Neumann-Lara conjectured in 1985 that every planar digraph with
digirth at least three is 2-colourable, meaning that the vertices can be
2-coloured without creating any monochromatic directed cycles. We
prove a relaxed version of this conjecture: every planar digraph of
digirth at least four is 2-colourable.

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1 Introduction

Let $D$ be an oriented graph (i.e., a digraph without cycles of length at most 2). A function $f : V(D) \to \{1, \ldots, k\}$ is a $k$-colouring of $D$ if the subdigraph induced by vertices of colour $i$ is acyclic for all $i$. We say that $D$ is $k$-colourable if it admits a $k$-colouring.

The following conjecture was proposed by Neumann-Lara [4] (and independently by Škrekovski, see [1]).

**Conjecture 1.1.** Every oriented planar graph is 2-colourable.

There seems to be a lack of methods to attack Conjecture 1.1 and only sporadic partial results are known.

The *digirth* of a digraph is the length of its shortest directed cycle. Harutyunyan and one of the authors [2] proved Conjecture 1.1 under additional assumption that the digirth of $D$ is at least five. Their proof is based on elaborate use of the (nowadays standard) discharging technique. However, it is unlikely that the same method can be pushed further when directed 4-cycles are allowed.

The main result of this note is the following theorem whose proof is based on a novel technique, at least when colourings of graphs are concerned.

**Theorem 1.2.** Every oriented planar graph with a vertex $v_0$ such that each directed cycle of length 3 uses $v_0$ is 2-colourable.

**Corollary 1.3.** Every planar digraph with digirth at least 4 is 2-colourable.

The rest of the paper is devoted to the proof of Theorem 1.2.

2 Proof of Theorem 1.2

The main tool that we will use in our proof is the notion of a Tutte path. This is a special kind of a path that was first used by Tutte [6, 7] in his proof that every 4-connected planar graph is hamiltonian. A version used in this paper is taken from [5].

Tutte paths are defined using the following notion of connectivity. If $G$ is a graph and $H$ is a subgraph of $G$, then an $H$-component $B$ of $G$ is either an edge $e \in E(G) \setminus E(H)$ with both ends in $H$ or it is a connected component $Q$ of $G - V(H)$ together with all edges from $Q$ to $H$ and all ends of these
edges. The vertices of $V(B) \cap V(H)$ are called the *vertices of attachment* of $B$. A bridge with $k$ vertices of attachment is said to be *$k$-attached*.

Let $G$ be a graph with cycle $C$ and let $u, v$ be two vertices in $G$. A path $P$ in $G$ from $v$ to $u$ is called a *Tutte path* with respect to $C$ if

(i) each $P$-component has at most three vertices of attachment and

(ii) each $P$-component containing an edge of $C$ has at most two vertices of attachment.

**Theorem 2.1** (Thomassen [5]). Let $G$ be a 2-connected plane graph with facial cycle $C$. Let $v$ and $e$ be a vertex and edge, respectively, of $C$ and let $u$ be any vertex of $G$ distinct from $v$. Then $G$ has a Tutte path with respect to $C$ from $u$ to $v$ that contains the edge $e$. 

**Lemma 2.2.** Let $D$ and $D'$ be digraphs, whose intersection is a tournament $T$. Then any colouring of $D$ and any colouring of $D'$ that agree on $V(T)$ form a colouring of $D \cup D'$. 

**Proof.** The only detail to verify is that the combined colouring does not produce a monochromatic cycle. Since the colourings of $D$ and $D'$ have no monochromatic cycles, such a directed cycle $C$ would be composed of $2r$ ($r \geq 1$) directed paths $P_1 \cup P_2 \cup \cdots \cup P_{2r}$, where each $P_i$ is a directed path completely contained in either $D$ or $D'$ (joining vertices $v_i$ and $v_{i+1}$ in $T$ ($i = 1, \ldots, r$, indices taken modulo $r$). Since none of these paths together with the arcs of the tournament on $T$ forms a directed cycle, $T$ contains the arcs $v_i v_{i+1}$, which all together form a monochromatic directed cycle in $T$ and hence in both $D$ and $D'$. This contradiction completes the proof. 

**Lemma 2.3.** Let $G$ be a triangulation of the plane with the outer face bounded by a triangle $T = abc$. Then for any orientation $D$ of $G$ with digirth at least 4 and any precolouring of $T$ with colours 1 and 2, there exists a 2-colouring of $D$ that extends the precolouring on $T$.

The first case of the proof of this lemma is similar to that of Wu for induced forests [8].

