Chromatic Number Via Turán Number

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Abstract

A Kneser representation KG(H) for a graph G is a bijective assignment of hyperedges of a hypergraph H to the vertices of G such that two vertices of G are adjacent if and only if the corresponding hyperedges are disjoint. In this paper, we introduce a colored version of the Turán number and use that to determine the chromatic number of some families of graphs, in particular, a family of matching graphs whose vertex set is the set of all matchings of a specified size of a graph and two vertices of a matching graph are adjacent if the corresponding matchings are edge-disjoint. This extends the well-known theorems of Lovász (1978) [16] and Schrijver (1978) [19]. Also, we determine the chromatic number of every Kneser multigraph KG(H), where the vertex set of H is the edge set of a multigraph G such that the multiplicity of each edge is greater than 1 and a hyperedge in H corresponds to a subgraph of G isomorphic to some graph in a fixed prescribed family of simple graphs.

Keywords: Chromatic Number, General Kneser Hypergraph, Turán Number.
Subject classification: 05C15

1 Introduction

In this paper, we investigate the chromatic number of graphs. It is a known fact that any graph G has a Kneser representation KG(H), i.e., there exists a bijective assignment of hyperedges of a hypergraph H to the vertices of G such that two vertices of G are adjacent if and only if the corresponding hyperedges are disjoint. In [1], in view of the Kneser representations of hypergraphs, the authors introduced a lower bound for their chromatic number. In this regard, the alternating chromatic number was defined and it was shown that it provides a tight lower bound for the chromatic number. As mentioned in [17], it is a hard problem to compute the alternating chromatic number of hypergraphs. Although, we show that one can evaluate the chromatic number of some families of graphs via their alternating chromatic number. A graph has various Kneser representations and a crux is to find an appropriate representation to obtain a good lower bound for the alternating chromatic number.

†The research of Hossein Hajiabolhassan is supported by ERC advanced grant GRACOL.
If we consider a Kneser representation $KG(H)$ for a graph, then one can present some lower and upper bounds for the chromatic number of this graph in terms of the generalized Turán number of hyperedges of $H$. It is known that if a hypergraph provides a Kneser representation for a graph, then the minimum number of vertices of the hypergraph which meet all of hyperedges is an upper bound for chromatic number of that graph. Several interesting results or conjectures related to the chromatic number of hypergraphs can be reformulated in terms of the generalized Turán number. In this paper, we introduce a colored version of the Turán number and use that to present a lower bound for the chromatic number of graphs. Moreover, we determine the chromatic number of some families of graphs in terms of the generalized Turán number. In this regard, we determine the chromatic number of some families of matching graphs, path graphs, and Kneser multigraphs which can be considered as a generalization of Kneser graphs, Schrijver graphs, and circular complete graphs.

This paper is organized as follows. In the first section, we set up notations and terminologies. In particular, we define the alternating generalized Turán number as a generalization of the generalized Turán number which provides a lower bound for chromatic number of graphs. Also, we introduce several Kneser representations for some well-known families of graphs. This motivates us to introduce matching graphs and path graphs which can be considered as a generalization of Schrijver and circular complete graphs, respectively. In the second section, first we introduce some lower and upper bounds for chromatic number in terms of the generalized Turán number of their Kneser representations. Next, we determine the chromatic number of a large family of matching graphs. Also, we determine the exact value of the chromatic number of every Kneser multigraph $KG(H)$ such that vertex set of $H$ is the edge set of a multigraph $G$ where the multiplicity of each edge is greater than 1 and hyperedges in $H$ correspond to all subgraphs of $G$ each isomorphic to some fixed prescribed simple graphs. Moreover, we show that the chromatic number of a family of path graphs lies between the lower bound and upper bound given in terms of the generalized Turán number.

1.1 Notations

First, in this section, we setup some notations and terminologies. Hereafter, the symbol $[n]$ stands for the set $\{1, \ldots, n\}$. A hypergraph $H$ is an ordered pair $(V(H), E(H))$, where $V(H)$ is a finite set, called the set of vertices of $H$, and $E(H)$ is a family of nonempty subsets of $V(H)$, called the set of hyperedges of $H$. The multiplicity of a hyperedge $e$ is the number of multiple hyperedges which contain the same vertices as $e$. Unless otherwise stated, we consider simple hypergraphs and for any hypergraph $H$, we suppose that $V(H) = [n]$, where $n$ is a positive integer. Assume that $N = (N_1, N_2, \ldots, N_r)$, where $N_i$’s are pairwise disjoint subsets of $[n]$. The induced hypergraph $H|_N$ has $\cup_{i=1}^r N_i$ and $\{A \in E(H) : \exists i; 1 \leq i \leq r, A \subseteq N_i\}$ as the vertex set and the hyperedge set, respectively. A hypergraph homomorphism from a hypergraph $H$ to a hypergraph $G$ is a map from the vertex set of $H$ to that of $G$ such that the image of any hyperedge of $H$ contains some hyperedges of $G$. A $t$-coloring of a hypergraph $H$ is a mapping $h : V(H) \rightarrow [t] = \{1, 2, \ldots, t\}$ such that every hyper-
edge is not monochromatic. For a hypergraph $H$, its chromatic number $\chi(H)$ is the least positive integer (the number of colors) such that there exists a $\chi(H)$-coloring for $H$. If $H$ has a hyperedge of size 1, then we define its chromatic number to be infinite. For any hypergraph $H = (V(H), E(H))$ and positive integer $r \geq 2$, the general Kneser hypergraph $KG^r(H)$ has $E(H)$ as its vertex set and the hyperedge set consisting of all $r$-tuples of pairwise disjoint hyperedges of $H$. For simplicity of notation, when $r = 2$, the Kneser graph $KG^2(H)$ is shown by $KG(H)$. It is known that for any graph $G$, there exists a hypergraph $H$ such that $G \simeq KG(H)$, i.e., $G$ is isomorphic to $KG(H)$.

A subset $S \subseteq [n]$ is s-stable if any two distinct elements of $S$ are at least “at distance $s$ apart” on the $n$-cycle, that is, $s \leq |i - j| \leq n - s$ for distinct $i, j \in S$. For a subset $A \subseteq [n]$, the symbols $\binom{A}{k}$ and $\binom{A}{r}$, stand for the set of all $k$-subsets of $A$ and the set of all $s$-stable $k$-subsets of $A$, respectively. If $H_1 = ([n], \binom{[n]}{k})$, $H_2 = ([n], \binom{[n]}{s})$, and $r \geq 2$ is a positive integer, then the hypergraphs $KG^r(H_1)$ and $KG^r(H_2)$ are denoted by $KG^r(n, k)$ and $KG^r(n, k)_{s-stab}$ and termed the Kneser hypergraph and the s-stable Kneser hypergraph, respectively. For simplicity of notation, when $r = 2$, the Kneser graph $KG^2(n, k)$ and the s-stable Kneser graph $KG^2(n, k)_{s-stab}$ are shown by $KG(n, k)$ and $KG(n, k)_{s-stab}$, respectively. Also, the generalized Kneser graph $KG(n, k, s)$ has $\binom{[n]}{s}$ as its vertex set and two vertices are adjacent if the size of intersection of corresponding sets is at most $s$. Also, the notations $C_n, P_n, K_n, K_{m,n}$, and $rK_2$ stand for the $n$-cycle, the path with length $n$, the complete graph with $n$ vertices, the complete bipartite graph, and the matching of size $r$, respectively.

The circular complete graph $K_{\frac{n}{d}}$ has $[n]$ as the vertex set and two vertices $i$ and $j$ are adjacent if $d \leq |i - j| \leq n - d$. Circular complete graphs can be considered as a generalization of complete graphs and they have been studied in the literature, see [23]. Assume that $m, n, r$ are positive integers, where $r \leq m, n$. For a subset $A \subseteq [m]$ and an injective map $f : A \rightarrow [n]$, the ordered pair $(A, f)$ is said to be an $r$-partial permutation [7]. Let $S_r(m, n)$ denote the set of all $r$-partial permutations. Two partial permutations $(A, f)$ and $(B, g)$ are said to be intersecting, if there exists an $x \in A \cap B$ such that $f(x) = g(x)$. Note that $S_n(n, n)$ is the set of all $n$-permutations. The permutation graph $S_r(m, n)$ has all $r$-partial permutations $(A, \sigma)$ as its vertex set and two $r$-partial permutations are adjacent if and only if they are not intersecting. The structure of maximum independent sets of $S_r(m, n)$ was studied in several papers, see [4, 8, 14].

1.2 Kneser Representations

A Kneser representation for a graph $G$ is an assignment of subsets of a ground set to the vertices of $G$ such that it assigns distinct subsets to the vertices of $G$ and it satisfies disjoint property, i.e., two vertices are adjacent if and only if the corresponding sets are disjoint. In this regard, a graph $G$ has $[n, k]$-Kneser representation, if it is isomorphic to an induced subgraph of the Kneser graph $KG(n, k)$. In other words, we assign distinct sets of size $k$ of $[n]$ to vertices of $G$ such that it satisfies disjoint property. The Kneser index of a graph which is the minimum $k$ for which there exists such a representation was studied in [10, 11].
For any hypergraph $H$ and a family of its subhypergraphs $\mathcal{F}$, we define the hypergraph $\left(\frac{E(H)}{\mathcal{F}}\right)$ (resp. $\left(\frac{V(H)}{\mathcal{F}}\right)$) as follows. This hypergraph has $E(H)$ (resp. $V(H)$) as its vertex set and the hyperedge set (vertex set) of any subhypergraph of $H$ isomorphic to a member of $\mathcal{F}$ forms a hyperedge. Hereafter, by abuse of notation, we show the general Kneser hypergraphs $KG^r(\left(\frac{E(H)}{\mathcal{F}}\right))$ and $KG^r(\left(\frac{V(H)}{\mathcal{F}}\right))$ by $KG^r(H, F)$ and $KG^r_v(H, F)$, respectively. Moreover, for simplicity of notation, when $r = 2$, these hypergraphs are shown by $KG(H, F)$ and $KG_v(H, F)$, respectively. Moreover, by abuse of notation, if $\mathcal{F} = \{F\}$, where $F$ is a hypergraph, then we write $KG(H, F)$ and $KG_v(H, F)$ instead of $KG(H, F)$ and $KG_v(H, F)$, respectively. In this terminology, one can see that the hypergraph $KG^r(H)$ is isomorphic to $KG^r(H, E(H))$. Assume that $F = (V(F), E(F))$ is a subhypergraph of $H$. Any vertex of $V(F) \setminus \cup_{T \in E(F)} T$ is termed an isolated vertex. If we set $V' = \cup_{T \in E(F)} T$, $E' = E(F)$, and $F' = (V', E')$, then one can see that the general Kneser hypergraphs $KG^r(H, F)$ and $KG^r(H, F')$ are isomorphic. Hence, for any general Kneser graph $KG(H, F)$, we assume that any $F \in \mathcal{F}$ does not contain any isolated vertex. In contrast, we allow the hypergraph $H$ to have isolated vertices. In fact, we show that the isolated vertices of a hypergraph $H$ may be useful to present an appropriate lower bound for the chromatic number of the general Kneser hypergraph $KG(H, \mathcal{F})$.

