BILINEAR KAKEYA-NIKODYM AVERAGES OF EIGENFUNCTIONS ON COMPACT RIEMANNIAN SURFACES

CHANGXING MIAO, CHRISTOPHER D. SOGGE, YAKUN XI, AND JIANWEI YANG

Abstract. We obtain an improvement of the bilinear estimates of Burq, Gérard and Tzvetkov [6] in the spirit of the refined Kakeya-Nikodym estimates [2] of Blair and the second author. We do this by using microlocal techniques and a bilinear version of Hörmander’s oscillatory integral theorem in [7].

1. Introduction

Let \((M, g)\) be a two-dimensional compact boundaryless Riemannian manifold with Laplacian \(\Delta_g\). If \(e_\lambda\) are the associated eigenfunctions of \(\sqrt{-\Delta_g}\) such that \(-\Delta_g e_\lambda = \lambda^2 e_\lambda\), then it is well known that

\[
\|e_\lambda\|_{L^4(M)} \leq C \lambda^{1/8} \|e_\lambda\|_{L^2(M)},
\]

which was proved in [9] using approximate spectral projectors \(\chi_\lambda = \chi(\lambda - \sqrt{-\Delta_g})\) and showing

\[
\|\chi_\lambda f\|_{L^4(M)} \leq C \lambda^{1/8} \|f\|_{L^2(M)}.
\]

If \(0 < \lambda \leq \mu\) and \(e_\lambda, e_\mu\) are two associated eigenfunctions of \(\sqrt{-\Delta_g}\) as above, Burq et al [6] proved the following bilinear \(L^2\)-refinement of (1.1)

\[
\|e_\lambda e_\mu\|_{L^2(M)} \leq C \lambda^{1/4} \|e_\lambda\|_{L^2(M)} \|e_\mu\|_{L^2(M)},
\]

as a consequence of a more general bilinear estimate on the reproducing operators

\[
\|\chi_\lambda f \chi_\mu g\|_{L^2(M)} \leq C \lambda^{1/4} \|f\|_{L^2(M)} \|g\|_{L^2(M)}.
\]

The bilinear estimate (1.3) plays an important role in the theory of nonlinear Schrödinger equations on compact Riemannian surfaces and it is sharp in the case when \(M = S^2\) endowed with the canonical metric and \(e_\lambda(x) = h_\mu(x)\), \(e_\mu(x) = h_\mu(x)\) are highest weight spherical harmonic functions of degree \(p\) and \(q\), concentrating along the equator

\[
\{x = (x_1, x_2, x_3) : x_1^2 + x_2^2 = 1, x_3 = 0\}
\]

with \(\lambda^2 = p(p+1), \mu^2 = q(q+1)\). Indeed, one may take \(h_k(x) = (x_1 + ix_2)^k\) to see \(\|h_k\|_2 \approx k^{-1/4}\) by direct computation.

Key words and phrases. Eigenfunctions; Bilinear estimates; Kakeya-Nikodym averages.

The first author was supported in part by the NSF of China. The second author was supported in part by the NSF grants DMS-1361476. The fourth author was supported by ERC Advanced Grant No. 291214BLOWDISOL.
In Section 2, we will construct a generic example to show the optimality of (1.4) and exhibit that the mechanism responsible for the optimality seems to be the existence of eigenfunctions concentrating along a tubular neighborhood of a segment of a geodesic. As observed in [10], (1.2) is saturated by constructing an oscillatory integral which highly concentrates along a geodesic. The dynamical behavior of geodesic flows on \( M \) accounts for the analytical properties of eigenfunctions exhibits the transference of mathematical theory from classical mechanics to quantum mechanics (see [12]).

That the eigenfunctions concentrating along geodesics yield sharp spectral projector inequalities leads naturally to the refinement of (1.1) in [11], where it is proved for an \( L^2 \) normalized eigenfunction \( e_\lambda \), its \( L^4 \)-norm is essentially bounded by a power of

\[
\sup_{\gamma \in \mathcal{H}} \frac{1}{|T_{\lambda-1/2}(\gamma)|} \int_{T_{\lambda-1/2}(\gamma)} |e_\lambda(x)|^2 \, dx,
\]

where \( \mathcal{H} \) denotes the collection of all unit geodesics and \( T_\delta(\gamma) \) is a tubular \( \delta \)-neighborhood about the geodesic \( \gamma \). This fact motivates the Kakeya-Nikodym maximal average phenomena measuring the size and concentration of eigenfunctions.

This result was refined by Blair and Sogge [2], where the authors proved for every \( 0 < \varepsilon \leq 1/2 \), there is a \( C = C(\varepsilon, M) \) so that

\[
\|e_\lambda\|_{L^4(M)} \leq C \lambda^{\frac{1}{2}} \|e_\lambda\|_{L^2(M)}^2 \times \left( \sup_{\gamma \in \mathcal{H}} \int_{T_{\lambda-\frac{1}{2}+\varepsilon}(\gamma)} |e_\lambda(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]

We shall assume throughout that our eigenfunctions are \( L^2 \)-normalized, but we shall formulate our main estimates as in (1.6) to emphasize the difference between the norms over all of \( M \) and over shrinking tubes.

As mentioned in [2] it would be interesting to see whether the \( \varepsilon \)-loss in (1.6) can be eliminated. Further results for higher dimensions are in [4].

Inspired by [11], we are interested in the bilinear version of the main result in [11], namely, searching for the essentially appropriate control of \( \|e_\lambda e_\mu\|_2 \) by means of Kakeya-Nikodym maximal averages. In fact, we will obtain a better result by establishing the microlocal version of Kakeya-Nikodym average in the spirit of [2], and our main result reads

**Theorem 1.1.** Assume \( 0 < \lambda \leq \mu \) and \( e_\lambda, e_\mu \) are two eigenfunctions of \( \sqrt{-\Delta_g} \) associated to the frequencies \( \lambda \) and \( \mu \) respectively. Then for every \( 0 < \varepsilon \leq \frac{1}{2} \), we have a \( C_\varepsilon > 0 \) such that

\[
\|e_\lambda e_\mu\|_{L^2(M)} \leq C_\varepsilon \lambda^{\frac{1}{2}} \|e_\mu\|_{L^2(M)} \|e_\lambda\|_{KN(\lambda, \varepsilon)},
\]

and

\[
\|e_\lambda e_\mu\|_{L^2(M)} \leq C_\varepsilon \lambda^{\frac{1}{2}} \|e_\lambda\|_{L^2(M)} \|e_\mu\|_{KN(\lambda, \varepsilon)},
\]

where the Kakeya-Nikodym norm is defined by

\[
\|f\|_{KN(\lambda, \varepsilon)} = \left( \sup_{\gamma \in \mathcal{H}} \lambda^{\frac{1}{4} - \varepsilon} \int_{T_{\lambda-\frac{1}{2}+\varepsilon}(\gamma)} |f(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]
Note also that we can reformulate our main estimates as follows
\[(1.7') \quad \|e\mu\|_{L^2(M)} \leq C e^\lambda \|\|e\mu\|_{L^2(M)} \times \left( \sup_{\gamma \in H} \int_{\lambda \sim \frac{1}{2} + \varepsilon} |\mu|^2 \, dx \right)^{\frac{1}{2}},\]

and
\[(1.8') \quad \|e\mu\|_{L^2(M)} \leq C e^\lambda \|\|e\mu\|_{L^2(M)} \times \left( \sup_{\gamma \in H} \int_{\lambda \sim \frac{1}{2} + \varepsilon} |\mu|^2 \, dx \right)^{\frac{1}{2}},\]

both of which are bilinear variants of (1.6). Also, by taking the geometric means of (1.7) and (1.8) one of course has that
\[(1.10) \quad \|e\mu\|_{L^2(M)} \leq C e^\lambda \|\|e\mu\|_{L^2(M)} \|e\mu\|_{C^2(K N(\lambda, \varepsilon))} \|e\mu\|_{C^2(K N(\lambda, \varepsilon))}.\]

Note that it is the geodesic tubes corresponding to the lower frequency that accounts for the optimal upper bound of \(\|e\mu\|_{L^2(M)}\). We point out that in (1.8) one cannot take the \(K N(\mu, \varepsilon)\)-norm of \(e\mu\). For on \(T^n \sim (\pi, \pi)^n\) if \(e\lambda = e^{ijx}, |j| = \lambda,\) and \(e\mu = e^{ikx}, |k| = \mu,\) the analog of (1.8) involving \(\|e\mu\|_{K N(\mu, \varepsilon)}\) is obviously false for small \(\varepsilon > 0\) if \(\mu > \lambda.\)

Note also that if \(e\mu\) is replaced by a subsequence, \(e_{\mu_{j_k}}\) of quantum ergodic eigenfunctions (see [12]) then (1.8') implies that \(e\|e\mu\|_{L^2(M)} \to 1\) as \(\mu_{j_k} \to \infty.\) This is another reason why it would be interesting to know whether the analog of (1.8') is valid with \(\varepsilon = 0\) there.

This paper is organized as follows. In Section 2, we construct an example to show the sharpness of (1.4). In Section 3, we introduce some basic preliminaries and reduce the proof of Theorem 1.1 to the situation, where the strategy in [2] can be applied. In Section 4, we employ the orthogonality argument to conclude the theorem by assuming a specific bilinear oscillatory integral inequality. Finally, we prove this inequality in Section 5 based on the instrument in [6], which provides a bilinear version of Hörmander’s oscillatory integral theorem [7]. We shall assume \(0 < \lambda \leq \mu\) throughout this paper.

2. A GENERIC EXAMPLE

In this section, we shall construct an example showing the optimality of the universal bounds (1.4). We will use approximate spectral projectors \(\chi(\lambda)\) and \(\chi(\mu)\) which reproduce eigenfunctions and can be written as proper Fourier integral operators up to a smooth error.

