HORIZONTAL LOOPS IN ENGEL SPACE

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Abstract. A simple proof is given of the following result first observed by J. Adachi: embedded circles tangent to the standard Engel structure on \(\mathbb{R}^4\) are classified, up to isotopy via such embeddings, by their rotation number.

1. INTRODUCTION

The standard Engel structure \(\mathcal{D}\) on \(\mathbb{R}^4\) is the maximally non-integrable 2-plane distribution defined, in terms of Cartesian coordinates \((x, y, z, w)\), by the equations

\[
\frac{dz - y}{dx} = 0 \quad \text{and} \quad \frac{dw - z}{dx} = 0.
\]

In other words, it is the tangent 2-plane field spanned by the vector fields

\[
e_1 := \partial_x + y\partial_z + z\partial_w \quad \text{and} \quad e_2 := \partial_y.
\]

A horizontal loop is an embedding \(\gamma : S^1 \to \mathbb{R}^4\) everywhere tangent to \(\mathcal{D}\). If we write \(\gamma(s) = (x(s), y(s), z(s), w(s))\), \(s \in S^1\), the condition for \(\gamma\) to be horizontal becomes

\[
z'(s) - y(s)x'(s) = 0 \quad \text{and} \quad w'(s) - z(s)x'(s) = 0 \quad \text{for all} \quad s \in S^1.
\]

A horizontal isotopy is an isotopy via horizontal loops. The rotation number \(\text{rot}(\gamma)\) of a horizontal loop \(\gamma\) is the number of complete turns of the velocity vector \(\gamma'(t) \in \mathcal{D}_{\gamma(t)}\), as we once traverse the loop in positive direction, relative to the trivialisation of \(\mathcal{D}\) given by \(e_1, e_2\). The rotation number is clearly invariant under horizontal isotopies, and it is easy to show by examples (see Section 3) that every integer can be realised as the rotation number of a horizontal loop.

The following theorem was proved by J. Adachi in [1]:

Theorem. Two horizontal loops are horizontally isotopic if and only if their rotation numbers agree.

Adachi proves this theorem by studying the image of \(\gamma\) under the projection \((x, y, z, w) \mapsto (x, w)\). (Beware that I have interchanged \(y\) and \(w\) in the definition of \(\mathcal{D}\) compared with Adachi’s notation. This is more in line with the usual conventions as regards the contact geometric aspects of our discussion.) He determines the ‘Reidemeister moves’ in this projection and then reduces the proof to the corresponding classification of topologically trivial Legendrian knots in standard contact 3-space, due to Ya. Eliashberg and M. Fraser [2].

The purpose of the present note is to show that a much shorter proof can be given by using the projection \((x, y, z, w) \mapsto (x, z)\) instead.
2. Horizontal Loops, Legendrian Immersions, and Fronts

Let \( \gamma(s) = (x(s), y(s), z(s), w(s)), s \in S^1 \), be a horizontal loop. Notice that
\[ x'(s) = 0 \implies z'(s) = w'(s) = 0 \]
and hence — \( \gamma \) being an embedding — \( y'(s) \neq 0 \).
This means that
\[ \bar{\gamma}(s) := (x(s), y(s), z(s)), \ s \in S^1, \]
defines a Legendrian immersion into \( \mathbb{R}^3 \) with its standard contact structure
\[ \xi := \ker(dz - y \, dx). \]

The rotation number \( \text{rot}(\gamma) \) — in the contact geometric sense — of such a
Legendrian immersion is defined as the number of complete turns made by \( \gamma'(s) \in \xi_{\bar{\gamma}(s)} \) as the loop \( \bar{\gamma} \) is traversed once in positive direction, relative to the trivialisation
of \( \xi \) given by the vector fields
\[ e_1 := \partial_{x} + y \partial_{z} \quad \text{and} \quad e_2 := \partial_{y}. \]

So it is obvious that \( \text{rot}(\gamma) = \text{rot}(\bar{\gamma}) \).

The rotation number \( \text{rot}(\bar{\gamma}) \) is invariant under Legendrian regular homotopies,
i.e. \( C^1 \)-homotopies via Legendrian immersions. Moreover, one can prove by elementary methods that the map \( \bar{\gamma} \mapsto \text{rot}(\bar{\gamma}) \) defines a one-to-one correspondence
between Legendrian regular homotopy classes of Legendrian immersions \( \bar{\gamma}: S^1 \to (\mathbb{R}^3, \xi) \) on the one hand, and the integers on the other; see [3, Thm. 6.3.10] or [4].

