On the predictive power of local scale invariance

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Abstract. Local scale invariance (LSI) is a theory for anisotropic critical phenomena designed in the spirit of conformal invariance. For a given representation of its generators this leads to non-trivial predictions about the form of universal scaling functions. In the past decade several representations have been identified, and the corresponding predictions were confirmed for various anisotropic critical systems. Such tests are usually based on a comparison of two-point quantities such as autocorrelation and response functions. The present work highlights a potential problem of the theory in that it may predict any type of two-point function. More specifically, it is argued that for a given two-point correlator it is possible to construct a representation of the generators which exactly reproduces this particular correlator. This observation calls for a critical examination of the predictive content of the theory.

Keywords: correlation functions, critical phenomena

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1. Introduction

Local scale invariance (LSI) is a theory developed by Henkel and collaborators which generalizes global scale invariance of anisotropic critical systems and ageing phenomena to a local space–time-dependent symmetry [1]. It is inspired by the success of conformal invariance applied to two-dimensional equilibrium systems [2, 3] and uses a similar terminology.

Local scale transformations for anisotropic systems are generated by an infinite set of generators $X_{-1}, X_0, X_1, X_2, \ldots$ and $Y_{-1/z}, Y_{-1/z+1}, Y_{-1/z+2}, \ldots$. These generators obey the commutation relations

\[ [X_n, X_m] = (n - m)X_{n+m}, \]
\[ [X_n, Y_m] = \left(\frac{n}{z} - m\right)Y_{n+m}, \]

where $z$ is the dynamical exponent which quantifies the degree of anisotropy.

Remarkably, the theory yields predictions of the specific form of universal scaling functions appearing in response or correlation functions while it yields no prediction about the values of critical exponents. In recent years the theory has been extended and successfully applied to a large variety of models [4]–[21]. This success raised the hope that LSI could be a generic symmetry of scale-invariant anisotropic systems. However,
some authors reported results which seem to be incompatible with the predictions from LSI \[22\]–[29] which provoked a debate concerning the applicability of the theory.

In a given model, local scale invariance can be established by choosing a suitable representation of the algebra (1) and (2) in such a way that the system under consideration is invariant under transformations generated by \(X_n\) and \(Y_m\). In some cases such a representation can be derived exactly from an underlying partial differential equation, whereas in other cases LSI is just assumed as a hypothetical symmetry, leading to certain predictions which can be tested via numerical simulations. Since LSI is a model-independent theory these predictions depend exclusively on the chosen representation of its generators.

The simplest representation can be derived from the Schrödinger equation which describes diffusing particles with \(z = 2\) [1]. This representation was generalized to the case \(z \neq 2\) and successfully tested for \(z = 4\) [21]. Recently, Baumann and Henkel [30] found two representations of the LSI algebra (1), (2) for arbitrary \(z\) which involve non-local fractional derivatives [31] (see appendix B). The present work generalizes this concept even further by considering representations with arbitrary non-local operators. It is shown that the resulting representations of the LSI algebra are so general that in principle any two-point correlation function can be reproduced by the theory. This has important consequences as regards the predictive power of the theory, as will be discussed at the end of this paper.

2. Geometrical interpretation of the generators

To gain some intuition as to how the generators \(X_n\) and \(Y_m\) work let us first recall the geometrical interpretation of the generated transformations. For simplicity, we will restrict to the 1+1-dimensional case. The generalization to higher dimensions is not difficult and requires us to replace \(Y_m\) by a vector operator \(Y^{(j)}_m\); see [1,32].

The simplest representation of the generators \(X_n\) and \(Y_m\), which describes the geometrical content of local scale transformations, is given by

\[
X_n = -t^{n+1} \frac{\partial}{\partial t} - \frac{n+1}{z} t^n r \frac{\partial}{\partial r},
\]

\[
Y_{k-1/z} = -t^k \frac{\partial}{\partial r},
\]

where we have adopted the convention of non-integral indices \(m = k - 1/z\) with \(k \in \mathbb{N}\). One can easily verify that these operators satisfy the commutation relations (1) and (2). Moreover, one can see that they carry the physical dimensions

\[
[X_n] = [\text{time}]^n,
\]

\[
[Y_{k-1/z}] = [\text{time}]^k [\text{length}]^{-1}.
\]

As shown in figure 1, each of these generators corresponds to a well-defined geometrical transformation in space–time. For example, the generator \(X_{-1} = -\partial_t\) generates translations in time while \(Y_{-1/z} = -\partial_r\) generates translations in space:

\[
\exp(\tau X_{-1}) f(t,r) = f(t-\tau,r),
\]

\[
\exp(sY_{-1/z}) f(t,r) = f(t,r-s).
\]
Likewise \( X_0 = -t\partial_t - (1/z)r\partial_r \) generates anisotropic dilatations
\[
\exp(\lambda X_0)f(t, r) = f(t/b, r/b^{1/z})
\]
by the factor \( b = e^\lambda \) and thus it can be identified as the generator of global scale transformations. Here the amount of anisotropy is controlled by the dynamical exponent \( z \).

