QUASI-FINITE REPRESENTATIONS, FREE FIELD REALIZATIONS, AND CHARACTER FORMULAE OF LIE SUPERALGEBRAS OF INFINITE RANK

NGAU LAM\textsuperscript{1} AND R. B. ZHANG\textsuperscript{2}

Abstract. We classify the quasi-finite irreducible highest weight modules over the infinite rank Lie superalgebras $\mathfrak{gl}\infty|\infty$, $\mathfrak{c}$ and $\mathfrak{d}$, and determine the necessary and sufficient conditions for quasi-finite irreducible highest weight modules to be unitarizable with respect to natural $\ast$-structures of the Lie superalgebras. The unitarizable irreducible modules are constructed in terms of Fock spaces of free quantum fields, and explicit formulae for their formal characters are also obtained by investigating Howe dualities between the infinite rank Lie superalgebras and classical Lie groups.

Key words: Infinite dimensional Lie superalgebras, quasi-finite representations, unitarizable representations, character formulae.

1. Introduction

Supersymmetry permeated many areas of mathematics in the last decade, producing deep results such as the Seiberg-Witten theory and mirror symmetry. In all applications, supersymmetry manifests itself as concrete representations of the relevant Lie superalgebras \cite{15}. Thus it is of central importance to develop the representation theory of Lie superalgebras in order to use supersymmetry as a tool to address problems in other areas.

In this paper we investigate the representation theory of the Lie superalgebra $\mathfrak{gl}\infty|\infty$ and its $osp$-type Lie sub superalgebras. These Lie superalgebras constitute a class of $1/2\mathbb{Z}$-graded infinite rank Lie superalgebras arising from central extensions of Lie superalgebras of complex matrices of infinite size. The Lie superalgebra $\mathfrak{gl}\infty|\infty$ and its $osp$-type Lie sub superalgebras featured very prominently in the study \cite{7} of the super $W_{1+\infty}$ algebra, i.e., the central extension of the superalgebra of differential operators on the super circle, which plays a fundamental role in conformal filed theory and the theory of superstrings. Also, it was demonstrated in \cite{3} that the representation theory of the infinite rank Lie superalgebras is intimately related to that of affine Kac-Moody superalgebras arising from central

\textsuperscript{1}Partially supported by NSC-grant 92-2115-M-006-016 of the R.O.C..
\textsuperscript{2}Supported by the Australian Research Council.

\textbf{2000 Mathematics Subject Classification:} 17B65, 17B10.
extensions of the loop algebras of finite dimensional simple Lie superalgebras. In this paper we shall focus on $\widehat{\mathfrak{gl}}_{\infty|\infty}$ and its subalgebras $\widehat{\mathfrak{c}}$ and $\widehat{\mathfrak{d}}$ (see Section 4 for their definitions). Aspects of a $\mathfrak{b}$ type subalgebra of $\widehat{\mathfrak{gl}}_{\infty|\infty}$ were studied in [7].

Recall that the infinite dimensional Lie algebra $\widehat{\mathfrak{gl}}_{\infty}$ and its various subalgebras were extensively studied in [17, 18, 24, 25] in relation to the $W_1^{\infty}$ algebra. In particular, the notion of quasi-finite modules [17] over infinite dimensional graded Lie (super)algebras were introduced. Such modules are close to finite dimensional representations of finite dimensional Lie (super)algebras in spirit. In our context, a $\frac{1}{2}\mathbb{Z}$-graded module $M = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} M_j$ over a $\frac{1}{2}\mathbb{Z}$-graded Lie superalgebra will be called quasi-finite if all its homogeneous subspaces $M_j$ are finite dimensional. One of our results in the present paper is the classification of all the quasi-finite irreducible highest weight modules over $\widehat{\mathfrak{gl}}_{\infty|\infty}$, $\widehat{\mathfrak{c}}$ and $\widehat{\mathfrak{d}}$.

It is well known that the energy of a quantum system is always bounded below. Also, the space of the physical states of the quantum system always admits a positive definite contravariant Hermitian form as required by the probabilistic interpretation of quantum theory. Therefore, the representations of Lie superalgebras, which are potentially useful in quantum physics, are the unitarizable highest weight (or lowest weight) representations. Another result of the present paper is the classification of the unitarizable quasi-finite irreducible highest weight modules over $\widehat{\mathfrak{gl}}_{\infty|\infty}$, $\widehat{\mathfrak{c}}$ and $\widehat{\mathfrak{d}}$ with respect to some natural $\mathbb{C}$-conjugate linear anti-involutions of these Lie superalgebras.

We analyse the unitarizable irreducible quasi-finite highest weight modules in some detail. The main results obtained are the following. We first realize these irreducible representations on Fock spaces of free quantum fields. We then prove generalized Howe dualities between the infinite rank Lie superalgebras and certain classical Lie groups. This way we are able to set up one to one correspondences between the unitarizable irreducible quasi-finite highest weight modules of the infinite rank Lie superalgebras and the finite dimensional irreducible representations of the associated classical groups. Finally we derive explicit formulae for the formal characters of the unitarizable quasi-finite irreducible highest weight modules over the Lie superalgebras $\widehat{\mathfrak{c}}$ and $\widehat{\mathfrak{d}}$. (We recall that the formal characters of the unitarizable irreducible modules over $\widehat{\mathfrak{gl}}_{\infty|\infty}$ were obtained in [3].)

The method used here for the construction of the character formulae is a generalization of that developed in [3, 8, 11], which relies in an essential way on Howe dualities [13, 14]. Howe dualities for Lie superalgebras were known in the original paper of Howe [13], and also in [22, 23], and were further investigated in [24, 25, 20, 21, 5, 6]. Recent investigations on Howe dualities led to a thorough understanding of the Segal-Shale-Weil representations of Lie superalgebras [8, 3], in particular, the construction of character formulae for them. In [12] and
Howe dualities were established respectively in the contexts of affine Kac-Moody algebras and infinite rank Lie algebras. The Howe dualities obtained in the present paper are generalizations of those studied by Wang in [24, 25].

The arrangement of the paper is as follows. Section 2 provides some background material on generalized partitions and unitarizable modules over Lie superalgebras. The material will be used throughout the paper. Section 3 examines central extensions of the Lie superalgebra $\mathfrak{gl}_{\infty|\infty}$ of infinite matrices and its $osp$-type subalgebras, and gives the definitions of the Lie superalgebra $\hat{\mathfrak{gl}}_{\infty|\infty}$, and its subalgebras $\hat{\mathfrak{A}}$, $\hat{\mathfrak{C}}$ and $\hat{\mathfrak{D}}$. The remaining three sections constitute the main body of the paper. In Section 4, we classify the quasi-finite irreducible highest weight modules over the Lie superalgebra $\hat{\mathfrak{gl}}_{\infty|\infty}$, and its subalgebras $\hat{\mathfrak{A}}$, $\hat{\mathfrak{C}}$ and $\hat{\mathfrak{D}}$. In Section 5, we classify the unitarizable quasi-finite irreducible highest weight modules over these Lie superalgebras with respect to specific $\ast$-structures, and construct Fock space realizations of the unitarizable irreducible modules. Generalized Howe dualities between the infinite rank Lie superalgebras and classical Lie groups will also be established in this section, which are used in Section 6 to derive character formulae for the unitarizable quasi-finite irreducible highest weight modules over $\hat{\mathfrak{C}}$ and $\hat{\mathfrak{D}}$.

2. Preliminaries

We work on the field $\mathbb{C}$ of complex numbers throughout the paper. For any vector space $V$, we shall denote its dual space by $V^*$.

2.1. Shifted Frobenius notation for generalized partitions. By a partition $\lambda$ of length $d$ we mean a non-increasing finite sequence of non-negative integers $(\lambda_1, \cdots, \lambda_d)$ and shall use $l(\lambda)$ to denote the length of $\lambda$. We will let $\lambda'$ denote the transpose of the partition $\lambda$. We define the rank of a partition $\lambda = (\lambda_1, \cdots, \lambda_d)$, denoted by $\text{rank}(\lambda)$, to be the largest integer $i$, for which $\lambda_i \geq i$. Note that $\text{rank}(\lambda) = \text{rank}(\lambda') \leq d$. For example, if $\lambda = (4, 3, 1, 0, 0)$, then $l(\lambda) = 5$, $\lambda' = (3, 2, 2, 1)$, and $\text{rank}(\lambda) = 2$. By a generalized partition of length $d$, we shall mean a non-increasing finite sequence of integers $(\lambda_1, \cdots, \lambda_d)$ and the length of $\lambda$ is also denoted by $l(\lambda)$. A generalized partition of $\lambda = (\lambda_1, \cdots, \lambda_d)$ is called a generalized partition of non-positive integers if $\lambda_i \leq 0$ for all $i$. Corresponding to each generalized partition $\lambda = (\lambda_1, \cdots, \lambda_d)$, we will define $\lambda^* := (-\lambda_d, \cdots, -\lambda_1)$. Then $\lambda^*$ is also a generalized partition. In particular, if $\lambda = (\lambda_1, \cdots, \lambda_d)$ is a generalized partition of non-positive integers, then $\lambda^* = (-\lambda_d, \cdots, -\lambda_1)$ is a partition. In this case, we define the rank, $\text{rank}(\lambda)$, of $\lambda$ by $\text{rank}(\lambda) := -\text{rank}(\lambda^*)$. We also set $\lambda'_j := -(\lambda^*)_j$ for all $j \in \{-1, -2, \cdots, \lambda_d\}$. 
Each generalized partition $\lambda = (\lambda_1, \cdots, \lambda_d)$ of length $d$ can be uniquely expressed as $\lambda = \lambda^+ + \lambda^-$, with
\[
\lambda^+ := (\max\{\lambda_1, 0\}, \cdots, \max\{\lambda_d, 0\}), \\
\lambda^- := (\min\{\lambda_1, 0\}, \cdots, \min\{\lambda_d, 0\}).
\]
Note that $\lambda^+$ is a partition of length $d$, while $\lambda^-$ is a generalized partition of non-positive integers of length $d$. Furthermore,
\[
(2.1) \quad \text{depth of } \lambda^+ + \text{depth of } (\lambda^-)^* \leq d,
\]
where the depth of a partition is the number of positive integers in it. (Note that the depth of a partition $\lambda$ equals $\lambda^1$.)

Now we will define the shifted Frobenius notation for generalized partitions (see [19]) which is very useful for describing the highest weights of unitarizable irreducible quasi-finite modules over $\hat{\mathfrak{gl}}_{\infty|\infty}$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ of length $d$ and $\text{rank}(\lambda) = r > 0$, we let $\xi_i := \lambda_i + \frac{1}{2} - i + \frac{1}{2}$ and $\xi_j := \lambda'_j - j$, for $i \in \frac{1}{2} + \mathbb{Z}_+$ with $\frac{1}{2} \leq i \leq r - \frac{1}{2}$, and $j \in \mathbb{N}$ with $1 \leq j \leq r$. We have
\[
(2.2) \quad \xi_1 > \xi_2 > \cdots > \xi_{r-\frac{1}{2}} > 0, \quad \xi_1 > \xi_2 > \cdots > \xi_r \geq 0.
\]
The shifted Frobenius notation for the partition $\lambda$ is given by
\[
\Lambda(\lambda) := (\xi_1, \xi_2, \cdots, \xi_{r-\frac{1}{2}} \mid \xi_1, \xi_2, \cdots, \xi_r).
\]
Clearly, we have
\[
(2.3) \quad \xi_1 + \min\{\xi_2, 1\} \leq d,
\]
When $\lambda = (0, 0, \cdots, 0)$, we define $\Lambda(0, 0, \cdots, 0) := (0, 0)$. Note that (2.2) and (2.3) implies $r \leq d$.

Conversely, if two finite sequences $\xi_1, \xi_2, \cdots, \xi_{r-\frac{1}{2}}$ and $\xi_1, \xi_2, \cdots, \xi_r$ of non-negative integers of length $d$ satisfy (2.2) and (2.3), we may regard them as the shifted Frobenius notation of a unique partition of length $d$, which we will denote by $F(\xi_1, \xi_2, \cdots, \xi_{r-\frac{1}{2}} \mid \xi_1, \xi_2, \cdots, \xi_r)$. We put $F(0, 0) := (0, 0, \cdots, 0)$. Thus we have a one-to-one correspondence between the set of all partitions of length $d$ and the set of all pairs of finite sequences of non-negative integers, $\xi_1, \xi_2, \cdots, \xi_{r-\frac{1}{2}}$ and $\xi_1, \xi_2, \cdots, \xi_r$ satisfying (2.2) and (2.3), and also the requirement that
\[
(2.4) \quad \xi_{r-\frac{1}{2}} = 0 \text{ if and only if } r = 1 \text{ and } \xi_{\frac{1}{2}} = \xi_1 = 0.
\]

Similarly, given any non-zero generalized partition $\lambda = (\lambda_1, \ldots, \lambda_d)$ of non-positive integers with $\text{rank}(\lambda) = s$, the shifted Frobenius notation for the generalized partition $\lambda$ of non-positive integers, also denoted by $\Lambda(\lambda)$, is defined by
\[
\Lambda(\lambda) := (\xi_{s+\frac{1}{2}}, \xi_{s+\frac{3}{2}}, \cdots, \xi_{-\frac{1}{2}} \mid \xi_{s+1}, \xi_{s+2}, \cdots, \xi_0)
\]
where \( \xi_i := \lambda_{i-1} - i \) and \( \xi_j := \lambda_{d + \frac{1}{2} + j} - j + \frac{1}{2} \) for all \( i \in \{0, -1, -2, \ldots, s+1\} \) and \( j \in \{-\frac{1}{2}, -1\frac{1}{2}, \ldots, s + \frac{1}{2}\} \). We also define \( \Lambda(0, 0, \cdots, 0) := (0, 0) \). Similarly we have a one-to-one correspondence between the set of all generalized partitions of non-positive integers of length \( d \) and the set of all pairs of finite sequences of non-positive integers, \( \xi_{s+\frac{1}{2}}, \xi_{s+\frac{3}{2}}, \cdots, \xi_{-\frac{1}{2}} \) and \( \xi_{s+1}, \xi_{s+2}, \cdots, \xi_0 \), satisfying the following conditions

\[
0 \geq \xi_{s+\frac{1}{2}} > \xi_{s+\frac{3}{2}} > \cdots > \xi_{-\frac{1}{2}}, \quad 0 \geq \xi_{s+1} > \xi_{s+2} > \cdots > \xi_0,
\]

\( \xi_{s+1} = 0 \) if and only if \( s = -1 \) and \( \xi_{-\frac{1}{2}} = \xi_0 = 0 \), and

\[
\xi_0 \leq d.
\]

Now we define the shifted Frobenius notation for the generalized partitions as follows. For a nonzero generalized partition \( \lambda \) of length \( d \), the shifted Frobenius notation for the generalized partition \( \lambda \), also denoted by \( \Lambda(\lambda) \), is defined by \( \Lambda(\lambda) := (\Lambda(\lambda^-)|\Lambda(\lambda^+)) \).

Similarly we have a one-to-one correspondence between the set of all generalized partitions of length \( d \) and the set of all quartets of finite sequences of integers, \( \xi_{s+\frac{1}{2}}, \xi_{s+\frac{3}{2}}, \cdots, \xi_{-\frac{1}{2}}; \xi_{s+1}, \xi_{s+2}, \cdots, \xi_0; \xi_{\frac{1}{2}}, \xi_{\frac{3}{2}}, \cdots, \xi_{-\frac{3}{2}} \) and \( \xi_1, \xi_2, \cdots, \xi_r \) satisfying (2.7) and

\[
\min\{\xi_{\frac{1}{2}}, 1\} + \xi_1 - \xi_0 \leq d.
\]

We will denote by

\[
F(\xi_{s+\frac{1}{2}}, \xi_{s+\frac{3}{2}}, \cdots, \xi_{-\frac{1}{2}}; \xi_{s+1}, \xi_{s+2}, \cdots, \xi_0; \xi_{\frac{1}{2}}, \xi_{\frac{3}{2}}, \cdots, \xi_{-\frac{3}{2}}; \xi_1, \xi_2, \cdots, \xi_r),
\]

the unique generalized partition corresponding to the quartet of finite sequences of integers \( \xi_{s+\frac{1}{2}}, \xi_{s+\frac{3}{2}}, \cdots, \xi_{-\frac{1}{2}}; \xi_{s+1}, \xi_{s+2}, \cdots, \xi_0; \xi_{\frac{1}{2}}, \xi_{\frac{3}{2}}, \cdots, \xi_{-\frac{3}{2}} \) and \( \xi_1, \xi_2, \cdots, \xi_r \) satisfying (2.7)

\[
(2.2), \ (2.4), \ (2.6) \quad \text{and} \quad (2.7).
\]

2.2. Unitarizable modules. Let us recall some basic facts about *-superalgebras and their unitarizable modules. A *-superalgebra is an associative superalgebra \( A \) together with an anti-linear anti-involution \( \omega : A \to A \). Here we should emphasize that for any \( a, b \in A \), we have \( \omega(ab) = \omega(b)\omega(a) \), where no sign factors are involved. A *-superalgebra homomorphism \( f : (A, \omega) \to (A', \omega') \) is a superalgebra homomorphism obeying \( f \circ \omega = \omega' \circ f \). Let \( (A, \omega) \) be a *-superalgebra, and let \( M \) be a \( \mathbb{Z}_2 \)-graded \( A \)-module. A Hermitian form \( \langle \cdot \mid \cdot \rangle \) on \( M \) is said to be contravariant if \( \langle av | v' \rangle = \langle v | \omega(a)v' \rangle \), for all \( a \in A, v, v' \in M \). An \( A \)-module \( M \) is called unitarizable if \( M \) admits a positive definite contravariant Hermitian form.

Let \( g \) be a Lie superalgebra together with an anti-linear anti-involution \( \omega \) (i.e. \( \omega \) is an anti-linear map satisfying \( \omega([x, y]) = [\omega(y), \omega(x)] \) for all \( x, y \in g \)). In this
case, we also call $\omega$ a $*$-structure of $\mathfrak{g}$). Let $M$ be a $\mathfrak{g}$-module. A Hermitian form $\langle \cdot | \cdot \rangle$ on $M$ is said to be contravariant if $\langle xv|v' \rangle = \langle v|\omega(x)v' \rangle$, for all $x \in \mathfrak{g}$, $v,v' \in M$. When the Hermitian form $\langle \cdot | \cdot \rangle$ is positive definite, we define $\|u\| := \sqrt{\langle u|u \rangle}$ for all $u \in M$. A $\mathfrak{g}$-module $M$ is called unitarizable if $M$ admits a positive definite contravariant Hermitian form. The anti-linear anti-involution $\omega$ can be naturally extended to an anti-linear anti-involution, also denoted by $\omega$, on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$, making $(\mathcal{U}(\mathfrak{g}),\omega)$ a $*$-superalgebra. Moreover, a $\mathfrak{g}$-module $M$ is unitarizable if and only if it is a unitarizable $\mathcal{U}(\mathfrak{g})$-module. (Throughout the paper, $\mathcal{U}(\mathfrak{s})$ stands for the universal enveloping algebra of the Lie superalgebra $\mathfrak{s}$.)

3. Lie superalgebras of infinite rank

We present here the infinite rank Lie superalgebras to be studied in this paper.

Consider the infinite-dimensional complex superspace $\mathbb{C}^{\infty|\infty}$ with a basis $\{e_j | j \in \mathbb{Z}\}$ for the even subspace, and a basis $\{e_r | r \in \frac{1}{2} + \mathbb{Z}\}$ for the odd subspace. We introduce a $\frac{1}{2}\mathbb{Z}$-gradation on $\mathbb{C}^{\infty|\infty}$ by setting the degree of $e_p$ equal to $-p$ for all $p \in \frac{1}{2}\mathbb{Z}$. For any $p,q \in \frac{1}{2}\mathbb{Z}$, let $e_{pq}$ be the endomorphism of $\mathbb{C}^{\infty|\infty}$ defined by $e_{pq}(e_r) = \delta_{qr} e_p$. Then $T$ is a homogeneous endomorphism on $\mathbb{C}^{\infty|\infty}$ of degree $p$ if and only if $T = \sum_{j \in \frac{1}{2}\mathbb{Z}} a_j e_{j-p,j}$, where $a_j \in \mathbb{C}$. Denote by $(M_{\infty|\infty})_p$ the set of all endomorphisms of $\mathbb{C}^{\infty|\infty}$ of degree $p$, and let $M_{\infty|\infty} := \oplus_{p \in \frac{1}{2}\mathbb{Z}} (M_{\infty|\infty})_p$.

Then $M_{\infty|\infty}$ is a $\frac{1}{2}\mathbb{Z}$-graded associative superalgebra, which also acquires a Lie superalgebra structure with the usual Lie super-bracket

\[
[A, B] := AB - (-1)^{\text{deg}(A)\text{deg}(B)} BA,
\]

where $\text{deg}(A)$ and $\text{deg}(B)$ are the degrees of $A$ and $B$ respectively. We shall denote this Lie superalgebra by $\text{gl}_{\infty|\infty} := \oplus_{p \in \frac{1}{2}\mathbb{Z}} (\text{gl}_{\infty|\infty})_p$. Note that the subspace $\text{gl}_{\infty|\infty}^f$ generated by $\{e_{ij} | i,j \in \frac{1}{2}\mathbb{Z}\}$ is a subalgebra of $\text{gl}_{\infty|\infty}$. By arranging the basis elements of $\mathbb{C}^{\infty|\infty}$ in strictly increasing order, any endomorphism of $\mathbb{C}^{\infty|\infty}$ may be written as an infinite-sized square matrix with coefficients in $\mathbb{C}$. Thus

\[
\text{gl}_{\infty|\infty} := \{(a_{ij})_i \in \frac{1}{2}\mathbb{Z} | a_{ij} = 0 \text{ for } |j - i| >> 0\}.
\]

The Lie superalgebra $\text{gl}_{\infty|\infty}$ has a central extension by a non-trivial two co-cycle. Let $J = \sum_{r \leq 0} e_{rr}$. Define

\[
\alpha(A, B) := \text{Str}([J, A]B), \quad A, B \in \text{gl}_{\infty|\infty},
\]

where $\text{Str}$ stands for the supertrace defined for a matrix $D = (d_{ij}) \in \text{gl}_{\infty|\infty}$ by $\text{Str}(D) = \sum_{r \in \frac{1}{2}\mathbb{Z}} (-1)^{2r} d_{rr}$, provided that the infinite sum is not divergent. Then $\alpha(A, B)$ is well behaved for all $A, B \in \text{gl}_{\infty|\infty}$, and indeed defines a two co-cycle.
We denote by \( \hat{gl}_{\infty|\infty} \) the central extension of \( gl_{\infty|\infty} \) by the even central element \( C \) associated with this two co-cycle. By setting the degree of \( C \) equal to 0, the Lie superalgebra \( \hat{gl}_{\infty|\infty} \) acquires a \( \frac{1}{2}\mathbb{Z} \)-gradation from that of \( gl_{\infty|\infty} \). Let

\[
\hat{gl}^f_{\infty|\infty} := \{ (a_{ij}) \in \hat{gl}_{\infty|\infty} \text{ finitely many of the } a_{ij} \text{ are non-zero } \} \oplus \mathbb{C} C.
\]

It is easy to see that \( \hat{gl}^f_{\infty|\infty} \) is a \( \frac{1}{2}\mathbb{Z} \)-graded subalgebra of \( \hat{gl}_{\infty|\infty} \).

Let us now introduce the Lie sub superalgebra \( A \) of \( gl_{\infty|\infty} \) defined by

\[
A := \{ (a_{ij}) \in gl_{\infty|\infty} \text{ } a_{ij} = 0 \text{ if } i = 0 \text{ or } j = 0 \}.
\]

It also admits a central extension by an even central element \( C \) associated with the two co-cycle \( \mathbb{R}^2 \). We shall denote the central extension of \( A \) by \( \hat{A} \). Then the Lie superalgebra \( \hat{A} \) also acquires a \( \frac{1}{2}\mathbb{Z} \)-gradation from that of \( \hat{gl}_{\infty|\infty} \) by declaring \( C \) to have degree 0.

An alternative way to describe the Lie superalgebra \( A \) is as follows. Consider the infinite-dimensional complex superspace \( C^\infty|\infty \) with even basis \( \{ e_j \mid j \in \mathbb{Z}^* \} \) and odd basis \( \{ e_r \mid r \in \frac{1}{2} + \mathbb{Z} \} \). Then \( A \) is the Lie superalgebra of graded endomorphisms of \( C^\infty|\infty \).

Let us now construct a Lie sub superalgebra \( \mathcal{C} \) of \( A \). Introduce a non-degenerate skew-supersymmetric bilinear form \( (\cdot|\cdot) \) on \( C^\infty|\infty \) defined by

\[
(e_i|e_j) = -(e_j|e_i) = \text{sgn}(i)\delta_{i,-j}, \quad i, j \in \mathbb{Z}^*;
\]

\[
(e_r|e_s) = (e_s|e_r) = \delta_{r,-s}, \quad r, s \in \frac{1}{2} + \mathbb{Z};
\]

\[
(e_i|e_r) = (e_r|e_i) = 0, \quad i \in \mathbb{Z}^*, r \in \frac{1}{2} + \mathbb{Z};
\]

where \( \text{sgn}(i) := +1 \) if \( i \in \frac{1}{2}\mathbb{N} \) and \( \text{sgn}(i) := -1 \) if \( i \in -\frac{1}{2}\mathbb{N} \). We define the Lie superalgebra \( \mathcal{C} = C_0 \oplus \hat{C}_1 \) to be the \( \frac{1}{2}\mathbb{Z} \)-graded Lie sub superalgebra of \( A \) preserving this form, i.e.

\[
\mathcal{C}_\epsilon = \{ A \in \mathcal{A}_\epsilon | (Av|w) = -(-1)^{\epsilon|v}(v|Aw) \}, \quad \epsilon = 0, 1,
\]

where \( |v| \) denotes the parity of \( v \in C^\infty|\infty \), namely, \( |v| = 0 \) (respectively 1) if \( v \) belongs to the even (respectively odd) homogeneous subspace of \( C^\infty|\infty \). Then \( \mathcal{C} \) is a Lie superalgebra of type \( SPO \). It is easy to see that the subspace \( \mathcal{C}^f \) spanned by the following elements is a subalgebra of \( \mathcal{C} \) \( (i, j \in \mathbb{Z}^*, r, s \in \frac{1}{2} + \mathbb{Z}) \):

\[
\hat{e}_{i,j} := -\hat{e}_{-j,-i} := e_{i,j} - e_{-j,-i}, \quad ij > 0 \text{ (i.e., } i, j > 0 \text{ or } i, j < 0);\]

\[
\hat{e}_{i,j} := \hat{e}_{-j,-i} := e_{i,j} + e_{-j,-i}, \quad ij < 0 \text{ (i.e., } i, j > 0 \text{ or } i, j < 0);\]

\[
\hat{e}_{r,s} := -\hat{e}_{-s,-r} := e_{r,s} - e_{-s,-r};\]

\[
\hat{e}_{i,r} := \hat{e}_{-r,-i} := e_{i,r} + e_{-r,-i}, \quad i > 0;\]

\[
\hat{e}_{i,r} := -\hat{e}_{-r,-i} := e_{i,r} - e_{-r,-i}, \quad i < 0.
\]
Note that $\mathcal{C}_0^f$ is a direct sum of an infinite dimensional symplectic Lie algebra and an infinite dimensional orthogonal Lie algebra. Let $\mathcal{C}$ denote the central extension of $\mathcal{C}$ by an even central element $C$ associated with the two-cocycle $(\ref{eq:2})$. By setting the degree of $C$ to zero, $\mathcal{C}$ becomes a $\mathbb{Z}_2$-graded Lie superalgebra, with the gradation compatible with that of $\mathcal{A}$.

