x^2}^{2} \right)^{1/2}$$

$$\lesssim T^{1/2} \left( \sum_{N, N' \geq 1} N_1^{2s_1} N'^{2s'} \left\| \sum_{N_2 \geq N, N'_2 \leq N' N_1 - N_2, N'_1 - N'} u_{N_2, N'_2} \right\|_{L_t^2 L_x^2}^{2} \right)^{1/2}$$

$$\lesssim T^{1/2} M \| n \|_{L_t^\infty H_x^{s_1, s'}} \lesssim T^{1/2} MR.$$

Now we estimate $J_{4,3,1}$. By $U_2 \hookrightarrow L_t^\infty L_x^2$, we have

$$J_{4,3,1} = \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \geq N, N'_2 \leq N' N_1 - N_2, N'_1 - N'} n_{N_1, N'_1} u_{N_2, N'_2} dt' \right\|_{L_t^2 L_x^2}^{2} \right)^{1/2}$$

$$\lesssim \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \geq N, N'_2 \leq N' N_1 - N_2, N'_1 - N'} n_{N_1, N'_1} u_{N_2, N'_2} \right\|_{L_t^2 L_x^2}^{2} \right)^{1/2}$$

$$\lesssim \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left\| \sup_{\| v \|_{V^2_2} = 1} \sum_{N_2 \geq N, N'_2 \leq N' N_1 - N_2, N'_1 - N'} \int_{R^{d+1}} 1_{[0, T]} n_{N_1, N'_1} u_{N_2, N'_2} \frac{U_{N, N'}}{\| U_{N, N'} \|_{L_t^\infty L_x^2}} dtdx \right\|_{L_t^2 L_x^2}^{2} \right)^{1/2}.$$ 

For $d \geq 3$, we apply Lemma 2.7 (2.12), $s_l > (d - 2)/2$ and the Cauchy-Schwarz inequality, then we have

$$\text{R.H.S. of (3.5)} \lesssim \left( \sum_{N, N' \geq 1} N^{2s_k} N'^{2s'} \left( \sum_{N_2 \geq N, N'_2 \leq N' N_1 - N_2, N'_1 - N'} T^{1/2} N_1^{(d-3)/2+} N_2^{1/2+} \| n_{N_1, N'_1} \|_{L_t^\infty L_x^2} \| u_{N_2, N'_2} \|_{V^2_2} \right)^{2}/2 \right)^{1/2}$$

$$\lesssim T^{1/2} \| u \|_{Y^{s_k, s'}_2} \| n \|_{L_t^\infty H_x^{s_1, s'}} \lesssim T^{1/2} JR.$$
For $d = 2$, by Lemma 2.7 (2.13) and the Cauchy-Schwarz inequality,

\[
(R.H.S. \text{ of } (3.3)) \lesssim \left( \sum_{N, N' \geq 1} N^{2s_k} N^{r_{2s'}} \left( \sum_{N_2 \sim N, N'_2 \lesssim N'} K_1 N_1 \ll N_2, N'_2 \right)^{1/2} \right)^{1/2} \lesssim T^{3/4} \left( ||u||_{Y_S^{s_k, s'}} ||n||_{L_t^\infty H_z^{s_k, s'}} \right) \lesssim T^{3/4} JR.
\]

Let us estimate $K$. We first estimate high frequency part of the Duhamel term and we only consider the case $|\xi_1| = \max_{1 \leq i \leq d-1} |\xi_i|$, namely

\[
\left( \sum_{N, N' \geq 1} N^{2(s_k+1/2)} N^{r_{2s'}} \right) \left[ P_{N, N'} \int_0^t S(t - t')(nu)(t') dt' \right]^{1/2}.
\]

The above term is bounded by

\[
\left( \sum_{N, N' \geq 1} N^{2(s_k+1/2)} N^{r_{2s'}} \right) \left[ P_{N, N'} \int_0^t S(t - t') \left( \sum_{N_2 \ll N} \sum_{N_1 \sim N} + \sum_{N_2 \gg N} \sum_{N_1 \sim N_2} + \sum_{N_2 \sim N} \sum_{N_1 \ll N_2} \right) (n_{N_1}u_{N_2})(t') \right]^{1/2} dt' \lesssim K_1 + K_2 + K_3,
\]

where

\[
K_1^2 := \sum_{N, N' \geq 1} N^{2(s_k+1/2)} N^{r_{2s'}} \left( P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \ll N} \sum_{N_1 \sim N} (n_{N_1}u_{N_2})(t') dt' \right)^2,
\]

\[
K_2^2 := \sum_{N, N' \geq 1} N^{2(s_k+1/2)} N^{r_{2s'}} \left( P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \gg N} \sum_{N_1 \sim N_2} (n_{N_1}u_{N_2})(t') dt' \right)^2,
\]

\[
K_3^2 := \sum_{N, N' \geq 1} N^{2(s_k+1/2)} N^{r_{2s'}} \left( P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \sim N} \sum_{N_1 \ll N_2} (n_{N_1}u_{N_2})(t') dt' \right)^2.
\]
Let us separate $K_1$ into the following three parts.

