THE SOLOW-SWAN MODEL WITH ENDOGENOUS POPULATION GROWTH

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Abstract. This paper presents a reformulation of the classical Solow-Swan growth model where a dynamic of the endogenous population is incorporated. In our model, the population growth rate continually depends on per capita consumption. We find that – as in the classic Solow-Swan model – there is a steady state for the capital-labour ratio, which is always lower than that deduced from the original model with zero population growth rate, but it is not necessarily unique. Under certain conditions, there is an odd amount, and only the smallest and the largest are locally stable. Finally, a study of comparative static of stationary states is performed by varying the total factor productivity, and the results are compared with those of the original model. It is found that the effects of exogenous variables on endogenous variables differ from the original model.

1. Introduction. Growth models point out, in broad terms, the factors that determine the long-term growth of an economy. The explicit objective is to describe the growth’s phenomenon, as well as its speed and determinants.

One of the assumptions used in the model, which is also used in standard growth models (Solow [22]-Swan [23], Ramsey [20]-Cass [12]-Koopmans [18], Romer [21], Mankiw-Romer-Weil [19]), is that the labour force (associated with the size of the population) grows at a constant rate \( n \geq 0 \). This implies that the population grows exponentially. While it can provide an adequate approximation during the initial period, it is clearly not realistic, because, as it grows exponentially, the population has an infinite limit when time \( t \) has an infinite limit. The assumption, which strongly conditions the results of the model, is clearly not realistic. The evolution of the population in the last century (United Nations, [24]) shows that it
has grown, but at a decreasing rate— with a decreasing rate that tends to zero, and even tends negative in some regions.

The main objective of this work is to introduce an endogenous population into the Solow model, where the dynamics of the population and, more specifically, its growth rate, are determined by a function that depends on per capita consumption.

Several studies focus on the reform of growth models by introducing alternative population laws to exponential law. These studies can be divided according to two alternative approaches. The first introduces population laws, which verify some properties that capture the main stylized facts of population dynamics (increasing, decreasing, and tending to zero) or population dynamics explained endogenously by the model. The articles of Accinelli and Brida [1,2,3], Bay [4], Brida and Maldonado [10], Cai [11] and Ferrara and Guerrini [14,15], among others, are part of the first approach. In these, under different hypotheses about population dynamics, growth models are reformulated and the resulting dynamics are analysed. Either using particular population laws, which offers the demography or other social sciences, or general laws that meet some properties such as variable but bounded population, increasing at a decreasing rate. All of them fulfill properties that conform to the observed behavior of the world population in the last hundred years. However, these are determined outside the economic system, the dynamics of the population is not influenced in any way by the evolution of the variables of the model, they are exogenous.

The second approach reformulates the growth models, assuming the interdependence between the variables of the model and the behavior of the population. There are several theoretical models that incorporate the relationship between one of the determinants of population evolution: the fertility rate and economic growth. Perhaps the most relevant models are those developed by Galor [17], Benhabib and Nishimura [9], and Becker, Murphy and Tamura [7], among others.

All these theoretical developments, from different perspectives (family economy, human capital theories, growth theories) and through different transmission mechanisms, explain how population dynamics and economic performance are closely related, whether through simple arithmetic (negative effect of population growth on per capita income), for differential wages between men and women (Galor and Weil [16]), for the costs of raising children, opportunity cost (Becker [5] and Becker and Lewis [8]) or the rising returns on human capital (Becker and Barro [6]). Regardless of the causal mechanism, in general, an inverse relationship between the population growth rate (mainly explained by fertility) and the level of income per capita is postulated. This relationship, on the other hand, is observed in the main modern economies, where high levels of income coexist with extremely low population growth rates in relation to those of economies with low income levels.

