Roughening Transition of Interfaces in Disordered Systems

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The behavior of interfaces in the presence of both lattice pinning and random field (RF) or random bond (RB) disorder is studied using scaling arguments and functional renormalization techniques. For the first time we show that there is a continuous disorder driven roughening transition from a flat to a rough state for internal interface dimensions $2 < D < 4$. The critical exponents are calculated in an $\epsilon$-expansion. At the transition the interface shows a superuniversal logarithmic roughness for both RF and RB systems. A transition does not exist at the upper critical dimension $D_c = 4$. The transition is expected to be observable in systems with dipolar interactions by tuning the temperature.

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The thermal roughening transition (RT) from a flat, localized to a rough, delocalized interface in the three dimensional Ising and related lattice models is one of the paradigms of condensed matter physics \cite{1}. It can be directly observed if a crystal surface undergoes a transition from a faceted to a smooth shape \cite{2,3}. At the RT temperature $T_R$ the free energy of a step on the surface vanishes. Apart from its shape also other physical properties are influenced by the presence of the RT, e.g., the (growth) velocity $u$ of the weakly driven interface (surface) changes dramatically at the RT from $u \sim \exp(-C/f)$ for $T < T_R$ to $u \sim f$ for $T > T_R$. Here $f$ denotes the driving force density, which is proportional to the magnetic field in the case of Ising magnets and to the difference $\Delta \mu$ of the chemical potentials of the crystal and its melt in the case of crystal growth, respectively. The thermal RT was shown to be in the same universality class as the metal-insulator transition of a 2D-Coulomb gas \cite{4}, the Kosterlitz-Thouless transition of a 2D XY-model \cite{5} (both with an inverted temperature axis) and the phase transition of the 2D sine-Gordon model \cite{6}. Interestingly, Kosterlitz \cite{5} and Forgacs et al. \cite{7} considered the thermal RT for $D \leq 2$ interface dimensions and found a superuniversal logarithmic roughness at the transition. Finally, the thermal RT disappears in $D = 1$ dimensions: 1-dimensional interfaces are rough at all finite $T$ \cite{1}. \textbf{Disorder} is an even more efficient source of interface roughening and physically important as well \cite{8}. Interface roughening due to disorder was considered to determine the lower critical dimension of the random field Ising model $\frac{13}{2}$ and the mobility of domain walls in disordered magnets $\frac{12,13,14}$. It was argued that in the presence of disorder interfaces of dimensions $D \leq 2$ are always rough, but undergo a roughening transition as a function of the disorder strength for $D > 2$ dimensions $\frac{10}$. Bouchaud and Georges \cite{15} considered explicitly the competition between lattice and impurity pinning in $2 \leq D \leq 4$ dimensions using a variational calculation. Surprisingly they found three phases: a weakly disordered flat, a glassy rough (GR) and an intermediate flat phase (GF) with strong glassy behavior. The transition between the GR and the GF phase turned out to be first order. For $D = 2$ only the glassy rough phase was expected to survive, but that shadows of the two other phases would still be seen in the short scale behavior. However, it should be kept in mind that the variational calculation is in general an uncontrolled approximation which is known to give spurious first order transitions $\frac{15}$. Flory-like exponents as indeed found in $\frac{17}$.

It is therefore the aim of the present paper to reconsider this problem by using a functional renormalization group calculation in $D = 4 - \epsilon$ interface dimensions. It turns out that the glassy flat phase found in $\frac{17}$ is replaced by a crossover region with logarithmic roughness around the RT. The model we consider is that of $\frac{17}$:

$$\mathcal{H} = \gamma \int d^{D} x \left[ \frac{1}{2} (\nabla \mathbf{z})^2 + V_L(z(x)) + V_R(z(x), \mathbf{x}) \right].$$ (1)

Here $z(x)$ denotes the displacement of the $D$ dimensional interface from a flat reference configuration, and $\gamma$ is an elastic stiffness constant. For simplicity, the lattice potential can be considered to be given by $V_L(z) = -va^2 \cos(2\pi z/a)$: higher harmonics could be taken into account but turn out to be irrelevant. Below, we will measure all transverse lengths in units of $a/2\pi$ and therefore put $a = 2\pi$ in the following. The random pinning potential $V_R(z(x), \mathbf{x})$ is assumed to be Gaussian distributed with a zero average value $\overline{V_R(z, \mathbf{x})} = 0$ and

$$\overline{V_R(z, \mathbf{x})V_R(z', \mathbf{x}')} = R(z - z')\delta(\mathbf{x} - \mathbf{x'}).$$ (2)

