The Moutard transformation of two-dimensional Dirac operators and conformal geometry of surfaces in the four-space

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1 Introduction

In [1] the Moutard transformation, for a two-dimensional Dirac operator with a real-valued potential, derived in [2], was related with conformal geometry of surfaces in the three-space. In this article we expand this picture for surfaces in the four-space, because every such a surface admits a Weierstrass representation related to a two-dimensional Dirac operator [3, 4]. Therewith we generalize the Moutard transformation from [2] onto Dirac operators with complex-valued potentials, i.e. for operators of the form

\[ D = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}, \]

where \( \partial = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) \) and \( \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}) \).

Let us briefly expose the main results. We consider the operator

\[ D^\vee = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \bar{U} & 0 \\ 0 & U \end{pmatrix}. \]

Let \( \psi \) and \( \varphi \) satisfy the equations

\[ D\psi = 0, \quad D^\vee \varphi = 0. \]
Then the matrix-valued functions

\[
\Psi = \begin{pmatrix}
\psi_1 & -\bar{\psi}_2 \\
\psi_2 & \bar{\psi}_1 \\
\end{pmatrix}, \quad \Phi = \begin{pmatrix}
\varphi_1 & -\bar{\varphi}_2 \\
\varphi_2 & \bar{\varphi}_1 \\
\end{pmatrix}
\]

satisfy the equations

\[
\mathcal{D}\Psi = 0, \quad \mathcal{D}^\vee\Phi = 0,
\]

which, in fact, means the solutions of (2) are invariant with respect to the transformations

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
-\bar{\psi}_2 \\
\bar{\psi}_1 \\
\end{pmatrix}, \quad \begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
-\bar{\varphi}_2 \\
\bar{\varphi}_1 \\
\end{pmatrix}.
\]

Here it is important that the potentials are complex conjugate to each other.

In this article we show that

1. every pair of solutions \(\psi\) and \(\varphi\) to (2) and every point \(x_0 \in \mathbb{R}^4\) define a transformation of the Moutard type of the operator \(\mathcal{D}\) to an operator of the same form;

2. geometrically the Moutard transformation is given by an action of composition of the inversion and the reflection, with respect to a line, on a surface in \(\mathbb{R}^4\). This surface is defined via the Weierstrass representation by vector functions (spinors) \(\psi\) and \(\varphi\) and \(x_0 \in \mathbb{R}^4\) and the potential \(U\) of the Dirac operator enters into this Weierstrass representation.

## 2 The Moutard transformation

Let us consider the quaternion algebra \(\mathbb{H}\), realized by matrices of the form

\[
\begin{pmatrix}
a & b \\
-\bar{b} & \bar{a} \\
\end{pmatrix}, a, b \in \mathbb{C}.
\]

To every vector function \(\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\end{pmatrix}\) we correspond a matrix valued function

\[
\Psi = \begin{pmatrix}
\psi_1 & -\bar{\psi}_2 \\
\psi_2 & \bar{\psi}_1 \\
\end{pmatrix}
\]

with the value in \(\mathbb{H}\).

To every pair \(\Phi\) and \(\Psi\) of \(\mathbb{H}\)-valued functions we correspond the 1-form \(\omega\):

\[
\omega(\Phi, \Psi) = \Phi^\top \Psi dy - i\Phi^\top \sigma_3 \Psi dx = -i \left(\Phi^\top \sigma_3 \Psi + \Phi^\top \Psi\right) dz - i \left(\Phi^\top \sigma_3 \Psi - \Phi^\top \Psi\right) d\bar{z},
\]

where \(\sigma_3\) is the Pauli matrix for spin 1/2.
and the function
\[ S(\Phi, \Psi)(z, \bar{z}, t) = \Gamma \int_0^z \omega(\Phi, \Psi), \tag{6} \]
where
\[ \Gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
\(\sigma_2\) and \(\sigma_3\) are the Pauli matrices.

The form \(\omega\) and the function \(S\) take values in \(\mathbb{H}\), and moreover \(S\) is defined up to integration constants, i.e. up to a constant matrix from \(\mathbb{H}\).

Here and in the sequel we define the transposition of \(X\) by \(X^T\).

