Simultaneous Orthogonal Planarity

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Abstract. We introduce and study the OrthoSEFE-k problem: Given k planar graphs each with maximum degree 4 and the same vertex set, do they admit an OrthoSEFE, that is, is there an assignment of the vertices to grid points and of the edges to paths on the grid such that the same edges in distinct graphs are assigned the same path and such that the assignment induces a planar orthogonal drawing of each of the k graphs? We show that the problem is NP-complete for k ≥ 3 even if the shared graph is a Hamiltonian cycle and has sunflower intersection and for k ≥ 2 even if the shared graph consists of a cycle and of isolated vertices. Whereas the problem is polynomial-time solvable for k = 2 when the union graph has maximum degree five and the shared graph is biconnected. Further, when the shared graph is biconnected and has sunflower intersection, we show that every positive instance has an OrthoSEFE with at most three bends per edge.

1 Introduction

The input of a simultaneous embedding problem consists of several graphs G1 = (V, E1),..., Gk = (V, Ek) on the same vertex set. For a fixed drawing style S, the simultaneous embedding problem asks whether there exist drawings Γ1,..., Γk of G1,..., Gk, respectively, in drawing style S such that for any i and j the restrictions of Γi and Γj to Gi ∩ Gj = (V, Ei ∩ Ej) coincide.

The problem has been most widely studied in the setting of topological planar drawings, where vertices are represented as points and edges are represented as

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pairwise interior-disjoint Jordan arcs between their endpoints. This problem is
called Simultaneous Embedding with Fixed Edges or SEFE-\(k\) for short, where \(k\) is the number of input graphs. It is known that SEFE-\(k\) is NP-complete
for \(k \geq 3\), even in the restricted case of sunflower instances [26], where every
pair of graphs shares the same set of edges, and even if such a set induces a
star [3]. On the other hand, the complexity for \(k = 2\) is still open. Recently,
efficient algorithms for restricted instances have been presented, namely when
(i) the shared graph \(G_\cap = G_1 \cap G_2\) is biconnected [19,1] or a star-graph [1],
(ii) \(G_\cap\) is a collection of disjoint cycles [13], (iii) every connected component of
\(G_\cap\) is either subcubic or biconnected [26,11], (iv) \(G_1\) and \(G_2\) are biconnected
and \(G_\cap\) is connected [14], and (v) \(G_\cap\) is connected and the input graphs have
maximum degree 5 [13]; see the survey by Bläsius et al. [12] for an overview.

For planar straight-line drawings, the simultaneous embedding problem is
called Simultaneous Geometric Embedding and it is known to be NP-hard
even for two graphs [17]. Besides simultaneous intersection representation for,
e.g., interval graphs [20,14] and permutation and chordal graphs [21], it is only
recently that the simultaneous embedding paradigm has been applied to other
fundamental planarity-related drawing styles, namely simultaneous level planar
drawings [2] and RAC drawings [4,8].

We continue this line of research by studying simultaneous embeddings in the
planar orthogonal drawing style, where vertices are assigned to grid points and
edges to paths on the grid connecting their endpoints [29]. In accordance with the
existing naming scheme, we define OrthoSEFE-\(k\) to be the problem of testing
whether \(k\) input graphs \((G_1,\ldots,G_k)\) admit a simultaneous planar orthogonal
drawing. If such a drawing exists, we call it an OrthoSEFE of \((G_1,\ldots,G_k)\).
Note that it is a necessary condition that each \(G_i\) has maximum degree 4 in
order to obtain planar orthogonal drawings. Hence, in the remainder of the
paper we assume that all instances have this property. For instances with this
property, at least when the shared graph is connected, the problem SEFE-2 can
be solved efficiently [14]. However, there are instances of OrthoSEFE-2 that
admit a SEFE but not an OrthoSEFE; see Fig. 1(a).

Unless mentioned otherwise, all instances of OrthoSEFE-\(k\) and SEFE-\(k\) we
consider are sunflower. Notice that instances with \(k = 2\) are always sunflower.
Let \((G_1 = (V,E_1),G_2 = (V,E_2))\) be an instance of OrthoSEFE-2. We define the
shared graph (resp. the union graph) to be the graph \(G_\cap = (V,E_1 \cap E_2)\)
(resp. \(G_\cup = (V,E_1 \cup E_2)\)) with the same vertex set as \(G_1\) and \(G_2\), whose edge
set is the intersection (resp. the union) of the ones of \(G_1\) and \(G_2\). Also, we call
the edges in \(E_1 \cap E_2\) the shared edges and we call the edges in \(E_1 \setminus E_2\) and in
\(E_2 \setminus E_1\) the exclusive edges. The definitions of shared graph, shared edges,
and exclusive edges naturally extend to sunflower instances for any value of \(k\).

One main issue is to decide how degree-2 vertices of the shared graph are
represented. Note that, in planar topological drawings, degree-2 vertices do not
require any decisions as there exists only a single cyclic order of their incident
edges. In the case of orthogonal drawings there are, however, two choices for a
degree-2 vertex: It can either be drawn straight, i.e., it is incident to two angles
Fig. 1. (a) A negative instance of OrthoSEFE-2. Shared edges are black, while exclusive edges are red and blue. The red edges require 270° angles on different sides of \( C \). Thus, the blue edge \((u,v)\) cannot be drawn. Note that the given drawing is a SEFE-2. (b) Examples of side assignments for the exclusive edges incident to degree-2 vertices of \( G_\cap \): orthogonality constraints are satisfied at \( v_4 \) and \( v_5 \), while they are violated at \( v_3 \).

of 180°, or bent, i.e., it is incident to one angle of 90° and to one angle of 270°. If \( v \) is a degree-2 vertex of the shared graph with neighbors \( u \) and \( w \), and two exclusive edges \( e, e' \), say of \( G_1 \), are incident to \( v \) and are embedded on the same side of the path \(uvw\), then \( v \) must be bent, which in turn implies that also every exclusive edge of \( G_2 \) incident to \( v \) has to be embedded on the same side of \(uvw\) as \( e \) and \( e' \). In this way, the two input graphs of OrthoSEFE-2 interact via the degree-2 vertices. It is the difficulty of controlling this interaction that marks the main difference between SEFE-\( k \) and OrthoSEFE-\( k \). To study this interaction in isolation, we focus on instances of OrthoSEFE-2 where the shared graph is a cycle for most of the paper. Note that such instances are trivial yes-instances of SEFE-\( k \) (provided the input graphs are all planar).

Contributions and Outline. In Section 2 we provide our notation and we show that the existence of an OrthoSEFE of an instance of OrthoSEFE-\( k \) can be described as a combinatorial embedding problem. In Section 3 we show that OrthoSEFE-3 is NP-complete even if the shared graph is a cycle, and that OrthoSEFE-2 is NP-complete even if the shared graph consists of a cycle plus some isolated vertices. This contrasts the situation of SEFE-\( k \) where these cases are polynomially solvable [1, 10, 19, 26]. In Section 4 we show that OrthoSEFE-2 is efficiently solvable if the shared graph is a cycle and the union graph has maximum degree 5. Finally, in Section 5 we extend this result to the case where the shared graph is biconnected (and the union graph still has maximum degree 5). Moreover, we show that any positive instance of OrthoSEFE-\( k \) whose shared graph is biconnected admits an OrthoSEFE with at most three bends per edge. We close with some concluding remarks and open questions in Section 6.

Full proofs can be found in the Appendix.

2 Preliminaries

We will extensively make use of the Not-All-Equal 3-Sat (NAE3Sat) problem [25, p.187]. An instance of NAE3Sat consists of a 3-CNF formula \( \phi \) with variables \( x_1, \ldots, x_n \) and clauses \( c_1, \ldots, c_m \). The task is to find a NAE truth as-
Assignment, i.e., a truth assignment such that each clause contains both a true and a false literal. NAE3SAT is known to be NP-complete [27]. The variable–clause graph is the bipartite graph whose vertices are the variables and the clauses, and whose edges represent the membership of a variable in a clause. The problem Planar NAE3SAT is the restriction of NAE3SAT to instances whose variable–clause graph is planar. Planar NAE3SAT can be solved efficiently [23,28].

Embedding Constraints. Let \( \langle G_1, \ldots, G_k \rangle \) be an OrthoSEFE-\( k \) instance. A SEFE is a collection of embeddings \( E_i \) for the \( G_i \) such that their restrictions on \( G_\cap \) are the same. Note that in the literature, a SEFE is often defined as a collection of drawings rather than a collection of embeddings. However, the two definitions are equivalent [22]. For a SEFE to be realizable as an OrthoSEFE it needs to satisfy two additional conditions. First, let \( v \) be a vertex of degree 2 in \( G_\cap \) with neighbors \( u \) and \( w \). If in any embedding \( E_i \) there exist two exclusive edges incident to \( v \) that are embedded on the same side of the path \( uvw \), then any exclusive edge incident to \( v \) in any of the \( E_j \neq E_i \) must be embedded on the same side of the path \( uvw \). Second, let \( v \) be a vertex of degree 3 in \( G_\cap \). All exclusive edges incident to \( v \) must appear between the same two edges of \( G_\cap \) around \( v \). We call these the orthogonality constraints. See Fig. 1(b).

Theorem 1. An instance \( \langle G_1, \ldots, G_k \rangle \) of OrthoSEFE-\( k \) has an OrthoSEFE if and only if it admits a SEFE satisfying the orthogonality constraints.

For the case in which the shared graph is a cycle \( C \), we give a simpler version of the constraints in Theorem 1 which will prove useful in the remainder of the paper. By the Jordan curve Theorem, a planar drawing of cycle \( C \) divides the plane into a bounded and an unbounded region – the inside and the outside of \( C \), which we call the sides of \( C \). Now the problem is to assign the exclusive edges to either of the two sides of \( C \) so that the following two conditions are fulfilled.

Planarity Constraints. Two exclusive edges of the same graph must be drawn on different sides of \( C \) if their endpoints alternate along \( C \).

Orthogonality Constraints. Let \( v \in V \) be a vertex that is adjacent to two exclusive edges \( e_i \) and \( e'_i \) of the same graph \( G_i \), \( i \in \{1, \ldots, k\} \). If \( e_i \) and \( e'_i \) are on the same side of \( C \), then all exclusive edges incident to \( v \) of all graphs \( G_1, \ldots, G_k \) must be on the same side as \( e_i \) and \( e'_i \).

Note that this is a reformulation of the general orthogonality constraints. Further, the orthogonality constraints also imply that if \( e_i \) and \( e'_i \) are on different sides of \( C \), then for each graph \( G_j \) that contains two exclusive edges \( e_j \) and \( e'_j \) incident to \( v \), with \( j \in \{1, \ldots, k\} \), \( e_j \) and \( e'_j \) must be on different sides of \( C \).

The next theorem follows from Theorem 1 and from the following two observations. First, for a sunflower instance \( \langle G_1, \ldots, G_k \rangle \) whose shared graph is a cycle, any collection of embeddings is a SEFE [22]. Second, the planarity constraints are necessary and sufficient for the existence of an embedding of \( G_1 \) [5].

Theorem 2. An instance of OrthoSEFE-\( k \) whose shared graph is a cycle \( C \) has an OrthoSEFE if and only if there exists an assignment of the exclusive edges to the two sides of \( C \) satisfying the planarity and orthogonality constraints.
3 Hardness Results

We show that OrthoSEFE-$k$ is NP-complete for $k \geq 3$ for instances with sunflower intersection even if the shared graph is a cycle, and for $k = 2$ even if the shared graph consists of a cycle and isolated vertices.

**Theorem 3.** OrthoSEFE-$k$ with $k \geq 3$ is NP-complete, even for instances with sunflower intersection in which (i) the shared graph is a cycle and (ii) $k - 1$ of the input graphs are outerplanar and have maximum degree 3.

**Proof sketch.** The membership in NP directly follows from Theorem 2. To prove the NP-hardness, we show a polynomial-time reduction from the NP-complete problem Positive Exactly-Three Nae3Sat [24], which is the variant of Nae3Sat in which each clause consists of exactly three unnegated literals.

Let $x_1, x_2, \ldots, x_n$ be the variables and let $c_1, c_2, \ldots, c_m$ be the clauses of a 3-CNF formula $\phi$ of Positive Exactly-Three Nae3Sat. We show how to construct an equivalent instance $\langle G_1, G_2, G_3 \rangle$ of OrthoSEFE-3 such that $G_1$ and $G_2$ are outerplanar graphs of maximum degree 3. We refer to the exclusive edges in $G_1$, $G_2$, and $G_3$ as red, blue, and green, respectively; refer to Fig. 2.

For each clause $c_j$, $j = 1, \ldots, m$, we create a clause gadget $C^j$ as in Fig. 2(a) (top). For each variable $x_i$, $i = 1, \ldots, n$, and each clause $c_j$, $j = 1, \ldots, m$, we create a variable-clause gadget $V_i^j$ as in Fig. 2(a) (bottom). Observe that the (dotted) green edge $\{w_i^j, r_i^j\}$ in a variable-clause gadget is only part of $V_i^j$ if $x_i$ does not occur in $c_j$. Otherwise, there is a green edge $\{w_i^j, y_i^j\}$ connecting $w_i^j$ to one of the three vertices $y_i^a$, $y_i^b$, or $y_i^c$ (dashed stubs) in the clause gadget. Observe that these three variable-clause edges per clause can be realized in such a way that there exist no planarity constraints between pairs of them. In Fig. 2(b) the variable-clause gadgets $V_4^1$, $V_5^1$, $V_4^2$ and $V_5^2$ contain edges $\{w_4^1, r_4^1\}$ and $\{w_5^2, r_5^2\}$, respectively.
The gadgets are ordered as indicated in Fig. 2(b). The variable-clause gadgets $V_i^j$, with $i = 1, \ldots, n$, always precede the clause gadget $V_j^j$, for any $j = 1, \ldots, m$. Further, if $j$ is odd, then the gadgets $V_i^j, \ldots, V_i^{j+1}$ appear in this order, otherwise they appear in reversed order $V_i^{j+1}, \ldots, V_i^j$. Finally, $V_i^j$ and $V_i^{j+1}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m − 1$, are connected by an edge $\{w_i^j, w_i^{j+1}\}$, which is blue if $j$ is odd and red if $j$ is even. We call these edges transmission edges.

