A Study of Legendre Polynomials Approximation for Solving Initial Value Problems

Oday I. Al-Shaher 1 and Mohammed S. Mechee 2

1 General Management of Education in Najaf, Najaf, Iraq.
2 Information Technology Research and Development Center (ITRDC) - University of Kufa, Najaf, Iraq.
Corresponding E-mail: mohammeds.abed@uokufa.edu.iq

Abstract. Legendre polynomials (LPs) basis on the interval I = [-1,1] have been introduced and their properties are studied. Legendre polynomials with Gaussian integration method (GIM) are used to solve the initial value problems (IVPs) of third-order ordinary differential equations (ODEs). The aim of this work is to study the numerical solution of IVP which is widely applicable in the fields of engineering and science using Legendre base. The implementations have been discussed to demonstrate the validity and applicability of the technique of collections and variational methods which compared with the numerical results that obtained by classical RK and modern direct RKD numerical methods. Finally, the numerical results show the high accuracy and efficiency of the proposed method.

1. Introduction

The differential equations (DEs) are the most significant tool in mathematical model for physical phenomena. DE is a key for the analysis of a wide range of real-world phenomena in the chemistry, physics, engineering, biology and economics. Nuclear and chemical reactions are written by set of DEs. A lot of mathematical models using DEs in real life are written in terms of functions and their derivatives. Furthermore, DEs may be classified as either IVPs or boundary-value problems (BVPs). During long time, the mathematicians have been challenged by the methods of approximations of the solutions of DEs. However, in the literature a lot of numerical and theoretical methods dealing with the solutions of such DEs of have developed. There are many modern and classical numerical or analytical integrators for solving some types of ODEs with their conditions. For this reason, the classical numerical or analytical methods have been developed for finding approximated or analytical solutions of ODEs with their conditions. The classical numerical methods for solving DEs are not efficient to solve all kinds of DEs or they can solve sevral of them not directly. This aim make the researchers to construct and derive more efficient direct or indirect numerical integratorss for solving DEs. For this propose, direct numerical methods: RKD, RKT, RKFD & RKM; have been derived and constructed by [1-15] for solving ODEs up to ninth-order. For this purpose, they generalized RK methods and implemented them in solving ODEs of orders up to ninth-order ODEs (RKD, RKT, RKFD & RKM). Legendre polynomials (LPs), or (Legendre functions (LFs)) have significate in applied mathematics. Legendre coefficients are the solutions to the
Legendre differential equations (LDEs). LPs which named by \( P_n(x) \) are illustrated in Figure 1 for \( x \) on interval \( I \) and \( n = 1, 2, ..., 5 \). LPs have been implemented in the Wolfram Language as Legendre \( P[n;x] \).

The orthogonal basis on the interval \( I \) generated by LPs on \( I \). The properties of orthogonal LPs with GIM are used to solve the IVPs of third-order ODEs. For the review of LPs, [16] has introduced an orthogonal basis on the square \( I \times I = [-1;1] \times [-1;1] \) which generated by LPs on interval \( I \) and they described some properties and derive a uniform convergence theorem, [17] studied the numerical solutions of nonlinear fractional DEs (FrDEs) where the Caputo fractional derivatives are used in constructing of efficient algorithms of variational iteration method (VIM) and homotopy analysis method (HAM) which can be used to study the approximation solutions of nonlinear FrDEs using LPs, [18] has studied the Legendre wavelets (LWs) for solving the IVPs of Bratu-type which the efficient and applicability of the modified method have been proved using some implementations while the approximated solutions have been compared with the analytical solutions, [19] has introduced the matrices of Legendre collocation method to solve the linear Fredholm integro-DEs of high-order with mixed conditions in terms of LPs and [18] also, has studied LWs for the solving the IVPs of Bratu-type, which are widely applicable in the transfer of heat. The properties of LWs together with GIM are used to solve the non-linear algebraic equations. This work aims to study the numerical solutions of the third-order ODEs which is widely applicable in science and engineering fields using Legendre base. Some of numerical implementations are examined to show the accuracy and the the efficiency of the developed algorithm.

