The nonstandard picture of a turbulent field is presented in the article. By the concepts of nonstandard mathematics, the picture describes the hierarchical structure of turbulence and shows the mechanism of fluctuation appearing in a turbulent field. The uncertainty of measurement is pointed out. And the fundamental equations of turbulence are given too. It is natural in this picture that the reasonable closure methods can be obtained and seem to be precise.

1 Some Concepts of Nonstandard Analysis

The nonstandard analysis theory of turbulence will be presented in this paper. The nonstandard analysis means the mathematic fundament of the nonstandard picture of turbulence. Therefore, some concepts of nonstandard analysis should be introduced in the first place.

On the basis of mathematic logic, A.Robinson \[\text{I}\] proved in the sixties of the last century that a real number system \( R \) can be expanded into the hyperreal number system \( R^* \). Besides all real numbers (i.e., standard numbers), \( R^* \) contains also hyperreal numbers (i.e., nonstandard numbers). The infinitesimal \( \varepsilon \) and infinite \( L \) are elementary nonstandard numbers. There are other nonstandard numbers, for example, \( \xi \pm \varepsilon, \xi \varepsilon, \xi L, \xi \pm L, \varepsilon \varepsilon, LL \), etc. (\( \xi \) stands for any real number).

It is known that real numbers one-to-one correspond to the points on a real line, but on the real line there are no such points which correspond
to nonstandard numbers. In other words nonstandard numbers can not be indicated by the points on a real line. The concept of monad needs to be presented to indicate the nonstandard numbers. For every real number $\xi$, there exists a monad that is composed of the numbers infinitely close to $\xi$ and $\xi$ belongs also to the monad. In fact, there exist infinite monads containing the real number $\xi$. The size of all these monads is infinitesimal $\varepsilon$. The numbers in this monad, except $\xi$, are all nonstandard numbers of which $\xi$ is the standard part.

So any point on a real line is expanded into a monad. The real number, represented by the point on the real line, is the standard part of the nonstandard numbers contained in the monad. And a real line, formed by the points which indicate only real numbers,is expanded into a hyperreal line composed of monads. The set of the standard parts of all these monads composes the real line mentioned above. Every monad on a hyperreal line is also called a point (i.e.,standard point). Clearly every such standard point is not the absolute point which has the length of absolute zero, but the monad whose length is infinitesimal $\varepsilon$.

Every monad must contain only one real number. If a monad contains the real number $\xi$, we call the monad as $\xi$-monad.

The physical world is in a hierarchical structure. This hierarchical structure can be described by the concepts of nonstandard mathematics. For example, there are three levels: level $\alpha$ (with infinitesimal $\varepsilon$, infinite $\varepsilon^{-1}$), level $\beta$ (with infinitesimal $\varepsilon^3$, infinite $\varepsilon$) and level $\gamma$ (with infinitesimal $\varepsilon^{-1}$, infinite $\varepsilon^{-3}$). Any monad on a hyperreal line in level $\alpha$ is also a whole hyperreal line in level $\beta$. However, the whole hyperreal line in level $\alpha$ is only a monad of the
hyperreal axis in higher level $\gamma$. From the viewpoint of hierarchical structure, the meanings of infinite and infinitesimal are not fixed. It is understandable in physics. Physicists, in fact, can even determine how large infinite and infinitesimal are in a specific physical problem, and often take a certain number as infinite or infinitesimal. In physics any number (not absolute zero) is taken as infinite, infinitesimal or finite in various specific condition.

The coordinate systems in two levels will be set up for the convenience of description. Let space coordinates be $(x_1, x_2, x_3)$, time $t$ in level $\alpha$ and space $(x'_1, x'_2, x'_3)$, time $t'$ in level $\beta$, the range of values of $(x'_1, x'_2, x'_3, t')$ be $[0, L_1), [0, L_2), [0, L_3)$ and $[0, T)$ respectively. Here $L_1, L_2, L_3$ and $T$ are the infinites of space and time, respectively, in level $\alpha$. For any time-space point $(\vec{x}, t)$ in level $\alpha$, the monad of this point is expressed by $(\vec{x}, t)$ and the dimensions of the monad are $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_t)$. Here $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_t$ are infinitesimals of space and time, respectively, in level $\alpha$. Moreover, assume that:

$$\varepsilon_1 L_1 = 1, \quad \varepsilon_2 L_2 = 1, \quad \varepsilon_3 L_3 = 1, \quad \varepsilon_t T = 1$$

(1)

The space-distance between any two points

$$\Delta l = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^3}$$

in level $\alpha$, and

$$\Delta l' = \sqrt{(\Delta x'_1)^2 + (\Delta x'_2)^2 + (\Delta x'_3)^2}$$

in level $\beta$. The actual length (i.e., the length observed from the angle of level $\alpha$) of $\Delta l'$ is:
\[
\sqrt{(\Delta x'_1\varepsilon_1^2) + (\Delta x'_2\varepsilon_2^2) + (\Delta x'_3\varepsilon_3^2)}^2
\]  \hspace{1cm} (2)

If \(\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon\) the actual length of \(\Delta l'\) is:

\[
\Delta l'\varepsilon^2
\]  \hspace{1cm} (3)

Similarly, the actual value (i.e., the value observed from level \(t\)) of \(\Delta t'\) is:

\[
\Delta t'\varepsilon^2_t
\]  \hspace{1cm} (4)

And the actual lengths of the whole space and time axes in level \(\beta\) are:

\[
L_1\varepsilon_1^2 = \varepsilon_1, \quad L_2\varepsilon_2^2 = \varepsilon_2, \quad L_3\varepsilon_3^2 = \varepsilon_3, \quad T\varepsilon_t^2 = \varepsilon_t
\]  \hspace{1cm} (5)

They are just the dimensions of a monad in level \(\alpha\).