Without loss of generality, we may assume the injectivity radius of \(M\) is sufficiently large. Take a Schwartz function \(\chi(\lambda) \in \mathcal{S}(\mathbb{R})\) with \(\chi(0) = 1\) and \(\chi\) supported in [1, 2], so that the spectral projectors are represented by
\[\chi(\lambda) f(x) = \lambda^{1/2} T(\lambda) f(x) + R(\lambda) f(x), \quad \chi(\mu) g(x) = \mu^{1/2} T(\mu) g(x) + R(\mu) g(x),\]

where
\[\|R(\lambda) g\|_{L^\infty(M)} \leq C N \lambda^{-N} \|f\|_{L^1(M)}, \quad \|R(\mu) g\|_{L^\infty(M)} \leq C N \mu^{-N} \|g\|_{L^1(M)},\]

for all \(N = 1, 2, \ldots,\) and the main terms read
\[(2.1) \quad T(\lambda) f(x) = \int_M e^{i\lambda d(x, y)} a(x, y, \lambda) f(y) \, dy,\]
\[(2.2) \quad T(\mu) g(x) = \int_M e^{i\mu d(x, z)} a(x, z, \mu) g(z) \, dz.\]
Here $d_g(x, y)$ is the geodesic distance between $x, y \in M$, and the amplitudes $a(x, y, \lambda), a(x, z, \mu) \in C^\infty$ have the following property

$|\partial^\alpha_{x,y} a(x, y, \lambda)| + |\partial^\alpha_{x,z} a(x, z, \mu)| \leq C_\alpha, \quad \text{for all } \alpha.$

Moreover $a(x, y, \lambda) = 0$ if $d_g(x, y) \not\in (1, 2)$ and likewise for $a(x, z, \mu)$. (See [10, Lemma 5.1.3].)

After applying a partition of unity, for small $\delta$ fixed, we may fix three points $x_0, y_0, z_0 \in M$ with $1 \leq d_g(x_0, y_0) \leq 2$, $1 \leq d_g(x_0, z_0) \leq 2$, and assume that $a(x, y, \lambda)$ vanishes outside the region $\{(x, y)| x \in B(x_0, \delta), y \in B(y_0, \delta)\}$, $a(x, z, \mu)$ vanishes outside the region $\{(x, z)| x \in B(x_0, \delta), z \in B(z_0, \delta)\}$. To see the sharpness of (1.4), we will prove the following result.

**Proposition 2.1.** There exist $f$ and $g$ such that for some $C > 0$,

$$\|T_\lambda f \, T_\mu g\|_{L^2} \geq C \lambda^{-1/4} \mu^{-1/2} \|f\|_{L^2} \|g\|_{L^2}. \quad (2.3)$$

We will choose suitable $f$ and $g$ concentrating along a segment of the geodesic $\gamma_0$ connecting $x_0$ and $y_0$ with appropriate oscillations. The explicit expression of $f$ and $g$ will yield automatically upper bounds on $\|f\|_{L^2} \|g\|_{L^2}$. On the other hand, we will see there is a strip region $\Omega_\mu$ containing $x_0$ such that $\|T_\lambda f T_\mu g\|_{L^2(\Omega_\mu)}$ is bounded below by $(\lambda \mu)^{-1/2}$ times the upper bound of $\lambda^{1/4} \|f\|_{L^2} \|g\|_{L^2}$.

Recall first the geodesic normal coordinate centered at $y_0$. Let $\{e_1, e_2\}$ be the orthonormal basis in $T_{y_0}M$ such that $e_1$ is the tangent vector of $\gamma_0$, pointing to $x_0$. The exponential map $\exp_{y_0}$ is a smooth diffeomorphism between the ball $\{Y \in T_{y_0}M : Y = Y_1 e_1 + Y_2 e_2, |Y| < 10\}$ and $B(y_0, 10)$. Let $\{\omega_1, \omega_2\}$ be the dual basis of $\{e_1, e_2\}$ and set $y_j = \omega_j \circ \exp_{y_0}^{-1}$ for $j = 1, 2$. Then $\{y_1, y_2\}$ is the Riemannian geodesic normal coordinates such that $y_0 = 0$ and

$$g_{ij}(0) = \delta_{ij}, \quad d_{g_{ij}}(0) = 0, \quad \text{for all } i, j = 1, 2.$$

In particular, $\Gamma^0_{ij}(0) = 0, \forall i, j, k = 1, 2$, and $dG(0) = 0$ with $G = \det(g_{ij})$. In this coordinate system, $\gamma_0$ is parameterized by $t \mapsto \{(t, 0)\}$.

**Lemma 2.2.** If we denote by $\phi(x, y) = d_g(x, y)$, then in these coordinates $\phi(x, 0) = |x|$. Moreover, if we set $x = (x_1, x_2)$, $y = (y_1, y_2)$ and assume $0 < y_1 < x_1$, then $\phi(x, y) = x_1 - y_1 + O((x_2 - y_2)^2)$.

**Proof.** See p. 144 in [10].

With Lemma 2.2 at hand, we are ready to prove Proposition 2.1.

**Proof of Proposition 2.1.** We work in the above coordinates and let

$$\Omega_\mu = \{ x : \delta/2C_0 \leq x_1 \leq 2C_0 \delta, |x_2| \leq \varepsilon_1 \mu^{-1/2}\}, \quad 0 < \varepsilon_1 \ll \delta,$$

where $C_0 > 0$ is chosen as on p. 144 in [10]. The region $\Omega_\lambda$ is defined similarly. Take $\alpha \in C^\infty_0((-1, 1))$ and set

$$f(y) = \alpha(y_1/\varepsilon_1) \alpha(\lambda^{1/2} y_2/\varepsilon_1) e^{i\lambda y_1},$$

$$g(z) = \alpha(z_1/\varepsilon_1) \alpha(\mu^{1/2} z_2/\varepsilon_1) e^{i\mu z_1}.$$
Denote by $\epsilon = \lambda / \mu$. Then similar to Chapter 5 in [10], we estimate
\[
\int_{\Omega_\mu} |T_\lambda f(x)T_\mu g(x)|^2 \, dx.
\]
Indeed, for $x \in \Omega_\mu$, we have
\[
|T_\lambda f(x)|^2 = \int_{\Omega_\mu^2} e^{i\lambda (d_\mu(x,y) - d_\mu(x,y')) - [(x_1 - y_1) - (x_1 - y_1')]_a(x,y,\lambda) \alpha(y_1/\epsilon_1) \alpha(\lambda^{1/2} y_2/\epsilon_1)}
\times \alpha(x,y',\lambda) \alpha(y'_1/\epsilon_1) \alpha(\lambda^{1/2} y'_2/\epsilon_1) \, dydy',
\]
Notice that by Lemma 2.2, the phase function equals $O(|x_2 - y_2|^2) + O(|x_2 - y'_2|^2)$. Since $|x_2| \leq \epsilon_1 \mu^{-1/2}$ and $|y_2|, |y'_2| \leq \epsilon_1 \lambda^{-1/2}$, we see that the phase in the exponent is of order $\epsilon_1^2$ on $\Omega_\mu$, and the oscillation is eliminated in the integrand by choosing $\epsilon_1$ small. Thus on $\Omega_\mu$
\[
|T_\lambda f(x)|^2 \gtrsim |\Omega_\lambda|^2 = \lambda^{-1}.
\]
Similarly,
\[
|T_\lambda g(x)|^2 \gtrsim |\Omega_\mu|^2 = \mu^{-1},
\]
Thus, $\|T_\lambda fT_\mu g\|_{L^2(\Omega_\mu)}$ is bounded below by $\mu^{-3/4} \lambda^{-1/2}$. On the other hand, $\|f\|_2 \|g\|_2 \leq c(\lambda \mu)^{-1/4}$ for $f$ and $g$ given by (2.4) (2.5), we have
\[
(\lambda \mu)^{1/2} \|T_\lambda fT_\mu g\|_2 / (\|f\|_2 \|g\|_2) \geq C_{\epsilon_1} \lambda^{1/4}.
\]
\[\square\]
This example exhibits the concentration of eigenfunctions along a tubular neighborhood of a geodesic leading to the sharpness of the bilinear spectral projector estimate (1.4), where our bilinear generalization of the main result in [11] is motivated.

**Remark 2.3.** Comparing this example with (1.10), one may suspect that (1.10) can be further refined. Indeed, one may observe that the example suggests the possibility of refining (1.10) by strengthening the $L^2$-norm of the eigenfunction corresponding to the higher frequency on the right side to a $\lambda^{-1/2}$-neighborhood of the same geodesic segment for the lower frequency eigenfunction. An interesting problem would be to see if the following refinement of (1.10) is valid:

\[
\|e_\lambda e_\mu\|_{L^2(M)} \leq C_{\epsilon_0} \lambda^{1/2} \sup_{\gamma \in \Pi} \left[ \left( \int_{T_\lambda^{-1/2+\epsilon_0}(\gamma)} |e_\mu(x)|^2 \, dx \right)^{1/2} \right].
\]

3. **Microlocal Kakeya-Nikodym averages**

3.1. **Basic notions.** In view of $\chi_\lambda e_\lambda = e_\lambda$ and $\chi_\mu e_\mu = e_\mu$, we are reduced to estimating $\|T_\lambda fT_\mu g\|_{L^2}$. By scaling, we may assume the injectivity radius of $M$ is large enough, say $\text{inj} M > 10$. We use partitions of unity on $M$ to reduce the $L^2$ integration of $T_\lambda fT_\mu g$ on the geodesic ball $B(x_0, \delta)$ with $\delta > 0$ small. In view of the property of $\text{supp} a$, we may apply partition of unity once more and assume $\text{supp} f \subset B(y_0, \delta)$ and $\text{supp} g \subset B(z_0, \delta)$ for some $y_0$ and $z_0$ satisfying
\[
1 \leq d_g(x_0, y_0), d_g(x_0, z_0) \leq 2.
\]
Next, we need to choose a suitable coordinate system to simplify the calculations on a larger ball \( B(x_0, 10) \). As in [11] and [4], we shall use Fermi coordinate system about the geodesic \( \gamma \) connecting \( x_0 \) and \( y_0 \). Let \( \gamma^\perp \) be the geodesic through \( x_0 \) perpendicular to \( \gamma \). The Fermi coordinates about \( \gamma \) is defined on the ball \( B(x_0, 10) \), where the image of \( \gamma^\perp \cap B(x_0, 10) \) in the resulting coordinate system is parameterized by \( s \mapsto \{(s, 0)\} \). All the horizontal segments are parameterized by \( s \mapsto (s, t_0) \) and we have

\[
d_g((s_1, t_0), (s_2, t_0)) = |s_1 - s_2|.
\]

Clearly, in our coordinate system, \( y_0 \) is on the 2nd coordinate axis, and \( z_0 \) is a point satisfying \( 1 \leq d_g(z_0, (0, 0)) \leq 2 \).