The implementations of examples show the efficient and applicability of collections and VIM the techniques and the numerical results have been compared with the classical RK numerical method and modern RKD numerical method.

2. Preliminary

The definition and the properties of LPs introduced in this section

**Definition 2.1 (Legendre Polynomials (LPs))**

The LP can be defined as follows:

\[
p_n(\xi) = \frac{1}{2^n n!} \frac{d^n}{d\xi^n} (\xi^2 - 1)^n,
\]

**Lemma 2.1. Properties of LP**

The LP has the following properties over the interval \( I \):

1. \( P_n(\xi) = \frac{1}{2^n n!} \frac{d^n}{d\xi^n} (\xi^2 - 1)^n \),
2. \( P_n(\xi) = \frac{\xi^n}{n!} \sum_{i=0}^{n} \binom{n+i}{i} (-\xi)^i \)
3. \( P_n(\xi) = \frac{2^{n-1}}{n} \xi \frac{P_{n-1}(\xi)}{n} - \frac{2}{n+1} \frac{P_{n-2}(\xi)}{n} \), for \( n = 2, 3, 4, ... \)

**Theorem 2.1. Orthogonality**

The base of LPs \( p = \{ P_0(\xi), P_1(\xi), ..., P_m(\xi) \} \) is orthogonal base (see Figure 1). That’s mean

\[
\int_{-1}^{1} P_n(\xi) P_m(\xi) d\xi = \frac{2}{2n + 1} \delta_{nm}
\]

2.1 Third-Order Initial Value Problems (IVPs)

The special ODEs of third-order has been defined by [1] as following:
\[ V'''(\zeta) = \Box(\zeta, V(\zeta)) ; \quad \zeta \geq \zeta_0, \]  

(1)

with ICs:

\[ V(\zeta_0) = \beta_0, V'(\zeta_0) = \beta_1, V''(\zeta_0) = \beta_2. \]

where,

\[ g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \]

\[ V(t) = [V_1(\zeta), V_2(\zeta), ..., V_n(\zeta)]. \]

\[ \Box(\zeta, V(\zeta)) = [\Box_1(\zeta, V(\zeta)), \Box_2(\zeta, V(\zeta)), ..., \Box_n(\zeta, V(\zeta))]. \]

\[ \beta_0 = [\beta_0^1, \beta_0^2, ..., \beta_0^n], \quad \beta_1 = [\beta_1^1, \beta_1^2, ..., \beta_1^n], \quad \beta_2 = [\beta_2^1, \beta_2^2, ..., \beta_2^n]. \]

### 2.2 General RKD & RKT Methods

[1] and [3] have written the public formulas of RKT and RKD integrators with \( \varphi \)-stage for solving special ODEs of third-order as follows:

\[ w_{n+1} = w_n + hw'_n + \frac{h^2}{2}w''_n + \sum_{i=1}^{\varphi} b_1 k_i \]  

(2)

\[ w'_{n+1} = w'_n + h^2 \sum_{i=1}^{\varphi} b'_i k_i \]  

(3)

and

\[ \acute{w}_{n+1} = \acute{w}_n + h \sum_{i=1}^{\varphi} b''_i k_i \]  

(4)

Where, \( h = \frac{b-a}{m} \), \( t_{n+1} = t_n + h, n = 0, 1, ..., m - 1 \)

and, \( k_i \) is defined as follow:

\[ k_1 = g(t_n, y_n) \]  

(5)

\[ k_i = g \left( t_n + c_i h, y_n + hc_i y'_n + \frac{(c_i h)^2}{2} y''_n + h^3 \sum_{j=1}^{i-1} a_{ij} k_j \right) \]  

