Duality Symmetries in Orbifold Models.

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ABSTRACT

We derive the duality symmetries relevant to moduli dependent gauge coupling constant threshold corrections, in Coxeter $\mathbb{Z}_N$ orbifolds. We consider those orbifolds for which the point group leaves fixed a 2-dimensional sublattice $\Lambda_2$, of the six dimensional torus lattice $\Lambda_6$, where $\Lambda_6$ cannot be decomposed as $\Lambda_2 \oplus \Lambda_4$. 
In the space of all known conformal field theories, orbifold models represent good candidates for a phenomenologically promising string compactification [1, 2]. The marginal deformations of the underlying conformal field theory of the orbifold are the moduli which parametrize, locally, the string background vacuum [3]. A peculiar feature in string compactifications, not shared with that of conventional point-particles, is the invariance of the spectrum under the action of some discrete group acting on the moduli [5, 6, 8, 9]. This group, the so-called target space duality which generalizes the well known \( R \to 1/R \) duality symmetry for circle compactification where \( R \) is the radius of the circle, can then be implemented to restrict the moduli space to a fundamental domain.

The duality groups for toroidal and orbifold compactification in lower dimensions have been considered in [3-10]. For two-dimensional toroidal compactification [4] one finds two copies of the modular group \( \text{PSL}(2, \mathbb{Z}) \) acting on the two complex moduli, \( T \) and \( U \) describing the target space. By comparison, for the two-dimensional \( \mathbb{Z}_3 \) orbifold, the \( U \) modulus is frozen (i.e. its value is fixed), and the duality group \( \Gamma_T \) associated with the complex \( T \) modulus is the modular group \( \text{PSL}(2, \mathbb{Z}) \). Based on these results, it was sometimes assumed in the literature that the modular group is realized as a duality group for each complex modulus associated with the three complex planes of the six-dimensional orbifold. However a counter example has been found in ref. [11], in which the duality group of the Coxeter \( \mathbb{Z}_7 \) orbifold with \( SU(7) \) lattice has been shown to be an overall \( \text{PSL}(2, \mathbb{Z}) \) for the three complex moduli \( T_i \) rather than \( \text{PSL}(2, \mathbb{Z})^3 \).

More recently in ref. [15], the moduli-dependent threshold corrections to the gauge coupling constants for some orbifold models, arising from the twisted sectors with one unrotated plane under the twist action [12, 13, 14], were found not to be invariant under the full modular group but rather under certain congruence subgroups of \( \text{PSL}(2, \mathbb{Z}) \). These are the only sectors yielding moduli dependent contributions to the threshold corrections, and are known as \( \mathcal{N} = 2 \) sectors as they possess two space-time supersymmetries. In ref. [13] it was demonstrated that provided the six-dimensional lattice can be decomposed into a direct sum of
a two-dimensional and a four-dimensional sub-lattices, \( \Lambda_6 = \Lambda_2 \bigoplus \Lambda_4 \), with the unrotated plane lying in \( \Lambda_2 \), the threshold corrections are then invariant under the full modular group. This is essentially consistent with the modular symmetry of those moduli associated with the plane lying in the two-dimensional lattice, \( \Lambda_2 \). However many orbifold lattices do not admit the above decomposition. Here we will investigate the duality group of the moduli of the invariant planes of these orbifolds, which is a symmetry of those sectors of Hilbert space contributing to gauge coupling threshold corrections.

This work is organized as follows. First, we briefly review toroidal and orbifold compactification, and the method of obtaining their duality symmetries. Next we concentrate on the two-dimensional case and show that the duality group is \( PSL(2, \mathbb{Z}) \) for all symmetric \( \mathbb{Z}_N \) orbifolds. This demonstrates that provided the lattice is a direct sum of three two-dimensional sub-lattices, the duality group of a six-dimensional orbifold is always a product of the modular group \( PSL(2, \mathbb{Z}) \), one for each complex modulus. Note that the \( U \) moduli are only present for \( \mathbb{Z}_2 \) planes. The relevance of 2-dimensional compactification to the study of threshold corrections will become clear in what follows.

