THE REDNER–BEN-AVRAHAM–KAHNG COAGULATION SYSTEM WITH CONSTANT COEFFICIENTS: THE FINITE DIMENSIONAL CASE

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Abstract. We study the behaviour as \( t \to \infty \) of solutions \((c_j(t))\) to the Redner–Ben-Avraham–Kahng coagulation system with positive and compactly supported initial data, rigorously proving and slightly extending results originally established in \([5]\) by means of formal arguments.

1. Introduction

In a recent paper \([2]\) we started the study of a coagulation model first considered in \([3, 5]\) which we have called the Redner–Ben-Avraham–Kahng cluster system (RBK for short). This is the infinite-dimensional ODE system

\[
\frac{dc_j}{dt} = \sum_{k=1}^{\infty} a_{j+k,k} c_{j+k} c_k - \sum_{k=1}^{\infty} a_{j,k} c_j c_k, \quad j = 1, 2, \ldots
\]

with symmetric positive coagulation coefficients \(a_{j,k}\). As with the discrete Smoluchowski’s coagulation system \([1]\) this is a mean-field model describing the evolution of a system given at each instant by a sequence \((c_j)\), such that \(c_j\) is the density of \(j\)-clusters for each integer \(j\), undergoing a binary reaction described by a bilinear infinite-dimensional vector field. However, while in the Smoluchowski’s coagulation model one \(k\)-cluster reacts with one \(j\)-cluster producing one \((j+k)\)-cluster, in RBK the interaction between such clusters produce one \(|k-j|\)-cluster.

If we assume that there is no destruction of mass, in the former model it makes sense to think of \(j\) as the size, or mass, of each \(j\)-cluster. However in RBK the situation is different since with the same interpretation there would be a loss of mass in each reaction. Hence, it makes more sense to think of \(j\) as the size of the cluster ‘active part’, being the difference between \((j+k)\) and \(|j-k|\) the size of the resulting cluster that becomes inactive for the reaction process. A pictorial illustration of this is presented in Figure 1.

For more on the physical interpretation of (1.1) see \([3, 5]\). The nonexistence of a mass conservation property in RBK model makes for one of the major differences with respect to the Smoluchowski’s model. Also, unlike in this one, in RBK a \(j\) and a \(k\)-cluster react to produce a \(j'\)-cluster with \(j' < \max\{j, k\}\), implying that to an initial condition with an upper bound \(N\) for the subscript values \(j\) for which \(c_j(0) > 0\) there corresponds a solution with the same property for all instants \(t \geq 0\). This is an invariance property rigorously stated
on Proposition 7.1 in [2]. In this work we will consider such solutions for a finite prescribed upper bound \( N \geq 3 \) and \( j \)-independent coagulation coefficients \( a_{j,k} = 1 \), for all \( j, k \). Then, if \( c_j(0) = 0 \), for all \( j \geq N + 1 \), then \( c_j(t) = 0 \) for \( t \geq 0 \) and for the same values of \( j \), while \((c_1(t), c_2(t), \ldots, c_N(t))\) satisfy the following \( N \)-dimensional ODE

\[
\frac{dc_j}{dt} = N - j \sum_{k=1}^{N-j} c_{j+k}c_k - c_j \sum_{k=1}^{N} c_k, \quad j \in \mathbb{N} \cap [1, N],
\]

where the first sum in the right-hand side is defined to be zero when \( j = N \).

In this work we study system (1.2) for nonnegative initial conditions at \( t = 0 \), from the point of view of the asymptotic behaviour of each component, \( c_j(t) \), \( j = 1, \ldots, N \), as \( t \to \infty \). This problem has already been addressed in [5], where the authors have used a formal approach. In Theorem 2.1, we obtain the result for the general case \( c_j(0) \geq 0 \), for \( j = 1, 2, \ldots, N \), proving rigorously that the result in [5] is correct for initial conditions such that \( c_N(0) > 0 \) and the greater common divisor of the subscript values \( j \) for which \( c_j(0) > 0 \) is 1.

2. The main result

Consider \( N \geq 3 \). We are concerned with nonnegative solutions of \((1.2)\). By applying the results we have proved in [2] in the more general context referred above, we can deduce that, for a solution \( c = (c_j) \) to \((1.2)\), if \( c_j(0) \geq 0 \), for \( j = 1, \ldots, N \), then it is defined for all \( t \in [0, \infty) \) and \( c_j(t) \geq 0 \), for \( j = 1, \ldots, N \), and all positive \( t \). Let \( P = \{ j \in \mathbb{N} \cap [1, N] \mid c_j(0) > 0 \} \) be the set of subscript values for which the components of the initial condition \( c(0) \) are positive, and let \( \gcd(P) \) be the greatest common divisor of the elements of \( P \). In this paper we prove the following:

**Theorem 2.1.** Let \( c = (c_j) \) be a solution of \((1.2)\) satisfying \( c_j(0) \geq 0 \) for all \( j = 1, \ldots, N \). If \( m := \gcd(P) \) and \( p := \sup P \), then, for each \( j = m, 2m, \ldots, p \), there exists \( e_j : [0, \infty) \to \mathbb{R} \) such that \( e_j(t) \to 0 \) as \( t \to \infty \), and

\[
c_j(t) = \frac{\bar{A}_j}{t(\log t)^{m-1}}(1 + e_j(t))
\]

where

\[
\bar{A}_j := \frac{(N-1)!}{(N-j/m)!}.
\]

For all other \( j \in \mathbb{N} \cap [1, N] \), \( c_j(t) = 0 \), for all \( t \geq 0 \).

