D4-D0 BRANES ON THE QUINTIC

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Abstract

It is proposed that the quantum mechanics of $N$ D4-branes and $M$ D0-branes on the quintic is described by the dimensional reduction of a certain $U(N) \times U(M)$ quiver gauge theory, whose superpotential encodes the defining quintic polynomial. It is shown that the moduli space on the Higgs branch exactly reproduces the moduli space of degree $N$ hypersurfaces in the quintic endowed with the appropriate line bundle, and that the cohomology growth reproduces the D4-D0 black hole entropy.
1. Introduction

BPS black holes in IIA compactification on a Calabi-Yau threefold $X$ are constructed by wrapping D-branes around even-dimensional cycles. In general one expects the brane quantum mechanics, which should be the large $N$ dual of the black hole, to be a kind of dimensional reduction of an $N = 1$, $d = 4$ quiver gauge theory, with one node for each type of cycle [1-9]. On the other hand, the microstates of many such black holes have been identified with moduli space cohomology of high degree subvarieties in $X$ [10,11]. Hence one expects a direct connection between the moduli spaces of quiver gauge theories and of subvarieties in $X$. It is the purpose of this paper to study this problem and, more generally, the dual relation between BPS black holes and quiver gauge theories.

We focus herein on the canonical example of the $N$ D4-branes wrapping the basic hypersurface in the quintic, denoted $P_1$, and $M$ D0-branes on the quintic. $P_1$ turns out not to be a spin manifold and hence a wrapped D4-brane is required to have a gauge field valued in half-integral cohomology and an induced D2 charge [13-16]. This system is relatively simple but still rich enough to describe macroscopic black holes. We analyze the field content, superpotential and D-term constraints of the low-energy D-brane gauge theory, which should be enough to capture the topological features of the theory. We

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1 Extensive analyses of exact D-brane boundary states, not considered here, can be found in [2,12].
propose that it is (the reduction from \( d = 4 \) to \( d = 1 \) of) a \( U(N) \times U(M) \) quiver gauge theory with 4 chiral multiplets in the adjoint of \( U(N) \), corresponding to the 4 deformations of \( P_1 \), 3 chiral multiplets in the adjoint of \( U(M) \), corresponding to the 3 motions of a D0 in the quintic, and 2 bifundamentals, together with a certain superpotential connecting all 3 types of chiral multiplets.

For \( M = 0 \), (no D0-branes) one expects that the moduli space of the Higgs branch should match the moduli space of degree \( N \) hypersurfaces, denoted \( P_N \). At first sight this seems impossible, because the dimension of the latter grows like \( N^3 \), while the gauge theory contains only order \( N^2 \) fields. However it is important to realize that the \( P_N \) obtained by combining \( N P_1 \)'s must come with a particular supersymmetric \( U(1) \) worldvolume gauge field \( F \) as required by D0 and D2 charge conservation. We show that the existence of a suitable such \( F \) is equivalent to the existence of a suitable holomorphic divisor on \( P_N \). Requiring the existence of such a divisor reduces the dimension of the moduli space from a number which grows like \( N^3 \) to one that grows like \( N^2 \). Indeed we find a remarkable match between the moduli space of such \( P_N \)'s and the moduli space of the gauge theory Higgs branch. We further study the problem of combining hypersurfaces of different degrees and gauge fields and find perfect agreement with the appropriate gauge theory Higgs branch.

We then show that the coupling of \( M \) D0-branes to the associated \( U(M) \) gauge sector leads to a factor in the Higgs branch moduli space involving the symmetric product of \( M \) copies of \( P_N \). This factor has a large cohomology, leading to a quantum ground state degeneracy which reproduces the Bekenstein-Hawking area-entropy law \( S = 2\pi \sqrt{5N^3M} \).

Some interesting features emerge from our analysis which we expect to carry over to a generic Calabi-Yau. The quiver gauge quantum mechanics is expected to be dual, at least at the level of topological data, to string theory on the corresponding attractor geometry. The Kähler moduli of the attractor geometry are fixed in terms of the black hole charges—in this case \((N, M)\)-while the complex moduli can be continuously varied. This is mirrored in the gauge theory, where the superpotential depends only on the complex structure and can be continuously varied, while the charges \((N, M)\) determine the gauge symmetry and field content.

