An analytical theory for a three-dimensional thick-disc thin-plate vibratory gyroscope

G T Sedebo*†, S V Joubert‡‡, and M Y Shatalov***
Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria, South Africa
E-mail: *SedeboGT@tut.ac.za, ‡‡JoubertSV@tut.ac.za; ***ShatalovM@tut.ac.za

Abstract. We consider a cylindrical vibratory gyroscope comprising a not necessarily thin-shelled annular disc with small-plate thickness, vibrating in the \( m \)-th vibration mode in-plane and in the \((m + 1)\)st vibration mode out-of-plane. We derive the equations of motion for this contrivance in the “force-to-rebalance regime” and show how a slow (three-dimensional) inertial rotation rate of the gyroscope can be calculated in terms of amplitudes of vibration and other constants, all of which can be measured experimentally or calculated when the eigenfunctions and eigenvalues of the system are known. By means of a concrete example, a numerical experiment demonstrates how varying the inner radius of the annulus as well as the thickness of the plate allows us to “tune” the vibration frequencies of the in-plane and out-of-plane vibrations so that they coincide (for all practical purposes), eliminating any frequency split. Conventionally, an array of at least three thin-shelled hemispherical (or thin-ring) vibratory (resonator) gyroscopes is used to measure any three-dimensional rotation of the craft to which the gyroscopes are fixed. With the design proposed here, the array can be reduced to a solitary, tuned, annular thick-disc thin-plate vibratory gyroscope, reducing both size and cost.

1. Introduction
A gyroscope is a sensor that measures the angular rate or the angle of rotation of an object (see [1]). Rotating wheel gyroscopes have many disadvantages such as bearing friction and wear. The hemispherical vibratory gyroscopes (HVGs), ring vibratory gyroscopes (RVGs), disc vibratory gyroscopes (DVG), et cetera, eliminate mechanical rotation and avoid bearing problems. In 1890, G.H. Bryan [2] laid the foundation for the development of vibratory gyroscopes (VGs) based on the Coriolis effect. He observed what we call Bryan’s law or effect [3]:

The vibration pattern of a revolving cylinder or bell revolves at a rate proportional to the inertial rotation rate of the cylinder or bell.

Bryan’s law has been used in the design and manufacture of numerous variations of VGs. Owing to their high performance, high reliability, and small size, VGs are used in many fields with great diversity. These include navigation of air, space and marine vehicles, tracking control and detection systems in motor vehicles.

Commercial three-dimensional VGs consist of “many body arrays” comprising at least, say, three HVGs, each with orthogonal axes of rotation. These arrays are expensive, and engineers have therefore designed various micro-VGs operating in single-axis, dual-axis and three-dimensional modes, with varying sensitivities and performance [4]. For instance, [5] used
Figure 1. The polar coordinates $r$ and $\varphi$ of the position of rest of particle $dm = \rho hr dr d\varphi$ vibrating in-plane and out-of-plane within the annular disc of axial thickness $h$, inner radius $a$ and outer radius $b$ rotating slowly at a rate $(\Omega_\xi, \Omega_\eta, \Omega_\zeta)^T$ in three dimensions.

a MEMS device consisting of three orthogonal yaw-axis gyroscopes integrated into a single microchip, yielding a “yaw-sensing, pitch-sensing and roll-sensing gyroscope.”

Below we tentatively discuss under various assumptions, the first approximation mathematical principles of a single “cylindrical” vibratory gyroscope comprising of a not necessarily thin-shelled annular disc for in-plane vibrations and a thin plate for out-of-plane vibrations that measures three-dimensional rotation (see figure 1). We call this gyroscope a tuned disc vibratory gyroscope (TDVG) for reasons that will become apparent. A TDVG eliminates the integration of separate proof masses for each orthogonal axis in the gyroscope body, replacing the need for an array of orthogonal symmetry axis VGs. Such a design should reduce both the size and cost of the gyroscope.

