FUZZY STABILITY OF AN AQCQ-FUNCTIONAL EQUATION IN MATRIX FUZZY NORMED SPACES

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Abstract. Using the fixed point method, we prove the Hyers-Ulam stability of an additive-quadratic-cubic-quartic functional equation in matrix fuzzy normed spaces.

1. Introduction and Preliminaries

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of matricially normed spaces \cite{64} implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see \cite{20}).

The proof given in \cite{64} appealed to the theory of ordered operator spaces \cite{13}. Effros and Ruan \cite{21} showed that one can give a purely metric proof of this important theorem by using a technique of Pisier \cite{54} and Haagerup \cite{28} (as modified in \cite{19}).

The stability problem of functional equations originated from a question of Ulam \cite{69} concerning the stability of group homomorphisms.

The functional equation

\[ f(x + y) = f(x) + f(y) \]

is called the \textit{Cauchy additive functional equation}. In particular, every solution of the Cauchy additive functional equation is said to be an \textit{additive mapping}. Hyers \cite{29} gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki \cite{2} for additive mappings and by Th.M.

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Rassias [58] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [27] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias’ approach.

In 1990, Th.M. Rassias [59] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [26] following the same approach as in Th.M. Rassias [58], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [26], as well as by Th.M. Rassias and Šemrl [63] that one cannot prove a Th.M. Rassias’ type theorem when $p = 1$ (cf. the books of P. Czerwik [17], D.H. Hyers, G. Isac and Th.M. Rassias [30]).

In 1982, J.M. Rassias [56] followed the innovative approach of the Th.M. Rassias’ theorem [58] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

The functional equation
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]
is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [68] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [14] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. Czerwik [15] proved the Hyers-Ulam stability of the quadratic functional equation.

In [33], Jun and Kim considered the following cubic functional equation
\[
(1.1) \quad f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).
\]
It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping.

In [43], Lee et al. considered the following quartic functional equation
\[
(1.2) \quad f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y).
\]
It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping. The stability problems of several
functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 24, 31, 34, 35, 39, 42, 44, 50, 66, 57], [60]–[62]).

The theory of fuzzy space has much progressed as developing the theory of randomness. Some mathematicians have defined fuzzy norms on a vector space from various points of view [4, 25, 37, 41, 47, 70]. Following Cheng and Mordeson [9], Bag and Samanta [4] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [40] and investigated some properties of fuzzy normed spaces [5].

We use the definition of fuzzy normed spaces given in [4, 47, 48] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy-Jensen functional equation in the fuzzy normed algebra setting.

**Definition 1.1** ([4, 47, 48, 49]). Let $X$ be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

1. $N(x, t) = 0$ for $t \leq 0$;
2. $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
3. $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
4. $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
5. $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$;
6. $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair $(X, N)$ is called a *fuzzy normed space*.

**Definition 1.2** ([4, 47, 48, 49]). (1) Let $(X, N)$ be a fuzzy normed space. A sequence $\{x_n\}$ in $X$ is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $N$-$\lim_{n \to \infty} x_n = x$.

(2) Let $(X, N)$ be a fuzzy normed space. A sequence $\{x_n\}$ in $X$ is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a **fuzzy Banach space**.

We say that a mapping $f : X \to Y$ between fuzzy normed spaces $X$ and $Y$ is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then
the sequence \( \{ f(x_n) \} \) converges to \( f(x_0) \). If \( f : X \to Y \) is continuous at each \( x \in X \), then \( f : X \to Y \) is said to be continuous on \( X \) (see [5]).

We introduce the concept of matrix fuzzy normed space.

**Definition 1.3.** Let \( (X, N) \) be a fuzzy normed space. (1) \( (X, \{ N_n \}) \) is called a matrix fuzzy normed space if for each positive integer \( n \), \( (M_n(X), N_n) \) is a fuzzy normed space and 
\[
N_k(A \cdot x, t) \geq N_n(x, \frac{t}{\|A\| \cdot \|B\|})
\]
for all \( t > 0 \), \( A \in M_{k,n}(\mathbb{R}) \), \( x = [x_{ij}] \in M_n(X) \) and \( B \in M_{n,k}(\mathbb{R}) \) with \( \|A\| \cdot \|B\| \neq 0 \).

(2) \( (X, \{ N_n \}) \) is called a matrix fuzzy Banach space if \( (X, N) \) is a fuzzy Banach space and \( (X, \{ N_n \}) \) is a matrix fuzzy normed space.