The lowest non-trivial operators, which mix space and time, are \( X_1 \) and \( Y_{1-1/z} \). As shown in figure 1, the operator \( Y_{1-1/z} \) generates a Galilei transformation
\[
\exp(cY_{1-1/z})f(t, r) = f(t, r - ct),
\]
which may be interpreted as a global shear transformation in space–time. Similarly, \( X_1 \) generates a dilatation with an elongation factor proportional to the actual time. By combining the action of all generators it is possible to generate the full group of local scale transformations. Note that these transformations obey causality, i.e., in figure 1 horizontal lines will always be mapped onto horizontal lines.

3. Local scale invariance of quasi-primary fields

As a next step, the geometric generators (3) and (4) have to be extended in order to describe the symmetry properties of a scale-free critical phenomenon. Within the framework of LSI it is assumed that the physical properties of such a system can be expressed in terms of so-called quasi-primary fields\(^1\), here denoted as \( \phi(t, r) \), which transform covariantly under the action of the generators. As we will see, this requires us to extend the representation of the generators with additional terms.

\(^1\) Currently it is not yet fully clear how quasi-primary fields can be characterized in anisotropic systems. Usually it is believed that order parameter fields are quasi-primary while their derivatives are not. For a discussion see [15].
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Figure 2. Action of the generator $Y_{-1/2}$ using the example of a simple diffusion equation $\partial_t \phi(r, t) = D \nabla^2 \phi(r, t)$. In this case the two-point response function is an ordinary Gaussian distribution $G(r, t) = (1/\sqrt{\pi Dt}) \exp(-r^2/4Dt)$ which is shown in the left panel. As can be seen, this function is not invariant under infinitesimal transformations generated by $Y_{1/2} = -t \partial_r$ since geometrical shear leads to a skewed function of the form $G(r, t)(1 + r \epsilon/2D)$. Therefore, the generator $Y_{1/2} = -t \partial_r$ is not a symmetry operator; rather it has to be extended with an additional term that compensates this tilt, restoring the original form of the function. For the diffusion equation this term takes the simple form $-Mr$ with the so-called mass $M = 1/2D$. As shown in the right panel, a subsequent application of this term restores the original non-tilted function up to order $O(\epsilon^2)$. This demonstrates the mechanism which makes the diffusion equation invariant under the action of the generator $Y_{1/2} = -t \partial_r - Mr$.

For simplicity let us assume that the system under consideration is translationally invariant in space and time, for example a critical kinetic Ising model in its stationary equilibrium state. This means that all quasi-primary fields are translationally invariant as well; hence the two operators $X_{-1} = -\partial_t$ and $Y_{-1/2} = -\partial_r$ are already symmetry operators and need no extension. The situation is different for global dilatations generated by $X_0$. Here a quasi-primary field changes its amplitude according to the scaling law

$$
\phi(t/b, r/b^{1/z}) = b^{x/z} \phi(t, r),
$$

where the exponent $x$ is the so-called scaling index associated with the field $\phi(t, r)$. Consequently, the bare generator $X_0$ defined in (9) is no longer a symmetry operator; rather it has to be extended with a suitable term that compensates the change in the field amplitude. Obviously the required term is just a constant, leading to the standard representation

$$
X_0 = -t \partial_t - \frac{1}{z} r \partial_r - \frac{x}{z}.
$$

Next, let us consider the operator $Y_{1-1/z}$ which generates global shear transformations. It is important to note that global shear itself is generally no symmetry transformation because it distorts the fields in a non-trivial way. This is demonstrated in figure 2 using the example of the diffusion equation where $z = 2$. As can be seen, global shear distorts the response function, leading to a skewed profile in space–time. Therefore, the generator
Y_{1/z} has to be extended with a suitable additional term that compensates this distortion. For the diffusion equation this term takes the particularly simple form $-Mr$, where $M$ is the so-called mass parameter. However, in general the required compensation terms may be much more complicated.

To be as general as possible, we therefore assume that $Y_{1/z}$ is extended with an arbitrary linear operator $B_{1/z}$:

$$Y_{1/z} = -t \partial_r - B_{1/z}. \quad (13)$$

Likewise we assume that all remaining generators, which generate combinations of shear and dilatations, are extended with certain linear operators as well:

$$X_n = -t^{n+1} \partial_t - \frac{n + 1}{z} t^n r \partial_r - \frac{x}{z} (n + 1)t^n - A_n \quad n = 1, 2, 3, \ldots, \quad (14)$$

$$Y_{k/z} = -t^k \partial_r - B_{k/z} \quad k = 1, 2, 3, \ldots. \quad (15)$$

The linear operators $A_n$ and $B_n$, by means of which the generators are extended, are of course not independent; rather they are constrained by the commutation relations. For example, the operator $A_2$ cannot be chosen freely; instead it is constrained by the commutation relations

