Some orthogonal polynomials arising from coherent states

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Abstract

We explore in this paper some orthogonal polynomials which are naturally associated with certain families of coherent states, often referred to as nonlinear coherent states in the quantum optics literature. Some examples turn out to be known orthogonal polynomials but in many cases we encounter a general class of new orthogonal polynomials for which we establish several qualitative results.

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1. Introduction

Coherent states are well-known objects, both in physics and mathematics (see, for example, [3] and references cited therein). Their use in physics goes back to the early days of quantum mechanics, starting with the 1926 paper by Schrödinger [37] and their rediscovery in the theory of optical coherence several decades later [19, 29]. In recent years, coherent states have been extensively used in many areas of physics, e.g., quantum optics, atomic and molecular physics, etc, (see, for example, [21] for a detailed account). Their mathematical properties, especially in the context of quantization theory, square-integrable group representation theory, symplectic geometry, etc, have also been extensively studied in recent decades. In this paper, we look at two sets of orthogonal polynomials that are naturally associated with a particular family of coherent states, known in the quantum optical literature as nonlinear coherent states (see [3] and [39] for an extensive discussion). One set of these polynomials, arising from the shift operators associated with these coherent states, have been known and studied before [10, 11, 34, 35]. Here we take another look at these polynomials, but study a different aspect of their structure. We also show how a second set of orthogonal polynomials can be obtained from
the measure which gives the resolution of the identity for the coherent states. The moments of this measure are obtained from $n$-term partial products of the terms of the sequence defining the shift operators. In this way, both sets of orthogonal polynomials are intimately related. While it would be our aim to eventually take a closer look at the relationship between these two sets of polynomials, in this paper we mainly focus on measures of the type which arise from the above-mentioned resolution of the identity and for them discuss the associated families of orthogonal polynomials.

We start out by introducing the notion of nonlinear coherent states mathematically and the derivation of their related polynomials, in the framework within which we would like to study them here.

1.1. Nonlinear coherent states

We ought to mention at the outset that the term nonlinear, as applied to coherent states, does not refer to any mathematical nonlinearity, but rather is a reflection of their appearance in nonlinear optics. Generally, a family of such coherent states is an overcomplete set of vectors in a Hilbert space, labelled by a continuous parameter $z$ which runs over a complex domain. The vectors are, in addition, subject to a resolution of the identity condition. More precisely, let $\mathfrak{H}$ be a (complex, separable, infinite-dimensional) Hilbert space, $\{|\phi_n\rangle\}_{n=0}^{\infty}$ an orthonormal basis of it and let $\{x_n\}_{n=0}^{\infty}$, $x_0 = 0$, be an infinite sequence of positive numbers. Let $\lim_{n \to \infty} x_n = L^2$, where $L > 0$ could be finite or infinite, but not zero. We shall use the notation $x_n^r = x_1 x_2 \cdots x_n$ and $x_0^r = 1$. For each $z \in \mathfrak{H}$ (some domain in $\mathbb{C}$), we define a nonlinear coherent state, i.e. a vector $\eta_z \in \mathfrak{H}$, in the manner

$$\eta_z = N(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{x_n}} \phi_n, \quad (1.1)$$

where the normalization constant $N(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^n}{x_n}$ is chosen so that $\|\eta_z\| = 1$. It is clear that the vectors $\eta_z$ are well defined for all $z$ for which the above sum, representing $N(|z|^2)$, converges, i.e. $\mathfrak{H} = \{z \in \mathbb{C} \mid |z| < L\}$. Furthermore, we require that there exists a measure $d\nu(z, \overline{z})$ on $\mathfrak{H}$ for which the resolution of the identity condition,

$$\int_{\mathfrak{H}} |\eta_z\rangle \langle \eta_z | N(|z|^2) \ d\nu(z, \overline{z}) = I_{\mathfrak{H}}, \quad (1.2)$$

holds.

It is easily seen that in order for (1.2) to be satisfied, $d\nu$ has to have the form

$$d\nu(z, \overline{z}) = \frac{1}{2\pi} \ d\theta \ d\lambda(r), \quad z = r e^{i\theta}, \quad (1.3)$$

where the measure $d\lambda$ is a solution of the moment problem,

$$\int_{0}^{L} r^n d\lambda(r) = x_n^r!, \quad n = 0, 1, 2, \ldots, \quad (1.4)$$

provided that such a solution exists (see, e.g., [1] for a discussion of the moment problem). In most of the cases that occur in practice, the support of the measure $d\nu$ is the whole of $\mathfrak{H}$, i.e. $d\lambda$ is supported on the entire interval $(0, L)$.

Below are some examples of the above general construction.

1.1.1. Canonical coherent states. Let $x_n = n$ so that $L = \infty$. In that case, the coherent states $\eta_z$, defined for all $z \in \mathbb{C}$, are the well-known canonical coherent states:

$$\eta_z = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n, \quad (1.5)$$
The moment problem becomes
\[ \int_0^\infty r^{2n} \, d\lambda_n(r) = n!, \quad n = 0, 1, 2, \ldots, \]
so that
\[ d\lambda_n(r) = 2r \, e^{-r^2} \, dr, \quad 0 \leq r < \infty. \] (1.7)

1.1.2. SU(1, 1) discrete series coherent states. This time \( x_n = \frac{n}{2j + n - 1} \) so that \( x_n! = \frac{n!}{(2j)_n} \), where we have used the shifted factorials \( (a)_n = a(a + 1)(a + 2) \cdots (a + n - 1) \), and \( j = 1, 1/2, 2, 3/2, 3, \ldots \), is a constant. Also, \( L = 1 \) and the associated coherent states, defined for all \( z \) in the open unit disc \( D = |z| < 1 \), are
\[ \eta_c = (1 - r^2)^j \sum_{n=0}^{\infty} \left[ \frac{(2j)_n}{n!} \right]^{1/2} \phi_n, \quad r = |z|. \] (1.8)

The corresponding moment problem is now
\[ \int_0^1 r^{2n} \, d\lambda_n(r) = \frac{n!}{(2j)_n}, \] (1.9)
which has the solution
\[ d\lambda_n(r) = 2(2j - 1)r(1 - r^2)^{2j - 2} \, dr, \quad 0 \leq r < 1. \] (1.10)

The coherent states, \( \eta_c \), above, arise from the discrete series representations (parametrized by \( j \)) of the SU(1, 1) group.

