Large-N Reduction, Master Field and Loop Equations in Kazakov–Migdal Model

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Abstract

I study the large-$N$ reduction $a la$ Eguchi–Kawai in the Kazakov–Migdal lattice gauge model. I show that both quenching and twisting prescriptions lead to the coordinate-independent master field. I discuss properties of loop averages in reduced as well as unreduced models and demonstrate those coincide in the large mass expansion. I derive loop equations for the Kazakov–Migdal model at large $N$ and show they are reduced for the quadratic potential to a closed set of two equations. I find an exact strong coupling solution of these equations for any $D$ and extend the result to a more general interacting potential.
1 Introduction

The purpose of this paper is a further study of the Kazakov–Migdal lattice gauge theory [1] which is defined by the partition function

$$Z_{KM} = \int \prod_{x,\mu} dU(x) \prod_{x} d\Phi(x) e^{\sum_{x} N \text{tr} \left( -V[\Phi(x)] + \sum_{\mu} \Phi(x) U_{\mu}(x) \Phi(x+\mu) U_{\mu}^\dagger(x) \right)}.$$  \hspace{1cm} (1.1)

Here the field $\Phi(x)$ takes values in the adjoint representation of the gauge group $SU(N)$ and the link variable $U_{\mu}(x)$ is an element of the group. The lowest coefficient of the expansion of the potential $V[\Phi]$ in the lattice spacing, $a$,

$$V[\Phi] = m_0^2 \Phi^2 + \ldots$$  \hspace{1cm} (1.2)

is identified with a bare mass parameter while the others play the role of couplings of self-interaction of the field $\Phi$.

The recent interest in the model (1.1) is due to the following reasons. As it is proposed in Ref. [1] the model (1.1) induces QCD in the continuum limit which should be obtained, as usually in lattice gauge theories, in the vicinity of a second order phase transition. This limit should be reached by approaching the couplings of the potential (1.2) to the corresponding critical values. As was proposed in Ref. [1] this critical point should be identified with the one separating the strong and weak coupling phases of the model (1.1). On the other hand the model (1.1) is potentially solvable in the large-$N$ limit with different distributions of eigenvalues of the matrix $\Phi$ in the two phases [1, 2]. To obtain a confining continuum limit, one should approach the critical point from the site of strong coupling phase while approaching from the weak coupling phase seems to result in deconfining Higgs phase.

However, soon after this scenario of inducing QCD had been proposed, it was pointed out by Kogan, Semenoff and Weiss [3] that the model (1.1) possesses an extra local $Z_N$ symmetry which leads in the strong coupling phase to local confinement when quarks cannot propagate even inside hadrons. To escape this physically unacceptable picture, it was assumed this $Z_N$ symmetry to be spontaneously broken at the point of the strong to weak coupling phase transition. A similar scenario has been proposed by Khokhlov and the author when normal confinement restores after the large-$N$ phase transition occurs before the one associated with continuum limit (i.e. within the strong coupling phase in our terminology). This conjectured phase looks similar to the weak coupling phase of standard lattice gauge theories. To avoid terminological confusions I shall refer it as the intermediate coupling phase.

The problem of whether the large-$N$ phase transition occurs according to this scenario is a dynamical one and can be studied using standard methods of lattice gauge theories. In particularly, the mean field method has been applied in Ref. [4]. The result for the case of the quadratic potential $V[\Phi]$ in negative — no first order phase transition occurs
in the strong coupling phase. This conclusion coincides with the one made on the basis of the exact solution found for the quadratic potential by Gross [5].

While the above scenario failed for the quadratic potential for this reason, a possibility of an alternative ‘stringy’ continuum limit has been conjectured for this case by Kogan, Morozov, Semenoff and Weiss [6] when the lattice spacing is taken to be $N$-dependent and should approach zero simultaneously with $N \to \infty$ in a special way. Such a ‘stringy’ large-$N$ limit differs from the ‘t Hooft topological expansion of QCD which is dictated by the known dependence of the coupling constant on $N$ prescribed by asymptotic freedom at small distances. For this reason, all the standard large-$N$ technology like factorization, saddle point equations, etc. are not applicable to the ‘stringy’ large-$N$ limit. In particularly, the exact $N = \infty$ solutions of Refs. [4, 5] can not be applied as well.

The latter exact solutions have been obtained by solving the ‘master field equation’ derived by Migdal [2] under the assumption [1] that the path integral over $\Phi(x)$ is saturated as $N \to \infty$ by a single $x$-independent saddle point configuration $\Phi_s$ — the master field. While it was pointed out [1] that such a master field does not contradict to all our knowledge about the large-$N$ limit, a mechanism of its appearance was mysterious.

In the present paper I consider this problem from the viewpoint of the large-$N$ reduction which had been first advocated by Eguchi and Kawai [7] for lattice gauge theories at $N = \infty$. The large-$N$ reduction states that the model on an infinite lattice is equivalent that at one point (a plaquette in the case of lattice gauge theory) so that the space-time degrees of freedom are eaten by the internal symmetry group. My idea would be to identify the master field $\Phi_s$ with a the saddle point configuration of a one-matrix model which appears after the reduction. I consider both quenching [8] and twisting [9] prescriptions of the large-$N$ reduction and argue in Section 2 that while they correctly reproduce the perturbative expansion of the Kazakov–Migdal model, any dependence on the quenched momenta can be absorbed by a (nonperturbative) gauge transformation.