$$[X_2, X_0] = 2X_2 \implies [A_2, X_0] = 2A_2, \quad (16)$$

$$[[X_2, X_-], X_-] = 6X_0 \implies [[A_2, X_-], X_-] = 0. \quad (17)$$

### 4. Iterative construction of the generators

The first point of this work is to show that any space–time representation of the LSI commutation relations is fully determined by the generator $X_2$ or, equivalently, by the operator $A_2$.\footnote{If one is only interested in the subalgebra \{X_{-1}, X_0, X_1, Y_{1/z}, Y_{-1/z}\} it even suffices to specify $A_1$.}

The construction starts by choosing the operator $A_2$ in such a way that it obeys the constraints (16). Once $A_2$ is specified, all other operators can be constructed iteratively as follows. The lowest generators $X_{-1}, X_0$ and $Y_{-1/z}$ are always given by their standard representation

$$X_{-1} := -\partial_t, \quad (18)$$

$$X_0 := -t \partial_t - \frac{1}{z} r \partial_r - \frac{x}{z}, \quad (19)$$

$$Y_{-1/z} := -\partial_r. \quad (20)$$

Moreover, the generator $X_1$ can be computed by setting

$$X_1 := \frac{1}{3} [X_2, X_{-1}], \quad (21)$$

$$[X_2, X_0] = 2X_2 \implies [A_2, X_0] = 2A_2, \quad (16)$$

$$[[X_2, X_-], X_-] = 6X_0 \implies [[A_2, X_-], X_-] = 0. \quad (17)$$
meaning that \( A_1 = \frac{1}{3}[\partial_t, A_2] \). Likewise, all other generators can be constructed recursively using

\[
X_n := \frac{1}{n - 2} [X_{n-1}, X_1] \quad n = 3, 4, 5, \ldots ,
\]

\[
Y_m := \frac{1}{m - 1/z - 1} [Y_{m-1}, X_1] \quad m = 1 - 1/z, 2 - 1/z, 3 - 1/z, \ldots .
\]

As shown in appendix A, any set of generators which is constructed in such a way satisfies all commutation relations in equations (1) and (2) automatically. Therefore, we can conclude that any representation is fully determined by a single linear operator, namely, \( A_2 \).

5. Representation with integral kernels

Given that the whole representation is determined by the linear operator \( A_2 \), one has to specify the most general form of this operator under the constraints (16). To this end it is convenient to represent \( A_n \) acting on some function \( \phi \) in the form of a convolution integral

\[
[A_n \phi](t,r) = \int dt' \int dr' A_n(t,r,t',r')\phi(t',r'),
\]

with a kernel \( A_n(t,r,t',r') \) which can be thought of as the ‘matrix elements’ of \( A_n \). Likewise, the operators \( B_m \), which appear in the generator \( Y_m \), can be written as convolution integrals:

\[
[B_m \phi](t,r) = \int dt' \int dr' B_m(t,r,t',r')\phi(t',r'),
\]

with a kernel \( B_m(t,r,t',r') \). The local contributions of these kernels, which appear as ordinary differential operators in the LSI representation, correspond to Dirac \( \delta \)-functions and their derivatives. However, in general the kernel may be non-local in space and time, including the recently discovered non-local representations involving fractional derivatives as special cases.

Because of the commutation relation \([X_n, X_0] = nX_n\) the kernel itself is a generalized homogeneous function under anisotropic dilatation by a factor \( b > 0 \):

\[
A_n(t/b, r/b^{1/z}, t'/b, r'/b^{1/z}) = b^{1/z + 1 - n} A_n(t, r, t', r').
\]

Moreover, the kernel \( A_2 \) is constrained by \([[[X_2, X_{-1}], X_{-1}] = 0\) in equation (16), tantamount to \([A_2, \partial_t, \partial_r] = 0\), which implies that \( A_2 \) has to be of the form

\[
A_2(t, r, t', r') = \frac{3}{2} (t + t') K(t - t', r, r') + L(t - t', r, r'),
\]

where \( K \) and \( L \) are functions which depend only on the time difference \( t - t' \). Because of equation (26), they are generalized homogeneous functions as well:

\[
K \left( \frac{t - t'}{b}, \frac{r}{b^{1/z}}, \frac{r'}{b^{1/z}} \right) = b^{1/z} K(t - t', r, r'),
\]

\[
L \left( \frac{t - t'}{b}, \frac{r}{b^{1/z}}, \frac{r'}{b^{1/z}} \right) = b^{1/z - 1} L(t - t', r, r').
\]
Hence any representation of the LSI algebra is determined by two homogeneous time-translation-invariant functions $K$ and $L$.

