Estimation of Parameters of Stable Distributions

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Abstract
In this paper, we propose a method based on GMM (the generalized method of moments) to estimate the parameters of stable distributions with $0 < \alpha < 2$. We don't assume symmetry for stable distributions.

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1 Introduction

The class of stable distributions is an important probability distribution class, which can be viewed as the sum of a large number of independent and identically distributed random variables with very small individual effects. The stable non-Gaussian distributions has the property that the second moment is infinite. Plenty of evidence suggested that some important economic variables such as stock price changes, interest rate changes, and price expectations etc. can be better described by stable distributions, in most cases, by stable non-Gaussian distributions, see Mandelbrot (1963a,b), Fama (1965) etc.. It was suggested by Bartels (1977), Koenker and Bassett (1978) that the distribution of the regression disturbance may also belong to the class of stable distributions.
One dimensional stable distributions with parameter $\alpha = 1$ and $\alpha = 2$ are Cauchy distribution and normal distribution respectively. Both are well studied.

For stable distributions with $0 < \alpha < 2$, since there are no closed form expressions for the density function in most cases, estimation of parameters encounters difficulties. Some important works in this area were done by Fama and Roll (1968, 1971) and others. Stable distributions were also used in tests for normality by Bera and McKenzie (1986).

Previous methods on estimation of parameters of stable distributions include the fractile method of Fama and Roll (1968, 1971), the improved version of McCulloch (1986), the approximate maximum likelihood theory developed by DuMouchel (1973a, b, 1975), and the iterative regression method of Koutrouvelis (1980, 1981) etc.. Akgiray and Lamoureux (1989) made a comparative study of the fractile method and iterative regression method.

In this paper we propose a method based on GMM (the generalized method of moments) to estimate simultaneously all the parameters of stable distributions with $1 < \alpha < 2$ and $0 < \alpha < 1$. We don’t assume the symmetry of stable distribution here.

Stable distributions with $0 < \alpha < 2$ and $\alpha \neq 1$ may be defined by the characteristic function

$$\hat{\mu} = \exp[-c|z|^\alpha (1 - i\beta \tan(\frac{\pi \alpha}{2}\text{sgn}z)) + i\tau z],$$

with $c > 0, \beta \in [-1, 1]$ and $\tau \in \mathbb{R}$, where $\tan$ is tangent function and $\text{sgn}$ is sign function. $\beta = 0$ corresponds to symmetric stable distributions. $\beta \neq 0$ corresponds to nonsymmetric stable distributions. Two special cases $\beta = 1$ and $\beta = -1$ correspond to one sided stable distributions. Here we focus on the cases that $-1 < \beta < 1$. $\tau$ is the drift term. For detailed account of stable distributions, see Sato (1999) and Zolotarev (1983).

We assume the random variable $x$ has the stable distribution with parameters $\alpha, \beta, \tau, \text{ and } c$.

In order to present infinite and finite series expressions of the density function of the stable distribution in Sato (1999), we introduce a new parameter $\tilde{c}$ which
satisfies
\[ c = \cos\left(\pi \beta \frac{2 - \alpha}{2}\right) \tilde{c}, \quad \text{if } 1 < \alpha < 2; \]
\[ c = \cos\left(\pi \beta \frac{\alpha}{2}\right) \tilde{c}, \quad \text{if } 0 < \alpha < 1. \]

Thus estimation of \( \tilde{c} \) is equivalent to estimation of \( c \). Consider the new variable
\[ u = \frac{x - \tau}{\tilde{c}}. \]

\( u \) has \( \cos\left(\pi \beta \frac{2 - \alpha}{2}\right) \) if \( 1 < \alpha < 2 \) and \( \cos\left(\pi \beta \frac{\alpha}{2}\right) \) if \( 0 < \alpha < 1 \) as a parameter which appears in the position of \( c \) in the characteristic function expression (1.1).

By Sato (1999), we have the following infinite and finite series expressions of the density function associated with variable \( u \).

1. When \( 1 < \alpha < 2 \), the convergent series expression for the density function is
\[
p_\alpha(u) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} (\sin(\pi n(1 - \beta \frac{2 - \alpha}{\alpha})) / 2) u^{n-1}, \quad \text{for } u \in \mathbb{R},
\]
(1.2)
and for \( u \to \infty \), the finite series expression is:
\[
p_\alpha(u) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} (\sin(\pi n(1 - \beta \frac{2 - \alpha}{\alpha})) / 2) u^{-n\alpha-1} + O(u^{-(N+1)\alpha-1}).
\]
(1.3)

