TURBULENCE AS STATISTICS OF VORTEX CELLS

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Abstract

We develop the formulation of turbulence in terms of the functional integral over the phase space configurations of the vortex cells. The phase space consists of Clebsch coordinates at the surface of the vortex cells plus the Lagrange coordinates of this surface plus the conformal metric. Using the Hamiltonian dynamics we find an invariant probability distribution which satisfies the Liouville equation. The violations of the time reversal invariance come from certain topological terms in effective energy of our Gibbs-like distribution. We study the topological aspects of the statistics and use the string theory methods to estimate intermittency.
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1 Introduction

The central problem of turbulence is to find the analog of the Gibbs distribution for the energy cascade. The mathematical formulation of this problem is amazingly simple. In inertial range we could neglect the viscosity and forcing and study the Euler dynamics of ideal incompressible fluid

\[ \partial_t v_\alpha = v_\beta \omega_{\alpha\beta} - \partial_\alpha h \]  

(1)

Here

\[ \omega_{\alpha\beta} = \partial_\alpha v_\beta - \partial_\beta v_\alpha \]  

(2)

is the vorticity field and

\[ h = p + \frac{1}{2} v_\alpha^2 \]  

(3)

is the enthalpy which is eliminated from the incompressibility condition

\[ \partial_\alpha v_\alpha = 0 \]  

(4)

The key point is that the Euler dynamics can be regarded as a Hamiltonian flow in functional phase space. This geometric view was first proposed by Arnold in 1966 (see his famous book [2]) and later developed by other mathematicians (see the references in the Moffatt’s lecture in the 1992 Santa Barbara conference proceedings [10]). Here we derive the Hamiltonian dynamics from scratch using the conventional physical terminology. All the necessary computations are presented in Appendix A. Some of these results are new.

The Hamiltonian here is just the fluid energy

\[ H = \int d^3r \frac{1}{2} v_\alpha^2 \]  

(5)

and the phase space corresponds to all the velocity fields subject to the incompressibility constraint. The Poisson brackets between the components of velocity field

\[ [v_\alpha(r_1), v_\beta(r_2)] = \int d^3r T_{\alpha\mu}(r_1 - r) T_{\beta\nu}(r_2 - r) \omega_{\mu\nu}(r) \]  

(6)

where

\[ T_{\alpha\beta}(r) = \delta_{\alpha\beta}\delta(r) + \partial_\alpha \partial_\beta \frac{1}{4\pi r} \]  

(7)

is the projection operator. The Euler equation can be written in a manifestly Hamiltonian form

\[ \partial_t v_\alpha = \int d^3r' T_{\alpha\beta}(r - r') \omega_{\beta\gamma}(r') v_\gamma(r') = [v_\alpha, H] \]  

(8)

The Liouville theorem of the phase space volume conservation applies here

\[ \frac{\delta}{\delta v_\alpha(r)} [v_\alpha(r), H] = 0 \]  

(9)

\[ (Dv) = \prod_r d^3v(r) = \text{const} \]  

(10)
The probability distributions $P[v]$ compatible with this dynamics, must also be conserved, i.e. it must commute with the Hamiltonian

$$[P[v], H] = 0$$

(11)

The Gibbs distribution

$$P[v] = \exp (-\beta H)$$

(12)

is the only known general solution of the Liouville equation (11). The above mentioned central problem of turbulence is to find another one.

This formulation of turbulence is significantly different from the popular formulation based on the so called Wyld diagram technique. There, the functional integral is in place from the very beginning but the problem is to find its turbulent limit (zero viscosity). This functional integral involves time, so it describes the kinetic phenomena in addition to the steady state we are studying here.

After trying for few years to do something with the Wyld approach I conclude that this is a dead end. The best bet here would be the renormalization group, which magically works in statistical physics. Those critical phenomena were close to Gaussian, which allowed Wilson to develop the $\epsilon$ expansion by rearranging the ordinary perturbation expansion.

There is no such luck in turbulence. The nonlinear effects are much stronger. The observed variety of vorticity structures with their long range interactions does not look like the block spins of critical phenomena. Moreover, there are notorious infrared divergencies, which make problematic the whole existence of the universal kinetics of turbulence. No! These old tricks are not going to work, we have to invent the new ones.

If we are looking for something pure and simple this might be the steady state distribution of vorticity structures. Here the geometric methods may allow us to go much further than the general methods of quantum field theory. Being regarded as a problem of fractal geometry rather than a nonlinear wave problem, turbulence may reveal some mathematical beauty to match the beauty of the Euler-Lagrange dynamics.

This dream motivated the geometric approach to the vortex sheet dynamics in, where the attempt was made to simulate turbulence as the stochastic motion of the vortex sheet. This project ran out of computer resources, as it happened before to many other projects of that kind. However, the geometric tools developed in that work proved to be useful and we are going to use them here.

Another false start: I tried to solve the Liouville equation variationally, with the Gaussian Anzatz with anomalous dimensions for the velocity field. The numbers came out too far from the experiment and it was hard to improve them. Similar attempts with the Gaussian Clebsch variables also failed to produce the correct numbers. It became clear to me that velocity field fluctuates too much to be used as a basic variable.
We encounter the same problem in QCD where the gauge field strongly fluctuates, and its correlations do not decrease with distance. The problem is not yet solved there, but some useful tools were developed. In particular, the Wilson loops (the averages of the ordered exponentials of the circulation of the gauge field) are known to be a better field variables. The Wilson loop is expected to be described by some kind of the string theory, though nobody managed to prove this.

The dynamical equations for the Wilson loops as functionals of the shape of the loop were derived, and studied for many years[4]. The QCD loop equations proved to be very hard to solve, because of the singularities at self intersections.

In my recent paper[12] I derived similar equations for the averages of the exponentials of velocity circulation in (forced) Navier-Stokes equation. These equations have no singularities at self intersections, in addition they are linear, unlike the loop equations of QCD. This raised the hopes to find exact solution in terms of the string functional integrals.

The theory developed below started as the solution the (Euler limit of the) loop equations. However, later I found how to derive it from the Liouville equation, without unjustified assumptions of the loop calculus. This is how I am presenting this theory here.

2 Loop functional

It is generally believed that vorticity is more appropriate than velocity for the description of turbulence. Vorticity is invariant with respect to Galilean transformations which shift the space independent part of velocity

\[ v_\alpha(r) \Rightarrow v_\alpha(r - ut) + u_\alpha; \quad \omega_{\alpha\beta}(r) \Rightarrow \omega_{\alpha\beta}(r - ut) \]  

(13)

The correlation functions of velocity field involve the infrared divergencies coming from this part. Say, in the two- point correlation function this would be the energy density

\[ \langle v_\alpha(r)v_\alpha(r') \rangle = \frac{2H}{V} - \frac{1}{2}(v_\alpha(r) - v_\alpha(r'))^2 \]  

(14)

which formally diverges as

\[ \frac{H}{V} \propto \int_{1/L}^{\infty} dk k^{-\frac{5}{3}} \sim L^{\frac{2}{3}} \]  

(15)

according to Kolmogorov scaling [4].

