$C^k$-SMOOTH APPROXIMATIONS OF LUR NORMS

PETR HÁJEK AND ANTONÍN PROCHÁZKA

Abstract. Let $X$ be a WCG Banach space admitting a $C^k$-smooth norm where $k \in \mathbb{N} \cup \{\infty\}$. Then $X$ admits an equivalent norm which is simultaneously, $C^1$-smooth, LUR, and the limit of a sequence of $C^k$-smooth norms. If $X = C([0, \alpha])$, where $\alpha$ is any ordinal, then the same conclusion holds true with $k = \infty$.

1. Introduction

The famous averaging result of Asplund [1] states that if a Banach space $X$ admits a norm which is locally uniformly rotund (LUR) as well as another norm whose dual norm is LUR, then $X$ admits a third norm which is simultaneously $C^1$-smooth and LUR. On the other hand, denoting by $\omega_1$ the first uncountable ordinal, the space $C([0, \omega_1])$ admits an equivalent LUR norm according to Troyanski [18], and, by Talagrand [17], $C([0, \omega_1])$ admits an equivalent $C^1$-smooth norm but does not admit any norm whose dual norm would be LUR or even strictly convex. Therefore the Asplund averaging cannot be applied. The natural question whether $C([0, \omega_1])$ admits an equivalent $C^1$-smooth and LUR norm has been raised e.g. in [19]. Here we answer the question in the affirmative.

Theorem 1.1. Let $\alpha$ be an ordinal. Then the space $C([0, \alpha])$ admits an equivalent norm which is $C^1$-smooth, LUR, and the limit (uniform on bounded sets) of a sequence of $C^\infty$-smooth norms.

The existence of a $C^\infty$-smooth norm on spaces $C([0, \alpha])$ has been known since Haydon’s improvement [14] of Talagrand’s renorming mentioned above. The new norm of Theorem 1.1 is in fact as good as one can hope since by [9] (2, Proposition V.1.3), a space admitting an LUR, $C^2$-smooth norm is superreflexive.

Let us comment on the fact that the new norm is a limit of $C^\infty$-smooth norms. It is a general open question whether it is possible to approximate (uniformly on bounded sets) all equivalent norms on a Banach space $X$ by $C^k$-smooth norms, provided that $X$ admits an equivalent $C^k$-smooth norm. Even in the separable case, the answer for $k \geq 2$ is not known in full generality, although the positive results in [3] and [4] are quite strong and apply to most classical Banach spaces. (Added in proof: the separable question has been settled in the affirmative by J. Talponen and the first-named author [13].) In the non-separable setting, no general result is

Received by the editors January 22, 2009 and, in revised form, April 4, 2011, May 3, 2012 and June 19, 2012.

2010 Mathematics Subject Classification. Primary 46B20, 46B03, 46E15.

Key words and phrases. LUR, smoothness, higher order smoothness, renorming.

This work was supported by grants GA CR Grant P201/11/0345, RVO: 67985840, and PHC Barrande 2012 26516YG.
known, with the small exception of [7]. In particular, it is unknown whether the $C^\infty$-smooth or even $C^1$-smooth norms on $C([0, \omega_1])$ are dense.

One of the open problems in [2] addressing these issues is whether on a given WCG Banach space with an equivalent $C^k$-smooth norm, there exists an equivalent LUR norm which is the uniform limit on bounded sets of a sequence of $C^k$-smooth norms. Such a result is of interest for several reasons. It can be used to obtain rather directly the uniform approximations of general continuous operators by $C^k$-smooth ones. Moreover, since LUR norms form a residual set in the metric space of all equivalent norms on an LUR renormable Banach space, a positive answer has been expected. The main result of our paper is the following theorem which implies both Theorem 1.1 and the positive answer to the problem above (see Corollary 3.3).

**Theorem 1.2.** Let $k \in \mathbb{N} \cup \{\infty\}$. Let $(X, \|\cdot\|)$ be a Banach space with a projectional resolution of the identity $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$ such that for each $\omega \leq \alpha < \mu$, the subspace $(P_{\alpha+1} - P_\alpha)X$ admits an equivalent norm which is $C^1$-smooth, LUR, and is the uniform limit on $B(P_{\alpha+1} - P_\alpha)X$ of $C^k$-smooth norms. Assume that $X$ admits an equivalent $C^k$-smooth norm $\|\cdot\|$.

Then $X$ admits an equivalent norm $\| \cdot \|$ which is $C^1$-smooth, LUR, and is the uniform limit on $B_X$ of a sequence of $C^k$-smooth norms.

A similar method is applicable to obtain the same kind of renorming on spaces of continuous functions on trees; see [12].

The paper is organized as follows. In Section 2 we introduce our notation and we present some auxiliary lemmata. We include some of the easy proofs for the reader’s convenience. The consequences of Theorem 1.2 are stated and proved in Section 3. The proof of Theorem 1.2 then occupies Section 4.

2. Preliminaries

Our notation is standard, e.g. as in [6]. The closed unit ball of a Banach space $(X, \|\cdot\|)$ is denoted by $B_X(\|\cdot\|)$, or $B_X$ for short. The open unit ball of $X$ is $B_X^0 = B_X^0(\|\cdot\|)$. Let $k \in \mathbb{N}$. A function $f$ is said to be $C^k$-smooth on an open set $U$ if the derivative $f^{(k)}$ of $f$ of order $k$ exists and is continuous on $U$. A function $f$ is said to be $C^\infty$-smooth on $U$ if it is $C^k$-smooth on $U$ for every $k \in \mathbb{N}$. We say that a norm is $C^k$-smooth ($k \in \mathbb{N} \cup \{\infty\}$) if it is $C^k$-smooth on $X \setminus \{0\}$. A norm $\|\cdot\|$ on a Banach space $X$ is called locally uniformly rotund (LUR) if the following implication is satisfied for all points $x, x_r \in X$, $r \in \mathbb{N}$:

$$\lim_{r} \|x_r - x\| = 0 \text{ whenever } \lim_{r} \left(2 \|x_r\|^2 + 2 \|x\|^2 - \|x_r + x\|^2\right) = 0.$$

We proceed to the definitions and lemmata that will be used in the proof of Theorem 1.2. By $\Gamma$ we denote an index set. If $A$ is a set, then $|A|$ stands for the cardinality of $A$.

**Definition 2.1.** We say that a function $f : \ell_\infty(\Gamma) \to \mathbb{R}$ locally depends on finitely many coordinates (LFC) at $x \in \ell_\infty(\Gamma)$ if there exists a neighborhood $U$ of $x$, a finite $M := \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ and a function $g : \mathbb{R}^{|M|} \to \mathbb{R}$ such that $f(y) = (g \circ \pi_M)(y)$ for each $y \in U$ (here $\pi_M(y) := (y(\gamma_1), \ldots, y(\gamma_n))$ is the natural projection from $\ell_\infty(\Gamma)$ onto $\mathbb{R}^{|M|}$).

Let $A \subset \ell_\infty(\Gamma)$. The function $f$ is called LFC on $A$ if $f$ is LFC at $x$ for each $x \in A$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
**Lemma 2.3.** Let $G$ be an open subset of a Banach space $X$ and let $h : G \to \ell_\infty(\Gamma)$ be a continuous mapping which is coordinatewise $C^k$-smooth, $k \in \mathbb{N} \cup \{\infty\}$. Let $\Omega$ be an open subset of $\ell_\infty(\Gamma)$ and let $f : \Omega \to \mathbb{R}$ be $C^k$-smooth and LFC on $\Omega$. If $h(G) \subset \Omega$, then $f \circ h$ is $C^k$-smooth on $G$.

**Proof.** Let $x \in G$ be fixed. Since $f$ is LFC at $h(x)$, there is a neighborhood $U \subset \Omega$ of $h(x)$, $M := \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ and $g : \mathbb{R}^{|M|} \to \mathbb{R}$ as in Definition 2.1.

The function $g$ is $C^k$-smooth on the open set $\pi_M(U)$ because $f$ is $C^k$-smooth and $g(y) = f((x - \pi_M x) + Ty)$ where $T : \mathbb{R}^{|M|} \to \ell_\infty(\Gamma)$ is defined as $T(y)\gamma_i = y_i$ for $i = 1, \ldots, n$ and $T(y)\gamma_i = 0$ if $\gamma_i \notin M$. As $h$ is continuous, there exists a neighborhood $V \subset G$ of $x$ such that $h(V) \subset U$. Since $h$ is coordinatewise $C^k$-smooth and $M$ is finite, it follows that $\pi_M \circ h$ is $C^k$-smooth from $G$ to $\mathbb{R}^{|M|}$. Finally, we have for each $y \in V$ that $(f \circ h)(y) = (g \circ \pi_M \circ h)(y)$ and the claim follows.

**Lemma 2.4.** Let $\Phi : \ell_\infty(\Gamma) \to \mathbb{R}$ be a function differentiable around a point $x \in \ell_\infty(\Gamma)$ and assume that

a) $\Phi$ is LFC at $x$, 

b) $\Phi'(x) x \neq 0$, 

c) $\Phi(\cdot)$ and $\Phi'(\cdot)$ are continuous at $x$.

Then there is a neighborhood $U$ of $x$ and a unique function $F : U \to \mathbb{R}$ which is continuous at $x$ and satisfies $F(x) = 1$ and $\Phi(\frac{y}{F(y)}) = 1$ for all $y \in U$. Moreover $F$ is LFC at $x$.

**Proof.** The first part of the assertion follows immediately from the Implicit Function Theorem. We will show that $F$ is LFC at $x$. From assumption a) we know that there is a neighborhood $V$ of $x$, $M = \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$, and $g : \mathbb{R}^n \to \mathbb{R}$ such that $\Phi(y) = (g \circ \pi_M)(y)$ for all $y \in V$. Since $g'(\pi_M x)\pi_M x = \Phi(x) x$, it is possible to apply the Implicit Function Theorem to the equation $g\left(\frac{y}{h(y)}\right) = 1$ to get $h : V' \to \mathbb{R}$ (for some neighborhood $V'$ of $\pi_M x$) such that $h(\pi_M x) = 1$ and $h$ is continuous at $\pi_M x$. There is a neighborhood $U' \subset U \cap V$ of $x$ such that $H : U' \to \mathbb{R}$ given by $H(y) := (h \circ \pi_M)(y)$ for $y \in U'$ is well defined. Then $H(x) = 1$ and $H$ is continuous at $x$. Also, $\Phi\left(\frac{y}{H(y)}\right) = g\left(\frac{\pi_M y}{H(\pi_M y)}\right) = 1$. The uniqueness of $F$ implies that $F = H$ on $U'$, so $F$ is LFC at $x$.

The following lemma is a variant of Fact II.2.3(i) in [2].

