Reconstruction of Small Inclusions in Electrical Impedance Tomography Problems

Xiaoping Fang\textsuperscript{1,2}, Youjun Deng\textsuperscript{3,*} and Xiaohong Chen\textsuperscript{2,4}

\textsuperscript{1}School of Mathematics and Statistics, Hunan University of Commerce, Changsha 410205, China.
\textsuperscript{2}Institute of Big Data and Internet Innovation, Hunan University of Commerce, Changsha 410205, China.
\textsuperscript{3}School of Mathematics and Statistics, Central South University, Changsha 410083, China.
\textsuperscript{4}School of Business, Central South University, Changsha 410083, China.

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Abstract. An inverse problem to recover small inclusions inside a two-layer structure is considered. Integral representations for the solution of two-layer inhomogeneous conductivity problem are derived and asymptotic expansions of a perturbed electrical field are obtained. Moreover, the uniqueness of the recovery of the locations and conductivities of small inclusions is proved.

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1. Introduction

Electrical impedance tomography (EIT) is a noninvasive type of medical imaging, where the electrical conductivity of a part of the body (inclusion) is inferred from surface electrode measurements. The current sources — e.g. electrodes, are placed on the surface and voltage is measured (or vice versa) at a few or all source positions. EIT has been widely used in monitoring the lung function [15], breast cancer imaging [4,19] and so on. In mathematical terms, the problem consists in the recovering of conductivity from surface currents and potentials. The unique recovering from infinitely many measurements or from the Newmann-to-Dirichlet map is considered in Refs. [5,7,13,16,20]. If only finitely many measurements are available, the uniqueness is related to recovery of the inclusion shape, whereas global uniqueness is only obtained for convex polyhedrons and balls in three-dimensional space and for polygons and disks in the plane — cf. [6,10–12,14,18].

*Corresponding author. Email addresses: fxp12220163.com (X. Fang), youjundeng@csu.edu.cn, dengyijun_001@163.com (Y. Deng), xiaohongchen@163.com (X. Chen)
We also refer the reader to works [8, 9, 21], where numerical methods in imaging are considered. Here, we deal with the recovery of small inclusions inside of an inhomogeneous body, containing inhomogeneous core and homogeneous outer layers. The small inclusions are buried in the homogeneous layer and we only need to use one measurement to recover the corresponding locations and conductivity.

Let $D$ be a central region generating an electric field and $\Omega \setminus \overline{D}$ be the surrounding body tissue. We assume that $D$ is compactly supported in $\Omega$ — i.e. $D \subset \subset \Omega$, and consider the following problem:

$$
\nabla \cdot \sigma \nabla u = f \quad \text{in} \quad \mathbb{R}^3,
$$

$$
u(x) = \theta(|x|^{-1}) \quad \text{as} \quad x \to \infty
$$

with the source term $f$ supported in $D$ and such that

$$
\int_{\mathbb{R}^3} f(x) dx = 0.
$$

If there are no small inclusions, the electric conductivity function has the form

$$
\sigma(x) = \sigma_0(x) \chi(D) + k_0 \chi(\Omega \setminus D) + \chi(\mathbb{R}^3 \setminus \overline{\Omega}),
$$

where $\chi$ denotes the characteristic function. Besides, we also assume that the background tissue contains a group of small inclusions. Let $l_0 \in \mathbb{N}^+$ and $D_l$ be small inclusions with the conductivities $k_l > 0$, $l = 1, 2, \cdots, l_0$. In the presence of small inclusions $(D_l; k_l)$, \( l = 1, 2, \cdots, l_0 \), the conductivity distribution can be described as

$$
\sigma(x) = \sigma_0(x) \chi(D) + k_0 \chi(\tilde{D}) + \sum_{l=1}^{l_0} k_l \chi(D_l) + \chi(\mathbb{R}^3 \setminus \overline{\Omega}),
$$

where

$$
\tilde{D} := \Omega \setminus \bigcup_{l=1}^{l_0} D_l \cup D.
$$

Let $u_0$ and $u$ be, respectively, the solutions of the conductivity problems (1.1), (1.2) and (1.1), (1.3). Here, we are looking for the solution of the following inverse problem:

$$
(u(x) - u_0(x)) \bigg|_{x \in \partial \Omega} \rightarrow \bigcup_{l=1}^{l_0} (D_l; k_l).
$$

More exactly, we want to recover small inclusions by monitoring the change of the electric field on the boundary $\partial \Omega$. Assume that

$$
D_l = \delta B + z_l, \quad l = 1, 2, \cdots, l_0,
$$

where $\delta \ll 1$ and $B$ is a simply-connected $C^{1,1}$ domain centered at the origin.

In this work we use asymptotic analysis, the layer potential technique and the unique continuation theorem to show the possibility of the unique recovery of small inclusions.
This method works for three- and two-dimensional cases with appropriate changes in fundamental solutions.

The rest of the paper is organised as follows. In Section 2, layer potential and boundary integral operators are introduced and a layer potential technique is used to establish an integral representation for the background electrostatic field \( u_0 \) and an asymptotic expansion for the perturbed electric field. Section 3 is devoted to the unique recovery of the positions and conductivities of small inclusions. Section 4 contains conclusion remarks.

2. Integral Representation and Asymptotic Analysis

In order to establish the integral representations of electric fields with small inclusions, we have to recall the properties of layer potentials [3, 17].

2.1. Layer potentials

Let \( \Gamma \) be the fundamental solution to the Laplace equation in \( \mathbb{R}^3 \) — i.e.

\[
\Gamma(x) = -\frac{1}{4\pi|x|}.
\]

If \( B \subset \mathbb{R}^3 \) is a bounded domain, then \( \mathcal{S}_B : H^{-1/2}(\partial B) \rightarrow H^1(\mathbb{R}^3 \setminus \partial B) \) refers to the single layer potential operator

\[
\mathcal{S}_B[\phi](x) := \int_{\partial B} \Gamma(x-y)\phi(y) \, ds_y,
\]

and \( \mathcal{K}^s_B : H^{-1/2}(\partial B) \rightarrow H^{-1/2}(\partial B) \) to the Neumann-Poincaré operator

\[
\mathcal{K}^s_B[\phi](x) := \text{p.v.} \int_{\partial B} \frac{\partial \Gamma(x-y)}{\partial \nu_x} \phi(y) \, ds_y,
\]

where p.v. means the Cauchy principle value. Here and in what follows, \( \nu \) denotes the exterior unit normal to the boundary.