Therefore, if we set \( y = (s, t) \), \( z = (s', t') \) in this coordinate system, we may write \( \mathcal{T}_x f \) and \( \mathcal{T}_y g \) locally as

\[
\mathcal{T}_x f(x) = \int_{\mathbb{R}^2} e^{i\lambda dx}(x, (s, t)) a(x, (s, t), \lambda) f(s, t) ds dt,
\]

\[
\mathcal{T}_y g(x) = \int_{\mathbb{R}^2} e^{i\mu dy}(x, (s', t')) a(x, (s', t'), \mu) g(s', t') ds' dt'.
\]

Moreover, by noting that \( 1 \leq d_g(x_0, y_0), d_g(x_0, z_0) \leq 2 \) and \( y \in B(y_0, \delta), z \in B(z_0, \delta) \), we shall assume

\[
\max\{|s|, |t - d_g(y_0, x_0)|, |d_g((s', t'), z_0)|\} \leq \delta.
\]

We remark that we are at liberty to take \( \delta \) to be small when necessary.

### 3.2. Preliminary reductions

First of all, we deal with the case when the angle between \( \gamma \) and the geodesic \( \gamma' \) connecting \( x_0 \) and \( z_0 \) is bounded below by some \( \varepsilon_2 > 0 \). To do this, we shall use the geodesic normal coordinates around \( x_0 \). Set \( \{e_1, e_2\} \) to be the orthonormal basis in \( T_{x_0} M \), where the metric \( g \) at \( x_0 \) is normalized, such that \( e_1 \) is the tangent vector of \( \gamma^\perp \) at \( x_0 \) and \( -e_2 \) is the tangent vector of \( \gamma \) at \( x_0 \) if \( \gamma \) is oriented from \( x_0 \) to \( y_0 \). Let \( \omega_1, \omega_2 \) be the dual basis of \( \{e_1, e_2\} \) and set \( \{x_j = \omega_j \circ \exp_{x_0}^{-1}\}_{j=1,2} \) to be the Riemannian geodesic normal coordinate system on \( B(x_0, 10) \), where \( x_0 = 0 \) and \( \gamma \) is parameterized by \( x_2 \mapsto \{(0, x_2)\} \), whereas \( \gamma^\perp \) is parameterized by \( x_1 \mapsto \{(x_1, 0)\} \) with \( |x_1| \leq 5 \). Let \( \theta_0 = \theta(z_0) \) be such that \( z_0 = d_g(x_0, z_0)(\cos \theta_0, \sin \theta_0) \), where the angular variable is oriented in clockwise direction. It follows that \( \gamma'^\perp \) is given by \( r \mapsto \exp_0((r \cos \varphi_0, r \sin \varphi_0)) \) with \( \varphi_0 = \theta_0 + \frac{\pi}{2} \) and \( |r| < 5 \).

Writing

\[
y = (r_1 \cos \theta_1, r_1 \sin \theta_1), \quad z = (r_2 \cos \theta_2, r_2 \sin \theta_2)
\]

in geodesic normal coordinates, we have

\[
\mathcal{T}_x f(x) = \int \int e^{i\lambda dx}(x, (r_1, \theta_1)) a(x, (r_1, \theta_1), \lambda) f(r_1, \theta_1) dr_1 d\theta_1,
\]

\[
\mathcal{T}_y g(x) = \int \int e^{i\mu dy}(x, (r_2, \theta_2)) a(x, (r_2, \theta_2), \mu) g(r_2, \theta_2) dr_2 d\theta_2.
\]

We recall the following fact.
Proposition 3.1. Let $\varepsilon_2 > 0$ be a small parameter. Assume $|\theta(z_0) + \frac{\lambda}{2}| \geq \varepsilon_2$ and $|\theta(z_0) - \frac{\lambda}{2}| \geq \varepsilon_2$. If we choose $\delta$ small enough depending on $\varepsilon_2$, there exists $C$ such that
\begin{equation}
\|T_{\varepsilon_2} g\|_2 \leq C(\lambda \mu)^{-1/2} \|f\|_2 \|g\|_2.
\end{equation}

Thus in order to prove Theorem 1.1, it suffices to consider either $|\theta(z_0) + \frac{\lambda}{2}| \leq \varepsilon_2$ or $|\theta(z_0) - \frac{\lambda}{2}| \leq \varepsilon_2$. This confines $z_0$ in a small neighborhood of the geodesic $\gamma$ by compressing $\gamma$ and $\gamma'$ to be almost parallel with each other.

Essentially, this proposition is proved in [6] based on the following lemma.

Lemma 3.2. Let $y = \exp_0 (r(\cos \theta, \sin \theta))$ and $\phi_r(x, \theta) = d_y(x, y)$. For every $0 < \varepsilon_2 < 1$, there exists $c > 0$, $\delta_1 > 0$ such that for every $|x| < \delta_1$,
\begin{equation}
|\det (\nabla_x \phi_r(x, \theta), \nabla_x \phi_r(x, \theta'))| \geq c,
\end{equation}
if $|\theta - \theta'| \geq \varepsilon_2$ and $|\theta + \pi - \theta'| \geq \varepsilon_2$. In addition, for every $\theta \in [0, 2\pi]$,
\begin{equation}
|\det (\nabla_x \phi_r(x, \theta), \nabla_x \phi_r(x, \theta))| \geq c.
\end{equation}

This is an immediate consequence of the following fact.

Lemma 3.3. Let $y \mapsto \kappa(y) = \exp_0^{-1}(y)$ be the geodesic normal coordinates vanishing at $x_0$, as described above. Then we have
\begin{equation}
\nabla_x d_y(x, y) \bigg|_{x=x_0} = \kappa(y)/|\kappa(y)|.
\end{equation}

Proof. Relation (3.9) is equivalent to Gauss’ lemma. See [11] and [6].

Remark 3.4. We see from this lemma that the set of points $\{\nabla_x d_y(x, y) : x = x_0, d_y(x_0, y) \in (1/2, 2)\}$ is exactly the cosphere at $x_0$, i.e.
\[S^*_{x_0} M = \left\{ \xi : \sum g^{jk}(x_0) \xi_j \xi_k = 1 \right\}, \quad g^{ij} = (g_{ij})^{-1}.\]

The map $y \mapsto \kappa(y)$ is a local radial isometry. See [11].

We sketch the proof of Proposition 3.1 briefly for completeness. In our situation, we have $\theta(0) = -\frac{\lambda}{2}$. Fixing a parameter $\varepsilon_2 > 0$, we assume $|\theta(z_0) + \frac{\lambda}{2}| \geq \varepsilon_2$ and $|\theta(z_0) - \frac{\lambda}{2}| \geq \varepsilon_2$. Since $y \in B(y_0, \delta), z \in B(z_0, \delta)$ given by (3.3), we may choose $\delta < \delta_1$. As a consequence, we have $|\theta_1 - \theta_2| \geq \varepsilon_2$ and $|\theta_1 + \pi - \theta_2| \geq \varepsilon_2$ with some $c > 0$. By Schur’s test, it suffices to show
\begin{equation}
|K(\theta_1, \theta_2, \theta'_1, \theta'_2)| \leq C(\mu |\theta_2 - \theta'_2| + \lambda |\theta_1 - \theta'_1|)^{-10},
\end{equation}
where
\begin{align*}
K(\theta_1, \theta_2, \theta'_1, \theta'_2) &= \int e^{i \Psi_{\lambda, \mu}(x; \theta_1, \theta_2, \theta'_1, \theta'_2)} A(x ; \theta_1, \theta'_1, \theta_2, \theta'_2) dx, \\
A(x ; \theta_1, \theta'_1, \theta_2, \theta'_2) &= a(x, (r_1, \theta_1), \lambda) a(x, (r_1, \theta'_1), \lambda) a(x, (r_2, \theta_2), \mu) a(x, (r_2, \theta'_2), \mu), \\
\Psi_{\lambda, \mu}(x; \theta_1, \theta_2, \theta'_1, \theta'_2) &= \lambda (\phi_{r_2}(x, \theta_1) - \phi_{r_1}(x, \theta'_1)) + \mu (\phi_{r_2}(x, \theta_2) - \phi_{r_2}(x, \theta'_2)).
\end{align*}

For all multi-index $\alpha, |\alpha| \leq 10$, Lemma 3.2 and the above formula give
\[|\nabla_x \Psi_{\lambda, \mu}| \geq C(\lambda |\theta_1 - \theta'_1| + \mu |\theta_2 - \theta'_2|), \quad |\partial_{\alpha}^{\alpha} \Psi_{\lambda, \mu}| \leq C(\lambda |\theta_1 - \theta'_1| + \mu |\theta_2 - \theta'_2|).
\]

Now (3.10) follows from integration by parts.
3.3. Decomposition of the phase space and microlocal Kakeya-Nikodym averages. We will employ the strategy introduced by [2], where a microlocal refinement of Kakeya-Nikodym averages are exploited. From now on, we shall always assume
\[ |\theta(z_0) + \frac{\pi}{2}| \leq \varepsilon_2 \ll 1 \]
where \( \frac{\pi}{2} = -\theta(y_0) \). Recall that we may write, modulo trivial errors,
\begin{align}
\chi_\lambda f(x) &\approx \lambda^\frac{1}{2} \int_{\mathbb{R}^2} e^{i\lambda d_\mu(x,y)} a_\lambda(x,y) f(y) \, dy, \\
\chi_\mu g(x) &\approx \mu^\frac{1}{2} \int_{\mathbb{R}^2} e^{i\mu d_\nu(x,z)} a_\mu(x,z) g(z) \, dz,
\end{align}
with \( \text{supp } f \subset B(y_0, \delta) \), \( \text{supp } g \subset B(z_0, \delta) \) and \( x \in B(0, \delta) \).

