THE ANOMALY-FREE QUANTIZATION
OF TWO-DIMENSIONAL RELATIVISTIC STRING. I

S.N. Vergeles

L. D. Landau Institute of Theoretical Physics, Russian Academy of Sciences.
142432 Chernogolovka, Russia.

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Abstract.

An anomaly-free quantum theory of a relativistic string is constructed in
two-dimensional space-time. The states of the string are found to be similar
to the states of a massless chiral quantum particle. This result is obtained
by generalizing the concept of an “operator” in quantum field theory.

1. Introduction

It has recently been asserted in a number of works (see, for example, Refs. [1]
and [2]) that anomaly-free quantization of some models of two-dimensional gravity is
possible. Specifically, Ref. [1] examined a model of two-dimensional gravity Ref. [3]
which in certain variables was described by the same constraints as those describing
a relativistic bosonic string in two-dimensional space-time:

\[ E = -E_0 + E_1 \approx 0 , \]

\[ E_0 = \frac{1}{2} \left( (\pi_0)^2 + (r^0)^2 \right) , \quad E_1 = \frac{1}{2} \left( (\pi_1)^2 + (r^1)^2 \right) , \]

\[ P = r^{a'} \pi_a = r^{0'} \pi_0 + r^{1'} \pi_1 \approx 0 \]  \hspace{1cm} (1.1a)

Dimensionless quantities are employed. Here \( r^a(x) \) and \( \pi_a(x) \), \( a = 0, 1 \), are canoni-
cally conjugated fields on a one-dimensional manifold, so that the nonzero commu-
tation relations are

\[ [r^a(x), \pi_b(y)] = i \delta^a_b \delta(x - y) \]  \hspace{1cm} (1.2)

A prime or overdot signifies a derivative \( \partial/\partial x \) or \( \partial/\partial t \), respectively.

The ground state of the theory is determined at this stage of quantization. This
makes it possible to perform normal ordering of the operator in the constraints (1.1).
The determined normal ordering in the constraints can lead in turn to anomalies
in the commutators of the constraints. These anomalies partially destroy the weak
equalities (1.1). To determine the ground state, the fields \( r^a \) and \( \pi_a \) are expanded
in the modes that arise when solving the Heisenberg equations

\[ i \dot{r}^a = [r^a, H] , \quad i \dot{\pi}_a = [\pi_a, H] , \]

\[ H = \int dx \mathcal{E} \]  \hspace{1cm} (1.3)

\[ e-mail: Vergeles@itp.ac.ru \]
The solutions of Eqs. (1.2) and (1.3) can be written in the form

\[
r^a(t, x) = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2|k|}} \left\{ c^a_k e^{-i(|k|x-tk)} + c^{a+}_k e^{i(|k|x-tk)} \right\},
\]

\[
\pi^a(t, x) = -i \int \frac{dk}{2\pi} \frac{|k|}{\sqrt{2}} \left\{ c^a_k e^{-i(|k|x-tk)} - c^{a+}_k e^{i(|k|x-tk)} \right\},
\]

\[
[c^a_k, c^{b+}_p] = 2\pi \eta^{ab} \delta(k-p), \quad [c^a_k, c^b_p] = 0 \quad (1.4)
\]

Here \(\eta^{ab}\) (below - \(\eta^{\mu\nu}\)) = \(\text{diag}(-1,1)\). We also have the following commutational relations

\[
[\mathcal{H}, c^a_k] = -|k| c^a_k, \quad [\mathcal{H}, c^{a+}_k] = |k| c^{a+}_k \quad (1.5)
\]

In conventional quantization the operators \(c^a_k\) are considered as annihilation operators, while their hermitian conjugated operators \(c^{a+}_k\) are considered as creation operators. The ground state \(|0\rangle\) satisfies the conditions

\[
c^a_0 |0\rangle = 0 \quad (1.6)
\]

Normal ordering of the operators \((c^{a+}_k, c^a_k)\) in the quantities (1.1) means that the creation operators stand to the left of all annihilation operators.

Let us consider the state

\[
|k, a\rangle = c^{a+}_k |0\rangle \quad (1.7)
\]

It follows immediately from the commutation relations (1.5) that

\[
\mathcal{H} |k, a\rangle = (|k| + E_0) |k, a\rangle, \quad (1.8)
\]

where \(E_0\) is the value of the operator \(\mathcal{H}\) for the ground state. The equality (1.8) means that the operator \(\mathcal{H}\) is positive-definite.