Now, we introduce Kneser representations of Kneser graphs, Schrijver graphs, circular complete graphs, generalized Kneser graphs, $s$-stable Kneser graphs, and permutation graphs as follows. One can check that the following representations are isomorphic to the corresponding graphs.

1. The Kneser graph $KG(n, k) \ (n \geq 2k)$ is isomorphic to $KG(nK_2, kK_2)$.
2. The Schrijver graph $SG(n, k) \ (n \geq 2k)$ is isomorphic to $KG(C_n, kK_2)$.
3. The circular complete graph $K_{d}^c \ (n \geq 2d)$ is isomorphic to $KG(C_n, P_d)$.
4. The generalized Kneser graph $KG(n, k, s) \ (n \geq k > s)$ is isomorphic to $KG(K_{n,s+1}, K_{k,s+1})$, where the complete hypergraph $K_{n,s}$ contains all $s$-subsets of $[n]$.
5. The $s$-stable Kneser graph $KG(n, k)_{s-stab}$ is isomorphic to $KG_v(K_{2}^s, K_k)$, where $n \geq sk$.
6. The permutation graph $S_r(m, n) \ (m, n \geq r)$ is isomorphic to $KG(K_{m,n}, rK_2)$.

1.3 Generalized Turán Number

Throughout this section, assume that $H$ is a finite (multi) hypergraph and $\mathcal{F}$ is a finite family of (multi) subhypergraphs of $H$. A hypergraph is called $\mathcal{F}$-free, if it does not contain any member of $\mathcal{F}$ as a subhypergraph. The maximum number of edges of an $\mathcal{F}$-free spanning subhypergraph of $H$ (a subhypergraph of $H$ with the same vertices as $H$) is denoted by $ex(H, \mathcal{F})$. Similarly, the maximum number of vertices of an $\mathcal{F}$-free induced subhypergraph of $H$ is denoted by $ex_v(H, \mathcal{F})$. A spanning subhypergraph of $H$ is called $\mathcal{F}$-extremal if it has $ex(H, \mathcal{F})$-edges and it is $\mathcal{F}$-free. We denote the family of all extremal $\mathcal{F}$-free subhypergraphs of $H$ with
EX(H, F). It is known that ex(K_n, K_3) = \frac{n^2}{4} + 1. It is usually a hard problem to
determine the exact value of ex(H, F).

Assume that σ = (e_1, e_2, ..., e_t) is an ordering of hyperedges of H, where t = |E(H)|. A 2-coloring of a subset T of hyperedges of H (with two colors blue and
red) is called an alternating coloring (with respect to σ), if we assign two colors alternatively to members of T with respect to the ordering σ. In other words,
we assign distinct colors to any two consecutive hyperedges of T. Also, if we have
an alternating coloring, then the number of colored hyperedges is termed the length
of this alternating coloring. More precisely, an alternating coloring of T ⊆ E(H)
with respect to an ordering, has length |T|. Assume that σ = (e_1, e_2, e_3, e_4, e_5, e_6)
and σ' = (e_2, e_1, e_3, e_4, e_5, e_6) are two orderings of the edges of the complete graph
H = K_4. Also, suppose that G is a subgraph of K_4, where E(G) = \{e_1, e_2, e_4, e_6\}.
One can check that the coloring f : E(G) → \{B, R\}, where f(e_1) = R, f(e_2) =
B, f(e_4) = R, and f(e_6) = B, is an alternating coloring of E(G) with respect to the
ordering σ. But this coloring is not an alternating coloring of E(G) with respect to the
ordering σ'. For a given 2-coloring of a subset of hyperedges of H, a spanning
subhypergraph of H whose hyperedge set contains all hyperedges such that we have
assigned color red (resp. blue) to them, is termed the red subhypergraph H_R (resp. the blue subhypergraph H^B). For instance, in the above-mentioned example, for the
alternating coloring with respect to the ordering σ, we have E(H_R) = \{e_1, e_4\} and
E(H^B) = \{e_2, e_6\}.

Suppose that σ is an ordering of E(H). The maximum number of hyperedges of
a spanning subhypergraph of H such that there exists an alternating coloring for the
hyperedges of this subhypergraph with respect to the ordering σ and that each of
the red subhypergraph and the blue subhypergraph (resp. the red subhypergraph or
the blue subhypergraph) forms an F-free subhypergraph is denoted by ex_{alt}(H, F, σ)
(resp. ex_{salt}(H, F, σ)). Set

ex_{alt}(H, F) = \min\{ex_{alt}(H, F, σ) : σ is an ordering of E(H)\}.
ex_{salt}(H, F) = \min\{ex_{salt}(H, F, σ) : σ is an ordering of E(H)\}.

Note that if we assign alternatively two colors red and blue to the hyperedge set
of a member of EX(H, F) with respect to an arbitrary ordering σ, then one can
conclude that ex(H, F) ≤ ex_{alt}(H, F, σ). Also, it is straightforward to see that
ex_{alt}(H, F, σ) ≤ 2ex(H, F). Accordingly,

ex(H, F) ≤ ex_{alt}(H, F) ≤ 2ex(H, F).

Similarly, one can see that ex(H, F) + 1 ≤ ex_{salt}(H, F) ≤ 2ex(H, F) + 1. In the
sequel, we presents several families such that the equality holds in the aforementioned
inequalities. Consider the complete graph K_4 and assume that E(K_4) = \{e_1, e_2, e_3, e_4, e_5, e_6\}. Moreover, let \{e_{2i-1}, e_{2i}\} form a matching for any 1 ≤ i ≤ 3.
One can see that for the path P_2, we have ex(K_4, P_2) = 2. Also, one can check
that for the ordering σ = (e_1, e_2, e_3, e_4, e_5, e_6), we have ex_{alt}(K_4, P_2, σ) = 2. Consequently, ex(K_4, P_2) = ex_{alt}(K_4, P_2) = 2.

The chromatic number of general Kneser graphs has been investigated in several
papers with different notations. It may be of interest to note that several interesting
results or conjectures related to the chromatic number of hypergraphs can be
reformulated in terms of the generalized Turán number. Here, we present some of them as follows.

1. The Kneser graph $KG(nK_2, kK_2)$ (Lovász [16]):

$$\chi(KG(nK_2, kK_2)) = |E(nK_2)| - 2ex(nK_2, kK_2) = n - 2k + 2.$$ 

2. The Schrijver graph $KG(C_n, kK_2)$ (Schrijver [19]):

$$\chi(KG(C_n, kK_2)) = |E(C_n)| - ex(C_n, kK_2) = n - 2k + 2.$$ 

3. The Kneser hypergraph graph $KG^r(nK_2, kK_2)$ (Alon, Frank, and Lovász [2]):

$$\chi(KG^r(nK_2, kK_2)) = \left\lceil \frac{|E(nK_2)| - r.ex(nK_2, kK_2)}{r - 1} \right\rceil = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$ 

4. The generalized Kneser graph $KG(K_n, K_k)$ (Frankl [6]):

$$\chi(KG(K_n, K_k)) = |E(K_n)| - ex(K_n, K_k) = (k - 1)\left(\frac{s}{2}\right) + rs,$$

where $n = (k - 1)s + r$, $0 \leq r < k - 1$, and $n$ is sufficiently large.

5. The $s$-stable Kneser graph $KG_v(K_n^s, K_k)$ (Jonson [12]). For $s \geq 4$ and sufficiently large $n$, we have

$$\chi(KG_v(K_n^s, K_k)) = |V(K_n)| - ex_v(K_n^s, K_k) = n - s(k - 1).$$

We should mention that the chromatic number of the generalized Kneser graph $KG(K_n, K_3)$ (i.e., $KG(n, 3, 1)$) was determined by Tort [20]. Also, Frankl [6] introduced the following conjecture about the chromatic number of generalized Kneser graphs.

**Conjecture A.** (Frankl [6]) Let $n, k$ and $s$ be positive integers, where $k > s \geq 2$ and $n \geq 2k - s + 1$. If $n$ is sufficiently large, then

$$\chi(KG(K_{n,s}, K_{k,s})) = |E(K_{n,s})| - ex(K_{n,s}, K_{k,s}),$$

where the complete hypergraph $K_{n,s}$ contains all of $s$-subsets of $[n]$.

Also, in [13], the authors introduced several conjectures and problems. Again, these problems can be reformulated in terms of the generalized Turán number as follows.

**Conjecture B.** [13] If $k$ is an odd integer and $n$ is sufficiently large, then

$$\chi(KG(K_n, C_k)) = |E(K_n)| - ex(K_n, C_k) = \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor.$$
Problem A. [13] Let $k$ be an even integer. Does
\[ \binom{n}{2} - O(n^{1 + \frac{2}{k}}) \leq \chi(KG(K_n, C_k)) \leq \binom{n}{2} - \Omega(n^{1 + \frac{1}{k}}) \]
hold?

