ANALYSIS OF A ONE-DIMENSIONAL PRESCRIBED MEAN CURVATURE EQUATION WITH SINGULAR NONLINEARITY

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Abstract. In this paper, the classical solution set \((\lambda, u)\) of the one-dimensional prescribed mean curvature equation

\[
- \left( \frac{u'}{\sqrt{1 + (u')^2}} \right)' = \frac{\lambda}{(1 - u)^2}, \quad -L < x < L; \quad u(-L) = u(L) = 0,
\]

for \(\lambda > 0\) and \(L > 0\), is analyzed via a time map. It is shown that the solution set depends on both parameters \(\lambda\) and \(L\) and undergoes two bifurcations. The first is a standard saddle node bifurcation, which happens for all \(L\) at \(\lambda = \lambda^*(L)\). The second is a splitting bifurcation, namely, there exists a value \(L^*\) such that as \(L\) transitions from greater than or equal to \(L^*\) to less than \(L^*\) the upper branch of the bifurcation diagram of problem \((\ast)\) splits into two parts. In contrast, the solution set of the semilinear version of problem \((\ast)\) is independent of \(L\) and exhibits only a saddle node bifurcation. Therefore, as this analysis suggests, the splitting bifurcation is a byproduct of the mean curvature operator coupled with the singular forcing.

Key words. Prescribed mean curvature, Splitting bifurcation, Nonlinear forcing, Time map, MEMS

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1. Introduction

The study of nonparametric surfaces of prescribed mean curvature, i.e., the study of solutions of

\[
- \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \lambda f(u), \quad x \in \Omega; \quad u = 0, \quad x \in \partial \Omega,
\]

(1.1)
go back to 1805 and 1806 when Thomas Young [64] and Pierre-Simon Laplace [42] separately looked at the properties of capillary surfaces, namely, where \(f\) is linear. Later the theory, which still focused on capillary surfaces, was put on a solid mathematical foundation by Gauss [30] and attracted the attention of many nineteenth century scientific luminaries. During the first half of the twentieth century, the problem fell out of vogue, taking a backseat to the study of constant mean curvature surfaces (c.f. [56]); however in the past few decades, due to mathematical advances and the miniaturization of technology, the study of problem (1.1) has come back into focus. Most modern authors have focused on the existence, nonexistence and multiplicity of positive solutions. The case where \(\Omega \subseteq \mathbb{R}^n\), for \(n \geq 2\), has been studied by numerous authors: Concus and Finn [19, 20, 21, 22, 23, 24, 25]; Giusti [32, 33, 34, 35]; Gilbarg and Trudinger [31]; Ni and Serrin [49, 50, 51]; Finn [28, 29] (and the references therein); Peletier and Serrin [61]; Atkinson, Peletier and Serrin [4, 3, 5]; Serrin [62]; Ishimura [38, 39]; Kusano and Swanson [41]; Nakao [48]; Noussair, Swanson and Jianfu [52]; Biaut-Véron [10]; Clément, Manásevich and Mitidieri [17]; Coffman and Ziemer [18]; Conti and Gazzola [26]; Amster and Marian [1]; Habets and Omari [36]; Le [43, 44]; Chang and Zhang [16]; del Pino and Guerra [27]; Moulton and Pelesko [46, 47]; Bereau, Jebelean and Mawhin [8, 9]; Obersnel and Omari [54, 55]; Brubaker and Pelesko [14]; Brubaker and Lindsay [13]. Also, the case where \(n = 1\) has been studied by numerous authors in a recent series of papers: Kusahara and Usami [40]; Benevieri, do Ó and

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de Medeiros [6, 7]; Bonheure, Habets, Obersnel and Omari [11, 12]; Habets and Omari [37];
Obersnel [53]; Pan [57]; Li and Liu [45]; Burns and Grinfeld [15]; Pan and Xing [59, 58]. One
of the fascinating aspects of problem (1.1) that these studies have revealed is the
disappearing solution behavior shown to be present for numerous choices of \( f \) (cf. [59], [58] and the references
therein). For example, Pan showed in [57] that the solution set of
\[
- \left( \frac{u'}{\sqrt{1 + (u')^2}} \right)' = \lambda e^u, \quad 0 < x < L; \quad u(0) = u(L) = 0,
\]
which is a quasilinear analogue of the semilinear Gelfand–Bratu equation, is fully characterized
by the following theorem for \( \lambda > 0 \) (see Figure 1 for the resulting bifurcation diagram).

**Theorem** (Pan [57]). Assume that \( \lambda > 0 \).

(i) If \( L < \pi \), there exists constants \( \lambda_* \) and \( \lambda^* \) such that (a) for any \( \lambda \in (0, \lambda_*) \cup \{ \lambda^* \} \) there
is a unique positive solution of (1.2); (b) for any \( \lambda \in [\lambda_*, \lambda^*] \), there exists exactly two
positive solutions of (1.2); (c) for any \( \lambda \in (\lambda^*, \infty) \), no positive solutions of (1.2) exist.

(ii) If \( L \geq \pi \), there exists a constant \( \lambda^* \) such that (a) for any \( \lambda \in (0, \lambda^*) \) there exists exactly
two positive solutions of (1.2); (b) for \( \lambda = \lambda^* \), there is a unique positive solution of (1.2);
(c) for any \( \lambda \in (\lambda^*, \infty) \), no positive solutions of (1.2) exist.

In this paper we continue to study the disappearing solution behavior of prescribed mean
curvature problem (1.1), with \( n = 1 \), by considering the case where \( f \) is an inverse square
nonlinearity that is singular at \( u = 1 \). Namely, we study the solution set of
\[
- \left( \frac{u'}{\sqrt{1 + (u')^2}} \right)' = \frac{\lambda}{(1 - u)^2}, \quad -L < x < L; \quad u(-L) = u(L) = 0,
\]
with \( u < 1 \) in \([-L, L]\), for positive \( \lambda \) and \( L \). Previously, these types of equations (i.e., one-
dimensional singular prescribed mean curvature equations) have been studied in a general
context by Bonheure et al. in [11]. In particular, applying their results to problem (1.3) gives
the following:

- if \( \lambda > 0 \) is sufficiently small, then there exists at least two classical solutions of problem
  (1.3) [11, Theorems 3.1 and 5.1];

\[\text{Figure 1. Bifurcation Diagrams of (1.2) for } L \leq \pi \text{ and } L \geq \pi. \text{ Note that in (b), the bifurcation diagram continues; in particular, } \lambda \to 0^+ \text{ as } \|u\|_\infty \to \infty [57].\]
there exists a \( \lambda^* > 0 \) such that for all \( \lambda > \lambda^* \), no (classical or non-classical) positive solutions of problem (1.3) exist [11, Theorem 6.1]. To build on their work, we use the method of time maps to give a description of the exact number of classical solutions, i.e., \( C^2((-L, L)) \cap C([-L, L]) \) solutions, when the parameters \( \lambda \) and \( L \) vary. In doing so, we prove the following theorem that fully characterizes the solution set of (1.3).

**Theorem 1.1.** Let \( L > 0 \) and \( L^* \) be defined by \( L^* := \max_{\lambda > 0} g(\lambda) \), where

\[
g(\lambda) = \begin{cases} 
\frac{\lambda}{(1 - \lambda^2)^{3/2}} \left( \log \frac{1 + \sqrt{1 - \lambda^2}}{\lambda} - \sqrt{1 - \lambda^2} \right), & \text{for } \lambda \in (0, 1), \\
1/3, & \text{for } \lambda = 1, \\
\frac{\lambda}{(\lambda^2 - 1)^{3/2}} \left( \sqrt{\lambda^2 - 1} - \sec^{-1}(\lambda) \right), & \text{for } \lambda \in (1, \infty).
\end{cases}
\]  

(i) If \( L < L^* \), then there exists three values \( \lambda_*, \lambda_{**} \) and \( \lambda^* \), which depend on \( L \), such that

(a) for \( \lambda \in (0, \lambda_*) \cup [\lambda_{**}, \lambda^*) \), (1.3) has exactly two positive solutions;
(b) for \( \lambda \in (\lambda_*, \lambda_{**}) \cup \{\lambda^*\} \), (1.3) has exactly one positive solution;
(c) for \( \lambda > \lambda^* \), (1.3) has no solutions.

(ii) If \( L \geq L^* \), then there exists a value \( \lambda^* \), which depend on \( L \), such that

(a) for \( \lambda \in (0, \lambda^*) \), (1.3) has exactly two positive solutions;
(b) for \( \lambda = \lambda^* \), (1.3) has exactly one positive solution;
(c) for \( \lambda > \lambda^* \), (1.3) has no solutions.