Consider the non-degenerate supersymmetric bilinear form $(\cdot|\cdot)$ on $\mathbb{C}\infty|\infty$ defined by

\[
(e_i|e_j) = (e_j|e_i) = \delta_{i,-j}, \quad i, j \in \mathbb{Z}^*;
\]
\[
(e_r|e_s) = -(e_s|e_r) = \text{sgn}(r)\delta_{r,-s}, \quad r, s \in \mathbb{Z};
\]
\[
(e_i|e_r) = (e_r|e_i) = 0, \quad i \in \mathbb{Z}^*, r \in \mathbb{Z}.
\]

We define the Lie superalgebra $\mathcal{D} = \mathcal{D}_0 \oplus \mathcal{D}_1$ to be the subalgebra of $\mathcal{A}$ preserving this form, i.e.

\[
\mathcal{D}_\epsilon = \{A \in \mathcal{A}_\epsilon| (Av|w) = -(\epsilon|v)(Aw)\}, \quad \epsilon = 0, 1.
\]

This is a Lie superalgebra of type $OSP$. It is easy to see that the subspace $\mathcal{D}^f$ spanned by the following elements is a subalgebra of $\mathcal{D}$ $(i, j \in \mathbb{Z}^*, r, s \in \mathbb{Z})$:

\[
\tilde{e}_{i,j} := -\tilde{e}_{-j,-i} := e_{i,j} - e_{-j,-i};
\]
\[
\tilde{e}_{r,s} := -\tilde{e}_{-s,-r} := e_{r,s} - e_{-s,-r}, \quad rs > 0 \text{ (i.e., } r, s > 0 \text{ or } r, s < 0);\]
\[
\tilde{e}_{r,s} := \tilde{e}_{-s,-r} := e_{r,s} + e_{-s,-r}, \quad rs < 0 \text{ (i.e., } r, -s > 0 \text{ or } -r, s < 0);\]
\[
\tilde{e}_{i,r} := -\tilde{e}_{-r,-i} := e_{i,r} + e_{-r,-i}, \quad r > 0;
\]
\[
\tilde{e}_{i,r} := -\tilde{e}_{-r,-i} := e_{i,r} - e_{-r,-i}, \quad r < 0.
\]

Note that $\mathcal{D}_0^f$ is a direct sum of an infinite dimensional symplectic Lie algebra and an infinite dimensional orthogonal Lie algebra. The Lie superalgebra $\mathcal{D}$ has a central extension by an even central element $C$ associated with the two-cocycle given in $(\ref{eq:2})$. We shall denote this central extension by $\hat{\mathcal{D}}$. The Lie superalgebra $\hat{\mathcal{D}}$ also has a $\mathbb{Z}_2$-graduation compatible with that of $\hat{\mathcal{A}}$ with $C$ being of degree zero.

Remark 3.1. Both $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$ are $\mathbb{Z}_2$-graded Lie sub superalgebras of $\hat{\mathcal{A}}$. Thus the triangular decomposition $\hat{\mathcal{A}} = \hat{\mathcal{A}}_+ \oplus \hat{\mathcal{A}}_0 \oplus \hat{\mathcal{A}}_-$ of $\hat{\mathcal{A}}$ leads to natural triangular decompositions of $\hat{\mathcal{C}}$ and $\hat{\mathcal{D}}$:

\[
\hat{\mathcal{C}} = \hat{\mathcal{C}}_+ \oplus \hat{\mathcal{C}}_0 \oplus \hat{\mathcal{C}}_-, \quad \hat{\mathcal{D}} = \hat{\mathcal{D}}_+ \oplus \hat{\mathcal{D}}_0 \oplus \hat{\mathcal{D}}_-,
\]

where $\hat{\mathcal{C}}_\varpi = \hat{\mathcal{C}} \cap \hat{\mathcal{A}}_\varpi$, $\hat{\mathcal{D}}_\varpi = \hat{\mathcal{D}} \cap \hat{\mathcal{A}}$, for $\varpi$ being $+$, $-$ and $0$. This will be of considerable importance when we develop the representation theory of these Lie superalgebras.
Remark 3.2. For \( \hat{\mathfrak{g}} \) being \( \hat{\mathfrak{A}}, \hat{\mathfrak{C}} \) or \( \hat{\mathfrak{D}} \), we shall use the notation \( \hat{\mathfrak{g}}^{f} \) to denote the \( \frac{1}{2}\mathbb{Z} \)-graded subalgebra \( \hat{\mathfrak{g}} \cap \hat{\mathfrak{g}}^{f}_{\infty|\infty} \).

4. Criterion for quasi-finiteness of modules

In this section we give a complete classification of all the quasi-finite irreducible highest weight modules over the Lie superalgebras discussed in Section 3.

4.1. Quasi-finite modules. Let \( \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_{j} \) (possibly \( \dim \mathfrak{g}_{j} = \infty \)) be a \( \frac{1}{2}\mathbb{Z} \)-graded Lie superalgebra, with the even subspace \( \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{j} \), and odd subspace \( \bigoplus_{j \in \frac{1}{2}+\mathbb{Z}} \mathfrak{g}_{j} \). We assume that \( \mathfrak{g}_{0} \) is abelian. We have the triangular decomposition

\[
\mathfrak{g} = \mathfrak{g}_{-} \oplus \mathfrak{g}_{0} \oplus \mathfrak{g}_{+}, \quad \text{with} \quad \mathfrak{g}_{\pm} = \bigoplus_{r>0} \mathfrak{g}_{r\pm}.
\]

A \( \mathfrak{g} \)-module \( M = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} M_{j} \) is graded if \( \mathfrak{g}_{j} M_{j} \subseteq M_{i+j} \). A vector \( v \in M \) is called homogeneous of degree \( j \) if \( v \in M_{j} \) for some \( j \in \frac{1}{2}\mathbb{Z} \). Following the terminology of Kac and Radul [17], we shall call \( M \) quasi-finite if \( \dim M_{j} < \infty \) for all \( j \in \frac{1}{2}\mathbb{Z} \).

A \( \mathfrak{g} \)-module \( M \) is called a highest weight module with highest weight \( \xi \in \mathfrak{g}_{0}^{*} \) if there is a nonzero vector \( v_{\xi} \in M \) satisfying the following conditions:

1. \( hv_{\xi} = \xi(h)v_{\xi} \), for all \( h \in \mathfrak{g}_{0} \),
2. \( \mathfrak{g}_{+} v_{\xi} = 0 \),
3. \( \mathfrak{U}(\mathfrak{g}_{-}) v_{\xi} = M \).

Then \( v_{\xi} \) is called a highest weight vector of \( M \). Note that by declaring the highest weight vector of the highest weight module \( M \) to be of degree zero, the module \( M \) is naturally \( \frac{1}{2}\mathbb{Z} \)-graded. More precisely, we have \( M = \bigoplus_{r \in \frac{1}{2}\mathbb{Z}} M_{-r} \) and \( M_{0} = \mathbb{C}v_{\xi} \).

A homogeneous nonzero vector \( v \) in the highest weight module \( M \) is said to be singular if \( \mathfrak{g}_{+} v = 0 \). A highest weight module is irreducible if and only if the space of singular vectors is 1-dimensional.

Associated with every \( \xi \in \mathfrak{g}_{0}^{*} \), there is a Verma module

\[
M(\mathfrak{g}, \xi) := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{g}_{0} \oplus \mathfrak{g}_{+})} \mathbb{C}v_{\xi},
\]

where \( \mathbb{C}v_{\xi} \) is regarded as a \( \mathfrak{U}(\mathfrak{g}_{0} \oplus \mathfrak{g}_{+}) \)-module by setting \( hv_{\xi} = \xi(h)v_{\xi} \) for all \( h \in \mathfrak{g}_{0} \) and \( \mathfrak{g}_{+} v_{\xi} = 0 \). Note that \( M(\mathfrak{g}, \xi) \) is a highest weight module and for every highest weight module \( M \) of highest weight \( \xi \), there is a natural epimorphism \( \varphi \) from \( M(\mathfrak{g}, \xi) \) onto \( M \) determined by \( \varphi(v_{\xi}) = u_{\xi} \), where \( u_{\xi} \) is a highest weight vector in \( M \). Thus every highest weight module \( M \) of highest weight \( \xi \) is a quotient of \( M(\mathfrak{g}, \xi) \). Moreover, \( M(\mathfrak{g}, \xi) \) contains a unique maximal proper submodule which is also graded. Hence, for any \( \xi \in \mathfrak{g}_{0}^{*} \), there is the unique irreducible highest weight module, denoted by \( L(\mathfrak{g}, \xi) \), which is isomorphic to the quotient of \( M(\mathfrak{g}, \xi) \) by the maximal proper graded submodule.

We recall the following criterion for quasi-finite highest weight modules.
Proposition 4.1. Let \( \mathfrak{g} = \oplus_{j \in \frac{1}{2} \mathbb{Z}} g_j \) be a \( \frac{1}{2} \mathbb{Z} \)-graded Lie superalgebra such that \( g_0 \) is abelian. Let \( M = \oplus_{j \in \frac{1}{2} \mathbb{Z}} M_j \) be a highest weight \( \mathfrak{g} \)-module with highest weight \( \xi \in g_0^* \). For any non-zero highest weight vector \( v_\xi \) in \( M \), the subspace \( g_j v_\xi \) is finite-dimensional for all \( j \) if and only if \( M \) is quasi-finite.

Let \( M \) be a \( \mathfrak{g} \)-module. For any \( \lambda \in g_0^* \), set

\[
M_\lambda = \{ v \in M \mid hv = \lambda(h)v, \text{ for all } h \in g_0 \}.
\]

When \( M_\lambda \neq 0 \), \( \lambda \) is called a weight of \( M \), and \( M_\lambda \) is called the weight space of weight \( \lambda \). We let \( P(M) \) denote the set of all weights of \( M \). A graded \( \mathfrak{g} \)-module \( M = \oplus_{j \in \frac{1}{2} \mathbb{Z}} M_j \) is called \( g_0 \)-diagonalizable if \( M \) satisfies the following conditions:

(i) \( M_\lambda \) is finite dimensional,

(ii) \( M_j = \oplus_{\lambda \in P(M)} (M_\lambda \cap M_j) \), for all \( j \in \frac{1}{2} \mathbb{Z} \).

For any \( \lambda \in g_0^* \), we also set

\[
g_\lambda = \{ x \in g \mid [h, x] = \lambda(h)x, \text{ for all } h \in g_0 \}.
\]

As all the infinite rank Lie superalgebras in Section 3 are \( \frac{1}{2} \mathbb{Z} \)-graded, the representation theoretical notions discussed above are all valid for them. The following easy lemma is also useful for the purpose of studying their representation theory.

Lemma 4.1. Let \( \pi \) be any transcendental real number over the field of rational numbers. For any integers \( j_1 < j_2 < \cdots < j_n \), we let \( v_i := (\pi^{j_1}, \pi^{j_2}, \cdots, \pi^{j_n}) \), for \( i = 1, 2, \cdots, n \). Then \( v_1, v_2, \cdots, v_n \) are linearly independent in \( \mathbb{C}^n \).

Proof. Let

\[
f(x) = \det \begin{pmatrix} x^{j_1} & x^{j_2} & \cdots & x^{j_n} \\ x^{2j_1} & x^{2j_2} & \cdots & x^{2j_n} \\ \vdots & \vdots & \cdots & \vdots \\ x^{nj_1} & x^{nj_2} & \cdots & x^{nj_n} \end{pmatrix}.
\]

Then \( f(x) \) is a nonzero Laurent polynomial with integral coefficients. Therefore \( f(\pi) \neq 0 \) and this implies the lemma.

4.2. Quasi-finite \( \widehat{g}_{\infty|\infty} \)-modules. For any \( k \in \frac{1}{2} \mathbb{Z} \) and \( N \in \frac{1}{2} \mathbb{Z}^+ \), we let

\[
(\widehat{g}_{\infty|\infty})_{k,N} := \{ x \in \widehat{g}_{\infty|\infty} \mid x = \sum_{\substack{|j| \geq N, \\ j \in \frac{1}{2} \mathbb{Z} \\ j \neq 0}} a_j e_{j,k,j}, \ a_j \in \mathbb{C} \}.
\]

The following lemma can be confirmed by a straightforward computation.

Lemma 4.2. Given any fixed positive integer or half integer \( N \), we have

\[
[\widehat{g}_{\infty|\infty}]_p \cap (\widehat{g}_{\infty|\infty})_{-k,k+N} \subset (\widehat{g}_{\infty|\infty})_{-(k-p),(k-p)+N},
\]

for all \( k, p \in \frac{1}{2} \mathbb{Z}^+ \) with \( p \leq k \).
Proposition 4.2. Let \( M = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}^+} M_{-j} \) be a highest weight \( \hat{\mathfrak{gl}}_{\infty|\infty} \)-module and \( v_0 \) a non-zero highest weight vector. If \( \hat{\mathfrak{gl}}_{\infty|\infty}^{-r}v_0 \) is finite dimensional for a fixed number \( r \in \frac{1}{2}\mathbb{N} \), then for every \( p \in \frac{1}{2}\mathbb{Z}^+ \) with \( p < r \), there exists \( N \in \mathbb{N} \) such that
\[
(\hat{\mathfrak{gl}}_{\infty|\infty})^{-p,N}v_0 = 0.
\]
In particular, \( (\hat{\mathfrak{gl}}_{\infty|\infty})^{-p}v_0 \) is finite dimensional for all \( p \leq r \).

Proof. Fixing a transcendental real number \( \pi \), we let \( w_i = \sum_{j \in \frac{1}{2} \mathbb{Z}} \pi^{2ij}e_{j+r,j} \) for each \( i \in \mathbb{N} \), which belong to \( (\hat{\mathfrak{gl}}_{\infty|\infty})^{-r} \). For any \( x = \sum_{i=1}^k \alpha_i w_i \), where \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are nonzero complex numbers with \( i_1 < i_2 < \cdots < i_k \), we can re-write it as \( x = \sum_{j \in \frac{1}{2} \mathbb{Z}} \beta_j e_{j+r,j} \). Then by applying Lemma 4.1 we easily show that the \( \beta_j \) are nonzero except for finitely many \( j \). Thus there always exists some positive integer \( N \) with \( N > r \) such that \( \beta_j \neq 0 \) for all \( j \) with \( |j| \geq N \). Since \( (\hat{\mathfrak{gl}}_{\infty|\infty})^{-r}v_0 \) is finite dimensional, we can always find nonzero complex numbers \( \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k} \) so that \( x = \sum_{i=1}^k \alpha_i w_i \) satisfies \( xv_0 = 0 \). We fix such an \( x \).

We shall prove the proposition by contradiction. Assume that there exists \( p \in \frac{1}{2}\mathbb{Z}^+ \) with \( p < r \) such that \( (\hat{\mathfrak{gl}}_{\infty|\infty})^{-p,q}v_0 \neq 0 \), for all \( q \in \mathbb{N} \). Then we can find \( y := \sum_{|j| \geq N} a_j e_{j+p,j} \in (\hat{\mathfrak{gl}}_{\infty|\infty})^{-p,N} \) such that \( yv_0 \neq 0 \). We claim that corresponding to each such \( y \), there exits a \( u = \sum_{j \in \frac{1}{2} \mathbb{Z}} b_j e_{j+p,j+r} \in (\hat{\mathfrak{gl}}_{\infty|\infty})^{-r} \) such that
\[
(4.1) \quad [u, x] = y.
\]
Indeed if we choose an element \( u \) with the coefficients \( b_j, -N - 2r < j < N + r \), given by
\[
b_j := \begin{cases} 
0, & \text{if } -N - r < j < N; \\
\frac{a_j}{\beta_j}, & \text{if } N \leq j < N + r; \\
\frac{(-1)^{4r(r-p)}a_j e_{j+p,r}}{\beta_j}, & \text{if } -N - 2r < j \leq -N - r;
\end{cases}
\]
and the \( b_j \) for \( j \leq -N - 2r \) or \( j \geq N + r \) given recursively by
\[
b_j := \begin{cases} 
\frac{a_j + (-1)^{4r(r-p)}\beta_{j+p-r} b_{j+r}}{\beta_j}, & \text{if } j \geq N + r; \\
\frac{(-1)^{4r(r-p)}(b_{j+r}\beta_{j+r}-a_{j+r})}{\beta_{j+p}}, & \text{if } j \leq -N - 2r,
\end{cases}
\]
then (4.1) holds true as can be shown by a direct computation. However, equation (4.1) leads to the obvious contradiction \( yv_0 = [u, x]v_0 = 0 \). This completes the proof. □
The following theorem is an obvious consequence of Proposition 4.1 and Proposition 4.2.

**Theorem 4.1.** Let $M = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_+} M_{-j}$ be a highest weight $\hat{\mathfrak{gl}}_{\infty|\infty}$-module with highest weight $\xi$ and $v_0$ a non-zero highest weight vector. Then $M$ is quasi-finite if and only if for every $r \in \frac{1}{2}\mathbb{Z}_+$, there exists $N \in \mathbb{N}$ such that $$(\hat{\mathfrak{gl}}_{\infty|\infty})_{-r,N}v_0 = 0.$$ In this case, $M = \mathcal{U}(\hat{\mathfrak{gl}}_{\infty|\infty})^f v_0$ and hence is $(\hat{\mathfrak{gl}}_{\infty|\infty})_0$-diagonalizable.

Denote by $\Lambda_0, \omega_s$ ($s \in \frac{1}{2}\mathbb{Z}$), the fundamental weights of $\hat{\mathfrak{gl}}_{\infty|\infty}$, where $\Lambda_0, \omega_s \in (\hat{\mathfrak{gl}}_{\infty|\infty})^*_0$, are defined by

$$\omega_s(\sum_{r \in \frac{1}{2}\mathbb{Z}} a_re_{rr} + dC) = a_s, \quad \Lambda_0(\sum_{r \in \frac{1}{2}\mathbb{Z}} a_re_{rr} + dC) = d,$$

for all $\sum_{r \in \frac{1}{2}\mathbb{Z}} a_re_{rr} + dC \in (\hat{\mathfrak{gl}}_{\infty|\infty})_0$.

**Theorem 4.2.** Let $M$ be an irreducible highest weight $\hat{\mathfrak{gl}}_{\infty|\infty}$-module and $v_\xi$ a non-zero highest weight vector. Then the following are equivalent:

(i) $M$ is quasi-finite,
(ii) $(\hat{\mathfrak{gl}}_{\infty|\infty})_{-\frac{1}{2}v_\xi}$ is finite dimensional,
(iii) there exists $N \in \mathbb{N}$ such that $\xi = \sum_{j \in \frac{1}{2}\mathbb{Z}} \xi_j \omega_j + d\Lambda_0$, where $\xi_j, d \in \mathbb{C}$.

**Proof.** It clearly follows from Proposition 4.2 that (i) implies (ii) and (ii) implies (iii). Now we show that (iii) implies (i). Assume that $\xi = \sum_{j \in \frac{1}{2}\mathbb{Z}} \xi_j \omega_j + d\Lambda_0$.

Then by Theorem 4.1, it is sufficient to show that for all $r \in \frac{1}{2}\mathbb{Z}_+$, there exists $N \in \mathbb{N}$ such that $(\hat{\mathfrak{gl}}_{\infty|\infty})_{-r,N}v_\xi = 0$. We shall prove it by induction. It is obviously true for $r = 0$ and we assume that it is also true for all $p \in \frac{1}{2}\mathbb{N}$ with $0 \leq p < r$. Choose $N_p \in \mathbb{N}$ such that $(\hat{\mathfrak{gl}}_{\infty|\infty})_{-p,N_p}v_\xi = 0$. Let $N = \max\{N_0 + 1, N_{1/2}, \ldots, N_{r-1/2}\}$. For all $p \in \frac{1}{2}\mathbb{N}$ with $p > r$, it is clear that $(\hat{\mathfrak{gl}}_{\infty|\infty})_p(\hat{\mathfrak{gl}}_{\infty|\infty})_{-r,N}v_\xi \subseteq (\hat{\mathfrak{gl}}_{\infty|\infty})_{p-r,N}v_\xi = 0$. By Lemma 4.2 we also have

$$(\hat{\mathfrak{gl}}_{\infty|\infty})_p(\hat{\mathfrak{gl}}_{\infty|\infty})_{-r,N}v_\xi \subseteq [(\hat{\mathfrak{gl}}_{\infty|\infty})_{p}, (\hat{\mathfrak{gl}}_{\infty|\infty})_{-r,N}]v_\xi$$

$$\subseteq (\hat{\mathfrak{gl}}_{\infty|\infty})_{-(r-p),N}v_\xi$$

$$= 0,$$
for all $p \in \frac{1}{2}\mathbb{N}$ with $0 < p \leq r$. Thus $(\widehat{g}_{\infty|\infty})_+(\widehat{g}_{\infty|\infty})_{-r,r+N}v_\xi = 0$. Similarly using Lemma 4.2 again,

$$(\widehat{g}_{\infty|\infty})_0(\widehat{g}_{\infty|\infty})_{-r,r+N}v_\xi$$

$$\subseteq (\widehat{g}_{\infty|\infty})_{-r,r+N}v_\xi + (\widehat{g}_{\infty|\infty})_{-r,r+N}v_\xi$$

Therefore $U((\widehat{g}_{\infty|\infty})_-(\widehat{g}_{\infty|\infty})_{-r,r+N}v_\xi)$ is a proper submodule of $M$. Thus we have $(\widehat{g}_{\infty|\infty})_{-r,r+N}v_\xi = 0$ since $M$ is irreducible.

We will let $\widehat{gl}_\infty$ denote the the $\mathbb{Z}$-graded subalgebra of $\widehat{g}_{\infty|\infty}$ defined by

$$\widehat{gl}_\infty := \{(a_{ij}) \in \widehat{g}_{\infty|\infty} \mid a_{ij} = 0 \text{ for } i = \frac{1}{2} \text{ or } j = \frac{1}{2} \} \oplus \mathbb{C}e_1.$$

Then $\widehat{gl}_\infty$ is a Lie algebra with the natural $\mathbb{Z}$-gradation induced from $\widehat{g}_{\infty|\infty}$. Therefore, the notions of highest weight modules, quasi-finite highest weight modules, etc. can also be defined for the Lie algebra $\widehat{gl}_\infty$. We also let $\Lambda_0, \omega_i, i \in \mathbb{Z}$, denote the fundamental weights of $\widehat{gl}_\infty$. That is, $\Lambda_0, \omega_i \in (\widehat{gl}_\infty)_0$ with $\omega_i(\sum_{j \in \mathbb{Z}} a_j e_{ij} + dC) = a_i$ and $\Lambda_0(\sum_{j \in \mathbb{Z}} a_j e_{ij} + dC) = d$, for all $\sum_{j \in \mathbb{Z}} a_j e_{ij} + dC \in \mathfrak{g}_0$.

Using arguments analogous to those in the proof of Theorem 4.2 we can prove the following theorem.

**Theorem 4.3.** Let $M$ be an irreducible highest weight $\widehat{gl}_\infty$-module and $v_\xi$ a non-zero highest weight vector. Then the following are equivalent:

(i) $M$ is quasi-finite,
(ii) $(\widehat{gl}_\infty)_- v_\xi$ is finite dimensional,
(iii) there exists $N \in \mathbb{N}$ such that $\xi = \sum_{|j| \leq N} \xi_j \omega_j + d\Lambda_0$, where $\xi_j, d \in \mathbb{C}$.

**4.3. Quasi-finite $\widehat{A}$-modules.** Results proved in the last subsection all generalize to the Lie superalgebra $\widehat{A}$. We shall summarize them here, but omit all the proofs, as they are the same as in the case of $\widehat{gl}_\infty$.

For any $k \in \frac{1}{2}\mathbb{Z}$ and $N \in \frac{1}{2}\mathbb{Z}_+$, we let

$$\widehat{A}_{k,N} := \{x \in \widehat{A} \mid x = \sum_{|j| \geq N, j \in \mathbb{Z}^2} a_j e_{j-k,j}, a_j \in \mathbb{C} \}.$$ 

**Proposition 4.3.** Let $M = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_+} M_{-j}$ be a highest weight $\widehat{A}$-module and $v_0$ be a non-zero highest weight vector in $M$. If $\widehat{A}_{-r}v_0$ is finite dimensional for a fixed
number \( r \in \frac{1}{2}\mathbb{N} \), then for every \( p \in \frac{1}{2}\mathbb{Z}_+ \) with \( p < r \), there exists \( N \in \mathbb{N} \) such that
\[
\hat{A}_{-p,N}v_0 = 0.
\]

In particular, \( \hat{A}_{-p}v_0 \) is finite dimensional for all \( p \leq r \).

The following theorem is an obvious consequence of Proposition 4.1 and Proposition 4.3.

**Theorem 4.4.** Let \( M = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_+} M_j \) be a highest weight \( \hat{A} \)-module and \( v_0 \) be a non-zero highest weight vector in \( M \). Then \( M \) is quasi-finite if and only if for every \( r \in \frac{1}{2}\mathbb{Z}_+ \), there exists \( N \in \mathbb{N} \) such that
\[
\hat{A}_{-r,N}v_0 = 0.
\]

In this case, \( M = \mathcal{U}(\hat{A}^f)v_0 \) and is \( \hat{A}_0 \)-diagonalizable.

We let \( \Lambda_0, \omega_s \in \hat{A}_0^* \) \((s \in \frac{1}{2}\mathbb{Z}^*, \mathbb{Z}^* := \mathbb{Z}\setminus\{0\})\), denote the fundamental weights of \( \hat{A} \), which are defined by \( \omega_s(\sum_{r \in \frac{1}{2}\mathbb{Z}} a_re^{rr} + dC) = a_s, \ \Lambda_0(\sum_{r \in \frac{1}{2}\mathbb{Z}} a_re^{rr} + dC) = d, \) for all \( \sum_{r \in \frac{1}{2}\mathbb{Z}} a_re^{rr} + dC \in \hat{A}_0 \).

**Theorem 4.5.** Let \( M \) be an irreducible highest weight \( \hat{C} \)-module and \( v_\xi \) a non-zero highest weight vector in \( M \). Then the following are equivalent:

(i) \( M \) is quasi-finite,

(ii) \( \hat{A}_{-\frac{1}{2}}v_\xi \) is finite dimensional,

(iii) there exists \( N \in \mathbb{N} \) such that \( \xi = \sum_{j \in \frac{1}{2}\mathbb{Z}^*} |j| \leq N, \ \xi_j \omega_j + d\Lambda_0, \) where \( \xi_j, d \in \mathbb{C} \).

### 4.4 Quasi-finite \( \hat{C} \)-modules

For any \( k \in \frac{1}{2}\mathbb{Z} \) and \( N \in \frac{1}{2}\mathbb{Z}_+ \), we let
\[
\hat{C}_{k,N} := \{ x \in \hat{C} \mid x = \sum_{j \geq N, j \in \frac{1}{2}\mathbb{Z}} a_j \tilde{e}_{j-k,j}, \ a_j \in \mathbb{C} \}.
\]

We have the following lemma, which can be confirmed by a direct computation.

