An interpolation problem in the Denjoy–Carleman classes

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Abstract
Inspired by some iterative algorithms useful for proving the real analyticity (or the Gevrey regularity) of a solution of a linear partial differential equation with real-analytic coefficients, we consider the following question. Given a smooth function defined on $[a, b] \subset \mathbb{R}$ and given an increasing divergent sequence $d_n$ of positive integers such that the derivative of order $d_n$ of $f$ has a growth of the type $M_{d_n}$, when can we deduce that $f$ is a function in the Denjoy–Carleman class $C^M([a, b])$? We provide a positive result and show that a suitable condition on the gaps between the terms of the sequence $d_n$ is needed.

KEYWORDS
Denjoy–Carleman classes, Gevrey functions, interpolation, quasi-analytic functions

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1 | INTRODUCTION AND STATEMENT OF THE RESULTS

A way to prove the local regularity of the solutions of linear elliptic partial differential equations with real-analytic coefficients consists in the so called $L^2$–methods, that is, an iterative procedure based on the use of the elliptic estimate (see, e.g., [1, p. 207]). This approach extends to degenerate elliptic equations, such as sums of squares of vector fields, satisfying a subelliptic estimate (see, e.g., [2, 3] and [4] for recent applications of this method). Then, a natural question is which derivatives one should control in order to conclude that a function is in a given Gevrey class (or, in particular, is real-analytic). This is the main motivation for the present paper. In order to put in evidence the essential points in our proofs, we state our results for the Denjoy–Carleman classes, and, for the sake of simplicity, we limit our considerations to functions of one variable. To be definite: let $M_0 = 1$, $M_1$, ... be a sequence of positive numbers and consider the Denjoy–Carleman class $C^M$

$$C^M([a, b]) = \{ f \in C^\infty([a, b]) \mid \exists K > 0 \text{ such that } |f^{(n)}(x)| \leq K^{n+1}M_n \text{ if } x \in [a, b], n = 0, 1, ... \}.$$ 

In what follows, we assume that the sequence $M_n$ is log-convex. That is, for $j < \ell < k$, we have that

$$M_\ell \leq M_j^{k-\ell}M_k^{\ell-j}. \tag{1.1}$$
We denote by \( \mathbb{N} \) the set of all the non-negative integers while \( \mathbb{N}_+ \) stands for the set of positive integers. Condition (1.1) can be equivalently stated as

\[
M_n^2 \leq M_{n-1}M_{n+1}, \quad \forall n \in \mathbb{N}_+,
\]

that is, \( M_{n+1}/M_n \) is an increasing sequence. In particular, if \( M_n \) is log-convex then

\[
M_n^{1/n} \quad \text{is an increasing sequence for } n \in \mathbb{N}_+.
\]

For the reader convenience, we provide the proofs of the assertions (1.1) \( \iff \) (1.2) and (1.2) \( \Rightarrow \) (1.3) in Appendix A.

We will assume the additional condition that there exists \( m_0 \geq 0 \) such that

\[
M_n \leq M_{n+1}^{1/m_0} M_{i}^{1/i}, \quad \text{for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k. \quad (1.4)
\]

**Example 1.1.**

(i) Our first example of a sequence satisfying (1.1) and (1.4) is \( M_n = n^{s_1}, n \in \mathbb{N}_+ \), for a suitable \( s \geq 1 \). This choice corresponds to the Gevrey class \( G^s \) (the set of all the real-analytic functions in the case \( s = 1 \)). We point out that (1.4) holds with \( m_0 = 1 \).

(ii) Our second example is \( M_0 = M_1 = 1, M_2 = \sqrt{M_3}, M_n = n^{s_1} (\log n)^{s_2} \) \( (n = 3, 4, ...), \) for suitable \( s_1, s_2 \geq 1 \). This sequence is log-convex and satisfies (1.4) with \( m_0 = e^2 \). By the Denjoy–Carleman theorem, \( C^M([a, b]) \) is a quasi-analytic class for \( s_1 = s_2 = 1 \). (We recall that a class \( C^M([a, b]) \) is called quasi-analytic if for every \( u \in C^M \) vanishing of infinite order at a point in \([a, b]\), it follows that \( u \) is identically zero on \([a, b]\).)

We observe that, for \( s_1 > 1 \) or \( s_2 > 1 \), \( C^M([a, b]) \) is a non-quasi-analytic class (other than Gevrey).

For the proof of the claims in (i) and (ii), we refer the interested reader to Appendix B.

Furthermore, we assume that the class \( C^M \) contains the real-analytic functions on \([a, b]\), that is, that there exists \( c > 0 \) such that

\[
M_n \geq c^n n^n.
\]

For a smooth function \( f : [a, b] \to \mathbb{R} \), we define

\[
F_n = \max_{x \in [a, b]} |f^{(n)}(x)|, \quad n \in \mathbb{N}.
\]

Let \( d_n > 0 \) be an increasing divergent sequence of integers such that for a suitable \( K > 0 \),

\[
F_{d_n} \leq K^{d_n+1} M_{d_n}, \quad \text{for every } n \in \mathbb{N}. \quad (1.7)
\]

We consider the following problem: given a function \( f \in C^\infty([a, b]) \) satisfying (1.7), under what condition on \( \{d_n\}_n \) can we conclude that \( f \) belongs to the class \( C^M \)?

