Ewa Tyszkowska

TOPOLOGICAL CLASSIFICATION OF CONFORMAL ACTIONS ON $p$-HYPERELLIPTIC AND $(q, n)$-GONAL RIEMANN SURFACES

Abstract. A compact Riemann surface $X$ of genus $g > 1$ is said to be $p$-hyperelliptic if $X$ admits a conformal involution $\rho$ for which $X/\rho$ has genus $p$. A conformal automorphism $\delta$ of prime order $n$ such that $X/\delta$ has genus $q$ is called a $(q, n)$-gonal automorphism. Here we study conformal actions on $p$-hyperelliptic Riemann surface with $(q, n)$-gonal automorphism.

Keywords: $p$-hyperelliptic Riemann surface, automorphism of a Riemann surface.

Mathematics Subject Classification: Primary: 30F20, 30F50; Secondary: 14H37, 20H30, 20H10.

1. INTRODUCTION

A compact Riemann surface $X$ of genus $g \geq 2$ is said to be $p$-hyperelliptic if $X$ admits a conformal involution $\rho$, called a $p$-hyperelliptic involution, such that $X/\rho$ is an orbifold of genus $p$. This notion has been introduced by H. Farkas and I. Kra in [17] where they also proved that for $g > 4p + 1$, $p$-hyperelliptic involution is unique and central in the group of all automorphisms of $X$. In [22] it has been proved that every two $p$-hyperelliptic involutions commute for $3p + 2 \leq g \leq 4p + 1$ and $X$ admits at most two such involutions if $g > 3p + 1$.

In the particular cases $p = 0, 1$, $X$ are called hyperelliptic and elliptic-hyperelliptic Riemann surfaces respectively. Hyperelliptic Riemann surfaces and their automorphisms have received a good deal of attention in the literature. In [2] and [12] the authors determined the full groups of conformal automorphisms of such surfaces which made it possible to classify symmetry types of such actions in [5]. The $p$-hyperelliptic ($p \geq 1$) surfaces at large have been studied in [7–11,13–15] and [23], where the most attention has been paid to a study of groups of automorphisms of such surfaces and their symmetries.
We say that a finite group $G$ acts on a topological surface $X$ if there exists a monomorphism $\varepsilon : G \to \text{Hom}^+(X)$, where $\text{Hom}^+(X)$ is the group of orientation-preserving homeomorphisms of $X$. Two actions of finite groups $G$ and $G'$ on $X$ are topological equivalent if the images of $G$ and $G'$ are conjugate in $\text{Hom}^+(X)$. There are two reasons for the topological classification of finite actions rather than just the groups of homeomorphisms. First, the equivalence classes of group actions are in $1-1$ correspondence to conjugacy classes of finite subgroups of the mapping class group and so such a classification gives some information on the structure of this group. Second, the enumeration of finite group actions is a principal component of the analysis of singularities of the moduli space of conformal equivalence classes of Riemann surfaces of a given genus since such space is an orbit space of Teichmüller space by a natural action of the mapping class group, see [4].

The classification of conformal actions up to topological conjugacy is a classical problem, which has been considered for surfaces of genera $g = 2, 3$ in [3] and $g = 4$ in [1]. In the case $p$-hyperelliptic Riemann surfaces it has been studied in [24, 20] and [21] for $p = 0, 1$ and 2, respectively.

Here we study conformal actions on $p$-hyperelliptic Riemann surface $X$ which admits a conformal automorphism $\delta$ of prime order $n > 2$ such that $X/\delta$ has genus $q$ [18]. The automorphism $\delta$ is called the $(q, n)$-gonal automorphism and in the case $q = 0$, $n$-gonality automorphism. $(q, n)$-gonal Klein surfaces have been considered in [16].

2. PRELIMINARIES

We shall approach the problem using Riemann uniformization theorem by which each compact Riemann surface $X$ of genus $g \geq 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group $\Gamma$. Furthermore, a group $G$ of automorphisms of a surface $X = \mathcal{H}/\Gamma$ can be represented as $G = \Lambda/\Gamma$ for another Fuchsian group $\Lambda$. Each Fuchsian group $\Lambda$ is given a signature $\sigma(\Lambda) = (g; m_1, \ldots, m_r)$, where $g, m_i$ are integers verifying $g \geq 0, m_i \geq 2$. The $g = 0$ in signature will be omitted and $m_i = m$ repeated $r$-times will be written $m^r$. The signature determines the presentation of $\Lambda$:

- generators: $x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g$,
- relations: $x_1^{m_1} = \ldots = x_r^{m_r} = x_1 \ldots x_r [a_1, b_1] \ldots [a_g, b_g] = 1$.

