Bott vanishing using GIT and quantization

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Outline

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5. The toric case
A smooth projective variety $Y$ is said to satisfy Bott vanishing if

$$H^i(Y, \Omega^j_Y \otimes L) = 0$$

for every $i > 0$, $j \geq 0$ and $L$ ample.

Stronger than Kodaira-Akizuki-Nakano vanishing.

Very restrictive property.

Example

Suppose $Y$ is Fano and satisfies Bott vanishing. Then

$$H^1(Y, T_Y) = H^1(Y, \Omega^{n-1}_Y \otimes K_Y) = 0.$$ 

In particular, $Y$ must be rigid.
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What is known

- $\mathbb{P}^n$ satisfies Bott vanishing (Bott, 1957).
- Toric varieties satisfy Bott vanishing (Danilov 1978, Batyrev-Cox 1993, Buch-Thomsen-Lauritzen-Mehta 1997, ...).
- Quintic del Pezzo surface (Totaro 2019).
- Bott vanishing holds for $K3$ surfaces of degree $\geq 24$, fails for $K3$ surfaces of degree $< 20$ (Totaro 2019).
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- Bott vanishing holds for K3 surfaces of degree $= 20$ or $\geq 24$, fails for K3 surfaces of degree $< 20$ (Totaro 2019).
The quintic del Pezzo is isomorphic to $\tilde{M}_{0,5}$ can be obtained as a GIT quotient $(\mathbb{P}^1)^5 \sslash O(2,2,2,2) PGL_2$. It parametrizes 5-tuples of points on $\mathbb{P}^1$ where no three of them coincide.
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From now on, we will work over $\mathbb{C}$. 
Let $Y$ be a GIT quotient $(\mathbb{P}^1)^n \sslash L PGL_2$ given by a linearization with no strictly semi-stable locus. Then $Y$ satisfies Bott vanishing.
In order to prove that theorem, we use:

- **Quantization.** This allows us to compute cohomology on $Y$ as cohomology on $X$ of a suitable object $F$.
- **Geometric syzygies.** The cohomologies of $F$ correspond to the Koszul resolution of certain locus in $X \times \mathbb{P}(g)$.
- **Gelfand-MacPherson correspondence.** This allows us to see global invariant sections as polynomials in the Plücker minors, and we characterize these as directed graphs.
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Definition (GIT quotient)

Let $X = \text{Proj } R$ be a variety with an action by a group $G$. Extend the action of $G$ to $R$. Then the GIT quotient $X // G$ is defined as $\text{Proj } R^G$. 

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There are two choices involved:

- The coordinate ring $R$. This amounts to specifying an ample line bundle $\mathcal{L}$, so that $R = \bigoplus_{k \geq 0} H^0(X, \mathcal{L} \otimes k)$.

- The action of $G$ on $R$. This amounts to extending the action of $G$ on $X$ to the total space of $\mathcal{L}$.
Example

Let $G$ act on a ring $R$. If we extend the action trivially to the trivial line bundle, then we get $\text{Spec } R \ltimes G = \text{Spec } R^G$. 

Example

Let $C^*$ act on $A^{n+1}$ by multiplication. Extend this action to $O$ and twist it by the character $t \mapsto t^r$, that is:

$$t \cdot p(x_0, \ldots, x_n) = t^r p(t^{-1} x_0, \ldots, t^{-1} x_n).$$

Then one obtains the GIT quotient $A^{n+1} \ltimes C^* = \mathbb{P}^n$. 

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Let $\mathbb{C}^*$ act on $\mathbb{A}^{n+1}$ by multiplication. Extend this action to $\mathcal{O}$ and twist it by the character $t \mapsto t$, that is: $t \cdot p(x_0, \ldots, x_n) = tp(t^{-1}x_0, \ldots, t^{-1}x_n)$. Then one obtains the GIT quotient $\mathbb{A}^{n+1} \sslash \mathbb{C}^* = \mathbb{P}^n$. 
Definition

Given an action of $G$ on $X$ and a $G$-linearized ample line bundle $\mathcal{L}$, the semi-stable locus is defined as

$$X^{ss} = \{ x \in X \mid \exists \sigma \in H^0(X, \mathcal{L}^k)^G, \sigma(x) \neq 0 \}$$

and then we have a quotient map

$$\pi : X^{ss} \to X \sslash \mathcal{L} G.$$
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We are interested in the cases when $X^{ss} = X^s$. 

