ZETA FUNCTION REGULARIZATION IN CASIMIR EFFECT CALCULATIONS AND J. S. DOWKER’s CONTRIBUTION

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A summary of relevant contributions, ordered in time, to the subject of operator zeta functions and their application to physical issues is provided. The description ends with the seminal contributions of Stephen Hawking and Stuart Dowker and collaborators, considered by many authors as the actual starting point of the introduction of zeta function regularization methods in theoretical physics, in particular, for quantum vacuum fluctuation and Casimir effect calculations. After recalling a number of the strengths of this powerful and elegant method, some of its limitations are discussed. Finally, recent results of the so called operator regularization procedure are presented.

Keywords: Zeta functions; zeta regularization; regularized determinant; Casimir effect; operator regularization.

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1. Introduction

We devote this introductory section to a historical account of some relevant contributions to the subject of operator zeta functions and their applications in theoretical physics. We start recalling the definition of the standard zeta function: the Riemann zeta function, \( \zeta(s) \), which is a function of a complex variable, \( s \). To define it, one considers the infinite series

\[
\sum_{n=1}^{\infty} \frac{1}{n^s}
\]  

which is absolutely convergent for all complex values of \( s \) such that \( \text{Re} \, s > 1 \), and then defines \( \zeta(s) \) as the analytic continuation, to the whole complex \( s \)-plane, of the function given for \( \text{Re} \, s > 1 \) by the sum of the preceding series. Actually, Leonhard Euler had already considered the above series in 1740, but only for positive integer values of \( s \) and some years later Chebyshev had extended the definition to \( \text{Re} \, s > 1 \). Riemann formulated his famous hypothesis on the non-trivial zeros of the zeta function.
function in 1859. It turns out that this is the only one in the famous list of twenty-three problems discussed in the address given in Paris by David Hilbert (on August 9, 1900), which has gone into the new list of seven Millennium Prize Problems, established by the Clay Mathematics Institute of Cambridge, Massachusetts (USA), and which were announced at a meeting in Paris (held on May 24, 2000) at the Collège de France.

In 1916, in their seminal paper “Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes”, Godfrey H. Hardy and John E. Littlewood did much of the earlier work concerning possible applications of the zeta function as a regularization procedure, by establishing the convergence and equivalence of series regularized with the heat kernel and zeta function regularization methods. Also very important in this respect was the appearance of Hardy’s book entitled Divergent Series. We should also note that Srinivasa I. Ramanujan had already found, working in isolation, the functional equation of the zeta function, independently of all this development, as Hardy could later certify.

Torsten Carleman, in 1935, in his work in French Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes obtained the zeta function encoding the eigenvalues of the Laplacian of a compact Riemannian manifold for the case of a compact region of the plane. This was an important first step towards extending the concept of zeta function as associated to the spectrum of a differential operator, which is actually the situation at issue here. And, as has been much more widely recognized in the literature, a decade and a half later Subbaramiah Minakshisundaram and Åke Pleijel, in their 1949 paper Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds, extended Carleman’s results explicitly showing that, if $A$ is the Laplacian of a compact Riemannian manifold, then the corresponding zeta function, $\zeta_A(s)$, converges and has an analytic continuation as a meromorphic function to all complex numbers, what is actually a very remarkable result.

Another milestone in this development was Robert Seeley’s seminal work, published in 1967, Complex powers of an elliptic operator. Seeley fully extended in this paper the above treatment to general elliptic pseudo-differential operators on compact Riemannian manifolds. He proved that, for all such operators, one can rigorously define a determinant using zeta function regularization. In 1971, Daniel B. Ray and Isadore M. Singer used Seeley’s theory in their famous paper entitled R-torsion and the Laplacian on Riemannian manifolds to define the determinant of a positive self-adjoint operator, $A$. Such operator is, in their explicit applications, the Laplacian of a Riemannian manifold; denoting its eigenvalues by $a_1, a_2, \ldots$, then its zeta function is formally given by the trace

$$\zeta_A(s) = \text{Tr} (A^{-s}).$$ (2)
The method defines also the (possibly divergent) infinite product as
\[
\prod_{n=1}^{\infty} a_n = \exp[-\zeta_A'(0)].
\] (3)

At this point we arrive, in our chronological survey, to the very important contribution of Stuart Dowker and Raymond Critchley. In their seminal work, published in 1976, *Effective Lagrangian and energy-momentum tensor in de Sitter space*\(^8\), these authors went definitely further in the application of the above procedures to physics: they actually proposed, for the first time, a fully-fledged zeta function regularization method for quantum physical systems. This paper has got high recognition, having gathered over 600 citations to present date. For the sake of completeness let us here reproduce in full its abstract:

*The effective Lagrangian and vacuum energy-momentum tensor \( < T^{\mu\nu} > \) due to a scalar field in a de Sitter space background are calculated using the dimensional-regularization method. For generality the scalar field equation is chosen in the form \((\Box^2 + \xi R + m^2)\phi = 0\). If \(\xi = 1/6\) and \(m = 0\), the renormalized \( < T^{\mu\nu} > \) equals \(g^{\mu\nu}(960\pi^2 a^4)^{-1}\), where \(a\) is the radius of de Sitter space. More formally, a general zeta-function method is developed. It yields the renormalized effective Lagrangian as the derivative of the zeta function on the curved space. This method is shown to be virtually identical to a method of dimensional regularization applicable to any Riemann space.*

One thing specialists often point out is that, in spite of the fact that, elaborating from the methods developed in this paper, it is true that a well defined and clear regularization prescription for a general case can be easily obtained, the authors actually described the method very briefly in this work, the uses and wide possibilities of the procedure not having been fully exploited or even anticipated there.