**Example 1.4.** Let \( (X, \{ \| \cdot \|_n \}) \) be a matrix normed space. Let 
\[
N_{n}(x, t) := \frac{t}{t + \|x\|_n}
\]
for all \( t > 0 \) and \( x = [x_{ij}] \in M_n(X) \). Then 
\[
N_k(A \cdot x, t) = \frac{t}{t + \|A\cdot B\|_k} \geq \frac{t}{t + \|A\| \cdot \|x\|_n \cdot \|B\|} = \frac{t}{\|A\| \cdot \|B\| + \|x\|_n}
\]
for all \( t > 0 \), \( A \in M_{k,n}(\mathbb{R}) \), \( x = [x_{ij}] \in M_n(X) \) and \( B \in M_{n,k}(\mathbb{R}) \) with \( \|A\| \cdot \|B\| \neq 0 \).
So \( (X, \{ N_n \}) \) is a matrix fuzzy normed space.

Let \( E, F \) be vector spaces. For a given mapping \( h : E \to F \) and a given positive integer \( n \), define \( h_n : M_n(E) \to M_n(F) \) by 
\[
h_n([x_{ij}]) = [h(x_{ij})]
\]
for all \( [x_{ij}] \in M_n(E) \).

Let \( X \) be a set. A function \( d : X \times X \to [0, \infty] \) is called a generalized metric on \( X \) if \( d \) satisfies

1. \( d(x, y) = 0 \) if and only if \( x = y \);
2. \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
3. \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in X \).

We recall a fundamental result in fixed point theory.

**Theorem 1.5** ([6, 18]). Let \( (X, d) \) be a complete generalized metric space and let \( J : X \to X \) be a strictly contractive mapping with Lipschitz constant \( \alpha < 1 \). Then for each given element \( x \in X \), either

\[
d(J^n x, J^{n+1} x) = \infty
\]
for all nonnegative integers \( n \) or there exists a positive integer \( n_0 \) such that

1. \( d(J^n x, J^{n+1} x) < \infty \), \( \forall n \geq n_0 \);
(2) the sequence \( \{J^n x\} \) converges to a fixed point \( y^* \) of \( J \);
(3) \( y^* \) is the unique fixed point of \( J \) in the set \( Y = \{y \in X \mid d(J^0 x, y) < \infty\} \);
(4) \( d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy) \) for all \( y \in Y \).

In 1996, G. Isac and Th.M. Rassias [32] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [7, 8, 36, 46, 51, 52, 55]).

Throughout this paper, let \((X, \{N_n\})\) be a matrix fuzzy normed space and \((Y, \{N_n\})\) a matrix fuzzy Banach space.

In this paper, we prove the Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation
\[
(1.3) \quad f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y)
\]
in matrix fuzzy normed spaces by using the fixed point method.

One can easily show that an odd mapping \( f : X \to Y \) satisfies (1.3) if and only if the odd mapping \( f : X \to Y \) is an additive-cubic mapping, i.e.,
\[
f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x).
\]
It was shown in [23, Lemma 2.2] that \( g(x) := f(2x) - 2f(x) \) and \( h(x) := f(2x) - 8f(x) \) are cubic and additive, respectively, and that \( f(x) = \frac{1}{5} g(x) - \frac{1}{2} h(x) \).

One can easily show that an even mapping \( f : X \to Y \) satisfies (1.3) if and only if the even mapping \( f : X \to Y \) is a quadratic-quartic mapping, i.e.,
\[
f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + 2f(2y) - 8f(y).
\]
It was shown in [22, Lemma 2.1] that \( g(x) := f(2x) - 4f(x) \) and \( h(x) := f(2x) - 16f(x) \) are quartic and quadratic, respectively, and that \( f(x) = \frac{1}{12} g(x) - \frac{1}{12} h(x) \).

2. **Hyers-Ulam Stability of the AQCQ-Functional Equation (1.3) in Matrix Fuzzy Normed Spaces: Odd Mapping Case**

In this section, we prove the Hyers-Ulam stability of the AQCQ-functional equation (1.3) in matrix fuzzy normed spaces for an odd mapping case.

We will use the following notations:
\( M_n(X) \) is the set of all \( n \times n \)-matrices in \( X \);
Lemma 2.1. Let \((X, \{N_n\})\) be a matrix fuzzy normed space.

1. \(N_n(E_{kl} \otimes x, t) = N(x, t)\) for all \(t > 0\) and \(x \in X\).
2. For all \([x_{ij}]\) \(\in M_n(X)\) and \(t = \sum_{i,j=1}^n t_{ij}\),
   \[
   N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\{N(x_{ij}, t_{ij}) : i, j = 1,2,\ldots, n\},
   \]
   \[
   N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\left\{N\left(x_{ij}, \frac{t_{ij}}{n^2}\right) : i, j = 1,2,\ldots, n\right\}.
   \]
3. \(\lim_{n \to \infty} x_n = x\) if and only if \(\lim_{n \to \infty} x_{ijn} = x_{ij}\) for \(x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)\).

