On the global existence for a fluid-structure model with small data

Igor Kukavica, Wojciech Ożański, and Amjad Tuffaha

Friday 29th October, 2021

Abstract

We address the system of partial differential equations modeling motion of an elastic body interacting with an incompressible fluid. The fluid is modeled by the incompressible Navier-Stokes equations while the structure is represented by a damped wave equation with no added stabilization terms. We prove global existence and exponential decay of strong solutions for small initial data in a suitable Sobolev space. The elastic displacement is controlled using a new almost divergence-free corrector in the fluid domain. Our approach allows for any superlinear perturbation of the wave equation.

Mathematics Subject Classification: 35R35, 35Q30, 76D05

Keywords: Navier-Stokes equations, fluid-structure interaction, long time behavior, global solutions, damped wave equation

1 Introduction

We are concerned with a model of an elastic body interacting with a viscous incompressible fluid. The elastic body occupies a time-dependent domain $\Omega_\varepsilon(t)$, and is described by a displacement function $w(x, t) := \eta(x, t) - x$, where $\eta(x, t)$ denotes the Lagrangian mapping of a particle located at $x \in \Omega_\varepsilon(0) =: \Omega_\varepsilon$. The displacement $w$ is governed by the linear damped equation

$$w_{tt} - \Delta w + \alpha w_t = 0$$

in $\Omega_\varepsilon \times (0, \infty)$. We note that all the results presented below extend to the case of an additional super-quadratic term $f(w)$, i.e., a smooth nonlinearity $f$ such that $f(x) = o(x)$ as $x \to 0$. For the boundary of $\Omega_\varepsilon(t)$, we suppose that $\partial \Omega_\varepsilon(t) = \Gamma_\varepsilon(t) \cup \Gamma_c(t)$. On $\Gamma_\varepsilon(t)$ we assume the zero displacement condition

$$w(x, t) = 0 \quad \text{for } x \in \Gamma_\varepsilon, \ t > 0,$$

and $\Gamma_c(t)$ is assumed to be a common boundary shared with a fluid described by the incompressible Navier-Stokes equations

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0,$$

$$\text{div } u = 0 \quad \text{in } \{(x, t) : t > 0, \ x \in \Omega(t)\}$$
(1.3)), where \( \Omega_f(t) \) denotes the time-dependent fluid domain, which, except for the common boundary \( \Gamma_c(t) \), has a free boundary \( \Gamma_f(t) \). We consider a simplified form \( \Omega_e = \Omega_e(0) \) and \( \Omega_f = \Omega_f(0) \) by assuming periodicity in the variables \( x_1, x_2 \), and that the fluid is located above the elastic solid. Namely, we take

\[
\Omega_e := \{ x = (x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{T}^2, h_1 \leq x_3 \leq h_2 \}, \\
\Omega_f := \{ x = (x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{T}^2, h_2 \leq x_3 \leq h_3 \};
\]

see Figure 1 for a sketch. We assume that \( h_1 = 0, h_2 = 1/2, h_3 = 1 \), for the sake for simplicity.

![Figure 1: The sketch of the fluid-structure interaction model.](image_url)

In order to state our main result, we first introduce the Lagrangian setting of the Navier-Stokes equations problem and specify the boundary conditions on \( \Gamma_c \) and \( \Gamma_f \).

The displacement \( \eta(\cdot, t) \) is defined in the fluid domain \( \Omega_f \) as a solution of the system

\[
\eta_t(x, t) = v(x, t), \\
\eta(x, 0) = x,
\]

for \( x \in \Omega_f \) and \( t > 0 \), where

\[
v(x, t) := u(\eta(x, t), t).
\]

The incompressible Navier-Stokes equations \( (1.3) \) in Lagrangian coordinates read

\[
\partial_t v_i - \partial_j (a_{jl} a_{kl} \partial_k v_i) + \partial_k (a_{ki} q) = 0 \quad \text{in} \quad \Omega_f \times (0, T), \quad i = 1, 2, 3, \\
a_{ki} \partial_k v_i = 0 \quad \text{in} \quad \Omega_f \times (0, T),
\]

where \( a := (\nabla \eta)^{-1} \) and \( q(x, t) := p(\eta(x, t), t) \) denotes the Lagrangian pressure. Except for the zero displacement condition \( (1.2) \) on the elastic boundary, we also assume the continuity of the velocities

\[
\partial_t w = v \quad \text{on} \quad \Gamma_c \times (0, T)
\]

and continuity of the stresses

\[
\partial_j w_i N_j = a_{jl} a_{kl} \partial_k v_i N_j - a_{ji} q N_j \quad \text{on} \quad \Gamma_c \times (0, T),
\]
for $i = 1, 2, 3$, where we apply the summation convention on repeated indices, and $N = e_3$ denotes the outward unit normal vector with respect to $\Omega_e$. On the outside fluid boundary $\Gamma_t$, we assume the non-slip boundary condition

$$v = 0 \text{ on } \Gamma_t \times (0, T). \quad (1.10)$$

Note that we use the strain tensor $\nabla v$ in (1.9) instead of the symmetric gradient matrix $\nabla v + \nabla v^T$ for the sake of simplicity.

We thus seek a solution $(v, w, q, a, \eta)$ to the damped fluid-structure system (1.1)–(1.10). Our main result is the following.

**Theorem 1** (Main result). Let $v, w, \eta, a \in C^3([0, \infty); H^3)$ be a smooth solution to (1.1)–(1.10), and let

$$Y(t)^2 := \|v\|_3^2 + \|v_t\|_3^2 + \|v_{tt}\|_3^2 + \|w\|_3^2 + \|w_t\|_2^2 + \|w_{tt}\|_1^2 + \|w_{ttt}\|_2^2.$$ 

Then there exists $C > 1$ and $\varepsilon > 0$ such that if $Y(0) \leq \varepsilon$ then

$$Y(t) \leq \varepsilon e^{-t/C},$$

for $t \geq 0$.

Here and throughout the paper we use the notation

$$\| \cdot \|_k := \| \cdot \|_{H^k(\Omega)}, \quad \| \cdot \| := \| \cdot \|_{L^2(\Omega)},$$

where $\Omega = \Omega_t$ or $\Omega = \Omega_e$.

As in [KLT3], the a priori estimate of Theorem 1 can be used to show global-in-time existence and exponential decay for (1.1)–(1.10) for sufficiently small initial data.

**Theorem 2** (Global-in-time well-posedness for small data). There exists $C > 1$ such that for every sufficiently small $\varepsilon > 0$ if

$$\|v_0\|_4, \|w_0\|_3, \|\partial_t w_0\|_2, \|\partial_t q_0\|_1, \|q_0\|_3 \leq \varepsilon,$$

with appropriate compatibility conditions, then there exists a unique solution $(v, w, q, a, \eta)$ of (1.1)–(1.10). Moreover the solution satisfies

$$\|v\|_3 + \|w\|_3 + \|w_t\|_2 + \|w_{tt}\|_1 + \|w_{ttt}\| + \|v_{tt}\| \leq \varepsilon e^{-t/C},$$

for $t \geq 0$.

Existence of local-in-time solutions to the system in question was first studied, without damping, by Coutand and Shkoller [CS1, CS2]. Other local-in-time well-posedness results were established for less regular initial data in [KT1, KT2, RV, BGT]. Solutions for an analogous model involving the
compressible Navier-Stokes equations were also considered in [BG1, BG2, KT3]; cf. also [ALT, AT1, B, BGLT1, BGLT2, BZ1, DEGL, DGHL, GGCC, IKLT1, MC1, MC2] for other results on local strong and global weak solutions for related models. Global in-time strong solutions under small initial data for the system with both boundary and interior damping were obtained in [IKLT2] and with interior damping only in [IKLT3].