In consequence of Eqs. (1.4) and (1.6) we have for the scalar product of the vectors (1.7)

\[
\langle k, a | p, b \rangle = 2\pi \eta^{ab} \delta(k-p) \quad (1.9)
\]

Hence it is seen that the metric in the whole state space is indefinite.

Next, let us calculate the commutator \([\mathcal{E}, \mathcal{P}]\). According to Eq. (1.1) it can be represented as a sum of two terms

\[
[\mathcal{E}(x), \mathcal{P}(y)] = -[\mathcal{E}_0(x), r^{0\prime} \pi_0(y)] + [\mathcal{E}_1(x), r^{1\prime} \pi_1(x)] \quad (1.10)
\]

In consequence of Eq. (1.2), both commutators on the right-hand side of Eq. (1.10) are identical up to a change of the index \(a\). These commutators are proportional (up to the ordering) to the quantities \(\mathcal{E}_0\) and \(\mathcal{E}_1\), respectively. As is well known, the normal ordering of the operators in these commutators leads to anomalies.

Indeed, it follows from the commutation relations (1.4) that the correspondences \(c^0_k \leftrightarrow c^{1+}_k\) and \(c^{0+}_k \leftrightarrow c^1_k\) establish an isomorphism of the Heisenberg algebras \(H_0\) and \(H_1\), whose generators are \((c^0_k, c^{0+}_k)\) and \((c^{1+}_k, c^1_k)\), respectively. In this case the normal ordering of the operators in the algebra \(H_1\) is transferred by the indicated isomorphism into antinormal ordering in the algebra \(H_0\).

As is known in such case the normal and antinormal orderings result in anomalies differing only in sign. Therefore the contribution of the first commutator on the right-hand side of the Eq. (1.10) to the anomaly will be \(-A\) and that of the second
will be $A$. But, since a minus sign stands in front of the first commutator in Eq. (1.10), the anomaly in Eq. (1.10) equals $-(-A) + A = 2A$.

Let us now examine the problem from a different point of view.

In Ref. [1] it is asserted that in the present theory the positive-definiteness condition (1.8) for the operator $\mathcal{H}$ is not necessary. The initial requirement of the theory is satisfaction of the weak equalities (1.1). Therefore we have the right to reject the quantization conditions (1.6) and replace them with the conditions

$$c_{0}^{0+}|0\rangle = 0, \quad c_{k}^{1}|0\rangle = 0$$

(1.11)

Under the quantization conditions (1.11) the basis of the whole Fock space of the theory consists of vectors of the type

$$c_{k_{1}}^{0} \cdots c_{k_{m}}^{0} c_{p_{1}^{+}}^{1} \cdots c_{p_{n}^{+}}^{1} |0\rangle$$

(1.12)

It follows from the commutation relations (1.4) that the scalar product of the states (1.12) is positive-definite. Moreover, there is no anomaly in the operator algebra (1.1).

Indeed, under the conditions (1.11) normal ordering consists of arranging the operators $(c_{k}^{0}, c_{k}^{1+})$ to the left of all operators $(c_{k}^{0+}, c_{k}^{1})$. This means that normal ordering occurs in both Heisenberg algebras $H_{0}$ and $H_{1}$. Now, with normal ordering both commutators in (1.10) make the same contribution, equal to $A$, to the anomaly. Taking account of the minus sign in front of the first commutator on the right-hand side of the equality (1.10), the total anomaly is $-A + A = 0$.

The absence of an anomaly in the operator algebra $(\mathcal{E}, \mathcal{P})$ makes it possible to satisfy all weak equalities $\mathcal{E} \approx 0$ and $\mathcal{P} \approx 0$. Two physical states which the operators $\mathcal{E}$ and $\mathcal{P}$ annihilate are presented in Ref. [1]:

$$\Psi_{gravity}(r^{a}) = \exp \pm i \frac{1}{2} \int dx \varepsilon_{ab} r^{a} r^{b}$$

In the present paper we shall likewise reexamine the quantization conditions for a relativistic string in two-dimensional space-time. In so doing, we shall determine the space of physical states with a positive-definite scalar product. The nonphysical states are not studied in the theory. The physical states annihilate all first class constraints, i.e. all Virasoro operators. The physical states are characterized by a continuous parameter, which has the meaning of momentum. However, in our theory not all dynamical variables are linear operators in the space of physical states. In the proposed theory the states of a relativistic string in two-dimensional space-time are found to be identical to the states of a massless chiral particle.

