GENERIC 1-CONNECTIVITY OF FLAG DOMAINS
IN HERMITIAN SYMMETRIC SPACES

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Abstract. A flag domain is an open real group orbit in a complex flag manifold. It has been shown that a flag domain is either pseudoconvex or pseudoconcave. Moreover, generically 1-connected flag domains are pseudoconcave. In this study, for flag domains contained in irreducible Hermitian symmetric spaces of type $A_{III}$ or $C_I$, we determine which pseudoconcave flag domain is generically 1-connected.

1. Introduction
Let $G$ be a connected complex semisimple Lie group, and let $G_0$ be a real form of $G$. An open $G_0$-orbit in a $G$-flag manifold is called a flag domain. For example, a Hermitian symmetric domain is a flag domain. According to [HHS18, HHL19], a flag domain is either pseudoconvex or pseudoconcave. A pseudoconvex flag domain, such as a Hermitian symmetric domain, possesses plenty of global functions. In contrast, any global function on a pseudoconcave flag domain is constant. In this study, we investigate pseudoconcave flag domains, focusing on generic 1-connectivity.

Let $K_0$ be a maximally compact subgroup of $G_0$. By the Matsuki duality [Mat79], $G_0$-orbits correspond to $K$-orbits with complexification $K$ of $K_0$. Through this correspondence, a flag domain $D$ contains a compact submanifold called the base cycle of $D$. The base cycle and its $G$-transformations play an important role in the study of pseudoconcave flag domains. Let us choose a base point $z$ in the base cycle. The isotropy subgroup $Q_z$ at $z$ is a parabolic subgroup, and we have a unique open $Q_z$-orbit in the ambient flag manifold $G(z)$. We say that $D$ is generically 1-connected if the open $Q_z$-orbit intersects with the base cycle. Huckleberry [Huc10] showed that a flag domain is pseudoconcave if it is generically 1-connected. He also showed that $D$ is generically 1-connected if $K$ is a simple Lie group. For example, all $SL(n, \mathbb{R})$-flag domains are generically 1-connected. However, the following problem is still open: are all pseudoconcave flag domains generically 1-connected? In this study, we provide an answer for flag domains contained in irreducible compact Hermitian symmetric spaces of type $A_{III}$ or $C_I$.

All flag domains contained in the Hermitian symmetric space under consideration correspond to the signature, and our results indicate that generic 1-connectivity depends on the numerical condition of the signature. In the case of type $C_I$, where $G_0 = Sp(2n, \mathbb{R})$, few flag domains are generically 1-connected: the Hermitian symmetric space contains $(n + 1)$ flag domains, of which $(n - 1)$ are pseudoconcave, and at most one of them is generically 1-connected. In contrast, in the case of type $A_{III}$, where $G_0 = SU(p, q)$, more than one flag domain can be generically 1-connected.

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almost half of the flag domains under consideration are generically 1-connected if $2p < q$. We prove these by using combinatorics of the Weyl groups and their action on the roots. Moreover, we consider the generic 1-connectivity of a certain type of flag domain fibered over the flag domains in the Hermitian symmetric spaces.

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2. Cycle Connectivity of Flag Domains

In this section, we review pseudoconcavity, cycle connectivity, and generic 1-connectivity. Subsequently, we present combinatorial conditions that are equivalent to generic 1-connectivity.

2.1. Pseudoconcavity.

Let $X$ be a connected complex manifold. Andreotti [And74] defined pseudoconcavity as follows:

**Definition 2.1.** $X$ is pseudoconcave if we can find a relatively compact open subset $Y \subset X$ such that at every point $z \in \text{bd}(Y)$, a holomorphic map $\rho$ on the unit disk $D$ to $\text{cl}(Y)$ satisfying $\rho(0) = z$ and $\text{bd}(\rho(D)) \subset Y$ exists.

This definition is weaker than the definition of $q$-pseudoconcavity in [AG62], where a smooth exhaustion is required for the definition. Similar to the finiteness theorem of [AG62] for higher cohomologies of $q$-pseudoconcave manifolds, we have a weaker version of the finiteness theorem:

**Proposition 2.2 ([And74]).** If $X$ is pseudoconcave, then any global function on $X$ is constant, and $\dim \mathbb{C} H^0(X, \mathcal{F}) < \infty$ for any coherent sheaf $\mathcal{F}$.

