Controlling charge and spin transport in an Ising-superconductor Josephson junction

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An in-plane magnetic field applied to an Ising superconductor converts spin-singlet Cooper pairs to spin-triplet ones. In this work, we study a Josephson junction formed by two Ising superconductors that are proximitized by ferromagnetic layers. This leads to highly tunable spin-triplet pairing correlations which allow to modulate the charge and spin supercurrents through the in-plane magnetic exchange fields. For a junction with a nonmagnetic barrier, the charge current is switchable by changing the relative alignment of the in-plane exchange fields, and a π-state can be realized. Furthermore, the charge and spin current-phase relations display a φ₀-junction behavior for a strongly spin-polarized ferromagnetic barrier.

Introduction.– The interplay between magnetism and superconductivity leads to a number of fascinating phenomena. However, it is nontrivial to observe since spin-singlet superconductivity is typically destroyed by strong magnetic fields through orbital [1] or Zeeman-induced pair breaking [2, 3]. Recently, superconductivity was experimentally realized in various two-dimensional transition-metal dichalcogenides [4–19]. In these materials, the orbital depairing effect from an in-plane magnetic field is suppressed due to their two-dimensional nature. For an odd-number-layer crystal, inversion symmetry is broken so that the spin-orbit interaction from the transition-metal atom leads to spin-valley locking, i.e., a valley-dependent Zeeman-like spin splitting [20]. Since the spins are polarized out of plane, this Zeeman-like field was termed Ising spin-orbit coupling (ISOC). Its presence makes the superconducting state resilient against the Zeeman effect from an in-plane magnetic field [21, 23] far beyond the Pauli paramagnetic limit [2, 3]. Thus, these so-called Ising superconductors provide an ideal laboratory to study the interplay between superconductivity and ferromagnetism. Furthermore, applying an in-plane magnetic field induces triplet correlations [19, 24–28], mirage gaps [28], or a two-fold rotational symmetry of the superconducting state [14, 15].

Very recently, van der Waals heterostructures consisting of Ising superconductors and ferromagnetic barriers have attracted a great deal of experimental interest [16–18]. In particular, ferromagnetic Josephson junctions have been fabricated and the coexistence of 0 and π states in the junction region has been demonstrated which can be used to construct φ-phase Josephson junctions [16, 17]. Transport properties of Ising superconductors have also been theoretically investigated in ferromagnet–Ising-superconductor junctions [29, 30] and Josephson junctions with a half-metal barrier [31]. These theoretical works focused on phenomena arising from spin-triplet Andreev reflection at the interfaces. So far, however, the influence of spin-triplet pairing correlations induced by ferromagnetism on the transport properties of Ising superconductors has not yet been discussed.

In this Letter, we study the implications of in-plane exchange field-induced triplet pairing correlations in a Josephson junction based on Ising superconductors [see Fig. 1(a)]. For a junction with a nonmagnetic barrier, we find that the charge supercurrent is switchable by changing the exchange fields between parallel and antiparallel alignments. At low temperatures and in the clean limit, a π-state charge supercurrent can be realized if the exchange-field magnitudes are larger than the ISOC and superconductivity is not fully destroyed. Noncollinear exchange fields give rise to a finite spin supercurrent. We also study the case of a ferromagnetic barrier and find that both the charge and spin current-phase relations can be tuned if the barrier is strongly polarized.

Model and formalism.– We consider an Ising superconductor with an s-wave pairing gap Δ and superconducting phase φα in contact with a ferromagnetic layer. The effective Bogoliubov-de Gennes Hamiltonian near one of the valleys can be written in the Nambu basis (c,p,↑,c,p,↓,c↑,−p,−s,↑,c↑,−p,−s,↓) as

\[
H(p) = \begin{bmatrix}
H_0(p,s) & \Delta e^{i\phi} i\sigma_y \\
-\Delta e^{-i\phi} i\sigma_y & -H_0^*(-p,-s)
\end{bmatrix},
\]

where \( p \) is the momentum deviation from the \( K \) or \( K' \) point and \( s = \pm \) denotes the valley index. The normal-state Hamiltonian \( H_0 \) is

\[
H_0(p,s) = \xi_p \sigma_0 + s \beta \sigma_σ - J \cdot \sigma,
\]

where the dispersion \( \xi_p = |p|^2/(2m) - \mu \) is measured from the chemical potential \( \mu \) and \( m \) is the electron mass. The Pauli matrices \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) act on spin space, with \( \sigma_0 \) being the corresponding unit matrix. By defining the out-of-plane direction along the \( z \) axis, the ISOC is taken into account via the term \( s \beta \sigma_z \). The in-plane exchange field \( J = J_x x + J_y y \) from the ferromagnetic layer gives rise to the Zeeman term \( -J \cdot \sigma \) and converts Cooper pairs from spin-singlet to spin-triplet [26, 28]. In NbSe₂, the ISOC strength is about 40 meV for a monolayer and is decreasing with the number of layers, while the Fermi energy is about 0.4 eV [6, 15].