Such set of generators is called the canonical set of generators and often, by abuse of language, the set of canonical generators. Geometrically $x_i$ are elliptic elements which correspond to hyperbolic rotations and the remaining generators are hyperbolic translations. The integers $m_1, m_2, \ldots, m_r$ are called the periods of $\Lambda$ and $g$ is the genus of the orbit space $\mathcal{H}/\Lambda$. Fuchsian groups with signatures $(g; -)$ are called surface groups and they are characterized among Fuchsian groups as these ones which are torsion free.

The group $\Lambda$ has associated to it a fundamental region whose area $\mu(\Lambda)$, called the area of the group, is:

$$\mu(\Lambda) = 2\pi \left( 2g - 2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) \right). \quad (2.1)$$


If $\Gamma$ is a subgroup of finite index in $\Lambda$, then we have the Riemann-Hurwitz formula which says that
\[ [\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}. \tag{2.2} \]

The number of fixed points of an automorphism of $X$ can be calculated by the following theorem of Macbeath [19].

**Theorem 2.1.** Let $X = H/\Gamma$ be a Riemann surface with the automorphism group $G = \Lambda/\Gamma$ and let $x_1, \ldots, x_r$ be elliptic canonical generators of $\Lambda$ with periods $m_1, \ldots, m_r$ respectively. Let $\theta : \Lambda \to G$ be the canonical epimorphism and for $1 \neq g \in G$ let $\varepsilon_i(g)$ be $1$ or $0$ according whether $g$ is or is not conjugate to a power of $\theta(x_i)$. Then the number $F(g)$ of points of $X$ fixed by $g$ is given by the formula
\[ F(g) = |N_G(g)| \sum_{i=1}^{r} \varepsilon_i(g)/m_i. \tag{2.3} \]

Let $G$ be a finite group acting on a surface $X$ of genus $g > 1$ such that the canonical projection $X \to X/G$ is ramified at $r$ points with multiplicities $m_1, \ldots, m_r$ and $s$ is the genus of $X/G$. Then a $(2s + r)$-tuple $(\tilde{a}_1, \ldots, \tilde{a}_s, b_1, \ldots, \tilde{b}_s, \tilde{x}_1, \ldots, \tilde{x}_r)$ of generators of $G$ such that $\tilde{x}_i$ has order $m_i$ for $i = 1, \ldots, r$, $\tilde{x}_1 \cdots \tilde{x}_r \prod_{i=1}^{s} [\tilde{a}_i, \tilde{b}_i] = 1$ and $2g - 2 = |G|(2s + \sum_{i=1}^{r} (1 - 1/m_i))$ is called a generating $(s; m_1, \ldots, m_r)$-vector.

For every generating $(s; m_1, \ldots, m_r)$-vector of $G$, there exists a Fuchsian group $\Lambda$ with the signature $(s; m_1, \ldots, m_r)$ and an epimorphism $\theta : \Lambda \to G$ defined by the assignment $\theta(a_i) = \tilde{a}_i, \theta(b_i) = \tilde{b}_i$ and $\theta(x_j) = \tilde{x}_j$. The kernel $\Gamma$ of $\theta$ is a surface Fuchsian group of orbit genus $g$ and $G$ acts as an automorphism group on a Riemann surface $X = \mathcal{H}/\Gamma$. If an involution $\rho$ appears in generating vector as an image of $k$ consecutive elliptic generators of $\Lambda$, then we shall write $\rho^{[k]}$ instead of $\rho, \ldots, \rho$.

3. **$p$-HYPERELLIPTIC RIEMANN SURFACE WITH $(q, n)$-GONAL AUTOMORPHISM**

In this section we study Riemann surfaces of genera $g > 1$ which are $p$-hyperelliptic and cyclic $(q, n)$-gonal simultaneously for a prime $n > 2$ and a natural $q$. If $g > 4p + 1$, then its $(q, n)$-gonal automorphism and $p$-hyperelliptic involution commute. The first theorem gives necessary and sufficient conditions on $p$ and $g$ for the existence of such surface.