Furthermore, in both cases, \( \lambda^* < \min \{L^{-1}, \pi^2/(27L^2)\} \).

**Remark 1.2.** Using (1.4), the value of \( L^* \) can easily be approximated: \( L^* \approx 0.3499676 \).

The results of this theorem are illustrated in Figure 2 for the two separate cases. Consequently, we see that the solutions set of (1.3) contains two bifurcations:

1. a saddle node bifurcation occurs, for all \( L \), at the point \( (\lambda^*(L), \|u(\cdot; \lambda^*, L)\|_\infty) \), where \( u(x; \lambda, L) \) is the solution of (1.3);
2. a splitting bifurcation occurs at \( L = L^* \). That is, when \( L \geq L^* \), the upper solution branch of the bifurcation diagram of (1.3) is continuous (see Figure 2(d)); however, when \( L < L^* \), the upper solution branch splits into two parts (see Figure 2(c)).

**Remark 1.3.** Equation (1.3) arises in the study of microelectromechanical systems, namely, [14] proposed that the steady state deflection, \( u \), of an elastic membrane due to an electrostatic forcing satisfies

\[
- \nabla \cdot \frac{\nabla u}{\sqrt{1 + \varepsilon^2 |\nabla u|^2}} = \frac{\mu}{(1-u)^2}, \quad x \in \Omega; \quad u < 1, \quad x \in \Omega; \quad u = 0, \quad x \in \partial \Omega,
\]

where \( 0 < \varepsilon \ll 1 \) is an aspect ratio of the device, \( \mu \) is a positive ratio of the reference electrostatic force over the reference elastic force and \( \Omega \) is a open, connected, bounded domain in \( \mathbb{R}^n \), for \( n = 1, 2 \). Therefore, when \( n = 1 \) (without loss of generality, we assume that the domain is centered at the origin), the change of variable \( y = \varepsilon^{-1} x \) yields (1.3), where \( \lambda := \varepsilon^2 \mu \) and \( L := 1/\varepsilon \).

**Remark 1.4.** The disappearing solutions behavior exhibited in case (i) of Theorem 1.1 is due to the mean curvature operator; specifically, from [60], we know that for the related semilinear problem

\[
- u'' = \frac{\lambda}{(1-u)^2}, \quad -L < x < L; \quad u(-L) = u(L) = 0,
\]  

(1.5)
there exists a $\lambda^* > 0$ such that for $\lambda \in (0, \lambda^*)$ there are exactly two solutions of (1.5), for $\lambda = \lambda^*$ there is a unique solution of (1.5) and for $\lambda > \lambda^*$ there are no solutions of (1.5). So, in other words, the bifurcation diagram of (1.5) depends on only a single parameter, which can easily be seen through the change of variable $y = \sqrt{\lambda} x$, and has same qualitative shape as the one shown in Figure 2(d). Hence the disappearance of solutions is a byproduct of the mean curvature operator coupled with the singularity.

We start the next section by proving some necessary conditions for the existence of classical solutions of (1.3). From this, we restrict the parameter space to where solutions may exist and use the inherent symmetry of (1.3) to reduce it to an equivalent problem. Then using these restrictions, in Section 3, we introduce and analyze a time map à la [57] and [63]. From this analysis, we then prove Theorem 1.1.
2. Necessary conditions for the existence of solutions

In this section, we derive some properties about the solutions of (1.3). Also, we show that solutions cannot exist under certain conditions.

**Lemma 2.1.** If \( u(\cdot; \lambda, L) \) is solution of (1.3) for fixed \( \lambda > 0 \) and \( L > 0 \), then \( u > 0 \) in \((L, L)\); furthermore, \( u \) is strictly concave down in this region, which implies that its maximum, \( \|u\|_\infty \in (0,1) \), is unique.

**Proof.** From carrying out the differentiation in differential equation of (1.3), we obtain \( u''(x) < 0 \) for all \( x \) in \((-L, L)\) and \( \lambda > 0 \), which gives the desired result. \( \Box \)

Next, it is easily verified that the ordinary differential equation given in (1.3) has the first integral

\[
\frac{1}{\sqrt{1+(u')^2}} - \frac{\lambda}{1-u} = E, \tag{2.1}
\]

where \( E \) is a conserved quantity of the system and can be determined by the maximum deflection of \( u \). That is, let \( c \) in \((-L, L)\) be the unique value such that \( u(c) = \|u\|_\infty \); hence, \( u'(c) = 0 \) and first integral (2.1) gives \( E = 1 - \lambda(1 - \|u\|_\infty)^{-1} \), which implies

\[
\frac{1}{\sqrt{1+(u')^2}} - \frac{\lambda}{1-u} = 1 - \frac{\lambda}{1-\|u\|_\infty}. \tag{2.2}
\]

Using this we can prove the following necessary condition for classical solutions.

**Theorem 2.2.** Let \( u(\cdot; \lambda, L) \) be a solution of (1.3) for \( \lambda > 0 \) and \( L > 0 \). Then \( \|u\|_\infty \leq 1/(1+\lambda) \).

**Proof.** For contradiction assume that \( \|u\|_\infty > (1+\lambda)^{-1} \). Hence,

\[
\|u\|_\infty > \frac{1-(1+\lambda)}{1-\lambda - \|u\|_\infty} > 0. \tag{2.3}
\]

From Lemma 2.1, we know that \( u \) attains its unique maximum at a value, say \( c \), in \((-L, L)\), i.e., \( u(c) = \|u\|_\infty > u(x) \) for all \( x \) in \((-L, c) \cup (c, L)\); therefore, inequality (2.3) implies that there exists an \( \hat{x} \) in \((-L, c) \cup (c, L)\) such that

\[
u(\hat{x}) = \frac{1-(1+\lambda)}{1-\lambda - \|u\|_\infty}.
\]

Now, from first integral (2.2), we know

\[
u'(x; \lambda, L)^2 = -1 + \frac{(1-u)^2(1-\|u\|_\infty)^2}{[(1-u)(1-\|u\|_\infty)-\lambda(\|u\|_\infty-u)]^2}, \quad x \neq \hat{x}.
\]

Since \( u' \) is continuous and bounded in \((-L, L)\), we take \( x \to \hat{x} \) so

\[
u'(\hat{x}; \lambda, L)^2 = -1 + \lim_{x \to \hat{x}} \frac{(1-u)^2(1-\|u\|_\infty)^2}{[(1-u)(1-\|u\|_\infty)-\lambda(\|u\|_\infty-u)]^2};
\]

however, the limit on the right-hand side diverges to infinity and we have a contradiction. \( \Box \)

Now, from this restriction, we may also find the value of \( x \) where the maximum deflection of \( u \) happens. To do so, we first have the following proposition.

**Proposition 2.3.** Assume that \( \lambda > 0 \) is fixed, \( \alpha \in (0,1/(1+\lambda)] \) and \( u(x) \) is a function whose range is contained in \((0,1)\). Then

\[
2(1-u)(1-\alpha) - \lambda(\alpha-u) > 0 \quad \text{and} \quad (1-u)(1-\alpha) - \lambda(\alpha-u) > 0. \tag{2.4a,b}
\]
From Lemma 2.1, we know that

\[2(1-u)(1-\alpha) - \lambda(\alpha-u) = (-2(1-u) - \lambda)\alpha + 2 - 2u + \lambda u,\]

which is linear in \(\alpha\). Therefore, since \(u \in (0,1)\), \(-2(1-u) - \lambda < 0\) and the function takes on its minimum at \(\alpha = 1/(1+\lambda)\). Hence,

\[2(1-u)(1-\alpha) - \lambda(\alpha-u) \geq \frac{\lambda(1-u+\lambda u)}{1+\lambda} > 0.\]

Similarly, one can show \((1-u)(1-\alpha) - \lambda(\alpha-u) > 0\).

\[\square\]

From this proposition, first integral (2.2) and Theorem 2.2, we have that the derivative of a solution \(u(\cdot; \lambda, L)\) of (1.3) must satisfy

\[u'(x; \lambda, L) = \begin{cases} \sqrt{\lambda(\|u\|_\infty - u)}\sqrt{2(1-u)(1-\|u\|_\infty) - \lambda(\|u\|_\infty - u)} & , \quad x \in (-L, c), \\ -\sqrt{\lambda(\|u\|_\infty - u)}\sqrt{2(1-u)(1-\|u\|_\infty) - \lambda(\|u\|_\infty - u)} & , \quad x \in (c, L), \end{cases}\]

(2.5)

where \(\|u\|_\infty \in (0, 1/(1+\lambda))\) and \(\lambda > 0\). Next, we use derivative (2.5) to prove that the solutions to (1.3) must be even.