Let \( \varphi = \varphi(x) \) be a function defined on \( B \cup (\mathbb{R}^3 \setminus (B \cup \partial B)) \) and \( t \in \partial B \). By \( \varphi_+ = \varphi_+(t) \) and \( \varphi_- = \varphi_-(t) \) we denote the limits of \( \varphi(x) \) when \( x \) tends to \( t \) from the domains \( B \) and \( \mathbb{R}^3 \setminus (B \cup \partial B) \), respectively. It is known that the single layer potential operator \( \mathcal{S}_B \) satisfies the trace formula

\[
\frac{\partial}{\partial \nu} \mathcal{S}_B[\phi] \big|_\pm = \left( \pm \frac{1}{2} I + \mathcal{K}^s_B \right)[\phi] \quad \text{on} \quad \partial B.
\]

We also note that the operators \( 1/2 \pm \mathcal{K}^s_B \) are invertible on the space \( H^{-1/2}_0(\partial B) \) — cf. [3], where \( H^{-1/2}_0(\partial B) \) is the subspace of \( H^{-1/2}(\partial B) \) with elements having zero average on \( \partial B \). Let \( B \) and \( D \) be bounded domains such that \( \partial B \cap \partial D = \emptyset \). We define the boundary integral operator \( \mathcal{K}_{D,B} : H^{-1/2}(\partial B) \rightarrow H^{-1/2}(\partial D) \) by

\[
\mathcal{K}_{D,B}[\phi](x) := \int_{\partial B} \frac{\partial \Gamma(x-y)}{\partial \nu_x} \phi(y) \, ds_y, \quad x \in \partial D,
\]
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and set

\[ \mathcal{K}_{D,D} := \mathcal{K}_D^e. \]  

(2.1)

Moreover, we will use the parameters

\[ \lambda_{k_0} := \frac{k_0 + 1}{2(k_0 - 1)}, \quad \lambda_{k_l} := \frac{k_l + k_0}{2(k_l - k_0)}, \quad l = 1, 2, \cdots, l_0. \]

2.2. The integral representation of \( u_0 \)

We now introduce an integral representation for the solution \( u_0 \) of the problem (1.1), (1.2). This problem is equivalent to the following transmission problem:

\[
\begin{align*}
\nabla \cdot \sigma_0 \nabla u &= f \quad \text{in} \; D, \\
\Delta u &= 0 \quad \text{in} \; (\Omega \setminus \overline{D}) \cup (\mathbb{R}^3 \setminus \overline{\Omega}), \\
u|_+ &= u|_- \quad \text{on} \; \partial D \cup \partial \Omega, \\
\sigma_0 \frac{\partial u}{\partial \nu}|_- &= k_0 \frac{\partial u}{\partial \nu}|_+ \quad \text{on} \; \partial D, \\
k_0 \frac{\partial u}{\partial \nu}|_- &= \frac{\partial u}{\partial \nu}|_+ \quad \text{on} \; \partial \Omega, \\
u_0(x) &= \sigma \left( |x|^{-1} \right) \quad \text{as} \; x \to \infty.
\end{align*}
\]

(2.2)

Recall that the Neumann-to-Dirichlet (NtD) map \( \Lambda_\sigma : H_{-1/2}^{-1}(\partial D) \to H_{1/2}^{1/2}(\partial D) \) is defined by

\[ \Lambda_\sigma[g] := u|_{\partial D}, \]

where \( u \) is the solution to the problem

\[
\begin{align*}
\nabla \cdot \sigma \nabla u &= 0 \quad \text{in} \; D, \\
\sigma \frac{\partial u}{\partial \nu} &= g \quad \text{on} \; \partial D \left( \int_{\partial D} u = 0 \right)
\end{align*}
\]

(2.3)

for \( g \in H_{-1/2}^{-1}(\partial D) \). Specially, the operator \( \Lambda_1 \) is the NtD map when \( \sigma = 1 \).

Let \( N_\sigma(x,y) \) be the Neumann function of the problem (2.3) — i.e. for any fixed \( y \in D \) the equations

\[
\begin{align*}
\nabla \cdot \sigma \nabla N(\cdot,y) &= -\delta_y(\cdot) \quad \text{in} \; D, \\
\sigma \nabla N(\cdot,y) \cdot \nu|_{\partial D} &= \frac{1}{|\partial D|}, \\
\int_{\partial D} N(x,y) \; ds_x &= 0
\end{align*}
\]

hold. Then the function

\[ u(x) := \mathcal{N}_{D,\sigma}[g](x) := \int_{\partial D} N_\sigma(x,y) g(y) \; ds_y, \quad x \in D \]
is the solution of the Eqs. (2.3), so that
\[
\Lambda_\sigma[g](x) = \mathcal{K}_{D,\sigma}[g](x), \quad x \in \partial D.
\]
We now define a function \( u_0 \) by
\[
u_0(x) = \begin{cases} 
\mathcal{K}_D[\phi_0](x) + \mathcal{K}_\Omega[\varphi_0](x), & x \in \mathbb{R}^3 \setminus D, \\
\mathcal{K}_{D,\sigma}[\psi_0](x) - \mathcal{K}_{D,\sigma}[f](x) + C, & x \in D,
\end{cases}
\tag{2.4}
\]
where \( \varphi_0 \in H^{-1/2}(\partial \Omega), (\phi_0, \psi_0) \in H^{-1/2}(\partial D) \times H_0^{-1/2}(\partial D) \) and \( \mathcal{K}_{D,\sigma}[f] \) is the volume potential,
\[
\mathcal{K}_{D,\sigma}[f](x) := \int_D N_\sigma(x,y)f(y)\,dy
\]
with the constant
\[
C := \frac{1}{|\partial D|} \int_{\partial D} \mathcal{K}_D[\phi_0] \, ds + \frac{1}{|\partial D|} \int_{\partial D} \mathcal{K}_\Omega[\varphi_0] \, ds.
\]
Lemma 2.1. The function \( u_0 \) of (2.4) is the unique solution of the problem (1.1), (1.2).

