Reflection Bootstrap Equations for Toda Field Theory

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Abstract

An algebraic approach to integrable quantum field theory with a boundary (a half line) is presented and interesting algebraic equations, Reflection equations (RE) and Reflection Bootstrap equations (RBE) are discussed. The Reflection equations are a consistent generalisation of Yang-Baxter equations for factorisable scatterings on a half line (or with a reflecting boundary). They determine the so-called reflection matrices. However, for Toda field theory and/or other theories with diagonal S-matrices, the Reflection-Bootstrap equations proposed by Fring and Köberle determine the reflection matrices, since the reflection equations and the Yang-Baxter equations become trivial in these cases. The explicit forms of the reflection matrices together with their symmetry properties are given for various Toda field theories, simply laced and non-simply laced.

1. Introduction

This is the second talk on Toda field theory in this Conference. The first part, given by Ed Corrigan, showed that Toda field theory is a simple and very good example of 1 + 1 dimensional quantum field theory that can be solved exactly. As is well known, the quantum field theory describes the behaviour of elementary particles, the fundamental building blocks of the matter. However, because of various difficulties, in particular, the infinite degrees of freedom involved, the complete solution of quantum field theory in 3 + 1 dimensions does not seem to be an easy goal to be achieved. In most cases our understanding is limited to the perturbative regime. Therefore, examples of completely solvable quantum field theories, like the Toda field theory, in spite of its 1+1 dimensionality,
are expected to clarify the structure of quantum field theory beyond perturbation. This is
the general motivation of our working in Toda field theory and its generalisation.

This second part deals with integrable quantum field theory on a half line, instead of
the entire line. Namely, with certain boundary conditions at the end of the half line. This
problem is also related with integrable statistical lattice models with non-trivial bound-
ary conditions, scattering of electrons by an impurity in solids (the Kondo problem) and
deformations of conformal field theory with boundaries. Here we would try to show that
Toda field theory again provides a very good theoretical laboratory.

Affine Toda field theory \[1\] is a massive scalar field theory with exponential interactions
in 1 + 1 dimensions described by the Lagrangian density
\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi^a \partial^\mu \phi^a - V(\phi)
\]
in which
\[
V(\phi) = \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \phi} \phi.
\]
Here \( \phi \) is an \( r \)-component scalar field, \( r \) is the rank of a compact semi-simple Lie algebra
\( g \) with \( \alpha_i; i = 1, \ldots, r \) being its simple roots. An additional root, \( \alpha_0 = -\sum_{i=1}^{r} n_i \alpha_i \) is an
integer linear combination of the simple roots, is called the affine root; it corresponds to
the extra spot on an extended Dynkin-Kac diagram for \( \hat{g} \) and \( n_0 = 1 \). When the term
containing the extra root is removed, the theory becomes conformally invariant (conformal
Toda field theory). The simplest affine Toda field theory, based on the simplest Lie algebra
\( a_1 \), the algebra of \( su(2) \), is called sinh-Gordon theory, a cousin of the well known sine-
Gordon theory. \( m \) is a real parameter setting the mass scale of the theory and \( \beta \) is a real
coupling constant, which is relevant only in quantum theory.

Toda field theory is integrable at the classical level due to the presence of an infinite
number of conserved quantities. Many beautiful properties of Toda field theory, both at
the classical and quantum levels, have been explained in some detail in Ed Corrigan’s
talk. In particular, it is firmly believed that the integrability survives quantisation. The
exact quantum S-matrices are known \[2-9\] for all the Toda field theories based on non-
simply laced algebras as well as those based on simply laced algebras. The singularity
structure of the latter S-matrices, which in some cases contain poles up to 12-th order
\[5\], is beautifully explained in terms of the singularities of the corresponding Feynman
diagrams \[10\], so called Landau singularities.
In this talk we need only the most rudimentary facts of Toda field theory, namely masses and three point couplings. Expanding the potential \( V(\phi) \) around the classical vacuum \( \phi \equiv 0 \) we get

\[
V(\phi) = \frac{m^2}{\beta^2} h + \frac{1}{2} (M^2)^{ab} \phi^a \phi^b + \frac{1}{3!} c^{abc} \phi^a \phi^b \phi^c + \cdots, \tag{1.3}
\]

where \( h = \sum_i n_i \) is the Coxeter number of \( g \), \( M^2 \) is the mass matrix \((M^2)^{ab} = m^2 \sum_i n_i \alpha_i^a \alpha_i^b \). We choose the representation of the simple roots such that the mass matrix becomes diagonal. Then we get the mass spectrum as the eigenvalues \( m_1^2, m_2^2, \ldots m_r^2 \).

