0. Introduction

We work over fields of characteristic 0.

Let $X$ be a variety of general type defined over a number field $K$. A well known conjecture of S. Lang \cite{L} states that the set of rational points $X(K)$ is not Zariski-dense in $X$. As noted in \cite{H}, this implies that if $X$ is a variety which only dominates a variety of general type then $X(K)$ is still not dense in $X$.

J. Harris proposed a way to quantify this situation \cite{H1}: define the Lang dimension of a variety to be the maximal dimension of a variety of general type which it dominates. Harris conjectured in particular that if the Lang dimension is 0 then for some number field $L \supset K$ we have that the set of $L$ rational points $X(L)$ is dense in $X$. The full statement of Harris’s conjecture will be given below (Conjecture 2.3).

The purpose of this note is to provide a geometric context for Harris’s conjecture, by showing the existence of a universal dominant map to a variety of general type, which we call the Lang map.

1. The Lang map

**Theorem 1.1.** Let $X$ be an irreducible variety over a field $k$, $\text{char}(k) = 0$. There exists a variety of general type $L(X)$ and a dominant rational map $L : X \dashrightarrow L(X)$, defined over $k$, satisfying the following universal property:

Given a field $K \supset k$ and a dominant rational map $f : X_K \dashrightarrow Z$ defined over $K$, where $Z$ is of general type, then there exists a unique dominant rational map $L(f) : L(X) \dashrightarrow Z$ such that $L(f) \circ L = f$.\footnote{Partially supported by NSF grant DMS-9503276.}
Definition 1.2. The universal dominant rational map $L : X \rightarrow L(X)$ is called the Lang map\footnote{I believe this name is appropriate since (1) $L$ is closely related to Lang’s conjecture, and (2) the construction resembles in many ways the constructions of Albanese, trace etc. in Lang’s book \cite{Lang:AV}. I’m also surprised that such a definition has not been made before. (Is that true?)}. The dimension $\dim L(X)$ is called the Lang dimension of $X$.

Lemma 1.3. Assume $f_i : X_K \rightarrow Z_i$ are dominant rational maps with irreducible general fiber, where $Z_i$ are varieties of general type, $i = 1, 2$. Then there exists a variety of general type $Z$ over $K$ and dominant rational maps $f : X \rightarrow Z$ and $g_i : Z \rightarrow Z_i$ such that $g_if = f_i$.

**Proof.** Let $Z = \text{Im}(f_1 \times f_2 : X \rightarrow Z_1 \times Z_2)$, and let $f : X \rightarrow Z$ be the induced map. The map $g_i : Z \rightarrow Z_i$ is dominant and has irreducible general fiber. We claim that $Z$ is a variety of general type. By Viehweg’s additivity theorem (\cite{Viehweg:ADP}, Satz III), it suffices to show that the generic fiber of $g_1$ is of general type. This follows since the fibers of $g_1$ sweep $Z_2$. (Specifically, let $d$ be the dimension of the generic fiber of $g_2$. Choose a general codimension-$d$ plane section $H \subset Z_1$, then $g_1^{-1}H \rightarrow Z_2$ is generically finite and dominant, therefore $g_1^{-1}H$ is of general type, therefore the generic fiber of $g_1^{-1}H \rightarrow Z_1$ is of general type.)

Lemma 1.4. Given a field extension $K \supset k$, let $l_K$ be the maximal dimension of a variety of general type $Y/K$ such that there exists a dominant rational map $L_K : X_K \rightarrow Y$ with irreducible general fiber. Let $l = \max_{K \supset k} l_K$, and let $K \supset k$ be an extension such that $l_K = l$. Then any such map $L_K$ is the Lang map of $X_K$.

**Proof.** Given an extension $E \supset K$ let $f_2 : X_E \rightarrow Z_2$ be a dominant rational map. By the lemma above with $L_K = f_1$ there exists a variety $Z$ and a dominant rational map with irreducible general fiber $f : X_E \rightarrow Z$ dominating both $Y_E$ and $Z_2$. By maximality $\dim Y = \dim Z_2$ and since the general fibers of $Z \rightarrow Y_E$ are irreducible, $Z \rightarrow Y_E$ is birational. The map $g_2 \circ g_1^{-1} : Y_E \rightarrow Z_2$ gives the required dominant rational map.

**Proof of the theorem.** Using Stein factorization we may restrict attention to maps with irreducible general fibers. As above, let $l = \max_{K \supset k} l_K$, and let $K \supset k$ be an extension such that $l_K = l$. We need to show that $L_K$ can be descended to $k$.