Proof. It follows from (2.4) that
\[
\nabla \cdot \sigma \nabla u_0 = f \quad \text{in} \quad D, \\
\Delta u_0 = 0 \quad \text{in} \quad (\Omega \setminus \overline{D}) \cup (\mathbb{R}^3 \setminus \overline{\Omega}), \\
u_0(x) = O(|x|^{-1}) \quad \text{as} \quad x \to \infty.
\]
The transmission condition on the boundaries \( \partial D \) and \( \partial \Omega \) yield
\[
\begin{align*}
-\mathcal{K}_D[\phi_0] - \mathcal{K}_\Omega[\varphi_0] + \Lambda_\sigma[\psi_0] &= \mathcal{K}_{D,\sigma}[f] \quad \text{on} \quad \partial D, \\
-k_0 \left( \frac{1}{2} + \mathcal{K}^s_D \right) [\phi_0] - k_0 \mathcal{K}_{D,\Omega}[\varphi_0] + \psi_0 &= 0 \quad \text{on} \quad \partial D, \\
(\lambda k_0 I - \mathcal{K}^s_D) [\varphi_0] - \mathcal{K}_{\Omega,0}[\phi_0] &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\tag{2.5}
\]
where
\[
\begin{align*}
\mathcal{K}_D[\phi_0] := \mathcal{K}_D[\phi_0] - \frac{1}{|\partial D|} \int_{\partial D} \mathcal{K}_D[\phi_0] \, ds \quad \text{on} \quad \partial D, \\
\mathcal{K}_\Omega[\varphi_0] := \mathcal{K}_\Omega[\varphi_0] - \frac{1}{|\partial D|} \int_{\partial D} \mathcal{K}_\Omega[\varphi_0] \, ds \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
It remains to establish the unique solvability of the system (2.5). Set
\[
\mathcal{A} := \begin{bmatrix} -\mathcal{K}_D & \Lambda_\sigma \\
-k_0 \left( \frac{1}{2}I + \mathcal{K}^s_D \right) & I
\end{bmatrix}.
\]
By [1, Lemma 3.1], the operator \( \mathcal{A} : H^{-1/2}(\partial D) \times H_0^{-1/2}(\partial D) \rightarrow H_0^{1/2}(\partial D) \times H^{-1/2}(\partial D) \) is invertible. Since \( (\lambda k_0 I - \mathcal{X}_\Omega^*) \) is invertible on \( H^{-1/2}(\partial \Omega) \), the third equation in (2.5) has unique solution

\[
\varphi_0 = (\lambda k_0 I - \mathcal{X}_\Omega^*)^{-1} \mathcal{X}_{\Omega,D}[\phi_0].
\]

Substituting it into the first two equations in (2.5) yields

\[
\begin{align*}
-\left( \tilde{\mathcal{F}}_D + \tilde{\mathcal{F}}_\Omega (\lambda k_0 I - \mathcal{X}_\Omega^*)^{-1} \mathcal{X}_{\Omega,D} \right) [\phi_0] + \Lambda_\sigma [\psi_0] &= \gamma_{D,o} [f] \quad \text{on} \quad \partial D, \\
-k_0 \left( \frac{1}{2} + \mathcal{X}_D^* + \mathcal{X}_{D,\Omega} (\lambda k_0 I - \mathcal{X}_\Omega^*)^{-1} \mathcal{X}_{\Omega,D} \right) [\phi_0] + \psi_0 &= 0 \quad \text{on} \quad \partial D.
\end{align*}
\]

(2.6)

We show that the system (2.6) has a unique \( (\phi_0, \psi_0) \in H^{-1/2}(\partial D) \times H_0^{-1/2}(\partial D) \). To this end, we define an operator \( \mathcal{B} \) by

\[
\mathcal{B} := \mathcal{A}^{-1} \begin{bmatrix} \tilde{\mathcal{F}}_\Omega (\lambda k_0 I - \mathcal{X}_\Omega^*)^{-1} \mathcal{X}_{\Omega,D} & 0 \\ k_0 \mathcal{X}_{D,\Omega} (\lambda k_0 I - \mathcal{X}_\Omega^*)^{-1} \mathcal{X}_{\Omega,D} & 0 \end{bmatrix}.
\]

(2.7)

Since the operator \( \mathcal{A} : H^{-1/2}(\partial D) \times H_0^{-1/2}(\partial D) \rightarrow H_0^{1/2}(\partial D) \times H^{-1/2}(\partial D) \) is invertible and \( \mathcal{B} - \mathcal{A} : H^{-1/2}(\partial D) \times H_0^{-1/2}(\partial D) \rightarrow H_0^{1/2}(\partial D) \times H^{-1/2}(\partial D) \) compact, we only have to show the injectivity of \( \mathcal{B} \). But this is the consequence of the unique solvability of the transmission problem (2.2). The uniqueness of the solution \( (\phi_0, \varphi_0, \psi_0) \) now follows from the inclusion \( \gamma_{D,o} [f] \in H_0^{1/2}(\partial D) \).

2.3. Integral representation and approximation

Consider the solution \( u \) of the problem (1.1), (1.3). Using the transmission conditions on \( \partial \Omega \) and \( \partial D_l, l = 1, 2, \ldots, l_0 \), we show that \( u \) also satisfies the transmission problem

\[
\begin{align*}
\nabla \cdot \sigma_0 \nabla u &= f \quad \text{in} \quad D, \\
\Delta u &= 0 \quad \text{in} \quad \bigcup_{l=1}^{l_0} D_l \cup \bar{D} \cup (\mathbb{R}^3 \setminus \bar{\Omega}), \\
[u]_+ &= [u]_- \quad \text{on} \quad \bigcup_{l=1}^{l_0} \partial D_l \cup \partial \Omega \cup \partial D, \\
k_0 \frac{\partial u}{\partial y}_+ &= \sigma_0 \frac{\partial u}{\partial y}_- \quad \text{on} \quad \partial D, \\
\frac{\partial u}{\partial y}_+ &= k_0 \frac{\partial u}{\partial y}_- \quad \text{on} \quad \partial \Omega, \\
k_0 \frac{\partial u}{\partial y}_+ &= k_l \frac{\partial u}{\partial y}_- \quad \text{on} \quad \partial D_l, \\
u(x) &= \Theta(|x|^{-1}) \quad \text{as} \quad x \to \infty.
\end{align*}
\]

(2.8)
Theorem 2.1. If $D$ write we identify the variable used to denote the points of the corresponding curve — e.g. we write $v_x$ if $x \in \partial D$.

