MORSE FUNCTIONS AND REAL LAGRANGIAN THIMBLES ON ADJOINT ORBITS

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Abstract. We compare Lagrangian thimbles for the potential of a Landau–Ginzburg model to the Morse theory of its real part. We explore Landau–Ginzburg models defined using Lie theory, constructing their real Lagrangian thimbles explicitly and comparing them to the stable and unstable manifolds of the real gradient flow.

Contents

1. Real and complex Morse functions 1
2. Symplectic Lefschetz fibrations 3
3. The gradient field of of $\text{Re} f_H$ 5
4. Lagrangian vanishing cycles 9
5. Real Lagrangian thimbles 12
6. The potential and graphs 15
7. Minimal semisimple orbits 19
8. Acknowledgements 24
References 24

1. Real and complex Morse functions

Given a real manifold $M$, a smooth function $f : M \to \mathbb{R}$ is called a Morse function if it has only nondegenerate critical points. Recall that a critical point $p$ of $f$ is nondegenerate if the Hessian matrix $\frac{\partial^2 f}{\partial x_i \partial x_j}(p)$ is nonsingular. Nondegenerate critical points are isolated, and the lemma of Morse tells us that, on a neighborhood of such a critical point, the Morse function can be written in local coordinates as

$$f = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

The integer $\lambda$ is called the index of $f$ at $p$. Morse theory tells us how to recover the topology of a compact manifold $M$ from the data of indices of critical points of $f$. In fact, $M$ has the homotopy type of a finite CW-complex with one cell of dimension $\lambda$ for each critical point of index $\lambda$, see [M].

Given a complex manifold $M$, a complex function $f : M \to \mathbb{C}$ is called a Landau–Ginzburg model and the function $f$ is called the superpotential. If in addition $df$ is surjective outside a finite number of points and $f$ is a holomorphic Morse function, that is, it has only nondegenerate critical points, then $f$ is called a Topological Lefschetz fibration. On a neighborhood
of a critical point, a Lefschetz fibration can be written in local coordinates as

\[ f = f(p) + z_1^2 + \cdots + z_n^2. \]

Note that changing the coordinate \( z_j \) to \( i z_j \) takes \( z_j^2 \) to \( -z_j^2 \) so that it is meaningless to talk about indices of critical points in the complex case. Furthermore, observe that if the complex manifold \( M \) is compact, then any holomorphic function \( f: M \to \mathbb{C} \) is constant. Hence, this type of Landau–Ginzburg model is interesting only in the case when \( M \) is non-compact. For compact manifolds the natural concept of Landau–Ginzburg model is \( f: M \to \mathbb{P}^1 \).

In algebraic geometry functions \( f: M \to \mathbb{P}^1 \) give rise to so-called pencils, which are studied extensively within the context of Picard–Lefschetz theory. Fibers of such pencils intersect in the base locus, and a topological Lefschetz fibration can be obtained by blowing up this base locus. If \( M_b \) is a regular fibre contained in a small neighborhood of a singular fibre \( M_o \), then there is a retraction \( M_b \to M_o \) which induces a surjection in homology \( H_*(M_b) \to H_*(M_o) \). The classes in the kernel of this surjection are called vanishing cycles and are important objects of study in Hodge theory, see [PS]. The fundamental theorem of Picard–Lefschetz theory describes the intersection theory of vanishing cycles in the case when \( M \) is a projective variety. In particular, for compact \( M \) this fundamental theorem implies that each critical point of \( f \) has a corresponding vanishing cycle. However, in the noncompact case existence of a vanishing cycle corresponding to each critical point is not guaranteed, see [S].

If \((M, \omega)\) is a symplectic manifold, then a topological Lefschetz fibration \( f: M \to \mathbb{C} \) is called a Symplectic Lefschetz fibration provided the symplectic form \( \omega \) is nondegenerate on the fibre \( M_x \) for all \( x \) in the sense that:

1. \( M_x \) is a symplectic submanifold of \( M \) for each regular value \( x \), and
2. for each critical point \( p \) the symplectic form \( \omega_p \) is non degenerate over the tangent cone of \( M_{f(p)} \) at \( p \).

For any symplectic fibration there exists a natural connection obtained by taking the symplectic orthogonal to the fibre. If \( o \) is a critical value of \( f \) and \( b \) is a regular value contained in a neighborhood of \( o \), then consider a path \( \lambda: [0, 1] \to \mathbb{C} \) from \( \lambda(0) = b \) to \( \lambda(1) = o \). Given a vanishing cycle \( \alpha \subset M_b \) we can use the connection to parallel transport the cycle \( \alpha \) along \( \lambda \) all the way to the corresponding critical point \( p \). For each \( t \) in \([0, 1]\) we obtain a cycle \( \alpha_t \subset M_{\lambda(t)} \) so that \( \alpha_0 = \alpha \) and \( \alpha_1 = p \). The object traced by the cycle \( \alpha \) on its way to \( p \) is topologically a closed disc \( D = \{ \cup_{t \in [0, 1]} \alpha_t \} \) with boundary \( \partial D = \alpha \) and is called a thimble.

Vanishing cycles live naturally in the middle homology of the regular fibre, hence \( \dim \alpha = \dim \mathbb{C} M_b \). Thus, it makes sense ask whether \( \alpha \) is a Lagrangian submanifold of \( M_b \), that is, if \( \omega \) vanishes on \( \alpha \), and in the affirmative case \( \alpha \) is called a Lagrangian vanishing cycle. If the corresponding thimble is a Lagrangian submanifold of \( M \) it is then called a Lagrangian thimble. Lagrangian thimbles are the objects that generate the so-called Fukaya–Seidel category of the fibration, and they are our main objects of study in this paper.
We explore symplectic Lefschetz fibrations on semisimple adjoint orbits, recalling the construction of the complex superpotential in section 2 and describing the gradient vector field of its real part in section 3. We then construct Lagrangian vanishing cycles in section 4 and Lagrangian thimbles of a preferred type which we name real Lagrangian thimbles (definition 18) obtained using the Morse theory of the real part of the superpotential.

Profiting from the knowledge of Lagrangian submanifolds of the adjoint orbits described in \([GGSM2]\) and existence of Lagrangian submanifolds inside their compactifications described in \([GSMV]\) we have existence of Lagrangian submanifolds \(V\) passing through any critical value \(c\) of the superpotential \(f = f_1 + i f_2\) and containing a real sphere that is a vanishing cycle for \(f_1\) constructed in section 4. Then, considering the restriction of the real part \(g_1 = f_1\mid V\) to the Lagrangian submanifold \(V\) of \(O(H_0)\) we are able to find out explicitly the desired real thimbles:

**Theorem (17).** Take \(c\) near \(f_1(x) = g_1(x)\). We have that

\[
\begin{align*}
g_1^{-1}[c, g_1(x)] &= f_1^{-1}[c, f_1(x)] \cap V \text{ in the negative definite case, or} \\
g_1^{-1}[g_1(x), c] &= f_1^{-1}[f_1(x), c] \cap V \text{ in the positive definite case}
\end{align*}
\]

is homeomorphic to a closed ball in \(\mathbb{R}^{\dim V}\). This ball is a Lagrangian thimble.

We provide examples that illustrate the behaviour of the superpotential over Lagrangian submanifolds obtained from graphs in section 6. Finally, exploring the graph of \(\Gamma(R_{w_0})\) of the right translation by the principal involution of the Weyl group, we explicitly describe examples of the relation between the Morse theory of the real part and the real Lagrangian thimbles of the superpotential, concluding this work with:

**Theorem (29).** The stable and unstable manifolds of \(\text{grad} \ (Re f_H)\) at the critical point \([e_j]\) are open in the graph \(\Gamma(m_j^\pm \circ R_{w_0})\). The real Lagrangian thimbles are closed balls contained in the graph \(\Gamma(m_j^\pm \circ R_{w_0})\).

The relation between real thimbles in symplectic Lefschetz fibrations and stable and unstable manifolds of the gradient flow is part of the folklore of the subject and is presumably well known to experts. Nevertheless, we were unable to find such relation explained in detail anywhere in the literature, and we believe that the explicit constructions given here are useful illustrations of the construction of Lagrangians.

### 2. Symplectic Lefschetz Fibrations

In this section we summarize the construction of symplectic Lefschetz fibrations on adjoint orbits, their compactifications and Lagrangian submanifolds discussed in \([GGSM1, GGSM2, BGGM]\).

Let \(G\) be a complex semisimple Lie group with Lie algebra \(\mathfrak{g}\) and denote by \((X,Y) := \text{tr}(\text{ad}(X),\text{ad}(Y))\) the Cartan–Killing form of \(\mathfrak{g}\). Fix a Cartan subalgebra \(\mathfrak{h} \subset \mathfrak{g}\) and a real compact form \(\mathfrak{u}\) of \(\mathfrak{g}\). Associated to these subalgebras are the subgroups \(T = \exp(\mathfrak{h}) = \exp\mathfrak{h}\) and \(U = \exp\mathfrak{u} = \exp\mathfrak{u}\). Denote by \(\tau\) the conjugation associated to \(\mathfrak{u}\) which is defined by \(\tau(X) = X\) if \(X \in \mathfrak{u}\) and \(\tau(Y) = -Y\) if \(Y \in i\mathfrak{u}\), that is, if \(Z = X + iY \in \mathfrak{g}\) with \(X, Y \in \mathfrak{u}\)
then $\tau(X + iY) = X - iY$. In this case we can define the Hermitian form $H_\tau: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ as
\[ H_\tau(X, Y) := -\langle X, \tau Y \rangle. \tag{1} \]
If we write the real and imaginary parts of $H_\tau$ as
\[ H_\tau(X, Y) = (X, Y) + i\Omega(X, Y), \]
it is well known that the real part $(\cdot, \cdot)$ is an inner product and the imaginary part $\Omega$ is a symplectic form on $\mathfrak{g}$. Indeed, we have
\[ 0 \neq iH(H, X, X) = H(iX, X) = i\Omega(iX, X), \]
that is, $\Omega(iX, X) \neq 0$ for all $X \in \mathfrak{g}$, which shows that $\Omega$ is nondegenerate. Moreover, $d\Omega = 0$ because $\Omega$ is a constant bilinear form. The fact that $\Omega(iX, X) \neq 0$ for all $X \in \mathfrak{g}$ guarantees that the restriction of $\Omega$ to any complex subspace of $\mathfrak{g}$ is also nondegenerate.

We denote by $O(H_0)$ the adjoint orbit $\text{Ad}G(H_0)$ of $H_0 \in \mathfrak{g}$. Denote by $\mathfrak{h}^*$ the dual vector space of $\mathfrak{h}$ and by $\Pi$ the set of all roots associated to the Cartan subalgebra $\mathfrak{h}$. An element $H \in \mathfrak{h}$ is called regular if $\alpha(H) \neq 0$ for all $\alpha \in \Pi$. As the restriction of Cartan–Killing form to $\mathfrak{h}$ is nondegenerate, the map $\varphi: \mathfrak{h} \to \mathfrak{h}^*$ defined by $\varphi(X) = \langle X, \cdot \rangle$ is a linear isomorphism. We denote by $\mathfrak{h}_R$ the real subspace of $\mathfrak{h}$ generated by $\varphi^{-1}(\Pi)$. The pullback of the symplectic form $\Omega$ by the inclusion $O(H_0) \hookrightarrow \mathfrak{g}$ defines a symplectic form on $O(H_0)$. With this choice of symplectic form, we have our construction of Symplectic Lefschetz fibrations via Lie theory as follows:

**Theorem 1.** [GGSM1, Thm. 2.2] Let $\mathfrak{h}$ be the Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Given $H_0 \in \mathfrak{h}$ and $H \in \mathfrak{h}_R$ with $H$ a regular element. The height function $f_H: O(H_0) \to \mathbb{C}$ defined by
\[ f_H(x) = \langle H, x \rangle \quad x \in O(H_0) \]
has a finite number $= |W|/|W_{H_0}|$ of isolated singularities and gives $O(H_0)$ the structure of a symplectic Lefschetz fibration.

