IMPROVED INGHAM-TYPE RESULT ON $\mathbb{R}^d$ AND ON CONNECTED, SIMPLY CONNECTED NILPOTENT LIE GROUPS

MITHUN BHOWMIK

ABSTRACT. In [3] we have characterized the existance of a non zero function vanishing on an open set in terms of the decay of it’s Fourier transform on the $d$-dimensional Euclidean space, the $d$-dimensional torus and on connected, simply connected two step nilpotent Lie groups. In this paper we improved these results on $\mathbb{R}^d$ and prove analogus results on connected, simply connected nilpotent Lie groups.

1. Introduction

It is a well known fact in harmonic analysis that if the Fourier transform of an integrable function on the real line is very rapidly decreasing then the function can not vanish on a nonempty open set unless it vanishes identically. For instance if, let $f \in L^1(\mathbb{R})$ and $a > 0$ be such that its Fourier transform satisfies the estimate

$$|\hat{f}(\xi)| \leq Ce^{-a|\xi|}, \quad \text{for all } \xi \in \mathbb{R}.$$

If $f$ vanishes on a nonempty open set then $f$ is identically zero. This initial observation motivates one to endeavour for a more optimal decay of the Fourier transform $\hat{f}$ for such a conclusion to hold. For instance we may ask: if $\hat{f}$ decays faster than $1/(1+|\cdot|)^n$ for all $n \in \mathbb{N}$ but slower than the function $e^{-a|\cdot|}$, can $f$ vanish on a nonempty open set without being identically zero? A more precise question could be: is there a nonzero integrable function $f$ on $\mathbb{R}$ vanishing on a nonempty open set such that its Fourier transform satisfies the following estimate

$$|\hat{f}(\xi)| \leq Ce^{-\frac{\theta|\xi|}{\log|\xi|}}, \quad \text{for large } |\xi|? \quad (1.1)$$

The answer to the above question is in the negative and follows from a classical result due to Ingham [7]. Analogous results were also obtained by Paley-Wiener, Levinson and Hirschman [11, 10, 9, 6]. In [3], we have obtained analogues results on $d$-dimensional Euclidean spaces, $d$-dimensional torus and on connected, simply connected two step nilpotent Lie groups. In the context of Euclidean space we have the following theorem [3]. For $f \in L^1(\mathbb{R}^d)$, we shall define its Fourier transform $\hat{f}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx, \quad \text{for } \xi \in \mathbb{R}^d.$$

For $1 < p \leq 2$, the definition of Fourier transform of $f \in L^p(\mathbb{R}^d)$ is extended in the usual way.

2010 Mathematics Subject Classification. Primary 22E25; Secondary 22E30, 43A80.

Key words and phrases. nilpotent Lie group.

This work was supported by Indian Statistical Institute, India (Research fellowship).
Theorem 1.1. Let $f \in L^p(\mathbb{R}^d)$, for some $p$ with $p \in [1, 2]$ and $\psi : [0, \infty) \to [0, \infty)$ be a locally integrable function. Suppose that the Fourier transform $\hat{f}$ of $f$ satisfies the following estimate
\begin{equation}
|\hat{f}(\xi)| \leq CP(\xi)e^{-\psi(\|\xi\|)},
\end{equation}
for almost every $\xi \in \mathbb{R}^d$, where $P(\xi)$ is a polynomial and we define
\begin{equation}
I = \int_{t}^{\infty} \psi(t) dt.
\end{equation}

(a) Let $\psi(t) = \theta(t)t$, for $t \in [0, \infty)$, where $\theta : [0, \infty) \to [0, \infty)$ is a decreasing function which decreases to zero as $t$ goes to infinity. If $f$ vanishes on any nonempty open set in $\mathbb{R}^d$ then $I$ is finite and it is infinite if $f$ is zero on $\mathbb{R}^d$. Conversely, if $I$ is finite, then given any positive real number $l$ there exists a nonzero even function $f \in C_c(\mathbb{R}^d)$ supported in ball $B(0, l)$ of radius $l$, centered at zero satisfying (1.2).

(b) Let $\psi$ be an increasing function on $[0, \infty)$. If $f$ vanishes on any nonempty open set in $\mathbb{R}^d$ then $I$ is infinite and it is finite if $f$ is zero on $\mathbb{R}^d$. Conversely, if $I$ is finite, given any positive real number $l$ there exists a nonzero $f \in C_c(\mathbb{R}^d)$ supported in $B(0, l)$ satisfying (1.2).

(c) If
\begin{equation}
\text{supp } f \subseteq \{ x \in \mathbb{R}^d \mid x \cdot \eta \leq s \},
\end{equation}
and $I$ is infinite then $f$ is zero on $\mathbb{R}^d$. Conversely, if $I$ is finite then there exists a nonzero $f \in L^2(\mathbb{R}^d)$ vanishing on the set $\{ x \in \mathbb{R}^d : x \cdot \eta \geq x_0 \}$, for some $x_0 \in \mathbb{R}$ and $\eta \in S^{d-1}$ satisfying (1.2).