According to potential theory, the solution $u$ of (2.8) may be represented in the form

$$u = \begin{cases} \mathcal{S}_D[\phi] + \mathcal{S}_\Omega[\varphi] + \sum_{l=1}^{l_0} \mathcal{S}_{D_l}[\phi_l] & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \mathcal{N}_{D,\sigma}[\psi] - \mathcal{G}_{D,\sigma}[f] + C_1 & \text{in } D, \end{cases} \tag{2.9}$$

where $\phi \in H^{-1/2}(\partial D)$, $\varphi \in H^{-1/2}(\partial \Omega)$ and $\phi_l \in H^{-1/2}(\partial D_l)$, $l = 1, 2, \ldots, l_0$ and

$$C_1 := \frac{1}{|\partial D|} \int_{\partial D} \left( \mathcal{S}_D[\phi] + \mathcal{S}_\Omega[\varphi] + \sum_{l=1}^{l_0} \mathcal{S}_{D_l}[\phi_l] \right) ds.$$

Lemma 2.2. There exists a unique vector

$$(\phi, \varphi, \phi_1, \ldots, \phi_{l_0}) \in H^{-1/2}(\partial D) \times H^{-1/2}(\partial \Omega) \times H^{-1/2}(\partial D_1) \times \cdots \times H^{-1/2}(\partial D_{l_0})$$

such that the corresponding function $u$ in (2.9) is the solution of the problem (2.8).

The proof of this lemma is moved to Appendix A.

Theorem 2.1. If $D_l$, $l = 1, 2, \ldots, l_0$ are the domains (1.4), then on the set $\mathbb{R}^3 \setminus \overline{\Omega}$ the solution $u$ of the problem (1.1), (1.3) can be represented as

$$u = u_0 - \delta^3 (h + u_1) - \delta^3 \sum_{l=1}^{l_0} \mathcal{S}_{\Omega_l} (\lambda_k I - \mathcal{N}^*)^{-1} \left[ \nu^\gamma \nabla^2 \Gamma(x - z_l) \right] M_l \nabla u_0(z_l) + O(\delta^4), \tag{2.10}$$

where

$$h := \sum_{l=1}^{l_0} \nabla \Gamma(x - z_l)^T M_l \nabla u_0(z_l), \tag{2.11}$$

$M_l$ is the polarisation tensor (PT) defined by

$$(M_l)_{ij} := \int_{\partial B} \bar{y} \left( \lambda_k I - \mathcal{N}^* \right)^{-1} \left[ (v_l)_j \right] (\bar{y}) d\bar{y}, \quad l = 1, 2, \ldots, l_0,$$

and $u_1$ is the solution of the problem

$$\nabla \cdot \sigma_0 \nabla u_1 = 0 \quad \text{in } D,$$

$$\Delta u_1 = 0 \quad \text{in } (\Omega \setminus D) \cup (\mathbb{R}^3 \setminus \overline{\Omega}),$$

$$u_1|_+ - u_1|_- = h, \quad k_0 \frac{\partial u_1}{\partial y} \bigg|_+ - \sigma_0 \frac{\partial u_1}{\partial y} \bigg|_- = \frac{\partial h}{\partial y} \quad \text{on } \partial D,$$

$$u_1|_+ = u_1|_-, \quad \frac{\partial u_1}{\partial y} \bigg|_+ = k_0 \frac{\partial u_1}{\partial y} \bigg|_- \quad \text{on } \partial \Omega,$$

$$u_1 = \mathcal{O}(|x|^{-1}) \quad \text{as } x \to \infty. \tag{2.12}$$
Before we start with the proof of Theorem 2.1, let us make a comment on the formula (2.10). It is known that in the presence of the small inclusions, electric field $u$ experiences a disturbance. Introducing the function

$$v := \sum_{l=1}^{l_0} \mathcal{S}_\Omega \left( \lambda_{k_0} I - \mathcal{X}_\Omega^* \right)^{-1} \left[ v^T \nabla^2 \Gamma(x - z_l) \right] M_l \nabla u_0(z_l),$$

we note that it is a solution of the problem

$$\begin{align*}
\Delta v &= 0 & \text{in } \Omega \cup (\mathbb{R}^3 \setminus \overline{\Omega}),
|v|_{+} - |v|_{-} &= 0, & \text{on } \partial \Omega, \\
\frac{\partial v}{\partial n}_{+} - k_0 \frac{\partial v}{\partial n}_{-} &= (1 - k_0) \frac{\partial h}{\partial n} & \text{on } \partial \Omega, \\
v &= O \left( |x|^{-1} \right) & \text{as } x \to \infty.
\end{align*}$$

(2.13)

The representation (2.10) shows that the main perturbation — i.e. the leading term, is combined with $h$ and with the solutions $u_1$ and $v$ of the transmission problems (2.12) and (2.13), respectively. Therefore, $u_1$ and $v$ can be considered as perturbations of the dipole source $h$.