The main idea of our approach is to derive a number of energy estimates related to various energy levels of solutions. Namely we consider differential operators

\[ S \in \{ \text{id}, \partial_t, \partial'_t, \partial''_t, \partial_{tt} \}, \]

where \( \partial', \partial'' \) denote any derivatives with respect to the horizontal variables, \( x_1 \) and \( x_2 \), of order 1 or 2, respectively. For each choice of \( S \) we derive an energy inequality (3.15) that expresses an energy coupling between the fluid and the elastic solid that also depends on a coupling between two energy levels (i.e., between (3.13) and (3.14)). Such energy inequalities can be obtained partially due to the fact that \( S \) involves only derivatives in the horizontal variables \( x_1, x_2 \) and time derivatives, and so it can be applied to the boundary conditions (1.8)–(1.9) on \( \Gamma_c \) for all times.

Using such energy inequalities we then select some quantities appearing in the inequalities to identify a total energy (3.20) of the system. We then estimate the behavior in time of all ingredients of the total energy (see Steps 1 to 3 in Section 3.3) in a way that enables us to employ an ODE-type lemma (Lemma 3) to deduce exponential decay of the total energy for small data.

Such approach, although fundamentally inspired by [IKLT2, IKLT3], involves a number of improvements and new ideas, such as an immense simplification of all estimates, a more refined use of the coupling of the fluid with the elastic solid (see Section 3.1), as well as an introduction of a new test function \( \phi \) (see (3.8)).

Moreover we note that the assumption of interior damping used by [IKLT2, IKLT3], that is an inclusion of an additional term \( \beta w \) on the left-hand side of (1.1) does not have physical justification. It was used by [IKLT2, IKLT3] for controlling the evolution of the 0-th order norm of \( w \). For example, it was not clear how to control a translational and rotational motions of the elastic solid without this additional term.

In this work, we do not assume such additional damping, but instead impose homogeneous Dirichlet constraint at the fluid boundary \( \Gamma_c \) (recall (1.2)). Although this does give some control on the 0-th order behavior of \( w \) via the Poincaré inequality, such control is not sufficient by itself. This is due to the coupling of the two levels of energy in each of our energy estimates pointed out above, and is related to the introduction of a scaling parameter which is denoted by \( \lambda \) below (see (3.15)–(3.16)). We discuss this issue in detail in Step 1 in Section 3.3.

In order to deal with this problem we introduce a certain “almost divergence-free” corrector of \( w \) which extends it to the fluid domain \( \Omega_f \), and which is a new way of coupling \( w \), defined on \( \Omega_c \), with...
v, defined on Ω. In fact the “almost divergence-free” property, namely (3.3), can be obtained by the divergence-free property (1.7) of v as well as the continuity boundary condition (1.8). Such extension then allows us to control the 0-th order behavior of w; see (3.28) for details.

The structure of the paper is as follows. In Section 2 we introduce some notation and discuss some properties of the Lagrangian map η as well as the Stokes estimates concerning (1.6). We then prove our main result, Theorem 1 in Section 3. The proof is based on the extension of w into the fluid domain, discussed in Section 3.1, and a number of energy estimates, given in Section 3.2, which lead to the definition (3.20) of the total energy X(t) in Section 3.3 where we also combine the energy estimates into an a priori bound (3.26) on X. The exponential decay of X is then established using an ODE-type result, Lemma 3, which is proven in Section 3.4.

2 Preliminary results

We denote by C any universal constant, the value of which may change from line to line. We apply the summation convention over repeated indices. We denote by ∂′ the gradient with respect to the variables x_1 and x_2 only, and by ∂′′ the matrix of all second order derivatives with respect to x_1 and x_2.

2.1 Estimates of the particle map

We note that, for each t > 0, we have

\( \nabla \eta = I + \int_0^t \nabla v, \)

due to (1.4), where I denotes the 3 × 3 identity matrix, and so in particular

\[ \| I - \nabla \eta(t) \|_2 \leq \| v \|_{L^1((0,t);H^3)}, \]

for every t > 0. Moreover, due to the incompressibility condition in (1.7), we have that det \( \nabla \eta = 1 \) for all times, which shows that \( a = (\nabla \eta)^{-1} \) is the corresponding cofactor matrix, that is

\[ a_{ij} = \frac{1}{2} \epsilon_{imn} \epsilon_{jkl} \partial_m \eta_k \partial_n \eta_l, \]

where \( \epsilon_{ijk} \) denotes the permutation symbol.

This shows that

\[ \| a_t \|_2 \lesssim \| \nabla \eta \|_2 \| v \|_3 \lesssim \| v \|_{L^\infty((0,t);H^3)} (1 + \| v \|_{L^1((0,t);H^3)}), \]

\[ \| a_{tt} \|_1 \lesssim \| \nabla \eta \|_2 \| v_t \|_2 + \| v \|_2^2, \]

and

\[ \| I - a \|_2 \leq \int_0^t \| a_t \|_2 \lesssim \int_0^t \| \nabla \eta \|_2 \| v \|_3 \lesssim \| v \|_{L^1((0,t);H^3)} (1 + \| v \|_{L^1((0,t);H^3)}), \]

for every t > 0, due to (2.1). Moreover,

\[ \| I - aa^T \|_2 \lesssim \| I - a \|_2 + \| a \|_2 \| I - a^T \|_2 \lesssim \| I - a \|_2 (1 + \| I - a \|_2), \]
2.2 Stokes-type estimates

We note that if \((2.5)\) holds for a sufficiently small \(\gamma > 0\), then the second and third terms on the left-hand side of the equation \((1.6)\) for \(v\) become approximately \(-\Delta v\) and \(\partial_t q\). A perturbation argument then allows one to obtain the following Stokes-type estimate,

\[
\|v\|_3 + \|\nabla q\|_1 \lesssim \|v_t\|_1 + \|v\|_{H^{1/2}(\partial \Omega)} \lesssim \|v_t\|_1 + \|\partial'' v\|_1 + \|v\|. \tag{2.7}
\]

Also, we have another Stokes estimate that avoids \(\|\partial'' v\|_1\) on the right-hand side,

\[
\|v\|_3 + \|\nabla q\|_1 \lesssim \|v_t\|_1 + \left\| \frac{\partial v}{\partial N} \right\|_{H^{1/2}(\Gamma_\gamma)} \lesssim \|v_t\|_1 + \|w_{tt}\|_1 + \|w_t\|_1 + \|w\|_1 + \|\partial'' w\|_1 \tag{2.8}
\]
as well as a Stokes-type estimate for \(v_t\),

\[
\|v_t\|_2 + \|\nabla q_t\| \lesssim \|v_{tt}\| + \left\| \frac{\partial v_t}{\partial N} \right\|_{H^{1/2}(\Gamma_\gamma)} + \|v\|_3(\|v_t\|_2 + \|q\|_1) \lesssim (1 + \|v\|_3(\|v_t\|_1 + \|w_{tt}\| + \|w_t\| + \|w\|_1) + \|\partial'' w\|_1 + \|\partial'' w\|_1) \tag{2.9}
\]
at each time \(t > 0\) such that \((2.5)\) holds with a sufficiently small \(\gamma\). We note that the boundary regularity of \(v_t\) has been transferred onto \(w\) in the first inequalities in \((2.8), (2.9)\) above, and, in the second inequalities, respectively, we used a trace estimate as well as \((1.1)\) to write

\[
\|w\|_3 \lesssim \|w_{tt}\|_1 + \|w_t\|_1 + \|\partial'' w\|_1 + \|w\|, \tag{2.10}
\]

The Stokes estimates \((2.7), (2.9)\) are well-known, and we refer the reader to (3.10)–(3.16) in \([IKLT3]\) for a detailed proof of \((2.8), (2.9)\), while \((2.7)\) follows by an easier estimate using standard Stokes estimate and the trace theorem.

