QCD Sum Rules for Exclusive Decays of Heavy Mesons

A. Khodjamirian\textsuperscript{a,1} and R. Rückl\textsuperscript{a,b}

\textsuperscript{a} Institut für Theoretische Physik, Universität Würzburg, D-97074 Würzburg, Germany
\textsuperscript{b} Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, D-80805 München, Germany

Abstract

Applications of QCD sum rules to exclusive decays of heavy mesons are reviewed. In detail, we discuss the calculations of the $B$ and $D$ decay constants, the heavy-to-light form factors $f^+(p^2)$ and $f^0(p^2)$, and the strong couplings $g_{B^*B\pi}$ and $g_{D^*D\pi}$. Predictions are presented for the semileptonic weak decays $B \to \pi l\bar{\nu}_l$ and $B \to \rho l\bar{\nu}_l$, and used to extract $V_{ub}$ from the measured decay widths. Results are also shown on $D \to \pi l\bar{\nu}_l$ and $D^* \to D\pi$. Furthermore, we reconsider the factorization hypothesis in nonleptonic two-body decays, and describe a sum rule estimate of the nonfactorizable contribution to the amplitude of $B \to J/\psi K$.

To appear in Heavy Flavours, 2nd edition, eds. A.J. Buras and M. Lindner (World Scientific, Singapore)

1 Introduction

During recent years a large amount of new experimental data on exclusive decays of charm and bottom hadrons has become available. In order to make optimal use of these data in determinations of fundamental parameters and tests of the standard model, one needs a quantitative understanding of the impact of strong interactions. This requires accurate calculations of hadronic matrix elements of weak operators in QCD beyond perturbation theory. Current approaches include lattice calculations, QCD sum rules, heavy quark effective theory (HQET), chiral perturbation theory (CHPT), and phenomenological quark models. Each of these approaches has advantages and disadvantages. For example, quark models are easy to use and good for intuition. However, their relation to QCD is unclear. On the other hand, lattice calculations are rigorous from the point of view of QCD, but they suffer from lattice artifacts and uncertainties connected with the necessary extrapolations to the physical quark masses. Furthermore, effective theories are usually applicable only to a restricted class of problems, and sometimes require substantial corrections which cannot be calculated within the same framework. For example, HQET is very powerful in treating

\textsuperscript{1} on leave of absence from the Yerevan Physics Institute, 375036 Yerevan, Armenia
b → c transitions, but a priori less suitable for b → u transitions, while CHPT is designed for processes involving soft pions and kaons.

In this review, we focus on the method of QCD sum rules. Proceeding from the firm basis of QCD perturbation theory this approach aims to incorporate nonperturbative elements of full QCD. Schematically, hadronic matrix elements of operators composed of quark and gluon fields are extracted from correlation functions of quark currents, rather than estimated directly in some models. One makes use of operator product expansion (OPE), the analyticity principle (dispersion relations), and S-matrix unitarity. In addition, one assumes the validity of quark-hadron duality in a rather strong form, sometimes called semi-local duality. The long-distance dynamics is parameterized in terms of vacuum condensates or light-cone wave functions. These new elements cannot yet be calculated rigorously in QCD, but are to be determined from experimental data in the one or other way. Nevertheless, because of the universal nature of the nonperturbative input, the sum rule approach retains its predictive power. Moreover, it is rather flexible and can therefore be applied to a large variety of problems in hadron physics.

The application of sum rules to heavy quark physics has always been particularly successful. The more recent development of sum rule techniques for exclusive heavy meson decays again looks very promising. We shall substantiate this assessment for leptonic, semileptonic as well as nonleptonic decays of B and D mesons. Although the two flavours could be treated in parallel, we shall usually refer to B mesons, for definiteness. In most cases, it is obvious how to obtain the corresponding results for D mesons.

The leptonic decay process B → ℓν involves the simplest hadronic matrix element, that is the B-meson decay constant f_B. We therefore have chosen the latter for our introductory study case. Furthermore, f_B is one of the important parameters of mixing and CP-violation in the B system, and a necessary input in order to eventually extract the CKM element V_{ub} from future measurements of B → τν_τ. Finally, f_B also enters more complicated QCD sum rules for other hadronic properties of B mesons, e.g., form factors.

The semileptonic decays B → πℓν and B → ρℓν have already been observed experimentally. These exclusive modes are considered to provide interesting alternatives to the determination of V_{ub} from inclusive b → u transitions. However, truly competitive results can only be expected if the calculation of heavy-to-light form factors can be improved significantly. From the theoretical point of view, the B → π and B → ρ form factors are excellent examples for the application of light-cone sum rules.

Exclusive nonleptonic decays are an even greater challenge to theory. Although over the years one has developed a qualitative understanding and even an amazingly consistent quantitative description of two-body decays, some features still lack a clear dynamical explanation. This concerns most of all the factorization of matrix elements of four-quark operators into products of matrix elements of quark currents. Since nonfactorizable amplitudes are a priori expected to be channel-dependent, the apparent universality of the effective coefficients a_1 and a_2 constitutes a major puzzle. The decay mode B → J/ψK containing two heavy quarks in the final state may play a special role in this respect and therefore shed some light on the factorization problem. We present this example also because it shows conceptual and technical limits of the sum rule approach.

The content of this review is as follows. In section 2, we describe the derivation of

---

2A comprehensive study is presented by M. Neubert and B. Stech in this volume, ref. [4].
the sum rule for \( f_B \) from a two-point correlation function and summarize the analogous calculations for other \( B \) and \( D \) mesons. Also shown is a comparison with lattice results and experimental data. The straightforward generalization of the above method to three-point correlation functions is discussed in section 3 and applied to heavy-to-light form factors. In section 4, we explain an alternative approach based on light-cone expansion and employ it in section 5 to obtain sum rules for the \( B \to \pi \) form factors \( f^+ \) and \( f^0 \). Recent results on the \( B \to \rho \) form factors are also mentioned here. Section 6 is devoted to sum rules for the strong \( B^*B\pi \) and \( D^*D\pi \) couplings. The estimates are used to normalize the single-pole approximation for \( f^+ \) and to calculate the width for \( D^* \to D\pi \). In section 7, we present the sum rule predictions on the decay widths and distributions for \( B \to \pi\bar{\nu}l \) and \( B \to \rho\bar{\nu}l \), and extract \( V_{ub} \) from the comparison with data. Results are also shown on \( D \to \pi\bar{\nu}l \). Section 8 deals with the heavy mass expansion of the sum rules for heavy-to-light form factors and couplings, and with the asymptotic scaling behaviour. Finally, in section 9 we discuss the factorization hypothesis for \( B \to J/\psi K \) and describe a first sum rule estimate of the nonfactorizable piece in the \( B \to J/\psi K \) amplitude.

\section{Decay constants of \( B \) and \( D \) mesons}

Leptonic decays of \( B \) mesons are induced by the weak annihilation process \( b\bar{u} \to l\bar{\nu}l \). The decay width
\[
\Gamma(B^- \to l^-\bar{\nu}_l) = \frac{G_F^2}{8\pi}|V_{ub}|^2m_Bm_l^2\left(1 - \frac{m_l^2}{m_B^2}\right)^2f_B^2
\]
is suppressed by three small parameters: the CKM matrix element \( V_{ub} \), the lepton mass \( m_l \) and the decay constant \( f_B \). While the factor \( m_l^2 \) is enforced by helicity conservation, \( f_B \) characterizes the size of the \( B \)-meson wave function at the origin. Usually, the decay constant is defined by the matrix element of the relevant weak axial-vector current:
\[
\langle 0 | \bar{q}\gamma_\mu\gamma_5 b | B \rangle = i f_B q_\mu ,
\]
\( \bar{q} \) carrying the flavour of the light quark constituent in the \( B \), and \( q_\mu \) being the \( B \) four-momentum. Later we shall use the equivalent definition in terms of the matrix element of the corresponding pseudoscalar current:
\[
m_b\langle 0 | \bar{q}i\gamma_5 b | B \rangle = m_B^2 f_B,
\]
\( m_b \) being the \( b \)-quark mass. Because of the strong suppression of leptonic \( B \) decays only the mode \( B \to \tau \nu_\tau \) has a chance to be measured. However, this task will not be easy, even at future \( B \) factories. For this and other reasons which will become clear in the course of this review, accurate theoretical predictions of \( f_B \) are indispensable.

The QCD sum rule estimate of \( f_B \) is based on an analysis of the two-point correlation function
\[
\Pi(q^2) = i \int d^4xe^{iqx} \langle 0 | T\{\bar{q}(x)i\gamma_5b(x),\bar{b}(0)i\gamma_5q(0)\} | 0 \rangle.
\]
By inserting a complete set of states with \( B \)-meson quantum numbers between the currents in (4) one obtains the hadronic representation
\[
\Pi(q^2) = \frac{\langle 0 | \bar{q}i\gamma_5 b | B \rangle \langle B | b i\gamma_5 q | 0 \rangle}{m_B^2 - q^2}.
\]
\[ + \sum_{h} \frac{<0 \mid \bar{q}i\gamma_5 b \mid h \rangle < h \mid \bar{b}i\gamma_5 q \mid 0 \rangle}{m_h^2 - q^2}. \]  

The decay constant \( f_B \) appears in the first term contributed by the ground state \( B \)-meson. The second term takes into account the higher resonances and non-resonant states. The result (5) can be rewritten in the form of a dispersion relation:

\[ \Pi(q^2) = \int_{m_B^2}^{\infty} \frac{\rho(s)ds}{s - q^2}, \]  

where the spectral density is given by

\[ \rho(s) = \delta(s - m_B^2) \frac{m_B^4 f_B^2}{m_b^2} + \rho^h(s) \Theta(s - s_0^h). \]  

Obviously, the \( \delta \)-function term on the r.h.s. of (7) represents the \( B \) meson, while \( \rho^h(s) \) and \( s_0^h \) are the spectral density and threshold energy squared of the excited resonances and continuum states, respectively. In general, the dispersion integral (6) is ultraviolet divergent. In order to make the integral finite, one subtracts a sufficient number of terms of the Taylor expansion of \( \Pi(q^2) \) at \( q^2 = 0 \). In the case at hand, two subtractions are needed:

\[ \Pi_f(q^2) = \Pi(q^2) - \Pi(0) - q^2 \Pi'(0) = q^4 \int_{m_B^2}^{\infty} \frac{\rho(s)ds}{s^2(s - q^2)}. \]  

Substitution of (7) in (8) yields

\[ \Pi_f(q^2) = \frac{m_B^4 f_B^2}{m_b^2(m_B^2 - q^2)} + \int_{s_0^h}^{\infty} \frac{\rho^h(s)ds}{s - q^2} - \Pi(0) - q^2 \Pi'(0). \]  

The subtraction terms can be removed by performing a Borel transformation with respect to \( q^2 \):

\[ \mathcal{B}_{M^2} \Pi(q^2) = \lim_{n \to M^2} \left( -\frac{q^2}{n} \right)^n \left( \frac{d}{dq^2} \right)^n \Pi(q^2) \equiv \Pi(M^2). \]  

With

\[ \mathcal{B}_{M^2} \left( \frac{1}{s - q^2} \right)^k = \frac{1}{(k - 1)!} \left( \frac{1}{M^2} \right)^{k-1} e^{-s/M^2} \]  

and

\[ \mathcal{B}_{M^2} (-q^2)^k = 0 \]  

for \( k \geq 0 \) one readily finds

\[ \Pi(M^2) = \Pi_f(M^2) = \frac{m_B^4 f_B^2}{m_b^2} e^{-m_b^2/M^2} + \int_{s_0^h}^{\infty} \rho^h(s) e^{-s/M^2} ds. \]  

We see that Borel transformation removes arbitrary polynomials in \( q^2 \) and suppresses the contributions from excited and continuum states exponentially relative to the ground-state contribution. The second point is actually the main motivation for this transformation.

At \( q^2 \ll m_b^2 \), below the poles and cuts associated with the resonances and continuum states, it is possible to calculate the correlation function \( \Pi(q^2) \) in terms of the quark and
gluon degrees of freedom in QCD. To this end, one expands the $T$-product of currents in \( (4) \) in a series of local operators $\Omega_d$ constructed from quark and gluon fields and normalized at the scale $\mu$:

$$
i \int d^4xe^{iqx} T\{\bar{q}(x)i\gamma_5b(x), \bar{b}(0)i\gamma_5q(0)\} = \sum_d C_d(q^2, \mu)\Omega_d(\mu)$$

(14)

with the sum running over the canonical dimensions $d$ of the operators. The lowest-dimensional operators with $d=0,3,4,5,6$ are given by

$$\Omega_d = 1, \bar{q}q, G^a_{\mu\nu}G^{a\mu\nu}, \frac{\lambda^a}{2} G^a_{\mu\nu}q, (\bar{q}\Gamma_r q)(\bar{q}\Gamma_s q) ,$$

(15)

respectively. Here, $\lambda^a$ are the usual $SU(3)$-colour matrices, and $G^a_{\mu\nu}$ is the gluon field strength tensor. $\Gamma_r$ denotes a certain combination of Lorentz and colour matrices. Note the absence of a colour-neutral, Lorentz-invariant operator with $d=2$.

The most important virtue of OPE is the possibility to separate long- and short-distance contributions to the correlation function

$$\Pi(q^2) = \sum_d C_d(q^2, \mu) \langle 0 \mid \Omega_d(\mu) \mid 0 \rangle .$$

(16)

While the strong interaction effects at momenta $k^2 > \mu^2$ are included in the coefficients $C_d(q^2, \mu)$, the effects at $k^2 < \mu^2$ are absorbed into the matrix elements of the operators $\Omega_d(\mu)$. Thus, if $\mu \gg \Lambda_{QCD}$ the Wilson coefficients depend only on short-distance dynamics, and can therefore be calculated in perturbation theory, while the long-distance effects are taken into account by the vacuum averages $\langle 0 \mid \Omega_d \mid 0 \rangle \equiv \langle \Omega_d \rangle$. These so-called condensates describe properties of the full QCD vacuum and are process-independent. At present, they can only be estimated in some crude approximations. For this reason, they are usually determined empirically by fitting sum rules to experimental data. It is essential for the whole approach that at $q^2 \ll m^2_q$ the expansion (16) can be cut off after a few terms. The reason is that the higher the dimension of $\Omega_d$, the more suppressed by inverse powers of $m^2_q - q^2$ is the corresponding Wilson coefficient $C_d$. Most applications in the past include the set of operators given in (15).

The short-distance coefficients $C_d$ are calculated from the diagrams depicted in Fig. 1. Specifically, the coefficient $C_0$ of the unit operator is obtained by contracting all quark fields in the correlation function \( (4) \), and inserting the free quark propagators

$$\langle 0 \mid T\{q_i(x)\overline{q}_j(0)\} \mid 0 \rangle = i\tilde{S}^\beta_{ij}(x) = \delta_{ij} \int \frac{d^4k}{i(2\pi)^4} e^{-ikx} \frac{\not{k} + m_q}{m^2_q - k^2} ,$$

(17)

where $i, j$ are colour indices. For light quarks we put $m_q = 0$. Diagrammatically, this approximation is represented by Fig. 1a. It constitutes the zeroth order approximation for the correlation function \( (4) \) in QCD perturbation theory. The result can be written in the form of a dispersion relation:

$$\Pi_{Fig.1a}(q^2) = C_0(q^2) = \frac{1}{\pi} \int_{m^2_q}^{\infty} ds \frac{\text{Im} C_0(s)}{s - q^2}$$

(18)
Figure 1: Diagrams determining the Wilson coefficients in the OPE of the two-point correlation function (4): $C_0$ (a), $C_3$ (b), $C_4$ (c,d), $C_5$ (b,e), $C_6$ (b,f). Solid lines denote quarks, dashed lines gluons, wavy lines external currents. Crosses indicate vacuum fields.

with

$$\text{Im} C_0(s) = \frac{3}{8\pi} \frac{(s - m_b^2)^2}{s}. \quad (19)$$

The subtraction procedure necessary to make the integral (18) finite is postponed in anticipation of the Borel transformation (10).

The coefficient $C_3$ of the $\bar{q}q$ operator in (16) is obtained by treating the light-quark fields in (4) as external vacuum fields and neglecting their momenta as compared to the momentum of the freely propagating off-shell $b$ quark. The corresponding diagram is shown in Fig. 1b. Substituting (17) in (4) one gets

$$\Pi_{\text{Fig. 1b}}(q^2) = \int d^4x e^{i qx} \langle 0 \mid \bar{q}(0)_{ia} q(0)_{j\beta} \mid 0 \rangle \{ \gamma_5 \tilde{S}_{ij}(x) \gamma_5 \} \alpha\beta, \quad (20)$$

where $\alpha, \beta$ are the spinor indices of the quark fields. Because of vanishing spin and colour of the vacuum,

$$\langle 0 \mid \bar{q}(0)_{ia} q(0)_{j\beta} \mid 0 \rangle = \frac{1}{12} \delta_{a\beta} \delta_{ij} \langle \bar{q}q \rangle. \quad (21)$$

Integration of (20) over $x$ then yields

$$\Pi_{\text{Fig. 1b}}(q^2) = C_3(q^2) \langle \bar{q}q \rangle \quad (22)$$

with

$$C_3(q^2) = m_b/(q^2 - m_b^2). \quad (23)$$

The more complicated calculation of the coefficients $C_{4,5,6}$ from the diagrams of Fig. 1 is discussed, for example, in [3].
Figure 2: Feynman diagrams of the $O(\alpha_s)$ correction to the Wilson coefficient $C_0(q^2)$.

The accuracy of the OPE can be further improved by including perturbative QCD corrections to the Wilson coefficients corresponding to hard gluon exchanges in the diagrams of Fig. 1. Most important are the $O(\alpha_s)$ effects on the coefficient $C_0$ \cite{3,6} given by the two-loop diagrams of Fig. 2. Adding these corrections to the imaginary part of $C_0$ given in \cite{19}, one has

$$\text{Im} C_0(s) = \frac{3}{8\pi} \left( \frac{s - m_b^2}{s} \right)^2 \left( 1 + \frac{4\alpha_s}{3\pi} f(s, m_b^2) \right),$$

(24)

where

$$f(s, m_b^2) = \frac{9}{4} + 2\text{Li}_2 \left( \frac{m_b^2}{s} \right) + \frac{s}{m_b^2} \ln \frac{s}{s - m_b^2} + \frac{3}{2} \ln \frac{m_b^2}{s - m_b^2} + \ln \frac{s}{s - m_b^2} + \frac{m_b^2}{s} \ln \frac{s - m_b^2}{m_b^2} + \frac{m_b^2}{s - m_b^2} \ln \frac{s}{s - m_b^2},$$

(25)

$\text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1 - t)$, and $m_b$ is the pole mass of the $b$ quark.