**Proof of Theorem 2.1.** It follows from (2.9) that

$$u = \mathcal{S}_D[\phi] + \mathcal{S}_\Omega[\varphi] + \sum_{l=1}^{l_0} \mathcal{S}_{D_l}[\varphi_l] \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega},$$

(2.14)

where $\phi$, $\varphi$ and $\varphi_l, l = 1, 2, \cdots, l_0$ are described in (A.8), (A.9), (A.12), (A.13) and (A.14). Substituting these representations into (2.14), we obtain

$$u = u_0 - \delta^3 h - \delta^3 \mathcal{S}_D[\phi_h] - \delta^3 \mathcal{S}_\Omega \left( \lambda_{k_0} I - \mathcal{X}_\Omega^* \right)^{-1} \mathcal{X}_{\Omega,D}[\phi_h]$$

$$- \delta^3 \mathcal{S}_\Omega \left( \lambda_{k_0} I - \mathcal{X}_\Omega^* \right)^{-1} \left[ \partial_\Omega \varphi + O(\delta^4) \right],$$

where $h$ is introduced in (2.11) and

$$\phi_h = (1, 0) \mathcal{B}^{-1} \left[ \frac{\hat{h}}{\partial h} \frac{\partial \varphi}{\partial \hat{\nu}} \right],$$

$$\hat{h} := h - 1/|\partial D| \int_{\partial D} h.$$

Following the proof of Lemma 2.1, we can show that the function

$$\mathcal{S}_D[\phi_h] + \mathcal{S}_\Omega \left( \lambda_{k_0} I - \mathcal{X}_\Omega^* \right)^{-1} \mathcal{X}_{\Omega,D}[\phi_h]$$

is the solution of the problem (2.12) in $\mathbb{R}^3 \setminus \overline{\Omega}$. \qed
3. Unique Recovery

We now consider the uniqueness for the recovery of small inclusions via the boundary measurements in the electrostatic system (1.1), (1.3). Let $D_l^{(1)}$ and $D_l^{(2)}$, $l = 1, 2, \cdots, l_0$, be two sets of small inclusions, satisfying the condition (1.4) with $z_l$ replaced by $z_l^{(1)}$ and $z_l^{(2)}$, respectively. Moreover, for $D_l^{(1)}$ and $D_l^{(2)}$, $l = 1, 2, \cdots, l_0$, $j = 1, 2$ the conductivities $k_l$ are denoted by $k_l^{(1)}$ and $k_l^{(2)}$, respectively. Let $u_l^{(1)}$ and $u_l^{(2)}$ be the solutions of the problem (1.1) and (1.3) with $D_l$ replaced by $D_l^{(1)}$ and $D_l^{(2)}$, respectively. The polarisation tensors for $D_l^{(1)}$ and $D_l^{(2)}$, $l = 1, 2, \cdots, l_0$ are, respectively, denoted by $M_l^{(1)}$ and $M_l^{(2)}$ and the function $h$ in (2.11) equipped with the corresponding parameters, is written as $h^{(1)}$ and $h^{(2)}$.

**Theorem 3.1.** Let $\Pi$ be an open subset of $\partial \Omega$. If

$$u_l^{(1)} = u_l^{(2)} \text{ on } \Pi,$$

then

$$z_l^{(1)} = z_l^{(2)}, \quad k_l^{(1)} = k_l^{(2)}, \quad l = 1, 2, \cdots, l_0. \tag{3.2}$$

**Proof.** First of all, we note that the relation (3.2) means that after an appropriate order rearrangement the sets of positions and conductivities coincide.

Since $u_l^{(1)}$ and $u_l^{(2)}$ are harmonic functions in $\mathbb{R}^3 \setminus \overline{\Omega}$, the Eq. (3.1) and the unique continuation theorem yield that

$$u_l^{(1)} = u_l^{(2)} \text{ in } \mathbb{R}^3 \setminus \overline{\Omega},$$

and, using the asymptotic expansion (2.10), we obtain

$$h_l^{(1)} + u_l^{(1)} + v^{(1)} = h_l^{(2)} + u_l^{(2)} + v^{(2)} \text{ in } \mathbb{R}^3 \setminus \overline{\Omega},$$

where $u_l^{(1)}$ and $u_l^{(2)}$ are the solutions of (2.12), with the corresponding terms $z_l$, $k_l$, $M_l$ with $z_l^{(1)}$, $k_l^{(1)}$, $M_l^{(1)}$ and $z_l^{(2)}$, $k_l^{(2)}$, $M_l^{(2)}$. Analogously, $v^{(1)}$ and $v^{(2)}$ are the solutions of the corresponding problems (2.13). It is easily seen that if

$$\Sigma^{(j)} := \bigcup_{l=1}^{l_0} z_l^{(j)}, \quad j = 1, 2,$$

then the function $w^{(j)} := u_l^{(j)} + v^{(j)}$, $j = 1, 2$ is the solution of the problem

$$\Delta w^{(j)} = 0 \quad \text{in } \left(\mathbb{R}^3 \setminus \overline{\Omega}\right) \cup \left(\Omega \setminus \overline{D}\right),$$

$$w^{(j)}|_+ - w^{(j)}|_- = 0, \quad \frac{\partial w^{(j)}}{\partial n}|_+ - k_0 \frac{\partial w^{(j)}}{\partial n}|_- = 0 \quad \text{on } \partial \Omega,$$

