Modelling the neutrino in terms of Cosserat elasticity

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The paper deals with the Weyl equation which is the massless Dirac equation. We study the Weyl equation in the stationary setting, i.e. when the spinor field oscillates harmonically in time. We suggest a new geometric interpretation of the stationary Weyl equation, one which does not require the use of spinors, Pauli matrices or covariant differentiation. We think of our 3-dimensional space as an elastic continuum and assume that material points of this continuum can experience no displacements, only rotations. This framework is a special case of the Cosserat theory of elasticity. Rotations of material points of the space continuum are described mathematically by attaching to each geometric point an orthonormal basis which gives a field of orthonormal bases called the coframe. As the dynamical variables (unknowns) of our theory we choose the coframe and a density. We choose a particular potential energy which is conformally invariant and then incorporate time into our action in the standard Newtonian way, by subtracting kinetic energy. The main result of our paper is the theorem stating that in the stationary setting our model is equivalent to a pair of Weyl equations. The crucial element of the proof is the observation that our Lagrangian admits a factorisation.

Keywords: neutrino; spin; torsion; Cosserat elasticity.

1. Our model

In this paper we work on a 3-manifold $M$ equipped with local coordinates $x^\alpha$, $\alpha = 1, 2, 3$, and prescribed positive metric $g_{\alpha\beta}$ which does not depend on time $x^0$. We view our 3-manifold $M$ as an elastic continuum whose material points can experience no displacements, only rotations, with rotations of different material points being independent. Rotations of material points of the elastic continuum are described mathematically by attaching to each geometric point of the manifold $M$ an orthonormal basis, which gives a field of orthonormal bases called the coframe.

The coframe $\vartheta$ is a triple of orthonormal covector fields $\vartheta^j$, $j = 1, 2, 3$, on the 3-manifold $M$. Each covector field $\vartheta^j$ can be written more explicitly as $\vartheta^j_\alpha$ where the tensor index $\alpha = 1, 2, 3$ enumerates the components. The orthonormality condition for the coframe can be represented as a single tensor identity

$$ g = \delta_{jk} \vartheta^j \otimes \vartheta^k. \quad (1) $$

We view the identity (1) as a kinematic constraint: the metric $g$ is given (prescribed) and the coframe elements $\vartheta^j$ are chosen so that they satisfy (1), which leaves us with three real degrees of freedom at every point of $M$.

As dynamical variables in our model we choose the coframe $\vartheta$ and a positive density $\rho$. These are functions of local coordinates $x^\alpha$ on $M$ as well as of time $x^0$.

At a physical level, making the density $\rho$ a dynamical variable means that we
view our continuum more like a fluid rather than a solid. In other words, we allow the material to redistribute itself so that it finds its equilibrium distribution.

The crucial element in our construction is the choice of potential energy. As a measure of deformations caused by rotations of material points we choose axial torsion, which is the 3-form given by the explicit formula $T^{ax} := \frac{1}{3} \delta_{jk} \vartheta^j \wedge d \vartheta^k$ where $d$ denotes the exterior derivative. We take the potential energy of our continuum to be $P(x^0) := \int_M \|T^{ax}\|^2 \rho \, dx^1 dx^2 dx^3$. It is easy to see that this potential energy is conformally invariant: it does not change if we rescale our coframe as $\vartheta^j \mapsto e^h \vartheta^j$ and our density as $\rho \mapsto e^{2h} \rho$ where $h : M \rightarrow \mathbb{R}$ is an arbitrary scalar function.

We take the kinetic energy of our continuum to be $K(x^0) := \int_M \|\dot{\vartheta}\|^2 \rho \, dx^1 dx^2 dx^3$ where $\dot{\vartheta}$ is the 2-form $\dot{\vartheta} := \frac{1}{3} \delta_{jk} \vartheta^j \wedge \partial_0 \vartheta^k$ with $\partial_0$ denoting the time derivative. The 2-form $\dot{\vartheta}$ is, up to a constant factor, the Hodge dual of the vector of angular velocity.

We now combine the potential energy and kinetic energy to form the action (variational functional) of our dynamic problem:

$$S(\vartheta, \rho) := \int_\mathbb{R} \frac{1}{2} \left( P(x^0) - K(x^0) \right) \, dx^0 = \int_{\mathbb{R} \times M} L(\vartheta, \rho) \, dx^0 dx^1 dx^2 dx^3 \quad (2)$$

where

$$L(\vartheta, \rho) := (\|T^{ax}\|^2 - \|\dot{\vartheta}\|^2) \rho \quad (3)$$

is our dynamic (time-dependent) Lagrangian density.