A natural question which arises is the relation of the quiver quantum mechanics discussed here and the superconformal quantum mechanics expected from \( AdS_2/CFT_1 \) – for example the one discussed in [17]. We don’t know the answer to this interesting question, though it may involve some kind of RG flow and integrating out of fields.
This paper is organized as follows. In section 2 we study the classical geometry of
degree $N$ hypersurfaces with gauge fields and curvatures inducing D0 and D2 charges,
and how several such cycles may be joined in a manner dictated by charge conservation.
In section 3 we show that this geometric structure is exactly reproduced by the Higgs
branches of appropriate quiver gauge theories. In section 4 we match the ground state
degeneracy of the quiver gauge theory to the black hole entropy. In section 5 we study the
geometry of joining a general set of D4-branes in a manner that is consistent with charge
conservation.

In the text we will consider both space-filling wrapped branes and those which are
pointlike in the noncompact space. While ultimately we are interested in the latter the two
are related by dimensional reduction and the former are often simpler and more familiar.
In either case we will refer to the brane by the real dimension of the cycle which it wraps.
We will also use symbols such as $F$, $J$, $P$ or $\Sigma_A$ to denote either a cycle or its dual form,
with the precise meaning hopefully clear from the context.

2. Geometry

2.1. Hypersurfaces in $X$

In this section we review some basic facts about single D4 branes wrapping a degree
$N$ hypersurface $P_N$ in the quintic $X$.

The Chern class of $X$ is

$$c(TX) = 1 + 10J^2 - 40J^3$$

with $J$ the hyperplane section and $\int_X J^3 = 5$. The Chern class of a degree $N$ divisor
$P = NJ$ is

$$c(TP) = 1 - NJ + (10 + N^2)J^2,$$

with $\int_P J^2 = 5N$. $P_N$ has Betti numbers

$$b_0 = 1, \quad b_1 = 0, \quad b_2^+ = \frac{5N^3 + 25N}{3} - 1, \quad b_2^- = \frac{10N^3 + 125N}{3} - 1$$

so that the Euler character is $\chi = 5N^3 + 50N$ and the signature is $\sigma = -(5N^3 + 100N)/3$.
The complex moduli space $\mathcal{M}(P)$ consists of the $\frac{1}{2}(b_2^+ - 1)$ complex deformations of $P_N$
in $X$. 
$P_N$ may also carry a two-form $U(1)$ field strength $F$. This and the curvature of $TP_N$ induce a D0 charge \[1\]

$$q_0 = - \int_{P_N} \left( \frac{1}{2} F^2 + \frac{1}{24} c(TP_N) \right) = k - \frac{5N^3 + 50N}{24},$$

where $k$ is the instanton number. There is also induced $D2$ charge

$$q_2 = \int_{P_N} F \wedge J. \quad (2.4)$$

Supersymmetry requires that $F$ take the form\[2\]

$$F = F^- + \frac{q_2}{5N} J, \quad (2.5)$$

with

$$F^- = - * F^- . \quad (2.7)$$

This is equivalent to saying $F^-$ is orthogonal to $J$ and is of type (1, 1). Furthermore, for $N$ even $F$ must be in integer cohomology, but for $N$ odd it must be shifted by $\frac{1}{2} J$, so that $q_2$ and $q_0$ are always nonzero and fractional \[13,16\]. Note that (2.7) depends on the complex structure of $P_N$ (as induced from $X$). Hence in general a solution of (2.7) does not remain a solution under deformations of $P_N$ in $X$ and moduli are frozen when $F^-$ is nonzero.

Suppose two surfaces of degrees $N'$ and $N''$ and instanton numbers $k'$ and $k''$ join into a single one of degree $N' + N'' = N$. D0 charge conservation requires the final instanton number

$$k = k' + k'' + \frac{5N'^2 N'' + 5N' N''^2}{8}. \quad (2.8)$$

2.2. $N = 2$

Now we try to combine two degree one $P_1$ surfaces to get a single degree 2 surface $P_2$. The initial surfaces have $F = \frac{1}{7} J$ so that the total charges are

$$2q_2(1) = 5, \quad 2q_0(1) = -\frac{35}{6}. \quad (2.9)$$

\[2\] For large $F$ these equations must be deformed \[18\]. We expect this will not qualitatively affect our discussion.
This implies that the degree 2 surface must have

\[ k = -\frac{1}{2} \int_{P_2} F^2 = 0, \]  

(2.10)

\[ \int_{P_2} F \wedge J = 5, \]  

(2.11)

with \( F = F^- + \frac{1}{2} J \) an integral form. Since \( F \) is also type \((1, 1)\) it can be represented by a holomorphic divisor, in which case (2.10) and (2.11) represent intersection numbers (we also use the symbol \( F \) to denote this divisor). At generic points there are no divisors other than \( J \), so we cannot make a generic degree 2 surface by combining two degree 1 surfaces. However consider surfaces described by quadratic equations of the form

\[ \det \Phi = 0, \]  

