Semi-Cartesian Squares
And The Snake Lemma

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À Claude Chevalley, dont le cours de DEA (1965-1966) m’a fait découvrir les carrés semi-cartésiens.

Résumé: On démontre le lemme du serpent de façon purement catégorique (§ 3). Aucun point n’apparaîtra, ni « points » au sens de Grothendieck ni pseudo-éléments (Guglielmetti & Zaganidis [2009]). En revanche, un fort usage sera fait des carrés semi-cartésiens (§ 2) introduits par Chevalley. Le paragraphe 1 est dévolu à quelques résultats de base sur les catégories abéliennes utilisés par la suite.

Abstract: The snake lemma is proved entirely within category theory (§ 3) without the help of “points with value in...” à la Grothendieck nor pseudo-elements (Guglielmetti & Zaganidis [2009]). Instead, we use consistently semi-cartesian squares (§ 2), promoted by Chevalley. Section 1 is devoted to a few basic results on abelian categories, for further use.

This paper is mainly intended to promote the semi-cartesian squares, introduced by Chevalley in a course given at the IHP, and is an example of their flexibility. The first two sections are extracted from this course. The third is a purely categorical proof of the snake lemma.

Categories are supposed to be known: objects and arrows between objects. Arrows are composed associatively and each object X has an identity arrow denoted $1_X$. The dual or opposite category has same objects but arrows are reversed. The class of arrows from X to Y is denoted by $\text{Hom}_C(X,Y)$. In a small category, arrows form a set (objects also, why?). An initial (final) object is an object with a unique arrow to (from) every object. By definition, monomorphisms are arrows that are simplifiable from the left ($mu = mv \Rightarrow u = v$) and epimorphisms arrows simplifiable from the right ($up = vp \Rightarrow u = v$). Categories will be denoted by bold upper case letters.

A functor is a mapping between categories $F : C \to D$ compatible with the identities and composition. Given two functors $F$ and $G$ from $C$ to $D$, a natural (or functorial) morphism $\phi : F \to G$ is a family of arrows of $D$ indexed by the objects of $C$, such that the squares of figure 1 are commutative for all arrows $X \to Y$ of $C$. If $C$ is a small category, functors from $C$ to $D$ are the objects of a category $\text{F}(C, D)$ the arrows of which are the natural morphisms.

Figure 1.

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1. Summary of abelian category

Given a small category I of indices, a functor \( A : I \to C \) can be seen as a commutative diagram of type I in C. With every object \( X \) of C is associated a constant diagram \((K_X, i) = X\) and \((K_X, j_1) = 1_X\). A projective (inductive) limit of a functor \( A : I \to C \) is a right (left) adjoint of functor \( K : X \to K_X \) i.e. \( \text{Hom}_{I,F(C)}(K_X, A) \cong \text{Hom}_{C}(X, \text{lim proj} A) \) (resp. \( \text{Hom}(A,K_X) \cong \text{Hom}(\text{lim ind} A, X) \)).

\[ \text{Hom}_{I,F(C)}(K_X, A) \cong \text{Hom}_{C}(X, \text{lim proj} A) \]

\[ \text{Lim}(A) \cong \text{Hom}(\text{lim ind} A, X) \]

Examples: 1) \( I = \emptyset \) the projective (inductive) limit of the empty set is the final (initial) object.
2) \( I = \{1,2\} \) the projective limit of \( A_1, A_2 \) is the product \( A_1 \times A_2 \) and the inductive limit is the sum \( A_1 + A_2 \).
3) \( I = \{2\} \) a constant diagram \((K_X, A) \cong \text{Hom}_{C}(X, \text{lim proj} A) \) is the kernel of \( q \) if \( A_u \cong A \) for every \( u \) of \( I \).
4) \( I = \{1 \to 0 \to 2\} \) the projective limit of \( A_1 \to A_0 \leftarrow A_2 \) is the cartesian square or fiber product built on these arrows (left square beneath). The inductive limit is got by reversing the arrows: it is a cocartesian square or amalgamated sum (right square beneath).

\[ \begin{array}{ccc}
P & \longrightarrow & A_2 \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & A_0 \\
\end{array} \]

\[ \begin{array}{ccc}
A_0 & \longrightarrow & A_1 \\
\downarrow & & \downarrow \\
A_2 & \longrightarrow & S \\
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Definition 1. A category is abelian if
1) it contains a null objet (i.e. initial and final) denoted by \( 0 \);
2) it accepts finite projective and inductive limits;
3) every monomorphism is a kernel and every epimorphism is a cokernel.

