EXISTENCE OF THE $\det^{S^2}$ MAP

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ABSTRACT. In this paper we show that for a vector space $V_d$ of dimension $d$ there exists a linear map $\det^{S^2} : V_d^{d(2d-1)} \to k$ with the property that $\det^{S^2}(\otimes_{1 \leq i < j \leq 2d}(v_{i,j})) = 0$ if there exists $1 \leq x < y < z \leq 2d$ such that $v_{x,y} = v_{x,z} = v_{y,z}$. The existence of such a map was conjectured in [4]. We present two applications of the map $\det^{S^2}$ to geometry and combinatorics.

1. Introduction

The determinant of a matrix plays an important role in several areas of mathematics. It captures quantitative information (like area of a region, volume of a solid), but also qualitative information (like linear dependence of $d$ vectors in a $d$-dimensional vector space). Heuristically, the best way to introduce the determinant of a linear transformation $T : V_d \to V_d$ is to consider the exterior algebra of the $d$-dimensional vector space $V_d$, and then define the determinant as the unique constant that determines the map $\Lambda(T) : \Lambda V_d[d] \to \Lambda V_d[d]$. Equivalently, one can show that the determinant is the unique (up to a scalar) nontrivial linear map $\det : V_d^{\otimes d} \to k$ with the property that $\det(\otimes_{1 \leq i < j \leq d}(v_i)) = 0$ if there exist $1 \leq x < y \leq d$ such that $v_x = v_y$.

The graded vector space $\Lambda_{V_d}^{S^2}$ was introduced in [4] as a generalization of the exterior algebra. It has properties similar with the ones of the exterior algebra, for example $\Lambda_{V_d}^{S^2}[n] = 0$ if $n > 2d$. It was conjectured in [4] that $\dim_k(\Lambda_{V_d}^{S^2}[2d]) = 1$. This conjecture is equivalent with the existence and uniqueness (up to a scalar) of a nontrivial linear map $\det^{S^2} : V_d^{\otimes d(2d-1)} \to k$ with the property that $\det(\otimes_{1 \leq i < j \leq 2d}(v_{i,j})) = 0$ if there exist $1 \leq x < y < z \leq 2d$ such that $v_{x,y} = v_{x,z} = v_{y,z}$. The conjecture was checked to be true in the case $d = 2$ ([4]) and $d = 3$ ([2]).

In this paper we show for every $d$ there exists a nontrivial map $\det^{S^2} : V_d^{\otimes d(2d-1)} \to k$ with the above mentioned property. For this we consider a system of $2d$ vector equations associate to $(v_{i,j})_{1 \leq i < j \leq 2d} \in V_d^{d(2d-1)}$. The corresponding matrix is of dimension $2d^2 \times d(2d - 1)$, but we can eliminate one of the vector equations to get a square $d(2d - 1) \times d(2d - 1)$ matrix. The determinant of this square matrix is nontrivial, and has the universality property we are looking for. In particular this shows that $\dim_k(\Lambda_{V_d}^{S^2}[2d]) \geq 1$. The uniqueness of the map $\det^{S^2}$ is still an open question.

As an application we give a geometrical interpretation of the condition $\det^{S^2}(\otimes_{1 \leq i < j \leq 2d}(v_{i,j})) = 0$. In particular, we show that if $p_i \in V_d$ for $1 \leq i \leq 2d$, and we take $v_{i,j} = p_j - p_i$ for all $1 \leq i < j \leq 2d$ then $\det^{S^2}(\otimes_{1 \leq i < j \leq 2d}(v_{i,j})) = 0$. This a generalization of a result proved in [5] for the case $d = 2$ and $d = 3$.

We also give an application to combinatorics. More precisely we show that $(\Gamma_1, \ldots, \Gamma_d)$ is cycle free $d$-partition of the complete graph $K_{2d}$ if and only if $\det^{S^2}(f_{(\Gamma_1, \ldots, \Gamma_d)}) \neq 0$ (where $f_{(\Gamma_1, \ldots, \Gamma_d)} \in V_d^{d(2d-1)}$ is a certain element associated to $(\Gamma_1, \ldots, \Gamma_d)$). The case $d = 2$ and $d = 3$ was proved in [2]. One can think about this result as a generalization of the fact that a $d \times d$ matrix that has $d$
entries equal to 1 and the rest of the entries equal to zero will have a nonzero determinant if and only if it has a nonzero entry in every row and every column.

2. Preliminary

In this paper $k$ is a field, $V_d$ is a $d$-dimensional vector space, and $B_d = \{e_1, \ldots, e_d\}$ is a fixed basis for $V_d$. We denote by $V_d \otimes^n$ the $m$-th tensor power of $V_d$.

The exterior algebra $\Lambda_{V_d}$ can be defined as the quotient of the tensor algebra $T_{V_d} = \oplus_n V_d \otimes^n$ by the ideal generated by elements of the form $u \otimes u$ where $u \in V_d$. It is well known that $\dim_k(\Lambda_{V_d}[d]) = 1$, in particular if $T : V_d \to V_d$ is a linear map then $\Lambda(T) : \Lambda_{V_d}[d] \to \Lambda_{V_d}[d]$ is the multiplication by a constant, which by definition is denoted by $\det(T)$. Alternatively, one can define the determinant as the unique nontrivial linear map $\det : V_d \otimes^d \to k$ with the property that $\det(\otimes_{1 \leq i \leq d}^n v_i) = 0$ if there exist $1 \leq x < y \leq d$ such that $v_x = v_y$.

Next we recall from [2], [4], and [5] a few results about $\Lambda_{V_d}^S$ and the $\det^S$ map. For every $n \geq 0$ we define

$$\Lambda_{V_d}^S[n] = \frac{T_{V_d}^S[n]}{E_{V_d}^S[n]},$$

where $T_{V_d}^S[n] = V_d \otimes^n \frac{n(n-1)}{2}$, and $E_{V_d}^S[n]$ is the subspace of $T_{V_d}^S[n]$ generated by those elements $\otimes_{1 \leq i \leq n}(v_{i,j}) \in V_d \otimes^n \frac{n(n-1)}{2}$ with the property that there exists $1 \leq x < y < z \leq n$ such that $v_{x,y} = v_{x,z} = v_{y,z}$. Notice that we use a slightly different notation from the one in [4], more precisely the grading of $\Lambda_{V_d}^S$ is shifted by 1 (i.e. $\Lambda_{V_d}^S[n] = \Lambda_{V_d}^S(n + 1)$). This is more consistent with the usual grading on the exterior algebra.

