Universality in Four-Boson Systems

Abstract. We report recent advances on the study of universal weakly bound four-boson states from the solutions of the Faddeev-Yakubovsky equations with zero-range two-body interactions. In particular, we present the correlation between the energies of successive tetramers between two neighbor Efimov trimers and compare it to recent finite range potential model calculations. We provide further results on the large momentum structure of the tetramer wave function, where the four-body scale, introduced in the regularization procedure of the bound state equations in momentum space, is clearly manifested. The results we are presenting confirm a previous conjecture on a four-body scaling behavior, which is independent of the three-body one. We show that the correlation between the positions of two successive resonant four-boson recombination peaks are consistent with recent data, as well as with recent calculations close to the unitary limit. Systematic deviations suggest the relevance of range corrections.

Keywords. Universality · Four-boson systems · Bound states · Contact potential

1 Introduction

Quantum few-body systems interacting with short-ranged forces is not shaped only by three-body properties \[ \hbar. \] Weakly-bound tetramers composed by identical bosons have a characteristic scale, which is independent of the trimmer one, for resonant pairwise interaction in the unitary limit (zero two-body binding \( E_2 = 0 \) or infinite scattering length \( a \)). Such property can be revealed if one considers the general case, not constrained by some specific strong short-range interaction. The existence of
limit cycles associated with the dependence of tetramer properties on the four-body scale. The calculations with Faddeev-Yakubovsky (FY) equations for a renormalized zero-range two-body interaction demands the introduction of two independent regularization scales. The debate on the theoretical evidences of the short-range four-body scales on real systems, as in cold atom traps, is underway (see e.g. [1, 3, 5–14]).

A universal scaling function relating the energies of two successive tetramer states, $E_4^{(N)}$ and $E_4^{(N+1)}$ (where for $N = 0$ we have the ground-state) and the corresponding trimer energy $E_3$ of an Efimov state [15, 16] was reported as a limit cycle at the unitary limit. This scaling function applies to the energies of two tetramers between two neighbor Efimov states. Calculations for tetramer energies with different potential models, local and nonlocal, with and without three-body forces, at the unitary limit [10, 13, 17] for tetramer states between excited Efimov trimers, are consistent with the scaling function, in the sense that they exhibit the characteristic dependence on the four-body scale, as the results obtained with the zero-range potential with two independent scales in the FY equations [3]. It is worth noting that, the $N + 1$ tetramer state emerges from the 3+1 threshold for a universal ratio $E_4^{(N)}/E_3 = 4.6$ for $E_2 = 0$, which does not depend on $N$.

It was reported in [3] that the tetramers move as the short-range four-body scale is changed and the existence of the new scale can be also revealed by a resonant atom-trimer relaxation. The resonant behavior arises when a tetramer becomes bound at the atom-trimer scattering threshold. Furthermore, the independent four-body scale implies in a family of Tjon lines in the general case [18]. Also, the positions of two successive resonant four-boson recombination peaks, show the effect of a four-boson short-range scale through a universal correlation, in a form of a scaling function, which is relevant for actual cold atom experiments and in fairly agreement with recent data [19, 20] and calculations with $s$–wave separable potentials [21], as will be presented in an updated plot. We will discuss the systematic deviations from this correlation, by other models and data, suggesting that range corrections is of some importance and it is of interest for future research.

Furthermore, we will present to some extent the momentum-space structure of the FY components of weakly-bound tetramers at the unitary limit. We show that both channels of the FY decomposition [trimer plus atom (T+A), or $K$–type and dimer plus dimer (D+D), or $H$–type] present high momentum tails, which reflects the short-range four-body scale. We also point out the relevance of the $H$ component to bring the dependence of the four-body short-range scale to the tetramer properties.

Our analysis gives further theoretical support in favor of the independent role of a four-body scale near a Feshbach resonance with implications for cold atom physics. The relevance of our study is related to the experimental possibilities to explore universal few-body properties with tunable two-body interactions.