To every pair of \(\mathbb{H}\)-valued functions \(\Phi\) and \(\Psi\) we correspond a matrix valued function
\[ K(\Phi, \Psi) = \Psi S^{-1}(\Phi, \Psi) \Gamma \Phi^\top \Gamma^{-1} = \begin{pmatrix} i\bar{W} & a \\ -\bar{a} & -iW \end{pmatrix}. \tag{7} \]

By straightforward computations it is proved that

**Theorem 1** Let \(\Psi_0\) and \(\Phi_0\) be solutions of the form (3) of the Dirac equations (2).

Then for every pair \(\Psi\) and \(\Phi\) of solutions of (2) the functions
\[ \tilde{\Psi} = \Psi - \Psi_0 S^{-1}(\Phi_0, \Psi_0) S(\Phi_0, \Psi), \]
\[ \tilde{\Phi} = \Phi - \Phi_0 S^{-1}(\Psi_0, \Phi_0) S(\Psi_0, \Phi) \tag{8} \]
satisfy the Dirac equations
\[ \tilde{D}\tilde{\Psi} = 0, \quad \tilde{D}^\lor \tilde{\Phi} = 0 \]
for the Dirac operators \(\tilde{D}\) and \(\tilde{D}^\lor\) with the potential
\[ \tilde{U} = U + W, \tag{9} \]
where \(W\) is defined by the formula (7) for \(K(\Phi_0, \Psi_0)\).

**Remarks.** 1) Due to matrix integration constants in (6) \(\tilde{\Psi}\) and \(\tilde{\Phi}\) are defined up to multiplication on \((\Psi_0 S^{-1}(\Phi_0, \Psi_0)) \cdot A\) and \((\Phi_0 S^{-1}(\Psi_0, \Phi_0)) \cdot B\), respectively, with \(A\) and \(B\) constant matrices from \(\mathbb{H}\).

2) The formulas (5) are the same as for the Moutard transformation of the Dirac operator with a real-valued potential \(U\) [1, 2]. The transformation from Theorem 1 reduces to it for a real-valued potential \(U\) and \(\Phi_0 = \Psi_0\. The
proof of Theorem 1 will follow to its analogue, for the case of a real-valued potential, given in [1].

**Proof.** 1) Let

\[ \tilde{\Psi}_0 = \Psi_0 S^{-1}(\Phi_0, \Psi_0), \quad \tilde{\Phi}_0 = \Phi_0 S^{-1}(\Psi_0, \Phi_0). \]

We show that \( \tilde{\Psi}_0 \) and \( \tilde{\Phi}_0 \) satisfy the Dirac equations

\[ \tilde{D} \tilde{\Psi} = D_0 \tilde{\Psi} + \begin{pmatrix} \tilde{U} & 0 \\ 0 & U \end{pmatrix} \tilde{\Psi} = 0, \quad \tilde{D}^\dagger \tilde{\Phi} = 0, \tag{10} \]

where

\[ D_0 = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix}, \]

with the potential \( \tilde{U} = U + W \) given by [1].

Let us apply the “Leibniz rule” [1]

\[ D_0(A \cdot B) = (D_0 A) \cdot B + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A \cdot \partial B + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} A \cdot \bar{\partial} B \tag{11} \]

to \( A = \Psi_0 \) and \( B = S^{-1} = S^{-1}(\Phi_0, \Psi_0): \)

\[ D_0(\Psi_0 S^{-1}(\Phi_0, \Psi_0)) = \]

\( (D_0 \Psi_0) S^{-1} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Psi_0 S^{-1} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \Psi_0 S^{-1} = \]

\[ = -\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \Psi_0 S^{-1} + i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Psi_0 S^{-1} \Gamma \Phi_0^\dagger \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \Psi_0 S^{-1} + \]

\[ + i \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \Psi_0 S^{-1} \Gamma \Phi_0^\dagger \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \Psi_0 S^{-1}. \tag{12} \]

It follows from \( S^{-1} S = 1 \) that

\( (S^{-1})_z = -S^{-1} S_z S^{-1}, (S^{-1})_{\bar{z}} = -S^{-1} S_{\bar{z}} S^{-1} \)

and, by the definition of \( S(\Phi_0, \Psi_0) \), we have

\( (S^{-1})_z = \frac{i}{2} S^{-1} \Gamma \Phi_0^\dagger (\sigma_3 + 1) \Psi_0 S^{-1}, \quad (S^{-1})_{\bar{z}} = \frac{i}{2} S^{-1} \Gamma \Phi_0^\dagger (\sigma_3 - 1) \Psi_0 S^{-1}. \tag{13} \)

In view of these identities the formula (12) takes the form

\[ D_0(\Psi_0 S^{-1}) = -\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} (\Psi_0 S^{-1}) + \]
\[ +i\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} G \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} G \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) (\Psi_0 S^{-1}) = \]
\[ = - \left( \begin{pmatrix} U + W & 0 \\ 0 & U + W \end{pmatrix} \right) (\Psi_0 S^{-1}), \]

where

\[ G = K(\Phi_0, \Psi_0)\Gamma^{-1} = \Psi_0 S^{-1}(\Phi_0, \Psi_0)\Gamma\Phi_0^\top = \begin{pmatrix} -a & i\bar{W} \\ iW & -\bar{a} \end{pmatrix}. \]

therefore we prove that \( \tilde{\Psi}_0 \) satisfies the first equation from \( (10) \). Analogously it is proved that \( \tilde{\Phi}_0 \) satisfies the second equation from \( (10) \).