**Lemma 2.5.** Let $\varphi : X \to \mathbb{R}$ be a convex non-negative function, $x_r, x \in X$, for $r \in \mathbb{N}$. Then the following conditions are equivalent:

(i) $\varphi^2(x_r) + \varphi^2(x) = \varphi^2(x + x_r) \to 0$ as $r \to \infty$,

(ii) $\lim_{r \to \infty} \varphi(x_r) = \lim_{r \to \infty} \varphi\left(\frac{x + x_r}{2}\right) = \varphi(x)$.

If $\varphi$ is homogeneous, the above conditions are also equivalent to

(iii) $2\varphi^2(x_r) + 2\varphi^2(x) - \varphi^2(x + x_r) \to 0$ as $r \to \infty$. 

Proof. Since \( \varphi \) is convex and non-negative, we have

\[
\frac{\varphi^2(x_r) + \varphi^2(x)}{2} - \varphi^2 \left( \frac{x + x_r}{2} \right) \\
\geq \frac{\varphi^2(x_r) + \varphi^2(x)}{2} - \left( \frac{\varphi(x) + \varphi(x_r)}{2} \right)^2
\]
\[
= \left( \frac{\varphi(x) - \varphi(x_r)}{2} \right)^2,
\]
which proves (i) \( \Rightarrow \) (ii). The implication (ii) \( \Rightarrow \) (i) is trivial and so is the equivalence (i) \( \Leftrightarrow \) (iii). \( \square \)

**Definition 2.6.** Let \( A \) be a convex subset of \( X \). A continuous convex function \( f : A \to \mathbb{R} \) is said to be **uniformly convex** if for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( x, y \in A \) the estimate

\[
\frac{f(x) + f(y)}{2} - f \left( \frac{x + y}{2} \right) < \delta
\]
implies that \( \| x - y \| < \varepsilon \). A convex function \( f \) is called **strictly convex** if the equality \( \frac{f(x) + f(y)}{2} = f \left( \frac{x + y}{2} \right) \) implies that \( x = y \).

Note that a uniformly (resp. strictly) convex norm on \( X \) is not a uniformly (resp. strictly) convex function, but its square already is. Clearly, a uniformly convex function is strictly convex. Also, if \( A \) is compact and \( f \) is strictly convex, then \( f \) is uniformly convex.

**Definition 2.7.** We say that a function \( f : \ell_\infty(\Gamma) \to [0, +\infty[ \) is **lattice on \( \Omega \subset \ell_\infty(\Gamma) \)** if \( f(x) \leq f(y) \) whenever \( x, y \in \Omega \) and \( |x(\gamma)| \leq |y(\gamma)| \) for all \( \gamma \in \Gamma \). We say that \( f \) is **lattice** if it is lattice on \( \ell_\infty(\Gamma) \). This property has been called “strongly lattice” in [7].

**Lemma 2.8.** Let \( g : \ell_\infty(\Gamma) \to \mathbb{R} \) be a continuous non-negative lattice function which is uniformly convex on a convex set \( A \subset \ell_\infty(\Gamma) \). Assume that \( y, y_r, u_r \in A \) and \( 0 \leq u_r \leq \frac{y + y_r}{2} \). If

\[
g(y) + g(y_r) - g(u_r) \to 0 \quad \text{as} \quad r \to \infty,
\]
then \( \lim_{r \to \infty} y_r = \lim_{r \to \infty} u_r = y \).

Proof. Since \( g \) is lattice it is immediate that \( \frac{g(y) + g(y_r)}{2} - g \left( \frac{y + y_r}{2} \right) \to 0 \). As \( g \) is uniformly convex, it follows that \( y_r \to y \), and so \( g(u_r) \to g(y) \). We have that \( u_r \leq \frac{y + y_r}{2} \leq \frac{y + y_r}{2} \), and so from the lattice property of \( g \) it follows that \( g \left( \frac{y + y_r}{2} \right) \to g(y) \). Using the uniform convexity of \( g \) we get \( \| u_r - \frac{y + y_r}{2} \| \to 0 \), which yields \( u_r \to y \). \( \square \)

**Lemma 2.9.** Let \( I \subset \mathbb{R} \) be an open interval and let \( f, g : I \to [0, +\infty) \) be twice differentiable convex functions. We define \( F : I^2 \to [0, +\infty) \) as \( F(x, y) := f(x)g(y) \) for \( x, y \in I \). The function \( F \) is convex on \( I^2 \) if

\[
(f'(x))^2(g'(y))^2 \leq f''(x)f(x)g''(y)g(y)
\]
for all \( (x, y) \in I^2 \). The function \( F \) will be strictly convex provided the above inequality is strict and \( f, g \) are supposed to be strictly convex.
Proof. If $f$ or $g$ is constant on $I$ the claim is trivially true, so we assume that neither $f$ nor $g$ is constant. The function $F$ is convex if and only if $F|_J$ is convex for every linear segment $J \subset F^2$. Let $J$ be any such segment. If it is vertical, then $F|_J$ is convex as $g$ is convex. Otherwise we may assume, without loss of generality, that $J$ is parametrized as $J(t) = (t, at + b), t \in [\alpha, \beta]$, for some fixed $a, b \in \mathbb{R}$ and $-\infty < \alpha < \beta < \infty$. Fix $t \in [\alpha, \beta]$ and denote $x = t$ and $y = at + b$. We have

$$(F \circ J)''(t) = f(x)g''(y)a^2 + 2f'(x)g'(y)a + f''(x)g(y).$$

If $g''(y) = 0$, then $g'(y) = 0$ by (1) because $f$ is not constant on $I$. Similarly, if $f(x) = 0$, then $f'(x) = 0$ by (1) because $g$ is not constant on $I$. In both cases $(F \circ J)''(t) = f''(x)g(y) \geq 0$. If $f(x)g''(y) \neq 0$, then $(F \circ J)''(t) \geq 0$ since the discriminant $(2f''(x)g'(y))^2 - 4f(x)g''(y)f''(x)g(y)$ of the quadratic term above is non-positive by our assumption (1).

The proof of the strict convexity of $f$ is an obvious modification of the one just presented.

\[\blacksquare\]

Lemma 2.10. Let $f : \ell_\infty(\Gamma) \to \mathbb{R}$ be convex and lattice on $\{x \in \ell_\infty(\Gamma) : x(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma\}$. Let $g : X \to \ell_\infty(\Gamma)$ be coordinatewise convex and coordinatewise non-negative. Then $f \circ g : X \to \mathbb{R}$ is convex.

The easy proof is left to the reader.

Lemma 2.11. Define $[\cdot] : \ell_\infty(\Gamma) \to \mathbb{R}$ by $[x] = \inf \{t ; \gamma ; |x(\gamma)| > t\}$ is finite}. Then $[\cdot]$ is a 1-Lipschitz, lattice seminorm on $(\ell_\infty(\Gamma), \|\cdot\|_\infty)$.

Proof. The direct proof is an easy exercise. Another proof comes immediately from the equality $[x] = \|q(x)\|_{\ell_\infty/c_0}$, where $q : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)/c_0(\Gamma)$ is the quotient map and $\|\cdot\|_{\ell_\infty/c_0}$ is the canonical norm on the quotient $\ell_\infty(\Gamma)/c_0(\Gamma)$.

The lattice property of $[\cdot]$ follows directly from the definition.

\[\blacksquare\]

Definition 2.12. Let $(X, \|\cdot\|)$ be a Banach space and let $\mu$ be the smallest ordinal such that $|\mu| = \text{dens}(X)$. A system $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$ of linear projections from $X$ into $X$ is called a projectional resolution of the identity (PRI) provided that, for every $\alpha \in [\omega, \mu)$, the following conditions hold true:

(a) $\|P_\alpha\| = 1$,
(b) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $\omega \leq \alpha \leq \beta \leq \mu$,
(c) $\text{dens}(P_\alpha X) \leq |\alpha|$,
(d) $\bigcup \{P_{\beta+1}X : \beta < \alpha\}$ is norm-dense in $P_\alpha X$,
(e) $P_\mu = \text{id}_X$.

If $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$ is a PRI on a Banach space $(X, \|\cdot\|)$, we use the following notation: $\Lambda := \{0\} \cup [\omega, \mu)$, $Q_\gamma := P_{\gamma+1} - P_\gamma$ for all $\gamma \in [\omega, \mu)$, while $Q_0 := P_\omega$.

Lemma 2.13. Let $(X, \|\cdot\|)$ be a Banach space with a PRI $\{P_\alpha\}_{\omega \leq \alpha \leq \mu}$. Then for each $x \in X$ and $\varepsilon > 0$ there exists a finite set $A \subset \Lambda$ such that

$$\|x - \sum_{\gamma \in A} Q_\gamma x\| < \varepsilon,$$

and $Q_\gamma x \neq 0$ for $\gamma \in A$. 
Proof. Let \( x \in X \) be fixed. We will show the following claim by a transfinite induction: for every \( \varepsilon > 0 \) and \( \alpha \in [\omega, \mu] \) there exists a finite set \( A_\varepsilon^\alpha \subset \Lambda \) such that
\[
\left\| P_\alpha x - \sum_{\gamma \in A_\varepsilon^\alpha} Q_\gamma x \right\| < \varepsilon.
\]
If \( \alpha = \omega \), then put \( A_\varepsilon^\omega := \{0\} \) for all \( \varepsilon > 0 \). If \( \alpha = \beta + 1 \) for some ordinal \( \beta \) and we have already found \( A_\varepsilon^\beta \), then put \( A_\varepsilon^\alpha := A_\varepsilon^\beta \cup \{\beta\} \) for all \( \varepsilon > 0 \). Finally, if \( \alpha \) is a limit ordinal, we will use the continuity of the mapping \( \gamma \mapsto P_\gamma x \) at \( \alpha \) [2, Lemma VI.1.2] to find \( \beta < \alpha \) such that \( \|P_\beta x - P_\alpha x\| < \varepsilon/2 \). Thus it is enough to set \( A_\varepsilon^\alpha := A_\varepsilon^{\beta+1} \).

Now, we just set \( A = A_\varepsilon^\mu \setminus \{\gamma \in \Lambda : Q_\gamma x = 0\} \).

\[ \square \]

3. The corollaries of Theorem 1.2

The first corollary of Theorem 1.2 is the already mentioned Theorem 1.1.

Proof of Theorem 1.1. By a result of Talagrand [17], resp. its strengthening by Haydon [14], the space \( C([0, \alpha]) \) admits an equivalent \( C^1 \)-smooth norm, resp. an equivalent \( C^{\infty} \)-smooth norm. On the other hand, when \( \alpha \) is an initial ordinal there is a natural PRI on \( C([0, \alpha]) \) defined as
\[
(P_\gamma x)(\beta) = \begin{cases} 
  x(\beta) & \text{if } \beta \leq \gamma, \\
  x(\gamma) & \text{if } \beta > \gamma.
\end{cases}
\]

Now \( Q_0 X = P_\omega X \) is isomorphic to \( c_0(\mathbb{N}) \), so it admits an equivalent norm which is \( C^1 \)-smooth, LUR, and is the uniform limit on \( B_{Q_0 X} \) of a sequence of \( C^{\infty} \)-smooth norms (see [2, Theorem V.1.5]). Finally, the space \( Q_\gamma X = (P_{\gamma+1} - P_\gamma) X \) is one-dimensional for each \( \gamma \in [\omega, \alpha] \), so Theorem 1.2 yields the conclusion.