Finally, motivated by the results of [15], we determine the symmetry group which leaves invariant the spectrum of the twisted sectors with only two rotated planes, i.e. those that possess \( \mathcal{N} = 2 \) supersymmetry. This spectrum is only sensitive to the geometry of the unrotated complex plane and independent of the moduli of the other two completely rotated complex planes. The symmetry groups obtained are those relevant to threshold corrections of the gauge coupling constants in these models. Only the cases where the invariant planes do not lie entirely in a two-dimensional sub-lattice of the 6-dimensional torus lattice, a la Dixon et al [13], are considered.

We begin with some aspects of duality transformations of closed string compactification on tori and orbifolds. A \( d \)-dimensional torus is defined as a quotient
of $\mathbb{R}^d$ with respect to a lattice $\Lambda$ defined by

$$\Lambda = \left\{ \sum_{i=1}^{d} a^i e_i, \quad a^i \in \mathbb{Z} \right\}. \quad (1)$$

In the absence of Wilson lines, the toroidal compactification [16, 17] is described by $d^2$ parameters given by an antisymmetric field $B_{ij}$, and a metric $G_{ij}$ defined as

$$G_{ij} = e_i.e_j \quad (2)$$

For the string coordinates compactified on the torus in the above background, the left and right momenta are given by

$$P_L = \frac{p}{2} + (G - B)w, \quad P_R = \frac{p}{2} - (G + B)w, \quad (3)$$

where $w$ and $p$, the windings and the momenta respectively, are $d$-dimensional integer valued vectors taking values on the lattice $\Lambda$ and its dual $\Lambda^*$. The zero modes of the compactified string coordinates taking values on the lattice $\Lambda$ and its dual $\Lambda^*$, the dependence on the geometry of the background has the contribution $H$ and $S$ to the scaling dimension and spin of the vertex operators given by

$$H = \frac{1}{2}(P_L^t G^{-1} P_L + P_R^t G^{-1} P_R) = \frac{1}{4} p^t G^{-1} p - p^t G^{-1} Bw - w^t B G^{-1} Bw + w^t G w,$$
$$S = \frac{1}{2}(P_L^t G^{-1} P_L - P_R^t G^{-1} P_R) = p^t w, \quad (4)$$

It is very convenient to write $H$ and $S$ in the following quadratic forms

$$H = \frac{1}{2} u^t \Xi u, \quad S = \frac{1}{2} u^t \eta u. \quad (5)$$

where

$$u = \begin{pmatrix} w \\ p \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}, \quad \Xi = \begin{pmatrix} 2(G - B)G^{-1}(G + B) & BG^{-1} \\ -G^{-1} B & \frac{1}{2} G^{-1} \end{pmatrix}. \quad (6)$$

Here the index $t$ denotes the transpose, $u$ is a $2d$ component integer vector, $\Xi$ and $\eta$ are $2d \times 2d$ dimensional matrices and $1_d$ denotes the identity matrix.
in \(d\) dimensions. Clearly the \(d^2\) moduli are all contained in the matrix \(\Xi\). The discrete target space duality symmetries are determined by searching for all integer-valued linear transformations of the quantum numbers which leaves the spectrum invariant. These linear transformations can be written as

\[
\Omega : u \rightarrow S_\Omega(u) = \Omega^{-1}u. \tag{7}
\]

To preserve the spin, the transformation matrix \(\Omega\) should satisfy the condition:

\[
\Omega^t \eta \Omega = \eta. \tag{8}
\]

Moreover, the invariance of \(H\) induces a transformation on the moduli. Such a transformation defines the action of the duality group given by

\[
\Xi \rightarrow S_\Omega(\Xi) = \Omega^t \Xi \Omega. \tag{9}
\]

The generalization of the above results to the orbifold case, without Wilson lines is straightforward. The orbifold is defined by the quotient of the torus by a group of automorphisms \(P\) of the lattice, also known as the point group [1]. This group acts on the quantum numbers by

\[
u \rightarrow u' = \mathcal{R} u, \quad \mathcal{R}^N = 1, \tag{10}
\]

where \(\mathcal{R}\) is given by the matrix

\[
\mathcal{R} = \begin{pmatrix} Q & 0 \\ 0 & (Q^t)^{-1} \end{pmatrix} \tag{11}
\]

and \(Q\) is an integer matrix specifying the orbifold point group. To insure that the point group is a lattice automorphism, the background fields must satisfy

\[
\mathcal{R}^t \Xi \mathcal{R} = \Xi, \quad \Rightarrow \quad Q^t G Q = G, \quad Q^t B Q = B. \tag{12}
\]