Similar to our TDVG is the three-dimensional VDG described in [6]. They use the FEM to verify their experimental observations, but “shell theory” (thin or otherwise) is never discussed and a complete system of gyroscopic equations of motion is not provided. In contrast, our tentative theoretical approach for the TDVG demonstrates that we are not limited to “thin-shell theory” for in-plane vibrations of the annulus, but that we are limited to “thin-plate theory” for out-of-plane vibrations. The natural frequencies of both modes of vibration are tuned and determined by selecting suitable in-plane and out-of-plane vibration modes and appropriate values of the plate thickness and radii of the disc. We analytically derive the gyroscopic equations of motion in terms of four generalised variables, defining Bryan’s factor and two auxiliary Bryan’s factors in terms of quotients of integrals of eigenfunctions. Because the eigenfunctions may be determined in terms of Bessel and Neumann functions, the three Bryan’s factors may be calculated. Two of these factors enable us to show exactly how to calculate the three orthogonal components of the slow inertial rotation rate in terms of measurable quantities such as amplitudes of vibration and linear damping coefficients.

2. Preliminaries

We say that a vibratory gyroscope exhibits ideal behaviour if it operates according to Bryan’s law. It is well known that a frequency split (vibration “beats”) causes deviation from the ideal behaviour of a vibratory gyroscope. Indeed, various schemes have been introduced to eliminate the frequency split caused by mass and/or stiffness and/or shape imperfections and/or anisotropic damping (see [3] and the by no means exhaustive list of citations therein). We assume that the TDVG described below has been manufactured from a substance that exhibits isotropic damping with perfect axial symmetry and that mass and/or stiffness imperfections have been eliminated for both in-plane vibration and out-of-plane vibration by first applying an optimised
mass-balancing action (see [7]) and then an appropriate parallel-plate capacitor control array (see [3]). Hence, when we determine the Euler-Lagrange equations of motion for the vibrating system, we assume ideal conditions, taking only light isotropic linear damping into consideration (and deal with isotropic nonlinear damping in a future study).

Because of the nature of the TDVG described below, a frequency split between the resonant frequency of in-plane and the resonant frequency of out-of-plane vibration might be inevitable. We describe below how to “tune” in-plane and out-of-plane resonant frequencies so that this cause of frequency split may be eliminated.

Consider two coordinate systems 0ξηζ and 0xyz respectively, where 0ξηζ is connected to the disc and rotates with respect to any inertial coordinate system, whereas 0xyz is located at an angle \( \varphi \) from the 0ξ-axis of the 0ξηζ system and characterises the location of the infinitesimal element \( dm = \rho hr dr d \varphi \) as shown in figure 1. Furthermore, assume that the 0ξ-axis of 0ξηζ and the 0η-axis of 0xyz system overlap and coincide with the axis of the disc, as shown in figure 1. Suppose the disc is subjected to slow three-dimensional rotation which has components \( \Omega_\xi, \Omega_\eta, \) and \( \Omega_\zeta \) with directions along the 0ξ-axis, 0η-axis, and 0ζ-axis, respectively. Assume that the annular disc (of axial plate thickness \( h \)) has an inner radius \( a \) and outer radius \( b \) with \( r \) the position of a vibrating particle in the disc, as shown in figure 1. Let \( E \) be Young’s modulus, with \( \rho \) and \( \mu \) the mass density and Poisson’s ratio respectively of the material of the disc. We assume that \( h = \min(a, b) \), and hence we use “thin-plane theory” to simplify matters. Furthermore, assume that the position vector of a point on the disc is given by \( \vec{R} = (r + u, v, w)^T \), where \( u, v, \) and \( w \) are the displacements in the radial, tangential and axial directions, respectively. It is clear that

\[
\vec{R} = (\Omega_\xi, \Omega_\eta, \Omega_\zeta)^T = \begin{pmatrix} \Omega_\xi \cos \varphi + \Omega_\eta \sin \varphi \\ -\Omega_\xi \sin \varphi + \Omega_\eta \cos \varphi \\ \Omega_\zeta \end{pmatrix}.
\]

Thus it follows that the absolute linear velocity of the particle element \( dm \) in figure 1 is given by Euler’s formula (see [8])

\[
\vec{V} = (\dot{u}, \dot{v}, \dot{w})^T + \vec{\Omega} \times \vec{R},
\]

where

\[
\vec{\Omega} \times \vec{R} = \begin{pmatrix} \Omega_\eta \cos \varphi - \Omega_\xi \sin \varphi \omega \xi \omega \zeta \\ \Omega_\xi (r + u) - (\Omega_\xi \cos \varphi + \Omega_\eta \sin \varphi) \omega \zeta \\ (\Omega_\xi \cos \varphi + \Omega_\eta \sin \varphi) \omega \zeta \end{pmatrix}.
\]

