Invariance properties of induced Fock measures for U(1) holonomies

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Abstract
We study invariance properties of the measures in the space of generalized U(1) connections associated to Varadarajan’s r-Fock representations.

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1 Introduction

Holonomies are the starting point for a rigorous approach to quantum gravity – often called ”loop quantum gravity” – carried throughout the last decade. It is based on Ashtekar’s formulation of general relativity as a gauge theory [As], loop variables [GT, RoSm], $C^*$-algebra techniques [AI2, Ba1] and integral and functional calculus in spaces of generalized connections [AL1, AL3, MM, Ba2] (an excellent review of both the fundamentals and the most recent developments in this field can be found in [I]). Since the early days of this approach, free Maxwell theory has been a preferred testing ground for new ideas, especially in what concerns the relation between background independent representations of holonomy algebras and the standard Fock representation for smeared fields [ARS, AL, AR]. Recently, Varadarajan revisited this subject, and proposed a family of representations for a kinematical Poisson algebra of $U(1)$ holonomies and certain functions of the electric fields [Va1, Va2]. Varadarajan’s work allowed the emergence of Fock states within the framework of generalized connections and is therefore a promising starting point to close the gap between non-perturbative loop quantum gravity states and low energy states [AL4] (see also [I] for a general discussion of the issue of semiclassical analysis in loop quantum gravity).

In the present work we study (quasi-)invariance and mutual singularity properties of the measures associated to Varadarajan’s representations. These are measures on the space $\mathcal{A}/\mathcal{G}$ of generalized $U(1)$ connections that can be obtained, by push-forward, from the standard Maxwell-Fock measure (see [AI2, AR, ARS] for previous work along these lines and also [AL4] for a projective construction of the measures). We will show that the measures are singular with respect to each other and are singular with respect to the measure $\mu_0$ of Ashtekar and Lewandowski. This implies, in particular, that the Fock states are not in $L^2(\mathcal{A}/\mathcal{G}, \mu_0)$ (but rather in an extension thereof [Va2]). It also follows that the measures are not quasi-invariant with respect to the
natural action of $\mathcal{A}/\mathcal{G}$ on itself, which is an obstruction to the quantization of the usual smeared electric fields. On the other hand, the measures on $\mathcal{A}/\mathcal{G}$ inherit quasi-invariance properties directly related to the electric operators considered by Varadarajan.

This work is organized as follows. In section 2 we review the Fock representation, whereas the loop approach is reviewed in section 3. Varadarajan’s measures are presented in section 4 and studied in section 5, which contains our main results. We conclude with a brief discussion, in section 6.

2 Smeared fields and Fock representation

This section briefly reviews some aspects of the Schrödinger representation of the usual Fock space for the Maxwell field, following [GV, ReSi, GJ, BSZ]. We use spatial coordinates $(x^a)$, $a = 1, 2, 3$, and units such that $c = \hbar = 1$. The Euclidean metric $\delta_{ab}$ in $\mathbb{R}^3$ is used to raise and lower indices whenever necessary.

As is well known, connections $A$ and electric fields $E$ do not give rise to well defined quantum operators. In the Fock framework, they are replaced by smeared versions $A(\epsilon) = \int A_a \epsilon^a d^3x$ and $E(\lambda) = \int \lambda_a E^a d^3x$, where $\epsilon$ belongs to the (nuclear) space $\mathcal{E}\_\infty$ of smooth and fast decaying transverse vector fields and $\lambda$ to the (nuclear) space $(\mathcal{A}/\mathcal{G})\_\infty$ of smooth and fast decaying transverse connections. The Poisson bracket between these basic observables is

$$\{A(\epsilon), E(\lambda)\} = \int \lambda_a \epsilon^a d^3x,$$

(1)

to which correspond the Weyl relations:

$$\mathcal{V}(\lambda) \mathcal{U}(\epsilon) = e^{i\int \lambda_a \epsilon^a d^3x} \mathcal{U}(\epsilon) \mathcal{V}(\lambda).$$

(2)