Finally, Corchón [13] reformulates the Solow model under a Malthusian approach, where the population growth rate is a growing function of real wages. With this specification, the model shows the possibility of multiple equilibria, depending on the population dynamics. In our article, the procedure used by Corchón [13] is followed, with two differences. On the one hand, we generalize the production function. This article does not make an explicit specification of the function. On the other hand, a positive population growth is assumed, where for a range of per capita consumption levels, it grows at an increasing rate, reflecting the Malthusian ideas regarding population dynamics and economic growth. It should be noted that making population growth depend on per capita consumption or real wages (as
Corchón [13] does) is equivalent. Due to the properties assumed from the production function. After a threshold, it grows at a decreasing rate and converges at a minimum level close to zero, fundamentally expressing Becker’s theoretical contributions. Therefore, our article generalizes the results of Corchón for a specific type of population growth. This paper is organized as follows. In the next section, we present a reformulation of the Solow-Swan model and characterize the solution. In section 3, a comparative static exercise is performed, and finally, section 4 presents our concluding remarks.

2. The model. The standard version of the model assumes an economy that produces a unique good $Y$, which can be consumed or used in the production of the good itself. The economy is endowed with a technology defined by a production function $F(K, L)$, where $K$ is the capital stock and $L$ is the labor, which meets the following properties:

$$ Y(t) = AF(K(t), L(t)) $$ (1)

where $A$ is the total factor productivity, and $F$ verifies the usual properties:

1. $\frac{\partial F(K, L)}{\partial K} > 0$, $\frac{\partial F(K, L)}{\partial L} > 0$, $\frac{\partial^2 F(K, L)}{\partial K^2} < 0$, $\frac{\partial^2 F(K, L)}{\partial L^2} < 0$.
2. $F(K, 0) = F(0, L) = 0; \forall K, L \in R^+.$
3. $F(\lambda K, \lambda L) = \lambda F(K, L); \forall \lambda, K, L \in R^+$
4. $\lim_{K \to 0} \frac{\partial F(K, L)}{\partial K} = \lim_{L \to 0} \frac{\partial F(K, L)}{\partial L} = +\infty$; $\lim_{K \to +\infty} \frac{\partial F(K, L)}{\partial K} = \lim_{L \to +\infty} \frac{\partial F(K, L)}{\partial L} = 0$

Capital depreciates at a constant rate $\delta$. The product can be used as a consumption good or as investment:

$$ Y(t) = AF(K(t), L(t)) = C(t) + \dot{K}(t) + \delta K(t), \delta \in (0, 1) $$ (2)

Consumers save a fixed fraction of income $s$. Thus, capital accumulation is:

$$ \dot{K}(t) = sY(t) - \delta K(t) $$ (3)

$$ \frac{\dot{K}(t)}{K(t)} = s \frac{Y(t)}{K(t)} - \delta = s \frac{Y(t)}{L(t)} \frac{L(t)}{K(t)} - \delta = s \frac{Af(k(t))}{k(t)} - \delta $$ (4)

where $k(t) = \frac{K(t)}{L(t)}$ is the capital-labor ratio and $Af(k(t)) = \frac{AF(K(t), L(t))}{L(t)}$ is the output per capita.

Furthermore, in terms per capita, the capital accumulation rate is:

$$ \frac{\dot{k}}{k} = \frac{K/L}{(K/L)} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} $$ (5)

2.1. Reformulation: Dynamics of the endogenous population. The population growth assumption is an innovation that incorporates our model with respect to the model proposed by Solow-Swan. The latter used a constant growth rate, which implies that the total population grows exponentially. This assumption would seem appropriate in the mid-twentieth century, although it seems unlikely that population dynamics have been observed since the mid-twentieth century. In the context where dynamic growth models, such as the Solow-Swan or Ramsey-Cass-Koopmans models, were developed, this assumption seems to be credible (see Figure 1). However, in our model (following Corchón [13]), population dynamics emerge from the
Malthus trap, and we seek to reflect the evolution of the world population from a historical perspective, reflecting the dynamics of the last two centuries.

**Figure 1.** Evolution of the world population. Period 1750-2100.

![Graph showing the evolution of the world population from 1750 to 2100.](image1)

Source: Max Roser and Esteban Ortiz-Ospina (2019) – “Future population growth”. Published online at OurWorldInData.org. Retrieved from: https://ourworldindata.org/world-population-growth [Online Resource].

The population evolution was not exponential. The world demographic transition led to a slowdown in population growth. According to estimates by the United Nations [24], the population will stop growing and stagnate at the end of the twenty-first century (assuming a variation in average fertility). This is reflected in figure 2, which shows an evolution in the rate of growth of the declining world population.