**Lemma 4.3.** Given any fixed positive integer or half integer \( N \), we have
\[
[\hat{C}_k, \hat{C}_{-k,k+N}] \subset \hat{C}_{-(k-p),(k-p)+N},
\]
for all \( k, p \in \frac{1}{2}\mathbb{Z}_+ \) with \( p \leq k \).

**Proposition 4.4.** Let \( M = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_+} M_j \) be a highest weight \( \hat{C} \)-module and \( v_0 \) a non-zero highest weight vector in \( M \). If \( \hat{C}_{-p}v_0 \) is finite dimensional for a fixed number \( r \in \frac{1}{2}\mathbb{N} \), then for every \( p \in \frac{1}{2}\mathbb{Z}_+ \) with \( p < r \), there exists \( N \in \mathbb{N} \) such that
\[
\hat{C}_{-p,N}v_0 = 0.
\]

In particular, \( \hat{C}_{-p}v_0 \) is finite dimensional for all \( p \leq r \).
Proof. The proof is quite similar to Proposition 4.2. Choosing a fixed transcendental real number $\pi$, we let $w_i = \sum_{j \in \frac{1}{2} \mathbb{N}} \pi^{2ij} \hat{e}_{j+r,j}$ be an element in $\hat{C}_{-r}$ for each $i \in \mathbb{N}$. We put $x = \sum_{l=1}^{k} \alpha_i w_i$, where $\alpha_i, \alpha_i_2, \cdots, \alpha_i_k$ are nonzero complex numbers. By Lemma 4.1 we have $x = \sum_{j \in \frac{1}{2} \mathbb{N}} \beta_j \hat{e}_{j+r,j}$ such that $\beta_j$ are nonzero except finitely many $j$. Choose a positive integer $N$ with $N > r$ such that $\beta_j \neq 0$ for all $j$ with $j \geq N$.

We shall prove by contradiction. Assume that there exists $p \in \frac{1}{2} \mathbb{Z}_+ \cup \{0\}$ with $p < r$ such that $\hat{C}_{-p,q} v_0 \neq 0$, for all $q \in \mathbb{N}$. Therefore, we can find $y := \sum_{j \geq N} \alpha_j \hat{e}_{j+p,j} \in \hat{C}_{-p,N}$ such that $y v_0 \neq 0$. Let $u = \sum_{j \in \frac{1}{2} \mathbb{N}} b_j \hat{e}_{j+p,j} + r \in \hat{C}_{r-p}$, where $b_j$ is defined as follows:

$$b_j := \begin{cases} 0, & \text{if } 0 < j < N; \\ \frac{\alpha_j}{\beta_j}, & \text{if } N \leq j < N + r; \end{cases}$$

and for $j \geq N + r$, $b_j$ is defined by the following recursive relations:

$$b_j := \frac{\alpha_j + (-1)^{4r(r-p)} \beta_j \hat{e}_{j+p,j} \hat{e}_{j-r}}{\beta_j}.$$ 

Direct computations show that $[u, x] = y$, and hence $ux v_0 = y v_0 \neq 0$. On the other hand, we can find nonzero complex numbers $\alpha_i, \alpha_i_2, \cdots, \alpha_i_k$ so that $x v_0 = \sum_{l=1}^{k} \alpha_i w_i v_0 = 0$ since $\hat{C}_{-r} v_0$ is finite dimensional, which contradicts $ux v_0 \neq 0$. \hfill $\square$

The following theorem is an immediate consequence of the Proposition 4.1 and Proposition 4.4.

**Theorem 4.6.** Let $M = \oplus_{j \in \frac{1}{2} \mathbb{Z}_+} M_{-j}$ be a highest weight $\hat{C}$-module with highest weight $\xi$ and $v_0$ a non-zero highest weight vector in $M$. Then $M$ is quasi-finite if and only if for every $r \in \frac{1}{2} \mathbb{Z}_+$, there exists $N \in \mathbb{N}$ such that $\hat{C}_{-r,N} v_0 = 0$.

In this case, $M = \mathcal{U}(\hat{\mathcal{C}}') v_0$ and is $\hat{C}_0$-diagonalizable.

Let $\Lambda_0, \omega_s, s \in \frac{1}{2} \mathbb{N}$, denote the fundamental weights of $\hat{C}$, that is, $\Lambda_0, \omega_s \in (\hat{C}_0)^*$ defined by $\omega_s \left( \sum_{r \in \frac{1}{2} \mathbb{N}} a_r \hat{e}_{rr} + d C \right) = a_s$, $\Lambda_0 \left( \sum_{r \in \frac{1}{2} \mathbb{N}} a_r \hat{e}_{rr} + d C \right) = d$, for all $\sum_{r \in \frac{1}{2} \mathbb{N}} a_r \hat{e}_{rr} + d C \in \hat{C}_0$. We have the following theorem, the proof of which will be omitted here, since it is similar to the proof of Theorem 4.2.

**Theorem 4.7.** Let $M$ be an irreducible highest weight $\hat{C}$-module with a non-zero highest weight vector $v_\xi$. Then the following are equivalent:

(i) $M$ is quasi-finite,

(ii) $\hat{C}_{-\frac{1}{2}} v_\xi$ is finite dimensional,
(iii) there exists $N \in \mathbb{N}$ such that $\xi = \sum_{j \in \frac{1}{2} \mathbb{N}} \xi_j \omega_j + d\Lambda_0$, where $\xi_j, d \in \mathbb{C}$.

4.5. Quasi-finite $\hat{D}$-modules. All theorems and propositions proved in the last subsection can be adapted to the Lie superalgebra $\hat{D}$. We summarize the results here, but omit their proofs as they are much the same as in the case of $\hat{C}$.

For any $k \in \frac{1}{2} \mathbb{Z}$ and $N \in \frac{1}{2} \mathbb{Z}^+$, we let

$$\hat{D}_{k,N} := \{ x \in \hat{D} \mid x = \sum_{j \geq N, \ j \in \frac{1}{2} \mathbb{N}} a_j \tilde{e}_{j-k,j}, \ a_j \in \mathbb{C} \}.$$ 

The following lemma can be proven by a direct computation.

**Lemma 4.4.** Given any fixed positive integer or half integer $N$, we have

$$[\hat{D}_p, \hat{D}_{-k,k+N}] \subset \hat{D}_{-(k-p),(k-p)+N},$$

for all $k, p \in \frac{1}{2} \mathbb{Z}^+$ with $p \leq k$.

**Proposition 4.5.** Let $M = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}^+} M_j$ be a highest weight $\hat{D}$-module and $v_0$ a non-zero highest weight vector in $M$. If $\hat{D}_{-r} v_0$ is finite dimensional for a fixed number $r \in \frac{1}{2} \mathbb{N}$, then for every $p \in \frac{1}{2} \mathbb{Z}^+$ with $p < r$, there exists $N \in \mathbb{N}$ such that $\hat{D}_{-p,N} v_0 = 0$.

In particular, $\hat{D}_{-p} v_0$ is finite dimensional for all $p \leq r$.

The following theorem is an obvious consequence of the Proposition 4.4 and Proposition 4.5.

**Theorem 4.8.** Let $M = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}^+} M_j$ be a highest weight $\hat{D}$-module with highest weight $\xi$ and $v_0$ a non-zero highest weight vector. Then $M$ is quasi-finite if and only if for every $r \in \frac{1}{2} \mathbb{Z}^+$, there exists $N \in \mathbb{N}$ such that $\hat{D}_{-r,N} v_0 = 0$.

In this case, $M = U(\hat{D}^0)v_0$ and is $\hat{D}_0$-diagonalizable.

Let $\Lambda_0, \omega_s, s \in \frac{1}{2} \mathbb{N}$, denote the fundamental weights of $\hat{D}$, that is, $\Lambda_0, \omega_s \in (\hat{D}_0)^*$ defined by $\omega_s(\sum_{r \in \frac{1}{2} \mathbb{N}} a_r \tilde{e}_{rr} + dC) = a_s$, $\Lambda_0(\sum_{r \in \frac{1}{2} \mathbb{N}} a_r \tilde{e}_{rr} + dC) = d$, for all $\sum_{r \in \frac{1}{2} \mathbb{N}} a_r \tilde{e}_{rr} + dC \in \hat{D}_0$.

**Theorem 4.9.** Let $M$ be an irreducible highest weight $\hat{D}$-module and $v_\xi$ a non-zero highest weight vector in $M$. Then the following are equivalent:

(i) $M$ is quasi-finite,

(ii) $\hat{D}_{-\frac{1}{2}} v_\xi$ is finite dimensional,
(iii) there exists $N \in \mathbb{N}$ such that \[ \xi = \sum_{j \leq N, j \in \frac{1}{2} \mathbb{N}} \xi_j \omega_j + d \Lambda_0, \] where $\xi_j, d \in \mathbb{C}$.

Remark 4.1. All results in this section can be restricted to the non-super case, leading to descriptions and classifications of the irreducible quasi-finite highest weight modules over $\hat{\mathfrak{gl}}_\infty$ and its $\mathbb{Z}$-graded subalgebras.

5. Unitarizable representations and their free field realizations

In this section we study in detail a particularly nice class of modules over the infinite rank Lie superalgebras $\hat{\mathfrak{gl}}_{\infty|\infty}, \hat{\mathfrak{A}}, \hat{\mathfrak{C}}$ and $\hat{\mathfrak{D}}$, namely, the quasi-finite irreducible highest weight modules, which are unitarizable with respect to natural choices of $\ast$-structures for these Lie superalgebras. Several results are obtained here, including (1). the classification of the unitarizable quasi-finite irreducible highest weight modules over the Lie superalgebras $\hat{\mathfrak{gl}}_{\infty|\infty}, \hat{\mathfrak{A}}, \hat{\mathfrak{C}}$ and $\hat{\mathfrak{D}}$; (2). realizations of these irreducible modules on Fock spaces; and (3). generalized Howe dualities between these infinite rank Lie superalgebras and classical Lie algebras. The generalized Howe dualities will provide the principal tool for constructing character formulae for the unitarizable quasi-finite irreducible highest weight modules in the next section.

5.1. Unitarizable $\hat{\mathfrak{gl}}_{\infty|\infty}$-modules and their Fock space realizations.

5.1.1. Free field realization of $\hat{\mathfrak{gl}}_{\infty|\infty}$ and $(\hat{\mathfrak{gl}}_{\infty|\infty}, \mathfrak{gl}_d)$-duality. Let $\mathfrak{gl}_d$ denote the Lie algebra of all complex $d \times d$ matrices. Let $\{e^1, \ldots, e^d\}$ be the standard basis for $\mathbb{C}^d$. Denote by $E_{ij}$ the elementary matrix with 1 in the $i$-th row and $j$-th column and 0 elsewhere. Then $\mathfrak{h}_d = \sum_{i=1}^d \mathbb{C} E_{ii}$ is a Cartan subalgebra, while $\mathfrak{b}_d = \sum_{1 \leq i < j \leq d} \mathbb{C} E_{ij}$ is the standard Borel subalgebra containing $\mathfrak{h}_d$. The weight of $e^i$ is denoted by $\tilde{\epsilon}_i$ for $1 \leq i \leq d$.

We regard a finite sequence $\lambda = (\lambda_1, \cdots, \lambda_d)$ of complex numbers as an element of the dual vector space $\mathfrak{h}_d^*$ of $\mathfrak{h}_d$ defined by $\lambda(E_{ii}) = \lambda_i$, for $i = 1, \cdots, d$. Denote by $V^\lambda_d$ the irreducible $\mathfrak{gl}_d$-module with highest weight $\lambda$ relative to the standard Borel subalgebra $\mathfrak{b}_d$.

Consider $d$ pairs of free fermionic fields $\psi_{i}^{\pm}(z)$ and $d$ pairs of free symplectic bosonic fields $\gamma_{i}(z)$, $i = 1, \cdots, d$, with the following mode expansions:

\[
\psi_{n}^{i}(z) = \sum_{n \in \mathbb{Z}} \psi_{n}^{i} z^{-n-1}, \quad \psi_{n}^{-i}(z) = \sum_{n \in \mathbb{Z}} \psi_{n}^{-i} z^{-n},
\]
\[
\gamma_{r}^{i}(z) = \sum_{r \in \frac{1}{2} \mathbb{Z}} \gamma_{r}^{i} z^{-r-1/2}, \quad \gamma_{r}^{-i}(z) = \sum_{r \in \frac{1}{2} \mathbb{Z}} \gamma_{r}^{-i} z^{-r-1/2},
\]
where the operators $\psi_{n,i}^\pm$ and $\gamma_{r,i}^\pm$ satisfy the usual (anti-)commutation relations with the non-trivial ones being given by

\[
\psi_{m,i}^+ \psi_{n,j}^- + \psi_{n,j}^- \psi_{m,i}^+ = \delta_{ij} \delta_{m+n,0}, \\
\gamma_{r,i}^+ \gamma_{s,j}^- - \gamma_{s,j}^- \gamma_{r,i}^+ = \delta_{ij} \delta_{r+s,0}.
\]

Denote by $A^d$ the associative superalgebra generated by these operators. $A^d$ admits a $^*$-structure $\omega : A^d \to A^d$ defined by

\[
\omega(\psi_{m,k}^+) = \psi_{m,-k}^-, \quad \omega(\psi_{m,-k}^-) = \psi_{m,k}^+, \\
\omega(\gamma_{r,k}^+) = \begin{cases} 
\gamma_{r,-k}^-, & \text{if } r > 0, \\
-\gamma_{r,-k}^+, & \text{if } r < 0,
\end{cases}, \quad \omega(\gamma_{r,-k}^-) = \begin{cases} 
-\gamma_{r,+k}^+, & \text{if } r > 0, \\
\gamma_{r,+k}^-, & \text{if } r < 0,
\end{cases}
\]

for all $m \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$, $k = 1, 2, \cdots, d$. It can be easily shown that $\omega$ indeed defines an anti-linear anti-involution on $A^d$.

We shall take the operators $\psi_{n,i}^+, \psi_{m,i}^-, \gamma_{r,i}^{\pm}$, with $i = 1, 2, \cdots, d$, $n \geq 0$, $m > 0$, $r > 0$, as annihilation operators, and the rest as creation operators. Let $\mathfrak{F}^d$ be the Fock space of $A^d$ generated by the vacuum vector $|0\rangle$, which is nullified by the annihilation operators, i.e.,

\[
\psi_{n,i}^+ |0\rangle = \psi_{m,i}^- |0\rangle = \gamma_{r,i}^{\pm} |0\rangle = 0,
\]

for all $i = 1, 2, \cdots, d$; $n \geq 0$; $m > 0$; $r > 0$.

Let $\langle \cdot | \cdot \rangle$ be the contravariant Hermitian form on the Fock space $\mathfrak{F}^d$ defined with respect to the anti-linear anti-involution $\omega$ given in (5.1) and (5.2). We normalize the form on the vacuum vector $|0\rangle$ so that $\langle 0 | 0 \rangle = 1$.

**Lemma 5.1.** The Fock space $\mathfrak{F}^d$ equipped with the contravariant Hermitian form $\langle \cdot | \cdot \rangle$ is a unitarizable $A^d$-module.

**Proof.** In the ‘particle number basis’ for the Fock space, the lemma can be confirmed by a straightforward calculation, details of which are omitted here. However, see Remark 5.1 below.

**Remark 5.1.** If we remove all the minus signs from (5.2), we still obtain a $^*$-structure for $A^d$. In fact this is the usual conjugation rule for symplectic bosons adopted in the physics literature. However, it is quite well known that the Fock space of symplectic bosons is not unitarizable with respect to the usual conjugation rule.
The Lie superalgebra \( \hat{\mathfrak{gl}}_{\infty|\infty} \) can be realized on the Fock space \( \mathcal{F}^d \) as follows:

\[
C = d, \\
e_{ij} = \sum_{p=1}^{d} : \psi_{i-p}^{+}\psi_{j-p}^{-} : , \\
e_{rs} = -\sum_{p=1}^{d} : \gamma_{r-p}\gamma_{s-p}^{-} : , \\
e_{is} = \sum_{p=1}^{d} : \psi_{i-p}\gamma_{s-p}^{-} : , \\
e_{rj} = -\sum_{p=1}^{d} : \gamma_{r-p}\psi_{j-p}^{-} : ,
\]

where \( i, j \in \mathbb{Z} \) and \( r, s \in \frac{1}{2} + \mathbb{Z} \). There also exists an action of \( GL_d \) on \( \mathcal{F}^d \), which is given by the formula

\[
E_{ij} = \sum_{n \in \mathbb{Z}} : \psi_{-n+i}^{+}\psi_{n-j}^{-} : - \sum_{r \in \frac{1}{2} + \mathbb{Z}} : \gamma_{r+i}\gamma_{r-j}^{-} : .
\]

Here and further \( :: \) denotes the normal ordering of operators. That is, if \( A \) and \( B \) are two operators, then \( : AB : = AB \), if \( B \) is an annihilation operator, while \( : AB : = (-1)^{|A||B|} BA \), otherwise, where \( |X| \) denotes the parity of \( X \).

The following result is the \( (\hat{\mathfrak{gl}}_{\infty|\infty}, \mathfrak{gl}_d) \)-duality of [7].

**Theorem 5.1.** [7] The Lie superalgebra \( \hat{\mathfrak{gl}}_{\infty|\infty} \) and \( \mathfrak{gl}_d \) form a dual pair on \( \mathcal{F}^d \) in the sense of Howe. Furthermore we have the following (multiplicity-free) decomposition of \( \mathcal{F}^d \) with respect to their joint action

\[
\mathcal{F}^d \cong \sum_{\lambda} L(\hat{\mathfrak{gl}}_{\infty|\infty}, \Lambda(\lambda)) \otimes V^\lambda_d,
\]

where the summation is over all generalized partitions of length \( d \).

Here the notation \( \Lambda(\lambda) \) requires some explanation. For a generalized partition \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d) \) of length \( d \), we have the shifted Frobenius notation (see Section 2.1)

\[
\Lambda(\lambda) = (\xi_{s+\frac{1}{2}}, \xi_{s+\frac{3}{2}}, \cdots, \xi_{s+\frac{d}{2}} | \xi_{s+1}, \xi_{s+2}, \cdots, \xi_0 | \xi_{1+\frac{1}{2}}, \xi_{1+\frac{3}{2}}, \cdots, \xi_{r-\frac{1}{2}} | \xi_1, \xi_2, \cdots, \xi_r).
\]

We identify \( \Lambda(\lambda) \) with an element of the dual space \( (\hat{\mathfrak{gl}}_{\infty|\infty})^*_0 \) of \( (\hat{\mathfrak{gl}}_{\infty|\infty})_0 \) defined by

\[
(5.4) \quad \Lambda(\lambda) := \sum_{s+\frac{1}{2} \leq j \leq r, \frac{1}{2} \in \xi_j} \xi_j \omega_j + d\Lambda_0.
\]

5.1.2. **Unitarizable \( \hat{\mathfrak{gl}}_{\infty|\infty} \)-modules.** Let us consider the anti-linear anti-involution on \( \hat{\mathfrak{gl}}_{\infty|\infty}^f \) defined by

\[
C \mapsto C, \quad e_{pq} \mapsto (-1)^{|p|+|q|} e_{qp}, \quad \forall p, q,
\]
where $[r] = 1$ if $r$ is a negative half integer, and $[r] = 0$ otherwise. This map naturally extends to an anti-linear anti-involution on $\hat{gl}_{\infty|\infty}$ with

$$C \mapsto C, \quad \sum_{p \in \frac{1}{2}\mathbb{Z}} a_p e_{p-k,p} \mapsto \sum_{p \in \frac{1}{2}\mathbb{Z}} (-1)^{|p|+|p-k|} \overline{a}_p e_{p,p-k},$$

for all $\sum_{p \in \frac{1}{2}\mathbb{Z}} a_p e_{p-k,p} \in (\hat{gl}_{\infty|\infty})_k$ and for all $k \in \frac{1}{2}\mathbb{Z}$. Here $\overline{a}$ denotes the complex conjugate of the complex number $a$. Abusing the notation, we shall also denote this map by $\omega$.

The realization (5.3) of $\hat{gl}_{\infty|\infty}$ in $\mathcal{A}^d$ defines an associative superalgebra homomorphism $\Phi : \mathcal{U}(\hat{gl}_{\infty|\infty}) \to \mathcal{A}^d$. By using (5.1) and (5.2) we can show by a direct computation that $\omega \Phi(x) = \Phi(\omega(x))$ for all $x \in \mathcal{U}(\hat{gl}_{\infty|\infty})$. Thus $\Phi$ is a $*$-superalgebra homomorphism.

Since the Fock space $\mathcal{F}^d$ equipped with the Hermitian form $\langle \cdot | \cdot \rangle$ is a unitarizable $\mathcal{A}^d$-module, it naturally restricts to a unitarizable module over $\hat{gl}_{\infty|\infty}$. Hence $\mathcal{F}^d$ forms a unitarizable $\hat{gl}_{\infty|\infty}$-module as can be easily seen by examining the action of $\hat{gl}_{\infty|\infty}$ on $\mathcal{F}^d$. By Theorem 5.1, for each generalized partition $\lambda$ of length $d$, the irreducible $\hat{gl}_{\infty|\infty}$-module $L(\hat{gl}_{\infty|\infty}, \Lambda(\lambda))$ is unitarizable. We shall show that such modules exhaust all the unitarizable irreducible quasi-finite highest weight $\hat{gl}_{\infty|\infty}$-modules with respect to the $*$-structure $\omega$.

**Theorem 5.2.** Let $M$ be an irreducible quasi-finite highest weight $\hat{gl}_{\infty|\infty}$-module with highest weight $\xi$. Then $M$ is unitarizable with respect to the $*$-structure $\omega$ if and only if $\xi = \Lambda(\lambda)$ for some generalized partition $\lambda$. In other words, $M$ is unitarizable if and only if

$$\xi = \sum_{s+\frac{1}{2}\leq j \leq r, j \in \frac{1}{2}\mathbb{Z}} \xi_j \omega_j + d\Lambda_0$$

such that $d \in \mathbb{Z}_+$, $-s, r \in \mathbb{N}$ and $\xi_j \in \mathbb{Z}$ for all $j$ satisfying the following conditions:

(i) $\xi_{\frac{1}{2}} > \xi_{\frac{3}{2}} > \cdots > \xi_{r-\frac{1}{2}} \geq 0$, $\xi_1 > \xi_2 > \cdots > \xi_r \geq 0$, and $\xi_{r-\frac{1}{2}} = 0$ if only if $r = 1$ and $\xi_{\frac{1}{2}} = \xi_1 = 0$,

(ii) $0 \geq \xi_{s+\frac{1}{2}} > \xi_{s+\frac{3}{2}} > \cdots > \xi_{\frac{1}{2}}$, $0 \geq \xi_{s+1} > \xi_{s+2} > \cdots > \xi_0$, and $\xi_{s+1} = 0$ if only if $s = -1$ and $\xi_{\frac{1}{2}} = \xi_0 = 0$,

(iii) $\min\{\xi_1, 1\} + \xi_1 - \xi_0 \leq d$. 

Proof. We already know that for any generalized partition \( \lambda \), the \( \mathfrak{gl}_{\infty|\infty} \)-module \( L(\mathfrak{g}_{\infty|\infty}, \Lambda(\lambda)) \) is unitarizable. Now we show that if \( M \) is a unitarizable irreducible quasi-finite highest weight \( \mathfrak{gl}_{\infty|\infty} \)-module with the highest weight \( \xi \), then \( \xi = \Lambda(\lambda) \) for some generalized partition \( \lambda \). Let \( \langle \cdot | \cdot \rangle \) be a positive definite contravariant Hermitian form on \( M \) and \( v_\xi \) a highest weight vector of \( M \) such that \( \langle v_\xi | v_\xi \rangle = 1 \). We put \( \xi(e_{i,j}) = \xi_i \) for all \( i \in \frac{1}{2} \mathbb{Z} \). By Theorem 4.2, there exists \( N \in \mathbb{N} \) such that \( \xi = \sum_{j \in \mathbb{Z}} \xi_j \omega_j + d \Lambda_0 \), where \( \xi_j, d \in \mathbb{C} \). For each \( i \in \mathbb{N} \), \( \{e_{i,i} - e_{i+1,i+1}, e_{i,i+1}, e_{i+1,i}\} \) forms a standard basis for the Lie algebra \( sl(2, \mathbb{C}) \) and \( \omega(e_{i,i+1}) = e_{i+1,i} \). Unitarizability of \( M \) with respect to this subalgebra requires (see, e.g., Theorem 11.7 in [13]) \( \xi_i - \xi_{i+1} = \xi(e_{i,i} - e_{i+1,i+1}) \in \mathbb{Z}_+ \). Since \( M \) is a quasi-finite highest weight \( \mathfrak{gl}_{\infty|\infty} \)-module, we have \( \xi_i \in \mathbb{Z}_+ \) for all \( i \in \mathbb{N} \) and

\[
\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \geq \xi_{n+1} = \xi_{n+2} = \cdots = 0 \quad \text{for some } n \in \mathbb{N}.
\]

Similarly, we have \( \xi_i \in \mathbb{Z}_+ \) for all \( i \in \frac{1}{2} + \mathbb{Z}_+ \) and

\[
\xi_{\frac{1}{2}} \geq \xi_{\frac{3}{2}} \geq \cdots \geq \xi_{r-\frac{1}{2}} \geq \xi_{r+\frac{1}{2}} = \xi_{r+\frac{3}{2}} = \cdots = 0 \quad \text{for some } r \in \mathbb{N}.
\]

Now we are going to show that for each \( i \in \frac{1}{2} \mathbb{N} \), either

\[
(5.5) \quad \xi_i > \xi_{i+1} \quad \text{or} \quad \xi_i = \xi_{i+\frac{1}{2}} = 0.
\]

It is sufficient to show that \( \xi_i = \xi_{i+1} \) implies \( \xi_i = \xi_{i+\frac{1}{2}} = 0 \). Assume that \( \xi_i = \xi_{i+1} \).

Then

\[
\langle e_{i+1,i} v_\xi | e_{i+1,i} v_\xi \rangle = \langle v_\xi | \omega(e_{i+1,i}) e_{i+1,i} v_\xi \rangle = \langle v_\xi | e_{i,i+1} e_{i+1,i} v_\xi \rangle = \xi_i - \xi_{i+1} = 0.
\]

Therefore, we have \( e_{i+1,i} v_\xi = 0 \). On the other hand, we have

\[
\langle e_{i+1,i} v_\xi | e_{i+1,i} v_\xi \rangle = \langle e_{i+\frac{1}{2},i+\frac{1}{2}} v_\xi | e_{i+\frac{1}{2},i+\frac{1}{2}} v_\xi \rangle = \langle e_{i+\frac{1}{2},i+\frac{1}{2}} v_\xi | e_{i+\frac{1}{2},i+\frac{1}{2}} v_\xi \rangle = \langle v_\xi | e_{i,i} + e_{i+\frac{1}{2},i+\frac{1}{2}} v_\xi \rangle = \xi_i + \xi_{i+\frac{1}{2}}.
\]

Thus \( \xi_i + \xi_{i+\frac{1}{2}} = 0 \) since \( e_{i+1,i} v_\xi = 0 \). Since \( \xi_i \geq 0 \) and \( \xi_{i+\frac{1}{2}} \geq 0 \), we have \( \xi_i = \xi_{i+\frac{1}{2}} = 0 \). By (5.5), there is \( r \in \mathbb{N} \) such that

\[
\xi_1 > \xi_2 > \cdots > \xi_r > \xi_{r+1} = \xi_{r+2} = \cdots = 0,
\]

\[
\xi_{\frac{1}{2}} > \xi_{\frac{3}{2}} > \cdots > \xi_{r-\frac{1}{2}} > \xi_{r+\frac{1}{2}} = \xi_{r+\frac{3}{2}} = \cdots = 0,
\]

or \( \xi_{\frac{1}{2}} = \xi_1 = 0 \).
By using similar argument as above (note that \(\omega(e_{i,-\frac{1}{2}}) = -e_{i,\frac{1}{2}} \) for all \(-i \in \mathbb{Z}_+\)), we have \(-\xi_i \in \mathbb{Z}_+\) for all \(-i \in \frac{1}{2}\mathbb{Z}_+\), and there is \(s \in \mathbb{N}\) such that

\[
0 = \cdots = \xi_{s-\frac{1}{2}} = \xi_{s-1} \geq \xi_{s+\frac{1}{2}} > \cdots > \xi_{-\frac{1}{2}},
\]

or \(\xi_{-\frac{1}{2}} = \xi_0 = \mathbf{0}\).