As discussed in the last section, we may choose \( \varepsilon_2 > 0 \) sufficiently small to make \( z_0 \) to be within an fixed small neighbourhood of \( \gamma \).

To decompose the phase space, we shall use the geodesic flow \( \Phi_\tau(y, \xi) \) on the cosphere bundle \( S^*M \), which starts from \( y \) in direction of \( \xi \in S^*_yM \). We use the Fermi coordinates around \( \gamma \) to write
\[(y(\tau), \xi(\tau)) = \Phi_\tau(y, \xi), \quad (y(0), \xi(0)) = (y, \xi),\]
where \( \xi(\tau) \) is the unit cotangent vector in \( T^*_yM \). Define \( \Theta : (y, \xi) \in S^*M \to \mathbb{R} \times \mathbb{R} \) by
\[\Theta(y, \xi) = \left( \Pi_{y_1} \Phi_{\tau_0}(y, \xi), \frac{\Pi_{\xi_1} \Phi_{\tau_0}(y, \xi)}{|\Pi_{\xi_1} \Phi_{\tau_0}(y, \xi)|} \right),\]
where \( \tau_0 \) is chosen so that \( y_2(\tau_0) = \Pi_{y_2} \Phi_{\tau_0}(y, \xi) = 0 \). By \( \Pi_\Theta \), we mean the projection to the component of \( \Theta \)-variable.

Remark 3.5. As in [2], we require \( |\xi_1| < \delta \) with \( \delta \) small enough with \( y \in B(y_0, C_0 \delta) \). Moreover, \( \Theta \) is constant on the orbit of \( \Phi \) and \( |\Theta(y, \xi) - \Theta(z, \eta)| \) can be used as a natural distance function between geodesics passing respectively through \( (y, \xi) \) and \( (z, \eta) \).

Next, we microlocalize \( \chi_\lambda f \) and \( \chi_\mu g \) by introducing smooth functions \( \alpha_1(y) \) and \( \alpha_2(z) \) adapted respectively to the ball \( B(y_0, 2\delta) \) and \( B(z_0, 2\delta) \) and setting
\begin{align}
Q^\nu(y, \xi) &= \alpha_1(y) \beta(\theta^{-1} \Theta(y, \xi) + \nu) \, \Upsilon(|\xi|/\lambda), \\
P^\nu(z, \eta) &= \alpha_2(z) \beta(\theta^{-1} \Theta(z, \eta) + \nu) \, \Upsilon(|\eta|/\mu),
\end{align}
where \( \lambda^{-1/2} \leq \theta \leq 1, \nu, \nu \in \mathbb{Z}^2 \), with \( \beta \) smooth such that
\[\sum_{\nu \in \mathbb{Z}^2} \beta(\cdot + \nu) = 1, \quad \text{supp } \beta \subset \{x \in \mathbb{R}^2 : |x| \leq 2\},\]
and \( \Upsilon \in C^\infty_0(\mathbb{R}) \) is supported in \([c, c^{-1}]\) for some \( c > 0 \).

Let us take a look at the symbols \( Q^\nu(y, \xi) \) and \( P^\nu(z, \eta) \). First, we define \( \beta(\theta^{-1} \Theta(y, \xi) + \nu) \) and \( \beta(\theta^{-1} \Theta(z, \eta) + \nu) \) on the cosphere bundle. Since these two functions are of degree zero in the cotangent variables, we then extend them homogeneously to the cotangent bundle with the zero section removed. The above
$Q^\nu_0(y, \xi)$ and $P^\nu_0(z, \eta)$ are well-defined for $\xi \neq 0$, $\eta \neq 0$. Given $\xi$, $\beta(\theta^{-1}\Theta(y, \xi) + \nu) = 0$ unless $y$ belongs to a tubular neighborhood of $\gamma_\nu$, where

$$
\gamma_\nu = \{ y(\tau) : -2 \leq \tau \leq 2, (y(\tau), \xi(\tau)) = \Phi_r(y, \xi), \Theta(y, \xi) + \theta \nu = 0 \}.
$$

Moreover, if we set $\nu = (\nu_1, \nu_2)$, the direction of $\gamma_\nu$ at $y(\tau_0)$ is determined by $\theta \nu_2$ and is independent of $\lambda$. Since $(y, \xi) = \Phi^{-1}_r(y(\tau_0), \xi(\tau_0))$ and $y(\tau_0) = (y_1(\tau_0), 0)$ with $y_1(\tau_0) = \theta \nu_1 + O(\theta)$, one easily finds that $y \in T_{C_1}(\gamma_\nu)$, for some $C_1 \geq 1$.

Similar statements hold for $P^\nu_0(z, \eta)$.

Let $Q^\nu_0(x, D)$, $P^\nu_0(x, D)$ be the pseudo-differential operators associated to the symbols defined in (3.13) and (3.14) respectively. We next record some properties of $Q^\nu_0(y, D)$ and $P^\nu_0(z, D)$. The first lemma indicates that these two kinds of operators provide a natural microlocal wave-packet decomposition in the phase space for 2-dimensional manifolds.

**Lemma 3.6.** If $\lambda^{-1/2+\varepsilon} \leq \theta \leq 1$ with $\varepsilon > 0$ fixed, the symbols $Q^\nu_0$ and $P^\nu_0$ belong to a bounded subset of $S^0_{1/2+\varepsilon, 1/2-\varepsilon}$. Then there is $C_\epsilon$ and $C_2 \geq C_1$ such that for $\lambda^{-1/2+\varepsilon} \leq \theta \leq 1$, we have

$$
\|Q^\nu_0(x, D) f\|_{L^2} \leq C_\epsilon \|f\|_{L^2(T_{C_2}(\gamma_\nu))} + C_N \lambda^{-N} \|f\|_2
$$

(3.16)

$$
\|P^\nu_0(x, D) g\|_{L^2} \leq C_\epsilon \|g\|_{L^2(T_{C_2}(\gamma_\nu))} + C_N \mu^{-N} \|g\|_2.
$$

(3.17)

Moreover, for any integer $N \geq 0$, one may write

$$
\chi_\lambda f = \sum_{\nu \in \mathbb{Z}^2} \chi_\lambda \circ Q^\nu_0(x, D) f + R_\lambda f, \quad \text{if supp } f \subset B(y_0, \delta),
$$

(3.18)

$$
\chi_\mu g = \sum_{\nu \in \mathbb{Z}^2} \chi_\mu \circ P^\nu_0(x, D) g + R_\mu g, \quad \text{if supp } g \subset B(z_0, \delta),
$$

(3.19)

with $\|R_\lambda\|_{L^2 \to L^\infty} \lesssim \lambda^{-N}$, $\|R_\mu\|_{L^2 \to L^\infty} \lesssim \mu^{-N}$.

**Proof.** That $Q^\nu_0(y, \xi) \in S^0_{1/2+\varepsilon, 1/2-\varepsilon}$ has already been proved in [2]. If we use $\mu \geq \lambda$, we get $\mu^{-1}\lambda^{1/2-\varepsilon} \leq \mu^{-1/2-\varepsilon}$, and the same calculation as for $Q^\nu_0$ yields that the $P^\nu_0(z, D)$ belong to a bounded subset of pseudodifferential operators of order zero and type $(1/2 + \varepsilon, 1/2 - \varepsilon)$. To see (3.16), one observes that the kernel $K^\nu_0(x, y)$ of the operator $Q^\nu_0$ is bounded by $O(\lambda^{-N})$ if $y$ does not belong to $T_{C_2}(\gamma_\nu)$ for some large $C_2 > C_1$ by using integration by parts. We can deduce (3.18) from (3.15). In fact, if we recall the process of constructing parametrix for the half wave operator $e^{itN^{-1/2}}$ in [10], we may use integration by parts to see that in (3.18), one may assume $\hat{f}(\xi) = 0$ if $|\xi| \geq c\lambda, C\lambda$ up to some terms of the form $R_\lambda f$. It suffices to see the difference of $f(x)$ and $\sum_{\nu} Q^\nu_0(x, D) f(x)$ is of the form $R_\lambda f(x)$. This is easy due to the fact that $\sum (|\xi|/\lambda) = 1$ on the support of $\hat{f}$ by choosing suitable $c, C$ and $(1 - \alpha_1(x))f(x) = 0$. Now (3.15) yields

$$
\alpha_1(x) f(x) = \sum_{\nu} Q^\nu_0(x, D) f(x).
$$

Similar argument yields (3.17) and (3.19) \(\square\).