First, we observe that Condition (1.7) yields some additional properties on the function \( f \) when \( C^M \) is the class of real-analytic functions, that is, \( M_k = k^k, k \in \mathbb{N}_+ \). Precisely, we have the following:

**Proposition 1.2.** Let \( f \in C^\infty([a, b]) \) and let \( C^F([a, b]) \) be the corresponding Denjoy–Carleman class defined by the sequence \( \{F_n\}_n \) in (1.6). Let \( d_n > 0 \) be an increasing divergent sequence of integers. Assume that \( F_{d_n} \leq K^{d_n+1} d_n^{d_n} \) for a suitable \( K > 0 \) and for every \( n \in \mathbb{N} \). Then \( f \) is identically zero, whenever \( f \) vanishes of infinite order at a point of \([a, b]\).

In other words, the class \( C^F([a, b]) \) is quasi-analytic (independently on the sequence \( d_n \)).
Remark 1.3. In general, we cannot deduce that the class $C^\infty([a, b])$, given by Proposition 1.2, is contained in the class of the real-analytic functions, this is a consequence of Theorem 1.6 with $M_n = n^n$.

If we make an additional assumption on the growth of $d_{n+1} - d_n$, we obtain the following

**Theorem 1.4.** Let $f \in C^\infty([a, b])$, assume (1.1), (1.4), (1.5), (1.7) and that there exists $c_0 > 0$ such that

$$d_{n+1}/d_n \leq c_0, \quad \text{for} \quad n \in \mathbb{N}. \quad (1.8)$$

Then, $f \in C^M([a, b])$.

In particular, for $f \in C^\infty$ to be in the Gevrey class $G^s$, it is enough to control only a set of derivatives $f^{(d_n)}$ where the sequence $\{d_n\}_n$ satisfies (1.8).

**Remark 1.5.**

(i) We point out that the above result applies to both quasi-analytic and non-quasi-analytic Denjoy–Carleman classes.

(ii) Theorem 1.4 is related to the Carleman problem (see [7] and [9]).

Finally, we show that Theorem 1.4 may fail in the absence of a condition on the sequence $\{d_n\}_n$.

**Theorem 1.6.** Let $\{M_n\}_n$ be an arbitrary sequence of positive numbers such that

$$\limsup_{n \to +\infty} \frac{M_n^n}{M^n} = +\infty. \quad (1.9)$$

Then, we can find an increasing, divergent, sequence of positive integers $\{d_n\}_n$, and a function $f \in C^\infty([a, b])$, such that (1.7) holds true but $f \notin C^M([a, b])$.

2 | PROOFS

2.1 | Proof of Proposition 1.2

The proof of the Proposition 1.2 is an elementary computation based on the Taylor expansion (see, e.g., [8]). Indeed, let $x_0 \in [a, b]$, then by the Taylor formula, we find that

$$|f(x)| \leq K^{d_n+1} \frac{d_n}{d_n!} |x - x_0|^{d_n} \leq K(K e |x - x_0|)^{d_n}, \quad \forall x \in [a, b],$$

and taking the limit as $n \to \infty$ in the formula above, we deduce that $f(x) = 0$, for every $x \in [a, b]$ with $|x - x_0| < 1/(K e)$. Then, the conclusion follows from finitely many interactions.

2.2 | Proof of Theorem 1.4

We want to show that $f \in C^M([a, b])$ if $f \in C^\infty([a, b])$ and $\{M_n\}_n$ is a log-convex sequence such that

(A) $M_j \leq M_k^{j/i} M_i^j$, for $i, k > m_0$ with $i < j$ and $j/i < k$, for a suitable $m_0 \geq 0$;

(B) $M_n \geq c^n n^n$, for a suitable $c > 0$;

(C) $F_{d_n} \leq K^{d_n+1} M_{d_n}$ (see (1.6)).
where \( \{d_n\}_n \) is an increasing divergent sequence of natural numbers satisfying the gap condition,

\[
\frac{d_{n+1}}{d_n} \leq c_0, \quad \forall n \in \mathbb{N},
\]

for a suitable positive integer \( c_0 \).

The idea of the proof consists in showing that the control of the derivatives of length \( d_n \) yields, by interpolation, an estimate for the intermediate derivatives (provided that the rescaled gap \( (d_{n+1} - d_n)/d_n \) is bounded, uniformly w.r.t. \( n \)).

For this purpose, we recall an estimate due to Cartan and Gorny (see, e.g., [5, 6] and [9]).