The infrared divergencies are absent in vorticity correlations. The corresponding Fourier integral is ultraviolet divergent due to extra factor of \(k^2\), but this is healthy. The

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1This does not mean that the simple Gaussian models in velocity or Clebsch variables could not explain observed turbulence in finite systems. We are talking about idealized problem of isotropic homogeneous turbulence with infinite Reynolds number.
physical observables involve the vorticity correlations at split points, where the integrals converge.

The complete set of such observables is generated by velocity circulations for various loops in the fluid

\[ \Gamma_C[v] = \oint_C dr_\alpha v_\alpha(r) \]  

The constant part of velocity drops here after the integration over the closed loop, thus the circulation is Galilean invariant. The loop gets translated

\[ C \Rightarrow C - ut \]  

but all the equal time correlations of the circulation stay invariant in virtue of translation symmetry.

One can express the circulation in terms of vorticity via the Stokes theorem

\[ \Gamma_C[v] = \sum_{\mu < \nu} \int_S dr_\mu \wedge dr_\nu \omega_{\mu\nu}(r) \]  

where \( S \) is an arbitrary surface bounded by \( C \). In particular, for infinitesimal loop we would get the local vorticity. The moments of the circulation

\[ \langle (\Gamma_C[v])^n \rangle = \int [Dv] P[v] (\Gamma_C[v])^n \]  

all converge in the ultraviolet as well as in the infrared domain. The infrared convergence is guaranteed since these are surface integrals of vorticity, and the ultraviolet one is guaranteed since these are line integrals of velocity.

This nice property suggests to study the distribution of the velocity circulation

\[ P_C[\Gamma] = \int [Dv] P[v] \delta (\Gamma - \Gamma_C[v]) \]  

It is more convenient to study the Fourier transform

\[ \Psi_C(\gamma) = \int_{-\infty}^{\infty} d\Gamma \exp (i \gamma \Gamma) P_C(\Gamma) = \int [Dv] P[v] \exp (i \gamma \Gamma_C[v]) \]  

We expect these functionals to exist in the turbulent limit unlike the distribution \( P[v] \).

The dynamical equation for these functionals (the loop equation) was derived in my previous work\[12\]. The time derivative of the circulation reads

\[ \partial_t \Gamma_C[v] = \oint_C d\alpha \omega_{\alpha\beta}(r)v_\beta(r) \]  

All the nonlocal terms reduced to the closed loop integrals of the total derivatives and vanished. Being averaged with appropriate measure, the remaining terms in time derivative must vanish according to the Liouville equation

\[ \langle \partial_t \Gamma_C[v] \exp (i g \Gamma_C[v]) \rangle = 0 \]
Note that this is not the Kelvin theorem of conservation of the circulation. The circulation is conserved in a Lagrange sense, at purely kinematical level
\[
\frac{d}{dt} \Gamma_C[v] = \partial_t \Gamma_C[v] + \oint d\theta \frac{\delta \Gamma_C[v]}{\delta C(\theta)} v_\beta (C(\theta)) = 0 \tag{24}
\]
In other words, the Euler derivative \((22)\) is exactly compensated by the shift of every point at the loop by local velocity. Now, according to the Liouville equation, each of these equal terms must vanish in average.

The formal derivation goes as follows
\[
\int [Dv] P[v] \exp(\im \gamma \Gamma_C[v]) \Gamma_C[v], H \tag{25}
\]
\[
= \int [Dv] P[v] \exp(\im \gamma \Gamma_C[v]) \Gamma_C[v], H
\]
\[
= - \int [Dv] [P[v], H] \exp(\im \gamma \Gamma_C[v]) = 0
\]
and the physics is obvious: the average of any time derivative with time independent weight must vanish.

Let us interpret in these terms the solution for \(\Psi_C(\gamma)\) found in the previous paper \([12]\)
\[
\Psi_C(\gamma) = f(\Sigma_{\alpha\beta}); \Sigma_{\alpha\beta} = \oint C dr \alpha r \beta \tag{26}
\]
In virtue of linearity of the Liouville equation it suffices to check the Fourier transform
\[
\exp(\im \gamma R_{\alpha\beta} \Sigma_{\alpha\beta}) \tag{27}
\]
In terms of the velocity field this corresponds to global rotation
\[
v_\alpha = R_{\alpha\beta} r \beta; \omega_{\alpha\beta} = 2R_{\alpha\beta} \tag{28}
\]
The corresponding Poisson brackets reads
\[
[\Gamma_C[v], H] = \oint_C dr_\alpha \omega_{\alpha\beta} v_\beta = 4 \Sigma_{\alpha\gamma} R_{\alpha\beta} R_{\beta\gamma} = 0 \tag{29}
\]
The sum over tensor indexes vanished by symmetry.

This solution is always present in fluid mechanics due to the conservation of the angular momentum. Unfortunately, it has nothing to do with turbulence, contrary to the hopes expressed in my previous work.

Let us also note, that the Gibbs solution does not apply here. One could formally compute the loop functional for the Gibbs distribution, but the result is singular:
\[
\Psi_C(\gamma) = \exp \left( -\frac{\gamma^2}{2\beta} \oint_C dr_\alpha \oint_C dr_\alpha' \delta^3(r - r') \right) \tag{30}
\]
\footnote{This is yet another advantage of the loop functional: the Gibbs solution for the loop functional does not exist, which forces us to look for alternative invariant distributions.}
Should we cut off the wavevectors at \( k \sim \frac{1}{r_0} \) this would become

\[
\Psi_C(\gamma) \approx \exp \left( -\frac{\gamma^2}{\beta r_0} \oint_C |dr| \right)
\] (31)

This is so called perimeter law, characteristic to the local vector fields. Clearly this is not the case in turbulence, as velocity field is highly nonlocal. Also, the odd correlations of velocity, such as the triple correlation, which are present due to the time irreversibility, would, in general make the loop functional complex.

## 3 Vortex dynamics

Let us study the dynamics of the vortex structures from the Hamiltonian point of view. We shall assume that vorticity is not spread all over the space but rather occupies some fraction of it. It is concentrated in some number of cells \( D_i \) of various topology moving in their own velocity field.

Clearly, this picture is an idealization. In the real fluid, with finite viscosity, there will always be some background vorticity between cells. In this case, the cells could be defined as the domains with vorticity above this background. The reason we are doing this is obvious: we would like to work with the Euler equation with its symmetries.

