Pineda-Villavicencio, G. (2021). A new proof of Balinski's theorem on the connectivity of polytopes. Discrete Mathematics, 344(7), 112408.

Which has been published in final form at:
https://doi.org/10.1016/j.disc.2021.112408
A NEW PROOF OF BALINSKI’S THEOREM ON THE CONNECTIVITY OF POLYTOPES

GUILLERMO PINEDA-VILLAVICENCIO

Centre for Informatics and Applied Optimisation, Federation University, Australia
School of Information Technology, Deakin University, Geelong, Australia

Abstract. Balinski (1961) proved that the graph of a $d$-dimensional convex polytope is $d$-connected. We provide a new proof of this result. Our proof provides details on the nature of a separating set with exactly $d$ vertices; some of which appear to be new.

1. Introduction

A (convex) polytope is the convex hull of a finite set $X$ of points in $\mathbb{R}^d$; the convex hull of $X$ is the smallest convex set containing $X$. The dimension of a polytope in $\mathbb{R}^d$ is one less than the maximum number of affinely independent points in the polytope; a set of points $\vec{p}_1, \ldots, \vec{p}_k$ in $\mathbb{R}^d$ is affinely independent if the $k-1$ vectors $\vec{p}_1 - \vec{p}_k, \ldots, \vec{p}_{k-1} - \vec{p}_k$ are linearly independent. A polytope of dimension $d$ is referred to as a $d$-polytope.

A polytope is structured around other polytopes, its faces. A face of a polytope $P$ in $\mathbb{R}^d$ is $P$ itself, or the intersection of $P$ with a hyperplane in $\mathbb{R}^d$ that contains $P$ in one of its closed halfspaces. A face of dimension 0, 1, and $d-1$ in a $d$-polytope is a vertex, an edge, and a facet, respectively. The set of vertices and edges of a polytope or a graph are denoted by $V$ and $E$, respectively. The graph $G(P)$ of a polytope $P$ is the abstract graph with vertex set $V(P)$ and edge set $E(P)$.

A graph with at least $d+1$ vertices is $d$-connected if removing any $d-1$ vertices leaves a connected subgraph. Balinski (1961) showed that the graph of a $d$-polytope is $d$-connected. His proof considers a hyperplane in $\mathbb{R}^d$ passing through a set of $d-1$ vertices of a $d$-polytope, and so do the proofs of Grünbaum (2003, Thm. 11.3.2), Ziegler (1995, Thm. 3.14), and Brøndsted (1983, Thm. 15.6). Such proofs yield a geometric structure of separators in the graph of the polytope (Lemma 7). A set $X$ of vertices in a graph $G$ separates two vertices $x, y$ if every path in $G$ between $x$ and $y$ contains an element of $X$, and $x, y \notin X$. And $X$ separates $G$ if it separates $x, y$.
two vertices of $G$. A separating set of vertices is a separator and a separator of cardinality $r$ is an $r$-separator.

**Lemma 1.** Let $P$ be a $d$-polytope in $\mathbb{R}^d$ and let $H$ be a hyperplane in $\mathbb{R}^d$. If $X$ is a proper subset of $H \cap V(P)$, then removing $X$ does not disconnect $G(P)$. In particular, a separator of $G(P)$ with exactly $d$ vertices must form an affinely independent set in $\mathbb{R}^d$.

Other proofs with a geometric flavour were given by Brøndsted & Maxwell (1989) and Barnette (1995). Our proof has a more combinatorial nature, relying on certain polytopal complexes in a polytope. Another combinatorial proof, based on a different idea, can be found in Barnette (1973).

The boundary complex of a polytope $P$ is the set of faces of $P$ other than $P$ itself. And the link of a vertex $x$ in $P$, denoted $\text{lk}(x)$, is the set of faces of $P$ that do not contain $x$ but lie in a facet of $P$ that contains $x$ (Fig. 1(b)). We require a result from Ziegler (1995).

**Proposition 2** (Ziegler 1995, Ex. 8.6). Let $P$ be a $d$-polytope. Then the link of a vertex in $P$ is combinatorially isomorphic to the boundary complex of a $(d - 1)$-polytope. In particular, for each $d \geq 3$, the graph of the link of a vertex is isomorphic to the graph of a $(d - 1)$-polytope.

