k-Dirac operator and parabolic geometries

Tomáš Salač*
Charles University, Prague
salac@karlin.mff.cuni.cz

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Abstract

The principal group of a Klein geometry has canonical left action on the homogeneous space of the geometry and this action induces action on the spaces of sections of vector bundles over the homogeneous space. This paper is about construction of differential operators invariant with respect to the induced action of the principal group of a particular type of parabolic geometry. These operators form sequences which are related to the minimal resolutions of the $k$-Dirac operators studied in Clifford analysis.

1 Introduction.

Let $\mathbb{R}^n$ be a Clifford algebra of $\mathbb{R}^n$ with an Euclidean scalar product and let $\{\varepsilon_j, 1 \leq j \leq n\}$ be the standard basis. The $k$-Dirac operator $\{\partial_1, \ldots, \partial_k\}$ is an over-determined system of first order differential operators. Let $f$ be a smooth $\mathbb{R}^n$-valued function on $\mathbb{R}^{kn}$. Then

$$\partial_i f = \sum_{1 \leq j \leq n} \varepsilon_j \partial_{ij} f,$$

(1)

where we identify $\mathbb{R}^{kn}$ with the space of the real matrices $M(n, k, \mathbb{R})$ of rank $n \times k$. Then $\partial_{ij}$ are the usual partial derivatives and $\varepsilon_j$ stands for the multiplication by the Clifford number $\varepsilon_j$.

The solutions of the $k$-Dirac equation ($\forall i : \partial_i f = 0$) are called monogenic functions in several Clifford variables. Monogenic functions share analogous properties as holomorphic functions in one complex variable and from this point of view, the $k$-Dirac operator can be viewed as a generalization of the Cauchy-Riemann operator.

Interesting behaviour of holomorphic functions in $n$ variables on domains in $\mathbb{C}^n$, such as Hartog’s paradox, can be characterized by the sheaf cohomology of holomorphic functions. The sheaf cohomology can be defined as the left derived functor to the functor of global sections. Thus one needs a suitable resolution of the sheaf of holomorphic functions. This is usually the Dolbeault resolution. In light of these facts, natural question is to find a resolution of the sheaf of monogenic functions.

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Let us first mention some results from Clifford analysis. From a general theorem for
differential operators with constant coefficients follows existence of a finite resolution
of the sheaf monogenic functions, see [CSSS]. However there is no direct way how
to translate this general theorem into explicit form of operators in the resolution.
Candidates for the minimal resolutions were computed in the papers [SSS], [SSSL]. If
\( n \geq 2k \), all the operators in the sequence are polynomial combinations of the operators
(1) and the formulas do not explicitly depend on \( n \). Thus we suppress the parameter \( n \)
and talk about the \( k \)-Dirac operator. This case is called the stable case. The unstable
range, i.e. \( n < 2k \), is more complicated since exceptional syzygies, which do not
come from the commutation relations of the operators (1), arise. Methods used in
these papers include mostly methods from partial differential equations, homological
and computer algebra. These methods run very quickly into very high computational
complexity.

For \( n = 4 \) the operator is called the \( k \)-Cauchy-Fueter operator. This operator was
studied in the papers [BAS] and [BS] from more geometric point of view, that is from
the point of view of symmetries of the operator using methods of representation theory.
The \( k \)-Cauchy-Fueter operator was studied in more geometric setting of quaternionic
manifolds in the paper [B]. In this paper we will study \( k \)-Dirac operator in similar
geometric setting as in the paper [B] so let us briefly recall some basic facts.

A quaternionic manifold \( M \) is a manifold of dimension \( 4n \) with a reduction of structure
group of the tangent bundle to \( Sp(1)Sp(n) \) admitting a compatible torsion free
connection. There is the unique normal parabolic geometry \((G, \omega)\) induced by the
quaternionic structure, see [CS]. To the principle fibre bundle \( P \) one can associate
a triple of spaces. The Penrose transform transfer cohomological data from the first
space, so called twistor space, over the second space to a sequence of differential op-
erators on the third space, i.e. on the quaternionic manifold \( M \), and to solutions of
the differential operators in the sequence. Vanishing of the cohomology groups on the
twistor space implies local exactness of the sequence on the manifold \( M \). In this way,
one gets a resolution of the \( k \)-Cauchy operator on quaternionic manifolds. For more
discussion on comparison between \( k \)-Cauchy-Fueter operators living in quaternionic
manifolds and Euclidean spaces see [CSS]. For more information about the Penrose
transform see [BE] and [WW].

Each operator in the resolutions of \( k \)-Cauchy-Fueter operators living in quater-
nionic manifolds is invariant with respect to the induced parabolic structure \((G, \omega)\).
The parabolic structure puts severe conditions on invariant operators and thus we get
rid of ambiguity which was present on the Euclidean space. Linear differential opera-
tors invariant with respect to parabolic structures has been studied intensively through
the last century. The operators of the first order were completely classified and charac-
terized in the paper [SS]. After reducing the structure group of a parabolic geometry
to its reductive part, any such operator is given by the derivation with the chosen Weyl
connection and a linear projection. Invariance of such operator gives equation for the
generalized conformal weight and if the equation is satisfied then the resulting formula
does not depend on the chosen Weyl connection. However situation with higher order
operators is far more complicated since there are usually more operators which can be
combined and finding the right combination is a non-trivial task.

The starting point for understanding differential operators invariant with respect
to some particular parabolic structure is understanding \( G \)-invariant operators on the
homogeneous (flat) space of the geometry. Let $G$ be a principal group of a parabolic geometry, i.e. $G$ is a semi-simple Lie group with a parabolic subgroup $P$. Let $(V, \rho)$ be a representation of the group $P$ and let $G \times_P V$ be the associated vector bundle. The space of sections $\Gamma(G \times_P V)$ can be identified with the space $C^\infty(G, V)^P$ of $V$-valued $P$-equivariant functions on the total space $G$. The action of the group $G$ on the space of smooth sections is, using the isomorphism, defined by

$$(g.f)(h) = f(g^{-1}h)$$

where $g, h \in G, f \in C^\infty(G, V)^P$. An operator

$$D : \Gamma(G \times_P V) \rightarrow \Gamma(G \times_P W)$$

is called $G$-invariant if

$$D(g.s) = g.(Ds).$$

for any $s \in \Gamma(G \times_P V)$ and for all $g \in G$. It is well known fact that there is a one-to-one correspondence between $G$-invariant operators and homomorphisms of generalized Verma modules.

In the regular character, there is complete classification of the homomorphisms and one gets so called BGG-sequences, see [CSlS]. Namely, the highest weights of generalized Verma modules are connected by the affine action of the Weyl group of the Lie algebra $g$ of the Lie group $G$ and comparable with respect to the partial order iff there is a non-zero homomorphism between generalized Verma modules such that the domain of the homomorphism is the Verma module whose highest weight is smaller or equal to the highest weight of the latter Verma module.

In the singular character holds only the if part. Nevertheless we have a necessary conditions on pairs of generalized Verma modules which admit a non-trivial homomorphism. To keep track on sequences of homomorphisms of generalized Verma modules in the singular character, we usually draw so called singular Hasse graphs, i.e. oriented graphs where the vertices are the highest weights of the generalized Verma modules and the arrows correspond to non-trivial homomorphism of generalized Verma modules which may or may not exist. In the picture of invariant differential operator, we know for which vector bundles we should look for.

In [F] were computed singular Hasse graphs which coincide with the graphs of minimal resolutions which were found by people in Clifford analysis. The singular Hasse graphs correspond to sequences of first and second order operators. Existence of the first order operators was proved but existence of the second order operators was in general only conjectured. Also local form of the operators was given only in some particular cases. In contrast to the $k$-Cauchy-Fueter operator, the geometry is no longer 1-graded but is 2-graded which brings some new difficulties.

The main goal of this paper is to fill some gaps into comparison of sequences of operators studied in [F] and the minimal resolutions found by people from Clifford analysis. The main result is construction of $G$-invariant second order operators conjectured in [F] in the stable range on the homogeneous space of the geometry.

**Remark 1.** In general, a $G$-invariant operator going between two fixed homogeneous vector bundles in one direction need not be unique up to a scalar multiple. However in this case $k = 2, 3$ uniqueness was proved in the paper [BC]. See also [H].
I have used a construction called splitting operators. The splitting operators are given by polynomials in Curved Casimir operators. The Curved Casimir operator was introduced in [CS]. A formula (33) gives a general second order operator as a linear combination of operators invariant with respect to the Levi factor of the parabolic subgroup. The coefficients are given in the theorem 5. This formula is given in a preferred Weyl structure and Cartan gauge over an open affine subset of the homogeneous space. An explicit formula is given in the case \( k = 2 \) in a local chart in (40). I have introduced some additional assumptions on the sections over the affine subset of the homogeneous space, this is related to non-trivial grading of the geometry. In this way, we can consider sections which have the same freedom as in the Euclidean setting. These additional assumptions are used in the last section in the case \( k = 3 \) where one can again verify directly that the sequence is a complex. In both cases we have that the sequences of symbols are exact when restricted to the distribution which rules the geometry. These are the theorems 6. and 7.

The Penrose transform is being used also by L. Krump in [K] in order to obtain more information about sequences which correspond to \( k \)-Dirac resolutions, in particular in the non-stable range. The Penrose transform might be hopefully the right tool to answer the question about local exactness of the operators given in the paper.

We close the introductory section with structure of the paper. First is introduced the parabolic geometry where the operators live together with some necessary notation. Then we recall fundamental results from [F] which we will need. Then is introduced the homogeneous vector bundle where all computations will be carries out with small hint how this bundle can be found out. Then we recall the definition of the Curved Casimir operator and give local formulas for the operator using a local adapted frame. We briefly recall how to get splitting operators from the Curved Casimir operator. Then we set preferred choices over the affine open set of the homogeneous space, in particular we will introduce some particular local adapted frame which turns out to be most useful for our purposes. Then we carry out computations in the preferred trivializations to obtain the formula for general second order operator and verify that this construction gives the operators conjectured in [F]. In the last two section, the general formula is given in to local form for \( k = 2 \) and \( k = 3 \). In the case \( k = 3 \), we will work only with the smaller set as mentioned in the previous paragraph.