Now we choose a large positive integer \(n\) such that \(\xi(e_{n,n}) = \xi(e_{-n,-n}) = 0\). Consider the subalgebra \(sl(2, \mathbb{C})\) spanned by \(\{e_{-n-n} - e_{n,n} + C, e_{-n,n}, e_{n,-n}\}\). Note that \(\omega(e_{-n,n}) = e_{n,-n}\). Using the standard trick on unitarizable modules again, we have \(d = \xi(C) = \xi(e_{-n,-n} - e_{n,n} + C) \in \mathbb{Z}_+\). Finally, we need to show

\[
\min\{\xi_{\frac{1}{2}}, 1\} + \xi_1 - \xi_0 = 0.
\]

Since \(\langle e_{1,0}v_{\xi} | e_{1,0}v_{\xi} \rangle \geq 0\) and \(\langle e_{1,0}v_{\xi} | (e_{0,0} - e_{1,1} + C)v_{\xi} \rangle = \xi_0 - \xi_1 + d\), we have \(\xi_0 - \xi_1 + d \geq 0\). If \(\xi_0 - \xi_1 + d > 0\), the proof of the theorem is completed. Otherwise, we have \(\xi_0 - \xi_1 + d = 0\) and \(e_{1,0}v_{\xi} = 0\). Therefore we have

\[
(5.6) \quad d + \xi_0 = \xi_1 \geq 0.
\]

On the other hand, by using (5.6) and \(e_{1,0}v_{\xi} = 0\), we have

\[
0 = \langle e_{\frac{1}{2},1}e_{1,0}v_{\xi} | e_{\frac{1}{2},1}e_{1,0}v_{\xi} \rangle = \langle e_{\frac{1}{2},0}v_{\xi} | e_{\frac{1}{2},0}v_{\xi} \rangle = \langle v_{\xi} | \omega(e_{1,0})e_{\frac{1}{2},0}v_{\xi} \rangle = \langle v_{\xi} | (e_{0,0} + e_{\frac{1}{2},\frac{1}{2}} + C)v_{\xi} \rangle = \xi_0 + \xi_1 + d = \xi_{\frac{1}{2}} + \xi_1.
\]

Since \(\xi_{\frac{1}{2}} \geq 0\) and \(\xi_1 \geq 0\), we have \(\xi_{\frac{1}{2}} = \xi_1 = 0\) and \(\min\{\xi_{\frac{1}{2}}, 1\} + \xi_1 - \xi_0 = -\xi_0 \leq d\) by (5.6) again. This completes the proof of the theorem.

Recall that in [24] Wang showed that there is a \(\hat{gl}_\infty \times gl_d\) duality on the subspace of the Fock space \(\mathfrak{F}^d\) generated by the fermionic operators. By modifying the arguments in the proof of Theorem 5.2, we can show that

**Theorem 5.3.** Let \(M\) be an irreducible quasi-finite highest weight \(\hat{gl}_\infty\)-module with the highest weight \(\xi\). Then \(M\) is unitarizable if and only if

\[
\xi = \sum_{s \leq j \leq r} \xi_j \omega_j + d\Lambda_0
\]

such that \(d \in \mathbb{Z}_+, r \in \mathbb{N}, -s \in \mathbb{Z}_+\) and \(\xi_j \in \mathbb{Z}\) for all \(j\) satisfying the following conditions:
\( (i) \) \( \xi_1 > \xi_2 > \cdots > \xi_r \geq 0, \)
\( (ii) \) \( 0 \geq \xi_s > \xi_{s+1} > \cdots > \xi_0, \)
\( (iii) \) \( \xi_1 - \xi_0 \leq d. \)

5.2. Unitarizable \( \hat{A} \)-modules and their Fock space realizations.

5.2.1. Free field realization of \( \hat{A} \) and \( (\hat{A}, gl_d) \)-duality. Consider \( d \) pairs of free fermions \( \tilde{\psi}^{\pm,i}(z) \) and \( d \) pairs of free symplectic bosons \( \gamma^{\pm,i}(z) \) (\( i = 1, \cdots, d \))

\[
\tilde{\psi}^{+,i}(z) = \sum_{n \in \mathbb{Z}^*} \psi^{+,i}_n z^{-n-1}, \quad \tilde{\psi}^{-,i}(z) = \sum_{n \in \mathbb{Z}^*} \psi^{-,i}_n z^{-n},
\]

\[
\gamma^{+,i}(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \gamma^{+,i}_r z^{-r-1/2}, \quad \gamma^{-,i}(z) = \sum_{r \in \frac{1}{2} + \mathbb{Z}} \gamma^{-,i}_r z^{-r-1/2}
\]

with the non-trivial anti-commutation relations \([\psi^{+,i}_m, \psi^{-,j}_n] = \delta_{ij}\delta_{m+n,0}\) and commutation relations \([\gamma^{+,i}_r, \gamma^{-,j}_s] = \delta_{ij}\delta_{r+s,0}\). We take \( \psi^{+,i}_m, \psi^{-,i}_m, \gamma^{+,i}_r, \gamma^{-,i}_r, m, r > 0 \), as annihilation operators, and \( \psi^{+,i}_m, \gamma^{+,i}_r, m, r < 0 \) as creation operators. Let \( \mathfrak{F}_d^d \) denote the corresponding Fock space generated by the vacuum vector \( |0) \) with \( \psi^{+,i}_m|0) = \gamma^{+,i}_r|0) = 0, \) for \( i = 1, 2, \cdots, d, m > 0 \) and \( r > 0. \)

We have an action of \( \hat{A} \) with central charge \( d \) on \( \mathfrak{F}_d^d \) given by \( (i, j \in \mathbb{Z}^* \) and \( r, s \in \frac{1}{2} + \mathbb{Z}) \)

\[
e_{ij} = \sum_{p=1}^d : \psi^{+,i}_p \psi^{-,j}_p : ; \quad e_{rs} = - \sum_{p=1}^d : \gamma^{+,i}_r \gamma^{-,j}_s : ;
\]

\[
e_{is} = \sum_{p=1}^d : \psi^{+,i}_p \gamma^{-,s}_p : ; \quad e_{rj} = - \sum_{p=1}^d : \gamma^{+,i}_r \psi^{-,j}_p : .
\]

There is also an action of \( gl_d \) on \( \mathfrak{F}_d^d \), which is given by the formula

\[
E_{ij} = \sum_{n \in \mathbb{Z}^*} : \psi^{+,i}_{-n} \psi^{-,j}_n : - \sum_{r \in \frac{1}{2} + \mathbb{Z}} : \gamma^{+,i}_r \gamma^{-,j}_r : .
\]

For \( j \in \mathbb{N} \) we define the matrices \( X^{-j} \) as follows:

\[
X^{-1} = \begin{pmatrix}
\gamma^{-d}_{-\frac{1}{2}} & \gamma^{-,d-1}_{-\frac{1}{2}} & \cdots & \gamma^{-,1}_{-\frac{1}{2}} \\
\psi^{-d}_{-1} & \psi^{-,d-1}_{-1} & \cdots & \psi^{-,1}_{-1} \\
\vdots & \vdots & \ddots & \vdots \\
\psi^{-d}_{-1} & \psi^{-,d-1}_{-1} & \cdots & \psi^{-,1}_{-1}
\end{pmatrix},
\]
The matrices $X^j$, for $j \in \mathbb{N}$, are defined similarly. Namely, $X^j$ is obtained from $X^{-j}$ by replacing $\psi^{-i}_i$ by $\psi^{+i-d-k+1}_i$ and $\gamma^{-d}_r$ by $\gamma^{+d-k+1}_r$. For $0 \leq r \leq d$ and $i \in \mathbb{Z}^*$, we let $X^i_r$ denote the first $r \times r$ minor of the matrix $X^i$. (We use the definition given in [5] for the determinant of a matrix with Grassmannian entries.)

Given a generalized partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ of length $d$, we have the shifted Frobenius notations $\Lambda(\lambda^+) = (\xi^+, \xi^+_{1\frac{1}{2}}, \xi^+_{r-\frac{1}{2}} | \xi^+_1, \xi^+_2, \ldots, \xi^+_r)$ and $\Lambda(\lambda^-) = (\xi^-, \xi^-_{1\frac{1}{2}}, \ldots, \xi^-_{s-\frac{1}{2}} | \xi^-_1, \xi^-_2, \ldots, \xi^-_s)$ for the partitions $\lambda^+$ and $\lambda^-$, respectively (see Section 2.1). We let $\Lambda^\hat{A}(\lambda)$ be an element of the dual space $(\hat{A}_0)^*$ of $\hat{A}_0$ defined by

$$\Lambda^\hat{A}(\lambda) := \sum_{j \leq r, j \in \frac{1}{2} \mathbb{N}} \xi^+_j \omega_j - \sum_{j \geq -s, j \in \frac{1}{2} \mathbb{N}} \xi^-_j \omega_j + d\Lambda_0.$$  

Let $\mathfrak{g}^d_0$ be the subalgebra of $\mathfrak{g}^d$ generated by those $\psi^{+i}_m, \psi^{-i}_m, \gamma^{+i}_r$ and $\gamma^{-i}_r$, $m \in \mathbb{Z}^*, r \in \begin{cases} \frac{1}{2} + \mathbb{Z}, i = 1, 2, \ldots, d \end{cases}$. Then $gl_d$ acts on $\mathfrak{g}^d_0$ by the usual commutator. This action lifts to an action of $GL_d$ on $\mathfrak{g}^d_0$ by conjugation. The $GL_d$ invariants of the associative algebra $\mathfrak{g}^d_0$ is generated by $\hat{A}^I$, hence the $gl_d$-action on $\mathfrak{f}_0^d$ commutes with the $\hat{A}$-action. Therefore we have the following result, which is analogous to the $(gl_\infty, gl_d)$-duality of [7].

**Theorem 5.4.** The Lie superalgebra $\hat{A}$ and $gl_d$ form a dual pair on $\mathfrak{f}_0^d$ in the sense of Howe. In particular, we have the following (multiplicity-free) decomposition of $\mathfrak{f}_0^d$ with respect to their joint action

$$\mathfrak{f}_0^d \cong \sum_\lambda L(\hat{A}, \Lambda^\hat{A}(\lambda)) \otimes V^\lambda_d,$$
where the summation is over all generalized partitions of length $d$. Furthermore, the joint highest weight vector of the $\lambda$-component is given by
\[
X^{\lambda_d}_{(\lambda_0 \cdots \lambda_d)} \cdots X^{-1}_{(\lambda^*-1)} \cdots X^1_{(\lambda^*+1)} \cdots X^{\lambda_1}_{(\lambda_1^*)} |0\rangle.
\]

5.2.2. Unitarizable $\hat{A}$-modules. The restriction of the anti-linear anti-involution $\omega$ on the Lie superalgebra $\mathfrak{g}_{\infty|\infty}$ to $\hat{A}$ gives an anti-linear anti-involution on $\hat{A}$, which will also be denoted by $\omega$. We have $\omega(C) = C$, and
\[
\omega(\sum_{p \in \frac{1}{2}Z^*} a_p e_{p-k,p}) = \sum_{p \in \frac{1}{2}Z^*} (-1)^{|p|+(p-k)} a_pe_{p,p-k},
\]
for all $\sum_{p \in \frac{1}{2}Z^*} a_p e_{p-k,p} \in \hat{A}_k$ and for all $k \in \frac{1}{2}Z$. It is clear that the Fock space $\mathfrak{F}_d$ is a subspace of the Fock space $\mathfrak{F}^d$ which is defined in the last subsection. Since $\mathfrak{F}^d$ is a unitarizable $\mathfrak{g}_{\infty|\infty}$-module with respect to the Hermitian form $\langle \cdot | \cdot \rangle$, the Fock space $\mathfrak{F}_d$ is a unitarizable $\hat{A}$-module with respect to the anti-linear anti-involution $\omega$ on $\hat{A}$. By Theorem 5.2, for each generalized partition $\lambda$ of length $d$, the irreducible $\hat{A}$-module $L(\hat{A}, \Lambda^\hat{A}(\lambda))$ is unitarizable. In fact they are all the unitarizable irreducible $\hat{A}$-modules with respect to $\omega$. We have the following theorem.

**Theorem 5.5.** Let $M$ be an irreducible quasi-finite highest weight $\hat{A}$-module with highest weight $\xi$. Then $M$ is unitarizable if and only if $\xi = \Lambda^\hat{A}(\lambda)$ for some generalized partition $\lambda$. In other words, $M$ is unitarizable if and only if
\[
\Lambda^\hat{A}(\lambda) := \sum_{j \in \frac{1}{2}N} \xi_j^+ \omega_j - \sum_{j \in -\frac{1}{2}N} \xi_j^- \omega_j + d\Lambda_0.
\]
such that $d \in \mathbb{Z}_+$, $s, r \in \mathbb{N}$ and $\xi_j \in \mathbb{Z}$ for all $j$ satisfying the following conditions:

(i) $\xi_{\frac{s}{2}}^+ > \xi_{\frac{s-1}{2}}^+ > \cdots > \xi_{\frac{r}{2}}^+ > \xi_{\frac{r-1}{2}}^+ = 0$, $\xi_{\frac{s}{2}}^- > \xi_{\frac{s-1}{2}}^- > \cdots > \xi_{\frac{r}{2}}^- = 0$, and $\xi_{\frac{r}{2}}^+ = 0$ if and only if $r = 1$ and $\xi_{\frac{r}{2}}^- = \xi_1^+ = 0$,

(ii) $\xi_{\frac{s}{2}}^- > \xi_{\frac{s-1}{2}}^- > \cdots > \xi_{\frac{s}{2}}^- = 0$, $\xi_{\frac{s-1}{2}}^- > \xi_{\frac{s-2}{2}}^- > \cdots > \xi_{\frac{r}{2}}^- = 0$, and $\xi_{\frac{s-1}{2}}^- = 0$ if and only if $s = 1$ and $\xi_{\frac{s}{2}}^- = \xi_1^- = 0$,

(iii) $\min\{\xi_{\frac{s}{2}}^+, 1\} + \min\{\xi_{\frac{s}{2}}^-, 1\} + \xi_{\frac{s}{2}}^+ + \xi_{\frac{s}{2}}^- \leq d$.

**Proof.** We already know that $L(\hat{A}, \Lambda^\hat{A}(\lambda))$ are unitarizable irreducible quasi-finite highest weight $\hat{A}$-modules for any generalized partition $\lambda$. Now we are going to show that if $M$ is a unitarizable irreducible quasi-finite highest weight $\hat{A}$-module with the highest weight $\xi$, then $\xi = \Lambda^\hat{A}(\lambda)$ for some generalized partition $\lambda$. Let $\langle \cdot | \cdot \rangle$ be a positive definite contravariant Hermitian form on $M$ and $v_\xi$ a highest weight vector of $M$ such that $\langle v_\xi | v_\xi \rangle = 1$. We put $\xi(e_{i,i}) = \xi_i$ for all
In the proof of Theorem 5.2, we have $\xi = \sum_{j<s} \xi_j^+ \omega_j - \sum_{j<s} \xi_j^- \omega_j + d\Lambda_0$, where $\xi_j^+$, $\xi_j^-$, $d \in \mathbb{C}$. By using similar argument as those in the proof of Theorem 5.2, we have $\xi_i^+$, $\xi_j^-$, $d \in \mathbb{Z}$ for $i = \frac{1}{2}, 1, \cdots, r - \frac{1}{2}, r$, $\frac{1}{2}, 1, \cdots, s - \frac{1}{2}, s$, and

$$\xi_1^+ > \xi_2^+ > \cdots > \xi_{r-\frac{1}{2}}^+ \geq 0,$$

and

$$\xi_{\frac{1}{2}}^- > \xi_{\frac{1}{2}}^- > \cdots > \xi_{s-\frac{1}{2}}^- \geq 0.$$  

Moreover, $\xi_{r-\frac{1}{2}}^+ = 0$ if and only if $r = 1$ and $\xi_{\frac{1}{2}}^+ = \xi_0 = 0$, and $\xi_{s-\frac{1}{2}}^- = 0$ if and only if $s = 1$ and $\xi_0^- = 0$.

Finally, we need to show $\min\{\xi_{\frac{1}{2}}^+, 1\} + \min\{\xi_0^-, 1\} + \xi_1^+ + \xi_1^- \leq d$. Since $\langle e_{1,0} \omega \xi | e_{1,0} \omega \xi \rangle \geq 0$ and

$$\langle e_{1,-1} \omega \xi | e_{1,-1} \omega \xi \rangle = \langle v_\xi | \omega(e_{1,-1}) e_{1,-1} \omega \xi \rangle$$

we have $-\xi_1^- - \xi_1^+ + d \geq 0$. If $-\xi_1^- - \xi_1^+ + d = 0$, then we have $e_{1,-1} \omega \xi = 0$. Therefore we have

$$d - \xi_1^- = \xi_1^+ \geq 0.$$

On the other hand, by using (5.9) and $e_{1,-1} \omega \xi = 0$, we have

$$0 = \langle e_{\frac{1}{2},1} e_{1,-1} \omega \xi | e_{\frac{1}{2},1} e_{1,-1} \omega \xi \rangle$$

Thus we have $\xi_{\frac{1}{2}}^+ = \xi_1^+ = 0$. Similarly, we also have

$$0 = \langle e_{-1,-\frac{1}{2}} e_{1,-1} \omega \xi | e_{-1,-\frac{1}{2}} e_{1,-1} \omega \xi \rangle = \langle e_{1,-1} \omega \xi | e_{1,-1} \omega \xi \rangle$$

Thus we have $\xi_{\frac{1}{2}}^- = \xi_1^- = 0$. Hence $\xi_{\frac{1}{2}}^+ = \xi_1^+ = \xi_{\frac{1}{2}}^- = \xi_1^- = 0$ and $\min\{\xi_{\frac{1}{2}}^+, 1\} + \min\{\xi_{\frac{1}{2}}^-, 1\} + \xi_1^+ + \xi_1^- = 0 \leq d$ when $-\xi_1^- - \xi_1^+ + d = 0$. 


Now we assume that $-\xi^{+}_1 - \xi^{+}_1 + d > 0$. If $\xi^{+}_2 = 0$, then $\min\{\xi^{+}_2, 1\} + \min\{\xi^{+}_2, 1\} + \xi^{+}_1 + \xi^{+}_1 \leq 1 + \xi^{+}_1 + \xi^{+}_1 \leq d$ and the proof of the theorem is completed. Otherwise, we may assume that $\xi^{+}_1 > 0$. Also we may assume that $-\xi^{+}_1 - \xi^{+}_1 + d = 1$, otherwise the proof is also completed. Now we consider

$$
\langle e_{1,-1}e_{1,\frac{1}{2}}v_\xi \mid e_{1,-1}e_{1,\frac{1}{2}}v_\xi \rangle = \langle e_{1,\frac{1}{2}}v_\xi \mid e_{-1,1}e_{1,\frac{1}{2}}v_\xi \rangle = \langle e_{1,\frac{1}{2}}v_\xi \mid (e_{-1,-1} - e_{1,1} + C)e_{1,\frac{1}{2}}v_\xi \rangle = (-\xi^{+}_1 - \xi^{+}_1 - 1 + d)\langle e_{1,\frac{1}{2}}v_\xi \mid e_{1,\frac{1}{2}}v_\xi \rangle = (-\xi^{+}_1 - \xi^{+}_1 - 1 + d)(\xi^{+}_2 + \xi^{+}_1) = 0.
$$

Thus we have $\langle e_{1,-1}e_{1,\frac{1}{2}}v_\xi \mid e_{1,-1}e_{1,\frac{1}{2}}v_\xi \rangle = 0$ and

$$
\langle e_{-1,\frac{1}{2}}e_{1,-1}e_{1,\frac{1}{2}}v_\xi \mid e_{-1,\frac{1}{2}}e_{1,-1}e_{1,\frac{1}{2}}v_\xi \rangle = \langle e_{-1,-1}e_{1,\frac{1}{2}}v_\xi \mid e_{-1,-1}e_{1,\frac{1}{2}}v_\xi \rangle = \langle e_{1,\frac{1}{2}}v_\xi \mid -e_{-1,-1}e_{1,\frac{1}{2}}v_\xi \rangle = \langle e_{1,\frac{1}{2}}v_\xi \mid (e_{-1,-1} + e_{1,1} - 1)e_{1,\frac{1}{2}}v_\xi \rangle = (-\xi^{+}_1 - \xi^{+}_1)(\xi^{+}_2 + \xi^{+}_1) = (-\xi^{+}_1 - \xi^{+}_1)(\xi^{+}_2 + \xi^{+}_1).
$$

Therefore, we have $\xi^{+}_2 = 0$ and $\min\{\xi^{+}_2, 1\} + \min\{\xi^{+}_2, 1\} + \xi^{+}_1 + \xi^{+}_1 \leq 1 + \xi^{+}_1 + \xi^{+}_1 \leq d$. This completes the proof of the theorem. \hfill $\Box$

### 5.3. Unitarizable $\hat{c}$-modules and their Fock space realizations.

#### 5.3.1. Free field realization of $\hat{c}$ and $(\hat{c}, Sp(2d))$-duality.

Let us first recall some facts about the complex symplectic group $Sp(2d)$ (see, e.g., [2][3][4]). Consider the non-degenerate skew-symmetric bilinear form on $\mathbb{C}^{2d}$ given by the $2d \times 2d$ matrix

$$
J_{2d} = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix},
$$

where $I_d$ is the $d \times d$ identity matrix. The symplectic group $Sp(2d)$ is the subgroup of $GL(2d)$ which consists of those $A \in GL(2d)$ with $A^tJ_{2d}A = J_{2d}$, where $A^t$ is the transpose of the matrix $A$. The Lie algebra of $Sp(2d)$ is $\mathfrak{sp}(2d)$ which consists of those $A \in gl(2d)$ with $A^tJ_{2d} + J_{2d}A = 0$. Denote by $e_{ij}$ the elementary matrix with 1 in the $i$-th row and $j$-th column and 0 elsewhere. Then $\mathfrak{h} := \sum_{1 \leq i < d} \mathbb{C}(e_{ii} - e_{d+i,d+i})$ is a Cartan subalgebra, while $\mathfrak{b} := \sum_{1 \leq i \leq d} \mathbb{C}(e_{ii} - e_{j+d,i+d}) + \sum_{i \leq j \leq d} \mathbb{C}(e_{i,j+d} + e_{j+d,i})$ is the standard Borel subalgebra containing $\mathfrak{h}$. Let $h_i = e_{ii} - e_{d+i,d+i}$.

We write an element $\lambda \in \mathfrak{h}^*$ as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ where $\lambda_i = \lambda(h_i)$ for $i = 1, 2, \ldots, d$. Let $V^\lambda_{sp(2d)}$ denote the irreducible $\mathfrak{sp}(2d)$-module with highest weight $\lambda \in \mathfrak{h}^*$ defined with respect to the standard Borel subalgebra. Then $V^\lambda_{sp(2d)}$ is finite-dimensional if and only if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ and $\lambda_i \in \mathbb{Z}_+$ for
\(i = 1, \ldots, d\). Furthermore every such representation lifts to a unique irreducible representation of \(Sp(2d)\), which is denoted by \(V^\lambda_{Sp(2d)}\) and so we obtain an obvious parametrisation of \(Sp(2d)\)-highest weights in terms of Young diagrams \(\lambda\) with \(l(\lambda) \leq d\).

We let \(\epsilon_i \in h^*\) so that \(\epsilon_i(h_j) = \delta_{ij}\). We put \(z_i = e^{\epsilon_i}\) when dealing with characters of \(Sp(2d)\).

Introduce the following operators on the Fock space \(\mathcal{F}_0^d\):

\[
E^+_{ij} = \sum_{n \in \mathbb{N}} \psi^{+,j}_n \psi^{+,j}_{-n} - \sum_{n \in -\mathbb{N}} \psi^{+,j}_{-n} \psi^{+,j}_n + \sum_{r \in 1/2 + \mathbb{Z}} \gamma^{+,j}_r \gamma^{+,j}_{-r}; \tag{5.10}
\]
\[
E^-_{ij} = \sum_{n \in \mathbb{N}} \psi^{-,j}_n \psi^{-,j}_{-n} - \sum_{n \in -\mathbb{N}} \psi^{-,j}_{-n} \psi^{-,j}_n - \sum_{r \in 1/2 + \mathbb{Z}} \gamma^{-,j}_r \gamma^{-,j}_{-r}; \tag{5.11}
\]

where \(1 \leq i, j \leq d\). It is clear that (5.10) and (5.11) together with (5.7) form a basis for the Lie algebra \(sp(2d)\). The action of the Lie algebra \(sp(2d)\) on the Fock space \(\mathcal{F}_0^d\) can be lifted to an action of Lie group \(Sp(2d)\). Moreover \(\mathcal{F}_0^d\) is a direct sum of finite dimensional irreducible \(Sp(2d)\)-modules.

On the other hand, \(Sp(2d)\) acts on \(\mathcal{A}_0^d\) by conjugation. It is not hard to see that the \(Sp(2d)\)-invariants of the associative algebra \(\mathcal{A}_0^d\) is generated by the following combinations of the elements of (5.5):

\[
C; \quad \tilde{e}_{r,s} = e_{r,s} - e_{-s,-r}; \tag{5.12}
\]
\[
\tilde{e}_{i,j} = e_{i,j} - e_{-j,-i}, \quad ij > 0; \quad \tilde{e}_{i,j} = e_{i,j} + e_{-j,-i}, \quad ij < 0; \tag{5.13}
\]
\[
\tilde{e}_{i,r} = \tilde{e}_{-r,-i} = e_{i,r} + e_{-r,-i}, \quad i > 0; \quad \tilde{e}_{i,r} = -\tilde{e}_{-r,-i} = e_{i,r} - e_{-r,-i}, \quad i < 0, \tag{5.14}
\]

where \(i, j \in \mathbb{Z}^*\) and \(r, s \in \frac{1}{2} + \mathbb{Z}\). Note that (5.12), (5.13) and (5.14) form the Lie superalgebra \(\widehat{\mathcal{C}}^\ell\) and hence \(Sp(2d)\) commutes with \(\widehat{\mathcal{C}}\). By a result of Howe \[13\], we have the following multiplicity-free decomposition

\[
\mathcal{F}_0^d \cong \sum_{\lambda} W^\lambda \otimes V^\lambda_{Sp(2d)}; \tag{5.15}
\]

of \(\mathcal{F}_0^d\) with respect to the joint action of \(\widehat{\mathcal{C}}\) and \(Sp(2d)\), where the summation is over a subset of all partitions of length \(d\). Here \(W^\lambda\) denotes an irreducible module over \(\widehat{\mathcal{C}}\).

