ON SEMIGROUP ORBITS OF POLYNOMIALS IN
SUBGROUPS

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Abstract. We study intersections of semigroup orbits in polynomial
dynamics with multiplicative subgroups, extending results of
Ostafe and Shparlinski (2010).

1. Introduction

Let $K$ be a field of characteristic 0 and $\overline{K}$ its algebraic closure. Let
$\mathcal{F} = \{\phi_1, ..., \phi_k\} \subset K[X]$ be a set of polynomials of degree at least 2,
let $x \in K$, and let
$$O_\mathcal{F}(x) = \{\phi_{i_n} \circ ... \circ \phi_{i_1}(x) | n \in \mathbb{N}, i_j = 1, ..., k\}$$
denote the forward orbit of $P$ under $\mathcal{F}$. We denote the $n$-dimensional
torus $G^n_m$ as $(\mathbb{Q}^*)^n$ endowed with the group law defined by the multi-
plication coordinate by coordinate.

For $S \subset K$ reasonably sparse and somehow unrelated to $\mathcal{F}$, it is
natural to study the intersection $O_\mathcal{F}(x) \cap S$. A generalisation of this
situation is to study the intersection of orbits generated by multiva riate
polynomials with higher dimensional algebraic varieties. This is known
as the dynamical Mordell-Lang conjecture, for which we refer [1]. In the
univariate case, when $S = \mathbb{U}$ is the set of roots of unity and the initial
points are defined over the cyclotomic closure $K^c = K(\mathbb{U})$ over an
algebraic number field, Ostafe [4] has proved finiteness for such points
that are preperiodic for the initial polynomial.

When $k = 1$ and $S \subset K$ has certain multiplicative properties in the
univariate case (e.g. a finitely generated group $\Gamma \subset K^*$) Ostafe and
Shparlinski [5] have provided results for the frequency of intersections
of polynomial orbits with such sets. Namely, they have proved that
$$\#\{n \leq N : f^{(n)}(x) \in \Gamma\} \leq \frac{(10 \log \deg f + o(1))N}{\log \log N}, \text{ as } N \to \infty,$$
for $f \in K[X], x \in K$.

In this paper we seek to generalise results of this sort when the
dynamical systems are generated as semigroups under composition by
several maps initially. Precisely, putting $\mathcal{F}_n = \{\phi_{i_n} \circ ... \circ \phi_{i_1} | 1 \leq i_j \leq k\}$
for the \( n \)-level set, and supposing that \( \{t_N\}_N^{+\infty} \) is a sequence of positive integers going to \( \infty \) satisfying that
\[
\#\{v \in \Gamma|v = f(u), f \in \mathcal{F}_n, n \leq N\} \geq ck^t_N
\]
for each \( N \), where \( c > 0 \) is a constant, we prove among other results that
\[
\#\{v \in \Gamma|v = f(u), f \in \mathcal{F}_n, n \leq N\} \leq \exp(\exp((10 \log d + o(1))t_N))
\]
as \( N \to \infty \), where \( d = \max_i \deg \phi_i \). Namely, if the number of orbit points of iteration order at most \( N \) falling on a finitely generated group is bigger than a multiple of the size of the complete \( k \)-tree of depth \( t_N - 1 \), then such pursued number grows slower than a sequence obtained by exponentiating twice a multiple of the sequence \( \{t_N\} \).

In particular, if our conditions are satisfied with
\[
t_N \sim \frac{1}{10 \log d + o(1)} \log \log \left(\frac{(10 \log d + o(1))N}{\log \log N}\right), \text{ as } N \to \infty,
\]
then we can generalize and recover the results of [5] under such conditions.

In Section 2 we set some notations and facts about heights, orbits and finitely generated groups. In Section 3 we recall some arithmetic and combinatoric results that are used to obtain results of frequency with orbits generated by a sequence of maps in Section 4. In Section 5 we state a necessary recent graph theory result that is used in Section 6 to obtain results about the frequency of intersection of polynomial semigroup orbits with sets.

2. Preliminary notations

Let \( K \) be a field of characteristic 0 and \( \overline{K} \) its algebraic closure. For \( x \in \overline{\mathbb{Q}} \), the naive logarithmic height \( h(x) \) is given by
\[
\sum_{v \in M_K} \frac{[K_v: \mathbb{Q}_v]}{[K: \mathbb{Q}]} \log(\max\{1, |x|_v\}),
\]
where \( M_K \) is the set of places of \( K \), \( M_K^\infty \) is the set of archimedean (infinite) places of \( K \), \( M_K^0 \) is the set of nonarchimedean (finite) places of \( K \), and for each \( v \in M_K \), \(|.|_v\) denotes the corresponding absolute value on \( K \) whose restriction to \( \mathbb{Q} \) gives the usual \( v \)-adic absolute value on \( \mathbb{Q} \). Also, we write \( K_v \) for the completion of \( K \) with respect to \(|.|_v\), and we let \( \mathbb{C}_v \) denote the completion of an algebraic closure of \( K_v \). To simplify notation, we let \( d_v = [K_v: \mathbb{Q}_v]/[K: \mathbb{Q}] \). Let \( \mathcal{F} = \{\phi_1, \ldots, \phi_k\} \subset K[X] \) be a set of polynomials of degree at least 2, let \( x \in K \), and let \( \mathcal{O}_\mathcal{F}(x) = \{\phi_{i_1} \circ \cdots \circ \phi_{i_k}(x) | n \in \mathbb{N}, i_j = 1, \ldots, k\} \) denote the forward orbit of \( P \) under \( \mathcal{F} \). We denote the \( n \)-dimensional torus
$\mathbb{G}_m^n$ as $(\mathbb{Q})^n$ endowed with the group law defined by the multiplication coordinate by coordinate.