Assume $(G_1, G_2, G_3)$ admits an OrthoSEFE. Planarity constraints and orthogonality constraints guarantee three properties: (i) If the edge $\{u_i^1, v_i^1\}$ is inside $C$, then so is $\{u_i^{2j}, v_i^{2j+1}\}, i = 1, \ldots, n, j = 1, \ldots, m − 1$. This is due to the fact that, by the planarity constraints, the two green edges incident to $u_i^j$ lie on the same side of $C$ and hence, by the orthogonality constraints, the two transmission edges incident to $u_i^j$ also lie in this side. We call $\{u_i^1, v_i^1\}$ the truth edge of variable $x_i$. (ii) Not all the three green edges $a = \{\alpha, \beta^j\}, b = \{\beta^j, \gamma^j\}$, and $c = \{\gamma^j, \delta^j\}$ lie on the same side of $C$. Namely, the two red edges of the clause gadget $C^j$ must lie on opposite sides of $C$ because of the interplay between the planarity and the orthogonality constraints in the subgraph of $C^j$ induced by the vertices between $\beta^j$ and $\gamma^j$. Hence, if edges $a$, $b$, and $c$ lie in the same side of $C$, then the orthogonality constraints at either $\beta^j$ or $\gamma^j$ are not satisfied. (iii) For each clause $c_j = (x_a, x_b, x_c)$, edge $a = \{\alpha, \beta^j\}$ lies in the same side of $C$ as the truth edge of $x_a$. This is due to the planarity constraints between each of these two edges and the variable-clause edge $\{w_{a_i}^j, y_{a_i}^j\}$. Analogously, edge $b$ (edge $c$) lies on the same side as the truth edge of $x_b$ (of $x_c$). Hence, setting $x_i = \text{true}$ ($x_i = \text{false}$) if the truth edge of $x_i$ is inside $C$ (outside $C$) yields a NAE3Sat truth assignment that satisfies $\phi$.

The proof for the other direction is based on the fact that assigning the truth edges to either of the two sides of $C$ according to the NAE3Sat assignment of $\phi$ also implies a unique side assignment for the remaining exclusive edges that satisfies all the orthogonality and the planarity constraints.

It is easy to see that $G_1$ and $G_2$ are outerplanar graphs with maximum degree 3, and that the reduction can be extended to any $k > 3$. \hfill \Box

In the following we describe how to modify the construction in Theorem 3 to show hardness of OrthoSEFE-2. We keep only the edges of $G_1$ and $G_3$. Variable-clause gadgets and clause gadgets remain the same, as they are composed only of edges belonging to these two graphs. We replace each transmission edge in $G_2$ by a transmission path composed of alternating green and red edges, starting and ending with a red edge. This transformation allows these paths to traverse the transmission edges of $G_1$ and the variable-clause edges of $G_3$ without introducing crossings between edges of the same color. It is easy to see that the properties described in the proof of Theorem 3 on the assignments of the exclusive edges to the two sides of $C$ also hold in the constructed instance, where transmission paths take the role of the transmission edges.

**Theorem 4.** OrthoSEFE-2 is NP-complete, even for instances $(G_1, G_2)$ in which the shared graph consists of a cycle and a set of isolated vertices.
Fig. 3. (a) Instance \( (G_1, G_2) \) satisfying the properties of Lemma 1, where the edges in \( E_2 \) belonging to the components \( \alpha, \beta, \gamma, \) and \( \delta \) of \( H \) have different line styles. (b) Polygons for the components of \( H \). (c) Graph \( \tilde{G} \). (d) Variable–clause graph \( G_\phi \).

4 Shared Graph is a Cycle

In this section we give a polynomial-time algorithm for instances of OrthoSEFE-2 whose shared graph is a cycle and whose union graph has maximum degree 5 (Theorem 5). In order to obtain this result, we present an efficient algorithm for more restricted instances (Lemma 1) and give a series of transformations (Lemma 2–3) to reduce any instance with the above properties to one that can be solved by the algorithm in Lemma 1.

Lemma 1. OrthoSEFE-2 is in \( P \) for instances \( (G_1, G_2) \) such that the shared graph \( C \) is a cycle and \( G_1 \) is an outerplanar graph with maximum degree 3.

Proof. The algorithm is based on a reduction to Planar Nae3Sat, which is in \( P \) [23,28]. First note that since \( G_1 \) is outerplanar there exist no two edges in \( E_1 \) alternating along \( C \). Hence, there are no planarity constraints for \( G_1 \).

We now define an auxiliary graph \( H \) with vertex set \( E_2 \setminus E_1 \) and edges corresponding to pairs of edges alternating along \( C \); see Fig. 3(a). W.l.o.g. we may assume that \( H \) is bipartite, since \( G_2 \) would not meet the planarity constraints otherwise [5]. Let \( B \) be the set of connected components of \( H \), and for each component \( B \in B \), fix a partition \( B_1, B_2 \) of \( B \) into independent sets (possibly \( B_2 = \emptyset \) in case of a singleton \( B \)). Note that in any inside/outside assignment of the exclusive edges of \( G_2 \) that meets the planarity constraints, for every \( B \in B \), all edges of \( B_1 \) lie in one side of \( C \) and all edges of \( B_2 \) lie in the other side.

Draw the cycle \( C \) as a circle in the plane. For a component \( B \in B \), let \( P_B \) be the polygon inscribed into \( C \) whose corners are the endvertices in \( V \) of the edges in \( E_2 \) corresponding to the vertices of \( B \); refer to Fig. 3(b). If \( B \) only contains one vertex (i.e., one edge of \( G_2 \)), we consider the digon \( P_B \) as the straight-line segment connecting the vertices of this edge. If \( B \) has at least two vertices, we let \( P_B \) be open along its sides, i.e. it will contain the corners and all inner points (in Fig. 3(b) we depict this by making the sides of \( P_B \) slightly concave). One can easily show that for any two components \( B, D \in B \), their polygons \( P_B, P_D \) may share only some of their corners, but no inner points. Hence the graph \( G \)
obtained by placing a vertex \( x \) inside the polygon \( P_B \), for \( B \in \mathcal{B} \), making \( x_B \) adjacent to each corner of \( P_B \) and adding the edges \( E_1 \), is planar; see Fig. 3(c).

We construct a formula \( \phi \) with variables \( x_B, B \in \mathcal{B} \), such that \( \phi \) is NAE-satisfiable if and only if \( \langle G_1, G_2 \rangle \) admits an inside/outside assignment meeting all planarity and orthogonality constraints. The encoding of the truth assignment will be such that \( x_B \) is true when the edges of \( B_1 \) are inside \( C \) and the edges of \( B_2 \) are outside, and \( x_B \) is false if the reverse holds. Every assignment satisfying the planarity constraints for \( G_2 \) defines a truth-assignment in the above sense.

Let \( e = (v, w) \) be an exclusive edge of \( E_1 \) and let \( e^1_v, e^2_v \) (\( e^1_w, e^2_w \)) be the exclusive edges of \( E_2 \) incident to \( v \) (to \( w \), respectively); we assume that all such four edges of \( E_2 \) exist, the other cases being simpler. Let \( B(u, i) \) be the component containing the edge \( e^i_u \), for \( u \in \{v, w\} \) and \( i \in \{1, 2\} \). Define the literal \( \ell_u \) to be \( x_B(u, i) \) if \( e^i_u \in B_1(u, i) \) and \( \neg x_B(u, i) \) if \( e^i_u \in B_2(u, i) \). With our interpretation of the truth assignment, an edge \( e^i_u \) is inside \( C \) if and only if \( \ell_u \) is true. Now, for the assignment to meet the orthogonality constraints, if \( \ell_u = \ell_v \), say both are true, then \( e \) must be assigned inside \( C \) as well, which would cause a problem if and only if \( \ell_u = \ell_w = \text{false} \). Hence the orthogonality constraints are described by NAE-satisfiability of the clauses \( c_e = (\ell^1_v, \ell^2_v, \neg \ell^1_w, \neg \ell^2_w) \), for each \( e \in E_1 \). To reduce to NAE3SAT, we introduce a new variable \( x_e \) for each edge \( e \in E_1 \setminus E_2 \) and replace the clause \( c_e \) by two clauses \( c'_e = (\ell^1_v, \ell^2_v, x_e) \) and \( c''_e = (\neg x_e, \neg \ell^1_w, \neg \ell^2_w) \). A planar drawing of the variable-clause graph \( G_\phi \) of the resulting formula \( \phi \) is obtained from the planar drawing \( \bar{\Gamma} \) of \( G \) (see Figs. 3(c) and 3(d)) by (i) placing each variable \( x_B \), with \( B \in \mathcal{B} \), on the point where vertex \( x_B \) lies in \( \bar{\Gamma} \), (ii) placing each variable \( x_e \), with \( e \in E_1 \), on any point of edge \( e \) in \( \bar{\Gamma} \), (iii) placing clauses \( c'_e \) and \( c''_e \), for each edge \( e = (v, w) \in E_1 \), on the points where vertices \( v \) and \( w \) lie in \( \bar{\Gamma} \), respectively, and (iv) drawing the edges of \( G_\phi \) as the corresponding edges in \( \bar{\Gamma} \). This implies that \( G_\phi \) is planar and hence we can test the NAE-satisfiability of \( \phi \) in polynomial time [23,28].

The next two lemmas show that we can use Lemma 1 to test in polynomial time any instance of OrthoSEFE-2 such that \( G_1 \) is a cycle and each vertex \( v \in V \) has degree at most 3 in either \( G_1 \) or \( G_2 \).

**Lemma 2.** Let \( \langle G_1, G_2 \rangle \) be an instance of OrthoSEFE-2 whose shared graph is a cycle and such that \( G_1 \) has maximum degree 3. It is possible to construct in polynomial time an equivalent instance \( \langle G'_1, G'_2 \rangle \) of OrthoSEFE-2 whose shared graph is a cycle and such that \( G'_1 \) is outerplanar and has maximum degree 3.

**Proof sketch.** We construct an equivalent instance \( \langle G'_1, G'_2 \rangle \) of OrthoSEFE-2 such that \( G'_1 \) is a cycle, \( G'_1 \) has maximum degree 3, and the number of pairs of edges in \( G'_1 \) that alternate along \( G'_1 \) is smaller than the number of pairs of edges in \( G_1 \) that alternate along \( G_1 \). Repeatedly applying this transformation yields an equivalent instance \( \langle G'_1, G'_2 \rangle \) satisfying the requirements of the lemma.

Consider two edges \( e = (u, v) \) and \( f = (w, z) \) of \( G_1 \) such that \( u, w, v, z \) appear in this order along cycle \( G_1 \) and such that the path \( P_{u,z} \) in \( G_1 \), between \( u \) and \( z \) that contains \( v \) and \( w \) has minimal length. If \( G_1 \) is not outerplanar, then the edges \( e \) and \( f \) always exist. Fig. 4 illustrates the construction of \( \langle G'_1, G'_2 \rangle \).
Fig. 4. Instances (left) $\langle G_1, G_2 \rangle$ and (right) $\langle G_1', G_2' \rangle$ for the proof of Lemma 2. Edges of $G_1$ ($G_2'$) are black. Exclusive edges of $G_1$ ($G_1'$) are red and those of $G_2$ ($G_2'$) are blue.

Fig. 5. Illustration of the transformation for the proof of Lemma 3 to reduce the number of vertices incident to two exclusive edges in $G_1$. Edges $e', f'$ of $G_2$ and $h'$ of $G_1$ (right) take the role of edges $e, f$ of $G_1$ and $h$ of $G_2$ (left), respectively. Thus, the orthogonality constraints at $v'$ are equivalent to those at $v$.

By the choice of $e$ and $f$, and by the fact that $G_1$ has maximum degree 3, there is no exclusive edge in $G_1$ with one endpoint in the set $H_2$ of vertices between $w$ and $v$, and the other one not in $H_2$. Further, observe that in an OrthoSEFE of $\langle G_1', G_2' \rangle$ edges $f$ and $f'$ (edges $e$ and $e'$) must be on the same side. Further, $e$ and $f$ must be in different sides of $G_1'$. It can be concluded that $\langle G_1', G_2' \rangle$ has an OrthoSEFE if and only if $\langle G_1, G_2 \rangle$ has an OrthoSEFE. □

The proof of the next lemma is based on the replacement illustrated in Fig. 5.

Afterwards, we combine these results to present the main result of the section.

**Lemma 3.** Let $\langle G_1, G_2 \rangle$ be an instance of OrthoSEFE-2 whose shared graph is a cycle and whose union graph has maximum degree 5. It is possible to construct in polynomial time an equivalent instance $\langle G_1^*, G_2^* \rangle$ of OrthoSEFE-2 whose shared graph is a cycle and such that graph $G_1^*$ has maximum degree 3.

**Theorem 5.** OrthoSEFE-2 can be solved in polynomial time for instances whose shared graph is a cycle and whose union graph has maximum degree 5.