Because of $X_1 = \frac{1}{3}[X_2, X_{-1}]$ the kernel $A_1 = -\frac{1}{3}[A_2, \partial_t]$ is essentially the temporal derivative of $A_2$, i.e.

$$A_1(t, r, t', r') = K(t - t', r, r').$$  \hspace{1cm} (30)

Because of equation (2), the operators $B_m$ are given by

$$B_{k-1} = \frac{z}{k+1} [A_k, \partial_t], \hspace{1cm} k = 0, 1, 2, \ldots, $$  \hspace{1cm} (31)

so that the kernel of the shear generator $Y_{1-1/z}$ reads

$$B_{1-1}(t, r, t', r') = \frac{z}{2} \left( \frac{\partial}{\partial_t} + \frac{\partial}{\partial_r} \right) K(t - t', r, r').$$  \hspace{1cm} (32)

Again this kernel is a generalized homogeneous function, i.e.,

$$B_{1-1}(\frac{t}{b}, \frac{r}{b^{1/z}}, \frac{t'}{b}, \frac{r'}{b^{1/z}}) = b^{2/z} B_{1-1}(t, r, t', r').$$ \hspace{1cm} (33)

Once all kernels have been determined, one can easily check them by testing the commutation relations $[X_n, X_{-1}] = (n+1)X_{n-1}$ and $[Y_n, X_{-1}] = (n+1/z)Y_{n-1}$, which can be translated into the differential equations

$$\left( \frac{\partial}{\partial_t} + \frac{\partial}{\partial_r} \right) A_n(t, r, t', r') = (n+1)A_{n-1}(t, r, t', r'),$$ \hspace{1cm} (34)

$$\left( \frac{\partial}{\partial_t} + \frac{\partial}{\partial_r} \right) B_n(t, r, t', r') = (n+1/z)B_{n-1}(t, r, t', r').$$ \hspace{1cm} (35)

6. Two-point correlation functions

Let us now investigate the properties of a correlation function

$$C(t, r, t', r') = \langle \phi_1(t, r)\phi_2(t', r') \rangle \hspace{1cm} (t > t')$$ \hspace{1cm} (36)

of two quasi-primary fields $\phi_1$ and $\phi_2$. For simplicity let us assume that the two fields carry the same scaling dimension $x_1 = x_2 = x$ and that the correlator respects causality, i.e., it is non-zero only for $t \geq t'$. Because of translational invariance in space and time this function depends only on the differences of the coordinates. Moreover, invariance under global scale transformations, as expressed by the condition $X_0C = 0$, implies the scaling form

$$C(t, r, t', r') = \Theta(t - t')(t - t')^{-2x/z} \Phi \left( \frac{r - r'}{(t - t')^{1/z}} \right),$$ \hspace{1cm} (37)

where $\Phi$ is a scaling function and $\Theta(t - t')$ is the Heaviside step function which accounts for causality.

In the following we first address the problem of how $\Phi$ can be determined for a given representation of LSI generators. Then we consider the inverse problem, i.e., for a given correlation function we ask for a suitable representation of the generators.

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6.1. Integro-differential equation for the scaling function

LSI is based on the postulate that correlation functions of primary fields are invariant under the action of the generators, i.e., $X_n C = Y_m C = 0$. These conditions lead to integro-differential equations which determine the form of the scaling function $\Phi$. It is important to note that in the LSI theory the representations of the generators acting on $\phi_1$ and $\phi_2$ are generally different. More specifically, it was argued that a correlator vanishes unless the two representations are related in a specific way, giving rise to so-called Bargmann superselection rules. For example, in the standard Schrödinger representation the ‘mass terms’ occurring in the generators are known to have different signs. For this reason we will work with two different representations, denoting the integral kernels acting on $\phi_1$ and $\phi_2$ with the superscripts $^{(1)}$ and $^{(2)}$, respectively.

Let us first consider the generator $X_1$, which acts on the two-point function as follows:

$$0 = [X_1 C](t, r, t', r')$$
$$= -\left( t^2 \partial_t + t'^2 \partial_{t'} + \frac{2tr}{z} \partial_r + \frac{2t'r'}{z} \partial_{r'} + \frac{2xt}{z} + \frac{2xt'}{z} \right) C(t, r, t', r')$$
$$- \int_{t'}^t dt'' \int_{-\infty}^{\infty} dr'' K^{(1)}(t - t'', r, r') C(t'', r'', t', r')$$
$$- \int_{t'}^t dt'' \int_{-\infty}^{\infty} dr'' K^{(2)}(t' - t'', r', r') C(t, r, t'', r''). \quad (38)$$