**Theorem 3.1.** There exists a $p$-hyperelliptic Riemann surface of genus $g \geq 2$ admitting $(q, n)$-gonal automorphism commuting with a $p$-hyperelliptic involution if and only if $p = n\gamma + b(n - 1)/2$ and $g = nq + a(n - 1)/2$ for some integers $\gamma, b, a$ such that
\[ b = -2 \text{ or } b \geq 0, \quad b \leq a \leq 2(b + 1), \quad 0 \leq \gamma \leq (q + 1)/2. \tag{3.1} \]
Furthermore, the $(q, n)$-gonal automorphism admits $a + 2$ fixed points.
Proof. Assume that a Riemann surface $X = \mathcal{H}/\Gamma$ admits $p$-hyperelliptic involution $\rho$ and $(q, n)$-gonal automorphism $\delta$. The groups $\langle \delta \rangle$ and $\langle \rho \rangle$ can be identified with $\Gamma_\delta/\Gamma$ and $\Gamma_\rho/\Gamma$, where $\Gamma_\delta$ and $\Gamma_\rho$ are Fuchsian groups containing $\Gamma$ as a normal subgroup of index $n$ and 2, respectively. By the Riemann-Hurwitz formula they have signatures

$$\sigma(\Gamma_\delta) = (q; n, \ldots), \quad \sigma(\Gamma_\rho) = (p; 2, \ldots, 2),$$

where $s = 2g + 2 - 4p$ and $r = 2 + (2g - 2nq)/(n - 1)$. Thus $g = nq + a(n - 1)/2$ for $a = r - 2$. If $\rho$ and $\delta$ commute then they generate the group $\mathbb{Z}_{2n}$ which can be represented by $\Lambda/\Gamma$ for a Fuchsian group $\Lambda$ with the signature

$$\langle \gamma; 2, k_1, 2, n, k_2, n, 2n, k_3, 2n \rangle.$$

By the Riemann-Hurwitz formula

$$2g - 2 = 4n\gamma - 4n + nk_1 + 2k_2(n - 1) + k_3(2n - 1)$$

and according to Theorem 2.1

$$nk_1 = s - k_3, \quad 2k_2 = r - k_3.$$ 

By substituting the last equalities to (3.4), we obtain $p = n\gamma + b(n - 1)/2$, for an integer $b$ such that $a = 2b + 2 - k_3$. Thus

$$k_1 = 2q + a - 4\gamma - 2b, \quad k_2 = a - b, \quad k_3 = 2 + 2b - a$$

are nonnegative integers if and only if the inequalities (3.1) are satisfied.

Conversely, assume that $g = nq + a(n - 1)/2$ and $p = n\gamma + b(n - 1)/2$ for some integers $a, b$ and $\gamma$ satisfying the inequalities (3.1). Then there exists a Fuchsian group $\Lambda$ with the signature (3.3). Let $\theta : \Lambda \rightarrow \langle \rho \rangle \oplus \langle \delta \rangle$ be an epimorphism which maps all hyperbolic generators of $\Lambda$ onto $\rho\delta$, the first $k_1$ of elliptic generators onto $\rho$ and the remaining in the following way:

$$\delta \ldots \delta \delta^{-1} \ldots \delta^{-1} \delta^{-2} \rho \delta \ldots \rho \delta \rho^{-1} \ldots \rho^{-1} \rho^{-2} \text{ if } k_2 \equiv 1 \pmod{2} \text{ and } k_3 \equiv 1 \pmod{2},$$

$$\delta \ldots \delta \delta^{-1} \ldots \delta^{-1} \delta^{-2} \rho \delta \ldots \rho \delta \rho^{-1} \ldots \rho^{-1} \rho^{-2} \text{ if } k_2 \equiv 1 \pmod{2} \text{ and } k_3 \equiv 0 \pmod{2},$$

$$\delta \ldots \delta \delta^{-1} \ldots \delta^{-1} \rho \delta \ldots \rho \delta \rho^{-1} \ldots \rho^{-1} \rho^{-2} \text{ if } k_2 \equiv 0 \pmod{2} \text{ and } k_3 \equiv 1 \pmod{2},$$

$$\delta \ldots \delta \delta^{-1} \ldots \delta^{-1} \rho \delta \ldots \rho \delta \rho^{-1} \ldots \rho^{-1} \rho^{-2} \text{ if } k_2 \equiv 0 \pmod{2} \text{ and } k_3 \equiv 0 \pmod{2}.$$ 

Then the kernel of $\theta$ is a surface Fuchsian group $\Gamma$ of genus $g$ while $\theta^{-1}(\rho)$ and $\theta^{-1}(\delta)$ are Fuchsian groups with the signatures (3.2). Thus $\mathcal{H}/\Gamma$ is a $p$-hyperelliptic Riemann surface admitting $(q, n)$-gonal automorphism. It is easy to notice that for $k_2 < 3$ or $k_3 < 3$, such an epimorphism does not exist if and only if $k_2 + k_3 + \gamma = 0$ or $k_2 + k_3 = 1$. The first equality is never satisfied since if $k_2 + k_3 = 0$ then $b = -2$ and $p = n(\gamma - 1) + 1$ what requires $\gamma \geq 1$. The second one occurs for $b = -1$ and therefore this value of $b$ is rejected. 