5 Shared Graph is Biconnected

We now study OrthoSEFE-$k$ for instances whose shared graph is biconnected. In Theorem 6 we give a polynomial-time Turing reduction from instances of OrthoSEFE-2 whose shared graph is biconnected to instances whose shared graph is a cycle. In Theorem 7 we give an algorithm that, given a positive instance of OrthoSEFE-$k$ such that the shared graph is biconnected together
with a SEFE satisfying the orthogonality constraints, constructs an OrthoSEFE
with at most three bends per edge.

We start with the Turing reduction, i.e., we develop an algorithm that takes as
input an instance \( \langle G_1, G_2 \rangle \) of OrthoSEFE-2 whose shared graph \( G \cap = G_1 \cap G_2 \)
is biconnected and produces a set of \( O(n) \) instances \( \langle G_1^1, G_2^1 \rangle, \ldots, \langle G_1^h, G_2^h \rangle \) of
OrthoSEFE-2 whose shared graphs are cycles. The output is such that \( \langle G_1, G_2 \rangle \)
is a positive instance if and only if all instances \( \langle G_1^i, G_2^i \rangle, i = 1, \ldots, h \), are
positive. The reduction is based on the SEFE testing algorithm for instances
whose shared graph is biconnected by Bläsius et al. \[10,11\], which can be seen
as a generalized and unrooted version of the one by Angelini et al. \[1\].

We first describe a preprocessing step. Afterwards, we give an outline of the
approach of Bläsius et al. \[11\] and present the Turing reduction in two steps. We
assume familiarity with SPQR-trees \[7,6\]; for formal definitions see Appendix A.

**Lemma 4.** Let \( \langle G_1, G_2 \rangle \) be an instance of OrthoSEFE-2 whose shared graph is
biconnected. It is possible to construct in polynomial time an equivalent instance
\( \langle G_1^*, G_2^* \rangle \) whose shared graph is biconnected and such that each endpoint of an
exclusive edge has degree 2 in the shared graph.

We continue with a brief outline of the algorithm by Bläsius et al. \[11\] and present the Turing reduction in two steps. We assume familiarity with SPQR-trees \[7,6\]; for formal definitions see Appendix A.

Let \( \langle G_1, G_2 \rangle \) be an instance of OrthoSEFE-2 whose shared graph is
biconnected. It is possible to construct in polynomial time an equivalent instance
\( \langle G_1^*, G_2^* \rangle \) whose shared graph is biconnected and such that each endpoint of an
exclusive edge has degree 2 in the shared graph.

We now describe the algorithm of Bläsius et al. \[11\] in more detail. Consider
a node \( \mu \) of \( T \). A part \( x \) of skel(\( \mu \)) is either a vertex of skel(\( \mu \)) or a virtual edge of
skel(\( \mu \)), which represents a subgraph of \( G \). An exclusive edge \( e \) has an attachment
in a part \( x \) of skel(\( \mu \)) if \( x \) is a vertex that is an endpoint of \( e \) or if \( x \) is a virtual edge
whose corresponding subgraph contains an endpoint of \( e \). An exclusive edge \( e \) of
\( G_1 \) or of \( G_2 \) is important for \( \mu \) if its endpoints are in different parts of skel(\( \mu \)).
It is not hard to see that, to obtain a SEFE, the embedding of the skeleton
skel(\( \mu \)) of each node \( \mu \) has to be chosen such that for each exclusive edge \( e \) the
parts containing the attachments of \( e \) share a face. It can be shown that any
embedding choice for P-nodes and R-nodes that satisfies these conditions can,
after possibly flipping it, be used to obtain a SEFE \[11\] Theorem 1]. The proof
does not modify the order of exclusive edges around degree-2 vertices of \( G_\cap \), and
therefore applies to OrthoSEFE-2 as well.

Now let \( \mu \) be an S-node. Let \( \varepsilon \) be a virtual edge of skel(\( \mu \)), \( G_\varepsilon \) be the subgraph
represented by \( \varepsilon \), and \( \nu \) be the corresponding neighbor of \( \mu \) in the SPQR-tree
of \( G \). An attachment of \( \nu \) with respect to \( \mu \) is an interior vertex of \( G_\varepsilon \) that is
incident to an important edge \( e \) for \( \mu \). If \( \nu \) has such an attachment, then it is
Fig. 6. (a) Skeleton of an S-node \(\mu\) in which the R-node \(\nu\) corresponding to the virtual edge \(\varepsilon = (u, v)\) is expanded to show its skeleton. (b) Replacing \(\varepsilon\) with cycle \(C_{\varepsilon}\). (c) Replacing \(C_{\varepsilon}\) with path \(P_{\varepsilon}\); vertices \(a_1, a_2, x_1, \ldots, x_4, b_1, b_2\) are green boxes.

a P- or R-node. It is a necessary condition on the embedding of \(G_{\varepsilon}\) that each attachment \(x\) with respect to \(\mu\) must be incident to a face incident to the virtual edge twin(\(\varepsilon\)) of skel(\(\nu\)) representing \(\mu\), and that their clockwise circular order together with the poles of \(\varepsilon\) is fixed up to reversal [11, Lemma 8].

For the purpose of avoiding crossings in skel(\(\mu\)), we can thus replace each virtual edge \(\varepsilon\) that does not represent a Q-node by a cycle \(C_{\varepsilon}\) containing the attachments of \(\varepsilon\) with respect to \(\mu\) and the poles of \(\varepsilon\) in the order \(O_{\varepsilon}\). We keep only the important edges of \(\mu\). Altogether this results in an instance \(\langle G_{\mu}^1, G_{\mu}^2 \rangle\) of SEFE modeling the requirements for skel(\(\mu\)); see Figs. 6(a) and 6(b).

**Lemma 5.** Let \(\langle G_1, G_2 \rangle\) be an instance of OrthoSEFE-2 whose shared graph is biconnected. Then \(\langle G_1, G_2 \rangle\) admits an OrthoSEFE if and only if all instances \(\langle G_{\mu}^1, G_{\mu}^2 \rangle\) admit an OrthoSEFE.

Next, we transform a given instance \(\langle G_{\mu}^1, G_{\mu}^2 \rangle\) of OrthoSEFE-2 as above into an equivalent instance \(\langle G_{\mu}^1, G_{\mu}^2 \rangle\) whose shared graph is a cycle. Let \(C_{\varepsilon}\), be the cycles corresponding to the neighbor \(\nu_i, i = 1, \ldots, k\) of \(\mu\) in \(\langle G_{\mu}^1, G_{\mu}^2 \rangle\). To obtain the instance \(\langle G_{\mu}^1, G_{\mu}^2 \rangle\), we replace each cycle \(C_{\varepsilon}\), with poles \(u\) and \(v\) by a path \(P_{\varepsilon}\) from \(u\) to \(v\) that first contains two special vertices \(a_1, a_2\) followed by the clockwise path from \(u\) to \(v\) (excluding the endpoints), then four special vertices \(x_1, \ldots, x_4\), then the counterclockwise path from \(u\) to \(v\) (excluding the endpoints), and finally two special vertices \(b_1, b_2\) followed by \(v\). In addition to the existing exclusive edges (note that we do not remove any vertices), we add to \(G_1\) the exclusive edges \((a_2, x_3), (x_1, x_3), (x_2, x_4), (x_2, b_1)\), and to \(G_2\) the exclusive edges \((a_1, x_3)\) and \((x_2, b_2)\) to \(G_2\); see Fig. 6(c).

The above reduction together with the next lemma implies the main result.

**Lemma 6.** \(\langle G_{\mu}^1, G_{\mu}^2 \rangle\) admits an OrthoSEFE if and only if \(\langle G_{\mu}^1, G_{\mu}^2 \rangle\) does.

**Theorem 6.** OrthoSEFE-2 when the shared graph is biconnected is polynomial-time Turing reducible to OrthoSEFE-2 when the shared graph is a cycle. Also, the reduction does not increase the maximum degree of the union graph.

**Corollary 1.** OrthoSEFE-2 can be solved in polynomial time for instances whose shared graph is biconnected and whose union graph has maximum degree 5.
Observe that, from the previous results it is not hard to also obtain a SEFE satisfying the orthogonality constraints, if it exists. We show how to construct an orthogonal geometric realizations of such a SEFE.

**Theorem 7.** Let \( \langle G_1, \ldots, G_k \rangle \) be a positive instance of OrthoSEFE-\( k \) whose shared graph is biconnected. Then, there exists an OrthoSEFE \( \langle \Gamma_1, \Gamma_2, \ldots, \Gamma_k \rangle \) of \( \langle G_1, \ldots, G_k \rangle \) in which every edge has at most three bends.

**Proof sketch.** We assume that a SEFE satisfying the orthogonality constraints is given. We adopt the method of Biedl and Kant [9]. We draw the vertices with increasing y-coordinates with respect to an s-t-ordering [15] \( v_1, \ldots, v_n \) on the shared graph. We choose the face to the left of \((v_1, v_n)\) as the outer face of the union graph. The edges will bend at most on y-coordinates near their incident vertices and are drawn vertically otherwise. Fig. 11 indicates, how the ports are assigned. We make sure that an edge may only leave a vertex to the bottom if it is incident to \( v_n \) or to a neighbor with a lower index. Thus, there are exactly three bends on \( \{v_1, v_n\} \). Any other edge \( \{v_i, v_j\} \), \( 1 \leq i < j \leq n \) has at most one bend around \( v_i \) and at most two bends around \( v_j \). □

### 6 Conclusions and Future Work

In this work we introduced and studied the problem OrthoSEFE-\( k \) of realizing a SEFE in the orthogonal drawing style. While the problem is already NP-hard even for instances that can be efficiently tested for a SEFE, we presented a polynomial-time testing algorithm for instances consisting of two graphs whose shared graph is biconnected and whose union graph has maximum degree 5. We have also shown that any positive instance whose shared graph is biconnected can be realized with at most three bends per edge.

We conclude the paper by presenting a lemma that, together with Theorem 6, shows that it suffices to only focus on a restricted family of instances to solve the problem for all instances whose shared graph is biconnected.

**Lemma 7.** Let \( \langle G_1, G_2 \rangle \) be an instance of OrthoSEFE-2 whose shared graph \( G_\cap \) is a cycle. An equivalent instance \( \langle G_1^*, G_2^* \rangle \) of OrthoSEFE-2 such that (i) the shared graph \( G_\cap^* \) is a cycle, (ii) graph \( G_1^* \) is outerplanar, and (iii) no two degree-4 vertices in \( G_1^* \) are adjacent, can be constructed in polynomial time.
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Appendix

A Definitions for the appendix

In Section 2 we already discussed how to assign the exclusive edges to either of the two sides of $C$. We formalise this assignment by means of a function $A : \bigcup_{i=1}^{k} E_i \setminus E(G) \to \{l, r\}$, where $A(e) = l$ (resp. $A(e) = r$) if edge $e$ lies to the left (resp. to the right) of $C$, according to an arbitrary orientation of $C$.

Connectivity and SPQR-trees. A graph $G = (V, E)$ is connected if there is a path between any two vertices. A cutvertex is a vertex whose removal disconnects the graph. A separating pair $\{u, v\}$ is a pair of vertices whose removal disconnects the graph. A connected graph is biconnected if it does not have a cutvertex and a biconnected graph is 3-connected if it does not have a separating pair.

We consider $st$-graphs with two special pole vertices $s$ and $t$. The family of $st$-graphs can be constructed in a fashion very similar to series-parallel graphs. Namely, an edge $st$ is an $st$-graph with poles $s$ and $t$. Now let $G_i$ be an $st$-graph with poles $s_i, t_i$ for $i = 1, \ldots, k$ and let $H$ be a planar graph with two designated adjacent vertices $s$ and $t$ and $k + 1$ edges $st, e_1, \ldots, e_k$. We call $H$ the skeleton of the composition and its edges are called virtual edges; the edge $st$ is the parent edge and $s$ and $t$ are the poles of the skeleton $H$. To compose the $G_i$’s into an $st$-graph with poles $s$ and $t$, we remove the edge $st$ from $H$ and replace each $e_i$ by $G_i$ for $i = 1, \ldots, k$ by removing $e_i$ and identifying the poles of $G_i$ with the endpoints of $e_i$. In fact, we only allow three types of compositions: in a series composition the skeleton $H$ is a cycle of length $k + 1$, in a parallel composition $H$ consists of two vertices connected by $k + 1$ parallel edges, and in a rigid composition $H$ is 3-connected.

For every biconnected planar graph $G$ with an edge $st$, the graph $G − st$ is an $st$-graph with poles $s$ and $t$. Much in the same way as series-parallel graphs, the $st$-graph $G − st$ gives rise to a (de-)composition tree $T$ describing how it can be obtained from single edges. The nodes of $T$ corresponding to edges, series, parallel, and rigid compositions of the graph are $Q$-, $S$-, $P$-, and $R$-nodes, respectively. To obtain a composition tree for $G$, we add an additional root $Q$-node representing the edge $st$. We associate with each node $\mu$ the skeleton of the composition and denote it by $\text{skel}(\mu)$. For a $Q$-node $\mu$, the skeleton consists of the two endpoints of the edge represented by $\mu$ and one real and one virtual edge between them representing the rest of the graph. For a node $\mu$ of $T$, the pertinent graph $\text{pert}(\mu)$ is the subgraph represented by the subtree with root $\mu$. For a virtual edge $\varepsilon$ of a skeleton $\text{skel}(\mu)$, the expansion graph of $\varepsilon$ is the pertinent graph $\text{pert}(\mu')$ of the neighbor $\mu'$ corresponding to $\varepsilon$ when considering $T$ rooted at $\mu$.

