SEMISTAR DIMENSION OF POLYNOMIAL RINGS AND PRÜFER-LIKE DOMAINS

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ABSTRACT. Let $D$ be an integral domain and $\star$ a semistar operation stable and of finite type on it. In this paper we define the semistar dimension (inequality) formula and discover their relations with strongly universally catenarian domains and $\star$-stably strong S-domains. As an application we give new characterizations of $\star$-quasi-Prüfer domains and UM$_t$ domains in terms of dimension inequality formula (and the notions of universally catenarian domain, stably strong S-domain, strong S-domain, and Jaffard domains). We also extend Arnold’s formula to the setting of semistar operations.

1. Insertion

All rings considered in this paper are (commutative integral) domains (with 1); throughout, $D$ denotes a domain with quotient field $K$. In [22], Okabe and Matsuda introduced the concept of a semistar operation. Let $D$ be an integral domain and $\star$ a semistar operation on $D$.

In [24] we defined and studied the $\tilde{\star}$-Jaffard domains and proved that every $\tilde{\star}$-Noetherian and $\star$-MD of finite $\tilde{\star}$-dimension is a $\tilde{\star}$-Jaffard domain. In [25] we defined and studied two subclasses of $\tilde{\star}$-Jaffard domains, namely the $\tilde{\star}$-stably strong S-domains and $\star$-universally catenarian domains and showed how these notions permit studies of $\star$-quasi-Prüfer domains in the spirit of earlier works on quasi-Prüfer domains. The next natural step is to seek a semistar analogue of dimension (inequality) formula [15]. In Section 2 of this paper we define the $\tilde{\star}$-dimension (inequality) formula and show that each $\tilde{\star}$-universally catenarian domain satisfies the $\tilde{\star}$-dimension formula and each $\tilde{\star}$-stably strong S-domain satisfies the $\tilde{\star}$-dimension inequality formula. In Section 3 we give new characterizations of $\star$-quasi-Prüfer domains and UM$_t$ domains in terms of the classical notions of dimension inequality formula, universally catenarian domain, stably strong S-domain, strong S-domain, and Jaffard domains. In the last section we extend Arnold’s formula to the setting of semistar operations (see Theorem [16]).

To facilitate the reading of the introduction and of the paper, we first review some basic facts on semistar operations. Denote by $\mathcal{F}(D)$ the set of all nonzero $D$-submodules of $K$, and by $\mathcal{F}(D)$ the set of all nonzero fractional ideals of $D$; i.e., $E \in \mathcal{F}(D)$ if $E \in \mathcal{F}(D)$ and there exists a nonzero element $r \in D$ with $rE \subseteq D$. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of $D$. Obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \mathcal{F}(D)$. As in [22], a semistar operation on $D$ is a map $\star : \mathcal{F}(D) \to \mathcal{F}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $x$.

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If \( \Delta = \emptyset \) is a set of prime ideals of a domain \( D \), then there is an associated semistar operation on \( D \), denoted by \( \star_{\Delta} \), defined as follows:

\[
E^{\star_{\Delta}} := \cap \{ ED_P | P \in \Delta \}, \quad \text{for each } E \in F(D).
\]

If \( \Delta = 0 \), let \( E^{\star_{\Delta}} := K \) for each \( E \in F(D) \). When \( \Delta := \text{QMax}^{*f} (D) \), we set \( \star := \star_{\Delta} \). This is the semistar operation of finite type if \( \star \) is of finite type. We say that a nonzero \( \Delta \) is a set of prime ideals of a domain \( D \) if \( I \cap \Delta = 1 \); a \( \star \)-prime (ideal of \( D \)), if \( I \) is a prime \( \star \)-ideal of \( D \); and a \( \star \)-maximal (ideal of \( D \)), if \( I \) is maximal in the set of all proper \( \star \)-ideals of \( D \). Each \( \star \)-maximal ideal of \( D \) is a \( \star \)-prime ideal. For each \( \star \)-prime \( P \) of \( D \), the \( \star \)-height of \( P \) (for short, \( \star \)-ht \( (P) \)) is defined to be the supremum of the lengths of the chains of \( \star \)-primes of \( D \), between prime ideal \( 0 \) (included) and \( P \). Obviously, if \( \star = d_D \) is the identity \( \star \)-operation on \( D \), then \( \star \)-ht \( (P) = \text{ht} (P) \), for each prime ideal \( P \) of \( D \). If the set of \( \star \)-primes of \( D \) is not empty, the \( \star \)-dimension of \( D \) is defined as follows:

\[
\star \text{-dim} (D) := \sup \{ \star \text{-ht} (P) | P \text{ is a } \star \text{-prime of } D \}.
\]

If the set of \( \star \)-primes of \( D \) is empty, then \( \star \text{-dim} (D) := 0 \). Thus, if \( \star = d_D \), then \( \star \text{-dim} (D) = \text{dim} (D) \), the usual (Krull) dimension of \( D \). It is known (see [12] Lemma 2.11) that

\[
\bar{\star} \text{-dim} (D) = \sup \{ \text{ht} (P) | P \text{ is a quasi-}\bar{\star} \text{-prime ideal of } D \}.
\]

Let \( \star \) be a semistar operation on a domain \( D \). Recall from [12] Section 3 that \( D \) is said to be a \( \star \)-Noetherian domain, if \( D \) satisfies the ascending chain condition on \( \star \)-ideals. Also recall from [16] that, \( D \) is called a \( \star \)-Prüfer domain (for short, a \( \star \)-MD) if each \( \star \)-invertible ideal of \( D \) is \( \star \)-invertible; i.e., if \( (II^{-1})^{\star} = D^* \) for all \( I \in f(D) \). When \( \star = v \), we recover the classical notion of \( \text{PM} \); when \( \star = d_D \), the identity \( \star \)-operation, we recover the notion of \( \text{Prüfer} \) domain. Finally recall from [7] that \( D \) is said to be a \( \star \)-Prüfer domain, in case, if \( Q \) is a prime ideal in \( D[X] \), and \( Q \subseteq P[X] \), for some \( P \in \text{QSpec}^* (D) \), then \( Q = (Q \cap D)[X] \). This notion is the semistar analogue of the classical notion of the \( \star \)-Prüfer domains. By [7] Corollary 2.4, \( D \) is a \( \star \)-Prüfer domain if and only if \( D \) is a \( \bar{\star} \)-Prüfer domain.
2. The $\ast$-dimension (inequality) formula