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Corollary 3.2. Let $X$ be a $p$-hyperelliptic Riemann surface of genus $g > 4p + 1$. Then for any prime $n \geq 3$,

(i) $X$ can be realized as $n$-sheeted covering of the Riemann sphere if and only if $p = 0$ and $g = n - 1$ or $g = (n - 1)/2$ and its $n$-gonality automorphism admits 4 or 3 fixed points, respectively.

(ii) $X$ can be realized as $n$-sheeted covering of an elliptic curve if and only if $p = 0$ and $g \in \{2n - 1, (3n - 1)/2, n\}$ or $p = (n - 1)/2$ and $g \in \{3n - 2, (5n - 3)/2\}$ and its $(1, n)$-gonal automorphism admits 4, 3, 2 or 6, 5 fixed points, respectively.

Corollary 3.3. Let $X = \mathcal{H}/\Gamma$ be a Riemann surface of genus $g \geq 2$ which admits $p$-hyperelliptic involution $\rho$ and $(q, n)$-gonal automorphism $\delta$ for $p < n$. If $\delta$ and $\rho$ commute then $p = b(n - 1)/2$, $g = nq + a(n - 1)/2$ for integers $a, b$ in range $0 \leq b \leq 2$ and $b \leq a \leq 2b + 2$ and a Fuchsian group $\Lambda$ such that $\langle \delta, \rho \rangle = \Lambda/\Gamma$ has a signature $(0, 2, 2^{q+a-2}, 2, n, a-b, n, 2n, 2^{b+a-2}, 2n)$. Furthermore, $\delta$ admits $a + 2 \leq 8$ fixed points.

Theorem 3.4. All group actions on a $p$-hyperelliptic and cyclic $n$-gonal Riemann surface are given in Table 1, up to topological conjugacy; four of them correspond to the full automorphism groups: $2, b, 3, a, 3, b$ and $5, c$.

Proof. Let $X = \mathcal{H}/\Gamma$ be a $p$-hyperelliptic Riemann surface of genus $g \geq 2$ admitting a $n$-gonality automorphism $\delta$. Then by Corollary 3.2, $X$ is hyperelliptic, $\delta$ admits 4 or 3 fixed points and its order is one of two possible prime orders greater than $g$, namely $n = g + 1$ or $n = 2g + 1$, respectively. The automorphism groups of hyperelliptic Riemann surfaces are given in [12] and we need to chose those which admit an automorphism satisfying the above conditions. The action of finite group $G$ on $X$ is determined by the signature of a Fuchsian group $\Lambda$ and an epimorphism $\theta : \Lambda \to G$ with kernel $\Gamma$. Let $x_1, \ldots, x_r$ be all elliptic generators of $\Lambda$. An element of $\Lambda$ has a fixed point in $\mathcal{H}$ if and only if it has a finite order and it is conjugate to some power of precisely one of elliptic generators $x_i$. Consequently an element of $G$ has a fixed point in $X$ if and only if it is conjugate to some power of the image of $x_i$ via homomorphism $\theta$. Since $\theta$ preserves orders, it follows that the order $n$ of the $n$-gonality automorphism divides one of periods $m_i$ in the signature of $\Lambda$. First we chose all signatures corresponding to group actions on a hyperelliptic Riemann surface of genus $g$ for which $g + 1$ or $2g + 1$ divides one of its periods. The authors of [12] denoted by $t$ the number of periods 2 in the signature of $\Lambda$ which correspond to elliptic generators mapped by $\theta$ on the hyperelliptic involution and expressed $t$ in terms of the genus $g$ and the the number $N = |G|/2$. Let us consider for example $\sigma(\Lambda) = (2, \ldots, 2, 2, 3, 3)$ with $t = (g + 1)/6$. The number 3 is the only prime integer greater that 2 which divides a period of $\Lambda$. Thus $\delta$ has order 3 and so $g = 2$. However $t$ is not integer for $t = 2$ and therefore this signature is not suitable. In the similar way we reject the remaining signatures except:

