GENERIC CUSPIDAL REPRESENTATIONS OF $U(2,1)$

SANTOSH NADIMPALLI

Abstract. Let $F$ be any non-Archimedean local field with a Galois involution $\sigma$ and $F_0$ be the fixed field for the action of $\sigma$. When the residue characteristic of $F_0$ is odd, using the explicit construction of cuspidal representations of classical groups by Stevens, we classify generic cuspidal representations of $U(2,1)(F/F_0)$.

1. Introduction

Let $F$ be a non-Archimedean local field with a non-trivial Galois involution $\sigma$, and let $F_0$ be the fixed field of $\sigma$. Let $V$ be a 3-dimensional $F$-vector space and $h : V \times V \to F$ be a non-degenerate hermitian form with

$$h(xv,yw) = x\sigma(y)\sigma(h(w,v)) \text{ for all } v, w \in V \text{ and } x, y \in F.$$ (1.1)

We assume that the space $(V,h)$ is isotropic and let $G$ be the unitary group $U(V,h)$. Let $\sigma_h$ be the adjoint anti-involution induced by $h$ on $\text{End}_F(V)$. In this article, we classify generic cuspidal representations of $G$ using the inducing data of cuspidal representations, i.e., using a skew semisimple stratum involved in the construction of a cuspidal representation.

Let $H$ be the group of $F_0$-rational points of a quasi-split reductive algebraic group defined over $F_0$. Let $U$ be the unipotent radical of an $F_0$-Borel subgroup of $H$. A generic representation of $H$ is an irreducible smooth representation which admits a certain $U$-equivariant linear functional (see Section 2.4). A cuspidal representation $(\pi,V)$ is a generic representation if and only if there exists a linear functional $l : V \to \mathbb{C}$ and a character $\Psi$ of $U$ such that

$$l(\pi(u)v) = \Psi(u)l(v), \text{ for all } u \in U, v \in V.$$ If such a linear functional exists, then the character $\Psi$ necessarily satisfies the non-degenerate condition on the pair $(U,\Psi)$. The functional $l$ is called a Whittaker linear functional. Moreover, for the group $G$, genericity of an irreducible smooth representation does not depend on the choice of $(U,\Psi)$.

A Whittaker linear functional on an irreducible smooth representation, if it exists, is unique up to scalars (see [Sha74] and [Rod73]). These functionals are first used by Jacquet–Langlands to define the local $L$ and $\epsilon$-factors for $\text{GL}_2(F_0)$ ([LJ70, Theorem 2.18]). These methods and their generalisations have played a fundamental role in the Langlands program, and especially in the theory of integral representations of $L$-functions (see [GS88]). Moreover, in the context of the local Langlands correspondence for a quasi-split group, generic representations are used to associate enhanced $L$-parameters, in a canonical way, inside a given tempered $L$-packet (see [Kal16, Section 1.2]). It is in this context that we are interested in classification of generic cuspidal representations.

Explicit results on the genericity of cuspidal representations have been studied in several works. To begin with, every cuspidal representation of $\text{GL}_n(F_0)$ is generic (see [BZ76, Chapter 3, 5.18]), but this is no longer true for classical groups. If the characteristic of $F_0$ is zero, $F/F_0$ is unramified and the cardinality of the residue field of $F_0$ is odd, the cuspidal generic representations of $G$ are classified by Murnaghan in the article [Mur95, Theorem 7.13]. The methods used in [Mur95] are based on character formulas for cuspidal representations—the Murnaghan–Kirillov formula—and using a local character expansion to relate with Shalika germs. Reeder and DeBacker also studied genericity of very cuspidal representations, arising from unramified torus, of an unramified $p$-adic group (see [DR10]). Blondel and Stevens using different techniques have classified generic cuspidal representations for $\text{Sp}_4(F_0)$, for a non-Archimedean local field.

Date: April 16, 2019.


We describe our results using stratum from the theory of types (see Section 3 and references in loc.cit).

Let \( \mathfrak{r} = [\Lambda, n, 0, \beta] \) be any skew semisimple stratum in \( \text{End}_F(V) \), in particular, \( \Lambda \) is a lattice sequence, \( n \) is a non-negative integer, and \( \beta \in \text{End}_F(V) \) with \( \sigma_\mathfrak{r}(\beta) = -\beta \) is an elliptic semisimple element of the Lie algebra of \( G \). Let \( \Pi_\mathfrak{r} \) be the set of cuspidal representation containing a type, in the sense of Bushnell–Kutzko, constructed from the stratum \( \mathfrak{r} \). Let \( \psi \) be a fixed additive character of \( F \), and let \( \psi_\beta \) be the function sending \( x \in \text{End}_F(V) \) to \( \psi(\text{tr}(\beta(x - 1))) \). Let \( \mathfrak{X}_\beta \) be the set consisting of Borel subgroups \( B \) of \( G \) such that \( \psi_\beta \) defines a character on the unipotent radical of \( B \).

The main result of this article is the following theorem:

**Theorem 1.0.1.** Let \( F \) be a non-Archimedean local field with odd residue characteristic. Let \( \mathfrak{r} = [\Lambda, n, 0, \beta] \) be any skew semisimple stratum with \( n > 0 \). All representations contained in \( \Pi_\mathfrak{r} \) are either generic or non-generic. If \( \mathfrak{X}_\beta \) is empty, then any cuspidal representation in the set \( \Pi_\mathfrak{r} \) is non-generic. Except when \( \beta \) has an eigenspace of dimension 2, a cuspidal representation in the set \( \Pi_\mathfrak{r} \) is generic if and only if \( \mathfrak{X}_\beta \) is non-empty. If \( \beta \) has an eigenspace of dimension 2, then any cuspidal representation in the set \( \Pi_\mathfrak{r} \) is non-generic.

The set \( \mathfrak{X}_\beta \) is the set of \( F_0 \)-rational points of a closed subvariety of the variety of Borel subgroups of the unitary group associated to \((V, h)\). Except when \( F/F_0 \) is ramified and \( \beta \) has 3-different eigenspaces it is not hard to discribe whether \( \mathfrak{X}_\beta \) is empty or non-empty. We hence prove a more precise form of Theorem 1.0.1 and for this we refer to Theorem 1.0.1. The genericity of depth-zero cuspidal representations \( G \) is well understood and see [DR09, Section 5]. However, we recall these results for giving a complete analysis of genericity of cuspidal representations of \( G \), especially, for those results not stated in the literature, for instance, when \( F/F_0 \) is ramified.

We briefly sketch the method of proof and the contents in each section. In general, the proof uses ideas in the line of Mackey decomposition and the explicit construction of cuspidal representations for classical groups by Stevens in the articles [Ste05] and [Ste08]. We are heavily influenced by the paper of Blondel–Stevens [BS09] in which they related the set \( \mathfrak{X}_\beta \) and genericity of cuspidal representations of \( \text{Sp}_4(F_0) \).

To begin with, the algebra \( F[\beta] \) is a direct sum of fields, say \( F[\beta] = \bigoplus_{i=1}^k F[\beta_i] \) with \( \beta = \sum_{i=1}^k \beta_i \). This decomposition of \( F[\beta] \) corresponds to an orthogonal decomposition of \( V = \bigoplus_{i=1}^k V_i \), where \( F[\beta] \) acts on \( V_i \) via its projection onto \( F[\beta_i] \), for \( 1 \leq i \leq k \). The above decomposition is unique and is determined by \( \beta \). In Sections 2 and 3 we recall some necessary results from the construction of cuspidal representations for classical groups. In Section 4 we consider the case where \( F[\beta] \) is a field. We prove that \( \mathfrak{X}_\beta \) is non-empty and from this we will show that any representation in \( \Pi_\mathfrak{r} \) is generic.

In Section 5 we consider the case where \( F[\beta] \) is a 3-dimensional algebra, \( F[\beta] = F[\beta_2] \oplus F[\beta_3] \), and \( \beta = \beta_2 + \beta_3 \) such that \( |F[\beta_2] : F| = 2 \). We will completely determine when \( \mathfrak{X}_\beta \) is non-empty. This depends only on the valuations of \( \beta_i \) in the field \( F[\beta_i] \), and on the isomorphism class of the hermitian space \((V_2, h)\). Then we will use these results to show that a representation in \( \Pi_\mathfrak{r} \) is generic if and only if \( \mathfrak{X}_\beta \) is non-empty. In this case we have to sometimes find a nice integral model of \( \mathfrak{X}_\beta \) and lift points from its special fibre.

In sections 6 and 7 we treat the cases where \( F[\beta] \) is a direct sum of two copies of \( F \) and three copies of \( F \) respectively. The strategy is similar to that of the previous sections. But, in Section 5 we will see examples when \( \mathfrak{X}_\beta \) is non-empty and yet any representation in the set \( \Pi_\mathfrak{r} \) is non-generic. We remind that \( \beta \) is not a regular semisimple element in this case. In section 7 we have \( \beta = \beta_1 + \beta_2 + \beta_3 \), with \( \beta_i \in F \) and \( \sigma(\beta_i) = -\beta_i \). When \( F/F_0 \) is unramified, the non-emptiness of the set \( \mathfrak{X}_\beta \) depends only the valuations of \( \beta_i \). However, when \( F/F_0 \) is ramified the information on the valuations of \( \beta_i \), for \( 1 \leq i \leq 3 \), is not enough to determine if the set \( \mathfrak{X}_\beta \) is empty or not. Nonetheless, we will show that any representation in \( \Pi_\mathfrak{r} \) is generic if and only if the set \( \mathfrak{X}_\beta \) is non-empty.

**Acknowledgements** I want to thank Maarten Solleveld for many useful discussions and clarifications during the course of this work. The author is supported by the NWO Vidi grant “A Hecke algebra approach to the local Langlands correspondence” (nr. 639.032.528). I want to thank Peter Badea for suggesting some useful references and to Geo Kam-Fei Tam for various discussions on the present subject.
2. Preliminaries

All representations in this article are defined over complex vector spaces. Let $G, H$ be two groups with $H \subset G$ and let $\rho$ be a representation of $H$. We denote by $\rho^g$ the representation of $g^{-1}Hg$ sending $h \in g^{-1}Hg$ to $\rho(ghg^{-1})$. The group $g^{-1}Hg$ is denoted by $H^g$.

For any real number $x$, we denote by $\lfloor x \rfloor$ the greatest integer less than or equal to $x$. Let $\lfloor x \rfloor$ be the smallest integer greater than or equal to $x$. Let $x^+$ be the smallest integer strictly bigger than $x$ and $x^-$ be the greatest integer strictly smaller than $x$.

2.1. For any non-Archimedean local field $K$, let $\mathfrak{o}_K$ be the ring of integers of $K$, let $\mathfrak{p}_K$ be the maximal ideal of $\mathfrak{o}_K$ and let $k_K$ be the residue field $\mathfrak{o}_K/\mathfrak{p}_K$. The cardinality of the residue field is denoted by $q_K$. Let $\nu_K$ be the normalised valuation of $K$. From now we assume that $q_K$ is odd.

Let $F$ be a non-Archimedean local field with a Galois involution $\sigma$. Let $F_0$ be the fixed field of $\sigma$. Let $\varpi$ be an uniformiser of $F$ such that $\sigma(\varpi) = (-1)^{e(F/F_0)-1}\varpi$. If $F/F_0$ is ramified, let $\varpi_0$ be the element $N_{F/F_0}(\varpi)$ and set $\varpi_0 = \varpi$ otherwise. The element $\varpi_0$ is an uniformiser of $F_0$. Let $\psi_0$ be a fixed additive character of $F_0$ with conductor $\mathfrak{p}_F$. The character $\psi_0 \circ \text{tr}_{F/F_0}$ is denoted by $\psi_F$. Let $F = F_0[\delta]$, where $\sigma(\delta) = -\delta$ and $\nu_F(\delta) = e(F/F_0) - 1$. We also set $\nu_{F/F_0}$ the valuation of $F$ extending the normalised valuation $\nu_{F_0}$ of $F_0$.

For any $F_0$ scheme $X$, we denote by $X$ the set of $F_0$-rational points of $X$. If $\mathbf{H}$ is a reductive algebraic group over $F_0$, then the group $H$ is considered as a topological group whose topology is induced from the non-Archimedean metric on $F$.

2.2. Let $V$ be a 3-dimensional $F$-vector space and $h$ be a non-degenerate hermitian form on $V$ as defined in (1.1). Let $\sigma_h$ be the adjoint anti-involution on $\text{End}_F(V)$ induced by the hermitian form $h$. We assume that $(V, h)$ is isotropic and the determinant of $(V, h)$ is the trivial class in $F_0^\times/N_{F/F_0}(F^\times)$. Let $G$ be the unitary $F_0$-group scheme associated to the pair $(V, h)$. The Lie algebra $\mathfrak{g}$ of $G$ is identified with the space

$$\{ X \in \text{End}_F(V) : \sigma_h(X) = -X \}.$$ 

Henceforth, the ring $\text{End}_F(V)$ is denoted by $A$.

2.3. A basis $(e_1, e_0, e_{-1})$ of $(V, h)$ is called as a Witt-basis if and only if $h(e_1, e_1) = h(e_{-1}, e_{-1}) = 0$, $h(e_1, e_{-1}) = 1$, and $e_0 \in (e_1, e_{-1})^\perp$ with $h(e_0, e_0) = 1$. A basis $(e_1, e_{-1})$, for a 2-dimensional hermitian space $(V', h')$, is called a Witt-basis if and only if $h(e_1, e_1) = h(e_{-1}, e_{-1}) = 0$, and $h(e_1, e_{-1}) = 1$. Let $B$ be any $F_0$-rational Borel subgroup of $G$ with unipotent radical $U$. Let $T$ be a maximal $F_0$-split torus contained in $B$. Let $U$ be the unipotent radical of the opposite Borel subgroup $B$ of $B$ with respect to $T$. Let $Z$ and $N$ be the centraliser and the normaliser of $T$ respectively. We denote by $W_G$ the Weyl group $N/Z$.

There exists a Witt-basis $(e_1, e_0, e_{-1})$ of $V$—giving an embedding of $G$ in $\text{GL}_3(F)$—such that $B$ stabilises the line $\langle e_1 \rangle$. The groups $T$ and $Z$ are identified with the groups

$$\{ \text{diag}(t, 1, t^{-1}) : t \in F_0^\times \} \text{ and } \{ \text{diag}(z, z', \sigma(z)^{-1}) : z, z' \in F^\times, \ z'\sigma(z') = 1 \}$$

respectively. The groups $U$ and $\tilde{U}$ are identified with the groups

$$\begin{align*}
\begin{cases}
u(c, d) := &\frac{1}{c} & d \\ 0 & 1 & -\sigma(c) \\ 0 & 0 & 1 \end{cases} : c, d \in F, \ c\sigma(c) + d + \sigma(d) = 0,
\end{align*}$$

$$\begin{align*}
\begin{cases}
u(c, d) := &\frac{1}{c} & 0 \\ 0 & 1 & 0 \\ d & -\sigma(c) & 1 \end{cases} : c, d \in F, \ c\sigma(c) + d + \sigma(d) = 0
\end{align*}$$

respectively. The derived groups of $U$ and $\tilde{U}$, denoted by $U_{\text{der}}$ and $\tilde{U}_{\text{der}}$ respectively, are identified with the groups $\{ u(0, d) : d \in F, d + \sigma(d) = 0 \}$ and $\{ \tilde{u}(0, d) : d \in F, d + \sigma(d) = 0 \}$ respectively. We define a filtration $\{ U_{\text{der}}(r) : r \in \mathbb{Z} \}$ of compact subgroups of $U_{\text{der}}$ as follows:

$$U_{\text{der}}(r) := \{ u(0, y) : y \in \delta\mathfrak{p}_F^r \}. \quad (2.1)$$

Similarly, set $\tilde{U}_{\text{der}}(r)$ to be the group $\{ \tilde{u}(0, y) : y \in \delta\mathfrak{p}_F^r \}$, for $r \in \mathbb{Z}$. 
2.4. Let $U$ be the unipotent radical of an $F_0$-Borel subgroup $B$ of $G$. An irreducible smooth representation $(\pi, W)$ of $G$ is called a generic representation if and only if there exists a linear functional $l : W \to \mathbb{C}$ and a non-trivial character $\Psi$ of $U$ such that

$$l(\pi(u)w) = \Psi(u)l(w) \quad \text{for all } u \in U, w \in W.$$  \hfill (2.2)

The group $N$ acts transitively on the set of non-trivial characters of $U$ and hence, the genericity of an irreducible smooth representation $(\pi, W)$ of $G$ does not depend on the pair $(U, \Psi)$. The linear functional $l : W \to \mathbb{C}$ is called a Whittaker linear functional. We have

$$\dim_{\mathbb{C}} \text{Hom}(\pi, \Psi) \leq 1,$$

for any irreducible smooth representation $\pi$ of $G$ and a non-trivial character $\Psi$ of $U$ (see \cite{Sha74} and \cite{Rod73}).

2.5. Let $\beta$ be any element in the algebra $A$. Let $\psi_\beta$ be the function on $A$ given by:

$$\psi_\beta(X) = \psi_F(\text{tr}(\text{id}_V - X)) \quad \text{for all } X \in A.$$  

Let $V_1 \subset V_2 \subset V$ be a complete flag of $F$-vector spaces, and let $P$ be the Borel subgroup of $\text{GL}_F(V)$ fixing this flag. Let $N$ be the unipotent radical of $P$. If $V_2 = V_1^\perp$, then $N \cap G$ is the unipotent radical of the Borel subgroup $P \cap G$ of $G$. The function $\psi_\beta$ is a character on $N$ if and only if

$$\beta V_1 \subset V_2.$$  \hfill (2.3)

Let $\mathcal{B}$ be the set of $F_0$-rational Borel subgroups of $G$. Let $\mathcal{X}_\beta$ be the following subset of $\mathcal{B}$:

$$\mathcal{X}_\beta = \{ B \in \mathcal{B} : \psi_\beta \text{ is a character of } R_u(B)(F_0) \}.\quad \hfill (2.4)$$

Here, $R_u(B)$ is the unipotent radical of a Borel subgroup $B$ of $G$. The unipotent radical of a Borel subgroup in $\mathcal{X}_\beta$ is said to be in good position with respect to $\beta$. Note that the set $\mathcal{X}_\beta$ is the set of $F_0$-rational points of a sub-variety of the variety of Borel subgroups of $G$.

2.6. The following lemma is frequently used to prove that certain cuspidal representations are non-generic. Let $(e_1, e_0, e_{-1})$ be a Witt-basis for $(V, h)$ and $B$ be the Borel subgroup of $G$ fixing the line $\langle e_1 \rangle$. Let $U$ be the unipotent radical of $B$.

**Lemma 2.6.1.** Let $g$ be an element of $G$, and let $r$ be an integer. The character $\psi_{\beta}^g$ of $U_{\text{der}}(r)$ is non-trivial if and only if

$$\nu_{F_0}(\delta h(ge_1, \beta ge_1)) \leq -r.$$  

Similarly, the character $\psi_{\bar{\beta}}^g$ of the group $\bar{U}_{\text{der}}(r)$, is non-trivial if and only if

$$\nu_{F_0}(\delta h(ge_{-1}, \beta ge_{-1})) \leq -r.$$  

**Proof.** We prove the lemma for $U_{\text{der}}(r)$ and the other case is similar. Let $X_{\text{der}}$ be the $3 \times 3$ matrix

$$\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$$

and we have the following equality:

$$\psi_\beta(gu(0, \delta d)g^{-1}) = \psi_0(d(\text{tr}_{F/F_0}(\delta \text{tr}(\beta gX_{\text{der}}g^{-1}))).$$

Note that $\text{tr}(\beta gX_{\text{der}}g^{-1})$ is equal to $\text{tr}(g^{-1}\beta gX_{\text{der}})$. From the choice of Witt-Basis $(e_1, e_0, e_{-1})$ for $V$, we get that $\text{tr}(g^{-1}\beta gX_{\text{der}}) = h(ge_1, \beta ge_1)$. This equality and the fact that $\beta$ is skew implies that

$$\text{tr}_{F/F_0}(\delta h(ge_1, \beta ge_1)) = 2\delta h(ge_1, \beta ge_1).$$

Hence, the character $\psi_{\beta}^g$ is trivial on $U_{\text{der}}(r)$ if and only if the character $d' \mapsto \psi_0(d'\delta h(ge_1, \beta ge_1))$ on the group $p_{F_0}^r$, is trivial. Since, the conductor of $\psi_0$ is equal to $p_{F_0}$, we get the required inequality. \hfill \square

3. Strata and cuspidal representations

In this section, we recall some structures in the construction of cuspidal representations via Bushnell–Kutzko’s theory. We refer to the the articles \cite{Ste05}, \cite{Ste08} and \cite{MST14} for more details.
3.1. An \(\mathfrak{o}_F\)-lattice sequence \(\Lambda\) on \(V\) is a function from \(Z\) to the set of \(\mathfrak{o}_F\)-lattices in \(V\) satisfying the following conditions:

1. \(\Lambda(n+1) \subseteq \Lambda(n)\), for all \(n \in Z\).
2. there exists a positive integer \(e(\Lambda)\) such that \(\Lambda(n+e(\Lambda)) = p_F\Lambda(n)\), for any \(n \in Z\).

Given any lattice \(L \subseteq V\), let \(L^\#\) be the lattice \(\{v \in V \mid h(v, L) \subseteq p_F\}\). For any lattice sequence \(\Lambda\), let \(\Lambda^\#\) be the lattice sequence defined as:

\[
\Lambda^\#(n) = \Lambda(-n)^\#, \quad \text{for all } n \in Z.
\]

A lattice sequence \(\Lambda\) is said to be self-dual if and only if there exists an integer \(d\) such that \(\Lambda^\#(n) = \Lambda(n+d)\), for all \(n \in Z\). Since, we would only concern ourselves with \(\mathfrak{o}_F\)-lattice sequences we will drop the notation \(\mathfrak{o}_F\).

Let \(W\) be a subspace of the vector space \(V\), and let \(\Lambda\) be a lattice sequence on \(V\). We denote by \(\Lambda \cap W\) the lattice sequence on \(W\) sending \(n\) to \(\Lambda(n) \cap W\).