The complete result for the Borel-transformed correlation function \cite{1} given in \cite{7} reads

$$\Pi_{QCD}(M^2) = \frac{3}{8\pi^2} \int_{m_b^2}^{\infty} ds \left( s - m_b^2 \right)^2 \left( 1 + \frac{4\alpha_s}{3\pi} f(s, m_b^2) \right) \exp \left( -\frac{s}{M^2} \right) \left[ -m_b \langle \bar{q}q \rangle + \frac{1}{12} \left( \frac{\alpha_s}{\pi} G_{\mu\nu}^a G^{a\mu\nu} \right) - \frac{m_b}{2M^2} \left( 1 - \frac{m_b^2}{2M^2} \right) \langle \bar{q}q \rangle \right. \left. + \frac{16\pi\alpha_s}{27M^2} \left( 1 - \frac{m_b^2}{4M^2} - \frac{m_b^4}{12M^4} \right) \langle \bar{q}q \rangle^2 \right] \exp \left( -\frac{m_b^2}{M^2} \right).$$

(26)

Here, use has been made of the conventional parametrization for the quark-gluon condensate density:

$$\langle \bar{q} \sigma_{\mu\nu} \frac{\lambda^a}{2} G^{a\mu\nu} q \rangle = m_b^2 \langle \bar{q}q \rangle,$$

(27)

and of vacuum saturation reducing four-quark condensates to squares of quark condensates with known coefficients \cite{1}:

$$\langle \bar{q} \Gamma_r \bar{q} \Gamma_s q \rangle = \frac{1}{(12)^2} \left\{ (Tr\Gamma_r)(Tr\Gamma_s) - Tr(\Gamma_r\Gamma_s) \right\} \langle \bar{q}q \rangle^2.$$

(28)

Now, by equating the hadronic representation of $\Pi(M^2)$ given in \cite{13} with the QCD result \cite{24} one obtains an interesting sum rule which, however, does not yet allow to determine $f_B$. This is due to the unknown spectral density $\rho^h(s)$ associated with the excited and
continuum states. In order to proceed, one assumes quark-hadron duality and substitutes, in (13), the perturbative spectral density:

\[ \rho^h(s) \Theta(s - s_0^h) = \frac{1}{\pi} \text{Im} C_0(s) \Theta(s - s_0^B) . \]  

(29)

considering \( \sqrt{s_0^B} \) as an effective threshold parameter of the order of the mass of the first excited \( B \) resonance. Subtraction of the integral in (13) from the corresponding term in \( \Pi_{QCD}(M^2) \) then amounts to a simple change of the upper limit of integration, and yields the following sum rule for \( f_B \):

\[ f_B^2 m_B^4 = \frac{3m_b^2}{8\pi^2} \int_{s_0^B} ds \frac{(s - m_b^2)^2}{s} \left( 1 + \frac{4\alpha_s}{3\pi} f(s, m_b^2) \right) \exp \left( \frac{m_B^2 - s}{M^2} \right) \]

\[ + m_b^2 \left\{ -m_b \langle \bar{q}q \rangle \left( 1 + \frac{m_b^2}{2M^2} \left( 1 - \frac{m_b^2}{2M^2} \right) \right) + \frac{1}{12} \left( \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{a\mu\nu} \right) \right. \]

\[ - \frac{16\pi \alpha_s}{27} \left( \frac{m_b^4}{M^2} - \frac{m_b^2}{12M^4} \right) \right\} \exp \left( \frac{m_B^2 - m_b^2}{M^2} \right) . \]  

(30)

The uncertainty due to the crude subtraction procedure should not be too harmful because of the suppression of excited and continuum states after Borel transformation. In principle, since the l.h.s. of (30) is a measurable quantity, the scale dependence of the quark masses, the condensates and the running coupling on the r.h.s. must cancel. In practice, this can only be achieved approximately.

The \( b \)-quark mass appearing in the perturbative coefficient \( C_0(q^2, \mu) \) is defined to be the pole mass, while the choice of \( m_b \) in the leading-order coefficients of the higher-dimensional terms in (30) is arbitrary. Following the usual procedure we take the pole mass

\[ m_b = 4.7 \pm 0.1 \text{ GeV} \]  

(31)

everywhere in (30). The interval (31) covers the current estimates obtained from bottomonium sum rules [8].

Turning to the condensates, it is important to note that the combinations \( m_b \langle \bar{q}q \rangle \) and \( \left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{a\mu\nu} \right\rangle \) are renormalization-group invariant. The numerical value of the quark condensate density at \( \mu = O(1 \text{ GeV}) \) is obtained from the PCAC relation [1, 9]:

\[ \langle \bar{q}q \rangle (1 \text{ GeV}) = - \frac{f_\pi^2 m_\pi^2}{2(m_u + m_d)} \simeq -(240 \text{ MeV})^3 . \]  

(32)

From that and the central value of (31) converted to the running mass at 1 GeV, one finds

\[ m_b \langle \bar{q}q \rangle \simeq -0.084 \text{ GeV}^4 . \]  

(33)

The gluon condensate density is determined from charmonium sum rules [11]:

\[ \left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{a\mu\nu} \right\rangle \simeq 0.012 \text{ GeV}^4 . \]  

(34)

For the remaining parameters we take

\[ m_0^2 (1 \text{ GeV}) \simeq 0.8 \text{ GeV}^2 . \]  

(35)
as extracted from sum rules for light baryons \[10\], and
\[
\alpha_s \langle \bar{q} q \rangle^2 = 8 \cdot 10^{-5} \text{ GeV}^6 ,
\]
(36)
as given in \[7\]. The scale dependence of \(m_0^2 \langle \bar{q} q \rangle\) and \(\alpha_s \langle \bar{q} q \rangle^2\) is negligible. The numerical uncertainties on the condensates vary from 10 % to about 50 % or more. However, they influence the final result on \(f_B\) rather little.

This leaves us with the choice of scale in the running coupling \(\alpha_s\). As the average virtuality of the quarks in the correlator \[4\] is characterized by the Borel parameter \(M^2\), it is reasonable to take
\[
\mu^2 = O(M^2) .
\]
(37)
Finally, we have to determine the values of \(M^2\) for which the sum rule \(30\) can be trusted. On the one hand, \(M^2\) has to be small enough such that the contributions from excited and continuum states are exponentially damped and the approximation \(29\) is satisfactory. On the other hand, the scale \(M^2\) must be large enough such that higher-dimensional operators are suppressed and the OPE converges sufficiently fast. Consequently, there is at best a finite range of \(M^2\) in which a given sum rule is valid. Moreover, in this range the numerical result should be stable under variations of \(M^2\). Whether or not these requirements can be met has to be investigated for each case separately. For the sum rule \(30\) we have checked that with
\[
3 \text{ GeV}^2 \leq M^2 \leq 5.5 \text{ GeV}^2
\]
the necessary conditions are fulfilled: the quark-gluon condensate contributes less than 15 %, the contributions from the gluon condensate and from the four-quark operators are negligibly small, and the excited states and continuum contribute less than 30 %. Moreover, the change of \(f_B\) due to variation of \(M^2\) in the range \(38\) is indeed very small.

The value of \(f_B\) derived from the sum rule \(30\) is given in Table 1. The value changes from 210 MeV to 150 MeV when \(m_b\) is varied in the range \(31\) from 4.6 to 4.8 GeV, and at the same time \(s^B_0\) from 37 to 33 GeV\(^2\). The correlation between \(m_b\) and \(s^B_0\) improves the stability of the sum rule under variation of \(M^2\). As compared to the above uncertainty, the uncertainties from the condensates are negligible. Note that the effect of the \(O(\alpha_s)\) correction \(24\) to \(C_0\) is sizeable. Without this correction the value for \(f_B\) is much lower:
\[
\overline{f}_B \equiv f_B(\alpha_s = 0) = 140 \pm 30 \text{ MeV} .
\]
(39)
Also shown in Table 1 is the corresponding prediction on \(f_D\). It follows from \(30\) after replacing \(m_b\) by the pole mass \(m_c = 1.3 \pm 0.1 \text{ GeV}\) of the \(c\) quark, and \(s^B_0\) by the effective threshold \(s^D_0 = 6 \pm 1 \text{ GeV}^2\) in the \(D\)-meson channel. The allowed interval in the Borel mass is found to be \(1 \text{ GeV}^2 < M^2 < 2 \text{ GeV}^2\). For later use, we again quote the result without the \(O(\alpha_s)\) correction:
\[
\overline{f}_D \equiv f_D(\alpha_s = 0) = 170 \pm 20 \text{ MeV}.
\]
(40)
In addition, Table 1 allows comparing QCD sum rule estimates with lattice results and with the available experimental data. The mutual agreement, within the uncertainties of the two theoretical methods, is satisfactory. On the other hand, the data are just beginning to challenge theory.
| Method               | Ref.          | \( f_B \)  | \( f_{B_s} \) | \( f_D \)  | \( f_{D_s} \)  |
|---------------------|---------------|-------------|---------------|-------------|---------------|
| QCD sum rules       | this review   | 180 ± 30    | –             | 190 ± 20    | –             |
|                     | [11] \(^a\)  | 175         | 210           | 180 ± 10    | 220 ± 10      |
| Lattice             | [13]          | 175 ± 25    | 200 ± 25      | 205 ± 15    | 235 ± 15      |
|                     | [14]          | 180 ± 32    | 205 ± 35      | 221 ± 17    | 237 ± 16      |
|                     | [13]          | 172^{+27}_{-31} | 196^{+30}_{-35} | 191^{+19}_{-28} | 206^{+18}_{-28} |
| Experiment          | [16]          | –           | –             | –           | < 310         |
|                     | [17] \(^b\)  | –           | –             | –           | 241 ± 21 ± 30 |

\(^a\) update of results of [3, 7, 12] taking \( m_b = 4.67 \) GeV, \( s_0^B = 35 \) GeV\(^2\); and \( m_c = 1.3 \) GeV, \( s_0^D = 5.5 \) GeV\(^2\).

\(^b\) world average

It should be mentioned that the \( B \)-meson decay constant has been derived also from a two-point sum rule in the heavy quark limit [18] using HQET and including \( 1/m_Q \) corrections [19, 20]. In this approach, the scale in \( \alpha_s \) is lower than (37), of the order of the reduced mass in the \( B \)-meson bound state. This leads to a somewhat larger value for \( f_B \), close to the upper end of the range obtained from the sum rule (30). For more details, one may consult the original papers quoted above.

Finally, for the calculation of the \( B^*B\pi \) and \( D^*D\pi \) couplings in section 6, we will need the \( B^* \) and \( D^* \) decay constants. Since they can be estimated from similar two-point sum rules as \( f_B \) and \( f_D \), we briefly summarize the result here. The decay constant of the \( B^* \) is defined by the matrix element

\[
\langle 0 \mid \bar{q}\gamma_\mu b \mid B^* \rangle = m_{B^*} f_{B^*} \epsilon_\mu ,
\]

\( \epsilon_\mu \) being the \( B^* \) polarization vector. From the correlation function of the vector currents \( \bar{q}\gamma_\mu b \) and \( \bar{b}\gamma_\mu q \) similar to (4), and with the same approximations as in (30), one derives the
following sum rule:

\[
f^2_{B^*} m^2_{B^*} = \frac{1}{8\pi^2} \int_{m_b^2}^{s_B^*} ds \frac{(s - m_b^2)^2}{s} \left(2 + m_b^2\right) \left(1 + \frac{4\alpha_s}{3\pi} f^*(s, m_b^2)\right) \exp\left(\frac{m^2_{B^*} - s}{M^2}\right) \\
+ \left\{ -m_b \langle \bar{q} q \rangle \left(1 - \frac{m_b^2 m_b^2}{4M^4}\right) + C_4(M^2) \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{a\mu\nu} \right. \\
+ C_6(M^2) \alpha_s \langle \bar{q} q \rangle^2 \right\} \exp\left(\frac{m^2_{B^*} - m_b^2}{M^2}\right). \tag{42}\]

The explicit expressions for the coefficient \( f^* \) of the \( O(\alpha_s) \) correction, as well as the coefficients \( C_4 \) and \( C_6 \) of the numerically insignificant contributions from the gluon and four-quark condensates are given in [3]. The corresponding sum rule for \( f_{D^*} \) can be directly inferred from \( \text{(12)} \). For the parameters one may take the same values as in the case of the pseudoscalars. For later use we give below the leading-order results in which the \( O(42) \). For the parameters one may take the same values as in the case of the pseudoscalars.

\[
\bar{f}_{B^*} = f_{B^*}(\alpha_s = 0) = 160 \pm 30 \text{ MeV}, \tag{43}\]
\[
\bar{f}_{D^*} = f_{D^*}(\alpha_s = 0) = 240 \pm 20 \text{ MeV}. \tag{44}\]

### 3 Three-point sum rules for heavy meson form factors

We next turn to the more complicated matrix elements of hadronic transitions induced by weak currents. An important example is the \( B \rightarrow \pi \) transition. The relevant matrix element is parametrized by two independent form factors:

\[
\langle \pi(q) | \bar{u} \gamma_\mu b | B(p + q) \rangle = 2f^+(p^2)q_\mu + \left(f^+(p^2) + f^-(p^2)\right)p_\mu \tag{45}\]

with \( p + q, q \) and \( p \) being the \( B \) and \( \pi \) momenta, and the momentum transfer, respectively.

Generalizing the method employed in section 2 one starts from the following vacuum correlation function of three currents:

\[
T_{\alpha\mu}(p, q) = - \int d^4x d^4y e^{i p x + i q y} \langle 0 | T \left\{ \bar{d}(y) \gamma_\alpha \gamma_5 u(y), \bar{u}(x) \gamma_\mu b(x), \bar{b}(0) i \gamma_5 d(0) \right\} | 0 \rangle, \tag{46}\]

where the \( B \) and \( \pi \) meson states of \( \text{(15)} \) are replaced by the corresponding generating currents. Insertion of the complete sets of hadronic states carrying \( B \) and \( \pi \) quantum numbers, respectively, yields the double dispersion relation

\[
T_{\alpha\mu}(p, q) = \frac{m_B^2 f_B f_B \pi \alpha a}{m_b (m_b^2 - (p + q)^2)(m_B^2 - q^2)} \\
\left. + \int \int_{\Sigma_1} ds_1 ds_2 \rho^h_{\alpha\mu}(s_1, s_2, p^2) \frac{\rho^h_{\alpha\mu}(s_1, s_2, p^2)}{(s_1 - (p + q)^2)(s_2 - q^2)} \right. \\
+ P_1(q^2, p^2) \int_{\Sigma_1} ds_1 \rho^h_{\alpha\mu}(s_1, p^2) + P_2((p + q)^2, p^2) \int_{\Sigma_2} ds_2 \rho^h_{\alpha\mu}(s_2, p^2) \tag{47}
\]
In the above, the matrix elements (3), (43), and
\[ \langle 0 \mid \bar{d} \gamma_\alpha \gamma_5 u \mid \pi(q) \rangle = i f_\pi q_\alpha \] (48)
have been used. While the first term on the r.h.s. of (47) is the the B and \( \pi \) ground state contribution, the integral over the double spectral density \( \rho_{\alpha \mu}^h \) takes into account the contributions of higher resonances and continuum states in the \( B \) and \( \pi \) channels. \( \Sigma_{12} \) denotes the integration region in the \((s_1, s_2)\) plane. \( \Theta \)-functions defining the actual thresholds are implicitly contained in \( \rho_{\alpha \mu}^h \). The additional single dispersion integrals multiplied by polynomials \( P_{1,2} \) arise from subtractions, similar to the polynomial in (4). They vanish after Borel transformation.

\[ \begin{array}{ccc}
\text{(a)} & \text{(b)} & \text{(c)} \\
\text{(d)} & \text{(e)}
\end{array} \]

Figure 3: Diagrams determining the Wilson coefficients of the OPE of the three-point correlation function (46). The symbols are as in Fig. 1.

At \((p + q)^2 \ll m_b^2, q^2 \ll 0, \) and \( p^2 \leq m_b^2 - 2m_b \chi, \) \( \chi \) being a \( m_b \)-independent scale of order \( \Lambda_{QCD} \), the correlation function \( T_{\alpha \mu}(p, q) \) can be approximated by the first few terms in the OPE of the \( T \)-product of currents in (46), in analogy to (16):
\[ T^{\alpha \mu}(p, q) = \sum_d C_d^{\alpha \mu}(p, q, \mu) \langle \Omega_d(\mu) \rangle. \] (49)

The restriction on \( p^2 \) is necessary in order to stay sufficiently far away from the physical states in the \( \bar{b}u \) channel, most notably the \( B^* \). For \( d \leq 6 \) the short-distance coefficients \( C_d^{\alpha \mu} \) can be calculated from the diagrams shown in Fig. 3. The calculation follows the
procedure outlined in the previous section. However, the kinematics is more complicated because of the presence of two independent external four-momenta. The explicit expressions can be found in \[21, 22, 23\]. Decomposing $C_{d}^{\alpha\mu}$ as well as the spectral densities $\rho_{\alpha\mu}^{h}$ in the independent tensor structures,

$$
C_{d}^{\alpha\mu}(p, q) = C_{d}((p + q)^{2}, q^{2}, p^{2})q^{\mu}(2q + p)^{\alpha} + \ldots ,
$$

(50)

$$
\rho_{\alpha\mu}^{h}(s_{1}, s_{2}, p^{2}) = \rho^{h}(s_{1}, s_{2}, p^{2})q_{\alpha}(2q + p)_{\mu} + \ldots .
$$

(51)

and equating the corresponding invariant functions in \[17\] and \[19\], one obtains

$$
\frac{m_{b}^{2}f_{B}f_{\pi}f^{+}(p^{2})}{m_{b}^{2}B - (p + q)^{2}(m_{\pi}^{2} - q^{2})} + \int \int_{\Sigma_{12}} ds_{1}ds_{2} \frac{\rho^{h}(s_{1}, s_{2}, p^{2})}{(s_{1} - (p + q)^{2})(s_{2} - q^{2})} + \ldots 
$$

$$
= \sum_{d} C_{d}((p + q)^{2}, q^{2}, p^{2}, \mu)\langle \Omega_{d}(\mu) \rangle,
$$

(52)

where the ellipses denote the subtraction terms. Since light quarks are taken massless, we put $m_{\pi} = 0$ for consistency.

Analogously to \[23\], the double spectral density $\rho^{h}$ is approximated by

$$
\rho^{h}(s_{1}, s_{2}, p^{2}) = \frac{1}{\pi^{2}} \text{Im} s_{1}\text{Im} s_{2} C_{0}(s_{1}, s_{2}, p^{2})\Theta(s_{1} - s_{0}^{B})\Theta(s_{2} - s_{0}^{\pi}) ,
$$

(53)

where $C_{0}$ is the perturbative coefficient calculated from Fig. 3a, and $s_{0}^{\pi}$ is the effective threshold parameter in the $\pi$-meson channel. After Borel transformation \[14\] with respect to the variables $(p + q)^{2}$ and $q^{2}$ the subtraction terms disappear from \[12\] and the contributions from excited and continuum states become exponentially suppressed. The final sum rule for $f^{+}$ takes the following schematical form \[21, 22, 23, 24\]:

$$
f^{+}(p^{2}) = \frac{m_{b}}{m_{b}^{2}f_{B}f_{\pi}}\exp \left( \frac{m_{B}^{2}}{M_{1}^{2}} + \frac{m_{\pi}^{2}}{M_{2}^{2}} \right) \times \left\{ \frac{1}{\pi^{2}} \int \int_{\Sigma_{12}(s_{0}^{B}, s_{0}^{\pi})} ds_{1}ds_{2}\text{Im} s_{1}\text{Im} s_{2} C_{0}(s_{1}, s_{2}, p^{2}, \mu) \exp \left( -\frac{s_{1}}{M_{1}^{2}} - \frac{s_{2}}{M_{2}^{2}} \right) \right. 
$$

$$
+ \left. \sum_{d=3}^{6} C_{d}(M_{1}^{2}, M_{2}^{2}, p^{2}, \mu)\langle \Omega_{d}(\mu) \rangle \right\}
$$

(54)

with $M_{1}^{2}$ and $M_{2}^{2}$ being the Borel parameters in the $B$ and $\pi$ channels, respectively. The appearance of the threshold parameters $s_{0}^{B}$ and $s_{0}^{\pi}$ reflects the continuum subtraction. Similarly as for the decay constants, there may be important perturbative QCD corrections to the short-distance coefficients, in particular, to $C_{0}$. To our knowledge, the two-loop diagrams corresponding to hard-gluon exchanges in Fig. 3a have not yet been calculated.

The numerical estimate \[25\] for $f^{+}(p^{2})$ obtained from \[14\] is plotted in Fig. 4 in comparison with other predictions. The values for $m_{b}$, $s_{0}^{B}$, and the vacuum condensates are already stated in the previous section; for $f_{B}$ the leading-order estimate \[13\] is used for consistency. Furthermore, the pion decay constant is $f_{\pi} = 132$ MeV, while the threshold parameter $s_{0}^{\pi} = 0.75$ GeV$^{2}$ is inferred from the two-point sum rule \[10\] for the correlation function $\langle 0 | T\{\bar{\psi}_{a}\gamma_{5}\psi(x), \bar{\psi}_{\beta}\gamma_{5}\psi(0)\} | 0 \rangle$. 