These axioms are due to P. Freyd [1964]. They are preserved by duality.

The null arrow \( A \to B \) is the composed arrow \( A \to 0 \to B \); the kernel of a single arrow \( u : A_1 \to A_2 \) is the kernel of \((u,0)\). It is easy to show that every kernel is a monomorphism and that every cokernel is an epimorphism. Condition (3) shows that these two notions coincide and more precisely:

Lemma 1. 1) If \( n \) is a kernel of an epimorphism \( q \), then \( q \) is a cokernel of \( n \). 2) If \( q \) is a cokernel of a monomorphism \( n \), then \( n \) is a kernel of \( q \).

Case 2 is dual of case 1. For case 1, \( q \) is a cokernel of an arrow \( f : qf = 0 \) and there exists therefore a unique arrow \( g \) such that \( f = ng \) because \( n \) is a kernel of \( q \). Suppose an arrow \( u \) such that \( a = 0 \). Then \( ung = uf = 0 \) and since \( q \) is a cokernel of \( f \), there exists a unique arrow \( v \) such that \( u = vq \), QED.

One can deduce the following decomposition of the arrows.

Proposition 1. Every arrow can be decomposed into \( f = mq \) where \( m \) is a monomorphism and \( q \) an epimorphism. This decomposition is unique up to a unique isomorphism.

Soit \( m \) the kernel d'une arrow cokernel \( p \) of \( f \). Then \( pf = 0 \iff (\exists! q) f = mq \). To show that \( q \) is an epimorphism, let us remark:

a) \( f \) epimorphism \( \iff p = 0 \iff m \) invertible (because \( pm = 0 \iff pnmr^\top = p = 0 \))

b) Decompose \( q \) like \( q: n \) is a kernel of a cokernel of \( q \). Considering a), it is enough to show that \( n \) is invertible. Now let \( r \) stisfying \( rnm = 0 \). Then \( rmns = rrf = 0 \). Since \( p \) is a cokernel of \( f \), \((\exists ! l) r = tp \). Hence \( p \) is a cokernel of \( mn \) and from lemma 1, \( mn \) is a kernel of \( p \). But \( m \) is also a kernel of \( p \), therefore \( n \) is invertible.
c) Let \( f = m'q' \) be an other decomposition. Since \( p \) is a cokernel of \( m'q' \) and \( q' \) is an epimorphism, \( p \) is a cokernel of \( m' \). By lemma 1, \( m' \) is a kernel of \( p \). Since \( m \) is another one, they are isomorphic, \( \text{QED} \).

**Corollary.** An arrow which is a monomorphism and an epimorphism is an isomorphism.

The decomposition of \( f \) into an epimorphism followed by a monomorphism is unique up to isomorphisms. But \( f \) has two such decompositions: \( 1f = f1 \). Therefore it is invertible, \( \text{QED} \).

**Proposition 2.** If \( q \) is an epimorphism, then \( mq \) and \( m \) have same cokernel. The converse is true if \( m \) is a monomorphism. If \( m \) is a monomorphism, then \( mq \) and \( m \) have same kernel. The converse is true if \( m \) is an epimorphism.

If \( q \) is an epimorphism, every arrow \( t \) satisfies \( tm = 0 \iff tmq = 0 \). Hence, \( m \) and \( mq \) have same cokernel. Conversely, decompose \( q \) into an epimorphism \( e \) and a monomorphism \( n \): The direct part shows that \( mne \) and \( mn \) have same cokernel. The hypothesis becomes: monomorphisms \( m \) and \( mn \) have same cokernel. They are therefore kernel of the same arrow; they are isomorphic and \( n \) is invertible: \( q \) is an epimorphism.

