Generalized Lieb’s theorem for non-Hermitian $n$-partite lattices

A. M. Marques$^{1}$ and R. G. Dias$^{1}$

$^{1}$Department of Physics & i3N, University of Aveiro, 3810-193 Aveiro, Portugal

Hermitian bipartite models are characterized by the presence of chiral symmetry and by the Lieb’s theorem, which derives the number of zero-energy flat bands of the model from the imbalance of sites between its two sublattices. Here, we introduce a class of non-Hermitian models with an arbitrary number of sublattices connected in a non-reciprocal and cyclical way, and show that the number of zero-energy flat bands of these models can be found from a generalized version of the Lieb’s theorem, involving the imbalance between each sublattice and the sublattice of lower dimension. Furthermore, these models are also shown to obey a generalized chiral symmetry, of the type found in the context of certain clock or parafermionic systems. The main results are illustrated with a simple toy model.

Introduction. Lieb’s theorem [1], initially formulated to demonstrate that the ground state magnetization at half filling of repulsive Hubbard bipartite lattices is directly proportional to the sublattice (SL) imbalance [2–4], is now understood in a more broad sense. Concretely, it states that the number of zero-energy flat bands (FBs) of a crystalline and bipartite tight-binding (TB) model, of which Lieb-type lattices are a prime example [5–9], is given by the SL imbalance. In real-space, the global sublattice imbalance of any bipartite system (including non-crystalline ones) indicates the total number of zero-energy states present there. Here, we introduce a certain class of non-Hermitian models with $n \geq 2$ SLs, which we call $n$-partite systems, and show that the number of zero-energy FBs of these models is given by a generalized version of the Lieb’s theorem.

Bipartite models are also characterized by the presence of chiral symmetry, which pairs eigenvalues with symmetric energies. Some extensions of the usual chiral symmetry, which pairs eigenvalues with symmetric energies, are discussed in [10], where the entries not shown are zeros, the momentum value. Models belonging to the class introduced here, on the other hand, are shown to obey the same generalized chiral symmetry as the one found in the generalized quantum Ising chains known as Baxter’s clock models [11–13]. A simple 1D toy model is introduced for the purpose of illustrating both the generalized Lieb’s theorem and the generalized chiral symmetry.

$n$-partite models. Let us consider a non-Hermitian bulk Hamiltonian with $n$ sublattices (SLs) of arbitrary sizes, each coupled to its nearest-neighbor in a directed fashion,

$$ H(k) = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_{n-1} \\ h_n & & & & \\ & h_{n-1} & & & \\ & & \ddots & & \\ & & & h_2 & \\ & & & & h_1 \end{pmatrix}, \quad (1) $$

where the entries not shown are zeros, the momentum vector reads as $k = (k_1, k_2, \ldots, k_D)$, with $D$ the dimensionality of the system, and $h_j = h_j(k)$, with $j = 1, 2, \ldots, n$, is a rectangular matrix of size $d_j \times d_{j+1}$, with $j = n+1 \to j = 1$ from the periodic boundaries conditions. This Hamiltonian describes a periodic model composed of non-reciprocal hopping terms from sites in $SL_j$ to sites in $SL_{j-1}$. Upon raising the Hamiltonian in (1) to the $n^{th}$-power one arrives at a diagonal matrix of the form

$$ H^n(k) = \text{diag}(H_1, H_2, \ldots, H_n), \quad (2) $$

$$ H_j = h_j h_{j+1} \cdots h_{n-1} h_j \quad (3) $$

Each diagonal block $H_j$ is a $d_j \times d_j$ square matrix, and the set $\{H_j\}$ represents all cyclic permutations of the ordered product of all the original $h_j$ matrices. We note that there is a recent study in driven systems [19] where the authors, by considering Floquet operators with formal properties similar to those of (1)-(3), were able to construct high-root [20–24] Floquet topological insulators of any order.

We assume for convenience that the SLs are ordered in a way that obeys $d_1 \leq d_j \leq 1$, such that $H_1$ in (2) is the smallest block (or in the set of smallest blocks) of dimension $d_1 \times d_1$. It can then be shown that the energy spectrum of $H_1$ is shared by all other $H_j$, such that it is $n$-fold degenerate in $H^n(k)$. The Schrödinger equation for the $H_1$ block is written as

$$ H_1 |u^s(k)\rangle = E_{1,s}(k) |u^s(k)\rangle, \quad s = 1, 2, \ldots, d_1, \quad (4) $$

where $|u^s(k)\rangle$ is the eigenstate with momentum $k$ of band $s$, and only has weight on the $d_1$ components of the first sublattice. Applying $h_n$ on both sides of (4) and using the identity $h_j H_{j+1} = H_j h_j$, derived from [3],...
leads to
\[ H_n(h_n | u_n^1(k)\rangle) = E_{1,a}(k)(h_n | u_n^2(k)\rangle), \] (5)
which, after defining the \(d_n\)-dimensional eigenvector \( |u_n^1(k)\rangle := h_n | u_n^2(k)\rangle \), becomes
\[ H_n | u_n^s(k)\rangle = E_{1,a}(k) | u_n^s(k)\rangle, \quad s = 1, 2, \ldots, d_1, \] (6)

Since \(d_n \geq d_1\), only accounts for the \(d_1\) energy bands that are proven to be degenerate with the equivalent ones coming from the diagonalization of \(H_1\). There are, however, extra \(d_n - d_1\) bands coming from \(H_n\) which do not belong to the shared spectrum. From a sequential application of \(h_{n-1}, h_{n-2}, \ldots, h_2\) to both sides of (6), one can generalize this proof to show that
\[ H_j | u_n^s(k)\rangle = E_{1,a}(k) | u_n^s(k)\rangle, \quad s = 1, 2, \ldots, d_1, \] (7)
where the \(d_j\)-dimensional eigenvectors are defined as \( |u_n^s(k)\rangle := h_j | u_n^{s+1}(k)\rangle \), which only have weight on \(S_L\).

Generalized chiral symmetry. The Hamiltonian in (1) obeys a generalized chiral symmetry, \(C_n\), whose effect on \(H(k)\) can be compactly written as
\[ C_n : \Gamma_n H(k) \Gamma_n^{-1} = \omega_n^{-1} H(k), \quad \omega_n = e^{i \frac{2\pi}{n}}, \] (8)
\[ \Gamma_n = \text{diag}({\mathbb{1}}_{d_1}, \omega_n {\mathbb{1}}_{d_2}, \ldots, \omega_n^{n-2} {\mathbb{1}}_{d_{n-1}}, \omega_n^{n-1} {\mathbb{1}}_{d_n}), \] (9)
where \({\mathbb{1}}_{d_j}\) is the identity matrix of dimension \(d_j\), \(\Gamma_n \Gamma_n^{-1} = \Gamma_n = \Gamma_n^\dagger\), with \(d_H = \sum_{j=1}^n d_j\) the dimension of \(H(k)\), and \(\omega_n^0 = 1\). For a bipartite system, \(n = 2\), \(\omega_n^0\) reduces to the usual chiral symmetry \(\Gamma_2 \Gamma_2^{-1} = -H(k)\). Therefore, \(\Gamma_n\) is the operator defining the chiral symmetry of an \(n\)-partite system, defined by the presence of \(n\) sublattices.

