DUAL SPACES OF GEODESIC CURRENTS

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Abstract. Every geodesic current on a hyperbolic surface has an associated dual space. If the current is a lamination, this dual embeds isometrically into a real tree. We show that, in general, the dual space is a hyperbolic metric tree-graded space, and express its Gromov hyperbolicity constant in terms of the geodesic current. In the case of geodesic currents with no atoms and full support, such as those coming from certain higher rank representations, we show the duals are homeomorphic to the surface. We also analyze the completeness of the dual and the properties of the action of the fundamental group of the surface on the dual. Furthermore, we equip the space of duals with a topology compatible with the natural topology of geodesic currents.

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1. Introduction

An R-tree is a geodesic metric space where any two points are connected by a unique arc isometric to a closed interval in R. A measured lamination \( \lambda \) on a surface \( X \) defines a (dual) \( \pi_1(X) \)-action on an R-tree as follows. Lift the lamination to the
universal cover $\tilde{X}$ and define the pseudo-distance between two points by considering the measure of the set of geodesics intersecting the geodesic segment connecting them. This turns out to define a 0-hyperbolic space $X_\lambda$, that embeds isometrically into an $\mathbb{R}$-tree, called the tree dual to the measured lamination. There are several equivalent formulations of this construction: see [BIPP21a], [MS91] and [Hub22] for a formulation using measured laminations, and [Wol98] for a formulation using foliations. See [Kap09] for a connection between both. See also Appendix A for a precise comparison of all the different approaches. It follows from the construction of the dual $X_\lambda$ of a measured lamination that the translation length of an element $g \in \pi_1(X)$ is equal to the intersection number of the measured lamination with $[g]$, the homotopy class represented by $g$.

First introduced by Bonahon in his seminal paper [Bon86], geodesic currents can be understood as an extension of measured laminations where the geodesics in the support of the measure are allowed to intersect each other.

The above construction of dual space of a lamination can be extended to any geodesic current $\mu$ on a compact hyperbolic surface $X$ (possibly with non-empty geodesic boundary).

**Definition (Dual space of a geodesic current).** A geodesic current $\mu$ induces a pseudo-distance on $\tilde{X}$ given by

$$d_\mu(x, \bar{y}) = \frac{1}{2} \{\mu(G(x, \bar{y})) + \mu(G(\bar{x}, \bar{y}))\}.$$  

where $G(x, \bar{y})$ denotes the set of hyperbolic geodesics of $\tilde{X}$ transverse to $[x, \bar{y}]$. The dual space of $\mu$, denoted by $X_\mu$, is defined as the metric quotient of on $\tilde{X}$ under this pseudo-distance. The set of all dual spaces of geodesic currents will be denoted by $D(X)$.

This space was introduced by Burger-Iozzi-Parreau-Pozzetti in [BIPP21a]. We note that $X_\mu$ depends on the choice of hyperbolic structure $X$ (see Subsection 3.4). $X_\mu$ might not, a priori, be 0-hyperbolic, but we show that, in fact, it is always $\delta$-hyperbolic for some $\delta \geq 0$ that can be described in terms of $\mu$. If $B$ denotes a box of geodesics in $\tilde{X}$ given by a product of two intervals on $\partial \tilde{X}$, $B^\perp$ denotes the opposite box given by the complementary intervals, and $\mathcal{B}$ the family of all boxes $B$, we prove the following result.

**Theorem A (Hyperbolicity).** Let $\mu$ be a geodesic current on $X$, and let

$$\delta_\mu = \sup_{B \in \mathcal{B}} \min \{\mu(B), \mu(B^\perp)\}.$$  

Then the dual space $X_\mu$ is a $\delta_\mu$-hyperbolic space, and $\delta_\mu$ is the optimal hyperbolicity constant.
It follows from this that $X_\mu$ is 0-hyperbolic if and only if $\mu$ is a measured lamination (see Corollary 6.14). Theorem A is stated as Theorem 6.13.

For the following, compare Figure 1.1.

Figure 1.1. The figure shows a sketch of a geodesic current $\mu$ with two components $\mu_1$ (type 1, blue) and $\mu_2$ (type 2, green), separated by a simple multi-curve of one single component (type 3, red).

Even though, by the above, $X_\mu$ is not in general isometric to an $\mathbb{R}$-tree, it has a structure that resembles that of an $\mathbb{R}$-tree, as follows. In [BIPP21a], Burger-Iozzi-Parreau-Pozzetti showed that there exists a simple multi-curve $m$ associated to $\mu$, called special multi-curve, given by disjoint simple closed geodesics $(s_j)_{j=1}^k$, that decompose $X$ into sub-surfaces with geodesic boundary $X_1, \ldots, X_n$ such that the geodesic current $\mu$ on $X$ can be written as a sum

$$\mu = \sum_{i=1}^n \mu_i + \sum_{j=1}^k a_j s_j,$$

where each $\mu_i$ is supported on $X_i$. Moreover, each $\mu_i$ is either a filling geodesic current within $X_i$ (type 1) or a non-discrete measured lamination (type 2). Filling here means $\mu_i$ intersects all geodesics within the interior of $X_i$, and a non-discrete measured lamination is one without closed components (see Subsection 2.2 for details, and Section 7 for more on decomposition).

Parallel to this result, we obtain a decomposition theorem for the dual space $X_\mu$ of $\mu$ as a (metric) tree graded space. A tree graded space $X$, in the sense of Drutu-Sapir [DS07], is a geodesic metric space together with a family of distinguished subsets $P$ called pieces. Intuitively, $X$ is assembled from $P$ by attaching them along an $\mathbb{R}$-tree $T$ that acts as a “central spine”, in such a way that any two pieces intersect each other at most at one point along $T$.

This notion was suggested to us by A. Parreau and B. Pozzetti. In fact, in [BIPP21c, Chapter 6] Burger, Iozzi, Parreau and Pozzetti relate $\mathbb{R}^2$-tree-graded spaces to the dual of a current given as a sum of two transverse measured laminations. Our dual
spaces are not endowed with a geodesic structure in general, so we introduce the more general notion of metric tree-graded space, and prove the following.

**Theorem B** (Dual structure theorem). *The dual space $X_\mu$ is a metric tree-graded space whose underlying tree is the dual tree of the special multi-curve $m$, and the pieces are the dual spaces $(X_i)_{\mu_i}$ of the currents $\mu_i$ on the surfaces $X_i$.*

See Theorem 7.11 and Section 7 for a precise statement and definitions. Compare Figure 1.2 for a part of a sketch of a geometric realization of the dual space $X_\mu$ and a hint of its tree graded structure, where $\mu$ is the current illustrated in Figure 1.1. The Figure shows three pieces of $X_\mu$, two peripheral ones corresponding to the current $\mu_2$, and one, central, corresponding to $\mu_1$, as well as two edges of the tree $T$.

![Figure 1.2](image.png)

**Figure 1.2.** The figure shows a sketch of the dual space $X_\mu$ corresponding to the current in Figure 1.1. The sketch is superimposed on the support of $\mu$ (the lifts of the geodesics on $X$ to the universal cover $\tilde{X}$), that has been faded out so that a (geometric realization of the) dual stands out.
Dual spaces come equipped with a natural action of \( \pi_1(X) \), induced from the action of such group in the universal cover \( \tilde{X} \). We relate the properties of the action to the properties of the geodesic current.

**Theorem C (Action).** Given any geodesic current \( \mu \) on a surface \( X \), the fundamental group \( \pi_1(X) \) acts by isometries on the dual space \( X_\mu \), and it does so:

1. Cobounded.
2. Properly if and only if \( \mu \) has no components of type 2 in its decomposition.
3. Freely if and only if \( \mu \) has only one component in its decomposition.

Theorem C is stated as a series of smaller results in Section 8. We also study the metric completeness of the dual spaces.

**Theorem D (Completeness).** Let \( \mu \) be a geodesic current on \( X \), the dual space \( X_\mu \) is metrically complete if and only if \( \mu \) has no components of type 2 in its decomposition.

Theorem D appears stated as Theorem 9.9, and its proof spans Section 9.9.

The dual space \( X_\mu \) comes equipped with the natural projection map \( \pi_\mu : \tilde{X} \to X_\mu \).

We study its continuity properties.

**Theorem E (Continuity of projection).** Given a geodesic current \( \mu \) on \( X \), the projection \( \pi_\mu : \tilde{X} \to X_\mu \) satisfies:

1. The projection \( \pi_\mu \) is continuous if and only if \( \mu \) has no atoms;
2. If \( \mu \) has no atoms and is filling, then \( \pi_\mu \) is closed;
3. If \( \mu \) has no atoms and has full support, then \( \pi_\mu \) is a homeomorphism.

Theorem E appears as a series of Propositions in Section 4. Theorem E, in conjunction with Theorem C, shows that \( X_\mu / \pi_1(X) \) is homeomorphic to \( X \) for geodesic currents coming from certain higher rank representations of \( \pi_1(X) \) (precisely, for those coming from positively ratioed representations in the sense of [MZ19]). In the case of real convex projective structures on \( X \), we use this result to induce an isometry between the dual space and the surface \( X \) equipped with the Hilbert metric (see Subsection 5.1.1).

We also study and relate two natural topologies in the space of geodesic currents \( \text{Curr}(X) \) and in the space of duals \( \mathcal{D}(X) \). The space \( \mathcal{D}(X) \) can be equipped with the equivariant Gromov-Hausdorff topology, first introduced and studied by Paulin [Pau88]. This is a variation of the Gromov-Hausdorff topology that bakes in the action of a group. On the other hand, the space of currents \( \text{Curr}(X) \) is naturally endowed with the weak*-topology. We prove

**Theorem F (Topologies).** The map \( \Psi : \text{Curr}(X) \to \mathcal{D}(X) \), defined by \( \mu \mapsto X_\mu \), is a homeomorphism.
Theorem \( F \) appears stated as Theorem 10.4. Our theorem extends a result of Paulin [Pau89] from the setting of \( \mathbb{R} \)-trees to more general \( \delta \)-hyperbolic spaces within the class \( D(X) \).

1.1. Outline. In Section 2 we introduce geodesic currents and give some examples whose duals we will study later on in Section 5. We also describe the weak*-topology on the space of currents and give a convenient family of neighborhoods that will play a role in Section 10.

In Section 3 we introduce the dual of a geodesic current and relate it to the notion of measured wall spaces [CD17].

In Section 4 we show that the natural projection map from the universal cover of the surface to the dual space is continuous when the current has no atoms, and a homeomorphism when the current is non-atomic and has full support. We also show that if \( \mu \) has atoms, the projection map is neither lower nor upper-semicontinuous.

In Section 5 we explore other natural examples of such duals other than the \( \mathbb{R} \)-trees coming from measured laminations (which are discussed in detail in the Appendix A). For example, we study the dual space of two intersecting measured laminations, and relate it to the concept of core of trees previously introduced by Guirardel in [Gui05]. We also show how for geodesic currents coming from certain Anosov representations, known as positively ratioed, such as strictly convex projective structures, the duals are homeomorphic to the surface \( X \).

In Section 6, we prove that duals are \( \delta \)-hyperbolic and moreover relate the optimal \( \delta \)-hyperbolicity constant to the geodesic current. Using this relation, we give inequalities between the \( \delta \)-hyperbolicity constants of \( X_\mu \) and the duals of the subcurrents \( \mu_i \) in its structural decomposition (in the sense of [BIPP21a]).

In Section 7, we prove that, using the decomposition theorem for geodesic currents proven in [BIPP21a], one can obtain a corresponding decomposition for the dual space as a tree graded space (in the sense of Drutu-Sapir [DS07]). Strictly speaking, we need to develop a notion of metric tree graded space, because of the lack of geodesic structure on \( X_\mu \).

In Section 8, we prove that the action of the fundamental group on the dual space is cobounded, it is proper if and only if the current has no filling measured lamination components (type 2), and it is free if and only if the current has only one component in its decomposition.

In Section 9 we prove that \( X_\mu \) is complete if and only if \( \mu \) has no components of type 2 in its decomposition.

In Section 10 we relate two natural topologies in the space of duals. One the one hand, the axis topology, given in terms of translation lengths, is directly related to the weak*-topology on currents. On the other hand, one can also consider the equivariant Gromov-Hausdorff topology, previously introduced by Paulin in [Pau88]. We show
these two are equivalent. Explicitly, we prove that the map sending a geodesic current to its dual is a homeomorphism when the space of currents is equipped with the weak*-topology and the space of duals is equipped with the equivariant Gromov-Hausdorff topology. This generalizes work of Paulin in [Pau89] for $\mathbb{R}$-trees within $D(X)$. We also discuss connections with recent work of Oregón-Reyes [OR22] and Jenya Sapir [Sap22].

Cantrell and Oregón-Reyes [COR22] fit the notion of dual spaces of geodesic currents in the general framework of boundary metric structures. These are left-invariant hyperbolic pseudo-metrics on a non-elementary hyperbolic group satisfying the bounded backtracking property. Such property has also been studied independently by Kapovich and the second author in upcoming work [KMG22], where they use it to construct an extension to geodesic currents for the stable length of such actions, as well as other natural notions of length, and relate it to the concept of small action of groups on $\mathbb{R}$-trees.

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2. Background

Table 1 outlines the main notation in this paper, unless we explicitly state otherwise.

In this section we introduce the basic concepts we will explore in this paper: the definition and basic properties of geodesic current, some examples of geodesic currents that will feature in this paper, and the weak*-topology for geodesic currents. This will motivate the central object of study, in Section 3, the dual of a geodesic current.

2.1. Geodesic Currents and the Intersection Form. Let $S$ be a compact connected orientable topological surface of negative Euler characteristic, genus $g$ with possibly non-empty boundary. We fix once for all a hyperbolic structure $X$ on $S$ with totally geodesic boundary, so that we can identify $X$ with the quotient $\tilde{X}/\Gamma$ for
| Notation | Meaning |
|----------|---------|
| $S$      | compact topological surface |
| $X$      | compact hyperbolic surface (with totally geodesic boundary) |
| $\Gamma, \pi_1(X)$ | deck transformation group of $X$ |
| $\tilde{X}$ | universal cover of $X$ |
| $X_\mu$ | dual of $\mu$ |
| $x, K$ | points/sets in $\tilde{X}$ (as opposed to in $X_\mu$) |
| $X_\lambda, T_\lambda$ | geometric realization of $X_\lambda$ |
| $\mathcal{G}(\tilde{X})$ | bi-infinite geodesics on $\tilde{X}$ |
| $\gamma$ | geodesic in $\mathcal{G}(\tilde{X})$ |
| $\lambda$ | measured lamination |
| $\mu$ | geodesic current |
| $s$ | simple closed curve |
| $c$ | closed curve |
| $\text{Curves}(X)$ | (weighted) multi-curves |
| $\mathcal{ML}(X)$ | measured laminations on $X$ |
| $\text{Curr}(X)$ | geodesic currents on $X$ |
| $\mathbb{P}\text{Curr}(X)$ | projective geodesic currents on $X$ |
| $\mathcal{D}(X)$ | space of duals |
| $G$ | general group (mostly hyperbolic) |

Table 1. Notation for the objects related to surfaces, curves, geodesic currents and duals.

Some $\Gamma = \pi_1(X) \leq \text{PSL}(2,\mathbb{R})$ Fuchsian subgroup. There are several equivalent definitions of geodesic currents. For an excellent account we refer the reader to [ES22, Chapter 3]. We will use the following.

**Definition 2.1** (Geodesic current). A *geodesic current* on $X$ is a positive $\Gamma$-invariant locally finite Radon measure on the set $\mathcal{G}(\tilde{X})$ of unoriented unparameterized bi-infinite geodesics $\tilde{X}$, which we identify by their endpoints in the boundary at infinity. Let

$$(\partial \tilde{X})^{(2)} = \{(x, y) \in (\partial \tilde{X})^2 : x \neq y\}.$$  

We define

$$\mathcal{G}(\tilde{X}) := (\partial \tilde{X})^{(2)}/\sim$$

where $(x, y) \sim (y, x)$. 

We say that an interval in $\partial \tilde{X}$ (or a geodesic segment $\tau$ in $\tilde{X}$) is *degenerate* if it is a singleton. We say it is *non-degenerate* otherwise.

We will consider two types of subsets of geodesics: boxes and transversals.

**Definition 2.2 (Box of geodesics).** Let $I_{a,b}$ denote a *generalized ordered interval* in $\partial \tilde{X}$, i.e., any of the following non-empty and possibly degenerate intervals: $(a,b), [a,b), (a,b], (a,b)$, where $a, b$ are ordered counter-clockwise in $\partial \tilde{X}$.

We define a *box of geodesics* as any subset of $G(\tilde{X})$ of the type $B = I_{a,b} \times I_{c,d}$. Let $\mathcal{B}$ denote the family of all boxes of geodesics.

**Definition 2.3 (Transversal of geodesics).** Given a geodesic segment $\tau$ in $\tilde{X}$ (which could be degenerate), $G(\tau)$ denotes the subset of geodesics in $G(\tilde{X})$ intersecting $\tau$ transversely, i.e., all $\gamma \in G(\tilde{X})$ so that $\gamma \cap \tau = \{p\}$, where $p \in \tau$. Given any subset $A$ of $\tilde{X}$ which contains at least one non-degenerate geodesic segment, $G(A)$ denotes the set of geodesics $\gamma \in G(\tilde{X})$ intersecting transversely at least one non-degenerate geodesic segment $\tau \subseteq A$. We will refer to these sets $G(\tau)$ and $G(A)$ as *transversals*.

Note that as subsets of geodesics, a transversal $G(\tau)$, for $\tau$ a geodesic segment, is not contained in one single box of geodesics, but it can be contained in a union of two boxes. Also, if $\tau$ is a non-degenerate geodesic segment, then $G(\tau)$ it contains a non-degenerate box of geodesics.

Geodesic currents provide a unifying generalization of notions such as simple closed curves, hyperbolic structures on $X$, and geodesic laminations, among many others. For an introduction on the subject we refer to the seminal paper [Bon88] by Bonahon.

Denote by $\mathcal{I} \subset G(\tilde{X}) \times G(\tilde{X})$ the open set consisting of pairs of transversely intersecting geodesics in $\tilde{X}$. We endow $\mathcal{I}$ with the subspace topology from $G(\tilde{X}) \times G(\tilde{X})$. Notice that the diagonal $\Gamma$-action on $G(\tilde{X}) \times G(\tilde{X})$ descends to a free, properly discontinuous and cocompact action on $\mathcal{I}$ [ES22, Page 44], and hence the projection $\pi: \mathcal{I} \rightarrow \mathcal{I}/\Gamma$ is a topological covering.

Given two geodesic currents $\mu, \nu \in \text{Curr}(X)$, they induce a $\Gamma$-invariant product measure $\mu \times \nu$ on $G(\tilde{X}) \times G(\tilde{X})$, and hence on $\mathcal{I}$. This measure descends to a measure on $\mathcal{I}/\Gamma$ via the covering $\pi: \mathcal{I} \rightarrow \mathcal{I}/\Gamma$. We will still indicate such measure with $\mu \times \nu$, for notation simplicity.

**Definition 2.4 (Intersection form).** The *intersection form* evaluated on the two currents $\mu, \nu \in \text{Curr}(X)$ is the total volume of $\mathcal{I}/\Gamma$ in the measure $\mu \times \nu$

$$i(\mu, \nu) := (\mu \times \nu)(\mathcal{I}/\Gamma)$$
2.2. **Boundaries and filling currents.** Although the reader can assume that $X$ is a closed hyperbolic surface throughout the text, all our results hold true also if $X$ is compact with boundary. Moreover, when working with dual spaces, we will need to refer to subsurfaces with boundary. Given a compact hyperbolic surface $X$ with geodesic boundary, the space of *internal geodesic currents* is the subspace $\text{Curr}_0(X) \subseteq \text{Curr}(X)$ consisting of currents not supported on lifts of boundary geodesics. If $\mu$ is only supported on lifts of boundary parallel geodesics, we say it is a *boundary geodesic current*. By the definition of intersection number of geodesic currents (see also [ES22, Exercise 3.11]), $i(\mu, \nu) = 0$ for every non-trivial $\nu \in \text{Curr}(X)$ if and only if $\mu$ is a boundary geodesic current. In fact, any geodesic current can be written uniquely as a sum of a boundary current and an internal current, the subspace of boundary currents is closed, and the subspace of internal currents is dense in $\text{Curr}(X)$ (see the proof of all these claims in [EM18a, Lemma 2.12]). We define a *filling geodesic current* as any internal current $\mu$ so that $i(\mu, \nu) > 0$ for every non-trivial internal current $\nu$.

Let $p: \tilde{X} \to X$ denote the universal covering projection, $Y$ a subsurface of $X$ with totally geodesic boundary, and suppose that $\mu$ is a geodesic current on $X$ so that $p(\text{supp}(\mu)) \subset Y - \partial Y$. We will say $\mu$ is *filling in a subsurface $Y$ of $X$* if for every non-trivial current $\nu \in \text{Curr}(X)$ so that $p(\text{supp}(\mu)) \subset Y - \partial Y$, we have $i(\mu, \nu) > 0$. An example of a filling geodesic current on $X$ is a *filling multi-curve*, i.e., a multi-curve $c$, so that $X - c$ is a disjoint union of topological disks and once-punctured disks. In fact, a multi-curve $c$ is a filling multi-curve if and only if its associated geodesic current is filling in the sense of geodesic currents [ES22, Exercise 3.13]. Another example, if $X$ is closed, is the Liouville current (see 2.10).

2.3. **Examples of geodesic currents.** In this section we recall some examples of geodesic currents that will appear in the forthcoming sections.

2.3.1. **Measured laminations.** A *measured geodesic lamination* $(\lambda, \Lambda)$ is a geodesic lamination $\Lambda$ together with a transverse measure $\lambda$. We will refer to it simply as $\lambda$. See Definition A.1 for details.

It is a well-known fact (see [Bon88, Proposition 17]) that measured laminations can be embedded into the space of geodesic currents. We collect the result here and sketch how the relation goes.

**Proposition 2.5** ([AL17, Lemma 4.4]). *Given a measured lamination $(\Lambda, \nu)$, there is an associated geodesic current $\eta_\nu$, and the map sending $\nu$ to $\eta_\nu$ is an embedding into geodesic currents, when the space of measured laminations is equipped with the weak*-topology of measured laminations and $\text{Curr}(X)$ with the weak*-topology of geodesic currents.*
Let us recall how one obtains a measured lamination \((\Lambda, \nu)\) from such a geodesic current \(\eta\). Since \(i(\eta, \eta) = 0\), follows that the support of \(\eta\) is consists of disjoint geodesics. Since the support is closed, we can put \(\Lambda := \text{supp}(\eta)\). Now we construct the transverse measure. Let \(J\) be a transverse arc to the lamination \(\Lambda\). We define \(\nu(J)\) to be the \(\eta\)-measure in \(G(\tilde{X})\) of the set of geodesics of the lamination which intersect \(J\) transversely. On the other hand, given a measured lamination \((\Lambda, \nu)\), we obtain a geodesic current as follows. Lift \((\Lambda, \nu)\) to the universal cover \(\tilde{X}\) obtaining a lamination \(\tilde{\Lambda}\) on \(\tilde{X}\) and a \(\Gamma\)-invariant transverse measure \(\tilde{\nu}\). For every box of geodesics \(B\), there exists a geodesic arc \(\gamma\) so that \(\gamma \in B\) if and only if \(\gamma\) intersects \(J\) transversely. Let \(\eta(B) := \nu(J)\) (see [Mar16, Proposition 8.3.7]).

In fact, the image of the above embeddings is characterized as those geodesic currents \(\eta\) so that \(i(\alpha, \alpha) = 0\).

**Proposition 2.6** ([Bon88, Proposition 14]). The image of the embedding \((\Lambda, \mu) \mapsto \eta_\mu\) consists of geodesic currents \(\eta\) so that \(i(\alpha, \alpha) = 0\).

When \(X\) has boundary, there are multiple types of measured laminations one can consider (see [PH92, 1.8], [Kap09, Chapter 11] for several treatments). In this paper, we will only consider measured laminations whose associated geodesic currents are internal currents, so they are in \(\text{Curr}_0(X)\). When working in a subsurface \(Y \subset X\), we will only consider internal measured laminations within that subsurface, in the sense that, for \(p: \tilde{X} \to X\) the universal covering projection, we have \(p(\text{supp}(\mu)) \subset Y - \partial Y\).

We say that a measured lamination is **discrete** if it is a simple multi-curve, i.e., all the leaves of the support of the lamination \(\Lambda\) are simple closed geodesics. We say it is a **non-discrete measured lamination** otherwise. Non-discrete measured laminations are equivalent to type 2 subcurrents in the structural decomposition theorem for geodesic currents (see Section 7). For any such measured lamination \(\lambda\), there is a minimal (with respect to inclusion) subsurface \(Y\) of \(X\) that contains \(p(\text{supp}(\lambda))\), and so that for every internal closed curve \(c\) in \(Y\), we have \(i(\lambda, c) > 0\). Some authors choose to call these measured laminations “filling”, but that would clash with our choice of “filling” for geodesic currents. Observe that a measured lamination is never filling in the sense of geodesic currents, since \(i(\lambda, \lambda) = 0\).

**2.3.2. Weighted multi-curves.** Given a weighted multi-curve \(c = \lambda_1c_1 + \cdots + \lambda_n c_n\), the corresponding \(\Gamma\)-orbit in \(G(\tilde{X})\) is discrete. The geodesic current corresponding to \(c\) is the weighted Dirac measure on \(G(\tilde{X})\) supported on the orbit \(\Gamma \tilde{c} \subseteq G(\tilde{X})\).