In [3] we have shown that the original statements of the results of Ingham, Levinson and Paley-Wiener on $\mathbb{R}$ are follows form the theorem above. We shall show in this paper that it is possible to prove analogues of Theorem 1.1 under the following variant of (1.2)
\begin{equation}
\int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^q e^{q\psi(\|\xi\|)}}{(1 + \|\xi\|)^N} d\xi < \infty,
\end{equation}
where $q \in [1, \infty]$ (with obvious modification for $q = \infty$) and $N$ is a positive real number. Our next main result in this paper is an analogue of Theorem 1.1 (c) on the connected, simply connected nilpotent Lie group $G$.

We shall use the following standard notation in this paper: supp $f$ denotes the support of the function $f$ and $C$ denotes a constant whose value may vary. For a finite set $A$ we shall use the symbol $\#A$ to denote the number of elements in $A$. $B(0, r)$ denotes the open ball of radius $r$ centered at 0 in $\mathbb{R}^d$ and $\overline{B}(0, r)$ denotes its closure. For $x, y \in \mathbb{R}^d$, we shall use $\|x\|$ to denote the norm of the vector $x$ and $x \cdot y$ to denote the Euclidean inner product of the vectors $x$ and $y$. A function $\psi : \mathbb{R}^d \to [0, \infty)$ is said to be radially increasing (or radially decreasing) if $\psi$ is radial and satisfies the condition $\psi(x) \geq \psi(y)$ (or $\psi(x) \leq \psi(y)$) whenever $|x| \geq |y|$. We shall often consider a radial function on $\mathbb{R}^d$ as an even function on $\mathbb{R}$ or equivalently, as a function on $[0, \infty)$.

2. Results of Ingham, Levinson and Paley-Wiener on $\mathbb{R}^d$

In this section our aim is to improve Theorem 1.1 in the following way. Instead of the pointwise decay (1.2) on the Fourier transform side we will consider the weighted $L^q$ estimate for $1 \leq q \leq \infty$ of the form (1.3). Using convolution techniques we will be able to show that it is enough to prove the result for $q = 1$ and $N = 0$ and rest of the proof goes like the proof of Theorem 1.1. First we state and prove an analogue of Theorem 1.1 (a) on $\mathbb{R}^d$. 
Theorem 2.1. Let $\theta : \mathbb{R}^d \to [0, \infty)$ be a radially decreasing measurable function with $\lim_{\|\xi\| \to \infty} \theta(\xi) = 0$ and

$$I = \int_{\|\xi\| \geq 1} \frac{\theta(\xi)}{\|\xi\|^q} d\xi.$$  

(a) Let $f \in L^p(\mathbb{R}^d)$, $p \in [1, 2]$, be such that

$$\int_{\mathbb{R}^d} |f(\xi)|^q e^{q\theta(\xi)} \|\xi\|^q \frac{d\xi}{1 + \|\xi\|^N} < \infty,$$

for some $q \in [1, \infty)$ and some $N \geq 0$ or

$$|f(\xi)| \leq C |P(\xi)| e^{-\theta(\xi)} \|\xi\|^q,$$

where $P$ is a polynomial. If $f$ vanishes on a nonempty open subset in $\mathbb{R}^d$ and $I$ is infinite, then $f$ is zero almost everywhere on $\mathbb{R}^d$.

(b) If $I$ is finite then there exists a nontrivial $f \in C^\infty_c(\mathbb{R}^d)$ satisfying the estimate (2.2) (or (2.3) if $q = \infty$).

Proof. We shall first prove the assertion (b). For $q = \infty$ it follows from Theorem 1.1 (a). For $q \in [1, \infty)$ we choose $\phi_1 \in C^\infty_c(\mathbb{R}^d)$ with supp $\phi_1 \subseteq B(0, l/2)$ and consider the function $f = f_0 * \phi_1$. Clearly, support of the function $f$ is contained in $B(0, l)$ and

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^q e^{q\theta(\xi)} \|\xi\|^q \frac{d\xi}{1 + \|\xi\|^N} \leq C \int_{\mathbb{R}^d} |\hat{\phi_1}(\xi)|^q d\xi < \infty.$$

This, in particular, proves (b). It now remains to prove (a).