**Definition 2.1.** A polynomial $F \in \mathbb{Q}[X,Y]$ is said to be special if it has a factor of the form $aX^mY^n - b$ or $aX^m - bY^n$ for some $a, b \in \mathbb{Q}$ and $m, n \geq 0$. Otherwise we call $F$ to be non-special.

**Definition 2.2.** For a finitely generated group $\Gamma \subset \mathbb{G}_m^n$, we define the division group $\overline{\Gamma}$ by

$$\overline{\Gamma} = \{ x \in \mathbb{G}_m^n | \exists t \in \mathbb{N} \text{ with } x^t \in \Gamma \}.$$

**Definition 2.3.** For $E, \epsilon \geq 0$ and a set $S \subset \mathbb{G}_m^n$, we define the sets

$$\mathcal{B}_n(S, E) = \{ x \in \mathbb{G}_m^n | \exists y, z \in \mathbb{G}_m^n \text{ with } x = yz, y \in S, h(z) \leq E \}$$

and

$$\mathcal{C}_n(S, \epsilon) = \{ x \in \mathbb{G}_m^n | \exists y, z \in \mathbb{G}_m^n \text{ with } x = yz, y \in S, h(z) \leq \epsilon(1 + h(y)) \}.$$

We also omit the subscript $n$ for $n = 1$ writing

$$\mathcal{B}(S, E) = \mathcal{B}_1(S, E) \text{ and } \mathcal{C}(S, \epsilon) = \mathcal{C}_1(S, \epsilon).$$

We also write $\mathcal{A}(K, H)$ for the set of elements in the field of height at most $H$, namely

$$\mathcal{A}(K, H) = \{ x \in \mathbb{K} | h(x) \leq H \}.$$

For $\mathcal{F} = \{ \phi_1, \ldots, \phi_k \}$, we set

$$J = \{ 1, \ldots, k \}, \quad W = \prod_{i=1}^{\infty} J_i \quad \text{and} \quad \Phi_{w} := (\phi_{w_j})_{j=1}^{\infty}$$

to be a sequence of polynomials from $\mathcal{F}$ for $w = (w_j)_{j=1}^{\infty} \in W$.

In this situation we let

$$\Phi_{w}^{(n)} = \phi_{w_n} \circ \cdots \circ \phi_{w_1} \quad \text{with } \Phi_{w}^{(0)} = \text{Id}, \text{ and also } \mathcal{F}_n := \{ \Phi_{w}^{(n)} | w \in W \}.$$

Precisely, we consider polynomials sequences $\Phi = (\phi_{i_j})_{j=1}^{\infty} = \prod_{i=1}^{\infty} \mathcal{F}_i$ and $x \in \overline{\mathcal{F}}$, denoting $\Phi(x) := \phi_{i_n}(\cdots(\phi_{i_1}(x)))$.

The set

$$\{ x, \Phi^{(1)}(x), \Phi^{(2)}(x), \Phi^{(3)}(x), \ldots \} = \{ x, \phi_{i_1}(x), \phi_{i_2}(\phi_{i_1}(x)), \phi_{i_3}(\phi_{i_2}(\phi_{i_1}(x))), \ldots \}$$

is called the forward orbit of $x$ under $\Phi$, denoted by $\mathcal{O}_\Phi(x)$.

The point $x$ is said to be $\Phi$-preperiodic if $\mathcal{O}_\Phi(x)$ is finite.

For a $x \in K$, the $\mathcal{F}$-orbit of $x$ is defined as

$$\mathcal{O}_F(x) = \{ \phi(x) | \phi \in \bigcup_{n \geq 1} \mathcal{F}_n \} = \{ \Phi_{w}^{(n)}(x) | n \geq 0, w \in W \} = \bigcup_{w \in W} \mathcal{O}_{\Phi_{w}}(x).$$

The point $x$ is called preperiodic for $\mathcal{F}$ if $\mathcal{O}_F(x)$ is finite.

For $S \subset K$ and an integer $N \geq 1$, we use $T_{x,\Phi}(N, S)$ to denote the number of $n \leq N$ with $\Phi^{(n)}(x) \in S$, namely,

$$T_{x,\Phi}(N, S) = \# \{ n \leq N | \Phi^{(n)}(x) \in S \}.$$
For $f = \sum_{i=0}^{d} a_i X^i \in \overline{K}[X]$ and $K$ a field containing all the coefficients of $f$, denote the weil height of $f$ by
\[ h(f) = \sum_{v \in M_K} d_v \log(\max_i |a_i|_v), \]
and for the system of polynomials $F = \{\phi_1, ..., \phi_k\}$, denote $h(F) = \max_i h(\phi_i)$.