\[
K_{1,1} := \left( \sum_{N, N' \geq 1} N^{2(s_k + 1/2)} N'^{2s'} \right) \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \leq N, N'_2 \leq N'} \sum_{N_1 \sim N, N'_1 \sim N'} n_{N_1, N'_1} u_{N_2, N'_2} dt' \right\|_{L^2_{x_1} L^2_{x_2} \ldots x_{d,t}}^2 \right)^{1/2},
\]

\[
K_{1,2} := \left( \sum_{N, N' \geq 1} N^{2(s_k + 1/2)} N'^{2s'} \right) \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \leq N, N'_2 \gg N'} \sum_{N_1 \sim N, N'_1 \sim N'} n_{N_1, N'_1} u_{N_2, N'_2} dt' \right\|_{L^2_{x_1} L^2_{x_2} \ldots x_{d,t}}^2 \right)^{1/2},
\]

\[
K_{1,3} := \left( \sum_{N, N' \geq 1} N^{2(s_k + 1/2)} N'^{2s'} \right) \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \leq N, N'_2 \sim N'} \sum_{N_1 \ll N, N'_1 \ll N'_2} n_{N_1, N'_1} u_{N_2, N'_2} dt' \right\|_{L^2_{x_1} L^2_{x_2} \ldots x_{d,t}}^2 \right)^{1/2}.
\]

Likewise, we have $K_i \lesssim \sum_{j=1}^3 K_{i,j}$, $i = 2, 3$. For simplicity, we only show the case $N'_2 \lesssim N' \sim N'_1$, namely $K_{i,1}$, $i = 1, 2, 3$. We estimate $K_{1,1}$ as follows. By $|\xi_1| = \max_{1 \leq i \leq d-1} |\xi_i|$, Proposition 2.4, (2.8), (3.3) and the estimate for $J_{4,1,1}$,

\[
K_{1,1} \lesssim \left( \sum_{N, N' \geq 1} N^{2(s_k + 1/2) - 2} N'^{2s'} \right) \left\| P_{N, N'} \left( \sum_{N_2 \leq N, N'_2 \leq N'} \sum_{N_1 \sim N, N'_1 \sim N'} n_{N_1, N'_1} u_{N_2, N'_2} \right) \right\|_{L^2_{x_1} L^2_{x_2} \ldots x_{d,t}}^2 \right)^{1/2}
\]

\[
\lesssim T^{1/2} MR.
\]

We estimate $K_{2,1}$. By Proposition 2.4, (2.8), (3.4) and the estimate for $J_{4,2,1}$,

\[
K_{2,1} := \left( \sum_{N, N' \geq 1} N^{2(s_k + 1/2)} N'^{2s'} \right) \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \gg N, N'_2 \leq N'} \sum_{N_1 \sim N_2, N'_1 \sim N'} n_{N_1, N'_1} u_{N_2, N'_2} dt' \right\|_{L^2_{x_1} L^2_{x_2} \ldots x_{d,t}}^2 \right)^{1/2}
\]

\[
\lesssim \left( \sum_{N, N' \geq 1} N^{2(s_k + 1/2) - 2} N'^{2s'} \right) \left\| P_{N, N'} \sum_{N_2 \gg N, N'_2 \leq N'} \sum_{N_1 \sim N_2, N'_1 \sim N'} n_{N_1, N'_1} u_{N_2, N'_2} \right\|_{L^2_{x_1} L^2_{x_2} \ldots x_{d,t}}^2 \right)^{1/2}
\]

\[
\lesssim T^{1/2} MR.
\]
We estimate $K_{3,1}$. From $|\xi_1| = \max_{1 \leq i \leq d-1} |\xi_i|$ and Remark 2.1, we have

$$K_{3,1} := \left( \sum_{N, N' \geq 1} N^{2(s_k + 1/2)} N'^{2s'} \right) \left( \sum_{N_2 \sim N, N_2' \sim N'} \sum_{N_1 \sim N} \sum_{N_1' \sim N'} S(t - t') \right) \left( \sum_{N_2 \sim N, N_2' \sim N'} \sum_{N_1 \sim N} \sum_{N_1' \sim N'} n_{N_1, N_1'} u_{N_2, N_2'} dt^t \left| \left( \int_0^t \right) \right| \right)^{1/2}$$

From (3.5) and the estimate for $J_{4,1,3,1}$, we obtain

$$K_{3,1} \lesssim \begin{cases} T^{1/2} JR & (d \geq 3), \\ T^{3/4} JR & (d = 2). \end{cases}$$

Let us estimate $M$. We only estimate the following high frequency part of the Duhamel term, namely

$$\left( \sum_{N, N' \geq 1} N^{2(s_k - (d-1)/2 - \varepsilon)} N'^{2s'} \right) \left( \sum_{N_2 \sim N} \sum_{N_1 \sim N} \sum_{N_2' \sim N_1' \sim N_2} S(t - t')(nu)(t') dt' \right)^{1/2}$$