For part (a), we first show that it suffices to prove the case $q = 1, N = 0$. Suppose $f \in L^p(\mathbb{R}^d)$ vanishes on an open set $U \subseteq \mathbb{R}^d$ and satisfies (2.2), for some $q > 1$ and $N \in \mathbb{N}$. Since the condition (2.2) is invariant under translation of $f$ we can assume that $f$ vanishes on $B(0, l)$, for some positive $l$. If we choose $\phi \in C^\infty_c(\mathbb{R}^d)$ supported in $B(0, l/2)$ then $f * \phi$ vanishes on the ball $B(0, l/2)$. In fact, if $\|x\| < l/2$ then

$$\int_{\mathbb{R}^d} f(x - y) \phi(y) dy = \int_{B(0, l/2)} f(x - y) \phi(y) dy = 0,$$

as $\|x - y\| \leq \|x\| + \|y\| < l$.

By using Hölder’s inequality we get

$$\int_{\mathbb{R}^d} |(f * \phi)(\xi)| e^{\theta(\xi)} \|\xi\|^q d\xi$$

$$\leq \left( \int_{\mathbb{R}^d} |\hat{f}(\xi)|^q e^{q\theta(\xi)} \|\xi\|^q \frac{d\xi}{1 + \|\xi\|^N} \right)^{\frac{1}{q'}} \|\phi(\xi)\|_{L^{q'}(\mathbb{R}^d)} < \infty,$$

as $N/q$ is smaller than $N$. Here $q'$ satisfies the relation

$$\frac{1}{q} + \frac{1}{q'} = 1.$$
Hence, by the case \( q = 1, N = 0 \) it follows that \( f \ast \phi \) vanishes identically. As \( \phi \in C_c^\infty (\mathbb{R}^d) \) we have that \( \hat{\phi} \) is nonzero almost everywhere. This implies that \( \hat{f} \) vanishes almost everywhere and so does \( f \). The same technique can be applied to reduce the case \( q = 1 \) and \( N \in \mathbb{N} \) to the case \( q = 1 \) and \( N = 0 \) by using Hölder’s inequality. For the case \( q = \infty \), we get from (2.3) that

\[
\int_{\mathbb{R}^d} |\hat{f} \ast \phi (\xi)| e^{\|\xi\| \|\xi\|} d\xi \leq \int_{\mathbb{R}^d} |P(\xi)| \|\hat{\phi} (\xi)\| d\xi < \infty.
\]

So, without loss of generality, we assume that \( f \in L^p (\mathbb{R}^d) \) such that \( \hat{f} \) satisfies the condition

\[
\int_{\mathbb{R}^d} |\hat{f} (\xi)| e^{\|\xi\| \|\xi\|} d\xi = C' < \infty.
\]

Rest of the proof goes similarly as the proof of Theorem 2.2 in [3].

As in Theorem 2.1 we will have the following version of Theorem 1.1, (b) on \( \mathbb{R}^d \).

**Theorem 2.2.** Let \( \psi : \mathbb{R}^d \to [0, \infty) \) be a radially increasing function and

\[
I = \int_{\|\xi\| \geq 1} \frac{\psi(\xi)}{\|\xi\|^{d+1}} \, d\xi.
\]

(a) Let \( f \in L^p (\mathbb{R}^d) \), \( p \in [1, 2] \), be a function satisfying the estimate

\[
\int_{\mathbb{R}^d} |\hat{f} (\xi)|^q e^{q \psi(\xi)} \frac{e^{q \psi(\xi)}}{(1 + \|\xi\|)^N} \, d\xi < \infty,
\]

for some \( q \in [1, \infty) \) and some \( N \geq 0 \) or

\[
|\hat{f} (\xi)| \leq C \|P(\xi)\| e^{-\psi(\xi)}, \quad \text{for almost every } \xi \in \mathbb{R}^d,
\]

where \( P \) is a polynomial. If \( f \) vanishes on a nonempty open subset in \( \mathbb{R}^d \) and \( I \) is infinite, then \( f \) is zero almost everywhere on \( \mathbb{R}^d \).

(b) If \( I \) is finite then there exists a nontrivial \( f \in C_c^\infty (\mathbb{R}^d) \) satisfying the estimate (2.5) (or (2.6) if \( q = \infty \)).

In the following, we shall improve Theorem 2.1 (c) which will be used in the next section to prove an analogous result for connected simply connected nilpotent Lie groups (see Theorem 3.1).

**Theorem 2.3.** Let \( \psi : [0, \infty) \to [0, \infty) \) be a locally integrable function and

\[
I = \int_1^\infty \frac{\psi(t)}{t^2} \, dt.
\]

(a) Let \( f \in L^p (\mathbb{R}^d) \), \( p \in [1, 2] \), be such that

\[
\text{supp } f \subseteq \{ x \in \mathbb{R}^d \mid x \cdot \eta \leq s \},
\]

for some \( \eta \in S^{d-1} \) and \( s \in \mathbb{R} \) and \( \hat{f} \) satisfies the estimate

\[
\int_{\mathbb{R}^d} |\hat{f} (\xi)|^q e^{q \psi(|\xi|)} \frac{e^{q \psi(|\xi|)}}{(1 + \|\xi\|)^N} \, d\xi < \infty.
\]

for some \( q \in [1, \infty) \) and some \( N \geq 0 \), or

\[
|\hat{f} (\xi)| \leq C (1 + \|\xi\|)^N e^{-\psi(\xi \cdot \eta)}, \quad \text{for almost every } \xi \in \mathbb{R}^d.
\]