We begin the proof of this theorem by reducing it to the case \( m = 1, p = N \). Consider, for each \( t \geq 0 \), \( \mathcal{J}(t) := \{ j \in \mathbb{N} \cap [1, N] \mid c_j(t) > 0 \} \), the set of subscript

![Diagram](https://via.placeholder.com/150)
values for which the components of the solution are positive at instant $t$. Obviously, $P = \mathcal{J}(0)$. The case $\#P = 1$ is an immediate consequence of Proposition 7.3 in [2] and its proof. Consider now the case $\#P > 1$. Then, according to Proposition 7.2 in [2], $\mathcal{J}(t) = mN \cap [1, p]$, for all $t > 0$. Let $\tilde{N} := p/m$ and, for $j = 1, 2, \ldots, \tilde{N}$, let us write $\tilde{c}_j := c_{jm}$. Then it is straightforward to check that (1.2) is again satisfied with $N$ and $c_j$, for $j = 1, 2, \ldots, N$, replaced by $\tilde{N}$ and $\tilde{c}_j$, for $j = 1, 2, \ldots, \tilde{N}$, respectively. From the definition of $\mathcal{J}(t)$, we also have that, for $j = 1, \ldots, \tilde{N}$ and for all $t > 0$, $\tilde{c}_j(t) > 0$. For $j = 1, \ldots, \tilde{N}$, if $j \notin m\mathbb{N} \cap [1, p]$, then $c_j(t) = 0$, for all $t > 0$. Hence, after having established the validity of Theorem 2.1 with the restrictions $m = 1$ and $p = N$, if we consider a solution $c(\cdot)$ with initial conditions for which $m > 1$, $p < N$ or both, we can apply that restricted version of the theorem to $\tilde{c}$ and then use the fact that, for $j = 1, \ldots, p$, $c_j(t) = \tilde{c}_{jm}(t)$. For the other subscript values, $c_j(t)$ identically vanishes.

In conclusion, it is sufficient to prove the above theorem for $m = 1$, $p = N$, in which case, as we have seen, $c_j(t) > 0$, for $j = 1, 2, \ldots, N$, and all $t > 0$. This is done in next section.

3. Long time behaviour of strictly positive solutions

Consider a solution $c(\cdot) = (c_j(\cdot))$ to (1.2) such that $c_j(t) > 0$ for all $j = 1, \ldots, \tilde{N}$ and all $t \geq 0$. By the above and the fact that the ODE is autonomous we will see that this does not imply a loss of generality. Let

$$\nu(t) := \sum_{j=1}^{N} c_j(t),$$

so that (1.2) can be rewritten as

$$\dot{c}_j(t) + c_j(t)\nu(t) = \sum_{k=1}^{N-j} c_{j+k}(t)c_k(t),$$

and, in particular,

$$\dot{c}_N(t) + c_N(t)\nu(t) = 0.$$  (3.2)

We start by following the procedure already used in [5] that consists in time rescaling (1.2) so that the resulting equations only retain the production terms. From (3.2)

$$c_N(t)/c_N(0) = \exp\left(-\int_{0}^{t} \nu(s) \, ds\right).$$

Since $e^{\int_{0}^{t} \nu \, ds}$ is an integrating factor of (3.1), we conclude that

$$\frac{d}{dt} \left( \frac{c_j(t)}{c_N(t)} \right) = \frac{1}{c_N(t)} \sum_{k=1}^{N-j} c_{j+k}(t)c_k(t).$$

(3.3)

Let $y(t) := \int_{0}^{t} c_N(s) \, ds$ and define functions $\phi_j(y)$, such that

$$c_j(t) = \phi_j(y(t))c_N(t),$$

(3.4)

for each $j = 1, \ldots, N$, and $t \geq 0$. Then, for $j = 1, \ldots, N - 1$, $\phi_j(y)$ is defined and is strictly positive for $y \in [0, \omega)$, where $\omega := \int_{0}^{\infty} c_N \in (0, +\infty]$. Let us denote by
\((\cdot)'\) the derivative with respect to \(y\). Then, from (3.3) we obtain

\[
\phi_j'(y) = \sum_{k=1}^{N-j} \phi_{j+k}(y) \phi_k(y), \quad j = 1, \ldots, N-1,
\]

\[
\phi_N(y) = 1,
\]

for \(0 \leq y < \omega\). Conversely, if \((\phi_j(y))\) is a solution of (3.5) in its maximal positive interval \((0, \omega^*)\) and if \(c_N(\cdot)\), and therefore \(y(\cdot)\), is given, then \(c_j(t) = c_N(t) \phi_j(y(t))\), for \(j = 1, \ldots, N\), solves (1.2) for \(t \in [0, \infty)\), so that \(\omega^* = \omega\).

In the next two lemmas we state some results about the asymptotic behaviour of \(\phi(y)\).