**Theorem 2.4.** If \(u(\cdot; \lambda, L)\) is a solution of (1.3) for \(\lambda > 0\) and \(L > 0\), then \(u\) is an even function in \(x\); moreover, it attains its unique maximum at \(x = 0\), which implies that \(u(0) = \|u\|_\infty\) and \(u'(0) = 0\).

**Proof.** From Lemma 2.1, we know that \(u\) attains its unique maximum at a value, say \(x = c\), in \((-L, L)\). To show that \(u\) is even, we use the fact that for every \(U \in (0, u(c))\) there exists two values \(x_1\) and \(x_2\) in \((-L, c)\) and \((c, L)\), respectively, such that \(u(x_1) = u(x_2) = U\). Thus, using (2.5), we obtain

\[\int_{-L}^{x_1} dx = \int_0^U \frac{(1-u)(1-\|u\|_\infty) - \lambda(\|u\|_\infty - u)}{\sqrt{\lambda(\|u\|_\infty - u)}\sqrt{2(1-u)(1-\|u\|_\infty) - \lambda(\|u\|_\infty - u)}} du \]

\[= -\int_0^U \frac{(1-u)(1-\|u\|_\infty) - \lambda(\|u\|_\infty - u)}{\sqrt{\lambda(\|u\|_\infty - u)}\sqrt{2(1-u)(1-\|u\|_\infty) - \lambda(\|u\|_\infty - u)}} du = \int_{x_2}^{L} dx,\]

which implies \(x_1 + L = L - x_2\); that is, \(x_1 = -x_2\), proving that \(u\) is an even function in \(x\); moreover, since it has only one maximum, \(c = 0\).

\[\square\]

We summarize the results of this section in the following theorem.

**Theorem 2.5.** The function \(u(\cdot; \lambda, L)\) is a solution of

\[-\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = \frac{\lambda}{(1-u)^2}, \quad 0 < x < L; \quad u < 1, \quad -L < x < L \quad u'(0) = u(L) = 0,\]

if and only if its even extension to \((-L, L)\) is a solution of (1.3). Also, no solutions of (1.3) exist, if \(u(0) > 1/(1+\lambda)\).
3. Time Map

In this section, to investigate the existence of solutions of (1.3), we define and analyze the time map of
\[- \left( \frac{u'}{\sqrt{1 + (u')^2}} \right)' = \frac{\lambda}{(1-u)^2}, \quad x > 0; \quad u(0) = \alpha, \quad u'(0) = 0, \quad (3.1)\]
for \( \alpha \in (0, 1/(1 + \lambda)] \).

**Definition 3.1.** For fixed \( \lambda > 0 \), we define the time map \( T(\cdot; \lambda) : (0, 1/(1 + \lambda)] \to \mathbb{R} \) of (3.1) as the function that takes in \( \alpha \) and returns the so-called “time” it takes for the solution of (3.1) to hit the line \( u = 0 \), i.e., it returns the smallest value \( L \) such that \( u(L; \lambda, \alpha) = 0 \), where \( u(\cdot; \lambda, \alpha) \) is the solution of (3.1).

From this definition we have the following theorem.

**Theorem 3.2.** For a given \( L \), a certain value of \( \lambda \) admits a solution of (2.6) if and only if we can find an \( \alpha \) in \((0, 1/(1 + \lambda)]\) such that \( T(\alpha; \lambda) = L \).

**Remark 3.3.** Note that the upper bound on \( \alpha \) comes from Theorem 2.5.

Because of this equivalent formulation for finding solutions of (2.6), it is beneficial to have an analytic expression for the time map. To this end, we deduce that similar to solutions of (1.3), solutions of (3.1) satisfy the first integral
\[
\frac{1}{\sqrt{1 + (u')^2}} - \frac{\lambda}{1-u} = 1 - \frac{\lambda}{1-\alpha},
\]
which implies
\[
u'(x; \lambda, \alpha) = -\frac{\sqrt{\lambda(\alpha-u)}\sqrt{2(1-u)(1-\alpha)-\lambda(\alpha-u)}}{(1-u)(1-\alpha)-\lambda(\alpha-u)}, \quad (3.2)
\]
for \( \alpha \in (0, 1/(1 + \lambda)] \) and \( \lambda > 0 \). Hence, in separating variables, integrating with respect to \( x \) from 0 to \( L \) and using the change of variables \( u = \alpha z \), we deduce from (3.2) that our time map is given by
\[
T(\alpha; \lambda) = \frac{\sqrt{\alpha}}{\sqrt{2}} \int_0^1 \frac{(1-\alpha\lambda)(1-z) - \lambda\alpha(1-z)}{\sqrt{1-z}\sqrt{2(1-\alpha\lambda)(1-\alpha) - \lambda\alpha(1-z)}} \, dz =: \int_0^1 K(\alpha, z; \lambda) \, dz. \quad (3.3)
\]

Next we prove that \( T(\cdot; \lambda) \) is well defined for \( \alpha \in (0, 1/(1 + \lambda)] \) and is in \( C^2(0, 1/(1 + \lambda)) \).

**Lemma 3.4.** Let \( \lambda > 0 \) be fixed. Then \( T(\alpha; \lambda) \) exists for each \( \alpha \in (0, 1/(1 + \lambda)] \). Moreover, \( T(\alpha; \lambda) \) is differentiable at each \( \alpha \in (0, 1/(1 + \lambda)] \) with its derivative given by the formula
\[
T'(\alpha; \lambda) = \int_0^1 \frac{\partial K}{\partial \alpha}(\alpha, z; \lambda) \, dz. \quad (3.4)
\]
Furthermore, \( T'(\alpha; \lambda) \) is differentiable at each \( \alpha \in (0, 1/(1 + \lambda)] \) with its derivative given by the formula
\[
T''(\alpha; \lambda) = \int_0^1 \frac{\partial^2 K}{\partial \alpha^2}(\alpha, z; \lambda) \, dz. \quad (3.5)
\]

**Proof.** First, \( T \) is well defined for \( \alpha \in (0, 1/(1 + \lambda)] \) since
\[
|T(\alpha; \lambda)| \leq \sqrt{\alpha} \int_0^1 \sqrt{1-z} \frac{1}{\sqrt{2(1-\alpha z)(1-\alpha) - \lambda\alpha(1-z)}} \, dz
\]
\[
\leq \frac{\sqrt{\alpha} \sqrt{1+\lambda}}{\lambda} \int_0^1 \frac{1}{\sqrt{1-z} \sqrt{1-\alpha z}} \, dz = \frac{\sqrt{1+\lambda}}{\lambda} \log \left( \frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} \right), \quad (3.6)
\]
where from inequality (2.4a) we have used the fact that
\[
\frac{1}{\sqrt{2(1 - \alpha z)(1 - \alpha)} - \lambda \alpha (1 - z)} \leq \frac{\sqrt{1 + \lambda}}{\sqrt{\lambda} \sqrt{1 - \alpha}} \tag{3.7}
\]
for \(\alpha \in (0, 1/(1 + \lambda)], \lambda > 0, z \in (0, 1)\).

Next,
\[
\frac{\partial K}{\partial \alpha}(\alpha, z; \lambda) = \frac{1}{2\sqrt{\lambda} \alpha} \frac{(1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z)}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha)} - \lambda \alpha (1 - z)}
\]
\[
- \frac{\alpha}{\sqrt{\lambda}} \frac{1 + (1 - 2\alpha) z + \lambda(1 - z)}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha)} - \lambda \alpha (1 - z)}
\]
\[
+ \frac{\alpha}{\lambda} \frac{((1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z))(2 + \lambda(1 - z) + (2 - 4\alpha) z)}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha)} - \lambda \alpha (1 - z))^{3/2}}
\]
\[
=: H_1(\alpha, z; \lambda) - H_2(\alpha, z; \lambda) + H_3(\alpha, z; \lambda),
\]
which is defined for \(\alpha \in (0, 1/(1 + \lambda)]\). Thus,
\[
\left| \frac{\partial K}{\partial \alpha}(\alpha, z; \lambda) \right| \leq |H_1| + |H_2| + |H_3|,
\]
so that for \(\alpha \in [a, 1/(1 + \lambda)], a > 0\), and \(z \in (0, 1)\),
\[
\left| \frac{\partial K}{\partial \alpha}(\alpha, z; \lambda) \right| \leq \frac{C_1}{\sqrt{z} \sqrt{1 - z}} + \frac{C_2}{(1 - (1 + \lambda))^{3/2} \sqrt{1 - z}} \in L^1(0, 1), \tag{3.8}
\]
where \(C_1\) and \(C_2\) are positive constants independent of \(z\) and \(\alpha\) (see A, equations (A.1)–(A.4)). Therefore, we have by Theorem 10.39 in [2] that \(T(\alpha; \lambda)\) is differentiable at each \(\alpha \in (0, 1/(1 + \lambda))\) with its derivative given by equation (3.4).