Given any lattice sequence \(\Lambda\) and integers \(a, b \in Z\), the lattice sequence \(a\Lambda + b\) is defined by setting

\[
(a\Lambda + b)(n) = \Lambda([n-b]/a), \quad \text{for all } n \in Z.
\]

The decreasing sequence of lattices \(\{a\Lambda + b \mid a, b \in Z\}\) is called the affine class of \(\Lambda\). For any self-dual lattice-sequence \(\Lambda\), we can find a lattice sequence \(\Lambda'\) in the affine class of \(\Lambda\) such that \(e(\Lambda')\) is an even integer, and \((\Lambda')^\# = \Lambda' - 1\). Henceforth, we assume that all self-dual lattice sequences satisfy these conditions.

3.2. Given any lattice sequence \(\Lambda\), and an integer \(n\), let \(\tilde{\alpha}_n(\Lambda)\) be the following sublattice of \(\text{End}_F(V)\):

\[
\tilde{\alpha}_n(\Lambda) = \{T \in \text{End}_F(V) \mid T \Lambda(i) \subseteq \Lambda(i+n) \quad \forall i \in Z\}.
\]

The decreasing sequence of lattices \(\{\tilde{\alpha}_n(\Lambda)\}_{n \geq 0}\) has trivial intersection. Given any element \(T \in \text{End}(V)\), we denote by \(\nu_\Lambda(T)\) the unique integer \(k\) such that \(T \subseteq \tilde{\alpha}_k(\Lambda)\) and \(T \not\subseteq \tilde{\alpha}_{k+1}(\Lambda)\). Let \(P_0(\Lambda)\) be the units in the ring \(\mathfrak{o}(\Lambda)\). For any positive integer \(n\), let \(P_n(\Lambda)\) be the compact open subgroup \(\text{id}_V + \tilde{\alpha}_n(\Lambda)\) of \(\text{GL}_F(V)\). For any self-dual lattice sequence \(\Lambda\), the lattices \(\tilde{\alpha}_n(\Lambda)\) are stable under \(\sigma_h\). For any non-negative integer \(n\), let \(P_n(\Lambda)\) be the compact open subgroup \(P(\Lambda) \cap G\) of \(G\).

The group \(P_0(\Lambda)/P_1(\Lambda)\) is the set of \(k_{F_0}\)-rational points of a not necessarily connected reductive algebraic group over \(k_{F_0}\), and let \(P(\Lambda)\) be the inverse image of the \(k_{F_0}\)-rational points of its connected component. The compact subgroup \(P(\Lambda)\) is called the parahoric subgroup associated to \(\Lambda\). If \(F/F_0\) is unramified, then \(P(\Lambda)\) is equal to \(P_0(\Lambda)\) and has index \(2\) in \(P_0(\Lambda)\) otherwise.

A stratum in \(\text{End}_F(V)\) is the data \([\Lambda, n, r, \beta]\) consisting of a lattice sequence \(\Lambda\) on \(V\), integers \(n \geq r \geq 0\), and an element \(\beta \in \text{End}_F(V)\) such that \(\beta \in \tilde{\alpha}_{-n}(\Lambda)\). Two strata \([\Lambda, n, r, \beta_1]\) and \([\Lambda, n, r, \beta_2]\) are said to be equivalent if and only if \(\beta_2 - \beta_1 \in \tilde{\alpha}_{-r}(\Lambda)\). A stratum \([\Lambda, n, r, \beta]\) is called a zero stratum if and only if \(n = r\) and \(\beta = 0\). For any \(n \geq r \geq n/2 > 0\), the set of equivalence classes of strata are in bijection with the characters of the group \(P_{r+1}(\Lambda)/P_{n+1}(\Lambda)\). The character corresponding to the equivalence class containing the stratum \([\Lambda, n, r, \beta]\) is given by

\[
\psi_\beta : 1 + X \mapsto \psi_F(\text{tr} \beta X).
\]

A stratum is called skew if the lattice sequence \(\Lambda\) is self-dual and \(\beta \in \mathfrak{g}\). We have the same notion of equivalence on skew strata. For any \(n \geq r \geq n/2 > 0\), an equivalence class of skew strata corresponds to a character on the group \(P_{r+1}(\Lambda)/P_{n+1}(\Lambda)\). The character corresponding to the skew stratum \([\Lambda, n, r, \beta]\) is given by \(\text{res}_{P_n(\Lambda)} \psi_\beta\).

3.3. Recall that a stratum \([\Lambda, n, r, \beta]\) is called a simple stratum if it satisfies the following conditions:

1. We have \(n \geq r \geq 0\).
2. the valuation of \(\beta\) with respect to \(\Lambda\), denoted by \(\nu_\Lambda(\beta)\), is equal to \(-n\).
3. the algebra \(F[\beta]\) is a field which normalizes the lattice sequence \(\Lambda\), and
4. we have \(r < -k_0(\Lambda, \beta)\), where \(k_0(\Lambda, \beta)\) is the critical constant defined in [Ste05], Section 1.2 2].

A stratum \([\Lambda, n, r, \beta]\) is called a semisimple stratum if it is either a zero stratum or if it satisfies the following conditions:
(1) we have \( n \geq r \geq 0 \) and \( \nu_{\Lambda}(\beta) = -n \).
(2) there exists a decomposition \( V = \oplus_{i=1}^k V_i \) for which \( \Lambda(k) = \oplus_{i=1}^k (\Lambda(k) \cap V_i) \), for all \( k \in \mathbb{Z} \).
(3) let \( 1_i \) be the projection of \( V \) onto \( V_i \) with kernel \( \oplus_{j \neq i} V_j \). The element \( \beta = \oplus_{i=1}^k 1_i \), where \( \beta_i = 1_i \beta_1 \), for \( 1 \leq i \leq k \),
(4) the stratum \( [\Lambda_i, \nu_{\beta_i}] \) with \( q_i = r \) if \( \beta_i = 0 \) and \( q_i = -\nu_{\Lambda}(\beta_i) \) otherwise--is either a zero stratum or a simple stratum, this data must satisfy the following crucial condition:
(5) the stratum \( [\Lambda_i + \Lambda_j, q, r, \beta_i + \beta_j] \), with \( q = \max\{q_i, q_j\} \), is non-equivalent to a zero stratum or a simple stratum, for \( 1 \leq i, j \leq k \) and \( i \neq j \).

The decomposition \( V = \oplus_{i=1}^k V_i \) is uniquely determined by the element \( \beta \), and will be called as the underlying splitting of the semisimple stratum \( [\Lambda, r, \beta] \). A semisimple stratum \( [\Lambda, n, r, \beta] \) is called a skew semisimple stratum if the decomposition \( V = \oplus_{i=1}^k V_i \) is an orthogonal decomposition with respect to the form \( h \) on \( V \) and \( \sigma_h(\beta_i) = -\beta_i \), for \( 1 \leq i \leq k \). Observe that the algebra \( F[\beta] \) is isomorphic to the algebra

\[
F[\beta_1] \oplus F[\beta_2] \oplus \cdots \oplus F[\beta_k].
\]

We use the notation \( \tau \) for a general skew semisimple stratum \([\Lambda, n, 0, \beta]\).

Let \( C_{\beta}(\Lambda) \) be the centraliser of \( F[\beta] \) in \( \text{End}_F(V) \). The group \( G \cap C_{\beta}(\Lambda) \) is denoted by \( G_{\beta} \). Let \( n \) be any integer and let \( b_n(\Lambda) \) and \( b_n(\Lambda) \) be the groups \( a_n(\Lambda) \cap C_{\beta}(\Lambda) \) and \( a_n(\Lambda) \cap C_{\beta}(\Lambda) \) respectively. For any non-negative integer \( n \), let \( \hat{P}_n(\Lambda) \) and \( P_n(\Lambda) \) be the groups \( \hat{P}_n(\Lambda) \cap C_{\beta}(\Lambda)^{\times} \) and \( P_n(\Lambda) \cap G_{\beta} \) respectively.

### 3.4. Stevens generalised the Bushnell–Kutzko construction of cuspidal representations of general linear groups to classical groups.

Starting from a skew semisimple stratum \( \tau = [\Lambda, n, 0, \beta] \) one constructs some special compact subgroups \( J^0(\Lambda, \beta) \) and \( H^0(\Lambda, \beta) \) of \( G \); and certain special representations of which are then induced to the group \( G \) to obtain cuspidal representations. We will not go into the details of this construction, however, we describe these groups as required in the later part of the article.

Let \( J^0(\Lambda, \beta) \) be the compact open subgroup \( J^0(\Lambda, \beta) \cap P_i(\Lambda) \), for any non-negative integer \( i \). For any skew-semisimple stratum \( \tau = [\Lambda, n, 0, \beta] \), the construction of cuspidal representations of \( G \), begins with a specific set of characters, called skew semisimple characters and denoted by \( C(\Lambda, 0, \beta) \), of the group \( H^1(\Lambda, \beta) \) (see [Ste05, Section 3.6] and the set \( C(\Lambda, 0, \beta) \) is denoted by \( \mathcal{C}_-(\Lambda, 0, \beta) \) there). The group \( P_{n(1/2)}(\Lambda) \) is contained in \( H^1(\Lambda, \beta) \) and we have \( \text{res} P_{n(1/2)}(\Lambda) \theta = \psi_{\beta} \), for any \( \theta \in C(\Lambda, 0, \beta) \). For any character \( \theta \in C(\Lambda, 0, \beta) \), the map sending \( g_1, g_2 \in J^1(\Lambda, \beta) \) to \( \theta([g_1, g_2]) \) induces a perfect alternating pairing

\[
\kappa_{\theta} : \frac{J^1(\Lambda, \beta)}{H^1(\Lambda, \beta)} \times \frac{J^1(\Lambda, \beta)}{H^1(\Lambda, \beta)} \to \mathbb{C}^{\times}.
\]

Using the theory of Heisenberg lifting, for any character \( \theta \in C(\Lambda, 0, \beta) \), there exists a unique representation \( \eta_{\theta} \) of \( J^1(\Lambda, \beta) \) such that \( \text{res} J^1(\Lambda, \beta) \eta_{\theta} \) is equal to a power of \( \theta \). There are special set of extensions of the representation \( \eta_{\theta} \) to the group \( J^0(\Lambda, \beta) \) called beta-extensions; these representations are denoted by \( \kappa \) (see [Ste08, Section 4]).

The group \( P_0(\Lambda) \) is contained in \( J^0(\Lambda, \beta) \). The inclusion of \( P_0(\Lambda) \) in \( J^0(\Lambda, \beta) \) induces an isomorphism

\[
P_0(\Lambda)/P_1(\Lambda) \simeq J^0(\Lambda, \beta)/J^1(\Lambda, \beta).
\]

The group \( P_0(\Lambda)/P_1(\Lambda) \) is the \( k_{F_0} \)-rational points of a reductive group over \( k_{F_0} \). Let \( \tau \) be a cuspidal representation of \( P_0(\Lambda)/P_1(\Lambda) \). If \( P^0(\Lambda) \) is a maximal parahoric subgroup of \( C_{\beta}(\Lambda) \cap G \), then the induced representation

\[
\text{ind}_{P^0(\Lambda, \beta)}^G(\kappa \otimes \tau)
\]

is irreducible and any supercuspidal representation of \( G \) arises in this way. The pair \( (J^0(\Lambda, \beta), \kappa \otimes \tau) \) is a Bushnell–Kutzko type for the Bernstein component containing the representation \( (3.1) \). Let \( \Pi_\kappa \) be the set of cuspidal representations of \( G \) containing a Bushnell–Kutzko type of the form \( (J^0(\Lambda, \beta), \kappa \otimes \tau) \), for some \( \kappa \) and \( \tau \) as above.
3.5. For the convenience of the reader we recall some frequently used results from [BS09]. Let us begin with the following lemma which is useful in calculating the group $H^1(\Lambda, \beta) \cap U$.

**Lemma 3.5.1** (Blondel–Stevens). Let $[\Lambda, n, 0, \beta]$ be a semisimple stratum in $\text{End}_F(V)$ such that $C_\beta(A)$ does not contain any nilpotent elements. Let $N$ be a maximal unipotent subgroup of $G$. For $k \geq m \geq 1$, we have

$$P_m(\Lambda_\beta)P_k(\Lambda) \cap N = P_k(\Lambda) \cap N.$$ 

We refer to [BS09] Section 6.3, Lemma 6.5] for a proof of the above lemma. The following result is proved in greater generality by Blondel and Stevens (see [BS09] Section 4, Corollary 4.2, Theorem 4.3), however, in the present context we will use the following simple version.

**Proposition 3.5.2** (Blondel–Stevens). Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum such that $X_\beta$ is non-empty. Assume that $J^0(\Lambda, \beta)/J^1(\Lambda, \beta)$ is anisotropic. Let $U$ be the unipotent radical of a Borel subgroup in $X_\beta$ and $\theta$ be a skew semisimple character of $H^1(\Lambda, \beta)$. If

$$\text{res}_{H^1(\Lambda, \beta)} \theta = \text{res}_{H^1(\Lambda, \beta)} \psi_\beta,$$

then any representation in $\Pi_\theta$ is generic.

**Proof.** The proof is identical to [BS09] Section 4, Corollary 4.2, Theorem 4.3. Since we use this in a crucial way to get genericity results, we briefly sketch the proof. Let $\tilde{H}^1$ be the group $(J^0(\Lambda, \beta) \cap U)H^1(\Lambda, \beta)$ and $\Theta_\beta$ be the character of $\tilde{H}^1$ defined by

$$\Theta_\beta(jh) = \psi_\beta(j)\theta(h)$$

for all $j \in J^0(\Lambda, \beta) \cap U$, $h \in H^1(\Lambda, \beta)$.

The group $\tilde{H}^1 \cap J^1(\Lambda, \beta)$ is equal to $(J^1(\Lambda, \beta) \cap U)H^1(\Lambda, \beta)$, and is a totally isotropic subspace for the pairing $\kappa_\beta$ on $J^1(\Lambda, \beta)/H^1(\Lambda, \beta)$. The representation $\eta_\theta$ is the induced representation from an extension of the character $\Theta_\beta$ to the inverse image in $J^1(\Lambda, \beta)$ of a maximal isotropic subspace of $J^1(\Lambda, \beta)/H^1(\Lambda, \beta)$. Hence, $\text{res}_{J^1(\Lambda, \beta) \cap U} \eta_\theta$ contains the character $\text{res}_{J^1(\Lambda, \beta) \cap U} \psi_\beta$. Since the group $J^0(\Lambda, \beta) \cap U$ is equal to $J^1(\Lambda, \beta) \cap U$, we prove the proposition. 

It is convenient to partition the set of skew-semisimple strata in $\text{End}_F(V)$ into four classes. The stratum $\pi = [\Lambda, n, 0, \beta]$ is of type (A) if $F[\beta]$ is a field, $\pi$ is called a stratum of type (B) if $F[\beta]$ is a direct sum of two fields with one of them a quadratic extension of $F$, (C) if the algebra $F[\beta]$ is a direct sum of two copies of $F$, and finally $\pi$ is a stratum of type (D) if the algebra $F[\beta]$ is a direct sum of three copies of $F$.

4. The simple case.

4.1. A skew semisimple strata $[\Lambda, n, 0, \beta]$ is of type (A) if it is simple, i.e., the algebra $F[\beta]$ is a field. We refer to [BK93] for the construction of the group $J^0(\Lambda, \beta)$, and their representations beta-extensions. We will not need the explicit description of these groups or their representations, the beta-extensions.

**Lemma 4.1.1.** If $[\Lambda, n, 0, \beta]$ is a skew simple strata of type (A), then the set $X_\beta$ is non-empty.

**Proof.** The involution $\sigma_\beta$ stabilizes the field $F[\beta]$. Let $D$ be the fixed field of the automorphism $\sigma_\beta$ on $F[\beta]$. Let $\lambda$ be a $\sigma_\beta$-equivariant non-zero $F$-linear form on $F[\beta]$. There exists a unique Hermitian form $h_1: V \times V \to F[\beta]$ such that

$$h(v, w) = \lambda((h_1(v, w)))$$

for all $v, w \in V$.

The set $X_\beta$ is non-empty if and only there exists a non-zero vector $v \in V$ such that

$$h(v, v) = 0 \text{ and } h(v, \beta v) = 0.$$ 

Observe that $h(v, \beta v) = \lambda(h_1(v, \beta v)) = \lambda(3h_1(v, v))$. The $F$-linear forms $\lambda$ and $\lambda \circ \beta$ are linearly independent, since $\beta$ does not stabilize a proper non-trivial subspace of $F[\beta]$. Let $W$ be the space $\ker(\lambda) \cap \ker(\lambda \circ \beta)$ and we have $\dim_F W = 1$. Since, $\sigma_\beta(\beta) = -\beta$, the space $W$ is stable under the action of $\sigma_\beta$. Note that $W$ is an $F$-vector space, and hence $\sigma_\beta(w) \neq \pm w,$ for all $w \in W$. Hence, the eigenvalues of $\sigma_\beta$ on $W$ are distinct and we get that $\dim_F(W \cap D) = 1$. The $F_0$-vector space $W \cap D$ is denoted by $W_0$ and we have $W_0 \otimes_{F_0} F = W$. 

The form $h_1(x, x)$ is equal to $x a \sigma_h(x)$, for some $a \in D$. Let $N_a$ be the set \{ $x a \sigma_h(x) \mid x \in F[\beta]^\times$ \}. Assume that the inclusion of $F_0^\times$ in $D^\times$ induces a surjection of $F_0^\times$ onto the quotient $D^\times/(\mathrm{Nr}_{F[\beta]/D}(F[\beta]^\times))$. Let $w$ be a non-zero vector in $W_0$. If $w \in N_a$, then we get our required solution. If $w \notin N_a$, then we may scale $w$ with some element $x \in F_0^\times$ such that $xw1 \in N_a$.

Let us prove that the inclusion of $F_0^\times$ in $D^\times$ induces a surjection of $F_0^\times$ onto the quotient $D^\times/(\mathrm{Nr}_{F[\beta]/D}(F[\beta]^\times))$.

We have the following diagram of fields:

$$
\begin{array}{c}
F[\beta] \\
\downarrow \\
F \\
\downarrow \\
D = F_0[\delta\beta] \\
\downarrow \\
F_0
\end{array}
$$

Fix a valuation $\nu : F[\beta]^\times \to 1/e[F[\beta] : F] \mathbb{Z}$ such that $\nu(\infty) = 1$. Assume that $\nu_0 = \mathrm{Nr}_{F[\beta]/D}(x)$, for some $x \in F[\beta]$. Hence, we have $\nu(\infty) = 2 \nu(x)$. If $F/F_0$ is unramified, then such an equality is impossible and therefore, we get that $\nu_0 \notin \mathrm{Nr}_{F[\beta]/D}(F[\beta]^\times)$. Consider the case where $F$ is a ramified extension of $F_0$. Let $x$ be an element of $\mathfrak{o}_{F_0}^\times$ such that $x \in k_{F_0}$ is not a square in $k_{F_0}$. The element $x$ clearly does not belong to $\mathrm{Nr}_{F[\beta]/D}(F[\beta]^\times)$: since the automorphism $\sigma_h$, having order $2$, must act trivially on the residue field of $k_{F[\beta]}$ (note that $[k_{F[\beta]} : k_{F_0}] = 3$).

Proposition 4.1.2. Let $\mathfrak{r} = [\Lambda, n, 0, \beta]$ be any skew simple strata of the type (A). Any cuspidal representation contained in $\Pi_\mathfrak{r}$ is generic.

Proof. Let $(\pi, V)$ be any representation in the set $\Pi_\mathfrak{r}$. Since $\dim_E V = 3$, we may twist the representation $(\pi, V)$, if necessary, and assume that $\mathfrak{r}$ is minimal. Let $(J^0(\Lambda, \beta), \kappa)$ be a Bushnell–Kutzko type contained in $\pi$. Now, the representation $\pi \simeq \text{ind}_F^{G(\Lambda, \beta)} \kappa$. Note that

$$
J^0(\Lambda, \beta) = \mathfrak{o}_{F[\beta]}^\times \mathbb{P}_{n/2}(\Lambda)
$$

and $\kappa$ is a beta-extension of the Heisenberg lift $\eta_\theta$ of a skew semisimple character $\theta$ of $H^1(\Lambda, \beta)$. Now, Lemma 4.1.1 implies that the set $\mathfrak{X}_{\beta}$ is non-empty. Let $U$ be the unipotent radical of a Borel subgroup of $\mathfrak{X}_{\beta}$. Using Lemma 3.5.4 we get that the intersection $J^0(\Lambda, \beta) \cap U$ is equal to $\mathbb{P}_{n/2}(\Lambda) \cap U$. This shows that

$$
\text{res}_{H^1(\Lambda) \cap U} \theta = \text{res}_{H^1(\Lambda, \beta) \cap U} \psi_\beta.
$$

Now using Proposition 3.5.2 we get that the representation $\pi$ is generic. \qed

5. The non simple type (B) strata

A skew semisimple stratum $\mathfrak{r} = [\Lambda, n, 0, \beta]$ is of type (B) if the underlying splitting of $\mathfrak{r}$ is given by $V = V_1 \perp V_2$, where $\dim_F(V_i) = i$, for $i \in \{1, 2\}$, and the algebra $F[\beta_2]$ is a quadratic extension of $F$, where $\beta_i = 1_i^{\beta_1} 1_1^{\beta_2}$, for $i \in \{1, 2\}$; we also have $\beta = \beta_1 + \beta_2$. If $F/F_0$ is unramified, then the extension $F[\beta_2]/F$ is ramified. If $F/F_0$ is ramified then the extension $F[\beta_2]/F$ is unramified. We recall the notations $q_i = -\nu_{\Lambda_i}(\beta_i)$, for $i \in \{1, 2\}$. Generality of a representation in the set $\Pi_\mathfrak{r}$ depends only on the isomorphism class of the hermitian space $(V_2, h)$ and the integers $q_1$ and $q_2$. We begin with some explicit description of the types involved. Since $\dim_F(V_2) = 2$, after twisting with a character, if necessary, we may (and do) assume that $\beta_2$ is minimal.

5.1. Lattice sequences. In this case the lattice sequence $\Lambda$ is uniquely determined by $\beta$. We first fix a representative for the $G$-conjugacy class containing $\Lambda$ and describe an explicit Witt-basis for $(V, h)$ which also provides a splitting for $\Lambda$. 


