Heisenberg scaling in relativistic quantum metrology

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We address the issue of precisely estimating small parameters encoded in a general linear transformation of the modes of a bosonic quantum field. Such Bogoliubov transformations frequently appear in the context of quantum optics. We provide a recipe for computing the quantum Fisher information for arbitrary pure initial states. We show that the maximally achievable precision of estimation is inversely proportional to the squared average particle number, and that such Heisenberg scaling requires non-classical, but not necessarily entangled states. Our method further allows to quantify losses in precision arising from being able to monitor only finitely many modes, for which we identify a lower bound.

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Introduction—Quantum metrology exploits distinctive quantum features, such as entanglement, to enhance the estimation precision of parameters governing the dynamical evolution of the probe systems beyond that achievable by classical means. This enhancement is manifested in the form of a scaling gap in precision with respect to the available resources (the number of probe systems or the average input energy) between the corresponding optimal quantum and classical strategies, and depends on the specific encoding of the parameter in the Hamiltonian describing the evolution. In the case where the parameter of interest is a multiplicative factor of a local Hamiltonian acting on N probes the optimal quantum strategy provides a quadratic scaling gap in N, known as the Heisenberg limit, over the best classical strategy [1, 2]. This quadratic improvement is also present for a class of quasi-local Hamiltonians [3, 4], whereas for Hamiltonians involving highly non-local interactions, even a super-Heisenberg scaling gap is possible [5–7].

Here, we are interested in determining the ultimate precision limits for a different type of parameter estimation task, namely where the parameter of interest is encoded non-trivially in the operator that describes the transformation of the (infinitely many) modes of a quantum field. This problem is of particular relevance in a large variety of physical systems from quantum optics [8], and condensed matter [9], to the description of quantum effects in curved spacetime [10], such as the Unruh effect [11] and the dynamical Casimir effect [12], or analogue realizations thereof [13–18]. Consequently, parameter estimation in relativistic scenarios has received much attention in recent years [19–26]. A paradigm example that highlights the challenges encountered in the context of relativistic quantum effects is the estimation of the acceleration of a non-uniformly moving cavity [22, 23]. While the parameter in question is small, the transformation couples all pairs of modes in nontrivial ways [27, 28]. Due to the resulting notoriously cumbersome perturbative calculations, the only known bounds on precision involve Gaussian input states. As a result the ultimate limits on how precisely one can determine small accelerations, as well as whether distinctive quantum features provide an improvement, are not known.

Here, we show that within such a perturbative approach, there exists a quadratic scaling gap between the optimal quantum and classical strategies for a fixed average input energy. We construct an optimal estimation strategy utilizing separable input states of fixed particle number and boson counting measurements, where we use the quantum Fisher information (QFI) as a figure of merit for the estimation precision. We provide simple formulas for the leading order contributions to the QFI for arbitrary pure states, both when all modes can be controlled and when only part of the spectrum is accessible, and show that, within the considered perturbative regime, Heisenberg scaling is the ultimate precision limit. Moreover, we identify the family of states that exhibit Heisenberg scaling and show that, while some of these states may be entangled, the crucial feature is their non-classicality rather than their correlations. Furthermore, we lower bound the loss in precision due to tracing out inaccessible modes and provide criteria for minimizing such losses.

Framework—In its most general form, our metrological protocol can be described as follows: We wish to estimate as precisely as possible a single real parameter, \( \theta \), that is encoded in a unitary transformation, \( U(\theta) \), acting on the Fock space of an arbitrary number of bosonic modes. Our probe system is a set of (in principle infinitely many) non-interacting harmonic oscillators with creation and annihilation operators \( a_\alpha^\dagger \) and \( a_\alpha \), respectively, where \( n = 1, 2, \ldots \) can be a multi-label distinguishing frequencies, polarizations, or other degrees of freedom. The operators satisfy the usual commutation relations

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relations \([a_m, a_n^\dagger] = \delta_{mn}\) and \([a_m, a_n] = 0\), and the ground state is annihilated by all \(a_n\), i.e., \(a_n|0\rangle = 0\) \(\forall n\).

Arbitrary pure states \(|\psi\rangle\) can be decomposed into superpositions of Fock states, e.g., \(|m_{k_1}\rangle|n_{k_2}\rangle\ldots|p_{k_N}\rangle\), with fixed numbers of excitations. These Fock states form a basis of the total Hilbert space and can be obtained from the vacuum by applying the appropriate creation operators, i.e., \(|n_k\rangle = (\hat{a}_k^\dagger)^n/\sqrt{n!}|0\rangle\).

