Orbital Fulde-Ferrell pairing state in moiré Ising superconductors

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In this work, we study superconducting moiré homobilayer transition metal dichalcogenides where the Ising spin-orbit coupling (SOC) is much larger than the moiré bandwidth. We call such non-centrosymmetric superconductors, moiré Ising superconductors. Due to the large Ising SOC, the depairing effect caused by the Zeeman field is negligible and the in-plane upper critical field ($B_{c2}$) is determined by the orbital effects. This allows us to study the effect of large orbital fields. Interestingly, when the applied in-plane field is larger than the conventional orbital $B_{c2}$, a finite-momentum pairing phase would appear which we call the orbital Fulde-Ferrell (FF) state. In this state, the Cooper pairs acquire a net momentum of $2q_B$ where $2q_B = eBd$ is the momentum shift caused by the magnetic field $B$ and $d$ denotes the layer separation. This orbital field-driven FF state is different from the conventional FF state driven by Zeeman effects in Rashba superconductors. Remarkably, we predict that the FF pairing would result in a giant superconducting diode effect under electric gating when layer asymmetry is induced. An upturn of the $B_{c2}$ as the temperature is lowered, coupled with the giant superconducting diode effect, would allow the detection of the orbital FF state.

Introduction.—Since the discovery of correlated insulating states and unconventional superconductivity in twisted bilayer graphene [1, 2], moiré superlattices have become important platforms for studying correlated physics, superconductivity, and topological states [3]. Recently, these studies have been extended to a new type of moiré materials based on transition metal dichalcogenides (TMD) [4–23]. Notably, the WSe$_2$ moiré superlattice further shows a possible signature of superconductivity, in which the resistance drops to zero at a critical temperature of about 1 to 3K [6, 16]. Importantly, superconductivity appears when the Fermi energy is near the valence band top of WSe$_2$ such that the Ising spin-orbit coupling (SOC) is exceedingly large (in the order of hundreds of meV [24]). The Ising SOC, which pins electron spins at opposite momentum to opposite (out-of-plane) directions [25, 27], strongly suppresses the effect of in-plane Zeeman field and enhances in-plane upper critical field $B_{c2}$ [26–41]. Due to the large Ising SOC, the Zeeman depairing effect of the magnetic field can be ignored and the superconductivity of the moiré bilayer can only be suppressed by the orbital effects.

In this work, we study the role of Ising SOC in superconducting TMDs with moiré bands. Specifically, we show that in-plane $B_{c2}$ of the superconducting states goes beyond the Pauli limit [42, 43] but the in-plane $B_{c2}$ is limited by the orbital effect instead of the Zeeman effect. Moreover, we show that the moiré Ising superconductor can be driven to a finite-momentum pairing state at low temperatures by the orbital effects of the magnetic field. Using realistic parameters of twisted bilayer TMDs, we find that the nature of this finite-momentum tends to be a $2q_B$-Fulde-Ferrell (FF) pairing state [44], in which Cooper pairs at both layers carry a finite-momentum around $2q_B$ perpendicular to applied fields. The phase transition from the conventional pairing to the finite-momentum pairing can be detected by the temperature dependence of the upper critical field. Interestingly, we predict a giant superconducting diode effect induced by the $2q_B$-FF pairing under electric gating. The combination of $B_{c2}$ and the diode effect would provide strong evidence of the novel orbital FF state.

Model.—To study the properties of moiré Ising superconductors, we start with a continuum model of twisted homobilayer TMD with Ising SOC and external magnetic fields [5]. We focus on homobilayer TMDs with AA stacking. The lattice constant of each monolayer is denoted by $a_0$. The top layer and the bottom layer are rotated by an angle of $\theta/2$ and $-\theta/2$ respectively with respect to one of the transition metal sites (see Fig. 1(a)). The crystal point group symmetry is $D_3$, which is generated by a two-fold rotation $C_{2y}$ along the $y$-axis and a three-fold rotation $C_{3z}$ along the $z$-axis. It is important to note that inversion symmetry is broken in the moiré bilayer TMD such that the superconducting state can be different from the centrosymmetric bilayer TMD studied in Ref. [5].

The moiré superlattice, which has a moiré lattice constant of $L_M = a_0/\sin \theta$, folds the energy bands and gives rise to the moiré Brillouin. The moiré bands under a finite in-plane magnetic field are described by the Hamiltonian

$$H_{\xi}(r) = \begin{pmatrix} h_{\xi}(r) & \hat{T}(r) \\ \hat{T}(r)^\dagger & h_{\xi}(r) \end{pmatrix}. \quad (1)$$

where $\xi = \pm$ is the valley index for $\pm K$ valley. Here the Hamiltonian of each individual layer is given by

$$h_{\xi}(r) = -\frac{1}{2m^*}(\hat{p} + q_B s_x - \xi K_t)^2 - \mu + \Omega^{I}_{\xi}(r) - \xi \beta_s \hat{s}_z + u_B B \cdot \hat{s}, \quad (2)$$
where \( l = t(b) \) labels the top (bottom) layer, \( m_s \) denotes the effective mass of valence band, \( \mu \) is the chemical potential, and \( \tau_i \) and \( s_i \) are Pauli matrices defined in layer and spin space, respectively. The \( \beta_{so} \) characterizes the strength of Ising SOC. The orbital effect of an external magnetic field introduces a momentum shift \( q_B = q_B + \frac{eB}{2} \) with \( q_B = eA \) and \( A = \frac{1}{2}dB \times \hat{z} \) as the chosen gauge potential, where \( B \) denotes the in-plane external magnetic field, \( d \) denotes the interlayer distance, \( e \) is the electron charge. The Zeeman effect of the external magnetic field is captured by the last term, where the \( g \) factor is taken to be 2 and \( u_B \) denotes the Bohr magneton. \( \Omega^{(+)}_q(\mathbf{r}) \) is the intralayer moiré potential, and \( \hat{T}(\mathbf{r}) \) is the interlayer moiré potential. The detailed form of moiré potentials and the model parameters adopted from Ref. 9 are presented in Supplementary Material (SM) Sec. I.

We describe the superconducting twisted homobilayer TMD by a mean-field Hamiltonian, which is written as

\[
H_{MF}(\mathbf{r}) = H(\mathbf{r}) + \sum_\xi \Psi_\xi^\dagger(\mathbf{r}) \Delta(\mathbf{r}) \Psi_\xi(\mathbf{r}) + \text{H.c.} \tag{3}
\]

Here the moiré Hamiltonian

\[
H(\mathbf{r}) = \sum_\xi \int d\mathbf{r} \Psi_\xi^\dagger(\mathbf{r}) \mathcal{H}_\xi(\mathbf{r}) \Psi_\xi(\mathbf{r}) \tag{4}
\]

and \( \Psi_\xi(\mathbf{r}) = (\psi_{\xi \xi \uparrow}, \psi_{\xi \downarrow \downarrow}, \psi_{\xi \uparrow \downarrow}, \psi_{\xi \uparrow \uparrow})^T \) denotes a four-component electron annihilation operator. The pairing matrix \( \Delta(\mathbf{r}) \) is represented in the layer and spin space. Due to the layered structure, we expect the pairings to be within electrons of the same layer which can be classified with irreducible representations of \( D_3 \) point group (see SM Sec. I 10). The favoured pairing form is determined by the microscopic interaction. In this work, for illustrative purposes, we consider the two conventional gapped intralayer pairings: \( \Delta_{A_1} = \Delta \tau_y s_y \) and \( \Delta_{A_2} = \Delta \tau_z s_y \), where \( A_1, A_2 \) label the irreducible representations of \( D_3 \). Here, we consider both \( A_1 \) and \( A_2 \) pairings as they would generally be mixed by in-plane magnetic fields in the case of finite-momentum pairings.