Here $W = \text{Nor}_G(\mathfrak{h})/\text{Cen}_G(\mathfrak{h})$ denotes the Weyl group. In the language used in Homological Mirror Symmetry the pair $(O(H_0), f_H)$ is called a Landau–Ginzburg model with superpotential $f_H$. See [BBGGS] for a discussion of the mirror of $O(H_0)$ in the case of $\mathfrak{sl}(2, \mathbb{C})$.

Given a regular element $H_0 \in \mathfrak{g}$, consider the set $\Theta$ of simple roots that have $H_0$ in their kernel. Let $\mathfrak{p}_\Theta$ be the parabolic subalgebra determined by $\Theta$, with corresponding parabolic subgroup $P_\Theta$. The quotient $F_\Theta := G/P_\Theta$ by the parabolic subgroup is the flag manifold determined by $H_0$. Another regular element in $\mathfrak{g}$ will correspond to the same flag manifold if it is annihilated by the same set of roots $\Theta$, so for questions regarding the isomorphism with $T^*F_\Theta$ we denote the regular element by $H_0$ instead of $H_0$.

The adjoint orbit of a regular element $H_\Theta$ is isomorphic to the cotangent bundle of the flag manifold $F_\Theta$ [GGSM2, Thm. 2.1]. The isomorphism $\iota: O(H_\Theta) \to T^*F_\Theta$ is obtained observing that
\[ O(H_\Theta) = \bigcup_{k \in K} \text{Ad}(k) \left( H_\Theta + \mathfrak{n}_\Theta^+ \right), \]
then taking for each $X \in \mathfrak{n}_\Theta^\perp$, the correspondence:

$$\text{Ad} \, (k) \left( \mathcal{H}_\Theta + X \right) \mapsto \langle \text{Ad} \, (k) \, X, \cdot \rangle$$

where $\text{Ad} \, (k) \mathfrak{n}_\Theta^\perp$ is identified with the tangent space $T_{\mathfrak{h}_\Theta^\perp} \mathfrak{F}_\Theta$, where $\mathfrak{b}_\Theta$ is the origin of the flag.

Let $\mu$ be the moment map of the action $a \colon G \times T^* \mathfrak{F}_\Theta \to T^* \mathfrak{F}_\Theta$. Then $\mu \colon T^* \mathfrak{F}_\Theta \to \text{Ad} \, (G) \, \mathcal{H}_\Theta$ is the inverse of the map $\iota \colon \text{Ad} \, (G) \, \mathcal{H}_\Theta \to T^* \mathfrak{F}_\Theta$, and satisfies

$$\mu^* \omega = \Omega,$$

where $\Omega$ is the canonical symplectic form of $T^* \mathfrak{F}_\Theta$ and $\omega$ the (real) Kirillov–Kostant–Souriau form on $\text{Ad} \, (G) \, \mathcal{H}_\Theta$.

We compactify the total space of $T^* \mathfrak{F}_\Theta$ to the trivial product $F_\Theta \times F_\Theta^\ast$ as:

$$\mathcal{O}(\mathcal{H}_\Theta) \sim \!\!\!\sim T^* \mathfrak{F}_\Theta \sim \!\!\!\sim \overline{T^* \mathfrak{F}_\Theta} = F_\Theta \times F_\Theta^\ast.$$

[BGGSM, Thm. 5.3] showed how to extend the potential $f_H$ to the compactification in the case of minimal adjoint orbits.

Let $w_0$ be the principal involution of the Weyl group $\mathcal{W}$, that is, the element of highest length as a product of simple roots. The right action $R_{w_0} : \mathbb{F}_{H_0^+} \rightarrow \mathbb{F}_{H_0^+}$ is anti-symplectic with respect to the Kähler forms on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^+}$ given by the Borel metric and canonical complex structures. We use graphs of anti-symplectic maps to construct Lagrangian submanifolds of $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H_0^+}$; these graphs will be used in the construction of Lagrangian thimbles of $\mathcal{O}(H_0)$ in section 5.

**Notation 2.** We will denote by $\Gamma(f)$ the graph of a map $f$.

**Remark 3.** $\Gamma(R_{w_0})$, that is, the graph of $R_{w_0}$ the right translation by the principal involution of $\mathcal{W}$, is the orbit of $K$ by the diagonal action. This orbit is the zero section of $T^* \mathbb{F}_{H_0}$ under the identification with $\mathcal{O}(H_0) \approx G \cdot (H_0, -H_0)$. Therefore, $\Gamma(R_{w_0})$ is a real Lagrangian submanifold of the product.

### 3. The Gradient Field of of $\text{Re}f_H$

The field $Z(x) = [x, [\tau x, H]]$ is defined over the whole algebra $\mathfrak{g}$ and is tangent to the adjoint orbits, since the tangent space to $\text{Ad} \, (G) \, x$ at $x$ is the image of $\text{ad} \, (x)$. Assume here that both $H$ and $H_0$ are regular and belong to the Weyl chamber $\mathfrak{h}_\mathbb{R}^+$. The field $Z$ is gradient, not with respect to the inner product coming from $\mathfrak{g}$ (the real part of $\mathfrak{h}$), but with respect to the Riemannian metric $m$ on the adjoint orbit $\mathcal{O}(H_0)$, which does not extend naturally to $\mathfrak{g}$.

The metric $m$ is defined as follows: the tangent space $T_x \mathcal{O}(H_0)$ is the image of $\text{ad} \, (x)$, which is the sum of the eigenspaces associated to the nonzero eigenvalues of $x$. This happens because $\text{ad} \, (x)$ is conjugate to $\text{Ad} \, (H_0)$ (the formula $\text{ad} \, (\phi_x) = \phi \circ \text{ad} \, (x) \circ \phi^{-1}$ holds true for any automorphism $\phi \in \text{Aut} \, (\mathfrak{g}$), in particular for $\phi = \text{Ad} \, (g)$, $g \in G$). Now, $\text{ad} \, (H_0)$ is diagonalizable and its image is the sum of the root spaces, which are the eigenspaces of the nonzero eigenvalues of $\text{ad} \, (H_0)$ (since $H_0$ is regular). By conjugation the same is true for $\text{ad} \, (x)$, $x \in \mathcal{O}(H_0)$. As a consequence, the restriction of $\text{ad} \, (x)$ to its image is an invertible linear transformation.
Taking this into account, define
\[ m_x(u, v) = (\text{ad}(x)^{-1} u, \text{ad}(x)^{-1} v) , \]
where \((\cdot, \cdot)\) is the inner product given by the real part of \(\mathcal{H}(\cdot, \cdot)\), and the inverse of \(\text{ad}(x)\) is just the inverse of its restriction to the tangent space. The form \(m_x(\cdot, \cdot)\) is a well defined Riemannian metric on \(O(H_0)\).

**Remark 4.** The realification \(g^R\) of \(g\) is a real semisimple Lie algebra. Its Cartan–Killing form \(\langle \cdot, \cdot \rangle^R\) is given by \(\langle \cdot, \cdot \rangle^R = 2\Re\langle \cdot, \cdot \rangle\) (see [SM]). Consequently, the inner product \((\cdot, \cdot)\) is given by
\[ (X, Y) = \frac{1}{2} \langle X, \tau Y \rangle^R \]
where \(\tau\) is conjugation with respect to \(u\), which is a linear transformation of \(g^R\) (over \(R\)).

Returning to the field \(Z(x)\), define the height function \(h_H: O(H_0) \to \mathbb{R}\) by
\[ h_H(x) = (x, H) . \]
Given \(A \in g\), the tangent vector \([A, x]\) is given by
\[ [A, x] = \frac{d}{dt} \bigg|_{t=0} \text{Ad} \left( e^{tA} \right) x. \]
Therefore,
\[ (dh_H)_x ([A, x]) = \frac{d}{dt} \bigg|_{t=0} \left( \text{Ad} \left( e^{tA} \right) x, H \right) = ([A, x], H) . \tag{2} \]

On one hand,
\[ m_x([A, x], Z(x)) = -m_x(\text{ad}(x) A, \text{ad}(x) [\tau x, H]) = - ([A, [\tau x, H]]) , \]
by definition of \(m_x\). On the other hand, by lemma 6 below,
\[ ([A, [\tau x, H]]) = (A, \text{ad}(\tau x) H) = - (\text{ad}(x) A, H) . \]
Thus,
\[ m_x([A, x], Z(x)) = (\text{ad}(x) A, H) = - ([A, x], H) . \]
Combining this with (2) we arrive at
\[ (dh_H)_x ([A, x]) = -m_x([A, x], Z(x)) . \]

In conclusion:

**Proposition 5.** \(Z(x) = - \nabla h_H \) with respect to the metric \(m_x\).

**Lemma 6.** Consider the inner product \((\cdot, \cdot) := \Re \mathcal{H}(\cdot, \cdot)\), then
- the conjugation \(\tau\) is an isometry for this inner product, and
\[ (\text{ad}(X) Y, Z) = - (Y, \text{ad}(\tau X) Z) . \]
- if \(\tau X = X\), that is, if \(X \in u\), then \(\text{ad}(X)\) is antisymmetric for \((\cdot, \cdot)\),
- if \(\tau Y = -Y\), that is, \(Y \in iu\), then \(\text{ad}(Y)\) is symmetric for \((\cdot, \cdot)\).
Proof. If $X \in \mathfrak{g}$ then $\mathcal{H}(\text{ad}(X)Y, Z) = -\mathcal{H}(Y, \text{ad}(\tau X)Z)$, and it follows that the same relation holds true for the inner product $\langle \cdot, \cdot \rangle$. In fact,

$$\mathcal{H}(\tau X, Y) = -\langle \tau X, \tau Y \rangle = -\langle \tau Y, \tau X \rangle = \mathcal{H}(\tau Y, X) = \mathcal{H}(X, \tau Y),$$

which means that $(\tau X, Y) = (X, \tau Y)$. For the second item:

$$\mathcal{H}([X,Y],Z) = -\langle [X,Y], \tau Z \rangle = \langle Y, [X,\tau Z] \rangle = \langle \tau Y, [\tau X, Z] \rangle = -\mathcal{H}(Y, [\tau X, Z]).$$

Remark 7. ($Z$ as a field on $\mathfrak{g}$) We show that considered on the entire vector space $\mathfrak{g}$ the vector field $Z(x) = [x, [\tau x, H]]$ is not gradient with respect to $(\cdot, \cdot)$. Take the differential form $\alpha_x(v) = (v, Z(x))$. Then $d\alpha(v,v) = v\alpha(w) - w\alpha(v) - \alpha[v,w]$, where the last term vanishes if $v$ and $w$ are regarded as constant vector fields on $\mathfrak{g}$. The expression for $Z$ then gives

$$(d\alpha)_x(v,w) = (w, [v, [\tau x, H]]) + (w, [x, [\tau v, H]]) - (v, [w, [\tau x, H]]) - (v, [x, [\tau w, H]]).$$

Evaluating this expression on $x = H_1 \in \mathfrak{h}$, we obtain

$$(d\alpha)_x(v,w) = (w, [H_1, [\tau v, H]]) - (v, [H_1, [\tau w, H]]) = (w, \text{ad}(H)\text{ad}(H_1)\tau v) - (v, \text{ad}(H)\text{ad}(H_1)\tau w).$$

We have

$$(w, \text{ad}(H)\text{ad}(H_1)\tau v) = \left(\tau w, \tau^{-1}\text{ad}(H)\text{ad}(H_1)\tau v\right) = (\tau w, \text{ad}(\tau H)\text{ad}(\tau H_1)v) = \left(\text{ad}(\tau H_1)\text{ad}(\tau H)\tau w, v\right) = (\text{ad}(\tau H)\text{ad}(\tau H_1)\tau w, v)$$

where the last equality comes from the fact that $\text{ad}(H)$ commutes with $\text{ad}(H_1)$. Therefore,

$$(d\alpha)_x(v,w) = (\text{ad}(\tau H)\text{ad}(\tau H_1)\tau w, v) - (\text{ad}(H)\text{ad}(H_1)\tau w, v).$$

But, setting $\tau H = -H$ and $\tau(H_1) = H_1$, then the right hand side becomes $-2\text{ad}(H)\text{ad}(H_1)\tau w, v)$ which does not vanish identically on $v, w$. Thus, $d\alpha \neq 0$, implying that the vector field is not gradient on $\mathfrak{g}$.