**Lemma 3.1.** Any solution of (3.5), say \(\phi(y) = (\phi_1(y), \ldots, \phi_{N-1}(y), 1)\), satisfying \(\phi_j(0) > 0\), for all \(j = 1, \ldots, N\), is defined for \(y \in [0, \omega)\) where \(\omega > 0\) is finite and moreover,

(i) \(\phi_j(y) \to +\infty\) as \(y \to \omega\), for all \(j = 1, 2, \ldots, N-1\);

(ii) \(\phi_j(y)/\phi_{j+1}(y) \to +\infty\) as \(y \to \omega\), for all \(j = 1, 2, \ldots, N-1\).

**Proof.** Let \((\phi_j(y))\) be a solution of (3.5) in its positive maximal interval of existence \([0, \omega)\) satisfying the hypothesis of the lemma. Then, for all \(j = 1, \ldots, N\), \(\phi_j(y) > 0\), for all \(y \in [0, \omega)\). Since,

\[
\phi_j'(y) \geq \phi_{j+1}(y) \phi_1(y),
\]

for \(j = 1, \ldots, N-1\) (with equality for \(j = N-1\)), and \(\phi_N(y) = 1\), by defining \(\tau(y) := \int_0^y \phi_1(s) \, ds\), and \(\psi_j(\tau)\), such that \(\phi_j(y) = \psi_j(\tau(y))\), we obtain,

\[
d \frac{d}{d\tau} \psi_j(\tau) \geq \psi_{j+1}(\tau),
\]

for \(j = 1, \ldots, N-1\) (with equality for \(j = N-1\)), \(\psi_N(\tau) = 1\), for \(0 \leq \tau < \int_0^\omega \phi_1\).

The \(N-1\) equation gives,

\[
\psi_{N-1}(\tau) = \tau + c_0.
\]

Then by successively integrating (3.7) for \(j = N-2, N-3, \ldots, 1\), and taking in account that \(\psi_j(0) \geq 0\) for \(j = 1, \ldots, N\), we obtain

\[
\psi_{N-k}(\tau) \geq \frac{\tau^k}{k!}, \quad k = 1, \ldots, N-1.
\]

In particular,

\[
\psi_1(\tau) \geq \frac{\tau^{N-1}}{(N-1)!},
\]

which is equivalent to

\[
\tau'(y) \geq \frac{\tau(y)^{N-1}}{(N-1)!}.
\]

Since, by hypothesis, \(N-1 > 1\), this clearly implies that \(\omega < +\infty\). By ODE fundamental theory, this in turn implies that for our solution, we have \(|\phi(y)| \to \infty\), as \(y \to \omega\). This, together with the monotonicity property of each \(\phi_j(y)\), implies that there is a \(y^* \in \{1, \ldots, N-1\}\) such that \(\phi_j(y) \to +\infty\) as \(y \to \omega\). We now prove the nontrivial fact that this is true for all \(j = 1, \ldots, N-1\). In order to derive such conclusion we first prove that, for \(j = 1, \ldots, N-1\), \(\phi_j(y)/\phi_{j+1}(y)\) is
bounded away from zero for \( y \) sufficiently close to \( \omega \). Specifically, we prove that for 
\( n = N - 1, N - 2, \ldots, 2, 1 \), there are \( \eta > 0 \), \( Y \in [0, \omega) \) such that

\[
(3.8) \quad \frac{\phi_j(y)}{\phi_{j+1}(y)} > \eta,
\]
for \( j = n, n + 1, \ldots, N - 1 \), and for all \( y \in [Y, \omega) \).

Consider \( n = N - 1 \). Then \( \phi_{N-1}(y) = \phi_1(y) \), so that \( \phi_{N-1}(y)/\phi_N(y) = \phi_{N-1}(0) + \int_0^y \phi_1 \) and, by the positivity of \( \phi_1 \) the result is obvious with \( \eta = \phi_{N-1}(Y) \) for any \( Y \in (0, \omega) \).

Suppose now that we have proved our claim for \( n+1 \), with \( n \in \{1, \ldots, N - 1\} \), that is, there are \( \eta > 0 \), \( Y \in [0, \omega) \) such that (3.8) is true, for \( j = n + 1, n + 2, \ldots, N - 2 \) and for \( y \in [Y, \omega) \). We prove the same holds for \( n \). Since, for \( y \in [Y, \omega) \)

\[
\frac{\phi_n'(y)}{\phi_{n+1}'(y)} = \frac{\sum_{k=1}^{N-n} \phi_{k+n}(y) \phi_k(y)}{\sum_{k=1}^{N-n-1} \phi_{k+n+1}(y) \phi_k(y)} \geq \frac{\sum_{k=1}^{N-n-1} \phi_{k+n+1}(y) \phi_k(y)}{\sum_{k=1}^{N-n-1} \phi_{k+n+1}(y) \phi_k(y)} \geq \eta,
\]
and therefore
\[
\phi_n'(y) \geq \eta \phi_{n+1}'(y),
\]
by integration we obtain
\[
\phi_n(y) - \phi_n(Y) \geq \eta \left( \phi_{n+1}(y) - \phi_{n+1}(Y) \right)
\]
or
\[
\frac{\phi_n(y)}{\phi_{n+1}(y)} \geq \frac{\phi_n(Y)}{\phi_{n+1}(Y)} + \eta \left( 1 - \frac{\phi_{n+1}(Y)}{\phi_{n+1}(y)} \right).
\]
Let \( \tilde{Y} \in (Y, \omega) \). Then, for \( y \in [\tilde{Y}, \omega) \),
\[
\phi_{n+1}(y) \geq \phi_{n+1}(\tilde{Y}) > \phi_{n+1}(Y)
\]
Define and defining
\[
\tilde{\eta} := \eta \left( 1 - \frac{\phi_{n+1}(Y)}{\phi_{n+1}(\tilde{Y})} \right)
\]
we conclude that, for \( y \in [\tilde{Y}, \omega) \),
\[
\frac{\phi_n(y)}{\phi_{n+1}(y)} \geq \tilde{\eta}.
\]
By redefining \( Y, \eta \) as \( \tilde{Y}, \tilde{\eta} \) we have proved (3.8) for \( n \). This completes our induction argument.