Proof. (1) Since \(E_{kl} \otimes x = e_k^* x e_l\) and \(\|e_k^*\| = \|e_l\| = 1\), \(N_n(E_{kl} \otimes x, t) \geq N(x, t)\).

Since \(e_k(E_{kl} \otimes x)e_l^* = x\), \(N_n(E_{kl} \otimes x, t) \leq N(x, t)\). So \(N(E_{kl} \otimes x, t) = N(x, t)\).

(2) \(N(x_{kl}, t) = N(e_k[x_{ij}]e_l^*, t) \geq N_n([x_{ij}], t) \geq \min\{N(E_{ij} \otimes x, t_{ij}) : i, j = 1,2,\ldots, n\}\).

where \(t = \sum_{i,j=1}^n t_{ij}\). So \(N_n([x_{ij}], t) \geq \min\{N\left(x_{ij}, \frac{t_{ij}}{n^2}\right) : i, j = 1,2,\ldots, n\}\).

(3) By \(N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\{N(x_{ij}, t_{ij}) : i, j = 1,2,\ldots, n\}\), we obtain the result. 

For a mapping \(f : X \to Y\), define \(Df : X^2 \to Y\) and \(Df_n : M_n(X^2) \to M_n(Y)\) by

\[
Df(a, b) := f(a + 2b) + f(a - 2b) - 4f(a + b) - 4f(a - b) + 6f(a) - f(2b) - f(-2b) + 4f(b) + 4f(-b),
\]

\[
Df_n([x_{ij}], [y_{ij}]) := f_n([x_{ij}] + 2[y_{ij}]) + f_n([x_{ij}] - 2[y_{ij}]) - 4f_n([x_{ij}] + [y_{ij}]) - 4f_n([x_{ij}] - [y_{ij}]) + 6f_n([x_{ij}]) - f_n(2[y_{ij}]) - f_n(-2[y_{ij}]) + 4f_n([y_{ij}]) + 4f_n(-[y_{ij}])
\]

for all \(a, b \in X\) and all \(x = [x_{ij}], y = [y_{ij}] \in M_n(X)\).
Theorem 2.2. Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( \alpha < 1 \) with

\[
\varphi(a, b) \leq \frac{\alpha}{2} \varphi(2a, 2b)
\]

for all \( a, b \in X \). Let \( f : X \to Y \) be an odd mapping satisfying

\[
N_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t + \sum_{i,j=1}^n \varphi(x_{ij}, y_{ij})}
\]

for all \( t > 0 \) and \( x = [x_{ij}], y = [y_{ij}] \in M_n(X) \). Then

\[
A(a) := N - \lim_{l \to \infty} 2^l \left( f \left( \frac{2a}{2^l} \right) - 8f \left( \frac{a}{2^l} \right) \right)
\]

exists for each \( a \in X \) and defines an additive mapping \( A : X \to Y \) such that

\[
N (f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{t}{(1 - \alpha)t + n^2 \alpha \sum_{i,j=1}^n (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))}
\]

for all \( t > 0 \) and \( x = [x_{ij}] \in M_n(X) \).

Proof. Let \( n = 1 \). Then (2.2) is equivalent to

\[
N(Df(a, b), t) \geq \frac{t}{t + \varphi(a, b)}
\]

for all \( t > 0 \) and \( a, b \in X \).

Letting \( b = a \) in (2.4), we get

\[
N (f(3b) - 4f(2b) + 5f(b), t) \geq \frac{t}{t + \varphi(b, b)}
\]

for all \( t > 0 \) and \( b \in X \).

Replacing \( a \) by \( 2b \) in (2.4), we get

\[
N (f(4b) - 4f(3b) + 6f(2b) - 4f(b), t) \geq \frac{t}{t + \varphi(2b, b)}
\]

for all \( t > 0 \) and \( b \in X \).

By (2.5) and (2.6),
\[(2.7) \quad N\left(f(4b) - 10f(2b) + 16f(b), t\right) \geq \min \left\{ N\left(4(f(3b) - 4f(2b) + 5f(b)), \frac{t}{2}\right), N\left(f(4b) - 4f(3b) + 6f(2b) - 4f(b), \frac{t}{2}\right) \right\} \]

\[= \min \left\{ N\left(f(3b) - 4f(2b) + 5f(b), \frac{t}{8}\right), N\left(f(4b) - 4f(3b) + 6f(2b) - 4f(b), \frac{t}{2}\right) \right\} \]

\[\geq \min \left\{ \frac{t/8}{t/8 + \varphi(b, b)}, \frac{t/2}{t/2 + \varphi(2b, b)} \right\} \]

\[= \min \left\{ \frac{t}{t + 8\varphi(b, b)}, \frac{t}{t + 2\varphi(2b, b)} \right\} \]