The form of $R(z)$ depends on the type of the disorder; in particular we have to distinguish between random field (RF) and random bond (RB) disorder. For RF’s, to begin with, $R(z) \sim -\Delta_0|z|$ for large $z$ where $\sqrt{\sum_0}$ denotes the strength of the random field $\frac{14}$. For RB’s the bare form of $R(z)$ is a smeared out $\delta$-function of width $\xi_0$ and
strength $\Delta_0 \xi_0^2$. For simplicity we assume that $\xi_0$ is of the order $a$.

We begin our calculation with some elementary considerations. For $\Delta_0 = 0$ a step of height $2\pi$ in the otherwise planar interface of linear dimension $L$ costs an energy $E_{\text{step}} \simeq \gamma \sqrt{\alpha} L^{D-1}$. The step itself has a width $w_{\text{step}} \simeq v^{-1/2}$. Here and below we omit all prefactors of order unity. Switching on RF disorder, we consider the energy of a hump of total height $z$ in the interface consisting of $z/2\pi$ such steps. This gives for large enough lattice pinning $v$

$$E_{\text{total}}/\gamma \simeq \sqrt{\alpha} L^{D-1} z - \sqrt{\Delta_0} L^D z.$$  \hfill (3)

The second part represents the interaction energy with the RF. \cite{21}. Minimizing \cite{21} we get $z \simeq \left(L/L_R\right)^{(2-D)}$ with $L_R \simeq \left(v/\Delta_0\right)^{1/(2-D)}$. For $D < 2$ and $L > L_R$ the interface is therefore rough even in the presence of the lattice potential. Since the total step free energy vanishes now on scales $L \simeq L_R$, the system is described on larger scales by an effective elastic Hamiltonian with a stiffness constant $\gamma_{\text{eff}}$. A rough estimate for $\gamma_{\text{eff}}$ follows from balancing the elastic and the bare step free energy on the scale $L_R$, which gives $\gamma_{\text{eff}} \simeq \gamma L_R/w_{\text{step}}^{2-D}$. On scales $L \gg L_R$ the hump height scales as $z \simeq \left(L/L_R\right)^{\xi}$ where $\xi = (4-D)/3$. \hfill (1)

To be meaningful, $L_R$ should be larger than the bare stepwidth, $L_R > w_{\text{step}}$, a condition which can be rewritten as $v > v_c \equiv \Delta_0^{2/(4-D)}$. $L_R$ can then be cast in the more conventional form of a Larkin-length $L_R = \left(v/v_c\right)^{1/(2-D)} \Delta_0^{1/(4-D)}$. If $v \rightarrow v_c +$, $\gamma_{\text{eff}}$ changes into $\gamma$. For $v < v_c$ we neglect the lattice pinning term in \cite{21} completely and $v$ has to be replaced by $v_c$ in the above formulas.

So far we have considered the case $D < 2$. For $D = 2$ it was shown by Binder \cite{12,19} that $L_R$ becomes exponentially large, $L_R \simeq \exp(C v/\Delta_0)$, with $C$ a numerical factor, and interfaces are again rough for $L > L_R$. Thus, there is no flat phase in $D \leq 2$ dimensions.

For $D > 2$ we have to expect from the previous considerations that the interface becomes rough only for $v < v_c$, i.e., that there is a RT as a function of $v$ (or $\Delta_0$) at $v \simeq v_c$. In the rough phase the mean square displacement of the interface can be estimated again from the energy term to describe the crossover from logarithmic to power law roughness happens at length scales $L \simeq \xi_0 \sim (v_c - v)^{-1/2\xi_0^2}$.