For any given partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)\) of length \(d\), we have the shifted Frobenius notation \(\Lambda(\lambda) = (\xi_{1, \frac{1}{2}}, \xi_{1, \frac{1}{2}}, \ldots, \xi_{r-\frac{1}{2}, \frac{1}{2}} \mid \xi_1, \xi_2, \ldots, \xi_r)\) (see Section 2.1). Let \(\Lambda^{\widehat{\mathcal{C}}}(\lambda)\) be an element of the dual space \(\widehat{\mathcal{F}}_0^d\) of \(\mathcal{F}_0^d\) defined by

\[
\Lambda^{\widehat{\mathcal{C}}}(\lambda) := \sum_{\substack{j \leq r \mid j \in \frac{1}{2} \mathbb{N}}} \xi_j \omega_j + d\Lambda_0. \tag{5.16}
\]
Let $\lambda$ be a partition of length $d$. Then the following vector
\begin{equation}
X^1_{\lambda_1} \cdot X^2_{\lambda_2} \cdots X^\lambda_{\lambda_1} |0\rangle
\end{equation}
is a highest weight vector of the Lie superalgebra $\hat{\mathfrak{c}}$, which is defined with respect to the Borel subalgebra $\hat{\mathfrak{c}}_0 \oplus \hat{\mathfrak{c}}_+$ of $\hat{\mathfrak{c}}$ (see Remark 3.1) and has weight $\Lambda^{\hat{\mathfrak{c}}} (\lambda)$. This follows from the fact that the vector is actually a highest weight vector of $\hat{\mathfrak{a}}$. It is easy to see that the vector given by (5.17) is also annihilated by (5.10). Thus it is a joint highest weight vector of $\text{Sp}(2d)$ and $\hat{\mathfrak{c}}$. Note that the weights of the vectors of the form (5.17) with respect to the Lie group $\text{Sp}(2d)$ are exactly the weights associated with all partitions of length $d$. By the decomposition of (5.15), we have the following theorem.

**Theorem 5.6.** The Lie superalgebra $\hat{\mathfrak{c}}$ and $\text{Sp}(2d)$ form a dual pair on $F^d_0$ in the sense of Howe. Furthermore we have the following (multiplicity-free) decomposition of $F^d_0$ with respect to their joint action
\[
F^d_0 \cong \sum_{\lambda} L(\hat{\mathfrak{c}}, \Lambda^{\hat{\mathfrak{c}}} (\lambda)) \otimes V^\lambda_{\text{Sp}(2d)},
\]
where the summation is over all partitions of length $d$. The joint highest weight vector of the $\lambda$-component is given by
\[
X^1_{\lambda_1} \cdot X^2_{\lambda_2} \cdots X^\lambda_{\lambda_1} |0\rangle.
\]

5.3.2. Unitarizable $\hat{\mathfrak{c}}$-modules. The restriction of the anti-linear anti-involution $\omega$ on the Lie superalgebra $\hat{\mathfrak{a}}$ to $\hat{\mathfrak{c}}$ gives rise to an anti-linear anti-involution on $\hat{\mathfrak{c}}$, which will also be denoted by $\omega$. It satisfies $\omega(C) = C$ and
\[
\omega \left( \sum_{p \in \frac{1}{2} \mathbb{Z}^*} a_p \tilde{e}_{-p,k} \right) = \sum_{p \in \frac{1}{2} \mathbb{Z}^*} (-1)^{|p| + |p-k|} \tau_{p, p-k} \tilde{e}_{p,p-k},
\]
for all $\sum_{p \in \frac{1}{2} \mathbb{Z}^*} a_p \tilde{e}_{-p,k} \in (\hat{\mathfrak{c}})_k$ and for all $k \in \frac{1}{2} \mathbb{Z}$. Since the Fock space $F^d_0$ is a unitarizable $\hat{\mathfrak{a}}$-module with the positive definite contravariant Hermitian form $( \cdot | \cdot )$, it is also a unitarizable $\hat{\mathfrak{c}}$-module with respect to the anti-linear anti-involution $\omega$. By Theorem 5.6 the irreducible $\hat{\mathfrak{c}}$-module $L(\hat{\mathfrak{c}}, \Lambda^{\hat{\mathfrak{c}}} (\lambda))$ is unitarizable with respect to $\omega$ for every partition $\lambda$ of length $d$. In fact these modules exhaust all the irreducible quasi-finite highest weight $\hat{\mathfrak{c}}$-modules, which are unitarizable with respect to $\omega$. We have the following theorem.

**Theorem 5.7.** Let $M$ be an irreducible quasi-finite highest weight $\hat{\mathfrak{c}}$-module with highest weight $\xi$. Then $M$ is unitarizable if and only if $\xi = \Lambda^{\hat{\mathfrak{c}}} (\lambda)$ for some
partition $\lambda$ of length $d$. In other words, $M$ is unitarizable if and only if

$$\Lambda^\hat{C}(\lambda) := \sum_{j \leq r, j \in \frac{1}{2}N} \xi_j \omega_j + d\Lambda_0$$

such that $d \in \mathbb{Z}_+, r \in \mathbb{N}$ and $\xi_j \in \mathbb{Z}$ for all $j$ satisfying the following conditions:

(i) $\frac{1}{2} > \xi_{\frac{1}{2}} > \cdots > \xi_{\frac{r-\frac{1}{2}}{2}} \geq 0$, $\xi_1 > \xi_2 > \cdots > \xi_r \geq 0$, and $\xi_{r-\frac{1}{2}} = 0$ if only if $r = 1$ and $\xi_{\frac{1}{2}} = \xi_1 = 0$,

(ii) $\min\{\xi_{\frac{1}{2}}, 1\} + \xi_1 \leq d$.

Proof. By the argument above, $L(\hat{C}, \Lambda^\hat{C}(\lambda))$ are unitarizable irreducible quasi-finite highest weight $\hat{C}$-modules for any partition $\lambda$. Now we are going to show that if $M$ is a unitarizable irreducible quasi-finite highest weight $\hat{C}$-module with the highest weight $\xi$, then $\xi = \Lambda^\hat{C}(\lambda)$ for some partition $\lambda$. Let $\langle \cdot | \cdot \rangle$ be a positive definite contravariant Hermitian form on $M$ and $v_\xi$ a highest weight vector of $M$ such that $\langle v_\xi | v_\xi \rangle = 1$. We put $\xi(i, i) = \xi_i$ for all $i \in \frac{1}{2}\mathbb{N}$. By Theorem 4.7 there exists $r, s \in \mathbb{N}$ such that $\xi = \sum_{j \leq r} \xi_j \omega_j + d\Lambda_0$, where $\xi_j, d \in \mathbb{C}$. Using similar arguments as in the proof of Theorem 5.2 we can show that $\xi_i \in \mathbb{Z}_+$ for $i = \frac{1}{2}, 1, \cdots, r - \frac{1}{2}, r$, and

$$\xi_{\frac{1}{2}} > \xi_{\frac{1}{2}} > \cdots > \xi_{\frac{r-\frac{1}{2}}{2}} \geq 0, \quad \xi_1 > \xi_2 > \cdots > \xi_r \geq 0.$$  

Moreover, we also have $\xi_{r-\frac{1}{2}} = 0$ if only if $r = 1$ and $\xi_{\frac{1}{2}} = \xi_1 = 0$.

Now we choose a large positive integer $n$ such that $\xi(\hat{e}_{n,n}) = 0$. Consider the subalgebra $sl(2, \mathbb{C})$ with standard basis $\{-\hat{e}_{n,n} + C, \frac{1}{2}\hat{e}_{-n,n}, \frac{1}{2}\hat{e}_{n,-n}\}$. Note that $\omega(\frac{1}{2}\hat{e}_{-n,n}) = \frac{1}{2}\hat{e}_{-n,-n}$, $\omega(\frac{1}{2}\hat{e}_{n,-n}) = \frac{1}{2}\hat{e}_{n,-n}$ and $\omega(-\hat{e}_{n,n} + C) = -\hat{e}_{n,n} + C$. A standard result on unitarizable $sl(2, \mathbb{C})$-modules (see, e.g., [16]) leads to $d = \xi(C) = \xi(-\hat{e}_{n,n} + C) \in \mathbb{Z}_+$.

Finally, we need to show $\min\{\xi_{\frac{1}{2}}, 1\} + \xi_1 \leq d$. Since $\langle \hat{e}_{1,-1} v_\xi | \hat{e}_{1,-1} v_\xi \rangle \geq 0$ and

$$\langle \frac{1}{2}\hat{e}_{1,-1} v_\xi | \frac{1}{2}\hat{e}_{1,-1} v_\xi \rangle = \langle v_\xi | \frac{1}{2}\omega(\frac{1}{2}\hat{e}_{1,-1})\hat{e}_{1,-1} v_\xi \rangle = \langle v_\xi | (-\hat{e}_{1,1} + C) v_\xi \rangle = d - \xi_1,$$

we have $d \geq \xi_1$. If $d > \xi_1$, the proof the theorem is completed. Otherwise, we assume $d - \xi_1 = 0$ and hence $\hat{e}_{1,-1} v_\xi = 0$. 


On the other hand, by using \( \tilde{e}_{1,-1}v_\xi = 0 \) and \( d - \xi_1 = 0 \), we have
\[
0 = \langle \tilde{e}_{1,1} \tilde{e}_{1,-1}v_\xi | \tilde{e}_{1,1} \tilde{e}_{1,-1}v_\xi \rangle \\
= \langle \tilde{e}_{2,-1}v_\xi | \tilde{e}_{2,-1}v_\xi \rangle \\
= \langle v_\xi | \omega(\tilde{e}_{1,1}) \tilde{e}_{2,-1}v_\xi \rangle \\
= \langle v_\xi | (2C - \tilde{e}_{1,1} + \tilde{e}_{2,-1})v_\xi \rangle \\
= 2d - \xi_1 + \xi_1 \\
= d + \xi_1.
\]
Hence we have \( d = \xi_1 = \xi_1 = 0 \) and \( \min\{\xi_1, 1\} + \xi_1 \leq d \). This completes the proof of the theorem. \qed

5.4. Unitarizable \( \hat{D} \)-modules and their Fock space realizations. In this subsection we will construct two types of free field realizations of \( \hat{D} \) which are respectively associated with the \((\hat{D}, O(2d))\) and \((\hat{D}, O(2d+1))\)-dualities.

5.4.1. Basic facts on representations of the complex orthogonal group. We start by recalling some facts about finite dimensional representations of the complex orthogonal group \( O(k) \) (see, e.g., [2, 10, 11]). We first consider the case when \( k = 2d \) is even. Consider the non-degenerate symmetric bilinear form on \( \mathbb{C}^{2d} \) given by the \( 2d \times 2d \) matrix
\[
K_{2d} = \begin{pmatrix}
0 & I_d \\
I_d & 0
\end{pmatrix},
\]
where \( I_d \) is the \( d \times d \) identity matrix. The orthogonal group \( O(2d) \) is the subgroup of \( GL(2d) \) which consists of those \( A \in GL(2d) \) with \( A^t K_{2d} A = K_{2d} \), where \( A^t \) is the transpose of the matrix \( A \). The Lie algebra of \( O(2d) \) is \( \mathfrak{so}(2d) \) which consists of those \( A \in gl(2d) \) with \( A^t K_{2d} + K_{2d} A = 0 \).

Denote by \( e_{ij} \) the elementary matrix with 1 in the \( i \)-th row and \( j \)-th column and 0 elsewhere. Let \( h_i := e_{ii} - e_{d+i,d+i}, E_{i,j}^{so} := e_{i,j+d} - e_{j,i+d} \) and \( E_{i,j}^{so} := e_{i+d,j} - e_{j+d,i} \) for \( 1 \leq i, j \leq d \). Then \( \mathfrak{h} := \sum_{1 \leq i \leq d} \mathfrak{h}_i \) is a Cartan subalgebra, while \( \mathfrak{b} := \sum_{1 \leq i \leq j \leq d} \mathbb{C}(e_{i,j} - e_{j+i,d+i}) + \sum_{1 \leq i, j \leq d} \mathbb{C}E_{i,j}^{so} \) is the standard Borel subalgebra containing \( \mathfrak{h} \).

Write an element \( \lambda \in \mathfrak{h}^* \) as \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \), where \( \lambda_i = \lambda(h_i) \), for \( i = 1, 2, \ldots, d \). Let \( V^\lambda_{\mathfrak{so}(2d)} \) denote the irreducible highest weight \( \mathfrak{so}(2d) \)-module with highest weight \( \lambda \in \mathfrak{h}^* \) defined with respect to the standard Borel subalgebra. Then \( V^\lambda_{\mathfrak{so}(2d)} \) is finite dimensional if and only if \( \lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_d| \) with either \( \lambda_i \in \mathbb{Z} \) or else \( \lambda_i \in \frac{1}{2} + \mathbb{Z} \) for all \( i = 1, \ldots, d \). Furthermore, the \( \mathfrak{so}(2d) \)-module \( V^\lambda_{\mathfrak{so}(2d)} \) lifts to an \( SO(2d) \)-module if and only if \( \lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_d| \) with \( \lambda_i \in \mathbb{Z} \) for all \( i = 1, \ldots, d \).

Let \( V \) be a finite-dimensional irreducible \( O(2d) \)-module. When regarded as an \( \mathfrak{so}(2d) \)-module we have the following possibilities:
There is a one to one correspondence between the finite dimensional irreducible \( \mathfrak{so}(2d) \)-representations and the partitions \( \lambda \) of length \( 2d \) satisfying the condition \( \lambda_1 + \lambda_2 \leq 2d \), we denote by \( \lambda \) the partition obtained from \( \lambda \) by replacing its first column by a column of length \( 2d - \lambda' \). Let \( V^\lambda_{O(2d)} := W^\lambda_{O(2d)} \otimes \text{det} \).

(i) \( V \) is a direct sum of two irreducible \( \mathfrak{so}(2d) \)-modules with integral highest weights \( (\lambda_1, \lambda_2, \cdots, \lambda_d) \) and \( (\lambda_1, \lambda_2, \cdots, \lambda_{k-1}, -\lambda_d) \) respectively, where \( \lambda_k > 0 \).

(ii) \( V \) is an irreducible \( \mathfrak{so}(2d) \)-module with integral highest weight of the form 
\[
(\lambda_1, \lambda_2, \cdots, \lambda_{d-1}, 0).
\]

In the first case, that is when \( V \) is the direct sum of the two irreducible \( \mathfrak{so}(2d) \)-modules, we denote \( V \) by \( W^\lambda_{O(2d)} \), where we let \( \lambda := (\lambda_1, \lambda_2, \cdots, \lambda_{d-1}, \lambda_d > 0) \). In the second case there are two possible choices of \( V \), which we denote by \( W^\lambda_{O(2d)} \) and \( W^\lambda_{O(2d)} \otimes \text{det} \), respectively. Recalling that \( O(2d) \) is a semidirect product of \( SO(2d) \) and \( \mathbb{Z}_2 \). Thus the \( O(2d) \)-modules \( W^\lambda_{O(2d)} \) and \( W^\lambda_{O(2d)} \otimes \text{det} \) restrict to isomorphic \( SO(2d) \)-modules. However as \( O(2d) \)-modules they differ by the determinant representation so that we may distinguish these two modules as follows: consider the element \( \tau \in O(2d) - SO(2d) \) that switches the basis vector \( e^d \) with \( e^{2d} \) and leaves all other basis vectors of \( \mathbb{C}^{2d} \) invariant. We declare \( W^\lambda_{O(2d)} \) to be the \( O(2d) \)-module on which \( \tau \) transforms an \( SO(d) \)-highest weight vector trivially. Note that \( \tau \) transforms an \( SO(2d) \)-highest weight vector in the \( O(2d) \)-module \( W^\lambda_{O(2d)} \otimes \text{det} \) by \( -1 \).

We may associate Young diagrams \( \lambda \) of length \( 2d \) to these integral highest weights of \( O(2d) \) as follows (cf. [14]). For \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0 \) with \( \lambda_i \in \mathbb{Z}_+ \) for all \( i \), we have an obvious Young diagram of length \( 2d \) by putting \( \lambda := (\lambda_1, \lambda_2, \cdots, \lambda_d, 0, \cdots, 0) \). When \( \lambda_d = 0 \), we associate to the highest weight of \( W^\lambda_{O(2d)} \) the usual Young diagram of length \( 2d \) by putting \( \lambda := (\lambda_1, \lambda_2, \cdots, \lambda_d, 0, \cdots, 0) \). We put \( V^\lambda_{O(2d)} := W^\lambda_{O(2d)} \). To the highest weight of \( W^\lambda_{O(2d)} \otimes \text{det} \) we associate the Young diagram \( \bar{\lambda} \) obtained from \( \lambda \) by replacing its first column by a column of length \( 2d - \lambda'_1 \). Let \( V^\bar{\lambda}_{O(2d)} := W^\bar{\lambda}_{O(2d)} \otimes \text{det} \).

Hereafter, we shall adopt the following convention. Given any partition \( \lambda \) of length \( 2d \) satisfying the condition \( \lambda_1 + \lambda_2 \leq 2d \), we denote by \( \bar{\lambda} \) the partition obtained from \( \lambda \) by replacing its first column by a column of length \( 2d - \lambda'_1 \). There is a one to one correspondence between the finite dimensional irreducible \( O(2d) \)-representations and the partitions \( \lambda \) of length \( 2d \) satisfying the condition \( \lambda_1 + \lambda_2 \leq 2d \).

Next consider the case when \( k = 2d + 1 \) is odd. Take the non-degenerate symmetric bilinear form on \( \mathbb{C}^{2d+1} \) given by the \( (2d + 1) \times (2d + 1) \) matrix
\[
K_{2d+1} = \begin{pmatrix}
0 & 0 & I_d \\
0 & 1 & 0 \\
I_d & 0 & 0
\end{pmatrix},
\]
where \( I_d \) is the \( d \times d \) identity matrix. The orthogonal group \( O(2d + 1) \) is the subgroup of \( GL(2d+1) \) which consists of those \( A \in GL(2d+1) \) with \( A^t K_{2d+1} A = K_{2d+1} \).
of $W$ obtained from $O$ the identity $(2\lambda_i)$.

In this case, we let $\lambda_i^-$ the elementary matrix with 1 in the $i$-th row and $j$-th column and 0 elsewhere. Let $h_i := e_{ii} - e_{d+1,i,d+1+i}$, $E_i^{so+} := e_{i,d+1} - e_{d+1,d+1+i}$, $E_i^{so-} := e_{d+1,j} - e_{j,d+1}$, $E_{ij}^{so} := e_{i,j+d+1} - e_{j,i+d+1}$ and $E_{ij}^{so-} := e_{i,j+1,d+1} - e_{j,j+1,i}$ for $1 \leq i,j \leq d$. Then $h := \sum_{1 \leq i \leq d} C h_i$ is a Cartan subalgebra, while $b := \sum_{1 \leq i,j \leq d} C(e_{ij} - e_{j+i+1,j+d+1}) + \sum_{1 \leq i \leq d} C E_i^{so+} + \sum_{1 \leq i,j \leq d} CE_{ij}^{so}$ is the standard Borel subalgebra containing $h$.

Write an element $\lambda \in h^*$ as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$, where $\lambda_i = \lambda(h_i)$, for $i = 1, 2, \ldots, d$. Let $V_{so(2d+1)}^\lambda$ denote the irreducible highest weight $so(2d+1)$-module with highest weight $\lambda \in h^*$ defined with respect to the standard Borel subalgebra. Then $V_{so(2d+1)}^\lambda$ is finite dimensional if and only if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ with either $\lambda_i \in \mathbb{Z}_+$ or $\lambda_i \in \frac{1}{2} + \mathbb{Z}_+$ for all $i = 1, \ldots, d$. Furthermore $V_{so(2d+1)}^\lambda$ lifts to a representation of $SO(2d+1)$ if and only if $\lambda_i \in \mathbb{Z}_+$.

Recall that $O(2d+1)$ is a direct product of $SO(2d+1)$ and $Z_2$. Thus any finite-dimensional irreducible representation of $O(2d+1)$, when regarded as an $SO(2d+1)$-module, remains irreducible. Conversely an irreducible representation of $SO(2d+1)$ gives rise to two non-isomorphic $O(2d+1)$-modules that differ from each other by the determinant representation det. We let $W_{O(2d+1)}^{\lambda}$ stand for the irreducible $O(2d+1)$-module corresponding to $\bar{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ on which the element $-I_{2d+1}$ transforms trivially, so that $\{\ W_{O(2d+1)}^{\lambda}, W_{O(2d+1)}^{\lambda} \otimes \text{det} \}$ with $\bar{\lambda}$ ranging over all partitions with length $d$ as above is a complete set of finite-dimensional non-isomorphic irreducible $O(2d+1)$-modules, where $I_{2d+1}$ is the identity $(2d+1) \times (2d+1)$ matrix.

Similarly as before we may associate Young diagrams of length $2d+1$ to these $O(2d+1)$-highest weights. For the highest weight $\bar{\lambda} = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_k \geq 0)$ of $W_{O(2d+1)}^{\lambda}$, we have an obvious Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d, 0, \cdots, 0)$ of length $2d+1$. Let $V_{O(2d+1)}^{\lambda} := W_{O(2d+1)}^{\bar{\lambda}}$.

To the highest weight of $W_{O(2d+1)}^{\lambda} \otimes \text{det}$ we associate the Young diagram $\bar{\lambda}$ obtained from $\lambda$ by replacing its first column by a column of length $2d+1 - \lambda'_1$.

In this case, we let $V_{O(2d+1)}^{\lambda} := W_{O(2d+1)}^{\bar{\lambda}} \otimes \text{det}$.

Hereafter, we adopt the convention that for any given any partition $\lambda$ of length $2d+1$ satisfying $\lambda'_1 + \lambda'_2 \leq 2d+1$, we denote by $\bar{\lambda}$ the partition of length $2d+1$ obtained from $\lambda$ by replacing its first column by a column of length $2d+1 - \lambda'_1$. There is a one to one correspondence between the finite dimensional irreducible representations of $O(2d+1)$ and the partitions $\lambda$ of length $2d+1$ satisfying $\lambda'_1 + \lambda'_2 \leq 2d+1$.

Let $\epsilon_i \in h^*$ so that $\epsilon_i(h_j) = \delta_{ij}$. We put $z_i = e^{\epsilon_i}$ when dealing with characters of $O(2d)$ and $O(2d+1)$.
5.4.2. Free field realization of \( \hat{\mathfrak{D}} \) and \( (\hat{\mathfrak{D}}, O(2d)) \)-duality. Let us consider the realization of \( \hat{\mathfrak{D}} \) on the Fock space \( \mathcal{F}^d_0 \) related to the \( (\hat{\mathfrak{D}}, O(2d)) \)-duality. Introduce the following operators on \( \mathcal{F}^d_0 \):

\[
E_{ij}^{so^+} = \sum_{n \in \mathbb{Z}^*} \psi^{+}_{-n} \psi^{+}_{n} : + \sum_{r \in 1/2 + \mathbb{Z}^*_+} \gamma^{+}_{-r} \gamma^{+}_{r} : - \sum_{r \in 1/2 - \mathbb{Z}^*_+} \gamma^{+}_{-r} \gamma^{+}_{r} :,
\]

\[
E_{ij}^{so^-} = \sum_{n \in \mathbb{Z}^*} \psi^{-}_{-n} \psi^{-}_{n} : - \sum_{r \in 1/2 + \mathbb{Z}^*_+} \gamma^{-}_{-r} \gamma^{-}_{r} : + \sum_{r \in 1/2 - \mathbb{Z}^*_+} \gamma^{-}_{-r} \gamma^{-}_{r} :,
\]

where \( 1 \leq i, j \leq d \). It is easy to see that (5.18) and (5.19) together with (5.7) form a basis for the Lie algebra \( so(2d) \). The action of the Lie algebra \( so(2d) \) on the Fock space \( \mathcal{F}^d_0 \) can be lifted to an action of Lie group \( SO(2d) \) and extend to the action of the Lie group \( O(2d) \). Moreover \( \mathcal{F}^d_0 \) is a direct sum of finite dimensional irreducible modules over \( O(2d) \).

On the other hand, \( O(2d) \) acts on \( \mathcal{A}^d_0 \) by conjugation. It is not hard to see that the \( O(2d) \)-invariants in the associative algebra \( \mathcal{A}^d_0 \) is generated by the following combinations of the elements of (5.3):

\[
C; \quad \tilde{e}_{i,j} = e_{i,j} - e_{-j,-i};
\]

\[
\tilde{e}_{r,s} = e_{r,s} - e_{-s,-r}, \quad rs > 0; \quad \tilde{e}_{r,s} = e_{r,s} + e_{-s,-r}, \quad rs < 0;
\]

\[
\tilde{e}_{i,r} = \tilde{e}_{-r,-i} = e_{i,r} + e_{-r,-i}, \quad r > 0; \quad \tilde{e}_{i,r} = -\tilde{e}_{-r,-i} = e_{i,r} - e_{-r,-i}, \quad r < 0,
\]

where \( i, j \in \mathbb{Z}^* \) and \( r, s \in \frac{1}{2} + \mathbb{Z} \). Note that (5.20), (5.21) and (5.22) form the Lie superalgebra \( \hat{\mathcal{D}} \) in \( \mathcal{A}^d_0 \). Therefore the \( \hat{\mathcal{D}} \)-action on \( \mathcal{F}^d_0 \) commutes with the \( O(2d) \)-action. Following the general reasoning of [14], we have the following multiplicity-free decomposition of \( \mathcal{F}^d_0 \) with respect to the joint action of \( \hat{\mathcal{D}} \) and \( O(2d) \):

\[
\mathcal{F}^d_0 \cong \bigoplus_{\lambda} W^\lambda \otimes V_{O(2d)}^\lambda,
\]

where the summation is over a subset of all partitions of length with \( \lambda_1 + \lambda_2 \leq 2d \), and \( W^\lambda \) is a certain irreducible module over \( \hat{\mathcal{D}} \).

For each \( j \in \{1, 2, \cdots, d\} \), we define the \( d \times d \) matrix \( \tilde{X}^j \) as follows:

\[
\tilde{X}^j := \begin{pmatrix}
\gamma_{\lambda_1 + 1}^j & \gamma_{\lambda_2 + 2}^j & \cdots & \gamma_{\lambda_1 + d - 1}^j & \gamma_{\lambda_2 - d}^j \\
\gamma_{\lambda_1 + 1}^j & \gamma_{\lambda_2 + 2}^j & \cdots & \gamma_{\lambda_1 + d - 1}^j & \gamma_{\lambda_2 - d}^j \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma_{\lambda_1 + 1}^j & \gamma_{\lambda_2 + 2}^j & \cdots & \gamma_{\lambda_1 + d - 1}^j & \gamma_{\lambda_2 - d}^j \\
\gamma_{\lambda_1 + 1}^j & \gamma_{\lambda_2 + 2}^j & \cdots & \gamma_{\lambda_1 + d - 1}^j & \gamma_{\lambda_2 - d}^j \\
\psi_{-j}^1 & \psi_{-j}^2 & \cdots & \psi_{-j}^{d-1} & \psi_{-j}^{-d} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\psi_{-j}^1 & \psi_{-j}^2 & \cdots & \psi_{-j}^{d-1} & \psi_{-j}^{-d} \\
\psi_{-j}^1 & \psi_{-j}^2 & \cdots & \psi_{-j}^{d-1} & \psi_{-j}^{-d}
\end{pmatrix}
\]
For any integer \( j \geq d \), \( \tilde{X}^j := \tilde{X}^d \). Note that the matrix \( \tilde{X}^j \) is obtained from \( X^j \) by replacing its last column by \((\gamma_{-d}^1, \gamma_{-d}^2, \cdots, \gamma_{-d}^d, \psi_{-j}^d \cdots \psi_{-j}^d)\). For \( 0 \leq r \leq d \) and \( i > 0 \), we let \( \tilde{X}^i_r \) denote the first \( r \times r \) minor of the matrix \( \tilde{X}^i \).

