Research article

Biharmonic submanifolds of Kaehler product manifolds

Yanlin Li\textsuperscript{1,*}, Mehraj Ahmad Lone\textsuperscript{2} and Umair Ali Wani\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Hangzhou Normal University, Hangzhou, 311121, China
\textsuperscript{2} Department of Mathematics, NIT Srinagar, J&K, 190006, India

\* Correspondence: Email: liyl@hznu.edu.cn.

Abstract: In this paper, the authors have established the necessary and sufficient conditions for the submanifolds of Kaehler product manifolds to be biharmonic. Moreover, the magnitude of scalar curvature for the hypersurfaces in a product of two unit spheres has been derived. Also, for the same product, the magnitude of the mean curvature vector for Lagrangian submanifolds has been estimated. Finally, the non-existence condition for totally complex Lagrangian submanifolds in a product of unit sphere and a hyperbolic space has been proved.

Keywords: biharmonic submanifolds; Kaehler product manifolds; totally real submanifolds; Lagrangian submanifolds; mean curvature

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1. Introduction

For any two manifolds (M,g) and (N,h), a harmonic map ψ is the critical point of the energy functional defined as

\[ E(ψ) = \frac{1}{2} \int_M |dψ|^2 dv_g. \]

The natural generalization of the harmonic maps was given by J. Eells and J. H. Sampson [1]. The established map ψ is called biharmonic if it is the critical point of energy functional

\[ E_2(ψ) = \frac{1}{2} \int_M |τ(ψ)|^2 dv_g. \]

with \( τ(ψ) = tr(∇dψ) \) as the vanishing tensor field for any harmonic map. For the above established \( E_2 \), the first and second variation was studied by G. Y. Jiang [2]. For the same bi-harmonic functional, the associated Euler-Lagrange equation is \( τ_2(ψ) = 0 \), where \( τ_2(ψ) \) is called bi-tension field and is defined as

\[ τ_2(ψ) = Δτ(ψ) - tr(R^N(dψ, τ(ψ)))dψ. \]
Similarly, the Riemannian curvature tensor $\hat{R}^p$ of $\hat{M}^p(c_2)$ is given by

$$
\hat{R}_p(X, Y)Z = \frac{1}{4}c_2[g_p(Y, Z)X - g_p(X, Z)Y] + \frac{1}{4}c_2[g_p(J_pY, Z)J_pX - g_p(J_pX, Z)J_pY + 2g_p(X, J_pY)J_pZ].
$$

For any generalized submanifold $M$ of any complex space form $N$, the almost complex structure $J$
induces the existence of four operators on \( M \), namely

\[
j : TM \rightarrow TM, k : TM \rightarrow NM, l : NM \rightarrow TM, m : NM \rightarrow NM,
\]
defined for all \( X \in TM \) (tangent bundle) and \( \zeta \in NM \) (normal bundle) by

\[
JX = jX + kX, \\
J\zeta = l\zeta + m\zeta.
\]

Since \( J \) is the almost complex structure, it satisfies \( J^2 = -Id \). For any \( X, Y \) tangent to \( N \), we also have \( g(JX, Y) = -g(X, JY) \). Using the above properties of \( J \), the relations for the operators, \( j, k, l \) and \( m \) are given as

\[
\begin{align*}
    f^2X + lkX + X &= 0, \\
m^2\zeta + kl\zeta + \zeta &= 0, \\
jl\zeta + lm\zeta &= 0, \\
kjX + mkX &= 0,
\end{align*}
\]

for all \( X \in \Gamma(TM) \) and \( \zeta \in \Gamma(NM) \). Also, \( j \) and \( m \) are skew-symmetric.