2 Parabolic geometry behind the \( k \)-Dirac operator.

Let \( G \) be a 4 : 1 cover of the connected component of the identity of the real group \( \text{SO}(n + k, k) \). The group \( G \) has a natural transitive action on the Grassmannian manifold \( V_0(k, n + 2k) \) of oriented maximal isotropic vector subspaces of dimension \( k \) in the vector space \( \mathbb{R}^{n+2k} \) with a \( G \)-invariant quadratic form of signature \( (n + k, k) \). A parabolic subgroup \( P \) is the stabilizer of a chosen maximal isotropic \( k \)-dimensional

1If \( k > 2 \), the fundamental group of \( \text{SO}(n + k, k) \) is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) and thus the Lie group \( G \) is the universal cover of \( \text{SO}(n + k, k) \). We assume throughout the paper that \( k > 2 \) and \( n \geq 2k \). The fundamental group of \( \text{SO}(n + 2, 2) \) is isomorphic to \( \mathbb{Z} \times \mathbb{Z}_2 \) and the group \( G \) is then the covering corresponding to the subgroup \( 2\mathbb{Z} \times 1 \subset \mathbb{Z} \times \mathbb{Z}_2 \).
where block matrices on the diagonal are square matrices with ranks equal to $k, n, k$. Thus we can think of the homogeneous space $G/P$ also as the Grassmannian $V_0(k, n + 2k)$. A choice of a complement of $L$ in $L^\perp$ gives a natural gradation

$$
 g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2,
$$

(4)
on the Lie algebra $g$ of $G$.

Let $G_0$ be the subgroup of $P$ whose adjoint action preserve the gradation $\{4\}$. The group $G_0$ is a maximal reductive subgroup of $P$ and its Lie algebra is isomorphic to $\mathfrak{gl}(k, \mathbb{R}) \oplus \mathfrak{so}(n)$. Let us denote by $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, \mathfrak{g}_0^i = \oplus_{j \geq i} \mathfrak{g}_j$. We denote the one dimensional center of $\mathfrak{g}_0$ by $E_0$.

There are isomorphisms of $G_0$-modules

$$
 \mathfrak{g}_1 \cong V \otimes E, \ \mathfrak{g}_2 \cong \Lambda^2V \otimes \mathbb{R},
$$

(5)
where $V$, resp. $E$, resp. $\mathbb{R}$, denotes the defining representation of $\mathfrak{gl}(k, \mathbb{R})$, resp. of $\mathfrak{so}(n)$, resp. the trivial representation of $\mathfrak{so}(n)$.

The Lie bracket $\mathfrak{g}_1 \otimes \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is the tensor product of anti-symmetrization and contraction with respect to the $\mathfrak{so}(n)$-invariant Euclidean scalar product $g$ on $E$, i.e.

$$
 \wedge \otimes Tr : (V \otimes E) \otimes (V \otimes E) \rightarrow \Lambda^2V \otimes \mathbb{R}
$$

(6)
with $v_1, v_2 \in V, \varepsilon_{\alpha}, \varepsilon_{\beta} \in E$.

### 2.1 Notation for $\mathfrak{g}_0$-modules.

Let $\mathfrak{h} = \{H \mid H = (h_{ij})\}$ is a trace free diagonal $(k \times k)$-matrix be a Cartan subalgebra of the algebra $\mathfrak{sl}(k, \mathbb{R})$. Let $\varepsilon_i, 1 \leq i \leq k$, be the element of the dual space $\mathfrak{h}^*$ defined by $\varepsilon_i(H) = h_{ii}$. The set $\{\varepsilon_i\}_{i=1}^k$ is a spanning set of the space $\mathfrak{h}^*$. Let $\lambda \in \mathfrak{h}^*$, then we can express $\lambda = \sum_{i=1}^k \lambda_i \varepsilon_i$ with $\lambda_i \in \mathbb{R}$. Then we write $\lambda = (\lambda_1, \ldots, \lambda_k)$ and we denote by $\lambda(ij)$ the weight $\lambda(ij) = (\lambda_1, \ldots, \lambda_{i-1} + 1, \ldots, \lambda_j + 1, \ldots, \lambda_k)$ with $1 \leq i < j \leq k$.

The weight $\lambda \in \mathfrak{h}^*$ is dominant integral iff $\lambda_i - \lambda_j \in \mathbb{N} \cup \{0\}$ for all $1 \leq i < j \leq k$. Let $V_\lambda$ be an irreducible complex finite dimensional representation of $\mathfrak{sl}(k, \mathbb{R})$ with a highest weight $\lambda$. We extend the action of $\mathfrak{sl}(k, \mathbb{R})$ to $\mathfrak{gl}(k, \mathbb{R})$ on $V_\lambda$ by $1_k \in \mathfrak{gl}(k, \mathbb{R}) \mapsto (\sum \lambda_i)Id_{V_\lambda} \in End(V_\lambda)$.

**Remark 2.** All considered representations are complex, if necessary we consider complexification of a real representation. Tensor products are over the field of complex numbers.

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2The matrix of the scalar product with respect to the preferred basis is

$$
\begin{pmatrix}
0 & 0 & 1_k \\
0 & 1_n & 0 \\
1_k & 0 & 0
\end{pmatrix},
$$

where block matrices on the diagonal are square matrices with ranks equal to $k, n, k$. Thus we get a preferred matrix realization of $\mathfrak{so}(n + k, k)$. The preferred maximal isotropic subspace $L$ is the one spanned by null vectors $e_i, i = 1, \ldots, k$ of the preferred basis. A natural candidate for a complement of $L$ in $L^\perp$ is then the subspace spanned by the vectors $e_{k+1}, \ldots, e_{k+n}$ of the preferred basis.
2.2 Sequences of differential operators related to minimal resolutions of the $k$-Dirac operator.

Let us fix $n, k$ such that $n \geq 2k \geq 4$ and let us suppose for simplicity that $n$ is even. The case $n$ is odd proceeds similarly with obvious modifications. We want to find a second order $G$-invariant differential operator $D$ which belongs to the sequence starting with the $k$-Dirac operator. From [F] we know that

$$D : \Gamma(G \times_P W(\lambda)) \to \Gamma(G \times_P W(\nu)) \quad (7)$$

such that highest weights of the $P$-modules dual to the irreducible modules $W(\lambda), W(\nu)$ lie either on the affine orbit of the weight $\frac{1}{2}(-n+1, \ldots, -n+1|1, \ldots, 1)$ or on the affine orbit of the weight $\frac{1}{2}(-n+1, \ldots, -n+1|1, \ldots, 1, -1)$. This implies that the module $W(\lambda)$, resp. $W(\mu)$ is isomorphic to the module $V_{\lambda} \otimes S_{\pm}$, resp. $V_{\mu} \otimes S_{\pm}$ where:

1. $V_{\lambda}$, resp. $V_{\mu}$ is the irreducible $\mathfrak{gl}(k, \mathbb{R})$-module with the highest weight $\lambda$, resp. $\mu$.

2. $S_{\pm}$ is isomorphic to the complex $\mathfrak{so}(n)$-module $S_{\pm}$ or to the module $S_{\mp}$.

3. $\nu = \lambda_{(ij)}$ for some $1 \leq i < j \leq k$.

4. The weights $\lambda - \frac{1}{2}(n-1, \ldots, n-1)$ and $\lambda_{(ij)} - \frac{1}{2}(n-1, \ldots, n-1)$ have the Young diagrams symmetric with respect to the reflection along the main diagonal.

2.3 Receipt for finding the operator (7).

In order to find the operator (7), we will use the Curved Casimir operator on a suitable homogeneous vector bundle. First we need to find the right homogeneous vector bundle. Let us consider the following facts.

Let us choose a Weyl structure. With this choice, we can write the operator (7) as a combination of $G_0$-invariant operators. It is reasonable to expect that the highest (second) order part of the operator is a combination of operators which are given by differentiating twice with vector fields lying only in the distribution $G \times_P (\mathfrak{g}^{-1}/\mathfrak{p})$ of the tangent bundle $TG/P \cong G \times_P \mathfrak{g}/\mathfrak{p}$ and algebraic $G_0$-equivariant projections. Thus we have to consider maps from the space of the sections of the homogeneous vector bundle associated to $V_{\lambda} \otimes S_{\pm}$ to the space of the sections of the bundle associated to $\mathfrak{g}_1 \otimes V_{\lambda} \otimes S_{\pm}$.

We notice that the multiplicity of the target module $V_{\lambda_{(ij)}} \otimes S_{\pm}$ in the $G_0$-module $\mathfrak{g}_1 \otimes V_{\lambda} \otimes S_{\pm}$ is equal to four and thus there are four second order $G_0$-invariant differential operators to combine. But there is, up to a scalar multiple, a unique combination which can be the highest order part of a $G$-invariant operator and the first aim is to find such linear combination.

Thus we need to find some $P$-module, which is completely reducible as $G_0$-module, containing the module $V_{\lambda} \otimes S_{\pm}$ and the target module $V_{\lambda_{(ij)}} \otimes S_{\pm}$, both with the multiplicity one. A natural choice is the minimal $P$-module $M_{\lambda}^{\pm}$ in the module $\mathfrak{g} \otimes V_{\lambda} \otimes S_{\pm}$ containing $G_0$-submodule $E_0 \otimes V_{\lambda} \otimes S_{\pm}$ where $E_0$ is the one-dimensional

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3Here $|$ separates the weight of $\mathfrak{gl}(k, \mathbb{R})$ from the weight of $\mathfrak{so}(n)$.
4See the remark
centre of \( g_0 \). It is easy to see that the P-module \( M_\lambda^\pm \) is, as the \( G_0 \)-module, isomorphic to

\[
M_\lambda^\pm \cong g_0 \ E_0 \otimes (V_\Lambda \otimes S_\pm) \oplus g_1 \otimes (V_\Lambda \otimes S_\pm) \oplus g_2 \otimes (V_\Lambda \otimes S_\pm).
\]

(8)

So we see that \( M_\lambda^\pm \) contains \( V_\Lambda \otimes S_\pm \) with multiplicity one. Now we prove that \( M_\lambda^\pm \) contains \( V_{\lambda^{(ij)}} \otimes S_\pm \) with multiplicity one.

**Lemma 1.** There is a unique \( G_0 \)-module in \( M_\lambda^\pm \) isomorphic to \( V_{\lambda^{(ij)}} \otimes S_\pm \) and this module is a submodule of \( M_\lambda^\pm \).