Problem B. [13] Is the following statement true?
\[ \chi(KG(K_n, C_4)) = \binom{n}{2} - \frac{1}{2} n^\frac{3}{2} + o(n^\frac{3}{2}). \]

It is known that $ex(K_n, C_4) = \frac{1}{2} n^\frac{3}{2} - o(n^\frac{3}{2})$. Hence, it may be of interest to know
whether the equality $\chi(KG(K_n, C_4)) = |E(K_n)| - ex(K_n, C_4)$ holds provided that $n$
is sufficiently large.

Problem C. [13] If $q$ is a prime power and $n = q^2 + q + 1$, does
\[ \chi(KG(K_n, C_4)) = |E(K_n)| - ex(K_n, C_4) = \left( \frac{q^2 + q + 1}{2} \right) - \frac{1}{2} q(q + 1)^2 \]
hold?

The chromatic number of the hypergraph $KG^r(n, k)_{s-stab}$ was studied in [1, 3, 18]. In particular, Alon et al. [3] studied the chromatic number of $r$-stable $r$-uniform
Kneser hypergraph to construct ideals in $\mathbb{N}$ which are not non-atomic but they have
the Nikodým property and presented a conjecture. Next, Meunier [18] strengthened
their conjecture as follows. One can check that $ex_v(K_\frac{n}{s}, K_k) = s(k - 1)$ provided
that $n \geq s(k - 1)$.

Conjecture C. [18] If $k, r, s$ and $n$ are positive integers, where $n \geq sk$ and $s \geq r \geq 2$, then
\[ \chi(KG^r_v(K_\frac{n}{s}, K_k)) = \left\lceil \frac{n - s(k - 1)}{r - 1} \right\rceil = \left\lceil \frac{|V(K_\frac{n}{s})| - ex_v(K_\frac{n}{s}, K_k)}{r - 1} \right\rceil. \]

In [12], it was shown that if $r = 2$ and $n$ is sufficiently large, then for $s \geq 4$ the
aforementioned conjecture holds. Then in [1], it was proved that for positive integers
$k, n$ and $r$, where $n \geq rk$, if $n \not\equiv k \pmod{r}$ or $r$ is an even integer, then
$\chi(KG^r(n, k)_{2-stab}) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil$. Also, it was proved that if $2^a|r$, then
$\chi(KG^r(n, k)_{2a-stab}) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil$.

1.4 Alternating Chromatic Number

Assume that $X = (x_1, x_2, \ldots, x_n)$ is a sequence of $\{-1, 0, +1\}$. The subsequence
$x_{j_1}, x_{j_2}, \ldots, x_{j_m}$ ($j_1 < j_2 < \cdots < j_m$) of $X$ is said to be an alternating sequence if
any two consecutive terms in this subsequence are different. We denote by $alt(X)$
the length of a longest alternating subsequence of nonzero terms in $X$. For instance,
if $X = (+1, -1, 0, -1, 0, +1, +1, -1)$, then $alt(X) = 4$. 
One can consider \(\{-1,0,+1\}^n \setminus \{(0,0,\ldots,0)\}\) as the set of all signed subsets of \([n]\), that is, the family of all \((X^+,X^-)\) of disjoint subsets of \([n]\). Precisely, for \(X = (x_1,x_2,\ldots,x_n) \in \{-1,0,+1\}^n \setminus \{(0,0,\ldots,0)\}\), we define
\[
X^+ = \{i \in [n] : x_i = +1\}, \quad X^- = \{i \in [n] : x_i = -1\}.
\]
Throughout this paper, for any \(X = (x_1,x_2,\ldots,x_n) \in \{-1,0,+1\}^n \setminus \{(0,0,\ldots,0)\}\), we use these representations interchangeably, i.e., \(X = (x_1,x_2,\ldots,x_n)\) or \(X = (X^+,X^-)\). For a linear ordering (or a permutation) \(\sigma = (i_1,i_2,\ldots,i_n)\) of \([n]\), define \(\sigma(j) = i_j\), where \(1 \leq j \leq n\). Also, for any \(X = \{-1,0,+1\}^n \setminus \{(0,0,\ldots,0)\}\), define
\[
X^+ = \{\sigma(i) : i \in [n] \& x_i = +1\}, \quad X^- = \{\sigma(i) : i \in [n] \& x_i = -1\},
\]
and \(X_\sigma = (X^+_,X^-)\). Note that for the natural ordering (the identity permutation) \(I\) of \([n]\), we have \(X_I^+ = X^+\) and \(X_I^- = X^-\). Moreover, for a hypergraph \(F \subseteq 2^\{n\}\) and a positive integer \(i\), set \(alt_i(F,i)\) to be the largest integer \(k\) such that there exists an \(X \in \{-1,0,+1\}^n \setminus \{(0,0,\ldots,0)\}\) with \(alt(X) = k\) and that the hypergraph \(F_{X^\sigma}\) contains at most \(i - 1\) pairwise disjoint hyperedges of \(F\). If for each \(X \in \{-1,0,+1\}^n \setminus \{(0,0,\ldots,0)\}\) either \(X^+_\sigma\) or \(X^-\sigma\) has some hyperedges of \(F\), then we define \(alt_\sigma(F,1) = 0\). In this terminology, one can see that \(alt_\sigma(F,1)\) is the largest integer \(k\) such that there exists an \(X \in \{-1,0,+1\}^n \setminus \{(0,0,\ldots,0)\}\) with \(alt(X) = k\) and that both of \(X^+_\sigma\) and \(X^-\sigma\) do not contain any hyperedge of \(F\). Set \(alt(F,i) = \min\{alt_\sigma(F,i) : \sigma \in S_n\}\). We define the \(i^{th}\) alternating chromatic number of a graph \(G\) as follows.
\[
\chi_{alt}(G,i) = \max_n \max_{F \subseteq 2^\{n\}} \{n - alt(F,i) + i - 1 : KG(F) \simeq G\}.
\]
Note that in [1], the present authors defined the \(i^{th}\) alternation number for hypergraphs as follows. For a hypergraph \(F \subseteq 2^\{n\}\), a positive integer \(i\), and a linear ordering \(\sigma\) of \([n]\), set \(alt^*_i(F,i)\) to be the largest integer \(k\) such that there is an \(X = (x_1,x_2,\ldots,x_n) \in \{-1,0,+1\}^n \setminus \{(0,0,\ldots,0)\}\) with \(alt((x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}) = k\) and that the chromatic number of hypergraph \(KG(F_{X^\sigma})\) is at most \(i - 1\). Define \(alt^*(F,i) = \min\{alt^*_\sigma(F,i) : \sigma \in S_n\}\).

In the sequel, we show that \(alt^*(F,i) \leq alt(F,i)\). In view of \(alt^*(-,i)\), one can present a better lower bound for chromatic number. Usually, it is not easy to determine the chromatic number of the hypergraph \(KG(F_{X^\sigma})\). Hence, in this paper, we determine \(alt(-,i)\) for several families of hypergraphs. Now, we introduce an equivalent definition for \(alt^*(F,i)\) to show that \(alt^*(F,i) \leq alt(F,i)\) as follows. Set \(alt^*_\sigma(F,i)\) to be the largest integer \(k\) such that there exists an \(X \in \{-1,0,+1\}^n \setminus \{(0,0,\ldots,0)\}\) with \(alt(X) = k\) and that the chromatic number of hypergraph \(KG(F_{X^\sigma})\) is at most \(i - 1\). Note that if there is no such positive integer \(k\), then we set \(alt^*_\sigma(F,i) = 0\). One can see that \(alt^*_\sigma(F,i) \leq alt_\sigma(F,i)\). Hence, if we show that \(alt^*(F,i) = alt^*(F,i)\), then there is nothing to prove.

**Lemma 1.** For any hypergraph \(F \subseteq 2^\{n\}\) and positive integer \(i\), we have \(alt^*(F,i) = alt^*(F,i)\).
Proof. Consider an arbitrary permutation $\sigma \in S_n$. Let $alt^*_\gamma(F, i) = k$. In view of the definition of $alt^*_\gamma(F, i)$, there is an $X = (x_1, x_2, \ldots, x_n) \in \{+1, 0, -1\}^n \setminus \{(0, 0, \ldots, 0)\}$ such that $alt(X) = k$ and the chromatic number of the hypergraph $KG(F|_{X_\sigma})$ is at most $i - 1$. Now, let

$$Y = (y_1, y_2, \ldots, y_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}).$$

Note that $(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) = X$ and so $alt((y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)})) = alt(X) = k$. Also, we have

$$Y^+ = \{i \in [n] : y_i = +1\} = \{i \in [n] : x_{\sigma^{-1}(i)} = +1\} = \{\sigma(j) \in [n] : x_j = +1\} = X^+_{\sigma}$$

and similarly, $Y^- = X^-_{\sigma}$. Therefore, we have $KG(F|_Y) = KG(F|_{X_\sigma})$ and so the chromatic number of the hypergraph $KG(F|_Y)$ is at most $i - 1$. It implies $alt'_\gamma(F, i) \geq alt^*_\gamma(F, i)$; and therefore, $alt'_\gamma(F, i) \geq alt^*(F, i)$.

Now, let $\gamma \in S_n$ be an arbitrary permutation of $[n]$. Assume that $alt'_\gamma(F, i) = k'$. In view of the definition of $alt'_\gamma(F, i)$, there exists an $X = (x_1, x_2, \ldots, x_n) \in \{+1, 0, -1\}^n \setminus \{(0, 0, \ldots, 0)\}$ such that $alt((x_{\gamma(1)}, x_{\gamma(2)}, \ldots, x_{\gamma(n)})) = k'$ and the chromatic number of the hypergraph $KG(F|_X)$ is at most $i - 1$. Now, let $Z = (z_1, z_2, \ldots, z_n) = (x_{\gamma(1)}, x_{\gamma(2)}, \ldots, x_{\gamma(n)})$. Note that $alt(Z) = k'$,

$$Z^+_{\gamma} = \{\gamma(j) : z_j = +1\} = \{\gamma(j) : x_{\gamma(j)} = +1\} = X^+_{\sigma},$$

and similarly $Z^-_{\gamma} = X^-_{\sigma}$. Therefore, we have

$$KG(F|_X) = KG(F|_{Z_\gamma}).$$

Thus, the chromatic number of the hypergraph $KG(F|_{X_\sigma})$ is at most $i - 1$.