The enhanced in-plane upper critical field \( B_{c2} \).— The in-plane \( B_{c2} \) of the moiré Ising superconductor can be obtained from the linearized gap equation

\[
U_0 \chi_s(\mathbf{q}, B, T) = 1. \tag{5}
\]

Here, \( U_0 \) denotes the interaction strength that stabilizes \( A_1(2)-\)pairing, \( q \) is to take account of the possible finite pairing momentum, \( T \) is the temperature and the superconducting susceptibility \( \chi_s(\mathbf{q}, B, T) \), in general, is given by the maximal eigenvalue of the pairing susceptibility matrix

\[
\tilde{\chi}(\mathbf{q}, B, T) = \left( \begin{array}{ccc}
\chi_{11}(\mathbf{q}, B, T) & \chi_{12}(\mathbf{q}, B, T) \\
\chi_{21}(\mathbf{q}, B, T) & \chi_{22}(\mathbf{q}, B, T)
\end{array} \right) . \tag{6}
\]

The susceptibility matrix is expressed in the \( \tilde{\Delta}(\mathbf{q}) = (\Delta_{A_1}(\mathbf{q}), \Delta_{A_2}(\mathbf{q}))^T \) space. More details of the calculations for the pairing susceptibility and the \( B_{c2} \) from the Hamiltonian \( H_0(p, B) \) can be found in SM Sec. VI. To be specific, we would fix the filling at \( \nu \approx -0.6 \) in our calculations and set the field direction along \( x \)-direction. In general, a three-fold anisotropy would be expected for the upper critical field. In the main text, we set the twist angle \( \theta = 5^\circ \), near where the possible signature of superconductivity would appear in the experiment [4].

The calculated in-plane \( B_{c2} \) of the zero-momentum pairing with \( q = 0 \) is shown in Fig. 2. Figure 2(a) displays the corresponding moiré energy bands at \( K \) valley, where the spin of the top moiré band that contributes to the superconductivity (in red) is fully polarized by the Ising SOC. In this case, the in-plane critical magnetic field \( B_{c2} \) (in the unit of the Pauli limit \( B_p \)) versus critical temperature \( T \) (in units of zero-field critical temperature \( T_c \)) curves are plotted in Fig. 2(b), where the orbital effects are present or absent according to Eq. 2. When the orbital effects are artificially turned off while the Zeeman effects are included, it can be seen that the superconducting critical temperature is almost insensitive to the external fields due to the strong Ising SOC. In contrast, the in-plane \( B_{c2} \) would ultimately be limited to several \( B_p \), when orbital effects are included (red line).

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**FIG. 1:** (a) The lattice structure of a twisted homobilayer TMD. The moiré unit cell is highlighted with \( L_M \) as the moiré lattice constant. (b) A schematic plot of the top moiré band of spin-up state and spin-down state at two valleys. Here, \( 2\beta_{so} \) labels the spin-splitting induced by the Ising SOC.

**FIG. 2:** (a) The moiré bands of a homobilayer TMD with twist angle \( \theta = 5^\circ \), where the top moiré bands are highlighted in red. (b) The in-plane upper critical field \( B_{c2} \) (in units of Pauli limit \( B_p \approx 1.86T_c \)) as a function of temperature (in units of the zero-field critical temperature) with (in red) and without (in blue) orbital effects of the magnetic field. We set \( T_c = 1 \) K and fix the chemical potential at \( \nu \approx -0.6 \) (the black dashed line in (a)) in (b).
This stands in sharp contrast to superconducting MoS\(_2\) and NbSe\(_2\) where the depairing due to the paramagnetic effect is dominant because of the much smaller Ising SOC at the Fermi energy in these materials.

To estimate the magnitude of the resulting orbital effect limited \(B_{c2}\), we can construct a phenomenological GL free energy theory by taking the order parameter of top and bottom layer to be \(\Delta c \equiv |\Delta c| e^{i\varphi c}\) and \(\Delta b \equiv |\Delta b| e^{i\varphi b}\), respectively. The Ginzburg- Landau (GL) free energy that captures our system can be written as (see SM Sec.V for more details):

\[
\mathcal{F}(|\Delta|) = -(\alpha_0 - \alpha_1(B))|\Delta|^2 + \frac{\beta_0}{2}|\Delta|^4 + \lambda_J(1 - \cos(\varphi_c - \varphi_b))|\Delta|^2. \tag{7}
\]

Here, \(\alpha_0 \propto (T_c - T)\) and \(\beta_0\) are the GL coefficients, \(\alpha_1(B) = Aq_B^2\). \(A\) depends on the electron effective mass and interlayer coupling. \(\lambda_J\) denotes the Josephson coupling strength between two layers. As expected, the critical field \(B_{c2}\) for zero-momentum pairing is determined by the \(A_1\) pairing, where \(\varphi_c = \varphi_b\) to minimize the Josephson coupling energy. According to the coefficient of \(|\Delta|^2\), the upper critical field is now estimated as

\[
B_c = \sqrt{\frac{4\alpha_0}{\varepsilon^2D^2\lambda}}. \tag{8}
\]

Therefore, the orbital effect limited \(B_{c2} \propto \sqrt{T_c - T}\) is mainly determined by the effective mass and thickness. Note that the effective mass strongly depends on the twist angle. As shown in SM Sec. III, the in-plane \(B_{c2}\) can be enhanced prominently when we artificially decrease the twist angle.

**Orbital Fulde-Ferrell pairing state.**— Next, we study the case of finite-momentum pairings with \(q \neq 0\) induced by the orbital effects of magnetic fields. The stabilized finite-momentum pairing is expected to be \(q = (0, q_x)\), as the orbital motion of electrons is perpendicular to the in-plane magnetic fields. To find the robust \(q\) driven by the in-plane magnetic fields, we display the critical field \(B_c\) as a function of \(q\) in Fig. 2(a) at various temperatures with \(T = (0.9, 0.5, 0.1)T_c\). Here, we have used the magnitude of the momentum shift \(q_B = |q_B|\) as defined in Eq. 2 as a natural unit for the pairing momentum \(q\). The robust finite-momentum pairing can be determined by the one with \(q\) that maximizes the critical field \(B_{c2}\). Notably, although the zero-momentum pairing \(q = 0\) is favored near the critical temperature, a prominent \(q \approx \pm 2q_B\) pairing becomes favorable at low temperatures. Figure 2(b) displays the superconducting pairing \(\chi_s\) versus \(B\) curve at \(q = 0\) and \(|q| = 2q_B\). It clearly shows that the finite-momentum pairing state with \(|q| = 2q_B\) exhibits a higher \(B_{c2}\) than the zero-momentum pairing state. This \(2q_B\) finite-momentum pairing can be understood from the momentum shift induced by orbital effects. The momentum of electrons at two opposite valleys, which would pair together, obtains the same \(q_B\) momentum shift according to Eq. 2.