5.1.1. The Unramified case. Let $F/F_0$ be an unramified extension and consider the case where $(V_2, h)$ is isotropic. Since $F/F_0$ is unramified, the extension $F[\beta]/F$ is ramified. Let $(e_1, e_{-1})$ be a Witt-basis for the space $V_2$, and a vector $e_0 \in V_1$ such that $h(e_0, e_0) = 1$. Let $\Lambda$ be a lattice sequence with period 4 and

\[
\Lambda(-1) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}, \quad \Lambda(0) = o_F e_1 \oplus o_F e_0 \oplus p_F e_{-1}, \\
\Lambda(1) = o_F e_1 \oplus p_F e_0 \oplus o_F e_{-1}, \quad \Lambda(2) = p_F e_1 \oplus p_F e_0 \oplus p_F e_{-1}.
\]

The group $P(\Lambda)$ is an Iwahori subgroup of $G$. Note that the filtration $\{P_m(\Lambda) : m > 0\}$ is not the standard filtration of an Iwahori subgroup of $GL_3(F)$. Any self-dual lattice sequence normalised by $\beta$ is of the above form.

Now, consider the case where $F/F_0$ is unramified and $(V_2, h)$ is anisotropic. There exists an orthogonal basis $(v_2, v_3)$ of $V_2$ such that $h(v_2, v_2) = 1$ and $h(v_3, v_3) = \varpi_0$. Let $v_1$ be any vector in $V_1$ such that $h(v_1, v_1) = \varpi_0$. The lattice sequence $\Lambda$ has period 2 and we have

\[
\Lambda(0) = o_F v_1 \oplus o_F v_3 \oplus o_F v_3 \quad \text{and} \quad \Lambda(1) = o_F v_1 \oplus p_F v_2 \oplus o_F v_3.
\]

\textbf{Lemma 5.1.1.} Let $(V_2, h)$ be anisotropic and $\Lambda$ be the lattice sequence in (5.1). There exists a Witt-basis $(e_1, e_{-1})$ for $(\langle v_1, v_3 \rangle, h)$ such that

\[
o_F v_1 \oplus o_F v_3 = p_F e_1 \oplus o_F e_{-1}.
\]

\textit{Proof.} Let $\epsilon \in F$ be an element with $\epsilon \sigma(\epsilon) = -1$; such an element exists because $F/F_0$ is unramified. The vectors $e_1 = \epsilon 1/2 v_1 + 1/2 v_3$ and $\varpi e_{-1} = -e_1 + v_3$ are isotropic and $h(e_1, e_{-1}) = 1$. This implies that $(e_1, e_{-1})$ is a Witt-basis for $(v_1, v_3)$, and since $\nu_F(\epsilon) = 0$, the tuple $(e_1, \varpi e_{-1})$ is a $o_F$-basis for the lattice $o_F v_1 \oplus o_F v_3$. \hfill \Box

When $(V_2, h)$ is anisotropic as above, let $e_0$ be the vector $v_2$ and the Witt-basis $(e_1, e_0, e_{-1})$, where $(e_1, e_{-1})$ is a Witt-basis for $(\langle v_1, v_3 \rangle, h)$ as in Lemma 5.1.1 provides a splitting for $\Lambda$. In the basis $(e_1, e_0, e_{-1})$ the lattice sequence $\Lambda$, having period 2, is given by

\[
\Lambda(0) = o_F e_1 \oplus o_F e_0 \oplus p_F e_{-1} \quad \text{and} \quad \Lambda(1) = o_F e_1 \oplus p_F e_0 \oplus p_F e_{-1}.
\]

5.1.2. Ramified case. Let $F/F_0$ be a ramified extension, and note that the extension $F[\beta]/F$ is an unramified extension. If $(V_2, h)$ is isotropic, then $\Lambda$ is a lattice sequence of period 2 and there exists a Witt-Basis for $(V_2, h)$ such that

\[
\Lambda(0) = o_F e_1 \oplus o_F e_0 \oplus p_F e_{-1} \quad \text{and} \quad \Lambda(1) = o_F e_1 \oplus p_F e_0 \oplus p_F e_{-1},
\]

where $e_0 \in V_1$ such that $h(e_0, e_0) = 1$. In the case where $(V_2, h)$ is anisotropic, there exists an orthogonal basis $(v_2, v_3)$ of $V_2$ and a non-zero vector $v_1 \in V_1$ such that: $h(v_1, v_1) = \lambda_i \in o_F^*$, for $1 \leq i \leq 3$, the space $\langle v_1, v_3 \rangle$ is isotropic and the lattice sequence $\Lambda$, has periodicity 2 with

\[
\Lambda_2(-1) = \Lambda_2(0) = o_F v_1 \oplus o_F v_2 \oplus o_F v_3.
\]

\textbf{Lemma 5.1.2.} Let $(V_2, h)$ be anisotropic. There exists a Witt-basis $(e_1, e_{-1})$ of $(\langle v_1, v_3 \rangle, h)$ such that

\[
o_F v_1 \oplus o_F v_3 = o_F e_1 \oplus o_F e_{-1}.
\]

\textit{Proof.} We fix an $\epsilon \in F$ such that $\epsilon \sigma(\epsilon) = -\lambda_3 \lambda_1^{-1}$. The vectors $e_1 = 1/2 v_1 + 1/2 v_3$ and $e_{-1} = (-e v_1 + v_3)\lambda_3^{-1}$ are isotropic and $h(e_1, e_{-1}) = 1$. Moreover, the vectors $e_1$ and $e_{-1}$ are a basis for the $o_F$-lattice $o_F v_1 \oplus o_F v_3$. \hfill \Box

In the basis $(e_1, e_0, e_{-1})$, with $(e_1, e_{-1})$ as in Lemma 5.1.2 and $e_0 = v_2$, the period 2 lattice sequence $\Lambda$ is given by

\[
\Lambda(-1) = \Lambda(0) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}.
\]
5.2. The integers $q_1$ and $q_2$ have some constraints: If $F/F_0$ is unramified and $(V_2, h)$ is isotropic, then $q_2 = 4m_2 + 2$ and $q_1 = 4m_1$ for some $m_1,m_2 \in \mathbb{Z}$. If $F/F_0$ is unramified and $(V_2, h)$ is anisotropic, then $q_2 = 2m_2 + 1$ and $q_1 = 2m_1$, for some $m_1,m_2 \in \mathbb{Z}$. If $F/F_0$ is ramified, then $q_2 = 4m_2$ and $q_1 = 4m_1 + 2$, for some $m_1,m_2 \in \mathbb{Z}$, since the image of

$$y_{\beta_2} = \varpi^{q_2/g} \beta_2^{\epsilon(\Lambda_2)/g} = \varpi^{q_2/2} \beta_2$$

(here $g = (q_2, e(\Lambda_2))$) in $k_{F[\beta_2]}$ must generate the degree 2 field extension $k_{F[\beta_2]}$ over $k_F$. Hence, in all the above cases $q_1 \neq q_2$.

5.3. We will need the structure of compact subgroups $J^0(\Lambda, \beta)$ and $H^0(\Lambda, \beta)$ of $G$. First assume that $q_2 < q_1$ and in this case the constant $k_0(\Lambda, \beta)$, defined in [Sted05, equation 3.6], is equal to $q_2$. The stratum $[\Lambda, n, q_2, \beta_1]$ is a skew semisimple stratum equivalent to the stratum $[\Lambda, n, q_2, \beta_1]$. We then have:

$$J^0(\Lambda, \beta) = P_0(\Lambda_\beta) P_{q_2/2}(\Lambda_\beta_1) P_{n/2}(\Lambda).$$

$$H^1(\Lambda, \beta) = P_1(\Lambda_\beta) P_{(q_2/2)}(\Lambda_\beta_1) P_{(n/2)}(\Lambda).$$

If $q_2 > q_1$, then we have

$$J^0(\Lambda, \beta) = P_0(\Lambda_\beta) P_{q_2/2}(\Lambda).$$

$$H^1(\Lambda, \beta) = P_1(\Lambda_\beta) P_{(n/2)}(\Lambda).$$

5.4. In each of the above cases, we fixed a Witt-basis $(e_1, e_0, e_{-1})$ splitting the lattice sequence $\Lambda$. Let $B$ be the Borel subgroup of $G$ such that $B$ fixes the space $(e_1)$. Let $T$ be the maximal $F_0$-split torus of $G$ such that $T$ stabilises the decomposition $V = (e_1) \oplus (e_0) \oplus (e_{-1})$. Let $B$ be the opposite Borel subgroup of $B$ with respect to $T$. Let $U$ (resp. $\bar{U}$) be the unipotent radical of $B$ (resp. $B$). We also recall the notations $u(c, d)$ and $\bar{u}(c, d)$ for elements in $U$ and $\bar{U}$ respectively. In the basis $(e_1, e_0, e_{-1})$, let $I$ be the Iwahori subgroup

$$\left( \begin{array}{ccc} o_F & o_F & o_F \\ p_F & o_F & o_F \\ p_F & p_F & o_F \end{array} \right) \cap G.$$

From the Iwasawa decomposition we get that

$$G = \coprod_{w \in W_G} I w B.$$

(5.2)

5.5. When $r$ is a type (B) stratum, we give a necessary and sufficient condition to show that $X_\beta$ is non-empty. Let $\lambda$ be the $F$-linear $\sigma_r$-equivariant linear functional $\lambda : F[\beta_2] \rightarrow F$ such that $\lambda(\beta_2) = \beta_1$ and $\lambda(1) = 1$. The field $F[\beta_2]$ is stable under the action of $\sigma_h$ and let $D$ be the fixed field of the automorphism $\sigma_h$. There exists an unique hermitian form $h' : V_2 \times V_2 \rightarrow F[\beta_2]$ such that

$$h'(xv, yw) = x\sigma_h(y)\sigma_h(h'(w, v)), \text{ for all } v, w \in V \text{ and } x, y \in F[\beta_2]$$

and

$$h(v, w) = \lambda(h'(v, v)), \text{ for all } v, w \in V_2.$$ 

Since $F_0[\beta]$ is a quadratic extension of $F_0$, the kernel of $\lambda$ is equal to

$$(\beta_2 - \beta_1) F_0 \oplus \delta(\beta_2 - \beta_1) F_0.$$ 

The set $X_\beta$ is non-empty if and only if there exists $v_1 \in V_1, v_2 \in V_2$, and $v_1 + v_2 \neq 0$ such that

$$h(v_1, v_1) + \lambda(h'(v_2, v_2)) = 0,$$

$$\beta_1 h(v_1, v_1) + \lambda(h'(v_2, v_2)) = 0.$$ 

For any two vectors $v_1$ and $v_2$ as above we must have $v_1 \neq 0$ and $v_2 \neq 0$. Note that $(\beta_1 - \beta_2) h'(v_2, v_2)$ is contained in the kernel of $\lambda$. This implies that $h'(v_2, v_2) \in F_0^\times$. If $d_1$ and $d_2$ are the determinants of $(V_2, h')$ and $(V_1, h)$, then we have

$$-d_1d_2^{-1} \in \text{Nr}_{F[\beta_2]/F}[F[\beta_2]^\times].$$

(5.4)

Conversely, if the above condition holds, then it is easy to see that the the set $X_\beta$ is non-empty.
5.6. **The case where** \((V_2, h)\) **is isotropic.** In this part we treat the case where \((V_2, h)\) is isotropic and let us begin with the case where \(q_2 > q_1\).

**Lemma 5.6.1.** Let \(F/F_0\) be an unramified extension and let \(\mathfrak{x}\) be a stratum of the type \((B)\). If \((V_2, h)\) is isotropic and \(q_2 > q_1\), then any representation contained in \(\Pi_\mathfrak{x}\) is non-generic. Moreover, the set \(\mathfrak{X}_\beta\) is the empty set.

**Proof.** Let \(\pi\) be a representation in the set \(\Pi_\mathfrak{x}\). There exists a beta-extension \(\kappa\) of \(J^0(\Lambda, \beta)\) such that \(\pi = \text{ind}^G_{J^0(\Lambda, \beta)} \kappa\). If \(\pi\) is genetic, then there exists a non-trivial character \(\Psi\) of \(U\), \(p \in I\), and \(w \in W_G\) such that

\[
\text{Hom}_{J^0(\Lambda, \beta)p \cap U^w}((\kappa^p, \Psi^w)) \neq 0.
\]

Using Iwahori decomposition of \(I\), we write \(p = p^+ u^-\) with \(p^+ \in B^w \cap I\) and \(u^- \in U^w \cap I\), where \(U^w\) is the unipotent radical of the opposite Borel subgroup of \(B^w\) containing \(T\). Now the equation (5.5) implies that

\[
\text{Hom}_{J^0(\Lambda, \beta)\nu \cap U^w}((\kappa^\nu, \Psi^\nu)) \neq 0,
\]

for some character \(\Psi'\) of \(U^w\). The group \(P_{(q_2/2)+}(\Lambda)\) is normalised by the element \(u^- = u(x, y)\) and hence \(P_{(q_2/2)+}(\Lambda)^w \cap U^w\) is equal to \(P_{(q_2/2)+}(\Lambda) \cap U^w\).

We set \(q_2 = 8k + 2r\), for some integer \(k\) and \(r \in \{1, 3\}\). For the following calculations it is convenient to refer to the appendix (10.3) for an explicit description of the filtration \(\{a_m(\Lambda) : m \in \mathbb{Z}\}\). We have \((q_2/2)+ = 4k + r + 1\) and the intersection \(P_{(q_2/2)+}(\Lambda) \cap U^w\) is given by:

\[
P_{(q_2/2)+}(\Lambda) \cap U^w = \begin{cases} U(k + \lfloor r/2 \rfloor), & \text{if } w = \text{id}, \\ U^w(k + 1 + \lfloor r/2 \rfloor), & \text{if } w \neq \text{id}. \end{cases}
\]

With \(e_w = w_{e_1}\) and \(e_{-w} = w_{e_{-1}}\), we have:

\[
h(u^- e_w, \beta u^- e_w) = \beta_1 x \bar{x} + h(e_w, \beta_2 e_w) + \frac{y + \bar{y}}{8} h(e_w, \beta_2 e_{-w}) + \frac{y + \bar{y}}{8} h(e_{-w}, \beta_2 e_{-w}),
\]

and the valuation \(\nu_F(\delta h(u^- e_w, \beta u^- e_w))\) is given by

\[
\nu_F(\delta h(u^- e_w, \beta u^- e_w)) = \begin{cases} -(2k + \lfloor r/2 \rfloor) & \text{if } w = \text{id}, \\ -(2k + \lfloor r/2 \rfloor + 1) & \text{if } w \neq \text{id}. \end{cases}
\]

From the equations (5.6) and (5.7), we get that

\[
\nu(h(u^- e_w, u^- e_w)) \leq -s,
\]

where \(U^w_{der}(s)\) is equal to \(P_{(q_2/2)+}(\Lambda) \cap U^w\). Now, Lemma 2.6.1 implies that the character \(\psi^{\Lambda^w}_{\beta}\) is non-trivial on the group \(P_{(q_2/2)+}(\Lambda) \cap U^w\) and we get a contradiction to (5.5). Using Iwahara decomposition, we get that \(\nu_F(h(ge_1, \beta g e_1)) \leq -s\), for all \(g \in G\). Hence \(h(ge_1, \beta g e_1) \neq 0\), for all \(g \in G\). This shows that the set \(\mathfrak{X}_\beta\) is empty.

**Lemma 5.6.2.** Let \(F/F_0\) be a ramified extension and \(\mathfrak{x}\) be a stratum of the type \((B)\) such that \((V_2, h)\) is isotropic. If \(q_2 > q_1\), then any representation in the set \(\Pi_\mathfrak{x}\) is non-generic and the set \(\mathfrak{X}_\beta\) is empty.

**Proof.** The group \(P^0(\Lambda)\) is a special parahoric subgroup of \(G\) and from Iwasawa decomposition we get that

\[
G = P(\Lambda)TU.
\]

Let \(\pi = \text{ind}^G_{P(\Lambda, \beta)} \kappa\) be a generic representation in the set \(\Pi_\mathfrak{x}\). Now, there exists a \(g \in P(\Lambda)\), and a character \(\Psi|U\) such that

\[
\text{Hom}_{P(\Lambda, \beta)U}((\kappa^g, \Psi^g)) \neq 0.
\]

We refer to the appendix (10.2) for an explicit description of the filtration \(\{a_m(\Lambda) : m \in \mathbb{Z}\}\). We set \(q_2 = 4k_2\) and \(q_1 = 4k_1 + 2\), for some integers \(k_1\) and \(k_2\). The group \(P(\Lambda)\) normalises \(P_{(q_2/2)+}(\Lambda)\) and hence, \(P_{(q_2/2)+}(\Lambda)^g \cap U_{der}\) is equal to \(P_{(q_2/2)+}(\Lambda) \cap U_{der}\). We have

\[
P_{(q_2/2)+}(\Lambda) \cap U_{der} = U_{der}([[(k_2 - 1)/2]]).
\]
Since, $\nu_{\beta}(e_1) = 1$, we have $ge_1 = ae_1 + \omega b e_0 + \omega c e_{-1}$, for some $a, b, c \in \mathfrak{a}_F$. We now try to estimate the valuation of $h(ge_1, \beta ge_1)$. Observe that $h(ge_1, \beta ge_1)$ is equal to

$$-\omega^2 \beta_1 \sigma(b) + a \sigma(a) h(e_1, \beta_2 e_1) + \omega c \sigma(c) h(e_{-1}, \beta_2 e_{-1})$$

Recall that $F[\beta_2]$ is the unramified quadratic extension of $F$ and the element $\tilde{\beta}_2 = \omega q_2/2 \beta_2$ belongs to $a_o(A_2)$. Note that $\nu_{\beta}(e_1) = 1$ and hence, both $a$ and $c$ cannot be in $\mathfrak{p}_F$. Since $\text{Nr}_{F[\beta_2]/F}(\beta_2) \notin \text{Nr}_{F/F_0}(F^\times)$, we have

$$a \sigma(a) h(e_1, 1/\omega \tilde{\beta}_2 e_1) - a \sigma(c) h(e_1, \tilde{\beta}_2 e_{-1}) + c \sigma(c) h(e_{-1}, \tilde{\beta}_2 e_{-1}) = 0 \pmod{\mathfrak{p}_F}.$$  

We observe that

$$\nu_{F/F_0}(\omega^2 \beta_1 \sigma(b)) \geq -k_1 + 1/2.$$  

Since $q_2 > q_1$ we have $-k_2 < -k_1 - 1/2$ and we deduce that $\nu_{F/F_0}(\delta(h(ge_1, \beta ge_1)))$ is equal to $-k_2 + 1$. From the equation (5.9), we get that

$$\nu_{F/F_0}(\delta(h(ge_1, \beta ge_1))) \leq -s,$$

where $P_{\nu(q_2/2)+}(\Lambda) \cap U_{\text{der}} = U_{\text{der}}(s)$ and from Lemma 5.6.1 we get a contradiction to (5.8). Hence, any representation $\pi$ in the set $\Pi_\beta$ is a non-generic representation. Using Iwasawa decomposition, we get that $\nu_F(h(ge_1, \beta ge_1)) \leq -s$, for all $g \in G$. Hence, $h(ge_1, \beta ge_1) \neq 0$, for all $g \in G$. This shows that the set $X_\beta$ is empty.

**Lemma 5.6.3.** Let $F/F_0$ be an unramified extension and let $\mathfrak{f}$ be a stratum of the type (B) such that $(V_2, h)$ is isotropic. If $q_1 > q_2$, then the set $X_\beta$ is non-empty.

**Proof.** We continue using notations introduced in [7.5]. Since $(V_2, h)$ is isotropic, we get that $\ker(\lambda)$ has non-trivial intersection with the set $\{h'(v, v) : v \in V_2, v \neq 0\}$. Since $F[\beta_2]$ is a ramified extension of $F$ and $F/F_0$ is unramified extension, we get that $F_0^\times$ is contained in $\text{Nr}_{F[\beta_2]/F}(F[\beta_2]^\times)$. This implies that the determinant of the form $(V_2, h')$ is the class in $D^\times/\text{Nr}_{F[\beta_2]/F}(F[\beta_2]^\times)$ containing $\delta(\beta_2 - \beta_1)$. Since, $q_1 > q_2$ we have

$$\nu_{F[\beta_2]}(\delta(\beta_2 - \beta_1)) = \nu_{F[\beta_2]}(\beta_1) \in 2\mathbb{Z}.$$  

From the observation that $d_1$, the determinant of $(V_1, h)$, belongs to $F_0$, we see that

$$-d_1 d_1^{-1} \in \text{Nr}_{F[\beta_2]/F}(F[\beta_2]^\times).$$

Now, using the criterion [5.4], we get that the set $X_\beta$ is non-empty. 

**Lemma 5.6.4.** Let $F/F_0$ be a ramified extension and let $\mathfrak{f}$ be a stratum of the type (B) such that $(V_2, h)$ is isotropic. If $q_1 \geq q_2$, then the set $X_\beta$ is non-empty.

**Proof.** We continue using notations introduced in [5.3]. Since the space $(V_2, h)$ is isotropic, the set $\{h'(v, v) : v \in V_2, v \neq 0\}$ has a non-trivial intersection with $\ker(\lambda)^\times$. Hence, the determinant of $(V, h')$ is equal to the coset in $D^\times/\text{Nr}_{F[\beta_2]/F}(F[\beta_2]^\times)$ containing the element $\delta(1 - \beta_2 \beta_1^{-1})$. We note that $(1 - \beta_2 \beta_1^{-1})$ belongs to $1 + \mathfrak{p}_{F[\beta_2]}$ and $\beta_1 \in \delta F_0$. Since $k_D$ is a quadratic extension of $k_{F_0}$, we get that $F_0^\times$ is contained in $\text{Nr}_{F[\beta_2]/F}(F[\beta_2]^\times)$. Since $\beta_1 \in F_0$, we get that $\delta(\beta_1 - \beta_2)$ belongs to $\text{Nr}_{F[\beta_2]/F}(F[\beta_2]^\times)$. Hence, using the criterion [5.4] we get that the set $X_\beta$ is non-empty.

**Lemma 5.6.5.** Let $F/F_0$ be a quadratic extension and let $\mathfrak{f}$ be a stratum of the type (B) such that $(V_2, h)$ is isotropic. If $q_1 > q_2$, then any representation in the set $\Pi_\beta$ is generic.