It implies that $alt^*_\gamma(F, i) \geq alt'_\gamma(F, i)$; and consequently, $alt^*(F, i) \geq alt'(F, i)$. \[\blacksquare\]

Now, we are in a position to introduce a lower bound for chromatic number of any graph $G$ in terms of the alternation number of its general Kneser hypergraph representations. The next theorem was expressed in terms of $alt'(F, i) = alt^*(F, i)$ in [1] and since $alt^*_\gamma(F, i) \leq alt^*(F, i)$ we have the following theorem as well.

**Theorem A.** [1] For any graph $G$ and any positive integer $i \leq \chi(G) + 1$, we have

$$\chi(G) \geq \chi_{alt}(G, i).$$

Throughout this paper, for a hypergraph $F \subseteq 2^{[n]}$, we set $alt_\sigma(F)$ (resp. $salt_\sigma(F)$) to be the largest integer $k$ such that there exists an $X \in \{-1, 0, +1\}^n \setminus \{(0, 0, \ldots, 0)\}$ with $alt(X) = k$ and that both (resp. at least one) of $X^+_{\sigma}$ and $X^-_{\sigma}$ do not contain any hyperedge of $F$. Note that $alt_\sigma(F) = alt_\sigma(F, 1)$ and $alt_\sigma(F, 2) \leq salt_\sigma(F)$. Also, $alt_\sigma(F) \leq alt_\sigma(F, 2) \leq salt_\sigma(F)$ and the equality can hold. For instance, one can see that for $k \geq 2$, $alt_1(\binom{n}{k}) = salt_1(\binom{n}{k}) = 2(k - 1) + 1$. Now, set $alt(F) = \min\{alt_\sigma(F) ; \sigma \in S_n\}$ and $salt(F) = \min\{salt_\sigma(F) ; \sigma \in S_n\}$. Note that for any hypergraph $F = (V(F), E(F))$, $ex_v(F, E(F)) \leq alt(F) \leq 2ex_v(F, E(F))$. Moreover, one can check that $\chi(KG(F)) \leq |V(F)| - ex_v(F, E(F))$. We define the
alternating chromatic number and strong alternating chromatic number for a graph $G$, respectively, as follows.

$$\chi_{alt}(G) = \max_F \{|V(F)| - alt(F) : KG(F) \simeq G\},$$

$$\chi_{salt}(G) = \max_F \{|V(F)| + 1 - salt(F) : KG(F) \simeq G\}.$$

Note that $\chi_{alt}(G) = \chi_{alt}(G, 1)$ and $\chi_{salt}(G) \leq \chi_{alt}(G, 2)$; consequently, in view of Theorem A, we have the following lower bound for the chromatic number of graphs.

**Theorem B.** [1] For any graph $G$, we have

$$\chi(G) \geq \max\{\chi_{alt}(G), \chi_{salt}(G)\}.$$

We can usually find an appropriate upper bound for $alt_2(F, 2)$ by computing $salt_2(F)$. Also, one can check that $salt((E(F)) = ex_{salt}(H, F)$. This enables us to specify the value of $\chi_{salt}(G)$ for some families of graphs. Consequently, in this paper, we determine the chromatic number of some families of graphs by applying Theorem B, although, Theorem A is stronger than Theorem B.

**Remark 1.** For any hypergraph $H$ on $n$ vertices, in view of the definition of $alt_\sigma(H)$ where $\sigma$ is an ordering of the vertex set of $H$, throughout this paper, we assume that $V(H)$ is identified with the set $[n]$. We may represent $V(H)$ with different sets, nevertheless we can consider any representation as a relabeling of the set $[n]$.

In view of Theorem B, we can consider the alternating chromatic number and the strong alternating chromatic number as tight lower bounds for chromatic number of graphs. For instance, in view of representation of the Kneser graph $KG(n, k)$ and the Schrijver graph $SG(n, k)$, one can see that $min\{\chi_{alt}(KG(n, k)), \chi_{salt}(KG(n, k))\} = n - 2k + 2$, $\chi_{alt}(SG(n, k)) \geq n - 2k + 1$, and $\chi_{salt}(SG(n, k)) = n - 2k + 2$. Note that a graph has several Kneser representations and different Kneser representations can lead us to different lower bounds for chromatic number. For instance, consider the five cycle. Set $F = (V(F), E(F))$ and $H = (V(H), E(H))$ as follows.

$$V(F) = \{1, 2, 3, 4, 5\} \& E(F) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\},$$

$$V(H) = [10] = \{1, 2, 3, 4, 5, a, b, c, d, e\}$$

and

$$E(H) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\}.$$

Note that the hypergraphs $F$ and $H$ have the same hyperedge set and the hypergraph $H$ has 5 isolated vertices. One can check that $alt(F) = 3$ and this shows that the chromatic number of five cycle is at least two. Although, one can check that $alt(H) = 7$ and this leads us to 3 as a lower bound for the chromatic number. To see this, consider the ordering

$$\sigma = (1, a, 2, b, 3, c, 4, d, 5, e).$$
On the contrary, suppose that \( \text{alt}_\sigma(H) \geq 8 \). This means that there exists an \( X = (x_1, x_2, \ldots, x_{10}) \in \{-1, 0, +1\}^{10} \setminus \{(0,0,\ldots,0)\} \) such that \( \text{alt}(X) = 8 \) and that both of \( X^\sigma_+ \) and \( X^\sigma_- \) do not contain any hyperedge of \( H \). We can assume that \( X \) has exactly 8 nonzero coordinates. Otherwise, suppose that \( x_{i_1}, x_{i_2}, \ldots, x_{i_8} \) is an alternating sequence of nonzero terms in \( X \), where \( i_1 < i_2 < \cdots < i_8 \). By changing the value of \( x_j \) to 0 for all \( j \notin \{i_1, i_2, \ldots, i_8\} \), one can obtain a \( Y \in \{-1, 0, +1\}^{10} \setminus \{(0,0,\ldots,0)\} \) such that \( \text{alt}(Y) = 8 \) and that both of \( Y^\sigma_+ \) and \( Y^\sigma_- \) do not contain any hyperedge of \( H \). Therefore, we can assume that \( \text{alt}(X) = 8 \) and \( X \) has exactly 8 nonzero coordinates. On the other hand, for any odd integer \( 1 \leq i \leq 7 \), at least one of \( x_i, x_{i+1}, \) or \( x_{i+2} \) is zero; since otherwise, one can conclude that either \( X^\sigma_+ \) or \( X^\sigma_- \) contains a hyperedge of \( H \). Also, if \( x_1 = x_9 = +1 \) or \( x_1 = x_9 = -1 \), then \( X^\sigma_+ \) or \( X^\sigma_- \) contains a hyperedge of \( H \). Now, by a double counting, one can prove the assertion.

2 General Kneser Graphs

In this section, we introduce some tight lower bounds for the chromatic number of general Kneser graphs. In fact, by presenting an upper bound for alternating Turán number (resp. strong alternating Turán number), we find a lower bound for chromatic number. In view of these bounds, we determine the chromatic number of some families of graphs.

2.1 Lower and Upper Bound

In this section, we introduce some upper bounds and lower bounds for chromatic number of \( KG(G,F) \). In fact, the chromatic number of some families of general Kneser graphs has been studied by several researchers with different notations. Frankl [6] determined the chromatic number of generalized Kneser graph \( KG(n,k,1) \) provided that \( n \) is sufficiently large. Note that \( KG(n,k,1) \) is isomorphic to \( KG(K_n,K_k) \). Also, the chromatic number of \( KG(K_n,C_m) \) was investigated in [13] and the authors independently obtained the same result of Tort [20] when \( m = 3 \) and \( n \geq 10 \). In fact, Tort has shown that \( \chi(KG(K_n,C_3)) = \binom{n}{2} - \text{ex}(K_n,C_3) = \left\lceil \frac{(n-1)^2}{4} \right\rceil \) for \( n \geq 5 \). Moreover, Katona and Tuza [13] introduced some interesting problems about chromatic number of \( KG(K_n,C_k) \) (see Conjecture B, and Problems A, B, and C).

Now, we introduce some bounds for the chromatic number of general Kneser hypergraphs.

**Lemma 2.** Let \( H \) be a hypergraph and \( F \) be a family of hypergraphs. For any positive integer \( r \geq 2 \),

\[
\frac{|E(H)| \times r \times \text{ex}(H,F)}{r-1} \leq \chi(KG^r(H,F)) \leq \left\lceil \frac{|E(H)| \times \text{ex}(H,F) + 1}{r-1} \right\rceil.
\]

**Proof.** First, we present a hypergraph homomorphism

\[
\phi : KG^r(|E(H)|, \text{ex}(H,F) + 1) \rightarrow KG^r(H,F),
\]
to show that the lower bound holds. Assume that $E(H) = \{e_1, \ldots, e_{|E(H)|}\}$. For any vertex $A = \{i_1, \ldots, i_l\}$, where $l = |E(H)| + 1$, set $\phi(A)$ equal to an arbitrary subhypergraph of $H$ whose hyperedge set is a subset of $\{e_{i_1}, \ldots, e_{i_l}\}$ and that it is isomorphic to a member of $\mathcal{F}$. In view of definition of $ex(H, \mathcal{F})$, it is easy to check that $\phi$ is a hypergraph homomorphism. Consequently,

$$\chi(KG^r(|E(H)|, ex(H, \mathcal{F}) + 1)) = \left\lceil \frac{|E(H)| - r \cdot ex(H, \mathcal{F})}{r - 1} \right\rceil \leq \chi(KG^r(H, \mathcal{F})).$$

Now, we prove the upper bound. Assume that $G \in EX(H, \mathcal{F})$. Also, let $E(H) \setminus E(G) = \{e_1, \ldots, e_t\}$, where $t = |E(H)| - ex(H, \mathcal{F})$. Assume that $S_1 \cup \cdots \cup S_q = \{e_1, \ldots, e_t\}$, where $S_i$’s are pairwise disjoint, $|S_1| = \cdots = |S_q - 1| = r - 1$, and $0 < |S_q| \leq r - 1$ ($q = \lceil \frac{|E(H)|}{r - 1} \rceil$). For any subhypergraph of $H$ which is isomorphic to a member of $\mathcal{F}$, assign the color $i$ to it, where $i$ is the smallest positive integer such that this subhypergraph has nonempty intersection with $S_i$. One can check that this assignment provides a proper coloring for $KG^r(H, \mathcal{F})$.