Finally the modular symmetries of the orbifold are those of the torus commuting with the twist matrix \(\mathcal{R}\) [18].
Before moving on to discuss modular symmetries of six dimensional $Z_N$ orbifolds, it is useful to describe the target space duality groups in 2-dimensional toroidal [4] and $Z_N$ orbifold compactifications. In the two-dimensional case, it is convenient to group the four real degrees of freedom parametrizing the background into two complex moduli [4] defined as

$$T = 2\left(B + i\sqrt{\det G}\right), \quad U = \left(\frac{G_{12}}{G_{11}} + i\frac{\sqrt{\det G}}{G_{11}}\right)$$

In this complex parametrization of the moduli, the duality group is given by two copies of the modular group $PSL(2, \mathbb{Z})$ acting on the moduli as

$$U \rightarrow \frac{a'd'U + b'}{c'd'U + d'}, \quad a'd' - b'c' = 1, \quad T \rightarrow \frac{aT + b}{cT + d}, \quad ad - bc = 1. \quad (14)$$

These transformations, respectively, are induced from the following transformations on the quantum numbers

$$\begin{pmatrix} w \\ p \end{pmatrix} \rightarrow \Omega_U^{-1} \begin{pmatrix} w \\ p \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & (M')^{-1} \end{pmatrix} \begin{pmatrix} w \\ p \end{pmatrix},$$

$$\begin{pmatrix} w \\ p \end{pmatrix} \rightarrow \Omega_T^{-1} \begin{pmatrix} w \\ p \end{pmatrix} = \begin{pmatrix} dI_2 & -cL \\ bL & aI_2 \end{pmatrix} \begin{pmatrix} w \\ p \end{pmatrix},$$

where

$$M = \begin{pmatrix} a' & -b' \\ -c' & d' \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (15)$$

Note that the spectrum is also invariant under the exchange $T \leftrightarrow U$ induced by the exchange $n_1 \leftrightarrow -m_1$ and “parity transformations” $T \leftrightarrow -\bar{T}, U \leftrightarrow -\bar{U}$ [4].

The symmetries of the two-dimensional $Z_N$ orbifold are those of the torus commuting with the matrix $\mathcal{R}$ defining the twist. In the case when $N = 2$, one obtains the same modular symmetries as for the toroidal case. However for $N \neq 2$, the twist freezes the $U$ moduli and it can be easily seen that the $PSL(2, \mathbb{Z})$ acting on $T$ still describes the duality group, since $\Omega_T^{-1}$ commutes with all the $\mathcal{R}$ matrices defining the twists of the various orbifolds.
It is now clear, from the results of the previous section, that in orbifold models for which the lattice is a product of three two-dimensional sublattices, and where each of the three complex planes of the orbifold lies entirely in each of these sublattices, the duality group is a product of \( PSL(2, \mathbb{Z}) \)'s one for each complex modulus. Such symmetry is demonstrated by studying the spectrum of the untwisted sectors of these theories. Also in these models, the twisted sectors with one unrotated complex plane have the modular group as a duality group. This is simply because the twisted spectrum depends on the moduli of the unrotated plane in the same way as that of the untwisted sectors. Thus the threshold corrections to the gauge coupling constants in these models are invariant under the modular group \( PSL(2, \mathbb{Z}) \) [12]. However, there are many orbifold models where the unrotated plane does not lie in a two-dimensional sub-lattice. Examples of such models are certain \( \mathbb{Z}_N \) Coxeter orbifolds [19]. It is our purpose in this section to study the duality symmetries of the threshold corrections in these models by investigating the symmetries of the spectra of their twisted sectors with one unrotated plane.

As an example, consider the orbifold \( \mathbb{Z}_6 - II \), with the twist defined by \( \theta = (2, 1, -3)/6 \) and an \( SU(6) \times SU(2) \) lattice. Clearly in this model the \( \theta^2 \) and \( \theta^3 \) sector, respectively, have the first and third planes unrotated. We would like to investigate the symmetry group for the moduli \( T_1 \) and \( (T_3, U_3) \) associated with the first and third complex planes respectively, which leaves the spectrum of the \( \theta^2 \) and \( \theta^3 \) twisted sectors invariant. The matrix \( Q \) defining the twist action on the quantum numbers is given by

\[
Q = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}.
\]