In this basis the cubic term in the field operators, \( c^{abc} = m^2 \beta \sum_{i=0}^r n_i \alpha_i^a \alpha_i^b \alpha_i^c \), describes the three point interaction in which particle \( a \), \( b \) and \( c \) meet at one space-time point and annihilate. The same term describes the other processes, \( a \) and \( b \) meet to produce \( \bar{c} \) (the anti-particle of \( c \)), \( b \) and \( c \) meet to produce \( \bar{a} \), etc. This is due to a simple fact in relativistic field theory that a field operator \( \phi_a \) annihilates a particle \( a \) and creates an anti-particle \( \bar{a} \). Crossing symmetry is the expression of this fact at the S-matrix level. Armed with these information one can go on to determine the S-matrices of affine Toda field theory.

Let us first look at the exact factorisable S-matrices in a broader context. The stage is the 2 dimensional Minkowski space, with the particles satisfying the mass-shell conditions:

\[
p^2 = (p_0)^2 - (p_1)^2 = m^2,
\]

which can be conveniently parametrised in terms of rapidity \( \theta \),

\[
p_0 = m \cosh \theta, \quad p_1 = m \sinh \theta.
\]

Most known examples of exact S-matrices are obtained as solutions of the Yang-Baxter equation, which is the necessary condition for the factorisability of the three-body elastic S-matrix into a product of three two-body S-matrices. Namely the Yang-Baxter equation requires the equality of the products of S-matrices in two different orders.

\[
S_{12}(\theta_1 - \theta_2)S_{13}(\theta_1 - \theta_3)S_{23}(\theta_2 - \theta_3) = S_{23}(\theta_2 - \theta_3)S_{13}(\theta_1 - \theta_3)S_{12}(\theta_1 - \theta_2). \tag{1.4}
\]

The S-matrices depend on the rapidity difference as required by the Lorentz invariance.

In the simplest case when the masses of the particles are all degenerate, say \( N \) fold, the S-matrices have \( N^2 \times N^2 \) entries. The explicit indices dependence was omitted because it would be messy and we are not going to use it in the Toda field theory context, either. In Toda field theory most of the particle masses are distinct, and in the less common cases of degeneracy, these particles are distinguished by the infinite number of conserved quantities. Therefore, the S-matrices are diagonal, to be denoted by \( S_{ab}(\theta) \) (\( a, b = 1, \ldots, r \) are the label of two incoming particles) which are complex numbers with unit modulus for real \( \theta \). And
the Yang-Baxter equations become trivial, since the diagonal matrices always commute. There is, however, another set of algebraic equations, called Bootstrap equations which govern the exact S-matrices for Toda field theory instead of the Yang-Baxter equations.

Suppose \( S_{ab}(\theta) \) has a pole at \( \theta = i\theta_{ab}^c (0 < \theta_{ab}^c < \pi) \), corresponding to a bound state particle \( c \) with the mass \( m_c^2 = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2m_am_b \cos \theta_{ab}^c \). In Toda field theory based on simply laced algebras, this happens when and only when the corresponding three point coupling \( c^{ab\bar{c}} \) ([1.3]) is non-vanishing. (For the generalised bootstrap principle for the non-simply laced theory, see [9].) Then on top of the pole (ie, \( \theta = i\theta_{ab}^c \)) the two particle state \( ab \) is completely dominated by the single particle state \( c \), which leads to the following equation among the S-matrices

\[
S_{cd}(\theta) = S_{ad}(\theta + i\theta_{ac})S_{bd}(\theta - i\theta_{bc}),
\]

in which \( d \) is an arbitrary particle and

\[
\theta_{ab}^c + \theta_{bc}^a + \theta_{ca}^b = 2\pi, \quad \bar{\theta} = \pi - \theta, \quad \bar{\theta}_{ab}^c + \bar{\theta}_{bc}^a + \bar{\theta}_{ca}^b = \pi.
\]

