Solution of the Three–Anyon Problem

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Abstract

We solve, by separation of variables, the problem of three anyons with a harmonic oscillator potential. The anyonic symmetry conditions from cyclic permutations are separable in our coordinates. The conditions from two-particle transpositions are not separable, but can be expressed as reflection symmetry conditions on the wave function and its normal derivative on the boundary of a circle. Thus the problem becomes one-dimensional. We solve this problem numerically by discretization. N-point discretization with very small N is often a good first approximation, on the other hand convergence as N → ∞ is sometimes very slow.


1 Introduction

Anyons are two-dimensional point particles with fractional statistics [1–3], which seem to be useful as models for the quasiparticle excitations in the fractional quantum Hall systems. For reviews see e.g. [6–10].

We use here what is known as the “anyonic” gauge, in which the vector potential associated with particle statistics vanishes, and the wave functions are many-valued, as follows. Go along a continuous curve in the $N$-anyon configuration space, starting and ending with the same configuration $A$, but such that two particles are interchanged in the counterclockwise direction, while no other particles are encircled or interchanged. Then the value of a wave function $\psi$ must change continuously from $\psi(A)$ to $e^{i\theta} \psi(A)$. The phase angle $\theta$ is a constant, independent of the configuration $A$ and the wave function $\psi$. It defines the statistics of the particles, it is 0 for bosons, $\pi$ for fermions, and may take any real value for anyons.

The energy spectrum for two anyons in a harmonic oscillator potential is easily found, and it interpolates continuously between the boson and fermion spectra when $\theta$ varies [1]. The energies vary linearly with $\theta$, sometimes with a discontinuous change in slope at the Bose points. A more general class of quadratic Hamiltonians, including the case of a constant magnetic field, can be treated just as easily.

The same problem for more than two anyons is much more difficult. Wu found a class of exact solutions that generalize the two-anyon solutions [11]. More general exact solutions have been found, but all have energies that depend linearly on $\theta$ [12–31,9]. Thus, even the three-anyon ground state close to Fermi statistics, which has an energy varying non-linearly with $\theta$, is still not known analytically. Even though the exact $\theta$-dependence is unknown for many energy levels, the spectral flow with $\theta$ from the boson to the fermion spectrum is known [28, 29].

Sen found an interesting “supersymmetry” in the three-anyon spectrum, excluding some of the states with linearly varying energy [22, 23]. His supersymmetry transformation transforms bosons into fermions and vice versa, and more generally transforms $\theta$ into $\pi - \theta$.

The lowest part of the complete energy spectrum of three or four anyons in a harmonic oscillator potential has been calculated numerically [24–36]. In the three-anyon problem, good analytical approximations to the wave functions corresponding to non-linear variation of energy are known [37]. More systematic approximation methods are perturbation theory, starting from the known boson and fermion spectra [38–40], and the Hartree–Fock approximation [41, 42].

One reason for the interest in the spectra of two- and three-particle systems in an external harmonic oscillator potential is that they can be used for calculating the second and third virial coefficients in the equation of state for an ideal gas of free particles, i.e. when there is no interaction apart from the statistics interaction. We will not discuss this problem here, but refer to the literature [43–50, 12, 22, 23, 55, 56].

We show here how to solve the three-anyon problem with a harmonic oscillator potential, by separation of variables in a suitable set of coordinates. The same method applies to a class of similar problems, as mentioned above. An anticlockwise cyclic interchange of the three particles gives a phase factor of $e^{i\theta}$ in the wave function, and this condition is compatible with the separation of variables. However, the interchange of only two particles gives a condition which can in general only be satisfied by a superposition of separated wave functions. Thus the separation is incomplete. We find general solutions numerically, but we hope that this method may also lead to some progress in the search for analytical solutions.
2 Formulation of the Problem

It is a well known procedure to describe a particle in the plane by a complex coordinate \( z \) and its complex conjugate \( \bar{z} \), which may be scaled so as to become dimensionless. In the three-anyon problem the key to the separation of variables is a transformation from the particle coordinates \( z_1, z_2, z_3 \) to the centre of mass coordinate \( Z \) and the relative coordinates \( u, v \), defined by

\[
Z = \frac{1}{\sqrt{3}} (z_1 + z_2 + z_3),
\]
\[
u = \frac{1}{\sqrt{3}} (z_1 + \eta z_2 + \eta^2 z_3),
\]
\[
u = \frac{1}{\sqrt{3}} (z_1 + \eta^2 z_2 + \eta z_3).
\]