2) Let us find a transformation of an arbitrary solution \( \Psi \) of \( (4) \) to a solution \( \tilde{\Psi} \) of \( (10) \). We will look for it in the form

\[ \tilde{\Psi} = \Psi + \tilde{\Psi}_0 N. \]

By \( (11) \), we have

\[ 0 = \tilde{D}\tilde{\Psi} = (\mathcal{D} + \begin{pmatrix} W & 0 \\ 0 & \bar{W} \end{pmatrix}) (\Psi + \tilde{\Psi}_0 N) = \mathcal{D}\Psi + \begin{pmatrix} W & 0 \\ 0 & \bar{W} \end{pmatrix} \Psi + (\tilde{D}\tilde{\Psi}_0) N + \]
\[ + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{\Psi}_0 \partial N + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \bar{\tilde{\Psi}}_0 \bar{\partial} N, \]

where \( W = \bar{U} - U \), and, since \( \tilde{D}\tilde{\Psi} = \mathcal{D}\Psi = 0 \), we will look for \( N \) such that

\[ \begin{pmatrix} W & 0 \\ 0 & \bar{W} \end{pmatrix} \Psi = - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{\Psi}_0 \partial N - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \bar{\tilde{\Psi}}_0 \bar{\partial} N. \]

However it follows from the formula for \( W \) that

\[ \begin{pmatrix} W & 0 \\ 0 & \bar{W} \end{pmatrix} \Psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{\Psi}_0 S_z(\Phi_0, \Psi) + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \bar{\tilde{\Psi}}_0 S_z(\Phi_0, \Psi), \]

therefore \( N \) is equal to

\[ N = -S(\Phi_0, \Psi) \]

up to a constant matrix from \( \mathbb{H} \) and, hence, the action of the Moutard transformation on \( \Psi \) takes the form pointed out by Theorem 1:

\[ \tilde{\Psi} = \Psi - \Psi_0 S^{-1}(\Phi_0, \Psi) S(\Phi_0, \Psi). \]

Analogously the transformation of \( \Phi \) is derived.

Theorem 1 is proved.
3 Geometry of the Moutard transformation

3.1 The Weierstrass representation of surfaces in \( \mathbb{R}^4 \)

The Weierstrass representation of surfaces in \( \mathbb{R}^4 \) correspond to solutions \( \psi \) and \( \varphi \) of (2) the surface defined by the formulas

\[
x^k(P) = x^k(P_0) + \int \left( x_z^k dz + \overline{x}_z^k d\overline{z} \right), \quad k = 1, 2, 3, 4,
\]

where

\[
x_1^z = \frac{i}{2}(\varphi_2 \overline{\psi}_2 + \varphi_1 \psi_1), \quad x_2^z = \frac{1}{2}(\varphi_2 \overline{\psi}_2 - \varphi_1 \psi_1),
\]

\[
x_3^z = \frac{1}{2}(\varphi_2 \psi_1 + \varphi_1 \overline{\psi}_2), \quad x_4^z = \frac{i}{2}(\varphi_2 \psi_1 - \varphi_1 \overline{\psi}_2),
\]

where the integral is taken along a path from the initial point \( P_0 \) to \( P \).

Therewith the induced metric is equal to

\[
e^{2\alpha} dz d\bar{z} = (|\psi_1|^2 + |\psi_2|^2)(|\varphi_1|^2 + |\varphi_2|^2) dz d\bar{z}
\]

and the mean curvature vector

\[
H = \frac{2x_z}{e^{2\alpha}}
\]

is related to \( U \) as follows

\[
|U| = \frac{|H| e^\alpha}{2}.
\]

These formulas for constructing surfaces in \( \mathbb{R}^4 \) were introduced in [5]. For \( \psi = \varphi, U = \bar{U} \) we have \( x^4 = \text{const} \) and these formulas reduce to analogous formulas for surfaces in \( \mathbb{R}^3 \).

These formulas have a local character and for their globalization it is necessary to consider vector functions \( \psi \) as sections of spinor bundles. Such a representation is constructed (up to a multiplication of \( \psi \) by \( \pm 1 \)) for every surface in \( \mathbb{R}^3 \) and therewith \( 4 \int U^2 dx dy \) coincides with the value of the Willmore functional [6].

