A CONSTRUCTIVE PROOF OF NC FEJÉR-RIESZ THEOREM

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Abstract. In this paper we present a constructive proof of Popescu’s non-commutative Fejér-Riesz theorem for non-commuting polynomials. We are considering non-commuting polynomial in left-creation and left-annihilation multi-Toeplitz operators.

1. Introduction

The classical Fejér-Riesz states the following: if a trigonometric polynomial

\[ w(e^{it}) = \sum_{j=-m}^{m} c_j e^{ijt} \]

is nonnegative for all real \( t \), then it is expressible in the form

\[ w(e^{it}) = |p(e^{it})|^2 \]

for some analytic polynomial \( p(z) = \sum_{j=0}^{m} a_j z^j \). For proof refer to Lemma 2 in [1]. There is also an operator version where the coefficients of \( w \) are matrices or operators ([2]). The Fejér-Riesz theorem can be reformulated as a statement about Toeplitz operators: the function \( w \) may be interpreted as the symbol of a Toeplitz operator \( T_w \); in particular if \( S \) denotes the unilateral shift on \( \ell^2(\mathbb{N}) \) then \( T_w \) is the operator defined by

\[ T_w = c_0 I + \sum_{k=1}^{m} c_k S^k + \sum_{k=1}^{m} c_{-k} S^{*k}, \]

and then the factorization \( w = |p|^2 \) is equivalent to the factorization of operators

\[ T_w = T_p^* T_p \]

where \( T_p = \sum_{k=0}^{m} a_k S^k = p(S) \). The equivalence of the two formulations follows easily from the fact that \( S \) is an isometry (\( S^* S = I \)). It turns out that this operator formulation admits a generalization, in the noncommutative setting, to so-called multi-Toeplitz operators, where the single isometry \( S \) is replaced by a row isometry, for example the \( d \) tuple of left shifts, \((L_1, L_2, \cdots, L_d)\) or \( d \) tuple of right shifts, \((R_1, R_2, \cdots, R_d)\). Precise definitions are given in the next section. Following an idea of Dritschel and Woerdeman [3], this paper develops a constructive proof of Riesz-Fejér theorem in the noncommutative setting. We have a nonnegative multi-Toeplitz polynomial operator

\[ T_Q := Q_0 \otimes I_{\mathbb{F}^d} + \sum_{0<|v| \leq n} Q_v \otimes L_v + \sum_{0<|v| \leq n} Q_v^* \otimes L_v^*. \]

We then find the multi-Toeplitz operator factorization of this polynomial,

\[ T_Q := T_F^* T_F \]

where \( T_F := \sum_{0 \leq |v| \leq n} F_v \otimes L_v \) [This is a slight rewording of the theorem 1.6 [4]].
2. Preliminaries

Let us recall a few definitions required for the following section:

**Definition 2.1.** Let $\mathcal{F}_d^+$ denote the word set which is a monoid formed from the letters, 1, 2, \cdots, d. We say that the **Fock space** is $\ell^2(\mathcal{F}_d^+)$. The Fock space, $\ell^2(\mathcal{F}_d^+)$ is the Hilbert space with orthonormal basis $\{\xi_w\}_{w \in \mathcal{F}_d^+}$.

**Definition 2.2.** The **left-creation operator** $L_j$ is defined as $L_j\xi_w = \xi_{jw}$ for $j = 1, \cdots, d$ and can be extended linearly. Similarly, the **left-annihilation operator** $L_j^*$ is defined as

$$L_j^*\xi_w = \begin{cases} \xi_v, & w = jv \\ 0, & \text{otherwise} \end{cases}$$

for $j = 1, \cdots, d$. Thus $\{L_j | j = 1, \cdots, d\}$ form a system of isometries with orthogonal ranges: $L_i^*L_j = \delta_{ij}I$.