In fact, a geodesic current \(\mu\) has atoms as a measure if and only if \(\mu = \nu + \delta\), where \(\delta\) is a non-trivial weighted multi-curve. First, we consider the case of “0-dimensional atoms”, i.e., atoms whose topological dimension in the space \(G(\tilde{X})\) is 0.
Lemma 2.7 ([Mar16, Proposition 8.2.7]). Let \( \mu \in \text{Curr}(X) \). If \( \mu(\{\gamma\}) > 0 \) for \( \gamma \in \mathcal{G}(\tilde{X}) \), then \( \gamma \) is a lift of a closed geodesic.

In fact, 1-dimensional atoms also come from weighted multi-curves. We prove the following characterization.

Lemma 2.8. A geodesic current \( \mu \) has an atom if and only if there exists \( z \in \partial X \) so that the pencil \( P(z) \) of geodesics at \( z \), satisfies \( \mu(P(z)) > 0 \).

Proof. If \( \mu \) has an atom \( \gamma \), then \( \mu(P(\gamma^+)) > 0 \). On the other hand, suppose that \( \mu \) has no atoms, but \( \mu(P(z)) > 0 \) for some \( z \in \partial X \). Then, by [Mar16, Proposition 8.2.8], there exists a closed geodesic \( c \) so that \( z = \gamma_+ \). Let \( P(z, [x, y]) \) be the set of geodesics with one endpoint at \( z \) and the other within \( [x, y] \subset \partial X \), where we assume that \( \gamma_- \in [x, y] \). Note that, by the north-south dynamics of \( \gamma \), \( P(z) = \bigcup_{n \geq 0} \gamma^n P(z, [x, y]) = \gamma^n P(z, [x, y]) \). Then, taking measures, by \( \pi_1(X) \)-invariance, we have \( \mu(P(z)) = \mu(\gamma^n P(z, [x, y])) = \mu(P(z, [x, y])) \). Thus, by assumption it follows \( \mu(P(z, [x, y])) > 0 \).

On the other hand, \( \{(\gamma_+, \gamma_-)\} = \bigcap_{n \geq 0} \gamma^{-n} P(z, [x, y]) \). Thus, by continuity of measures from below ([Hal50, Theorem D], \( \mu(\gamma) = \lim_n \mu(\gamma^{-n} P(z, [x, y])) \). By assumption, \( \mu(\gamma) = 0 \), and by \( \pi_1(X) \)-invariance, \( \lim_n \mu(\gamma^{-n} P(z, [x, y])) = \mu(P(z, [x, y])) \). Thus, it follows \( \mu(P(z, [x, y])) = 0 \), a contradiction. \( \square \)

Finally, any geodesic current can be approximated by a sequence of weighted multi-curves.

Proposition 2.9 ([Bon86, Proposition 4.4]). The subset of geodesic currents coming from weighted multi-curves is dense with respect to the weak*-topology of currents.

We discuss the weak*-topology in Subsection 2.4.

2.3.3. Liouville current.

Example 2.10. Given a box \( B = [a, b] \times [c, d] \subset \mathcal{G}(\tilde{X}) \), the Liouville current can be explicitly defined as follows. Consider the hyperbolic cross ratio on \( \mathbb{H}^2 \), defined by taking, for any box of geodesics \( B = [a, b] \times [c, d] \in \mathcal{G}(\tilde{X}) \), the expression

\[
L(B) = \left\lvert \log \frac{|a-c||b-d|}{|a-d||b-c|} \right\rvert.
\]

Let \( [(Y, \varphi)] \in \text{Teich}(X) \), where \( Y \) is some hyperbolic structure on \( X \), a priori distinct from \( X \). Since \( Y \) is a hyperbolic structure, we have a \( \pi_1(Y) \)-invariant isometry \( I: \mathbb{H}^2 \to \tilde{Y} \). We consider the following measure on \( \mathcal{G}(\tilde{Y}) \), given by \( \mathcal{L}_Y := I_*(L) \).

From the definition of \( \mathcal{L}_Y \) and the intersection number of geodesic currents, one can check the following property

\[
i(\mathcal{L}_Y, c) = \ell_Y(g)
\]
where $\ell_Y$ denotes the hyperbolic length (see [Bon88, Proposition 14]). In fact, $\mathcal{L}_Y$ is characterized by this property, by [Ota90, Théorème 2]. We will call this the intersection property.

A stronger property of $\mathcal{L}_Y$, which also follows from its definition, and fully characterizes $\mathcal{L}_Y$ (see [Mar16, Proposition 8.1.12]), is the following. Let $x, y \in \tilde{Y}$, then

$$\mathcal{L}_y(G[x, y]) = d_{\tilde{Y}}(x, y).$$

We will call this the Crofton property, since it's a special case of the Crofton formula in integral geometry [San04, 19].

Now, since $\varphi: X \to Y$ is a quasi-conformal marking from the base hyperbolic structure $X$ to another hyperbolic structure $Y$, we can define a geodesic current $\mathcal{L}_X^Y$ in $\text{Curr}(X)$ as follows. Since the marking $\varphi$ induces a $\pi_1(X)$-equivariant homeomorphism $\varphi: \tilde{X} \to \tilde{Y}$, we put

$$\mathcal{L}_X^Y := \varphi^{-1}_* \mathcal{L}_Y = \varphi^{-1}_* \circ I_*(L).$$

The geodesic current $\mathcal{L}_X^Y$ has full support and has no atoms, and it is defined as the Liouville current associated to $[(Y, \varphi)]$ (see [Bon88, Page 145]).

Otal, in [Ota90, Page 155], extended the construction of Liouville current $\mathcal{L}_Y$ to any negatively curved Riemannian metric $Z$ on $X$ (not necessarily of constant curvature $-1$). Otal’s current, $\mathcal{L}_Z$, also satisfies the Crofton property

$$\mathcal{L}_Z(G[x, y]) = \ell_Z([x, y])$$

for any $Z$-geodesic segment $[x, y]$ in $\tilde{Z}$, where here $G[x, y]$ denotes the set of $Z$-geodesics intersecting $[x, y]$ transversely.

### 2.3.4. Geodesic currents coming from Anosov representations.

Let $G$ be a real, connected, non-compact, semisimple, linear Lie group. Let $K$ denote a maximal compact subgroup of $G$, so that $V = G/K$ is the Riemannnian symmetric space of $G$. Let $[P]$ be the conjugacy class of a parabolic subgroup $P \subset G$. Then there is a notion of $[P]$-Anosov representation $\rho: \pi_1(X) \to G$; see, for example, Kassel’s notes [Kas18, Section 4]. When $\text{rank}_{\mathbb{R}}(G) = 1$ there is essentially one class $[P]$, so we can simply refer to them as Anosov representations, and they can be defined as those injective representations $\rho: \pi_1(X) \to G$ where $\Gamma := \rho(\pi_1(X))$ preserves and acts co-compactly on some nonempty convex subset of $X$. Examples of these are Fuchsian and quasi-fuchsian representations. In general rank, the conjugacy classes of parabolic subgroups of $G$ correspond to subsets $\theta$ of the set of restricted simple roots $\Delta$ of $G$. For a given $[P]$-Anosov representation and each $\alpha \in \theta$, one can define a curve functional on oriented curves

$$\ell_\alpha^p: \mathcal{C}(X) \to \mathbb{R}_{\geq 0}$$

by considering the log of the diagonal matrix of eigenvalues of $\rho(g)$ and composing it with $\alpha + i(\alpha), \alpha \in \Delta$ is a root, and $i(\alpha)$ denotes the root obtained by acting
by the negative of the largest element in the Weyl group. See [MZ19, Section 2] for
details. Martone and Zhang show that for a certain subset of Anosov representations
called *positively ratioed* [MZ19, Definition 2.21], there exists a geodesic current $\mu_\rho$ so that
\[ i(\mu_\rho, [g]) = \ell_\alpha^\rho(g) \]
for all $g \in \pi_1(X)$. The construction goes through interpreting geodesic currents as
generalized positive cross-ratios, an observation that was already used by Hamend-
staedt [Ham97, Lemma 1.10], [Ham99, Section 2]. This class includes two types
of representations of interest: Hitchin representations and maximal representations.
In this paper we will only consider *Hitchin representations*, i.e., a representation
$\rho: \pi_1(X) \to SL(n, \mathbb{R})$ which may be continuously deformed to a composition of the
irreducible representation of $PSL(2, \mathbb{R})$ into $PSL(n, \mathbb{R})$ with a discrete faithful repre-
sentation of $\pi_1(X)$ into $PSL(2, \mathbb{R})$.

Continuity of the cross-ratio is crucial in Martone-Zhang’s construction of positive
cross-ratios. From the geodesic current viewpoint, it translates into the fact that their
associated geodesic currents have no atoms. In fact, the following can be extracted
from [MZ19, Page 17]).

**Lemma 2.11.** For $\rho: \pi_1(X) \to G$ a positively ratioed Anosov representation, the
associated geodesic current $\mu_\rho$ is non-atomic and has full support.

Recently, Burger-Iozzi-Parreau-Pozzeti [BIPP21b, Proposition 4.3] have lifted the
continuity assumption in the generalized cross-ratio, thus extending the construction
of such currents beyond positively ratioed representations: see [BP21a] and [BP21b].
Their associated currents can, in general, have atoms.

In Subsection 5.1.1, we discuss in more detail the case of Hitchin representations
for $SL(3, \mathbb{R})$, their connection to convex projective structures and their associated
dual spaces.

2.4. *Weak*-topology of currents. As a space of Radon measures on $G(\tilde{X})$, it is
natural to endow the space of geodesic currents $\text{Curr}(X)$ with the *weak*-topology on
geodesic currents, defined by the family of semi-norms
\[ |\alpha|_\xi = \int_{G(\tilde{X})} \xi \alpha \]
for $\alpha \in \text{Curr}(X)$, as $\xi$ ranges over all continuous function $\xi: G(\tilde{X}) \to \mathbb{R}$ with com-
 pact support. $\text{Curr}(X)$ is second countable and completely metrizable (see [ES22,
Proposition A.9]). Thus, the topology can be specified via sequential convergence.

The intersection number $i: \text{Curr}(X) \times \text{Curr}(X) \to \mathbb{R}$ is continuous with respect
to this topology (see [Bon86, Proposition 4.5]).
In fact, the weak* topology coincides with the topology of intersection numbers, by [DLR10, Theorem 11], which essentially follows by work of Otal in [Ota90, Théorème 2].

**Theorem 2.12.** A sequence of geodesic currents \((\mu_i)\) converges \(\mu_i \to \mu\) in the weak* topology if and only if \(i(\mu_i, c) \to i(\mu, c)\).

In Section 10 we will use the following family of basis of neighborhoods for the weak* topology of geodesic currents.

**Proposition 2.13.** Denote by \(W_{\tilde{X}}\) the family of sets

\[
W(C, \mu, \varepsilon) = \{\nu \in \text{Curr}(X) : |\nu(G[\bar{x}, \bar{y}]) - \mu(G[\bar{x}, \bar{y}])| < \varepsilon : \bar{x}, \bar{y} \in C\}
\]

as \(C \subset \tilde{X}\) ranges over all finite subsets so that \(\mu(\partial G(\bar{x}, \bar{y})) = 0\) and \(\nu(\partial G(\bar{x}, \bar{y})) = 0\) for all currents \(\nu \in \text{Curr}(X)\), for all \(\bar{x}, \bar{y} \in C\). Then \(W_{\tilde{X}}\) is a subbasis generating the weak* topology in \(\mathcal{G}(\tilde{X})\).

**Proof.** We relate this family of neighborhoods to the topology generated by flow boxes for geodesic currents as in measures on the projective tangent bundle \(PT(X)\) of \(X\), as described, for example, in [AL17, Lemma 3.4.4] or [Bon86]. We note that, in the flow box topology of \(PT(X)\), an \(H\)-shape \((\tau_L, \gamma, \tau_R)\) is determined by a pair of arcs \(\tau_L, \tau_R\) on \(X\), that we will furthermore assume to be geodesic, as well as another geodesic segment \(\gamma\) on \(X\), transverse to both \(\tau_L\) and \(\tau_R\), with one endpoint on the first and the other in the second. An \(H\)-shape then consists of all geodesic arcs on \(X\) homotopic to \(\gamma\) and transverse to \(\tau_L\) and \(\tau_R\). An \(H\)-shape \(H\) defines a subset of \(PT(X)\) by considering \(B_H\), the set of lifts to \(PT(X)\) of geodesic segments on \(H\). A lift of \(B_H\) to \(PT(\tilde{X})\) is then given by the set of lifts to \(PT(\tilde{X})\) of \(B_H\), i.e., a set of geodesic segments on \(\tilde{X}\) with endpoints on lifts \(\bar{\tau}_L\) and \(\bar{\tau}_R\). These lifts can be uniquely extended to bi-infinite geodesics, obtaining \(G(\tau_L, \tau_R)\), the set of geodesics intersecting both \(\tau_L\) and \(\tau_R\). This is the set of bi-infinite geodesics \(G_H = G(\tau_L) \cap G(\tau_R) \subset \mathcal{G}(\tilde{X})\).

By [Bon86, Lemma 4.3], it follows that the family of sets

\[
W(\mu, G_H, \varepsilon) = \{|\nu(G_H) - \mu(G_H)| < \varepsilon : G_H \subset \mathcal{G}(\tilde{X})\}
\]

as \(\varepsilon > 0\) ranges over all positive values, and \(\mathcal{G}\) ranges over all finite sets of geodesics \(G_H\) determined by flow boxes \(B_H\) so that \(\nu(\partial B_H) = 0\) for all geodesic currents \(\nu \in \text{Curr}(X)\), is a subbasis of the weak*-topology. This condition can be phrased in terms of the set of geodesics \(G_H\) by saying that \(\nu(\partial G_H) = 0\). We finally note that \(W(C, \mu, \varepsilon)\) is a finite intersection of sets of type \(W(\mu, G_H, \varepsilon)\), and thus also a neighborhood. \(\square\)
2.5. **Systole of a geodesic current.** Given a geodesic current $\mu$ on $X$, we define the *systole of $\mu$* as

$$\text{sys}(\mu) := \inf \{ i(\mu, c) : c \in \text{Curves}(X) \}$$

We point out that, as a function on geodesic currents with the weak$^*$-topology, sys is a continuous function (see [BIPP21a, Corollary 1.5(1)]).

Given a subsurface $Y$ of $X$ with totally geodesic boundary and $\mu$ a geodesic current with $\text{supp}(\mu) \subset Y$, we define the *systole of $\mu$ relative to $Y$*, as follows,

$$\text{sys}_Y(\mu) := \inf \{ i(\mu, c) : c \in \text{Curves}(Y - \partial Y) \}.$$ 

### 3. Dual space of a geodesic current

In this section we define and prove the basic properties of the current dual.

We start by recalling some facts about pseudo-metric spaces. A pseudo-metric on a set $Y$ is a map $d: Y \times Y \to \mathbb{R}$ satisfying the symmetry and triangle inequalities. Points $x \neq y$ with $d(x, y) = 0$ are allowed. A pseudo-metric space $Y$ with pseudo-metric $d$ has a canonical metric quotient $Y/\sim$. It is given by the equivalence classes for the equivalence relation identifying $x$ and $y$ in $Y$ if and only if $d(x, y) = 0$, and endowed with the metric $\overline{d}(x, y) := d(x, y)$. We call $X/\sim$ equipped with $\overline{d}$ the *metric quotient* of $Y$. In most of this paper, we will let $Y = \tilde{X}$ be the universal cover of the hyperbolic surface $X$ equipped with the pullback hyperbolic metric on $X$, and the pseudo-distance will be defined from a geodesic current as defined below. We will decorate the points in $\tilde{X}$ with an overline, as in $\overline{x}$.

#### 3.1. The dual space of a geodesic current.

A geodesic current $\mu \in \text{Curr}(X)$ induces a pseudo-distance on $\tilde{X}$ given by

$$d_\mu(\overline{x}, \overline{y}) = \frac{1}{2} \left\{ \mu(G[\overline{x}, \overline{y}]) + \mu(G(\overline{x}, \overline{y})) \right\}.$$ 

Note that the pseudo-distance $d_\mu$ is *straight* (see [BIPP21a, Proposition 4.1], in the sense that it is additive on hyperbolic geodesic lines. Precisely, let $\overline{x}, \overline{y}, \overline{z} \in \gamma$ be three points lying in the order $\overline{x} < \overline{y} < \overline{z}$ on a hyperbolic geodesic $\gamma \subseteq \tilde{X}$, then we have

$$d_\mu(\overline{x}, \overline{z}) = d_\mu(\overline{x}, \overline{y}) + d_\mu(\overline{y}, \overline{z}).$$

A non-straight version of this pseudo-distance was first considered by Glorieux in [Glo17].

**Definition 3.1** (Dual space of a geodesic current). For $\overline{x}, \overline{y} \in \tilde{X}$, consider the equivalence under the pseudo-metric, $\overline{x} \sim \overline{y}$ if and only if $d_\mu(\overline{x}, \overline{y}) = 0$. The metric quotient $X_\mu := \tilde{X}/\sim$ will be called the *dual space of the geodesic current* $\mu$. 

When $\mu$ is a measured lamination, then it is known ([MS91]) that $X_\mu$ is a 0-hyperbolic space. Hence, it can be isometrically embedded in a unique $\mathbb{R}$-tree $\tilde{X}_\mu$ (see also Appendix A). It follows that $X_\mu$ can be endowed with a geodesic structure via such embedding $X_\mu \hookrightarrow \tilde{X}_\mu$.

**Remark 3.2.** In the remaining of the paper, when $\mu$ is a measured lamination, we will often denote $\tilde{X}_\mu$ simply with $T(\mu)$, to emphasise that it is an $\mathbb{R}$-tree.

3.2. **Geodesic structure.** In this subsection we explore how to define a geodesic structure on $X_\mu$ when $\mu$ is not necessarily a measured lamination, i.e. when the support of $\mu$ is allowed to have intersections. In particular, we will construct an isometric embedding $X_\mu \hookrightarrow \tilde{X}_\mu$ when $\mu$ is a purely atomic multi-curve with self intersections, and when $\mu$ has no atoms at all. Finally, we will suggest possible strategies for the mixed case, i.e., when the current has both atomic and non-atomic parts, that we will fully flesh out in a sequel to this project with Anne Parreau.

3.2.1. **Purely atomic case.** Let now $\mu$ be a discrete current, i.e. a current whose support consists of the lifts of a collection of closed geodesics, not necessarily disjoint. In this case, the dual space $X_\mu$ is in general not an $\mathbb{R}$-tree, but it can still be isometrically embedded in a graph, hence in a geodesic space.

Let $v_1, v_2 \in X_\mu$. They correspond to two regions $R_1 = \pi_\mu^{-1}(v_1), R_2 = \pi_\mu^{-1}(v_1)$ of $\tilde{X}$, where $\pi_\mu: \tilde{X} \to X_\mu$ denotes the natural projection $\pi_\mu: \tilde{X} \to X_\mu$.

We say that $v_1$ and $v_2$ are **adjacent** if there exists a hyperbolic geodesic arc $\gamma$ from the region $R_1$ to the region $R_2$ which intersects transversely the support of $\mu$ only in one geodesic.

If $v_1$ and $v_2$ are adjacent, we add an edge $e$ between them of length $i(\mu, c)$. This embeds $X_\mu$ isometrically into a connected graph $\tilde{X}_\mu$, and hence a geodesic space. The following illustrates a concrete example.

**Example 3.3.** Consider the 1-punctured torus $X = X_{1,1}$ and the current $\mu = \alpha + \beta$ whose support is given by the two intersecting simple closed geodesics $\alpha$ and $\beta$.

In order to visualize the dual space $X_\mu$ we lift $\alpha$ and $\beta$ in $\tilde{X}$. Looking at the regions in Figure 3.1, we see that the regions $A$ and $B$ are adjacent, while the regions $A$ and $C$ are not. The graph in which we embed $X_\mu$ is obtained by adding an edge between points of $X_\mu$ corresponding to adjacent regions, as in Figure 3.2.

3.2.2. **Non-atomic case.** Assuming $\mu$ has no atoms, we will show $X_\mu$ is a geodesic space.
Figure 3.1. The orbits $\Gamma(\alpha_-, \alpha_+)$ and $\Gamma(\beta_-, \beta_+)$ are denoted in green and yellow, respectively. The support of $\mu$ is precisely the union of the two orbits.

Figure 3.2. A sketch of $\hat{X}_\mu$ when $\mu$ is a union of two intersecting simple closed curves in a once-punctured torus.

A metric space $(X, d)$ is called Menger convex if, for every $x, y \in X$, there exists $z \in X$ so that $d(x, z) = d(y, z) = \frac{1}{2}d(x, y)$. The following Lemma can be found in [Pap14, Theorem 2.6.2].
Lemma 3.4. Let $X$ be a proper metric space. $X$ is geodesic if and only if it is Menger convex.

Proposition 3.5. If $\mu$ is a geodesic current without atoms, then $X_\mu$ is a geodesic metric space.

Proof. Since $\mu$ has no atoms, by Proposition 4.12, $\pi_\mu$ is continuous. Furthermore, by Proposition 8.13, $X_\mu$ is proper. Thus, it suffices to check Menger convexity, by Lemma 3.4. Given any two points $x,y \in X_\mu$, let $\overline{x} \in \pi_\mu^{-1}(x)$ and $\overline{y} \in \pi_\mu^{-1}(y)$, and let $I$ be the hyperbolic geodesic segment connecting $\overline{x}$ and $\overline{y}$. $\pi_\mu(I)$ is connected, and thus there exists $z \in I$ so that $z = \pi_\mu(\overline{z})$ satisfies the condition of the statement. □

It would be interesting to see which conditions to impose on $\mu$ in order to obtain sharper convexity properties. We say that a $\delta$-hyperbolic with midpoints is convex if, for every triple of points $p,q,r$, if $m_1$ denotes the midpoint between $p$ and $q$ and $m_2$ denotes the midpoint between $p$ and $r$, then we have $d(m_1, m_2) \leq \frac{1}{2}d(q, r)$. What are conditions on $\mu$ ensuring that $X_\mu$ is convex? Convexity properties of this type (and stronger) are useful to guarantee sequential pre-compactness in the setting of the equivariant Gromov-Hausdorff topology (see Section 10 and [Pau88]).

3.3. Mixed case. If a geodesic current $\mu$ decomposes as a sum of a non-trivial non-atomic current and a non-trivial multi-curve with intersecting supports, then equipping $X_\mu$ with a geodesic structure is more subtle. We will explore this case further in a subsequent paper. We outline some possible strategies, here. One approach could consist of getting rid of atoms via a so-called blow-up construction (see [Kap09, Section 11.12] for the case of currents corresponding to measured laminations). However, this approach gets into problems as soon as the geodesics in the support of $\mu$ are allowed to intersect. It would be interesting to understand what happens when the geodesic currents have atoms. In a sequel to this project with Anne Parreau, we are exploring the construction of an isometric embedding of $X_\mu$ into a geodesic blow-up space $\widehat{X}_\mu$, so that: (1) $\widehat{X}_\mu$ is homeomorphic to a surface, even if $\mu$ has atoms and (2) if $\mu$ has no atoms, then $X_\mu$ is isometric to $\widehat{X}_\mu$.

We also refer the reader to [BIPP21c, Definition 3.4] for an example of the blow-up construction in the case of a filling sum of two measured laminations.

We note also note that the construction of embedding the $0$-hyperbolic $X_\lambda$ for $\lambda$ a measured lamination into its $\mathbb{R}$-tree $\widehat{X}_\lambda$ is a specific example of a more general phenomenon, the so-called injective hull (see [Lan13, Proposition 1.3]). We thank Eduardo Oregón-Reyes for pointing out this notion to us.

Definition 3.6 (Injective hull). There exists a functor, called injective envelope, in the category of Gromov hyperbolic spaces, assigning, to every $\delta$-hyperbolic space $(X,d)$, another $\delta$-hyperbolic Gromov space $(\hat{X},\hat{d})$ (with the same $\delta$-hyperbolicity
constant) which is moreover geodesic, and there is an isometric embedding \( \iota : X \to \hat{X} \). Moreover, every isometry of \( X \) extends uniquely to an isometry of \( \hat{X} \). We will call \((\hat{X}, \hat{d})\) the injective hull of \((X, d)\).

In principle, one could define the geometric realization of \( X_\mu \) to be its injective hull, however, one would need to check if the injective hull construction preserves the tree graded structure in the sense of Section 7 (see Remark 7.9 for further comments on this point).

Finally, we comment that a priori one should expect the geometric realization to be dependent on the choice of hyperbolic structure \( X \) on \( S \). For a fixed (unweighted) multi-curve \( \mu \), however, the dual spaces \( X_\mu \), for different choices of hyperbolic structure \( X \), embed as 2-dimensional faces of a \( \text{CAT}(0) \)-cube complex \( \mathcal{S}(\mu) \) independent on the choice of hyperbolic structure: the Sageev complex (see [AG17, 3] for a description in this setting, and [CN05],[Sag95],[Sag14] in bigger generality. It would be interesting to see if there is an analogous construction of such complex for an arbitrary geodesic current.

### 3.4. Dependence on hyperbolic structure

The start by proving the following Lemma, which will be very useful in the sequel.

**Lemma 3.7.** Given a geodesic current \( \mu \), and a homeomorphism \( f : X \to X \), let \( \tilde{f} \) denote the induced homeomorphism on geodesics. Then

\[
\tilde{f}_*^{-1} \mu(G[x, y]) = \mu(G[\tilde{f}(x), \tilde{f}(y)]).
\]

where \( \tilde{f}_* \) denotes the induced pushforward of geodesic currents.