This is maybe the main reason why Stephen Hawking’s 1977 extremely influential paper (it has got over 1100 citations up to date) entitled *Zeta function regularization of path integrals in curved spacetime*\(^9\) is considered by many to be the actual seminal reference where the zeta function regularization method was defined, with all its computational power and possible physical applications, which were very clearly identified there. Needless to say, the title of the paper is absolutely explicit. Again, let us reproduce, for comparison, its abstract:

*This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to \(n\) dimensions by adding extra flat dimensions. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fifth dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations*
of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This electromagnetic tensor has an anomalous trace.

In my view, after investigating the case in some detail, it is fair to conclude that the priority of Dowker and Critchley in this matter has been sufficiently well established in the literature I have consulted (with some really incredible exceptions, however, as the running Wikipedia article on “Zeta function regularization”, where not the least reference to Dowker and Critchley is done!). Further to this, considering the number of citations collected by each one of the two papers, taking then into account the enormous mediatic impact of S.W. Hawking to modern physics (and well beyond it), and also the careful analysis of both the abstracts and the whole papers themselves, the ratio of citations to both works seems fair enough. But this is just to be taken as my personal opinion, of course.

To continue this account further would require a very hard work and would end in a very long report, at least book size, what is not the purpose here. Let us finish this short report here, at the point when, as already mentioned, the zeta function regularization method is considered to have been clearly defined and its usefulness for physics undoubtedly established. A large number of very interesting research articles in several directions have been published on these matters during the last 35 years. I will just mention a few references [10], which by no means are meant to constitute an optimized list. In the next section a short description of the main basic concepts involved in any rigorous formulation of the procedure of zeta function regularization will be given, together with some results originally obtained by the author.

2. Zeta function of a pseudodifferential operator and determinant

2.1. The zeta function

The zeta function \( \zeta_A \) of \( A \), a positive-definite elliptic pseudodifferential operator (ΨDO) of positive order \( m \in \mathbb{R} \) (acting on the space of smooth sections of \( E \), an \( n \)-dimensional vector bundle over a closed \( n \)-dimensional manifold, \( M \)) is defined as

\[
\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} \equiv s_0.
\]

(4)

being \( s_0 = \dim M/\text{ord } A \) the abscissa of convergence of \( \zeta_A(s) \). It can be proven that \( \zeta_A(s) \) has a meromorphic continuation to the whole complex plane \( \mathbb{C} \) (regular at \( s = 0 \)), provided the principal symbol of \( A (a_m(x, \xi)) \) admits a spectral cut: \( L_\theta = \{ \lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2 \}, \text{Spec } A \cap L_\theta = \emptyset \) (Agmon-Nirenberg condition [11]). This definition of \( \zeta_A(s) \) depends on the position of the cut \( L_\theta \). The only possible singularities of \( \zeta_A(s) \) are simple poles at \( s_k = (n-k)/m, \; k = 0, 1, 2, \ldots, n-1, n+1, \ldots \). M. Kontsevich and S. Vishik have managed to extend this definition to the case when \( m \in \mathbb{C} \) (no spectral cut exists) [11].
2.2. Zeta regularized determinant

Let $A$ be a $\Psi$DO operator with a spectral decomposition: $\{\varphi_i, \lambda_i\}_{i \in I}$, with $I$ some set of indices. The definition of determinant starts by trying to make sense of the product $\prod_{i \in I} \lambda_i$, which can be easily transformed into a “sum”: $\ln \prod_{i \in I} \lambda_i = \sum_{i \in I} \ln \lambda_i$. From the definition of the zeta function of $A$: $\zeta_A(s) = \sum_{i \in I} \lambda_i^{-s}$, by taking the derivative at $s = 0$: $\zeta_A'(0) = -\sum_{i \in I} \ln \lambda_i$, we arrive to the following definition of determinant of $A$ [12]:

$$\det_\zeta A = \exp \left[ -\zeta_A'(0) \right]. \quad (5)$$

An older definition (due to Weierstrass) is obtained by subtracting in the series above (when it is such) the leading behavior of $\zeta_A$ as $i \to \infty$, until the series $\sum_{i \in I} \ln \lambda_i$ is made to converge [13]. The shortcoming—for physical applications—is here that these additional terms turn out to be non-local and, thus, are non-admissible in a renormalization procedure.