It was shown in [4] that $\Lambda_{V_d}^S[2] = 0$ if $n > 2d$. It was conjectured in the same paper that $\dim_k(\Lambda_{V_d}^S[2d]) = 1$. This conjecture is equivalent with the existence and uniqueness (up to a constant) of a nontrivial linear map $\det^S : V_d \otimes^d \to k$ such that $\det^S(\otimes_{1 \leq i \leq d}^n v_{i,j}) = 0$ if there exist $1 \leq x < y < z \leq 2d$ such that $v_{x,y} = v_{x,z} = v_{y,z}$. Notice the similitude with the determinant map.

The conjecture was checked for $d = 2$ in [4], and for $d = 3$ in [2]. In particular for $d = 2$ and $d = 3$ there exists a map $\det^S : V_d \otimes^d \to k$ with the above mentioned property. In this paper we show $\dim_k(\Lambda_{V_d}^S[2d]) \geq 1$, i.e. we prove the existence of a nontrivial map $\det^S$ for any $d$. The uniqueness is still an open question for $d > 3$.

We denote by $E_d$ the element $\otimes_{1 \leq i \leq 2d}^n (e_{i,j}) \in V_d \otimes^d$ determined by

$$e_{i,j} = \begin{cases} 
  e_t & \text{if } i < 2t - 1, \text{ is odd and } j = 2t - 1, \\
  e_t & \text{if } i < 2t, \text{ is even and } j = 2t, \\
  e_t & \text{if } i = 2t - 1, \text{ } j > 2t - 1 \text{ and } j \text{ is even,} \\
  e_t & \text{if } i = 2t, \text{ } j > 2t \text{ and } j \text{ is odd.} \end{cases} \quad (2.1)$$

It was shown in [2] that if $d = 2$ or $d = 3$ then $\det^S(E_d) = 1$ (in particular this shows that the map $\det^S$ is nontrivial in those two cases).

Remark 2.1. When $d = 2$ it was shown in [5] that $\det(\otimes_{1 \leq i \leq j \leq 4}^n v_{i,j}) = 0$ if and only if there exist $p_1, p_2, p_3, p_4 \in V_2$, and $\lambda_{i,j} \in k$ not all trivial such that $\lambda_{i,j} v_{i,j} = p_j - p_i$. A similar but partial result is also true when $d = 3$; more precisely if $q_1, q_2, q_3, q_4, q_5, q_6 \in V_3$ and we take $w_{i,j} = q_j - q_i$ then $\det^S(\otimes_{1 \leq i < j \leq 2}^n w_{i,j}^d) = 0$. Later in the paper we will generalize this result.

Next we recall from [2] a few definitions and examples of $d$-partitions of the complete graph $K_{2d}$.
Definition 2.2. A $d$-partition of the complete graph $K_{2d}$ is an ordered collection $(\Gamma_1, \Gamma_2, \ldots, \Gamma_d)$ of sub-graphs $\Gamma_i$ of $K_{2d}$ such that:
1) $V(\Gamma_i) = V(K_{2d})$ for all $1 \leq i \leq d$,
2) $E(\Gamma_i) \cap E(\Gamma_j) = \emptyset$ for all $i \neq j$,
3) $\bigcup_{i=1}^{n} E(\Gamma_i) = E(K_{2d})$.

We say that the $d$-partition $(\Gamma_1, \Gamma_2, \ldots, \Gamma_d)$ in homogeneous if $|E(\Gamma_i)| = |E(\Gamma_j)|$ for all $1 \leq i < j \leq d$. We say that the partition $(\Gamma_1, \Gamma_2, \ldots, \Gamma_d)$ is cycle-free if each $\Gamma_i$ is cycle-free.

Let $\mathcal{B}_d = \{e_1, e_2, \ldots, e_d\}$ be a basis for the vector space $V_d$. It was noticed in [2] that the set
$$\mathcal{G}_{\mathcal{B}_d}[2d] = \{ \otimes_{1 \leq i < j \leq 2d}(v_{i,j}) \in V_d^\otimes(2d-1) \mid v_{i,j} \in \mathcal{B}_d \},$$
(which is a basis for $V_d^\otimes(2d-1)$) is in bijection with the set $\mathcal{P}_d(K_{2d})$ of $d$-partitions of the complete graph $K_{2d}$. Indeed, if $f = \otimes_{1 \leq i < j \leq 2d}(v_{i,j}) \in \mathcal{G}_{\mathcal{B}_d}[2d]$ we consider the sub-graphs $\Gamma_i(f)$ of $K_{2d}$ constructed as follows: for every $1 \leq i \leq d$ we take $V(\Gamma_i(f)) = \{1, 2, \ldots, 2d\}$ and $E(\Gamma_i(f)) = \{(s,t)|v_{s,t} = e_i\}$. One can easily see that $\Gamma(f) = (\Gamma_1(f), \ldots, \Gamma_d(f))$ is a $d$-partition of $K_{2d}$.

Moreover, the map $f \mapsto \Gamma(f)$ is a bijection from the set $\mathcal{G}_{\mathcal{B}_d}[2d]$ to the set of $d$-partitions of $K_{2d}$. We will denote by $f_{(\Gamma_1, \ldots, \Gamma_d)}$ the element in $\mathcal{G}_{\mathcal{B}_d}[2d]$ corresponding to the partition $(\Gamma_1, \ldots, \Gamma_d)$.

Example 2.3. Let
$$E_3 = \left(\begin{array}{cccccc}
1 & e_1 & e_2 & e_3 & e_1 \\
1 & e_1 & e_2 & e_1 & e_1 \\
1 & e_2 & e_1 & e_3 & e_2 \\
1 & e_3 & e_2 & e_1 & e_1 \\
1 & e_3 & e_2 & e_1 & e_1 \\
1 & e_3 & e_2 & e_1 & e_1
\end{array}\right) \in V_3^\otimes 15,$$
then the corresponding 3-partition $\Gamma(E_3)$ of $K_6$ is given in Figure 1. Notice that $\Gamma(E_3)$ is homogeneous and cycle free. See [2] for more examples.

![Figure 1. $\Gamma(E_3) = (\Gamma_1, \Gamma_2, \Gamma_3)$ the 3-partition associated to $E_3$](image)

The following results were proved in [2].

Lemma 2.4. Take $(\Gamma_1, \ldots, \Gamma_d)$ a $d$-partition of $K_{2d}$ that is not cycle-free. Then $\hat{f}_{(\Gamma_1, \ldots, \Gamma_d)} = 0 \in \Lambda^S_{\mathcal{B}_d}[2d]$.