The usual Fock representation can be realized in the Hilbert space $L^2(\mathcal{E}\_\infty^*, \mu_\Delta)$, where $\mathcal{E}\_\infty^*$ is the space of tempered distributional 1-forms (the topological dual
of $E_\infty$) and $\mu_\Delta$ is the Gaussian measure defined by:

$$\int_{E_\infty} e^{i\phi(\epsilon)} d\mu_\Delta(\phi) := \exp \left( -\frac{1}{4} \int \delta_{ab} \epsilon^a (-\Delta)^{-1/2} \epsilon^b d^3x \right), \quad (3)$$

where $\Delta$ is the Laplacian in $\mathbb{R}^3$. It is well known that $\mu_\Delta$ is quasi-invariant with respect to the action of $(\mathcal{A}/\mathcal{G})_\infty$ on $E_\infty^*$:

$$E_\infty^* \ni \phi \mapsto \phi + \lambda \quad \lambda \in (\mathcal{A}/\mathcal{G})_\infty, \quad (4)$$

where $\phi + \lambda \in E_\infty^*$ is defined by

$$(\phi + \lambda)(\epsilon) := \phi(\epsilon) + \int \lambda_a \epsilon^a d^3x, \quad \forall \epsilon \in E_\infty. \quad (5)$$

(Recall that a measure $\mu$ is quasi-invariant with respect to a group of transformations $T$ if the push-forward measure $T_*\mu$ has the same zero measure sets as $\mu$, $\forall T$, i.e. $(T_*\mu)(B) = 0$ if and only if $\mu(B) = 0$.) One therefore has an unitary representation $V$ of the abelian group $(\mathcal{A}/\mathcal{G})_\infty$ as translations:

$$(V(\lambda) \psi)(\phi) = \sqrt{\frac{d\mu_{\Delta,\lambda}(\phi)}{d\mu_\Delta(\phi)}} \psi(\phi - \lambda), \quad \psi \in L^2(E_\infty^*, \mu_\Delta), \quad (6)$$

where $\mu_{\Delta,\lambda}$ is the push-forward of the measure $\mu_\Delta$ by the map $\lambda$ and $d\mu_{\Delta,\lambda}/d\mu_\Delta$ is the Radon-Nikodym derivative. (The existence of both the Radon-Nikodym derivative and of its inverse is equivalent to quasi-invariance.)

A representation of the Weyl relations $A$ is achieved with the following representation $U$ of $E_\infty$:

$$(U(\epsilon) \psi)(\phi) = e^{-i\phi(\epsilon)} \psi(\phi). \quad (7)$$

Since both representations $U$ and $V$ are continuous, the quantized fields $\hat{A}(\epsilon)$ and $\hat{E}(\lambda)$ can be identified with the generators of the one-parameter groups $U(t\epsilon)$ and $V(t\lambda)$, respectively.
This section briefly reviews the loop approach to the Maxwell field, following in essence the general framework for gauge theories with a compact (not necessarily abelian) group (see e.g. [T] and references therein). Notice, however, that the presentation of the uniform measure $\mu_0$ [AL1] and the quantization of electric fields [AL3] are considerably simpler in the $U(1)$ case.

In the loop approach the configuration variables are (traces of) holonomies rather than smeared connections. Let us then consider $U(1)$ holonomies

$$T_\alpha(A) := e^{i \oint \alpha A_a dx^a} \quad (8)$$

associated with piecewise analytic loops on $\mathbb{R}^3$. It is convenient to eliminate redundant loops, i.e. one identifies two loops $\alpha$ and $\beta$ such that $T_\alpha(A) = T_\beta(A) \forall A$. Such classes of loops are called hoops. The set $HG$ of all $U(1)$ hoops is an abelian group under the natural composition of loops.

The set of holonomy functions $T_\alpha$, $\alpha \in HG$, is an abelian $*$-algebra. The $C^*$ completion in the supremum norm is called the $U(1)$ holonomy algebra $HA$ [AI2, AL1]. It turns out that $HA$ is isomorphic to the algebra of continuous functions on the space $\overline{A/G}$ of generalized connections, where $\overline{A/G}$ is the set of all group morphisms from the hoop group $HG$ to $U(1)$. In order to describe the isomorphism, let us consider the functions $\Psi_\alpha: \overline{A/G} \to U(1)$:

$$\overline{A} \mapsto \Psi_\alpha(\overline{A}) := \overline{A}(\alpha), \quad (9)$$

where $\alpha \in HG$ and $\overline{A}(\alpha)$ denotes evaluation. The space $\overline{A/G}$ is compact in the weakest topology such that all functions $\Psi_\alpha$ are continuous. It is a key result that $\overline{A/G}$ is homeomorphic to the spectrum of $HA$ [MM, AL1, AL2], with the functions $\Psi_\alpha$ corresponding to $T_\alpha$.