1.1.3. Barut–Girardello coherent states. There is a second set of coherent states associated with the SU(1, 1) group which are constructed using the ladder operators appearing in the Lie algebra of the group. These are known as the Barut–Girardello coherent states [6]. For these coherent states, \( X = \mathbb{C} \), \( x_n = n(2j + n - 1) \) and \( x_n! = n!(2j)_n \) with \( j = 1, 1/2, 2, 3/2, 3, \ldots \), as before. The coherent states are
\[ \eta_c = \frac{|z|^{2j-1}}{\sqrt{I_{2j-1}(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!(2j)_n}} \phi_n, \quad z \in \mathbb{C}, \] (1.11)
where \( I_n(x) \) denotes the order-\( n \) modified Bessel function of the first kind. These coherent states satisfy the resolution of the identity,
\[ \frac{2}{\pi} \int_0^\infty |\eta_c(z)|^2 I_{2j-1}(2r) I_{2j-1}(2r) \, r \, dr \, d\theta = I, \quad z = re^{i\theta}, \] (1.12)
where again \( K_n(x) \) is the order-\( n \) modified Bessel function of the second kind.

1.1.4. Coherent states from analytic functions. All three examples above could be seen as special cases of a more general construction. Let \( f(z) \) be an analytic function which has a Taylor expansion (around the origin) of the type
\[ f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\rho(n)}, \quad 0 < \rho(n) < \infty, \quad \rho(0) = 1, \] (1.13)
and let \( L = \lim_{n \to \infty} \frac{\rho(n+1)}{\rho(n)} > 0 \) be its radius of convergence. Then
\[ \mathcal{N}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\rho(n)^2}. \]
converges in the disc $|z| < L$. Defining $x_n = \left[ \dfrac{\rho(n+1)}{\mu(n+1)} \right]^2$, we can construct vectors $\eta_n$ in the Hilbert space $H$ following (1.1), which will be well defined in this disc and will constitute a family of coherent states provided the moment problem (1.4) has a solution. It is clear that a large number of hypergeometric functions will lead to families of coherent states in this manner.

1.2. Two families of orthogonal polynomials

As mentioned earlier, there are two sets of orthogonal polynomials, naturally associated with a family of nonlinear coherent states, that we now present.

1.2.1. Polynomials orthogonal with respect to $d\lambda$. The first of these sets is determined by the measure $d\lambda$. This measure can be extended to an even positive measure

$$d\mu(t) = \frac{1}{2} d\lambda(|t|),$$

(1.14) on the symmetric interval $[-L, L]$, with moments

$$\mu_{2n} = 2 \int_0^\infty t^{2n} d\mu(t) = x_n, \quad \mu_{2n+1} = 2 \int_0^\infty t^{2n+1} d\mu(t) = 0, \quad n = 0, 1, 2, \ldots$$

(1.15)

With these we can build a set of monic polynomials $P_n(x)$, $n = 0, 1, 2, \ldots$, orthogonal with respect to the measure $d\mu$, in the usual manner [24] using the Hankel determinant

$$D_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_n \\ \mu_1 & \mu_2 & \ldots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \ldots & \mu_{2n} \end{vmatrix}, \quad P_n(x) = \frac{1}{D_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-1} \\ 1 & x & \ldots & x^n \end{vmatrix}. \quad (1.16)$$

For example, for the canonical coherent states with measure (1.7), these would be Laguerre polynomials in the variable $x^2$. For the $SU(1, 1)$ discrete series case, with measure (1.10), the associated polynomials are related to the Jacobi polynomials as explained in section 3.3, with $\alpha = \frac{1}{2}$ and $\beta = 2j - 2$. For case of the Barut–Girardello coherent states, the measure is

$$d\lambda(r) = \frac{2}{\pi} K_{2j-1}(2r) r^{-2j} \, dr. \quad (1.17)$$

The corresponding polynomials $P_n$ can of course be computed using (1.16), but are not among the well-known polynomials.

1.2.2. Polynomials generated by shift operators. To obtain the second set of polynomials, let us go back to definition (1.1) and define the formal shift operators

$$a\phi_n = \sqrt{\lambda_n} \phi_{n-1}, \quad a^* \phi_n = 0, \quad a^* \phi_n = \sqrt{\lambda_{n+1}} \phi_{n+1}, \quad n = 0, 1, 2, \ldots$$

(1.18)

(A general treatment of shift operators may be found in [33].) Then, if $\sum_{n=0}^\infty \frac{1}{\sqrt{\lambda_n}} = \infty$, the operator $Q = a^* a^\dagger$ is essentially self-adjoint and hence has a unique self-adjoint extension [10, 35], which we again denote by $Q$. It acts on the basis vectors $\phi_n$ in the manner

$$Q \phi_n = \sqrt{\lambda_n} \phi_{n-1} + \sqrt{\lambda_{n+1}} \phi_{n+1}. \quad (1.19)$$

From the general theory of self-adjoint operators, there is a Hilbert space $L^2(\mathbb{R}, dw)$ (where $dw$ is an even measure) on which $Q$ acts as the operator of multiplication and the $\phi_n$ are
functions in this space. Transforming to this space, we may then rewrite the above recurrence relation as

$$xφ_n(x) = \sqrt{\frac{x_n}{2}} φ_{n-1}(x) + \sqrt{\frac{x_{n+1}}{2}} φ_{n+1}(x), \quad x \in \mathbb{R},$$

(1.20)

which is a set of two-term recurrence relations for a family of orthogonal polynomials, $φ_n(x)$. The polynomials can be computed successively, assuming the initial conditions $φ_{-1} = 0$, $φ_0 ≡ 1$. The measure $dw$ comes from the spectral family of projectors, $E_x$, $x \in \mathbb{R}$, of the operator $Q$, in the manner $dw(x) = d⟨φ_0 \mid E_x \phi_0⟩$.