We shall use two types of tensor and vector indexes. The Euler (fixed space) tensors will be denoted by Greek subscripts such as \( r_\alpha \). The Lagrange tensors (moving with fluid) will be denoted by latin subscripts such as \( \rho_a \). The field \( X_\alpha(\rho) \) describes the instant position of the point with initial coordinates \( \rho_a \). The transformation matrix from the Lagrange to Euler frame is given by \( \partial_a X_\alpha(\rho) \).

The contribution of each cell \( D \) to the net velocity field can be written as follows

\[
v_\alpha(r) = -\epsilon_{\alpha\beta\gamma} \partial_\beta \int_D d^3\rho \frac{\Omega_\gamma(\rho)}{4\pi |r - X(\rho)|}
\] (32)

where

\[
\Omega_\gamma(\rho) = \Omega^a(\rho) \partial_a X_\gamma(\rho)
\] (33)

is the vorticity vector in the Euler frame. The vorticity vector \( \Omega^a(\rho) \) in the Lagrange frame is conserved

\[
\partial_t \Omega^a(\rho) = 0
\] (34)

The physical vorticity tensor \( \omega_{\alpha\beta} = \partial_\alpha v_\beta - \partial_\beta v_\alpha \) inside the cell can be readily computed from velocity integral. The gradients produce the \( \delta \) function so that we get

\[
\omega_{\alpha\beta}(X(\rho)) = \left( \frac{\partial (X_1, X_2, X_3)}{\partial (\rho_1, \rho_2, \rho_3)} \right)^{-1} e_{\alpha\beta\gamma} \Omega_\gamma(\rho)
\] (35)
or in Euler frame, inverting $\rho = X^{-1}(r)$

$$\omega_{\alpha\beta}(r) = \frac{1}{6} e_{\lambda\mu\nu} e_{abc} \partial X^\alpha \partial X^\beta \partial X^\gamma \Omega^f(\rho) \partial_f X_\gamma = e_{abc} \partial_{\alpha} \rho^a(\rho) \partial_{\beta} \rho^b(\rho) \Omega^c(\rho(\rho)) \quad (36)$$

The inverse matrix $\partial_{\alpha} \rho^\alpha = (\partial_a X_\alpha)^{-1}$ relates the Euler indexes to the Lagrange ones, as it should. These relations between the Euler and Lagrange vorticity are equivalent to the conservation of the vorticity 2-form

$$\Omega = \sum_{\alpha<\beta} \omega_{\alpha\beta}(X) dX_\alpha \wedge dX_\beta = \sum_{a<b} \Omega_{ab}(\rho) d\rho^a \wedge d\rho^b; \quad \Omega_{ab} = e_{abc} \Omega^c \quad (37)$$

which is the Kelvin theorem of the conservation of velocity circulation around arbitrary fluid loop.

The field $X_\alpha(\rho)$ moves with the flow (the Helmholtz equation)

$$\partial_t X_\alpha(\rho) = v_\alpha(X(\rho)) \quad (38)$$

In Appendix B we derive the Euler equation from the Helmholtz equation and study the general properties of the former. We show that this is equivalent to the Hamiltonian dynamics with the (degenerate) Poisson brackets

$$[X_\alpha(\rho), X_\beta(\rho')] = -\delta^3(\rho - \rho') e_{\alpha\beta\gamma} \Omega^\gamma(\rho) \quad (39)$$

The Hamiltonian is given by the same fluid energy, with the velocity understood as functional of $X(.)$ The degeneracy of the Poisson brackets reflects the fact that there are only two independent degrees of freedom at each point. This leads to the gauge symmetry which we discuss below.

The Hamiltonian variation reads

$$\frac{\delta H}{\delta X_\alpha(\rho)} = e_{\alpha\beta\gamma} v_\beta(X(\rho)) \Omega_\gamma(\rho) \quad (40)$$

This variation is orthogonal to velocity, which provides the energy conservation. It is also orthogonal to vorticity vector which leads to the gauge invariance. The gauge transformations

$$\delta X_\alpha(\rho) = f(\rho) \Omega_\alpha(\rho) \quad (41)$$

leave the Hamiltonian invariant. These transformations reparametrize the coordinates

$$\delta \rho^a = f(\rho) \Omega^a(\rho) \quad (42)$$

The vorticity density transforms as follows

$$\delta \Omega^a = -\Omega^a \partial_b(f \Omega^b) + f \Omega^b \partial_b \Omega^a + \Omega^b \partial_b(f \Omega^a) = 2 f \Omega^b \partial_b \Omega^a \quad (43)$$
The first term comes from the volume transformation, the second one - from the argument transformation and the third one - from the vector index transformation of $\Omega$ so that

$$d^3 \rho \Omega^a \partial_a = \text{inv}$$

(44)

The identity

$$\partial_a \Omega^a = 0$$

(45)

was taken into account. We observe that these gauge transformations leave invariant the whole velocity field, not just the Hamiltonian.

The vorticity 2-form simplifies in the Clebsch variables

$$\Omega^a = e^{abc} \partial_b \phi_1(\rho) \partial_c \phi_2(\rho)$$

(46)

$$\Omega = d\phi_1 \wedge d\phi_2 = \text{inv}$$

(47)

The Clebsch variables provide the bridge between the Lagrange and the Euler dynamics. By construction they are conserved, as they parametrize the conserved vorticity. The Euler Clebsch fields $\Phi_i(r)$ can be introduced by solving the equation $r = X(\rho)$

$$\Phi_i(r) = \phi_i \left( X^{-1}(r) \right)$$

(48)

However, unlike the vorticity field, the Clebsch variables cannot be defined globally in the whole space. The inverse map $\rho = X^{-1}(r)$ is defined separately for each cell, therefore one cannot write $v_a = \Phi_1 \partial_a \Phi_2 + \partial_a \Phi_3$ everywhere in space. Rather one should add the contributions from all cells to the Poisson integral, as we did before.

This explains the notorious helicity paradox. The conserved helicity integral

$$H = \int d^3 \rho v_c(\rho) \Omega^c(\rho)$$

(49)

where

$$v_a(\rho) = \phi_1 \partial_a \phi_2 + \partial_a \phi_3; \partial^2 \phi_3 = -\partial_a (\phi_1 \partial_a \phi_2)$$

(50)

is the initial velocity field (to be distinguished from the physical velocity field $v_a(r)$ which cannot be paramatrized globally by the Clebsch variables).