Proof: It is easy to see that a submodule of \( M_\lambda^\pm \) isomorphic to \( V_{\lambda^{(ij)}} \otimes S_\pm \) must be a submodule of \( g_2 \otimes V_\Lambda \otimes S_\pm \). First let us notice that \( g_2 \otimes V_\Lambda \otimes S_\pm \cong \Lambda^2 V \otimes C \otimes V_\Lambda \otimes S_\pm \cong \Lambda^2 V \otimes V_\Lambda \otimes S_\pm \). Thus we need to show that \( \Lambda^2 V \otimes V_\Lambda \) contains the representation \( V_{\lambda^{(ij)}} \) with multiplicity one.

**Lemma 2.** Let \( \lambda = (\lambda_1, \ldots, \lambda_k) \) and \( \lambda^{(ij)} = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_j + 1, \ldots, \lambda_k) \) be integral dominant weights for \( sl(k) \). Then the module \( V_{\lambda^{(ij)}} \) appears with multiplicity one in the module \( \Lambda^2 V \otimes V_\Lambda \).

The lemma can be proved by the Klimyk lemma. First we need following notation. Let \( g' \) be a semi-simple Lie algebra, let \( V' \) be a \( g' \)-module and \( \sigma \) be an element of the weight lattice of \( g' \). Let \( m^V_\sigma \) be the dimension of the weight space of the weight \( \sigma \) in the module \( V' \). For a regular weight \( \sigma \), let \( \eta_\sigma \) be the determinant of \( S \) where \( S \) is the unique element of the Weyl group such that \( S(\sigma) \) is a dominant weight \( [\sigma] \) in the orbit of \( \sigma \) under the action of the Weyl group and set \( \eta_\sigma = 0 \) for a singular weight.

**Lemma 3 (Klimyk lemma).** Let \( g' \) be a semi-simple Lie algebra. Let \( \nu, \nu', \nu'' \) be integral dominant weights for \( g' \), let \( \rho \) be the lowest weight of \( g' \). Then the multiplicity \( n_\nu \) of the irreducible representation \( V_\nu \) with the highest weight \( \nu \) in \( V_{\nu'} \otimes V_{\nu''} \) is equal to

\[
\sum m^V_{\sigma'\sigma''\rho} \eta_{\sigma + \nu' + \rho} \text{ where the sum is taken over all the weights } \sigma \text{ the representation } V_{\nu'}, \text{ for which } [\sigma + \nu'' + \rho] = \nu + \rho.
\]

Proof: See [1].

Proof of the lemma In the formulation of lemma we have \( \nu' = (1, 1, 0, \ldots), \nu'' = \lambda, \nu = \lambda^{(ij)}, \rho = (k - 1, k - 2, \ldots, 1, 0) \). The weights of \( \Lambda^2 V \) are of the form \( \{\sigma_{\alpha\beta}|1 \leq \alpha < \beta \leq k\} \) where \( \sigma_{\alpha\beta} \) has 1 at \( \alpha \)-th and \( \beta \)-th entry and 0 otherwise. Moreover \( \rho + \lambda + \rho' \) and \( \lambda^{(ij)} + \rho \) are strictly dominant and for any \( \alpha, \beta \) the weight \( \rho + \lambda + \sigma_{\alpha\beta} \) is dominant (not necessarily strictly). Since the action of the Weyl group of \( sl(k) \) only permutes the entries of a weight, the only dominant weight in the orbit of \( \rho + \lambda + \sigma_{\alpha\beta} \) is the weight itself. Thus the only solution to \( [\sigma_{\alpha\beta} + \rho] = \lambda^{(ij)} + \rho \) is \( [\sigma_{\alpha\beta} + \rho] = [\sigma_{ij} + \lambda + \rho] = \sigma_{ij} + \lambda + \rho \). Thus the formula for the multiplicity \( n_{\lambda^{(ij)}} \) reduces to \( n_{\lambda^{(ij)}} = m_{\sigma_{ij}} det(Id) = 1 \).

We have proved that there is a unique \( G_0 \)-module \( N \) in \( g_2 \otimes V_\Lambda \otimes S_\pm \) such that \( g_2 \otimes V_\Lambda \otimes S_\pm \cong V_{\lambda^{(ij)}} \otimes S_\pm \otimes N \). Moreover the module \( N \) is a P-module of \( M_\lambda^\pm \). Let us denote the P-module \( M_\lambda^\pm / N \) by \( U_\lambda^\pm \).

From now on we will consider the case \( S_\pm = S_+ \). Let us denote by \( T \) the twistor representation of \( Spin(n) \). The final piece of information is that

\[
(V \otimes E) \otimes (V_\Lambda \otimes S_{\pm}) \cong (V_{\lambda^{(ij)}} \otimes S_{-}) \oplus (V_{\lambda^{(ij)}} \otimes S_{+}) \oplus (V_{\lambda^{(ij)}} \otimes T) \oplus (V_{\lambda^{(ij)}} \otimes T)
\]

(9)

as \( G_0 \)-modules.

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5The other case \( S_+ = S_- \) goes through by simultaneously substituting everywhere \( S_- \) for \( S_+ \) and \( S_+ \) for \( S_- \).
2.4 Associated Homogeneous Bundle.

We denote associated vector bundles by the curly letters. A choice of a Weyl structure gives a reduction of the structure group $P$ of the associated vector bundle $U^+_\lambda = G \times_P U^+_\lambda$ to the reductive subgroup $G_0$. From the formulas (5), (8), (9) follows that the bundle $U^+_\pm \lambda$ decomposes into $G_0$-subbundles

$$\begin{pmatrix} \mathcal{E}_0 \otimes \mathcal{E} \otimes (\mathcal{V} \otimes \mathcal{S}_+) \\ \Lambda^2 \mathcal{V} \otimes (\mathcal{V}_\lambda \otimes \mathcal{S}_+) \end{pmatrix} \cong \begin{pmatrix} \mathcal{V}_\lambda \otimes \mathcal{S}_+ \\ \mathcal{V}_{\lambda(ij)} \otimes \mathcal{S}_+ \\ \mathcal{V}_{\lambda(ij)} \otimes \mathcal{T} \otimes \mathcal{V} \otimes \mathcal{S}_+ \end{pmatrix} \otimes \mathcal{V}_\lambda \otimes \mathcal{S}_+$$ (10)

There is a natural bundle map, which is in the abstract index notation given by

$$\Gamma \left( \mathcal{V} \otimes \mathcal{E} \otimes (\Lambda^2 \mathcal{V} \otimes \mathcal{S}_+) \right) \otimes \Gamma(U^+_0) \rightarrow \Gamma(U^+_\lambda)$$ (11)

$$\pi_{ij} : \Gamma(\Lambda^2 \mathcal{V} \otimes \mathcal{S}_+) \rightarrow \Gamma(\mathcal{V}_{\lambda(ij)} \otimes \mathcal{S}_+)$$ (12)

3 The Curved Casimir operator.

We will recall the invariant definition of the Curved Casimir operator from [CS].

**Definition 1.** Let $(G \rightarrow M, \omega)$ be a parabolic geometry of type $(G, P)$. The bundle $\mathcal{AM} = G \times_P g$ is called the adjoint tractor bundle and let $\mathcal{A}^* \mathcal{M}$ be the dual bundle. Let $\mathcal{W} \mathcal{M}$ be the vector bundle induced by a $P$-module $W$. The Curved Casimir operator $C$ is invariantly defined as the composition

$$C : \Gamma(\mathcal{W} \mathcal{M}) \xrightarrow{D^2} \Gamma(\otimes^2 \mathcal{A}^* \mathcal{M} \otimes \mathcal{W} \mathcal{M}) \xrightarrow{B} \Gamma(\mathcal{W} \mathcal{M}) ,$$

where $B$ is the pairing induced on $\mathcal{A} \mathcal{M}$ and thus also on $\mathcal{A}^* \mathcal{M}$ by the Killing form of $\mathfrak{g}$ and $D$ is the fundamental derivative.

The $\mathfrak{g}$-valued Cartan form $\omega$ trivializes the bundle $T \mathcal{G}$ and identifies sections of $\mathcal{A} \mathcal{M}$ with $P$-invariant vector fields on $\mathcal{G}$. Let $s \in \Gamma(\mathcal{A} \mathcal{M})$ be a section and let $X$ be the corresponding vector field on $\mathcal{G}$. Let $\psi \in \Gamma(\mathcal{W} \mathcal{M})$ be a section of the induced vector bundle as in the definition and let $f$ be the corresponding equivariant function. Then the fundamental derivative is a pairing

$$D : \Gamma(\mathcal{A} \mathcal{M}) \otimes \Gamma(\mathcal{W} \mathcal{M}) \rightarrow \Gamma(\mathcal{W} \mathcal{M})$$

$$(s, \psi) \mapsto D_s \psi,$$
where $D_s \psi \in \Gamma(WM)$ is the section which correspond to the $P$-equivariant function $X.f$ where $X.f$ is just the derivation of the function $f$ with respect to the vector field $X$.

Now we give the definition (13) of $C$ in a local trivialization. Let us denote by $\mathcal{A}^i M = G \times_P g^i$ the vector subbundles of the adjoint tractor bundle corresponding to the $P$-submodules $g^i = \oplus_{j \geq i} g_j$ of the module $g$. Let $U \subset M$ be an open set over which the adjoint tractor bundle $\mathcal{A} M$ is a trivial vector bundle. Let $A U$ be the pullback of the adjoint tractor bundle under the inclusion $U \to M$. Over the set $U$, we can choose sections $s_i, t_j \in \Gamma(A U)$, $i, j = 1, \ldots, \dim(g_-)$ such that:

1. the sections $s_i$ trivialize the vector bundle $A U / A^0 U$.
2. the sections $t_j$ trivialize the vector bundle $A^1 U$.
3. the sections $s_i, t_j$ are dual with respect to the pairing $B$, i.e. $B(s_i, t_j) = \delta_{ij}$ at any point $x \in U$.

**Theorem 1.** Let $\rho \in \Gamma(WM)$ and let us use the notation as above. Then on the set $U$ we have

$$C(\rho)|_U = -2 \sum_i t_i \cdot D_s \rho + c^{\rho_0} \rho,$$

where $c^{\rho_0}$ is a zero order operator computable from representation data of the module $W$ and $\bullet$ denotes the algebraic action of vertical vector fields.