Inserting the scaling form (37) and using the homogeneity condition (28) one obtains an integro-differential equation. As we have assumed the correlator to be translationally invariant, this equation is generally overdetermined unless the kernels $K^{(1,2)}$ obey specific constraints, referred to as Bargmann superselection rules. A straightforward calculation shows that one obtains an autonomous integro-differential equation for the scaling function $\Phi$ if and only if the two kernels have the form

$$K^{(1)}(t - t', r, r') = \frac{1}{4}(r + r')^2 f(t - t', r - r') + \frac{1}{2}(r + r') g(t - t', r - r') + h(t - t', r - r'), \quad (39)$$
$$K^{(2)}(t - t', r, r') = -\frac{1}{4}(r + r')^2 f(t' - t, r' - r) + \frac{1}{2}(r + r') g(t' - t, r' - r) - h(t' - t, r' - r), \quad (40)$$

where $f, g, h$ are certain functions which depend on only two parameters. Because of equation (28) these functions are homogeneous; hence they obey the scaling form

$$f(\tau, \xi) = \tau^{-3/z} F(\xi \tau^{-1/z}),$$
$$g(\tau, \xi) = \tau^{-2/z} G(\xi \tau^{-1/z}),$$
$$h(\tau, \xi) = \tau^{-1/z} H(\xi \tau^{-1/z}). \quad (41)$$
With these kernels and the scaling form (37) the integral equation (38) turns into an integro-differential equation for the scaling function \( \Phi(\xi) \):

\[
\Phi'(\xi) + z \int_{-\infty}^{\infty} d\tilde{\xi} \int_{0}^{1} d\mu \mu^{(1-2x)/z} \left[ \xi(1-\mu)^{-3/z} \mu^{1/z} F\left( \frac{\xi - \tilde{\xi}}{(1-\mu)^{1/z}} \right) \right] + (1-\mu)^{-2/z} G\left( \frac{\xi - \tilde{\xi}}{(1-\mu)^{1/z}} \right) \Phi(\tilde{\xi}) = 0.
\]

(42)

As can be seen, the kernel function \( H \) drops out, meaning that it does not influence the form of the two-point functions. Knowing the kernels \( F, G \), this is the integro-differential equation which determines the scaling function \( \Phi \) of the two-point correlation function.

Turning to the generator of shear transformations \( Y_{1-1/z} \), one can show that the invariance condition \([Y_{1-1/z}] (t, r, t', r') = 0 \) leads exactly to the same integral equation and thus does not provide any new information.

6.2. Inverse problem

Let us now consider the inverse problem which can be formulated as follows: for a given two-point function characterized by \( x, z, \) and \( \Phi \) we would like to determine appropriate kernel functions \( K^{(1,2)} \) such that \( \Phi \) is a solution of equation (42). Since these kernel functions determine all LSI generators through recursion relations, solving the inverse problem would mean constructing a representation that renders exactly a given two-point function. In the following we outline how this problem may be solved.

For simplicity let us restrict to the case \( G = H = 0 \), i.e. we want to determine \( F \) for a given \( \Phi \). With the substitution \( \xi \to \mu^{-1/z} \xi \) the integro-differential equation (42) turns into a convolution product

\[
\Phi'(\xi) + z \int_{-\infty}^{\infty} d\tilde{\xi} \int_{0}^{1} d\mu \mu^{-2x/z}(1-\mu)^{-3/z} \tilde{\xi} F\left( \frac{\xi - \tilde{\xi}}{(1-\mu)^{1/z}} \right) F\left( \frac{\xi - \tilde{\xi}}{(1-\mu)^{1/z}} \right) \Phi(\mu^{-1/z}\tilde{\xi}) = 0.
\]

By introducing the Fourier transforms

\[
F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk e^{i\alpha k} \hat{F}(k),
\]

(43)

\[
\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk e^{ik\xi} \hat{\Phi}(k)
\]

(44)

one is led to

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk e^{ik\xi} ik \hat{\Phi}(k) = -\frac{z}{2\pi} \int_{0}^{1} d\mu \mu^{(1-2x)/z}(1-\mu)^{-2/z} \int_{-\infty}^{+\infty} \tilde{\xi} \int_{-\infty}^{+\infty} dk_{1} \int_{-\infty}^{+\infty} dk_{2} \hat{F}(\mu^{1/z}k_{1}) \hat{F}(\mu^{1/z}k_{2}) \frac{\partial}{\partial k_{2}} \Phi(\mu^{1/z}k_{1}) \frac{1}{\mu^{1/z}k_{1}} e^{ik_{1}(\xi - \tilde{\xi}) + ik_{2} \xi} = 0.
\]

(45)

After integrating by parts in \( k_{2} \) one can carry out the integration over \( \tilde{\xi} \). Finally another Fourier transformation of the entire equation yields

\[
\frac{k \hat{\Phi}(k)}{\sqrt{2\pi}} + z \int_{0}^{1} d\mu \mu^{2-2x/z}(1-\mu)^{-2/z} \hat{F}(1-\mu)^{1/z} k) \hat{\Phi}(\mu^{1/z}k) = 0.
\]