The above term is bounded by

$$\left( \sum_{N, N' \geq 1} N^{2(s_k - (d-1)/2 - \varepsilon)} N'^{2s'} \right) \left( \sum_{N_2 \sim N} \sum_{N' \sim N} \sum_{N_2' \sim N_1' \sim N_2} S(t - t')(nu)(t') dt' \right)^{1/2} + \sum_{N_2 \sim N} \sum_{N_1 \sim N_2} (n_{N_1} u_{N_2})(t') dt' \left| \left( \int_0^t \right) \right| \right)^{1/2}$$

$$\lesssim M_1 + M_2 + M_3.$$
where

\[
M_1 := \left( \sum_{N, N' \geq 1} N^{2(s_k-(d-1)/2-\varepsilon)} N'^{2s'} \right) \left\| P_{N, N'} \int_0^t S(t - t') \right\|_{L^2_{x_1} L^{\infty}_{x_2 \cdots x_{d-1}, t} L^2_{x_d}}^2,
\]
\[
M_2 := \left( \sum_{N, N' \geq 1} N^{2(s_k-(d-1)/2-\varepsilon)} N'^{2s'} \right) \left\| P_{N, N'} \int_0^t S(t - t') \right\|_{L^2_{x_1} L^{\infty}_{x_2 \cdots x_{d-1}, t} L^2_{x_d}}^2,
\]
\[
M_3 := \left( \sum_{N, N' \geq 1} N^{2(s_k-(d-1)/2-\varepsilon)} N'^{2s'} \right) \left\| P_{N, N'} \int_0^t S(t - t') \right\|_{L^2_{x_1} L^{\infty}_{x_2 \cdots x_{d-1}, t} L^2_{x_d}}^2.
\]

From Proposition \[2.5\] (2.10) and \(|\xi_1| = \max_{1 \leq i \leq d-1} |\xi_i|\), \(M_1\) is bounded by

\[
\left( \sum_{N, N' \geq 1} N^{2(s_k-1/2)} N'^{2s'} \right) \left\| P_{N, N'} \sum_{N_2 \lesssim N, N_1 \sim N} n_{N_1} u_{N_2} \right\|_{L^1_{x_1} L^2_{x_2 \cdots x_{d-1}, t}}^2 \right)^{1/2}.
\]

Then by \(s_k - s_l = 1/2\), \([3.6]\) is bounded by \(M_{1,i}, i = 1, 2, 3\),

\[
M_1 \lesssim \left( \sum_{N, N' \geq 1} N^{2s_k N'^{2s'}} \right) \left( \sum_{N_2 \lesssim N, N_2' \lesssim N'} \sum_{N_1' \sim N'} \sum_{N_2' \lesssim N, N_2' \sim N'} \sum_{N_2 \lesssim N, N_2' \sim N'} \right) P_{N, N'} \left( n_{N_1, N_1'} (n_{N_2, N_2'}) \right) \left\| \left\| P_{N, N'} \left( n_{N_3, N_3'} \right) \right\|_{L^1_{x_1} L^2_{x_2 \cdots x_{d-1}, t}}^2 \right)^{1/2},
\]

\[
\lesssim M_{1,1} + M_{1,2} + M_{1,3},
\]

where

\[
M_{1,1}^2 := \sum_{N, N' \geq 1} N^{2s_k N'^{2s'}} \left\| P_{N, N'} \left( \sum_{N_2 \lesssim N, N_2' \lesssim N'} \sum_{N_1' \sim N'} n_{N_1, N_1'} (n_{N_2, N_2'}) \right) \right\|_{L^1_{x_1} L^2}^2,
\]
\[
M_{1,2}^2 := \sum_{N, N' \geq 1} N^{2s_k N'^{2s'}} \left\| P_{N, N'} \left( \sum_{N_2 \lesssim N, N_2' \lesssim N'} \sum_{N_1' \sim N'} n_{N_1, N_1'} (n_{N_2, N_2'}) \right) \right\|_{L^1_{x_1} L^2}^2,
\]
\[
M_{1,3}^2 := \sum_{N, N' \geq 1} N^{2s_k N'^{2s'}} \left\| P_{N, N'} \left( \sum_{N_2 \lesssim N, N_2' \lesssim N'} \sum_{N_1' \sim N'} n_{N_1, N_1'} (n_{N_2, N_2'}) \right) \right\|_{L^1_{x_1} L^2}^2.
\]

From \([3.3]\) and the estimate for \(J_{4,1,1}\), we obtain \(M_{1,1} \lesssim T^{1/2} MR\). Similarly we can check \(M_{1,i} \lesssim T^{1/2} MR, i = 2, 3\). For the estimate of \(M_2\), we separate \(M_2\) into three parts \(M_{2,i}, i = 1, 2, 3\) as \(M_{1,i}\) above, then we can obtain \(M_{2,i} \lesssim T^{1/2} MR, i = 1, 2, 3\).
For instance, the estimate for $M_{2,1}$ is derived by (3.4) and the estimate for $J_{4,2,1}$. Now we estimate $M_3$. From Proposition 2.2 (2.4), $M_3$ is bounded by

$$
\left( \sum_{N, N' \geq 1} N^{2s_k} N^{2s'} \left\| P_{N, N'} \int_0^t S(t - t') \sum_{N_2 \sim N} \sum_{N_1 \ll N_2} (n_{N_1} u_{N_2})(t') \right\|_{U^3_N}^2 \right)^{1/2}
$$