**Proof.** Let $U$ be the unipotent radical of a Borel subgroup of $G$ in $X_\beta$. Let $\theta$ be any skew semisimple character in the set $C(\Lambda, 0, \beta)$. We will first check that

$$\text{res}_{H^1(\Lambda, \beta)} U \theta = \psi_\beta.$$  

The group $H^1(\Lambda, \beta)$ is equal to

$$P_1(\Lambda, \beta)P_{(q_2/2)+}(\Lambda, \beta)P_{(q_1/2)+}(\Lambda).$$
We define $H'$ to be the subgroup $P_1(\Lambda_\beta)P_{(q_2/2)+}(\Lambda)$. Now, using Lemma [8.5.1], we get that $H' \cap U$ is equal to $P_{(q_2/2)+}(\Lambda) \cap U$. Since $H^1(\Lambda, \beta) \cap U$ is equal to $H^1(\beta, \Lambda) \cap (H' \cap U)$, we get that $H^1(\Lambda, \beta) \cap U$ is equal to $P_{(q_2/2)+}(\Lambda, \beta_1)P_{(q_1/2)+}(\Lambda) \cap U$. Let $g_1g_2 \in U$ for some $g_1 \in P_{(q_2/2)+}(\Lambda F_{[\beta_1]})$ and $g_2 \in P_{(q_1/2)+}(\Lambda)$. Let $v = v_1 + v_2$ be a non-trivial vector fixed by $U$. We recall the notations defined in 5.3. If $v_1 = 0$, then we have $\lambda(h'(v_2, v_2)) = 0 = \lambda(\beta_2 h'(v_2, v_2))$. This implies that $\beta_2$ stabilises the kernel of $\lambda$ and this absurd. Hence, we get that $v_1 \neq 0$. We have

$$g_2(v_1 + v_2) = g_1(v_1 + v_2) = xv_1 + g_1v_2.$$ 

Now, comparing both sides we get that $x \in F^X \cap P_{(q_1/2)+}(\Lambda)$. This implies that $1_1g_11_1 \in F^X \cap P_{(q_1/2)+}(\Lambda)$. Since, the determinant of $g_2g_1$ is equal to 1, we get that determinant of $1_2g_11_2$ belongs to $F^\times \cap P_{(q_1/2)+}(\Lambda)$. From the definition of simple character $\theta$ we get (5.10). The Lemma is now a consequence of Proposition 5.7.2.

5.7. The case where $(V_2, h)$ is anisotropic. As in the case where $(V_2, h)$ is isotropic, the genericity of a cuspidal representation in the set $\Pi_\mathfrak{F}$ depends only on the integers $q_1$ and $q_2$. However, the condition for genericity becomes the opposite to the case where $(V_2, h)$ is isotropic, i.e., the inequality $q_2 > q_1$ is necessary and sufficient condition for genericity of a cuspidal representation in $\Pi_\mathfrak{F}$.

Lemma 5.7.1. Let $F/F_0$ be an unramified extension and let $x$ be a stratum of the type (B) such that $(V_2, h)$ is anisotropic and $q_1 > q_2$. Any representation contained in the set $\Pi_\mathfrak{F}$ is non-generic and the set $\mathfrak{X}_\beta$ is empty.

Proof. The group $P(\Lambda)$ is a special parahoric subgroup of $G$ and from Iwasawa decomposition we get that

$$G = P(\Lambda)B.$$ 

Let $\pi = \text{Ind}_{\mathfrak{P}(\Lambda, \beta)}^G \kappa$ be a generic representation in the set $\Pi_\mathfrak{F}$. Then there exists a $g \in P(\Lambda)$ and a character $\Psi$ of $U$ such that

$$\text{Hom}_{\mathfrak{P}(\Lambda, \beta) \cap U}(\kappa, \Psi) \neq 0.$$ (5.11)

Let $q_1 = 4m_1 + 2r$ and $q_2 = 2m_2 + 1$, for some integers $m_1, m_2$ and $r \in \{0, 1\}$. Note that $P_{(q_1/2)+}(\Lambda) \cap U_\text{der}$ is equal to $U_\text{der}(m_1)$ (see 5.1.1 and 10.2) for an explicit description of the lattice sequence $\Lambda$ and the induced filtrations on $\text{End}_F(V)$. Since $\nu_\Lambda(e_1) = 1$ we get that $ge_1 = av_1 + w_0b_2 + c(v_3)$, where $a, b, c \in O_F$. Since, $e_1$ is isotropic we get that

$$a \sigma(a) + w_0b_2 \sigma(b) + c \sigma(c) = 0.$$ 

Since $\nu_\Lambda(e_1) = 1$, the above equality implies that $a, c \in O_F^\times$. Note that $\nu_F[\beta_2](\beta_2^{-1}) > 0$, and in the basis $(v_2, v_3)$ for $V_2$, the element $\beta_2^{-1}$ belongs to $\text{End}_F(V)$ belongs to the following lattice of $\text{End}_F(V)$:

$$\begin{pmatrix} \mathfrak{O}_F & \mathfrak{P}_F \\ \mathfrak{O}_F & \mathfrak{O}_F \end{pmatrix}.$$ 

Hence, we get that

$$\nu_F(h(ge_1, \beta ge_1)) = \nu_F(\beta_1 a \sigma(a) + h(w_0b_2 + c(v_3), \beta_2(w_0b_2 + c(v_3)))) = -2m_2 - r \leq -m_1.$$ 

From this, we get that the character $\psi^\beta_f$ is non-trivial on the group $P_{(q_1/2)+}(\Lambda) \cap U_\text{der}$. This contradicts the assumption (5.11) and hence any representation $\pi$ in the set $\Pi_\mathfrak{F}$ is non-generic. Using Iwasawa decomposition, we get that $\nu_F(h(ge_1, \beta ge_1))$ is bounded above, for all $g \in G$. Hence, $h(ge_1, \beta ge_1) = 0$ for all $g \in G$. This shows that the set $\mathfrak{X}_\beta$ is empty. □

Lemma 5.7.2. Let $F/F_0$ be a ramified extension and let $x$ be a stratum of the type (B) such that $(V_2, h)$ is anisotropic. If $q_1 > q_2$, then any representation in the set $\Pi_\mathfrak{F}$ is non-generic and the set $\mathfrak{X}_\beta$ is empty.

Proof. The group $P_0(\Lambda)$ is a special parahoric subgroup of $G$, and using Iwasawa decomposition we get that

$$G = P(\Lambda)B.$$ 

Let $\pi = \text{Ind}_{\mathfrak{P}(\Lambda, \beta)}^G \kappa$ be a generic representation in $\Pi_\mathfrak{F}$. Then there exists a $g \in P(\Lambda)$ and a character $\Psi$ of $U$ such that

$$\text{Hom}_{\mathfrak{P}(\Lambda, \beta) \cap U}(\kappa, \Psi) \neq 0.$$ (5.12)
SANTOSH NADIMPALLI

Let \( q_1 = 4m_1 + 2 \) and \( q_2 = 4m_2 \), for some integers \( m_1 \) and \( m_2 \). Observe that \( P_{(q_1/2)^+}(\Lambda)^g \cap U_{der} \) is equal to \( U_{der}(\mathbb{Z}[1/2]) \). Since \( \nu_\alpha(e_1) = 0 \), we get that \( ge_1 = av_1 + bv_2 + cv_3 \) for some \( a, b, c \in \mathfrak{o}_F \). As the vector \( e_1 \) is isotropic we get that

\[
\lambda_1 a \sigma(a) + \lambda_2 \beta \sigma(b) + \lambda_3 \beta \sigma(c) = 0.
\]

Now, the space \((V_2, h)\) is anisotropic and this implies that \( a \in \mathfrak{o}_F^\times \). Now, the valuation \( \nu_F(\delta h(ge_1, \beta ge_1)) \) is given by:

\[
\nu_F(\delta h(ge_1, \beta ge_1)) = \nu_F(\beta_1 a \sigma(a) + h(bv_2 + cv_3, \beta_2(bv_2 + cv_3)))
\]

From the inequality \( \nu_F(\delta \beta_1) = -m_1 \leq -\lfloor m_1/2 \rfloor \), we get that \( \psi_\beta^3 \) is a non-trivial character on \( P_{(q_1/2)^+}(\Lambda)^g \cap U_{der} \). This is a contradiction to the assumption \( \nu_F(h(ge_1, \beta ge_1)) \) is bounded above, for all \( g \in G \). Hence, \( h(ge_1, \beta ge_1) \neq 0 \), for all \( g \in G \). This shows that the set \( \mathfrak{x}_\beta \) is empty.

In the case where \((V_2, h)\) is anisotropic, we could not use the criterion in 5.5. However, the following observation motivates the fact that \( \mathfrak{x}_\beta \) is non-empty in the case where \( q_2 > q_1 \). We suppose that \( F/F_0 \) is unramified. In the basis \((v_1, v_2, v_3)\) consider a skew element \( \beta \) of the form

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \beta_2 \\
\beta_2 & 0 & 0
\end{pmatrix}.
\]

Let \( e_1 \) be an isotropic vector in \((v_1, v_3)\). We see that \( e_1 \) is a solution for \( \mathfrak{x}_\beta \). Now, the class of \( \beta \) in \( -n(\Lambda)/a_{1, n}(\Lambda) \), represented in the matrix form, is given by a matrix as in (5.13). This suggests that we may lift a point from the special fibre of an integral model of \( \mathfrak{x}_\beta \), and we do this in the following lemma.

**Lemma 5.7.3.** Let \( F/F_0 \) be a quadratic extension and let \( \mathfrak{r} \) be a stratum of the type \((B)\) such that \((V_2, h)\) is anisotropic. If \( q_2 > q_1 \), then the set \( \mathfrak{x}_\beta \) is non-empty.

**Proof.** We will lift a point from the special fibre of a smooth model for \( \mathfrak{x}_\beta \). We have a basis \((v_1, v_2, v_3)\) for \( V \) as defined in 5.1.1 if \( F/F_0 \) is unramified, and in 5.1.2 otherwise. Let \( \beta = (\beta_{ij}) \) be the matrix representation of \( \beta \) in the basis \((v_1, v_2, v_3)\). We have \( \beta_{11} = \beta_1 \). First consider the case where \( F/F_0 \) is unramified. A Borel subgroup, fining the line spanned by \( xv_1 + yv_2 + zv_3 \), belongs to \( \mathfrak{x}_\beta \) if and only if \((x, y, z)\) satisfy the following equations:

\[
wx^\sigma(x) + y\sigma(y) + wz\sigma(z) = 0
\]

and

\[
\beta_1 wx^\sigma(x) + \beta_2 y\sigma(y) + \beta_3 wz\sigma(z) + \beta_3 y\sigma(y) + wz\sigma(z) = 0.
\]

Changing \( y \) to \( xy \) and rescaling the second equation by \( w^{-2} \), with \( \sigma = -\nu_F(\beta_{32}) \), we get the following set of equations:

\[
wx^\sigma(x) + xy\sigma(y') + wz\sigma(z) = 0
\]

and

\[
wx^\sigma(x) + \beta_2 xy\sigma(y') + \beta_3 wz\sigma(z) + \beta_3 y\sigma(y') + wz\sigma(z) = 0.
\]

Note that the coefficients of (5.15) are integral and the two equations (5.14) and (5.15) define a flat closed subscheme \( \mathcal{X}_\beta \) of \( \mathfrak{p}_F^5 \) such that the generic fibre is \( \mathfrak{x}_\beta \). The special fibre is given by the set of equations

\[
wx^\sigma(x) + wz\sigma(z) = 0
\]

and

\[
C_1(y'\sigma(z) - z\sigma(y')) + C_2 x^\sigma(x) + C_3 z\sigma(z) = 0,
\]

where \( C_1 = \frac{w^{-1} \beta_{23}}{\beta_{23}} \), \( C_2 = \frac{w^{-1} \beta_{11}}{\beta_{11}} \), and \( C_3 = \frac{w^{-1} \beta_{33}}{\beta_{33}} \). Note that \( C_1 \neq 0 \), and therefore, the special fibre \( \mathfrak{x}_\beta \) is smooth. Since \( \mathcal{X}_\beta \) is flat we get that \( \mathfrak{x}_\beta \) is smooth over \( \mathfrak{o}_F \). The special fibre has a rational point and hence by Hensel's lemma the set \( \mathfrak{x}_\beta \) is non-empty.
Now, consider the case where $F/F_0$ is a ramified extension. A Borel subgroup $B$, fixing the line spanned by $av_1 + bv_2 + cv_3$, belongs to $\mathfrak{X}_\beta$ if and only if:
\[\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) = 0,\]
and
\[\lambda_1 \beta_1 a\sigma(a) + \lambda_2 \beta_2 b\sigma(b) + \lambda_3 \beta_3 c\sigma(c) + b\sigma(c)\beta_3 + \lambda_3 \beta_2 c\sigma(b) = 0.\]
In the present case $F[\beta_2]$ is an unramified extension of $F$. After rescaling with a power of $\pi$, if necessary, we may assume that we may assume that
\[\beta_2 = \begin{pmatrix} \beta_{22} & \beta_{23} \\ \beta_{32} & \beta_{33} \end{pmatrix} \in \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{o}_F & \mathfrak{o}_F \end{pmatrix},\]
and $\beta_1 \in \mathfrak{o}_F$. Because $\beta_1$ is skew, we get that $\beta_1 \in \mathfrak{p}_F$. By a change of variable $b$ to $\pi b'$, the above set of equations become:
\[\lambda_1 a\sigma(a) + \lambda_2 \pi b'\sigma(b') + \lambda_3 c\sigma(c) = 0\]
and
\[\pi^{-1} \lambda_1 \beta_1 a\sigma(a) - \pi \lambda_2 \beta_2 b\sigma(b) + \pi^{-1} \lambda_3 \beta_3 c\sigma(c) + \lambda_3 \beta_2 c\sigma(b) - \lambda_3 \beta_2 c\sigma(b) = 0.\]
Since $\beta$ is skew, we get that $\sigma(\beta_{22}) + \beta_{22} = 0$, $\sigma(\beta_{33}) + \beta_{33} = 0$, and $\lambda_1 \beta_{23} = -\lambda_3 \sigma(\beta_{32})$. Hence, the above two equations have integral coefficients. The above two equations define a flat projective sub-variety $X_\beta$ in $\mathbb{P}^5_{\mathfrak{p}_F}$ with generic fibre $\mathfrak{X}_\beta$. The special fibre is given by
\[\lambda_1 a^2 + \lambda_3 c^2 = 0,\]
and
\[C_1 a^2 + C_2 c^2 + C_3 bc = 0,\]
where $C_1 = \pi^{-1} \lambda_1 \beta_1$, $C_2 = \pi^{-1} \lambda_3 \beta_{33}$, and $C_3 = \pi \lambda_2 \beta_{23}$. Clearly $C_3 \neq 0$ as the element $\beta_2$ is minimal. The special fibre is smooth and hence, $X_\beta$ is a smooth model for $\mathfrak{X}_\beta$ such that the generic fibre has a rational point. Using Hensel’s lemma, we get that the set $X_\beta$ is non-empty. □

Lemma 5.7.4. Let $F/F_0$ be a quadratic extension and let $T$ be a stratum of the type (B) such that $(V_2, h)$ is anisotropic. If $q_2 > q_1$, then any representation in the set $\Pi_T$ is generic.

Proof. Let $U$ be the unipotent radical of a Borel subgroup in the set $\mathfrak{X}_\beta$. Note that the group $H^1(\Lambda, \beta)$ is equal to $P_1(\Lambda_\beta)P_{(q_2/2)+}(\Lambda)$ and it follows from Lemma 3.5.1 that $H^1(\Lambda, \beta) \cap U$ is equal to $P_{(q_2/2)+}(\Lambda) \cap U$. This implies that
\[\text{res}_{H^1(\Lambda, \beta) \cap U} \theta = \psi_\beta,\]
where $\theta$ is any skew semisimple character of $H^1(\Lambda, \beta)$. Now, genericity is a consequence of Proposition 3.5.2. □

6. Non simple type (C) strata

A skew semisimple stratum $T = [\Lambda, n, 0, \beta]$ is of type (C) if the underlying splitting of $T$ is of the form $V = V_1 \perp V_2$ with $\dim_F V_i = i, \beta = \beta_1 + \beta_2$ such that $\beta_i \in F$ and $\sigma(\beta_i) = -\beta_i$, for $i \in \{1, 2\}$. Recall that $\beta_i$ is equal to $1_i \beta_1$, for $i \in \{1, 2\}$. We will show that any representation contained in the set $\Pi_T$ is non-generic. We begin with some preliminaries.

6.1. Lattice sequences. We will describe the lattice sequences up to $G_\beta$-conjugacy. Note that $\Lambda$ is a lattice sequence on $V$ such that $P^{\beta}(\Lambda_\beta)$ is a maximal parahoric subgroup of $G_\beta$. We will also have to fix a Witt-basis of $(V, h)$ which gives a splitting of these lattice sequences.
6.1.1. The unramified case. Consider the case where \( F/F_0 \) is unramified and \( (V_2, h) \) is isotropic. We fix a Witt-basis \( (e_1, e_0, e_{-1}) \) for \( (V, h) \) such that \( e_1, e_{-1} \in V_2 \). The lattice sequence, up to \( G_{\beta}\)-conjugation, is given by one of the following period 2 lattice sequences:

\[
\Lambda(-1) = \Lambda(0) = \sigma_F e_1 \oplus \sigma_F e_0 \oplus \sigma_F e_{-1}
\]

or

\[
\Lambda(0) = \sigma_F e_1 \oplus \sigma_F e_0 \oplus \sigma_F e_{-1} \quad \text{and} \quad \Lambda(1) = \sigma_F e_1 \oplus \sigma_F e_0 \oplus \sigma_F e_{-1}.
\]

If \( F/F_0 \) is an unramified extension and \( (V_2, h) \) is anisotropic, we fix vectors \( v_1 \in V_1, v_2, v_3 \in V_2 \) such that \( (v_1, v_2, v_3) \) is an orthogonal basis for \( V \) and

\[
h(v_1, v_1) = h(v_3, v_3) = \varpi \quad \text{and} \quad h(v_2, v_2) = 1.
\]

The lattice sequence \( \Lambda \), up to \( G_{\beta}\)-conjugation, is given by the following period 2 lattice sequence:

\[
\Lambda(0) = \sigma_F v_1 \oplus \sigma_F v_2 \oplus \sigma_F v_3 \quad \text{and} \quad \Lambda(1) = \sigma_F v_1 \oplus \sigma_F v_2 \oplus \sigma_F v_3.
\]

There exists a Witt-basis \( (e_1, e_0, e_{-1}) \) of \( V \) with \( e_1, e_{-1} \in \langle v_1, v_3 \rangle \) such that \( (e_1, e_0, e_{-1}) \) provides a splitting for the lattice sequence \( \Lambda \), and we have

\[
\Lambda(0) = \sigma_F e_1 \oplus \sigma_F e_0 \oplus \sigma_F e_{-1}, \quad \text{and} \quad \Lambda(1) = \sigma_F e_1 \oplus \sigma_F e_0 \oplus \sigma_F e_{-1}.
\]

6.1.2. The ramified case. If \( F/F_0 \) is a ramified extension and \( (V_2, h) \) is isotropic, we fix a Witt-basis \( (e_1, e_{-1}) \) for \( (V_2, h) \) and \( e_0 \in V_1 \) be a non-zero vector such that \( (e_1, e_0, e_{-1}) \) is a Witt-basis for \( (V, h) \). The lattice sequence \( \Lambda \), up to \( G_{\beta}\)-conjugation, is given by the following period 2 lattice sequence:

\[
\Lambda_1(0) = \sigma_F e_1 \oplus \sigma_F e_0 \oplus \sigma_F e_{-1} \quad \text{and} \quad \Lambda_1(1) = \sigma_F e_1 \oplus \sigma_F e_0 \oplus \sigma_F e_{-1}.
\]

Now assume that \( (V_2, h) \) is anisotropic, and fix an orthogonal basis \( (v_1, v_2, v_3) \) of \( (V, h) \) such that \( v_1 \in V_1 \) and \( v_2, v_3 \in V_2 \), we may (and do) assume that \( h(v_1, v_1) = \lambda_1 \in \sigma_F^\times \). Up to \( G_{\beta}\)-conjugation, the lattice sequence \( \Lambda \) is given by the following period 2 lattice sequence:

\[
\Lambda(-1) = \Lambda(0) = \sigma_F v_1 \oplus \sigma_F v_2 \oplus \sigma_F v_3.
\]

There exist a Witt-basis \( (e_1, e_0, e_{-1}) \) for the space \( (V, h) \) with \( e_1, e_{-1} \in \langle v_1, v_3 \rangle \) such that

\[
\Lambda(-1) = \Lambda(0) = \sigma_F e_1 \oplus \sigma_F e_0 \oplus \sigma_F e_{-1}.
\]

The groups \( J^0(\Lambda, \beta) \) and \( H^1(\Lambda, \beta) \) are given by

\[
P_0(\Lambda, \beta)P_{(n/2)}(\Lambda)
\]

and

\[
P_1(\Lambda, \beta)P_{(n/2)+}(\Lambda)
\]

respectively.

6.2. (\( V_2, h \)) is anisotropic. In this part we consider the case where \( (V_2, h) \) is anisotropic. To show that any representation \( \pi \in \Pi_\xi \) is non-generic, we will show that the character \( \psi_{\beta}^\pi \) is non-trivial on \( U_{\text{der}} \cap P_{(n/2)+}(\Lambda) \), for all \( g \in P(\Lambda) \).

Lemma 6.2.1. Let \( F/F_0 \) be any quadratic extension and let \( \xi \) be a strata of the type (C) such that \( (V_2, h) \) is anisotropic. Any representation contained in the set \( \Pi_\xi \) is non-generic.