Now, we recall the microlocal Kakeya-Nikodym norm in [2], corresponding to frequency $\lambda$ and $\theta_0 = \lambda^{-1/2+\varepsilon_0}$

$$
\|f\|_{MKN(\theta_0, \varepsilon_0)} = \sup_{\theta_0 \leq \theta \leq 1} \left( \sup_{\nu \in \mathbb{Z}^2} \|Q^\nu_0(x, D) f\|_{L^2(R^2)} \right) + \|f\|_{L^2(R^2)}.
$$

(3.20)
As pointed out in [2], the maximal microlocal concentration of \( f \) about all unit geodesics in the scale of \( \theta \) amounts to the quantity
\[
\sup_{\nu \in \mathbb{Z}^2} \theta^{-1/2} \|Q_\theta'(x, D) f\|_{L^2(\mathbb{R}^2)}.
\]
From Lemma 3.6, one can prove \( \|f\|_{MKN(\lambda, \epsilon_0)} \leq C_{\epsilon_0} \|f\|_{KN(\lambda, \epsilon_0)} \). We refer to [2] for more details. Similarly, for the same \( \theta_0 \), we can define
\[
(3.21) \quad \|g\|_{MKN'(\lambda, \epsilon_0)} = \sup_{\theta_0 \leq \theta \leq \frac{1}{2}, \nu \in \mathbb{Z}^2} \left( \sup_{\nu \in \mathbb{Z}^2} \theta^{-1/2} \|P_\theta'(x, D) f\|_{L^2(\mathbb{R}^2)} \right) + \|g\|_{L^2(\mathbb{R}^2)},
\]
again by Lemma 3.6, we see that \( \|g\|_{MKN'(\lambda, \epsilon_0)} \leq C_{\epsilon_0} \|g\|_{KN(\lambda, \epsilon_0)} \).

We will use the following fact in the next section.

**Lemma 3.7.** For any \( \epsilon > 0 \), there exists some \( C_\epsilon > 0 \) such that for all \( \lambda^{-1/2+\epsilon} \leq \theta \leq 1 \),
\[
(3.22) \quad \left\| \sum_{\nu} (Q_\theta'(x))^* \circ Q_\theta' f \right\|_{L^2} \leq C_\epsilon \|f\|_{L^2}, \quad \left\| \sum_{\nu} (P_\theta'(x))^* \circ P_\theta' g \right\|_{L^2} \leq C_\epsilon \|g\|_{L^2}.
\]

**Proof.** The \( L^2 \)-estimates (3.22) are valid thanks to (3.15) and the classical calculus of pseudo-differential operators of type \( (1/2 + \epsilon, 1/2 - \epsilon) \) with \( \epsilon > 0 \).

We describe next the kernels of the operators \( \chi_\lambda Q_\theta'(x, D) = (\chi_\lambda \circ Q_\theta')(x, D) \) and \( \chi_\mu P_\theta'(x, D) = (\chi_\mu \circ P_\theta')(x, D) \) following [2].

**Lemma 3.8.** Denote by \( (\chi_\lambda Q_\theta')(x, y) \) and \( (\chi_\mu P_\theta')(x, z) \) the kernels of the pseudodifferential operators \( \chi_\lambda Q_\theta'(x, D) \) and \( \chi_\mu P_\theta'(x, D) \) respectively. Assume \( \theta \in [C_0 \theta_0, 1] \) with \( \theta_0 = \lambda^{-1/2+\epsilon} \) and \( C_0 \gg 1 \). We can find a uniform constant \( C \) so that for each \( N = 1, 2, 3, \ldots \), we have
\[
(3.23) \quad \|\chi_\lambda Q_\theta'(x, y)\|_{L^\infty} \leq C \lambda^{-N}, \quad \text{if} \ x \notin T_{C \theta_0}(\gamma_\nu) \text{ or } y \notin T_{C \theta_0}(\gamma_\nu),
\]
and
\[
(3.24) \quad \|\chi_\mu P_\theta'(x, z)\|_{L^\infty} \leq C \mu^{-N}, \quad \text{if} \ x \notin T_{C \theta_0}(\gamma_\nu) \text{ or } z \notin T_{C \theta_0}(\gamma_\nu).
\]
Furthermore,
\[
(3.25) \quad (\chi_\lambda Q_\theta')(x, y) = \lambda^{1/2} e^{i\lambda d_\nu(x, y)} a_{\nu, \theta}(x, y) + O_N(\lambda^{-N}),
\]
\[
(3.26) \quad (\chi_\mu P_\theta')(x, z) = \mu^{1/2} e^{i\mu d_\nu(x, z)} b_{\nu, \theta}(x, z) + O_N(\mu^{-N}),
\]
where we have the uniform bounds
\[
(3.27) \quad |(\nabla^\lambda_x)^\alpha a_{\nu, \theta}(x, y)| \leq C \theta^{-|\alpha|}, \quad |(\nabla^\mu_x)^\alpha b_{\nu, \theta}(x, z)| \leq C \theta^{-|\alpha|},
\]
and
\[
(3.28) \quad |\partial^\ell_\nu a_{\nu, \theta}(x, x_\nu(t))| \leq C \ell, \quad x \in \gamma_\nu = \{x_\nu(t)\},
\]
\[
(3.29) \quad |\partial^\ell_\nu b_{\nu, \theta}(x, x_\nu(t))| \leq C \ell, \quad x \in \gamma_\nu = \{x_\nu(t)\},
\]
where \( \nabla^\lambda_x \) denotes the directional derivative along the direction perpendicular to the geodesics \( \{x_\nu(t)\} \) with \( \nu = \nu \) or \( v \) and
\[
\gamma_\nu = \{z_\nu(\tau): -2 \leq \tau \leq 2, (z_\nu(\tau), \eta_\nu(\tau)) = \Phi_\tau(z_\nu, \eta_\nu), \theta^{-1} \Theta(z_\nu, \eta_\nu) + \nu = 0\}.
\]

**Proof.** The properties for \( (\chi_\lambda Q_\theta')(x, y) \) are exactly the same as in [2], and the proof is identical to that of Lemma 3.2 in [2]. Since \( \theta \geq \mu^{-1/2+\epsilon} \), the properties for \( (\chi_\lambda P_\theta')(x, z) \) follows from the same proof.
On account of the above lemma, we have the following fact which will be used in the next section.

Lemma 3.9. Assume \( \theta \geq \theta_0 \) and \( N_1 \) is fixed. Then there exists \( C_0 \gg 1 \), when \( |\nu - \bar{\nu}| + |v - \bar{v}| \geq C_0 \) and \( |\nu - v|, |\bar{\nu} - \bar{v}| \leq N_1 \), we have

\[
\left| \int \chi\lambda Q_\theta^\nu h_1(x) \chi\mu P_\theta^\nu h_2(x) \chi\lambda Q_\theta^\nu h_3(x) \chi\mu P_\theta^\nu h_4(x) \, dx \right| \leq C_N \mu^{-N} \prod_{j=1}^{4} \|h_j\|_2.
\]

Proof. To get \( O_N(\mu^{-N}) \) decay as claimed, we need to split into two cases depending on the size of \( \mu \). Assume first \( \mu \geq \lambda^2 \).

It suffices to consider the kernel

\[
K(y, z, \bar{y}, \bar{z}) = \int \chi\lambda Q_\theta^\nu(x, y) \chi\mu P_\theta^\nu(x, z) \chi\lambda Q_\theta^\nu(x, \bar{y}) \chi\mu P_\theta^\nu(x, \bar{z}) \, dx.
\]

Indeed, by Lemma 3.8, up to a \( O_N(\mu^{-N}) \) error, we can restrict the domain of integration here to \( \Omega = TC_\theta(\gamma_\nu) \cap TC_\theta(\gamma_\bar{\nu}) \).

Plugging (3.26) into the expression of \( K(y, z, \bar{y}, \bar{z}) \), we get

\[
K(y, z, \bar{y}, \bar{z}) = \mu \int \Omega b(x, y, z, \bar{y}, \bar{z}) e^{i\mu(d_\theta(x, z) - d_\theta(x, \bar{z}))} \, dx + O_N(\mu^{-N}),
\]

where

\[
b(x, y, z, \bar{y}, \bar{z}) = \chi\lambda Q_\theta^\nu(x, y) \chi\lambda Q_\theta^\nu(x, \bar{y}) \, b_\theta,\theta(x, z) \, b_\theta,\theta(x, \bar{z}).
\]

It is easy to see that \( b(x, y, z, \bar{y}, \bar{z}) \) satisfies

\[
|\nabla_x b(x, y, z, \bar{y}, \bar{z})| \leq C\lambda^{|\alpha|+1}.
\]

Now we consider the phase function

\[
\mu(d_\theta(x, z) - d_\theta(x, \bar{z})).
\]

The gradient reads

\[
\mu \nabla_x (d_\theta(x, z) - d_\theta(x, \bar{z})).
\]

We claim that for \( C_0 \) big enough, there exists some \( c_0 > 0 \), such that

\[
|\nabla_x (d_\theta(x, z) - d_\theta(x, \bar{z}))| \geq c_0 \theta,
\]

then our lemma follows from simple integration by parts argument.

Indeed, since \( x \in TC_\theta(\gamma_\nu) \cap TC_\theta(\gamma_\bar{\nu}) \), \( z \in TC_\theta(\gamma_\nu) \) and \( \bar{z} \in TC_\theta(\gamma_\bar{\nu}) \), we see that

\[
|\nabla_x (d_\theta(x, z) - d_\theta(x, \bar{z}))| \geq |\nu - \bar{\nu}| \theta,
\]

noticing that

\[
|\nu - \bar{\nu}| \geq |\nu - v| - |\bar{\nu} - \bar{v}| - |\nu - \bar{\nu}| - |\nu - \bar{\nu}| - 2N_1,
\]

thus for \( C_0 \) big enough,

\[
|\nu - \bar{\nu}| \geq \frac{1}{2} (|\nu - v| + |\nu - \bar{\nu}|) - N_1 \geq \frac{1}{2} C_0 - N_1 \geq c_0,
\]

finishes the proof for the case \( \mu \geq \lambda^2 \).