$$w^{(j)} = \sigma \left(|x|^{-1}\right) \quad \text{as } x \to \infty.$$
Using the unique continuation theorem again, we find out that
\[ h^{(1)} + w^{(1)} = h^{(2)} + w^{(2)} \quad \text{in} \quad \mathbb{R}^3 \setminus \left( \Sigma^{(1)} \cup \Sigma^{(2)} \cup \overline{D} \right), \]
or
\[ F := h^{(1)} - h^{(2)} + w^{(1)} - w^{(2)} = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \left( \Sigma^{(1)} \cup \Sigma^{(2)} \cup \overline{D} \right). \tag{3.3} \]
The analytic continuation of the harmonic function above shows that
\[ F = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}. \]
It follows from (2.11) and Taylor expansion that
\[
(h^{(1)} - h^{(2)})(x) = \sum_{i=1}^{l_0} \nabla \Gamma \left( x - z^{(1)}_i \right) ^T M^{(1)}_i \nabla u_0(z^{(1)}_i) - \sum_{i=1}^{l_0} \nabla \Gamma \left( x - z^{(2)}_i \right) ^T M^{(2)}_i \nabla u_0(z^{(2)}_i)
\]
\[= \sum_{i=1}^{l_0} \left( \nabla \Gamma \left( x - z^{(1)}_i \right) ^T - \nabla \Gamma \left( x - z^{(2)}_i \right) ^T \right) M^{(1)}_i \nabla u_0(z^{(1)}_i)
\]
\[+ \sum_{i=1}^{l_0} \nabla \Gamma \left( x - z^{(2)}_i \right) ^T \left( M^{(1)}_i \nabla u_0(z^{(1)}_i) - M^{(2)}_i \nabla u_0(z^{(2)}_i) \right)
\]
\[= \sum_{i=1}^{l_0} \left( \nabla^2 \Gamma_0 \left( x - z_i \right) \left( z^{(1)}_i - z^{(2)}_i \right) \right) ^T M^{(1)}_i \nabla u_0(z^{(1)}_i)
\]
\[+ \nabla \Gamma_0 \left( x - z^{(2)}_i \right) ^T \left( M^{(1)}_i \nabla u_0(z^{(1)}_i) - M^{(2)}_i \nabla u_0(z^{(2)}_i) \right). \tag{3.4} \]
Considering the poles of the functions in the right-hand side of (3.3) and using the Eq. (3.4), we obtain
\[ \nabla^2 \Gamma_0 \left( x - z_i \right) \left( z^{(1)}_i - z^{(2)}_i \right) ^T M^{(1)}_i \nabla u_0(z^{(1)}_i) = 0, \quad x \in \mathbb{R}^3 \setminus \overline{D}, \]
\[\nabla \Gamma_0 \left( x - z^{(2)}_i \right) ^T \left( M^{(1)}_i \nabla u_0(z^{(1)}_i) - M^{(2)}_i \nabla u_0(z^{(2)}_i) \right) = 0, \quad x \in \mathbb{R}^3 \setminus \overline{D}. \]
Therefore,
\[ z^{(1)}_i = z^{(2)}_i, \quad h^{(1)} = h^{(2)}, \]
and the uniqueness of the harmonic continuation implies
\[ u^{(1)} = u^{(2)} \quad \text{in} \quad \mathbb{R}^3 \setminus \bigcup_{l=1}^{l_0} D_l \cup \overline{D}. \]
It follows that
\[ \Delta \left( u^{(1)} - u^{(2)} \right) = 0 \quad \text{in} \quad D_l, \]
\[ u^{(1)} - u^{(2)} = 0 \quad \text{on} \quad \partial D_l, \]
which yields
\[ u^{(1)} = u^{(2)} \quad \text{in} \quad D_l, \quad l = 1, 2, \ldots, l_0. \]
It remains to recall the transmission condition on \( \partial D_l \) and obtain \( k^{(1)}_i = k^{(2)}_i \). \( \square \)
4. Conclusion

We derived integral representations for the solution of two-layer inhomogeneous conductivity problem and obtained asymptotic expansions of a perturbed electrical field with respect to the size of small inclusions. Moreover, we established the uniqueness of the recovery of locations and conductivities of small inclusions. These properties can be used in the development of efficient numerical reconstruction schemes to detect body abnormalities.

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A. Uniqueness of Solution

Let us start with auxiliary results. We consider the operators $\mathcal{M}_{D_l,\Omega,D} : H^{-1/2}(\partial D) \to H^{-1/2}(\partial D_l)$ defined by

$$\mathcal{M}_{D_l,\Omega,D} := \mathcal{K}_{D_l,D} + \mathcal{K}_{D_l,\Omega}(\lambda_l I - \mathcal{K}_{\Omega}^s)^{-1} \mathcal{K}_{\Omega, D}, \quad l = 1, 2, \ldots, l_0. \quad (A.1)$$

and similar operators $\mathcal{M}_{D_l,\Omega, D_l'}$, $l, l' = 1, 2, \ldots, l_0$.

**Lemma A.1** (cf. [2, 3]). Assuming that the domains $D_l$, $l = 1, 2, \ldots, l_0$ satisfy the condition (1.4) for $\delta << 1$ and for $x \in \partial D_l$, we set $\tilde{x} := \delta^{-1}(x - z_l)$ and $\tilde{\phi}_l(x) := \phi_l(x)$. If $\phi_l \in H^{-1/2}_0(\partial D_l)$, then

$$\mathcal{K}_{D_l}^s[\phi_l] = \mathcal{K}_{D_l}^s[\tilde{\phi}_l], \quad (A.2)$$

and

$$\|\mathcal{K}_{D_l}^s[\phi_l]\|_{H^{-1/2}(\partial D)} \leq C \delta^3 \|\tilde{\phi}_l\|_{H^{-1/2}(\partial D_l)},$$

$$\|\mathcal{K}_{\Omega, D_l}[\phi_l]\|_{H^{-1/2}(\partial \Omega)} \leq C \delta^3 \|\tilde{\phi}_l\|_{H^{-1/2}(\partial D)},$$

$$\|\mathcal{K}_{D_l,\Omega}[\phi_l]\|_{H^{-1/2}(\partial D_l)} \leq C \delta^3 \|\tilde{\phi}_l\|_{H^{-1/2}(\partial D_l)},$$

$$\|\mathcal{K}_{D_l, D_l'}[\phi_l]\|_{H^{-1/2}(\partial D_l')} \leq C \delta^2 \|\tilde{\phi}_l\|_{H^{-1/2}(\partial D_l')}, \quad (A.3)$$

where $l' \neq l$.

**Lemma A.2.** If $\phi_l$ and $\tilde{\phi}_l$ are the functions defined in Lemma A.1, then

$$\mathcal{M}_{D_l,\Omega, D}[\phi_l] = \mathcal{K}_{D_l}^s[\tilde{\phi}_l] + O(\delta^2 \|\tilde{\phi}_l\|_{H^{-1/2}(\partial D_l)}), \quad (A.4)$$
Moreover, if \( l \neq l' \), then

\[
\mathcal{M}_{D_l \Omega, D_{l'}} [\phi_l] = \mathcal{O} \left( \delta^2 \| \tilde{\phi}_l \|_{H^{-1/2}(\partial B)} \right).
\]  

(A.5)

**Proof.** It follows from (2.1) that

\[
\mathcal{M}_{D_l \Omega, D_{l'}} [\phi_l] = \mathcal{X}_{D_l}^+ [\phi] + \mathcal{X}_{D_l \Omega} \left( \lambda_k I - \mathcal{X}_{\Omega}^+ \right)^{-1} \mathcal{X}_{\Omega, D_l} [\phi],
\]

and taking into account the Eq. (A.2) and the inequalities (A.3), we obtain the relation (A.4). The proof of (A.5) is similar but one has to use (A.1) and (A.3).