\[\geq \frac{t}{t + 8\varphi(b, b) + 2\varphi(2b, b)} \]

for all \(t > 0\) and \(b \in X\). Replacing \(b\) by \(\frac{a}{2}\) and letting \(g(a) := f(2a) - 8f(a)\) in (2.7), we get

\[(2.8) \quad N\left(g(a) - 2g\left(\frac{a}{2}\right), t\right) \geq \frac{t}{t + 8\varphi\left(\frac{a}{2}, \frac{a}{2}\right) + 2\varphi\left(a, \frac{a}{2}\right)} \]

\[\geq \frac{t}{t + \alpha(4\varphi(a, a) + \varphi(2a, a))} \]

for all \(t > 0\) and \(a \in X\).

Consider the set

\[S := \{ g : X \to Y \}\]

and introduce the generalized metric on \(S\):

\[d(g, h) = \inf \left\{ \mu \in \mathbb{R}_+ : N(g(a) - h(a), \mu t) \geq \frac{t}{t + 4\varphi(a, a) + \varphi(2a, a)}, \forall a \in X, \forall t > 0 \right\}, \]

where, as usual, \(\inf \varphi = +\infty\). It is easy to show that \((S, d)\) is complete (see the proof of [45, Lemma 2.1]).

Now we consider the linear mapping \(J : S \to S\) such that

\[Jg(a) := 2g\left(\frac{a}{2}\right) \]

for all \(a \in X\).
Let \( g, h \in S \) be given such that \( d(g, h) = \varepsilon \). Then

\[
N\left(g(a) - h(a), \varepsilon t\right) \geq \frac{t}{t + 4\nu(a, a) + \nu(2a, a)}
\]

for all \( a \in X \) and \( t > 0 \). Hence

\[
N\left(Jg(a) - Jh(a), \alpha \varepsilon t\right) = N\left(2g\left(\frac{\alpha}{2}\right) - 2h\left(\frac{\alpha}{2}\right), \alpha \varepsilon t\right) = N\left(g\left(\frac{\alpha}{2}\right) - h\left(\frac{\alpha}{2}\right), \frac{\alpha \varepsilon t}{t}\right)
\]

\[
\geq \frac{\frac{\alpha t}{2} + 4\nu\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right) + \nu\left(a, \frac{\alpha}{2}\right)}{t} \geq \frac{\frac{\alpha t}{2} + \frac{\alpha t}{2}(4\nu(a, a) + \nu(2a, a))}{t + 4\nu(a, a) + \nu(2a, a)}
\]

for all \( a \in X \) and \( t > 0 \). So \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq \alpha \varepsilon \). This means that

\[
d(Jg, Jh) \leq \alpha d(g, h)
\]

for all \( g, h \in S \).

It follows from (2.8) that \( d(g, Jg) \leq \alpha \).

By Theorem 1.5, there exists a mapping \( A : X \rightarrow Y \) satisfying the following:

1. \( A \) is a fixed point of \( J \), i.e.,

\[
A\left(\frac{a}{2}\right) = \frac{1}{2}A(a)
\]

for all \( a \in X \). The mapping \( A \) is a unique fixed point of \( J \) in the set

\[
M = \{g \in S : d(f, g) < \infty\}.
\]

2. \( d(J^lg, A) \rightarrow 0 \) as \( l \rightarrow \infty \). This implies the equality

\[
N- \lim_{l \rightarrow \infty} 2^lg\left(\frac{a}{2^l}\right) = A(a)
\]

for all \( a \in X \).

3. \( d(g, A) \leq \frac{1}{1 - \alpha} d(g, Jg) \), which implies the inequality

\[
d(g, A) \leq \frac{\alpha}{1 - \alpha}.
\]

By (2.2),

\[
N\left(2^lg\left(\frac{a}{2^l}, \frac{b}{2^l}\right), 2^lt\right) \geq \frac{t}{t + 2\nu\left(\frac{2a}{2^l}, \frac{2b}{2^l}\right) + 16\nu\left(\frac{a}{2^l}, \frac{b}{2^l}\right)}
\]

for all \( a, b \in X \) and \( t > 0 \). So

\[
N\left(2^lg\left(\frac{a}{2^l}, \frac{b}{2^l}\right), t\right) \geq \frac{t}{t^2 + 2^l\nu\left(\frac{2a}{2^l}, \frac{2b}{2^l}\right) + 16^l\nu\left(\frac{a}{2^l}, \frac{b}{2^l}\right)}
\]
for all \(a, b \in X\) and \(t > 0\). Since 
\[
\lim_{l \to \infty} \frac{t}{2^l + 2^n \alpha^2 \varphi(2a, 2b) + 16 \alpha^2 t \varphi(a, b)} = 1
\]
for all \(a, b \in X\) and \(t > 0\), 
\[
N(DA(a, b), t) = 1
\]
for all \(a, b \in X\) and \(t > 0\). Thus \(DA(a, b) = 0\) for all \(a, b \in X\). So the mapping 
\(A : X \to Y\) is additive.