2. Relativistic bosonic string
   in two-dimensional space - time

Let $X^{\mu}, \mu = 0, 1$, be coordinates in two-dimensional Minkowski space. Let us examine the Nambu action for a bosonic string

$$S = -\frac{1}{l^{2}} \int \sqrt{-g} \, d^{2}\theta = \int d\tau \mathcal{L}$$

(2.1)

Here $\theta^{a} = (\tau, \phi)$ are the parameters of the world sheet of the string and

$$g = \text{Det} \, g_{ab}, \quad g_{ab} = \eta_{\mu\nu} \frac{\partial X^{\mu}}{\partial \theta^{a}} \frac{\partial X^{\nu}}{\partial \theta^{b}}$$
The parameter $\tau$ is timelike and $\phi$ is spatial. The partial derivatives $\partial/\partial \tau$ and $\partial/\partial \phi$ will be denoted below by an overdot and a prime, respectively. It is easy to show that the generalized momenta $\pi_\mu = \partial \mathcal{L}/\partial X^\mu$ satisfy the conditions

$$\mathcal{E} = \frac{l^2}{2} \pi_\mu \pi^\mu + \frac{1}{2 l^2} X^{\mu'} X'_\mu \approx 0,$$

$$\mathcal{P} = X^{\mu'} \pi_\mu \approx 0 \quad (2.2)$$

The quantities $\mathcal{E}(\phi)$ and $\mathcal{P}(\phi)$ exhaust all the first class constraints. The Hamiltonian of the system

$$\mathcal{H} = \int d\phi \pi_\mu \dot{\phi}^\mu - \mathcal{L} = 0$$

is also equal to zero. For this reason, following Dirac, we must employ a generalized Hamiltonian which is an arbitrary linear combination of the first class constraints (2.2)

$$\mathcal{H}_T = \int d\phi (v \mathcal{P} + w \mathcal{E}) \quad (2.3)$$

The equations of motion can be obtained from the variational principle

$$\delta \mathcal{S} = \delta \left\{ \int d\tau \left( \int d\phi \pi_\mu \dot{X}^\mu - \mathcal{H}_T \right) \right\} = 0 \quad (2.4)$$

In the case of an open string, when the parameter $\phi$ varies from 0 to $\pi$, the variational principle (2.4) gives, besides the equations of motion, the boundary conditions

$$(v \pi_\mu + w \frac{1}{l^2} X'_\mu) \big|_{\phi=0,\pi} = 0 \quad (2.5)$$

which ordinarily are replaced by the conditions

$$v|_{\phi=0,\pi} = 0, \quad X'_\mu|_{\phi=0,\pi} = 0 \quad (2.6)$$

For a closed string instead of the boundary condition there is the periodicity condition.

Let us study an open string next.

The first step in the quantization process is to postulate the commutation relations for the generalized coordinates and momenta:

$$[ X^\mu(\phi), \pi^\nu(\phi') ] = i \eta^{\mu\nu} \delta(\phi - \phi') \quad (2.7)$$

The commutation relations (2.7) and the boundary conditions (2.6) are satisfied if

$$X^\mu(\phi) = \frac{l}{\sqrt{\pi}} \left( x^\mu + i \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n \cos n\phi \right),$$

$$\pi^\mu(\phi) = \frac{1}{\sqrt{\pi} l} \sum_n \alpha^\mu_n \cos n\phi \quad (2.8)$$

and the elements $(x^\mu, \alpha^\mu_n)$ satisfy the commutation relations

$$[ x^\mu, \alpha^\nu_n ] = i \eta^{\mu\nu} \delta_n, \quad [ x^\mu, x^\nu ] = 0,$$

$$[ \alpha^\mu_m, \alpha^\nu_n ] = m \eta^{\mu\nu} \delta_{m+n} \quad (2.9)$$
Since the quantities (2.8) are real,
\[ x^{\mu^+} = x^\mu, \quad \alpha^{\mu^+}_n = \alpha^\mu_{-n} \] (2.10)

The constraints (2.2) can be represented as
\[ (\mathcal{E} \pm \mathcal{P})(\phi) = \frac{1}{2} (\xi_\pm^\mu(\phi))^2, \] (2.11)

where
\[ \xi_\pm^\mu(\phi) = \frac{1}{\sqrt{\pi}} \sum_n \alpha_n^\mu \exp \mp in\phi \] (2.12)

Hence it is seen that \( \mathcal{E} - \mathcal{P} \) differs from \( \mathcal{E} + \mathcal{P} \) by the replacement \( \phi \) by \(-\phi\). This simplifies the analysis, since on the interval \(-\pi \leq \phi \leq \pi\) the quantity \( \mathcal{E} + \mathcal{P} \) contains all information about the quantities \( \mathcal{E} \pm \mathcal{P} \) on the interval \( 0 \leq \phi \leq \pi \). Therefore the Fourier components
\[ L_n = \frac{1}{2} \int_{-\pi}^\pi d\phi (\mathcal{E} + \mathcal{P}) \exp in\phi \] (2.13)
are equivalent to the set of quantities (2.2) for \( 0 \leq \phi \leq \pi \). According to Eqs. (2.11)-(2.13), we have
\[ L_n = \frac{1}{2} : \sum_m \alpha^\mu_{n-m} \alpha^\mu_m : \] (2.14)

The sense of the ordering operation in Eq. (2.14) is determined by the quantization method.