In Sect. 2, we present the homogeneous set of coupled FY equations for the zero-range interaction, properly regularized to permit to calculate the tetramer bound states with a fixed trimer spectrum. In Sect. 3 we discuss the concept of scaling functions for the four-boson system, with particular attention to two of them: (i) the correlation among the energies of successive tetramer states between two neighbor Efimov states, and (ii) the correlation of the position of successive peaks in the four-boson recombination. Both scaling functions are due to the freedom in the four-body regularization scale, and the comparison with other model calculations shows their universal validity for short-range forces in the Efimov limit. In Sect. 4 we present results for the tetramer structure as given by their associated FY amplitudes. Our main conclusions are given in Sect. 5.

### 2 Subtracted FY Coupled Equations - Regularization

#### 2.1 Formalism

The FY integral equations for the contact interaction, which has a separable $s$–wave form, have a simplified form (see e.g. [4]). The FY components can be written in terms of reduced functions, where the dependence on one of the relative two-particle momentum drops out, and they depend only on the other two independent Jacobi momentum. In the case of the $K$ components the associated reduced
\( K_{ij,k} \) function depends on the relative momentum of the particle \( k \), in respect to the center of mass of the pair \((ij)\), and the on the relative momentum of particle \( l \) to the center of mass of the remaining three-particle subsystem. In the case of the \( H \) components the associated reduced \( H_{ij,kl} \) function depends on the relative momentum of the pair \((kl)\), and on the relative momentum of subsystem \((ij)\) in respect to \((kl)\).

Fig. 1 Diagrammatic representation of the Faddeev-Yakubovsky integral equations, where the black blobs represent two-body T-matrices for the zero-range potential (4). The reduced FY amplitudes \( K \) and \( H \) do not depend on the relative momentum of the pair connected by the thick black line. The regularizations of the FY equations with the subtracted form of the free four-body Green’s function are indicated by the vertical dotted and dashed lines, where the respective regulator energy parameters are \( \mu_2^3 \) and \( \mu_4^3 \), as shown in Eq. (3).

The coupled set of integral equations for the reduced FY amplitudes of a four-boson system are represented diagrammatically in Fig. 1. These integral equations have to be regularized and the four-body intermediate state propagation should have an ultraviolet cutoff. The kernel part labeled with (a) induces the Thomas collapse of the three-body system if not regularized. One should recognize that if the interaction between the particle \( l \) and the rest of the system is turned off, only the three-body term will survive. Then, a momentum regularization is chosen to keep the three-boson binding energy at a given value. This procedure is implemented by a subtraction scheme of the kernel at some energy scale \( \mu_2^3 \) [22], which is indicated by the vertical dotted line in the diagram. However, without regularization of the remaining terms [(b), (c), (d) and (e)] shown in Fig. 1 the tetramer will collapse, as already discussed in Ref. [1]. Therefore, another subtraction is introduced together with the intermediate four-boson propagators in diagrams (b)-(e), indicated by the vertical dashed lines. The new subtraction point is in principle a free parameter and, by exploring this freedom, the tetramer and trimer energies become independent. While in momentum space this procedure looks straightforward, in configuration space a short-range four-body force will play the role to separate the trimer and tetramer energies. Another possibility is to use different short-ranged three-body forces chosen together with on-shell equivalent two-body potentials, by keeping the trimer energy fixed.

The reduced FY amplitudes for the four-boson bound states are the solutions of the set of coupled integral equations (see Fig. 1) given by:

\[
| K_{ij,k}^{l} \rangle = 2 \tau(\epsilon_{ij,k}) \left[ G^{(3)}(\epsilon_{ij,k}) \left[ K_{ik,j}^{l} \right] + G^{(4)}(\epsilon_{ij,k}) \left[ H_{ik,j}^{l} \right] \right],
\]