Lemma 3. For any hypergraph $H$ and a family $\mathcal{F}$ of hypergraphs,

$$|E(H)| - ex_{alt}(H, \mathcal{F}) \leq \chi(KG(H, \mathcal{F})) \leq |E(H)| - ex(H, \mathcal{F}),$$

$$|E(H)| + 1 - ex_{salt}(H, \mathcal{F}) \leq \chi(KG(H, \mathcal{F})) \leq |E(H)| - ex(H, \mathcal{F}).$$

Proof. It is easy to check that $alt(\binom{E(H)}{r}) = ex_{alt}(H, \mathcal{F})$ and $salt(\binom{E(H)}{r}) = ex_{salt}(H, \mathcal{F})$. Now, in view of Theorem B and Lemma 2, the assertion holds.

In view of Lemma 3, if $ex(H, \mathcal{F}) = ex_{alt}(H, \mathcal{F})$ or $ex(H, \mathcal{F}) + 1 = ex_{salt}(H, \mathcal{F})$, then

$$\chi(KG(H, \mathcal{F})) = |E(H)| - ex(H, \mathcal{F}).$$

Here, we present several examples to show that the upper bound mentioned in Lemma 2 is sharp. We have seen that $ex(K_4, P_2) = ex_{alt}(K_4, P_2) = 2$. Consequently, $\chi(KG(K_4, P_2)) = |E(K_4)| - ex(K_4, P_2) = 4$. Furthermore, it is known that if $n$ is sufficiently large, then the Turán number of $K_k$, i.e. $ex(K_n, K_k)$, is $\binom{n}{2} - (k - 1)\binom{s}{2} - rs$, where $n = (k - 1)s + r$, $0 \leq r < k - 1$. Hence, in view of the result of Frankl [6], one can see that there exists an integer $n_k$ such that for $n \geq n_k$, we have $\chi(KG(K_n, K_k)) = \binom{n}{2} - ex(K_n, K_k)$. Also, we can reformulate Conjecture B, Problems A and C in terms of the Turán number. In fact, these problems are about the equality of $\chi(KG(K_n, C_k)) = |E(K_n)| - ex(K_n, C_k)$. Also, the following example confirms that the lower bound mentioned in Lemma 2 is sharp. One can see that if $G = C_n$ and $\mathcal{F}$ is all of subgraphs of $C_n$ with exactly $k$-edges, then $\chi(KG(G, \mathcal{F})) = n - 2k + 2 = |E(G)| - 2ex(G, \mathcal{F})$.

It is known that $\Omega(n^{1+\frac{1}{k}}) \leq ex(K_n, C_k) \leq O(n^{1+\frac{2}{k}})$, so in view of Lemma 2 we have the following proposition which gives an affirmative answer to Problem A.

Proposition 1. Let $k$ be an even integer. We have

$$\binom{n}{2} - O(n^{1+\frac{2}{k}}) \leq \chi(KG(K_n, C_k)) \leq \binom{n}{2} - \Omega(n^{1+\frac{1}{k}}).$$
2.2 Matching Graphs

In this section, we investigate the chromatic number of matching graph $KG(G, rK_2)$ which can be considered as a generalization of Kneser, Schrijver, and permutation graphs. In fact, as we mentioned before, $KG(nK_2, rK_2)$, $KG(C_n, rK_2)$, and $KG(K_{m,n}, rK_2)$, are isomorphic to Kneser, Schrijver, and permutation graphs, respectively. Hence, as a generalization of Lovász’s Theorem [16] and Schrijver’s Theorem [19], it would be of interest to determine the chromatic number of matching graph $KG(G, rK_2)$. It seems that for any graph $G$, we usually have $\chi(KG(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2)$, but the assertion is not true when $G$ is not a connected graph. For instance, note that $\chi(KG(K_2, rK_2)) = |E(nK_2)| - 2\text{ex}(nK_2, rK_2)$. In the sequel, we introduce some sufficient conditions such that the equality $\chi(KG(G, rK_2)) = |E(G)| - \text{ex}(G, rK_2)$ holds.

A famous generalization of Tutte’s Theorem by Berge in 1985, says that the largest number of vertices saturated by a matching in $G$ is $\min \{|V(G)| - o(G - S) + |S|\}$, where $o(G - S)$ is the number of odd components in $G - S$. For a bipartite graph, we define its odd girth to be infinite.

**Theorem 1.** Assume that $r \geq 2$ is an integer and $G$ is a connected graph with odd girth at least $g$, vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and degree sequence $\deg_G(v_1) \geq \deg_G(v_2) \geq \cdots \geq \deg_G(v_n)$. Moreover, suppose that $r \leq \max\left\{\frac{g}{2}, \frac{\deg_G(v_{r-1})+1}{4}\right\}$ and $\{v_1, \ldots, v_{r-1}\}$ forms an independent set. If every $\deg_G(v_{r-1})$ is an even integer or $\deg_G(v_{r-1}) > \deg_G(v_r)$, then $\chi(KG(G, rK_2)) = |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i)$.

**Proof.** Consider the subgraph of $G$ containing of exactly all edges incident to some $v_i$, for $1 \leq i \leq r - 1$. This subgraph does not have any $r$-matching; consequently, we have $\chi(KG(G, rK_2)) \leq |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i)$. Hence, it is sufficient to show that $G$ satisfies $\chi(KG(G, rK_2)) \geq |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i)$. Set $s$ to be the number of $v_i$’s such that $\deg_G(v_i)$ is an odd integer, where $1 \leq i \leq r - 1$. If $G$ is an even graph, set $H = G$; otherwise, add a new vertex $w$ and join it to any odd vertex of $G$ to obtain the graph $H$. Now, $H$ has an Eulerian tour $e_1', e_2', \ldots, e_m'$, where if $G$ is not an even graph, we start the Eulerian tour with $w$; otherwise, it starts with $v_n$. Consider the ordering $(e_1', e_2', \ldots, e_m')$ and remove all new edges incident with $w$ from this ordering to obtain the ordering $\sigma$ for the edge set of $G$. In other words, if we traverse the edge $e_i$ before than the edge $e_j$ in the Eulerian tour, then in the ordering $\sigma$ we have $e_i < e_j$. Now, assume that we have an alternating coloring (with colors blue and red) of edges of $G$ with respect to the ordering $\sigma$ of length $t$, where if $s \neq 0$, then $t = 1 + \sum_{i=1}^{r-1} \deg_G(v_i)$; otherwise, $t = 2 + \sum_{i=1}^{r-1} \deg_G(v_i)$. Consider the red (resp. blue) subgraph $G^R$ (resp. $G^B$), i.e., the spanning subgraph of $G$ whose edge set consists of all red (resp. blue)
edges. We show that if \( s = 0 \), then each of \( G^R \) and \( G^B \) has an \( r \)-matching; and consequently, \( \text{ex}_{\text{alt}}(G, rK_2, \sigma) \leq 1 + \sum_{i=1}^{r-1} \deg_G(v_i) \). Also, we show that, if \( s \neq 0 \), then either \( G^R \) or \( G^B \) has an \( r \)-matching; and consequently, \( \text{ex}_{\text{alt}}(G, rK_2, \sigma) \leq \sum_{i=1}^{r-1} \deg_G(v_i) \). These imply \( \chi(K_G(rK_2)) = |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i) \). In view of the ordering \( \sigma \), one can see that for any vertex \( x \) of \( G \) (except at most the first vertex of the Eulerian tour), color red (resp. blue) can be assigned to at most half of edges incident to it. This amount can be increased by at most one when \( t \) is an even integer, then any color can be assigned to at most half of edges incident to it. This amount can be increased by at most one when \( t \) is an odd integer. Now, the proof falls into two parts.

a) First, we show that if \( r \leq \frac{s}{2} \), then the assertion holds. In fact, if \( s \neq 0 \), then we determine the chromatic number by evaluating the alternating chromatic number. Otherwise, we show that the strong alternating chromatic number is equal to chromatic number. Assume that \( j \in \{R, B\} \). If \( G^j \) does not have any \( r \)-matching, then in view of the Tutte-Berge Formula, there exists an \( S^j \subseteq V(G^j) = V(G) \) such that \(|V(G^j)| - o(G^j - S^j) + |S^j| \leq 2r - 2 \). Suppose that \( O^j_1, O^j_2, \ldots, O^j_{t_j} \) are the components of \( G^j - S^j \). One can check that for each \( 1 \leq i \leq t_j \), \(|V(O^j_i)| \leq 2r - |S^j| - 1 \leq g - 1 \). Therefore, every component \( O^j_i \) does not have any odd cycle and so it would be a bipartite graph. Assume that \( O^j_i = O^j_i(X^j_i, Y^j_i) \) such that \(|X^j_i| \leq |Y^j_i| \) (\( X^j_i \) may be an empty set). Set \( X^j = \bigcup_{i=1}^{t_j} X^j_i \). Note that

\[
|X^j| = \sum_{i=1}^{t_j} \left[ \frac{|V(O^j_i)|}{2} \right] \leq \frac{|V(G^j)| - |S^j|}{2} + \frac{|V(G^j)| + |S^j| - 2r + 2}{2} = r - |S^j| - 1.
\]