For Ising superconductors, the superconducting gap and ISOC strength are much smaller than the Fermi energy, and we can employ the quasiclassical Green’s function formalism [32, 33] that provides expressions of the charge and spin currents [35–37]. The structure of the quasiclassical Green’s function, which is both valley and energy \( \varepsilon \) dependent, can be
In Eq. (4), the tunnel barrier separating the superconductors can be either nonmagnetic or ferromagnetic. The charge current $I_c$ and spin current $I_s$ as a function of $\phi$ and $\theta$ for the junction with a nonmagnetic barrier. In (b) and (c), $\beta_{so} = 5\Delta_0$, $J = 4\Delta_0$, and $T = 0.0T_0$ where $\Delta_0$ and $T_0$ are, respectively, the zero-temperature gap and transition temperature in the absence of an external field. (d) The critical charge current versus angle $\theta$. The black curve is the vertical line-cut along the dashed line of panel (b) at $\phi = \pi/2$. The green curve with $J = 6\Delta_0$ is multiplied by 10 and shows a 0-\pi transition at $\theta \approx 3\pi/4$. The charge and spin currents are, respectively, in units of $G_t\Delta_0/e$ and $hG_t\Delta_0/(2e^2)$ where $G_t$ is the tunnel conductance.

written as [34, 38]

$$\hat{g}(s, \varepsilon) = \left[ \begin{array}{c} g_0\sigma_0 + g_1\sigma_1 + g_2\sigma_2 - \hat{g}f_0\sigma_0 + f_0\sigma_0 + f_1\sigma_1 + f_2\sigma_2 \end{array} \right] i\sigma_y,$$

(3)

where the bar operation is defined as $\bar{g}(s, \varepsilon) = g(-s, -\varepsilon)^*$. With $q \in \{g_0, f_0, g, f\}$. The anomalous Green’s functions $f_0$ and $f$ characterize the singlet and the triplet pairings, respectively. As we show in the Supplemental Material [39], all components of $\hat{g}$ can be obtained from the Eilenberger equation [28, 32, 34]

$$\varepsilon\tau_3\sigma_0 - \Delta - \hat{\nu} - \hat{\Sigma}(\varepsilon) = 0,$$

(4)

together with $\text{tr}(\hat{g}) = 0$ and the normalization condition $\hat{g}^2 = 1$ [28]. In Eq. (4), $\tau_3$ is the third Pauli matrix acting on Nambu space,

$$\Delta = \Delta \left[ \begin{array}{cc} 0 & e^{-i\phi_0}i\sigma_y \\ e^{i\phi_0}i\sigma_y & 0 \end{array} \right],$$

(5)

and $\hat{\nu} = -\text{diag}(\mathbf{J} \cdot \sigma, \mathbf{J} \cdot \sigma^*) + s_0\alpha\tau_3\sigma_z$. Nonmagnetic intervalley impurity scattering is taken into account via $\hat{\Sigma}(\varepsilon) = -i\Gamma(\hat{g}(s, \varepsilon))$ where $\Gamma$ is the impurity-scattering rate and $(\cdots)$ denotes averaging over all Fermi-momentum directions.

In particular, we find $f = a(\varepsilon\mathbf{J} + 18\beta_{so}x \times \mathbf{J})$ in the clean limit and the factor $a$ can be fixed by the normalization condition [28, 39]. The second term in $\mathbf{f}$ originates from the commutator between the Zeeman term $\mathbf{J} \cdot \sigma$ and the ISOC term $s_0\beta_{so}\sigma_z$ in the Green’s function. The retarded counterpart of $\hat{g}(s, \varepsilon)$ is obtained by replacing $\varepsilon$ with $\varepsilon + i\eta$, where $\eta$ is the Dynes broadening parameter [40].

We now turn to the Josephson junction which is schematically shown in Fig. 1(a). The in-plane exchange fields in the two Ising superconductors, which are denoted as $s1$ and $s2$, originate from the corresponding ferromagnetic layers. The two exchange fields, with a relative orientation defined by the angle $\theta$, are assumed to have the same magnitude $J = |\mathbf{J}|$.

The magnetization direction of a ferromagnetic layer can be controlled by an external magnetic field. For a junction in which the two ferromagnetic layers have different thicknesses, an applied magnetic field will predominantly tune the magnetization of the thinner layer leading to a controlled relative alignment between the exchange fields [41]. The Josephson phase is denoted by $\phi = \phi_{s1} - \phi_{s2}$. The central barrier separating the two superconductors can be either nonmagnetic or ferromagnetic with out-of-plane magnetization and spin polarization $\mathbf{P}$. We consider a tunnel junction characterized by the conductance $G_t$ including the valley degree of freedom. The expressions of the charge current $I_c$ and the z-polarized spin current $I_s$ in superconductor $s1$ are, respectively, given by [37, 39]

$$I_c = \frac{G_t}{8e} \int_{-\infty}^{\infty} d\varepsilon \text{tr} (\tau_3\sigma_0\hat{I}),$$

(6)

$$I_s = \frac{hG_t}{16e^2} \int_{-\infty}^{\infty} d\varepsilon \text{tr} (\tau_3\sigma_z\hat{I}),$$

where

$$\hat{I} = \frac{1}{2} Re \left[ (1 + \sqrt{1 - P^2})\hat{g}_{s1} + \mathbf{P}(\hat{g}, \hat{g}_{s2}) + (1 - \sqrt{1 - P^2})\hat{\nu}\hat{g}_{s2} \hat{\nu} \right] \text{tanh}[\varepsilon/(2k_BT)],$$

(7)

with $\hat{\nu} = \text{diag}(\sigma_z, \sigma_z)$, $k_B$ the Boltzmann constant, and $T$ the temperature. Here, $\hat{g}_{s1}$ is the retarded counterpart of $\hat{g}$ in Eq. (3) for superconductor $s = s1, s2$.