(Cyclic) representations of $HA$ are in 1-1 correspondence with positive linear functionals on $HA$. By the above isomorphism, those are in turn in
1-1 correspondence with Borel measures in $\overline{\mathcal{A}/G}$. Given a measure $\mu$, one thus has a representation $\Pi$ of $\mathcal{H}_A$ in the Hilbert space $L^2(\overline{\mathcal{A}/G},\mu)$:

$$ (\Pi(T_\alpha)\psi)(\bar{A}) = \Psi_\alpha(\bar{A})\psi(\bar{A}), \quad \forall \psi \in L^2(\overline{\mathcal{A}/G},\mu). \quad (10) $$

The associated positive linear functional $\varphi$ is defined by:

$$ \varphi(T_\alpha) = \langle 1, \Pi(T_\alpha)1 \rangle = \int_{\mathcal{A}/G} \Psi_\alpha d\mu. \quad (11) $$

In the $U(1)$ case, $\overline{\mathcal{A}/G}$ is a topological group $[\text{AL1}, \text{Ma}]$ with multiplication

$$ (\bar{A}'\bar{A})(\alpha) = \bar{A}'(\alpha)\bar{A}(\alpha), \quad \bar{A}', \bar{A} \in \overline{\mathcal{A}/G}, \quad \alpha \in \mathcal{H}G, \quad (12) $$

and inverse $\bar{A}^{-1}(\alpha) = \bar{A}(\alpha^{-1})$. Let us consider the Haar measure $\mu_0$ and the associated representation $\Pi_0$ of $\mathcal{H}_A [\text{AL1}]$. Since $\mu_0$ is invariant, we also have an unitary representation $V_0$ of the group $\overline{\mathcal{A}/G}$ in $L^2(\overline{\mathcal{A}/G},\mu_0)$:

$$ (V_0(\bar{A}')\psi)(\bar{A}) = \psi(\bar{A}'\bar{A}), \quad \forall \psi \in L^2(\overline{\mathcal{A}/G},\mu_0). \quad (13) $$

The representation $V_0$ leads to smeared electric operators, as follows. For $\lambda \in (\mathcal{A}/G)_\infty$, let $\bar{A}_\lambda$ denote the element of $\overline{\mathcal{A}/G}$ defined by holonomies, i.e. $\bar{A}_\lambda(\alpha) := T_\alpha(\lambda), \quad \forall \alpha \in \mathcal{H}G$. Restricting $V_0$ to elements $\bar{A}_\lambda$ and the representation $\Pi_0$ to the functions $T_\alpha$, one obtains the commutation relations:

$$ V_0(\lambda)\Pi_0(T_\alpha) = e^{i\oint_\alpha a dx^a}\Pi_0(T_\alpha)V_0(\lambda), \quad (14) $$

where $V_0(\lambda) := V_0(\bar{A}_\lambda)$. The action of $V_0(\lambda)$ is particularly simple for the dense space of finite linear combinations of functions $\Psi_\alpha$:

$$ V_0(\lambda)\Psi_\alpha = e^{i\oint_\alpha a dx^a}\Psi_\alpha. \quad (15) $$

Let us consider the one-parameter unitary group $V_0(t\lambda), \quad t \in \mathbb{R}$, and let $dV_0(\lambda)$ be its self-adjoint generator. From (14) one finds the commutator:

$$ [dV_0(\lambda),\Pi_0(T_\alpha)] = \left( \oint_\alpha a^a dx^a \right) \Pi_0(T_\alpha), \quad (16) $$
showing that the operators $\Pi_0(T_\alpha)$ and $dV_0(\lambda)$ give a quantization of the Poisson algebra of holonomies $T_\alpha$ and smeared electric fields $E(\lambda)$, in $L^2(\mathcal{A}/\mathcal{G}, \mu_0)$. In this representation, the states $\Psi_\alpha$ describe one-dimensional excitations of the electric field along loops, or electric flux "quanta", and are therefore called loop states \[ GT, RoSm \]. These type of excitations are, of course, absent in Fock space. On the other hand, nor are the familiar Fock $n$-particle states or coherent states obviously related to loop states.

