Tadpole-Improved Perturbation Theory for Heavy-Light Lattice Operators

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Abstract
Lattice calculations of matrix elements involving heavy-light quark bilinears are of interest in calculating a variety of properties of $B$ and $D$ mesons, including decay constants and mixing parameters. A large source of uncertainty in the determination of these properties has been uncertainty in the normalization of the lattice-regularized operators that appear in the matrix elements. Tadpole-improved perturbation theory, as formulated by Lepage and Mackenzie, promises to reduce these uncertainties below the ten per cent level at one-loop. In this paper we study this proposal as it applies to lattice-regularized heavy-light operators. We consider both the commonly used zero-distance bilinear and the distance-one point-split operator. A self-contained section on the application of these results is included. The calculation reduces the value of $f_B$ obtained from lattice calculations using the heavy quark effective theory.

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1. Introduction

For a heavy quark such as the $b$ quark, a variety of approaches are available for lattice Monte Carlo calculations, including non-relativistic QCD (NRQCD) [2], the heavy quark effective theory (HQET) [3], extrapolation from Wilson fermion results [4], and formulations which interpolate continuously between these actions [5]. Whatever approach is used, the relationship of the lattice operators to the operators coming from the continuum electroweak theory must be calculated in order to make use of the lattice results. While these short-distance strong interaction corrections are in principle perturbatively calculable, in practice, the one-loop corrections are sometimes so large as to be of questionable reliability [6]–[9]. It is possible that two-loop corrections are so large as to make existing calculations very inaccurate at realistically attainable values of $\beta = 6/g_0^2$.

It is disturbing that these large corrections exist and they seem to point to a deficiency in our understanding of lattice perturbation theory. Recently Lepage and Mackenzie [1] have suggested a reorganization of perturbation theory, the root of which is a nonperturbatively determined renormalization of the basic operators in the lattice action. These redefinitions remedy the problem of large renormalizations arising from lattice tadpole graphs.

Another problem with lattice perturbation theory is that if one uses $\beta$ to determine the perturbative coupling, $\alpha_{\text{lat}}$, one-loop perturbative corrections in quantities such as the mass renormalization for Wilson fermions are consistently underestimated. These perturbation theory problems are due to the fact that $\alpha_{\text{lat}}$ is a poor choice of expansion parameter. For example at an inverse lattice spacing of 2 GeV, Lepage and Mackenzie’s tadpole-improved expansion parameter is $\alpha_V = 0.16$, which is twice as large as $\alpha_{\text{lat}} = 0.08$. The reason that $\alpha_{\text{lat}}$ is so different from $\alpha_V$ is that $\alpha_{\text{lat}}$ receives large all orders corrections from the tadpoles present in lattice QCD. One is led to the new expansion parameter after one redefines the basic fields in the lattice action and follows the tadpole improvement program referred to in the previous paragraph. Lepage and Mackenzie argue that the best way to arrive at $\alpha_V$ is from a non-perturbative lattice determination of a perturbatively calculable quantity, such as the gauge field plaquette expectation value. After the reorganization of perturbation theory and the determination of a suitable expansion
parameter, Lepage and Mackenzie reanalyze several lattice quantities noted for their poor agreement with perturbative calculations, and they find that the results of their program are in good agreement with the Monte Carlo results.

The scope of the present paper is a study of the Lepage-Mackenzie prescription as it applies to the calculation of quantities determined using the heavy quark effective action, particularly $f_B$. In the following section, we will review the effect of tadpole improvement on the heavy and light quark actions. To keep the explanation simple, discussions of one-loop perturbative corrections are postponed to Section 3. Then in Section 4, we determine the coupling and the scale for which the coupling should be evaluated as it appears in the expressions of Section 3. In Section 5, we complete the lattice-to-continuum matching.

Our main results are in Tables 4a and 4b of Section 6. This concluding section has been made as self-contained as possible, and readers who just want to know how to apply our results can turn directly to it. There we incorporate the results of additional continuum calculations of the next-to-leading running and matching from the continuum effective theory at the lattice scale to the full theory at the scale $m_b$ [10] which is not otherwise discussed here. More detail on the application of those results can be found in Ref. [11]. The results are tabulated in a form in which they can be easily used to obtain physical predictions from lattice results. Throughout the paper, it is assumed that the light quark action is the standard Wilson fermion action. Both staggered fermion and naive fermion results correspond to the case $r = 0$ [7]. However the conventional normalization of the Wilson fermion field is singular as $r \to 0$, and a different convention is used in that case. This is discussed in the conclusions.