12
Figure 4: $B \to \pi$ form factor $f^+$ calculated in different approaches: three-point sum rule [23] (dashed), quark model [26] (dotted), and light-cone sum rule [41, 43] (solid).

Similar three-point sum rules have been applied to a variety of hadronic problems including the pion form factor [27], radiative charmonium transitions [3, 28, 29], and rare $B$ decays [30, 31, 32]. However, there are theoretical difficulties which put some doubts on the reliability of this method. Most disturbing in the case of heavy-to-light form factors is the breakdown of the OPE in the heavy mass limit. More specifically, the coefficients of the subleading quark and quark-gluon condensate terms grow faster with $m_b$ than the coefficient of the leading unity operator, that is the perturbative contribution. In contrast, the heavy mass limit of the two-point sum rules for decay constants discussed in the previous section is completely well-behaved. A more detailed discussion on this issue can be found in [33, 34].

Although three-point sum rules still remain a useful calculational tool for selected problems, the above difficulties call for more consistent and reliable sum rule methods. One important development is explained in the next section.

4 Light-cone expansion

For processes where a light meson such as a $\pi$, $K$, or $\rho$ is involved, there is an interesting alternative to the short-distance OPE of vacuum-vacuum correlation functions in terms of condensates, namely the expansion of vacuum-meson correlators near the light-cone in terms of meson wave functions [35, 36, 37]. The latter are functions of the light-cone momentum fractions carried by the constituents of a given meson. Similarly as the deep-inelastic structure functions, the wave functions can be classified by the twist of the corresponding composite operators. While the light mesons are taken on mass shell, the heavy meson channels are treated in the usual way: choice of a generating current, contraction of the
heavy quark fields, dispersion relation, Borel transformation and continuum subtraction. The light-cone variant of QCD sum rules suggested in [38, 39, 40] allows to incorporate additional information about the Euclidean asymptotics of correlation functions in QCD for arbitrary external momenta. Moreover, it avoids the problems of the three-point sum rules mentioned at the end of the previous section.

Here, we explain the general idea using the \( B \to \pi \) transition element (45) as an example. The detailed derivation of the light-cone sum rules for \( f^+ \) and \( f^- \) [41, 42, 43] is postponed to the next section. The starting point is the vacuum-pion correlation function

\[
F_{\mu}(p, q) = i \int d^4xe^{ipx} \langle \pi(q)| T\{\bar{u}(x)\gamma_\mu b(x), \bar{b}(0)i\gamma_5d(0)\} | 0 \rangle \tag{55}
\]

Since the pion is on-shell, \( q^2 = m_\pi^2 \) vanishes in the chiral limit adopted throughout this discussion. For the momenta in the \( \bar{b}d \) and \( \bar{u}b \) channels we respectively require \( (p+q)^2 \ll m_b^2 \) and \( p^2 \leq m_b^2 - 2m_b\chi \), as before. Contracting the \( b \)-quark fields in (55) and inserting the free \( b \)-quark propagator (17), one gets

\[
F^{(a)}_{\mu}(p, q) = i \int \frac{d^4x d^4k}{(2\pi)^4(m_b^2-k^2)}e^{i(p-k)x} \left( m_b\langle \pi(q)|\bar{u}(x)\gamma_\mu\gamma_5d(0)|0\rangle + k^\nu\langle \pi(q)|\bar{u}(x)\gamma_\mu\gamma_\nu\gamma_5d(0)|0\rangle \right) . \tag{56}
\]

This contribution is diagrammatically depicted in Fig. 5a.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Diagrammatic representation of the correlation function (55): terms involving two-particle (a) and three-particle (b) wave functions. Solid lines represent quarks, dashed lines gluons, wavy lines external currents, and ovals light-cone wave functions of the pion.}
\end{figure}

Let us first focus on the matrix element

\[
\langle \pi(q)|\bar{u}(x)\gamma_\mu\gamma_5d(0)|0\rangle , \tag{57}
\]

and expand the bilocal quark-antiquark operator around \( x = 0 \):

\[
\bar{u}(x)\gamma_\mu\gamma_5d(0) = \sum_r \frac{1}{r!}\bar{u}(0)(\overleftrightarrow{D}\cdot x)^r\gamma_\mu\gamma_5d(0) . \tag{58}
\]
The matrix elements of the local operators can be written in the form
\[ \langle \pi(q) | \bar{u} \overset{\longrightarrow}{D}_{\alpha_1} \overset{\longrightarrow}{D}_{\alpha_2} \cdots \overset{\longrightarrow}{D}_{\alpha_r} \gamma_\mu \gamma_5 d | 0 \rangle = (i)^r q_\mu q_{\alpha_1} q_{\alpha_2} \cdots q_{\alpha_r} M_r + \ldots, \] (59)
where the ellipses stand for additional terms containing various combinations of the metric tensor \( g_{\alpha_i \alpha_k} \). Note that the lowest twist \( \frac{3}{2} \) in (59) is equal to 2. Substituting (58) and (59) in the first term of (56), integrating over \( x \) and \( k \), and comparing the result with (55), one obtains
\[ F(p^2, (p + q)^2) = i \frac{m_b}{m_b^2 - p^2} \sum_{r=0}^\infty \xi^r M_r, \] (60)
with
\[ \xi = \frac{2(p \cdot q)}{m_b^2 - p^2} = \frac{(p + q)^2 - p^2}{m_b^2 - p^2}. \] (61)

Now, one immediately encounters the following problem. If the ratio \( \xi \) is finite one must keep an infinite series of matrix elements in (60). All of them give contributions of the same order in the heavy quark propagator \( 1/(m_b^2 - p^2) \), differing only by powers of the dimensionless parameter \( \xi \). In other words, the expansion (58) is useful only in the case \( \xi \to 0 \), i.e., when \( p^2 \approx (p + q)^2 \), or equivalently when the momentum of the light meson vanishes. Under this condition, the series in (60) can be truncated after a few terms involving only a manageable number of unknown matrix elements \( M_r \). However, generally, when \( p^2 \neq (p + q)^2 \) one has to sum up the infinite series of matrix elements of local operators in some way.

One possible solution is provided by expanding the bilocal operator (58) near the light-cone. For the matrix element (57) the first term of this expansion is given by
\[ \langle \pi(q) | \bar{u}(x) \gamma_\mu \gamma_5 d(0) | 0 \rangle_{x^2=0} = -i q_\mu f_\pi \int_0^1 du \, e^{iuq \cdot x} \varphi_\pi(u) . \] (62)
The function \( \varphi_\pi(u) \) is known as the twist 2 light-cone wave function of the pion. It is normalized to unity, as can be seen by putting \( x = 0 \) in (62). As already mentioned, \( \varphi_\pi \) represents the distribution in the fraction of the light-cone momentum \( q_0 + q_3 \) of the pion carried by a constituent quark. The path-ordered gluon operator
\[ P exp \left\{ ig_s \int_0^1 d\alpha \, x_\mu A^\mu(\alpha x) \right\} \] (63)
necessary for gauge invariance of the matrix element (57), is unity in the light-cone gauge, \( x_\mu A^\mu = 0 \), assumed here. Substitution of (62) in (54) and integration over \( x \) and \( k \) yields
\[ F(p^2, (p + q)^2) = m_b f_\pi \int_0^1 \frac{du \, \varphi_\pi(u)}{m_b^2 - (p + uq)^2}. \] (64)
We see that the infinite series of matrix elements of local operators encountered before in (50) is effectively replaced by a wave function. If one expands (64) in \( q \),
\[ F(p^2, (p + q)^2) = m_b f_\pi \sum_{r=0}^\infty \frac{(2p \cdot q)^r}{(m_b^2 - p^2)^{r+1}} \int_0^1 du \, u^r \varphi_\pi(u), \] (65)

\[ \text{Twist is defined as the difference between the canonical dimension and the spin of a traceless and totally symmetric local operator.} \]
and compares the above expression with (60), one can directly read off the relation between the matrix elements $M_r$ defined in (59) and the moments of $\varphi_\pi(u)$:

$$M_r = -i f_\pi \int_0^1 du \; u^r \varphi_\pi(u).$$  \hspace{1cm} (66)

It should be noted that the pion wave function $\varphi_\pi(u)$ is a universal function. It is the same quantity which enters, for example, the $\pi^0\gamma^*\gamma^*$ form factor [35]. This universality is essential for the whole approach, similarly as the universality of the vacuum condensates for the short-distance sum rules.

Including the next-to-leading terms in $x^2$, the light-cone expansion of the matrix element (57) reads

$$\langle \pi(q)|\bar{u}(x)\gamma_\mu\gamma_5 d(0)|0\rangle = -iq_\mu f_\pi \int_0^1 du \; e^{iuqx}\left(\varphi_\pi(u) + x^2 g_1(u)\right)$$

$$+ f_\pi \left(\gamma_\mu \frac{x^2 q_\mu}{q^2} - 1\right) \int_0^1 du \; e^{iuqx} g_2(u),$$  \hspace{1cm} (67)

where $g_1$ and $g_2$ are twist 4 wave functions. Proceeding to the second term in (56) and using the relation

$$\gamma_\mu\gamma_\nu\gamma_5 = g_{\mu\nu}\gamma_5 - i\sigma_{\mu\nu}\gamma_5,$$  \hspace{1cm} (68)

one encounters two further matrix elements:

$$\langle \pi(q)|\bar{u}(x)i\gamma_5 d(0)|0\rangle = f_\pi \int_0^1 du \; e^{iuqx} \varphi_\pi^0(u)$$  \hspace{1cm} (69)

and

$$\langle \pi(q)|\bar{u}(x)\sigma_{\mu\nu}\gamma_5 d(0)|0\rangle = i(q_\mu x_\nu - q_\nu x_\mu) \frac{f_\pi \mu_\pi}{6} \int_0^1 du \; e^{iuqx} \varphi_\sigma(u)$$  \hspace{1cm} (70)

with $\mu_\pi = m_\pi^2/(m_u + m_d)$. Only the leading terms in the expansion are considered here. They have twist 3 and are parameterized by the wave functions $\varphi_\pi$ and $\varphi_\sigma$.

Beyond twist 2, one should also take into account the higher-order terms [14] resulting from the contraction of the $b$-quark fields in the correlator (55):

$$\langle 0|T\{b(x)\bar{b}(0)\}|0\rangle = i\hat{S}_{b}(x) - ig_s \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \int_0^1 dv \left[ \frac{1}{2} \left( \frac{k}{m_b^2 - k^2} \right) G^{\mu\nu}(vx)\sigma_{\mu\nu} + \frac{1}{m_b^2 - k^2} \right]$$

$$+ \frac{1}{m_b^2 - k^2} v x_\mu G^{\mu\nu}(vx)\gamma_\nu \right],$$  \hspace{1cm} (71)

where $\hat{S}_{b}$ is the free propagator given in (47), and $G_{\mu\nu} = G_{\mu\nu}^{c}\frac{\Lambda_c}{2}$ with $\text{tr}(\lambda^a\lambda^b) = 2\delta^{ab}$. Insertion of the gluonic term in the correlation function (59) yields the contribution represented by the diagram Fig. 5b:

$$F_{\mu}^{(b)}(p, q) = ig_s \int \frac{d^4 k d^4 x d v}{(2\pi)^4 (m_b^2 - k^2)} e^{i(p-k)x} \langle \pi|\bar{u}(x)\gamma_\mu \left(v x_\rho G^{\rho\lambda}(vx)\gamma_\lambda + \frac{1}{2} \left( \frac{k}{m_b^2 - k^2} \right) \right) \rangle \gamma_5 d(0)|0\rangle .$$  \hspace{1cm} (72)
Use of the identities
\[ \gamma_{\mu}\sigma_{\rho\lambda} = i(g_{\mu\rho}\gamma_\lambda - g_{\mu\lambda}\gamma_\rho) + \varepsilon_{\mu\rho\lambda\sigma}\gamma^\sigma\gamma_5 \] (73)
and
\[ \gamma_{\mu}\gamma_{\nu}\sigma_{\rho\lambda} = (\sigma_{\mu\lambda}g_{\nu\rho} - \sigma_{\mu\rho}g_{\nu\lambda}) + i(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \\
- \varepsilon_{\mu\nu\rho\lambda}\gamma_5 - i\varepsilon_{\nu\rho\lambda\sigma}g^{\alpha\beta}\sigma_{\mu\beta}\gamma_5 \] (74)
leads to the following matrix elements of quark-antiquark-gluon operators which bring three-particle wave functions into the game:

\[ \langle \pi|\bar{u}(x)\gamma_{\rho}\gamma_5g_{\alpha\beta}(vx)d(0)|0 \rangle = \int D\alpha_i \phi_{3\pi}(\alpha_i)e^{iqx(\alpha_1 + \alpha_2 + \alpha_3)} \]
(75)

\[ \langle \pi|\bar{u}(x)\gamma_{\mu}\gamma_5g_{\alpha\beta}(vx)d(0)|0 \rangle = \int D\alpha_i \phi_{2\perp}(\alpha_i)e^{iqx(\alpha_1 + \alpha_2 + \alpha_3)} \]
(76)

\[ \langle \pi|\bar{u}(x)\gamma_{\mu}\gamma_{\rho}\gamma_5g_{\alpha\beta}(vx)d(0)|0 \rangle = \int D\alpha_i \phi_{2\parallel}(\alpha_i)e^{iqx(\alpha_1 + \alpha_2 + \alpha_3)} \]
(77)

with \( \tilde{G}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\sigma\tau}G^{\sigma\tau} \), and \( D\alpha_i = d\alpha_1 d\alpha_2 d\alpha_3 (1 - \alpha_1 - \alpha_2 - \alpha_3) \). The wave function \( \phi_{3\pi}(\alpha_1, \alpha_2, \alpha_3) \) has twist 3, while \( \phi_{2\perp}, \phi_{2\parallel}, \tilde{\phi}_{2\perp} \) and \( \tilde{\phi}_{2\parallel} \) are all of twist 4. Gluons emitted from the light quark lines in Fig. 5a are effectively taken care of by the twist 3 and 4 wave functions as was shown in [39, 45, 46]. Components of the pion wave function with two extra gluons, or with an additional \( \bar{q}q \) pair are neglected.

Finally, a comment is in order concerning the perturbative corrections to the correlation function (75) from hard gluon exchanges. The first-order diagrams are shown in Fig. 6. Returning to the twist 2 approximation (64) of the invariant function \( F \), one can more generally write it as a convolution of a hard scattering amplitude \( T \) with the twist 2 wave function:

\[ F(p^2, (p + q)^2) = f_\pi \int_0^1 du \phi_{\pi}(u, \mu) T(p^2, (p + q)^2, u, \mu) \]
(78)

where \( T \) is given by

\[ T(p^2, (p + q)^2, u, \mu) = T_0(p^2, (p + q)^2, u) + \frac{\alpha_s C_F}{4\pi} T_1(p^2, (p + q)^2, u, \mu) + O(\alpha_s^2) \]
(79)
As well known, the renormalization procedure and the factorization of the collinear logarithms generated by gluon radiation induce scale dependences. In (78) we have chosen a common scale $\mu$, for simplicity.

In leading-order approximation (LO), the lowest-order amplitude

$$T_0(p^2, (p + q)^2, u) = \frac{m_b}{m_b^2 - p^2(1 - u) - (p + q)^2u}$$

is convoluted with the wave function obeying the Brodsky-Lepage evolution equation [35]:

$$d\varphi_\pi(u, \mu)/d\ln \mu = \int_0^1 dw V(u, w)\varphi_\pi(w, \mu)$$

with

$$V(u, w) = \frac{\alpha_s(\mu)C_F}{\pi} \left[ \frac{1 - u}{1 - w} \left( 1 + \frac{1}{u - w} \right) \Theta(u - w) + \frac{u}{w} \left( 1 + \frac{1}{w - u} \right) \Theta(w - u) \right]_+ ,$$

and

$$R(u, w)_+ = R(u, w) - \delta(u - w) \int_0^1 R(v, w)dv .$$
This equation effectively sums up the leading logarithms to all orders. It can be solved by expanding \( \varphi_\pi(u, \mu) \) in terms of Gegenbauer polynomials in which case the coefficients are multiplicatively renormalizable \[35\].

The next-to-leading order approximation (NLO) is given by the hard amplitude \( [39] \) including the first-order term \( T_1 \) \[47, 48\], together with the wave function \( \varphi_\pi(u, \mu) \) solving the evolution equation in next-to-leading order \[49\]. To date, the NLO program is completed only for the twist 2 contribution to the invariant amplitude \( F \). The relevant wave function parameters and correction terms are given in Appendices 1 and 2.

5 Light-cone sum rules for heavy-to-light form factors

In order to determine the \( B \to \pi \) form factors \( f^+(p^2) \) and \( f^-(p^2) \) from the correlation function \( [52] \), we employ a QCD sum rule with respect to the \( B \)-meson channel following essentially the same steps as in the derivation of the sum rule for \( f_B \) in section 2. The hadronic representation of \( (55) \) is obtained by inserting a complete set of intermediate states with \( B \)-meson quantum numbers:

\[
F_\mu(p, q) = \frac{\langle \pi | \bar{u} \gamma_\mu b | B \rangle \langle B | \bar{b} i \gamma_5 d | 0 \rangle}{m_B^2 - (p + q)^2} + \sum_h \frac{\langle \pi | \bar{u} \gamma_\mu b | h \rangle \langle h | \bar{b} i \gamma_5 d | 0 \rangle}{m_h^2 - (p + q)^2}.
\]  

Using the matrix elements \( [3] \) and \( [45] \), and representing the sum over excited and continuum states by a dispersion integral with the spectral density \( \rho^h \) \( \rho^h(p^2, s) \) \( \Theta(s - s_0^h) = \frac{1}{\pi} \Im F_{QCD}(p^2, s) \Theta(s - s_0^B) \), one obtains the following relations for the invariant amplitudes \( F \) and \( \tilde{F} \):

\[
F(p^2, (p + q)^2) = \frac{2m_B^2 f_B f^+(p^2)}{m_b (m_B^2 - (p + q)^2)} + \int_{s_0^B}^{\infty} \frac{\rho^h(p^2, s) ds}{s - (p + q)^2},
\]

\[
\tilde{F}(p^2, (p + q)^2) = \frac{m_B^2 f_B (f^+(p^2) + f^-(p^2))}{m_b (m_B^2 - (p + q)^2)} + \int_{s_0^B}^{\infty} ds \frac{\tilde{\rho}^h(p^2, s)}{s - (p + q)^2}.
\]

Similarly as in \( [29] \), the integrals over \( \rho^h \) and \( \tilde{\rho}^h \) are again approximated by integrals over the corresponding spectral densities calculated from the light-cone expansion of \( (55) \) using

\[
\rho^h(p^2, s) \Theta(s - s_0^h) = \frac{1}{\pi} \Im F_{QCD}(p^2, s) \Theta(s - s_0^B),
\]

and the analogous relation for \( \tilde{\rho}_h \) and \( \Im \tilde{F}_{QCD} \). The calculation of \( F_{QCD} \) and \( \tilde{F}_{QCD} \) has already been outlined in the preceding section. With \( (77) \) it is now straightforward to subtract the contribution of the excited and continuum states from the corresponding integrals on the l.h.s. of \( (85) \) and \( (86) \). After performing the Borel transformation in \( (p + q)^2 \), one finally arrives at the sum rules

\[
f_B f^+(p^2) = \frac{m_b}{2\pi m_B^2} \int_{s_0^B}^{\infty} \Im F_{QCD}(p^2, s) \exp \left( \frac{m_B^2 - s}{M^2} \right) ds,
\]

19
\begin{equation}
f_B(f^+(p^2) + f^-(p^2)) = \frac{m_b}{\pi m_B^2} \int_0^s \text{Im} \tilde{F}_{QCD}(p^2, s) \exp \left( \frac{m_B^2 - s}{M^2} \right) ds.
\end{equation}

From the above, the form factor $f^-$ follows directly by subtraction, while the scalar form factor $f^0$ is given by

\begin{equation}
f^0(p^2) = f^+(p^2) + \frac{p^2}{m_B^2 - m_\pi^2} f^-(p^2).
\end{equation}