Now the limit of tending to zero (\(\Delta x_i \to 0\)) in level \(\alpha\) is not tending to absolute zero but infinitesimal \(\varepsilon\). Here \(\Delta x_i\) can not go into the inner of the zero monad. In the inner of the zero monad in level \(\alpha\) exists \(\Delta x'_i\) instead of \(\Delta x_i\). In other words, the process of tending to zero does not mean tending to absolute zero but the dimension of a monad in the same level. The standard part of any number \(\omega\) is denoted by \(St\omega\). To show clearly the meanings of \(\Delta x_i \to 0\) and \(\Delta t \to 0\) mentioned above, we do not use the notation \(\Delta x_i \to 0\) and \(\Delta t \to 0\). Instead, we use

\[
St\Delta x_i \to 0, \quad St\Delta t \to 0
\]  \hspace{1cm} (6)

i.e.,

\[
\lim_{St\Delta x_1 \to 0} \Delta x_1 = \varepsilon_1, \quad \lim_{St\Delta x_2 \to 0} \Delta x_2 = \varepsilon_2, \quad \lim_{St\Delta x_3 \to 0} \Delta x_3 = \varepsilon_3, \quad \lim_{St\Delta t \to 0} \Delta t = \varepsilon_t
\]  \hspace{1cm} (7)
In other words, the definitions of \((\Delta x_i \to \varepsilon_i)\) and \((\Delta t \to \varepsilon_t)\) are, respectively, \((St\Delta x_i \to 0)\) and \((St\Delta t \to 0)\). Similarly, the limit of tending to infinite does not mean tending to absolute infinite but the nonstandard number \(L\), i.e.,

\[
\lim_{St\Delta x_i \to \infty} \Delta x_1 = L_1, \quad \lim_{St\Delta x_2 \to \infty} \Delta x_2 = L_2, \quad \lim_{St\Delta x_3 \to \infty} \Delta x_3 = L_3, \quad \lim_{St\Delta t \to \infty} \Delta t = T
\]  

(8)

Moreover, in standard case, a function is expressed by \(f(x_1, x_2, x_3, t)\) which represents the function value at the standard point \((x_1, x_2, x_3, t)\). However, in the nonstandard analysis, a point \((x_1, x_2, x_3, t)\) is a monad within which there are still infinite nonstandard points \((x'_1, x'_2, x'_3, t')\). Then a function should be expressed by \(f(x_1, x_2, x_3, t, x'_1, x'_2, x'_3, t')\) which is the function value at the nonstandard point \((x'_1, x'_2, x'_3, t')\) contained in the monad \((x_1, x_2, x_3, t)\). Still we define the partial derivatives in form:

\[
\frac{\partial f}{\partial x'_i} = \lim_{St\Delta x'_i \to 0} \frac{f(x'_1 + \Delta x'_i) - f(x'_1)}{\Delta x'_i}, \quad \frac{\partial f}{\partial t'} = \lim_{St\Delta t' \to 0} \frac{f(t' + \Delta t') - f(t')}{\Delta t'}
\]  

(9)

\[
\frac{\partial f}{\partial x_i} = \lim_{St\Delta x_i \to 0} \frac{f(x_1 + \Delta x_i) - f(x_i)}{\Delta x_i}, \quad \frac{\partial f}{\partial t} = \lim_{St\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}
\]  

(10)

\[(i = 1, 2, 3)\]

Definition (9)-(10) is only a definition without the ordinary meaning of derivative in the standard case. The difference of \(f(x_i + \Delta x_i) - f(x_i)\) in (10) is the abbreviation of \(f(x_i + \Delta x_i, x_j, x_k, t, x'_i, x'_j, x'_k, t') - f(x_i, x_j, x_k, t, x'_i, x'_j, x'_k, t')\), the increment of the function value between two monads of the points \((x_i + \Delta x_i)\)
and \((x_i)\). The distance between the points
\[
\lim_{\Delta x_i \to 0} \left( x_i + \Delta x_i, x_i' \right)
\]
and
\[
(x_i, x_i')
\]
is infinite from the angle of level \(\beta\). Therefore, \(\frac{\partial f}{\partial x_i}\) and \(\frac{\partial f}{\partial t}\) in (10) have the meanings of an average. If one insists on the concept of ordinary derivative in standard case, \(\frac{\partial f}{\partial x_i}\) and \(\frac{\partial f}{\partial t}\) in (10) have no sense.

2 Nonstandard Model Of Turbulence

On the basis of the above-mentioned concepts of nonstandard analysis a nonstandard description of turbulent field can be presented as follows.

Assumption 1: Global turbulent field is composed of standard points, and every standard point is yet a monad. Each monad possesses the internal structure, namely a monad is also composed of infinite nonstandard points (so called interior points of the monad).

It is well known that a space is formed from the points. In fluid mechanics, fluid mechanists always take the fluid element (fluid particle) as point. In fact, a fluid particle has the volume not to be absolute zero but to be a micro volume, which is taken as infinitesimal from macroscopic viewpoint and infinite from microscope. People generally think of the whole fluid particle as uniform. Namely the fluid particle is taken to be equal to a point in reality. Now, in the case of turbulence, the motion of fluid varies so fast that a fluid particle can not be uniform as a whole but becomes a monad. From now on in this paper, the meanings of a fluid particle, a monad on a whole and a standard
point are identical with each other and so are the meanings of a fluid particle in the lower level, an interior point of a monad and a nonstandard point.

Since the point of the turbulent field becomes a monad with internal structure, the motion features of different interior points of the monad are different. So it is obvious that there is a flow or flow field in the inner of the monad. Hence, a fluid particle (monad) is made up of nonstandard points. The volume of these nonstandard points is not absolute zero, and these nonstandard points, in fact, are the fluid particles in the lower level. Now the particles in the lower level are thought to be uniform. They are points (nonstandard points) actually. Moreover, a lot of fluid molecules are included still in the fluid particle of the lower level.

Therefore, two kinds of flow fields exist in turbulence. One is the global turbulent field and the other is the flow field in a monad (called monad field in this paper). They are flow fields in two different levels. Now the global turbulent field is not composed of the points in which there are no structures, but of monad fields. So there are three levels in turbulent field: molecular movement, monad field and global field.

Assumption 2: The flows in monad fields are controlled by the Navier-Stokes equations..

Then the motion-equations of the flows in monad fields can be written as follows (for incompressible fluid, unsteadiness and three dimensions):

\[
\frac{\partial U_i}{(\varepsilon^2 \partial x')} = 0 \quad (11)
\]
\[
\frac{\partial U_i}{\varepsilon t \partial t'} + \frac{\partial U_i U_j}{(\varepsilon^2 \partial x')_j} = \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x'_j}
\] (12)

Here \( U_i \) is velocity component in \( i \)-direction, \( \rho \) density of fluid, \( \sigma_{ij} \) stress tensor and \( \sigma_{ij} = \sigma_{ji} \). The independent variables of these functions in (11)-(12) are \((x_1, x_2, x_3, t, x'_1, x'_2, x'_3, t')\). \( \varepsilon_t, \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are infinitesimals (i.e., the linear dimensions of a monad) of time \( t \), space \( x_1, x_2 \) and \( x_3 \), respectively. Let \( \varepsilon_t = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon \) here and later, then the governing equations becomes:

\[
\frac{\partial U_i}{\partial x'_i} = 0 \quad \text{(13)}
\]

\[
\frac{\partial U_i}{\partial t'} + \frac{\partial U_i U_j}{\partial x'_j} = \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x'_j} 
\] (14)

The governing equations (11)-(12) show the internal structure of a monad.