Now, let us consider the Kaehlerian product manifold \( \tilde{M}^n(c_1) \times \tilde{M}^p(c_2) \) denoted by \( \tilde{M} \). If \( P \) and \( Q \)
denote projection operators of the tangent spaces of \( \tilde{M}^n(c_1) \) and \( \tilde{M}^p(c_2) \), then we always have \( P^2 = P \), \( Q^2 = Q \) and \( PQ = QP \). If we put \( F = P - Q \), the properties of \( P \) and \( Q \) establish \( F^2 = I \). This \( F \) is almost product structure of \( \tilde{M}^n(c_1) \times \tilde{M}^p(c_2) \). Moreover, we define a Riemannian metric \( g \) on \( \tilde{M} \) as

\[
g(X, Y) = g_n(PX, PY) + g_p(QX, QY).
\]

Where \( X \) and \( Y \) are vector fields on \( \tilde{M} \). It further follows, \( g(FX, Y) = g(X, FY) \). If we put \( JX = J_nPX + J_pQX \), we get \( J_nP = PJ, J_pQ = QJ, FJ = JF, g(JX, JY) = g(X, Y) \), \( \tilde{\nabla}J = 0 \). Thus \( J \) is the Kaehlerian structure on \( \tilde{M} \). The Riemannian curvature tensor \( \tilde{R} \) of the product manifold \( \tilde{M} \) is given as [21]

\[
\tilde{R}(X, Y)Z = \frac{c_1 + c_2}{16}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X, JY)JZ + g(FY,Z)FX - g(FX,Z)FY + g(FY,Z)FX - g(FJX,Z)FJY + g(FJY,Z)JX - g(FJX,Z)JY + 2g(FX, JY)JZ + 2g(X, JY)JFZ].
\]

The product structure \( F \) induces the existence of four operators:

\[
j : TM \rightarrow TM, h : TM \rightarrow NM, s : NM \rightarrow TM and t : NM \rightarrow NM,
\]
defined for all $X \in TM$ (tangent bundle) and $\zeta \in NM$ (normal bundle) by

$$FX = fX + hX, \quad F\zeta = s\zeta + t\zeta. \quad (2.8)$$

These four operators follow the following relations

$$f^2 X + shX = X, \quad (2.9)$$

$$\hat{f} \zeta + hs\zeta = \zeta, \quad (2.10)$$

$$fs\zeta + st\zeta = 0, \quad (2.11)$$

$$hfX + thX = 0, \quad (2.12)$$

$$g(hX, \zeta) = g(X, s\zeta). \quad (2.13)$$

for all $X \in \Gamma(TM)$ and $\zeta \in \Gamma(NM)$. Also, $f$ and $t$ are symmetric.

3. Results

The first theorem gives necessary and sufficient condition for the manifold to be biharmonic.

**Theorem 3.1.** Let $M$ be a $u$-dimensional submanifold of the Kaehler product manifold $\hat{M} = \hat{M}^p(c_1) \times \hat{M}^q(c_2)$ with $A$, $B$ and $H$, respectively denoting the shape operator, second fundamental form and mean curvature vector. Then, this submanifold is biharmonic if and only if the following equations are satisfied:

$$-\nabla^\perp H + tr(B(., A_H, .)) + \frac{c_1 + c_2}{16}[-uH + 3klH + hsH - tr(f)tH + 2(hjflH + tkflH + hjsmH + tksmH) - tr(fj + sk)(hH + tH)] + \frac{c_1 - c_2}{16}[-tr(f)H] - uhH + 3(kf\ell + ksmH) - tr(fj + sk)(mH) + 3(hj\ell + tk\ell) = 0. \quad (3.1)$$

$$\frac{u}{2} \text{grad}|H|^2 + 2tr(A_{\perp H}(., .)) + \frac{c_1 + c_2}{8}[3jlH + fsH - tr(f)sH + 2(fjflH + skflH + fsH) - tr(fj + sk)(flH + smH) + \frac{c_1 - c_2}{8}[sH - usH + 3(jjflH + jsmH) - tr(fj + sk)(lH) + 3(fj\ell + sk\ell)] = 0. \quad (3.2)$$