In algebraic QFT, to write down an action in operator language one needs a functional that replaces integration. For the Yang-Mills theory this is the Dixmier trace, which is the unique extension of the usual trace to the ideal $L^{1,\infty}$ of the compact operators $T$ such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum: $\sigma_N(T) \equiv \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N)$, $\mu_0 \geq \mu_1 \geq \cdots$. The definition of the Dixmier trace of $T$ is: $\text{Dtr } T = \lim_{N \to \infty} \frac{1}{\ln N} \int_1^\lambda f(u) \frac{\mu_s}{\mu_s - 1} du$. Then, the Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator $T^{-1}$ at $s = 1$ (see Connes [14]): $\text{Dtr } T = \lim_{s \to 1^+} (s - 1) \zeta_{T^{-1}}(s)$.

The Wodzicki (or noncommutative) residue [15] is the only extension of the Dixmier trace to the $\Psi$DOs which are not in $L^{1,\infty}$. It is the only trace one can define in the algebra of $\Psi$DOs (up to a multiplicative constant), its definition being: $\text{Res}_{s=0} \text{ tr } (A\Delta^{-s})$, with $\Delta$ the Laplacian. It satisfies the trace condition: $\text{Res } (AB) = \text{Res } (BA)$. A very important property is that it can be expressed as an integral (local form) $\text{Res } A = \int_{S^* M} a_{\text{res}}(x, \xi) \, d\xi$, with $S^* M \subset T^* M$ the co-sphere bundle on $M$ (some authors put a coefficient in front of the integral: Adler-Manin residue[16]).

If $\dim M = n = -\text{ord } A$ (M compact Riemann, $A$ elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and one has $\text{Res}_{s=1} \zeta_A(s) = \frac{1}{n} \text{ res } A^{-1}$. The Wodzicki residue continues to make sense for $\Psi$DOs of arbitrary order and, even if the symbols $a_j(x, \xi)$, $j < m$, are not invariant under coordinate choice, their integral is, and defines a trace. All residua at poles of the zeta function of a $\Psi$DO can be easily obtained from the Wodzicki residue[17].

2.3. Multiplicative anomaly

Given $A$, $B$ and $AB$ $\Psi$DOs, even if $\zeta_A$, $\zeta_B$ and $\zeta_{AB}$ exist, it turns out that, in general, $\text{det}_\zeta(AB) \neq \text{ det}_\zeta A \text{ det}_\zeta B$. The multiplicative (or noncommutative, or determinant)
anomaly is defined as:

$$\delta(A, B) = \ln \left( \frac{\det\zeta(AB)}{\det\zeta A \cdot \det\zeta B} \right) = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0).$$

(6)

Wodzicki’s formula for the multiplicative anomaly:

$$\delta(A, B) = \frac{\text{res}\left\{ \ln [\sigma(A, B)]^2 \right\}}{2^{\text{ord } A \cdot \text{ord } B}} \cdot \sigma(A, B) := A^{\text{ord } B} B^{-\text{ord } A}. \quad (7)$$

At the level of Quantum Mechanics (QM), where it was originally introduced by Feynman, the path-integral approach is just an alternative formulation of the theory. In QFT it is much more than this, being in many occasions the actual formulation of QFT [19]. In short, consider the Gaussian functional integration

$$\int [d\Phi] \exp \left\{ - \int d^D x \left( \Phi^\dagger(x) (\cdots) \Phi(x) + \cdots \right) \right\} \rightarrow \det \left( \cdots \right)^{\pm 1}, \quad (8)$$

(the sign ± depends on the spin-class of the integration fields) and assume that the operator matrix has the following simple structure (being each $A_i$ an operator on its own):

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \rightarrow \begin{pmatrix} A \\ B \end{pmatrix}, \quad (9)$$

where the last expression is the result of diagonalizing the operator matrix. A question now arises. What is the determinant of the operator matrix: $\det(AB)$ or $\det A \cdot \det B$? This has been very much on discussion during the last months [20]. There is agreement in that: (i) In a situation where a superselection rule exists, $AB$ has no sense (much less its determinant), and then the answer must be $\det A \cdot \det B$. (ii) If the diagonal form is obtained after a change of basis (diagonalization process), then the quantity that is preserved by such transformations is the value of $\det(AB)$ and not the product of the individual determinants (there are counterexamples supporting this viewpoint [21]).

3. Explicit calculation of $\zeta_A$ and $\det\zeta A$

A fundamental property of many zeta functions is the existence of a reflection formula, also known as the functional equation by mathematicians. For the Riemann zeta function: $\Gamma(s/2)\zeta(s) = \pi^{s-1/2} \Gamma(1-s/2)\zeta(1-s).$ For a generic zeta function, $Z(s)$, it is $Z(\omega - s) = F(\omega, s)Z(s)$, and allows for its analytic continuation in an easy way —what is, as advanced above, the whole story of the zeta function regularization procedure (at least the main part of it). But the analytically continued expression thus obtained is just another series, again with a slow convergence behavior, of power series type [22] (actually the same that the original series had, in its own domain of validity). S. Chowla and A. Selberg found a formula, for the Epstein zeta function in the two-dimensional case [23], that yields exponentially quick convergence, and not only in the reflected domain. They were extremely proud of
that formula —as one can appreciate just reading the original paper (where actual- 
ly no hint about its derivation was given, see [23]). In Ref. [24], I generalized 
this expression to inhomogeneous zeta functions (most important for physical ap-
plications), but staying always in two dimensions, for this was commonly believed 
to be an unsurmountable restriction of the original formula (see, e.g., Ref. [25]). 
Later I obtained an extension to an arbitrary number of dimensions [26], both in the 
inhomogeneous (quadratic form) and non-homogeneous (quadratic plus affine form) 
cases.