2. Tadpole Improvement of the Heavy and Light Quark Actions

Using tadpole improvement of the Wilson action for quarks on the lattice as a guide, one can perform tadpole improvement of the heavy quark action, and this has been done by Bernard in Ref. [12]. Instead of discretizing

$$S = \int d^4x \, b^\dagger (i\partial_0 + gA_0) \, b$$

(2.1)
as
\[ S = i a^3 \sum_n b^\dagger(n) \left( b(n) - U_0(n-\hat{0})^\dagger b(n-\hat{0}) \right). \]  \hspace{0.5cm} (2.2)

the analog of the Lepage and Mackenzie prescription for the Wilson action is
\[ S_{\text{tadpole-improved}} = i a^3 \sum_n b^\dagger(n) \left( b(n) - \frac{1}{u_0} U_0(n-\hat{0})^\dagger b(n-\hat{0}) \right), \]  \hspace{0.5cm} (2.3)

where \( u_0 \) is defined as
\[ u_0 \equiv \langle \frac{1}{3} \text{Tr} U_{\text{plaq}} \rangle^{1/4}. \]  \hspace{0.5cm} (2.4)

The combination \( U_\mu(x)/u_0 \) more closely corresponds to the continuum field \((1 + iagA_\mu(x))\), than does \( U_\mu(x) \) itself.

Since Monte-Carlo calculations have been performed with the former of the two actions in Eqns. (2.2) and (2.3), we now derive the relationship between matrix elements determined with the two actions. Consider the matrix element
\[ \langle J_5^0 \dagger(n_0, 0) J_5^0(0) \rangle = \frac{1}{a^3} G_B(n_0), \]  \hspace{0.5cm} (2.5)

where
\[ J_5^0(n) \equiv b^\dagger(n)(1 0) \gamma^0 \gamma_5 q(n). \]  \hspace{0.5cm} (2.6)

The two-by-four matrix \((1 0)\) is the upper two rows of the projection operator \((1 + \gamma_0)/2\). In general our notation for heavy quark fields and heavy-light currents follows that of Ref. [8]. As one finds by performing the \( b \) and \( b^\dagger \) integrations in the lattice-regularized functional integral in a fixed gauge field configuration \( \{ U_\mu(n) \} \), the heavy quark propagator is just the product
\[ U_0(n_0 - 1, 0)^\dagger...U_0(0, 0)^\dagger. \]  \hspace{0.5cm} (2.7)

With the tadpole-improved action, there is an additional factor of \( 1/u_0 \) for each gauge field link in the product. Thus
\[ [G_B(n_0)]_{\text{tadpole-improved}} = \frac{G_B(n_0)}{u_0 a^3}. \]  \hspace{0.5cm} (2.8)

The \( B \) meson decay constant \( f_B \) is usually extracted from numerical simulations by fitting \( G_B(n_0) \) to
\[ \frac{(f_Bm_B)^2}{2m_B} \exp[-Cn_0a] \]  \hspace{0.5cm} (2.9)
As discussed in Ref. [6], for perturbative corrections, this necessitates using the reduced value of the heavy quark wave function renormalization, the constant part of which at one-loop is, \( c = 4.53 \).

From Eqns. (2.8) and (2.9), we see that the tadpole improvement procedure has no effect on the fitted value of \( f_B \). Its only effect is the change

\[
C \rightarrow C + \frac{\ln u_0}{a},
\]

(2.10)

that is, a linearly divergent mass renormalization.

Interestingly, if one uses the other fitting procedure discussed in Ref. [6], the application of the Lepage-Mackenzie prescription leads one to introduce factors of \( u_0 \) which led to a final result identical to the above. Thus the reduction of \( c \) from 24.48 to 4.53 discussed in Ref. [6] is the same reduction caused by the reorganization of the tadpole contributions [12].