To justify this reduction procedure, I derive in Section 3 loop equations for the Kazakov–Migdal model. I show that in addition to the adjoint Wilson loop

$$W_A(C) = \left\langle \frac{1}{N^2} \left( |\text{tr} U(C)|^2 - 1 \right) \right\rangle \quad (1.3)$$

where the average is understood with the same measure as in Eq. (1.1), the objects of a new kind

$$G(C_{xy}) = \left\langle \frac{1}{N} \text{tr} (\Phi(x)U(C_{xy})\Phi(y)U(C_{yx})) \right\rangle \quad (1.4)$$

emerges in the loop equations. This has a meaning of the average for an open loop which is made gauge invariant by attaching scalar fields at the ends. For the quadratic potential I obtain in the large-$N$ limit the closed set of two equations.

In Section 4, I discuss properties of the loop averages both in reduced and in unreduced cases and show explicitly of how they coincide to the leading order of the large mass expansion. I speculate as well on the properties of the intermediate phase and argue that it should resemble the standard Wilson lattice gauge theory.
An exact solution of the loop equations in the strong coupling phase is found in Section 4 for the quadratic potential at any number of dimensions $D$. An explicit formula for the average (1.4) is given by Eq. (5.7) below. While this solution agrees with that by Gross [5], I do not make any assumptions about the master field to find it. Moreover it is a first example of exact calculations of extended objects in Kazakov–Migdal model. I extend the solution to a more general potential

$$N \text{ tr } V[\Phi] = m_0^2 N \text{ tr } \Phi^2 + N^2 f\left(\frac{1}{N} \text{ tr } \Phi^2\right), \quad (1.5)$$

where $f$ is an arbitrary function, of the type studied recently in the matrix models [10] and show that no large-$N$ phase transition occurs in the strong coupling phase for this potential as well.

2 Large-$N$ reduction

2.1 Scalar field

The idea of large-$N$ reduction was put forward by Eguchi and Kawai [7] who showed the Wilson lattice gauge theory on a $D$-dimensional hypercubic lattice to be equivalent at $N = \infty$ to the one on a hypercube with periodic boundary conditions. This construction is based on an extra $(Z_N)^D$-symmetry which the latter theory possesses to each order of the strong coupling expansion but is broken in the weak coupling region. To cure the construction at weak coupling, the quenching prescription was proposed by Bhanot, Heller and Neuberger [8] and elaborated by many authors (for a review, see [11]). An elegant alternative reduction procedure based on twisting prescription was advocated by Gonzalez-Arroyo and Okawa [9]. An extension of the quenched Eguchi–Kawai model to the case of hermitian matrices was proposed by Parisi [12] end elaborated by Gross and Kitazawa [13] while that of the twisting prescription was advocated by Eguchi and Nakayama [14] and has been discussed recently by Alvaréz-Gaume and Barbón [10] in the context of $D > 1$ strings. Let me first briefly review these results which allow to reduce the partition function of self-interacting matrix scalar field on the infinite $D$-dimensional lattice at $N = \infty$ to a hermitian one-matrix model in an external field.

For a pure scalar theory whose partition function is defined by the path integral similar to (1.1) but without gauging:

$$Z = \int \prod_x d\Phi(x) e^{\sum_x N \text{ tr } \left(-V[\Phi(x)] + \sum_{\mu} \Phi(x)\Phi(x+\mu)\right)}, \quad (2.1)$$

the quenched momentum prescription is formulated as follows. One substitutes

$$\Phi(x) \to S(x)\Phi S^\dagger(x) \quad (2.2)$$

where

$$[S(x)]_{ij} = e^{ik^\mu_{x+\mu} \delta_{ij}} \quad (2.3)$$
is a unitary matrix which eats the coordinate dependence. The averaging of a functional $F[\Phi(x)]$ with the same weight as in Eq. (2.1) can be calculated at $N = \infty$ by

$$
\langle F[\Phi(x)] \rangle \to \int_{-\pi}^{\pi} \prod_{\mu=1}^{D} \prod_{i=1}^{N} \frac{dk_{\mu}^{i}}{2\pi} \langle F[S(x)\Phi S^{\dagger}(x)] \rangle_{\text{Reduced}}
$$

(2.4)

where the average on the r.h.s. is calculated for the quenched reduced model defined by the partition function $\text{Z}_{\text{QRM}}$:

$$
\text{Z}_{\text{QRM}} = \int d\Phi e^{-N \text{tr} V[\Phi] + N \sum_{ij} \Phi_{ij}^{2} \left(D - \sum_{\mu} \cos (k_{\mu}^{i} - k_{\mu}^{j})\right)}
$$

(2.5)

which can be obtained from the one (2.1) by the substitution (2.2).