When \( \alpha \) is not an initial ordinal we just take an initial ordinal \( \beta \) such that \( |\beta| > |\alpha| \). Now \( C([0, \alpha]) \) inherits the desired norm from \( C([0, \beta]) \).

\[ \square \]

Remark 3.1. Note that the new norm \( \| \cdot \| \) on \( C([0, \omega_1]) \) is not lattice since, by [7], there exists no lattice and Gâteaux smooth norm on the subspace \( C_0([0, \omega_1]) \) of \( C([0, \omega_1]) \).

Clearly, \( \| \cdot \| \) is not LFC, as it is LUR. Moreover, its derivative cannot be locally uniformly continuous (and in particular \( \| \cdot \| \) cannot be \( C^2 \)-smooth), since then the restriction of \( \| \cdot \| \) to any subspace of \( C([0, \omega_1]) \) isomorphic to \( c_0(\omega_1) \) would be strictly convex and \( C^1 \)-smooth with locally uniformly continuous derivative which is impossible according to [10].

Theorem 3.2. Let \( k \in \mathbb{N} \cup \{\infty\} \). Let \( P \) be a class of Banach spaces such that every \( X \in P \)

- admits a PRI \( \{P_\alpha\}_{\omega \leq \alpha \leq \mu} \) such that \( (P_{\alpha+1} - P_\alpha) X \in \mathcal{P} \) for every \( \omega \leq \alpha < \mu \), and
- admits an equivalent \( C^k \)-smooth norm.

Then each \( X \in \mathcal{P} \) admits an equivalent norm which is LUR, \( C^1 \)-smooth, and is the uniform limit on \( B_X \) of a sequence of \( C^k \)-smooth norms.

Proof. We will carry out induction on the density of \( X \). Let \( X \in \mathcal{P} \) be separable. Then we get the result from the theorem of McLaughlin, Poliquin, Vanderwerff and Zizler [15]; see [2, Theorem V.1.7].
Now, let us assume that \( \aleph \) is an uncountable cardinal and that the conclusion of our theorem was proved for every \( Y \in \mathcal{P} \) with \( \text{dens}(Y) < \aleph \). Let \( X \in \mathcal{P} \) and \( \text{dens}(X) = \aleph \) and let \( \mu \) be the smallest ordinal of cardinality \( \aleph \). Let \( \{ P_\alpha \}_{\omega \leq \alpha \leq \mu} \) be a PRI on \( X \) such that \( Q_\alpha X \in \mathcal{P} \) for each \( \alpha \in \Lambda \). Then \( \text{dens}(Q_\alpha X) \leq |\alpha + 1| = |\alpha| < \aleph \). Thus the inductive hypothesis enables us to use Theorem 1.2. \( \square \)

The above theorem has immediate corollaries for each \( \mathcal{P} \)-class (see [11] for this notion). The following corollary solves in the affirmative Problem 8.8 (s) in [8] (see also Problem VIII.4 in [2]).

**Corollary 3.3.** Let \( X \) admit a \( C^k \)-smooth norm for some \( k \in \mathbb{N} \cup \infty \). If \( X \) is weakly Lindelöf determined (WLD) or \( C(K) \) where \( K \) is a Valdivia compact, then \( X \) admits an equivalent norm which is \( C^1 \)-smooth, LUR, and is the uniform limit on \( B_X \) of a sequence of \( C^k \)-smooth norms.

**Proof.** We put \( \mathcal{P} = \{ X : X \text{ is WLD and admits } C^k \text{-smooth norm} \} \); then \( \mathcal{P} \) satisfies the assumptions of Theorem 3.2 (see [11, Chapter 5]). Similarly for \( C(K) \) with \( K \) Valdivia compact. For the definition of the Valdivia compacta and the fact that the spaces of continuous functions on them form a \( \mathcal{P} \)-class, see [2, Section VI.7]. \( \square \)

### 4. Proof of Theorem 1.2

The desired norm \( \| \cdot \| \) on the space \( X \) will be constructed with the help of a sequence of convex functions on \( X \). Each of these functions factors through some \( \ell_\infty(\Gamma) \). We start by describing the component which goes from \( \ell_\infty(\Gamma) \) to \([0, \infty)\).

#### 4.1. Norms on \( \ell_\infty(\Gamma) \) that are LUR for large coordinates.

Let \( \{ \phi_\eta \}_{0 < \eta < 1} \) be a system of functions satisfying

1. \( \phi_\eta : [0, +\infty) \to [0, +\infty) \), for \( 0 < \eta < 1 \), is a convex \( C^\infty \)-smooth function such that \( \phi_\eta \) is strictly convex on \([1 - \eta, +\infty)\), that \( \phi_\eta([0, 1 - \eta]) = \{0\} \) and that \( \phi_\eta(1) = 1 \).
2. \( \phi_\eta \) is increasing and strictly convex on \([0, +\infty)\), e.g. \( \phi_\eta(0) = 0 \) if \( x \leq 0 \) and

\[
\phi(x) = \int_0^x \exp(-\frac{1}{t}) \, dt / \int_0^1 \exp(-\frac{1}{t}) \, dt \quad \text{ if } x > 0.
\]

We define \( \phi_\eta(x) := \phi(\frac{x - (1 - \eta)}{\eta}) \) for all \( \eta \in (0, 1) \) and \( x \in [0, +\infty) \). Now the system \( \{ \phi_\eta \} \) satisfies (ii) since \( \eta \mapsto x - (1 - \eta) \) is increasing for every \( x \in [0, 1) \), while the validity of (i) follows from properties of \( \phi \).

We define an (l.s.c.) function \( \Phi_\eta : \ell_\infty(\Gamma) \to [0, +\infty] \) by

\[
\Phi_\eta(y) = \sum_{\gamma \in F} \phi_\eta(|y(\gamma)|) := \sup \left\{ \sum_{\gamma \in F} \phi_\eta(|y(\gamma)|) : F \text{ is a finite subset of } \Gamma \right\}.
\]

Let us define \( Z_\eta : \ell_\infty(\Gamma) \to [0, +\infty) \) as the Minkowski functional of the set \( M_\eta = \{ y \in \ell_\infty(\Gamma) : \Phi_\eta(y) \leq 1 \} \). This formula as well as Lemma 4.3 are inspired by [2, Theorem V.1.5] (which is an adaptation of [16]).
Lemma 4.1. Let $0 < \eta < 1$ be fixed. Then $Z_\eta$ is a lattice norm such that $(1 - \eta)Z_\eta(\cdot) \leq \|\cdot\|_\infty \leq Z_\eta(\cdot)$ and $Z_\eta$ is LFC and $C^\infty$-smooth on the set

$$A_\eta(\Gamma) := \{y \in \ell_\infty(\Gamma) : \|y\|_\infty < (1 - \eta) \|y\|_\infty\}.$$  

Moreover $(1 - \eta)Z_\eta(y) < \|y\|_\infty$ for all $y \in A_\eta(\Gamma)$.

Proof. The definition of $\Phi_\eta$ yields that $(1 - \eta)B_{\ell_\infty(\Gamma)} \subset M_\eta \subset B_{\ell_\infty(\Gamma)}$.

Let $A'_\eta(\Gamma) := \{y \in \ell_\infty(\Gamma) : \|y\|_\infty < 1 - \eta\}$. This set is convex and open since $[\cdot]$ is a continuous and convex (seminorm). The function $\Phi_\eta$ is, locally on $A'_\eta(\Gamma)$, the sum of finitely many convex $C^\infty$-smooth LFC functions, thus it is a convex function which is LFC and $C^\infty$-smooth on $A'_\eta(\Gamma)$.

Let us fix $y_0 \in A'_\eta(\Gamma)$ such that $\Phi_\eta(y_0) = 1$. Then, since $\phi_\eta$ is increasing at the points where it is not zero, we get $Z_\eta(y_0) = 1$ and $\Phi'_\eta(y_0)y_0 > 0$. As is usual, we consider the equation $\Phi_\eta\left(\frac{y}{Z_\eta(y)}\right) = 1$. By the Implicit Function Theorem (see e.g. [5]), this equation locally redefines $Z_\eta$ and proves that $Z_\eta$ is $C^\infty$-smooth on some neighborhood $U$ of $y_0$ since $\Phi_\eta$ is. Moreover, Lemma 2.4 guarantees that $Z_\eta$ is LFC at $y_0$.

To prove that $Z_\eta$ is LFC and $C^\infty$-smooth on $A_\eta(\Gamma)$ it is enough to show that for each $y \in A_\eta(\Gamma)$ there is $\lambda > 0$ such that $\lambda y \in A'_\eta(\Gamma)$ and $\Phi_\eta(\lambda y) = 1$, and then use the homogeneity of $Z_\eta$. Let $y \in A_\eta(\Gamma)$. Then $\left[\frac{y}{\|y\|_\infty}\right] < 1 - \eta$ and, since $A'_\eta(\Gamma)$ is convex, it follows that $[0, \frac{y}{\|y\|_\infty}] \subset A'_\eta(\Gamma)$. We further have that $+\infty \geq \Phi_\eta\left(\frac{y}{\|y\|_\infty}\right) \geq 1$, $\Phi_\eta(0 \cdot y) = 0$ and the mapping $\lambda \mapsto \Phi_\eta(\lambda y)$ is continuous for $\lambda \in [0, \frac{1}{\|y\|_\infty}]$. Hence there must exist $\lambda \in (0, \frac{1}{\|y\|_\infty}]$ such that $\lambda y \in A'_\eta(\Gamma)$ and $\Phi_\eta(\lambda y) = 1$.

We continue by showing that $Z_\eta$ is lattice. First observe that $\Phi_\eta$ is lattice as $\phi_\eta$ is non-decreasing. Also observe that $\left\{y \in \ell_\infty(\Gamma) : Z_\eta(y) = 1\right\} = \left\{y \in \ell_\infty(\Gamma) : \Phi_\eta(y) \vee \frac{\|y\|}{1 - \eta} = 1\right\}.$

Let $|y| \leq |z|$ and $Z_\eta(y) = 1$. Then $[\|y\|] = 1 - \eta$ or $\Phi_\eta(y) = 1)$. Since both functions $[\cdot]$ and $\Phi_\eta$ are lattice, we infer $([z] \geq 1 - \eta$ or $\Phi_\eta(z) \geq 1)$, which in turn implies that $Z_\eta(z) \geq 1$. For a general $y$ we employ the homogeneity of $Z_\eta$, so $Z_\eta$ is lattice.