The SPQR-tree of $G$ with respect to the edge $st$, originally introduced by Di Battista and Tamassia [16], is the (unique) smallest decomposition tree $T$ for $G$. Using a different edge $s't'$ of $G$ and a composition of $G − s't'$ corresponds to rerooting $T$ at the node representing $s't'$. It thus makes sense to say that
\( \mathcal{T} \) is the SPQR-tree of \( G \). The SPQR-tree of \( G \) has size linear in \( G \) and can be computed in linear time [13]. Planar embeddings of \( G \) correspond bijectively to planar embeddings of all skeletons of \( \mathcal{T} \); the choices are the orderings of the parallel edges in P-nodes and the embeddings of the R-node skeletons, which are unique up to a flip. When considering rooted SPQR-trees, we assume that the embedding of \( G \) is such that the root edge is incident to the outer face, which is equivalent to the parent edge being incident to the outer face in each skeleton.

We remark that in a planar embedding of \( G \), the poles of any node \( \mu \) of \( \mathcal{T} \) are incident to the outer face of \( \text{pert}(\mu) \). Hence, in the following we only consider embeddings of the pertinent graphs with their poles lying on the same face.

### B Omitted or Sketched Proofs from Section 2

**Theorem 1.** An instance \( \langle G_1, \ldots, G_k \rangle \) of OrthoSEFE-\( k \) has an OrthoSEFE if and only if it admits a SEFE satisfying the orthogonality constraints.

**Proof.** For the if part, let \( \mathcal{E} \) be the embedding of \( G_\cap \) determined by the SEFE \( \langle \mathcal{E}_1, \ldots, \mathcal{E}_k \rangle \) of \( \langle G_1, \ldots, G_k \rangle \). Observe that the orthogonality constraints at each vertex define (i) whether a degree 2 vertex of \( G_\cap \) has to be drawn straight or bent, and (ii) which face incident to a degree 3 vertex of \( G_\cap \) has to be assigned the 180° angle. It is not hard to see that a planar orthogonal drawing \( \Gamma \) of \( G_\cap \) in which the embedding of \( G_\cap \) is \( \mathcal{E} \) satisfying such requirements can be constructed. We draw the exclusive edges in each \( E_i \) as orthogonal polylines in \( \Gamma \) inside the face of \( \mathcal{E}_i \) determined by the SEFE. The fact that the exclusive edges of each \( E_i \) can be drawn in \( \Gamma \) without introducing any crossings descends from the fact that \( \mathcal{E}_i \) is a planar embedding of \( G_i \).

For the only if part, let \( v \) be a vertex in \( G_\cap \) such that the orthogonality constraints are not satisfied at \( v \). If \( v \) has exactly two neighbors \( u \) and \( w \) in \( G_\cap \), then we need to assign a port to two exclusive edges of the same graph (one for each of these edges) on one side of the path \( uvw \) and a port to at least one exclusive edge on the other side of the path \( uvw \). If \( v \) has degree 3 in \( G_\cap \), then we need to assign a port to an exclusive edge between a pair of edges of \( G_\cap \) and a port to an exclusive edge between a different pair of edges of \( G_\cap \). Hence, in both cases we need at least five ports, which is not possible on the grid. \( \square \)

### C Omitted or Sketched Proofs from Section 3

**Theorem 3.** OrthoSEFE-3 is NP-complete, even for instances \( \langle G_1 = (V, E_1), G_2 = (V, E_2), G_3 = (V, E_3) \rangle \) with sunflower intersection in which (i) the shared graph \( G_\cap = (V, E_1 \cap E_2 \cap E_3) \) is a cycle and (ii) \( G_1 \) and \( G_2 \) are outerplanar graphs with maximum degree 3.

**Proof.** The membership in NP directly follows from Theorem 2, since an assignment \( A \), which is a certificate for our problem, can be easily verified in polynomial time to satisfy all the planarity and the orthogonality constraints.
To prove that the problem is NP-hard, we show a reduction from the NP-complete
problem \textsc{Positive Exactly-Three NaE3Sat} \cite{24}, which is the variant
of NaE3Sat in which each clause consists of exactly three unnegated literals. See
Fig. 2.

Let \( x_1, x_2, \ldots, x_n \) be the variables and let \( c_1, c_2, \ldots, c_m \) be the clauses of a
3-CNF formula \( \phi \) of \textsc{Positive Exactly-Three NaE3Sat}. We show how to
construct an equivalent instance \( G_1 = (V, E_1), G_2 = (V, E_2), G_3 = (V, E_3) \) of
\textsc{OrthoSEFE-3}; refer to Fig. 2(b). Assume, without loss of generality, that the
literals in each clause \( c_j = (x'_j, x''_j, x'''_j) \) are such that \( a > b > c \), if \( j \) is odd, and
\( a < b < c \), otherwise.

A \textit{variable-clause gadget} \( V^i_j \) for a variable \( x_i \) belonging to a clause \( c_j \) is a subgraph of \( G_\cup \) defined as follows. Gadget \( V^i_j \) contains a path \((s^j_i, u^j_i, v^j_i, v'^j_i, z^j_i, r^j_i, t^j_i)\)
belonging to \( G_\cap \), and edges \( \{u^j_i, v^j_i\} \) and \( \{w^j_i, z^j_i\} \) belonging to \( E_3 \); see Fig. 2(a).

The \textit{clause gadget} \( C^j \) for a clause \( c_j \) is a subgraph of \( G_\cup \) defined as follows. Gadget \( C^j \) contains a path \((s^j, \alpha^j, y^j_1, \beta^j, y^j_2, d^j_1, \ldots, d^j_s, \gamma^j, \delta^j, t^j)\) belonging to
\( G_\cap \), and edges \( \{\alpha^j, \beta^j\}, \{\beta^j, \gamma^j\}, \{\gamma^j, \delta^j\}, \{d^j_1, d^j_2\}, \{d^j_2, d^j_3\}, \{d^j_s, d^j_1\}, \{d^j_1, d^j_2\}, \{d^j_3, d^j_4\} \)
belonging to \( E_3 \), and edges \( \{\beta^j, d^j_2\} \) and \( \{d^j_2, \gamma^j\} \) belonging to \( E_1 \); see Fig. 2(a).

Initialize \( G_\cup \) to the union of \( V^i_j \), for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \), and of \( C^j \), for \( j = 1, \ldots, m \). Then, for \( j = 1, \ldots, m \) and for \( i = 1, \ldots, n \), identify vertex
\( t^i_j \) with vertex \( s^i_{j+1} \), if \( j \) is odd, or identify vertex \( t^i_{j+1} \) with vertex \( s^i_j \), otherwise.

Further, for \( j = 1, \ldots, m \) (where \( m + 1 = 1 \)), identify vertex \( t^i_j \) with vertex \( s^i_j \)
and vertex \( s^i_{j+1} \), if \( j \) is odd, or identify vertex \( t^i_{j+1} \) with vertex \( s^i_j \)
and vertex \( s^i_{j+1} \), otherwise.

To complete the construction of \( (G_1, G_2, G_3) \) we add to \( G_\cup \) exclusive edges as follows. For \( i = 1, \ldots, n \) and for \( j = 1, \ldots, m - 1 \), we add an edge \( \{w^j_i, w^j_{i+1}\} \)
to \( E_2 \), if \( j \) is odd, or to \( E_1 \), otherwise. We call these edges \textit{transmission edges}.

Further, for \( i = 1, \ldots, n \) and for \( j = 1, \ldots, m \), we add an edge \( \{w^j_i, y^j_1\} \), if \( x_i \in c_j \),
or an edge \( \{w^j_i, r^j_1\} \), otherwise.

Clearly, the construction of instance \( (G_1, G_2, G_3) \) can be completed in polynomial
time.

Graph \( G_\cap \) is a cycle, as we already observed. Also, the transmission edges in
\( E_1 \) (in \( E_2 \)) do not alternate along \( G_\cap \), since the variable-clause gadgets appear along \( G_\cap \) in the order \( V^1_1 \), \ldots, \( V^1_m \), if \( j \) is odd, or in the order \( V^m_1 \), \ldots, \( V^m_1 \), otherwise. Also, no transmission edge in \( E_1 \) alternates with edges \( (\beta^j, d^j_2) \) and
\( (d^j_4, \gamma^j) \), for any \( j \), and such edges do not alternate with each other by construction. Hence, \( G_1 \) and \( G_2 \) are outerplanar. The fact that \( G_1 \) and \( G_2 \) have maximum
degree 3 also directly follows from the construction.

Given a positive instance \( \phi \) of \textsc{Positive Exactly-Three NaE3Sat}, we show that
\( (G_1, G_2, G_3) \) is a positive instance of \textsc{OrthoSEFE-3}. Given a satisfying
truth assignment \( T : X \rightarrow \{\text{true}, \text{false}\} \) where \( X \) denotes the set of variables
in \( \phi \), we construct an assignment \( A \) of the exclusive edges of \( E_1 \cup E_2 \cup E_3 \) to the
two sides of \( G_\cap \) satisfying all the planarity and the orthogonality constraints.

For \( i = 1, \ldots, n \) and for \( j = 1, \ldots, m \), we set \( A(e) = l \), for each exclusive edge
e \( E_1 \cup E_2 \cup E_3 \) incident to \( w^j_i \), if \( T(x_i) = \text{true} \), or \( A(e) = r \), otherwise. For i =
1, \ldots, n and for \( j = 1, \ldots, m \), we set \( A(u^j_i, v^j_i) = r \), if \( T(x_i) = \text{true} \), or \( A(u^j_i, v^j_i) = l \), otherwise. For each clause \( c_j = (x^j_1, x^j_2, x^j_3) \), we set \( A(\alpha^j, \beta^j) = l \), if \( T(x_a) = \text{false} \), or \( A(\alpha^j, \beta^j) = r \), otherwise; we set \( A(\beta^j, \gamma^j) = l \), if \( T(x_b) = \text{false} \), or \( A(\beta^j, \gamma^j) = r \), otherwise; and we set \( A(\gamma^j, \delta^j) = l \), if \( T(x_c) = \text{false} \), or \( A(\gamma^j, \delta^j) = r \), otherwise. Finally, for each clause \( c_j = (x^j_1, x^j_2, x^j_3) \), consider the literal \( x_o \) with \( o \in \{a, c\} \) such that \( T(x_o) = T(x_b) \), if any, otherwise let \( x_o = x_a \). Suppose that \( x_o = x_a \); set \( A(d^j_1, d^j_3) = A(d^j_2, d^j_4) = A(\beta^j, \gamma^j) = r \) and set \( A(d^j_2, d^j_3) = A(d^j_1, d^j_4) = A(\beta^j, \gamma^j) = l \), if \( T(x_o) = \text{false} \), or set \( A(d^j_1, d^j_4) = A(d^j_2, d^j_3) = A(\beta^j, \gamma^j) = l \) and set \( A(d^j_2, d^j_3) = A(d^j_1, d^j_4) = A(\beta^j, \gamma^j) = r \), otherwise. Suppose that \( x_o = x_c \); set \( A(d^j_1, d^j_3) = A(d^j_2, d^j_4) = A(\beta^j, \gamma^j) = l \) and set \( A(d^j_2, d^j_3) = A(d^j_1, d^j_4) = A(\beta^j, \gamma^j) = r \), if \( T(x_o) = \text{false} \), or set \( A(d^j_2, d^j_3) = A(d^j_1, d^j_4) = A(\beta^j, \gamma^j) = l \) and set \( A(d^j_2, d^j_3) = A(d^j_1, d^j_4) = A(\beta^j, \gamma^j) = r \), otherwise.

We show that \( A \) satisfies the planarity constraints. First observe that the planarity constraints for the edges in \( E_1 \) and \( E_2 \) are trivially satisfied by \( A \) since \( G_1 \) and \( G_2 \) are outerplanar. As for the edges in \( E_3 \), we have that the only pairs of edges that alternate along \( G_1 \) are \( \{(u^j_i, v^j_i), (u^j_i, z^j_i)\} \), for \( i = 1, \ldots, n \) and for \( j = 1, \ldots, m \), pairs \( \{(u^j_i, y^j_i), (\alpha^j, \beta^j)\}, \{(u^j_i, y^j_i), (\beta^j, \gamma^j)\}, \) and \( \{(u^j_i, z^j_i), (\gamma^j, \delta^j)\} \), for \( j = 1, \ldots, m \), and the edges incident to the dummy vertices \( d^j_1, \ldots, d^j_6 \), for \( j = 1, \ldots, m \). However, it is easy to verify that \( A \) assigns alternating edges to different sides of \( G_1 \).

We show that \( A \) satisfies the orthogonality constraints at every vertex. For all the vertices except for \( u^j_i, \beta^j, d^j_1, d^j_2, \) and \( \gamma^j, \) for \( i = 1, \ldots, n \) and for \( j = 1, \ldots, m \), this is true since they have only one incident exclusive edge. For vertices \( u^j_i, d^j_1, d^j_3, \) and \( d^j_2, \) with \( i = 1, \ldots, n \) and for \( j = 1, \ldots, m \), this is true since all the edges incident to \( u^j_i \) are assigned to the same side of \( G_1 \) by \( A \), by construction. For vertex \( \beta_j \) and \( \gamma_j \), we distinguish two cases based on whether there exists a \( o \in \{a, c\} \) with \( T(x_b) = T(x_a) \): (i) If this is case, let \( o = c \) without loss of generality; the case \( o = a \) can be shown analogously. Then, \( A(\beta^j, \gamma^j) = A(\gamma^j, \delta^j) = A(d^j_1, d^j_4), \) by construction, and hence the orthogonality constraints are satisfied at \( \gamma^j \). To prove that they are also satisfied at \( \beta^j \), it suffices to show that the two edges of \( E_3 \) incident to \( \beta^j \) are assigned to different sides of \( G_1 \), given that \( \beta^j \) has degree 3 in \( G_1 \) and degree 2 in \( G_2 \). Namely, due to the fact that \( T \) is a NAE3SAT truth assignment we have that \( T(x_a) \neq T(x_b) \), and hence \( A(\alpha^j, \beta^j) \neq A(\beta^j, \gamma^j) \). (ii) In the second case, \( T(x_a) \neq T(x_b) \neq T(x_c) \), hence we have that \( A(\alpha^j, \beta^j) \neq A(\beta^j, \gamma^j) \neq A(\gamma^j, \delta^j) \). Since vertices \( \beta^j \) and \( \gamma^j \) have degree 4 in \( G_3 \), degree 3 in \( G_1 \), and degree 2 in \( G_2 \), this implies that the orthogonality constraints are satisfied at \( \beta^j \) and at \( \gamma^j \).