We now consider the point-split operator. A variety of operators with the same continuum limit as in Eq. (2.6) can be constructed. Consider the set of distance-one bilinears which respects the remnants of the \( O(3) \) rotational group that is present in the lattice heavy quark effective theory. As discussed in Ref. [8], the only operator in this set that one needs to consider is

\[
J_{ps5}^0(n) \equiv \frac{1}{6} \sum_i \left[ b^\dagger(n+i)U_i(n)(1 0)\gamma^0\gamma_5q(n) + b^\dagger(n-i)U_i(n-i)(1 0)\gamma^0\gamma_5q(n) \right].
\]

(2.11)

The index \( i \) runs only over the three spatial directions. The sum of six terms has been chosen to respect the remnants of the \( O(3) \) rotational group. The tadpole improvement procedure for this operator is to multiply it by a single factor of \( 1/u_0 \) since it contains a single gauge field link.

So far in this section, we have seen that tadpole improvement does not affect the extraction of \( f_B \) as it is generally done in lattice Monte Carlo calculations using the zero-distance bilinear, and that the effect of tadpole improvement on the distance-one bilinear receives a contribution of \( 1/u_0 \). However we must still take into account the effect of tadpole improvement of the light quark action, and this will involve some additional factors.

As conventionally defined in lattice Monte Carlo calculations the lattice operator \( J_5^0 \) involved in calculating \( f_B \) is renormalized by a factor \( \sqrt{2\kappa_c Z} \), where
κc is the critical value of the hopping parameter for the light quarks. The tree level value of κc = 1/(8r). What Lepage and Mackenzie are advocating is a reorganization of perturbation theory so that the κc factors are included in Z and the renormalizing factor becomes \( \tilde{Z}/(2\sqrt{r}) \). To the extent that the non-perturbatively calculated value of κc agrees with its perturbatively calculated counterpart, there is no difference in these two prescriptions. In the next section we will see how to proceed with this procedure at one-loop order.

3. Tadpole Improvement of Heavy-Light Operators at One-Loop Order

This section will present in parallel the analysis for the zero-distance and point-split lattice discretizations of \( J_0^0 \). To summarize the previous section, we saw that rather than multiplying the zero-distance operator by \( \sqrt{2\kappa_c}Z \), it should be multiplied by \( \tilde{Z}/(2\sqrt{r}) \), and that rather than multiplying the distance-one operator by \( \sqrt{2\kappa_c}Z_{ps} \), it should be multiplied by \( \tilde{Z}_{ps}/(2u_0\sqrt{r}) \). In this section we will calculate and explain the relationship between Z and \( \tilde{Z} \) (and \( Z_{ps} \) and \( \tilde{Z}_{ps} \)) at one-loop order.

Suppose that calculations of Z and \( Z_{ps} \) have been carried out to one-loop order and that the result is

\[
Z = 1 + \frac{\alpha_s}{3\pi} \left[ \int \frac{d^4q}{\pi^2} g(q) + \frac{3}{2} \ln(\mu a)^2 \right],
\]

\[
Z_{ps} = 1 + \frac{\alpha_s}{3\pi} \left[ \int \frac{d^4q}{\pi^2} g_{ps}(q) + \frac{3}{2} \ln(\mu a)^2 \right].
\]

Suppose further that a one-loop calculation of \( 8\kappa_c \) has been performed, and that the result is

\[
\frac{1}{8r\kappa_{c\,one-loop}} = 1 - \frac{\alpha_s}{3\pi} \int \frac{d^4q}{\pi^2} h(q) \tag{3.2}
\]

(we will reserve the unadorned symbols κc and u0 for the non-perturbative lattice Monte Carlo values, and qualify them with the subscript “one-loop” when referring to their perturbative values). Finally, suppose that a one-loop calculation of \( u_0 \) gives,

\[
u_0\,_{one-loop} = 1 + \frac{\alpha_s}{3\pi} \int \frac{d^4q}{\pi^2} j(q). \tag{3.3}
\]

† This is actually a slight generalization of the Lepage-Mackenzie prescription to the case \( r \neq 1 \).
Then any quantity obtained from a lattice calculation, for which short-distance corrections have been correctly included at one-loop order can be multiplied by

\[
\left( \frac{8r\kappa_{c\,\text{one-loop}}}{8r\kappa_c} \right)^p \left( \frac{u_0\,\text{one-loop}}{u_0} \right)^s
\]

where \( p \) and \( s \) are any small numbers, and the result remains correct at one-loop order.