**Proof.** The equations of biharmonicity have been already established in [12, 22, 23]. Projection of the equation $\tau(\psi) = 0$ on both tangential and normal bundles establishes the following equations...
\[-\nabla_{H}^{\perp} + \text{tr}(B(., A_{H} .)) + \text{tr}(\bar{R}(., H.)^{\perp}) = 0,\]
\[
\frac{\mu}{2} \text{grad}[H]^{2} + 2\text{tr}(AV_{H}(.)) + 2\text{tr}(\bar{R}(., H.)^{\top}) = 0. \tag{3.3}
\]

Suppose that \(\{X_{i}\}_{i=1}^{u}\) is a local orthonormal frame for TM, then by using the Eq 2.7 of curvarture tensor \(\bar{R}\), we have
\[
\text{tr}(\bar{R}(., H.) = \sum_{i=1}^{u} \bar{R}(X_{i}, H.)X_{i}, \tag{3.4}
\]
\[
\implies \text{tr}(\bar{R}(., H.) = \sum_{i=1}^{u} \frac{c_{+c_{i}}}{16} [g(H, X_{i})X_{i} - g(X_{i}, X_{i})H + g(JH, X_{i})JX_{i}
\]
\[
- g(JX_{i}, X_{i})JH + 2g(X_{i}, JH)JX_{i} + g(FH, X_{i})FX_{i} - g(FX_{i}, X_{i})FH
\]
\[
+ g(FJH, X_{i})FJX_{i} - g(FJX_{i}, X_{i})FJH + g(FX_{i}, JH)FJX_{i}
\]
\[
+ \frac{c_{-c_{i}}}{16} [g(FH, X_{i})X_{i} - g(FX_{i}, X_{i})H + g(H, X_{i})FX_{i} - g(X_{i}, X_{i})FH
\]
\[
+ g(FJH, X_{i})JX_{i} - g(FJX_{i}, X_{i})JH + g(JH, X_{i})FJX_{i} - g(JX_{i}, X_{i})FJH
\]
\[
+ 2g(FX_{i}, JH)JX_{i} + 2g(X_{i}, JH)JFX_{i}].
\]

Introducing the established sets of four operators, \(j, k, l\) and \(m\) and \(f, h, s\) and \(t\) for \(J\) and \(F\) respectively, we get the simplified equation as
\[
\text{tr}(\bar{R}(., H.) = \frac{c_{+c_{i}}}{16} [-uH + \sum_{i=1}^{u} g(lH, X_{i})JX_{i} + \sum_{i=1}^{u} 2g(X_{i}, lH)JX_{i}
\]
\[
+ F(H)^{\top} - \text{tr}(f)FH + FJ(FJH)^{\top} - \text{tr}(fj + sk)FJH + FJ(FJH)^{\top}
\]
\[
+ \frac{c_{-c_{i}}}{16} [(FH)^{\top} - \text{tr}(f)H - uFH + J(FJH)^{\top} - \text{tr}(fj + sk)JH
\]
\[
+ \sum_{i=1}^{u} g(lH, X_{i})FJX_{i} + 2J(FJH)^{\top} + \sum_{i=1}^{u} 2g(X_{i}, lH)JFX_{i}],
\]
or \[
\text{tr}(\bar{R}(., H.) = \frac{c_{+c_{i}}}{16} [-uH + 3JlH + f sH + hsH - \text{tr}(f)sH - \text{tr}(f)tH
\]
\[
+ 2FJ(flH + smH) - \text{tr}(fj + sk)FJH]
\]
\[
+ \frac{c_{-c_{i}}}{16} [sH - \text{tr}(f)H - uFH + J(flH + smH) - \text{tr}(fj + sk)JH +
\]
\[
3FJlH + 2J(flH + smH)],
\]
\[
\implies \text{tr}(\bar{R}(., H.) = \frac{c_{+c_{i}}}{16} [-uH + 3jlH + 3klH + f sH + hsH - \text{tr}(f)sH - \text{tr}(f)tH
\]
\[
+ 2(f jf lH + h jf lH + skflH + tkflH + f jsmH + hjsmH + sksmH + tksmH)
\[ \nabla^2 H + \text{tr}(B(\cdot, A_{H_H} \cdot)) + \frac{c_1 + c_2}{16} [-uH + 3klH + hsH - \text{tr}(f)tH] \\
+ 2(tkflH + tksmH) - \text{tr}(sk)(flH + tmH)] + \frac{c_1 - c_2}{16} [-\text{tr}(f)H] \\
- utH + 3(kfjH + kmH) - \text{tr}(sk)(mH) + 3(tkH) = 0. \]