Finally, if $y \in A_\eta(\Gamma)$, then the above reasoning implies that $\Phi_\eta\left(\frac{y}{Z_\eta(y)}\right) = 1$. This is possible only if there is some $\gamma \in \Gamma$ such that $\frac{y(\gamma)}{Z_\eta(y)} > 1 - \eta$, and the "moreover" claim follows. \hfill \Box

Lemma 4.2. Let $0 < \eta_1 \leq \eta_2 < 1$. Then $Z_{\eta_1}(y) \leq Z_{\eta_2}(y)$ for every $y \in A_{\eta_2}(\Gamma)$.

Proof. First of all, if $y \in A_{\eta_2}(\Gamma)$, then $y \in A_{\eta_1}(\Gamma)$. So the equivalence $Z_{\eta_1}(\lambda y) = 1 \Leftrightarrow \Phi_{\eta_1}(\lambda y) = 1$ holds for both $i = 1, 2$. Let us assume that $Z_{\eta_1}(\lambda y) = 1$ for some $\lambda > 0$. Then $\|\lambda y\|_{\infty} \leq 1$ and the ordering of functions $\phi_\eta$ yields $1 = \Phi_{\eta_1}(\lambda y) \leq \Phi_{\eta_2}(\lambda y)$, which results in $Z_{\eta_2}(\lambda y) \geq 1$. \hfill \Box

Lemma 4.3. Let $0 < \eta < 1$ be given and let $y_r, y \in A_\eta(\Gamma)$, $r \in \mathbb{N}$, be coordinatewise non-negative and such that

$$2Z_\eta^2(y) + 2Z_\eta^2(y_r) - Z_\eta^2(y + y_r) \to 0 \quad \text{as} \quad r \to \infty.$$
Then

(i) \( \lim_{r \to \infty} y_r(\gamma) = y(\gamma) \) for any \( \gamma \in \Gamma \) such that \( y(\gamma) > Z_\eta(y)(1-\eta) \),

(ii) \( \limsup_{r \to \infty} y_r(\gamma) \leq Z_\eta(y)(1-\eta) \) uniformly on \( \{ \gamma \in \Gamma : y(\gamma) \leq Z_\eta(y)(1-\eta) \} =: \Gamma' \).

\textbf{Proof.} (i) The assumption and Lemma \textbf{2.5} yield

\begin{equation}
Z_\eta(y_r) \to Z_\eta(y) \quad \text{and} \quad Z_\eta\left(\frac{y + y_r}{2}\right) \to Z_\eta(y).
\end{equation}

Let us put \( \tilde{y} := \frac{y}{Z_\eta(y)} \) and \( \tilde{y}_r := \frac{y_r}{Z_\eta(y_r)} \). We get from \textbf{(3)} that

\[ 2Z_\eta^2(\tilde{y}) + 2Z_\eta^2(\tilde{y}_r) - Z_\eta^2(\tilde{y} + \tilde{y}_r) \to 0. \]

Since \( Z_\eta(\tilde{y}) = Z_\eta(\tilde{y}_r) = 1 \), the above implies that

\[ \lambda_r := Z_\eta(\tilde{y} + \tilde{y}_r) \to 2. \]

We deduce from \( y, y_r \in A_\eta(\Gamma) \) that \( \Phi_\eta(\tilde{y}) = 1 = \Phi_\eta(\tilde{y}_r) \) for all \( r \in \mathbb{N} \). Also, \( \Phi_\eta(\lambda_r^{-1}(\tilde{y} + \tilde{y}_r)) = 1 \) for all but finitely many \( r \in \mathbb{N} \). Indeed, if \( \Phi_\eta(\lambda_r^{-1}(\tilde{y} + \tilde{y}_r)) \neq 1 \), then \( \lambda_r^{-1}(\tilde{y} + \tilde{y}_r) = 1 - \eta \). As \( \tilde{y}_r \in A_\eta(\Gamma) \), there is \( \xi > 0 \) such that \[ [\tilde{y}] + \xi < 1 - \eta. \]

By the same reasoning \( [\tilde{y}_r] < 1 - \eta \). By the convexity of \( \lfloor . \rfloor \) and these estimates one has

\[ [\tilde{y} + \tilde{y}_r] \leq [\tilde{y}] + [\tilde{y}_r] < 2(1 - \eta) - \xi. \]

Finally, \( \lambda_r < \frac{2(1-\eta)-\xi}{1-\eta} \), which can happen only for finitely many \( r \) since \( \lambda_r \to 2. \)

As \( \Phi_\eta \) is continuous at \( \tilde{y} \) and \( \lambda_r \to 2 \), it follows that

\[ \Phi_\eta((\lambda_r - 1)^{-1}\tilde{y}) \to 1. \]

Consequently

\begin{equation}
(1 - \lambda_r^{-1})\Phi_\eta((\lambda_r - 1)^{-1}\tilde{y}) + \lambda_r^{-1}\Phi_\eta(\tilde{y}_r) - \Phi_\eta(\lambda_r^{-1}(\tilde{y} + \tilde{y}_r)) \to 0 \quad \text{as} \ r \to \infty.
\end{equation}

The convexity and \textbf{(4)} imply for any \( \gamma \in \Gamma \) that

\[ (1 - \lambda_r^{-1})\phi_\eta((\lambda_r - 1)^{-1}\tilde{y}(\gamma)) + \lambda_r^{-1}\phi_\eta(\tilde{y}_r(\gamma)) - \phi_\eta(\lambda_r^{-1}(\tilde{y}(\gamma) + \tilde{y}_r(\gamma))) \to 0 \quad \text{as} \ r \to \infty. \]

Here, as \( (\tilde{y}_r)_{r \in \mathbb{N}} \) is a bounded sequence, the uniform continuity of \( \phi_\eta \) on bounded intervals yields that

\[ \frac{1}{2}\phi_\eta(\tilde{y}(\gamma)) + \frac{1}{2}\phi_\eta(\tilde{y}_r(\gamma)) - \phi_\eta\left(\frac{1}{2}(\tilde{y}(\gamma) + \tilde{y}_r(\gamma))\right) \to 0 \quad \text{as} \ r \to \infty. \]

Take any \( \gamma \in \Gamma \) such that \( y(\gamma) > (1 - \eta)Z_\eta(y) \). Then \( \tilde{y}_r(\gamma) \) is eventually contained in some compact interval \( I \subset (1 - \eta, +\infty) \). The uniform convexity of \( \phi_\eta \) on \( I \) guarantees that \( \tilde{y}_r(\gamma) \to \tilde{y}(\gamma) \) as \( r \to \infty \). This together with \textbf{(3)} yields the result.

(ii) Take a subsequence of \( (y_r) \) if necessary and assume that there is some \( \varepsilon > 0 \) such that for every \( r \in \mathbb{N} \) there is \( \gamma_r \in \Gamma' \) satisfying \( y_r(\gamma_r) > Z_\eta(y)(1-\eta) + \varepsilon \). Then using \textbf{(3)} we get

\[ \frac{y_r(\gamma_r)}{Z_\eta(y_r)} > 1 - \eta + \frac{\varepsilon}{Z_\eta(y)}. \]
for \( r \) large enough. It follows that \( \phi_{\eta}(y_{r}(\gamma)) > \varepsilon' \) for some \( \varepsilon' > 0 \). But then we have from part (i) and from the finiteness of \( \Gamma \setminus \Gamma' \) that
\[
1 = \lim_{r \to \infty} \Phi_{\eta}(y_{r}) = \lim_{r \to \infty} \left( \sum_{\gamma \in \Gamma \setminus \Gamma'} \phi_{\eta}(y_{r}(\gamma)) + \sum_{\gamma \in \Gamma'} \phi_{\eta}(y_{r}(\gamma)) \right) \geq \Phi_{\eta}(y) + \varepsilon' = 1 + \varepsilon',
\]
which is the desired contradiction. \( \square \)

Next we present an analogue of Fact II.2.3 (ii) in \[2\] which will help us to pass the LUR hypothesis from the values of \( Z_{\eta} \) onto the “large” coordinates of \( y \). See also our Lemma 2.8.

**Lemma 4.4.** Let \( 0 < \eta < 1 \). Suppose that \( y_{r}, y, u_{r} \in A_{\eta}(\Gamma) \), \( r \in \mathbb{N} \), are coordinatewise non-negative and satisfy \( 0 \leq u_{r} \leq \frac{y_{r} + y}{2} \). If
\[
\frac{Z_{\eta}^{2}(y_{r}) + Z_{\eta}^{2}(y)}{2} - Z_{\eta}^{2}(u_{r}) \to 0 \quad \text{as} \quad r \to \infty,
\]
then
\[
u_{r}(\gamma) \to y(\gamma) \quad \text{and} \quad y_{r}(\gamma) \to y(\gamma) \quad \text{as} \quad r \to \infty
\]
for any \( \gamma \in \Gamma \) such that \( y(\gamma) > Z_{\eta}(y)(1 - \eta) \).

**Proof.** As \( Z_{\eta} \) is convex and lattice, and since \( u_{r} \leq \frac{y_{r} + y}{2} \), condition (5) implies
\[
\frac{Z_{\eta}^{2}(y_{r}) + Z_{\eta}^{2}(y)}{2} - Z_{\eta}^{2}\left(\frac{y_{r} + y}{2}\right) \to 0 \quad \text{as} \quad r \to \infty.
\]
It follows from Lemma 2.8 that
\[
\lim_{r \to \infty} Z_{\eta}(y_{r}) = \lim_{r \to \infty} Z_{\eta}\left(\frac{y_{r} + y}{2}\right) = \lim_{r \to \infty} Z_{\eta}(u_{r}) = Z_{\eta}(y).
\]

Let \( \tilde{y}, \tilde{y}_{r} \) be as in the proof of Lemma 4.3 and let us define \( \tilde{u}_{r} := u_{r}/Z_{\eta}(u_{r}) \). As \( y \in A_{\eta}(\Gamma) \) and \( \|\tilde{y}\|_{\infty} \leq Z_{\eta}(\tilde{y}) = 1 \), we get \[\|\tilde{y}\| < (1 - \eta) \|\tilde{y}\|_{\infty} \leq (1 - \eta)\]. Let us take \( \delta \in (0, (1 - \eta - \|\tilde{y}\|)/3) \) so small that \( \Gamma = \Gamma_{1} \cup \Gamma_{2} \cup \Gamma_{3} \) for the three disjoint sets \( \Gamma_{1} := \{ \gamma : \tilde{y}(\gamma) > (1 - \eta) \} \), \( \Gamma_{2} := \{ \gamma : \tilde{y}(\gamma) = (1 - \eta) \} \) and \( \Gamma_{3} = \{ \gamma : \tilde{y}(\gamma) < (1 - \eta - 3\delta) \} \). Notice that \( \Gamma_{1} \cup \Gamma_{2} \) is finite.