### 3.1 Algebraic action of the Curved Casimir operator.

The algebraic action $c^{\rho_0}$ of $C$ on $U^+_{\Lambda}$ is

$$c^{\rho_0} \left( \begin{array}{c} \varphi^S_i \\ \varphi^S_j \\ \varphi^T_i \\ \varphi^T_j \\ \varphi_{ij} \end{array} \right) = \left( \begin{array}{cc} c^{S_0}_i \varphi^S_i \\ c^{S_0}_j \varphi^S_j \\ c^{S_0}_i \varphi^T_i \\ c^{S_0}_j \varphi^T_j \\ c^{\rho_0}_{\Lambda ij} \varphi_{ij} \end{array} \right),$$

where we write section with respect to the decomposition on the right side of (10). In particular $c^{\rho_0}$ acts on each irreducible $G_0$-subbundle as a multiple of the identity. Formulas for the constants $c^{*}_\Lambda$ are explicitly given in the proof of the theorem 4. Let us denote by $\alpha_{\Lambda} := c_\Lambda - c^{*}_{\Lambda}$ and $\alpha_{ij} = c_\Lambda - c_{\Lambda ij}$.

### 3.2 Splitting operator from the Curved Casimir operator.

Let $\pi : \Gamma(U^+_{\Lambda}) \to \Gamma(V_\Lambda \otimes S_+)$ be the canonical projection. A splitting operator $S$ for $\pi$ is a differential operator

$$S : \Gamma(V_\Lambda \otimes S_+) \to \Gamma(U^+_{\Lambda})$$

such that $\pi \circ S$ is a scalar multiple of the identity operator on $\Gamma(V_\Lambda \otimes S_+)$.

**Theorem 2.** Let $i : \Gamma(V_\Lambda \otimes S_+) \to \Gamma(U^+_{\Lambda})$ be the inclusion of $G_0$-subbundle given by some Weyl structure. Let us denote by $E := \prod_{s=a}(C - c^{*}_s)$, where the product is running over all $G_0$-submodules in $U^+_{\Lambda}$ except the submodule $E_0 \otimes V_\Lambda \otimes S_+$. The operator $E \circ i$ is a splitting operator for $\pi$. 

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Proof: See [CS]. □

The fact that $S$ is a splitting operator is equivalent to the following: if we substitute in (16) $e^{\theta_0}$ by $E$, then the right hand side will depend only on $\varphi$ but not on $\varphi^*_i, \varphi_{ij}$. We will use the operator $S := E \circ i$ to get the operator (7). It remains to compute the differential part of the Curved Casimir operator.

4 Affine subset of the homogeneous space $G/P$.

Details and proofs, which has been omitted in this paragraph, can be found in [Ko].

Let $\pi: G \to G/P$ be the homogeneous space of the parabolic geometry. There is a distribution $\mathcal{H} \subset TG/P$ which is the projection of the distribution spanned by the vector fields on $G$ corresponding to $g^1$ under the left trivialization of $TG$. This distribution is non-integrable and the Lie bracket is given by the Lie bracket on the Lie algebra $g$.

Let $G_-$ be the closed analytic subgroup $G_- := \exp(g_-)$ of $G$. Then $\mathcal{U} := \pi(G_-)$ is an open dense subset of the homogeneous space and $\pi$ is a diffeomorphism between $G_-$ and $\mathcal{U}$. We will denote the $P$-principal fibre bundle $\pi^{-1}(\mathcal{U})$ by $\mathcal{G}$.

4.1 Preferred trivialization of principle bundles over $\mathcal{U}$ and flat Weyl structure.

Let $\mu: \mathcal{U} \to \mathcal{G}$ be the section determined by

$$\mu(\pi(\exp(X))) = \exp(X) \quad \text{(18)}$$

for all $X \in g_-$. Let $P_+ := \exp(p_+)$ and let us denote by $\mathcal{G}_0 := \mathcal{G}/P_+$. The quotient $\mathcal{G}_0$ is a principal $G_0$-bundle over $\mathcal{U}$. Let $\pi': \mathcal{G} \to \mathcal{G}_0$ be the canonical projection. Let us denote by $\rho$ the gauge

$$\rho: \mathcal{U} \to \mathcal{G}_0 \quad \text{(19)}$$

$$\rho(x) = \pi' \circ \mu(x).$$

Let $\sigma$ be the Weyl structure, i.e. a $G_0$-equivariant section, determined by the commutativity of the following diagram

$$\begin{diagram}
\node{\mathcal{G}} \arrow{e} \node{\mathcal{G}_0} \arrow{s, \sigma} \\
\node{\mathcal{U}} \arrow{n, \mu} \arrow{e, \rho} \node{\mathcal{G}_0} 
\end{diagram} \quad \text{(20)}$$

The Weyl structure $\sigma$ pulls back the Maurer-Cartan form $\omega$ to $\mathcal{G}_0$ and the $g_0$-part of $\sigma^*\omega$ is principal connection on the $\mathcal{G}_0$ which we denote by $\omega_0$. The principal connection $\omega_0$ induces connection on any associated vector bundle and in particular the induced linear connection on $T\mathcal{U}$ is flat. Moreover, in the gauge $\rho$, the connection form, i.e. Christoffel symbols, $\rho^*\omega_0$ is zero.
4.2 Preferred local adapted frame over $\mathcal{U}$.

Let $(W,\rho)$ be a $\mathcal{P}$-module and let $w \in W$ be a vector. Let us define a $\mathcal{P}$-equivariant function $f_w$ on $\mathcal{G}$ by the formula

$$f_w : \mathcal{G} \to W$$

$$f_w(g) = \rho^{-1}(p)w,$$

where $g \in \mathcal{G}$ is written uniquely as $g = \exp(X)p$ with $X \in \mathfrak{g}_-, p \in \mathcal{P}$.

In the special case of the adjoint representation $W = \mathfrak{g}$, we denote by $\hat{X}$ the $\mathcal{P}$-equivariant vector field on $\mathcal{G}$ corresponding to the section $f_X \in \Gamma(\mathcal{A}\mathcal{U}), X \in \mathfrak{g}$. The value of the vector field $\hat{X}$ at a point $\exp(Y)p \in \mathcal{G}$, with $Y \in \mathfrak{g}_-, p \in \mathcal{P}$, is the vector tangent to the curve $\exp(Y) \exp(tX)p, t \in (-\epsilon, \epsilon)$ for $\epsilon > 0$ sufficiently small at $t = 0$.

Later on we will need following observations.

**Lemma 4.** Let $X \in \mathfrak{g}_-, Z \in \mathfrak{g}$. Then the derivation of the function $f_Z$ with respect to the vector field $\hat{X}$ is equal to

$$\hat{X}.f_Z = 0$$

on $\mathcal{U}$.

**Proof:**

$$\hat{X}.f_Z(\exp(Y)p) = \left. \frac{d}{dt} \right|_0 f_Z(\exp(Y)\exp(tX)p) = \left. \frac{d}{dt} \right|_0 f_Z(\exp(Y + tX + \frac{1}{2}[Y,tX])p) = 0,$$

where we use the Baker-Campbell-Hausdorff formula for $\mathfrak{g}_-$.

**Lemma 5.** Let $Z \in \mathfrak{g}, Z' \in \mathfrak{p}_+$. Then the derivation of the function $f_Z$ with respect to the vector field $\hat{Z}'$ is equal to

$$\hat{Z}'.f_Z = f_{-\left[Z',Z\right]} = f_{\left[Z,Z'\right]}.$$

**Proof:** As in the formula (22) we have

$$\hat{Z}'.f_Z(\exp(Y)p) = \left. \frac{d}{dt} \right|_0 f_Z(\exp(Y)\exp(tZ')p) = \left. \frac{d}{dt} \right|_0 p^{-1}e^{-tZ'}Ze^{tZ'}p = f_{-\left[Z',Z\right]} = f_{\left[Z,Z'\right]}.$$

Let $p : \mathcal{A}\mathcal{U} \to T\mathcal{U}$ be the canonical projection. Let $\xi_X \in \mathfrak{X}(\mathcal{U})$ be the vector field $\xi_X = p_\ast(\hat{X})$.

**Lemma 6.** Let $X, Y \in \mathfrak{g}_-$ and let $\xi_X, \xi_Y \in \mathfrak{X}(\mathcal{U})$ be vector fields on the open set $\mathcal{U}$ defined above. Then the composition

$$\mathfrak{g}_- \longrightarrow \Gamma(\mathcal{A}\mathcal{U}) \longrightarrow \mathfrak{X}(\mathcal{U})$$

$$X \mapsto \hat{X} \mapsto p(\hat{X})$$

is a homomorphism of Lie algebras, i.e. $[\xi_X, \xi_Y]_{\mathfrak{X}(\mathcal{U})} = p([X,Y]_{\mathfrak{g}})$.

**Proof:** See [Ko].
4.3 Functions on $\mathcal{U}$ and sections.

Let $W$ be a $P$-representation. Let us use $\mu$ to define an isomorphism
\[
\beta : \mathcal{C}^\infty(\mathcal{G}, W) \rightarrow \mathcal{C}^\infty(\mathcal{U}, W)
\]
\[
f \rightarrow f \circ \mu.
\]
We will write for simplicity $\beta(f) = \tilde{f}$. The function $f_w$ in (21) is then the constant function $\tilde{f}_w(x) = w$ for all $x \in \mathcal{U}$ and we will denote it for the sake of brevity only as $w$. The inverse map $\beta^{-1}$ we will need only in the simple form $\beta^{-1}(\tilde{f}) = f = f_w$.

In particular for any $X \in \mathfrak{g}_-$ and $f \in \mathcal{C}^\infty(\mathcal{G}, W)^P$, we have that
\[
\beta(\hat{X}.f) = \xi_X.\tilde{f}.
\]
(26)

For more see [Ko].

4.4 Restriction on sections over $\mathcal{U}$.

The freedom of $P$-equivariant functions on $G$ is the set $G_-$. To compare the operators living in the Euclidean and parabolic setting, we consider functions on the affine subset of the homogeneous functions which satisfy some additional restrictions. We will do that in the last section where are given formulas for the case $k = 3$. The case $k = 2$ is given without this restriction. Here are some preliminary notations which will be used later.