For \( u \to -\infty \), since the dual distribution \( \tilde{u} = -u \) has the density \( \tilde{p}_\alpha(u) = p_\alpha(-u) \), the finite series expression is:
\[
p_\alpha(u) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} (\sin(\pi n(1 + \beta \frac{2 - \alpha}{\alpha})) / 2) |u|^{-n\alpha-1} + O(|u|^{-(N+1)\alpha-1}).
\]
(1.4)

2. When \( 0 < \alpha < 1 \), for \( u > 0 \), the convergent series expression for the density function is
\[
p_\alpha(u) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} (\sin(\pi n(1 + \beta) / 2\alpha)) u^{-n\alpha-1},
\]
(1.5)
and for $u < 0$, the convergent series expression for the density function is

$$p_\alpha(u) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} (\sin(\pi n(1-\beta)/(2\alpha))) |u|^{-n\alpha - 1},$$

(1.6)

which is obtained by duality argument similar to the above.

For $u \to 0$, the finite series expression is:

$$p_\alpha(u) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} (\sin(\pi n(1+\beta)/(2))) u^{n-1} + O(u^N).$$

(1.7)

In the following two sections, we discuss a method based on GMM to estimate parameters $\alpha, \beta, \tau, \hat{c}$. For convenience, we denote the vector $(\alpha, \beta, \tau, \hat{c})$ by $\theta$ and the vector $(\alpha_0, \beta_0, \tau_0, \hat{c}_0)$ by $\theta_0$. In section 2, we look at the case when $1 < \alpha < 2$. In section 3, we study the case when $0 < \alpha < 1$.

2 $1 < \alpha < 2$

Let us consider the following moment functions:

$$f_k(y; \theta) = \begin{cases} d_1 \left( \frac{y-\tau}{c} \right)^{1+1/(k+1)}, & \text{if } y \geq R_1; \\ d_2 \left( \frac{\tau-y}{c} \right)^{1+1/(k+1)}, & \text{if } y \leq R_2, \end{cases}$$

(2.1)

for $k = 1, 2, ..., m$ with $m > 4$, where the exponent $2(1 + \frac{1}{2(k+1)})$ means first take square and then take exponent $1 + \frac{1}{2(k+1)}$. $R_1$ and $R_2$ are constants such that $R_2 < \tau < R_1$ with $\frac{R_1-\tau}{c}$ and $\frac{\tau-R_2}{c}$ not large. The reason for this requirement is that the convergent formula (1.2) of the density function is appropriate for small $\tau$ and the finite series expressions (1.3) and (1.4) are appropriate for $\tau$ is large. This phenomenon was pointed out by Famma and Roll (1968) for symmetric stable distribution. When we solve the optimization problem resulted by GMM, if the solution for $\tau$ is very close to $R_1$ or $R_2$, it means that we may not get the globally
optimization yet. We have to make $R_1$ bigger or $R_2$ bigger, and do optimization again with the solution from previous step as the initial point. About $d_1$ and $d_2$ in the above expression, they are constants satisfying

$$d_1 = ((R_1 - \tau)/\tilde{c})^2 \quad \text{and} \quad d_2 = ((\tau - R_2)/\tilde{c})^2.$$ 

We can see that $f_k(y; \theta), k = 1, ..., m$, are continuous functions.

Let $x$ be the random variable of stable distribution with parameters $\theta$. Then

$$E(f_k(x; \theta)) = \int_{-\infty}^{\infty} f_k(y; \theta)p_\alpha(y - \tau\tilde{c})\frac{1}{\tilde{c}}dy,$$  \hspace{1cm} (2.2)

where $p_\alpha$ is the density function in (1.1), (1.2) and (1.3).

Let $x_1, x_2, ..., x_n$ be independent stable distributions with the same parameter vector $\theta_0$. Since $E|f_k(x_i; \theta)| < \infty, \ i = 1, ..., n, k = 1, ..., m$, by strong law of large numbers, we have

$$\frac{\sum_{i=1}^{n} f_k(x_i; \theta_0)}{n} \to E(f_k(x; \theta_0)) \ \text{almost surely,} \ k = 1, ..., m,$$

as $n \to \infty$.

Denote the sample moment $\frac{\sum_{i=1}^{n} f_k(x_i; \theta_0)}{n}$ by $E_n f_k(x; \theta), k = 1, ..., m$.

We can see that $E_n f_k(x; \theta)$ and $Ef_k(x; \theta)$ have first derivatives (vector) with respect to $\theta = (\alpha, \beta, \tau, \tilde{c})$. Define

$$V = c_1[E_n f_1(x; \theta_0) - Ef_1(x; \theta)]^2 + \cdots + c_m[E_n f_m(x; \theta_0) - Ef_m(x; \theta)]^2,$$  \hspace{1cm} (2.3)

where $c_1, ..., c_m$ are positive constants. Then by GMM, the minimization solution $\hat{\theta}$ of $V$ converges in distribution to $\theta_0$.