Since $N \to \infty$ it is not necessary to integrate over the quenched momenta in Eq. (2.4). The integral should be recovered if $k_{\mu}^{i}$’s would be uniformly distributed in a $D$-dimensional hypercube. Moreover, a similar property holds for the matrix integral over $\Phi$ as well which can be substituted by its value at the saddle point configuration $\Phi_{s}$:

$$
\langle F[\Phi(x)] \rangle \to F[S(x)\Phi_{s} S^{\dagger}(x)] .
$$

(2.6)

This saddle point configuration was referred as the master field [15].

An alternative reduction procedure is based on the twisting prescription. One performs again the unitary transformation (2.2) with the matrices $S(x)$ being expressed via a set of $D$ (unitary) $N \times N$ matrices $\Gamma_{\mu}$ by the path-dependent factors

$$
S(x) = P \prod_{l \in C_{x\infty}} \Gamma_{\mu} .
$$

(2.7)

The path-ordered product in this formula runs over all links $l = (z, \mu)$ forming a path $C_{x\infty}$ from infinity to the point $x$. The matrices $\Gamma_{\mu}$ are explicitly constructed in Ref. [3] and commute by

$$
\Gamma_{\mu} \Gamma_{\nu} = Z_{\mu\nu} \Gamma_{\nu} \Gamma_{\mu}
$$

(2.8)

with $Z_{\mu\nu} = Z_{\nu\mu}^{\dagger}$ being elements of $Z_{N}$.

Due to Eq. (2.8), changing the form of the path multiplies $S(x)$ by the abelian factor

$$
Z(C) = \prod_{\square \in S : \partial S = C} Z_{\mu\nu}(\square)
$$

(2.9)

where $(\mu, \nu)$ is the orientation of the plaquette $\square$. The product runs over any surface spanned by the closed loop $C$ which is obtained by passing the original path forward and the new path backward. Due to the Bianchi identity

$$
\prod_{\square \in \text{cube}} Z_{\mu\nu}(\square) = 1
$$

(2.10)

where the product goes over six plaquettes forming a 3-dimensional cube on the lattice, the product on the r.h.s. of Eq. (2.9) does not depend on the form of the surface $S$ and is a functional of the loop $C$. 

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It is easy to see now that under this change of the path one gets
\[ (S(x))_{ij}(S^\dagger(x))_{kl} \rightarrow |Z(C)|^2 (S(x))_{ij}(S^\dagger(x))_{kl} \] (2.11)
and the path-dependence is cancelled because \(|Z(C)|^2 = 1\). This is a general property which holds for the twisting reduction prescription of any even \((\text{i.e. invariant under the center } Z_N)\) representation of \(SU(N)\).

For definitiveness one can choose
\[ S(x) = \Gamma_1^x \Gamma_2^x \Gamma_3^x \Gamma_4^x \] (2.12)
where the coordinates of the (lattice) vector \(x_\mu\) are measured in the lattice units.

For the twisting reduction prescription, Eq. (2.4) is valid providing the average on the r.h.s. is calculated for the \textit{twisted reduced model} which is defined by the partition function \[ Z_{TRM} = \int d\Phi e^{-N \text{tr} V[\Phi] + N \sum_\mu \text{tr} \Gamma_\mu \Phi \Gamma_\mu^\dagger \Phi} . \] (2.13)
Now the perturbation theory for the unreduced model (2.1) is recovered due to an explicit dependence of \(\Gamma_\mu\) on momenta.

### 2.2 Kazakov–Migdal model

Analogous quenching and twisting reduction prescriptions hold for the Kazakov–Migdal model as well. To make consideration similar to that of the previous section, let us introduce for the Itzykson–Zuber–Metha integral the notation
\[ I[\Phi, \Psi] = \int d\Phi e^{-N \text{tr} V[\Phi] + N \sum_\mu \text{tr} \Gamma_\mu \Phi \Gamma_\mu^\dagger \Phi} . \] (2.14)
where \(dU\) is the Haar measure on \(SU(N)\) while \(\Phi_i\) and \(\Psi_j\) stand for eigenvalues of the matrices \(\Phi\) and \(\Psi\), respectively, with \(\Delta[\Phi] = \prod_{i<j}(\Phi_i - \Phi_j)\) being the Vandermonde determinant. This formula implies \([4]\) the following representation of the partition function (1.1) in terms of the path integral over \(\Phi(x)\):
\[ Z_{KM} = \int \prod_x d\Phi(x) e^{-\sum_x N \text{tr} V[\Phi(x)]} \prod_{x,\mu} I[\Phi(x), \Phi(x + \mu)] . \] (2.15)
It is instructive to refer Eq. (2.13) as the \textit{matrix model} representation of the partition function (1.1).

