A remark on the convergence of Betti numbers in the thermodynamic regime

Trinh Khanh Duy

Abstract

The convergence of the expectations of Betti numbers of Čech complexes built on binomial point processes in the thermodynamic regime is established.

Keywords: Čech complex; Betti number; binomial point process; thermodynamic regime;

AMS MSC 2010: 55N05, 60F99

1 Terminologies and main results

Definition 1.1 (Čech complex). Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a collection of points in \( \mathbb{R}^d \). The Čech complex \( C(X, r) \), for \( r > 0 \), is constructed as follows.

(i) The 0-simplices (vertices) are the points in \( X \).

(ii) A \( k \)-simplex \([x_{i_0}, \ldots, x_{i_k}]\) is in \( C(X, r) \) if \( \bigcap_{j=0}^{k} B_{r/2}(x_{i_j}) \neq \emptyset \).

Here \( B_r(x) = \{ y \in \mathbb{R}^d : \|y - x\| \leq r \} \) denotes a ball of radius \( r \) and center \( x \), and \( \|x\| \) is the Euclidean norm of \( x \). The Čech complex can be also constructed from an infinite collection of points.

Let \( X_1, X_2, \ldots, \) be a sequence of i.i.d. (independent identically distributed) random variables with common probability density function \( f(x) \). Define the induced binomial point processes as \( X_n = \{X_1, \ldots, X_n\} \). The object here is the Čech complex \( C(X_n, r_n) \) built on \( X_n \), where the radius \( r_n \) also varies with \( n \). Denote by \( \beta_k(K) \) the \( k \)th Betti number, or the rank of the \( k \)th homology group, of a simplicial complex \( K \). The limiting behaviour of Betti numbers \( \beta_k(C(X_n, r_n)) \) in various regimes has been studied recently by many authors. See [1] for a brief survey. This aim of this paper is to refine a limit theorem in the thermodynamic regime, a regime that \( n^{1/d_{r_n}} \to r \in (0, \infty) \).

In the thermodynamic regime, the expectations of the \( k \)th Betti numbers, for \( 1 \leq k \leq d - 1 \), grow linearly in \( n \), that is, \( c_1 n \leq \mathbb{E}[\beta_k(C(X_n, r_n))] \leq c_2 n \) as \( n \to \infty \). After centralizing, the strong law of large number holds,

\[
\frac{1}{n} \left( \beta_k(C(X_n, r_n)) - \mathbb{E}[\beta_k(C(X_n, r_n))] \right) \to 0 \text{ a.s. as } n \to \infty,
\]

provided that the density function \( f \) has compact, convex support and that on the support of \( f \), it is bounded both below and above [7, Theorem 4.6]. A remaining problem is to describe the exact limiting behaviour of the expected values of the Betti numbers. This paper gives a solution to that problem. Note that the 0th Betti number which counts connected components in a random geometric graph was completely described [3, Chapter 13].
Betti numbers are tightly related to the number of \( j \)-simplices in \( C(X,r) \), denoted by \( S_j(C(X,r)) \) or simply by \( S_j(X,r) \), which can be expressed as

\[
S_j(X,r) = \frac{1}{j+1} \sum_{x \in X} \xi(x,X),
\]

where \( \xi(x,X) \) is the number of \( j \)-simplices containing \( x \). Note that \( \xi(x,X) \) is a local function in the sense that it depends only on points near \( x \). Then in the thermodynamic regime, the weak and strong laws of large numbers for \( S_j(C(X_n,r_n)) \) hold as a consequence of general results in [4, 5],

\[
\frac{S_j(X_n,r_n)}{n} \to \hat{S}_j \text{ a.s. as } n \to \infty.
\]

The limit \( \hat{S}_j \) can be expressed explicitly. However, Betti numbers do not have expression like the above form, and hence those general results can not be applied.

To establish a limit theorem for Betti numbers, we exploit the following two properties. The first one is the nearly additive property of Betti numbers that was used in [7] to study Betti numbers of the Čech complex built on stationary point processes. The second one is the property that binomial point processes behave locally like a homogeneous Poisson point process. The latter property is also a key tool to establish the law of large numbers for local geometric functionals [4, 5].