The presence of \(C_n\) imposes a constraint on the complex energy spectrum of \(H(k)\). Let us consider a finite energy eigenstate of the system,
\[ H(k) | \psi_0(k)\rangle = E | \psi_0(k)\rangle, \] (10)
then, by iteratively applying \(C_n\) on both sides and using (8) at each iteration, one arrives at
\[ H(k) | \psi_l(k)\rangle = \omega_l^{n} E | \psi_l(k)\rangle, \] (11)
where \(|\psi_l(k)\rangle := \Gamma_l^n | \psi_0(k)\rangle, \quad l = 1, 2, \ldots, n - 1\), and \(\langle \psi_l(k)|\psi_j(k)\rangle = \delta_{lj}\). This tells us that if \(E\) is a finite eigenvalue of an eigenstate of the system, then all its rotated versions, given by the presence of \(C_n\)-symmetric Hamiltonian models [50, 52], and shown to pin the higher-order corner modes at zero energy. However, it has been recently demonstrated that the \(C_3\) symmetry of the model fails to protect the corner modes against certain perturbations that preserve it [31]. On the basis of the energetic constraints imposed by \(C_n\) symmetry on the specific non-Hermitian Hamiltonians of the form of (1), and encapsulated in (11), we argue that the Hamiltonian models obeying \(C_3 [16, 30, 32, 33]\) or \(C_4 [34, 55]\) symmetry studied so far fail to reveal the relevant consequences of the generalized chiral symmetry detailed here (a more expanded discussion can be found in Appendix A). That is because \(C_\pi\)-symmetric Hamiltonian models require \(n\) applications of the generalized chiral symmetry in order to recover the original Hamiltonian, whereas in our case the Hamiltonian is recovered, up to a global phase factor [see (8)], after each application of the symmetry transformation, i.e., acting with \(\Gamma_n\) on the Hamiltonian changes its chiral color. The class of models introduced here, namely non-Hermitian \(n\)-partite models with non-reciprocal hopping terms between adjacent SLs defined in a cyclic fashion, should be regarded as the first example of a fully \(C_n\)-symmetric class of TB models.

It should be noted, however, that the same kind of generalized chiral symmetry has already been addressed in a different context, namely that of generalized quantum Ising chains known as Baxter’s clock model [17, 39], where the “spin” or clock internal degree of freedom at each site can take any value \(\omega_j^j\), with \(j = 0, 1, \ldots, n - 1\) (this model can be reframed in a parafermionic language, as shown, e.g., in [18, 57]). A brief introduction to Baxter’s clock model is provided in Appendix B along with the analogies that can be drawn between this model and the one introduced in this paper.

It is convenient to introduce the phase commutator between matrices \(A\) and \(B\), which we define as
\[ [A, B]_\theta := AB - e^{-i\theta} BA, \quad \theta \in [0, 2\pi), \] (16)
reducing to the commutation relation for \([A, B]_0 = [A, B]\), and to the anti-commutation relation for \(\theta = \pi\). The compact expression for \(C_n\) in (9) can be restated, through (16), as a phase commutator of the form
\[ [\Gamma_n, H(k)]_{\phi_n} = 0, \] (17)
with \(\phi_3 \rightarrow \phi_3 \rightarrow \phi_3 \rightarrow \phi\), from where it can be seen that
\[ H_3 + H_0 + H_1 = 0, \] (15)
that is, and in more general terms, the \(n\) chiral colors of a given \(C_n\)-symmetric Hamiltonian sum to zero (or, alternatively, the \(n\) colors sum to white).

The operator of the generalized chiral symmetry was first introduced in the context of the tripartite Hermitian breathing kagome model [30], and shown to pin the higher-order corner modes at zero energy. However, it has been recently demonstrated that the \(C_3\) symmetry of the model fails to protect the corner modes against certain perturbations that preserve it [31]. On the basis of the energetic constraints imposed by \(C_n\) symmetry on the specific non-Hermitian Hamiltonians of the form of (1), and encapsulated in (11), we argue that the Hamiltonian models obeying \(C_3 [16, 30, 32, 33]\) or \(C_4 [34, 55]\) symmetry studied so far fail to reveal the relevant consequences of the generalized chiral symmetry detailed here (a more expanded discussion can be found in Appendix A). That is because \(C_\pi\)-symmetric Hamiltonian models require \(n\) applications of the generalized chiral symmetry in order to recover the original Hamiltonian, whereas in our case the Hamiltonian is recovered, up to a global phase factor [see (8)], after each application of the symmetry transformation, i.e., acting with \(\Gamma_n\) on the Hamiltonian changes its chiral color. The class of models introduced here, namely non-Hermitian \(n\)-partite models with non-reciprocal hopping terms between adjacent SLs defined in a cyclic fashion, should be regarded as the first example of a fully \(C_n\)-symmetric class of TB models.
commutation relation $[\Gamma_1, H(k)] = 0$, since $\Gamma_1 = \mathbb{1}$. In the context of Baxter’s clock model analyzed in Appendix B, the phase commutator in [17] can be viewed as the analog of the “$\omega$-commutator” [17, 18] of the generalized Clifford algebra [19] involving the local operators with which the Hamiltonian of this model is constructed.

![Diagram](image)

**Figure 1.** Unit cell of the toy model for $n = 3$. The hopping terms are non-reciprocal and follow the direction of the arrows. Open site 1 at the right belongs to the adjacent unit cell. The flow between sublattices is depicted at the top left.

**Generalized Lieb’s theorem.** The combination of the results above leads to another important result.

(i) On the one hand, from the discussion leading to (7), we found that the finite energy spectrum of the smallest diagonal block of $H^n(k)$, which we set as $H_1$, is $n$-fold degenerate. On the other hand, this translates in the original model $H(k)$, through (11), as an $n$-fold degeneracy of all absolute finite energy values, which also leads to an $n$-fold degeneracy of $E^n$. As a result, whenever $d_1 \leq d_{j>1}$, the extra bands of $H_j$, in relation to $H_1$, must be zero-energy FBs, otherwise their finite energies would have to be $n$-fold degenerate, that is, shared also by $H_1$, which is not the case.