Proof. Let \( \pi \) be a representation in the set \( \Pi_\xi \) and we have \( \pi \simeq \text{ind} P^{\Lambda}_0(\Lambda, \beta) \kappa, \) where \( (J^0(\Lambda, \beta), \kappa) \) is a Bushnell–Kutzko type contained in \( \pi \). Note that the group \( P^\beta(\Lambda) \) is a special parahoric maximal compact subgroup of \( G \) and we have \( G = P(\Lambda)TU \). Here, \( U \) is the unipotent radical of the Borel subgroup fixing the line \( \langle e_1 \rangle \). Now, assume that there exists a \( g \in P(\Lambda) \) and a non-trivial character \( \Psi \) of \( U \) such that

\[
\text{Hom}_{P^\beta(\Lambda)\cap U}(\kappa^\beta, \Psi) \neq 0.
\]

(6.1)

First consider the case where \( F/F_0 \) is a ramified extension. Let \( ge_1 = av_1 + bv_2 + cv_3 \), for some \( a, b, c \in \sigma_F \). Since \( e_1 \) is isotropic, we get that

\[
\lambda_1 a \sigma(a) + \lambda_2 b \sigma(b) + \lambda_3 c \sigma(c) = 0.
\]
This implies that the vector \( a \in \mathfrak{g}_\mathfrak{g}^\times \), since \(-\lambda_2 \lambda_2^{-1} \not\in \text{Nr}_{F/F_0}(F^\times)\). Let \( \nu_\Lambda(\beta) = -n = -(4m + 2r) \), for some integer \( m \) and \( r \in \{0, 1\} \). Since \( \sigma(\beta) = -\beta \), we get that \( r = 1 \). Now, \( (n/2)^+ = 2m + 2 \) and we have \( P_{(n/2)^+}(\Lambda) \cap U_{\text{der}} \) is equal to \( U_{\text{der}}(\mathbb{A}) \). Then we observe that

\[
\nu_{F/F_0}(\delta h(ge_1, \beta ge_1)) = \nu_{F/F_0}(\delta(\beta - \beta_2)\lambda_2 \sigma(a)) = -m.
\]

As \( \nu_{F/F_0}(h(ge_1, ge_1)) \leq -(m + 1)/2 \), for any \( m \geq 1 \), we get that \( \psi_\beta \) is a non-trivial character on the group \( P_{(n/2)^+}(\Lambda) \cap U_{\text{der}} \). This is a contradiction to \([6.1.1]\). Hence, any representation in the set \( \Pi_\tau \) is non-generic.

Consider the case where \( F/F_0 \) is unramified. Note that \( e_1 \in \Lambda(1) \) and hence we get that \( ge_1 = av_1 + bv_2 + cv_3 \), for some \( a, c \in \mathfrak{p}_F \), \( b \in \mathfrak{g}_F \) and

\[
\varpi a \sigma(a) + b \sigma(b) + \varpi c \sigma(c) = 0.
\]

By a change of variable: \( b' = \varpi b \), we have

\[
a \sigma(a) + \varpi b' \sigma(b') + c \sigma(c) = 0.
\]

From the above equality, we get that \( a, c \in \mathfrak{g}_F^\times \), since \( \nu_\Lambda(e_1) = 1 \). Now, we have

\[
\nu_F(h(ge_1, \beta ge_1)) = \nu_F(\beta_1 \varpi a \sigma(a) + \beta_2 (b \sigma(b) + \varpi c \sigma(c))) = \nu_F((\beta_1 - \beta_2)) + 1
\]

We note that \( 2\nu_F(\beta_1) = \nu_{\lambda_1}(\beta_1) \) and \( \nu_F(\beta_2) = \nu_{\lambda_2}(\beta_2) \) with the condition that

\[
-n = \nu_\Lambda(\beta) = \min\{\nu_{\lambda_1}(\beta_1), \nu_{\lambda_2}(\beta_2)\}.
\]

Assume that \(-n = \nu_{\lambda_1}(\beta_1) \leq \nu_{\lambda_2}(\beta_2)\). In this case, \( n = 4m + 2r \), where \( m \) is an integer and \( r \in \{0, 1\} \). Now, we have \((n/2)^+ = 2m + r + 1 \) and the group \( P_{(n/2)^+}(\Lambda) \cap U_{\text{der}} \) is equal to \( U_{\text{der}}(\mathbb{A}) \). We may have two possibilities either \( \nu_{\lambda_2}(\beta_2) \leq \nu_F(\beta_1) = -(2m + r) \) or \( \nu_{\lambda_2}(\beta_2) \geq \nu_F(\beta_1) = 2m + r \). In the first case, we have

\[
\nu_F(\beta_1 - \beta_2) + 1 = \nu_F(\beta_2) + 1 \leq -(2m + r) + 1 \leq -m.
\]

In the second case, we have

\[
\nu_F(\beta_1 - \beta_2) + 1 = \nu_F(\beta_1) + 1 = -(2m + r) + 1 \leq -m.
\]

Hence, the character \( \psi_\beta \) on \( P_{(n/2)^+}(\Lambda) \) is non-trivial on \( P_{(n/2)^+}(\Lambda) \cap U_{\text{der}} \) and we obtain a contradiction to \([6.1.1]\).

Assume that \(-n = \nu_{\lambda_1}(\beta_2) \leq \nu_{\lambda_1}(\beta_1) \) and set \( n = 4m + r \), for some integer \( m \) and \( 0 \leq r \leq 3 \). In this case, the group \( P_{(n/2)^+}(\Lambda) \cap U_{\text{der}} \) is equal to \( U_{\text{der}}(\mathbb{A}) \). Since \( \nu_{\lambda_1}(\beta) \leq \nu_F(\beta_1) \), we get that

\[
\nu_F(\beta_1 - \beta_2) + 1 = \nu_F(\beta_2) = -(4m + r) + 1 \leq -m.
\]

The above inequality implies that the character \( \psi_\beta \) is non-trivial on the group \( P_{(n/2)^+}(\Lambda) \cap U_{\text{der}} \) and hence we get a contradiction to the assumption \([6.1.1]\). \( \square \)

### 6.3. \((V_2, h)\) is isotropic

In this part we assume that \((V_2, h)\) is isotropic. Note that the set \( \mathfrak{X}_\beta \) is non-empty, however, it turns out that any representation in \( \Pi_\tau \) is non-generic. The essential reason being that \( \tau \) is a cuspidal representation of \( P_0(\Lambda_\beta)/P_1(\Lambda_\beta) \). The group \( P_0(\Lambda_\beta)/P_1(\Lambda_\beta) \) is equal to \( U(1, 1)(k_F/k_{F_0}) \times U(1)(k_F/k_{F_0}) \), when \( F/F_0 \) is unramified and is equal to \( \text{SL}_2(k_F) \times \{\pm 1\} \), when \( F/F_0 \) is ramified.

Let \( B \) be the Borel subgroup fixing the subspace \( \langle e_1 \rangle \) and \( U \) be the unipotent radical of \( B \). Although \( P^0(\Lambda) \) is a special parahoric subgroup of \( G \), it is convenient to use the decomposition

\[
G = I_{W_G} B
\]

for Mackey-decompositions. Here \( I \) is the Iwahori subgroup contained in the subgroups \( P(\Lambda) \), where \( \Lambda \) varies over the two (representatives for \( G_\beta \) conjugacy classes) lattice sequences in \([6.1.1]\) and \([6.1.2]\) when \( F/F_0 \) is unramified and ramified respectively.
6.3.1. **Shallow elements.** We need to define a measure of shallowness, relative to the group \( P_{n/2}^+(\Lambda) \), of the elements \( u(x, y) \) and \( \bar{u}(x, y) \) in \( I \cap U \) and \( I \cap U^w \) respectively (here \( w \) is the non-trivial element in \( W_G \)). This is achieved by defining an integer \( d(x, w, x) \). The main purpose of the definition of \( d(x, w, x) \) becomes apparent in Lemma \[6.3.3\].

Let \( n = 4m + 2r \), for some positive integer \( m \) and \( r \in \{0, 1\} \). Consider the case where \( F/F_0 \) is unramified. For any \( x, y \in F \) such that \( x\sigma(x) + y + \sigma(y) = 0 \), \( w \in W_G \) and \( \Lambda \) a lattice sequence defined in \[6.1.1\] or in \[6.1.2\], we set \( d(x, w, x) \) to be

\[
d(x, w, x) = \begin{cases} 
\max\{1, m + 1 - \nu_F(x)\}, & \text{if } \Lambda(0) \cap V_2 = 0_F e_1 \oplus 0_F e_{-1}, \\
\max\{1, 0, m + r - \nu_F(x)\}, & \text{if } \Lambda(0) \cap V_2 = 0_F e_1 \oplus p_F e_{-1}, w = \text{id}, \\
\max\{2, m + r + 1 - \nu_F(x)\}, & \text{if } \Lambda(0) \cap V_2 = 0_F e_1 \oplus p_F e_{-1}, w \neq \text{id}.
\end{cases}
\]

(6.2)

If \( F/F_0 \) is a ramified extension, then we have one \( G_{F[\beta]} \) conjugacy class of lattice sequences \( \Lambda \)-defined in \[6.3.2\]—such that \( P^0(\Lambda_{F[\beta]}) \) is a maximal parahoric subgroup in \( G_{F[\beta]} \).

Note that \( d(x, w, x) \) is a constant for \( \nu_F(x) > 0 \), and we denote this constant by \( d(x, w) \). For example, when \( F/F_0 \) is unramified, \( \Lambda(0) = 0_F e_1 \oplus p_F e_{-1} \), and \( w \neq \text{id} \) we have \( d(x, w) = 2 \). At the same time when \( F/F_0 \) is ramified, we have \( d(x, w) = 1 \), for \( w \neq 1 \).

6.3.2. With these preliminaries we are ready to prove that any representation in the set \( \Pi \) is non-generic. Let \( T \) be the unipotent radical of the opposite Borel subgroup of \( B^w \) with respect to the torus \( T \), for all \( w \in W_G \). Recall that \( T \) is the maximal \( F_0 \)-split torus such that \( T \) stabilises the decomposition \( \{e_i\} \oplus \{e_0\} \oplus \{e_{-1}\} \).

**Lemma 6.3.1.** Let \( F/F_0 \) be any quadratic extension and let \( x \) be a skew semisimple strata of the type \( (C) \) such that \( (V_2, h) \) is isotropic. Let \( u^- = \bar{u}(x, y) \) be an element of \( I \cap U^w \) then we have

\[
U^w_{\text{der}}(d(x, w, x)) \subseteq H^1(\Lambda, \beta)^{u^-} \cap U^w_{\text{der}}.
\]

**Proof.** We assume that \( w = \text{id} \) and the case where \( w \neq \text{id} \) is entirely similar. We have to show that \( \{U_{\text{der}}(d(x, w, x))\}^{u^-} \) is contained in the group \( H^1(\Lambda, \beta) \). We have the matrix identity

\[
\bar{u}(x, y)u(0, a)\bar{u}(-x, -y - x\sigma(x)) = \begin{pmatrix} 1 - a(x\sigma(x) + y) & ax(\sigma(x)) & a \\
ax(-y - x\sigma(x)) & 1 + ax\sigma(x) & ax \\
-ay(y + x\sigma(x)) & ax(\sigma(x)) & ay + 1 \end{pmatrix}.
\]

(6.4)

For any element \( u(0, a) \in U_{\text{der}}(d(x, w, x)) \), the element \( \bar{u}(x, y)u(0, a)\bar{u}(-x, -y - x\sigma(x)) \) belongs to the group \( P_1(\Lambda_{F[\beta]})P_{(n/2)+}^+(\Lambda) \subseteq H^1(\Lambda, \beta) \).

\[ \square \]

**Lemma 6.3.2.** With the same assumptions and notations as in Lemma \[6.3.1\] for any skew semisimple character \( \theta \in C(\Lambda, 0, \beta) \) defined on the group \( H^1(\Lambda, \beta) \), we have

\[
\text{res}_{U^w_{\text{der}}(d(x, w, x)} \theta^{u^-} = \psi_{\beta_{-1}}.
\]

**Proof.** Let \( u^+ = u(0, a) \) be an element in \( U_{\text{der}}(d(x, w, x)) \) and \( u^- = \bar{u}(x, y) \) be any element as in Lemma \[6.3.1\] Assume that \( u^- u^+(u^-)^{-1} = g_1 g_2 \), where \( g_1 \in P_1(\Lambda_{F[\beta]}) \) and \( g_2 \in P_{(n/2)+}^+(\Lambda) \). From the matrix identity \[6.3.1\] and from the definition of \( d(x, w, x) \), the constant \( 1 + ax\sigma(x) \) belongs to \( F^x \cap P_{(n/2)+}^+(\Lambda) \). Hence, we get that \( 1_1 g_1 1_1 \in F^x \cap P_{(n/2)+}^+(\Lambda) \). This implies that the determinant of the element \( g_2 g_1 g_2 \) is contained in \( F^x \cap P_{(n/2)+}^+(\Lambda) \). Now, from the definition of a simple character, we get that \( \theta(u^- u^+(u^-)^{-1}) \) is equal to \( \psi_{\beta_{-1}}(u^- u^+(u^-)^{-1}) \) and this completes the lemma.

\[ \square \]

**Lemma 6.3.3.** With the same assumptions in the lemma \[6.3.1\] the representation \( \text{res}_{\text{der}}(\Lambda, \beta) \cap U^w_{\text{der}}(\kappa \otimes \tau) \) is a direct sum of non-trivial characters.
Proof. We essentially follow ideas from [PS08, Theorem 2.6] and [BS09, Theorem 4.3]. We prove this lemma in the case where \( w = \text{id} \) and the other case is similar. Note that \( P_0(\Lambda)/P_1(\Lambda) \) is isomorphic to \( U(1,1)(k_F/k_{F_0}) \times U(1)(k_F/k_{F_0}) \) if \( F/F_0 \) is unramified, and is isomorphic to \( \operatorname{SL}_2(k_F) \times \{ \pm 1 \} \) otherwise. Let \( \tilde{J}_1 \) be the group \( (J^0(\Lambda, \beta) \cap U)J^1(\Lambda, \beta) \) and observe that we have \( \tilde{J}_1 \) is equal to \( (J^0(\Lambda, \beta) \cap U_{\text{der}})J^1(\Lambda, \beta) \). Hence, the image of \( J^0(\Lambda, \beta) \cap U \) in the quotient \( P_0(\Lambda)/P_1(\Lambda) \) is a p-Sylow subgroup. Similarly, let \( \tilde{H}_1 \) be the group \( (J^0(\Lambda, \beta) \cap U)H^1(\Lambda, \beta) \).

Note that \( \psi_\beta \) defines the trivial character on \( U \). Let \( \tilde{J}_1 \) be the group \( (J^0(\Lambda, \beta) \cap U)J^1(\Lambda, \beta) \). Let \( g \in P_{(n/2)+}(\Lambda) \). Let \( g_1 \in P_1(\Lambda_{\beta_1}) \cap U(V_2, h) \), and \( g_1' \in P_1(\Lambda_{\beta_1}) \cap U(V_1, h) \). We have \( g_2 g_1 g_1' e_1 = e_1 \) and hence, \( g_2 g_1 \in U \) and \( g \) is unramified. Now, the determinant of \( g_2^{-1} \) and \( g_1' \) are the same. With this observation, we get that \( \text{res}_{H^1(\Lambda, \beta) \cap U} \eta \psi_\beta = \text{id} \). Let \( \theta \) be a skew semisimple character of \( H^1(\Lambda, \beta) \) and \( \eta \) be a Heisenberg lift of \( \theta \) to the group \( J^1(\Lambda, \beta) \). Let \( \tilde{\theta} \) be the following character of \( \tilde{H}_1 \):

\[
\tilde{\theta}(jh) = \theta(h) \quad \text{for all} \quad j \in J^0(\Lambda, \beta) \cap U, h \in H^1(\Lambda, \beta).
\]

Using [PS08, Lemma 2.5], we get that the representation \( \text{ind}_{\tilde{H}_1 \cap J^1(\Lambda, \beta)}^{J^1(\Lambda, \beta)} \tilde{\theta} \) is isomorphic to \( \eta \).

Note that the group \( J^1(\Lambda, \beta) \cap U_{\text{der}} \) is equal to \( H^1(\Lambda, \beta) \cap U_{\text{der}} \) and hence

\[
\text{res}_{J^1(\Lambda, \beta) \cap U_{\text{der}}} \eta \simeq \text{res}_{H^1(\Lambda, \beta) \cap U_{\text{der}}} \eta
\]

and the later is a trivial representation. This implies that \( \eta \) extends as a representation of \( \tilde{J}_1 \) such that \( J^1(\Lambda, \beta) \cap U_{\text{der}} \) acts trivially on this extension; let us denote this extension by \( \tilde{\eta} \). By Frobenius reciprocity we get a map

\[
\text{ind}_{\tilde{H}_1}^{\tilde{J}_1} \tilde{\theta} \to \tilde{\eta}, \quad (6.5)
\]

The representation \( \eta \) is irreducible. The dimension of the representation \( \text{ind}_{\tilde{H}_1}^{\tilde{J}_1} \tilde{\theta} \) is equal to \( [J^1(\Lambda, \beta) : H^1(\Lambda, \beta)]^{1/2} \) and hence the map \( (6.5) \) is an isomorphism.

Let \( [\lambda^m, n_1, 0, \beta] \) be a skew semisimple stratum such that \( \tilde{J}_1 \) is equal to \( (P_1(\Lambda_{\beta}^m) \cap U_{\text{der}})J^1(\Lambda, \beta) \). Let \( \theta_m \) be the skew semisimple character of \( H^1(\Lambda_{\beta}^m, \beta) \) obtained as a transfer from the skew semisimple character \( \theta \) of \( H^1(\Lambda, \beta) \). We note that the groups \( \tilde{H}_1 \) and \( H^1(\Lambda_{\beta}^m, \beta) \) have the Iwahori decomposition with respect to \( (B, T) \). We then get that

\[
\text{Hom}_{\tilde{H}_1 \cap H^1(\Lambda_{\beta}^m, \beta)}(\tilde{\theta}, \theta_m) \neq 0.
\]

This implies that

\[
\text{ind}_{\tilde{J}_1}^{P_1(\Lambda_{\beta}^m)} \tilde{\eta} \simeq \text{ind}_{\tilde{J}_1}^{P_1(\Lambda_{\beta}^m)} \eta_m.
\]

From the uniqueness properties of beta-extensions, we get that \( \text{res}_{J^1(\Lambda, \beta) \cap U_{\text{der}}} \kappa \simeq \tilde{\eta} \). This shows that the representation

\[
\text{res}_{J^0(\Lambda, \beta) \cap U_{\text{der}}} \kappa \otimes \tau
\]

is a direct sum of non-trivial characters. \( \square \)

Lemma 6.3.4. Let \( F/F_0 \) be any quadratic extension and let \( x, y \in \mathfrak{o}_F \) such that \( x \sigma(x) + y + \sigma(y) = 0 \). If \( d(x, w, x) > d(x, w) \), then we have

\[
\nu_F(\delta(x_1 - \beta_2) x \sigma(x)) \leq -d(x, w, x).
\]

Proof. Assume that \( F/F_0 \) is unramified and \( \Lambda(0) \cap V_2 = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_{-1} \). If \( d(x, w, x) > d(x, w) \), then we get that \( \nu_F(x) < m \) and this implies that

\[
2\nu_F(x) - 2m - r \leq -(m + 1) + \nu_F(x).
\]

Now, consider the case where \( \Lambda(0) \cap V_2 = \mathfrak{o}_F e_1 \oplus \mathfrak{o}_F e_{-1} \) and assume that \( d(x, w, x) > d(x, w) \). If \( r = 0 \) and \( w = \text{id} \), then we get that \( \nu_F(x) < m \) which implies that

\[
2\nu_F(x) - 2m \leq -m + \nu_F(x).
\]
If \( r = 0 \) and \( w \neq \text{id} \), then we get that \( \nu_F(x) < m - 1 \) and hence
\[
2\nu_F(x) - 2m \leq -(m - 1) + \nu_F(x).
\]
If \( r = 1 \) and \( w = \text{id} \), then we have \( \nu_F(x) < m + 1 \) and hence we get that
\[
2\nu_F(x) - (2m + 1) \leq -(m + 1) + \nu_F(x).
\]
Finally, we consider the case where \( r = 1 \) and \( w \neq \text{id} \); we then have \( \nu_F(x) < m \). Hence, we get that
\[
2\nu_F(x) - (2m + 1) \leq -(m + 1) + \nu_F(x).
\]
Now, assume that \( F/F_0 \) is a ramified extension and \( d(x, w, x) > d(x, w) \). Note that \( \nu_F(\beta_1 - \beta_2) = -(2m + 1) \), for some \( m \in \mathbb{Z} \). If \( w = \text{id} \), then we have \( \nu_F(x) < m/2 \) and hence
\[
2\nu_F(x) - (2m + 1)/2 + 1/2 < -(m/2 + \nu_F(x)) \leq -d(x, w, x).
\]
If \( w \neq \text{id} \), then we have \( \nu_F(x) < (m - 1)/2 \) and we have
\[
2\nu_F(x) - (2m + 1)/2 + 1/2 < -(m + 1)/2 + \nu_F(x) \leq -d(x, w, x).
\]
From the above inequalities, in all exhaustive cases, gives the required inequality:
\[
\nu_F(\delta(\beta_2 - \beta_1)x\sigma(x)) \leq -d(x, w, x)
\]

\[\square\]

**Lemma 6.3.5.** Let \( F/F_0 \) be an unramified extension and \( x \) be a strata of the type \( (C) \) such that \( (V_2, h) \) is isotropic. Any representation contained in the set \( \Pi_x \) is non-generic.