The remaining task is to complete the calculation of $F_{QCD}$ and $\tilde{F}_{QCD}$, and to determine the imaginary parts. We proceed along the lines explained in section 4. The contribution of two-particle wave functions is derived from the diagram Fig. 5a inserting the matrix elements (67), (69), and (70) in (56). This results in

\begin{equation}
F^{(a)}_{QCD}(p^2, (p + q)^2) = f_\pi \int_0^1 \frac{du}{m_\pi^2 - (p + uq)^2} \left\{ m_\pi \varphi_\pi(u) + \mu_\pi \left[ u \varphi_p(u) + \frac{1}{6} \left( 2 + \frac{m_\pi^2}{m_\pi^2 - (p + uq)^2} \right) \varphi_\sigma(u) \right] + m_b \left[ \frac{2uq_2(u)}{m_\pi^2 - (p + uq)^2} - \frac{8m_b^2}{m_\pi^2 - (p + uq)^2} \right] \left( g_1(u) - \int_0^u dv g_2(v) \right) \right\},
\end{equation}

\begin{equation}
\tilde{F}^{(a)}_{QCD}(p^2, (p + q)^2) = f_\pi \int_0^1 \frac{du}{m_\pi^2 - (p + uq)^2} \left\{ \mu_\pi \varphi_\pi(u) + \frac{\mu_\pi \varphi_\sigma(u)}{6u} \right\} \times \left[ 1 - \frac{m_\pi^2 - p^2}{m_\pi^2 - (p + uq)^2} + \frac{2m_b q_2(u)}{m_\pi^2 - (p + uq)^2} \right].
\end{equation}

Furthermore, the three-particle contribution is obtained from Fig. 5b using (72) and the matrix elements (75) to (77):

\begin{equation}
F^{(b)}_{QCD}(p^2, (p + q)^2) = \int_0^1 dv \int D\alpha_i \left\{ \frac{4f_3 \varphi_3 \varphi_3 (\alpha_i) \psi (pq)}{[m_\pi^2 - (p + (\alpha_1 + v\alpha_3)q)]^2} \right\} + m_b f_\pi \frac{2\varphi_\perp (\alpha_i) - \varphi_\parallel (\alpha_i) + 2\varphi_\perp (\alpha_i) - \varphi_\parallel (\alpha_i)}{[m_\pi^2 - (p + (\alpha_1 + v\alpha_3)q)]^2},
\end{equation}

\begin{equation}
\tilde{F}^{(b)}_{QCD}(p^2, (p + q)^2) = 0.
\end{equation}

It is interesting to note that there are no contributions from twist 2 and three-particle wave functions to $\tilde{F}_{QCD}$, and hence to $f^+ + f^-$. Including all operators up to twist 4 one has in total

\begin{equation}
F_{QCD}(p^2, (p + q)^2) = F^{(a)}_{QCD}(p^2, (p + q)^2) + F^{(b)}_{QCD}(p^2, (p + q)^2),
\end{equation}

\begin{equation}
\tilde{F}_{QCD}(p^2, (p + q)^2) = \tilde{F}^{(a)}_{QCD}(p^2, (p + q)^2).
\end{equation}

We see that every power of $x^2$ in the light-cone expansion of the integrand in (53) leads to an additional power of the denominator $m_b^2 - (p + uq)^2$ in $F_{QCD}$ and $\tilde{F}_{QCD}$. This justifies
the neglect of higher-twist operators, provided \((p + q)^2 \ll m_b^2\) and \(p^2 < m_b^2\) by a few GeV\(^2\). More definitely, numerical studies indicate that contributions beyond twist 4 can safely be neglected.

The imaginary parts of the two-particle functions \(F_{QCD}^{(a)}\) and \(\tilde{F}_{QCD}^{(a)}\) are relatively easy to find. Terms proportional to \(1/(m_b^2 - (p + uq)^2)\) can be directly converted into dispersive integrals with respect to \((p + q)^2\) by changing the variable \(u\) into \(s = (m_b^2 - p^2)/u + p^2\).

Subtraction of the continuum simply shifts the lower limit of integration in (91), and in (92) from 0 to \(\Delta = (m_b^2 - p^2)/(s_0^B - p^2)\). Terms proportional to higher powers of \(1/(m_b^2 - (p + uq)^2)\) need to be partially integrated. After continuum subtraction, this leads to additional surface terms which are not completely negligible. A similar procedure is possible in the case of the three-particle function \(F_{QCD}^{(b)}\). However, since this contribution is anyway only a small correction, we neglect the corresponding surface term, for simplicity.

In summary, (88), (91), (93), and (95) yield the following sum rule for \(f^+\), or more precisely, for the product \(f_B f^+\)

\[
f_B f^+(p^2) = \frac{f_\pi m_b^2}{2m_B^2} \exp \left( \frac{m_B^2}{M^2} \right) \left\{ \int \frac{du}{u} \exp \left[ - \frac{m_b^2 - p^2(1 - u)}{uM^2} \right] \right. \\
\times \left( \varphi_\pi(u) + \frac{\mu_\pi}{m_b} u \varphi_p(u) + \frac{\varphi_\sigma(u)}{3} \left( 1 + \frac{m_b^2 + p^2}{2uM^2} \right) \right) \\
- \frac{4m_b^2 g_1(u)}{u^2 M^4} + \frac{2}{uM^2} \int_0^u g_2(v) dv \left( 1 + \frac{m_b^2 + p^2}{uM^2} \right) \right\} + t^+(s_0^B, p^2, M^2) \\
+ f_G^+(p^2, M^2) + \frac{\alpha_s C_F}{4\pi} \delta^+(p^2, M^2) \right\}. \tag{97}
\]

Here, \(t^+\) denotes the surface term just mentioned, \(f_G^\pm\) the contribution from the three-particle wave functions, and \(\delta^+\) the \(O(\alpha_s)\) correction to the leading twist 2 term. Explicit expressions can be found in Appendix 2. Similarly, the sum rule for \(f^+ + f^-\)

is obtained from (89) and (90):

\[
f_B(f^+(p^2) + f^-(p^2)) = \frac{f_\pi m_b^2}{m_B^2} \exp \left( \frac{m_B^2}{M^2} \right) \left\{ \int \frac{du}{u} \exp \left[ - \frac{m_b^2 - p^2(1 - u)}{uM^2} \right] \right. \\
\times \left[ \varphi_p(u) + \frac{\varphi_\sigma(u)}{6u} \left( 1 - \frac{m_b^2 - p^2}{uM^2} \right) + \frac{2m_b g_2(u)}{\mu_\pi uM^2} \right] + t^\pm(s_0^B, p^2, M^2) \right\} \tag{98}
\]

with the surface term \(t^\pm\) being given in Appendix 2. Perturbative QCD corrections to this sum rule are still missing.

In the following, we shall illustrate the above results numerically. For this purpose, it suffices to work in LO. Strictly speaking, in the absence of a complete NLO analysis this is also more consistent. Furthermore, for \(m_b\) and \(f_B\) we choose the values given in (91) and (93), respectively, for the threshold parameter we take \(s_0^B = 35 \pm 2\) GeV\(^2\). The pion wave functions are listed in Appendix 1. With this input we have checked that for \(M^2 = 10 \pm 2\) GeV\(^2\) the twist 4 corrections are very small, and the contributions from excited
and continuum states do not exceed 30%. Up to \( p^2 \simeq 18 \text{ GeV}^2 \), the sum rules are also quite stable with respect to a variation of \( M^2 \). However, at larger momentum transfer the sum rules become unstable, and the twist 4 contribution grows rapidly. This is not surprising because the light-cone expansion and the sum rule method are expected to break down as \( p^2 \) approaches \( m_b^2 \). If not stated otherwise, the numerical results given below are obtained from the central values of the input parameters, which we refer to as our nominal choice.

The momentum dependence of the form factors \( f^+ \) and \( f^+ + f^- \) can be seen in Fig. 7. Specifically, at zero momentum transfer we predict

\[
\begin{align*}
    f^+(0) &= 0.30, \\
    f^+(0) + f^-(0) &= 0.06. \\
\end{align*}
\]

If the known \( O(\alpha_s) \) corrections are included in the sum rules for \( f_B f^+(p^2) \) and \( f_B \), one gets instead

\[
\begin{align*}
    f^+(0) &= 0.27. \\
\end{align*}
\]

Of course, a most important question concerns the reliability of these predictions. Below we comment on the main sources of uncertainties. Allowing the parameters to vary within the ranges given above and in Appendix 1 we use the deviations of \( f^+ \) from the value obtained with the nominal choice of parameters as an uncertainty estimate.

(a) Borel mass parameter

The dependence of \( f^+ \) on \( M^2 \) is illustrated in Fig. 8a. In the allowed interval of \( M^2 \), \( f^+ \) varies by only \( \pm(3 \text{ to } 5) \% \), depending on \( p^2 \).

(b) \( b \)-quark mass and continuum threshold

Fig. 8b shows the variation of \( f^+ \) with \( m_b \) keeping all other parameters except \( f_B \) fixed. The value of \( f_B \) is taken from the sum rule (30) dropping the \( O(\alpha_s) \) corrections. The analogous

\[p^2[\text{GeV}^2]
\]

**Figure 7:** \( B \to \pi \) form factors obtained from light-cone sum rules in LO.
Figure 8: Variation of the sum rule prediction on $f^+(p^2)$ with the Borel parameter (a), the $b$-quark mass (b), and the threshold parameter $s_0^B$ (c). Considered are three typical values of momentum transfer: $p^2 = 0$ (solid), $p^2 = 8 \text{ GeV}^2$ (long-dashed), and $p^2 = 16 \text{ GeV}^2$ (short-dashed).

test for $s_0^B$ is performed in Fig. 8c. If $s_0^B$ and $m_b$ are varied simultaneously such that one achieves maximum stability of the sum rule for $f_B$, the change in $f^+$ is negligible at small $p^2$ rising to about $\pm 3\%$ at large $p^2$.

(c) higher-twist contributions

No reliable estimates exist for wave functions beyond twist 4. Therefore, we use the magnitude of the twist 4 contribution to $f^+$ as an indicator for the uncertainty due to the neglect higher-twist terms. From Fig. 9a we see that the impact of the twist 4 components is comfortably small, less than $2\%$ at low $p^2$ and about $5\%$ at large $p^2$. This suggests a conservative estimate of $\pm 5\%$ due to unknown higher-twist.

(d) light-cone wave functions

The asymptotic wave functions and the scale dependence of the nonasymptotic coefficients are given in perturbative QCD. However, the values of these coefficients at a certain input
Figure 9: $B \rightarrow \pi$ form factor $f^+$: (a) individual contributions of twist 2 (dashed), 3 (dash-dotted), and 4 (dotted), and the total sum (solid); (b) wave functions with (solid) and without (dashed) nonasymptotic corrections.
scale $\mu_0$ have to be determined empirically. They are presently only known with considerable uncertainties. In order to clarify the sensitivity of $f^+$ to nonasymptotic effects, we put the latter to zero and compare the result in Fig. 9b with the nominal prediction. The change amounts to about $-10\%$ at small $p^2$ and $+10\%$ at large $p^2$, while the intermediate region around $p^2 \approx 10 \text{ GeV}^2$ is very little affected. Since this exercise is rather extreme, a more careful estimate should give a smaller number. Note that the sum rule being dominated by convolutions of smooth amplitudes with normalized wave functions, is actually expected to be relatively insensitive to the precise shape of the latter.

![Figure 10: $B \to \pi$ form factor $f^+(p^2)$ in leading twist 2 approximation: LO (dashed) in comparison to NLO (solid).](image)

(e) perturbative corrections

The $O(\alpha_s)$ corrections to the leading twist 2 term in the sum rule (97) for $f_B f^+$ and to the sum rule (30) for $f_B$, both being about 30%, cancel in the ratio. The net effect is the unusually small correction to $f^+$ shown in Fig. 10. This result [47, 48] eliminates one main uncertainty. Unfortunately, perturbative corrections to the higher-twist terms have not yet been calculated. Considering that the twist 3 terms contribute about 50% to the sum rule for $f_B f^+$, and assuming again radiative corrections of about 30% but no cancellation, one has to face a remaining uncertainty of about 15%. This, however, may be an overestimate. More importantly, with some major effort this deficiency can be cured.

In summary, the present total uncertainty in $f^+(p^2)$ is estimated to be about 25%, if the uncertainties (a) to (e) with the exception of (d) are added up linearly, and about 17% if they are added in quadrature as is often done in the literature. Once the perturbative QCD correction to the twist 3 term is calculated, this uncertainty, which mainly concerns the normalization, can be reduced to 10%. In addition, there is a shape-dependent uncertainty from (d) of another 10% at low and high $p^2$. However, in the integrated width the latter
averages out almost completely. The uncertainties from (a) to (d) on \( f^+ + f^- \) are of comparable size, while the effect of radiative corrections is still unknown.

The sum rules (97) and (98) for the \( D \to \pi \) form factors are formally converted into sum rules for the \( D \to \pi \) form factors by replacing \( b \) with \( c \) and \( B \) with \( D \). The input parameters are taken over from the calculation of \( f_B \) in section 2. In addition, one has to rescale the wave functions from \( \mu_b \simeq 2.4 \text{ GeV} \) to \( \mu_c \simeq 1.3 \text{ GeV} \) as specified in Appendix 1. The allowed Borel mass window is \( 3 \text{ GeV}^2 \leq M^2 \leq 5 \text{ GeV}^2 \). With this choice, the \( D \to \pi \) form factor at \( p^2 = 0 \) is predicted to be \( 0.68 \).

\[ O(\alpha_s) \text{ corrections are not included here. The momentum dependence of } f^+ \text{ in the range } 0 \leq p^2 \leq m_c^2 - O(1 \text{ GeV}^2) \text{ is shown in the next section, together with an extrapolation to higher } p^2. \]

Other important applications of light-cone sum rules include the estimate (31) of the \( B \to K \) form factor which determines the factorizable part of the \( B \to J/\psi K \) amplitude, the prediction (34) of the matrix element of the electromagnetic penguin operator for \( B \to K^* \gamma \), and the more recent calculation (33) of the \( B \to \rho \) form factors. In the latter work, essentially the same procedure is applied as outlined above. However, the relevant weak currents and light-cone wave functions are very different. For illustration, one has to deal with the following matrix elements (33, 34, 37):

\[
\langle \rho^+(q, \lambda) | \bar{u}(0) f_{\mu} d(x) | 0 \rangle = -i f^+_{\rho}(\lambda)_{\mu} q_{\nu} - \epsilon(\lambda)_{\nu} q_{\mu} \int_0^1 du e^{iux} \phi_{\perp}(u),
\]

\[
\langle \rho^+(q, \lambda) | \bar{u}(0) \gamma_\mu d(x) | 0 \rangle = q_{\mu} \epsilon(\lambda) x f_{\rho} m_{\rho} \int_0^1 du e^{iux} \phi_{\parallel}(u)
\]

\[
+ \left( \epsilon(\lambda)_{\mu} - q_{\mu} \frac{\epsilon(\lambda) x}{q x} \right) f_{\rho} m_{\rho} \int_0^1 du e^{iux} g_{\perp}^{(v)}(u),
\]

\[
\langle \rho^+(q, \lambda) | \bar{u}(0) \gamma_\mu \gamma_5 d(x) | 0 \rangle = \frac{1}{4} \epsilon_{\rho\sigma\tau\nu} \epsilon(\lambda)^{\nu} q^\rho x^\sigma f_{\rho} m_{\rho} \int_0^1 du e^{iux} g_{\perp}^{(a)}(u),
\]

Here, \( \phi_{\perp} \) and \( \phi_{\parallel} \) are twist 2 wave functions of transversely and longitudinally polarized \( \rho \) mesons, respectively, while \( g_{\perp}^{(v)} \) and \( g_{\perp}^{(a)} \) are associated with both twist 2 and twist 3 operators. Higher-twist components and three-particle wave functions are still missing, as are perturbative corrections.

There are four independent \( B \to \rho \) form factors:

\[
\langle \rho(\lambda) | \bar{u} \gamma_\mu (1 - \gamma_5) b | B \rangle = -i(m_B + m_\rho) A_1(t) \epsilon(\lambda)_\mu + \frac{i A_2(t)}{m_B + m_\rho} (\epsilon(\lambda) p_B)(p_B + p_\rho)_\mu
\]

\[
+ \frac{i A_3(t)}{m_B + m_\rho} (\epsilon(\lambda) p_B)(p_B - p_\rho)_\mu
\]

\[
+ \frac{2V(t)}{m_B + m_\rho} \epsilon_\mu^\alpha \beta^\gamma (\epsilon(\lambda)_{\alpha p_B \beta p_\rho})_{\gamma}.
\]

\[ f^+(0) = 0.68. \]
with \( t = (p_B - p_\rho)^2 \). Only \( V(t) \), \( A_1(t) \) and \( A_2(t) \) contribute to the semileptonic decay \( B \to \rho l \nu_l \) for \( l = e, \mu \). These form factors are shown in Fig. 11. They all are predicted to rise with momentum transfer \( t \), contrary to some earlier claims based on three-point sum rules. This contradiction was studied in detail in [33] with the conclusion that the three-point sum rules are not reliable for reasons indicated in section 4.

6 \( B^*B\pi \) and \( D^*D\pi \) couplings

The strong couplings of \( B \) and \( D \) mesons to pions have been studied with different variants of sum rules and in a variety of quark models. We shall use the following definition of the \( B^*B\pi \) coupling constant:

\[
\langle \bar{B}^{*0} \pi^- | B^- \rangle = -g_{B^*B\pi}(q \cdot \epsilon) \quad (106)
\]

The couplings of the different charge states are related by isospin symmetry:

\[
g_{B^*B\pi} \equiv g_{B^{*0}B^-\pi^0} = -\sqrt{2}g_{B^*B_-\pi^0} = \sqrt{2}g_{B^*-B^0\pi^0} = -g_{B^*-B^0\pi^-} \quad (107)
\]

In [42], a light-cone sum rule for \( g_{B^*B\pi} \) has been suggested which is derived from the same correlation function [54] as the corresponding \( B \to \pi \) form factors and which depends on the same nonperturbative input. The key idea is to write a double dispersion integral for the invariant function \( F(p^2, (p + q)^2) \). Inserting in (105) complete sets of intermediate hadronic states carrying \( B \) and \( B^* \) quantum numbers, respectively, and using the matrix elements (9), (11) and (106), one obtains

\[
F(p^2, (p + q)^2) = \frac{m_B^2 m_{B^*} f_{B^*} f_{B^{-}B^*-\pi^0}}{m_b(p^2 - m_{B^*}^2)((p + q)^2 - m_B^2)} + \int \frac{\rho^h(s_1, s_2)ds_1ds_2}{(s_1 - p^2)(s_2 - (p + q)^2)} \quad (108)
\]

\[
+ (\text{subtractions}) .
\]

Again, the first term is the ground-state contribution and contains the \( B^*B\pi \) coupling, while the spectral function \( \rho^h(s_1, s_2) \) represents higher resonances and continuum states in the \( B^* \) and \( B \) channels. The integration region in the \((s_1, s_2)\) - plane is denoted by \( \Sigma \). The subtraction terms are polynomials in \( p^2 \) and/or \((p + q)^2\) which vanish by Borel transformation of (108) with respect to both variables \( p^2 \) and \((p + q)^2\). The transformed hadronic representation of \( F \) is given by

\[
F(M_1^2, M_2^2) \equiv B_{M_1}B_{M_2}F(p^2, (p + q)^2) = \frac{m_B^2 m_{B^*} f_{B^*} f_{B^{-}B^*-\pi^0}}{m_b} \exp \left[ \frac{m_{B^*}^2 - m_B^2}{M_1^2 - M_2^2} \right] \\
+ \int_{\Sigma_{12}} \exp \left[ -\frac{s_1}{M_1^2} - \frac{s_2}{M_2^2} \right] \rho^h(s_1, s_2)ds_1ds_2 , \quad (109)
\]

where \( M_1^2 \) and \( M_2^2 \) are the Borel parameters associated with \( p^2 \) and \((p + q)^2\), respectively. The same transformation is also applied to (105) yielding \( F_{QCD}(M_1^2, M_2^2) \). Then, the sum rule for \( g_{B^*B\pi} \) results from the equality of \( F \) and \( F_{QCD} \), and continuum subtraction. This last step deserves some more explanations.