The occurrence of the four-body collapse can be avoided by different approaches. One way, for example, is by regularizing the remaining terms with the same parameter considered at the three-body level. However, in this way the possible independent behavior of the four-body scale in relation to the three-body one is not being explored. Another way is by considering a different (four-body) scaling parameter (or by parameterizing the four-body interaction with a new scale), such that a possible independent behavior can be manifested in relation to the three-body one. This second procedure has to be done with care when considering local interactions in order to avoid killing any possible independent behavior by introducing some strong short range repulsion, as it happens naturally in the specific case of nuclear forces. As observed by Tjon [18], in nuclear physics the sensitivity on the number of short range scales stops at the three-body level, because the nuclear interaction is dominated by strong repulsive two-body forces, evidenced by the correlation between the triton and the \(^4\)He binding energies when considering local two-body interactions - the well-known Tjon-line.
\[ |H_{ij,kt}\rangle = \tau(\epsilon_{ij,kt}) G_{ij,kt}^{(4)} \left[ 2 |K_{kl,ij}\rangle + |H_{kt,ij}\rangle \right]. \tag{2} \]

The free 4-body propagators in Fig. 1 are subtracted, and indicated by the dashed vertical lines. The projected 4-body free Green’s function operator \( G^{(N)} \) for \( N = 3 \) or 4 are subtracted at the energies \(-\mu_3^2\) and \(-\mu_4^2\):

\[
G_{ij,ik}^{(N)} = \langle \chi_{ij} | \frac{1}{E-H_0} - \frac{1}{-\mu_N^2-H_0} | \chi_{ik} \rangle
\tag{3}
\]

where the form factors of the contact interaction of the pair \((ij)\) is \((p_{ij} |\chi_{ij}) = 1\) with \(p_{ij}\) the relative momentum.

In the coupled equations for the reduced FY components the two-boson T-matrix of the pair \((ij)\), represented by the dark blob in Fig. 1 is given by

\[
t_{ij}(\epsilon) = |\chi_{ij}| \tau_{ij}(\epsilon) \langle \chi_{ij} |, \quad \tau_{ij}^{-1}(\epsilon) = 2\pi^2 \left( \frac{1}{a} - \sqrt{-\epsilon} \right).	ag{4}
\]

The energies of the virtual two-body subsystem \((ij)\) appearing as arguments of the two-boson scattering amplitude in Eqs. 1 and 2, are \(\epsilon_{ij,k}\) and \(\epsilon_{ij,kt}\). The former is the energy of the virtual pair, in the 3+1 partition, while the latter in the 2+2 partition.

We emphasize that \(G_{ij,ik}^{(3)}\) and \(G_{ij,ik}^{(4)}\) are matrix elements of the free-propagators of the four-boson systems, regularized with different momentum parameters \(\mu_3\) and \(\mu_4\), respectively, as shown in Eq. 3. Here, it should be clear that, instead of regularizing the integral equations with some given momentum cutoffs, the regularization procedure is done by subtracting the propagator at given scales, such that one replaces the four-body free Green’s function by their subtracted form in the FY equations. The resolvent carrying the subtraction energy \(\mu_3^2\), i.e., the first term on the right-hand side of Eq. 1, keeps the trimer properties fixed. After a close inspection of this term, together with the left hand side, one can realize that the homogeneous trimer equation is recovered, if the subtraction scale is the one used originally to regularize the zero-range model for the trimer. The extra terms brought by the four-particle FY decomposition, in 1 and 2, are regularized with an independent momentum parameter \(\mu_4\). Within this procedure, we are sure that the three-body subtraction scale fixes three-body observables correctly. Note that, if we drop the extra terms brought by the interaction with the fourth particle (say \(G_{ij,ik}^{(4)}\)), we go back to the regularized zero-range three-boson bound state Faddeev equation. In short, the introduction of the three-body regularization parameter \(\mu_3\) avoids the Thomas-collaps, fixing a three-body observable, whereas the regularizing parameter \(\mu_4\) fixes the four-body scale within a more general perspective of four particle interactions 1.

2.2 Scattering Cuts and S-matrix Poles in Four-Boson Systems

The variation of the subtraction parameter \(\mu_4\) for fixed \(\mu_3\), moves the tetramer energies and these states can only emerge from the atom-trimer threshold in the unitary limit. Two possible situations (a) and (b) are represented in Fig. 2 for increasing \(\mu_4\) in the complex energy plane. The thresholds and cuts associated with the atom-trimer scattering are shown in the figure for the ground and first excited trimer.