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Example

Let $\mathcal{L} = \mathcal{O}(d_1, \ldots, d_n)$ be a $PGL_2$-linearized ample line bundle in $X = (\mathbb{P}^1)^n$. The semi-stable (resp. stable) locus consists of tuples $(z_1, \ldots, z_n) \in (\mathbb{P}^1)^n$ such that whenever $\sum_{i \in I} d_i > \sum_{i \in I^c} d_i$ (resp. $\geq$) for some $I \subset \{1, \ldots, n\}$, the coordinates $\{z_i, i \in I\}$ do not all coincide.

The GIT quotient $Y = (\mathbb{P}^1)^n \sslash_{\mathcal{L}} PGL_2$ parametrizes such configurations of $n$ points up to projective equivalence.
We are interested in computing cohomologies $H^i(Y, F)$, for certain vector bundles $F$. Under certain circumstances, this can be computed as $H^i(X, F)^G$ for some suitable object $F$. 

**Definition**

We say that a $G$-linearized chain complex $F$ of vector bundles on $X$ descends to $F$ if $F|_{X^{ss}} \sim \pi^* F$.

**Quantization Theorem (Teleman, Halpern-Leistner)**

Suppose $F$ descends to $F$. Take a Kempf-Ness stratification of the unstable locus $X \setminus X^{ss} = \bigsqcup S_\alpha$. If all the weights of $F$ on $S_\alpha$ are $< \eta_\alpha$, then $H^i(Y, F) = H^i(X, F)^G$. 

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$$H^i(Y, F) = H^i(X, \mathcal{F})^G.$$
In our case, we have $X = (\mathbb{P}^1)^n$, $G = PGL_2$ and let $g = sl_2$ be the Lie algebra. The action of $G$ induces a map of sheaves $\Omega_X \to g^\vee$. 

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**Definition**

We denote by $L_X$ the $PGL_2$-linearized two-step chain complex $[\Omega_X \rightarrow \mathfrak{g}^\vee]$. 
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**Remark**

$L_X$ descends to $\Omega_Y$. This is because of the following short exact sequence

$$0 \to \pi^*\Omega_Y \to \Omega_X^{ss} \to g^\vee \to 0.$$
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Similarly, $\land^j L_X$ descends to $\Omega_Y^j$, where $\land^j L_X$ is the chain complex

$$0 \to \Omega_X^j \to \Omega_X^{j-1} \otimes \mathfrak{g}^\vee \to \cdots \to S^j \mathfrak{g}^\vee \to 0.$$
Lemma

Let $X = (\mathbb{P}^1)^n$, $G = \text{PGL}_2$ and $\mathcal{L} = \mathcal{O}(d_1, \ldots, d_n)$ a $\text{PGL}_2$-linearized ample line bundle such that $X^{ss} = X^s$. Then $\mathcal{N}^i L_\chi \otimes \mathcal{L}$ satisfies the hypotheses of the Quantization theorem, and so

$$H^i(Y, \Omega^j_Y \otimes \mathcal{L}) = H^i(X, \mathcal{N}^i L_\chi \otimes \mathcal{L})^{\text{PGL}_2}$$

where $Y = (\mathbb{P}^1)^n \sslash_{\mathcal{L}} \text{PGL}_2$ and $\mathcal{L}$ is the descent of $\mathcal{L}$. 
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Lemma

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where $Y = (\mathbb{P}^1)^n \sslash \mathcal{L} \text{PGL}_2$ and $\mathcal{L}$ is the descent of $\mathcal{L}$.

In fact, to show Bott vanishing on $Y$, it suffices to check that

$$H^i(X, \wedge^j L_{\mathcal{X}} \otimes \mathcal{L})^{\text{PGL}_2} = 0$$

for $i > 0$. 
How to prove Bott vanishing for $Y = (\mathbb{P}^1)^n \sslash L PGL_2$

Recall $\Lambda^j L_X \otimes L$ is the complex

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**Lemma**

$H^i(X, \wedge^j L_x \otimes \mathcal{L})$ can be computed as the cohomology of the complex of global sections

$$0 \to H^0(\Omega^j_X \otimes \mathcal{L}) \to H^0(\Omega^{j-1}_X \otimes \mathcal{L}) \otimes g^\vee \to \cdots \to H^0(\mathcal{L}) \otimes S^j g^\vee \to 0.$$
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**Lemma**

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For example, for $j = 1$, showing that $H^1(Y, \Omega_Y \otimes L) = 0$ is equivalent to showing that the map of invariant global sections

$$H^0(X, \Omega_X \otimes \mathcal{L})^{PGL_2} \to (H^0(X, \mathcal{L}) \otimes \mathfrak{g}^\vee)^{PGL_2}$$

is surjective.
How to prove Bott vanishing for $Y = (\mathbb{P}^1)^n \sslash L PGL_2$