**Figure 2.** World population growth rate. Period 1950-2100.

![Graph showing the annual growth rate of the world population from 1950 to 2100.](image2)

Source: United State Census Bureau (September 2018) and United Nations (2019).
This evidence allows us to argue that these types of models, particularly the Solow-Swan model, have a strong weakness in terms of this assumption, which appears to be little discussed, except in the aforementioned cases. It could be said, then, that little has been discussed and questioned about this assumption in the theoretical literature.

These observations motivated us to modify the formulation, ultimately incorporating a different evolution of the population growth rate over time. As we show in Figure 2, growth rate is not constant, but it is positive at the rate of increase in the graph from 1960 through to the date of writing. This is formalized below, giving it energy through the capital accumulated in the economy. It is assumed that the number of workers and consumers coincides, and grows at a rate that depends on per capita consumption:

\[
\frac{\dot{L}(t)}{L(t)} = n \left[ \frac{C(t)}{L(t)} \right] = n [c(t)] , n(c) \geq 0, \forall c \geq 0
\] (6)

with \( n(.) \) continuously differentiable and exist \( c_0 \) such that \( n'(c) > 0, \forall c < c_0 \), and \( n'(c) \leq 0, \forall c \geq c_0 \). Also: \( \lim_{c \to +\infty} n(c) = 0 \)

The per capita consumption is fixed fraction of income \( 1 - s \).

\[
c(t) = (1 - s)Af(k(t))
\] (7)

Note that a constant population growth rate as in the Solow model is a particular case, since \( n(c) = n, \forall c \)

The population dynamics are endogenized, unlike the traditional Solow-Swan model. As for the case of Corchón, a variant is added. The dynamics of the population depend on the per capita consumption. In this way, the population is linked in a similar way to the Corchón model, in terms of the model’s economic variables.

2.2. Capital accumulation. Incorporating the population assumption and using (6), (7) and (4), the right-hand side equality of (5) becomes the equation of motion for the model, which describes how capital-labour ratio varies over time:

\[
g(k) = \frac{\dot{k}}{k} = s \frac{Af(k)}{k} - \delta - n((1 - s)Af(k))
\] (8)

2.3. Equilibria and stability: Qualitative analysis. In this section, we will study the dynamics of the model with endogenous population growth. While the Solow-Swan model suggests that there is only one capital per capita of equilibrium to which the economy inevitably converges, in our model, this solution is not necessarily unique. From the interactions between population growth and capital accumulation, new results emerge compared to the Solow-Swan model. In the first place, we find an odd number of stationary states, although not necessarily unique. Additionally, the solution of the Solow model is a specific case of ours. Ultimately, under an unrestricted condition, the balances will be stable.

2.3.1. The positive steady state. This section investigates the dynamic behaviour of the model’s solution \( k(t) \)

**Proposition 1.** The model has at least one Steady state.
Proof. It is clear that the function \( g_k = \frac{sAf(k)}{k} - \delta - n((1 - s)Af(k)) \) tends to \(+\infty\) when \( k \to 0 \). Being a continuous function, it takes positive values in an interval close to zero.

When \( k \to +\infty \) the function tends to \(-\delta - n((1 - s)Af(k)) < 0\) which for \( k \) large enough is negative since \( n(c) \geq 0, \forall c \).

The function \( g_k = \frac{sAf(k)}{k} - \delta - n((1 - s)Af(k)) \) is continuous and change sign in \((0, +\infty)\). Bolzano’s theorem allows us to affirm that he has at least one root. 

Let \( k_s \) be the equilibrium value of capital per capita that is deduced from the Solow model with a zero population growth rate, that is, which meets the condition:

\[
s\frac{Af(k_s)}{k_s} = \delta
\]

**Proposition 2.** The value of the per capita equilibrium capital of the steady state \( k^* \) is always less than or equal to \( k_s \).