We return to the study of the singularities of the gradient field $Z$ on the orbit $\mathcal{O}(H_0)$. We have verified that the set of such singularities is $\mathcal{O}(H_0) \cap \mathfrak{h}$, which is the orbit of $H_0 \in \mathfrak{h}$ by the Weyl group. We now recall the proof that these singularities are nondegenerate. To see this, let $x = wH_0$ be one of the singularities. Then the differential of $Z$ at $x$ is given by

$$dZ_x(v) = [v, [\tau x, H]] + [x, [\tau v, H]] = [x, [\tau v, H]].$$

The tangent space to $\mathcal{O}(H_0)$ at $x$ is

$$T_x\mathcal{O}(H_0) = \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha = \sum_{\alpha > 0} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$
If \( v = \sum_{\alpha \in \Pi} a_\alpha X_\alpha \) then \( \tau v = -\sum_{\alpha \in \Pi} a_\alpha X_\alpha \). Consequently,

\[
dZ_x (v) = \text{ad} (x) \text{ad} (H) \left( \sum_{\alpha \in \Pi} a_\alpha X_\alpha \right)
\]

\[
= \sum_{\alpha \in \Pi} a_\alpha (x) \alpha (H) X_\alpha.
\]

In particular, let \( \alpha \) be a \textit{positive} root. Then, \( g_\alpha + g_{-\alpha} \) (regarded as a real vector space) is invariant by \( dZ_x \). Furthermore, with respect to the basis \( \{ X_\alpha, X_{-\alpha}, iX_\alpha, iX_{-\alpha} \} \), the restriction of \( dZ_x \) to this subspace is given by the matrix

\[
\alpha (x) \alpha (H) \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

which has eigenvalues \( \pm \alpha (x) \alpha (H) \) with associated eigenspaces

\[
V_{-\alpha (x) \alpha (H)} = \text{span} \mathbb{R} \{ X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha}) \} = (g_\alpha + g_{-\alpha}) \cap u, \\
V_{\alpha (x) \alpha (H)} = \text{span} \mathbb{R} \{ X_\alpha + X_{-\alpha}, i(X_\alpha - X_{-\alpha}) \} = (g_\alpha + g_{-\alpha}) \cap iu.
\]

Therefore, \( T_x \mathcal{O} (H_0) = \sum_{\alpha \in \Pi} g_\alpha \) decomposes into \( T_x \mathcal{O} (H_0) = V_x^+ \oplus V_x^- \), where \( V_x^+ \) (unstable space) is the sum of eigenspaces with positive eigenvalues and \( V_x^- \) (stable space) is where \( dZ_x \) has negative eigenvalues. The dimension of \( T_x \mathcal{O} (H_0) \) over \( \mathbb{R} \) is \( 2|\Pi| \), whereas \( \text{dim}_{\mathbb{R}} V_x^\pm = |\Pi| \).

**Proposition 8.** The subspaces \( V_x^+ \) and \( V_x^- \) are Lagrangian with respect to the symplectic form \( \Omega = \mathcal{S} \mathcal{H} \).

**Proof.** \( V_{\alpha (x) \alpha (H)} \) and \( V_{-\alpha (x) \alpha (H)} \) are isotropic subspaces, since they are contained in either \( u \) or \( iu \) and both are subspaces where the Hermitian form \( \mathcal{H} \) takes real values. On the other hand, if \( \alpha \neq \beta \) are positive roots, then \( g_\alpha + g_{-\alpha} \) is orthogonal to \( g_\beta + g_{-\beta} \) with respect to the Cartan–Killing form, and since these subspaces are \( \tau \)-invariant they are also orthogonal with respect to \( \mathcal{H} \). Therefore, \( \mathcal{H} \) assumes real values on \( V_x^+ \) and on \( V_x^- \) as well, hence these subspaces are isotropic, and by dimension count they are Lagrangian. \( \square \)

The subspaces \( V_x^+ \) and \( V_x^- \) are the tangent subspaces to the unstable and stable submanifolds of \( Z \) with respect to the fixed point \( x \). These submanifolds are denoted by \( V_x^+ \) and \( V_x^- \), respectively.

We will investigate the stable manifold of \( Z \) for the case \( x = H_0 \). If \( \alpha > 0 \) then \( \alpha (H_0), \alpha (H) \) and \( \alpha (H_0) \alpha (H) \) are all positive. It follows that

\[
V_{H_0}^+ = iu \cap \sum_{\alpha > 0} (g_\alpha + g_{-\alpha}) \qquad V_{H_0}^- = u \cap \sum_{\alpha > 0} (g_\alpha + g_{-\alpha}).
\]

Observe that \( u \cap \sum_{\alpha > 0} (g_\alpha + g_{-\alpha}) \) is a \textit{real} vector space with basis

\[
\{ A_\alpha = X_\alpha - X_{-\alpha}, iS_\alpha = i(X_\alpha + X_{-\alpha}) : \alpha > 0 \}.
\]

Moreover, for \( H \in \mathfrak{h} \) and \( \alpha > 0 \) the following relations hold:

\[
\bullet [H, A_\alpha] = \alpha (H) (X_\alpha + X_{-\alpha}).
\]

\[
\bullet [H, S_\alpha] = \alpha (H) i(X_\alpha - X_{-\alpha}).
\]
• \( \langle A_\alpha, S_\alpha \rangle = 0 \), and \( \langle A_\alpha, A_\alpha \rangle = \langle A_\alpha, A_{-\alpha} \rangle = 2 \) since, \( \langle X_\alpha, X_{-\alpha} \rangle = 1 \).

• If \( \beta \neq \alpha \) then \( \langle A_\alpha, A_\beta \rangle = \langle S_\alpha, S_\beta \rangle = \langle A_\alpha, S_\beta \rangle = \langle S_\alpha, A_\beta \rangle = 0 \).

**Lemma 9.** For \( x = z + y \) with \( y \in u \) and \( z \in h_{\mathbb{R}}^\perp \), we have that \( \mathcal{H}_\tau (Z (x), y) \) is real and negative.

**Proof.** We have \( \tau x = y - z \). Thus, \( Z (x) = [y + z, [y - z, H]] = [y + z, [y, H]] \). Consequently,

\[
(Z (x), y) = ([y + z, [y, H]], y) = ([y, H], [z, y]) = - ([H, y], [z, y]).
\]

Set \( y = \sum_{\alpha > 0} (a_\alpha S_\alpha + b_\alpha A_\alpha) \) with \( a_\alpha, b_\alpha \in \mathbb{R} \). Then, by the above relations

\[
[H, y] = \sum_{\alpha > 0} \alpha (H) (a_\alpha (X_\alpha - X_{-\alpha}) + b_\alpha i (X_\alpha + X_{-\alpha})),
\]

and similarly with \( z \) in place of \( H \). Still using the above relations, we obtain

\[
\langle [H, y], [z, y] \rangle = 2 \sum_{\alpha > 0} \alpha (H) \alpha (H_0) \left( a_\alpha^2 + b_\alpha^2 \right)
\]

which is \( > 0 \) because \( \alpha (H), \alpha (H_0) > 0 \) and \( a_\alpha, b_\alpha \in \mathbb{R} \). This finishes the proof, since \( (Z (x), y) = - ([H, y], [z, y]) \).

\[
\square
\]

4. **Lagrangian vanishing cycles**

We construct Lagrangian spheres inside regular fibres, which are our candidates for vanishing cycles. The correct dimension of the desired spheres is \( n - 1 \) real, that is, half of the dimension of the regular fibre. Here \( n \) is the complex dimension of the adjoint orbit, and the real dimension of the flag \( F_\Theta \) where \( \Theta = \Theta (H_0) = \{ \alpha \in \Sigma : \alpha (H_0) = 0 \} \). The number of Lagrangian spheres to be found equals \( |W| \), that is, the number of singularities.

Here we assume that \( H_0 \in \text{cla}^+ \) and that \( H \in a^+ \), hence \( H \) is regular. Recall that the symplectic form \( \Omega \) on the orbit \( O (H_0) \) is the restriction of the imaginary part of the Hermitian form of \( g \)

\[
\mathcal{H}_\tau (X, Y) = - (X, \tau Y).
\]

On the other hand, the real part is the inner product defined by

\[
B_\tau (X, Y) = - \text{Re} \langle X, \tau Y \rangle = - \frac{1}{2} \langle X, \tau Y \rangle^R,
\]

where \( \langle \cdot, \cdot \rangle^R \) is the Cartan–Killing form of the realification of \( g \). Thus,

\[
\mathcal{H}_\tau (X, Y) = B_\tau (X, Y) + i \Omega (X, Y)
\]

and the equality \( \Omega (X, Y) = B_\tau (X, iY) \) holds since \( \mathcal{H}_\tau (X, Y) \) is Hermitian.

We can search for an isotropic submanifold by taking a subspace \( V \subset g \) where \( \mathcal{H}_\tau \) takes real values, and then check whether the intersection \( V \cap g \) is indeed a submanifold.

Two examples of subspaces where \( \mathcal{H}_\tau \) takes real values are: i) the compact real form \( u \), where \( \mathcal{H}_\tau \) is negative definite and ii) the symmetric part \( i u \), where \( \mathcal{H}_\tau \) is positive definite.

The intersection \( u \cap O (H_0) \) is empty because the eigenvalues of \( \text{ad} (X) \), for \( X \in O (H_0) \) are real whereas those of \( \text{ad} (Y) \), for \( Y \in u \) are imaginary. The latter happens because \( \text{ad} (Y) \) is anti-symmetric with respect to the
Cartan–Killing form of \( \mathfrak{u} \), see lemma 6. On the other hand, the intersection \( i\mathfrak{u} \cap \mathcal{O}(H_0) \) is the flag \( \mathbb{F}_0 \) itself, since it is the orbit of the compact group \( U = \exp i\mathfrak{u} \).