Suppose that \( \langle G_1 = (V, E_1), G_2 = (V, E_2), G_3 = (V, E_3) \rangle \) is a positive instance of ORTHOSEFE-3 and let \( A \) be the corresponding assignment of the exclusive edges to the sides of \( G_1 \). We show how to construct a NAE3SAT truth assignment \( T \) that satisfies \( \phi \). For \( i = 1, \ldots, n \), we set \( T(x_i) = \text{true} \) if and only if \( A(u^j_i, z^j_i) = l \). We start by proving that, for each \( i = 1, \ldots, n \), all the edges incident to \( u^j_i \), with \( 1 \leq j \leq m \), are assigned to the same side of \( G_1 \).

Observe that, for each \( i = 1, \ldots, n \) and for each for each \( j = 1, \ldots, m \), the
two edges in \( G_3 \) incident to \( w_i^j \) both alternate with edge \((w_i^j, v_i^j)\) along \( G_\cap \), and hence are assigned to the same side of \( G_\cap \) by the planarity constraints. Hence, by the orthogonality constraints at \( w_i^j \), all the exclusive edges in \( E_1 \cup E_2 \) incident to \( w_i^j \) lie on the same side of \( G_\cap \) as \((w_i^j, z_i^j)\). Further, since any two vertices \( w_i^j \) and \( w_i^{j+1} \), are connected by a transmission edge in either \( E_1 \) or in \( E_2 \), the statement follows. This property allows us to focus on each clause separately. Let \( c_j = (x_i^j, x_i^j, x_i^j) \) be a clause in \( \phi \), with \( 1 \leq j \leq m \). We show that \( T(x_i^j) = T(x_i^j) = T(x_i^j) \) does not hold. First, we show that \( A(\beta^j, d_i^j) \neq A(\gamma^j). \) Namely, by the planarity constraints, \( A(d_i^j, d_i^j) = A(d_i^j, d_i^j) = A(d_i^j, d_i^j) \); then, by the orthogonality constraints at \( d_i^j \) and at \( d_i^j \), we have that \( A(\beta^j, d_i^j) = A(d_i^j, d_i^j) = A(d_i^j, d_i^j) \) and that \( A(d_i^j, \gamma^j) = A(d_i^j, d_i^j) = A(d_i^j, d_i^j), \) and the statement follows. Second, \( A(\alpha^j, \beta^j) = A(\beta^j, \gamma^j) = A(\gamma^j, \delta^j) \) does not hold, since \( A(\beta^j, d_i^j) \neq A(d_i^j, \gamma^j) \) and by the orthogonality constraints at \( \beta^j \) and at \( \gamma^j \). This implies that \( A(w_i^j, y_i^j) = A(w_i^j, y_i^j) = A(w_i^j, y_i^j) \) does not hold, and hence \( A(w_i^j, z_i^j) = A(w_i^j, z_i^j) = A(w_i^j, z_i^j) \) does not hold, since all the edges incident to \( w_i^j \), with \( 1 \leq j \leq m \), are assigned to the same side of \( G_\cap \). This concludes the proof that \( T(x_i^j) = T(x_i^j) = T(x_i^j) \) does not hold.

It is easy to see that the reduction can be performed in polynomial time and that it can be extended to any \( k > 3 \) by subdividing two edges of \( G_\cap \) for each additional graph \( G_i \) and by introducing an exclusive edge between these vertices only belonging to \( G_i \).

\[ \square \]

### D Omitted or Sketched Proofs from Section 4

**Lemma 2** Let \( \langle G_1, G_2 \rangle \) be an instance of OrthoSEFE-2 such that \( G_\cap = (V, E_1 \cup E_2) \) is a cycle and \( G_1 \) has maximum degree 3. It is possible to construct in polynomial time an equivalent instance \( \langle G_1', G_2' \rangle \) of OrthoSEFE-2 such that \( G_\cap = (V, E_1 \cap E_2) \) is a cycle and \( G_1' \) is an outerplanar graph with maximum degree 3.

**Proof.** We describe how to construct an equivalent instance \( \langle G_1', G_2' \rangle \) of OrthoSEFE-2 such that \( G_\cap' \) is a cycle, \( G_1' \) has maximum degree 3 and the number of pairs of edges in \( G_1' \) that alternate along \( G_\cap' \) is smaller than the number of pairs of edges in \( G_1 \) that alternate along \( G_\cap \). Note that repeatedly performing this transformation eventually yields an equivalent instance \( \langle G_1', G_2' \rangle \) satisfying the requirements of the lemma.

Consider two edges \( e = (u, v) \) and \( f = (w, z) \) of \( G_1 \) such that \( u, w, v, z \) appear in this order along cycle \( G_\cap \) and such that the path \( P_{u, z} \) in \( G_\cap \) between \( u \) and \( z \) that contains \( v \) and \( w \) has minimal length. If \( G_1 \) is not outerplanar, edges \( e \) and \( f \) always exist.

Initialize \( G_\cap = G_\cap \). Replace path \( P_{u, z} \) in \( G_\cap \) by a path \( P'_{u, z} \), as follows; refer to Fig. 8 Let \( H_1, H_2, \) and \( H_3 \) be the sets of vertices between \( u \) and \( w \), between \( w \) and \( v \), and between \( v \) and \( z \) in \( G_\cap \). Path \( P'_{u, z} \) contains \( u \), then the vertices in \( H_1 \), then a dummy vertex \( w' \), then the vertices in \( H_2 \), then a dummy vertex
z', then three dummy vertices x₁, x₂, x₃, then v, then four dummy vertices x₄, x₅, x₆, x₇, then w, then three dummy vertices x₈, x₉, x₁₀, then three dummy vertices u', x₁₁, and u', then the vertices in H₃, and finally z. Note that G'_₁ contains all the vertices of G₁, plus a set of dummy vertices. We now describe the exclusive edges in E'_₁ and E'_₂. Initialize E'_₁ = E₁ and E'_₂ = E₂. Add edges e' = (u', v') and f' = (w', z') to E'_₁. Also, add edges (z', x₂), (x₁, v), (x₃, x₄), (v, x₆), (x₅, w), (x₇, x₈), (w, x₁₀), (x₈, u'), and (x₉, w') to E'_₂. Finally, replace in E'_₂ each edge (x, w) incident to w by an edge (x, w') and each edge (x, v) incident to v with an edge (x, v').

Before proving the statement, we observe an important property that will be used in the following, namely that there exists no exclusive edge in E₁, and hence in E'_₁, with an endpoint in H₂ and the other one not in H₂. In fact, there exists no edge connecting a vertex of H₂ to any of u, v, w, z, since these vertices are already incident to edges e and f, respectively, and since G₁ has maximum degree 3. Also, there exists no exclusive edge g connecting a vertex of H₂ to a vertex of H₁ (of H₃), since in this case g would alternate with f (with e), hence contradicting the minimality of path P_u,z. Finally, the existence of an exclusive edge connecting a vertex of H₂ to any other vertex in V would immediately make the instance negative, since G₁ would not be planar.

We now prove that ⟨G'_₁, G'_₂⟩ satisfies the required properties. First, graph G'_₁ is a cycle by construction. Second, G'_₁ has maximum degree 3, since (i) every vertex in V ∩ V' is incident to the same edges in E'_₁ as in E₁, (ii) dummy vertices xᵢ, with i = 1, ..., 11, have degree 2, and (iii) dummy vertices w', z', u', and v' have degree 3. Third, the number of pairs of alternating edges in ⟨G'_₁, G'_₂⟩ is smaller than in ⟨G₁, G₂⟩. In fact, (i) edge e' does not alternate with any edge of E'_₁, since x₁₁ is not incident to any exclusive edge in E'_₁, (ii) edge f' does not alternate with any edge in E'_₁, since there exists no exclusive edge in E'_₁ with an endpoint in H₂ and the other one not in H₂, and (iii) all pairs of edges in E₁ ∩ E'_₁ that alternate along G'_₁ also alternate along G₁, except for edges e and f, which alternate along G₁ but not along G'_₁.

We now prove that ⟨G'_₁, G'_₂⟩ is equivalent to ⟨G₁, G₂⟩.

Suppose that ⟨G₁, G₂⟩ admits an OrthoSEFE ⟨Γ₁, Γ₂⟩. By Theorem 2, ⟨Γ₁, Γ₂⟩ determines an assignment A of the exclusive edges of E₁ and of E₂ to the two sides of G₁ satisfying all the planarity and the orthogonality constraints. We

Fig. 8. Instances (a) ⟨G₁, G₂⟩ and (b) ⟨G'_₁, G'_₂⟩ for the proof of Lemma 2. Edges of the shared graph G₁ are black. Exclusive edges of G₁ are red and those of G₂ are blue.
show how to construct an assignment $A'$ of the exclusive edges of $E'_1$ and of $E'_2$ to the two sides of $G'_1$ satisfying all the constraints.

For each exclusive edge $g \in E_1 \cap E'_1$, set $A'(g) = A(g)$. Also, set $A'(e') = A(e)$ and $A'(f') = A(f)$. For each exclusive edge $g \in E_2 \cap E'_2$, set $A'(g) = A(g)$. Also, for each edge $(x, w')$ (resp. $(x, v')$) incident to $w'$ (resp. to $v'$), set $A'(x, w') = A(x, w)$ (resp. $A'(x, v') = A(x, v)$). Further, set $A'(x_1, v) = A'(v, x_6) = A'(x_7, x_8) = A'(x_8, u') = A'(x_9, u') = A(e)$ and set $A'(z', x_2) = A'(z', x_3) = A'(x_3, x_4) = A'(x_5, w) = A'(w, x_{10}) = A(f)$.

The planarity constraints for the edges of $G'_1$ are satisfied since any pair of edges that alternate along $G'_1\cap G'_2$ also alternate in $G_1$ and since their assignment in $A'$ and in $A$ are the same, by construction.

We prove that the planarity constraints for the edges of $G'_2$ are satisfied by $A'$. For the edges that are not incident to any dummy vertex, this is true for the same reason as for the edges of $G'_1$. For each edge $(x, w')$ incident to $w'$, this is true since $A'(x, w') = A(x, w)$, and since $(x, w')$ alternates with an edge $g \in E'_2$ along $G'_1$ if and only if edge $(x, w)$ alternates with an edge $g^*$ along $G_1$, where $g^* = g$ if $g$ is not incident to $v'$, while $g^* = (y, v)$ if $g = (y, v')$. Analogous arguments hold for each edge $(x, v')$ incident to $v'$. Finally, the fact that the planarity constraints for each edge incident to two dummy vertices are satisfied by $A'$ can be easily verified; recall that $A(e) \neq A(f)$.

We now prove that the orthogonality constraints are satisfied by $A'$ at every vertex of $V'$. For the vertices in $V' \cap V \setminus \{w, v\}$, this is true since they are satisfied by $A$ and since for every exclusive edge $g \in E'_1 \cup E'_2$ incident to these vertices, we have that $g \in E_1 \cup E_2$, by construction, and $A'(g) = A(g)$. For vertex $w$, this is true since $A'(x_5, w) = A'(w, x_{10}) = A(f) = A'(f)$. For vertex $v$, this is true since $A'(x_1, v) = A'(v, x_6) = A(e) = A'(e)$. For vertex $v'$, this is true since $A'(x_8, u') = A'(x_9, u') = A'(e') = A(e)$. For vertex $z'$, this is true since $A'(z', x_2) = A'(z', x_3) = A'(f') = A(f)$. For vertex $w'$, assume there exist two exclusive edges $e^a_w, e^b_w \in E'_2$ that are incident to $w'$, the case in which there exists only one or none of them being trivial. Since $A'(e^a_w) = A(e^a_w), A'(e^b_w) = A(e^b_w)$, and $A'(f') = A(f)$, and since the orthogonality constraints at $w$ are satisfied by $A$, the orthogonality constraints at $w'$ are satisfied by $A'$. Analogously, the orthogonality constraints at $v'$ between edges $e^a_v, e^b_v \in E'_2$, if any, and edge $e' \in E'_1$ are satisfied by $A'$ since the same constraints at $v$ between edges $e^a_v, e^b_v \in E_2$ and $e \in E_1$ are satisfied by $A$. Since vertices $x_i$, with $i = 1, \ldots, 11$, have degree 2 in $G'_1$, this concludes the proof that $A'$ satisfies the orthogonality constraints.

Suppose that $\langle G'_1, G'_2 \rangle$ admits OrthoSEFE $\langle \Gamma'_1, \Gamma'_2 \rangle$, and let $A'$ be the corresponding assignment of the exclusive edges of $E'_1$ and of $E'_2$ to the two sides of $G'_1$. We show how to construct an assignment $A$ of the exclusive edges of $E_1$ and of $E_2$ to the two sides of $G_1$ satisfying all the planarity and the orthogonality constraints.