(2.12)

where \( \Phi = \Phi_A z^A \) is a \( 2 \times 2 \) matrix of linear polynomials in \( z^A, A = 1, \ldots, 5 \). There is a 13-dimensional moduli space of such surfaces. They admit a divisor \( F \) described by the non-transverse equations

\[ \Phi_{11} = 0, \quad \Phi_{12} = 0. \]  

(2.13)

To compute (2.11) we note that \( J \) and (2.13) intersect at 5 points in the quintic. Since these 5 points automatically lie in the surface (2.12) this is also the intersection of \( F \) with \( J \). To compute the self intersection of \( F \), we note that (2.13) may be deformed to

\[ \Phi_{11} = \epsilon \Phi_{21}, \quad \Phi_{12} = \epsilon \Phi_{22}. \]  

(2.14)

This clearly has no intersection with the original curve, in agreement with (2.10).

This agrees with the number of Higgs branch moduli of a \( U(2) \) gauge theory with four chiral multiplets in the adjoint, one for each deformation of \( P_1 \) in \( X \). On the Higgs branch with \( U(2) \to U(1) \), three chiral multiplets are eaten by the Higgs mechanism. The remaining 16-3=13 neutral chiral multiplets will be found in section 3 to match the 13 moduli of the degree 2 surface with \( F \) described above.
2.3. \( N = 3 \)

Next we combine three degree 1 surfaces into a degree 3 surface \( P_3 \). The initial surfaces again have \( F = \frac{1}{2} J \) and the total charges are

\[
3q_2(1) = \frac{15}{2}, \quad 3q_0(1) = -\frac{35}{4}.
\] (2.15)

This implies that the degree 3 surface must have

\[
k = -\frac{1}{2} \int_{P_3} F^2 = \frac{25}{8},
\] (2.16)

\[
\int_{P_3} F \wedge J = \frac{15}{2},
\] (2.17)

with \( F = F^- + \frac{1}{2} J \). For \( N \) odd, \( F^- \) (rather than \( F \)) is a type \((1,1)\) integral form and can be represented by a divisor of \( P_3 \) on a subvariety of the moduli space. This subvariety is described by cubic equations of the form

\[
det \Phi = 0,
\] (2.18)

where \( \Phi \) is now a \( 3 \times 3 \) matrix of linear polynomials in \( z^A \). There is a 28-dimensional moduli space of such surfaces. They admit a divisor \( C \) described by the non-transverse equations

\[
C : \quad \tilde{\Phi}_{11} = 0, \quad \tilde{\Phi}_{12} = 0, \quad \tilde{\Phi}_{13} = 0,
\] (2.19)

where

\[
\tilde{\Phi}_{kk'} = \frac{1}{2} \epsilon_{ik'} \epsilon_{j'k} \Phi_{ii'} \Phi_{jj'}
\] (2.20)

is the minor associated to \( \Phi_{kk'} \). \( C \) intersects \( J \) at 15 points, as we show below in section 2.4. \( C \) may be continuously deformed to

\[
v^j \tilde{\Phi}_{jk} = 0
\] (2.21)

for any nondegenerate vector \( v^k \). To compute the self intersection of \( C \) take the deformed surface defined by \( v = (0,0,1) \). This intersects (2.19) at

\[
\Phi_{21} = 0, \quad \Phi_{22} = 0, \quad \Phi_{23} = 0,
\] (2.22)

which consists of 5 points in the quintic.
C is not anti-self-dual since it intersects J. An anti-self-dual form is defined by
\[ F^- = C - J. \] (2.23)
so that
\[ F = C - \frac{1}{2}J. \] (2.24)
It is readily checked that F obeys (2.16) and (2.17) as required.

This concurs with our expectation from a U(3) gauge theory with four chiral multiplets in the adjoint. On the Higgs branch with \( U(3) \to U(1) \), 8 chiral multiplets are eaten by the Higgs mechanism. The remaining \( 36-8=28 \) neutral chiral multiplets match the 28 moduli of the degree 3 surface with \( F \) described above.