Therefore, \( \mathbb{F}_0 \) is an isotropic submanifold, in fact Lagrangian, and any submanifold of \( \mathbb{F}_0 \) is isotropic as well. Moreover, the function \( f_H(x) = \langle H, x \rangle \) takes real values on \( \mathbb{F}_0 = i\mathfrak{u} \cap \mathcal{O}(H_0) \). Since by hypothesis \( H \) is regular, it follows that the restriction \( f_H^\mathbb{F}_0 \) to \( \mathbb{F}_0 \) is a Morse function. The origin \( H_0 \) is a singularity and the hypothesis that \( H_0 \in \mathfrak{cl}^+ \) implies that \( H_0 \) is an attractor (with negative definite Hessian). Therefore, the levels \( \left(f_H^\mathbb{F}_0\right)^{-1}\left(f_H^\mathbb{F}_0(x)\right) \) of \( f_H^\mathbb{F}_0 \) around \( H_0 \) are codimension 1 spheres in \( \mathbb{F}_0 \). These levels are isotropic submanifolds. Clearly \( \left(f_H^\mathbb{F}_0\right)^{-1}\left(f_H^\mathbb{F}_0(x)\right) \subset \left(f_H\right)^{-1}\left(f_H(x)\right) \) and since \( \dim \left(f_H\right)^{-1}\left(f_H(x)\right) = \dim \mathbb{F}_0 - 2 \), it follows that for \( x \) around \( H_0 \) the spheres \( \left(f_H^\mathbb{F}_0\right)^{-1}\left(f_H^\mathbb{F}_0(x)\right) \) are Lagrangian cycles at the levels \( \left(f_H\right)^{-1}\left(f_H(x)\right) \) of the Lefschetz fibrations.

We can now carry out the analogous construction around other critical points \( wH_0, w \in \mathcal{W} \). We use the following notation

(i) Given a root \( \alpha > 0 \), let

\[
u_\alpha = (\mathfrak{g}_a \ominus \mathfrak{g}_{-a}) \cap \mathfrak{u} \quad \text{and} \quad i\nu_\alpha = (\mathfrak{g}_a \ominus \mathfrak{g}_{-a}) \cap i\mathfrak{u}.
\]

Taking a Weyl basis \( X_\beta \in \mathfrak{g}_\beta \), with \( \beta \) a root, these subspaces are generated by:

- \( \nu_\alpha = \operatorname{span}_\mathbb{R}\{A_\alpha = X_\alpha - X_{-\alpha}, iS_\alpha = i(X_\alpha + X_{-\alpha})\} \) and
- \( i\nu_\alpha = \operatorname{span}_\mathbb{R}\{iA_\alpha = i(X_\alpha - X_{-\alpha}), S_\alpha = X_\alpha + X_{-\alpha}\} \).

(ii) For \( w \in \mathcal{W} \), let \( \Pi_w = \Pi^+ \cap w^{-1}\Pi^- \) be the set of positive roots that are taken to negative roots by \( w \).

(iii) For \( w \in \mathcal{W} \) define the real vector subspace

\[
V_w = \mathfrak{h}_\mathbb{R} \oplus \sum_{\alpha \in \Pi_w} \nu_\alpha \oplus \sum_{\alpha \in \Pi^+ \setminus \Pi_w} i\nu_\alpha.
\]

When \( w = 1 \) the subspace \( V_1 = i\mathfrak{u} \), since \( \Pi_1 = \emptyset \). The subspaces \( V_w \), \( 1 \neq w \in \mathcal{W} \), will replace \( i\mathfrak{u} \) in the constructions of spheres around the critical points \( wH_0 \).

**Lemma 10.** \( \mathcal{H}_\tau \) and the Cartan–Killing form \( \langle \cdot, \cdot \rangle \) take real values in \( V_w \).

**Proof.** Both \( \mathcal{H}_\tau \) and \( \langle \cdot, \cdot \rangle \) are real in each of the components of \( V_w \) (positive definite in \( \mathfrak{h}_\mathbb{R} \) and \( \sum_{\alpha \in \Pi^+ \setminus \Pi_w} i\nu_\alpha \) and negative definite in \( \sum_{\alpha \in \Pi_w} \nu_\alpha \)). Moreover \( \mathfrak{h}_\mathbb{R}, \nu_\alpha \) and \( \nu_\beta \) are orthogonal with respect to \( \mathcal{H}_\tau \) and to \( \langle \cdot, \cdot \rangle \) if \( \alpha \neq \beta \).

Consequently, the restriction of the imaginary part \( \Omega \) of \( H_\tau \) to \( V_w \) vanishes identically. On the orbit \( \mathcal{O}(H_0) \) we define a distribution \( \Delta_w \) by

\[
\Delta_w(x) = V_w \cap T_x \mathcal{O}(H_0).
\]

By lemma 10, the subspaces \( \Delta_w \) are isotropic with respect to the symplectic form \( \Omega \) (restricted to the orbit). The goal is to prove that this distribution is integrable (at least around the singularity \( wH_0 \)). Once this
MORSE FUNCTIONS AND REAL LAGRANGIAN THIMBLES ON ADJOINT ORBITS

is accomplished, the integral submanifold passing through $wH_0$ will be a Lagrangian submanifold (for $\Omega$). Consequently, a ball around the singularity $wH_0$, inside the integral submanifold will be our candidate to a Lagrangian thimble.

Remark 11. A priori a distribution obtained by intersecting a fixed subspace with the tangent spaces of an embedded submanifold (such as our distribution $\Delta_w$) might not even be continuous (that is, admit local parametrizations by continuous fields). As an example, consider the case of the circle $S^1 = \{ x \in \mathbb{R}^2 : |x| = 1 \}$. The horizontal line $\{(t, 0) : t \in \mathbb{R}\}$ contains the tangent space at $(0, 1)$, however, it intersects in dimension zero the tangent spaces of points near $(0, 1)$. For a continuous distribution the dimension does not decrease around a point.

At the singularity $wH_0$ (or any other singularity) the distribution is

$$\Delta_w (wH_0) = \sum_{\alpha \in \Pi_w} u_{\alpha} \oplus \sum_{\alpha \in \Pi^+ \setminus \Pi_w} iv_{\alpha}.$$  

This is due to the fact that the tangent space at $wH_0$ is given by $T_{wH_0} \mathcal{O}(H_0) = \sum_{\alpha \in \Pi} \mathfrak{g}_\alpha$, which intersects $V_w$ at $u_{\alpha}$ and $iv_{\alpha}$, showing that

$$\dim_{\mathbb{R}} \Delta_w (wH_0) = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{O}(H_0).$$

Hence, $\Delta_w (wH_0)$ is a Lagrangian subspace. It follows that $\dim_{\mathbb{R}} \Delta_w (wH_0) \geq \dim \Delta_w (x)$, $x \in \mathcal{O}(H_0)$, since the subspaces $\Delta_w (x)$ are isotropic for $\Omega$.

We will parametrize $\Delta_w$ around $wH_0$ by Hamiltonian fields. Given $X \in \mathfrak{g}$ define the real height function (with respect to the inner product $B_\tau$) $f_X : \mathcal{O}(H_0) \rightarrow \mathbb{R}$ by

$$f_X (x) = B_\tau (X, x).$$

Denote by $\text{ham} f_X$ the Hamiltonian field of $f_X$ with respect to $\Omega$ and by $\text{grad} f_X$ its gradient with respect to $B_\tau$ (both $\Omega$ and $B_\tau$ are restricted to $\mathcal{O}(H_0)$). By definition, if $v \in T_x \mathcal{O}(H_0)$ then

$$(df_X)_x (v) = \Omega (v, \text{ham} f_X (x)) = B_\tau (v, \text{grad} f_X (x)).$$

The formula $\Omega (X, Y) = B_\tau (X, iY)$, guaranties that

$$\Omega (v, \text{ham} f_X (x)) = B_\tau (v, i \text{ham} f_X (x)) = B_\tau (v, \text{grad} f_X (x)).$$

Since this equality holds for all $v \in T_x \mathcal{O}(H_0)$ it follows that

$$\text{ham} f_X (x) = -i \text{grad} f_X (x),$$

for all $x \in \mathcal{O}(H_0)$.

A basis for $\Delta_w (wH_0)$ is given by $\text{ham} f_X (iH_0)$ with $X$ belonging to

$$\{ iA_\alpha, S_\alpha : \alpha \in \Pi_w \} \cup \{ A_\alpha, iS_\alpha : \alpha \in \Pi^+ \setminus \Pi_w \}$$

where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = X_\alpha + X_{-\alpha}$.

Moreover, these Hamiltonian fields are tangent to the distribution $\Delta_w$. 
5. Real Lagrangian thimbles

Let $u$ be the Lie algebra of $K$. We first we recall the following result.

**Proposition 12.** [GGSM2, Prop 6.2] The tangent space to $\Gamma (k \circ R_{u_0})$ at $(x, y) = (x, k \circ R_{u_0} (x))$ is given by

$$\{(A, \text{Ad} (k) A) \sim (x, k \circ R_{u_0} (x)) : A \in u\}$$

where $(A, \text{Ad} (k) A) \sim$ is the vector field on $F_{H_0} \times F_{H_0}^* = F_{(H_0, H_0^*)}$ induced by $(A, \text{Ad} (k) A) \in u \times u$.

For Lefschetz fibrations on an adjoint orbit $O (H_0)$ we can obtain Lagrangian submanifolds as graphs of symplectic maps on the corresponding flag. The idea of our construction is based on the following generalities. Let $f : N \to \mathbb{C}$ be a Lefschetz fibration where the total space $N$ is a Hermitian manifold with Hermitian metric $h$, complex structure $J$, and Kähler form $\Omega$. Set $f = f_1 + if_2$ and let $V$ be a Lagrangian submanifold which contains a critical point $x$ of $f$. Let $g = g_1 + ig_2$ be the restriction of $f$ to $V$. We define following gradient vector fields

$$F_1 = \text{grad} f_1, \quad F_2 = \text{grad} f_2, \quad G_1 = \text{grad} g_1, \quad \text{and} \quad G_2 = \text{grad} g_2.$$ 

Since $f$ is a holomorphic function, $df (Jv) = idf (v)$ for all $v \in TN$. This means

$$df_1 (Jv) + idf_2 (Jv) = idf_1 (v) - df_2 (v),$$

hence $df_2 (v) = - df_1 (Jv)$. That is, $h (F_2, v) = h (F_1, Jv) = h (JF_1, v)$ which shows that

$$F_2 = JF_1 \quad F_1 = JF_2.$$ 

From these equalities it follows that $x$ is a critical point of $f$ if and only if $x$ is a critical point of both $f_1$ and $f_2$. The Hessians of $f_1$ and $f_2$ at the critical point $x$ are related as follows. If $A$ and $B$ are vector fields, then

$$\text{Hess} f_1 (A, B) = BA f_1 = Bh (F_1, A) = Bh (F_2, JA) = B (JA) f_2 = \text{Hess} f_2 (JA, B).$$

If $f$ has isolated critical points, then both $f_1$ and $f_2$ are Morse functions. The relation between the Hessians shows that at every critical point of $f$ the number of positive eigenvalues of the Hessian equals the number of negative eigenvalues. In fact, if $\text{Hess} f_1 (A, A) > 0$ then

$$\text{Hess} f_1 (JA, JA) = \text{Hess} f_2 (-A, JA) = - \text{Hess} f_2 (JA, A) = - \text{Hess} f_1 (A, A)$$

hence the positive definite and negative definite parts have the same dimension.