$$
\lesssim \left( \sum_{N, N' \geq 1} N^{2s_k} N^{2s'} \sup_{\|v\|_{V^3_N} = 1} \left( \sum_{N_2 \sim N, N' \ll N'} \sum_{N_1 \ll N_2, N_1' \ll N'} + \sum_{N_2 \sim N, N_2' \gg N'} \sum_{N_1 \ll N_2, N_1' \ll N_2'} \right) \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1} n_{N_2} n_{N_2'} u_{N_2} u_{N_2'} V_{N, N'} dx dt \right)^{1/2}
$$

$$
\lesssim M_{3,1} + M_{3,2} + M_{3,3},
$$

where

$$
M^2_{3,1} := \sum_{N, N' \geq 1} N^{2s_k} N^{2s'} \sup_{\|v\|_{V^3_N} = 1} \left( \sum_{N_2 \sim N, N' \ll N'} \sum_{N_1 \ll N_2, N_1' \ll N'} \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1} u_{N_2} u_{N_2'} V_{N, N'} dx dt \right)^2,
$$

$$
M^2_{3,2} := \sum_{N, N' \geq 1} N^{2s_k} N^{2s'} \sup_{\|v\|_{V^3_N} = 1} \left( \sum_{N_2 \sim N, N_2' \gg N'} \sum_{N_1 \ll N_2, N_1' \ll N_2'} \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1} n_{N_2} n_{N_2'} u_{N_2} u_{N_2'} V_{N, N'} dx dt \right)^2,
$$

$$
M^2_{3,3} := \sum_{N, N' \geq 1} N^{2s_k} N^{2s'} \sup_{\|v\|_{V^3_N} = 1} \left( \sum_{N_2 \sim N, N_2' \sim N'} \sum_{N_1 \ll N_2, N_1' \ll N_2'} \int_{\mathbb{R}^{d+1}} 1_{[0, T]} n_{N_1} n_{N_2} n_{N_2'} u_{N_2} u_{N_2'} V_{N, N'} dx dt \right)^2.
$$

From (3.5) and the estimate for $J_{4,3,1}$, we have

$$
M_{3,1} \lesssim \begin{cases} 
T^{1/2} JR & (d \geq 3), \\
T^{3/4} JR & (d = 2).
\end{cases}
$$

Similarly, we can check $M_{3,2}, M_{3,3}$ is bounded by the right-hand side of (3.7). We estimate the low frequency part for the Schrödinger equation below. From Proposition 2.3 (2.5), Proposition 2.4 (2.7) and Proposition 2.5 (2.9), we have

$$
\left\| P_0 \int_0^t S(t - t')(nu)(t') dt \right\|_{L^\infty_t L^2_x \cap L^\infty_t L^2_{x_1} \cap L^\infty_t L^2_{x_2} \cap \ldots \cap L^\infty_t L^2_{x_d}} \lesssim \left\| P_0 (nu) \right\|_{L^1_t L^2_{x_1} \cap \ldots \cap \ldots \cap L^1_t L^2_{x_d}}
$$

$$
\lesssim T^{1/2} \left\| n \right\|_{L^\infty_t L^2_{x_1} H^s_{x_d}} \left\| u \right\|_{L^\infty_t L^2_{x_2} \cap \ldots \cap \ldots \cap L^2_{x_d}} \lesssim T^{1/2} MR.
$$

(3.8)
Moreover, we have
\[
\left\| P_0 \int_0^t S(t-t')(nu)(t') \, dt' \right\|_{V_2} \lesssim \sup_{\|v\|_{L^2} = 1} \left| \int_{\mathbb{R}^{d+1}} 1_{[0,T]} nu \overline{P_0v} \, dx \right| + \left| P_0 \int_0^t S(t-t')(nu)(t') \, dt' \right|_{L^\infty_t L^2_x}.
\]
From (3.8), the second term of the right-hand side of the above inequality is bounded by \( T^{1/2} MR \). The first term is estimated by the Hölder inequality and the Sobolev inequality
\[
\left| \int_{\mathbb{R}^{d+1}} 1_{[0,T]} nu \overline{P_0v} \, dx \right| \lesssim T \|n\|_{L^\infty_t L^2_x} \|u\|_{L^\infty_t L^2_x} \|(\nabla \tilde{x})^{(d-1)/2} + (\partial_{x_d})^{1/2} + P_0 v\|_{L^\infty_t L^2_x}
\lesssim T \|n\|_{Z_{\tilde{W}, \pm}^{s_k, \epsilon'}} \|u\|_{Y_{\tilde{W}, \pm}^{s_k, \epsilon'}} \|P_0 v\|_{V_2}.
\]
Hence by \( U_2 \hookrightarrow V_2 \) we have
\[
\sup_{\|v\|_{L^2} = 1} \left| \int_{\mathbb{R}^{d+1}} 1_{[0,T]} nu \overline{P_0 v} \, dx \right| \lesssim TJR.
\]