Now let us get into more detail to state the main result of the paper. We begin with the definition of a homogeneous Poisson point processes. Let \( N \) be the set of all counting measures on \( \mathbb{R}^d \) which are finite on any bounded Borel set and for which the measure of a point is at most 1. Define \( \mathcal{N} \) as the \( \sigma \)-algebra generated by sets of the form

\[
\{ \mu \in N : \mu(A) = k \},
\]

where \( A \) is a Borel set and \( k \) is an integer. Then a point process \( \Phi \) is a measurable mapping from some probability space into \( (N,\mathcal{N}) \). For a Borel set, let \( \Phi(A) \) denote the number of points in \( A \). By definition of the \( \sigma \)-algebra \( \mathcal{N} \), \( \Phi(A) \) is a random variable. A homogeneous Poisson point process is defined as follows. For some basic properties of point processes, see [2], for example.

**Definition 1.2** (Homogeneous Poisson point process). The point process \( \mathcal{P} \) is said to be a Poisson point process with density \( \lambda > 0 \) if

(i) for disjoint Borel sets \( A_1, \ldots, A_k \), the random variables \( \mathcal{P}(A_1), \ldots, \mathcal{P}(A_k) \) are independent;

(ii) for any bounded Borel set \( A \), the number of points in \( A \) has Poisson distribution with parameter \( |A| \), \( \mathcal{P}(A) \sim \text{Pois}(|A|) \), that is,

\[
\mathbb{P}(\mathcal{P}(A) = k) = e^{-|A|} \frac{|A|^k}{k!}, \quad k = 0,1,\ldots,
\]

where \( |A| \) denotes the Lebesgue measure of \( A \).

For homogeneous Poisson point processes, the following law of large numbers for Betti numbers was established in [7]. Let \( \mathcal{P}(\lambda) \) be a homogeneous Poisson point process on \( \mathbb{R}^d \) with density \( \lambda > 0 \). Denote by \( \mathcal{P}_A(\lambda) \) the restriction of \( \mathcal{P}(\lambda) \) on a Borel set \( A \). For a window of the form \( W_L = [-L^{1/d} \frac{1}{2}, L^{1/d} \frac{1}{2}]^d \), we write \( \mathcal{P}_L(\lambda) \) instead of \( \mathcal{P}_{W_L}(\lambda) \). For \( \lambda = 0 \),
we mean a trivial point process with no point and all functions are assumed to be zero at \( \lambda = 0 \). Then for \( 1 \leq k \leq d - 1 \), there is a constant \( \hat{\beta}_k(\lambda, r) \) such that \([7, \text{Theorem 3.5}]\),

\[
\frac{\beta_k(C(\mathcal{P}_L(\lambda), r))}{L} \to \hat{\beta}_k(\lambda, r) \text{ a.s. as } L \to \infty.
\]

Now we can state our main result.

**Theorem 1.3.** Assume that the common probability density function \( f(x) \) has bounded support, is bounded and Riemann integrable. Then as \( n \to \infty \) with \( n^{1/d} r_n = r \in (0, \infty) \),

\[
\frac{\mathbb{E}[\beta_k(C(\mathcal{X}_n, r_n))]}{n} \to \int_{\mathbb{R}} \hat{\beta}_k(f(x), r) dx.
\]

For the proof, we need a Poissonized version of the binomial processes. Let \( N_n \) be a random variable which is independent of \( \{X_n\}_{n \geq 1} \) and has Poisson distribution with parameter \( n \). Let

\[
\bar{P}_n = \{X_1, X_2, \ldots, X_{N_n}\}.
\]

Then \( \bar{P}_n \) becomes a non-homogeneous Poisson point process with intensity function \( nf(x) \).

Here a non-homogeneous Poisson point process is defined as follows.

**Definition 1.4 (Non-homogeneous Poisson point process).** Let \( f(x) \geq 0 \) be a locally integrable function on \( \mathbb{R}^d \). The point process \( \mathcal{P} \) is said to be a (non-homogeneous) Poisson point process with intensity function \( f(x) \) if

(i) for mutually disjoint Borel sets \( A_1, \ldots, A_k \), the random variables \( \mathcal{P}(A_1), \ldots, \mathcal{P}(A_k) \) are mutually independent;

(ii) for any bounded Borel set \( A \), \( \mathcal{P}(A) \sim \text{Pois}(\int_A f(x) dx) \).

As proved later, Theorem 1.3 is equivalent to the following result.