**Proof.** If \( M \) is a totally real submanifold, then we know that for any \( X \in \Gamma(TM) \), we have \( JX = kX \).

In other words, \( jX = 0 \). Using this fact in Theorem 3.1, we get the required equations. \( \square \)

**Corollary 3.3. a)**: If \( M \) is any hypersurface of the Kaehler product manifold \( \tilde{M} = \tilde{M}^p(c_1) \times \tilde{M}^p(c_2) \).

Then, \( M \) is biharmonic if and only if the following equations are satisfied

\[ \begin{align*}
\nabla^2 H + \text{tr}(B(\cdot, A_{H_H} \cdot)) + \frac{c_1 + c_2}{16} [-uH + 3klH + hsH - \text{tr}(f)tH] \\
+ 2(tkflH + tksmH) - \text{tr}(sk)(flH + tmH)] + \frac{c_1 - c_2}{16} [-\text{tr}(f)H] \\
- utH + 3(kfjH + kmH) - \text{tr}(sk)(mH) + 3(tkH) = 0. \end{align*} \]
Then, M is biharmonic if and only if the following equations are satisfied:

\[ - \nabla^\perp H + tr(B(.,A_{H,.}))+ \frac{c_1 + c_2}{16}[-(n-2)H + hsH - tr(f)H + 2(tkfH) = 0. \]

**b): If M is any totally real hypersurface of the Kaehler product manifold**

\[ \tilde{M} = \tilde{M}^{\sigma}(c_1) \times \tilde{M}^{\sigma}(c_2). \]

Then, M is biharmonic if and only if the following equations are satisfied:

\[ \begin{align*}
&- \nabla^\perp H + tr(B(.,A_{H,.}))+ \frac{c_1 + c_2}{16}[-(n-2)H + hsH - tr(f)H + 2(tkfH) + (n-1)H - 3sH] = 0. \\
&+ \frac{c_1 - c_2}{16} \left[ f sH - tr(f)sH + 2(skfH) - tr(sk)(flH) \right].
\end{align*} \]

**Proof.** a): For any hypersurface M, J maps normal vectors to tangent vectors as such \( m = 0. \) Using this fact with the Eqs 2.3 and 2.4 for H, we get the required equations from Theorem 3.1.

b): For any totally real hypersurface M, we have \( j = 0 \) and \( m = 0. \) \( \square \)

**Corollary 3.4.** If M is a \( u \)-dimensional Lagrangian manifold of the Kaehler product manifold

\[ \tilde{M} = \tilde{M}^{\sigma}(c_1) \times \tilde{M}^{\sigma}(c_2). \]

Then, M is biharmonic if and only if the following equations are satisfied

\[ \begin{align*}
&- \nabla^\perp H + tr(B(.,A_{H,.}))+ \frac{c_1 + c_2}{16}[-(n-2)H + hsH - tr(f)H + 2(tkfH) + (n-1)H - 3sH] = 0. \\
&+ \frac{c_1 - c_2}{16} \left[ f sH - tr(f)sH + 2(skfH) - tr(sk)(flH) \right].
\end{align*} \]

**Proof.** If M is a Lagrangian manifold, then \( j = 0 \) and \( m = 0. \) Using this fact with Eq 2.3, we get the required equations from Theorem 3.1. \( \square \)

