PHASE METHODS FOR MEASURING THE SPATIAL ORIENTATION OF OBJECTS USING SATELLITE NAVIGATION EQUIPMENT

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Abstract. This paper discusses phase methods for measuring the spatial orientation of objects using satellite navigation equipment. An analysis of methods for resolving phase ambiguity is presented in the paper. The effectiveness and applicability of the single-step methods are discussed in detail. The authors have shown that applying the exhaustive method requires a constellation of 5 – 6 visible satellites. The measurement of signals, having a base length of 1 m transmitted from 8 satellites, demonstrates that an unambiguous solution is achieved practically in all cases.

1. Introduction. Professional development of navigational equipment in terms of increased accuracy is associated with the application of phase measurement methods in navigation signals [1]. Their application in survey-grade equipment enables determining the relative coordinates of an object within an accuracy of a few millimeters. Another application of phase methods is the determination of the spatial orientation of objects within GLONASS / GPS signal range.

The angular position of an object in space using satellite signals can be determined by measuring the difference in the path of signals between antennas located at the ends of vector bases. In order to improve the accuracy of spatial orientation, interferometers having a distance of several meters between the antennas (base length) are used. The main problem of phase measurements is phase ambiguity; this is a result of the short wavelength of the measured signals – about 19 cm. This signal is much less than the length of the baselines of the interferometer.

Currently, the majority of angle measuring and geodetic equipment for resolving ambiguities applies the LAMBDA method. This method requires at least two measurements considering that the phase ambiguity of the measurements does not vary. Thus, single-step methods for resolving ambiguity are of particular interest. Single-step methods based on the

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maximum likelihood use the redundancy of the system of equations; they can be obtained through the excess constellation of navigation satellites.

In a single-step method for resolving ambiguities an exhaustive search method is used in the interferometer. The solution is selected through the maximum likelihood. The constellation for implementing the exhaustive search method must contain at least 5 – 6 observable navigation satellites. When measuring signals from 8 navigation satellites with a base length of 1 m, almost all cases have a unique solution.

Modern satellite radio-navigation systems can determine the actual coordinates of an object within an accuracy of 3 – 5 m and a velocity vector within an accuracy of 0.1 m/s. Time-frequency binding equipment on the basis of the GLONASS / GPS radio-navigation systems is widely applied, enabling the synchronization of the onboard time scale with the Universal Time Coordinated (UTC) scale within an accuracy of 100 ns.

Further development of radio-navigation equipment is directly related to the application of phase methods for measuring parameters of navigation signals. The application of phase methods in geodetic equipment enables to determine the relative coordinates of an object within centimeter and millimeter accuracy.

Another application of phase methods consists in determining the spatial orientation of an object using GLONASS/GPS signals. Unlike magnetic sensors and inertial systems for measuring the spatial orientation of an object, satellite goniometric equipment determines the angles of course, pitch, and roll with respect to the true meridian – in this case the measured parameters are free of drift. Additionally, satellite goniometric equipment requires less initialization time and in comparison to inertial sensors is more cost efficient.

SRNS signals can determine the angular position of an object in space by measuring the difference of the navigation satellite signal course between the antennas situated at the endpoints of base-vectors. To determine spatial orientation only two non-collinear base-vectors (three antennas) are required.

The relationship between the phase shift of a navigation satellite signal, received by two spaced antennas and the cosine of the angle between the base-vector and the vector directed towards the navigation satellite (Fig. 1) is

$$\cos \alpha = \frac{\lambda \cdot \varphi}{2\pi B}$$

where $\lambda$ is the wave length of the satellite signal, $\varphi$ is the phase shift, $B$ is the base length, $\alpha$ is the angle between the base-vector and the vector directed towards the satellite. This expression is a single-base interferometer equation, which is widely applied in the theory of phase position-finders and antenna arrays.

Figure 1. Single-base interferometer.
The coordinates of a base-vector can be determined from an equation on the basis of the scalar product of vectors:

\[ k_x x + k_y y + k_z z = \Delta R = \lambda \frac{\Phi}{2\pi} \]  

where \( k_x, k_y, k_z \) are the direction cosines of the vector directed towards a navigation satellite, \( x, y, z \) are base-vector coordinates, \( \Delta R \) is path difference, \( \Phi \) is phase shift of signals, \( \lambda \) is wave length.