Let us consider the twist 2 expression for \( F \) given in (104). In order to write it in the form of a double dispersion relation,

\[
F_{QCD}(p^2, (p + q)^2) = \int_{m_b^2}^{\infty} \frac{ds_1}{s_1 - p^2} \int_{m_b^2}^{\infty} \frac{ds_2}{s_2 - (p + q)^2} \rho^{QCD}(s_1, s_2) , \quad (110)
\]
Figure 11: The $B \rightarrow \rho$ form factors: predictions from light-cone sum rules (solid) in comparison to lattice results [13]. The dashed curves indicate the uncertainties in the sum rule results (from [33]).
we change \( u \) to \( s = (m_b^2 - p^2)/u + p^2 \), and get

\[
F_{QCD}(p^2, (p + q)^2) = m_b f_\pi \int_{m_b^2}^{\infty} \frac{ds}{s - p^2} \frac{\varphi_\pi(s)}{(s - (p + q)^2)} .
\] (111)

In general, the wave function \( \varphi_\pi(u) \) can be expressed as a power series in \((1 - u)\):

\[
\varphi_\pi(u) = \sum_k a_k (1 - u)^k = \sum_k a_k \left( \frac{s - m_b^2}{s - p^2} \right)^k .
\] (112)

Substituting (112) in (111) and introducing formally two variables \( s_1 \) and \( s_2 \) instead of \( s \), one reproduces the double integral (110) with

\[
\rho^{QCD}(s_1, s_2) = m_b f_\pi \sum_k \frac{(-1)^k a_k}{\Gamma(k + 1)} (s_1 - m_b^2)^k \delta^{(k)}(s_1 - s_2) .
\] (113)

Now we Borel-transform (110),

\[
F_{QCD}(M_1^2, M_2^2) = m_b f_\pi \sum_k \int_{m_b^2}^{\infty} ds_1 \int_{m_b^2}^{\infty} ds_2 \frac{(-1)^k a_k}{\Gamma(k + 1)} (s_1 - m_b^2)^k \times \delta^{(k)}(s_1 - s_2) \exp \left[ -\frac{s_1}{M_1^2} - \frac{s_2}{M_2^2} \right] .
\] (114)

introduce again new variables \( s = s_1 + s_2 \) and \( v = s_1/s_2 \), and use the \( \delta \)-function to integrate over \( v \). The result is

\[
F_{QCD}(M_1^2, M_2^2) = m_b f_\pi \sum_k \frac{a_k}{2^{k+1} k!} \int_{2m_b^2}^{\infty} ds \left( \frac{d}{dv} \right)^k \left( v - \frac{m_b^2}{s} \right)^k \times \exp \left[ -s v M_2^2 + s(1 - v) M_1^2 \right] \left( M_1^2 M_2^2 \right) \right]_{v=1/2} .
\] (115)

For \( M_1^2 = M_2^2 = 2M^2 \) the \( v \)-dependence of the exponent disappears, and the differentiation of the bracket gives a factor \( k! \). As a consequence, (113) reduces to

\[
F_{QCD}(M^2, M^2) = m_b f_\pi \sum_k \frac{a_k}{2^{k+1} k!} \int_{2m_b^2}^{\infty} ds \, e^{-s M^2} = m_b f_\pi \varphi_\pi(u_0) M^2 e^{-m_b^2/4M^2} .
\] (116)

with \( u_0 = 1/2 \). For arbitrary values of \( M_1^2 \) and \( M_2^2 \) a similar expression is obtained, with \( u_0 = M_1^2/(M_1^2 + M_2^2) \) and \( M^2 = M_1^2 M_2^2/(M_1^2 + M_2^2) \).

With these manipulations, and the replacement of the integral over \( \rho^h \) in (109) by a corresponding integral over \( \rho^{QCD} \), it is straightforward to subtract the contributions from excited and continuum states from (113). The remaining integral is restricted to the region below a given boundary in \((s_1, s_2)\). For the latter one may take

\[
(s_1)^a + (s_2)^a \leq (s_0)^a .
\] (117)

For \( a = 1 \), the duality region is a triangle in the \((s_1, s_2)\) - plane, while for \( a \to \infty \) it is a square. Since the spectral density (113) vanishes everywhere except at \( s_1 = s_2 \), it is actually
irrelevant which form of the boundary one chooses provided the length of the duality interval at $s_1 = s_2$ is the same. For example, the triangle with $s_0 = 2s_0^B$ is equivalent to the square with $s_0 = s_0^B$. Here, we take $s_0^B$ as the effective threshold in both the $B$ and $B^*$ channels. Repeating the steps following (114) one obtains an expression similar to (115), but with the upper limit of integration in $s$ lowered to $2s_0^B$ and with the addition of surface terms. The latter disappear for $M_1^2 = M_2^2$, in which case the subtracted invariant function replacing (116) is simply given by

$$F_{\text{subtr}}(M^2, M^2) = m_b f_\pi \sum_k \left( \frac{a_k}{2k+1} \right) \int_{2m_b^2}^{2s_0^B} ds \ e^{-\frac{s}{2M^2}}$$

$$= m_b f_\pi \varphi_\pi(u_0) M^2 \left[ e^{-\frac{m_b^2}{M^2}} - e^{-\frac{s_0^B}{M^2}} \right]. \quad (118)$$

However, in general, the proportionality of $F_{\text{QCD}}(M_1^2, M_2^2)$ to the wave function $\varphi_\pi$ at $u_0 = M_1^2/(M_1^2 + M_2^2)$ found in (116) is destroyed by continuum subtraction. For this reason we adopt the particular choice $M_1^2 = M_2^2$ ($u_0 = 1/2$) in what follows.

So far we have solved the problem only for the twist 2 component of the sum rule. Unfortunately, the subtraction procedure explained above does not work for terms in $F_{\text{QCD}}$ which contain higher powers of $1/(m_b^2 - (p + uq)^2)$ (see (111)). The main problem is that the corresponding double spectral densities are not concentrated near $s_1 = s_2$, making the continuum subtraction rather complicated in these cases. For further discussion we refer the reader to [50]. On the other hand, this difficulty only concerns higher-twist terms which contribute relatively little to the sum rule. Hence, to a good approximation one may disregard the continuum subtraction in these terms.

Applying these recipes to $F_{\text{QCD}}$ as given in (95) one ends up with the following light-cone sum rule for the $B^*B\pi$ coupling:

$$f_B f_{B^*} g_{B^*B\pi} = \frac{m_b^2 f_\pi}{m_B m_{B^*}} e^{-\frac{m_b^2 + m_{B^*}^2}{2M^2}} \left\{ M^2 \left[ e^{-\frac{m_b^2}{M^2}} - e^{-\frac{s_0^B}{M^2}} \right] \right.$$

$$\times \left[ \varphi_\pi(u_0) + \frac{\mu_{B^*}}{m_b} \left( u_0 \varphi_\rho(u_0) + \frac{1}{6} \varphi_\sigma(u_0) + \frac{1}{6} u_0 \frac{d\varphi_\pi}{du}(u_0) + \frac{2f_\pi}{m_b f_\pi} I_3^G(u_0) \right) \right.$$

$$+ e^{-\frac{m_b^2}{M^2}} + \frac{\mu_{B^*} m_b}{3} \varphi_\sigma(u_0) + 2u_0 g_2(u_0) - \frac{4m_b^2}{M^2} \left( g_1(u_0) - \int_0^{u_0} g_2(v) dv \right) + I_4^G(u_0) \right\} \bigg|_{u_0=1/2}, \quad (119)$$

where

$$I_3^G(u_0) = \int_0^{u_0} d\alpha_1 \left[ \varphi_{3\pi}(\alpha_1, 1 - u_0, u_0 - \alpha_1) - \int_{u_0 - \alpha_1}^{1 - \alpha_1} d\alpha_3 \varphi_{3\pi}(\alpha_1, 1 - \alpha_1 - \alpha_3, \alpha_3) \right] \quad (120)$$

and

$$I_4^G(u_0) = \int_0^{u_0} d\alpha_1 \int_{u_0 - \alpha_1}^{1 - \alpha_1} d\alpha_3 \left[ 2\varphi_\perp(\alpha_i) - \varphi_\parallel(\alpha_i) + 2\bar{\varphi}_\perp(\alpha_i) - \bar{\varphi}_\parallel(\alpha_i) \right] \quad (121)$$

are the contributions from the three-particle amplitude (95). As a further simplification, $G$-parity implies $g_2(u_0) = \frac{d\varphi_\pi}{du}(u_0) = 0$ at $u_0 = 1/2$, whence these terms can be dropped in the above sum rule.
The wave function \( \varphi_\pi(u_0) \) at the symmetry point \( u_0 = 1/2 \) also enters the sum rules for other hadronic couplings involving the pion. We take the value
\[
\varphi_\pi(u = 1/2) = 1.2 \pm 0.2
\] (122)
obtained from the light-cone sum rule for the pion-nucleon coupling \([39]\). This value is consistent with the choice of the coefficients \( a_{2,4} \) given in (13). For the remaining parameters we use the same values as in the calculation of the form factor \( f^+ \) in section 5. With this choice the sum rule (119) yields
\[
f_B f_{B^*} g_{B^*B\pi} = 0.64 \pm 0.06 \text{GeV}^2 ,
\] (123)
or
\[
g_{B^*B\pi} = 29 \pm 3 ,
\] (124)
where \( f_B \) and \( f_{B^*} \) have been substituted from (39) and (43), respectively. The uncertainties quoted above are to be interpreted as the range of values corresponding to the allowed window in the Borel mass, \( 6 \leq M^2 \leq 12 \text{ GeV}^2 \), in which the excited and continuum states contribute less than 30 % and the twist 4 corrections do not exceed 10 %. Also these predictions can and should be improved by calculating the radiative gluon corrections.

The sum rule (119) for \( g_{B^*B\pi} \) is translated into a sum rule for \( g_{D^*D\pi} = g_{D^*D^0\pi^-} \) by formally changing \( b \) to \( c \), \( \bar{B} \) to \( D \), and \( \bar{B}^* \) to \( D^* \). With (122) and the input parameters from the calculation of the \( D \to \pi \) form factor, one finds
\[
f_D f_{D^*} g_{D^*D\pi} = 0.51 \pm 0.05 \text{GeV}^2 ,
\] (125)
and using the LO estimates for \( f_D \) and \( f_{D^*} \) from [10] and [14], respectively,
\[
g_{D^*D\pi} = 12.5 \pm 1.0 .
\] (126)
The variation of the numerical results with \( M^2 \) in the allowed interval \( 2 \leq M^2 \leq 4 \text{ GeV}^2 \) is again quoted as an uncertainty.

From (126) one can calculate the width for the decay \( D^* \to D\pi \). The predicted value,
\[
\Gamma(D^* \to D^0\pi^+) = \frac{g_{D^*D\pi}}{24\pi m_D^2} |\vec{q}|^3 = 32 \pm 5 \text{keV} ,
\] (127)
lies well below the current experimental upper limit,
\[
\Gamma(D^+ \to D^0\pi^+) < 89 \text{keV} ,
\] (128)
which is derived from the upper limit \( \Gamma_{tot}(D^*) < 131 \text{keV} \) and from the branching ratio \( BR(D^* \to D^0\pi^+) = (68.3 \pm 1.4)\% \) [71]. Predictions for other charge combinations are readily obtained from (127) and isospin relations analogous to (107). Accounting also for the differences in phase space, one expects
\[
\Gamma(D^* \to D^0\pi^+) = 2.2 \Gamma(D^+ \to D^+\pi^0) = 1.44 \Gamma(D^{*0} \to D^0\pi^0) .
\] (129)
Table 2: *Theoretical estimates of the strong $B^*B\pi$ and $D^*D\pi$ couplings (from [42]).*

| Reference | $\hat{g}$     | $g_{B^*B\pi}$ | $g_{D^*D\pi}$ | $\Gamma(D^{*+} \rightarrow D^0\pi^+)$ (keV) |
|-----------|----------------|---------------|---------------|------------------------------------------|
| [42]      | 0.32 ± 0.02    | 29 ± 3        | 12.5 ± 1.0    | 32 ± 5                                    |
| [42]      | –              | 28 ± 6        | 11 ± 2        | –                                         |
| [54]      | –              | 32 ± 6        | –             | –                                         |
| [55]      | 0.2 ÷ 0.7      | –             | –             | –                                         |
| [56]      | 0.39 ± 0.16    | 20 ± 4        | 9 ± 1         | –                                         |
| [56]      | 0.21 ± 0.06    | 15 ± 4        | 7 ± 1         | 10 ± 3                                    |
| [57]      | 0.7            | –             | –             | –                                         |
| [58]      | –              | 64            | –             | –                                         |
| [59]      | 0.75 ÷ 1.0     | –             | –             | 100 ÷ 180                                 |
| [60]      | 0.6 ÷ 0.7      | –             | –             | 61 ÷ 78                                   |
| [61]      | 0.4 ÷ 0.7      | –             | –             | –                                         |
| [62]      | 0.3            | –             | –             | –                                         |
| [63]      | –              | –             | 16.2          | 53.4                                      |
| [64]      | –              | –             | 19.5 ± 1.0    | 76 ± 7                                    |
| [65]      | –              | –             | 16.2 ± 0.3    | 53.3 ± 2.0                                |
| [66]      | –              | –             | 8.9           | 16                                        |
| [67]      | –              | –             | 8.2           | 13.8                                      |
| [68]      | –              | –             | < 21          | < 89                                      |

\(^a\) QCD sum rules in external axial field or soft pion limit.
\(^*\) including perturbative correction to the heavy meson decay constants.
\(^b\) Quark model + chiral HQET.
\(^c\) Chiral HQET with experimental constraints on $D^*$ decays.
\(^d\) Extended NJL model + chiral HQET.
\(^e\) Quark Model + scaling relation.
\(^f\) Relativistic quark model.
\(^g\) Bag model.
\(^h\) SU(4) symmetry.
\(^i\) Reggeon quark-gluon string model.
\(^k\) Experimental limits
In Table 2, we summarize the numerical results discussed above and compare them with other estimates. One observes significant differences. Some of the predictions are rather close to the experimental upper limit, some even violate it. It would be very interesting to have more precise data.

In contrast to $g_{D^* D \pi}$, the coupling constant $g_{B^* B \pi}$ cannot be measured directly, since the corresponding decay $B^* \to B \pi$ is kinematically forbidden. However, the $B^* B \pi$ on-shell vertex is of great importance for the understanding of the behaviour of the $B \to \pi$ form factors at large momentum transfer. Near the kinematic limit the form factor $f^+$ is expected to be dominated by the $B^*$ pole. The single-pole approximation given by

$$f^+(p^2) = \frac{f_{B^*} g_{B^* B \pi}}{2m_{B^*}(1 - p^2/m_{B^*}^2)}$$

is illustrated in Fig. 12 taking $g_{B^* B \pi}$ from (124) and $f_{B^*}$ from (13). Extrapolation of the single-pole model to smaller $p^2$ matches quite well with the direct estimate from the light-cone sum rule (97) at intermediate momentum transfer $p^2 = 15$ to 20 GeV$^2$. This provides us with a consistent and complete theoretical description of $f^+$. The extrapolation for the $D \to \pi$ form factor using the analogous single-pole formula with $g_{D^* D \pi}$ from (126) and $f_D$ from (14) is shown in Fig. 13. Also in this case we find the direct sum rule result and the pole model to match nicely at $p^2 \approx 0.7$ GeV$^2$.

Finally, we refer the reader to [52] for a similar calculation of scalar and axial $B$ meson couplings yielding

$$\Gamma(B(0^{++}) \to B \pi) \simeq \Gamma(B(1^{++}) \to B^* \pi) \simeq 360 \text{ MeV},$$

and to [53] for an investigation of the $B^* B \rho$ coupling.
7  Exclusive semileptonic decays, $V_{ub}$, and all that

With the $B \to \pi$, $D \to \pi$, and also $B \to \rho$ form factors at hand, we are now in the position to predict the widths and differential distributions for exclusive semileptonic decays. In the case of $f^+(p^2)$, the sum rules (97) and (119) together with the single-pole approximation (130) provide a complete description as illustrated in Fig. 12 and 13. For convenience, we have fitted parametrizations of the form

$$f^+(p^2) = \frac{f^+(0)}{1 - ap^2/m_P^2 + bp^4/m_P^4},$$

(132)

$m_P$ denoting the $B$ and $D$ meson mass, to the theoretical results plotted in these figures. For the $B \to \pi$ form factor we get

$$f^+(0) = 0.27, \quad a = 1.50, \quad b = 0.52.$$  

(133)

Here, the NLO correction to the twist 2 contribution has been included, and $f^+(0)$ has been fixed at the value given in (100). At $p^2 < 17 \text{ GeV}^2$ the fit reproduces the prediction of the light-cone sum rule (97), while at $p^2 > 20 \text{ GeV}^2$ it coincides with the single-pole approximation (130). In Fig. 14 the interpolation (132) is shown in comparison with recent lattice results. The agreement is very encouraging. The analogous fit of (132) to the LO $D \to \pi$ form factor plotted in Fig. 13 yields

$$f^+(0) = 0.68, \quad a = 1.16, \quad b = 0.32,$$

(134)

$^4$For a comprehensive review see the article by J.M. Flynn and C.T. Sachraudna in this volume, ref. [68].
Figure 14: The sum rule prediction for the $B \to \pi$ form factor $f^+$ in comparison to lattice results [13].

where $f^+(0)$ has been kept fixed at the value given in (101).

The distribution of the momentum transfer squared in $B \to \pi l \nu_l$ is given by

$$
\frac{d\Gamma}{dp^2} = \frac{G^2|V_{ub}|^2}{24\pi^3} \frac{(p^2 - m_l^2)^2 \sqrt{E_\pi^2 - m_\pi^2}}{p^4 m_B^2} \left\{ \left( 1 + \frac{m_l^2}{2p^2} \right) m_B^2 (E_\pi^2 - m_\pi^2) [f^+(p^2)]^2 
+ \frac{3m_l^2}{8p^2} (m_B^2 - m_\pi^2)^2 [f^0(p^2)]^2 \right\}
$$

with $E_\pi = (m_B^2 + m_\pi^2 - p^2)/2m_B$ being the pion energy in the $B$ rest frame, and the form factors being as defined in (13) and (90). For $l = e$ or $\mu$, the form factor $f^0$ plays a negligible role because of the smallness of the electron and muon masses. Another interesting observable is the distribution of the charged lepton energy $E_l$ in the $B$ rest frame:

$$
\frac{d\Gamma}{dE_l} = \frac{G^2|V_{ub}|^2}{64\pi^3 m_B} \int_{p_{min}^2}^{p_{max}^2} dp^2 \left\{ \left[ 8E_l (m_B^2 - m_\pi^2 + p^2) 
- 4m_B (p^2 + 4E_l^2) + \frac{m_l^2}{m_B} (8m_B E_l - 3p^2 + 4m_\pi^2) - \frac{m_l^4}{m_B^2} \right] [f^+(p^2)]^2 
+ 2m_l^2 \left[ 2m_B^2 + p^2 - 2m_\pi^2 - 4m_B E_l + m_l^2 \right] f^+(p^2) f^-(p^2) 
+ \frac{m_l^2}{m_B} (p^2 - m_l^2) [f^-(p^2)]^2 \right\}
$$

with the integration limits $p_{max}^2 = m_B(E_l \pm \sqrt{E_l^2 - m_\pi^2}) + O(m_\pi^2)$. The terms proportional to the pion mass squared not shown explicitly are taken into account in the numerical
calculations. Here, terms involving the form factor $f^-$ are suppressed by the lepton mass if $l = e, \mu$.

\[ d\Gamma/dp^2[|V_{ub}|^2 \text{ps}^{-1}\text{GeV}^{-2}] \]

\[ p^2[\text{GeV}^2] \]

Figure 15: Distribution of the momentum transfer squared in $B \to \pi \bar{l}\nu_l$ ($l = e, \mu$). The dashed curves indicate the theoretical uncertainty discussed in detail in the text.