Nonmagnetic barrier.– We first discuss the case of a tunnel junction with a nonmagnetic barrier. The charge current $I_c$ and the spin current $I_s$ can be expressed as

$$I_c = (I_{c0} + I_1 \cos \theta) \sin \phi,$$

(8)

$$I_s = (h/2e)(I_{s0} + I_1 \cos \phi) \sin \theta,$$

(9)

where $I_{c0}$, $I_{s0}$, and $I_1$ are derived in the Supplemental Material [39]. Equations (8) and (9) were previously obtained in Ref. [42] for an SNS-junction with a diffusive normal metal. The charge and spin currents obey the relation

FIG. 1. (a) Schematic plot of the Ising-superconductor Josephson junction. The ferromagnetic (FM) layers provide in-plane magnetic exchange fields in the two Ising superconductors which are labeled as $s1$ and $s2$. The angle between the exchange fields is $\theta$ and the superconducting phase difference is $\phi$. The tunnel barrier separating the superconductors can be either nonmagnetic or ferromagnetic. (b) Charge current $I_c$ and (c) spin current $I_s$ as a function of $\phi$ and $\theta$ for the junction with a nonmagnetic barrier. In (b) and (c), $\beta_{so} = 5\Delta_0$, $J = 4\Delta_0$, and $T = 0.0T_0$ where $\Delta_0$ and $T_0$ are, respectively, the zero-temperature gap and transition temperature in the absence of an external field. (d) The critical charge current versus angle $\theta$. The black curve is the vertical line-cut along the dashed line of panel (b) at $\phi = \pi/2$. The green curve with $J = 6\Delta_0$ is multiplied by 10 and shows a 0-\pi transition at $\theta \approx 3\pi/4$. The charge and spin currents are, respectively, in units of $G_t\Delta_0/e$ and $hG_t\Delta_0/(2e^2)$ where $G_t$ is the tunnel conductance.

The magnetization direction of a ferromagnetic layer can be controlled by an external magnetic field. For a junction in which the two ferromagnetic layers have different thicknesses, an applied magnetic field will predominantly tune the magnetization of the thinner layer leading to a controlled relative alignment between the exchange fields [41]. The Josephson phase is denoted by $\phi = \phi_{s1} - \phi_{s2}$. The central barrier separating the two superconductors can be either nonmagnetic or ferromagnetic with out-of-plane magnetization and spin polarization $\mathbf{P}$. We consider a tunnel junction characterized by the conductance $G_t$ including the valley degree of freedom. The expressions of the charge current $I_c$ and the z-polarized spin current $I_s$ in superconductor $s1$ are, respectively, given by [37, 39]
\[ \frac{\partial I_c}{\partial \theta} = \frac{2e}{\hbar} \frac{\partial I_s}{\partial \phi} \]

The terms proportional to \( I_{c0} \) and \( I_s \), respectively, result from singlet and triplet pairing correlations. The phase-independent spin current component \( \langle h/2e \rangle I_{s0} \sin \theta \) is due to the noncollinear in-plane magnetizations induced by the exchange fields. Both \( I_1 \) and \( I_{s0} \) vanish if one of the exchange fields is absent. Thus, the spin current can be modulated by changing the orientation \( \theta \) or the magnitude \( J \) of the in-plane exchange fields. Moreover, there is a pure spin current at \( \phi = 0 \) or \( \phi = \pi \) for which the charge current vanishes. The phase transition is shown in Fig. 2(b) and can be used to control the charge supercurrent, and we will call this phenomenon the switch effect. The phase-independent part of the spin current \( hI_{s0}/(2e) \) is negligible compared to the contribution proportional to \( I_1 \) as can be seen from Fig. 1(c) at \( \phi = \theta = \pi/2 \).

For a clean Ising superconductor, there is an upturn for the critical magnetic field at low temperatures \([5, 26, 44, 45]\) so that the critical field can be even larger than the ISOC. In this situation, the magnitude of \( I_1 \) can be larger than that of \( I_{c0} \) at \( J \geq \beta_{s0} \) so that the critical charge current can be negative, and a \( \pi \)-state Josephson junction is realized. In Fig. 1(d), the green curve shows the critical charge current versus the angle \( \theta \) at \( J > \beta_{s0} \). It can be seen that the critical charge current changes sign at \( \theta \approx 3\pi/4 \). Thus, a 0-\( \pi \) transition can be achieved by changing the magnitudes or the relative orientation of the exchange fields for weak impurity scattering and low temperature. Typically, \( \pi \)-states are realized in Josephson junctions with a ferromagnetic barrier between two superconductors \([31, 46-54]\). Here, its appearance is due to the interplay of the supercurrents induced by singlet and triplet pairing correlations.

In the following, we discuss the dependence of the switch effect of the charge current on the temperature and exchange fields. Figure 2(a) shows the critical charge currents versus the exchange-field magnitude at different temperatures. The critical currents for the parallel and antiparallel exchange fields are shown in solid and dash-dotted lines, respectively. On increasing the exchange-field magnitude \( J \), the critical charge currents for both the parallel and antiparallel configurations decrease, which results from the suppression of the superconducting gap \( \Delta \). Meanwhile, the triplet pairing correlations first increase and then decrease due to the interplay between the increase of the exchange field and the decrease of the superconducting gap. This is reflected in the difference of the critical charge currents between the parallel and antiparallel configurations with increasing \( J \). The spin current at \( \phi = 0 \) exhibits the same nonmonotonic behavior with respect to \( J \) as shown in Fig. 2(c).

To study the switch effect quantitatively, we define the ratio of the critical charge currents between the parallel and antiparallel configurations,

\[ \mathcal{R} = \left. \frac{I_c(\theta = 0)}{I_c(\theta = \pi)} \right|_{\phi = \pi/2} = \frac{I_{c0} + I_1}{I_{c0} - I_1} \]  

The switch ratio versus the exchange-field magnitude is shown in Fig. 2(b) where the vertical lines indicate the critical fields at different temperatures. We see that the switch ratio increases with increasing the exchange-field magnitude and can achieve large values at low temperatures. The gray line shows the approximation \([39]\)

\[ \mathcal{R} \approx \left( \frac{\beta_{s0} + J^2}{\beta_{s0}^2 - J^2} \right), \]

which neglects the pair-breaking effect. In this approximation, the switch ratio diverges near \( J = \beta_{s0} \) and can be negative for
$J > \beta_{\alpha}$ indicating a $\pi$-state for the antiparallel configuration. This behavior can be seen in the inset of Fig. 2(a) with the temperature being close to zero.

We have assumed that the ISOCs have the same sign on both sides of the junction. If the signs are opposite, there is a quantitative change: the sign of $I_1$ is reversed so that the critical charge current for the antiparallel configuration is larger than that for the parallel configuration [39]. This does not affect the existence of the switch effect and the $0-\pi$ transition.