How to think of $PGL_2$-invariant global sections of a line bundle $\mathcal{O}(d_1, \ldots, d_n)$ on $(\mathbb{P}^1)^n$?
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**Gelfand-MacPherson correspondence**

Consider the action of the torus $\mathbb{C}^*$ on the Grassmannian $\text{Gr}(2, n)$. Let $\mathcal{O}(1)$ be the ample line bundle on $\text{Gr}(2, 5)$ given by the Plücker embedding. We endow it with a $\mathbb{C}^*$-linearization by choosing the character $(d_1, \ldots, d_n)$. Then

$$
\bigoplus_{k \geq 0} H^0((\mathbb{P}^1)^n, \mathcal{O}(kd_1, \ldots, kd_n))^{PGL_2} = \bigoplus_{k \geq 0} H^0(\text{Gr}(2, n), \mathcal{O}(k))^{(\mathbb{C}^*)^n}
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$$\bigoplus_{k \geq 0} H^0((\mathbb{P}^1)^n, \mathcal{O}(kd_1, \ldots, kd_n))^{PGL_2} = \bigoplus_{k \geq 0} H^0(Gr(2, n), \mathcal{O}(k))^{(\mathbb{C}^*)^n}$$

In particular, $PGL_2$-invariant global sections of $\mathcal{O}(d_1, \ldots, d_n)$ can be found in the coordinate ring of the Grassmannian.
How to prove Bott vanishing for $Y = (\mathbb{P}^1)^n \sslash L PGL_2$

In fact, $H^0((\mathbb{P}^1)^n, \mathcal{O}(d_1, \ldots, d_n))^{PGL_2}$ consists of polynomials in $x_i y_j - x_j y_i$ having homogeneous degree $d_1, \ldots, d_n$ in the variables $x_1, y_1; \ldots; x_n, y_n$, subject to the Plücker equivalence relations.
In fact, $H^0((\mathbb{P}^1)^n, \mathcal{O}(d_1, \ldots, d_n))^{PGL_2}$ consists of polynomials in $x_i y_j - x_j y_i$ having homogeneous degree $d_1, \ldots, d_n$ in the variables $x_1, y_1; \ldots; x_n, y_n$, subject to the Plücker equivalence relations.

Each such section can be described as a linear combination of (directed) graphs having $n$ vertices, $v_1, \ldots, v_n$, each of them of degree $\text{deg } v_i = d_i$. 
In fact, $H^0((\mathbb{P}^1)^n, \mathcal{O}(d_1, \ldots, d_n))^{PGL_2}$ consists of polynomials in $x_iy_j - x_jy_i$ having homogeneous degree $d_1, \ldots, d_n$ in the variables $x_1, y_1; \ldots; x_n, y_n$, subject to the Plücker equivalence relations.

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The Plücker relations can be depicted as follows:

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This way one can show that $H^0(X, \Omega_X \otimes \mathcal{L})^{PGL_2} \to (H^0(X, \Omega_X) \otimes g^\vee)$ and so

$$H^1(Y, \Omega_Y \otimes L) = H^1(X, L_X \otimes \mathcal{L})^{PGL_2} = 0$$
This way one can show that $H^0(X, \Omega_X \otimes \mathcal{L})^{PGL_2} \rightarrow (H^0(X, \Omega_X) \otimes g^\vee)$ and so

$$H^1(Y, \Omega_Y \otimes L) = H^1(X, L_X \otimes \mathcal{L})^{PGL_2} = 0$$

Using similar techniques (plus an argument with Koszul complexes), one can also show that

$$H^i(Y, \Omega_Y \otimes L) = H^i(X, \Lambda^j L_X \otimes \mathcal{L})^{PGL_2} = 0, \quad i > 0, j \geq 0$$

and so $Y$ satisfies Bott vanishing, as long as the linearization does not admit strictly semi-stable locus.
Interestingly, quantization can also be applied successfully towards toric varieties. In fact, a smooth projective toric variety $Y$ can be written as a GIT quotient $Y = \mathbb{A}^d \left/ (\mathbb{C}^*)^{d-n} \right.$, where $(\mathbb{C}^*)^{d-n} = \text{Hom}(\text{Pic } Y, \mathbb{C}^*)$, and the action is free on the semi-stable locus.
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Using quantization and similar techniques, we recover yet another proof of the following well-known result.

**Theorem**

A smooth projective toric variety satisfies Bott vanishing.
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Thanks!