**Proof.** By definition of pushforward of measures, we have

\[
\tilde{f}_*^{-1} \mu(G[x, y]) = \mu(\tilde{f}(G[x, y])).
\]

We show that

\[
\mu(\tilde{f}(G[x, y])) = \mu(G[\tilde{f}(x), \tilde{f}(y)]).
\]

We say that a geodesic \( \gamma \) in \( G(\tilde{X}) \) separates two points \( x, y \in \tilde{X} \), if \( x \) and \( y \) are in different connected components of \( \tilde{X} - \gamma \). Since \( f \) is a homeomorphism, \( \gamma \) in \( G(\tilde{X}) \) separates two points \( x, y \in \tilde{X} \) if and only if \( \tilde{f}(\gamma) \) in \( G(\tilde{X}) \) separates \( \tilde{f}(x), \tilde{f}(y) \in \tilde{X} \). Finally, \( G[x, y] \) is equal to the set of geodesics \( \gamma \) separating \( x \) and \( y \). It then follows

\[
\tilde{f}(G[x, y]) = G[\tilde{f}(x), \tilde{f}(x)].
\]

The claim follows. \( \square \)

To define the space of geodesic currents \( \text{Curr}(X) \), we fixed a hyperbolic structure \( X \). Given a homeomorphism between two different hyperbolic structures \( f : X \to X' \), we get a homeomorphism \( \tilde{f} : \partial \tilde{X} \to \partial \tilde{X}' \) by [CB88, Lemma 3.7] extending
$f$ to the boundary. This induces a homeomorphism between the corresponding spaces of geodesics and, by pushforward of measures, induces a homeomorphism $\tilde{f}_\ast: \text{Curr}(X) \to \text{Curr}(X')$. In particular, this induces an action of the mapping class group on $\text{Curr}(X)$. For any two hyperbolic structures $X$, we have a commutative diagram of bijections

$$
\begin{array}{ccc}
\text{Curr}(X) & \longrightarrow & \mathcal{D}(X) \\
\downarrow & & \downarrow \\
\text{Curr}(X') & \longrightarrow & \mathcal{D}(X').
\end{array}
$$

The left vertical arrow is a homeomorphism from the above discussion. From Theorem 10.4, it will follow that, with respect to the equivariant Gromov-Hausdorff topology on $\mathcal{D}(X)$, the two horizontal maps are homeomorphisms. Thus, we have a commutative diagram of homeomorphisms. Moreover, by Lemma 3.7, the map $\tilde{f}$ induces an isometry between $X_\mu$ and $X'_{\tilde{f}_\ast(\mu)}$.

3.5. Median spaces. We show that dual spaces are examples of measured wall spaces. We start by explaining what median spaces are. For details we commend the reader to, for example, [CD17] or [Fio20].

Definition 3.8 (Chain). Let $(Y,d)$ be a pseudo-metric space. For $a \in Y$, and $r \geq 0$, $B(a,r)$ denotes the closed ball of radius $r$ centered at $a$, i.e., the set $\{b \in Y : d(a,b) \leq r\}$, and given a subset $Z \subset Y$, $N_r(Z)$ is the closed $r$-neighborhood of $Z$ in $Y$, i.e., $\{b \in Y : d(a,Z) \leq r\}$. For $a,b,c \in Y$, we say that $b$ is between $a$ and $c$ if $d(a,b) + d(b,c) = d(a,c)$. Let $I(a,b)$ be the set of points between $a$ and $b$. A chain is a finite sequence of points $(a_1,a_2,\ldots,a_n)$ in $Y$. A straight chain is a chain that satisfies

$$d(a_0,a_n) = d(a_0,a_1) + \cdots + d(a_{n-1},a_n).$$

For example, $(a,b,c)$ is a straight chain if and only if $b \in I(a,b)$.

Definition 3.9 (Median point). Let $a,b,c$ be three points in a pseudo-metric space $(X,d)$. Denote the intersection $I(a,b) \cap I(b,c) \cap I(a,c)$ by $M(a,b,c)$, the set of median points for $a,b,c$.

Definition 3.10 (Median spaces). A median (pseudo-metric) space is a (pseudo-metric) space $X$ so that for any three points $x,y,z$, the set $M(x,y,z)$ is non-empty and so that any two median points are at pseudo-distance 0. For example, if $X$ is a metric space, it is median if and only if $M(a,b,c)$ consists of one and only one median point.
3.6. Measured wall spaces. Given a set $X$, a wall of $X$ is a partition $X = h \cup h^c$ where $h$ is any subset of $X$ and $h^c$ denotes its complement. A collection $\mathcal{H}$ is called a collection of half-spaces if for every $h \in \mathcal{H}$ the complementary subset $h^c$ is also in $\mathcal{H}$. Let $W_\mathcal{H}$ denote the collection of pairs $w = (h, h^c)$ with $h \in \mathcal{H}$. We say that $h$ and $h^c$ are the two half-spaces bounding the wall $w$. We say that a wall $w = (h, h^c)$ separates two disjoint subsets $A, B$ in $X$ if $A \subset h$ and $B \subset h^c$ or vice-versa and denote by $W(A|B)$ the set of walls separating $A$ and $B$. In particular, $W(\emptyset|\emptyset)$ is the set of walls $w = (h, h^c)$ such that $A \subset h$ or $A \subset h^c$, hence $W(\emptyset|\emptyset) = W$. We use $W(x|y)$ to denote $W(\{x\}|\{y\})$.

**Definition 3.11** (Space with measured walls). A space with measured walls is a 4-tuple $(X, W, A, \mu)$ where $W$ is a collection of walls, $A$ is a $\sigma$-algebra of subsets in $W$ and $\mu$ is a measure on $A$ so that for every two points $x, y \in X$, the set of separating walls $W(x|y)$ is in $A$ and has finite measure. We let $d_\mu(x, y) = \mu(W(x|y))$, and we call it the wall-pseudo metric.

**Lemma 3.12.** Given a geodesic current $\mu$, and $d_\mu$ defined as in Definition 3.1, the pseudo-metric space $(\widetilde{X}, d_\mu)$ is a measured wall space.

**Proof.** Let $X = \widetilde{X}$. Let $\mathcal{H}$ denote the set of open and closed half-planes for every geodesic $\ell \in \text{supp } \mu$, and $W$ be the set of walls of $\mathcal{H}$ (i.e., the support of $\mu$). Given a geodesic, there are two pairs of hyperplanes that determine the same wall: one where $h$ is open and $h^c$ closed and viceversa. Thus, given $W \subset W$, the measure on walls $\mu'$ is given by $\mu'(W) = \frac{1}{2}\mu(W)$. This normalization factor is introduced to account for the fact that we get two copies of each geodesic in the support of the current in the space of walls.

Suppose $h$ is closed and $h^c$ is open. If $x \in \partial h$, and $y \in h^c$, we have that the wall $w = (h, h^c)$ separates $x$ and $y$, then the geodesic determined by $w$ is in $G[x]$. Suppose $h$ is open, $h^c$ is closed, $x \in h$ and $y \in \partial h^c$, then $w \in G[y]$. Whenever neither $x$ nor $y$ are in $\partial h$ or $\partial h^c$, then in both cases the two walls are in $G(x, y)$. Thus, we have

$$\mu'(W(x|y)) = \frac{1}{2}\mu(W(x|y)) = \frac{1}{2}\mu(G[x]) + \frac{1}{2}\mu(G[y]) + \mu(G(x, y)).$$

□

It would be interesting to see if an analog of geodesic currents can be defined for $\text{CAT}(0)$ spaces of higher rank, other than the hyperbolic plane (or $\mathbb{H}^n$). Indira Chatterji pointed out to us that in [PSZ22] the notion of curtain has been introduced, which can be seen as a generalization of walls in higher rank.
4. Continuity of the projection

In this section we study the semicontinuity and continuity properties of the natural metric quotient projection map $\pi_\mu : \tilde{X} \to X_\mu$.

4.1. Measure theory results. We recall some measure theory results that will be used in this section. Given a sequence of subsets $(A_n)$, $A_n \subset X$, we have the following definitions:

(4.1) \[
\liminf_n A_n := \bigcup_{n \geq 1} \bigcap_{j \geq n} A_j
\]
and

(4.2) \[
\limsup_n A_n := \bigcap_{n \geq 1} \bigcup_{j \geq n} A_j.
\]

The Morgan laws immediately give the following identity

(4.3) \[
\liminf_n A_n = \left( \limsup_n A_n^c \right)^c.
\]

The following result is standard and can be found, for example, in [Hal50, Theorem D] and [Hal50, Theorem E].

**Lemma 4.4.** If $(A_n)$ is a sequence of measurable sets in $X$, so that $A_{n+1} \subseteq A_n$, then for any measure $\mu$ on $X$, we have

\[
\lim n \mu(A_n) = \mu(\cap_n A_n).
\]

This property is called continuity of measures from below. If $(A_n)$ is a sequence of measurable sets in $X$, so that $A_n \subseteq A_{n+1}$ so that, for some $N > 0$, $\mu(A_n) < \infty$ for $n \geq N$, then for any measure $\mu$ on $X$, we have

\[
\lim n \mu(A_n) = \mu(\cup_n A_n).
\]

This property is called continuity of measures from above.

**Lemma 4.5.** Let $\mu$ be a (non-necessarily finite) measure on a measurable space $X$, and let $(A_n)$ be a sequence of measurable sets. Then

\[
\liminf_n \mu(A_n) \geq \mu(\liminf_n A_n).
\]

Moreover, if for some $n$, $\mu(\cup_{j \geq n} A_j)$ is finite, we have

\[
\limsup_n \mu(A_n) \leq \mu(\limsup_n A_n).
\]
Proof. For the first inequality, let $B_n := \cap_{j \geq n} A_j$. Note that $B_n \subset B_{n+1}$, and thus by continuity of measures from below 4.4, we get
\[
\mu(\lim_n B_n) = \lim_n \mu(B_n).
\]
Note that
\[
\lim_n B_n = \cup_{n \geq 1} B_n = \lim \inf N \geq 1 \cap n \geq N A_n.
\]
Since $\cap_{j \geq n} A_n \subset A_k$ for $k \geq n$, we have
\[
\mu(\cap_{j \geq n} A_n) \subset \inf \{\mu(A_k) : k \geq n\}.
\]
Thus, letting $n \to \infty$, we get
\[
\mu(\lim \inf A_n) \leq \lim \inf \mu(A_n).
\]
The other inequality follows from continuity of measures from above 4.4, where the hypothesis of $\mu(\cup_{j \geq n} A_j)$ being finite for some $n$ is crucial.

Proposition 4.6. If $\mu$ is a geodesic current without atoms, then the function
\[
f(\cdot, \cdot) := d_{\mu}(\pi_{\mu}(\cdot), \pi_{\mu}(\cdot)) : \tilde{X} \times \tilde{X} \to \mathbb{R}
\]
is a continuous function.

Proof. Let $((x_n, y_n))$ be a sequence in $\tilde{X} \times \tilde{X}$ converging to $(x, y)$. We first show that
\[
\lim_n d_{\mu}(\pi_{\mu}(x_n), \pi_{\mu}(y_n)) \geq d_{\mu}(\pi_{\mu}(x), \pi_{\mu}(y))
\]
Note that, since $\mu$ has no atoms, we have
\[
d_{\mu}(\pi_{\mu}(x_n), \pi_{\mu}(y_n)) = \mu(G(x_n, y_n)).
\]
for all $n$.

Claim 4.8. If $\gamma \in G(x, y)$, then $\gamma \in \cup_{N \geq 1} \cap_{n \geq N} G(x_n, y_n)$.

Proof. Since $\gamma \in G(x, y)$, $\gamma$ intersects $(x, y)$ transversely at some $z \in (x, y)$. Since $(x_n, y_n)$ is converging in Hausdorff distance to $(x, y)$, we have that for $n_0$ large enough, $\gamma$ must intersect transversely all $(x_n, y_n)$ for all $n \geq n_0$.

From Claim 4.8 and Lemma 4.5, it follows that
\[
\mu(G(x, y)) \leq \mu(\cup_{N \geq 1} \cap_{n \geq N} G(x_n, y_n)) \leq \lim \inf \mu(G(x_n, y_n))
\]
from which Equation 4.7 follows. We will now prove
\[
\lim sup_n d_{\mu}(\pi_{\mu}(x_n), \pi_{\mu}(y_n)) \leq d_{\mu}(\pi_{\mu}(x), \pi_{\mu}(y)).
\]
Claim 4.10. If $\gamma$ is a geodesic disjoint from $(x, y)$, then $\gamma \notin \lim sup_n G(x_n, y_n)$. 

Proof. Let $A_n = G(x_n, y_n)$. We are assuming $\gamma$ is a geodesic disjoint from $(x, y)$, and we want to show $\gamma \notin \limsup_n A_n$, which, by Equation 4.3 is equivalent to showing $\gamma \in \liminf_n A_n^c$, i.e., to showing that there exists $N > 0$ so that for all $n \geq N$, $\gamma \notin A_n$. This follows, since $\gamma$ and $(\bar{x}, \bar{y})$ are a definite hyperbolic distance apart, and $(\bar{x}_n, \bar{y}_n)$ is converging to $(\bar{x}, \bar{y})$ in the Hausdorff hyperbolic distance of $\bar{X}$. Thus, for $n_0$ large enough, $(\bar{x}_n, \bar{y}_n)$ is also disjoint from $\gamma$, for all $n \geq n_0$.

A geodesic which is not in $G(\bar{x}, \bar{y})$ is either disjoint from $(\bar{x}, \bar{y})$, or it is the geodesic $\gamma_{x,y}$ determined by the points $x$ and $y$. Claim 4.10 can be phrased as follows:

$$\limsup_n A_n \subset G(\bar{x}, \bar{y}) - \{\gamma_{x,y}\}.$$ 

Since $\mu$ has no atoms, we have

$$\limsup_n \mu(A_n) \leq \mu(\limsup_n A_n) \leq \mu(G(\bar{x}, \bar{y}) - \{\gamma_{x,y}\}) = \mu(G(\bar{x}, \bar{y})), $$

where we have used the second equation in Lemma 4.5 in the first inequality, and the last equation plus the monotonicity of $\mu$ in the second inequality. It remains to explain why the application of that Lemma is justified. Since $\bar{x}_n \to \bar{x}$ and $\bar{y}_n \to \bar{y}$, there exists $n_0$ so that for $n \geq n_0$, $(\bar{x}_n, \bar{y}_n) \subset B$ for some compact ball $B$. Then

$$\cup_{n \geq n_0} A_n \subset G(B).$$

Since $B$ is compact, $\mu(G(B))$ is finite, and thus the application of the Lemma is justified. This shows Equation 4.9 and finishes the proof. \qed

**Proposition 4.11.** Let $\mu \in \text{Curr}(X)$. If $\mu$ has atoms, then $f(\cdot, \cdot)$ is neither lower semicontinuous nor upper semicontinuous.

**Proof.** Suppose $\mu$ is a geodesic current with atoms. By Lemma 2.7 and Lemma 2.8, the only 0-dimensional and 1-dimensional atoms of a geodesic current are concentrated on lifts of closed geodesics and pencils containing lifts of closed geodesics, respectively. Let then $\gamma$ be a lift of a closed geodesic in the support of $\mu$. Let $[\bar{x}, \bar{y}]$ be a subsegment of $\gamma$, short enough so that no other lift of a closed geodesic intersects it transversely. Let $\bar{y}_n$ be a sequence of points outside of $\gamma$ converging to $\bar{y}$. Then, if we denote $\mu(G[\bar{x}, \bar{y}_n]) = \varepsilon_1$, we have $\varepsilon_1 > 0$ for all $n$, while $\mu(G[\bar{x}, \bar{y}]) = \varepsilon_2$; where $\varepsilon_1 > \varepsilon_2 \geq 0$. This shows that

$$\limsup_n f(\bar{x}_n, \bar{y}_n) > f(\bar{x}, \bar{y})$$

so $f$ is not upper semicontinuous. Observe that this construction can also be adapted for an open interval $(\bar{x}, \bar{y})$, by considering open segments $(\bar{x}_n, \bar{y}_n)$ given by sequences of points $(\bar{x}_n)$ and $(\bar{y}_n)$, where $\bar{x}_n \to \bar{x}$ and $\bar{y}_n \to \bar{y}$ from distinct sides of $\gamma$. This ensures that $(\bar{x}_n, \bar{y}_n)$ crosses $\gamma$ transversely for all $n$, while $\gamma$ does not intersect $(\bar{x}, \bar{y})$ transversely. Let’s show how lower semicontinuity fails. Let $\gamma$ be again a lift of
a closed geodesic in the support of $\mu$. Suppose that $\gamma$ intersects a segment $[x, y]$ transversely at $x$, and let $x_n$ be a sequence converging to $x$ so that $[x_n, y] \subset [x_{n+1}, y]$ for all $n$. Then, $\mu(G[x_n, y]) \leq \varepsilon_1$ for all $n$ and for some $\varepsilon_1 \geq 0$, while, if $\mu(G[x, y]) = \varepsilon_2$, we have that $0 \leq \varepsilon_1 < \varepsilon_2$. This shows that

$$\lim inf_n f(x_n, y_n) < f(x, y),$$

so $f$ is not lower semicontinuous. \qed

Combining both Propositions 4.6 and 4.11, we get the following.

**Proposition 4.12.** Let $\mu \in \text{Curr}(X)$. Then $\pi_\mu : \tilde{X} \rightarrow X_\mu$ is continuous if and only if $\mu$ is has no atoms.

**Proof.** It follows directly from Propositions 4.6 and 4.11. Namely, for $\tilde{x} \in \tilde{X}$, consider the continuous function $g_{\tilde{x}}(\cdot) := f(\tilde{x}, \cdot)$. Since this function is continuous at $\tilde{x}$, for every $\varepsilon > 0$, we can find $\delta_{\varepsilon, \tilde{x}}$ so that if $d_{\tilde{X}}(\tilde{x}, \tilde{y}) < \delta$, we have $g_{\tilde{x}}(\tilde{y}) < \varepsilon$, i.e., $d_\mu(\pi_\mu(\tilde{x}), \pi_\mu(\tilde{y})) < \varepsilon$, which proves continuity of $\pi_\mu$ at $\tilde{x}$. \qed

Note that Proposition 4.1 has the following curious consequence. In general, $f : X \rightarrow \mathbb{R}$ is a continuous function, if and only if $f$ is both upper and lower semicontinuous. However, for $\pi_\mu$ we have the following.

**Proposition 4.13.** $\pi_\mu$ is continuous if and only if it is lower semicontinuous or upper semicontinuous.

In fact, we can combine the above results with the results in Section 8, to obtain the following.

**Proposition 4.14.** Given a geodesic current $\mu$ on $X$, the projection $\pi_\mu : \tilde{X} \rightarrow X_\mu$ satisfies:

1. If $\mu$ has no atoms and no subcurrents of type 2 in its decomposition then $\pi_\mu$ is a $\pi_1(X)$-equivariant closed map.
2. If $\mu$ has no atoms and has full support, then $\pi_\mu$ is a $\pi_1(X)$-equivariant homeomorphism.

**Proof.** Since $\mu$ has no atoms, by Proposition 4.6, $\pi_\mu$ is continuous and by Proposition 8.13, $X_\mu$ is a proper metric space. Therefore, by Lemma 9.6, it is locally compact. By Lemma 8.9 the action of $\pi_1(X)$ on $X_\mu$ is cocompact and by Lemma 8.18, the action is proper. Hence, the quotient $X_\mu/\pi_1(X)$ is Hausdorff (and compact). By $\pi_1(X)$ equivariance of $\pi_\mu$, the descended map $\overline{\pi}_\mu : X \rightarrow X_\mu/\pi_1(X)$ is a well-defined continuous surjective map from the compact surface $X$ to the Hausdorff space $X_\mu/\pi_1(X)$, so it is a closed map. If moreover $\mu$ has full support, then $\pi_\mu$ is injective, and hence it is a homeomorphism. By equivariance, this implies the same results for $\pi_\mu$. \qed
5. **Examples**

In this section we study examples of dual spaces associated to the geodesic currents discussed in Subsection 2.3.

5.1. **Guirardel core: a filling sum of two measured laminations on the surface.** Let $X$ be a compact hyperbolic surface (possibly with boundary), and consider $\alpha, \beta$ two measured laminations so that $\alpha + \beta$ is filling as a geodesic current. For concreteness, the reader might want to assume that $\alpha$ and $\beta$ are simple closed curves, so that the multi-curve $\mu = \alpha \cup \beta$ is filling (this can always be achieved, see [FM12, 1.3.2] for the argument in genus 2).

Given a finitely generated group $G$ and two $\mathbb{R}$-trees $T_1, T_2$ equipped with isometric actions, the Guirardel’s core $C$ is the smallest subset of $T_1 \times T_2$ which is

1. $\pi_1(X)$-invariant
2. closed and connected
3. For every $x_0 \in T_1$ and every $x_1 \in T_1$, both $C \cup \{ x_0 \} \times T_2$ and $T_1 \times \{ x_1 \}$ are convex.

We commend the reader to the paper [Gui05] for more details. When $G$ is the fundamental group of a closed surface, and $T_1, T_2$ are the $\mathbb{R}$-trees $\hat{X}_\alpha$ and $\hat{X}_\beta$, we claim that $X_\mu$ embeds isometrically into the Guirardel’s core $C$ of the product of trees $\hat{X}_\alpha$ and $\hat{X}_\beta$. In this case, the core can be described as follows. Let $\pi_\alpha: \hat{X} \to \hat{X}_\alpha$ and $\pi_\beta: \hat{X} \to \hat{X}_\beta$ be the composition of the natural projection maps with the isometric embedding of the dual into their $\mathbb{R}$-tree, and define $f: \hat{X} \to \hat{X}_\alpha \times \hat{X}_\beta$ by $(\pi_\alpha(x), \pi_\beta(x))$. Guirardel’s core corresponds to $C = f(\hat{X})$. We define a map from $X_\mu$ to $C$, by, to every $x \in X_\mu$, picking a $\bar{x} \in \pi^{-1}_\mu(x)$, and defining $\bar{f}(x) := f(\bar{x})$. We show that $\bar{f}$ is well-defined. Indeed, suppose that $x \sim y$, i.e., $d_\mu(x, y) = 0$. This happens in the following cases:

1. $x, y$ are the same point
2. $x, y$ are in the same complementary region of $\mu$.
3. $x, y$ are in the same lift of $\alpha$ and not separated by any lift of $\beta$, or on the same lift of $\beta$ and not separated by any lift of $\alpha$.

In any of these cases, we see that $\bar{f}(x) = \bar{f}(y)$. In fact, one can check that those conditions also characterize the set of pairs $(x, y)$ so that $\bar{f}(x) = \bar{f}(y)$. $\bar{f}$ is also continuous with respect to the topology of $X_\mu$ (although not with respect to the topology of $\hat{X}$, since $\alpha + \beta$ might have atoms as a geodesic current if any of $\alpha$ or $\beta$ contain simple closed curves in their support.

**Remark 5.1.** In his work, Guirardel goes on to prove that $i(\alpha, \beta)$ is equal to the volume of $\mathcal{C}/\pi_1(X)$, where volume is defined by taking the supremum, over finite trees
(convex hull of finitely many points) $K_1, K_2$ of $T_1, T_2$ of the product of Lebesgue measures $\mu_{K_1} \times \mu_{K_2}$ (a finite tree is simplicial, and the Lebesgue measure of a simplicial tree is induced from the Lebesgue measure on the edges, since each edge is isometric to an interval of $\mathbb{R}$). On the other hand, $i(\alpha, \beta) = \frac{1}{2} i(\alpha + \beta, \alpha + \beta) = \frac{1}{2} i(\mu, \mu)$. It would be interesting to see if one can recover the self-intersection number of $\mu$ as some sort of volume of $X_{\mu}/\pi_1(X)$. Compare this with Example 5.1.2, where we show that when $\mu$ is the Liouville current of $X$, $X/\pi_1(X)$ is isometric to the hyperbolic surface $X$, and, on the other hand, $i(\mu, \mu) = \pi^2 |\chi(X)|$ (by [Bon88, Proposition 15]), so the volume on the dual is not quite the hyperbolic area of $X$, but rather a multiple of it. We refer to [BIPP21c, Definition 3.4] as well as [Ouy19, Section 6.5] for a construction of a space on which our space embeds isometrically (provided one makes a choice of $\ell_1$ metric on the product).

5.1.1. Duals for positively ratioed Anosov representations. In this example we assume $X$ is a closed hyperbolic surface and we go back to the class of geodesic currents introduced in Subsection 2.3.4 namely, positively ratioed Anosov representations. First, we have the following immediate observation.

**Lemma 5.2.** Let $\rho: \pi_1(X) \to G$ be a positively ratioed representation and $\mu_\rho$ its associated geodesic current. Then $X_{\mu_\rho}/\pi_1(X)$ is homeomorphic to $X$.

**Proof.** By Lemma 2.11, geodesic currents associated to positively ratioed Anosov representations have no atoms and full support. Thus, by Proposition 4.14, the result follows. \qed

**Remark 5.3.** Note that there are points in the boundary of the Hitchin component corresponding to geodesic currents that might have atoms or might not have full support (see [BIPP21a] and [BIPP21b]). In fact, as we mentioned in Subsection 2.3.4, in the paper [BIPP21b] the notion of positively ratioed representations is generalized to representations whose cross-ratios need not be continuous (and, thus, whose associated geodesic currents might have atoms). In the sequel to this project with Anne Parreau, we will endow the dual spaces coming from these representations with a geodesic structure.

In what follows we specialize to the case $G = SL(3, \mathbb{R})$, and consider only Hitchin representations. By work of Choi-Goldman [CG93], to every $SL(3, \mathbb{R})$ Hitchin representation corresponds, in a one-to-one fashion, a convex real projective structure $\Omega_\rho$.

