IS THERE A RAMSEY-HINDMAN THEOREM?

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Abstract. We show that there does not exist a joint generalization of the theorems of Ramsey and Hindman, or more explicitly, that the property of containing a symmetric IP-set is not divisible.

1. IP AND SIP SETS

We will use $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ to stand for the sets of integers, nonnegative integers and positive integers, respectively.

For $F$ a finite subset of $\mathbb{Z}$, we denote by $\sigma_F \in \mathbb{Z}$ the sum of the elements of $F$ with the convention that $\sigma_\emptyset = 0$. Of course, if $F$ is a nonempty subset of $\mathbb{N}$, then $\sigma_F \in \mathbb{N}$.

Call a subset $A$ of $\mathbb{Z}$ symmetric if $-A = A$ where $-A = \{-a : a \in A\}$. For any subset $A$ of $\mathbb{Z}$ let $A_+ = A \cup -A$ so that $A_+$ is the smallest symmetric set which contains $A$. On the other hand, let $A_+ = A \cap \mathbb{N}$, the positive part of $A$. Note that if $A$ is symmetric then $A = A_+$ and $A \setminus \{0\} = (A_+)_\pm$.

For subsets $A_1, A_2$ of $\mathbb{Z}$ we let $A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$ and $A_1 - A_2 = A_1 + (-A_2)$. If $A_2 = \{n\}$ we write $A_1 - n$ for $A_1 - A_2$.

Let $A$ be a nonempty subset of $\mathbb{Z}$. We set

\[
D(A) = \{a_1 - a_2 : a_1, a_2 \in A\} = A - A
\]

\[
IP(A) = \{\sigma_F : F \text{ a finite subset of } A\}
\]

\[
SIP(A) = D(IP(A)) = IP(A) - IP(A)
\]

Clearly, $0 \in D(A)$ and $D(A)$ is symmetric and so the same is true of $SIP(A)$. If $0 \in A$ then $A \subset D(A)$. In general, $D(A) \cup A \cup -A = D(A \cup \{0\})$.

In particular, $0 = \sigma_\emptyset \in IP(A)$ implies $IP(A) \subset SIP(A)$. 

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If $A \subset \mathbb{N}$ then $IP(A) = \{0\} \cup IP(A)_+ \text{ since } 0 = \sigma_0 \in IP(A).$  
$IP(A) = IP(A \cup \{0\}) = IP(A \setminus \{0\}).$

**Lemma 1.1.** If $B$ is a nonempty subset of $\mathbb{N}$ then 

$$ SIP(B) = IP(B_\pm). $$

If $A$ is a symmetric subset of $\mathbb{Z}$ with $A \setminus \{0\}$ nonempty then 

$$ SIP(A_+) = IP(A). $$

**Proof:** If $F \subset B_\pm$ then 

$$ \sigma_F = \sigma_{F \cap B} + \sigma_{F \cap -B} = \sigma_{F \cap B} - \sigma_{(-F) \cap B} $$

Hence, $IP(B_\pm) \subset SIP(B)$

For the reverse inclusion, let $F_1, F_2$ be finite subsets of $B.$

$$ \sigma_{F_1} - \sigma_{F_2} = \sigma_{F_1 \cup -F_2} $$

since $F_1$ and $-F_2$ are disjoint.

If $A$ is symmetric and $A \setminus \{0\}$ is nonempty then $A_+$ is nonempty and the previous result applied to $B = A_+$ yields the second equation since $(A_+)_\pm = A \setminus \{0\}$ and $IP(A) = IP(A \setminus \{0\}).$

$\square$

We say that a subset $B \subset \mathbb{N}$ is

- a **difference set** if there exists an infinite subset $A$ of $\mathbb{N}$ such that $D(A)_+ \subset B.$
- an **IP set** if there exists an infinite subset $A$ of $\mathbb{N}$ such that $IP(A)_+ \subset B.$
- an **SIP set** if there exists an infinite subset $A$ of $\mathbb{N}$ such that $SIP(A)_+ \subset B.$

Since $A \setminus \{0\} = (A_+)_{\pm}$ if $A$ is a symmetric subset of $\mathbb{Z}$ it follows from Lemma 1.1 that $B$ is an SIP set iff there exists an infinite symmetric subset $A$ of $\mathbb{Z}$ such that $IP(A)_+ \subset B.$

We next recall the statements of two -now classical- combinatorial theorems (see [4] and [5]):

**Theorem 1.2.** [Ramsey] Let $A$ be an infinite subset of $\mathbb{N}.$ If one colors the set $D(A)_+$ in finitely many colors then there exists an infinite subset $L \subset A$ such that $D(L)_+$ is monochromatic.

**Theorem 1.3.** [Hindman] Let $A$ be an infinite subset of $\mathbb{N}.$ If one colors the set $IP(A)_+$ in finitely many colors then there exists an infinite subset $L \subset \mathbb{N}$ such that $IP(L)_+ \subset IP(A)_+$ and $IP(L)_+$ is monochromatic.
In view of these two famous and basic theorems it is natural to pose the following question. Suppose a finite coloring of a set of the form $SIP_+(A) = D(IP(A))_+$ is given, is there an infinite subset $L \subset \mathbb{N}$ such that $SIP_+(L) \subset SIP_+(A)$ and $SIP_+(L)$ is monochromatic?

In other words the question is: is there a combined Ramsey-Hindman theorem?

In this paper we will show, as expected, that the answer to this question is negative. We show that it fails in a strong sense and, in the process, raise some related dynamics questions. For more details and background see [2] and [1]. We thank Benjy Weiss for his very helpful advice.