Now we assume \( \mu \leq \lambda^2 \), then again by Lemma 3.8, up to a \( O_N(\mu^{-N}) = O_{2N}(\lambda^{-2N}) \) error, we can further restrict the domain of integration in this case to \( \Omega' = TC_\theta(\gamma_\nu) \cap TC_\theta(\gamma_\bar{\nu}) \cap TC_\theta(\gamma_\nu) \cap TC_\theta(\gamma_\bar{\nu}) \).
Similarly as above, by plugging \((3.25)\) and \((3.26)\) into the expression of \(K(y, z, \tilde{y}, \tilde{z})\), we see that the resulting phase function is given by

\[
\lambda(d_g(x, y) - d_g(x, \tilde{y})) + \mu(d_g(x, z) - d_g(x, \tilde{z})).
\]

The gradient reads

\[
\lambda \nabla_x (d_g(x, y) - d_g(x, \tilde{y})) + \mu \nabla_x (d_g(x, z) - d_g(x, \tilde{z})).
\]

Let us denote \(\nabla_x (d_g(x, y)) = Y\), here \(Y\) is a unit vector in \(T_x M\), similarly denote \(\nabla_x (d_g(x, \tilde{y})) = \tilde{Y}\), \(\nabla_x (d_g(x, z)) = Z\) and \(\nabla_x (d_g(x, \tilde{z})) = \tilde{Z}\). By the separation conditions we have, it is easy to see that \(\angle(Y, Z), \angle(\tilde{Y}, \tilde{Z}) \leq N_1 \theta\) and \(\angle(Y, \tilde{Y}) + \angle(Z, \tilde{Z}) \geq C_0 \theta\).

We Claim that

\[
\angle(Y + \frac{\mu}{\lambda} Z, \tilde{Y} + \frac{\mu}{\lambda} \tilde{Z}) \geq \angle(Y, \tilde{Y}) - 2N_1 \theta,
\]

which implies the desired result using integration by parts. Indeed, it suffices to show that \(\angle(Y + \frac{\mu}{\lambda} Z, \tilde{Y} + \frac{\mu}{\lambda} \tilde{Z})\) is bounded below by some uniform constant times \(\theta\). Note that \(\angle(Y + \frac{\nu}{\lambda} Z, Y), \angle(\tilde{Y}, \tilde{Y} + \frac{\nu}{\lambda} \tilde{Z}), \angle(Y + \frac{\nu}{\lambda} Z, Z), \angle(\tilde{Z}, \tilde{Y} + \frac{\nu}{\lambda} \tilde{Z}) \leq N_1 \theta\), we have

\[
\angle(Y + \frac{\mu}{\lambda} Z, \tilde{Y} + \frac{\mu}{\lambda} \tilde{Z}) \geq \angle(Y, \tilde{Y}) - 2N_1 \theta,
\]

similarly,

\[
\angle(Y + \frac{\mu}{\lambda} Z, \tilde{Y} + \frac{\mu}{\lambda} \tilde{Z}) \geq \angle(Z, \tilde{Z}) - 2N_1 \theta.
\]

Thus for \(C_0\) large enough,

\[
\angle(Y + \frac{\mu}{\lambda} Z, \tilde{Y} + \frac{\mu}{\lambda} \tilde{Z}) \geq \frac{1}{2} (\angle(Y, \tilde{Y}) + \angle(Z, \tilde{Z})) - 2N_1 \theta \geq \frac{1}{2} C_0 \theta - 2N_1 \theta \geq c_0 \theta,
\]

finishes the proof.

\[\square\]

4. Proof of the main theorem I: Orthogonality

In this section, we use orthogonality argument to reduce the proof of Theorem 1.1 to a specific bilinear estimate. We use Lemma 3.6 and Minkowksi’s inequality to estimate \(\|\chi_\lambda f \chi_\mu g\|_2\) by

\[
(4.1) \quad \left\| \sum_{|\nu - \ell| \leq M} \chi_\lambda Q^\nu_{b_0} f \chi_\mu P^\ell_{b_0} g \right\|_2
\]

\[
(4.2) \quad + \sum_{\ell = \log \lambda / \log 2} \left\| \sum_{2^\ell \leq |\nu - \ell| < 2^{\ell+1}} \chi_\lambda Q^\nu_{b_0} f \chi_\mu P^\ell_{b_0} g \right\|_2,
\]

for certain dyadic \(M\) large enough. The square of \((4.1)\) is estimated by

\[
(4.3) \quad \left[ \sum_{|\nu - \nu'| + |\nu - \ell| \leq C_0} + \sum_{|\nu - \nu'| + |\nu - \ell| \geq C_0} \right] \int \chi_\lambda Q^\nu_{b_0} f(x) \chi_\mu P^\ell_{b_0} g(x) \overline{\chi_\lambda Q^{\nu'}_{b_0} f(x) \chi_\mu P^{\ell'}_{b_0} g(x)} dx,
\]

where \(|\nu - \ell|, |\nu - \nu'| \leq M\).

By Lemma 3.9, the second term of \((4.3)\) is negligible by choosing \(C_0\) sufficiently large.
We can estimate the contribution of the first term as
\[ \sum_{\nu \in \mathbb{Z}^2} \left\| \lambda^{\frac{d}{2}} Q^\nu_{\theta_0} f \chi_{\mu} P^\nu_{\theta_0} g \right\|_2^2. \]

If we use the bilinear estimate (1.4), we can estimate this sum by
\[ \lambda^{\frac{d}{2}} \sum_{\nu \in \mathbb{Z}^2} \left\| P^\nu_{\theta_0} g \right\|_2 \sum_{\nu \in \mathbb{Z}^2} \left\| Q^\nu_{\theta_0} f \right\|_2. \]

By the $L^2$-orthogonality, we see the contribution of (4.1) is
\[ \lambda^{\frac{d}{2}} \| g \|_2 \times \left( \lambda^{\frac{d}{2} - \varepsilon_0} \sup_{\nu} \| Q^\nu_{\theta_0} f \|_2 \right)^{\frac{1}{2}}, \]
which corresponds to (1.7). Similarly, since the sum is symmetric, we can also bound (4.1) by
\[ \lambda^{\frac{d}{2}} \| f \|_2 \times \left( \lambda^{\frac{d}{2} - \varepsilon_0} \sup_{\nu} \| P^\nu_{\theta_0} g \|_2 \right)^{\frac{1}{2}}, \]
which corresponds to (1.8).

**The second microlocalization.** For the off diagonal part (4.2), we will reduce the matters to a bilinear oscillatory integrals as in [2]. Fixing $\ell \geq \log M/\log 2$, we see that if $2^\ell \leq |\nu - \nu| < 2^{\ell+1}$, then the distance between $\gamma_\nu$ and $\gamma_{\nu}$ in the sense of Remark 3.5 is approximately $2^\ell \theta_0$. To explore this and use orthogonality argument, one naturally employs wider tubes to collect thinner tubes by making use of the second microlocalization. Precisely, up to some negligible terms, we may write for $\theta_\ell = 2^\ell \theta_0$ with $c_0$ to be specified later
\[ \chi_{\lambda} Q^\nu_{\theta_0} f(x) \approx \sum_{\sigma_1 \in \mathbb{Z}^2} \left( \chi_{\lambda} Q^{\sigma_1}_{c_0 \theta_\ell} \right) \circ Q^\nu_{\theta_0} f(x), \quad \chi_{\mu} P^\nu_{\theta_0} g(x) \approx \sum_{\sigma_2 \in \mathbb{Z}^2} \left( \chi_{\mu} P^{\sigma_2}_{c_0 \theta_\ell} \right) \circ P^\nu_{\theta_0} g(x). \]

Noting that the kernels of the operators $(\chi_{\lambda} Q^{\sigma_1}_{c_0 \theta_\ell}) \circ Q^\nu_{\theta_0}$ and $(\chi_{\mu} P^{\sigma_2}_{c_0 \theta_\ell}) \circ P^\nu_{\theta_0}$ decrease rapidly unless $T_{c_1 \theta_\ell}(\sigma_1) \cap T_{c_2 \theta_\ell}(\sigma_2) \neq \emptyset$ and $T_{c_1 \theta_\ell}(\gamma_\nu) \cap T_{c_2 \theta_\ell}(\gamma_{\nu}) \neq \emptyset$, we have by choosing $M$ large enough, there are $N_0 = N_0(c_0, M)$ and $N_1$ such that up to some negligible terms
\[ (4.4) \sum_{2^\ell \leq |\nu - \nu| < 2^{\ell+1}} \chi_{\lambda} Q^\nu_{\theta_0} f(x) \chi_{\mu} P^\nu_{\theta_0} g(x) \]
\[ = \sum_{\sigma_1, \sigma_2 \in \mathbb{Z}^2, N_0 \leq |\sigma_1 - \sigma_2| \leq N_1} \sum_{2^\ell \leq |\nu - \nu| < 2^{\ell+1}} \left( \chi_{\lambda} Q^{\sigma_1}_{c_0 \theta_\ell} \right) \circ Q^\nu_{\theta_0} f(x) \left( \chi_{\mu} P^{\sigma_2}_{c_0 \theta_\ell} \right) \circ P^\nu_{\theta_0} g(x). \]
Moreover, we may find a $C_3 > 0$ having the property that for every $\sigma_1$ and $\sigma_2$, there are $\nu(\sigma_1)$ and $\nu(\sigma_2)$ such that $|\nu - \nu(\sigma_1)|, |\nu - \nu(\sigma_2)| \geq C_3 2^\ell$ implies
\[ \left\| \left( \chi_{\lambda} Q^{\sigma_1}_{c_0 \theta_\ell} \right) \circ Q^\nu_{\theta_0} f \right\|_{L^\infty} \lesssim_{N} \lambda^{-N}, \quad \left\| \left( \chi_{\mu} P^{\sigma_2}_{c_0 \theta_\ell} \right) \circ P^\nu_{\theta_0} g \right\|_{L^\infty} \lesssim_{N} \mu^{-N}. \]
for all $N = 1, 2, \ldots$. Therefore, we may estimate (4.4) as follows
\[ \left\| \sum_{2^\ell \leq |\nu - \nu| < 2^{\ell+1}} \chi_{\lambda} Q^\nu_{\theta_0} f \chi_{\mu} P^\nu_{\theta_0} g \right\|_2^2 \]
\[ \lesssim \sum_{N_0 \leq |\sigma_1 - \sigma_2| \leq N_1} \int_{|\sigma_1 - \theta_\ell| + |\sigma_2 - \theta_\ell| \leq C} \left| T^{\sigma_1, \sigma_2}_{\lambda, \mu, \theta_\ell} F(x) T^{\sigma_1, \sigma_2}_{\lambda, \mu, \theta_\ell} \overline{F(x)} \right| dx \]
where $N_0$ can be sufficiently large by choosing $c_0$ small and

$$T_{\lambda, \mu, \theta_i}^{\sigma_1, \sigma_2} F(x) = \int \left( \chi_\lambda \circ Q_{c_0 \theta_i}^{\sigma_1} \right)(x, y) \left( \chi_\mu \circ P_{c_0 \theta_i}^{\sigma_2} \right)(x, z) F(y, z) \, dy \, dz,$$

with $F(y, z) = 0$ if $(y, z) \notin B(y_0, C_0 \delta) \times B(z_0, C_0 \delta)$. It follows again from Lemma 3.9 that if we choose $C$ large enough, the second term in the expression preceding (4.5) is negligible.