It follows from (6) and from Lemma 4.3 (i) that \( y_{r}(\gamma) \to y(\gamma) \) for every \( \gamma \in \Gamma_{1} \). So \( \limsup_{r \to \infty} u_{r}(\gamma) \leq y(\gamma) \). This yields
\[
\limsup_{r \to \infty} \tilde{u}_{r}(\gamma) \leq \tilde{y}(\gamma) \quad \text{for} \quad \gamma \in \Gamma_{1},
\]
since \( Z_{\eta}(u_{r}) \to Z_{\eta}(y) \). It remains to show that \( \liminf_{r \to \infty} \tilde{u}_{r}(\gamma) \geq \tilde{y}(\gamma) \) for \( \gamma \in \Gamma_{1} \).

We will need some auxiliary observations. Lemma 4.3 (ii) together with (7) imply that there is some \( r_{0} \in \mathbb{N} \) such that for all \( \gamma \in \Gamma_{3} \) and all \( r > r_{0} \) we can estimate \( \tilde{y}_{r}(\gamma) < 1 - \eta + \delta \). It follows that
\[
\frac{\tilde{y}(\gamma) + \tilde{y}_{r}(\gamma)}{2} < 1 - \eta - \delta \quad \text{for} \quad \gamma \in \Gamma_{3}.
\]
Since \( u_{r} \leq (y + y_{r})/2 \) we obtain from (9) and from (7) that
\[
\tilde{u}_{r}(\gamma) < 1 - \eta \quad \text{for} \quad \gamma \in \Gamma_{3}
\]
for all \( r \) larger than some \( r_1 > r_0 \). In particular the values \( \Phi_\eta(\tilde{u}_r) \) for \( r > r_1 \) depend only on the indices in \( \Gamma_1 \cup \Gamma_2 \). We further get from Lemma 4.3 (ii) and from (6) that

\[
\limsup_{r \to \infty} \tilde{u}_r(\gamma) \leq 1 - \eta \quad \text{for} \quad \gamma \in \Gamma_2.
\]

In order to prove that \( \liminf \tilde{u}_r(\gamma) \geq \tilde{y}(\gamma) \) for \( \gamma \in \Gamma_1 \), we proceed by contradiction: let us assume that \( \liminf \tilde{u}_r(\gamma) < \tilde{y}(\gamma) \) for some \( \gamma \in \Gamma_1 \). This means that \( \phi_\eta(\liminf \tilde{u}_r(\gamma)) < \phi_\eta(\tilde{y}(\gamma)) \); thus, taking into account (9), (10) and (11), there exists \( r \in \mathbb{N} \) such that \( \Phi_\eta(\tilde{u}_r) < \Phi_\eta(\tilde{y}) \). But the latter is impossible since \( \Phi_\eta(\tilde{y}) = 1 = \Phi_\eta(\tilde{y}) \). \( \square \)

We have proved all the properties that we need for the component going from \( \ell_\infty(\Gamma) \) to \([0, \infty)\), so we can proceed to the component which maps \( X \) to \( \ell_\infty(\Gamma) \).

4.2. Mapping the open ball of \( X \) into \( A_\eta(\Gamma) \). We recall that the assumptions of Theorem 1.2 furnish \( X \) with two norms. The norm \(|\cdot|\) is the original norm on \( X \) with respect to which there exists the PRI \( \{P_\alpha\}_{\omega_\alpha \leq \mu} \); in particular, \(|P_\alpha| = 1\) for all \( \alpha \in [\omega, \mu] \). The norm \( \|\cdot\| \) is the \( C^k \)-smooth norm on \( X \). We may and do assume that they satisfy

\[
|\cdot| \leq \|\cdot\| \leq C|\cdot|
\]

for some \( C > 1 \). This implies that \( \|Q_\alpha\| \leq 2C \) for all \( \alpha \in \Lambda \). We recall that \( \Lambda \) stands for \( \{0\} \cup [\omega, \mu] \).

The following system of functions on \( \mathbb{R}^2 \) is at the heart of our construction. Basically we just shift a well chosen function in the direction of the \( x \)-axis and we stretch it in both directions, depending on the parameters \( n, m, l \in \mathbb{N} \). The usefulness of this will become evident in the proofs of Lemma 4.7 and Lemma 4.8 where the key idea is to order sums of finitely many \( (l) \) coordinates according to their magnitudes but penalizing sums with larger \( l \).

Let \( g : \mathbb{R}^2 \to [0, \infty) \) be defined by \( g(t, s) = 0 \) when \( t \leq 0 \) and

\[
g(t, s) := \exp\left( -\frac{10}{t} \right) \left( \frac{s^2}{100} + \frac{s}{10} + 1 \right) \quad \text{when} \quad t > 0.
\]

The function \( g \) is \( C^\infty \)-smooth on \( \mathbb{R}^2 \) and 1-Lipschitz on the strip \((-\infty, 1] \times [0, 1]\) (equipped with \( \|\cdot\|_\infty \)); Lemma 2.9 yields that \( g \) is convex on \((-\infty, 1] \times [0, 1]\) and uniformly convex on compact subsets of \((0, 1] \times [0, 1]\). We define the system \( \{g_{n,m,l} : n, m, l \in \mathbb{N}, l \leq n \} \) of shifts and stretches of \( g \) by

\[
g_{n,m,l}(t,s) := g\left( \frac{t - l/n}{1 + 2nC}, \frac{s}{1 + 2nC} \right)
\]

where the constants \( \theta_{n,m} \in (0, 1) \) will be chosen later. In what follows, for each \( n, m, l \in \mathbb{N}, l \leq n \), the function \( g_{n,m,l} \) will be evaluated only in the domain \( D_{n,l} := \left[ 0, 2nC - \frac{n - l}{n} \right] \times [0, 1 + 2nC] \). We collect the (mostly obvious) properties of the system \( \{g_{n,m,l}\} \) in the following lemma.

**Lemma 4.5.** Let \( n, m, l \in \mathbb{N}, l \leq n \). Then \( g_{n,m,l} \) is \( C^\infty \)-smooth on \( \mathbb{R}^2 \) and

\[
\begin{align*}
(A1) \quad g_{n,m,l}(t_1,s_1) & \leq g_{n,m,l}(t_2,s_2) \quad \text{if} \quad t_1 \leq t_2 \quad \text{and} \quad s_1 \leq s_2 \quad \text{and} \quad (t_i,s_i) \in D_{n,l} \quad \text{for} \quad i = 1,2, \\
(A2) \quad \sup \{g_{n,m,l}(t,s) : (t,s) \in D_{n,l}\} & \leq g(1,1) < 1 \quad \text{and} \quad g_{n,m,l} \quad \text{is} \quad \frac{1}{1+2nC} \quad \text{-Lipschitz on} \quad (D_{n,l}, \|\cdot\|_\infty).
\end{align*}
\]
Suppose that \((t,s) \in D_{n,l}\). Then \(g_{n,m,l}(t,s) = 0\) if \((t,s) \in \left[0, \frac{l}{n} \right] \times [0, 1 + 2nC]\) = \(N_{n,l}\).

The function \(g_{n,m,l}\) is convex on \(D_{n,l}\) and uniformly convex on the compact subsets of \(D_{n,l} \setminus N_{n,l}\).

Let \(l < n\) and \((t,s) \in D_{n,l} \setminus N_{n,l}\). Then \(g_{n,m,l}(t,s) < g_{n,m,l+1}(t + \delta, s)\), provided \(\delta > 1/n\).

Further,

(A7) for every \(n \in \mathbb{N}\) there exists \(\rho_n > 0\) such that for all \(m, l \in \mathbb{N}\), \(l < n\), we have

\[
g_{n,m,l}(t,s) \geq g_{n,m,l+1}(t,s) + \rho_n \quad \text{whenever} \quad (t,s) \in D_{n,l} \setminus N_{n,l+1}.
\]

Moreover \(\rho_n \downarrow 0\) as \(n \to \infty\).

Finally there is an appropriate choice of constants \(\theta_{n,m} \in (0, 1)\), \(n, m \in \mathbb{N}\), such that

(A8) for each \(n, m \in \mathbb{N}\) there exists \(\kappa_{n,m} \in (0, \rho_n)\) such that if \(l \in \mathbb{N}\), \(l \leq n\) and \((t,0) \in D_{n,l} \setminus N_{n,l}\), then \(s \mapsto g_{n,m,l}(t,s)\) is strictly increasing on \([0, 1+2nC]\) and

\[
g_{n,m,l}(t,1+2nC) - g_{n,m,l}(t,0) \leq \kappa_{n,m}.
\]

Moreover for every \(n \in \mathbb{N}\) we have \(\kappa_{n,m} \downarrow 0\) as \(m \to \infty\).

Proof. The fact that \(g_{n,m,l}\) is \(C^\infty\)-smooth and properties (A1)-(A6) follow directly from the corresponding properties of the function \(g\) and/or from the definition of \(g_{n,m,l}\). Note that we have only used \(\theta_{n,m} < 1\) to obtain the estimate of the Lipschitz constant.

In order to satisfy (A7), we define \(\rho_n\) as

\[
\rho_n := \inf \{ g_{n,m,l}(t,s) - g_{n,m,l+1}(t,s) : l, m \in \mathbb{N}, l < n, (t,s) \in D_{n,l} \setminus N_{n,l+1} \}.
\]

Having in mind (A1) we see that \(\rho_n = g_{n,1,1}(2/n,0) = \exp(-10n(1+2nC)) \downarrow 0\) as \(n \to \infty\). See also Figure 1. Notice that this \(\rho_n\) does not depend on the choice of \(\theta_{n,m}\).

![Figure 1](https://www.ams.org/journal-terms-of-use)
In order to fulfill (A8), \( \kappa_{n,m} \) can be defined as
\[
\kappa_{n,m} := \sup \{ g_{n,m,l}(t, 1 + 2nC) - g_{n,m,l}(t, 0) : l \leq n, (t, 0) \in D_{n,l} \},
\]
which (according to (A1)) evaluates as \( \kappa_{n,m} = g_{n,m,n}(2nC, 1 + 2nC) - g_{n,m,n}(2nC, 0) \)
\[
g\left(\frac{2nC-1}{1+2nC}, \theta_{n,m}\right) - g\left(\frac{2nC-1}{1+2nC}, 0\right).
\]
It is clear that for each \( n \in \mathbb{N} \), the number \( \theta_{n,1} \in (0, 1) \) can be chosen so that \( \kappa_{n,1} < \rho_n \). In order to have \( \kappa_{n,m} \rightarrow 0 \) as \( m \rightarrow \infty \), it is enough to take any \( \theta_{n,m} \) with \( \theta_{n,m} \rightarrow 0 \) as \( m \rightarrow \infty \).

**Remark 4.6.** When \( k < \infty \) in Theorem 1.2 then we only need that the functions \( g_{n,m,l} \) are \( C^k \)-smooth, and so it is enough to define \( g \) as \( g(t, s) = 0 \) for \( t \leq 0 \) and
\[
g(t, s) = \left(\frac{t}{10}\right)^{\max\{3,k+1\}} \left(\frac{s^2}{100} + \frac{s}{10} + 1\right) \quad \text{for} \quad t > 0.
\]
The exponent above must be larger than 2, as otherwise the function \( g \) would not be convex in \([0, 1] \times [0, 1] \). We thank a referee for this observation.