2. Families of Sets

For an infinite set $Q$ a family $\mathcal{F}$ on $Q$ is a collection of subsets of $Q$ which is hereditary upwards. That is, $\mathcal{F} \subset \mathcal{P}$, where $\mathcal{P}$ is the power set of $Q$ and $A \in \mathcal{F}$ and $A \subset B$ implies $B \in \mathcal{F}$. For any collection $\mathcal{F}_1$ of subsets of $Q$, the family $\mathcal{F} = \{B : A \subset B \text{ for some } A \in \mathcal{F}_1\}$ is the family generated by $\mathcal{F}_1$.

The dual family $\mathcal{F}^*_1 = \{B : B \cap A \neq \emptyset \text{ for all } A \in \mathcal{F}_1\}$ is indeed a family, and when $\mathcal{F}_1$ is a family we have $\mathcal{F}^*_1 = \{B : Q \setminus B \notin \mathcal{F}_1\}$.

A family $\mathcal{F}$ is proper when $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$. The dual of a proper family is proper and $\mathcal{F}^{**} = \mathcal{F}$.

Given families $\mathcal{F}_1, \mathcal{F}_2$ the join $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{A_1 \cap A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$. By the heredity condition $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}_1 \cdot \mathcal{F}_2$. Clearly, $\mathcal{F}_1 \cdot \mathcal{F}_2$ is proper iff $\mathcal{F}_2 \subset \mathcal{F}_1^*$. We say that two proper families meet when the join is proper.

It is easy to check that for families $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$

- $(\mathcal{F}^*)^* = \mathcal{F}$.
- $\mathcal{F}_1 \subset \mathcal{F}_2$ implies $\mathcal{F}_2^* \subset \mathcal{F}_1^*$. More generally, $\mathcal{F} \cdot \mathcal{F}_1 \subset \mathcal{F}_2$ implies $\mathcal{F} \cdot \mathcal{F}_2^* \subset \mathcal{F}_1^*$.
- $\mathcal{F}$ is proper if $\mathcal{F}$ is proper.

If $\mathcal{P}_+$ is the collection of all nonempty subsets of $Q$ then $\mathcal{P}_+$ is the largest proper family with dual $(\mathcal{P}_+)^* = \{Q\}$, the smallest proper family. The collection $\mathcal{B}$ of all infinite subsets of $Q$ is a proper family and the dual $\mathcal{B}^*$ is the family of all cofinite subsets of $Q$. 
A family is a **filter** when it is proper and closed under finite intersection. That is, \( A_1, A_2 \in \mathcal{F} \) implies \( A_1 \cap A_2 \in \mathcal{F} \). Equivalently, \( \mathcal{F} \cdot \mathcal{F} \subset \mathcal{F} \) and so \( \mathcal{F} \cdot \mathcal{F} = \mathcal{F} \). Thus, \( \mathcal{F} \) is a filter iff \( \mathcal{F} \cdot \mathcal{F}^* \subset \mathcal{F}^* \). In particular, if \( \mathcal{F} \) is a filter, then \( \mathcal{F} \subset \mathcal{F}^* \).

The dual of a filter is called a **filterdual**. It is sometimes called a **divisible family**. A family is a filterdual when it satisfies what Furstenberg dubbed the **Ramsey Property**:

\[
A_1 \cup A_2 \in \mathcal{F} \implies A_1 \in \mathcal{F} \quad \text{or} \quad A_2 \in \mathcal{F}.
\]

A family \( \mathcal{F} \) on \( Q \) is **full** if it is proper and \( B \in \mathcal{F} \) implies \( B \setminus F \in \mathcal{F} \) for any finite \( F \subset Q \) and so \( \mathcal{F} \) is full if \( \mathcal{F} \cdot \mathcal{B}^* = \mathcal{F} \). A filter \( \mathcal{F} \) is full iff \( \mathcal{B}^* \subset \mathcal{F} \). In particular, \( \mathcal{B}^* \) is the smallest full filter while \( \mathcal{P}_+ = \{ Q \} \) is the smallest filter.

If a family \( \mathcal{F} \) is full then

\[
\mathcal{F}^* = \{ B : B \cap A \text{ is infinite, for all } A \in \mathcal{F} \},
\]

and \( \mathcal{F}^* \) is full.

If \( \mathcal{F} \) is a filterdual then, by induction, for all positive integers \( k \), \( A_1 \cup \cdots \cup A_k \in \mathcal{F} \) implies \( A_i \in \mathcal{F} \) for some \( i = 1, \ldots, k \). We can interpret this in terms of colorings. If one colors a set \( A \in \mathcal{F} \) in finitely many colors then there exists \( B \in \mathcal{F} \) with \( B \subset A \) and \( B \) is monochromatic.

Thus, the Ramsey Theorem [1.2] implies that the family of difference sets is a filterdual on \( \mathbb{N} \) and the Hindman Theorem [1.3] says exactly that the family of IP sets is a filterdual on \( \mathbb{N} \).

If \( A \subset \mathbb{N} \) is infinite and \( K \) is a positive integer then \( A \setminus [1, K] \) is infinite and \( IP(A \setminus [1, K]) \) is disjoint from \([1, K]\) and is contained in \( IP(A) \). It follows that \( \mathcal{I} \) the family of IP sets is a full family. If \( A = \{ a_k : k = 1, 2, \ldots \} \subset \mathbb{N} \) with \( a_{k+1} > a_k + K \) for all \( k \) then \( D(A)_+ \) is disjoint from \([1, K]\). Since any infinite set contains such a subsequence it follows that the family of difference sets is full as well. We will see below that the family \( \mathcal{S} \) of SIP sets is also full.