We revisit the following bound calculated in other works, for example, [2, Proposition 3.3].

**Proposition 2.4.** Let $F = \{\phi_1, ..., \phi_k\}$ be a finite set of polynomials over $K$ with $\deg \phi_i = d_i \geq 2$, and $d := \max_i d_i$. Then for all $n \geq 1$ and $\phi \in F_n$, we have
\[ h(\phi) \leq \left(\frac{d^n - 1}{d - 1}\right) h(F) + d^2 \left(\frac{d^{n-1} - 1}{d - 1}\right) \log 8 = O(d^n(h(F) + 1)). \]

The following is an easy consequence of [5, Corollary 2.3].

**Proposition 2.5.** Let $K$ be an number field and $F = \{\phi_1, ..., \phi_k\} \subseteq K[X]$ a dynamical system of polynomials. Let also $g \in k[X]$ be such that $g, g \circ \phi_1, ..., g \circ \phi_k$ have at least two distinct roots in $\overline{Q}$. Then, for every finitely generated subgroup $\Gamma \subseteq K^*$, $x \in \overline{Q}$, $E > 0$, we have that
\[ \mathcal{O}_F(x) \cap g^{-1}(\mathcal{B}(\Gamma, E)) \]
is finite.

## 3. Some preliminar results

We define the height of $x = (x, y) \in \mathbb{G}_m^2$ by $h(x) = h(x) + h(y)$.

For $F \in \overline{Q}[X, Y]$ an absolutely irreducible polynomial of degree $d$ and height $h$, which is not special, we use the notation $\Delta = \deg_X F + \deg_Y F$.

For $\Gamma$ a finitely generated subgroup of $\mathbb{G}_m^2$ of rank $r > 0$, we take $K$ to be the smallest number field containing all coefficients of $F$ and the group $\Gamma$, so that
\[ F \in K[X, Y] \text{ and } \Gamma \subseteq (K^*)^2. \]

Letting $\mathcal{C} \subseteq \overline{Q}^2$ be the zero set of the above polynomial, we state the following technical result.

**Lemma 3.1.** [5, Lemma 4.5] Let $K, \Gamma, \mathcal{C}, \Delta$ and $h$ as above with $\Delta \geq 2$. Then there is a constant $c_0(K, \Gamma)$ depending only on $K$ and the generators of $\Gamma$, such that for $\zeta$ defined by
\[ \zeta^{-1} = c_0(K, \Gamma) \exp(2\Delta^2) \Delta^{7r+22}(\Delta + h)(\log \Delta)^6, \]
where $r$ is the rank of $\Gamma$, we have that
\[ \#(\mathcal{C} \cap \mathcal{G}_2(\overline{\Gamma}, \zeta)) \leq \exp((h + 1) \exp((2 + o(1))\Delta^2)). \]
In other side and more generally, if $K$ is an algebraically closed field of characteristic zero, we consider polynomials $F \in K[X]$ that are not monomials. If one denotes $A(n, r) = (8n)^{4n^t(n+r+1)}$, we quote the following counting result.

**Lemma 3.2.** [5, Lemma 4.7] Let $F \in K[X]$ be a polynomial of degree $D$ which is not a monomial and let $\Gamma \subset K^*$ be a multiplicative subgroup of rank $r$. Then
\[
\#\{ (u, v) \in \Gamma^2 | F(u) = v \} < D.A(D + 1, r) + D.2^{D+1}.
\]

We also will make use of the combinatorial statement below, which has been used and proved in a number of works.

**Lemma 3.3.** [5, Lemma 4.8] Let $2 \leq T < N/2$. For any sequence $0 \leq n_1 < \ldots < n_T \leq N$, there exists $r \leq 2N/T$ such that $n_{i+1} - n_i = r$ for at least $T(T-1)/4N$ values of $i \in \{1, \ldots, T-1\}$.

The following result for more general fields is a direct application of the previous lemma.

**Proposition 3.4.** Let $K$ be an arbitrary field, $x \in K$ and let $S \subset K$ be an arbitrary subset of $K$. Suppose there exist a real number $0 < \tau < 1/2$, and also $\Phi$ a sequence of polynomials contained in $\mathcal{F} = \{ \phi_1, \ldots, \phi_k \} \subset K[X]$ such that
\[
T_x, \Phi(N, S) = \tau N \geq 2.
\]

Then there exists an integer $t \leq 2\tau^{-1}$ such that
\[
\#\{ (u, v) \in S^2 | \exists \psi \in \mathcal{F}_t \text{ with } \psi(u) = v \} \geq \frac{\tau^2 N}{8}.
\]

**Proof.** Letting $T := T_x, \Phi(N, S)$, we consider all the values $1 \leq n_1 < \ldots < n_T \leq N$ such that $\Phi^{(n_i)}(x) \in S$, $i = 1, \ldots, T - 1$.