The above decay distributions are shown in Fig. 15 and 16 for $l = e, \mu$. For the integrated width one obtains

\[ \Gamma(B \to \pi \bar{l}\nu_l) = (7.5 \pm 2.5) |V_{ub}|^2 \text{ps}^{-1}, \] (137)

where the theoretical uncertainty reflects the uncertainties in $f^+$ (added in quadrature) which have been discussed in detail in section 5.

Contrary to the semileptonic decays into $e$ and $\mu$, the decay $B \to \pi \bar{\tau}\nu_\tau$ is quite sensitive to the scalar form factor $f^0$. Unfortunately, the single-pole approximation used to extrapolate the sum rule result on $f^+$ to maximum $p^2$ cannot be applied to $f^0$. This is because the scalar $B$ ground state which should be about 500 MeV heavier than the pseudoscalar $B$ lies too far above the kinematical endpoint $p^2 = (m_B - m_\pi)^2$ of the $B \to \pi$ transition in order to dominate the form factor. Nearby excited resonances and nonresonant states are expected to give comparable contributions. For illustrative purposes [43], we extrapolate the form factor $f^0$ linearly from the maximal value $p^2 = 15$ GeV$^2$ at which the sum rules (97) and (98) still hold to the value at $p^2 \approx m_B^2$ dictated by the Callan-Treiman limit [69]:

\[ \lim_{p^2 \to m_B^2} f^0(p^2) = \frac{f_B}{f_\pi} = 1.1 \text{ to } 1.6, \] (138)

where we have used the conservative estimate $f_B = 150$ to 210 MeV. This rough extrapolation of the sum rule result is plotted in Fig. 17. Also shown are lattice data. They are systematically lower than our expectation. It should however be noted that in contrast to $f^+$, NLO effects are still missing in the sum rule for $f^0$. 

36
Fig. 16 shows the resulting distribution of $E_\tau$. As anticipated, the spectrum is quite sensitive to the large $p^2$ behaviour of $f_0$, and may thus allow to determine, or at least constrain the scalar form factor experimentally. The integrated partial width is

$$\Gamma(B \to \pi \bar{\tau} \nu_\tau) = (6.1 \pm 0.4) |V_{ub}|^2 \text{ ps}^{-1}, \quad (139)$$

yielding, together with (137), the ratio

$$\frac{\Gamma(B \to \pi \bar{\tau} \nu_\tau)}{\Gamma(B \to \pi \bar{e} \nu_e)} = 0.75 \text{ to } 0.85. \quad (140)$$

This ratio is independent of $V_{ub}$, and less sensitive to uncertainties in the sum rule parameters than the widths themselves. The range quoted above corresponds to the variation of $f_0$ within the two extrapolations considered in Fig. 17.

![Figure 16](image)

Figure 16: Distribution of the lepton energy in $B \to \pi \bar{l} \nu_l$ for $l = e, \mu$ (solid) and $l = \tau$ (dashed). In the latter case the two curves correspond to the two extrapolations of $f_0$ shown in Fig. 17.

Recently, the CLEO collaboration [70] has reported the first observation of the semileptonic decays $B \to \pi \bar{l} \nu_l$ and $B \to \rho \bar{l} \nu_l$ ($l = e, \mu$). From the measured branching fraction $BR(B^0 \to \pi^- l^+ \nu_l) = (1.8 \pm 0.4 \pm 0.3 \pm 0.2) \cdot 10^{-4}$, and the world average of the $B^0$ lifetime [51], $\tau_{B^0} = 1.56 \pm 0.06$ ps, one derives

$$\Gamma(B^0 \to \pi^- l^+ \nu_l) = (1.15 \pm 0.35) \cdot 10^{-4} \text{ ps}^{-1}, \quad (141)$$

where the errors have been added in quadrature. Comparison of (141) with (137) yields

$$|V_{ub}| = 0.0039 \pm 0.0006 \pm 0.0006. \quad (142)$$
Here, the first (second) error corresponds to the current experimental (theoretical) uncertainty.

A similar analysis can be performed for $B \rightarrow \rho \bar{\ell} \nu_\ell$. Fig. 18 shows the distribution of the momentum transfer squared obtained from the form factors plotted in Fig. 11\cite{33}. The integrated width predicted in \cite{33} is

$$
\Gamma(B \rightarrow \rho \bar{\ell} \nu_\ell) = (13.5 \pm 4) |V_{ub}|^2 \text{ ps}^{-1}.
$$

(143)

Comparison with the CLEO result,

$$
\Gamma(B^0 \rightarrow \rho^- l^+ \nu_\ell) = (1.60 \pm 0.6) \cdot 10^{-4} \text{ ps}^{-1},
$$

(144)

derived from the measured branching ratio $BR(B^0 \rightarrow \rho^- l^+ \nu_\ell) = (2.5 \pm 0.4^{+0.5}_{-0.7} \pm 0.5) \cdot 10^{-4}$ and the $B^0$ lifetime also used in (141), gives

$$
|V_{ub}| = 0.0034 \pm 0.0006 \pm 0.0005.
$$

(145)

Within errors, the values of $|V_{ub}|$ extracted from the two exclusive semileptonic decays are nicely consistent with each other, and also coincide with the inclusive determination of $V_{ub}$.

A further test of the consistency between theory and experiment is provided by comparing the ratio ($l = e, \nu$)

$$
\frac{\Gamma(B^0 \rightarrow \rho^- l^+ \nu_\ell)}{\Gamma(B^0 \rightarrow \pi^- l^+ \nu_\ell)} = 1.8 \pm 0.7
$$

(146)

calculated from (143) and (137) with the ratio

$$
\frac{BR(B^0 \rightarrow \rho^- l^+ \nu_e)}{BR(B^0 \rightarrow \pi^- l^+ \nu_e)} = 1.4^{+0.6}_{-0.4} \pm 0.3 \pm 0.4
$$

(147)
observed by CLEO. Although the test is passed by the present estimate, the uncertainties
on both sides are still too big to draw any firm conclusion.

\[ d\Gamma/dt [|V_{ub}|^2 \text{ps}^{-1}\text{GeV}^{-2}] \]

\[ B \rightarrow \rho e\nu \]

Figure 18: Distribution of the momentum transfer squared in \( B \rightarrow \rho e\nu_e \). The dashed curves show the theoretical uncertainty (from [33]).

Turning to semileptonic \( D \) decays we show, in Fig. 19, the distribution of the momentum transfer squared in \( D \rightarrow \pi \bar{l}\nu_l \). For the integrated width we get

\[ \Gamma(D \rightarrow \pi \bar{l}\nu_l) = 0.16 |V_{cd}|^2 \text{ps}^{-1} = 8.0 \cdot 10^{-3} \text{ps}^{-1} \]  \hspace{1cm} (148)

using \( |V_{cd}| = 0.224 \pm 0.016 \) [51]. This prediction should be compared with the experimental result

\[ \Gamma(D^0 \rightarrow \pi^- e^+ \nu_e) = (9.2^{+2.9}_{-2.4}) \cdot 10^{-3} \text{ps}^{-1} \]  \hspace{1cm} (149)

derived from the branching ratio \( BR(D^0 \rightarrow \pi^- e^+ \nu_e) = (3.8^{+1.2}_{-1.0}) \cdot 10^{-3} \) and the lifetime \( \tau_{D^0} = 0.415 \pm 0.004 \text{ ps} \) [22]. The present theoretical uncertainty is estimated to be of the order of the experimental error. In other words, the CKM-suppressed exclusive semileptonic \( D \) decays are not yet measured precisely enough to really challenge theory. In addition, there is a further demand for better data. One may use a precise measurement of \( D \rightarrow \pi \bar{l}\nu_l \) to constrain the light-cone sum rule for the \( D \rightarrow \pi \) form factor. Since the evolution of the pion wave function and other scale-dependent input quantities from the charm to the bottom scale is well under control, this is a promising way to considerably improve the sum rule predictions on \( B \rightarrow \pi \) transitions.

8 Heavy quark limit

The light-cone sum rules described in section 5 provide a unique possibility to investigate the heavy-mass dependence of the \( B \rightarrow \pi \) form factors. To this end, one employs the following
scaling relations for mass parameters and decay constants:

\[ m_B = m_b + \bar{\Lambda}, \quad s_0^B = m_b^2 + 2m_b\omega_0, \quad M^2 = 2m_b\tau, \quad (150) \]

\[ f_B = \hat{f}_B/\sqrt{m_b}, \quad f_{B*} = \hat{f}_{B*}/\sqrt{m_b}, \quad (151) \]

where in the heavy quark limit \( \bar{\Lambda}, \omega_0, \tau, \hat{f}_B \) and \( \hat{f}_{B*} \) are \( m_b \)-independent quantities. With these substitutions it is straightforward to expand the sum rules (97) and (98) in \( m_b \). In both cases, the light-cone expansion in terms of wave functions with increasing twist is consistent with the heavy mass expansion, that is the higher-twist contributions either scale with the same power of \( m_b \) as the leading-twist term, or they are suppressed by extra powers of \( m_b \).

We find that the asymptotic scaling of the form factors differs sharply at small \( 5 \) and large momentum transfer \([43]\). At \( p^2 = 0 \)

\[ f^+(0) = f^0(0) \sim m_b^{-3/2}, \quad (152) \]

\[ f^+(0) + f^-(0) \sim m_b^{-3/2}, \quad (153) \]

whereas at \( p^2 = m_b^2 - 2m_b\chi, \chi \) being independent of \( m_b \),

\[ f^+(p^2) \sim m_b^{1/2}, \quad (154) \]

\[ f^+(p^2) + f^-(p^2) \simeq f^0(p^2) \sim m_b^{-1/2}. \quad (155) \]

The sum rules thus nicely reproduce the asymptotic dependence of the form factors \( f^+ \) and \( f^- \) on the heavy quark mass \( m_b \) derived in \([58, 63]\) for small pion momentum in the rest

\[ ^5 \text{see also ref. [40]} \]
frame of the $B$ meson. In addition, the sum rules allow to investigate the opposite region of maximum pion momentum where neither HQET nor the single-pole model can be trusted. In particular, from the sum rule point of view it is expected that the excited and continuum states become more and more important as $p^2 \to 0$. This is reflected in the change of the asymptotic mass dependence. Claims in the literature which differ from (152) to (155) are often based on the pole model and therefore incorrect in our opinion. A similar analysis has been carried out in [33] with essentially the same conclusions.

Furthermore, the sum rule (119) for the $B^*B\pi$ coupling constant suggests

$$g_{B^*B\pi} \sim m_b ,$$

in agreement with the expectation from HQET [57, 58, 71]. At the expense of additional parameters such as $\Lambda$, $\omega_0$, etc. light-cone sum rules can also be used to determine the $1/m_Q$ corrections. For $g_{B^*B\pi}$ and $g_{D^*D\pi}$, a simple quantitative estimate of the latter is obtained by fitting the numerical results (124) and (126) to the form

$$g_{B^*B\pi} = \frac{2m_B}{f_\pi} \cdot \hat{g} \left( 1 + \frac{\Delta}{m_B} \right)$$

(157)

and the analogous expression for $g_{D^*D\pi}$. This yields [42]

$$\hat{g} = 0.32 \pm 0.02 , \quad \Delta = (0.7 \pm 0.1) \text{ GeV} .$$

In Table 2, the above value of the reduced coupling constant $\hat{g}$ is compared to the results in other approaches. The next-to-leading term in the heavy mass is sizeable, increasing from about 15% for $g_{B^*B\pi}$ to 40% for $g_{D^*D\pi}$.

Finally, in the heavy quark limit the ratio

$$r = \frac{g_{B^*B\pi}f_{B^*}\sqrt{m_D}}{g_{D^*D\pi}f_{D^*}\sqrt{m_B}}$$

(159)

is expected to approach unity. Moreover, it has been shown [72] that $r$ is subject to $1/m_Q$ corrections only in next-to-leading order. Indeed, the ratio $r = 0.92$ derived from the light-cone sum rules deviates surprisingly little from unity, in agreement with the HQET expectation.

9 The nonfactorizable amplitude for $B \to J/\psi K$

As a final example for applications of QCD sum rules to exclusive heavy meson decays we consider nonleptonic two-body decays. This class of processes is theoretically much more complicated than the (semi)leptonic decays discussed so far. As compared to the latter, effects from (a) hard gluon exchange at short distances, (b) soft interactions of quarks and gluons including nonspectator effects, (c) hadronization, and (d) final state interactions among the hadronic decay products change things considerably. Up to now, only the hard-gluon effects can be systematically taken into account in the framework of improved QCD perturbation theory. The result is an effective weak Hamiltonian at the physical scale $\mu \simeq m_Q \ll m_W$, given by a sum of local operators with renormalized Wilson coefficients [6].

6See, e.g., ref. [4].
We restrict our discussion to the decay $B \to J/\psi K$ which will play an important role at future $B$-factories and which brings the main theoretical difficulties to light. The piece of the effective Hamiltonian relevant for this decay mode may be written in the form

$$H_W = \frac{G}{\sqrt{2}} V_{cb} V_{cs}^\ast \{ (c_2 + \frac{c_1}{3}) O_2 + 2 c_1 \tilde{O}_2 \},$$  

(160)

where

$$O_2(\mu) = (\bar{c} \Gamma^\rho c) (\bar{s} \Gamma_\rho b), \quad \tilde{O}_2(\mu) = (\bar{c} \Gamma^\rho \lambda^a c) (\bar{s} \Gamma_\rho \lambda^a b)$$  

(161)

with $\Gamma_\rho = \gamma_\rho (1 - \gamma_5)$. The Wilson coefficients $c_i(\mu)$ contain the effects from QCD interactions at short distances below the scale set by the inverse $b$-quark mass. The hadronic matrix elements of the four-quark operators (161) are supposed to incorporate the long-distance effects (b) to (d). The problem of calculating these matrix elements is extremely demanding and still far from a satisfactory solution.

In a radical first approximation, one may factorize the matrix elements of $H_W$ for $B \to J/\psi K$ into products of hadronic matrix elements of the currents that compose $H_W$. Strong interactions at scales lower than $\mu$ between quarks entering different currents as well as nonspectator effects are thereby completely neglected. Moreover, the matrix element of the operator $\tilde{O}_2$ vanishes because of colour conservation so that

$$\langle J/\psi K \mid H_W \mid B \rangle = \frac{G}{\sqrt{2}} V_{cb} V_{cs}^\ast \left( c_2(\mu) + \frac{c_1(\mu)}{3} \right) \langle J/\psi K \mid O_2(\mu) \mid B \rangle.$$  

(162)

The factorized matrix element of the operator $O_2$ is given by

$$\langle J/\psi K \mid O_2(\mu) \mid B \rangle = \langle J/\psi \mid \bar{c} \Gamma^\rho c \mid 0 \rangle \langle K \mid \bar{s} \Gamma_\rho b \mid B \rangle = 2 f_\psi f_{B \to K}^+ m_\psi (e^\psi \cdot q),$$  

(163)

where

$$f_\psi = 405 \text{ MeV}$$  

(164)

is the decay constant determined by the leptonic width $\Gamma(J/\psi \to l^+ l^-) = 5.26 \pm 0.37 \text{ keV},$ and

$$f_{B \to K}^+ = 0.55 \pm 0.05$$  

(165)

is the $B \to K$ form factor at the momentum transfer $p^2 = m_\psi^2$ estimated from a light-cone sum rule similar to the one for the $B \to \pi$ form factor given in (97). Obviously, $e^\psi$ denotes the $J/\psi$ polarization vector, and $q$ the $K$ four-momentum.

Already at this point one encounters a principal problem: since the matrix elements of quark currents in (163) are scale-independent, the $\mu$-dependence of $\langle J/\psi K \mid O_2(\mu) \mid B \rangle$ which is supposed to cancel the $\mu$-dependence of the Wilson coefficients in (162) in order to give a physically sensible result, is lost. Hence, the above approximation can at best be valid at a particular value of $\mu$, which could be called the factorization scale $\mu_F$. The conventional assumption is $\mu_F = O(m_b)$.

Using the next-to-leading order coefficients $c_{1,2}(\mu)$ in the HV scheme with $\Lambda^{(5)}_{\overline{MS}} = 225$ MeV from (73) and taking $\mu = m_b \simeq 5 \text{ GeV}$, one has

$$c_2(\mu) + \frac{c_1(\mu)}{3} = 0.155.$$  

(166)
Together with (164) and (165), this yields
\[ BR(B \to J/\psi K) = 0.025\% , \]
(167)
a branching ratio which is considerably smaller than the measurements [51, 74]
\[ BR(B^- \to J/\psi K^-) = (0.101 \pm 0.014)\% , \]
(168)
\[ BR(B^0 \to J/\psi K^0) = (0.075 \pm 0.021)\% . \] (169)

The quantitative failure and the scale problem pointed out above imply that naive factorization of matrix elements does not work. Factorization has to be accompanied by a reinterpretation of the Wilson coefficients. For decays such as \( B \to J/\psi K \), the short-distance coefficient \( c_2(\mu) + c_1(\mu)/3 \) is substituted by an effective coefficient \( a_2 \) which is supposed to incorporate possible nonfactorizable contributions. Phenomenologically [26], \( a_2 \) is treated as a free parameter to be determined from experiment. From (162), (163), and (168), the most precise of the two measurements, one finds
\[ |a^{B\psi K}_2| = 0.31 \pm 0.02 , \] (170)
where the quoted error is purely experimental. The sign of \( a^{B\psi K}_2 \) remains undetermined. The above value is close to the outcome of a comprehensive analysis [47, 79] of nonleptonic two-body \( B \)-decays.

The big difference between the short-distance and effective coefficient (166) and (170), respectively, points at the existence of sizeable nonfactorizable contributions. The latter are also needed to cancel or at least soften the strong \( \mu \)-dependence of (166). A deeper study shows that the dominant nonfactorizable effects should arise from the matrix element of the operator \( \tilde{O}_2 \). Writing the latter in the convenient parametrization
\[ \langle J/\psi K \mid \tilde{O}_2(\mu) \mid B \rangle = 2f_\psi \tilde{f}_{B\psi K}(\mu)m_\psi (\epsilon_\psi \cdot q) , \] (171)
and adding it to the factorized matrix element (163) of \( O_2 \), one gets for the complete decay amplitude:
\[ \langle J/\psi K \mid H_W \mid B \rangle = \sqrt{2} G V_{cb} V^*_{cs} a^{B\psi K}_2 f_\psi f_{B \to K}^{+} m_\psi (\epsilon_\psi \cdot q) \] (172)
with the effective coefficient [75]
\[ a^{B\psi K}_2 = c_2(\mu) + \frac{c_1(\mu)}{3} + 2c_1(\mu) \frac{\tilde{f}_{B\psi K}(\mu)}{f_{B \to K}^{+}} . \] (173)

In Fig. 20, we show the partial width for \( B \to J/\psi K \) as a function of the parameter \( \tilde{f}_{B\psi K} \) associated with \( \langle \tilde{O}_2 \rangle \). Note that \( \tilde{f}_{B\psi K} = 0 \) corresponds to naive factorization, while the fitted value of \( a^{B\psi K}_2 \) given in (170) implies
\[ \tilde{f}_{B\psi K}(\mu = m_b) = +0.04 \text{ or } -0.12 . \] (174)
For a smaller scale the value of \( \tilde{f}_{B\psi K} \) is slightly shifted to the right on the real axis, e.g.,
\[ \tilde{f}_{B\psi K}(\mu = \frac{1}{2} m_b) = +0.06 \text{ or } -0.09 . \] (175)
Figure 20: The partial width for $B \to J/\psi K$ as a function of $\tilde{f}_{B\psi K}$ parameterizing the nonfactorizable contribution to the decay amplitude. The horizontal dotted lines represent the average of the experimental widths given in the text. The solid (dashed) curves show the theoretical expectation for $\mu = m_b/2$ ($\mu = m_b$). The arrows indicate the QCD sum rule estimate.