(46)
This equation holds for all \( k \in \mathbb{R} \). As \( \Phi \) and \( F \) are symmetric, let us restrict to \( k > 0 \). Substituting \( \lambda = \mu^{1/z} k \) one obtains an inhomogeneous Volterra integral equation of the first kind for \( \tilde{F}(\lambda) \):
\[
\tilde{\Phi}(k) = \int_0^k d\lambda \Psi(k, \lambda) \tilde{F}(\lambda), \tag{47}
\]
with the kernel
\[
\Psi(k, \lambda) = \sqrt{2\pi z^2 k^{1-z} \lambda^{3-z} \left[ 1 - \lambda^z k^{-z} \right]^{(2-2x)/z} \tilde{\Phi}'([1 - \lambda^z k^{-z}]^{1/z} k)}. \tag{48}
\]
Hence for given \( x, z, \) and \( \Phi \) the solution of the inverse problem amounts to performing the following non-trivial steps:

(i) compute the Fourier transform of \( \Phi \) in equation (44);
(ii) plug \( \tilde{\Phi} \) into equation (48), compute the kernel \( \Psi(k, \lambda) \), and solve the integral equation (47) to obtain \( \tilde{F} \);
(iii) compute the inverse Fourier transform of \( \tilde{F} \), plug it into equations (41) and (39) and determine the kernel \( K \); this kernel establishes the representation of \( X_1 \);
(iv) determine the LSI generators by recursion, setting e.g. \( L = 0 \).

7. Discussion

The success of LSI applied to various exactly solvable systems raised the hope that this might be a generic feature of anisotropic scale-free phenomena, including systems which cannot be solved exactly. To verify this expectation, various authors measured two-point autocorrelation and response functions numerically and compared them with the predictions from LSI using a suitable representation of its generators. For some systems, most notably the kinetic Ising model and the contact process, deviations were found, leading the authors to the conclusion that those systems are probably not invariant under local scale transformations.

However, such a conclusion would be premature. The results of the preceding section lead to the conjecture that for any physically meaningful two-point function with arbitrary \( x \) and \( z \) it is possible to construct a representation of LSI generators which precisely reproduces the desired two-point function. On the one hand, this means that numerical discrepancies do not necessarily falsify LSI; instead they could also come from choosing the wrong representation. On the other hand, the space of possible representations is so huge that LSI itself can probably not be used to predict the form of two-point functions; hence on this level it cannot be used to set up a classification scheme of anisotropic critical phenomena.

For the sake of simplicity the present work was restricted to stationary situations in 1+1 dimensions, where correlation functions depend only on differences of the coordinates. It would be interesting to investigate recent applications of LSI to ageing \([4, 5, 9, 19, 21]\), \([33]–[38]\) along similar lines. Here the algebra is relaxed by giving up time-translational invariance, probably reducing the predictive power of the theory even further.

Why is LSI less predictive than conformal invariance in two dimensions? In my opinion this issue can be traced back to different symmetry properties. Conformal invariance \([2]\) generalizes global dilatations combined with rotations to a local symmetry.
Similarly, LSI generalizes global dilatations combined with shear transformations to a local symmetry. However, rotations and shear transformations are very different in character. In the case of conformal invariance, an isotropic equilibrium model is expected to be rotationally invariant in itself. Contrarily, as demonstrated in figure 2, an anisotropic process is not automatically invariant under shear in itself; rather this invariance has to be established manually by adding suitable terms to the generators. At this point the theory of LSI requires an input of extra information which is not needed in conformally invariant systems. The message of this paper is that the ambiguity caused by this additional information reduces the predictive power of LSI. More specifically, it is suggested that this information can be expressed in terms of functions with a single parameter, providing so many degrees of freedom that any two-point function can be reproduced using the theory.

The present findings make it easy to see why LSI was applied successfully to many exactly solvable systems while it continued to fail for certain non-integrable systems. In exactly solvable systems one usually arrives at a partial differential equation which determines the two-point function. This allows one to derive suitable LSI generators in a closed form. For a non-integrable system such as directed percolation [26] the scaling function \( \Phi \) is a complex object which involves loop corrections to all orders of the underlying field theory. The corresponding LSI generators may exist, but apparently it is impossible to write them down in a closed form. In such cases the attempt to guess a suitable representation and to confirm it numerically by comparing two-point functions is likely to fail.

The results of the present work do not rule out LSI having some predictive power on the level of three-point functions. In my opinion this is one of the key issues to address in the future.

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Appendix A. Consistency of the construction scheme

In this appendix it is shown that the generator \( X_2 \), constrained by equation (16), determines all generators iteratively in such a way that all commutation relations are satisfied.