Furthermore,
\[
\frac{\partial^2 K}{\partial \alpha^2}(\alpha, z; \lambda) = \frac{1}{\alpha^{3/2} \sqrt{\lambda}} \frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z))^{5/2}}}, \tag{3.9}
\]
which is defined for each \(\alpha \in (0, 1/(1 + \lambda))\) and where
\[
a_0 := -1 + \alpha(1 - \lambda), \quad a_1 := \alpha(1 + 5\alpha - 10\alpha^2 + \lambda + 2\alpha^3(2 + \lambda)),
\]
\[
a_2 := -\alpha^3(10 - 17\alpha + \alpha^2(7 + 3\lambda)), \quad a_3 := \alpha^4(4 + 3\alpha^2 - 2\lambda + \alpha(-7 + 3\lambda)). \tag{3.10}
\]
Thus, for \(\alpha \in [a, 1/(1 + \lambda)], a > 0\), and \(z \in (0, 1)\),
\[
\left| \frac{\partial^2 K}{\partial \alpha^2}(\alpha, z; \lambda) \right| \leq \frac{|a_0| + |a_1| + |a_2| + |a_3|}{\alpha^{3/2} \sqrt{\lambda} \sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z))^{5/2}}
\]
\[
\leq \frac{(|a_0| + |a_1| + |a_2| + |a_3|)(1 + \lambda)^{5/2}}{a^{3/2} \lambda^3} \frac{1}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha))^{5/2}}
\]
\[
\leq \frac{B}{\sqrt{1 - z} \sqrt{2(1 - (1 + \lambda))}^{5/2}}
\]
where \(B\) is a positive constant that is independent of \(z\) and \(\alpha\). Since \((1 - z)^{-1/2}(1 - az)^{-5/2} \in L^1(0, 1)\) for \(\alpha \in (0, 1)\), by Theorem 10.39 in [2] we have that \(T'(\alpha; \lambda)\) is differentiable at each \(\alpha \in (0, 1/(1 + \lambda))\) with its derivative given by equation (3.5).

For fixed \(\lambda > 0\), we can now differentiate \(T\) by taking the derivative inside the integral, which makes it is easier to analyze the behavior of \(T\) on \((0, 1/(1 + \lambda))\).

**Proposition 3.5.** Let \(\lambda > 0\) be fixed. Then
(i) for all \( \alpha \in (0, 1/(1 + \lambda)] \), \( T(\alpha; \lambda) > 0 \);
(ii) \( T(\alpha; \lambda) \to 0 \) as \( \alpha \to 0^+ \);
(iii) \( T'(\alpha; \lambda) \to +\infty \) as \( \alpha \to 0^+ \);
(iv) \( \lim_{\alpha \to (1/\sqrt{\lambda})^-} T'(\alpha; \lambda) < 0 \);
(v) there exists a value \( \alpha^* \in (0, 1/(1+\lambda)] \) such that \( T(\alpha^*; \lambda) = \max \{T(\alpha; \lambda): \alpha \in (0, 1/(1+\lambda)]\} \).

**Proof.**

(i) This follows from definition (3.3) and Proposition 2.3.

(ii) Since \( \lim_{\alpha \to 0^+} \log \left( \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \right) = 0 \), by inequality (3.6), we have \( T(\alpha; \lambda) \to 0 \) as \( \alpha \to 0^+ \).

(iii) From inequality (3.7), we have
\[
\left| \int_0^1 H_2(\alpha, z; \lambda) \, dz \right| \leq \int_0^1 |H_2(\alpha, z; \lambda)| \, dz \leq (2 + \lambda) \frac{\sqrt{1 + \lambda}}{\lambda^2} \int_0^1 \frac{1}{\sqrt{1 - z}} \frac{1}{\sqrt{1 - \alpha z}} \, dz
\]
\[
= \frac{(2 + \lambda) \sqrt{1 + \lambda}}{\lambda^2} \log \left( \frac{1 + \sqrt{\lambda}}{1 - \sqrt{\lambda}} \right) \to 0 \quad \text{as} \quad \alpha \to 0^+;
\]
which implies that \( \int_0^1 H_2(\alpha, z; \lambda) \, dz \to 0 \) as \( \alpha \to 0^+ \). Similarly, from inequality (A.3),
\[
\left| \int_0^1 H_3(\alpha, z; \lambda) \, dz \right| \leq \int_0^1 |H_3(\alpha, z; \lambda)| \, dz \leq \frac{\sqrt{\alpha}(1 + \lambda)^{3/2}(4 + \lambda)}{2\alpha^2} \int_0^1 \frac{1}{(1 - \alpha z)^{3/2}} \frac{1}{\sqrt{1 - z}} \, dz
\]
\[
= \frac{(1 + \lambda)^{3/2}(4 + \lambda)}{\alpha^2} \frac{\sqrt{\alpha}}{1 - \alpha} \to 0 \quad \text{as} \quad \alpha \to 0^+;
\]
and \( \int_0^1 H_3(\alpha, z; \lambda) \, dz \to 0 \) as \( \alpha \to 0^+ \). But
\[
\int_0^1 H_1(\alpha, z; \lambda) \, dz \geq \frac{1}{2\sqrt{2\lambda \alpha}} \int_0^1 ((1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z)) \, dz
\]
\[
= \frac{1}{2\sqrt{2\lambda \alpha}} \left( 1 - 3\alpha + \frac{\alpha^2}{2} - \frac{\lambda \alpha}{2} \right) \to +\infty \quad \text{as} \quad \alpha \to 0^+.
\]
Therefore, from equation (3.4) and inequalities (3.11)–(3.13), \( \lim_{\alpha \to 0^+} T'(\alpha; \lambda) = +\infty \).

(iv) From inequality (3.8) and Theorem 10.38 of [2], we have
\[
\lim_{\alpha \to (1/\sqrt{\lambda})^-} T'(\alpha; \lambda) = \int_0^1 \frac{\partial K}{\partial \alpha} \left( \frac{1}{1 + \lambda}, z; \lambda \right) \, dz.
\]

Now,
\[
\frac{\partial K}{\partial \alpha} \left( \frac{1}{1 + \lambda}, z; \lambda \right) = \frac{3\lambda(z - 1) - (z - 1)^2 - \lambda^2(z^2 - 2z + 3) + \lambda^3(z^2 + z - 1)}{\lambda \sqrt{1 + \lambda} \sqrt{1 - z} (1 - z + \lambda(1 + z))^{3/2}}.
\]

**Case I:** If \( \lambda = 1 \), then integrating by parts gives
\[
\int_0^1 \frac{\partial K}{\partial \alpha} \left( \frac{1}{1 + \lambda}, z; \lambda \right) \, dz = \int_0^1 \frac{-8 + 8z - z^2}{4\sqrt{1 - z}} \, dz = -\frac{8}{5}, \quad \text{for} \quad \lambda = 1.
\]

**Case II:** If \( \lambda \neq 1 \), then
\[
\int_0^1 \frac{\partial K}{\partial \alpha} \left( \frac{1}{1 + \lambda}, z; \lambda \right) \, dz = \frac{1 + \lambda^2 - 2\lambda^4 + 6\lambda^2 \sqrt{\lambda^2 - 1} \arctan \sqrt{1 + \lambda}}{\lambda(1 + \lambda)(\lambda - 1)^3}.
\]
(a) If \( \lambda < 1 \), we have \( \sqrt{\lambda^2 - 1} = i\sqrt{1 - \lambda^2} \) and
\[
\arctan \frac{\lambda - 1}{1 + \lambda} = \arctan i \sqrt{\frac{1 - \lambda}{1 + \lambda}} = \frac{i}{2} \log \left( \frac{\sqrt{1 + \lambda} + \sqrt{1 - \lambda}}{\sqrt{1 + \lambda} - \sqrt{1 - \lambda}} \right),
\]
which implies from equation (3.16) that
\[
\int_0^1 \frac{\partial K}{\partial \alpha} \left( \frac{1}{1 + \lambda}, z; \lambda \right) \, dz = \frac{1 + \lambda^2 - 2\lambda^4 - 3\lambda^2 \sqrt{1 - \lambda^2} \log \left( \frac{\sqrt{1 + \lambda} + \sqrt{1 - \lambda}}{\sqrt{1 + \lambda} - \sqrt{1 - \lambda}} \right)}{\lambda(1 + \lambda)(\lambda - 1)^3} < 0, \quad \text{for } \lambda < 1. \tag{3.17}
\]