The switch effect in this work is due to the interplay between ISOC and exchange fields. This differs from the effect described by Bergeret et al. [55] in which the critical current of the antiparallel configuration is larger than that of the parallel configuration.

**Ferromagnetic barrier**—The switch ratio can be increased by reducing the supercurrent carried by the singlet Cooper pairs. This can be achieved by replacing the nonmagnetic barrier with a ferromagnetic one. We consider the case where the magnetization of the ferromagnetic barrier points out of plane with spin polarization $P$. The charge current $I_c$ and spin current $I_s$ are, respectively, expressed as [39]

$$I_c = (\sqrt{1 - P^2} I_{c0} + I_1 \cos \theta) \sin \phi + P I_1 \sin \theta \cos \phi, \quad (12)$$

$$I_s = (\hbar/2e)[(I_{s0} + I_1 \cos \phi) \sin \theta + P I_1 \sin \phi \cos \theta], \quad (13)$$

where $I_{c0}$, $I_{s0}$ and $I_1$ are the same coefficients as in Eqs. [8] and [9] that are derived in the Supplemental Material [39]. The charge current carried by the singlet Cooper pairs becomes $\sqrt{1 - P^2} I_{c0}$ and is reduced compared to that for a nonmagnetic barrier. This enhances the switch effect of the critical charge current between $\theta = 0$ and $\theta = \pi$. The $\pi$-state can also be realized even at $J < \beta_{\alpha}$. Moreover, the ferromagnetic barrier leads to a phase shift to the supercurrent induced by the triplet pairing correlations.

The charge and spin current-phase relations at different angles $\theta$ are shown in Fig. 3. We use the same parameters as in Figs. 1(b) and 1(c) apart from the finite spin polarization of the barrier. A $\pi$-state of the charge current arises for the antiparallel configuration ($\theta = \pi$) [see Fig. 3(a)]. Moreover, as can be seen from Fig. 3, both the charge and spin current-phase relations can be controlled by the angle $\theta$ between the exchange fields. This tunability depends on the spin polarization of the ferromagnetic barrier. When the ferromagnetic barrier is fully polarized ($P = \pm 1$), we get $I_c = I_1 \sin (\phi \pm \theta)$ and $I_s = (\hbar/2e)[I_{s0} \sin \theta + I_1 \sin (\theta \pm \phi)]$ indicating that the phases of both the charge and spin currents can be arbitrarily tuned. Thus, the present system is a realization of a so-called $\phi_0$-junction for charge and spin supercurrents [37].

There are further interesting implications of Eqs. (12) and (13). At Josephson phase $\phi = 0$ or $\phi = \pi$, the supercurrent is only carried by the triplet Cooper pairs and both the charge and spin currents depend on the angle between the exchange fields in a sinusoidal way. Moreover, for the spin current, we have

$$I_s(\theta = 0) = -I_s(\theta = \pi) = (\hbar/2e) P I_1 \sin \phi, \quad (14)$$

$$I_s(\theta = \pi/2) = -I_s(\theta = 3\pi/2) = (\hbar/2e)(I_{s0} + I_1 \cos \phi), \quad (15)$$

which can be also seen from Fig. 3(b). Equations (14) and (13) show that the spin current-phase relation can be switched between sine and cosine by changing the relative orientation of the exchange fields.

**Discussion and conclusion**—For our investigation we only considered Ising superconductors in the clean limit. Since the induced triplet pairing correlations are more sensitive to intervalley scattering than the singlet pairing correlations [26, 28], the switch effect of the charge current is suppressed by impurity scattering [39].

In the presence of an in-plane exchange field, finite-energy pairing correlations emerge in an Ising superconductor accompanied by mirage gaps [28]. In fact, for the present discussion of the Josephson effect, the contribution from the mirage gaps to the supercurrents is negligible compared to that from the main superconducting gap [39].

On the experimental side, Josephson junctions with ferromagnetic barriers based on NbSe$_2$ with Ising superconductivity have been fabricated and studied [16–18]. Moreover, the magnetic proximity effect in van der Waals heterostructures consisting of the Ising superconductor NbSe$_2$ and a two-dimensional magnet such as CrBr$_3$ [15].

To conclude, we have studied the transport properties of a van der Waals Josephson junction consisting of Ising superconductors and ferromagnets. The ferromagnetic layers provide in-plane magnetic exchange fields that induce controllable triplet pairing correlations in the superconductors. As a result, both the charge and spin currents can be modulated by the strength and relative directions of the exchange fields. In particular, we have described a switch effect for the charge current and a $0-\pi$ transition. Furthermore, both the charge and spin current-phase relations are tunable if the barrier is strongly spin-polarized. Our predictions show that Josephson...
junctions based on Ising superconductors exhibit rich transport properties, and they confirm the great potential in van der Waals superconducting heterostructures.

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Supplemental Material for “Controlling charge and spin transport in an Ising-superconductor Josephson junction”

QUASICLASSICAL GREEN’S FUNCTION

We consider an Ising superconductor with a singlet \( s \)-wave pairing gap \( \Delta \) and superconducting phase \( \phi_s \).

The effective Bogoliubov-de Gennes Hamiltonian near one of the valleys can be written in the Nambu basis \( (c_{p,s,\uparrow}, c_{p,s,\downarrow}, c_{-p,-s,\uparrow}, c_{-p,-s,\downarrow}) \) as

\[
H = \begin{pmatrix}
H_0(p, s) & \Delta e^{i\phi_s} i\sigma_y \\
-\Delta e^{-i\phi_s} i\sigma_y & -H_0^*(-p, -s)
\end{pmatrix},
\]

where \( H_0(p, s) = \xi_p \sigma_0 + s\beta_{so} \sigma_z - J \cdot \sigma \),

where the dispersion \( \xi_p = |p|^2/(2m) - \mu \) is measured form the chemical potential \( \mu \) and \( m \) is the electron mass. The Pauli matrices \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) acts on spin space with \( \sigma_0 \) being the corresponding unit matrix. The strength of the ISOC which pins the electron spins in the out-of-plane direction is denoted by \( \beta_{so} \).