Assume that the components of displacement of the vibrating particle of figure 1 are given by

\[
\begin{align*}
    u &= U(r)[C(t) \cos(m \varphi) + S(t) \sin(m \varphi)], \\
    v &= V(r)[C(t) \sin(m \varphi) - S(t) \cos(m \varphi)], \\
    w &= W(r)[A(t) \cos(m + 1) \varphi + B(t) \sin(m + 1) \varphi],
\end{align*}
\]

where \( m \) is a vibration mode number, \( U(r) \) and \( V(r) \) are eigenfunctions of the in-plane vibrations and \( W(r) \) is the eigenfunction of the out-of-plane vibration. These eigenfunctions may be calculated in terms of known functions using well-known standard techniques. For instance, using the theory of thin shells to investigate out-of-plane vibration of the disc ([9] or [10]), if \( \omega_{\text{out}} \) is the angular frequency of the \( n^{\text{th}} \) mode of out-of-plane vibration, then

\[
W(r) = c_1 J_n(\beta r) + c_2 Y_n(\beta r) + c_3 I_n(\beta r) + c_4 K_n(\beta r),
\]

where \( J_n(\beta r) \) (resp. \( Y_n(\beta r) \)) is a Bessel function of the first (resp. second) kind of order \( n \), \( I_n(\beta r) \) (resp. \( K_n(\beta r) \)) is a modified Bessel function of the first (resp. second) kind of order \( n \),

\[
\beta^4 = \frac{12\rho(1 - \nu^2)}{Eh^2} \omega_{\text{out}}^2
\]
and the constants \( c_1, c_2, c_3, c_4 \) determine the mode shape and are determined using the chosen boundary conditions. For the in-plane vibration pattern, \( C(t) \) and \( S(t) \) may respectively be regarded as the “cosine” and “sine” channels of a vibratory gyroscope connected to a two-channel oscilloscope; the same applies to the \( A- \) and \( B- \) channels for out-of-plane vibration (see [11]). However, the functions of time \( C(t), S(t), A(t), \) and \( B(t) \) may readily be determined numerically once the equations of motion (see (16) to (19) below) have been established.

3. Energy

We assume that the size of the angular rate of rotation \( \Omega \) is substantially smaller than the lowest eigenvalue \( \omega_0 \) of either in-plane or out-of-plane vibration. Consequently, we neglect terms containing \( \Omega^2, \Omega \xi \), \( \Omega \Omega \xi, \Omega \Omega \xi \), \( \xi \xi \Omega \xi \), \( \xi \xi \Omega \xi \). Neglecting the effects of rigid body motion, from (2) we obtain the the kinetic energy \( E_k \) of the vibrating disc as

\[
E_k = \frac{\rho h}{2} \int_0^{2\pi} \int_a^b (\vec{V} \cdot \vec{V}) r \, dr \, d\varphi \approx \pi \left[ I_0(\dot{C}^2 + \dot{S}^2) + 2\Omega \xi(\dot{C}S - \dot{C}S)I_1 + I_3\Omega \xi(AC - AC + B\dot{S} - \dot{B}\dot{S}) + I_3\Omega \xi(AS - \dot{A}\dot{S} + \dot{B}C - B\dot{C}) + I_4(\dot{A}^2 + \dot{B}^2) \right],
\]

where

\[
I_0 = \frac{\rho h}{2} \int_a^b (U^2 + V^2) r \, dr, \quad I_1 = \rho h \int_a^b UV \, r \, dr, \quad I_3 = \frac{\rho h}{2} \int_a^b (U + V) W \, r \, dr, \quad I_4 = \frac{\rho h}{2} \int_a^b W^2 \, r \, dr.
\]