For a given input state \(\rho\) and a suitable measurement whose outcomes are used to estimate \(\theta\), the precision of the estimation of \(\theta\), quantified by the variance \(\Delta \theta\) of the corresponding (unbiased) estimator, is lower-bounded by the inverse of the (quantum) Fisher information (QFI), \(\mathcal{I}(\rho(\theta))\) [29], via the (quantum) Cramér-Rao inequality [30, 31] \(\Delta \theta \geq 1/\sqrt{\mathcal{F}(\rho(\theta))}\), where \(\nu\) is the number of repetitions. Intuitively, the QFI quantifies how well small changes of the parameter in question may be inferred from measurements of the final state \(\rho(\theta)\) after the dynamical evolution of the probes. One may express \(\mathcal{I}(\rho(\theta))\) as

\[
\mathcal{I}(\rho(\theta)) = \lim_{d\theta \to 0} \frac{1}{8} \left( 1 - \sqrt{\mathcal{F}(\rho(\theta), \rho(\theta + d\theta))} \right),
\]

where \(\mathcal{F}(\rho(\theta), \rho(\theta + d\theta)) = (\text{Tr}\sqrt{\rho(\theta)\rho(\theta + d\theta)\sqrt{\rho(\theta)}})^2\) is the Uhlmann fidelity between the states \(\rho(\theta)\) and \(\rho(\theta + d\theta)\). If \(\rho(\theta) = |\psi(\theta)\rangle\langle \psi(\theta)|\) one simply recovers \(\mathcal{F}(\rho(\theta), \rho(\theta + d\theta)) = |\langle \psi(\theta)| \psi(\theta + d\theta)\rangle|^2\). The latter expression applies for the case where all modes can be controlled and measured. Expanding \(|\psi(\theta + d\theta)\rangle\) in powers of \(d\theta\) one obtains

\[
\mathcal{I}(\psi(\theta)) = 4 \left( |\langle \psi(\theta)| \dot{\psi}(\theta)\rangle - |\langle \dot{\psi}(\theta)| \psi(\theta)\rangle|^2 \right),
\]

where \(|\dot{\psi}(\theta)\rangle = \frac{\partial}{\partial \theta} |\psi\rangle\rangle\). When some of the modes are not accessible and are traced out, Eq. (2) provides an upper bound on the QFI. To allow for more specific statements about the QFI, additional information about the initial states or transformations encoding \(\theta\) are required.

We now consider cases where \(\theta\) is very small and close to zero. For this case the transformation \(U(\theta)\) is amenable to a perturbative approach and has a Taylor-Maclaurin expansion of the form

\[
U(\theta) = U^{(0)} + \theta U^{(1)} + \theta^2 U^{(2)} + \mathcal{O}(\theta^3),
\]

where \(U^{(0)}\) is a unitary that encodes the free time evolution of the non-interacting bosonic modes, while for \(i > 0\) the \(U^{(i)}\) represent (non-unitary) perturbations. Consequently the final state can be written as

\[
|\tilde{\psi}\rangle = |\tilde{\psi}^{(0)}\rangle + \theta |\tilde{\psi}^{(1)}\rangle + \theta^2 |\tilde{\psi}^{(2)}\rangle + \mathcal{O}(\theta^3),
\]

where \(|\tilde{\psi}^{(i)}\rangle = U^{(i)}|\psi\rangle\) \((i = 0, 1, 2, \ldots)\). Note that \(|\tilde{\psi}\rangle\) and \(|\tilde{\psi}^{(0)}\rangle\) are normalized, but the vectors \(|\tilde{\psi}^{(i)}\rangle\) \((i > 0)\) generally are not. Substituting (4) into the expression for the QFI in (2) we arrive at

\[
\mathcal{I}(\tilde{\psi}) = 4 \left| \langle \tilde{\psi}^{(1)}| \tilde{\psi}^{(0)}\rangle \right|^2 + \mathcal{O}(\theta). \tag{5}
\]

Hence, given that all the bosonic modes are accessible, the QFI can be straightforwardly computed for any initial (pure) state from just its linear perturbations.

As all our results utilize the perturbative approach, a few important remarks concerning the applicability of the perturbative approach are in order. In the latter we assume that all higher order terms in \(\theta\) are small, and are therefore neglected. To ensure such reasoning is justified, we regard our approach as valid so long as the perturbation to the state \(|\psi^{(0)}\rangle\) remains small, such that \(1 - \mathcal{F}(|\psi^{(0)}\rangle, |\psi\rangle) \ll 1\). This implies that \(\theta^2 \mathcal{I}(\psi)/4 \ll 1\), and all optimizations will be performed with this constraint in mind.