**Note** that same holds for the right shift operators as well. From the above definition we get that $(L_1, L_2, \cdots, L_d)$ and $(R_1, R_2, \cdots, R_d)$ are row-isometries. For any $w \in \mathcal{F}_d^+, w = i_1i_2\cdots i_n$ we denote $L_w = L_{i_1}L_{i_2}\cdots L_{i_n}$. So for any $w = i_1i_2\cdots i_n$ and $v = j_1j_2\cdots j_m$ in $\mathcal{F}_d^+$, $L_wL_v = L_{i_1}^*\cdots L_{i_n}^*L_{j_1}L_{j_2}\cdots L_{j_m}$. Thus

$$L_w^*L_v = \begin{cases} L_x, & v = wx \\ L_y, & v = xy \\ 0, & \text{otherwise} \end{cases}$$

**Definition 2.3.** In the classical setting, $T$ is said to be a **Toeplitz operator** if $S^*TS = T$ where $S$ is unilateral shift. An operator $T$ is a **$L$-multi-Toeplitz** if $L_j^*TL_j = \delta_{ij}T$ where $L_j$ is left-creation operator. Similarly, $T$ is called a **$R$-multi-Toeplitz** if $R_j^*TR_j = \delta_{ij}T$ where $R_j$ is right-creation operator.

**Example 1.** Any left-creation operator $L_w$ is $R$-multi-Toeplitz. Since $L_i$ and $R_j$ commute with each other, $R_j$ commutes with $L_w$ for all $w$ and thus we have

$$R_i^*L_wR_j = R_i^*R_jL_w = \delta_{ij}L_w$$

Similarly, $L_v^*$ is $R$-multi-Toeplitz for any word $v$. Therefore for any non commutative polynomials $f, g$, we have that $f(L)^* + g(L)$ is $R$-multi-Toeplitz.

Next let us consider a $R$-multi-Toeplitz operator say,

$$T := \sum_{0 \leq |v| \leq n} q_v L_v + \sum_{0 < |v| \leq n} q_v^* L_v^*$$

Then corresponding to the Fock space basis $\{\xi_v\}_{v \in \mathcal{F}_d^+}$ we get its matrix representation which is a multi-Toeplitz matrix:
Here we have used the lexicographic ordering for ordering the elements of the word set, \( \mathcal{F}_d^- \).

Now we do some relabeling of the indexes here and define for \( d = 1, \ldots, n \):

\[
q_k := \text{col}(q_w)_{w \in \mathcal{F}_d^+; |w|=k} \quad \text{and} \quad q_{-k} := \text{row}(q_w^*)_{w \in \mathcal{F}_d^+; |w|=k}
\]

and also identifying, \( q_1 \otimes I_d := (q_{11} \cdots 0) \) and so on.

Thus we get the following compact form of \( T \), which makes it easier to see multi-Toeplitz form of the matrix:

\[
T =
\begin{bmatrix}
q_0 & q_{-1} & q_{-2} & \cdots & q_{-n} & 0 & 0 & \cdots \\
q_1 & q_0 \otimes I_d & q_{-1} \otimes I_d & \cdots & q_{-(n-1)} \otimes I_d & q_{-n} \otimes I_d & 0 & \cdots \\
q_2 & q_1 \otimes I_d & q_0 \otimes I_d \otimes I_d & \cdots & q_{-(n-2)} \otimes I_d \otimes I_d & q_{-(n-1)} \otimes I_d \otimes I_d & q_{-n} \otimes I_d \otimes I_d & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
q_n & q_{n-1} \otimes I_d & q_{n-2} \otimes I_d \otimes I_d & \cdots & q_0 \otimes I_d \otimes \cdots \otimes I_d & q_{-1} \otimes I_d \otimes \cdots \otimes I_d & q_{-2} \otimes I_d \otimes \cdots \otimes I_d & \cdots \\
0 & q_n \otimes I_d & q_{n-1} \otimes I_d \otimes I_d & \cdots & q_1 \otimes I_d \otimes \cdots \otimes I_d & q_0 \otimes I_d \otimes (n+1) & q_{-1} \otimes I_d \otimes (n+1) & \cdots \\
0 & 0 & q_n \otimes I_d \otimes I_d & \cdots & q_2 \otimes I_d \otimes \cdots \otimes I_d & q_1 \otimes I_d \otimes (n+1) & q_{-1} \otimes I_d \otimes (n+2) & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]
We are going to use a Schur complement technique from Dritschel and Woerdeman [3] in the proof of the main theorem. So let us define the following:

**Definition 2.4.** If \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are Hilbert spaces and

\[
M = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2
\]

is a positive semidefinite operator then there exists a unique contraction \( G : \text{ran}(C) \rightarrow \text{ran}(A) \) such that \( B = A^{1/2}GC^{1/2} \). The **Schur complement of \( M \)** supported on \( \mathcal{H}_1 \) is defined to be positive semidefinite operator \( A^{1/2}(1 - GG^*)A^{1/2} \).