From the previous lemma, there exists $t \leq 2\tau^{-1}$ such that the number of $i = 1, \ldots, T - 1$ with $n_{i+1} - n_i = t$ is at least
\[
\frac{T(T - 1)}{4N} = \frac{T^2}{4} \left(1 - \frac{1}{T}\right) = \frac{\tau^2 N}{4} \left(1 - \frac{1}{T}\right) \geq \frac{\tau^2 N}{8}.
\]

Moreover, if $J := \{ 1 \leq j \leq T - 1 | n_{j+1} - n_j = t \}$, then for each $j \in J$, $\Phi^{(n_j)}(x) \in S$ and $\Phi^{(n_{j+1})}(x) = \psi(\Phi^{(n_j)}(x)) \in S$, where $\psi \in \mathcal{F}_t$.

and hence
\[
\#\{ (u, v) \in S^2 | \psi(u) = v \text{ for some } \psi \in \mathcal{F}_t \} \geq \frac{\tau^2 N}{8}.
\]
4. Orbits in sets

Definition 4.1. We say that an orbit $O_F(x)$ of an element $x \in K$ under a semigroup generated by a finite set $F$ intersects a family of sets $S = \{S_N\}_{N \in \mathbb{N}}$ with low frequency if

$$\lim_{N \to \infty} \frac{\max \Phi_{sequence of F} T_{x, \Phi}(N, S_N)}{N} = 0$$

In the particular case that $S$ is a set with $S := S_1 = S_2 = \ldots$ in the limit above, we say that $O_F(x)$ intersects the set $S$ with low frequency.

Now we give a result for the frequency of intersection of orbits of semigroups of polynomials with the set $\mathcal{C}(\Gamma, \epsilon)$ for a finitely generated subgroup $\Gamma \subset \mathbb{G}_m$.

Theorem 4.2. Let $K$ be an number field and $F = \{\phi_1, ..., \phi_k\} \subset K[X]$ a finite set of polynomials that are not monomials with $\deg \phi_i = d_i \geq 2$, and $d = \max_i d_i$. Suppose that, for a finitely generated subgroup $\Gamma \subset K^*$ of rank $r$, $x \in \overline{\mathbb{Q}}$, and $\theta_N = (\log N)^{-2}(\log \log N)^{-7r/2-12}$, we have that $O_F(x)$ intersects the family of sets $\{\mathcal{C}(\Gamma, \theta_N)\}_{N \in \mathbb{N}}$ with low frequency. Then

$$\max_{\Phi_{sequence of F}} T_{x, \Phi}(N, \mathcal{C}(\Gamma, \theta_N)) \leq \frac{(4 \log d + o(1))N}{(\log \log \log N)}$$

as $N \to \infty$.

Proof. For each $\Phi$, we define $\tau_\Phi$ by

$$\tau_\Phi = T_{x, \Phi}(N, \mathcal{C}(\Gamma, \theta_N))/N.$$ 

We can assume that

$$\tau_\Phi \geq \frac{4 \log d}{\log \log \log N} \geq \frac{2}{N}$$

for some $\Phi$, for otherwise there is nothing to be proved.

For $N$ large enough, Proposition 3.4 shows that there exists

$$t_\Phi \leq 2\tau_\Phi^{-1} \leq \frac{\log \log \log N}{2 \log d}$$

such that

$$\#\{(u, v) \in \mathcal{C}(\Gamma, \theta_N)^2 | \psi(u) = v \text{ for some } \psi \in F_{t_\Phi}\} \geq \frac{\tau_\Phi^2 N}{8}.$$
For any \( \psi \in \mathcal{F}_{t_\Phi} \), we denote by \( C_\psi \) the curve defined by the zero set of the polynomial \( \psi(X) - Y = 0 \). Then

\[
\sum_{\psi \in \mathcal{F}_{t_\Phi}} \#(C_\psi \cap \mathcal{C}(\Gamma, \theta_N)^2)
= \sum_{\psi \in \mathcal{F}_{t_\Phi}} \# \{(u, v) \in \mathcal{C}(\Gamma, \theta_N)^2 | \psi(u) = v\}
= \# \{(u, v) \in \mathcal{C}(\Gamma, \theta_N)^2 | \psi(u) = v \text{ for some } \psi \in \mathcal{F}_{t_\Phi}\}
\geq \frac{\tau_{\Phi}^2 N}{8}.
\]

The set \( \{(u, v) \in \mathcal{C}(\Gamma, \theta_N)^2 | \psi(u) = v\} \) is the intersection of the curve \( C_\psi \) with the set \( \mathcal{C}(\Gamma, \theta_N)^2 \).

We will define a \( \zeta_\Phi \) as in Lemma 3.1 with parameters \( \Delta_\Phi = d_{t_\Phi} + 1 \) and \( h = h(\mathcal{F}_{t_\Phi}) \). By Proposition 2.4, we have that

\[
h \leq O(d_{t_\Phi}(h(\mathcal{F}) + 1)) = O(\Delta_\Phi),
\]

where the referred constant does not depend on \( \Phi \) satisfying (4.1), but only on \( \mathcal{F} \).