(b) If \( \lambda > 1 \), we have \( 0 < \sqrt{\lambda - 1} < \sqrt{\lambda + 1} \), which implies \( 0 < \arctan \frac{\lambda - 1}{1 + \lambda} \), and hence, from equation (3.16),
\[
\int_0^1 \frac{\partial K}{\partial \alpha} \left( \frac{1}{1 + \lambda}, z; \lambda \right) \, dz = \frac{1 + \lambda^2 - 2\lambda^4 + 6\lambda^2 \sqrt{\lambda^2 - 1} \arctan \frac{\sqrt{\lambda - 1}}{\sqrt{\lambda + 1}}}{\lambda(1 + \lambda)(\lambda - 1)^3} < 0, \quad \text{for } \lambda > 1. \tag{3.18}
\]

Therefore, by equations (3.15), (3.17) and (3.18) the integral \( \int_0^1 \frac{\partial K}{\partial \alpha} \left( (1 + \lambda)^{-1}, z; \lambda \right) \, dz < 0 \), for \( \lambda > 0 \), which implies from equation (3.14) that
\[
\lim_{\alpha \to (1 + \lambda)^{-1}} T'(\alpha; \lambda) < 0.
\]

(v) If we extend \( T \) to be defined on \([0, 1/(1 + \lambda)]\) such that \( T(0; \lambda) = 0 \), then by part (ii) \( T(\alpha; \lambda) \) is continuous on a compact set, \([0, 1/(1 + \lambda)]\) and it attains its supremum in \([0, 1/(1 + \lambda)]\). That is, by part (i), there exists a value \( \alpha^* \in (0, 1/(1 + \lambda)] \) such that \( T(\alpha^*; \lambda) = \max_{\alpha \in [0, 1/(1 + \lambda)]} T(\alpha; \lambda) \). 

\( \square \)

A consequence of this lemma is that there exists at least one critical point of \( T(\cdot; \lambda) \) in \((0, 1/(1 + \lambda)]\). Next, we prove there exists only one.

**Proposition 3.6.** Let \( \lambda > 0 \) be a fixed value. Then there is exactly one critical point of \( T(\cdot; \lambda) \) in \((0, 1/(1 + \lambda)]\).

**Proof.** By Proposition 3.5, we know there exists at least one critical point of \( T \) in \((0, 1/(1 + \lambda)]\). Thus, to complete the proof we only need to prove that there exists at most one critical point of \( T \) in \((0, 1/(1 + \lambda)]\). To do so, we will show that \( T''(\alpha; \lambda) < 0 \) for all \( \alpha \in (0, 1/(1 + \lambda)] \).

From Proposition 2.3 and equation (3.9) we see that the sign of \( \partial^2 K/\partial \alpha^2 \) for \( \alpha \in (0, 1/(1 + \lambda)] \) depends on the numerator
\[
p(z, \alpha; \lambda) := a_0(\alpha; \lambda) + a_1(\alpha; \lambda)z + a_2(\alpha; \lambda)z^2 + a_3(\alpha; \lambda)z^3,
\]
where the coefficients are defined in (3.10). Therefore, in rearranging we have
\[
p(z, \alpha; \lambda) = b_0 + b_1 \lambda,
\]
where
\[
b_0 := -1 + 5a^2z + 3a^6z^3 + \alpha(1 + z) - 10a^3(1 + z)z - 7a^5(1 + z)z^2 + \alpha^4(1 + 4z)(z + 4)z,
b_1 := -\alpha(1 - z) - 3a^5(1 - z)x^2 - 2a^4(1 - z^2)z.
\]
After a long computation it can be shown that \( b_0 \leq 0 \) and \( b_1 \leq 0 \) for all \( \alpha \) and \( z \) in \((0, 1)\), which implies that \( p(z, \alpha; \lambda) \leq 0 \) for all \( \alpha \) and \( z \) in \((0, 1)\). Therefore, \( \partial^2 K/\partial \alpha^2 < 0 \) for \((\alpha, z) \in (0, 1) \times (0, 1)\) which implies that \( \partial^2 K/\partial \alpha^2 < 0 \) for \((\alpha, z) \in (0, 1/(1 + \lambda)) \times (0, 1)\). Thus, by equation (3.5), \( T''(\alpha; \lambda) < 0 \) for all \( \alpha \in (0, 1/(1 + \lambda)) \). \( \square \)
From Proposition 3.5 and 3.6, we now have that the graph of \( T(\alpha; \lambda) \) looks like that of Figure 3. Upon inspection, we see that the number of solutions of \( T(\alpha; \lambda) = L \) depends on two values: the value of \( T \) at the end point \( \alpha = 1/(1+\lambda) \) and the maximum value of \( T \) for \( \alpha \in (0, 1/(1+\lambda)) \).

With this as motivation we define the functions
\[
M(\lambda) := \max \{T(\alpha; \lambda) : \alpha \in (0, 1/(1+\lambda))\}, \quad \text{for } \lambda \in (0, \infty),
\]
\[
g(\lambda) := T(1/(1+\lambda); \lambda), \quad \text{for } \lambda \in (0, \infty),
\]
and examine their properties.

![Figure 3. Plot of the time map \( T(\alpha; \lambda) \).](image)

**Lemma 3.7.** Let \( g(\lambda) \) be defined as in (3.20). Then \( g'(\lambda) \) and \( g''(\lambda) \) exist for each \( \lambda > 0 \). Moreover, \( g'(\lambda) \) and \( g''(\lambda) \) are given by
\[
g'(\lambda) = \int_0^1 \frac{\partial \phi}{\partial \lambda}(z, \lambda) \, dz,
\]
\[
g''(\lambda) = \int_0^1 \frac{\partial^2 \phi}{\partial \lambda^2}(z, \lambda) \, dz,
\]
respectively, where
\[
\phi(z, \lambda) := K(1/(1+\lambda), z; \lambda) = \frac{\lambda}{(1+\lambda)^{3/2}} \frac{z}{\sqrt{1-z} \sqrt{1-z + \lambda(1+z)}}.
\]

**Proof.** First, \( 1-z+\lambda(1+z) = 1+\lambda+(-1+\lambda)z \), which is linear in \( z \), and hence,
\[
1-z+\lambda(1+z) \geq \min \{1+\lambda; 2\lambda\} > 0.
\]
Then, from the definition of \( g(\lambda) \), we have
\[
0 < g(\lambda) = \frac{\lambda}{(1+\lambda)^{3/2}} \int_0^1 \frac{z}{\sqrt{1-z} \sqrt{1-z + \lambda(1+z)}} \, dz \leq \frac{\lambda}{(1+\lambda)^{3/2}} \frac{1}{\min \{1+\lambda; 2\lambda\}} \int_0^1 \frac{1}{\sqrt{1-z}} \, dz < \infty,
\]
for each \( \lambda > 0 \). Moreover, for \( \lambda > 0 \),
\[
\frac{\partial \phi}{\partial \lambda} = -\frac{(\lambda^2 - 1)z + (\lambda^2 - \lambda + 1)z^2}{(1+\lambda)^{5/2} \sqrt{1-z} (1-z + \lambda(1+z))^{3/2}}
\]
and
\[
\frac{\partial^2 \phi}{\partial \lambda^2} = \frac{(2\lambda^3 - 6\lambda - 4)z + (4\lambda^3 - 4\lambda^2 - 2\lambda + 6)z^2 + (2\lambda^3 - 4\lambda^2 + 7\lambda - 2)z^3}{(1+\lambda)^{7/2} \sqrt{1-z} (1-z + \lambda(1+z))^{5/2}}.
\]
Thus, from inequality (3.21), we obtain for each λ in [a, b], where 0 < a < b,
\[ |\partial \phi / \partial \lambda| \leq \frac{2b^2 + b + 2}{(1+a)^{3/2}(\min\{1+a, 2a\})^{3/2} \sqrt{1-z}} \leq \frac{C_1}{\sqrt{1-z}} \]
and
\[ |\partial^2 \phi / \partial \lambda^2| \leq \frac{8b^3 + 8b^2 + 15b + 12}{(1+a)^{7/2}(\min\{1+a, 2a\})^{5/2} \sqrt{1-z}} \leq \frac{C_2}{\sqrt{1-z}}. \]
Here \( C_1 \) and \( C_2 \) are positive constants independent of \( \lambda \) and \( z \). Since \((1-z)^{-1/2} \in L^1(0,1)\), by Theorem 10.39 in [2] we have our result. \( \square \)