(6)

for \( i = 2, 3, ..., \varphi \). The coefficients of the RKD and RKT methods in equations (2)-(6) are \( b_v, a_{ij}, b'_i \) and \( b''_i \), for \( i, j = 1, 2, ..., \varphi \) have been assumed to be real. The RKD method is an explicit method if \( a_{i,j} = 0 \), for \( i \leq j \) and otherwise its implicit method. The coefficients of the RKD and RKT methods are listed in Butcher Tableaus (BT) 1-2.
Table 1: BT of RKD Method

| $\xi$ | BT of RKD Method |
|-------|------------------|
| 0     | 0                |
| $\frac{1}{2} - \frac{\sqrt{3}}{10}$ | $\frac{2}{130} - 3\frac{\sqrt{3}}{2000}$ |
| $\frac{1}{2}$ | $\pm \frac{1}{90} + \frac{\sqrt{3}}{480}$ $\pm \frac{1}{32} - \frac{\sqrt{3}}{480}$ |
| $\frac{1}{2} + \frac{\sqrt{3}}{10}$ | $-\frac{1}{600} + \frac{\sqrt{3}}{500}$ $\frac{\sqrt{3}}{30}$ $\frac{3}{30} - \frac{\sqrt{3}}{150}$ | 0 |

| $\xi$ | BT of RKT Method |
|-------|------------------|
| 0     | 0                |
| $\frac{1}{18} + \frac{\sqrt{3}}{72}$ | $\frac{1}{18}$ $\frac{1}{18} - \frac{\sqrt{3}}{72}$ |
| 0     | $\frac{5}{36} + \frac{\sqrt{3}}{36}$ $\frac{2}{9}$ $\frac{5}{36} - \frac{\sqrt{3}}{36}$ |
| 0     | $\frac{5}{18}$ $\frac{4}{9}$ $\frac{5}{18}$ |

Table 2: BT of RKT Method

3. Analysis of Proposed Method.
In this section, we introduce the analysis of proposed method

3.1 Proposed Collection Method

To approximate the function $w(\xi)$ over the domain $[a,b]$ using the base $P$, we suppose the following:

$$ w(\xi) = \sum_{i=0}^{\infty} C_i P_i(\xi). \quad (7) $$

where, $c_i = (2l + 1) \int f(\xi) p_i(\xi) d\xi$

Hence, at $\xi = \xi_i$ we have

$$ w(\xi_i) = \sum_{i=0}^{\infty} C_i P_i(\xi_i). \quad (8) $$

and
\[ w'(\zeta_i) = \sum_{i=0}^{\infty} C_i p_i'(\zeta_i), \quad w''(\zeta_i) = \sum_{i=0}^{\infty} C_i p_i''(\zeta_i), \]
\[ w'''(\zeta_i) = \sum_{i=0}^{\infty} C_i p_i'''(\zeta_i), \quad (9) \]

and so on, we have \( w^{(j)}(\zeta_i) = \sum_{i=0}^{\infty} C_i p_i^{(j)}(\zeta_i) \), for \( j = 0, 1, 2, 3, \ldots \)

### 3.1.1. Algorithm of Proposed Collection Method

The algorithm of the proposed collection method can be listed as follows:

1. Consider the nodes \( \zeta_i = \zeta_{i-1} + h \) in the domain of Equation (1)
2. Put the nodes \( \zeta = \zeta_i \) in Equation (7) to obtain Equation (8)
3. Put the form in Equation (8) and Equation (9) in Equation (1) to obtain the system of equations
\[ \sum_{i=0}^{\infty} C_i p_i''''(\zeta) = f(\zeta), \quad \sum_{i=0}^{\infty} C_i p_i'(\zeta), \quad \sum_{i=0}^{\infty} C_i p_i''(\zeta) \]
4. Solve the system in step 3 to obtain \( \{ C_1, C_2, C_3, \ldots, C_m \} \)