In short, for the following zeta functions (corresponding to the general quadratic 
—plus affine— case and to the general affine case, in any number of dimensions, d) 
explicit formulas of the CS type were obtained in [26], namely,

$$
\zeta_1(s) = \sum_{\vec{n} \in \mathbb{Z}^d} [Q(\vec{n}) + A(\vec{n})]^{-s}
$$

and

$$
\zeta_2(s) = \sum_{\vec{n} \in \mathbb{N}^d} A(\vec{n})^{-s},
$$

where \(Q\) is a non-negative quadratic form and \(A\) a general affine one, in \(d\) dimensions 
giving rise to Epstein and Barnes zeta functions, respectively. Moreover, expressions for the more difficult cases when the summation ranges are interchanged, that is:

$$
\zeta_3(s) = \sum_{\vec{n} \in \mathbb{N}^d} [Q(\vec{n}) + A(\vec{n})]^{-s}
$$

and

$$
\zeta_4(s) = \sum_{\vec{n} \in \mathbb{Z}^d} A(\vec{n})^{-s}
$$

have been given in [26].

3.1. Extended Epstein zeta function in \(p\) dimensions

The starting point is Poisson’s resummation formula in \(p\) dimensions, which 
arises from the distribution identity \(\sum_{\vec{n} \in \mathbb{Z}^p} \delta(\vec{x} - \vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^p} e^{2\pi i \vec{m} \cdot \vec{x}}\). (We shall indistinctly write \(\vec{m} \cdot \vec{x} \equiv \vec{m}^T \vec{x}\) in what follows.) Applying this identity to the function \(f(\vec{x}) = \exp\left(-\frac{1}{2} \vec{x}^T A \vec{x} + \vec{b}^T \vec{x}\right)\), with \(A\) an invertible \(p \times p\) matrix, and integrating over \(\vec{x} \in \mathbb{R}^p\), we get: \(\sum_{\vec{n} \in \mathbb{Z}^p} \exp\left(-\frac{1}{2} \vec{n}^T A \vec{n} + \vec{b}^T \vec{n}\right) = \frac{(2\pi)^{p/2}}{\sqrt{\det A}} \sum_{\vec{m} \in \mathbb{Z}^p} \exp\left[\frac{1}{4} \left(\vec{b} + 2\pi i \vec{m}\right)^T A^{-1} \left(\vec{b} + 2\pi i \vec{m}\right)\right]\). Consider now the following zeta function (Re \(s > p/2\)):

$$
\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} \left[\frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q\right]^{-s} = \sum_{\vec{n} \in \mathbb{Z}^p} \left[Q(\vec{n} + \vec{c}) + q\right]^{-s}
$$

(the prime on the summation signs mean that the point $\vec{n} = \vec{0}$ is to be excluded from the sum). The aim is to obtain a formula giving (the analytic continuation of) this multidimensional zeta function in terms of an exponentially convergent multiseries and which is valid in the whole complex plane, explicitly exhibiting the singularities (simple poles) of the meromorphic continuation and the corresponding residues. The only condition on the matrix $A$ is that it corresponds to a (non negative) quadratic form, which we call $Q$. The vector $\vec{c}$ is arbitrary, while $q$ will (for the moment) be a positive constant. Use of the Poisson resummation formula yields

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s-p/2)}{\Gamma(s)} \frac{2^{s/2+p/4+1/2}q^{-s/2+p/4}}{\sqrt{\det A} \Gamma(s)} \sum_{\vec{m} \in \mathbb{Z}^p_{\nu=1/2}}' \cos(2\pi \vec{m} \cdot \vec{c}) \left(\vec{m}^T A^{-1} \vec{m}\right)^{s/2-p/4} K_{p/2-s} \left(2\pi \sqrt{2q \vec{m}^T A^{-1} \vec{m}}\right),$$

where $K_{\nu}$ is the modified Bessel function of the second kind, and the subindex $1/2$ in $\mathbb{Z}^p_{\nu=1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ intervene in the sum. That is, if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ (as simple criterion one can, for instance, select those vectors in $\mathbb{Z}^p \setminus \{\vec{0}\}$ whose first non-zero component is positive). Eq. (14) fulfills all the requirements demanded before. It is notorious to observe how the only pole of this inhomogeneous Epstein zeta function appears explicitly at $s = p/2$, just where it should, its residue being $\text{Res}_{s=p/2} \zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\sqrt{\det A} \Gamma(p/2)}$. It is relatively simple to obtain the limit of expression (14) as $q \to 0$.