To prove this finiteness theorem, the maximum principle works essentially.

**Remark 2.3.** Higher cohomologies of a pseudoconcave flag domain have a significant meaning in several aspects. In Hodge theory, Green et al. [GGK13] studied them with specific Mumford-Tate domains in connection with automorphic cohomology. In representation theory, higher cohomologies give a geometric realization of Zuckerman derived functor modules, see [Kob98] and references therein.

2.2. Cycle connectivity.

Let $G$ be a connected complex Lie group. For a $G$-flag manifold $Z$, we fix a base point $z \in Z$. Then, $Z \cong G/Q_z \cong G(z)$, where $Q_z$ is the parabolic subgroup that stabilizes $z$. Let $\mathfrak{g}_0$ be a real form of the Lie algebra $\mathfrak{g}$ of $G$, and let $\tau$ be the associated complex conjugation. The $\tau$-invariant complex subspace $\mathfrak{q}_z \cap \tau \mathfrak{q}_z$ contains a $\tau$-stable Cartan subalgebra $\mathfrak{h}$, where $\mathfrak{q}_z$ is the Lie algebra of $Q_z$. For a $\mathfrak{h}$-root system $\Sigma$ of $\mathfrak{g}$, we choose a positive root system $\Sigma^+$ such that $\mathfrak{q}_z$ contains the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{f}_\alpha$. Let $\Psi$ be the simple root system corresponding to $\Sigma^+$. For a subset $\Phi \subset \Psi$, we define

$$\Phi_\tau = \{ \sum_{\psi \in \Psi} \epsilon_\psi \psi \in \Sigma \mid \epsilon_\psi = 0 \text{ whenever } \psi \not\in \Phi \}$$

$$\Phi^\pm_n = \{ \alpha \in \pm \Sigma^+ \mid \alpha \not\in \Phi_\tau \}.$$
We may choose $\Phi$ such that $\Sigma(q_\Phi) = \{\alpha \in \Sigma \mid g_0 \subset q_\alpha\} = \Phi_+ \cup \Phi_+^\perp$. Here $\Phi_+$ (resp. $\Phi_+^\perp$) is reductive (resp. nilpotent) part of $\Sigma(q_\Phi)$.

Let $G_0$ be the real form of $G$ corresponding to $g_0$. By [Aom66, Wol69], $G_0$-orbits in $Z$ are finitely many, and there is an open orbit. An open $G_0$-orbit is called a flag domain. Suppose that $D = G_0(z)$ is a flag domain. Let $\theta$ be a Cartan involution that commutes with $\tau$. Then, we may assume that $\mathfrak{h}$ and $\Sigma^+$ satisfy the following conditions (see [Wol69, Theorem 4.5]):

- $\mathfrak{h}_0 = \mathfrak{h} \cap g_0$ is a $\theta$-stable maximally compact Cartan subalgebra of $g_0$;
- $\tau \Sigma^+ = -\Sigma^+$.

For the compact subgroup $K_0 = G_0^0$ and its complexification $K$, $K_0$-orbit $K_0(z)$ coincides with the $K$-orbit, and $C_0 = K_0(z) = K(z)$ is a complex compact manifold (see [FHW06, Theorem 4.3.1]). Here $C_0$ is called the base cycle.

For any point $x, y \in D$, we write $x \sim y$ if there exists $C_i = g_i(C_0) \subset D$ with $g_i \in G$ such that $x \in C_i$ and $y \in C_N$, where the chain $C_1 \cup \cdots \cup C_N$ is connected. The relation $\sim$ is an equivalence relation, and $D/\sim$ is classified into two types:

**Proposition 2.4 ([Hu10]).** $D/\sim$ is either a Hermitian symmetric domain or a point. In the former case, $D$ is pseudoconvex. In the latter case, we say $D$ is cycle connected.

Because a holomorphic function on $D$ is factored as $D \to D/\sim \to C$, the flag domain $D$ is cycle connected if and only if any global function on $D$ is constant. Moreover, by Proposition 2.2, pseudoconcave flag domains are cycle connected. Rather, the following theorem holds.