If \( \mathcal{F} \) is a family on \( \mathbb{N} \) then \( \mathcal{F} \) is **invariant** if \( A \in \mathcal{F} \) implies \( (A + n)_+ = (A + n) \cap \mathbb{N} \in \mathcal{F} \) for all \( n \in \mathbb{Z} \). A proper, invariant family is full since \( A \setminus [1, n] = ((A - n)_+ + n)_+ \). If \( \mathcal{F} \) is a family of infinite sets, i. e. \( \mathcal{F} \subset \mathcal{B} \), then we let

\[
\gamma \mathcal{F} = \{ (A + n)_+ : A \in \mathcal{F}, n \in \mathbb{Z} \},
\]

\[
\tilde{\gamma} \mathcal{F} = \{ A : (A + n)_+ \in \mathcal{F}, \text{ for all } n \in \mathbb{Z} \}.
\]

That is, \( \gamma \mathcal{F} \) is the smallest invariant family containing \( \mathcal{F} \) and \( \tilde{\gamma} \mathcal{F} \) is a largest invariant family contained in \( \mathcal{F} \).
It is easy to see that the dual of an invariant family is invariant from which it follows that for any family $\mathcal{F}$
\[(\gamma \mathcal{F})^* = \tilde{\gamma}(\mathcal{F}^*).\]

Also, one observes that if $\mathcal{F}$ is a filter then $\tilde{\gamma} \mathcal{F}$ is an invariant filter contained in $\mathcal{F}$ and so $\gamma(\mathcal{F}^*)$ is an invariant filterdual containing the filterdual $\mathcal{F}^*$.

3. Dynamics

We call $(X, T)$ a dynamical system when $X$ is a compact metric space and $T$ is a homeomorphism on $X$. We review some well-known facts about such systems.

If $A, B \subset X$ then the hitting time set is
\[N(A, B) = \{ n \in \mathbb{N} : T^n(A) \cap B \neq \emptyset \} = \{ n \in \mathbb{N} : A \cap T^{-n}(B) \neq \emptyset \}.\]

If $A = \{x\}$ then we write $N(x, B)$ for $N(A, B)$. Observe that for $k \in \mathbb{N}$
\[(3.1)\]
\[N(A, T^k(B)) = N(A, B) + k,\]
\[N(A, T^{-k}(B)) \cup (N(A, B) - k)_+ \supset N(A, T^{-k}(B)) \setminus [1, k].\]

The system $(X, T)$ is topologically transitive if whenever $U, V \subset X$ are nonempty and open, $N(U, V)$ is nonempty. In that case, all such $N(U, V)$'s are infinite. A point $x \in X$ is called a transitive point if $N(x, U)$ is nonempty for every open and nonempty $U$ in which case, again, the $N(x, U)$'s are infinite. We denote by $\text{Trans}_T$ the set of transitive points in $X$. The system is topologically transitive iff $\text{Trans}_T$ is nonempty in which case it is a dense $G_\delta$ subset of $X$. The system is minimal when $\text{Trans}_T = X$.

**Proposition 3.1.** If $U, V \subset X$ are nonempty and open and $x$ is a transitive point for $(X, T)$, then
\[(3.2)\]
\[N(U, V) = (N(x, V) - N(x, U))_+.\]

**Proof:** If $n > m$ and $T^n(x) \in V, T^m(x) \in U$ then $T^{n-m}(T^m(x)) = T^n(x)$ implies $n - m \in N(U, V)$. On the other hand, suppose that $k \in N(U, V)$. Then $U \cap T^{-k}(V)$ is a nonempty open set and so there exists $m \in \mathbb{N}$ such that $T^m(x) \in U \cap T^{-k}(V)$. Hence, $T^m(x) \in U$ and $T^n(x) \in V$ with $n - m = k$. \[\square\]
Proposition 3.2. Let $U$ be an open set with $x \in U$ where $x$ is a transitive point for $(X, T)$. The hitting time set $N(x, U)$ is an IP set.

Assume, in addition, that there exists an involution $J$ on $X$ which maps $T$ to $T^{-1}$ and fixes $x$. That is, $J^2 = id_X, J \circ T = T^{-1} \circ J$ and $J(x) = x$. In that case, $N(x, U)$ is an SIP set.

Proof: We assume $J$ exists as described above. By intersecting $U$ and $J(U)$ we can assume that $U$ is $J$ invariant.

Suppose that $F \in N$ of cardinality $N$ such that $SIP(F_N) \subset N(x, U)$. That is, for every $n \in SIP(F_N), T^n(x) \subset U$. Since $J$ fixes $x$ and $U$ and maps $T$ to $T^{-1}$ it follows that $T^{-n}(x) \subset U$ for all such $n$ as well. That is, $T^n(x) \subset U$ for all $n$ in the symmetric finite set $SIP(F_N)$. Let $V = \bigcap_{n \in SIP(F_N)} T^{-n}(U)$. By symmetry, $J(V) = V$ and $V$ is a nonempty open set containing $x$. Since $N(x, V)$ is infinite, there exists $m \in N(x, V)$ which is larger than any element of $SIP(F_N)$. By construction $T^n(x) \subset U$ for all $n \in SIP(F \cup \{m\})$.

It follows that $F_{N+1} = F_N \cup \{m\} \subset N(x, U)$.

Let $F = \bigcup_{n \in N} \{F_n\}$. Since we go from $F$ to $IP(F)$ via finite sums, it follows that $SIP(F) = \bigcup_{n \in N} \{SIP(F_n)\}$. Hence, $SIP(F) \subset N(x, U)$.

The inductive construction for the more general IP result is similar, but easier, as the dance using symmetry is not required.

\[\square\]

Corollary 3.3. If $(X, T)$ is topologically transitive and $U, V \subset X$ are nonempty and open then $N(U, U)$ is an SIP set and $N(U, V)$ is the translation of an SIP set.

Proof: Let $x$ be a transitive point contained in $U$. By Proposition 3.1 $N(U, U) = (N(x, U) - N(x, U))_+$ and by Proposition 3.2 $N(x, U)$ is an IP set.