We define the \( 2d \times 2d \) matrix \( \Gamma \) as follows:

\[
\Gamma := \begin{pmatrix}
\gamma_{-d}^1 & \gamma_{-d}^2 & \cdots & \gamma_{-d}^d & \gamma_{-d}^1 & \cdots & \gamma_{-d}^d \\
\psi_{-1}^d & \psi_{-1}^d & \cdots & \psi_{-1}^d & \psi_{-1}^d & \cdots & \psi_{-1}^d \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\psi_{-1}^1 & \psi_{-1}^2 & \cdots & \psi_{-1}^d & \psi_{-1}^1 & \cdots & \psi_{-1}^d
\end{pmatrix}.
\]

For \( r > d \), we let \( \Gamma_r \) denote the first \( r \times r \) minor of the matrix \( \Gamma \).

Given a partition \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{2d}) \) of length \( 2d \) with \( \lambda_1 + \lambda_2 \leq 2d \), we have the shifted Frobenius notation \( \Lambda(\lambda) = (\xi_1^1, \xi_1^2, \cdots, \xi_r^d \mid \xi_1, \xi_2, \cdots, \xi_r) \) for the partition \( \lambda \) (see Section 2.1). Let \( \Lambda^\tilde{D}(\lambda) \) an element of the dual space \( \tilde{D}_0^* \) of the vector space \( \tilde{D}_0 \) defined by

\[
\Lambda^\tilde{D}(\lambda) := \sum_{j \in \mathbb{N}} \xi_j \omega_j + d \Lambda_0.
\]

By using similar arguments as in [7] (see also [24]), we can explicitly construct the joint highest weight vectors of \( \tilde{D} \) and \( SO(2d) \) appearing in (5.23). The following theorem is an easy consequence of the decomposition (5.23) and the description of the joint highest weight vectors.

**Theorem 5.8.** The Lie superalgebra \( \tilde{D} \) and \( O(2d) \) form a dual pair on \( \mathfrak{f}_d^d \) in the sense of Howe, namely, we have the following multiplicity-free decomposition of \( \mathfrak{f}_d^d \) with respect to their joint action

\[
\mathfrak{f}_d^d \cong \sum_\lambda L(\tilde{D}, \Lambda^\tilde{D}(\lambda)) \otimes V_\lambda^{O(2d)},
\]

where the summation is over all partitions of length \( 2d \) with \( \lambda_1 + \lambda_2 \leq 2d \). Furthermore the \( \tilde{D} \times \mathfrak{so}(2d) \) joint highest weight vector of the \( \lambda \)-component can be described in the following way.

(i) When \( \lambda_1 < d \),

\[
X_{\lambda_1}^1 \cdot X_{\lambda_2}^2 \cdots X_{\lambda_1}^{\lambda_1} |0\rangle
\]

is a joint highest weight vector of \( \tilde{D} \times \mathfrak{so}(2d) \) with the joint highest weight

\[
\Lambda^\tilde{D}(\lambda) + \sum_{i=1}^d \lambda_i \epsilon_i \in \tilde{D}_0^* \oplus \mathfrak{h}^*.
\]

(ii) When \( \lambda_1 = d \),

\[
X_{\lambda_1}^1 \cdot X_{\lambda_2}^2 \cdots X_{\lambda_1}^{\lambda_1} |0\rangle
\]
and
\[ \tilde{X}_{X_1}^1 \cdot \tilde{X}_{X_2}^2 \cdots \tilde{X}_{X_{\lambda_1}}^{\lambda_1} |0\rangle \]
are joint highest weight vectors of \( \hat{D} \times \mathfrak{so}(2d) \) with the joint highest weights
\[ \sum_{i=1}^{d} \lambda_i \epsilon_i + \Lambda_{\hat{D}}(\lambda) \] and \[ \sum_{i=1}^{d-1} \lambda_i \epsilon_i - \lambda_d \epsilon_d + \Lambda_{\hat{D}}(\lambda), \] respectively.

(iii) When \( \lambda_1' > d \),
\[ \Gamma_{\lambda_1'} \cdot X_{\lambda_2}^2 \cdots X_{\lambda_{\lambda_1'}}^{\lambda_{\lambda_1'}} |0\rangle \]
is a joint highest weight vector of \( \hat{D} \times \mathfrak{so}(2d) \) with the joint highest weight
\[ \sum_{i=1}^{2d-\lambda_1'} \lambda_i \epsilon_i + \Lambda_{\hat{D}}(\lambda). \]

**Remark 5.2.** By examining the embedding of \( \hat{D} \) in \( \hat{A} \) (also recalling Remark 3.1), we can see that the restriction of the anti-linear anti-involution \( \omega \) on the Lie superalgebra \( \hat{A} \) to \( \hat{D} \) gives an anti-linear anti-involution also denoted by \( \omega \) on \( \hat{D} \) such that \( \omega(C) = C \) and
\[ \omega\left( \sum_{p \in \frac{1}{2}Z^*} a_p \tilde{e}_{p-k,p} \right) = \sum_{p \in \frac{1}{2}Z^*} (-1)^{[p]} \cdot [p-k] \tilde{e}_{p-k,p}, \]
for all \( \sum_{p \in \frac{1}{2}Z^*} a_p \tilde{e}_{p-k,p} \in \hat{D}_k \), \( k \in \frac{1}{2}Z \). Since the Fock space \( \mathfrak{F}_0^d \) is a unitarizable \( \hat{A} \)-module with the positive contravariant Hermitian form \( \langle \cdot | \cdot \rangle \), it is also a unitarizable \( \hat{D} \)-module with respect to the anti-linear anti-involution \( \omega \). By Theorem 5.8 for every partition \( \lambda \) of length \( 2d \) satisfying \( \lambda_1' + \lambda_2' \leq 2d \), the irreducible \( \hat{D} \)-module \( L(\hat{D}, \Lambda_{\hat{D}}(\lambda)) \) is unitarizable.

5.4.3. **Free field realization of \( \hat{D} \) and \( (\hat{D}, O(2d + 1)) \)-duality.** Now we turn to the free field realization of \( \hat{D} \) associated with the \( (\hat{D}, O(2d + 1)) \)-duality. Introduce a free fermionic field \( \phi(z) := \sum_{n \in \mathbb{Z}^*} \phi_n z^{-n-1} \) and a free bosonic field \( \chi(z) := \sum_{r \in \frac{1}{2}Z} \chi_r z^{-r-1/2} \) with the non-trivial anti-commutation relations \( [\phi_m, \phi_n] = \delta_{ij} \delta_{m+n,0} \) and commutation relations \( [\chi_r, \chi_s] = \delta_{ij} \delta_{r+s,0} \) for \( r > 0 \). We shall denote by \( \mathfrak{A}_0^{d+\frac{1}{2}} \) the associative superalgebra generated by the modes of all the quantum fields \( \tilde{\psi}_i^{\pm,i}(z), \gamma_i^{\pm,i}(z), i = 1, \cdots, d, \phi(z), \) and \( \chi(z) \). Let \( \mathfrak{A}_0^{d+\frac{1}{2}} \) denote the Fock space of the quantum fields generated by the vacuum vector \( |0\rangle \), where \( \psi_i^{\pm,i}|0\rangle = \gamma_i^{\pm,i}|0\rangle = \phi_m|0\rangle = \chi_i|0\rangle = 0 \), for \( i = 1, 2 \cdots, d, m > 0 \) and \( r > 0 \).

Introduce an anti-linear anti-involution \( \omega \) on \( \mathfrak{A}_0^{d+\frac{1}{2}} \) in the following way. It is defined by (5.1) and (5.2) on all the \( \tilde{\psi}_i^{\pm,i} \) and \( \gamma_i^{\pm,i} \), and
\[ \omega(\phi_i) = \phi_{-i}, \text{ for all } i; \quad \omega(\chi_r) = \chi_{-r}, \text{ for all } r. \]
(5.24)
The Fock space \( \mathfrak{A}_0^{d+\frac{1}{2}} \) admits a positive definite contravariant Hermitian form with respect to this \( \ast \)-structure of \( \mathfrak{A}_0^{d+\frac{1}{2}} \). As usual, we shall normalize the form on the vacuum vector \( |0\rangle \) so that \( \langle 0 | 0 \rangle = 1 \).
We have an action of $\hat{D}$ of central charge $d + \frac{1}{2}$ on $\mathfrak{so}_0^{d+\frac{1}{2}}$ given by $(i, j \in \mathbb{Z}^*$ and $r, s \in \frac{1}{2} + \mathbb{Z})$

\[
\tilde{e}_{ij} := \sum_{p=1}^{d} :\psi_{-i}^+ \psi_{j}^- : - \sum_{p=1}^{d} :\psi_{j}^+ \psi_{-i}^- : + :e_{-i}^+ e_{j}^- :
\]

\[
\tilde{e}_{rs} := -\tilde{e}_{-s,-r} := \sum_{p=1}^{d} :\gamma_{-r}^+ \gamma_{s}^- : - \sum_{p=1}^{d} :\gamma_{s}^+ \gamma_{-r}^- : + :e_{-r}^+ e_{s}^- ,
\]

Therefore the realization of $\hat{D}$ in $\mathfrak{so}_0^{d+\frac{1}{2}}$ given above defines a $*$-superalgebra homomorphism from $\mathcal{U}(\hat{D})$ to $\mathfrak{so}_0^{d+\frac{1}{2}}$. Note that the $*$-structure on $\hat{D}$ is the restriction of that described in Remark 5.2.

Introduce the following operators on the Fock space $\mathfrak{so}_0^{d+\frac{1}{2}}$:

\begin{align*}
\tilde{E}^{+}_{i} &:= \sum_{n \in \mathbb{Z}^*} :\phi_{-n} \psi_{n}^{+,i} : - \sum_{r \in 1/2 + \mathbb{Z}_+} :\chi_{-r} \gamma_{r}^{+,i} : + \sum_{r \in -1/2 - \mathbb{Z}_+} :\chi_{-r} \gamma_{r}^{+,i} :, \\
\tilde{E}^{-}_{j} &:= \sum_{n \in \mathbb{Z}^*} :\phi_{-n} \psi_{n}^{-,j} : + \sum_{r \in 1/2 + \mathbb{Z}_+} :\chi_{-r} \gamma_{r}^{-,j} :,
\end{align*}

where $1 \leq i, j \leq d$. Note that (5.7), (5.18) and (5.19) can be extended to actions on the Fock space $\mathfrak{so}_0^{d+\frac{1}{2}}$. It is easy to see that (5.7), (5.18) and (5.19) together with (5.25) and (5.26) form a basis for the Lie algebra $\mathfrak{so}(2d + 1)$. The action of the Lie algebra $\mathfrak{so}(2d + 1)$ on the Fock space $\mathfrak{so}_0^{d+\frac{1}{2}}$ can be lifted to an action of the Lie group $SO(2d + 1)$, which can further be extended to an action of the Lie
group $O(2d + 1)$. Moreover $\mathfrak{s}_0^{d + \frac{1}{2}}$ is a direct sum of finite dimensional irreducible representations of $O(2d + 1)$.

Using similar arguments as before, we can show that the Lie superalgebra $\hat{D}$ and $O(2d + 1)$ form a dual pair on $\mathfrak{s}_0^{d + \frac{1}{2}}$ in the sense of Howe [12]. We have the following multiplicity-free decomposition of $\mathfrak{s}_0^{d + \frac{1}{2}}$ with respect to the joint action of $\hat{D}$ and $O(2d + 1)$:

\begin{equation}
\mathfrak{s}_0^{d + \frac{1}{2}} \cong \sum_\lambda W^\lambda \otimes V^\lambda_{O(2d+1)},
\end{equation}

where the summation is over a subset of all partitions of length with $\lambda_1 + \lambda_2 \leq 2d + 1$, and $W^\lambda$ is a certain irreducible module over $\hat{D}$.

We define the $(2d + 1) \times (2d + 1)$ matrix $\Gamma$ as follows:

\[
\Gamma := \begin{pmatrix}
\gamma_0^+ & \gamma_1^+ & \ldots & \gamma_d^+ & \chi_{-\frac{1}{2}} & \gamma_{-\frac{1}{2}}^{-d} & \gamma_{-\frac{1}{2}}^{-d-1} & \ldots & \gamma_{-\frac{1}{2}}^{-1} \\
\phi_0^- & \phi_1^- & \ldots & \phi_d^- & \phi_{-1}^- & \phi_{-1}^- d & \ldots & \phi_{-1}^- 1 \\
\phi_0^- & \phi_1^- & \ldots & \phi_d^- & \phi_{-1}^- & \phi_{-1}^- d & \ldots & \phi_{-1}^- 1 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
\phi_0^- & \phi_1^- & \ldots & \phi_d^- & \phi_{-1}^- & \phi_{-1}^- d & \ldots & \phi_{-1}^- 1
\end{pmatrix}
\]

For any nonnegative integer $r$, we let $\Gamma_r$ denote the first $r \times r$ minor of the matrix $\Gamma$.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2d+1})$ of length $2d + 1$ with $\lambda_1 + \lambda_2 \leq 2d + 1$, we have the shifted Frobenius notations $\Lambda(\lambda) = (\xi_{\frac{1}{2}}, \xi_{-\frac{1}{2}}, \xi_{-1} \mid \xi_1, \xi_2, \ldots, \xi_r)$ for the partition $\lambda$ (see Section 2.1). Let $\Lambda^\hat{D}(\lambda)$ be an element of the dual space $\hat{D}_0^*$ of the vector space $\hat{D}_0$ defined by

\begin{equation}
\Lambda^\hat{D}(\lambda) := \sum_{j=1}^{2d+1} \xi_j \omega_j + \frac{2d + 1}{2} \Lambda_0.
\end{equation}

The following theorem is a consequence of the decomposition (5.27) and the explicit description of the $\hat{D} \times \mathfrak{so}(2d+1)$ joint highest weight vectors.

**Theorem 5.9.** The Lie superalgebra $\hat{D}$ and $O(2d + 1)$ form a dual pair on $\mathfrak{s}_0^{d + \frac{1}{2}}$ in the sense of Howe. In particular, we have the following (multiplicity-free) decomposition of $\mathfrak{s}_0^{d + \frac{1}{2}}$ with respect to their joint action

\[
\mathfrak{s}_0^{d + \frac{1}{2}} \cong \sum_\lambda L(\hat{D}, \Lambda^\hat{D}(\lambda)) \otimes V^\lambda_{O(2d+1)},
\]

where the summation is over all partitions of length $2d + 1$ with $\lambda_1 + \lambda_2 \leq 2d + 1$. Furthermore, the joint highest weight vector of the $\lambda$-component with respect to
The $\hat{D} \times SO(2d+1)$ is given
\[ \Gamma_{\lambda_1} \cdot \sum_{\mu} X_{\mu}^2 \cdots X_{\lambda_{2d+1}}^{\lambda_1} |0\).

**Remark 5.4.** By recalling Remark 5.3, we can easily show that all the irreducible $\hat{D}$-modules appearing in the above theorem are unitarizable with respect to the anti-linear anti-involution $\omega$ described in Remark 5.2. That is, every irreducible module $L(\hat{D}, \Lambda^{\hat{D}}(\lambda))$ associated with a partition $\lambda$ of length $2d + 1$ satisfying $\lambda_1 + \lambda_2 \leq 2d + 1$ is unitarizable.

5.4.4. **Unitarizable $\hat{D}$-modules.** Now we classify the unitarizable irreducible quasi-finite highest weight $\hat{D}$-modules with respect to the anti-linear anti-involution $\omega$ described in Remark 5.2. We have the following result.

**Theorem 5.10.** Let $M$ be an irreducible quasi-finite highest weight $\hat{D}$-module with highest weight $\xi$. Then $M$ is unitarizable if and only if there exists a non-negative integer or half integer $k$ such that $\xi = \Lambda^{\hat{D}}(\lambda)$ for some partition $\lambda$ of length $2k$ with $\lambda_1 + \lambda_2 \leq 2k$. In other words, $M$ is unitarizable if and only if
\[ \Lambda^{\hat{D}}(\lambda) := \sum_{j \leq k} \xi_j \omega_j + k \Lambda_0 \]
such that $k \in \frac{1}{2} \mathbb{Z}_+$, $r \in \mathbb{N}$ and $\xi_j \in \mathbb{Z}$ for all $j$ satisfying the following conditions:

(i) $\xi_{\frac{1}{2}} > \xi_{\frac{3}{2}} > \cdots > \xi_{r-\frac{1}{2}} \geq 0$, $\xi_1 > \xi_2 > \cdots > \xi_r \geq 0$, and $\xi_{r+\frac{1}{2}} = 0$ if and only if $r = 1$ and $\xi_{\frac{1}{2}} = \xi_1 = 0$,

(ii) $\xi_1 + \xi_2 + l_{1,2}(\xi_{\frac{1}{2}}) + \min\{\xi_{\frac{1}{2}}, 1\} \leq 2k$,

where $l_{1,2}$ is a function from non-negative integers to itself with $l_{1,2}(0) = 0$, $l_{1,2}(1) = 1$ and $l_{1,2}(x) = 2$ if $x \geq 2$.

**Proof.** We have already pointed out in Remarks 5.2 and 5.4 that the $L(\hat{D}, \Lambda^{\hat{D}}(\lambda))$ are unitarizable irreducible quasi-finite highest weight $\hat{D}$-modules for all partitions $\lambda$. Now we are going to show that if $M$ is a unitarizable irreducible quasi-finite highest weight $\hat{D}$-module with the highest weight $\xi$, then $\xi = \Lambda^{\hat{D}}(\lambda)$ for some partition $\lambda$. Let $\langle \cdot | \cdot \rangle$ be a positive definite contravariant Hermitian form on $M$ and $v_\xi$ a highest weight vector of $M$ such that $\langle v_\xi | v_\xi \rangle = 1$. We put $\xi(\hat{e}_{i,i}) = \xi_i$ for all $i \in \frac{1}{2} \mathbb{N}$. By Theorem 4.9, there exists $r \in \mathbb{N}$ such that $\xi = \sum_{j \leq k} \xi_j \omega_j + k \Lambda_0$, where $\xi_j$, $k \in \mathbb{C}$. By using similar arguments as in the proof of Theorem 5.2, we can show that $\xi_i \in \mathbb{Z}_+$ for $i = \frac{1}{2}, 1, \cdots, r - \frac{1}{2}, r$, and
\[ \xi_{\frac{1}{2}} > \xi_1 > \cdots > \xi_{r-\frac{1}{2}} \geq 0, \quad \xi_1 > \xi_2 > \cdots > \xi_r \geq 0. \]
Moreover, $\xi_{r+\frac{1}{2}} = 0$ if and only if $r = 1$ and $\xi_{\frac{1}{2}} = \xi_1 = 0$.

Now we choose a large positive integer $n$ such that $\xi(\hat{e}_{n,n}) = \xi(\hat{e}_{n+1,n+1}) = 0$. Consider the subalgebra $sl(2, \mathbb{C})$ with the standard basis $\{2C - \hat{e}_{n,n} - \hat{e}_{n+1,n+1}, \hat{e}_{n,n}, \hat{e}_{n+1,n+1}, \hat{e}_{n,n+1}, \hat{e}_{n+1,n} \}$. 
\[ \bar{e}_{-n-1,n}, \bar{e}_{n,-n-1}. \]
Note that \( \omega(2C - \bar{e}_{n,n}) - \bar{e}_{n+1,n+1} = 2C - \bar{e}_{n,n} - \bar{e}_{n+1,n+1}, \)
\( \omega(\bar{e}_{n,-n-1}) = \bar{e}_{-n-1,n} \) and \( \omega(-\bar{e}_{-n-1,n}) = -\bar{e}_{n,-n-1}. \) Unitarizability with respect to this \( sl_2 \) subalgebra requires \( 2k = \xi(2C) = \xi(2C - \bar{e}_{n,n} - \bar{e}_{n+1,n+1}) \in \mathbb{Z}_+. \) Hence we have \( k \in \frac{1}{2}\mathbb{Z}_+. \)

Finally, we need to show \( \xi_1 + \xi_2 + l_{1,2}(\xi_\frac{1}{2}) + \min\{\xi_\frac{3}{2}, 1\} \leq 2k. \) We may assume that \( \xi_\frac{1}{2} > 0. \) Otherwise, we have \( \xi_1 + \xi_2 + l_{1,2}(\xi_\frac{1}{2}) + \min\{\xi_\frac{3}{2}, 1\} = 0 \leq 2k. \) Direct computations show that \( \langle \bar{e}_{1,\frac{1}{2}}v_\xi | \bar{e}_{1,\frac{1}{2}}v_\xi \rangle = \xi_\frac{1}{2} + \xi_1 > 0 \) and

\[
0 \leq \|\bar{e}_{2,-1}\bar{e}_{1,\frac{1}{2}}v_\xi\|^2 = \langle \bar{e}_{1,\frac{1}{2}}v_\xi | \bar{e}_{-1,2}\bar{e}_{1,\frac{1}{2}}v_\xi \rangle = \langle \bar{e}_{1,\frac{1}{2}}v_\xi | (2C - \bar{e}_{1,1} - \bar{e}_{2,2})\bar{e}_{1,\frac{1}{2}}v_\xi \rangle = (2k - \xi_1 - \xi_2 - 1)(\xi_\frac{1}{2} + \xi_1). \tag{5.29}
\]

Thus we have \( 2k - \xi_1 - \xi_2 - 1 \geq 0 \) since \( \xi_\frac{1}{2} + \xi_1 > 0. \) When \( \xi_\frac{1}{2} = 1, \) we have \( \xi_\frac{3}{2} = \xi_2 = 0. \) Thus \( \xi_1 + \xi_2 + l_{1,2}(\xi_\frac{1}{2}) + \min\{\xi_\frac{3}{2}, 1\} = \xi_1 + \xi_2 + 1 + 0 \leq 2k. \) Now we assume that \( \xi_\frac{3}{2} \geq 2. \) We compute

\[
\|\bar{e}_{-1,1}e_{2,-1}\bar{e}_{1,\frac{1}{2}}v_\xi\|^2 = \|\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi\|^2 = \langle \bar{e}_{1,\frac{1}{2}}v_\xi | \bar{e}_{-1,2}\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi \rangle = \langle \bar{e}_{1,\frac{1}{2}}v_\xi | (\bar{e}_{-1,\frac{3}{2}} + \bar{e}_{2,2})\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi \rangle = (\xi_\frac{1}{2} + \xi_2 - 1)(\xi_\frac{1}{2} + \xi_1) > 0. \tag{5.30}
\]

Thus \( \|\bar{e}_{2,-1}\bar{e}_{1,\frac{1}{2}}v_\xi\|^2 > 0 \) and hence we have \( \xi_1 + \xi_2 + l_{1,2}(\xi_\frac{1}{2}) + \min\{\xi_\frac{3}{2}, 1\} = \xi_1 + \xi_2 + 2 \leq 2k \) for \( \xi_\frac{1}{2} \geq 2 \) and \( \xi_\frac{3}{2} = 0 \) by using (5.29). Eventually we assume that \( \xi_\frac{1}{2} \geq 2 \) and \( \xi_\frac{3}{2} > 0. \) Then

\[
\|\bar{e}_{-1,1}\bar{e}_{2,-1}\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi\|^2 = \|\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi\|^2 = \langle \bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi | (\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi) \rangle = (\xi_\frac{1}{2} + \xi_2)(\xi_\frac{1}{2} + \xi_1)(\xi_\frac{1}{2} + \xi_2) > 0.
\]

Therefore we have \( \bar{e}_{2,-1}\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi \neq 0 \) and

\[
0 < \|\bar{e}_{2,-1}\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi\|^2 = \langle \bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi | \bar{e}_{-1,2}\bar{e}_{2,-1}\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi \rangle = \langle \bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi | (2C - \bar{e}_{1,1} - \bar{e}_{2,2})\bar{e}_{2,\frac{3}{2}}\bar{e}_{1,\frac{1}{2}}v_\xi \rangle = (2k - \xi_\frac{1}{2} - \xi_2 - 2)(\xi_\frac{3}{2} + \xi_1) > 0.
\]

Therefore we have \( \xi_\frac{1}{2} + \xi_2 + 3 \leq 2k. \) Hence \( \xi_1 + \xi_2 + l_{1,2}(\xi_\frac{1}{2}) + \min\{\xi_\frac{3}{2}, 1\} = \xi_\frac{3}{2} + \xi_2 + 3 \leq 2k \) and the proof is completed.
6. CHARACTER FORMULAS FOR UNITARIZABLE IRREDUCIBLE MODULES

In this section we derive explicit formulae for the formal characters of the unitarizable quasi-finite irreducible highest weight modules over the infinite rank Lie algebras \( \widehat{\mathfrak{g}} \) and \( \widehat{D} \). The method employed here is a generalization of that developed in [3, 8, 4], which makes essential use of Howe dualities. We mention that the character formulae for the unitarizable irreducible modules over \( \widehat{\mathfrak{gl}}_{\infty|\infty} \) (and hence \( \widehat{A} \)) were obtained in [8].

6.1. Character formula for \( \widehat{\mathfrak{g}} \). The central result of this subsection is Theorem 6.1, which gives the character formula for the unitarizable quasi-finite irreducible highest weight \( \widehat{\mathfrak{g}} \)-modules. In order to establish the result, we need some basic facts on characters of the symplectic group (see [10, 13, 14]), which we recall here.

When we deal with characters of modules over \( \mathfrak{sp}(2m) \), we put \( \tilde{h}_i := -h_{m-i+1} \) and \( x_i = e^{-t_{m-i+1}} \) for \( i = 1, 2, \ldots, m \). That is \( x_i = z_{m-i+1}^{-1} \). Recall that \( e_{ij}(h_j) = \delta_{ij} \) and \( z_i = e^{t_i} \) where \( e_i \in \mathfrak{h}^* \) such that \( e_i(h_j) = \delta_{ij} \) (see Section 5.3). The definitions of \( x_i \)'s and \( \tilde{h}_i \)'s are somewhat nonstandard, but they allow us to deal with only polynomials instead of Laurent polynomials when considering characters of certain representations of \( \mathfrak{sp}(2m) \). For each finite sequence of complex numbers \( \lambda = (\lambda_1, \ldots, \lambda_m) \), we let \( W^\lambda_{\mathfrak{sp}(2m)} := V^\lambda_{\mathfrak{sp}(2m)} \) where \( \lambda^* = (-\lambda_m, \ldots, -\lambda_1) \). Note that \( \lambda^*(\tilde{h}_i) = \lambda_i \) for \( i = 1, \ldots, m \). \( W^\lambda_{\mathfrak{sp}(2m)} \) is a finite-dimensional irreducible representation if and only if \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \) and \( \lambda_i \in -\mathbb{Z}_+ \) for \( i = 1, \ldots, m \).