For actual turbulence the possible linear dimensions of flow fields in two levels could be estimated roughly as follows:

**Global turbulent field:**
- infinite \( L = \varepsilon^{-1} \sim 10^{1.5} \text{cm} \)
- infinitesimal \( \varepsilon \sim 10^{-1.5} - 10^{-2} \text{cm} \)

(Linear dimension of a monad)

**Monad field:**
- infinite \( \varepsilon \sim 10^{-1.5} - 10^{-2} \text{cm} \)
- infinitesimal \( \varepsilon^3 \sim 10^{-4.5} - 10^{-5} \text{cm} \)

(Linear dimension of the particle of lower level)

According to this estimation, one space monad of the turbulent field contains approximately \( 10^9 - 10^{10} \) nonstandard points (the particles in the lower level). If the fluid is gas, there are approximately \( 10^5 - 10^6 \) gas molecules in one nonstandard point of a monad field. This number of gas molecules is enough
for stable mean values in the statistical average.

**Assumption 3: Turbulent field is continuous.**

Here the meanings of continuity are: firstly the global turbulent field is composed continuously of standard points (i.e., monad fields); secondly the flows in the monad fields are continuous; and thirdly the physical quantities on the interface between two infinitely close monads are continuous as well.

The monads of

\[
\lim_{St\Delta x_i \to 0} (x_i + \Delta x_i, x_j, x_k, t)
\]

and

\[
(x_i, x_j, x_k, t)
\]

are taken as two infinitely close monads in space. And so are in time. The mathematic expression of the third is

\[
\lim_{St\Delta x \to 0} U(x + \Delta x, 0) = U(x, L), \quad \lim_{St\Delta t \to 0} U(t + \Delta t, 0) = U(t, T)
\]  

(15)

There is an even more important meaning of the continuity. That will be stated in Assumption 6.

### 3 Uncertainty Of Measurement

To make a physical measurement, some time-space point \((x_1, x_2, x_3, t)\) has to be assigned first, at which the measurement is carried out. The results of the measurement are taken as the measuring data of this point \((x_1, x_2, x_3, t)\). According to the standpoint of nonstandard analysis, the point \((x_1, x_2, x_3, t)\) is not an absolute point but a monad, in which there are infinite nonstandard points. Generally speaking, the motion features of different nonstandard points
are different. So the meaning of saying “measuring at point \((x_1, x_2, x_3, t)\)” becomes ambiguous. Obviously, if there are fields in two different levels in the system being studied, there must exist the new meaning of a measurement at any point of the field in higher level.

Assumption 4: When a measurement at any point (monad) \((x_1, x_2, x_3, t)\) in a physical field is taken, the operation of the measurement will act randomly on one interior point (nonstandard point) of the point \((x_1, x_2, x_3, t)\).

This assumption states clearly that one measurement operation at some standard point (monad) in a field will always take place at an interior point of such a monad. But the measurer can not determine which interior point is acted on by the measurement operation. What actually happens is random. The measurer knows only the standard point (monad) in which the interior point acted on by one measurement operation is contained. It is surprising that when a measurement is carried out there is no way to determine the exact position of the measured object. That is the uncertainty of measurement. At first sight it seems that the uncertainty of measurement stems from the rough measuring skill and that the circumstance conditions of the measurement are not exactly controllable etc.. After further thinking, it is comprehended that the uncertainty rises essentially from limited ability and means of the description of the physical world in hierarchical structures. Apparently, in their quest of the laws of the universe, people need not (and can not in fact) describe all motion states of every molecule and electron in the universe. Conversely, even though an exact and perfect description of the motion states of every molecule
and electron in the universe is given, the laws of the motion of the whole universe are still not understood. It is clearly that because the physical world has a hierarchical structure, therefore any description and measurement of physical phenomena should be related with the hierarchical structure. When the effect of hierarchical structure can not be ignored, the uncertainty of measurement occurs. The measurement and observation of turbulence can be taken for examples of this case.

An operation of measurement will act randomly on various interior points in a monad. By what probability does the measurement operation act on each interior point of a monad? In turbulence it is thought that one operation of measurement will take place in equiprobability at each interior point of a monad.

Assumption 5: When a measurement at any point (monad) of a turbulent field is made, the operation of the measurement will act in equiprobability on various interior points of the monad. This assumption is called the equiprobability assumption.

So there are two kinds of averages. One is that one measurement is just taking average over a large number of molecules contained in some interior point of a monad, while the other average over the motions of all interior points in a monad has to be taken in order to get the mean values of the physical quantities over the standard point (monad) of the turbulent field. According to the equiprobability assumption, the second average formula can be given. Let \( U(\vec{x}, t, \vec{x}', t') \) represent some physical quantity at an interior point \((\vec{x}', t')\) of the monad \((\vec{x}, t)\) and \( \tilde{U}(\vec{x}, t) \) express the average of \( U(\vec{x}, t, \vec{x}', t') \) over all interior
points of the monad. Then it follows that

$$\tilde{U}(\vec{x}, t) = \frac{1}{T} \int_0^T dt' \frac{1}{L^3} \int_0^L dx'_1 \int_0^L dx'_2 \int_0^L dx'_3 U(\vec{x}, t, \vec{x'}, t') \quad (16)$$

Where the monad \((\vec{x}, t)\) \(\equiv (x_1, x_2, x_3, t)\), the interior point of the monad \((\vec{x'}, t')\) \(\equiv (x'_1, x'_2, x'_3, t')\), and \(T, L\) are infinites of time and space respectively. Let \(T = L_1 = L_2 = L_3 = L\) here and later. Obviously, \(\tilde{U}\) is only a function of the standard point \((\vec{x}, t)\).

If the monad fields of a global flow field are uniform, there is no difference between saying “one measurement acting on the interior point of a monad” and “one measurement acting on the point (monad)”. Then the global field is laminar. Exactly, the essential feature of turbulence is that its particle (standard point) has internal structure, i.e., internal flow field (monad field). That could be taken as the definition of turbulence.

Because of the uncertainty of measurement, one measurement at the point \((\vec{x}, t)\) will act randomly on any interior point. Therefore, the results of many measurements carried out at the same point (monad) under constant circumstance conditions will indicate randomly the motion states of various interior points of the monad. Surely fluctuation of data of the measurements must appear, provided that there are considerable different states among various interior points of the monad. This is the mechanism of fluctuation appearing in a turbulent field.

Thus, a turbulent field is composed of monads with interior structure (monad field). The flows in the monad fields are controlled by the governing equations of (13)-(14). Still continuity of turbulent field exists. Therefore,
turbulence is also regular flow. The appearance of the fluctuation of a turbu-
ulent flow stems from the uncertainty of measurement and observation in the
turbulent field.