**Theorem 2.5 ([HHS18, HHL19]).** A flag domain $D$ is cycle connected if, and only if, $D$ is pseudoconcave.

### 2.3. Generic 1-connectivity.

Let $W = W(G, H)$ be the Weyl group with the Cartan subgroup $H = \exp(\mathfrak{h})$, and let $W_\Phi$ be the subgroup generated by the simple reflections associated with $\Phi$. By the Bruhat decomposition, $Q_\Phi$-orbits in $Z$ are parameterized by $W_\Phi \backslash W/W_\Phi$, and there is a unique open $Q_\Phi$-orbit $\mathcal{O}$.

**Definition 2.6.** A flag domain $D$ is generically 1-connected if $C_0 \cap \mathcal{O} \neq \emptyset$.

The preimage of the base point under $D \to D/\sim$ contains an open subset if $D$ is generically 1-connected. Then, by Proposition 2.4, we have the following corollary:

**Corollary 2.7.** Generically 1-connected flag domains are cycle connected (or equivalently pseudoconcave).

The above corollary implies generic 1-connectivity is a kind of cycle connectivity. While cycle connectivity guarantees that any two points are connected by a chain of cycles of finite length, generic 1-connectivity ensures that any point in $\mathcal{O}$ is connected to the base point by a chain of length 1. In fact, if $D$ is generically 1-connected, any point in $\mathcal{O}$ is written as $g(z')$ with $z' \in C_0 \cap \mathcal{O}$ and $g \in Q_\Phi$. Then both $z$ and $g(z')$ are contained in $g(C_0)$.

Now $K$ is a reductive subgroup of $G$. Because $C_0$ is a projective variety, $Q_\Phi^K = K \cap Q_\Phi$ is a parabolic subgroup. Then $C_0$ can be decomposed into the disjoint union of $Q_\Phi^K$-orbits. Let $\mathfrak{h}_0 = \mathfrak{t}_0 + \mathfrak{a}_0$ be the $\theta$-stable decomposition, where $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{t}_0$ and $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$ with the Cartan decomposition $g_0 = \mathfrak{t}_0 + \mathfrak{p}_0$. Because $\mathfrak{h}_0$ is maximally compact, $\mathfrak{t}_0$ is a maximal abelian subalgebra of $\mathfrak{t}_0$. Let $W_K = W(K_0, T_0)$ be the
Weyl group with the maximal torus $T_0 = \exp(t_0)$. Then, each $Q^K_z$-orbit in $C_0$ is the orbit at $w(z)$ with some $w \in W_K$. Let $w^K_0$ be the longest element in $W_K$ with respect to the simple root system corresponding to the Borel subgroup contained in $Q^K_z$. Then the $Q^K_{z}$-orbit at $w^K_0(z)$ is open.

**Proposition 2.8.** The following conditions are equivalent:

1. $D$ is generically 1-connected;
2. $w^K_0(z) \in \mathcal{O}$;
3. $w^K_0(\Phi^-) \cap \Phi^- = \emptyset$.

**Proof.** First, we show the equivalence between (1) and (2). If $D$ is not generically 1-connected, $\mathcal{O} \cap C_0 = \emptyset$, and thus $w(z) \not\in \mathcal{O}$ for all $w \in W_K$. Contrastingly, if $D$ is generically 1-connected, $\mathcal{O} \cap C_0$ is an $Q^K_z$-invariant subset that must contain the open $Q^K_z$-orbit in $C_0$. Hence $w^K_0(z) \in \mathcal{O}$.

Next, we show the equivalence between (2) and (3). We write $z' = w^K_0(z)$. Since $\Sigma = w^K_0(\Sigma(q) \cup \Phi^-) = \Sigma(q) \cup w^K_0(\Phi^-)$, we have

$$
\dim (Q_z(z')) = |\Sigma(q) \cap w^K_0(\Phi^-)| = |w^K_0(\Phi^-)| - |\Phi^- \cap w^K_0(\Phi^-)| = \dim D - |\Phi^- \cap w^K_0(\Phi^-)|.
$$

Then $Q_z(z')$ is open if, and only if, $w^K_0(\Phi^-) \cap \Phi^- = \emptyset$. \hfill \Box

For the longest element $w_0 \in W$, the $Q_z$-orbit at $w_0(z)$ is open. Moreover, any element $w \in W$ such that $w(z)$ contained in $\mathcal{O}$ is written as $w = w_1w_0w_2$ with $w_1, w_2 \in W_\Phi$. If $h_0$ is compact, that is, $h_0 = t_0$, then $W_K$ is a subset of $W$. In this case, $w^K_0$ is written as $w^K_0 = w_1w_0w_2$ if and only if $w^K_0(z) \in \mathcal{O}$. By Proposition 2.8, we have the following corollary:

**Corollary 2.9.** In the case where $h_0$ is compact, $D$ is generically 1-connected if, and only if, there exists $w_1, w_2 \in W_\Phi$ such that $w_1w^K_0w_2$ is the longest element in $W$.