Without some kind of additional consideration there is no reason to prefer one of these one-loop results over another. In terms of the various quantities defined above, the tadpole improvement requires that for the \( Z \) factor of the distance-zero operator, \( p = 1/2 \) and \( s = 0 \). In other words the relationship between \( \tilde{Z} \) and \( Z \) is

\[
\tilde{Z} = \sqrt{8r\kappa_{c\,\text{one-loop}}} Z .
\]

Thus if an expression for \( \tilde{Z} \) analogous to the expression for \( Z \) in Eqn. (3.1) were defined, we would have

\[
\tilde{g}(q) = g(q) + h(q)/2 .
\]

Similarly, for the point-split operator, \( p = 1/2 \) and \( s = 1 \), implying

\[
\tilde{Z}_{ps} = u_0\,\text{one-loop} \sqrt{8r\kappa_{c\,\text{one-loop}}} Z_{ps}
\]

and

\[
\tilde{g}_{ps}(q) = g_{ps}(q) + h(q)/2 + j(q) .
\]

For clarity we emphasize that the relationship between \( Z \) and \( \tilde{Z} \) is not \( \tilde{Z} = \sqrt{8r\kappa_c}Z \) and \( \tilde{Z}_{ps} = u_0\sqrt{8r\kappa_c}Z_{ps} \). In that case, tadpole improvement would be without content, since we would be renormalizing by the exact same factor in both cases.

The functions \( g(q), h(q), g_{ps}(q), \) and \( j(q) \) have already been calculated, and we now assemble the results. For various frequently appearing functions of \( q \), we will use the notation \( \Delta_i \) as defined in Ref. [13]. (Note that \( \Delta_6 \) defined in Ref. [9] conflicts with \( \Delta_6 \) defined in Ref. [13].) We also define \( \Delta_i^{(3)} \) to be the same as \( \Delta_i \).
but with \( q_0 = 0 \). From Refs. [6] and [7], we have

\[
g(q) = -\frac{1}{4\Delta_1 \Delta_2} + \frac{\theta(1-q^2)}{q^4} + \frac{1}{16\Delta_2} - \frac{r}{4\Delta_2^{(3)}} + \frac{1}{16\pi^2} \\
- \frac{1}{2} \left( -\frac{1}{8\pi^2} + \frac{2}{16\Delta_1^2} - \frac{2\theta(1-q^2)}{q^4} - \frac{1}{32\Delta_1} \right) \\
- \frac{1}{2} \left( \frac{1}{32\pi^2} + \frac{1}{8\Delta_1} + \frac{\theta(1-q^2)}{q^4} - \frac{\Delta_4 + \Delta_5}{16\Delta_1^2 \Delta_2} + \frac{r^2(2-\Delta_1)}{4\Delta_2} \right) .
\]  

(3.9)

The second and third groups are \(-c/2\) (\(c\) is the reduced heavy quark wave function renormalization discussed in Section 2) and \(-(f-F)/2\) (\(f-F\) is the constant part of the wave function renormalization for a Wilson fermion), respectively.

The critical hopping parameter \( \kappa_c \) is related to the critical bare mass by the expression

\[
\frac{1}{8r\kappa_c} = 1 + \frac{m_c a}{4r} .
\]  

(3.10)

Thus for massless Wilson quarks only the linearly divergent part of the correction to \( m_c \) survives in the continuum limit. At one-loop, in terms of \( h(q) \) defined in (3.2), the result is [13][14],

\[
h(q) = \frac{1}{4} \left( -\frac{1}{2\Delta_1} - \frac{1}{4\Delta_1 \Delta_2} [2\Delta_1 \Delta_6 + \Delta_4] \right) .
\]  

(3.11)

The first term comes from the tadpole graph and has the larger value after integration.

From Ref. [8], we find

\[
g_{ps}(q) = g(q) + \frac{4\Delta_1 - \Delta_1^2 + 2\Delta_4 + 12r^2 \Delta_1^2}{24\Delta_1 \Delta_2} - \frac{r \Delta_1^{(3)}}{6\Delta_2^{(3)}} .
\]  

(3.12)

where \( g(q) \) is defined in Eq. (3.9). Finally,

\[
j(q) = -\frac{1}{16} .
\]  

(3.13)

The integral of \( j(q) \) is determined by the coefficient of the first term in Eq. (4.1).