For surfaces in \( \mathbb{R}^4 \) the situation is more complicated [3]: every surface in \( \mathbb{R}^4 \) is also given by such formulas, however \( \psi \) are \( \varphi \) are defined by a factorization of the Gauss map and are defined not uniquely but up to gauge transformations

\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} e^h \psi_1 \\ e^h \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} e^{-h} \varphi_1 \\ e^{-h} \varphi_2 \end{pmatrix},
\]

(16)
where $h$ is an arbitrary function on the universal covering of the surface. Among $\psi$ and $\varphi$, constructed from the Gauss map, we can find such functions which satisfy the Dirac equations (2). The spinors $\psi$ and $\varphi$, satisfying (2), again are not uniquely defined but up to gauge transformations (16) where $h$ is a holomorphic function on the universal covering. Therewith the phase of the potential is also changed:

$$U \rightarrow e^{(\bar{h}-h)}U.$$  

Let us clarify the relation of $U$ to the mean curvature vector. For a surface $M \subset \mathbb{R}^4$ at every point $x \in M$ there exists a two-dimensional space $\nu M_x$ formed by all tangent vectors to $\mathbb{R}^4$ which are normal to the surface. Given a Weierstrass representation of the surface, let us choose in $\nu M_x$ the basis $n_1$ and $n_2$ in the form

$$n_1 = e^{-\alpha}(-\text{Im}(\psi_2\bar{\varphi}_1 - \bar{\psi}_1\varphi_2), -\text{Re}(\bar{\psi}_1\varphi_2 + \psi_2\varphi_1),$$

$$\text{Re}(\psi_2\bar{\varphi}_2 - \bar{\psi}_1\varphi_1), -\text{Im}(\bar{\psi}_1\varphi_1 + \psi_2\varphi_2)),

n_2 = e^{-\alpha}(\text{Re}(\psi_2\varphi_1 - \bar{\psi}_1\bar{\varphi}_2), -\text{Im}(\bar{\psi}_1\varphi_1 + \psi_2\varphi_1),$$

$$\text{Im}(\psi_2\bar{\varphi}_2 - \bar{\psi}_1\varphi_1), \text{Re}(\bar{\psi}_1\varphi_1 + \psi_2\bar{\varphi}_2)).$$

Define a complex-valued vector

$$p = e^{\alpha}(n_1 + in_2).$$  

By straightforward computations, it is shown the the potential $U$ of the Weierstrass representation takes the form

$$U = \frac{1}{2}\langle H, p \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean (not Hermitian) scalar product. Different Weierstrass representations of the same surface with a fixed conformal parameter are related by a gauge transformation (16) and, therefore, the vectors $n_1$ and $n_2$, constructed from the representations, as well as $U$ are related by (17).

3.2 The inversion of $\mathbb{R}^4$ and the Moutard transformation

By the Liouville theorem, for $n \geq 3$ the group formed by all orientation-preserving conformal transformations of $S^n = \mathbb{R}^n \cup \{\infty\}$ is generated by translations, rotations of $\mathbb{R}^n$, and the inversion.
The inversion of $\mathbb{R}^4$ has the form

$$T : \mathbf{x} \rightarrow \frac{\mathbf{x}}{|\mathbf{x}|^2}, \quad \mathbf{x} = (x^1, \ldots, x^4) \in \mathbb{R}^4.$$ 

In [1] in the definition of the inversion the right-hand side of the analogous formula was taken with the opposite sign to preserve the orientation.

Let $u$ be a vector tangent to $\mathbb{R}^4$ at $\mathbf{x}$: $u \in T_\mathbf{x}\mathbb{R}^4$ and let $\mathbf{x} \neq 0$. By straightforward computations we derive the formula

$$T^* u = \frac{u}{|\mathbf{x}|^2} - 2\mathbf{x}\frac{\langle \mathbf{x}, u \rangle}{|\mathbf{x}|^4}.$$ 

This implies that

$$\langle T^* u, T^* v \rangle = \frac{\langle u, v \rangle}{|\mathbf{x}|^4}, \quad u, v \in T_\mathbf{x}\mathbb{R}^4.$$ 

Let us consider an immersed surface $r : U \rightarrow \mathbb{R}^4$ with a conformal parameter $z$. The inversion maps it into the surface $\tilde{r} = T \cdot r : U \rightarrow \mathbb{R}^4$, on which $z$ is also a conformal parameter and the conformal factors of the metrics satisfy the equality

$$e^{3\alpha(z, \bar{z})} = \frac{e^{\alpha(z, \bar{z})}}{|r(z, \bar{z})|^2}, \quad e^{2\alpha} = \frac{1}{2} \langle \tilde{r}_z, \tilde{r}_{\bar{z}} \rangle, \quad e^{2\alpha} = \frac{1}{2} \langle r_z, r_{\bar{z}} \rangle.$$ 

Let $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ define the surface $r : U \rightarrow \mathbb{R}^4$ via the Weierstrass representation.