### 3. Flag Domains in Hermitian Symmetric Spaces

In this section, we suppose that $Z$ is an irreducible Hermitian symmetric space of compact type. We then have the dual Hermitian symmetric domain $G_0/K_0$. To state our result, we review the root structure of $\mathfrak{g}$. Let $h_0$ be a maximal abelian subalgebra of $\mathfrak{z}_0$. We can choose a simple $\mathfrak{h}$-root system $\{\psi_1, \ldots, \psi_n\}$ such that only one root is noncompact and compact otherwise. We suppose $\psi_m$ is noncompact. Then, the set $\Sigma$ of roots can be decomposed into $\Sigma = \Sigma_c \cup \Sigma^+_n \cup \Sigma^-_n$, where

$$
\Sigma_c = \{ \sum \epsilon_i \psi_i \mid \epsilon_m = 0 \}, \quad \Sigma^+_n = \{ \sum \epsilon_i \psi_i \mid \epsilon_m = \pm 1 \}.
$$

Let $p^\pm = \sum_{\alpha \in \Sigma^\pm} g_\alpha$ and $P^\pm = \exp(p^\pm)$. Then, we have $Z \cong G/KP^+$, and the Hermitian symmetric domain $G_0/K_0$ is regarded as the $G_0$-orbit at the identity coset $z_0 \in Z$.

In the Hermitian symmetric space $Z$, all $G_0$-orbits are related by the Cayley transforms. Choosing a maximal set $\Xi = \{\xi_1, \ldots, \xi_r\} \subset \Sigma^+_n$ of strongly orthogonal roots with $r = \text{rank}_Z \mathfrak{g}_0$, the partial Cayley transform $c_\xi$ and the product $c_T = \prod_{\xi \in \Gamma} c_\xi$ is constructed from $\xi \in \Gamma \subset \Xi$ (see [Wol69, WZ97] for this construction). For disjoint subsets $\Gamma, \Delta \subset \Xi$, we define $z_{\Gamma, \Delta} = c_T c^2_\Delta(z_0)$. By [Wol69, WZ97], the following properties hold:
generic 1-connectivity of flag domains in Hermitian symmetric spaces

- Every $G_0$-orbit on $Z$ is written as $G_0(z_{\Gamma, \Delta})$ with some $\Gamma, \Delta \subset \Xi$;
- $G_0(z_{\Gamma, \Delta}) = G_0(z_{\Gamma', \Delta'})$ if and only if $|\Gamma| = |\Gamma'|$ and $|\Delta| = |\Delta'|$;
- $G_0(z_{\Gamma, \Delta})$ is open if and only if $\Gamma = \emptyset$.

Then, any flag domain in $Z$ is written as $G_0(z_{\emptyset, \Delta})$, and it depends on the cardinality of $\Delta$.

We choose $\Delta$ as the set $\{\xi_1, \ldots, \xi_a\}$ with $1 \leq a \leq r$. The square $c_{\Delta}^2$ of the partial Cayley transform is $s_\Delta = \prod_{1 \leq i \leq a} s_i$ with the reflection $s_i$ with respect to $\xi_i \in \Delta$. We set $z_a = s_\Delta(z_0)$ as a base point. Then the $G_0$-orbit $D_a = G_0(z_a)$ is a flag domain in $Z$. We denote by $q_a$ the parabolic subalgebra at $z_a$. Here, $q_a = s_\Delta(\mathfrak{t} + \mathfrak{p}_+)$, which contains the Borel subalgebra corresponding to the simple root system $\Psi = \{s_\Delta(\psi_1), \ldots, s_\Delta(\psi_n)\}$. The set $\Sigma(q_a)$ of roots is decomposed as

$$
\Sigma(q_a) = \Phi_\ast \cap \Phi_+^n \text{ with } \Phi = \Psi \setminus \{s_\Delta(\psi_m)\}.
$$