In general, the measure $dw$ is different from $dμ$ and so are the sets of polynomials $P_n(x)$ in (1.16) and $φ(x)$ above. For example, in the case of the canonical coherent states in section 1.1.1, the polynomials $φ_n(x)$ are the well-known Hermite polynomials, while for the $SU(1, 1)$ discrete series coherent states, they are the Polaczeck polynomials mentioned at the end of section 3.3, for $β = 2j - 2$. For the Barut–Girardello CS in section 1.1.3, $x_n = n(2j + n - 1)$. However, again the corresponding polynomials, which can be computed using (1.20), are not well known.

In all the above examples, the sequence $x_n$ is strictly increasing and its limit is either a positive number or infinity. Dickinson, Pollack and Wannier [16] studied a class of orthogonal polynomials of the form in (1.20) but with $x_n$ replaced by $x_{n+μ}$ and requiring the condition $[x_n] \to 0$. Thus, our condition (see (2.3) below) is the exact opposite of their condition. The polynomials in the Dickinson–Pollack–Wannier class are orthogonal with respect to a discrete measure supported on a countable set whose only limit point is $x = 0$. Later Goldberg [22] corrected the claim in [16] that $x = 0$ does not support a mass by showing that $x = 0$ may actually support a positive mass of the orthogonality measure.

There is a simple way to compute the monic versions of the polynomials $φ_n(x)$. To see this, note first that in virtue of (1.19), the operator $Q$ can be represented in the $φ_n$ basis as the infinite tridiagonal matrix,

$$Q = \begin{pmatrix}
0 & b_1 & 0 & 0 & 0 & \ldots \\
b_1 & 0 & b_2 & 0 & 0 & \ldots \\
0 & b_2 & 0 & b_3 & 0 & \ldots \\
0 & 0 & b_3 & 0 & b_4 & \ldots \\
0 & 0 & 0 & b_4 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}, \quad b_n = \sqrt{\frac{x_n}{2}}. \tag{1.21}$$

Let $Q_n$ be the truncated matrix consisting of the first $n$ rows and columns of $Q$ and $I_n$ the $n \times n$ identity matrix. Then,

$$xI_n - Q_n = \begin{pmatrix}
 x & -b_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
-b_1 & x & -b_2 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & -b_2 & x & -b_3 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & -b_3 & x & -b_4 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & -b_4 & x & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & x & -b_{n-2} & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & -b_{n-2} & x & -b_{n-1} \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & -b_{n-1} & x \\
\end{pmatrix}. \tag{1.22}$$

It now follows that the monic polynomial $q_n$, associated with $φ_n(x)$, is just the characteristic polynomial of $Q_n$:

$$q_n(x) = \det[xI_n - Q_n]. \tag{1.23}$$
These polynomials are related to the $\phi_n(x)$ via

$$q_n(x) = \left[\frac{x!}{2^n}\right]^{1/2} \phi_n(x),$$

and satisfy the recurrence relations

$$q_{n+1}(x) = xq_n(x) - \frac{x_n}{2}q_{n-1}(x).$$

(1.24)

2. Generalities about orthogonal polynomials

To make the paper self-contained, we collect here a few preliminary results on orthogonal polynomials.

2.1. Recurrence relations and orthogonality measures

Every sequence of monic polynomials $\{P_n(x)\}$ orthogonal with respect to a positive measure $\mu$ satisfies a three-term recurrence relation of the form [24]

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad n \geq 0, \quad P_0(x) = 1, \quad \beta_0 P_{-1}(x) := 0.$$  

(2.1)

(Note that for the recurrence relations (1.24), this amounts to taking $\alpha_n = 0$ and $\beta_n = x_n/2$.) To avoid additional assumptions, we consider only polynomials having an orthogonality measure supported on an infinite set (i.e. the support does not consist of a finite set of points). In the general theory [1, 24], the measures are normalized to have unit total mass.

We start with an even positive measure $\mu$ supported on $[-L, L]$, $L \leq \infty$, and whose moments are $\mu_n, n = 0, 1, \ldots$. Further, normalize $\mu$ by $\mu_0 = 1$. Thus, $\mu_{2n+1} = 0$ for all $n$ and $\mu_{2n} > 0$. We now assume that $\mu_{2n}$ factors as follows:

$$\mu_{2n} = 2 \int_0^L t^{2n} \, d\mu(t) = x_1 x_2 \cdots x_n = x_n!, \quad n > 0.$$  

(2.2)

(Note that this is always possible by writing $x_n = x_n/2$.) However, we shall only work with cases where $x_n$ has convenient forms, such as being expressible as rational functions of the integer variable $n$.

Note that $2\mu$ has total mass $= 1$ on $[0, L]$. Representation (2.2) clearly implies

$$x_n > 0, \quad \text{for } n > 0.$$  

(2.3)

From here we introduce a family of orthogonal polynomials, of type (1.20) and generated by

$$\phi_0(x) := 1, \quad \phi_1(x) = \frac{2}{x_1}, \quad x \phi_n(x) = \sqrt{\frac{x_n+1}{2}} \phi_{n+1}(x) + \sqrt{\frac{x_n}{2}} \phi_{n-1}(x).$$  

(2.4)

We shall mainly explore in this paper the consequences of expressing the moments $\mu$ in the form (2.2) and the resulting polynomials of the above type.

2.2. Additional background material

It is easy to see that the quadratic form $\sum_{j,k=0}^{n} \mu_{j+k} y_j y_k$ is positive definite, see [24] for example; hence, all the Hankel determinants $D_n$,

$$D_n := \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix},$$  

(2.5)

are positive, by the Sylvester criterion.
The following theorem is important in recovering the absolutely continuous component of an orthogonality measure from the large-degree behaviour of the orthonormal polynomials.