Let us identify $\mathbb{R}^4$ with the Lie algebra $u(2)$ (or, which that same, with the matrix realization of quaternions) by the mapping

$$\mathbf{x} = (x^1, x^2, x^3, x^4) \rightarrow \mathbf{X} = \begin{pmatrix} ix^3 + x^4 & -x^1 - ix^2 \\ x^1 - ix^2 & -ix^3 + x^4 \end{pmatrix}.$$ 

By straightforward computation, we derive

**Proposition 1** In this representation the map

$$\mathbf{X} \rightarrow \mathbf{X}^{-1}$$

is a composition of the inversion $\mathbf{x} \rightarrow \frac{\mathbf{x}}{|\mathbf{x}|^2}$ and the reflection

$$(x^1, x^2, x^3, x^4) \rightarrow (-x^1, -x^2, -x^3, x^4).$$
We have

**Proposition 2** The formula \( (6) \) gives an immersion into \( u(2) = \mathbb{R}^4 \) of the surface defined by the spinors \( \psi \) and \( \varphi \) via the Weierstrass representation.

**Proof.** By (15) and (5),

\[
S(\Phi_0, \Psi_0)(P) = \Gamma \int_{P_0}^P -\frac{i}{2} \left( \Phi_0^\top (\sigma_3 + 1) \Psi_0 dz + \Phi_0^\top (\sigma_3 - 1) \Psi_0 \right) d\bar{z}) = \\
= i \int_0^P \left( \begin{array}{c} \psi_1 \bar{\varphi}_2 \\ \psi_1 \varphi_1 \\ \bar{\psi}_2 \varphi_2 \\ -\bar{\psi}_2 \varphi_1 \end{array} \right) dz + \left( \begin{array}{cc} \psi_2 \bar{\varphi}_1 & \bar{\psi}_1 \bar{\varphi}_1 \\ \bar{\psi}_2 \varphi_2 & -\psi_1 \varphi_2 \end{array} \right) d\bar{z} = (21) \\
= \int_0^P d \left( \begin{array}{cc} ix^3 + x^4 & -x^1 - ix^2 \\ x^1 - ix^2 & -ix^3 + x^4 \end{array} \right) \in u(2),
\]

i.e. \( S \) is the surface determined by the spinors \( \psi \) and \( \varphi \) via the Weierstrass representation. Proposition is proved.

The following theorem demonstrates the geometrical meaning of the Moutard transformation from Theorem 1.

**Theorem 2** Let a surface

\[
S = S(\Phi_0, \Psi_0) : \mathcal{U} \to \mathbb{R}^4
\]

with a conformal parameter \( z \in \mathcal{U} \subset \mathbb{C} \) is defined by the spinors \( \Psi_0 \) and \( \Phi_0 \) via the Weierstrass representation. Then the surface

\[
S^{-1} : \mathcal{U} \to \mathbb{R}^4 \cup \{ \infty \},
\]

obtained from \( S \) by applying the composition of the inversion and the reflection (see Proposition 1) is defined by the spinors

\[
\tilde{\Psi}_0 = \Psi_0 S^{-1}(\Phi_0, \Psi_0), \quad \tilde{\Phi}_0 = \Phi_0 S^{-1}(\Psi_0, \Phi_0)
\]

via the Weierstrass representation.

**Proof.** Let \( \tilde{\Psi} \) and \( \tilde{\Phi} \) define the surface \( S^{-1}(\Phi_0, \Psi_0) \) via the Weierstrass representation. The formula (13) implies the equality

\[
\tilde{S}_z = -\frac{i}{2} \Gamma \tilde{\Phi}_0^\top (1 + \sigma_3) \tilde{\Psi}_0 = \frac{i}{2} S^{-1}(\Phi_0, \Psi_0) \Gamma \Phi_0^\top (1 + \sigma_3) \Psi_0 S^{-1}(\Phi_0, \Psi_0),
\]

which is simplified up to the form

\[
\tilde{\Phi}_0^\top (1 + \sigma_3) \tilde{\Psi}_0 = -\Gamma^{-1} S^{-1} \Gamma \Phi_0^\top (1 + \sigma_3) \Psi_0 S^{-1}.
\]
It is easy to check the following identity

$$-\Gamma^{-1}S^{-1}(\Phi_0, \Psi_0)\Gamma = \Gamma S^{-1}(\Phi_0, \Psi_0)\Gamma = (S^{-1}(\Psi_0, \Phi_0))^\top,$$

which together with the preceding equality imply

$$D^\top (1 + \sigma_3)C = (1 + \sigma_3) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad (22)$$

for

$$C = \Psi_0 S^{-1}(\Phi_0, \Psi_0)\hat{\Psi}_0^{-1}, \quad D = \Phi_0 S^{-1}(\Psi_0, \Phi_0)\hat{\Phi}_0^{-1}.$$ 

Analogously, by considering $\hat{S}_z$ and $S_z$, we conclude that

$$D^\top (\sigma_3 - 1)C = (\sigma_3 - 1) = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}. $$