To simplify \( g \), we have from the definition
\[ g(\lambda) = \frac{\lambda}{(1+\lambda)^{3/2}} \int_0^1 \frac{z}{\sqrt{1-z} \sqrt{1-z+\lambda(1+z)}} \, dz. \]
Thus,
\[ g(1) = \frac{1}{4} \int_0^1 \frac{z}{\sqrt{1-z}} \, dz = \frac{1}{3}. \] (3.23)

For \( \lambda \neq 1 \), the function
\[ g(\lambda) = \frac{\lambda \sqrt{1-\lambda}}{(1+\lambda)^{3/2}} \int_0^1 \frac{z}{\sqrt{(1-\lambda)z - 1} - \lambda \sqrt{z - 1}} \, dz \]
\[ = -\frac{\lambda}{(1-\lambda^2)^{3/2}} \int_1^\lambda \frac{-s + 1}{\sqrt{s^2 - \lambda^2}} \, ds \]
\[ = -\frac{\lambda}{(1-\lambda^2)^{3/2}} \left\{ \begin{array}{ll}
\int_1^\lambda (s-1)(s^2 - \lambda^2)^{-1/2} \, ds, & \text{for } \lambda \in (0,1), \\
\int_0^1 s \int_0^s (1-s)(s^2 - \lambda^2)^{-1/2} \, ds, & \text{for } \lambda \in (1,\infty).
\end{array} \right. \] (3.24)
and we have by equations (3.23)–(3.24) that
\[ g(\lambda) = \begin{cases} 
\lambda(1-\lambda)^{-3/2} \left( \log[(1+\sqrt{1-\lambda^2})/\lambda] - \sqrt{1-\lambda^2} \right), & \text{for } 0 < \lambda < 1, \\
1/3, & \text{for } \lambda = 1, \\
\lambda(\lambda^2 - 1)^{-3/2} \left( \sqrt{\lambda^2 - 1} - \arcsin \lambda \right), & \text{for } 1 < \lambda < \infty.
\end{cases} \] (3.25)

Now, we may prove the following properties of \( g(\lambda) \).

**Proposition 3.8.** Let \( g(\lambda) \) be defined as in (3.20). Then
(i) \( g(\lambda) > 0 \) for all \( \lambda \);
(ii) \( g(\lambda) \to 0 \) as \( \lambda \to 0^+ \);
(iii) \( g(\lambda) \to 0 \) as \( \lambda \to +\infty \);
(iv) \( g'(\lambda) \to +\infty \) as \( \lambda \to 0^+ \);
(v) \( g'(\lambda) \to 0^- \) as \( \lambda \to +\infty \);
(vi) there exists exactly one critical point \( c \) of \( g \) for \( \lambda > 0 \). Moreover, \( c \in (0,1) \) and \( g(c) = L^* < \infty \), where
\[ L^* := \max_{\lambda > 0} g(\lambda). \] (3.26)

**Proof.** (i) Follows from inequality (3.22).
(ii) From inequality (3.22), we see \( g(\lambda) \to 0 \) as \( \lambda \to 0^+ \).
(iii) From inequality (3.22), we see \( g(\lambda) \to 0 \) as \( \lambda \to +\infty \).
(iv) From equation (3.25),
\[
g'(\lambda) = \frac{-(2 + \lambda^2)(1 - \lambda^2 + \sqrt{1 - \lambda^2}) + (1 + 2\lambda^2)(1 + \sqrt{1 - \lambda^2}) \log[(1 + \sqrt{1 - \lambda^2})/\lambda]}{(1 - \lambda^2)^{5/2}(1 + \sqrt{1 - \lambda^2})},
\]
for \( \lambda \in (0, 1) \), which, because \( \log[(1 + \sqrt{1 - \lambda^2})/\lambda] \to +\infty \) as \( \lambda \to 0^+ \), implies \( g'(\lambda) \to +\infty \) as \( \lambda \to 0^+ \).

(v) From equation (3.25),
\[
g'(\lambda) = (1 + 2\lambda^2)\arcsin \frac{\lambda - \sqrt{\lambda^2 - 1} (2 + \lambda^2)}{\sqrt{\lambda^2 - 1}^{5/2}} = -\frac{1}{\lambda^2} + O(\lambda^{-3}) \text{ as } \lambda \to +\infty,
\]
for \( \lambda > 1 \), which implies \( \lim_{\lambda \to +\infty} g'(\lambda) = 0 \).

(vi) From equation (3.25), we deduce that \( g' < 0 \) on \( (1, \infty) \). Furthermore, by Lemma 3.7,
\[
g'(1) = \lim_{\lambda \to 1^+} \frac{(1 + 2\lambda^2)\arcsin \frac{\lambda - \sqrt{\lambda^2 - 1} (2 + \lambda^2)}{\sqrt{\lambda^2 - 1}^{5/2}}}{(1 - \lambda^2)^{5/2}} = -\frac{1}{15} < 0,
\]
and we have \( g' < 0 \) on \([1, \infty)\). Therefore, \( g' \) cannot have a critical point in \([1, \infty)\). Next, from part (iv) of this proposition, inequality (3.27) and Lemma 3.7, we have by the intermediate value theorem for derivatives that there exists a \( c \) in \((0, 1)\) such that \( g'(c) = 0 \). Then using equation (3.25) it can be shown that \( g''(\lambda) < 0 \) for \( \lambda \in (0, 1) \), which implies that \( c \) corresponds to a local max and must be unique. Also, since \( g(\lambda) > 0 \) and \( g'(\lambda) < 0 \) for \( \lambda \in [1, \infty) \), \( g(c) > g(1) \geq g(\lambda) > 0 \), and \( c \) corresponds to a global max, i.e., \( g(c) = \max_{\lambda > 0} g(\lambda) \).

From this analysis we have that the graph of \( g(\lambda) \) is given by Figure 4. Furthermore, with the help of Proposition 3.8 we can numerically estimate \( L^* \) and find \( L^* \approx 0.3499676 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph.png}
\caption{Graph of the right end point, \( g(\lambda) \), of the time map \( T(\alpha; \lambda) \).}
\end{figure}

Next, we look at the properties of \( M(\lambda) \).

**Proposition 3.9.** Assume that \( M(\lambda) \) is defined as in (3.19). Then

(i) \( M(\lambda) \) is well-defined and continuous for \( \lambda > 0 \);
(ii) \( M(\lambda) \to 0^+ \) as \( \lambda \to +\infty \);
(iii) \( M(\lambda) \to +\infty \) as \( \lambda \to 0^+ \).
Proof. (i) By Proposition 3.5(v), we have that \( M(\lambda) \) is well defined for each \( \lambda > 0 \). Furthermore, by the definition it is easy to see that \( T(\alpha; \lambda) \) is continuous for each point in \((0, 1/(1 + \lambda)] \times (0, \infty)\) which implies that \( M(\lambda) \) is continuous on \((0, \infty)\).

(ii) For any \( \lambda > 0 \) and \( \alpha \in (0, 1/(1 + \lambda)] \) we have by inequality (3.6) and part (i) of Proposition 3.5 that

\[
0 \leq T(\alpha; \lambda) \leq \frac{\sqrt{\alpha \sqrt{1 + \lambda}}}{\lambda} \int_0^1 \frac{1}{\sqrt{1 - z}} \frac{1}{\sqrt{1 - (1 + \lambda)^{-1}z}} \, dz \leq \frac{2\sqrt{1 + \lambda}}{\lambda} \log \left( \frac{1 + \sqrt{1 + \lambda}}{\sqrt{\lambda}} \right),
\]

which implies

\[
0 \leq M(\lambda) \leq \frac{2\sqrt{1 + \lambda}}{\lambda} \log \left( \frac{1 + \sqrt{1 + \lambda}}{\sqrt{\lambda}} \right), \quad \text{for } \lambda > 0.
\]

Now, the right hand side has the far-field behavior \( 2/\lambda + \mathcal{O}(\lambda^{-2}) \) as \( \lambda \to +\infty \). Therefore, \( M(\lambda) \to 0^+ \) as \( \lambda \to +\infty \).