To understand the nature of this 2\(q_B\)-finite-momentum pairing, we can check the finite-momentum pairing susceptibility \(\chi_{ij}(q = 2q_B)\) versus \(B\). The stabilized pairing form could be obtained from the pairing susceptibility matrix Eq. 6, which can be written as

\[
\Delta(r) = \sum_q \Delta_q \left(\cos\frac{\theta_q}{2} + \sin\frac{\theta_q}{2} \tau_z\right) i\sigma_y e^{iqr}. \tag{9}
\]

Here, \((\cos\frac{\theta_q}{2}, \sin\frac{\theta_q}{2})^T\) represents the corresponding eigenvector of \(\chi_s\) with \(\theta_q = \arcsin\frac{x_{12}}{\sqrt{(x_{11} - x_{22})^2 + 4x_{12}^2}}\). Due to the presence of finite interlayer coupling, the resulting finite-momentum pairing susceptibility \(x_{11} - x_{22} \gg x_{12}\) so that \(\theta_q \approx 0\) (SM Sec. III). As a result, according to Eq. 9, the stabilized pairing form behaves as a FF pairing, which can be parameterized as \(\Delta(r) = |\Delta| e^{iq\cdot r}\) or \(\Delta(r) = |\Delta| e^{-iq\cdot r}\) with \(q = (0, 2q_B)\) (see an illustration in the inset of Fig. 3). We denote these two pairings as \(\pm 2q_B\)-FF pairings. Note that although these two pairings with opposite pairing momentum are nearly degenerate, the mixing of them is not favorable according to the GL free energy analysis up to the fourth order (see SM Sec. V). Moreover, according to a phenomenological GL analysis in Sec. V, the interlayer coupling would increase the kinetic energy of
we obtained a layer-antisymmetric FF pairing analyti-
cal, which would saturate (Δc = 0) for (a) and (b), re-
spectively. In (a), the δF(q) function at (Δc = 0) is the 
normal state free energy with respect to Δc. Here, we 
consider the current direction to be along y-direction so 
that we can denote q = (0, q).

The landscape of the minimized free energy f(q) (blue 
line) and the corresponding supercurrent j(q) (red line) 
in the case without displacement fields (D = 0) and 
with displacement fields (D = 5 meV) are plotted in 
Fig. 4. Here a large in-plane magnetic field (B/Bp = 3 
and B/Bp = 2.5 for (a) and (b), respectively), and a 
temperature T = 0.1Tc are adopted so that the system 
is deep in the FF pairing state. It is important to note that 
Ising SOC is very essential here. Without Ising SOC, 
the superconductivity could have been killed by the para-
magnetic effect before reaching the FF state. Without 
dispacement fields (Fig. 4(a)), the free energy of q near 
±2qB is lower than q = 0 under a large B. In other 
words, ±2qB-FF pairing would be stabilized, being 
consistent with the previous linearized gap equation calcu-
lation. However, the diode effect is absent (Δjc = 0) in 
this case (Fig. 4(a)). As shown in Fig. 4(b), the diode 
effect becomes finite at finite displacement fields (D = 5 
meV). Notably, the resulting Δjc ≈ 53% is much larger 
than the one proposed in superconductors with Rashba 
SOC. This giant superconducting diode effect origi-
nates from the lifting of the degeneracy between 2qB- 
FF pairing and −2qB-FF pairing by the displacement 
field, which enables a highly asymmetric free energy 
configuration as shown in Fig. 4(b). The implementation 
of an electric gate-tunable superconducting diode effect 
is generally difficult in previous systems 47, 53, 54, as 
the high electron density hinders the gate-controllability. 
The giant gate-tunable superconducting diode effect in 
the present system is potentially useful for dissipationless 
electronics, superconducting circuits and superconduct-

ing computing devices.

Discussion.— It is worth noting that the pairing form 
Δ(τ) can be changed if the interlayer coupling strength 
can be tuned. For example, as shown in SM Sec. VI, 
we obtained a layer-antisymmetric FF pairing analyti-
cally, where Δc = |Δ|e^{-iqc} and Δb = |Δ|e^{-iqb} with 
q = (0, 2qB), in the case without twisting and in the 
weak interlayer coupling limit. We note that this ex-
otic pairing has been proposed in centrosymmetric AB

The superconductor under in-plane magnetic fields due 
to the canonical momentum mixing between the two 
layers. On the other hand, the 2qB-FF pairing would lower 
this energy, which could make it more favorable than the 
zero-momentum pairing.

To obtain the B−T phase diagram, we plot the critical 
Bc−T (left axis, solid blue) and the corresponding stabilized 
q = (0, q) (right axis, red) as a function of temperature 
T in Fig. 3(c). The finite-momentum pairing (q > 0) is 
seen to emerge at temperature T ≈ 0.75Tc, near where the 
Bc−T curve exhibits an upturn at the phase transi-
tion. Notably, the momentum shifts q would saturate 
and the previously discussed 2qB-FF pairings emerge at low 
temperatures T ≲ 0.5Tc. The finite-momentum pair-
ing phase region, the boundary of which is roughly given 
by the Bc−T curve with q = 0 and finite q, is highlighted in 
Fig. 3(c). It can be seen that the 2qB-FF state can be 
stabilized with a temperature T ≲ 0.5Tc and a magnetic 
field B roughly higher than 2Bp.

Finally, we point out that the degeneracy between 
+2qB-FF pairing and −2qB-FF pairing can be lifted ex-
trinsically by out-of-plane displacement fields D, which 
induces layer asymmetry. As shown in Fig. 3(d), when an 
out-of-plane displacement field D = 5 meV is ap-
plicated, the Bc−T of 2qB finite-momentum pairing becomes 
much higher than the Bc−T of the −2qB pairing, imply-
ng that +2qB-FF pairing would be the favorable finite-
momentum pairing under a large in-plane magnetic field. 
Note that in the experiment, superconductivity of twisted 
bilayer TMDs occurs in the presence of a displacement 
field.

Gate-tunable superconducting diode effect.— Next, we 
demonstrate a gate-tunable superconducting diode ef-
fect based on the proposed 2qB-FF pairing in moiré 
Ising superconductors. The superconducting diode ef-
fect is characterized by the critical current difference be-
 tween currents flowing in opposite directions: \( J_c = \frac{\langle J_{c+} - J_{c-} \rangle}{\langle J_{c+} + J_{c-} \rangle} \), where the + and − 
signs denote the opposite current directions respectively.