It follows from the last equality and from (22) that the matrices $C$ and $D$ are diagonal and $C = D^{-1}$. Since $\Psi_0, S^{-1}, \Psi_0, \Phi_0, \hat{\Phi}_0 \in \mathbb{H}$, we have $C, D \in \mathbb{H}$, therefore,

$$C = D^{-1} = \begin{pmatrix} e^h & 0 \\ 0 & e^{-h} \end{pmatrix}$$

and we infer that

$$\hat{\Psi}_0 = \begin{pmatrix} e^h & 0 \\ 0 & e^{-h} \end{pmatrix} \Psi_0 S^{-1}(\Phi_0, \Psi_0), \quad \hat{\Phi}_0 = \begin{pmatrix} e^{-h} & 0 \\ 0 & e^h \end{pmatrix} \Phi_0 S^{-1}(\Psi_0, \Phi_0),$$

i.e. the spinors $(\hat{\Psi}_0, \hat{\Phi}_0)$ are obtained from $(\Psi_0, S^{-1}(\Phi_0, \Psi_0), \Phi_0, S^{-1}(\Psi_0, \Phi_0))$ by the gauge transformation (16) and define the same surface. Theorem 2 is proved.

For the completeness of exposition let us compute the function $W = \tilde{U} - U$ in terms of the Weierstrass representation.

**Proposition 3**

$$W = \frac{\langle r, p \rangle}{|r|^2} = \frac{e^\alpha}{|r|^2} \langle r, n_1 + in_2 \rangle,$$

where $r : U \to \mathbb{R}^4$ is a surface in $\mathbb{R}^4$, the vector $p$ has the form (18), $e^{2\alpha}$ is the conformal factor of the metric and $(n_1, n_2)$ is a basis of the normal bundle.
Proof. Let us compute the function $K_{22} = -iW$ given by (7). We have

$$K = \Psi_0 S^{-1}(\Phi_0, \Psi_0) \Gamma \Phi_0^\top \Gamma^{-1} = \Psi_0 S^{-1} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} \varphi_1 & \varphi_2 \\ -\bar{\varphi}_2 & \bar{\varphi}_1 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) =$$

$$= \frac{1}{|r|^2} \left( \begin{array}{cc} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \psi_1 \end{array} \right) \left( \begin{array}{cc} -ix^3 + x^4 & x^1 + ix^2 \\ -x^1 + ix^2 & ix^3 + x^4 \end{array} \right) \left( \begin{array}{cc} \varphi_1 & \varphi_2 \\ -\bar{\varphi}_2 & \bar{\varphi}_1 \end{array} \right),$$

where $|r|^2 = \sum_{k=1}^{3} (x_k)^2$, and conclude that, by (18),

$$K_{22} = \frac{1}{|r|^2} (x^1(\psi_2\varphi_1 - \bar{\psi}_1\bar{\varphi}_2) + ix^2(\psi_2\varphi_1 + \bar{\psi}_1\bar{\varphi}_1) + ix^3(\bar{\psi}_1\varphi_1 - \psi_2\bar{\varphi}_2) +$$

$$+ x^4(\bar{\psi}_1\varphi_1 + \psi_2\bar{\varphi}_2)) = -\frac{i}{|r|^2} \langle r, p \rangle.$$

Proposition is proved.

4 An integrable example of “conformal” transformations of the spectral curve and of the Floquet functions

Let the potential $U$ is double-periodic:

$$U(z + \lambda) = U(z), \quad \lambda \in \Lambda \approx \mathbb{Z}^2 \subset \mathbb{C}.$$

A solution $\psi$ of (2) is called the Floquet function (on the zero energy level) of $D$, if there exist constants $\mu_1$ and $\mu_2$ (the Floquet multipliers) such that

$$\psi(z + \lambda_k) = \mu_k \psi(z), \quad k = 1, 2,$$

where $\lambda_1$ and $\lambda_2$ generate the period lattice $\Lambda$. The Floquet functions are parameterized by the spectral curve $\Gamma$ of $D$ [7] (see, also, [4]), which was first introduced in [8] for the two-dimensional Schrödinger operator.

In [9] it was proved that the actions of conformal transformations of $\mathbb{R}^4$ on tori preserve the Floquet multipliers of the Dirac operators coming into their Weierstrass representations. The proof consists in the following:

1) the identity map and the inversion are connected by a smooth curve $\gamma(t)$ in the space of conformal transformations;

2) to the torus $\Sigma \subset \mathbb{R}^2$ with a fixed conformal parameter $z$ was applied the conformal transformation $\gamma(t)$;
3) on the constructed torus $\Sigma_t = \gamma(t) \cdot \Sigma$ the parameter $z$ is also conformal and its Weierstrass representation has the potential $U(z, \bar{z}, t)$;

4) the derivatives in $t$ of the Floquet functions are computed and, therewith, it is proved that the derivatives of the multipliers vanish.