An alternative way to define the Schur complement of \( M \) supported on \( \mathcal{H}_1 \) is via

\[
\langle Sf, f \rangle = \inf \left\{ \left\langle \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle : g \in \mathcal{H}_2 \right\}
\]

that is, \( S : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \) is the largest positive semidefinite operator which may be subtracted from \( A \) in \( M \) such that the resulting operator matrix remains positive semidefinite.

**Remark.** Consider any positive semidefinite operator matrix, \( M \), say

\[
M = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2
\]

and let \( S_M \) be the Schur complement of \( M \) supported on \( \mathcal{H}_1 \). Then for a positive semidefinite matrix, \( M \otimes I_d \), the Schur complement supported on \( \mathcal{H}_1 \otimes \mathbb{C}^d \) is \( S_{M \otimes I_d} = S_M \otimes I_d \).

If \( Q \) is an operator from \( \mathcal{H} \) to \( \mathcal{H} \) for some Hilbert space \( \mathcal{H} \), then \( Q \otimes I_d \) takes values from \( \mathcal{H} \otimes \mathbb{C}^d \) and outputs in \( \mathcal{H} \otimes \mathbb{C}^d \).

Let us denote \( \mathcal{H}_i = \mathcal{H} \otimes (\mathbb{C}^d)^\otimes i \) for \( i \geq 0 \).

We make use of the following notation for the next section from [3]: Typically we will index rows and columns of an \( n \times n \) matrix with \( 0, \ldots, n - 1 \). For \( \Lambda \subseteq \{0, \ldots, n - 1\} \) and an \( n \times n \) matrix \( M \), we write \( S(M; \Lambda) \), or \( S(\Lambda) \) when there is no chance of confusion, for the Schur complement supported on the rows and columns labeled by elements of \( \Lambda \). It is usual to view \( S(\Lambda) \) as an \( m \times m \) matrix, where \( m = \text{card} \Lambda \), however it is often useful to take \( S(\Lambda) \) as an \( n \times n \) matrix by padding rest of the entries in this \( n \times n \) matrix with zeros. For notational convenience we have used \( S(m) \) for Schur complement supported on rows and columns labeled by \( \{0, \ldots, m\} \).

## 3. MAIN THEOREM

The following Theorem 3.1 and Corollary 3.2 are multi-Toeplitz versions of Proposition 3.1 and corollary 3.2 from [3].

**Theorem 3.1.** Consider the positive semidefinite multi-Toeplitz operator matrix

\[
T_Q = \begin{bmatrix} Q_0 & Q_{-1} & Q_{-2} & \cdots & \cdots \\ Q_1 & Q_0 \otimes I_d & Q_{-1} \otimes I_d & \cdots & \cdots \\ Q_2 & Q_1 \otimes I_d & Q_{0} \otimes I_d \otimes I_d & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}
\]

acting on \( \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \). Then Schur complement of \( T_Q \) satisfies the recurrence relation:

\[
S(m) = \begin{bmatrix} A & B^* \\ B & S(m - 1) \otimes I_d \end{bmatrix}
\]
for appropriate choice of \( A : \mathcal{H}_0 \to \mathcal{H}_0 \) and \( B^* : \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \to \mathcal{H} \). When \( Q_j = 0 \) for \( j \geq m + 1 \), then \( A = Q_0 \) and \( B = \text{col}(Q_i)_{i=1}^m \).

**Remark.** Given

\[
T_Q = \begin{bmatrix}
Q_0 & Q_{-1} & Q_{-2} & \cdots & \cdots \\
Q_1 & Q_0 \otimes I_d & Q_{-1} \otimes I_d & \cdots & \cdots \\
Q_2 & Q_1 \otimes I_d & Q_0 \otimes I_d \otimes I_d & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix},
\]

we observe that \( T_Q \) can be identified with

\[
T_Q = \begin{bmatrix}
Q_0 & \text{row}(Q_{-j})_{j \geq 1} \\
\text{col}(Q_j)_{j \geq 1} & T_Q \otimes I_d \\
\end{bmatrix}.
\]