Now $\Phi_\ast$ is decomposed as $\Phi_\ast = (\Phi_\ast \cap \Sigma_c) \cup (\Phi_\ast \cap \Sigma_{nc})$. Using the longest element $w^0_K$, we have $w^0_K(\Phi_\ast \cap \Sigma_c) \subset \Sigma_c$ and $w^0_K(\Phi_\ast \cap \Sigma_{nc}) \cap (\Phi_\ast \cap \Sigma_{nc}) = \emptyset$. Then, to show generic 1-connectivity, Proposition 2.8 is simplified as follows:

**Corollary 3.1.** $D_a$ is generically 1-connected if and only if $w^0_K(\Phi_\ast \cap \Sigma_c) \cap (\Phi_\ast \cap \Sigma_{nc}) = \emptyset$.

The Hermitian symmetric space with a classical group $G$ can be classified into four types (see [Hel01, Chapter VII] for details): AIII, DIII, BDI, and CI. We consider the 1-connectivity of $D_a$ in the cases of type AIII and CI.

**3.1. Case for type CI.** We fix a symplectic form $\omega$ on $\mathbb{R}^{2n}$, and let $G_0 = Sp(2n, \mathbb{R})$ be the subgroup of $SL(2n, \mathbb{R})$, leaving invariant this form. We have a basis $\{f_i\}_i$ that satisfies $\sqrt{-1}\omega(v, w) = \sum_{i \leq n} (v_i w_i - v_{n+i} w_{n+i})$ for $v = \sum v_i f_i$ and $w = \sum w_i f_i$. Using this basis, we may regard $G_0$ as $U(n, n) \cap G$.

The Grassmannian $Z$ of $\omega$-isotropic $n$-planes in $\mathbb{C}^{2n}$ is a $G$-flag manifold, and all open $G_0$-orbits correspond to pairs of numbers of positive and negative signatures of the associated Hermitian form. Let $z_a = \text{Span} \{f_i \mid a < i \leq n + a\} \in Z$

for $0 \leq a \leq n$, where $\text{sgn}(z_a) = (n - a, a)$. Then, any flag domain is written as $D_a = G_0(z_a)$, and both $D_0$ and $D_a$ are Hermitian symmetric domains, that is, each one is the Siegel upper (or lower) half space.

**Theorem 3.2.** An $Sp(2n, \mathbb{R})$-flag domain $D_a$ in the Hermitian symmetric space is generically 1-connected if and only if $2a = n$.

For this proof, we consider the root structure and the Weyl group action. We choose $\mathfrak{k}_0 = \mathfrak{u}(2n) \cap \mathfrak{g}_0 \cong \mathfrak{u}(n)$, and let $\mathfrak{h}_0 \subset \mathfrak{u}(n)$ be the maximal torus consisting of diagonal matrices. We define $e_i \in \mathfrak{h}^+$ by $e_i(X) = a_i$ for $X = \text{diag}(a_1, \ldots, a_n) \in \mathfrak{h}$. Then, we may choose the simple root system $\{\psi_1, \ldots, \psi_n\}$, where $\psi_i = e_i - e_{i+1}$ for $i < n$ and $\psi_n = 2e_n$, and we can write

$$
\Sigma_c = \{\sum \epsilon_i \psi_i \mid \epsilon_n = 0\} = \{\epsilon_i - \epsilon_j \mid i \neq j\},
$$

$$
\Sigma_{nc}^\pm = \{\sum \epsilon_i \psi_i \mid \epsilon_n = \pm 1\} = \{\pm(\epsilon_i + \epsilon_j) \mid 1 \leq i \leq j \leq n\}.
$$