A **convex real projective surface** is a quotient $Z = \Omega/\Gamma$ where $\Omega \subseteq \mathbb{RP}^2$ is a convex domain of the real projective plane, and $\Gamma < SL(3, \mathbb{R})$ is a discrete group of projective transformations acting properly on $\Omega$. 


Let $X$ be a closed oriented hyperbolic surface of genus $g \geq 2$, a convex $\mathbb{R}^2$-structure on $X$ is a geometric structure (see [Thu97, 3.3]), that makes $X$ into a convex $\mathbb{R}^2$-surface. There is an induced $\pi_1(X)$ invariant developing map $\text{dev}: \tilde{X} \to \Omega$, defined by the usual analytic continuation of paths. The associated holonomy homomorphism $h: \pi_1(X) \to \text{SL}(3, \mathbb{R})$ satisfies the relation

$$\text{dev} \circ g = h(g) \circ \text{dev}.$$ 

Thus, the set of projective equivalence classes of convex $\mathbb{RP}^2$-structures on $X$ can be identified as a subspace of $\text{Hom}(\pi_1(X), \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$.

We will now define a proper path metric $d_\Omega$ which is invariant under $\text{Aut}(\Omega)$, called the Hilbert metric associated to a convex $\mathbb{RP}^2$-structure.

Let $p, q \in \Omega$, the projective geodesic through $p$ and $q$ in $\Omega$ defines a pair of points $a$ and $b$ on $\partial \Omega$. We set

$$d_\Omega(p, q) := \log \text{cr}(a, b, p, q)$$

where

$$\text{cr}(a, b, p, q) = \frac{|a - c| \cdot |b - d|}{|a - d| \cdot |b - c|}.$$ 

When $\Omega$ is an ellipse, i.e. the image of a Euclidean disc under an affine map, then $d_\Omega$ coincides with the hyperbolic distance.

This cross-ratio also induces a geodesic current $\mu_\Omega$ with the property that $\ell_H^X(c) = i(\mu_\Omega, c)$ for every closed curve $c \in \text{Curves}(X)$, and

$$\ell_H^X : \Gamma \to \mathbb{R} : c \mapsto \frac{1}{2} \log \frac{\lambda_1(\rho(c))}{\lambda_3(\rho(c))},$$

where $\lambda_i(\rho(c))$ is the $i$-th eigenvalue of $\rho(c) \in \text{SL}(3, \mathbb{R})$. By work of Martone-Zhang [MZ19, Section 3.1], this $\mu_\Omega$ is the same as the geodesic current $\mu_\rho$ coming from the Hitchin representation associated to the convex projective structure $\Omega$.

We give here a third incarnation of this geodesic current, in terms of the so-called “Crofton measure”, that recovers Hilbert length of a geodesic segment as the measure of geodesics intersecting it transversely. This will show that $\mu_\Omega$ satisfies the Crofton property in the sense introduced in Subsection 2.10.

**Definition 5.4 (Projective Crofton measure).** A Crofton Measure on $\Omega$ is a signed measure $\mu_C$ on the set of geodesic lines $\mathcal{G}(\Omega)$ on $\Omega$ that satisfies

$$\ell_\Omega(I) = \int_{\mathcal{G}(\Omega)} \text{card}(L \cap I) \mu_C(L)$$

where $L \in \mathcal{G}(\Omega)$ is a projective geodesic in $\Omega$, and card denotes the cardinality of the set.
This is another instance of Crofton’s formula [San04, 19].

In particular, for every $I = [x, y]$ geodesic segment in $\Omega$ we have that a projective line $L$ has either empty intersection with $[x, y]$, or only one point in common. Hence the right hand side is the integral of the indicator function of $G[x, y]$, i.e., the set of projective geodesics intersecting $[x, y]$ transversely. Thus,

$$\ell_\Omega([x, y]) = \mu_C(G[x, y]).$$

The existence and uniqueness of such measure $\mu$ is discussed in [Sch06].

The Crofton measure $\mu_C$ defined on the geodesic lines $G(\Omega)$ induces a measure on $(\partial \Omega \times \partial \Omega) - \Delta)/\mathbb{Z}_2$. This follows from the fact that geodesics in $\Omega$ are given by segments which are intersection of projective lines with $\Omega$. Since $\Omega$ is a strictly convex domain, each such line defines uniquely a pair of point on $\partial \Omega$. For details see [KP14, §6]. In other words, for any strictly convex projective structure $\rho$ on $X$, we obtain a geodesic current on $\Omega$ that realizes the Hilbert length as in Equation 5.5.

Let us now fix a Hitchin representation $\rho: \Gamma \to SL(3, \mathbb{R})$. Martone-Zhang show in [MZ19] that $\rho$ induces a geodesic current $\mu_\rho$ on $G(X)$ such that

$$i(\mu_\rho, c) = \ell^H_\rho(c)$$

for every $c \in Curves(X)$. Moreover, the Hitchin representation $\rho$ induces a boundary map

$$\psi: \partial \widetilde{X} \to \partial \Omega.$$

This boundary map is the extension to the boundary at infinity of the developing map of the strictly convex real projective structure.

**Proposition 5.6.** Given $X$ a closed hyperbolic surface and a Hitchin representation $\rho: \pi_1(X) \to SL(3, \mathbb{R})$, let $dev: \widetilde{X} \to \Omega$ be the developing map of the associated convex projective structure. Let $\psi$ denote the extension to the boundary at infinity of this homeomorphism

$$\psi: \partial \widetilde{X} \to \partial \Omega.$$

If $g \in \pi_1(X)$, then $\psi(\gamma^\pm) = (\rho(g))^\pm$.

**Proof.** $\Omega$ is $\delta$-hyperbolic if and only if it is a strictly convex divisible domain [Ben04, Theorem 4.5]. Thus, the extension of $dev$ to the boundaries at infinity is well-defined. The property follows by the definition of developing map via analytic continuation. \qed

**Lemma 5.7.** The current $\mu_C$ is the push-forward of $\mu_\rho$ via $\psi$, the extension to the boundary of the developing map.

**Proof.** By the rigidity of currents respect to the intersection with closed curves ([Ota90, Théorème 2]) we need to check the claim only on $i(\cdot, c)$ for $c \in Curves(X)$. Let $\gamma_-$ and $\gamma_+$ be the repelling and attractive fixed points in $\partial \widetilde{X}$ of the deck transformation
corresponding to \( \gamma \). By Proposition 5.6, we have that \( \gamma_+ = \psi(\gamma_+) \) corresponds to the attracting point of \( \rho(\gamma) \), and \( \gamma_- = \psi(\gamma_-) \) corresponds to the repelling point of \( \rho(\gamma) \).

Then we have, on the one hand, if \( \gamma_-, \gamma_+ \) denotes the projective line determined by the points at infinity \( \gamma_- \) and \( \gamma_+ \), and \( x \in \gamma_- \gamma_+ \), we have

\[
\ell^H_\rho(c) = \ell^H_\rho(G[x, \gamma x]) = \mu_C(G[x, \gamma x]) = i(\mu_C, \gamma) = \mu_C([\gamma_-, \gamma_+] \times [z, \gamma' z])
\]

Where the last equality follows from \cite[Lemma 4.4(i)]{MZ19}. On the other hand, by Lemma 3.7 applied to \( \psi \) and \( \pi_1(X) \)-equivariance of \( \psi \), we get

\[
\psi_* \mu_\rho([\gamma_-, \gamma_+] \times [z, \gamma' z]) = \mu_\rho([\gamma_-, \gamma_+] \times [\psi^{-1}(z), \psi^{-1}(\gamma' z)]) = i(\mu_\rho, \gamma) = \ell^H_\rho(\gamma).
\]

Therefore from \( i(\mu_\rho, c) = i(\psi_* \mu_C, c) \) as we wanted to show.

In fact, a similar proof yields the following.

**Proposition 5.8.** The map \( \text{dev} \circ \pi_1^{-1} \) defines a \( \pi_1(X) \)-equivariant isometry from \( (X_{\mu_\rho}, d_{\mu_\rho}) \) to \( (\Omega, d_\Omega) \).

**Proof.** Since \( \mu_\rho \) has no atoms and full support, \( \pi_{\mu_\rho} : \tilde{X} \to X_{\mu_\rho} \), by Proposition 4.14, \( \pi_{\mu_\rho} \) is a \( \pi_1(X) \)-equivariant homeomorphism. We can thus define the \( \pi_1(X) \)-equivariant homeomorphism \( \varphi := \text{dev} \circ \pi_{\mu_\rho}^{-1} \). We show that this map is, in fact, an isometry.

For any points \( x, y \in X_{\mu_\rho} \), let \( x = \pi_\rho(\bar{x}) \) and \( y = \pi_\rho(\bar{y}) \), for \( \bar{x}, \bar{y} \in \tilde{X} \). Note that

\[
d_{\mu_\rho}(x, y) = d_{\mu_\rho}(\pi_\rho(\bar{x}), \pi_\rho(\bar{y})) = \mu_\rho(G[\bar{x}, \bar{y}]) = \psi_*^{-1} \mu_C(G[\bar{x}, \bar{y}]) = \mu_C(G[\text{dev}(\bar{x}, \text{dev}(\bar{y}))]) = \mu_C(G[\text{dev}(\pi_{\mu_\rho}^{-1}(x), \text{dev}(\pi_{\mu_\rho}^{-1}(y))]) = d_\Omega(\varphi(x), \varphi(y)),
\]

where we have used Lemma 3.7 in the third equality, and the property of the Crofton measure \( \mu_C \) in the fourth equality. This shows \( \varphi \) is an isometry. \( \square \)

**5.1.2. Duals for hyperbolic/negatively curved Riemannian Liouville current.** In this subsection, we assume that \( X \) is closed. Recall that the geodesic current \( \mathcal{L}_X^Y \) has full support and has no atoms, and it is has been defined as the Liouville current associated to \( [(Y, \varphi)] \in \text{Teich}(X) \) (see Subsection 2.10 for details).

**Lemma 5.9.** Given \( [(Y, \varphi)] \in \text{Teich}(X) \), the map \( \psi := \varphi \circ \pi_{\mathcal{L}_X^Y}^{-1} \) induces a \( \pi_1(X) \)-equivariant isometry between \( (\tilde{X}_{\mathcal{L}_X^Y}, d_{\mathcal{L}_X^Y}) \) and \( (\tilde{Y}, d_{\tilde{Y}}) \).
Proof. Note that since $L^X$ has no atoms and full support, it follows from Proposition 4.14 that $\pi_{L^X}$ is a $\pi_1(X)$-equivariant homeomorphism. Let $x, y \in X_{L^X}$, and set $\bar{x} = \pi_{L^X}^{-1}(x), \bar{y} = \pi_{L^X}^{-1}(y)$. We have

\[
d_{\bar{Y}}(\psi(x), \psi(y)) = L_Y(G[\psi(x), \psi(y)]) = L_Y(G[\varphi \circ \pi_{L^X}^{-1}(x), \varphi \circ \pi_{L^X}^{-1}(y)]) = \varphi_*^{-1}L_Y(G[\bar{x}, \bar{y}]) = L^X_Y(G[\bar{x}, \bar{y}]) = d_{L^X}([\bar{x}, \bar{y}]).\]

□

From Lemma 5.9, the following identity follows immediately.

**Corollary 5.10.** The following two spaces are identical

$$(\tilde{X}_{L^X}, d_{L^X}) = (\tilde{X}, d_{\tilde{X}})$$

We end this example by remarking that the same argument for a negatively curved Riemannian metric $Z$, and its associated geodesic current $L^Z$ as defined by Otal (see Subsection 2.10), yield

$$(\tilde{X}_{L^Z}, d_{L^Z}) = (\tilde{Z}, d_{\tilde{Z}}).$$

For example, by [OT21, Proposition 4.2], the Blaschke metric induced by cubic differentials is a negatively curved metric, with a negatively curved Liouville current whose dual space is isometric to $X$ equipped with said metric. One can obtain similar equivalences for other geodesic currents satisfying the Crofton property associated to non-positively curved metrics.

### 6. Hyperbolicity

In this section we prove that the dual spaces $X_\mu$ are $\delta$-hyperbolic metric spaces.

Recall (see Definition 2.2) that $I_{a,b}$ denotes a generalized ordered interval in $\partial \tilde{X}$, and a box of geodesics was defined as any subset of $G(\tilde{X})$ of the type $B = I_{a,b} \times I_{c,d}$. Recall also $B$ denotes the family of all boxes of geodesics.

**Definition 6.1** (opposite box). Given a box of geodesics $B = I_{a,b} \times I_{c,d}$, its opposite box is defined as $B^\perp = I_{d,a} \times I_{b,c}$, so that the intervals $I_{a,b}, I_{c,d}, I_{d,a}, I_{b,c}$ partition $\partial \tilde{X}$. See Figure 6.1.

The following lemma is straightforward and not new. The second equation in Lemma 6.2 appears in work of Otal [Ota90, Page 154], without proof. We provide a proof here, for completeness.

The setup for the lemma is the following. Let $k, k' \subset \tilde{X}$ be two geodesic segments contained in the geodesics $\gamma$ and $\gamma'$, respectively. Let $m$ and $n$ be the (unbounded)
geodesic segments such that the concatenation $mkn$ is $\gamma$ and similarly let $m',n'$ be the geodesics segments such that $m'k'n' = \gamma'$. Finally, let $l$ be the geodesic segment joining the start point of $k$ to the end point of $k'$, and $l'$, similarly, the geodesic segment joining the endpoint of $k$ to the start point of $k'$. The situation is illustrated in Figure 6.2.

We introduce the following notation. Given two geodesics segments $k$ and $k'$, let $G(k,k')$ be the set of geodesics intersecting both $k$ and $k'$ transversely. Let $G(k|k')$ be the set of geodesics separating $k$ from $k'$, i.e., $\gamma \in G(k|k')$ if $k$ and $k'$ lie in distinct connected components of $\tilde{X} - \gamma$. We note that the two types of sets $G(k|k')$ and $G(m,m')$ are related. Indeed, let $x,y,z,w \in \tilde{X}$ appear counter-clockwise as vertices of an embedded geodesic quadrilateral on $\tilde{X}$, $k = [x,y)$, $k' = [z,w)$, $m = [y,z)$, and $m' = [w,x)$. Then

$$G(k|k') = G(m,m').$$

We will refer to sets of the type $G(m,m')$ as double transversals.

**Lemma 6.2.** Given the setup described above, for any $\mu$ geodesic current we have

$$\mu(G(\gamma|\gamma')) = \mu(G(k|k')) - \mu(G(m,n')) - \mu(G(m',n));$$

$$2\mu(G(k|k')) = \mu(G(l)) + \mu(G(l')) - \mu(G(k)) - \mu(G(k')).$$
Proof. We have the following partitions:

\begin{align}
(6.3a) \quad G(k|k') &= G(\gamma|\gamma') \cup G(m, n') \cup G(m', n) \\
& \quad \cup G(k, [a, b]) \cup G(k', [c, d]); \\
(6.3b) \quad G(l) &= G(\gamma|\gamma') \cup G(k, k') \cup G(m, n') \cup G(m', n) \cup G(k', n) \cup G(k, m') \\
& \quad \cup G(k, [a, b]) \cup G(k', [c, d]); \\
(6.3c) \quad G(l') &= G(\gamma|\gamma') \cup G(k, k') \cup G(m, n') \cup G(m', n) \cup G(k', m) + G(k', n) \\
& \quad \cup G(k', [a, b]) \cup G(k, [c, d]); \\
(6.3d) \quad G(k) &= G(k, k') \cup G(k, m') \cup G(k, [a, b]) \cup G(k, n') \cup G(k, [c, d]); \\
(6.3e) \quad G(k') &= G(k, k') \cup G(k', m) \cup G(k', [a, b]) \cup G(k', n) \cup G(k', [c, d]).
\end{align}

We now applying the measure \( \mu \) to all the equations, add the equations resulting from Equation 6.3b and 6.3c, and subtract this from the equations resulting from Equation 6.3d and 6.3e, we get the result. \( \square \)

**Definition 6.4** (double transversals and boxes for 4-tuples). In what follows, compare Figure 6.3 for illustrations. Given \( x, y, z, w \) four distinct points in \( \tilde{X} \). Up to relabeling, we assume that they appear as vertices of an embedded geodesic quadrilateral ordered counter-clockwise on \( \tilde{X} \). Consider the oriented hyperbolic geodesic \( \gamma_1 \) connecting \( x \) to \( y \), and the oriented hyperbolic geodesic \( \gamma_2 \) connecting \( w \) to \( z \). Let \( \delta_1 \) be the oriented hyperbolic geodesic connecting \( z \) to \( y \) and \( \delta_2 \) be the oriented geodesic connecting \( w \) to \( x \). We define four sets of geodesics associated to the tuple \( (x, y, z, w) \).
Let \( b_{x,y,z,w} \) be the box of geodesics defined by \([\gamma_2^-, \gamma_1^-) \times [\gamma_1^+, \gamma_2^+]\).

Let \( G_{x,y,z,w} \) denote a double transversal, defined as the set of geodesics intersecting both \([w, x)\) and \([y, z)\) (or, equivalently, separating the segments \([z, w)\) and \([x, y)\) along \(\gamma_1\) and \(\gamma_2\)). Let, also \( G^\perp \) denote the set of geodesics intersecting \([z, w)\) and \([x, y)\) (equivalently, separating the segments \([w, x)\) and \([y, z)\) contained in \(\delta_2\) and \(\delta_1\), respectively).

Let \( B_{x,y,z,w} \) be the box of geodesics defined by \([\delta_2^-, \delta_1^-) \times [\delta_1^+, \delta_2^+]\).

The following result follows directly from Definition 6.4.

**Proposition 6.5.** Given the setting as described above, dropping the subscripts, we have
\[
b \subseteq G \subseteq B.
\]
Moreover, we have
\[
B^\perp \subseteq G^\perp \subseteq b^\perp.
\]

We define the following two quantities.

**Definition 6.6** (\(\delta_\mu\) with boxes). For a given geodesic current \(\mu\), define
\[
\delta_\mu^B = \sup_{B \in B} \min\{\mu(B), \mu(B^\perp)\}
\]

We observe that \(B \times B^\perp \subset \mathcal{I}\) where \(\mathcal{I}\) is the subset of \(\mathcal{G}(\tilde{X}) \times \mathcal{G}(\tilde{X})\) consisting of transversely intersecting geodesics, used in the definition of intersection number of geodesic currents (see Definition 2.4). Thus, \(\delta_\mu\) is giving another measure, related to intersection number, of ‘how far is \(\mu\) from being a measured lamination’.

**Definition 6.7** (\(\delta_\mu\) with double transversals). For a given geodesic current \(\mu\), define
\[
\delta_\mu^G = \sup_{G} \min\{\mu(G), \mu(G^\perp)\}
\]
where \(G\) ranges over all \(G_{x,y,z,w}\) with \(x, y, z, w\) all distinct.

**Remark 6.8.** Note that the definition of both \(\delta_\mu^B\) (resp. \(\delta_\mu^G\)) can be restricted to boxes \(B\) (resp. sets \(G\)) that either \(\mu(B) > \mu(B^\perp)\) (resp. \(\mu(G) > \mu(G^\perp)\)) or \(\mu(B) < \mu(B^\perp)\) (resp. \(\mu(G) < \mu(G^\perp)\)).

**Lemma 6.9.** For any geodesic current \(\mu\),
\[
\delta_\mu^B = \delta_\mu^G.
\]
Proof. We restrict the definitions to boxes $B$ so that $\mu(B) > \mu(B^\perp)$ and sets $G$ so that $\mu(G) > \mu(G^\perp)$. Moreover, by the inequalities above we have that, for fixed $x, y, z, w$,

$$b^\perp \supseteq G^\perp \supseteq B^\perp.$$ 

Since ranging over all $x, y, z, w$, $B$ and $b$ exhaust all possible boxes (and same for $B^\perp, b^\perp$), we have, taking measure $\mu$ and supremum over all distinct $x, y, z, w$, that

$$\delta^B \geq \delta^G \geq \delta^B,$$

as we wanted to show. \hfill $\square$

![Figure 6.3](image_url)

Since $\delta^G$ and $\delta^B$ are the same quantity, we will simply refer to it as $\delta_\mu$. For some proofs it will be easier to use one viewpoint or the other.

**Proposition 6.10.** $\delta_\mu$ is finite.

Proof. Suppose $\delta_\mu$ is infinite. This implies there exists a sequence of boxes $B_n$ so that $\mu(B_n) \geq n$ and $\mu(B_n^\perp) \geq n$. Up to taking subsequences, we can furthermore assume that $B_n \subset B_{n+1}$ for all $n$, and thus $B_{n+1}^\perp \subset B_n^\perp$ for all $n$. We note that since a geodesic current $\mu$ is locally finite, the only way for a nested sequence of boxes $B_n$ to have the property that $\mu(B_n) \to \infty$ is if the sequence $(B_n)$ escapes every compact subset of $G(X)$, i.e., whenever $B_n = I_{a_n,b_n} \times I_{c_n,d_n}$, $d_n - a_n \to 0$ or
\( b_n - c_n \to 0 \). Say \( a_n - d_n \to 0 \). Then the perpendicular box is \( B_n^\perp = I_{d_n,a_n} \times I_{b_n,c_n} \), and \( \cap_n B_n \subset P(a) \), where \( P(a) \) is a pencil of geodesics based at some \( a \in I_{d_n,a_n} \). By lower continuity of measures, we have \( \lim_n \mu(B_n^\perp) = \mu(\cap_n B_n^\perp) \leq \mu(P(a)) \). Finally, we note that \( \mu(P(a)) = 0 \) unless \( P(a) \) contains a lift of a closed geodesic in the support of \( \mu \), in which case \( \mu(P(a)) < \infty \), since it can at most contain one closed lift. See [Mar16, 8.2.4] and Lemma 2.8 for these last two claims. But this contradicts that \( \mu(B_n^\perp) \to \infty \), so it follows that \( \delta_\mu \) is finite. □

Given a geodesic current \( \mu \), we say a box of geodesics \( B \) is \( \mu \)-generic if \( \mu(\partial B) = 0 \).

Let \( B^\mu \subset B \) denote the subset of \( \mu \)-generic geodesic boxes.

The following is an easy but crucial observation.

**Lemma 6.11.** In the definition of \( \delta_\mu \) we can restrict to \( \mu \)-generic boxes, i.e.,

\[
\delta_\mu = \sup_{B \in B^\mu} \min\{\mu(B), \mu(B^\perp)\}
\]

where \( B^\mu \subset B \) consists of boxes \( B \) so that \( B \) and \( B^\perp \) are \( \mu \)-generic.

**Proof.** Suppose that \( B = [a,b) \times [c,d) \), and \( \mu(P(a)) > 0 \). By Lemma 2.7, there exists a lift of a closed geodesic in the support. For every \( \varepsilon > 0 \), there exists a point \( a' \in \partial X \), so that the box \( K = [a',a) \times [c,d) \) has no atoms and \( \mu(B') < \varepsilon \). Indeed, it has no atoms, since atoms are concentrated on the set \( C_\mu \) of lifts of closed geodesics in the support of \( \mu \), and this set \( C_\mu \) is discrete, by local finiteness of \( \mu \). By choosing \( a' \) close enough to \( a \), we can ensure \( K \) has no atoms. Otherwise, some \( K \) would contain infinitely many elements in \( C_\mu \), but this would violate local finiteness of \( \mu \). Note that then the box \( B' := K \cup B \) is \( \mu \)-generic, and by taking \( a' \) closer to \( a \), we can guarantee \( \mu(B') - \mu(B) < \varepsilon \). Moreover, \( \partial B^\perp \) has the same atoms as \( \partial B^\perp \). By repeating the same argument with \( B^\perp \), we can guarantee that \( B^\perp \) is also \( \mu \)-generic.

The following is a restatement of the Gromov 4-point condition for \( \delta \)-hyperbolic spaces [BH11, Page 410].

**Definition 6.12 (\( \delta \)-hyperbolicity).** A metric space \((X, d)\) is \( \delta \)-hyperbolic if and only if for any 4-tuple of points \( x, y, z, w \in X \), among the following three quantities

- \( d(x,y) + d(z,w) \)
- \( d(x,z) + d(y,w) \)
- \( d(y,z) + d(x,w) \)

the two largest of them are within \( 2\delta \) of each other. If \( \delta = 0 \), then it means that the maximum appears at least twice.

**Theorem 6.13.** If \( \mu \) is a geodesic current then \( X_\mu \) is a \( \delta_\mu \)-hyperbolic space in the sense of Definition 6.12, and \( \delta_\mu \) is the optimal \( \delta \)-hyperbolicity constant.
Proof. We prove that $X_\mu$ satisfies the $\delta_\mu$-hyperbolic 4-point condition. Let $x, y, z, w \in X_\mu$ be four arbitrary points. Assume that $d(x, z) + d(y, w) > d(x, y) + d(z, w) + 2\delta$. We want to show that $d(x, z) + d(y, w) \leq d(y, z) + d(x, w) + 2\delta$. Notice how the first inequality is equivalent to

$$\mu(G[x, z]) + \mu(G[y, w]) - \mu(G[x, y]) - \mu(G[z, w]) > 2\delta$$

which is equivalent to

$$\mu(G^\bot) > \delta$$

where $G = G_{x, w, y, z}$, by Lemma 6.2. On the other hand, $d(x, z) + d(y, w) \leq d(y, z) + d(x, w) + 2\delta$, is equivalent to

$$\mu(G[x, z]) + \mu(G[y, w]) - \mu(G[y, z]) - \mu(G[x, w]) \leq 2\delta,$$

i.e., $\mu(G) \leq \delta$, by Lemma 6.2.