(ii) Furthermore, if $H_1$ is itself bipartite, i.e., if there are $\#_{H_1}^{FB}$ zero-energy FBs in the spectrum of $H_1$ coming from sublattice imbalance within $SL_1$, then the same number of extra FBs appears in the other $n - 1$ diagonal blocks $H_{j>1}$, meaning that the degenerate block spectra given by (7) actually remains valid for zero-energy FBs, that is, when $E_{1,s}(k) = 0$. The same reasoning of (i) can be applied here to prove the negative is impossible. Let us suppose that we construct $\#_{H_1}^{FB}$ dispersive bands in each of the $H_{j>1}$ blocks, with global $(n-1)$-fold degeneracy for band $E_α(k)$, with $α = 1, 2, \ldots, \#_{H_1}^{FB}$ the band index. Then, due to the $C_n$-symmetry of the original Hamiltonian, the finite energies appear in groups of $n$ elements of the form $\{\sqrt{E_α(k)}, \omega^n \sqrt{E_α(k)}, \ldots, \omega^{n-1} \sqrt{E_α(k)}\}$. However, $E_α(k)$ was assumed to be $(n-1)$-fold degenerate, since it is absent from $H_1$, and therefore cannot originate the $n$ elements for each $k$ mentioned above for the original Hamiltonian and, as a consequence, the extra $(n-1)\#_{H_1}^{FB}$ bands of the spectrum are also zero-energy FBs, that is, degenerate with the $\#_{H_1}^{FB}$ FBs present in $H_1$.

Since the number of zero-energy FBs is the same for $H^n(k)$ and $H(k)$, the results of this section can be summarized in the following formula that generalizes Lieb’s theorem for a $C_n$-symmetric $n$-partite system,

$$\#_{FB} = \sum_{j=2}^{n} (d_j - d_1) + n\#_{H_1}^{FB}, \quad (18)$$

that is, the total number of zero-energy FBs is given by the sum of imbalances between each $SL_{j>1}$ and the smallest sublattice $SL_1$ [the first term on the right, coming from point (i) above], plus $n$ times the number of zero-energy FBs already present in the smallest $H_1$ block [the second term on the right, coming from point (ii) above]. It should be noted that the second term should be included already for non-Hermitian systems with $n = 2$, as we illustrate in Appendix C with an example, showing that the Lieb’s theorem can be generalized also for the (bipartite) lattices for which it was formulated. As a corollary, we can also infer that if $H_1$ has a real energy spectrum, all $H_{j>1}$ have real spectra, given that their extra bands must be zero-energy FBs, such that they are pseudo-Hermitian Hamiltonians [39] obeys $H_j^\dagger = \eta H_j \eta^{-1}$, with $\eta$ a positive definite unitary matrix [40] that reduces to the identity for Hermitian Hamiltonians.

One should be reminded, at this point, that $H(k)$ in (1) is non-Hermitian and therefore can be defective, that is, the number of linearly independent eigenstates (LIEs) of $H(k)$ can be lower than its dimensionality $d_H$, if $H(k)$ falls into exceptional points or lines of the parameter space [41]. Regarding the eigenstates, [18] should be interpreted as giving the maximum possible number of LIEs within the set of zero-energy FBs of $H(k)$. However, defective models can have less LIEs in this set than $\#_{FB}$, down to a minimum given by

$$\#_{\text{LIEs}}^{\min} = \sum_{j=2}^{n} \text{Max}(d_j - d_{j-1}, 0), \quad (19)$$

which we derive in Appendix D where an explicit example of a defective system is also provided.

**Toy model.** In order to illustrate the results above, we introduce the simple 1D $n$-partite model ($k \rightarrow k$), with a bulk Hamiltonian of the form of (4), whose entries are explicitly given by

$$h_1 = t(1 + e^{-ik})J_1 \times 2, \quad (20)$$

$$h_j = tJ_{j \times j+1}, \quad j = 2, 3, \ldots, n - 1, \quad (21)$$

$$h_n = t(1 + e^{ik})J_n \times 1, \quad (22)$$

where $J_{i \times j}$ is a matrix of ones of size $i \times j$, the lattice spacing was set to $a = 1$ here and everywhere below, and $t$ is the magnitude of the non-reciprocal hopping terms, set as the energy unit henceforth. The unit cell of this model for $n = 3$ is depicted in Fig. 1. When raised to the $n^{th}$-power, this Hamiltonian has the form of (2), with the
diagonal blocks reading as

\[ H_j = 2^{n_j!} \frac{\ell^n}{j!} \cos(k_j) J_{j\times j}, \]

with \( j = 1, 2, \ldots, n \), such that \( H_1 \) is a matrix of size \( j \times j \), that is, \( d_j = j \) is the number of sites in \( SL_j \). In particular, the smallest block is already diagonal and has the form \( H_1 = 2\ell^n (1 + \cos k) \), which models a simple uniform and Hermitian linear chain with hopping strength \( n!\ell^n \) and an overall \( 2n!\ell^n \) energy shift. The energy band characterizing the spectrum of \( H_1 \) is \( n \)-fold degenerate in \( H^n(k) \) through (7), since it is common to all \( H_j \) blocks.

The energy spectrum of the model defined through (20)-(22) is shown for different \( n \) in Figs. 2(a)-(c). The presence of the respective \( \mathcal{C}_n \) symmetry is apparent in all three cases, as the energy spectra are manifestly invariant under \( \phi_n \) rotations about the \( k \)-axis. At the same time, the degeneracy of the zero-energy FBs agrees with the generalized Lieb’s theorem expressed in (18). In Fig. 2(d), we show the normalized energy spectrum of \( H^n(k) \), whose diagonal blocks are given by (23). Notice that this spectrum is purely real since the smallest block \( H_1 \) has a real spectrum, and that the degeneracy of zero-energy FB reconfirms the generalized Lieb’s theorem, which can also be checked against the independent diagonalization of all \( H_{j>1} \) blocks and counting the total number of FBs each of them generates.

**Conclusions.** We introduced a class of non-Hermitian TB models characterized by the presence of \( n \geq 2 \) SLs coupled in a cyclic fashion through non-reciprocal couplings. When the Hamiltonian of these models is raised to the \( n \)-th power it becomes block diagonal, with the energy spectrum of the smallest block, corresponding to the \( H_1 \) block in (2) by construction, being a common feature of all blocks. The excess energy bands of the blocks of higher dimensionality, in relation to the smallest one, were proved to be zero-energy FBs. The same total number of these zero-energy FBs is also present at the original Hamiltonian, which enabled us to generalize Lieb’s theorem [1], originally only applicable to Hermitian bipartite systems, to account for the total number of these bands in the non-Hermitian \( n \)-partite models we considered.