We see that a nonfactorizable amplitude of 10 to 20% of the size of the factorizable one is sufficient to conciliate expectation with experiment.

The theoretical calculation of $a_{2B\psi K}$ and of the analogous coefficients for other two-body decays \cite{26} is one of the most important tasks in heavy flavour physics. As a first step in this direction, we have undertaken a rough estimate of $\tilde{f}_{B\psi K}$ using again QCD sum rule methods. Following the general idea put forward in \cite{76}, we choose the four-point correlation function

$$\tilde{\Pi}(p, q) = \int d^4x\, d^4y\, d^4z\, e^{ipx + ipy} \langle 0 | T\{j^K_{\mu\rho}(x)j^\psi_{\nu\rho}(y)\tilde{O}_2(z)j_B^{(0)}(0)\} | 0 \rangle$$

(176)

with $j_B^5 = \bar{b}i\gamma_5 u$, $j^\psi_\nu = \bar{c}\gamma_\nu c$, and $j^K_{\mu\rho} = \bar{u}\gamma_\mu\gamma_5 s$ being the generating currents of the mesons participating in the decay $B \to J/\psi K$, and $p + q$, $p$ and $q$ being the respective four-momenta.

Generalizing the procedure applied in section 2 and 3 to the two- and three-point correlation functions (4) and (46), respectively, one writes a dispersion relation for (176) in terms of intermediate hadronic states in the $B$, $J/\psi$, and $K$ channel. The ground state

---

\footnote{We refer here to the so-called new model for the heavy-to-light form factors.}

\footnote{This work was done in collaboration with B. Lampe. Results were already reported in \cite{75}.}
The scale \( \mu \) representation (177) to the OPE result (178) for various values of the Borel mass in addition to \( \tilde{O} \) states. In total, in this approximation the correlator (177) contains three free parameters. The allowed range turns out to be given by \( M \gtrsim \sqrt{m_{B}^{2} - m_{B}^{2}} \gtrsim \frac{1}{2} m_{b} \simeq 2.4 \text{ GeV} \). Substituting (179) in (173), and evaluating the short-distance coefficients \( c_{1,2}(\mu) \) also at \( \mu = M \), one gets

\[
a_{2}^{B\psi K} = -0.29 + 0.38 - (0.19 \text{ to } 0.31) = -(0.10 \text{ to } 0.22),
\]  

Because of the restriction to short distances it suffices to keep only the operators with low dimensions given in (15). We have included all relevant operators up to \( d = 6 \). The corresponding coefficients \( \tilde{C}_{d}^{\mu
u}(p,q,\mu) \) have been calculated from the diagrams shown in Fig. 21. By equating (177) and (178) one can derive a sum rule for the matrix element \( \langle J/\psi K | \tilde{O}_{2} | B \rangle \).

There are two complications that are not present in the two- and three-point sum rules discussed above. One problem is the presence of a light continuum in the \( B \) channel below the pole of the ground state \( B \)-meson. This contribution to (177) can be associated with processes of the type \( B \rightarrow "D^{*}D_{s}" \rightarrow J/\psi K \), where an intermediate state carrying \( D^{*}D_{s} \) quantum numbers rescatters into the final \( J/\psi K \) state. Formally, in (176) it is created from the vacuum by the combined action of the operator product \( \tilde{O}_{2}j_{5}^{B} \). As a reasonable solution we suggest to cancel this unwanted piece against those terms in the OPE (178) with the quark content \( c\bar{c}s\bar{q} \) which develop a nonzero imaginary part at \( (p+q)^{2} \gtrsim 4m_{c}^{2} \).

In the approximation considered this is the case for the four-quark condensate contribution represented by Fig. 21c.

The second problem concerns the subtraction of contributions from excited resonances and continuum states in the remaining sum rule. A closer look at the \( J/\psi \) channel of the correlation function (174) reveals that the higher charmonium resonances contribute with alternating signs. Therefore, the usual subtraction procedure in which the dispersion integral over the excited and continuum states is approximated by its perturbative counterpart is not reliable here. In order to proceed we employ explicit, although rough models for the hadronic spectral functions. Since the number of additional parameters has to be manageable, only the first excited resonances are included in each channel besides the \( B, J/\psi \) and \( K \) ground states. In total, in this approximation the correlator (177) contains three free parameters in addition to \( \tilde{f}_{B\psi K} \).

We then Borel transform (177) and (178) in the \( B \)-meson channel and take moments in the charmonium channel. In the \( K \)-meson channel, \( q^{2} \) is kept spacelike. Fitting the hadronic representation (177) to the OPE result (178) for various values of the Borel mass \( M \) and \( q^{2} \), and for several moments, we find

\[
\tilde{f}_{B\psi K} = -(0.045 \text{ to } 0.075).
\]  

The scale \( \mu \) implicit in this estimate is set by the Borel mass. The central value in the allowed range turns out to be given by \( M \simeq \sqrt{m_{B}^{2} - m_{b}^{2}} \simeq \frac{1}{2} m_{b} \simeq 2.4 \text{ GeV} \). Substituting (179) in (173), and evaluating the short-distance coefficients \( c_{1,2}(\mu) \) also at \( \mu = M \), one gets

\[
a_{2}^{B\psi K} = -0.29 + 0.38 - (0.19 \text{ to } 0.31) = -(0.10 \text{ to } 0.22),
\]
where the three terms in the first relation refer to the three terms in (173) in the same order. Interestingly, the sum rule approach seems to favour the negative solution for $\tilde{f}_{B\psi K}$. Although in comparison with (175) our estimate falls somewhat short, the gap between theory and experiment is narrowed considerably as can be seen from Fig. 20.

![Figure 21: Diagrams determining the Wilson coefficients of the OPE of the correlation function $\bar{\Pi}_{\mu\nu}$](image)

Several comments are in order. Firstly, the nonfactorizable matrix element (171) is small as compared to the factorizable one given in (163), numerically, $|\tilde{f}_{B\psi K}/f_{B\to K}(m_\psi^2)| \simeq 0.1$. Nevertheless, it has a strong quantitative impact on $a_{B\psi K}$ because of the large coefficient $|2c_1/(c_2 + c_1/3)| \simeq 20$ to 30. Secondly, the factorizable amplitude proportional to $c_1/3$ and the nonfactorizable one proportional to $\tilde{f}_{B\psi K}$ are opposite in sign and hence tend to cancel. In fact, if $|\tilde{f}_{B\psi K}|$ is taken at the upper end of the estimated range, the cancellation is almost complete resulting in a considerable enhancement of the branching ratio (167). Note that both terms are nonleading in $1/N_c$. This is exactly the scenario anticipated by the $1/N_c$ – rule for $D$ decays [77] which has found theoretical support by the global sum rule analysis of [76]. In [78, 79], a similar trend was claimed for $B \to D\pi$. Thirdly, our estimate yields a negative overall sign for $a_{2}^{B\psi K}$ in contradiction to naive factorization (first two terms in (173)), and also to a global fit of the factorized decay amplitudes with two universal coefficients $a_1$ and $a_2$ to the data [4]. It should be stressed, however, that in this fit the positive sign of $a_2$ actually results from the channels $B^- \to D^0\pi^-, D^0\rho^-$, $D^{*0}\pi^-$, and $D^{*0}\rho^-$, and is then assigned also to the $J/\psi K$ channel. This assignment may
not be correct. Certainly, the sum rule approach described above provides no justification for such an assumption. On the contrary, diagrams of the kind shown in Fig. 21 suggest some channel-dependence of the nonfactorizable matrix elements. For class II processes involving $a_2 = c_2 + c_1/3 + 2c_1 f/ f^+$ (see (173)), the channel-dependence is enhanced by the large coefficient $2c_1$, while the factorized matrix elements come with the small coefficient $c_2 + c_1/3$. A concrete numerical example is provided by (180). The opposite is the case for class I processes involving $a_1 = c_1 + c_2/3 + 2c_2 f/f^+$: the nonfactorizable contributions are damped by the small coefficient $c_2$, while the factorized term has a large coefficient. Therefore, $a_1$ is indeed expected to be universal to a good approximation, but $a_2$ should exhibit some channel-dependence, in particular when comparing decays with very different final states such $D\pi$ and $J/\psi K$.

As a last remark, from the sum rule point of view there is also no simple relation between $B$ and $D$ decays. In [4], arguments are presented which support a change of sign in $a_2$ when going from $D$ to $B$ decays. However, it is not obvious that these arguments hold independently of the particle composition of the final state. In the sum rule approach, for example, one has significant differences in the OPE of the correlation functions such as (176) for $B \to D\pi$, $B \to J/\psi K$, and $D \to K\pi$ as can be imagined from Fig. 21. From this point of view at least, the relation of $B$ and $D$ decays is expected to involve more than just a change of mass scales.

10 Conclusion

QCD sum rule techniques have proved to be very useful in calculating hadronic matrix elements for exclusive decays of $B$ and $D$ mesons. In this review, applications are discussed to decay constants, form factors, and amplitudes of nonleptonic two-body decays, as well as to the $B^*B\pi$ and $D^*D\pi$ couplings. We have not considered exclusive radiative decays such as $B \to K^*\gamma$ and $B \to \rho\gamma$, but we at least want to mention that here sum rules have been employed to determine the matrix elements of the leading magnetic penguin operator [30, 31, 32, 34] and to estimate long-distance effects [80, 81, 82]. The above examples are by no means exhaustive.

Using the sum rule results we have presented predictions on decay distributions and integrated widths for $B \to \pi l\bar{\nu}_l$, $B \to \rho l\bar{\nu}_l$, and $D \to \pi l\bar{\nu}_l$. Comparison with the CLEO measurements [70] of exclusive semileptonic $B^0$ decays yields values for $V_{ub}$ in good agreement with each other and with the determination from inclusive data [84]. Agreement between expectation and measurement is also found for the Cabibbo-suppressed semileptonic $D^0$ decay [51].

From the sum rule estimate of the $D^*D\pi$ coupling we have calculated the decay width for $D^* \to D\pi$ and compared our result with other estimates. The experimental upper limit [51] is still about three times larger than the expected width.

Furthermore, we have described an attempt to estimate nonfactorizable matrix elements of nonleptonic decay amplitudes with the help of sum rule techniques. Using $B \to J/\psi K$ as a prototype example, it is argued that nonfactorizable effects play an essential role. With the sum rule estimate of the nonfactorizable contributions included the effective coefficient $d_2^{B\psi K}$ is found to be consistent with the value extracted from the experimental branching ratio [51]. However, the sign of $d_2^{B\psi K}$ predicted by the sum rule analysis is opposite to the
sign determined from data, if channel-independent, universal coefficients $a_{1,2}$ are assumed [4]. We have explained why in the sum rule approach universality can be expected for $a_1$, but not for $a_2$. This issue certainly requires further clarification. As long as one is not able to actually calculate the effective coefficients, one cannot claim a complete theoretical understanding of the exclusive nonleptonic decays.

Among the different variants of sum rules we have put particular emphasis on the light-cone sum rules which provide a powerful tool in problems involving a pion, kaon or $\rho$ meson such as heavy-to-light form factors and couplings. In this approach, the light hadrons are described on mass-shell by a set of wave functions with different twist and quark-gluon multiplicity, representing distribution amplitudes in the fraction of the hadron light-cone momentum carried by the constituents. This avoids the notorious model-dependence of extrapolations from Euclidean to physical momenta in light channels. The light-cone sum rules also seem to be fully consistent with the heavy mass expansion in contrast to some of the sum rules based on short-distance expansion. Moreover, the derivation of light-cone sum rules is often technically easier than conventional sum rule calculations.

We have addressed in some detail the present theoretical uncertainties, and the prospects for improvement. On the theoretical side, we see room for it by determining the nonasymptotic features of the light-cone wave functions more accurately, and by including higher-order perturbative effects in the sum rules. Work is under way in both directions: the re-analysis of the twist 2 pion wave function in [85], the determination of $\rho$ wave functions in [86], and the calculations of the $O(\alpha_s)$ correction to the $B \to \pi$ form factor $f^+$ in [17, 18] are recent examples. On the experimental side, the advent of new and more precise measurements at future $B$ and tau-charm factories should allow to tightly constrain the input parameters and to test the reliability of the sum rule approach in a very challenging way. Because of the universality of the nonperturbative input it appears conceivable to decrease the uncertainties from presently 20 to 30 % to about 10 %.

Finally, we have pointed out the encouraging agreement of lattice and sum rule calculations in the case of $f_B$, $f_D$, and the $B \to \pi$ and $B \to \rho$ form factors. This agreement should be enough motivation to join efforts. We believe that it would be very fruitful to combine the flexibility of the sum rule method with the rigorous nature of the lattice approach.

Acknowledgements

We are thankful to V.M. Belyaev, V.M. Braun, B. Lampe, Ch. Winhart, S. Weinzierl, and O. Yakovlev for collaboration on various subjects of this review and for useful discussions. This work was supported by the Bundesministerium für Bildung, Wissenschaft, Forschung und Technologie, Bonn, Germany, Contract 05 7WZ91P (0).

Appendix 1

Here, we collect the formulae for the light-cone wave functions of the pion and specify the parameters. It is important to note that the asymptotic form of these functions and the scale dependence are given by perturbative QCD [37, 46].

The twist 2 wave function $\varphi_\pi$ is expressed as an expansion in Gegenbauer polynomials:

$$\varphi_\pi(u, \mu) = 6u(1 - u)\left[1 + a_2(\mu)C_2^{3/2}(2u - 1) + a_4(\mu)C_4^{3/2}(2u - 1) + \ldots\right], \quad (A1)$$
where

\[
C_2^{3/2}(2u - 1) = \frac{3}{2}[5(2u - 1)^2 - 1],
\]

\[
C_4^{3/2}(2u - 1) = \frac{15}{8}[21(2u - 1)^4 - 14(2u - 1)^2 + 1].
\] (A2)

The normalization is such that

\[
\int_0^1 du \varphi_\pi(u, \mu) = 1.
\] (A3)

The nonperturbative effects are contained in the coefficients \( a_n \). In LO, they are multiplicatively renormalizable and have the following scale dependence:

\[
a_n(\mu) = a_n(\mu_0) \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\gamma_n/b},
\] (A4)

where \( b = 11 - 2n_f/3 \) is the LO coefficient of the QCD beta function, \( n_f \) being the number of active flavours, and

\[
\gamma_n = C_F \left[ -3 - \frac{2}{(n + 1)(n + 2)} + 4 \left( \sum_{k=1}^{n+1} \frac{1}{k} \right) \right]
\] (A5)

are the anomalous dimensions [37]. We see that \( a_n(\mu) \), \( n \geq 2 \) vanishes for \( \mu \to \infty \). Therefore, these terms describe nonasymptotic features of the wave function (A1).

The initial values of the nonasymptotic coefficients can be estimated from two-point sum rules [37] for the moments \( \int u^n \varphi_\pi(u, \mu) du \) at low \( n \). The nonperturbative information encoded in the quark and gluon condensates is thereby transmuted into the long-distance properties of the wave function. Alternatively, one can determine the coefficients directly from light-cone sum rules for known hadronic quantities such as the \( \pi NN \) and \( \omega \rho \pi \) couplings.

For the numerical results shown in this review we have used the following estimate at \( \mu_0 = 0.5 \) GeV [39]:

\[
a_2(\mu_0) = \frac{2}{3}, \quad a_4(\mu_0) = 0.43.
\] (A6)

On the basis of the approximate conformal symmetry of QCD it has been shown [46] that the expansion (A1) converges sufficiently fast so that the terms with \( n > 4 \) are negligible.

The scale \( \mu \) to be used in the light-cone sum rules (97) and (98) for the \( B \to \pi \) form factors, and in the sum rule (119) for the \( B^* B \pi \) coupling is somewhat ambiguous, in particular in LO approximation. As a reasonable choice, we take

\[
\mu_b = \sqrt{m_B^2 - m_b^2} \simeq 2.4 \text{ GeV}.
\] (A7)

The analogous choice for the \( D \) meson form factors and coupling is

\[
\mu_c = \sqrt{m_D^2 - m_c^2} \simeq 1.3 \text{ GeV}.
\] (A8)

These scales characterize the typical virtuality of the \( b \), respectively \( c \) quark, and coincide within a factor of two also with the respective value of the Borel mass \( M \). With (A4) and (A6), one gets for the evolved coefficients

\[
a_2(\mu_b) = 0.35, \quad a_4(\mu_b) = 0.18,
\]
\[ a_2(\mu) = 0.41, \quad a_4(\mu) = 0.23. \quad (A9) \]

The evolution of \( a_\nu(\mu) \) is also known in NLO, where mixing effects occur \cite{[19]}. Using the same input values \( \mu_0 \) and \( \mu_b \) from \( \mu_c \), one finds \cite{[17]}
\[ a_2(\mu_b) = 0.22, \quad a_4(\mu_b) = 0.084. \quad (A10) \]

The twist 3 two-particle wave functions \( \varphi_\nu \) and \( \varphi_3 \) are related to the three-particle wave function \( \varphi_{3\pi} \) by equation of motion. It is therefore sufficient to specify the latter \cite{[37, 15, 16]}. Including the first three nonasymptotic terms, which is consistent with retaining \( a_{2,4} \) in \( \varphi_3 \), one has
\[ \varphi_{3\pi}(\alpha_i, \mu) = 360\alpha_1\alpha_2\alpha_3^2 \left( 1 + \omega_{1,0}(\mu) \frac{1}{2}(7\alpha_3 - 3) \right. \]
\[ + \left. \omega_{2,0}(\mu)(2 - 4\alpha_1\alpha_2 - 8\alpha_3 + 8\alpha_3^2) + \omega_{1,1}(\mu)(3\alpha_1\alpha_2 - 2\alpha_3 + 3\alpha_3^2) \right) \]
(yielding
\[ \varphi_\nu(u, \mu) = 1 + B_2(\mu)\frac{1}{2}(3(u - \bar{u})^2 - 1) + B_4(\mu)\frac{1}{8}(35(u - \bar{u})^4 - 30(u - \bar{u})^2 + 3) \]
(A11)
and
\[ \varphi_\sigma(u, \mu) = 6u\bar{u}\left[ 1 + C_2(\mu)\frac{3}{2}(5(u - \bar{u})^2 - 1) + C_4(\mu)\frac{15}{8}(21(u - \bar{u})^4 - 14(u - \bar{u})^2 + 1) \right] \]
(A12)
with \( \bar{u} = 1 - u \) and
\[ B_2 = 30\frac{f_{3\pi}}{\mu_p f_\pi}, \quad B_4 = \frac{3}{2}\frac{f_{3\pi}}{\mu_p f_\pi}(4\omega_{2,0} - \omega_{1,1} - 2\omega_{1,0}), \]
\[ C_2 = \frac{f_{3\pi}}{\mu_p f_\pi}(5 - \frac{1}{2}\omega_{1,0}), \quad C_4 = \frac{1}{10}\frac{f_{3\pi}}{\mu_p f_\pi}(4\omega_{2,0} - \omega_{1,1}). \]
(A13)

The parameter \( f_{3\pi}(\mu) \) and the coefficients \( \omega_{i,k}(\mu) \) have again been estimated from sum rules \cite{[33]}:
\[ f_{3\pi}(1\text{GeV}) = 0.0035 \text{ GeV}^2, \]
\[ \omega_{1,0}(1\text{GeV}) = -2.88, \quad \omega_{2,0}(1\text{GeV}) = 10.5, \quad \omega_{1,1}(1\text{GeV}) = 0. \quad (A14) \]

After renormalization \cite{[37, 16]} to the relevant scales \( \mu_b \) and \( \mu_c \) one has
\[ f_{3\pi}(\mu_b) = 0.0026 \text{ GeV}^2, \quad \omega_{1,0}(\mu_b) = -2.18, \quad \omega_{2,0}(\mu_b) = 8.12, \quad \omega_{1,1}(\mu_b) = -2.59, \]
\[ f_{3\pi}(\mu_c) = 0.0032 \text{ GeV}^2, \quad \omega_{1,0}(\mu_c) = -2.63, \quad \omega_{2,0}(\mu_c) = 9.62, \quad \omega_{1,1}(\mu_c) = -1.05. \quad (A15) \]