By Lemma 2.1 and (2.9),
\[
N(g_n([x_{ij}]) - A_n([x_{ij}]), t) 
\geq \min \left\{ N\left( g(x_{ij}) - A(x_{ij}), \frac{t}{n^2} \right) : i, j = 1, 2, \cdots, n \right\}
\geq \min \left\{ \frac{(1 - \alpha)t}{(1 - \alpha)t + n^2 \alpha(4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))} : i, j = 1, 2, \cdots, n \right\}
\geq \frac{(1 - \alpha)t}{(1 - \alpha)t + 2n^2 \alpha} \sum_{i,j=1}^n (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))
\]
for all \(x = [x_{ij}] \in M_n(X)\). Thus \(A : X \to Y\) is a unique additive mapping satisfying (2.3), as desired. 

\[\square\]

**Corollary 2.3.** Let \(r, \theta\) be positive real numbers with \(r < 1\). Let \(f : X \to Y\) be an odd mapping satisfying
\[
N(g_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{t}{t + \sum_{i,j=1}^n \theta(||x_{ij}||^r + ||y_{ij}||^r)}
\]
for all \(t > 0\) and \(x = [x_{ij}], y = [y_{ij}] \in M_n(X)\). Then
\[
A(a) := N - \lim_{l \to \infty} 2^l \left( f\left( \frac{2a}{2^l} \right) - 8f\left( \frac{a}{2^l} \right) \right)
\]
exists for each \(a \in X\) and defines an additive mapping \(A : X \to Y\) such that 
\[
N(f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}]), t) \geq \frac{(2 - 2r)t}{(2 - 2r)t + n^2 \cdot 2^r (2^r + 9) \sum_{i,j=1}^n \theta ||x_{ij}||^r}
\]
for all \(t > 0\) and \(x = [x_{ij}] \in M_n(X)\).

**Proof.** The proof follows from Theorem 2.2 by taking \(\varphi(a, b) = \theta(||a||^r + ||b||^r)\) for all \(a, b \in X\). Then we can choose \(\alpha = 2^{r-1}\) and we get the desired result. \[\square\]

**Theorem 2.4.** Let \(f : X \to Y\) be an odd mapping satisfying (2.2) for which there exists a function \(\varphi : X^2 \to [0, \infty)\) such that there exists an \(\alpha < 1\) with 
\[
\varphi(a, b) \leq 2\alpha^2 \varphi\left( \frac{a}{2}, \frac{b}{2} \right)
\]
for all $a, b \in X$. Then $A(a) := N\lim_{l \to \infty} \frac{1}{2l} (f(2^{l+1}a) - 8f(2^la))$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N (f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}], t) \geq \frac{t}{(1 - \alpha)t + n^2 \sum_{i,j=1}^{n} (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.8) that

$$N \left( g(a) - \frac{1}{2} g(2a), t \right) \geq t \bigg/ t + 4\varphi(a,a) + \varphi(2a,a)$$

for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2.

Corollary 2.5. Let $r, \theta$ be positive real numbers with $r > 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.10). Then $A(a) := N\lim_{l \to \infty} \frac{1}{2l} (f(2^{l+1}a) - 8f(2^la))$ exists for each $a \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N (f_n(2[x_{ij}]) - 8f_n([x_{ij}]) - A_n([x_{ij}], t) \geq \frac{(2r - 2)t}{(2r - 2)t + n^2 \cdot 2r(2r + 9) \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(a,b) = \theta(||a||^r + ||b||^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{-r}$ and we get the desired result.

Theorem 2.6. Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with

$$\varphi(a,b) \leq \frac{\alpha}{8} \varphi(2a,2b)$$

for all $a, b \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then $C(a) := N\lim_{l \to \infty} 8^l \left( f \left( \frac{3a}{2^l} \right) - 2f \left( \frac{a}{2^l} \right) \right)$ exists for each $a \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N (f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}], t) \geq \frac{4(1 - \alpha)t}{4(1 - \alpha)t + n^2 \alpha \sum_{i,j=1}^{n} (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.
Proof. Let \((S,d)\) be the generalized metric space defined in the proof of Theorem 2.2.

Replacing \(b\) by \(\frac{a}{2}\) and letting \(g(a) := f(2a) - 2f(a)\) in (2.7), we get

\[
N \left( g(a) - 8g \left( \frac{a}{2} \right), t \right) \geq \frac{t}{t + 8\varphi \left( \frac{a}{2}, \frac{a}{2} \right) + 2\varphi \left( a, \frac{a}{2} \right)} \geq \frac{t}{t + \frac{a}{4}(4\varphi(a,a) + \varphi(2a,a))}
\]

for all \(t > 0\) and \(a \in X\).