If respectively, \( \sigma_r, \sigma_\varphi, \sigma_r \varphi \) (resp. \( \epsilon_r, \epsilon_\varphi, \epsilon_r \varphi \)) are the usual tensile radial, tensile tangential and shear stresses (resp. strains) for in-plane vibrations, then the potential energy of in-plane vibrations is (see [12] and [13] (8) and (17))

\[
E_{p,\text{in}} = \frac{Eh}{2(1 - \mu^2)} \int_0^{2\pi} \int_a^b \left[ \sigma_r \epsilon_r + \sigma_\varphi \epsilon_\varphi + \sigma_r \varphi \epsilon_r \varphi \right] r \, dr \, d\varphi = \pi I_2(C^2 + S^2),
\]

where the definite integral \( I_2 \) is given by [13] (20). Using the assumptions of “thin-plate theory,” with

\[
\chi_1 = -\frac{\partial^2 w}{r \partial \varphi^2}, \quad \chi_2 = \frac{1}{r} \frac{\partial w}{\partial r} - \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2}, \quad \tau = \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \varphi} + \frac{1}{r^2} \frac{\partial w}{\partial \varphi},
\]

the potential energy for out-of-plane vibrations is (see Novozhilov [9] (9.12))

\[
E_{p,\text{out}} = \frac{Eh^3}{24(1 - \mu^2)} \int_0^{2\pi} \int_a^b \left[ \chi_1^2 + \chi_2^2 + 2\mu \chi_1 \chi_2 + 2(1 - \mu) r^2 \right] r \, dr \, d\varphi = \pi I_5(A^2 + B^2),
\]

where

\[
I_5 = \frac{Eh^3}{24(1 - \mu^2)} \int_a^b \left\{ \frac{2\mu WW''}{r} - \frac{2(\mu - 1)(m + 1)^2 - 1}{r^2} - \frac{2\mu(m + 1)^2 WW''}{r^2} + \frac{(2\mu - 3)(m + 1)^2 WW'}{r^3} + \frac{m^2(m^2 + 4m + 8 - 2\mu) - 2(\mu - 2)(2m + 1) - 1}{r^4} \right\} r \, dr.
\]

Note that the definite integral \( I_2 \) ([13] (20)) is given similarly in terms of \( E, \mu, \) and an integrand containing sums of products and/or powers of \( U, U', V, V' \).
4. Equations of motion

We assume that there is ever-present light, isotropic, linear damping, with damping factor \( \delta \) for both in-plane and out-of-plane damping and define Rayleigh’s dissipation functions \( R \) (see [14])

\[
R = \pi I_0 \delta (\dot{C}^2 + \dot{S}^2). \tag{15}
\]

By “light damping” we mean that \( \delta \) is substantially smaller than the lowest eigenvalue \( \omega_0 \) of either in-plane or out-of-plane vibration. As demonstrated in [14], we assume that the rotation rates of vibrating patterns are not affected by light isotropic damping.

Using (9), (11), and (13), we determine Lagrangian \( L = E_k - (E_{p,\text{in}} + E_{p,\text{out}}) + E_e \), where \( E_e \) is the electrical potential energy introduced to achieve electronic control of the gyroscope. We choose \( E_e \) in such a manner that it also produces a so-called force-to-rebalance regime (see [15] for a similar setup) that drives only the \( C\)- and \( S\)-channels (not the \( A\)- and \( B\)-channels) of the gyroscope at an operational angular frequency \( \omega \) close to the natural angular frequencies, respectively, \( \omega_{\text{in}} \) and \( \omega_{\text{out}} \) of in-plane and out-of-plane vibrations. As in [3] (12.33), we apply the Euler-Lagrange equations to the generalised coordinates \( C, S, A, B \) in the Lagrangian \( L \), obtaining the coupled system of ODE of gyroscopic motion appropriate to our TDVG:

\[
\dot{C} + 2\delta \dot{C} + 2\beta_1 \Omega_c \dot{S} + \beta_1 \dot{\Omega}_c S + \omega_{\text{in}}^2 C + \beta_2 [2(\Omega_\eta \dot{A} - \dot{\Omega}_c B)] = f \cos(\omega t + \psi), \tag{16}
\]

\[
\dot{S} + 2\delta \dot{S} - 2\beta_1 \Omega_c \dot{C} - \beta_1 \dot{\Omega}_c C + \omega_{\text{in}}^2 S + \beta_2 [2(\Omega_\eta \dot{A} + \dot{\Omega}_c B)] = f \cos(\omega t + \Psi), \tag{17}
\]

\[
\ddot{A} + 2\delta \ddot{A} + \omega_{\text{out}}^2 A - \beta_3 [2(\Omega_\eta \dot{C} + \dot{\Omega}_c S)] + (\dot{\Omega}_\eta C + \dot{\Omega}_c S) = 0. \tag{18}
\]

\[
\ddot{B} + 2\delta \ddot{B} + \omega_{\text{out}}^2 B + \beta_3 [2(\Omega_\eta \dot{C} - \dot{\Omega}_c S)] + (\dot{\Omega}_\eta C - \dot{\Omega}_c S) = 0. \tag{19}
\]