The second assertion is the dual of the first, \( \text{QED} \).

**Proposition 3.** If \( ba \) is a kernel of \( e \) and \( b \) is a monomorphism, then \( a \) is a kernel of \( cb \).

Let \( t \) be an arrow with \( cbt = 0 \). Since \( ba \) is a kernel of \( e \), there exists a unique arrow \( s \) such that \( bt = bas \). Since \( b \) is a monomorphism, \( t = as \), \( \text{QED} \).

The following notion is a well-known generalisation of the notion of kernel and cokernel.

**Definition 2.** Let two successive morphisms \( f = mq \) and \( g = np \), decomposed into epimorphisms followed by monomorphisms. The sequence \((f, g)\) is exact when \( m \) is a kernel of \( p \).

Equivalently (prop. 2), one can require that \( m \) be a kernel of \( g \), or that \( p \) be a cokernel of \( m \), or that \( p \) be a cokernel of \( f \).

Finally, recall that in an abelian category, there exists an isomorphism from the sum to the product and that the insertions \( i : A \to A+B \) and \( j : B \to A+B \), and the projections \( p : A\times B \to A \) and \( q : A\times B \to B \) satisfy

\[
pi = 1, \quad qi = 0, \\
pj = 0, \quad qj = 1, \\
iq + jq = 1
\]

These equalities characterize the direct sum of \( A \) and \( B \), which will be denoted by \( A + B \).

2. Semi-cartesian squares

What does one get by composing cartesian and cocartesian squares? Semi-cartesian squares in the following sense.

**Proposition 1.** Let \( ca = db \) be a commutative square as in Fig. 2. Let \( B + C \) be the direct sum with insertions \((i,j)\) and projections \((p,q)\); construct the fiber product \((P, f, g)\) of \((c,d)\) with the kernel \( n \) of
cp – dq, and the amalgamated sum \((S, r,s)\) of \((a,c)\) with the cokernel \(t\) of \(ia + jb\). Then there exist unique arrows \(e : A \rightarrow P\) and \(m : S \rightarrow C\) making commutative the obvious triangles of figure 2. Then the following conditions are equivalent:

(i) \(e\) is an epimorphism,
(ii) \(m\) is a monomorphism,
(iii) the sequence \(0 \rightarrow P \overset{p}{\rightarrow} B + C \overset{t}{\rightarrow} S \overset{0}{\rightarrow}\) is exact,
(iv) the sequence \(A \overset{ia + jb}{\rightarrow} B + C \overset{cp – dq}{\rightarrow} D\) is exact.

When constructing \(P\) and \(S\) we defined
\[
\begin{align*}
f &= pn, & g &= qn, \\
r &= ti, & s &= - tj.
\end{align*}
\]
So there exists a unique \(e\) with \(a = fe\) and \(b = ge\); and a unique \(m\) with \(e = mr\) and \(d = ms\).

Composing on the right \(1_S = ip + jq\) with \(ne\), one finds \(ne = ia + jb\). Therefore \(t\) is a cokernel of \(ne\).

(i) \(\iff\) (iii): For the sequence in (iii) to be exact, it is necessary and sufficient that \(t\) be a cokernel of \(n\), that is, that \(e\) be an epimorphism.

(iii) \(\iff\) (ii): Condition (iii) is preserved by duality and is thus is equivalent to (ii) which is dual of (i).

(iii) \(\iff\) (iv): because \(n\) is a monomorphism and \(q\) an epimorphism.

(iv) \(\iff\) (iii): because the cokernel \(t\) of \(ne\) is cokernel of a kernel of \(mt = cp – dq\), that is of \(n\), hence (iii), QED.

**Definition.** — A commutative square is semi-cartesian if it satisfies the conditions of proposition 1.

For instance, a cartesian square \((e\) is invertible), or a cocartesian square \((m\) is invertible), is semi-cartesian. Next is a partial converse in which notations are those of figure 2.

**Proposition 2.** — In a semi-cartesian square \(ca = db\), if \(a\) is a monomorphism, then \(d\) is a monomorphism and the square is cartesian. Si \(d\) is an epimorphism, then \(a\) is an epimorphism and the square is cocartesian.