Our field equations (Euler–Lagrange equations) are obtained by varying the action (2) with respect to the coframe $\vartheta$ and density $\rho$. Varying with respect to the density $\rho$ is easy: this gives the field equation $\|T^{ax}\|^2 = \|\dot{\vartheta}\|^2$ which is equivalent to $L(\vartheta, \rho) = 0$. Varying with respect to the coframe $\vartheta$ is more difficult because we have to maintain the kinematic constraint (1).

### 2. Switching to the language of spinors

The technical difficulty mentioned above can be overcome by switching to a different dynamical variable. It is known that in dimension 3 a coframe $\vartheta$ and a (positive) density $\rho$ are equivalent to a nonvanishing spinor field $\xi$ modulo the sign of $\xi$. The explicit formulae relating coframes and spinors are given in Appendix C of Ref. 1. The advantage of switching to a spinor field $\xi$ is that there are no kinematic constraints on its components, so the derivation of field equations becomes straightforward.

Switching to spinors in formula (3) we arrive at the following self-contained explicit spinor representation of our dynamic Lagrangian density

$$L(\xi) = \frac{4}{9 \xi^c \sigma_{a \dot{c}} \xi^d} \left( [i(\tilde{\xi}^a \sigma^a \dot{\xi}^b - \xi^b \sigma^a \dot{\xi}^a)]^2 \right. \left. - [i(\tilde{\xi}^a \sigma_{a \dot{a} \dot{b}} \partial_0 \xi^b - \xi^b \sigma_{a \dot{a} \dot{b}} \partial_0 \tilde{\xi}^a) \| \right. \left. \sqrt{\det g} \right) \quad (4)$$

where the $\sigma$ are Pauli matrices and the $\nabla$ are covariant derivatives.
3. Separating out time

We write down the dynamic (containing time derivatives) field equation for the Lagrangian density $L$ and seek solutions of the form

$$\xi(x^0, x^1, x^2, x^3) = e^{-ip_0 x^0} \eta(x^1, x^2, x^3)$$

where $p_0 \neq 0$ is a real number. We call solutions of the form (5) stationary.

It turns out that despite the fact that our dynamic field equation is nonlinear, time $x^0$ can be separated out and stationary solutions do indeed make sense. The underlying group-theoretic reason for our nonlinear dynamic field equation admitting separation of variables is the fact that our model is U(1)-invariant, i.e. it is invariant under the multiplication of a spinor field by a complex constant of modulus 1.

Our problem has been reduced to the study of the stationary (time-independent) Lagrangian density

$$L(\eta) = \frac{16}{9\bar{\eta}^0 \sigma_{abcd} \eta^d} \left[ \frac{i}{2} (\bar{\eta}^a \sigma^a_{\alpha \beta} \eta^b - \eta^b \sigma^a_{\alpha \beta} \nabla^a \eta^a) \right]^2 - (p_0 \bar{\eta}^a \sigma_{0ab} \eta^b)^2 \sqrt{\det g}$$

which is our dynamic Lagrangian density (4) with time $x^0$ separated out.

4. Main result

Our main result is the following

**Theorem 4.1.** A nonvanishing time-independent spinor field $\eta$ is a solution of the field equation for our stationary Lagrangian density (6) if and only if it is a solution of one of the two stationary Weyl equations

$$\pm p_0 \sigma^0_{ab} \eta^b + i \sigma^a_{\alpha \beta} \nabla^a \eta^b = 0.$$  

**Proof.** Observe that our stationary Lagrangian density (6) factorises as

$$L(\eta) = -\frac{32p_0}{9} \frac{L_+(\eta) L_-(\eta)}{L_+(\eta) - L_-(\eta)}$$

where $L_{\pm}(\eta) := \frac{i}{2} (\bar{\eta}^a \sigma^a_{\alpha \beta} \eta^b - \eta^b \sigma^a_{\alpha \beta} \nabla^a \eta^a) \pm p_0 \bar{\eta}^a \sigma_{0ab} \eta^b \sqrt{\det g}$ are the Lagrangian densities for the stationary Weyl equations (7). It is easy to see that the latter posses the property of scaling covariance:

$$L_{\pm}(e^h \eta) = e^{2h} L_{\pm}(\eta)$$

where $h : M \to \mathbb{R}$ is an arbitrary scalar function. In fact, the Lagrangian density of any formally selfadjoint (symmetric) linear first order partial differential operator has the scaling covariance property (9).

The abstract argument presented in Section 6 of Ref. 1 shows that properties (8) and (9) imply the statement of Theorem 4.1.}

References

1. O. Cheryova and D. Vassiliev, The stationary Weyl equation and Cosserat elasticity, preprint [http://arxiv.org/abs/1001.4726](http://arxiv.org/abs/1001.4726)