**Proof.** Let us write

\[
S(m) = \begin{bmatrix}
A & B^* \\
B & C \\
\end{bmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_m \to \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_m.
\]

By definition of Schur complement we have that

\[
T_Q - \begin{bmatrix}
S(m) & 0 \\
0 & 0 \\
\end{bmatrix} \geq 0.
\]

That is, we have that

\[
(2) \quad \begin{bmatrix}
Q_0 & \text{row}(Q_{-j})_{j \geq 1} \\
\text{col}(Q_j)_{j \geq 1} & T_Q \otimes I_d \\
\end{bmatrix} - \begin{bmatrix}
A & 0 & 0 \\
B & C & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \geq 0.
\]

Then leaving out 0th row and column in (2) we get,

\[
T_Q \otimes I_d - \begin{bmatrix}
0 & 0 \\
C & 0 \\
0 & 0 \\
\end{bmatrix} \geq 0.
\]

So we have from theorem 3.1, \( C \leq S(m - 1) \otimes I_d \).

Now leaving out the rows and columns 1, \ldots, m in (2), we get

\[
\begin{bmatrix}
Q_0 & \text{row}(Q_{-j})_{j \geq m+1} \\
\text{col}(Q_j)_{j \geq m+1} & T_Q \otimes I_d^{m+1} \\
\end{bmatrix} \geq 0.
\]

That is,

\[
A \leq S \left( \begin{bmatrix}
Q_0 & \text{row}(Q_{-j})_{j \geq m+1} \\
\text{col}(Q_j)_{j \geq m+1} & T_Q \otimes I_d^{m+1} \\
\end{bmatrix} ; 0 \right) := \tilde{A}.
\]

Note that when \( Q_j = 0, j \geq m + 1 \) then \( \tilde{A} = Q_0 \).

Again considering the following operator matrix

\[
(3) \quad \begin{bmatrix}
Q_0 - \tilde{A} & X \\
X^* & (Q_{-j} \otimes I_d)_{i=1,j=1}^m \otimes (Q_{i,j} \otimes I_d)_{i=1,j=1}^{m+1,\infty} - S(m-1) \otimes I_d \\
\text{col}(Q_j)_{j \geq m+1} \otimes I_d & (Q_{-j} \otimes I_d)_{i=1,j=1}^{\infty,m} \otimes I_d \otimes I_d^{m+1} \\
\end{bmatrix}
\]
The existence of an operator $X$ making this into a positive semidefinite matrix is a variant of a standard operator matrix completion problem, (see Theorem XVI.3.1 in [5]) so there always exists such an $X$. Note that when $\hat{A} = Q_0$ we have necessarily that $X = 0$. We fix such an $X$. Now (3) is positive semidefinite, we obtain that
\[
\begin{bmatrix}
\hat{A} & \text{row}(Q_j^*)_{j=1}^m - X \\
\text{col}(Q_j)_{j=1}^m - X & S(m-1) \otimes I_d
\end{bmatrix} \leq S(m) = \begin{bmatrix}A & B^* \\ B & C\end{bmatrix}.
\]

This implies that $\hat{A} \leq A$ and $S(m-1) \otimes I_d \leq C$. From above we also have $A \leq \hat{A}$ and $C \leq S(m-1) \otimes I_d$, thus the equalities $A = \hat{A}$ and $C = S(m-1) \otimes I_d$ follow.

Moreover, if $Q_j = 0$ for $j \geq m + 1$, we have that $\hat{A} = Q_0$ and $X = 0$, and thus $B = \text{col}(Q_j)_{j=1}^m$.

\begin{proof}
We will prove this by induction on $m$.

\textbf{Base step:} $S(0)$ being a positive semidefinite operator, we can write $S(0) = F_0^* F_0$ where $F_0 = (S(0))^{1/2}$.

\textbf{Induction hypothesis:} Let us assume that the result holds for $S(m-1)$.

By [8], Proposition 3.1, we have that $(S(m))_{m,m} = (S(m-1))_{m-1,m-1} \otimes I_d = F_0^* F_0 \otimes I_d^{\otimes (m-1)} \otimes I_d = F_0^* F_0 \otimes I_d^{\otimes m}$.