For each exclusive edge $g \in E_1$, set $A(g) = A'(g)$. For each exclusive edge $g \in E_2 \cap E'_2$, set $A(g) = A'(g)$. Also, for each edge $(x, w)$ (resp. $(x, v)$) incident to $w$ (resp. to $v$), set $A(x, w) = A'(x, w')$ (resp. $A(x, v) = A'(x, v')$).
We prove that the planarity constraints for the edges of \( G_1 \) are satisfied by \( A \). For each pair \((e_1, e_2)\) of exclusive edges in \( E_1 \) such that \( \{e_1, e_2\} \neq \{e, F\} \), this is true since \( e_1 \) and \( e_2 \) alternate along \( G_{\gamma} \) if and only if they alternate along \( G'_{\gamma} \), by construction. For pair \((e, f)\), this is true for the following reason. By planarity constraints, we have \( A'(x_1, v) = A'(v, x_6) \neq A'(x_3, x_4) \); hence, by orthogonality constraints at vertex \( v \), we have \( A'(e) = A'(x_1, v) = A'(v, x_6) \). Analogously, we have \( A'(f) = A'(x_3, w) = A'(w, x_{10}) \neq A'(x_7, x_8) \). Since, by planarity constraints, \( A'(v, x_6) \neq A'(x_3, w) \), we have \( A'(e) \neq A'(f) \) and hence \( A(e) \neq A(f) \). We prove that the planarity constraints for the edges of \( G_2 \) are satisfied by \( A \). For each pair \((e_1, e_2)\) of exclusive edges in \( E_2 \) such that neither \( e_1 \) nor \( e_2 \) is incident to either of \( w \) and \( v \), this is true since \( e_1 \) and \( e_2 \) alternate along \( G_{\gamma} \) if and only if they alternate along \( G'_{\gamma} \), by construction. For each edge \((x, w)\) incident to \( w \), this is true since \( A(x, w) = A'(x, w') \), and since \((x, w)\) alternates with an edge \( g \in E_2 \) along \( G_{\gamma} \), if and only if edge \((x, w)\) alternates with an edge \( g' \) along \( G'_{\gamma} \), where \( g' = g \) if \( g \) is not incident to \( v \), while \( g' = (g, v') \) if \( g = (g, v) \). Analogous arguments hold for each edge \((x, v)\) incident to \( v \).

We now prove that the orthogonality constraints are satisfied by \( A \) at every vertex of \( V \). For vertices in \( V \setminus \{w, v\} \), this is true since they are satisfied by \( A' \) and since for every exclusive edge \( g \in E_1 \cup E_2 \) incident to these vertices, \( g \in E'_1 \cup E'_2 \), by construction, and \( A(g) = A'(g) \). In order to prove that the constraints are satisfied also at \( w \) and \( v \), we first argue that \( A(f) = A'(f') \) and \( A(e) = A'(e') \). Namely, by planarity constraints, \( A'(z', x_2) \neq A'(x_1, v) \neq A'(z', x_3) \), and hence \( A'(z', x_2) = A'(z', x_3) \). Similarly, \( A'(x_5, w) \neq A'(x_7, x_8) \neq A'(w, x_{10}) \), and hence \( A'(x_5, w) = A'(w, x_{10}) \). Then, by using the longer chain of alternating edges we get \( A'(z', x_3) \neq A'(x_1, v) \neq A'(x_3, x_4) \neq A'(v, x_6) \neq A'(x_5, w) \). Finally, by orthogonality constraints at \( x' \) and \( w \), we get \( A'(f') = A'(z', x_3) \) and \( A'(f) = A'(x_5, w) \). Since \( A(f) = A'(f) \) we conclude \( A(f) = A'(f') \). The equality \( A(e) = A'(e') \) follows symmetrically. We now prove that the orthogonality constraints are satisfied at \( w \) and \( v \). For vertex \( w \), assume there exist two exclusive edges \( e^a_w, e^b_w \in E_2 \) incident to \( w \), the case in which there exists only one or none of them being trivial. Since \( A(e^a_w) = A'(e^a_w), A(e^b_w) = A'(e^b_w) \), and \( A(f) = A'(f') \), and since the orthogonality constraints at \( w' \) between \( e^a_w, e^b_w \) and \( f' \) are satisfied by \( A' \), the orthogonality constraints at \( w \) between \( e^a_w, e^b_w \) and \( f \) are satisfied by \( A \). Analogously, the orthogonality constraints at \( v \) between edges \( e^a_v, e^b_v \in E_2 \), if any, and edge \( e \in E_1 \) are satisfied by \( A \) since the same constraints at \( v' \) between \( e^a_v, e^b_v \in E'_2 \) and \( e' \in E'_1 \) are satisfied by \( A' \). This concludes the proof of the lemma.

**Lemma 3.** Let \( \langle G_1, G_2 \rangle \) be an instance of OrthoSEFE-2 such that \( G_{\gamma} = (V, E_1 \cap E_2) \) is a cycle and each vertex \( v \in V \) has degree at most 3 in either \( G_1 \) or \( G_2 \). It is possible to construct in polynomial time an equivalent instance \( \langle G'_1, G'_2 \rangle \) of OrthoSEFE-2 such that \( G'_{\gamma} = (V', E'_1 \cap E'_2) \) is a cycle and graph \( G'_1 \) has maximum degree 3.

**Proof.** We describe how to construct an equivalent instance \( \langle G'_1, G'_2 \rangle \) of OrthoSEFE-2 such that \( G'_{\gamma} \) is a cycle, each vertex \( v \in V' \) has degree at most 3 in either \( G'_1 \).
or $G'_2$, and the number of degree-4 vertices in $G'_1$ is smaller than the number of degree-4 vertices in $G_1$. Note that repeatedly performing this transformation eventually yields an equivalent instance $\langle G'_1, G'_2 \rangle$ satisfying the requirements of the lemma.

Consider a vertex $v \in V$ such that there exists two edges $e = (v, u_e), f = (v, u_f) \in E_1$ incident to $v$. Assume without loss of generality that $u_e, v, u_f$ appear in this order along $G \cap$. Suppose that there exists an edge $h = (v, u_h) \in E_2$ incident to $v$, the other case being simpler. We describe the construction for the case in which vertices $u_e, v, u_f, u_h$ appear in this order along $G \cap$; the other cases are analogous.

Initialize $G'_\cap = G'_\cap$; refer to Fig. 9. Replace $v$ in $G'_\cap$ by a path $P_v = x_1, x_2, v_e, x_3, y_1, y_2, v_f, y_3, v', z_3, z_1, z_2, v_h, z_3$ composed of dummy vertices.

We now describe the exclusive edges in $E'_1$ and $E'_2$. Set $E'_i$, with $i = 1, 2$, contains all the exclusive edges in $E_i$ that are not incident to $v$. Also, $E'_1$ contains edges $e'' = (v_e, u_e), f'' = (v_f, u_f)$, and $h' = (v_h, v')$. Finally, $E'_2$ contains edges $(x_1, v_e), (x_2, x_3), (y_1, v_f), (y_2, y_3), (z_1, v_h), (z_2, z_3)$, and edges $e' = (v_e, v'), f' = (v_f, v')$, and $h'' = (v_h, u_h)$.

We prove that $\langle G'_1, G'_2 \rangle$ satisfies the required properties. First, graph $G'_\cap$ is a cycle by construction. Second, the degree of the vertices in $V \setminus V'$ is the same in $G'_1$ (resp. $G'_2$) as in $G_1$ (resp. as in $G_2$), while all the dummy vertices have degree at most 3 in $G'_1$. Hence, every vertex in $V'$ has degree at most 3 in either $G'_1$ or $G'_2$; also, the number of degree-4 vertices in $G'_1$ is smaller than the number of degree-4 vertices in $G_1$, since $v \notin G'_1$.

We now prove that $\langle G'_1, G'_2 \rangle$ is equivalent to $\langle G_1, G_2 \rangle$.

Suppose that $\langle G_1, G_2 \rangle$ admits an OrthoSEFE $\langle I'_1, I'_2 \rangle$, and let $A$ be the corresponding assignment of the exclusive edges of $E_1$ and of $E_2$ to the two sides of $G\cap$, which exists by Theorem 2. We show how to construct an assignment $A'$ of the exclusive edges of $E'_1$ and of $E'_2$ to the two sides of $G'_\cap$ satisfying all the constraints.

For each exclusive edge $g \in E_1 \cup E_2$ incident to $v$, set $A'(g) = A(g)$. Also, set $A'(e'') = A(e), A'(f'') = A(f), \text{ and } A'(h') = A(h)$. Finally, set $A'(x_1, v_e) = A'(e') = A(e)$ and $A'(x_2, x_3) \neq A(e)$; set $A'(y_1, v_f) = A'(f') = A(f)$ and $A'(y_2, y_3) \neq A(f)$; and set $A'(z_1, v_h) = A'(h'') = A(h)$ and $A'(z_2, z_3) \neq A(h)$.
We prove that the planarity constraints for the edges of $G'_1$ are satisfied by $A'$. Note that, by construction, edge $h'$ does not alternate with any edge of $G'_1$ along $G'_\cap$. Also, edges $e''$ and $f''$ do not alternate with each other along $G'_\cap$. Further, if edge $e''$ (edge $f''$) alternates with an edge $g \in G'_1$ along $G'_\cap$, then edge $e$ (edge $f$) alternates with $g$ along $G_\cap$. Finally, any two edges not incident to any dummy vertex that alternate along $G'_\cap$ also alternate along $G_\cap$. In all the described cases, the planarity constraints are satisfied by $A'$ since they are satisfied by $A$.

We prove that the planarity constraints for the edges of $G'_2$ are satisfied by $A'$. Note that, by construction, edges $e'$, $f'$, $(x_1, v_e)$, $(x_2, x_3)$, $(y_1, v_f)$, $(y_2, y_3)$, $(z_1, v_h)$, and $(z_2, z_3)$ do not alternate with any edge of $G'_2$ that is not incident to a dummy vertex along $G'_\cap$; it easy to verify that $A'$ satisfies the planarity constraints among these edges. Also, edge $h''$ alternates with $(z_2, z_3)$, but $A'(h'') \neq A'(z_2, z_3)$ by construction. Further, if edge $h''$ alternates with an edge $g \neq (z_2, z_3) \in G'_2$ along $G'_\cap$, then edge $h$ alternates with $g$ along $G_\cap$. Finally, any two edges not incident to any dummy vertex that alternate along $G'_\cap$ also alternate along $G_\cap$. In all these cases, the planarity constraints are satisfied by $A'$ since they are satisfied by $A$.

We now prove that the orthogonality constraints are satisfied by $A'$ at every vertex in $V'$. For non-dummy vertices, this is true since they are satisfied by $A$ and since all the edges incident to them have the same assignment in $A$ as in $A'$. For vertices $x_i, y_i,$ and $z_i$, with $i = 1, 2, 3$, this is true since they have degree 2 in $G'_1$. For vertex $v_e$, this is true since $A'(e'') = A'(x_1, v_e) = A'(e') = A(e$); similar arguments apply for vertices $v_f$ and $v_h$. Finally, for vertex $v'$, this is true since (i) $A'(e') = A(e), A'(f') = A(f), and A'(h') = A(h)$, (ii) $e$, $f$, and $h$ are incident to $v$ in $G_2$, and (iii) $A$ satisfies the orthogonality constraints. This concludes the proof that $A'$ satisfies the orthogonality constraints.

Suppose that $\langle G'_1, G'_2 \rangle$ admits an OrthoSEFE $\langle \Gamma'_1, \Gamma'_2 \rangle$, and let $A'$ be the corresponding assignment of the exclusive edges of $E'_1$ and of $E'_2$ to the two sides of $G'_\cap$. We show how to construct an assignment $A$ of the exclusive edges of $E_1$ and of $E_2$ to the two sides of $G_\cap$ satisfying all the planarity and the orthogonality constraints.

For each exclusive edge $g \in E_1 \cup E_2$ not incident to $v$, set $A(g) = A'(g)$. Also, set $A(e) = A'(e''), A(f) = A'(f'')$, and $A(h) = A'(h'')$.

We prove that the planarity constraints for the edges of $G_1$ and of $G_2$ are satisfied by $A$. Consider any pair of edges $\langle g_1, g_2 \rangle$ of the same graph $G_i$, with $i = 1, 2$, that alternate along $G_\cap$. If none of $g_1$ and $g_2$ is incident to $v$, then they also alternate along $G'_\cap$. Hence, the planarity constraints are satisfied by $A$ since they are satisfied by $A'$. Otherwise, assume $g_1$ is incident to $v$; note that $g_2$ is not incident to $v$, since $g_1$ and $g_2$ alternate along $G_\cap$. If $g_1 = e$ (if $g_1 = f$; if $g_1 = h$), then edge $e''$ (edge $f''$; edge $h''$) alternates with $g_2$ along $G'_\cap$. Further, $A(e) = A'(e''), A(f) = A'(f''), A(h) = A'(h'')$, and $A(g_2) = A(g_2)$. Hence, the planarity constraints for these edges are satisfied by $A$ since they are satisfied by $A'$.
We finally prove that the orthogonality constraints are satisfied by $A$. For the vertices in $V \setminus \{v, u_e, u_f, u_h\}$, this is true since they are satisfied by $A'$ and since for every exclusive edge $g \in E_1 \cup E_2$ incident to these vertices, we have that $g \in E'_1 \cup E'_2$, by construction, and $A(g) = A'(g)$. For vertex $u_e$, this is true since for each edge $g$ incident to $u_e$ different from $e$, it holds that $A(g) = A'(g)$, since $A(e) = A'(e'')$, and since the orthogonality constraints at $u_e$ are satisfied by $A'$. Analogous arguments hold for vertices $u_f$ and $u_h$. To prove that this is true for $v$, we first argue that $A(e) = A'(e'')$, and that $A(h) = A'(h')$: Namely, by planarity constraints, we get $A'(e') = A'(x_1, v_3)$ since they both alternate with $(x_2, x_3)$; hence, by orthogonality constraints at $v_3$, we get $A'(e'') = A'(e')$. Since $A(e) = A'(e'')$, by construction, we conclude $A(e) = A'(e')$. The equalities $A(f) = A'(f')$ and $A(h) = A'(h')$ follow symmetrically. Hence, the orthogonality constraints at $v$ are satisfied by $A$ since they are satisfied at $v'$ by $A'$. This concludes the proof. \(\square\)

**Theorem 5** OrthoSEFE-2 can be solved in polynomial time for instances whose shared graph is a cycle and whose union graph has maximum degree 5.