(iii) For any \( \lambda > 0 \) and \( \alpha \in (0, 1/(1 + \lambda)] \) we have

\[
M(\lambda) \geq T \left( \frac{1}{2(1 + \lambda)}; \lambda \right)
= \frac{1}{4\sqrt{\lambda}(1 + \lambda)^{3/2}} \int_0^1 \frac{2 + 4\lambda - z + 2\lambda^2(1 + z)}{\sqrt{1 - z} \sqrt{2 - \lambda(-5 + z) - z + \lambda^2(3 + z)}} \, dz
\geq \frac{1 + 4\lambda + 2\lambda^2}{2\sqrt{(1 + \lambda)^{3/2}} \sqrt{2 + 5\lambda + 4\lambda^2}} \to +\infty \quad \text{as } \lambda \to 0^+.
\]

Now, we can prove the following lemma:

**Lemma 3.10.** Let \( T \) and \( L^* \) be defined as (3.3) and (3.26), respectively. Then

(i) for fixed \( \alpha \), \( T(\alpha; \lambda) \) is strictly decreasing with respect to \( \lambda \), which implies that \( M \) is strictly decreasing;

(ii) if \( 0 < L < L^* \), then there exist two unique constants \( \lambda_* \) and \( \lambda_{**} \), where \( \lambda_* < \lambda_{**} \), such that \( g(\lambda_*) = g(\lambda_{**}) = L \);

(iii) for each \( L \) in \((0, \infty)\), there exists a unique \( \lambda^* \) such that \( M(\lambda^*) = L \);

(iv) if \( 0 < L < L^* \), then \( \lambda_* < \lambda_{**} < \lambda^* \), where the values \( \lambda_* \), \( \lambda_{**} \) and \( \lambda^* \) are from parts (ii) and (iii).

Proof. (i) By definition (3.3),

\[
T(\alpha; \lambda) = \sqrt{\frac{\alpha}{\lambda}} \int_0^1 \frac{(1 - \alpha z)(1 - \alpha) - \lambda\alpha(1 - z)}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha) - \lambda\alpha(1 - z)}} \, dz.
\]

For fixed values of \( \alpha \) and \( z \), the integrand is decreasing with respect to \( \lambda \), which yields the desired result.

(ii) From parts (ii) and (vi) of Proposition 3.8 and \( 0 < L < L^* = g(\alpha) \), we have by the intermediate value theorem that there exists a \( \lambda_* \in (0, \alpha) \) such that \( g(\lambda_*) = L \). Furthermore, by part (vi) of Proposition 3.8, \( \alpha \) is unique and \( g(\alpha) \) is a maximum. This implies that \( g \) is increasing on \((0, \alpha)\), and \( \lambda_* \) is unique.

Next, from part (iii) of Proposition 3.8, there exists an \( N > 0 \), such that \( \lambda \geq N \) implies \( L > g(\lambda) \), and hence, \( g(\lambda) \neq L \) for \( \lambda \geq N \). Furthermore, since \( g \) is continuous on \([\alpha, N]\) and \( g(N) < L < L^* \), we have by the intermediate value theorem that there exists a \( \lambda_{**} \in (\alpha, N) \) such that \( g(\lambda_{**}) = L \). Again using part (vi) of Proposition 3.8, we find that \( g \) is decreasing on \((\alpha, N)\) which implies that \( \lambda_{**} \) is unique.

By construction, \( \lambda_* < \alpha < \lambda_{**} \), and claim (ii) holds.
(iii) From Proposition 3.9(ii), we know there exists \( b > 0 \) such that \( \lambda \geq b \) implies \( M(\lambda) < L \). Similarly, from Proposition 3.9(iii), we know that there exists an \( a > 0 \) (and \( a < b \)), such that \( \lambda \leq a \) implies \( M(\lambda) > L \). Thus, since \( M \) is continuous on \([a, b]\), and \( M(b) < L < M(a)\), by the intermediate value theorem we have that there exists an \( \lambda^* \in [a, b] \subset (0, \infty) \) such that \( M(\lambda^*) = L \).

By part (i), \( M(\lambda) \) is strictly decreasing and thus \( \lambda^* \) is unique.

(iv) From part (ii), \( \lambda_* < \lambda_{**} \). Furthermore, \( M(\lambda^*) = L = g(\lambda_{**}) < M(\lambda_{**}) \) where the last inequality is from the definition of \( M \) and \( g — (3.19) \) and (3.20), respectively—and Proposition 3.5(iv). Thus, since \( M \) is strictly decreasing, \( \lambda_{**} < \lambda^* \).

\[ \square \]

Now, we are setup to prove the Theorem 1.1 which characterizes the solution set of (1.3).

**Proof.** Let \( T(\alpha; \lambda) \) be defined by (3.3). Then, by Theorems 2.5 and 3.2, finding solutions to (1.3) is equivalent to finding \( \alpha \in (0, 1/(1+\lambda)] \), \( \lambda > 0 \), such that

\[ T(\alpha; \lambda) = L. \]  

(3.28)

Specifically, \( \alpha \) is a solution of \( T(\alpha; \lambda) = L \) if and only if \( \alpha = u(0; \lambda, L) \), where \( u(\cdot; \lambda, L) \) is a solution of (1.3), and the numbers of solutions of (1.3) and (3.28) are the same. Therefore, we look at the solutions of (3.28) in the two situations: (i) \( L < L^* \); (ii) \( L \geq L^* \). The following analysis is illustrated in Figure 5.

(i) Let \( L < L^* \). By Lemma 3.10, we know there exist constants \( \lambda_* \), \( \lambda_{**} \) and \( \lambda^* \) such that \( \lambda_* < \lambda_{**} < \lambda^* \) and \( g(\lambda_*) = g(\lambda_{**}) = M(\lambda^*) = L \). (a) If \( 0 < \lambda \leq \lambda_* \), then \( \lambda < \lambda^* \) which, since \( M \) is strictly decreasing implies \( M(\lambda) > M(\lambda^*) = L \). Furthermore, by the proof of (ii) of Lemma 3.10, \( g \) is increasing on \([0, \lambda_*] \) and \( g(\lambda) \leq g(\lambda_*) \); hence, \( g(\lambda) \leq L < M(\lambda) \), and by Proposition 3.5 and 3.6 there exists two solutions of \( T(\alpha; \lambda) = L \). (b) If \( \lambda_* < \lambda < \lambda_{**} \), then by Proposition 3.8 and Lemma 3.10, \( g(\lambda) \in (L, g(\alpha)] \). Thus, \( g(\lambda) > L \) and \( M(\lambda) > L \), and by Propositions 3.5 and 3.6 there exists one solution of \( T(\alpha; \lambda) = L \). (c) If \( \lambda_{**} < \lambda < \lambda^* \), then \( M(\lambda) > M(\lambda_{**}) = L \); also by the proof of (ii) of Lemma 3.10, \( g \) is decreasing on \([\lambda_{**}, \lambda^*] \) which implies \( g(\lambda) \leq g(\lambda_{**}) = L \). Hence, \( g(\lambda) \leq L < M(\lambda) \), and by Proposition 3.5 and 3.6 there exists two solutions of \( T(\alpha; \lambda) = L \). (d) If \( \lambda = \lambda^* \), then \( M(\lambda) = L \), and there exists only one solution of \( T(\alpha; \lambda) = L \). (e) If \( \lambda > \lambda^* \), then \( M(\lambda) < M(\lambda^*) = L \), and \( T(\alpha; \lambda) \neq L \).

(ii) Let \( L \geq L^* \) which implies that \( g(\lambda) \leq L \) for \( \lambda > 0 \). Also, by Lemma 3.10, we know there exists a constant \( \lambda^* \) such that \( M(\lambda^*) = L \). (a) If \( 0 < \lambda < \lambda^* \), then \( M(\lambda) > M(\lambda^*) = L \). Thus, \( g(\lambda) \leq L < M(\lambda) \), and we have by Proposition 3.5 and 3.6 that there exists two solutions of \( T(\alpha; \lambda) = L \). (b) If \( \lambda = \lambda^* \), then \( M(\lambda) = L \), and there exists only one solution of \( T(\alpha; \lambda) = L \). (c) If \( \lambda > \lambda^* \), then \( M(\lambda) < M(\lambda^*) = L \), and \( T(\alpha; \lambda) \neq L \).