Now let $n \in N(U, V)$ and let $U_0 = U \cap T^{-n}(V)$. $N(U, T^{-n}(V))$ contains the SIP set $N(U_0, U_0)$. Since $S$ is a full family, (3.1) implies that $N(U, V)$ is the translate of an SIP set.

\[\square\]

It is possible to get SIP recurrence under much more general circumstances but this result will take care of what we need. We use it to produce a dynamic example which will prove the following:

Theorem 3.4. The family $S$ of SIP sets is not a filter dual.
Theorem 3.5. We consider the case where $X$ is the circle $\mathbb{R}/\mathbb{Z}$ and with $a$ a fixed irrational let $T(x) = x + a$, the irrational rotation on the circle. This is a minimal system and so every point is a transitive point.

We can regard the circle as $X = [-\frac{1}{2}, \frac{1}{2}]$ with $-\frac{1}{2} = \frac{1}{2}$. The involution $J$ on $X$ given by $x \mapsto -x$ fixes 0 and maps $T$ to $T^{-1}$. Hence, if $U$ is a open set containing 0 then $N(0, U)$ is an SIP set by Proposition 3.2. If $b \in X$ then the translation $x \mapsto x + b$ commutes with $T$ and maps 0 to $b$. It follows that if $U$ is an open set which contains $b$ then $N(b, U)$ is an SIP set.

Let $U = \left(\frac{1}{8}, \frac{1}{3}\right)$, $U_+ = [0, \frac{1}{8})$, $U_- = (-\frac{1}{8}, 0]$. The SIP set $N(0, U)$ is the union $N(0, U_+) \cup N(0, U_-)$ and we will show that neither $N(0, U_+)$ nor $N(0, U_-)$ is an SIP set. Replacing $a$ by $-a$ interchanges the two sets and so it suffices to focus on $N(0, U_+)$. We have to show that there is no infinite subset $A$ of $\mathbb{N}$ such that $SIP(A)_+ \subset N_T(0, U_+)$.

Assume such $A$ exists. Let

$$M = \sup \{T^t(0) = ta : t \in SIP(A)_+\}.$$ 

Thus, $0 < M \leq \frac{1}{8}$. Given any $\epsilon > 0$ there is a finite subset $F \subset A_+$ with $0 < \sigma_F$ and such that $M - \epsilon < T^{\sigma_F}(0) = \sigma_F a \leq M \leq \frac{1}{8}$. Since $A$ is infinite, there exists $t \in A$ larger than all the elements of $SIP(F)_+$ and so with $t > \sigma_F$. Thus, $t - \sigma_F, t, t + \sigma_F \in SIP(A)_+ \subset N_T(0, U_+)$. Thus, $0 < (t - \sigma_F)a \leq \frac{1}{8}$. Since $2\sigma_F a \leq 2M \leq \frac{1}{4}$, we have $(t + \sigma_F)a = (t - \sigma_F)a + 2\sigma_F a > 2(M - \epsilon)$. If $\epsilon$ is chosen less than $\frac{M}{2}$ then $t + \sigma_F \in SIP(A)_+$ with $(t + \sigma_F)a > M$. This contradicts the definition of $M$.

However, the dynamics suggests a further conjecture.

A dynamical system $(X, T)$ is called $\mathcal{F}$ topologically transitive for a proper family $\mathcal{F}$ of subsets of $\mathbb{N}$ if for all $U, V \subset X$ open and nonempty $N(U, V) \in \mathcal{F}$. From (3.1) it follows that every translate of $N(U, V)$ is also in $\mathcal{F}$ and so $N(U, V) \in \mathcal{\gamma F}$. That is, an $\mathcal{F}$ topologically transitive family is automatically a $\mathcal{\gamma F}$ topologically transitive family.

A system $(X, T)$ is called mild mixing if it is $S^*$ topologically transitive. Glasner and Weiss [3, Theorem 4.11, page 614] (and also, independently, Huang and Ye [6]) prove the following.

**Theorem 3.5.** $(X, T)$ is mild mixing iff for every topologically transitive system $(Y, S)$ the product system $(X \times Y, T \times S)$ is topologically transitive.
Proof: Suppose $U_1, V_1 \subset X$ and $U_2, V_2 \subset Y$ are open and nonempty. Fix $n \in N(U_2, V_2)$ so that $U_3 = U_2 \cap S^{-n}(V_2) \subset Y$ is open and nonempty. By Corollary 3.3 $N(U_2, S^{-n}(V_2)) \supset N(U_3, U_3)$ is an SIP set. Because $(X, T)$ is mild mixing $N(U_1, T^{-n}(V_1))$ is an SIP set. Because $S$ is a full family the intersection is infinite. The intersection is $N(U_1 \times U_2, V_1 \times V_2)$ and so by (3.1), $N(U_1 \times U_2, V_1 \times V_2)$ is infinite. Thus, the product is topologically transitive.

If $(X, T)$ is not mild mixing then there exist $U, V \subset X$ open and nonempty and an SIP set $A \subset N$ such that $N(U, V) \cap A = \emptyset$. The result then follows a construction of Glasner-Weiss which shows

Theorem 3.6. If $A \subset N$ is an SIP set then there exists a topologically transitive system $(Y, S)$ and $G \subset Y$ a nonempty open set such that $N(G, G) \subset A$.

\[\Box\]

Corollary 3.7. The product of any collection of mild mixing systems is mild mixing.

Proof: If $T_1$ and $T_2$ are mild mixing homeomorphisms and $S$ is topologically transitive, then $T_2 \times S$ is transitive and so $T_1 \times T_2 \times S$ is transitive. Hence, $T_1 \times T_2$ is mild mixing.

By induction a finite product of mild mixing systems is mild mixing.

An infinite product times $S$ is the inverse limit of finite products times $S$ and the inverse limit of transitive systems is transitive. It follows that the infinite product is mild mixing.

\[\Box\]

Let $M$ be the family on $N$ generated by \{ $N(U, V) : (X, T)$ mild mixing and $U, V \subset X$ open and nonempty \}. From Corollary 3.7 it follows that $M$ is a filter. Because $S^*$ transitivity implies $\tilde{\gamma}(S^*)$ transitivity it follows that $M \subset \tilde{\gamma}(S^*)$.