### 2.4. General \( N \)

Consider the complex hypersurface in the quintic defined by
\[ P_N : \quad \det \Phi = 0 \] (2.25)
where \( \Phi \) is an \( N \times N \) matrix of linear combinations of the \( z^A \)'s. Since a complexified \( SU(N) \) transformation of \( \Phi \) leaves the hypersurface unchanged, such hypersurfaces have a moduli space of complex dimension \( 3N^2 + 1 \). This contrasts with general degree \( N \) hypersurfaces, which have a dimension \( \frac{5N^3 + 25N}{6} - 1 \). We note that the moduli space of hypersurfaces (2.25), unlike the more general case, does not depend on the defining polynomial for the quintic. These hypersurfaces are also special in that they admit a holomorphic curve \( C \) defined by the vanishing of the first (for example) row of minors of \( \Phi \)
\[ C : \quad \tilde{\Phi}_{1i} = 0, \quad 1 \leq i \leq N. \] (2.26)
We could consider curves defined by other linear combinations of rows of minors, but they are homologous to \( C \) since they can be continuously deformed to \( C \).\(^3\) Note that (2.26) is a set of non-transverse equations.

Let us compute the intersection number \( C \cdot J \). We can represent \( J \) by the hyperplane
\[ H(z) = \sum h_A z^A = 0. \] Setting the first row of minors of \( \Phi \) to zero is equivalent to the \( N-1 \)
\(^3\) We could also consider a column of minors, which gives a surface homologous to \( C \) plus a multiple of \( J \).
rows \((\Phi_{ik})_{1 \leq k \leq N}\) being linearly dependent for \(i = 2, \ldots, N\). Now it amounts to counting the number of solutions to the set of equations
\[
U(z^A) = 0, \quad H(z) = 0,
\sum_{2 \leq i \leq N} c_i \Phi_{ij}(z) = 0, \quad 1 \leq j \leq N,
\]
(2.27)
on \(\mathbb{P}^4 \times \mathbb{P}^{N-2}\), where \(c_i\)'s are the homogeneous coordinates on \(\mathbb{P}^{N-2}\), and \(U\) is the defining degree 5 polynomial for the quintic. Since the curve given in (2.26) is not a complete intersection, we have introduced a set of auxiliary variables \(c_n\) in \(\mathbb{P}^{N-2}\), and a set of equations which define the curve as a complete intersection. Now we can use standard techniques to compute the intersection \(C \cdot J\).

\[
\int_{\mathbb{P}^4 \times \mathbb{P}^{N-2}} (1 + x)(1 + 5x)(1 + x + y)^N = 5 \binom{N}{2}.
\]
(2.28)

Next we will compute the self-intersection \(C \cdot C\). This amounts to setting the first two rows of minors of \(\Phi\) to zero, which is equivalent to having the \(N - 2\) rows \((\Phi_{ik})_{1 \leq k \leq N}\) being linear dependent, \(i = 3, \ldots, N\). So we can describe the points in \(C \cdot C\) by the set of equations
\[
U(z^A) = 0,
\sum_{3 \leq i \leq N} c_i \Phi_{ij}(z) = 0, \quad 1 \leq j \leq N.
\]
(2.29)
The number of solutions is given by
\[
\int_{\mathbb{P}^4 \times \mathbb{P}^{N-3}} (1 + 5x)(1 + x + y)^N = 5 \binom{N}{3}.
\]
(2.30)

We shall identify the curve \(C\) with an integral \((1, 1)\) harmonic form on \(P_N\). We can construct an anti-self-dual form
\[
F^- = C - \frac{N - 1}{2} J
\]
(2.31)
and the flux
\[
F = \frac{J}{2} + F^-
\]
(2.32)

---

The calculation amounts to counting the zeros of a section (i.e. the Euler class) of the vector bundle \(V = i_1^* H_1 \oplus i_1^* H_1^{\otimes 5} \oplus (i_1^* H_1 \otimes i_2^* H_2)^{\oplus N}\), where \(H_1\) and \(H_2\) are the hyperplane bundles over \(\mathbb{P}^4\) and \(\mathbb{P}^{N-2}\) respectively, \(i_1\) and \(i_2\) are the natural projection maps from their product.
The total induced D2-brane charge on the D4-brane wrapped on $P_N$ is
\[ q_2 = F \cdot J = \frac{5}{2} N, \]  
(2.33)
and the total induced D0-brane charge is
\[ q_0 = -\int_{P_N} \frac{F^2}{2} + \frac{c_2(TP_N)}{24} \]
\[ = -\frac{35}{12} N \]  
(2.34)
These are precisely $N$ times the charges of a D4-brane wrapped on $P_1$ with $F = J/2$.

3. Gauge Theory

3.1. $N$ D4-branes

In this subsection we consider a stack of $N$ D4-branes all wrapped on the same hypersurface $P_1$, given by the linear equation $\phi_A z^A = 0$ in the quintic, with the minimal required flux $F = \frac{1}{2} J$ on each. This is described at low energies in the noncompact spacetime by the dimensional reduction (to one dimension) of a four-dimensional $\mathcal{N} = 1$ $U(N)$ gauge theory. In general this theory contains an infinite number of higher dimension operators and cannot be described exactly. However here we are interested only in the topological properties which are encoded in the $F$ and $D$ terms. On general grounds \cite{[7]} it is expected that the $F$ terms depend only on the complex structure while the $D$ terms depend only on the Kähler class.