To obtain the relation between $F_1$ and $G_1$ we observe that, since $V$ is a Lagrangian submanifold, the tangent space to $N$ at a point $y \in V$ decomposes into

$$T_y N = T_y V \oplus JT_y V \quad y \in V,$$

as $JT_y V$ is the orthogonal complement with respect to the Hermitian metric $M (\cdot, \cdot)$, of $T_y V$. Indeed, if $u, v \in T_y V$ then

$$M (u, Jv) = - \Omega (u, v) = 0.$$
therefore the subspaces $T_y V$ and $JT_y V$ are orthogonal and have the same dimension, thus are complementary. Consequently, the following relation between $F_1$ and $G_i$ holds on points of $V$.

**Proposition 13.** If $y \in V$ then $F_1(y) = G_1(y) - JG_2(y)$ and $F_2(y) = G_2(y) - JG_1(y)$.

**Proof.** For the case of $F_1$ take the decomposition $F_1(y) = u + Jv \in T_y V \oplus JT_y V$. Since $g_1$ is the restriction of $f_1$ to $V$, it follows that $(df_1)_y(w) = (dg_1)_y(w)$ if $w \in T_y V$. Therefore, for $w \in T_y V$

\[
(dg_1)_y(w) = (df_1)_y(w) = M(F_1(y), w) = M(u + v, w) = M(u, w)
\]

and we see that $u = G_1(y)$. Now take $Jw \in JT_y V$. So,

\[
(df_1)_y(Jw) = - (df_2)_y(w) = -(dg_2)_y(w) = -M(G_2(y), w)
\]

and $M(F_1(y), Jw) = -M(G_2(y), w)$, that is,

\[
M(v, w) = M(Jv, Jw) = -M(G_2(y), w).
\]

Since $w$ is arbitrary, it follows that $v = -G_2(y)$. Thus, for $y \in V$

\[
F_2 = -JF_1 = -J(G_1 - JG_2) = G_2 - JG_1.
\]

□

The expressions from proposition 13 show that if $G_2 = 0$, then $F_1 = G_1$ and, consequently $F_1$ is tangent to $V$. It follows that

**Corollary 14.** If the imaginary part is constant on the Lagrangian subvariety $V$, then $\text{grad} f_1$ is tangent to $V$.

Consequently, we obtain the following method of constructing stable and unstable manifolds of grad $f_1$ at a critical point $x$ (in the case of Morse functions).

**Proposition 15.** Let $V$ be a Lagrangian submanifold that contains a critical point $x$ of the function $f = f_1 + i f_2$ that defines a Lefschetz fibration. Suppose that $f_2$ is constant on $V$ and that the restriction of the Hessian $\text{Hess}(f)(x)$ to the tangent subspace $T_x V$ is negative definite (respectively positive definite). Then, the stable (respectively unstable) manifold of $g_1$ in $V$ is an open subset of the stable (respectively unstable) manifold of $f_1$.

**Proof.** The Hessian of $g_1$ is the restriction to $T_x V$ of the Hessian of $f_1$. The hypothesis guaranties that the fixed point $x$ is an attractor (respectively repeller) of $G_1 = \text{grad} g_1$. Consequently, in the negative definite case, the stable manifold of $G_1$ is an open subset $V$ that contains $x$. In this open set $F_1$ coincides with $G_1$, since by hypothesis $f_2$ is constant on $V$, that is, $G_2 = 0$. Therefore, the stable manifold of $G_1$ is contained in the stable manifold of $F_1$. A similar argument handles the positive definite case. □

Since the levels of a Morse function in the neighborhood of an attracting or repelling singularity are spheres (by the Morse lemma), this proposition has the following consequence.
Corollary 16. In the setup of proposition 15 consider a level \( g_1^{-1}\{c\} = f_1^{-1}(c) \cap V \) with \( c \) near \( g_1(x) = f_1(x) \) such that \( c < g_1(x) \) in the negative definite case and \( c > g(x) \) in the positive definite case. Then, \( g_1^{-1}\{c\} \) is a sphere of dimension \( \dim V - 1 \).

The sphere \( g_1^{-1}\{c\} \) in this corollary is a Lagrangian submanifold of the level \( f^{-1}\{c\} \) (since in proposition 15 we took the hypothesis that \( g_2 \) is constant).

The next goal is to construct a Lagrangian thimble having as boundary the sphere \( g_1^{-1}\{c\} \) contained in the Lagrangian submanifold \( V \). For this observe that for any \( y \in N \), the symplectic orthogonal of the fibre \( F_y = f^{-1}\{f(y)\} \) is generated by \( F_1 = \text{grad} f_1 \) and \( J \text{grad} f_1 = -\text{grad} f_2 = -F_2 \). In fact, \( F_1(y) \) is the metric orthogonal of \( T_y \Phi_y \) since it is gradient. However, \( \Phi_y \) is a complex submanifold, thus \( \Omega(F_1(y), v) = M(F_1(y), Jv) = 0 \) if \( v \in T_y \Phi_y \), which shows that \( F_1(y) \) and \( JF_1(y) \) generate the symplectic orthogonal to \( \Phi_y \).

Consequently we obtain the following Lagrangian thimble for \( f \).

Theorem 17. In the setup of proposition 15 take \( c \) near \( f_1(x) = g_1(x) \). We have that

\[
g_1^{-1}[c, g_1(x)] = f_1^{-1}[c, f_1(x)] \cap V \quad \text{in the negative definite case, or}
\]

\[
g_1^{-1}[g_1(x), c] = f_1^{-1}[f_1(x), c] \cap V \quad \text{in the positive definite case}
\]

is homeomorphic to a closed ball in \( \mathbb{R}^{\dim V} \). This ball is a Lagrangian thimble.

Proof. In the negative definite case \( g_1^{-1}[c, g_1(x)] \) is the Lagrangian thimble obtained by parallel transport of the Lagrangian sphere \( g^{-1}\{c\} \) along the line segment \([c, g(x)] \subset \mathbb{R} \). In fact, if \( s \in [c, g(x)] \) and \( z \in g^{-1}\{s\} \) then the horizontal lift of the vector \( d/dt \) is a multiple of \( F_1(z) \). This happens because the horizontal lift is a vector \( W = aF_1(z) + bJF_1(z) \), \( a, b \in \mathbb{R} \), which satisfies \( df_2(W) = (df_1)_z(W) + i(df_2)_z(W) = d/dt \), thus, \( df_2(W) \) is real and therefore coincides with \((df_1)_z(W) \). This implies \((df_2)_z(W) = 0\), so,

\[
0 = M(F_2, W) = -M(JF_1(z), aF_1(z) + bJF_1(z))
\]

consequently, \( b = 0 \). In the negative definite case we have the coefficient \( a > 0 \), since \( f_1 \) grows in the direction of \( F_1 \).

Therefore, the parallel transport of a point of \( g_1^{-1}\{c\} \) along the segment \([c, g_1(x)] \) follows the trajectories of \( F_1 \) (reparametrized). Such trajectories converge to \( x \), thus, the union of parallel transports of \( s \in [c, g(x)] \) is the ball \( g_1^{-1}[c, g_1(x)] = f_1^{-1}[c, f_1(x)] \cap V \).

The same argument works in the positive definite case, with \(-F_1 \) in place of \( F_1 \). \qed

Definition 18. A Lagrangian thimble inside a stable or unstable of submanifold constructed as in theorem 17 is called a real Lagrangian thimble, since it is obtained by lifting of a real horizontal curve.
6. The potential and graphs

The goal in this section is to analyze the behavior of the potential given by the height function \( f_H (x) = \langle x, H \rangle \) on Lagrangian graphs. The cases of interest here are the graphs of the composites \( m \circ R_w \) with \( m \) in the torus \( T = \exp (i \mathfrak{h}_\mathbb{R}) \). Such graphs all pass through the critical points of \( f_H \). In fact, in the product \( \mathbb{F}_{H_0} \times \mathbb{F}_{H_0}^* \) these critical points are given by \((wH_0, w w_0 H_0^*) = (wH_0, -wH_0)\). Since
\[
m \circ R_w (wH_0) = \text{Ad} (m) (w w_0 H_0^*) = w w_0 H_0^*
\]
we see that these pairs belong to \( \Gamma (m \circ R_w) \).

The Hessian of \( f_H \) at a critical point is calculated considering everything from the point of view of the adjoint orbit \( O (H_0) = \text{Ad} (G) H_0 \). In this case the field \( \tilde{A} \) induced by \( A \in \mathfrak{g} \) is linear \( \tilde{A} = \text{ad} (A) \). Therefore \( \tilde{A} f_H (x) = \langle [A, x], H \rangle \) and the second derivative is \( B \tilde{A} f_H (x) = \langle [A, [B, x]], H \rangle \). Thus, if \( x = wH_0 \) is a critical point, then
\[
\text{Hess} (f_H) \left( \tilde{A} (x), \tilde{B} (x) \right) = -\langle [B, wH_0], [A, H] \rangle = -\langle wH_0, B \rangle, [H, A] \rangle. \tag{3}
\]

The goal now is to find the restriction of this Hessian to the tangent spaces to the graphs \( \Gamma (m \circ R_w), m \in T \) at the critical points. These tangent spaces were described in proposition 12 using the realization of the homogenous space as an orbit inside the product \( \mathbb{F}_{H_0} \times \mathbb{F}_{H_0}^* = \mathbb{F}_{(H_0, H_0)} \).

Such description must be translated to the viewpoint where the homogenous space is the adjoint orbit \( O (H_0) = \text{Ad} (G) H_0 \). This translation will be made in the next proposition. First recall that from the point of view of the open orbit \( G \cdot (H_0, -H_0) \subset \mathbb{F}_{H_0} \times \mathbb{F}_{H_0}^* \) the critical points are \((wH_0, w w_0 H_0^*) = (wH_0, -wH_0), w \in \mathcal{W} \).

**Proposition 19.** Let \( m \in T = \exp (i \mathfrak{h}_\mathbb{R}) \) and consider \( \Gamma (m \circ R_w) \) as a Lagrangian submanifold of \( O (H_0) = \text{Ad} (G) H_0 \). Then the tangent space to \( \Gamma (m \circ R_w) \) at the critical point \( wH_0, w \in \mathcal{W} \), is generated by the vectors
\begin{enumerate}
  \item \( \tilde{X}_\alpha (wH_0) = \text{Ad} (m) \tilde{X}_\alpha (wH_0) = [X_\alpha, wH_0] - [\text{Ad} (m) X_\alpha, wH_0] \) with \( \alpha (wH_0) < 0 \) and
  \item \( i \tilde{X}_\alpha (wH_0) + \text{Ad} (m) i \tilde{X}_\alpha (wH_0) = i [X_\alpha, wH_0] + i [\text{Ad} (m) X_\alpha, wH_0] \) with \( \alpha (wH_0) < 0 \).
\end{enumerate}

**Proof.** By proposition 12 the tangent space to \( \Gamma (m \circ R_w) \) at the critical point \( (wH_0, -wH_0) \) (seen as the orbit in the product) is generated by
\[
(A, \text{Ad} (m) A)^\sim (wH_0, -wH_0) = \left( \tilde{A} (wH_0), \text{Ad} (m) \tilde{A} (-wH_0) \right)
\]
with \( A \in \mathfrak{u} \).