With the notations of Fig. 2, since \(a\) is a monomorphism, \(e\) is also a monomorphism. Since it is an epimorphism, it is invertible and the given square is cartesian. Let \(k : N \rightarrow C\) be a kernel of \(d\) and \(o : N \rightarrow B\) the null arrow. There exists a unique arrow \(h : N \rightarrow A\) such that \(k = bh\) and \(o = ah\). But \(a\) is a monomorphism, therefore \(h = 0\), hence \(k = 0\), QED.

Contrary to arrows, squares will be written in the same order as they are drawn.

**Proposition 3.** —
1) Suppose \(K\) is cocartesian. Then \(KL\) semi-cartesian \(\iff\) \(L\) semi-cartesian.
2) Suppose \(L\) is cartesian. Then \(KL\) semi-cartesian \(\iff\) \(K\) semi-cartesian.
3) \(K\) and \(L\) semi-cartesians \(\Rightarrow\) \(KL\) semi-cartesian.

1) Let \((r; s)\) be an amalgamated sum of \((c, v)\) and \(m\) the unique arrow such that \(w = mr\) and \(d = ms\).

The square \(r(ca) = (sb)u\) is composed of cocartesian squares and therefore is cocartesian. then \(KL\) semi-cartesian \(\iff\) \(m\) monomorphism \(\iff\) \(L\) semi-cartesian (see Fig. 3).
5) Consider figure 4; set \( S = B +_A C \), with a unique monomorphism \( n : S \to D \); set \( T = S +_B E \). The square ACTE is then cocartesian (composition of cocartesian squares) hence a unique arrow \( m : T \to F \) making the diagram commutative. Since \( L \) is semi-cartesian and BSTE cocartesian, square SDFT is semi-cartesian from (1) and \( m \) is a monomorphism from proposition 2, QED.

This proposition shows that a semi-cartesian square remains a semi-cartesian square when is removed a cartesian square on the right or a cocartesian square on the left; and also that semi-cartesian squares are got by composing cartesian and cocartesian squares. This is always the case, as shown by the corollary of the following proposition.

**Proposition 4.** — Let \( KL \) be a semi-cartesian square. If \( K \) is an epimorphism, then \( L \) is semi-cartesian. If \( L \) is a monomorphism, then \( K \) is semi-cartesian.

Let us prove the first assertion; the second is dual.

Let \((r,s)\) be an amalgamated sum of \((u,ca)\); since \( KL \) is semi-cartesian, there exists a unique monomorphism \( m \) with \( mr = db \). Since \( b \) is an epimorphism, it is a cokernel of some \( z \), and \( rz = 0 \) (compose on the left with \( m \) monomorphism). This implies a unique \( t \) such that \( r = tb \); and \( tv = sc \) (compose on the right with epimorphism \( a \)).

Let square \( tv = sc \) is cocartesian; if \( xc = yv \), a fortiori \( xca = yva = ybu \) and since \((ru = s(ca))\) is a cocartesian square, there exists a unique \( n \) such that \( x = ns \) and \( yb = nr = ntb \). Since \( b \) is an epimorphism, one deduces \( y = nt \). Since \( m \) is a monomorphism, square \( L \) is semi-cartesian, QED.
In other terms, in the class of semi-cartesian squares, one can simplify by epimorphisms on the left and by monomorphisms on the right. Beware that the converse is false: a semi-cartesian square (for instance the identity) preceded by an epimorphism is not necessarily semi-cartesian (there exists epimorphisms that are not semi-cartesian).

**Corollary.** — *Semi-cartesian squares can be decomposed into a cocartesian epimorphism followed by a cartesian monomorphism.*

Decompose the square into an epimorphism followed by a monomorphism. Proposition 4 ensures that they are also semi-cartesian squares. From proposition 2, the first one is cocartesian and the second one is cartesian.

**Proposition 5.** — Consider two successive commutative squares $K$ and $L$ as in figure 6.

1) Suppose that $K$ is a kernel of $L$. Then:
   a) $w$ monomorphism $\Rightarrow$ $K$ cartesian.
   b) $L$ semi-cartesian $\Rightarrow$ $u$ epimorphism.