**Proof.** First apply Lemma 3 to obtain an equivalent instance $\langle G_1', G_2' \rangle$ such that $G_1'$ is a cycle and graph $G_2'$ has maximum degree 3. Then, apply Lemma 2 to obtain an equivalent instance $\langle G_1'', G_2'' \rangle$ such that $G_1''$ is a cycle and $G_2''$ is an outerplanar graph with maximum degree 3. Finally, apply Lemma 1 to test in polynomial time whether $\langle G_1'', G_2'' \rangle$, and hence $\langle G_1, G_2 \rangle$, is a positive instance. \(\square\)

### E Omitted or Sketched Proofs from Section 5

**Lemma 4** Let $\langle G_1, G_2 \rangle$ be an instance of OrthoSEFE-2 whose shared graph is biconnected. It is possible to construct in polynomial time an equivalent instance $\langle G_1^*, G_2^* \rangle$ whose shared graph is biconnected and such that each endpoint of an exclusive edge has degree 2 in the shared graph.

**Proof.** We start with a simplification step that removes certain edges. An exclusive edge $e = uv$ of $G_1$ or of $G_2$ is an intra-pole edge if its endpoints are adjacent in some skeleton of the SPQR-tree of the shared graph $G_{\tau}$. If neither $u$ nor $v$ is incident to other exclusive edges, then $uv$ is isolated. Let $E_1'$ and $E_2'$ be the isolated intra-pole edges of $G_1$ and $G_2$, respectively.

We claim that the instance $\langle G_1 - E_1', G_2 - E_2' \rangle$ admits an OrthoSEFE if and only if $\langle G_1, G_2 \rangle$ does. The if part is clear since we can simply remove the isolated intra-pole edges from an OrthoSEFE of $\langle G_1, G_2 \rangle$ to obtain an OrthoSEFE. Conversely, Angelini et al. \(1\) show that the intra-pole edges can be reinserted into any SEFE, and thus also into an OrthoSEFE of $\langle G_1 - E_1', G_2 - E_2' \rangle$ without crossings, i.e., the planarity constraints are satisfied for $\langle G_1, G_2 \rangle$. Since the edges in $E_1' \cup E_2'$ are isolated also the orthogonal constraints are trivially satisfied, and we obtain an OrthoSEFE of $\langle G_1, G_2 \rangle$. This finishes the proof of the claim.
In the following, we assume that \( \langle G_1, G_2 \rangle \) has been preprocessed in this way, and it hence does not contain isolated intra-pole edges.

Consider an exclusive edge \( e = uv \) in \( G_1 \) or \( G_2 \), say in \( G_1 \), such that \( u \) has degree 3 in the shared graph. Assume that \( ux \) is an edge of \( G \) incident to \( u \) such that, in every OrthoSEFE of \( \langle G_1, G_2 \rangle \) the edge \( uv \) is embedded in a face of \( G \) incident to \( ux \) (we describe how to determine such an edge later). We perform the following transformation. We subdivide \( ux \) by three vertices \( w_1, w_2, w_3 \) and add the edge \( w_1w_3 \). We further replace \( uv \) by \( w_2v \) and also, if it exists, the (unique) exclusive edge \( e' = wv' \) (from \( G_1 \)) by \( w_2v' \); see Fig. 10. Call the resulting instance \( \langle G'_1, G'_2 \rangle \). It is not difficult to see that \( \langle G_1, G_2 \rangle \) admits an OrthoSEFE if and only if \( \langle G'_1, G'_2 \rangle \) does. If \( \langle G'_1, G'_2 \rangle \) admits an OrthoSEFE, we can contract the vertices \( w_1, w_2, w_3 \) onto \( u \) to obtain an OrthoSEFE of \( \langle G_1, G_2 \rangle \). Note that the orthogonal constraint at \( u \) is satisfied since the triangle \( w_1w_2w_3 \) ensures that the exclusive edges incident to \( u \) are embedded in the same face of \( G' \), and hence in \( G \). Conversely, given an OrthoSEFE of \( \langle G_1, G_2 \rangle \), due to the orthogonality constraints at \( u \) all exclusive edges incident to \( u \) are embedded in the same face of \( G_r \), and hence the replacement can be carried out locally without creating crossings. Note that, after the transformation, there are fewer endpoints of exclusive edges that have degree 3 in the shared graph. We iteratively apply this transformation to obtain the instance \( \langle G'_1, G'_2 \rangle \).

Fig. 10. Moving exclusive edges from a vertex with degree 3 in the shared graph to a new vertex with degree 2 in the shared graph.

It remains to show that there always exists a suitable edge \( ux \). Let \( T \) denote the SPQR-tree of the shared graph \( G_r \). Since \( u \) has degree 3, there is exactly one node \( \mu \) of \( T \) whose skeleton contains \( u \) and where the degree of \( u \) in \( \text{skel}(\mu) \) is 3. Note that \( \mu \) is either a P-node or an R-node.

First assume \( \mu \) is an R-node and consider the position of \( v \) inside \( \text{skel}(\mu) \), where it is either a vertex of \( \text{skel}(\mu) \) or it is contained in a virtual edge \( \epsilon_v \) of \( \text{skel}(\mu) \). Since \( \mu \) is an R-node, \( u \) and \( v \) (and \( \epsilon_v \)) share at most two faces, both of which are incident to a virtual edge \( \epsilon \) incident to \( u \). We choose \( ux \) as the (unique) edge incident to \( u \) that is contained in the subgraph represented by \( \epsilon \).

Second, assume \( \mu \) is a P-node. If the other endpoint \( v \) of \( e \) is not a pole of \( \mu \), then \( v \) is contained in a virtual edge \( \epsilon_v \) of \( \text{skel}(\mu) \), and we can proceed as in the previous case; see Fig. 10. Now assume that \( v \) is the other vertex of \( \text{skel}(\mu) \), i.e., \( e \) is an intra-pole edge. Since \( e \) cannot be isolated (due to the simplification step at the beginning), there exists an exclusive edge \( e' \) in \( G_2 \) incident to \( u \) or \( v \). Since \( e \neq e' \), the edge \( e' \) has an endpoint \( v' \) that is contained in a subgraph represented by a virtual edge \( \epsilon \) of \( \text{skel}(\mu) \). It follows that in every planar embedding of \( G_2 \), the edge \( e' \) is embedded in a face incident to \( \epsilon \). By the orthogonality constraints

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at the vertex shared by \( e \) and \( e' \), \( e \) also has to be embedded in a face incident to \( \varepsilon \) in any OrthoSEFE. We thus choose \( ux \) as the (unique) edge incident to \( u \) contained in the subgraph represented by \( \varepsilon \).

**Lemma 5** Let \( \langle G_1, G_2 \rangle \) be an instance of OrthoSEFE-2 such that the shared graph \( G_\cap \) is biconnected. Then \( \langle G_1, G_2 \rangle \) admits an OrthoSEFE if and only if all instances \( \langle G_{1\mu}, G_{2\mu} \rangle \) admit an OrthoSEFE.

**Proof.** It is not hard to see that each \( \langle G_{1\mu}, G_{2\mu} \rangle \) can be obtained from \( \langle G_1, G_2 \rangle \) by removing some vertices and edges and suppressing subdivision vertices. Thus, if \( \langle G_1, G_2 \rangle \) admits an OrthoSEFE, so does each \( \langle G_{1\mu}, G_{2\mu} \rangle \).

Conversely, assume that each \( \langle G_{1\mu}, G_{2\mu} \rangle \) admits an OrthoSEFE. Recall that we have fixed a reference embedding for each skeleton of the SPQR-tree of the shared graph \( G_\cap \) up to a flip. We fix the flips of all reference embeddings as follows. For each S-node \( \mu \) and each neighbor \( \nu \), represented by a virtual edge \( \varepsilon \) in \( \text{ske}(\mu) \), we consider the flips of the cycle \( C_\varepsilon \) in the OrthoSEFE of \( \langle G_{1\mu}, G_{2\mu} \rangle \) with respect to the ordering \( O_\varepsilon \) of the attachments of the subgraph represented by \( \varepsilon \). If the reference embedding is used, we label the edge \( \mu \nu \) with label 1, otherwise we label it \(-1\). Finally, we choose an arbitrary root \( \mu_0 \) of the augmented SPQR-tree for which we fix the reference embedding. For each skeleton \( \text{ske}(\mu), \mu \neq \mu_0 \), we choose the reference embedding if and only if the product of the labels on the (unique) path from \( \mu_0 \) to \( \mu \) is 1, and its flip otherwise. We denote the planar embedding of \( G_\cap \) obtained in this way by \( E \).

It remains to determine the embeddings of \( G_1 \) and \( G_2 \). After suitably flipping the given OrthoSEFEs, we can assume that their embeddings can be obtained from \( E \) by removing vertices and edges, and by contracting edges. We now determine the embeddings of \( G_1 \) and \( G_2 \) as follows. Recall that every vertex that is incident to exclusive edges has degree 2 in the shared graph. For each vertex \( v \) that is incident to exclusive edges of \( G_1 \) (of \( G_2 \)), we consider the unique S-node \( \mu \) whose skeleton contains \( v \), and we choose the edge ordering as in the given OrthoSEFE of \( \langle G_{1\mu}, G_{2\mu} \rangle \). We claim that this results in an OrthoSEFE \( \langle E_1, E_2 \rangle \) of \( \langle G_1, G_2 \rangle \). Refer to Figs. 6(a) and 6(b).

First observe that the orthogonality constraints are satisfied, since the edge ordering of each vertex is chosen according to one of the given OrthoSEFEs. It remains to show that the embeddings also satisfy the planarity constraints. Due to the construction of the embeddings, all the exclusive edges are embedded in faces of \( E \); otherwise we would observe crossings in the skeletons of the (augmented) SPQR-tree. Consider two exclusive edges \( uv \) and \( u'v' \) from the same graph that cross. Since \( uv \) and \( u'v' \) cross, there exists a node \( \mu \) of the (augmented) SPQR-tree such that for each of the two edges the endpoints are in different parts of \( \text{ske}(\mu) \). If \( \mu \) is a P-node or an R-node and all four parts containing these endpoints are distinct, then the parts containing the endpoints of these edges alternate around a face of \( \text{ske}(\mu) \). This contradicts the planarity of the corresponding input graph \( G_1 \) or \( G_2 \). Thus, in this case at least two attachments are contained in the same virtual edge \( \varepsilon \) of \( \text{ske}(\mu) \). Let \( \nu \) be the S-node of the augmented SPQR-tree corresponding to \( \varepsilon \). Clearly, in \( \text{ske}(\nu) \), the endpoints
of each of the two edges are distinct parts of skel(ν). It follows that the endpoints of the two edges alternate around the two faces of ⟨G′_1, G′_2⟩ corresponding to the two faces of skel(μ). By construction of ⟨E_1, E_2⟩ this contradicts the assumption that the given drawing of ⟨G′_1, G′_2⟩ is an OrthoSEFE. □

**Lemma 6** ⟨G′_1, G′_2⟩ admits an OrthoSEFE if and only if ⟨G″_1, G″_2⟩ does.

**Proof.** We simply show that, in terms of embeddings, the path P_ε replacing C_ε behaves the same as C_ε. First, observe that the edge (a_2, x_3) ensures that all exclusive edges of G_1 incident to the clockwise uv-path of C_ε are embedded on the same side of the path P_ε. Similarly, (x_2, b_1) ensures that all exclusive edges of G_2 incident to the counterclockwise uv-path of C_ε are embedded on the same side of the path P_ε. Moreover, since the endpoints of the edges (a_2, x_3) and (x_2, b_1) alternate along P_ε, they are embedded on different sides of P_ε. Thus, the exclusive edges of G_1 incident to the clockwise and counterclockwise uv-path of C_ε cannot be embedded on the same side of P_ε. Similarly, the exclusive edges (a_1, x_1) and (x_2, b_2) ensure that the exclusive edges of G_2 incident to the clockwise uv-path are all on one side of P_ε and the exclusive edges of G_2 incident to the counterclockwise uv-path are on the other side of P_ε. Finally, due to the alternation with (a_2, x_3), the edges (x_2, x_4) and (x_2, b_1) must be embedded on the same side of P_ε. By the orthogonality constraint at x_2, the edge (x_2, b_2) must be also embedded on the same side as (x_2, x_4). Thus, (a_2, x_3) and (a_1, x_3) are on the same side of P_ε and likewise for (x_2, b_1) and (x_2, b_2). This ensures that the exclusive edges of G_1 and G_2 incident to the clockwise uv-path of C_ε are embedded on the same side of C_ε and likewise for those incident to the counterclockwise uv-path. □

**Theorem 7** Let ⟨G_1, ..., G_k⟩ be a positive instance of OrthoSEFE-k whose shared graph is biconnected. Then, there exists an OrthoSEFE ⟨Γ_1, Γ_2, ..., Γ_k⟩ of ⟨G_1, ..., G_k⟩ in which every edge has at most three bends.

**Proof.** We assume that a cyclic order of the edges of the union graph around each vertex is given such that (a) it induces a planar embedding on each G_i, i = 1, ..., k, and (b) we can assign the incident edges around a vertex to at most four ports such that at most one edge of each G_i is assigned to the same port.