**Lemma 2.1.** Let \( g \) be a function \( m \)-times differentiable on the closed interval \([a, b]\) and set \( G_\ell = \max_{x \in [a, b]} |g^{(\ell)}(x)| \), for every \( \ell \in \mathbb{N} \). Then, for every \( m \in \mathbb{N} \), with \( m \geq 2 \), and \( k \in \{1, \ldots, m-1\} \), one has

\[
G_k \leq 2 \left( \frac{e^m}{k} \right)^{1 - \frac{k}{m}} \left( \max \left\{ m! G_0 \left( \frac{2}{b-a} \right)^m, G_m \right\} \right)^{\frac{k}{m}}. \tag{2.2}
\]

Let \( d_n < \ell < d_{n+1} \) and choose \( g = f^{(d_n)} \), \( m = d_{n+1} - d_n \) and \( k = \ell - d_n \) in (2.2). It turns out that

\[
\frac{1}{p} := 1 - \frac{k}{m} = \frac{d_{n+1} - \ell}{d_{n+1} - d_n}, \quad \frac{1}{q} := \frac{k}{m} = \frac{\ell - d_n}{d_{n+1} - d_n},
\]

and

\[
\frac{d_n}{p} + \frac{d_{n+1}}{q} = \ell. \tag{2.3}
\]

Whence (2.2) reads as

\[
F_\ell \leq 2 \left( \frac{e^2(d_{n+1} - d_n)}{\ell - d_n} \right)^{\ell - d_n} \max \left\{ \left( \frac{d_{n+1} - d_n}{\ell - d_n} \right)^{1 - \frac{1}{q}} F_{d_n}^{-\ell d_n}, \frac{1}{q} \left( \frac{d_{n+1} - d_n}{\ell - d_n} \right)^{1 - \frac{1}{q}} F_{d_{n+1}}^{-\ell d_n} \right\}. \tag{2.4}
\]

Now, in view of (2.1), we have that

\[
\left( \frac{e^2(d_{n+1} - d_n)}{\ell - d_n} \right)^{\ell - d_n} = e^{2(1 + \frac{d_{n+1} - \ell}{\ell - d_n})^{\ell - d_n}} \leq e^{2(\ell - d_n) + d_{n+1} - \ell} \leq e^{\ell + d_n \left( \frac{d_{n+1} - d_n}{\ell - d_n} - 2 \right)} \leq e^{(c_0-1)}.
\]

Furthermore,

\[
\left( \frac{d_{n+1} - d_n}{\ell - d_n} \right)^{1 - \frac{1}{q}} \leq \left( \frac{d_{n+1} - d_n}{\ell - d_n} \right)^{\frac{d_{n+1} - d_n}{q}} \leq \left( \frac{2d_n(c_0 - 1)}{b-a} \right)^{\ell - d_n},
\]

and, since \( F_{d_n} \leq K^{d_{n+1}} M_{d_n} \), we obtain that

\[
F_\ell \leq \max \left\{ 2e^{(c_0-1)} \left( \frac{2d_n(c_0 - 1)}{b-a} \right)^{\ell - d_n} K^{d_{n+1}} M_{d_n}, 2e^{(c_0-1)} K^{d_{n+1}} M_{d_n}^{\frac{1}{p}} K^{d_{n+1}} M_{d_{n+1}}^{\frac{1}{q}} \right\}. \tag{2.5}
\]

We observe that by (2.3) and the fact that \( d_n < \ell \), we have that

\[
K^{\frac{d_{n+1}}{p}} K^{\frac{d_{n+1}+1}{q}} = K^{\ell + 1}, \quad \text{and} \quad K^{d_{n+1}} \leq K^{\ell+1} \tag{2.6}
\]
Moreover, by Assumption (A) above\(^1\) and by the gap condition (2.1), we see that
\[
M_{d_n}^{\frac{1}{p}} M_{d_{n+1}}^{\frac{1}{q}} \leq M_{d_n}^{\frac{1}{q}} M_{d_{n+1}}^{\frac{1}{p}} M_{c_0}^{\frac{c}{q}}.
\]
Whence, as a consequence of the log-convexity of \(\{M_n\}_n\), (1.3), and (2.3), we find that
\[
M_{d_n}^{\frac{1}{p}} M_{d_{n+1}}^{\frac{1}{q}} \leq M_{e_{d_n}}^{\frac{1}{p}} M_{e_{d_{n+1}}}^{\frac{1}{q}} M_{c_0}^{\frac{c}{q}} \leq M_{e} M_{c_0}^{\frac{c}{q}}.
\] (2.7)
Assumption (B) yields that
\[
d_n^{\frac{c}{q} - d_n} \leq \left( \frac{M_{d_n}}{c} \right)^{\frac{c}{q} - d_n}.
\] (2.8)
Hence, taking together (2.5), (2.6), (2.7), (2.8) and using the log-convexity of \(\{M_n\}_n\), (1.3), we deduce that
\[
F_{\ell} \leq \max \left\{ 2 e^{\ell (c_0 - 1)} \left( \frac{2M_{d_n} (c_0 - 1)}{c(b - a)} \right)^{\frac{\ell}{d_n}} M_{d_n}, 2 e^{\ell (c_0 - 1)} M_{\ell} M_{c_0}^{\frac{c}{d_n}} K^{\ell+1} \right\}
\]
\[
\leq \max \left\{ 2 e^{\ell (c_0 - 1)} \left( \frac{2(c_0 - 1)}{c(b - a)} \right)^{\frac{\ell}{d_n}} M_{c_0}^{\frac{c}{d_n}}, 2 e^{\ell (c_0 - 1)} M_{c_0}^{\frac{c}{d_n}} \right\} K^{\ell+1} M_{\ell} \leq C_1 K^{\ell+1} M_{\ell},
\]
where
\[
C_1 = 2 e^{\ell (c_0 - 1)} \left( \frac{2(c_0 - 1)}{c(b - a)} + M_{c_0}^{\frac{c}{d_n}} \right).
\]
Then, we conclude that there exists \(K_1 > 0\) such that
\[
F_n \leq K_1^{n+1} M_n, \quad \text{for every } n \in \mathbb{N},
\]
that is, \(f \in \mathcal{C}^M([a, b])\). This completes our proof of Theorem 1.4.