The case (a) in Fig. 2 corresponds to calculations performed for stable tetramers below the trimer ground state, and shows a possibly trajectory for the tetramer excited state coming from a virtual four-body state. The suggestion comes in close analogy to the behavior of an excited Efimov state, which appears from the atom-dimer continuum threshold, as the ultraviolet regulator parameter \(\mu_3\) is increased for a fixed scattering length see also 2, 24). Such pattern is the same as the one already found long ago for the triton virtual state calculated with separable nucleon-nucleon potentials, when the deuteron binding is decreased 22. The case (b) in Fig. 2 is an illustration of the trajectory followed by a tetramer resonance between two successive Efimov trimers. The resonance state emerges from the elastic cut of the atom and first-excited trimer, passing through the threshold as the four-body subtraction scale is increased. Then, the state becomes a resonance entering the second energy sheet associated with the atom and ground-trimmer elastic cut. From this pattern it follows that the
Fig. 2 Schematic representation of the scattering cuts in a four-boson system at the unitary limit in the complex energy plane. The dashed lines with indicative arrows show different possibilities of movement of the position of the tetramer state in the complex energy plane when \( \mu_4/\mu_3 \) increases. The position of the trimer ground and first excited states are indicated in the diagram by \( E_3^{(0)} \) and \( E_3^{(1)} \), respectively. In the case of a stable tetramer (a), the arrows on the dashed line indicate the trajectory of the corresponding state as \( \mu_4/\mu_3 \) increases, moving from a virtual to a bound state. The sketch of a possible trajectory for a resonant tetramer state is shown in (b).

tetramers close to the trimer should have a smaller width, going to zero when the real part of the tetramer resonance energy is at the threshold. Indeed calculations of the tetramer resonance with s-wave separable potentials shows that the tetramer closest to the threshold has a smaller width in comparison to the other one below it [13]. The proposed picture has yet to be checked quantitatively within our model.

3 Four-Boson Scaling Functions

The four-boson observables in the universal regime are determined by few quantities. The physical information contained in a short range potential can be parameterized using a two, three and four-body scales, which corresponds to the scattering length, the trimer bound state energy and one tetramer energy. The set of coupled FY integral equations (1) and (2) for the zero-range interaction, is solvable once the scattering length, and the subtraction scale \( \mu_3 \) and \( \mu_4 \) are given. Therefore, the tetramer energies are functions of these parameters. This dependence can be translated to a dependence on physical quantities, the reference trimer energy and one tetramer energy, then we write that:

\[
E_4^{(N+1)} = E_4^{(N)} F_4\left(\sqrt{E_3/E_4^{(N)}}, \pm \sqrt{E_2/E_3}\right),
\]

in units of \( \hbar = m = 1 \), with \( m \) being the boson mass. The signs \( \pm \) distinguish between a two-body bound (+) or virtual (−) state. The function \( F \) is a scaling function. On purpose we have presented no dependence on \( N \) in the scaling function, that in principle could be the case, but it is not [3].

The use of the concept of scaling functions to study tetramer properties comes from the successful applications to describe Efimov trimers in the situation where they can be bound, virtual or resonant when the range of the interaction vanishes (see e.g. [2]). In this case only the physical scales corresponding to energies of the dimer and a trimer are enough to determine the successive trimer state. Remarkably, all Efimov spectrum collapses in only one single curve, which represents the correlation between successive trimer energies. Thus, a trimer universal scaling function gives the energy of an excited Efimov state, \( E_3^{(N+1)} \), as a function of \( E_3^{(N)} \), which is written as:

\[
E_3^{(N+1)} = E_3^{(N)} F_3\left(\pm \sqrt{E_2/E_3^{(N)}}\right),
\]