Here \( \eta = \exp(2i\pi/3) = (-1 + i\sqrt{3})/2 \) is a cube root of unity, with \( \eta^2 = \eta^{-1} = \bar{\eta} \) and \( 1 + \eta + \eta^2 = 0 \). Hence,

\[
u = \frac{1}{\sqrt{3}} (\eta (z_2 - z_1) + \eta^2 (z_3 - z_1)),
\]
\[
u = \frac{1}{\sqrt{3}} (\eta^2 (z_2 - z_1) + \eta (z_3 - z_1)).
\]

This coordinate transformation is a discrete Fourier transformation, which means that a cyclic interchange of particle positions,

\[
(z_1, z_2, z_3) \mapsto (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) = (z_2, z_3, z_1),
\]

becomes diagonal,

\[
(Z, u, v) \mapsto (\tilde{Z}, \tilde{u}, \tilde{v}) = (Z, \eta^2 u, \eta v).
\]

The interchange of particles 2 and 3 is just an interchange of \( u \) and \( v \). Note that these two permutations generate the whole symmetric group \( S_3 \). A similar treatment of permutation symmetry is known in nuclear physics [49, 50] (we thank Alex Lande for pointing this out).

Three particles in the plane define a triangle. The ratio \( s = (z_3 - z_1)/(z_2 - z_1) \) is real when the triangle is degenerate so that the particles lie on a straight line. We define the orientation of a non-degenerate triangle as positive or negative depending on whether the imaginary part of \( s \) is positive or negative. We have that

\[
\frac{|u|}{|v|} = \frac{|\eta + \eta^2 s|}{|\eta^2 + \eta s|} = \frac{|s + \eta^2|}{|s + \eta|}.
\]

Hence \(|u| = |v|\) if \( s \) is real, \(|u| < |v|\) if the orientation of the triangle is positive, and \(|u| > |v|\) if the orientation is negative.

The quantization of the centre of mass motion is trivial, and the problem we want to discuss here is the simultaneous diagonalization of the dimensionless relative Hamiltonian \( H_{rel} \) and relative angular momentum \( L_{rel} \), defined by

\[
H_{rel} = -\frac{\partial^2}{\partial u \partial \bar{u}} - \frac{\partial^2}{\partial v \partial \bar{v}} + u \bar{u} + v \bar{v},
\]
\[
L_{rel} = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} - \bar{u} \frac{\partial}{\partial \bar{u}} - \bar{v} \frac{\partial}{\partial \bar{v}}.
\]
3 Separation of Variables

The three-particle configuration is completely described by a total scale factor \( r > 0 \), a relative scale factor \( q \geq 0 \), and two angles \( \varphi_1 \) and \( \varphi_2 \) such that
\[
  u = \frac{r q e^{i \varphi_1}}{\sqrt{2(1 + q^2)}} , \quad v = \frac{r e^{i \varphi_2}}{\sqrt{2(1 + q^2)}} .
\] (6)

These are the hyperspherical coordinates of Kilpatrick and Larsen [51], except that they use \( \varphi_1 \pm \varphi_2 \) instead of \( \varphi_1 \) and \( \varphi_2 \). We now have that
\[
  H_{\text{rel}} = -\frac{1}{2 r^3} \frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} - \frac{1}{2 r^2} \left( \frac{1 + q^2}{q} \frac{\partial}{\partial q} q \frac{\partial}{\partial q} + \frac{1}{q^2} \frac{\partial^2}{\partial \varphi_1^2} + \frac{\partial^2}{\partial \varphi_2^2} \right) + \frac{r^2}{2} ,
\] (7)

and
\[
  L_{\text{rel}} = -i \left( \frac{\partial}{\partial \varphi_1} + \frac{\partial}{\partial \varphi_2} \right) .
\] (8)

Assume that the wave function of the relative motion is
\[
  \psi = \psi(r, q, \varphi_1, \varphi_2) = f(r) g(q) e^{i(j \varphi_1 + k \varphi_2)} .
\] (9)

Then the eigenvalue equation \( H_{\text{rel}} \psi = E \psi \) separates into an angular eigenvalue equation,
\[
  (1 + q^2) \left( -\frac{1 + q^2}{q} \frac{d}{dq} q \frac{d}{dq} + \frac{j^2}{q^2} + k^2 \right) g = \lambda g ,
\] (10)

with \( \lambda \) as eigenvalue, and a radial equation,
\[
  -\frac{1}{2} f''(r) - \frac{3}{2r} f'(r) + \left( \frac{\lambda}{2 r^2} + \frac{r^2}{2} \right) f(r) = Ef(r) .
\] (11)

A general wave function can be written as a linear combination of such separated wave functions. As shown in the next section, we need linear combinations, where \( \lambda \) and \( j + k \) are constant but \( j - k \) varies, in order to satisfy the anyonic boundary conditions.