**Theorem 1.5.** Assume that the common probability density function \( f(x) \) has bounded support, is bounded and Riemann integrable. Then as \( n \to \infty \) with \( n^{1/d} r_n = r \in (0, \infty) \),

\[
\frac{\mathbb{E}[\beta_k(C(\bar{P}_n, r_n))]}{n} \to \int_{\mathbb{R}} \hat{\beta}_k(f(x), r) dx.
\]

### 2 Proofs of main theorems

We will use the following two important properties of Poisson point processes. Denote by \( \mathcal{P}(f(x)) \) the non-homogeneous Poisson point process with intensity function \( f(x) \).

(i) Scaling property. For any \( \theta > 0 \) and \( t \in \mathbb{R}^d \),

\[
\theta(\mathcal{P}(f(x)) - t) \overset{d}{=} \mathcal{P}(\theta^{-d} f(t + \theta^{-1} x)),
\]

where \( \overset{d}{=} \) denotes the equality in distribution. In particular, \( \theta(\mathcal{P}(\lambda) - t) \overset{d}{=} \mathcal{P}(\theta^{-d} \lambda) \).

(ii) Coupling property. Let \( \mathcal{P}(g(x)) \) be a Poisson point process with intensity function \( g(x) \) which is independent of \( \mathcal{P}(f(x)) \). Then

\[
\mathcal{P}(f(x)) + \mathcal{P}(g(x)) \overset{d}{=} \mathcal{P}(f(x) + g(x)).
\]

Here \( \overset{+}{=} \) means the superposition of two point processes.
We begin with a result for the simplices counting function.

**Lemma 2.1 (cf. [7 Lemma 3.2]).** Let \( S_j(\lambda, r; L) \) be the number of \( j \)-simplices in \( C(\mathcal{P}_L(\lambda), r) \). Then for fixed \( r > 0 \),

\[
\frac{\mathbb{E}[S_j(\lambda, r; L)]}{L} \to \hat{S}_j(\lambda, r) \text{ as } L \to \infty, \text{ uniformly for } 0 < \lambda \leq \Lambda.
\]

In addition, for fixed \( r \), the limit \( \hat{S}_j(\lambda, r) \) is a continuous function of \( \lambda \) on \([0, \infty)\).

**Proof.** For convenience, let \( A_l(\lambda) := S_j(\lambda, r; l^d) = S_j(C(\mathcal{P}_{V_l(\lambda)}(r)), \text{ where } V_l = [-\frac{l}{2}, \frac{l}{2})^d. \)

Our aim now is to show that

\[
\frac{\mathbb{E}[A_l(\lambda)]}{l^d} \text{ uniformly converges as } l \to \infty,
\]

and that \( \mathbb{E}[A_l(\lambda)] \) is continuous for \( \lambda \in [0, \infty) \). Let us first show the continuity of \( \mathbb{E}[A_l(\lambda)] \).

For \( 0 \leq \lambda < \mu \), we use the coupling \( \mathcal{P}(\mu) = \mathcal{P}(\lambda) + \mathcal{P}(\mu - \lambda) \). Here \( \mathcal{P}(\lambda) \) and \( \mathcal{P}(\mu - \lambda) \) are two independent Poisson point processes with density \( \lambda \) and \( (\mu - \lambda) \), respectively. Let \( N_\lambda \) (resp. \( N_{\mu; \lambda} \)) be the number of points of \( \mathcal{P}(\lambda) \) (resp. \( \mathcal{P}(\mu - \lambda) \)) in \( V_l \), which has Poisson distribution with parameter \( \lambda l^d \) (resp. \( (\mu - \lambda) l^d \)). Then the continuity follows from a trivial estimate

\[
0 \leq A_l(\mu) - A_l(\lambda) \leq N_{\mu; \lambda}(N_{\mu; \lambda} + N_\lambda)^j.
\]

Next, we show the uniform convergence. The proof here is similar to that of the pointwise convergence ([7 Lemma 3.2]). Define the function

\[
h(\mathcal{P}(\lambda)) := \frac{1}{j + 1} \sum_{x \in \mathcal{P}_l(\lambda)} \# \{ j \text{-simplices in } C(\mathcal{P}(\lambda), r) \text{ containing } x \}.
\]

Then for \( l > 2r + 1 \),

\[
\sum_{z \in \mathbb{Z}^d \cap V_{l-2r-1}} h(\mathcal{P}(\lambda) - z) \leq A_l(\lambda) \leq \sum_{z \in \mathbb{Z}^d \cap V_{l+2r+1}} h(\mathcal{P}(\lambda) - z).
\]