At least three equations are required to determine all unknowns. Considering that the coordinates of the base-vector are interrelated and having an acquainted base length \( B \) the system of equations is

\[
\begin{align*}
\begin{cases}
    k_{x,j} x + k_{y,j} y + k_{z,j} z = \Phi_j, \\
    x^2 + y^2 + z^2 = B^2,
\end{cases}
\end{align*}
\]

where \( j = 1, \ldots, N \) is the visual satellite ordinal number.

Phase ambiguity is a major issue in phase measurements. Interferometers with a distance between antennas (the base length) exceeding several meters are used to improve the accuracy of determining spatial orientation. The ambiguity of phase shift measurement is caused by the fact that the wave length of the measured signals is rather small – approximately 19 cm. This is significantly smaller than the length of interferometer bases.

Methods for resolving phase ambiguity can be subdivided into two categories – single-step methods (apply the results of each measurement [2, 3]) and methods based on filtration (require the measurements of phase shifts during some time interval [4, 5])

\[ k_x x + k_y y + k_z z = \Phi_j + n_j \lambda_j \]  

where \( n_j \) is the integer phase ambiguity.

Currently, most goniometric and geodetic equipment utilizes the LAMBDA method for resolving phase ambiguity [6, 7]. At the first stage, according to this method, the integer ambiguity \( n \) is assumed to be an additional unknown without considering its integer value. Then each measurement gives one unknown \( n_j \). In result of each separate measurement for \( N \) navigational satellites, the system of equations will include \( N + 3 \) unknowns, which requires more equations. Therefore the LAMBDA method requires at least two measurements, considering that phase ambiguity does not vary when measured. However, because the position of the satellite changes slowly, having a motionless object for each new measurement in (2) intensely correlates them with the previous ones. The system of equations degenerates, despite its redundancy. Decorrelation is applied to solve the ill-condition system of equations – this enables to assess part of the unknowns \( n_j \) with a certain degree of
accuracy. On the second stage, the obtained values of phase ambiguity are reduced to the integer form (generally by rounding); further, the initial system of equations is solved.

2. Single-step method for resolving phase ambiguity. Single-step methods for resolving phase ambiguity [8-10] are of specific interest. Based on the maximum likelihood, these methods utilize the redundancy from the system of equations, which can be obtained by using redundant navigation satellite constellations.

The single-step method utilizes the exhaustive method for phase ambiguity resolution in a single-base interferometer. A solution is selected, using the maximum likelihood criterion. When receiving signals of \(N\) navigation satellites, the likelihood function (LF) is

\[
W(\Phi_1, \Phi_2, \ldots, \Phi_N | x, y, z) = \prod_{j=1}^{N} \left[ \frac{1}{\sigma_j \sqrt{2\pi}} \right] \cdot \exp \left[ -\sum_{j=1}^{N} \left[ \Phi_j + n_j \lambda_j - \left( k_{x,j} x + k_{y,j} y + k_{z,j} z \right) \right] \right] \]

(5)

with the additional condition

\[x^2 + y^2 + z^2 = B^2\]

(6)

Function (5) possesses local minima for every combination of ambiguities \(n_j\). In order to minimize the likelihood function with respect to all possible values of \(n_j\) an exhaustive search is conducted. The main disadvantage of this method is the large number of combinations of ambiguities \(n_j\). When receiving signals from \(N\) satellites, the number of combinations of ambiguities is \(n_{max}^N = \text{int}(2B/\lambda + 1)\) where \(\text{int}(s)\) is the integer part of a real \(s\). For example, having a base length \(B = 1\) m the ambiguity \(n_j\) for each navigation satellite may have 11 values (from 5 to 5). The total number of combinations of ambiguities is \(11^3 = 1331\) for three navigation satellite measurements; \(11^4 = 14641\) for four satellites; and \(\approx 2 \cdot 10^8\) for eight satellites. A local minimum of (5) corresponds to each combination of \(n_j\). For a large number of \(n_j\) combinations, the values of local minima approach a global minimum. This increases the risk of an error. To reduce the number of calculations for resolving phase ambiguity, the base length can be decreased. However, this results in a poor accuracy of angular measurements.