Lemma 2.5. Take $(\Gamma_1, \ldots, \Gamma_d)$ a cycle-free homogeneous $d$-partition of $K_{2d}$, and $1 \leq x < y < z \leq 2d$. Then, there exist $(\Lambda_1, \ldots, \Lambda_d)$, a unique cycle-free homogeneous $d$-partition of $K_{2d}$ such that the two partitions $(\Gamma_1, \ldots, \Gamma_d)$ and $(\Lambda_1, \ldots, \Lambda_d)$ coincide on every edge of $K_{2d}$ except on the edges $(x,y)$, $(x,z)$, and $(y,z)$ where they are different on at least two edges. We will denote $(\Lambda_1, \ldots, \Lambda_d)$ by $(\Gamma_1, \ldots, \Gamma_d)^{(x,y,z)}$. With the above notations we have
$$\hat{f}_{(\Gamma_1, \ldots, \Gamma_d)} = -\hat{f}_{(\Gamma_1, \ldots, \Gamma_d)^{(x,y,z)}} \in \Lambda^S_{\mathcal{B}_d}[2d].$$
Remark 2.6. Lemma 2.4 and Lemma 2.5 were used in [2] to show existence of the map $\det S^2$ when $d = 3$. In that case the map $\det S^2$ can be written as a sum over all cycle-free $d$-partitions of the complete graph $K_{2d}$. The approach in this paper is different, but the above two lemmas provide an intriguing connection between linear algebra behind $\Lambda_{V_a} S^2 [2d]$, and the combinatorics of $\mathcal{P}_d (K_{2d})$. We will discuss this connection in the last section.

Finally, one should notice that (when $d = 2$ or $d = 3$) the map $\det S^2$ is invariant under the action of the group $SL_d (k)$ on $V_d$, and so, by general results from invariant theory ([6]), it can be written as a sum of products of determinants of matrices with columns the vectors $v_{i,j}$ (see [5]).

3. The Existence of the $\det S^2$ Map

Take $V_d$ a vector space of dimension $d$, and fix $v_{i,j} \in V_d$ for all $1 \leq i < j \leq 2d$. For each $1 \leq k \leq 2d$ consider the vector equation $E_k ((v_{i,j})_{1 \leq i < j \leq 2d})$ defined as

$$
\sum_{s=1}^{k-1} (-1)^{s-1} \lambda_{s,k} v_{s,k} + \sum_{t=k+1}^{2d} (-1)^{t} \lambda_{k,t} v_{k,t} = 0.
$$

We denote by $S((v_{i,j})_{1 \leq i < j \leq 2d})$ the system of vector equations $E_k ((v_{i,j})_{1 \leq i < j \leq 2d})$ for all $1 \leq k \leq 2d$.

Notice that the $2d$ vector equations are dependent. Indeed, each vector $v_{i,j}$ appears twice in our system, first time in equation $E_i ((v_{i,j})_{1 \leq i < j \leq 2d})$ with coefficient $(-1)^i \lambda_i$, and second time in the equation $E_j ((v_{i,j})_{1 \leq i < j \leq 2d})$ with coefficient $(-1)^{i-1} \lambda_{i,j}$. This means that

$$
\sum_{k=1}^{2d} (-1)^{k-1} E_k ((v_{i,j})_{1 \leq i < j \leq 2d}) = 0,
$$

and so, when studying this system it is enough to consider any $2d - 1$ of the $2d$ vector equations.

Definition 3.1. Let $v_{i,j} \in V_d$ for all $1 \leq i < j \leq 2d$. We denote by $A((v_{i,j})_{1 \leq i < j \leq 2d})$ the $d(2d) \times d(2d - 1)$ matrix of the system $S((v_{i,j})_{1 \leq i < j \leq 2d})$. We denote by $A_k((v_{i,j})_{1 \leq i < j \leq 2d})$ the $d(2d - 1) \times d(2d - 1)$ matrix corresponding to the system obtained after eliminating equation $E_k ((v_{i,j})_{1 \leq i < j \leq 2d})$. Finally, we denote by $M_k((v_{i,j})_{1 \leq i < j \leq 2d})$ the $d \times d(2d - 1)$ matrix of the vector equation $E_k ((v_{i,j})_{1 \leq i < j \leq 2d})$.

When the vectors $v_{i,j}$ are clear from the context, in the interest of shortening the notation, we will suppress the vectors $v_{i,j}$ and write $A$, $A_k$ and $M_k$ respectively. Notice that matrix $A((v_{i,j})_{1 \leq i < j \leq 2d})$ can be obtained by staking all the $M_k((v_{i,j})_{1 \leq i < j \leq 2d})$’s on top of each other, i.e.

$$
A((v_{i,j})_{1 \leq i < j \leq 2d}) = \begin{pmatrix}
M_1((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_2((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{2d}((v_{i,j})_{1 \leq i < j \leq 2d})
\end{pmatrix}.
$$

Similarly, we have

$$
A_k = \begin{pmatrix}
M_1((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_2((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{k-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_{k+1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{2d-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_{2d}((v_{i,j})_{1 \leq i < j \leq 2d})
\end{pmatrix}.
$$
Remark 3.2. Using the matrices $M_k((v_{i,j})_{1 \leq i < j \leq 2d})$, the dependence between the equations $\mathcal{E}_k$ (i.e. equation 3.2) can be rewritten as
\[
\sum_{k=1}^{2d}(-1)^{k-1}M_k((v_{i,j})_{1 \leq i < j \leq 2d}) = 0 \in M_{d\times d(2d-1)}.
\] (3.5)