\begin{align*}
2.a : \quad & \sigma(\Lambda) = (2, \ldots, 2, N, N), \quad t = (2g + 2)/N, \\
2.b : \quad & \sigma(\Lambda) = (2, \ldots, 2, N, 2N), \quad t = (2g + 1)/N,
\end{align*}
3.a : \[ \sigma (\Lambda) = (2, \ldots, 2, 2, 2, 2, N/2), \quad t = (2g + 2)/N, \]
3.b : \[ \sigma (\Lambda) = (2, \ldots, 2, 2, 4, N/2), \quad t = (2g + 2)/N - 1/2, \]
3.c : \[ \sigma (\Lambda) = (2, \ldots, 2, 4, 4, N/2), \quad t = (2g + 2)/N - 1, \]
4.d : \[ \sigma (\Lambda) = (2, \ldots, 2, 4, 3, 3), \quad t = (g - 2)/6, \]
5.c : \[ \sigma (\Lambda) = (2, \ldots, 2, 2, 3, 8), \quad t = (g - 2)/12, \]

In the case 2.a, \( G = \langle z : z^2 \rangle \oplus \langle x : x^N \rangle \) and \( z \) is the hyperelliptic involution. The order \( n \) of \( \delta \) divides a period of \( \Lambda \) if and only if \( n = g + 1 \) and \( N \) has one of values \( 2g+2 \) or \( g+1 \). Thus \( \langle \delta \rangle = \langle x^2 \rangle \) or \( \langle x \rangle \), respectively and we shall denote these two possibilities by 2.a and 2.a’ in Table 1. With the help of Macbeath’s theorem we check that in both cases \( \delta \) has 4 fixed points as required. Using the pair of automorphisms \( (\text{id}_\Lambda, \varphi) \), where \( \varphi (x) = xz \) and \( \varphi (z) = z \) if necessary, we can show that any generating vector is equivalent to \( v = (z, \ldots, z, xz^t, x^{-1}) \). A similar consideration of the all signatures listed above provides the remaining results in Table 1.

\[ \begin{array}{|c|c|c|c|}
\hline
\text{Case} & \sigma (\Lambda) & G = \Lambda/\Gamma \text{ of order } 2N & N \text{ gen. vector } \delta \\
\hline
2.a & [2, N, N] & \langle z : z^2 \rangle \oplus \langle x : x^N \rangle & 2g + 2 & (z, zx, x^{-1}) \quad x^2 \\
2.a’ & [2, 2, N, N] & \langle z : z^2 \rangle \oplus \langle x : x^N \rangle & g + 1 & (z, z, x, x^{-1}) \quad x \\
2.b & [2, N, 2N] & \langle x : x^{2N} \rangle & 2g + 1 & (x^N, x^2, x^{N-2}) \quad x^2 \\
3.a & [2, 2, 2, N] & \langle z : z^2 \rangle \oplus \langle x, y : x^2, y^2, (xy)^N \rangle & 4g + 1 & (xy^{-1}, x, y) \quad y^2 \\
3.b & [2, 4, N/2] & \langle x, y : x^4, y^{N/2}, (xy)^2, (x^{-1}y)^2 \rangle & 4g + 1 & (xy, x, y^{-1}) \quad xy \\
3.c & [2, 4, 4, N] & \langle x, y : x^4, x^2y^2, (xy)^N \rangle & g + 1 & (x, x, y^{-1}) \quad y \\
4.d & [3, 3, 3] & \langle x, y : x^4, y^3, (xy)^3, yx^2y^{-1}x^2 \rangle & 24 & (x, y, (xy)^{-1}) \quad xyyx \\
5.c & [2, 2, 3, 3] & \langle x, y : x^2, y^3, (xy)^4(xy)^2, (xy)^8 \rangle & 24 & (x, y, (xy)^{-1}) \quad xyyx \\
\hline
\end{array} \]