4 \textit{$r$-Fock measures}

In this section we present Varadarajan’s $r$-Fock representations of the $U(1)$ holonomy algebra $\mathcal{HA}$ from the measure theoretic point of view.

Let us start with hoop form factors \[ ARS, AR, AP \]. Given a hoop $\alpha$, the form factor $X_\alpha$ is the transverse distributional vector field such that

$$\int X_\alpha^a(x)A_a(x)d^3x = \oint_\alpha A_a dx^a, \ \forall A.$$ \hspace{1cm} (17)

Consider the one-parameter family of functions on $\mathbb{R}^3$:

$$f_r(x) = \frac{1}{2\pi^{3/2}r^3} e^{-x^2/2r^2},$$ \hspace{1cm} (18)

where $r > 0$. The smeared form factors are smooth and fast decaying transverse vector fields, i.e. elements of $\mathcal{E}_\infty$, defined by:

$$X_\alpha^a, r(x) := \int f_r(y - x)X_\alpha^a(y)d^3y.$$ \hspace{1cm} (19)

One thus has, for each $r$, a map $\alpha \mapsto X_{\alpha, r}$ from hoops to $\mathcal{E}_\infty$. Notice that the composition of hoops is preserved, i.e. $X_{\alpha, \beta, r} = X_{\alpha, r} + X_{\beta, r}$ \[ AR \].

Smeared form factors can be used to define measurable maps from the space of distributional connections $\mathcal{E}_\infty^*$ to $\mathcal{A}/\mathcal{G}$. Consider then the family of maps $\Theta_r : \mathcal{E}_\infty^* \rightarrow \mathcal{A}/\mathcal{G}$ given by $\phi \mapsto \tilde{A}_{\phi, r}$, where

$$\tilde{A}_{\phi, r}(\alpha) := e^{i\phi(X_{\alpha, r})} \ \forall \alpha \in \mathcal{H}\mathcal{G}.$$ \hspace{1cm} (20)
Since the \(\sigma\)-algebra of measurable sets in \(\overline{\mathcal{A}/\mathcal{G}}\) is the smallest one such that all functions \(\Psi_\alpha\) are measurable, one sees that \(\Theta_r\) is measurable if and only if the maps \(\Psi_\alpha \circ \Theta_r : \mathcal{E}_\infty^* \to U(1)\) are measurable for all \(\alpha \in \mathcal{HG}\), which is clearly true, since they can be obtained as a composition of measurable maps: \(\phi \mapsto \phi(X_{\alpha,r}) \mapsto e^{i\phi(X_{\alpha,r})}\).

One can now use the maps \(\Theta_r\) to push-forward the Fock measure \(\mu_\Delta\), thus obtaining a family of measures \(\mu_r := (\Theta_r)_* \mu_\Delta\) on \(\overline{\mathcal{A}/\mathcal{G}}\). By definition

\[
\mu_r(B) = \mu_\Delta(\Theta_r^{-1}B) \quad \forall \text{ measurable set } B \subset \overline{\mathcal{A}/\mathcal{G}}.
\]

Each of the measures \(\mu_r\) provides us with a Hilbert space \(L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_r)\) and a representation \(\Pi_r\) of \(\overline{\mathcal{HA}}\). The associated positive linear functional \(\varphi_r\) is:

\[
\varphi_r(T_\alpha) = \int_{\overline{\mathcal{A}/\mathcal{G}}} \Psi_\alpha(\overline{A}) \, d\mu_r(\overline{A}) = \int_{\mathcal{E}_\infty^*} e^{i\phi(X_{\alpha,r})} \, d\mu_\Delta(\phi).
\]

Expression (22) shows that the representation \(\Pi_r\) is the \(r\)-Fock representation considered by Varadarajan in [Va1], \(\mu_r\) being the \(r\)-Fock measure in \(\overline{\mathcal{A}/\mathcal{G}}\) whose existence was proved in [Va2].