**Theorem 2.1.** Assume that $P_n$ satisfies (2.1) with $\alpha_{n-1} \in \mathbb{R}$ and $\beta_n > 0$ for all $n > 0$. In addition, assume that
\[
\sum_{n=0}^{\infty} \left[ \sqrt{\beta_n} - \frac{1}{2} + |\alpha_n| \right] < \infty; \tag{2.6}
\]
then the orthogonality measure $\mu$ has an absolutely continuous component $\mu'$ supported on $[-1, 1]$. Furthermore if $\mu$ has a discrete part, then it will lie outside $(-1, 1)$. In addition, the limiting relation
\[
\limsup_{n \to \infty} \left[ \sqrt{1-x^2} P_n(x) \sqrt{\zeta_n} - \frac{2\sqrt{1-x^2}}{\pi \mu'(x)} \sin((n+1)\theta - \varphi(\theta)) + \varepsilon_n \right] = 0 \tag{2.7}
\]
holds, with $x = \cos \theta \in (-1, 1)$ and where $\varepsilon_n \to 0$ as $n \to \infty$. In (2.7), $\varphi(\theta)$ does not depend on $n$ and $\zeta_n = \beta_1 \beta_2 \cdots \beta_n$.

This theorem is due to Nevai [32]. Observe that Nevai’s theorem then relates the asymptotics of general polynomials to those of the Chebyshev polynomials. Note also that Nevai’s theorem involves a term (the second term within the square brackets in (2.7)) which has the form of a scattering amplitude (depending on the orthogonality measure), possibly also including a phase shift $\varphi(\theta)$.

**Theorem 2.2.** Let $P_n(x)$ be generated by (2.1). Then the zeros of the polynomial $P_n(x)$ lie in $(A, B)$, where
\[
B = \max \xi_j : 0 < j < n, \quad A = \min \eta_j : 0 < j < n,
\]
where $\eta_j \leq \xi_j$ and
\[
\xi_j, \eta_j = \frac{1}{2}(\alpha_j + \alpha_{j-1}) \pm \frac{1}{2} \sqrt{\left(\alpha_j - \alpha_{j-1}\right)^2 + 16\beta_j}, \quad 1 \leq j < n. \tag{2.8}
\]

Theorem 2.2 is the special case $c_n = 1/4$ of a result due to Ismail and Li in [26]. The full result is also stated and proved in [24, theorem 7.2.7].

The zeros of orthogonal polynomials are real and simple, so we shall follow the standard notation in [24] or [38] and arrange the zeros $x_{n,j}, 1 \leq j \leq n$, as
\[
x_{n,1} > x_{n,2} > \cdots > x_{n,n}. \tag{2.9}
\]

**Theorem 2.3.** Assume that $\{1/x_n\}$ is a Hausdorf moment sequence, that is, there is a probability measure $\nu$ supported on $[0, 1]$ such that $1/x_n = \int_0^1 t^n \, d\nu(t)$. Then there is a probability measure $\xi$ supported on $[0, \infty)$ such that $x_1 x_2 \cdots x_n = \int_0^\infty t^n \, d\xi(t)$, that is, $\{x_1 x_2 \cdots x_n\}$ is a Stieltjes moment sequence.

Many examples and applications are in [7, 8]. In the theory of coherent states, we start with the sequence $\{x_n\}$ and construct the measure $\lambda$, so theorem 2.3 gives a sufficient condition for the existence of the measure $\lambda$.

### 3. Results and examples

We enunciate a few results in this section and work out some examples.
Therefore,

\[ x_8 \]

Our first result is the monotonicity of \( \{x_n\} \).

**Theorem 3.1.** The sequence \( \{x_n : n = 1, 2, \ldots\} \) is strictly increasing. If \( L < \infty \), then \( x_n < L^2 \).

**Proof.** Clearly

\[
0 < D_2 = \left| \begin{array}{ccc}
1 & 0 & x_1 \\
0 & x_j & 0 \\
x_1 & 0 & x_1, x_2
\end{array} \right| = x_1^2 (x_2 - x_1).
\] (3.1)

Therefore, \( x_2 > x_1 \). We note that \( x^2 d\mu \) is a positive even measure supported on \([-L, L]\) and its \( D_2 \) is

\[
\begin{bmatrix}
\mu_{2n} & 0 & \mu_{2n+2} \\
0 & \mu_{2n+2} & 0 \\
\mu_{2n+2} & 0 & \mu_{2n+4}
\end{bmatrix} = \mu_{2n+2}[\mu_{2n}]^2 [x_{n+1} x_{n+2} - x_{n+1}^2].
\] (3.2)

which implies \( x_{n+2} > x_{n+1} \), for \( n = 0, 1, \ldots \). The same conclusion also follows from

\[
0 < 2 \int_0^L t^{2n} (t^2 - x_{n+1}^2)^2 d\mu(t) = \mu_{2n+4} - 2x_{n+1} \mu_{2n+2} + x_{n+1}^2 \mu_{2n}
\]

If \( L \) is finite, we use the definition of \( \mu_{2n} = x_1 \cdots x_n \) and conclude that

\[
x_1 \cdots x_n L^2 = x_1 \cdots x_{n+1} = 2 \int_0^L t^{2n} (L^2 - t^2) d\mu(x) > 0.
\]

This implies \( x_{n+1} < L^2 \) for all \( n \geq 0 \). \( \square \)

A partial converse to the above theorem is the following.

**Theorem 3.2.** If \( L = \infty \), then the sequence \( \{x_n\} \) is unbounded.

**Proof.** Assume \( L = \infty \) and \( x_n \leq M \), and \( M > 1 \). Then for every \( A > 0 \), the integral \( \int_{[A, \infty)} d\mu(t) > 0 \). It is clear that

\[
M^n > x_1, x_2, \cdots, x_n = 2 \int_{[0, \infty)} t^{2n} d\mu(t) \geq 2 \int_{[2M, \infty)} t^{2n} d\mu(t) \geq 2 [2M]^{2n} \int_{[2M, \infty)} d\mu(t).
\]

Therefore, \( M > 4M^2 [2 \int_{[2M, \infty)} d\mu(t)]^{1/n} \) which is impossible for sufficiently large \( n \). \( \square \)

Note that if \( L \leq 1 \), then \( x_n < 1 \) and \( 1/x_n \) is never a Hausdorff moment sequence. In these cases, the assumptions of the Berg–Durand theorem, theorem 2.3, are not satisfied.