Based on the mass spectrum [4], analyticity (in \( \theta \)),

Unitarity \( S_{ab}(\theta)S_{ab}(-\theta) = 1, \quad (1.6) \)

which simply means \(|S_{ab}(\theta)| = 1\) for real \( \theta \) and

Crossing Symmetry \( S_{ab}(i\pi - \theta) = S_{a\bar{b}}(\theta), \quad (1.7) \)

together with certain conditions imposed by the Langangian ([1,4]), these over-determined set of Bootstrap equations determine the Toda S-matrices. They are \( 2\pi i \) periodic function of \( \theta \) and have symmetry properties due to \( C, P \) and \( T \) invariance

\( S_{ab}(\theta) = S_{ba}(\theta) = S_{\bar{a}\bar{b}}(\theta). \quad (1.8) \)

The basic building block of the S-matrix satisfying the unitarity may be taken to be of the form

\[
(x) = (x)_\theta = \frac{\sinh \left( \frac{\theta}{2} + \frac{i\pi x}{2\pi} \right)}{\sinh \left( \frac{\theta}{2} - \frac{i\pi x}{2\pi} \right)},
\]

\( 2 \) See, for example [3].
where $h$ is the Coxeter number. In order to incorporate the coupling constant dependence it is advantageous to introduce the modified building block

\[ \{ x \} = \{ x \} \theta = \frac{(x + 1)(x - 1)}{(x + 1 - B)(x - 1 + B)}, \]  

(1.10)

where $B = B(\beta) = \frac{\beta^2/2\pi}{1 + \beta^2/4\pi}$ gives the coupling dependence [11]. We usually suppress the $\theta$ dependence of $(x)$ and $\{x\}$ unless otherwise stated. Here we list the S-matrices of $a_n^{(1)}$ and $d_n^{(1)}$ Toda field theories in the above notation:

\[ a_n^{(1)} : \quad S_{ab} = \{ a + b - 1 \} \{ a + b - 3 \} \ldots \{ |a - b| + 1 \} = \prod_{\substack{|a-b|+1 \\text{step 2}}}^{a+b-1} \{ p \}, \]  

(1.11)

\[ d_n^{(1)} : \quad S_{ab} = \prod_{\substack{|a-b|+1 \\text{step 2}}}^{a+b-1} \{ p \} \{ h - p \}, \quad S_{s'a} = S_{sa} = S_{\bar{s}a} = \prod_{\substack{0 \\text{step 2}}}^{2a-2} \{ n - a + p \}, \]  

(1.12)

\[ d_{\text{even}}^{(1)} : \quad S_{ss} = S_{s's'} = \prod_{\substack{1 \\text{step 4}}}^{h-1} \{ p \}, \quad S_{ss'} = \prod_{\substack{3 \\text{step 4}}}^{h-3} \{ p \}, \]  

(1.13)

\[ d_{\text{odd}}^{(1)} : \quad S_{ss} = \prod_{\substack{1 \\text{step 4}}}^{h-3} \{ p \}, \quad S_{\bar{s}s} = \prod_{\substack{3 \\text{step 4}}}^{h-1} \{ p \}. \]  

(1.14)

For the S-matrices of $e_n$ series and non-simply laced theories, see [2-9].

2. Reflection equations and Reflection Bootstrap equation

Next let us consider the factorisable scattering on a half line $x \leq 0$. This problem was first formulated by Cherednik [12]. Here we have an additional element, the reflection at the boundary $x = 0$, denoted by the K-matrix, $K(\theta)$ (an $N \times N$ matrix for an $N^2 \times N^2$ S-matrix). The K-matrices should satisfy the so-called Reflection equations encoding the equality of the two alternative ways of reflection

\[ S(\theta_1 - \theta_2)K_1(\theta_1)S(\theta_1 + \theta_2)K_2(\theta_2)K_1(\theta_1)S(\theta_1 - \theta_2) = K_2(\theta_2)S(\theta_1 + \theta_2)K_1(\theta_1)S(\theta_1 - \theta_2). \]  

(2.1)

At this point some general remarks are in order. It is well known that the Yang-Baxter equations can be generalised into quadratic algebras or exchange algebras, which are sometimes called quantum groups. Their rich and beautiful mathematical structures
play important roles. The Reflection equations can also be generalised to define quadratic algebras (REA, Reflection equation algebras) associated with quantum groups. The Reflection equations and REA inherit many nice properties from Yang-Baxter equations and quantum groups [13].