Example 3.3. When $d = 2$ and $v_{i,j} = \begin{pmatrix} \alpha_{i,j} \\ \beta_{i,j} \end{pmatrix}$ for all $1 \leq i < j \leq 4$, the system $\mathcal{S}((v_{i,j})_{1 \leq i < j \leq 4})$ becomes
\[
\begin{pmatrix}
\alpha_{1,2} & -\alpha_{1,3} & 0 & \alpha_{1,4} & 0 & 0 \\
\beta_{1,2} & -\beta_{1,3} & 0 & \beta_{1,4} & 0 & 0 \\
\alpha_{1,2} & 0 & -\alpha_{2,3} & 0 & \alpha_{2,4} & 0 \\
\beta_{1,2} & 0 & -\beta_{2,3} & 0 & \beta_{2,4} & 0 \\
0 & \alpha_{1,3} & -\alpha_{2,3} & 0 & 0 & \alpha_{3,4} \\
0 & \beta_{1,3} & -\beta_{2,3} & 0 & 0 & \beta_{3,4} \\
0 & 0 & 0 & \alpha_{1,4} & -\alpha_{2,4} & \alpha_{3,4} \\
0 & 0 & 0 & \beta_{1,4} & -\beta_{2,4} & \beta_{3,4}
\end{pmatrix}
= 0,
\] (3.6)
with $A((v_{i,j})_{1 \leq i < j \leq 4})$ being the $8 \times 6$ matrix of the system. When $k = 2$ we have
\[
A_2((v_{i,j})_{1 \leq i < j \leq 4}) = \begin{pmatrix}
\alpha_{1,2} & -\alpha_{1,3} & 0 & \alpha_{1,4} & 0 & 0 \\
\beta_{1,2} & -\beta_{1,3} & 0 & \beta_{1,4} & 0 & 0 \\
0 & \alpha_{1,3} & -\alpha_{2,3} & 0 & 0 & \alpha_{3,4} \\
0 & \beta_{1,3} & -\beta_{2,3} & 0 & 0 & \beta_{3,4} \\
0 & 0 & 0 & \alpha_{1,4} & -\alpha_{2,4} & \alpha_{3,4} \\
0 & 0 & 0 & \beta_{1,4} & -\beta_{2,4} & \beta_{3,4}
\end{pmatrix},
\] (3.7)
\[
M_2((v_{i,j})_{1 \leq i < j \leq 4}) = \begin{pmatrix}
\alpha_{1,2} & 0 & -\alpha_{2,3} & 0 & \alpha_{2,4} & 0 \\
\beta_{1,2} & 0 & -\beta_{2,3} & 0 & \beta_{2,4} & 0
\end{pmatrix}.
\] (3.8)

Remark 3.4. Computing the determinant of the matrix $A_1((v_{i,j})_{1 \leq i < j \leq 4})$ one recovers the formula for $\text{det}^{S^2}$ from [4] (see also [5]). A different approach to get the same result is to show that $\text{det}(A_1((v_{i,j})_{1 \leq i < j \leq 4}))$ satisfies the universality property of the map $\text{det}^{S^2}$.

Indeed, first notice that
\[
\text{det}(A_1((v_{i,j})_{1 \leq i < j \leq 4})) = \text{det}(A_k((v_{i,j})_{1 \leq i < j \leq 4})),
\]
for all $1 \leq k \leq 4$. This essentially follows from Equation 3.5. For example, in order to show that $\text{det}(A_1((v_{i,j})_{1 \leq i < j \leq 4})) = \text{det}(A_2((v_{i,j})_{1 \leq i < j \leq 4}))$ we use elementary transformations and the fact that $M_2 = M_1 + M_3 - M_4$. More precisely, in the matrix
\[
A_2((v_{i,j})_{1 \leq i < j \leq 4}) = \begin{pmatrix}
M_1((v_{i,j})_{1 \leq i < j \leq 4}) \\
M_3((v_{i,j})_{1 \leq i < j \leq 4}) \\
M_4((v_{i,j})_{1 \leq i < j \leq 4})
\end{pmatrix},
\]
we add rows $M_3 - M_4$ to $M_1$ (i.e. $R_3 - R_5$ to $R_1$, respectively $R_4 - R_6$ to $R_2$) to get
\[
\text{det}(A_1((v_{i,j})_{1 \leq i < j \leq 4})) = \text{det}(M_1((v_{i,j})_{1 \leq i < j \leq 4}) + M_3((v_{i,j})_{1 \leq i < j \leq 4}) - M_4((v_{i,j})_{1 \leq i < j \leq 4}),
\]
\[
= \text{det}(M_2((v_{i,j})_{1 \leq i < j \leq 4}),
\]
and so $\text{det}(A_1((v_{i,j})_{1 \leq i < j \leq 4})) = \text{det}(A_2((v_{i,j})_{1 \leq i < j \leq 4})).$
Next, notice that if \( v_{2,3} = v_{2,4} = v_{3,4} \) then column three, five and six of the matrix \( A \) are linearly dependent because their sum is zero. And so, the corresponding columns of matrix \( A_1 \) are linearly dependent, which means that \( \det(A_1) = 0 \). Similarly, if \( v_{1,3} = v_{1,4} = v_{3,4} \) then \( \det(A_2) = 0 \), if \( v_{1,2} = v_{1,4} = v_{2,4} \) then \( \det(A_3) = 0 \), and if \( v_{1,2} = v_{1,3} = v_{2,3} \) then \( \det(A_4) = 0 \).

Finally, one can check that if \( e_{1,2} = e_{1,4} = e_{2,3} = e_1 \) and \( e_{1,3} = e_{2,4} = e_{3,4} = e_2 \) then \( \det(A_1((e_{i,j})_{1 \leq i < j \leq 4})) = 1 \), which combined with the uniqueness of the map \( \det^{S^2} \) proved in [4] for \( d = 2 \), it shows that \( \det(A_1((e_{i,j})_{1 \leq i < j \leq 4})) = \det^{S^2}((e_{i,j})_{1 \leq i < j \leq 4}) \).

Remark 3.5. The system in Remark 3.4 gives a geometrical interpretation for the condition \( \det^{S^2}((v_{i,j})_{1 \leq i < j \leq 4}) = 0 \) that is equivalent with the one in [5]. Notice however that the setting here is slightly different, as we use a different set of vector equations. Later in the paper we will come back to this geometrical interpretation and discuss the general case. In particular, we will strengthen the result for \( d = 3 \) from [5].

Next we show the existence of a nontrivial map \( \det^{S^2} \) map for any \( d \).

Theorem 3.6. Let \( d \geq 2 \) and \( V_d \) be a \( d \)-dimensional vector space. Define \( \det^{S^2} : V_d^{d(2d-1)} \to k \) determined by \( \det^{S^2}((v_{i,j})_{1 \leq i < j \leq 2d}) = \det(A_1((v_{i,j})_{1 \leq i < j \leq 2d})) \). Then \( \det^{S^2} \) is a nontrivial multilinear map with the property that \( \det^{S^2}((v_{i,j})_{1 \leq i < j \leq 2d}) = 0 \) if there exist \( 1 \leq x < y < z \leq 2d \) such that \( v_{x,y} = v_{x,z} = v_{y,z} \).