When $q = 0$ there is no way to use the Poisson formula on all $p$ indices of $\vec{n}$. However, one can still use it on some of the $p$ indices $\vec{n}$ only, say on just one of them, $n_1$. Poisson’s formula on one index reduces to the celebrated Jacobi identity for the $\theta_3$ function, which can be written as $\sum_{n=-\infty}^{\infty} e^{-(n+z)^2} = \sqrt{\pi} \left[1 + \sum_{n=1}^{\infty} e^{-\pi^2 n^2 / t} \cos(2\pi nt)\right].$ Here $z$ and $t$ are arbitrary complex numbers, $z,t \in \mathbb{C}$, with the only restriction that $\text{Re} t > 0$. Applying this last formula to the first component, $n_1$, we obtain the following recurrent formula (for the sake of simplicity we set $\vec{c} = \vec{0}$, but the result can be easily generalized to $\vec{c} \neq \vec{0}$):

$$\zeta_{A,\vec{0},q}(s) = \zeta_{A,\vec{0},q}(s) + \sqrt{\frac{\pi}{a}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta_{A_{p-1},\vec{0},q}(s-1/2) + \frac{4\pi^s}{a^{s/2+1/4} \Gamma(s)} \sum_{n_2 \in \mathbb{Z}^{p-1} \setminus \{0\}}' \cos\left(\frac{\pi n_1}{a} \vec{n}_2^T \vec{n}_2\right) n_1^{s-1/2} \left(\vec{n}_2^T A_{p-1} \vec{n}_2 + q\right)^{1/4-s/2} K_{s-1/2} \left(\frac{2\pi n_1}{\sqrt{a}} \sqrt{\vec{n}_2^T A_{p-1} \vec{n}_2 + q}\right).$$

This is a recurrent formula in $p$, the number of dimensions, the first term of the recurrence being (see e.g., the 6th reference in [10])

$$\zeta_{A,\vec{0},q}(s) = 2 \sum_{n=1}^{\infty} \left(an^2 + q\right)^{-s} = q^{-s} + \sqrt{\frac{\pi}{a}} \frac{\Gamma(s-1/2)}{\Gamma(s)} q^{1/2-s}$$

$$+ \frac{4\pi^s}{\Gamma(s)} a^{-1/4-s/2} q^{1/4-s/2} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2} \left(2\pi n \sqrt{\frac{q}{a}}\right).$$
To take in these expressions the limit \( q \to 0 \) is immediate:

\[
\zeta_{A,0,0}(s) = 2a^{-s}\zeta(2s) + \sqrt{\frac{\pi}{a}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta_{\Delta_{p-1,0},0}(s-1/2) + \frac{4\pi^s}{a^{s/2+1/4} \Gamma(s)} \sum_{\vec{n}_2 \in \mathbb{Z}^{p-1}}', \cos \left( \frac{\pi n_1}{a} \vec{b} \cdot \vec{n}_2 \right) n_1^{s-1/2} (n_2^T \Delta_{p-1,0} n_2)_{1/4-s/2} K_{s-1/2} \left( \frac{2\pi n_1}{\sqrt{a}} \sqrt{n_2^T \Delta_{p-1} n_2} \right). \tag{17}
\]

In the above formulas, \( A \) is a \( p \times p \) symmetric matrix \( A = (a_{ij})_{i,j=1,2,\ldots,p} = A^T \), \( A_{p-1} \) the \((p-1) \times (p-1)\) reduced matrix \( A_{p-1} = (a_{ij})_{i,j=2,\ldots,p} \), \( a \) the component \( a = a_{11}, \vec{b} \) the \((p-1) - 1\) vector \( \vec{b} = (a_{21},\ldots,a_{p1})^T = (a_{12},\ldots,a_{1p})^T \), and \( \Delta_{p-1} \) is the following \((p-1) \times (p-1)\) matrix: \( \Delta_{p-1} = A_{p-1} - \frac{1}{4a} \vec{b} \otimes \vec{b} \).

It turns out that the limit as \( q \to 0 \) of Eq. (14) is again the recurrent formula (17). More precisely, what is obtained in the limit is the reflected formula, which one gets after using Epstein zeta function’s reflection \( \Gamma(s)Z(s;A) = \frac{\zeta_{s-p/2}^s}{\sqrt{\text{det} A}} \Gamma(p/2 - s)Z(p/2 - s;A^{-1}) \), being \( Z(s;A) \) the Epstein zeta function. This result is easy to understand after some thinking. Summing up, we have thus checked that Eq. (14) is valid for any \( q \geq 0 \), since it contains in a hidden way, for \( q = 0 \), the recurrent expression (17).