We begin with the following definition. Recall that if $D \subseteq T$ are domains, then $\text{tr. deg}_D(T)$ is defined as the transcendence degree of the quotient field of $T$ over the quotient field of $D$. If $P$ is a prime ideal of $D$, then $\mathbb{K}(P)$ is denoted the residue field of $D$ in $P$, i.e., $D_P/PD_P$, which is canonically isomorphic to the field of quotients of the integral domain $D/P$.

**Definition 2.1.** Let $D \subseteq T$ be an extension of domain and $\ast$ and $\ast'$ are semistar operation on $D$ and $T$ respectively. We say that $D \subseteq T$ satisfies the $(\ast, \ast')$-dimension formula (resp. $(\ast, \ast')$-dimension inequality formula) if for all $Q \in \text{QSpec}^{\ast'}(T)$ such that $(Q \cap D)^\ast \subseteq D^\ast$, $\ast'$-ht$(Q) + \text{tr. deg}_{\mathbb{K}(Q)}(\mathbb{K}(Q)) = \ast$-ht$(Q \cap D) + \text{tr. deg}_D(T)$.

(3) \hspace{1cm} $D_M$ satisfy the dimension formula for each $M \in \text{QMax}^{\ast}(D)$ (resp. dimension inequality formula).

**Proposition 2.2.** Let $D$ be a domain and $\ast$ a semistar operation on $D$. Then the following conditions are equivalent:

1. $D$ satisfy the $\tilde{\ast}$-dimension formula (resp. $\tilde{\ast}$-dimension inequality formula);
2. $D_P$ satisfy the dimension formula for each $P \in \text{QSpec}^{\ast}(D)$ (resp. dimension inequality formula);
3. $D_M$ satisfy the dimension formula for each $M \in \text{QMax}^{\ast}(D)$ (resp. dimension inequality formula).

**Proof.** We only prove the case of dimension formula and the other case is the same.

1. $\Rightarrow$ (2) Let $P \in \text{QSpec}^{\ast}(D)$. Let $T$ be a finitely generated domain over $D_P$. So that there exist finitely many elements $\theta_1, \ldots, \theta_r \in T$ such that $T = D_P[\theta_1, \ldots, \theta_r]$. Set $T' = D[\theta_1, \ldots, \theta_r]$. Then $T = T'_{D/P}$ and $T'$ is a finitely generated domain over $D$. Let $Q$ be a prime ideal of $T$ and set $qD_P := Q \cap D_P$, where $q(\subseteq P)$ be a prime ideal of $D$. Thus there exists a prime ideal $Q'$ of $T'$ such that $Q' \cap (D \setminus P) = \emptyset$ and $Q' = Q\cap T'_{D/P}$. Thus $Q' \cap D = q$. Since $q \subseteq P$, we have $q$ is a quasi-$\tilde{\ast}$-prime ideal of $D$. Since $\tilde{\ast}$-ht$(q) = \text{ht}(q)$, then by the hypothesis we have:

$$\text{ht}(Q') + \text{tr. deg}_{\mathbb{K}(q)}(\mathbb{K}(Q')) = \text{ht}(q) + \text{tr. deg}_D(T').$$

Since $\text{ht}(Q') = \text{ht}(Q)$ we see that

$$\text{ht}(Q) + \text{tr. deg}_{\mathbb{K}(q)}(\mathbb{K}(Q)) = \text{ht}(q) + \text{tr. deg}_{D_P}(T).$$

2. $\Rightarrow$ (3) is trivial.

3. $\iff$ (1) Suppose that $T$ is a finitely generated domain over $D$. Let $Q \in \text{Spec}(T)$ and set $P := Q \cap D$ such that $P^\ast \subseteq D^\ast$. Thus $P \in \text{QSpec}^{\ast}(D) \cup \{0\}$. Let $M$ be a quasi-$\tilde{\ast}$-maximal ideal of $D$ containing $P$. Note that $T_{D \setminus M}$ is a finitely generated domain over $D_M$ and that $Q \cap (D \setminus M) \neq \emptyset$. Thus $Q T_{D \setminus M} \in \text{Spec}(T_{D \setminus M})$ and $PD_M = Q T_{D \setminus M} \cap D_M$. Therefore by the (3), we have

$$\text{ht}(Q T_{D \setminus M}) + \text{tr. deg}_{\mathbb{K}(PD_M)}(\mathbb{K}(Q T_{D \setminus M})) = \text{ht}(PD_M) + \text{tr. deg}_{D_M}(T_{D \setminus M}).$$
Now since \( \tilde{\text{ht}}(Q) = \text{ht}(Q) = \text{ht}(QT_{D,M}) \), \( \text{ht}(P) = \text{ht}(PD_M) \), \( \text{tr.deg}_{K(P)}(K(Q)) = \text{tr.deg}_{K(D_M)}(K(QT_{D,M})) \) and \( \text{tr.deg}_D(T) = \text{tr.deg}_{D_M}(T_{D,M}) \) the proof is complete.