It is important that \(\|q\|\) is estimated by employing the boundary condition \((1.9)\). Thus we have

\[
\|q\|_{L^2(\Gamma_\gamma)} \leq \|(I - a)q\|_{L^2(\Gamma_\gamma)} + \|a_3 q\|_{L^2(\Gamma_\gamma)} \leq \gamma \|q\|_{L^2(\Gamma_\gamma)} + \|\partial_3 w_3\|_{L^2(\Gamma_\gamma)} + \|a_3 a_3 k \partial k v_3\|_{L^2(\Gamma_\gamma)} \leq \gamma \|q\|_{L^2(\Gamma_\gamma)} + C \|w\|_2 + C(1 + \gamma) \|v\|_2,
\]

where we used \((1.9)\) and \((2.5)\) in the second line. Hence, for \(\gamma < 1/2\),

\[
\|q\| \lesssim \|\nabla q\| + \|q\|_{L^2(\Gamma_\gamma)} \lesssim \|\nabla q\| + \|w\|_2 + \|v\|_2. \tag{2.11}
\]
In a similar way we can control \( q_t \). Indeed taking \( \partial_t \) of (1.9) gives
\[
\partial_t (a_{33}q) = \partial_t (a_{31}a_{kl}\partial_k v_3) - \partial_t \partial_3 w_3
\]
on \( \Gamma_c \), and so
\[
\|q_t\|_{L^2(\Gamma_c)} \leq \| (I - a)q_t \|_{L^2(\Gamma_c)} + \|a_t q\|_{L^2(\Gamma_c)} + \|\partial_t (a_{31}a_{kl}\partial_k v_3)\|_{L^2(\Gamma_c)} + \|\partial_3 w_3\|_{L^2(\Gamma_c)}
\]
\[
\leq \gamma \|q_t\|_{L^2(\Gamma_c)} + C \left( \|a_t\|_1 \|q\|_1 + \|a_t\|_2 \|a\|_2 \|v\|_2 + \|a\|_2^2 \|v_t\|_1 + \|w_t\|_2 \right).
\]
Thus, by absorbing the first term on the right-hand side, we obtain
\[
\|q_t\| \lesssim \|\nabla q_t\| + \|a_t\|_2 \|q\|_1 + \|a_t\|_2 \|a\|_2 \|v\|_2 + \|a\|_2^2 \|v_t\|_1 + \|w_t\|_2.
\] (2.12)

We emphasize that the implicit constants in (2.7)–(2.12) are independent of \( \gamma \), but they hold only when \( \gamma \) is sufficiently small. Note that the norms involving \( w \) are taken over \( \Omega_e \), while the norms of \( v, q, \) and \( a \) are taken over \( \Omega_f \). We continue with this convention throughout.

## 3 Proof of the main result

Here we prove our main result, Theorem 1. We shall consider
\[
S \in \{ \text{id}, \partial_t, \partial' \partial_t, \partial'', \partial_{tt} \},
\]
and for each \( S \) we derive an energy estimate in Section 3.2. In Section 3.3 we then use such individual energy estimates to define the total energy of the system and estimate it in the form of an ODE-type inequality. Then, using the ODE-type Lemma 3, which we prove in Section 3.4, we obtain the required a priori bound for the total energy.

### 3.1 Extension of \( w \) into the fluid domain \( \Omega_f \)

Here, for a fixed \( \tau \geq 0 \), we discuss the notion of extension \( \tilde{Sw} \) of \( Sw \) to the fluid domain \( \Omega_f \) for \( S \in \{ \text{id}, \partial'' \} \) such that
\[
\tilde{Sw}(\tau) \mid_{\Gamma_c} = Sw(\tau) \mid_{\Gamma_c}
\]
\[
\tilde{Sw}(\tau) \mid_{\Gamma_f} = 0.
\] (3.1)

As discussed in the introduction, this turns out to be an important concept enabling us to couple the dynamics of the fluid with the dynamics of the elastic body.

For \( S = \partial'' \) we choose any extension \( \tilde{Sw} \) satisfying (3.1) such that
\[
\|\tilde{Sw}(\tau)\|_1 \lesssim \|Sw(\tau)\|_1.
\] (3.2)

Such \( \tilde{Sw} \) exists due to the Sobolev extension and trace theorems.

The case \( S = \text{id} \) turns out to be more difficult, as in this case we need to require that \( \tilde{w} \) is “almost divergence-free”, namely that
\[
\|\text{div} \ \tilde{w}\| \lesssim \int_0^T \|I - a\|_2 \|v\|_1.
\] (3.3)
In fact, according to (2.3), the right-hand side of (3.3) is at least quadratic in \( v \), which shall be crucial for us below in (3.28). We note that (3.3) is related to the fact that our elasticity model (1.1) does not include the zeroth order term \( w \), which was included in previous works [IKLT2, IKLT3], as discussed in the introduction.

In order to define \( \tilde{w} \), we first note that

\[
\tilde{w}_i(\tau) = \int_0^\tau \partial_t w_i = \int_0^\tau v_i = \int_0^\tau (\delta_{ik} - a_{ik}) v_k + \int_0^\tau a_{ik} v_k =: \tilde{w}_i(\tau) + \bar{w}_i(\tau)
\]

on \( \Gamma_c \), where we used the fact that \( w = 0 \) at time 0 in the first equality and (1.8) in the second. We now extend both \( \tilde{w} \) and \( \bar{w} \) to functions defined in \( \Omega_f \) such that

\[
\| \tilde{w} \|_1 \lesssim \| \tilde{w} \|_{H^{1/2}(\Gamma_c)} \lesssim \| \tilde{w} \|_{H^{1/2}(\Gamma_c)} \lesssim \| \tilde{w} \|_{H^{1/2}(\Gamma_c)} \lesssim \int_0^\tau \| a \|_2 \| v \|_1.
\]

The existence of such extensions \( \tilde{w}, \bar{w} \) follows from the Sobolev extension and trace theorems. Moreover, by linearity of the extension operator we also have that

\[
\| \tilde{w} + \bar{w} \|_1 \lesssim \| w(\tau) \|_{H^1(\Gamma_c)} \lesssim \| w(\tau) \|_1.
\]

We now let \( b \in H^1_0(\Omega_f) \) be a solution of the divergence problem

\[
\text{div } b = \text{div } \bar{w} \quad \text{in } \Omega_f,
\]

\[
\| b \|_1 \lesssim \| \text{div } \bar{w} \|,
\]

where \( H^1_0(\Omega_f) \) is the closure of the space of smooth vector fields that are periodic in \( x_1, x_2 \) and have compact support in \( x_3 \in [0, 1/2] \). Such \( b \) can be constructed using the standard Bogovski˘ı lemma (see [Bo1, Bo2] or (III.3.8) in [Ga]), since

\[
\int_{\Omega_f} \text{div } \bar{w} = \int_{\Gamma_c} \bar{w}_3 = \int_{\Omega_f} \int_0^\tau a_{3k} v_k = \int_0^\tau \int_{\Omega_f} \partial_j(a_{jk} v_k) = 0,
\]

due to the Fubini theorem and the divergence-free condition (1.7). Letting

\[
\tilde{w}(\tau) := \tilde{w} + \bar{w} - b
\]

we obtain an extension \( \tilde{w} \) that satisfies (3.2) up to an error that is quadratic in \( v \), i.e.,

\[
\| \tilde{w}(\tau) \|_1 \lesssim \| \tilde{w} + \bar{w} \|_1 + \| b \|_1 \lesssim \| \tilde{w} \|_1 + \| \bar{w} \|_1 + \| w(\tau) \|_1 + \int_0^\tau \| I - a \|_2 \| v \|_1,
\]

where we applied the triangle inequality in \( \| b \|_1 \lesssim \| \bar{w} \|_1 \lesssim \| \tilde{w} + \bar{w} \|_1 + \| \bar{w} \|_1 \) and (3.4)–(3.5). Moreover \( \| \text{div } \tilde{w} \| = \| \text{div } \tilde{w} \| \lesssim \int_0^\tau \| I - a \|_2 \| v \|_1 \), which gives (3.3), as required.
3.2 Energy estimates

Here we derive an energy inequality for each \( S \in \{ \text{id}, \partial_t, \partial^\prime \partial_t, \partial^\prime \partial^\prime_t, \partial_t^\prime \} \).