2) Dually suppose that $L$ is a cokernel of $K$. Then:
   c) $u$ epimorphism $\Rightarrow$ $L$ cocartesian.
   d) $K$ semi-cartesian $\Rightarrow$ $w$ monomorphism.

Assertions (a),(b) are dual of (c),(d).

Let us show (a). Consider figure 7, in which $(s, t)$ verifies only $sc = tv$. Then $0 = sca = tva = tbu$ hence $tb = 0$ since $u$ is an epimorphism. Since $d$ is cokernel of $b$, there exists a unique arrow $z$ such that $t = zd$. One checks $s = zw$ by composing with epimorphism $c$ on the right.

Let us show (b). After decomposing $L$ with the help of the above corollary, one may assume that $L$ is a cocartesian epimorphism. Let $u = me$ be the decomposition of $u$ into an epimorphism followed by a monomorphism. If one shows that $d$ is a cokernel of $bm$, then $bm$ will be a kernel of $d$ (cf. lemma 1 § 1) as is $b$, and therefore $m$ will be invertible. Now let $t$ be such that $tbm = 0$. It follows $tbme = 0 = tva$. Since $c$ is a cokernel of $a$, there is a unique arrow $s$ such that $sc = tv$. Since $L$ is cocartesian, there exists a unique arrow $z$ such that $t = zd$ (and $s = zw$), QED.

**5. The snake lemma**

The snake lemma constructs an exact sequence connecting kernels and cokernels.

**Proposition 1.** — Suppose two successive squares $K$ and $L$, where $L$ is semi-cartesian. If $(a, c)$ is exact and $db = 0$, then $(b, d)$ is exact. Dually, supposing $K$ semi-cartesian, then if $(b, d)$ is exact and $ca = 0$, then $(a, c)$ is exact.
Let \( Q \) be the cokernel of \( K \) so that \( L = KI \) (see Fig. 7). Since \( ca = 0 \) and \( db = 0 \), there exist unique arrows \( i \) and \( j \) such that \( c = ip \) and \( d = jq \). Then \( I \) is semi-cartesian (prop. 4 § 2). Since the sequence \((a,c)\) is exact, \( i \) is a monomorphism. From proposition 2 § 2, \( j \) also is a monomorphism (and \( I \) is cartesian): \((b,d)\) is exact, \( \text{q.e.d.} \)

For each arrow \( u \) one selects a kernel arrow of \( u \) and denotes its source by \( \text{Ker}(u) \). In this way, \( \text{Ker}(u) \) becomes a functor.

**Proposition 2.** — Kernel functors are left-exact; cokernel functors are right-exact.

Kernels are (finite) projective limits. Therefore, they commute with projective limits. Dually, cokernel functors \( \text{Coker} \) are right-exact.

The following lemma, called the snake lemma, connects these two functors.

**Lemma.** — Given a diagram like Fig. 7, in which \( i, j, k \) are kernels of \( u, v, w \), and \( p, q, r \) are their cokernels, in which \( c \) is a cokernel of \( a \) and \( b \) is a kernel of \( d \), there exists an arrow \( \delta \) such that the following sequence is exact:

\[
\text{Ker}(u) \to^s \text{Ker}(v) \to^t \text{Ker}(w) \to^\delta \text{Coker}(u) \to^r \text{Coker}(v) \to^\gamma \text{Coker}(w).
\]

Decompose \( a = me \) into an epimorphism \( e \) followed by a monomorphism \( m \) as in figure 7. There exists arrows \( i' \) and \( u' \) such that \( js = mi' \) and \( vm = bu' \), because \( s, m \) and \( b \) are the respective kernels of \( t, c \) and \( d \). And since the functor \( \text{Ker} \) is left exact, \( i' \) is a kernel of \( u' \). In this way, changing notations, one may assume that \( a \) is a kernel of \( c \) and dually that \( d \) is a cokernel of \( b \).