Proof. First noticed that for all \( 1 \leq t \leq 2d \) we have

\[
\det(A_1((v_{i,j})_{1 \leq i < j \leq 2d})) = \det(A_t((v_{i,j})_{1 \leq i < j \leq 2d})).
\]

Indeed, from Equation 3.5 we know that

\[
(-1)^t M_t((v_{i,j})_{1 \leq i < j \leq 2d}) = \sum_{k=1, k \neq t}^{2d} (-1)^{k-1} M_k((v_{i,j})_{1 \leq i < j \leq 2d}).
\]

In the matrix

\[
A_t((v_{i,j})_{1 \leq i < j \leq 2d}) = \begin{pmatrix}
M_1((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_2((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{t-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_t((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{2d-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_{2d}((v_{i,j})_{1 \leq i < j \leq 2d})
\end{pmatrix},
\]

we add \( \sum_{k=2, k \neq t}^{2d} (-1)^{k-1} M_k \) to \( M_1 \) and to get

\[
B_t((v_{i,j})_{1 \leq i < j \leq 2d}) = \begin{pmatrix}
(-1)^t M_t((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_2((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{t-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_t((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{2d-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_{2d}((v_{i,j})_{1 \leq i < j \leq 2d})
\end{pmatrix}.
\]
Using properties of determinants and elementary transformations, we get

\[
det(A_t)((v_{i,j})_{1 \leq i < j \leq 2d}) = det(B_t)((v_{i,j})_{1 \leq i < j \leq 2d})
\]

\[
= (-1)^{td}det(
\begin{pmatrix}
M_t((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_2((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{t-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_{t+1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{2d-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_{2d}((v_{i,j})_{1 \leq i < j \leq 2d})
\end{pmatrix}
\]

\[
= (-1)^{td}(-1)^{d^2(t-2)}det(
\begin{pmatrix}
M_2((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{t-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_1((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_{t+1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
\vdots \\
M_{2d-1}((v_{i,j})_{1 \leq i < j \leq 2d}) \\
M_{2d}((v_{i,j})_{1 \leq i < j \leq 2d})
\end{pmatrix}
\]

\[
= (-1)^{(d^2+d)-2d^2}det(A_1((v_{i,j})_{1 \leq i < j \leq 2d}))
\]

\[
= det(A_1((v_{i,j})_{1 \leq i < j \leq 2d})).
\]

Next we will show if there exist \(1 \leq x < y < z \leq 2d\) such that \(v_{x,y} = v_{x,z} = v_{y,z} = w\) then \(det(A_1((v_{i,j})_{1 \leq i < j \leq 2d})) = 0\). Chose \(1 \leq t \leq 2d\) such that \(x \neq t \neq y, t \neq z\). We know from the above remarks that \(det(A_1((v_{i,j})_{1 \leq i < j \leq 2d})) = det(A_t((v_{i,j})_{1 \leq i < j \leq 2d}))\). Notice that in the matrix \(A_t((v_{i,j})_{1 \leq i < j \leq 2d})\) the columns \((x,y), (x,z)\) and \((y,z)\) are dependent. Indeed, since \(t\) is distinct from \(x, y\) and \(z\), when removing equation \(E_t\) we do not affect matrices \(M_x, M_y\) and \(M_z\). Which gives us the following matrix

\[
A_t((v_{i,j})_{1 \leq i < j \leq 2d}) =
\begin{pmatrix}
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\cdots & (-1)^y v_{x,y} & \cdots & (-1)^z v_{x,z} & \cdots & 0 & \cdots \\
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\cdots & (-1)^{x-1} v_{x,y} & \cdots & 0 & \cdots & (-1)^z v_{y,z} & \cdots \\
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\cdots & 0 & \cdots & (-1)^{x-1} v_{x,z} & \cdots & (-1)^{y-1} v_{y,z} & \cdots \\
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots 
\end{pmatrix}
\]

\[
= \
\begin{pmatrix}
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\cdots & (-1)^y w & \cdots & (-1)^z w & \cdots & 0 & \cdots \\
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\cdots & (-1)^{x-1} w & \cdots & 0 & \cdots & (-1)^z w & \cdots \\
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\cdots & 0 & \cdots & (-1)^{x-1} w & \cdots & (-1)^{y-1} w & \cdots \\
\cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots 
\end{pmatrix}
\]

\[
(3.9)
\]

\[
(3.10)
\]

If we denote by \(c_{x,y}\) the column corresponding to the pair \((x,y)\) then

\[
(-1)^x c_{x,y} - (-1)^y c_{x,z} + (-1)^z c_{y,z} = 0,
\]
which proves that if \( v_{x,y} = v_{x,z} = v_{y,z} \) for some \( 1 \leq x < y < z \leq 2d \) then \( \det(A_t((v_{i,j}))_{1 \leq i < j \leq 2d})) = 0 \).

Finally, we want to show that \( \det(A_1(E_d)) \neq 0 \) where \( E_d = (e_{i,j})_{1 \leq i < j \leq 2d} \) is defined by Equation 2.1. It is enough to show that the system \( S(E_d) \) has only the trivial solution. We will prove this by induction. When \( d = 2 \) this was checked in Remark 3.4.

Notice that if \( (\lambda_{i,j})_{1 \leq i < j \leq 2d} \) is a solution for \( S(E_d) \) then \( \lambda_{i,2d-1} = 0 = \lambda_{i,2d} \) for all \( 1 \leq i \leq 2d-2 \), and \( \lambda_{2d-1,2d} = 0 \).

Indeed, if \( i \) is odd and \( i \leq 2d-2 \) then \( e_{i,2d-1} = e_d \) and so equation \( E_i(E_d) \) becomes

\[
\sum_{s=1}^{i-1} (-1)^{s-1} \lambda_{s,i} e_{s,i} + \sum_{t=i+1, t \neq 2d-1}^{2d} (-1)^t \lambda_{i,t} e_{i,t} + (-1)^{2d-1} \lambda_{2d-1} e_d = 0.
\]  

(3.11)

Using the definition of \( E_d \), notice that the vectors \( e_{s,i} \) and \( e_{i,t} \in \{e_1, e_2, \ldots, e_{d-1}\} \), for all \( 1 \leq s \leq i-1 \), and all \( i+1 \leq t \leq 2d, t \neq 2d-1 \). Since \( \{e_1, \ldots, e_d\} \) is a basis, we get \( \lambda_{i,2d-1} = 0 \) if \( i \) is odd.

Similarly, if \( i \) is even and \( i \leq 2d-2 \) we have \( e_{i,2d} = e_d \) and so equation \( E_i(E_d) \) becomes

\[
\sum_{s=1}^{i-1} (-1)^{s-1} \lambda_{s,i} e_{s,i} + \sum_{t=i+1, t \neq 2d}^{2d} (-1)^t \lambda_{i,t} e_{i,t} + (-1)^{2d} \lambda_{2d-1} e_d = 0.
\]  

(3.12)

Notice that the vectors \( e_{s,i} \) and \( e_{i,t} \in \{e_1, e_2, \ldots, e_{d-1}\} \) for all \( 1 \leq s \leq i-1 \), and all \( i+1 \leq t \leq 2d-1 \). Since \( \{e_1, \ldots, e_d\} \) is a basis, we get \( \lambda_{i,2d} = 0 \) if \( i \) is even.