In the usual way we find the solutions of the radial equation to be
\[
  f(r) = r^\mu e^{-r^2/2} \sum_{m=0}^{n_r} a_m r^{2m} ,
\] (12)

with \( n_r = 0, 1, 2, \ldots \), with \( \mu > -2 \) for normalizability, \( \lambda = \mu(\mu + 2) \), and
\[
  E = 2 + \mu + 2n_r .
\] (13)

The coefficients in the polynomial satisfy the recursion formula
\[
  2(m + 1)(m + 2 + \mu) a_{m+1} = (2m + 2 + \mu - E) a_m .
\] (14)

The angular equation has two asymptotic solutions \( q^{\pm j} \) in the limit \( q \to 0 \). We exclude the singular solution (for \( j = 0 \) the singularity is logarithmic). In fact there is no reason for any singularity at \( q = 0 \), where the configuration is an equilateral triangle. This leaves a
solution which is unique up to a multiplicative constant, and with a convenient normalization it may be written as

\[ g(q) = q^{|j|} (1 + q^2)^\kappa \frac{a b}{c} F(a, b; c; -q^2) . \]  

(15)

The constant \( \kappa \) here may be chosen in one of two ways,

\[ \kappa = \frac{\mu}{2} + 1 \quad \text{or} \quad \kappa = -\frac{\mu}{2}, \]  

(16)

giving two different representations of the same solution. The constants

\[ a = |j| + |k| + \kappa, \quad b = |j| - |k| + \kappa, \quad c = 1 + |j|, \]  

(17)
define the hypergeometric series

\[ F(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!} , \]  

(18)

where, e.g., \((a)_0 = 1, (a)_{n+1} = (a)_n (a + n)\). The convergence radius for this series is 1. There exist however other representations,

\[ g(q) = q^{|j|} (1 + q^2)^{\kappa-a} F\left(a, c - b; c; \frac{q^2}{1 + q^2}\right) \]

\[ = q^{|j|} (1 + q^2)^{\kappa-b} F\left(c - a, b; c; \frac{q^2}{1 + q^2}\right), \]  

(19)

where the series converge for all real \( q \).

A useful parameter, as will be seen below, is the logarithmic derivative of \( g \) at \( q = 1 \),

\[ \beta = \frac{g'(1)}{g(1)} = |j| + \kappa - \frac{2ab F(a + 1, b + 1; c + 1; -1)}{c F(a, b; c; -1)}. \]  

(20)

4 The Anyonic Boundary Conditions

For three identical particles there is a six-fold identification of points in the Euclidean relative space. We will restrict the wave functions to the region \( 0 \leq q \leq 1 \), which corresponds to all the positively oriented triangles, but is still a three-fold covering of the true configuration space. The boundary conditions defining the particles to be anyons are of two types, since there are two classes of non-trivial permutations. The first class contains the three-particle cyclic permutations, which leave \( q \) invariant. The second class contains the two-particle interchanges, which transform \( q \) into \( 1/q \), and so give boundary conditions at \( q = 1 \).

Consider first a continuous, counterclockwise and cyclic deformation of the configuration, as defined in eq. (3), with no extra overall rotation of the triangle. It gives a phase factor \( e^{2i\theta} \) in the wave function, where \( \theta = \nu \pi \) is the statistics parameter. If we deform while keeping all the time \( |u| < |v| \), the phase of \( v \) increases continuously from \( \varphi_2 \) to \( \varphi_2 + (2\pi/3) \), whereas the phase of \( u \) changes from \( \varphi_1 \) to \( \varphi_1 - (2\pi/3) + 2m' \pi \), where \( m' \) is any integer. The corresponding boundary condition on the wave function is, therefore,

\[ \psi\left(r, q, \varphi_1 \frac{2\pi}{3} + 2m' \pi, \varphi_2 + \frac{2\pi}{3}\right) = e^{2i\theta} \psi(r, q, \varphi_1, \varphi_2) . \]  