If the signature of \( \Lambda \) does not appear in the first column of the Tables 1.5.1 or 1.5.2 in [25] then \( \Lambda \) can be chosen to be maximal [25] and so \( G \) can be assumed to be the full group of automorphisms of \( X \). In the other case \( \Lambda \) is always contained in a Fuchsian group \( \Lambda' \) and the signature of of such a group is given in the second column of the corresponding row, what we shall denote by \( \sigma (\Lambda) \subset \sigma (\Lambda') \). By inspecting the signatures from Table 1 we obtain: \( [2, 2g + 2, 2g + 2] \subset [2, 4, 2g + 2], [2, 2, g + 1, g + 1] \subset [2, 2, 2, g + 1], [4, 4, g + 1] \subset [2, 4, 2g + 2], [4, 3, 3] \subset [2, 3, 8] \) and \( [2, N, 2N] \subset 2, 3, 2N \). In each of these cases except the last one, there exists a group \( G' \) acting on a hyperelliptic Riemann surface of genus \( g \), group embeddings \( i : \Lambda \hookrightarrow \Lambda' \), \( j : G \hookrightarrow G' \) and an epimorphism \( \theta' : \Lambda' \to G' \) such that \( [\Lambda' : \Lambda] = [G' : G] \) and \( \theta' i = j \theta \). In the last case the genus of a surface on which \( G' \) acts is different from \( g \). Consequently \( G \) is the full automorphism group of a hyperelliptic Riemann surface only in cases 2.b, 3.a, 3.b and 5.c.

Using Corollary 3.2, Macbeath’s theorem and group actions on hyperelliptic, elliptic-hyperelliptic and 2-hyperelliptic Riemann surfaces given, up to topological conjugacy, in [12, 20] and [21], we obtain the next theorems. Their proofs are similar to the previous one and so we omit them. \( \square \)
Theorem 3.5. A p-hyperelliptic Riemann surface of genus $g > 4p + 1$ can be realized as cyclic $3$-sheeted covering of an elliptic curve if and only if $p = 0$ and $g = 3, 4, 5$ or $p = 1$ and $g = 6, 7$ while the topologically non-equivalent group actions on such surfaces are listed in Table 2.

Theorem 3.6. A p-hyperelliptic Riemann surface of genus $g > 4p + 1$ can be realized as cyclic $5$-sheeted covering of an elliptic curve if and only if $p = 0$ and $g = 5, 7, 9$ or $p = 2$ and $g = 11, 13$ while the topologically non-equivalent group actions on such surfaces are listed in Table 3.

Theorem 3.7. For any prime $n > 5$, a hyperelliptic $(1,n)$-gonal Riemann surface has genus $2n - 1$, $(3n - 1)/2$ or $n$ and the finite group actions on such surfaces are given in Table 4.

| $g$ | $\sigma(\Lambda)$ | $G = \Lambda/\Gamma$ | gen. vector | $\rho$ | $\delta$ |
|-----|------------------|-----------------|-------------|-------|--------|
| 3   | $[2^2, 6^2]$    | $(x : x^6)$     | $(\rho^2, x, x^{-1})$ | $x^3$ | $x^2$  |
|     | $[2, 6^2]$      | $(z : z^2) \oplus (x, y : x^2, y^3, (xy)^3)$ | $(x, \delta \rho, (x \delta)^{-1} \rho)$ | $z$   | $y$    |
|     | $[2, 6, 4]$     | $(z : z^2) \oplus (x, y : x^2, y^3, (xy)^4)$ | $(x, \delta \rho, (x \delta)^{-1} \rho)$ | $z$   | $y$    |
|     | $[2, 12^2]$     | $(x : x^{12})$  | $(\rho, x^7, x^{-1})$ | $x^6$ | $x^4$  |
| 4   | $[2^2, 6]$      | $(x, y : x^2, y^3, (xy)^6)$ | $(\rho, x, y, \rho \delta)$ | $(xy)^3$ | $(xy)^2$ |
|     | $[2^2, 6]$      | $(x, y : x^2, y^9, (xy)^6)$ | $(\rho, x, y, \rho \delta)$ | $(xy)^3$ | $(xy)^2$ |
|     | $[2^4, 6]$      | $(x, y : x^2, y^6, (xy)^6)$ | $(x, (y x)^{-1}, y)$ | $x^2$ | $y^2$  |
|     | $[2^4, 6]$      | $(x, y : x^2, y^9, (xy)^6)$ | $(x, (y x)^{-1}, y)$ | $x^2$ | $y^2$  |
| 5   | $[2^3, 3]$      | $(x : x^6)$     | $(\rho^3, \delta, x)$ | $x^3$ | $x^2$  |
|     | $[2, 9, 18]$    | $(x : x^{18})$  | $(\rho, x^7, x^7)$ | $x^9$ | $x^6$  |
| 6   | $[2^3, 3^2, 6]$ | $(z : z^2) \oplus (c : c^3)$ | $(\rho^3, \delta, \delta^2, \rho \delta)$ | $z$   | $c$    |
|     | $[2, 4, 13, 12]$| $(c : c^{12})$  | $(\rho, c^3, \delta, \rho \delta)$ | $c^6$ | $c^4$  |
| 7   | $[2^3, 3^2, 6]$ | $(x, y, z, c : z^2, c^2, y^2, z, x^2 z, [x, y], [z, c])$ | $(c^3 x, c^2 y, c, c^3 y)$ | $z$   | $c^4$  |
|     | $[2^3, 3^2, 6]$ | $(z : z^2) \oplus (c : c^6)$ | $(\rho^2, c^3, \delta, y^2, \delta \rho)$ | $z$   | $c^6$  |
| 8   | $[2^3, 3^2, 6]$ | $(z : z^2) \oplus (c : c^3)$ | $(\rho^4, c^4, \delta, \delta^2)$ | $z$   | $c$    |
| 9   | $[2, 3^2, 6]$   | $(z : z^2) \oplus (y : y^3) \oplus (c : c^3)$ | $(\rho^4, \delta, c^2 y, \delta \rho)$ | $z$   | $c$    |
| 10  | $[2^2, 3^2, 12]$| $(x, y, c : c^{12})$ | $(c^3 x, c^2 y, c)$ | $c^6$ | $c^4$  |
| 11  | $[3^2, 6]$      | $(x, y, c, z : z^2, c^3, y^6, [x, y], [x, z], x^2 y^2, y xy^2, xy x^2 y^2)$ | $(c^4/3, c^2 y, c)$ | $z$   | $c$    |
Table 3. Actions on $p$-hyperelliptic cyclic $(1, 5)$-gonal Riemann surfaces