A.1. Commutators \( [X_n, X_m] = (n - m)X_{n+m} \)

To prove these commutation relations by induction, we first notice that

\[
[X_0, X_{-1}] = X_{-1}, \quad [X_2, X_{-1}] = 3X_1, \\
[X_2, X_0] = 2X_2, \quad [X_1, X_{-1}] = 2X_0.
\]  
\[(A.1)\]

The first relation is always fulfilled, while the second one was used to define \( X_1 \) in equation (21). The third and the fourth relation have been used as constraints for \( X_2 \) in equation (16) and thus they are satisfied as well. Moreover, the generators \( X_n \) have been constructed according to equation (22); hence the commutators

\[
[X_n, X_1] = (n - 1)X_{n+1} \quad n = 2, 3, \ldots
\]  
\[(A.2)\]

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are valid by construction. Anchored at these relations, the remaining commutation relations $[X_n, X_m] = (n - m)X_{n+m}$ can be proven by induction. To this end let us assume that these commutation relations hold for $n + m = N - 1$ and show that they also hold for $n + m = N$:

$$[X_n, X_m] = \frac{1}{n-2}[[X_{n-1}, X_1], X_m]$$

$$= \frac{1}{n-2} \left( [[X_m, X_1], X_{n-1}] + [[X_{n-1}, X_m], X_1] \right)$$

apply (22)

induction

$$= \frac{1}{n-2} \left( (m - 1)[X_{m+1}, X_{n-1}] + (n - m - 1)[X_{n+m-1}, X_1] \right)$$

apply (22)

$$= \frac{1}{n-2} \left( (m - 1)[X_{m+1}, X_{n-1}] + (n - m - 1)(n + m - 2)X_{n+m} \right).$$

(A.3)

This is again a recursion relation with fixed $N$ for an inductive step from $(n - 1, m + 1)$ to $(n, m)$. More specifically, if the commutation relation $[X_{m+1}, X_{n-1}] = (m - n + 2)X_{m+n}$ is known to be valid, the above equation implies that the commutator

$$[X_n, X_m] = \frac{(1-m)(n-m-2) - (m-n+1)(n+m-2)}{n-2} X_{n+m}$$

$$= (n-m)X_{n+m}.$$  

(A.4)

The same recursion works also in a different direction as an inductive step from $(n+1, m-1)$ to $(n, m)$. This twofold recursion scheme allows one to check all commutators of the form $[X_n, X_m] = (n-m)X_{n+m}$ iteratively.

**A.2. Commutators** $[X_n, Y_m] = (n/z - m)Y_{n+m}$

To prove these commutation relations, we first show by induction that

$$[X_n, Y_{1/z}] = \frac{n+1}{z} Y_{n-1/z}.   \quad (A.5)$$

The induction is anchored at $n = -1$ since $[X_{-1}, Y_{1/z}] = [-\partial_t, -\partial_x] = 0$. Assuming that the commutation relations are satisfied for $n = 1$, i.e.

$$[X_{n-1}, Y_{1/z}] = \frac{n}{z} Y_{n-1-1/z},$$

(A.6)

the same relations hold for $n$ because

$$[X_n, Y_{1/z}] = \frac{1}{n-2} [[X_{n-1}, X_1], Y_{1/z}]$$

$$= \frac{1}{n-2} \left( [X_1, Y_{1/z}, X_{n-1}] + [X_{n-1}, [X_1, Y_{1/z}]] \right)$$

apply (A.6)

apply (23)

$$= \frac{1}{n-2} \left( \frac{n}{z} [X_1, Y_{n-1-1/z}] + \frac{2}{z} [X_{n-1}, Y_{1-1/z}] \right)$$

apply (23)

apply (A.6)

$$= -\frac{(n/z)(2/z - n + 1) + (2/z)(n/z - 1)}{n-2} Y_{n-1/z} = \frac{n+1}{z} Y_{n-1/z}.   \quad (A.7)$$

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Next, let us assume that the commutation relations
\[ [X_n, Y_{m-1}] = (n/z - m - 1)Y_{n+m-1} \] (A.8)
hold for all \( n \) and a given \( m \), anchored at \( m = -1/z \) by equation (A.5). Then we can prove the remaining relations through a another induction:
\[
[X_n, Y_m] = \frac{1}{1/z + 1 - m} [X_n, [X_1, Y_{m-1}]] \\
= \frac{1}{1/z + 1 - m} \left( [Y_{m-1}, X_n], X_1 \right) + [X_n, X_1, Y_{m-1}] \\
= \frac{1}{1/z + 1 - m} \left( [Y_{m-1}, X_n], X_1 \right) + (n - 1) [X_{n+1}, Y_{m-1}] \\
= \frac{(n/z - m + 1)(1/z - m - n + 1) + (n - 1)((n + 1)/z - m + 1)}{1/z + 1 - m} Y_{n+m} \\
= \left( \frac{n}{z} - m \right) Y_{n+m}.
\] (A.9)

Appendix B. The most important representations

This appendix demonstrates that the most important representations of LSI derived so far can be described within the unified framework of generating kernel functions.