To keep the concepts mentioned above in mind, the order of magnitude
of fluctuation in a turbulent field can be estimated roughly. Let $\Delta x'$ be finite
length in the coordinate system of a monad, $U \sim 0(1)$ any physical quantity
and its fluctuation $u \sim U(x' + \Delta x') - U(x') \sim \frac{\partial U}{\partial x'} \Delta x'$. The actual rate of
change (i.e., observed from the angle of the global field level) of function $U$
is $\frac{\partial U}{\varepsilon^2 \partial x'}$. In one case $\frac{\partial U}{\partial x'} \sim 0(\varepsilon^2)$, $u \sim 0(\varepsilon^2)$, no obvious fluctuation occurs.
In another case $\frac{\partial U}{\partial x'} \sim 0(1)$, and $\frac{\partial U}{\varepsilon^2 \partial x'} \sim 0(\frac{1}{\varepsilon^2})$ showing that there is excess
rate of shear strain in the field. This will lead to instability in the flow field.

So the case should be excluded. Then in the third case $\frac{\partial U}{\partial x'} \sim 0(\varepsilon)$ only, and
$\frac{\partial U}{\varepsilon^2 \partial x'} \sim 0(\frac{1}{\varepsilon})$ with fluctuation $u \sim 0(\varepsilon)$. So there is notable fluctuation in
order of magnitude $u \sim 0(\varepsilon)$ in turbulence provided that there exists any set
of interior points in a monad, and on the set $\frac{\partial U}{\partial x'} \sim 0(\varepsilon)$.

4 Conservation Equations

Generally speaking, in fluid mechanics a small volume $\Delta x_1 \Delta x_2 \Delta x_3$ is taken
and the flux of conservative physical quantity through the boundary planes of
the volume is computed. Then let $\Delta x_1 \rightarrow 0, \Delta x_2 \rightarrow 0, \Delta x_3 \rightarrow 0$, the small
volume will tend to the point $(x_1, x_2, x_3)$. Thus, conservation equations at
the point $(x_1, x_2, x_3)$ are obtained. The boundary planes of the small volume
are absolute planes without thickness. However, in nonstandard case a small
volume is still taken. By Use of the same method as in standard case but
let $St \Delta x_1 \rightarrow 0, St \Delta x_2 \rightarrow 0, St \Delta x_3 \rightarrow 0$, the small volume $\Delta x_1 \Delta x_2 \Delta x_3$ will
become the monad of \((x_1, x_2, x_3)\). Now the boundary planes of the small volume are not absolute planes but very thin layers with thickness of \(\varepsilon\) (the linear dimension of the monad). Also conservation equations at this monad are obtained. Surely the equations are those of mean quantities over the monad in which infinite interior points are contained.

Now these conservation equations are given in the following.

Let “\(\sim\)” express the average operation over \((x'_1, x'_2, x'_3, t')\), “\(\sim^j\)” over \(x'_i\), “\(\sim\)” over \(t'\), etc., in a monad.

Through the pair of the planes perpendicular to \(x_1\) axis the mass inflow per unit time is:

\[
\bar{\Delta x_2 \Delta x_3} (x_1) \Delta x_2 \Delta x_3
\]

the mass outflow

\[
\bar{\Delta x_2 \Delta x_3} (x_1 + \Delta x_1) \Delta x_2 \Delta x_3
\]

the net inflow

\[
- \frac{\partial}{\partial x_1} \bar{\Delta x_2 \Delta x_3} \Delta x_1 \Delta x_2 \Delta x_3
\]

Through the other two pairs of the planes perpendicular to the two other axes the net mass inflow per unit time are, respectively,

\[
- \frac{\partial}{\partial x_2} \bar{\Delta x_1 \Delta x_3} \Delta x_1 \Delta x_2 \Delta x_3
\]

and
Because of the law of mass conservation, it follows that

$$\frac{\partial}{\partial x_1} \bar{\bar{U}}_1^{x_1} \Delta x_2 \Delta x_3 + \frac{\partial}{\partial x_2} \bar{\bar{U}}_2^{x_1} \Delta x_1 \Delta x_3 + \frac{\partial}{\partial x_3} \bar{\bar{U}}_3^{x_1} \Delta x_1 \Delta x_2 = 0$$

Here the average of $\sim \Delta x_1 \Delta x_2$ is taken over the plane of $\Delta x_1 \Delta x_2$, which tends to $\sim 12$ when $St \Delta x_1 \to 0, St \Delta x_2 \to 0$. And so are the others. Let $St \Delta x_1 \to 0, St \Delta x_2 \to 0, St \Delta x_3 \to 0$, then it is obtained that

$$\frac{\partial}{\partial x_1} \bar{\bar{U}}_1^{123} + \frac{\partial}{\partial x_2} \bar{\bar{U}}_2^{123} + \frac{\partial}{\partial x_3} \bar{\bar{U}}_3^{123} = 0$$

i.e.,

$$\frac{\partial \bar{U}_1}{\partial x_1} + \frac{\partial \bar{U}_2}{\partial x_2} + \frac{\partial \bar{U}_3}{\partial x_3} = 0 \quad (17)$$

Similarly, the equations of momentum conservation can be obtained too.

Through the pair of the planes perpendicular to $x_1$ axis, the net inflow of the momentum in $x_1$ direction per unit time is

$$\rho \bar{\bar{U}}_1^{x_1} \Delta x_2 \Delta x_3 - \rho \bar{\bar{U}}_1^{x_1} \Delta x_2 \Delta x_3 = -\rho \frac{\partial}{\partial x_1} \bar{\bar{U}}_1^{x_1} \Delta x_1 \Delta x_2 \Delta x_3$$

Through the other two pairs of the planes perpendicular to the other two axes, the net inflows of the momentum in $x_1$ direction per unit time are

$$-\rho \frac{\partial}{\partial x_2} \bar{\bar{U}}_2^{x_1} \Delta x_1 \Delta x_3$$
and

\[ -\rho \frac{\partial}{\partial x_3} \bar{U}_3 \bar{U}_1 \Delta x_1 \Delta x_2 \Delta x_3 \]

respectively.

The force in \( x_1 \) direction exerting on the pair of the planes perpendicular to \( x_1 \) axis is:

\[ -\frac{\bar{\sigma}_{11} \Delta x_2 \Delta x_3}{\bar{\sigma}_{11}} (x_1) \Delta x_2 \Delta x_3 + \frac{\bar{\sigma}_{11} \Delta x_2 \Delta x_3}{\bar{\sigma}_{11}} (x_1 + \Delta x_1) \Delta x_2 \Delta x_3 \]

\[ = \frac{\partial}{\partial x_1} \frac{\bar{\sigma}_{11} \Delta x_2 \Delta x_3}{\bar{\sigma}_{11}} \Delta x_1 \Delta x_2 \Delta x_3 \]

The forces in \( x_1 \) direction exerting on the other two pairs of the planes perpendicular to the other two axes are

\[ \frac{\partial}{\partial x_2} \frac{\bar{\sigma}_{21} \Delta x_1 \Delta x_3}{\bar{\sigma}_{21}} \Delta x_1 \Delta x_2 \Delta x_3 \]

and

\[ \frac{\partial}{\partial x_3} \frac{\bar{\sigma}_{31} \Delta x_1 \Delta x_2}{\bar{\sigma}_{31}} \Delta x_1 \Delta x_2 \Delta x_3 \]

respectively.