Now to get the first bound on \( \lambda^* \), we first let \((\lambda, u)\) be a solution pair of (1.3) and consider the eigenvalue problem

\[- \left( \frac{\varphi'}{\sqrt{1 + (u')^2}} \right)' = \mu \varphi, \quad -L < x < L; \quad \varphi(-L) = \varphi(L) = 0, \]  

(3.29)

for \( \varphi \in H^1_0[-L, L] \). After multiplying (1.3) and (3.29) by \( \varphi \) and \( u \), respectively, and integrating over \((-L, L)\), we obtain

\[ \int_{-L}^{L} \varphi' u' \sqrt{1 + (u')^2} \, dx = \int_{-L}^{L} \frac{\varphi \lambda^2}{(1 - u)^2} \, dx, \quad \int_{-L}^{L} \frac{\varphi' u'}{\sqrt{1 + (u')^2}} \, dx = \int_{-L}^{L} \mu u \varphi \, dx, \]
which yields the following solvability condition for $u$:

$$0 = \int_{-L}^{L} \varphi \left[ \frac{\lambda}{(1-u)^2} - \mu u \right] \, dx,$$

for all eigenpairs, $(\mu, \varphi)$, of problem (3.29). In particular, it holds for the first eigenvalue $\mu_1$ of problem (3.29), which is positive and simple, and its corresponding eigenvalue $\varphi_1$, which can—and will—be chosen to be strictly positive in $[-L, L]$. In choosing $\varphi_1(x) > 0$ for all $x$ in $[-L, L]$, the term $\lambda(1 - u)^{-2} - \mu_1 u$ must be identically zero or change sign. It is clear that it is not zero; hence, there must be a value of $u$, say $\hat{u} \in (0, 1)$, where $\lambda(1 - \hat{u})^{-2} = \mu_1 \hat{u}$. Since this expression must be true, we obtain $0 < \lambda \leq 4\mu_1/27$. However, using Rayleigh’s formula, we also have

$$\mu_1 = \inf \left\{ \int_{-L}^{L} \frac{|\varphi'|^2}{\sqrt{1 + |u'|^2}} \, dx : \varphi \in H^1_0[-L, L], \|\varphi\|_2 = 1 \right\}$$

$$< \inf \left\{ \int_{-L}^{L} |\varphi'|^2 \, dx : \varphi \in H^1_0[-L, L], \|\varphi\|_2 = 1 \right\} = \kappa_1,$$

where $\kappa_1 = \pi^2/(4L^2)$ is the first eigenvalue of $-\partial^2/\partial x^2$ on $[-L, L]$ with homogeneous Dirichlet boundary conditions and $\| \cdot \|_2$ is the standard $L^2$ norm on $[-L, L]$. Therefore, the value $\lambda \leq 4\mu_1/27 < 4\kappa_1/27 = \pi^2/(27L^2)$.

To get the second bound we note that if $u$ is a solution to (1.3), then $(1-u)^{-2} > 1$. Therefore, integrating the differential equation in (1.3) from 0 to $x$ and using the aforementioned inequality gives

$$- \frac{u'(x)}{\sqrt{1 + u'(x)^2}} > \lambda x, \quad 0 < x \leq L,$$

where we have used that fact that $u'(0) = 0$. From Lemma 2.1, we know that $-u'(x) = |u'(x)|$; thus,

$$\lambda x < \frac{|u'(x)|}{\sqrt{1 + |u'(x)|^2}} < 1,$$

which upon taking $x \to L^-$ yields $\lambda < L^{-1}$.

Hence, $\lambda^* < \min\{L^{-1}, \pi^2/(27L^2)\}$. 

\[\square\]

**Figure 5.** Graphical illustration of Theorem 1.1
4. Conclusion

In this work, we have analyzed the solution set of the one-dimensional prescribed mean curvature problem

\[- \left( \frac{u'}{\sqrt{1 + (u')^2}} \right)' = \frac{\lambda}{(1-u)^2}, \quad -L < x < L; \quad u(-L) = u(L) = 0, \quad (1.3)\]

with \( u < 1 \) in \([-L, L]\), for positive \( L \) and positive \( \lambda \). In particular, we have shown that the solution set undergoes two bifurcations: a saddle node bifurcation and codimension 2 splitting bifurcation, which depends on the parameters \( L \) and \( \lambda \). As a result of this analysis, we speculate that the latter, which is not present in the corresponding semilinear problem \((-\Delta u = \lambda^{-1}(1-u)^{-2}\) with the same boundary conditions), is due to the interplay of the mean curvature operator and finite-singularity forcing. This is because while other one-dimensional prescribed mean curvature equations have exhibited a disappearing solutions behavior (c.f. [57]), none with a continuous forcing have exhibited this splitting bifurcation. With this in mind it would be desirable to fully characterize the solution set of the one-dimensional prescribed mean curvature equation with \( f(u) = \lambda(1-u)^{-p} \), for \( p > 0 \), and see if the splitting bifurcation generalizes to this class of problems.

Appendix. Bounding \( H_i \), for \( i = 1, 2, 3 \).

By Proposition 2.3 for \( \alpha \in [a, 1/(1 + \lambda)] \), \( a > 0 \), and \( z \in (0, 1) \), we deduce

\[
0 \leq H_1(\alpha, z; \lambda) = \frac{1}{2\sqrt{\lambda} \alpha} \left( \frac{1 - \alpha z(1 - \alpha) - \lambda \alpha (1 - z)}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z)}} \right) \leq \frac{1}{2\sqrt{\lambda} \alpha} \frac{1}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z)}},
\]

and

\[
0 \leq |H_2(\alpha, z; \lambda)| = \sqrt{\frac{\alpha}{\lambda}} \frac{|1 + (1 - 2\alpha) z + \lambda(1 - z)|}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z)}} \leq \sqrt{\frac{\alpha}{\lambda}} \frac{1 + |1 - 2\alpha| + \lambda}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z)}} \leq \sqrt{\frac{\alpha}{\lambda}} \frac{2 + \lambda}{\sqrt{1 - z} \sqrt{2(1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z)}}.
\]

Moreover from Proposition 2.3,

\[
2(1 - \alpha z)(1 - \alpha) - \lambda \alpha (1 - z) \geq \frac{\lambda}{1 + \lambda}(1 - \alpha z + \lambda \alpha z) \geq \frac{\lambda^2 \alpha}{1 + \lambda} z > 0,
\]

which implies

\[
0 \leq H_1 < \frac{\sqrt{1 + \lambda}}{2\lambda^{3/2} \alpha} \frac{1}{\sqrt{2} \sqrt{1 - z}} < \frac{\sqrt{1 + \lambda}}{2\lambda^{3/2} a} \frac{1}{\sqrt{2} \sqrt{1 - z}} \quad (A.1)
\]

and

\[
0 \leq |H_2| \leq (2 + \lambda) \frac{\sqrt{1 + \lambda}}{\lambda^{3/2}} \frac{1}{\sqrt{2} \sqrt{1 - z}}.
\]
Again, from Proposition 2.3 for \( \alpha \in [a, 1/(1 + \lambda)] \), \( a > 0 \), and \( z \in (0, 1) \), we have
\[
0 \leq |H_3(\alpha, z; \lambda)| = \sqrt{\frac{\alpha}{\lambda} \left( (1 - \alpha z)(1 - \alpha) - \alpha(1 - z) \right) + (2 + \lambda(1 - z) + (2 - 4\alpha)z)} \\
\leq \sqrt{\frac{\alpha}{\lambda} \left( \frac{2 + \lambda + 2|1 - 2\alpha|}{2\sqrt{1 - z} (2(1 - \alpha z)(1 - \alpha) - \alpha(1 - z))^{3/2}} \right)} \\
\leq \sqrt{\frac{\alpha}{\lambda} \left( \frac{4 + \lambda}{2\sqrt{1 - z} (2(1 - \alpha z)(1 - \alpha) - \alpha(1 - z))^{3/2}} \right)}.
\]

Also, from Proposition 2.3,
\[
2(1 - \alpha z)(1 - \alpha) - \alpha(1 - z) \geq \frac{\lambda}{1 + \lambda} (1 - \alpha z + \alpha z) \geq \frac{\lambda}{1 + \lambda} (1 - \alpha z) > 0,
\]
which implies
\[
0 \leq |H_3| \leq \sqrt{\frac{\alpha}{\lambda} \left( \frac{(1 + \lambda)^{3/2}(4 + \lambda)}{2\lambda^2} \right)} \left( \frac{1}{(1 - \alpha z)^{3/2} \sqrt{1 - z}} \right) \leq \frac{1}{2\lambda^2} \left( \frac{1}{1 + \lambda} \right) \left( \frac{1}{(1 - (1 + \lambda)^{-1}z)^{3/2} \sqrt{1 - z}} \right).
\]

Furthermore, for \( \lambda > 0 \)
\[
\int_0^1 \frac{1}{\sqrt{z}\sqrt{1 - z}} \, dz = \pi, \quad \int_0^1 \frac{1}{(1 - (1 + \lambda)^{-1}z)^{3/2} \sqrt{1 - z}} \, dz = 2(1 + \lambda^{-1}),
\]
which means that \( 1/(\sqrt{z}\sqrt{1 - z}) \) and \( (1 - (1 + \lambda)^{-1}z)^{-3/2}/\sqrt{1 - z} \) are in \( L^1(0, 1) \).

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ANALYSIS OF A ONE-DIMENSIONAL PRESCRIBED MEAN CURVATURE EQUATION

19

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