**Construction of diagram 9.** Let \((m,f)\) be the fiber product of \((k,c)\). The square \( hf = cm \) is cartesian and since \( e \) is an epimorphism, so is \( f \) and the square is cocommutative (prop. 5 § 2). Let \( z \) be a kernel of \( f \). Since \( cmz = kfx = 0 \) and \( a \) is a kernel of \( c \), there exists a unique arrow \( l \) such that \( al = mz \). The square thus built is a kernel of the square built over \( m \) and \( k \); since \( k \) is a monomorphism, this square is cartesian (prop. 5 § 2).
Dually, one builds the amalgamated sum \((n, g)\) of \((p, b)\). This square is cocartesian and since \(b\) is a monomorphism, so is \(g\) and the square is cartesian (prop. 2 § 2). Similarly, one builds the cokernel \(h\) of \(g\) and one completes the square over \(h\) and \(d\), which is a cokernel of the square built over \(g\) and \(b\); since \(p\) is an epimorphism, this square is cocartesian (prop. 5(c) § 2).

Proof. Arrow \(nvm\) satisfies \((nvm)z = g(pu)l = 0\) and since \(f\) is a cokernel of \(z\), there exists a unique arrow \(\theta\) such that \(nvm = \theta f\). Now \(h\theta\) is nul parce que \(h\theta f = 0\) and \(f\) is an epimorphism. Therefore \(\theta\) factorises through the kernel of \(h\), that is \(g\): there exists a unique arrow \(\delta\) such that \(\theta = g\delta\). This terminates the construction of \(\delta\).

There remains to show that the sequence \((t, \delta)\) is exact or again, since \(g\) is a monomorphism, that \((t, \theta)\) is exact; by the duality property \((d, x)\) will also be exact. It is already clear that \(nvt = 0\). Let us show that the sequence \((t, \theta)\) is exact.

**Step 1:** Notice that \(nva = gpu = 0\) implies that \(nv\) factorizes through the cokernel \(c\) of \(a\): \(nv = k\) for a unique arrow \(k\). Moreover, \(k\) and \(\theta\) are two monomorphisms.

Decompose \(n\) into an epimorphism \(\varepsilon\) followed by a monomorphism \(\zeta\). Since \((va, n)\) is an exact sequence, \(\varepsilon\) is a cokernel of \(va\); since \(c\) is a cokernel of \(a\), there exists a unique arrow \(\rho\) such that \(\rho c = \varepsilon v\) (see Fig. 11).

**Step 2:** The sequence \((va, n)\) is exact. Indeed, in figure 10, the sequence \((u, p)\) is exact, \(n(va) = gpυ = 0\) and square C is cocartesian by construction. Proposition 1 ensures that the sequence \((va, n)\) is exact. Since \((s, t)\) is exact, \(\sigma\) is a cokernel of \(s\). Since \(c\) is a cokernel of \(a\), there exists a unique arrow \(\nu\) such that \(\nu c = \varepsilon j\). And \(\nu = k\tau\) is a monomorphism, since \(k\) and \(\tau\) are two monomorphisms.

Decompose \(n\) into an epimorphism \(\varepsilon\) followed by a monomorphism \(\zeta\). Since \((va, n)\) is an exact sequence, \(\varepsilon\) is a cokernel of \(va\); since \(c\) is a cokernel of \(a\), there exists a unique arrow \(\rho\) such that \(\rho c = \varepsilon v\) (see Fig. 11).

**Step 3:** Now, square IV in fig. 11 is a cokernel of square III, and since \(l\) is an epimorphism, IV is cocartesian (prop. 5(c) § 2). Further, the sequence \((j, v)\) is exact and \(nv = 0\), as can be checked if we precede it with the epimorphism \(\sigma\): \(\rho v = \varepsilon vj = 0\). From proposition 1, \((v, \rho)\) is exact. Since \(v\) is a monomorphism, it is a kernel of \(\rho\), and also of \(\zeta\) since \(\zeta\) is a monomorphism. Proposition 3 of § 1 terminates the proof: \(\nu = k\tau\) is a kernel of \(\kappa\), therefore \(\tau\) is a kernel of \(kk = \theta\), QED.

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