In addition to a $U(N)$ gauge multiplet the theory contains 4 adjoint chiral matter fields, corresponding to the 4 complex deformations of the surface $P_1$ in the quintic. The latter live in the projective space $\mathbb{P}^4$, with projective coordinates $\Phi_A$, promoted to $N \times N$ matrices. These are modded out by a $GL(N, \mathbb{C})$ action

\[ \Phi_A \rightarrow g \Phi_A, \quad g \in GL(N, \mathbb{C}), \]  
(3.1)
which reduces the 5 $\Phi_A$ to 4 physical fields. It is easiest to restrict to an affine patch, on which $\Phi_1$ can be set to the identity matrix and the remaining $\Phi_A \ A = 2, \ldots, 5$ are independent $N \times N$ matrices. The moduli space of the Higgs branch is determined by the D-term constraint

\[ G^{A\bar{B}}[\Phi_A, (\Phi^1)^{\bar{B}}] = 0, \]  
(3.2)
which consists of $N^2 - 1$ real equations. The precise form of the moduli space metric $G^{AB}$ will not be needed. Modding out by the $SU(N)$ gauge action (the $U(1)$ acts trivially) leads to a $3N^2 + 1$ dimensional complex moduli space.
3.2. Adding one D0-brane

Now we add a single D0-brane. This gives a $U(N) \times U(1)$ quiver gauge theory, with bifundamental matter $\phi_{04}, \phi_{40}$ coming from 0-4 and 4-0 strings, and neutral matter $Z^A$ associated to the coordinates of the D0-brane in the CY. Again the $Z^A$'s are naturally projective coordinates, but we will work on the affine patch and set $Z^5 = 1$. There is a superpotential

$$W = \phi_{04} \Phi_A \phi_{40} Z^A + \Lambda U(Z^A)$$

(3.3)

which involves only the complex structure. Here we have introduced a chiral superfield Lagrange multiplier $\Lambda$ to impose the condition that the $Z^A$'s lie on the quintic defined by $U(Z^A) = 0$. The non-zero D2 brane charge $q_2 = 5N/2$ leads to a $U(1)$ Fayet-Illiopoulos term and associated D-term constraint.

$$\phi_{40}^\dagger \phi_{40} - \phi_{04} \phi_{04}^\dagger = 5N\zeta.$$  

(3.4)

The D-term constraint (3.2) is also deformed. (3.4) forces a 4-0 vev and breaks the $U(1) \times U(1)$ symmetry (associated to the D4 and D0 centers of mass) down to $U(1)$. Once these vevs are nonzero, generic values of the D0-brane coordinate $Z^A$ no longer minimize the potential. This corresponds to the fact that the presence of D2 charge on the D4 leads to an attractive force between the D4's and the D0-brane.

We must also ensure that the superpotential is stationarized. This requires that $\phi_{40}$ and $\phi_{04}$ are zero eigenvectors of $\Phi_A Z^A$. This is possible if and only if the latter has a zero eigenvalue, i.e.

$$\det[\Phi_A Z^A] = 0.$$  

(3.5)

(3.3) is obeyed. Hence we precisely recover the hypersurface equation derived earlier from geometry in (2.23), from the D and F flatness conditions of the quiver gauge theory!

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5 It is likely there is a more elegant formulation following the Landau-Ginzburg construction of Calabi-Yau sigma-models [13,20,21]. For now we take a more pedestrian approach.

6 This is discussed in a slightly different language in [10]. For a surface $p^A \Sigma_A$ with flux $F = f^A \Sigma_A$ on a more general Calabi-Yau with Kahler class $J^A \Sigma_A$, the right hand side would be proportional to $D_{ABC} p^A \tilde{f}^A f^A$. This involves only even classes, as befits a $D$-term.