To evaluate the first term there, we are reduced to estimating

$$\sum_{N_0 \leq |\sigma_1 - \sigma_2| \leq N_1} \left\| T_{\lambda, \mu, \theta_i}^{\sigma_1, \sigma_2} F \right\|_{L^2(B(0, \delta))}^2,$$

We shall need the following proposition whose proof is postponed to the next section.

**Proposition 4.1.** Let

$$T_{\lambda, \mu, \theta_i}^{\sigma_1, \sigma_2} F(x) = \int \left( \chi_\lambda \circ Q_{c_0 \theta_i}^{\sigma_1} \right)(x, y) \left( \chi_\mu \circ P_{c_0 \theta_i}^{\sigma_2} \right)(x, z) F(y, z) \, dy \, dz.$$

Assume as before that $\delta > 0$ is sufficiently small and $\theta_i$ is larger than a fixed positive constant times $\theta_0$. Then if $N_0$ is sufficiently large and $N_1 > N_0$ is fixed, there exists a positive constant $C = C_{c_0}$ such that

$$\left\| T_{\lambda, \mu, \theta_i}^{\sigma_1, \sigma_2} F \right\|_{L^2(B(0, \delta))} \leq C \theta^{-1/2} \| F \|_2, \quad \text{if} \ N_0 \leq |\sigma_1 - \sigma_2| \leq N_1.$$

Assuming (4.9), we can now complete the proof of Theorem 1.1. In fact, we have

$$\left\| \sum_{2^k \leq |v - w| < 2^{k+1}} \chi_\lambda Q_{c_0 \theta_0}^w f \chi_\mu P_{c_0 \theta_0}^v g \right\|_2^2 \leq C (2^k \theta_0)^{-1} \sum_{N_0 \leq |\sigma_1 - \sigma_2| \leq N_1} \int \int \left| \sum_{2^k \leq |v - w| < 2^{k+1}} Q_{c_0 \theta_0}^w f(y) P_{c_0 \theta_0}^v g(z) \right|^2 \, dy \, dz.$$
to get

\[ Q \lesssim \lambda \]

Thanks to the fact that we are allowed to have an extra small power of \( \lambda \), we may sum over \( 1 \leq \ell \leq \log \lambda \) to finish the proof of (1.7). To get (1.8), one notes that the above sum is again symmetric, thus we may interchange the role of \( Q_0^\nu f \) and \( P_0^\nu g \) to get

\[ \left\| \sum_{2^\ell \leq |\nu-v| < 2^{\ell+1}} \chi_\lambda Q_0^\nu f(x) \chi_\mu P_0^\nu g(x) \right\|_2^2 \leq C (2^\ell \theta_0)^{-1} \left\| f \right\|_2^2 \sup_{\nu \in \mathbb{Z}^2} \left\| P_2^\nu \theta_0 g \right\|_2^2, \]

summing over \( \ell \) finishes the proof of (1.8).

5. PROOF OF THE MAIN THEOREM II: BILINEAR OSCILLATORY INTEGRAL ESTIMATES

In this section, we take \( \theta = c_0 \theta_0 \) and prove Proposition 4.1. We work in the geodesic normal coordinates about a fixed point \( \tilde{x} \in T_{C_0}(\gamma_{\sigma_1}) \cap T_{C_0}(\gamma_{\sigma_2}) \). Without loss of generality, we may assume \( \tilde{x} \in \gamma_{\sigma_1} \) and the geodesic \( \gamma_{\sigma_1} \) is parameterized by \( \{0, s) : |s| \leq 2\} \). In the following, we denote by \( \phi(x,y) = d_g((x_1, x_2), (y_1, y_2)) \) the geodesic distance between \( x \) and \( y \).

In order to estimate the \( L^2(B(0, \delta)) \) norm of

\[ T_{\lambda, \mu, \theta}^{\sigma_1, \sigma_2} F(x) = \int (\chi_\lambda \circ Q_0^\nu)(x,y) \left( \chi_\mu \circ P_0^\nu \right)(x,z) F(y,z) \, dy \, dz, \]

we shall need the following lemma to further restrict the domain of \( x, y, z \).

**Lemma 5.1.** There exists a constant \( C \), such that if we set \( \Omega_1 = T_{C_0}(\gamma_{\sigma_1}) \) and \( \Omega_2 = T_{C_0}(\gamma_{\sigma_2}) \) we have

\[ \left\| \int_{y \in \Omega_1} (\chi_\lambda \circ Q_0^\nu)(x,y) \left( \chi_\mu \circ P_0^\nu \right)(x,z) F(y,z) \, dy \, dz \right\|_{L^2(B(0, \delta))} \leq C N \lambda^{-N} \left\| f \right\|_2 \left\| g \right\|_2, \]

and

\[ \left\| T_{\lambda, \mu, \theta}^{\sigma_1, \sigma_2} F \right\|_{L^2(B(0, \delta) \setminus \Omega_1)} \leq C N \lambda^{-N} \left\| f \right\|_2 \left\| g \right\|_2. \]

Similarly, we have

\[ \left\| \int_{y \notin \Omega_2} (\chi_\lambda \circ Q_0^\nu)(x,y) \left( \chi_\mu \circ P_0^\nu \right)(x,z) F(y,z) \, dy \, dz \right\|_{L^2(B(0, \delta))} \leq C N \mu^{-N} \left\| f \right\|_2 \left\| g \right\|_2, \]

and

\[ \left\| T_{\lambda, \mu, \theta}^{\sigma_1, \sigma_2} F \right\|_{L^2(B(0, \delta) \setminus \Omega_2)} \leq C N \mu^{-N} \left\| f \right\|_2 \left\| g \right\|_2. \]

**Proof.** Since we know there are at most \( O(\lambda^2) \) many terms in the sum

\[ F(y,z) = \sum_{\nu \leq |\nu-v| \leq 2^{\ell+1}} \sum_{\nu \leq |\nu-v| \leq 2^{\ell+1}} Q_0^\nu f(y) P_0^\nu g(z), \]
it suffices to show the $L^2(B(0,\delta))$ norm of
\[
\iint (\chi_\lambda \circ Q_\sigma^1)(x, y) (\chi_\mu \circ P_\theta^2)(x, z) f(y) g(z) \, dydz
\]
satisfies our claim.

Indeed, by Lemma 3.8, we can find $C$ such that if $x \notin T_{C\theta}(\gamma_{\sigma_1})$ or $y \notin T_{C\theta}(\gamma_{\sigma_1}),$
\[
|\langle \chi_\lambda Q_\theta^\mu \rangle(x, y)| \leq C_N \lambda^{-N}.
\]
Thus
\[
\left\| \int (\chi_\lambda \circ Q_\sigma^1)(x, y) f(y) \, dy \right\|_{L^\infty(dx)} \leq C_N \lambda^{-N} \|f\|_{L^2},
\]
while we know $\chi_\mu$ has $L^2 \to L^2$ norm 1, so
\[
\left\| \int (\chi_\mu \circ P_\theta^2)(x, y) g(z) \, dz \right\|_{L^2(dx)} \leq \|g\|_{L^2}.
\]

Therefore
\[
\left\| \iint (\chi_\lambda \circ Q_\sigma^1)(x, y) (\chi_\mu \circ P_\theta^2)(x, z) f(y) g(z) \, dydz \right\|_{L^2} \leq C_N \lambda^{-N} \|f\|_{L^2} \|g\|_{L^2}
\]
as claimed.

The second part of our lemma follows from the exact same proof. \hfill \Box

\[\text{Figure 1}\]
Remark 5.2. By the above lemma, we see that we can assume in (5.1), $y \in T_{C\theta}(\gamma_{\sigma_1})$, $z \in T_{C\theta}(\gamma_{\sigma_2})$, and $x \in T_{C\theta}(\gamma_{\sigma_1}) \cap T_{C\theta}(\gamma_{\sigma_2})$. Moreover, if $N_0 \leq |\sigma_1 - \sigma_2| \leq N_1$, then we may assume the angle $\text{Ang}(x,y,z)$ between the geodesic connecting $x$ and $y$ and the one connecting $x$ and $z$ belongs to $[\theta, \hat{C}\theta]$. This geometric assumption yields $x,y,z \in T_{C\theta}(\gamma_{\sigma})$ for some large constant $C_4$. Moreover, we also have $\angle(\gamma_{\sigma_1}, \gamma_{\sigma_2}) \geq N_0\theta$. Noticing that $d_y(x,y)$ and $d_y(x,z)$ are comparable to 1, we claim that for $N_0$ sufficiently large, we can find $c > 0$ such that

$$|y_1 - z_1| > c\theta.$$  

Indeed, it is easy to see that $|y_1| \leq C\theta$ and $d_y(z, \gamma_{\sigma_2}) \leq C\theta$. Since the constant $C$ here is a uniform constant as in Lemma 3.8, we can choose $N_0 > C$. Then we have $|z_1| \geq N_0\theta - C\theta$, see Figure 1. Therefore $|y_1 - z_1| \geq N_0\theta - 2C\theta \geq c\theta$ as claimed.