The formulas here can be considered as generalizations of the Chowla-Selberg series formula. All share the same properties that are so much appreciated by number-theorists as pertaining to the CS formula. In a way, these expressions can be viewed as improved reflection formulas for zeta functions; they are in fact much better than those in several aspects: while a reflection formula connects one region of the complex plane with a complementary region (with some intersection) by analytical continuation, the CS formula and the formulas above are valid on the whole complex plane, exhibiting the poles of the zeta function and the corresponding residues explicitly. Even more important, while a reflection formula is intended to replace the initial expression of the zeta function—a power series whose convergence can be extremely slow—by another power series with the same type of convergence, it turns out that the expressions here obtained give the meromorphic extension of the zeta function, on the whole complex \( s \)-plane, in terms of an exponentially decreasing power series (as was the case with the CS formula, that one being its most precious property). Actually, exponential convergence strictly holds under the condition that \( q \geq 0 \). However, the formulas themselves are valid for \( q < 0 \) or even complex. What is not guaranteed for general \( q \in \mathbb{C} \) is the exponential convergence of the series. Those analytical continuations in \( q \) must be dealt with specifically. The physical example of a field theory with a chemical potential falls clearly into this class.
3.2. Generalized Epstein zeta function in $d = 2$

For completeness, let us write down the corresponding series when $p = 2$ explicitly. They are, with $q > 0$,

$$
\zeta_E(s; a, b, c; q) = -q^{-s} + \frac{2\pi q^{1-s}}{(s-1)\sqrt{\Delta}} + \frac{4}{\Gamma(s)} \left( \frac{q}{a} \right)^{1/4} \left( \frac{\pi}{\sqrt{qa}} \right)^s \\
\times \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2} \left( 2\pi n \sqrt{\frac{q}{a}} \right) + \sqrt{\frac{q}{a}} \left( \frac{2\pi}{\sqrt{\Delta}} \right)^s \sum_{n=1}^{\infty} n^{s-1} K_{s-1} \left( 4\pi n \sqrt{\frac{aq}{\Delta}} \right) \\
+ \frac{2^{s+5/2} \pi^s}{\Gamma(s) \Delta^{s-2-1/4} \sqrt{\Delta}} \sum_{n=1}^{\infty} n^{s-1/2} \sigma_1(n) \cos(\pi nb/a) K_{s-1/2} \left( \frac{\pi n}{a} \sqrt{\Delta} \right),
$$

where $\Delta = 4ac - b^2 > 0$, and, with $q = 0$, the CS formula has seldom (if ever) been properly addressed in the literature. As an example, let us consider the following series in one dimension:

$$
\zeta_G(s; a, b; c) = 2\zeta(2s) a^{-s} + \frac{2^{2s+5/2} \pi^s}{\Gamma(s) \Delta^{s-2-1/4} \sqrt{\Delta}} \sum_{n=1}^{\infty} n^{s-1/2} \sigma_1(n) \cos(\pi nb/a) K_{s-1/2} \left( \frac{\pi n}{a} \sqrt{\Delta} \right).
$$

3.3. Truncated Epstein zeta function in $d = 2$

The most involved case in the family of Epstein-like zeta functions corresponds to having to deal with a truncated range. This comes about when one imposes boundary conditions of the usual Dirichlet or Neumann type. Jacobi’s theta function identity and Poisson’s summation formula are then useless and no expression in terms of a convergent series for the analytical continuation to values of $\Re s$ below the abscissa of convergence can be obtained. The method must use then the zeta function regularization theorem and the best one gets is an asymptotic series. The issue of extending the CS formula, or the most general expression we have obtained before, to this situation is not an easy one (see, however, Ref. [26]). This problem has seldom (if ever) been properly addressed in the literature.

As an example, let us consider the following series in one dimension:

$$
\zeta_G(s; a, b; c) = 2\zeta(2s) a^{-s} + \frac{2^{2s+5/2} \pi^s}{\Gamma(s) \Delta^{s-2-1/4} \sqrt{\Delta}} \sum_{n=1}^{\infty} n^{s-1/2} \sigma_1(n) \cos(\pi nb/a) K_{s-1/2} \left( \frac{\pi n}{a} \sqrt{\Delta} \right).
$$

In this case the Jacobi identity is of no use. The way to proceed is employing specific techniques of analytic continuation of zeta functions. There is no place to describe
them here in detail. The usual method involves three steps \[28\]. The first step is easy: to write the initial series as a Mellin transform \[ \sum_{n=0}^{\infty} [a(n+c)^2 + q]^{-s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \int_0^\infty dt t^{s-1} \exp \{ -[a(n+c)^2 + q] \} \]. The second, to expand in power series part of the exponential, leaving a converging factor: \[ \sum_{n=0}^{\infty} [a(n+c)^2 + q]^{-s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \int_0^\infty dt \sum_{m=0}^{\infty} \frac{(-a)^n \Gamma(m+s)}{m!} (n+c)^{2m} t^{s+m-1} e^{-qt} \]. The third, and most difficult, step is to interchange the order of the two summations — with the aim to obtain a series of zeta functions — what means transforming the second series into an integral along a path on the complex plane, that has to be closed into a circuit (the sum over poles inside reproduces the original series), with a part of it being sent to infinity. Usually, after interchanging the first series and the integral, there is a contribution of this part of the circuit at infinity, what provides in the end an additional contribution to the trivial commutation (given by the zeta function regularization theorem \[28\]). More important, what one obtains in general through this process is not a convergent series of zeta functions, but an asymptotic series (see e.g., the 4th and 6th references in \[10\]).