Let us fix, for each \( n \in \mathbb{N} \), some \( C^\infty \)-smooth, convex function \( \xi_n \) from \([0, +\infty) \) to \([0, +\infty) \) which satisfies \( \xi_n(0, \frac{1}{n}) = \{0\} \), \( \xi_n(t) > 0 \) for \( t > \frac{1}{n} \) and \( \xi_n(t) = t - \frac{2}{n} \) for \( t \geq \frac{3}{n} \). Such a function can be constructed e.g. as follows: let \( b : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^\infty \)-smooth, non-negative bump such that \( \text{supp}(b) \subset [0, 1] \), that \( b(\frac{1}{2} + t) = b(\frac{1}{2} - t) \) for all \( t \in \mathbb{R} \), and that \( \int_{\mathbb{R}} b(t) \, dt = 1 \). Then for \( \beta(t) := \int_{-\infty}^{t} b(s) \, ds \) we have
\[
\beta(\frac{1}{2} + t) + \beta(\frac{1}{2} - t) = 1 \quad \text{for all} \quad t \in \mathbb{R}.
\]
It follows that \( \int_{-\infty}^{1} \beta(s) \, ds = 1/2 \). Put \( \xi(t) = \int_{-\infty}^{t} \beta(s) \, ds \). Then \( \xi_n(t) := \frac{2}{n} \xi\left(\frac{nt}{2} - \frac{1}{2}\right) \) has the desired properties.

Recall that \( \Lambda = \{0\} \cup [\omega, \rho] \). It follows from the assumptions of Theorem 1.2 that, for each \( \gamma \in \Lambda \), there is a \( C^1 \)-smooth, LUR norm \( \|\cdot\|_{\gamma} \) on \( Q_{\gamma}X \) such that
\[
\|x\|_{\gamma} \leq \|x\|
\]
for all \( x \in Q_{\gamma}X \). The basic properties of PRI \([2, \text{Lemma VI.1.2}] \) and \( (12) \) yield \( \|Q_{\gamma}x\|_{\gamma} \leq \|x\| \) for all \( x \in X \).

**Lemma 4.7.** Let \( n, m \in \mathbb{N} \), put \( F_n = \{(A, B) : \emptyset \neq B \subset A \subset \Lambda, |A| \leq n\} \), and define \( H_{n,m} : B_{(X,\|\cdot\|)}^O \rightarrow \mathbb{R}^{F_n} \) by
\[
H_{n,m}x(A, B) := g_{n,m,n}(A) \left(\sum_{\gamma \in A} \xi_n(\|Q_{\gamma}x\|_{\gamma}), \xi_n\left(\left\|x - \sum_{\gamma \in B} Q_{\gamma}x\right\|\right)\right).
\]
Then \( H_{n,m} \) is a mapping from \( B_{(X,\|\cdot\|)}^O \) to \( \ell_\infty(F_n) \) which is \( 1 \)-Lipschitz, coordinatewise convex and coordinatewise \( C^1 \)-smooth, and for each \( x \in B_{(X,\|\cdot\|)}^O \) we have \( H_{n,m}x \in A_{\rho_n - \kappa_{n,m}}(F_n) \cup \{0\} \) (see \([2]\)).

**Proof.** When \( \|x\| < 1 \), then \( (12) \) yields
\[
\left(\sum_{\gamma \in A} \xi_n(\|Q_{\gamma}x\|_{\gamma}), \xi_n\left(\left\|x - \sum_{\gamma \in B} Q_{\gamma}x\right\|\right)\right) \in [0, 1 + 2 |A| C] \times [0, 2 |A| C] \subset D_{n,|A|}.
\]
So, for each \( (A, B) \in F_n \), the function \( x \mapsto H_{n,m}x(A, B) \) is \( C^1 \)-smooth as the composition of \( C^1 \)-smooth mappings. Using \( \|Q_{\gamma}x\|_{\gamma} \leq \|Q_{\gamma}x\| < 2C \|x\| \) (\( x \in X \), \( \gamma \in \Lambda \)) and Lemma 4.3 (A2) we obtain that \( x \mapsto H_{n,m}x(A, B) \) is \( 1 \)-Lipschitz for each \( (A, B) \in F_n \), \( n, m \in \mathbb{N} \). Therefore \( H_{n,m} \) is \( 1 \)-Lipschitz for each \( n, m \in \mathbb{N} \). Each
\[
x \mapsto H_{n,m}(A, B)
\]
is convex by application of Lemma 2.10 since \( g_{n,m,|A|} \) is convex and lattice on \( D_{n,|A|} \). Lemma 4.5 (A2) also yields \( \|H_{n,m}x\|_\infty < 1 \).

Let \( x \in B_{(X, \|\cdot\|)}^0 \) be fixed. We are going to prove that either \( \|H_{n,m}x\|_\infty = 0 \) or \( [H_{n,m}x] < \|H_{n,m}x\|_\infty (1 - \rho_n + \kappa_{n,m}) \). Let \( \Lambda_0 = \{ \gamma \in \Lambda : \|Q_\gamma x\|_\gamma > \frac{1}{n} \} \) and define \( E = \{ (A, B) \in F_n : A \cap \Lambda_0 \} \). Since \( E \) is finite, we have

\[
[H_{n,m}x] = [H_{n,m}x \mid_{F_n \setminus E}] \leq \sup \{ H_{n,m}x(A, B) : (A, B) \in F_n \setminus E \}.
\]

If there is no \( (A, B) \in F_n \setminus E \) such that \( H_{n,m}x(A, B) > 0 \), then \( [H_{n,m}x] = 0 \) and our claim is trivially true. We proceed assuming that \( H_{n,m}x(A, B) > 0 \) for some \( (A, B) \in F_n \setminus E \). Then, by (A3) in Lemma 4.5, we always have

\[
\sum_{\gamma \in A \cap \Lambda_0} \xi_n(\|Q_\gamma x\|_\gamma), \xi_n(\|x - \sum_{\gamma \in B} Q_\gamma x\|_\gamma) \notin N_{n,|A|},
\]

which can happen only if \( A \cap \Lambda_0 \neq \emptyset \). Since \( (A, B) \notin E \), we have \( 1 \leq |A \cap \Lambda_0| < |A| \). Still, we have

\[
\sum_{\gamma \in A} \xi_n(\|Q_\gamma x\|_\gamma) = \sum_{\gamma \in A \cap \Lambda_0} \xi_n(\|Q_\gamma x\|_\gamma)
\]

by the definition of \( \Lambda_0 \) and a property of \( \xi_n \). The first inequality in the next estimate follows by the application of (15) and Lemma 4.5 (A7) (the possibility of using it is guaranteed by (14)); the second one follows by the application of Lemma 4.5 (A8):

\[
g_{n,m,|A|}(\sum_{\gamma \in A} \xi_n(\|Q_\gamma x\|_\gamma), \xi_n(\|x - \sum_{\gamma \in B} Q_\gamma x\|_\gamma)) \leq g_{n,m,|A \cap \Lambda_0|}(\sum_{\gamma \in A \cap \Lambda_0} \xi_n(\|Q_\gamma x\|_\gamma), \xi_n(\|x - \sum_{\gamma \in B} Q_\gamma x\|_\gamma)) - \rho_n
\]

\[
\leq g_{n,m,|A \cap \Lambda_0|}(\sum_{\gamma \in A \cap \Lambda_0} \xi_n(\|Q_\gamma x\|_\gamma), \xi_n(\|x - \sum_{\gamma \in D} Q_\gamma x\|_\gamma)) - \rho_n + \kappa_{n,m}
\]

for any \( \emptyset \neq D \subset A \cap \Lambda_0 \). Therefore, as \( \|H_{n,m}x\|_\infty < 1 \), we have

\[
H_{n,m}x(A, B) \leq \|H_{n,m}x\|_\infty - \rho_n + \kappa_{n,m} < (1 - \rho_n + \kappa_{n,m}) \|H_{n,m}x\|_\infty
\]

for any \( (A, B) \in F_n \setminus E \). This together with (13) gives \( [H_{n,m}x] < \|H_{n,m}x\|_\infty (1 - (\rho_n - \kappa_{n,m})) \).

Notice that, by the definition of \( \kappa_{n,m} \) in Lemma 4.5, we always have \( \rho_n - \kappa_{n,m} > 0 \). We will use the notation \( \eta_{n,m} := \rho_n - \kappa_{n,m} \).

**Lemma 4.8.** Let \( 0 \neq x \in B_{(X, \|\cdot\|)}^0 \) and let \( A \) be a finite subset of \( \Lambda \) such that \( Q_\gamma x \neq 0 \) when \( \gamma \in A \). Then for all \( n \in \mathbb{N} \) sufficiently large there exist \( N_n \in \mathbb{N} \) and a finite \( C_n \subset A \), with \( |C_n| \leq n \), such that

1. \( A \subset C_n \), and
2. \( H_{n,m}x(C_n, A) > (1 - \eta_{n,m})Z_{\eta_{n,m}}(H_{n,m}x) \) for all \( m > N_n \).

**Proof.** We start by defining \( A^* := \{ \gamma \in \Lambda : \|Q_\gamma x\|_\gamma \geq \min_{\alpha \in A} \|Q_\alpha x\|_\alpha \} \) and we set out to find \( C_n \) so that in fact \( A^* \subset C_n \) for all large \( n \in \mathbb{N} \).
Let us investigate the mapping $L_n : B^O_{\{X, \|\cdot\|\}} \to \ell_\infty(F_n)$ defined as

$$L_n y(D, E) := g_{n,1,|D|} \left( \sum_{\gamma \in D} \xi_n(\|Q_\gamma y\|_\gamma), 0 \right), \quad (D, E) \in F_n.$$  

By the same argument as in the proof of Lemma 4.7 we get that either $L_n x = 0$ or $|L_n x| < (1 - \rho_n) \|L_n x\|_\infty$. Hence $L_n x \in A_{\rho_n}(F_n) \cup \{0\}$. If $n$ is large enough, necessarily $L_n x \neq 0$. It follows that $\|L_n x\|_\infty$ is realized at some coordinate. For $n \in \mathbb{N}$, let $C_n \subset \Lambda$ be such that $|C_n| \leq n$ and $L_n x(C_n, D) = \|L_n x\|_\infty$ for some (actually for all) non-empty $D \subset C_n$. We claim that, for $n$ sufficiently large, $A^* \subset C_n$. 

Let us denote $b := \min \left\{ \|Q_\gamma x\|_\gamma : \gamma \in A^* \right\} - \max \left\{ \|Q_\gamma x\|_\gamma : \gamma \in \Lambda \setminus A^* \right\}$. Since $Q_\gamma x \neq 0$ for all $\gamma \in A$, and since $\|Q_\gamma x\|_\gamma \in c_0(\Lambda)$, it follows that $b > 0$. Notice that

$$\xi_n \left( \min \left\{ \|Q_\gamma x\|_\gamma : \gamma \in A^* \right\} \right) - \xi_n \left( \max \left\{ \|Q_\gamma x\|_\gamma : \gamma \in \Lambda \setminus A^* \right\} \right) = b$$

for large $n \in \mathbb{N}$, say $n > n_0$.