The helicity integral for the finite cell could be finite. It can be written in invariant terms of the vorticity forms

$$H_D = \int_D d^3 \rho e^{abc} \partial_a \phi_1 \partial_b \phi_2 \partial_c \phi_3 = \int_D d\phi_1 \wedge d\phi_2 \wedge d\phi_3 = \int_{\partial D} \phi_3 d\phi_1 \wedge d\phi_2$$

(51)

from which representation it is clear that it is conserved

$$H_D = \int_{\partial D} \Phi_3 d\Phi_1 \wedge d\Phi_2$$

(52)

Our gauge transformation leave the Clebsch field invariant

$$\delta \phi_i(\rho) = f(\rho) \Omega^a(\rho) \partial_a \phi_i(\rho) = 0$$

(53)
The velocity integral in Clebsch variables reduces to the 3-form

\[ v_\alpha(r) = -e_{\alpha\beta\gamma} \partial_\beta \int_D \frac{dX_\gamma \wedge d\phi_1 \wedge d\phi_2}{4\pi|r - X|} \]  \hspace{1cm} (54)

This gauge invariance is less than the full diffeomorphism group which involves arbitrary function for each component of \( \rho \). This is a subtle point. The field \( \Omega^a(\rho) \) has no dynamics, it is conserved. However, the initial values of \( \Omega^a \) can be defined only modulo diffeomorphisms, as the physical observables are parametric invariant. So, we could as well average over reparametrizations of these initial values, which would make the parametric invariance complete.

Another subtlety. The equations of motion, which literally follow from above Poisson brackets describe the motion in the direction orthogonal to the gauge transformations, namely

\[ \partial_t X_\alpha(\rho) = \left( \delta_{\alpha\beta} - \frac{\Omega_\alpha(\rho)\Omega_\beta(\rho)}{\Omega_\mu(\rho)} \right) v_\beta(X(\rho)) = v_\alpha(X(\rho)) + f(\rho)\Omega_\alpha(\rho) \]  \hspace{1cm} (55)

The difference is unobservable, due to gauge invariance. We could have defined the Helmholtz equation this way from the very beginning. The conventional Helmholtz dynamics represents so called generalized Hamiltonian dynamics, which cannot be described in terms of the Poisson brackets. The formula (40) cannot be solved for the velocity field because the matrix

\[ \Omega_{\alpha\beta}(\rho) = e_{\alpha\beta\gamma}\Omega_\gamma(\rho) \]  \hspace{1cm} (56)

cannot be inverted (there is the zero mode \( \Omega_\beta(\rho) \)). The physical meaning is the gauge invariance which allows us to perform the gauge transformations in addition to the Lagrange motion of the fluid element.

Formally, the inversion of the \( \Omega \)–matrix can be performed in a subspace which is orthogonal to the zero mode. The inverse matrix \( \Omega^{\beta\gamma} \) in this subspace satisfies the equation

\[ \Omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta_{\alpha\gamma} - \frac{\Omega_\alpha\Omega_\gamma}{\Omega_\mu} \]  \hspace{1cm} (57)

which has the unique solution

\[ \Omega^{\beta\gamma} = -e_{\alpha\beta\gamma}\frac{\Omega_\alpha}{\Omega_\mu} \]  \hspace{1cm} (58)

This solution leads to our Poisson brackets.

Now we see how the correct number of degrees of freedom is restored. The Hamiltonian vortex dynamics locally has only two degrees of freedom, those orthogonal to the gauge transformations. We could have obtained the same Poisson brackets by canonical transformations in the vorticity form from the Clebsch variables to the \( X \)–variables. This canonical transformation describes the surface in the \( X \)–space. The vorticity form at this surface can be treated as a degenerate form in the 3-dimensional space.
One may readily check the conservation of the volume element of the cell

\[ \partial_t \frac{\partial (X_1, X_2, X_3)}{\partial (\rho_1, \rho_2, \rho_3)} = \frac{\partial (X_1, X_2, X_3)}{\partial (\rho_1, \rho_2, \rho_3)} \partial_\alpha v_\alpha (X) = 0 \]  \hspace{1cm} (59)

Using the formulas for the variations of the velocity field derived in Appendix B one may prove the stronger statement of the phase volume element conservation (Liouville theorem)

\[ \frac{\delta v_\alpha (X(\rho))}{\delta X_\alpha (\rho)} = 0 \]  \hspace{1cm} (60)

\[ (DX) = \prod_\rho d^3X(\rho) = \text{const} \]  \hspace{1cm} (61)

4 Vortex statistics

Let us recall the foundation of statistical mechanics. The Gibbs distribution can be formally derived from the Liouville equation plus extra requirement of multiplicativity. In general, any additive conserved functional \( E \) could serve as energy in the Gibbs distribution.

The physical mechanism is the energy exchange between the small subsystem under study and the rest of the system (thermostat). The conditional probability for a subsystem is obtained from the microcanonical distribution \( d\Gamma' d\Gamma \delta (E' + E - E_0) \) for the whole system by integrating out the configurations \( d\Gamma' \) of the thermostat.

The corresponding phase space volume \( e^{S(E')} = \int d\Gamma' d\Gamma \delta (E' + E - E_0) \) of the thermostat depends upon its energy \( E' = E_0 - E \) where the contribution \(-E\) from the subsystem represents the small correction. Expanding \( S(E_0 - E) = S(E_0) - \beta E \) we arrive at the Gibbs distribution.

In case of the vortex statistics we may try the same line of arguments. The important addition to the general Gibbs statistics is the parametric invariance. Reparametrizations, or gauge transformations, are part of the dynamics, as have been seen above.\(^3\) So, the Gibbs distribution should be both gauge invariant and conserved.

The net volume of the set of vortex cells \( V = \sum_i V(D_i) \) where

\[ V(D) = \int_D d^3\rho \frac{\partial (X_1, X_2, X_3)}{\partial (\rho_1, \rho_2, \rho_3)} = \int_D dX_1 \wedge dX_2 \wedge dX_3 \]  \hspace{1cm} (62)

is the simplest term in effective energy of the Gibbs distribution. It is gauge invariant, additive and positive definite. It is bounded from above by the volume of the system, so that there could be no infrared divergencies.\(^4\)

\(^3\)This makes so hard the numerical simulation of the Lagrange motion. The significant part of notorious instability of the Lagrange dynamics is the reparametrization of the volume inside the vortex cell.

\(^4\)The Hamiltonian does not exist in the turbulent flow because the energy spectrum diverges at small wavevectors. The net Hamiltonian grows faster than the volume of the system which is unacceptable for the Gibbs distribution.
The mechanism leading to the thermal equilibrium is quite transparent here. The volume of a little cell surrounded by the large amount of other cells, would fluctuate due to exchange with neighbors. These are the viscous effects, in the same way as the energy exchange mechanisms in the ideal gas were the effects of interaction. The relaxation time is inversely proportional to the strength of interaction (viscosity in our case).

We must take these effects into account in kinetics, but the resulting statistical distribution involves only the energy of the ideal system. This was the most impressive part of the achievement of Gibbs and Boltzman. They found the shortcut from mechanics to statistics, avoiding the kinetics. All the interactions are hidden in the temperature and chemical potentials.