Now $G_0$ is the split real form, i.e. $\text{rank}_\mathbb{R} \mathfrak{g}_0 = \dim \mathfrak{h}_0 = n$. The maximal set of noncompact orthogonal roots is $\{\xi_1, \ldots, \xi_n\}$ with $\xi_i = 2e_i$. Then $s_{\Delta}(e_i) = -e_i$ if $i < a$, and $s_\Delta(e_i) = e_i$ otherwise. Since $-s_\Delta(\psi_i) \in \Sigma_c \cup \Sigma_{nc}^+$, i.e. $-\psi_i \in s_\Delta(\Sigma_c \cup \Sigma_{nc}^+)$,
for \( 1 \leq i \leq n - 1 \), \( \Sigma(q_a) = s_\Delta(\Sigma_c \cup \Sigma_w^+) \) contains the set \( \{-\psi_i\}_{i \neq n} \) of simple roots of \( \Sigma_c \). By using these simple roots, \( W_K \) is the symmetry group \( S_n \) in terms of the permutations of the indices of \( e_1, \ldots, e_n \), and the longest element in \( W_K \) is

\[
(3.1) \quad w^K_0 = \left( \frac{1}{n} \frac{2}{n-1} \cdots \frac{n-1}{1} \right).
\]

**Proof of Theorem 3.2.** We apply Corollary 3.1. Since \( \Phi^-_n = s_\Delta(\Sigma_{nc}) \), we have

\[
\Phi^-_n \cap \Sigma_{nc} = \{ e_i + e_j \mid 1 \leq i \leq j \leq a \} \cup \{ -e_i - e_j \mid a + 1 \leq i \leq j \leq n \}.
\]

Then, \( w^K_0(\Phi^-_n \cap \Sigma_{nc}) \cap (\Phi^-_n \cap \Sigma_{nc}) = \emptyset \) if and only if \( 2a = n \). \( \square \)

Next, we consider the generic 1-connectivity of a certain type of flag domain fibered over \( D_a \) with \( 0 < a < n \). Let \( Z_m \) be the flag manifold consisting of sequences \( V_1 \subset V_2 \) of \( \omega \)-isotropic subspaces with \( 0 < \dim V_1 = m \leq a < \dim V_2 = n \). We set

\[
(3.2) \quad z_{m,0} = (F_m \subset F_n) \in Z_m \quad \text{where} \quad F_r = \text{Span} \{ f_i \}_{i \leq r}.
\]

We continue to assume \( \Delta = \{ \xi_1, \ldots, \xi_a \} \) and set \( z_{m,a} = s_\Delta(z_{m,0}) \) as the base point. Then, \( D_{m,a} = G_0(z_{m,a}) \) is the flag domain in \( Z_m \), determined by

\[
\text{sgn}(V_1) = (0, m), \quad \text{sgn}(V_2) = (n - a, a).
\]

**Proposition 3.3.** The flag domain \( D_{m,a} \) is not generically 1-connected.

**Proof.** We denote by \( q_{m,a} \) the parabolic subalgebra at \( z_{m,a} \). Then, \( q_{m,a} \) contains the Borel subalgebra corresponding to the simple root system \( \Psi = \{ s_\Delta(\psi_1), \ldots, s_\Delta(\psi_n) \} \), where \( \Sigma(q_{m,a}) = \Phi \cup \Phi^+ \) with \( \Phi = \Psi \setminus \{ s_\Delta(\psi_m), s_\Delta(\psi_n) \} \). We have

\[
e_1 + e_n = s_\Delta(-e_1 + e_n) = -\sum_{i=1}^{n-1} s_\Delta(\psi_i) \in \Phi^-_n.
\]

Moreover, \( \Sigma(q_{m,a}) \) contains the set \( \{-\psi_i\}_{i \neq n} \) of simple roots of \( \Sigma_c \), and the longest element \( w^K_0 \) is defined in (3.1). Then, \( w^K_0(e_1 + e_n) = e_1 + e_n \); hence, \( D_{m,a} \) is not generically 1-connected by Proposition 2.8. \( \square \)

### 3.2. Case for type AIII

We fix a Hermitian form \( \langle \cdot, \cdot \rangle \) on \( \mathbb{C}^{p+q} \), and let \( G_0 = SU(p, q) \) be the subgroup of \( SL(p + q, \mathbb{C}) \), leaving invariant this form. We may assume \( p \leq q \). We have a basis \( \{ f_i \} \) that satisfies \( \langle v, w \rangle = \sum_{i \leq p} v_i w_i - \sum_{j > p} v_j w_j \) for \( v = \sum v_i f_i \) and \( w = \sum w_i f_i \).