First, we may assume that $K$ is finitely generated over $k$, since both $Y$ and $L_K$ require only finitely many coefficients in their defining equations.
Next, we descend $L_K$ to an algebraic extension of $k$. Choose a model $B$ for $K$, and a model $Y \to B$ for $Y$. We have a dominant rational map $X_B \dasharrow Y$ over $B$. There exists a point $p \in B$ with $[k(p) : k]$ finite, such that $Y_p$ is a variety of general type of dimension $l$ and such that the rational map $X_p \dasharrow Y_p$ exists. The lemma above shows that $(Y_p)_K$ is birational to $Y$. Alternatively, this step follows since by theorems of Maehara (see [Mor]) and Kobayashi - Ochiai (see [MD-LM]) the set of rational maps to varieties of general type $X \dasharrow Z$ is discrete, therefore each $f : X_K \dasharrow Z$ is birationally equivalent to a map defined over a finite extension of $k$.

We may therefore replace $K$ by an algebraic Galois extension of $k$, which we still call $K$. Let $\text{Gal}(K/k) = \{\sigma_1, \ldots, \sigma_m\}$. For any $1 \leq i \leq m$ we have a rational map $(f_1 \times \sigma_i \circ f_1) : X \dasharrow Y \times Y^{\sigma_i}$. Applying lemma 1.3 we obtain a birational map $Y \dasharrow Y^{\sigma_i}$. There are open sets $U_i \subset Y^{\sigma_i}$ over which these maps are regular isomorphisms, giving rise to descent data for $U_i$ to $k$.

Is there a way to describe the fibers of the Lang map $X \dasharrow L(X)$? A first approximation is provided by the following:

**Proposition 1.5.** The generic fiber of the Lang map has Lang dimension 0.

**Proof.** Let $\eta \in L(X)$ be the generic point and let $X_\eta \dasharrow L(X_\eta)$ be the lang map of the generic fiber. Let $M \to L(X)$ be a model of $L(X_\eta)$. By definition, the generic fiber $M_\eta = L(X_\eta)$ of $M$ is of general type, therefore by Viehweg’s additivity theorem $M$ is of general type, and by definition $M$ is birational to $L(X)$.

**Question 1.6.** Is there an open set in $X$ where the Lang map is defined and the fibers have Lang dimension 0?

We will see that the answer is yes, if one assumes the following inspiring conjecture of higher dimensional classification theory:

**Conjecture 1.7** (see Conjecture 1.24 of [Ko:FlAb]).

1. Let $X$ be a variety in characteristic 0. Then either $X$ is uniruled, or $\text{Kod}(X) \geq 0$.
2. If $\text{Kod}(X) \geq 0$ then there is an open set in $X$ where the fibers of the Iitaka fibration have Kodaira dimension 0.

This conjecture allows us to “construct” the Lang map “from above”:

**Proposition 1.8.** Assume that conjecture 1.7 holds true. Then there is a finite sequence of dominant rational maps

$$X \dasharrow X_1 \dasharrow \cdots \dasharrow X_n = L(X)$$

where each map $X_i \dasharrow X_{i+1}$ is either an MRC fibration (see [Ko-Mi-Mo], 2.7) or an Iitaka fibration. In particular, the answer to question 1.6 is “yes”.
The proof is obvious. We remark that since \[1.7\] is known when the fibers have dimension \(\leq 2\). In particular, \[1.8\] is known unconditionally when \(\dim X \leq 3\).

2. Harris’s conjecture

As mentioned above, we define the Lang dimension of a variety \(X\) to be \(\dim L(X)\), and Lang’s conjecture implies that if \(K\) is a number field, and if \(X/K\) has positive Lang dimension, then \(X(K)\) is not Zariski-dense in \(X\). In [H1], J. Harris proposed a complementary statement:

Conjecture 2.1 (Harris’s conjecture, weak form). Let \(X\) be a variety of Lang dimension 0 defined over a number field \(K\). Then for some finite extension \(E \supset K\) the set of \(E\)-rational points \(X(E)\) is Zariski dense in \(X\).

It is illuminating to consider the motivating case of an elliptic surface of positive rank.

Let \(\pi_0 : X_0 \to \mathbb{P}^1\) be a pencil of cubics through 9 rational points in \(\mathbb{P}^2\). By choosing the base points in general position we can guarantee that the pencil has 12 irreducible singular fibers which are nodal rational curves. The Mordell-Weil group of \(\pi_0\) has rank 8. The relative dualizing sheaf \(\omega_{\pi_0} = \mathcal{O}_{X_0}(F_0)\) where \(F_0\) is a fiber. Let \(f : \mathbb{P}^1 \to \mathbb{P}^1\) be a map of degree at least 3. Let \(\pi : X \to \mathbb{P}^1\) be the pull-back of \(X_0\) along \(f\). Then \(\omega_\pi = \mathcal{O}_X(3F)\), therefore \(\omega_X = \mathcal{O}_X(F)\) and \(X\) has Kodaira dimension 1. The Iitaka fibration is simply \(\pi\). The elliptic surface \(X\) still has a Mordell-Weil group of rank 8 of sections. By applying these sections to rational points on \(\mathbb{P}^1\) we see that the set of rational points \(X(\mathbb{Q})\) is dense in \(X\).