Returning to $T_{\lambda, \mu, \theta}^{\alpha_1, \alpha_2} F(x)$, we have from Cauchy-Schwarz

$$\|T_{\lambda, \mu, \theta}^{\alpha_1, \alpha_2} F\|_2^2$$

$$\lesssim \lambda \mu \int \int e^{i\mu \Phi(x; (y_1, y_2), (z_1, z_2))} \sigma_{\lambda, \mu, \theta}^{\alpha_1, \alpha_2}(x,y,z) F(y_1, z_1) dy_1 dz_1 \|_2^2$$

where $\epsilon = \lambda / \mu$ and

$$\Phi(x; y, z) = \epsilon \phi(x, y) + \phi(x, z),$$

$$a_{\lambda, \mu, \theta}^{\alpha_1, \alpha_2}(x,y,z) = a_{\alpha_1, \phi}(x,y) b_{\alpha_2, \phi}(x,z).$$

Fix $y_2$ and $z_2$, it suffices to prove

$$\int e^{i\mu \Phi(x; (y_1, y_2), (z_1, z_2))} \sigma_{\lambda, \mu, \theta}^{\alpha_1, \alpha_2}(x,y,z) G(y_1, z_1) dy_1 dz_1 \leq C (\lambda \mu \theta)^{-1} \|G\|_{L^2}^2,$$

uniformly with respect to $y_2, z_2$ where we set $G(y_1, z_1) = F(y, z)$ for brevity.

Squaring the left side of (5.3) shows that we need to estimate

$$\int e^{i\mu \Psi(x, y_1, y_1', z_1')} A^{\alpha_1, \alpha_2}_{\lambda, \mu, \theta}(x; y_1, y_1', z_1') G(y_1, z_1) G(y_1', z_1') \|_2^2$$

where

$$A^{\alpha_1, \alpha_2}_{\lambda, \mu, \theta}(x; y_1, y_1', z_1') = a^{\alpha_1, \alpha_2}_{\lambda, \mu, \theta}(x, (y_1, y_2), (z_1, z_2)),$$

$$\Psi = \Psi_{x, y_2, z_2}(x; y_1, y_1', z_1) = \Phi(x; y_1, y_2, z_1) - \Phi(x; y_1', y_2, z_1).$$

Set

$$K^{\alpha_1, \alpha_2}_{\lambda, \mu, \theta}(y_1, y_1'; z_1, z_1') = \int e^{i\mu \Psi(x, y_1, y_1', z_1')} A^{\alpha_1, \alpha_2}_{\lambda, \mu, \theta}(x; y_1, y_1', z_1') dx.$$

Then by Schur test, we are reduced to proving

$$\sup_{y_1, z_1} \int |K^{\alpha_1, \alpha_2}_{\lambda, \mu, \theta}(y_1, y_1'; z_1, z_1')| dy_1 dz_1 \leq C / \lambda \mu \theta.$$

By symmetry, we shall only deal with the first one.

By Remark 5.2, we have
This would allow us to study the oscillatory integral (5.5) using the strategy of [11] and a change of variables argument similar to the one in p. 217-218 of [6]. In fact, if we let \( \psi(x, y_1) = \phi(x, (y_1, y_2)) \), then \( \psi \) is a Carleson-Sjölin phase for fixed \( y_2 \), i.e.

\[
\det \begin{pmatrix} \psi''_{x_1 y_1} & \psi''_{x_2 y_1} \\ \psi''_{x_1 y_1 y_1} & \psi''_{x_2 y_1 y_1} \end{pmatrix} \neq 0,
\]

see [10, 11]. Changing variables \((y_1, z_1) \mapsto (\tau, \tau'), (y_1', z_1') \mapsto (\tilde{\tau}, \tilde{\tau}')\) by

\[
\begin{align*}
\tau &= \frac{1}{2\mu} (y_1 - z_1)^2, \\
\tau' &= z_1 + \frac{1}{\mu} y_1, \\
\tilde{\tau} &= \frac{1}{\mu} (y_1' - z_1')^2, \\
\tilde{\tau}' &= z_1' + \frac{1}{\mu} y_1',
\end{align*}
\]

where we may assume \( y_1 > z_1 \) by symmetry. It is clear that the above bijective mapping sends variables from \( \{(y_1 - z_1 \geq c\theta) \} \) to \( \{ (\tau, \tau') : \tau \geq c\lambda\theta^2/2\mu \} \), whose Jacobian reads

\[
\frac{D(\tau, \tau')}{D(y_1, z_1)} = (1 + \epsilon)^{(2\epsilon\tau)^{1/2}}.
\]

The phase function in (5.5) goes to

\[
\tilde{\Psi}(x; \tau, \tilde{\tau}, \tau', \tilde{\tau}') = \Psi(x; y_1, y_1', z_1, z_1'),
\]

under the change of variables. The Carleson-Sjölin condition allows us to obtain as in [6]

\[
|\nabla_x \tilde{\Psi}(x; \tau, \tilde{\tau}, \tau', \tilde{\tau}')| \approx |\tau - \tilde{\tau}| + |\tau' - \tilde{\tau}'|,
\]

\[
|\partial^{\alpha}_x \tilde{\Psi}(x; \tau, \tilde{\tau}, \tau', \tilde{\tau}')| \leq C_\alpha(|\tau - \tilde{\tau}| + |\tau' - \tilde{\tau}'|), |\alpha| \leq 5.
\]

In view of integration by parts and relation (5.6), we have for fixed \((y_1', z_1')\) hence fixed \((\tilde{\tau}, \tilde{\tau}')\), thus

\[
\int_{y_1 - z_1 \geq c\theta} |K_{\lambda, \mu, \delta}(y_1, y_1'; z_1, z_1')| \, dy_1 \, dz_1 
\leq C \int_{\tau \geq c\lambda\theta^2/2\mu} (1 + \mu|\tau - \tilde{\tau}| + \mu|\tau' - \tilde{\tau}'|)^{-5} \left( \frac{\lambda}{\mu} \right)^{-1/2} \, d\tau \, d\tau'
\leq C / \lambda \mu \theta.
\]

finishes the proof.

**Remark 5.3.** As mentioned before, it would be interesting to see that if one could get rid of the \( \epsilon \)-loss that appears in Theorem 1.1. In the earlier work [11] of the second author on the linear case, there is no \( \epsilon \)-loss in his result, while the power of \( \| e_\lambda \|_{KN(\lambda)} \) is less favorable. Thus it is natural to consider that if one could apply the strategies presented in [11] to get a \( \epsilon \)-loss free version of Theorem 1.1. However, it seems more difficult to use the microlocal decomposition if we want to get rid of the \( \epsilon \). Without the help of microlocal techniques, the separation of \((y_0, \delta)\) and \((z_0, \delta)\) in the radial direction becomes problematic, and it seems difficult to get around by simply applying ideas in [11].
REFERENCES

[1] M. Blair and C. D. Sogge, On Kakeya-Nikodym averages, $L^p$-norms and lower bounds for nodal sets of eigenfunctions in higher dimensions, J. Eur. Math. Soc. **17** (2015), 2513-2543.

[2] M. Blair and C. D. Sogge, Refined and microlocal Kakeya-Nikodym bounds for eigenfunctions in two dimensions, Anal. PDE **8** (2015), 747-764.

[3] M. Blair and C. D. Sogge, Concerning Toponogov’s Theorem and logarithmic improvement of estimates of eigenfunctions, arXiv:1510.07726.

[4] M. Blair and C. D. Sogge, Refined and Microlocal Kakeya-Nikodym Bounds of Eigenfunctions in Higher Dimensions, arXiv:1510.07724.

[5] J. Bourgain, Geodesic restrictions and $L^p$ estimates for eigenfunctions of Riemannian surfaces, Linear and Complex analysis: dedicated to V. P. Havin on the occasion of his 75th birthday, Amer. Math. Soc. Transl, Advances in the Mathematical sciences (2009), 27-35.

[6] N. Burq, P. Gérard and N. Tzvetkov, Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces, Invent. Math. 2005,159:187-223.

[7] L. Hörmander, Oscillatory integrals and multipliers on $L^p$, Ark Math. II (1973), 1-11.

[8] G. Mockenhaupt, A. Seeger and C.D.Sogge, Local smoothing of Fourier integral operators and Carleson-Sjölin estimates, J. Amer. Math. Soc. 6 (1993), 65-130.

[9] C. D. Sogge, Concerning the $L^p$ norm of spectral clusters for second-order elliptic operators on compact manifolds, J. Funct. Anal. **77** (1988), 1231-38.

[10] C. D. Sogge, Fourier integrals in classical analysis, Cambridge Univ. Press, 1993.

[11] C. D. Sogge, Kakeya-Nikodym average and $L^p$-norms of eigenfunctions, Tohoku Math. J. (2) 63 (2011), no.4, 519-538.

[12] C. D. Sogge, Hangzhou lectures on eigenfunctions of the Laplacian, Princeton Univ. Press, 2014.

[13] C. D. Sogge and S. Zelditch, On eigenfunction restriction estimates and $L^4$-bounds for compact surfaces with nonpositive curvature, Advances in analysis: the legacy of Elias M. Stein, 447–461, Princeton Math. Ser., 50, Princeton Univ. Press, Princeton, NJ, 2014.

INSTITUTE OF APPLIED PHYSICS AND COMPUTATIONAL MATHEMATICS, BEIJING, 100088, CHINA
E-mail address: miao_changxing@iapcm.ac.cn

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA
E-mail address: sogge@jhu.edu

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA
E-mail address: ykxi@math.jhu.edu

LAGA (UMR 7539), INSTITUT GALILÉE, UNIVERSITÉ PARIS 13, SORBONNE PARIS CITÉ, ÎLE-DE-FRANCE.

BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, BEIJING 100871, CHINA
E-mail address: geeweyoung@pku.edu.cn