Results—Thus far we have assumed that all bosonic modes are accessible and can be controlled. This is an unrealistic assumption in practice, as only a finite subset \(\mathcal{N} = \{k_1, k_2, \ldots, k_N\}\) of the modes can be addressed simultaneously. Moreover, as the estimation strategy is to be optimized at a fixed investment of energy, it appears unwise to initially populate modes in the complementary subset \(-\mathcal{N}\), which cannot be measured. We therefore assume that the preparation of non-trivial initial states is also limited to \(\mathcal{N}\). Hence, the input states that we consider are of the form \(|\psi\rangle = |\psi\rangle_{\mathcal{N}}|0\rangle_{-\mathcal{N}}\). The unitary \(U(\theta)\), on the other hand, acts on all modes such that the final state is \(|\tilde{\psi}\rangle = U(\theta)|\psi\rangle\), but only the reduced state \(\tilde{\rho}_k = \text{Tr}_{-\mathcal{N}} |\tilde{\psi}\rangle\langle \tilde{\psi}|\) is accessible for the estimation of \(\theta\), see Fig. 1. Thus, Eq. (5) provides an upper bound, i.e., \(\mathcal{I}(\tilde{\rho}_k(\theta)) \leq \mathcal{I}(|\psi\rangle)\) for the precision with which \(\theta\) can be estimated. However, a precise expression for the losses incurred by tracing can be established as we will show now.

Now, let us expand the reduced state, \(\tilde{\rho}_k(\theta)\) in powers of \(\theta\),

\[
\tilde{\rho}_k(\theta) = \tilde{\rho}_k^{(0)} + \theta \tilde{\rho}_k^{(1)} + \theta^2 \tilde{\rho}_k^{(2)} + \mathcal{O}(\theta^3), \tag{6}
\]

where the leading order is \(\tilde{\rho}_k^{(0)} = \text{Tr}_{-\mathcal{N}} (|\psi^{(0)}\rangle\langle \psi^{(0)}|)\), and the corrections are given by

\[
\tilde{\rho}_k^{(1)} = \text{Tr}_{-\mathcal{N}} (|\tilde{\psi}^{(0)}\rangle\langle \tilde{\psi}^{(0)}| + |\tilde{\psi}^{(1)}\rangle\langle \tilde{\psi}^{(1)}|) \tag{7}
\]

\[
\tilde{\rho}_k^{(2)} = \text{Tr}_{-\mathcal{N}} (|\tilde{\psi}^{(0)}\rangle\langle \tilde{\psi}^{(2)}| + |\tilde{\psi}^{(1)}\rangle\langle \tilde{\psi}^{(1)}| + |\tilde{\psi}^{(1)}\rangle\langle \tilde{\psi}^{(1)}|). \tag{8}
\]
As the modes are non-interacting, the free evolution $U^{(0)}$ is a local operation that leaves the vacuum invariant, and we may write $|\tilde{\psi}(0)\rangle = U^{(0)}|\psi\rangle_k |0\rangle_{-k} = |\tilde{\psi}(0)\rangle_k |0\rangle_{-k}$. It then easily follows that $\tilde{\rho}_k^{(0)} = |\tilde{\psi}(0)\rangle_k \langle \tilde{\psi}(0)|$ is a pure state. In a similar way we may expand $\tilde{\rho}_k(\theta + d\theta)$ as

$$\tilde{\rho}_k(\theta + d\theta) = \tilde{\rho}_k(\theta) + d\theta \frac{\partial \tilde{\rho}_k(\theta)}{\partial \theta} + \frac{d\theta^2}{2} \frac{\partial^2 \tilde{\rho}_k(\theta)}{\partial \theta^2} + O(d\theta^3) = \tilde{\rho}_k^{(0)} + d\theta \tilde{\rho}_k^{(1)} + d\theta^2 \tilde{\rho}_k^{(2)} + O(\theta, d\theta^3).$$

Inserting (6) and (9) into Eq. (1), and taking into account the unitarity of the transformation, we arrive at the central result

$$\mathcal{I}(\tilde{\rho}_k(\theta)) = \mathcal{I}(\tilde{\psi}) - 4 \sum_{i\neq 0} |k\langle \tilde{\psi}(0)|\tilde{\psi}(0)\rangle_k - k|\tilde{i\langle U^{(1)}(i)\rangle|\tilde{\psi}|^2 + O(\theta) = \mathcal{I}(\tilde{\psi}) - \Delta_{tr}(|\tilde{\psi}\rangle_k) + O(\theta),$$

where $\mathcal{I}(\tilde{\psi})$ is given as in Eq. (5) and the sum runs over all basis states of the Fock space of $-k$ that are orthogonal to the vacuum state $|0\rangle_{-k}$. Note that Eq. (11) suggests that in order to compute the QFI to leading order in $\theta$, one requires only leading order corrections to the initial state, whereas Eq. (10) above implied that second order corrections are needed. However, as we show in detail in the Appendix, the requirement that the transformation is unitary implies that the second order correction in Eq. (10) can be expressed in terms of the linear order corrections to the state.