Now let \( K := \{ j = 1, \ldots, N - 1 \mid \phi_j(y) \rightarrow \infty \text{ as } y \rightarrow \omega \} \). We already know that \( K \neq \emptyset \), so that we can define \( J := \max K \). Then, from (3.8) we get
\[
\phi_j(y) \rightarrow \infty \text{ as } y \rightarrow \omega, \quad \text{for all } j = 1, \ldots, J.
\]
It is then sufficient to prove that, in fact, \( J = N - 1 \). This is based on the integral version of (3.8), namely

\[
(3.9) \quad \phi_j(y) - \phi_j(Y) = \int_Y^y \phi_{j+1} \phi_1 + \int_Y^y \phi_{j+2} \phi_2 + \ldots + \int_Y^y \phi_{N-j-1} \phi_{N-1} + \int_Y^y \phi_{N-j},
\]
for \( j = 1, \ldots, N - 1 \). Now, by absurd, suppose that \( J < N - 1 \). Then, for
\( j = J + 1, \ldots, N - 1 \), \( \phi_j(y) \) is bounded for \( y \in [Y, \omega) \). But then, since (3.9) implies that
\[
\phi_j(y) - \phi_j(Y) = \int_Y^y \phi_{N-j} \mathrm{d}y,
\]
we conclude that \( \int_Y^y \phi_j \) must be bounded for \( j = 1, 2, \ldots, N - J - 1 \) and \( y \in [Y, \omega) \).
Therefore, for all \( y \in [Y, \omega) \),
\[
\phi_j(y) - \phi_j(Y) \leq \phi_{j+1}(y) \int_Y^y \phi_1 \, \mathrm{d}y + \phi_{j+2}(y) \int_Y^y \phi_2 \, \mathrm{d}y + \ldots
\]
\[
\ldots + \phi_{N-1}(y) \int_Y^y \phi_{N-J-1} + \int_Y^y \phi_{N-J}
\]
\[
\leq M + \int_Y^y \phi_{N-J},
\]
for some positive constant \( M \). Since \( \phi_j(y) \to \infty \) as \( y \to \omega \), this bound forces
\( \int_Y^y \phi_{N-J} \to \infty \) as \( y \to \omega \). But again by (3.8), we have, for \( y \in [Y, \omega) \),
\[
\phi_1(y) \geq \eta \phi_2(y) \geq \eta^2 \phi_3(y) \geq \ldots \geq \eta^{N-J-1} \phi_{N-J}(y),
\]
implying that
\[
\int_Y^y \phi_1 \geq \eta^{N-J-1} \int_Y^y \phi_{N-J}
\]
and so
\[
\int_Y^y \phi_1 \to \infty \quad \text{as} \quad y \to \omega.
\]
From (3.9) we have, for all \( j = 1, 2, \ldots, N - 1 \), and \( y \in [Y, \omega) \),
\[
\phi_j(y) - \phi_j(Y) > \phi_{j+1}(Y) \int_Y^y \phi_1.
\]
We are lead to the conclusion that, for all \( j = 1, 2, \ldots, N - 1 \), \( \phi_j(y) \to \infty \) as \( y \to \omega \),
thus contradicting the assumption that \( J < N - 1 \). This proves that \( J = N - 1 \).

It remains to be proved assertion (ii). For \( j = N - 1 \) is trivial, since
\[
\frac{\phi_{N-1}(y)}{\phi_N(y)} = \phi_{N-1}(y) \to +\infty \quad \text{as} \quad y \to \omega,
\]
as we have seen before. Suppose we have proved (ii) for \( j = N - 1, N - 2, \ldots, n + 1 \) for some \( n \in \{1, 2, \ldots, N - 2\} \). We prove that the same holds for \( j = n \). We consider again, for \( y \) close to \( \omega \), the quotient
\[
\frac{\phi_n'(y)}{\phi_{n+1}'(y)} = \frac{\sum_{k=1}^{N-n} \phi_{k+n}(y) \phi_k(y)}{\sum_{k=1}^{N-n-1} \phi_{k+n+1}(y) \phi_k(y)} = \frac{\sum_{k=1}^{N-n} \phi_{k+n+1}(y) \phi_k(y)}{\sum_{k=1}^{N-n-1} \phi_{k+n+1}(y) \phi_k(y)}
\]
\[
> \phi_1(y) \frac{1 + \sum_{k=2}^{N-n-1} \eta^{-k+1} \phi_{k+n+1}(y) \phi_k(y)}{\phi_{n+1}(y) \phi_1(y)} \to +\infty,
\]
as \( y \to \omega \). Then, we know by the Cauchy rule, that
\[
\lim_{y \to \omega} \frac{\phi_n(y)}{\phi_{n+1}(y)} = \lim_{y \to \omega} \frac{\phi_n'(y)}{\phi_{n+1}'(y)} = +\infty,
\]
Lemma 3.2. In the conditions of the previous lemma, for each \( j = 1, \ldots, N - 1 \), there is \( \rho_j : [0, \omega) \to \mathbb{R} \) such that \( \rho_j(y) \to 0 \) as \( y \to \omega \), and

\[
\phi_j(y) = \frac{A_j}{(\omega - y)^{\alpha_j}}(1 + \rho_j(y)),
\]

such that

\[
\alpha_j := \frac{N - j}{N - 2}, \quad A_j := \frac{1}{(N - j)!} \left( \frac{(N - 1)!}{N - 2} \right)^{\alpha_j}.
\]