If the integral \( I \) is infinite then \( f \) is the zero function.
(b) If $\psi$ is non-decreasing and $I$ is finite then there exists a nontrivial $f \in C_c(\mathbb{R}^d)$ satisfying the estimate (2.7), for some $q \in [1, \infty)$ and all $\eta \in S^{d-1}$ or (2.8), for $q = \infty$ and all $\eta \in S^{d-1}$.

Proof. We shall first prove a) for the case $p = 2$. Then for all $p \in [1, 2)$ the result will follow by reducing it to the case $p = 2$ as was done in the proof of Theorem 1.1, (c). As in the proof of Theorem 2.1 it suffices to prove the case $q = 1$ and $N = 0$. Now, by using translation and rotation of the function $f$, we can assume without loss of generality that $\eta = e_1 = (1, 0, \cdots, 0)$ and $s = 0$. Then, by writing $\xi = (\xi_1, \xi_2, \cdots, \xi_d)$ the hypothesis (2.7) becomes

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)| e^{\psi(|\xi_1|)} \, d\xi < \infty. \tag{2.9}$$

For $y \in \mathbb{R}^{d-1}$, we now define $g_y$ by

$$g_y(x) = F_{d-1}f(x, y), \quad \text{for almost every } x \in \mathbb{R}. \tag{2.10}$$

It then follows that for almost every $y \in \mathbb{R}^{d-1}$, $g_y \in L^2(\mathbb{R})$ with

$$\text{supp } g_y \subseteq \{ x \in \mathbb{R} \mid x \leq 0 \},$$

and by (2.9)

$$\int_{\mathbb{R}} |\hat{g}_y(t)| e^{\psi(|t|)} \, dt < \infty. \tag{2.10}$$

As $y$ varies over a set of full $(d-1)$ dimensional Lebesgue measure, we just need to prove that $g_y$ is the zero function. By the Paley-Wiener theorem (Theorem 2.7, [3]) it suffices to show that

$$\int_{\mathbb{R}} \frac{|\log(|\hat{g}_y(t)|)|}{1 + t^2} \, dt = \infty. \tag{2.11}$$

If

$$\int_{\mathbb{R}} \frac{|\log(|\hat{g}_y(t)||e^{\psi(|t|)}|)}{1 + t^2} \, dt < \infty \tag{2.11}$$

then

$$\int_{\mathbb{R}} \frac{|\log(|\hat{g}_y(t)||e^{\psi(|t|)}|)}{1 + t^2} \, dt \geq \int_{\mathbb{R}} \frac{|\log(|\hat{g}_y(t)||e^{\psi(|t|)}|)}{1 + t^2} \, dt - \int_{\mathbb{R}} \frac{|\log(|\hat{g}_y(t)||e^{\psi(|t|)}|)}{1 + t^2} \, dt.\tag{2.11}$$

As $I$ is infinite, it follows from (2.11) that

$$\int_{\mathbb{R}} \frac{|\log(|\hat{g}_y(t)||e^{\psi(|t|)}|)}{1 + t^2} \, dt$$

is divergent. Hence, by the Paley-Wiener theorem (Theorem 2.7, [3]) it follows that $g_y$ is the zero function.