For the remainder of this paper, we shall work with Eq. (11) to investigate how tracing losses can be minimized, and optimal scaling can be achieved. For this task, we will specialize our discussion to Bogoliubov transformations (see, e.g., Ref. [32] for a pedagogical introduction), that is, unitary transformations on the Fock space that can be viewed as linear transformations of the mode operators, i.e.,

$$a_m \mapsto \tilde{a}_m = \sum_n (\alpha_{mn}^* a_n - \beta_{mn}^* a_n^\dagger),$$

where unitarity places restrictions on the complex coefficients $\alpha_{mn}$ and $\beta_{mn}$. Another way to view such transformations is as unitaries that are generated by quadratic combinations of the mode operators. In the perturbative regime that we consider here, the Bogoliubov coefficients are expanded as

$$\alpha_{mn} = \alpha_{mn}^{(0)} + \alpha_{mn}^{(1)} \theta + \alpha_{mn}^{(2)} \theta^2 + O(\theta^3),$$

$$\beta_{mn} = \beta_{mn}^{(1)} \theta + \beta_{mn}^{(2)} \theta^2 + O(\theta^3),$$

where $\alpha_{mn}^{(0)} = \delta_{mn} G_n = \delta_{mn} \exp(i \phi_n)$ encodes the phases $\phi_n$ that are accumulated due to the free time evolution $U^{(0)}$. The coefficients $\alpha_{mn}$ and $\beta_{mn}$ are responsible for the shifts of single excitations between modes $m$ and $n$, or the creation and annihilation of pairs of particles in these modes, respectively. Any initial state of the Fock space may simply be written in terms of linear combinations of products of creation operators acting on the vacuum. The operators transform according to Eq. (12), whereas the vacuum transforms as

$$|0\rangle \mapsto |0\rangle - \theta \sum_{p,q} G_{q,p}^{* \beta_{pq}^{(1)}} a_p^\dagger a_q^\dagger |0\rangle + O(\theta^2),$$

where $G_{q,p}^{* \beta_{pq}^{(1)}} = G_{p,q}^{* \beta_{pq}^{(1)}}$ due to unitarity (see, e.g., Ref. [32, pp. 93]). We now determine the tracing losses $\Delta_{tr}(|\tilde{\psi}\rangle_k)$ and the scaling of $\mathcal{I}(\tilde{\psi})$ with the average energy of the initial state in the above scenario.

The linear order operation $U^{(1)}$ can be (uniquely) separated into (all) terms that leave the particle content of $k$ or $-k$ invariant, and a term that correlates both sets via the creation or exchange of excitations, i.e.,

$$U^{(1)} = U^{(1)}_k \otimes U^{(0)}_{-k} + U^{(0)}_k \otimes U^{(1)}_{-k} + U^{(1)}_{k,-k}.$$ 

From Eq. (11) it is obvious that $U^{(1)}_k \otimes U^{(0)}_{-k}$ does not contribute to $\Delta_{tr}(|\tilde{\psi}\rangle_k)$. The other two terms generally provide nonzero contributions. However, to linear order, the operation $U^{(1)}_{k,-k}$ populates $|0\rangle_{-k}$ with pairs of particles, whereas $U^{(1)}_k$ may either create one particle each in $k$ and $-k$, or shift one excitation from $k$ to $-k$. As a result, for the same fixed state $|i\rangle_{-k}$, which contains at most 2 excitations, $\langle -k| i\langle U^{(1)}_k| 0\rangle_{-k}$ and $\langle k| \tilde{\psi}(0)\rangle_{-k} |i\langle U^{(1)}_{-k}| \tilde{\psi}|^2$ cannot both be nonzero, and hence contribute separately to the tracing loss. The latter term can be avoided by choosing $|\tilde{\psi}\rangle_k$ such that superpositions of states that differ only by one excitation are excluded. The former term, on the other hand, cannot be avoided as it is independent of the state $|\tilde{\psi}\rangle_k$. It can, however, straightforwardly be computed using the vacuum state of Eq. (15), for which $k|0\rangle_{-k} |i\langle U^{(1)}_{-k}| 0\rangle_{-k} = 0$. For this case we find (see Appendix)

$$\Delta_{tr}(|\tilde{\psi}\rangle_k) \geq \Delta_{tr}(|0\rangle_k) = 2 \sum_{p,q \neq k} |\beta_{pq}^{(1)}|^2.$$ 

Thus, by choosing our initial state $|\tilde{\psi}\rangle_k$ such that superpositions of states that differ only by one excitation are excluded, the inequality in Eq. (17) becomes a strict equality and the QFI can be precisely determined to linear order in a straightforward way using Eq. (11). We shall restrict our discussion to such states from now on.