Proof. If \( n((1 - s)Af(k_s)) = 0 \) is true that \( k^* = k_s \) because \( s\frac{Af(k_s)}{k_s} = \delta \). If \( n((1 - s)Af(k_s)) > 0 \) is true that the function \( g \) it takes positive values for values of \( k \) close to zero and a negative value in \( k_s \) (\( g(k_s) < 0 \)) then it can be affirmed that exists \( k^* \) such that \( g(k^*) = 0 \) in addition, \( k^* \in (0, k_s) \) this is: \( 0 < k^* < k_s \) 

**Proposition 3.** If \( k^* \) is unique it is locally stable

Proof. If \( k^* \) is unique \( \Rightarrow g(k) = 0 \Leftrightarrow k = k^* \) and using the same arguments as in proposition 1 is met:

\[
\frac{\dot{k}}{k} = g(k) \geq 0, \forall k \leq k^*, \text{ namely } \frac{\dot{k}}{k} \geq 0, \Rightarrow k(t) \text{ is increasing } \forall k \leq k^* \text{ and } \frac{\dot{k}}{k} = g(k) < 0, \forall k > k^*, \text{ namely } k \text{ is decreasing } \forall k \leq k^*. 
\]

**Remark 1.** Given the assumption of dynamics of the population, the pattern and speed of convergence could differ from that derived from the Solow-Swan model. An economy in its path of balanced growth could go through prolonged periods of accelerated growth before reaching steady state. This possibility cannot occur in the Solow-Swan model. As the economy approaches its steady state its growth rate is decreasing (see Figure 3). Economies with less capital per capita may not tend to grow faster in per capita terms.

As we can see in the Figure 4 there is a particular case where there is not an odd amount of equilibria. For these reason we consider the following restriction to ensure that there are an odd number off equilibria.

**Proposition 4.** If the condition is met:

\[
\frac{sA}{k^*2} [f(k^*) - k^* f'(k^*)] \neq (s - 1)Af'(k^*)n'((1 - s)Af(k^*)) \quad (9)
\]

there is an odd number of equilibria, the minor and the major alternate and are locally stable.

Proof. It is immediate following the same reasoning as in the first proposition. The condition ensures that \( g \) changes sign in each of the roots, there is an odd amount of equilibrium, minor and the major are stable and stable and unstable alternate.
The condition 9 is not restrictive. To show it, we consider the Cobb-Douglas function for the production. In this case we have:

$$sAf(k^*) = sAk^\alpha$$

(10)

Then, the condition is operative if and only if, the following restriction is met:

$$\frac{1 - \alpha}{\alpha n'(.)} + 1 \neq s$$

(11)

If we consider a usual value for $\alpha = 1/3$, and $s = 0.20$, the only slope value of $n'$ is -0.375. For this reason, the probability of achieve this value tends to zero.

**Remark 2.** Unlike the Solow-Swan model, our model supports the possibility of multiple equilibrias. An economy could be trapped in a stable steady state, there being another where the product per capita is greater (see Figure 5a).
For example, if we consider the next function (Normal distribution) that complies the properties of \( n(c) \):

\[
f(x, \mu, \sigma) = \frac{e^{-(c(x)-\mu)^2/(2\sigma^2)}}{\sigma \sqrt{2\pi}} \tag{12}
\]

where \( \mu = 9 \) and \( \sigma = 4 \). Furthermore, we consider a typical Cobb-douglas function with \( \alpha = 1/3 \). So we have the following dynamic equation:

\[
g(k) = \frac{\dot{k}}{k} = sAk^{\alpha-1} - \delta - \frac{e^{-(c(k)-9)^2/(2\cdot4^2)}}{5\sqrt{2\pi}} \tag{13}
\]

Where: \( c(k) = (1-s)Ak^\alpha \) and \( a > 0 \) that reflect the fact that the function \( n(c) \) start in \( c = 0 \) with a positive value. Considering the following parameters values:

- \( A = 5 \)
- \( s = 0.2 \)
- \( \delta = 0.04 \)

As can be seen in Figure 6, there are three steady states for the model calibrated with the values of the aforementioned parameters. These three \( k^* \), take the following values: \( k^* = 23.02 \), \( k^* = 58.37 \) and \( k^* = 106.63 \). So, if we consider a Cobb-Douglas function, there are two possible gross domestic product (GDP) per capita stable steady-states. One reaches a value of 23.71, the other a value of 14.22. This mean that with the same characteristics in each economy, one country could achieve a product 66% bigger than the other. Furthermore, the population growth rate reaches plausible values. In the poor balance, the rate is 8.3%, 2.6% in the middle and 0.4% in the highest.