The constant part of the one-loop corrections for any value of the Wilson mass term coefficient \( r \) can readily be obtained by numerically integrating the preceding expressions for \( g(q) \), \( h(q) \), \( g_{ps}(q) \), and \( j(q) \). However, the scale of \( \alpha_S(q) \) still has to be set. We turn to this in the next section.
\[ -\ln \langle \frac{1}{3} \text{Tr} U_{\text{plaq}} \rangle = \frac{4\pi}{3} \alpha_V(3.41/a)[1 - \alpha_V(3.41/a)(1.19 + 0.025 n_f) + O(\alpha_V^2)]. \] (4.1)

The coefficient of \( n_f \) in Eq. (4.1) is for dynamical Wilson fermions with \( r = 1 \). The coefficient is \( r \)-dependent and is also different for staggered fermions [15]. However, in the application to the quenched approximation, we set \( n_f = 0 \), and the value of the coefficient does not enter our results. The \( \Lambda \) value for this coupling is denoted \( \Lambda_V \). The results of this determination are summarized in Table 1. The values of \( a^{-1} \) are those given by the charmonium scale [16], and the values of the plaquette expectation value are taken from Ref. [1].

It remains to fix the scale \( q^* \), at which \( \alpha_V(q) \) is evaluated, Lepage and Mackenzie propose to do that by calculating the expectation value of \( \ln(q^2) \) in

| \( \beta \) | \( a^{-1}(\text{GeV}) \) | \( \langle \frac{1}{3} \text{Tr} U_{\text{plaq}} \rangle \) | \( \alpha_V(3.41/a) \) | \( \Lambda_V a \) |
|---|---|---|---|---|
| 5.7 | 1.15(8) | 0.549 | 0.1830 | 0.294 |
| 5.9 | 1.78(9) | 0.582 | 0.1595 | 0.198 |
| 6.1 | 2.43(15) | 0.605 | 0.1450 | 0.144 |

Table 1. \( a^{-1}, \alpha_V(3.41/a) \) and \( \Lambda_V \) obtained from Monte Carlo calculations of the charmonium spectrum [16] and the plaquette expectation value [1] at various \( \beta \).
the one-loop perturbative lattice correction. In equations,

\[
Y \equiv \int d^4 q \, \tilde{g}(q) \\
\langle \ln(qa)^2 \rangle \equiv \frac{\int d^4 q \, \tilde{g}(q) \ln(q^2a^2)}{Y} \\
q^* a \equiv \exp[\langle \ln(qa)^2 \rangle/2]
\]  

(4.2)

Similar expressions for \(Y_{ps}\) and \(Y_{ps}\langle \ln (qa)^2 \rangle\) are defined with \(\tilde{g}_{ps}(q)\) replacing \(\tilde{g}(q)\). The only ambiguity in finding \(q^*\) in either case is how to deal with the constants which do not arise from four-dimensional lattice integrals, and some momentum-space integrations involving the heavy quark field which were three-dimensional. Fortunately these terms are generally small relative to the lattice contributions. We write all such constants \(C\) as

\[
C = \int d^4 q \frac{C}{(2\pi)^4},
\]

(4.3)

and similarly take the \(q_0\) dependence of the three-dimensional integrals to be constant. The values of the quantities just defined are tabulated in Table 2a and 2b for various \(r\). These quantities were evaluated using the Monte Carlo integration routine VEGAS [17]. Errors on \(Y\) and \(Y\langle \ln(q^2a^2)\rangle\) are at most \(O(1)\) in the last decimal place.

5. Application of Results to Lattice-Continuum Matching

In this section, we illustrate the application of the results of the foregoing analysis to the lattice-to-continuum part of the matching for a couple of cases of interest.