It is not hard to modify this example to obtain a varying family of elliptic surfaces which has a dense collection of sections. Let \(B\) be a curve and let \(g : B \times \mathbb{P}^1 \to \mathbb{P}^1\) be a family of rational functions on \(\mathbb{P}^1\) which varies in moduli (such families exists as soon as the degree is at least 3). Let \(Y\) be the pullback of \(X\) to \(B \times \mathbb{P}^1\). Then \(p : Y \to B\) is a family of elliptic surfaces, of variation \(Var(p) = 1\), and relative kodaira dimension 1. By composing sections of \(E\) with \(g\) and arbitrary rational maps \(B \to \mathbb{P}^1\), we see that \(p\) has a dense collection of sections.

Harris’s weak conjecture for elliptic surfaces is attributed to Manin. In case of surfaces over \(\mathbb{P}^1_{\mathbb{Q}}\), it has been related to the conjecture of Birch and Swinnerton-Dyer: let \(\pi : X \to \mathbb{P}^1\) be an elliptic surface defined over \(\mathbb{Q}\). In [Man], E. Manduchi shows that under certain assumptions on the behavior of the \(j\) function, the set of points in \(\mathbb{P}^1(\mathbb{Q})\) where the fiber has root number \(-1\) is dense (in the classical topology). According to the conjecture of Birch and Swinnerton-Dyer, the root number gives the parity of the Mordell-Weil rank. It is likely
that some of Manduchi’s conditions (at least condition (1) in Theorem 1 of [Man]) can be relaxed once one passes to a number field.

What can be said in case $0 < \dim L(X) < \dim X$? In [H2], Harris proposed the following definition:

**Definition 2.2.** The diophantine dimension, $D\dim(X)$ is defined as follows:

$$D\dim(X) := \min_{\emptyset \neq U \subset X \text{ open}} \max_{|E:K| < \infty} \dim(U(E))$$

Harris proceeded to propose the following:

**Conjecture 2.3** (Harris’s conjecture). For any variety $X$ over a number field,

$$D\dim(X) + \dim L(X) = \dim X.$$  

I do not know whether or not Harris himself believes this conjecture. This does not really matter. What is appealing in this conjecture, apart from its “tightness”, is that any evidence, either for or against it, is likely to be of much interest.

For lack of any better results, we just note that proposition [L8] directly implies the following:

**Proposition 2.4.** Assuming [L8], Lang’s conjecture together with the weak form of Harris’s conjecture [2.1] implies Harris’s conjecture [2.3].

**ACKNOWLEDGEMENTS** I would like to thank D. Bertrand, J. Harris, J. Kollár, K. Matsuki, D. Rohrlich and J. F. Voloch for discussions related to this note.

**References**

[N] D. Abramovich, Uniformité des points rationnels des courbes algébriques sur les extensions quadratiques et cubiques, C.R. Acad. Sc. Paris, t. 321, Sér. I, p. 755-758, 1995.

[H1] J. Harris, *Lang dimension?* letter to D.A., L. Caporaso and B. Mazur. Jan. 2, 1995.

[H2] J. Harris, *Lang on steroids*, letter to D.A., L. Caporaso and B. Mazur. Jan. 3, 1995.

[Ko:FlAb] J. Kollár et al., *Flips and abundance for algebraic threefolds*. Astérisque 211, 1992.

[Ko-Mi-Mo] J. Kollár, Y. Miyaoka and S. Mori, *Rationally Connected Varieties*, J. Alg. Geom. 1 (1992) p. 429-448.

[L] S. Lang, *Hyperbolic diophantine analysis*, Bull. A.M.S. 14 (1986) p. 159-205.

[L:AV] S. Lang, *Abelian Varieties*. Interscience, New York 1959.

[Man] E. Manduchi, *Root numbers of fibers of elliptic surfaces*, Comp. Math. 99 (1995) p. 33-58.

[MD-LM] M. Martin-Deschamps and R. Lewin-Ménégux, *Applications rationnelles séparables dominantes sur une variété de type général*, Bull. S. M. France 106 (1978) no. 3, p. 279-287.

[Maz] B. Mazur, *The topology of rational points*, J. Experimental Math. 1 (1992) p. 35-45.

[Mor] A. Moriwaki, *Remarks on S. Lang’s conjecture over function fields*, preprint.

[http://xxx.lanl.gov/e-print/alg-geom/9412021]

[Roh] D. E. Rohrlich, *Variation of the root number in families of elliptic curves*, Comp. Math. 87 (1993) p. 119-151.

[V1] E. Viehweg, *Die Additivität der Kodaira Dimension für projektive Fasserräume über Varietäten des allgemeinen Typs*, Jour. reine und angew. Math. 330 (1982), 132-142.