Now, suppose

$$\int_{\mathbb{R}} \frac{|\log(|\hat{g}_y(t)||e^{\psi(|t|)}|)}{1 + t^2} \, dt = \infty. \tag{2.12}$$
For a measurable function $F$ on $\mathbb{R}^d$, we define
\[
\log^+ |F(x)| = \max\{\log |F(x)|, 0\}
\]
\[
\log^- |F(x)| = -\min\{\log |F(x)|, 0\},
\]
and hence
\[
|\log |F(x)|| = \log^+ |F(x)| + \log^- |F(x)|.
\]
As $\log^+ |F(x)|$ is always smaller than $|F(x)|$ we get that
\[
\int_{\mathbb{R}} \frac{\log^+ (|\hat{g}_y(t)|e^{\psi(|t|)})}{1 + t^2} dt \leq \int_{\mathbb{R}} \frac{|\hat{g}_y(t)|e^{\psi(|t|)}}{1 + t^2} dt < \infty,
\]
by (2.10). From (2.12) we now conclude that
\[
\int_{\mathbb{R}} \frac{\log^- (|\hat{g}_y(t)|e^{\psi(|t|)})}{1 + t^2} dt = \infty.
\]
But
\[
\int_{\mathbb{R}} \frac{\log^- (|\hat{g}_y(t)|e^{\psi(|t|)})}{1 + t^2} dt = \int_{\{t \mid |\hat{g}_y(t)|e^{\psi(|t|)} \leq 1\}} \frac{\log^- (|\hat{g}_y(t)|e^{\psi(|t|)})}{1 + t^2} dt
\]
\[
\leq \int_{\mathbb{R}} \frac{\log^- |\hat{g}_y(t)|}{1 + t^2} dt,
\]
as on the set $\{t \mid |\hat{g}_y(t)|e^{\psi(|t|)} \leq 1\}$ we have
\[
|\hat{g}_y(t)| \leq |\hat{g}_y(t)|e^{\psi(|t|)} \leq 1,
\]
and hence
\[
\log^- (|\hat{g}_y(t)|) \geq \log^- (|\hat{g}_y(t)|e^{\psi(|t|)}) .
\]
Therefore the integral
\[
\int_{\mathbb{R}} \frac{\log^- (|\hat{g}_y(t)|)}{1 + t^2} dt
\]
is divergent. Hence, the integral
\[
\int_{\mathbb{R}} \frac{|\log (|\hat{g}_y(t)|)|}{1 + t^2} dt
\]
is divergent. By the Paly-Wiener theorem (Theorem 2.7, [3]) it now follows that $g_y$ is the zero function. This completes the proof of part a). To prove Part b) we observe that since $\psi$ is non decreasing
\[
\psi(||\xi||) \leq \psi(||\eta||), \quad \text{for } \xi \in \mathbb{R}^d.
\]
Therefore the proof follows from Theorem 2.2 (b). \qed

Remark 2.4. (1) It is clear from the statement of Theorem 2.3 that if $f$ is compactly supported continuous function then $\eta \in S^{d-1}$ can be taken to be arbitrary.

(2) If $\psi$ is assumed to be a radial increasing function then Theorem 1.1 (c) follows from the case $p = \infty$ of Theorem 2.3 and also from Theorem 2.4.

3. NILPOTENT LIE GROUPS

In this section, our aim is to prove an analogue of Theorem 2.3 in the context of connected, simply connected nilpotent Lie groups.
3.1. Preliminaries on Nilpotent Lie Groups. In this section, we shall discuss the preliminaries and notation related to connected, simply connected nilpotent Lie groups. These are standard and can be found, for instance, in [1, 2, 5].

Let $G$ be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector space of real-valued linear functionals on $\mathfrak{g}$. In this case, the exponential map $\exp : \mathfrak{g} \to G$ becomes an analytic diffeomorphism, which enables us to identify $G$ with $\mathfrak{g} \cong \mathbb{R}^d$, where $d = \dim \mathfrak{g}$. Let

\[(3.1) \quad \{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_d = \mathfrak{g}\]

be a Jordan-Hölder series of ideals of the nilpotent Lie algebra $\mathfrak{g}$ such that $\dim \mathfrak{g}_j = j$, for $j = 0, \cdots, d$ and $\text{ad}(X)\mathfrak{g}_j \subseteq \mathfrak{g}_{j-1}$, for $j = 1, \cdots, d$ and for all $X \in \mathfrak{g}$ ([5], Theorem 1.1.9). We choose from this sequence a Jordan-Hölder basis

\[(3.2) \quad \{X_1, \cdots, X_d\}, \quad \text{where} \quad X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}, \]

for $j = 1, \cdots, d$. In particular, $\mathbb{R}X_1$ is contained in the center of $\mathfrak{g}$. Let $\{X_1^*, \cdots, X_d^*\}$ be the basis of $\mathfrak{g}^*$ dual to $\{X_1, \cdots, X_d\}$. We define coordinates on $G$ by

\[(3.3) \quad \|x\| = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}, \quad x = \exp(x_1X_1 + \cdots + x_dX_d) \in G, \quad x_j \in \mathbb{R}.

The composite map

\[\mathbb{R}^d \longrightarrow \mathfrak{g} \longrightarrow G, \quad (x_1, \cdots, x_d) \mapsto \sum_{j=1}^{d} x_jX_j \mapsto \exp(\sum_{j=1}^{d} x_jX_j), \]

is a diffeomorphism and maps the Lebesgue measure on $\mathbb{R}^d$ to a Haar measure on $G$. We shall also identify $\mathfrak{g}^*$ with $\mathbb{R}^d$ via the map

\[\nu = (\nu_1, \cdots, \nu_d) \rightarrow \sum_{j=1}^{d} \nu_jX_j^*, \quad \nu_j \in \mathbb{R}, \]

where $\{X_1^*, \cdots, X_d^*\}$ is the dual basis of $\mathfrak{g}^*$. On $\mathfrak{g}^*$ we define the Euclidean norm relative to this basis by

\[\|\sum_{j=1}^{d} \nu_jX_j^*\| = (\nu_1^2 + \cdots + \nu_d^2)^{\frac{1}{2}} = \|\nu\|.