We estimate the wave part. We need to estimate the following.
\[
\left( \sum_{N_1, N'_1 \geq 1} N_1^{2s_1} N'_1^{2s'} \sup_{\|n\|_{V_2} = 1} \left| \int_{\mathbb{R}^{d+1}} 1_{[0,T]} u \overline{v} \omega n N_1, N'_1 \, dx \right| \right)^{1/2}.
\]
The above term is bounded by
\[
\left( \sum_{N_1, N'_1 \geq 1} N_1^{2s_1} N'_1^{2s'} \sup_{\|n\|_{V_2}} \left| \left( \sum_{N_2 \ll N} \sum_{N \sim N_1} + \sum_{N_2 \gg N} \sum_{N_1 \ll N_2} + \sum_{N_2 \sim N_1} \sum_{N \ll N_2} \right) \int_{\mathbb{R}^{d+1}} 1_{[0,T]} u N_2 \overline{v} \omega n N_1, N'_1 \right. \right)^{1/2} \lesssim R_1 + R_2 + R_3,
\]
where
\[
R_1^2 := \sum_{N_1, N'_1 \geq 1} N_1^{2s_1} N'_1^{2s'} \sup_{\|n\|_{V_2}} \left| \left( \sum_{N_2 \ll N} \sum_{N_1 \sim N_2} \int_{\mathbb{R}^{d+1}} 1_{[0,T]} u N_2 \overline{v} \omega n N_1, N'_1 \right. \right|^2,
\]
\[
R_2^2 := \sum_{N_1, N'_1 \geq 1} N_1^{2s_1} N'_1^{2s'} \sup_{\|n\|_{V_2}} \left| \left( \sum_{N_2 \gg N} \sum_{N \sim N_2} \int_{\mathbb{R}^{d+1}} 1_{[0,T]} u N_2 \overline{v} \omega n N_1, N'_1 \right. \right|^2,
\]
\[
R_3^2 := \sum_{N_1, N'_1 \geq 1} N_1^{2s_1} N'_1^{2s'} \sup_{\|n\|_{V_2}} \left| \left( \sum_{N_2 \sim N_1} \sum_{N_1 \ll N_2} \int_{\mathbb{R}^{d+1}} 1_{[0,T]} u N_2 \overline{v} \omega n N_1, N'_1 \right. \right|^2.
\]
$R_1$ is bounded by

$$\left( \sum_{N_1, N_1' \geq 1} N_1^{2s_t + 2} N_1'^{r_2s'} \sup_{\|n\|_{V_\pm}^2 = 1} \left| \sum_{N_2 \ll N, N_2' \ll N_1'} \sum_{N \sim N_1, N' \sim N_1'} \sum_{N_2 \ll N, N_2' \geq N_1'} \right| 1_{[0, T]} u_{N_2, N_2'} v_{N, N_1', N_1'} dxdt \right)^2 \right)^{1/2} \lesssim R_{1, 1} + R_{1, 2} + R_{1, 3},$$

where

$$R_{1, 1}^2 := \sum_{N_1, N_1' \geq 1} N_1^{2s_t + 2} N_1'^{r_2s'} \sup_{\|n\|_{V_\pm}^2 = 1} \left| \sum_{N_2 \ll N, N_2' \ll N_1'} \sum_{N \sim N_1, N' \sim N_1'} \sum_{N_2 \ll N, N_2' \geq N_1'} \right| 1_{[0, T]} u_{N_2, N_2'} v_{N, N_1', N_1'} dxdt \right|^2,$$

$$R_{1, 2}^2 := \sum_{N_1, N_1' \geq 1} N_1^{2s_t + 2} N_1'^{r_2s'} \sup_{\|n\|_{V_\pm}^2 = 1} \left| \sum_{N_2 \ll N, N_2' \ll N_1'} \sum_{N \sim N_1, N' \sim N_1'} \sum_{N_2 \ll N, N_2' \geq N_1'} \right| 1_{[0, T]} u_{N_2, N_2'} v_{N, N_1', N_1'} dxdt \right|^2,$$

$$R_{1, 3}^2 := \sum_{N_1, N_1' \geq 1} N_1^{2s_t + 2} N_1'^{r_2s'} \sup_{\|n\|_{V_\pm}^2 = 1} \left| \sum_{N_2 \ll N, N_2' \ll N_1'} \sum_{N \sim N_1, N' \sim N_1'} \sum_{N_2 \ll N, N_2' \geq N_1'} \right| 1_{[0, T]} u_{N_2, N_2'} v_{N, N_1', N_1'} dxdt \right|^2.$$