Mechanically, the cells avoid each other as well as themselves and preserve their topology but the implicit viscous interactions would lead to fluctuations. Even if we start from one spherical cell it would inevitably touch itself in course of the time evolution. At the vicinity of the touching point the viscous effects show up, which break the topological conservation laws of the Euler dynamics. The result could be a handle, or the splitting into two cells. After long evolution we would end up with the ensemble of cells $D_i$ with various number $h_i$ of handles.

The related subject is the vorticity vector field $\Omega^a(\rho)$ inside the cells. In the Euler dynamics it is conserved, but the viscosity-generated interaction would lead to fluctuations. The invariant measure is

$$\langle D\Omega \rangle = \prod_\rho d^3\Omega(\rho)\delta[\partial_a\Omega^a(\rho)]$$

(63)

In terms of the Clebsch variables (46) the measure is simply

$$\langle D\phi \rangle = \prod_\rho d^2\phi(\rho)$$

(64)

as these are canonical hamiltonian variables. As discussed above, there are no global Clebsch variables in Euler sense. These Clebsch variables are defined separately in each cell and the net velocity field is a sum of contributions from all cells rather than the single Clebsch-parametrized expression $\Phi_1 \partial_a \Phi_2 + \partial_a \Phi_3$.

What could be an effective energy for the vorticity? The helicity integral is excluded as a pseudoscalar, besides, it is nonlocal, like the Hamiltonian. We insist on parametric invariance and locality in a sense that the cell splitting and joining do not change this energy. The Clebsch variables are defined modulo additive constants, they could be multivalued in complex topology, therefore we have to use the vorticity field itself. The generic $\Omega$–invariant in $d$ dimensions is

$$\int_D d^d\rho \sqrt{\det \Omega_{ab}}$$

(65)

$^5$The volume, as well as any other local functional of the cell does not change at splitting/joining, therefore these processes go with significant probability. For the nonlocal functionals, such as the hamiltonian, there are long range interactions, which makes the exchange process less probable.
In even dimensions $d = 2k$ this invariant reduces to the Pfaffian

$$\int_D d^2k \rho \sqrt{\det \Omega_{ab}} = \frac{1}{k!} \int_D \Omega \wedge \Omega \ldots \wedge \Omega = \int_D \frac{\Omega^k}{k!}$$  \hspace{1cm} (66)$$

In two dimensions it would be simply the net vorticity of the cell

$$\int_D d^2 \rho \Omega_{12}$$  \hspace{1cm} (67)$$

However, in odd dimensions it vanishes so that there is no $\Omega$–invariant. We see that there is a significant difference in the vortex statistics in even and odd dimensions.

Another interesting comment. For odd $k = \frac{d}{2}$ the $\Omega$–invariant is an odd functional of $\Omega$ which breaks time reversal invariance. The vorticity stays invariant under space reflection but changes sign at time reversal. This simple local mechanism of the time irreversibility is present only at $d = 4k + 2, k = 0, 1, \ldots$. In three dimensions we live in it is absent.

Let us turn to the boundary terms. The boundary of the cell $S = \partial D$ is described by certain parametric equation

$$S : \rho_a = R_a(\xi_1, \xi_2)$$  \hspace{1cm} (68)$$

Clearly, in our case this is a self-avoiding surface. Here we could add the following local surface terms to the energy

$$E_\phi = \sum_i \int_{\partial D_i} d^2 \xi \left( a \sqrt{g} + b \sqrt{g} g^{ij} \partial_j \phi_k \partial_j \phi_k \right)$$  \hspace{1cm} (69)$$

where

$$g_{ij}(\xi) = \partial_i R_a(\xi) \partial_j R_a(\xi)$$  \hspace{1cm} (70)$$

is the induced metric.

There is also a topological term for each non-contractible loop $L$ of $\partial D$

$$E_\Theta = \Theta \int_L \phi_1 d\phi_2$$  \hspace{1cm} (71)$$

This is simply the velocity circulation around such loop. Assuming continuity (i.e. vanishing) of vorticity at the boundary of the cell, this circulation can be also written as the integral in the external space. Then it is obvious that this integral does not depend upon the shape of the loop $L$, it is given by the invariant vorticity flux through the cross section $\Sigma$ of the corresponding handle

$$E_\Theta = \Theta \int_{\Sigma} \Omega; \ L = \partial \Sigma$$  \hspace{1cm} (72)$$

This term breaks the time reversal! This is the only possible source of the irreversibility in this theory in three dimensions.
The following observation leads to crucial simplifications. The only term which depends upon the Lagrange field $X_\alpha(\rho)$ is the volume term, which in fact is the functional of the bounding surface $X(\partial D)$

$$V = \int_D dX_1 \wedge dX_2 \wedge dX_3 = \int_{\partial D} X_3 dX_1 \wedge dX_2$$

(73)

The rest of the terms in the effective energy also depending only upon the boundary, this lowers the dimension of our effective field theory. We are dealing with the theory of self-avoiding random surfaces rather that the 3-d Lagrange dynamics. With some modifications the methods of the string theory can be applied to this problem.

The invariant distance in the $X(\xi)$ functional space is

$$||\delta X||^2 = \int_{\partial D} d^2 \xi \sqrt{g} (\delta X_\alpha(\xi))^2$$

(74)

where $g$ as usual stands for the determinant of the metric tensor. One may check that the corresponding volume element

$$(DX) = \prod_{\xi \in \partial D} \delta X$$

(75)

is conserved in the Euler-Helmholtz dynamics as well as the complete volume element. The key point in this extension of the Liouville theorem is the observation that the matrix trace of the functional derivative vanishes

$$\frac{\delta v_\alpha(X(\rho))}{\delta X_\alpha(\rho')} = 0$$

(76)

for arbitrary $\rho, \rho'$, including the boundary points. The metric tensor $g_{ij}$ does not introduce any complications, as it is $X$ independent.

This metric is the motion invariant in the vortex dynamics. In the statistics, according to the general philosophy these invariants become variables. We see that the field $R_a(\xi)$ enter only via the induced metric, which allows us to introduce the latter as a collective field variable.

One has to introduce the functional space of all metric tensors with the Polyakov distance

$$||\delta g||^2 = \int_S d^2 \xi \sqrt{\hat{g}} (\delta g_{ij} \delta g_{kl}) \left( A g^{ij} g^{kl} + B g^{ik} g^{jl} \right)$$

(77)

The parametric invariance can be most conveniently fixed by the conformal gauge

$$g_{ij}(\xi) = \hat{g}_{ij}(\xi) e^{\alpha \varphi(\xi)}$$

(78)

where $\hat{g}$ is some background metric parametrizing the surface with given topology. Unlike the internal metric, the background metric does not fluctuate. We have the freedom to choose any parametrization of the background metric.
The effective energy which emerges after all computations of the functional jacobians associated with the gauge fixing reads

\[ E_\phi = \frac{1}{4\pi} \int d^2 \xi \sqrt{\hat{g}} \left( \frac{1}{2} \hat{g}^{ij} \partial_i \phi \partial_j \phi - Q \hat{R} \phi + \mu e^{\alpha \phi} \right) \] (79)

where \( \hat{R} \) is the scalar curvature in the background metric.