Let $n \geq \max \{ |A^*|, n_0 \}$ be so large that $\frac{1}{n} < \xi_n \left( \min \left\{ \|Q_\gamma x\|_\gamma : \gamma \in A^* \right\} \right)$. Assume that there exists $\gamma_1 \in A^* \setminus C_n$. If $|C_n| < n$, then we define $\tilde{C}_n := \{ \gamma_1 \} \cup C_n$. By our choice of $n$, we have that $\xi_n(\|Q_{\gamma_1} x\|_{\gamma_1}) > \frac{1}{n}$, and so by property (A6) in Lemma 4.5 we get that

$$g_{n,1,|C_n|} \left( \sum_{\gamma \in C_n} \xi_n(\|Q_\gamma x\|_\gamma), 0 \right) < g_{n,1,|\tilde{C}_n|} \left( \sum_{\gamma \in \tilde{C}_n} \xi_n(\|Q_\gamma x\|_\gamma), 0 \right),$$

contradicting the fact that $\|L_n x\|_\infty = L_n x(C_n, D)$. If $|C_n| = n$, then there exists $\gamma_2 \in C_n \setminus A^*$ and we define $\tilde{C}_n := \{ \gamma_1 \} \cup C_n \setminus \{ \gamma_2 \}$. Our choice of $n$ yields that $\xi_n(\|Q_{\gamma_2} x\|_{\gamma_2}) - \xi_n(\|Q_{\gamma_2} x\|_{\gamma_2}) \geq b > 0$, so the fact that $g_{n,1,n}(\cdot, 0)$ is increasing when non-zero implies

$$g_{n,1,n} \left( \sum_{\gamma \in C_n} \xi_n(\|Q_\gamma x\|_\gamma), 0 \right) \geq g_{n,1,n} \left( \sum_{\gamma \in C_n} \xi_n(\|Q_\gamma x\|_\gamma), 0 \right),$$

once again contradicting the fact that $\|L_n x\|_\infty = L_n x(C_n, D)$. Therefore $A^* \subset C_n$.

We proved that for all $n \in \mathbb{N}$ large enough we have $(C_n, A) \in F_n$ and

$$[L_n x] < (1 - \rho_n) \|L_n x\|_\infty = (1 - \rho_n) L_n x(C_n, A).$$

Fix any such $n$. The “moreover” part of Lemma 4.11 says that $L_n x(C_n, A) > (1 - \rho_n) Z_{\rho_n}(L_n x)$. Recalling that $\eta_{n,m} = \rho_n - \kappa_{n,m}$ and that $\kappa_{n,m} \to 0$ as $m \to \infty$, we can find $N_n \in \mathbb{N}$ so large that

$$H_{n,m} x(C_n, A) > (1 - \eta_{n,m}) Z_{\rho_n}(H_{n,m} x) \quad \text{whenever} \quad m > N_n.$$

Here we used the fact that $H_{n,m} x \to L_n x$ in $\ell_\infty(F_n)$ as $m \to \infty$, which is easily verified using Lemma 4.5 (A8) and (A5). Now, Lemma 4.2 gives the desired inequality.
4.3. The definition and properties of the new norm on $X$. First, we define $J_{j,n,m} : B^O_{(X,\|\cdot\|)} \to \mathbb{R}$ for $j, n, m \in \mathbb{N}$ as

$$J_{j,n,m}(x) := \xi_j(Z_{n,m}(H_{n,m}x)).$$

Next, let $J : B^O_{(X,\|\cdot\|)} \to \mathbb{R}$ be defined as

$$J^2(x) := \|x\|^2 + \sum_{j,n,m \in \mathbb{N}} \frac{1}{2j+n+m} J_{j,n,m}^2(x),$$

and finally, let $\| \cdot \| : X \to \mathbb{R}$ be the Minkowski functional of $\{x \in X : J(x) \leq 1/2\}$.

**Lemma 4.9.** The function $\| \cdot \|$ is an equivalent norm on $X$ which is $C^1$-smooth away from the origin.

**Proof.** By the Implicit Function Theorem, it is sufficient to show that for each $x \in X$ such that $J(x) = 1$, the function $J$ is Fréchet differentiable on some neighborhood of $x$ and $J'(x) \neq 0$.

It follows from Lemmas [4.7], [4.1] and [2.3] that each $Z_{n,m} \circ H_{n,m}$ is $C^1$-smooth on the open set $\{x \in B^O_{(X,\|\cdot\|)} : H_{n,m}x \neq 0\}$. Thus for every $j \in \mathbb{N}$, the composition $J_{j,n,m}$ is $C^1$-smooth on the open ball $B^O_{(X,\|\cdot\|)}$. Further we claim that there is a constant $K' > 0$ such that each $J_{j,n,m}$ is $K'$-Lipschitz on $B^O_{(X,\|\cdot\|)}$. Indeed, $H_{n,m}$ is 1-Lipschitz on $B^O_{(X,\|\cdot\|)}$ for all $n, m \in \mathbb{N}$ (Lemma [4.7]); the norm $Z_\eta$ is 2-Lipschitz for each $0 < \eta < 1/2$, and $\xi_j$ is 1-Lipschitz for each $j \in \mathbb{N}$.

It follows that $J$ is $K$-Lipschitz on $B^O_{(X,\|\cdot\|)}$ for some $K > 0$. The calculus rules lead to the conclusion that $J$ is Fréchet differentiable on a neighborhood of any $x \in X$ such that $\|x\| < 1$; then the convexity of all terms implies that $J'(x)x \geq \|x\|$, $x = 1$. In particular, if $x \in X$ is such that $J(x) = 1$, then $\|x\| \leq 1/2$ and the premise of the Implicit Function Theorem is satisfied.

Finally, $2\|x\| \leq J(x) \leq 2K\|x\|$ where the second inequality follows from the fact that $J$ is $K$-Lipschitz and $J(0) = 0$.

**Proposition 4.10.** The norm $\| \cdot \|$ is locally uniformly rotund.

**Proof.** Let $x, x_r \in X, r \in \mathbb{N}$ be such that

$$2\|x\|^2 + 2\|x\|^2 - \|x + x_r\|^2 \to 0 \text{ as } r \to \infty$$

and let $\varepsilon > 0$. We may assume, without loss of generality, that $\|x\| = 1$. Let $A \subset \Lambda$ be a set, whose existence is ensured by Lemma [2.13] that satisfies

$$\left\| x - \sum_{\gamma \in A} Q_\gamma x \right\| < \varepsilon$$

and where $Q_\gamma x \neq 0$ for $\gamma \in A$. We claim that $\limsup_{r \to \infty} \left\| x_r - \sum_{\gamma \in A} Q_\gamma x_r \right\| < \varepsilon$ and

$$\lim_{r \to \infty} \left\| \sum_{\gamma \in A} Q_\gamma (x - x_r) \right\| = 0.$$ This yields that $\limsup_{r \to \infty} \|x_r - x\| < 2\varepsilon$ and proves the proposition.

Let us prove the claim. It follows from (16) and from the uniform continuity of $J$ on bounded sets that

$$\frac{J^2(x_r) + J^2(x)}{2} - J^2 \left( \frac{x + x_r}{2} \right) \to 0 \text{ as } r \to \infty.$$
By the convexity of the terms in the definition of $J^2$, we get that
\[
\frac{J^2_{j,n,m}(x_r) + J^2_{j,n,m}(x)}{2} - J^2_{j,n,m}\left(\frac{x + x_r}{2}\right) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty
\]
or, equivalently,
\[
\lim_{r \to \infty} J_{j,n,m}(x_r) = J_{j,n,m}(x) = \lim_{r \to \infty} J_{j,n,m}\left(\frac{x + x_r}{2}\right)
\]
for each $j, n, m \in \mathbb{N}$.

Let us borrow the notation $L_n x$ from the proof of Lemma 4.8. Let us recall that $H_{n,m} x \geq L_n x \geq 0$ (in the lattice $\ell_\infty(F_n)$) for all $m \in \mathbb{N}$. There is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that $L_n x \neq 0$. Hence $Z_{\eta,m}(H_{n,m} x) \geq Z_{\eta,m}(L_n x) > 0$ for $n \geq n_0$ and $m \in \mathbb{N}$. Therefore for each $n \geq n_0$ there exists $j_n \in \mathbb{N}$ such that for all $j \geq j_n$ and all $m \in \mathbb{N}$ one has $J_{j,n,m}(x) > 0$. Since $\xi_j \mid (1/j, +\infty)$ has a continuous inverse, it follows from (18) that
\[
\lim_{r \to \infty} Z_{\eta,m}(H_{n,m} x_r) = Z_{\eta,m}(H_{n,m} x) = \lim_{r \to \infty} Z_{\eta,m}\left(H_{n,m}\left(\frac{x + x_r}{2}\right)\right)
\]
for all $n \geq n_0$ and $m \in \mathbb{N}$.

Let us fix $n \geq n_0$ and $m \in \mathbb{N}$ both large enough in the sense of Lemma 4.8. We also require that $\frac{1}{n} < \varepsilon$ and $\frac{1}{n} < \|Q_y x\|_\gamma$ for all $\gamma \in A$. By the application of Lemma 4.8, we obtain a set $C_n$ such that $\gamma := (C_n, A) \in F_n$ satisfies the assumptions of Lemma 4.4 for $y := H_{n,m} x, y_r := H_{n,m} x_r, \text{ and } x_r := H_{n,m}\left(\frac{x + x_r}{2}\right)$. Thus, using this lemma, we infer that
\[
\lim_{r \to \infty} H_{n,m}(x_r) = \lim_{r \to \infty} H_{n,m}(x) (C_n, A) = H_{n,m}(x)(C_n, A).
\]
For better clarity we will write
\[
\alpha(z) := \sum_{\gamma \in C_n} \xi_n(\|Q_\gamma z\|_\gamma) \quad \text{and} \quad \beta(z) := \xi_n\left(\left\|z - \sum_{\gamma \in A} Q_\gamma z\right\|\right).
\]
Then $H_{n,m}(z) = g_{n,m,C_n}(\alpha(z), \beta(z))$. Since $g_{n,m,C_n}(\alpha(x), \beta(x)) \neq 0$, property (A4) in Lemma 4.5 allows us to use Lemma 2.8 with $\Gamma := \{1, 2\}$. Let us fix $\alpha(x), \beta(x), y := (\alpha(x), \beta(x))$ and $x_r := (\alpha(x), \beta(x))$. Thus a) $\lim_{r \to \infty} \alpha(x_\gamma) = \lim_{r \to \infty} \alpha(\frac{x + x_r}{2}) = \alpha(x)$ and b) $\lim_{r \to \infty} \beta(x_r) = \beta(x)$. The convexity of the terms in the definition of $\alpha$ and $\beta$ imply that, for every $\gamma \in C_n$, we have
\[
\lim_{r \to \infty} \xi_n(\|Q_\gamma x_r\|_\gamma) = \lim_{r \to \infty} \xi_n\left(\left\|Q_\gamma \frac{x_r + x}{2}\right\|_\gamma\right) = \xi_n(\|Q_\gamma x\|_\gamma).
\]
By our choice of $n$ and by the definition of $\xi_n$ we get for every $\gamma \in A$ that
\[
\lim_{r \to \infty} \|Q_\gamma x_r\|_\gamma - \frac{2}{n} = \lim_{r \to \infty} \frac{2}{n} = \|Q_\gamma x\|_\gamma - \frac{2}{n}.
\]
Now the LUR property of the norm $\|\cdot\|_\gamma$ implies that $Q_\gamma x_r \rightarrow Q_\gamma x$ for every $\gamma \in A$. Finally, b) and the definition of $\xi_n$ imply that $\limsup_{r \to \infty} \left\|x_r - \sum_{\gamma \in A} Q_\gamma x_r\right\| \leq \max\left\{\frac{1}{n}, \left\|x - \sum_{\gamma \in A} Q_\gamma x\right\|\right\} < \varepsilon$. \qed
Lemma 4.11. The norm \( \| \cdot \| \) is the uniform limit on \( B_X \) of a sequence of \( C^k \)-smooth norms.