Consequently, by the stationarity of the Poisson point process \( \mathcal{P}(\lambda) \),

\[
(l - 2r - 2)^d \mathbb{E}[h(\mathcal{P}(\lambda))] \leq \mathbb{E}[A_l(\lambda)] \leq (l + 2r + 2)^d \mathbb{E}[h(\mathcal{P}(\lambda))].
\]

Note that \( \mathbb{E}[h(\mathcal{P}(\lambda))] \) is non-decreasing in \( \lambda \) and for any \( \lambda > 0 \),

\[
\mathbb{E}[h(\mathcal{P}(\lambda))] \leq \mathbb{E}[\mathcal{P}(\lambda; V_{1+2r})]^j + 1] < \infty.
\]

Here \( \mathcal{P}(\lambda; V_{1+2r}) \) is the number of points of \( \mathcal{P}(\lambda) \) in \( V_{1+2r} \). Therefore uniformly for \( 0 \leq \lambda \leq \Lambda \),

\[
\frac{\mathbb{E}[A_l(\lambda)]}{l^d} \to \mathbb{E}[h(\mathcal{P}(\lambda))] \text{ as } l \to \infty.
\]

The proof is complete. \( \Box \)

The following estimate for Betti numbers is a key tool to derive the convergence of Betti numbers from that of simplices counting functions. Recall that \( \beta_k(K) \) denotes the \( k \)th Betti number of the simplicial complex \( K \).
Lemma 2.2 ([7] Lemma 2.2). Let $K_1, K_2$ be two finite simplicial complexes such that $K_1 \subset K_2$. Then for every $k \geq 1$,

$$|\beta_k(K_1) - \beta_k(K_2)| \leq \sum_{j=k}^{k+1} \# \{ j \text{-simplices in } K_2 \setminus K_1 \}. $$

For the sake of simplicity, we denote by $\beta_k(\lambda, r; L)$ the $k$th Betti number of the Čech complex $\mathcal{C}(P_{W_L}(\lambda), r)$, where $W_L$ is any rectangle of the form $x + [\frac{-L^{1/d}}{2}, \frac{L^{1/d}}{2}]^d$.

Lemma 2.3. For fixed $r > 0$, uniformly for $0 \leq \lambda \leq \Lambda$,

$$\frac{E[\beta_k(\lambda, r; L)]}{L} \rightarrow \hat{\beta}_k(\lambda, r) \text{ as } L \to \infty.$$ 

The limit $\hat{\beta}_k(\lambda, r)$ has the following scaling property,

$$\hat{\beta}_k(\lambda, r) = \frac{1}{\theta} \hat{\beta}_k(\lambda \theta, \frac{r}{\theta^{1/d}}), \text{ for any } \theta > 0.$$ 

In particular, $\hat{\beta}_k(\lambda, r) = \lambda \hat{\beta}_k(1, \lambda^{1/d} r)$ is a continuous function in both $\lambda$ and $r$, and $\hat{\beta}(\lambda, r) > 0$, if $\lambda > 0$ and $r > 0$.

Proof. For fixed $r > 0$ and fixed $\Lambda > 0$, the convergence of the expectations of Betti numbers was shown in [7] Lemma 3.3. The positivity is a consequence of [6] Theorem 4.2. Here we will show the uniform convergence for $0 \leq \lambda \leq \Lambda$. We use the following criterion for the uniform convergence on a compact set, which is related to the Arzelà–Ascoli theorem. The sequence of continuous functions $\{a_L(\lambda)\}_{L>0}$ converges uniformly on $[0, \Lambda]$ if and only if it converges pointwise and is equicontinuous, that is, for any $\varepsilon > 0$, there are $\delta > 0$ and $L_0 > 0$ such that

$$|a_L(\lambda_1) - a_L(\lambda_2)| < \varepsilon \text{ for all } \lambda_1, \lambda_2 \in [0, \Lambda], |\lambda_1 - \lambda_2| < \delta, \text{ and all } L > L_0.$$ 