In the Toda field theory context or in any theory with diagonal S-matrices, the Reflection equation is trivial. When a particle $a$ hits the end point $x = 0$ with rapidity $\theta$ it is reflected elastically ($\theta \rightarrow -\theta$) and acquires a factor $K_a(\theta)$. It is expected that the analog of the Bootstrap equations, the so-called Reflection-Bootstrap equations (RBE) determine the K-matrices. This problem was first clarified by Fring and Köberle [14]

$K_c(\theta) = K_a(\theta + i\bar{\theta}_{ac}^b)K_b(\theta - i\bar{\theta}_{bc}^a)S_{ab}(2\theta + i\bar{\theta}_{ac}^b - i\bar{\theta}_{bc}^a). \quad (2.2)$

The unitarity conditions are given by

Unitarity $K_a(\theta)K_a(-\theta) = 1, \quad (2.3)$

and the crossing conditions are derived by Ghoshal and Zamolodchikov [13]

Crossing Condition $K_a(\theta)K_{\bar{a}}(\theta - i\pi) = S_{aa}(2\theta) = S_{\bar{a}\bar{a}}(2\theta). \quad (2.4)$

It should be remarked that the crossing conditions (2.4) need not be imposed independently. They are built in automatically as the consistency conditions among the Reflection Bootstrap equation (2.2) when all the possible fusings are taken into account, namely $ab \rightarrow c$, $b\bar{c} \rightarrow \bar{a}$ and $c\bar{a} \rightarrow \bar{b}$. The crossing symmetry at the field operator level requires the latter two fusings ($b\bar{c} \rightarrow \bar{a}$ and $c\bar{a} \rightarrow \bar{b}$) when the former $ab \rightarrow c$ exists. However, the converse is not true. There is no guarantee for $K$’s satisfying the Crossing conditions (2.4) to satisfy the Reflection Bootstrap equations (2.2).

Here we list some important properties of the Reflection Bootstrap equations. First by comparing the Bootstrap equation and the Reflection Bootstrap equation

$S_{cd}(\theta) = S_{ad}(\theta + i\bar{\theta}_{ac}^b)S_{bd}(\theta - i\bar{\theta}_{bc}^a), \quad (2.5)$

it is easy to see a new solution $K'_c(\theta)$ can be obtained from an old one

$K'_c(\theta) = K_c(\theta)(S_{cd}(\theta))^{\pm 1}, \quad d: \text{ arbitrary.} \quad (2.6)$
In fact, any quantity satisfying the Bootstrap equation will do the same job. (For example, the non-minimal part of the Toda S-matrix with an arbitrary coupling dependence.) Because of the C, P, T invariance, \(K'_a(\theta) = K_a(\theta)\), \(a = 1, \ldots, r\) is also a solution. Similarly

\[
K'_a(\theta) = K_a(\theta - i\pi) = S_{aa}(2\theta)/K_a(\theta),
\]

is also a solution. The second equality follows from the crossing condition (2.4).