Moreover,

\[
\Delta_{t_\Phi} = d_{t_\Phi} + 1 \leq (\log \log N)^{1/2} + 1,
\]

and thus

\[
\zeta_\Phi^{-1} : = \exp(2\Delta_\Phi^2 + O(1)) \Delta_\Phi^{7r+23}(\log \Delta_\Phi)^6
= O((\log N)^2(\log \log N)^{7r+23}(\log \log \log N)^6),
\]

for \( N \) sufficiently large, with the referred constant not depending on \( \Phi \) satisfying (4.1) again.

For our choice of \( \theta_N \), we have that \( \theta_N \leq \zeta_\Phi/2 \) for any \( N \) large enough, and so

\[
\mathcal{C}(\Gamma, \theta_N)^2 \subset \mathcal{C}_2(\Gamma \times \Gamma, \zeta_\Phi).
\]

By the previous calculations, this implies that

\[
\sum_{\psi \in \mathcal{F}_{t_\Phi}} \#(C_\psi \cap \mathcal{C}_2(\Gamma \times \Gamma, \zeta)) \geq \frac{\tau_{\Phi}^2 N}{8}.
\]
Using Lemma 3.1 to obtain upper bounds for the \( \#(C_\psi \cap \mathcal{C}_2(\Gamma \times \Gamma, \zeta)) \), knowing that \( t_\Phi \leq 2\tau_\Phi^{-1} \), we will have, as \( \tau_\Phi \to 0, N \to \infty \), that

\[
N \leq 8\tau_\Phi^{-2}k^{t_\Phi} \exp(h \exp((2 + o(1)\Delta_\Phi^2))) \\
\leq 8\tau_\Phi^{-2}k^{t_\Phi} \exp(\exp((2 \log d + o(1)t_\Phi))) \\
\leq 8\tau_\Phi^{-2}k^{t_\Phi} \exp(\exp((4 \log d + o(1)\tau_\Phi^{-1}))) \\
\leq \{\exp(\exp(\exp((4 \log d + o(1)\tau_\Phi^{-1})))\}) ,
\]

and then

\[
\tau_\Phi \leq \frac{(4 \log d + o(1))}{\log \log \log N} ,
\]

and hence

\[
\tau_\Phi \leq \max_\Phi \tau_\Phi \leq \frac{(4 \log d + o(1))}{\log \log \log N}
\]

as wanted when \( N \to \infty \). □

**Corollary 4.3.** Under the conditions of Theorem 4.2 we have that

\[
\#\{y \in \mathcal{C}(\Gamma, \theta_N) | y = f(x), f \in \mathcal{F}_n, n \leq N\} \leq k^N\frac{(4 \log d + o(1))N}{(\log \log \log N)}
\]

as \( N \to \infty \).

**Proof.** Given \( N \) very large, the set \( \mathcal{F}_N \) contains \( k^N \) polynomials. For each \( f \in \mathcal{F}_N \), we can choose a sequence \( \Phi \) of terms in \( \mathcal{F} \) whose \( \Phi^{(N)} = f \), obtaining \( k^N \) sequences representing the elements of \( \mathcal{F}_N \). For each sequence \( \Phi \) chosen, when \( N \) is large,

\[
\#\{n \leq N | \Phi^{(N)}(x) \in \mathcal{C}(\Gamma, \theta_N)\} \leq \frac{(4 \log d + o(1))N}{(\log \log \log N)}
\]

uniformly for any \( \Phi \) by the previous theorem, or in other words, for each path in the \( N \)-tree \( \mathcal{F}_N \). Since there are \( k^N \) paths(polynomials, sequences) in the \( n \)-tree \( \mathcal{F}_N \), this yields

\[
\#\{y \in \mathcal{C}(\Gamma, \theta_N) | y = f(x), f \in \mathcal{F}_n, n \leq N\} \leq k^N\frac{(4 \log d + o(1))N}{(\log \log \log N)}
\]

as \( N \to \infty \). □

**Theorem 4.4.** Let \( K \) be a field of characteristic zero and \( \mathcal{F} = \{\phi_1, ..., \phi_k\} \subset K[X] \) a finite set of polynomials that are not monomials with \( \deg \phi_i = d_i \geq 2 \) and \( d = \max_i d_i \). Then, for a finitely generated subgroup \( \Gamma \subset K^* \) of rank \( r \), \( x \in K \) such that \( \mathcal{O}_\mathcal{F}(x) \) intersects \( \Gamma \) with low frequency, we have that
\[
\max_{\Phi \text{ sequence in } \mathcal{F}} T_{x,\Phi}(N, \Gamma) \leq \frac{(10 \log d + o(1))N}{(\log \log N)}, \text{ as } N \to \infty.
\]