\section{Properties of the \(r\)-Fock measures}

The present section contains our main results. We show that the \(r\)-Fock measures \(\mu_r\) are all mutually singular, and are singular with respect to the Haar measure \(\mu_0\). We study also (quasi-)invariance properties of the \(r\)-Fock measures \(\mu_r\) and their relation to the quantization of certain twice smeared electric fields introduced in [Va1].

Let \(\text{Diff}\) be the group of (analytic) diffeomorphisms of \(\mathbb{R}^3\). The natural action of \(\text{Diff}\) on the (piecewise analytic) curves of \(\mathbb{R}^3\) induces an action on the hoop group \(\mathcal{HG}\):

\[
\mathcal{HG} \times \text{Diff} \ni (\alpha, \varphi) \mapsto \varphi \alpha.
\]
and therefore one has an action of $\text{Diff}$ in $\mathcal{A}/\mathcal{G}$, given by

$$(\varphi^* \bar{A})(\alpha) = \bar{A}(\varphi \alpha), \quad \varphi \in \text{Diff}, \quad \bar{A} \in \mathcal{A}/\mathcal{G}, \quad \alpha \in \mathcal{H}\mathcal{G}.$$  \hfill (24)

It can be seen that the maps $\varphi^* : \mathcal{A}/\mathcal{G} \to \mathcal{A}/\mathcal{G}$ are continuous \cite{AL1, AL2, Ba1}. The Haar measure $\mu_0$ is invariant under the action of $\text{Diff}$, since no background geometrical structure is used in its definition \cite{AL1}. The induced measures $\mu_r$, on the other hand, are not invariant, due to the appearance of the Euclidean metric $\delta_{ab}$ in the construction of the Fock measure $\mu_{\Delta}$.

From now on we will restrict our attention to the Euclidean group, i.e., the subgroup of $\text{Diff}$ of transformations that preserve the Euclidean metric. It is clear that the measures $\mu_r$ are invariant under these transformations, given the well known Euclidean invariance of the Fock measure.

Besides being invariant, the Fock measure is moreover ergodic with respect to the action of the Euclidean group (see e.g. \cite{BSZ, Ve2}), which means that the only invariant functions in $L^2(\mathcal{E}_{\Delta}^\infty, \mu_{\Delta})$ are the constant functions. This ergodic property is shared by the measures $\mu_r$, since if an invariant and non-constant function $\psi$ were to exist in $L^2(\mathcal{A}/\mathcal{G}, \mu_r)$, then the pull-back $\psi \circ \Theta_r$ would define an invariant and non-constant function in $L^2(\mathcal{E}_{\Delta}^\infty, \mu_{\Delta})$.

An important fact is that the Haar measure $\mu_0$ on $\mathcal{A}/\mathcal{G}$ is also ergodic under the action of the Euclidean group, as follows from more general results proven in \cite{MTV}. Thus, all measures $\mu_r$, $r \in \mathbb{R}^+$, and $\mu_0$ are invariant and ergodic under the action of the same group. From well known results in measure theory (see e.g. \cite{Ya}), this is only possible if all these measures are mutually singular, meaning that each measure of the set $\{\mu_r, \ r \in \mathbb{R}^+\} \cup \mu_0$ is supported on a subset of $\mathcal{A}/\mathcal{G}$ which has zero measure with respect to all the other measures (recall that a subset $X$ of a space $M$ is said to be a support for the measure $\mu$ on $M$ if any measurable subset $Y$ on the complement, $Y \subset X^c$, has measure zero). It is thus proven that

**Theorem 1** The measures in the set $\{\mu_r, \ r \in \mathbb{R}^+\} \cup \mu_0$ are all singular with respect to each other.
Theorem 1 leads to the conclusion that none of the measures $\mu_r$ is quasi-invariant under the action of $\mathcal{A}/\mathcal{G}$ on itself. This follows from the fact that $\mathcal{A}/\mathcal{G}$ is a compact group, which implies that any quasi-invariant measure is in the equivalence class of the Haar measure, meaning that it must have the same zero measure sets (see e.g. [Ki, 9.1]). Thus