Let us also write out expressions for the momentum and angular momentum of a string:
\[ P^\mu = \int_0^\pi d\phi \pi^\mu, \quad J^{\mu\nu} = \int_0^\pi d\phi (X^\mu \pi^\nu - X^\nu \pi^\mu) \] (2.15)

With the help of Eqs. (2.6) and (2.7) we immediately verify that
\[ [P^\mu, \mathcal{H}_T] = 0, \quad [J^{\mu\nu}, \mathcal{H}_T] = 0 \]
This means that the momentum and angular momentum of the string are conserved.

In the currently conventional quantization the ground state \(|0\rangle\) satisfies the conditions
\[ \alpha^\mu_m |0\rangle = 0, \quad m \geq 0 \] (2.16)

The whole space of states has the orthogonal basis
\[ \alpha^\mu_{m_1} \ldots \alpha^\mu_{m_s} |0\rangle, \quad m_i < 0 \] (2.17)

Therefore all \( \alpha^\mu_m \) are linear operators in the whole space of states. From Eqs. (2.9) and (2.6) it follows that the metric in the state space (2.17) is indefinite. The ordering in Eq. (2.14) means that the operators \( \alpha^\mu_m \) with \( m < 0 \) are arranged to the left of all operators \( \alpha^\mu_n \) for \( n \geq 0 \). With this ordering the commutators of Virasoro operators contain anomalies
\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{1}{12} D (n^3 - n) \] (2.18)

Here \( D \) is the dimension of the \( x \)-space, which in our case is 2. Therefore the maximum that can be achieved is annihilation of the operators \( L_n \) with \( n \geq 0 \). As
a result the theory is consistent only for \( D = 26 \). A detailed study of the problems arising under the quantization (2.16) can be found in Ref. [4].

We shall now present the path proposed here for quantization of a two-dimensional string that leads to a self-consistent theory of a string in a space of two dimensions. Our method of quantization of a string is similar to Dirac’s method of quantization of the electromagnetic field (see Ref. [5], and also Appendix).

Let

\[
x_{\pm} = x^0 \pm x^1,
\]

\[
\alpha_m^{(\pm)} = \alpha_{m+1}^{(\pm)}
\]  (2.19)

From Eq. (2.9) we obtain:

\[
[\alpha_m^{(+)} + \alpha_n^{(-)}] = [\alpha_m^{(-)} + \alpha_n^{(+)}] = 0,
\]

\[
[\alpha_m^{(+)}, \alpha_n^{(-)}] = -2m \delta_{m+n}
\]

\[
[x_+, x_-] = 0,
\]

\[
[x_+, \alpha_n^{(+)}] = [x_-, \alpha_n^{(-)}] = 0,
\]

\[
[x_+, \alpha_n^{(-)}] = -2i \delta_n,
\]

\[
[x_-, \alpha_n^{(+)}] = -2i \delta_n
\]  (2.20)

Let us write the Virasoro operators in the variables \( \alpha^{(\pm)} \):

\[
L_n = -\frac{1}{2} : \sum_m \alpha_{n-m}^{(+)} \alpha_m^{(-)} :
\]  (2.21)

By definition, the ordering operation in Eq. (2.21) means that either the elements \( \alpha^{(+)} \) are arranged to the left of all elements \( \alpha^{(-)} \) or the elements \( \alpha^{(-)} \) are arranged to the left of all elements \( \alpha^{(+)} \). Both orderings are equivalent. Indeed,

\[
\sum_m \alpha_m^{(-)} \alpha_m^{(+)} = \sum_m \alpha_m^{(+)} \alpha_m^{(-)} + 2 \sum_m m,
\]

It can be assumed that the last sum is zero, since it can be written as \( \zeta(-1) - \zeta(-1) \), where \( \zeta(s) \) is the Riemann zeta-function. It is known that the zeta-function

\[
\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s}
\]

possesses a unique analytical continuation to the point \( s = -1 \) and \( \zeta(-1) = -1/12 \).

For definiteness, let us choose the same ordering as in Eq. (2.21).