7 For generic values of $Z^A$, $\det \Phi_A Z^A \neq 0$ and the first term in (3.3) gives masses to all the 0-4 and 4-0 strings. This corresponds to the fact that the if the D0 and D4 are not coincident these strings are stretched and massive. Hence (3.3) may also be viewed, along the lines of [24] as the locus of massless strings connected to a D0 probe.
3.3. M D0-branes

More generally we can consider having $M$ D0-branes and $N$ D4-branes wrapped on the cycle $J$. Then we have a $U(N) \times U(M)$ gauge theory with D0-brane coordinates $Z^A$ in the adjoint of $U(M)$, D4-brane moduli $\Phi_A$ in the adjoint of $U(N)$. The key terms (up to higher order commutators) in the superpotential are \[ W = \text{Tr} \left[ \phi_{04} \Phi_A \phi_{40} Z^A \right] + \text{Tr} \Lambda U(Z^A) + \text{Tr} \Omega_{ABC} Z^A[Z^B, Z^C], \] (3.6)

with $\Omega$ the holomorphic three form.\footnote{This can be written as the pullback from $P^4$ of $\epsilon_{ABCD} \frac{\partial z}{\partial U}$ for $A, B = 1, \ldots, 4$.} We are interested in the branch on which all the D0-branes coincide with the D4-brane world volume, corresponding to \[ \det_N [\Phi_A Z^A] = 0, \] (3.7)

where the determinant is only taken with respect to the $U(N)$ indices of $\Phi_A$, and with $\phi_{04} = 0$, and $\phi_{40}$ a zero eigenvector of $\Phi$.\footnote{There may be other branches — in fact some might be expected corresponding to D0 branes dissolving into $F^-$.

Note that despite the ordering ambiguities in (3.6), the condition (3.8) consistently solves the supersymmetry constraints since all $Z^A$'s commute with one another. Together with (3.7) and the D-term constraints, these define the Hilbert scheme of $M$ points on $P_N$, corresponding to the moduli space of $M$ pointlike instantons on the D4-brane world volume.

4. D4-D0 Black Hole Entropy

Given the preceding construction of the classical moduli space of the quiver gauge theory, the problem of counting the ground states to leading order and matching to the black hole entropy is essentially equivalent to previous analysis \cite{10,11,17}. Nevertheless we give a brief review for completeness. The large entropy for large $(M, N)$ arises from the D0’s. For fixed $P_N$, their BPS states are in one-to-one correspondence with the cohomology
classes of the symmetric product orbifold $\text{Sym}^M(P_N)$. These can be constructed from the basic cohomology classes of $P_N$ and their counterparts appearing in the $M$ twisted sectors of the orbifold. The cohomology classes can be organized as

$$\prod_{i=1}^{k} \alpha_{-n_i}^{A_i} |0\rangle.$$ (4.1)

The $n_i$ take values from 1 to $M$ (labeling the twisted sector) with the restriction that the sum equals $M$:

$$\sum_{i=1}^{k} n_i = M.$$ (4.2)

The index $A = 0, 1, \ldots, 5N^3 + 50N$ runs over the $\chi(P)$ cohomology classes with $A = 0$ corresponding to $H^0(P)$. The state

$$(\alpha_{-1}^0)^M |0\rangle$$ (4.3)

corresponds to $H^0(\text{Sym}_M(P_N))$. The counting of such states is equivalent to the counting of left-moving states of a $c_L = \chi(P_N)$ 2D CFT at level $M$. We accordingly find the asymptotic formula for the entropy

$$S = 2\pi \sqrt{5N^3 M}.$$ (4.4)

This agrees with the Bekenstein-Hawking area law for the corresponding extremal charge $(N, M)$ black hole.

5. D4-branes on Surfaces of Higher Degrees

In this section we consider the geometry of D4-branes wrapped on higher degree surfaces, with the proper fluxes turned on. As in section 2, when the D4-branes join together, charge conservation requires a certain integral $(1,1)$ flux that exists only on a special class of hypersurfaces. In the case when all the D4-branes are wrapped on the same cycle, we find a gauge theory description whose moduli space matches that of the above mentioned hypersurfaces with flux. For the more general case of D4-branes wrapped on different 4-cycles, we propose a geometric construction whose moduli space is expected to match that of an appropriate quiver gauge theory, although we have not determined this gauge theory in the present paper.
5.1. Joining like cycles

Let us put together \( N \) D4-branes wrapped on degree \( n \) surfaces. We propose that the resulting degree \( nN \) surface \( P_{N|n} \) is given by

\[
\det \Phi^{(n)}(z^A) = 0
\]

where \( \Phi^{(n)} \) is an \( N \times N \) matrix of degree \( n \) polynomials in \( z^A \). This geometry is described by the Higgs branch of an \( \mathcal{N} = 1 \) \( U(N) \) gauge theory with adjoint matter \( \Phi_{(A_1 \cdots A_n)} \).

The geometry of the D4-brane can be understood in terms of a probe D0-brane, which introduces a superpotential of the form

\[
W = \phi_0 \Phi_{(A_1 \cdots A_n)} \phi_4 z^{A_1} \cdots z^{A_n}
\]

Again, there are massless 0-4 open string modes when the D0-brane coincides with the D4-brane, giving rise to the condition (5.1).