Here \( \psi \) and \( \Psi \) are phase angles with

\[
\beta_1 = \frac{I_1}{I_0}, \quad \beta_2 = \frac{I_3}{2I_0}, \quad \beta_3 = \frac{I_3}{2I_4}, \quad \omega_{\text{in}} = \sqrt{\frac{I_2}{I_0}}, \quad \omega_{\text{out}} = \sqrt{\frac{I_5}{I_4}}. \tag{20}
\]

The constant \( \beta_1 \) is Bryan’s factor (see [3] (12.8)), while \( \beta_2 \) and \( \beta_3 \) are auxiliary Bryan’s factors. Recall that the quantity \( \omega_{\text{in}} \) (resp. \( \omega_{\text{out}} \)) is the eigenvalue or angular rate of the in-plane (resp. out-of-plane) vibration.

5. Bryan’s factors and inertial rotation rates

The control or excitation force \( f \cos(\omega t + \psi) \) in (16) is used to keep the vibration amplitude of the \( C\)-channel constant, with the phase shift \( \psi \) chosen in such a way that \( C(t) = C_0 \cos(\omega t) \). With an appropriate phase shift \( \Psi \), the rebalance force \( F \cos(\omega t + \Psi) \) in (17) is used to yield \( S(t) = 0 \).

To demonstrate how \( \beta_1 \) and \( \beta_3 \) are related to rotation rates, we use the force-to-rebalance electronic array discussed above and assume that a steady-state solution for (16) to (19) is of the form

\[
C(t) = C_0 \cos(\omega t), \tag{21}
\]

\[
S(t) = 0, \tag{22}
\]

\[
A(t) = A_0 \cos(\omega t) + A_\delta \sin(\omega t), \tag{23}
\]

\[
B(t) = B_0 \cos(\omega t) + B_\delta \sin(\omega t). \tag{24}
\]

Owing to the slow inertial rotation rate \( \dot{\Omega} \), the amplitudes \( A_0, A_\delta, B_0, B_\delta \) are slowly varying time-dependent quantities, which we assume are constant in order to determine simplified expressions for angular rates. The exact nature of these amplitudes is left for future study, while good average estimations of these amplitudes should be measurable by experiment.
Substituting (21) to (24) into (16) to (19) yields equations for the angular velocities $\Omega_\zeta$, $\Omega_\eta$, $\Omega_\xi$ and angular accelerations $\dot{\Omega}_\zeta$, $\dot{\Omega}_\eta$, $\dot{\Omega}_\xi$ as follows:

$$
\dot{\Omega}_\zeta = -\frac{F \sin \Psi}{2 \omega C_c \beta_1}, \quad \dot{\Omega}_\eta = \frac{1}{2} \frac{A_c \delta \omega + A_s \omega^2 - A_s \omega^2_{\text{out}}}{C_c \beta_3 \omega}, \quad \dot{\Omega}_\xi = \frac{1}{2} \frac{2 B_c \delta \omega + B_s \omega^2 - B_s \omega^2_{\text{out}}}{C_c \beta_3 \omega}, \quad (25)
$$

$$
\ddot{\Omega}_\zeta = -\frac{F \cos \Psi}{C_c \beta_1}, \quad \ddot{\Omega}_\eta = -\frac{A_c \omega^2 - A_s \omega^2_{\text{out}} - 2 A_c \delta \omega}{C_c \beta_3}, \quad \ddot{\Omega}_\xi = \frac{B_c \omega^2 - B_s \omega^2_{\text{out}} - 2 B_c \delta \omega}{C_c \beta_3}. \quad (26)
$$