According to Eq. (2.20), we have

\[
[\alpha_n^{(-)}] = -n \alpha_n^{(-)}
\]  (2.22)

One can see from Eqs. (2.20) and (2.22) that the weak equalities \( \alpha_n^{(-)} \approx 0 \) and \( L_n \approx 0 \) are algebraically consistent. For this reason, we determine the physical states as the states satisfying the conditions

\[
\alpha_n^{(-)} | \rangle = 0 , \quad n = 0, \pm 1, \ldots
\]  (2.23)

It follows immediately from Eqs. (2.23) and (2.21) that

\[
L_n | \rangle = 0 , \quad n = 0, \pm 1, \ldots
\]  (2.24)

for any physical states. The equalities (2.24) mean that for the quantization (2.23) the Virasoro algebra has no anomalies:

\[
[L_n, L_m] = (n-m) L_{n+m}
\]  (2.25)
The latter formula can also be easily obtained by direct calculation of the commutators, provided that the ordering is assumed to be the same as in Eq. (2.21). The quantization conditions (2.23) are precisely analogous to the quantization conditions (A.8) used by Dirac to quantize the electromagnetic field [5].

We call attention to the fact that states of the form
\[ \alpha_n^{(+)} | \rangle, \quad n \neq 0 \] (2.25)
are not considered in this theory, since these states do not satisfy the conditions (2.23). For this reason, the matrix elements of the quantities \( \alpha_n^{(+)} \) with \( n \neq 0 \) with respect to the physical states (2.26) cannot be calculated. Therefore the quantities \( \alpha_n^{(+)} \) with \( n \neq 0 \) cannot be operators in the space of physical states. Hence it follows that observables cannot depend on the elements \( \alpha_n^{(+)} \) with \( n \neq 0 \). In other words, observables must commute with all operators \( \alpha_n^{(-)} \). According to Eq. (2.20), there are two such quantities: \( x_- \) and \( p_+ \) \( (p_\pm \equiv \alpha_0^{(\pm)}) \). Both are real.

Thus we can see that the quantities \( \alpha_n^{(\mu)} \) with \( n \neq 0 \) are not, generally speaking, linear operators in state space in the conventional sense. Here we adhere to the concept formulated and applied by Dirac in Ref. [5]. According to this concept, in quantum field theory linear operators acting in certain linear spaces are replaced by so-called \( q \) numbers, which form an associative noncommutative algebra with an involution over the complex numbers. Here we shall formulate Dirac’s concept using the conventional mathematical terminology.

Let \( \mathcal{A} \) be an associative noncommutative involutive algebra with an identity over the complex numbers. Associativity means that for any elements \( u, v, \) and \( w \) of the algebra \( \mathcal{A} \) and any number \( c \) the equalities
\[
(u v) w = u (v w), \quad (c u) v = u (c v) = c (u v)
\]
hold. The involution property of the algebra means that there exists a mapping \( u \mapsto u^+ \) from \( \mathcal{A} \) into \( \mathcal{A} \) such that
\[
(u^+)^+ = u, \quad (c_1 u + c_2 v)^+ = \bar{c}_1 u^+ + \bar{c}_2 v^+, \quad (u v)^+ = v^+ u^+
\]
for any \( u, v \in \mathcal{A} \) and any numbers \( c_1 \) and \( c_2 \). An overbar signifies complex conjugation. If \( u^+ = u \), the element \( u \) is said to be hermitian.

It is also assumed that the algebra \( \mathcal{A} \) has a system of generators \( \{\alpha_p\} \) for which all relations are limited by the form of the commutators
\[
[\alpha_p, \alpha_{p'}] = c_{pp'}
\]
Here \( c_{pp'} \) are complex \( c \)-numbers (in the Dirac sense).

The definition of involutive algebras (or algebras with involution) and other mathematical definitions presented here can be found in Refs. [6] and [7].

Let \( V \) be a vector space with elements \( |\Lambda\rangle, |\Sigma\rangle, \ldots \) over the complex numbers and let \( V^+ \) be the conjugated space, whose elements are denoted by \( \langle \cdot | \). There is a one-to-one correspondence between the elements of the spaces \( V \) and \( V^+ \) such that \( c |\Lambda\rangle \leftrightarrow \langle \Lambda| \bar{c} \).