Proof. By (ii) of the previous lemma, we know that, for \( j = 1, \ldots, N - 1 \),

\[
\sum_{k=1}^{N-j} \frac{\phi_{j+k}(y)\phi_k(y)}{\phi_{j+1}(y)\phi_1(y)} = 1 + \sum_{k=2}^{N-j} \frac{\phi_{j+k}(y)}{\phi_{j+1}(y)} \cdot \frac{\phi_k(y)}{\phi_1(y)} \to 1 \quad \text{as} \quad y \to \omega.
\]

Hence, we can write, for \( j = 1, \ldots, N - 1 \), and \( y \in (0, \omega) \)

\[
(3.10) \quad \psi_j(y) = \phi_1(y)\phi_1(y)(1 + r_j(y))
\]

such that \( r_j(y) \to 0 \), as \( y \to \omega \). We now perform the same change of variables as in the beginning of the proof of the previous lemma, this time giving, for \( \tau \geq 0 \),

\[
(3.11) \quad \frac{d}{d\tau} \psi_j(\tau) = \psi_{j+1}(\tau)(1 + \hat{r}_j(\tau)),
\]

such that \( \hat{r}_j(\tau) \to 0 \), as \( \tau \to \infty \). We now prove that, for \( j = 1, \ldots, N - 1 \),

\[
(3.12) \quad \psi_j(\tau) = \frac{\tau^{N-j}}{(N - j)!}(1 + \hat{\rho}_j(\tau))
\]

where \( \hat{\rho}_j(\tau) \to 0 \) as \( \tau \to \infty \). For \( j = N - 1 \), taking in account that \( \hat{r}_{N-1}(\tau) \equiv 0 \), the result easily follows:

\[
\psi_{N-1}(\tau) = \tau + c_0 = \tau(1 + c_0 \tau^{-1}).
\]

Now suppose we have verified (3.12) for \( j = n + 1 \), for some \( n = 1, \ldots, N - 2 \). We prove the same holds for \( j = n \). Defining \( \delta(\tau) \) by

\[
\delta(\tau) = (1 + \hat{\rho}_{n+1}(\tau))(1 + \hat{r}_n(\tau)) - 1,
\]

we have \( \delta(\tau) \to 0 \) as \( \tau \to \infty \), and by (3.11) and (3.12),

\[
\frac{d}{d\tau} \psi_n(\tau) = \frac{\tau^{N-n-1}}{(N - n - 1)!}(1 + \delta(\tau)),
\]

and therefore, upon integration,

\[
\psi_n(\tau) - \psi_n(0) = \frac{\tau^{N-n}}{(N - n)!} + \frac{1}{(N - n - 1)!} \int_0^\tau s^{N-n-1} \delta(s) \, ds,
\]

which can be written as

\[
\psi_n(\tau) = \frac{\tau^{N-n}}{(N - n)!}(1 + \hat{\rho}_n(\tau))
\]

where

\[
\hat{\rho}_n(\tau) := \frac{(N - n)! \psi_n(0)}{\tau^{N-n}} + \frac{N - n}{\tau^{N-n}} \int_0^\tau s^{N-n-1} \delta(s) \, ds.
\]
If the integral in the right hand side stays bounded for $\tau \geq 0$, then the last term converges to 0 as $\tau \to \infty$. If it is unbounded, since its integrand is positive then the integral tends to $+\infty$, as $\tau \to \infty$. In this case we can apply Cauchy rule since
\[
\left(\int_0^\tau \frac{s^{N-n-1} \delta(s)}{(\tau s)^n} \right)' = \frac{\delta(\tau)}{N-n} \to 0, \quad \text{as} \quad \tau \to \infty,
\]
thus proving that also in this case, the last term converges to 0 as $\tau \to \infty$. Either way we have $\hat{\rho}_j(\tau) \to 0$ as $\tau \to \infty$, thus proving assertion (3.12) for $j = n$. Our induction argument is complete.

In particular,
\[
\psi_1(\tau) = \frac{\tau^{N-1}}{(N-1)!} (1 + \hat{\rho}_1(\tau))
\]
which is equivalent to
\[
\tau'(y) = \frac{\tau(y)^{N-1}}{(N-1)!} (1 + \hat{\rho}_1(\tau(y)))
\]
for $y \in (0, \omega)$.