Applying \( S \) to (1.11) and testing against \( Sw \) gives
\[
\frac{d}{dt} \int_{\Omega_t} \partial_t Sw_iSw_i - \int_{\Omega_t} \partial_t Sw_i\partial_t Sw_i + \int_{\Omega_t} \partial_j Sw_i\partial_j Sw_i + \frac{\alpha}{2} \frac{d}{dt} \int_{\Omega_t} |Sw|^2 - \int_{\Gamma_c} \partial_j Sw_iSw_iN_jd\sigma(x) = 0,
\]
or, equivalently,
\[
\frac{d}{dt} \left( \frac{\alpha}{2} \|Sw\|^2 + \int_{\Omega_t} \partial_t Sw_iSw_i \right) + \|\nabla Sw\|^2 = \|Sw_t\|^2 + \int_{\Gamma_c} \partial_j Sw_iSw_iN_jd\sigma(x). \tag{3.7}
\]

For a fixed \( \tau > 0 \) and for \( t \geq \tau \), we consider a test function
\[
\phi(t) := Sw(t) - S\eta(t) + \tilde{Sw}(\tau) \quad \text{on } \Omega_t. \tag{3.8}
\]

We note that the last two terms in (3.8) appear only in the cases where \( S \) does not involve time derivative, that is only for \( S \in \{ \text{id}, \partial^\prime \} \). Note that
\[
\phi(t) \big|_{\Gamma_c} = Sw(t) \big|_{\Gamma_c} = \tilde{Sw}(t)|_{\Gamma_c}, \tag{3.9}
\]
where the first equality follows from \( w_t = \eta = \eta_t \) on \( \Gamma_c \) (recall (1.8)) and then integrating in \( t \). Applying \( S \) to the velocity equation (1.6) and testing it with \( \phi \) gives
\[
\int_{\Omega_t} S\partial_t v_i\phi_i - \int_{\Gamma_c} \partial_j S(\alpha_ja_{ki}\partial_k v_i)\phi_i + \int_{\Omega_t} \partial_k S(a_{ki}q)\phi_i = 0,
\]
from where
\[
\frac{d}{dt} \int_{\Omega_t} Sv_i\phi_i - \int_{\Omega_t} |Sv|^2 + \int_{\Omega_t} S(a_ja_{ki}\partial_k v_i)\partial_j\phi_i + \int_{\Gamma_c} S(a_{ki}q)\partial_k \phi_i - \int_{\Gamma_c} S(a_{ki}q)\phi_iN_kd\sigma(x) = 0. \tag{3.10}
\]

We rewrite the third term as
\[
\int_{\Omega_t} S(a_ja_{ki}\partial_k v_i)\partial_j\phi_i = \int_{\Omega_t} \nabla Sv : \nabla \phi + \int_{\Omega_t} S((\alpha_ja_{ki} - \delta_{jk})\partial_k v_i)\partial_j\phi_i
\]
\[
= \int_{\Omega_t} \nabla S\partial_t(\eta - \eta(\tau)) : \nabla S(\eta - \eta(\tau)) + \int_{\Omega_t} \nabla S\partial_t(\eta - \eta(\tau)) : \nabla S\tilde{w}(\tau)
\]
\[
+ \int_{\Omega_t} S((\alpha_ja_{ki} - \delta_{jk})\partial_k v_i)\partial_j\phi_i
\]
\[
= \frac{1}{2} \frac{d}{dt} \|\nabla S(\eta - \eta(\tau))\|^2 + \int_{\Omega_t} \nabla S v : \nabla S\tilde{w}(\tau) + \int_{\Omega_t} S((\alpha_ja_{ki} - \delta_{jk})\partial_k v_i)\partial_j\phi_i, \tag{3.11}
\]
where \( A : B = A_{ij}B_{ij} \), and the fifth term in (3.10) as
\[
- \int_{\Omega_t} S(a_{ki}q)\partial_k \phi_i dx = - \int_{\Omega_t} S\eta \text{div } \phi + \int_{\Omega_t} S((\delta_{ki} - a_{ki})q)\partial_k \phi_i. \tag{3.12}
\]
Applying (3.11) and (3.12) in (3.10) and adding the resulting equation to (3.7), we observe that the integrals over $\Gamma$ cancel due to (3.9) and the boundary condition (1.9). We obtain
\[
\frac{d}{dt} \left( \frac{\alpha}{2} \|Sw\|_t^2 + \int_{\Omega_t} \partial_t S w_t S w_{\tau} + \frac{1}{2} \|\nabla S(\eta - \eta(\tau))\|_t^2 + \int_{\Omega_t} S v_t \phi_i \right) + \|\nabla S w\|_t^2 \\
= \|S w_t\|_t^2 + \|S v\|_t^2 - \int_{\Omega_t} \nabla S : \nabla S w(\tau) + \int_{\Omega_t} S((\delta_{jk} - a_{ji} a_{kl}) \partial_k v_i) \partial_j \phi_i \\
+ \int_{\Omega_t} S q \text{div} \phi - \int_{\Omega_t} S((\delta_{ki} - a_{ki}) q) \partial_k \phi_i.
\] (3.13)

Next, we derive the energy inequality for the differentiated system. We apply $S$ to (1.6) to get
\[S \partial_t v_i - \partial_j(a_{ji} a_{kl} \partial_k S v_i) + a_{ki} \partial_k q = (S \partial_j(a_{ji} a_{kl} \partial_k v_i) - \partial_j(a_{ji} a_{kl} \partial_k S v_i)) - (S(a_{ki} \partial_k q) - a_{ki} \partial_k S q) \] in $\Omega_t \times (0, T)$, \quad $i = 1, 2, 3$,

and we test it with $S v_i$. We also apply $S$ to (1.1) and test it with $\partial_t S w_i$. Summing the resulting equations we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|S v\|_t^2 + \|S w_t\|_t^2 + \|S \nabla w\|_t^2 \right) + \|\nabla S v\|_t^2 \leq \int_{\Omega_t} (S \partial_j(a_{ji} a_{kl} \partial_k v_i) - \partial_j(a_{ji} a_{kl} \partial_k S v_i)) S v_i - \int_{\Omega_t} (S(a_{ki} \partial_k q) - a_{ki} \partial_k S q) S v_i \]
\[+ \int_{\Omega_t} (\delta_{jk} - a_{ji} a_{kl}) \partial_k S v_i \partial_j S v_i + \int_{\Omega_t} S q \partial_k(a_{ki} S v_i) a_{ki} \partial_k S v_i .
\] (3.14)