Since we don’t have closed form expression for the density of stable distributions, we need approximate finite series expressions for $Ef_k(x; \theta), k = 1, ..., m$. When we minimize the function $V$ with respect to the parameter vector $\theta$, we use the approximate finite series expressions for $Ef_k(x; \theta)$ instead of $Ef_k(x; \theta), k = 1, ..., m$ themselves.

In the following, we will show the existence of approximate finite series expressions for $Ef_k(x; \theta), k = 1, ..., m$. 

5
For positive integer $N > 0$, define $p_{1,N}$, $p_{2,N}$ and $p_{3,N}$ as follows:

\[ p_{1,N}(u) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n \alpha + 1)}{n!} (\sin(\pi n (1 - \frac{2 - \alpha}{\alpha})) / 2) \sqrt{\frac{2 - \alpha}{\alpha}} u^{-n} - 1, \quad \text{for } u > 0, \]

\[ p_{2,N}(u) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n \alpha + 1)}{n!} (\sin(\pi n (1 + \frac{2 - \alpha}{\alpha})) / 2) u^{-n} - 1, \quad \text{for } u < 0, \]

\[ p_{3,N}(u) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} (\sin(\pi n (1 - \frac{2 - \alpha}{\alpha})) / 2) u^{-n} - 1, \quad \text{for } u \in \mathbb{R}, \]

with $p_{1,N}$, $p_{2,N}$ and $p_{3,N}$ corresponding to (1.3), (1.4) and (1.2) respectively.

By (1.3), we have

\[
\int_{R_{1}}^{\infty} f_{k}(y; \theta) |p_{\alpha}((y - \tau) / \tilde{c}) - p_{1,N}((y - \tau) / \tilde{c})|^{1/\tilde{c}} \, dy \\
\leq M_{1} \int_{R_{1} - \tau}^{\infty} u^{1/\tilde{c}} u^{-(N+1)\alpha - 1} \, du \\
\leq M_{1} \int_{R_{1} - \tau}^{\infty} u^{-(N+1)\alpha - 1 + \frac{1}{\tilde{c} + 1}} \, du \\
\leq O(((R_{1} - \tau) / \tilde{c})^{-(N+1)\alpha + \frac{1}{\tilde{c} + 1}}),
\]

which goes to 0 as $N \to \infty$, when $(R_{1} - \tau) / \tilde{c} > 1$.

Similarly, by (1.4), we have

\[
\int_{-\infty}^{R_{2}} f_{k}(y; \theta) |p_{\alpha}((y - \tau) / \tilde{c}) - p_{2,N}((y - \tau) / \tilde{c})|^{1/\tilde{c}} \, dy \\
\leq O(((\tau - R_{2}) / \tilde{c})^{-(N+1)\alpha + \frac{1}{\tilde{c} + 1}}),
\]

which goes to 0 as $N \to \infty$ when $(\tau - R_{2}) / \tilde{c} > 1$.

By (1.2), we have

\[
p_{1,N}((y - \tau) / \tilde{c}) \to p_{\alpha}((y - \tau) / \tilde{c}) \quad \text{pointwisely for } y \in [R_{2}, R_{1}], \quad \text{as } N \to \infty,
\]

and the convergence is uniform when both $(R_{1} - \tau) / \tilde{c}$ and $(\tau - R_{2}) / \tilde{c}$ are less than 1.
In fact, we can show that for given \( \theta = (\alpha, \beta, \tau, \tilde{c}) \), given \( R_1 \) and \( R_1 \), without the assumption that \( ((R_1 - \tau)/\tilde{c}) \) and \( ((\tau - R_2)/\tilde{c}) \) are less than 1,

\[
p_{1,N}(y - \tau)/\tilde{c}) \to p_\alpha((y - \tau)/\tilde{c}) \quad \text{uniformly for all } y \in [R_2, R_1], \quad \text{as } N \to \infty \tag{2.9}
\]

To show this statement, we need the following lemma:

**Lemma 2.1** for given \( M > 0, \ L > 1 \), there exists a positive integer \( n_0 \), which depends on \( L, M \) and \( 1 < \alpha < 2 \), such that when \( n \geq n_0 \),

\[
\frac{\Gamma(n/\alpha + 1)}{n!} M^{n-1} \leq L^{-n}. \tag{2.10}
\]

**Proof.** We know that

\[
\Gamma(n/\alpha + 1) \leq C_0 ([n/\alpha] + 1)!, \tag{2.11}
\]

where \([n/\alpha]\) is the greatest integer less than \( n/\alpha \) and \( C_0 \) is a positive constant.