The universal scaling function describes all Efimov spectrum, once the dimer and one reference trimer energy is known. It is also useful to parameterize the Efimov excited energy showing the state starting as virtual for \( \alpha > 0 \) then become bound and then becomes a resonance at a critical value of \( \alpha < 0 \) [26]. For a finite range potential the function \( F \) exist in the limit of \( N \to \infty \) when \( |\alpha| \to \infty \), however, in practice it converges fast for a zero-range model calculation.
The concept of scaling functions is also used to present results for other tetramer observables, as in the case of the values of the negative scattering length where the tetramer binding energy goes to zero, in the Brunnian region \[27, 28\], where the Borromean trimer has already turned into a continuum resonance \[26\]. In trapped cold atoms, both the positions or values of the scattering length where a trimer or a tetramer cross zero, corresponds to a resonant recombination, which produces a peak in trap losses, and it was observed in several experiments \[29–34\]. The experimental positions of resonant recombination losses, for the trimer and the two successive shallow tetramers, were compared with the scaling plot for the correlation between positions of successive four boson recombination peaks in \[3\]. In the following we will present it compared to recent experiments \[19\] and calculations \[21\].

### 3.1 Tetramer Energies Close to the Unitary Limit

The correlation of the energies of successive tetramer states appearing between two consecutive Efimov trimers in the unitary limit is given by the scaling function evaluated at \(E_2 = 0\):

\[
E_4^{(N+1)} = E_4^{(N)} F_4 \left( \sqrt{\frac{E_3}{E_4^{(N)}}, 0} \right),
\]

which is presented in the scaling plot shown in Fig. 3 in the form of the ratio \(\sqrt{(E_4^{(N+1)} - E_3)/E_4^{(N)}}\).

We have performed calculations for \(N = 0\) and \(N = 2\), i.e., up to three tetramers below the ground state trimer \[3, 4\]. The curves for \(N = 0\) and \(N = 1\) are not distinguishable in the scale of the figure, i.e., a limit cycle has been achieved fast. For tetramers between two successive Efimov trimers, the...
plot should be used above $\sqrt{E_3/E_1^{(4)}} > 1/22.7$. With this constraint up to three tetramers between two Efimov trimers are possible, with their energies changing according to the value of the short-range four-boson scale, parameterized in our model by $\mu_4$.

The validity of the scaling function to any tetramer is verified by the consistency with other theoretical results obtained not only for stable tetramers but for tetramers attached to excited trimers \cite{10,13} in the unitary limit. This is shown in Fig. 3 where the thicker solid line shows the unitary limit (infinite $a$). We also consider corrections due to finite scattering lengths, as indicated inside the figure, in order to verify how sensitive are the results with respect to small positive and negative changes of $a$. We observe that the binding energies of excited tetramers are favored in case of $a > 0$, with the curves opening to the right side. The same behavior is verified within effective field theory calculations without range corrections and by considering the leading-order three-body potential \cite{7}. In this case, the unitary limit was found by extrapolation from finite (positive and negative) values of $a$. Other calculations with finite range potentials are shown in the figure, where one should note that all of them are on the right side, above the unitary limiting curve. Such results are indicative that range corrections (which certainly are being considered in these other model calculations) tend to open the scaling function. This is also the case of calculations with realistic potentials for the $^4$He trimers and tetramers performed in Ref. \cite{37}.

3.2 Position of Four-Body Resonant Losses

The scaling function corresponding to the correlation between the negative scattering lengths where two successive tetramers cross the continuum threshold and produces a four-atom resonant recombination, is written as \cite{3}:

$$a_{N_3,N+1}^T = a_{N_3}^- A \left( a_{N_3,N}^T / a_{N_3}^- \right) ,$$

(8)

where $a_{N_3}^-$ is the position of the three-atom resonant recombination for $a < 0$, and $a_{N_3,N}^T$ is the scattering length for which the excited $N$-th tetramer touch the branch point of the four-body continuum cut.