The parameter \( \mu_\pi = m_\pi^2/(m_u + m_d) \) can be inferred from the PCAC relation \cite{[32]}:
\[ \mu_\pi(1\text{ GeV}) = 1.65 \text{ GeV}, \quad \mu_\sigma(\mu_c) = 1.76 \text{ GeV}, \quad \mu_\pi(\mu_b) = 2.02 \text{ GeV}. \quad (A16) \]

For convenience, we also list the complete set of twist 4 wave functions given in ref. \cite{[16]}. It includes four three-particle wave functions specified by two parameters:
\[ \varphi_\perp(\alpha_i, \mu) = 30\delta^2(\mu)(\alpha_1 - \alpha_2)\alpha_3^2 \left[ \frac{1}{3} + 2\varepsilon(\mu)(1 - 2\alpha_3) \right], \]
\[ \varphi_\parallel(\alpha_i, \mu) = 120\delta^2(\mu)\varepsilon(\mu)(\alpha_1 - \alpha_2)\alpha_1\alpha_2\alpha_3, \]
\[ \tilde{\varphi}_\perp(\alpha_i, \mu) = 30\delta^2(\mu)\alpha_3^2(1 - \alpha_3) \left[ \frac{1}{3} + 2\varepsilon(\mu)(1 - 2\alpha_3) \right], \]
\[ \tilde{\varphi}_\parallel(\alpha_i, \mu) = -120\delta^2\alpha_1\alpha_2\alpha_3 \left[ \frac{1}{3} + \varepsilon(\mu)(1 - 3\alpha_3) \right]. \quad (A18) \]
and two two-particle wave functions related to the former by equations of motion:

\[ g_1(u, \mu) = \frac{5}{2} \delta^2(\mu) u^2 u^2 + \frac{1}{2} \varepsilon(\mu) \delta^2(\mu) [\bar{u} u (2 + 13 \bar{u} u) + 10 u^3 \ln u (2 - 3 u + \frac{6}{5} u^2)] , \]

\[ g_2(u, \mu) = \frac{10}{3} \delta^2(\mu) \bar{u} u (u - \bar{u}) . \]  

(A19)

The parameter \( \delta^2 \) is actually defined by the matrix element

\[ \langle \pi | g_s \bar{d} G_{\alpha \mu} \gamma^\alpha u | 0 \rangle = i \delta^2 f_{\pi} q_\mu . \]  

(A20)

Renormalizing the values

\( \delta^2(1\text{GeV}) = 0.2 \text{ GeV}^2 \)  

(A21)

and

\( \varepsilon(1\text{GeV}) = 0.5 \)  

(A22)

obtained from sum rule estimates to the relevant scales \( \mu_c \) and \( \mu_b \), one finds

\[ \delta^2(\mu_c) = 0.19 \text{ GeV}^2, \quad \varepsilon(\mu_c) = 0.45 , \]

\[ \delta^2(\mu_b) = 0.17 \text{ GeV}^2, \quad \varepsilon(\mu_b) = 0.36 . \]  

(A23)

**Appendix 2**

Here, we give the explicit expressions for various subdominant contributions to the light-cone sum rule (97):

the surface term \( t^+ \)

\[ t^+(s^B, p^2, M^2) = \exp \left( - \frac{s^B}{M^2} \right) \left\{ \frac{\mu_\pi(m^2_b + p^2)}{6m_b(m^2_b - p^2)} \Phi_1(\Delta) \right\} \]

\[ - \frac{4m_b^2}{(m^2_b - p^2)^2} \left( 1 + \frac{s^B}{M^2} \right) g_1(\Delta) + \frac{4m_b^2}{(s^B - p^2)(m^2_b - p^2)} \int_0^\Delta \frac{dg_1(\Delta)}{du} \]

\[ + \frac{2}{m^2_b - p^2} \left[ 1 + \frac{m^2_b + p^2}{m^2_b - p^2} \left( 1 + \frac{s^B}{M^2} \right) \right] \int_0^\Delta g_2(v) dv - \frac{2(m^2_b + p^2)}{(m^2_b - p^2)(s^B - p^2)} g_2(\Delta) \} , \]  

(B1)

the contribution of three-particle wave functions \( f^+_G \)

\[ f^+_G(p^2, M^2) = - \int_0^1 u du \int \frac{d\alpha_i \Theta(\alpha_1 + u \alpha_3 - \Delta)}{(\alpha_1 + u \alpha_3)^2} \]

\[ \times \exp \left[ - \frac{m^2_b - p^2 (1 - \alpha_1 - u \alpha_3)}{(\alpha_1 + u \alpha_3) M^2} \right] \Phi_3(u, \alpha_i, M^2, p^2) , \]  

(B2)
where
\[
\Phi_3 = \frac{2f_{3\pi}}{f_{\pi} m_b} \varphi_{3\pi}(\alpha_i) \left[ 1 - \frac{m_b^2 - p^2}{(\alpha_1 + u\alpha_3) M^2} \right] - \frac{1}{uM^2} \left[ 2\varphi_{\perp}(\alpha_i) - \varphi_{\parallel}(\alpha_i) + 2\tilde{\varphi}_{\perp}(\alpha_i) - \tilde{\varphi}_{\parallel}(\alpha_i) \right],
\]
(B3)

and the perturbative correction \(\delta^+(p^2, M^2)\)
\[
\delta^+(p^2, M^2) = \frac{1}{\pi} \int_0^1 du \varphi_{\pi}(u, \mu) \int_{m_b^2}^{s_0^B} ds \text{Im}T_1 \left( \frac{p^2}{m_b^2}, \frac{s}{m_b^2}, u, \mu \right) \exp \left( -\frac{s}{M^2} \right),
\]
(B4)

where
\[
\frac{1}{\pi} \text{Im}T_1(r_1, r_2, u, \mu) = \delta(1 - \rho) \left[ \pi^2 - 6 + 3 \ln \frac{m_b^2}{\mu^2} - 2\text{Li}_2(r_1) \right] + 2\text{Li}_2(1 - r_2) - 2 \left( \ln \frac{r_2 - 1}{1 - r_1} \right)^2 + 2 \left( \ln r_2 + \frac{1 - r_2}{r_2} \right) \left( 2 \ln(r_2 - 1) - \ln(1 - r_1) \right) \]
\[+ \theta(\rho - 1) \left[ 8 \frac{\ln(\rho - 1)}{\rho - 1} \right] + 2 \left( \ln r_2 + \frac{1}{r_2} - 2 - 2 \ln(r_2 - 1) + \ln \frac{m_b^2}{\mu^2} \right) \frac{1}{\rho - 1}, \]
\[- 2 \frac{r_2 - 1}{(r_1 - r_2)(\rho - r_1)} \left( \ln \rho - 2 \ln(\rho - 1) + 1 - \ln \frac{m_b^2}{\mu^2} \right) \]
\[+ \frac{1 - r_1}{(r_1 - r_2)(r_2 - \rho)} \left( \ln \frac{\rho}{r_2} - 2 \ln \frac{\rho - 1}{r_2 - 1} \right) - 4 \ln \frac{\rho}{r_2} - 1 + 2 \frac{1}{r_2 - \rho} \left( \frac{1 - \frac{1}{r_2}}{\rho} \right) + 1 - \frac{1}{\rho} \]
\[+ \theta(1 - \rho) \left[ 2 \left( \ln r_2 + \frac{1}{r_2} - 2 \ln(r_2 - 1) + \ln \frac{m_b^2}{\mu^2} \right) \frac{1}{\rho - 1} \right] + \frac{1 - r_1}{(r_1 - r_2)(r_2 - \rho)} \left( \ln r_2 + 1 - 2 \ln(r_2 - 1) - \ln \frac{m_b^2}{\mu^2} \right) - 2 \frac{1 - r_2}{r_2 - \rho} \]
(B5)

with
\[
\rho = r_1 + u(r_2 - r_1), \quad \int d\rho f(\rho) \frac{1}{1 - \rho} = \int d\rho (f(\rho) - f(1)) \frac{1}{1 - \rho},
\]
(B6)

The surface term \(t^\pm\) appearing in the sum rule (B8) is given by
\[
t^\pm(s_0^B, p^2, M^2) = \exp \left( -\frac{s_0^B}{M^2} \right) \left[ -\frac{\varphi_0(\Delta)}{6\Delta} + \frac{2m_b g_2(\Delta)}{\mu_\pi (m_b^2 - p^2)} \right].
\]
(B7)
References

[1] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B147 (1979) 385, 448.

[2] Vacuum Structure and QCD Sum Rules, ed. M.A. Shifman (North-Holland, Amsterdam, 1992).

[3] L.J. Reinders, H.L. Rubinstein and S. Yasaki, Phys. Rep. 127 (1985) 1.

[4] M. Neubert and B. Stech, hep-ph/9705292 and this volume.

[5] V.A. Novikov, L.B. Okun, M.A. Shifman, A.I. Vainshtein, M.B. Voloshin and V.I. Zakharov, Phys. Rev. Lett. 38 (1977) 626.

[6] D.J. Broadhurst, Phys. Lett. B101 (1981) 423;
   D.J. Broadhurst and S.C. Generalis, preprint OUT-4102-8/R , 1982 (unpublished).

[7] T.M. Aliev and V.L. Eletsky, Sov. J. Nucl. Phys. 38 (1983) 936.

[8] M.B. Voloshin, Int. J. Mod. Phys. A10 (1995) 2865;
   M. Jamin and A. Pich, hep-ph/9702276.

[9] J. Gasser and H. Leutwyler, Phys. Rep. 87 (1982) 77.

[10] V.M. Belyaev and B.L. Ioffe, Sov. Phys. JETP 56 (1982) 493.

[11] C.A. Dominguez, Talk at 3rd Workshop on the Tau-Charm Factory, Marbella, Spain,
    1993, hep-ph/9309260.

[12] C.A. Dominguez and N. Paver, Phys. Lett. B197 (1987) 423;
    S. Narison, Phys. Lett. B198 (1987) 104;
    L.J. Reinders, Phys. Rev. D38 (1988) 423.

[13] J.M. Flynn, Proc. of 28th ICHEP, Warsaw, ed. Z. Ajduk and A.K. Wroblewski (World
    Scientific, Singapore, 1996) p. 335, hep-lat/9611016.

[14] C.R. Allton, L. Conti, M. Crisafulli, L. Giusti, G. Martinelli and F. Rapuano, Phys.
    Lett. B405 (1997) 133.

[15] H. Wittig, Int. J. Mod. Phys. A12 (1997) 4477.

[16] J. Adler et al. (MARK III Collab.), Phys. Rev. Lett. 60 (1988) 1375.

[17] J.D. Richman, Proc. of 28th ICHEP, Warsaw, ed. Z. Ajduk and A.K. Wroblewski
    (World Scientific, Singapore, 1996) p. 143.

[18] E. Shuryak, Nucl. Phys. B198 (1982) 83.

[19] M. Neubert, Phys. Rev. D45 (1992) 2451; ibid D46 (1992) 1076.

[20] E. Bagan, P. Ball, V.M. Braun and H.G. Dosch, Phys. Lett. B278 (1992) 457.
[21] T.M. Aliev, V.L. Eletsky and Ya.I. Kogan, Sov. J. Nucl. Phys. 40 (1984) 527.

[22] P. Ball, V.M. Braun and H.G. Dosch, Phys. Rev. D44 (1991) 3567.

[23] P. Ball, Phys. Rev. D48 (1993) 3190.

[24] C.A. Dominguez and N. Paver, Z. Phys. C41 (1988) 217;
   A.A. Ovchinnikov, Phys. Lett. B229 (1989) 127;
   S. Narison, Phys. Lett. B283 (1992) 384.

[25] P. Ball, V.M. Braun and H.G. Dosch, Phys. Lett. B273 (1991) 316.

[26] M. Wirbel, B. Stech and M. Bauer, Z. Phys. C29 (1985) 637;
   M. Bauer, B. Stech and M. Wirbel, Z. Phys. C34 (1987) 103.

[27] B.L. Ioffe and A.V. Smilga, Phys. Lett. B114 (1982) 353;
   A.V. Nesterenko and A.V. Radyushkin, Phys. Lett. B115 (1982) 410.

[28] A. Khodjamirian, Phys. Lett. 90B (1980) 460; Sov. J. Nucl. Phys. 39 (1984) 614.

[29] V.A. Beilin and A.V. Radyushkin, Sov. J. Nucl. Phys. 39 (1984) 800;
   Nucl. Phys. B260 (1985) 61.

[30] C.A. Dominguez, N. Paver and Riazuddin, Phys. Lett. B214 (1988) 459;
   T.M. Aliev, A.A. Ovchinnikov and V.A. Slobozhenik, Phys. Lett. B237 (1990) 569;
   P. Colangelo, C.A. Dominguez, G. Nardulli and N. Paver, Phys. Lett. B317 (1993) 183;
   S. Narison, Phys. Lett. B327 (1994) 354.

[31] P. Ball, preprint hep-ph/9308244 (unpublished).

[32] P. Colangelo, F. De Fazio, P. Santorelli and E. Scrimieri, Phys. Rev. D53 (1996) 3672.

[33] P. Ball and V.M. Braun, Phys. Rev. D55 (1997) 5561.

[34] A. Ali, V.M. Braun and H. Simma, Z. Phys. C63 (1994) 437.

[35] V.L. Chernyak and A.R. Zhitnitsky, JETP Lett. 25 (1977) 510; Yad. Fiz. 31 (1980) 1053.
   A.V. Efremov and A.V. Radyushkin, Phys. Lett. B94 (1980) 245; Teor. Mat. Fiz. 42 (1980) 147.
   G.P. Lepage and S.J. Brodsky, Phys. Lett. B87 (1979) 359; Phys. Rev. D22 (1980) 2157.

[36] S.J. Brodsky and G.P. Lepage, in: Perturbative Quantum Chromodynamics, ed. A.H. Mueller (World Scientific, Singapore, 1989) p. 93.

[37] V.L. Chernyak and A.R. Zhitnitsky, Phys. Rep. 112 (1984) 173.

[38] I.I. Balitsky, V.M. Braun and A.V. Kolesnichenko, Sov. J. Nucl. Phys. 44 (1986) 1028;
   Nucl. Phys. B312 (1989) 509.

[39] V.M. Braun and I.B. Filyanov, Z. Phys. C44 (1989) 157.
[40] V.L. Chernyak and I.R. Zhitnitsky, Nucl. Phys. B345 (1990) 137.

[41] V.M. Belyaev, A. Khodjamirian and R. Rückl, Z. Phys. C60 (1993) 349.

[42] V.M. Belyaev, V.M. Braun, A. Khodjamirian and R. Rückl, Phys. Rev. D51 (1995) 6177.

[43] A. Khodjamirian, R. Rückl and C. Winhart, in preparation.

[44] I.I. Balitsky and V.M. Braun, Nucl. Phys. B311 (1988) 541.

[45] A.S. Gorsky, Sov. J. Nucl. Phys. 41 (1985) 1008; ibid. 45 (1987) 512; ibid. 50 (1989) 498.

[46] V.M. Braun and I.B. Filyanov, Z. Phys. C48 (1990) 239.

[47] A. Khodjamirian, R. Rückl, S. Weinzierl and O. Yakovlev, hep-ph/97006303, to appear in Phys. Lett. B.

[48] E. Bagan, P. Ball and V.M. Braun, preprint NORDITA-97-59, hep-ph/9709243.

[49] F.M. Dittes and A.V. Radyushkin, Phys. Lett. B134 (1984) 359; M.H. Sarmadi, Phys. Lett. B143 (1984) 471; S.V. Mikhailov and A.V. Radyushkin, Nucl. Phys. B254 (1985) 89.

[50] P. Ball and V.M. Braun, Phys. Rev. D49 (1994) 2472.

[51] Particle Data Group, Phys. Rev. D54 (1996) 1.

[52] P. Colangelo, F. De Fazio, G. Nardulli, N. Di Bartolomeo and R. Gatto, Phys. Rev. D52 (1995) 6422.

[53] T.M. Aliev, D.A. Demir, E. Iltan and N.K. Pak, Phys. Rev. D53 (1996) 355.

[54] A.A. Ovchinnikov, Sov. J. Nucl. Phys. 50 (1989) 519.

[55] A.G. Grozin and O.I. Yakovlev, preprint BUDKERINP–94–3, hep-ph/9401267 (unpublished).

[56] P. Colangelo, G. Nardulli, A. Deandrea, N. Di Bartolomeo, R. Gatto and F. Feruglio, Phys. Lett. B339 (1994) 151.

[57] S. Nussinov and W. Wetzel, Phys. Rev. D36 (1987) 130.

[58] N. Isgur and M.B. Wise, Phys. Rev. D41 (1990) 151; ibid. D42 (1990) 2388.

[59] T.-M. Yan, H.-Y. Cheng, C.-Y. Cheung, G.-L. Lin, Y.C. Lin and H.-L. Yu, Phys. Rev. D46 (1992) 1148, Erratum, ibid. D55 (1997) 5851.

[60] P. Cho and H. Georgi, Phys. Lett. B296 (1992) 408.

[61] J.F. Amundson, C.G. Boyd, E. Jenkins, M. Luke, A.V. Manohar, J.L. Rosner, M.J. Savage and M.B. Wise, Phys. Lett. B296 (1992) 415.
[62] W.A. Bardeen and C.T. Hill, Phys. Rev. D49 (1994) 409.
[63] E. Eichten, K. Gottfried, T. Kinoshita, K.D. Lane and T.M. Yan, Phys. Rev. D21 (1980) 203.
[64] P.J. O’Donnell and Q.P. Xu, Phys. Lett. B336 (1994) 113.
[65] G.A. Miller and P. Singer, Phys. Rev. D37 (1988) 2564.
[66] R.L. Thews and A.N. Kamal, Phys. Rev. D32 (1985) 810.
[67] A.B. Kaidalov and A.V. Nogteva, Sov. J. Nucl. Phys. 47 (1988) 321.
[68] J.M. Flynn and C.T. Sachraida, [hep-lat/9710057](hep-lat/9710057)and this volume.
[69] M.B. Voloshin, Sov. J. Nucl. Phys. 50 (1989) 105.
[70] J.P. Alexander et al. (CLEO Collab.), Phys. Rev. Lett. 77 (1996) 5000.
[71] M.B. Wise, Phys. Rev. D45 (1992) R2188.
[72] G. Burdman, Z. Ligeti, M. Neubert and Y. Nir, Phys. Rev. D49 (1994) 2331.
[73] A.J. Buras, Nucl. Phys. B434 (1995) 606.
[74] M.S. Alam et al. (CLEO Collab.), Phys. Rev. D50 (1994) 43.
[75] A. Khodjamirian and R. Rückl, Nucl. Instr. and Methods A368 (1995) 28.
[76] B.Yu. Blok and M.A. Shifman, Sov. J. Nucl. Phys. 45 (1987) 135, 301, 522.
[77] A.J. Buras, J.-M. Gerard and R. Rückl, Nucl. Phys. B268 (1986) 16.
[78] B. Blok and M. Shifman, Nucl. Phys. B389 (1993) 534.
[79] I. Halperin, Phys. Lett. B349 (1995) 548.
[80] A. Khodjamirian, G. Stoll and D. Wyler, Phys. Lett. B358 (1995) 129.
[81] A. Ali and V. Braun, Phys. Lett. B359 (1995) 223.
[82] A. Khodjamirian, R. Rückl, G. Stoll and D. Wyler, Phys. Lett. B402 (1997) 167.
[83] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B237 (1984) 525; V.L. Chernyak, A.R. Zhitnitsky and I.R. Zhitnitsky, Sov. J. Nucl. Phys. 38 (1983) 645.
[84] I. Bigi, M. Shifman and N. Uraltsev, [hep-ph/9703290](hep-ph/9703290); M. Neubert, [hep-ph/9702377](hep-ph/9702377)and this volume.
[85] V.M. Belyaev and M.B. Johnson, preprint SPhT-t97/032, [hep-ph/9703244](hep-ph/9703244).
[86] P. Ball and V.M. Braun, Phys. Rev. D54 (1996) 2182.