Now, we multiply (3.13) with $\lambda$, where $0 < \lambda \leq 1/C$, and add to (3.14) in order to obtain
\[
\frac{d}{dt} E_S(t, \tau) + D_S(t, \tau) \leq L_S(t, \tau) + N_S(t, \tau) + C_S(t, \tau).
\] (3.15)

where the pointwise energy $E_S$, the dissipation energy $D_S$, the linear part $L_S$, the nonlinear part $N_S$, and the commutator part $C_S$ are defined by
\[
E_S(t, \tau) := \frac{1}{2} \|S v(t)\|_t^2 + \frac{1}{2} \|S w_t(t)\|_t^2 + \frac{1}{2} \|\nabla S w(t)\|_t^2 + \frac{\lambda \alpha}{2} \|S w\|_t^2 + \frac{\lambda}{2} \|\nabla S(\eta - \eta(\tau))\|_t^2 \\
+ \lambda \int_{\Omega_t} \partial_t S w_t S w_i + \lambda \int_{\Omega_t} S v_i S \phi_i,
\]
\[
D_S(t, \tau) := \frac{1}{2} \|\nabla S v\|_t^2 + \frac{1}{C} \|S v\|_t^2 + (\alpha - \lambda) \|S w_t\|_t^2 + \frac{\lambda}{2} \|\nabla S w\|_t^2 + \frac{\lambda}{C} \|S w\|_t^2,
\]
\[
L_S(t, \tau) := -\lambda \int_{\Omega_t} \nabla S v : \nabla S \tilde{w}(\tau) + \lambda \int_{\Omega_t} S q \text{div} \phi =: L_{S, 1}(t, \tau) + L_{S, 2}(t, \tau),
\]
\[
N_S(t, \tau) := \lambda \int_{\Omega_t} S((\delta_{jk} - a_{ji} a_{kl}) \partial_k v_i) \partial_j \phi_i - \lambda \int_{\Omega_t} S((\delta_{ki} - a_{ki}) q) \partial_k \phi_i + \int_{\Omega_t} (\delta_{jk} - a_{ji} a_{kl}) \partial_k S v_i \partial_j S v_i,
\]
\[
C_S(t, \tau) := \int_{\Omega_t} (S \partial_j(a_{ji} a_{kl} \partial_k v_i) - \partial_j(a_{ji} a_{kl} \partial_k S v_i)) S v_i - \int_{\Omega_t} (S(a_{ki} \partial_k q) - a_{ki} \partial_k S q) S v_i \\
+ \int_{\Omega_t} S q (a_{ki} \partial_k S v_i - S(a_{ki} \partial_k v_i)),
\] (3.16)
where \( \phi = S\eta - S\eta(\tau) + \tilde{S}w(\tau) \) (recall \(3.8\)), \( C > 1 \) is a constant, and we used the Poincaré inequality \( \| Sv \|^2 \leq C \| \nabla Sv \|^2 /2 \), and similarly for \( Sw \), to include the terms \( \| Sv \|^2 \) and \( \| Sw \|^2 \) in \( D_S \).

By applying Young’s inequality \( \lambda ab \leq a^2 /2 + \lambda^2 b^2 /2 \) and the Poincaré inequality \( \| S(\eta - \eta(\tau)) \| \leq \| \nabla S(\eta - \eta(\tau)) \| \) to the last three terms of \( E_S \), we see that \( E_S \) may be bounded from above and below as

\[
E_S(t, \tau) \sim \| Sv(t) \|^2 + \| Sw(t) \|^2 + \| \nabla Sw(t) \|^2 + \lambda \alpha \| Sw \|^2 + \lambda \| \nabla S(\eta - \eta(\tau)) \|^2, \tag{3.17}
\]

where \( a \sim b \) means \( a \lesssim b \) and \( a \gtrsim b \).

### 3.3 The total energy of the system

In this section we combine the energy estimates \([3.15]\) from the previous section to define the total energy \( X(t) \) of the system \([1.1] - [1.10]\) and derive an a priori estimate for it.

We choose \( \gamma \in (0, 1] \) sufficiently small so that the Stokes estimates \([2.7]\) and \([2.9]\) hold, and that

\[
\gamma \leq \frac{1}{2C}, \tag{3.18}
\]

where \( C \) is the constant from the definition \([3.16]\) of \( D_S \).

Before defining \( X \), we note that we aim to obtain an ODE-type estimate of the form

\[
X(t) + \int_\tau^t X \leq C \left( 1 + \lambda(t - \tau) \right) X(\tau) + C\lambda(t - \tau)^2 \int_\tau^t X + \text{(small terms)}, \tag{3.19}
\]

for all \( \tau \geq 0 \) and \( t \geq \tau \), where the small terms are at least cubic in \( X^{3/2} \). This way we can obtain an exponential decay of \( X(t) \) for a sufficiently small \( X(0) \); see Lemma \([3]\) below for details. The question therefore becomes which terms appearing in the energy estimates \([3.15]\), where \( S \in \{ \text{id}, \partial_t, \partial'\partial_t, \partial''\partial_t \} \), should be included in the total energy \( X(t) \).

To this end we observe that \([3.19]\) requires that, given \( \tau \geq 0 \), we must be able to control each such term for every \( t \geq \tau \) as well as its integral from \( \tau \) to \( t \). This can be ensured by choosing only terms that appear in both \( E_S \) and \( D_S \), for some \( S \), i.e., we take

\[
X(t) := \sum_{S \in \{ \text{id}, \partial_t, \partial'\partial_t, \partial''\partial_t \}} \left( \| Sv(t) \|^2 + \| Sw(t) \|^2 + \| \nabla Sw(t) \|^2 + \| Sw(t) \|^2 \right). \tag{3.20}
\]

With this definition of \( X \), the Stokes estimates \([2.7]\)–\([2.10]\) and the 0-th order pressure estimates \([2.11]\)–\([2.12]\) give

\[
\| v \|_3, \| q \|_2, \| v_t \|_2, \| q_t \|_1, \| w \|_3, \| w_t \|_2 \lesssim X^{3/2} \left( 1 + X^{3/2} \right). \tag{3.21}
\]

We note that this is the only place in the paper where we make use of \( S = \partial'\partial_t \), which is essential to estimate \( \| \partial'w_t \|_1 \) (in \([2.9]\)) by \( X^{1/2} \).

We now note that some terms appear in \( E_S \), but not in \( D_S \), and vice versa. For example, \( \| \nabla S(\eta - \eta(\tau)) \| \) appears in \( E_S \) (recall \([3.17]\)), but we shall not use this term; we simply omit it after integration of \([3.15]\) from \( \tau \) to \( t \). On the other hand \( \| \nabla Sv \|^2 \), appearing in \( D_S \), is one of the main dissipation terms,
3.3 The total energy of the system

and we make full use of it in estimating some linear parts $L_S$, as well as $N_{\partial_t}$. In fact, we show in Step 1 below that

$$L_{\partial''} + L_{id} \lesssim \delta (D_{\partial''} + D_{\partial_t} + D_{\partial'} + D_{id}) + C_5 \lambda^2 \left( X(\tau) + (t - \tau) \int_{\tau}^{t} X + O(X^{\frac{3}{2}}) \right)$$

(3.22)

and in Step 2 below that

$$\int_{\tau}^{t} (N_{\partial_t} + L_{\partial_t}) \lesssim (\delta + \gamma) \int_{\tau}^{t} \| \nabla_{\partial_t} v \|^2 + C_5 O(X^{\frac{3}{2}}),$$

(3.23)

for all $\delta > 0$, where

$$O(X^{\frac{3}{2}})$$

denotes any finite sum of products of any power of $(t - \tau)$

with at least $k$ factors of the form $X(t)^{\frac{3}{2}}, X(\tau)^{\frac{3}{2}}$ or $(\int_{0}^{t} X^{\frac{3}{2}})^{\frac{1}{3}},$

(3.24)

for $k \geq 0$.

Furthermore we show in Step 3 below that all the other terms appearing on the right hand sides of (3.15) for $S \in \{id, \partial_t, \partial' \partial_t, \partial'' \}$ are cubic in $X^{\frac{3}{2}}$, namely that

$$\int_{\tau}^{t} \left( L_{\partial_t} + L_{\partial} + \sum_{S \in \{id, \partial_t, \partial' \partial_t, \partial'' \}} N_S + \sum_{S \in \{id, \partial_t, \partial' \partial_t, \partial'' \}} C_S \right) \lesssim O(X^{\frac{3}{2}}).$$

(3.25)

Thus, integrating each of the energy estimates (3.15) over the time interval $(\tau, t)$, summing over $S \in \{id, \partial_t, \partial' \partial_t, \partial'' \}$ and using (3.18) together with a choice of sufficiently small $\delta$ to absorb the first terms on the right-hand side of (3.22) and of (3.23), respectively, we obtain

$$X(t) + \int_{\tau}^{t} X \leq C \left( 1 + \lambda (t - \tau) \right) X(\tau) + C \lambda (t - \tau)^2 \int_{\tau}^{t} X + \frac{C}{\lambda} O(X^{\frac{3}{2}}),$$

(3.26)

for all $\lambda \in (0, 1], \tau \geq 0$, and $t \geq \tau$.