The task now is to show that the sequence $\{L^{-1} E[\beta_k(\lambda, r; L)]\}$ is equicontinuous. Let $\lambda < \mu$. By using the coupling $P(\mu) = P(\lambda) + P(\mu - \lambda)$, the Čech complex $\mathcal{C}(P_L(\lambda), r)$ becomes a sub-complex of $\mathcal{C}(P_L(\mu), r)$. Thus, by Lemma 2.2,

$$|\beta_k(\mu, r; L) - \beta_k(\lambda, r; L)| \leq \sum_{j=k}^{k+1} \# \{ j \text{-simplices in } \mathcal{C}(P_L(\mu), r) \setminus \mathcal{C}(P_L(\lambda), r) \}$$

$$= \sum_{j=k}^{k+1} (S_j(\mu, r; L) - S_j(\lambda, r; L)).$$

Therefore

$$\left| \frac{E[\beta_k(\mu, r; L)]}{L} - \frac{E[\beta_k(\lambda, r; L)]}{L} \right| \leq \sum_{j=k}^{k+1} \left( \frac{E[S_j(\mu, r; L)]}{L} - \frac{E[S_j(\lambda, r; L)]}{L} \right).$$

The sequence $\{L^{-1} E[S_j(\lambda, r; L)]\}$ converges uniformly on $[0, \Lambda]$ by Lemma 2.1 and hence, is equicontinuous, which then implies the equicontinuity of the sequence $\{L^{-1} E[\beta_k(\lambda, r; L)]\}$.

By observing that $\theta^{-1/d}P(\lambda)$ has the same distribution with $P(\lambda \theta)$, we obtain the scaling property of $\hat{\beta}_k(\lambda, r)$. It then follows from the scaling property that $\hat{\beta}_k(\lambda, r)$ is continuous in both $\lambda$ and $r$. The lemma is proved. \qed
Let us now consider the scaled Poissonized version \( P_n = \{ n^{1/d}X_1, n^{1/d}X_2, \ldots, n^{1/d}X_{N_n} \} \). Recall that \( N_n \) is independent of \( \{ X_n \} \) and has Poisson distribution with parameter \( n \). Then \( P_n = n^{1/d}P_n \) is a non-homogeneous Poisson point process with the intensity function \( f_n(x) := f(x/n^{1/d}) \). It is clear that \( C(P_n, r) = C(P_n, r_n) \) because \( n^{1/d}r_n = r \). Thus Theorem 1.5 can be rewritten as follows.

**Theorem 2.4.** Assume that the common probability density function \( f(x) \) has bounded support, is bounded and Riemann integrable. Then for fixed \( r > 0 \), as \( n \to \infty \),

\[
\frac{\mathbb{E}[\beta_k(C(P_n, r))]}{n} \to \int_{\mathbb{R}} \hat{\beta}_k(f(x), r) dx = \int_{\mathbb{R}} \hat{\beta}_k(1, f(x)^{1/d}) f(x) dx.
\]

**Lemma 2.5.** Assume that \( f(x), g(x) \leq \Lambda \) in \( W_L \), where \( W_L \subset \mathbb{R}^d \) is a set of volume \( L \). Then there exists a constant \( c = c(k, \Lambda L) \) such that

\[
\left| \mathbb{E}[\beta_k(C(P_{W_L}(f(x)), r))] - \mathbb{E}[\beta_k(C(P_{W_L}(g(x)), r))] \right| \leq c \int_{W_L} |f(x) - g(x)| dx.
\]

**Proof.** By considering \( f(x) := f(x)|_{W_L} \) and \( g(x) := g(x)|_{W_L} \), we omit the subscript \( W_L \) in formulae. Let \( h(x) = \max\{f(x), g(x)\} \). A key idea here is the following coupling

\[
P(h(x)) = P(f(x)) + P(h(x) - f(x)).
\]

Let \( t = \int (h(x) - f(x)) dx = \int (g(x) - f(x))^+ dx \) and \( N_t \) be the number of points of \( P(h(x) - f(x)) \) in \( W_L \). Then \( N_t \) has Poisson distribution with parameter \( t \). The total number of points of \( P(h(x)) \) is bounded by \( N_t + N_{\Lambda L - t} \), where \( N_{\Lambda L - t} \) has Poisson distribution with parameter \( (\Lambda L - t) \) which is independent of \( N_t \). It now follows from Lemma 2.2 that

\[
\left| \beta_k(C(P(f(x)), r)) - \beta_k(C(P(h(x)), r)) \right| \leq \sum_{j=k}^{k+1} S_j \left( C(P(h(x)), r) \setminus C(P(f(x)), r) \right)
\]

\[
\leq 2N_t(N_t + N_{\Lambda L - t})^{k+1},
\]

and hence,

\[
\left| \mathbb{E}[\beta_k(C(P(f(x)), r))] - \mathbb{E}[\beta_k(C(P(h(x)), r))] \right| \leq 2\mathbb{E}[N_t(N_t + N_{\Lambda L - t})^{k+1}].
\]