For $\nu \in \mathfrak{g}^*$ we have a natural bilinear map

\[(X, Y) \mapsto B_\nu(X, Y) = \nu([X, Y]). \]

This map is antisymmetric and the radical of this map is

\[\{X \in \mathfrak{g} \mid B_\nu(X, Y) = 0, \quad \text{for all} \quad Y \in \mathfrak{g}\} = r_\nu.\]
Clearly, $B_{
u}$ descends to nondegenerate, antisymmetric bilinear map $\tilde{B}_{\nu} : g/\nu \times g/\nu \to \mathbb{R}$. Hence, $r_{\nu}$ is of even codimension in $g$ (5, Lemma 1.3.2). Since the coadjoint orbit $O_{\nu}$ of $\nu$ is diffeomorphic to $G/R_{\nu}$, it follows that $O_{\nu}$ is even dimensional. An index $j \in \{1, 2, \ldots, d\}$ is called a jump index for $\nu \in g^*$ if

$$r_{\nu} + g_{j} \neq r_{\nu} + g_{j-1},$$

where the Jordan-Hölder sequence $g_{j}$ are as in (3.2) (see [1]). Let $e(\nu)$ be the set of jump indices for $\nu \in g^*$. This set contains $\dim(O_{\nu})$ indices, which is an even number. Moreover, there are disjoint sets of indices $P$ and $Q$ such that $P \cup Q = \{1, \ldots, d\}$ and a $G$-invariant nonempty Zariski open set $U \subseteq g^*$ such that $e(\nu) = P$, for all $\nu \in U$ (5, Theorem 3.1.2; [1]). The elements of $U$ are called generic linear functionals. For $\nu \in U$, we define

$$B_{\nu,P} = \nu([X_{i}, X_{j}])_{i,j \in P}.$$

Then the Pfaffian $Pf(\nu)$ is given by

$$(3.4) \quad |Pf(\nu)|^2 = \det B_{\nu,P}, \quad \text{for all } \nu \in U.$$

We define

$$V_{P} = \text{span } \{X_{i}^* : i \in P\}, \quad \text{and } V_{Q} = \text{span}\{X_{i}^* : i \in Q\}.$$

Let $d\nu$ be the Lebesgue measure on $V_{Q}$ normalized so that the unit cube spanned by $\{X_{i}^* : i \in Q\}$ has volume 1. Then $g^* = V_{Q} \oplus V_{P}$ and $W = U \cap V_{Q}$ is a cross-section for the coadjoint orbits through points in $U$.

For $\phi \in L^{1}(G)$ we define its operator-valued Fourier transform by

$$\pi(\phi) = \int_{G} \phi(g)\pi(g^{-1}) \,dg, \quad \text{for } \pi \in \hat{G},$$

where $dg$ denotes the Haar measure of $G$ and $\hat{G}$ is the unitary dual of $G$. If $\phi \in L^{1}(G) \cap L^{2}(G)$ then $\pi(\phi)$ is a Hilbert-Schmidt operator and $||\pi(\phi)||_{HS}$ will denote the Hilbert-Schmidt norm of $\pi(\phi)$. If $d\nu$ denotes the element of Lebesgue measure on $W$, then $\mu$ is the Plancherel measure for $\hat{G}$, where $d\mu = |Pf(\nu)|d\nu$.

The Plancherel formula is thus given by

$$||\phi||^{2}_{2} = \int_{G} |\phi(g)|^{2} \,dg = \int_{W} \|\pi_{\nu}(\phi)\|^{2}_{HS} \,d\mu(\nu), \quad \text{for all } \phi \in L^{1}(G) \cap L^{2}(G).$$

We shall now state and prove the following analogue of Theorem 1.1 (c) on a connected, simply connected nilpotent Lie group $G$.

**Theorem 3.1.** Let $\psi : [0, \infty) \to [0, \infty)$ be a locally integrable function and

$$I = \int_{1}^{\infty} \frac{\psi(t)}{t^{2}} \,dt.$$

Suppose $f \in C_{c}(G)$ is such that

$$(3.5) \quad \int_{W} \|\pi_{\nu}(f)\|^{2}_{HS} \,e^{2\psi(|\nu|)} \,|Pf(\nu)| \,d\nu < \infty,$$

where $\nu_{1} = \nu(X_{1})$ and $X_{1}$ is given by (3.2). If $I$ is infinite then $f$ vanishes identically on $G$. Conversely, if $\psi$ is nondecreasing and $I$ is finite, then there exists a nontrivial $f \in C_{c}(G)$ such that (3.3) holds.
To prove the theorem we need some more notation. If \( r \) is the dimension of coadjoint orbits \( O_\nu, \nu \in U \), then \( Q \) has \( d - r \) elements and so \( V_Q \) can be identified with \( \mathbb{R}^{d-r} \). In an abuse of notation we write \( V_Q = \mathbb{R}X_1^* \oplus \mathbb{R}^{d-r-1} \) and let