(17)
The constant background fields compatible with the twist are obtained using (12) and are given as

\[
G = \begin{pmatrix}
  r^2 & x & l^2 & R^2 & l^2 & u^2 \\
x & r^2 & x & l^2 & R^2 & -u^2 \\
l^2 & x & r^2 & x & l^2 & u^2 \\
R^2 & l^2 & x & r^2 & x & -u^2 \\
l^2 & R^2 & l^2 & x & r^2 & u^2 \\
u^2 & -u^2 & u^2 & -u^2 & u^2 & y \\
\end{pmatrix}, \tag{18}
\]

\[
B = \begin{pmatrix}
  0 & -\beta & -\delta & 0 & \delta & -\gamma \\
  \beta & 0 & -\beta & -\delta & 0 & \gamma \\
  \delta & \beta & 0 & -\beta & -\delta & -\gamma \\
  0 & \delta & \beta & 0 & -\beta & \gamma \\
  -\delta & 0 & \delta & \beta & 0 & -\gamma \\
  \gamma & -\gamma & \gamma & -\gamma & \gamma & 0 \\
\end{pmatrix}, \tag{19}
\]

with \( R^2 = 2l^2 - r^2 - 2x \). Consider the \( \theta^3 \) sector first. Here the twisted states have left and right momenta, \( P_L \) and \( P_R \), characterized by the winding and momenta \( w \) and \( p \) satisfying \( Q^3 w = w \) and \( ((Q^T)^{-1})^3 p = p \). Therefore, they are given by

\[
w = \begin{pmatrix}
  n_1 \\
n_2 \\
0 \\
n_1 \\
n_2 \\
0 \\
\end{pmatrix}, \quad p = \begin{pmatrix}
m_1 \\
m_2 \\
-m_1 - m_2 \\
m_1 \\
m_2 \\
0 \\
\end{pmatrix}. \tag{20}
\]

In order to study the duality group of \( T_1 \), it is convenient to recast the geometry dependent scaling and spin \( H_1, S_1 \) associated with the vertex operators creating the \( \theta^3 \) twisted states in a quadratic form similar to (5). After some algebraic
calculations it turns out that one can write

\[ H_1 = \frac{1}{2} V_1^T \left( \begin{array}{ccc} 2(G_1 - B_1)G_1^{-1}G_1 + B_1 & B_1G_1^{-1} \\ -G_1^{-1}B_1 & \frac{1}{2}G_1^{-1} \end{array} \right) V_1, \quad S_1 = \frac{1}{2} V_1^T \eta V_1 \quad (21) \]

where

\[ V_1 = \begin{pmatrix} w \\ 2p \end{pmatrix}, \quad G_1 = 2 \begin{pmatrix} -2(t^2 + x) & t^2 + x \\ t^2 + x & -2(t^2 + x) \end{pmatrix}, \quad B_1 = 2 \begin{pmatrix} 0 & -\beta + \delta \\ -\beta + \delta & 0 \end{pmatrix}. \quad (22) \]

Clearly the transformation \( T_1 = \frac{aT_1 + b}{cT_1 + d} \) leaves the theory invariant provided that one transforms \( V_1 = \begin{pmatrix} w \\ p' \end{pmatrix}, \quad (p' = 2p) \), as in eq. (15),

\[ \begin{pmatrix} w \\ p' \end{pmatrix} \rightarrow \begin{pmatrix} dI_2 & -cL \\ bL & aI_2 \end{pmatrix} \begin{pmatrix} w \\ p' \end{pmatrix}. \quad (23) \]

In order that \( p \) transforms as integers, \( b \) must be even. Therefore the modular group \( \Gamma_{T_1} \) associated with the \( T_1 \) moduli is \( \Gamma^0(2) \). In general the group \( \Gamma^0(n) \) is represented by the following set of matrices

\[ \Gamma^0(n) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad ad - bc = 1, \quad b = 0 \mod n. \quad (24) \]

Similarly one can repeat the same analysis for the \( \theta^2 \) twisted sector, here the twisted states have left and right momenta, \( P_L \) and \( P_R \), characterized by the winding and momenta \( w \) and \( p \) satisfying \( Q^2w = w \) and \( ((Q^T)^{-1})^2 p = p \). Therefore, they are given by

\[ w = \begin{pmatrix} n_1 \\ 0 \\ n_1 \\ 0 \\ n_1 \\ n_2 \end{pmatrix}, \quad p = \begin{pmatrix} m_1 \\ -m_1 \\ m_1 \\ -m_1 \\ m_1 \\ m_2 \end{pmatrix}. \quad (25) \]