Since

\[
\frac{n!}{m!(n-m)!} \geq 1, \quad \text{for } m = 0, 1, ..., n, \tag{2.12}
\]

letting \( m = [n/\alpha] + 1 \), then

\[
\frac{([n/\alpha] + 1)!}{n!} \leq \frac{1}{(n - [n/\alpha] - 1)!}, \tag{2.13}
\]

which means

\[
\frac{\Gamma(n/\alpha + 1)}{n!} M^{n-1} \leq C_0 \frac{([n/\alpha] + 1)!}{n!} M^{n-1} \leq C_0 \frac{1}{(n - [n/\alpha] - 1)!} M^{n-1}. \tag{2.14}
\]

By Stirling’s formula,

\[
(n - [n/\alpha] - 1)! \sim (n - [n/\alpha] - 1)^{n-[n/\alpha]-1} e^{-n-[n/\alpha]-1} \sqrt{2\pi(n-[n/\alpha]-1)}, \tag{2.15}
\]

as \( n \to \infty \).

Thus, in order to show the lemma, it is sufficient to show that there exists \( n_0 > 0 \) such that for any \( n \geq n_0 \),

\[
C_0 M^{n-1} L^n < (n - [n/\alpha] - 1)^{(n-[n/\alpha]-1)} e^{-(n-[n/\alpha]-1)} \sqrt{2\pi(n-[n/\alpha]-1)}. \tag{2.16}
\]

It is clear that

\[
\log\left\{(n - [n/\alpha] - 1)^{(n-[n/\alpha]-1)} e^{-(n-[n/\alpha]-1)} \sqrt{2\pi(n-[n/\alpha]-1)}\right\}
\]

\[
= (n - [n/\alpha] - 1) \log(n - [n/\alpha] - 1) - (n - [n/\alpha] - 1) + \frac{1}{2} \log(2\pi) \tag{2.17}
\]

\[
+ \frac{1}{2} \log(n - [n/\alpha] - 1),
\]
where \((n - \lfloor n/\alpha \rfloor - 1) \log(n - \lfloor n/\alpha \rfloor - 1)\) becomes the dominant term when \(n\) is large enough.

We can see that when \(n\) is large enough,

\[
\log C_0 + (n - 1) \log M + n \log L < (n - \lfloor n/\alpha \rfloor - 1) \log(n - \lfloor n/\alpha \rfloor - 1),
\]
which means that there exists \(n_0 > 0\) such that for any \(n \geq n_0\), (2.16) holds. Therefore we approved the lemma. \(\square\)

Let \(M = \min((R_1 - \tau)/\tilde{c}, (\tau - R_2)/\tilde{c})\). Substituting \(\frac{y - \tau}{\tilde{c}}\) for \(u\) in (1.2), applying Lemma 2.1 and using the following fact,

\[
\sum_{n=N}^{\infty} L^{-n} = \frac{L^{-N}}{1 - \frac{1}{L}},
\]
we have (2.9) holds.

One thing we need to pay attention is that although we have uniform convergence in (2.9), from the proof of Lemma 1, we can see that the convergence speed depends on \((R_1 - \tau)/\tilde{c}, (\tau - R_2)/\tilde{c}\) and \(\alpha\). When \(\alpha\) is close to 1 or either \(R_1\) or \(R_2\) is very large, the convergence speed will be slow. Since we cannot change the parameter \(\alpha\), \(\tau\) and \(\tilde{c}\), the only way to obtain good convergence speed is to keep \(R_1\) and \(R_2\) close to \(\tau\).

Clearly (2.9) implies that

\[
\int_{R_1}^{R_2} f_k(y; \theta)p_\alpha((y - \tau)/\tilde{c}) - p_{\alpha,N}((y - \tau)/\tilde{c})\frac{1}{\tilde{c}}dy \to 0,
\]

as \(N \to \infty\).

Define

\[
T_{k,1,N} = \int_{R_1}^{\infty} f_k(y; \theta)p_{1,N}((y - \tau)/\tilde{c})\frac{1}{\tilde{c}}dy,
\]

\[
T_{k,2,N} = \int_{-\infty}^{R_2} f_k(y; \theta)p_{2,N}((y - \tau)/\tilde{c})\frac{1}{\tilde{c}}dy,
\]

\[
T_{k,3,N} = \int_{R_1}^{R_2} f_k(y; \theta)p_{3,N}((y - \tau)/\tilde{c})\frac{1}{\tilde{c}}dy,
\]

for \(k = 1, \ldots, m\).
Combining (2.7, (2.8) and (2.19), we have
\[
T_{k,1,N} + T_{k,2,N} + T_{k,3,N} \to E(f_k(x; \theta)),
\]
(2.21)
for \(k = 1, \ldots, m\), as \(N\) goes to \(\infty\). This means the sum of \(T_{k,1,N}, T_{k,2,N},\) and \(T_{k,3,N}\) is the approximate finite series expression we wanted.