Then the increments of the momentum in \( x_1 \) direction in the small volume per unit time are

\[ \rho \left\{ \bar{U}_1 \Delta x_1 \Delta x_2 \Delta x_3 (t + \Delta t) - \bar{U}_1 \Delta x_1 \Delta x_2 \Delta x_3 (t) \right\} \frac{\Delta x_1 \Delta x_2 \Delta x_3}{\Delta t} \]

\[ = \rho \frac{\partial}{\partial t} \bar{U}_1 \Delta x_1 \Delta x_2 \Delta x_3 \Delta x_1 \Delta x_2 \Delta x_3 \]

Because of the law of momentum conservation, it follows that
\[
\frac{\partial}{\partial t} \overline{U_1} + \frac{\partial}{\partial x_1} \overline{U_1 U_1} + \frac{\partial}{\partial x_2} \overline{U_2 U_1} + \frac{\partial}{\partial x_3} \overline{U_3 U_1} = \frac{1}{\rho} \frac{\partial}{\partial x_1} \overline{\sigma_{11}} + \frac{1}{\rho} \frac{\partial}{\partial x_2} \overline{\sigma_{21}} + \frac{1}{\rho} \frac{\partial}{\partial x_3} \overline{\sigma_{31}}
\]

Let \( St \Delta x_1 \rightarrow 0, St \Delta x_2 \rightarrow 0, St \Delta x_3 \rightarrow 0 \), so \( \sim \Delta x_1 \Delta x_2 \Delta x_3 \) tend to \( \sim 123 \), etc. Then it follows that

\[
\frac{\partial}{\partial t} \overline{U_1} + \frac{\partial}{\partial x_1} \overline{U_1 U_1} + \frac{\partial}{\partial x_2} \overline{U_2 U_1} + \frac{\partial}{\partial x_3} \overline{U_3 U_1} = \frac{1}{\rho} \frac{\partial}{\partial x_1} \overline{\sigma_{11}} + \frac{1}{\rho} \frac{\partial}{\partial x_2} \overline{\sigma_{21}} + \frac{1}{\rho} \frac{\partial}{\partial x_3} \overline{\sigma_{31}}
\]

i.e.,

\[
\frac{\partial}{\partial t} \overline{U_1} + \frac{\partial}{\partial x_1} \overline{U_1 U_1} + \frac{\partial}{\partial x_2} \overline{U_2 U_1} + \frac{\partial}{\partial x_3} \overline{U_3 U_1} = \frac{1}{\rho} \left( \frac{\partial}{\partial x_1} \overline{\sigma_{11}} + \frac{\partial}{\partial x_2} \overline{\sigma_{21}} + \frac{\partial}{\partial x_3} \overline{\sigma_{31}} \right)
\]

Similarly, the momentum equations in \( x_2 \) and \( x_3 \) directions can be written too. The general form of momentum equations is:

\[
\frac{\partial}{\partial t} \overline{U_i} + \frac{\partial}{\partial x_1} \overline{U_i U_1} + \frac{\partial}{\partial x_2} \overline{U_i U_2} + \frac{\partial}{\partial x_3} \overline{U_i U_3} = \frac{1}{\rho} \frac{\partial}{\partial x_1} \overline{\sigma_{1i}} + \frac{1}{\rho} \frac{\partial}{\partial x_2} \overline{\sigma_{2i}} + \frac{1}{\rho} \frac{\partial}{\partial x_3} \overline{\sigma_{3i}}
\]  

The equations (17) – (18) are about the mean quantities over a monad and called the conservation equations. Here no special mean-methods are assigned. But the mean \( \sim \) will always be taken as the mean-method showed in (16) in this paper.
5 Fundamental Equations of Turbulent Flows

Still instantaneous quantities $U_i$ and $P$ are decomposed into two parts such that

\[ U_i = \bar{U}_i + u_i, \quad P = \bar{P} + p, \quad \bar{u}_i = 0, \quad \bar{p} = 0 \]  

(19)

Here the quantities with “∼” are called mean quantities; in this regard, we will adopt the mean-method of (16) i.e.,

\[ \bar{U} = \frac{1}{T} \int_0^T dt' \frac{1}{L^3} \int_0^L dx'_1 \int_0^L dx'_2 \int_0^L dx'_3 U, \quad \bar{U} = \frac{1}{T} \int_0^T dt' U, \quad \bar{U}^i = \frac{1}{L} \int_0^L dx'_i U \]  

(20)

\[ \bar{U}^{ij} = \frac{1}{L^2} \int_0^L dx'_i \int_0^L dx'_j U, \quad \bar{U}^{ijk} = \frac{1}{L^3} \int_0^L dx'_i \int_0^L dx'_j \int_0^L dx'_k U \]

The relations of (19) are the definitions of $u_i$ and $p$. The $u_i$ and $p$ have the same the order of magnitude as the turbulent fluctuation mentioned in Section 3. We will call the $u_i$ and $p$ in (19) as fluctuant quantities. To note that the $\bar{U}_i$ and $\bar{P}$ are mean quantities over a monad, and $u_i$ and $p$ are meaningful to a monad (or in a monad). They have different meanings with those in the existent theory of turbulence.

Before derivation of the fundamental equations of turbulent flows, it is necessary to present the assumption of close property between two monads when they are infinitely close to each other, and analyze the mutual relation of the two monads in motion characteristics.

Assumption 6: In both the value and structure of function, physical function, defined on the interior points of the monads of a turbulent field, is very close between two monads, when these two monads are infinitely close to each other.
Considering any physical quantity $U_i \sim 0(1)$, its fluctuation $u_i$, the mathematical expressions of this assumption are as follows:

\[(a) \delta u_i \sim 0(\epsilon^2), \quad (b) \delta (u_i u_j - \bar{u}_i \bar{u}_j) \sim 0(\epsilon^4) \tag{21}\]

\[(0 \leq x'_i < L, \quad 0 \leq t' < T)\]

Here the operator $\delta$ is defined as

\[
\delta_i U = \lim_{\delta i \to 0} \left[ U(x_i + \Delta x_i, x'_i) - U(x_i, x'_i) \right], \quad \delta_t U = \lim_{\delta t \to 0} \left[ U(t + \Delta t, t') - U(t, t') \right] \tag{22}\]

It can be got from (a) in (21) that

\[
\delta \left[ \frac{\partial U}{\epsilon^2 \partial x'_i} - \frac{\partial U}{\partial x_i} \right] \sim 0(\epsilon^2) \tag{23}\]

Where $\frac{\partial U}{\epsilon^2 \partial x'_i}$ is the actual rate of change or figure slope of the function $U$ at the nonstandard point $x'_i$, and $\frac{\partial U}{\partial x_i}$ is the mean value of $\frac{\partial U}{\epsilon^2 \partial x'_i}$ over $x'_i$ (see(25)).