From [3] Corollary 2.3, we have that $S(m-1) = S(S(m); m-1)$. Thus applying Lemma 2.1 from [3] to
\begin{equation}
P = \begin{bmatrix}
F_0 & F_0 \otimes I_d \\
F_1 & F_1 \otimes I_d \\
\vdots & \vdots \\
F_{m-1} & F_{m-2} \otimes I_d \\
F_{m-2} & F_{m-1} \otimes I_d
\end{bmatrix}, \quad R = F_0 \otimes I_d^{\otimes m}
\end{equation}
there exist $(G_m \cdots G_1)$ so that
\[
S(m) = \begin{bmatrix}
F_0^* & F_0^* & F_2^* & \cdots & G_m^* \\
F_0^* \otimes I_d & F_1^* \otimes I_d & \cdots & G_{m-1}^* \\
F_0^* \otimes I_d^{\otimes 2} & F_2^* \otimes I_d^{\otimes 2} & \cdots & G_{m-2}^* \\
\vdots & \vdots & \vdots & \ddots \\
F_0^* \otimes I_d^{\otimes m} & \cdots & \cdots & \cdots & G_m \quad G_{m-1} \quad G_{m-2} \quad \cdots \quad F_0 \otimes I_d^{\otimes m}
\end{bmatrix}.
\]
and \( \text{ran}(G_m \cdots G_1) \subseteq \text{ran}F_0 \otimes (\mathbb{C}^d)^{\otimes m} \). Comparing with \( S(m) = \begin{bmatrix} A & B^* \\ B & S(m-1) \otimes I_d \end{bmatrix} \) along with the induction hypothesis yields, \( S(m-1) \otimes I_d \) factors into

\[
\begin{bmatrix}
F_0 \otimes I_d & F_1 \otimes I_d & \cdots & F_{m-2} \otimes I_d & G_{m-1} \\
F_0 \otimes I_d^{\otimes 2} & F_1 \otimes I_d^{\otimes 2} & \cdots & F_{m-2} \otimes I_d^{\otimes 2} & G_{m-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_0 \otimes I_d^{\otimes (m-1)} & F_1 \otimes I_d^{\otimes (m-1)} & \cdots & F_{m-2} \otimes I_d^{\otimes (m-1)} & G_m
\end{bmatrix}
\]

and thus we have

\[
F_0 \otimes I_d^{\otimes m} \quad (G_{m-1} \quad G_{m-2} \quad \cdots \quad G_1) = F_0 \otimes I_d^{\otimes m} \quad (F_{m-1} \otimes I_d \quad F_{m-2} \otimes I_d^{\otimes 2} \quad \cdots \quad F_1 \otimes I_d^{\otimes (m-1)})
\]

As

\[
\text{ran}(G_{m-1} \quad G_{m-2} \quad \cdots \quad G_1) \subseteq \text{ran}F_0 \otimes (\mathbb{C}^d)^{\otimes m}
\]

and

\[
\text{ran}(F_{m-1} \otimes I_d \quad F_{m-2} \otimes I_d^{\otimes 2} \quad F_{m-3} \otimes I_d^{\otimes 3} \quad \cdots \quad F_1 \otimes I_d^{\otimes (m-1)}) \subseteq \text{ran}F_0 \otimes (\mathbb{C}^d)^{\otimes m}
\]

it follows that \( G_j = F_j \otimes I_d^{\otimes (m-j)} \) for \( j = 1, \cdots, m-1 \). By setting \( F_m := G_m \), we have our result. \( \Box \)

**Theorem 3.3.** If \((L_1, L_2, \cdots, L_d)\) is the left \(d\)-shift, \( \mathcal{H} \) a Hilbert space and \( \{Q_w\}_{|w| \leq n} \) are operators \( Q_w : \mathcal{H} \to \mathcal{H} \) such that the operator \( T_Q : \mathcal{H} \otimes \ell^2(\mathcal{F}_d^+) \to \mathcal{H} \otimes \ell^2(\mathcal{F}_d^+) \) given by

\[
T_Q := Q_0 \otimes I_{\ell^2(\mathcal{F}_d^+)} + \sum_{0 < |v| \leq n} Q_v \otimes L_v + \sum_{0 < |v| \leq n} Q_v^* \otimes L_v^*
\]

is positive, then there exist operators \( F_0, \cdots, F_w : \mathcal{H} \to \mathcal{H}, \ |w| \leq n \), such that for \( T_F := \sum_{0 \leq |w| \leq n} F_v \otimes L_v \) we have \( T_Q = T_F^* T_F \).