**Proof.** The proof is by induction, with the base of induction corresponding to the case when the triangulation is 4-connected. We consider the dual graph $H$ of $G$. Note that $H$ is a cubic graph and that it is cyclically 4-edge-connected, i.e., any 3-edge-cut in $H$ isolates a single vertex from the rest of the graph. We distinguish two cases, depending on whether the precolouring on $T$ uses just one or both colours (see Figure 1).
4-connected – two colours case. Suppose first that not all vertices of $T$ are precoloured the same. We may assume that $a$ is coloured 1 and that $b, c$ are coloured 2. Let us denote the vertices of $H$ corresponding to $T$ and to the faces surrounding $T$ by $t^*, a^*, b^*, c^*$, where the face $a^*$ of $G$ contains vertices $b, c$ but not $a$, etc. Let $P^*$ be a Tutte path in $H$ with respect to the facial cycle of $H$ containing $t^*, a^*, b^*, c^*$ (i.e., the face dual to the vertex $a$) that connects $t^*$ and $b^*$ and passes through the edge $e = t^*e^*$. This path, whose existence follows from Theorem 2.1, together with the edge $b^*t^*$ forms a cycle $C^*$ in $H$. The cycle crosses the edges $ac$ and $ab$ of $T$. We may assume that the exterior of $C^*$ contains $b$ and $c$, while $a$ is in its interior. Now we colour the vertices of $D$ in the interior of the cycle by using colour 1, and those in the exterior by 2.

We claim that the above gives a 2-colouring of $D$. To see this, assume that the process gives rise to a monochromatic directed cycle $R$. Then we may assume that the whole cycle $R$ lies in the interior of $C^*$. Since this is also a cycle in $G$, its interior contains a vertex of $H$, but since $C^*$ is a Tutte path, any such cycle separates a $C^*$-component with at most 3 attachments. Since $G$ is 4-connected, this corresponds to a vertex in $H$ and the cycle $R$ is a triangle. But $D$ has digirth at least 4, so $R$ is not a directed cycle. This contradiction proves the claim.
4-connected – one colour case. Suppose now that all three vertices of T have the same colour. In this case we proceed in the similar way except that we consider the graph H obtained from the dual of G by deleting the vertex \( t^* \). The resulting graph is 2-connected and vertices \( a^*, b^*, c^* \) all lie on the outer facial cycle \( C^* \). Let \( u \) be a neighbor of \( a^* \) on \( C^* \) and \( w \) a neighbor of \( c^* \) in \( C^* \). Now we take a Tutte path with respect to \( C^* \) from \( a^* \) to \( u \) that passes through the edge \( e = c^*w \). Together with the edge \( ua^* \) we obtain a cycle \( R \), and we colour the vertices of \( D \) inside this cycle differently from the colour used on \( a, b, c \); and the vertices outside this cycle the same as \( a, b, c \).

The above gives an extension of the colouring of T, as desired. We only need to argue that there are no monochromatic directed cycles of D. As before, the cycles of G that are monochromatic are triangles that are dual to vertices in 3-attached R-components. All of these correspond to facial triangles in G that are not directed cycles in D by the assumption on the digirth. The only possible difference from the first case might be an R-component containing the vertex \( b^* \) (when \( b^* \notin V(R) \)). By property (ii) of Tutte cycles, this R-component in H is 2-attached. But in the dual graph of G, it is adjacent to \( t^* \) and thus gives a monochromatic subgraph of D that contains \( a, b, c \) and the vertex \( b' \neq b \) forming the second facial triangle with the edge \( ac \) of G. However, since all edges incident with \( a^* \) and \( c^* \) are in R, this subgraph only consists of four vertices. The two triangles in this subgraph are not directed cycles in D, which implies that also the 4-cycle in this subgraph cannot be a directed cycle. This shows that D has no monochromatic directed cycles.

Non-4-connected case. If G is not 4-connected, then there is a triangle \( T' = xyz \) that separates the graph, one component being the subgraph \( D_0 \) in the exterior of \( T' \) (including \( T' \), which now becomes a facial triangle) and the other one, \( D_1 \), formed from \( T' \) and the vertices and edges in the interior of \( T' \). Now we first extend the precolouring of T to \( D_0 \) (by using the induction hypothesis), and then apply the induction hypothesis to \( D_1 \) to extend the colouring of \( T' \) obtained in the first step. By Lemma 2.2 the combined colouring is a 2-colouring of D.

Proof of Theorem 1.2. We may assume the underlying undirected graph G is triangulated as otherwise, we can add a vertex inside each face of D adjacent to all vertices of that face in G and direct all edges towards the new vertices in D. This creates no new directed cycles.
The proof is now essentially the same as the proof of Lemma 2.3 with one difference the we will explain below. If \( G \) has a separating triangle, the induction step is the same as in Lemma 2.3 (by applying either the lemma or this theorem inductively). So, it suffices to consider the 4-connected case, and we are not bound with any precolouring.

The faces of \( G \) containing \( v_0 \) form a cycle \( C^* \) in the dual graph \( G^* \). Let us choose three consecutive vertices \( u^*, v^*, w^* \) on \( C^* \). A Tutte path with respect to \( C^* \) joining \( u^* \) and \( v^* \) and containing the edge \( v^*w^* \) forms a cycle \( R \) together with the edge \( u^*v^* \). By the connectivity of \( G^* \), each \( R \)-component with two attachments is an edge. So all edges of \( C^* \) are either an edge of \( C^* \) or an entire \( P \)-component by property (ii) of Tutte paths. In particular, no vertex of \( C^* \) is in a 3-attached \( R \)-component and hence their duals, triangles containing \( v_0 \), all have a vertex inside \( R \) and a vertex outside \( R \). Thus, using colour 1 for all vertices inside \( R \) and using colour 2 outside \( R \) gives an acyclic colouring of \( D \).

\[ \square \]

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