Next, we use equation \( E_{2d-1}(E_d) \) to get

\[
\sum_{s=1}^{2d-2} (-1)^{s-1} \lambda_{s,2d-1} e_{s,2d-1} + (-1)^{2d} \lambda_{2d-1} e_{2d-1} = 0.
\]  

(3.13)

Since \( \lambda_{i,2d-1} = 0 \) if \( i \) is odd, \( e_{2j,2d-1} = e_j \) for all \( 1 \leq j \leq d-1 \), and \( e_{2d-1,2d} = e_d \) we get

\[
\sum_{s=1}^{d-1} (-1)^{2s-2} \lambda_{2s,2d-1} e_s + (-1)^{2d} \lambda_{2d-1} e_d = 0,
\]  

(3.14)

which obviously means that \( \lambda_{2j,2d-1} = 0 \) for all \( 1 \leq j \leq d-1 \), and \( \lambda_{2d-1,2d} = 0 \).

Similarly, using equation \( E_{2d}(E_d) \), and the fact that \( \lambda_{2j-1,2d} = 0 \) for all \( 1 \leq j \leq d \) one can show that \( \lambda_{2j,2d} = 0 \) for all \( 1 \leq j \leq d-1 \).

To summarize, if \( (\lambda_{i,j})_{1 \leq i < j \leq 2d} \) is a solution for \( S(E_d) \) then \( \lambda_{i,2d-1} = 0 = \lambda_{i,2d} \) for all \( 1 \leq i \leq 2d-2 \) and \( \lambda_{2d-1,2d} = 0 \). This means that \( (\lambda_{i,j})_{1 \leq i < j \leq 2d-1} \) is a solution for \( S(E_{d-1}) \), and so by induction we get \( \lambda_{i,j} = 0 \) for all \( 1 \leq i < j \leq 2d \).

\[ \square \]

**Corollary 3.7.** Let \( d \geq 2 \) and \( V_d \) be a \( d \)-dimensional vector space. Then there exists a nontrivial linear map \( \det^{S^2} : V_d^{2d(2d-1)} \to k \) with the property that \( \det^{S^2}(\otimes_{1 \leq i < j \leq 2d}(v_{i,j})) = 0 \) if there exist \( 1 \leq x < y < z \leq 2d \) such that \( v_{x,y} = v_{x,z} = v_{y,z} \).

**Corollary 3.8.** Let \( d \geq 2 \) and \( V_d \) be a \( d \)-dimensional vector space. Then \( \dim_k(\Lambda^{S^2}_{V_d}(2d)) \geq 1 \).

**Remark 3.9.** By abuse of notation we will denote \( \det^{S^2} \) both, the multilinear map on \( V_d^{2d(2d-1)} \), and the linear map on \( V_d^{2d(2d-1)} \). This should not create any confusion.

4. Applications

4.1. Geometrical Application. We have the following geometrical interpretation for the condition \( \det^{S^2} = 0 \) that generalizes the results from [5].
Proposition 4.1. Let $d \geq 2$, $V_d$ be a $d$-dimensional vector space, and take $(v_{i,j})_{1 \leq i < j \leq 2d} \in V_d^{d(2d-1)}$. The following are equivalent:

1. $\det^{S^2}((v_{i,j})_{1 \leq i < j \leq 2d}) = 0$.
2. There exists $\lambda_{i,j} \in k$ for all $1 \leq i < j \leq 2d$ not all zero such that

$$\sum_{s=1}^{k-1}(-1)^{s-1}\lambda_{s,k}v_{s,k} + \sum_{t=k+1}^{2d}(-1)^{t}\lambda_{k,t}v_{k,t} = 0. \quad (4.1)$$

for all $1 \leq k \leq 2d$.

Proof. It follows from the definition of map $\det^{S^2}$.

In [5] it was proved that if $p_i \in V_3$ for all $1 \leq i \leq 6$, and we define $v_{i,j} = p_j - p_i$ then $\det^{S^2}((v_{i,j})_{1 \leq i < j \leq 6}) = 0$. It was conjectured that a similar result is true in general. Indeed we have the following.

Corollary 4.1. Let $d \geq 2$ and $V_d$ be a $d$-dimensional vector space. Take $p_i \in V_d$ for $1 \leq i \leq 2d$ and define $v_{i,j} = p_j - p_i$ for all $1 \leq i < j \leq 2d$. Then $\det^{S^2}((v_{i,j})_{1 \leq i < j \leq 2d}) = 0$.

Proof. We will show that the system \( S((v_{i,j})_{1 \leq i < j \leq 2d}) \) has a nontrivial solution and so by Proposition 4.1, we get that $\det^{S^2}((v_{i,j})_{1 \leq i < j \leq 2d}) = 0$.

Consider the vectors $v_{1,j} = p_j - p_1$ for $2 \leq j \leq 2d$. Since $\dim_k(V_d) = d < 2d - 1$ then there exist $\lambda_j$ for $2 \leq j \leq 2d$ not all zero such that

$$\sum_{t=2}^{2d}(-1)^{t}\lambda_tv_{1,t} = 0. \quad (4.2)$$

Case I. Assume that $\Lambda = \sum_{t=2}^{2d}(-1)^{t}\lambda_t \neq 0$. For $1 \leq i < j \leq 2d$ take

$$\lambda_{i,j} = \begin{cases} \lambda_j & \text{if } i = 1, \\ \lambda_t \lambda_i & \text{if } i > 1. \end{cases} \quad (4.3)$$

We will check that $\{\lambda_{i,j}\}_{1 \leq i < j \leq 2d}$ is a nontrivial solution for the system $S((v_{i,j})_{1 \leq i < j \leq 2d})$.