Let \( D \) be an integral domain with quotient field \( K \), let \( X, Y \) be two indeterminates over \( D \) and let \( * \) be a semistar operation on \( D \). Set \( D_1 := D[X], K_1 := K(X) \) and take the following subset of \( \text{Spec}(D_1) \):

\[
\Theta_1^* := \{ Q_1 \in \text{Spec}(D_1) | Q_1 \cap D = (0) \text{ or } (Q_1 \cap D)^* \subseteq D^* \}.
\]

Set \( \Theta_1^* : = S(\Theta_1^*) := D_1[Y] \setminus (\bigcup \{ Q_1[Y] | Q_1 \in \Theta_1^* \}) \) and:

\[
E^{\Theta_1^*} := E[Y]_{\Theta_1^*} \cap K_1, \text{ for all } E \in \overline{\mathcal{F}}(D_1).
\]

It is proved in \cite{24} Theorem 2.1] that the mapping \( *[X]\) := \( \cap_{E^{\Theta_1^*}} \mathcal{F}(D_1) \rightarrow \mathcal{F}(D_1), \)

\[
E \mapsto E*[X]
\]

is a stable semistar operation of finite type on \( D[X] \), i.e., \( *[X] = *[Y]. \)

It is also proved that \( \tilde{\star}[X] = \star[X] = \tilde{\star}[X], \) \( d_D[X] = d_D[X] \) and \( Q\text{Spec}^{\tilde{\star}}(D[X]) = \Theta_1^* \setminus \{0\} \).

If \( X_1, \ldots, X_r \) are indeterminates over \( D, \) for \( r \geq 2, \) we let

\[
\star[X_1, \ldots, X_r] := \{ \star[X_1, \ldots, X_{r-1}] \}[X_r],
\]

where \( \star[X_1, \ldots, X_{r-1}] \) is a stable semistar operation of finite type on \( D[X_1, \ldots, X_{r-1}] \).

For an integer \( r, \) put \( \star[r] \) to denote \( \star[X_1, \ldots, X_r] \) and \( D[r] \) to denote \( D[X_1, \ldots, X_r] \).

Following \cite{24}, the domain \( D \) is called \( \star \)-catenary, if for each pair \( P \subset Q \) of quasi-\( \star \)-prime ideals of \( D, \) any two saturated chain of quasi-\( \star \)-prime ideals between \( P \) and \( Q \) have the same finite length. If for each \( n \geq 1, \) the polynomial ring \( D[n] \) is \( \star[n] \)-catenary, then \( D \) is said to be \( \star \)-universally catenarian. Every \( \star \)-MD which is \( \tilde{\star} \)-LFD (that is \( \text{ht}(P) < \infty \) for all \( P \in \text{QSpec}^{\tilde{\star}}(D) \)), is \( \tilde{\star} \)-universally catenarian by \cite{25} Theorem 3.4.

**Corollary 2.3.** Let \( D \) be an \( \tilde{\star} \)-universally catenarian domain. Then \( D \) satisfies the \( \tilde{\star} \)-dimension formula.

**Proof.** Let \( P \in \text{QSpec}^{\tilde{\star}}(D). \) Hence \( D_P \) is a universally catenarian domain by \cite{25} Lemma 3.3. Thus by \cite{24} Corollary 4.8, \( D_P \) satisfies the dimension formula. Now Proposition 2.2 completes the proof. \( \square \)

The domain \( D \) is called a \( \star \)-strong \( S \)-domain, if each pair of adjacent quasi-\( \star \)-prime ideals \( P_1 \subset P_2 \) of \( D \), extend to a pair of adjacent quasi-\( \star \)[X]-prime ideals \( P_1[X] \subset P_2[X] \) of \( D[X] \). If for each \( n \geq 1, \) the polynomial ring \( D[n] \) is \( \star[n] \)-strong \( S \)-domain, then \( D \) is said to be an \( \star \)-stably strong \( S \)-domain. Every \( \tilde{\star} \)-Noetherian, \( \tilde{\star} \)-quasi-Prüfer or \( \tilde{\star} \)-universally catenarian domain is \( \star \)-stably strong \( S \)-domain by Corollaries 2.6 and 3.6.

**Corollary 2.4.** Let \( D \) be an \( \tilde{\star} \)-stably strong \( S \)-domain. Then \( D \) satisfies the \( \tilde{\star} \)-dimension inequality formula.

**Proof.** Use \cite{24} Proposition 2.5 and \cite{20} Theorem 1.6 and the same argument as proof of Corollary 2.3 \( \square \)

A valuation overring \( V \) of \( D \) is called a \( \star \)-valuation overring of \( D \) provided \( F^* \subseteq FV, \) for each \( F \in f(D) \). Following \cite{24}, the \( \star \)-valuative dimension of \( D \) is defined as:

\[
\star\text{-dim}_V(D) := \sup\{\text{dim}(V) | V \text{ is } \star \text{-valuation overring of } D\}.
\]
Although Example 4.4 of [24] shows that \( \star \)-dim \((D)\) is not always less than or equal to \( \star \)-dim \(\overline{D}\), but it is observed in [24] that \( \overline{\star} \)-dim \(D\) \(\leq\) \( \star \)-dim \(\overline{D}\). We say that \(D\) is a \( \overline{\star} \)-Jaffard domain, if \( \overline{\star} \)-dim \(D\) \(=\) \( \star \)-dim \(\overline{D}\) \(<\) \(\infty\). When \(\star = d\) the identity operation then \(d\)-Jaffard domain coincides with the classical Jaffard domain (cf. [1]). It is proved in [24], that \(D\) is a \( \overline{\star} \)-Jaffard domain if and only if
\[
\star [X_1, \cdots, X_n] \cdot \dim(D[X_1, \cdots, X_n]) = \overline{\star} \text{-dim}(D) + n,
\]
for each positive integer \(n\).

**Lemma 2.5.** For each domain \(D\), \(\overline{\star} \text{-dim}_v(D) = \sup\{\dim_v(D_P) | P \in \text{QSpec}^\overline{\star}(D)\}\).