Clearly the condition (23) and (a) in (21) show that the figure shape of the function $U$ has very little difference between two infinitely close monads. Similarly the condition (b) in (21) shows that the figure shape of the function $u_i u_j$ has very little difference between two infinitely close monads.

Assumption 6 could be called “Continuous Assumption”. It is well known that with a few exceptions, physical quantities vary usually continuously with time and space. In the case of standard analysis, a function is defined on the standard points. It is characterized only by its value. The continuity of a physical quantity only requires that the function representing the physical quantity be continuous in mathematics. On the other hand, in the nonstandard case a point becomes a monad and a function is defined at the interior point of the monad. There exists a function-structure on the whole monad. So
the function is characterized by not only its value but also its structure on the monad. The continuity of a physical function will mean that in both the values and figure-shape of the function, the variation of the physical function with monads is continuous. Thus, there is very little difference between two infinitely close monads in both value and structure of the physical function. In other words, the Assumption 6 is the nonstandard expansion of the physical function continuity in the case of standard analysis. Moreover, the Assumption 6 is reasonable and natural, because the circumstance conditions imposed on two infinitely close monads are very close to each other.

In a monad, we consider that stress tensor for Newtonian fluid is

$$\sigma_{ij} = -P\delta_{ij} + \mu \left( \frac{\partial U_i}{\partial x_j'} + \frac{\partial U_j'}{\partial x_i} \right) \frac{1}{\varepsilon^2} \quad (24)$$

Using (20) and (17), (15) (Assumption 3) we can give the following computation:

$$\frac{1}{\varepsilon^2} \frac{\partial \tilde{U}_i^j}{\partial x_j'} = \left( \frac{\partial U_i}{\partial x_j} \right)_{x_j'=0} \quad (25)$$

Then

$$\tilde{\sigma}_{ij} = -\tilde{P}\delta_{ij} + \mu \left( \frac{\partial \tilde{U}_i^{kl}}{\partial x_j} \right)_{x_j'=0} + \mu \left( \frac{\partial \tilde{U}_j^{mn}}{\partial x_i} \right)_{x_j'=0}$$

$$\frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} = -\frac{\partial \tilde{P}}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \left( \frac{\partial \tilde{U}_i^{kl}}{\partial x_j} \right)_{x_j'=0} + \mu \frac{\partial}{\partial x_j} \left( \frac{\partial \tilde{U}_j^{mn}}{\partial x_i} \right)_{x_j'=0}$$

$$= -\frac{\partial \tilde{P}}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \frac{\partial \tilde{U}_i}{\partial x_j} + \mu \frac{\partial}{\partial x_j} \left( \frac{\partial \tilde{\sigma}_i^{kl}}{\partial x_j} \right)_{x_j'=0}$$

By use of (a) in (21) it follows that

$$\left( \frac{\partial^2 \tilde{U}_i^{kl}}{\partial x_j^2} \right)_{x_j'=0} \sim 0(\varepsilon) \quad (26)$$
To omit them, it is obtained that

$$\frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} = -\frac{\partial \tilde{P}}{\partial x_i} + \mu \frac{\partial^2 \tilde{U}_i}{\partial x_j^2}$$  \hspace{1cm} (27)

Then the equations (18) can be written as

$$\frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial \tilde{U}_i \tilde{U}_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i, \quad \nabla^2 = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j}$$  \hspace{1cm} (28)

(17) is written as

$$\frac{\partial \tilde{U}_i}{\partial x_i} = 0$$  \hspace{1cm} (29)

Here $\nu = \frac{\mu}{\rho}$ the kinematic viscosity.

Note that now the mean method in the equations (28) and (29) is defined by (20). Then by use of the relation (19), the decomposition of (29) and (28) can be done as

$$\frac{\partial U_i}{\partial x_i} - \frac{\partial u_i}{\partial x_i} = 0$$  \hspace{1cm} (30)

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} - \frac{\partial u_i}{\partial t} - \frac{\partial (U_i U_j)_{fl}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 U_i - \nu \nabla^2 u_i$$  \hspace{1cm} (31)

Here $u_i, p$ and $(U_i U_j)_{fl}$ are the fluctuations of the velocity, pressure and $(U_i U_j)$ respectively. The independent variables of the functions in the equations (30)-(31) are $(x_1, x_2, x_3, t, x'_1, x'_2, x'_3, t')$. Based on Assumption 6 the order of magnitude of the terms concerned with fluctuation quantities is $\sim 0(\varepsilon)$, but the order of magnitude of the terms concerned with instantaneous quantities is $\sim 0(1)$. So when (30)-(31) are split into two parts having different order of magnitude, the following can be obtained:
Equations, in which terms are in the order of magnitude $\sim 0(1)$, of instantaneous quantities are:

$$\frac{\partial U_i}{\partial x_i} = 0 \quad \text{(32)}$$

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i \quad \text{(33)}$$

Equations, in which terms are in the order of magnitude $\sim 0(\varepsilon)$, of fluctuation quantities are:

$$\frac{\partial u_i}{\partial x_i} = 0 \quad \text{(34)}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial (U_i U_j)_{fl}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \quad \text{(35)}$$

Where $(U_i U_j)_{fl}$ in (35) is:

$$(U_i U_j)_{fl} = U_i U_j - \tilde{U}_i \tilde{U}_j$$

$$= \tilde{U}_i u_j + \tilde{U}_j u_i + u_i u_j - \tilde{u}_i \tilde{u}_j$$

Substitution of the relations into (35) produces

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial \tilde{U}_i}{\partial x_j} + \tilde{U}_j \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i u_j}{\partial x_j} - \frac{\partial \tilde{u}_i \tilde{u}_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \quad \text{(36)}$$

Equations (28)(29),(32)(33) and (34)(35) are the governing equations about mean, instantaneous and fluctuant quantities respectively. They can be called as the fundamental equations of turbulent flows.

It is found easily that these equations have the same form as those in the existent theory of turbulent flows. But there are essentially differences between the two. These differences are:
1. Different methods to get them. In the existent theory of turbulence the mean equations are obtained by the average of the instantaneous equations (i.e., the Navier-Stokes equations). Then the mean equations are subtracted from the instantaneous equations to get the fluctuation equations. But in this paper, the mean equations will FIRST be obtained in view of the physical conservation laws, and then by virtue of the close property assumption, the instantaneous and fluctuant equations are obtained respectively.