\[
p^*: V_Q \rightarrow \mathbb{R}X_1^*, \quad \nu \mapsto \nu_1 X_1^*
\]
denote the canonical projection. As \( W \) is a Zariski open set in \( V_Q \), \( p^*(W) = \mathcal{O} \) is also a nonempty Zariski open subset of \( \mathbb{R} \). In particular, this is a subset in \( \mathbb{R} \) with full Lebesgue measure. So, it will be convenient to write elements \( \nu \in W \) as \((\nu_1, \nu')\), where \( \nu_1 \in \mathcal{O} \) and \( \nu' \) is in the set \( \mathcal{W}_{\nu_1} \), defined by

\[
\mathcal{W}_{\nu_1} = \{ \nu' \in \mathbb{R}^{n-d-1} : (\nu_1, \nu') \in W \}.
\]

It turns out that \( \mathcal{W}_{\nu_1} \) is also a Zariski open subset of \( \mathbb{R}^{n-d-1} \), for each fixed \( \nu_1 \in \mathcal{O} \). Given \( f \in C_c(G) \) and the Jordan-Hölder basis \( \{X_1, \cdots, X_d\} \) given in \( \text{(3.2)} \) we define for each \( y \in \mathbb{R}^{d-1} \)

\[
fy(t) = f \left( \exp \left( tX_1 + \sum_{j=2}^{d} y_j X_j \right) \right), \quad \text{for } t \in \mathbb{R},
\]

and \( f_y^*(t) = \overline{fy(-t)} \). We now consider the function

\[
g(t) = \int_{\mathbb{R}^{d-1}} (fy \ast f_y^*)(t) dy, \quad \text{for } t \in \mathbb{R}.
\]

Since \( f \in C_c(G) \) it follows that \( g \in C_c(\mathbb{R}) \). In particular, \( g \in L^1(\mathbb{R}) \). We now have the following lemma.

**Lemma 3.2.** \([2], \text{Lemma 3.2} \) For all \( \nu_1 \in \mathcal{O} \),

\[
\hat{g}(\nu_1) = \int_{\mathcal{W}_{\nu_1}} \|\pi_\nu(f)\|_{HS}^2 |Pf(\nu)| \, d\nu',
\]

where \( \hat{g} \) is the one dimensional Fourier transform of \( g \).

We now present the proof of the theorem.

**Proof of Theorem 3.1.** It follows from the definition of \( g \) that

\[
\hat{g}(\xi) = \int_{\mathbb{R}^{d-1}} |\overline{\hat{f}_y}(\xi)|^2 dy, \quad \xi \in \mathbb{R},
\]

and consequently \( \hat{g} \geq 0 \) with

\[
\int_{\mathbb{R}} \hat{g}(\xi) = \int_{\mathbb{R}^d} |\hat{f}_y(\xi)|^2 dy \, d\xi = \|f\|_2^2.
\]

Therefore the function \( f \) is identically zero on \( G \) if and only if \( g \) is zero on \( \mathbb{R} \). We now complete the proof by showing that \( g \) vanishes identically. In order to do this we shall apply Theorem 2.3 to the function \( g \).

By Lemma 3.2 it follows that

\[
\int_{\mathbb{R}} \hat{g}(\nu_1) e^{\psi(|\nu_1|)} \, d\nu_1 = \int_{\mathcal{O}} \int_{\mathcal{W}_{\nu_1}} \|\pi_\nu(f)\|_{HS}^2 e^{\psi(|\nu_1|)} |Pf(\nu)| \, d\nu' \, d\nu_1
\]

\[
= \int_{\mathcal{W}} \|\pi_\nu(f)\|_{HS}^2 e^{\psi(|\nu_1|)} |Pf(\nu)| \, d\nu.
\]