We estimate $R_{1, 1}$. From (2.14), $s_k - s_l = 1/2$, $V_{W_\pm}^2 \hookrightarrow L_t^\infty L_x^2$ and the Cauchy-Schwarz inequality we have

$$R_{1, 1} \lesssim T^{1/2} \left( \sum_{N, N' \geq 1} N_2^{2(s_k + 1/2)} N_2'^{r_2s'} \| u_{N, N'} \|_{L_t^\infty L_x^{2, ..., x_d}}^2 \left( \| P_0 u \|_{L_t^2 L_x^\infty L_{x_{d-1}}^{2, ..., x_d}}^2 \right)^{1/2} + \left( \sum_{1 \leq N_2 \ll N, 1 \leq N_2' \ll N'} N_2^{2(s_k - (d-1)/2 - \varepsilon)} N_2'^{r_2s'} \| u_{N_2, N_2'} \|_{L_t^2 L_x^\infty L_{x_{d-1}}^{2, ..., x_d}}^2 \right)^{1/2} \right)^{1/2} \lesssim T^{1/2} K M.$$
and the Cauchy-Schwarz inequality we have
\[
R_{2,1} = \left( \sum_{N_1,N_1' \geq 1} N_1^{2s_k+2} N_1'^{2s'} \sup_{\|n\|_V^2 = 1} \sum_{N_2 \geq 1, N_2' \leq N_1, N_2' \sim N_2, N' \sim N_1'} \int_{\mathbb{R}^{d+1}} 1_{[0,T]} u_{N_2,N_2'} u_N^n dtdx \right)^{1/2} \leq T^{1/2} \sum_{N_2 \geq 1, N_2' \sim N_2, N' \sim N_1} \left( \sum_{N_1 \leq N_2, N_1' \sim N'} N_1^{2(s_k+1/2+(d-3)/2)} N_1'^{2s'} N_2'^{1+}
\right)^{1/2} \leq T^{1/2} \|u\|_{Y^k}^2 \|v\|_{Y^k}^2 \leq T^{1/2} J^2.
\]

For \( d = 2 \), Lemma 2.7 (2.13), \( V_{2,1}^2 \hookrightarrow L_t^\infty L_x^2 \), the Cauchy-Schwarz inequality leads
\[
R_{2,1} \leq T^{3/4} \sum_{N_2 \geq 1, N_2' \sim N_2} \sum_{N_1 \leq N_2, N_1' \sim N'} N_1^{2(s_k+1/2)} N_1'^{2s'} N_2'^{1+} \|u_{N_2,N_2'}\|_{V_S}^2 \|v_{N,N'}\|_{Y_S}^2 \leq T^{3/4} J^2.
\]

Similar to the estimate for \( R_{2,1} \), we can obtain the estimate for \( R_{2,2}, R_{2,3} \). By symmetry, the estimate for \( R_3 \) is obtained in a similar way to \( R_1 \). Finally we estimate the low frequency part of the Duhamel term for the wave equation.
\[
\|P_0 \int_0^t W_{\pm}(t-t')\omega(u\tilde{v})(t') dt\|_{V_{2,1}^2} \leq \sup_{\|n\|_V^2 = 1} \left| \int_{\mathbb{R}^{d+1}} 1_{[0,T]} u\tilde{v} \omega n dtdx \right| \tag{3.9}
\]

By the Hölder inequality and \( \|u\|_{L_t^4 L_x^{2(d-1)/(d-2)} L_{2,d}^2} \leq \|u\|_{Y^k} \), the right-hand side of (3.9) is bounded by
\[
T^{1/2} \sup_{\|n\|_V^2 = 1} \|u\|_{L_t^4 L_x^{2(d-1)/(d-2)} L_{2,d}^2} \|v\|_{L_t^4 L_x^{2(d-1)/(d-2)} L_{2,d}^2} \|P_0\omega n\|_{L_t^\infty L_x^{d-1} L_{2,d}^d} \leq T^{1/2} J^2
\]

for \( d \geq 3 \). For \( d = 2 \), we see
\[
\sup_{\|n\|_V^2 = 1} \left| \int_{\mathbb{R}^2} 1_{[0,T]} u\tilde{v} \omega P_0\omega n dtdx \right| \leq T^{3/4} \sup_{\|n\|_V^2 = 1} \|u\|_{L_t^4 L_x^2} \|v\|_{L_t^\infty L_x^2} \|P_0\omega n\|_{L_t^\infty L_x^2} \leq T^{3/4} J^2.
\]
Therefore for $0 < T < 1$, we have
\[
\|G u\|_{X^s} \lesssim \|u_0\|_{H_{xk,s'}^k} + T^{1/2}\|n\|_{Z_{W^\pm_s}^{s',d}} \|u\|_{X^s},
\]
\[
\|\tilde{G} n\|_{Z_{W^\pm_s}^{s',d}} \lesssim \|n_0\|_{H_{xk,s'}^k} + T^{1/2}\|u\|_{X^s},
\]
\[
\|G u - \tilde{G} v\|_{X^s} \lesssim T^{1/2}\|n - m\|_{Z_{W^\pm_s}^{s',d}} \|u\|_{X^s} + \|m\|_{Z_{W^\pm_s}^{s',d}} \|u - v\|_{X^s},
\]
\[
\|\tilde{G} n - \tilde{G} m\|_{Z_{W^\pm_s}^{s',d}} \lesssim T^{1/2}\|u - v\|_{X^s} (\|u\|_{X^s} + \|v\|_{X^s}).
\]

This shows the existence and the uniqueness of a local solution $u, n_\pm$ in $X^s, Z_{W^\pm_s}^{s',d}$ with $T = T(\|u_0\|_{H_{xk,s'}^k}, \|n_0\|_{H_{xk,s'}^k})$ small enough.