Next we give the expressions of \(T_{k,1,N}, T_{k,2,N},\) and \(T_{k,3,N}\) without integrals inside. It is easy to see

1. \(T_{k,1,N}\)
\[
T_{k,1,N} = \int_{R_1}^{\infty} f_k(y; \theta)p_{1,N}(y-\tau)\frac{1}{\tilde{c}}dy
\]
\[= \frac{(R_1-\tau)^2}{\tilde{c}^2 \pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} \left( \sin(\pi n (1 - \beta/\alpha))/2\alpha \right) \]
\[
\cdot \int_{\frac{R_1-\tau}{\tilde{c}}}^{\infty} u^{\frac{1}{k+1}} u^{-n\alpha-1} du,
\]
where
\[
\int_{\frac{R_1-\tau}{\tilde{c}}}^{\infty} u^{\frac{1}{k+1}} u^{-n\alpha-1} du
\]
\[= \int_{\frac{R_1-\tau}{\tilde{c}}}^{\infty} u^{-n\alpha-1+\frac{1}{k+1}} du
\]
\[= \frac{1}{-n\alpha + \frac{1}{k+1} + \frac{1}{k+1}} (\frac{R_1-\tau}{\tilde{c}})^{-n\alpha+\frac{1}{k+1}}.
\]

2. \(T_{k,2,N}\)
\[
T_{k,2,N} = \int_{-\infty}^{R_2} f_k(y; \theta)p_{2,N}(y-\tau)\frac{1}{\tilde{c}}dy
\]
\[= \frac{(\tau - R_2)/\tilde{c}}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} \left( \sin(\pi n (1 + \beta/\alpha))/2\alpha \right) \]
\[
\cdot \int_{-\infty}^{\frac{R_2-\tau}{\tilde{c}}} |u|^{\frac{1}{k+1}} |u|^{-n\alpha-1} du,
\]
where

\[
\int_{-\infty}^{R_2 - \tau/\tilde{c}} |u|^{1/\alpha} |u|^{-n\alpha - 1} du = \int_{-\infty}^{\infty} u^{-(n+1) + \frac{1}{k+1}} du = \frac{1}{-n\alpha + \frac{1}{k+1}} ((\tau - R_2)/\tilde{c})^{-n\alpha + \frac{1}{k+1}).
\] (2.25)

3.

\[T_{k,3,N} = \int_{R_1}^{R_2} f_k(y; \theta)p_3, N((y - \tau)/\tilde{c}) \frac{1}{\tilde{c}} dy = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \Gamma(n/\alpha + 1) \frac{1}{n!} (\sin(\pi n(1 - \beta - \frac{2}{\alpha})/2)) \cdot \int_{R_1 - \tau}^{R_1} u^{2(1 + \frac{1}{2(k+1)})} u^{n-1} du,
\] (2.26)

where

\[\int_{R_2 - \tau}^{R_1 - \tau} u^{2(1 + \frac{1}{2(k+1)})} u^{n-1} du = \begin{cases} A_1 + A_2, & \text{if } n \text{ is odd;} \\ A_1 - A_2, & \text{if } n \text{ is even,} \end{cases}
\] (2.27)

with

\[A_1 = \int_{0}^{R_1 - \tau} u^{2(1 + \frac{1}{2(k+1)})} u^{n-1} du = \int_{0}^{R_1 - \tau} u^{n-1 + 2(1 + \frac{1}{2(k+1)})} du = \frac{1}{n + 2(1 + \frac{1}{2(k+1)})} ((R_1 - \tau)/\tilde{c})^{n+2(1 + \frac{1}{2(k+1)})},
\] (2.28)

and

\[A_2 = \int_{0}^{\tau - R_2} u^{2(1 + \frac{1}{2(k+1)})} u^{n-1} du = \int_{0}^{\tau - R_2} u^{n-1 + 2(1 + \frac{1}{2(k+1)})} du = \frac{1}{n + 2(1 + \frac{1}{2(k+1)})} ((\tau - R_2)/\tilde{c})^{n+2(1 + \frac{1}{2(k+1)})}.
\] (2.29)
Thus we obtain the expressions of $T_{k,1,N}$, $T_{k,2,N}$, and $T_{k,3,N}$ without integrals inside. By (2.21), we can substitute the sum of $T_{k,1,N}$, $T_{k,2,N}$ and $T_{k,3,N}$ for $E(f_k(x; \theta))$ in the expression of $V$ of (2.3) and then do optimization to get the estimate $\hat{\theta}$ for $\theta_0$.