(21)
That is,
\[ j \left( -\frac{2\pi}{3} + 2m'\pi \right) + k \frac{2\pi}{3} = 2(n' + \nu)\pi , \]
for some integer \( n' \). Since \( m' \) is an arbitrary integer, \( j \) must be an integer. Then
\[ k = j + 3(n + \nu) , \]
where \( n = n' - jm' \) is an arbitrary integer, and the eigenvalue of the relative angular momentum \( L_{\text{rel}} \) is
\[ \ell = j + k = 2j + 3(n + \nu) . \]

These relations take care of the cyclic permutations of all three particles. What remains is only to take care of one of the three cases where two particles are interchanged, for example \( z_2 \leftrightarrow z_3 \), or equivalently, \( u \leftrightarrow v \). This is the same as \( q \leftrightarrow 1/q \) and \( \varphi_1 \leftrightarrow \varphi_2 \), if we define angles so that \( u = v \) corresponds to \( \varphi_1 = \varphi_2 \). To be more precise, we consider a continuous interchange, with \( q = 1 \) at the beginning and end, and \( q < 1 \) during the interchange. The interchange should be anticlockwise, which means that we start with \( \varphi_1 > \varphi_2 \), and end with \( \varphi_1 < \varphi_2 \). There is one further restriction, that \( |\varphi_1 - \varphi_2| < (2\pi/3) \) when \( q = 1 \), meaning that the particle position \( z_1 \) must not be encircled.

Thus, the boundary condition on \( \psi \) at \( q = 1 \) is
\[ \psi(r, 1, \varphi_2, \varphi_1) = e^{i\theta} \psi(r, 1, \varphi_1, \varphi_2) , \]
for \( 0 < \varphi_1 - \varphi_2 < (2\pi/3) \). However, this condition is incomplete, because the general condition is
\[ \psi(r, 1/q, \varphi_2, \varphi_1) = e^{i\theta} \psi(r, q, \varphi_1, \varphi_2) . \]

Since the Schrödinger equation is second order in the \( q \) derivative, we need boundary conditions at \( q = 1 \) both for the wave function \( \psi \) and its normal derivative \( \psi_q = \partial\psi/\partial q \). The derivative condition is easily deduced,
\[ \psi_q(r, 1, \varphi_2, \varphi_1) = -e^{i\theta} \psi_q(r, 1, \varphi_1, \varphi_2) . \]

The boundary conditions for \( \psi \) and \( \psi_q \) can not be satisfied by a wave function which is separable in \( q, \varphi_1 \) and \( \varphi_2 \). But we may quantize the relative angular momentum \( \ell \), and according to eq. (24) \( \ell - 3\nu = 2j + 3n \) is an integer, either even or odd. Let \( \nu' = \nu \) if \( n = 2m \) and \( \nu' = \nu + 1 \) if \( n = 2m + 1 \), with \( m \) integer. Then
\[ j = \frac{\ell}{2} - 3 \left( m + \frac{\nu'}{2} \right) , \quad k = \frac{\ell}{2} + 3 \left( m + \frac{\nu'}{2} \right) . \]

Let \( g_m(q) \) be the function \( g(q) \) as given by eq. (15). Introducing \( \varphi = (\varphi_1 + \varphi_2)/2 \), and summing over \( m \), including an as yet undetermined coefficient \( C_m \) for each \( m \), we get the following angular wave function,
\[ \Omega(q, \varphi_1, \varphi_2) = \sum_{m=-\infty}^{\infty} C_m g_m(q) e^{i(j\varphi_1+k\varphi_2)} = e^{i\ell\varphi} \sum_{m=-\infty}^{\infty} C_m g_m(q) e^{-3i(m+(\nu'/2))(\varphi_1-\varphi_2)} . \]
It is natural to call $\Omega$ an anyonic spherical harmonic function, whenever it satisfies the anyonic boundary conditions.