### 2.3 Proof of Theorem 1.6

The proof is based on the following result, which ensures the existence, in every class \(\mathcal{C}^N\), of a function that attains the bounds \(N_j\). Although different formulations are already present in the literature (see, e.g., [10] for the case of complex-valued functions), for the sake of completeness we provide its proof.

**Lemma 2.2.** Let \(\{N_j\}_j\) be a positive sequence satisfying (1.1). Then, there exists \(f \in \mathcal{C}^N([a, b])\) such that
\[
\left| f^{(j)} \left( \frac{a + b}{2} \right) \right| \geq N_j, \quad \forall j \in \mathbb{N}.
\] (2.9)

**Proof.** Set \(m_j = N_{j+1}/N_j\) and observe that, by (1.1), it is an increasing sequence. We note that
\[
\left( \frac{1}{m_k} \right)^{k-j} \leq \frac{N_j}{N_k} \quad \forall k, j \in \mathbb{N}.
\] (2.10)
Indeed, (2.10) trivially holds in the case of $j = k$. Moreover, if $j < k$, we have that

$$\left( \frac{1}{m_k} \right)^{k-j} \leq \frac{1}{m_{k-1} \ldots m_{k-(k-j)}} = \frac{N_j}{N_k}.$$ 

Finally, if $j > k$, we obtain

$$m_{j-k} \leq m_k \ldots m_{j-1} = \frac{N_j}{N_k}.$$ 

Summing up, (2.10) holds true.

Let us define

$$g(x) = \sum_{k=0}^{\infty} \frac{N_k}{(2m_k)^k} (\cos(2m_k x) + \sin(2m_k x)).$$

We notice that $g \in C^N(I)$ for every interval $I \subset \mathbb{R}$. Indeed, in light of (2.10), we get

$$|g^{(n)}(x)| \leq \sum_{k=0}^{\infty} \frac{N_k}{(2m_k)^k} (2m_k)^n 2 \leq N_n \sum_{k=0}^{\infty} \frac{2^{n+1}}{2^k} \leq 2^{n+2} N_n.$$ 

Furthermore, we have that

$$|g^{(n)}(0)| = \left| \left( \sum_{k=0}^{\infty} \frac{N_k}{(2m_k)^k} \right) (\cos^{(n)}(0) + \sin^{(n)}(0)) \right|,$$

and, since

$$\begin{cases} 
\cos^{(n)}(0) = (-1)^{n/2} \text{ and } \sin^{(n)}(0) = 0, & \text{ for } n \text{ even}, \\
\cos^{(n)}(0) = 0 \text{ and } \sin^{(n)}(0) = (-1)^{n/2}, & \text{ for } n \text{ odd}, 
\end{cases}$$

we deduce that

$$|g^{(n)}(0)| = \sum_{k=0}^{\infty} \frac{N_k}{(2m_k)^k} \geq N_n.$$ 

The conclusion follows by taking

$$f(x) := g \left( x - \frac{a + b}{2} \right), \quad (x \in [a, b]).$$

Then, the proof of Theorem 1.6 reduces to show that given a positive sequence $\{M_n\}_n$ satisfying (1.9), there exist

- a sequence $\{N_n\}_n$ satisfying (1.1),
- two divergent sequences of positive integers $\{d_n\}_n$ and $\{i_n\}_n$,

so that

$$N_{d_n} = M_{d_n}, \quad n = 0, 1, ...$$

and

$$\frac{N_{i_n}}{M_{i_n}} = 2^{2^n}. \quad (2.11)$$
We point out that once the existence of such a sequence \( \{N_n\}_n \) is established, Lemma 2.2 yields the existence of a function \( f \in C^N([a,b]) \) such that \( N_n d_n \leq M d_n \). On the other hand, due to (2.11), we have that \( f \notin C^M([a,b]) \). In other words, in general (if the gap \((d_{n+1} - d_n)/d_n\) is suitably large), the interpolation between \( d_n \) and \( d_{n+1} \) does not provide the bounds on the derivatives of \( f \) ensuring that \( f \in C^M([a,b]) \).