One can apply now to Eq. (2.15) at \(N = \infty\) the reduction prescription described in the previous section which results in Eq. (2.4) (or (2.6)) with the reduced models defined by the formulas like (2.5) for the quenching prescription and or like (2.13) for the twisting prescription. However, since the Itzykson–Zuber–Metha integral depends only on the eigenvalues of \(\Phi(x)\) and \(\Phi(x + \mu)\), it does not depend actually on \(S(x)\) and \(S(x + \mu)\):
\[ I[S(x)\Phi S^\dagger(x), S(x + \mu)\Phi S^\dagger(x + \mu)] = I[\Phi, \Phi] . \] (2.16)
Therefore, the dependence on the quenched momenta or on the matrix $\Gamma_\mu$ which is constructed from momenta is cancelled.

The averages in the Kazakov–Migdal model can now be calculated according to Eq. (2.6) via the master field. For the free energy itself, one gets

$$\frac{1}{\text{Vol.}} \log Z_{KM} = 2 \log (\Delta[\Phi_s]) - N \text{tr} V[\Phi_s] + D \log I[\Phi_s, \Phi_s]$$

(2.17)

where $\text{Vol.}$ stands for the number of sites on the lattice.

Some comments concerning the proposed reduction procedure of the Kazakov–Migdal model are now in order:

- While the reduced model involves no explicit dependence on the momenta entering either $S(x)$ or $\Gamma_\mu$, this dependence survives when calculating averages of $x$-dependent quantities according to Eq. (2.6). The point is that the invariance which remains after the reduction is solely global $SU(N)$: $\Phi_s \rightarrow \Omega \Phi_s \Omega^\dagger$, which can cancel the matrix $S(x)$ at only one point (this corresponds to the translation invariance of averages in the unreduced model (1.1)).

- Eq. (2.16) is a consequence from the invariance of the Haar measure in Eq. (2.14) under multiplication by a unitary matrix from the left and from the right separately, i.e. $S(x)$ and $S(x+\mu)$ which appear after the substitution (2.2) can be absorbed by a gauge transformation. While only $SU(N)$ gauge transformations are allowed, this is enough for our purposes since the $U(1)$ part is again cancelled in the bilinear expressions of the form $[S(x)]_{ij}[S^\dagger(x)]_{kl}$.

It is instructive to see of how the planar graphs of the original model (1.1) are reproduced by the proposed reduction procedure employing the known results for the reduction of scalar field reported in the previous section. To this aid, let us calculate the partition function (1.1) first integrating over scalar field:

$$Z_{KM} = \int \prod_{x,\mu} dU_\mu(x) \ e^{-S_{ind}[U_\mu(x)]},$$

(2.18)

where the \textit{induced action} for the gauge field $U_\mu(x)$ is defined by the integral over $\Phi(x)$ in Eq. (1.1):

$$e^{-S_{ind}[U_\mu(x)]} = \int \prod_x d\Phi(x) \ e^{\sum_x \text{tr} \left( -V[\Phi(x)] + \sum_\mu \Phi(x) U_\mu(x) \Phi(x+\mu) U_\mu^\dagger(x) \right)}.$$  

(2.19)

I shall refer Eqs. (2.18) and (2.19) as the \textit{gauge field} representation of the Kazakov–Migdal model.

The required planar graphs of the model (1.1) can then be obtained in two steps. The first one is to calculate the induced action employing the reduction prescription to the integral over scalar field on the r.h.s. of Eq. (2.19). The second step would be to average
over the gauge field according to Eq. (2.18). Since I do not apply the reduction procedure for the gauge field, it would be enough to show that the two induced actions coincides perturbatively.

For the r.h.s. of Eq. (2.19), one can apply the formulas of the previous section which gives for the quenched momentum prescription:

\[
S_{\text{ind}}[U_{\mu}(x)] = N \sum_{x,\mu} \text{tr} \left( \Phi_s S^\dagger(x) U_{\mu}(x) S(x + \mu) \Phi_s S^\dagger(x + \mu) U_{\mu}^\dagger(x) S(x) \right). \tag{2.20}
\]

Expanding the matrices \(U_{\mu}(x)\) around the unity and using standard rules \[1\] to obtain diagrams for the scalar field, one recovers the correct diagrammatic expansion for the induced action \(S_{\text{ind}}[U_{\mu}(x)]\).

Notice that when one integrates over \(U_{\mu}(x)\) as discussed above, these diagrams are absorbed into gauge degrees of freedom. This looks precisely like the mechanism of recovering planar graphs by the master field which has been proposed in Ref. \[1\] on the basis on the conjecture about the translationally invariant master field. I have related therefore this scenario to the large-\(N\) reduction phenomenon and the master field of Ref. \[1\] to the saddle point matrix \(\Phi_s\).

Up to now I did not discuss what equation determines \(\Phi_s\). This equation can be derived by averaging the (quantum) equation of motion for \(\Phi\) field in the Kazakov–Migdal model, written in the form (2.15), according to the prescription (2.6). Introducing the spectral density \(\rho_s(\lambda)\) which describes the distribution of eigenvalues of the matrix \(\Phi_s\) and using the remaining global gauge invariance, one gets

\[
2 \int dx \frac{\rho_s(x)}{\lambda - x} = \frac{\partial V(\lambda)}{\partial \lambda} - \frac{D}{N^2} \frac{\partial}{\partial \lambda} \frac{\delta \log I[\rho, \rho_s]}{\delta \rho(\lambda)} \bigg|_{\rho=\rho_s} \tag{2.21}
\]

which coincides with the saddle point equation for the coordinate-independent master field of Ref. \[1\].