The in-plane exchange field and the ISOC are included in \( \nu \) as

\[
\nu = \left[ (\nu_+ + \nu_-) \cdot \sigma \right] 0 \quad 0 \quad (\nu_+ - \nu_-) \cdot \sigma^* \]

where \( \nu_+ = -J \) and \( \nu_- = (0, 0, s\beta_{so}) \). The off-diagonal terms of the Eilenberger equation in Nambu space are

\[
\begin{align*}
\varepsilon & f_0 - \nu_+ \cdot f + \Delta g_{00} = 0, \\
\varepsilon & f - \nu_+ f_0 - i\nu_\times f + \Delta g_{+-} = 0, \\
\varepsilon & f_0 + \nu_+ \cdot f + \Delta g_{00} = 0, \\
\varepsilon & f_0 + \nu_- f_0 + i\nu_\times f - \Delta g_{--} = 0,
\end{align*}
\]

where the phase factors of \( (f_0, f) \) and \( (f_0, \tilde{f}) \) are dropped out for simplicity. Equations \((25)\) and \((26)\) can be written explicitly as

\[
\begin{bmatrix}
\varepsilon & J_x & J_y \\
J_x & \varepsilon & is\beta_{so} \\
J_y & -is\beta_{so} & \varepsilon
\end{bmatrix} \begin{bmatrix}
f_0 \\
f_x \\
f_y
\end{bmatrix} + \Delta \begin{bmatrix}
g_{00} \\
g_{0x} \\
g_{0y}
\end{bmatrix} = 0. \tag{29}
\]

The fact that \( f_x = 0 \) leads to \( g_{-x} = g_{-y} = g_{+z} = 0 \) from Eqs. \((29)\) and \((27)\).

By neglecting the superconducting phase factors, we obtain \( \tilde{f}_0 = \tilde{f}_0 \). Combining this with Eqs. \((25)\) and \((27)\), we have

\[
J_x(f_x + \tilde{f}_x) + J_y(f_y + \tilde{f}_y) = 0. \tag{30}
\]

By combining Eqs. \((26)\) and \((28)\), we have

\[
\varepsilon[J_y(f_x + \tilde{f}_x) - J_x(f_y + \tilde{f}_y)] = is\beta_{so} [J_y(\tilde{f}_y - f_x) + J_x(f_x - \tilde{f}_x)]. \tag{31}
\]

Using the requirement \( f(s, \varepsilon) = f(-s, -\varepsilon)^* \), one can get

\[
\begin{align*}
f_x &= a (\varepsilon J_x - is\beta_{so} J_y), & f_x &= -f_x^*, \\
f_y &= a (\varepsilon J_y + is\beta_{so} J_x), & f_y &= -f_y^*.
\end{align*} \tag{32}
\]

where \( a \) is to be fixed by the normalization condition. The above relations can alternatively be expressed as

\[
\begin{align*}
f &= a (\varepsilon J \times \sigma_z), & f &= -f^*.
\end{align*} \tag{34}
\]
as provided in the main text. Consequently, Eq. (29) is rewritten as
\[ \varepsilon f_0 + a \varepsilon J^2 + \Delta g_0 = 0, \]  
(35)
\[ J_x f_0 + a J_x (\varepsilon^2 - \beta_{so}^2) + \Delta g_{x,x} = 0, \]  
(36)
\[ J_y f_0 + a J_y (\varepsilon^2 - \beta_{so}^2) + \Delta g_{y,y} = 0. \]  
(37)
Combining Eqs. (35)-(37) with Eq. (20), which can be written as
\[ g_0 g_{x,x} = a \varepsilon J_x f_0, \quad g_0 g_{y,y} = a \varepsilon J_y f_0, \]  
(38)
we obtain \( g_0 \) and \( f_0 \) as
\[ g_0 = a \varepsilon / (2\Delta), \quad f_0 = -a (J^2 + c/2), \]  
(39)
where
\[ c = \varepsilon^2 - \beta_{so}^2 - J^2 - \Delta^2 + u, \]  
(40)
with
\[ u = \sqrt{\left( \varepsilon^2 - \beta_{so}^2 - J^2 - \Delta^2 \right)^2 - 4J^2 \Delta^2}. \]  
(41)
The term \( g_{x,x} \) is obtained from Eq. (21) with
\[ g_0 g_{x,x} = a^2 \varepsilon J^2 s_{\beta_{so}}. \]  
(42)
The coefficient \( a \) is fixed by Eq. (19), which is
\[ g_0^2 + g_{x,x}^2 + g_{y,y}^2 - f_0 f_0 + f_x f_x + f_y f_y = 1, \]  
(43)
so that
\[ a^2 (4J^2 \Delta^2 - c^2) \left[ \Delta^2 (2J^2 + c) - c^2 \varepsilon^2 + 4J^2 \beta_{so}^2 \Delta^2 \right] = 4c^2 \Delta^2, \]  
(44)
which can be simplified as
\[ a^2 = \frac{c \Delta^2}{u^2 c (\varepsilon^2 - \Delta^2) - 2 \Delta^2 J^2}. \]  
(45)
We should be reminded that \((f_0, f)\) and \((\bar{f}_0, \bar{f})\) contain the phase factors \( e^{i\phi} \) and \( e^{-i\phi} \), respectively, which are neglected in the derivation above for simplicity. The phase factors are important when considering a Josephson junction. To this end, we obtain all the components of the quasiclassical Green’s function \( \hat{g} \). The retarded and advanced counterparts of \( \hat{g}(s, \varepsilon) \) are obtained, respectively, by replacing the real \( \varepsilon \) in \( \hat{g}(s, \varepsilon) \) with \( \varepsilon + i\eta \) and \( \varepsilon - i\eta \), where \( \eta \) is an infinitesimal positive number.