Optimality—We now show that the optimally achievable precision in this setup is Heisenberg scaling, and how it can be realized even with non-entangled states. To determine the optimal states, the exact values of the Bogoliubov coefficients $\alpha_{mn}$ and $\beta_{mn}$ have to be known. This makes the procedure highly task-specific. Nonetheless, it is possible to identify families of states that scale
optimally with the input energy under some minimal assumptions about the Bogoliubov coefficients. First, consider a single mode Fock state with $n$ excitations. Using Eqs. (12)-(15) this state transforms to

$$|n_k\rangle \rightarrow G_k^n |n_k\rangle - \theta G_k^n \left( \frac{1}{2} \sqrt{n(n-1)} G_k \beta_{kk}^{(1)} |n-2k\rangle - n G_k \alpha_{kk}^{(1)} |n_k\rangle \right)$$
$$- \sum_{p \neq k} \left( \sqrt{n} G_k \alpha_{kp}^{(1)} |n_k\rangle - \frac{1}{2} \sqrt{(n+1)(n+2)} G_k \beta_{kk}^{(1)} |n+2k\rangle - \sqrt{n+1} G_k \beta_{kk}^{(1)*} |n+1k\rangle |1_p\rangle \right)$$
$$+ \frac{1}{2} \sum_{p,q \neq k} G_k^* \beta_{kk}^{(1)*} |n_k\rangle |1_p\rangle |1_q\rangle + \frac{1}{2} \sum_{p \neq k} G_p^* \beta_{pp}^{(1)*} |n_k\rangle |2_p\rangle + \mathcal{O}(\theta^2). \quad (18)$$

As $|\alpha_{kp}^{(1)}| = |\alpha_{pk}^{(1)}|$, one immediately obtains the QFI

$$\mathcal{I}(|n_k\rangle) = 2n(n+1)|\beta_{kk}^{(1)}|^2 + 4n \sum_{p \neq k} (|\alpha_{pk}^{(1)}|^2 + |\beta_{pk}^{(1)}|^2)$$
$$+ \mathcal{I}(|0\rangle) + \mathcal{O}(\theta). \quad (19)$$

Notice that, here, the coefficient $\alpha_{kk}^{(1)}$, which leaves the occupation number of the mode $k$ unchanged, does not contribute to the QFI, but it may do so for superpositions of different particle numbers. However, when $\beta_{kk}^{(1)} \neq 0$, which corresponds to single-mode squeezing transformations, terms that shift the population of the mode $k$ by two excitations yield a QFI that scales quadratically with $n$, i.e., Heisenberg scaling. On the other hand, classical states, which in the present context are all coherent states, yield a QFI that scales linearly with the average particle number [23].

When the diagonal linear coefficients $\beta_{kk}^{(1)}$ vanish, as in the transformation of field modes of non-uniformly accelerating rigid cavities [27, 28], excitations in a single mode are not sufficient to obtain Heisenberg scaling, and one requires initial states with at least two occupied modes. For instance, when the modes $k$ and $k'$ can be controlled and $\beta_{kk}^{(1)} = \beta_{kk}^{(1)*} = 0$, the QFI for the state $|n_k\rangle|m_{k'}\rangle$ is (see Appendix)

$$\mathcal{I}(|n_k\rangle|m_{k'}\rangle) = 8mn(|\alpha_{kk}^{(1)}|^2 + |\beta_{kk}^{(1)}|^2) + 4n \sum_{p \neq k} (|\alpha_{pk}^{(1)}|^2 + |\beta_{pk}^{(1)}|^2) + 4m \sum_{p \neq k'} (|\alpha_{pk'}^{(1)}|^2 + |\beta_{pk'}^{(1)}|^2) + \mathcal{I}(|0\rangle) + \mathcal{O}(\theta). \quad (20)$$

The factor $mn$ allows the QFI to scale optimally with the average number of excitations. Indeed, inspection of the first order expansion of $|n_k\rangle|m_{k'}\rangle$ reveals that Heisenberg scaling is the ultimate achievable precision scaling within our approach (see Appendix).

**Discussion**—Whereas the scaling with respect to the average energy cannot be better than quadratic, one can still improve the constant pre-factor in this scaling. For example, assuming $\beta_{kk}^{(1)} = 0$, the particle numbers $n$ and $n \pm 2$ in the state $(|n_k\rangle|n_{k'}\rangle + |n_k\rangle|n_{k'}-2\rangle + |n_k\rangle|n_{k'}\rangle)/\sqrt{3}$, guarantee additional optimally scaling terms in the QFI, whilst avoiding losses originating from $\langle |\psi^{(0)}\rangle |\psi^{(1)}\rangle \rangle^2$. The latter is nonzero whenever $|\psi\rangle$ contains superpositions of Fock states that can be converted into each other by changes of at most two excitations, such as $|n_k\rangle|n+1k\rangle$ and $|n+1k\rangle|n_k\rangle$, or $|n_k\rangle|n_k\rangle$ and $|n+1k\rangle|n+1k\rangle$.