By the Hindman Theorem, $I$ the family of IP sets is a filterdual and so $I^*$ is a filter. It then follows that $\tilde{\gamma}(I^*) = (\gamma(I))^*$ is a filter.

We know from the above example that $S^*$ is not a filter, but it might still be true that $\tilde{\gamma}(S^*)$ is a filter. This would be true if $M = \tilde{\gamma}(S^*)$. In that case, $\gamma S$ would be a filterdual. In the example itself, $N(0, U_+)$ is not an SIP set, but if $T^k(0) \in (-\frac{1}{2}, 0)$ then 0 is in the interior of $T^k(U_+)$ and so $N(0, T^k(U_+))$ is an SIP set. From (3.1) it follows that $N(0, U_+)$ is the translate of an SIP set.

It is to this question that we now turn. As we will see this conjecture fails as well.
4. SIP Sets and Their Refinements

Let \( e \in \mathbb{N} \) and \( b = 2e + 1 \) so that \( b \) is an odd number greater than 1. Define \( \alpha_b : \mathbb{N} \to \mathbb{N} \) by \( \alpha_b(n) = b^n - 1 \). The \( b \) expansion of an integer \( t \) is the sum \( \sum_{n \in \mathbb{N}} \epsilon_n \alpha_b(n) = t \) such that:

- \( |\epsilon_n| \leq e \) for all \( n \in \mathbb{N} \).
- \( \epsilon_n = 0 \) for all but finitely many \( n \).

**Proposition 4.1.** Every integer in \( \mathbb{Z} \) has a unique \( b \) expansion.

**Proof:** By the Euclidean Algorithm every integer \( t \) can be expressed uniquely as \( \epsilon + bs \) with \( |\epsilon| \leq e \). It follows by induction that every integer \( t \) with \( |t| < \frac{1}{2}(b^k - 1) \) has an expansion with \( \epsilon_n = 0 \) for \( n \geq k \). There are \( b^k \) such integers and the same number of expansions. So by the pigeonhole principle the expansions are unique.

\( \square \)

We will only need the \( b = 3 \) expansions with \( e = 1 \) so that each \( \epsilon_n = -1, +1 \) or 0. We will write \( \alpha \) for \( \alpha_3 \) so that \( \alpha(n) = 3^n - 1 \). From Proposition 4.1 we obviously have \( \mathbb{Z} = \text{SIP}(\alpha(\mathbb{N})). \)

The length \( r(t) \) of \( t \) is the number of nonzero \( \epsilon_i \)'s in the expansion of \( t \). With \( r = r(t) \) we let \( j_1(t), ..., j_r(t) \) be the corresponding indices written in increasing order. That is,

- \( j_1(t) < \cdots < j_r(t) \) and \( \epsilon_{j_i(t)} = \pm 1 \) for \( i = 1, \ldots, r = r(t) \).
- \( t = \sum_{i=1}^{r} \epsilon_{j_i(t)} \alpha(j_i(t)). \)

We call this representation the **reduced expansion** and \( j_1(t), \ldots, j_r(t) \) the **indices** of \( t \).

Notice that 0 has length 0 and equals the empty sum.

Because \( 3^{n+1} > 1 + 3 + \cdots + 3^n \) it follows that

\[
(4.1) \quad t > 0 \quad \Leftrightarrow \quad \epsilon_{j_r(t)} = 1.
\]

**Definition 4.2.** Assume that \( j_1(t), \ldots, j_r(t) \) and \( j_1(s), \ldots, j_r(s) \) are the indices of the reduced expansions for \( t, s \in \mathbb{Z} \).

(a) Call \( t \) of **positive type** (or of **negative type**) if \( \epsilon_{j_1(t)} \epsilon_{j_r(t)} \) is positive (resp. is negative). So \( t \) is of positive type if coefficients of its first and last indices have the same sign. By convention we will say that 0 is of positive type.

(b) We will write \( t \succ s \) if \( j_1(t) > j_r(s) \), that is, the indices for \( t \) are larger than all of the indices of \( s \). We will say that \( t \) is **beyond** \( s \) when \( t \succ s \).
If \( t > 0 \) then \( \epsilon_{j_1(t)} = 1 \), and so \( t \) is of positive type (or negative type) if \( \epsilon_{j_0(t)} \) is positive (resp. \( \epsilon_{j_1(t)} \) is negative). Notice that if \( j_r(s) = n + 1 \), then
\[
(4.2) \quad t \succ s \iff t \succ 3^n \iff t \equiv 0 \pmod{3^{n+1}}.
\]

Now we turn to SIP sets.

**Definition 4.3.** (a) We call a strictly increasing function \( k : \mathbb{N} \to \mathbb{N} \) a +function.

(b) If \( k_1 \) and \( k_2 \) are +functions we say that \( k_2 \) directly refines \( k_1 \) if \( k_2(\mathbb{N}) \subset k_1(\mathbb{N}) \). We say that \( k_2 \) refines \( k_1 \) if \( IP(k_2(\mathbb{N})) \subset IP(k_1(\mathbb{N})) \).

Clearly, direct refinement implies refinement and each relation is transitive.

For an infinite subset \( A \subset \mathbb{N} \) there is a unique +function \( k_A \) such that \( k_A(\mathbb{N}) = A \), i.e. the function which counts the elements of \( A \) in increasing order.

**Lemma 4.4.** If \( k \) is a +function, then for any \( N \in \mathbb{N} \) there exists a +function \( k_1 \) such that
- \( k_1 \) refines \( k \).
- \( k_1(n) > 3^{N-1} \) for all \( n \in \mathbb{N} \).