Moreover in [9] it was shown that the evolution in $t$ of the Floquet functions has the form of a nonlinear equation of the Melnikov type. In [9] it was pointed out that under such a deformation the spectral curve may become singular due to creation of double points. We demonstrate that below by using explicit analytical formulas.

For tori in $\mathbb{R}^3$ the preservation of the multipliers, conjectured by us, was proved in [10]. The question on the preservation of the spectral curve was not discussed in [10].

Let us present an explicit example of such a deformation of a potential $U(z, \bar{z}, t)$ and of the corresponding Floquet functions.

This example is related to the Clifford torus $\Sigma$, which is defined by the equations

\[(x^1)^2 + (x^2)^2 = \frac{1}{2}, \quad (x^3)^2 + (x^4)^2 = \frac{1}{2}\]

and is parameterized as follows $x^1 = \frac{1}{\sqrt{2}} \cos x$, $x^2 = \frac{1}{\sqrt{2}} \sin x$, $x^3 = -\frac{1}{\sqrt{2}} \cos y$, $x^4 = -\frac{1}{\sqrt{2}} \sin y$. Its Weierstrass representation is given by the potential

\[U_{\text{clifford}} = -i \sqrt{8}\]

(23)

and by the spinors

\[\psi_0 = \frac{e^{-i(x+y)/2}}{\sqrt{2}} \left( e^{i\frac{3\pi}{8}} \right), \quad \varphi_0 = \frac{e^{i(y-x)/2}}{2} \left( -e^{-i\frac{3\pi}{8}} \right)\]

(in [4] we used the potential $U = 1+i/4$ which is related to (23) by a gauge transformation (17)).

The basis of the Floquet functions of the Dirac operator (1) with a constant potential $U$ may be taken in the form

\[\psi(z, \bar{z}, \lambda) = \exp \left( \lambda z - \frac{|U|^2}{\lambda} \bar{z} \right) \left( \frac{1}{-U} \right),\]

where $\lambda \in \mathbb{C} \setminus \{0\}$ and the (compactified) spectral curve is the Riemann sphere: $\Gamma = \mathbb{C} \cup \{\infty\}$. For the potential (23) of the Clifford this basis takes the form

\[\psi_{\text{Clifford}}(z, \bar{z}, \lambda) = \exp \left( \lambda z - \frac{1}{8\lambda} \bar{z} \right) \left( \frac{1}{i \sqrt{8}\lambda} \right).\]

(24)
Let us consider the family of surfaces $\Sigma_t$ obtained from the Clifford tours by translations by $t$ along the $Ox^4$ where $t \in \mathbb{R}$:

$$(x^1, x^2, x^3, x^4) \rightarrow (x^1, x^2, x^3, x^4 + t),$$

and apply to each torus form this family the mapping (19), i.e. a composition of the inversion with the center at the origin and the reflection (20). The obtained tori we denote by $\tilde{\Sigma}_t$. The potentials $U(z, \bar{z}, t)$ of their Weierstrass representations are explicitly computed by using Theorem 1 and are as follows:

$$U(z, \bar{z}, t) = -\frac{i}{\sqrt{8}} + \frac{\sqrt{2i} - (1 + i)t \sin y}{2(t^2 - \sqrt{2t} \sin y + 1)}.$$ 

The tori $\tilde{\Sigma}_t$ are defined by the spinors

$$\tilde{\psi}_0 = \frac{e^{i\frac{3\pi}{8}}}{\sqrt{2(t^2 - \sqrt{2t} \sin y + 1)}} \begin{pmatrix} e^{i\frac{\pi}{4}} \exp \left(\frac{i(y-x)}{2}\right) + t \exp \left(-\frac{i(x+y)}{2}\right) \\ \exp \left(\frac{i(y-x)}{2}\right) - t e^{i\frac{\pi}{4}} \exp \left(-\frac{i(x+y)}{2}\right) \end{pmatrix}$$ 

$$\tilde{\varphi}_0 = \frac{e^{i\frac{\pi}{4}}}{2(t^2 - \sqrt{2t} \sin y + 1)} \begin{pmatrix} - \exp \left(-\frac{i(x+y)}{2}\right) - t e^{i\frac{\pi}{4}} \exp \left(\frac{i(y-x)}{2}\right) \\ e^{i\frac{\pi}{4}} \exp \left(-\frac{i(x+y)}{2}\right) - t \exp \left(\frac{i(y-x)}{2}\right) \end{pmatrix}.$$ 

We derive from these formulas that

1. for $t = 0$ the Clifford torus is mapped into itself and the potential is mapped into a gauge equivalent potential:

$$U_{\text{Clifford}} = -\frac{i}{\sqrt{8}} \rightarrow U(z, \bar{z}, 0) = \frac{i}{\sqrt{8}};$$