From now on, the authors will consider the ambient space to be product of two 2-spheres of same radius (for simplicity radius equals 1 unit). The reason for taking 2-sphere follows from [24] as it is the only sphere which accepts Kaehler structure. In the following equations, we will have

\[ \frac{c_1 + c_2}{16} = \frac{c_1}{8} = \frac{1}{8} \text{ and } \frac{c_1 - c_2}{8} = b = 0. \]

To estimate the magnitude of mean curvature vector and scalar curvature, the authors will further assume the cases where F will map the whole of tangent bundle or normal vectors to respective bundles only. The reason being the equations involve the product of almost complex structure J and product structure F. As such it isn’t possible to get simpler equations involving dimensions of submanifolds and mean curvature vector only.
Proposition 3.5. Let $M$ be any hypersurface of $S^2 \times S^2$ with non-zero constant mean curvature such that $FX \in \Gamma(TM^\perp)$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$. Then $M$ is biharmonic if we have

$$|B|^2 = \frac{1}{8} \left[ 1 + \frac{1}{|H|^2} \text{tr}(sk)(FJH, H) \right].$$  \hspace{1cm} (3.13)

Proof. By the established hypothesis on $F$, we have $f = 0$ and $t = 0$. Using these equations along with Eqs 2.9 and 2.10 in Eq 3.7, we get

$$-\nabla^\perp H + \text{tr}(B(., A_{H,.})) - \frac{1}{8} [H + \text{tr}(sk)(hlH)] = 0,$$  \hspace{1cm} (3.14)

Since $M$ is a hypersurface, the above equation becomes,

$$\text{tr}(B(., A_{H,.})) - \frac{1}{8} [H + \text{tr}(sk)(hlH)] = 0,$$  \hspace{1cm} (3.15)

Since $\text{tr}(B(., A_{H,.})) = |B|^2 H$, on further simplifying, we get,

$$|B|^2 H^2 = \frac{1}{8} \left[ H^2 + \text{tr}(sk)(hlH) \right],$$  \hspace{1cm} (3.16)

or

$$|B|^2 = \frac{1}{8} \left[ 1 + \frac{1}{|H|^2} \text{tr}(sk)(FJH, H) \right].$$  \hspace{1cm} (3.17)

Remark 3.6. It can be easily concluded from above proposition that there doesn’t exist any hypersurface of $S^2 \times S^2$ when $FX \in \Gamma(TM^\perp)$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$ for

$$\text{tr}(sk)(FJH, H) + |H|^2 \leq 0.$$

The above proposition can be used to derive the value of scalar curvature for biharmonic hypersurface $M$ when $FX \in \Gamma(TM^\perp)$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$.

Proposition 3.7. Let $M$ be any proper-biharmonic hypersurface of $S^2 \times S^2$ with non-zero constant mean curvature such that $FX \in \Gamma(TM^\perp)$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$. Then the scalar curvature $\tau$ of $M$ is given by

$$\tau_M = \frac{1}{8}[5 + \text{tr}(sk)^2 - \frac{1}{|H|^2} \text{tr}(sk)(FJH, H)] + 3|H|^2.$$  \hspace{1cm} (3.18)

Proof. By the equation of Gauss, we have,

$$\tau_M = \sum_{i,j=1}^{n-1} \langle \hat{R}(X_i, X_j)X_j, X_i \rangle - |B|^2 + (n-1)|H|^2,$$

The curvature tensor $\hat{R}$ for $S^2 \times S^2$ is given by Eq 2.7 with

$$\frac{c_1 + c_2}{16} = \frac{c_1}{8} = \frac{1}{8} \text{ and } \frac{c_1 - c_2}{8} = 0.$$
And,
\[
\langle \hat{R}(X_i, X_j)X_j, X_i \rangle = \frac{1}{8}[1 + \langle FX_j, X_j \rangle \langle FX_i, X_i \rangle - \langle FX_i, X_j \rangle^2 + \langle FJX_j, X_j \rangle \langle FX_i, X_i \rangle],
\]
(3.18)