**Proof.** Note that this operator, \( T_Q \) is \( R \) multi-Toeplitz operator. Let us consider the matrix representation of this operator, \( T_Q \) corresponding to basis, \( \beta = \{e_i \otimes \xi^v\}_{i,v} \) where \( \{e_i\}_i \) is some orthonormal basis for \( \mathcal{H} \) and \( v \in \mathcal{F}_d^+ \). It is a multi-toeplitz operator matrix (corresponding to polynomial (1)) as was defined on page 3 with operator entries \( Q_j \) for \( j \in \mathcal{F}_d^+ \) instead. Then as done before, after relabeling the indexes we get the multi-Toeplitz matrix:
In this case we have that \( Q_j = 0 \) for \( |j| \geq n + 1 \).

Now consider the Schur complement, \( S(n) = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \) (that is, supported on first \( n + 1 \) rows and columns of \( T_Q \)).

Thus from theorem 3.1 we get that

\[
A = Q_0 \quad \text{and} \quad B = \text{col}(Q_i)_{i=1}^n.
\]

Comparing the first row of \( S(n) \) above with the first row of the product in corollary 3.2 factorization we get that:

\[
Q_0 = \sum_{j=0}^{n} F_j^* F_j;
\]

\[
Q_{-j} = \sum_{k=j}^{n} F_k^* (F_{k-j} \otimes I_d^{\otimes j}) \quad \text{for} \quad 1 \leq j \leq n.
\]

These \( F_j \) for \( i = 0, \cdots, n \) are known from corollary 3.2.

Due to the Toeplitz structure and self-adjointness of the matrix, \( T_Q \), the above equations give us the following factorization:

\[
T_Q = T_F^* T_F
\]

where

\[
T_F = \begin{bmatrix}
Q_0 & Q_1 & Q_2 & \cdots & Q_{n-1} & Q_{n} \\
Q_1 & Q_0 \otimes I_d & Q_1 \otimes I_d & \cdots & Q_{(n-1)} \otimes I_d & Q_{n} \otimes I_d \\
Q_2 & Q_1 \otimes I_d & Q_0 \otimes I_d \otimes I_d & \cdots & Q_{(n-2)} \otimes I_d \otimes I_d & Q_{(n-1)} \otimes I_d \otimes I_d & Q_{n} \otimes I_d^{\otimes 2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
Q_n & Q_{n-1} \otimes I_d & Q_{n-2} \otimes I_d \otimes I_d & \cdots & Q_1 \otimes I_d^{\otimes n} & Q_0 \otimes I_d^{\otimes (n+1)} & Q_{n} \otimes I_d^{\otimes (n+2)} \\
0 & Q_n \otimes I_d & Q_{n-1} \otimes I_d \otimes I_d & \cdots & Q_2 \otimes I_d^{\otimes n} & Q_1 \otimes I_d^{\otimes (n+1)} & Q_{n} \otimes I_d^{\otimes (n+2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
F_n & F_{n-1} \otimes I_d & F_{n-2} \otimes I_d^{\otimes 2} & \cdots & Q_0 \otimes I_d^{\otimes n} & Q_{n} \otimes I_d^{\otimes (n+1)} & Q_{n} \otimes I_d^{\otimes (n+2)} \\
0 & F_n \otimes I_d & F_{n-1} \otimes I_d^{\otimes 2} & \cdots & F_1 \otimes I_d^{\otimes n} & F_0 \otimes I_d^{\otimes (n+1)} & F_{n} \otimes I_d^{\otimes (n+2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots 
\end{bmatrix}
\]
In compact representation of $T_F$, recall $F_i = \text{col}(F_v)_{|v|=i}$. Thus the operator corresponding to the above matrix is,

$$T_F = \sum_{0 \leq |v| \leq n} F_v \otimes L_v.$$

Therefore, we have a multi-toeplitz factorization for a multi-toeplitz positive semidefinite matrix, $T_Q$ which is of the form $T_F^* T_F$.

\[\Box\]

**Remark:** The above polynomial operator, $T_Q$ (since its a polynomial in the left shifts, $L$) is $R$-multi-Toeplitz operator. Analogously, we have similar result for the polynomial operators in right shifts.

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