**Remark.** It is not clear that we can say much about the case \( L = \infty \) so we will assume that \( L \) is bounded throughout the rest of this section. Therefore, \( \{x_n\} \) is a monotone sequence converging to \( M \), say. One can derive nonlinear inequalities satisfied by the \( x_n \)s. For example, using

\[
0 < 2 \int_0^L t^{2n} (t^2 - x_{n+1}^2)^2 (t^2 - x_{n+2}^2) d\mu(t)
\]

one obtains

\[
x_{n+3} (x_{n+4} - x_{n+2}) + x_{n+1} (x_{n+2} - x_{n+1}) > x_{n+2} (x_{n+3} - x_{n+2}) + 2x_{n+1} (x_{n+3} - x_{n+2}).
\] (3.3)
Of course one can integrate other factions like $\alpha^2 (t^2 - x_{n+1})^2$ or $\beta^2 (t^2 - x_{n+1})^2$ and obtain other inequalities. On the other hand, by expanding the determinant, Hankel determinant,

\[
\begin{vmatrix}
\mu_{2n} & 0 & \mu_{2n+2} & 0 & \mu_{2n+4} \\
0 & \mu_{2n+2} & 0 & \mu_{2n+4} & 0 \\
\mu_{2n+2} & 0 & \mu_{2n+4} & 0 & \mu_{2n+6} \\
0 & \mu_{2n+4} & 0 & \mu_{2n+6} & 0 \\
\mu_{2n+4} & 0 & \mu_{2n+6} & 0 & \mu_{2n+8}
\end{vmatrix} > 0,
\]

and after deleting the positive terms, we obtain the necessary condition

\[
2\alpha_1 x_{n+1} + 2\alpha_2 x_{n+3} + \alpha_{n+2} x_{n+4} > x_{n+1} x_{n+1}^2 + x_{n+2} x_{n+3} + x_{n+1} x_{n+3} x_{n+4}. \quad (3.4)
\]

**Theorem 3.3.** Let $L$ be finite and let $M := \lim_{n \to \infty} x_n$. If

\[
\sum_{n=1}^{\infty} |\sqrt{x_n} - \sqrt{M}| \text{ converges},
\]

the orthogonality measure of $P_n(x)$ has an absolutely continuous component supported on the interval $[-2\sqrt{M}, 2\sqrt{M}]$. Moreover, all the zeros of the polynomials lie in the interval $(-2\sqrt{M}, 2\sqrt{M})$; hence, the discrete part of the orthogonality measure is either empty or has two discrete masses (bound states) at $x = \pm 2\sqrt{M}$.

**Proof.** Let $x = y\sqrt{2M}$ and put $\phi_n(x) = \psi_n(y)$ and apply theorem 2.1 to see that $\psi_n$s are orthogonal with respect to a measure whose absolutely continuous component is supported on $[-1, 1]$. Thus, the orthogonality measure of $\phi_n$ has an absolutely continuous component supported on $[-2\sqrt{M}, 2\sqrt{M}]$. Next apply theorem 2.2. In the present case $\alpha_\nu = 0$, $\beta_\nu = \frac{x_n}{2}$; hence, $\xi_j, \eta_j = \pm \sqrt{2x_j}$. The monotonicity of the $x_n$s shows that the zeros of $\phi_n(x)$ belong to $(-\sqrt{x_n}, \sqrt{x_n})$. From theorem 2.1, we conclude that the discrete part of the orthogonality measure is outside $(-2\sqrt{M}, 2\sqrt{M})$. If $[A, B]$ is the smallest interval containing the support of the orthogonality measure, then the largest and smallest zeros of $\phi_n$ converge to $B$ and $A$, respectively. Thus, $[A, B] = [-2\sqrt{M}, 2\sqrt{M}]$. This shows that the discrete part may only occur at $\pm 2\sqrt{M}$. \(\square\)

### 3.2 Example 1

Consider the ultraspherical polynomials $\{C_n^\nu(x)\}$ where

\[
d\mu(x) = \frac{\Gamma(v + 1)(1 - x^2)^{v-1/2}}{\sqrt{\pi} \Gamma(v + 1/2)} \, dx, \quad x \in [-1, 1], \quad v > -1/2.
\]

Now

\[
\begin{align*}
x_1 x_2 \cdots x_n &= \frac{2\Gamma(v + 1)}{\sqrt{\pi} \Gamma(v + 1/2)} \int_0^1 x^{n}(1 - x^2)^{v-1/2} \, dx \\
&= \frac{\Gamma(v + 1)\Gamma(n + 1/2)}{\sqrt{\pi} \Gamma(v + n + 1)} = (1/2)_n = (v + 1)_n.
\end{align*}
\]

Therefore

\[
x_n = \frac{n - 1/2}{v + n}.
\]

Here is an interesting point to show how sharp the monotonicity of the $x_n$s is. An easy calculation is to show that $x_n < x_{n+1}$ is equivalent to the integrability of the weight function,
namely \( v > -1/2 \). The monic recurrence relation for the family of polynomials in (2.4), after replacing \( x \) by \( 2x \), is

\[
2xu_n(x) = u_{n+1}(x) + \frac{n - 1/2}{v + n} u_{n-1}(x),
\]

which is not a standard polynomial. It is a special case of the associated Pollaczek polynomials, see [24, chapter 5]. We know the absolutely continuous component of its orthogonality measure but we do not know whether \( x = \pm 1 \) support any discrete masses.

### 3.3. Example 2

This is more general than example 1. Consider the absolutely continuous measure

\[
d\mu(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta + 3/2) x^{2\alpha} (1 - x^2)^\beta}{\Gamma(\alpha + 1/2) \Gamma(\beta + 1)} \, dx,
\]

where \( \alpha > -1/2, \beta > -1 \). This is essentially the measure \( \lambda \) in (1.4). The polynomials in this case are defined according to their parity. The polynomials of even degree are constant multiples of the Jacobi polynomials \( P_n^{(\alpha-1/2, \beta)}(1 - 2x^2) \) while the odd-degree ones are constant multiples of the Jacobi polynomials \( xP_n^{(\alpha+1/2, \beta)}(1 - 2x^2) \).