The right hand side is a polynomial of \( t \) whose smallest order is 1 and note that \( t \leq \Lambda L \), thus it is bounded by \( c(k, \Lambda L)t \), where the constant \( c(k, \Lambda L) \) depends only on \( k, \Lambda L \), namely we have

\[
\left| \mathbb{E}[\beta_k(C(P(f(x)), r))] - \mathbb{E}[\beta_k(C(P(h(x)), r))] \right| \leq c \int (g(x) - f(x))^+ dx.
\]

An analogous estimate holds when we compare the \( k \)th Betti number of \( C(P(g(x)), r) \) and \( C(P(h(x)), r) \). The proof is complete.

**Proof of Theorem 2.4.** Let \( S \) be the support of \( f \) and \( \Lambda := \sup f(x) \). Divide \( \mathbb{R}^d \) according to the lattice \( \left( L/n \right)^{1/d} \mathbb{Z}^d \) and let \( \{ C_i \} \) be the cubes which intersect with \( S \). Since we also consider the Poisson point process with density \( 0 \), we may assume that \( S = \bigcup_i C_i \).

Let \( W_i \) be the image of \( C_i \) under the map \( x \mapsto n^{1/d}x \). Then \( W_i \) is a cube of length \( L^{1/d} \). Let \( \beta_k(W_i, r) \) be the \( k \)th Betti number of \( C(P_{n_{\mid W_i}}(x), r) \). We now compare the \( k \)th
Betti number of $C(P_n, r)$ and that of $\cup_i C(P_n | W_i, r)$ by using Lemma 2.2:

$$\left| \beta_k(C(P_n, r)) - \beta_k(\bigcup_i C(P_n | W_i, r)) \right| \leq \sum_{j=k}^{k+1} S_j \left( C(P_n, r) \setminus \bigcup_i C(P_n | W_i, r) \right)$$

$$\leq \sum_{j=k}^{k+1} S_j(P_n, r; \cup_i (\partial W_i)^{(r)}) \ . \ (1)$$

Here $S_j(P_n, r; A)$ is the number of $j$-simplices in $C(P_n, r)$ which has a vertex in $A$, $\partial A$ denotes the boundary of the set $A$ and $A^{(r)}$ is the set of points with distance at most $r$ from $A$. The second inequality holds because any simplex in $C(P_n, r) \setminus \cup_i C(P_n | W_i, r)$ must have a vertex in $\cup_i (\partial W_i)^{(r)}$.

Finally, by the coupling $\mathcal{P}(\Lambda) = \mathcal{P}_n + \mathcal{P}(\Lambda - f(x/n^{1/d}))$, it follows that for any bounded Borel set $A$,

$$E[S_j(P_n, r; A)] \leq E[S_j(\mathcal{P}(\Lambda), r; A)] \leq E\left[ \sum_{x \in \mathcal{P}(\Lambda) \cap A} \mathcal{P}(A; B_r(x)) \right] =: \mu_{\Lambda, r, j}(A) < \infty.$$ 

Here $\mu_{\Lambda, r, j}$ becomes a translation invariant measure on $\mathbb{R}^d$ which is finite on bounded Borel sets. Thus $\mu_{\Lambda, r, j}(A) = c(\Lambda, r, j) |A|$ for some constant $c(\Lambda, r, j)$ depending only on $\Lambda, r$ and $j$. Now by taking the expectation in (1), we get

$$\left| \mathbb{E}[\beta_k(C(P_n, r))] - \sum_i \mathbb{E}[\beta_k(C(P_n | W_i, r))] \right| \leq c \sum_i |(\partial W_i)^{(r)}| \leq c' \frac{|S|}{L^{(d-1)/d}} = c' \frac{|S|}{L^{1/d}},$$

where $c$ and $c'$ are constants which do not depend on $n$ and $L$. Therefore,

$$\limsup_{n \to \infty} \left| \frac{\mathbb{E}[\beta_k(C(P_n, r))]}{n} - \frac{1}{n} \sum_i \mathbb{E}[\beta_k(W_i, r)] \right| \leq c' \frac{|S|}{L^{1/d}} . \ (2)$$