I did not presented a direct proof of the reduction prescription (2.7) with \(\Phi_s\) given by Eq. (2.21). The above perturbative arguments work only in the weak coupling region where the perturbative expansion is relevant and are not applicable in the strong coupling region which is separated by a phase transition. I hope that loop equations which are derived in the next section could be useful for this purpose.

## 3 Loop equations

As is pointed out in Ref. \[4\], the simplest gauge invariant objects in the Kazakov–Migdal model are the adjoint Wilson loops which are defined by Eq. (1.3) where the average is understood with the same measure as in Eq. (1.1). The nonabelian phase factor \(U(C_{xx})\) which is associated with a parallel transport from a point \(x\) along a closed loop \(C_{xx}\) is
defined by the path ordered product

\[ U(C_{xx}) = P \prod_{l \in C_{xx}} U_\mu(z) \]  

(3.1)

where \( l \) stands for the link \((z, \mu)\).

One can derive the set of loop equations satisfied by these quantities quite similarly to the case of standard lattice gauge theory (for a review, see [16]) performing an infinitesimal shift

\[ U_\mu(x) \to (1 + i \epsilon_\mu(x) U_\mu(x)) \]  

(3.2)

of the link variable \( U_\mu(x) \) at the link \((x, \mu)\) with \( \epsilon \) being an infinitesimal hermitian matrix which leaves the Haar measure invariant. Since the plaquette term is absent in the action of the Kazakov–Migdal model, it does not appear on the l.h.s. of the loop equation. However, a new term associated with the interaction between gauge and scalar fields arises on the l.h.s.. The r.h.s. of the loop equation satisfied by \( W_A(C) \) looks at large \( N \) very similar to the standard one. The equation reads schematically

\[
\left\langle \frac{1}{N} \mathrm{tr} \left( \Phi(x) U(C_{xx}) U_\mu(x) \Phi(x + \mu) U_\mu^\dagger(x) U_\mu^\dagger(C_{xx}) \right) \right\rangle -
\left\langle \frac{1}{N} \mathrm{tr} \left( \Phi(x) U_\mu^\dagger(C_{xx}) U_\mu(x) \Phi(x + \mu) U_\mu^\dagger(x) U(C_{xx}) \right) \right\rangle =
\sum_{l \in C_{xx}} \tau_\mu(l) \delta_{xz} W_A(C_{xz}) W_A(C_{zx})
\]  

(3.3)

where \( \tau_\mu(l) \) stands for a unit vector in the direction of the link \( l \).

A conceptual difference between Eq. (3.3) and the standard loop equation of manycolor QCD is that the former one is not closed. While the r.h.s. of Eq. (3.3) involves the same quantity \( W_A \), a new gauge invariant object of a generic type \([1.4]\) emerges on the l.h.s.. One should derive, therefore, an equation satisfied by this quantity.

The corresponding equation results from the invariance of the measure over \( \Phi \) under an infinitesimal shift

\[ \Phi(x) \to \Phi(x) + \xi(x) \]  

(3.4)

of \( \Phi(x) \) at the given site \( x \) with \( \xi(x) \) being an infinitesimal hermitian matrix. While to close the set of equations one has to consider averages of the type \([1.4]\) with arbitrary powers of \( \Phi(x) \) and \( \Phi(y) \), drastic simplifications occur for the quadratic potential when the resulting equation reads

\[
2m_0^2 G(C_{xy}) - \sum_{\mu=-D}^{D} G(C_{(x+\mu)x}C_{xy}) = \delta_{xy} W_A(C_{xy})
\]  

(3.5)

Here the path \( C_{(x+\mu)x}C_{xx} \) is obtained by attaching the link \((x, \mu)\) to the path \( C_{xx} \) at the end point \( x \) as is depicted in Fig. 1. Notice that \( W_A \) for a closed loop enters the r.h.s. of Eq. (3.3) due to the presence of the delta-function. Therefore, the set of Eqs. (3.3) and (3.5) is closed.
4 Properties of reduced and unreduced averages

4.1 Reduced loop averages

A prescription to calculate \( W_A(C) \) and \( G(C_{xy}) \) in the reduced theory can be obtained from Eq. (2.6). The crucial role in this construction is played by the following integral over the unitary group:

\[
I^{ab}[\Phi, \Psi] = \frac{1}{I[\Phi, \Psi]} \int dU e^{N \text{tr} \Phi U \Psi U^\dagger} \frac{1}{N} \text{tr} t^a U t^b U^\dagger ,
\]

where \( t^a \) for \( a = 1, \ldots, N^2 - 1 \) are (hermitian) generators of \( SU(N) \) which are normalized by

\[
\frac{1}{N} \text{tr} t^a t^b = \delta^{ab} , \quad \text{and} \quad [t^a]_{ij}[t^a]_{kl} = N \delta_{il} \delta_{kj} - \delta_{ij} \delta_{kl}
\]

and \( I \) stands for the integral \((2.14)\). Notice that the \( U(1) \)-part is cancelled in this formula so that the integrals over \( U(N) \) and \( SU(N) \) coincide.