We adopt the method of Biedl and Kant [9]. First, we compute in linear time [15] an s-t-ordering on the shared graph, i.e., we label the vertices v_1, ..., v_n such that {v_1, v_n} is an edge of the shared graph and, for each i = 2, ..., n - 1, there are j < i < k such that {v_j, v_i} and {v_i, v_k} are edges of the shared graph. We choose the face to the left of (v_1, v_n) as the outer face of the union graph.

We now draw the union graph by adding the vertices in the order in which they appear in the s-t-ordering while respecting the given order of the edges around each vertex. The edges will bend at most on y-coordinates near their incident vertices and are drawn vertically otherwise. We draw the edges around v_1 as indicated in Fig. [11(a)] where some of the incident edges might actually indicate several exclusive edges – at most one from each graph.
For $i = 2, \ldots, n - 1$, an edge may only leave $v_i$ to the bottom if it is incident to a neighbor with a lower index. Again, some of the ports might host several exclusive edges, even one to a vertex with a lower index and one to a vertex with a higher index. Special cases occur when the ordering around $v_i$ is such that four exclusive edges of two distinct graphs must be assigned to two consecutive ports. In particular, an edge leaving $v_i$ to a vertex with a lower index might bend twice around $v_i$ (see, e.g., the two small circles in Fig. 11(b)).

Finally, the edges around $v_n$ are placed such that the edge $\{v_1, v_n\}$ enters it from the left. Thus, there are exactly three bends on $\{v_1, v_n\}$; see Fig. 11(c).

For any other edge, there is at most one bend around the endvertex with lower index and at most two bends around the endvertex with higher index. \hfill \qed

F Omitted or Sketched Proofs from Section 6

Lemma 7. Let $\langle G_1, G_2 \rangle$ be an instance of OrthoSEFE-2 whose shared graph $G \cap$ is a cycle. It is possible to construct in polynomial time an equivalent instance $\langle G^*_1, G^*_2 \rangle$ of OrthoSEFE-2 such that (i) the shared graph $G^*_\cap$ is a cycle, (ii) graph $G^*_1$ is outerplanar, and (iii) no two degree-4 vertices in $G^*_1$ are adjacent to each other.

Proof. The reduction works in two steps. In the first step, we construct an instance $\langle G'_1, G'_2 \rangle$ satisfying properties (i) and (iii) that is equivalent to $\langle G_1, G_2 \rangle$; then, in the second step we construct the final instance $\langle G^*_1, G^*_2 \rangle$ equivalent to $\langle G'_1, G'_2 \rangle$, which also satisfies property (ii).

For the first step, we show how to construct an instance $\langle G'_1, G'_2 \rangle$ of OrthoSEFE-2 equivalent to $\langle G_1, G_2 \rangle$ such that $G'_\cap$ is a cycle and the number of vertices with degree 4 in $G'_1$ not satisfying the condition of property (iii) is smaller than the number of vertices with degree 4 in $G_1$ not satisfying this condition. Repeatedly performing this transformation eventually yields the required instance $\langle G^*_1, G^*_2 \rangle$.

Consider a vertex $v$ with degree 4 in $G_1$ not satisfying the condition of property (iii). Let $e = (u, v)$ and $f = (v, w)$ be the two exclusive edges of $G_1$ incident to $v$. Assume that $u$, $v$, and $w$ appear in this order along $G_\cap$, the other cases being analogous.

Initialize $G'_\cap = G_\cap$; refer to Fig. 12. Replace $v$ in $G'_\cap$ by a path $P_v$ composed of dummy vertices $x_1, x_2, v_\alpha, x_3, \ldots, x_8, w', x_9, x_{10}$, of vertex $v$, and of dummy vertices $y_1, y_2, w', y_3, \ldots, y_8, v_\beta, y_9, y_{10}$. Note that $G'_\cap$ contains all the vertices of
$G_\cap$, plus a set of dummy vertices. We now describe the exclusive edges in $E_1'$ and $E_2'$. Set $E_i'$, with $i = 1, 2$, contains all the exclusive edges in $E_i$, except for $e$ and $f$. Also, $E_1'$ contains edges $e' = (u, v_a)$, $e'' = (u', v)$, $f'' = (v, w')$, and $f' = (v_b, w)$. Finally, $E_2'$ contains edges $(x_1, v_a)$, $(x_2, x_3)$, $(v_a, v_b)$, $(x_4, x_7)$, $(x_6, u')$, $(x_8, x_9)$, $(u', x_{10})$, $(y_1, u')$, $(y_2, y_3)$, $(w', y_5)$, $(y_6, v_b)$, $(y_8, y_9)$, and $(v_b, y_{10})$.

We prove that $\langle G_1', G_2' \rangle$ satisfies the required properties. First, graph $G_\cap$ is a cycle by construction. Second, the number of vertices of degree 4 in $G_1'$ not satisfying the condition of property (iii) is smaller than the number of such vertices in $G_1$. In fact, any vertex $x \neq v$ with degree 4 in $G_1'$ satisfies the required condition if and only if it satisfies the same condition in $G_1$. On the other hand, vertex $v$ does not satisfy the condition in $G_1$, by hypothesis, but it satisfies the condition in $G_1'$, since $u'$ and $w'$ have degree 3 in $G_1'$ and the path between them along $G_\cap$ containing $v$ only contains dummy vertices $x_9$, $x_{10}$, $y_1$, and $y_2$, which are not incident to any exclusive edge of $G_1'$, by construction.

We now prove that $\langle G_1', G_2' \rangle$ is equivalent to $\langle G_1, G_2 \rangle$.

Suppose that $\langle G_1, G_2 \rangle$ admits an OrthoSEFE $\langle \Gamma_1, \Gamma_2 \rangle$, and let $A$ be the corresponding assignment of the exclusive edges of $E_1$ and of $E_2$ to the two sides of $G_\cap$, which exists by Theorem 2. We show how to construct an assignment $A'$ of the exclusive edges of $E_1'$ and of $E_2'$ to the two sides of $G_\cap$, satisfying all the constraints.

For each exclusive edge $g \in E_1 \cap E_1'$, set $A'(g) = A(g)$. Also, set $A'(e') = A'(e''') = A(e)$ and $A'(f') = A'(f''') = A(f)$. For each exclusive edge $g \in E_2 \cap E_2'$, set $A'(g) = A(g)$. Also, set $A'(x_1, v_a) = A'(v_a, x_3) = A'(x_6, u') = A'(u', x_{10}) = A'(y_1, u') = A'(w', y_5) = A'(v_b, y_{10}) = A(e)$ and set $A'(x_2, x_3) = A'(x_4, x_7) = A'(x_8, x_9) = A'(y_2, y_3) = A'(y_4, y_7) = A'(y_8, y_9) = A(f)$.

We prove that the planarity constraints for the edges of $G_1'$ are satisfied by $A'$. Note that, by construction, edges $e''$ and $f'''$ do not alternate with any edge of $G_1'$ along $G_\cap$. Also, edges $e'$ and $f'$ do not alternate with each other. Further, if edge $e'$ (edge $f'$) alternates with an edge $g \in E_1'$ along $G_\cap$, then edge $e$ (edge $f$) alternates with $g$ along $G_\cap$. Finally, any two edges different from $e'$, $e''$, $f'$, $f'''$ that alternate along $G_\cap$, also alternate along $G_\cap$. In all the described cases, the planarity constraints are satisfied by $A'$ since they are satisfied by $A$.

We prove that the planarity constraints for the edges of $G_2'$ are satisfied by $A'$. Note that, by construction, edges in $E_2' \cap E_2$ do not alternate with any edge...
incident to a dummy vertex along \( G'_{v_1} \), and alternate with each other along \( G'_{v_2} \) if only if they alternate with each other along \( G'_{v_2} \). Hence, the planarity constraints for these edges are satisfied by \( A' \) since they are satisfied by \( A \). On the other hand, it is easily verified that the planarity constraints are satisfied by \( A' \) also for the edges incident to dummy vertices.

We now prove that the orthogonality constraints are satisfied by \( A' \) at every vertex in \( V' \). For the non-dummy vertices in \( V' \setminus \{u, v, w\} \), this is true since they are satisfied by \( A \) and since the edges incident to these vertices have the same assignment in \( A \) as in \( A' \). For vertex \( u \), this is true since they are satisfied by \( A \), since \( A'(e') = A(e) \), and since the other edges incident to \( u \) have the same assignment in \( A \) and in \( A' \). Analogously, for \( v \) this is true since they are satisfied by \( A \), since \( A'(e'') = A(e) \), and since the other edges have the same assignment in \( A \) and in \( A' \). For \( v_a \), this is true since \( A'(v_1, v_a, x_5) = A'(e') = A(e) \). For \( u' \), this is true since \( A'(y_6, u') = A'(u', x_{10}) = A'(e'') = A(e) \). For \( w \), this is true since \( A'(y_6, v_b) = A'(v_b, y_{10}) = A'(f') = A(f) \). Since all the other dummy vertices have degree 2 in \( G'_{v_1} \), this concludes the proof that \( A' \) satisfies the orthogonality constraints.

Suppose that \( \langle G'_{v_1}, G'_{v_2} \rangle \) admits OrthoSEFE \( \langle \Gamma'_1, \Gamma'_2 \rangle \), and let \( A' \) be the corresponding assignment of the exclusive edges of \( E'_1 \) and of \( E'_2 \) to the two sides of \( G'_{v_i} \). We show how to construct an assignment \( A \) of the exclusive edges of \( E_1 \) and of \( E_2 \) to the two sides of \( G_{v_i} \) satisfying all the planarity and the orthogonality constraints.

For each exclusive edge \( g \in E_1 \), set \( A(g) = A'(g) \). Also, set \( A(e) = A'(e') \) and \( A(f) = A'(f') \). Finally, for each exclusive edge \( g \in E_2 \cap E'_2 \), set \( A(g) = A'(g) \).

We prove that the planarity constraints for the edges of \( G_1 \) are satisfied by \( A \). Note that \( e \) and \( f \) do not alternate with each other since they are incident to the same vertex \( v \). Also, if edge \( e \) (edge \( f \)) alternates with an edge \( g \in E_1 \) along \( G_{v_1} \), then edge \( e' \) (edge \( f' \)) alternates with \( g \) along \( G'_{v_1} \). Finally, any two edges different from \( e \) and \( f \) that alternate along \( G_{v_1} \) also alternate along \( G'_{v_1} \). In all these cases, the planarity constraints are satisfied by \( A \) since they are satisfied by \( A' \).

The planarity constraints for the edges of \( G_2 \) are satisfied by \( A \) since any two of these edges alternate along \( G_{v_2} \) if and only if they alternate along \( G'_{v_2} \), and since the planarity constraints are satisfied by \( A' \).

We finally prove that the orthogonality constraints are satisfied by \( A \) at every vertex in \( V \). For the vertices in \( V \setminus \{v\} \), this is true since they are satisfied by \( A' \) and since for every exclusive edge \( g \in E_1 \cup E_2 \) incident to these vertices, we have \( g \in E'_1 \cup E'_2 \), by construction, and \( A(g) = A'(g) \). To prove that this is true also for \( v \), we first argue that \( A(e) = A'(e'') \) and \( A(f) = A'(f'') \): By planarity constraints, we get \( A'(x_1, v_a) = A'(v_a, x_5) = A'(x_6, u') = A'(u', x_{10}) \), since they belong to a sequence of alternating edges; hence, by orthogonality constraints at \( v_a \) and \( u' \), we get \( A'(e') = A'(x_1, v_a) = A'(v_a, x_5) = A'(x_6, u') = A'(u', x_{10}) = A'(y_6, v_b) = A'(v_b, y_{10}) = A'(f') = A(f) \).
\[ A(e') \]; since \( A(e) = A'(e') \), by construction, we conclude \( A(e) = A'(e'') \). The equality \( A(f) = A'(f'') \) follows symmetrically. Hence, the orthogonality constraints at \( v \) are satisfied by \( A \) since they are satisfied at \( v \) by \( A' \). This concludes the proof that \( \langle G_1^+, G_2^+ \rangle \), which satisfies properties (i) and (iii), is equivalent to \( \langle G_1, G_2 \rangle \).

In order to construct an instance \( \langle G_1^*, G_2^* \rangle \) equivalent to \( \langle G_1^+, G_2^+ \rangle \) that also satisfies property (ii), we observe that the proof of Lemma 2 can be easily extended so that it can be applied to \( \langle G_1^+, G_2^+ \rangle \). This lemma, in fact, holds for instances \( \langle G_1, G_2 \rangle \) satisfying property (i) and a property that is stronger than (iii), namely that \( G_1 \) has degree at most 3. This stronger condition, however, is only used to ensure that there exists no exclusive edge in \( E_1 \) with an endpoint in \( H_2 \) and the other one not in \( H_2 \); refer to Fig. 8. In particular, it is used to ensure that there exists no edge connecting a vertex of \( H_2 \) to any of \( u, v, w, z \). However, it is possible to prove that property (iii) is already sufficient to ensure the absence of these edges. Namely, suppose that there exists an edge in \( E_1 \) connecting a vertex \( x \) of \( H_2 \) to vertex \( v \), the other cases being analogous. This implies that \( v \) has degree 4 in \( G_1 \), since it is also adjacent to \( u \). However, any path in cycle \( G_1 \) containing \( u, x, v \) and \( v \) also contains either \( w \) or \( z \), since \( e \) and \( f \) alternate along \( G_1 \); this is a contradiction to property (iii), since each of \( w \) and \( z \) is incident to an exclusive edge of \( G_1 \), namely \( f \). This concludes the proof of the lemma. \( \Box \)