Proof. As before, we again define \(\tau_\Phi = T_{x,\Phi}(N, \Gamma)/N\) and assume \(\tau_\Phi \geq 2/N\). Again from Proposition 3.4, for \(N\) large, there exists \(t_\Phi \leq 2\tau_\Phi^{-1}\) such that

\[
\frac{\tau_\Phi^2 N}{8} \leq \sum_{\psi \in \mathcal{F}_\Phi} \#\{(u, v) \in \Gamma^2 | \psi(u) = v\},
\]

which by Lemma 3.2, as \(\deg \psi \leq d_{t_\Phi}\), is upper bounded by

\[
k^{2\tau_\Phi^{-1}} \left( d^{2\tau_\Phi^{-1}} A(d^{2\tau_\Phi^{-1}} + 1, r) + d^{2\tau_\Phi^{-1}} 2^{d^{2\tau_\Phi^{-1}} + 1} \right).
\]

Therefore

\[
\frac{\tau_\Phi^2 N}{8} \leq k^{2\tau_\Phi^{-1}} d^{2\tau_\Phi^{-1}} A(d^{2\tau_\Phi^{-1}} + 1, r) + k^{2\tau_\Phi^{-1}} d^{2\tau_\Phi^{-1}} 2^{d^{2\tau_\Phi^{-1}} + 1}.
\]

When \(N \to \infty\) (\(\tau_\Phi \to 0\) uniformly on \(\Phi\)), we have

\[
N \leq 8\tau_\Phi^{-2} \left( k^{2\tau_\Phi^{-1}} d^{2\tau_\Phi^{-1}} A(d^{2\tau_\Phi^{-1}} + 1, r) + k^{2\tau_\Phi^{-1}} d^{2\tau_\Phi^{-1}} 2^{d^{2\tau_\Phi^{-1}} + 1} \right)
\]

bounded by

\[
N \leq \exp \left( \exp ((10 \log d + o(1))\tau_\Phi^{-1}) \right),
\]

from where the result follows.

And as in Corollary 4.3, the following is proven in an analogous way, working for more general fields of characteristic zero.

Corollary 4.5. Under the conditions of Theorem 4.4 we have that

\[
\#\{y \in \Gamma | y = f(x), f \in \mathcal{F}_n, n \leq N\} \leq kN \frac{(10 \log d + o(1))N}{(\log \log N)},
\]

as \(N \to \infty\).

5. A graph theory result

Here we present a graph theory result of Mérai and Shparlinski [3] that will be used later in proofs.

Let \(\mathcal{H}\) be a directed graph with possible multiple edges. Let \(\mathcal{V}(\mathcal{H})\) be the set of vertices of \(\mathcal{H}\). For \(u, v \in \mathcal{V}(\mathcal{H})\), let \(d(u, v)\) be the distance from \(u\) to \(v\), that is, the length of a shortest (directed) path from \(u\) to \(v\). Assume, that all the vertices have the out-degree \(k \geq 1\), and the edges from all vertices are labeled by \(\{1, ..., k\}\).
For a word $\omega \in \{1, ..., k\}^*$ over the alphabet $\{1, ..., k\}$ and $u \in V(\mathcal{H})$, let $\omega(u) \in V(\mathcal{H})$ be the end point of the walk started from $u$ and following the edges according to $\omega$.

Let us fix $u \in V(\mathcal{H})$ and a subset $\mathcal{A} \subset V(\mathcal{H})$. Then for words $\omega_1, ..., \omega_l$ put

$$L_N(u, \mathcal{A}; \omega_1, ..., \omega_l) = \# \{v \in V(\mathcal{H}) : d(u, v) \leq N, d(u, \omega_i(v)) \leq N, \omega_i(v) \in \mathcal{A}, i = 1, ..., l\}.$$ 

To state the results, for $k, t \geq 1$, let $B(k, t)$ denote the size of the complete $k$-tree of depth $t - 1$, that is

$$B(k, t) = \begin{cases} t & \text{if } k = 1, \\ \frac{k^t - 1}{k - 1} & \text{otherwise}. \end{cases}$$

**Lemma 5.1.** Let $u \in V(\mathcal{H})$, and $t, l \geq 1$ be fixed. If $\mathcal{A} \subset V(\mathcal{H})$ is a subset of vertices with $\# \{v \in \mathcal{A} : d(u, v) \leq N\} \geq \max \{3B(k, t), \frac{3l}{t} \# \{v \in V(\mathcal{H}) : d(u, v) \leq N\}\}$, then there exist words $\omega_1, ..., \omega_l \in \{1, ..., k\}^*$ of length at most $t$ such that

$$L_N(u, \mathcal{A}; \omega_1, ..., \omega_l) \gg \frac{t}{B(k, t)^{l+1}} \# \{v \in V(\mathcal{H}) : d(u, v) \leq N\},$$

where the implied constant depend only on $l$.