![Graph](image_url)

**Fig. 4** (Color on-line) Positions of four-atom recombination peaks ($a < 0$) where two successive tetramers become unbound. The results presented by the solid line, reported in Ref. \cite{3}, are compared with calculations done by Stecher et al. \cite{10} (triangles) and Deltuva \cite{21} (crosses). The experimental results are from Refs. \cite{19,20,33,34}, as indicated in the figure.
In our calculation we have only obtained results for tetramers below the ground state trimer, and in Fig. 4 we show our results. The scaling plot is compared to experimental data from [10, 32, 34] and to other recent theoretical results obtained for tetramers [10, 21], also attached to excited trimers. The converged calculation of ref. [21], obtained for tetramers attached to a highly excited trimer is agreement with the plot, while other calculations are systematically below the curve. Such behavior suggests that range corrections, that are certainly more relevant for the tetramers attached to less excited trimers, may be the cause of such systematic deviation. Range corrections will be pursued in future works. The distortion provided by the effective range in the universal properties of three-body systems has already been considered within effective field theory (see e.g. [38]).

4 Tetrimer Structure

The dependence on the four-body scale in the solution of the FY equations (1) and (2), comes with the coupling of the $K$ and $H$ components. For instance, if $H$ is dropped out and only $K$ is considered to solve the FY equations, the dependence of the tetramer binding energy on $\mu_4$ is not relevant. Thus, the dependence on the four-body scale comes from the coupling between the two FY components. The structure of the tetramer, presented in terms of these components, reveals how the dependence on the four-body scale affects both components, and remarkably when the tetramer is strongly bound the $H$ amplitude can be larger than $K$ even at low momentum.

Fig. 5 Definition of the four-body Jacobi momenta corresponding to the $K$– and $H$–type fragmentations.

The Jacobi momenta for $K$–type and $H$–type configurations are depicted in Fig. 5. The reduced FY amplitudes have momentum dependence given as: $\langle u_2, u_3| K_{12,3} \rangle \equiv K(u_2, u_3)$ and $\langle v_2, v_3| H_{12,34} \rangle \equiv H(v_2, v_3)$. The four-body regulator scale, $\mu_4$ is carried by momenta $u_3$ and $v_2$ in the $K$ and $H$ configurations, respectively. Therefore, one should expect a longer tail associated with $\mu_4 >> \mu_3$ in the momentum variable $u_3$ in $K$ and in $v_2$ in $H$, in comparison with the dependence in $u_2$ and $v_3$. This property can be verified by the results presented in Figs. 6 and 7 for the $s$–wave projected FY equations with a scale ratio $\mu_4/\mu_3 = 200$. The results are presented by considering the normalization such that $\int_{0}^{\infty} du_2 u_2^3 \int_{0}^{\infty} du_3 u_3^3\ K^2(u_2, u_3) + \int_{0}^{\infty} dv_2 v_2^3 \int_{0}^{\infty} dv_3 v_3^3\ H^2(v_2, v_3) = 1$. The effect of the four-body short-range scale appears in all three tetramer states that exist for this scale ratio, as one can verify from the results presented in Fig. 6. It is clearly observed that $K(0, u_3)$ and $H(v_2, 0)$ extend over much larger momentum than $K(u_2, 0)$ and $H(0, v_3)$, for all states. In Fig. 6 we choose to plot $K$ and $H$ in each frame by comparing the two different momenta in respect to the dependence on the three- and four-body short-range scales. Then one can appreciate the difference between the momentum tails, showing the dominance of two momentum scales well separated. The ground state results are presented in frames (a) and (d), the first excited state in (b) and (e), and the second excited state in (c) and (f). Noteworthy to observe that the $H$ reduced amplitude can be larger than the $K$ one in the low momentum region, as shown for the case of the ground state. Even for the shallowest and second excited tetrimer state the $H$ and $K$ components are comparable as shown in frame (c). This example points on the importance of the $H$ configuration in bringing the dependence on the short-range four-body scale to the tetramer physics. We remark that if the $H$ amplitude is set to zero in (11) and the uncoupled equation for $K$ is solved, the regularization parameter $\mu_4$ can be dropped off, stressing the role of the $H$ configuration in bringing the dependence of the short-range four-body scale to the tetramer properties.