---

**Lemma A.3.** If \( \phi_l \) and \( \tilde{\phi}_l \) are the functions defined in Lemma A.1, then

\[
\mathcal{K}_{D_l} [\phi_l] = -\delta^3 \nabla \Gamma(x - z_l) \int_{\partial B} \tilde{\phi}_l(y) d\gamma_y + \mathcal{O} \left( \delta^4 \| \tilde{\phi}_l \|_{H^{-1/2}(\partial B)} \right) \quad \text{on} \quad \partial D, \]

(A.6)

\[
\mathcal{X}_{D_{l} D_{l}} [\phi_l] = -\delta^3 \psi^T \nabla^2 \Gamma(x - z_l) \int_{\partial B} \tilde{\phi}_l(y) d\gamma_y + \mathcal{O} \left( \delta^4 \| \tilde{\phi}_l \|_{H^{-1/2}(\partial B)} \right) \quad \text{on} \quad \partial D,
\]

where \( \psi \) denotes the operation of transposition.

**Proof.** Let us consider the relation (A.6). Using Taylor expansions, we obtain

\[
\Gamma(x - y) = \Gamma(x - z_l) - \nabla \Gamma(x - z_l)^T (y - z_l) + \frac{1}{2} (y - z_l)^T \nabla^2 \Gamma(x - z_l) (y - z_l), \quad x \in \partial D, \quad y \in \partial D_l,
\]

where \( x' = \theta (x - z_l) + (1 - \theta)(y - z_l) \). Since \( y - z_l = \delta y \) and \( \tilde{\phi}_l \in H_0^{-1/2}(\partial B) \), the estimate (A.6) follows.

**Proof of Lemma 2.2.** The function \( u \) defined in (2.9) satisfies the first, second and last equations in (2.8). Moreover, third, fourth and fifth equations in (2.8) yield

\[
-\mathcal{K}_{D} [\phi] - \mathcal{K}_{D_l} [\psi] - \sum_{l' = 1}^{l_0} \mathcal{K}_{D_{l'}} [\phi_{l'}] + \lambda_k \mathcal{K}_{D,D_l} [\psi] = \mathcal{V}_{D, \Omega} [f] \quad \text{on} \quad \partial D,
\]

\[
-k_0 \left( \frac{I}{2} + \mathcal{X}_{\Omega}^+ \right) [\phi] - k_0 \mathcal{X}_{D, \Omega} [\varphi] - \sum_{l' = 1}^{l_0} \mathcal{X}_{D_{l}, D_{l'}} [\phi_{l'}] + \psi = 0 \quad \text{on} \quad \partial D,
\]

\[
-\mathcal{X}_{D, \Omega} [\phi] + \left( \lambda_k I - \mathcal{X}_{\Omega}^+ \right) [\varphi] - \sum_{l' = 1}^{l_0} \mathcal{X}_{\Omega, D_{l'}} [\phi_{l'}] = 0 \quad \text{on} \quad \partial \Omega,
\]

\[
-\mathcal{X}_{D_{l}, D_{l}} [\phi] - \mathcal{X}_{D_{l}, D_{l}} [\varphi] - \sum_{l' \neq l}^{l_0} \mathcal{X}_{D_{l}, D_{l'}} [\phi_{l'}] + \left( \lambda_k I - \mathcal{X}_{D_{l}}^+ \right) [\phi_{l}] = 0 \quad \text{on} \quad \partial D_{l}. \quad (A.7)
\]

Applying the boundary integral on both sides of (A.7), we obtain that \( \phi_{l} \in H_0^{-1/2}(\partial D_{l}) \), \( l = 1, 2, \cdots, l_0 \). If \( \mathcal{B} \) is the operator (2.7), then the considerations similar to the proof of
Lemma 2.1 show that

\[
\begin{pmatrix}
\phi \\
\psi
\end{pmatrix}
= \mathcal{B}^{-1}
\begin{bmatrix}
\mathcal{Y}_{D,\sigma}[f] & 0 \\
0 & \mathcal{Q}_D
\end{bmatrix}
+ \mathcal{B}^{-1}
\begin{bmatrix}
\sum_{l'=1}^{l_0} \mathcal{J}_{D,l'}[\phi_{l'}] \\
\sum_{l'=1}^{l_0} \mathcal{K}_{D,l'}[\phi_{l'}]
\end{bmatrix}_{\partial D}
= \begin{pmatrix}
\phi^{(0)} \\
\psi^{(0)}
\end{pmatrix}
+ \begin{pmatrix}
\phi^{(1)} \\
\psi^{(1)}
\end{pmatrix},
\tag{A.8}
\]

and the Eqs. (2.6) yield

\[
\phi^{(0)} = \phi_0, \quad \psi^{(0)} = \psi_0.
\tag{A.9}
\]

Substituting (A.8) and \(\varphi\) in (A.7), we arrive at the equation

\[-\mathcal{M}_{D_i,\Omega,D}[\phi^{(1)}] - \sum_{l'=1}^{l_0} \mathcal{M}_{D_i,\Omega,D}[\phi_{l'}] + \lambda_{k_i} \phi_{l_i} = \mathcal{M}_{D_i,\Omega,D}[\phi_0] \quad \text{on} \quad \partial D_i, \tag{A.10}\]

and (A.1), (2.4) imply

\[
\mathcal{M}_{D_i,\Omega,D}[\phi_0] = \frac{\partial u_0}{\partial \nu_i} \quad \text{on} \quad \partial D_i. \tag{A.11}\]

Taking into account Lemmas A.1, A.3 and the Eq. (A.8), we write

\[
\phi^{(1)} = -\delta^3(1,0) \sum_{l'=1}^{l_0} \mathcal{B}^{-1}
\begin{bmatrix}
\nabla^2 \Gamma(x - z_{l'})^T \\
\nabla^4 \Gamma(x - z_{l'})
\end{bmatrix}
\times \int_{\partial D_i} \tilde{\gamma} \tilde{\psi}_{l'}(\tilde{y}) \, ds_y + \mathcal{O}\left(\delta^4 \sum_{l'=1}^{l_0} \left\| \tilde{\psi}_{l'} \right\|_{H^{-1/2}(\partial B)}\right), \tag{A.12}\]

where

\[
\nabla^2 \Gamma(x - z_{l'}) = \nabla \Gamma(x - z_{l'}) - \frac{1}{|\partial D|} \int_{\partial D} \nabla \Gamma(x - z_{l'}) \, dx.
\]