Let $G_2$ be the closed subgroup $G_- := \exp(\mathfrak{g}_-)$ of $G_-$. Let $\tilde{G}$ be the quotient of $G$ by the natural left action of $G_2$. Then
\[
q : G \rightarrow \tilde{G}
\]
is a principal $G_2$-bundle. Let us denote by $\tilde{\mathcal{U}}$ the quotient space $G_2 \backslash \mathcal{U}$ by the induced action of $G_2$ on $\mathcal{U}$. The projection $\pi$ is diffeomorphism between the right coset space $G_2 \backslash G_-$ and $\tilde{\mathcal{U}}$. Moreover $\pi$ descends to $P$-principal bundle $\tilde{\pi} : \tilde{G} \rightarrow \tilde{\mathcal{U}}$.

Let $G_1$ be the subset $G_1 := \exp(\mathfrak{g}_1)$ of $G_-$. Let $X \in \mathfrak{g}_-$ be a vector, then there are unique vectors $X^1 \in \mathfrak{g}_1, X^2 \in \mathfrak{g}_2$ such that $X = X^1 + X^2$. Mapping $\exp(X) \mapsto \exp(X^1)$ gives isomorphism of the right coset space $G_2 \backslash G_-$ with $G_1$ and thus also identifies $G_1$ with $\tilde{\mathcal{U}}$.

5 Construction of the operator (7).

In this section we derive explicit formula for the first term in the theorem 1. The operator $S$ given in (32) is a polynomial combination of Curved Casimir operators, in particular it is an operator of degree five in the Curved Casimir operators. However, since the algebraic action of vertical vector fields is compatible with the gradation on the bundle (10), i.e. the algebraic action of vector fields corresponding to $p_+$ raises the homogeneity of sections as can be seen in (11), in the final formula (33) will appear at most second order operator given by differentiating with vector fields in the distribution given by $\mathfrak{g}_1/p$. We will need in this section the formulas (22), (23) and the fact that $p_+$ acts trivially on irreducible $P$-representations.
Let us first choose a basis \{X_i|i=1,\ldots,\dim(g_-)\} of \frak{g}_- consisting of homogeneous elements. Let \{Z_i|i=1,\ldots,\dim(p_+)\} be the basis of \frak{p}_+ dual to the basis \{X_i|i=1,\ldots,\dim(g_-)\} with respect to the Killing form of \frak{g}. The homogeneity of elements will be encoded by upper index, for example \(X^1 \in g_{-1}, Z^1 \in g_1\) etc.

Any \(P\)-equivariant function of \(G \times \mathbb{P}\) \(U_1\) can be written as \(\sum_i f_i \otimes f_i + f_{e_0} \otimes f_0\). Then

\[
\begin{align*}
\sum_{k=1}^{\dim(g_-)} 2 \sum_{i,k} \hat{Z}_k \cdot \hat{X}_k \cdot (\sum_i f_i) & f_i + f_{e_0} \otimes f_0) \\
&= 2 \sum_{i,k} (\hat{Z}_k \cdot f_i \otimes \hat{X}_k \cdot f_i + \hat{Z}_k \cdot f_{e_0} \otimes \hat{X}_k \cdot f_0) \\
&= -2 \sum_{i,k} (f_i \cdot \hat{Z}_k \cdot f_k + f_i \cdot f_{e_0} \otimes \hat{X}_k \cdot f_0)
\end{align*}
\]

In the gauge \(\mu\), the formula (28) is

\[
\begin{align*}
2 \sum_{k=1}^{\dim(g_-)} Z_k \cdot X_k \cdot (\sum_i Z_i \otimes f_i) = & \begin{pmatrix}
2 \sum_k Z_k \otimes X_k \
\sum_j Z_j \otimes f_j
\end{pmatrix}
& \begin{pmatrix}
\pi_{11} \sim 1 \\
\pi_{12} \sim 2 \sum_k Z_k \otimes X_k \otimes f_0 \\
\pi_{21} \sim 2 \sum_k Z_k \otimes X_k \otimes f_0 \\
\pi_{22} \sim -2 \sum_k Z_k \otimes X_k \otimes f_0
\end{pmatrix}
\end{align*}
\]

Let us set \(D^1 := \sum_{i=1}^{\dim(g_+)} Z_i \cdot X_i \). Then \(D^1\) is first order differential operator on the set \(U\). Let us denote by \(D^2\) the second order operator on \(U\) given by

\[
D^2_* : C^\infty(U, V_\lambda \otimes S_-) \xrightarrow{\pi^W_0 D^1} C^\infty(U, V_\lambda \otimes W) \xrightarrow{\pi \otimes D^1} C^\infty(U, V_\lambda \otimes S_+)
\]

where \(\ast \in \{i, j\}\) and \(W\) stands for the representation \(S_-\) or \(T\). The symbol \(\pi^W_0\) is the algebraic projection \(\pi^W_0 : (V \otimes E) \otimes (V_\lambda \otimes S_-) \to V_\lambda \otimes W\). In particular, the second map in (30) is the composition

\[
C^\infty(U, V_\lambda \otimes S_-) \xrightarrow{D^1} C^\infty(U, (V \otimes E) \otimes (V \otimes E) \otimes V_\lambda \otimes S_-) \xrightarrow{\rho} C^\infty(U, (V \otimes S_+) \otimes (V_\lambda \otimes S_-)) \xrightarrow{\circ \omega} C^\infty(U, V_\lambda \otimes S_+).
\]

In particular \(\rho = (\wedge \otimes \operatorname{Tr}) \otimes \operatorname{Id}_{V_\lambda \otimes S_-}\) is a tensor product of the Lie bracket \(\wedge \otimes \operatorname{Tr}\) given in [3] with the identity map on \(V_\lambda \otimes S_-\). The map \(\pi_{ij}\), as in the formula (11), is the natural projection.

With this notation we can give formula for the operator

\[
E := (C - c_{\lambda\gamma})(C - c_{\lambda\gamma})(C - c_{\lambda\gamma})(C - c_{\lambda\gamma})(C - c_{\lambda\gamma})
\]
where the coefficients are $1 \leq i < j$.

Let $\lambda \in \mathbf{R}$ such that the highest weights of the dual modules lie either on the affine orbit of the weight $1$ or on the affine orbit of the weight $2$.

Theorem 4. Let $n \geq 2k \geq 4$ and suppose that $n$ is even. Let $V_\lambda \otimes S_\pm$ and $V_\nu \otimes S_\pm$ be two irreducible $P$-modules such that the highest weights of the dual modules lie either on the affine orbit of the weight $\frac{1}{2}(1-n, \ldots, 1-n)$ or on the affine orbit of the weight $\frac{1}{2}(1-n, \ldots, 1-n, 1, \ldots, 1, -1)$ and moreover suppose that $\nu = \lambda_{ij}$ for some $1 \leq i < j \leq k$. Then the equation (34) holds.

Theorem 3. Suppose that (34) holds. Then the second order differential operator $D$ given in the last row in the formula (33) is $G$-invariant operator.

...
Proof: Let $W$ be an irreducible $G_0$-module with the lowest weight $-\mu$, the highest weight $\nu$ and let $\delta$ be the lowest weight of $g$. The (algebraic) action of the curved Casimir operator on the sections of the bundle induced by $W$ is equal to $<\mu, \mu + 2\delta>$. Then for the module $W$ we have that

$$v = (\mu_1, \ldots, \mu_k | \mu_{k+1}, \ldots, \mu_{k+n}) \leftrightarrow -\mu = (\mu_k, \ldots, \mu_1 | -\mu_{k+1}, \ldots, -\mu_{k+n}).$$

Thus the difference $c_\lambda - c_{\lambda_{ij}}$ is equal to

$$(n + 2k - 2)(c_\lambda - c_{\lambda_{ij}}) = -\lambda_i(-\lambda_i + 2(n/2 + i - 1) - \lambda_j(-\lambda_j + 2(n/2 + j - 1))$$

$$= \left(\lambda_i - 1\right)(-\lambda_i - 1 + 2(n/2 + i - 1)) - \left(\lambda_j - 1\right)(-\lambda_j - 1 + 2(n/2 + j - 1))$$

$$= -2\lambda_i - 1 + 2(n/2 + i - 1) - 2\lambda_j - 1 + 2(n/2 + j - 1)$$

$$= -2\lambda_i - 2\lambda_j + 2n + 2i + 2j - 6. \quad (36)$$

The weight $\lambda$ can be written as $\lambda = \lambda' + (\lambda - \lambda')$ where $\lambda' = \frac{1}{2}(n - 1, \ldots, n - 1)$ and moreover the partition $\lambda - \lambda'$ has the Young diagram symmetric with respect to the main diagonal. From the symmetry we get that

$$\lambda_j - \frac{1}{2}(n - 1) = i - 1; \lambda_i - \frac{1}{2}(n - 1) = j - 1. \quad (37)$$

Plugging (37) into (36) we get that

$$\alpha_{i,j} = c_\lambda - c_{\lambda_{ij}} = \frac{2n - 2(n - 1) - 2}{n + 2k - 2} = 0.$$

**Theorem 5.** The coefficients of the operator $c^S_{\alpha}$ are equal to $\alpha^S_i = \frac{2(\lambda_j - \lambda_i)}{n + 2k - 2}, \alpha^S_j = \frac{2(\lambda_i - \lambda_j)}{n + 2k - 2}.$

Proof: Let us compute for example $c_\lambda - c_{\lambda_{ij}}$. Then we have

$$(n + 2k - 2)c_\lambda - c^S_{\lambda_{ij}} = -2\lambda_i - 1 + 2(n/2 + i - 1) = -2\lambda_i + n + 2i - 3$$

$$= -2\lambda_i + (n - 1) + 2i - 2 = -2\lambda_i + 2\lambda_j = 2(\lambda_j - \lambda_i),$$

where we have used (37). Similarly we get $c_\lambda - c^S_{\lambda_{ij}} = 2(\lambda_i - \lambda_j)$ and similarly for the remaining coefficients.

### 6 Local formulas of the operators.

Let $\{e_1, e_2, \ldots, e_k\}$ be the standard basis of $\mathbb{R}^k$ and let $\{e^1, e^2, \ldots, e^k\}$ be the dual basis of the SL($k, \mathbb{R}$)-module $(\mathbb{R}^k)^*$. Let $\{e_\alpha, \alpha = 1, \ldots, n\}$ be an orthonormal basis of $\mathbb{R}^n$. We denote the $\mathfrak{so}(n)$-invariant product on $\mathbb{R}^n$ by $g_{\alpha \beta}$. Let $B$ the G-invariant scalar product on $g$ as in the remark 3.