The real compact form \( u \) is generated by \( i \mathfrak{h}_\mathbb{R}, A_\alpha = X_\alpha - X_\alpha \) and \( Z_\alpha = i (X_\alpha + X_\alpha) \) with \( \alpha \) running through all roots. The field induced by an element of \( i \mathfrak{h}_\mathbb{R} \) vanishes at the critical point \( wH_0 \) hence it suffices to consider the fields induced by \( A_\alpha \) and \( Z_\alpha \).

Choose a root \( \alpha \) such that \( \alpha (wH_0) < 0 \). Then, in \( \mathbb{F}_{H_0}, \tilde{A}_\alpha (wH_0) = \tilde{X}_\alpha (wH_0) \) and \( \tilde{Z}_\alpha (wH_0) = i \tilde{X}_\alpha (wH_0) \) since \( \tilde{X}_\alpha (wH_0) = 0 \).

On the other hand, \( \text{Ad} (m) A_\alpha = \text{Ad} (m) X_\alpha - \text{Ad} (m) X_\alpha \) and \( \text{Ad} (m) Z_\alpha = \text{Ad} (m) i X_\alpha + \text{Ad} (m) i X_\alpha \) given that both \( \text{Ad} (m) X_\pm \alpha \) and \( \text{Ad} (m) i X_\pm \alpha \) belong to \( \mathfrak{g}_\pm \alpha \) since \( \text{Ad} (m) \mathfrak{g}_\pm \alpha = \mathfrak{g}_\pm \alpha \) (because \( m \in T \)).
Taking now the induced field on $\mathbb{F}_{H^*}$ and using the fact that $\alpha(wH_0) < 0$ we obtain that $\text{Ad}(m) \tilde{X}_\alpha(-wH_0) = 0$ on $\mathbb{F}_{H^*}$ (since $\alpha(-wH_0) > 0$). Therefore $\text{Ad}(m) \tilde{A}_\alpha(-wH_0) = -\text{Ad}(m) \tilde{X}_\alpha(-wH_0)$ and $\text{Ad}(m) \tilde{Z}_\alpha(-wH_0) = i\text{Ad}(m) \tilde{X}_\alpha(-wH_0)$.

Now, the isomorphism between $G \cdot (H_0, -H_0)$ and $\mathcal{O}(H_0)$ takes a field induced by an element of $\mathfrak{u}$ to an induced field. Moreover, the isomorphism associates $(wH_0, -wH_0) \in \mathbb{F}_{H^*} \times \mathbb{F}_{H^*}$ to $wH_0 \in \mathcal{O}(H_0)$. This way, the isomorphism takes $\tilde{A}(wH_0), \text{Ad}(m) \tilde{A}(-wH_0)$ to $\tilde{A}(wH_0) + \text{Ad}(m) \tilde{A}(wH_0)$ (where the former $\tilde{\cdot}$ means the field induced on $\mathbb{F}_{H_0}$ and $\mathbb{F}_{H^*_0}$ whereas the latter the one induced on $\mathcal{O}(H_0)$).

Hence the tangent space at the critical point $wH_0 \in \mathcal{O}(H_0)$ is generated by $\tilde{X}_\alpha(wH_0) - \text{Ad}(m) \tilde{X}_\alpha(wH_0)$ and $i\tilde{X}_\alpha(wH_0) + i\text{Ad}(m) \tilde{X}_\alpha(wH_0)$. □

The generators of the tangent space at $\Gamma(m \circ R_{w_0})$ of the previous proposition can also be described in the following simpler manner. Take $H_1 \in h_\mathbb{R}$ such that $m = e^{iH_1}$. Then, $\text{Ad}(m) X_\alpha = e^{i\alpha(H_1)} X_\alpha$. This way, the vector fields that provide the generators at $wH_0$ become

- $\tilde{X}_\alpha - \text{Ad}(m) \tilde{X}_\alpha = \tilde{X}_\alpha - e^{-i\alpha(H_1)} \tilde{X}_\alpha$ with $\alpha(wH_0) < 0$ and
- $i\tilde{X}_\alpha + \text{Ad}(m) i\tilde{X}_\alpha = i\tilde{X}_\alpha + e^{-i\alpha(H_1)} i\tilde{X}_\alpha$ with $\alpha(wH_0) < 0$.

It is now possible to calculate $\text{Hess}(f_H)$ at $wH_0$ using formula (3). The elements of $\mathfrak{g}$ which define the generating fields belong to $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. Hence the Hessian vanishes at a pair of generators coming from distinct roots, since, with respect to the Cartan-Killing form, $\mathfrak{g}_{\pm\alpha}$ is orthogonal to $\mathfrak{g}_{\pm\beta}$ if $\beta \neq \pm\alpha$. For the fields provided by a root $\alpha$ with $\alpha(wH_0) < 0$, we obtain (in $wH_0$):

- $\text{Hess}(f_H) \left( \tilde{X}_\alpha - e^{-i\alpha(H_1)} \tilde{X}_\alpha, \tilde{X}_\alpha - e^{-i\alpha(H_1)} \tilde{X}_\alpha \right) = -\langle [wH_0, X_\alpha - e^{-i\alpha(H_1)} X_\alpha], [H, X_\alpha - e^{-i\alpha(H_1)} X_\alpha] \rangle$.

The second term equals

$-\langle \alpha(wH_0) X_\alpha + e^{-i\alpha(H_1)} \alpha(wH_0) X_\alpha, \alpha(H) X_\alpha + e^{-i\alpha(H_1)} \alpha(H) X_\alpha \rangle$.

Now, $\langle X_\alpha, X_\alpha \rangle = \langle X_\alpha, X_{-\alpha} \rangle = 0$ and $\langle X_\alpha, X_{-\alpha} \rangle = 1$ (Weyl basis) therefore, the Hessian becomes

$-2\alpha(wH_0) \alpha(H) e^{-i\alpha(H_1)}$.

- $\text{Hess}(f_H) \left( i\tilde{X}_\alpha + e^{-i\alpha(H_1)} i\tilde{X}_\alpha, i\tilde{X}_\alpha + e^{-i\alpha(H_1)} i\tilde{X}_\alpha \right) = -\langle [wH_0, iX_\alpha + e^{-i\alpha(H_1)} iX_\alpha], [H, iX_\alpha + e^{-i\alpha(H_1)} iX_\alpha] \rangle$.

That is,

$\langle \alpha(wH_0) X_\alpha - e^{-i\alpha(H_1)} \alpha(wH_0) X_\alpha, \alpha(H) X_\alpha - e^{-i\alpha(H_1)} \alpha(H) X_\alpha \rangle$.

Thus the Hessian equals

$-2\alpha(wH_0) \alpha(H) e^{-i\alpha(H_1)}$.

- $\text{Hess}(f_H) \left( \tilde{X}_\alpha - e^{-i\alpha(H_1)} \tilde{X}_\alpha, i\tilde{X}_\alpha + e^{-i\alpha(H_1)} i\tilde{X}_\alpha \right) = -\langle [wH_0, X_\alpha - e^{-i\alpha(H_1)} X_\alpha], [H, iX_\alpha + e^{-i\alpha(H_1)} iX_\alpha] \rangle$. 


That is,
\[-i\langle \alpha (wH_0) X_\alpha + e^{-i\alpha(H_1)} \alpha (wH_0) X_{-\alpha}, \alpha (H) X_\alpha - e^{-i\alpha(H_1)} \alpha (H) X_{-\alpha} \rangle = 0.\]

Summing up,

**Proposition 20.** The Hessian of \(f_H\) restricted to the tangent space
\[T_{wH_0} \left( \Gamma \left( e^{H_1} \circ R_{w_0} \right) \right)\]
is diagonalizable in the basis
\[\{(X_\alpha - e^{-i\alpha(H_1)} X_{-\alpha}) (wH_0), (iX_\alpha + e^{-i\alpha(H_1)} iX_{-\alpha}) (wH_0) : \alpha (wH_0) < 0 \}\].
The diagonal elements are given by
\[-2\alpha (wH_0) \alpha (H) e^{-i\alpha(H_1)}.\]

For example, the orbit of the compact group (the zero section in the identification with the cotangent bundle) is \(\Gamma (R_{w_0})\). If \(w = 1\) and \(H_1 = 0\), then
\[-2\alpha (wH_0) \alpha (H) e^{-i\alpha(H_1)} = -2\alpha (H_0) \alpha (H)\] which is \(< 0\) since \(\alpha (H_0) < 0\) implies that \(\alpha < 0\) and consequently \(\alpha (H) < 0\). That is, the Hessian is negative definite, which was to be expected given that the critical point \(H_0\) is a maximum of \(\operatorname{Re} f_H\) on the zero section.

We now consider graphs in the compactification \(\mathbb{F}_{H_\mu} \times \mathbb{F}_{H_\mu}^*\). The isomorphism between the open orbit in \(\mathbb{F}_{H_\mu} \times \mathbb{F}_{H_\mu}^*\) (diagonal action) and the orbit \(G \cdot (v_0 \otimes \varepsilon_0)\) of \(v_0 \otimes \varepsilon_0 \in V \otimes V^*\) (representation of \(G\)) leads to a convenient description of the intersection of graphs of anti-holomorphic functions \(\mathbb{F}_{H_\mu} \to \mathbb{F}_{H_\mu}^*\) with the open orbit.

We return to the anti-holomorphic functions \(m \circ R_{w_0} : \mathbb{F}_{H_\mu} \to \mathbb{F}_{H_\mu}^*\) with \(m \in T\), the maximal torus. The submanifold determined by graph \((R_{w_0})\) in \(R_{w_0}\) on \(\mathbb{F}_{H_\mu} \times \mathbb{F}_{H_\mu}^*\) is the orbit of the compact group \(K\) through \((v_0, \varepsilon_0)\). This orbit stays inside \(G \cdot (v_0, \varepsilon_0)\) and is identified with the \(K\)-orbit of \(v_0 \otimes \varepsilon_0\) in \(V \otimes V^*\) (by equivariance). The isomorphism with the adjoint orbit \(\operatorname{Ad}(G) H_\mu\) associates this \(K\)-orbit inside \(V \otimes V^*\) with the intersection \(\mu \cap \operatorname{Ad}(G) H_\mu\) (the Hermitian matrices in the case of \(\mathfrak{sl}(n+1, \mathbb{C})\) or else the zero section of \(T^* \mathbb{F}_{H_\mu}\)). This set is formed by the elements \(v \otimes \varepsilon \in G \cdot (v_0 \otimes \varepsilon_0)\) such that \(\ker \varepsilon = v^\perp\) (with respect to the \(K\)-invariant Hermitian form \(\langle \cdot , \cdot \rangle^K\), since \(u \in K\) is an isometry of \(\langle \cdot , \cdot \rangle^K\) and \(\ker \varepsilon_0 = v_0^\perp\). The converse is true as well: if \(v \otimes \varepsilon \in G \cdot (v_0 \otimes \varepsilon_0)\) and \(\ker \varepsilon = v^\perp\) then \(v \otimes \varepsilon \in \Gamma (R_{w_0})\). In fact, if \(\ker \varepsilon = v^\perp\) and \(X \in u\) then \(\rho_\mu (X)\) is anti-Hermitian, thus \((\rho_\mu (X) v, v)^K\) is purely imaginary and since \(\ker \varepsilon = v^\perp\), then \(\varepsilon (\rho_\mu (X) v)\) is purely imaginary as well. Therefore, \(\langle M (v \otimes \varepsilon), X \rangle = \varepsilon (\rho_\mu (X) v)\) is imaginary for arbitrary \(X \in u\), which implies that \(M (v \otimes \varepsilon) \in iu\).