**Proof:** By induction we can assume that \( k(n) > 3^{N-2} \) for all \( n \in \mathbb{N} \) (the condition is vacuous when \( N = 1 \)). This means that \( j_1(k(n)) \geq N \) for all \( n \in \mathbb{N} \).

Case 1: There is an infinite set \( A \subset \mathbb{N} \) such that \( j_1(k(n)) > N \) for all \( n \in A \). Let \( k_1 \) be the +function with \( k_1(\mathbb{N}) = k(A) \). Then \( k_1 \) is a direct refinement of \( k \) and \( j_1(k_1(n)) \geq N + 1 \) for all \( n \in \mathbb{N} \), i.e. \( k_1(n) > 3^{N-1} \) for all \( n \).

Case 2: There is an infinite set \( A \subset \mathbb{N} \) such that \( j_1(k(n)) = N \) and \( \epsilon_{j_1(k(n))} = \delta = -1 \) for all \( n \in A \), or there is an infinite set \( A \subset \mathbb{N} \) such that \( j_1(k(n)) = N \) and \( \epsilon_{j_1(k(n))} = \delta = +1 \) for all \( n \in A \).

Let \( \tilde{k} \) be the +function with \( \tilde{k}(\mathbb{N}) = k(A) \), a direct refinement of \( k \) and \( \tilde{k}(n) \equiv \delta 3^{N-1} \pmod{3^N} \) for all \( n \in \mathbb{N} \).

Now define
\[
k_1(n) = \tilde{k}(3n - 2) + \tilde{k}(3n - 1) + \tilde{k}(3n).
\]

Clearly, \( IP(k_1(\mathbb{N})) \subset IP(\tilde{k}(\mathbb{N})) \subset IP(k(\mathbb{N})) \) and so \( k_1 \) refines \( k \). For all \( n \), \( k_1(n) \equiv \delta 3^{N-1} \equiv 0 \pmod{3^N} \). Thus, \( k_1(n) > 3^{N-1} \) for all \( n \).

\( \square \)

**Theorem 4.5.** If \( A \subset \mathbb{N} \) is a translate of an SIP set then there exists \( t_0 \in A \) and +function \( k \) such that
(i) $k(1) \succ t_0$.
(ii) $k(n + 1) \succ k(n)$ for all $n \in \mathbb{N}$.
(iii) Either $k(n)$ is of positive type for all $n \in \mathbb{N}$, or else $k(n)$ is of negative type for all $n \in \mathbb{N}$.
(iv) $t_0 + SIP(k(\mathbb{N}))_+ = (t_0 + SIP(k(\mathbb{N})))_+ \subset A$.

**Proof:** There exists $u \in \mathbb{Z}$ and a $+$-function $k_0$ such that $(SIP(k_0(\mathbb{N})))_+ + u_+ \subset A$.

For sufficiently large $N_0$, $t_0 = u + \Sigma_{n=1}^{N_0} k_0(n) > 0$ and so lies in $A$.

Let $k_0^+$ be the direct refinement of $k_0$ with $k_0^+(\mathbb{N}) = k_0([N_0 + 1, \infty))$. Hence,

$$(4.3) \quad (t_0 + SIP(k_0^+(\mathbb{N})))_+ \subset A.$$

Now we repeatedly apply Lemma 4.4. Let $N_1 > j_r(t_0)$.

Choose $k_1$ a $+$-function which refines $k_0^+$ and with $k_1(n) \succ 3^{N_1}$ for all $n \in \mathbb{N}$. In particular, $k_1(1) \succ t_0$. So from (4.3) we have

$$(4.4) \quad (t_0 + SIP(k_1(\mathbb{N})))_+ \subset A.$$

Let $k_1^+$ be the direct refinement of $k_1$ with $k_1^+(\mathbb{N}) = k_1([2, \infty))$.

Let $N_2 > j_r(k_1(1))$ and choose $k_2$ a $+$-function which refines $k_1^+$ and with $k_2(n) \succ 3^{N_2}$ for all $n \in \mathbb{N}$. In particular, $k_2(1) \succ k_1(1)$. Furthermore,

$$(4.5) \quad IP[\{k_1(1)\} \cup IP(k_2(\mathbb{N}))] \subset IP[\{k_1(1)\} \cup IP(k_1^+(\mathbb{N}))] = IP(k_1(\mathbb{N})).$$

Inductively, let $k_q^+$ be the direct refinement of $k_q$ with $k_q^+(\mathbb{N}) = k_q([2, \infty)$ and let $N_{q+1} > j_r(k_q(1))$. Choose $k_{q+1}$ a refinement of $k_q^+$ with $k_{q+1}(n) \succ 3^{N_{q+1}}$ for all $n \in \mathbb{N}$. Hence, $k_{q+1}(1) \succ k_q(1)$ and

$$(4.6) \quad IP[\{k_q(1)\} \cup IP(k_{q+1}(\mathbb{N}))] \subset IP[\{k_q(1)\} \cup IP(k_q^+(\mathbb{N}))] = IP(k_q(\mathbb{N})),
\quad IP[\{k_1(1), \ldots, k_q(1)\} \cup IP(k_{q+1}(\mathbb{N}))] \subset IP(k_1(\mathbb{N})).$$

Now define $\tilde{k}(n) = k_n(1)$ for $n \in \mathbb{N}$. Either $\tilde{k}(n)$ is of positive type infinitely often or of negative type infinitely often (or both). So we can choose a direct refinement $k$ of $\tilde{k}$ so that, (i), (ii) and (iii) hold. In addition,

$$IP(k(\mathbb{N})) \subset IP(\tilde{k}(\mathbb{N})) \subset IP(k_1(\mathbb{N})).$$

Clearly, $k(n+1) \succ k(n) \succ t_0$ and from (4.6) it follows that $IP(\tilde{k}(\mathbb{N})) \subset IP(k_1(\mathbb{N}))$ and so from (4.4) $[t_0 + SIP(k(\mathbb{N})))_+ \subset A$.