The number of calculations may be considerably reduced provided that an initial constellation with a minimal number of satellites – a nonredundant constellation – is selected. Analyzing all possible combinations of phase ambiguity and solving the problem of phase ambiguity for these values, an initial set of solutions is obtained. Further each solution from the initial set is verified within the entire constellation. Additionally, false solutions are rejected by the maximum likelihood criterion or, equally, by an acceptable total disparity for a solution of the least squares minimum (LSM).
The likelihood function can be used to analyze the potential of the exhaustive methods. The angular position of a base-vector for a known length can be defined by two parameters – the course angle $K$ and the pitch angle $\Psi$. Hence, the likelihood function is two-dimensional. The course and pitch angles are related with the rectangular coordinates through the following expressions

$$X = B \cdot \cos K \cdot \cos \Psi ; \quad Y = B \cdot \sin K \cdot \cos \Psi ; \quad Z = B \cdot \sin \Psi . \tag{7}$$

When resolving phase ambiguity, the probability of gross errors – cases where phase ambiguity is determined with errors – is of special interest. Gross errors arise when the likelihood function has subordinate maxima which are comparable in magnitude with the main maximum, which corresponds to a correct solution. In Fig. 2 the likelihood function is given for one satellite. It isn’t possible to resolve phase ambiguity with a separate measurement for one base of each navigation satellite because the likelihood function has extreums in whole areas, making errors indistinguishable from correct solutions.

Increasing the number of visible satellites transforms the total disparity to a sum of wave-like functions, obtained through measurements for each satellite and being a result of their interference. In Fig. 3 we can see the likelihood function for four visible satellites. The main and subordinate maxima can be clearly distinguished.

**Figure 2.** LF with measurements for one navigation satellite.
Since the likelihood function is complicated for analyzing there is demand for a parameter that enables estimating the probability of not making a correct solution and the probability of gross errors, i.e., the probability of making a false solution. The index of the likelihood function can be taken for this parameter, being the total residual of the solution of the LSM, which is equal to the sum of squares of residuals for all satellites or the square root of this value.

The residuals have two components: the result of residuals in subordinate maxima after a false solution and the result of dispersion in the measured phase shifts. A subordinate maximum of the LF provides a false solution. This makes the system of equations incompatible even for zero values of phase shift measurement errors in case of redundancy. In the absence of measurement noise, the magnitude of a residual in a subordinate maximum of the LF depends on the shape of the satellite and on the values of phase ambiguity. Hence, this value can be accepted as the mathematical expectation of residuals. The noise error of the phase shift measurements has a normal distribution. Thus, when solving (3), residuals for each satellite in the main and subordinate maxima of the LF are distributed by the normal law with mathematical expectation equal to residuals arising in the absence of measurement noise.

We shall consider the probability distribution function for the total residual. Provided that the mathematical expectations of the quantities $x_j$ are zero and their dispersions are equal to one another, the quantity $z = x_1^2 + x_2^2 + ... + x_n^2$ is distributed by the $\chi^2$ law with $n$ degrees of shift [10]. This takes place in the main maximum of the LF for the phase shift measurements with the same accuracy. Mathematical expectation in a subordinate maximum is not equal to zero and the $\chi^2$ distribution law in the classical form cannot be applied.

The distribution function for the total residual can be obtained as follows. First, the probability density for the square of one random variable is obtained. Then, using addition rules for sums of random variables, the necessary probability density is obtained. Characteristic functions are used to calculate the distribution function for the square of the total residual [12].

The characteristic function of a squared random variable with nonzero mathematical expectation is
\[
\Theta(v) = \frac{1}{\sqrt{1 - 2i\sigma^2 v}} \cdot \exp\left(\frac{iv}{1 - 2i\sigma^2 v}\right).
\]

The characteristic function of the sum of squares of independent normal random variables with nonzero averages is equal to the product of characteristic functions

\[
\Theta_\mathbf{v}(v) = \left(1 - 2i\sigma^2 v\right)^{\frac{1}{2}} \cdot \exp\left(\frac{iv}{1 - 2i\sigma^2 v} \sum_k m_k^2\right).
\]

Equation (9) implies a property of the characteristic function: it depends not on mathematical expectations of initial random variables but on the sum of their squares \(m^2 = \sum_k m_k^2\). The distribution function for the sum of the squares of residuals must have the same property.

The probability density can be obtained by the inverse Fourier transform of the characteristic function (9)

\[
P_s(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta_s(v) \cdot e^{-ivx} dv
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(1 - 2i\sigma^2 v\right)^{-\frac{1}{2}} \cdot \exp\left(\frac{iv}{1 - 2i\sigma^2 v} \sum_k m_k^2\right) \cdot e^{-ivx} dv
\]

The graphs of probability density for different values of \(m\) for five observed satellites can be observed in Fig. 4. The graph proves that a false solutions for \(m > 5\sigma\) can be rejected.