**Proof.** We can assume that \(\overline{\star} \text{-dim}_v(D)\) is a finite number. Suppose that \(n = \overline{\star} \text{-dim}_v(D)\). Then there exists a \(\overline{\star}\)-valuation overring \(V\), with maximal ideal \(N\), of \(D\) such that \(\dim(V) = n\). Set \(P := N \cap D\). So that \(V\) is a valuation overring of \(D_P\). Hence \(n = \dim(V) \leq \dim_v(D_P) \leq \overline{\star} \text{-dim}_v(D) = n\), where the second inequality is true since each valuation overring of \(D_P\) is a \(\overline{\star}\)-valuation overring of \(D\) ([17 Theorem 3.9]).

In [1 Page 174] it is proved that a finite-dimensional domain satisfying the dimension inequality formula is a Jaffard domain. In the following result we give the semistar analogue of the mentioned result.

**Theorem 2.6.** Let \(D\) be a domain of finite \(\overline{\star}\)-dimension. If \(D\) satisfies the \(\overline{\star}\)-dimension inequality formula, then \(D\) is a \( \overline{\star} \)-Jaffard domain.

**Proof.** Let \(P \in \text{QSpec}^\overline{\star}(D)\). Then \(D_P\) is a finite dimensional domain and satisfies the dimension inequality formula by Proposition 2.2. Consequently \(D_P\) is a Jaffard domain by [1]. Thus using Lemma 2.5 we have
\[
\overline{\star} \text{-dim}(D) = \sup\{\dim(D_P) | P \in \text{QSpec}^\overline{\star}(D)\}
\]
\[
= \sup\{\dim_v(D_P) | P \in \text{QSpec}^\overline{\star}(D)\}
\]
\[
= \overline{\star} \text{-dim}_v(D).
\]
Thus \(D\) is a \( \overline{\star} \)-Jaffard domain. \(\square\)

Therefore we have the following implications for finite \(\overline{\star}\)-dimensional domains:

\[
\begin{align*}
\overline{\star}\text{-Noetherian or } \overline{\star}\text{-quasi-Prüfer} &\quad \Downarrow \\
\overline{\star}\text{-stably strong S-domain} &\quad \Downarrow \\
\overline{\star}\text{-dimension inequality formula} &\quad \Downarrow \\
\overline{\star}\text{-Jaffard} &\quad \Downarrow \\
\overline{\star}\text{-MD} &\quad \Downarrow \\
\overline{\star}\text{-quasi-Prüfer} &\quad \Downarrow \\
\overline{\star}\text{-dimension formula} &\quad \Downarrow \\
\end{align*}
\]

Let \(D\) be a domain with quotient field \(K\), let \(X\) be an indeterminate over \(D\), let \(\star\) be a semistar operation on \(D\), and let \(P\) be a quasi-\(\star\)-prime ideal of \(D\) (or \(P = 0\)). Set
\[
S_P := (D/P)[X] \setminus \bigcup \{[(Q/P)[X] | Q \in \text{QSpec}^\overline{\star}(D) \text{ and } P \subseteq Q}\}.
\]
Clearly, \(S_P\) is a multiplicatively closed subset of \((D/P)[X]\).

For all \(E \in \mathcal{F}(D/P)\), set
\[
E^C S_P := E(D/P)[X] S_P \cap (D_P/PD_P).
\]
It is proved in [10] Theorem 3.2] that the mapping $\star/P := \mathcal{F}(D/P) \to \mathcal{F}(D/P)$, $E \mapsto E^{\mathcal{F}P}$, is a stable semistar operation of finite type on $D/P$; i.e., $\star/P = \star/P$, $Q\text{Max}^{\mathcal{F}P}(D/P) = \{Q/P \in \text{Spec}(D/P) \mid Q \in Q\text{Max}^{\mathcal{F}f}(D)$ and $P \subseteq Q\}, \mathcal{F}P = \mathcal{F}_f/P = \mathcal{F}_f$ and $d_P/P = d_{D/P}$.

**Lemma 2.7.** A domain $D$ is $\mathfrak{f}$-universally catenarian if and only if $D/P$ is $(\star/P)$-universally catenarian for each $P \in \text{QSpec}(D)$.

**Proof.** $(\Rightarrow)$ Let $P \in \text{QSpec}^\mathfrak{f}(D)$. By [10] Theorem 3.2 (a), $\star/P = \mathcal{F}_f/P$. Hence, by [25] Proposition 3.2 and Lemma 3.3, $D/P$ is $(\mathcal{F}_f/P)$-universally catenarian if and only if $(D/P)_M$ is a universally catenarian domain for each $M \in Q\text{Max}^{\mathcal{F}f}(D/P)$, that is (by [10] Theorem 3.2 (b)], if and only if $D_M/PD_M$ is a universally catenarian domain whenever $P$ is a subset of $M \in Q\text{Max}^\mathfrak{f}(D)$. But by [25] Proposition 3.2 and Lemma 3.3, $D_M$ is a universally catenarian domain for all $M \in Q\text{Max}^\mathfrak{f}(D)$. This, in turn, is immediate since any factor domain of a universally catenarian domain must be a universally catenarian domain.

$(\Leftarrow)$ It is enough to consider $P = 0$, since we have $\star/0 = \mathfrak{f}$. \hfill $\square$

In [21] Corollary 14.D it is proved that a Noetherian domain $D$ is an universally catenarian domain if and only if $D$ is catenary and $D/P$ satisfies the dimension formula for each $P \in \text{Spec}(D)$. In the following result we give the semistar analogue of this result.

**Theorem 2.8.** Let $D$ be a $\mathfrak{f}$-Noetherian domain. Then $D$ is an $\mathfrak{f}$-universally catenarian domain if and only if $D$ is $\mathfrak{f}$-catenary and $D/P$ satisfies the $(\star/P)$-dimension formula for each $P \in \text{QSpec}(D)$.