Suppose that $\mu(G) > \delta$. Then,

$$\min\{\mu(G), \mu(G^\bot)\} > \delta = \sup_G \min(\mu(G), \mu(G^\bot))$$

which is a contradiction. Since $\delta\mu$ is defined in terms of a supremum, it follows it is the optimal $\delta$-hyperbolicity constant. \hfill \Box

Corollary 6.14. $X_\mu$ is 0-hyperbolic if and only if $\mu$ is a measured lamination.

Proof. By [BIPP21b, Proposition 2.1], $\mu$ is a measured lamination if and only if, for every box $B \subset G(X)$, we have $\mu(B)\mu(B^\bot) = 0$. This last equality is true if and only if $\min\{\mu(B)\mu(B^\bot)\} = 0$. If $\mu$ is a measured lamination, then $\delta\mu = 0$, and thus by Theorem 6.13, $X_\mu$ is 0-hyperbolic. If $X_\mu$ is 0-hyperbolic, since $\delta\mu$ is the smallest hyperbolicity constant, we must have $\delta\mu = 0$. Since $\delta\mu$ is defined in terms of a supremum, this implies that for all boxes $B \subset G(\tilde{X})$, $\mu(B)\mu(B^\bot) = 0$, $\mu$ must be a lamination. \hfill \Box

As a consequence, we recover the following well-known result.

Corollary 6.15. If $\mu$ is the Liouville current $\mathcal{L}_X$, then $X_\mu$ is equal to $\tilde{X}$, and its optimal hyperbolicity constant is $\delta_\mu = \log(2)$.

Proof. Note that $\tilde{X} = X_\mu$ as metric spaces. By [Bon88, Theorem 13], a geodesic current $\mu$ is a Liouville current $\mathcal{L}_X$ for some hyperbolic structure $X$, if and only if $e^{-\mu(B)} + e^{-\mu(B^\bot)} = 1$. Maximizing the function $\delta_\mu$ subject to this relation, yields $\log(2)$. Thus, by Theorem 6.13, the result follows. \hfill \Box

Remark 6.16. Recall that when $\mu$ is a hyperbolic Liouville current, then $X_\mu$ is isometric to the hyperbolic plane by Example 5.1.2. Observe that $\log(2)$ is the optimal $\delta$-hyperbolicity constant which was computed for example in [Nv16, Corollary 5.4]. Corollary 6.15 recovers this result.
Proposition 6.17. \( \delta_\mu \) is a lower semi-continuous function on geodesic currents.

Proof. Let \( \mu_i \to \mu \) in the weak\(^*\)-topology. Recall that, by Lemma 6.11, in the definition of \( \delta_\mu \), we can restrict to \( \mu \)-generic boxes \( B \subset G(X) \) without affecting the supremum. For any such \( B \), let \( f_B(\mu) := \min\{\mu(B), \mu(B^\perp)\} \). By the Portmanteau theorem [Bau01, Theorem 30.12], \( f_B(\mu_i) \to f_B(\mu) \), so \( f_B \) is a continuous functions on geodesic currents. Since \( \delta_\mu = \sup_B f_B(\mu) \) is a supremum of continuous functions, by [vRS82, Theorem 10.3] it must be lower semi-continuous. \( \square \)

Example 6.18. We discuss an example. Take a sequence \( \mu_i \) of scaled hyperbolic Liouville currents \( \mu_i := a_i L_{X_i} \) converging to a measured lamination \( \lambda \) in the weak\(^*\)-topology. Then, we must have \( a_i \to 0 \) (see argument at the bottom of [Bon88, Page 152]). By Remark 6.16, the dual \( X_{\mu_i} \) is isometric to \( \tilde{X} \), but the measure is scaled by \( a_i \) and thus \( \delta_{\mu_i} = a_i \log(2) \) for all \( i \in \mathbb{N} \), and thus \( \lim_i a_i \log(2) = 0 \). On the other hand \( \delta_\lambda = 0 \), also.

To finish this section, we show a few inequalities between the \( \delta \)-hyperbolicity constants of \( X_\mu \) and the ones of its subcurrents according to the decomposition theorem for geodesic currents Theorem 7.2. In Section 7 we will see that \( X_\mu \) decomposes as a graph of spaces with vertices the duals of its subcurrents \( \mu_i \). The following inequalities relate the \( \delta \)-hyperbolicity constants of the components of the space to those of its pieces.

Lemma 6.19. Let \( \mu, \mu_1, \mu_2 \) be geodesic currents so that \( \mu = \mu_1 + \mu_2 \), and let \( B \subseteq G(X) \).

(1) We have
\[
\min\{\mu(B), \mu(B^\perp)\} \geq \min\{\mu_1(B), \mu_1(B^\perp)\} + \min\{\mu_2(B), \mu_2(B^\perp)\}.
\]

(2) If, furthermore \( \mu_1 \perp \mu_2 \), then
\[
\min\{\mu(B), \mu(B^\perp)\} = \min\{\mu_1(B), \mu_1(B^\perp)\} + \min\{\mu_2(B), \mu_2(B^\perp)\}
\]

Proof. First note that, by the definition of sum of measures
\[
\min\{\mu(B), \mu(B^\perp)\} = \min\{\mu_1(B) + \mu_2(B), \mu_1(B^\perp) + \mu_2(B^\perp)\}.
\]
Then, it is just a consideration of the different cases.

(1) Assume, without lost of generality, that \( \mu_1(B) + \mu_2(B) < \mu_1(B^\perp) + \mu_2(B^\perp) \), so that
\[
\min\{\mu(B), \mu(B^\perp)\} = \mu_1(B) + \mu_2(B).
\]
Then consider two cases:
- \( \mu_1(B) < \mu_1(B^\perp) \). Consider two subcases:
\[-\mu_2(B) < \mu_2(B^\perp). \text{ Then}\]
\[
\min\{\mu_1(B), \mu_1(B^\perp)\} = \mu_1(B)
\]
and
\[
\min\{\mu_2(B), \mu_2(B^\perp)\} = \mu_2(B)
\]

\[-\mu_2(B) \geq \mu_2(B^\perp). \text{ Then}\]
\[
\min\{\mu_1(B), \mu_1(B^\perp)\} = \mu_1(B)
\]
and
\[
\min\{\mu_2(B), \mu_2(B^\perp)\} = \mu_2(B^\perp)
\]

\[\bullet \mu_1(B) \geq \mu_1(B^\perp). \text{ Consider two subcases:}\]
\[-\mu_2(B) < \mu_2(B^\perp). \text{ Then}\]
\[
\min\{\mu_1(B), \mu_1(B^\perp)\} = \mu_1(B^\perp)
\]
and
\[
\min\{\mu_2(B), \mu_2(B^\perp)\} = \mu_2(B)
\]
\[-\mu_2(B) \geq \mu_2(B^\perp). \text{ Then}\]
\[
\min\{\mu_1(B), \mu_1(B^\perp)\} = \mu_1(B^\perp)
\]
and
\[
\min\{\mu_2(B), \mu_2(B^\perp)\} = \mu_2(B^\perp)
\]

In any case, we get the result.

(2) Since \(\mu_1\) and \(\mu_2\) are orthogonal, it means we cannot have \(g\) and \(h\) intersecting hyperbolic geodesics so that \(g \in \text{supp}(\mu_1)\) and \(h \in \text{supp}(\mu_2)\). Thus, we have that the following two statements hold simultaneously:

(a) \(\mu_1(B) = 0\) or \(\mu_2(B^\perp) = 0\), and

(b) \(\mu_1(B^\perp) = 0\) or \(\mu_2(B) = 0\).

From here, we get the following cases.

| \(\mu_1(B)\) | \(\mu_2(B)\) | \(\mu_1(B^\perp)\) | \(\mu_2(B^\perp)\) |
|---|---|---|---|
| \(\mu_1(B)\) | \(\mu_2(B)\) | 0 | 0 |
| \(\mu_1(B)\) | 0 | \(\mu_1(B^\perp)\) | 0 |
| 0 | \(\mu_2(B)\) | 0 | \(\mu_2(B^\perp)\) |
| 0 | 0 | \(\mu_1(B^\perp)\) | \(\mu_2(B^\perp)\) |

In any case, the result follows.
Proposition 6.20. Let \( \mu \) be a geodesic current which decomposes according to the structural Theorem 7.2, \( \mu = \sum_i \nu_i + \sum_j \lambda_j + \sum_k a_k s_k \), where \( s_k \) is a geodesic current supported on a simple closed curve, \( \lambda_j \) is a non-discrete measured lamination, and \( \nu_i \) is a geodesic current which is filling in a subsurface, and all the currents in the decomposition have orthogonal supports (in the sense that \( \langle \text{supp}(\alpha), \text{supp}(\beta) \rangle = 0 \), according to [BIPP21a, Proposition 3.2]). Then, we have

\[
\min\{\mu(B), \mu(B^\perp)\} = \sum_i \min\{\nu_i(B), \nu_i(B^\perp)\}
\]

and, thus, for every \( i \), we have

\[
\delta_{\nu_i} \leq \delta_{\mu} \leq \sum_i \delta_{\nu_i}
\]

Proof. By the structural theorem [BIPP21a, Proposition 3.2] all the currents in the decomposition are pairwise orthogonal. Thus, equation 6.21 follows by Proposition 6.19, and considering [BIPP21a, Proposition 2.1], it follows that

\[
\min\{\lambda_j(B), \lambda_j(B^\perp)\} = 0
\]

and

\[
\min\{s_k(B), s_k(B^\perp)\} = 0.
\]

The first inequality follows from the fact that, by Equation 6.21, \( \min\{\mu(B), \mu(B^\perp)\} \geq \min\{\nu_i(B), \nu_i(B^\perp)\} \) for every \( i \), and for any two real valued functions \( f, g \) so that \( f \leq g \), we have \( \sup f \leq \sup g \). The second inequality in Equation 6.22 follows by Equation 6.21 and the fact that for two real valued functions \( f, g \), we have \( \sup f + g \leq \sup f + \sup g \). \( \square \)

7. Decomposition theorem

We begin by defining the set of special geodesics, as introduced in [BIPP21a]. Given a geodesic current \( \mu \) on a compact hyperbolic surface \( X \), let

\[
E_\mu := \{ c \subset X : c \text{ closed geodesic such that } \langle \mu, c \rangle = 0 \text{ and for every } c' \subset X \text{ closed geodesic, } \langle \mu, c' \rangle > 0 \text{ whenever } \langle c, c' \rangle > 0 \}
\]

The set \( E_\mu = \{s_1, \ldots, s_n\} \) is a finite set of pairwise disjoint simple closed geodesics, which decomposes \( X \) in subsurfaces with geodesic boundary

\[
X = \bigcup_i X_i
\]

Given a current \( \mu \) on \( X \), recall that the systole of \( \mu \) relative to \( X_i \) is

\[
sys_{X_i}(\mu) := \inf\{i(\mu, c) : c \in \text{Curves}(X_i - \partial X_i)\}.
\]
We now state the decomposition theorem for geodesic currents, as proven in [BIPP21a, Theorem 1.2].

**Theorem 7.2 (Decomposition Theorem for Geodesic Currents).** Any current \( \mu \) on \( X \) decomposes as

\[
\mu = \sum_{i=1}^{n} \mu_i + \sum_{j=1}^{m} a_js_j
\]

where each \( \mu_i \) is a geodesic current supported on \( X_i \) and \( \sum_{j=1}^{m} a_js_j \) is a weighted simple multi-curve.

Moreover, for each \( \mu_i \) we either have

- (type 1) \( \text{sys}_{X_i}(\mu_i) > 0 \);
- (type 2) \( \mu_i \) is a measured lamination compactly supported on the interior of \( X_i \) and intersecting every curve in \( X_i \).

**Remark 7.4.** For the remaining of this paper we will refer to currents which fall into the case (1) as subcurrents of type 1, the ones falling into case (2) will be referred as subcurrents of type 2, and the weighted simple curves in the special simple multi-curve will be referred as subcurrents of type 3.

Each component \( \mu_i \) of \( \mu \) is a geodesic current itself, supported on the subsurface \( X_i \). On \( \tilde{X}_i \subseteq \tilde{X} \) we can define the pseudo-distance \( d_{\mu_i} \) in the same way as for \( d_\mu \). The sub-dual space \( X_{\mu_i} \) is the quotient space \( \tilde{X}_i / \{d_{\mu_i} = 0\} \) endowed with the \( \pi_1(X_i) \)-action.

In order to precisely describe the dual \( X_\mu \) in terms of the sub-duals \( X_{\mu_i} \) we use the notion of tree-graded space.

For the standard definition of tree-graded space when \( (X,d) \) is a geodesic metric space we refer to the work by Drutu-Sapir [DS05] and [DS07].

**Definition 7.5 (Tree-graded space).** A geodesic metric space \( (X,d) \) is said to be tree-graded with respect to a collection of geodesic subspaces \( \mathcal{P} \), called pieces, if

1. axiom pieces. Given two distinct pieces \( P_1, P_2 \in \mathcal{P} \) the intersection \( P_1 \cap P_2 \) contains at most one point;
2. axiom triangles. Any simple geodesic triangle in \( (X,d) \) is contained in a piece.

The following can be thought as a local to global principle for geodesics in a tree graded space.

**Definition 7.6 (Piece-wise geodesic).** Let \( (X,\mathcal{P}) \) be a tree graded space. Suppose that the pieces \( P_k \in \mathcal{P} \) are geodesics with respect to the restricted metric. Let \( \gamma = \gamma_1\gamma_2\cdots\gamma_{2m} \) be a curve in the tree-graded space \( (X,\mathcal{P}) \) which is a composition...
Figure 7.1. The upper figure shows a sketch of a genus 3 surface with a geodesic current $\mu$ whose special multi-curve $m$ consists of one single geodesic (red curve, separating the left two handles from the right handle), and yields two subsurfaces. The left one, genus 2, supports a filling geodesic current $\mu_1$ (in blue) within that subsurface. The right one, of genus 1, supports a non-discrete measured lamination $\mu_2$ (in green). The lower figure shows a part of the support of the geodesic current $\mu$ in the universal cover. The lifts of $m$ separate $\tilde{X}$ into countably many regions. In the figure, three are depicted, the central one is a region corresponding to the support of $\mu_2$, whereas the upper and lower ones correspond to the support of $\mu_1$.

of geodesics $\gamma_k$ in $X$. Suppose that all geodesics $\gamma_{2k}$ with $k \in \{1, \cdots, m - 1\}$ are non-trivial and for every $k \in \{1, \cdots, m\}$ the geodesic $\gamma_{2k}$ is contained in a piece $P_k$ while for every $k \in \{0, \cdots, m - 1\}$ the geodesic $\gamma_{2k+1}$ intersects $P_k$ and $P_{k+1}$ only in
its respective endpoints. In addition assume that if $\gamma_{2k+1}$ is empty then $M_k \subset M_{k+1}$. We call this $\gamma$ a piece-wise geodesic.

The next proposition is Lemma [DS07, Lemma 2.28].

**Proposition 7.7.** A curve $\gamma$ in a tree graded space $(X, \mathcal{P})$ is a geodesic if and only if it is a piece-wise geodesic.

Geodesics in a tree graded space can be then thought of as concatenations of geodesics within pieces and geodesics in the transversal trees $T_x$. Here $T_x$ denotes the set of points $y \in X$ that can be connected to $x$ by a geodesic intersecting each piece at most once (see [DS05, Lemma 2.14] to see why these are trees).

Our goal is to show that a dual space $X_\mu$ is a tree graded space where its pieces can be isometrically identified with the duals $X_\mu_i$ of the subcurrents of $\mu$.

However, the dual spaces have not been endowed with a canonical geodesic structure in general (see Subsection 3.3). Because of that, we will give a more general definition of tree-graded space that is not assumed to be geodesic, but coincides with the usual definition of tree graded space when the underlying space $X$ is assumed to be geodesic.

**Definition 7.8 ((metric) tree-graded space).** A metric space $(X, d)$ is said to be a (metric) tree-graded space with respect to a collection of subspaces $\mathcal{P}$, called pieces, if

1. **axiom pieces.** Given two distinct pieces $P_1, P_2 \in \mathcal{P}$ the intersection $P_1 \cap P_2$ contains at most one point;

2. **axiom transversals.** For every $x \in X$, there exists a $0$-hyperbolic metric subspace $T_x \subset X$ containing $x$ and intersecting each piece at most in one point with the following properties:
   - (a) for every $y \in X$, if $y \in T_x$, then $T_x = T_y$. Every intersection of a piece $P$ with a $T_x$ is called a point of contact of $P$.
   - (b) All points of $T_x$ are cut points except (possibly) the points of contact.

3. **axiom contact chain triangle.** We say that two contact points are adjacent if they are contained in the same $T_x$ or in the same piece. Let a straight contact chain be an ordered sequence of adjacent contact points $(y_1, \ldots, y_n)$ where no two consecutive points in the sequence are equal. We say that two points $x, y \in X$ are connected by a straight contact chain if there is a straight contact chain $(x, \ldots, y)$. An interior point of that chain is any point of the chain different from $x$ and $y$. A contact chain triangle $\Delta = xyz$ is a set of three points in $X$ and three straight contact chains connecting the points, one per side. The axiom contact chain triangle says that if a straight contact chain triangle is contained in more than one piece, an interior point of one of the side chains must be shared with another chain.
To motivate the role of the 0-hyperbolic subspaces $T_x$, note that these will be the transversal trees when $X$ is geodesic. For the purposes of the tree graded metric structure of our dual spaces $X_\mu$ might seem unnecessarily abstract. However, it is in our opinion of independent interest to have such a definition, since tree graded metric spaces appear as natural examples when one considers actions of hyperbolic groups on hyperbolic spaces with the bounded backtracking property, as studied by the second author in [KMG22] (see Definition 10.5). We note that in [COR22, Lemma 6.10], Cantrell–Oregón-Reyes show that the action of $\pi_1(X)$ on a dual space satisfies the bounded backtracking property. In any case, the following Remark gives the intuitive picture the reader should probably keep in mind.

**Remark 7.9 (Geometric realization of the $T_x$).** Let $\mu = \sum_{i=1}^n \mu_i + \sum_{j=1}^m a_j s_j$ as in theorem 7.2, with set of special geodesics $\{s_1, \ldots, s_n\}$. Two regions $R_i$ and $R_j$ bounded by the lift of a special geodesic $c$ correspond in the dual to two pieces $P_i$ and $P_j$ joined in a single point $x := [c]$. Let $a_{i,j}$ be the weight of $c$. Hence every arc joining the piece $P_i$ to the piece $P_j$ will pass through $x$ and since there is an $a_{i,j}$ measure accumulated at the point $x_{i,j}$, every such arc will gain an $a_{i,j}$ contribution to its length as soon as it passes by through the point $x$. It follows that we may imagine, loosely speaking, that the pieces $P_i$ and $P_j$ are joined by an edge of length $a_{i,j}$, and any arc joining $P_i$ to $P_j$ passes through such arc. Let us now make this idea precise. Compare Figure 7.2 for what follows.

Given two adjacent pieces $P_i$ and $P_j$ as above, consider their disjoint sum $P_i \amalg P_j$, and attach an edge $e_{i,j}$ of length $a_{i,j}$ joining the point $x_i := [c] \in P_i$ and $x_j := [c] \in P_j$. At the level of the surface, this corresponds to pinching the special geodesics to points, obtaining a noded surface, and then replacing the nodes by edges of length equal to the weight of the component of the especial multi-curve.

More generally, for a metric tree graded space, one can define a projection map $p: X \to T$, where $T$ is a 0-hyperbolic space obtained by collapsing each piece to a point. This space is 0-hyperbolic because it consists of the union of the subspaces $T_x$ equipped with the restriction of the distance on $X$. By Section A, this 0-hyperbolic space can be isometrically embedded into an $\mathbb{R}$-tree. This induces a $\mathbb{R}$-tree structure on the $T_x$.

It would be interesting to see if the injective hull construction of Definition 3.6 preserves the tree graded structure. Precisely: if $X_\mu$ is a metric graded space in the sense of Definition 7.8, is the injective hull of $X_\mu$ a metric space in the sense of Definition 7.5)? This relates to the comment in Subsection 3.3.

Under the geodesic hypothesis, both definitions above are equivalent. See Proposition B.1.
Figure 7.2. The figure shows a sketch of the result of collapsing the special multi-curve $m$ of the geodesic current of Figure 7.1. That is how one can go about putting a geodesic structure on the transversals $T_x$: by pinching $m$ to obtain nodal surfaces, and add edges between them.

Figure 7.3. This figure shows a schematic of a triangle composed of segments of type $\alpha$ (green, dashed) and $\beta$ (red, solid), and the corresponding straight chains for its sides, labelled by $y_i$ (red solid dots). It is useful to visualize axiom straight chain triangle, as well as the proof of implication from Definition 7.5 to Definition 7.8 (in Appendix B).

**Theorem 7.10.** The current dual $X_\mu$ is a (metric) tree-graded space with respect to $\mathcal{P}$.

**Proof.** (1) *axiom pieces.* We firstly recall that supp$(m)$ is a pairwise disjoint union of geodesics in $\tilde{X}$. The fact that two pieces $P_1$ and $P_2$ have at most one
a continuous injective pushforward of geodesic currents i.e. the piece $\pi_{\mathcal{C}}$ of geodesic current in $\mathcal{C}$ the subsurface $X_i$ boundary, and its universal cover is isometric to $\mu_i$ on $X_i$. Let $P_1 = R_1/(d_\mu = 0)$ and $P_2 = R_2/(d_\mu = 0)$, then by construction if $R_1 \cap R_2 \neq \emptyset$ it means they are adjacent, i.e. there is a special geodesic $r \in R_1 \cap R_2$ bounding both. Notice moreover that if $R_1 \cap R_2 = \emptyset$, then $P_1 \cap P_2 = \emptyset$ as well. Hence assuming the the two pieces $P_1$ and $P_2$ have non-empty intersection amounts to say that $R_1 \cap R_2 = \{r\}$, therefore we need to show that in the quotient the line $r$ is a single point, which amounts to show that for all $x, y \in r$ we have $d_\mu(x, y) = 0$. Assume by contradiction $d_\mu(x, y) > 0$, then there exists a line $l \in \text{supp}(\mu)$ that intersects transversely $r$, contradicting the fact that $r$ is special (i.e. doesn’t intersect any other line in $\text{supp}(\mu)$), as wanted.

(2) **axiom transversals.** Recall $\mathcal{R}$ denotes the family of complementary regions determined by the lifts of the special multi-curve $m$. For every $x \in X_\mu$, let $T_x = \{x\}$ if $x$ is in the interior of $R_i$, for $R_i \in \mathcal{R}$; and, otherwise, let $T_x$ be the subset of $X_\mu$ given by $T_x = (\tilde{X} - \cup_i \tilde{R}_i)/(d_\mu = 0)$ equipped with the restriction of the metric $d_\mu$. Since $T_x$ is isometric to $X_m$, $T_x$ is $0$-hyperbolic. Moreover, the transversal condition is trivially satisfied.

(3) **axiom straight chain triangle.** A straight chain in this setting corresponds to a sequence $(y_1, \cdots, y_n)$, where $y_i$ are the points obtained by collapsing lifts of simple closed curves in $m$. If a straight chain triangle is contained in more than one piece, we show that the interior point of one of the chains must intersect one of the points in another side chain. Indeed, let the straight chain triangle be composed of side chains $C_1, C_2, C_3$, and suppose that the endpoints of $C_1$, $x$ and $y$, are in distinct pieces $P_1, P_2$. Then, there is an interior point of $C_1$, say $y_i$, that is the equivalence class of a lift of a special geodesic $\tilde{m}_i$, separating regions $R_1, R_2$ corresponding to the pieces $P_1, P_2$. Since $\tilde{m}_i$ separates $\tilde{X}$ into two components, the straight chain $C_2 \cup C_3$, which also connects $x$ and $y$, must also contain $y_i$. 

Let $P = R/(d_\mu = 0)$ be a piece corresponding to a region $R$ which is a lift of the subsurface $X_i \subseteq X$. The subsurface $X_i$ is a hyperbolic surface with geodesic boundary, and its universal cover is isometric to $R$. Moreover the pseudo-distance $d_\mu$ on $R = \tilde{X}_i$ coincides with the restriction of $d_\mu$ on $R$. It follows that $P = (X_i)_{\mu_i}$, i.e. the piece $P$ is the dual space of the current $\mu_i \in \text{Curr}(X_i)$, understood as a geodesic current in $\text{Curr}(X_i)$. Precisely, there is an inclusion $\iota: X_i \hookrightarrow X$ inducing a continuous injective pushforward of geodesic currents $\iota_*: \text{Curr}(X_i) \to \text{Curr}(X)$

\[ \]
(see [EM18a, Section 4.2]), which sends $\mu_i$ as a current on $X_i$ to $\mu_i$ as a current on $X$. This allows us to restate the previous Theorem 7.10 as follows.

**Theorem 7.11.** The dual space $X_\mu$ is a metric tree-graded space where the underlying tree is the dual tree of the special multi-curve $m$ and the pieces are the dual spaces of the subcurrents $\mu_i$ of $\mu$ on the subsurfaces $X_i$.

We conclude this subsection by showing that the pseudo-distance $d_\mu$ on $X_\mu$ can be computed from the pseudo-distances $d_{\mu_i}$ on the pieces $P_i$.

In this setting, we define a *chain* between two points $x, y \in \tilde{X}$ is a sequence of points $C = (x_0, x_1, \ldots, x_{n+1})$ with $x_0 = x$ and $x_{n+1} = y$ such that any two consecutive points $x_i$ and $x_{i+1}$ are in the same piece $P_i$, with $P_i \neq P_{i+1}$.