The Grassmannian \( Z \) of the \( p \)-planes in \( \mathbb{C}^{p+q} \) is a \( G \)-flag manifold, and all open \( G_0 \)-orbits correspond to pairs of numbers of positive and negative signatures. Let

\[
z_a = \text{Span} \{ f_i \mid a < i \leq p, p + q - a < i \leq p + q \}
\]

for \( 0 \leq a \leq p \), where \( \text{sgn}(z_a) = (p-a, a) \). Then, any flag domain is written as \( D_a = G_0(z_a) \), and \( D_0 \) is the Hermitian symmetric domain \( \{ X \in M_{p,q}(\mathbb{C}) \mid I - \bar{X}X > 0 \} \).

**Theorem 3.4.** An \( SU(p, q) \)-flag domain \( D_a \) in the Hermitian symmetric space is generically 1-connected if and only if \( p \leq 2a \leq q \).

As in type CI, we consider the root structure and Weyl group action. We choose \( \mathfrak{t}_0 = su(p, q) \cap \mathfrak{u}(p + q) \), and let \( \mathfrak{h}_0 \subset \mathfrak{t}_0 \) be the maximal torus consisting of diagonal matrices. Then, we may choose the simple root system \( \{ \psi_1, \ldots, \psi_{p+q-1} \} \), where
\( \psi_i = e_i - e_{i+1} \). The simple root \( \psi_i \) is compact if \( i = p \) and is noncompact otherwise. Then we can write

\[
\Sigma_c = \{ \sum e_i \psi_i \mid e_p = 0 \} = \{ e_i - e_j \mid i, j \leq p \text{ or } p < i, j \},
\]

\[
\Sigma_{\pm}^{nc} = \{ \sum e_i \psi_i \mid e_p = \pm 1 \} = \{ \pm (e_i - e_j) \mid i \leq p < j \}.
\]

We set the maximal set \( \{ \xi_1, \ldots, \xi_p \} \) of strongly orthogonal noncompact roots with \( \xi_i = e_i - e_{p+q+1-i} \). The reflection \( s_i \) with respect to \( \xi_i \) is the permutation of the indices of \( e_1, \ldots, e_{p+q} \), which exchanges \( i \) and \( p+q+1-i \). Then \( s_\Delta = \prod_{i=1}^p s_i \) is the permutation

\[
(3.3) \quad s_\Delta = (\underbrace{1 \cdots \cdots a}_{p+q} \cdots \cdots \overbrace{a+1 \cdots \cdots p+q-a}^{p+q-a+1} \cdots \cdots \overbrace{p+q-a}^{p+q-a} \cdots \cdots \overbrace{a+1 \cdots \cdots 1}^{a} \cdots \cdots \overbrace{1}^{p+q+1}).
\]

Therefore \( \Sigma(q_a) = s_\Delta(\Sigma_c \cup \Sigma^{nc}_\pm) \) contains the set \( \{ -\psi_i \}_{i \neq p} \) of simple roots of \( \Sigma_c \).

By using these simple roots, \( W \cong S_{p+q} \) and \( W_K \cong S_p \times S_q \). The longest element in \( W_K \) is the permutation

\[
(3.4) \quad w_0^K = (\underbrace{1 \cdots \cdots p}_{p} \cdots \cdots \overbrace{p+q}^{p+q+1}).
\]

**Proof of Theorem 3.4.** Similar to the proof of Theorem 3.2, we have

\[
(3.5) \quad \Phi_n^- \cap \Sigma_{nc} = \{ e_i - e_j \mid 1 \leq i \leq a, \quad p + q - a + 1 \leq j \leq p + q \}
\]

\[
\cup \{ -e_i + e_j \mid a + 1 \leq i \leq p, \quad p + 1 \leq j \leq p + q - a \}.
\]

Then, it is immediate to verify that \( w_0^K(\Phi_n^- \cap \Sigma_{nc}) \cap (\Phi_n^+ \cap \Sigma_{nc}) = \emptyset \) if and only if \( p \leq 2a \leq q \). Hence Theorem 3.4 follows from Corollary 3.1.