Summing up, we obtain the following description of \(\Gamma (R_{w_0})\) regarded as a subset of \(G \cdot (v_0 \otimes \varepsilon_0)\). Consider \(\Phi^{-1} (\Gamma (R_{w_0})) \subset G \cdot (v_0 \otimes \varepsilon_0)\), which, abusing notation, we also denote by \(\Gamma (R_{w_0})\). We have:

**Proposition 21.** \(\Gamma (R_{w_0}) = \{ v \otimes \varepsilon \in G \cdot (v_0 \otimes \varepsilon_0) : \ker \varepsilon = v^\perp \}\).

Consider now the graph of \(m \circ R_{w_0} : \mathbb{F}_{H_\mu} \to \mathbb{F}_{H_\mu}^*\) with \(m \in T\). In general \(\Gamma (m \circ R_{w_0}) \subset \mathbb{F}_{H_\mu} \times \mathbb{F}_{H_\mu}^*\) is not contained in the open orbit and, consequently, intercepts this orbit in a noncompact subset. In either case, take
the subgroup
\[ U^m = \left\{ (u, mum^{-1}) \in U \times U : u \in U \right\} \]
The graph \( \Gamma (m \circ R_{w_0}) \) is the orbit of \( K^m \) through \((v_0, \varepsilon_0)\). This happens because, if \( x = u \cdot v_0 \in F_{H_\mu} \) then \( R_{w_0} (x) = u \cdot \varepsilon_0 \) therefore
\[ (x, m \circ R_{w_0} (x)) = (x, m \cdot u \varepsilon_0) \]
This means that \( \Gamma (m \circ R_{w_0}) \) is formed by elements of the form \((x, my)\) with \((x, y) \in \Gamma (R_{w_0})\), that is,
\[ \Gamma (m \circ R_{w_0}) = m_2 \left( \Gamma (R_{w_0}) \right) \]
where \( m_2 (x, y) = (y, mx) \). Passing to the realization inside \( V \otimes V^* \) we obtain a geometric realization of \( \Phi^{-1} (\Gamma (m \circ R_{w_0})) \), also denoted by \( \Gamma (m \circ R_{w_0}) \):

**Proposition 22.** \( \Gamma (m \circ R_{w_0}) = \{ v \otimes \rho^*_\mu (m) \varepsilon \in G \cdot (v_0 \otimes \varepsilon_0) : \ker \varepsilon = v^\perp \} \).

Now we have the setup to prove that \( f_H \) is real on \( \Gamma (m \circ R_{w_0}) \). This is essential to obtain real Lagrangian thimbles. With the realization of \( G/Z_\mu \) as an orbit in \( V \otimes V^* \) the proof that \( f_H \) is real greatly simplifies. Actually, this is not only true for elements \( m \in T \), but for more general linear transformations of \( V \) (or more precisely, of \( V^* \)).

Before stating the result, observe that the function \( f_H \) is a priori defined on the orbit \( G \cdot (v_0 \otimes \varepsilon_0) \) and is given by \( f_H (v \otimes \varepsilon) = \varepsilon (\rho_\mu (H) v) \). From this expression we see that \( f_H \) extends to a linear functional of \( V \otimes V^* \), that is, it is defined on points outside the orbit \( G \cdot (v_0 \otimes \varepsilon_0) \) as well.

**Proposition 23.** Let \( D : V \to V \) be a linear transformation that is diagonalizable on a basis adapted to the root subspaces and consider the set
\[ D_2 (\Gamma (R_{w_0})) = \{ v \otimes D^* \varepsilon \in V \otimes V^* : \ker \varepsilon = v^\perp \} \]
where \( D^* \varepsilon = \varepsilon \circ D \). Suppose that \( D \) has real eigenvalues. Then, \( f_H \) assumes real values on \( D_2 (\Gamma (R_{w_0})) \).

**Proof.** If \( v \otimes D \varepsilon \in D_2 (\Gamma (R_{w_0})) \) then
\[ f_H (v \otimes D^* \varepsilon) = \varepsilon (D \rho_\mu (H) v) = \text{tr} ((v \otimes \varepsilon) D \rho_\mu (H)) \]
On a basis adapted to the root subspaces, \( D \rho_\mu (H) \) is diagonal with real eigenvalues. If this basis is orthonormal, then \( v \otimes \varepsilon \) has a Hermitian matrix, and therefore real diagonal entries. Hence, the last term of the equality above is real, and \( f_H (v \otimes D^* \varepsilon) \) is real as well. \( \square \)

**Corollary 24.** If \( m \in T \) satisfies \( m^2 = 1 \) then \( f_H \) is real on \( \Gamma (m \circ R_{w_0}) \).

**Proof.** In fact, if \( m^2 = 1 \) then the eigenvalues of \( m \) are \( \pm 1 \) and since \( m \in T \), \( \rho_\mu (m) = \pm \text{id} \) on the root spaces. \( \square \)

Further properties of Lagrangian submanifolds inside products of flags and their intersection numbers are described in [GSMV].
7. Minimal semisimple orbits

We now focus our attention on the case of minimal semisimple orbits, by considering the orbits of $\mathfrak{s}(n+1, \mathbb{C})$ of smallest dimension. The corresponding flag manifolds are $\mathbb{F}_{H_0} = \mathbb{P}^n$ and $\mathbb{F}_{H_0^*} = \text{Gr}_n(n+1) = \mathbb{P}^{n^*}$ for $H_0 = \left( \begin{array}{cc} n & 0 \\ 0 & -1 \end{array} \right)$ and $H_0^* = \left( \begin{array}{cc} 1_{n \times n} & 0 \\ 0 & -n \end{array} \right)$. These are dual to each other, so it suffices to consider the case of $\mathbb{P}^n$.

For minimal flags it is possible to describe real Lagrangian thimbles of $f_H$ for all singularities via the graphs graph($m \circ R_{w_0}$) with $m \in T$. This happens because for each of these singularities there are elements $m \in T$ such that Hess($f_H$) restricted to $\Gamma(m \circ R_{w_0})$ is either positive definite or negative definite. Together with the fact (proved below) that the imaginary part of $f_H$ is constant over the corresponding graphs, we obtain the stable and unstable manifolds of grad(Re$f_H$) and consequently also real Lagrangian thimbles.

Our general construction specializes to this situation as follows:

- The diagonal action of $\text{Sl}(n+1, \mathbb{C})$ on $\mathbb{F}_{H_0} \times \mathbb{F}_{H_0^*} = \mathbb{P}^n \times \mathbb{P}^{n^*}$ has 2 orbits. An open dense one formed by pairs of transversal vectors $(V, W) \in \mathbb{P}^n \times \mathbb{P}^{n^*}$ with $V \cap W = \{0\}$; and another orbit formed by vectors $(V, W) \in \mathbb{P}^n \times \mathbb{P}^{n^*}$ with $V \subset W$.

- The diffeomorphism between the open orbit and the adjoint orbit $O(H_0)$ associates to a pair $(V, W) \in \mathbb{P}^n \times \mathbb{P}^{n^*}$ with $V \cap W = \{0\}$ the linear transformation $T : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ with $Tv = nv$ if $v \in V$ and $Tv = -v$ if $v \in W$.

- The map $R_{w_0} : \mathbb{P}^n \to \mathbb{P}^{n^*}$ associates to a subspace of dimension 1 of $\mathbb{C}^{n+1}$ its orthogonal complement with respect to the canonical Hermitian form of $\mathbb{C}^{n+1}$.

- $W$ is the group of permutation of $n+1$ elements.

- The set of critical points of the potential in $\mathbb{P}^n$ (orbit of the Weyl group at the origin) has $n+1$ elements which are the subspaces $[e_j]$, $j = 1, \ldots, n+1$, generated by the vectors of the canonical basis of $\mathbb{C}^{n+1}$. The origin is given by $[e_1]$, so that under the identifications this origin gets identified to $H_0$; whereas $[e_j]$ gets identified to $wH_0$ for any permutation $w$ such that $w(1) = j$.

- The roots are $\alpha_{ij}$ with $i \neq j$, with corresponding eigenspaces generated by the elementary matrices $X_{\alpha_{ij}} = X_{ij}$ (with 1 in the position $ij$ and 0 elsewhere).

- The roots $\alpha$ with $\alpha(H_0) < 0$ are given by $\alpha_{j1}$ with $2 \leq j \leq n+1$ and consequently the tangent space at the origin is identified with the space of column matrices $\left( \begin{array}{cc} 0 & 0 \\ \ast & 0 \end{array} \right)$.

- The tangent spaces at the other critical points are obtained via permutation: let $w$ be the permutation such that $w(1) = j$, then $\alpha(wH_0) < 0$ if and only if $\alpha = \alpha_{ij}$ with $i \neq j$. Thus, the tangent space at $[e_j] \approx wH_0$ is formed by matrices whose nonzero entries belong to the $j$-th column and that have a zero on entry $jj$.

Assume now, once and for all that $n$ is even, hence we are working in $\mathfrak{s}(n+1)$ with matrices that have an odd number of diagonal entries.
DEFINITION 25. Given a singularity \([e_j]\) define \(m_j^\pm \in T = \exp (i\mathbb{H}_R)\) as follows:

\[ m_j^\pm = \mp (-1)^j \text{diag}(1, \ldots , 1, \pm 1_j, -1, \ldots , -1) \]

(the subindex indicates position \(j\)).

The fact that the number of diagonal entries is odd, guaranties that in all cases \(\det m_j^\pm = 1\) and, therefore, \(m_j^\pm\) indeed belongs to \(T\). (Although in the first and last cases this is also true for even \(n\).)

PROPOSITION 26. Consider the singularity \([e_j]\approx wH_0\). The restriction of \(\text{Hess}(f_H)\) to the tangent space \(T_{[e_j]} \Gamma (m \circ R_{w_0})\) is positive definite if \(m = m_j^+\) and negative definite if \(m = m_j^-\).

Proof. By proposition 20 the restriction of \(\text{Hess}(f_H)\) is diagonal in the basis given by roots \(\alpha (wH_0) < 0\). The diagonal entries associated to the 2 dimensional subspace corresponding to the root \(\alpha\) are:

\[-2\alpha (wH_0) \alpha (H) e^{-i\alpha(H)}\] and \(\alpha (wH_0) < 0\)

where \(H_1\) is such that \(m = \exp iH_1\).

As mentioned earlier, if \(wH_0 \approx [e_j]\) then the roots \(\alpha\) such that \(\alpha (wH_0) < 0\) are given by \(\alpha_k j\) with \(k \neq j\). Also, recall that from the start \(H\) was taken in the positive Weyl chamber \(h_R^+\). Therefore, for these roots we have

- \(\alpha_k j (H) > 0\) if \(k < j\), since \(\alpha_k j > 0\), and
- \(\alpha_k j (H) < 0\) if \(k > j\), since \(\alpha_k j < 0\).