We construct inductively the sequences \( \{N_n\}_n, \{d_n\}_n \), and \( \{i_n\}_n \). For this purpose, it is useful to define

\[
m_n = \frac{N_{n+1}}{N_n}.
\]

We observe that \( N_n \) satisfies (1.1) if and only if \( m_n \) is an increasing sequence; furthermore, without loss of generality, we may assume that \( M_0 = N_0 = 1 \). Clearly, we have

\[
m_0 \cdot \ldots \cdot m_{j-1} = \frac{N_j}{N_0} = N_j.
\]

Let us fix an arbitrary positive integer \( i_0 \). We claim that there exists a positive integer \( d_0 > i_0 \) so that setting

\[
\begin{align*}
m_0 = \ldots = m_{i_0-1} &= (2^{i_0} M_{i_0})^{\frac{1}{i_0}}, \\
m_{i_0} = \ldots = m_{d_0-1} &= \left( \frac{M_{d_0}}{2^{i_0} M_{i_0}} \right)^{\frac{1}{d_0-i_0}}
\end{align*}
\]

we have

\[
m_j \leq m_{j+1}, \quad \text{for } j = 0, \ldots, d_0 - 2.
\]

We observe that (2.13) is equivalent to the inequality

\[
(2^{2 i_0} M_{i_0})^{\frac{1}{i_0}} \leq \left( \frac{M_{d_0}}{2^{2 i_0} M_{i_0}} \right)^{\frac{1}{d_0-i_0}} \iff (2^{2 i_0} M_{i_0})^{\frac{1}{i_0}} \leq M_{d_0}^{\frac{1}{d_0}}.
\]

In view of (1.9), for every \( c > 0 \) there exists \( d_0 \) such that \( M_{d_0}^{\frac{1}{d_0}} \geq c \). Hence, we find that

\[
M_{d_0}^{\frac{1}{d_0}} \geq c \geq (2^{2 i_0} M_{i_0})^{\frac{1}{i_0}},
\]

provided that \( c > (2^{2 i_0} M_{i_0})^{\frac{1}{i_0}} \), that is, (2.13) holds true.

We point out that as a consequence of our construction, we have

\[
\begin{cases}
N_j = (2^{2 i_0} M_{i_0})^{\frac{j}{i_0}}, & \text{for } j = 0, \ldots, i_0, \\
N_j = 2^{2 i_0} M_{i_0} \left( \frac{M_{d_0}}{2^{2 i_0} M_{i_0}} \right)^{\frac{j-i_0}{d_0-i_0}}, & \text{for } j = i_0 + 1, \ldots, d_0,
\end{cases}
\]

\[
\frac{N_{i_0}}{M_{i_0}} = 2^{2 i_0}, \quad \text{and} \quad N_{d_0} = M_{d_0}.
\]
Let us show that there exist two positive integers $d_1 > i_1 > d_0$ so that setting

\[
\begin{align*}
  m_{d_0} = \cdots = m_{i_1-1} &= \left( \frac{2^{2i_1} M_{i_1}}{M_{d_0}} \right)^{1 \over i_1 - d_0}, \\
  m_{i_1} = \cdots = m_{d_1-1} &= \left( \frac{M_{d_1}}{2^{2i_1} M_{i_1}} \right)^{1 \over d_1 - i_1},
\end{align*}
\]  

we have that

\[
m_j \leq m_{j+1}, \quad \text{for } j = 0, \ldots, d_1 - 2.
\]  

We observe that (2.17) is equivalent to

\[
\begin{align*}
  \left( \frac{2^{2i_1} M_{i_1}}{M_{d_0}} \right)^{1 \over i_1 - d_0} &\geq \left( \frac{M_{d_0}}{2^{2i_0} M_{i_0}} \right)^{1 \over d_0 - i_0}, \\
  \left( \frac{M_{d_1}}{2^{2i_1} M_{i_1}} \right)^{1 \over d_1 - i_1} &\geq \left( \frac{2^{2i_1} M_{i_1}}{M_{d_0}} \right)^{1 \over i_1 - d_0}.
\end{align*}
\]  

As for the first inequality in (2.18), it is enough to show that

\[
\left( \frac{2^{2i_1} M_{i_1}}{M_{d_0}} \right)^{1 \over i_1 - d_0} \geq \left( M_{d_0} \right)^{1 \over d_0 - i_0},
\]  

which is equivalent to

\[
2^{2i_1} M_{i_1} \geq M_{d_0}^{d_0 - i_0}.
\]

Once more, by (1.9), this last inequality can be satisfied provided $i_1 > d_0$ is chosen suitably large. Furthermore, the second inequality in (2.18) can be rewritten as

\[
M_{d_1} \geq \left( \frac{2^{2i_1} M_{i_1}}{M_{d_0}} \right)^{d_1 - d_0 \over d_1 - i_1} \frac{d_1 - d_0}{d_1 - i_1} \frac{d_0}{i_1 - d_0},
\]

which can be fulfilled, due to (1.9), provided that $d_1 > i_1$ is large enough.

Summing up, we have constructed a sequence $N_j$, $j = 0, \ldots, d_1$, such that $m_j = \frac{N_j}{N_j}$ is an increasing sequence for $j = 0, \ldots, d_1$, and

\[
N_{d_0} = M_{d_0}, \quad N_{d_1} = M_{d_1}, \quad N_{i_0}/M_{i_0} = 2^{2i_0} \quad \text{and} \quad N_{i_1}/M_{i_1} = 2^{2i_1},
\]

for suitable positive integers $0 < i_0 < d_0 < i_1 < d_1$.