The rest of the proof is similar to the proof of Theorem 2.2. \(\Box\)

**Corollary 2.7.** Let \(r, \theta\) be positive real numbers with \(r < 3\). Let \(f : X \to Y\) be an odd mapping satisfying (2.10). Then \(C(a) := N \lim_{l \to \infty} \frac{1}{l} \left( f \left( \frac{2a}{l} \right) - 2f \left( \frac{a}{l} \right) \right)\) exists for each \(a \in X\) and defines a cubic mapping \(C : X \to Y\) such that

\[
N \left( f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t \right) \geq \frac{(8 - 2^r)t}{(8 - 2^r)t + n^2 \cdot 2^r(2^r + 9) \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r}
\]

for all \(t > 0\) and \(x = [x_{ij}] \in M_n(X)\).

Proof. The proof follows from Theorem 2.6 by taking \(\varphi(a,b) = \theta(\|a\|^r + \|b\|^r)\) for all \(a,b \in X\). Then we can choose \(\alpha = 2r^{-3}\) and we get the desired result. \(\Box\)

**Theorem 2.8.** Let \(f : X \to Y\) be an odd mapping satisfying (2.2) for which there exists a function \(\varphi : X^2 \to [0, \infty)\) such that there exists an \(\alpha < 1\) with

\[
\varphi(a,b) \leq 8\alpha \varphi \left( \frac{a}{2}, \frac{b}{2} \right)
\]

for all \(a,b \in X\). Then \(C(a) := N \lim_{l \to \infty} \frac{1}{l} \left( f \left( \frac{2^{l+1}a}{l} \right) - 2f \left( \frac{2^la}{l} \right) \right)\) exists for each \(a \in X\) and defines a cubic mapping \(C : X \to Y\) such that

\[
N \left( f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t \right) \geq \frac{4(1 - \alpha)t}{4(1 - \alpha)t + n^2 \sum_{i,j=1}^{n} (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))}
\]

for all \(t > 0\) and \(x = [x_{ij}] \in M_n(X)\).

Proof. Let \((S,d)\) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.11) that

\[
N \left( g(a) - \frac{1}{8}g(2a), t \right) \geq \frac{t}{t + \frac{a}{4}(4\varphi(a,a) + \varphi(2a,a))}
\]
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for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.9. Let $r, \theta$ be positive real numbers with $r > 3$. Let $f : X \rightarrow Y$ be an odd mapping satisfying (2.10). Then $C(a) := N\lim_{n \rightarrow \infty} \frac{1}{8^n} (f(2^{n+1}a) - 2f(2^na))$ exists for each $a \in X$ and defines a cubic mapping $C : X \rightarrow Y$ such that

$$N (f_n(2[x_{ij}]) - 2f_n([x_{ij}]) - C_n([x_{ij}]), t) \geq \frac{(2^r - 8)t}{(2^r - 8)t + n^2 \cdot 2^r (2^r + 9) \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 2.8 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{3-r}$ and we get the desired result. □

3. HYERS-ULAM STABILITY OF THE AQCQ-FUNCTIONAL EQUATION (1.3) IN MATRIX FUZZY NORMED SPACES: EVEN MAPPING CASE

In this section, we prove the Hyers-Ulam stability of the AQCQ-functional equation (1.3) in matrix fuzzy normed spaces for an even mapping case.

Theorem 3.1. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $\alpha < 1$ with

$$\varphi(a, b) \leq \frac{\alpha}{4} \varphi(2a, 2b)$$

for all $a, b \in X$. Let $f : X \rightarrow Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then $Q(a) := N\lim_{n \rightarrow \infty} 4^n (f(2^n a) - 16f(\frac{a}{2^n}))$ exists for each $a \in X$ and defines a quadratic mapping $Q : X \rightarrow Y$ such that

$$N (f_n(2[x_{ij}]) - 16f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \frac{2(1 - \alpha)t}{2(1 - \alpha)t + n^2 \alpha \sum_{i,j=1}^{n} (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof. Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

Letting $b = a$ in (2.4), we get

$$N (f(3b) - 6f(2b) + 15f(b), t) \geq \frac{t}{t + \varphi(b, b)}$$

for all $t > 0$ and $b \in X$. 

Replacing $a$ by $2b$ in (2.4), we get

\[(2.4) \quad N(\frac{f(4b) - 4f(3b) + 4f(2b) + 4f(b)}{t + \varphi(2b, b)}, t) \geq t \]

for all $t > 0$ and $b \in X$.