### 6. More Results of Orbits in Sets

**Theorem 6.1.** Let $K$ be a field of characteristic zero and $\mathcal{F} = \{\phi_1, ..., \phi_k\} \subset K[X]$ a finite set of polynomials that are not monomials, with $\deg \phi_i = d_i \geq 2$ and $d = \max_i d_i$. Suppose that $\Gamma \subset K^*$ is a finitely generated subgroup of rank $r$, and $u \in K$. Let also $t, l \geq 1$ be integers such that $t \geq 3l$ and $\# \{v \in \Gamma | v = f(u), f \in \mathcal{F}, n \leq N\} \geq 3B(k, t)$. Then

$$\# \{v \in \Gamma | v = f(u), f \in \mathcal{F}, n \leq N\} \ll \frac{B(k, t)^{l+1}}{t} (d^t A(d^t + 1, r) + d^t 2^{d^t+1}).$$

**Proof.** We consider the directed graph with the elements of $\Gamma$ as vertices, and edges $(x, \phi_i(x))$ for $i = 1, ..., k$ and $x \in \Gamma$. With the notation of Section 5 and Lemma 5.1, we let $\Gamma$ take the place of $\mathcal{H}$ and $\mathcal{A}$. By hypothesis, $l \leq t/3$ and $\# \{v \in \Gamma, d(u, v) \leq N\} \geq 3B(k, t)$. From
Lemma 5.1, there exist words \( \omega_1, ..., \omega_l \in \{1, ..., k\}^\ast \) of length at most \( t \), and therefore degree at most \( d^t \), such that

\[
(6.1) \quad L_N(u, \Gamma; \omega_1, ..., \omega_l) \gg \frac{t}{B(k, t)^{l+1}} \#\{ v \in \mathcal{V}(\Gamma) : d(u, v) \leq N \}.
\]

By Lemma 3.2, we compute

\[
L_N(u, \Gamma; \omega_1, ..., \omega_l) = \#\{ v \in \mathcal{V}(\Gamma) : d(u, v), d(u, \omega_i(v)) \leq N, \omega_i(v) \in \Gamma \}
\]
\[
\leq \sum_{i \leq l} \#\{ (x, y) \in \Gamma^2 : y = \omega_i(x) \}
\]
\[
\leq \sum_{i \leq l} (d^t A(d^t + 1, r) + d^t 2^{d^t + 1})
\]
\[
= l(d^t A(d^t + 1, r) + d^t 2^{d^t + 1}).
\]

Gathering this with (6.1), we conclude that

\[
\#\{ v \in \mathcal{V}(\Gamma) : d(u, v) \leq N \} \ll \frac{B(k, t)^{l+1}l}{t} (d^t A(d^t + 1, r) + d^t 2^{d^t + 1}),
\]

as desired.

\[ \square \]

Corollary 6.2. Let \( K \) be a field of characteristic zero and \( \mathcal{F} = \{ \phi_1, ..., \phi_k \} \subset K[X] \) a finite set of polynomials that are not monomials, with \( \deg \phi_i = d_i \geq 2 \) and \( d = \max_i d_i \). Suppose that \( \Gamma \subset K^\ast \) is a finitely generated subgroup of rank \( r \), \( u \in K \), and \( \{ t_N \}_N \) is a sequence of positive integers that goes to \( \infty \) as \( N \to \infty \) and that satisfies

\[
\#\{ v \in \Gamma \mid v = f(u), f \in \mathcal{F}_n, n \leq N \} \geq 3B(k, t_N)
\]

for each \( N \). Then

\[
\#\{ v \in \Gamma \mid v = f(u), f \in \mathcal{F}_n, n \leq N \} \leq \exp((10 \log d + o(1))t_N)),
\]

as \( N \to \infty \).

Proof. In the proof of the previous result, we can choose \( l \geq 1 \) an arbitrary integer and \( N \) big enough so that \( t_N \geq 3l \). For each of these \( t_N \)'s, we can apply the previous theorem, obtaining that

\[
\#\{ v \in \Gamma \mid v = f(u), f \in \mathcal{F}_n, n \leq N \} \ll \frac{B(k, t_N)^{l+1}}{t_N}(d^{t_N} A(d^{t_N} + 1, r) + d^{t_N} 2^{d^{t_N} + 1})
\]

Moreover, since \( t_N \to \infty \) as \( N \to \infty \), it yields

\[
B(k, t_N)^{l+1}(d^{t_N} A(d^{t_N} + 1, r) + d^{t_N} 2^{d^{t_N} + 1}) = \exp((10 \log d + o(1))t_N)),
\]

from where the result follows. \[ \square \]
Remark 6.3. If the hypothesis of Corollary 6.2 are satisfied with
\[ t_N \sim \frac{1}{10 \log d + o(1)} \log \log \left( \frac{(10 \log d + o(1)) N}{\log \log N} \right), \text{ as } N \to \infty, \]
then we recover and generalize Corollary 4.5, as well as Theorem 3.1 of [5], under our referred conditions.