**Proof.** $(\Rightarrow)$ Let $P \in \text{QSpec}(D)$. Then $D/P$ is $(\mathcal{F}_f/P)$-universally catenarian by Lemma 2.7. Hence $D/P$ satisfies the $(\mathcal{F}_f/P)$-dimension formula by Corollary 2.3.

$(\Leftarrow)$ Let $M \in Q\text{Max}^\mathfrak{f}(D)$. It is enough to show that $D_M$ is a universally catenarian domain. To this end let $PD_M$ be a prime ideal of $D_M$. Thus $P$ is a quasi-$\mathcal{F}$-prime ideal of $D$. Since $D/P$ satisfies the $(\mathcal{F}_f/P)$-dimension formula, then $(D/P)_M/P = D_M/PD_M$ satisfies the dimension formula by Theorem 2.2. On the other hand $D_M$ is a Noetherian domain by [12] Proposition 3.8 and catenary by [25] Proposition 3.2. Consequently $D_M$ is a universally catenarian domain by [21] Corollary 14.D]. \hfill $\square$

Recall that the celebrated theorem of Ratliff [23] Theorem 2.6] says that a Noetherian ring $R$ is universally catenarian if and only if $R[X]$ is catenarian. On the other hand it is proved in [6] Theorem 1] that the Noetherian assumption in Ratliff’s theorem can be replaced with going-down condition by proving that: for a going-down domain $D$, we have $D$ is universally catenarian if and only if $D[X]$ is catenarian if and only if $D$ is an LFD strong S-domain. As a semistar analogue in [25] Theorem 3.7 we proved that: suppose that $D$ is $\mathfrak{f}$-Noetherian. Then $D[X]$ is $\mathfrak{f}[X]$-catenary if and only if $D$ is $\mathfrak{f}$-universally catenarian. In the last theorem of this section we treat the second case.

Let $D \subseteq T$ be an extension of domains. Let $\mathcal{F}$ and $\mathcal{F}'$ be semistar operations on $D$ and $T$, respectively. Following [9], we say that $D \subseteq T$ satisfies $(\mathcal{F}, \mathcal{F}')$-GD if, whenever $P_0 \subset P$ are quasi-$\mathcal{F}$-prime ideals of $D$ and $Q$ is a quasi-$\mathcal{F}'$-prime ideal of $T$ such that $Q \cap D = P$, there exists a quasi-$\mathcal{F}'$-prime ideal $Q_0$ of $T$ such that $Q_0 \subseteq Q$.
and \( Q_0 \cap D = P_0 \). The integral domain \( D \) is said to be a \(*\)-going-down domain (for short, a \(*\)-GD domain) if, for every overring \( T \) of \( D \) the extension \( D \subseteq T \) satisfies \((*, d_T)\)-GD. These concepts are the semistar versions of the “classical” concepts of going-down property and the going-down domains (cf. [8]). It is known by [9, Propositions 3.5 and 3.2(e)] that every \( P\)-MD and every integral domain \( D \) with \(*\)-dim\((D) = 1 \) is a \(*\)-GD domain.

**Theorem 2.9.** Let \( D \) be a \( \sim \)-GD domain. The following statements are equivalent:

1. \( D \) is a \( \sim \)-LFD \( \sim \)-strong S-domain.
2. \( D \) is \(*\)-universally catenarian.
3. \( D[X] \) is \( \sim [X] \)-catenarian.

*Proof.* (1) \( \Rightarrow \) (2) holds by [25, Theorem 4.1] and (2) \( \Rightarrow \) (3) is trivial. For (3) \( \Rightarrow \) (1) let \( P \in \text{QSpec}^\sim(D) \). Then \( D_P \) is a going-down domain by [10, Proposition 2.5] and \( D_P[X] \) is catenarian by [24, Lemma 3.3]. Thus \( D_P \) is a LFD strong S-domain by [6, Theorem 1]. Hence \( D \) is a \( \sim \)-LFD \( \sim \)-strong S-domain by [24, Proposition 2.4]. \( \square \)

### 3. Characterizations of \(*\)-quasi-Prüfer domains

In this section we give some characterization of \( \sim \)-quasi-Prüfer domains. We need to recall the definition of \((*,*,')\)-linked overrings. Let \( D \) be a domain and \( T \) an overring of \( D \). Let \( \sim \) and \( \sim' \) be semistar operations on \( D \) and \( T \), respectively. One says that \( T \) is \((*,*,')\)-linked to \( D \) (or that \( T \) is a \((*,*,')\)-linked overring of \( D \)) if \( F^* = D^* \Rightarrow (FT)^{*'} = T^{*'} \), when \( F \) is a nonzero finitely generated ideal of \( D \) (cf. [11]). In particular we are interested in the case \( *' = d_T \). We first recall the following characterization of \( \sim \)-quasi-Prüfer domains.

**Theorem 3.1.** ([25, Theorem 4.3]) Let \( D \) be an integral domain. Suppose that \( \sim\text{-dim}(D) \) is finite. Consider the following statements:

1. Each \((*,*,')\)-linked overring \( T \) of \( D \) is an \( \sim' \)-universally catenarian domain.
2. Each \((*,*,')\)-linked overring \( T \) of \( D \) is an \( \sim' \)-stably strong S-domain.
3. Each \((*,*,')\)-linked overring \( T \) of \( D \) is an \( \sim' \)-Jaffard domain.
4. Each \((*,*,')\)-linked overring \( T \) of \( D \) is an \( \sim' \)-quasi-Prüfer domain.
5. \( D \) is an \( \sim \)-quasi-Prüfer domain.

Then (1') \( \Rightarrow \) (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4) \( \Leftrightarrow \) (5).