Next we write down the Bootstrap equations for \(d = c, a\) and \(b\),

\[
S_{cc}(\theta) = S_{ac}(\theta + i\bar{\theta}_ac)S_{bc}(\theta - i\bar{\theta}_bc),
\]

\[
S_{ac}(\theta) = S_{aa}(\theta + i\bar{\theta}_ac)S_{ab}(\theta - i\bar{\theta}_bc),
\]

\[
S_{bc}(\theta) = S_{ab}(\theta + i\bar{\theta}_ac)S_{bb}(\theta - i\bar{\theta}_bc),
\]

and substitute the second and the third equations into the first to get (with replacement \(\theta \to 2\theta\))

\[
S_{cc}(2\theta) = S_{aa}(2(\theta + i\bar{\theta}_ac))S_{bb}(2(\theta - i\bar{\theta}_bc))S_{ab}^2(2\theta + 2i\bar{\theta}_ac - 2i\bar{\theta}_bc).
\]

It is easy to see that

\[
K_a(\theta) = \sqrt{S_{aa}(2\theta)}, \quad a = 1, \ldots, r,
\]

formally satisfies the Reflection Bootstrap equations in all the Toda field theories. However, in the rest of this talk we will consider only the meromorphic solutions.

3. Solutions of Reflection Bootstrap equations for Toda theory

Here we discuss some explicit solutions of Reflection Bootstrap equations (RBE) for various Toda field theories. At first some preparations are in order. Since we need \(S_{ab}(2\theta)\) etc, it is useful to introduce a new block corresponding to the ordinary block \(\{x\\} \) at \(2\theta\);

\[
\{x\}_{2\theta} = [x/2]\theta/[(h - x)/2] = [x/2]/[h - x/2],
\]

in which a new (half) block \(\lfloor x \rfloor_{\theta} = [x]\) is given in terms of the elementary block \((x)\),

\[
\lfloor x \rfloor_{\theta} = \frac{(x - \frac{1}{2})(x + \frac{1}{2})}{(x - \frac{1}{2} + \frac{B}{2})(x + \frac{1}{2} - \frac{B}{2})}.
\]

The fusing angles, eg. \(\theta_{ab}^c\) etc, in Toda theory are integer multiples of \(\pi/h\) (in non-simply laced case, \(\pi/H\)). So we denote \(x\pi/h\) simply \(x_h\). Among many possible solutions of RBE
we mainly discuss the ‘minimal’ solutions after [15]. The ‘minimal’ solutions have minimal set of (say, coupling independent) zeroes and poles in the physical strip \((0 < \text{Im} \theta < \pi)\). In other words we define the degree \(D\) of the solution \(\{K_a\}\),

\[
D = \sum_{a=1}^{r} (\text{number of poles and zeros of } K_a).
\] (3.3)

3.1. \(a_1^{(1)}\) case

The simplest Toda field theory \((h = 2)\), the sinh-Gordon theory, consists of one neutral particle and it has no three point coupling. Therefore the Reflection Bootstrap equation is void. But the Crossing condition \((2.4)\) makes sense. The ‘minimal’ solutions are

\[
K_1(\theta) = [1/2], \quad [3/2]^{-1}, \quad \text{for } S_{11}(2\theta) = [1/2]/[3/2].
\] (3.4)

The above two solutions are related by \((2.7)\).

3.2. \(a_2^{(1)}\) case

It has particles 1 and 2 which are hermitian conjugate to each other, \(2 = \bar{1}\) and \(h = 3\). The RBE reads

\[
K_2(\theta) = K_1(\theta + i1_h)K_1(\theta - i1_h)S_{11}(2\theta),
\]

\[
K_1(\theta) = K_2(\theta + i1_h)K_2(\theta - i1_h)S_{22}(2\theta).
\] (3.5)

Eliminating \(K_2\) in terms of \(K_1\) and using the bootstrap properties of \(S\)-matrices, we get

\[
K_1(\theta + i2_h)K_1(\theta)K_1(\theta - i2_h) = 1.
\] (3.6)

The ‘minimal’ solutions are \((D = 2)\)

\[
K_1(\theta) = 1, \quad K_2(\theta) = S_{11}(2\theta) = [1/2]/[5/2]; \quad K_1(\theta) = S_{11}(2\theta), \quad K_2(\theta) = 1.
\] (3.7)

There are charge conjugation \textit{even} solutions \((D = 6)\) related by \((2.7)\),

\[
K_1 = K_2 = [1/2][3/2], \quad ([3/2][5/2])^{-1}.
\] (3.8)

3.3. \(a_3^{(1)}\) case

It has a pair of complex particles 1 and 3 = \(\bar{1}\) and a neutral particle 2 with \(h = 4\).