That is, in our example, \( \sum_{n=0}^{\infty} [a(n+c)^2 + q]^{-s} \sim \sum_{m=0}^{\infty} \frac{(-a)^m \Gamma(m+s)}{m!} \zeta_H(-2m, c) + \text{additional terms} \). Being more precise, as outcome of the whole process we obtain the following result for the analytic continuation of the zeta function \[29\]

\[
\zeta_G, (s; a; c; q) \sim \left( \frac{1}{2} - c \right) q^{-s} + \frac{q^{-s}}{\Gamma(s)} \sum_{m=1}^{\infty} \frac{(-1)^m \Gamma(m+s)}{m!} \left( \frac{q}{a} \right)^{-m} \zeta_H(-2m, c) \tag{21}
\]

\[ + \sqrt{\frac{\pi}{a}} \frac{\Gamma(s-1/2)}{2 \Gamma(s)} a^{1/2-s} + \frac{2\pi^s}{\Gamma(s)} a^{-1/4-s/2} q^{1/4-s/2} \sum_{n=1}^{\infty} n^{s-1/2} \cos(2\pi n c) K_{s-1/2}(2\pi n \sqrt{q/a}). \]

(Note that this expression reduces to Eq. \[10\] in the limit \( c \to 0 \).) The first series on the rhs is asymptotic \[28\]. Observe, on the other hand, the singularity structure of this zeta function. Apart from the pole at \( s = 1/2 \), there is a whole sequence of poles at the negative real axis, for \( s = -1/2, -3/2, \ldots \), with residua: \( \text{Res}_{s=1/2-j} \zeta_G, (s; a; c; q) = (2j-1)! q^j \), \( j = 0, 1, 2, \ldots \). The generalization of this to \( p \) dimensions can be found in Ref. \[29\].

4. Direct physical applications

4.1. Calculation of the Casimir energy density

An application of the procedure is the calculation of the Casimir energy density corresponding to a massless \[31\] or massive \[32\] scalar field on a general, \( d \) dimensional toroidal manifold. In the spacetime \( \mathcal{M} = \mathbb{R} \times \Sigma \), with \( \Sigma = [0, 1]^d / \sim \), which is topologically equivalent to the \( d \) torus, the Casimir energy density for a massive scalar field is given directly by Eq. \[33\] at \( s = -1/2 \), with \( q = m^2 \) (mass of the field), \( \vec{b} = 0 \), and \( A \) being the matrix of the metric on \( \Sigma \), the general \( d \)-torus: \( E^c_{\mathcal{M}, m} = \zeta_G, 0, m^2 (s = -1/2) \). The components of \( g \) are, in fact, the coefficients of the different terms of the Laplacian, which is the relevant operator in the Klein-Gordon
field equation. The massless case is also obtained, with the same specifications, from the corresponding formula Eq. (17). In both cases no extra calculation needs to be done, and the physical results follow from a mere identification of the components of the matrix \( A \) with those of the metric tensor of the manifold in question. Very much related with this application but more involved and ambitious is the calculation of vacuum energy densities corresponding to spherical configurations and the bag model (see [33, 34] and references therein).

4.2. Effective action

Another application consists in calculating the determinant of a differential operator, say the Laplacian on a general \( p \)-dimensional torus. A very important problem related with this issue is that of the multiplicative anomaly discussed before. To this end the derivative of the zeta function at \( s = 0 \) has to be obtained. From Eq. (14), we get

\[
\zeta'_A,\bar{c},q(0) = \frac{4(2q)^{p/4}}{\sqrt{\det A}} \sum_{\bar{m} \in \mathbb{Z}_p^{p/2}} \frac{\cos(2\pi \bar{m} \cdot \bar{c})}{(\bar{m}^T \bar{A}^{-1} \bar{m})^{p/4}} K_{p/2} \left( 2\pi \sqrt{2q \bar{m}^T \bar{A}^{-1} \bar{m}} \right)
\]

\[
+ \begin{cases} 
(2\pi)^{p/2} \Gamma(-p/2)q^{p/2} / \sqrt{\det A}, & p \text{ odd,} \\
\frac{(-1)^k (2\pi)^{k} q^{k}}{k! \sqrt{\det A}} \left[ \Psi(k + 1) + \gamma - \ln q \right], & p = 2k \text{ even,}
\end{cases}
\]

and, from here, \( \det A = \exp -\zeta'_A(0) \). For \( p = 2 \), we have explicitly:

\[
\det A(a, b, c; q) = e^{2\pi(q - \ln q)/\sqrt{\Delta}} \left( 1 - e^{-2\pi \sqrt{q/a}} \right) \exp \left\{ -4 \sum_{n=1}^{\infty} \frac{1}{n} \left[ \sqrt{a} \right] K_1 \left( 4\pi n \sqrt{aq/\Delta} \right) \right\}.
\]