*Proof.* The implication (1') \( \Rightarrow \) (1) holds by [25, Corollary 3.6] and (1) \( \Rightarrow \) (2) is trivial. For (2) \( \Rightarrow \) (5) see proof of [25, Theorem 4.3] part (3) \( \Rightarrow \) (6). The implication (5) \( \Rightarrow \) (1) holds by [25, Corollary 2.6]. For (4) \( \Leftrightarrow \) (5) \( \Leftrightarrow \) (6) see [24, Theorem 4.14]. \( \square \)

Now we have the following theorem; a result reminiscent of the well-known result of Ayache, Cahen and Echi [4] (see also [15, Theorem 6.7.8]) for quasi-Prüfer domains.

**Theorem 3.2.** Let \( D \) be an integral domain. Suppose that \( \sim\text{-dim}(D) \) is finite. Then the following statements are equivalent:

1. Each \((*,d_T)\)-linked overring \( T \) of \( D \) is a stably strong S-domain.
2. Each \((*,d_T)\)-linked overring \( T \) of \( D \) is a strong S-domain.
3. Each \((*,d_T)\)-linked overring \( T \) of \( D \) is a Jaffard domain.
(4) Each \((\ast, d_T)\)-linked overring \(T\) of \(D\) is a quasi-Prüfer domain.

(5) \(D\) is an \(\tilde{T}\)-quasi-Prüfer domain.

Proof. We only prove the equivalence of \((1) \iff (5)\) and the proofs of \((2) \iff (5)\), \((3) \iff (5)\), and \((4) \iff (5)\), are the same. The implication \((5) \implies (1)\) holds by Theorem 3.11. For \((1) \implies (5)\) let \(P \in \text{QSpec}^{\ast}(D)\). It is enough for us to show that \(D_P\) is a quasi-Prüfer domain by [7, Theorem 2.16]. To this end let \(T\) be an overring of \(D_P\). Then \(T_{D_P} = T\) and therefore \(T\) is \((\ast, d_T)\)-linked overring of \(D\) by [11, Example 3.4 (1)]. Thus by the hypothesis we have \(T\) is a stably strong \(S\)-domain. Therefore \(D_P\) is a quasi-Prüfer domain by [15, Theorem 6.7.8].

Theorem 3.3. Let \(D\) be an integral domain. Suppose that \(\tilde{T}\text{-dim}(D)\) is finite. Then the following statements are equivalent:

1. \(D\) is an \(\tilde{T}\)-quasi-Prüfer domain.
2. For each \((\ast, \ast')\)-linked overring \(T\) of \(D\), every extension of domains \(T \subseteq S\), satisfies the \((\ast', \ast'')\)-dimension inequality formula, where \(\ast'\) and \(\ast''\) are semistar operations on \(T\) and \(S\) respectively.

Proof. \((1) \implies (2)\) If \(D\) is an \(\tilde{T}\)-quasi-Prüfer domain and \(T\) is \((\ast, \ast')\)-linked to \(D\), then \(T\) is a \(\ast'\)-Jaffard domain by [24, Theorem 4.14]. Let \(Q \in \text{QSpec}^{\ast'}(S)\) such that \((Q \cap T)^{\ast'} \subseteq T^{\ast'}\) and set \(q := Q \cap T\). Then we have \(q \in \text{QSpec}^{\ast'}(T) \cup \{0\}\). Set \(P := q \cap D\). Thus we have \(P \in \text{QSpec}^{\ast'}(D) \cup \{0\}\). Therefore \(D_P\) and hence \(T_q\), are quasi-Prüfer domains by [7, Theorem 1.1]. In particular \(T_q\) is a Jaffard domain. So that we have

\[
\dim(S_Q) + \text{tr. deg}_{\mathbb{K}(Q)}(\mathbb{K}(Q)) \leq \dim_v(S_Q) + \text{tr. deg}_{\mathbb{K}(Q)}(\mathbb{K}(Q)) \\
\leq \dim_v(T_q) + \text{tr. deg}_{\mathbb{K}(T)}(S),
\]

where the first inequality holds since \(\dim(S_Q) \leq \dim_v(S_Q)\) and the second one is by [15, Lemma 6.7.3]. The conclusion follows easily from the fact that \(\dim(T_q) = \dim_v(T_q)\).

\((2) \implies (1)\) Let \(T\) be an overring of \(D\) and \(\ast'\) be a semistar operation on \(T\) such that \(T\) is \((\ast, \ast')\)-linked to \(D\). Let \((V, N)\) be any \(\ast'\)-valuation overring of \(T\). Then \(V\) is \((\ast', d_T)\)-linked to \(T\) by [12, Lemma 2.7]. Set \(Q := N \cap T\). Then by assumption we have

\[
\dim(V) \leq \dim(T_Q) - \text{tr. deg}_{\mathbb{K}(Q)}(\mathbb{K}(N)).
\]

In particular \(\dim(V) \leq \dim(T_Q) \leq \ast'-\dim(T)\), and hence \(\ast'-\dim_v(T) = \ast'-\dim(T)\), that is \(T\) is a \(\ast'\)-Jaffard domain. Thus \(D\) is an \(\tilde{T}\)-quasi-Prüfer domain by [24, Theorem 4.14].

Corollary 3.4. Let \(D\) be an integral domain. Suppose that \(\tilde{T}\text{-dim}(D)\) is finite. Then the following statements are equivalent:

1. \(D\) is an \(\tilde{T}\)-quasi-Prüfer domain.
2. For each \((\ast, d_T)\)-linked overring \(T\) of \(D\), every extension of domains \(T \subseteq S\), satisfies the dimension inequality formula.

Proof. \((1) \implies (2)\) holds by Theorem 3.3. For \((2) \implies (1)\) let \(P \in \text{QSpec}^{\ast}(D)\). It is enough for us to show that \(D_P\) is a quasi-Prüfer domain by [7, Theorem 2.16]. To this end let \(T\) be an overring of \(D_P\). Then \(T_{D_P} = T\) and therefore \(T\) is \((\ast, d_T)\)-linked overring of \(D\) by [11, Example 3.4 (1)]. If \(T \subseteq S\) is any extension of domains,
Theorem 4.2. J. Arnold [2, Theorem 6].