The parameters \( Q, \alpha \) should be found from the self-consistency requirements. In case of the ordinary string theory in \( d \) dimensional space the requirement of cancellation of conformal anomalies yields

\[ \alpha = \frac{\sqrt{1 - d - \sqrt{25 - d}}}{2\sqrt{3}}; \quad Q = \frac{\sqrt{25 - d}}{3} \] (80)

In three dimensions \( \alpha \) is a complex number, which is fatal for the string theory. Fortunately this formula does not apply to turbulence, because the dynamics of the \( X \) field is completely different here. Later we speculate about the values of these parameters.

To summarize, the total phase space volume element of our string theory

\[ d\Gamma = \prod_{\text{cells}} (DX)(D\phi)(D\phi) \] (81)

and the total effective energy

\[ E = \sum_{\text{cells}} \left( V + E_\phi + E_\phi + \sum_{\text{loops}} E_\phi \right) \] (82)

The grand partition function

\[ Z = \sum_{N,h} \exp \left( -\mu N - \lambda \sum_{i=1}^{N} h_i \right) \int d\Gamma \exp \left( -\beta E \right) \] (83)

The interaction between cells comes from the excluded volume effect.

The vorticity correlations are generated by the loop functional

\[ \Psi_C(\gamma) = \langle \exp (i \Gamma_C[v]) \rangle \] (84)

where

\[ \Gamma_C[v] = \oint_C dr_\alpha v_\alpha(r) \] (85)

is the velocity circulation. Our solution for the loop functional reads

\[ \Psi_C(\gamma) = Z^{-1} \sum_{N,h} \exp \left( -\mu N - \lambda \sum_{i=1}^{N} h_i \right) \int' d\Gamma \exp \left( -\beta E + i \gamma \Gamma_C[v] \right) \] (86)

\[ ^6 \text{We cut some angles here. This effective measure was obtained with some extra locality assumptions in addition to the honest calculations. These assumptions were never rigorously proven, but they are known to work. All the results which were obtained with this measure coincide with those obtained by the mathematically justified method of dynamical triangulations and matrix models.} \]
where \( f' \) implies that the cells also avoid the loop \( C \).

There are now various topological sectors. In the trivial sector, the loop can be contracted to a point without crossing the cells. Clearly, circulation vanishes in this sector. In the nontrivial sectors, there is one or more handles encircled by the loop, so that the circulation is finite. Here is an example of such topology

![Diagram of a topology](image)

The circulation can be reduced to the Stokes integral of the vorticity 2-form over the surface \( S_C \) encircled by the loop \( C \). Only the parts \( s_i = S_C \cap D_i \) passing through the cells contribute. In each such part the vorticity 2-form can be transformed to the Clebsch coordinates which reduces it to the sum of the topological terms

\[
\Gamma_C = \sum_i \int_{s_i} \Omega = \sum_i \int_{s_i} d\phi_1 \wedge d\phi_2 = \sum_i \int_{\partial s_i} \phi_1 d\phi_2 \tag{87}
\]

In presence of the same terms in the effective energy the loop functional would be complex, as the positive and negative circulations would be weighted differently. This is manifestation of the time irreversibility.

The interesting thing is that the interaction between the \( X \) field and the rest of field variables originates in the topological restrictions. The circulation reduces to the net flux from the handles encircled by the loop in physical space (this is a restriction on the \( X \) field). Also, the requirement that the loop is avoided by cells imposes another restriction on the \( X \) field. There are no explicit interaction terms in the energy. The whole dependence of the loop functional on the shape of the loop \( C \) comes from the excluded volume effect. The implications of this remarkable property are yet to be understood.

Let us now check the Liouville equation. The time derivative of the circulation reads (see (22))

\[
\partial_t \Gamma_C[v] = \oint_C dr \omega_{\alpha\beta}(r)v_\beta(r) \tag{88}
\]

But the vorticity vanishes at the loop, because the cells avoid it! Therefore

\[
\langle \partial_t \Gamma_C[v] \exp(i\gamma C[v]) \rangle = 0 \tag{89}
\]

which is the Liouville equation.
For the readers of the previous paper[12] let us briefly discuss the loop equation. The essence of the loop equation is the representation of the vorticity as the area derivative acting on the loop functional

\[ \hat{\omega}_{\alpha\beta}(r) = -\frac{i}{\gamma} \frac{\delta}{\delta \sigma_{\alpha\beta}(r)} \]  

(90)

The formal definition of the area derivative in terms of the ordinary functional derivatives was discussed before [4, 12]. The geometric meaning is simple: add the little loop to the original one and find the term linear in the area \( \delta \sigma_{\alpha\beta} \) enclosed by this little loop. The parametric invariant functionals like this one can always be regularized so that the variation would start from \( \delta \sigma_{\alpha\beta} \). The area derivatives of the length and the minimal area inside the loop were derived in [4].

The velocity operator is related to vorticity by the Poisson integral

\[ \hat{v}_\alpha(r) = -\partial_\beta \int d^3R \frac{\hat{\omega}_{\alpha\beta}(R)}{4\pi|r - R|} \]  

(91)

The geometric meaning is as follows. The vorticity operator adds the little loop \( \delta C_{R,R} \) at the point \( R \) off the original loop \( C \). By adding the couple of straight line integrals

\[ W(r, R) = \exp \left( i \gamma \int_{L_{r,R}} dr_\alpha v_\alpha(r) \right) = W^{-1}(R, r) \]  

(92)

we reduce this to one loop of the singular shape

\[ \tilde{C} = \{ C_{r,r}, L_{r,R}, \delta C_{R,R}, L_{R,s} \} \]  

(93)

\[ \Gamma_C + \Gamma_{\delta C_{R,R}} = \Gamma_{\tilde{C}} \]  

(94)

which can be obtained from original one as certain variation[12]. Then the loop equation is simply

\[ \int_C dr_\alpha \hat{\omega}_{\alpha\beta}(r) \hat{v}_\beta(r) \Psi_C(\gamma) = 0 \]  

(95)

It was shown in [12] that these two operators commute as they should.

In our case this equation is satisfied in a trivial way. Regardless the subtleties in the definition of \( \hat{\omega}, \hat{v} \), as long as these operators act on the vortex sheet circulation as they should, they insert the vorticity and velocity at the loop. The rest of the argument is the same as in the Liouville equation.