Note that the factor $\lambda^{-1}$ in the last term comes from the fact that the terms $\| \nabla S w \|^2$ and $\| S w \|^2$, which are parts of $X$, are included in $D_S$ with the weight $\lambda$. We also recall (2.5), which reminds us that (3.26) is only valid for $t$ such that

$$C \left( \| v \|_{L^1(0,t;H^3)} + \| v \|_{L^\infty(0,t;H^3)} \right) \left( 1 + \| v \|^3_{L^1(0,t;H^3)} \right) \leq \gamma,$$

where $\gamma \in (0, 1]$ was fixed in (3.18) above. This is not a problem, since we can use (3.26) to show that $X(t)$ remains small and decays exponentially if $X(0)$ is sufficiently small. Indeed, the ODE-type result of Lemma 3 (see Section 3.4) shows that after choosing $\lambda$ sufficiently small we can find $\varepsilon > 0$ such that $X(0) \leq \varepsilon$ implies that $X(t) \leq 30 C \varepsilon e^{-t/2 C}$ for all $t > 0$, where $C \geq 1$ is from (3.26) above. This concludes the proof of Theorem 1.

It remains to prove (3.22), (3.23), and (3.25).

**Step 1.** We prove (3.22).
We first comment that in this step we need to control \( \|v\|_3 + \|\nabla q\|_1 \) by a sum of all dissipation terms \( D_S \). In fact, this can be achieved trivially using (3.21), since all terms included in the definition (3.20) are also included in the dissipation terms. Such approach, however, would result in an additional factor of \( \lambda^{-1} \), since terms \( \|Sw\|^2 \) and \( \|\nabla Sw\|^2 \) appear with coefficient \( \lambda \) in the definition (3.16) of \( D_S \).

Thus such control would be insufficient, as it would consequently give an ODE-type estimate on \( X \) of the form of (3.19), but without one of the \( \lambda \)'s on the right-hand side, and so would make it impossible to apply Lemma 3 to conclude the proof.

Instead, we control \( \|v\|_3 + \|\nabla q\|_1 \) using the Stokes estimate (2.7) directly by proving

\[
\|v\|_3 + \|\nabla q\|_1 \lesssim (D_{\theta^\nu} + D_{\partial_\mu} + D_{\partial_\nu} + D_{id}) \frac{1}{2}.
\]

We note that this is the only place in the paper where we use the operator \( S = \partial_t \) which is essential to bound \( \|v_t\|_1 \) (from (2.7)) by \( D_{\partial_t}^{\frac{1}{2}} \) without employing (2.9).

We can now prove the claim. For \( S = \partial^\nu \) we have that, for each time from \((\tau, t)\),

\[
L_{S,1} \lesssim \lambda \|\nabla Sv\| \|w(\tau)\|_3 \lesssim \delta D_S + C_\delta \lambda^2 \|w(\tau)\|_3^2 \lesssim \delta D_S + C_\delta \lambda^2 X(\tau),
\]

\[
L_{S,2} \lesssim \lambda \|Sq\| (\|\nabla S(\eta - \eta(\tau))\| + \|w(\tau)\|_3) \lesssim \delta \|Sq\|^2 + C_\delta \lambda^2 (\|\nabla S(\eta - \eta(\tau))\|^2 + \|w(\tau)\|_3^2)
\]

\[
\lesssim \delta (D_{\theta^\nu} + D_{\partial_\mu} + D_{\partial_\nu} + D_{id}) + C_\delta \lambda^2 \left( \left( \int_\Omega X^\frac{1}{2} \right)^2 + X(\tau) + O(X^\frac{3}{2}) \right),
\]

where we used (3.2) to estimate \( \|\nabla \tilde{w}(\tau)\| \lesssim \|w(\tau)\|_3 \), and (3.27) to obtain the terms with \( \delta \)'s, as well as the fact \( \|\nabla S(\eta - \eta(\tau))\| \lesssim \int_\tau^t \|\nabla Sv\| \lesssim \int_\tau^t X^\frac{1}{2} \) and (3.2), (3.21) to obtain the terms involving \( X \).

As for \( S = id \),

\[
L_{id} \lesssim \lambda \int_\Omega |\nabla v| |\nabla \tilde{w}(\tau)| + \lambda \int_\Omega |\nabla \eta| |\nabla \tilde{w}(\tau)| + \int_\Omega |\nabla \eta| |\nabla \tilde{w}(\tau)|
\]

\[
\lesssim \lambda \int_\Omega |\nabla v| |\nabla \tilde{w}(\tau)| + \lambda \int_\Omega |\nabla \tilde{w}(\tau)| + \lambda \int_\Omega |\nabla \tilde{w}(\tau)|
\]

\[
\lesssim \delta \|\nabla v\|^2 + C_\delta \lambda^2 |\tilde{w}(\tau)|_2^2 + \lambda \|q\| \int_\tau^t \|I - a\|_2 \|v\|_1
\]

\[
\lesssim \delta \|\nabla v\|^2 + C_\delta \lambda^2 \left( X(\tau) + \lambda^{-1} O(X^\frac{3}{2}) \right),
\]

where we used (1.4) and the divergence-free condition (1.7) in the second line, Young’s inequality and (3.3) in the third line, as well as (3.6), (3.21), and (2.3) in the last line.

We note that this is the only place in the paper where we use the construction of “almost-divergence” free extension \( \tilde{w} \) from Section 3.1. In fact, without (3.3) one could estimate \( \|\nabla \tilde{w}\| \) only using an inequality \( \|\tilde{w}\|_1 \leq \|w\|_1 \) (as in (3.2)), which would give us a term of the form \( \lambda \|q\| \|w\|_1 \). The problem with this, except for the resulting term being merely linear in \( X \), is that it cannot be absorbed using the dissipation term \( D_{id} \) as both \( \|\nabla Sw\|^2 \) and \( \|\nabla w\|^2 \) appear in the definition of \( D_S \) with the weight \( \lambda \), which would be fatal for the same reason as pointed out in the beginning of this step.
Step 2. We prove (3.23).

For $N_{\partial_t}$ we first note that applying the bound (3.21) in (2.1), (2.2), (2.3) and (2.4) gives

$$\|\nabla \eta\|_2, \|a\|_2 \lesssim O(1),$$

$$\|I - a\|_2, \|I - aa^T\|_2, \|a_t\|_2 \lesssim O(X^{\frac{3}{2}}).$$

Thus

$$\|\partial_t((I - a)q)\|_1 \lesssim \|a_t\|_2 \|q\|_2 + \|I - a\|_2 \|a_t\|_1 \lesssim O(X)$$

due to (3.21). Hence,

$$\int_\tau^t N_{\partial_t} = \lambda \int_\tau^t \int_{\Omega_t} \partial_t((\delta_{jk} - a_j a_k)\partial_t v)\partial_j v - \lambda \int_\tau^t \int_{\Omega_t} \partial_t((\delta_{ki} - a_{ki})q)\partial_k \partial_t v_i$$

$$+ \int_\tau^t \int_{\Omega_t} (\delta_{jk} - a_j a_k)\partial_t \partial_j \partial_t v_i$$

$$\leq \lambda \int_\tau^t \|\nabla \partial_t v\| \left(\|\partial_t((I - aa^T)\nabla v)\| + \|\partial_t((I - a)q)\|\right) + \int_\tau^t \|I - aa^T\|_{L^\infty} \|\nabla \partial_t v\|^2$$

$$\lesssim O(X^{\frac{3}{2}})$$

$$\leq (\delta + \gamma) \int_\tau^t \|\nabla \partial_t v\|^2 + C_0 O \left(X^{\frac{3}{2}}\right),$$

where, in the first inequality, we have integrated by parts in time in the first two terms.