Proof. It follows from the assumptions of Theorem 1.2 that, for each \( \gamma \in \Lambda \), there are \( C^k \)-smooth norms \( \| \cdot \|_{\gamma, i}, i \in \mathbb{N} \), on \( Q_\gamma X \) such that

\[
(1 - \frac{1}{i}) \| x \|_{\gamma} \leq \| x \|_{\gamma, i} \leq \| x \|_{\gamma}
\]

for all \( x \in Q_\gamma X \). Thus a natural choice for the approximating norms \( \| \cdot \|_{i}, i \in \mathbb{N} \), is as follows. Let us define

\[
H^i_{n,m}(A, B) := g_{n, m, l}(\sum_{\gamma \in A} \xi_n(\| Q_\gamma x \|_{\gamma, i}), \xi_n(\| x - \sum_{\gamma \in B} Q_\gamma x \|))
\]

for \( (A, B) \in F_n, x \in B^O_{(X, \| \cdot \|)} \), and

\[
J^i_{n,m,i}(x) := \xi_j(Z_{\eta_{n,m}}(H^i_{n,m})),
\]

\[
J^2_i(x) := \| x \|^2 + \sum_{1 \leq j,n,m \leq i} \frac{1}{2^{j+n+m}} J^2_{j,n,m,i}(x),
\]

and let \( \| \cdot \|_i \) be the Minkowski functional of \( \{ x \in X : J_i(x) \leq 1/2 \} \). As the sum of finitely many \( C^k \)-smooth functions, \( J_i \) is \( C^k \)-smooth. The Implicit Function Theorem implies the same for \( \| \cdot \|_i \). Moreover \( 2 \| x \| \leq \| x \|_{i} \leq 2K \| x \| \) as in the proof of Lemma 4.9. Let \( \varepsilon > 0 \) be given. We will show that there is an index \( i_0 \in \mathbb{N} \) such that \( |J^2_i(x) - J^2_i(x)| < \varepsilon \) whenever \( \| x \| < 1 \) and \( i \geq i_0 \). Putting together (20) and (12) we get for all \( n, i \in \mathbb{N}, A \subset \Lambda, |A| \leq n \) and all \( x \in B^O_{(X, \| \cdot \|)} \) that

\[
0 \leq \sum_{\gamma \in A} \xi_n(\| Q_\gamma x \|_{\gamma, i}) - \sum_{\gamma \in A} \xi_n(\| Q_\gamma x \|_{\gamma, i}) \leq \frac{2C |A|}{i}.
\]

It follows from Lemma 4.5 (A2) that \( \| H_{n,m}x - H^i_{n,m}x \|_\infty \leq \frac{1}{i} \) for all \( n, m, i \in \mathbb{N}, x \in B^O_{(X, \| \cdot \|)} \), and therefore \( |J^2_{n,m,i}(x) - J^2_{n,m,i}(x)| \leq \frac{\varepsilon}{i} \) since \( Z_{\eta_{n,m}} \) is 2-Lipschitz and since \( J_{n,m,i}(x) \) and \( J_{n,m,i}(x) \) are smaller than 2 when \( \| x \| < 1 \).

We find \( i_0 \in \mathbb{N} \) so large that \( \frac{8}{i_0} < \varepsilon/2 \) and

\[
\sum_{\max\{j,n,m\} \geq i_0} \frac{4}{2^{j+n+m}} < \varepsilon/2.
\]

Then for each \( i \geq i_0 \),

\[
|J^2_i(x) - J^2_i(x)| \leq \sum_{1 \leq j,n,m \leq i} \frac{1}{2^{j+n+m}} |J^2_{j,n,m}(x) - J^2_{j,n,m}(x)|
\]

\[
+ \sum_{\max\{j,n,m\} \geq i_0} \frac{1}{2^{j+n+m}} J^2_{j,n,m}(x)
\]

\[
< \sum_{1 \leq j,n,m \leq i} \frac{1}{2^{j+n+m}} \left( \frac{8}{i} \right) + \varepsilon/2 < \varepsilon.
\]

This proves that \( J_i \to J \) uniformly on \( B^O_{(X, \| \cdot \|)} \).
Now let us observe that, since \( J \) is convex and \( J(0) = 0 \), we have the estimate
\[
\frac{1}{2} |\lambda - 1| \leq \left| \frac{1}{2} - J(\lambda x) \right|
\]
for all \( x \in X \) such that \( J(x) = \frac{1}{2} \) and for all \( \lambda > 0 \).

In order to prove the uniform convergence \( \| \cdot \|_i \to \| \cdot \| \) on bounded sets we assume that there is a bounded sequence \( (x_i) \) in \( X \setminus \{0\} \) such that \( \lambda_i = \frac{\|x_i\|}{\|\lambda_i x_i\|} \to 1 \).
Therefore, after taking a subsequence, there is some \( \varepsilon > 0 \) such that \( |\lambda_i - 1| > 2\varepsilon \) for all \( i \in \mathbb{N} \). On the other hand, since \( \left\| \frac{x_i}{\|x_i\|} \right\| \leq \frac{1}{2} \) and since \( J_i \to J \) uniformly on \( B^O_{(X,\|\cdot\|)} \), we get that
\[
\left| J(\lambda_i \frac{x_i}{\|x_i\|}) - \frac{1}{2} \right| = \left| J(\frac{x_i}{\|x_i\|}) - J_i(\frac{x_i}{\|x_i\|}) \right| \leq \varepsilon
\]
for large enough. Thus, having in mind (21), we obtain \( |\lambda_i - 1| \leq 2\varepsilon \). As a result of this contradiction we can conclude that \( \| \cdot \|_i \to \| \cdot \| \) uniformly on bounded sets. \( \square \)

Acknowledgement

The authors thank the referee for several helpful suggestions, which greatly enhanced the clarity and presentation of this paper.

References

[1] Edgar Asplund, *Averaged norms*, Israel J. Math. 5 (1967), 227–233. MR0222610 (36 #5660)

[2] Robert Deville, Gilles Godefroy, and Václav Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow, 1993. MR1211634 (94d:46012)

[3] Robert Deville, Vladimir Fonf, and Petr Hájek, *Analytic and \( C^k \) approximations of norms in separable Banach spaces*, Studia Math. 120 (1996), no. 1, 61–74. MR1398174 (97h:46012)

[4] Robert Deville, Vladimir Fonf, and Petr Hájek, *Analytic and polyhedral approximation of convex bodies in separable polyhedral Banach spaces*, Israel J. Math. 105 (1999), 139–154, DOI 10.1007/BF02780326. MR1639743 (99h:46006)

[5] Pavel Drábek and Jaroslav Milota, *Methods of nonlinear analysis*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2007. Applications to differential equations. MR2323436 (2008i:47134)

[6] Marián Fabian, Petr Hájek, and Václav Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 8, Springer-Verlag, New York, 2001. MR1831176 (2002f:46001)

[7] Marián Fabian, Petr Hájek, and Václav Zizler, *A note on lattice renormings*, Comment. Math. Univ. Carolin. 38 (1997), no. 2, 263–272. MR1455493 (98e:46008)

[8] Marián Fabian, Vicente Montesinos, and Václav Zizler, *Smoothness in Banach spaces. Selected problems*, RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 100 (2006), no. 1-2, 101–125 (English, with English and Spanish summaries). MR2267403 (2007g:46026)

[9] M. Fabián, J. H. M. Whitfield, and V. Zizler, *Nons with locally Lipschitzian derivatives*, Israel J. Math. 44 (1983), no. 3, 262–276, DOI 10.1007/BF02760975. MR693663 (84i:46028)

[10] Petr Hájek, *On convex functions in \( c_0(\omega_1) \)*, Collect. Math. 47 (1996), no. 2, 111–115. MR1402064 (97b:46069)

[11] Petr Hájek, Vicente Montesinos Santalucía, Jon Vanderwerff, and Václav Zizler, *Biorthogonal systems in Banach spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 26, Springer, New York, 2008. MR2359536 (2008k:46002)

[12] P. Hájek, A. Procházka, *Smooth version of Deville’s master lemma*, preprint (2009).

[13] P. Hájek, J. Talponen, *Smooth approximations of norms in separable Banach spaces*, arXiv:1105.6046v1, 2011.

[14] Richard Haydon, *Smooth functions and partitions of unity on certain Banach spaces*, Quart. J. Math. Oxford Ser. (2) 47 (1996), no. 188, 455–468, DOI 10.1093/qmath/47.4.455. MR1460234 (2000c:46080)
[15] D. McLaughlin, R. Poliquin, J. Vanderwerff, and V. Zizler, *Second-order Gateaux differentiable bump functions and approximations in Banach spaces*, Canad. J. Math. 45 (1993), no. 3, 612–625, DOI 10.4153/CJM-1993-032-9. MR1222519 (94g:46025)

[16] J. Pechanec, J. H. M. Whitfield, and V. Zizler, *Norms locally dependent on finitely many coordinates*, An. Acad. Brasil. Ciênc. 53 (1981), no. 3, 415–417. MR663236 (83h:46025)

[17] Michel Talagrand, *Renormages de quelques $C(K)$*, Israel J. Math. 54 (1986), no. 3, 327–334, DOI 10.1007/BF02764961 (French, with English summary). MR853457 (88d:46041)

[18] S. L. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. 37 (1970/71), 173–180. MR0306873 (46 #5995)

[19] Václav Zizler, *Nonseparable Banach spaces*, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1743–1816, DOI 10.1016/S1874-5849(03)80048-7. MR1999608 (2004g:46030)