Note that this trick would not solve the Hopf equation for the velocity generating functional

\[ H[J] = \langle \exp \left( \int d^3r J_\alpha(r) v_\alpha(r) \right) \rangle \]  

(96)
The Hopf equation would require

\[
\langle \int d^3 r' T_{\alpha\beta} (r - r') \omega_{\beta\gamma}(r') v_{\gamma}(r') \exp \left( \int d^3 r J_\alpha (r) v_\alpha (r) \right) \rangle = 0 \tag{97}
\]

which we cannot satisfy since the volume integral \( \int d^3 r' \) would inevitably overlap with cells where vorticity is present.

Apparently we found invariant probability distribution for vorticity but not for the velocity. The velocity distribution may not exist in the infinite system, due to the infrared divergencies. In practice this would mean that the velocity distribution would depend of the details of the large scale energy pumping, but vorticity distribution would be universal.

The correlation functions of vorticity field can be obtained from the multiloop functionals by contracting loops to points. In case when the loops encircle one or more handles there would be nonvanishing correlation function. Here is an example of the topology with the two point correlation which also has nontrivial helicity because of the knotted handle.

5 Discussion

So far we do not have much to tell to the engineers. Even if this statistics of the vortex structures will be confirmed by further study, we shall face the formidable task of computing the correlation functions. Let us speculate what could come out of these computations.

The qualitative picture of intermittent distribution of vorticity will be the same as in the multifractal models\[10\]. In fact these models inspired our study to some extent. Our basic idea is that the part \( V \) of the volume occupied by vorticity fluctuate. In numerical and real experiments\[11\] the high vorticity structures were clearly seen. The cells take
the shape of long sausages rather than spheres, which does not contradict our general philosophy but still lacks an explanation.

Let us try to estimate the intermittency effects in the vortex cells statistics. Let us contract the loop to a point $r$ around some handle. What we get in the limit can be expressed in terms of the *vertex operator* of the string theory

$$\Gamma_C \propto \int_S d^2 \xi_0 \sqrt{g} \int_S d^2 \eta_0 \sqrt{g} \delta^3(X(\xi) - r)\delta^3(X(\eta) - r'); \ r' \to r$$  \hspace{1cm} (98)

The points $\xi_0$ and $\eta_0$ are mapped to the same point $r' \to r$ in physical space: this is the handle strangled by the loop. The important detail here is the factor $\sqrt{g} = e^{\alpha \phi}$ corresponding to the metric tensor at the surface.

The properties of such metric tensors were studied in the string theory\[3, 5\]. The moments of $\Gamma_C$ would behave as

$$\langle \Gamma_C^n \rangle \propto r_0^{-\Delta(n)}; \Delta(n) = \frac{1}{2} n \alpha (n \alpha + Q)$$  \hspace{1cm} (99)

where $r_0$ is the short-distance cutoff. The parameters $\alpha, Q$ are to be found from the selfconsistency conditions.

In case of the turbulence theory we know that the third moment has no anomalous dimension, due to the Kolmogorov’s $\frac{4}{5}$ law. This implies that

$$Q = -3\alpha$$  \hspace{1cm} (100)

after which we exactly reproduce the anomalous dimensions of so called Kolmogorov-Obukhov intermittency model.

They assumed in 1961 the log-normal distribution of the energy dissipation rate as a modification of the Kolmogorov scaling. Various multifractal models generalizing this distribution, and the corresponding experimental and numerical data are discussed in proceedings of the 1991 conference in Santa Barbara [10].

The physics of the fractal dimensions in our theory is almost the same as in the Kolmogorov-Obukhov model. The local energy dissipation rate at the edge can be estimated as $\Gamma_C^n \propto e^{3\alpha \varphi}$. This quantity is, indeed, an exponential of the gaussian fluctuating variable. The variance is proportional to $\log r$ since it is the two dimensional field with the logarithmic propagator (how could they have guessed that!).

Strictly speaking, the dimensions of powers of $\Gamma_C$ were never measured. People considered the moments of velocity differences instead. In principle, the potential part of $v$ which is present in these moments may change the trajectory, so we must be cautious.

The point is, the potential part has completely different origin in our theory. It comes from the nonlocal effects, involving all the scales, including the energy pumping scales. In short, this part is infrared divergent. The vorticity part which we compute, is ultraviolet divergent, it is determined by the small scale fluctuations of the vortex sheet, represented by the Liouville field.
In absence of direct evidence we may try to stretch the rules and estimate our intermittency exponents from the moments of velocity. I would expect this to be an upper estimate, as intermittency tends to decrease with the removal of the large scale effects. In my opinion, there is still no direct evidence for the anomalous dimensions in vorticity correlators. It would be most desirable to fill this gap in real or numerical experiments.

Let us stress once again, that above speculations do not pretend to be a theory of turbulence. Still they may give us an idea how to build one.

In statistical mechanics the Gibbs distribution was the beginning, not the end of the theory. If this approach is correct, which remains to be seen, then all the work is also ahead of us in the string theory of turbulence. The string theory methods should be fitted for this unusual case, and, perhaps, some scaling and area laws could be established. I appeal to my friends in the string community. Look at the turbulence, this is a beautiful example of the fractal geometry with extra advantage of being guaranteed to exist!

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A Poisson brackets for the Euler equation

One may derive the Poisson brackets for the Euler equation by comparing the right side of this equation with

$$[v_\alpha(r), H] = \int d^3r' \left[ v_\alpha(r), v_\beta(r') \right] \frac{\delta H}{\delta v_\beta(r')} = \int d^3r' \left[ v_\alpha(r), v_\beta(r') \right] v_\beta(r')$$

Comparing this with the Euler equation we find

$$[v_\alpha(r), v_\beta(r')] = T_{\alpha\nu}(r-r') \omega_{\nu\beta}(r') + \ldots$$

where dots stand for the $\partial_\beta$ terms which drop in the above integral. These terms should be restored in such a way that the Poisson brackets would become skew symmetric plus
they must be divergenceless in $r'$ as well as in $r$

$$\frac{\partial}{\partial r} [v_\alpha(r), v_\beta(r')] = \frac{\partial}{\partial r'} [v_\alpha(r), v_\beta(r')] = 0 \quad (103)$$

The unique solution is given by the formula in the text.

It is also worth noting that one could derive the Poisson brackets for velocity field using Clebsch variables, which are known to be an ordinary $(p, q)$ pair. The velocity field is represented as an integral

$$v_\alpha(r) = \int d^3 r' T_{\alpha\beta}(r-r') \Phi_1(r') \partial_\beta \Phi_2(r') \quad (104)$$

and the corresponding vorticity is local

$$\omega_{\alpha\beta}(r) = \epsilon^{ij} \partial_\alpha \Phi_i \partial_\beta \Phi_j \quad (105)$$

The Euler equations are equivalent to the following equations

$$\dot{\Phi}_i + v_\beta \partial_\beta \Phi_i = 0 \quad (106)$$

which simply state that the Clebsch fields are passively advected by the flow. These equations have an explicit Hamiltonian form

$$\dot{\Phi}_i = -\epsilon^{ij} \frac{\delta H}{\delta \Phi_j} \quad (107)$$

where it is implied that the Hamiltonian is the same, with velocity expressed in Clebsch fields.