**Proof.** Let \( \pi \in \Pi_x \) be a generic representation. Then there exists a \( g \in \mathcal{I} \), an element \( w \in W_G \), and a character \( \Psi \) of \( U^w \) such that
\[
\text{Hom}_{F(\Lambda, \beta) \cap U^w}((\kappa \otimes \tau)^\beta, \Psi) \neq 0.
\]
We write \( g = p^+ u^- \) such that \( p^+ \in B^w \cap \mathcal{I} \) and \( u^- \in \overline{U^w} \cap \mathcal{I} \), where \( \overline{U^w} \) is the unipotent radical of the opposite Borel subgroup of \( B^w \) which contains the torus \( T \). From the expression \( 6.6 \) we get that
\[
\text{Hom}_{F(\Lambda, \beta) \cap U^w}((\kappa \otimes \tau)^u^-, \Psi) \neq 0,
\]
for some character \( \Psi' \) of \( U^w \). We set \( e_w = we_1 \) and \( e_{-w} = w e_{-1} \), then we have
\[
h(u^- e_w, \beta u^- e_w) = \beta_1 x\sigma(x) + \beta_2 h(e_w + ye_{-w}, e_w + ye_{-w}) = (\beta_1 - \beta_2)x\sigma(x).
\]
Hence, \( \nu_F(\delta h(u^- e_w, \beta u^- e_w)) \) is equal to \( \nu_F(\delta(\beta_1 - \beta_2)x\sigma(x)) \). If \( d(x, w, x) > d(x, w) \), then Lemma 6.3.4 implies that
\[
\nu_F(\delta(\beta_1 - \beta_2)x\sigma(x)) \leq -d(x, w, x)
\]
which then implies that the character \( \theta \) on \( U_{der}(d(x, w, x)) \) equal to \( \psi^\beta_3 \) is non trivial on the group \( U_{der}(d(x, w, x)) \). But, this is a contradiction to the assumption in \( 6.6 \). Hence, we obtain \( d(x, w, x) = d(x, w) \) and this implies that \( u^- \in H^1(\Lambda, \beta) \). We may as well assume that \( u^- = \text{id} \). The lemma now follows from Lemma 6.3.3. \( \square \)

7. **Non simple type (D) strata**

7.1. **Inducing data.** A skew semisimple stratum \( x = [\Lambda, n, 0, \beta] \) is of type \( (D) \) if the underlying splitting is of the form \( V = V_1 \perp V_2 \perp V_3 \), with \( \dim_{\mathbb{Q}} V_i = 1 \), for \( 1 \leq i \leq 3 \). We also use the notation \( W_i \) for the space \( \otimes_{j \neq i} V_j \), for \( 1 \leq i \leq 3 \). If \( F/F_0 \) is unramified, let \( v_i \in V_i \) be a non-zero vector such that \( = \nu_F_0(h(v_i, v_i)) \in \{0, 1\} \), for all \( 1 \leq i \leq 3 \). If \( F/F_0 \) is ramified, let \( v_i \in V_i \) be a non-zero vector such that \( \nu_F_0(h(v_i, v_i)) = 0 \), for \( 1 \leq i \leq 3 \). Let \( \lambda_i = h(v_i, v_i) \), for \( 1 \leq i \leq 3 \). We have \( \beta = \beta_1 + \beta_2 + \beta_3 \), where \( \beta_i = 1, \beta_1 \), for \( 1 \leq i \leq 3 \). The lattice sequence \( \Lambda \) is uniquely determined by the element \( \beta \) and has period 2. Let \( \Lambda_i \) be the \( \sigma_F \)-lattice sequence \( \Lambda \cap V_i \), for \( 1 \leq i \leq 3 \). Without loss of generality we assume that
\[
-n = \nu_{\Lambda_1}(\beta_1) \leq \nu_{\Lambda_2}(\beta_2) \leq \nu_{\Lambda_3}(\beta_3) \leq 0.
\]

(7.1)
The period of $\mathfrak{o}_F$-lattice sequence $\Lambda_i = \Lambda \cap V_i$ is 2, for $1 \leq i \leq 3$, and we have:

\[
\begin{align*}
\Lambda_i(-1) &= \Lambda_i(0) = \mathfrak{o}_F v_i \text{ if } \nu_{F_0}(\lambda_i) = 0, \\
\Lambda_i(0) &= \Lambda_i(1) = \mathfrak{o}_F v_i \text{ if } \nu_{F_0}(\lambda_i) = 1.
\end{align*}
\]

As the lattice sequence $\Lambda$ depends on various possibilities on $V_i$, we will describe these lattice sequences and a Witt-basis giving a splitting $\Lambda$, as required in each individual case. However, the group $J^0(\Lambda, \beta)$ and $H^1(\Lambda, \beta)$ are given by

\[
\begin{align*}
P_0(\Lambda, \beta)P_{(q_2/2)}(\Lambda, \beta_1)P_{(n/2)}(\Lambda)
\end{align*}
\]

and

\[
\begin{align*}
P_1(\Lambda, \beta)P_{(q_2/2)}(\Lambda, \beta_1)P_{(n/2)}(\Lambda)
\end{align*}
\]

respectively.

7.2. **Criterion for non-emptiness of the set $X_\beta$.** If $F/F_0$ is unramified, the non-emptiness of the set $X_\beta$ depends only on the integers $\{\nu_F(\beta_i), \nu_{F_0}(\lambda_i) : 1 \leq i \leq 3\}$. This will be made precise in the following lemmas. However, if $F/F_0$ is ramified, then one requires more information on $\{\beta_1, \beta_2, \beta_3\}$ to determine when $X_\beta$ is non-empty. In the case where $F/F_0$ is ramified we will not make these conditions explicit, but we will show that a cuspidal representation in $\Pi_\kappa$ is generic if and only if $X_\beta$ is non-empty.

**Lemma 7.2.1.** Let $F/F_0$ be an unramified extension and let $\mathfrak{r}$ be a stratum of the type (D) such that $(W_i, h)$ is isotropic, for all $1 \leq i \leq 3$. The set $X_\beta$ is non-empty if and only if $\nu_F(\beta_1) - \nu_F(\beta_2)$ is even.

**Proof.** In this case we may assume that $\lambda_i = 1$, for $1 \leq i \leq 3$. There exists a maximal unipotent subgroup of $G$ in good position with respect to the stratum $\mathfrak{r} = [\Lambda, n, 0, \beta]$ if and only if the equations

\[
\begin{align*}
a \sigma(a) + b \sigma(b) + c \sigma(c) &= 0 \quad \text{and} \quad \beta_1 a \sigma(a) + \beta_2 b \sigma(b) + \beta_3 c \sigma(c) = 0
\end{align*}
\]

have a non-trivial common solution. From the ordering (7.1) we have:

\[

\nu_F(\beta_1) \leq \nu_F(\beta_2) \leq \nu_F(\beta_3) \leq 0.
\]

Since, $\mathfrak{r}$ is skew semisimple stratum of type (D), we get that $\beta_1 \neq 0$, $(1 - \beta_2\beta_1^{-1}) \neq 0$, and $(1 - \beta_3\beta_2^{-1}) \neq 0$. The set of equations (7.2) imply that

\[

a \sigma(a) = -c \sigma(c)\beta_2\beta_1^{-1}(1 - \beta_3\beta_2^{-1})(1 - \beta_2\beta_1^{-1})^{-1}.
\]

Hence, a non-zero common solution for the set of equations (7.2) exists if and only if $\nu_F(\beta_2) - \nu_F(\beta_1) = 2m$ for some $m \in \mathbb{Z}$. \hfill \square

**Lemma 7.2.2.** Let $F/F_0$ be an unramified extension and let $\mathfrak{r}$ be a stratum of the type (D) such that $(W_i, h)$ is anisotropic, for some $1 \leq i \leq 3$. The set $X_\beta$ is non-empty if and only if $\nu_F(\lambda_2) = \nu_F(\lambda_3) = 1$ and $\nu_F(\beta_1) - \nu_F(\beta_2)$ is odd.

**Proof.** Recall that the determinant of $(V, h)$ is the trivial class in $F_0^*/N_{F/F_0}(F^*)$. With the hypothesis on the spaces $W_i$, there exists an unique $i \in \{1, 2, 3\}$ such that $\nu_F(\lambda_i) = 0$. The set $X_\beta$ is non-empty if and only if the following equations

\[
\begin{align*}
\lambda_1 a \sigma(a) + \lambda_2 b \sigma(b) + \lambda_3 c \sigma(c) = 0 \quad \text{and} \quad \beta_1 \lambda_1 a \sigma(a) + \beta_2 \lambda_2 b \sigma(b) + \beta_3 \lambda_3 c \sigma(c) = 0
\end{align*}
\]

have a non-trivial simultaneous solution. Since $\mathfrak{r}$ is a skew semisimple stratum of type (D) and from our convention on the valuations of $\beta_i$, we get that $\beta_1 \neq 0$. Assume that there exists a non-trivial simultaneous solution. Hence, we get that

\[
(1 - \beta_2\beta_1^{-1})\lambda_2 b \sigma(b) + (1 - \beta_3\beta_1^{-1})\lambda_3 c \sigma(c) = 0.
\]

Since, $\mathfrak{r}$ is a skew semisimple strata, we get $1 - \beta_2\beta_1^{-1} \not\in \mathfrak{p}_F$ and $1 - \beta_3\beta_1^{-1} \not\in \mathfrak{p}_F$. Hence, $1 - \beta_2\beta_1^{-1}$ and $1 - \beta_3\beta_1^{-1}$ are units in $\mathfrak{o}_F$. This implies that $\nu_F(\lambda_2) = \nu_F(\lambda_3)$. From the assumption on the spaces $W_i$, for $1 \leq i \leq 3$, we get that $\nu_F(\lambda_2) = \nu_F(\lambda_3) = 1$ and $\nu_F(\lambda_1) = 0$. From the equation (7.3) we have

\[
\lambda_1 a \sigma(a) = \lambda_3 c \sigma(c)\beta_2\beta_1^{-1}(1 - \beta_3\beta_2^{-1})(1 - \beta_2\beta_1^{-1}).
\]

Hence, $X_\beta$ is non-empty if and only if $(W_1, h)$ is isotropic and $\nu_F(\beta_2) - \nu_F(\beta_1) = 2m - 1$ for some $m \in \mathbb{Z}$. \hfill \square
7.3. Estimating the valuation of $h(gv, \beta gv)$. As observed in the previous sections, our approach to show non-genericity is by showing an appropriate inequality on the function sending $g \in P(\Lambda)$ to $\nu_F(h(gv, gv))$, where $v$ is a well chosen isotropic vector with respect to $P(\Lambda)$. Hence, we need some technical lemmas to understand the growth of this function.

**Lemma 7.3.1.** Let $F/F_0$ be an unramified extension and let $\mathfrak{t}$ be a skew semisimple stratum of the type $(D)$ such that $\mathfrak{x}_\beta$ is the empty set. Let $v$ be any isotropic vector in $(V, h)$ and let $g \in G$. Assume that $gv = av_1 + bv_2 + cv_3$ for some $a, b, c \in F$. We have

$$\nu_F(h(gv, \beta gv)) = \min\{\nu_F(\beta_1 a\sigma(a)), \nu_F(\beta_2 b\sigma(b) + \beta_3 c\sigma(c))\}.$$

**Proof.** Before we begin the proof, it is useful to recall that the determinant of $(V, h)$ is the trivial class in $F^*_0/\text{Nr}_{F/F_0}(F^*)$. Since $v$ is an isotropic vector we get that

$$\lambda_1 a\sigma(a) + \lambda_2 b\sigma(b) + \lambda_3 c\sigma(c) = 0.$$

Note that rescaling the constants $a, b, c$ does not affect the lemma. Hence, rescaling $a, b$ and $c$, if necessary, we assume that $a, b, c \in \mathfrak{o}_F$ and the $\mathfrak{o}_F$-ideal $(a, b, c)$ is equal to $\mathfrak{o}_F$.

We first consider the case where $(W_i, h)$ is isotropic, for $1 \leq i \leq 3$. In this case we have $\nu_{F_0}(\lambda_i) = 0$, for $1 \leq i \leq 3$ and $\nu_F(\beta_1) - \nu_F(\beta_2)$ is an odd integer. Note that $\nu_F(a\sigma)$ is always an even integer. If $\nu_F(a) = 0$, then we note that

$$\nu_F(\beta_1 a\sigma(a) + \beta_2 b\sigma(b) + \beta_3 c\sigma(c)) = \nu_F(\beta_1(1 + 1) + \beta_2 b\sigma(b) + \beta_3 c\sigma(c))$$

and we prove the lemma in this case. Consider the case where $a \in \mathfrak{p}_F$ and in this case $b, c \in \mathfrak{o}_F^\times$. Now we have

$$\nu_F(\beta_1(1 + 1) + \beta_2 b\sigma(b) + \beta_3 c\sigma(c)) = \nu_F(\beta_2 b\sigma(b) + \lambda_3(\beta_2^{-1} c\sigma(c))).$$

and if $\nu_F(\beta_2^{-1} c\sigma(c))$ is non-zero, then $\nu_F(\beta_2 b\sigma(b) + \lambda_3(\beta_2^{-1} c\sigma(c)))$ is zero. Even if $\nu_F(\beta_2^{-1} c\sigma(c)) = 0$, we still have $\nu_F(\beta_2 b\sigma(b) + \lambda_3(\beta_2^{-1} c\sigma(c)) = 0$, because $\mathfrak{t}$ is a semisimple stratum which implies $1 - \beta_2^{-1} \notin \mathfrak{p}_F$. Hence in all cases we conclude that

$$\nu_F(\beta_1 a\sigma(a)) \neq \nu_F(\beta_2 b\sigma(b) + \beta_3 c\sigma(c))$$

which implies the lemma in this case.

Assume that $(W_i, h)$ is anisotropic for some $1 \leq i \leq 3$. Using Lemma 7.2.2 we get $\mathfrak{x}_\beta$ is empty in either of the following cases: case $(I)$ where $\nu_F(\lambda_2) \neq \nu_F(\lambda_3)$, in case $(II)$ where $\nu_F(\lambda_2) = \nu_F(\lambda_3)$ and $\nu_F(\beta_1) - \nu_F(\beta_2)$ is an even integer. We first assume that $\nu_{F_0}(\lambda_2) = 0$ and this implies that $\nu_{F_0}(\lambda_1) = \nu_{F_0}(\lambda_3) = 1$. We may as well assume that $\lambda_2 = 1$ and $\lambda_1 = \lambda_3 = \infty$. We observe that $b \in \mathfrak{p}_F$ and set $b = \infty b'$. Since we have

$$\omega a\sigma(a) + \omega^2 b\sigma(b) + \omega c\sigma(c) = 0,$$

we get that $a, c \in \mathfrak{o}_F^\times$. We now have

$$\nu_F(h(ge_1, ge_1)) = \nu_F(\beta_1) + \nu_F((1 - \beta_2^{-1})\omega^2 b\sigma(b') + (1 - \beta_3^{-1})\omega c\sigma(c))$$

and note that

$$\nu_F((1 - \beta_2^{-1})\omega^2 b\sigma(b') + (1 - \beta_3^{-1})\omega c\sigma(c)) = 1$$

from which we get that $\nu_F(h(ge_1, ge_1)) = \nu_F(\beta_1) + 1$. Note that $\nu_F(\beta_2 \omega^2 b\sigma(b') + \beta_3 c\sigma(c))$ is equal to $\nu_F(\beta_2) + 1$ and hence lemma follows, in case $(I)$, from the observation that

$$\nu_F(\beta_1 a\sigma(a)) = \nu_F(\beta_2 \omega a\sigma(a)) = \nu_F(\beta_1) + 1 \leq \nu_F(\beta_2) + 1.$$ 

The case where $\nu_F(\lambda_2) = 1$–in which case $\nu_F(\lambda_1) = \nu_F(\lambda_3) = 1$ and $\nu_F(\lambda_3) = 0$–is entirely similar.

Assume that we are in case $(II)$. In this case we may assume that $\lambda_2 = \lambda_3 = \infty$ and $\lambda_1 = 1$. Then the equation

$$\omega a\sigma(a) + \omega^2 b\sigma(b) + \omega c\sigma(c) = 0$$

implies that $b, c \in \mathfrak{o}_F^\times$ and $a \in \mathfrak{p}_F$. We now have

$$h(gv, \beta gv) = \beta_1(1 + 1) + \beta_2 b\sigma(b) + \beta_3 c\sigma(c)).$$
Since $\mathfrak{r}$ is a skew semisimple strata and $b, c \in \mathfrak{o}_F^\circ$, we get that $\nu_F((b_1a_1) + \beta_3\beta_2^{-1}c_3(c)) = 0$. The lemma now follows from the observation that the integer $\nu_F(\lambda_1 a_1)$ is even and the integer $\nu_F(\beta_2 b_1 + \beta_3\beta_2^{-1}c_3(c))$ is always odd.

**Lemma 7.3.2.** Let $F/F_0$ be a ramified extension and let $\mathfrak{r}$ be a stratum of the type (D) such that $\mathfrak{X}_\beta$ is the empty set. Let $v$ be any isotropic vector in $(V, h)$ and let $g \in G$. Assume that $gv = av_1 + bv_2 + cv_3$ for some $a, b, c \in F$. Then we have

$$\nu(h(gv, \beta gv)) = \min\{\nu_F(\beta_1 a \sigma(a)), \nu_F(\beta_2 b_2 \sigma(b) + \beta_3 c_3 \sigma(c))\}.$$ 

**Proof.** Since $v$ is an isotropic vector we get that

$$\lambda_1 a \sigma(a) + \lambda_2 b \sigma(b) + \lambda_3 c \sigma(c) = 0.$$  \hspace{1cm} (7.4)

Rescaling the constants $a, b, c$, if necessary, we assume that $a, b, c \in \mathfrak{o}_F$ and the $\mathfrak{o}_F$-ideal $(a, b, c)$ is equal to $\mathfrak{o}_F$. The definition of skew semisimple strata necessarily implies that $(1 - \beta_1 \beta_2) \in \mathfrak{o}_F^\circ$, for $1 \leq j < i \leq 3$. The set $\mathfrak{X}_\beta$ is empty in one of the two cases

(1) The case where

$$-(1 - \beta_2 \beta_1^{-1})(1 - \beta_3 \beta_2^{-1}) \lambda_2 \lambda_3^{-1} \notin \text{Nr}_F(F^\times)$$

(2) The case where

$$-(1 - \beta_2 \beta_1^{-1})(1 - \beta_3 \beta_2^{-1}) \lambda_2 \lambda_3^{-1} \in \text{Nr}_F(F^\times)$$

and

$$\lambda_3/\lambda_2 \beta_2 \beta_1^{-1}(1 - \beta_3 \beta_2^{-1})(1 - \beta_2 \beta_1^{-1}) \in \text{Nr}_F(F^\times).$$  \hspace{1cm} (7.5)

In Case (1) we note that

$$(1 - \beta_2 \beta_1^{-1}) \lambda_2 b \sigma(b) + (1 - \beta_3 \beta_2^{-1}) \lambda_3 c \sigma(c) \notin \mathfrak{p}_F.$$  \hspace{1cm} (7.6)

unless $b, c \in \mathfrak{p}_F$. Hence, we have

$$\nu_F(h(gv, \beta gv)) = \min\{\nu_F(\beta_1 a \sigma(a)), \nu_F(\beta_2 b_2 \sigma(b) + \beta_3 c_3 \sigma(c))\}.$$  \hspace{2cm} (7.7)

Now consider Case (2) and assume that

$$(1 - \beta_2 \beta_1^{-1}) \lambda_2 b \sigma(b) + (1 - \beta_3 \beta_2^{-1}) \lambda_3 c \sigma(c) \in \mathfrak{p}_F;$$

if the condition $\mathfrak{p}_F$ is false we get the lemma immediately. If $\nu_F(\beta_1) = \nu_F(\beta_2)$, then we have

$$-(1 - \beta_2 \beta_1^{-1}) \beta_2^{-1} \beta_1 a \sigma(a) + (1 - \beta_3 \beta_2^{-1}) \lambda_3 c \sigma(c) \in \mathfrak{p}_F.$$  \hspace{2cm} (7.8)

But, the above equality is a contradiction to the equation (7.4), the second condition in case (2). Hence we get that $\nu_F(\beta_1) \neq \nu_F(\beta_2)$. If $a \in \mathfrak{o}_F^\circ$, the valuation of $h(gv, \beta gv)$ is equal to

$$\nu_F(\beta_1) + \nu_F(\beta_1 a \sigma(a) + \beta_2 \beta_1^{-1} \lambda_2 b \sigma(b) + \beta_3 \beta_2^{-1} \lambda_3 c \sigma(c)) = \nu_F(\beta_1).$$

Note that the lemma follows from the observation that

$$\nu_F(\beta_1 a \sigma(a)) \leq \nu_F(\beta_2 b \sigma(b) + \beta_3 c \sigma(c)).$$

We consider the case where $a \in \mathfrak{p}_F$ and this implies that $b, c \in \mathfrak{o}_F^\circ$. Thus we have $\lambda_2 b \sigma(b) + \beta_3 \beta_2^{-1} \lambda_3 c \sigma(c) \in \mathfrak{p}_F$. Suppose

$$\lambda_1 a \sigma(a) + \beta_2 \beta_1^{-1} \lambda_2 b \sigma(b) + \beta_3 \beta_2^{-1} \lambda_3 c \sigma(c) \in \mathfrak{p}_F^{\nu_F(a) + 1},$$

then we have

$$\beta_1 \beta_2 \beta_1^{-1} \lambda_1 a \sigma(a) + \lambda_2 b \sigma(b) + \beta_3 \beta_2^{-1} \lambda_3 c \sigma(c) \in \mathfrak{p}_F.$$  \hspace{2cm} (7.9)

Using the equality (7.3), we get that

$$(1 - \beta_1 \beta_2^{-1}) \lambda_1 a \sigma(a) + (1 - \beta_3 \beta_2^{-1}) \lambda_3 c \sigma(c) \in \mathfrak{p}_F.$$  \hspace{2cm} (7.10)

This is a contradiction to the condition (7.5). Hence, we obtain

$$\nu(h(gv, \beta gv)) = \min\{\nu_F(\beta_1 a \sigma(a)), \nu_F(\beta_2 b \sigma(b) + \beta_3 c \sigma(c))\}.$$  \hspace{2cm} (7.11)
7.4. Generic cuspidal representations of type (D).

Lemma 7.4.1. Let $F/F_0$ be any quadratic extension and let $\xi$ be a stratum of the type (D) such that $\mathbb{X}_\beta$ is non-empty. Any representation contained in the set $\Pi_\xi$ is generic.

Proof. Let $\theta$ be any semisimple character in $C-(\Lambda,0,\beta)$ and $U$ be the unipotent radical of a Borel subgroup in $\mathbb{X}_\beta$. We will first show that

$$\text{res}_{H^1(\Lambda,\beta)\cap U} \theta = \psi_\beta.$$

If $q_1 = q_2$, then the group $H^1(\Lambda,\beta)$ is equal to $P_{1}(\Lambda_\beta)P_{q_1/2}(\Lambda)$ and from Lemma 3.5.1, we get that $H^1(\Lambda,\beta) \cap U$ is equal to $P_{q_1/2}(\Lambda) \cap U$. In the case where $q_1 > q_2$, the group $H^1(\Lambda,\beta)$ is equal to $P_{1}(\Lambda_\beta)P_{q_1/2} \cap (\Lambda_{\beta}) P_{q_1/2}(\Lambda)$.