| $g$ | $\sigma(\Lambda)$ | $G = \Lambda/\Gamma$ | gen. vector | $\rho$ | $\delta$ |
|-----|-------------------|----------------------|-------------|--------|--------|
| 5   | $[2^2, 10^2]$    | $\langle x : x^{10} \rangle$ | $(\rho^{[2]}, x, x^{-1})$ | $x^5$ | $x^2$ |
|     | $[2, 20^2]$      | $\langle x : x^{20} \rangle$ | $(\rho, px, x^{-1})$ | $x^{10}$ | $x^4$ |
|     | $[2^3, 10]$      | $\langle x, y : x^2, y^2, (xy)^{10} \rangle$ | $(\rho, px, y, (xy)^{-1})$ | $(xy)^5$ | $(xy)^2$ |
|     | $[4^2, 10]$      | $\langle x, y : x^2y^5, y^{10}, x^{-1}xy \rangle$ | $(x, (yx)^{-1}, y)$ | $x^2$ | $y^2$ |
| 7   | $[2^3, 5, 10]$   | $\langle x : x^{10} \rangle$ | $(px, y, \delta^2 \rho)$ | $z$ | $(xy)^2$ |
|     | $[2, 15, 30]$    | $\langle x : x^{30} \rangle$ | $(\rho, x, x^2 \delta^2)$ | $x^{15}$ | $x^6$ |
| 9   | $[2^4, 5^3]$     | $\langle z : z^2 \rangle \oplus \langle x : x^5 \rangle$ | $(\rho^{[4]}, \delta, \delta^{-1})$ | $z$ | $x$ |
|     | $[2^2, 10^2]$    | $\langle z : z^2 \rangle \oplus \langle x : x^{10} \rangle$ | $(\rho^{[2]}, x, x^{-1})$ | $z$ | $x^2$ |
|     | $[2, 20^2]$      | $\langle z : z^2 \rangle \oplus \langle x : x^{20} \rangle$ | $(\rho, px^{-1}, x)$ | $z$ | $x^4$ |
|     | $[2^4, 5]$       | $\langle z : z^2 \rangle \oplus \langle x, y : x^2, y^2, (xy)^5 \rangle$ | $(\rho^{[2]}, x, y, \delta^{-1})$ | $z$ | $xy$ |
|     | $[2, 4^2, 5]$    | $\langle x, y : x^4, x^2y^2, (xy)^5 \rangle$ | $(\rho, px, y, \delta^{-1})$ | $x^2$ | $xy$ |
|     | $[4^2, 10]$      | $\langle x, x : x^4, x^2y^2, (xy)^{10} \rangle$ | $(x, (xy)^{-1})$ | $x^2$ | $(xy)^2$ |
|     | $[2, 6, 5]$      | $\langle z : z^2 \rangle \oplus \langle x, y : x^2, y^3, (xy)^3 \rangle$ | $(px, y, \rho, \delta^{-1})$ | $z$ | $xy$ |
| 11  | $[10, 5^2, 2^3]$ | $\langle z : z^2 \rangle \oplus \langle x : x^5 \rangle$ | $(\delta \rho, \delta, \delta^3, \rho^{[3]})$ | $z$ | $x$ |
|     | $[4, 5, 20, 2]$  | $\langle x : x^{20} \rangle$ | $(\delta x, \delta, x, \rho)$ | $x^{10}$ | $x^4$ |
| 13  | $[5^3, 2^4]$     | $\langle z : z^2 \rangle \oplus \langle x : x^5 \rangle$ | $(\delta^2, \delta, \delta^5, \rho^{[4]})$ | $z$ | $x$ |
|     | $[2, 5, 10, 2^2]$ | $\langle z : z^2 \rangle \oplus \langle x : x^{10} \rangle$ | $(\delta^2 x, \delta^2, x, \rho^{[2]})$ | $z$ | $x^2$ |