By the hypothesis \( \text{(3.5)} \) the right-hand side integral is finite. Therefore by Theorem 2.3 we conclude that \( g \) is identically zero on \( \mathbb{R} \) and hence \( f \) vanishes identically.
Conversely, if $\psi$ is non decreasing and the integral $I$ is finite then by Theorem 1.1 (b), for a given $\delta$ positive there exists $g \in C_c(\mathbb{R})$ with $\text{supp } g \subset [-\delta, \delta]$ satisfying the estimate

$$|\widehat{g}(y)| \leq C e^{-\psi(|y|)}, \quad y \in \mathbb{R}. \quad (3.6)$$

Let $Z = \exp(\mathbb{R}X_1)$ denote a central subgroup of $G$ and we identify $Z$ with $\mathbb{R}$. Let $h \in C_c(G)$ with $\text{supp } h \subset V$ where $V$ is a symmetric neighbourhood of the identity element in $G$. We now define a function $f$ on $G$ by

$$f(x) = \int_Z g(t) \ h(t^{-1}x) \ dt, \quad x \in G.$$ 

Clearly, $f$ is continuous. Moreover, $f \in C_c(G)$, which follows from the fact that $\|t^{-1}x\| \geq \|x\| - \|t\|$ as $t \in Z$. It can be shown (31, P. 490) that

$$\|\pi_{\nu}(f)\|^2_{HS} = |\widehat{g}(\nu_1)|^2 \|\pi_{\nu}(h)\|^2_{HS}, \quad \text{for all } \nu \in W. \quad (3.7)$$

Therefore, from (3.6) and (3.7) it follows that

$$\int_W \|\pi_{\nu}(f)\|^2_{HS} e^{2\psi(\|\nu\|)} \ |Pf(\nu)| \ d\nu 
\quad = \int_W |\widehat{g}(\nu_1)|^2 \|\pi_{\nu}(h)\|^2_{HS} e^{2\psi(\|\nu_1\|)} |Pf(\nu)| \ d\nu 
\quad \leq C \int_W \|\pi_{\nu}(h)\|^2_{HS} |Pf(\nu)| \ d\nu 
\quad = C \|h\|_{L^2(G)} < \infty.$$

This completes the proof. \hfill \Box

If the function $\psi$ is assumed to be nondecreasing then we have the following corollary.

**Corollary 3.3.** Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function and

$$I = \int_1^\infty \frac{\psi(t)}{t^2} \ dt.$$ 

Suppose $f \in C_c(G)$ is such that

$$\int_W \|\pi_{\nu}(f)\|^2_{HS} e^{2\psi(\|\nu\|)} |Pf(\nu)| \ d\nu < \infty.$$ 

If the integral $I$ is infinite then $f$ vanishes identically on $G$.

**Proof.** Since $\psi$ is nondecreasing $\psi(\|\nu\|) \geq \psi(\|\nu_1\|)$. Therefore, the result follows from Theorem 3.1. \hfill \Box

**Acknowledgement.** We would like to thank Swagato K. Ray and Suparna Sen for the many useful discussions during the course of this work.
REFERENCES

[1] Baklouti, A; Ben Salah, Nour. On theorems of Beurling and Cowling-Price for certain nilpotent Lie groups, Bull. Sci. Math. 132 (2008), no. 6, 529-550. MR2445579 (2009f:22007)

[2] Baklouti, A; Thangavelu, S. Variants of Miyachi’s theorem for nilpotent Lie groups, J. Aust. Math. Soc. 88 (2010), no. 1, 1-17. MR2770922 (2012a:22012)

[3] Bhowmik, M.; Ray, S. K.; Sen, S. Around theorems of Ingham-type regarding decay of Fourier transform on $\mathbb{R}^n$, $\mathbb{T}^n$ and two step nilpotent Lie Groups, to appear in Bull. Sci. Math.

[4] Bochner, S. Quasi-analytic functions, Laplace operator, positive kernels Ann. of Math. (2) 51 (1950), 68-91. MR0032708 (11,334g)

[5] Corwin, L. J.; Greenleaf, F. P. Representations of nilpotent Lie groups and their applications Cambridge University Press, Cambridge, 1990. MR1070979 (92b:22007)

[6] Hirschman, I. I. On the behaviour of Fourier transforms at infinity and on quasi-analytic classes of functions Amer. J. Math. 72 (1950), 200-213. MR0032816 (11,350f)

[7] Ingham, A. E. A Note on Fourier Transforms J. London Math. Soc. 9 (1934) no. 1, 29-32. MR1574706

[8] Kaniuth, E.; Kumar, A. Hardy’s theorem for simply connected nilpotent Lie groups Math. Proc. Cambridge Philos. Soc. 131 (2001), no. 3, 487-494. MR1866390 (2002g:22007)

[9] Levinson, N. Gap and Density Theorems American Mathematical Society Colloquium Publications, v. 26. American Mathematical Society, New York, 1940. MR0003208 (2,180d)

[10] Paley, R. E. A. C.; Wiener, N. Fourier transforms in the complex domain (Reprint of the 1934 original) American Mathematical Society Colloquium Publications, 19. American Mathematical Society, Providence, RI, 1987. MR1451142 (98a:01023)

[11] Paley, R. E. A. C.; Wiener, N. Notes on the theory and application of Fourier transforms. I, II. Trans. Amer. Math. Soc. 35 (1933), no. 2, 348-355. MR1501688

Stat-Math Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata - 700108, India.
E-mail address: mithunbhowmik123@gmail.com