Now the Poisson brackets for the velocity fields can be computed in a standard way

$$[v_\alpha(r_1), v_\beta(r_2)] = \int d^3 r' e^{ij} \frac{\delta v_\alpha(r_1)}{\delta \Phi_i(r)} \frac{\delta v_\beta(r_2)}{\delta \Phi_j(r)} \quad (108)$$

which again yields the formula in the text. This derivation is not as general as the previous one, since there are some flows which cannot be globally described in Clebsch variables.

**B Helmholtz vs Euler dynamics**

Let us study the variation of the vortex cell velocity field. Our first objective would be to show that the time variation according to the Helmholtz equation

$$\delta X_\alpha(\rho) = dt \; v_\alpha(X(\rho)) \quad (109)$$

reproduces the Euler equation

$$\delta v_\alpha(r) = dt \; (v_\beta(r) \omega_{\alpha\beta}(r) - \partial_\alpha h(r)) \quad (110)$$
Simple calculation of the variation of the initial definition with integration by parts yields

$$\delta v_\alpha(r) = (e_{\alpha\beta\nu} \partial_\gamma - e_{\alpha\beta\gamma} \partial_\nu) \partial_\beta \int_D d^3\rho \frac{\delta X_\nu(\rho) \Omega_\gamma(\rho)}{4\pi |r - X(\rho)|}$$  \hfill (111)

Now we use the identity

$$e_{\alpha\beta\nu} \partial_\gamma - e_{\alpha\beta\gamma} \partial_\nu = e_{\beta\gamma\nu} \partial_\alpha - e_{\alpha\gamma\nu} \partial_\beta$$ \hfill (112)

(the difference between the left and the right sides represents the completely skew symmetric tensor of the fourth rank in three dimensions, which must vanish). This gives us the following two terms in velocity variation

$$\delta v_\alpha(r) = -\partial_\alpha \delta h(r) + e_{\alpha\gamma\nu} \int d^3\rho \frac{\delta X_\nu(\rho) \Omega_\gamma(\rho) \delta^3(r - X(\rho))}{4\pi |r - X(\rho)|}$$ \hfill (113)

where

$$\delta h(r) = e_{\gamma\beta\nu} \partial_\beta \int d^3\rho \frac{\delta X_\nu(\rho) \Omega_\gamma(\rho)}{4\pi |r - X(\rho)|}$$ \hfill (114)

The first term is purely potential, it can be reconstructed from the second term by solving $\partial_\alpha \delta v_\alpha(r) = 0$. Now we see that the Helmholtz variation reproduces the Euler equation with the enthalpy

$$h(r) = e_{\gamma\beta\nu} \partial_\beta \int d^3\rho \frac{v_\nu(X(\rho)) \Omega_\gamma(\rho)}{4\pi |r - X(\rho)|}$$ \hfill (115)

At the same time we see that the Helmholtz variation is defined modulo gauge transformation

$$\delta X_\nu(\rho) \Rightarrow \delta X_\nu(\rho) + f(\rho) \Omega_\nu(\rho)$$ \hfill (116)

which leaves the velocity variation invariant.

Note that the functional derivative

$$\frac{\delta v_\alpha(r)}{\delta X_\nu(\rho)} = (e_{\alpha\beta\nu} \partial_\gamma - e_{\alpha\beta\gamma} \partial_\nu) \partial_\beta \frac{\Omega_\gamma(\rho)}{4\pi |r - X(\rho)|}$$ \hfill (117)

is a traceless tensor in $\alpha, \nu$. This is sufficient for the volume conservation

$$\partial_t (DX) \propto \int d^3\rho \frac{\delta v_\alpha(X(\rho))}{\delta X_\alpha(\rho)} = 0$$ \hfill (118)

which is the Liouville theorem for the vortex dynamics.

We found the following Poisson bracket for the $X$ field

$$[X_\alpha(\rho), X_\beta(\rho')] = -\delta^3(\rho - \rho') e_{\alpha\beta\gamma} \frac{\Omega_\gamma(\rho)}{\Omega^2_\rho(\rho)}$$ \hfill (119)

The equations of motion corresponding to these Poisson brackets read

$$\partial_t X_\alpha(\rho) = [X_\alpha(\rho), H] = \int d^3\rho' [X_\alpha(\rho), X_\beta(\rho')] \frac{\delta H}{\delta X_\beta(\rho')}$$ \hfill (120)
Let us compare these equations with the usual Helmholtz dynamics. The variation of the Hamiltonian reads

\[ \delta H = \int d^3rv_\alpha(r)\delta v_\alpha(r) \tag{121} \]

Substituting here the velocity variation \(\delta h\) we could drop the \(\partial_\alpha \delta h\) term as it vanishes after integration by parts. As a result we find

\[ \frac{\delta H}{\delta X_\alpha(\rho)} = e_{\alpha\beta\gamma}v_\beta(X(\rho))\Omega_\gamma(\rho) \tag{122} \]

Finally, in the equation of motion we have

\[ \begin{align*}
\partial_t X_\alpha(\rho) &= -\int d^3\rho' \delta^3(\rho - \rho')e_{\alpha\beta\gamma}\frac{\Omega_\gamma(\rho)}{\Omega_\mu^2(\rho)}e_{\beta\mu\nu}v_\mu(X(\rho))\Omega_\nu(\rho) \\
&= \left(\delta_{\alpha\beta} - \frac{\Omega_\alpha(\rho)\Omega_\beta(\rho)}{\Omega_\mu^2(\rho)}\right)v_\beta(X(\rho))
\end{align*} \tag{123} \]

So, the Poisson brackets correspond to the motion in transverse direction to the gauge transformations. Or, to put it in different terms, we could modify the Helmholtz dynamics by adding the time dependent gauge transformations \(\delta \rho(t)\) to the time shift of the vortex sheet

\[ \partial_t X_\alpha(\rho) = v_\alpha(X(\rho)) + \partial_\alpha X_\alpha(\rho)\partial_t \delta \rho^a \tag{124} \]

where

\[ \partial_t \delta \rho^a = -\frac{\Omega_\alpha(\rho)\Omega_\beta(\rho)v_\beta(X(\rho))}{\Omega_\mu^2(\rho)} \tag{125} \]

Should we insist on the unmodified Helmholtz dynamics, we would have to admit that this cannot be achieved by any Poisson brackets. This is so called generalized Hamiltonian dynamics, where the formula (122) for the variation of the Hamiltonian cannot be uniquely solved for the velocity. The terms corresponding to the gauge transformation remain unspecified. In our opinion, this difference is immaterial, as the Helmholtz dynamics is indistinguishable from the one with the Poisson brackets.
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