Since \( FX_i \in \Gamma(TM^\perp) \) and \( f = 0 \). We have
\[
\sum_{i,j=1}^{n-1} \langle \hat{R}(X_i, X_j)X_j, X_i \rangle = \frac{1}{8}[6 + tr(sk)^2].
\]
(3.19)

Using the value of \(|B|^2\) gives the required equation. \( \square \)

**Proposition 3.8.** Let \( M \) be any totally complex-hypersurface of \( S^2 \times S^2 \) with non-zero constant mean curvature such that \( FX \in \Gamma(TM^\perp) \) and \( FN \in \Gamma(TM) \) for any \( X \in \Gamma(TM) \) and \( N \in \Gamma(TM^\perp) \). Then for trivially biharmonic \( M \), we have
\[
|B|^2 = \frac{1}{8}.
\]
(3.20)

**Proof.** By the established hypothesis on \( F \), we have \( f = 0 \) and \( t = 0 \). Using these equations along with Eqs 2.9 and 2.10 in Theorem 3.1, we get
\[
-\nabla H + tr(B(., A_H).) - \frac{1}{8}H = 0,
\]
(3.21)

Since \( M \) is a hypersurface, the above equation becomes
\[
tr(B(., A_H).) - \frac{1}{8}H = 0.
\]
(3.22)

Since \( tr(B(., A_H).) = |B|^2 H \). On further simplifying, we get the required equation. \( \square \)

**Proposition 3.9.** Let \( M \) be any proper-biharmonic totally complex-hypersurface of \( S^2 \times S^2 \) with non-zero constant mean curvature such that \( FX \in \Gamma(TM^\perp) \) and \( FN \in \Gamma(TM) \) for any \( X \in \Gamma(TM) \) and \( N \in \Gamma(TM^\perp) \). Then the scalar curvature \( \tau \) of \( M \) is given as
\[
\tau_M = \frac{1}{8}[5 + tr(sk)^2] + 3|H|^2.
\]
(3.23)

**Proof.** By the equation of Gauss, we have
\[
\tau_M = \sum_{i,j=1}^{n-1} \langle \hat{R}(X_i, X_j)X_j, X_i \rangle - |B|^2 + (n - 1)|H|^2,
\]
The curvature tensor \( \hat{R} \) for \( S^2 \times S^2 \) is given by Eq 2.7 with
\[
\frac{c_1 + c_2}{16} = \frac{c_1}{8} = \frac{1}{8} \quad \text{and} \quad \frac{c_1 - c_2}{8} = 0.
\]
Then,
\[
\langle \hat{R}(X_i, X_j)X_j, X_i \rangle = \frac{1}{8} \left( 6 + \text{tr}(sk) \right).
\]
(3.24)

Since \( FX_i \in \Gamma(TM^\perp) \) and \( f = 0 \). We have
\[
\sum_{i,j=1}^{n-1} \langle \hat{R}(X_i, X_j)X_j, X_i \rangle = \frac{1}{8} \left( 6 + \text{tr}(sk) \right).
\]
(3.25)

Using the value of \(|B|^2\) gives the required equation.

\[\Box\]

Corollary 3.10. Let \( M \) be \( u \)-dimensional Lagrangian submanifold of \( S^2 \times S^2 \) with non-zero constant mean curvature such that \( FX \in \Gamma(TM^\perp) \) and \( FN \in \Gamma(TM) \) for any \( X \in \Gamma(TM) \) and \( N \in \Gamma(TM^\perp) \). Let us further assume \( \text{tr}(sk)(FJH, H) \geq 0 \). Then we have

a): If \( M \) is a proper-biharmonic, then
\[ 0 < |H|^2 \leq \frac{u^2 + 2}{8u}. \]

b): If \( |H|^2 = \frac{u^2 + 2}{8u} \), then \( M \) is biharmonic if and only if it is pseudo-umbilical manifold, \( \nabla^\perp H = 0 \) and \( \text{tr}(sk) = 0 \).