\[
K_2(\theta) = K_1(\theta + i1_h)K_1(\theta - i1_h)S_{11}(2\theta),
\]

\[
K_3(\theta) = K_2(\theta + i1_h)K_1(\theta - i2_h)S_{12}(2\theta - i1_h),
\]

\[
K_2(\theta) = K_3(\theta + i1_h)K_3(\theta - i1_h)S_{33}(2\theta),
\]

\[
K_1(\theta) = K_2(\theta - i1_h)K_3(\theta + i2_h)S_{23}(2\theta + i1_h).
\] (3.9)
We eliminate $K_2$ and $K_3$ from the first two equations in favour of $K_1$ then the next two equations give the same nonlinear algebraic equation for $K_1$,

$$K_1(\theta + i3h)K_1(\theta + i1h)K_1(\theta - i1h)K_1(\theta - i3h)S_{11}(2\theta)/S_{13}(2\theta) = 1.$$ (3.10)

This reflects the fact that all the particles of $a_n^{(1)}$ Toda theory can be obtained as bound states of the ‘elementary’ particle 1 or $\bar{1}$. The ‘minimal’ solutions are charge conjugation even

$$K_1 = K_3 = [1/2], ([7/2]^{-1}), \quad K_2 = [3/2]/[7/2], ([1/2]/[5/2]), \quad D = 5,$$ (3.11)

which can be obtained by solving a much simpler Crossing condition (2.4) for the ‘elementary’ particle 1, $\bar{1}$ with $K_1 = K_\bar{1}$,

$$K_1(\theta)K_1(\theta - i\pi) = S_{11}(2\theta),$$ (3.12)

whose solutions always satisfy the full equation (3.10). The full equation (3.10) has both charge conjugation even and non-even solutions. The situation in $a_{odd}^{(1)}$ ($a_{even}^{(1)}$) theory is about the same as in $a_3^{(1)}$ ($a_2^{(1)}$). The lack of ‘minimal’ charge conjugation even solutions in $a_{even}^{(1)}$ ($h = odd$) theory could be ‘understood’ by the fact that the two points corresponding to particle $a$ and $\bar{a}$ have different colours when the spots of the Dynkin diagram are bi-coloured.

3.4. $d_n^{(1)}$ theory

In $d_n^{(1)}$ theory the ‘elementary’ particles are $s$ and $s'$ ($\bar{s}$) corresponding to the (anti-) spinor representations. In contrast to the $a_n^{(1)}$ case, both $s$ and $s'$ ($\bar{s}$) are necessary to make all the particles as bound states. Therefore in order to get nonlinear equations containing only one unknown variable (like (3.6) or (3.10)) we need certain assumption, for example, $K_s = K_{s'} = K_{\bar{s}}$. Solutions not satisfying the assumption must be considered separately. Of course ‘non-minimal’ solutions not satisfying the assumption can be easily obtained by means of (2.6).

For concreteness, let us consider $d_4^{(1)}$, which has three mass degenerate neutral particles 1, $s$ and $s'$ and a neutral particle 2 corresponding to the central spot of the Dynkin diagram ($h = 6$). With an assumption $K_1 = K_s = K_{s'}$, the Reflection Bootstrap equation reduce to

$$K_2(\theta) = K_1(\theta + i1h)K_1(\theta - i1h)S_{11}(2\theta),$$
$$K_1(\theta + i3h)K_1(\theta - i3h)S_{11}(2\theta) = 1,$$ (3.13)
\[ K_1(\theta) = K_1(\theta + i2\hbar)K_1(\theta - i2\hbar)S_{1s}(2\theta). \]  

(3.14)

The first eq. simply defines \( K_2 \) in terms of \( K_1 \) and the second eq. is the crossing condition for \( K_1 \). The third eq. gives a non-trivial condition. Solutions are

\[
K_1 = [1/2][5/2], \quad (([7/2][11/2])^{-1}),
\]

\[
K_2 = [3/2]^2[5/2]/[11/2], \quad ([1/2]/[7/2][9/2]^2).
\]

(3.15)

Not all the solutions of the crossing condition satisfy (3.14).