The section $[19]$ gives an isomorphism $\phi : g_{-1} \cong (\mathbb{R}^k)^* \otimes \mathbb{R}^n$. Let $\{X_{\alpha a} \sqrt{n} + \sum X_{\alpha a} = \phi^{-1}(e^\alpha \otimes e_\alpha), i = 1, \ldots, k; \alpha = 1, \ldots, n\}$ be a preferred basis of $g_{-1}$ and let $Z_{\alpha \beta}, j = 1, 2, \beta = 1, \ldots, n$ be the dual elements in $g_1$ with respect to $B$. Then $\{X_{ij} = -X_{ji} | \sqrt{n} + \sum [X_{\alpha a}, X_{\beta b}] = \delta_{\alpha \beta} X_{ij}, 1 \leq i, j \leq k, \alpha, \beta \leq n\}$ is a preferred basis of $g_{-2} \cong \Lambda^2(\mathbb{R}^*)^k$. Let $\{Z_{ij} | 1 \leq
\[ \{ i, j \leq k \} \text{ be the basis of } \mathfrak{g}_2 \text{ dual with respect to } B \text{ to the basis } \{ X_{ij} | 1 \leq i, j \leq k \} \text{ of } \mathfrak{g}_{-2}, \text{ then we have that } \sqrt{n + 2} [Z_{ir}, Z_{j\beta}] = -\delta_{\alpha\beta} Z_{ij}. \]

Let us write the canonical coordinates on \( \mathfrak{g}^- \) given by the preferred basis \( \{ X_{i\alpha}, X_{ij} \} \) by \( (x_{i\alpha}, x_{ij}) \). We may use these coordinates also on the set \( U \) and let \( \partial_{i\alpha} \) and \( \partial_{ij} \) be the coordinate vector fields. The left invariant vector fields are then

\[
\xi_{X_{\mu}}(x_{i\alpha}, x_{ln}) = \partial_{k\mu} - \frac{1}{2} \sum_i x_{i\mu} \partial_{ki},
\]

\[
\xi_{X_{rs}}(x_{i\alpha}, x_{ln}) = \partial_{rs}.
\]

The left invariant vector fields \( \xi_{X_{k\nu}} \) span the distribution \( \mathcal{H} \) on \( U \). With this notation, the commutator is

\[
[\xi_{X_{j\mu}}, \xi_{X_{k\nu}}](x_{i\alpha}, y_{mn}) = (\partial_{j\mu} - \frac{1}{2} \sum_i x_{i\mu} \partial_{ji})(\partial_{k\nu} - \frac{1}{2} \sum_i x_{i\nu} \partial_{ki})
\]

\[
- (\partial_{k\nu} - \frac{1}{2} \sum_i x_{i\nu} \partial_{ki})(\partial_{j\mu} - \frac{1}{2} \sum_i x_{i\mu} \partial_{ji}) = g_{\mu\nu} \left( \frac{-1}{2} \sum \partial_{kj} + \frac{1}{2} \partial_{jk} \right) = \frac{g_{\mu\nu}}{\sqrt{n + 2}} \partial_{jk}. \quad (39)
\]

For \( i = 1, \ldots, k \), let us denote by \( \partial_i \) the first order differential operators \( \partial_i = \sum_{\alpha=1}^{n} \varepsilon_{\alpha} \xi_{X_{i\alpha}} \) where \( \varepsilon_{\alpha} \) denotes the multiplication with the Clifford number \( \varepsilon_{\alpha} \). These operators are analogues of the \( k \)-Dirac operators defined in (1) in the parabolic setting.

### 6.1 The sequence for \( k = 2 \).

A regular parabolic geometry of this type is a contact geometry, in particular \( \mathfrak{g}_{-2} \) is one-dimensional. There are three differential operators in the sequence starting with the 2-Dirac operator. The first operator is the 2-Dirac operator, which we denote by \( D_1 \), the second one is a second order operator \( D_2 : \Gamma(V_{\lambda_2} \otimes S_+) \to \Gamma(V_{\lambda_3} \otimes S_+) \) (40)

where the weights are \( \lambda_2 = \frac{1}{2}(n + 1, n - 1) \), \( \lambda_3 = \frac{1}{2}(n + 3, n + 1) \) and the sequence closes with a first order operator \( D_3 \). The operators \( D_1, D_2 \) can be computed easily using the Curved Casimir operator. Here we will focus on the operator (40).

Before giving the formula for the operator, let us first give more explicitly the formula (11). The projections are

\[
E \otimes S_+ \rightarrow T \otimes S_-
\]

\[
\varepsilon \otimes \varphi \rightarrow (\varepsilon \otimes \varphi + \frac{1}{n} \sum_{\alpha} \varepsilon_{\alpha} \otimes \varepsilon_{\alpha} \varepsilon \varphi) \oplus \left( -\frac{1}{n} \sum_{\alpha} \varepsilon_{\alpha} \otimes \varepsilon_{\alpha} \varepsilon \varphi \right),
\]

where the sums are running over the standard basis of the defining module \( E \) and \( \varepsilon \in E, \varphi \in S_+ \). The sign is coming from the defining relation of the Clifford algebra, i.e. \( \varepsilon_{\alpha} \varepsilon_{\beta} + \varepsilon_{\beta} \varepsilon_{\alpha} = -2g_{\alpha\beta} \). Let us work in the gauge \( \mu \) as in (18). Let \( \varphi_{\ast} \) be spinor valued function on \( U \) and let us keep the notation introduced below the formula (25).
The formula (41) is in this case

\[
- \left( \sum_{\alpha} e_{\alpha} \otimes \varphi_{\alpha} \right) \cdot \left( \sum_{\mu} e_{\mu} \otimes \varphi_{\mu} \right)
\]

\[
= \left( \sum_{\alpha} e_{\alpha} \otimes e_{\alpha} \otimes (\varphi_{\alpha} \otimes \varphi_{\alpha}) \cdot \sum_{\beta} \sum_{\alpha} e_{\alpha} \otimes e_{\beta} \otimes e_{\alpha} \otimes e_{\beta} \otimes \varphi_{\alpha} \otimes \varphi_{\beta} \right)
\]

\[
= \left( \sum_{\alpha} e_{\alpha} \otimes e_{\alpha} \otimes (\varphi_{\alpha} \otimes \varphi_{\alpha}) \right) - \left( \sum_{\beta} \sum_{\alpha} e_{\alpha} \otimes e_{\beta} \otimes e_{\alpha} \otimes e_{\beta} \otimes \varphi_{\alpha} \otimes \varphi_{\beta} \right)
\]

With all these preliminary results we can finally give the simplest second order operator explicitly. The operator (40) is

\[
\begin{pmatrix}
 e_1 \otimes \phi_1 \\
 e_2 \otimes \phi_2
\end{pmatrix}
\]

\[
\mapsto \begin{pmatrix}
 e_1 \otimes e_1 \otimes (\partial_1 \partial_1 \phi_2 - \partial_2 \partial_1 \phi_1 + \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \partial_1 \phi) \\
 e_1 \otimes e_2 \otimes e_2 \otimes (\partial_1 \partial_2 \phi_2 - \partial_2 \partial_2 \phi_1 + \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \partial_2 \phi)
\end{pmatrix}.
\]

The formula (42) remains unchanged if we swap \(S_+\) and \(S_-\). Since the operator (40) with \(S_-\) is invariant for the same generalized conformal weight we can take \(S^\pm = S_+ \oplus S_-\) instead of just \(S_+\). We do this in the remaining paragraphs.

### 6.2 The sequence for \(k = 2\) is a complex.

The composition \(D_2 \circ D_1\) is equal to

\[
\phi \mapsto \begin{pmatrix}
 e_1 \otimes \partial_1 \phi \\
 e_2 \otimes \partial_2 \phi
\end{pmatrix}
\]

\[
\mapsto \begin{pmatrix}
 e_1 \otimes e_1 \otimes (\partial_1 \partial_1 \phi_2 - \partial_2 \partial_1 \phi_1 + \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \partial_1 \phi) \\
 e_1 \otimes e_2 \otimes e_2 \otimes (\partial_1 \partial_2 \phi_2 - \partial_2 \partial_2 \phi_1 + \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \partial_2 \phi)
\end{pmatrix}.
\]

(43)

From the lemma 6, follows that

\[
\partial_1 \partial_1 \phi_2 - \partial_2 \partial_1 \phi_1 = \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \partial_1 \phi, \quad \partial_1 \partial_2 \phi_2 - \partial_2 \partial_2 \phi_1 = \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \partial_2 \phi.
\]

(44)

Plugging (44) into (43) gives

\[
\frac{2}{\sqrt{n+2}} \begin{pmatrix}
 e_1 \otimes e_1 \otimes (\xi_{X_{12}} \partial_1 \phi + \xi_{X_{12}} \partial_1 \phi) \\
 e_1 \otimes e_2 \otimes e_2 \otimes (\xi_{X_{12}} \partial_2 \phi + \xi_{X_{12}} \partial_2 \phi)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

(45)

The composition of the last two operators is equal to

\[
\begin{pmatrix}
 e_1 \otimes \phi_1 \\
 e_2 \otimes \phi_2
\end{pmatrix}
\]

\[
\mapsto \begin{pmatrix}
 e_1 \otimes e_1 \otimes (\partial_1 \partial_1 \phi_2 - \partial_2 \partial_1 \phi_1 + \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \partial_1 \phi_1) \\
 e_1 \otimes e_2 \otimes e_2 \otimes (\partial_1 \partial_2 \phi_2 - \partial_2 \partial_2 \phi_1 + \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \partial_2 \phi_2)
\end{pmatrix}
\]

\[
\mapsto \frac{2}{\sqrt{n+2}} \begin{pmatrix}
 e_1 \otimes e_1 \otimes e_1 \otimes e_2 \otimes (\partial_1 (\partial_2 \phi_2 - \partial_2 \phi_2) + \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \phi_1) \\
 - \partial_2 (\partial_1 \phi_2 - \partial_2 \phi_1 + \partial_2 \phi_2) + \frac{2}{\sqrt{n+2}} \xi_{X_{12}} \phi_1)
\end{pmatrix}
\]

The equations for \(\phi_1\) and for \(\phi_2\) in (46) are identical to (45). Thus the sequence for \(k = 2\) is a complex.
6.3 The symbol sequence for \( k = 2 \).