Therefore, for \( s = 0 \) and any \( j \in \{R, B\} \), if \( G^j \) does not have any \( r \)-matching, then

\[
1 + \sum_{i=1}^{r-1} \frac{\deg_G(v_i)}{2} = |E(G^j)|
\]

\[
\leq \sum_{x \in S^j} \left[ \frac{\deg_G(x)}{2} \right] + \sum_{x \in X^j} \left[ \frac{\deg_G(x)}{2} \right]
\]

\[
\leq \sum_{i=1}^{r-1} \frac{\deg_G(v_i)}{2}
\]

which is impossible. This means that \( \text{ex}_{\text{alt}}(G, rK_2, \sigma) \leq 1 + \sum_{i=1}^{r-1} \deg_G(v_i) \). Accordingly, \( \chi(K_G(rK_2)) = |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i) \). Now, assume that \( s \neq 0 \).
Also, suppose that neither $G^R$ nor $G^B$ has any r-matching. In view of the assumption, either $\deg_G(v_{r-1})$ is an even integer or $\deg_G(v_{r-1}) > \deg_G(v_r)$. Hence,

$$1 + \sum_{i=1}^{r-1} \deg_G(v_i) = |E(G^R)| + |E(G^B)|$$

$$\leq \sum_{j} \sum_{x \in S_j} \deg_{G^j}(x) + \sum_{j} \sum_{x \in X^j} \deg_{G^j}(x)$$

$$\leq \sum_{i=1}^{r-1} \deg_G(v_i)$$

which is a contradiction. This means that $ex_{alt}(G, rK_2, \sigma) \leq \sum_{i=1}^{r-1} \deg_G(v_i)$.

Accordingly, $\chi(KG(G, rK_2)) = |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i)$.

b) Now, we show that if $r \leq \frac{\deg_G(v_{r-1})+1}{4}$, then the assertion holds. Assume that $j \in \{R, B\}$. If $G^j$ does not have any r-matching, in view of the Tutte-Berge Formula, there exists an $S^j \subseteq V(G^j)$ such that $|V(G^j)| - o(G^j - S^j) + |S^j| \leq 2r - 2$. Assume that $O^j_1, O^j_2, \ldots, O^j_{t_j}$ are all components of $G^j - S^j$, where $t_j \geq o(G^j - S^j)$. We consider two different cases $s = 0$ and $s > 0$. First assume that $s = 0$ and so $t = 2 + \sum_{i=1}^{r-1} \deg_G(v_i)$. Now, we show that each of $G^R$ and $G^B$ has a matching of size $r$. Assume that $G^R$ (resp. $G^B$) does not have any matching of size $r$. Therefore,

$$|E(G^R)| \leq \sum_{x \in S^R} \deg_{G^R}(x) + \sum_{i=1}^{t_R} \left( \frac{|V(O^R_i)|}{2} \right)$$

$$\leq \sum_{x \in S^R} \deg_{G^R}(x) + \left( \sum_{i=1}^{t_R} |V(O^R_i)| - (t_R - 1) \right)$$

$$\leq \sum_{x \in S^R} \deg_{G^R}(x) + \left( 2r - 2|S^R| - 1 \right)$$

$$\leq \frac{1}{2} \sum_{i=1}^{r-1} \deg_G(v_i)$$

which is impossible. Now, assume that $s \neq 0$. We show that $G^R$ or $G^B$ has a matching of size $r$. On the contrary, suppose that both $G^R$ and $G^B$ do not have any matching of size $r$. First, suppose that $|S^R| \neq r - 1$ or $|S^B| \neq r - 1$. 

\[ \]
Note that $1 + \sum_{i=1}^{r-1} \deg_G(v_i) = t = |E(G^R)| + |E(G^B)|$. On the other hand,

$$t \leq \sum_j \sum_{x \in S^j} \deg_{G^j}(x) + \sum_j \sum_{i=1}^{t_j} \left( \frac{|V(O^j_i)|}{2} \right)$$

$$\leq \sum_j \sum_{x \in S^j} \deg_{G^j}(x) + \sum_j \left( \sum_{i=1}^{t_j} \frac{|V(O^j_i)|}{2} - (t_j - 1) \right)$$

$$\leq \sum_j \sum_{x \in S^j} \deg_{G^j}(x) + \min \left( \frac{r - 1}{2}, \frac{|S^R| + |S^B|}{2} \right) + \sum_j \left( 2r - 2|S^j| - 1 \right)$$

$$\leq \sum_j \sum_{x \in S^j} \deg_{G^j}(x) + \sum_j \left( r - |S^j| - 1 \right) \frac{\deg_G(v_{r-1})}{2}$$

$$\leq \sum_{i=1}^{r-1} \deg_G(v_i)$$

which is impossible. If $|S^R| = |S^B| = r - 1$, then each connected component of $G^j - S^j$ is a single vertex. Hence,

$$1 + \sum_{i=1}^{r-1} \deg_G(v_i) \leq |E(G^R)| + |E(G^B)| \leq \sum_j \sum_{x \in S^j} \deg_{G^j}(x) \leq \sum_{i=1}^{r-1} \deg_G(v_i),$$

a contradiction. Consequently, $\text{ex}_{alt}(G, rK_2, \sigma) \leq \sum_{i=1}^{r-1} \deg_G(v_i)$; and accordingly,

$$\chi(KG(G, rK_2)) = |E(G)| - \sum_{i=1}^{r-1} \deg_G(v_i).$$

Note that $KG(C_n, rK_2) \simeq \text{SG}(n, r)$. Hence, the aforementioned theorem can be considered as a generalization of Schrijver’s Theorem [19].

**Theorem C.** [19] For any positive integers $n$ and $r$, where $n \geq 2r$, we have $\chi(\text{SG}(n, r)) = n - 2r + 2$.

**Corollary 1.** Assume that $G$ is a connected non-bipartite $k$-regular graph with odd girth at least $g$, where $k$ is an even integer. For any $r \leq \frac{g}{2}$, we have $\chi(KG(G, rK_2)) = |E(G)| - k(r-1)$.

**Proof.** Let $C$ be a minimal odd cycle in $G$. Note that $C$ is an induced subgraph of $G$ and $|V(C)| \geq g$. Consequently, it contains an independent set of size $\left\lfloor \frac{g}{2} \right\rfloor$. Therefore, in view of Theorem 1, the proof is completed. ■

Assume that $s \geq t$ are positive integers and $G = G(X, Y)$ is a connected $(s, t)$-regular connected graph. Theorem 1 implies that if $s$ is an even integer, then for any $r \leq |X|$ we have $\chi(KG(G, rK_2)) = s(|X| - r + 1)$.
2.3 Kneser Multigraph

As we have seen, there are several open problems which are about the equality of $\chi(KG(G,F))$ and $|E(G)| - \text{ex}(G,F)$. Here, we show that when $G$ is a multigraph and $F$ is a family of simple graphs, then the equality always holds.

**Theorem 2.** Assume that $G$ is a multigraph such that the multiplicity of each edge of $G$ is at least two. If $F$ is a family of simple subgraphs of $G$, then we have

$$\chi(KG(G,F)) = \chi_{alt}(KG(G,F)) = |E(G)| - \text{ex}(G,F).$$

In particular, if the multiplicity of each edge is an even integer, then

$$\chi(KG(G,F)) = \chi_{salt}(KG(G,F)).$$

**Proof.** First, we show that $\chi(KG(G,F)) = \chi_{alt}(KG(G,F))$. In view of Lemma 3, it is sufficient to show that $\text{ex}_{alt}(G,F) = \text{ex}(G,F)$. Assume that $E(G) = I_1 \cup \cdots \cup I_t$ is a partition of $E(G)$, where for any $1 \leq i \leq t$, there are two distinct vertices $u$ and $v$ such that $I_i$ consists of all edges incident with both of $u$ and $v$. Since the multiplicity of each edge is at least two, we have for any $1 \leq i \leq t$, $|I_i| \geq 2$. Consider an ordering $\sigma$ for the edge set of $G$ such that all edges of each $I_j$ appear consecutively in the ordering (they form an interval in this ordering). Now, we claim that $\text{ex}_{alt}(G,F,\sigma) = \text{ex}(G,F)$. To see this, consider an alternating coloring of a subset of edges of $G$ of length more than $\text{ex}(G,F)$ with respect to the ordering $\sigma$. In view of the ordering $\sigma$, for any $1 \leq j \leq t$ or both colors are assigned to some edges of $I_j$, or exactly one edge of $I_j$ is colored, or no color is assigned to members of $I_j$. Assume that $C_1 = \{I_{j_1}, I_{j_2}, \ldots, I_{j_{k_1}}\}$ is the set of all $I_{j_i}$’s such that both colors are assigned to some edges of $I_{j_i}$, $C_2 = \{I_{i_1}, I_{i_2}, \ldots, I_{i_{k_2}}\}$ is the set of all $I_{i_j}$’s such that just red color is assigned to some edges of $I_{i_t}$, and $C_3 = \{I_{t_1}, I_{t_2}, \ldots, I_{t_{k_3}}\}$ is the set of all $I_{t_i}$’s such that just blue color is assigned to some edges of $I_{t_i}$. Assume that $|C_2| = k_2 \geq k_3 = |C_3|$. Denote the spanning subgraph containing all the red edges (resp. the blue edges) by $G^R$ (resp. $G^B$). We claim that the subgraph $G^R$ contains some member of $F$. On the contrary, suppose that this is not true. Therefore, the subgraph $H$ consisting of all edges in $(\cup_{i=1}^{k_1} I_{j_i}) \cup (\cup_{i=1}^{k_2} I_{i_t})$ does not contain any members of $F$ and so $|E(H)| \leq \text{ex}(G,F)$. But, we have

$$|E(G^R)| + |E(G^B)| \leq \sum_{i=1}^{k_1} |I_{j_i}| + k_2 + k_3$$

$$\leq \sum_{i=1}^{k_1} |I_{j_i}| + 2k_2$$

$$\leq \sum_{i=1}^{k_1} |I_{j_i}| + \sum_{i=1}^{k_2} |I_{i_t}|$$

$$= |E(H)| \leq \text{ex}(G,F)$$

which is a contradiction. In fact, the number of colored edges, i.e., $|E(G^R)| + |E(G^B)|$, is more than $\text{ex}(G,F)$. 