The front projection \( \bar{\gamma}_F \) of the Legendrian immersion \( \bar{\gamma} \) is the curve
\[ \bar{\gamma}_F(s) := (x(s), z(s)), \ s \in S^1. \]

Generically, this is an immersed curve with semi-cubical cusps, but without vertical
tangencies [3, Section 3.2]. We call a planar curve of this type a front.

The coordinate \( y(s) \) can be recovered as the slope of the front projection:
\[ y(s) = \frac{z'(s)}{x'(s)} = \frac{dz}{dx}(s). \]
The coordinate \( w(s) \) can be recovered as an ‘area integral’:
\[ w(s) - w(s_0) = \int_{s_0}^{s} z(\sigma)x'(\sigma) \, d\sigma = \int z \, dx. \]

This equation defines the horizontal lift of a Legendrian immersion \( \bar{\gamma} \) even when
its front projection \( \bar{\gamma}_F \) is singular, e.g. during the first Legendrian Reidemeister move [4, Figure 7].

Thus, the condition for an arbitrary front \( \bar{\gamma}_F \) to lift to a horizontal immersion
\( \gamma: S^1 \to (\mathbb{R}^4, \mathcal{D}) \) is that
\[ \int_{\bar{\gamma}_F} z \, dx = 0. \]

In order for the lifted curve to be an embedding, we need to require in addition that
for any two distinct points \( s_0, s_1 \in S^1 \) where the front has a self-tangency, i.e.
\[ x(s_0) = x(s_1), \quad z(s_0) = z(s_1), \quad \text{and} \quad \frac{dz}{dx}(s_0) = \frac{dz}{dx}(s_1) \]
— where, in other words, the lifted Legendrian immersion \( \bar{\gamma} \) has a self-intersection —,
we have
\[ \int_{\bar{\gamma}_F(s_0)}^{\bar{\gamma}_F(s_1)} z \, dx \neq 0. \]
3. Proof of the Theorem

Figure 1 shows a front $\gamma_F$ in the $xz$-plane whose lift $\gamma: S^1 \to (\mathbb{R}^3, \xi)$ is a Legendrian embedding with $\text{rot}(\gamma) = \pm 3$, depending on the choice of orientation, for the rotation number can be computed from the front projection via the formula

$$\text{rot}(\gamma) = \frac{1}{2}(c_+ - c_-),$$

with $c_{\pm}$ denoting the number of cusps oriented upwards or downwards, respectively [3, Prop. 3.5.19].

![Figure 1](image1)

Figure 1. Front of a horizontal loop with $\text{rot} = \pm 3$.

Furthermore, the area condition discussed in the preceding section being satisfied, this lifts to a horizontal loop $\gamma: S^1 \to (\mathbb{R}^4, \mathcal{D})$ with $\text{rot}(\gamma) = \pm 3$. For other non-zero rotation numbers, the picture is entirely analogous; a front corresponding to a horizontal loop with rotation number zero is depicted in Figure 2.

![Figure 2](image2)

Figure 2. Front of a horizontal loop with $\text{rot} = 0$.

Now consider two horizontal loops $\gamma_0, \gamma_1: S^1 \to (\mathbb{R}^4, \mathcal{D})$ with $\text{rot}(\gamma_0) = \text{rot}(\gamma_1)$. The corresponding Legendrian immersions $\overline{\gamma}_0, \overline{\gamma}_1: S^1 \to (\mathbb{R}^3, \xi)$ satisfy $\text{rot}(\overline{\gamma}_0) = \text{rot}(\overline{\gamma}_1)$, so there is a Legendrian regular homotopy between them. As the proof in [3, p. 312] or [4, Prop. 4] shows, this can be realised by a homotopy of fronts $\overline{\gamma}_{t,F}$, $t \in [0, 1]$, where at finitely many times $t_i \in [0, 1]$ the front $\overline{\gamma}_{t_i,F}$ has either a single self-tangency or a singular point (during a first Legendrian Reidemeister move). The fronts $\overline{\gamma}_{0,F}$ and $\overline{\gamma}_{1,F}$ — being the projections of horizontal loops — satisfy the area conditions, and one can easily adjust the homotopy of fronts such that all $\overline{\gamma}_{t,F}$, $t \in [0, 1]$, satisfy these conditions, possibly at the cost of creating further tangencies. Then the lifted curves $\gamma_t: S^1 \to (\mathbb{R}^4, \mathcal{D})$ give the desired horizontal isotopy.
References

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