At the same time, we showed how these models obey a generalized chiral symmetry \( \mathcal{C}_n \) of the type introduced in [30]. On the basis of the different action of this symmetry on our TB models and on those appearing in recent literature [16, 31, 32, 33], we argued that only the former can be properly characterized as \( \mathcal{C}_n \)-symmetric models, expanding their class beyond the generalized spin systems for which this symmetry was originally proposed [17, 18]. A toy model was introduced to illustrate the appearance of zero-energy FBs whose cardinality is in agreement with the generalized Lieb’s theorem, and to manifest the \( n \)-fold rotation symmetry of the complex energy spectrum (see Fig. 2), which is a direct consequence of the \( \mathcal{C}_n \) symmetry.

Since the \( n \)-th root of the energy spectrum of the \( H_1 \) block was shown to be \( n \)-fold degenerate, in absolute value, at the level of the original Hamiltonian, the development of techniques to control the exact form of \( H_1 \) can open up interesting perspectives. To name only one, if \( H_1 \) is dressed with topological features by appropriately designing the original model \( H(k) \), then the latter will inherit its topological characterization directly from the former. In other words, one can use this method to construct \( \mathcal{C}_n \)-symmetric \( n \)-root topological insulators, which are currently limited to \( 2^n \)-root systems [20, 22], therefore extending to non-driven systems the recent results obtained for Floquet insulators [19]. These results are being finalized and will be the subject of a forthcoming article [42].

Concerning the experimental realization of the non-Hermitian \( n \)-partite models studied here, the main challenge relates to the implementation of non-reciprocal hopping terms. In this regard, electrical circuits appear to be in a prominent position to realize these systems [43, 47], since unidirectional capacitance couplings can be designed with the use of impedance converters with current inversion.

Figure 2. Complex energy spectrum as a function of the momentum obtained from diagonalizing the Hamiltonian defined in (20)-(22) for (a) \( n = 3 \), (b) \( n = 4 \), and (c) \( n = 5 \). (d) Normalized energy spectrum as a function of the momentum of the model in (a)-(c) raised to the \( n \)-th-power, which is purely real. In all plots, \((j)\) indicates the \( j \)-fold degeneracy of the respective band, \( N = \sum_{i=1}^{n} i \) is the total number of bands, and only the zero-energy FB is degenerate in (a)-(c).
ACKNOWLEDGMENTS

This work was developed within the scope of the Portuguese Institute for Nanostructures, Nanomodelling and Nanofabrication (i3N) projects No. UIDB/50025/2020 and No. UIDP/50025/2020 and funded by FCT - Portuguese Foundation for Science and Technology through the project PTDC/FIS-MAC/29291/2017. AMM acknowledges financial support from the FCT through the work Contract No. CDL-CTTRI-147-ARH/2018.

Appendix A: Comments on the generalized chiral symmetry

We start by considering a Hamiltonian of the form

\[ H_0(k) = \begin{pmatrix} 0 & h_{12} & \omega_3 h_{13} \\ h_{12}^* & 0 & \omega_2^* h_{23} \\ \omega_3 h_{13}^* & \omega_2^* h_{23} & 0 \end{pmatrix}, \]

(A1)

where \( \omega_3 = e^{i \frac{2 \pi}{3}} \) and all \( h_{ij} = h_{ji}(k) \) are scalars. This model was introduced in [33] to model Z₃ clock parafermions in a breathing kagome lattice. This Hamiltonian can be decomposed as (the momentum dependence is omitted henceforth)

\[ H_0 = H_\bigcirc + H_\bigtriangleup, \]

(A2)

\[ H_\bigcirc = \begin{pmatrix} 0 & h_{12} & 0 \\ h_{12}^* & 0 & \omega_2^* h_{23} \\ 0 & \omega_2^* h_{23} & 0 \end{pmatrix}, \]

(A3)

\[ H_\bigtriangleup = \begin{pmatrix} 0 & 0 & \omega_3 h_{13} \\ 0 & 0 & \omega_3 h_{13}^* \\ \omega_3 h_{13}^* & \omega_3 h_{13} & 0 \end{pmatrix}. \]

(A4)

The generalized chiral symmetry \( \mathcal{C}_3 \) defined in [8] reads here as

\[ \Gamma_3 H_0 \Gamma_3^{-1} = \omega_3^{-1} H_\bigcirc + \omega_3 H_\bigtriangleup := H_1. \]

(A5)

The action of \( \Gamma_3 \) is therefore to produce two counter-propagating \( \phi_3 \) rotations on the Hamiltonian terms, one clockwise for \( H_\bigcirc \) and another counterclockwise for \( H_\bigtriangleup \). Only \( H_\bigcirc \) or \( H_\bigtriangleup \) independently possess the generalized chiral symmetry \( \mathcal{C}_3 \) in the precise sense of [8], while \( H_0 \) does not. The original \( H_0 \) is recovered after three consecutive \( \Gamma_3 \) operations \( (H_0 = \Gamma_3^3 H_0 \Gamma_3^{-3}) \). Thus, if we define

\[ H_2 := \Gamma_3 H_1 \Gamma_3^{-1} = \omega_3^{-2} H_\bigcirc + \omega_2^2 H_\bigtriangleup, \]

(A6)

it follows that \( H_0 = \Gamma_3 H_2 \Gamma_3^{-1} \) and

\[ H_0 + H_1 + H_2 = 0, \]

(A7)

which formally replicates [15], although \( H_0, H_1 \) and \( H_2 \) are not different chiral colors of the same Hamiltonian, that is, they do not relate to each other by multiples of \( \omega_3 \) as in [12]-[14]. From the cyclic property of the trace of a matrix product we have

\[ \text{Tr}(H_1) = \text{Tr}(\Gamma_3 H_0 \Gamma_3^{-1}) = \text{Tr}(H_0), \]

(A8)

and similarly \( \text{Tr}(H_2) = \text{Tr}(H_0) \). From applying the trace to both sides of (A7) we conclude that \( \text{Tr}(H_0) = 0 \), that is, the eigenvalues of \( H_0 \) sum to zero.