It is convenient to consider \( I^{ab} \) as a \((N^2 - 1) \times (N^2 - 1)\) matrix and

\[
\Phi^a = \frac{1}{N} \text{tr} t^a \Phi
\]

as a \((N^2 - 1)\)-dimensional vector. Then, the counterparts of \( W_A \) and \( G \) in the reduced model read

\[
W_A^{\text{Reduced}} = I^L[\Phi_s, \Phi_s]^{aa}
\]

and

\[
G^{\text{Reduced}} = \Phi_s^a I^L[\Phi_s, \Phi_s]^{ab} \Phi_s^b
\]

where \( L \) is the length (in lattice units) of the appropriate contour.

It is easy to see that \( W_A^{\text{Reduced}} \) and \( G^{\text{Reduced}} \) given by Eqs. (4.4) and (4.5) satisfy for an arbitrary potential the same loop equation \((1.3)\) as \( W_A(C) \) and \( G(C_{xy}) \) defined by Eqs. (1.3) and (1.4) in the original unreduced model. It is trivial to see, employing the representation \((1.1)\) in terms of the integral over \( U \), that the equation coming from the shift \((3.2)\) of \( U \) coincides with Eq. \((3.3)\). The point is that we shift \( U \) only in one integral of the chain (what is an analog of the shift at one link) since one did not reduce \( U \)'s \( (\text{i.e. identify them})\).

The situation is different, however, for Eq. \((3.3)\) which results from the shift \((3.4)\) of \( \Phi \). Now it is not possible to make any conclusion since the corresponding equation satisfied by \( G^{\text{Reduced}} \) can not be derived by a straightforward variation w.r.t. \( \Phi_s \). Fortunately Eq. \((3.3)\) can be exactly solved in the strong coupling region (see Section 5) and the result can be compared with the large mass expansion of \( G^{\text{Reduced}} \).
4.2 Large mass expansion

The above conjecture about the equivalence of the reduced and original unreduced models can be tested by comparing the loop averages at large values of the mass $m_0^2$. To calculate the large mass expansion, one needs the following expansion of $I^{ab}$ defined by the integral (4.1):

$$I^{ab} = \Phi^a \Psi^b + \ldots,$$

where $\ldots$ stands for the terms of higher powers in $\Phi$ and $\Psi$ which correspond to higher order of the large mass expansion.

Applying this formula to the unreduced quantity (1.4), one gets

$$G(C_{xy}) = \prod_{z \in C_{xy}} \left( \frac{\int d\Phi(z) e^{-m_0^2 N \text{tr} \Phi^2(z)} \frac{1}{N} \text{tr} \Phi^2(z)}{\int d\Phi(z) e^{-m_0^2 N \text{tr} \Phi^2(z)}} \right) + \ldots = (G_0)^{L+1} + \ldots$$

while the reduced prescription (4.5) gives

$$G^{\text{Reduced}} = \left( \frac{1}{N} \text{tr} \Phi_s^2 \right)^{L+1} + \ldots$$

which coincides with (1.7) since $\frac{1}{N} \text{tr} \Phi_s^2 = G_0$. Therefore, one has explicitly demonstrated, in particular, the reduction to the leading order of the large mass expansion.

A similar analyses of the adjoint Wilson loops leads to a different result. Let us now estimate the average on the r.h.s. of Eq. (1.3) for a closed loop with $L \geq 4$ which does not contain parts that are passed back and forth. Exploiting the completeness condition (1.2) and Eq. (4.6), the result can be represented in the same form as (1.7) but with an extra factor $1/N^2$:

$$W_A(C) = \frac{1}{N^2} (G_0)^L + \ldots.$$ 

This expression is proportional to $1/N^2$ and vanishes in the large-$N$ limit which is in agreement with general arguments of Ref. [4]. The corresponding average in the reduced model can be calculated similarly to Eq. (4.8). The result is given by the r.h.s. of Eq. (4.9) and now holds independently of whether the loop is closed or open.

4.3 Intermediate coupling region

When $m_0^2$ is decreased, the system undergoes phase transitions. According to the scenario of Ref. [4] which is discussed in Section 1, a first order large-$N$ phase transition should occur before the one associated with the continuum limit in order for the Kazakov–Migdal model to induce continuum QCD. While the strong and weak coupling phases always exist (the weak coupling phase is associated as usual with the perturbative expansion) the existence of such an intermediate phase has been only conjectured. Let us discuss

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2The formulas for calculating the corresponding integrals over the Haar measure as $N \to \infty$ can be found in Ref. [17].
consequences from the phenomenon of large-$N$ reduction in the intermediate coupling region.

In the weak coupling region where the perturbative expansion is applicable the arguments of Section 2.2 about the large-$N$ reduction work.