To demonstrate this, we can calculate the supercurrent 
\( j(q) \) from the free energy \( f_s(\Delta, q) = \frac{\Delta^2}{U_0} - \frac{1}{\beta} \sum_{k,n} \ln(1 + e^{-\beta \epsilon_{nq}(k)}) \) (10)
where \( \beta = 1/k_BT \), \( \epsilon_{nq}(k) \) is the quasi-particle energy of the finite-momentum Bogoliubov-de Gennes (BdG) 
Hamiltonian (see SM Sec. IV [46] for more details). The supercurrent \( j(q) \) can be obtained by \( j(q) = \frac{2\delta f(q)}{\delta q} \), 
where \( f(q) \) is the lowest free energy at each pairing mo-
mentum \( q \) and is given by minimizing the free energy 
\( f_s(\Delta, q) - f_n \) (note \( f_n \equiv f_s(\Delta = 0) \) is the normal state free energy) with respect to \( \Delta \). Here, we consider 
the current direction to be along y-direction so that we can 
denote \( q = (0, q) \).

FIG. 4: (a) and (b) show the free energy \( f(q) \) and super-
current \( j(q) \) normalized to \([-1, 1]\) without displacement fields 
\( (D = 0 \text{ meV}) \) and with a finite displacement field 
\( (D = 5 \text{ meV}) \) respectively. Here the temperature \( T = 0.1T_c \), and 
\( B = 3B_p \) and \( B = 2.5B_p \) for (a) and (b), respectively. In (a), 
the maximum magnitudes of \( j(q) \) are the same in the positive 
and negative directions. This indicates the absence of the 
superconducting diode effect. In (b), the maximum magnitudes 
of \( j(q) \) are different for currents flowing in opposite directions, 
indicating the presence of the superconducting diode effect.
stacked bilayer TMDs without twisting previously. This layer-antisymmetric FF pairing is energetically not favored in our case due to the stronger interlayer coupling strength, which increases the Josephson coupling energy. The orbital FF pairings we find would not afford such Josephson coupling energy and are particularly allowed by noncentrosymmetric superconductors.

In conclusion, we have proposed an intriguing non-centrosymmetric superconductor—moiré Ising superconductor, in which the Ising SOC is dominant over moiré bandwidth and can be readily realized in superconducting moiré TMDs. We have highlighted that moiré Ising superconductors are wonderful platforms for exploring novel superconducting effects, including orbital magnetic field-driven finite-momentum pairing state and gate-tunable superconducting diode effects. In principle, our theory for the orbital FF pairing state can also be applied to some other non-twisted superconducting materials with inversion broken and giant Ising SOC.

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Supplementary Material for
“Orbital Fulde–Ferrell pairing state in Moiré Ising superconductors”

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I. DETAILS FOR THE MOIRÉ POTENTIAL AND MODEL PARAMETERS

In this supplementary material section, we present the detailed form of the moiré potential and the adopted model parameters for the twisted homobilayer TMD in the main text. The intralayer moiré potential is given by

$$\Omega^{(l)}(r) = V_{\xi s} \sum_{j=1,3,5} e^{i\xi (g_j \cdot r + \psi_{\xi s})} + h.c.,$$

where $V_{\xi s}$ and $\psi_{\xi s}$ ($\xi, s$ are valley and spin indices), respectively, characterize the amplitude and the shape of intralayer moiré potential, and the moiré reciprocal lattice vectors $g_i = \frac{4\pi}{\sqrt{3}L_{xy}} (\cos \frac{(i-1)\pi}{3}, \sin \frac{(i-1)\pi}{3})$. The interlayer tunneling moiré potential $\hat{T}(r)$ in main text is given by

$$\hat{T}(r) = \left( \begin{array}{cc} u\xi_{\uparrow} & u\xi_{\downarrow} \\ u\xi_{\downarrow} & u\xi_{\uparrow} \end{array} \right) + \left( \begin{array}{cc} u\xi_{\uparrow} \uparrow \omega_{\xi} & u\xi_{\downarrow} \uparrow \omega_{\xi} \\ u\xi_{\downarrow} \uparrow \omega_{\xi} & u\xi_{\uparrow} \downarrow \omega_{\xi} \end{array} \right) e^{-i\xi g \cdot r}$$

$$+ \left( \begin{array}{cc} u\xi_{\uparrow} \downarrow \omega_{\xi} & u\xi_{\downarrow} \uparrow \omega_{\xi} \\ u\xi_{\downarrow} \uparrow \omega_{\xi} & u\xi_{\uparrow} \downarrow \omega_{\xi} \end{array} \right) e^{-i\xi g \cdot r},$$

with $\omega = e^{i2\pi/3}$.

To roughly capture the relevant energy scale of twisted bilayer TMD, in the calculation, we adopt the model parameters given in ref. [5] in the main text: $1/2m^* = 510.45$ meV, $\beta_{so} = 110.25$ meV. $(V_{\uparrow \uparrow}, V_{\uparrow \downarrow}, u_{\uparrow \uparrow}) = (8 \text{ meV}, -89.8^\circ, -8.5 \text{ meV}), (V_{\downarrow \downarrow}, V_{\downarrow \uparrow}, u_{\downarrow \uparrow}) = (7.7 \text{ meV}, -88.35^\circ, -6.5 \text{ meV})$ and $u_{\uparrow \downarrow} = -i5.6 \text{ meV}$, which are obtained by fitting the first-principle band structure of homobilayer MoTe2. With the reversal symmetry operation, we have $V_{\uparrow \downarrow} = V_{\downarrow \uparrow}, \xi_{\uparrow \uparrow} = \psi_{\uparrow \downarrow}, u_{\uparrow \uparrow} = u_{\downarrow \downarrow}$.

II. PAIRING CLASSIFICATIONS FOR TWISTED BILAYER TMD

In the absence of displacement fields and external fields, the twisted bilayer TMD respects $\hat{T} \times D_3$ symmetry. Here, $\hat{T} = is_y K$ with $K$ as complex conjugate denotes time-reversal symmetry, and $D_3$ point group symmetry is generated by a three-fold rotational symmetry $C_{3z} = e^{-i\frac{2\pi}{3}}$ and an in-plane two-fold rotational symmetry $C_{2y} = -i\tau_x s_y$.

The continuum Hamiltonian for the superconducting part can be written as

$$H_{SC}(r) = \int dr \sum_{\xi \xi', s, s'} \psi_{\xi s}^\dagger(r) \Delta_{\xi s', s}^{(r)}(r) \psi_{\xi' s'}(r) + H.c.$$}

Here, $l = t, b$ is the layer indices. We can classify all possible pairings with the irreducible representations of the $D_3$ point group. This classification is done by noting (i) the pairing matrix transforms as

$$\hat{T} : \Delta(r) \rightarrow s_y \Delta^*(r) s_y$$

$$g : \Delta(r) \rightarrow U^T(g) \Delta(g r) U(g),$$

where $g$ denotes the symmetry operation $C_{3z}$ and $C_{2y}$, $U(g)$ is the matrix representation of the symmetry operation $g$. (ii) Due to the antisymmetric requirement of Cooper pair wave functions, the pairing matrices must satisfy $\Delta^T(r) = -\Delta(r)$.