Let us consider a general 4 \( \times \) 4 Hamiltonian of the form

\[ H = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}, \]

(A9)

where \( \{a, b, \ldots, p\} \in \mathbb{C}^{16} \). We can decompose \( H \) as

\[ H = H_A + H_B + H_C + H_D + H_E, \]

(A10)

\[ H_A = \begin{pmatrix} a & f \\ k & p \end{pmatrix}, \]

\[ H_B = \begin{pmatrix} b & g \\ m & l \end{pmatrix}, \]

\[ H_C = \begin{pmatrix} e & d \\ j & o \end{pmatrix}, \]

\[ H_D = \begin{pmatrix} c & h \\ i & n \end{pmatrix}, \]

(A11)

where all entries not shown are zeros. The generalized chiral symmetry operator acts in this case as

\[ \Gamma_4 H_4^{-1} = H_A + \omega_4^{-1} H_B + \omega_4 H_C + \omega_4^{-2} H_D + \omega_4^2 H_E, \]

(A12)

which obeys

\[ \sum_{j=0}^{3} \Gamma_4^j H_4^{-j} = 4 H_A, \]

(A13)

with \( \Gamma_4^0 = \Gamma_4^4 = \mathbb{1} \). Note that \( \omega_4 = e^{i \frac{2 \pi}{4}} \) is imposed by both \( \Gamma_4 H_B \Gamma_4^{-1} = \omega_4^{-1} H_B \) and \( \Gamma_4 H_C \Gamma_4^{-1} = \omega_4 H_C \). Taking the trace on both sides of (A13) leads to

\[ \text{Tr}

(\sum_{j=0}^{3} \Gamma_4^j H_4^{-j}) = 4 \text{Tr}(H_A) = 4 \text{Tr}(H), \]

(A14)

which, when \( H_A \) is traceless, \( \text{Tr}(H_A) = a + f + k + p = 0 \), implies that the eigenvalues of \( H \) sum to zero [30] [34].
When \( H_A = O_{4 \times 4} \), where \( O_{i \times j} \) is the \( j \times j \) zero matrix, then \( \{A13\} \) further reduces to the generalized chiral symmetry proposed in \[51\] \[33\]. The decomposition in \( \{A10\} \) can be straightforwardly generalized to a Hamiltonian of any dimension such that, under the action of \( \Gamma_n \), the different components rotate by multiples of \( \omega_n \), as illustrated in \( \{A12\} \) for \( n = 3 \). Therefore, \( n \) successive applications of \( \Gamma_n \) will retrieve the original Hamiltonian, implying that all Hamiltonians with a trivial diagonal component obey

\[
\sum_{j=0}^{n-1} \Gamma_n^j H \Gamma_n^{-j} = 0. \quad \{A15\}
\]

However, only a small subset of these Hamiltonians, which includes at least the chiral colored ones with the form of \( \{4\} \) (and evidently also their transposed versions), can be said to enjoy \( \mathcal{C}_n \)-symmetry in the more stringent sense of \( \{5\} \) with \( \omega_n^{-1} \rightarrow \omega_n \) for the transpose version \( H(k) \rightarrow H^T(k) \), in light of which it can be described as an extension of chiral symmetry to \( n \)-partite lattices, since it reduces to the usual chiral symmetry for a bipartite (\( n = 2 \)) model.

### Appendix B: Analogy with Baxter’s clock model

In order to highlight the parallel than can be drawn between the non-Hermitian \( n \)-partite models we are considering and the generalized version of the Ising model, we will follow closely below the systematic analysis provided by Fendley \[18\], to which we refer the reader for further details. Baxter’s clock model \[17, 36, 48\] can be viewed as an extension of the 1D Ising chain where, instead of having “up” and “down” as the internal spin degree of freedom at each site, the “spin” or clock value at each site can take the value \( \omega_n^j = e^{\frac{2\pi ij}{n}} \), with \( j = 0, 1, \ldots, n-1 \). Its Hamiltonian for an \( L \) sites chain reads as

\[
H_{BC} = \sum_{l=1}^{L} t_{2l-1} \tau_l + \sum_{l=1}^{L-1} t_{2l} \sigma_l^+ \sigma_{l+1}, \quad \{B1\}
\]

where \( \{t_i\} \) is a set of \( 2L-1 \) arbitrary complex coefficients and \( \sigma_j = 1_1 \otimes \cdots \otimes 1_n \otimes \sigma \otimes 1_n \otimes \cdots \) and \( \tau_j = 1_n \otimes \cdots 1_n \otimes \tau \otimes 1_n \cdots \) are operators acting at site \( l = 1, 2, \ldots, L \) through the local operators \[50\]

\[
\sigma = \begin{pmatrix}
1 & \omega_n & \cdots & \omega_n^{n-1} \\
\omega_n^{-1} & \omega_n & \cdots & \omega_n^{n-2} \\
& \ddots & \ddots & \ddots \\
& & \omega_n^{-1} & \omega_n & \cdots & \omega_n^{n-2} \\
\end{pmatrix}, \quad \tau = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots \\
\end{pmatrix}, \quad \{B2\}
\]

where all entries not shown are zeros and the local basis at site \( l \) spans \( \{\omega_n^j\} \), with \( j = 0, 1, \ldots, n-1 \). Clearly, \( \sigma \) measures the clock value, \( \sigma |\omega_n^j\rangle = \omega_n^j |\omega_n^j\rangle \), and \( \tau \) is the shifting operator acting as \( \tau |\omega_n^j\rangle = |\omega_n^{j+1}\rangle \). The Ising model is recovered for \( n = 2 \), where \( \sigma \) and \( \tau \) reduce to the \( \sigma_2 \) and \( \sigma_2 \) Pauli matrices, respectively. For \( n > 2 \), the one-site term of \( H_{BC} \) in \( \{B1\} \) generalizes the spin flipping term, while the two-site term represents a generalized nearest-neighbor interaction. Baxter’s model also enjoys a generalized chiral symmetry defined as

\[
\mathcal{C}_n : \quad \Sigma_n H_{BC} \Sigma_n^{-1} = \omega_n H_{BC}, \quad \{B3\}
\]

\[
\Sigma_n = \prod_{j=1}^{L} \sigma_j \prod_{i=1}^{L} \tau_i^{-1}, \quad \{B4\}
\]

with \( \Sigma_n \) a unitary operator whose action cycles the chiral colors of \( H_{BC} \) \[51\]. The presence of \( \mathcal{C}_n \)-symmetry similarly imposes that the spectrum of \( H_{BC} \) be formed by sequences of the form \( \{E, \omega_n E, \omega_n^2 E, \cdots, \omega_n^{n-1} E\} \), each summing to zero [see discussion below \{11\}]. Notice that \( \{B3\} \) is equivalent to \( \{8\} \) if the direction of all non-reciprocal hopping terms is switched, such that \( H(k) \rightarrow H^T(k) \) in \( \{1\} \).