In order to eliminate the frequency split introduced by a difference between the in-plane angular rate and the out-of-plane angular rates, the gyroscope may be “tuned” by varying the radial and/or axial thickness, enabling us to achieve $\omega_{\text{in}} = \omega_{\text{out}} = \omega$ for all practical purposes. We have verified that this “tuning” is possible by conducting a numerical experiment as discussed below. This “tuning” yields the following simplified equations

$$
\dot{\Omega}_\zeta = -\frac{F \sin \Psi}{2 \omega C_c \beta_1}, \quad \dot{\Omega}_\eta = \frac{A_c \delta}{C_c \beta_3}, \quad \dot{\Omega}_\xi = -\frac{B_c \delta}{C_c \beta_3}, \quad (27)
$$

$$
\ddot{\Omega}_\zeta = -\frac{F \cos \Psi}{C_c \beta_1}, \quad \ddot{\Omega}_\eta = \frac{2 A_c \delta \omega}{C_c \beta_3}, \quad \ddot{\Omega}_\xi = -\frac{2 B_c \delta \omega}{C_c \beta_3}. \quad (28)
$$

The first equation in (27) demonstrates that the axial inertial rotation rate $\Omega_\zeta$ is directly proportional to the amplitude $F$ of the rebalance force. This result is essentially the same as that reported for the axial rotation rate of an HVG operating under the force-to-rebalance regime (see [15] (6)).

The amplitude $F$ and phase angle $\Psi$ of the rebalance force, the operational angular frequency $\omega$ as well as the amplitudes $A_c$, $B_c$, and $C_c$ should all be experimentally measurable. Once the eigenfunctions $U(r)$, $V(r)$, and $W(r)$ have been calculated (as outlined immediately above (8)), Bryan’s factors $\beta_1$ and $\beta_3$ can be calculated from the formulae in (20). Consequently the angular rate and accelerations of each orthogonal axis can be calculated from (27).

6. Tuning

Ultimately, we provide a numerical experiment, which for this first approximation theory, demonstrates that it may well be possible to “tune” the TDVG once it has been manufactured. Consider an aluminium disc with Young’s modulus $E = 7 \times 10^{11}$ N/m$^2$, density $\rho = 2700$ kg/m$^3$, and Poisson’s ratio $\mu = 0.33$. With reference to figure 1, the thickness and outer radius are kept constant at $h = 4.824$ mm and $b = 100$ mm, respectively, while the inner radius $a$ is varied.

Considering mode number $m = 2$ for in-plane vibration and mode number $m + 1 = 3$ for out-of-plane vibration, and assuming free boundaries on the TDVG, the eigenfunctions are calculated as outlined immediately above (8). Tables 1 and 2 were generated using (20) and the computer algebra system Maple 18 at standard working precision.

In table 1, with $h = 4.824$ mm and $b = 100$ mm, the inner radius in each case is approximately $a = 40$ mm. Hence the thickness $T$ of the annular disc is

$$
T = b - a \approx 60 \text{ mm}
$$

for each inner radius, and so the value of the ratio

$$
\frac{T}{h} \approx 12
$$

lies in the interval defined for a “thin plate” in Section 1.1.3 of [10]. Consequently, the values used in table 1 are commensurate with the theory developed above. With an inner radius
Table 1. Eigenfrequencies and frequency split for an aluminium vibratory disc gyroscope for a fixed outer radius $b = 100$ mm and thickness $h = 4.824$ mm for various values of inner radius $a$.

| $a$ (mm) | $f_{in} = \frac{\omega_p}{2\pi}$ (kHz) | $f_{out} = \frac{\omega_p}{2\pi}$ (kHz) | $\Delta f = |f_{in} - f_{out}|$ (kHz) |
|----------|----------------------------------------|----------------------------------------|-------------------------------------|
| 39.51    | 6.1778                                 | 5.8329                                 | 0.3449                              |
| 40.51    | 6.0341                                 | 5.8049                                 | 0.2293                              |
| 41.51    | 5.8922                                 | 5.7777                                 | 0.1145                              |
| 42.51    | 5.7521                                 | 5.7518                                 | 0.0003                              |

Table 2. Bryan’s factor and auxiliary Bryan’s factors for an aluminium vibratory disc gyroscope for a fixed outer radius $b = 100$ mm and thickness $h = 4.824$ mm, for various values of inner radius $a$.

| $a$ (mm) | $\beta_1$ | $\beta_2$ | $\beta_3$ |
|----------|------------|------------|------------|
| 39.51    | -0.7513    | -0.1037    | -0.0060    |
| 40.51    | -0.7513    | -0.1057    | -0.0050    |
| 41.51    | -0.7514    | -0.1067    | -0.0040    |
| 42.51    | -0.7516    | -0.1064    | -0.0032    |

of 42.51 mm, it appears that it is feasible to “tune” the gyroscope and achieve a frequency split $\Delta f \approx 0$ kHz (to four significant figures of accuracy).