Notice that in all of the examples considered so far the optimal states are not entangled, but can be regarded as highly squeezed, and hence non-classical. The final state may become entangled due to the transformation, but, as seen from (18) and (19), optimal scaling may arise from terms (here $\beta_{kk}^{(1)} |n-2k\rangle$) that do not produce entanglement. Indeed, entanglement of the initial state is not necessary for optimality (a feature that was also noted in a related context in [33]) although one may find entangled states that do admit Heisenberg scaling in precision such as the state $(|n+1k\rangle|n-1k\rangle + |n-1k\rangle|n+1k\rangle)/\sqrt{2}$ which also has minimal tracing loss. For the entangled state above homodyne measurements would be required for optimal scaling, whereas the measurements for the previous examples need to be number resolving. The choice of input state and measurement is thus also a matter of what is practical in a given experimental set-up.

**Conclusion**—We have shown that the ultimate limits for estimating small parameters encoded in the linear transformations of the mode operators describing a set of non-interacting harmonic oscillators scales inversely proportional to the average input energy of the modes. Using a perturbative approach for these Bogoliubov transformation, applicable when the parameter of interest is small, e.g., as in the case of non-uniformly moving cavities, we have derived analytical formulas for the QFI, which apply to initial overall pure states of all modes. We have provided a lower bound on the losses when some of these modes cannot be controlled. Finally, we have identified the class of states that yield optimally scaling precision, whilst exhibiting minimal tracing losses. This provides a significant advancement beyond previous analysis of this problem [23], which was limited to pure Gaussian states. An investigation of mixed initial states, which will certainly be of future interest, goes beyond the scope of this paper, but is currently being undertaken [34].

Our results open up the possibility to explore optimality in a range of specific applications, where information about the Bogoliubov coefficients is available. Examples include (but are not limited to) field modes in non-uniformly moving cavities [27, 28], analogue gravity phenomena [13], and effects in superconducting materials [14, 16, 35].

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Appendix: Explicit calculations

In this appendix, we will present explicit proofs for the result of Eqs. (10), (11) and (20), and discuss the optimality of Heisenberg scaling. We start with Eq. (10), where we first consider the Uhlmann fidelity $\mathcal{F}(\hat{\rho}_k(\theta), \hat{\rho}_k(\theta + d\theta))$. Using Eqs. (6) and (9) we can write

$$\sqrt{\hat{\rho}_k(\theta)} \hat{\rho}_k(\theta + d\theta) \sqrt{\hat{\rho}_k(\theta)} = \sqrt{\hat{\rho}_k(\theta)} \hat{\rho}_k(\theta + d\theta^2) \sqrt{\hat{\rho}_k(\theta)} + O(\theta, d\theta^3).$$

Next, we shall give a detailed derivation of Eq. (11).

We have used the fact that $\rho_k = |\psi^{(0)}\rangle_k k |\psi^{(0)}\rangle_k$ is pure. Now, we can take a closer look at the term linear in $d\theta$, i.e.,

$$k |\psi^{(0)}\rangle_k k |\psi^{(0)}\rangle_k = k |\psi^{(0)}\rangle_k Tr_{\sim k} (|\psi^{(1)}\rangle \langle \psi^{(1)}| + H. c.) |\psi^{(0)}\rangle_k,$$

which correlates them, as shown in Eq. (16).

For the first term we can use the unitarity of the transformation from Eq. (A.3), such that

$$\langle \psi | U^{(2)}U^{(0)} + U^{(0)}U^{(2)} | \psi \rangle = - \langle \psi | U^{(1)}U^{(1)} | \psi \rangle = - \langle \psi^{(1)} | \psi^{(1)} \rangle. \tag{A.7}$$

In the remaining part, the vacuum state $|0\rangle_k$ may be extracted from the sum, such that

$$\sum_i |k \langle \psi^{(0)} | - k |\psi^{(0)}\rangle |^2 = |k \langle \psi^{(0)} | - k |\psi^{(0)}\rangle |^2 + \sum_{i \neq 0} |k \langle \psi^{(0)} | - k |\psi^{(0)}\rangle |^2 = |\langle \psi^{(0)} | \psi^{(1)} \rangle|^2 + \sum_{i \neq 0} |k \langle \psi^{(0)} | - k |\psi^{(0)}\rangle |^2. \tag{A.8}$$

Combining (A.6)-(A.8) and comparison with Eq. (5), we then immediately arrive at the result of Eq. (11).

To evaluate the tracing loss for the vacuum, for which this loss is minimal, note that the correction $U^{(1)}$ can be split up into terms $U_k^{(1)}$ and $U_{\sim k}^{(1)}$, which act trivially on the subspaces $k$ or $\sim k$, respectively, and the term $U_{k \sim k}^{(1)}$ which correlates them, as shown in Eq. (16).