An even more direct analogy can be made with the “\( \omega \)-commutation” relation \[17\] between the local \( \sigma \) and \( \tau \) operators in \( \{B2\} \), which we write here in a slightly different fashion for comparison purposes,

\[
\sigma \tau^T \sigma^{-1} = \omega_n^{-1} \tau^T, \quad \{B5\}
\]

which can be rewritten as a phase commutator of the form of \( \{10\} \), as \( [\sigma, \tau^T]_{\phi_n} = 0 \). Notice that \( \tau^T \) has the same general form of \( H(k) \) in \( \{4\} \), where the ones are replaced by the \( h_j \) rectangular matrices of different sizes. Similarly, \( \sigma \) closely resembles \( \Gamma_n \) in \( \{4\} \), where each diagonal power of \( \omega_n^j \), with \( l = 0, 1, \ldots, n-1 \), is enlarged into a diagonal square block of dimension \( d_l+1 \), that is, \( \omega_n^j \rightarrow \omega_n^j \bigotimes d_l+1 \). In the interest of keeping up with this analogy, the bulk Hamiltonian of the non-Hermitian \( n \)-partite models with the form of \( \{1\} \) can be viewed as a generalized version of the shift operator of Baxter’s clock model, where each SL corresponds to a different clock value, and the dimension \( d_j \) of \( \text{SL}_j \), i.e., the number of sites at \( \text{SL}_j \), can be viewed as counting the internal degrees of freedom of each clock value, connected between adjacent SLs in a non-reciprocal cyclic fashion.

It should also be stressed that, when mapping \( \tau^T \rightarrow H(k) \), a new ingredient is added to the system that is at the heart of the generalized Lieb’s theorem of \[18\]. Namely, the finite scalar entries of \( \tau^T \) are converted into rectangular matrices \( h_j \) with different dimensionalities in general, such that diagonal square blocks \( H_j \) of different sizes \( d_j \) are obtained for \( H^n(k) \), whereas one trivially gets \( (\tau^T)^n = \Sigma_n \). From \[18\], a nontrivial number of zero-energy FBs is precisely the combined result of finite sublattice imbalances (first term) and/or the existence of zero-energy FBs in the \( H_1 \) block (second term), both of which vanish for \( (\tau^T)^n \). At this point the analogy stops, since the generalized Lieb’s theorem cannot be revealed by the mathematical structure of Baxter’s clock model, but can be derived from the properties of the non-Hermitian \( n \)-partite models we considered.
Appendix C: Comments on the Lieb’s theorem

Here, we show that the Lieb’s theorem, which states that the number of zero-energy FBs in a bipartite system is given by the sublattice imbalance, while correct for Hermitian systems, fails to account for the extra zero-energy FBs that appear in certain non-Hermitian lattices. We show below an example of such a model, further illustrating the validity of the generalized Lieb’s theorem expressed in (18) already at the bipartite (n = 2) level.

The model considered here, with the unit cell depicted in Fig. 3 at the left, can be viewed as a non-Hermitian 

\[ H^2(k, \delta) = t^2 \begin{pmatrix} O_{3 \times 3} & h(k, \delta) \\ h^T(-k, \delta) & O_{3 \times 3} \end{pmatrix}, \]

where \( H^2(k, \delta) \) models the diamond chain with the unit cell depicted at the blue SL\(_1\) at the right-hand side of Fig. 3, which is known to host a flat band with the energy of its diagonal term \[ E_{FB} = \delta^2 - 1 \], that is, \( E_{FB} = \delta^2 - 1 \). The full expression of the pseudo-Hermitian block \( H_2(k, \delta) \), modeling a chain of the form of the gray SL\(_2\) at the right-hand side of Fig. 3, is omitted here for simplicity. In the language of [21], it corresponds to a topologically featureless residual block with shared spectral properties with the relevant \( H_1(k, \delta) \) block.

In Figs. 3(a)-(c), we plot the complex energy spectrum of \( H(k, \delta) \) in (C1) for three decreasing values of \( \delta \). For all three cases there are two zero-energy FBs originating from sublattice imbalance, in accordance with the Lieb’s theorem. However, two extra FBs with symmetric energies, directly obtained by taking the square-root of the diagonal term in (C4), i.e., \( E_{FB, \pm} = \pm \sqrt{\delta^2 - 1} \), are present in the spectra and can be seen to coalesce with the other two FBs as \( \delta \to 1 \) (they evolve in the imaginary energy axis for \( |\delta| < 1 \). Therefore, in Fig. 3(c) the system displays two extra zero-energy FBs not accounted by the Lieb’s theorem. Their appearance comes from the fact that, for the squared Hamiltonian \( H^2(k, 1) \), whose energy spectrum is shown in Fig. 3(d), the smaller \( H_1(k, 1) \) block itself has a zero-energy FB \( \#H_1 = 1 \) due to the sublattice imbalance within SL\(_1\)), since its diagonal term vanishes at \( \delta = 1 \), which must be shared by the \( H_2(k, 1) \) block also due to the isospectral properties (up to the zero-energy FBs already accounted for by the sublattice imbalance) between the diagonal blocks, as discussed in the main text. For \( \delta = 1 \), the total number of zero-energy FBs obtained from the generalized Lieb’s theorem is

\[ \#FB = d_2 - d_1 + 2 \#H_1 = 5 - 3 + 2 = 4. \] (C5)

There is a simple reason why the term proportional to \( \#H_1 \) is absent from Lieb’s theorem. It relates to the fact that it applies to Hermitian systems, where no zero-energy FBs can be present in \( H_1 \), apart from the trivial case where decoupled sites are present within the unit cell, which can allways be chosen to belong to the larger sublattice (notice, e.g., that site \( 8 \) becomes decoupled for the left model of Fig. 3 when \( \delta = 1 \), at which point one can ascribe it to either sublattice). When finite Hermitian couplings between sites in SL\(_1\) and SL\(_2\) are considered for a bipartite system, the diagonal terms of the squared Hamiltonian are necessarily positive [21][22][50]. If \( H_1 \) is itself bipartite, then it has \( \#H_1 \) coming from its sublattice imbalance, which are replicated in \( H_2 \), but with a finite energy given by its diagonal term \( c \) [with, e.g., \( c = \delta^2 - 1 \) for the \( H_1 \) block in (C4)]. In the original model, this translates in the appearance of \( \#H_1 \)-fold degenerate FBs at \( E = \pm \sqrt{c} \). These FBs are pushed in pairs to zero energy as \( c \to 0 \). The only way this can be achieved is by adding negative contributions to the diagonal term \( c \) of the squared model which, in turn, requires the inclusion of non-Hermitian hopping terms in the original model, as we exemplified in (C1) by considering non-Hermitian Peierls phases at some couplings. This demonstrates, in short, that the Lieb’s theorem needs to be generalized, not only for the \( n \)-partite systems considered in the main text, with \( n > 2 \), but also already for non-Hermitian bipartite systems, where extra zero-energy FBs originate from those that may be present in the \( H_1 \) squared block.
Appendix D: Comments on defective Hamiltonians