For $L_{\partial_t}$, we first note that

$$\|\partial_t((I - a)\nabla v)\|_1 \lesssim \|a_t\|_2 \|v\|_3 + \|I - a\|_2 \|v_t\|_2 \lesssim O(X)$$

$$\|\partial_t((I - a)\nabla v)\| \lesssim \|I - a\|_2 \|\nabla v_t\| + \|a_t\|_2 \|v_t\|_1 + \|a_{tt}\| \|v_t\|_3$$

$$\lesssim \|\nabla v_t\| O \left(X^{\frac{3}{2}}\right) + O(X),$$

as in (3.30), where we used (2.2) to estimate $\|a_{tt}\| \lesssim O(X^{\frac{3}{2}})$ and (3.29). Thus

$$\int_\tau^t L_{\partial_t} = \lambda \int_\tau^t \int_{\Omega_t} q_t \partial_t \eta \partial_t v = \lambda \int_\tau^t \int_{\Omega_t} q_t \partial_t(\delta_{jk} \partial_j v_k)$$

$$= -\lambda \int_\tau^t \int_{\Omega_t} q_t \partial_t(\delta_{jk} - a_{jk}) \partial_j v_k + \lambda \int_\tau^t \int_{\Omega_t} q_t \partial_t((\delta_{jk} - a_{jk}) \partial_j v_k)$$

$$\lesssim \int_\tau^t \|\nabla v_t\| O(X) + O \left(X^{\frac{3}{2}}\right) \lesssim \delta \int_\tau^t \|\nabla v_t\|^2 + C_0 O \left(X^{\frac{3}{2}}\right).$$

Step 3. We prove (3.25).

For $L_{\partial^\nu \partial_t}$ we use (3.21) and (3.29) to obtain

$$L_{\partial^\nu \partial_t} = \lambda \int_{\Omega_t} \partial^\nu q_t \partial^\nu v = \lambda \int_{\Omega_t} \partial^\nu q_t \partial^\nu(\delta_{jk} \partial_j v_k) = \lambda \int_{\Omega_t} \partial^\nu q_t \partial^\nu((\delta_{jk} - a_{jk}) \partial_j v_k)$$

$$\lesssim \|q_t\| \|I - a\|_2 \|v\|_2 \lesssim O(X^{\frac{3}{2}}).$$
The case of \( L_\partial \) gives \( O(X^{3/2}) \) in a similar way.

For the nonlinear terms, we have that

\[
N_S = \lambda \frac{\int_{\Omega_t} S(\delta_{jk} - a_j a_k) \partial_k \partial_t v_i \partial_j \phi - \lambda \frac{\int_{\Omega_t} S(\delta_{ki} - a_k) q \partial_k S \phi + \int_{\Omega_t} (\delta_{jk} - a_j a_k) \partial_k S v_i \partial_j S v_i}{\|I - a a^T\|_2 \|v\|_3 \|\phi\|_3 + \|I - a\|_2 \|q\|_3 + \|I - a a^T\|_{L^\infty} \|v\|_3^2 \leq O\left(X^2\right)}}{
\text{for } S \in \{\partial^\nu, \text{id}\}, \text{ where we used } (3.21) \text{ and } (3.29).}
\]

Moreover,

\[
N_{\partial^\nu \partial_t} = \frac{\int_{\Omega_t} \partial^\nu \partial_t \left(\delta_{jk} - a_j a_k\right) \partial_k \partial_t v_i \partial_j \phi - \lambda \frac{\int_{\Omega_t} \partial^\nu \partial_t \left(\delta_{ki} - a_k\right) q \partial_k \partial_t v_i + \int_{\Omega_t} \left(\delta_{jk} - a_j a_k\right) \partial_k \partial_t v_i \partial_j \phi}{\|\partial_t \left(I - a a^T\right) \nabla v\|_2 \|v\|_2 + \|\partial_t \left((I - a) q\right)\|_1 \|v\|_2 + \|I - a a^T\|_2 \|v\|_2 \leq O\left(X^{3/2}\right)}}{
\text{due to } (3.21), (3.29), \text{ and } (3.30).}
\]

In a similar way we show that \( N_{\partial_t} \leq O\left(X^{3/2}\right) \).

For the commutator terms \( C_S \) we note that \( C_{\text{id}} = 0 \), and that all terms involved in \( C_{\partial^\nu \partial_t}, C_{\partial^\nu \nu}, C_{\partial_{\partial_t}} \)

are of order \( O\left(X^{3/2}\right) \). Indeed, each of such terms involve a derivative of \( a \), which is of order \( O\left(X^{3/2}\right) \) due to \( (3.29) \), as well as two factors involving \( v \) or \( q \) that are of order \( O\left(X^{3/2}\right) \) each, due to \( (3.21) \). Altogether we obtain \( O\left(X^{3/2}\right) \), as required.

### 3.4 The ODE-type Lemma

In this section we prove the ODE-type result which is used to obtain global bounds from \( (3.26) \). In what follows “\( C \)” denotes a constant that does not change its value from line to line.

**Lemma 3** (An ODE-type lemma). Given \( C \geq 1, \gamma \in (0, 1], \text{ and } \lambda \in (0, 1/800C^4] \) there exists \( \varepsilon > 0 \) with the following property. Suppose that \( f: [0, \infty) \to [0, \infty) \) is a continuous function satisfying

\[
f(t) + \int_{\tau}^{t} f \leq C(1 + \lambda(t - \tau))f(\tau) + C\lambda(t - \tau)^2 \int_{\tau}^{t} f + \frac{C}{\lambda} O(f^{3/2})
\]

for all times \( t > 0 \) such that

\[
f(t) + \int_{0}^{t} f \leq \gamma,
\]

and all \( \tau \in [0, t] \). Then the condition \( f(0) \leq \varepsilon \) implies that

\[
f(t) \leq A\varepsilon e^{-t/2C}
\]

for all \( t \geq 0 \), where \( A := 30C \).

As in \( (3.24) \), we denote by \( O\left(f^{3/2}\right) \) any finite sum of products of any power of \( (t - \tau) \) and at least 3 factors of the form \( f(t), f(\tau)^{1/2}, \text{ or } (\int_{0}^{t} f)^{1/3} \).

**Proof.** First we note that, for some \( \varepsilon > 0 \), the claim is true for \( t \in [0, T] \) for some \( T > 0 \) by continuity. Suppose that the claim is false and let \( t > 0 \) be the first time such that

\[
f(t) = A\varepsilon e^{-t/2C}.
\]
Note that (3.32) remains valid on $[0, t]$ if $\varepsilon$ is taken sufficiently small, and so (3.31) holds on $(0, t)$. Let $\tau \in [0, t]$ be the last time such that

$$f(\tau) = 2\varepsilon e^{-\tau/2C}.$$ 

Note that $\tau > 0$ by assumption.

We first claim that

$$t - \tau \leq 4C. \tag{3.33}$$

If the claim is false then applying the assumption (3.31) on $(\tau, \tau + 4C)$, and ignoring the first term on the left-hand side, gives

$$4C\varepsilon e^{-\frac{\tau}{2C}} (1 - e^{-2}) = 2\varepsilon \int_{\tau}^{\tau + 4C} e^{-\frac{s}{2C}} ds \leq \int_{\tau}^{\tau + 4} f \leq 2C\varepsilon (1 + 4C\lambda)e^{-\frac{\tau}{2C}} + 64C^4\lambda A\varepsilon e^{-\frac{\tau}{2C}} + \tilde{C}(A\varepsilon)^{\frac{3}{2}} e^{-\frac{\tau}{2C}},$$

where the constant $\tilde{C} > 0$ depends only on $C$, $\lambda$, and the form of the cubic term $O(f^{3/2})$.