Equation $E_1$ is satisfied because of the definition of $\lambda_{1,j}$ (see Equation 4.2). Take $2 \leq k \leq 2d$, notice that $v_{s,k} = p_k - p_s = v_{1,k} - v_{1,s}$ if $1 \leq s \leq k-1$, and $v_{k,t} = p_t - p_k = v_{1,t} - v_{1,k}$ if $k+1 \leq t \leq 2d$. So we have:

$$\sum_{s=1}^{k-1}(-1)^{s-1}\lambda_{s,k}v_{s,k} + \sum_{t=k+1}^{2d}(-1)^{t}\lambda_{k,t}v_{k,t} = \lambda_k v_{1,k} + \sum_{s=2}^{k-1}(-1)^{s-1}\frac{\lambda_k \lambda_s}{\Lambda} (v_{1,k} - v_{1,s}) + \sum_{t=k+1}^{2d}(-1)^{t}\frac{\lambda_k \lambda_t}{\Lambda} (v_{1,t} - v_{1,k})$$

$$= \lambda_k v_{1,k} + \sum_{s=2}^{k-1}(-1)^{s-1}\frac{\lambda_k \lambda_s}{\Lambda} v_{1,k} + \sum_{t=k+1}^{2d}(-1)^{t-1}\frac{\lambda_k \lambda_t}{\Lambda} (v_{1,t} - v_{1,k})$$

$$= \lambda_k v_{1,k} + \sum_{s=2}^{k-1}(-1)^{s-1}\frac{\lambda_k \lambda_s}{\Lambda} v_{1,k} + \sum_{t=k+1}^{2d}(-1)^{t-1}\frac{\lambda_k \lambda_t}{\Lambda} v_{1,t}$$

$$= \lambda_k v_{1,k} + \sum_{s=2}^{k-1}(-1)^{s-1}\frac{\lambda_k \lambda_s}{\Lambda} v_{1,k} + \sum_{t=k+1}^{2d}(-1)^{t-1}\frac{\lambda_k \lambda_t}{\Lambda} v_{1,t}$$
\[
\frac{k-1}{2}\sum_{s=2}^{k-1} (-1)^s \frac{\lambda_s \lambda_k}{\Lambda} v_{1,s} + \frac{(\lambda_k - \lambda_k \frac{\Lambda}{\Lambda}) v_{1,k}}{2d} + \frac{(\frac{2d}{t=k+1} (-1)^t \lambda_k \lambda_t)}{\Lambda} v_{1,t} = 0,
\]

and so equation \( E \) is satisfied.

**Case II.** Assume that \( \Lambda = \sum_{t=2}^{2d} (-1)^t \lambda_t = 0 \). Take \( 2 \leq a \leq 2d - 1 \) with the property that \( \lambda_a \neq 0 \) and \( \lambda_i = 0 \) for all \( 2 \leq i < a \). For all \( 1 \leq i < j \leq 2d \) we define \( \lambda_{i,j} = \begin{cases} 0 & \text{if } i < a, \\ \frac{\lambda_i \lambda_j}{\lambda_a} & \text{if } i \geq a. \end{cases} \) (4.4)

We will check that \( \{\lambda_{i,j}\}_{1 \leq i < j \leq 2d} \) is a solution for the system \( S((v_{i,j})_{1 \leq i < j \leq 2d}) \).

If \( k < a \) then the equation \( E_k \) is trivially satisfied since when \( k < a \) we have \( \lambda_{i,k} = 0 \) for all \( 1 \leq i < k \), and \( \lambda_{k,j} = 0 \) for all \( k < j \leq 2d \).

Next assume that \( k = a \). We use the fact that \( v_{i,j} = p_j - p_i = v_{1,j} - v_{1,i} \) for \( 1 \leq i < j \leq 2d \), \( \lambda_{s,a} = 0 \) for all \( 1 \leq s < a - 1 \), and \( \lambda_{a,t} = \frac{\lambda_s \lambda_k}{\lambda_k} \) for all \( a < t \leq 2d \) to get

\[
\frac{a-1}{2d} \sum_{s=1}^{a-1} (-1)^{s-1} \lambda_{s,a} v_{s,a} + \frac{2d}{2d} (-1)^t \lambda_{a,t} v_{a,t} = \frac{2d}{2d} \sum_{t=a+1}^{2d} (-1)^t \lambda_{t} \frac{\lambda_a}{\lambda_a} (v_{1,t} - v_{1,a})
\]

\[
= \frac{2d}{2d} \sum_{t=a+1}^{2d} (-1)^t \lambda_{t} v_{1,t} - (\sum_{t=a+1}^{2d} (-1)^t \lambda_{t} v_{1,a})
\]

\[
= (\sum_{t=a+1}^{2d} (-1)^{a-1} \lambda_{a} v_{1,a} - (\sum_{t=a+1}^{2d} (-1)^{a-1} \lambda_{a} v_{1,a}) = 0.
\]

Where the next to last equality follows because \( \sum_{t=a}^{2d} (-1)^t \lambda_{t} v_{1,t} = 0 \), and \( \sum_{t=a}^{2d} (-1)^t \lambda_{t} = 0 \).

Finally, assume that \( k > a \). We use the fact that \( v_{i,j} = p_j - p_i = v_{1,j} - v_{1,i} \) for \( 2 \leq i < j \leq 2d \), that \( \lambda_{i,k} = 0 \) for all \( 1 \leq i < a - 1 \), \( \lambda_{s,k} = \frac{\lambda_s \lambda_k}{\lambda_k} \) for all \( a \leq s < k - 1 \), and \( \lambda_{k,t} = \frac{\lambda_s \lambda_k}{\lambda_k} \) for all \( k < t \leq 2d \) to get

\[
\frac{k-1}{2d} \sum_{s=1}^{k-1} (-1)^{s-1} \lambda_{s,k} v_{s,k} + \frac{2d}{2d} \sum_{t=k+1}^{2d} (-1)^t \lambda_{k,t} v_{k,t} = \frac{k-1}{2d} \sum_{s=a}^{k-1} (-1)^{s-1} \lambda_{s,k} \frac{\lambda_k}{\lambda_k} (v_{1,k} - v_{1,s}) + \frac{2d}{2d} \sum_{t=k+1}^{2d} (-1)^t \lambda_{k,t} \frac{\lambda_k}{\lambda_k} (v_{1,t} - v_{1,k})
\]
we get that
\[ 2 \leq 0. \]

Indeed, from the definition of \( \det \) we have
\[ \det \to \text{map} \]

\[ 1 \text{ cycle-free} \]

which is obviously a contradiction since \( B \) is cycle-free it follows that \( \Phi \) is nontrivial solution \( (v_1, \ldots, v_d) \) has a nontrivial solution \( (\mu_{1,2d}) \) is not empty. The partition \( (\Gamma_1, \ldots, \Gamma_d) \) is inducing a partition \( (\Phi_1, \ldots, \Phi_d) \) of \( G \). Take \( x \in \{ 1, \ldots, d \} \) such that \( E(\Phi_x) \neq \emptyset \) (such an \( x \) exists since \( E(G) \neq \emptyset \)). Because \( \Gamma_x \) is cycle-free it follows that \( \Phi_x \) is also cycle free, and so we can find a vertex \( a \in \{ 1, \ldots, 2d \} \) such that the degree of the vertex \( a \) in the graph \( \Phi_x \) is one (i.e. there exists exactly one other vertex \( b \) with the property that the edge \( (a, b) \in E(\Phi_x) \)).