Define $\xi = 3(\varphi_1 - \varphi_2)$. The two boundary conditions that must hold for $0 < \xi < 2\pi$ are

$$\sum_{m=-\infty}^{\infty} C_m g_m(1) e^{im\xi} = e^{i(\nu\pi - \nu'\xi)} \sum_{m=-\infty}^{\infty} C_m g_m(1) e^{-im\xi},$$

$$\sum_{m=-\infty}^{\infty} C_m g'_m(1) e^{im\xi} = -e^{i(\nu\pi - \nu'\xi)} \sum_{m=-\infty}^{\infty} C_m g'_m(1) e^{-im\xi}.$$

(30)

Recall that $g_m(1)$ and $g'_m(1)$ depend on the three parameters $\mu = E - 2 - 2n_r$, $\ell$ and $m$. We claim that for each given $\ell$, the parameter $\mu$, which determines the energy $E$, may be adjusted so that the above boundary conditions have non-trivial solutions for the coefficients $C_m$. For each $\ell$ there will be many solutions, possibly many with the same $\mu$, and this procedure should give the complete set of anyonic spherical harmonics. A complete proof would involve a study of the anyonic spherical harmonics, which we have not yet done, but we can give a plausibility argument, which also indicates one possible way to solve the problem numerically.

We may regard $\gamma_m = C_m g_m(1)$ and $\tilde{\gamma}_m = C_m g'_m(1)$ as the Fourier components of two functions

$$\gamma(\xi) = \sum_{m=-\infty}^{\infty} C_m g_m(1) e^{im\xi},$$

$$\tilde{\gamma}(\xi) = \sum_{m=-\infty}^{\infty} C_m g'_m(1) e^{im\xi},$$

(31)

periodic in $\xi$ with period $2\pi$. We may restrict their regions of definition to be the interval $[0, 2\pi]$. There is a natural scalar product between any two functions $\phi = \phi(\xi)$ and $\chi = \chi(\xi)$, with Fourier components $\phi_m$ and $\chi_m$ as above,

$$(\phi, \chi) = \int_0^{2\pi} d\xi (\phi(\xi))^* \chi(\xi) = 2\pi \sum_{m=-\infty}^{\infty} (\phi_m)^* \chi_m.$$

(32)

Define the linear operator $A$ by

$$[A \phi](\xi) = e^{i(\nu\pi - \nu'\xi)} \phi(2\pi - \xi),$$

(33)

for $0 < \xi < 2\pi$. Then $A$ is Hermitean with respect to the natural scalar product, and $A^2 = I$, the identity operator. Note that $A$ is a somewhat singular operator, unless $\nu$ is an integer, since the factor $e^{i(\nu\pi - \nu'\xi)}$, extended by periodicity outside the interval $[0, 2\pi]$, is discontinuous at every integer multiple of $2\pi$.

Define another operator $B$ such that $B \gamma = \tilde{\gamma}$. It is also Hermitean, since it is diagonalized by the Fourier transformation, and its eigenvalues $\beta_m = g'_m(1)/g_m(1)$ are real. The two boundary conditions are simply $\gamma = A \gamma$, $B \gamma = -AB \gamma$, equivalent to two simultaneous eigenvalue equations,

$$A \gamma = \gamma, \quad (I + A)B(I + A) \gamma = 0.$$

(34)

These two equations are compatible, since the operators $A$ and $B_\pm = (I + A)B(I + A)$ commute, and $B_+$ has a complete set of real eigenvalues, since it is Hermitean. If we select
one eigenvalue of $B_+$ on the subspace where $A\gamma = \gamma$, and set it equal to zero, this is one real equation for one real parameter $\mu$. By equating different eigenvalues to zero we should obtain a complete set of solutions.

There is one minor problem with this argument in that $B$ is undefined if an eigenvalue $\beta_m = g_m'(1)/g_m(1)$ is infinite. There is of course a similar argument using $B^{-1}$ instead of $B$, but in principle it might happen that $B$ and $B^{-1}$ are undefined simultaneously.

5 Numerical Solution

To find a numerical solution for the coefficients $C_m$ satisfying the boundary conditions in eq. (30), we must truncate to a finite number $N$ of coefficients, and in order to use the fast Fourier transform, we choose $N$ as a power of 2. Then we impose eq. (30) at the $N$ discrete points

$$\xi_k = \frac{(2k - 1)\pi}{N}, \quad \text{for} \quad k = 1, 2, \ldots, N. \quad (35)$$

It is a remarkable feature of our method that for very small $N$, say $N = 4$, it can give many energy levels with good accuracy. This is so when the low Fourier components, i.e. with small $m$, dominate. On the other hand, the convergence as $N \to \infty$ is sometimes very slow. This is clearly related to the fact that the wave functions for non-integer $\nu$ have non-integer power behaviour at $\xi = 0$, where two particles meet. Hence our approximations by finite Fourier series converge slowly. Note that eq. (30) at $\xi = 0$ gives that

$$(1 - e^{i\nu\pi})\gamma(0) = (1 + e^{i\nu\pi})\tilde{\gamma}(0) = 0, \quad (36)$$

assuming that the wave function is continuous. Thus $\gamma(0) = 0$, except in the boson case, and $\tilde{\gamma}(0) = 0$, except in the fermion case. For our numerical solutions these conditions at $\xi = 0$ are not imposed and will hold only in the asymptotic limit $N \to \infty$.