Now, let us suppose that we have already defined $d_0 < i_0 < \ldots < i_n < d_n$, and $N_j$, $j = 1, \ldots, n$, such that

\[
\begin{align*}
  m_j &= \frac{N_j}{N_j} \quad \text{are increasing for } j = 0, \ldots, d_n - 1, \\
  N_{d_k} &= M_{d_k}, & k = 0, \ldots, n, \\
  \frac{N_{i_k}}{M_{i_k}} &= 2^{2k}, & k = 0, \ldots, n.
\end{align*}
\]
In order to complete the proof of Theorem 1.6 it suffices to show that we can find $i_{n+1}, d_{n+1}$ and $N_j$, $j = d_n + 1, \ldots, d_{n+1}$ such that

1. $d_n < i_{n+1} < d_{n+1}$,
2. $m_j = \frac{N_{j+1}}{N_j}$ are increasing for $j = 0, \ldots, d_{n+1}$,
3. $N_{d_{n+1}} = M_{d_{n+1}}$ and $\frac{N_{i_{n+1}}}{M_{i_{n+1}}} = 2^{i_{n+1}}$.

Set

\[
\begin{cases}
  m_d = \cdots = m_{i_{n+1}-1} = \left( \frac{2^{i_{n+1}} M_{i_{n+1}}}{M_d} \right)^{\frac{1}{i_{n+1} - d_n}}, \\
  m_{i_{n+1}} = \cdots = m_{d_{n+1}-1} = \left( \frac{M_{d_{n+1}}}{2^{d_{n+1}} M_{i_{n+1}}} \right)^{\frac{1}{d_{n+1} - i_{n+1}}}.
\end{cases}
\]  

(2.19)

we have that

\[ m_j \leq m_{j+1}, \quad \text{for } j = 0, \ldots, d_{n+1} - 2. \]  

(2.20)

We observe that (2.20) reduces to verifying

\[
\begin{cases}
  \left( \frac{2^{i_{n+1}} M_{i_{n+1}}}{M_d} \right)^{\frac{1}{i_{n+1} - d_n}} \geq m_{d_n-1}, \\
  \left( \frac{M_{d_{n+1}}}{2^{d_{n+1}} M_{i_{n+1}}} \right)^{\frac{1}{d_{n+1} - i_{n+1}}} \geq \left( \frac{2^{i_{n+1}} M_{i_{n+1}}}{M_d} \right)^{\frac{1}{i_{n+1} - d_n}}.
\end{cases}
\]  

(2.21)

We observe that the first inequality in (2.21) is equivalent to

\[ M_{i_{n+1}} \geq \frac{M_{d_n} m_{i_{n+1} - d_n}}{2^{2^{i_{n+1}}}} \]

which, by (1.9), can be satisfied provided that $i_{n+1} > d_n$ is large enough. Finally, the second inequality in (2.21), can be rewritten as

\[ M_{d_{n+1}} \geq \left( \frac{2^{i_{n+1}} M_{i_{n+1}}}{M_d} \right)^{\frac{d_{n+1} - d_n}{i_{n+1} - d_n}} \]

which, once more by (1.9), can be satisfied provided that $d_{n+1} > i_{n+1}$ is large enough. Finally, as a consequence of (2.19) we have:

\[
\begin{align*}
  \frac{N_{i_{n+1}}}{N_{d_n}} &= m_d \cdot m_{d_{n}+1} \cdot \cdots \cdot m_{i_{n+1}-1} = \frac{2^{i_{n+1}} M_{i_{n+1}}}{M_d}, \\
  \frac{N_{d_{n+1}}}{N_{d_n}} &= m_d \cdot m_{d_{n}+1} \cdot \cdots \cdot m_{d_{n+1}-1} = \frac{M_{d_{n+1}}}{M_d}.
\end{align*}
\]

Since $N_{d_n} = M_{d_n}$, this completes our proof of Theorem 1.6. \qed
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CONFLICT OF INTEREST
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ENDNOTE
1 We observe that, without loss of generality, we can take $c_0$ and $d_n$ larger than the number $m_0$ in (1.4).

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APPENDIX A
Let $M_0 = 1, M_1, M_2, ...$ be a sequence of positive numbers. In this section, we consider the following assertions

(A) For $j < \ell < k$, $M_\ell \leq M_j^{\frac{k-\ell}{k-j}} M_k^{\frac{\ell-j}{k-j}}$;
(B) $M_n^2 \leq M_{n-1} M_{n+1}$, for $n \in \mathbb{N}_+$;
(C) $M_n^n$ is an increasing sequence for $n \in \mathbb{N}_+$.

We show that (A) $\iff$ (B) and (B) $\implies$ (C).

We observe that (B) can be rephrased by requiring that $M_n/M_{n-1}$ is an increasing sequence for $n \in \mathbb{N}_+$. Indeed, dividing both sides of the inequality in (B) by $M_{n-1} M_n$, we deduce that (B) holds if and only if $M_n/M_{n-1}$ is an increasing sequence for $n \in \mathbb{N}_+$.