3 $0 < \alpha < 1$

In order to ensure integrability, we consider the following moment functions, which are different from those when $1 < \alpha < 2$:

$$f_k(y; \theta) = \begin{cases} d_1(y-\tau)^{-1} \tilde{c}^{1+1}, & \text{if } y \geq \tilde{R}_1; \\ (y-\tau)^{2(1-\frac{1}{2(k+1)})} \tilde{c}, & \text{if } \tilde{R}_2 < y < \tilde{R}_1; \\ d_2(\tau-y)^{-1} \tilde{c}^{1+1}, & \text{if } y \leq \tilde{R}_2, \end{cases}$$

(3.1)

for $k = 1, 2, ..., m$ with $m > 4$, where the exponent $2(1 - \frac{1}{2(k+1)})$ means first take square and then take exponent $1 - \frac{1}{2(k+1)}$. $\tilde{R}_1$ and $\tilde{R}_2$ are constants such that $\tilde{R}_2 < \tau < \tilde{R}_1$ with $\frac{\tilde{R}_1-\tau}{\tilde{c}}$ and $\frac{\tau-\tilde{R}_2}{\tilde{c}}$ are small. The reason for this requirement is that when $u$ is small, we have the finite series expression (1.7) for the density function. $\tilde{d}_1$ and $\tilde{d}_2$ in the above expression are constants satisfying

$$\tilde{d}_1 = ((\tilde{R}_1 - \tau)/\tilde{c})^2 \text{ and } \tilde{d}_2 = ((\tau - \tilde{R}_2)/\tilde{c})^2.$$

It is clear that $\tilde{f}_k(y; \theta)$, $k = 1, ..., m$, are continuous functions.

Let $x$ be the random variable of stable distribution with parameters $\theta$. Then

$$E(f_k(x; \theta)) = \int_{-\infty}^{\infty} \tilde{f}_k(y; \theta)p_{\alpha}(\frac{y-\tau}{\tilde{c}}) \frac{1}{\tilde{c}} dy,$$

(3.2)

where $p_{\alpha}$ is the density function in (1.5), (1.6) and (1.7).

Similar to the argument in section 2, for independent stable distributions $x_1, x_2, ..., x_n$ with the same parameter vector $\theta_0$, denote the sample moment $\frac{\sum f_k(x; \theta)}{n}$ by $E_n f_k(x; \theta)$, $k = 1, ..., m$. Since $E|f_k(x_i; \theta)| < \infty$, $i = 1, ..., n$, $k = 1, ..., m$, by strong law of large numbers, we have

$$E_n f_k(x; \theta_0) \rightarrow E(f_k(x; \theta_0)) \text{ almost surely, } k = 1, ..., m,$$
as \( n \to \infty \).

Both \( E_n\bar{f}_k(x; \theta) \) and \( E\tilde{f}_k(x; \theta) \) have first derivatives (vector) with respect to \( \theta = (\alpha, \beta, \tau, \tilde{c}) \). We define

\[
V = \bar{c}_1[E_n\bar{f}_1(x; \theta_0) - E\tilde{f}_1(x; \theta_0)]^2 + \cdots + \bar{c}_m[E_n\bar{f}_m(x; \theta_0) - E\tilde{f}_m(x; \theta_0)]^2, \tag{3.3}
\]

where \( \bar{c}_1, ..., \bar{c}_m \) are positive constants. Then by GMM, the minimization solution \( \hat{\theta} \) of \( \bar{V} \) converges in distribution to \( \theta_0 \).

Because of no closed form expressions for the density of stable distributions, we substitute its approximate finite series expression (if they exist) for \( E\tilde{f}_k(x; \theta) \), \( k = 1, ..., m \) in the expression of \( \bar{V} \) of (3.3).