For simplicity, we only take momentum independent pairings into account. There are only six matrices: $s_y$, $\tau_x s_y$, $\tau_y s_y$, $\tau_y s_x$, $\tau_y s_z$ can couple to the momentum independent pairings. The possible momentum-independent pairings in the layer and spin space can be classified in Table S1. The $\Delta_{A_{1,1}}, \Delta_{A_2}$ are intralayer singlet pairings and the $\Delta_{A_{1,2}}, \Delta_{A_{1,3}}$ are interlayer singlet pairings, while the $\Delta_{E_1}, \Delta_{E_2}$ belonging to a two-dimensional irreducible representation is interlayer triplet pairings.
Supplementary Table 1: Classification of possible s-wave pairings for twisted bilayer TMD with the irreducible representations of $D_3$ point group in layer and spin space. Here $s, \tau$ are Pauli matrices defining in spin and layer space.

| Rep. | Matrix form $\gamma$ | Explicit form |
|------|----------------------|---------------|
| $A_1$ | $\Delta A_{1,1}$    | $\psi_{-\zeta-k,\uparrow}\psi_{\xi+k,\downarrow} + \psi_{-\zeta-k,\downarrow}\psi_{\xi+k,\uparrow}$ |
|      | $\Delta A_{1,2}$    | $\psi_{-\zeta-k,\uparrow}\psi_{\xi+k,\downarrow} + \psi_{-\zeta-k,\downarrow}\psi_{\xi+k,\uparrow}$ |
|      | $\Delta A_{1,3}$    | $\tau y s_x \psi_{-\zeta-k,\uparrow}\psi_{\xi+k,\downarrow} - \psi_{-\zeta-k,\downarrow}\psi_{\xi+k,\uparrow}$ |
| $A_2$ | $\Delta A_2$        | $i\tau z s_y \psi_{-\zeta-k,\uparrow}\psi_{\xi+k,\downarrow} - \psi_{-\zeta-k,\downarrow}\psi_{\xi+k,\uparrow}$ |
| $E \{\Delta E_1, \Delta E_2\}$ | $\{\tau y, \tau y s_z\}$ | ... |

III. EXTENDED FIGURES

FIG. S1: Pairing susceptibility for $A_1$ pairing $\chi_{11}$, $A_2$ pairing, $\chi_{22}$ and their mixing $\chi_{12}$. (a) The zero-momentum pairing susceptibility $\chi_{ij}(q = 0)$ versus in-plane magnetic fields $B$. The critical field $B_{c2}$ is highlighted. (b) The finite-momentum pairing susceptibility $\chi_{ij}(q = 2q_B)$ versus $B$. Other parameters are the same as the main text Fig. 2.

FIG. S2: The in-plane upper critical field $B_{c2}$ versus $T$ (with orbital effects) at various twist angle $\theta$. 
IV. THE LINEARIZED GAP EQUATION AND FREE ENERGY FOR THE FINITE MOMENTUM PAIRING

As presented in the main text, the mean-field Hamiltonian is written as

$$H_{MF}(r) = \sum_\xi \int dr \Psi_\xi^\dagger(r) \mathbf{H}_\xi(r) \Psi_\xi(r) + \sum_\xi (\Psi_\xi^\dagger(r) \mathbf{D}(r) \Psi_\xi(r) + H.c.).$$  \hspace{1cm} (S6)

Here, the four-component annihilation operator $\Psi_\xi(r) = (\psi_{\xi\uparrow}, \psi_{\xi\downarrow}, \psi_{\xi\uparrow}, \psi_{\xi\downarrow})^T$. By directly transforming the continuum superconducting Hamiltonian into momentum space, we obtain

$$H_{MF} = \frac{1}{A} \sum_p \Psi_\xi^\dagger(p) H_0(p) \Psi_\xi(p) + \frac{1}{A} \sum_{p', q} (\Psi_\xi^\dagger(p + \frac{q}{2}) \mathbf{D}(q) \Psi_\xi(q - p + \frac{q}{2}) + H.c.),$$  \hspace{1cm} (S7)

where $A$ is the area of the moiré unit cell, $p$ is the momentum within the first moiré Brillouin zone, $H_0(p)$ is the moiré Hamiltonian that can be represented a plane wave basis, where the elements can be given by $\langle p + mg_2 + ng_3 | H_\xi(r) | p + m'g_2 + n'g_3 \rangle$ with $m, n$ as integers, $g_j$ as moiré wave vectors defined in the main text. The moiré bands are obtained by diagonalizing the moiré Hamiltonian with a finite cut-off on $m, n$.

The linearized gap equation. We can decompose the pairings into the different channels $\Delta(q) = \sum_{ij} \Delta_{ij} \gamma_{ij}$ with $\gamma_{ij}$ denoting the representation matrix defined in layer and spin space, and the linearize gap equation is given by

$$\Delta_{ij}(q) = V_i \sum_{j\nu} \chi_{ij,\mu\nu}(q) \Delta_{j\nu}(q),$$  \hspace{1cm} (S8)

where $i, j$ label the representation, $\mu, \nu$ label the component in this representation, $V_i$ denotes the strength of attractive interaction, $q$ denotes the finite-momentum pairing of Cooper pairs. The superconductivity susceptibility is given by

$$\chi_{ij,\mu\nu}^{(2)}(q) = -\frac{1}{\beta} \sum_{\omega_n} Tr (\gamma_{ij} G_e(p + q/2, i\omega_n) \gamma_{j\nu} G_h(p - q/2, i\omega_n)), \hspace{1cm} (S9)$$

where $\beta = 1/k_B T$, the single-particle Green’s functions for electrons $G_e(p, i\omega_n) = (i\omega_n - H_0(p))^{-1}$ and holes $G_h(p, i\omega_n) = (i\omega_n + H_0^*(p))^{-1}$ with $H_0(p)$ as the moiré Hamiltonian.

By utilizing the eigenstates of $H_{0}(p)$: $H_0(p) | u_{ap} \rangle = E_a(p) | u_{ap} \rangle$, $H_0(p) | \nu_{bp} \rangle = E_b(p) | \nu_{bp} \rangle$ ($a$ and $b$ are band indices), we can further simplify Eq. (S9) as

$$\chi_{ij,\mu\nu}^{(2)}(q) = \int_p \sum_{a, b} O_{a, b}(p, q) O_{a, b}^\dagger(p, q) \xi_{ab}(p, q, B)$$  \hspace{1cm} (S10)
Here, we present the detailed process. To be convenient, we perform a gauge transform for the mean-field Hamiltonian to describe the FF pairing:

\[ H_{\text{BdG}}(p, q) = \left( \frac{H_0(p + \frac{q}{2})}{(\Delta_\sigma_y)_{\uparrow}} - H_0^T(-p + \frac{q}{2}) \right) \]

The free energy at every finite momentum \( q \) can then be calculated with

\[ F(q) = \frac{|\Delta|^2}{U_0} - \frac{1}{\beta} \sum_{p,n} \ln(1 + e^{-\beta \epsilon_{p,n}(q)}). \]

Here, \( \epsilon_{p,n}(q) \) are the eigenenergies of \( H_{\text{BdG}}(p, q) \), the attractive interaction strength \( U_0 \) can be fixed by the critical temperature \( T_c \).