By (3.1) and (3.2),

\[(3.3) \quad N(f(4b) - 20f(2b) + 64f(b), t) \]

\[\geq \min \left\{ N\left( \frac{4(f(3b) - 6f(2b) + 15f(b))}{t}, \frac{t}{2} \right), \left( f(4b) - 4f(3b) + 6f(2b) - 4f(b), \frac{t}{2} \right) \right\} \]

\[= \min \left\{ N\left( f(3b) - 4f(2b) + 5f(b), \frac{t}{8} \right), \left( f(4b) - 4f(3b) + 4f(2b) + 4f(b), \frac{t}{2} \right) \right\} \]

\[\geq \min \left\{ \frac{t/8}{t/8 + \varphi(b, b)}, \frac{t/2}{t/2 + \varphi(2b, b)} \right\} \]

\[= \min \left\{ \frac{t}{t + 8\varphi(b, b) + 2\varphi(2b, b)} \right\} \]

\[\geq \frac{t}{t + 8\varphi(b, b) + 2\varphi(2b, b)} \]

for all $t > 0$ and $b \in X$. Replacing $b$ by $a$ and letting $g(a) := f(2a) - 16f(a)$ in (3.3), we get

\[(3.4) \quad N\left( g(a) - 4g\left( \frac{a}{2} \right), t \right) \]

\[\geq \frac{t}{t + 8\varphi\left( \frac{a}{2}, \frac{a}{2} \right) + 2\varphi\left( a, \frac{a}{2} \right)} \]

\[\geq \frac{t}{t + \frac{a}{2}(4\varphi(a, a) + \varphi(2a, a))} \]

for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. \qed

**Corollary 3.2.** Let $r, \theta$ be positive real numbers with $r < 2$. Let $f : X \rightarrow Y$ be an even mapping satisfying (2.10). Then $Q(a) := N\lim_{n \rightarrow \infty} 4^l \left( f\left( \frac{a}{2} \right) - 16f\left( \frac{a}{2} \right) \right)$ exists for each $a \in X$ and defines a quadratic mapping $Q : X \rightarrow Y$ such that

\[N(f_n(2[x_{ij}]) - 16f_n([x_{ij}]), Q_n([x_{ij}]), t) \geq \frac{2(4 - 2^r)t}{2(4 - 2^r)t + n^2 \cdot 2^r(2^r + 9)\sum_{i,j=1}^n \theta \|x_{ij}\|^r} \]

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.  


Proof. The proof follows from Theorem 3.1 by taking \( \varphi(a,b) = \theta(||a||^r + ||b||^r) \) for all \( a, b \in X \). Then we can choose \( \alpha = 2^{r-2} \) and we get the desired result.

\[ \square \]

**Theorem 3.3.** Let \( f : X \to Y \) be an even mapping satisfying \( f(0) = 0 \) and (2.2) for which there exists a function \( \varphi : X^2 \to [0,\infty) \) such that there exists an \( \alpha < 1 \) with

\[ \varphi(a,b) \leq 4\alpha \varphi \left( \frac{a}{2}, \frac{b}{2} \right) \]

for all \( a, b \in X \). Then \( Q(a) := \lim_{l \to \infty} \frac{1}{4l} \left( f(2^{l+1}a) - 16f(2^la) \right) \) exists for each \( a \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that

\[ N(f_n(2[x_{ij}]) - 16f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \frac{t}{2(1-\alpha)l + n^2 \sum_{i,j=1}^n (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))} \]

for all \( t > 0 \) and \( x = [x_{ij}] \in M_n(X) \).

Proof. Let \( (S,d) \) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (3.4) that

\[ N \left( g(a) - \frac{1}{4} g(2a), t \right) \geq \frac{t}{t + \frac{1}{2} \varphi(a,a) + \varphi(2a,a)} \]

for all \( t > 0 \) and \( a \in X \).

The rest of the proof is similar to the proof of Theorem 2.2.

\[ \square \]

**Corollary 3.4.** Let \( r, \theta \) be positive real numbers with \( r > 2 \). Let \( f : X \to Y \) be an even mapping satisfying (2.10). Then \( Q(a) := \lim_{l \to \infty} \frac{1}{2} \left( f(2^la) - 16f(2^la) \right) \) exists for each \( a \in X \) and defines a quadratic mapping \( Q : X \to Y \) such that

\[ N(f_n(2[x_{ij}]) - 16f_n([x_{ij}]) - Q_n([x_{ij}]), t) \geq \frac{2^{2r-4}t}{2(2^r - 4)t + n^2 \cdot 2^r(2^r + 9) \sum_{i,j=1}^n \theta ||x_{ij}||^r} \]

for all \( t > 0 \) and \( x = [x_{ij}] \in M_n(X) \).