Next, we consider the generic 1-connectivity of a certain type of flag domain fibered over \( D_a \) with \( p \leq 2a \leq q \). We define sequences \( s : s_1 \leq \cdots \leq s_{a+1} \) and \( t : t_1 \geq \cdots \geq t_{a+1} \) with

\[
s_i = \begin{cases} i & \text{if } i \leq a \\ p & \text{if } i = a + 1, \end{cases} \quad t_i = \begin{cases} p + q - i & \text{if } i \leq a \\ p & \text{if } i = a + 1. \end{cases}
\]

Let \( Z_a \) (resp. \( Z_t \)) be a flag manifold consisting of sequences \( V_1 \subset \cdots \subset V_{a+1} \) with \( \dim V_i = s_i \) (resp. \( V_1 \supset \cdots \supset V_{a+1} \) with \( \dim V_i = t_i \)). We let

\[
z_{s,0} = (F_1 \subset \cdots \subset F_a \subset F_p), \quad z_{t,0} = (F_{p+q-1} \supset \cdots \supset F_{p+q-a} \supset F_p)
\]

where \( F_i \) is defined as in (3.2). We continue to assume \( \Delta = \{ \xi_1, \ldots, \xi_a \} \) and set \( z_s = s_\Delta(z_{s,0}) \) and \( z_t = s_\Delta(z_{t,0}) \). Then, \( D_a = G_0(z_s) \) (resp. \( D_t = G_0(z_t) \)) is the flag domain in \( Z_a \) (resp. \( Z_t \)) determined by

\[
\sgn(V_i) = \begin{cases} (0, i) & \text{if } i \leq a \\ (p - a, a) & \text{if } i = a + 1. \end{cases} \quad \text{resp. } \sgn(V_i) = \begin{cases} (p - i, q) & \text{if } i \leq a \\ (p - a, a) & \text{if } i = a + 1. \end{cases}
\]

**Proposition 3.5.** The flag domains \( D_a \) and \( D_t \) are generically 1-connected.

**Proof.** We denote by \( q_a \) the parabolic subalgebra at \( z_s \). Then, \( q_a \) contains the Borel subalgebra corresponding to the simple root system \( \Psi = \{ s_\Delta(\psi_i) \}_{1 \leq i \leq p+q+1} \), and \( \Sigma(q_a) = \Phi_n \cup \Phi_n^+ \) with \( \Phi = \Psi \setminus \{ s_\Delta(\psi_1), \ldots, s_\Delta(\psi_a), s_\Delta(\psi_p) \} \). By the permutation (3.3), we have

\[
\Phi_n^- \cap \Sigma_{nc} = \{ \pm (e_i - e_j) \mid a + 1 \leq i \leq p, \quad p + q - a + 1 \leq j \leq p + q \}.
\]
Moreover, $q_\alpha$ contains the set $\{-\psi_i\}_{i \neq p}$ of simple roots of $\Sigma_c$, and the longest element is the permutation $(3.4)$. Then, we have $w_\Phi^k(\Phi^-_n \cap \Sigma_{nc}) \cap (\Phi^-_n \cap \Sigma_{nc}) = \emptyset$ and $D_\Phi$ is generically 1-connected.

For the proof of $D_t$, the set $\Phi$ is replaced by $\Psi\{s_{\Delta(\psi_p + q - 1)}, \ldots, s_{\Delta(\psi_p + q - a)}\}$.

Then, $\Phi^-_n \cap \Sigma_{nc}$ is written as

$$\text{(the right hand side of (3.5)) } \cup \{e_i - e_j \mid 1 \leq i \leq a, p + 1 \leq j \leq p + q - a\}.$$  

Therefore, $D_t$ is generically 1-connected similarly as in the case for $D_\Phi$.

Let $s'$ (resp. $t'$) be a subsequence of $s$ (resp. $t$), and we define the subsequences $z_{s'}$ (resp. $z_{t'}$) of $z_\Phi$ (resp. $z_t$), as described above. Then, we have the flag domain $D_{s'} = G_0(z_{s'})$ and $D_{t'} = G_0(z_{t'})$, which compose the fibration $D_\Phi \rightarrow D_{s'} \rightarrow D_a \leftarrow D_{t'} \leftarrow D_t$.

Because the parabolic subalgebra at $z_{s'}$ (resp. $z_{t'}$) contains $q_\alpha$ (resp. $q_\Phi$), the above proposition and Proposition 2.8 imply the following corollary:

**Corollary 3.6.** The flag domains $D_{s'}$ and $D_{t'}$ are generically 1-connected.

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