Since $k(n) \succ t_0$ for all $n$ it follows from (4.2) that $t \succ t_0$ for all $t \in SIP(k(\mathbb{N}))$. Hence, if $t \in SIP(k(\mathbb{N}))$ is negative then $t_0 + t$ is
negative. Thus, \( t_0 + \text{SIP}(k(\mathbb{N}))_+ = t_0 + [\text{SIP}(k(\mathbb{N}))]_+ \), completing the proof of (iv).

\[ \square \]

**Remark:** Since \( N_1 \) can be chosen arbitrarily large it follows that \( \text{SIP}(k(\mathbb{N}))_+ \) and hence \( t_0 + \text{SIP}(k(\mathbb{N}))_+ \) can be chosen disjoint from an arbitrary finite subset of \( \mathbb{N} \). This shows that \( S \) and \( \gamma S \) are full families.

For two distinct numbers \( n, m \in \mathbb{Z} \setminus \{0\} \) define

\[
(4.7) \quad \delta(n, m) = \begin{cases} 
0 & \text{if } nm > 0, \\
1 & \text{if } nm < 0.
\end{cases}
\]

Now we define the **sign change count** to be the function \( z : \mathbb{N} \to \mathbb{Z}_+ \) so that if \( t \in \mathbb{N} \) has reduced expansion with indices \( j_1(t), \ldots, j_{r(t)}(t) \) then

\[
(4.8) \quad z(t) = \Sigma_{i=1}^{r(t)-1} \delta(\epsilon_{j_i}, \epsilon_{j_{i+1}}).
\]

In particular, if the length is one then the sum is empty and so \( z(3^{n-1}) = 0 \) for all \( n \in \mathbb{N} \).

For a positive integer \( K \) let \( \pi_K : \mathbb{Z} \to \mathbb{Z}/K\mathbb{Z} \) be the quotient map mod \( K \).

**Theorem 4.6.** If \( A \subset \mathbb{N} \) is a translate of an SIP set then for every odd number \( K \), \( \pi_K \circ z : A \to \mathbb{Z}/K\mathbb{Z} \) is surjective.

**Proof:** Fix \( K \). Since it is odd, 2 and \( -2 \) generate the cyclic group \( \mathbb{Z}/K\mathbb{Z} \).

By Theorem 4.5 we can choose \( t_0 \in A \) and a +function \( k \) which satisfies the four conditions of the theorem.

Let \( s_0 = t_0 + \sum_{n=1}^{2K+1} k(n) \). Since \( k(n+1) \succ k(n) \succ t_0 \) for all \( n \) we can regard the sequence \( \{k(n) : n \in \mathbb{N}\} \) as a sequence of disjoint ascending blocks in \( IP(\alpha) \).

Since each \( k(n) \) is positive, each \( \epsilon_{j_{rn}}(k(n)) \) is positive, where \( r_n = r(k(n)) \). For \( i = 1, \ldots, K-1 \) let

\[
s_i = t_0 + \sum_{n=1}^{2i} (-1)^{n+1} k(n) + \sum_{n=2i+1}^{2K+1} k(n).
\]

That is, moving from \( s_{i-1} \) to \( s_i \) we reverse the sign of the block \( k(2i) \) keeping the remaining blocks fixed. Clearly \( s_i \in A \) for \( i = 1, \ldots, K-1 \).

**Case 1-** Every \( k(n) \) is of positive type. Each \( \epsilon_{j_1(k(n))} \) is positive.

Moving from \( s_{i-1} \) to \( s_i \) increases \( z \) by exactly 2 because the ++ transition from \( j_{r_{2i-1}}(k(2i-1)) \) to \( j_1(k(2i)) \) is replaced by a +− transition...
and the ++ transition from $j_{r_2}(k(2i))$ to $j_1(k(2i+1))$ is replaced by a $-+$ transition. Thus, $\pi_K(z(s_i)) = \pi_K(z(s_{i-1})) + 2 \mod K$. Since 2 generates the cyclic group, $\pi_K \circ z$ is surjective.

Case 2- Every $k(n)$ for $n > 1$ is of negative type. Each $\epsilon_{j_1(k(n))}$ is negative. This time the $+-$ transition from $j_{r_2-1}(k(2i-1))$ to $j_1(k(2i))$ is replaced by a $++$ transition and the $+-$ transition from $j_{r_2}(k(2i))$ to $j_1(k(2i+1))$ is replaced by a $-+$ transition. Thus, in this case, $\pi_K(z(s_i)) = \pi_K(z(s_{i-1})) - 2 \mod K$. Again $\pi_K \circ z$ is surjective.

We can now deduce the following:

**Theorem 4.7.** If $A$ is any SIP subset of $\mathbb{N}$ (including $\mathbb{N}$ itself), then $A$ can be partitioned by two sets neither of which contains a translate of an SIP set. Thus, the family of translated SIP sets in $\mathbb{N}$ is not a filter dual.

**Proof:** With $K = 3$, the sign count map $z : \mathbb{N} \to \mathbb{Z}/3\mathbb{Z}$ determines a coloring of $\mathbb{N}$ and in any translated SIP set contained in $IP(k)_+$ all three colors occur.

In particular, let

\begin{align*}
A_0 &= \{ t \in \mathbb{N} : z(t) \equiv 0 \pmod{3} \}, \\
A_1 &= \mathbb{N} \setminus A_0 = \{ t \in \mathbb{N} : z(t) \not\equiv 0 \pmod{3} \}.
\end{align*}

Neither $A_0$ nor $A_1$ contains a translate of an SIP set. It follows that both $A_0$ and $A_1$ are elements of the dual $(\gamma S)^* = \tilde{\gamma}(S^*)$.

\[ \square \]

In general the congruence classes $\mod K$ of $z(t)$ (for $K$ odd) define a decomposition of $\mathbb{N}$ into $K$ elements, each a member of $\tilde{\gamma}(S^*) \subset S^*$. Thus, $S^*$ and $\tilde{\gamma}(S^*)$ fail to be filters in a very strong way.