### 3.2 Variational Method (VIM)

The main idea of VIM, from variational theory, is to approximate the solutions of DEs \( u(t) \) over the domain of Equation (1) by using an iteration formula of correctional functional which involves Lagrange multiplier. By applying Lagrange multiplier (LM), it can be determined sequence of approximated solutions. The iteration is initiated by a simple function such as a linear function. To illustrate the main concept of this VIM, we consider the following system of DEs:

\[ T u(t) = g(t) \]

where, \( T \) is a differential operator (DO) that acts on a sufficiently smooth function \( u(t) \) defined on such an interval \( I \subseteq R \). The given function \( g(t) \) is also defined in \( I \subseteq R \). Initially, split \( T \) in Equation (10) into its linear and non-linear part, namely

\[ T u(t) = L u(t) + N u(t) = g(t) \]

where \( L \) and \( N \) are linear and non-linear DO respectively. A correctional functional for Equation (11) is then defined iteratively as follows:

\[ u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t, s) (L u_n(s) + N u_n(s) - g(s)) ds \]

where \( \lambda(t, s) \) is LM, \( u_n \) is the \( n \)th order approximated solution of the problem, and \( \dot{u} \) is the variation of restricted, so that \( \dot{u} n = 0 \). By choosing such an simple initial function \( u_0 \), iterations are performed until it converges to a fixed point, under a condition where \( u_{n+1}(t) = u_n(t) \) when this condition is reached, then we obtain the following:

\[ \int_0^t \lambda(t, s) (L u_n(s) + N u_n(s) - g(s)) ds = 0 \]

which is equivalent to the condition in Equation (11). This means that \( u_{n+1}(t) \) can be considered as an approximated solution for Equation (12).

The iterations have been performed until they converge to a fixed points by choosing such an simple initial function \( u_0 \), under this condition, the approximation is reached, then we obtain sequence of approximated solutions of ODE in Equation (1). It means that an approximated solution for Equation (1) can be considered as \( u_n(t) + \tau(t) \).
3.2.1 Algorithm of Variational Method

The algorithm of the variational method can be listed as follows:
1. Split the differential operator $T$ in Equation (10) into its linear and non-linear part, as in Equation (11)
2. Find the correct functional of the problem yields the stationary condition as in Equation (12)
3. Using the initial conditions, approximate the sequence of iterations
4. Using the conditions of accuracy, you stop the iterations of approximations

4. Numerical Implementation

This section introduced set of two IVPs including third-order ODEs which are solved by using four numerical methods: Direct RKD method, classical RK method, collocation method and variational method (VIM). Then, the same set of IVPs are converted to two systems of first-order ODEs and then, they solved using classical RK method. In Tables 3-4, the numerical results using the four methods of the implementation are compared to indicate the total time against max error.

The notations that are used as following:
- **Step**: Step size used.
- **RKD**: The direct RKD method of third-order
- **RK**: Classical fourth-order RK method as given by Dorm and (1996).
- **Total Time(TT)**: The total computational time for approximated the problem in the seconds.
- **Max Error**: Max $|w(x_n) - w_n|$ is the maximum of absolute errors of the analytical and approximated solutions.

Example 4.1 (Non constant coefficients)

$$w^{'''(t)} = 4t(3 - 2t^2)w(t), \quad 0 < t \leq b$$

Initial conditions ,
$$w(0) = 1, w'(0) = 0, w''(0) = -2.$$  

Exact solution: $e^{-t^2}, b=1.$

Example 4.2 (Non homogeneous)

$$w^{'''(t)} = 2(e^t \cos(t) - w(t)), \quad 0 < t \leq b$$

Initial conditions,
$$w(0) = 0, w'(0) = 1, w''(0) = 2.$$  

Exact solution: $w(t) = e^t \sin(t), \quad b = 1.$

Table 3: Comparison of Total Time and Max Error for RKD, RK, Collecation and Variational Methods for Example 4.1
### Table 4: Comparison of Total Time and Max Error for RKD, RK, Collocation and Variational Methods for Example 4.2