Now that we have \(\Lambda_V\) and \(\alpha_V(3.41a^{-1})\) we use the two-loop formula with zero quark flavors \((n_f = 0)\), to obtain \(\alpha_V(q^*)\),

\[
\alpha_V(q) = [\beta_0 \ln(q^2/\Lambda_V^2) + (\beta_1/\beta_0) \ln \ln(q^2/\Lambda_V^2)]^{-1}
\]

(5.1)

where \(\beta_0 = 11/(4\pi)\) and \(\beta_1 = 102/(4\pi)^2\). Setting \(n_f = 0\) is appropriate for lattice calculations done in the quenched approximation. The explicit dependence on the value of \(a^{-1}\) drops out of the ratio \(q^*/\Lambda_V\). Hence, the only way we have used results from lattice Monte Carlo calculations so far is for the expectation value of the plaquette. In Eq. (3.1), which gives the lattice-to-continuum matching,
Table 2a. $Y$, $\langle \ln(q^2) \rangle$, and $q^*$ for the zero distance lattice representation of $J_5^0$ defined in Eq. (2.6).

| $r$  | $Y$  | $Y\langle \ln(q^2a^2) \rangle$ | $q^*a$ |
|------|------|---------------------------------|--------|
| 1.00 | -13.93 | -21.76                         | 2.18   |
| 0.75 | -12.76 | -21.12                         | 2.29   |
| 0.50 | -10.74 | -18.70                         | 2.39   |
| 0.25 | -6.58  | -11.02                         | 2.31   |
| 0.00 | 2.21   | 9.65                           | 8.88   |

Table 2b. $Y_{ps}$, $\langle \ln(q^2) \rangle$, and $q^*$ for the distance-one lattice representation of $J_5^0$ defined in Eq. (2.11).

| $r$  | $Y_{ps}$ | $Y_{ps}\langle \ln(q^2a^2) \rangle$ | $q^*a$ |
|------|----------|---------------------------------|--------|
| 1.00 | -7.20    | -10.87                         | 2.13   |
| 0.75 | -5.20    | -8.10                          | 2.18   |
| 0.50 | -2.13    | -2.81                          | 1.93   |
| 0.25 | 2.98     | 7.70                           | 3.64   |
| 0.00 | 8.65     | 20.44                          | 3.26   |

we put $\mu = q^*$, thus $\ln \mu a$ becomes $\ln q^* a$. Thus the Monte Carlo calculation of the plaquette, and the perturbative calculations are sufficient by themselves to determine the lattice-to-continuum matching. These results are summarized in Tables 3a and 3b.

It still remains to multiply $\tilde{Z}$ and $\tilde{Z}_{ps}$ by the additional factors summarized at the beginning of Section 3, then match the continuum heavy quark effective theory at the lattice scale to the full continuum theory at $m_b^*$. We leave this for the conclusions, to which we now turn.

6. Conclusions

In the preceding section, we illustrated the application of our one-loop results to obtain the factors $\tilde{Z}$ and $\tilde{Z}_{ps}$, for the local and point-split heavy-light bilinears. From these $\tilde{Z}$’s we will obtain the factors one multiplies a lattice calculation by in order to obtain a physical number.

First, we must multiply by the two-loop running from $q^*$ to $m_b^*$ and the matching that takes us from the continuum heavy quark effective theory to the
continuum full theory. The continuum results are explained in detail in Refs. [10] and [11]. It is in this step that the determination of the scale $a^{-1}$ is finally used. The resulting factor, $Z_{cont}$, is $q^*$- and $a$-dependent. The result after multiplying by this factor is tabulated in Tables 4a and 4b. Table 4a corresponds to using the zero-distance representation of the axial current given in Eq. (2.6), and Table 4b corresponds to using the distance-one representation given in Eq. (2.11). Now that perturbation theory has been reorganized to include tadpole corrections to all orders, we expect our one-loop calculation of the renormalization factor to be accurate to about 7%. This estimate of the magnitude of the two-loop correction was obtained simply by squaring the largest one-loop correction ($0.27^2 = 0.07$) in Tables 4a and 4b.

Second, we must multiply by the factors discussed in Section 3. We multiply $\tilde{Z}$ and $\tilde{Z}_{ps}$ by $1/(2\sqrt{r})$, and $1/(2u_0\sqrt{r})$, respectively. The value of $u_0$ is obtained from Eq. (2.4) and Table 1. We find $u_0 = 0.861$, 0.873, and 0.882 for $\beta = 5.7$, 5.9, and 6.1, respectively.