In the homogeneous case (CS formula) we obtain for the determinant:

\[
\det A(a, b, c) = \frac{1}{a} \exp \left[ -4\zeta'(0) - \frac{\pi \sqrt{\Delta}}{6a} - 4 \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} \cos(\pi nb/a) e^{-\pi n \sqrt{\Delta}/a} \right],
\]

or, in terms of the Teichmüller coefficients, \( \tau_1 \) and \( \tau_2 \), of the metric tensor (for the metric, \( A \), corresponding to the general torus in two dimensions):

\[
\det A(\tau_1, \tau_2) = \frac{\tau_2}{4\pi^2 |\tau|^2} \exp \left[ -4\zeta'(0) - \frac{\pi \tau_2}{3|\tau|^2} - 4 \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} \cos \left( \frac{2\pi n \tau_1}{|\tau|^2} \right) e^{-\pi n \tau_2/|\tau|^2} \right].
\]
5. Future Perspectives: Operator Regularization

The Operator Regularization (OR) approach, due originally to D. G. C. McKeon and T. N. Sherry, is considered as a genuine generalization of the zeta regularization approach. Its main aim is to extend zeta regularization, so effective at one-loop order, to higher loops. It has a distinct advantage over other competing procedures, in that it can be used with formally non-renormalizable theories, as shown in [38, 39]. A further feature of this approach is that divergences are not reabsorbed, each one is removed and replaced by an arbitrary factor. Indeed, operator regularization (OR) does not cure the non-predictability problem of non-renormalizability, but an advantage of the method is that the initial Lagrangian does not need to be extended with the addition of extra terms. The OR scheme is governed by the identity:

$$H - m = \lim_{\epsilon \to 0} \frac{d^n}{d\epsilon^n} \left[ 1 + \left( 1 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \ldots + \alpha_n \epsilon^n \right) \right],$$

(26)

where the \(\alpha_i\)'s are arbitrary, and it is enough that the degree of regularization is equal to the loop order, \(n\).

Two separate aspects of the procedure are, first the regularization itself and, second, the analytical continuation, where divergences are replaced by arbitrary factors. Thus, the effect of OR is in the end replace the divergent poles by arbitrary constants, as

$$\frac{1}{\epsilon^n} \to \alpha_n,$$

(27)

to yield the finite expression

$$H^{-m} = \alpha_n c_{-n} + \cdots + \alpha_1 c_{-1} + c_0.$$  

(28)

5.0.1. Generalization and further extensions

The OR method can be generalized to multiple operators, as in multi-loop cases

$$H^{-m_1} \cdots H^{-m_r} = \lim_{\epsilon \to 0} \frac{d^n}{d\epsilon^n} \left[ 1 + \left( 1 + \alpha_1 \epsilon + \alpha_2 \epsilon^2 + \ldots + \alpha_n \epsilon^n \right) \right] \times \frac{\epsilon^n}{n!} H^{-\epsilon-m_1} \cdots H^{-\epsilon-m_r}.$$  

(29)

Further extensions of the procedure have been proposed. Let us recall that OR was first introduced in the context of the Schwinger approach,

$$\ln H = - \lim_{\epsilon \to 0} \frac{d^n}{d\epsilon^n} \left( \frac{\epsilon^{n-1}}{n!} H^{-\epsilon} \right),$$

(30)

which is known to be equivalent to the Feynman one

$$H^{-m} = \lim_{\epsilon \to 0} \frac{d^n}{d\epsilon^n} \left( \frac{\epsilon^n}{n!} H^{-\epsilon-m} \right).$$

(31)
The Schwinger form can be transformed into the Feynman one, as

\[ H^{-m} = \frac{(-1)^{m-1}}{(m-1)!} \frac{d^m}{dH^m} \ln H \]  

Equivalence with dimensional regularization can be established in many cases, but not always. Problems, the main one being unitarity, may appear (see [40]). To start with, its naive application to obtain finite amplitudes breaks unitarity.

A definite advantage of the procedure is that, actually, no symmetry-breaking regulating parameter is ever inserted into the initial Lagrangian. One can use Bogoliubov’s recursion formula in order to show how to construct a consistent OR operator, and unitarity is upheld by employing a generalized evaluator consistently including lower-order quantum corrections to the quantities of interest. Unitarity requirements lead to unique expressions for quantum field theoretic quantities, order by order in \( \hbar \). This fact has been proven in many cases (as for the \( \Phi^4 \) theory at two-loop order, etc.). But I should say that, to my knowledge, a universal proof of this issue is actually still missing.

A final comment is in order. Using a BPHZ-like scheme, as the above one turns out to be, in the end, essentially reintroduces counterterms into the procedure, since they are actually hidden in the subtractions taking place at each step. In this way, the simplicity of the original zeta function regularization procedure, as described in the previous sections, and which is one of its main characteristics, is absent in the extended, operator regularization method.

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