| Step  | Method                | Total Time     | Max Error       |
|-------|-----------------------|----------------|-----------------|
| 4<sup>-1</sup> | Direct Runge-Kutta | 4.274314570105241e-004 | 1.696170139402597e-003 |
|       | Runge-Kutta           | 3.847693409695685e-003 | 3.1979871856566e-003 |
|       | Collocation           | 4.274314570105241e-004 | 1.696170139402597e-003 |
|       | Variational           | 3.847693409695685e-003 | 3.1979871856566e-003 |
| 4<sup>-2</sup> | Direct Runge-Kutta | 1.559820956860356e-004 | 3.118178203409538e-005 |
|       | Runge-Kutta           | 1.423367099260070e-002 | 5.461989905497511e-005 |
|       | Collocation           | 4.274314570105241e-004 | 1.696170139402597e-003 |
|       | Variational           | 3.847693409695685e-003 | 3.1979871856566e-003 |
| 4<sup>-3</sup> | Direct Runge-Kutta | 4.630034777930113e-003 | 5.125586057252995e-007 |
|       | Runge-Kutta           | 4.618123417895886e-002 | 8.644464797808956e-007 |
|       | Collocation           | 4.274314570105241e-004 | 1.696170139402597e-003 |
|       | Variational           | 3.847693409695685e-003 | 3.1979871856566e-003 |
| 4<sup>-4</sup> | Direct Runge-Kutta | 1.777871772183490e-002 | 8.116378680256275e-009 |
|       | Runge-Kutta           | 1.978740248808047e-001 | 1.353732428103408e-008 |
|       | Collocation           | 4.274314570105241e-004 | 1.696170139402597e-003 |
|       | Variational           | 3.847693409695685e-003 | 3.1979871856566e-003 |
| 4<sup>-5</sup> | Direct Runge-Kutta | 8.145749670209283e-002 | 1.272442151645237e-010 |
|       | Runge-Kutta           | 7.602624064512574e-001 | 2.116287145526030e-010 |
|       | Collocation           | 4.274314570105241e-004 | 1.696170139402597e-003 |
|       | Variational           | 3.847693409695685e-003 | 3.1979871856566e-003 |
Conclusion and Discussion

The orthogonal LPs basis defined on the square interval $I$ has been introduced. The properties of LPs with the Gaussian integration (GI) method are used to solve special class of third-order ODEs with the initial value conditions (IVCs). This work aims to study the numerical solutions of this problem which is widely applicable in the field of engineering and science using Legendre base. Numerical implementations have been proved the efficiency, validity and applicability of the collection method (CM) and variational iteration method (VIM). The numerical results which obtained using CM and VIM have been compared with the numerical results which obtained by classical RK and modern RKD methods.

Acknowledgements

Al-Shahey and Mechee thanks University of Kufa for supporting this paper

References

[1] M. Mechee, F. Ismail, N. Senu, and Z. Siri, “Directly solving special second order delay differential equations using runge-kutta-nystrom method,” Mathematical Problems in Engineering, vol. 2013, 2013.
[2] M. Mechee, N. Senu, F. Ismail, B. Nikouravan, and Z. Siri, “A three-stage fifth-order rungekutta method for directly solving special third-order differential equation with application to thin film flow problem,” Mathematical Problems in Engineering, vol. 2013, 2013.
[3] X. You and Z. Chen, “Direct integrators of runge–kutta type for special third-order ordinary differential equations,” Applied Numerical Mathematics, vol. 74, pp. 128–150, 2013.
[4] M. Mechee, F. Ismail, N. Senu, and Siri, “A third-order direct integrators of runge-kutta type for special third-order ordinary and delay differential equations,” Journal of Applied Sciences, vol. 2.
no. 6, 2014.