We proved Proposition 2 in Bui et al. (2018, Prop. 12) and exemplified it in Fig. 1. In this paper, we prove the following. The part about links appears to be new.

**Theorem 3.** For $d \geq 1$, the graph of $d$-polytope $P$ is $d$-connected. Besides, for each $d \geq 3$, each vertex $x$ in a $d$-separator $X$ of $G(P)$ lies in the link of every other vertex of $X$, and the set $X \setminus \{x\}$ is a separator of the link of $x$. 

![Figure 1. The link of a vertex in the four-dimensional cube, the convex hull of the $2^4$ vectors $(\pm 1, \pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^4$. (a) The four-dimensional cube with a vertex $x$ highlighted. (b) The link of the vertex $x$ in the cube. (c) The link of the vertex $x$ as the boundary complex of the rhombic dodecahedron (Proposition 2).](image-url)
As a corollary, we get a known result on $d$-separators in simplicial $d$-polytopes (Goodman et al. 2017, p. 509); see Corollary 4. A polytope is simplicial if all its facets are simplices, and a $d$-simplex is a $d$-polytope whose $d + 1$ vertices form an affinely independent set in $\mathbb{R}^d$. An empty $(d - 1)$-simplex in a $d$-polytope $P$ is a set of $d$ vertices of $P$ that does not form a face of $P$ but every proper subset does. An empty $(d - 1)$-simplex is also called a missing $(d - 1)$-simplex.

**Corollary 4.** Let $P$ be a simplicial $d$-polytope with $d \geq 2$. A $d$-separator of $G(P)$ forms an empty $(d - 1)$-simplex of $P$.

We remark that the paragraph after Balinski’s theorem in Goodman et al. (2017, p. 509) is meant to concern only simplicial $d$-polytopes, and not $d$-polytopes in general. While it is true that a $d$-separator of the graph of a $d$-polytope must form an affinely independent set in $\mathbb{R}^d$, it is not true that it must form an empty simplex. Take, for instance, the neighbours of a vertex in a $d$-dimensional cube (Fig. 1a).

We follow Diestel (2017) for the graph theoretical terminology that we have not defined.

2. **Proofs of Theorem 3 and Corollary 4**

A path between vertices $x$ and $y$ in a graph is an $x - y$ path, and two $x - y$ paths are independent if they share no inner vertex. For a path $L := x_0 \ldots x_n$ and for $0 \leq i \leq j \leq n$, we write $x_i L x_j$ to denote the subpath $x_i \ldots x_j$. We require a theorem of Whitney (1932) and one of Menger (1927).

**Theorem 5 (Whitney 1932).** Let $G$ be a graph with at least one pair of nonadjacent vertices. Then there is a minimum separator of $G$ disconnecting two nonadjacent vertices.

**Theorem 6 (Menger 1927).** Let $G$ be a graph, and let $x$ and $y$ be two nonadjacent vertices. Then the minimum number of vertices separating $x$ from $y$ in $G$ equals the maximum number of independent $x - y$ paths in $G$.

**Proof of Theorem 3.** Let $P$ be a $d$-polytope and let $G$ be its graph. Then $G$ has at least $d + 1$ vertices. If $G$ is a complete graph, there is nothing to prove, and suppose otherwise. In this case, $G$ has at least one pair of nonadjacent vertices. For $d = 2$, $G$ is $d$-connected. And so induct on $d$, assuming that $d \geq 3$ and that the theorem is true for $d - 1$. Let $X$ be a separator in $G$ of minimum cardinality, and let $y$ and $z$ be vertices separated by $X$. Then $y, z \notin X$. According to Whitney’s theorem (Theorem 5), there is a minimum separator of $G$ disconnecting two nonadjacent vertices. Hence we may assume that $y$ and $z$ are nonadjacent, and by Menger’s theorem (Theorem 6), that there are $|X|$ independent $y - z$ paths in $G$, each containing precisely one vertex from $X$. Let $L$ be one such $y - z$ path and let $x$ be the vertex in $X \cap V(L)$; say that $L = u_1 \ldots u_m$ such that $y = u_1$, $u_j = x$, and $u_m = z$. 