As before, the divisor (5.1) has a nice holomorphic curve \( C \) given by the vanishing of a row of minors in the matrix. Again we should compute the intersection numbers \( C \cdot C \) and \( C \cdot J \). The computation proceeds exactly as above, resulting in the integrals

\[
C \cdot J = \int_{\mathbf{P}^4 \times \mathbf{P}^{N-2}} (1 + x)(1 + 5x)(1 + nx + y)^N = 5n^2 \binom{N}{2},
\]

and

\[
C \cdot C = \int_{\mathbf{P}^4 \times \mathbf{P}^{N-3}} (1 + 5x)(1 + nx + y)^N = 5n^3 \binom{N}{3}.
\]

The D4-branes wrapped on the degree \( n \) surfaces with a gauge field \( F = \kappa J \) (where \( \kappa \) is integral or half integral according to the parity of \( n \)) have the following charges

\[
q_2 = 5\kappa n, \quad q_0 = -\frac{5}{2} n\kappa^2 - \frac{5}{24} (n^3 + 10n).
\]

The D4-brane wrapped on the degree \( nN \) determinantal surface with gauge field

\[
F = C - \frac{n(N - 1)}{2} J + \kappa J
\]

has charges

\[
q_2 = 5n\kappa N, \quad q_0 = -\left[\frac{5}{2} n\kappa^2 + \frac{5}{24} (n^3 + 10n)\right] N.
\]

in perfect agreement.
5.2. Generic cycles

Now we would like to analyze the case of \( N \) D4-branes wrapping cycles of degree \( n_i \), \( i = 1, \ldots N \), and carrying flux \( F = \kappa_i J \). We propose that a bound state of these D-branes is again described by a cycle of the form

\[
\det \Phi(z) = 0 \quad (5.8)
\]

where now the degree of the matrix components is \( \deg \Phi_{ij} = \frac{1}{2}(n_i + n_j) + \kappa_i - \kappa_j \). This 4-cycle has again a nontrivial, integral, anti-self-dual form \( F^- \), and the composite D4-brane carries a flux \( F = F^- + \kappa J \), where \( \kappa = \sum_i \kappa_i n_i \). If \( \kappa_i \neq \kappa_j \) the component D4-branes would generically preserve different supersymmetries, so the existence of a supersymmetric bound state - that is, of a supersymmetric vacuum for the \( U(1)^N \) gauge theory - depends on the Kähler moduli of the Calabi-Yau and is encoded in the D-term constraints. A (piece of) convincing evidence for the correctness of this construction is the nontrivial agreement in the D0-brane charge as computed for the single D4-branes and for the bound state. It would be interesting to understand the degrees of \( \Phi_{ij} \) in terms of \( 4 - 4' \) open string modes.

As before, the holomorphic curve \( C_1 \) is defined by setting the first row of minors of \( \Phi_{ij} \) to zero. Similar curves \( C_i \), defined by setting the \( i^{th} \) row of minors to zero, differ from \( C_1 \) by a multiple of \( J \). The computation of the intersection numbers \( C_1 \cdot J \) and \( C_1 \cdot C_2 \) is slightly more subtle than before.

Let us first calculate \( C_1 \cdot J \). We need to count solutions of equations

\[
U(z) = 0, \quad H(z) = 0,
\]

\[
\sum_{2 \leq i \leq N} a_i \Phi_{ij}(z) = 0, \quad 1 \leq j \leq N \quad (5.9)
\]

with the identifications \( a_i \sim \lambda a_i \) and \( (z^A, a_i) \sim (\mu z^A, \mu^{-\kappa_i - \frac{1}{2} n_i} a_i) \). These identifications define a bundle \( W \) of \( \mathbb{P}^{N-2} \) fibered over \( \mathbb{P}^4 \), which can also be thought of as the projectivization of the vector bundle \( V = \bigoplus_{i=2}^N \mathcal{O}(-\kappa_i - \frac{1}{2} n_i) \). The cohomology ring has two generators, given by \( x = \pi^* H_{\mathbb{P}^4} \) and \( y = -c_1(S) \) where \( S \) is the universal subbundle of \( \pi^* V \), and \( \pi \) is the projection map of the bundle. There are cohomology ring relations \( x^5 = 0 \) and \( y^{N-1} + \sum_k c_k(V) y^{N-1-k} = 0 \). Using

\[
\int_W x^4 y^{N-2} = 1 \quad (5.10)
\]
we can derive the following relations

\[
\int_W x^3 y^{N-1} = -c_1(V),
\]
\[
\int_W x^2 y^N = -c_2(V) + c_1(V)^2, \tag{5.11}
\]
\[
\int_W x y^{N+1} = -c_3(V) + 2c_1(V)c_2(V) - c_1(V)^3.
\]