For simplicity we will assume that \( a < b \) (the case \( a > b \) is similar). Consider the equation \( E_a((w_{i,j})_{1 \leq i < j \leq 2d}) \) where \( (w_{i,j})_{1 \leq i < j \leq 2d} = f(\Gamma_1, \ldots, \Gamma_d) \). We have
\[
\sum_{s=1}^{a-1} (-1)^{s-1} \mu_{s,a} w_{s,a} + \sum_{t=a+1}^{2d} (-1)^{t} \mu_{a,t} w_{a,t} = 0. \tag{4.5}
\]

Notice that \( w_{s,a}, w_{a,t} \in \{ e_1, \ldots, e_d \} \). Moreover \( w_{s,a} \neq e_x \) for all \( 1 \leq s \leq a-1 \), and for \( a+1 \leq t \leq 2d \) we have \( w_{a,t} = e_x \) if and only if \( t = b \). Finally, since the coefficient of \( w_{a,b} = e_x \) is \( \mu_{a,b} \neq 0 \), we get that equation \( E_a \) gives a nontrivial linear dependence relation among the vectors \( \{ e_1, \ldots, e_d \} \), which is obviously a contradiction since \( B_d \) is a basis. This means that the system \( S(f(\Gamma_1, \ldots, \Gamma_d)) \) has only the trivial solution, and so \( det^{S^2} (f(\Gamma_1, \ldots, \Gamma_d)) \neq 0 \).

For the converse we prove a more general statement. Take \( D : V_d^{\otimes d(2d-1)} \to k \) to be a linear map with the property that \( D(\otimes_{1 \leq i < j \leq 2d}(v_{i,j})) = 0 \) if there exist \( 1 \leq x < y < z \leq 2d \) such \( v_{x,y} = v_{x,z} = v_{y,z} \). It follows from results proved in [2] (see Lemma 2.4), that if the \( d \)-partition \( (\Gamma_1, \ldots, \Gamma_d) \) is not cycle-free, then that \( D(f(\Gamma_1, \ldots, \Gamma_d)) = 0 \). However, this exact statement was not made explicitly in [2], so for completeness sake, we recall the main steps of that proof.

Let \( (\Gamma_1, \ldots, \Gamma_d) \) be a \( d \)-partition of \( K_{2d} \) and \( f(\Gamma_1, \ldots, \Gamma_d) = \otimes_{1 \leq i < j \leq 2d}(f_{i,j}) \in \mathcal{G}_{B_d}[2d] \). If \( (\Gamma_1, \ldots, \Gamma_d) \) is not cycle-free then there exists \( 1 \leq k \leq d \) such that \( \Gamma_k \) has a cycle. If \( \Gamma_k \) has cycle of length

3, then there exist $1 \leq x < y < z \leq 2d$ such that $(x, y)$, $(x, z)$ and $(y, z) \in E(\Gamma_k)$, and so $f_{x,y} = f_{x,z} = f_{y,z} = e_k$. By the property of the map $D$ this means that $D(f_{(\Gamma_1, ..., \Gamma_d)}) = 0$.

If the length of the cycle in $\Gamma_k$ is $l > 4$, then it was shown in [2] (see Lemma 3.5), that there exist two $d$-partition $(\Gamma_1^{(1)}, ..., \Gamma_d^{(1)})$ and $(\Gamma_1^{(2)}, ..., \Gamma_d^{(2)})$ of $K_{2d}$ such that $\Gamma_k^{(1)}$ and $\Gamma_k^{(2)}$ have cycles of length $l - 1$, and

$$D(f_{(\Gamma_1, ..., \Gamma_d)}) + D(f_{(\Gamma_1^{(1)}, ..., \Gamma_d^{(1)})}) + D(f_{(\Gamma_1^{(2)}, ..., \Gamma_d^{(2)})}) = 0.$$  

From here, the statement follows by induction. \hfill \Box

**Remark 4.3.** The result in this section is consistent with conjecture from [2] where it was proposed that the $det^{S^2}$ map can be written as a sum over the set of homogeneous cycle free $d$-partition of the graph $K_{2d}$

$$det^{S^2}(\otimes_{1 \leq i < j \leq 2d}(v_{i,j})) = \sum_{(\Gamma_1, ..., \Gamma_d) \in P_d(K_{2d})} \varepsilon_d^2((\Gamma_1, ..., \Gamma_d))^{2} M(\Gamma_1, ..., \Gamma_d)((v_{i,j})_{1 \leq i < j \leq 2d}),$$

(4.6)

where $\varepsilon_d^2 : P_d(K_{2d}) \to \{-1, 1\}$, and $M(\Gamma_1, ..., \Gamma_d)((v_{i,j})_{1 \leq i < j \leq 2d})$ is a monomial associated to $(\Gamma_1, ..., \Gamma_d)$ and to the element $\otimes_{1 \leq i < j \leq 2d}(v_{i,j}) \in V_d^{\otimes 2d}$ (see [2] for more details). In order to prove this formula for every $d$, it would be enough to show that the involutions $(\Gamma_1, ..., \Gamma_d) \to (\Gamma_1, ..., \Gamma_d)^{(x,y,z)}$ (described in Lemma 2.5) act transitively on $P_d(K_{2d})$. This would also imply the uniqueness up to a constant of the map $det^{S^2}$.

**Remark 4.4.** The construction of $det^{S^2}$ is somehow similar with the definition the resultant of two polynomials (see [1]). It might be interesting to understand if there is a more general construction that cover both examples.

**Remark 4.5.** $A_{V_d}^{S^3}$ was introduced in [3] as another generalization of the exterior algebra. When $d = 2$, it was shown that there exists a linear map $det^{S^3} : V_2^{\otimes 20} \to k$ with the property that $det^{S^3}(\otimes_{1 \leq i < j < k \leq 6}(v_{i,j,k})) = 0$ if there exist $1 < x < y < z < t \leq 6$ such that $v_{x,y,z} = v_{x,y,t} = v_{x,z,t} = v_{y,z,t}$. We expect that the construction presented in this paper can be adapted to define a map $det^{S^3}$ for any $d$.

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