We then have

\[
x_1 x_2 \cdots x_n = \frac{\Gamma(\alpha + \beta + 3/2)}{\Gamma(\alpha + 1/2) \Gamma(\beta + 1)} \int_0^1 x^{2\alpha + 2\beta} (1 - x^2)^\beta \, dx
\]

\[
= \frac{(\alpha + 1/2)_n}{(\alpha + \beta + 3/2)_n}.
\]

This gives

\[
x_n = \frac{\alpha + n - 1/2}{\alpha + \beta + n + 1/2}.
\]

The case \( \alpha = 1/2 \) gives the Pollaczek polynomials with parameters \( \lambda = (\beta + 1)/2, a = (\beta + 1)/2, b = 0 \), see sections 5.4 and 5.5 of [24]. If \( \alpha \neq 1/2 \), we obtain the associated Pollaczek polynomials. They are given at the end of [14].

For completeness, it may be of interest to say something about the Pollaczek polynomials. They can be defined by the recurrence relation [15, 24]

\[
(n + 1)P_{n+1}^\alpha(x; a, b) = 2[(n + \lambda + a) x + b]P_n^\alpha(x; a, b)
\]

\[
- (n + 2\lambda - 1) P_{n-1}^\alpha(x; a, b), \quad n > 0,
\]

and the initial conditions

\[
P_0^\alpha(x; a, b) = 1, \quad P_1^\alpha(x; a, b) = 2(\lambda + a)x + 2b.
\]

Their hypergeometric representation, orthogonality relation and generating functions can be found in section 5.3 of [24]. The orthogonality restricts \( \lambda, a, b \) to be in a certain subset of \( \mathbb{R} \). The parameter domain is further divided into subsets according to the nature of their orthogonality measure. The measure always has an absolutely continuous component supported on \([-1, 1]\). In addition it may have an empty, finite or infinite discrete part depending on where \( \lambda, a, b \) lies in the parameter domain. This is described in detail in [14], see also section 5.3 in [24]. The Pollaczek polynomials also appeared in the J matrix method for discretization of the continuum where the energy parameter \( E \) for the hydrogen atom is related to \( x \) in the Pollaczek polynomials via \( x = (E - 1/8)/(E + 1/8) \). The details are in [41, 25], see also [24, section 5.8]. The latter reference records the explicit form of the measure in different parts of the parameter domain. It is interesting to note that the Pollaczek polynomials also appear in the relativistic Coulomb problem as in the work of Alhaidari [2] and Munger [31].
3.4. Further examples

We consider two additional examples of measures, with moments written in the form (2.2). Consider the integral [17, (27), p 51]

$$\int_0^\infty K_{2\nu}(\beta t)\mu^{2\mu-1}dt = 2^{2\mu-2}\beta^{-2\mu}\Gamma(\mu+v)\Gamma(\mu-v).$$  \hspace{1cm} (3.9)

where (\mu \pm v) > 0, \Re \beta > 0. The weight function

$$w(x) = \frac{2^{1-2\mu}\beta^{2\mu}}{\Gamma(\mu+v)\Gamma(\mu-v)}K_{2\nu}(\beta|\mu|)|\mu|^{2\mu-1}, \ \ x \in \mathbb{R}. $$ (3.10)

Therefore,

$$\mu_{2n} = 4^\mu\beta^{-2n}(\mu+v)(\mu-v)n,$$

and

$$x_n = (4/\beta^2)(\mu + v + n - 1)(\mu - v + n - 1).$$  \hspace{1cm} (3.11)

In this case, the polynomials generated by (2.4) are the associated Meixner–Pollaczek polynomials. Weight functions for these polynomials have been computed in [14].

Consider the integral

$$\int_0^\infty e^{-it}K_\nu(\beta t)t^{\nu-1}dt = \frac{\sqrt{\pi} (2\beta)^\nu\Gamma(\mu+v)\Gamma(\mu-v)}{\Gamma(\mu+1/2)(\alpha + \beta)^{\nu+v}}$$

$$\times {}_2F_1\left(\frac{\mu + v, v+1/2}{\mu+1/2}; \frac{a-\beta}{a+\beta}\right),$$

valid for \Re(\mu \pm v) > 0, \Re(\alpha + \beta) > 0. This is (26), p 50, of [17].

**Example 1.** To sum the $_2F_1$ we are forced to take \(a = \beta\); hence, there is no loss of generality in choosing \(a = \beta = 1\). Consider the even normalized weight function

$$w(t) := \frac{\Gamma(\mu+1/2)2^\mu}{\sqrt{\pi} \Gamma(\mu+v)\Gamma(\mu-v)}e^{-t^2}K_\nu(t^2)|t|^{2\mu-1}, \ \ t \in \mathbb{R}. $$ (3.13)

Thus,

$$\mu_{2n} = (\mu+v)_n(\mu-v)_n$$

and we find that \(x_n\) in (2.2) is

$$x_n = \frac{(\mu + v + n - 1)(\mu - v + n - 1)}{2(\mu + n - 1/2)}.$$  \hspace{1cm} (3.14)

The \(x_n\)s are unbounded as expected. Nothing is known about the polynomials generated by (2.4) with \(x_n\) defined by (3.14).