As an example, consider the distance-zero operator at $\beta = 6.1$. From Table 4a, we find $\tilde{Z} \times Z_{cont}$ to be 0.75. We multiply by $1/(2\sqrt{r}) = 1/2$ to obtain the final result, 0.37. It is worthwhile to compare this with the widely used value of $Z_A$.

| $r$ | $\beta = 5.7$ | $\beta = 5.9$ | $\beta = 6.1$ |
|-----|---------------|---------------|---------------|
| 1.00 | 0.88          | 0.90          | 0.91          |
| 0.75 | 0.93          | 0.94          | 0.95          |
| 0.50 | 1.00          | 1.00          | 1.00          |
| 0.25 | 1.13          | 1.11          | 1.10          |
| 0.00 | 1.24          | 1.21          | 1.19          |

Table 3b. $\tilde{Z}_{ps}$ for various values of $r$ and $\beta$. 

| $r$ | $\beta = 5.7$ | $\beta = 5.9$ | $\beta = 6.1$ |
|-----|---------------|---------------|---------------|
| 1.00 | 0.73          | 0.77          | 0.80          |
| 0.75 | 0.76          | 0.80          | 0.82          |
| 0.50 | 0.82          | 0.84          | 0.86          |
| 0.25 | 0.91          | 0.92          | 0.93          |
| 0.00 | 1.13          | 1.11          | 1.11          |

Table 3a. $\tilde{Z}$ for various values of $r$ and $\beta$. 

continuum full theory. The continuum results are explained in detail in Refs. [10] and [11]. It is in this step that the determination of the scale $a^{-1}$ is finally used. The resulting factor, $Z_{cont}$, is $q^*$- and $a$-dependent. The result after multiplying by this factor is tabulated in Tables 4a and 4b. Table 4a corresponds to using the zero-distance representation of the axial current given in Eq. (2.6), and Table 4b corresponds to using the distance-one representation given in Eq. (2.11). Now that perturbation theory has been reorganized to include tadpole corrections to all orders, we expect our one-loop calculation of the renormalization factor to be accurate to about 7%. This estimate of the magnitude of the two-loop correction was obtained simply by squaring the largest one-loop correction ($0.27^2 = 0.07$) in Tables 4a and 4b.

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| 0.50 | 1.00          | 1.00          | 1.00          |
| 0.25 | 1.13          | 1.11          | 1.10          |
| 0.00 | 1.24          | 1.21          | 1.19          |
of 0.8 [6], which does not benefit from tadpole improvement. To make a head-to-head comparison it is necessary to multiply $Z_A = 0.8$ by $\sqrt{2}\kappa_c$ which is 0.56 at $\beta = 6.1$ [18]. The product is 0.45. Consequently, tadpole improvement results in a reduction of the physical value of $f_B$ by a factor of 0.37/0.45, i.e., a reduction of 18%. The change is similar to what one would have obtained using non-tadpole-improved perturbation theory with a larger, “boosted” value of $\alpha$, such as $\alpha_{\overline{\text{MS}}}$ evaluated at the scale of the inverse lattice spacing. The direction of the change reduces the expected magnitude of the $1/m$ corrections. These corrections must be included to fit heavy quark effective field theory calculations and calculations that extrapolate to the $b$-quark mass using propagating fermions with various moderately heavy masses. For a discussion of this extrapolation, see Ref. [4]. Note that in that work a boosted coupling is used in the determination of $Z$.

As mentioned in the introduction, both staggered and naive fermion results correspond to the $r = 0$ case [7]. The normalization convention for the Wilson fermion field is such that the coefficient of $\bar{\psi}\psi(n)$ in the action is unity. This term is not present when $r = 0$, and it is necessary to choose a different normalization. If one normalizes the field so that in the naive continuum limit $\bar{\psi}\phi\psi$ has coefficient one, then the factor of $1/(2\sqrt{r})$ does not need to be included. The factor of $1/u_0$
remains necessary in the point-split case.

We have applied the Lepage-Mackenzie tadpole improvement program to the case of the heavy-light quark current necessary to determine $f_B$ from lattice calculations. We considered both the zero-distance and distance-one point-split lattice representations of $J_5^0$. Our analysis included a slight generalization of the Lepage-Mackenzie prescription to the case $r \neq 1$. The results can be applied whether the light quark is treated as a Wilson, staggered, or naive fermion.

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