4. Appendix

In this section we show the bilinear Strichartz estimate, namely Proposition 4.1. For the Zakharov system, see [3] and [4]. For dyadic numbers $N, N', L$, we set
\[
P_{N, N'} := \{(\xi, \xi') \in \mathbb{R}^{d-1} \times \mathbb{R} | N/2 \leq |\xi| \leq 2N, N'/2 \leq |\xi'| \leq 2N'\};
\]
\[
P_{N, 0} := \{(\xi, \xi') \in \mathbb{R}^{d-1} \times \mathbb{R} | N/2 \leq |\xi| \leq 2N, |\xi'| \leq 2\},
\]
\[
P_{0, N'} := \{(\xi, \xi') \in \mathbb{R}^{d-1} \times \mathbb{R} | |\xi| \leq 2, N'/2 \leq |\xi'| \leq 2N'\},
\]
\[
P_0 = P_{0, 0} := \{(\xi, \xi') \in \mathbb{R}^{d-1} \times \mathbb{R} | |\xi| \leq 2, |\xi'| \leq 2\},
\]
\[
W_{L^\pm}^\pm := \{(\tau, \xi, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} | L/2 \leq |\tau \pm |\xi|| \leq 2L\},
\]
\[
W_0^\pm := \{(\tau, \xi, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} | |\tau \pm |\xi|| \leq 2\},
\]
\[
S_L := \{(\tau, \xi, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} | L/2 \leq |\tau + |\xi|^2 + |\xi'| \leq 2L\},
\]
\[
S_0 := \{(\tau, \xi, \xi') \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} | |\tau + |\xi|^2 + |\xi'| \leq 2\}.
\]

**Proposition 4.1.** Let $d \geq 2$. (i - a) Let $u, v \in L^2(\mathbb{R}^{1+d})$ be such that
\[
\supp \mathcal{F} u \subset W_{L_1}^\pm \cap (\mathbb{R} \times ((C \times \mathbb{R}) \cap P_{N_1, N'_1})), \quad \supp \mathcal{F} v \subset S_L \cap (\mathbb{R} \times P_{N_2, N'_2})
\]
for dyadic numbers $L_i, N_i, N'_i$ ($i = 1, 2$) and a cube $C \subset \mathbb{R}^{d-1}$ of side length $\varepsilon$. If $N_1 \leq N_2, N_2 \gg 1$ and $N'_1 \geq N'_2$, it holds that
\[
\|uv\|_2 \lesssim N_2^{-1/2}N_1^{(d-2)/2}N_2^{-1/2}L_1^{1/2}L_2^{-1/2}\|u\|_2\|v\|_2.
\]

(i - b) Let $u, v \in L^2(\mathbb{R}^{1+d})$ be such that
\[
\supp \mathcal{F} u \subset W_{L_1}^\pm \cap (\mathbb{R} \times ((C \times \mathbb{R}) \cap P_{N_1, N'_1})), \quad \supp \mathcal{F} v \subset S_L \cap (\mathbb{R} \times P_{N_2, N'_2})
\]
for dyadic numbers \( L_i, N_i, N_i' \) (\( i = 1, 2 \)) and a cube \( C \subset \mathbb{R}^{d-1} \) of side length \( N \). If \( N_1 \sim N_2 \gg N, N_2 \gg 1 \) and \( N_1' \sim N_2' \gg N' \), it holds that

\[
\|P_{N, N'}(uv)\|_2 \lesssim N_2^{-1/2}N_1^{-1/2}N_1'N_2' N_3^{1/2}L_1^{1/2}L_2^{1/2}\|u\|_2\|v\|_2.
\]

(ii) Let \( u, v \in L^2(\mathbb{R}^{1+d}) \) be such that

\[
\text{supp } Fu \subset S_{L_3} \cap (\mathbb{R} \times P_{N_4, N_4'}), \quad \text{supp } Fv \subset S_{L_4} \cap (\mathbb{R} \times P_{N_4, N_4'})
\]

for dyadic numbers \( L_i, N_i, N_i' \) (\( i = 3, 4 \)). If \( N_3 \ll N_4, N_4 \gg 1 \) and \( N_3' \sim N_4' \), it holds that

\[
\|uv\|_2 \lesssim N_4^{-1/2}N_3^{-1/2}N_4 N_3^{1/2}L_3^{1/2}L_4^{1/2}\|u\|_2\|v\|_2.
\]

**Proof.** Let \( f := \hat{u}, \ g := \hat{v} \). By the Cauchy-Schwarz inequality, we have

\[
\left\| \int f(\tau_1, \xi_1, \xi'_1) g(\tau - \tau_1, \xi - \xi_1, \xi' - \xi'_1) \ d\tau_1 d\xi_1 d\xi'_1 \right\|_{L^2(\mathbb{T}, \mathbb{T}, \mathbb{T})} \lesssim \sup_{\tau, \xi, \xi'} |E(\tau, \xi, \xi')|^{1/2}\|f\|_2\|g\|_2
\]

where

\[
E(\tau, \xi, \xi') = \{ (\tau_1, \xi_1, \xi'_1) \in \text{supp } f \mid (\tau - \tau_1, \xi - \xi_1, \xi' - \xi'_1) \in \text{supp } g \} \subset \mathbb{R}^{1+d}.
\]