The intermediate coupling region looks pretty similar to the weak coupling phase of Wilson lattice gauge theory. My arguments are based on the gauge field representation (2.18) of the Kazakov–Migdal model. Let us represent the induced action in the form

$$S_{ind}[U] = -\frac{1}{2} \sum_C \beta_A(C) |\text{tr} U(C)|^2$$

(4.10)

where $\beta_A(C)$ are some loop-dependent couplings which are determined by the potential $V[\Phi]$. For the quadratic potential, one gets

$$\beta_A(C) = \frac{1}{l(C)m_0^{2\ell(C)}}$$

(4.11)

with $l(C)$ being the length of the loop $C$.

At $N = \infty$ the action (4.10) is equivalent to the following action

$$S_F[U] = -N \sum_C \bar{\beta}(C) \Re \text{tr} U(C)$$

(4.12)

with the couplings $\bar{\beta}(C)$ being determined by the self-consistency conditions

$$\bar{\beta}(C) = \beta_A(C)W_F(C; \{\bar{\beta}\})$$

(4.13)

where $W_F(C; \{\bar{\beta}\})$ is the fundamental Wilson loop average

$$W_F(C; \{\bar{\beta}\}) = \langle \frac{1}{N} \text{tr} U(C) \rangle$$

(4.14)

and the averaging is taken with the action (4.12). This procedure extends the one advocated by Khokhachev and the author [18] for the single-plaquette adjoint action

$$S_A = -\frac{\beta_A}{2} \sum_\square |\text{tr} U(\square)|^2$$

(4.15)

with the self-consistency condition given by Eq. (4.13) with $C = \square$.

In the intermediate coupling region Eqs. (4.13) possess a nontrivial solution and the model can be described in terms of the fundamental induced action (4.12). On the other hand since the area law holds for the Wilson loop averages, I expect that the contribution of long loops to the induced action (4.12) is suppressed according to Eq. (4.13). Therefore, properties of the intermediate coupling region should be similar to the standard lattice gauge theory.
5 An exact strong coupling solution

5.1 Quadratic potential

A drastic simplification of the loop equations (3.3) and (3.5) is due to the fact [4] that in the strong coupling region (before the large-$N$ phase transition) the adjoint Wilson loops vanish in the large-$N$ limit except closed ones with vanishing area $A_{\text{min}}(C)$ of the minimal surface (i.e. contractable to a point owing to unitarity of $U$’s):

$$W_A(C) = \delta_{0A_{\text{min}}(C)} + O\left(\frac{1}{N^2}\right).$$  

(5.1)

It can be shown that (5.1) is consistent with loop equations.

Given this behavior of adjoint Wilson loops entering the r.h.s. of Eq. (3.5), Eq. (3.5) admits a solution $G(C_{xy}) = G_L$ depending on the single parameter $L$ which is defined as the algebraic length of the loop (see [16]), i.e. the length after all possible contractions of the paths passing back and forth are made. For such an ansatz, Eq. (3.5) can be written as

$$2m_0^2G_L - G_{L-1} - (2D - 1)G_{L+1} = 0 \quad \text{for} \quad L \geq 1,$n

$$2m_0^2G_0 - 2DG_1 = 1.$$  

(5.2)

Before solving this equation let me show of how it is satisfied in $D = 1$ where the exact result for $G_L$ is given by the Laplace transform

$$G_L^{D=1} = \frac{1}{2} \int_0^{\infty} d\alpha e^{-m_0^2\alpha} I_L(\alpha) = \frac{\left(m_0^2 - \sqrt{m_0^4 - 1}\right)^L}{2\sqrt{m_0^4 - 1}},$$  

(5.3)

with $I_L(\alpha)$ being modified Bessel functions. Eq. (5.2) can now be viewed, say, as the recurrent relation

$$2I_L'(\alpha) - I_{L-1}(\alpha) - I_{L+1}(\alpha) = 0$$  

(5.4)

which is integrated according to Eq. (5.3) while 1 on the r.h.s. of the $L = 0$ equation results from the fact that $I_0(0) = 1$.

Eq. (5.2) can be solved by introducing the generating function

$$G(\lambda) = \sum_{L=0}^{\infty} G_L\lambda^L$$  

(5.5)

with the result being expressed via the initial data, $G_0$, by

$$G(\lambda) = \frac{1}{D} \left( m_0^2G_0 + D - \frac{1}{2} \right) \lambda - (2D - 1)G_0, \quad 2m_0^2\lambda - \lambda^2 + 1 - 2D$$  

(5.6)
While \( G_0 \) is defined as an average of \( \frac{1}{N} \text{tr} \Phi^2(x) \) in the Kazakov–Migdal model, one can determine it from Eq. (5.6) by imposing the analytic structure of \( G(\lambda) \) as a function of the spectral parameter \( \lambda \). For an arbitrary \( G_0 \) the denominator in Eq. (5.6) has two roots and \( G(\lambda) \) has therefore two poles. The exact solution at \( D = 1 \) given by Eq. (5.3) as well as the large mass expansion described in Section 4.2 lead to a \( G(\lambda) \) possessing only one pole. This can be achieved in Eq. (5.6) by choosing \( G_0 \) in a proper way. This requirement unambiguously determines \( G(\lambda) \) to be

\[
G(\lambda) = \frac{(2D-1)^2}{2 \left( m_0^2(D-1) + D\sqrt{m_0^4 + 1 - 2D} \right) \left[ \sqrt{m_0^4 + 1 - 2D} - m_0^2 \right] \lambda + 2D - 1}.
\]

while \( G_0 \) is fixed to be

\[
G_0 = \frac{D - \frac{1}{2}}{m_0^2(D-1) + D\sqrt{m_0^4 + 1 - 2D}}.
\]

I have checked that the solution (5.7) correctly reproduces the \( D = 1 \) solution (5.3) as well as the leading order of the large mass expansion (4.7) at any \( D \).