For the initial vacuum state, $|\psi \rangle_k = |0\rangle_k$, only the term $U_k^{(2)} \otimes U_{\sim k}$ contributes to the tracing loss, which we identify from the vacuum transformation as

$$U_k^{(2)} \otimes U_{\sim k} = - \frac{1}{2} \sum_{p,q \neq k} G^\beta_{pq} \rho_k \alpha_p^\dagger \alpha_q \tag{A.9}$$

With this, we may write the tracing loss for the initial vacuum state as

$$\Delta_{tr}(|0\rangle_k) = 4 \sum_{i \neq 0} |k \langle i | U_k^{(1)} \otimes U_{\sim k} | 0 \rangle_k|^2,$$

where we note that, as opposed to the sum over $i$, the sum over all $p$ and $q$ counts every basis vector twice, and a factor of $\frac{1}{2}$ therefore has to be included in the conversion. Since unitarity requires $G^\beta_{pq} \rho_k = G^\beta_{pq} \rho_{k'}$ (see Ref. [32, pp. 93]) and $\langle 0 | \rho_k \alpha_p \alpha_q \alpha_p^\dagger \alpha_q^\dagger | 0 \rangle_k = \delta_{pp'} \delta_{qq'} + \delta_{pq} \delta_{pp'}$, we arrive at

$$\Delta_{tr}(|0\rangle_k) = 2 \sum_{p,q \neq k} |G^\beta_{pq}|^2, \tag{A.11}$$

the lower bound on the tracing loss from Eq. (17).

We now give the explicit expression for the state transformation of $|n_k\rangle_m \rho_k$ and use the result to calculate the
corresponding QFI. From Eqs. (12)-(15) we can write
\[ |n_k\rangle|m_{k'}\rangle \rightarrow G^m_k G^m_{k'} |n_k\rangle|m_{k'}\rangle + \theta G^m_k G^m_{k'} \]
\[ + n G^m_k G^{(i)}_{k'k} |n_k\rangle|m_{k'}\rangle - \frac{1}{2} \sqrt{n(n-1)} G^m_k G^{(i)}_{k'k} |n-2k\rangle|m_{k'}\rangle \]
\[ - \frac{1}{2} \sqrt{(n+1)(n+2)} G^\ast_k \beta^{(i)}_{k'k} |n+2k\rangle|m_{k'}\rangle \]
\[ + m G^\ast_{k'} \alpha^{(i)}_{k'k} |n_k\rangle|m_{k'}\rangle - \frac{1}{2} \sqrt{m(m-1)} G^m_{k'} G^{(i)}_{k'k} |n_k\rangle|m-2k\rangle \]
\[ - \frac{1}{2} \sqrt{(m+1)(m+2)} G^m_{k'} \beta^{(i)}_{k'k} |n_k\rangle|m+2k\rangle \]
\[ + \sqrt{m(n+1)} G^m_{k'} \beta^{(i)}_{k'k} |n+1k\rangle|m-1k\rangle \]
\[ + \sqrt{n(m+1)} G^m_k \alpha^{(i)}_{k'k} |n+1k\rangle|m+k\rangle \]
\[ - \sqrt{mn} G^m_k \beta^{(i)}_{k'k} |n-1k\rangle|m+1k\rangle \]
\[ - \frac{1}{2} \sum_{p\neq k,k'} \left( \sqrt{n} G^m_k \alpha^{(i)}_{kp} |n-1k\rangle|m_{k'}\rangle |1_p\rangle \right) \]
\[ + \frac{1}{2} \sum_{p\neq k,k'} G^m_k \beta^{(i)}_{pq} |n_k\rangle|m_{k'}\rangle |1_p\rangle |1_q\rangle \]
\[ \frac{1}{2} \sum_{p\neq k,k'} G^m_{k'} \beta^{(i)}_{pp} |n_k\rangle|m_{k'}\rangle |2_p\rangle \]
\[ + \mathcal{O}(\theta^2), \quad (A.12) \]

where \(G^m_r \alpha^{(i)}_{mn} + G^m_r \alpha^{(i)}_{nm} = 0\), and \(G^m_m \beta^{(i)}_{mn} = G^m_n \beta^{(i)}_{nm}\) due to unitarity. For the initial state \(|\psi\rangle = |n_k\rangle|m_{k'}\rangle\) we find
\[ \langle \tilde{\psi}^{(i)} | \tilde{\psi}^{(i)} \rangle = \frac{1}{2} n(n+1) |\beta^{(i)}_{kk}'|^2 + \frac{1}{2} m(m+1) |\beta^{(i)}_{kk}'|^2 \]
\[ + n^2 |\alpha^{(i)}_{kk}'|^2 + m^2 |\alpha^{(i)}_{kk}'|^2 + 2mn (|\alpha^{(i)}_{kk}'|^2 + |\beta^{(i)}_{kk}'|^2) \]
\[ + m \sum_{p\neq k} (|\alpha^{(i)}_{pk}'|^2 + |\beta^{(i)}_{pk}'|^2) + n \sum_{p\neq k} (|\alpha^{(i)}_{kp}'|^2 + |\beta^{(i)}_{kp}'|^2) \]
\[ + \frac{1}{2} \sum_{p,q} |\beta^{(i)}_{pq}'|^2, \quad (A.13) \]