For any two vectors \( |\Lambda\rangle \) and \( |\Sigma\rangle \) there exist two complex mutually-conjugated \( c \)-number quantities \( \langle \Lambda | \Sigma \rangle \) and \( \langle \Sigma | \Lambda \rangle \). It is assumed that in the space \( V \) there exists a basis \( \{|\Lambda\rangle\} \) such that
\[
\langle \Lambda | \Sigma \rangle = \delta_{\Lambda \Sigma}
\] (2.27)
If the indices $\Lambda$ and $\Sigma$ run a continuous set, then in Eq. (2.27) $\delta_{\Lambda\Sigma}$ must be interpreted as a delta-function. The space $V$ is the space of physical states of the theory.

Let $B \subset A$ be a noncommutative involutive subalgebra with the identity element. The elements of the subalgebra $B$ are linear operators in the spaces $V$ and $V^+$ and, as usual,

$$ (u | \Lambda \rangle)^+ = \langle \Lambda | u^+, \quad u \in B $$

The observables correspond to certain hermitian elements from $B$. If $u \in A$ and $u \notin B$, then the action of the element $u$ on vectors from the spaces $V$ and $V^+$, generally speaking, is not defined. This distinguishes the Dirac theory from the conventional quantum field theory.

In the theories under study all vectors of the space $V$ are, ordinarily, annihilated by a series of operators of the subalgebra $B$. Therefore the conditions

$$ u_N | \rangle = 0, \quad u_{N'} | \rangle = 0, \ldots, \quad | \rangle \in V $$

(2.28) take place. The indices $N, N', \ldots$ in Eq. (2.28) run a certain set $J$ of indices. The conditions (2.28) must be algebraically consistent, i.e. the equalities

$$ [u_N, u_{N'}] = \sum_{N''} k_{NN',N''} u_{N''} $$

where $N, N', N'' \in J$ and $k_{NN',N''}$ can be any elements of the algebra $A$, must hold. Evidently, the operators $u_N$ in Eq. (2.28) do not include the identity operator. We denote by $N \subset B$ the subalgebra without identity with the generators $\{u_N\}$, where $N \in J$. Thus subalgebra $N$ annihilates the space of physical states $V$.

Let us now examine the set of elements of the form $ru$, where $r \in A$ and $u \in N$. We denote this set as $N'$. It is evident from the definition that $N'$ is a left $A$-module and a subalgebra in $A$, but $N'$ is not a subalgebra in $B$. Nonetheless, the action of the subalgebra $N'$ on the space $V$ is defined since it is trivial: $N'$ annihilates the space $V$. We note that the commutant $[N', N']$ is contained in $N'$. Indeed, if $r, s \in A$ and $u, v \in N$, then

$$ [ru, sv] = \{[ru, s]v + s[r, v]u + sr[u, v]\} \in N' $$

since $[u, v] \in N$. Then the conditions $N'V = 0$ are algebraically consistent.

Concrete theories can also contain other elements of the algebra $A$, which are not contained in either $B$ or $N$ and are linear operators on the space $V$.

A distinguishing feature of the Dirac theory is the fact that nonphysical state vectors that do not satisfy the conditions (2.28) are not considered in it. Moreover, in the Dirac theory an indefinite metrics in the state space is absent. This circumstance can radically alter the theory.

Let us return to the discussion of string theory. In the theory proposed here for a two-dimensional string the algebra $A$ has generators $\{x_\pm, \alpha^{(\pm)}_m\}$, while the subalgebras $B$ and $N$ have generators $\{x_-, p_+, \alpha^{(-)}_m\}$ and $\{\alpha^{(-)}_m\}$, respectively. The Virasoro operators $L_n$ are contained in the subalgebra $N'$. We note that the algebra of operators $L_n$ is an involutive subalgebra in $N'$, and since $L^+_n = L_{-n}$, the action of the operators $L_n$ is defined in both spaces $V$ and $V^+$.

From the definitions (2.15) we obtain the following formulas:

$$ (\exp i\omega J^{01}) \alpha^{(\pm)}_m (\exp -i\omega J^{01}) = (\exp \pm\omega) \alpha^{(\pm)}_m, $$
\[(\exp i \omega J^{01}) x_\pm (\exp -i \omega J^{01}) = (\exp \pm \omega) x_\pm \quad (2.29)\]

and
\[(\exp i a_\mu P^\mu) x_\pm (\exp -i a_\mu P^\mu) = X_\pm + \sqrt{\frac{\pi}{l}} a_\pm \quad (2.30)\]

Here \(\omega\) and \(a^\mu\) are arbitrary real numbers. It is evident from Eqs. (2.29) and (2.30) that translations and Lorentz transformations conserve the condition (2.23).