To illustrate the convergence we have tabulated the approximate energy $E_N$ as a function of $N$, in three cases. Table 1 is for the state which becomes the bosonic ground state at $\nu = 0$. It is exactly known and has energy $E = 2 + 3\nu$ and angular momentum $\ell = 3\nu$. Table 2 is for the state which becomes the fermionic ground state at $\nu = 1$. It has angular momentum $\ell = -3 + 3\nu$ and an energy depending on $\nu$ in an unknown way. Table 3 is for the state connected to the fermionic ground state by the supersymmetry transformation of Sen [22, 23]. It has angular momentum $\ell = -2 + 3\nu$. The energy eigenvalues in Tables 2 and 3 should be the same, except that $\nu$ in one table corresponds to $1 - \nu$ in the other.

It appears from the tables that the leading correction term for finite $N$ is of order $N^{-2\nu}$. Using two different $N$ one may therefore extrapolate to $N = \infty$, and this improves the convergence considerably. Another point to note is that one may take advantage of the supersymmetry in order to get more accurate energy levels.

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Table 1: The bosonic ground state energy

| $E_{\text{exact}}$ | $\nu = 0.2$ | $\nu = 0.4$ | $\nu = 0.6$ | $\nu = 0.8$ |
|-------------------|-------------|-------------|-------------|-------------|
| $N = 2$           | 2.3279      | 2.97208     | 3.66606815  | 4.34550634  |
| $N = 4$           | 2.3786      | 3.06317     | 3.74358644  | 4.38428582  |
| $N = 8$           | 2.4228      | 3.11916     | 3.77566477  | 4.39500104  |
| $N = 16$          | 2.4599      | 3.15274     | 3.78942368  | 4.39836691  |
| $N = 32$          | 2.4902      | 3.17256     | 3.79539568  | 4.39946267  |
| $N = 64$          | 2.5146      | 3.18414     | 3.79799523  | 4.39982287  |
| $E_\infty$ from fit to $E_N = E_{\text{exact}} + AN^{-\gamma}$ |
| $N = 2, 4$        | 2.5375      | 3.18609     | 3.80333554  | 4.4037554   |
| $N = 4, 8$        | 2.5611      | 3.19470     | 3.80038992  | 4.40027574  |
| $N = 8, 16$       | 2.5759      | 3.19805     | 3.80002870  | 4.40002381  |
| $N = 16, 32$      | 2.5851      | 3.19931     | 3.79999873  | 4.40000207  |
| $N = 32, 64$      | 2.5909      | 3.19976     | 3.79999891  | 4.40000018  |
Table 2: The fermionic ground state energy

| N   | ν = 0.2     | ν = 0.4     | ν = 0.6     | ν = 0.8     |
|-----|-------------|-------------|-------------|-------------|
| 2   | 4.6455      | 4.380322    | 4.1813475   | 4.0485533   |
| 4   | 4.6356      | 4.379612    | 4.1818967   | 4.0479552   |
| 8   | 4.6446      | 4.394153    | 4.1895454   | 4.0497399   |
| 16  | 4.6552      | 4.405332    | 4.1940494   | 4.0506380   |
| 32  | 4.6646      | 4.412454    | 4.1962485   | 4.0509943   |
| 64  | 4.6723      | 4.416732    | 4.1972586   | 4.0511243   |
| 128 | 4.6786      | 4.419241    | 4.1977105   | 4.0511700   |

\[ \gamma \text{ from fit to } E_N = E_\infty + AN^{-\gamma} \]

| N   | 8, 16, 32   | 16, 32, 64  | 32, 64, 128 |
|-----|-------------|-------------|-------------|
| 1/2 | 0.171       | 0.650       | 1.034       |
|     | 0.276       | 0.735       | 1.122       |
|     | 0.319       | 0.770       | 1.160       |
|     | 0.319       | 0.770       | 1.160       |
|     | 0.319       | 0.770       | 1.160       |
|     | 0.319       | 0.770       | 1.160       |