When we deal with characters of \( \mathcal{Sp}(2d) \)-modules, we put \( z_i = e^{t_i} \) where \( e_i \in \mathfrak{h}^* \) such that \( e_i(h_j) = e_{ij}(e_{jj} - e_{d+i,d+j}) = \delta_{ij} \) (see Section 5.3). For each partition \( \lambda \) of length \( d \) and each decreasing sequence of non-positive integers \( \nu \) of length \( m \), we write \( \chi^\lambda_{\mathcal{Sp}(2d)}(z) = \chi^\lambda_{\mathcal{Sp}(2d)}(z_1, \ldots, z_d) \) and \( \tilde{\chi}_{\mathcal{Sp}(2m)}^\nu(x) = \tilde{\chi}_{\mathcal{Sp}(2m)}^\nu(x_1, \ldots, x_m) \) for the characters of the \( \mathcal{Sp}(2d) \)-module \( V^\lambda_{\mathcal{Sp}(2d)} \) and the \( \mathfrak{sp}(2m) \)-module \( W^\nu_{\mathfrak{sp}(2m)} \), respectively. It is clear that \( \chi^\lambda_{\mathcal{Sp}(2d)}(z) = \chi^\lambda_{\mathcal{Sp}(2m)}(z_1^{-1}, \ldots, z_m^{-1}) \) for any partition \( \lambda \) of length \( m \) since \( \chi^\lambda_{\mathfrak{sp}(2m)}(z_1, \ldots, z_m) = \chi^\lambda_{\mathfrak{sp}(2m)}(z_1^{-1}, \ldots, z_m^{-1}) \).

By the \((\mathcal{Sp}(2d), \mathfrak{sp}(2m))\)-duality on the exterior algebra \( \Lambda(\mathbb{C}^{2d} \otimes \mathbb{C}^m) \) with \( m \geq d \) (13, 14), also see [5], we have the following identity:

\[
(6.1) \hspace{1cm} (x_1 \cdots x_m)^{-d} \prod_{i=1}^{d} \prod_{j=1}^{m} (1 + x_j z_i)(1 + x_j z_i^{-1}) = \sum_{\lambda} \chi^\lambda_{\mathcal{Sp}(2d)}(z) \tilde{\chi}_{\mathcal{Sp}(2m)}^{\lambda'-d(1^m)}(x).
\]

Here \( \lambda \) is summed over all partitions of length \( d \) with \( \lambda_i \leq m \), and for any \( k \in \mathbb{N} \), \((1^k)\) stands for the \( k \)-tuple \((1, 1, \ldots, 1)\). Note that the partition \( \lambda' \) is considered as a partition of length \( m \) and \( \lambda' - d(1^m) \) is a decreasing sequence of non-positive integers of length \( m \).
We let
\[ E'_r(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}) = E_r(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}) - E_{r-2}(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}), \]

where \( E_r \) is the \( r \)-th elementary symmetric polynomial for \( r \geq 0 \) and \( E_r = 0 \) for \( r < 0 \). For any partition \( \mu \) of length \( l \) (which is an arbitrary positive integer), we define
\[ |E'_\mu| = |E'_\mu(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1})| \]
by the determinant of the \( l \times l \) matrix
\[
\begin{pmatrix}
E'_{\mu_1} & E'_{\mu_1+1} + E'_{\mu_1-1} & E'_{\mu_1+2} + E'_{\mu_1-2} & \cdots & E'_{\mu_1+l-1} + E'_{\mu_1-l+1} \\
E'_{\mu_2-1} & E'_{\mu_2} + E'_{\mu_2-2} & E'_{\mu_2+1} + E'_{\mu_2-3} & \cdots & E'_{\mu_2+l-2} + E'_{\mu_2-l} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
E'_{\mu_l-2} & E'_{\mu_l-1} + E'_{\mu_l-i+1} & E'_{\mu_l-3} + E'_{\mu_l-i+2} & \cdots & E'_{\mu_l-i+l} + E'_{\mu_l-i-l+2} \\
E'_{\mu_l-l+1} & E'_{\mu_l-l} + E'_{\mu_l-l} & E'_{\mu_l-l+1} + E'_{\mu_l-l+1} & \cdots & E'_{\mu_l} + E'_{\mu_l-2l+2}
\end{pmatrix}
\]

Then the character of the irreducible \( \mathfrak{sp}(2m) \)-module \( V^\lambda_{\mathfrak{sp}(2m)} \) is given \[ \ref{10} \] by \( |E'_{\chi'}| = |E'_{\chi'}(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1})| \), where \( \lambda \) is any partition of length \( m \).

From the above result it is not very difficult to work out that the character \( \chi_{\mathfrak{sp}(2m)}(x) \) of the finite-dimensional irreducible \( \mathfrak{sp}(2m) \)-module \( W^\lambda_{\mathfrak{sp}(2m)} \) for every partition \( \lambda \) of length \( d \) with \( \lambda_1 \leq m \) is given by the determinant of the \( d \times d \) matrix whose \( i \)-th row is
\[
(E'_{m-\lambda_{d-i+1}-i+1} E'_{m-\lambda_{d-i+1}-i+2} + E'_{m-\lambda_{d-i+1}-i+3} + E'_{m-\lambda_{d-i+1}-i-1} \cdots E'_{m-\lambda_{d-i+1}-i+d} + E'_{m-\lambda_{d-i+1}-i-d+2}).
\]

On the other hand, for each \( r \in \mathbb{Z} \),
\[
E_{m-r}(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1})
= \sum_{i=0}^{m-r} E_i(x_1, \ldots, x_m) E_{m-r-i}(x_1^{-1}, \ldots, x_m^{-1})
= (x_1 \cdots x_m)^{-1} \sum_{i=0}^{m-r} E_i(x_1, \ldots, x_m) E_{r+i}(x_1, \ldots, x_m)
= (x_1 \cdots x_m)^{-1} \sum_{i=0}^{\infty} E_i(x_1, \ldots, x_m) E_{r+i}(x_1, \ldots, x_m).
\]
For $r \in \mathbb{Z}$, we define
\[
\tilde{E}_r := \sum_{i=0}^{\infty} E_i(x_1, \ldots, x_m) E_{r+i}(x_1, \ldots, x_m),
\]
\[
\tilde{E}'_r := \tilde{E}_r - \tilde{E}_{r-2}.
\]
Then we have
\[
E'_{m-r}(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}) = (x_1 \cdots x_m)^{-1} \tilde{E}'_r.
\]
Hence for each partition $\lambda$ of length $d$, $(x_1 \cdots x_m)^d \lambda \sp(\text{sp}(2m)) (x)$ equals the determinant of the $d \times d$ matrix whose $i$-th row is
\[
(\tilde{E}'_{\lambda_d-i+1+i-1} \tilde{E}'_{\lambda_d-i+1+i-2} + \tilde{E}'_{\lambda_d-i+1+i-3} + \tilde{E}'_{\lambda_d-i+1+i+1} \cdots \tilde{E}'_{\lambda_d-i+1+i-d} + \tilde{E}'_{\lambda_d-i+1+i+d-2}).
\]
For each partition $\lambda$ of length $d$ satisfying $\lambda_1 \leq m$ and $d \leq m$, the symplectic Schur polynomial of weight $d$ of $m$ variables, denoted by
\[
S_{\lambda}^{\text{sp},d}(x) = S_{\lambda}^{\text{sp},d}(x_1, \ldots, x_m),
\]
is defined by the determinant of the $d \times d$ matrix whose $i$-th row is
\[
(\tilde{E}'_{\lambda_d-i+1+i-1} \tilde{E}'_{\lambda_d-i+1+i-2} + \tilde{E}'_{\lambda_d-i+1+i-3} + \tilde{E}'_{\lambda_d-i+1+i+1} \cdots \tilde{E}'_{\lambda_d-i+1+i-d} + \tilde{E}'_{\lambda_d-i+1+i+d-2}).
\]
Thus we have
\[
\tilde{\lambda}^{\text{sp}(2m)}(x_1, \ldots, x_m) = (x_1 \cdots x_m)^{-d} S_{\lambda}^{\text{sp},d}(x_1, \ldots, x_m).
\]
Hence we can rewrite the combinatorial formula (6.1) as follows:
\[
\prod_{i=1}^{d} \prod_{j=1}^{m} (1 + x_j z_i) (1 + x_j z_i^{-1}) = \sum_{\lambda} \lambda_{\sp(2d)}(\mathbf{z}) S_{\lambda}^{\text{sp},d}(\mathbf{x}).
\]
Here $\lambda$ is summed over all partitions of length $d$ with $\lambda_1 \leq m$.

Now, for every partition $\lambda$ of length $d$, we still use
\[
S_{\lambda}^{\text{sp},d}(x) = S_{\lambda}^{\text{sp},d}(x_1, \ldots, x_m, \ldots)
\]
to denote the symplectic Schur function of weight $d$ of infinitely many variables, which is the determinant of the $d \times d$ matrix whose $i$-th row is
\[
(\tilde{c}'_{\lambda_d-i+1+i-1} \tilde{c}'_{\lambda_d-i+1+i-2} + \tilde{c}'_{\lambda_d-i+1+i-3} + \tilde{c}'_{\lambda_d-i+1+i+1} \cdots \tilde{c}'_{\lambda_d-i+1+i-d} + \tilde{c}'_{\lambda_d-i+1+i+d-2}),
\]
where $\tilde{c}'_r = \tilde{e}_r - \tilde{e}_{r-2}$ and $\tilde{e}_r = \sum_{i=0}^{\infty} e_i(x_1, x_2, \cdots) e_{r+i}(x_1, x_2, \cdots)$. Here $e_i(x_1, x_2, \cdots)$ stands for the $i$-th elementary symmetric function of infinitely many variables. Therefore, the symplectic Schur function $S_{\lambda}^{\text{sp},d}(x_1, x_2, \cdots)$ is the inverse
The limit of the symplectic Schur polynomials $S^{\text{sp},d}_\lambda(x_1, \cdots, x_m)$, and we thus have $S^{\text{sp},d}_\lambda(x_1, \cdots, x_m) = S^{\text{sp},d}_\lambda(x_1, \cdots, x_m, 0, 0, \cdots)$.

For any given partition $\lambda$ of length $d$, we let $D S^{\text{sp},d}_\lambda(x) = D S^{\text{sp},d}_\lambda(x_1, x_2, \cdots)$ denote the skew symplectic Schur function of weight $d$ of infinitely many variables, which is defined by the determinant of the $d \times d$ matrix whose $i$-th row is

\[
(\tilde{H}^\prime_{\lambda_d-i+1+i-1} \quad \tilde{H}^\prime_{\lambda_d-i+1+i-2} + \tilde{H}^\prime_{\lambda_d-i+1+i} + \tilde{H}^\prime_{\lambda_d-i+1+i-3} + \tilde{H}^\prime_{\lambda_d-i+1+i+1} \\
\cdots \quad \tilde{H}^\prime_{\lambda_d-i+1+i-d} + \tilde{H}^\prime_{\lambda_d-i+1+i+d-2}),
\]

where

\[
\tilde{H}^\prime_r := \tilde{H}_r - \tilde{H}_{r-2}, \\
\tilde{H}_r := \sum_{i=0}^\infty H_i(x_1, x_2, \cdots) H_{r+i}(x_1, x_2, \cdots),
\]

and $H_i(x_1, x_2, \cdots)$ are the complete symmetric functions of infinitely many variables. Note that $(x_1 \cdots x_m)^{-d} D S^{\text{sp},d}_\lambda(x_1^{-1}, \cdots, x_m^{-1}, 0, 0, \cdots)$ is the character of some infinite dimensional unitarizable module over the Lie algebra $\mathfrak{so}(2m)$.

Analogous to hook Schur functions (see [1] and [3]), for each partition $\lambda$ of length $d$, we define the symplectic hook Schur function $H S^{\text{sp},d}_\lambda(x, y) = S^{\text{sp},d}_\lambda(x_1, x_2, \cdots, y_1, y_2, \cdots)$ of weight $d$ in infinitely many variables by

\[
H S^{\text{sp},d}_\lambda(x, y) := \sigma(S^{\text{sp},d}_\lambda(x, y)),
\]

where $\sigma$ is the involution of the ring of symmetric functions (see for example [19]), which sends the elementary symmetric functions of $y_j$’s to the complete symmetric functions of $y_j$’s. Recall that the hook Schur function (cf. [3])

\[
H S_\lambda(x, y) = \sigma(S_\lambda(x, y))
\]

is a symmetric function of the variables $x$ and the variables $y$ separably where $\lambda$ is a partition. Hereafter $S_\lambda$ stands for the Schur function associated with the partition $\lambda$. Then we have

\[
H S_{(1^d)}(x, y) = \sigma(e_d(x, y)) = \sum_{i=0}^d e_i(x)e_{d-i}(y).
\]

The symplectic hook Schur function can be written in terms of hook Schur functions as follows.
Proposition 6.1. Let \( x = \{x_1, x_2, \cdots \} \) and \( y = \{y_1, y_2, \cdots \} \) be two infinite sets of variables. For each partition \( \lambda \) of length \( d \), the symplectic hook Schur function \( HS_{Sp}^\lambda(x, y) \) of weight \( d \) equals the determinant of the following \( d \times d \) matrix whose \( i \)-th row is
\[
(f^\prime_{\lambda_{d-i+1}+i} + \bar{f}^\prime_{\lambda_{d-i+1}+i-2} + f^\prime_{\lambda_{d-i+1}+i-3} + \bar{f}^\prime_{\lambda_{d-i+1}+i-4} + \cdots + \bar{f}^\prime_{\lambda_{d+i-1}+i-2} + \bar{f}^\prime_{\lambda_{d+i-1}+i-1}),
\]
where \( \bar{f}^\prime_r = \bar{f}_{r-2} - \bar{f}_r \) and \( \bar{f}_r = \sum_{i=0}^\infty HS_{(1^i)}(x, y)HS_{(1^r+i)}(x, y) \).

Proof. Let \( \sigma \) denote the involution of the ring of symmetric functions, which sends the elementary symmetric functions of variables \( y \) to the complete symmetric functions of variables \( y \). The proposition follows by applying the involution \( \sigma \) to the determinant of the following \( d \times d \) matrix whose \( i \)-th row is
\[
(c^\prime_{\lambda_{d-i+1}+i} + c^\prime_{\lambda_{d-i+1}+i-2} + c^\prime_{\lambda_{d-i+1}+i-3} + c^\prime_{\lambda_{d-i+1}+i-4} + \cdots + c^\prime_{\lambda_{d+i-1}+i-2} + c^\prime_{\lambda_{d+i-1}+i-1}),
\]
where \( \bar{c}^\prime_r = \bar{c}_r - \bar{c}_{r-2} \) and \( \bar{c}_r = \sum_{i=0}^\infty e_i e_{r+i} \) and \( e_i(x, y) \) are the elementary symmetric functions of infinitely many variables \( x \) and \( y \). □

Proposition 6.2. Let \( x = \{x_1, x_2, \cdots \} \) be an infinite set of variables and \( z = \{z_1, z_2, \cdots, z_d \} \) be \( d \) variables. Then
\[
(6.4) \quad \prod_{i=1}^d \prod_{j=1}^{\infty} (1 + x_j z_i)(1 + x_j z_i^{-1}) = \sum_{\lambda} \chi_{Sp(2d)}^{\lambda}(z) S_{\lambda}^{Sp,d}(x),
\]
where \( \lambda \) is summed over all partitions of length \( d \), and
\[
(6.5) \quad \prod_{i=1}^d \prod_{j=1}^{\infty} (1 - x_j z_i)^{-1}(1 - x_j z_i^{-1})^{-1} = \sum_{\lambda} \chi_{Sp(2d)}^{\lambda}(z) DS_{\lambda}^{Sp,d}(x),
\]
where \( \lambda \) is summed over all partitions of length \( d \).

Proof. The first identity follows from \((6.3)\) by putting \( m \to \infty \). Let \( \sigma \) denote the involution of the ring of symmetric functions, which sends the elementary symmetric functions of \( x_j \)'s to the complete symmetric functions of \( x_j \)'s. By applying the involution \( \sigma \) to both sides of the first identity of the proposition, we obtain the second identity. □

Proposition 6.3. Let \( x = \{x_1, x_2, \cdots \} \) and \( y = \{y_1, y_2, \cdots \} \) be two infinite sets of variables and \( z = \{z_1, z_2, \cdots, z_d \} \) be \( d \) variables. Then
\[
(6.6) \quad \prod_{i=1}^d \prod_{j=1}^{\infty} \prod_{k=1}^{1/2} (1 + x_j z_i)(1 + x_j z_i^{-1}) = \sum_{\lambda} \chi_{Sp(2d)}^{\lambda}(z) HS_{\lambda}^{Sp,d}(x, y),
\]
where \( \lambda \) is summed over all partitions of length \( d \).
Proof. By Proposition 6.2, we have

\[ \prod_{i=1}^{d} \prod_{j=1}^{\infty} \prod_{k=1}^{\frac{1}{2}} (1 + x_j z_i)(1 + x_j z_i^{-1})(1 + y_k z_i)(1 + y_k z_i^{-1}) = \sum_{\lambda} \chi^{\lambda}_{Sp(2d)}(z) S^{\lambda}_{\mu}(x, y), \]

where \( \lambda \) is summed over all partitions of length \( d \). The proposition follows by applying to both sides of equation (6.6) the involution of the ring of symmetric functions, which sends the elementary symmetric functions of \( y \) to the complete symmetric functions of \( y \).

We need the following lemma to prove the main result of this subsection.

Lemma 6.1. Suppose that \( f^\lambda(x) \) and \( g^\lambda(x) \) are power series in the variables \( x \) and suppose that

\[ \sum_{\lambda} f^\lambda(x) \chi^{\lambda}_{Sp(2d)}(z) = \sum_{\lambda} g^\lambda(x) \chi^{\lambda}_{Sp(2d)}(z), \]

where \( \lambda \) is summed over all partitions of length \( d \). Then \( f^\lambda(x) = g^\lambda(x) \), for all \( \lambda \).

Proposition 6.4. Let \( x = \{x_1, x_2, \cdots\} \) and \( y = \{y_1, y_2, \cdots\} \) be two infinite sets of variables. For each partition \( \lambda \) of length \( d \), we have

\[ HS^{\lambda}_{\mu}(x, y) := \sum_{\mu\nu} c^{\lambda}_{\mu\nu}(z) S^{\lambda}_{\mu}(x) D^{\lambda}_{\nu}(y) \]

where \( \mu \) and \( \nu \) are summed over all partitions of length \( d \). Here the non-negative integers \( c^{\lambda}_{\mu\nu} \) are the multiplicity of \( V^\lambda_{Sp(2d)} \) in the tensor product decomposition of \( V^\mu_{Sp(2d)} \otimes V^\nu_{Sp(2d)} \).

Proof. By Proposition 6.2, we have

\[ \prod_{i=1}^{d} \prod_{j=1}^{\infty} \prod_{k=1}^{\frac{1}{2}} (1 + x_j z_i)(1 + x_j z_i^{-1})(1 - y_k z_i)(1 - y_k z_i^{-1}) = \sum_{\mu} \chi^{\lambda}_{Sp(2d)}(z) S^{\lambda}_{\mu}(x) D^{\lambda}_{\nu}(y), \]

where \( \mu \) and \( \nu \) are summed over all partitions of length \( d \). On the other hand,

\[ \chi^{\lambda}_{Sp(2d)}(z) \chi^{\nu}_{Sp(2d)}(z) = \sum_{\lambda} c^{\lambda}_{\mu\nu} \chi^{\lambda}_{Sp(2d)}(z), \]
where the summation is over all partitions of length $d$. Thus

$$
\sum_\mu \chi_{Sp(2d)}^\mu(z)S_{\mu}^{sp,d}(x) \sum_\nu \chi_{Sp(2d)}^\nu(z)D_{\nu}^{sp,d}(y) = \sum_\lambda \chi_{Sp(2d)}^\lambda \left( \sum_{\mu \nu} c_{\mu \nu}^\lambda(z)S_{\mu}^{sp,d}(x)D_{\nu}^{sp,d}(y) \right),
$$

where $\lambda, \mu$ and $\nu$ are summed over all partitions of length $d$. Therefore we have

$$
(6.7) \quad \prod_{i=1}^d \prod_{j=1}^\infty \prod_{k=\frac{1}{2}}^{\infty} \frac{(1 + x_j z_i)(1 + x_j z_i^{-1})}{(1 - y_k z_i)(1 - y_k z_i^{-1})} = \sum_\lambda \chi_{Sp(2d)}^\lambda \left( \sum_{\mu \nu} c_{\mu \nu}^\lambda(z)S_{\mu}^{sp,d}(x)D_{\nu}^{sp,d}(y) \right),
$$

where $\lambda, \mu$ and $\nu$ are summed over all partitions of length $d$. Now the proposition follows from (6.7), Proposition 6.3 and Lemma 6.1.

Now we turn to the computation of the formal character of $\mathfrak{g}_0^d$ with respect to the abelian algebra $\sum_{s \in \frac{1}{2} \mathbb{N}} \mathbb{C}e_{ss} \oplus \sum_{i=1}^d \mathbb{C}E_{ii}$. We need the following commutation relations: for $i \in \mathbb{N}$, $r \in \frac{1}{2} + \mathbb{Z}_+$,

$$
[\hat{e}_{ii}, \psi_{-n}^{+p}] = \delta_{in} \psi_{-n}^{+p}, \quad [\hat{e}_{ii}, \psi_{-n}^{-p}] = \delta_{in} \psi_{-n}^{-p},
$$

$$
[\hat{e}_{rr}, \gamma_{-s}^{+p}] = \delta_{rs} \gamma_{-s}^{+p}, \quad [\hat{e}_{rr}, \gamma_{-s}^{-p}] = \delta_{rs} \gamma_{-s}^{-p},
$$

$$
[\hat{e}_{rr}, \psi_{-n}^{\pm p}] = [\hat{e}_{ii}, \gamma_{\pm p}] = 0.
$$

Furthermore for $i = 1, \ldots, d$, we have

$$
[E_{ii}, \psi_{-n}^{+p}] = \delta_{ip} \psi_{-n}^{+p}, \quad [E_{ii}, \psi_{-n}^{-p}] = -\delta_{ip} \psi_{-n}^{-p},
$$

$$
[E_{ii}, \gamma_{-r}^{+p}] = \delta_{ip} \gamma_{-r}^{+p}, \quad [E_{ii}, \gamma_{-r}^{-p}] = -\delta_{ip} \gamma_{-r}^{-p}.
$$

Let $e$ be a formal indeterminate. For $j \in \mathbb{N}$, $r \in \frac{1}{2} + \mathbb{Z}_+$, $i = 1, \ldots, d$, set

$$
z_i = e^{\epsilon_i}, \quad x_j = e^{\omega_j}, \quad y_r = e^{\omega_r},
$$

where $\epsilon_1, \ldots, \epsilon_d$ and $\omega_n$ are the respective fundamental weights of $Sp(2d)$ and $\hat{C}$ introduced earlier. By using the commutation relations established above, we can easily show that the formal character of $\mathfrak{g}_0^d$ with respect to the abelian algebra $\sum_{s \in \frac{1}{2} \mathbb{N}} \mathbb{C}e_{ss} \oplus \sum_{i=1}^d \mathbb{C}E_{ii}$, is given by

$$
(6.8) \quad \text{ch}_{\mathfrak{g}_0^d} = \prod_{i=1}^d \prod_{j=1}^\infty \prod_{k=\frac{1}{2}}^{\infty} \frac{(1 + x_j z_i)(1 + x_j z_i^{-1})}{(1 - y_k z_i)(1 - y_k z_i^{-1})}.
$$
By Proposition 6.3 we can rewrite (6.8) as
\begin{align}
\text{ch} \tilde{\Sigma}_0^d &= \sum_{\lambda} \chi_{\text{Sp}(2d)}^\lambda(z) H_{\lambda}^{\text{sp},d}(x, y),
\end{align}
where \( \lambda \) is summed over all partitions of length \( d \). On the other hand, Theorem 5.6 implies that
\begin{align}
\text{ch} \tilde{\Sigma}_0^d &\cong \sum_{\lambda} \text{ch} L(\widehat{\mathcal{C}}, \Lambda\widehat{\mathcal{C}}(\lambda)) \chi_{\text{Sp}(2d)}^\lambda(z),
\end{align}
where \( \lambda \) is summed over all partitions of length \( d \). Using (6.10) and (6.9) together with Lemma 6.1 we have the following character formula.

**Theorem 6.1.** For each partition \( \lambda \) of length \( d \), the formal character of the irreducible \( \widehat{\mathcal{C}} \)-module \( L(\widehat{\mathcal{C}}, \Lambda\widehat{\mathcal{C}}(\lambda)) \) is given by
\[ \text{ch} L(\widehat{\mathcal{C}}, \Lambda\widehat{\mathcal{C}}(\lambda)) = H_{\lambda}^{\text{sp},d}(x, y). \]

### 6.2. Character formula for \( \widehat{D} \)

We now construct a character formula for the unitarizable irreducible quasi-finite highest weight \( \widehat{D} \)-modules. Let us start by recalling some results on formal characters of finite dimensional representations of the orthogonal group (see [10], [13], [14]).

When we deal with characters of modules over \( \mathfrak{so}(2m) \), we will put \( \tilde{h}_i := -h_{m-i+1} \) and \( x_i = e^{-\epsilon m_{i+1}} \) for \( i = 1, 2, \ldots, m \). That is \( x_i = z_{m-i+1}^{-1} \).

Recall that \( \epsilon_i(h_j) = \delta_{ij} \) and \( \epsilon_i = e^{\epsilon_i} \) where \( \epsilon_i \in \mathfrak{h}^* \) such that \( \epsilon_i(h_j) = \delta_{ij} \) (see Section 5.4). For each finite sequence of complex numbers \( \lambda = (\lambda_1, \ldots, \lambda_m) \), we let \( W_{\mathfrak{so}(2m)}^\lambda := V_{\mathfrak{so}(2m)}^{\lambda} \) where \( \lambda^* = (-\lambda_m, \ldots, -\lambda_1) \). Note that \( \lambda(\tilde{h}_i) = \lambda_i \) for \( i = 1, \ldots, m \).

\( W_{\mathfrak{so}(2m)}^{\lambda} \) is a finite-dimensional irreducible representation if and only if \( -|\lambda_1| \geq \lambda_2 \geq \cdots \geq \lambda_m \) with either \( \lambda_i \in \mathbb{Z} \) or \( \lambda_i \in \mathbb{Z} + \frac{1}{2} \) for \( i = 2, \ldots, m \).

When considering characters of \( O(n) \)-modules, we put \( z_i = e^{\epsilon_i} \) with \( \epsilon_i \in \mathfrak{h}^* \) such that \( \epsilon_i(h_j) = \delta_{ij} \) for \( n = 2d \), and \( \epsilon_i(h_j) = \epsilon_i(e_{jj} - e_{d+j,j} - e_{d+j+1,d+j+1}) = \delta_{ij} \) for \( n = 2d + 1 \) (see Section 5.8). For each partition \( \lambda \) of length \( n \) and each sequence of complex numbers \( \nu \) of length \( m \), we write \( \chi_{O(n)}^\lambda(z) = \chi_{O(n)}^\lambda(z_1, \ldots, z_d) \) and \( \tilde{\chi}_{\mathfrak{so}(2m)}^\nu(x) = \tilde{\chi}_{\mathfrak{so}(2m)}^\nu(x_1, \ldots, x_m) \) for the character of \( O(n) \)-module \( V_{O(n)}^\lambda \) and \( \mathfrak{so}(2m) \)-module \( W_{\mathfrak{so}(2m)}^\nu \) to stress their dependence on the variables \( z_1, \ldots, z_d \) and \( x_1, \ldots, x_m \), respectively.