**Corollary 1** The measures $\mu_r$, $r \in \mathbb{R}^+$, are not quasi-invariant.

We saw in section 3 how the quantization of smeared electric fields can be obtained from an unitary representation of the group $\mathcal{A}/\mathcal{G}$ in the Hilbert space $L^2(\mathcal{A}/\mathcal{G}, \mu_0)$. From the corollary we conclude that such an unitary representation of $\mathcal{A}/\mathcal{G}$ is not available in the Hilbert spaces $L^2(\mathcal{A}/\mathcal{G}, \mu_r)$. One should thus look for the quantization of different functions of the electric fields.

Varadarajan showed in [Va1] that certain "Gaussian-smeared smeared" electric fields can be consistently quantized in the $r$-Fock representations. In the remaining we will relate the quantization of these functions to quasi-invariance properties of the $r$-Fock measures $\mu_r$. We will start by establishing the quasi-invariance properties, which, as expected, follow from the quasi-invariance of the Fock measure under the action (4).

Let us consider the restriction of the maps $\Theta_r$ (20) to $(\mathcal{A}/\mathcal{G})\infty$, i.e., we consider the maps

$$(\mathcal{A}/\mathcal{G})\infty \ni \lambda \mapsto \tilde{A}_{\lambda,r} \in \overline{\mathcal{A}/\mathcal{G}}$$

such that

$$\tilde{A}_{\lambda,r}(\alpha) = \exp \left( i \int \lambda_a(x)X_a^{\alpha,r}(x) d^3x \right), \quad \forall \alpha \in \mathcal{H}\mathcal{G}.$$  \hspace{1cm} (26)

It is clear that $\tilde{A}_{\lambda+\lambda',r} = \tilde{A}_{\lambda,r} \tilde{A}_{\lambda',r}$, and therefore the group $(\mathcal{A}/\mathcal{G})\infty$ acts on the space $\overline{\mathcal{A}/\mathcal{G}}$ as a subgroup of the full group $\mathcal{A}/\mathcal{G}$. Let us denote this action by $\Xi_r$:

$$(\mathcal{A}/\mathcal{G})\infty \times \overline{\mathcal{A}/\mathcal{G}} \ni (\lambda, \tilde{A}) \mapsto \tilde{A}_{\lambda,r} \tilde{A}.$$ \hspace{1cm} (27)
For any given \( \lambda \in (\mathcal{A}/\mathcal{G})_\infty \), let \( \mu_{\lambda,r} \) denote the push-forward of the measure \( \mu_r \) by the map \( \bar{A} \mapsto \bar{A}_{\lambda,r} \bar{A} \). The measure \( \mu_{\lambda,r} \) is completely determined by the integrals of continuous functions \( F(\bar{A}) \):

\[
\int_{\mathcal{A}/\mathcal{G}} F(\bar{A}) d\mu_{\lambda,r}(\bar{A}) = \int_{\mathcal{A}/\mathcal{G}} F(\bar{A}_{\lambda,r} \bar{A}) d\mu_r(\bar{A}).
\]

(28)

We need only to consider the functions \( \Psi_\alpha \) (9), and therefore the measure \( \mu_{\lambda,r} \) is determined by the following map from \( \mathcal{H}\mathcal{G} \) to \( \mathbb{C} \):

\[
\alpha \mapsto \int_{\mathcal{A}/\mathcal{G}} (\bar{A}_{\lambda,r} \bar{A})(\alpha) d\mu_r(\bar{A}).
\]

(29)

One gets from (12), (22) and (26):

\[
\int_{\mathcal{A}/\mathcal{G}} (\bar{A}_{\lambda,r} \bar{A})(\alpha) d\mu_r(\bar{A}) = \exp \left( i \int \lambda_a(x) X_{\alpha,r}^a(x) d^3x \right) \int_{\mathcal{E}_\infty^*} e^{i\phi(X_{\alpha,r})} d\mu_\Delta(\phi).
\]

(30)