The minimum number of LIEs within the set of zero-energy FBs follows from a simple argument. The adjacency graph of $H(k)$ is a directed graph with only outgoing links from $\text{SL}_j$ to $\text{SL}_{j-1}$, where $j = 1, \ldots, n$ and $j = 0 \rightarrow j = n$ from the periodicity. Assuming that the number of dispersive bands of $H^n(k)$ is given by the dimension of the smallest block $H_1$, that is, it is $nd_1$ [where $n$ reflects the $n$-fold degeneracy of $H^n(k)$], then the incoming hopping terms to each site $i_1$ in $\text{SL}_1$ determine a particular state $|\psi_i^{(2)}\rangle$ in $\text{SL}_2$. We can construct a basis for $\text{SL}_2$ applying a Gram-Schmidt orthonormalization to the states $\{|\psi_i^{(2)}\rangle\}$ (note this set spans a subspace of $\text{SL}_2$ of dimension $d_1$) and choosing an arbitrary set of orthonormal basis states (between themselves and to the set $\{|\psi_i^{(2)}\rangle\}$) that completes the basis of $\text{SL}_2$. If we draw the adjacency graph of $H(k)$ in this basis, the latter set of nodes will have no outgoing links and that will generate $d_2 - d_1$ zero-energy FBs with LIEs.

This argument can be extended to any Hamiltonian block between a pair of consecutive sublattices. Three situations can occur: (i) $d_j > d_{j-1}$; (ii) $d_j = d_{j-1}$; (iii) $d_j < d_{j-1}$. For the two latter cases, no nodes of $\text{SL}_j$ without outgoing links can be obtained with the procedure described above. When $d_j > d_{j-1}$, then the same reasoning will generate $d_j - d_{j-1}$ nodes in $\text{SL}_j$ without outgoing links. So the minimum number of LIEs in the set of zero-energy FBs is

$$\#_{\text{LIEs}}^{\text{min}} = \sum_{j=2}^{n} \max(d_j - d_{j-1}, 0).$$

(D1)

Basically, the argument above states that is possible to rotate the basis within each sublattice in such a way that the number of sites in the shortest section (smallest sublattice) in closed loops of the adjacency graph gives the number of dispersive bands and the number of endpoints of open paths gives the number of LIEs of the set of zero-energy FBs, while the total number of FBs in this set is always given by the generalized Lieb’s theorem in [18]. In the example of Fig. 5, while the model, through [18], has $\#_{\text{FB}} = 3$ zero-energy FBs, this set only counts $\#_{\text{LIEs}}^{\text{min}} = \max(d_2 - d_1, 0) + \max(d_3 - d_2, 0) = 2 + 0 = 2$.

(D2)

The defectiveness of the model comes in this case from the decoupled cluster within the unit cell involving sites 3 and 5, which yields two FBs but only one LIE, while the loop accounts for the three dispersive bands and decoupled site 4 for the other zero-energy FB. This defectiveness can be viewed as the skin effect that takes place for the decoupled cluster within each unit cell, which together form a set of decoupled non-reciprocal dimers in real-space.

[1] E. H. Lieb, Two theorems on the hubbard model, Phys. Rev. Lett. 62, 1201 (1989).

[2] S.-Q. Shen, Z.-M. Qiu, and G.-S. Tian, Ferrimagnetic long-range order of the hubbard model, Phys. Rev. Lett.
J. Gouveia and R. Dias, Magnetic phase diagram of the Hubbard model in the Lieb lattice, Journal of Magnetism and Magnetic Materials 382, 312 (2015).

Y. Zhang, Y. Zhang, H. Zhong, C. Li, Z. Zhang, Y. Zhang, and M. Belic, New edge-centered photonic square lattices with flat bands, Annals of Physics 382, 160 (2017).

J. Gouveia and T. D. Colson, Enhanced localization and protection of topological edge states due to geometric frustration, Phys. Rev. B 100, 045104 (2019).

R. G. Dias and A. M. Marques, One-dimensional topological insulators with non-centered inversion symmetry axis, Phys. Rev. B 100, 041104(R) (2019).

X. Ni, J. Yan, and F. Liu, Electronic structures of a diagonally striped lattice: Multiple (n−1)-fold degenerate flat bands, Phys. Rev. B 102, 235117 (2020).

T. Kawarabayashi, H. Aoki, and Y. Hatsugai, Lattice realization of the generalized chiral symmetry in two dimensions, Phys. Rev. B 94, 235307 (2016).

T. Kawarabayashi and Y. Hatsugai, Bulk-edge correspondence with generalized chiral symmetry, Phys. Rev. B 103, 205306 (2021).

A. M. Marques and R. G. Dias, One-dimensional topological insulators with noncentered inversion symmetry axis, Phys. Rev. B 100, 041104(R) (2019).

R. G. Dias and A. M. Marques, Long-range hopping and indexing assumption in one-dimensional topological insulators, Phys. Rev. B 105, 035102 (2022).

A. M. Marques and R. G. Dias, Analytical solution of open crystalline linear 1d tight-binding models, Journal of Physics A: Mathematical and Theoretical 53, 075303 (2020).

A. Anastasiadis, G. Styliearis, R. Chaumosi, G. Theocharis, and F. K. Diakonos, Bulk-edge correspondence in the trimer su-schrieffer-heeger model (2022), arXiv:2202.13789.

R. Baxter, A simple solvable zn hamiltonian, Physics Letters A 140, 155 (1989).

P. Fendley, Free parafermions, Journal of Physics A: Mathematical and Theoretical 47, 075001 (2014).

L. Zhou, R. W. Bomantara, and S. Wu, nth-root non-hermitian floquet topological insulators (2022), arXiv:2203.09838.

R. G. Dias and A. M. Marques, Matryoshka approach to sine-cosine topological models, Phys. Rev. B 103, 245112 (2021).

A. M. Marques, L. Madail, and R. G. Dias, One-dimensional 2n-root topological insulators and superconductors, Phys. Rev. B 103, 235425 (2021).

A. M. Marques and R. G. Dias, 2n-root weak, chern, and higher-order topological insulators, and 2n-root topological semimetals, Phys. Rev. B 104, 165410 (2021).

R. W. Bomantara, Square-root floquet topological phases and time crystals (2021), arXiv:2111.14327.

W. Deng, T. Chen, and X. Zhang, Nth power root topological phases in hermitian and non-hermitian systems (2022), arXiv:2204.05714 [cond-mat.mes-hall].

C. Yin, H. Jiang, L. Li, R. Liu, and S. Chen, Geometrical meaning of winding number and its characterization of topological phases in one-dimensional chiral non-hermitian systems, Phys. Rev. A 97, 052115 (2018).

S. Yao and Z. Wang, Edge states and topological invariants of non-hermitian systems, Phys. Rev. Lett. 121, 086803 (2018).