Table 4. Actions on a $p$-hyperelliptic cyclic $(1, n)$-gonal Riemann surface for $n > 5$

| $g$   | $\sigma(\Lambda)$ | $G = \Lambda/\Gamma$ | gen. vector | $\rho$ | $\delta$ |
|-------|-------------------|----------------------|-------------|--------|--------|
| $2n - 1$ | $[2^4, n^2]$    | $\langle z : z^2 \rangle \oplus \langle x : x^n \rangle$ | $(\rho^{[4]}, \delta, \delta^{-1})$ | $z$ | $x$ |
|       | $[2^2, (2n)^2]$  | $\langle z : z^2 \rangle \oplus \langle x : x^{2n} \rangle$ | $(\rho^{[2]}, x, x^{-1})$ | $z^2$ | $x^4$ |
|       | $[2, (4n)^2]$    | $\langle z : z^2 \rangle \oplus \langle x : x^{4n} \rangle$ | $(\rho, px^{-1}, x)$ | $z$ | $x^4$ |
|       | $[2^4, n]$       | $\langle z : z^2 \rangle \oplus \langle x, y : x^2, y^2, (xy)^n \rangle$ | $(\rho^{[2]}, x, y, \delta^{-1})$ | $z$ | $xy$ |
|       | $[4^2, 2n]$      | $\langle x, y : x^4, x^2y^2, (xy)^n \rangle$ | $(\rho, px, y, \delta^{-1})$ | $x^2$ | $xy$ |
|       | $[4^2, 2n]$      | $\langle x, y : x^4, x^2y^2, (xy)^{2n} \rangle$ | $(x, (xy)^{-1})$ | $x^2$ | $(xy)^2$ |
| $3n-1$ | $[2^3, n, 2n]$   | $\langle x : x^{2n} \rangle$ | $(\rho^{[3]}, \delta, x^n\delta^{-1})$ | $x^n$ | $x^2$ |
| $\frac{3n-1}{2}$ | $[2^3, n, 2n]$   | $\langle x : x^{2n} \rangle$ | $(\rho^{[3]}, x^n\delta^{-1})$ | $x^n$ | $x^2$ |
|       | $[2, 3n, 6n]$    | $\langle x : x^{6n} \rangle$ | $(\rho, x^2, px^{-2})$ | $x^{3n}$ | $x^6$ |
| $n$   | $[2^2, (2n)^2]$  | $\langle x : x^{2n} \rangle$ | $(\rho^{[2]}, x, x^{-1})$ | $x^n$ | $x^2$ |
|       | $[2, (4n)^2]$    | $\langle x : x^{4n} \rangle$ | $(\rho, px, x^{-1})$ | $x^{2n}$ | $x^4$ |
|       | $[2^3, 2n]$      | $\langle x, y : x^2, y^2, (xy)^{2n} \rangle$ | $(\rho, x, y, (xy)^{n-1})$ | $(xy)^n$ | $(xy)^2$ |
|       | $[4^2, 2n]$      | $\langle x, y : x^{2n}y^n, y^{2n}, x^{-1}xy \rangle$ | $(x, (xy)^{-1}, y)$ | $x^2$ | $y^2$ |

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Ewa Tyszkowska
ewa.tyszkowska@math.univ.gda.pl

University of Gdańsk
Institute of Mathematics
ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

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