It follows from the definition that given $P_j$ and $P_{j+1}$, the point $x_{j+1}$ is on the common geodesic boundary $c_j$. We call a chain *straight* if it does not go ‘back and forth’, i.e., if $c_j \neq c_{j+1}$ for $j = 1, \ldots, n - 1$. If a chain $C$ is straight, then the corresponding ordered sequence of boundary geodesics $(c_1, \ldots, c_n)$ are precisely the geodesics separating $x$ from $y$. Each piece $P_i$ is naturally endowed with the induced pseudo-distance $d_{\mu_i}$. We define the length of a chain to be

$$l(C) = \inf_{C} \sum_{i=1}^{n+1} d_{\mu_i}(x_i, x_{i+1}),$$

where the inf is taken over all chains $C$ joining $x$ to $y$. It is enough to consider the inf over straight chains $C$. Moreover, since the curves $c_i$ are special geodesics, there is no line in the support of $\mu$ intersecting them, and hence the distance between any two points on the same special geodesic $c_i$ is zero. It follows that $d(x_j, x_{j+1})$ does not depend on the choice of $x_j$ on $c_j$. This means that all straight chains from $x$ to $y$ have the same length, and hence we may as well define $d(x, y)$ as the length of any straight chain. These notions coincide with the notion of chain in Definition 3.8).

**Lemma 7.12.** Let $C = (x_0, \ldots, x_{n+1})$ be a straight chain from $x$ to $y$ in $\tilde{X}$. The pseudo-distance $d_\mu$ on $\tilde{X}$ can be expressed as

$$d_\mu(x, y) = \sum_{i=1}^{n+1} d_{\mu_i}(x_i, x_{i+1}).$$

**Proof.** Let $\gamma$ be the hyperbolic geodesic joining $x$ to $y$.

Without loss of generality we can consider $C$ to be the chain such that $x_{j+1} = \gamma \cap c_j$. Since the pseudo-distance $d_\mu$ is straight, it follows that

$$d_\mu(x, y) = \inf_{C} \sum_{i=0}^{n} d_{\mu_i}(x_i, x_{i+1}) = \sum_{i=0}^{n} d_{\mu_i}(x_i, x_{i+1}),$$
where the last equality follows from the fact that $x_i$ and $x_{i+1}$ belong to the piece $P_i$. \qed

7.1. Properties of the decomposition. We recall the definition of graph of groups, as in [Kap09, 10.2]. Let $Y$ be a finite graph where each edge is oriented. We assume that to each vertex $v$ of $Y$ is assigned a vertex group $v$ and to each edge $e$ is assigned an edge group $G_e$. Each inclusion $v \hookrightarrow e$ of a vertex into an edge (as the initial of terminal vertex) corresponds to a monomorphism $h_{ev}: G_e \to G_v$. The collection

$$(Y, \{G_e, G_v, h_{ev}: \text{where } e, v \text{ are edges and vertices of } Y\})$$

is called a graph of groups $(\mathcal{G}, Y)$, where $\mathcal{G}$ is the data of all vertex groups, edge groups and monomorphisms, and $Y$ is the underlying graph. We denote with $\mathcal{G}^0$ and $\mathcal{G}^1$ the set of vertices and edges of $Y$, respectively. When we don’t need to specify the underlying graph $Y$, we will refer to the graph of groups simply as $\mathcal{G}$.

The fundamental group $\pi_1(\mathcal{G})$ of a graph of groups $\mathcal{G}$ is defined as

$$\pi_1(\mathcal{G}) = \langle G_v, t_v : v \in \mathcal{G}^0, e \in \mathcal{G}^1 | t_v t_\tau = 1, t_e^{-1} h_e(g)t_e = h_{\tau}(g) \text{ for all } g \in G_e, e \in \mathcal{G}^1 \rangle$$

We have seen that any geodesic current $\mu$ decomposes as in 7.2, where each component $\mu_i$ is supported on a subsurface $X_i$. The decomposition of $X$ in the subsurfaces $X_i$ is given by the family of so-called special geodesics $\{c_1, \ldots, c_n\}$. Note that a special geodesic does not need to be separating.

Given $\mu \in \text{Curr}(X)$ we define a graph of groups $(\mathcal{G}, Y)$ as follows. For each subsurface $X_i$ we define a vertex $v_i$, and for each special curve $c$ we define an edge $e_{i,j}$ between the vertices $v_i$ and $v_j$ if the curve $c$ is the boundary of $X_i$ and $X_j$. Note that we may have $i = j$ when $c$ is not separating, and $e_{i,j}$ is in this case a loop based at $v_i = v_j$.

For each edge $e \in \mathcal{G}^1$ we put $G_e := \mathbb{Z}$, and for each $v \in \mathcal{G}^0$ we put $G_v := \pi_1(X_i)$. Let $e \in \mathcal{G}^1$ be an edge joining $v_i$ to $v_j$, and let $c$ be the special geodesic bounding $X_i$ and $X_j$. The monomorphisms $h_{v_i,e} : G_e \to G_{v_i}$ is given by $\iota_e : \mathbb{Z} \to \pi_1(X_i)$, i.e. the induced map from the inclusion $c \hookrightarrow X_i$ at the level of fundamental groups.

The following result is obtained by simply invoking classical Bass-Serre theory (see [Ser03]).

Proposition 7.13. The fundamental group $\pi_1(X)$ has a graph of groups decomposition. In particular, $\pi_1(X)$ is isomorphic to $\pi_1(\mathcal{G}, Y)$. Furthermore, the fundamental group $\pi_1(\mathcal{G})$ acts on the simplicial tree $T = T(m)$ dual tree of the special multi-curve $m = \sum_j a_j s_j$. The factor graph $T/\pi_1(\mathcal{G})$ is isomorphic to $Y$, and for such action we have

1. $\text{stab}_{\pi_1(\mathcal{G})}(v) \cong G_v$ for all $v \in T^0$;
2. $\text{stab}_{\pi_1(\mathcal{G})}(e) \cong G_e$ for all $e \in T^1$. 

8. Actions

The dual spaces $X_\mu$ are naturally equipped with a surface group action. We study the properties of this action.

Recall that $d_\mu$ denotes both the pseudo-distance in $\tilde{X}$ as well as the induced distance on $X_\mu$. We will distinguish them by writing points in $\tilde{X}$ with the overline notation $\overline{x} \in \tilde{X}$, and points in $X_\mu$ without it, $x \in X_\mu$.

**Definition 8.1** (Action). Let $\pi_\mu: \tilde{X} \to X_\mu$ be the natural quotient projection. Given $x \in X_\mu$, let $\overline{x} \in \pi_\mu^{-1}(x)$. Let $g \in \pi_1(X)$. Define $g \cdot x = \pi_\mu(g(\overline{x}))$.

**Lemma 8.2.** $\pi_1(X)$ acts by isometries on $X_\mu$.

**Proof.** First, we show that the action in Definition 8.1 is well-defined. Indeed, suppose that $\overline{x}, \overline{y} \in \pi_\mu^{-1}(x)$, i.e., $d_\mu(\overline{x}, \overline{y}) = 0$, i.e. $\frac{1}{2}\mu(G[\overline{x}, \overline{y}]) + \frac{1}{2}\mu(G(\overline{x}, \overline{y})) = 0$. Since $\mu$ is $\pi_1(X)$-invariant, we have

$$0 = \frac{1}{2}\mu(gG[\overline{x}, \overline{y}]) + \frac{1}{2}\mu(gG(\overline{x}, \overline{y})) =$$

$$\frac{1}{2}\mu(G[g \cdot \overline{x}, g \cdot \overline{y}]) + \frac{1}{2}\mu(G(g \cdot \overline{x}, g \cdot \overline{y})) =$$

$$d_\mu(g \cdot \overline{x}, g \cdot \overline{y}),$$

which shows that $g \cdot \overline{y} \in \pi_\mu^{-1}(g \cdot y)$, and thus shows that the action is well-defined. The same computation for two arbitrary points $x, y \in X_\mu$ with lifts $\overline{x}, \overline{y}$ shows that the action is by isometries. $\square$

**Definition 8.3** (Translation length). For $g \in \pi_1(X)$, we define

$$\ell_\mu(g) := \inf_{x \in X_\mu} d_\mu(x, g \cdot x).$$

**Lemma 8.4.** For $g \in \pi_1(X)$,

$$\ell_{X_\mu}(g) = i(\mu, g)$$

and thus

$$\ell_{X_\mu}(g^n) = n\ell_{X_\mu}(g).$$

**Proof.** The first can be found in [BIPP21a, Lemma 4.7]. The second result follows from general properties about Bonahon’s intersection number. $\square$
8.1. Coboundedness.

Definition 8.5 (Cobounded/cocompact). Let $X$ be a metric space. An action $(G, X)$ is said to be \textit{cobounded} if there exists a bounded set $B \subset X$ so that $X = GB$, i.e., $X = \cup_g g(B)$. An action $(G, X)$ is said to be \textit{cocompact} if there exists a compact set $K \subset X$ so that $X = GK$, i.e., $X = \cup_g g(K)$.

Let $X, Y$ be metric spaces and $f: X \to Y$ be map (not necessarily continuous). We say that $f$ is \textit{bornologous} if for every $R > 0$, there is $S > 0$ so that if $d_X(x, y) < R$ then $d_Y(f(x), f(y)) < S$. We say that $f$ is \textit{large scale Lipschitz} if there exist constants $c > 0$ and $A$ so that

$$d_Y(f(x), f(y)) \leq c \cdot d_X(x, y) + A.$$ 

$f$ is a \textit{quasi-isometric embedding} if there exist constants $c > 1$ and $A$ so that

$$1/c \cdot d_X(x, y) - A \leq d_Y(f(x), f(y)) \leq c \cdot d_X(x, y) + A.$$ 

$f$ is \textit{coarsely surjective} if there exists a constant $C$ so that for every $y \in Y$ there is $z \in g(X)$ so that $d_Y(z, y) < C$. We say that $f$ is a \textit{quasi-isometry} if it is a coarsely surjective quasi-isometric embedding.

Lemma 8.6 ([Roe03, Lemma 1.10]). Let $X$ be a length space and $Y$ a metric space. $f$ is bornologous if and only if it is large scale Lipschitz.

Proposition 8.7. Let $\mu \in \text{Curr}(X)$. Then $\pi_\mu: \tilde{X} \to X_\mu$ is large scale Lipschitz.

Proof. Given $R > 0$, we will find $S_R > 0$ so that if $d_{\tilde{X}}(x, y) < R$, then $d_{X_\mu}(\pi_\mu(x), \pi_\mu(y)) < S_R$. Let $K \subset \tilde{X}$ be a compact fundamental domain of the action of $\pi_1(X)$ on $\tilde{X}$, and let $D$ be the hyperbolic diameter of $K$. If $d_{\tilde{X}}(x, y) \leq D$, then there exist $g_1, g_2 \in \pi_1(X)$ (depending on $x, y$), so that $x \in g_1K$ and $x \in g_1g_2(K)$ (where $g_1, g_2$ could be the identity), and $g_2(K)$ is adjacent to $K$. Indeed, otherwise $d_{\tilde{X}}(x, y) > D$, contradicting the choice of $D$. Thus, $x, y \in g_1(K) \cup g_1g_2(K)$. Let $G(K)$ be the set of geodesics intersecting $g_1(K) \cup g_1g_2(K)$. We have, by subadditivity of $\mu$ and $\pi_1(X)$ invariance, that

$$\mu(G(K)) = \mu(G(g_1(K)) \cup G(g_1g_2(K))) \leq \mu(G(g_1(K)) \cup G(g_1g_2(K))) = 2\mu(G(K))$$

is finite (and independent of $x, y$). Furthermore, $\mu(G(K))$ is finite, since $K$ is compact. Thus, we have proven that if $d_{\tilde{X}}(x, y) \leq D$, then $d_{X_\mu}(\pi_\mu(x), \pi_\mu(y)) \leq 2\mu(G(K))$. If $D < d_{\tilde{X}}(x, y) \leq 2D$, the same proof now using three consecutively adjacent fundamental regions will yield that $d_{\tilde{X}}(x, y) \leq 3\mu(G(K))$. An induction argument then proves the general case.

Lemma 8.8. Let $\mu$ be any geodesic current. The action of $\pi_1(X)$ on $X_\mu$ is cobounded.
Proof. By Proposition 8.7, there exist constants \(c > 0\) and \(A \geq 0\) so that for all \(\overline{x}, \overline{y} \in \tilde{X}\),

\[
d_{\mu}(\pi_{\mu}(\overline{x}), \pi_{\mu}(\overline{y})) \leq c \cdot d_{\tilde{X}}(\overline{x}, \overline{y}) + A.
\]

Thus, for every \(x \in X_{\mu}\), and \(r > 0\), we have

\[
B_{\mu}(\overline{x}, r) \subseteq B_{\tilde{X}}(\overline{x}, c \cdot r + A),
\]

where \(\overline{x} \in \pi_{\mu}^{-1}(x)\). Since \(\pi_{1}(X)\) acts cocompactly on \(\tilde{X}\), there exist a compact subset \(K \subseteq \tilde{X}\) so that \(\tilde{X} = \cup_{g} gK\). Take a hyperbolic ball \(B_{\tilde{X}}(\overline{x}, r)\) so that

\[
K \subseteq B_{\tilde{X}}(\overline{x}, r)
\]

By the above,

\[
K \subseteq B_{\tilde{X}}(\overline{x}, c \cdot r + A).
\]

Thus, by equivariance of \(\pi_{\mu}\), we have

\[
X_{\mu} = \bigcup_{g \in \pi_{1}(X)} gB_{\tilde{X}}(\overline{x}, c \cdot r + A),
\]

as we wanted to see. \(\square\)

If \(\mu\) has no atoms, we can moreover show the action is cocompact.

Lemma 8.9. If \(\mu\) is a current with no atoms, then the action of \(\pi_{1}(X)\) on \(X_{\mu}\) is cocompact.

Proof. Since \(\pi_{1}(X)\) acts cocompactly on \(\tilde{X}\), there exist a compact subset \(K \subseteq \tilde{X}\) so that \(\tilde{X} = \cup_{g} gK\). Let \(K' = \pi_{\mu}(K)\), which is compact, by continuity of \(\pi_{\mu}\). Thus, by equivariance of \(\pi_{\mu}\), we have

\[
X_{\mu} = \bigcup_{g \in \pi_{1}(X)} gK'
\]

as we wanted to see. \(\square\)

8.2. Boundedness and compactness of balls. We show that a geodesic current \(\mu\) is filling if and only if all balls in \(X_{\mu}\) are bounded. Moreover, we show that if \(\mu\) has no atoms and it is filling, then \(X_{\mu}\) is a proper metric space.

The following proof is adapted from [Glo17, Proposition 2.8]. We claim no originality, but supply details and point out a gap in that proof.

Proposition 8.10. Let \(\overline{x} \in \tilde{X}\) and \(\mu \in \text{Curr}(X)\). The following are equivalent:

1. The current \(\mu\) is filling;
2. for every \(\overline{x} \in \tilde{X}\), \(B_{\mu}(\overline{x}, 0)\) is bounded;
3. for every \(\overline{x} \in \tilde{X}\) and every \(r \geq 0\), \(B_{\mu}(\overline{x}, r)\) is bounded.
Proof. 3. $\Rightarrow$ 2. is obvious.

2. $\Rightarrow$ 1. Assume $\mu$ is not filling, then there exists a geodesic line $g \in \mathcal{G}(\tilde{X})$ which does not intersect any line in $\text{supp}(\mu)$. It follows that $d_{\mu}(\vec{x}, \vec{y}) = 0$ for any $\vec{x}, \vec{y}$ on the geodesic $\gamma$. In particular $\gamma \subset B_{\mu}(\vec{x}, 0)$, and hence $B_{\mu}(\vec{x}, 0)$ is not bounded, contradiction.

1. $\Rightarrow$ 3. Now we assume that $\mu$ is filling. By Lemma 8.14, there exists $R > 0$ so that $B_{\mu}(\vec{x}, r) \subset B_{\mu}(\vec{x}, R)$. Since $B_{\mu}(\vec{x}, R)$ is bounded, it follows that $B_{\mu}(x, \varepsilon)$ is bounded.

\[\square\]

**Lemma 8.11.** Let $\mu$ be a filling geodesic current. There exist a constant $C(\mu) > 0$ and a constant $\varepsilon(C, \mu) > 0$, so that if $d_{\tilde{X}}(\vec{x}, \vec{y}) > C + 1$ then $d_{\mu}(x, y) > \varepsilon$.

**Proof.** The surface $X$ has been endowed with a fixed hyperbolic structure. Let $\phi_t$ denote the geodesic flow on $T^1X$ and $\pi : T^1X \to X$ the canonical projection. Define $r : T^1X \to \mathbb{R}$ to be the first return time of $\phi_t$ to the support of $\mu$.

\[r(v) := \inf\{t \in \mathbb{R} : \pi(\phi_v(t)) \cap \text{supp}\mu \neq \emptyset\}\]

In other words, $r(v)$ is the first time when the geodesic emanating from $v \in T^1X$ intersects the support of $\mu$ transversely. Notice that $r(v)$ is finite since $\mu$ is filling.

Since the function $r$ is upper semi-continuous, it follows that it admits an upper bound $C > 0$ on the compact set $T^1X$. By lifting $r$ to $\tilde{X}$ we have the upper bound $r(v) \leq C$ for all $v \in T^1\tilde{X}$.

Now let us fix $\vec{x}, \vec{y} \in \tilde{X}$ such that $d_{\tilde{X}}(\vec{x}, \vec{y}) \geq C + 1$. Then the segment $[\vec{x}, \vec{y}]$ must intersect transversely some geodesic line in the support of $\mu$, and therefore $\mu(G[\vec{x}, \vec{y}]) > 0$. If for every $n$, we could find $\vec{x}_n, \vec{y}_n \in \tilde{X}$ so that $d_{\tilde{X}}(\vec{x}_n, \vec{y}_n)$ and $\mu(G[\vec{x}_n, \vec{y}_n]) < \frac{1}{n}$, then compactness of $X$ would yield points $\vec{x}_\infty, \vec{y}_\infty \in \tilde{X}$ with $d_{\tilde{X}}(\vec{x}_\infty, \vec{y}_\infty) \geq C + 1$ but $\mu(G[\vec{x}_\infty, \vec{y}_\infty]) = 0$, which would contradict the choice of $C$.

Thus, there must exist a uniform lower bound $\varepsilon > 0$ so that for every $\vec{x}, \vec{y} \in \tilde{X}$ such that $d_{\tilde{X}}(\vec{x}, \vec{y}) > C + 1$, we have $\mu(G[\vec{x}, \vec{y}]) > \varepsilon > 0$. This shows that $B_{\mu}(\vec{x}, \varepsilon) \subset B_{\tilde{X}}(\vec{x}, C + 1)$, and thus $B_{\mu}(\vec{x}, \varepsilon)$ is bounded. \[\square\]

**Remark 8.12.** Note that in [Glo17, Proposition 2.8] it is claimed that $B_{\mu}(\vec{x}, \varepsilon)$ is compact. We contest this: in fact, it is not true in general. Observe that since, by Lemma 8.11, $B_{\mu}(\vec{x}, \varepsilon)$ is bounded, its compactness is equivalent to $B_{\mu}(\vec{x}, \varepsilon)$ being closed in $\tilde{X}$. At the same time,

\[B_{\mu}(\vec{x}, \varepsilon) = \{\vec{y} \in \tilde{X} : d_{\mu}(\vec{y}, \vec{x}) \leq \varepsilon\}\]
which is closed if and only if $d_\mu(y, \cdot)$ is lower semicontinuous. However, we know from Propositions 4.6 and 4.11 this happens if and only if $\mu$ has no atoms. We collect this in the next proposition.

A metric space $(X, d)$ is \textit{proper} if closed balls are compact.

**Proposition 8.13.** Given a geodesic current $\mu$ on $X$. If $\mu$ is filling and has no atoms, then $X_\mu$ is proper.

**Proof.** Assume $\mu$ is filling and has no atoms. Let $x \in X_\mu$ and $B = B_\mu(x, r)$ be a closed $d_\mu$-ball. The preimage $\pi^{-1}_\mu(B)$ is the closed $d_\mu$-ball in $\tilde{X}$ (in the pseudo-metric $d_\mu$), which by Proposition 8.11 is bounded. If $\mu$ has no atoms, then $\pi_\mu$ is continuous, thus $\pi^{-1}_\mu(B)$ is closed. Altogether, this shows $\pi^{-1}_\mu(B)$ is compact, and continuity of $\pi_\mu$ again implies $B$ is compact, so $X_\mu$ is proper. $\Box$

**Lemma 8.14.** Consider a filling geodesic current $\mu$ on a closed hyperbolic surface $X$. There exists a constant $\varepsilon(\mu, X) > 0$, so that for all $r \geq \varepsilon$, there exists a constant $R(r) > 0$, so that for all $x \in \tilde{X}$,

$$B_\mu(x, r) \subset B_\mu(x, R)$$

**Proof.** Let $C$ and $\varepsilon$ be the constants given by Lemma 8.11. Let $\pi, \bar{y} \in \tilde{X}$ with $d_\tilde{X}(\pi, \bar{y}) \geq C + 1$ and and $d_\mu(\pi, \bar{y}) \geq \varepsilon$. In other words, we have

$$B_\mu(\pi, \varepsilon) = \{\bar{y} \in \tilde{X}: d_\mu(\pi, \bar{y}) \leq \varepsilon\} \subseteq B_\mu(\pi, C + 1)$$

Now we proceed by induction on the distance $d_\tilde{X}(\pi, \bar{y}) = N \cdot (C + 1)$, for any $N \in \mathbb{N}$. The induction basis has been proven already. Assume that the following implication holds

$$d_\tilde{X}(\pi, \bar{y}) \geq (N - 1) \cdot (C + 1) \Rightarrow d_\mu(\pi, \bar{y}) \geq (N - 1) \cdot \varepsilon$$

assume that $d_\tilde{X}(\pi, \bar{y}) \geq N \cdot (C + 1)$. We want to show that $d_\mu(\pi, \bar{y}) \geq N \cdot \varepsilon$.

Let $\gamma \in \mathcal{G}(\tilde{X})$ be the geodesic line joining $\pi$ to $\bar{y}$ in $\tilde{X}$. Since $d_\tilde{X}(\pi, \bar{y}) \geq N(C + 1)$, then there must exist $\bar{z} \in \gamma$ such that $d_\tilde{X}(\pi, \bar{z}) \geq (N - 1)(C + 1)$ and $d_\tilde{X}(\bar{z}, \bar{y}) \geq C + 1$, and hence, by induction hypothesis $d_\mu(\pi, \bar{z}) \geq (N - 1)(C + 1)$ and $d_\mu(\bar{z}, \bar{y}) \geq \varepsilon$. Since the pseudo distance $d_\mu$ is straight, we have

$$d_\mu(\pi, \bar{y}) = d_\mu(\pi, \bar{z}) + d_\mu(\bar{z}, \bar{y}) \geq (N - 1)(C + 1) + (C + 1) = N(C + 1)$$

as wanted. $\Box$

**Proposition 8.15.** Let $\mu \in \text{Curr}(X)$. If $\text{sys}(\mu) > 0$, then $\pi_\mu : \tilde{X} \to X_\mu$ is a quasi-isometry.
Proof. First, note that for every \( x \in X_\mu \), \( \pi_\mu^{-1}(x) \) is a bounded subset of \( \bar{X} \). Indeed, its hyperbolic diameter is upper bounded by the maximum of the return time, as in the proof of Lemma 8.11. Making an arbitrary choice of \( z_x \in \pi_\mu^{-1}(x) \) for each \( x \in X_\mu \), we define a map \( g : X_\mu \to \bar{X} \), given by \( x \mapsto z_x \). This map is bornologous by Lemma 8.14. It follows then by Lemma 8.6, that there exist constants \( c > 0 \) and \( A \) so that

\[
d_{\bar{X}}(x, y) \leq c \cdot d_\mu(\pi_\mu(x), \pi_\mu(y)) + A.
\]

for every \( x, y \in g(X_\mu) \). This, together with Lemma 8.7 shows that \( \pi_\mu \) is a quasi-isometric embedding. We note that \( g \) is a coarsely surjective: the upper bound \( C \) of the return time satisfies the property that, for every \( x \in \bar{X} \), there is \( y \in g(X_\mu) \) so that

\[
d_{\bar{X}}(x, y) < C.
\]

From this it follows \( \pi_\mu \) is a quasi-isometry. \( \square \)

The following definition can be found in [BH11, Definition I.8.2].

**Definition 8.16** (Proper action). Let \( G \) a group acting by isometries on a metric space \( X \). The action is said to be proper if for each \( x \in X \), there exists \( r > 0 \), so that the set \( \{ g \in G : gB(x, r) \cap B(x, r) \neq \emptyset \} \) is finite, where \( B(x, r) \) denotes an open ball of radius \( r \) centered at \( x \).

The action \( \pi_1(X) \) on \( X_\mu \) is not proper in general. In this section we characterize when this happens.

**Proposition 8.17.** If \( \nu \) is a geodesic current with a subcurrent of type 2 in its decomposition, then the action of \( \pi_1(X) \) on \( X_\mu \) is not proper.