The geometry dependent scaling and spins \( H_3 \) and spin \( S_3 \) associated with the
vertex operators creating the $\theta^2$ twisted states are given by

$$H_3 = \frac{1}{2} V_3^T \begin{pmatrix} 2(G_3 - B_3)G_3^{-1}(G_3 + B_3) & B_3G_3^{-1} \\ -G_3^{-1}B_3 & \frac{1}{2}G_3^{-1} \end{pmatrix} V_3,$$

$$S_3 = \frac{1}{2} V_3^T \eta V_3 \quad (26)$$

where

$$V_3 = \begin{pmatrix} w \\ p' \end{pmatrix}, \quad p' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} p, \quad G_3 = \begin{pmatrix} 6l^2 + 3r^2 & 3u^2 \\ 3u^2 & y \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}.$$

Using (13), $T_1$ is expressed in terms of $(G_1, B_1)$, while the moduli $T_3$ and $U_3$ corresponding to the third plane are expressed in terms of $(G_3, B_3)$.

Now turning to the duality symmetries for the third plane, $H_3$ and $S_3$ remain invariant under the transformations

$$U_3 \rightarrow \frac{d'U_3 + b'}{c'U_3 + d'}, \quad a'd' - b'c' = 1, \quad T_3 \rightarrow \frac{aT_3 + b}{cT_3 + d}, \quad ad - bc = 1, \quad (27)$$

provided that $V_3$ transform as in eq. (15). Again in order for the $p$ to transform as integers, the following constraints are obtained

$$b = 0 \pmod{3}, \quad c' = 0 \pmod{3}. \quad (28)$$

Thus the duality group in this case is $\Gamma_{T_3} \times \Gamma_{U_3} = \Gamma_0(3) \times \Gamma_0(3)$. The group $\Gamma_0(n)$ is represented by the following set of matrices

$$\Gamma_0(n) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad ad - bc = 1, \quad c = 0 \pmod{n}. \quad (29)$$

Now consider the same orbifold model but with the lattice $SU(6) \times SU(2)$. This model has been investigated in [15] with regard to the threshold corrections to the gauge coupling constants. In this case, the first plane, unrotated by $\theta^2$ lies, entirely in the sub-lattice $SU(3)$ and hence the states in the $\theta^2$ twisted sector have winding
and momenta taking values in the $SU(3)$ lattice and its dual respectively. Clearly, the spectrum of these states is invariant under the full modular group acting on the $T_1$ moduli. The third complex planes is unrotated under the $\theta^3$ action. The $\theta^3$ twisted states have the following geometry-dependent scale and spin which can be written in the following quadratic forms,

$$H_3 = \frac{1}{2} V_3^T \begin{pmatrix} 2(G_3 - B_3)G_3^{-1}(G_3 + B_3) & B_3G_3^{-1} \\ -G_3^{-1} B_3 & \frac{1}{2} G_3^{-1} \end{pmatrix} V_3, \quad S_1 = \frac{1}{2} V_3^T \eta V_3$$

(30)

where

$$V_3 = \begin{pmatrix} w \\ p' \end{pmatrix}, \quad p' = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} w \\ p \end{pmatrix}$$

(31)

Here $G_3$ and $B_3$ are the background sub-matrices defining the moduli $T_3$ and $U_3$ [15].

$H_3$ and $S_3$ remains invariant under the transformation

$$T_3 \rightarrow \frac{aT_3 + b}{cT_3 + d}, \quad ad - bc = 1,$$

provided that the momenta quantum numbers transform by

$$p' \rightarrow bLw + ap', \quad \Rightarrow p \rightarrow \frac{-b}{3} \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} w + ap$$

(32)

In order for the momenta to transform as integers, $b$ must be a multiple of 3. Therefore $\Gamma_{T_3} = \Gamma^0(3)$. Also, $H_3$ and $S_3$ remains invariant under the transformation

$$U_3 \rightarrow \frac{a'U_3 + b'}{c'U_3 + d'}, \quad a'd' - b'c' = 1,$$

(33)

provided that the momenta quantum numbers transform by

$$p' \rightarrow (M^t)^{-1} p', \quad \Rightarrow p \rightarrow \frac{1}{3} \begin{pmatrix} 2d' + 2c' + b' + a' & 2d' - 4c' + b' - 2a' \\ d' + c' - b' - a' & d' - 2c' + b' - 2a' \end{pmatrix} p.$$