Let $H'$ be the group $P_{1}(\Lambda_\beta)P_{q_1/2}(\Lambda)$. From Lemma 3.5.1, we get that the group $H' \cap U$ is equal to $P_{q_1/2}(\Lambda) \cap U$ and hence $H^1(\Lambda,\beta) \cap U$ is equal to $P_{(q_1/2) +}(\Lambda_{\beta}) P_{q_1/2}(\Lambda) \cap U$.

Let $v = av_1 + bv_2 + cv_3$ be the isotropic vector fixed by $U$. Since $\xi$ is a skew semisimple stratum, we note that $a \neq 0$. Assume that $g_1g_2 \in U$, for some $g_1 \in P_{(q_1/2) +}(\Lambda_{\beta})$ and $g_2 \in P_{(q_1/2) +}(\Lambda)$. From the equality

$$g_1(a v_1 + b v_2 + c v_3) = a' v_1 + g_1(b v_2 + c v_3) = g_2(a v_1 + b v_2 + c v_3)$$

we get that

$$1_{v_1} g_1 1_{v_1} = a' \in F^\infty \cap P_{q_1/2}(\Lambda).$$

Which implies that the determinant of $1_{v_1} g_1 1_{v_1}$ belongs to $F^\infty \cap P_{(q_1/2) +}(\Lambda)$. Hence, the simple character $\theta(g_1g_2)$ is equal to $\psi_\beta(g_1g_2)$. Now, the lemma follows from Proposition 3.5.2.

7.5. Non-generic cuspidal representations of type (D). We will show that $\pi \in \Pi_\xi$ is non-generic if and only if $\mathbb{X}_\beta$ is empty. We will devide the proof into several cases beginning with the easier case where $(W_1,h)$ is anisotropic; in which case we will show that $\psi_\beta$ is non-trivial on $P_{(n/2) +}(\Lambda) \cap U_{\text{der}}$, for all $g \in P(\Lambda)$. In the case where $(W_1,h)$ is isotropic, the method of proof is more involved and we had to deal with conjugation of some shallow elements in the group $P(\Lambda)$.

7.5.1. The case where $(W_1,h)$ is anisotropic.

Lemma 7.5.1. Let $F/F_0$ be a quadratic extension and let $\xi$ be a skew semisimple stratum of type (D) such that $(W_1,h)$ is anisotropic. Then any representation in the set $\Pi_\xi$ is non-generic.

Proof. Let us begin with the case where $F/F_0$ is ramified. In this case the period 2 lattice sequence $\Lambda$ is given by

$$\Lambda(-1) = \Lambda(0) = o_{F}v_1 \oplus \varphi_{F}v_2 \oplus \varphi_{F}v_3.$$ 

It is possible that none of the spaces $(W_2,h)$ and $(W_3,h)$ is isotropic. However, there exists a $\varphi_F$-basis $(\tilde{v}_2, \tilde{v}_3)$ for the space $(W_1,h)$ such that $\langle v_1, \tilde{v}_2 \rangle$ is isotropic. This implies that we can choose a Witt-basis $(e_1, e_{-1}) \in \{v_1, \tilde{v}_2\}$ and $e_0 \in (v_1, \tilde{v}_2)^{\perp}$ such that $h(e_0, e_0) = 1$ and the Witt-basis $(e_1, e_0, e_{-1})$ provides a splitting for the lattice sequence $\Lambda$.

Let $U$ be the unipotent radical of the Borel subgroup fixing the line $\langle e_1 \rangle$. Let $n = -\nu_\Lambda(\beta) = 4m + 2$, for some integer $m$. Assume that $\pi \in \Pi_\xi$ is a generic representation. Let $(J^\rho(\Lambda, \beta, \kappa))$ be a Bushnell–Kutzko type contained in $\pi$. There exists a $g \in P(\Lambda)$ such that

$$\text{Hom}_{J^\rho(\Lambda, \beta, \kappa) \cap U}(\kappa^g, \Psi) \neq 0,$$

for some non-trivial character $\Psi$ of $U$. Since, $g$ normalises $P_r(\Lambda)$, for $r > 0$, we get that the group $P_{(n/2) +}(\Lambda) \cap U_{\text{der}}$ is equal to $U_{\text{der}}(\lfloor m/2 \rfloor)$. Let $g e_1 = av_1 + bv_2 + cv_3$ for some $a, b, c \in \varphi_F$. We then have

$$\lambda_1a\sigma(a) + \lambda_3b\sigma(b) + \lambda_3c\sigma(c) = 0.$$

If $a \in \varphi_F$, then $b, c \in \varphi_F$ as $(W_1,h)$ is anisotropic. Hence, we get that $a \in \varphi_F^*$. This shows that $\nu_F(\delta h(ge_1, \beta ge_1))$ is equal to $\nu_F(\beta_1) + 1/2 = m + 1$. This implies that

$$\nu_F(h(ge_1, \beta ge_1)) \leq -\lfloor m/2 \rfloor.$$
Now, Lemma 2.6.1 implies that the character $\psi_\beta$ is non-trivial on the group $P_{(n/2)}(\Lambda) \cap U_{\text{der}}$ and hence we get a contradiction to the assumption that $\pi$ is generic.

Let us consider the case where $F/F_0$ is unramified. In this case we may assume that $\lambda_1 = \varpi$ and $(\lambda_2, \lambda_3) \in \{ (\varpi, 1), (1, \varpi) \}$. So we define $\tilde{v}_3$ to be the vector in the set $\{ v_2, v_3 \}$ with $h(\tilde{v}_3, \tilde{v}_3) = \varpi$ and the remaining vector is denoted by $\tilde{v}_2$. The notation $\beta_i$ will be used for $1_i, \beta_1 \tilde{v}_i$, for $i \in \{ 2, 3 \}$. The period 2 lattice sequence $\Lambda$ is given by

$$
\Lambda(0) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1} \text{ and } \Lambda(1) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}.
$$

Let $e_0$ be the vector $\tilde{v}_2$. The space $\langle v_1, \tilde{v}_3 \rangle$ is isotropic, and there exists a Witt-basis $(e_1, e_{-1})$ for the space $\langle v_1, \tilde{v}_3 \rangle$ such that

$$
\Lambda(0) = o_F e_1 \oplus o_F e_0 \oplus 0_F e_{-1} \text{ and } \Lambda(1) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}.
$$

Let $U$ be the unipotent radical of the Borel subgroup fixing the line $\langle e_1 \rangle$. Assume that $\pi \in \Pi_r$ containing a Bushnell–Kutzko type $(J^0(\Lambda, \beta), \kappa)$, is a generic representation. There exists a $g \in P(\Lambda)$ and a non-trivial character $\Psi$ of $U$ such that

$$
\text{Hom}_{J^0(\Lambda, \beta) \cap U}(\kappa^g, \Psi) \neq 0.
$$

Note that $\nu_\Lambda(e_1) = 1$ and hence we get that $ge_1 = a v_1 + \varpi b \tilde{v}_1 + c \tilde{v}_3$, for some $a, b, c \in o_F$. Since $e_1$ is an isotropic vector, we get that

$$
a \sigma(a) + \varpi b \sigma(b) + c \sigma(c) = 0.
$$

From the above equality and the fact that $\nu_\Lambda(e_1) = 1$, we get that $a, c \in o_F^\times$. We set $n = 4m + 2r$ for some integer $m$ and $r \in \{0, 1\}$. The valuation of $h(ge_1, \beta_2 \beta_1)$ is equal to

$$
\nu_F(\beta_1) + \nu_F(a \sigma(a) + \beta_2 \beta_1^{-1} \bar{b} \sigma(b) + \beta_3 \beta_1^{-1} c \sigma(c)) = \nu_F(\beta_1) = -(2m + r).
$$

We observe that $g$ normalises the group $P_{(n/2)}(\Lambda)$ and hence we have $P_{(n/2)}(\Lambda)^g \cap U_{\text{der}}$ is equal to $U_{\text{der}}(m)$. Since $\nu_F(h(ge_1, \beta_2 \beta_1)) \leq -m$, we get that the character $\psi_\beta^g$ is non-trivial on the group $P_{(n/2)}(\Lambda)^g \cap U_{\text{der}}$. Thus we obtain a contradiction to the assumption on the genericity of the representation $\pi \in \Pi_r$. \hfill $\square$

7.5.2. The case where $(W_1, h)$ is isotropic. We begin with recalling the description of the lattice sequence $\Lambda$. Let $(e_1, e_{-1})$ be a Witt-basis for the space $(W_1, h)$ and $e_0$ be a vector in $V_1$ such that $h(e_0, e_0) = 1$. Let $B$ be the Borel subgroup of $G$ fixing $\langle e_1 \rangle$ and $U$ be the unipotent subgroup of $B$. If $F/F_0$ is unramified and $h(v_2, v_2) = h(v_3, v_3) = \varpi$, then the lattice sequence $\Lambda$ has period 2 and

$$
\Lambda(0) = o_F e_1 \oplus o_F e_0 \oplus 0_F e_{-1} \text{ and } \Lambda(1) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}.
$$

If $F/F_0$ is unramified and $h(v_2, v_2) = h(v_3, v_3) = 1$, then the lattice sequence $\Lambda$ has period 2 and

$$
\Lambda(-1) = \Lambda(0) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}.
$$

If $F/F_0$ is ramified, then the lattice sequence $\Lambda$ has period 2 and

$$
\Lambda(-1) = \Lambda(0) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}.
$$

Let $I$ be the $G$ stabiliser of the following period 3 lattice sequence given by:

$$
\Lambda_{12}(0) = o_F e_1 \oplus 0_F e_0 \oplus 0_F e_{-1}, \Lambda_{12}(1) = o_F e_1 \oplus 0_F e_0 \oplus 0_F e_{-1} \text{ and } \Lambda_{12}(2) = o_F e_1 \oplus 0_F e_0 \oplus 0_F e_{-1}.
$$

The parahoric subgroup attached to the lattice sequence $\Lambda_{12}$ is an Iwahori subgroup of $G$.

Recall the notation $q_i$ for $\nu_\Lambda(\beta_i)$, for $1 \leq i \leq 3$. The results in this subsection pertain to the case where $X_3$ is empty, and in the present case the $(W_1, h)$ is isotropic hence, we work with the assumption that $q_2 < q_1 = -n$. We set $q_1 = 4m_1 + 2r_1$ and $q_2 = 4m_2 + 2r_2$, for some integers $m_1, m_2$ and $r_1, r_2 \in \{0, 1\}$. 

\end{document}
Lemma 7.5.4. Hence, the determinant of non-empty. Any representation contained in the set $\Pi$ element in the group $\Gamma$.

Proof. We prove this in the case where $\Lambda = \{0\}$ and the same notations as in Lemma 7.5.2, and for any skew semisimple character $\theta$ such that $\theta(0) = 0$. Let $u^- = \hat{u}(x,y)$ be an element of $\mathcal{I}_\Lambda \cap \mathcal{U}$, where $\mathcal{U}$ is the unipotent radical of the opposite parabolic subgroup of $B^w$ with respect to $T$. We have

$$U_{\text{der}}(d(\hat{x},w,x)) \subseteq H^1(\Lambda, \beta)_{u^-} \cap U_{\text{der}}.$$  

Proof. We will prove the above lemma assuming that $w = \text{id}$ and the proof is entirely similar in the case where $w \neq \text{id}$. The group $H^1(\Lambda, \beta)$ is equal to

$$P_1(\Lambda F[\beta])P_{(q_2/2)+}(\Lambda F[\beta])P_{(n/2)+}(\Lambda).$$

Let $u^- = \hat{u}(x,y)$ be an element of $\mathcal{I}_\Lambda \cap \mathcal{U}$. Using the identity $\hat{u}(x,y) = (1 - a(x\sigma(x) + y))\hat{u}(-x, -y - x\sigma(x)) = \begin{pmatrix} 1 & a(x\sigma(x) + y) & a \sigma(x) & a \\ ax(-y - x\sigma(x)) & 1 + ax\sigma(x) & ax \\ -ay(y + x\sigma(x)) & a\sigma(x)y & ay + 1 \end{pmatrix}$. (7.10)

and from the definition of $d(\hat{x},w,x)$, we see that the group $\{U_{\text{der}}(d(\hat{x},w,x))\}_{u^-}$ is contained in the group $H^1(\Lambda, \beta)$. □

Lemma 7.5.3. With the same notations as in Lemma 7.5.2 and for any skew semisimple character $\theta$ in $C_-(\Lambda, 0, \beta)$ we have

$$\text{res}_{\Lambda \cap \mathcal{U}}(\theta_{\Lambda \cap \mathcal{U}}) \theta_{\Lambda \cap \mathcal{U}} = \theta_{\Lambda \cap \mathcal{U}}.$$

Proof. We prove this in the case where $\Lambda \cap \mathcal{U}$ is an element of $\mathcal{U}$ and let $u^- = \hat{u}(0,a)$ be an element in the group $U_{\text{der}}(d(\hat{x},w,x))$. Using the identity (7.10) we get that the element $u^+u^-u^-(u^+)^{-1}$ is of the form $g_1g_2$, where $g_1 \in P_{(q_2/2)+}(\Lambda F[\beta])$ and $g_2 \in P_{(n/2)+}(\Lambda)$. Now from the definition of the integer $d(\hat{x},w,x)$, we get that $1 + ax\sigma(x) \in F^+ \cap P_{(n/2)+}(\Lambda)$. This implies that $1_1g_1$ belongs to $F^+ \cap P_{(n/2)+}(\Lambda)$. Hence, the determinant of $1_2g_11_2$ belongs to $F^+ \cap P_{(n/2)+}(\Lambda)$. This shows the lemma. □

Lemma 7.5.4. Let $F/F_0$ be a any quadratic extension and $\Lambda$ be a strata of the type (D) such that $X_\beta$ is non-empty. Any representation contained in the set $\Pi \subset \Lambda$ is non-generic.

Proof. Let $\pi \in \Pi$ be a generic representation. Now there exists a $w \in \mathcal{U}$, an element $u^- = \hat{u}(x,y) \in \mathcal{I}_\Lambda \cap \mathcal{U}$, and a non-trivial character $\Psi$ of $U$ such that

$$\text{Hom}_{H^1(\Lambda, \beta)_{u^-} \cap \mathcal{U}}(\theta_{u^-}, \text{id}) \neq 0.$$  

(7.11)

where $\theta$ is the skew semisimple character contained in $\text{res}_{H^1(\Lambda, \beta)}$. Consider the case where $F/F_0$ is any quadratic extension and $\Lambda \cap \mathcal{U}$ is equal to $\mathcal{U}$ in this case we have $\nu_F(e_w) = 0$ and hence $u^-e_w = b_2 + ax_1 + cv_3$, for some $b, c \in \mathcal{U}$. If $\nu_F(x) = 0$, then we have $\nu_F(h(u^-e_w, \beta u^-e_w))$ is equal to $\nu_F(\beta_1)$. Now, if $\nu_F(x) > 0$, then we have $b, c \in \mathcal{U}$. Now, the element
(1−β_2β_3^{-1}) is an unit in \( \mathfrak{o}_F \) and from the assumption that \( a \in \mathfrak{p}_F \) we have \( \lambda_2 \sigma(b) + \lambda_3 \sigma(c) \in \mathfrak{p}_F \). Hence, we get that \( \nu_F(\lambda_2 \sigma(b) + \beta_3 \beta_3^{-1} \lambda_3 \sigma(c)) \) is an unit in \( \mathfrak{o}_F^\times \). Using Lemmas 7.3.1 and 7.3.2 we get that

\[
\nu_F(h(u^{-1}e_w, \beta u^{-1}e_w)) = \min\{\nu_F(\beta_1) + 2\nu_F(x), \nu_F(\beta_2) + \nu_F(\lambda_2 \sigma(b) + \beta_3 \beta_3^{-1} \lambda_3 \sigma(c))\} = \min\{\nu_F(\beta_1) + 2\nu_F(x), \nu_F(\beta_2)\}.
\]

Consider the case where \( F/F_0 \) is unramified and \( \Lambda \cap W_1 \) is equal to \( \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1} \). In this case, we have \( \nu_\Lambda(e_{\pm 1}) = \pm 1 \) and we have \( u^{-1}e_w = xv_1 + bv_2 + cv_3 \), where \( b, c \in \mathfrak{o}_F \), if \( w = \text{id} \) and \( b, c \in \mathfrak{p}_F^1 \), if \( w \neq \text{id} \). If \( w = \text{id} \), we observe that \( b \sigma(b) + \sigma(c) \in \mathfrak{p}_F \); which together with \( \nu_F(e_1) = 1 \) implies that \( b, c \in \mathfrak{o}_F^\times \). Since, \( \mathfrak{r} \) is skew semisimple we have get that \( b \sigma(b) + \beta_3 \beta_3^{-1} \sigma(c) \) is an unit in \( \mathfrak{o}_F \). From Lemma 7.3.1 we get that

\[
\nu_F(h(u^{-1}e_1, u^{-1}e_1)) = \min\{\nu_F(\beta_1) + 2\nu_F(x), \nu_F(\beta_2) + 1\}.
\]

If \( w \neq \text{id} \), then similar arguments as above imply that \( \nu_F(b) = \nu_F(c) = -1 \), and \( \nu_F(b \sigma(b) + \beta_3 \beta_3^{-1} \sigma(c)) = -2 \). Hence, using lemma we get that

\[
\nu_F(h(u^{-1}e_1, u^{-1}e_1)) = \min\{\nu_F(\beta_1) + 2\nu_F(x), \nu_F(\beta_2) - 1\}.
\]

We claim that \( \nu_\Lambda(\mathfrak{r} h(u^{-1}e_w, u e_w)) \) is less than or equal to \( -d(\mathfrak{r}, w, x) \). Using the Lemma 2.6.1 implies that the character \( \psi_{\mathfrak{r}}^{u^{-1}} \) is non-trivial on the group \( U_{\text{der}}(d(\mathfrak{r}, w, x)) \) and this is a contradiction in the equation (7.11), thus prove the lemma.

Let us begin the verification of our claim in the case where \( F/F_0 \) is unramified and \( \Lambda(0) \cap W_1 \) is equal to \( \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1} \). The integer \( \nu_F(h(u^{-1}e_1, u^{-1}e_1)) \) is equal to

\[
\min\{-(2m_2 + r_2), -(2m_1 + r_1) + 2\nu_F(x)\}.
\]

Recall that \( -d(\mathfrak{r}, w, x) \) is equal to \( \min\{m_2 - 1 - (r_1 + r_2)/2, m_1 - 1 + (r_1 + r_2)/2\} \). We should first note that \( -(2m_2 + r_2) \leq -m_2 - 1 \), unless \( m_2 + r_2 = 0 \). If this happens, we have \( \nu_F(\beta_2) = 0 \) and hence \( \nu_F(\beta_3) = 0 \); then we get that \( \mathfrak{r} \) is a strata of the type (C). Assume that \( -(2m_2 + r_2) \) is equal to \( \min\{-(2m_2 + r_2), -(2m_1 + r_1) + 2\nu_F(x)\} \). Then we have

\[
\nu_F(x) \geq m_1 - m_2 + (r_1 - r_2)/2 \geq -(2m_2 + r_2) + (m_2 - 1 + (r_1 + r_2)/2).
\]

Since, \( r_1 - r_2 \) is odd, we get that \( m_2 - 1 + (r_1 + r_2)/2 \geq -1/2 \). Hence, we get that

\[
\nu_F(x) - m_1 - 1 \geq -(2m_2 + r_2). \tag{7.12}
\]

Assume that \( -(2m_1 + r_1) + 2\nu_F(x) \) is equal to \( \min\{-(2m_2 + r_2), -(2m_1 + r_1) + 2\nu_F(x)\} \). We have

\[
\nu_F(x) \leq m_1 - m_2 + (r_1 - r_2)/2 \leq m_1 + r_1 - 1 - (m_2 + (r_1 + r_2)/2 - 1). \tag{7.13}
\]

Since, \( r_1 + r_2 \) is odd, we get that \( m_2 + (r_1 + r_2)/2 - 1 \geq -1/2 \). Hence, we the inequality \( \nu_F(x) \leq m_1 + r_1 - 1 \), which implies that \( -(2m_1 + r_1) + 2\nu_F(x) \leq -(m_1 + 1) + \nu_F(x) \). Finally, we observe the inequality

\[
-(2m_1 + r_1) + 2\nu_F(x) \leq -(2m_2 + r_2) \leq -(m_2 + 2).
\]

This shows our claim in the present case.