Now, let us show that (B) $\implies$ (C). (B) implies that

$$\frac{M_{n+1}}{M_n} \geq \frac{M_n}{M_{n-1}} \geq \frac{M_{n-1}}{M_{n-2}} \geq ... \geq \frac{M_1}{M_0} = M_1.$$
In particular, since $M_{n+1}/M_n$ is greater or equal than each of the factors $M_n/M_{n-1}, M_{n-1}/M_{n-2}, \ldots, M_1$, we find that it is greater or equal than the geometrical mean of these factors, that is,

$$\frac{M_{n+1}}{M_n} \geq \left( \frac{M_n}{M_{n-1}} \frac{M_{n-1}}{M_{n-2}} \ldots M_1 \right)^{\frac{1}{n}} = M_1^{\frac{1}{n}}.$$  

Then, we find that $M_{n+1} \geq M_n^{\frac{n+1}{n}}$, that is, $M_n^\frac{1}{n}$ is an increasing sequence. This completes the proof of $(B) \Rightarrow (C)$.

Now, let us consider the implication $(A) \Rightarrow (B)$.

Choosing $j = n - 1$, $\ell = n$ and $k = n + 1$, the inequality in $(A)$ can be rewritten as $M_n \leq M_{n-1}^{\frac{1}{2}} M_{n+1}^{\frac{1}{2}}$. Then, taking the square of both sides of the above inequality we deduce that $(A) \Rightarrow (B)$.

In order to show that $(B) \Rightarrow (A)$, we need the following

**Lemma A.1.** Let $M_0 = 1$, $M_1, M_2, \ldots$ be a sequence of positive numbers and assume that Condition (B) above holds. Then, for every $n \in \mathbb{N}$, the sequence

$$\mathbb{N}_+ \ni h \mapsto \left( \frac{M_{n+h}}{M_n} \right)^{\frac{1}{h}}$$

is increasing.

**Remark A.2.** The geometrical content of the above result is that a function is convex if and only if its slope is increasing.

**Proof.** The proof of the lemma proceeds by induction on $h$. We observe that

$$\left( \frac{M_{n+h+1}}{M_n} \right)^{\frac{1}{n+1}} \geq \left( \frac{M_{n+h}}{M_n} \right)^{\frac{1}{n}}$$

is equivalent to the inequality

$$M_{n+h} \leq M_n^{\frac{1}{n+1}} M_{n+h+1}^{\frac{1}{h+1}},$$  \hspace{1cm} (A.1)

for every $h \in \mathbb{N}_+$.

For $h = 1$, (A.1) reduces to (B). Now, let us suppose that the inequality (A.1) is satisfied for a suitable natural number $h$ and let us show that it is satisfied for $h + 1$. Indeed, (B) implies that

$$M_{n+h+1} \leq M_{n+h}^{\frac{1}{h}} M_{n+h+2}^{\frac{1}{h+1}}.$$ \hspace{1cm} (A.2)

Furthermore, by the inductive assumption, we have that

$$M_{n+h}^{\frac{1}{h}} \leq M_n^{\frac{1}{n+1}} M_{n+h+1}^{\frac{1}{h+1}}$$

and plugging the above inequality in (A.2), we find

$$M_{n+h+1} \leq M_n^{\frac{1}{n+1}} M_{n+h+1}^{\frac{1}{h+1}} M_{n+h+2}^{\frac{1}{h+2}}.$$  

This last inequality yields that

$$M_{n+h+1} \leq M_n^{\frac{1}{n+1}} M_{n+h+1}^{\frac{n}{h+1}} M_{n+h+2}^{\frac{1}{h+2}},$$

that is,

$$M_{n+h+1} \leq M_n^{\frac{1}{n+1}} M_{n+h+2}^{\frac{n+1}{h+2}}$$

and the proof of the lemma is completed. \qed
We observe that \((A)\) can be rewritten as
\[
(A') \quad \text{for } j < \ell < k, \log M_\ell \leq \frac{k-\ell}{k-j} \log M_j + \frac{\ell-j}{k-j} M_k.
\]

Now, Lemma A.1 implies that for every \(n \in \mathbb{N}\) the sequence
\[
\mathbb{N}_+ \ni h \mapsto \frac{\log(M_{n+h}) - \log(M_n)}{h}
\]
is increasing.

Then for \(j < \ell < k\), we find that
\[
\frac{\log(M_\ell) - \log(M_j)}{\ell - j} \leq \frac{\log(M_k) - \log(M_j)}{k - j},
\]
that is, \((A')\) holds. This completes the proof of the equivalence \((A) \iff (B)\).

**APPENDIX B**

In this section, we provide all the computations needed to justify the claims done in Example 1.

(i) Let \(s \geq 1\) and \(M_n = n^{s_1} \), for \(n \in \mathbb{N}_+\). In order to show that \(M_n\) is log-convex it suffices to show that the function \(f(x) = xs \log x\) is convex for \(x \geq 1\) (it is clear that, due to the limitation \(s \geq 1\), the factor \(s\) here is immaterial). Since \(f''(x) = s/x > 0\), for \(x > 0\), we deduce that \(M_n\) is log-convex.