**Proof.** It is a Corollary of Theorem 9.9, proven in the next section. \( \square \)

**Proposition 8.18.** If \( \text{sys}(\mu) > 0 \), then \( \pi_1(X) \) acts properly on \( X_\mu \).

**Proof.** Suppose not, i.e., suppose there exists \( x \) so that for all \( r > 0 \), the set \( \{ g \in G : gB_\mu(x, r) \cap B_\mu(x, r) \neq \emptyset \} \) is infinite. Consider a sequence \( (g_n) \) of elements in \( \pi_1(X) \). Choose \( r > \varepsilon \), where \( \varepsilon \) is the constant in Lemma 8.14, so that \( B_\mu(\overline{x}, r) \cap g_i(B_\mu(\overline{x}, r)) \neq \emptyset \). By Lemma 8.14, we have,

\[
B_\mu(\overline{x}, r) \cap g_i(B_\mu(\overline{x}, r)) \subset B_{\bar{X}}(\overline{x}, R) \cap g_i(B_{\bar{X}}(\overline{x}, R)).
\]

This contradicts that the action of \( \pi_1(X) \) on \( \bar{X} \) is proper. \( \square \)

The following theorem follows immediately from the previous propositions.

**Theorem 8.19.** The action of \( \pi_1(X) \) on \( X_\mu \) is proper if and only if \( \mu \) has no type 2 subcurrents in its decomposition.

**Proof.** If \( \text{sys}(\mu) > 0 \), then \( \mu \) has no type 2 subcurrents in its decomposition. Proposition 8.18 implies the action of \( \pi_1(X) \) on \( X_\mu \) is proper. If \( \text{sys}(\mu) = 0 \) then either \( \mu \) has a subcurrent of type 2 in its decomposition, in which case Proposition 8.17
implies the action is not proper, or \( \mu \) has no components of type 2 in its decomposition. In this case, by the graph of groups decomposition in Proposition 7.13, all the pieces in the tree graded space structure of \( X_\mu \) correspond to duals of subcurrents of type 1, and thus of positive systole \( \mu_i \) in their respective subsurfaces \( X_i \). Then, by Proposition 8.18 it follows the action of \( \iota_*(\pi_1(X_i)) \subset X \) on \( X_\mu \) is proper. Hence, by the graph of groups decomposition of \( \pi_1(X) \), it follows that the action of \( \pi_1(X) \) on \( X_\mu \) is proper. □

8.3. Freeness. The action of \( \pi_1(X) \) on \( X_\mu \) is not always free. In this section we characterize when this happens.

The following lemma relates stabilizers of points of \( X_\mu \) and the topology of the support of a measured lamination \( \mu \), and it’s just an observation.

**Lemma 8.20.** Let \( \mu \) be a non-trivial geodesic current. Let \( C \) be the set of connected components of \( \tilde{X} - \text{supp} \mu \).

1. If \( x \in C \in C \), then the stabilizer of \( p(x) \) is equal to the (set-wise) stabilizer of \( C \).
2. If \( x \in \gamma \in \text{supp} \mu \), so that \( \gamma \) doesn’t intersect any geodesic in \( \text{supp} \mu \), then the stabilizer of \( p(x) \) in \( X_\mu \) is the stabilizer of \( \gamma \) in \( \tilde{X} \).
3. If \( x \in \gamma \in \text{supp} \mu \), so that \( \gamma \) intersects some geodesic in \( \text{supp} \mu \), then the stabilizer of \( p(x) \) in \( X_\mu \) is trivial.

**Proof.**  
(1) Clear.
(2) Clear.
(3) Let \( \gamma' \in \text{supp}(\mu) \) so that \( \gamma \cap \gamma' \neq \emptyset \). Then \( \pi_{\mu}^{-1}(x) \) is a proper geodesic subsegment \( \eta \) of \( \gamma \), and the (set-wise) stabilizer of \( \eta \) is trivial. □

**Lemma 8.21.** Let \( \mu \) be a non-trivial geodesic current. Then \( \text{sys}(\mu) > 0 \) if and only if \( \mu \) consists of one and only one type 1 component.

**Proof.** Since \( \mu \) is non-trivial, then it has at least one component. If \( \mu \) has more than one type 1 component, then, by the decomposition theorem 7.2, it has at least one type 3 component \( c \). But then \( i(\mu, c) = 0 \), so \( \text{sys}(\mu) = 0 \). If \( \mu \) has no type 1 component, since \( \mu \) is non-trivial, it must have some non-trivial type 2 or type 3 component. A type 3 component yields a non-trivial stabilizer as before. A unique type 2 component means, by Theorem [BIPP21a, Theorem 1.7], that \( \text{sys}(\mu) = 0 \). □

**Lemma 8.22.** Let \( \mu \) be a geodesic current. Then the action of \( \pi_1(X) \) on \( X_\mu \) is free if and only if \( \mu \) has only one component in its decomposition.

**Proof.** Suppose that \( \mu \) has more than one component in its decomposition. Then, it has at least one component of type 3, which is a simple closed curve not intersected
by any other geodesic of the support of $\mu$. By Lemma 8.20, the stabilizer of $p(\gamma) \in X_\mu$ is infinite cyclic, so the action of $\pi_1(X)$ on $X_\mu$ is not free. Now, suppose $\mu$ has only one component in its decomposition. We show that the action is free. Assume first $\text{sys}(\mu) > 0$. If $g \in \pi_1(X)$ had a fixed point, i.e., $g(x) = x$ for some $x \in X_\mu$. Then $\ell_X(g) = 0$, i.e., $i(\mu, [g]) = 0$, contradicting that the systole is positive. Assume, to finish, that $\text{sys}(\mu) = 0$. If $g \in \pi_1(X)$ had a fixed point, i.e., $g(x) = x$ for some $x \in X_\mu$. Then $\ell_X(g) = 0$, i.e., $i(\mu, [g]) = 0$. If $c = [g]$ is a closed curve, then $i(\mu, c) = 0$, contradicting that $\mu$ is a type 2 subcurrent (and thus filling as a measured lamination within its support subsurface).

Remark 8.23. We give here a brief comment on isometry types and axes, which will not be used in the sequel. Recall that for Gromov hyperbolic spaces, there is an analogous classification of isometries in three types: hyperbolic, parabolic and elliptic. The classification can be described also in terms of the number of fixed points at infinity (see [CDP90, Chapter 10]). Observe that from the previous results, it follows that $g \in \pi_1(X)$ acts as an elliptic or hyperbolic isometry. Let $g \in \pi_1(X)$. If $i(\mu, [g]) > 0$ then $\ell_{X_\mu}(g) > 0$, and thus $g$ is hyperbolic (by [Fuj15, 2.2]). Suppose $i(\mu, [g]) = 0$. If $[g]$ is a special geodesic in the decomposition of $\mu$, then it follows it has a fixed point, as in the proof of Lemma 8.22. Otherwise, it’s in one of the subsurfaces $X_i$ of the decomposition of $X$. By the same argument as in Lemma 8.22, it must be in a subsurface that does not contain any projections of leaves in the support of $\mu$, and so $g$ has a fixed point. Similarly, one can define a notion of axis of a hyperbolic element $g$ as

$$T_g = \{ x \in X_\mu : d_\mu(x, g(x)) = \ell_{X_\mu}(g) \}.$$ 

By [MZ19, Proposition 4.4], it follows that the hyperbolic axis $A_g$ of $g$ is contained in $\pi_1^{-1}(T_g)$. In general, $T_g$ is geodesic (whenever $X_\mu$ is equipped with a geodesic structure) and corresponds to the sequence of complementary regions $R_i$ in $\widetilde{X}$ traversed by $A_g$, as in the setup of the decomposition theorem of Section 7. Finally, if two hyperbolic elements $g, h$ in $X_\mu$ have intersecting hyperbolic axes $A_g, A_h$ in $X$, then their axes $T_g$ and $T_h$ in the dual also intersect.

Theorem 8.24. If $\text{sys}(\mu) > 0$, then $\pi_1(X)$ acts properly, coboundedly and freely on $X_\mu$.

Proof. It follows by Lemma 8.8, Lemma 8.18 and Lemma 8.22. □

9. Completeness

Recall that a metric space $(X, d)$ is complete if every Cauchy sequence converges. In this section we characterize when a dual $X_\mu$ is complete in terms of the structural
decomposition of $\mu$ (see Theorem 7.2). The main theorem of this section, Theorem 9.9, shows that $X_\mu$ is complete if and only if it has no type 2 components in its structural decomposition, i.e., components which are non-discrete measured laminations. We split the proof into two subsections. In the first one, we analyze the case of measured laminations (type 2 and type 3 components), and in the second, the case of components with positive systole (type 1 components).

9.1. Case of measured laminations. In this subsection we prove the following result.

**Theorem 9.1.** Let $\mu \in \mathcal{ML}(X) \subset \text{Curr}(X)$ be a measured lamination. Then $X_\mu$ is complete if and only if $\mu$ is a simple multi-curve.

Recall that by Lemma 2.7 0-dimensional atoms of geodesic currents are on lifts of closed geodesics.

9.1.1. *The dual tree of a lamination.* In Definition 3.1 we introduced the dual of geodesic currents. By Theorem 6.13, $X_\mu$ is a 0-hyperbolic space in the sense of Gromov, and hence can be isometrically embedded in a unique real tree $\hat{X}_\mu$. We call such tree the *dual tree* of $\mu$ (see Theorem A.2 for the details).

Geometrically, the dual tree in the case of a simple multi-curve is the dual to the planar infinite graph given by the lifts of the lamination, and can be constructed explicitly as follows. Let $\tilde{\mu}$ be the lift to $\tilde{X}$ of the lamination $\mu$. A *complementary region* is a connected component of the complement $\tilde{X} \setminus \tilde{\mu}$. Let $C$ be a complementary region. Note that for any two points $x, y \in C$ one has $d_{\tilde{\mu}}(x, y) = 0$, and hence every complementary region is collapsed to one point in the quotient $X_\mu$. The same happens for any two points $x, y$ on the same leaf of the lamination. Hence every leaf of $\tilde{\mu}$ also gets collapsed to a point in $\hat{X}_\mu$. If $C_1, C_2$ are two distinct complementary regions, and $x \in C_1, y \in C_2$, then we construct the dual tree by adding a segment of length $d_{\mu}(x, y)$ between $x$ and $y$.

The following lemma is immediate, since $X_\mu$ is a discrete set of points and $\hat{X}_\mu$ is a simplicial tree.

**Lemma 9.2.** Let $\mu$ be a lamination induced by a multi-curve, then the dual $X_\mu$ (as well as the dual tree $\hat{X}_\mu$) are complete.

We will now show that if $\mu$ is a non-discrete measured lamination, then $X_\mu$ is not complete.

We will make use of the notion of *exotic ray* of a lamination as defined by T. Torkaman and Y. Zhang in [TZ21].

**Definition 9.3 (Geodesic ray).** A *geodesic ray* $r$ on a complete hyperbolic surface of finite area $X = \tilde{X}/\Gamma$ is a geodesic isometric immersion $r : [0, \infty) \to X$. 
Given a lamination $\Lambda$ on $X$ which is not a multi-curve, the intersection $i(r, \Lambda)$ between the ray and the lamination is generically infinite. There are two obvious cases when a ray has finite intersection with $\Lambda$:  

1. The ray $r$ is asymptotic to a leaf of $\Lambda$;  
2. The ray $r$ is eventually disjoint from $\Lambda$.

Nevertheless, these are not the only two possibilities. Namely, there exist geodesic rays such that $i(r, \Lambda) < \infty$ but are neither asymptotic to a leaf of $\Lambda$, nor eventually disjoint. Such rays are called exotic rays.

**Theorem 9.4 ([TZ21, Theorem 1.1]).** Let $\Lambda$ be a non-multi-curve lamination. Then there exist exotic rays for $\Lambda$.

We will make use of the above result to prove a characterization of metric completeness for dual trees of currents of lamination type.

**Lemma 9.5.** Let $\mu$ be a geodesic current corresponding to a measured lamination which is not a simple multi-curve. Then the current dual space $X_\mu$ is incomplete.

**Proof.** Let $r: [0, \infty) \to X$ be an exotic ray for $\mu$. Let $M := i(\mu, r) < \infty$. Now we define a nested family of sub-arcs $\tau_n: [0, 1] \to X$ of $r$. Let $\tau_1: [0, 1] \to X$ be any isometrically embedded sub-arc of $r$ such that $\tau_1(0) = r(0) = x_0$, and $0 < i(\tau_1, \mu) < \frac{M}{2}$. Similarly, we define $\tau_2$ such that $\tau_2(0) = r(0) = x_0$ and $\frac{M}{2} < i(\tau_2, \mu) < \frac{2}{3}M$. In general, we define $\tau_n$ such that $\tau_n(0) = r(0) = x_0$ and $\frac{n-1}{n}M < i(\tau_n, \mu) < \frac{n+1}{n}M$.

Note that $\tau_1 \subseteq \tau_2 \subseteq \ldots \tau_n \subseteq \ldots$ is a nested increasing sequence of sub-arcs of $r$, and $\bigcup_{n=1}^{\infty} \tau_i = r$.

We define $\overline{x}_n := \tau_n(1)$. Note that the sequence $(\overline{x}_n)$ exits all compact sets in $\tilde{X}$. Denote with $(x_n)$ the image sequence in the current dual space $X_\mu$ by the projection $\pi_\mu: \tilde{X} \to X_\mu$. Our goal is to show that the sequence $(\overline{x}_n)$ is Cauchy and does not converge in $X_\mu$.

In order to show that $(\overline{x}_n)$ is Cauchy we estimate $d_\mu(x_n, x_{n+1})$.

$$d_\mu(x_n, x_{n+1}) = \frac{1}{2} \{ \mu(G([\overline{x}_n, \overline{x}_{n+1}])) + \mu(G[\overline{x}_n, \overline{x}_{n+1}]) \}$$
\[ \leq i(\mu, [\overline{x}_n, \overline{x}_{n+1}]) \leq i(\mu, \tau_{n+1}) - i(\mu, \tau_n) \]
\[ \leq \frac{n+M}{n} - \frac{n-1}{n}M = \frac{2}{n^2 + 2n}M \to 0 \text{ for } n \to \infty \]

We are left to show that $(\overline{x}_n)$ does not converge. Assume by contradiction $x_n \to x \in X_\mu$. Then pick any $\overline{x} \in \pi^{-1}_\mu(x) \subseteq \tilde{X}$ and a small ball $B_\varepsilon(\overline{x})$ around it. It follows that there exists $N > 0$, so that $\overline{x}_n \notin B_\varepsilon(\overline{x})$ for all $n > N$, which is absurd as $(\overline{x}_n)$ exits all compact sets in $\tilde{X}$. \qed
9.2. General Case. The following result is elementary.

**Lemma 9.6.** If \((X, d)\) is a proper metric space, then it is complete and locally compact.

We start with a lemma.

**Lemma 9.7.** Let \(\mu\) be a geodesic current, and \(\gamma \in \text{supp} \mu\). The geodesic \(\gamma\) is not an atom of \(\mu\) if and only if for all \(\varepsilon > 0\), there exists \(\delta > 0\) so that \(\mu(N_\delta(\gamma)) < \varepsilon\), where \(N_\delta\) is the \(\delta\)-neighborhood of \(\gamma\) in \(G(\tilde{X})\).

**Proof.** If \(\gamma\) is an atom, then the condition is obviously violated.

If \(\gamma\) is not an atom, denote with \(\gamma_-\) and \(\gamma_+\) its endpoints, and consider the pencil \(P = P(\gamma_- (\gamma_- - \varepsilon, \gamma_+ + \varepsilon))\). We know from Lemma 2.8 that \(\mu(P) > 0\) if and only if it contains the axis of some non-zero element of \(\Gamma\) which projects to a closed geodesic in \(X\). Suppose that this is the case, i.e. there exists a geodesic line \(l \in P\) such that \(p(l)\) is a closed geodesic in \(X\). Then \(l\) is an atom for \(\mu\), contradicting the fact that \(\gamma\) is in the support of \(\mu\), and is asymptotic to \(l\). It follows that \(\mu(P(\gamma_- (\gamma_- - \varepsilon, \gamma_+ + \varepsilon))) = 0\).

Now we define a sequence of boxes as follows: start with

\[B_1 = N_\varepsilon(\gamma) = (\gamma_- - \varepsilon, \gamma_- + \varepsilon) \times (\gamma_+ - \varepsilon, \gamma_+ + \varepsilon)\]

and define

\[B_n = (\gamma_- - \varepsilon/n, \gamma_- + \varepsilon/n) \times (\gamma_+ - \varepsilon, \gamma_+ + \varepsilon)\]

by pinching one of the intervals of the corresponding box of geodesics so that \(\cap_{n=1}^\infty B_n = P(\gamma_- (\gamma_- - \varepsilon, \gamma_+ + \varepsilon))\). By continuity of measures from below, we have \(\lim_n \mu(\cap_{i=1}^n B_i) \rightarrow \mu(P(\gamma_- (\gamma_- - \varepsilon, \gamma_+ + \varepsilon))) = 0\). For any \(\varepsilon > 0\), if we take \(\delta\) so that \(N_\delta(\gamma) \subset B_n\), and \(\mu(B_n) < \varepsilon\), then we have \(\mu(N_\delta(\gamma)) < \varepsilon\), as wanted.

**Proposition 9.8.** If \(\text{sys}(\mu) > 0\), then \(X_\mu\) is complete.

**Proof.** Suppose \((x_n)\) is a Cauchy sequence in \(X_\mu\). Then, for \(N\) large, we can assume \(x_n \in B\) for \(n \geq N\), for some closed ball \(B = B_\mu(x, r)\). By Lemma 8.14, \(\pi_\mu^{-1}(B)\) is bounded. For each \(n \geq N\), let \(\overline{x_n} \in \pi_\mu^{-1}(x_n)\). The sequence \((\overline{x_n})\) is contained in \(\pi_\mu^{-1}(B)\), and thus it has a convergent subsequence \(\overline{x_{n_k}}\). Since \((x_n)\) is Cauchy, we can take \(r\) to be arbitrarily small, at the expense of taking \(N\) larger. Thus, by Lemma 9.7 there cannot be any atoms of \(\mu\) crossing \(\pi_\mu^{-1}(B)\) transversely. Hence, by the same argument as in the proof of Proposition 4.6, \(\pi_\mu\) restricted to \(\pi_\mu^{-1}(B)\) is continuous. Hence, \(x_{n_k} = \pi_\mu(\overline{x_{n_k}})\) is thus a convergent subsequence of \(x_n\). Since \(x_n\) is Cauchy, it must be convergent, and thus \(X_\mu\) is complete.

We can now conclude by invoking the structure theorem for geodesic currents, Theorem 7.2.

We are ready to prove the main theorem in this section.
Theorem 9.9. Let $\mu$ be a geodesic current on a closed surface $X$. $X_\mu$ is incomplete if and only if $\mu$ has a component of type 2 in its structural decomposition.

Proof. We use the structural decomposition theorem for the dual to study component by component. Prove that a graph of spaces is complete if and only if all the spaces are complete. By Proposition 9.8 and Proposition 9.5, $X_\mu$ is complete if and only if none of the components of $\mu$ is of type 2. $\square$

10. Topology

In this section we prove that the map sending a geodesic current to its dual space is a homeomorphism onto its image when the space of geodesic currents is equipped with the natural weak$^*$-topology and the space of duals is equipped with the also natural equivariant Gromov-Hausdorff topology introduced by [Pau88].

10.1. Topologies in the space of duals. We start by describing a topology on the space of duals. Let $\mathcal{Z}$ denote the space of Gromov hyperbolic spaces $Z$ with a cobounded action of $\pi_1(X)$ by isometries. One topology we can equip it with is the equivariant Gromov Hausdorff topology, introduced by Paulin [Pau88]. We start by defining the notion of $\varepsilon$-relation.

Definition 10.1 ($\varepsilon$-relation). Given $Z \in \mathcal{Z}$, $K \subset Z$ finite, and $P \subset \pi_1(X)$ a finite subset, we say that $(Z, K)$ is $\varepsilon$-related to $(Z', K')$ if

1. There exists a compact $K' \subset Z'$, and a relation $R$ between $K$ and $K'$ so that for all $x, y \in K$ and $x', y' \in K'$, if $xRx'$ and $yRy'$, then $|d(x, y) - d(x', y')| < \varepsilon$; and
2. For every $x \in K$, $x' \in K'$, and $g \in P$, if $g(x) \in K$, and $xRx'$, then $g(x') \in K'$ and $g(x)Rg(x')$.

We will think of the relation $R$ as a bijection $\varphi: K \to K'$, and write $(Z', K') \sim_{\varphi, \varepsilon} (Z, K)$ to denote that $(Z', K')$ is $\varepsilon$-related to $(Z, K)$ via the relation $\varphi$. We will also write $(Z', K') \sim_{\varepsilon} (Z, K)$ when we do not need to be explicit about the relation.

We now define a family of neighborhoods that yields the equivariant Gromov-Hausdorff topology.

Definition 10.2 (Standard neighborhoods of Gromov-Hausdorff topology). Now, given $Z \in \mathcal{Z}$, $K \subset Z$ finite subset and $\varepsilon > 0$, we define the subset $W(\mu, \varepsilon)$ to be

$$\{Z' \in \mathcal{Z} : (Z', K') \sim_{\varepsilon} (Z, K)\}.$$
constitutes a neighborhood subbasis of $\mu$, by Proposition 2.13.

Let $D(X) \subset Z$ be a subset of dual spaces of geodesic currents of $X$ (it is a subset of $Z$ by Theorem 6.13), equipped with the subspace topology inherited from the equivariant Gromov-Hausdorff topology.

**Remark 10.3.** Note that, strictly speaking, our definition of equivariant Gromov-Hausdorff differs from Paulin’s definition of equivariant Gromov-Hausdorff [Pau88]. We choose to work only with sets $K$ finite, whereas he allows $K$ to be compact. However, it follows from [Pau89, Proposition 4.1], that the two topologies are equivalent if the action of $\pi_1(X)$ is cocompact. This is the case, by [Pau89, Proposition 2.5], when one considers minimal actions of a finitely generated group on $\mathbb{R}$-trees. Thus, the topologies agree in the subspace $D_{ML}(X)$, i.e., the subspace of dual spaces coming from measured laminations. See Subsection 10.2 for more details. For the same reason, the two topologies agree when one restricts to the subspace of duals of currents without atoms, by Lemma 8.9.

**Theorem 10.4.** The map $\Psi : \text{Curr}(X) \to D(X)$ given by $\mu \mapsto X_\mu$ is a homeomorphism.

**Proof.** (1) Continuity of $\Psi$. Let $X = X_\mu$, and $K = \{x_1, \ldots, x_n\} \subset X$, $P = \{g\}$. Given a standard neighborhood of the equivariant Gromov-Hausdorff topology $U(X, K, P, \varepsilon)$, we find a neighborhood of the weak$^*$-topology $W$ so that $\Psi(W) \subset U$. Let $\alpha_1, \ldots, \alpha_n$ denote all the geodesic arcs of the type $[\bar{x}, \bar{y}] \subset \tilde{X}$, where $x, y \in K$, $x \neq y$, and $\pi_\mu(\bar{x}) = x$ and $\pi_\mu(\bar{y}) = y$. Note that, if needed, we can add more points to $K$ so that we get at least two distinct $\alpha_i$’s (we need at least 3 distinct points in $K$). Let $W = \cap_{i=1}^n W_\mu(\alpha_i, \varepsilon)$. We can then write

$$\cap_i G(\alpha_i) = \cap_{i \neq j} G(\alpha_i, \alpha_j)$$

where $G(\alpha_i, \alpha_j)$ denotes the set of geodesics intersecting $\alpha_i$ and $\alpha_j$ transversely, as in Proposition 2.13. Thus, we can write $W = \cap_{i \neq j} W_\mu(\alpha_i, \alpha_j, \varepsilon)$, where

$$W_\mu(\alpha_i, \alpha_j, \varepsilon) = \{\mu' \in \text{Curr}(X) : |\mu(G(\alpha_i, \alpha_j)) - \mu'(G(\alpha_i, \alpha_j))| < \varepsilon\}.$$

Let $\mu' \in W$. For every $x \in K \subset X_\mu$, pick one $\bar{x} \in \pi_\mu^{-1}(x) \subset \tilde{X}$, and let $\overline{K}$ be the finite set consisting of one $\overline{x} \in \pi^{-1}(x)$ for each $x \in K$. Finally, let $K' = \pi_\mu'(\overline{K})$, where we denote $\pi_\mu(\overline{x}) = x'$. We claim this defines an $\varepsilon$-relation between $x \in K$ and $x' \in K'$. We start by proving the first condition of Definition 10.1.

Note that

$$|\mu'(\bar{x}, \bar{y}) - \mu(\bar{x}, \bar{y})| < \varepsilon$$
can be written as
\[ |d_{\mu'}(x', y') - d_{\mu}(x, y)| < \varepsilon. \]

Thus, the first condition of the \( \varepsilon \)-relation is satisfied. As for the second condition, \( g(x) \in K \) means, by equivariance of \( \pi_\mu \), that \( g(\pi_\mu(\tau)) = \pi_\mu(g\tau) \in K \). Thus,
\[ g\tau \in \pi_\mu^{-1}(K), \]
and hence, by equivariance of \( \pi_{\mu'} \), we have
\[ g(x') = \pi_{\mu'}(g\tau) \in K' \]
which shows that \( g(x)Rg(x') \) and \( g(x') \in K' \), as we wanted. Thus, \( X_{\mu'} \in U(X, K, P, \varepsilon) \). This shows continuity of \( \Psi \).

(2) Injectivity of \( \Psi \). If \( X_\mu = X_{\mu'} \), then in particular \( \ell_{\mu'}([g]) = \ell_{\mu'}([g]) \) for all \( g \in \pi_1(X) \), i.e., by Lemma 8.4, and thus, we have \( \mu = \mu' \).