![Figure 4. Probability density for the total residual for different values of \(m\).](image)

The probability density (10) cannot be expressed by elementary or tabulated functions making it complicated to calculate the probability of a given quantity in a specific area. An integral distribution function can be applied for this purpose

\[
w_s(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta(v) \cdot e^{-ivx} dv = \frac{1}{2\pi} \int_{1}^{x} \Theta(v) \cdot e^{-iv} dv
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ivx}}{1 - 2i\sigma^2 v} \cdot \exp\left(\frac{im^2}{1 - 2i\sigma^2 v}\right) dv.
\]
In order to determine the probability of a false solution, given the probability of accepting a correct solution, it is necessary to determine the threshold value for which a correct solution is placed on the list of possible solutions with a given probability. The threshold value can be selected using the integral distribution function (11) with \( m = 0 \).

Fig. 5 illustrates the dependence of the probability of making a false solution on the ratio of the total mathematical expectation of residuals to their mean-square deviation (MSD) for different numbers of observed satellites. The probability of a false solution is characterized by a minimal mathematical expectation of the total residual in subordinate maxima. The efficiency of rejecting a false solution is achieved for \( m > (5 \cdots 6)\sigma \).

![Figure 5](image)

**Figure 5.** Dependence of probability of a false solution on the ratio of the total mathematical expectation of residuals to their MSD.

An exhaustive search of possible solutions results in a set of residuals produced by incorrect phase ambiguity resolution. The components are deterministic variables and represent mathematical expectation \( m \) of residuals. They can be calculated a priori for each combination of phase ambiguities. The minimum value of these quantities is of special interest since in this case probability of a false solution is maximal and with increasing \( m \) probability of a false solution decreases rapidly. However, the calculation of residuals for each specific case presents considerable difficulties, mainly because of a large number of combinations of phase ambiguity which take place in exhaustive search of all alternatives.

In the analysis mathematical expectations of residuals in subordinate maxima (for zero errors of the phase shift measurements) can be considered as random variables. According to the obtained data, the distribution of mathematical expectations of residuals does not depend on the shape of the constellation of navigation satellites, a space position, and the length of the base-vector. However, the minimal value of the total residual decreases with increasing the length of base. This dependence is explained by a quadratic increase of the number of possible positions of a base-vector with increasing its length. The square root of the sum of squares of residuals (the total residual) is reasonably described by the normal distribution and the mean-square deviation does not depend on the number of satellites in a constellation and equals 28 mm. The considered number of satellites in a constellation varies from 4 to 13 for different positions and lengths of base-vectors. The case with the measurements for 4 satellites is an exception. Distributions for \( n = 4 \) and \( n = 9 \) are shown in Fig. 6. The normal distribution for a large number of satellites can be explained by a consequence of the central limit theorem. If the number of satellites exceeds 5, the average value of the total residual depends linearly on the number of satellites in a constellation.
The probability of making a false solution is mainly defined by the minimal residual in subordinate maxima. Using (11) for the integral distribution function we can determine the probability of an incorrect solution. In Figs. 7 and 8 the probabilities of gross error depending on the error of the phase shift measurements are illustrated for the minimal total residual with a base length of 1 m and 10 m.

**Figure 6.** Distribution of the total residual for four (a) and nine (b) observed satellites.

**Figure 7.** Probability of gross error for a base length of (a) 1 m and (b) 10 m.
3. Conclusion. The following conclusions can be made from the research:

1. The efficiency of the exhaustion method for the phase ambiguity resolution depends on the number of the observed satellites and the base length. For the length base of 1 m the exhaustion method can work successfully for 5 observed satellites; the noise error of the phase shift measurements equal to 5°. However, in a base length of 10 m for the same error of the phase shift measurements – 7 to 8 observed satellites are required.

2. The exhaustion method for the phase ambiguity resolution can be applied for an interferometer base length of up to 3 m and the limit MSD error of the phase shift measurements from 15 to 20°.

3. The constellation for implementing the exhaustion method should consist of at least 5 – 6 observed satellites. In order to measure the signals of 8 satellites and for a base length of 1 m we apply a unique solution for most cases.

It should be noted that the single-step exhaustion method for one base is applied to form an initial set of solutions. Hence, the probability of rejecting of a correct solution (which is defined by the threshold value of the likelihood function) is an important aspect. The presence of errors in the initial set does not mean there is a gross error, provided that a correct solution is involved as well. Further rejections of false solutions can be carried out by the filtering solutions of an initial set or by using of a multi-base antenna system.

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