As was first shown in \cite{21}, the proper treatment of interface roughening requires a functional RG approach due to the occurrence of infinitely many relevant operators. A finite lattice pinning potential $V_L(z)$ breaks the continuous translational symmetry and thus a functional RG treatment of both pinning potentials generates also contributions to the elastic stiffness. Balents and Kardar \cite{21} have developed a RG approach in terms of an eigenfunction expansion of the interactions for a similar case. For the periodic lattice potential, this expansion is simply a Fourier expansion. This simplifies the analysis of the initial functional form of the RG flow since higher harmonics $\cos mz$, $m > 1$, are strongly irrelevant at the RT with scaling dimension $\lambda_m = 2 - (2 + \epsilon/4)m^2$. Therefore only the lowest harmonic $\cos z$ is taken into account.

We choose to keep $\gamma$ fixed by rewriting the renormalization of $\gamma$ as an additional renormalization of $T$, $R(z)$ and $v$. The resulting RG flow equations, valid to lowest order in $\epsilon$, read

$$\frac{dT}{dl} = \left[2 - D - 2\xi - \frac{1}{4} v^2 \Delta\right] T = -\theta T, \hfill (5)$$

$$\frac{dR}{dl} = \left[4 - D - 4\xi - \frac{1}{2} v^2 \Delta\right] R(z) + \xi z R'(z) + \frac{1}{2} \left|R''(z)\right|^2 - R''(0) R''(z), \hfill (6)$$

$$\frac{dv}{dl} = \left[2 - \frac{1}{2} \Delta\right] v + \frac{1}{8} \Delta v^3. \hfill (7)$$

Here $l = \ln(1/q)$ and $\Delta = -R''(0)\cos^{2\xi}$. We have set the ultraviolet cutoff to 1 and made the substitution $K_D R(z) \rightarrow R'(z)$ with $K_D^{-1} = 2^{D-1}\pi^{D/2} \Gamma(D/2)$. The factor $\exp(2\xi l)$ in the definition of $\Delta$ stems from the rescaling of the periodicity of the lattice pinning potential. Here $\exp(\xi l)$ has to be read as the abbreviation for $\exp(\bar{\xi} dl)$ with $l$ dependent $\xi$.

First, we remark that for $D > 2$ thermal fluctuations turn out to be irrelevant such that $T^* = 0$ at all fixed points of interest. For $v = 0$ three locally stable fixed points of Eq. \cite{4} are known, each with their own basin of attraction. These are the RF, the RB and the charge density wave fixed point. \cite{12,21}. By inspection of Eq. \cite{4} it is clear that a finite $v$ does not change the functional structure of the flow of $R(z)$ since $v^2 \Delta$ approaches a finite value at the fixed point as we will see below. Therefore the functional form of the fixed point $R^*(z)$ does not depend on $v$ and is given by a dimensionless function $r(u)$ with a characteristic scale of order unity. It is determined by the fixed point condition $\alpha r(u) + \beta u r'(u) + r''(u)(r''(u)/2+1) = 0$ with $r''(0) = -1$, where $\alpha$ and $\beta$ are numerical coefficients which differ for RF and RB disorder, respectively. They can be determined from the conditions that $r(u) \sim |u|$ at large $u$ for
RFG’s and that \( r(u) \) decays exponentially for RFG’s. For RFG’s we obtain \( \alpha/\beta = -1 \), which is sufficient to determine the exponents. For RF disorder \( r(0) > 0 \) and we can choose \( r(0) = 1 \) leading to \( \alpha = 1/2 \) by evaluating the above fixed point condition at \( u = 0 \). \( \beta = 0.6244 \) is determined numerically from the condition that the function \( r(u) \) vanishes exponentially for \( u \to \infty \) and \( r(0) = 1 \).

To obtain the critical behavior near the RT, we assume that \( R(z) \) has already approached its functional fixed point form, i.e., we make the Ansatz

\[
R(z) = \Delta(l)e^{2l}e^{-4cl}r\left(z/\xi(l)e^{-c'l}\right),
\]

(8) which is justified on sufficiently large length scales. Inserting this Ansatz into Eq. (5) and using the fixed point condition for \( r(u) \) we obtain an equation of the form \( Ar(u) + Bar'(u) = 0 \) where \( A \) and \( B \) depend on \( \Delta, \xi \) and its derivatives with respect to \( l \). This equation is fulfilled for all \( u \) if and only if \( A = B = 0 \), which is equivalent to the following flow equations for \( \Delta \) and \( \xi \),

\[
\frac{d\Delta}{dl} = \epsilon\Delta - \left[\frac{\alpha + 2\beta}{\xi^2} + \frac{1}{2}\right] \Delta^2, \tag{9}
\]

\[
\frac{d\xi}{dl} = \beta \Delta/\xi. \tag{10}
\]