Let us recall that we have denoted by \( \mathcal{H} \) the distribution on \( TU \) corresponding to \( g_{-1} \). The highest order parts of the operators \( D_1, D_2, D_3 \) belong to \( \mathcal{H} \) so the symbol of these operators is determined by its restriction to \( \mathcal{H} \). We have that \( \mathcal{H}^* \) is a quotient of \( T^* U \).

Let \( x \in U, v \in \mathcal{H}^*_x \). Then for the vector \( v \) the symbol sequence is

\[
V_{\lambda_1} \otimes S^\pm \xrightarrow{\sigma_{(x,v)}(D_1)} V_{\lambda_2} \otimes S^\pm \xrightarrow{\sigma_{(x,v)}(D_2)} V_{\lambda_3} \otimes S^\pm \xrightarrow{\sigma_{(x,v)}(D_3)} V_{\lambda_4} \otimes S^\pm \tag{46}
\]

where \( \lambda_2, \lambda_3 \) are given below [10] and \( \lambda_1 = \frac{1}{2}(n-1, n-1), \lambda_4 = \frac{1}{2}(n+3, n+3) \). Since the sequence of operators is complex also the sequence \([46]\) is a complex. Thus it suffices to show that, at each point in the sequence, the dimension of the image has maximal possible dimension. We can consider the symbol map up to a scalar multiple.

Let us denote \( \sum_{a}(\xi_{X_a, f})(x)e_i \otimes e_a \) for \( x \in U \) by \( f_i \in \mathcal{H}^*_x \cong V \otimes E \subset V \otimes \text{End}(S^\pm) \). The symbol of the first operator \( D_1 \) is

\[
\sigma_{(x,v)}(D_1) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{47}
\]

which is an injective map since if \( f_i \neq 0 \) then \( f_i \) is an isomorphism of spinor spaces.

The symbol \( \sigma_{(x,v)}(D_2) \) is

\[
\sigma_{(x,v)}(D_2) = \begin{pmatrix} -f_2f_1 & f_1^2 \\ -f_2^2 & f_2f_1 \end{pmatrix} \tag{48}
\]

If \( f_i \neq 0 \) then \( f_i^2 \) is a multiple of the identity matrix \( I \in \text{End}(S^\pm) \) and thus the rank of the matrix is equal to \( \text{dim}(S^\pm) \). The symbol of the last operator \( D_3 \) is

\[
\sigma_{(x,v)}(D_3) = \begin{pmatrix} -f_2 & f_1 \end{pmatrix} \tag{49}
\]

and so the last symbol is surjective when restricted to \( \mathcal{H}^* \).

Although we have computed symbols and checked that the sequence is a complex for a particular choice of Weyl structure over the open set \( U \), both statements remain true if we choose different Weyl structure over the set \( U \). Instead of the set \( U \), we can consider open covering \( \{ \pi(\exp(g_-)g) | g \in G \} \) of the homogeneous space. All constructions work over these sets as well and thus both statements hold on \( G/P \).

**Theorem 6.** The sequence of operators for \( k = 2 \) is a complex and the symbol sequence restricted to \( \mathcal{H} \) is exact.

7 Sequence for \( k = 3 \).

In this section we will consider smooth \( P \)-equivariant function on \( \pi^{-1}(U) \) which are in the image of the pullback \( q^* : C^\infty(\pi^{-1}(U), W)^P \to C^\infty(\pi^{-1}(U), W)^P \) where \( q \) is as in [27]. Any \( P \)-equivariant function on \( \pi^{-1}(U) \) is determined by its values on \( G_- \) and any smooth function \( f \in \text{Im}(q^*) \) is determined by its values on \( G_{-1} \). A smooth function \( f \) is in the image \( \text{Im}(q^*) \) of \( q^* \) iff \( f \) is constant on the orbits under the right action of \( G_{-2} \) on \( \pi^{-1}(U) \). Using the flat Weyl structure [20], we see that this is equivalent to
\[ \xi_X f = 0 \text{ for all } X \in g_{-2}. \] Since \( G_{-1} \cong \mathbb{R}^{n+k} \), we have an analogy with the operators appearing in the resolutions of the operator \( (\square) \). The sequence for \( k = 3 \) looks like

\[
\begin{align*}
\Gamma(V_{\lambda_1} \otimes S_\pm) & \xrightarrow{D_1} \Gamma((V_{\lambda_2} \otimes S_\pm) \xrightarrow{D_2} \Gamma((V_{\lambda_3} \otimes S_\pm) \xrightarrow{D_3} \Gamma((V_{\lambda_4} \otimes S_\pm) \xrightarrow{D_4} \Gamma((V_{\lambda_5} \otimes S_\pm) \xrightarrow{D_5} \Gamma((V_{\lambda_6} \otimes S_\pm) \xrightarrow{D_6} \Gamma((V_{\lambda_7} \otimes S_\pm)) \xrightarrow{D_7} \Gamma((V_{\lambda_8} \otimes S_\pm)) \xrightarrow{D_8} )
\end{align*}
\]

\[ \Gamma((V_{\lambda_{10}} \otimes S_\pm)) \]

where

\[
\begin{align*}
\lambda_1 &= \frac{1}{2} (n - 1, n - 1, n - 1) \\
\lambda_2 &= \frac{1}{2} (n + 1, n - 1, n - 1) \\
\lambda_3 &= \frac{1}{2} (n + 3, n + 1, n - 1) \\
\lambda_4 &= \frac{1}{2} (n + 3, n + 3, n - 1) \\
\lambda_5 &= \frac{1}{2} (n + 5, n + 1, n + 1) \\
\lambda_6 &= \frac{1}{2} (n + 5, n + 3, n + 1) \\
\lambda_7 &= \frac{1}{2} (n + 5, n + 5, n + 3) \\
\lambda_8 &= \frac{1}{2} (n + 5, n + 5, n + 5).
\end{align*}
\]

The operators \( D_2, D_4, D_6, D_7 \) are second order operators, others are first order operators. Considering only function belonging to the \( \text{Im}(q^*) \), one can verify directly that the sequence is complex, i.e. in the middle box one can show that \( D_6 \circ D_3 + D_5 \circ D_4 = 0 \), and that the symbol sequence is exact on \( \mathcal{H} \).

**Theorem 7.** The sequence of operators for \( k = 2 \) is a complex and the symbol sequence restricted to \( \mathcal{H} \) is exact.

Now we will give explicit formulas of the operators. Let \( \{e_1, e_2, e_3\} \) be a basis of \( V_{\lambda_2} \). Then

\[
D_1(\phi) = \begin{pmatrix}
  e_1 \otimes \partial_1 \phi \\
  e_2 \otimes \partial_2 \phi \\
  e_3 \otimes \partial_3 \phi
\end{pmatrix}.
\]  

Let

\[
\begin{align*}
h_{ij} &= e_i \wedge e_j \wedge e_i, i \neq j, 1 \leq i, j \leq 3, \\
w_1 &= \frac{1}{5} e_1 \wedge e_2 \wedge e_3, \\
w_2 &= \frac{1}{5} e_1 \wedge e_2 \wedge e_3.
\end{align*}
\]

be a basis of \( V_{\lambda_3} \). With respect to this basis the operator \( D^2 \) is equal to

\[
\begin{pmatrix}
  e_1 \otimes \phi_1 \\
  e_2 \otimes \phi_2 \\
  e_3 \otimes \phi_3
\end{pmatrix} \mapsto \begin{pmatrix}
  h_{ij} \otimes (\partial_i \partial_j \phi - \partial_j \partial_i \phi) \\
  w_1 \otimes (\{\partial_2, \partial_3\} \phi_1 - \partial_1 \partial_3 \phi_2 - \partial_1 \partial_2 \phi_3) \\
  w_2 \otimes (\{\partial_1, \partial_3\} \phi_2 - \partial_2 \partial_3 \phi_1 - \partial_2 \partial_1 \phi_3)
\end{pmatrix}.
\]
For brevity we omit the $\otimes$ symbol. Let
\[
\begin{align*}
v_{220} &= e_1 e_1 \wedge e_2 e_2 - e_2 e_1 \wedge e_2 e_1, \\
v_{202} &= e_1 e_1 \wedge e_3 e_3 - e_3 e_1 \wedge e_3 e_1, \\
v_{022} &= e_2 e_2 \wedge e_3 e_3 - e_3 e_2 \wedge e_3 e_2, \\
v_{211} &= e_1 e_1 \wedge e_3 e_2 + e_1 e_1 \wedge e_2 e_3 - e_3 e_1 \wedge e_2 e_1 - e_2 e_1 \wedge e_3 e_1, \\
v_{121} &= e_3 e_1 \wedge e_2 e_2 + e_3 e_1 \wedge e_2 e_2 - e_2 e_3 \wedge e_2 e_1 - e_2 e_1 \wedge e_2 e_3, \\
v_{112} &= e_3 e_1 \wedge e_3 e_2 + e_1 e_3 \wedge e_3 e_2 - e_3 e_3 \wedge e_2 e_1 - e_2 e_1 \wedge e_3 e_3
\end{align*}
\]
be a basis of $V_{\lambda_3}$. With respect to this basis the operator $D_3$ is equal to
\[
\begin{pmatrix}
h_{ij} \otimes \psi_{ij} \\
w_{1} \otimes \psi_{1} \\
w_{2} \otimes \psi_{2}
\end{pmatrix} \mapsto \frac{1}{4}
\begin{pmatrix}
2v_{220} \otimes (-\partial_1 \psi_{21} - \partial_2 \psi_{12}) \\
2v_{202} \otimes (-\partial_1 \psi_{31} - \partial_3 \psi_{13}) \\
2v_{022} \otimes (-\partial_3 \psi_{23} - \partial_2 \psi_{32}) \\
v_{211} \otimes (-\partial_2 \psi_{13} - \partial_3 \psi_{12} - \partial_1 \psi_{1}) \\
v_{121} \otimes (-\partial_1 \psi_{23} - \partial_3 \psi_{21} - \partial_2 \psi_{2}) \\
v_{112} \otimes (\partial_2 \psi_{31} + \partial_1 \psi_{32} - \partial_3 \psi_{1} - \partial_3 \psi_{2})
\end{pmatrix}.
\]