2. The terms in the equations of the existent theory are the limits, such as

\[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}
\]

But the terms in the equations of the nonstandard theory are the partial derivatives defined in (10). The equations in two theories are conceptually different. The former is based on, so called, the frame of $\delta - \varepsilon$, the latter is out of the frame of $\delta - \varepsilon$. Therefore, the equation in nonstandard picture of turbulence is the new kind of equation.

3. The physical quantities in the equations of the existent theory, such as velocity, pressure etc., are all functions of the standard point $(\vec{x}, t)$. However, those in the nonstandard case are all functions of the point $(\vec{x}, t, \vec{x'}, t')$, actually functions of the nonstandard point $(\vec{x'}, t')$ with $(\vec{x}, t)$ showing which monad contains it. It is reasonable that the physical quantities are taken as the functions of the standard point $(\vec{x}, t)$ in laminar flow, but not in turbulent flows, because a standard point becomes a monad in which there is an interior structure (i.e., the monad flow field) in the case of turbulence. In fact, there is no definition of instantaneous and fluctuant quantities of a whole monad, but there is the definition of instantaneous and fluctuant quantities of the interior
points of the monad. So for the global field, the Navier-Stokes equations in
existent meanings do not exist in principle. Therefore, though (32)-(33) are
the same as the Navier-Stokes equations in forms, they are different. The fun-
damental equations obtained in this paper are governing equations about the
physical quantities defined at those interior points of various monads. These
interior points have the same nonstandard coordinates \((\vec{x}', t')\). But the fun-
damental equations in the existent turbulent theory are governing equations
about the physical quantities defined at the standard points of the global flow
field. There is an essential difference between them.

4. The average in the existent theory usually means the ensemble aver-
age, but the distribution function can not be given. So the closure problem
continues to exist. Otherwise, it is definite in the nonstandard model to take
average over all interior points of a monad by the equiprobability assumption.

5. It should be pointed out that the function values at some standard
points are given for the initial and boundary conditions of the fundamental
equations in the existent theory. However, only the function values at some
nonstandard points can be taken as the initial and boundary conditions for the
instantaneous and fluctuation quantities in the fundamental equations in the
nonstandard case. Because a turbulent field always starts and develops from
walls or laminar flows and evolves from laminar flows or static fluids, the func-
tion values on walls or laminar flows can be taken as the boundary conditions,
and the conditions of static fluids or laminar flows can be taken as the initial
conditions. No slip condition is still kept. Thus, the initial and boundary con-
ditions of the fundamental equations obtained from the nonstandard analysis
can be given in principle.
6 On Closure Problem

Substitution of the relation \( U_i = \tilde{U}_i + u_i \) into (28) produces

\[
\frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial \tilde{U}_i \tilde{U}_j}{\partial x_j} + \frac{\partial u_i \tilde{u}_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i
\]

(37)

This is the same in form as famous Reynolds equation. It is not closure, and neither is equation (36) in which there is term \( \tilde{u}_i \tilde{u}_j \). The closure problem stemming from nonlinearity is very difficult in turbulence research. Now in the nonstandard theory, the turbulence is composed of monad fields, and the point-average is adopted. Then by the close property between two infinitely close monads, we can give the reasonable closure methods as follows.

By the relation (21)(Assumption 6) and the order of magnitude of fluctuation \( u_i \sim 0(\varepsilon) \), it follows that

\[
\delta(u_i u_j) = u_i \delta u_j + u_j \delta u_i \sim 0(\varepsilon^3)
\]

\[
\delta \tilde{u}_i \tilde{u}_j \sim 0(\varepsilon^3)
\]

\[
\frac{\partial \tilde{u}_i \tilde{u}_j}{\partial x_j} \sim 0(\varepsilon^2)
\]

and

\[
\delta(u_i u_j - \tilde{u}_i \tilde{u}_j) \sim 0(\varepsilon^4)
\]

also, it follows that

\[
\frac{\partial(u_i u_j - \tilde{u}_i \tilde{u}_j)}{\partial x_j} \sim 0(\varepsilon^3)
\]

(38)

There are three choices for the closure of equations:
(A) Choice one:

The term $\frac{\partial u_i u_j}{\partial x_j} \sim 0(\varepsilon^2)$ is neglected from equations (37). The closed equations of the mean quantities will be obtained as follows.

$$\frac{\partial \tilde{U}_i}{\partial x_i} = 0 \quad (39)$$

$$\frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial \tilde{U}_i \tilde{U}_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i \quad (40)$$

(B) Choice two:

To note that

$$\frac{\partial (u_i u_j - \tilde{u}_i \tilde{u}_j)}{\partial x_j} \sim 0(\varepsilon^3) \quad (38)$$

and the other terms in equation (36) have the order of magnitude of $\sim 0(\varepsilon)$. Hence, the term (38) can be neglected from (36). Then the equation (36) becomes

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial \tilde{U}_i}{\partial x_j} + \tilde{U}_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \nabla^2 u_i \quad (41)$$

With $\tilde{U}_i = U_i - u_i$, this equation changes to

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} - 2u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \quad (42)$$

Under certain initial and boundary conditions, the equations (32)-(33) are solved and $U_i$ are obtained. Then the equations (42) with (34) becomes also closed, the fluctuations $u_i$ and $p$ can be obtained after the equations (34)(42) are solved. Finally the mean quantities $\tilde{U}_i = U_i - u_i$ and $\tilde{P} = P - p$.

(C) Choice three:
Hence, the term (38) can be neglected from (36) and (37). Then it follows
that
\[
\frac{\partial \tilde{U}_i}{\partial x_i} = 0 \tag{43}
\]
\[
\frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial \tilde{U}_i \tilde{U}_j}{\partial x_j} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i \tag{44}
\]
\[
\frac{\partial u_i}{\partial x_i} = 0 \tag{45}
\]
\[
\frac{\partial u_i}{\partial t} + \tilde{U}_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \tilde{U}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i \tag{46}
\]
Obviously, the equations (43)-(46) are closed. We can obtain \( \tilde{U}_i, \tilde{P}, u_i, p, \)
from solving these equations.

Surely if \( u_i = 0, p = 0 \), i.e., no fluctuations occur, the equations (28)(29)
and (32)(33) are reduced to ordinary Navier-Stokes equations and the field
becomes a laminar field.