The rough phase of the interface is described by a stable fixed point with \( v^* = 0 \), \( \Delta^*/\xi^* = \epsilon/(\alpha + 4/\beta) \). In this phase, RB and RF systems are characterized by a roughness exponent which reads in our parameter approach \( \zeta = \epsilon/(4 + \alpha/\beta) \). With the above values for \( \alpha \) and \( \beta \) the known results \( \zeta^{RB} = \epsilon/3 \) and \( \zeta^{RF} = 0.2083\epsilon \) [20] are exactly reproduced. For \( D = 4 \) we find that the interface roughness grows only sub-logarithmically with the system size \( L \), \( \langle z^2 \rangle \sim \ln^{2} (L/a) \) with \( \sigma = 2/3 \) for RF’s and \( \sigma = 0.4166 \) for RB’s.

For finite \( v \) there is a new fixed point with \( \Delta^* = 4 + \epsilon/2 \), \( \xi^* = \sqrt{\epsilon/2} \) and \( \xi^* = \infty \). Notice that the perturbative RG is still justified at this fixed point since \( v \) is small. Linearizing around this fixed point we obtain in lowest order in \( \epsilon \) the eigenvalues \( \lambda_{\pm} = \pm 2\sqrt{\epsilon} \). Since the fixed point has an unstable direction, it has to be associated with the RT. In the flat phase the size of typical excursions of the interface from the preferred minimum of the lattice potential defines the longitudinal correlation length \( \xi_{\parallel} \sim |v - v_c|^{-\nu_{||}} \) with \( \nu_{||} = 1/2\sqrt{\epsilon} \). Interestingly \( \nu_{||} \) does not dependent on \( \alpha, \beta \) and is therefore universal for RF and RB disorder. The same expression for \( \nu_{||} \) has been obtained previously for the thermal roughening transition but with \( \epsilon = 2 - D \) [3].

The interface roughness \( K(x) = \langle [z(x) - z(0)]^2 \rangle \) can be obtained in terms of the effective and non-rescaled parameters \( \Delta = \Delta e^{-cl}, \tilde{v} = v e^{-2l} \) using \( l = \ln(1/q) \).

Sufficiently close to the transition where \( \xi_{\parallel} > |x| > L_R \), the interface roughness increases logarithmically, \( K(x) \sim \ln(|x|/L_R) \) on both sides of the RT, see Fig. 1(d). Beyond \( \xi_{\parallel} \) the roughness crosses over to the power law \( K(x) \sim (|x|/\xi_{\parallel})^2 \) in the rough phase. On the flat side of the transition \( K(x) \) saturates on scales larger than \( \xi_{\parallel} \) at a finite value \( \sim \ln(\xi_{\parallel}/L_R) \).

The RG calculation also confirms the result \( v_c(\Delta_0) = A(\epsilon)\Delta_0^{-\nu_c} \) already mentioned above. To obtain the coefficient \( A(\epsilon) \) we integrate the flow equations (9), (10) by neglecting the non-linear terms in \( v \). The effect of the non-linear terms is subsequently included by matching the resulting solution \( v(l) \) and \( v^* \) at the length scale \( L_R \), which yields \( A(\epsilon) = \frac{\xi}{2(z\epsilon/4)1/\zeta - 2 - (4\epsilon)^{-1/\zeta}} \) in good agreement with numerical solutions of the RG flow. Since \( \zeta < \epsilon/2 \), the function \( A(\epsilon) \) vanishes rapidly for \( \epsilon \to 0 \). Notice that in \( D = 4 \) an arbitrarily small \( v > 0 \) leads to a flat phase due to the sub-logarithmic roughness for \( v \equiv 0 \). The schematic RG flow for \( 2 < D < 4 \) is shown together with that expected for \( D \leq 2 \) and \( D \geq 4 \) in Fig. 1.

The fluctuations in the free energy grows with length as \( L^\theta \) where \( \theta \) is the violation of hyperscaling exponent. Whereas one has exactly \( \theta = D - 2 + 2\zeta \) in the rough phase, the relevance of the lattice potential destroys the tilt symmetry of (1) leading to \( \theta_c = 2 - \epsilon/2 \) at the transition.