Using the similar argument as we did in section 2, it is not hard to show the existence of approximate finite series expression for \( E\tilde{f}_k(x; \theta) \), \( k = 1, ..., m \). For \( N > 0 \), we denote it by the sum of \( \bar{T}_{k,1,N}, \bar{T}_{k,2,N}, \) and \( \bar{T}_{k,3,N} \), where

\[
\bar{T}_{k,1,N} = \int_{R_1}^{\infty} \bar{f}_k(y; \theta)\bar{p}_{1,N}((y-\tau)/\bar{c})\frac{1}{\bar{c}}dy,
\]

\[
\bar{T}_{k,2,N} = \int_{-\infty}^{R_2} \bar{f}_k(y; \theta)\bar{p}_{2,N}((y-\tau)/\bar{c})\frac{1}{\bar{c}}dy, \tag{3.4}
\]

\[
\bar{T}_{k,3,N} = \int_{R_1}^{R_2} \bar{f}_k(y; \theta)\bar{p}_{3,N}((y-\tau)/\bar{c})\frac{1}{\bar{c}}dy,
\]

for \( k = 1, ..., m \), with \( \bar{p}_{1,N}, \bar{p}_{2,N} \) and \( \bar{p}_{3,N} \) being:

\[
\bar{p}_{1,N}(u) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} (\sin(\pi n(1 + \beta)/2\alpha)) u^{-n\alpha - 1}, \quad \text{for } u > 0, \tag{3.5}
\]

\[
\bar{p}_{2,N}(u) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} (\sin(\pi n(1 - \beta)/2\alpha)) |u|^{-n\alpha - 1}, \quad \text{for } u < 0, \tag{3.6}
\]

\[
\bar{p}_{3,N}(u) = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{n!} (\sin(\pi n(1 + \beta)/2\alpha)) u^{n-1}, \quad \text{for } u \in \mathbb{R}, \tag{3.7}
\]

We can see that \( \bar{p}_{1,N}, \bar{p}_{2,N} \) and \( \bar{p}_{3,N} \) corresponding to (1.5), (1.6) and (1.7) respectively.

Next we give the expressions of \( \bar{T}_{k,1,N}, \bar{T}_{k,2,N}, \) and \( \bar{T}_{k,3,N} \) without integrals inside. It is easy to see
1. 

\[ T_{k,1,N} = \int_{R_1}^{\infty} f_k(y; \theta) \delta_1(N((y - \tau)/\tilde{c}) \frac{1}{\tilde{c}} dy) \]

\[ = \frac{(R_1 - \tau)^2}{\tilde{c}^2 \pi} \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \Gamma(n\alpha + 1) \frac{1}{n!} (\sin(\pi n(1 + \beta)/2\alpha)) \]

\[ \cdot \int_{\tilde{R}_1 - \tau}^{\infty} u^{\kappa+1} u^{-n\alpha-1} du, \]  

where

\[ \int_{\tilde{R}_1 - \tau}^{\infty} u^{\kappa+1} u^{-n\alpha-1} du = \int_{\tilde{R}_1 - \tau}^{\infty} u^{-n\alpha-1} \frac{1}{\kappa+1} du \]

\[ = \frac{1}{\kappa+1} ((\tilde{R}_1 - \tau)/\tilde{c})^{-n\alpha-1}. \]  

2. 

\[ T_{k,2,N} = \int_{-\infty}^{\tilde{R}_2} f_k(y; \theta) \delta_2(N((y - \tau)/\tilde{c}) \frac{1}{\tilde{c}} dy) \]

\[ = \frac{((\tau - \tilde{R}_2)/\tilde{c})^2}{\pi} \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \Gamma(n\alpha + 1) \frac{1}{n!} (\sin(\pi n(1 - \beta)/2\alpha)) \]

\[ \cdot \int_{-\infty}^{\tilde{R}_2 - \tau} \left| u \right|^{\kappa+1} \left| u \right|^{-n\alpha-1} du, \]  

where

\[ \int_{-\infty}^{\tilde{R}_2 - \tau} \left| u \right|^{\kappa+1} \left| u \right|^{-n\alpha-1} du = \int_{-\infty}^{\tau} u^{-(n+1)-\frac{1}{\kappa+1}} du \]

\[ = \frac{1}{\kappa+1} ((\tau - \tilde{R}_2)/\tilde{c})^{-n\alpha-1}. \]
3. \[ \tilde{T}_{k,3,N} \]
\[ = \int_{\bar{R}_3}^{\bar{R}_2} \tilde{f}_k(y; \theta)p_{3,N}((y - \tau)/\tilde{c}) \frac{1}{\tilde{c}}dy \]
\[ = \frac{1}{\pi} \sum_{n=1}^{N} (-1)^{n-1} \frac{\Gamma(n/\alpha + 1)}{n!} \left( \sin(\pi n (1 + \beta)/2) \right) \cdot \int_{\bar{R}_2 - \tau}^{\bar{R}_1 - \tau} u^{2(1 - \frac{1}{2(k+1)})} u^{n-1} du, \]