(3) Continuity of \( \Psi^{-1} : \Psi(\text{Curr}(X)) \to \text{Curr}(X) \). Let \( \mu \) be a geodesic current, \( X_\mu \) its dual space, \( K \) a finite subset of \( X_\mu \). For every \( x \in K \), let \( \tau \) be one element in \( \pi_\mu^{-1}(x) \), and call \( K \) the resulting finite set of points. Recall that \( W = W(K, \mu, \varepsilon) \) denotes the following open set of geodesic currents in the weak*-topology: the set of geodesic currents \( \beta \) so that for every \( x, y \in K \),
\[ |\beta(G[x, y]) - \mu(G[x, y])| < \varepsilon. \]
Moreover, these family of sets forms a subbasis for the weak*-topology (see Proposition 2.13).

Let \( U = U(X, K, P, \varepsilon) \) the standard neighborhood in the equivariant Gromov-Hausdorff topology. If \( X_{\mu'} \) is in \( U \), consider the corresponding \( \varepsilon \)-relation \( \varphi : K \to K' \), a bijective map satisfying
\[ |d_{\mu}(x, y) - d_{\mu'}(\varphi(x), \varphi(y))| < \varepsilon \]
for all \( x, y \in K \). For each \( x \in K \), consider one preimage \( \tau \in \pi_{\mu'}^{-1}(\varphi(x)) \), obtaining a finite set of points \( K' \subset \tilde{X} \). We can construct a homeomorphism \( f \) of \( \tilde{X} \) that sends each \( \tau \) to \( \tau' \). Since \( \overline{K} \cup \overline{K'} \) is a finite set of points, thus contained in a compact subset \( C \), we can assume \( f \) is the identity outside of \( C \). Hence, \( f \) is homotopic to the identity. Hence, the extension of \( f \) to the boundary \( \tilde{f} : \partial \tilde{X} \to \partial \tilde{X} \) is the identity. We then have, for every \( x, y \in K \), by Lemma 3.7,
\[ \mu'(G[\tau', \varphi']) = \mu'(G[f(\tau), f(\varphi)]) = f_*^{-1}\mu'(G[\tau, \varphi]) = \mu'(G[\tau, \varphi]). \]
Therefore, we have

|\mu(G[\bar{x}, \bar{y}]) - \mu'(G[\bar{x}, \bar{y}])| = \\
|\mu(G[\bar{x}', \bar{y}']) - \mu'(G[\bar{x}', \bar{y}'])| = \\
|d_\mu(\pi_\mu(\bar{x}), \pi_\mu(\bar{y})) - d_{\mu'}(\pi_{\mu'}(\bar{x}'), \pi_{\mu'}(\bar{y}'))| < \varepsilon.

where the last inequality follows because \(X_\mu\) and \(X_{\mu'}\) are \(\varepsilon\)-related. Thus, \(\mu' \in W(K, \mu, \varepsilon)\). This proves that \(\Psi^{-1}(U) \subset W\).

\(\square\)

10.2. Relation to Paulin’s work on \(\mathbb{R}\)-trees. Theorem 10.4 can be seen as a generalization of Paulin’s result [Pau89, Main Theorem].

Let \(G\) be a finitely generated group. The action of \(G\) on an \(\mathbb{R}\)-tree \(T\) is said to be minimal if the only invariant subtrees are \(\emptyset\) and \(T\). An end of an \(\mathbb{R}\)-tree \(T\) is an equivalence class of rays in \(T\), with two rays identified if their intersection is a ray. The action of \(G\) on \(T\) is said to be irreducible if there is no end of \(T\) fixed by every element of \(G\). The action is said to be reducible when it is not irreducible. Let \(T(G)\) the set of equivalence classes of \(\mathbb{R}\)-trees, containing more than one point, endowed with a minimal irreducible action of \(G\), where two \(\mathbb{R}\)-trees are identified whenever there is an isometry from one onto the other commuting with the actions.

Paulin defines the axial topology (or translation length topology) on \(T(G)\) by the family of neighborhoods

\[V(P, T, \varepsilon) = \{T' \in T : |\ell_T(g) - \ell_{T'}(g)| \text{ for all } g \in P\},\]

as \(P\) ranges over all finite subsets of \(G\), and \(\varepsilon > 0\). One can define a similar topology for any collection of metric spaces \(\mathcal{D}\) acted on by a fixed group of isometries \(G\). One can also endow \(T(G)\) with the equivariant Gromov-Hausdorff topology. When \(G = \pi_1(X)\) is a surface group, and one restricts furthermore to small actions of \(G\), Skora showed [Sko90, Theorem 3.3] (see also [Hub22, 13.6]) that \(T(G)\) corresponds to \(\mathcal{D}_{\mathcal{ML}(X)}\), the subset of dual spaces \(X_\lambda\) where \(\lambda \in \mathcal{ML}(X)\). Therefore, by Proposition 8.4 and [Pau89, Main Theorem], it follows that the restriction of our map \(\Psi\) to \(\mathcal{ML}(X)\),

\[\Psi: \mathcal{ML}(X) \rightarrow \mathcal{D}_{\mathcal{ML}(X)},\]

is a homeomorphism onto its image, where \(\mathcal{ML}(X)\) is equipped with the subspace weak*-topology, and \(\mathcal{D}_{\mathcal{ML}(X)}\) is equipped with the equivariant Gromov-Hausdorff topology. Our result is a generalization of Paulin’s work in this sense. On the other hand, Paulin’s result is more general than ours, since it applies to all irreducible and minimal \(G\)-actions on \(\mathbb{R}\)-trees, for any finitely generated group \(G\). For example, one can ask if this result is true for more general actions of hyperbolic groups on a hyperbolic uniformly quasi-geodesic space.
**Definition 10.5** (bounded backtracking property). Let $G$ be a non-elementary Gromov hyperbolic group acting on a hyperbolic uniformly quasi-geodesic space $Y$. Suppose, furthermore, that there exists $C > 0$, so that for any uniform quasi-geodesic $\gamma = [x, y]$ on $X = \text{Cay}(G)$, the orbit map $f : X \to Y$ satisfies that

$$f(\gamma) \in N_C([f(x), f(y)])$$

where $N_C$ denotes the $C$-neighborhood in the Hausdorff distance. We say then that $G$ acts with *bounded backtracking property* on $Y$.

Cantrell and Oregón-Reyes [COR22] fit the notion of dual spaces of geodesic currents in the general framework of *boundary metric structures*. These are left-invariant hyperbolic pseudo-metrics on a non-elementary hyperbolic group satisfying the bounded backtracking property. Such property has also been studied independently by Kapovich and the second author in upcoming work [KMG22], where they use it to construct an extension to geodesic currents for the stable length of such actions, as well as other natural notions of length. They also prove that for small actions of one-ended hyperbolic groups on $\mathbb{R}$-trees satisfy the bounded backtracking property.

In particular, we conjecture the following.

**Conjecture 10.6.** Let $G$ be a non-elementary hyperbolic group. Consider all cobounded actions of $G$ with bounded backtracking property on metric spaces, which we denote $\mathcal{D}(G)$ (as in [COR22]). The axial topology on $\mathcal{D}(G)$ is equivalent to the equivariant Gromov-Hausdorff topology.

10.3. **Relation to Cantrell–Oregón-Reyes and Sapir’s work.** Our map $\Psi$ descends to a homeomorphism

$$\mathbb{P} \Psi : \mathbb{P} \text{Curr}(X) \to \mathbb{P} \mathcal{D}(X).$$

Here $\mathbb{P} \mathcal{D}(X)$ denotes the projectivization of $\mathcal{D}(X)$, where $X_\mu$ and $X_\nu$ are equivalent if and only if there exists a constant $C > 0$, so that

$$d_\mu(\pi_\mu(x), \pi_\mu(y)) = C \cdot d_\mu'(\pi_\mu'(x), \pi_\mu'(y))$$

for all $x, y \in \tilde{X}$. The subspace $\mathbb{P} \text{Curr}_{\text{fill}}(X) \subseteq \mathbb{P} \text{Curr}(X)$ consisting of filling geodesic currents can be equipped with a distance, by recent work of Sapir [Sap22], defined as follows

$$d_{\text{fill}}([\mu], [\nu]) := \sup_{c \in \text{Curves}(X)} \log \frac{i(\mu, c)}{i(\nu, c)} + \sup_{c \in \text{Curves}(X)} \log \frac{i(\nu, c)}{i(\mu, c)}.$$ 

This distance coincides with the symmetrization of the Thurston distance on Teichmüller space, when restricted to $\text{Curr}_{\text{Teich}}(X)$, i.e., the embedded image of Teichmüller space in $\mathbb{P} \text{Curr}(X)$ by [Thu98, Theorem 8.5]. On the other hand, there is also
a natural distance \( d_D \) on \( \mathbb{P}D_{\text{fill}}(X) \), defined in recent work of Oregón-Reyes [OR22, Definition 1.2], which can also be expressed, by [OR22, Lemma 3.5], as

\[
d_D([\mu], [\nu]) := \sup_{g \in \pi_1(X)} \frac{\ell_{X_\mu}(g)}{\ell_{X_\mu}(g)} + \sup_{g \in \pi_1(X)} \frac{\ell_{X'_\mu}(g)}{\ell_{X'_\mu}(g)}.
\]

Indeed, by Proposition 8.15, \( \mathbb{P}D_{\text{fill}}(X) \) corresponds to a subspace of the space \( D(\pi_1(X)) \) (this also follows from [COR22, Theorem 1.11]), and one can endow it with the restriction of that metric. By [OR22, Lemma 3.5], and Proposition 8.4, it follows that the restriction of \( \Psi \) to \( \mathbb{P}\text{Curr}_{\text{fill}} \),

\[\mathbb{P}\Psi: \mathbb{P}\text{Curr}_{\text{fill}} \to \mathbb{P}D_{\text{fill}}(X)\]

is an isometry with respect to the metrics \( d_{\text{fill}} \) and \( d_D \). Furthermore, the topology induced by \( d_{\text{fill}} \) coincides with the subspace weak*-topology, by Theorem 2.12. The topology induced by \( d_D \) coincides with the topology of translation lengths or axial topology on \( \mathbb{P}D_{\text{fill}}(X) \) by [OR22, Lemma 3.5]. Since by [EM18b, Lemma 3.5] and [EM18b, Corollary 3.8] \( \mathbb{P}\text{Curr}_{\text{fill}}(X) \) is dense and open in \( \mathbb{P}\text{Curr}(X) \), then the extension of \( \mathbb{P}\Psi|_{\mathbb{P}\text{Curr}_{\text{fill}}(X)} \) to the closure is the same as \( \Psi \).

**Appendix A. Dual tree of a lamination**

In this section we relate the existing definitions of dual \( \mathbb{R} \)-tree of a measured lamination existing in the literature.

In section 3 we have defined the dual space \( X_\mu \) of a geodesic current \( \mu \) and in Corollary 6.14 we have showed that if \( \mu \) is a measured lamination, then \( X_\mu \) is 0-hyperbolic. However, it doesn’t come equipped with a geodesic structure, but we can endow it with one in an essentially unique way by embedding it isometrically and uniquely (up to equivariant isometry), into an \( \mathbb{R} \)-tree, by a result of Morgan-Shalen.

In this appendix, we recall this construction and relate it to other notions that appear in the literature. We begin by defining measured geodesic laminations. We follow [Mar16, Ch. 8].

**Definition A.1** (Measured geodesic lamination). A geodesic lamination \( \Lambda \) is a set of disjoint simple complete geodesics in \( X \), whose union is a closed subset of \( X \). A transverse measure for \( \Lambda \subset X \) is a family \( \lambda \) of locally finite Borel measures \( \lambda_\alpha \) on each arc \( \alpha \subset X \) transverse to \( \lambda \), such that

1. For every \( \alpha \) transverse arc, the support of \( \lambda_\alpha \) is \( \alpha \cap \Lambda \);
2. If \( \alpha' \subset \alpha \) is a sub-arc of \( \alpha \), then the measure \( \lambda_\alpha' \) is the restriction of \( \lambda_\alpha \);
3. For every \( \alpha \) transverse arc, the measure \( \lambda_\alpha \) is invariant through isotopies of transverse arcs.
A measured geodesic lamination is a geodesic lamination together with a transverse measure.

In section 3 we have defined the dual space $X_\mu$ of a geodesic current $\mu$.

Given a measured lamination $\lambda$ one can canonically associate to $\lambda$ an $\mathbb{R}$-tree called the dual tree of the lamination (see [MS91]). Let us recall the construction.

Let $X$ be a compact hyperbolic surface and let $(\lambda, \mu)$ be a measured lamination on $X$ with support $\lambda = \text{supp}(\mu)$ and transverse measure $\mu$. Denote with $(\tilde{\lambda}, \tilde{\mu})$ the lifted measured lamination on $\tilde{X}$, and define $\mathcal{C}$ the set of connected components of $\tilde{X} \setminus \text{supp} \lambda$, each of which is called a complementary region of $\lambda$.

Let $c_0, c_1 \in \mathcal{C}$. We define a metric on $\mathcal{C}$ as follows:

$$d_{(\lambda, \mu)}(c_0, c_1) = \inf \mu(\gamma)$$

where the inf is taken over all quasi-transverse arcs $\gamma$ such that $\gamma(0) = x_0 \in c_0$ and $\gamma(1) = x_1 \in c_1$. A quasi-transverse arc is an arc intersecting transversely each leaf of the lamination at most once.

Morgan and Shalen proved the following

**Theorem A.2** ([MS91] Lemma 5). Denote with $\mathcal{C}$ the set of complementary regions of $\tilde{\mu}$. There exists an $\mathbb{R}$-tree $\mathcal{T}(\mu)$ and an isometric embedding $\psi : \mathcal{C} \hookrightarrow \mathcal{T}(\mu)$ such that:

1. $\psi(\mathcal{C})$ spans $\mathcal{T}(\mu)$.
2. Any point $x \in \mathcal{T}(\mu) \setminus \psi(\mathcal{C})$ is an edge point, i.e. separates $\mathcal{T}(\mu)$ in two connected components.
3. The action $\pi_1(X)$ on $\mathcal{C}$ extends uniquely to an action by isometries of $\pi_1(X)$ on $\mathcal{T}(\mu)$.

Moreover, if $T$ and $T'$ are two $\mathbb{R}$-trees satisfying the above properties, then there exists an equivariant isometry $T \to T'$ with respect to the $\pi_1(X)$-action.

The proof of the theorem amounts to prove that the metric space $\mathcal{C}$ is 0-hyperbolic.

On the other hand, recall that measured geodesic laminations on $X$ are fully characterised as the geodesic currents on $X$ with zero self-intersection (Proposition 2.6).

Hence in particular, given a measured geodesic lamination $\lambda$, it makes sense to talk about the dual current space $X_\lambda$ as defined in section 3.

In section 6 we show that $X_\lambda$ is 0-hyperbolic as well, and hence embeds uniquely in an $\mathbb{R}$-tree $T$. We often refer with a slight abuse of notation to $X_\lambda$ as the dual tree $T$ of $\lambda$.

The two 0-hyperbolic spaces $\mathcal{C}$ and $X_\lambda$ are essentially the same, with the only difference being that $X_\lambda$ contains some extra points corresponding to atoms of $\lambda$. In fact, let $c_1$ and $c_2$ be two complementary regions of $\lambda$ separated by an atom $l_\alpha$. Let
\(x_1 \in c_1, x_2 \in c_2\) and \(x_3 \in l_\alpha\). In \(X_\lambda = \tilde{X}/\sim\) we have that \([x_1]_\sim, [x_2]_\sim\) and \([x_0]_\sim\) are three distinct points:

\[d_\mu(x_1, x_2) = 1\text{ and } d_\mu(x_0, x_1) = d_\mu(x_0, x_2) = \frac{1}{2}\]

In other words, the 0-hyperbolic space \(\mathcal{C}\) constructed by Morgan Shalen is simply the set of complementary regions. On the other side, the 0-hyperbolic space \(X_\mu\) contains a point for each complementary region, plus an additional point for each atom. Metrically, these new points are like midpoints between regions separated by the atom.

Let \(T_{BIPP}\) and \(T_{MS}\) the two \(\mathbb{R}\)-trees in which \(X_\lambda\) and \(\mathcal{C}\) isometrically embed, respectively. It is almost tautological that \(T_{BIPP}\) and \(T_{MS}\) are isometric.

**Lemma A.3.** The trees \(T_{BIPP}\) and \(T_{MS}\) are isometric.

**Proof.** The isometry is the obvious one: send each vertex \(v_c \in T_{BIPP}\) corresponding to the complementary region \(c\), to vertex of \(T_{MS}\) corresponding to the same complementary region. Given an atom \(l\) of \(\lambda\), let \(c_i\) and \(c_j\) be the adjacent complementary region. Send the edge \([v_{c_i}, v_{c_j}]\) to the edge corresponding to the same complementary regions in \(T_{MS}\). \(\square\)

From the previous results, we obtain a geodesic realization of our 0-hyperbolic space which is moreover unique up to isometry.

**Proposition A.4.** Given \(\lambda\) a measured lamination, and \(X_\lambda\) its dual space, there exists an \(\mathbb{R}\)-tree \(\hat{X}_\lambda\) and an isometric embedding \(\iota: X_\lambda \to \hat{X}_\lambda\).

There is a third equivalent way of defining the dual tree of a lamination \((\lambda, \mu)\), as for example in [Kap09, Chapter 11]. In this definition the dual tree \(T_K\) is also defined as a quotient of \(\tilde{X}\) by a pseudo-distance, but in this construction we firstly get rid of all atomic leaf with a procedure known as the blow-up.

Let \((\lambda, \mu)\) be a measured geodesic lamination on \(X\). For each isolated leaf \(\gamma\) of \(\lambda\) consider a small tubular neighbourhood \(U_\gamma\), such that \(U_\gamma \cap \Lambda = \{\gamma\}\). We foliate \(U_\gamma\) with curves of constant distance from \(\gamma\).

We endow \(U_\gamma\) with Lebesgue transversal measure \(\mu'_\gamma\) with total mass \(\mu(J)\), where \(J\) is a small arc intersecting every leaf on \(U_\gamma\) once, and no other leaf of \(\lambda\). We obtain, after blowing up every isolated leaf, a new measured lamination \((\lambda', \mu')\), where \(\lambda' = \lambda \cup \{U_\gamma\}_\gamma\) atomic leaf and

\[
\mu' = \begin{cases} 
\mu & \text{away from every } U_\gamma \\
\mu'_\gamma & \text{on } U_\gamma 
\end{cases}
\]
Such a new lamination $(\lambda', \mu')$ is referred as the blow-up of $(\lambda, \mu)$. Let $(\tilde{\lambda}', \tilde{\mu}')$ be the lifted lamination on $\tilde{X}$. We can define now a pseudo-distance on $\tilde{X}$ for $x, y \in \tilde{X}$

\[ d'(x, y) = \inf \tilde{\mu}'(\alpha) \]

where the inf ranges over all piece-wise geodesics $\alpha$ joining $x$ to $y$.

**Lemma A.5.** The infimum in $d'$ is always realized.

*Proof.* See [Kap09, p. 248]. □

We finally define $T_K = \tilde{X}/\{d' = 0\}$. Let us show that $T_K$ is indeed isometric to the dual tree of $(\lambda, \mu)$, as defined above.

**Lemma A.6.** The tree $T_K$ is isometric to $T_{MS}$.

*Proof.* First of all, we note that the set of complementary regions of $(\tilde{\Lambda}', \tilde{\mu}')$ is the same as $(\tilde{\Lambda}, \tilde{\mu})$, as well as the distance between any two complementary regions. Therefore repeating the same argument in [MS91, Lemma 5] we obtain isometric embeddings $\psi_1: \mathcal{C} \hookrightarrow T_K$ and $\psi_2: \mathcal{C} \hookrightarrow T_{MS}$ such that $\psi_1(\mathcal{C})$ spans $T_K$ and $\psi_2(\mathcal{C})$ spans $T_{MS}$. By the uniqueness part of [MS91, Lemma 5] we conclude that $T_K$ is isometric to $T_{MS}$. □

There is one last notion of dual tree defined by M. Wolf in [Wol95], which arises from the notion of measured foliation. It is known that measure foliations and measured laminations are essentially the same object, in the sense that there is an equivariant homeomorphism

\[ \Phi: \mathcal{MF}(X) \rightarrow \mathcal{ML}(X). \]

Given a measured foliation $(F, \nu)$, one can construct the dual tree $T_{(F, \nu)}$ of the foliation $F$. It is a technical observation that the dual tree of a lamination $\lambda$ is isometric to the dual tree of the corresponding foliation $\psi(\lambda)$.

**APPENDIX B. EQUIVALENCE OF NOTIONS OF TREE GRADED SPACES**

In this section we show that, under the assumption that $X$ is geodesic, the Drutu-Sapir definition of tree graded space as in Definition 7.5 and our Definition 7.8 are equivalent.

**Proposition B.1.** If $(X, d)$ is a geodesic metric space, then Definition 7.8 is equivalent to Definition 7.5.

*Proof.* Assume throughout that $(X, d)$ is geodesic metric space and also that it has a collection of pieces that intersect in at most one point, i.e., axiom pieces. Let $\Delta = xyz$ be a geodesic triangle with geodesic sides $\gamma^1 = \gamma_{xy}$, $\gamma^2 = \gamma_{yz}$ and $\gamma^3 = \gamma_{xz}$. Suppose $\Delta$ is contained in more than one piece. We want to show $\Delta$ is non-simple.
We can decompose each $\gamma^i$, according to Proposition 7.7, into a piece-wise geodesic consisting of geodesic segments $\alpha_j$ contained in pieces (we will call them $\alpha$-segments) and geodesic segments $\beta_k$ each contained in a transversal $T_x$ (we will call them $\beta$-segments), for $i = 1, 2, 3$. The endpoints of the $\beta$ for each $i$, give corresponding straight contact chains $(y^i_1, \ldots, y^i_\ell)$. Since $\Delta$ is contained in more than one piece, at least one of the $\beta$ segments are non-degenerate. By axiom straight chain triangle, one of the straight chains shares an interior point with another side chain say $y_1^1 = y_3^3$. This means that their corresponding subsegments $\beta_1$ and $\beta_3$ share an endpoint, and thus $\Delta$ is non-simple.

**Figure B.1.** From triangles to straight chain triangles: Proof of implication from Definition 7.5 to Definition 7.8. From a straight chain we generate a triangle with simple subsegments. The axiom triangles forces self-intersection of the triangles, giving two cases depending on which types of subsegments intersect. The left figure exhibits the cases of two $\beta$-segments intersecting, and the right one the case of two $\alpha$-segments intersecting.

Now we show that Definition 7.5 implies Definition 7.8. Given axiom triangles, the (axiom transversals) is [DS05, Lemma 2.12] and [DS05, Lemma 2.13].

We will see now that the axiom contact chain triangle also follows from axiom triangles.

Let $\Delta = xyz$ be a contact chain triangle, where $x, y, z$ are three points in $X$ contained in at least two distinct pieces. Suppose, say, that $x \in P_1$, and let $y \in P_2$ and $z \in P_3$, where $P_2$ and $P_3$ could potentially be the same, but both distinct from
Denote by $C_i := (y_{i1}, \ldots, y_{i\ell_i})$ the straight chains connecting the points $x, y, z$ pairwise, for $i = 1, 2, 3$. We can connect the contact points in these chains by simple geodesic segments that we will call, as before, $\alpha^i_{jk}$ and $\beta^i_{jk}$, depending on whether their interiors are in pieces or in $T_x$, respectively. Let $\hat{C}_i$ denote the corresponding piecewise geodesic realizing the straight chain $C_i$. They together form a corresponding geodesic triangle $\hat{\Delta}$ which is contained in more than one piece, so it must be non-simple. Thus, there must be a point of intersection $z$ between, say, the interior of $\hat{C}_1$ and a point in $\hat{C}_3$. The geodesics $[z, y_1^3]$ both intersect the piece $P_1$ only at their endpoints $y_1^3$ and $y_1^1$, and thus, by [DS05, Lemma 2.4], we have $y_1^3 = y_1^1$. Similarly, the geodesics $[z, y_1^1]$ and $[z, y_2^3]$ both intersect, say $P_2$, at $y_1^1$ and $y_2^3$, so $y_1^1 = y_2^3$. Then, a similar argument shows that $y_1^3 = y_2^1$. Thus, since $\alpha^i_j$ are all simple geodesics, it follows that $\alpha^1_1$, $\alpha^1_2$ and $\alpha^1_3$ must be singletons and equal to the points of contact $y_1, y_2, y_3$ at $P_1, P_2$ and $P_3$, respectively. Then, the $\beta$ segments adjacent to each $y_i$, since they share $y_i$, must be contained in the same $T_x$. If $z$ is in a $\beta$ segment, a similar argument using geodesics emanating from $z$ and Lemma [DS05, Lemma 2.4], as before, shows that all other $\beta$ segments are contained in the same $T_x$, and thus $\Delta = \Delta_1 \cup A$, where $\Delta_1$ is a triangle made of $\beta$ segments and contained in $T_x$ and $A$ is a disjoint union of $\alpha$-segments. If $z$ is in an $\alpha$ segment, a similar argument shows that $\Delta = \Delta_1 \cup A \cup B$, where $\Delta_1$ is a triangle made of $\beta$ segments contained in a single $T_x$, $A$ is a union of $\alpha$ segments, and $B$ is a union of $\beta$ segments. In any of the two cases, since $T_x$ is 0-hyperbolic and $\Delta_1$ is contained in $T_x$, the triangle $\Delta_1$ must be non-simple, and thus there must be two $\beta$ subsegments corresponding to different sides of $\Delta$ sharing an endpoint, which shows that one of the corresponding straight contact chains shares an interior point with another, thus contradicting the axiom contact chain triangle.

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