Some comments about the exact strong coupling solution (5.7) are now in order:

- Eq. (5.8) precisely coincides with the result by Gross [3] obtained by another method.
- While the solution (5.7) looks very simple, it corresponds in the language of the Kazakov–Migdal model to tedious calculations of integrals over unitary matrices with a subsequent averaging over \( \Phi \). In the reduced language (4.5) it remains still to calculate the integral (4.1) over unitary group and then substitute the value of the master field \( \Phi_s \). It would be very interesting to reproduce the result by this method.
- The solution (5.7) is the first example of calculations of extended objects (since \( G_L \) is the average (1.4) for the loop of the length \( L \) in the lattice units) in the strong coupling region of the Kazakov–Migdal model while the results of Refs. [1, 2, 5] refer to local objects like the spectral density.
- The very possibility to find such a simple exact solution to loop equations in the strong coupling region is related to a very simple form (5.1) of the Wilson loops. In particular, while the path \( C_{xy} \) looks like a string in the strong coupling expansion of the standard lattice gauge theory, fluctuations of its shape are now suppressed. This is why the result depends only on \( L \).
- The solution (5.7) at \( D = 1 \) could be interesting from the viewpoint of matrix models of 2D gravity.
5.2 More general potentials

While the solution (5.7) has been obtained for the quadratic potential, an analogous solution can be found for the more general potential

\[ N \, \text{tr} \, V[\Phi] = m_0^2 N \, \text{tr} \Phi^2 + g( \text{tr} \Phi^2 )^2 \]  

where \( g \sim 1 \) as \( N \to \infty \) to provide self-interaction of scalar field. This kind of potential was used \([10]\) in the matrix models in the context of \( D > 1 \) strings.

The Kazakov–Migdal model with the quartic potential (5.9) can be solved in the large-\( N \) limit for the following reasons. The model with the potential (5.9) is equivalent as \( N \to \infty \) to the one with quadratic potential whose mass parameter \( \bar{m}^2 \) is defined by the self-consistency relation

\[ \bar{m}^2 = m_0^2 + 2g \, G_0|_{\bar{m}^2} \]  

where \( G_0|_{\bar{m}^2} \) is given by Eq. (5.8) with \( m_0^2 \) replaced by \( \bar{m}^2 \). Eq. (5.10) can be obtained naively replacing one \( \text{tr} \Phi^2 \) by the average value due to factorization. A rigorous proof of Eq. (5.10) can be done using loop equations similarly to the proof \([18]\) of the reduction of the adjoint action to the Wilson action which is discussed in Section 4.3.

Eq. (5.10) can be used to study whether the large-\( N \) phase transition occurs for the potential (5.9). Such a phase transition were occur if \( \bar{m}^2 \) would depend on \( m_0^2 \) nonmonotonically. A similar idea for the large-\( N \) phase transition to occur in lattice gauge theories was advocated in Ref. \([19]\). For the scalar model of the type (5.9), it was employed in Ref. \([10]\) to obtain \( \gamma_{\text{string}} > 0 \).

Given Eqs. (5.10) and (5.8) it is easy to calculate the derivative

\[ \frac{\partial m_0^2}{\partial \bar{m}^2} = 1 + \frac{g(2D - 1)(D - 1 + \frac{\bar{m}^2 D}{\sqrt{m^4 + 1 - 2D}})}{(\bar{m}^2(D - 1) + D\sqrt{m^4 + 1 - 2D})^2} \]  

and see that it is positive for \( \bar{m}^2 > D \) where the gaussian model is stable. Therefore I conclude that, similarly to the case of the quadratic potential, there is no first order large-\( N \) phase transition for the Kazakov–Migdal model with the potential (5.9).

An analogous study can be performed for the potential (5.5) with an arbitrary function \( f \) when the second term on the r.h.s. of Eq. (5.11) is multiplied by \( \frac{1}{2} f''(G_0) \). A conclusion is that one has to look for a more complicated potential.

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Fig. 1  The graphic representation for \( \Phi(C_{xy}) \) (a) and \( \Phi(C_{(x+\mu)x} C_{xy}) \) (b) entering Eq. (3.5). The bold points represent \( \Phi(x) \) and \( \Phi(x + \mu) \). The (oriented) solid lines represent the path-ordered products \( U(C_{xy}) \) and \( U(C_{(x+\mu)x} C_{xy}) \). The color indices are contracted according to the arrows.