Now, if \(m = \exp iH_1 = \text{diag} \{\varepsilon_1, \ldots , \varepsilon_n\}\), then \(e^{-i\alpha_k j (H_1)} = \varepsilon_k \varepsilon_j\). Hence,

- for \(m_j^+ = \exp iH_1\), we have
  \[ e^{-i\alpha_k j (H_1)} = \begin{cases} 
  1 & \text{if } k < j \\
  -1 & \text{if } k > j 
\end{cases} \]

- for \(m_j^- = \exp iH_1\), we have
  \[ e^{-i\alpha_k j (H_1)} = \begin{cases} 
  -1 & \text{if } k < j \\
  1 & \text{if } k > j 
\end{cases} \]

Combining the signs of \(e^{-i\alpha_k j (H_1)}\) and of \(\alpha_k j (H)\) we see that the Hessian \(\text{Hess}(f_H)\) is positive definite on \(T_{[e_j]} \Gamma (m_j^+ \circ R_{w_0})\) and negative definite on \(T_{[e_j]} \Gamma (m_j^- \circ R_{w_0})\). \(\square\)

The goal now is to show that the imaginary part of \(f_H\) is constant on \(\Gamma (m_j^+ \circ R_{w_0})\). These graphs intercept the zero section \(\Gamma (R_{w_0})\) where \(f_H\) is real. So, we wish to show that \(f_H\) is real on \(\Gamma (m_j^\pm \circ R_{w_0})\).

For a transversal pair \((V,W)\in \mathbb{P}^n \times \mathbb{P}^n^*\) denote by \(\Phi (V,W)\) the linear transformation in \(\mathcal{O}(H_0)\) corresponding to the pair. As mentioned earlier, \(\Phi (V,W) v = nw\) if \(v \in V\) and \(\Phi (V,W) w = -w\) then \(w \in W\). To show that \(f_H\) is real on the graph \(\Gamma (m_j^\pm \circ R_{w_0})\) we will prove that the diagonal of \(\Phi (V,W)\) has real entries.

The following calculations work for any \(m \in T\) such that \(m^2 = 1\). So, fix once and for all \(m \in T\) such that \(m^2 = 1\). Take \([u] \in \mathbb{P}^n\). Then \(R_{w_0} [u] = [u]^\perp\) and \(\Phi ([u], [u]^\perp)\) is a Hermitian matrix whose diagonal entries
are real. Since, $m \circ R_{w_0} [u] = m [u]^\perp$, we ought to show that $\Phi \left( [u], m [u]^\perp \right)$ has real diagonal entries. Here assume that $[u]$ and $m [u]^\perp$ are transversal, that is $(u, mu) \neq 0$.

Suppose $|u| = 1$ and take an orthonormal basis $\{v_1, \ldots, v_n\}$ of $[u]^\perp$ with respect to the canonical Hermitian form $(\cdot, \cdot)$ of $\mathbb{C}^{n+1}$. The basis $\{u, v_1, \ldots, v_n\}$ is orthonormal in $\mathbb{C}^{n+1}$ and since $m \in \text{SU} (n)$ the basis $\beta = \{mu, mv_1, \ldots, mv_n\}$ is orthonormal as well, whereas the basis $\gamma = \{u, mv_1, \ldots, mv_n\}$ is not orthonormal. By the definition of $\Phi$ the matrix of $\Phi \left( [u], m [u]^\perp \right)$ on the basis $\gamma$ is given by

$$\left[ \Phi \left( [u], m [u]^\perp \right) \right]_\gamma = \text{diag} \{n, -1, \ldots, -1\}.$$ 

The matrices for the change of basis between $\beta$ and $\gamma$ are

$$[I]_\gamma^\beta = \begin{pmatrix} (u, mu) & 0 & \cdots & 0 \\ (u, mv_1) & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ (u, mv_n) & 0 & \cdots & 1 \end{pmatrix}$$

with inverse

$$[I]_\beta^\gamma = \begin{pmatrix} 1/(u, mu) & 0 & \cdots & 0 \\ -(u, mv_1)/(u, mu) & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ -(u, mv_n)/(u, mu) & 0 & \cdots & 1 \end{pmatrix}.$$ 

Therefore, $\left[ \Phi \left( [u], m [u]^\perp \right) \right]_\beta = [I]_\gamma^\beta \left[ \Phi \left( [u], m [u]^\perp \right) \right]_\gamma [I]_\gamma^\beta$ is given by

$$\left[ \Phi \left( [u], m [u]^\perp \right) \right]_\beta = \begin{pmatrix} n & 0 & \cdots & 0 \\ \nu (u, mv_1) & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \nu (u, mv_n) & 0 & \cdots & -1 \end{pmatrix},$$

where we have set $\nu := (n+1)/(u, mu)$.

We now claim that the diagonal elements of $\Phi \left( [u], m [u]^\perp \right)$ are given by

$$\left( \Phi \left( [u], m [u]^\perp \right) e_j, e_j \right)$$

where $e_j$ is an element of the canonical basis. To see this, take an arbitrary $x \in \mathbb{C}^{n+1}$ and write $\left[ \Phi \left( [u], m [u]^\perp \right) \right]_\beta = A + B$ with

$$A = \begin{pmatrix} n & 0 & \cdots & 0 \\ \nu (u, mv_1) & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \nu (u, mv_n) & 0 & \cdots & 0 \end{pmatrix}.$$
and

\[ B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & -1 \end{pmatrix}. \]

In coordinates \( x = (x, mu) + (x, mv_1) + \cdots + (x, mv_n) \). So,

\[
[Ax]_\beta = \begin{pmatrix} n & 0 & \cdots & 0 \\ \nu(u, mv_1) & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \nu(u, mv_n) & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} (x, mu) \\ (x, mv_1) \\ \vdots \\ (x, mv_n) \end{pmatrix} = \begin{pmatrix} n(x, mu) \\ (x, mv_1) \\ \vdots \\ (x, mv_n) \end{pmatrix} = \begin{pmatrix} \nu(u, mv_1)(x, mu) \\ \vdots \\ \nu(u, mv_n)(x, mu) \end{pmatrix}.
\]

and

\[
[Bx]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} (x, mu) \\ (x, mv_1) \\ \vdots \\ (x, mv_n) \end{pmatrix} = \begin{pmatrix} 0 \\ -(x, mv_1) \\ \vdots \\ -(x, mv_n) \end{pmatrix}.
\]

Since \( \beta \) is an orthonormal basis, we have that \( \Phi \left( [u], m [u]^{-1} \right) x, x \) is given by the sum of

- \( (Ax, x) = n |(x, mu)|^2 + \sum_{j=1}^{n} (u, mu) \sum_{j=1}^{n} (u, mv_j) (x, mv_j) \) and
- \( (Bx, x) = -\sum_{j=1}^{n} |(x, mv_j)|^2 \).

In this sum, the only part that is not evidently real is

\[
\frac{n}{(u, mu)} (x, mu) \sum_{j=1}^{n} (u, mv_j) (x, mv_j).
\]

To analyze this part, start up observing that \( (u, mu) \in \mathbb{R} \) since \( m \) is an isometry of \( (\cdot, \cdot) \). Thus, \( (u, mu) = (mu, u) = (u, mu) \). So, we ought to verify that

\[
(x, mu) \sum_{j=1}^{n} (u, mv_j) (x, mv_j) \in \mathbb{R}
\]

(4)

whenever \( x \) is an element of the canonical basis. This works specifically for an element \( x \) of the canonical basis, because in this case \( mx = \pm x \), that is, \( x \) belongs to an eigenspace of \( m \). The sum in (4) can be rewritten as \( \sum_{j=1}^{n} (mu, v_j) (mx, v_j) \), because \( m \in SU(n) \). It can also be expressed as

\[
\sum_{j=1}^{n} (mu, mx, v_j) v_j = \left( mu, \sum_{j=1}^{n} (mx, v_j) v_j \right).
\]

The second factor is the orthogonal projection \( \operatorname{proj} (mx) \) of \( mx \) over \( [v_1, \ldots, v_n] = [u]^\perp \). Since \( (mu, \operatorname{proj} (mx)) = (\operatorname{proj} (mu), mx) \) it then follows that the sum in (4) is

\[
(mx, u) \left( \sum_{j=1}^{n} (mu, v_j) v_j, mx \right).
\]
Finally, since $mx = \pm x$ (as it happens for the elements of the canonical basis) the previous expression becomes

$$ (x, u) \left( \sum_{j=1}^{n} (mu, v_j) v_j, x \right).$$

We can now prove this expression is real.

**Lemma 27.** Expression (5) is real.

**Proof.** Let $E_{\pm}$ be the eigenspaces associated to the eigenvalues $\pm 1$ of $m$ (since $m^2 = 1$), and write $u = u^+ + u^- \in E_+ \oplus E_-$ (that is, $u^+ = 1/2 (u + mu)$ and $u^- = 1/2 (u - mu)$. Then, for each index $j$, $0 = (u, v_j) = (u^+, v_j) + (u^-, v_j)$, that is,

$$ (u^-, v_j) = - (u^+, v_j).$$

It follows that $(mu, v_j) = (u^+ - u^-, v_j) = 2 (u^+, v_j) = -2 (u^-, v_j)$. Suppose, for example, that $x \in E_+$. Then, $(x, u) = (x, u^+)$, since $E_+$ is orthogonal to $E_-$, given that $m$ is unitary. Thus (5) can be rewritten as

$$ \sum_{j=1}^{n} (x, u) (mu, v_j) (v_j, x) = 2 \sum_{j=1}^{n} (x, u^+) (u^+, v_j) (v_j, x)$$

$$ = 2 \sum_{j=1}^{n} (x, u^+) (u^+, (v_j) v_j)$$

$$ = 2 (x, u^+) \sum_{j=1}^{n} (u^+, (v_j) v_j)$$

$$ = 2 (x, u^+) \left( u^+, \sum_{j=1}^{n} (v_j) v_j \right).$$

The sum inside the Hermitian form is

$$ \sum_{j=1}^{n} (x, v_j) v_j = (x, u) u + \sum_{j=1}^{n} (x, v_j) v_j - (x, u) u = x - (x, u) u$$

since $\{u, v_1, \ldots, v_n\}$ is an orthonormal basis. So, the last term is given by

$$ 2 (x, u^+) (u^+, x) - 2 (x, u^+) (u^+, (x, u) u)$$

$$ = 2 (x, u^+) (u^+, x) - 2 (x, u^+) (u, x) (u^+, u).$$

To see that this is real observe that the first term of the right hand side is $| (x, u^+) |^2$. As for the second term, $(u, x) = (u^+, x)$ is $(u^+, u) = (u^+, u^+)$, so, the second term is $2 | (x, u^+) |^2 (u^+, u^+)$ which is also real. □

Summing up, we have obtained:

**Proposition 28.** If $m \in T$ with $m^2 = 1$ and $V \in \mathbb{P}^m$ is such that $V$ does not belong to $m \circ R_{w_0} (V) = mV^\perp$ then the matrix of $\Phi (V, m \circ R_{w_0} V)$ has real diagonal entries (in the canonical basis).

In conclusion:
Theorem 29. Let \( m_j^\pm \) be as in definition 25. Then, the stable and unstable manifolds of \( \text{grad} (Re f_H) \) at the critical point \([e_j]\) are open in the graph \( \Gamma(m_j^\pm \circ R_{w_0}) \). The real Lagrangian thimbles are closed balls contained in the graph \( \Gamma(m_j^\pm \circ R_{w_0}) \).

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