Let $0 < y < y_1 < \omega$. Then, the integration of the previous inequality in $[y, y_1]$ yields
\[
\tau(y)^{2-N} - \tau(y_1)^{2-N} = \frac{N-2}{(N-1)!} \left[ y_1 - y + \int_y^{y_1} \hat{\rho}_1(\tau(s)) \, ds \right].
\]
Define $R(y, y_1) := \frac{1}{y_1-y} \int_y^{y_1} \hat{\rho}_1(\tau(s)) \, ds$. Then,
\[
\tau(y) = \left[ \tau(y_1)^{2-N} + \frac{N-2}{(N-1)!} (y_1 - y) (1 + R(y, y_1)) \right]^{-\frac{1}{N-2}}.
\]
(3.13) \[
\tau(y) = \left[ \frac{N-2}{(N-1)!} (\omega - y) (1 + R_0(y)) \right]^{-\frac{1}{N-2}}.
\]
with
\[
R_0(y) = \frac{1}{\omega - y} \int_y^\omega \hat{\rho}_1(\tau(s)) \, ds \to 0 \quad \text{as} \quad y \to \omega,
\]
by Cauchy rule and the fact that $\hat{\rho}_1(\tau(y)) \to 0$ as $y \to \omega$.

For $j = 1, \ldots, N-1$, define
\[
\rho_j(y) := (1 + R_0(y))^{-\frac{N-j}{N-2}} (1 + \hat{\rho}_j(\tau(y))) - 1.
\]
so that $\rho_j(y) \to 0$, as $y \to \omega$. By (3.12) and (3.14), for $j = 1, \ldots, N-1$ and $y \in (0, \omega)$,
\[
\phi_j(y) = \psi_j(\tau(y)) = \frac{1}{(N-j)!} \left( \frac{(N-1)!}{N-2} \right)^{\frac{N-j}{N-2}} (\omega - y)^{-\frac{N-j}{N-2}} (1 + \rho_j(y)).
\]
and the proof is complete. \hfill \Box

The following lemma is a weaker version of Theorem 2.1 which will be used to complete the proof of the full result:
Lemma 3.3. If $c_j(0) > 0$, for $j = 1, \ldots, N$, then, for each such $j$, there exists $e_j : [0, \infty) \to \mathbb{R}$ such that $e_j(t) \to 0$ as $t \to \infty$, and

$$c_j(t) = \frac{\tilde{A}_j}{t(\log t)^{j-1}}(1 + e_j(t))$$

where

$$\tilde{A}_j := \frac{(N - 1)!}{(N - j)!}.$$

Proof. From the expression (3.4) defining $\phi_j$ we have $c_j(t) = \phi_j(y(t))c_N(t)$ and thus, in order to prove the lemma we need to determine the asymptotic behaviour of $c_N(t)$ and $y(t)$ and apply Lemma 3.2 observing that the hypothesis of both lemmas are equivalent.

Let us start by the study of the behaviour of $c_N(t)$.

From (3.2) we have, after integration in $[T, t]$,

$$c_N(t) = c_N(T) \exp \left(- \int_T^t \nu(s)ds \right),$$

where

$$\nu(t) = \sum_{j=1}^N c_j(t) = c_N(t) \sum_{j=1}^N \phi_j(y(t))$$

$$= \phi_1(y(t))c_N(t) \left(1 + \sum_{j=2}^N \frac{\phi_j(y(t))}{\phi_1(y(t))} \right)$$

$$= \phi_1(y(t))c_N(t) \left(1 + \sum_{j=2}^N \prod_{\ell=1}^{j-1} \frac{\phi_{\ell+1}(y(t))}{\phi_{\ell}(y(t))} \right)$$

$$\leq \phi_1(y(t))c_N(t) \left(1 + \sum_{j=2}^N \varepsilon^{j-1} \right)$$

$$\leq \phi_1(y(t))c_N(t) \left(1 + \frac{\varepsilon}{1-\varepsilon} \right)$$

$$=: \phi_1(y(t))c_N(t)(1 + \varepsilon)$$

where the first inequality arises from Lemma 3.1(ii), by putting $\phi_{\ell+1}/\phi_{\ell} \leq \varepsilon$, which is true for all the (finite number of) values of $\ell$ and all $t > T$, for sufficiently large $T$. The last equality is the definition of $\varepsilon$ in terms of $\bar{\varepsilon}$. As we clearly have $\nu(t) = \sum_{j=1}^N c_j(t) \geq \phi_1(y(t))c_N(t)$, we can write the bounds

$$\exp \left(- \int_T^t (1 + \varepsilon)\phi_1(y(s))c_N(s)ds \right) \leq \frac{c_N(t)}{c_N(T)} \leq \exp \left(- \int_T^t \phi_1(y(s))c_N(s)ds \right).$$

Now observe that, due to the definition of $y$, namely $y(t) = \int_0^t c_N(s)ds$, we can write

$$\int_T^t \phi_1(y(s))c_N(s)ds = \int_{yt}^y \phi_1(\bar{y})d\bar{y} = \int_{yt}^y \frac{A_1}{(\omega - \bar{y})^{\alpha_1}}(1 + \rho(\bar{y}))d\bar{y}.$$
Let $\varepsilon > 0$ be arbitrary. There exists a $T > 0$ and a corresponding $y_T \in (0, \omega)$ such that $|\rho(\bar{y})| < \varepsilon$ for all $\bar{y} > y_T$. Thus,

\[(3.16) \quad (1 - \varepsilon) \int_{y_T}^y \frac{A_1}{(\omega - \bar{y})^{\alpha_1}} d\bar{y} \leq \int_t^t \phi_1(y(s)) c_N(s) ds \leq (1 + \varepsilon) \int_{y_T}^y \frac{A_1}{(\omega - \bar{y})^{\alpha_1}} d\bar{y},\]

and upon integration we have

\[\int_{y_T}^y \frac{A_1}{(\omega - \bar{y})^{\alpha_1}} d\bar{y} = \frac{A_1}{\alpha_1 - 1} (\omega - y)^{1-\alpha_1} - \frac{A_1}{\alpha_1 - 1} (\omega - y_T)^{1-\alpha_1}.\]