Noting that $1 - e^{-2} \geq 3/4$, and that $\lambda$ is sufficiently small so that $4C\lambda \leq 1/4$ and $64C^4\lambda A \leq C/4$ we can divide both sides of the above inequality by $\varepsilon e^{-\tau/2C}$ to obtain

$$3C \leq \frac{5}{2}C + \frac{C}{4} + \tilde{C}A^{\frac{3}{2}}\varepsilon^{\frac{1}{2}},$$

which gives a contradiction if $\varepsilon = \varepsilon(C) > 0$ is chosen sufficiently small. Thus (3.33) holds.

Applying the assumption (3.31) on $(\tau, t)$ and ignoring the second term on the left-hand side gives

$$A\varepsilon e^{-\frac{\tau}{2C}} = f(t) \leq 2C\varepsilon (1 + 4C\lambda)e^{-\frac{\tau}{2C}} + 16C^3\lambda A\varepsilon e^{-\frac{\tau}{2C}} + \tilde{C}(A\varepsilon)^{\frac{3}{2}} e^{-\frac{\tau}{2C}}.$$ 

Since $e^{-\tau/2C} = e^{-t/2C} e^{(t-\tau)/2C} \leq e^{-t/2C} e^{2} \leq 10e^{-t/2C}$ and the smallness of $\lambda$ gives that $16C^3\lambda A \leq C/10$, we can divide both sides by $\varepsilon e^{-t/2C}$ to obtain

$$30C \leq 25C + \tilde{C}A^{\frac{3}{2}}\varepsilon^{\frac{1}{2}},$$

which gives a contradiction for $\varepsilon = \varepsilon(C) > 0$ sufficiently small. □

Acknowledgments

IK was supported in part by the NSF grant DMS-1907992, while WSO was supported in part by the Simons Foundation.

References

[ALT] G. Avalos, I. Lasiecka, and R. Triggiani, Higher regularity of a coupled parabolic-hyperbolic fluid-structure interactive system, Georgian Math. J. 15 (2008), no. 3, 403–437.
REFERENCES

[AT1] G. Avalos and R. Triggiani, *The coupled PDE system arising in fluid/structure interaction. I. Explicit semigroup generator and its spectral properties*, Fluids and waves, Contemp. Math., vol. 440, Amer. Math. Soc., Providence, RI, 2007, pp. 15–54.

[Bo1] M. E. Bogovski˘ı, *Solution of the first boundary value problem for an equation of continuity of an incompressible medium*, Dokl. Akad. Nauk SSSR, 248(5):1037–1040, 1979.

[Bo2] M. E. Bogovski˘ı, *Solutions of some problems of vector analysis, associated with the operators div and grad*, In “Theory of cubature formulas and the application of functional analysis to problems of mathematical physics”, volume 1980 of Trudy Sem. S. L. Soboleva, No. 1, pages 5–40, 149. Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 1980.

[B] M. Boulakia, *Existence of weak solutions for the three-dimensional motion of an elastic structure in an incompressible fluid*, J. Math. Fluid Mech. 9 (2007), no. 2, 262–294.

[BG1] M. Boulakia and S. Guerrero, *Regular solutions of a problem coupling a compressible fluid and an elastic structure*, J. Math. Pures Appl. (9) 94 (2010), no. 4, 341–365.

[BG2] M. Boulakia and S. Guerrero, *A regularity result for a solid-fluid system associated to the compressible Navier-Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 3, 777–813.

[BGLT1] V. Barbu, Z. Gruji´c, I. Lasiecka, and A. Tuffaha, *Existence of the energy-level weak solutions for a nonlinear fluid-structure interaction model*, Fluids and waves, Contemp. Math., vol. 440, Amer. Math. Soc., Providence, RI, (2007), 55–82.

[BGLT2] V. Barbu, Z. Gruji´c, I. Lasiecka, and A. Tuffaha, *Smoothness of weak solutions to a nonlinear fluid-structure interaction model*, Indiana Univ. Math. J. 57 (2008), no. 3, 1173–1207.

[BGT] M. Boulakia, S. Guerrero, and T. Takahashi, *Well-posedness for the coupling between a viscous incompressible fluid and an elastic structure*, Nonlinearity 32 (2019), no. 10, 3548–3592.

[BZ1] L. Bociu and J.-P. Zol´esio, *Sensitivity analysis for a free boundary fluid-elasticity interaction*, Evol. Equ. Control Theory 2 (2013), no. 1, 55–79.

[CS1] D. Coutand and S. Shkoller, *Motion of an elastic solid inside an incompressible viscous fluid*, Arch. Ration. Mech. Anal. 176 (2005), no. 1, 25–102.

[CS2] D. Coutand and S. Shkoller, *The interaction between quasilinear elastodynamics and the Navier-Stokes equations*, Arch. Ration. Mech. Anal. 179 (2006), no. 3, 303–352.

[DEGL] B. Desjardins, M.J. Esteban, C. Grandmont, and P. Le Tallec, *Weak solutions for a fluid-elastic structure interaction model*, Rev. Mat. Complut. 14 (2001), no. 2, 523–538.

[DGHL] Q. Du, M.D. Gunzburger, L.S. Hou, and J. Lee, *Analysis of a linear fluid-structure interaction problem*, Discrete Contin. Dyn. Syst. 9 (2003), no. 3, 633–650.

[Ga] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems*, Springer Monographs in Mathematics. Springer, New York, second edition, 2011.

[GGCC] G. Guidoboni, R. Glowinski, N. Cavallini, and S. Canic, *Stable loosely-coupled-type algorithm for fluid-structure interaction in blood flow*, J. Comput. Phys. 228 (2009), no. 18, 6916–6937.

[IKLT1] M. Ignatova, I. Kukavica, I. Lasiecka, and A. Tuffaha, *On well-posedness for a free boundary fluid-structure model*, J. Math. Phys. 53 (2012), no. 11, 115624, 13 pp.

[IKLT2] M. Ignatova, I. Kukavica, I. Lasiecka, and A. Tuffaha, *On well-posedness and small data global existence for an interface damped free boundary fluid-structure model*, Nonlinearity 27 (2014), no. 3, 467–499.

[IKLT3] M. Ignatova, I. Kukavica, I. Lasiecka, and A. Tuffaha, *Small data global existence for a fluid-structure model*, Nonlinearity 30 (2017), 848–898.
REFERENCES

[KT1] I. Kukavica and A. Tuffaha, Solutions to a fluid-structure interaction free boundary problem, Discrete Contin. Dyn. Syst. 32 (2012), no. 4, 1355–1389.

[KT2] I. Kukavica and A. Tuffaha, Regularity of solutions to a free boundary problem of fluid-structure interaction, Indiana Univ. Math. J. 61 (2012), no. 5, 1817–1859.

[KT3] I. Kukavica and A. Tuffaha, Well-posedness for the compressible Navier-Stokes-Lamé system with a free interface, Nonlinearity 25 (2012), no. 11, 3111–3137.

[MC1] B. Muha and S. Čanić, Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls, Arch. Ration. Mech. Anal. 207 (2013), no. 3, 919–968.

[MC2] B. Muha and S. Čanić, Existence of a weak solution to a fluid-elastic structure interaction problem with the Navier slip boundary condition, J. Differential Equations 260 (2016), no. 12, 8550–8589.

[RV] J.-P. Raymond and M. Vanninathan, A fluid-structure model coupling the Navier-Stokes equations and the Lamé system, J. Math. Pures Appl. (9) 102 (2014), no. 3, 546–596.

[Te1] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, second ed., Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, 1997.

[Te2] R. Temam, Navier-Stokes equations, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis, Reprint of the 1984 edition.