V. GINZBURG-LANDAU FREE ENERGY FOR A BILAYER SUPERCONDUCTOR UNDER IN-PLANE ORBITAL MAGNETIC FIELDS

Phenomenologically, the Ginzburg-Landau (GL) free energy for a bilayer system under an in-plane orbital magnetic field can be written as

\[ F = F_c + F_k + F_J \]

\[ F_c = \frac{1}{2} A \left[ \int dr \sum_i (-\alpha_0)|\Delta_i(r)|^2 + \frac{\beta}{2} |\Delta_i(r)|^4 \right] \]

\[ F_k = \frac{1}{2} A \left[ \int dr \left\{ \frac{1}{2m} \sum_i |\Pi_i \Delta_i(r)|^2 - \Gamma (|\Pi_i \Delta_i|^2 - \Pi_i \Delta_i)^* (\Pi_i \Delta_i) + (\Pi_i \Delta_i)^* (\Pi_i \Delta_i) \right\} \right] \]

\[ F_J = \frac{\lambda_J}{2} A \left[ \int dr |\Delta_i(r) - \Delta_b(r)|^2 \right] \]

where \( \Delta_i(r) \) is the order parameter layer \( i \), the canonical momentum \( \Pi_i = (-i \nabla + 2eA_i) \) with \( A_i = \frac{l_\parallel}{2} B \times \hat{z} \), \( m = 2m^* \) is the mass of Cooper pairings, \( \lambda_J \propto N(0) t_c^2 \) \( (N(0) \) are the density of states near Fermi energy, \( t_c \) represents the coupling strength) is the Josephson coupling energy between the two layers, \( A \) is sample area. Here, \( F_c \) is the free energy saved by forming Cooper pairing, \( F_k \) contains kinetic energy arising from the intralayer canonical momentum and interlayer canonical momentum mixing of Cooper pairs, \( \Gamma \) denotes the canonical momentum mixing strength between Cooper pair mixing layers, \( F_J \) describes the Josephson term which captures the interlayer pairing mixing. Note that due to the giant Ising SOC, we have neglected the paramagnetic free energy.

Next, we simplify the free energy form in the following cases:

(i) The case where the amplitude of the order parameter in each layer has no spatial dependence. In this case, the order parameter becomes \( \Delta_i(r) \equiv |\Delta_i| e^{i\varphi_i} \) and \( \Delta_b(r) \equiv |\Delta_b| e^{i\varphi_b} \), and the free energy is simplified as

\[ F(|\Delta|) = -\alpha_0 |\Delta|^2 + \lambda_B \phi_B^2 |\Delta|^2 + \frac{\beta_0}{2} |\Delta|^4 + \lambda_J (1 - \cos(\varphi_i - \varphi_b)) |\Delta|^2, \]

which is presented in the main text as Eq. (10). Here, \( \lambda = (4\Gamma + \frac{1}{m^*}) \) The \( A_1 \) pairing with \( \varphi_i = \varphi_b \) is thus more favorable so that

\[ F(|\Delta|) = -\alpha_0 |\Delta|^2 + \lambda_B \phi_B^2 |\Delta|^2 + \frac{\beta_0}{2} |\Delta|^4. \]
For the $A_1$ pairing, we estimate the critical magnetic field as

$$q_B^2 = \frac{\alpha_0}{\lambda}.$$  

(S20)

(ii) The case with layer-antisymmetric FF pairing where $\Delta_s = |\Delta|e^{iq\cdot r}$ and $\Delta_b = |\Delta|e^{-iq\cdot r}$. Here, we have set $|\Delta_s| = |\Delta_b|$ to save $F_J$. As discussed in the main text, the favored $q = (0, 2q_B)$. In this case, the free energy becomes

$$F(|\Delta|) = -\alpha_0|\Delta|^2 + \frac{\beta_0}{2}|\Delta|^4 + \lambda_J|\Delta|^2.$$  

(S21)

It can be seen that the layered FF pairing would not pay kinetic energy but exhibit a finite Josephson energy $\lambda_J|\Delta|^2$.

(iii) The case with layer-symmetric FFLO pairing $\Delta_s = \Delta_b = |\Delta_+|e^{iq\cdot r} + |\Delta_-|e^{-iq\cdot r}$ with $q = (0, 2q_B), |\Delta_+|^2 + |\Delta_-|^2 = |\Delta|^2$. Note that the pairing within the two layers is identical. In this case, the free energy is deduced as

$$F(|\Delta|) = -\alpha_0|\Delta|^2 + \frac{\beta_0}{2}|\Delta|^4 + 2|\Delta|^2|\Delta|^2 + \frac{2q_B^2}{m^*}|\Delta|^2.$$  

(S22)

As $\beta_0 > 0$, the free energy is minimized with $|\Delta_+| = |\Delta|, |\Delta_-| = 0$ or $|\Delta_-| = |\Delta|, |\Delta_+| = 0$. Hence, up to the fourth order of the free energy, the favored pairing can only take $\Delta(r) = |\Delta|e^{iq\cdot r}$ or $\Delta(r) = |\Delta|e^{-iq\cdot r}$, which is the $2q_B$-FF pairing we study in the main text. Then, the free energy of this pairing is simplified as

$$F(|\Delta|) = -\alpha_0|\Delta|^2 + \frac{\beta_0}{2}|\Delta|^4 + \frac{2q_B^2}{m^*}|\Delta|^2.$$  

(S23)

Notably, this $2q_B$-FF pairing exhibits more intralayer kinetic energy but would not exhibit any kinetic energy from the Cooper canonical momentum mixing between two layers. The critical magnetic field of the $2q_B$-FF pairing is now given by

$$q_B^2 = m^*\alpha_0.$$  

(S24)

Therefore, this $2q_B$-FF pairing could survive at a higher magnetic field than the uniform pairings if

$$4m^*\Gamma > 1.$$  

(S25)

We can also infer the FF pairing would be more favorable than the LO pairing in the weak coupling where $\lambda_J \ll 2q_B^2/m^*$. We clarify here that the phenomenological free energy we present is to give a qualitative understanding of the results of the main text. Some relevant terms in the free energy can be different or some higher-order terms could play a role in the realistic model of twisted bilayer TMDs.