The following statements are equivalent:

1. Each \((t_D, d_T)\)-linked overring \(T\) of \(D\) is a stably strong \(S\)-domain.
2. Each \((t_D, d_T)\)-linked overring \(T\) of \(D\) is a strong \(S\)-domain.
3. Each \((t_D, d_T)\)-linked overring \(T\) of \(D\) is a Jaffard domain.
4. Each \((t_D, d_T)\)-linked overring \(T\) of \(D\) is a quasi-Prüfer domain.
5. For each \((t_D, d_T)\)-linked overring \(T\) of \(D\), every extension of domains \(T \subseteq S\), satisfies the dimension inequality formula.
6. \(D\) is a UM\(t\) domain.

4. ARNOLD’S FORMULA

In the last section we extend some results of J. Arnold of the dimension of polynomial rings to the setting of the semistar operations. First we wish to give the following lemma which is a new property of semistar valuative dimension.

Lemma 4.1. (see [24, Theorem 4.2]) Let \(D\) be an integral domain and \(n\) be an integer. Then the following statements are equivalent:

1. Each \((*, d_T)\)-linked overring \(T\) of \(D\) has dimension at most \(n\).
2. Each \(*\)-valuation overring \(T\) of \(D\) has dimension at most \(n\).

Proof. The implication \((1) \Rightarrow (2)\) is trivial. For \((2) \Rightarrow (1)\) let \(T\) be a \((*, d_T)\)-linked overring of \(D\) and \(V\) be a valuation overring of \(T\). Then it is easy to see that \(V\) is \((*, d_V)\)-linked overring of \(D\). Thus by [12, Lemma 2.7], \(V\) is an \(*\)-valuation overring of \(D\). Hence \(\text{dim}(V) \leq n\). Consequently \(\text{dim}(T) \leq \text{dim}_v(T) \leq n\) as desired.

When \(* = d_D\), the equivalence of \((1)\) and \((3)\) of the following theorem is due to J. Arnold [2, Theorem 6].

Theorem 4.2. Let \(D\) be an integral domain, and \(n\) be an integer. Then the following statements are equivalent:

1. \(\text{dim}_v(D) = n\).
2. \(\text{dim}(D[n]) = 2n\).
3. \(\text{dim}([r]) = r + n\) for all \(r \geq n - 1\).
4. Each \((*, d_T)\)-linked overring \(T\) of \(D\) has dimension at most \(n\), and \(n\) is minimal.

Proof. The equivalence \((1) \Leftrightarrow (2)\) follows from [24, Theorem 4.5], and \((3) \Rightarrow (2)\) is trivial. For \((1) \Rightarrow (3)\) suppose that \(\text{dim}_v(D) = n\). Then For all \(r \geq n\) we have \(\text{dim}(D[r]) = \text{dim}_v(D[r]) = r + \text{dim}_v(D) = r + n\). For \([r] \ni \text{ideal} \) of \(D\) such that \(n = \text{dim}_v(D_M)\), by [24, Corollary 4.7 and Theorem 4.8]. Now assume that \(r = n - 1\). Since \(\text{dim}_v(D) = n\), there exists a quasi-\(*\)-prime ideal \(M\) of \(D\) such that \(n = \text{dim}_v(D_M)\), by Lemma 2.5. So that by [2, Theorem 6] we have \(\text{dim}(D_M[r]) = r + n\). Let \(\mathcal{P} \in QSpec^*[r](D[r])\) be such
that \( \star[n]\)-\(\dim(D[n]) = \height(P) \). Set \( P := \mathcal{P} \cap D \). Then by [24, Remark 2.3] we have \( P \in \text{QSpec}^\star(D) \cup \{0\} \). Thus

\[
\begin{align*}
  r + n & \leq \star[r]\cdot \dim(D[r]) = \height(P) \\
  & = \dim(D[r], P) = \dim(D_P[r, D_P[r]]) \\
  & \leq \dim(D_P[r]) \leq \dim(D_M[r]) = r + n,
\end{align*}
\]

where the first inequality holds by [24, Theorem 3.1]. Hence \( \star[r]\cdot \dim(D[r]) = r + n \) for all \( r \geq n - 1 \).

The equivalence (1) \( \iff \) (4) follows from Lemma [11] \( \square \)

As an immediate consequence we have:

**Corollary 4.3.** \( \tilde{\star}\cdot \dim_e(D) = \sup\{\dim(T)|T \text{ is } (\star, d_T)-\text{linked overring of } D\} \).

One of the famous formulas in the dimension theory of commutative rings is the Arnold’s formula [2, Theorem 5] which states as

\[
\dim(D[n]) = n + \sup\{\dim(D[\theta_1, \cdots, \theta_n])|\{\theta_i\}_n \subseteq K\}.
\]

Now we prove the semistar analogue of Arnold’s formula.

**Lemma 4.4.** Let \( D \) be an integral domain and \( n \) be an integer. Then

\[
\star[n]\cdot \dim(D[n]) = \sup\{\dim(D_M[n])|M \in \text{QMax}^\star(D)\}.
\]

**Proof.** If \( P \) is a quasi-\( \tilde{\star}\)-prime ideal of \( D \), and if \( QD_P[n] \) is a non-zero prime ideal of \( D_P[n] = D[n]_{D_P} \), then \( Q \cap D \subseteq P \) and hence \( Q \in \text{QSpec}^{\star[n]}(D[n]) \) by [24, Remark 2.3]. So that the inequality \( \geq \) is true. Now let \( Q \in \text{QMax}^{\star[n]}(D[n]) \) be such that \( \star[n]\cdot \dim(D[n]) = \height(Q) \), and set \( P := Q \cap D \). Then by [24, Remark 2.3] we have \( P \in \text{QSpec}^\star(D) \cup \{0\} \). So that

\[
\begin{align*}
\star[n]\cdot \dim(D[n]) &= \height(Q) = \dim(D[n]_Q) \\
& = \dim(D_P[n]_{QD_P[n]}) \leq \dim(D_P[n]) \\
& \leq \star[n]\cdot \dim(D[n]).
\end{align*}
\]