7 Some Remarks on the Numerical Calculations

To take the \( \frac{\partial U}{\partial t} \) for example, in the standard case we have
\[
\left( \frac{\partial U}{\partial t} \right)_{st} = \lim_{\Delta t \to 0} \frac{U(x_1, x_2, x_3, t + \Delta t) - U(x_1, x_2, x_3, t)}{\Delta t}
\]
but in the nonstandard case by the definition of \( \left( \frac{\partial U}{\partial t} \right)_{nst} \) in this paper, it follows
that
\[
\left( \frac{\partial U}{\partial t} \right)_{nst} = \lim_{\Delta t \to 0} \frac{U(x_1, x_2, x_3, t + \Delta t, x'_1, x'_2, x'_3, t') - U(x_1, x_2, x_3, t, x'_1, x'_2, x'_3, t')}{\Delta t} \tag{47}
\]
In form, there is no difference between the two cases when the discretization
of time-space is taken for the numerical computation. However within a grid,
obtained from the discretization of \( \left( \frac{\partial U}{\partial t} \right)_{st} \), there is not structure (i.e., the whole grid should be uniform). But the interior structure is permitted through the grid obtained from the discretization of \( \left( \frac{\partial U}{\partial t} \right)_{nst} \). This difference between two cases is very important.

Surely in the nonstandard case, \( U(x_1, x_2, x_3, t+\Delta t, x'_1, x'_2, x'_3, t') \) and \( U(x_1, x_2, x_3, t, x'_1, x'_2, x'_3, t') \) are the values of physical quantity \( U \) at the interior point \( (x'_1, x'_2, x'_3, t') \) of two different monads in time, that is monad of \( (t + \Delta t) \) and monad of \( (t) \). Though the fundamental equations work for any interior point \( (x'_1, x'_2, x'_3, t') \) of every monad, it is better that let \( x'_1 = x'_2 = x'_3 = t' = 0 \) because the point \( (x_1, x_2, x_3, t, 0, 0, 0, 0) \) is located on the boundary of the time-space of a turbulent field, when \( (x_1, x_2, x_3, t) \) is the boundary monad of the field.

Moreover, \( U(x_1, x_2, x_3, t+\Delta t, 0, 0, 0, 0) \) and \( U(x_1, x_2, x_3, t, 0, 0, 0, 0) \) are hyperreal numbers. In the numerical calculations, their standard parts will be taken.

Therefore, after the discretization, \( \frac{\partial U}{\partial t} \) becomes:

\[
\frac{StU(x_1, x_2, x_3, t + \Delta t, 0, 0, 0, 0) - StU(x_1, x_2, x_3, t, 0, 0, 0, 0)}{\Delta t}
\]

For simplicity it can also be written as

\[
\frac{U(x_1, x_2, x_3, t + \Delta t, 0, 0, 0, 0) - U(x_1, x_2, x_3, t, 0, 0, 0, 0)}{\Delta t}
\] (48)

provided it is kept in mind that the \( U \) in (48) represents the standard parts of them.

There are similar discussions for the space partial derivatives of physical quantities.

After the \( U(x_1, x_2, x_3, t, 0, 0, 0, 0) \) and \( u(x_1, x_2, x_3, t, 0, 0, 0, 0) \), the instan-
taneous and fluctuant quantities respectively, are obtained, then

\[ \bar{U}(x_1, x_2, x_3, t) = U(x_1, x_2, x_3, t, 0, 0, 0, 0) - u(x_1, x_2, x_3, t, 0, 0, 0, 0) \]  (49)

That is the mean value over the small volume of \( \Delta x_1 \Delta x_2 \Delta x_3 \Delta t \), but it is taken as the mean value over the point \((x_1, x_2, x_3, t)\).

Moreover, the mean value \( \bar{U} \) obtained by the methods mentioned above, is the mean over the point (monad) and still random oscillatory. The random oscillation of the mean values \( \bar{U} \) ought to stem from the unavoidable random disturbances in real turbulent fields. If the average of these mean values over a finite time period (or a finite space range) is taken once again, the final average results could be compared with the measured average values of physical quantities over corresponding time period (or space range).

8 Conclusions

In a view of the concepts of the nonstandard analysis and the fact that the physical world has a hierarchical structure, one nonstandard picture on turbulent flows has been presented in this paper. The key points of this picture are as follows:

1. There exist two kinds of fields in different levels. One is the global turbulent field composed of the standard points (monads), and the other is the monad field. And as a whole a monad is not already uniform. Now the interior structure occurs in a monad; namely, different interior points of the monad possess different characteristics of motion.

2. The flows in a monad field are governed by the Navier-Stokes equations.

3. If there are two kinds of fields in different levels, one operation of
measurement at any point (actually a monad) of the field in a higher level will act randomly on one interior point of the monad field. The measurer can determine the monad, rather than its interior point on which the measurement operation acts.

The position of measured object can not be determined exactly, but over a range. Evidently there exists some uncertainty of measurement. That uncertainty stems from the description and observation of the physical phenomena in a hierarchical structure.

4. The flows in a turbulent field are also continuous and regular, while they are seen intuitively in disorderly and irregular form. Fundamentally, the disorder, irregularity and fluctuation occur because of the uncertainty of measurement. Hence, we can probably say that the disorder of turbulence stems from two sources: the first is the fluctuation because of the uncertainty of measurement. This kind of fluctuation is thought to be “real turbulent fluctuation”. The second is the violent unsteadiness and unhomogeneity of the field. The values of $\tilde{U}_i$ and $\tilde{P}$ vary fast with time and space. It is due to the flow-instability as the Reynolds number is very high. In this case, the small random disturbances will be amplified.

5. There are two kinds of averages. The values of the measurement acting on one interior point of a monad, merely stand for the mean results over a lot of fluid molecules contained in this interior point. The other is the average over all interior points in a monad. The second average results indicate the mean values of the physical quantities over the monad (the standard point) in the global field. By the equiprobability assumption, the mean formula of the second can be given in (20).
6. As the expansion of the fact that the physical functions are continuous in the standard case, the assumption of close property between two infinitely close monads shows that the variation of interior structure as well as values of the physical function $U(\vec{x}, t, \vec{x}', t')$ with $(\vec{x}, t)$ (Monad) is continuous too. In other words, there exists only very little difference in characteristics of the function $U(\vec{x}, t, \vec{x}', t')$ between two infinitely close monads. According to this assumption and the conservation laws, the governing equations of mean, instantaneous and fluctuant quantities in turbulence (i.e., the fundamental equations) are obtained, and the reasonable closure methods are given too. It should be noted that the point-average is used, instead of the Reynolds average, therefore the closure problem is easily overcome in the nonstandard case.

7. The new kind of equations of turbulent motion is presented in this paper. These equations are out of the frame of $\delta - \varepsilon$ and valid for the points, which are permitted to be not uniform. These points, in fact, are monads.

References

[1] A.Robinson, *Nonstandard Analysis* (North-Holland, Amsterdam, 1974), Chapter 1,2,10.