3.5. \( g_2^{(1)}d_4^{(3)} \) case

This is the simplest example of non-simply laced Toda theory [8-9]. It has two neutral particles, 1 and 2 with masses

\[ m_1 = \sin \pi/H \quad m_2 = \sin 2\pi/H, \]

(3.16)

up to an overall factor \((2\sqrt{2m})\) which is ignored. The parameter \( H \) floats in the range \( 6 \leq H \leq 12 \), ie between the Coxeter numbers of the partners in the pair. It has fusing 112 and self-couplings 111 and 222. This is a good example to understand the structure of the Reflection Bootstrap equation. The RBE reads

\[
K_2(\theta) = K_1(\theta + i1\hbar)K_1(\theta - i1\hbar)S_{11}(2\theta),
\]

\[
K_1(\theta) = K_2(\theta + i\pi - i2\hbar)K_1(\theta - i1\hbar)S_{12}(2\theta + i\pi - i3\hbar),
\]

\[
K_1(\theta) = K_1(\theta + i\pi/3)K_1(\theta - i\pi/3)S_{11}(2\theta),
\]

\[
K_2(\theta) = K_2(\theta + i\pi/3)K_2(\theta - i\pi/3)S_{22}(2\theta),
\]

(3.17)

The first eq. defines \( K_2 \) in terms of \( K_1 \), that of the ‘elementary’ particle. The consistency of the first two eqs. (both derived from the same fusing 112) gives rise to the crossing condition for \( K_1 \) as mentioned before. The crossing condition for \( K_2 \) is guaranteed by that of \( K_1 \). The third eq. implies the fourth eq. as well as the crossing condition for \( K_1 \). Thus we only need solve the third eq., with solutions

\[ K_1 = [1/2][H/2 + 1/2][3H/4]_{1/2}^{-1}, \quad [H/4]_{1/2}[H/2 - 1/2][H - 1/2]^{-1}. \]

(3.18)

Both are factors of

\[ S_{11}(2\theta) = \frac{[1/2][H/2 - 1/2][H/4]_{1/2}}{[H - 1/2][H/2 + 1/2][3H/4]_{1/2}}, \]
as in other cases. Here $[H/4]_{1/2}$ and $[3H/4]_{1/2}$ are the half blocks corresponding to a new building block $\{x\}_{1/2}$ necessary for the non-simply laced S-matrices [9].

$$\{x\}_\nu = \frac{(x - \nu B - 1)(x + \nu B + 1)}{(x + \nu B + B - 1)(x - \nu B - B + 1)}.$$  \hspace{1cm} (3.19)

The bracket notation has been adjusted slightly for the non-simply laced case and is now defined by

$$\langle x \rangle = \frac{\sinh \left( \frac{\theta}{2} + \frac{x \pi i}{2H} \right)}{\sinh \left( \frac{\theta}{2} - \frac{x \pi i}{2H} \right)}.$$  \hspace{1cm} (3.20)

In $g_2^{(1)}$-$d_4^{(3)}$ case $H = 6 + 3B$, and $0 \leq B \leq 2$ but this parametrisation is not intended to imply $B$ has the form given before for the simply laced theories. The S-matrix is given

$$S_{11}(\theta) = \{1\}\{H - 1\}\{H/2\}_{1/2}.$$  

4. Summary

An algebraic approach to integrable quantum field theory with a boundary (a half line) was presented with interesting algebraic equations, Reflection equations (RE) and Reflection Bootstrap equations (RBE). Many explicit solutions are found for the latter based on the known exact S-matrices for Toda field theory, simply laced and non-simply laced. There are, however, many interesting points I did not touch in this talk. For example: a unified description of the solutions of RBE. Lagrangian (space-time) description of the reflection matrices and the corresponding boundary conditions and the related problems of translational non-invariance etc. Explanation/interpretation of singularities and zeroes of the reflection matrices. Conserved quantities on a half line and relationship with deformed conformal field theory with boundaries [15]. Relationship with integrable lattice lattice models with non-trivial boundary conditions, etc. After returning from China, we found a second paper of Fring and Köberle [16] which has some overlap with the present paper.

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