and
\[ \langle \tilde{\psi}^{(0)} | \tilde{\psi}^{(i)} \rangle = n^2 |\alpha^{(i)}_{kk}'|^2 + m^2 |\alpha^{(i)}_{kk}'|^2. \quad (A.14) \]

We hence obtain the QFI for the state \(|n_k\rangle|m_{k'}\rangle\) to be
\[ \mathcal{I}(|n_k\rangle|m_{k'}\rangle) = 2m(n+1) |\beta^{(i)}_{kk}'|^2 + 2m(m+1) |\beta^{(i)}_{kk}'|^2 \]
\[ + 8mn (|\alpha^{(i)}_{kk}'|^2 + |\beta^{(i)}_{kk}'|^2) + 4n \sum_{p\neq k} (|\alpha^{(i)}_{pk}'|^2 + |\beta^{(i)}_{pk}'|^2) \]
\[ + 4m \sum_{p\neq k'} (|\alpha^{(i)}_{pk}'|^2 + |\beta^{(i)}_{pk}'|^2) + 2 \sum_{p,q} |\beta^{(i)}_{pq}'|^2 + \mathcal{O}(\theta), \quad (A.15) \]

which yields the expression in (20) for \(\beta^{(i)}_{kk} = \beta^{(i)}_{k'k'} = 0\).

**Optimal scaling.** Finally, we show that, within the perturbative approach we pursue here, no better scaling of the QFI is possible. To see this, note that Heisenberg scaling arises from the linear order correction terms in Eq. (A.12) that are proportional to square roots of, at most, quadratic combinations of \(m\) and \(n\). Each of these terms either corresponds to a shift of excitations between the modes \(k\) and \(k'\), indicated by coefficients \(\alpha^{(i)}_{kk'}\), or the creation (annihilation) of a pair of particles in these modes, corresponding to the coefficients \(\beta^{(i)}_{kk'}, \beta^{(i)}_{kk'}\) and \(\beta^{(i)}_{kk'}\). In other words, terms in \(|\tilde{\psi}^{(i)}\rangle\) that scale as \(n\) are the result of changing the occupation number of two highly excited modes by one excitation each, or the occupation of one such mode twice. To linear order in \(\theta\), no more than these two excitations may be changed. No initial state may hence have a linear correction \(|\tilde{\psi}^{(i)}\rangle\) that grows faster than linearly in \(n\).

This still leaves the option to consider superpositions of states with different occupation numbers, such that the average number of excitations remains fixed, for instance, the state \(\langle n_k\rangle|n_{k'}\rangle + |n_{k}\rangle|n-2k\rangle + |n_k\rangle|n+2k\rangle\)/\(\sqrt{3}\). The cross terms appearing in \(\langle \tilde{\psi}^{(i)} | \tilde{\psi}^{(i)} \rangle\) may increase (or decrease) the overall QFI. However, to linear order in \(\theta\) the corrections to every fixed-excitation state in the superposition are non-orthogonal only to the linear corrections to states that differ by no more than two excitations. For example, the linear corrections to both \(|n_k\rangle|n_{k'}\rangle\) and \(|n_k\rangle|n-2k\rangle\) contain a term proportional to \(|n-1k\rangle|n-1k\rangle\), whereas the linear correction to \(|n_k\rangle|n+2k\rangle\) cannot contain such a term, as can easily be verified from Eq. (A.12). The possible gain in the number of such cross-terms that is obtained from adding more terms to the superposition is therefore at most linear in the number of terms, which is compensated by the requirement of normalization. Consequently, the leading order correction (in \(\theta\)) to the QFI is at most quadratic in the average occupation number, as we have claimed.

The higher order corrections may feature higher powers of \(n\). However, the perturbative nature of the calculations means that the results presented here are only valid as long as the higher order corrections are negligibly small with respect to the leading order. Therefore, the perturbative approach is not uniformly valid in the average occupation number. In a regime where \(n\) is large enough such that \(\theta^2 \mathcal{I}(\tilde{\psi})/4 \approx 1\) the perturbative calculation no longer yields a reliable result. We can therefore conclude that, within the perturbative regime, Heisenberg scaling is indeed optimal.