**JOSEPHSON JUNCTION WITH NONMAGNETIC BARRIER**

We now consider a Josephson junction consisting of two Ising superconductors separated by a nonmagnetic barrier. The corresponding ferromagnetic layer induces an in-plane exchange field in each Ising superconductor through magnetic proximity effect. We assume that the exchange fields applied to the superconductors have the same magnitude. The superconducting phase difference is denoted as \( \phi = \phi_{s1} - \phi_{s2} \) and the relative angle between the two in-plane exchange fields as \( \theta \). The transparency of the junction is denoted as \( D \). From the quantum circuit theory [16], the expressions of the charge current \( I_c \) and the z-polarized spin current \( I_s \) are, respectively, given by
\[ I_c = \frac{G_0}{8e} \int_{-\infty}^{\infty} d\varepsilon \, \text{tr} (\tau_3 \sigma_0 \hat{I}^k), \]  
(46)
\[ I_s = \frac{hG_0}{16e^2} \int_{-\infty}^{\infty} d\varepsilon \, \text{tr} (\tau_3 \sigma_2 \hat{I}^k), \]  
(47)
with \( G_0 = 2e^2/h \) the conductance quantum. In above, \( \hat{I}^k \) is the Keldysh component of the matrix current given by
\[ \hat{I}^k = 2(\hat{A}^r \hat{X}^k + \hat{A}^i \hat{X}^a) \]  
(48)
with
\[ \hat{A} = 2D [\hat{g}_{s2}, \hat{g}_{s1}], \quad \hat{X} = \left[ 4 - D \left( 2 - \{ \hat{g}_{s2}, \hat{g}_{s1} \} \right) \right]^{-1}. \]  
(49)
We have included an additional factor of 2 in Eq. (48) to take the valley degree of freedom into account. Here, \( g_\alpha \) with \( \alpha = s1, s2 \) is in Keldysh space and has the structure
\[ g_\alpha = \left( \begin{array}{c} \hat{g}_\alpha^r \\ \hat{g}_\alpha^i \end{array} \right), \]  
(50)
where \( \hat{g}_\alpha^r \) and \( \hat{g}_\alpha^i \) are, respectively, the retarded and advanced counterparts of \( g_\alpha(s, \varepsilon) \). The Keldysh component \( \hat{g}_\alpha^r \) is obtained via the relation \( \hat{g}_\alpha^r = (\hat{g}_\alpha^r - \hat{g}_\alpha^i) \tanh [\varepsilon/(2k_B T)] \), where \( k_B \) is the Boltzmann constant and \( T \) is the temperature.

In the tunneling limit, i.e., \( D \ll 1 \), one has
\[ \hat{I}^k = 2D \text{Re} \left( [\hat{g}_{s2}, \hat{g}_{s1}^r] \right) \tanh [\varepsilon/(2k_B T)], \]  
(51)
and the expressions of the charge current \( I_c \) and the z-polarized spin current \( I_s \) are, respectively, given by
\[ I_c = \frac{G_t}{8e} \int_{-\infty}^{\infty} d\varepsilon \, \text{tr} (\tau_3 \sigma_0 \hat{I}), \quad I_s = \frac{hG_t}{16e^2} \int_{-\infty}^{\infty} d\varepsilon \, \text{tr} (\tau_3 \sigma_2 \hat{I}), \]  
(52)
with the tunnel conductance \( G_t = 2DG_0 \) and
\[ \hat{I} = \text{Re} \left( [\hat{g}_{s2}^r, \hat{g}_{s1}^r] \right) \tanh [\varepsilon/(2k_B T)]. \]  
(53)
By expressing the retarded Green’s function as
\[ g_\alpha^r(s, \varepsilon) = \left[ \begin{array}{c} g_{0,0} \sigma_0 + g_\alpha \cdot \sigma^r + (f_{0,0} \sigma_0 + f_\alpha \cdot \sigma) \sigma_0 \sigma_y \\ g_{0,0} \sigma_0 + g_\alpha \cdot \sigma^* \sigma_y \end{array} \right], \]  
(54)
where the superscripts “*” in all the components have been dropped out, we have
\[ \text{tr} (\tau_3 \sigma_0 [\hat{g}_{s2}^r, \hat{g}_{s1}^r]) = 4 \left( f_{0,0} \bar{f}_{0,0} - f_{0,0} \bar{f}_{0,0} - f_{0,1} \cdot \bar{f}_{0,1} + f_{0,1} \cdot \bar{f}_{0,1} \right), \]  
(55)
By considering the case where the ISOC strengths of both superconductors are the same at the same valley, we have in the clean limit the expressions:

\[
\text{Re}[\text{tr}(\tau_3 \sigma_3 [\hat{g}^{+}_{s2}, \hat{g}^{-}_{s1}])]
= -8 \text{Im}[f_0 f_\phi + a^2 (\varepsilon^2 + \beta_{so}^2) J^2 \cos \theta] \sin \phi \quad (57)
\]

and

\[
\text{Re}[\text{tr}(\tau_3 \sigma_3 [\hat{g}^{+}_{s2}, \hat{g}^{-}_{s1}])]
= -8 \text{Im}[-g^2_+ + a^2 (\varepsilon^2 + \beta_{so}^2) J^2 \cos \phi] \sin \theta. \quad (58)
\]