**Remark 3.** This result contradicts the original Solow model. In this model, the result obtained was an irreversible stable state, in which the economy would reach a constant per capita capital, as well as the rest of the variables that explain the economy in the model. In our model, the economy will also end up stagnant, however, the result and its implications are profoundly different. The economy will have two stable equilibria, one “good” and the other “bad”. This means that two economies with the same characteristics in all their parameters could have qualitatively and quantitatively different results. Therefore, our model explains why countries with similar characteristics might have different results. One would be stuck in a poverty trap, while another would obtain superior relative per capita wealth. This difference is simply explained by the different initial endowments of capital per capita, which determine divergent long-term results between the two economies due to population dynamics.
Comparative static. In this section, some comparative static exercises are performed from the steady state equilibrium. Incorporating a dynamic of the endogenous population causes some results to be very different from the original Solow model.

In the traditional Solow model, the effect of technology determines an increase in capital per capita. In our model, this effect is not direct. There is a threshold that determines whether this effect is positive or negative. The mechanism that acts to understand this dynamic is the following: on the one hand, technological changes generate an increase in production, while on the other, it induces an increase in consumption, which can determine an increase in the population growth rate. The results are similar to those found in the Solow model: an increase in $A$ and a change in $n$, where in one case $n$ increases and in the other, it decreases. In our model the result of the effect that will be found will depend on the speed with which the population grows, the capital per capita of the economy and the level of consumption that it has.

Proposition 5. From a stable equilibria a small increase in $A$ increases $k^*$ if

$$n'(k^*)k^* < \frac{s}{1 - s}$$

Proof.

$$dg_k = s \left[ \frac{Af'(k^*)k^* - Af(k^*)}{k^*} \right] dk^* - n'(1 - s)Af(k^*)Af'(k)(1 - s)dk^*$$

$$+ \frac{s f(k^*)}{k^*} dA - n'(1 - s)Af(k^*)f(k^*)dA = 0 \quad (15)$$

$$\frac{dk^*}{dA} = \frac{n'(1 - s)Af(k^*)Af'(k^*) - sf(k^*)}{s \left[ \frac{Af'(k^*)k^* - Af(k^*)}{k^*} \right] - n'(1 - s)Af(k^*)Af'(k^*)(1 - s)} \quad (16)$$

If the equilibrium is stable, the denominator is negative and the sign depends on the numerator.
\[
n'(k^*)(1-s)f(k^*) - \frac{sf(k^*)}{k^*} < 0
\]
\[
\Leftrightarrow \quad n'(k^*)k^* < \frac{s}{1-s}
\]

Note that the effect varies according a threshold given by the savings rate of the economy. For a decreasing length of the slope, per capita capital of equilibrium increases for positive savings rates. Then, for the growing stretch of the population curve the effect depends on the interaction with the stock of capital per capita in the equilibrium.

**Remark 4.** The result implies that an increase in \( A \) does not necessarily increase the equilibrium capital. The dynamics of the population can determine that in some cases it decreases, unlike the Solow model where it always increases.

**Proposition 6.** From a stable equilibrium, \( k^* \) a small increase in \( A \) increases the product per capita.

*Proof.* Totally differentiating \( y^* = A.f(k^*) \)

\[
\frac{dy^*}{dA} = f(k^*) + Af'(k^*)\frac{dk^*}{dA}
\]

\[
\frac{dy^*}{dA} = f(k^*) + Af'(k^*)\frac{dk^*}{dA} - n'(1-s)Af(k^*)\frac{df(k^*)}{dk^*}(1-s)
\]

\[
\frac{dy^*}{dA} = f(k^*)\left[1 + \frac{n'(1-s)Af(k^*)\frac{df(k^*)}{dk^*}}{s\left[\frac{f(k^*)}{k^*} - f(k^*)\right] - n'(1-s)Af(k^*)\frac{df(k^*)}{dk^*}(1-s)}\right]
\]

Again, our model shows how the effect of an increase in the savings rate does not necessarily determine increases in the level of capital per capita equilibrium. An increase in this rate decreases the level of consumption, which affects the population growth rate. Depending on the section of the population growth curve in which the economy is located, the effect of the savings rate will be different, since the decrease in consumption will have positive effects on the decreasing stretch and negative effects on the growing stretch.