VI. THE LAYER ANTI-SYMMETRIC $2q_B$ FF PAIRING IN AA STACKING BILAYER TMD IN WEAK INTERLAYER COUPLING LIMIT

A. Model

For the AA stacking bilayer TMD without twisting under external magnetic fields $B$, the effective low-energy Hamiltonian for valence bands is given by

$$H_0(k + eK) = -\frac{\hbar k + eAx_z}{2m^*} - \mu + \epsilon\beta_{so}s_z + t\tau_x.$$  

(S26)

where $\epsilon = \pm$ denote valley indices, $m^*$ is the effective mass of the valence bands of the monolayer TMD, $\mu$ is the chemical potential, $\beta_{so}$ is the Ising SOC strength, $t$ is the coupling strength between two TMD layers, and Pauli matrices $s_i, \tau_i$ operate on the spin-, layer-space, respectively. Notice this Hamiltonian breaks the inversion symmetry since the Ising SOC term $\epsilon\beta_{so}s_z$ is mapped to $-\epsilon\beta_{so}s_z$ under inversion operation. The Zeeman effect from external fields is omitted by assuming a giant Ising SOC $\beta_{so} \gg u_BB$, while the orbital effect from external fields is included in the gauge potential $A = \frac{d}{2}(B \times \hat{z}) = \frac{d}{2}(\sin \chi \hat{x} - \cos \chi \hat{y})$, which is opposite for two layers. Here, $d$ denotes the
interlayer separation, $\chi$ characterizes the direction of the magnetic field. Inserting the chosen gauge potential into Hamiltonian (S26), we obtain

$$H_0(k + \epsilon K) = -\frac{\hbar^2 k^2}{2m^*} - \frac{\hbar^2}{m^*} (k_B \sin \chi k_x - k_B \cos \chi k_y) \tau_z - \mu' + \epsilon \beta_{so} \tau_z + t \tau_x$$

(S27)

with $l_0 = \sqrt{\hbar/eB}$, $k_B = d/2l_0^2$, $\mu' = \mu + \hbar^2 k_B^2/2m^*$. Then the eigenenergies are

$$E_{\epsilon,s,\tau}(k) = -\frac{\hbar^2 k^2}{2m^*} - \mu' + \tau \sqrt{k^2 + \frac{\hbar^2 k_B}{2m^*}^2 (k_0^2 \sin^2 \chi + k_y^2 \cos^2 \chi - k_x k_y \sin 2\chi)} + \epsilon \beta_{so}.$$

(S28)

To be specific, we set the magnetic field to be along the x direction ($\chi = 0$). The giant SOC can push some bands away from Fermi energy, and in this case, only the top valence bands $E_{\epsilon,+,\tau}(k)$, $E_{\epsilon,-,\tau}(k)$ matter. By projecting the states on these top valence bands, we obtain an effective Hamiltonian as

$$H_0(k + \epsilon K) = -\frac{\hbar^2 k^2}{2m^*} - \mu - V(k_y) \tau_z + t \tau_x,$$

(S29)

where the chemical potential $\mu$ is measured from the valence band top, the orbital field induced term $V(k_y) = \hbar v_B k_y$ with $v_B = \hbar k_B/m^*$. Notice the spin and valley are locked in this case: the $K$ valley is locked as spin-up, while $-K$ valley is locked as spin-down. With the Hamiltonian (S29), we can obtain single-particle Green’s functions for normal states:

$$G_e(k, \omega_n) = (i \omega_n - H_0(k))^{-1} = G_+(k, \omega_n) + G_-(k, \omega_n) - \frac{V \tau_z}{\sqrt{\epsilon^2 + V^2}},$$

(S30)

$$G_h(k, \omega_n) = (i \omega_n + H_0(-k))^{-1} = -G_e^T(-k, -\omega_n),$$

(S31)

where $V \equiv V(k_y)$ for the compact of notation, $G_e(h)(k, \omega_n)$ is electron (hole) Green’s function, and

$$G_{\pm}(k, \omega) = \frac{1}{2 \omega - \epsilon_\pm(k)} \pm \frac{1}{\omega - \epsilon_\pm(k)}$$

(S32)

with $\xi_{\pm}(k) = \xi_k \pm \sqrt{\nu^2 + t^2}$, $\xi_k = -\frac{\hbar^2 k^2}{2m^*} - \mu$, the Matsubara frequency $\omega_n = (2n + 1)\pi k_B T$, and $T$ denotes the temperature.

### B. Zero-momentum pairing

Within in the Nambu basis $(\psi_{k+K,t}, \psi_{k+K,b}, \psi_{-k-K,t}^\dagger, \psi_{-k-K,b}^\dagger)$, it can be found the BDG Hamiltonian is written as

$$H_{BDG}(k) = (\xi_k + t \tau_z) \rho_z - V(k_y) \tau_z + \Delta \rho_x.$$  

(S33)

Let us first consider the usual BCS zero-momentum pairing. For zero-momentum pairing, the superconductivity susceptibility is

$$\chi^{(2)}(q = 0) = -\frac{1}{\beta_{so}} \sum_{n,k} \text{Tr}(G_+(k, \omega_n) G_h(k, \omega_n) G_-(k, \omega_n))$$

$$= \frac{2}{\beta_{so}} \sum_{n,k} (G_+(k, \omega_n) G_+(\mathbf{-k}, \mathbf{-i} \omega_n) + \frac{t^2 - V^2}{t^2 + V^2} G_+(k, \mathbf{i} \omega_n) G_-(\mathbf{-k}, \mathbf{-i} \omega_n))$$

$$= -\frac{1}{2 \beta_{so}} \sum_{n,k} \left(1 + \frac{t^2 - V^2}{t^2 + V^2}\right) \left(\frac{1}{i \omega_n - \xi_k - D} i \omega_n + \xi_k + D\right) + \frac{1}{i \omega_n - \xi_k + D} i \omega_n + \xi_k - D +$$

$$\left(1 - \frac{t^2 - V^2}{t^2 + V^2}\right) \left(\frac{1}{i \omega_n - \xi_k + D} i \omega_n + \xi_k + D\right) + \frac{1}{i \omega_n - \xi_k - D} i \omega_n + \xi_k - D$$

(S34)

Here $\beta_{so} = 1/k_B T$, $D = \sqrt{\nu^2 + t^2}$. It can be shown

$$\chi_0 = -\frac{1}{\beta_{so}} \sum_{n,k} \frac{1}{i \omega_n - \xi_k + A} i \omega_n + \xi_k - B$$

$$= N_0 \log(\frac{2 e^\gamma \omega_D}{\pi k_B T}) + N_0 \psi(\frac{1}{2}) - \frac{N_0}{2} \psi(\frac{1}{2} + \frac{i(A - B)}{4 k_B T}) + \psi(\frac{1}{2} + \frac{i(A - B)}{4 k_B T}),$$

(S35)
where \( \omega_D \) is the Debye frequency, \( N_0 \) denote the density of states. With Eq. [S33] and Eq. [S34] can be simplified as

\[
\chi^{(2)} = 2N_0 \log\left(\frac{2e^2}{\pi k_B T} \right) + \frac{2}{t^2 + \langle V^2 \rangle} \left[ N_0 \psi\left( \frac{1}{2} \right) - \frac{N_0}{2} \left( \psi\left( \frac{1}{2} - \frac{i \sqrt{\langle V^2 \rangle + t^2}}{2\pi k_B T} \right) + \psi\left( \frac{1}{2} + \frac{i \sqrt{\langle V^2 \rangle + t^2}}{2\pi k_B T} \right) \right) \right].
\] (S36)

Here \( \langle V^2 \rangle = \frac{1}{2} \hbar^2 v_B^2 k_f^2 \), \( \langle \ldots \rangle \) denotes the averaging over Fermi surface. Therefore, the linearized gap equation is

\[
\log\left( \frac{T}{T_c} \right) = \frac{\langle V^2 \rangle}{t^2 + \langle V^2 \rangle} \left[ \psi\left( \frac{1}{2} \right) - \frac{1}{2} \left( \psi\left( \frac{1}{2} - \frac{i \sqrt{\langle V^2 \rangle + t^2}}{2\pi k_B T} \right) + \psi\left( \frac{1}{2} + \frac{i \sqrt{\langle V^2 \rangle + t^2}}{2\pi k_B T} \right) \right) \right].
\] (S37)

When \( \sqrt{\langle V^2 \rangle + t^2} \ll T_c \), near \( T_c \), Eq. [S37] gives

\[
B_c = \frac{8\pi k_B T_c}{\pi T} \left( 1 - \frac{T}{T_c} \right)^{1/2},
\] (S38)

where the Fermi velocity \( v_f = \hbar k_f / m^* \).