Here, we have used the fact \( g_+^2 \) = \( g_{s2}^2 \). Then we arrive at

\[
I_c = (I_{c0} + I_1 \cos \phi) \sin \phi, \quad (59)
\]

\[
I_s = (h/2e)(I_{s0} + I_1 \cos \phi) \sin \theta, \quad (60)
\]

as presented in Eqs. (8) and (9) in the main text, where the critical currents read

\[
I_{c0} = -\frac{G_t}{e} \int_{-\infty}^{\infty} d\varepsilon \text{Im}(f_0 f_\phi) \tanh[\varepsilon/(2k_B T)], \quad (61)
\]

\[
I_{s0} = \frac{G_t}{e} \int_{-\infty}^{\infty} d\varepsilon \text{Im}(g^2_+) \tanh[\varepsilon/(2k_B T)], \quad (62)
\]

\[
I_1 = -\frac{G_t}{e} \int_{-\infty}^{\infty} d\varepsilon b(\varepsilon^2 + \beta_{so}^2) \tanh[\varepsilon/(2k_B T)], \quad (63)
\]

with \( b = \text{Im}(a^2) J^2 \). It can be seen that the term of \( I_1 \) originates from the triplet correlation functions. The expressions of \( x \) and \( y \)-polarized spin currents can be obtained by replacing \( \tau_3 \sigma_x \) with \( \tau_3 \sigma_y \) and \( \tau_0 \sigma_y \), respectively. One can find that both the \( x \) and \( y \)-polarized spin currents vanish by taking the contributions from the two different valleys into account.

In the tunneling limit, we define the ratio of the charge currents between the parallel and the antiparallel configurations as

\[
\mathcal{R} = \frac{I_c(\theta = 0)}{I_c(\theta = \pi)} = \frac{I_{c0} + I_1}{I_{c0} - I_1}. \quad (64)
\]

From Eq. (59), one has

\[
\mathcal{R} = \frac{\int_{-\infty}^{\infty} d\varepsilon \text{Im}[f_0 f_\phi + a^2 J^2 (\varepsilon^2 + \beta_{so}^2)] \tanh[\varepsilon/(2k_B T)]}{\int_{-\infty}^{\infty} d\varepsilon \text{Im}[f_0 f_\phi - a^2 J^2 (\varepsilon^2 + \beta_{so}^2)] \tanh[\varepsilon/(2k_B T)]}. \quad (65)
\]

Since \( \Delta \) is small compared to \( J \) and \( \beta_{so} \), Eq. (41) can be approximated as \( u \approx \varepsilon^2 - \beta_{so}^2 - J^2 - \Delta^2 \) so that the term \( f_0 \) in Eq. (59) becomes

\[
f_0 \approx -a(\varepsilon^2 - \beta_{so}^2 - \Delta^2). \quad (66)
\]

In the tunneling limit, the Andreev bound states are localized around \( \varepsilon = \pm \Delta \) so that

\[
\mathcal{R} \approx \frac{\beta^4_{so} + J^2 (\Delta^2 + \beta_{so}^2)}{\beta^4_{so} - J^2 (\Delta^2 + \beta_{so}^2)}, \quad (67)
\]

where we have ignored the contribution from the mirage gaps [28]. With \( \Delta \ll \beta_{so} \), we arrive at

\[
\mathcal{R} \approx (\beta^4_{so} + J^2)/(\beta^4_{so} - J^2), \quad (68)
\]

as presented in Eq. (11) in the main text.

When the ISOC fields of two superconductors are opposite in sign at the same valley, \( I_1 \) in Eqs. (59) and (60) become

\[
I_1 = -\frac{G_t}{e} \int_{-\infty}^{\infty} d\varepsilon b(\varepsilon^2 - \beta_{so}^2) \tanh[\varepsilon/(2k_B T)]. \quad (69)
\]

The charge currents for this case are shown in Fig. 4(b). For comparison, we also show the case that the ISOC fields have the same sign at both sides of the junction in Fig. 4(a). Being different from Fig. 4(a), the critical charge current in Fig. 4(b) is maximal (minimal) for the configuration of antiparallel (parallel) exchange fields.

**THE CONTRIBUTION FROM MIRAGE GAPS**

We define the spectral charge current \( j_c(\varepsilon) \) through \( I_c = \int_{-\infty}^{\infty} d\varepsilon j_c(\varepsilon) \). In Fig. 5(a), the spectral charge current \( j_c(\varepsilon) \) is shown against energy \( \varepsilon \) and Josephson phase \( \phi \) at \( \theta = 0 \). We can see the appearance of Andreev bound states at energies larger than the superconducting gap. This is due to the presence of the mirage gaps where the finite-energy pairings occur [28]. In Fig. 5(b), we compare the charge currents between the scenarios with and without the mirage gaps being considered. It can be seen that the influence of the mirage gaps on the charge current is small.
FIG. 5. (a) Spectral charge current \( j_c(\epsilon) \), in units of \( G_{\text{f}}/e \), against energy \( \epsilon \) and Josephson phase \( \phi \) at \( \theta = 0 \). (b) Comparison of the charge currents between the cases with (solid lines) and without (dashed lines) the mirage gaps being considered. The solid and dashed lines coincide with each other at \( \theta = 0 \). For the case without considering the contribution from the mirage gaps, the integration region to get the charge current is \( \epsilon \in [-4\Delta_0, 4\Delta_0] \). Here, \( \beta_{so} = 5\Delta_0 \), \( J = 4\Delta_0 \), and \( T = 0.1T_0 \).

FIG. 6. (a) Switch ratio \( R \) versus exchange-field magnitude at different intervalley scattering rates \( \Gamma \) in the tunneling limit. The gray line is the approximation in Eq. (68). The inset shows the charge currents for the parallel configuration. (d) Spin current at \( \phi = 0 \) versus exchange-field magnitude. Here, \( \beta_{so} = 5\Delta_0 \) and \( T = 0.1T_0 \).