**Proposition 7.** From a stable equilibrium, \( k^* \) a small increase in \( s \) increases \( k^* \)

\[
\Leftrightarrow \quad \frac{1}{k^*} + n'((1-s)f(k^*)) > 0.
\]

*Proof.*

\[
dk^* = s\left[\frac{Af'(k^*)k^* - Af(k^*)}{k^*} - n'((1-s)f(k^*))Af'(k^*)\right]dk^*(1-s)
\]

\[
+ \frac{Af(k^*)}{k^*}ds - n'(-Af(k^*))ds = 0
\]

\[
\frac{dk^*}{ds} = \frac{-Af(k^*) + n'(-Af(k^*))}{s\left[\frac{Af'(k^*)k^* - Af(k^*)}{k^*} - n'((1-s)f(k^*))Af'(k^*)\right](1-s)}
\]
simplifying:

\[
\frac{dk^*}{ds} = \frac{-Af(k^*) \left\{ \frac{1}{k^*} + n' \right\}}{\left\{ s \left[ \frac{Af'(k^*)}{k^*} - Af(k^*) \right] - n'(1 - s)Af(k^*)Af'(k^*)(1 - s) \right\}}
\]  

(25)

So the effect of a change in the savings rate is only positive, if it is fulfilled that \( \frac{1}{k^*} + n' > 0 \).

Remark 5. The result implies that an increase in \( s \) does not necessarily increase the equilibrium capital. Furthermore, we can say that an increase in the savings rate increases the equilibrium capital if there is an increase in the population growth rates \( n'(1 - s)Af(k^*) > 0 \).

4. Concluding remarks. In this article, we reformulated the classical Solow-Swan growth model incorporating a dynamic of the endogenous population. Instead of assuming a constant and independent population growth rate of model variables, the population growth depends on per capita consumption. Following the Corchón model, we were able to generalize the results using less restrictive assumptions and we obtained an improvement in the model. Because the production function is no longer limited to a Cobb-Douglas function.

We find that as in the original model, there is a stable steady state for the \( K/L \) ratio, but unlike the original model, it is not necessarily unique. Furthermore, under certain conditions (proposition 4) there is an odd number of equilibria, with instability only in even equilibria. As a result of this finding, the model admits that the economy could be trapped in “poverty traps”. In line with the previous result, the predictions that emerge from the Solow model regarding the effects on the variation in the exogenous parameters are no longer the same. Unlike the original model, an increase in the technological parameter \( A \) does not necessarily cause an increase in the per capita equilibrium capital, and it may even imply a reduction, since it increases the population growth rate. The increase in the coefficient associated with technological change does not always have a positive effect on the capital per capita of the economy, as the traditional Solow-Swan model did. Population dynamics now play a fundamental role in the solution and in the results of comparative static. The technical progress generates an effect of efficiency that allows us to increase the product and the consumption. By affecting consumption, technical progress generates changes in the population growth rate. Something similar occurs with the effect of the savings rate.

In summary, despite the simplicity of the Solow model, this is and has been widely used not only in teaching in economics, but also in theoretical modelling and as a framework for empirical research. Therefore our model demonstrates – and warns – that small modifications in the assumptions have relevant consequences in the conclusions that can be reached.

Future research will analyse the modified model in discrete time. Empirical studies based on the Solow-Swan model implicitly assume that the continuous time dynamic properties are the same as in discrete time, but the logistics equation is a classic example that this is not always the case. If the dynamic is different in a model or another, the conclusions and policy recommendations will also be different. In a second line of research, we will try to incorporate new simulations using this model, but in a discrete time. On the other hand, we will seek to test whether the results predicted by this simple model contrast with the empirical evidence of the world economy in the last 50 years.
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