Recalling the action (4) of \( (\mathcal{A}/\mathcal{G})_\infty \) on \( \mathcal{E}_\infty^* \), one gets further:

\[
\int_{\mathcal{A}/\mathcal{G}} (\bar{A}_{\lambda,r} \bar{A})(\alpha) d\mu_r(\bar{A}) = \int_{\mathcal{E}_\infty^*} e^{i\phi(X_{\alpha,r})} d\mu_\Delta(\phi)
\]

\[= \int_{\mathcal{E}_\infty^*} e^{i\phi(X_{\alpha,r})} d\mu_{\Delta,\lambda}(\phi),
\]

(31)

where the measure \( \mu_{\Delta,\lambda} \) is the push-forward of \( \mu_\Delta \) by the map \( \phi \mapsto \phi + \lambda \) (4).

Recalling also the arguments of section [4], one sees easily that the measure \( \mu_{\lambda,r} \) coincides with \( (\Theta_r)_\ast \mu_{\Delta,\lambda} \), the push-forward of \( \mu_{\Delta,\lambda} \) by the map \( \Theta_r \) (27). Since the Fock measure \( \mu_\Delta \) is quasi-invariant under the action of \( (\mathcal{A}/\mathcal{G})_\infty \), this is sufficient to prove that, for any \( r \in \mathbb{R}^+ \), the measure \( \mu_r \) is quasi-invariant with respect to the action \( \Xi_r \) (27). For if \( B \subset \mathcal{A}/\mathcal{G} \) is such that \( \mu_r(B) = 0 \) we then have \( \mu_\Delta(\Theta_r^{-1}B) = 0 \), by definition of the push-forward measure. The quasi-invariance of \( \mu_\Delta \) then shows that \( \mu_{\Delta,\lambda}(\Theta_r^{-1}B) = 0, \forall \lambda \in (\mathcal{A}/\mathcal{G})_\infty \), which in turn is equivalent to \( \mu_{\lambda,r}(B) = 0 \). Thus

**Theorem 2**  The measure \( \mu_r \) is quasi-invariant with respect to the action \( \Xi_r \), for any given \( r \).
Using this result, we define, for any $r$, a natural unitary representation $V_r$ of $(\mathcal{A}/\mathcal{G})_\infty$ in $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_r)$:

$$(V_r(\lambda)\psi)(\bar{A}) = \sqrt{\frac{d\mu_{\lambda,r}}{d\mu_r}} \psi(\bar{A}_{\lambda,r} \bar{A}), \quad \lambda \in (\mathcal{A}/\mathcal{G})_\infty, \quad \psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_r), \quad (32)$$

where $d\mu_{\lambda,r}/d\mu_r$ is the Radon-Nikodym derivative.

One can easily work out the commutation relations between $V_r(\lambda)$ and the $r$-Fock representation $\Pi_r(T_\alpha)$ of holonomies:

$$V_r(\lambda)\Pi_r(T_\alpha) = \exp \left( i \int \lambda_a(x) X_{a,r}^a(x) d^3x \right) \Pi_r(T_\alpha) V_r(\lambda). \quad (33)$$

Let us consider the self-adjoint generator $dV_r(\lambda)$ of the one-parameter unitary group $V_r(t\lambda)$, $t \in \mathbb{R}$. Notice that the existence of $dV_r(\lambda)$, or the continuity of the one-parameter group $V_r(t\lambda)$, follows from the continuity of the representation $V$ in $L^2(\mathcal{E}_\infty^*, \mu_\Delta)$. From (33) one obtains the following commutator:

$$[dV_r(\lambda), \Pi_r(T_\alpha)] = \left( \int \lambda_a(x) X_{a,r}^a(x) d^3x \right) \Pi_r(T_\alpha). \quad (34)$$