### C. 2qB layer-antisymmetric FF pairing in weak coupling limit

Let us consider the finite-momentum pairing case. We show that the layer antisymmetric pairing momentum with \( q = (0, 2k_B) \) at one layer and \( q = (0, -2k_B) \) at the other layer is more favorable in the weak coupling limit.

We assume the intra-layer pairing is dominant. As shown in Table S1, there are intra-layer pairing channels: \( A_1 \) pairing and \( A_2 \) pairing. In the presence of an in-plane magnetic field, \( \Delta_{A1,1} \) and \( \Delta_{A2,1} \) will couple with each other. The Landau free energy, up to the second order, is given by

\[
F = \frac{1}{2} \sum_q \left( \Delta_{A1,1}^* \Delta_{A2,1} \right) \left( \begin{array}{cc}
\frac{U_0}{\beta} - \chi_{11}(q) & -\chi_{12}(q) \\
-\chi_{21}(q) & \frac{U_0}{\beta} - \chi_{22}(q)
\end{array} \right) \left( \begin{array}{c}
\Delta_{A1,1} \\
\Delta_{A2,1}
\end{array} \right),
\] (S39)

where \( U_0 \) denotes the intra-layer interaction strength and the superconductivity susceptibility \( \chi \) can be written as

\[
\chi_{ij}(q) = -\frac{1}{\beta_{so}} \sum_{n,k} \mathrm{Tr}\left[ \tau_i G_n(k + q/2) \tau_j G_{n+}(k - q/2, -\omega_n) \right]
\] (S40)

with \( \tau_1 = \tau_0, \tau_2 = \tau_2 \). Substitute the single-particle Green’s function Eq. [S30] and Eq. [S31] the superconductivity susceptibility can further written as

\[
\chi_{11} = \frac{2}{\beta_{so}} \sum_{n,k} \left[ G_+(k + q/2, \omega_n) G_+(-k + q/2, -\omega_n) + \frac{V^2 + t^2}{V^2 + t^2} G_-(k + q/2, \omega_n) G_-(k - q/2, -\omega_n) \right],
\] (S41)

\[
\chi_{12} = \frac{2}{\beta_{so}} \sum_{n,k} \frac{V}{\sqrt{V^2 + t^2}} \left[ G_+(k + q/2, \omega_n) G_-(-k + q/2, -\omega_n) \right] - \left[ G_-(k + q/2, \omega_n) G_+(-k - q/2, -\omega_n) \right],
\] (S42)

\[
\chi_{22} = \frac{2}{\beta_{so}} \sum_{n,k} \left[ G_+(k + q/2, \omega_n) G_+(-k + q/2, -\omega_n) \right] - \left[ G_-(k + q/2, \omega_n) G_-(k + q/2, -\omega_n) \right].
\] (S43)

After some direct calculations, we obtain

\[
\chi_{11}(q) \approx 2N_0 \log\left(\frac{2e^2}{\pi k_B T} \right) + \frac{CN_0 v_B^2 q^2}{32\pi^2 k_B^4 T^2} + \frac{CN_0 V^2}{4\pi^2 k_B^4 T^2},
\] (S44)

\[
\chi_{12}(q) \approx -\frac{CN_0 v_B \sqrt{v_f} k_f q_0}{8\pi^2 k_B^2 T^2},
\] (S45)

\[
\chi_{22}(q) \approx 2N_0 \log\left(\frac{2e^2}{\pi k_B T} \right) + \frac{CN_0 v_B^2 q^2}{32\pi^2 k_B^4 T^2} + \frac{CN_0 V^2 + t^2}{4\pi^2 k_B^4 T^2},
\] (S46)

where \( \langle V^2 \rangle = \frac{1}{2} v_f^2 k_f^2 \), \( C = \psi^{(2)}(\frac{1}{2}) \).
In the weak coupling limit \( t \to 0 \), the Landau free energy is

\[
F \approx \sum_q \left( \Delta_{A_1,1}^* \Delta_{A_2,1}^* \right) \left[ (N_0 \log \left( \frac{T}{T_c} \right) - \frac{CN_0 v_B^2 q^2}{64 \pi^2 k_B^2 T^2} - \frac{CN_0 v_B^2 k_f^2}{16 \pi^2 k_B^2 T^2} ) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{CN_0 v_B v_f k_f q_y}{16 \pi^2 k_B^2 T^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_{A_1,1} \\ \Delta_{A_2,1} \end{pmatrix} \right],
\]

where \( \Delta_{1,2} = \Delta_{A_1,1} \pm \Delta_{A_2,1} \), \( \lambda_{1,2} = N_0 \log \left( \frac{T}{T_c} \right) - \frac{CN_0 v_B^2 q^2}{64 \pi^2 k_B^2 T^2} - \frac{CN_0 v_B^2 k_f^2}{16 \pi^2 k_B^2 T^2} \pm \frac{CN_0 v_B v_f k_f q_y}{16 \pi^2 k_B^2 T^2} \). The critical temperature is given by \( \min(\lambda_1, \lambda_2) = 0 \). This gives

\[
N_0 \log \left( \frac{T}{T_c} \right) - \frac{CN_0}{16 \pi^2 k_B^2 T_c^2} \left[ (\frac{q_y v_f}{2} - \text{sgn}(q_y) v_B k_f)^2 + \frac{q_x^2 v_f^2}{4} \right] = 0.
\]

The critical temperature \( T_c \) is maximized when \( q_x = 0, q_y = \text{sgn}(q_y) 2 v_B k_f / v_f \). It is easy to see \( q_y = \pm 2 k_B \) with \( v_B = \hbar k_B / m, v_f = \hbar k_f / m \). And hence, in weak coupling limit \( t \to 0 \), the finite-momentum pairing \( \Delta_1(q) = \left< c_{K+k+q/2,+}^+ c_{-K-k+q/2,-} \right> \) with \( q = (0, 2 k_B) \) and \( \Delta_2(q) = \left< c_{K+k+q/2,-}^+ c_{-K-k+q/2,+} \right> \) with \( q = (0, -2 k_B) \) are stabilized.