In table 2, the negative Bryan’s factor $\beta_1$ indicates that, for in-plane-vibrations, the vibration pattern is rotating in the opposite direction to the direction of the inertial axial rotation rate $\Omega_\zeta$. As the annulus decreases in thickness (inner radius increases), so Bryan’s factor $\beta_1$ and the auxiliary Bryan’s factor $\beta_2$ appear to increase in size, whereas the auxiliary Bryan’s factor $\beta_3$ decreases in size.

Conclusions
We have developed a first approximation theory for an essentially ideal tuning disc vibratory gyroscope (TDVG) rotating at a slow inertial angular rate in three dimensions. Under various mild assumptions, we have derived the equations of motion of a TDVG. Under a force-to-rebalance regime, assuming a steady-state solution, we have derived approximate expressions for the angular rotation rate along each orthogonal rotation axis as well as the corresponding angular accelerations. Our expression for the axial inertial angular rate agrees with known results for a single axis of rotation, hemispherical vibratory gyroscope. We have conducted a numerical experiment using our first approximation theory, which illustrates how it might be feasible to build and tune a three-dimensional TDVG.

Acknowledgments
The authors thank the the National Research Foundation (NRF) of South Africa (grant reference numbers IFR160211157784 and 106308) as well as the Tshwane University of Technology (TUT) for financial support.
References

[1] Akar C and Shkel A 2009 MEMS Vibratory Gyroscopes: Structural Approaches to Improve Robustness (New York: Springer)

[2] Bryan G H 1890 On the beats in the vibrations of a revolving cylinder or bell Proc. Camb. Phil. Soc. Math. Phys. Sci. 7 101–14

[3] Joubert S V, Shatalov M Y, and Spoelstra H 2017 On electronically restoring an imperfect vibratory gyroscope to an ideal state Mechanics for Materials and Technologies ed Altenbach H, Goldstein R V, and Murashkin E (Cham: Springer) pp 231–56

[4] Tsai N-C and Sue C-Y 2010 Experimental analysis and characterization of electrostatic-drive tri-axis micro-gyroscope Sensors Actuat. A: Phys. 158 (2) 231–9

[5] Sung W 2013 High-frequency Tri-axial Resonant Gyroscopes Ph.D. thesis Georgia Institute of Technology Atlanta-Georgia, USA

[6] Johari H, Shah J, and Ayazi F 2008 High frequency XYZ single-disk silicon gyroscope In Proc. IEEE Conf. on MEMS, Tucson, AZ, USA (2008) 856–9

[7] Shatalov M Y, Joubert S V, and Spoelstra H 2015 Modelling and controlling imperfections in vibratory gyroscopes In Int. Conf. on Sound and Vibration (ICSV22), Florence, Italy 1–8

[8] Spiegel M 1967 Theoretical Mechanics (New York: Schaum Publishing Co.)

[9] Novozhilov V 1970 The Theory of Thin Shells (Groningen: Wolters-Noordhoff.)

[10] Ventsel E and Krauthammer T 2001 Thin Plates and Shells: Theory, Analysis and Applications 1st ed (New York: Marcel Dekker, Inc.) ISBN 0-8247-0575-0

[11] Shatalov M Y, Joubert S V, and Coetzee C E 2011 The influence of mass imperfections on the evolution of standing waves in slowly rotating spherical bodies J. Sound Vibr. 330 127–35

[12] Redwood M 1960 Mechanical Waveguides (Oxford: Pergamon Press)

[13] Joubert S V, Shatalov M Y, and Fay T H 2009 Rotating structures and Bryan’s effect Am. J. Phys. 77 (6) 520–5

[14] Joubert S V, Shatalov M Y, and Manzhirov A V 2013 Bryan’s effect and isotropic nonlinear damping J. Sound. Vibr. 332 (23) 6169–76

[15] Wang X, Wu W, Luo B, et al. 2011 Force to rebalance control of HRG and suppression of its errors on the basis of FPGA Sensors 11 11761–73