Recalling Lemma A.2 and substituting (A.11) and (A.12) into (A.10) yields

\[
(\lambda_{k_i} I - \mathcal{K}_{D_i}^\ast + \mathcal{O}(\delta^2))[\phi_{l_i}] = \frac{\partial u_0}{\partial \nu_i} \quad \text{on} \quad \partial D_i.
\]

This and the Taylor expansion

\[
u_0(y) = u_0(z_i) + \delta \nabla u_0(z_i)^T \tilde{y} + \mathcal{O}(\delta^2), \quad y \in \partial D_i
\]

of the function \(u_0\) lead to the representation

\[
\tilde{\phi}_{l_i} = (\lambda_{k_i} I - \mathcal{K}_{D_i}^\ast)^{-1} [\nu_i \cdot \nabla u_0(z_i)] + \mathcal{O}(\delta). \tag{A.13}
\]

By substituting (A.13) into the third equation in (A.7), one then has

\[
\varphi = (\lambda_{k_0} I - \mathcal{K}_{\Omega}^\ast)^{-1} \mathcal{K}_{\Omega,D}[\phi] + (\lambda_{k_0} I - \mathcal{K}_{\Omega}^\ast)^{-1} \sum_{l'=1}^{l_0} \mathcal{K}_{\Omega,D,l'}[\phi_{l'}].
\]
\[= (\lambda_{k_0} I - \mathcal{K}^s_{\Omega})^{-1} \mathcal{K}_{\Omega,D} \left[ \phi^{(0)} + \phi^{(1)} \right] - \delta^3 (\lambda_{k_0} I - \mathcal{K}^s_{\Omega})^{-1} \]
\[
\times \sum_{l' = 1}^{l_0} v^T \nabla^2 \Gamma(x - z_{l'}) \int_{\partial B} \tilde{\phi}_{l'}(\tilde{y}) \, ds_{\tilde{y}} + O(\delta^4)
\]
\[
= \varphi_0 + (\lambda_{k_0} I - \mathcal{K}^s_{\Omega})^{-1} \mathcal{K}_{\Omega,D} \left[ \phi^{(1)} \right] - \delta^3 (\lambda_{k_0} I - \mathcal{K}^s_{\Omega})^{-1} \]
\[
\times \sum_{l' = 1}^{l_0} v^T \nabla^2 \Gamma(x - z_{l'}) \int_{\partial B} \tilde{\phi}_{l'}(\tilde{y}) \, ds_{\tilde{y}} + O(\delta^4),
\]
(A.14)
and the proof is completed. \qed

References

[1] H. Ammari, Y. Deng, H. Kang and H. Lee, Reconstruction of inhomogeneous conductivities via the concept of generalized polarization tensors, Ann. Inst. H. Poincaré Anal. Non Linéaire 31, 877–897 (2014).
[2] H. Ammari, R. Griesmaier and M. Hanke, Identification of small inhomogeneities: asymptotic factorization, Math. Comp. 76, 1425–1448 (2007).
[3] H. Ammari and H. Kang, Polarization and Moment Tensors With Applications to Inverse Problems and Effective Medium Theory, Applied Mathematical Sciences (2007).
[4] H. Ammari, O. Kwon, J.K. Seo and E.J. Woo, Anomaly detection in T-scan trans-admittance imaging system, SIAM J. Appl. Math. 65, 252–266 (2004).
[5] K. Astala, L. Paivärinta and M. Lassas, Calderón’s inverse problem for anisotropic conductivity in the plane, Comm. Partial Differential Equations 30, 207–224 (2005).
[6] B. Barceló, E. Fabes and J.K. Seo, The inverse conductivity problem with one measurement: uniqueness for convex polyhedra, Proc. Amer. Math. Soc. 122, 183–189 (1994).
[7] R. Brown and G. Uhlmann, Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions, Comm. Partial Differential Equations 22, 1009–1027 (1997).
[8] J. Chen and G. Huang, A direct imaging method for inverse electromagnetic scattering problem in rectangular waveguide, Commun. Comput. Phys. 23, 1415–1433 (2018).
[9] Z. Chen and G. Huang, Phaseless imaging by reverse time migration: acoustic waves, Numer. Math. Theory Methods Appl. 10, 1–21 (2018).
[10] A. Friedman and V. Isakov, On the uniqueness in the inverse conductivity problem with one measurement, Indiana Univ. Math. J. 38, 553–579 (1989).
[11] V. Isakov and J. Powell, On the inverse conductivity problem with one measurement, Inverse Problems 6, 311–318 (1990).
[12] H. Kang and J.K. Seo, Inverse conductivity problem with one measurement: Uniqueness of balls in R^3, SIAM J. Appl. Math. 59, 1533–1539 (1990).
[13] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements, II. Interior results, Comm. Pure Appl. Math. 38, 643–667 (1985).
[14] H. Liu and J. Zou, Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers, Inverse Problems 22, 515-524 (2006).
[15] T. Luecke, F. Corradi and P. Pelosi, Lung imaging for titration of mechanical ventilation, Curr. Opin. Anaesth. 25 (2), 131–140 (2012).
[16] A. I. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Ann. of Math. (2) 143, 71–96 (1996).
[17] J. C. Nédélec, *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*, Springer-Verlag (2001).

[18] J.K. Seo, *A uniqueness result on inverse conductivity problem with two measurements*, J. Fourier Anal. Appl. 2, 227-235 (1996).

[19] J.K. Seo, O. Kwon, H. Ammari and E.J. Woo, *Mathematical framework and anomaly estimation algorithm for breast cancer detection using TS2000 configuration*, IEEE Trans. Biomedical Engineering 51, 1898–1906 (2004).

[20] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. (2) 125 (1), 153–169 (1987).

[21] M. Zhang and J. Lv, *Numerical Method of Profile Reconstruction for a Periodic Transmission Problem from Single-Sided Data*, Commun. Comput. Phys. 24, 435–453 (2018).