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To prove the second part of theorem, in view of Lemma 3, it is sufficient to show that \( \text{ex}_{salt}(G, F) = \text{ex}(G, F) + 1 \). To see this, consider an alternating coloring of a subset of edges of \( G \) of length more than \( \text{ex}(G, F) + 1 \) with respect to the mentioned ordering \( \sigma \). Consider \( C_1 \), \( C_2 \), and \( C_3 \) as defined in the previous part. We claim that the red subgraph \( G^R \), and the blue subgraph \( G^B \), each of them contains some member of \( F \). Note that \( \text{ex}(G, F) \) is an even integer. Also, both of \( G^R \) and \( G^B \) have at least \( 1 + \frac{\text{ex}(G, F)}{2} \) edges. On the contrary, suppose that \( G^R \) does not have any members of \( F \). Therefore, the subgraph \( H \) consisting of all edges in \((\cup_{i=1}^{k_1}I_j) \cup (\cup_{i=1}^{k_2}I_i)\) does not have any members of \( F \) and so \( |E(H)| \leq \text{ex}(G, F) \). Since all multiplicities are even, for any \( i \), red color can be assigned to at most \( \frac{|I_i|}{2} \) edges of \( I_i \). This implies that

\[
1 + \frac{\text{ex}(G, F)}{2} \leq |E(G^R)| \leq \frac{1}{2} \sum_{i=1}^{k_1} |I_j| + k_2 \leq \frac{|E(H)|}{2} \leq \frac{\text{ex}(G, F)}{2},
\]

which is a contradiction. Similarly, the subgraph \( G^B \) has some member of \( F \) and this implies \( \chi(KG(G, F)) = \chi_{salt}(KG(G, F)) \).

**Theorem 3.** Let \( H \) be a simple graph and \( F \) be a family of subgraphs of \( H \). Assume that \( G \) is obtained from \( H \) by giving the same multiplicity \( r \geq 2 \) to some edges of \( H \). If the subgraph of \( H \) corresponding to the edges of \( G \) with multiplicity \( r \) has an \( F \)-free subgraph with \( \text{ex}(H, F) \) edges, then

\[
\chi(KG(G, F)) = \chi_{alt}(KG(G, F)) = |E(G)| - \text{ex}(G, F).
\]

In particular, if \( r \) is an even integer, then

\[
\chi(KG(G, F)) = \chi_{salt}(KG(G, F)).
\]

**Proof.** First, note that \( \text{ex}(G, F) = r \cdot \text{ex}(H, F) \); and therefore, in view of Lemma 3, it is sufficient to show that \( \text{ex}_{alt}(G, F) = r \cdot \text{ex}(H, F) \). Assume that \( E(G) = I_1 \cup \cdots \cup I_t \cup I_{t+1} \cup \cdots \cup I_m \) is a partition of \( E(G) \), where for any \( 1 \leq i \leq m \), there are two distinct vertices \( u \) and \( v \) such that \( I_i \) consists of all edges incident with both of \( u \) and \( v \), and moreover, \( |I_i| = r \geq 2 \) for any \( 1 \leq i \leq t \); otherwise, \( |I_i| = 1 \). Consider an ordering \( \sigma \) for the edge set of \( G \) such that all edges of each \( I_j \) appear consecutively in the ordering (they form an interval in this ordering) and the edges of \( I_{t+1}, \ldots, I_m \) are located at the end of this ordering. Now, we show that \( \text{ex}_{alt}(G, F, \sigma) = \text{ex}(G, F) \). To see this, consider an alternating coloring of a subset of edges of \( G \) of length more than \( \text{ex}(G, F) \) with respect to the ordering \( \sigma \). In view of the ordering \( \sigma \), for any \( 1 \leq j \leq m \), both colors are assigned to some edges of \( I_j \), or exactly one edge of \( I_j \) is colored, or no color is assigned to any member of \( I_j \). Denote the number of \( j \)'s such that both colors are assigned to some edges of \( I_j \) by \( k_1 \), the number of \( j \)'s such that just red color is assigned to some edges of \( I_j \) by \( k_2 \), and the number of \( j \)'s such that just blue color is assigned to some edges of \( I_j \) by \( k_3 \). Without loss of generality, assume that \( k_2 \geq k_3 \). Now, we have \( k_2 + k_3 > \text{ex}(G, F) - rk_1 = r \cdot \text{ex}(H, F) - rk_1 \geq 2(\text{ex}(H, F) - k_1) \). Therefore, \( k_1 + k_2 > \text{ex}(H, F) \) and so the red subgraph should have some member of \( F \), which completes the proof.
To prove the second part, in view of Lemma 3, it is sufficient to show that \(\text{ex}_{\text{alt}}(G, \mathcal{F}) = r \cdot \text{ex}(H, \mathcal{F}) + 1\). To see this, consider an alternating coloring of a subset of edges of \(G\) of length more than \(\text{ex}(G, \mathcal{F}) + 1\) with respect to the mentioned ordering \(\sigma\). Consider \(k_1, k_2,\) and \(k_3\), as defined in the previous part. Since \(r\) is an even integer, we have

\[
\frac{r}{2} k_1 + k_2 \geq \frac{r}{2} \cdot \text{ex}(H, \mathcal{F}) + 2 = \frac{r}{2} \cdot \text{ex}(H, \mathcal{F}) + 1.
\]

We need to show that \(k_1 + k_2 \geq \text{ex}(H, \mathcal{F}) + 1\) and \(k_1 + k_3 \geq \text{ex}(H, \mathcal{F}) + 1\). We just prove the first equality, because of similarity. On the contrary, suppose \(k_1 + k_2 \leq \text{ex}(H, \mathcal{F})\). Therefore,

\[
\frac{r}{2} \cdot \text{ex}(H, \mathcal{F}) + 1 \leq \frac{r}{2} k_1 + k_2 \leq \frac{r}{2} (\text{ex}(H, \mathcal{F}) - k_2) + k_2 = k_2 (1 - \frac{r}{2}) + \frac{r}{2} \cdot \text{ex}(H, \mathcal{F})
\]

which is a contradiction. ■

2.4 Path Graphs

We know that the general Kneser graph \(KG(C_n, P_d)\) is isomorphic to the circular complete graph \(K_{n,d}\). Hence, this motivates us to study the chromatic number of the path graph \(KG(G, P_d)\). Also, in view of Lemma 2, one can see that for any general Kneser graph \(KG(G, \mathcal{F})\), we have \(|E(G)| - 2 \cdot \text{ex}(G, \mathcal{F}) \leq \chi(KG(G, \mathcal{F})) \leq |E(G)| - \text{ex}(G, \mathcal{F})\). We introduced several families of graphs whose chromatic numbers attain the lower or upper bound. Hence, it may be of interest to present some general Kneser graphs whose chromatic numbers lie strictly between the upper bound and the lower bound. Now, by determining the chromatic number of the path graph \(KG(G, P_2)\), we show that this graph has such a property provided that \(G\) is a dense graph.

**Lemma 4.** If a graph \(G\) has \(n\) vertices and \(e\) edges, then it has at least \(\frac{2e}{n} (e - \frac{n}{2})\) subgraphs isomorphic to \(P_2\).

**Proof.** One can see that the number of subgraphs of \(G\) isomorphic to \(P_2\) is exactly \(\sum_{i=1}^{n} \binom{\text{deg}(v_i)}{2}\). In view of Jensen’s inequality, we have \(\sum_{i=1}^{n} \binom{\text{deg}(v_i)}{2} \geq n \left( \frac{\sum_{i=1}^{n} \text{deg}(v_i)}{2} \right)^2 \) which completes the proof. ■

Assume that \(S\) is an independent set of general Kneser graph \(KG(G, \mathcal{F})\). We call the independent set \(S\) an *intersecting* independent set, if there is an edge \(e\) of \(G\) appeared in each member of \(S\). Otherwise, it is termed a *non-intersecting* independent set of \(KG(G, \mathcal{F})\). One can easily check that a non-intersecting independent set in \(G(n, P_2)\) has at most three members.

**Theorem 4.** Assume that \(G\) is a connected graph with \(n\) vertices. Also, suppose that \(G\) has a spanning subgraph \(H\) whose connected components are \(H_1, H_2, \ldots, H_p\), where for any \(1 \leq i \leq p - 1\), \(H_i\) is a triangle and \(H_p \in \{K_2, K_3, 2K_2\}\). We have \(\chi(KG(G, P_2)) = |E(G)| - \left\lfloor \frac{2}{3} n \right\rfloor\).
**Proof.** First, we show that there exists a proper coloring for $KG(G, P_2)$ using $|E(G)| - \left\lfloor \frac{2n}{3} \right\rfloor$ colors. Set

$$T = \{e_1, e_2, \ldots, e_l\} = E(G) \setminus \cup_{i=1}^{p} E(H_i).$$

For any $P_2$ subgraph of $G$, it either has a nonempty intersection with $T$ or is a subgraph of some $H_i$, for $1 \leq i \leq p$. Note that if $P_2$ is a subgraph of $H_p$, then $H_p = K_3$. If a $P_2$ subgraph of $G$, say $H$, is a subgraph of $H_i$, then we assign it the color $l + i$. Otherwise, assume that $j$ is the least integer such that $e_j \in E(H)$ and assign the color $j$ to $H$. One can check that this coloring is a proper coloring with $|E(G)| - \left\lfloor \frac{2n}{3} \right\rfloor$ colors.