As discussed in the introduction, the RT disappears in the physically interesting case \( D > 2 + \). However, the RT is expected to be seen even for \( D = 2 \) in systems with dipolar interaction. Indeed, for magnetic or ferroelectric domain walls with the magnetization direction \( \rho \) parallel to the wall the elastic energy has the form [23, 24]

\[
E_{el} = \frac{g}{2} \int \frac{d^2q}{(2\pi)^2} q^2 \left( 1 + g q^2 q^2 q^{D+1} \right) \sim q^{-2}.
\]

where \( g \) measures the relative strength of the dipolar interaction. Our results can also be applied to interfaces
with this elastic interaction. Repeating the RG analysis for this case, we see that the calculated exponents remain valid to order $\epsilon$ if we replace $\epsilon = 4 - D$ by $\epsilon = 3(3 - D)/2$. Thus the upper critical dimension is shifted to $D = 3$, and there is a RT for 2D dipolar interfaces which is described by our results with $\epsilon = 3/2$. Note that 2D dipolar interfaces do not show a thermal RT.

The threshold condition $v_c(\Delta_0)\Delta_0^{-2/\epsilon} = A(\epsilon)$ and hence the RT can be reached by changing the temperature $T$ since $v_0$ and $\Delta_0^{2/\epsilon}$ depend in general in a different way on $T_c - T$. Here $T_c$ denotes the condensation temperature of the system, e.g., the Curie temperature for magnets. The precise $T$-dependence is model dependent but close to $T_c$ where $\xi_0/\alpha \gg 1$ we expect a very weak influence of the lattice potential, e.g., for Peierls barriers $v_0 \sim e^{-C\xi_0}$. Thus, for not too large disorder, we can expect to see this RT by increasing $T$.

Under the influence of a small driving force density $f$, the interface motion is dominated by jumps between neighboring metastable states in the rough phase and between adjacent minima of the lattice potential in the flat phase. In both cases the creep velocity $u$ is exponentially small \([4], u(f) \sim \exp(-E_c/[T(f_c/f)^\mu R])\). In the rough phase, $\mu = (2\zeta + D - 2)/(2 - \zeta)$, $E_c \sim \gamma_0^2 L^D R^{-2}$, and $f_c \sim \gamma_0^2 L^D R^{-2}$ whereas in the flat phase $\mu = D - 1$, $E_c \sim \gamma^2 u^{(2-D)/2}$ and $f_c \sim \gamma v$ for an interface with short range interactions. For dipolar systems, we obtain for a two dimensional interface in the rough phase $\mu_{RF} = 1$, $\mu_{RR} = 0.6666$ and for a flat interface $\mu = 1$ independent of the type of disorder. Therefore, in the RB case the phase of the interface can be determined by measuring $\mu$. A very recent experiment on driven domain walls \([24]\) shows that the exponent $\mu$ can be measured very accurately.

To summarize, the interplay between a random potential and lattice pinning for an interface leads for $2 < D < 4$ to a continuous disorder-driven roughening transition between a rough and a flat interface, in contrast to earlier results of Bouchaud and Georges for RB systems \([17]\) which are based on a variational calculation. However, there is a qualitative similarity in the phase diagram if we identify the GF phase of \([17]\) with the logarithmically rough crossover region of our results. Both separate the GR from the simple flat phase \([20]\). Moreover, we have obtained the exponent $v_D$ which has to lowest order in $\epsilon$ the same form as that of the thermal RT if $\epsilon = 2 - D$. For both random field and random bond systems, the interface shows a superuniversal logarithmic roughness at the transition. At the upper critical dimension $D = 4$ an arbitrary small lattice pinning produces a flat interface due to sub-logarithmic roughness in the absence of lattice effects. In contrast to the 2D thermal RT a diverging correlation length $\xi_0^\parallel$ appears on both sides of the transition. On the rough side $\xi_\parallel$ sets the length scale for the crossover from logarithmic to algebraic roughness. We have demonstrated that the disorder-driven RT will appear for two dimensional interfaces in dipolar systems.

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\[26\] The generalization of our results to a N component displacement yields also a continuous transition as long as $N < \infty$ and a first order transition for $N = \infty$ where the variational approach is expected to become exact.