Recall that when \( n = 2d + 1 \), the irreducible \( O(n) \)-modules \( V_{O(n)}^\lambda \) and \( V_{O(n)}^\lambda \) restrict to isomorphic \( SO(n) \)-modules. To distinguish these \( O(n) \)-representations at the level of characters, we let \( \epsilon \) be the eigenvalue of \( -I_n \in O(n) \) so that \( \epsilon^2 = 1 \).

Denote by \( \chi_{O(n)}^\lambda(\epsilon, z) \) the character of \( V_{O(n)}^\lambda \) (with \( \lambda'_1 + \lambda'_2 \leq n \)) with respect to the Cartan subalgebra \( \sum \mathbb{C} h_i \) together with \(-I_n\). It is easy to see that
\[ \chi_{O(n)}^\lambda(\epsilon, z) = e^{i|\lambda'|} \chi_{SO(n)}^\lambda(z) \]
where $|\lambda|$ is the size of $\lambda$.

By the classical $(\mathfrak{so}(2m), O(n))$-duality on the exterior algebra $\Lambda(\mathbb{C}^n \otimes \mathbb{C}^m)$ with $m \geq n$ (13, 14, also see 8), we have the following identities for $n = 2d$ and $n = 2d + 1$ respectively:

\begin{equation}
(x_1 \cdots x_m)^{-\frac{\nu}{2}} \prod_{i=1}^d \prod_{j=1}^m (1 + x_jz_i)(1 + x_jz_i^{-1}) = \sum_{\lambda} \chi_{\lambda}^{\mu}(\epsilon, z) \chi_{\lambda, \mu}(\tilde{x}_{\mathfrak{so}(2m)}(x), \tilde{x}_{\mathfrak{so}(2m)}(y)).
\end{equation}

The summations on the right hand sides of both equations range over all partitions of length $n$ with $\lambda'_1 + \lambda'_2 \leq n$ and $\lambda'_1 \leq m$. Note that the partition $\lambda'$ is considered as a partition of length $m$ and $\frac{\nu}{2} \geq \lambda'_2 \geq \cdots \geq \lambda'_m$ together with $n \geq \lambda'_1$.

Recall that $E_r := \det E_r(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1})$, where $E_r$ is the $r$-th elementary symmetric polynomial for $r \geq 0$ and $E_r = 0$ for $r < 0$ (see Section 6.1). For each partition $\mu$ of length $l$, we let $|E_{\mu}|$ denote the determinant of the $l \times l$ matrix with $i$-th row

\begin{equation}
(E_{\mu_i \cdot i+1} E_{\mu_i \cdot i+2} + E_{\mu_i \cdot i} E_{\mu_i \cdot i+3} + E_{\mu_i \cdot i-1} \cdots E_{\mu_i \cdot i+l} E_{\mu_i \cdot i+l+2}).
\end{equation}

For any partition $\nu$ of length $m$ with $\nu_m = 0$, the formal character of the finite-dimensional irreducible $\mathfrak{so}(2m)$-module $V_{\mathfrak{so}(2m)}^{\nu}$ equals $|E_{\nu}|$. On the other hand, if the partition $\nu = (\nu_1, \ldots, \nu_m)$ is of length $m$ with $\nu_m \neq 0$, the characters of the finite-dimensional irreducible representations $V_{\mathfrak{so}(2m)}^{\nu}$ and $V_{\mathfrak{so}(2m)}^{(\nu_1, \ldots, \nu_m-1, -1)}$ are respectively equal to (see 10)

\begin{align*}
\frac{1}{2}|E_{\nu}| + \frac{1}{2} \left( \prod_{i=1}^m (x_i - x_i^{-1}) \right) |E_{(\nu_1 - 1)^m}|, \\
\frac{1}{2}|E_{\nu}| - \frac{1}{2} \left( \prod_{i=1}^m (x_i - x_i^{-1}) \right) |E_{(\nu_1 - 1)^m}|,
\end{align*}

where $|E_{(\nu_1 - 1)^m}|$ is the determinant of the $(\nu_1 - 1) \times (\nu_1 - 1)$ matrix defined by (6.2). Note that $\nu - (1^m)$ is a partition, and its transpose partition has length $\nu_1 - 1$. Also for any partition $\nu = (\nu_1, \cdots, \nu_m)$, the characters of the finite-dimensional
irreducible modules \( V^{\nu_1+\frac{1}{2}}_{\text{so}(2m)} \) and \( V^{(\nu_1+\frac{1}{2}, \ldots, \nu_{m-1}+\frac{1}{2}, -\nu_m-\frac{1}{2})}_{\text{so}(2m)} \) respectively equal to

\[
\frac{1}{2} \left( \prod_{i=1}^{m} (x_i^{\frac{1}{2}} + x_i^{-\frac{1}{2}}) \right) |M^+_{\nu}| + \frac{1}{2} \left( \prod_{i=1}^{m} (x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}}) \right) |M^-_{\nu}|,
\]

\[
\frac{1}{2} \left( \prod_{i=1}^{m} (x_i^{\frac{1}{2}} + x_i^{-\frac{1}{2}}) \right) |M^+_{\mu}| - \frac{1}{2} \left( \prod_{i=1}^{m} (x_i^{\frac{1}{2}} - x_i^{-\frac{1}{2}}) \right) |M^-_{\mu}|.
\]

Hereafter, we let

\[ |M^-_{\mu}| := |M^-_{\mu}(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1})| \]

and

\[ |M^+_{\mu}| := |M^+_{\mu}(x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1})| \]

denote the determinants of the \( l \times l \) matrices respectively having the \( i \)-th rows

\[
(E_{\mu_i-1} - E_{\mu_i-1}) E_{\mu_i-2} - E_{\mu_i-1} \cdots E_{\mu_i-1-n} - E_{\mu_i-1-n+1}
\]

and

\[
(E_{\mu_i-1} + E_{\mu_i-1}) E_{\mu_i-2} + E_{\mu_i-1} \cdots E_{\mu_i-1+n} + E_{\mu_i-1+n+1},
\]

where \( \mu = (\mu_1, \ldots, \mu_l) \) is any partition of length \( l \).

Let \( \mathbf{x} = \{x_1, x_2, \cdots\} \) be an infinite set of variables. Analogous to the symplectic Schur polynomials, we also have the orthogonal Schur function of weight \( \frac{n}{2} \):

\[
S^{\text{so}, \frac{n}{2}}_{\lambda}(x) = S^{\text{so}, \frac{n}{2}}_{\lambda}(x_1, x_2, \cdots),
\]

defined for each partition \( \lambda \) of length \( n = 2d \) with \( \lambda'_1 + \lambda'_2 \leq 2d \) by

\[
S^{\text{so}, \frac{n}{2}}_{\lambda}(x) := \begin{cases} 
|\tilde{e}_{\lambda}|, & \text{if } \lambda'_1 = \frac{n}{2}; \\
\frac{1}{2} |\tilde{e}_{\lambda}| + \frac{1}{2} \left( \sum_{i=0}^{\infty} e_i \right) \left( \sum_{i=0}^{\infty} (-1)^i e_i \right) |\tilde{e}_{\lambda-(1^m)}|, & \text{if } \lambda'_1 < \frac{n}{2}. 
\end{cases}
\]

where \( |\tilde{e}_{\lambda}| \) denotes the determinant of the \( d \times d \) matrix whose \( i \)-th row is

\[
(\tilde{e}_{\lambda_{d-i+1}+i-1} \tilde{e}_{\lambda_{d-i+1}+i} \tilde{e}_{\lambda_{d-i+1}+i-3} + \tilde{e}_{\lambda_{d-i+1}+i+1} \cdots \tilde{e}_{\lambda_{d-i+1}+i+d-2}),
\]

\( |\tilde{e}_{\lambda}^\circ| \) denotes the determinant of the \((d-1) \times (d-1)\) matrix whose \( i \)-th row is

\[
(\tilde{e}'_{\lambda_{d-i}+i-1} \tilde{e}'_{\lambda_{d-i}+i} \tilde{e}'_{\lambda_{d-i}+i-3} + \tilde{e}'_{\lambda_{d-i}+i+1} \cdots \tilde{e}'_{\lambda_{d-i}+i+d-2}).
\]

Here \( \bar{e}_r = \sum_{i=0}^{\infty} e_i(\mathbf{x}) e_{r+i}(\mathbf{x}) \), \( \bar{e}'_r = \tilde{e}_r - \tilde{e}_{r-2} \) and \( e_i(\mathbf{x}) \) is the \( i \)-th elementary symmetric function in the infinite set of variables \( \mathbf{x} \). Recall that \( \underline{\lambda} \) is a partition
of length \( n \) obtained from the Young diagram of \( \lambda \) by replacing its first column by a column of length \( n - \lambda'_1 \) (see Section 5.3).

Similarly, for each partition \( \lambda \) of length \( n = 2d + 1 \) with \( \lambda'_1 + \lambda'_2 \leq 2d + 1 \), the orthogonal Schur function of weight \( \frac{n}{2} \)

\[
S_{\lambda}^{so, \frac{n}{2}}(x) = S_{\lambda}^{so, \frac{n}{2}}(x_1, x_2, \cdots)
\]
in infinitely many variables is defined by

\[
S_{\lambda}^{so, \frac{n}{2}}(x) := \begin{cases} 
\frac{1}{2} \left( \sum_{i=0}^{\infty} c_i |\tilde{m}_{\lambda}^+| \right) + \frac{1}{2} \left( \sum_{i=0}^{\infty} (-1)^i c_i |\tilde{m}_{\lambda}^-| \right) & \text{if } \lambda'_1 \leq \frac{n}{2}; \\
\frac{1}{2} \left( \sum_{i=0}^{\infty} c_i |\tilde{m}_{\lambda}^+| \right) - \frac{1}{2} \left( \sum_{i=0}^{\infty} (-1)^i c_i |\tilde{m}_{\lambda}^-| \right) & \text{if } \lambda'_1 > \frac{n}{2},
\end{cases}
\]
where \( |\tilde{m}_{\lambda}^+| \) denotes the determinant of the \( d \times d \) matrix whose \( i \)-th row is

\[
(\tilde{e}_{\lambda_{d-i+1}+i-1} + \tilde{e}_{\lambda_{d-i+1}+i} \quad \tilde{e}_{\lambda_{d-i+1}+i-2} + \tilde{e}_{\lambda_{d-i+1}+i+1} \\
\cdots \\
\cdots)
\]
and \( |\tilde{m}_{\lambda}^-| \) denotes the determinant of the \( d \times d \) matrix whose \( i \)-th row is

\[
(\tilde{e}_{\lambda_{d-i+1}+i-1} - \tilde{e}_{\lambda_{d-i+1}+i} \quad \tilde{e}_{\lambda_{d-i+1}+i-2} - \tilde{e}_{\lambda_{d-i+1}+i+1} \\
\cdots \\
\cdots)
\]
We put \( S_{\lambda}^{so,d}(x_1, \cdots, x_m) = S_{\lambda}^{so,d}(x_1, \cdots, x_m, 0, 0, \cdots) \). Using similar arguments as for \( \mathfrak{sp}(2m) \) in Section 6.4 we have

\[
(x_1 \cdots x_m)^{\frac{n-d(1^m)}{2}} S_{\lambda}^{so, \frac{n}{2}}(x_1, \cdots, x_m) = S_{\lambda}^{so,d}(x_1, \cdots, x_m)
\]
for any partition \( \lambda \) of length \( n \) with \( \lambda'_1 + \lambda'_2 \leq n \) and \( \lambda_1 = m \). Now we can rewrite the combinatorial formulae (6.11) and (6.12) respectively as follows (for \( m \geq n \)):

\[
\prod_{i=1}^{d} \prod_{j=1}^{m} (1 + x_j z_i)(1 + x_j z_i^{-1}) = \sum_{\lambda} \chi^\lambda_D(n)(z) S_{\lambda}^{so, \frac{n}{2}}(x_1, \cdots, x_m), \tag{6.13}
\]
for even integer \( n = 2d \) and

\[
\prod_{i=1}^{d} \prod_{j=1}^{m} (1 + e x_j z_i)(1 + e x_j z_i^{-1})(1 + e x_j) = \sum_{\lambda} \chi^\lambda_D(n)(e, z) S_{\lambda}^{so, \frac{n}{2}}(x_1, \cdots, x_m), \tag{6.14}
\]
for odd integer \( n = 2d + 1 \). In both equations the summations over \( \lambda \) range over all partitions of length \( n \) satisfying \( \lambda'_1 + \lambda'_2 \leq n \) and \( \lambda_1 \leq m \).

Let \( x = \{ x_1, x_2, \cdots \} \) and \( y = \{ y_1, y_1^+, \cdots \} \) be two infinite sets of variables. For each partition \( \lambda \) of length \( n \), the skew orthogonal Schur function of weight \( \frac{n}{2} \) of infinitely many variables denoted by \( D_{\lambda}^{so, \frac{n}{2}}(x) = D_{\lambda}^{so, \frac{n}{2}}(x_1, x_2, \cdots) \) is defined by \( \sigma(D_{\lambda}^{so, \frac{n}{2}}(x)) \) where \( \sigma \) is the involution of the ring of symmetric functions sending the elementary symmetric functions of \( x_j \)'s to the complete symmetric functions
of $x_j$'s. Also, for each partition $\lambda$ of length $n$, the hook orthogonal Schur function of weight $\frac{n}{2}$ of infinitely many variables denoted by $HS^{\frac{n}{2}}(x, y)$ is defined by $\sigma(S^{\frac{n}{2}}(x, y))$ where $\sigma$ is the involution of the ring of symmetric functions sending the elementary symmetric functions of $y_j$'s to the complete symmetric functions of $y_j$'s.

By using (6.13), (6.14) and similar arguments as in Section 6.1, we can prove the following two propositions.

**Proposition 6.5.** Let $x = \{x_1, x_2, \cdots \}$ be an infinite set of variables and $z = \{z_1, z_2, \cdots, z_d\}$ be $d$ variables.

(i) When $n = 2d$, we have

$$\prod_{i=1}^{d} \prod_{j=1}^{\infty} (1 + x_j z_i)(1 + x_j z_i^{-1}) = \sum_{\lambda} \chi^\lambda_\varnothing(n)(z) S^{\frac{n}{2}}_\varnothing(x),$$

(ii) When $n = 2d + 1$, we have

$$\prod_{i=1}^{d} \prod_{j=1}^{\infty} (1 - x_j z_i)^{-1}(1 - x_j z_i)^{-1} = \sum_{\lambda} \chi^\lambda_\varnothing(n)(z) D^{\frac{n}{2}}_\varnothing(x),$$

where the summations on the right hand sides of both equations range over all partitions of length $n$ satisfying $\lambda_1 + \lambda_2 \leq n$.

**Proposition 6.6.** Let $x = \{x_1, x_2, \cdots \}$ and $y = \{y_{\frac{1}{2}}, y_{\frac{1}{2}}, \cdots \}$ be two infinite sets of variables and $z = \{z_1, z_2, \cdots, z_d\}$ be $d$ variables.

(i) When $n = 2d$, we have

$$\prod_{i=1}^{d} \prod_{j=1}^{\infty} \prod_{k=\frac{1}{2}}^{\infty} \frac{(1 + x_j z_i)(1 + x_j z_i^{-1})}{(1 - y_k z_i)(1 - y_k z_i^{-1})} = \sum_{\lambda} \chi^\lambda_\varnothing(n)(z) H S^{\frac{n}{2}}(x, y),$$

where $\lambda$ is summed over all partitions of length $n$.

(ii) When $n = 2d + 1$, we have

$$\prod_{i=1}^{d} \prod_{j=1}^{\infty} \prod_{k=\frac{1}{2}}^{\infty} \frac{(1 + \epsilon x_j z_i)(1 + \epsilon x_j z_i^{-1})(1 + \epsilon x_j)}{(1 - \epsilon y_k z_i)(1 - \epsilon y_k z_i^{-1})(1 - \epsilon y_k)} = \sum_{\lambda} \chi^\lambda_\varnothing(n)(\epsilon, z) H S^{\frac{n}{2}}(x, y),$$
Lemma 6.2. Let \( f^\lambda(y) \) and \( g^\lambda(y) \) be power series in the variables \( y \).

(i) Suppose that \( n \) is odd and

\[
\sum_{\lambda} f^\lambda(y) \chi_{O(n)}^\lambda(\epsilon, x) = \sum_{\lambda} g^\lambda(y) \chi_{O(n)}^\lambda(\epsilon, x),
\]

where the summation is over all partitions of length \( n \). Then \( f^\lambda(y) = g^\lambda(y) \), for all \( \lambda \).

(ii) Suppose that \( n \) is even and

\[
\sum_{\lambda} f^\lambda(y) \chi_{O(n)}^\lambda(x) = \sum_{\lambda} g^\lambda(y) \chi_{O(n)}^\lambda(x),
\]

where the summation is over all partitions of length \( n \). Then \( f^\lambda(y) + f^\lambda(y) = g^\lambda(y) + g^\lambda(y) \).

Proposition 6.7. Let \( x = \{x_1, x_2, \ldots\} \) and \( y = \{y_1, y_2, \ldots\} \) be two set of infinitely many variables. Let \( n \) be a fixed non-negative integer and \( b_{\mu \nu} \) denote the multiplicity of \( V_{O(n)}^\lambda \) in the tensor product decomposition of \( V_{O(n)}^{\mu} \otimes V_{O(n)}^{\nu} \).

(i) Suppose that \( n \) is odd, for any partition \( \lambda \) of length \( n \) with \( \lambda_1 + \lambda_2 \leq n \), we have

\[
HS_{\lambda}^{60, \frac{\mu}{2}}(x, y) = \sum_{\mu,\nu} b_{\mu \nu}^\lambda(z) S_{\mu}^{S_{\frac{\mu}{2}}} (x) D_{\nu}^{S_{\frac{\nu}{2}}} (y)
\]

where \( \mu \) and \( \nu \) are summed over all partitions of length \( n \).

(ii) Suppose that \( n \) is even, for any partition of length \( n = \lambda_1 + \lambda_2 \leq n \), we have

\[
HS_{\lambda}^{60, \frac{\mu}{2}}(x, y) + HS_{\lambda}^{60, \frac{\mu}{2}}(x, y) = \left( \sum_{\mu,\nu} b_{\mu \nu}^\lambda(z) S_{\mu}^{S_{\frac{\mu}{2}}} (x) D_{\nu}^{S_{\frac{\nu}{2}}} (y) \right) + \left( \sum_{\mu,\nu} b_{\mu \nu}^\lambda(z) S_{\mu}^{S_{\frac{\mu}{2}}} (x) D_{\nu}^{S_{\frac{\nu}{2}}} (y) \right),
\]

where \( \mu \) and \( \nu \) are summed over all partitions of length \( n \) satisfying \( \mu_1 + \mu_2 \leq n \) and \( \nu_1 + \nu_2 \leq n \), respectively.

Proof. We shall only prove the case with \( n \) being even, as the odd case is analogous. Let \( n = 2d \). By Proposition 6.5 we have

\[
\prod_{i=1}^{d} \prod_{j=1}^{\infty} \prod_{k=1/2}^{\infty} \frac{(1 + x_j z_i)(1 + x_j z_i^{-1})}{(1 - y_k z_i)(1 - y_k z_i^{-1})}
\]

\[
= \sum_{\mu} \chi_{O(n)}^{\mu}(z) S_{\mu}^{S_{\frac{\mu}{2}}}(x) \sum_{\nu} \chi_{O(n)}^{\nu}(z) D_{\nu}^{S_{\frac{\nu}{2}}}(y),
\]
where μ and ν are summed over all partitions of length n satisfying \( \lambda'_1 + \lambda'_2 \leq n \) and \( \mu'_1 + \mu'_2 \leq n \), respectively. On the other hand,

\[
\chi_{O(n)}^\mu(z)\chi_{O(n)}^\nu(z) = \sum_\lambda b_{\mu\nu}^\lambda \chi_{O(n)}^\lambda(z),
\]

where the summation is over all partitions of length n with \( \lambda'_1 + \lambda'_2 \leq n \). Thus

\[
\sum_\mu \chi_{O(n)}^\mu(z)S_{\mu}^{\frac{\alpha_0}{\alpha}}(x)\sum_\nu \chi_{O(n)}^\nu(z)D_{\nu}^{\frac{\alpha_0}{\alpha}}(y) = \sum_\lambda \chi_{O(n)}^\lambda(z)\left( \sum_{\mu,\nu} b_{\mu\nu}^\lambda (z)S_{\mu}^{\frac{\alpha_0}{\alpha}}(x)D_{\nu}^{\frac{\alpha_0}{\alpha}}(y) \right),
\]

where \( \lambda, \mu \) and \( \nu \) are summed over all partitions of length n satisfying \( \lambda'_1 + \lambda'_2 \leq n \), \( \mu'_1 + \mu'_2 \leq n \) and \( \nu'_1 + \nu'_2 \leq n \), respectively. Therefore we have

\[
(6.19) \prod_{i=1}^{d} \prod_{j=1}^{\infty} \prod_{k=1}^{\frac{1}{2}} \left( 1 + x_j z_i \right) \left( 1 + x_j z_i^{-1} \right) \left( 1 - y_k z_i \right) \left( 1 - y_k z_i^{-1} \right)
\]

where \( \mu \) and \( \nu \) are summed over all partitions of length n satisfying \( \mu'_1 + \mu'_2 \leq n \) and \( \nu'_1 + \nu'_2 \leq n \), respectively.

Now we turn to the computation of the formal characters of the Fock spaces. Let \( d = \left\lceil \frac{n}{2} \right\rceil \), that is, \( n = 2d \) if \( n \) is even and \( n = 2d + 1 \) if \( n \) is odd. Let \( e \) be a formal indeterminate and set for \( j \in \mathbb{N}, r \in \frac{1}{2} + \mathbb{Z}_+, i = 1, \ldots, d \)

\[
z_i = e^{\epsilon_i}, \quad x_j = e^{\omega_j}, \quad y_r = e^{\omega_r},
\]

where \( \epsilon_1, \ldots, \epsilon_d \) and \( \omega_s \) are the respective fundamental weights of \( O(n) \) and \( \widehat{D} \) introduced earlier. By using similar arguments as in Section 6.1 we can easily show that the character of \( \mathfrak{H}^d_0 \), with respect to the abelian algebra \( \sum_{s \in \frac{1}{2}N} \mathbb{C} \hat{c}_{ss} \oplus \sum_{i=1}^{d} \mathbb{C} E_{ii} \), is given by

\[
(6.20) \quad \text{char}_{\mathfrak{H}^d_0} = \prod_{i=1}^{d} \prod_{j=1}^{\infty} \prod_{k=1}^{\frac{1}{2}} \left( 1 + x_j z_i \right) \left( 1 + x_j z_i^{-1} \right) \left( 1 - y_k z_i \right) \left( 1 - y_k z_i^{-1} \right).
\]
Similarly, the character of $\mathfrak{d}_0^{d+\frac{1}{2}}$, with respect to the abelian algebra $\sum_{s \in \frac{1}{2} \mathbb{N}} C \mathfrak{e}_s \oplus \sum_{i=1}^l C E_{ii}$, is given by

\begin{equation}
\text{ch} \mathfrak{d}_0^{d+\frac{1}{2}} = \prod_{i=1}^d \prod_{j=1}^{\infty} \prod_{k=1}^{\infty} \frac{(1 + \epsilon x_j z_i)(1 + \epsilon x_j z_i^{-1})(1 + \epsilon x_j)}{(1 - \epsilon y_k z_i)(1 - \epsilon y_k z_i^{-1})(1 - \epsilon y_k)}.
\end{equation}

By Proposition 6.6, we can rewrite (6.20) as

\begin{equation}
\text{ch} \mathfrak{d}_0^{n+\frac{1}{2}} = \sum \chi^\lambda_\mathfrak{O}(n)(z) HS^{\mathfrak{so}_1}_\lambda(x, y),
\end{equation}

where $\lambda$ is summed over all partitions of length $n$ with $\lambda_1' + \lambda_2' \leq n$. On the other hand, Theorem 5.8 implies that

\begin{equation}
\text{ch} \mathfrak{d}_0^{n+\frac{1}{2}} = \sum \chi^\lambda_\mathfrak{O}(n)(z) HS^{\mathfrak{so}_1}_\lambda(\hat{\mathfrak{D}}, \hat{\mathfrak{D}})(x, y),
\end{equation}

where $\lambda$ is summed over all partitions of length $n$ with $\lambda_1' + \lambda_2' \leq n$. Combining (6.23) with (6.22), we arrive at

\begin{equation}
\sum \chi^\lambda_\mathfrak{O}(n)(z) HS^{\mathfrak{so}_1}_\lambda(z) = \sum \chi L(\hat{\mathfrak{D}}, \hat{\mathfrak{D}})(\lambda) \chi^\lambda_\mathfrak{O}(n)(z).
\end{equation}

In a similar way, we can also derive the following equation for $n$ odd:

\begin{equation}
\sum \chi^\lambda_\mathfrak{O}(n)(\epsilon, z) HS^{\mathfrak{so}_1}_\lambda(z) = \sum \chi L(\hat{\mathfrak{D}}, \hat{\mathfrak{D}})(\lambda) \chi^\lambda_\mathfrak{O}(n)(\epsilon, z).
\end{equation}

Applying Lemma 6.7 to these equations, we obtain the following character formulae:

**Theorem 6.2.** For each partition $\lambda$ of length $n = 2d + 1$ with $\lambda_1' + \lambda_2' \leq n$, we have

\begin{equation}
\text{ch} L(\hat{\mathfrak{D}}, \hat{\mathfrak{D}})(\lambda) = HS^{\mathfrak{so}_1}_\lambda(x, y).
\end{equation}

For each partition $\lambda$ of length $n = 2d$ with $\lambda_1' + \lambda_2' \leq n$, we have

\begin{equation}
\text{ch} L(\hat{\mathfrak{D}}, \hat{\mathfrak{D}})(\lambda) + \text{ch} L(\hat{\mathfrak{D}}, \hat{\mathfrak{D}})(\bar{\lambda}) = HS^{\mathfrak{so}_1}_\lambda(x, y) + HS^{\mathfrak{so}_1}_{\bar{\lambda}}(x, y).
\end{equation}

**Remark 6.1.** From Theorem 6.1 and Theorem 6.2, we can easily extract character formulae for the unitarizable quasi-finite irreducible highest weight modules over the $\mathfrak{so}$ and $\mathfrak{sp}$ type subalgebras of $\hat{\mathfrak{g}}_{1,\infty}$.

**Acknowledgements.** We thank Shun-Jen Cheng for discussions. Financial support from the Australian Research Council and the National Science Council of the Republic of China is gratefully acknowledged.
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DEPARTMENT OF MATHEMATICS, NATIONAL CHENG KUNG UNIVERSITY, TAINAN, TAIWAN 701
E-mail address: nlam@mail.ncku.edu.tw

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, NEW SOUTH WALES 2006, AUSTRALIA
E-mail address: rzhang@maths.usyd.edu.au