In particular, the expected longer tail associated with $\mu_4 >> \mu_3$ in the momentum variable $u_3$ in $K$ and in $v_2$ in $H$, in comparison with the dependence in $u_2$ and $v_3$, is further evidenced in Fig. 6 which corresponds to frames (a) and (d) of Fig. 5 related to the ground-state tetramer. As one can
Fig. 6 The reduced Yakubovsky components $K$ and $H$, as functions of the Jacobi momenta for scale ratio $\mu_4/\mu_3 = 200$, when two four-body excited states exist. In the left frames we fix $(u_3, v_3) = 0$; whereas in the right frames, $(u_2, v_2) = 0$. (a) and (d) are for the ground state; (b) and (e), for the first excited state; and (c) and (f), for the second excited state.

Fig. 7 Comparison of the reduced Yakubovsky components $K$ (solid lines) and $H$ (dashed lines) for the tetramer ground-state for $\mu_4/\mu_3 = 200$. From Fig. 6, we should note that the Jacobi momenta directly affected by the four-body short-range scale are $u_3$ for $K$ and $v_2$ for $H$. In the left frame we are comparing the components as functions of $(u_2, v_3)$ when the ones related to the four-body scale are set to zero $(u_3, v_2) = 0$; and, in the right frame, we show the components as functions of $(u_3, v_2)$ when $(u_2, v_3) = 0$, in order to pin-down the relevance of the four-body scale in $K$ and $H$ components.

see from the other frames of Fig. 6 we have similar behaviors of the excited-state FY components. The large momentum tails of $K$ and $H$, manifesting the importance of the four-body scale in these amplitudes, can be seen in the right frame of Fig. 7. In the left frame the FY reduced amplitudes drop much faster, because these momenta are related to the three-body scale, which in this extreme case is 200 times smaller than the four-body one. Under the conjecture that the dependence on the four-body scale is irrelevant, the large momentum tail of the reduced FY amplitudes should keep only the same short-range characteristic scale, i.e., the three-body one. Such conjecture is clearly excluded by the results displayed in the two frames of Fig. 7.
5 Conclusions

Here we report recent results on the study of the universal aspects of four-boson systems [3, 4], as well as present some complementary considerations. Our main remark is that the limiting cycle of the scaling plot correlating the energies of successive states previously found in the case of three-boson systems close to the Efimov limit (leading to the well-known Efimov effect), can also be verified in the general case of four-boson systems. In particular, we mean the correlation of the energies of successive tetramers between two neighbor Efimov trimers. The relevance of such results, obtained within a renormalized zero-range model applied to Faddeev-Yakubovsky equations, is that the four-boson scale can be driven near the Feshbach resonance by induced four-body forces [1].

The scaling plot showing the universal correlation between energies of two successive tetramer states between two Efimov trimers [3] is compared with an updated collection of other potential model calculations. Particularly in the present work we include a qualitative discussion on the trajectory of the tetramer S-matrix poles, coming out from the cuts through the branching points, as the ratio between the three and four-body scales is decreased.

Another scaling function, relevant for actual cold atom experiments, correlates the positions of two successive resonant four-boson recombination peaks [3]. It is consistent with recent data [19] and with new calculations using $s$-wave separable potentials [21]. Systematic deviations of potential model results from these two scaling curves suggest that range corrections are of some importance and should be considered.

We have provided further insights about the dependence on the tetramer properties on the four-body scale by analyzing the structure of the reduced $K$ and $H$ Faddeev-Yakubovsky amplitudes for the contact potential. It is shown that, in the reduced $H$ amplitude, the dependence on the four-body short-range scale is evident in the behavior of the Jacobi momenta at large values corresponding to the relative motion between two-boson subsystems in the associated $2+2$ partition. Numerically, we have confirmed that the dependence on the short-range four-body scale can be dropped out if one considers the solution of the uncoupled equation for the reduced $K$ amplitude, i.e., setting the $H$-component to zero.

Our final remark is that a challenge remains to identify situations in real four-boson systems where the independence between the three- and four-body scales can be evidenced.

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