5. Dynamics again

We defined the family $\mathcal{M}$ generated by the sets $\mathcal{N}(U, V)$ with $(X, T)$ mild mixing and $U, V \subset X$ open and nonempty. We saw that $\mathcal{M}$ is an invariant filter contained in $\tilde{\gamma}(S^*)$. Now that we know that the latter is not a filter, we see that the inclusion is proper. Can we find another possible description of the sets in $\mathcal{M}$?
For a proper family $\mathcal{F}$ on an infinite set $Q$ we define the \textit{sharp dual} $\mathcal{F}^\#$ by
\begin{equation}
\mathcal{F}^\# = \{ A \subset Q : A \cap B \in \mathcal{F} \text{ for all } B \in \mathcal{F} \}.
\end{equation}

\textbf{Proposition 5.1.} Let $\mathcal{F}$ be a proper family on an infinite set $Q$.

(a) $\mathcal{F}^\#$ is a filter contained in $\mathcal{F} \cap (\mathcal{F}^*)$. It is full if $\mathcal{F}$ is full.
(b) $\mathcal{F}^\# = (\mathcal{F}^*)^\#$.
(c) If $\mathcal{F}$ is a filter, then $\mathcal{F}^\# = \mathcal{F}$. In particular, $(\mathcal{F}^\#)^\# = \mathcal{F}^\#$.
(d) If $\mathcal{F}$ is a filterdual then $\mathcal{F}^\# = \mathcal{F}^*$.

\textbf{Proof:} (a) Since $\mathcal{F}$ is proper, $\emptyset \notin \mathcal{F}$ and so $A \in \mathcal{F}^\#$ implies $A \cap B \neq \emptyset$ for all $B \in \mathcal{F}$. Thus, $A \in \mathcal{F}^\#$. Also, $Q \in \mathcal{F}$ and so $A = A \cap Q \in \mathcal{F}$.

If $A_1, A_2 \in \mathcal{F}^\#$ and $B \in \mathcal{F}$ then $(A_1 \cap A_2) \cap B = A_1 \cap (A_2 \cap B) \in \mathcal{F}$.

Thus, $A_1 \cap A_2 \in \mathcal{F}$.

If $A$ is a cofinite set and $B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ since $\mathcal{F}$ is full. Thus, $A \in \mathcal{F}^\#$. That is, $\mathcal{B}^\# \subset \mathcal{F}^\#$. Since the latter is a filter, it is full.

(b) If $A \in \mathcal{F}^\#$, $B_1 \in \mathcal{F}^*$, $B \in \mathcal{F}$ then $(A \cap B_1) \cap B = (A \cap B) \cap B_1 \neq \emptyset$.

Since $B$ was arbitrary, $A \cap B_1 \in \mathcal{F}^*$. Since $B_1$ was arbitrary, $A \in (\mathcal{F}^*)^\#$. The reverse inclusion follows from $(\mathcal{F}^*)^* = \mathcal{F}$.

(c) If $\mathcal{F}$ is a filter, then $\mathcal{F} \cdot \mathcal{F} = \mathcal{F}$ and so $\mathcal{F} \subset \mathcal{F}^\#$. From (a) it follows that $\mathcal{F}^\# \subset \mathcal{F}$.

(d) This follows from (b) and (c).

\hfill $\square$

\textbf{Theorem 5.2.} $\mathcal{M} \subset (\gamma S)^\# = (\tilde{\gamma}(S^*))^\#$.

\textbf{Proof:} The equation follows from Proposition 5.1 (b).

Now let $(X, T)$ be mild mixing and $U, V \subset X$ be open and nonempty. Let $A \subset \mathbb{N}$ be a translation of an SIP set. We show that $N(U, V) \cap A$ is the translation of an SIP set.

By the Glasner Weiss construction Theorem 3.6 there exists a topologically transitive system, $(Y, S)$, $G \subset Y$ open and nonempty and $n \in \mathbb{Z}$ so that $N(G, S^{-n}(G)) \setminus [1, |n|]$ is contained in $A$. It follows that $N(U, V) \cap A$ contains $N(U, V) \cap N(G, T^{-n}(G)) \setminus [1, |n|]$. Because $(X \times Y, T \times S)$ is topologically transitive, $N(U, V) \cap N(G, T^{-n}(G)) = N(U \times G, V \times T^{-n}(G))$ is the translation of an SIP set by Corollary 3.3. As $\gamma S$ is a full family, it follows that $N(U, V) \cap A$ is in $\gamma S$.

\hfill $\square$

Our final conjecture is that $\mathcal{M} = (\gamma S)^\#$. 

IS THERE A RAMSEY-HINDMAN THEOREM?

REFERENCES

1. E. Akin, *Recurrent in topological dynamical systems: Furstenberg families and Ellis actions*, Plenum Press, New York, 1997.
2. H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton university press, Princeton, N.J., 1981.
3. E. Glasner and B. Weiss, *On the interplay between measurable and topological dynamics*, Handbook of dynamical systems. Vol. 1B, 597–648, Elsevier B. V., Amsterdam, 2006.
4. R. Graham, B. L. Rothchild and J. H. Spencer, *Ramsey theory*, John Wiley & sons, 1980.
5. N. Hindman, *Finite sums from sequences within cells of a partition of N*, J. Combinatorial Theory, Ser. A 17, (1974), 1–11.
6. W. Huang and X. Ye, *Topological complexity, return times and weak disjointness*, Ergodic Theory Dynam. Systems 24, (2004), 825–846.

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