Let us consider the case where \( F/F_0 \) is unramified and we have \( \Lambda(0) \cap W_1 \) is equal to \( \mathfrak{o}_F e_1 \oplus \mathfrak{p}_F e_{-1} \). In this case we note that \( r_1 = r_2 \). Assume that \( w = \text{id} \), and we have \( -d(\mathfrak{r}, \text{id}, x) \) is equal to \( \min\{-(2m_2 + r_2) + 1, -(2m_1 + r_1) + 2\nu_F(x)\} \). Assume that \( -(2m_2 + r_2) + 1 \) is equal to \( \min\{-(2m_2 + r_2) + 1, -(2m_1 + r_1) + 2\nu_F(x)\} \), which implies that

\[
\nu_F(x) - m_1 - r_1 \geq -(m_2 - (r_1 + r_2)/2 + 1/2 \geq -(2m_2 + r_2) + 1 + (m_2 + (r_2 - r_1)/2 - 1/2). \tag{7.14}
\]

We note that \( m_2 + (r_2 - r_1)/2 - 1/2 \geq -1/2 \) and \( \nu_F(x) - m_1 - r_1 \) is greater than or equal to \( -(2m_2 + r_2) + 1 \). Now, assume that \( -(2m_1 + r_1) + 2\nu_F(x) \) is equal to \( \min\{-(2m_2 + r_2) + 1, -(2m_1 + r_1) + 2\nu_F(x)\} \), and this implies that

\[
\nu_F(x) \leq m_1 - m_2 + 1/2 \leq m_1 + 1/2. \tag{7.15}
\]

Since \( \nu_F(x) \) is an integer, we get that \( \nu_F(x) \leq m_1 \) and hence we get the inequality

\[
-(2m_1 + r_1) + 2\nu_F(x) \leq -(m_1 + r_1) + \nu_F(x).
\]
Which concludes that
\[ \min\{-2m_2 + r_2, -(2m_1 + r_1) + 2\nu_F(x)\} \leq -d(x, w, x). \]

Let us continue with the case considered in the previous paragraph but with \( w \neq \id \). We have \(-d(x, \id, x)\) is equal to \( \min\{-(2m_2 + 2), -(m_1 + r_1 + 1) + \nu_F(x)\} \). The value of \( \nu_F(h(u^{-e_{-1}}, u^{-e_{-1}})) \) is equal to \( \min\{-2m_2 + r_2 - 1, -(2m_1 + r_1 + 2\nu_F(x))\} \). Assume that \(-2m_2 + r_2 - 1\) is equal to \( \min\{-2m_2 + r_2 - 1, -(2m_1 + r_1) + 2\nu_F(x)\}\), and we get that \( \nu_F(x) \) is greater than or equal to \( m_1 - m_2 - 1/2 \). From this we get that \( \nu_F(x) \geq m_1 \), and hence
\[ -(m_1 + r_1 + 1) + \nu_F(x) \geq -(2m_2 + r_2) - 1. \]
Assume that \(-2m_1 + r_1 + 2\nu_F(x)\) is equal to \( \min\{-2m_2 + r_2 - 1, -(2m_1 + r_1) + 2\nu_F(x)\}\). Then we get that \( \nu_F(x) \) is less than or equal to \( m_1 - m_2 - 1/2 \), since \( \nu_F(x) \) is an integer we get that \( \nu_F(x) \) is less than or equal to \( m_1 - 1 \). We then get that
\[ -(2m_1 + r_1) + 2\nu_F(x) \leq -(m_1 + r_1 + 1) + \nu_F(x), \]
and we get the desired inequality that
\[ \min\{-2m_2 + r_2 - 1, -(2m_1 + r_1) + 2\nu_F(x)\} \leq \min\{-2m_2 + 2, -(m_1 + r_1 + 1) + \nu_F(x)\} = -d(x, w, x). \]

Let us consider the case where \( F/F_0 \) is a ramified extension, and we have \( \Lambda(0) \cap W' \) is equal to \( \sigma_F e_1 \oplus \sigma_F e_{-1} \). The integer \( \nu_F(\delta h(u^{-e_w}, u^{-e_w})) \) is equal to \( \min\{-m_2, -(m_1 + 2\nu_F(x))\} \), and we have \(-d(x, w, x) \) is equal to \( \min\{-m_2/2, -[m_1/2, -\nu_F(x)]\} \). We clearly have \(-m_2 \leq -[m_2/2], \) for \( m_2 \geq 0 \). Now, assume that \(-m_2 \) is equal to \( \min\{-m_2, -(m_1 + 2\nu_F(x))\} \), and we have
\[ m_2 \geq m_2/2 \geq m_1/2 - \nu_F(x). \]
From the above inequality we get that \( m_2 \geq [m_1/2 - \nu_F(x)] \). Now, assume that \(-m_1 + 2\nu_F(x)\) is equal to \( \min\{-m_2, -(m_1 + 2\nu_F(x))\} \), which implies that \( \nu_F(x) \leq m_1/2 - m_2/2 \). Hence, we have \( \nu_F(x) \leq m_1/2 \)
and this is equivalent to the inequality
\[ -m_1 + 2\nu_F(x) \leq -[m_1/2 - \nu_F(x)]. \]
We conclude that
\[ \min\{-m_2, -(m_1 + 2\nu_F(x))\} \leq \min\{-[m_2/2], -[m_1/2, -\nu_F(x)]\} = -d(x, w, x) \]
which implies the claim in this case. With this we verify the claim in all possible cases. \( \Box \)

8. The depth-zero case.

When \( F/F_0 \) is unramified the classification of generic depth-zero cuspidal representations of \( G \) can be deduced from the general work of DeBacker–Reeder for unramified groups in \cite{DR09} Section 6. From their results, generic cuspidal representations are precisely the representations of the form
\[ \text{ind}_{P_0(\lambda)}^G \sigma, \]
where \( P_0(\lambda) \) is a parahoric subgroup such that \( P_0(\lambda)/P_1(\lambda) \) is isomorphic to \( U(2,1)(k_F/k_{F_0}) \) and \( \sigma \) is the inflation of a cuspidal generic representation of \( P_0(\lambda)/P_1(\lambda) \). Now, we assume that \( F/F_0 \) is a ramified extension and consider a cuspidal representation of \( G \), isomorphic to
\[ \text{ind}_{P_0(\lambda)}^G \sigma, \quad (8.1) \]
where \( P_0(\lambda) \) is a maximal parahoric subgroup of \( G \) and \( \sigma \) is the inflation of a cuspidal representation of \( P_0(\lambda)/P_1(\lambda) \).

If \( F/F_0 \) is ramified, the groups \( P_0(\lambda)/P_1(\lambda) \) is the \( k_F \)-rational points of a disconnected reductive group over \( k_F \); the inverse image of \( k_F \)-rational points of the connected component, the parahoric subgroup, is denoted by \( P_0(\lambda) \). An irreducible representation \( \sigma \) of \( P_0(\lambda)/P_1(\lambda) \) is called a cuspidal representation if \( \text{res}_{P_0(\lambda)/P_1(\lambda)} \sigma \) is a direct sum of cuspidal representations. An irreducible representation \( \sigma \) of \( P_0(\lambda)/P_1(\lambda) \) is called generic if and only if its restriction to a \( p \)-Sylow subgroup, say \( H \), contains a non-trivial character of \( H \).
Let \((e_1, e_0, e_{-1})\) be any Witt-basis for \((V, h)\) then up to \(G\) conjugation there are two lattice sequences \(\Lambda_1\) and \(\Lambda_2\) such that \(P^0(\Lambda_i)\) is a maximal parahoric subgroup, for \(i \in \{1, 2\}\). The lattice sequences \(\Lambda_1\) and \(\Lambda_2\) have period 2 and are given by:

\[
\Lambda_1(-1) = \Lambda_1(0) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}
\]

and

\[
\Lambda_2(0) = o_F e_1 \oplus o_F e_0 \oplus p_F e_{-1} \quad \text{and} \quad \Lambda_2(1) = o_F e_1 \oplus p_F e_0 \oplus p_F e_{-1}.
\]

Let \(B\) be the Borel subgroup of \(G\) fixing the line \((e_1)\), and let \(U\) be the unipotent radical of \(B\). The groups \(P^0(\Lambda_1)\) and \(P^0(\Lambda_2)\) are special maximal compact subgroups of \(G\) and hence \(G = P_0(\Lambda_i)B\). The representation of the form \([8.1]\) is generic if and only if

\[
\Hom_{P_0(\Lambda_i) \cap U}(\sigma, \Psi) \neq 0,
\]

for some character \(\Psi\) of \(U\).

**Lemma 8.0.1.** Let \(F/F_0\) be a ramified quadratic extension. A depth-zero cuspidal representation \(\pi\) of \(G\) is generic if and only if

\[
\pi \simeq \Ind_{P_0(\Lambda_1)}^G \sigma,
\]

where \(\sigma\) is a generic cuspidal representation of \(P_0(\Lambda_1)/P_1(\Lambda_1)\).

**Proof.** Let \(\pi\) be a depth zero cuspidal representation isomorphic to \(\Ind_{P_0(\Lambda_1)}^G \sigma\). The image of the group \(U \cap P_0(\Lambda_1)\) in \(P_0(\Lambda_1)/P_1(\Lambda_1)\) is the pro-p Sylow subgroup of \(P_0(\Lambda_1)/P_1(\Lambda_1)\). Note that \(P_0(\Lambda_1) \cap U\) is equal to

\[
\{ u(x, y) : x, y \in F, y + \sigma(y) + x\sigma(x) = 0, \nu_F(y) \geq 0 \},
\]

and the group \(P_1(\Lambda_1) \cap U\) is equal to

\[
\{ u(x, y) : x, y \in F, y + \sigma(y) + x\sigma(x) = 0, \nu_F(y) \geq 1/2 \}.
\]

The quotient \((P_0(\Lambda_1) \cap U)/(P_1(\Lambda_1) \cap U)\) is isomorphic to \(\{ u(x, -x^2/2) : x \in k_F \}\). Let \(\Psi\) be any non-trivial character of \(U\) such that \(\res_{P_0(\Lambda_i) \cap U} \Psi \neq \id\) and \(\res_{P_1(\Lambda_i) \cap U} \Psi = \id\). For such a character \(\Psi\), the space

\[
\Hom_{P_0(\Lambda_i) \cap U}(\sigma, \Psi) \neq 0
\]

if and only if \(\sigma\) is the inflation of a generic cuspidal representation of \(P_0(\Lambda_1)/P_1(\Lambda_1)\). Hence, \(\pi\) is generic if and only if \(\sigma\) is generic.

Let \(\pi\) be a depth-zero cuspidal representation of the form \(\pi \simeq \Ind_{P_0(\Lambda_2)}^G \sigma\). If \(\pi\) is a generic, we get that

\[
\Hom_{P_0(\Lambda_2) \cap U_{der}} (\sigma, \id) \neq 0. \quad (8.2)
\]

The image of the group \(P_0(\Lambda_2) \cap U_{der}\) in the quotient \(P_0(\Lambda_2)/P_1(\Lambda_2) \simeq \SL_2(k_F) \times \{ \pm 1 \}\), is the pro-p Sylow subgroup and let the image be denoted by \(H\). Note that \(\res_H \sigma\) is a direct sum of non-trivial characters of \(H\). Thus we get a contradiction to the condition \([8.2]\). \(\square\)

9. Main theorem

In this section we combine the results obtained so far in the following theorem.

**Theorem 9.0.1.** Let \(F\) be any quadratic extension over a non-Archimedean local field of odd residue characteristic. Let \(Y = [\Lambda, n, 0, \beta]\) be a skew semisimple stratum and let \(\Pi_Y\) be the set of cuspidal representations containing a Bushnell–Kutzko type of the form \([P^0(\Lambda, \beta, \lambda)\)\]. Then we have:

1. If \(Y\) is a skew simple strata, then any representation contained in \(\Pi_Y\) is generic. In this case \(X_\beta\) is non-empty.

2. Let \(Y\) be a skew semisimple strata with underlying splitting \(V = V_1 \perp V_2\) such that \(\text{dim}_F V_i = i\), for \(i \in \{1, 2\}\). Assume that \(\beta = \beta_1 + \beta_2\) such that \([F][\beta_2] : F = 2\) and set \(q_i = \nu_{\lambda_i}(\beta_i)\). If \((V_2, h)\) is isotropic, then any representation in \(\Pi_Y\) is generic if and only if \(q_1 > q_2\). If \((V_2, h)\) is anisotropic, then any representation in \(\Pi_Y\) is generic if and only if \(q_2 > q_1\). In this case \(\Pi_Y\) contains a generic representation if and only if \(X_\beta\) is non-empty.
(3) Let \( r \) be a skew semisimple strata with underlying splitting \( V = V_1 \perp V_2 \) such that \( \dim_F V_i = i \), for \( i \in \{1, 2\} \). Assume that \( \beta = \beta_1 + \beta_2 \) such that \( \beta_i \in F \), \( \sigma(\beta_i) = -\beta_i \), for \( 1 \in \{1, 2\} \). Any representation in the set \( \Pi_r \) is non-generic.

(4) If \( r \) is a skew semisimple strata with the underlying splitting \( V = V_1 \perp V_2 \perp V_3 \), then any representation in the set \( \Pi_r \) is generic if and only if \( X_\beta \) is non-empty.

In the case where \( F/F_0 \) is unramified and \( r \) is a skew semisimple strata—\( \perp \) the underlying splitting \( V = V_1 \perp V_2 \perp V_3 \)—it is fairly easy to determine the conditions on \( \beta \) such that \( X_\beta \) is non-empty. This condition just depends on the set of integers \( \{ \nu_F(\beta_1), \nu_F(\beta_2), \nu_F(\beta_3) \} \). We refer to Lemmas 7.2.1 and 7.2.2 for these results. However, in the case where \( F/F_0 \) is ramified one requires more information on \( \beta \).

10. APPENDIX: FILTRATION OF \( U_{\text{der}} \) INDUCED BY LATTICE SEQUENCES

In this section we fix some representatives for \( G \)-conjugacy classes of self-dual lattice sequences on \( V \) and describe \( a_n(\Lambda) \), for \( n \in \mathbb{Z} \). Then we use them to determine \( U_{\text{der}} \cap a_n(\Lambda) \), for \( n \in \mathbb{Z} \). These calculations are used in showing certain representations are non-generic.

10.1. The unramified case: We begin with the case where \( F/F_0 \) is unramified. Let \( \Lambda_1 \) be the lattice sequence of periodicity 2 and

\[
\Lambda_1(-1) = \Lambda_1(0) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}.
\]

The filtration \( \{a_n(\Lambda) \mid n \in \mathbb{Z}\} \) of \( \text{End}_F(V) \) is given by

\[
a_{2m-1}(\Lambda_1) = \varpi^m \begin{pmatrix} o_F & o_F & o_F \\ o_F & o_F & o_F \\ o_F & o_F & o_F \end{pmatrix} \cap \mathfrak{g} \quad \text{and} \quad a_{2m}(\Lambda_1) = \varpi^m \begin{pmatrix} o_F & o_F & o_F \\ o_F & o_F & o_F \\ o_F & o_F & o_F \end{pmatrix} \cap \mathfrak{g},
\]

for all \( m \in \mathbb{Z} \). Let \( \Lambda_2 \) be a period 2 lattice sequence given by

\[
\Lambda_2(0) = o_F e_1 \oplus o_F e_0 \oplus p_F e_{-1} \quad \text{and} \quad \Lambda_2(1) = o_F e_1 \oplus p_F e_0 \oplus p_F e_{-1}.
\]

The filtration \( \{a_n(\Lambda_2) \mid n \in \mathbb{Z}\} \) is given by:

\[
a_{2m}(\Lambda_2) = \varpi^m \begin{pmatrix} o_F & o_F & p_F^{-1} \\ p_F & o_F & o_F \\ p_F & p_F & o_F \end{pmatrix} \cap \mathfrak{g} \quad \text{and} \quad a_{2m+1}(\Lambda_2) = \varpi^m \begin{pmatrix} p_F & o_F & o_F \\ p_F & p_F & o_F \\ p_F & p_F & p_F \end{pmatrix} \cap \mathfrak{g},
\]

for all \( m \in \mathbb{Z} \). Let \( \Lambda_3 \) be the lattice sequence of period 4 given by

\[
\Lambda_3(-1) = o_F e_1 \oplus o_F e_0 \oplus o_F e_{-1}, \quad \Lambda_3(0) = o_F e_1 \oplus o_F e_0 \oplus p_F e_{-1}, \\
\Lambda_3(1) = o_F e_1 \oplus p_F e_0 \oplus p_F e_{-1}, \quad \Lambda_3(2) = p_F e_1 \oplus p_F e_0 \oplus p_F e_{-1}.
\]

The filtration \( \{a_n(\Lambda_3) \mid n \in \mathbb{Z}\} \) on \( \mathfrak{g} \) is given by:

\[
a_{4m+r}(\Lambda_3) = \begin{cases} \\
\varpi^m \begin{pmatrix} o_F & o_F & o_F \\ p_F & o_F & o_F \\ p_F & p_F & o_F \end{pmatrix} \cap \mathfrak{g} & \text{if } r = 0, \\
\varpi^m \begin{pmatrix} o_F & o_F & o_F \\ p_F & o_F & o_F \\ p_F & p_F & o_F \end{pmatrix} \cap \mathfrak{g} & \text{if } r = 1, \\
\varpi^m \begin{pmatrix} o_F & o_F & o_F \\ p_F & p_F & o_F \\ p_F & p_F & p_F \end{pmatrix} \cap \mathfrak{g} & \text{if } r = 2, \\
\varpi^m \begin{pmatrix} o_F & o_F & o_F \\ p_F & p_F & p_F \\ p_F & p_F & p_F \end{pmatrix} \cap \mathfrak{g} & \text{if } r = 3.
\end{cases}
\]
Although, there is a lattice sequence, say \(\Lambda_3\), with period 6, we do not need to write it down explicitly.

This corresponds to the type \((A)\) strata and in this case all representations are generic. The filtration \(\{U_{\text{der}} \cap a_n(\Lambda_1) \mid n \in \mathbb{Z}\}\) is given by:

\[
U_{\text{der}} \cap a_{2m-1}(\Lambda_1) = U_{\text{der}} \cap a_{2m}(\Lambda_1) = U_{\text{der}}(m),
\]

for \(m \in \mathbb{Z}\). The filtration \(\{U_{\text{der}} \cap a_n(\Lambda_2) \mid n \in \mathbb{Z}\}\) is given by

\[
U_{\text{der}} \cap a_{2m}(\Lambda_2) = U_{\text{der}}(m-1) \quad \text{and} \quad U_{\text{der}} \cap a_{2m+1}(\Lambda_2) = U_{\text{der}}(m).
\]

The filtration \(\Lambda_3\) is given by

\[
U_{\text{der}} \cap a_{4m+r}(\Lambda_2) = \begin{cases} 
U_{\text{der}}(m) & \text{if } r = 0, \\
U_{\text{der}}(m) & \text{if } r = 1, \\
U_{\text{der}}(m) & \text{if } r = 2, \\
U_{\text{der}}(m+1) & \text{if } r = 3.
\end{cases}
\]

(10.4)

### 10.2. The ramified case:

Now, assume that \(F/F_0\) is a ramified extension and \(\Lambda_1\) and \(\Lambda_2\) be the lattice sequence of period 2 given by

\[
\Lambda_1(-1) = \Lambda(0) = \mathcal{o}_F e_1 \oplus \mathcal{o}_F e_0 \oplus \mathcal{o}_F e_{-1}.
\]

and

\[
\Lambda_2(0) = \mathcal{o}_F e_1 \oplus \mathcal{o}_F e_0 \oplus \mathcal{p}_F e_{-1} \quad \text{and} \quad \Lambda_2(1) = \mathcal{o}_F e_1 \oplus \mathcal{p}_F e_0 \oplus \mathcal{p}_F e_{-1}
\]

The filtration \(\{a_n(\Lambda_1) \mid n \in \mathbb{Z}\}\) is similar to the filtration in \([10.1]\). The filtration \(\{a_n(\Lambda_2) \mid n \in \mathbb{Z}\}\), in this case, is similar to the filtration in \([10.2]\). We will not require to write the filtrations \(\{a_n(\Lambda') \mid n \in \mathbb{Z}\}\) for which \(P^0(\Lambda')\) is an Iwahori subgroup of \(G\). The filtration \(\{U_{\text{der}} \cap a_n(\Lambda_1) \mid n \in \mathbb{Z}\}\) is given by

\[
U_{\text{der}} \cap a_{2m-1}(\Lambda_1) = U_{\text{der}} \cap a_{2m}(\Lambda_1) = U_{\text{der}}([m/2]),
\]

for all \(m \in \mathbb{Z}\). The filtration \(\{U_{\text{der}} \cap a_n(\Lambda_2) \mid n \in \mathbb{Z}\}\) is given by

\[
U_{\text{der}} \cap a_{2m-1}(\Lambda_2) = U_{\text{der}} \cap a_{2m}(\Lambda_2) = U_{\text{der}}([(m-1)/2]),
\]

for any \(m \in \mathbb{Z}\).

### References

[BK93] Colin J. Bushnell and Philip C. Kutzko. *The admissible dual of GL(n) via compact open subgroups*, volume 129 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993.

[BS09] Corinne Blondel and Shaun Stevens. Genericy of supercuspidal representations of \(p\)-adic Sp\(_4\). *Compos. Math.*, 145(1):213–246, 2009.

[BZ76] I. N. Bernštejn and A. V. Zelevinskii. Representations of the group \(GL(n, F)\), where \(F\) is a local non-Archimedean field. *Uspekhi Mat. Nauk*, 31(3(189)):5–70, 1976.

[DR09] Stephen DeBacker and Mark Reeder. Depth-zero supercuspidal \(L\)-packets and their stability. *Ann. of Math. (2)*, 169(3):705–901, 2009.

[DR10] Stephen DeBacker and Mark Reeder. On some generic very cuspidal representations. *Compos. Math.*, 146(4):1029–1055, 2010.

[GS88] Stephen Gelbart and Freydoon Shahidi. *Automorphic properties of automorphic \(L\)-functions*, volume 6 of *Perspectives in Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[JL70] H. Jacquet and R. P. Langlands. *Automorphic forms on GL(2)*. Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.

[Ka16] Taslo Kaletha. The local Langlands conjectures for non-quasi-split groups. In *Families of automorphic forms and the trace formula*, Simons Symp., pages 217–257. Springer, [Cham], 2016.

[MS14] Michitaka Miyawaki and Shaun Stevens. Semisimple types for \(p\)-adic classical groups. *Math. Ann.*, 358(1-2):257–288, 2014.

[Mu95] Fiona Murnaghan. Local character expansions for supercuspidal representations of \(U(3)\). *Canad. J. Math.*, 47(3):606–649, 1995.

[PS08] Vytautas Paskunas and Shaun Stevens. On the realization of maximal simple types and epsilon factors of pairs. *Canad. J. Math.*, 60(5):1211–1261, 2008.

[Rod73] François Rodier. Whittaker models for admissible representations of reductive \(p\)-adic split groups. *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, William Collier, WilliamsTown, Mass., 1972)*, pages 425–430, 1973.

[Sha74] J. A. Shalika. The multiplicity one theorem for \(GL(n)\). *Ann. of Math. (2)*, 100:171–193, 1974.

[Ste05] Shaun Stevens. Semisimple characters for \(p\)-adic classical groups. *Duke Math. J.*, 127(1):123–173, 2005.
Shaun Stevens. The supercuspidal representations of p-adic classical groups. *Inventiones mathematicae*, 172(2):289–352, May 2008.

Santosh Nadimpalli, IMAPP, Radboud Universiteit Nijmegen, Heyendaalseweg 135, 6525AJ Nijmegen, The Netherlands. nvrnsantosh@gmail.com, Santosh.Nadimpalli@ru.nl.