The graph $G_x$ of the link of $x$ in $P$ is isomorphic to the graph of a $(d-1)$-polytope (Proposition 2), and by the induction hypothesis it is $(d-1)$-connected. The neighbours of $x$ are all part of $\text{lk}(x)$, and so $u_{j-1}, u_{j+1} \in G_x$. Again, from Menger’s theorem follows the existence of at least $d-1$ independent $u_{j-1} - u_{j+1}$ paths in $G_x$. We must have that $X \setminus \{x\}$ separates $u_{j-1}$ from $u_{j+1}$ in $G_x$, since $X$ separates $y$ from $z$. Hence $|X\setminus \{x\}| \geq d-1$, which establishes that $G$ is $d$-connected.

Finally, let $d \geq 3$ and suppose $X$ is a $d$-separator of $G$. As stated above, the set $X \setminus \{x\}$, of cardinality $d-1$, separates $G_x$, implying that $X \setminus \{x\} \subseteq V(G_x)$. The aforementioned path $L$ was arbitrary among the $y-z$ paths separated by $X$, and each such path contains a unique vertex of $X$. It follows that every vertex in $X$ is in the link of every other vertex of $X$, which concludes the proof of the theorem. □

Proof of Corollary. Let $P$ be a simplicial $d$-polytope and let $G$ be its graph. Suppose that $X$ is a $d$-separator of $G$, that $x$ is a vertex of $X$, and that $G_x$ is the graph of the link of $x$ in $P$.

A simplicial 2-polytope is a polygon and a 2-separator in it satisfies the corollary. So assume that $d \geq 3$. From Theorem 3, it follows that every vertex in $X$ is in the link of every other vertex of $X$, and that $X \setminus \{x\}$ is a $(d-1)$-separator of $G_x$. Consequently, the subgraph $G[X]$ of $G$ induced by $X$ is a complete graph, as the set of neighbours of each vertex in $X$ coincides with the vertex set of the link of the vertex.

If $d = 3$, then, from $G[X]$ being a complete graph, it follows that it is an empty 2-simplex. And so an inductive argument on $d$ can start. Assume that $d \geq 4$. From the definition of a link and Proposition 2 we obtain that $\text{lk}(x)$ is combinatorially isomorphic to the boundary complex of a simplicial $(d-1)$-polytope.

By the induction hypothesis on $\text{lk}(x)$, every proper subset of $X \setminus \{x\}$ forms a face $F$ of $\text{lk}(x)$. And from the definition of $\text{lk}(x)$, that face $F$ lies in a facet of $P$ containing $x$, a $(d-1)$-simplex containing $x$. As a consequence, if $F$ is a face of dimension $k$, then the set $\text{conv}(F \cup \{x\})$ is a face of $P$ of dimension $k+1$. Since the vertex $x$ of $X$ was taken arbitrarily, the corollary ensues. □

References

Balinski, M. L. (1961). On the graph structure of convex polyhedra in $n$-space. *Pacific J. Math.*, 11, 431–434.

Barnette, D. W. (1973). Graph theorems for manifolds. *Israel J. Math.*, 16, 62–72.

Barnette, D. W. (1995). A short proof of the $d$-connectedness of $d$-polytopes. *Discrete Math.*, 137(1-3), 351–352.

Brøndsted, A. (1983). *An introduction to convex polytopes*, volume 90 of Graduate Texts in Mathematics. New York: Springer-Verlag.

Brøndsted, A. & Maxwell, G. (1989). A new proof of the $d$-connectedness of $d$-polytopes. *Can. Math. Bull.*, 32(2), 252–254.

Bui, H. T., Pineda-Villavicencio, G., & Ugon, J. (2018). The linkedness of cubical polytopes. [arXiv:1802.09230](http://arxiv.org/abs/1802.09230).
Diestel, R. (2017). *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Berlin: Springer-Verlag, 5th edition.

Goodman, J. E., O’Rourke, J., & Tóth, C. D., Eds. (2017). *Handbook of discrete and computational geometry*. Chapman & Hall/CRC, Boca Raton, FL, 3rd edition.

Grünbaum, B. (2003). *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, 2nd edition. Prepared and with a preface by V. Kaibel, V. Klee and G. M. Ziegler.

Menger, K. (1927). Zur allgemeinen kurventheorie. *Fundamenta Mathematicae*, 10(1), 96–115.

Whitney, H. (1932). Congruent Graphs and the Connectivity of Graphs. *Amer. J. Math.*, 54(1), 150–168.

Ziegler, G. M. (1995). *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.