This commutator is indeed the quantization of a given classical Poisson bracket, as realized by Varadarajan [Va1]. Consider then, for each $r$, the following functions of the electric field, parametrized by elements of $(\mathcal{A}/\mathcal{G})_\infty$:

$$E^a(x) \mapsto E_r(\lambda) := \int \lambda_a(x) \left( \int f_r(x - y) E^a(y) d^3y \right) d^3x, \quad (35)$$

where $f_r$ is given by (18). The functions $E_r(\lambda)$ are referred to as "Gaussian-smeread smeared electric fields" in [Va1]. The Poisson bracket between these functions and the holonomies is

$$\{T_\alpha, E_r(\lambda)\} = i \left( \int \lambda_a(x) X_{a,r}^a(x) d^3x \right) T_\alpha, \quad (36)$$

showing that $dV_r(\lambda)$ can be seen as the quantization in $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_r)$ of the classical function $E_r(\lambda)$. 
6 Discussion

Varadarajan’s use of smeared form factors allowed an embedding of distributional connections into $\mathcal{A}/\mathcal{G}$, overcoming the fact that the natural embedding of connections is not extensible to distributions. This step is very welcome, since $\mathcal{A}/\mathcal{G}$ can be seen as a common measurable space from which both Fock states and loop states can be defined. In particular, the $r$-Fock measures $\mu_r$ give natural images $L^2(\mathcal{A}/\mathcal{G}, \mu_r)$ of the Fock space [Val]. The Fock states, however, cannot be regarded as elements of $L^2(\mathcal{A}/\mathcal{G}, \mu_0)$, the kinematical Hilbert space for loop states, as a consequence of the mutual singularity between the Haar measure and the $r$-Fock measures, which we have proven. Nevertheless, Fock states can be exhibited within the framework of loop states. In fact, Varadarajan [Va2] showed that the Fock states can be realized as elements of a natural extension of $L^2(\mathcal{A}/\mathcal{G}, \mu_0)$, e.g. the dual $Cyl^*$ of the space of cylinder functions in $\mathcal{A}/\mathcal{G}$. Notice that such an extension from $L^2(\mathcal{A}/\mathcal{G}, \mu_0)$ to $Cyl^*$ is already required in loop quantum gravity, in order to solve the diffeomorphism constraint [ALMMT]. A suitable generalization of Varadarajan’s work to quantum gravity is, therefore, expected to produce an embedding of Minkowskian Fock-like states (describing ”gravitons” of a semiclassical or low energy effective theory) into the space $Cyl^*$ of non-perturbative physical loop states. These issues are currently under investigation (see [AL4] and also [I] for a more general approach to semiclassical analysis). In these efforts, measure theory in $\mathcal{A}/\mathcal{G}$ plays a relevant role, e.g. in the definition of quantum operators and in the analysis of the physical contents of the states. A good understanding of the $r$-Fock measures and their relation to the Haar measure may therefore be important to further developments.

In order to complement our measure theoretical results, we would like to conclude with a brief comment regarding topological aspects. Although the $r$-Fock measures are supported in irrelevant sets with respect to the
Haar measure, it can be shown on the topological side that every conceivable support of a $r$-Fock measure is dense in $\mathcal{A}/\mathcal{G}$. The $r$-Fock measures $\mu_r$ are therefore faithful just like the Haar measure $\mu_0$ \cite{AL1}. (Recall that a Borel measure is said to be faithful if every non empty open set has non zero measure, which is readily seen to be equivalent to the denseness of every conceivable support. Turning to representations, a measure in $\mathcal{A}/\mathcal{G}$ is faithful if and only if the corresponding representation of the holonomy algebra $\mathcal{H}$ is faithful.) The fact that the Haar measure and the $r$-Fock measures are all faithful and mutually singular means that one can find a family of mutually disjoint dense sets, each of which supports a different measure. Notice finally that dense sets in $\mathcal{A}/\mathcal{G}$ that do not contribute to the Haar measure were already known, e.g. the set of smooth connections \cite{MM} or a considerable extension of it given in \cite{MTV}. In the present case, however, one has new measures, living on $\mu_0$-irrelevant sets.

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\footnote{The crucial result, pointed out by M. Varadarajan, is that $\Theta_r(\mathcal{E}_\infty^*)$ is dense. The denseness of smaller supports follows from the faithfulness of the Fock measure $\mu_\Delta$ in $\mathcal{E}_\infty^*$ and the continuity of $\Theta_r$ with respect to an appropriate topology in $\mathcal{E}_\infty^*$.}

\footnote{An independent proof using projective arguments is given in \cite{AL4}.}
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