Assume that there exists a proper coloring of $KG(G, P_2)$ with $\chi(KG(G, P_2))$ colors which has $t$ intersecting color classes and $s$ non-intersecting color classes. If $t \geq |E(G)| - \left\lfloor \frac{2n}{3} \right\rfloor$, then there is nothing to prove. Therefore, we can assume that $t < |E(G)| - \left\lfloor \frac{2n}{3} \right\rfloor$. For each intersecting color class, remove an edge which appears in each member of this class to obtain $\bar{G}$. The graph $\bar{G}$ has $|E(G)| - t$ edges; and therefore, in view of Lemma 4, it has at least $\frac{2(|E(G)| - t)(|E(G)| - t - \frac{4}{3})}{n}$ subgraphs isomorphic to $P_2$. Since every non-intersecting class has at most 3 members, we have

$$\frac{2(|E(G)| - t)(|E(G)| - t - \frac{4}{3})}{3n} + t \leq \chi(KG(G, P_2)).$$

Now, set $x = |E(G)| - t > \left\lfloor \frac{2n}{3} \right\rfloor$. Thus, we have $p(x) = \frac{2}{3n} x(x - \frac{4}{3}) + |E(G)| - x \leq \chi(KG(G, P_2))$. One can check that $p(x)$ takes its minimum in $x = n$, which is $|E(G)| - \frac{2}{3}n$. 

Hajnal and Szemerédi [9] showed that for a graph $G$ if its minimum degree is at least $(1 - \frac{1}{3})n$, then $G$ contains $\left\lfloor \frac{n}{2} \right\rfloor$ vertex-disjoint copies of $K_r$. Corrádi and Hajnal [5] investigated the maximum number of vertex-disjoint cycles in a graph. They showed that if $G$ is a graph of order at least $3k$ with minimum degree at least $2k$, then $G$ contains $k$ vertex-disjoint cycles. In particular, when the order of $G$ is exactly $3k$, then $G$ contains $k$ vertex-disjoint triangles. Next, this result was extended as follows.

**Theorem D.** [21] Let $G$ be a graph of order at least $3k$, where $k$ is a positive integer. If for any pair of nonadjacent vertices $x$ and $y$ of $G$, we have $\deg_G(x) + \deg_G(y) \geq 4k - 1$, then $G$ contains $k$ vertex-disjoint cycles.

Assume that $G$ is a graph such that $\deg_G(x) + \deg_G(y) \geq 4k - 1$ for any pair of nonadjacent vertices $x$ and $y$ of $G$. In the previous theorem, if we set $k = \left\lfloor \frac{n}{2} \right\rfloor$, then $n = 3k + r$, where $0 \leq r \leq 2$. If $r = 0$, then $G$ has $k$ vertex-disjoint cycles. Obviously, these cycles should be triangles.

**Corollary 2.** Assume that $G$ is a graph with $n$ vertices. If for any two nonadjacent vertices $x$ and $y$, we have $\deg_G(x) + \deg_G(y) \geq \frac{4n}{3} - 1$, then $\chi(KG(G, P_2)) = |E(G)| - \left\lfloor \frac{2}{3}n \right\rfloor$.
Proof. We show that $G$ has vertex-disjoint subgraphs $H_1, H_2, \ldots, H_p$ such that $V(G) = \bigcup_{i=1}^{p} V(H_i)$ and for any $1 \leq i \leq p - 1$, $H_i$ is a triangle and $H_p \in \{K_2, K_3, 2K_2\}$.

Assume that $n = 3k + r$, where $0 \leq r \leq 2$. In view of Theorem D, $G$ has $k$ vertex-disjoint cycles. We consider three different cases.

Case 1: If $r = 0$, then $n = 3k$ and $G$ has $p = k$ vertex-disjoint cycles $H_1, H_2, \ldots, H_p$ and all of these cycles should be triangles. Also, clearly we have $V(G) = \bigcup_{i=1}^{p} V(H_i)$.

Case 2: For $r = 1$, the graph $G$ has $p = k$ vertex-disjoint cycles where $k - 1$ of these cycles, say $H_1, H_2, \ldots, H_{k-1}$, are triangles and the other cycle, say $C$, can be a $C_4$ or a triangle.

If $C$ is a $C_4$, then remove two nonadjacent edges of it to obtain $H_k \simeq 2K_2$. Otherwise, if $C$ is a triangle, then assume that $z$ is a vertex that is not in $V(C) \cup (\bigcup_{i=1}^{k-1} V(H_i))$. Consider $u \in V(G)$ such that $uz \in E(G)$. Without loss of generality, we can assume that $V(C) = \{u, v, w\}$. Now, consider $H_k \simeq 2K_2$ with the vertex set $\{z, u, v, w\}$ and the edge set $\{zu, vw\}$. One can check that $V(G) = \bigcup_{i=1}^{p} V(H_i)$.

Case 3: If $r = 2$, we add a new vertex $z$ to $G$ and join it to all of vertices of $G$ to obtain the graph $G'$. The graph $G'$ has $3(k+1)$ vertices and for any two nonadjacent vertices $x$ and $y$, $\deg_{G'}(x) + \deg_{G'}(y) \geq 4k_2 - 1 + 2 \geq 4(k+1) - 1$. Therefore, by Theorem D, $G'$ has $k + 1 = p$ vertex-disjoint triangles. By removing the vertex $z$, $G$ has a spanning subgraph $H$ such that it has $k + 1$ connected components where $k$ of them are triangles and one of them is $K_2$.

Now, in view of Theorem 4, the assertion follows. $\blacksquare$

The degree condition mentioned in the aforementioned corollary cannot be dropped. To see this, one can check that $\chi(KG(C_5, P_2))) = 3 \neq |E(C_5)| - \lfloor \frac{10}{3} \rfloor$.

2.5 Concluding Remarks

In Lemma 2, we introduced a lower and upper bound for the chromatic number of graphs in terms of the generalized Turán number. It can be of interest to find necessary and sufficient conditions to know when the equality holds in both cases. A subgraph $H$ of $G$ is locally dense in $G$, if there exists a set $A$ of vertices of $H$ such that any edge of $H$ is incident to some vertex of $A$ and that almost all edges incident to the vertices of $A$ are included in $H$. For instance, one can see that $K_{1,n-1}$ is locally dense in $K_n$. For a graph $G$ and a family of its subgraphs $\mathcal{F}$, if any member of $\text{EX}(G, \mathcal{F})$ is locally dense in $G$, then in view of the ordering obtained by an Eulerian tour, the alternating Turán number is almost the same as the generalized Turán number. Consequently, in this case it seems more likely, we have $\chi(KG(G, F)) = |E(G)| - \text{ex}(G, F)$.

**Question 1.** Let $G$ be a graph and $\mathcal{F}$ be a family of its subgraphs. When does the equality $\chi(KG(G, F)) = |E(G)| - \text{ex}(G, F)$ hold?
Also, note that we can generalize the definition of Kneser representation by considering labeled hypergraphs, i.e., we assign some labels to the hyperedges or vertices of $H$ and $F$. In this terminology, the hypergraph $\left( E(H) \right)_{F}$ (resp. $\left( V(H) \right)_{F}$) has $E(H)$ (resp. $V(H)$) as its vertex set and each subhypergraph of $H$ isomorphic to a member of $F$ forms a hyperedge, where any isomorphism should preserve the labels.

One can define the hypergraphs $KG^r(H, F)$ and $KG_v(H, F)$ similar to unlabeled ones. This new definition helps us to introduce some appropriate representations for some families of graphs. For instance, here we consider the Cartesian sum of Kneser graphs.

The Cartesian sum $G \oplus H$ of two graphs $G$ and $H$ has the vertex set $V(G) \times V(H)$ and two vertices $(u, v)$ and $(a, b)$ are adjacent, if either $u$ is adjacent to $a$ or $v$ is adjacent to $b$. The chromatic number of Cartesian sum of graphs has been studied in several papers, see [15, 22]. In general, it seems that it is not easy to evaluate the chromatic number of the Cartesian sum of two graphs. One can check that $\chi(G \oplus H) \leq \chi(H)\chi(G)$. In [15], this bound was improved and also the chromatic number of Cartesian sum of circular complete graphs was determined. As a natural question, it can be of interest to know the chromatic number of Cartesian sum of Kneser graphs. In particular, we are interested in finding the chromatic number of $\chi(KG(m, n), K(p, q))$. Note that $KG(m, p) \oplus KG(n, p)$ is isomorphic to $KG(K_{m,n}, K_{p,p})$. The problem of finding the chromatic number of $KG(K_{m,n}, K_{p,q})$ can be considered as a twin of Problem B. In view of Lemma 2, we have

$$mn - 2ex(K_{m,n}, K_{p,p}) \leq \chi(KG(m, p) \oplus KG(n, p)) \leq mn - ex(K_{m,n}, K_{p,p}).$$

One can see that the graph $KG(K_{m,n}, K_{p,q})$ is isomorphic to $KG(m, p) \oplus KG(n, q)$, where the labeled graph $K_{m,n}$ is obtained from the complete bipartite graph $K_{m,n}$ by assigning the label one to the vertices of the part with size $m$ and the label two to the others (or by assigning the same direction to all edges form the part of size $m$ to the other part). Note that if $p \neq q$ and $\min\{m, n\} \geq \max\{p, q\}$, then the graph $KG(K_{m,n}, K_{p,q})$ is not isomorphic to the graph $KG(K_{m,n}, K_{p,q})$.

**Question 2.** Assume that $m, n, p,$ and $q$ are positive integers. What are the values of $\chi(KG(K_{m,n}, K_{p,q}))$ and $\chi(KG(K_{m,n}, K_{p,q}))$ ?

**Acknowledgement:** The authors gratefully acknowledge for many stimulating conversations and the many helpful suggestions of Professor Carsten Thomassen during the preparation of the paper. Also, they wish to thank Dr. Saeed Shaebani for his useful comments. Furthermore, they would like to thank Skype for sponsoring their endless conversations in two countries.

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