Both observables \(x_-\) and \(p_+ = \alpha_0^+\) are real, and \([x_-, p_+] = -2i\). For this reason, we assume that the physical states are eigenstates of the operator \(p_+\):
\[p_+ | k \rangle = 2k | k \rangle \quad (2.31)\]

Here \(k\) is a continuous real parameter. According to Eq. (2.29)
\[p_+ (\exp -i \omega J^{01}) = (\exp \omega) (\exp -i \omega J^{01}) p_+ \quad (2.32)\]

Let us formally act with the equality (2.32) on the state \(| k \rangle\). As a result of Eq. (2.31) we obtain
\[p_+ (\exp -i \omega J^{01}) | k \rangle = 2k e^{\omega} (\exp -i \omega J^{01}) | k \rangle \quad (2.33)\]

The last equality makes it possible to determine the action of the operators \((\exp -i \omega J^{01})\) on the physical states as follows:
\[(\exp -i \omega J^{01}) | k \rangle = f_\omega | e^{\omega} k \rangle \quad (2.34)\]

Here \(f_\omega\) is a complex number different from zero. If the scalar product on physical state vectors is defined in a Lorentz-invariant manner as
\[\langle k | k' \rangle = k \delta(k - k')\]

then \(|f_\omega| = 1\). From Eq. (2.34) it is evident that one can assume
\[k > 0 \quad (2.35)\]

The angular momentum operator can be represented in the form
\[J^{01} = \frac{1}{2} (x_- p_+ - x_+ p_-) + \frac{i}{4} \sum_{n \neq 0} \frac{1}{n} (\alpha_n^{(-)} \alpha_n^{(+)} - \alpha_n^{(+)} \alpha_n^{(-)}) \quad (2.36)\]

We can see that although the expression (2.36) does not belong to either the subalgebra \(B\) or the subalgebra \(N'\), the action of the quantities \((\exp i \omega J^{01})\) on the space of physical states is nonetheless correctly determined.

According to Eqs. (2.8) and (2.15)
\[P^\mu = \sqrt{\frac{\pi}{l}} \alpha_0^\mu = \sqrt{\frac{\pi}{2l}} \{(\delta_0^\mu + \delta_1^\mu) p_+ + (\delta_0^\mu - \delta_1^\mu) p_-\}\]

Therefore from (2.23) and (2.31) we obtain:
\[P^\mu | k \rangle = \sqrt{\frac{\pi}{l}} k^\mu | k \rangle, \quad k^\mu = (k, k) \quad (2.37)\]
Thus, as a result of the above-described procedure of quantization of two-dimensional string there arises a system similar to a massless chiral quantum particle in two-dimensional space-time.

3. The conclusion

Let us note the differences of the main properties of string theory quantized in the conventional manner from those of the string theory proposed in the present paper. In the conventional quantization there exists a state which is invariant under Lorentz transformations. This state is the ground state. In this respect the conventional string theory is similar to the standard quantum field theory of point objects. In such field theories the ground state usually is Lorentz-invariant. Conversely, in our approach there does not exist a state that is invariant under Lorentz transformations. For this reason, the quantum-string theory proposed above is analogous to a quantum theory of a single relativistic particle. Once again there does not exist a Lorentz-invariant quantum state of a single relativistic particle. In order for a Lorentz-invariant state to exist in our theory we would have to introduce a string field and second-quantize the string field. In such a theory the ground state would be Lorentz-invariant, since there would be no real strings in the ground state.

In closing, we note that the quantization method proposed here can be applied to a $D$-dimensional string. This assertion is based on the fact that in string theory there exists an infinite set of so-called $DDF$-operators Ref. [4] which commute with all Virasoro operators. The $DDF$-operators describe almost all (with the exception of the total momentum of the string) physical degrees of freedom of the string. The independence of Virasoro operators from $DDF$-operators means that Virasoro operators can be put into the form (2.21), after which the quantization scheme which we have proposed above can be applied. However, the theory is much more complicated in the $D$-dimensional case because there exists an infinite set of physical degrees of freedom, contained in the $DDF$-operators.

APPENDIX

We shall describe the quantization of a free electromagnetic field following Dirac’s ideology Ref. [5], which is formulated in Sec. 2. The quantization which we propose for a two-dimensional string is performed in accordance with Dirac’s ideology.