Let

$$w(t) := \frac{\Gamma(\mu+1/2)2^\mu}{\sqrt{\pi} \Gamma(\mu+v)\Gamma(\mu-v)}e^{-|t|^2}K_\nu(|t|^2)|t|^{2\mu-1}, \ \ t \in \mathbb{R}. $$ (3.15)

Thus,

$$\mu_{2n} = \frac{\mu+v + n - 2)(\mu + v + 2n - 1)(\mu - v + 2n - 2)(\mu) - v + 2n - 1)}{4(\mu + 2n - 3/2)(\mu + 2n - 1/2)}.$$  \hspace{1cm} (3.16)

At first glance, these polynomials seem to be symmetric continuous Hahn polynomials, \([30, (1.4.2)]\) with \(a = c\) and \(b = d\). A closer examination however shows that this is not the case and the polynomials generated by (2.4) with the above \(x_n\)s are new.
3.5. Completely monotonic functions

Bustoz and Ismail [13] proved that the function
\[ f(x; a, b) := \frac{\Gamma(x)\Gamma(x + a + b)}{\Gamma(x + a)\Gamma(x + b)}, \quad a, b \geq 0, \]  
(3.17)
is completely monotonic, that is, \((-1)^n \frac{d^n}{dx^n} f(x, a, b) \geq 0\) on \((0, \infty)\). Therefore, the function
\[ g(x; a, c) := \frac{\Gamma(c)\Gamma(a + b)}{\Gamma(a + b + c)\Gamma(x + a + b + c)} \]  
(3.18)
is completely monotonic for \(a \geq c, b \geq c, c \geq 0\). When \(c > 0\), \(g\) is completely monotonic on \([0, \infty)\). By Bernstein’s theorem [40], there is a unique probability measure \(\alpha(x)\) supported on a subset of \([0, \infty)\) such that
\[ g(x; a, b, c) = \int_0^\infty e^{-xt} \, d\alpha(t). \]  
(3.19)

In fact Bustoz and Ismail [12] proved that the corresponding probability distribution is infinitely divisible [18]. Now the measure \(\mu(u) := \frac{1}{2\pi} \alpha(-2 \ln |u|)\) is an even probability measure on \(\mathbb{R}\) and its 2nth moment is
\[ \int_\mathbb{R} u^{2n} \, d\mu(u) = 2 \int_0^1 u^{2n} \, d\mu(u) = g(n; a, b, c) = x_1x_2 \cdots x_n. \]
This gives
\[ x_n = \frac{(c + n - 1)(a + b - c + n - 1)}{(a + n - 1)(b + n - 1)}. \]
The polynomials generated by (2.4) when \(c = 1, a = 1 + b = 0\) are the orthonormal ultraspherical polynomials
\[ \frac{n!(n + v)}{(2v)_n} C_n^v(x/\sqrt{2}), \]
[36, 17, 24]. In general we choose \(a = v + 1, b = v + c\) and keep \(c\) as an association parameter. The polynomials become constant multiples of orthonormal associated ultraspherical polynomials at \(x/\sqrt{2}\) [12].

Let \(0 < q < 1\). The \(q\)-shifted factorials are
\[ (a; q)_0 = 1, \quad (a; q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}), \quad n = 1, 2, \ldots, \text{ or } \infty. \]  
(3.20)
The \(q\)-Gamma function \(\Gamma_q(x)\) is [4, 20]
\[ \Gamma_q(x) = (1 - q)^{1-x} \prod_{k=0}^{\infty} \frac{1 - q^{k+1}}{1 - q^{x+k}}. \]  
(3.21)
It satisfies the functional equation
\[ \Gamma_q(x + 1) = \frac{1 - q^x}{1 - q} \Gamma_q(x). \]
It also has the initial values \(\Gamma_q(1) = \Gamma_q(2) = 1\).

We now consider the function
\[ h(x; a, c) := \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(c)\Gamma_q(a + b - c)} \frac{\Gamma_q(x + c)\Gamma_q(x + a + b - c)}{\Gamma_q(x + a)\Gamma_q(x + b)}. \]  
(3.22)
A special case of a result of Ismail and Muldoon [28] is that \(h(x; a, c) = e^{-H(x)}\) and \(H\) is completely monotonic on \([0, \infty)\) for \(a \geq c, b \geq c, c \geq 0\).
Now let $\beta$ be the probability measure defined by
\[
h(x; a, c) = \int_0^\infty e^{-u} \, d\beta(t).
\] (3.23)

As in the case of (3.19), we let $v(u) := \frac{1}{2} \beta(-2 \ln |u|)$ and set
\[
\int_{\mathbb{R}} u^{2n} \, d\mu(u) = 2 \int_0^\infty u^{2n} \, d\mu(u) = g(n; a, b, c) = x_1 x_2 \cdots x_n.
\]

With $A = q^a$, $B = q^b$, $C = q^c$, $D = q^d$, we have
\[
x_n = \frac{(1 - Cq^{n-1})(1 - ABCq^{n-1})}{(1 - Aq^{n-1})(1 - Bq^{n-1})}.
\]

When $C = q$, $A/q = B = \beta$, the polynomials are the $q$-ultraspherical polynomials of Askey and Ismail [5, 27],
\[
\sqrt{(q;q)_n(1-q^n)^{(\beta^2; q)_n}} C_n(x/\sqrt{2}, \beta|q).
\]

A complete treatment of the $q$-ultraspherical polynomials is available in [24]. If $C \neq q$, we obtain the associated $q$-ultraspherical polynomials of [12] or associated symmetric $q$-Pollaczek polynomials [14].

Many quotients of products of Gamma (respectively $q$-Gamma) functions are known to be completely monotonic, see for example [23] and [28]. Each combination gives rise to the orthogonal polynomials $\phi_n(x)$ of the type generated by (2.4), where $x_n$ is a quotient of two monic polynomials of $n$ (respectively of $q^n$) of the same degree. Therefore, we can always generate many cases which are not in the literature but to which the results of this section apply. We conclude this section with few examples of completely monotonic functions and write down the corresponding sequence $\{x_n\}$.

In order to state the more general results alluded to above, we need some additional notation. Let $S_n$ be the set (group) of all permutations on $n$ symbols, $a_1, a_2, \ldots, a_n$. Let $O_n$ and $E_n$ be the sets of odd and even permutations over $n$ symbols, respectively. Moreover, let $P_{r,k}$, $1 \leq k \leq n$, be the set of all vectors $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ such that $1 \leq m_r < m_s \leq n$ for $1 \leq r < s \leq k$, and $P_{n,0}$ is defined as the empty set.