K. Yokomizo and S. Murakami, Non-bloch band theory of non-hermitian systems, Phys. Rev. Lett. 123, 066404 (2019).

K. Lee, R. Melendez, A. Pal, and H. J. Changlani, Exact three-colored quantum scars from geometric frustration, Phys. Rev. B 101, 214111 (2020).

E. Chertkov and B. K. Clark, Motif magnetism and quantum many-body scars, Phys. Rev. B 104, 104410 (2021).

X. Ni, M. Weiner, A. Alù, and A. B. Khanikaev, Observation of higher-order topological acoustic states protected by generalized chiral symmetry, Nature Materials 18, 113 (2019).

G. van Miert and C. Ortix, On the topological immunity of corner states in two-dimensional crystalline insulators, npj Quantum Materials 5, 63 (2020).

S. N. Kempkes, M. R. Slot, J. J. van den Broeke, P. Capiod, W. A. Benalcazar, D. Vanmaekelbergh, D. Bercioux, I. Swart, and C. Morais Smith, Robust zero-energy modes in an electronic higher-order topological insulator, Nature Materials 18, 1292 (2019).

M. Ezawa, Topological parafermion corner states in clock-symmetric non-hermitian second-order topological insulator (2021), arXiv:2103.10645.

Z.-X. Li, Y. Cao, X. R. Wang, and P. Yan, Second-order topological solitonic insulator in a breathing square lattice of magnetic vortices, Phys. Rev. B 101, 184404 (2020).

Y.-Z. Li, Z.-F. Liu, X.-W. Xu, Q.-P. Wu, X.-B. Xiao, M.-R. Liu, L.-L. Chang, and R.-L. Zhang, New Journal of Physics 23, 043010 (2021).

R. J. Baxter, Superintegrable chiral pots model: Thermodynamic properties, an "inverse" model, and a simple associated hamiltonian, Journal of Statistical Physics 57, 1 (1989).

J. Alicea and P. Fendley, Topological phases with parafermions: Theory and blueprints, Annual Review of Condensed Matter Physics 7, 119 (2016). https://doi.org/10.1146/annurev-conmatphys-031115-011336.

When #_{FB}^{n} = 2, e.g., one could believe it would be possible to construct, from the 2n−2 remaining bands from the other H_{j>1} blocks, an n-fold degenerate finite energy band plus an (n−2)-fold degenerate zero-energy FB. However, the eigenstates of a finite energy band cannot vanish at any SL, which includes SL_{1} (it can be easily checked that, if there are nodes at all sites of one SL, then, through the TB equations, these nodes propagate sequentially to all sublattices). As such, if the eigenstates of this finite energy band have finite weight on some sites of SL_{1}, then it would have to be present in H_{1} also, which is assumed not to be the case. Therefore, all 2n−2 extra
bands are also zero-energy FBs.

[39] A. Mostafazadeh, Pseudo-hermiticity versus pt symmetry: The necessary condition for the reality of the spectrum of a non-hermitian hamiltonian, *Journal of Mathematical Physics* **43**, 205 (2002), https://doi.org/10.1063/1.1418246.

[40] X. Zhang, K. Xu, C. Liu, X. Song, B. Hou, R. Yu, H. Zhang, D. Li, and J. Li, Gauge-dependent topology in non-reciprocal hopping systems with pseudo-hermitian symmetry, *Communications Physics* **4**, 166 (2021).

[41] E. J. Bergholtz, J. C. Budich, and F. K. Kunst, Exceptional topology of non-hermitian systems, *Rev. Mod. Phys.* **93**, 015005 (2021).

[42] A. M. Marques and R. G. Dias, to be submitted.

[43] T. Hofmann, T. Helbig, C. H. Lee, M. Greiter, and R. Thomale, Chiral voltage propagation and calibration in a topolectrical chern circuit, *Phys. Rev. Lett.* **122**, 247702 (2019).

[44] S. Liu, R. Shao, S. Ma, L. Zhang, O. You, H. Wu, Y. J. Xiang, T. J. Cui, and S. Zhang, Non-hermitian skin effect in a non-hermitian electrical circuit, *Research* **2021**, 5608038 (2021).

[45] D. Zou, T. Chen, W. He, J. Bao, C. H. Lee, H. Sun, and X. Zhang, Observation of hybrid higher-order skin-topological effect in non-hermitian topolectrical circuits, *Nature Communications* **12**, 7201 (2021).

[46] R.-L. Zhang, Q.-P. Wu, M.-R. Liu, X.-B. Xiao, and Z.-F. Liu, Complex-real transformation of eigenenergies and topological edge states in square-root non-hermitian topolectrical circuits, *Annalen der Physik* , 2100497 (2022).

[47] Q.-B. Zeng and R. Lü, Evolution of spectral topology in one-dimensional long-range nonreciprocal lattices, *Phys. Rev. A* **105**, 042211 (2022).

[48] P. Fendley, *Journal of Statistical Mechanics: Theory and Experiment* **2012**, P11020 (2012).

[49] G. Albertini, B. M. McCoy, and J. H. Perk, Commensurate-incommensurate transition in the ground state of the superintegrable chiral Potts model, *Physics Letters A* **135**, 159 (1989).

[50] L. Mittag and M. J. Stephen, Dual transformations in many-component ising models, *Journal of Mathematical Physics* **12**, 441 (1971), https://doi.org/10.1063/1.1665606.

[51] This symmetry was labeled as a generalized charge conjugation symmetry (a.k.a. particle-hole symmetry) in [18]. However, charge conjugation is defined through an anti-unitary operator, whereas $\Sigma$ is unitary and therefore is rather the operator describing a generalized chiral symmetry.

[52] G. Pelegrí, A. M. Marques, R. G. Dias, A. J. Daley, V. Ahufnger, and J. Mompart, Topological edge states with ultracold atoms carrying orbital angular momentum in a diamond chain, *Phys. Rev. A* **99**, 023612 (2019).

[53] G. Pelegrí, A. M. Marques, R. G. Dias, A. J. Daley, J. Mompart, and V. Ahufinger, Topological edge states and aharonov-bohm caging with ultracold atoms carrying orbital angular momentum, *Phys. Rev. A* **99**, 023613 (2019).

[54] M. Kremer, I. Petrides, E. Meyer, M. Heinrich, O. Zilberberg, and A. Szameit, A square-root topological insulator with non-quantized indices realized with photonic aharonov-bohm cages, *Nature Communications* **11**, 907 (2020).

[55] G. Pelegrí, A. M. Marques, V. Ahufinger, J. Mompart, and R. G. Dias, Interaction-induced topological properties of two bosons in flat-band systems, *Phys. Rev. Research* **2**, 033267 (2020).

[56] M. Ezawa, Systematic construction of square-root topological insulators and superconductors, *Phys. Rev. Research* **2**, 033397 (2020).