(34)

In order for the momenta to transform as integers, the following constrains must
be imposed,

\[(d' + c') - (a' + b') = 0 \pmod{3}. \quad (35)\]

However, in [15], the threshold correction for this model, when expressed in terms of \( U'_3 = U_3 + 2 \), is invariant under \( \Gamma_{U'_3} = \Gamma^0(3) \). This symmetry can be explained as follows. By using (33) and (34), we get the following transformation

\[
U'_3 \rightarrow \frac{AU'_3 + B}{CU'_3 + D}; \quad AD - BC = 1, \quad B = 0 \pmod{3}, \quad (36)
\]

with

\[
A = d' - 2c', \quad B = -b' - 2d' + 4c' + 2a', \\
C = -c', \quad D = a' + 2c'. \quad (37)
\]

The above procedure has been applied for all Coxeter \( Z_N \) orbifolds to study the duality symmetries of the twisted sectors with one unrotated plane which does not lie in a two-dimensional sub-lattice. The results are summarized in the following table

| Orbifold | \( \theta \) | Lattice | Duality group |
|----------|-------------|---------|---------------|
| \( Z_4 - a \) | \( 1/4(1,1,-2) \) | \( SU(4) \times SU(4) \) | \( \Gamma_{T_3}=\Gamma^0(2), \Gamma_{U_3}=PSL(2,Z) \) |
| \( Z_4 - b \) | \( 1/4(1,1,-2) \) | \( SU(4) \times SO(5) \times SU(2) \) | \( \Gamma_{T_3}=\Gamma^0(2), \Gamma_{U_3}=\Gamma_0(2) \) |
| \( Z_6 - II - a \) | \( (2,1,-3)/6 \) | \( SU(6) \times SU(2) \) | \( \Gamma_{T_3}=\Gamma^0(3), \Gamma_{U_3}=\Gamma_0(3), \Gamma_{T_1}=\Gamma^0(2) \) |
| \( Z_6 - II - b \) | \( (2,1,-3)/6 \) | \( SU(3) \times SO(8) \) | \( \Gamma_{T_3}=\Gamma^0(3), \Gamma_{U_3+2}=\Gamma^0(3) \) |
| \( Z_6 - II - c \) | \( (2,1,-3)/6 \) | \( SU(3) \times SO(7) \times SU(2) \) | \( \Gamma_{T_3}=\Gamma^0(3), \Gamma_{U_3}=\Gamma_0(3) \) |
| \( Z_8 - II - a \) | \( (1,3,-4)/8 \) | \( SU(2) \times SO(10) \) | \( \Gamma_{T_3}=\Gamma^0(2), \Gamma_{U_3}=\Gamma_0(2) \) |
| \( Z_{12} - I - a \) | \( (1,-5,4)/12 \) | \( E_6 \) | \( \Gamma_{T_3}=\Gamma^0(2) \) |

As can be seen from the various examples in the above table, the duality group is different for different lattice choices. The symmetries of the threshold corrections in the examples considered in [15] are in agreement with our results. The duality
symmetries of string loop threshold corrections of other cases in the table, also agree with the duality symmetries obtained from the spectra of states [20].

In conclusion, we have calculated the symmetry groups associated with the moduli of the $\mathbb{Z}_N$ Coxeter orbifolds with planes that do not entirely lie in a two-dimensional sub-lattice of the full torus lattice. The duality groups of these models is always a congruence subgroup of $PSL(2,\mathbb{Z})$. The form of the lattice plays an important role in the determination of the modular group. It should be noted that we only considered background fields with no Wilson lines, i.e., $(2,2)$ models with unbroken $E_6$ gauge symmetry. To make contact with the low-energy physics, one should consider $(2,0)$ models with the inclusion of the appropriate Wilson lines. These Wilson lines will appear in the expression of the momenta of the twisted sectors with one unrotated plane. An important question for string phenomenology is the determination of the target space duality symmetry in the presence of Wilson lines and the calculation of the threshold corrections for $(2,0)$ orbifolds.

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