Let us verify that \(M_n\) satisfies (1.4), that is, there exists \(m_0 \geq 0\) such that
\[
M_j \leq M_k^{\frac{j}{k}} M_1^{\frac{i-j}{k-j}}, \quad \text{for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k.
\]

In this case, (1.4) can be written as
\[
j^{s_1} \leq k^{s_1} j^{s_1}, \quad \text{for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k,
\]
that is,
\[
j \leq ki, \quad \text{for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k
\]
which trivially holds true with \(m_0 = 1\).

(ii) Let \(s_1, s_2 \geq 1\), \(M_0 = M_1 = 1\), \(M_2 = \sqrt{M_3}\), and \(M_n = n^{s_1} (\log n)^{s_2} \) \((n = 3, 4, \ldots)\). We claim that \(M_n\) is log-convex. Recalling that the log-convexity of \(M_n\) is equivalent to the fact that the sequence \(M_{n+1}/M_n\) is increasing, in order to prove that \(M_n\) is log-convex it suffices to verify that \(M_0'' = M_1'' = 1\), \(M_2'' = 3^{s_1}/2\), \(M_3'' = n^{s_2} \) \((n = 3, 4, \ldots)\) and \(M_4'' = M_1'' = 1\), \(M_2'' = (\log 3)^{3s_1}/2\), \(M_3'' = (\log n)^{s_2} \) \((n = 3, 4, \ldots)\) are separately log-convex. In light of (i) above, \(M_n''\) is log-convex and the proof reduces to show that \(M_n''\) has the same regularity.

We have that
\[
\frac{M_1''}{M_0''} \leq \frac{M_2''}{M_1''} \iff 1 \leq (\log 3)^{3s_1},
\]
\[
\frac{M_2''}{M_1''} \leq \frac{M_3''}{M_2''} \iff M_3'' \leq M_4'',
\]
\[
\frac{M_1''}{M_2''} \leq \frac{M_3''}{M_1''} \iff M_3'' = (\log 3)^9 \leq M_4'' = (\log 4)^8.
\]

In order to prove this last inequality it suffices to show that
\[
\log 3 \leq \log 4/ \log 3. \tag{B.1}
\]
Using the concavity of the logarithm, we have that

$$\log 3 \leq \log e + \frac{1}{e}(3 - e) = \frac{3}{e} \leq \frac{3}{2 + 1/2} = \frac{6}{5},$$

(B.2)

and

$$\log 3 \leq \log 4 - \frac{1}{4} \iff \frac{\log 4}{\log 3} \geq 1 + \frac{1}{4\log 3}. $$

The inequality above and (B.2) yield

$$\frac{\log 4}{\log 3} \geq 1 + \frac{5}{24} \left(\geq \frac{1 + \frac{1}{5}}{\geq \log 3}\right),$$

and (B.1) follows. It remains to show that \((\log n)^{s_1 s_2}\) is log-convex for \(n \geq 3\). Once more, this fact can be reduced to a one-variable problem: it suffices to verify that the function \(f(x) = x s_2 \log(\log(x))\) is convex for \(x \geq 3\). Now, \(f^{(2)}(x) = s_2 (\log x - 1)/[(\log x)^2 x] > 0\), for \(x > e\), and the log-convexity of \(M_n^{s_2}\) (and \(M_n\)) follows.

It remains to show that \(M_n\) satisfies (1.4). Clearly, it suffices to verify that \(n^{s_1}\) and \((\log n)^{s_2}\) satisfy separately (1.4). The factor \(n^{s_1}\) was already treated in (i) above. Then the proof reduces to show that

\((\log j)^{s_2} \leq (\log k)^{s_2}(\log i)^{s_2}, \text{ for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k,\)

that is,

\(\log j \leq \log k \log i, \text{ for } i, k > m_0 \text{ with } i < j \text{ and } j/i < k.\)

(B.3)

Now, the inequality \(j/i < k\) and the monotonicity of the logarithm imply that

\(\log j < \log i + \log k.\)

Now, taking \(m_0 = e^2\), we have that

\(\log i + \log k \leq \log i \log k, \text{ for } i, k > m_0,\)

then (B.3), and this completes the proof that \(M_n\) satisfies (1.4).

Finally, the fact that the class \(C^M([a, b])\) is quasi-analytic for \(s_1 = s_2 = 1\) while it is non-quasi-analytic for \(s_1 > 1\) or \(s_2 > 1\), follows by the Denjoy–Carleman theorem (see, e.g., [8]). Indeed, since \(M_n^{s_1}\) is increasing (in view of the log-convexity of \(M_n\)), the Denjoy–Carleman theorem can be stated as follows:

the class \(C^M([a, b])\) is quasi-analytic if and only if

\[ \sum_{n=1}^{\infty} \frac{1}{M_n^{s_1}} = +\infty. \]  (B.4)

Hence, since \(\sum_{n=1}^{\infty} \frac{1}{M_n^{s_1}}\) behaves as

\[ \sum_{n=3}^{\infty} \frac{1}{n^{s_1}(\log n)^{s_2}} , \quad (s_1, s_2 \geq 1), \]

using the generalized integral associated with the series above, we conclude that the series in (B.4) diverges if and only if \(s_1 = s_2 = 1\).

We point out that, by Lemma 2.2, the class \(C^M([a, b])\) is a proper sub-class of \(G^{s_1 + \varepsilon}\), for every \(\varepsilon > 0\).