2. only for $t = \pm 1$ the spinors $\tilde{\psi}$ and $\tilde{\varphi}$ are proportional:

$$\tilde{\psi} = \mp \sqrt{2} \tilde{\varphi},$$

or, which is equivalent, the surface lies in the three-dimensional hyperplane. Indeed, only in these cases the torus $\Sigma_t$ passes through the origin and by the inversion is mapped into the hyperplane $x^4 = \text{const}$. In this hyperplane the surface $\tilde{\Sigma}_t$ is the Clifford torus (in $\mathbb{R}^3$) on which the Willmore functional attains its minimum among all tori in $\mathbb{R}^3$. The potentials of these tori are equal to

$$U(z, \bar{z}, \pm 1) = \pm \frac{\sin y}{2\sqrt{2(\sqrt{2} \mp \sin y)}};$$

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3. by Theorem 1, the Floquet functions of the operator with the potential $U(z, \bar{z}, t)$ are obtained from the functions [24] by the Moutard transformation and take the form

$$\tilde{\psi} = \exp \left( \lambda z - \frac{1}{8\lambda} \tilde{z} \right) \begin{pmatrix} 1 - \frac{2i + 2t(2\sqrt{2}\lambda \cos y + e^{-\frac{i}{2}\sin y})}{(8\lambda^2 + i)(t^2 - \sqrt{2t \sin y + 1)}} \\ i \left( \frac{1}{\sqrt{8\lambda}} - \frac{4\sqrt{2}\lambda + 2t \cdot (\cos y + 2\sqrt{2}\lambda e^{-\frac{i}{2}\sin y})}{(8\lambda^2 + i)(t^2 - \sqrt{2t \sin y + 1})} \right) \end{pmatrix}.$$ 

By multiplying that by $\lambda^8 + i$ to get rid of the appearing poles in $\lambda$ and by saving, for brevity, the notations, we derive for $u = \frac{1 + i}{4}$, $t = 1$ that

$$\tilde{\psi}(u) = \begin{pmatrix} i\sqrt{2} e^{\frac{\sqrt{2} u}{2} + (1-i)e^{\frac{\sqrt{2} u}{2}}} \\ (i-1) e^{-i\sqrt{2}y} + i\sqrt{2} e^{\frac{\sqrt{2} y}{2}} \end{pmatrix} \sqrt{2 - \sin y}, \quad \tilde{\psi}(-\bar{u}) = \begin{pmatrix} (1+i) e^{\frac{\sqrt{2} u}{2} - i\sqrt{2} e^{\frac{\sqrt{2} u}{2}}} \\ i\sqrt{2} e^{\frac{\sqrt{2} y}{2} + (1+i)e^{\frac{\sqrt{2} y}{2}}} \end{pmatrix} \sqrt{2 - \sin y},$$

$$\tilde{\psi}(-u) = \begin{pmatrix} i\sqrt{2} e^{-i\frac{\sqrt{2} u}{2} - (1-i)e^{-i\frac{\sqrt{2} u}{2}}} \\ (1-i) e^{i\sqrt{2}y} + i\sqrt{2} e^{-\frac{\sqrt{2} y}{2}} \end{pmatrix} \sqrt{2 - \sin y}, \quad \tilde{\psi}(\bar{u}) = \begin{pmatrix} -i\sqrt{2} e^{-i\frac{\sqrt{2} u}{2} - (1+i)e^{-i\frac{\sqrt{2} u}{2}}} \\ i\sqrt{2} e^{-i\frac{\sqrt{2} y}{2} - (1+i)e^{-i\frac{\sqrt{2} y}{2}}} \end{pmatrix} \sqrt{2 - \sin y},$$

which implies the following equalities

$$\tilde{\psi}(u) = \frac{1 + i}{\sqrt{2}} \tilde{\psi}(-\bar{u}), \quad \tilde{\psi}(-u) = -\frac{1 + i}{\sqrt{2}} \tilde{\psi}(\bar{u}).$$

Hence, for $t = 1$ the Floquet functions are uniquely parameterized by points of the singular curve $\mathbb{C} \setminus \{0\} \setminus \{u \sim -\bar{u}, -u \sim \bar{u}\}$, and the spectral curve $\Gamma_1$, compactified by a pair of “infinities” $\lambda = 0$ and $\lambda = \infty$, is a rational curve with a pair of double points. These finite gap integration data for the Clifford torus were obtained in [12].

From the explicit formulas for the Floquet functions it is easy to notice that for small $t$ the spectral curve of the operator with the potential $U(z, \bar{z}, t)$ is preserved and stays smooth, and for $t = 1$ on it appear a pair of double points. Therewith the Floquet multipliers are preserved.

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