Proof. The proof follows from Theorem 3.3 by taking \( \varphi(a,b) = \theta(||a||^r + ||b||^r) \) for all \( a, b \in X \). Then we can choose \( \alpha = 2^{2-r} \) and we get the desired result.

\[ \square \]

**Theorem 3.5.** Let \( \varphi : X^2 \to [0,\infty) \) be a function such that there exists an \( \alpha < 1 \) with

\[ \varphi(a,b) \leq \frac{\alpha}{16} \varphi(2a,2b) \]
for all \(a, b \in X\). Let \(f : X \to Y\) be an even mapping satisfying \(f(0) = 0\) and (2.2). Then \(R(a) := N \lim_{l \to \infty} 16^l \left( f \left( \frac{2a}{l} \right) - 4f \left( \frac{a}{l} \right) \right)\) exists for each \(a \in X\) and defines a quartic mapping \(R : X \to Y\) such that

\[
N \left( f_n(2[x_{ij}]) - 4f_n([x_{ij}]) - R_n([x_{ij}]), t \right) \geq \frac{8(1 - \alpha)t}{8(1 - \alpha)t + n^2\alpha \sum_{i,j=1}^{n} (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))}
\]

for all \(t > 0\) and \(x = [x_{ij}] \in M_n(X)\).

Proof. Let \((S, d)\) be the generalized metric space defined in the proof of Theorem 2.2.

Replacing \(b\) by \(a^2\) and letting \(g(a) := f(2a) - 4f(a)\) in (3.3), we get

\[
N \left( g(a) - 16g \left( \frac{a}{2} \right), t \right) \geq \frac{t}{t + \frac{8}{16}(4\varphi(a, a) + \varphi(2a, a))}
\]

for all \(t > 0\) and \(a \in X\).

The rest of the proof is similar to the proof of Theorem 2.2. \(\square\)

**Corollary 3.6.** Let \(r, \theta\) be positive real numbers with \(r < 4\). Let \(f : X \to Y\) be an even mapping satisfying (2.10). Then \(R(a) := N \lim_{l \to \infty} 16^l \left( f \left( \frac{2a}{l} \right) - 4f \left( \frac{a}{l} \right) \right)\) exists for each \(a \in X\) and defines a quartic mapping \(R : X \to Y\) such that

\[
N(f_n(2[x_{ij}]) - 4f_n([x_{ij}]) - R_n([x_{ij}]), t) \geq \frac{8(16 - 2^r)t}{8(16 - 2^r)t + n^2 \cdot 2r(2^r + 9) \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r}
\]

for all \(t > 0\) and \(x = [x_{ij}] \in M_n(X)\).

Proof. The proof follows from Theorem 3.5 by taking \(\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)\) for all \(a, b \in X\). Then we can choose \(\alpha = 2^{-r-4}\) and we get the desired result. \(\square\)

**Theorem 3.7.** Let \(f : X \to Y\) be an even mapping satisfying \(f(0) = 0\) and (2.2) for which there exists a function \(\varphi : X^2 \to [0, \infty)\) such that there exists an \(\alpha < 1\) with

\[
\varphi(a, b) \leq 16\alpha \varphi \left( \frac{a}{2}, \frac{b}{2} \right)
\]
for all $a, b \in X$. Then $R(a) := N \lim_{l \to \infty} \frac{1}{16} (f (2^{l+1}a) - 4f (2^l a))$ exists for each $a \in X$ and defines a quartic mapping $R : X \to Y$ such that

$$N (f_n(2[x_{ij}]) - 4f_n([x_{ij}]) - R_n([x_{ij}]), t) \geq \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2 \sum_{i,j=1}^n (4\varphi(x_{ij}, x_{ij}) + \varphi(2x_{ij}, x_{ij}))}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

**Proof.** Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (3.5) that

$$N \left( g(a) - \frac{1}{16} g (2a), t \right) \geq \frac{t}{t + \frac{1}{8} (4\varphi(a, a) + \varphi(2a, a))}$$

for all $t > 0$ and $a \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 3.8.** Let $r, \theta$ be positive real numbers with $r > 4$. Let $f : X \to Y$ be an even mapping satisfying (2.10). Then $R(a) := N \lim_{l \to \infty} \frac{1}{16} (f (2^{l+1}a) - 4f (2^l a))$ exists for each $a \in X$ and defines a quartic mapping $R : X \to Y$ such that

$$N(f_n(2[x_{ij}]) - 4f_n([x_{ij}]) - R_n([x_{ij}]), t) \geq \frac{8(2^r - 16)t}{8(2^r - 16)t + n^2 \cdot 2^r(2^{r+9}) \sum_{i,j=1}^n \theta \|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

**Proof.** The proof follows from Theorem 3.7 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $\alpha = 2^{4-r}$ and we get the desired result. □

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