Theorem 6.4. Let \( K \) be an algebraic number field and \( \mathcal{F} = \{ \phi_1, ..., \phi_k \} \subset K[X] \) a finite set of polynomials that are not monomials, with \( \deg \phi_i = d_i \geq 2 \) and \( d = \max_i d_i \). Suppose that \( \Gamma \subset K^* \) is a finitely generated subgroup of rank \( r \), \( u \in K \), and \( \{ t_N \}_N \) is a sequence of positive integers that goes to \( \infty \) as \( N \to \infty \) and that satisfies
\[ \#\{ v \in \mathcal{C}(\overline{\Gamma}, \theta_N) | v = f(u), f \in \mathcal{F}_n, n \leq N \} \geq 3B(k, t_N) \]
for each \( N \), with \( \theta_N \leq \left( \exp(d^{-2t_N})d^{O(7r-24)} \right) \). Then
\[ \#\{ v \in \mathcal{C}(\overline{\Gamma}, \theta_N) | v = f(u), f \in \mathcal{F}_n, n \leq N \} \leq \exp\left(\exp\left(\exp(4\log d + o(1)t_N)\right)\right), \]
as \( N \to \infty \).

Proof. We consider the directed graph with the elements of \( \mathcal{C}(\overline{\Gamma}, \theta) \) as vertices, and edges \( (x, \phi_i(x)) \) for \( i = 1, ..., k \) and \( x \in \mathcal{C}(\overline{\Gamma}, \theta) \). With the notation of Section 5 and Lemma 5.1, we let \( \mathcal{C}(\overline{\Gamma}, \theta) \) take the place of \( \mathcal{H} \) and \( \mathcal{A} \). By hypothesis, we can choose \( l \geq 1 \) an arbitrary integer and \( N \) big enough so that \( t_N \geq 3l \) and \( \#\{ v \in \mathcal{C}(\overline{\Gamma}, \theta), d(u, v) \leq N \} \geq 3B(k, t_N) \). From Lemma 5.1, for each \( N \), there exist words \( \omega_1, ..., \omega_l \in \{ 1, ..., k \}^* \) of length at most \( t_N \), and therefore degree at most \( d^{t_N} \), such that
\[ (6.2) \quad L_N(u, \mathcal{C}(\overline{\Gamma}, \theta); \omega_1, ..., \omega_l) \gg \frac{t_N}{B(k, t_N)^l + 1} \#\{ v \in \mathcal{V}(\mathcal{C}(\overline{\Gamma}, \theta)) : d(u, v) \leq N \}. \]

Putting \( \Delta t_N = d^{t_N} + 1 \) and \( h_N = h(\mathcal{F}_{t_N}) \), we have \( h = O(\Delta t_N) \) by Proposition 2.4. Defining \( \zeta_N \) as in Lemma 3.1 with parameters \( h_N, \Delta t_N \) we have that
\[ \zeta^{-1} = \exp(2\Delta_{t_N}^2 + O(1)) \Delta_{t_N}^{7r-23}(\log \Delta_{t_N})^6 = O(\exp(d^{2t_N})d^{t_N(7r-23)}(t_N \log d)^6). \]

As \( \theta_N \leq \zeta_N/2 = O \left( (\exp(d^{2t_N})d^{t_N(7r-23)}(t_N \log d)^6)^{-1} \right) \), for \( N \) large enough, it is true that
\[ \mathcal{C}(\overline{\Gamma}, \theta_N)^2 \subset \mathcal{C}_2(\overline{\Gamma} \times \overline{\Gamma}, \zeta). \]
By Lemma 3.1, we compute
\[
L_N(u, C(\Gamma, \theta_N); \omega_1, \ldots, \omega_l)
= \# \{ v \in V(C(\Gamma, \theta_N)) : d(u, v), d(u, \omega_i(v)) \leq N, \omega_i(v) \in C(\Gamma, \theta), i = 1, \ldots, l \}
\leq \sum_{i \leq l} \# \{ v \in V(C(\Gamma, \theta_N)) : d(u, v), d(u, \omega_i(v)) \leq N, \omega_i(v) \in C(\Gamma, \theta_N) \}
\leq \sum_{i \leq l} \# \{ (x,y) \in C(\Gamma, \theta_N)^2 : y = \omega_i(x) \}
\leq \sum_{i \leq l} \# \{ (x,y) \in C_2(\Gamma \times \Gamma, \zeta_N) : y = \omega_i(x) \}
\leq l \exp((h_N + 1) \exp((2 + o(1))\Delta_{t_N}^2)).
\]
Gathering this with (6.2), it follows that
\[
\# \{ v \in V(C(\Gamma, \theta_N)) : d(u, v) \leq N \}
\leq \frac{B(k,t_N)^{l+1}}{t_N} \exp((h_N + 1) \exp((2 + o(1))\Delta_{t_N}^2))
\leq \exp(\exp(\exp((4 \log d + o(1))t_N)))
\]
as we wanted to show.

\[
\square
\]

Remark 6.5. If the hypothesis of Theorem 6.4 are satisfied with
\[
t_N \sim \frac{1}{4 \log d + o(1) \log \log \log \left( \frac{(4 \log d + o(1))N}{\log \log \log N} \right)}, \text{ as } N \to \infty,
\]
then we recover and generalize Corollary 4.3, as well as Theorem 2.4 of [5], under our referred conditions.

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