Let $f_i^* := \sup_{x \in C_i} f(x)$ and $\beta_k(f_i^*, r)$ be the $k$th Betti number of the Čech complex built on a homogeneous Poisson point process $\mathcal{P}_{W_i}(f_i^*)$ with density $f_i^*$ restricted on $W_i$. Then by Lemma 2.3,

$$\left| \mathbb{E}[\beta_k(W_i, r)] - \mathbb{E}[\beta_k(f_i^*, r)] \right| \leq c(k, \Lambda L) \int_{W_i} (f_i^* - f(x/n^{1/d})) \, dx$$

$$= c(k, \Lambda L) \int_{C_i} (f_i^* - f(x)) \, dx .$$

Here $c(k, \Lambda L)$ is a constant depending only on $k$ and $\Lambda L$. Consequently,

$$\left| \frac{1}{n} \sum_i \mathbb{E}[\beta_k(W_i, r)] - \frac{1}{n} \sum_i \mathbb{E}[\beta_k(f_i^*, r)] \right| \leq c(k, \Lambda L) \sum_i \int_{C_i} (f_i^* - f(x)) \, dx \to 0 \text{ as } n \to \infty,$$

because the function $f(x)$ is assumed to be Riemann integrable.

Next by comparing $\mathbb{E}[\beta_k(f_i^*, r)]$ with the limit $\hat{\beta}_k(\lambda, r)$, we get

$$\left| \frac{1}{n} \sum_i \mathbb{E}[\beta_k(f_i^*, r)] - \frac{L}{n} \sum_i \hat{\beta}_k(f_i^*, r) \right| \leq \frac{L}{n} \# \{C_i\} \sup_{0 \leq \lambda \leq \Lambda} \left| \frac{\mathbb{E}[\beta_k(\lambda, L)]}{L} - \hat{\beta}_k(\lambda, r) \right|$$

$$= |S| \sup_{0 \leq \lambda \leq \Lambda} \left| \frac{\mathbb{E}[\beta_k(\lambda, L)]}{L} - \hat{\beta}_k(\lambda, r) \right| .$$
Note that for fixed $L$, as $n \to \infty$,
\[ \sum_i \hat{\beta}_k(f_i^*, r) \frac{L}{n} \to \int_S \hat{\beta}_k(f(x), r)dx. \]

Therefore
\[
\limsup_{n \to \infty} \left| \frac{1}{n} \sum_i \mathbb{E}[\beta_k(W_i, r)] - \int_S \hat{\beta}_k(f(x), r)dx \right| \leq |S| \sup_{0 \leq \lambda \leq \Lambda} \left| \frac{\mathbb{E}[\beta_k(\lambda, L)]}{L} - \hat{\beta}_k(\lambda, r) \right|. \tag{3}
\]

Combining two estimates (2) and (3) and then let $L \to \infty$, we get the desired result. The proof is complete.

The result for binomial point processes will follow from Theorem 1.5 and the following result.

**Lemma 2.6.** As $n \to \infty$,
\[
\left| \frac{\mathbb{E}[\beta_k(C(P_n, r_n))]}{n} - \frac{\mathbb{E}[\beta_k(C(X_n, r_n))]}{n} \right| \to 0.
\]

**Proof.** By Lemma 2.2 again, we have,
\[
\left| \beta_k(C(P_n, r_n)) - \beta_k(C(X_n, r_n)) \right| \leq \sum_{j=k}^{k+1} \left| S_j(C(P_n, r_n)) - S_j(C(X_n, r_n)) \right|.
\]

The right hand side, divided by $n$, converges to 0 as a consequence of general results in [4, 5] applied to $S_j$. Here we will give an easy proof.

For any $m$, let
\[
S_j(m, n) = |S_j(C(X_m, r_n)) - S_j(C(X_n, r_n))|.
\]

Since the probability density function $f(x)$ is bounded, in the regime that $nr_n^d \to r^d$, the probability that $\{X_1 \in B_x(r_n)\}$ is bounded by
\[
\mathbb{P}(X_1 \in B_x(r_n)) \leq \frac{c}{n},
\]
for some constant $c$ which does not depend on $n$.