Let
\[
\begin{align*}
(3, 1, 1) : w_{311} &= e_1 \wedge e_3 e_1 \wedge e_2 e_1 - e_1 \wedge e_2 e_1 \wedge e_3 e_1 \\
(1, 3, 1) : w_{131} &= e_2 \wedge e_3 e_1 \wedge e_2 e_2 - e_1 \wedge e_2 e_2 \wedge e_3 e_2 \\
(1, 1, 3) : w_{113} &= e_2 \wedge e_3 e_1 \wedge e_3 e_3 - e_1 \wedge e_3 e_2 \wedge e_3 e_3 \\
(2, 2, 1) : w_{221} &= e_2 \wedge e_3 e_1 \wedge e_2 e_1 + e_1 \wedge e_3 e_1 \wedge e_2 e_2 - e_1 \wedge e_2 e_2 \wedge e_3 e_1 - e_1 \wedge e_2 e_1 \wedge e_3 e_2 \\
(2, 1, 2) : w_{212} &= e_2 \wedge e_3 e_1 \wedge e_3 e_1 + e_1 \wedge e_3 e_1 \wedge e_2 e_3 - e_1 \wedge e_3 e_2 \wedge e_3 e_1 - e_1 \wedge e_2 e_1 \wedge e_3 e_3 \\
(1, 2, 2) : w_{122} &= e_2 \wedge e_3 e_1 \wedge e_3 e_2 + e_2 \wedge e_3 e_1 \wedge e_3 e_3 - e_1 \wedge e_3 e_2 \wedge e_3 e_2 - e_1 \wedge e_2 e_2 \wedge e_3 e_3
\end{align*}
\]
be a preferred basis of $V_{\lambda_5}$. With respect to this basis the operator $D_4$ is equal to
\[
\begin{pmatrix}
h_{ij} \otimes \psi_{ij} \\
w_{1} \otimes \psi_{1} \\
w_{2} \otimes \psi_{2}
\end{pmatrix} \mapsto \frac{1}{8}
\begin{pmatrix}
2w_{311} \otimes (\partial_1 \partial_1 (\psi_{1} + 2 \psi_{2}) + (2 \partial_2 \partial_1 + \partial_1 \partial_2) \psi_{13} - (\partial_1 \partial_1 + 2 \partial_5 \partial_1) \psi_{12}) \\
2w_{131} \otimes (-\partial_2 \partial_2 (2 \psi_{1} + \psi_{2}) + (\partial_5 \partial_2 + 2 \partial_5 \partial_2) \psi_{21} - (\partial_2 \partial_2 + 2 \partial_5 \partial_2) \psi_{23}) \\
2w_{113} \otimes (\partial_5 \partial_5 (\psi_{1} - \psi_{2}) + (\partial_5 \partial_1 + 2 \partial_5 \partial_1) \psi_{32} - (\partial_5 \partial_2 + 2 \partial_5 \partial_2) \psi_{31}) \\
w_{221} \otimes (\partial_1 (\partial_2 \partial_2 - 2 \partial_1 \partial_2) + \partial_2 (-\partial_1 \psi_{1} + 3 \partial_2 \psi_{13}) \\
+ (\partial_1 \partial_1 + 2 \partial_5 \partial_1) \psi_{21} - (\partial_2 \partial_2 + 2 \partial_5 \partial_2) \psi_{12}) \\
w_{212} \otimes (\partial_1 (\partial_3 \psi_{1} + \partial_3 \psi_{2} + 3 \partial_1 \partial_3) + \partial_1 (\partial_1 \psi_{1} - \partial_3 \psi_{12}) \\
- (\partial_1 \partial_2 + 2 \partial_5 \partial_1) \psi_{31} + (\partial_3 \partial_2 + 2 \partial_5 \partial_2) \psi_{13}) \\
w_{122} \otimes (\partial_1 (\partial_3 \psi_{1} - \partial_3 \psi_{2} - 3 \partial_2 \partial_3) + \partial_5 (-\partial_2 \psi_{2} + 3 \partial_3 \psi_{21}) \\
+ (\partial_2 \partial_1 + 2 \partial_5 \partial_2) \psi_{32} - (\partial_5 \partial_1 + 2 \partial_5 \partial_1) \psi_{23})
\end{pmatrix}.
\]

Let us denote by $\{h_{ij}^*, w_1^*, w_2^*\}$ the basis of $V_{\lambda_6}^*$ dual to the basis of $V_{\lambda_3}$. With respect

20
to this basis we can write the operator $D_5$ as

$$
\begin{pmatrix}
  w_{311} \otimes \psi_{311} \\
  w_{131} \otimes \psi_{131} \\
  w_{113} \otimes \psi_{113} \\
  w_{221} \otimes \psi_{221} \\
  w_{212} \otimes \psi_{212} \\
  w_{122} \otimes \psi_{122}
\end{pmatrix}
\mapsto \frac{1}{24}
\begin{pmatrix}
  h^*_{12} \otimes (\partial_3 \psi_{122} - \partial_2 \psi_{113}) \\
  h^*_{21} \otimes (\partial_1 \psi_{212} - \partial_1 \psi_{121}) \\
  h^*_{13} \otimes (\partial_1 \psi_{122} - \partial_3 \psi_{131}) \\
  h^*_{31} \otimes (\partial_2 \psi_{221} - \partial_1 \psi_{211}) \\
  h^*_{23} \otimes (\partial_1 \psi_{212} - \partial_3 \psi_{231}) \\
  h^*_{32} \otimes (\partial_1 \psi_{221} - \partial_2 \psi_{311}) \\
  w^*_1 \otimes (\partial_1 \psi_{122} - \partial_2 \psi_{212}) \\
  w^*_2 \otimes (\partial_2 \psi_{212} - \partial_3 \psi_{221})
\end{pmatrix}.
$$

The operator $D_6$ is given by

$$
\begin{pmatrix}
  v_{220} \otimes \varphi_{220} \\
  v_{202} \otimes \varphi_{202} \\
  v_{022} \otimes \varphi_{022} \\
  v_{211} \otimes \varphi_{211} \\
  v_{121} \otimes \varphi_{121} \\
  v_{112} \otimes \varphi_{112}
\end{pmatrix}
\mapsto \frac{1}{36}
\begin{pmatrix}
  h^*_{12} \otimes (\partial_3 \partial_1 - 2 \partial_1 \partial_3) \varphi_{022} - (\partial_3 \partial_2 + 2 \partial_2 \partial_3) \varphi_{112} + 3 \partial_3 \partial_3 \varphi_{121}) \\
  h^*_{21} \otimes (\partial_3 \partial_2 + 2 \partial_2 \partial_3) \varphi_{022} + (\partial_3 \partial_1 + 2 \partial_1 \partial_3) \varphi_{112} - 3 \partial_3 \partial_3 \varphi_{211}) \\
  h^*_{13} \otimes (2 \partial_1 \partial_2 + 2 \partial_2 \partial_1) \varphi_{022} - (\partial_2 \partial_3 + 2 \partial_3 \partial_2) \varphi_{121} + 3 \partial_2 \partial_2 \varphi_{112}) \\
  h^*_{23} \otimes (\partial_2 \partial_3 + 2 \partial_3 \partial_2) \varphi_{022} + (\partial_2 \partial_1 + 2 \partial_1 \partial_2) \varphi_{121} - 3 \partial_2 \partial_2 \varphi_{211}) \\
  h^*_{31} \otimes (\partial_3 \partial_2 - 2 \partial_2 \partial_3) \varphi_{022} - (\partial_3 \partial_1 + 2 \partial_1 \partial_3) \varphi_{112} + 3 \partial_3 \partial_3 \varphi_{121}) \\
  h^*_{32} \otimes (\partial_3 \partial_3 + 2 \partial_3 \partial_3) \varphi_{022} + (\partial_3 \partial_3 + 2 \partial_3 \partial_3) \varphi_{121} - 3 \partial_3 \partial_3 \varphi_{211}) \\
  w^*_1 \otimes (\partial_3 \partial_1 \varphi_{022} - 2 \partial_3 \partial_2 \varphi_{022} + \partial_3 \partial_3 \varphi_{220} - \partial_3 \partial_1 \varphi_{112}) - 3 \partial_3 \partial_3 \varphi_{211}) \\
  w^*_2 \otimes (2 \partial_3 \partial_1 \varphi_{202} - \partial_3 \partial_2 \varphi_{202} - \partial_3 \partial_3 \varphi_{220} + \partial_3 \partial_1 \varphi_{112} - \partial_3 \partial_1 \varphi_{121} + (\partial_3 \partial_2 + 2 \partial_2 \partial_3) \varphi_{211}) \\
\end{pmatrix}.
$$

The operator $D_7$ is given by

$$
\begin{pmatrix}
  h^*_{12} \otimes \varphi_{12} \\
  h^*_{21} \otimes \varphi_{21} \\
  h^*_{13} \otimes \varphi_{13} \\
  h^*_{31} \otimes \varphi_{31} \\
  h^*_{23} \otimes \varphi_{23} \\
  h^*_{32} \otimes \varphi_{32} \\
  w^*_1 \otimes \varphi_{1} \\
  w^*_2 \otimes \varphi_{2}
\end{pmatrix}
\mapsto \begin{pmatrix}
  e^*_1 \otimes (\partial_2 \partial_2 \varphi_{12} - \partial_1 \partial_2 \varphi_{12} + \partial_3 \partial_3 \varphi_{31} - \partial_1 \partial_1 \varphi_{13} + \partial_2 \partial_2 \varphi_{12}) \\
  e^*_2 \otimes (\partial_2 \partial_2 \varphi_{12} - \partial_1 \partial_2 \varphi_{22} + \partial_3 \partial_3 \varphi_{32} - \partial_1 \partial_2 \varphi_{23} + \partial_3 \partial_3 \varphi_{21} - \partial_3 \partial_3 \varphi_{21}) \\
  e^*_3 \otimes (\partial_1 \partial_1 \varphi_{13} - \partial_1 \partial_1 \varphi_{21} - \partial_3 \partial_2 \varphi_{32} - \partial_3 \partial_2 \varphi_{21} - \partial_1 \partial_2 \varphi_{22})
\end{pmatrix}.
$$

The operator $D_8$ is given by

$$
\begin{pmatrix}
  e^*_1 \otimes \psi_1 \\
  e^*_2 \otimes \psi_2 \\
  e^*_3 \otimes \psi_3
\end{pmatrix}
\mapsto (\partial_1 \psi_1 + \partial_2 \psi_2 + \partial_3 \psi_3).
$$

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