Now, taking $T$ large enough and $\varepsilon$ small enough so that both (3.15) and (3.16) hold simultaneously, we can write

\[\int_{y_T}^y \frac{A_1}{(\omega - \bar{y})^{\alpha_1}} d\bar{y} = \frac{A_1}{\alpha_1 - 1} \left(\omega - y_T\right)^{1-\alpha_1} - \frac{A_1}{\alpha_1 - 1} \left(\omega - y_T\right)^{1-\alpha_1}.\]

Let us first consider the upper bound (3.18).

By the definition of $\omega$ we know that $\omega(\bar{y}) = c_N(t)$. Let us denote $x := \omega - y$ so that $\dot{x} = -\dot{y} = -c_N(t)$. Then, (3.18) can be written as

\[(3.21) \quad \frac{dx}{dt} \geq -R_N(T) \exp \left( -\frac{A_1}{\alpha_1 - 1} (\omega - y_T)^{1-\alpha_1} \right),\]

and, after integration and a number of algebraic manipulations, we obtain

\[(3.22) \quad \int_0^\theta \hat{\theta}^{-N} \hat{e}^{\hat{\theta}} d\hat{\theta} \leq \bar{R}_N(T, \varepsilon)(t - t_0),\]

where

\[(3.23) \quad \theta := \frac{A_1}{\alpha_1 - 1} (1 - \varepsilon)x^{1-\alpha_1},\]

(analogously for $\theta_0$), and

\[(3.24) \quad \bar{R}_N(T, \varepsilon) := R_N(T) \left( \frac{(\alpha_1 - 1)^{\alpha_1}}{(1 - \varepsilon)A_1} \right)^{\frac{1}{\alpha_1 - 1}}.\]

Now observe that, from the fact that $\theta^{1-N}e^{\theta} \to \infty$ as $\theta \to \infty$, and

\[\frac{d}{d\theta} \left( \int_0^\theta \hat{\theta}^{-N} \hat{e}^{\hat{\theta}} d\hat{\theta} \right) = \frac{\theta^{1-N} e^{\theta}}{\theta^{1-N} e^{\theta} + (1 - N)\theta^{-N} e^{\theta}} \to 1 \quad \text{as} \quad \theta \to \infty,
\]

we can apply L'Hôpital's rule to conclude that

\[(3.25) \quad \int_0^\theta \hat{\theta}^{-N} \hat{e}^{\hat{\theta}} d\hat{\theta} = \theta^{1-N} e^{\theta} (1 + o(1)) \quad \text{as} \quad \theta \to \infty,
\]

and thus (3.22) can be written as

\[(3.26) \quad \theta^{1-N} e^{\theta} \leq t \bar{R}_N(T, \varepsilon) (1 + o(1)),\]
when $t, \theta \to \infty$. Remembering that, if $y = \theta^{1-N}e^\theta$, then, as $\theta \to \infty$, the following holds $\theta = \log y + (N-1)(\log \log y)(1 + o(1))$ (see [4]), and using the fact that the logarithm is a monotone increasing function, we deduce that (3.26) implies
\begin{equation}
\theta \leq \left[ \log t \bar{R}_N + (N-1) \log \log t \bar{R}_N \right] (1 + o(1)), \quad \text{as } t, \theta \to \infty.
\end{equation}

To obtain the lower bound we proceed in a similar way, starting with the lower bound (3.17). The inequality correspondent to (3.21) is now
\begin{equation}
\frac{dx}{dt} \leq -L_N(T) \exp \left( -\frac{A_1}{\alpha_1 - 1} x^{1-\alpha_1} (1 + \varepsilon)^2 \right).
\end{equation}
Integrating this differential inequality we obtain the following inequality, analogous to (3.22),
\begin{equation}
\int_{\xi_0}^{\xi} \tilde{\xi}^{1-N}e^{\tilde{\xi}} d\tilde{\xi} \geq \bar{L}_N(T, \varepsilon)(t - t_0),
\end{equation}
where
\begin{equation}
\xi := \frac{A_1}{\alpha_1 - 1} (1 + \varepsilon)^2 x^{1-\alpha_1} = \frac{(1 + \varepsilon)^2 \theta}{1 - \varepsilon}
\end{equation}
(and analogously for $\xi_0$), and
\begin{equation}
\bar{L}_N(T, \varepsilon) := L_N(T) \left( \frac{(\alpha_1 - 1)^{\alpha_1}}{(1 + \varepsilon)^2 A_1} \right)^{\frac{1}{\alpha_1 - 1}},
\end{equation}
and repeating the approach described above we have
\begin{equation}
\xi \geq \left[ \log t \bar{L}_N + (N-1) \log \log t \bar{L}_N \right] (1 + o(1)), \quad \text{as } t, \xi \to \infty,
\end{equation}
or, equivalently,
\begin{equation}
\theta \geq \left[ \log t \bar{L}_N + (N-1) \log \log t \bar{L}_N \right] \frac{1 - \varepsilon}{(1 + \varepsilon)^2} (1 + o(1)), \quad \text{as } t, \theta \to \infty.
\end{equation}

Having the inequalities (3.33) and (3.27), we can deduce bounds for $c_N$.