where
\[ \int_{\bar{R}_2 - \tau}^{\bar{R}_1 - \tau} u^{2(1 - \frac{1}{2(k+1)})} u^{n-1} du = \begin{cases} \tilde{A}_1 + \tilde{A}_2, & \text{if } n \text{ is odd;} \\ \tilde{A}_1 - \tilde{A}_2, & \text{if } n \text{ is even}, \end{cases} \]

with
\[ \tilde{A}_1 = \int_{0}^{\bar{R}_1 - \tau} u^{2(1 - \frac{1}{2(k+1)})} u^{n-1} du \]
\[ = \int_{0}^{\bar{R}_1 - \tau} u^{n+1+2(1 - \frac{1}{2(k+1)})} du \]
\[ = \frac{1}{n + 2(1 - \frac{1}{2(k+1)})} \left( (\bar{R}_1 - \tau)/\tilde{c} \right)^{n+2(1 - \frac{1}{2(k+1)})}, \]

and
\[ \tilde{A}_2 = \int_{\tau - \bar{R}_2}^{\bar{R}_2} u^{2(1 - \frac{1}{2(k+1)})} u^{n-1} du \]
\[ = \int_{0}^{\tau - \bar{R}_2} u^{n+1+2(1 - \frac{1}{2(k+1)})} du \]
\[ = \frac{1}{n + 2(1 - \frac{1}{2(k+1)})} \left( (\tau - \bar{R}_2)/\tilde{c} \right)^{n+2(1 - \frac{1}{2(k+1)})}. \]

Now we have the expressions of \( \tilde{T}_{k,1,N}, \tilde{T}_{k,2,N}, \) and \( \tilde{T}_{k,3,N} \) without integrals inside. We can substitute the sum of \( \tilde{T}_{k,1,N}, \tilde{T}_{k,2,N} \) and \( \tilde{T}_{k,3,N} \) for \( E(\tilde{f}_k(x; \theta)) \), \( k = 1, \ldots, m \) in the expression of \( \tilde{V} \) of (3.3) and then do optimization to get the estimate \( \hat{\theta} \) for \( \theta_0 \).

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References

[1] Akgiray, V. and Lamoureux, C. G., Estimation of stable-law parameters: a comparative study, *Journal of Business and Economic Statistics* 7(1989), 85-93.

[2] Bartels, R., On the use of limit theorems in economic statistics, *Amer. Statist.* 31(1977), 85-87.

[3] Basset, G. W. and Koenker, R., An empirical quantile function for linear models with i.i.d. errors, *J. Amer. Statist. Assoc.* 77(1982), 407-415.

[4] Bera, A. K. and Mckenzie, C. R., Tests for normality with stable alternatives, *J. Statist. Comput. Simul.* 25(1986), 37-52.

[5] DuMouchel, W. H., On the asymptotic normality of the maximum-likelihood estimate when sampling from a stable distribution, *The Annals of Statistics* 1(1973a), 948-957.

[6] DuMouchel, W. H., Stable distribution in statistical inference: 1. symmetric stable distributions compared to other symmetric long tailed distributions, *J. Amer. Statist. Assoc.* 68(1973b), 469-477.

[7] DuMouchel, W. H., Stable distribution in statistical inference: 2. Information from stably distributed samples, *J. Amer. Statist. Assoc.* 70(1975), 386-393.

[8] Fama, E. F. and Roll, R., The Behavior of Stock-Market Prices, *Journal of Business* XXXVIII(1965), 34-105.

[9] Fama, E. F. and Roll, R., Some properties of symmetric stable distributions, *J. Amer. Statist. Assoc.* 63(1968), 817-836.

[10] Fama, E. F. and Roll, R., Parameters estimates for symmetric stable distributions, *J. Amer. Statist. Assoc.* 66(1971), 331-338.

[11] Koenker, R. and Basset, G. W., Regression quantiles, *Econometrica* 46(1982), 33-50.

[12] Koutrouvelis, I., Regression-type estimation of the parameters of stable laws, *J. Amer. Statist. Assoc.* 75(1980), 918-928.

[13] Koutrouvelis, I., An iterative procedure for the estimation of the parameters of stable laws, *Communications in Statistics-Simulation and Computation* 10(1981), 17-28.

[14] Mandelbrot, B., The stable paretoian income distribution when the apparent exponent is near two, *International Economic Review* IV(1963a), 111-14.

[15] Mandelbrot, B., The variation of certain speculative prices, *Journal of Business* XXXVI(1963b), 394-419.

[16] McCulloch, J. H., Simple consistent estimators of stable distribution parameters, *Communications in Statistics-Simulation and Computation* 15(1986), 1109-1136.

[17] Sato, K., *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.

[18] Zolotarev, V. M., *One-Dimensional Stable Distribution*, Amer. Math. Soc., Providence, RI. [Russian original 1983].