Now the problem of computing \( C_1 \cdot J \) amounts to counting the zeros of a section of the vector bundle \( L^5_x \oplus L_x \oplus \bigoplus_{i=1}^N (L_x^{\frac{1}{2}n_i-\kappa_i} \otimes L_y) \), where \( L_x \) and \( L_y \) are the line bundles over \( W \) dual to \( x \) and \( y \) respectively. The answer is given by the Euler class

\[
C_1 \cdot J = \int_W (1 + 5x)(1 + x) \prod_{i=1}^N (1 + (-\kappa_i + \frac{1}{2}n_i)x + y)
= 5 \left[ \sum_{2 \leq i < j \leq N} n_i n_j + \sum_{j=2}^N n_j \left( \kappa_j - \kappa_1 + \frac{n_1 + n_j}{2} \right) \right], \tag{5.12}
\]

where we have applied (5.11). Similarly we can compute \( C_1 \cdot C_2 \), using the vector bundle \( V' = \bigoplus_{i=3}^N \mathcal{O}(-\kappa_i - \frac{1}{2}n_i) \) over \( \mathbb{P}^4 \). The problem again reduces to computing the Euler class of \( L^5_x \oplus \bigoplus_{i=1}^N (L_x^{\frac{1}{2}n_i-\kappa_i} \otimes L_y) \), given by

\[
C_1 \cdot C_2 = \int_W (1 + 5x) \prod_{i=1}^N (1 + (-\kappa_i + \frac{1}{2}n_i)x + y)
= 5 \left[ \sum_{3 \leq i < j < k \leq N} n_i n_j n_k + \sum_{3 \leq i < j \leq N} n_i n_j \left( -\kappa_1 - \kappa_2 + \frac{n_1 + n_2}{2} \right) \right.
+ \sum_{3 \leq i < j \leq N} n_i n_j \left( \frac{n_i + n_j}{2} + \kappa_i + \kappa_j \right) + \sum_{j=3}^N n_j \left( \kappa_j + \frac{n_j}{2} \right) \left( -\kappa_1 - \kappa_2 + \frac{n_1 + n_2}{2} \right) \]
\[
+ \sum_{j=3}^N n_j (-\kappa_1 + \frac{n_1}{2})(-\kappa_2 + \frac{n_2}{2}) + \sum_{j=3}^N n_j \left( \kappa_j + \frac{n_j}{2} \right)^2 \right] \tag{5.13}
\]

where \( W' \) is the projectivization of \( V' \).

In fact, the homology classes of all the \( C_i \)'s are related to the same curve \( C \) by

\[
C_i = C - (\kappa_i + \frac{n_i}{2})J, \quad 1 \leq i \leq N. \tag{5.14}
\]
$C$ can be constructed as

$$C : \sum_i v_i(z) \bar{\Phi}_{ij}(z) = 0, \quad 1 \leq j \leq N,$$  \hspace{1cm} (5.15)

where $v_i(z)$ are homogeneous polynomials of degree $\frac{1}{2} n_i + \kappa_i$. In particular one can deform $C$ continuously so that only one of the $v_i$’s is nonzero, and hence prove (5.14). Indeed one can check that the intersection numbers (5.12) and (5.13) are consistent with the relation (5.14). $C$ has intersection numbers

$$C \cdot J = 5 \left[ \frac{1}{2} \left( \sum n_i \right)^2 + \sum n_i \kappa_i \right]$$

$$C \cdot C = 5 \left[ \frac{1}{6} \left( \sum n_i \right)^3 + \frac{1}{12} \sum n_i^3 + \sum n_i \sum n_j \kappa_j + \sum_i n_i \kappa_i^2 \right]$$  \hspace{1cm} (5.16)

We shall consider the flux

$$F = C - \frac{1}{2} \left( \sum n_i \right) J.$$  \hspace{1cm} (5.17)

With $F$ turned on, the D4-brane wrapped on the surface defined by (5.8) has induced D2 and D0-brane charges

$$q_2 = \sum n_i \kappa_i,$$

$$q_0 = -\frac{5}{24} \sum n_i^3 - \frac{5}{2} \sum n_i \kappa_i^2 - \frac{25}{12} \sum n_i$$  \hspace{1cm} (5.18)

which are precisely as expected from the $N$ D4-branes wrapped on degree $n_i$ cycles.

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