**Theorem 3.4** [23]. Let $a_1 > a_2 > \cdots > a_n \geq 0$ and define
\[
F(x) = \frac{\prod_{\sigma \in E_n} [\Gamma(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \cdots + (n-1)a_{\sigma(n)})]}{\prod_{\sigma \in O_n} [\Gamma(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \cdots + (n-1)a_{\sigma(n)})]}.
\] (3.24)

Then $F(x - a_2 - 2a_3 - \cdots - (n-1)a_n) = e^{-H(x)}$ and $H$ is completely monotonic; hence, $F$ is completely monotonic. The same conclusion holds for the function
\[
F(x, q) = \frac{\prod_{\sigma \in E_n} [\Gamma_q(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \cdots + (n-1)a_{\sigma(n)})]}{\prod_{\sigma \in O_n} [\Gamma_q(x + a_{\sigma(2)} + 2a_{\sigma(3)} + \cdots + (n-1)a_{\sigma(n)})]}.
\] (3.25)

Note that
\[
\sum_{k=1}^n (k-1)(a_{\sigma(k)} - a_k) \geq 0
\]
holds for any permutation $\sigma$ when $a_1 > a_2 > \cdots > a_n > 0$.

**Theorem 3.5.** The function
\[
F_n(x) = \frac{\Gamma(x) \prod_{k=1}^{[x/2]} \left[ \prod_{m \in P_{2k,0}} \Gamma(x + \sum_{j=1}^{2k-1} a_{m_j}) \right]}{\prod_{k=1}^{[x+1/2]} \left[ \prod_{m \in P_{2k-1,0}} \Gamma(x + \sum_{j=1}^{2k-1} a_{m_j}) \right]}
\] (3.26)
is of the form $e^{-H(x)}$ and $H'(x)$ is completely monotonic; hence, $F_n$ is completely monotonic.
Theorem 3.5 is due to Grinshpan and Ismail [23]. In particular, the case \( n = 3 \), after shifting \( x \) by \( a_0 \), says that the function
\[
\frac{\Gamma(x + a_0) \Gamma(x + a_0 + a_2)}{\Gamma(x + a_0 + a_1 + a_2) \Gamma(x + a_0 + a_2 + a_3)}
\]
is completely monotonic for \( a_1 \geq a_2 \geq a_3 \geq 0 \), and \( a_0 \geq 0 \). Choosing \( a_0 = 1 \) and dividing the above function by its value at \( x = a_0 = 1 \), we find that the corresponding \( x_n \) are given by
\[
x_n = \frac{n(n + a_1 + a_2)(n + a_1 + a_3)(n + a_2 + a_3)}{(n + a_1)(n + a_2)(n + a_3)(n + a_1 + a_2 + a_3)}.
\]
(3.27)
In this case \( x_n \to 1 \) and \( |\sqrt{x_n} - 1| = O(1/n^2) \), as \( n \to \infty \); hence \( \sum_{n=1}^{\infty} |\sqrt{x_n} - 1| < \infty \). If we replace \( x \) by \( \sqrt{2x} \) in (2.4) and introduce the rescaled polynomials \( \psi_n(x) = \phi_n(\sqrt{2x}) \), then \( \beta_n = x_n/4 \) and the conclusions of Nevai’s theorem hold for these polynomials.

Remark 3.6. Let \( \mu \) be a probability measure such that
\[
F_r(x + a_0)/F_r(a_0) = \int_0^\infty e^{-x t} \, d\mu(t) = \int_0^1 u^{2n} \, dv(u)
\]
where we performed the change of variables \( e^{1/2} = u \). Therefore,
\[
F_r(n + a_0)/F_r(a_0) = \int_0^1 u^{2n} \, dv(u) = \int_{-1}^1 u^{2n} \left( \frac{1}{2} \nu([u]) \right)
\]
where \( \nu \) is extended as an even measure, so that \( \nu/2 \) is now a probability measure on \([-1, 1]\).
It is clear that we can define \( x_n \) by
\[
x_n = F_r(n + a_0)/F_r(n + a_0 - 1), \quad n \geq 1.
\]
(3.29)
We will show below that the \( x_n \)'s defined this way have the property \( |\sqrt{x_n} - 1| = O(1/n^2) \), \( n \to \infty \), so Nevai’s theorem is applicable.

We now show that the \( x_n \)'s defined by (3.29) satisfy the conditions in Nevai’s theorem. Observe that the number of terms in the numerator in (3.26) is
\[
1 + \binom{s}{2} + \binom{s}{4} + \cdots + \binom{s}{2\lfloor s/2 \rfloor} = 2^{s-1},
\]
while the number of terms in the denominator in (3.26) is
\[
\binom{s}{1} + \binom{s}{3} + \binom{s}{5} + \cdots + \binom{s}{2\lfloor s/2 \rfloor + 1} = 2^{s-1}.
\]
So the number of terms in the numerator and denominator in (3.26) is the same. Now the sum of the arguments of the Gamma functions in the numerator and denominator in (3.26) is
\[
2^{s-1} + 2 \binom{s}{2} + 4 \binom{s}{4} + \cdots + 2 \lfloor s/2 \rfloor \binom{s}{2\lfloor s/2 \rfloor}, \quad \text{and}
\]
\[
2^{s-1} + \binom{s}{1} + 3 \binom{s}{3} + 5 \binom{s}{5} + \cdots + 2 \lfloor (s + 1)/2 \rfloor \binom{s}{2\lfloor (s + 1)/2 \rfloor},
\]
respectively. But both are equal to \( 2^{s-1} + s2^r \). After discarding the \( s \), each sum of the remaining terms is \( s2^{s-2} \). But both the numerator and denominator sums are symmetric functions of the \( a_j \); hence, each \( a_j \) appears \( 2^{s-2} \) times. This and the monotonicity of \( F_r \) show that \( 1 > x_n = 1 - O(1/n^2) \).

It would be of great interest to study the coherent states arising from the above weight functions and associated sequences \( \{x_n\} \). Many of these are expected to have physical significance.
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