Put \( l := \min\{L_1, L_2\}, L := \max\{L_1, L_2\} \). By the Fubini theorem,

\[
|E(\tau, \xi, \xi')| \leq LN_2' \{ \xi_1 \mid \tau \pm |\xi| + |\xi - \xi'| + \xi' | \leq L, \xi_1 \in C, |\xi| \sim N_1, \}
\]

\[
|\xi'_1| \sim N_1', |\xi - \xi| \sim N_2, |\xi' - \xi'_1| \sim N_2' \}
\]

In the right-hand side of the above inequality, the subset of the \( \xi_1 \) is contained in a cube of side length \( m \), where \( m \sim \min\{e, N_1 \} \sim N_1 \). For some \( i \in \{1, ..., d-1\} \), we set \( |(\xi - \xi_1)_i| \sim N_2 \), where \( (\xi - \xi_1)_i \) denotes the \( i \)-th component of \( \xi - \xi_1 \). We compute

\[
|\partial_{\xi_1,i}(\tau \pm |\xi| + |\xi - \xi'| + \xi')| = \left| \pm \frac{\xi_1,i}{|\xi_1|} - 2(\xi - \xi_1)_i \right|,
\]

where \( \xi_{1,i} \) be the \( i \)-th component of \( \xi_1 \). Since \( |\xi_{1,i}| \leq |\xi_1| \) and \( |(\xi - \xi_1)_i| \sim N_2 \),

\[
\text{(R.H.S. of (4.1))} \sim N_2.
\]

Therefore,

\[
|\partial_{\xi_1,i}(\tau \pm |\xi| + |\xi - \xi'| + \xi')| \sim N_2.
\]
Hence by (4.2) and the mean value theorem, we have
\[ |\{\xi_1 \mid |\tau \pm |\xi_1| + |\xi - \xi_1|^2 + \xi'| \lesssim \tilde{T}, \xi_1 \in C, |\xi_1| \sim N_1, |\xi'| \sim N_1'| \]
\[ |\xi - \xi_1| \sim N_2, |\xi' - \xi_1'| \sim N_2' | \]
\[ \lesssim N_2^{-1} m^{d-2} \tilde{T}. \]

From \( m \sim N_1 \), we have
\[ |E(\tau, \xi, \xi')|^{1/2} \lesssim L^{1/2} N_2^{-1/2} N_2 m^{(d-2)/2} \tilde{T}^{1/2} \sim N_2^{-1/2} N_1^{(d-2)/2} N_2^{n/2} L_1^{1/2} L_2^{1/2}. \]

Thus, we obtain (i - a). (i - b) is proved by the same manner as for (i - a), hence we omit the proof. (ii) follows from the similar argument as the estimate for the case (i - a). Indeed in this case, we estimate \( |E(\tau, \xi, \xi')| \) as follows.
\[ |E(\tau, \xi, \xi')| \leq L N_3^3 \{ |\tau + |\xi - \xi_3|^2 + |\xi_3|^2 + \xi'| \lesssim \tilde{T}, |\xi_3| \sim N_3, \]
\[ |\xi'_3| \sim N_3', |\xi - \xi_3| \sim N_4, |\xi' - \xi'_3| \sim N_4' \}, \]
where \( L := \min\{L_3, L_4\}, \tilde{T} := \max\{L_3, L_4\} \). For some \( i \in \{1, \ldots, d - 1\} \), we set \(|(\xi - \xi_3)_i| \gtrsim N_4 \) where \((\xi - \xi_3)_i\) denotes the \( i \)-th component of \( \xi - \xi_3 \). Then by \( N_4 \gg N_3 \), we have
\[ |\partial_{\xi_3,i}(\tau + |\xi - \xi_3|^2 + |\xi_3|^2 + \xi')| = | -2(\xi - \xi_3)_i + 2\xi_3,i| \gtrsim N_4, \]
where \( \xi_3,i \) be the \( i \)-th component of \( \xi_3 \). Thus, we have
\[ |\{ |\xi_3 \mid |\tau + |\xi - \xi_3|^2 + |\xi_3|^2 + \xi'| \lesssim \tilde{T}, |\xi_3| \sim N_3, \]
\[ |\xi'_3| \sim N_3', |\xi - \xi_3| \sim N_4, |\xi' - \xi'_3| \sim N_4' \} | \lesssim N_4^{-1} N_3^{d-2} \tilde{T}. \]

Therefore, we obtain the desired result. \( \square \)

**Remark 4.1.** If we assume \( N_1 \gg N_2 \) instead of \( N_1 \lesssim N_2 \) in Proposition 4.1 (i - a), it holds that
\[ \|uv\|_2 \lesssim N_2^{(d-3)/2} N_2^{n/2} L_1^{1/2} L_2^{1/2} \|u\|_2 \|v\|_2. \]

The proof of the above inequality is obtained by the same way, hence we omit it.

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