[5] M. Mechee, F. Ismail, Z. Siri, N. Senu, and Senu, “A four stage sixth-order rkd method for directly solving special third order ordinary differential equations,” Life Science Journal, vol. 11, no. 3, 2014.

[6] M. Mechee and M. Kadhim, “Direct explicit integrators of rk type for solving special fourth order ordinary differential equations with an application,” Global Journal of Pure and Applied Mathematics, vol. 12, no. 6, pp. 4687–4715, 2016.

[7] M. S. Mechee and M. A. Kadhim, “Explicit direct integrators of rk type for solving special fifth-order ordinary differential equations,” American Journal of Applied Sciences, vol. 13, pp.1452–1460, 2016.

[8] M. Mechee, F. Ismail, Z. Hussain, and Z. Siri, “Direct numerical methods for solving a class of third-order partial differential equations,” Applied Mathematics and Computation, vol. 247, pp. 663–674, 2014.

[9] M. S. Mechee, Z. M. Hussain, and H. R. Mohammed, “On the reliability and stability of direct explicit runge-kutta integrators,” Global Journal of Pure and Applied Mathematics, vol. 12, no. 4, pp. 3959–3975, 2016.

[10] M. S. Mechee and Y. Rajihy, “Generalized rk integrators for solving ordinary differential equations: A survey & comparison study,” Global Journal of Pure and Applied Mathematics, vol. 13, no. 7, pp. 2923–2949, 2017.

[11] M. S. Mechee, G. A. Al-Juaifri, and A. K. Joohy, “Modified homotopy perturbation method for solving generalized linear complex differential equations,” Applied Mathematical Sciences, vol. 11, no. 51, pp. 2527–2540, 2017.

[12] M. S. Mechee and J. K. Mshachal, “Derivation of direct explicit integrators of rk type for solving class of seventh-order ordinary differential equations,” Karbala International Journal of Modern Science, vol. 5, no. 3, p. 8, 2019.

[13] M. S. Mechee, “Generalized rk integrators for solving class of sixth-order ordinary differential equations,” Journal of Interdisciplinary Mathematics, vol. 22, no. 8, pp. 1457–1461, 2019.

[14] M. S. Mechee and K. B. Mussa, “Generalization of rk integrators for solving a class of eighth-order ordinary differential equations with applications,” Advanced Mathematical Models & Applications, vol. 5, no. 1, pp. 111–120, 2020.

[15] M. S. Mechee, H. M. Wali, and K. B. Mussa, “Developed rk method for solving ninth-order ordinary differential equations with applications,” in Journal of Physics: Conference Series, vol. 1664, no. 1. IOP Publishing, 2020, p. 012102.

[16] N. Liu and E.-B. Lin, “Legendre wavelet method for numerical solutions of partial differential equations,” Numerical Methods for Partial Differential Equations: An International Journal, vol. 26, no. 1, pp. 81–94, 2010.

[17] Z. Odibat, “On legendre polynomial approximation with the vim or ham for numerical treatment of nonlinear fractional differential equations,” Journal of Computational and Applied Mathematics, vol. 235, no. 9, pp. 2956–2968, 2011.

[18] S. Venkatesh, S. Ayyaswamy, and S. R. Balachandar, “The legendre wavelet method for solving initial value problems of bratu-type,” Computers & Mathematics with Applications, vol. 63, no. 8, pp. 1287–1295, 2012.

[19] S. Yalçınbas, M. Sezer, and H. H. Sorkun, “Legendre polynomial solutions of high-order linear fredholm integro-differential equations,” Applied Mathematics and Computation, vol. 210, no. 2, pp. 334–349, 2009.

[20] J. R. Dormand, Numerical methods for differential equations: a computational approach.CRC Press, 1996, vol. 3. vol. 2013, 2013. Surname A and Surname B 2009 Journal Name 23 544