Prove. By the given hypothesis for \( F \), we have \( f = 0 \) and \( t = 0 \).
Implementing the above conditions along with Eq 2.9 in Corollary 3.4 a), we get
\[
- \Delta^\perp H + \text{tr}(B(., A_H)) - \frac{1}{8} [(u + 2)|H|\text{tr}(sk)(hlH)] = 0.
\]
(3.26)

By taking the inner product with \( H \), we get
\[
- \langle \Delta^\perp H, H \rangle + |A_H|^2 - \frac{1}{8} [(u + 2)|H|^2 + \text{tr}(sk)(FJH, H))] = 0,
\]
(3.27)

where \( A_H \) is the shape operator associated with mean curvature vector \( H \).

Using Bochner formula, we get
\[
\frac{1}{8} (u + 2)|H|^2 = |A_H|^2 + |\nabla^\perp H|^2 + \frac{1}{8} \text{tr}(sk)(FJH, H)).
\]
(3.28)

By the Cauchy-Schwarz inequality, we have \( |A_H|^2 \geq u|H|^4 \). Using this fact, we have
\[
\frac{1}{8} (u + 2)|H|^2 \geq u|H|^4 + |\nabla^\perp H|^2 + \frac{1}{8} \text{tr}(sk)(FJH, H)) \geq u|H|^4.
\]
(3.29)

Since \( H \) is a non-zero constant, we have
\[ 0 < |H|^2 \leq \frac{u^2 + 2}{8u}. \]

If \( |H|^2 \leq \frac{u^2 + 2}{8u} \) and \( M \) is proper-biharmonic, all of the above inequalities become equalities. Thus, we have \( \nabla^\perp H|^2 = 0 \) and \( \text{tr}(sk) = 0 \) as \( FJ \) is an isometry. Since the Cauchy-Schwarz inequality becomes equality, we have \( M \) as pseudo-umbilical.

\[\Box\]
Remark 3.11. The cases for which $FX \in \Gamma(TM)$ and $FN \in \Gamma(TM^\perp)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$ establish the results comparable to those established in this paper. The proofs of all those results follow a similar procedure; thus, they haven’t been discussed here.

Finally, we discuss a non-existence case for the product of a unit sphere and a hyperbolic space. Out of all the discussed cases, the non-existence result can be found only for totally-complex Lagrangian submanifolds. Same has been discussed here:

**Proposition 3.12.** There doesn’t exist any proper biharmonic totally complex Lagrangian submanifold (dimension $\geq 2$) with parallel mean curvature in $S^2 \times H^{n-2}$ such that $FX \in \Gamma(TM^\perp)$ and $FN \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$.

**Proof.** Since mean curvature $H$ is parallel and not identically zero. Therefore, $FH$ isn’t zero identically. $M$ is trivially biharmonic, according to Theorem 3.1, we have

$$\frac{u}{2} \text{grad}|H|^2 + 2 \text{tr}(A\nabla H) + \frac{c_1 + c_2}{8} [f sH - \text{tr}(f)sH]$$

$$+ \frac{c_1 - c_2}{8} [sH - usH - 3(sH)] = 0.$$

For the above equation, we have $c_1 + c_2 = 0$ and $c_1 - c_2 = 2$,

or

$$\frac{u}{2} \text{grad}|H|^2 + 2 \text{tr}(A\nabla H) + \frac{1}{4} [-(u + 2)sH] = 0.$$

Using the hypothesis, we have $sH = 0$ or $FH = 0$, which isn’t possible. $\square$

4. Conclusions

We established the necessary and sufficient conditions for the submanifolds of Kaehler product manifolds to be biharmonic. And we derived the magnitude of scalar curvature for the hypersurfaces in a product of two unit spheres. Also, for the same product, the magnitude of the mean curvature vector for Lagrangian submanifolds has been estimated. Finally, we proved the non-existence condition for totally complex Lagrangian submanifolds in a product of unit sphere and a hyperbolic space.

**Conflict of interest**

The authors declare no conflict of interest.

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