(a) Schrödinger representation

The standard Schrödinger form for diffusive systems with \( z = 2 \) is defined by [39]–[41]
\[
X_n = -t^{n+1} \partial_t - \frac{n+1}{2} t^n r \partial_r - \frac{x}{2} (n + 1) t^n - \frac{n(n + 1)}{4} \mathcal{M} t^{n-1} r^2, \quad (B.1)
\]
\[
Y_m = -t^{m+1/2} \partial_r - (m + 1/2) \mathcal{M} t^{m-1/2} r, \quad (B.2)
\]
where \( \mathcal{M} \) is the so-called mass parameter. As can be verified easily, this representation is generated by the kernel functions
\[
K(t - t', r, r') = \frac{1}{2} \mathcal{M} r^2 \delta(r - r') \delta(t - t'), \quad L(t - t', r, r') = 0. \quad (B.3)
\]

(b) Local representation type (i) for arbitrary \( z \)

This representation is given by [1]
\[
X_n = -t^{n+1} \partial_t - \frac{n+1}{z} t^n r \partial_r - \frac{x}{z} (n + 1) t^n - \frac{n(n + 1)}{2} B_{10} t^{n-1} r^z, \quad (B.4)
\]
\[
Y_{k-1/z} = -t^k \partial_r - \frac{z^2}{2} k B_{10} t^{k-1} r^{z-1}, \quad (B.5)
\]
and reduces to the Schrödinger representation for \( z = 2 \). The corresponding kernel reads
\[
K(t - t', r, r') = B_{10} r^z \delta(r - r') \delta(t - t'), \quad L(t - t', r, r') = 0. \quad (B.6)
\]
(c) Extended local representation type (ii) for $z = 2$

The representation [1]

$$X_n = -t^{n+1} \partial_t - \frac{n+1}{2} t^n r \partial_r - \frac{x}{2} (n+1) t^n - \frac{n(n+1)}{2} \frac{B_{10}}{A_{10}} t^{n-1} r^2 - \frac{(n^2-1)n}{6} \frac{B_{20}}{A_{20}} t^{n-2} r^4,$$

(B.7)

$$Y_{k-1/z} = -t^k \partial_r - 2 k B_{10} t^{k-1} r - \frac{4}{3} k (k-1) B_{20} t^{k-2} r^3$$

involves both kernels $K$ and $L$:

$$K(t-t', r, r') = B_{10} r^2 \delta(r-r') \delta(t-t'),$$

(B.9)

$$L(t-t', r, r') = B_{20} r^4 \delta(r-r') \delta(t-t').$$

(B.10)

(d) Local representation type (iii) for $z = 1$

The ‘conformally invariant’ representation [1]

$$X_n = -t^{n+1} \partial_t - A_{10}^{-1} [(t + A_{10} r)^{n+1} - t^{n+1}] \partial_r - (n+1) x t^n - \frac{n+1}{2} \frac{B_{10}}{A_{10}} [(t + A_{10} r)^n - t^n],$$

(B.11)

$$Y_{k-1} = -(t + A_{10} r)^k \partial_r - \frac{k}{2} B_{10} (t + A_{10} r)^{k-1} r$$

(B.12)

corresponds to

$$K(t-t', r, r') = \delta(t-t') \left[ A_{10} r^2 \delta'(r-r') + B_{10} r \delta(r-r') \right],$$

(B.13)

$$L(t-t', r, r') = \delta(t-t') \left[ A_{10}^2 r^2 \delta'(r-r') + \frac{2}{2} B_{10} A_{10} r^2 \delta(r-r') \right].$$

(B.14)

(e) Temporally non-local representation for arbitrary $z$

This representation, called type I in [30], is given by

$$X_1 = -t^2 \partial_t - \frac{2}{z} t r \partial_r - \frac{2x}{z} t - (\beta + \gamma) r^2 \partial_r^{2-z} - 2 \gamma (2-z) r \partial_r^{1-z} - \gamma (2-z)(1-z) \partial_r^{-z},$$

(B.15)

$$Y_{1-1/z} = -t \partial_r - (\beta + \gamma) z r \partial_r^{2-z} - \gamma z (2-z) \partial_r^{1-z}.$$  

(B.16)

It is non-local in space and involves fractional derivatives, depending on $z$ even with negative powers. Fractional derivatives can always be expressed with integral kernels. As one can see, this would determine the kernel $K$ while the kernel $L$ vanishes.
\( X_1 = -t^2 \partial_t - \frac{2}{z} t r \partial_r - \frac{2 x}{z} t - \alpha r^2 \partial^2_t z^{-1}, \) \hspace{1cm} (B.17)

\( Y_{1-1/z} = -t \partial_r - \alpha r^2 \partial^2_t z^{-1}. \) \hspace{1cm} (B.18)

As it involves temporal fractional derivatives, this representation is non-local in time. Again this would determine the form of the kernel \( K \) while the kernel \( L \) vanishes.

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