Therefore the proof is complete. \( \square \)

**Corollary 4.5.** Let \( D \) be an integral domain and \( n \) be an integer. Then there exist a quasi-\( \tilde{\star}\)-maximal ideal \( M \) of \( D \) and a quasi-\( \star[n]\)-maximal ideal \( Q \) of \( D[n] \) such that \( M = Q \cap D \) and

\[
\star[n]\cdot \dim(D[n]) = \height(Q) = n + \height(M[n]).
\]

**Proof.** By Lemma [11] there exists a quasi-\( \tilde{\star}\)-maximal ideal \( M \) of \( D \) such that \( \star[n]\cdot \dim(D[n]) = \dim(D_M[n]) \). Thus there exists a prime ideal \( Q \) of \( D[n] \) such that \( Q \cap (D \setminus M) = \emptyset \), \( \dim(D_M[n]) = \height(QD_M[n]) \) and that \( QD_M[n] \) is a maximal ideal of \( D_M[n] \). Since \( Q \cap D \subseteq M \) we have \( Q \) is a quasi-\( \star[n]\)-prime of \( D[n] \) by [24, Remark 2.3], and since \( \star[n]\cdot \dim(D[n]) = \height(Q) \), we have \( Q \) is a quasi-\( \star[n]\)-maximal ideal of \( D[n] \). Set \( PD_M := QD_M[n] \cap D_M \) for some \( P \in \text{QSpec}^\star(D) \). Then by [8, Corollary 2.9] we have \( \height(QD_M[n]) = n + \height(PD_M[n]) \) and that \( PD_M \) is a maximal ideal of \( D_M \). Thus we have \( P = M \) and

\[
\star[n]\cdot \dim(D[n]) = \height(QD_M[n]) = n + \height(MD_M[n]) = n + \height(M[n]),
\]

which ends the proof. \( \square \)
We are ready to prove the semistar analogue of Arnold’s formula.

**Theorem 4.6.** Let $D$ be an integral domain and $n$ be a positive integer. Then
\[ \ast[n] \cdot \dim(D[n]) = n + \sup \{ \bar{\ast}_r \cdot \dim(D[\theta_1, \cdots, \theta_n]) | \{ \theta_1, \cdots, \theta_n \} \subseteq K \} \]
where $\iota$ is the inclusion map of $D$ in $D[\theta_1, \cdots, \theta_n]$.

**Proof.** Let $M \in \text{QMax}^\ast(D)$ and $\{ \theta_i \}^n_1 \subseteq K$. Let $Q$ be a maximal ideal of $D_M[\theta_1, \cdots, \theta_n]$ such that $\dim(D_M[\theta_1, \cdots, \theta_n]) = \text{ht}(Q)$. Let $Q_0$ be a prime ideal of $D[\theta_1, \cdots, \theta_n]$ such that $Q_0 \cap D = \emptyset$ and $Q = Q_0D_M[\theta_1, \cdots, \theta_n]$. Thus $Q_0$ is a quasi-$\ast_r$-prime ideal of $D[\theta_1, \cdots, \theta_n]$ since $Q_0 \cap D \subseteq M$ ([24] Remark 2.3]). Hence we obtain that $\dim(D_M[\theta_1, \cdots, \theta_n]) = \text{ht}(Q) = \text{ht}(Q_0) \leq \bar{\ast}_r \cdot \dim(D[\theta_1, \cdots, \theta_n])$. Using Lemma [44] and Arnold’s formula [2] Theorem 5], we have
\[ \ast[n] \cdot \dim(D[n]) = n + \sup \{ \dim(D_M[\theta_1, \cdots, \theta_n]) \} \]
where the supremum is taken over $M \in \text{QMax}^\ast(D)$ and $\{ \theta_i \}^n_1 \subseteq K$. So that
\[ \ast[n] \cdot \dim(D[n]) \leq n + \sup \{ \bar{\ast}_r \cdot \dim(D[\theta_1, \cdots, \theta_n]) | \{ \theta_1, \cdots, \theta_n \} \subseteq K \} \]
Now choose $M \in \text{QMax}^\ast(D)$ and $\{ \theta_i \}^n_1 \subseteq K$ such that $\ast[n] \cdot \dim(D[n]) = n + \dim(D_M[\theta_1, \cdots, \theta_n])$. Let $Q'$ be a quasi-$\ast_r$-prime ideal of $D[\theta_1, \cdots, \theta_n]$ such that $\bar{\ast}_r \cdot \dim(D[\theta_1, \cdots, \theta_n]) = \text{ht}(Q')$ and set $P' := Q' \cap D$. Thus
\[ \bar{\ast}_r \cdot \dim(D[\theta_1, \cdots, \theta_n]) = \text{ht}(Q') = \dim(D[\theta_1, \cdots, \theta_n]Q') = \dim(D_{P'}[\theta_1, \cdots, \theta_n]Q'_{D_{P'}[\theta_1, \cdots, \theta_n]}) \leq \dim(D_{P'}[\theta_1, \cdots, \theta_n]) \leq \dim(D_M[\theta_1, \cdots, \theta_n]) \]
Hence by the first part of the proof $\dim(D_M[\theta_1, \cdots, \theta_n]) = \bar{\ast}_r \cdot \dim(D[\theta_1, \cdots, \theta_n])$. Thus we have $\ast[n] \cdot \dim(D[n]) = n + \bar{\ast}_r \cdot \dim(D[\theta_1, \cdots, \theta_n])$ to complete the proof. \(\square\)

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