For $m > n \geq j$, since each $j$-simplices in $C(X_m, r_n) \setminus C(X_n, r_n)$ must contain at least one vertex in $\{X_{n+1}, \ldots, X_m\}$, we have
\[
\mathbb{E}[S_j(m, n)] \leq (m - n) \mathbb{E}[\#\{j\text{-simplices in } C(X_m, r_n) \text{ containing } X_m\}] \leq (m - n) \binom{m}{j} \mathbb{P}(X_1 \in B_{X_m}(r_n), \ldots, X_j \in B_{X_m}(r_n)) \leq (m - n) \frac{m!}{j!(m - j)!} \left( \frac{c}{n} \right)^j \leq c_1 (m - n) \left( \frac{m}{n} \right)^j.
\]

When $j \leq m < n$, we change the role of $m$ and $n$ to get
\[
\mathbb{E}[S_j(m, n)] \leq (n - m) \binom{n}{j} \left( \frac{c}{n} \right)^j \leq c_2 (n - m).
\]
Combining two estimates, we have
\[ \mathbb{E}[S_j(m,n)] \leq c_3|m-n| \left[ 1 + \left( \frac{m}{n} \right)^j \right]. \]

Therefore,
\[
\begin{align*}
\mathbb{E} \left[ |S_j(C(\mathcal{P}_n, r_n)) - S_j(C(X_n, r_n))| \right] & \leq c_3 \mathbb{E} \left[ |N_n - n| \left( 1 + \frac{(N_n)^j}{n^j} \right) \right] \\
& \leq c_3 \mathbb{E}[(N_n - n)^2]^{1/2} \mathbb{E} \left[ \left( 1 + \frac{(N_n)^j}{n^j} \right)^2 \right]^{1/2}.
\end{align*}
\]

Here in the last inequality, we have used the Cauchy–Schwarz inequality. Note that \( \mathbb{E}[(N_n)^j] \) is a polynomial in \( n \) of degree \( j \). Thus the second factor in the above estimate remains bounded as \( n \to \infty \). Note also that
\[ \mathbb{E}[(N_n - n)^2] = \text{Var}[N_n] = n. \]

Therefore,
\[
\frac{\mathbb{E} \left[ |S_j(C(\mathcal{P}_n, r_n)) - S_j(C(X_n, r_n))| \right]}{n} \leq \frac{c_4}{n^{1/2}} \to 0 \text{ as } n \to \infty.
\]

The theorem is proved. \( \square \)

### 3 Concluding remarks

Together with the law of large numbers in [7], we have the following result. Assume that
the support of \( f \) is compact and convex and that
\[
0 < \inf_{x \in \text{supp } f} f(x) \leq \sup_{x \in \text{supp } f} f(x) < \infty.
\]

Assume further that \( f \) is Riemann integrable. Then for \( 1 \leq k \leq d - 1 \),
\[
\frac{\beta_k(C(X_n, r_n))}{n} \to \int_{\mathbb{R}} \beta_k(f(x), r)dx \text{ a.s. as } n \to \infty.
\]

A result for the Vietoris-Rips complex also holds.

### References

[1] Omer Bobrowski and Matthew Kahle, *Topology of random geometric complexes: a survey*, Topology in Statistical Inference, the Proceedings of Symposia in Applied Mathematics. (to appear)

[2] Ronald Meester and Rahul Roy, *Continuum percolation*, Cambridge Tracts in Mathematics, vol. 119, Cambridge University Press, Cambridge, 1996.

[3] Mathew Penrose, *Random geometric graphs*, Oxford Studies in Probability, vol. 5, Oxford University Press, Oxford, 2003.

[4] Mathew D. Penrose, *Laws of large numbers in stochastic geometry with statistical applications*, Bernoulli 13 (2007), no. 4, 1124–1150.
[5] Mathew D. Penrose and J. E. Yukich, *Weak laws of large numbers in geometric probability*, Ann. Appl. Probab. 13 (2003), no. 1, 277–303.

[6] D. Yogeshwaran and Robert J. Adler, *On the topology of random complexes built over stationary point processes*, Ann. Appl. Probab. 25 (2015), no. 6, 3338–3380.

[7] D. Yogeshwaran, Eliran Subag, and Robert J. Adler, *Random geometric complexes in the thermodynamic regime*, Probab. Theory Relat. Fields (2015), 1–36.

Trinh Khanh Duy  
Institute of Mathematics for Industry  
Kyushu University  
Fukuoka 819-0395, Japan  
e-mail: trinh@imi.kyushu-u.ac.jp