Let us start by the upper bound. Using (3.23), (3.33), and the expressions for $A_1$ and $\alpha_1$, the inequality (3.18) becomes
\begin{equation}
c_N(t) \leq R_N(T)e^{-\theta}
\end{equation}
\begin{equation}
\leq \frac{R_N(T)}{L_N^{\frac{1}{1+\varepsilon}} \left( \frac{1}{t(\log t)^N - 1} \right)^{\frac{1-\varepsilon}{(1+\varepsilon)^2}}} (1 + o(1))
\end{equation}
\begin{equation}
= ((N-1)! + O(\varepsilon)) \left( \frac{1}{t(\log t)^N - 1} \right)^{\frac{1-\varepsilon}{(1+\varepsilon)^2}} (1 + o(1)).
\end{equation}
The lower bound can be obtained in a similar way: using (3.30), (3.27), and again the expressions for $A_1$ and $\alpha_1$, (3.17) becomes

$$c_N(t) \geq L_N(T) e^{-\xi} \geq \frac{L_N(T)}{R_N(t)} \left( \frac{1}{t(\log t)^N} \right)^{\frac{(1+\varepsilon)^2}{1+\varepsilon}} \left( (N-1)! + O(\varepsilon) \right) \frac{1}{(t(\log t)^N)^N(1+o(1))}.$$  

(3.35)

From (3.34), (3.35), and the arbitrariness of $\varepsilon$ it follows that, as $t \to \infty$,

$$c_N(t) = ( (N-1)! ) \frac{1}{t(\log t)^N(1+o(1))}.$$  

(3.36)

Now we can use the expression defining $\phi_j$, namely,

$$c_j(t) = \phi_j(y(t)) c_N(t),$$  

(3.37)

and the result of Lemma 3.2 and (3.36) to complete the proof: we have

$$c_j(t) = \frac{A_j}{(\omega - y(t))^{\alpha_j}} ((N-1)! \frac{1}{t(\log t)^N(1+o(1))} \text{ as } t \to \infty.$$  

From $x = \omega - y$, using (3.23) and the definitions of $A_1$ and $\alpha_1$, we have

$$(\omega - y)^{-\alpha_j} = x^{-\alpha_j} = \left( \frac{N-2}{(N-1)!} \right)^{\frac{N-2}{(N-1)!}} (1 - \varepsilon)^-(N-j) \theta^{N-j}.$$  

Since (3.27) and (3.33) imply that, as $t \to \infty$, $\theta = (\log t)(1+o(1))$, we conclude that

$$c_j(t) = \frac{(N-1)!}{(N-j)!} \frac{1}{t(\log t)^j(1+o(1))} \text{ as } t \to \infty,$$  

(3.38)

as we wanted to prove. 

□

Now, consider the case $c_j(0) \geq 0$, for $j = 1, \ldots, N$, with $m = \gcd(P) = 1$ and $p = \sup P = N$, thus implying that $J(t) = N \cap [1, p]$ for all $t > 0$. Since (1.2) is an autonomous ODE, then, given a small $\varepsilon > 0$, for $t \geq \varepsilon$, $c(t) = c_\varepsilon(t - \varepsilon)$, where $c_\varepsilon(\cdot)$ is the solution of (1.2) satisfying the initial condition $c_\varepsilon(0) = c(\varepsilon)$. Therefore, the conditions of Lemma 3.3 apply to $c_\varepsilon(\cdot)$. Then, it is easy to see that the asymptotic results that we conclude with respect to $c_\varepsilon(t)$ also apply to $c(t)$, allowing us to state the following:

**Lemma 3.4.** Let $c = (c_j)$ be a solution satisfying $c_j(0) \geq 0$, with $m = 1$ and $p = N$. Then the conclusions of Lemma 3.3 apply.

This is, in fact, the particular case of Theorem 2.4 from which the full case follows as stated at the end of section 2.
4. Final remarks

A natural question to ask is: what is the asymptotic behaviour of the solutions of (1.2) in the infinite dimensional case \((N = \infty)\)? It is clear that Theorem 2.1 by itself is insufficient to answer this question since the passage to the limit, \(N \to \infty\), is not allowed without results on the uniformity of the various limits involved, which seems to be a hard task. Also it is far from clear how to rebuild the proofs of the lemmas in section 3 in this more general case since they heavily rely on the fact that there is a 'last equation', the \(N\)-component equation, that can be integrated by the reduction method we have used, being the asymptotic behaviour of the other components deduced in a 'backwards' manner. Such procedure is obviously impossible in an infinite dimensional setting. In fact, that the situation can be very different for \(N = \infty\) from the one displayed by Theorem 2.1 is shown by the existence of the self-similar solutions given by,

\[
c_j(t) = (\kappa + t)^{-1}(1 - \alpha^2)^{j-1}, \quad j = 1, 2, \ldots, \quad t \geq 0,
\]

with constants \(\kappa > 0\) and \(\alpha \in (0, 1)\) (see [2]), in which case, \(tc_j(t) \to (1 - \alpha^2)^{j-1}\), as \(t \to \infty\), for \(j = 1, 2, \ldots\). Further work will be devoted to fully understand this problem.

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