The quantization of an electromagnetic field is presented in the form

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2k}} \{a_\mu(k) e^{ikx} + a^+\mu(k) e^{-ikx}\} \quad (A1)$$

Here $\mu, \nu, \ldots = 0, 1, 2, 3$, $kx = k_\mu x^\mu = -k^0 x^0 + \vec{k} \cdot \vec{x}$, $k^0 = |k|$ and $\{a_\mu(k), a^+\mu(k)\}$ are some generators of an associative involutive algebra $A$ with an identity element (see Sec. 2). The nonzero commutation relations between these generators have the form

$$[a_\mu(k), a^+\nu(\vec{p})] = (2\pi)^3 \eta_{\mu\nu} \delta^{(3)}(\vec{k} - \vec{p}) \quad (A2)$$

One can see from the expansion (A1) that the set of elements $\partial_\mu A^\mu(x)$ is linearly equivalent to the set of elements $k^\mu a_\mu(k)$ and $k^\mu a^+\mu(k)$ from the algebra $A$. Let
Let \( a_i^T(\vec{k}) \) and \( a_i^{T+}(\vec{k}) \) be two independent elements (for fixed \( \vec{k} \)) satisfying the conditions

\[
\sum_{i=1}^{3} k_i a_i^T(\vec{k}) = 0,
\]

\[
[a_i^T(\vec{k}), a_j^{T+}(\vec{p})] = (2\pi)^3 \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \delta^{(3)}(\vec{k} - \vec{p})
\]  
(A3)

Eqs. (A1) and (A2) imply the following commutation relations \( (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu) \):

\[
[F_{\mu\nu}(x), k^\lambda a_\lambda(\vec{k})] = [F_{\mu\nu}(x), k^\lambda a_\lambda^+(\vec{k})] = 0,
\]

\[
[k^\mu a_\mu(\vec{k}), p^\nu a_\nu^+(\vec{p})] = 0
\]  
(A4)

It is obvious, that

\[
[a_i^T, k^\mu a_\mu(\vec{k})] = [a_i^{T+}, k^\mu a_\mu^+(\vec{k})] = 0
\]  
(A6)

Dirac quantization presupposes that the conditions

\[
A_i^T(\vec{k}) | 0 \rangle = 0
\]

are imposed on the ground state and the conditions

\[
k^\mu a_\mu(\vec{k}) | \rangle = 0, \quad k^\mu a_\mu^+(\vec{k}) | \rangle = 0
\]  
(A8)

are imposed on all states. As a result of Eqs. (A5) and (A6) the conditions (A7) and (A8) are algebraically consistent. The states satisfying the conditions (A8) are called physical. The Fock space of all physical states is constructed with the help of the creation operators \( a_i^{T+}(\vec{k}) \) from the ground state satisfying the conditions (A7) and (A8). As a result of Eq. (A6) any state of the Fock space constructed satisfies the conditions (A8). Following the terminology introduced in Sec. 2, this Fock space is designated by the symbol \( V \), the set of elements \( \{a_i^T, a_i^{T+}, k^\mu a_\mu(\vec{k}), k^\mu a_\mu^+(\vec{k})\} \) is a system of generators of the subalgebra \( B \) and the set of elements \( \{k_\mu a_\mu(\vec{k}), k_\mu a_\mu^+(\vec{k})\} \) is a system of generators of the subalgebra \( \mathcal{N} \).

Let \( k^\mu_\pm = (-k^0, \vec{k}) \). We find from Eq. (A2)

\[
[k^\mu a_\mu(\vec{k}), p^\nu a_\nu^+(\vec{p})] = 2 k^2 (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{p})
\]  
(A9)

The relations (A4) and (A9) mean that the observables \( F_{\mu\nu} \) do not depend on the generators \( \{k^\mu a_\mu(\vec{k}), k^\mu a_\mu^+(\vec{k})\} \), but rather they are linear combinations of the generators of the subalgebra \( B \). Therefore all matrix elements of the form \( \langle \Lambda | F_{\mu\nu} | \Sigma \rangle \), where \( |\Lambda\rangle, |\Sigma\rangle \in V \), are determined.

We note that as a result of Eqs. (A3) and (A7) the scalar product in the space \( V \) is positive-defined provided that \( \langle 0 | 0 \rangle = 1 \). We call attention to the fact that the action of the generators \( k^\mu_\pm a_\mu(\vec{k}) \) and \( k^\mu_\pm a_\mu^+(\vec{k}) \) on the physical states is not determined in Dirac quantization, and therefore these generators of the algebra \( A \) are not linear operators in the space of physical states \( V \).

In closing, we call attention to an analogy between the generators \( \{k^\mu_\pm a_\mu(\vec{k}), k^\mu_\pm a_\mu^+(\vec{k})\} \) and \( \{k^\mu a_\mu(\vec{k}), k^\mu a_\mu^+(\vec{k})\} \) in quantum electrodynamics and the generators \( \{\alpha_n^{(+)}\} \) and \( \{\alpha_n^{(-)}\} \) in string theory, respectively.

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