LOWER BOUND FOR THE ESCAPE PROBABILITY IN THE LORENTZ MIRROR MODEL ON $\mathbb{Z}^2$

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ABSTRACT. We show that in the Lorentz mirror model, at any density of mirrors,
$$P(0 \leftrightarrow \partial Q(n)) \geq \frac{1}{2n+1}.$$ 

Let $0 \leq p \leq 1$. Designate each vertex $x \in \mathbb{Z}^2$ a double-sided mirror with probability $p$. For every vertex the designations are independent and identically distributed. Vertices which are designated a mirror, with probability $1/2$ obtain a north-west mirror, otherwise they obtain north-east mirror. A ray of light traveling along the edges of $\mathbb{Z}^2$ is reflected when it hits a mirror (see image on the right) and keeps its direction unchanged at vertices which are not designated a mirror. The question is if for all values of $0 < p \leq 1$ the orbits of the ray of light are periodic, or, otherwise, for some values of $p < 1$ there is a positive probability that the light travels to infinity. For $p = 1$, a simple argument, see for instance [2, §13.3], shows that this question is equivalent to the question of the existence of an infinite open path in the bond percolation model on $\mathbb{Z}^2$ at the parameter value $1/2$, which is, due to the seminal result of [3], known to have negative answer. If $p = 1$ and mirrors are oriented with probabilities $p_{NW} \neq p_{NE}$ we have the same conclusion, see [4, p. 54]. No similar result is known for $p < 1$. We are ready to state our theorem. By $[0 \leftrightarrow A]$ we denote the event that ray of light starting at the origin reaches set $A \subset \mathbb{Z}^2$, and let $Q(n) = [-n, n]^2$.

**Theorem.** In the mirror model at any density $0 < p \leq 1$ of mirrors,
$$P_p(0 \leftrightarrow Q(n)^c) \geq \frac{1}{2n+1}.$$ 

**Proof.** Examine the mirror model on an infinite cylinder $\mathbb{Z} \times S_{2n+1}$ of odd width $2n+1$. We first note that, deterministically, there must exist an infinite path. Indeed, examine the paths crossing the $2n+1$ horizontal edges whose left-end vertex has the coordinate $0$. Then each finite path must cross an even number of edges, since each crossing moves it from the left half of the cylinder or back. Since the total number of edges is odd, at least one cannot belong to a finite path, hence it belongs to an infinite path.

Hence the expected number of edges that belong to an infinite path is $\geq 1$. Since the cylinder is invariant under rotations, we get that the probability that it crosses any given vertex is equal to $\frac{1}{2n+1}$ of this expectation. So it is $\geq \frac{1}{2n+1}$. 
Finally, since the path cannot tell the difference between being in the cylinder and in $\mathbb{Z}^2$ before getting to distance $n$, we get the desired claim on $\mathbb{Z}^2$. □

Remarks
(1) The argument works for reasonable high-dimensional analogs of the mirror model.
(2) The argument does not apply to the periodic Manhattan model (see [1, p. 13]) because the Manhattan model has the evenness built into it, and cannot be defined on a cylinder with odd width consistently. Thus we have avoided a contradiction as the result is not true for the the Manhattan model. It is not difficult to convince oneself that the path of the ray is contained inside the vacant $*$-cluster of the origin. Therefore when the density of obstacles is bigger than the critical value for site percolation on $\mathbb{Z}^2$, one has that $\mathbb{P}_p(0 \leftrightarrow Q(n)^c)$ decays exponentially.
(3) On the other hand, the argument does apply to the randomly oriented Manhattan model. In this model the orientations of the “streets” and “avenues” are random and i.i.d. Here there is no problem to define the model on a cylinder with odd width and the proof carries through literally.
(4) Rotating mirrors. In this model mirrors are changing their position deterministically after each interaction with the beam of light by flipping by 90° degrees. If $p = 1$, it is easy to see that the path of the ray of light is unbounded. However it is not known to the authors if in this case it is recurrent.

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