**DISORDER EFFECT**

In the presence of nonmagnetic impurities, Eq. (22) becomes

\[
\varepsilon \tau_3 \sigma_0 - \hat{\Delta} - \hat{\nu} - \hat{\Sigma}(\varepsilon), \hat{g} = 0,
\]

with \( \hat{\Sigma}(\varepsilon) = -i\Gamma \langle \hat{g}(\varepsilon, \varepsilon) \rangle \), where \( \Gamma \) is the impurity-scattering rate and \( \langle \cdots \rangle \) denotes averaging over all Fermi-momentum directions.

We consider the effects of nonmagnetic intervalley scattering in Fig. 6. It can be seen that the intervalley scattering reduces the maximal switch ratio and also the spin-current magnitude.

**JOSEPHSON JUNCTION WITH FERROMAGNETIC BARRIER**

We turn to the scenario with a ferromagnetic barrier which is characterized by the magnetization vector \( \mathbf{m} \) with \( |\mathbf{m}| = 1 \) and the spin polarization \( \mathcal{P} \). In the tunneling limit, \( \hat{I} \) in Eq. (52) is

\[
\hat{I} = \frac{1}{2} \text{Re} \left[ (1 + \sqrt{1 - \mathcal{P}^2}) \hat{g}^r_{s2} + \mathcal{P} \{ \hat{k}, \hat{g}^r_{s2} \} + (1 - \sqrt{1 - \mathcal{P}^2}) \hat{\kappa}_{s2} \hat{k}, \hat{g}^r_{s1} \} \right],
\]

with the spin matrix

\[
\hat{\kappa} = \text{diag}(\mathbf{m} \cdot \sigma, \mathbf{m} \cdot \sigma^*).
\]

Using the relations

\[
\{ \hat{k}, \hat{g}^r \} = 2 \left[ (\mathbf{m} \cdot \mathbf{g})\sigma_0 + g_0 \mathbf{m} \cdot \sigma \quad i(m \times f) \cdot \sigma \sigma_y \right] -i(\mathbf{m} \times \mathbf{f}) \cdot \sigma^* i\sigma_y \quad (m \cdot \mathbf{g})\sigma_0 + \bar{g}_0 \mathbf{m} \cdot \sigma^* \quad (73)
\]

and

\[
\hat{\kappa} \hat{g}^r \hat{\kappa} = \left[ -[\hat{f}_0 \sigma_0 + 2(m \cdot f) \mathbf{m} \cdot \sigma^* - \hat{f} \cdot \sigma^*] i\sigma_y \right],
\]

separately, we obtain

\[
\text{tr} \left( \tau_3 \sigma_0 \left[ \{ \hat{k}, \hat{g}^r_{s2} \}, \hat{g}^r_{s1} \} \right] \right) = 4 \left[ f_{0,s2} f_{0,s1} - f_{0,s1} f_{0,s2} - f_{s2} f_{s1} + f_{s2} f_{s1} + 2(m \cdot f_{s2}) (m \cdot f_{s1}) \right].
\]

Below, we consider the case where the magnetization of the barrier points out of plane with \( \mathbf{m} = (0, 0, 1) \). In the clean
limit, we have
\[
\text{Re}[\text{tr}(\tau_3\sigma_0[\{\hat{\kappa}, \hat{g}_{s2}\}, \hat{g}_{s1}])]
= -16 \text{Im}(a^2)(\varepsilon^2 + \beta_{\text{so}}^2) J^2 \sin \theta \cos \phi + \cdots, \quad (77)
\]
and
\[
\text{Re}[\text{tr}(\tau_3\sigma_0[\hat{\kappa}\hat{g}_{s2}\hat{\kappa}, \hat{g}_{s1}])]
= 8 \text{Im}[f_0 \bar{f}_0 - a^2(\varepsilon^2 + \beta_{\text{so}}^2) J^2 \cos \theta \sin \phi]. \quad (78)
\]
The terms which are odd in valley index in Eq. (77) are not shown. They do not contribute to the supercurrents since there is a sum over both valleys. Then the charge current is obtained as
\[
I_c = (\sqrt{1 - P^2} I_{c0} + I_1 \cos \theta) \sin \phi + P I_1 \sin \theta \cos \phi, \quad (79)
\]
as provided in Eq. (12) in the main text.

For the spin current in superconductor $s_1$, we have
\[
\text{tr}(\tau_3\sigma_z[\{\hat{\kappa}, \hat{g}_{s2}\}, \hat{g}_{s1}])
= 4[i g_{0,s2} m \times g_{-,s1} + 2(m \times f_{s2}) \times f_{s1} - 2(m \times f_{s2}) \times \bar{f}_{s1}]_z
= 8[(m \times \bar{f}_{s2}) \times f_{s1} - (m \times f_{s2}) \times \bar{f}_{s1}]_z, \quad (80)
\]
and
\[
\text{tr}(\tau_3\sigma_z[\hat{g}_{s2}\hat{g}_{s1}])
= 4i[-2(m \cdot f_{s2})(m \times f_{s1}) - 2(m \cdot f_{s2})(m \times f_{s1})
+ 2g_{+,s2} \times g_{+,s1} + f_{s2} \times f_{s1} - f_{s1} \times f_{s2}]_z
= \text{tr}(\tau_3\sigma_z[\hat{g}_{s2}, \hat{g}_{s1}]), \quad (81)
\]
where the second equality in each of the two equations above are obtained using $m = (0, 0, 1)$. Since
\[
\text{Re}[\text{tr}(\tau_3\sigma_z[\{\hat{\kappa}, \hat{g}_{s2}\}, \hat{g}_{s1}])]
= -16 \text{Im}(a^2)(\varepsilon^2 + \beta_{\text{so}}^2) J^2 \cos \theta \sin \phi + \cdots, \quad (82)
\]
where the terms which are odd in valley index are not shown, the spin current is expressed as
\[
I_s = (\hbar/2e)[(I_{s0} + I_1 \cos \phi) \sin \theta + P I_1 \sin \theta \cos \phi], \quad (83)
\]
as provided in Eq. (12) in the main text.