| N   | 8, 16, 32   | 16, 32, 64  | 32, 64, 128 |
|-----|-------------|-------------|-------------|
| 1/3 | 4.7394      | 4.424958    | 4.1983467   |
|     | 4.7092      | 4.423167    | 4.1981166   |
|     | 4.7038      | 4.422799    | 4.1980764   |
|     | 4.7038      | 4.422799    | 4.1980764   |
|     | 4.7038      | 4.422799    | 4.1980764   |
|     | 4.7038      | 4.422799    | 4.1980764   |

| N   | 2, 4        | 4, 8        | 8, 16       | 16, 32      | 32, 64      | 64, 128     |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| 1/2 | 4.6046      | 4.4378652   | 4.1823200   | 4.0476609   |
|     | 4.6726      | 4.413775    | 4.1954408   | 4.0506184   |
|     | 4.6883      | 4.420416    | 4.1975209   | 4.0510800   |
|     | 4.6940      | 4.422065    | 4.1979435   | 4.0511697   |
|     | 4.6966      | 4.422505    | 4.1980371   | 4.0511884   |
|     | 4.6981      | 4.422582    | 4.1980588   | 4.0511924   |
Table 3: The supersymmetric partner of the fermionic ground state

| $N$  | $\nu = 0.2$   | $\nu = 0.4$   | $\nu = 0.6$   | $\nu = 0.8$   |
|------|----------------|----------------|----------------|----------------|
| 2    | 3.8436         | 3.95456       | 4.2494885      | 4.62714752     |
| 4    | 3.8814         | 4.05114       | 4.3486537      | 4.67801375     |
| 8    | 3.9144         | 4.11177       | 4.3911581      | 4.69295572     |
| 16   | 3.9423         | 4.14784       | 4.4091615      | 4.69770346     |
| 32   | 3.9653         | 4.16898       | 4.4168475      | 4.69924411     |
| 64   | 3.9840         | 4.18127       | 4.4201526      | 4.69974811     |

$\gamma$ from fit to $E_N = E_\infty + AN^{-\gamma}$

| $N$  | $\gamma$ = 0.194 | $\gamma$ = 0.672 | $\gamma$ = 1.222 | $\gamma$ = 1.767 |
|------|-----------------|-----------------|-----------------|-----------------|
| 2, 4, 8 | 0.194          | 0.672           | 1.222           | 1.767           |
| 4, 8, 16 | 0.246          | 0.749           | 1.239           | 1.654           |
| 8, 16, 32 | 0.275          | 0.772           | 1.228           | 1.623           |
| 16, 32, 64 | 0.299         | 0.782           | 1.218           | 1.614           |

$E_\infty$ from fit to $E_N = E_\infty + AN^{-\gamma}$

| $N$  | $E_\infty$ = 4.1441 | $E_\infty$ = 4.21400 | $E_\infty$ = 4.4230430 | $E_\infty$ = 4.69917053 |
|------|-------------------|--------------------|----------------------|------------------------|
| 2, 4, 8 | 4.1441            | 4.21400            | 4.4230430            | 4.69917053            |
| 4, 8, 16 | 4.0925            | 4.20086            | 4.4223905            | 4.69991463            |
| 8, 16, 32 | 4.0749            | 4.19887            | 4.4225732            | 4.69998554            |
| 16, 32, 64 | 4.0652            | 4.19836            | 4.4226461            | 4.69999228            |

$E_\infty$ from fit to $E_N = E_\infty + AN^{-2\nu}$

| $N$  | $E_\infty$ = 3.9997 | $E_\infty$ = 4.18146 | $E_\infty$ = 4.4250877 | $E_\infty$ = 4.70305333 |
|------|-------------------|--------------------|----------------------|------------------------|
| 2, 4 | 3.9997            | 4.18146            | 4.4250877            | 4.70305333            |
| 4, 8 | 4.0178            | 4.19357            | 4.4239194            | 4.70031110            |
| 8, 16 | 4.0295            | 4.19652            | 4.4230381            | 4.70004061            |
| 16, 32 | 4.0374            | 4.19749            | 4.4227717            | 4.70000341            |
| 32, 64 | 4.0426            | 4.19786            | 4.4227001            | 4.69999592            |