STRUCTURED RANDOM SKETCHING FOR PDE INVERSE PROBLEMS

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Abstract. For an overdetermined system $Ax \approx b$ with $A$ and $b$ given, the least-square (LS) formulation
\[
\min_x \|Ax - b\|_2
\]
is often used to find an acceptable solution $x$. The cost of solving this problem depends on the
dimensions of $A$, which are large in many practical instances. This cost can be reduced by the use of random
sketching, in which we choose a matrix $S$ with many fewer rows than $A$ and $b$, and solve the sketched LS problem
\[
\min_x \|S(Ax - b)\|_2
\]to obtain an approximate solution to the original LS problem. Significant theoretical and
practical progress has been made in the last decade in designing the appropriate structure and distribution
for the sketching matrix $S$. When $A$ and $b$ arise from discretizations of a PDE-based inverse problem, tensor
structure is often present in $A$ and $b$. For reasons of practical efficiency, $S$ should be designed to have a structure
consistent with that of $A$. Can we claim similar approximation properties for the solution of the sketched LS
problem with structured $S$ as for fully-random $S$? We give estimates that relate the quality of the solution of the
sketched LS problem to the size of the structured sketching matrices, for two different structures. Our results
are among the first known for random sketching matrices whose structure is suitable for use in PDE inverse
problems.

1. Introduction

In overdetermined linear systems (in which the number of linear conditions exceeds the number of unknowns),
the least-squares (LS) solution is often used as an approximation to the true solution when the data contains
noise. Given the system $Ax = b$ where $A \in \mathbb{R}^{n \times p}$ with $n \gg p$, the least-squares solution $x^*$ is obtained by
minimizing the $l^2$-norm discrepancy between the $Ax$ and $b$, that is,
\[
\min_x \|Ax - b\|_2, \quad \Rightarrow \quad x^* = A^\dagger b,
\]where $A^\dagger \triangleq (A^\top A)^{-1}A^\top$. (1)
The matrix $A^\dagger$ is often called the pseudoinverse (more specifically the Moore-Penrose pseudoinverse) of $A$.

The LS method is ubiquitous in statistics and engineering, but large problems can be expensive to solve. Aside
from the cost of preparing $A$, the cost of solving for $x^*$ is $O(np^2)$ flops for general (dense) $A$ is prohibitive
in large dimensions.

We can replace the LS problem with a smaller approximate LS problem by using sketching. Each row of
the sketched system is a linear combination of the rows of $A$, together with the same linear combination of the
elements of $b$. This scheme amounts to defining a sketching matrix $S \in \mathbb{R}^{r \times n}$ with $r \ll n$, and replacing the
original LS problem by
\[
\min_x \|SAx - Sb\|_2, \quad \Rightarrow \quad x_s^* = (SA)^\dagger Sb.
\]For appropriate choices of $S$, the solutions of (1) and (2) are related in the sense that
\[
\|b - Ax^*\| \text{ is not too much greater than } \|b - Ax_s^*\|.
\]Usually one does not design $S$ directly, but rather draws its entries from a certain distribution. In such a setup,
we can ask whether (3) holds with high probability.

Many theoretical results and numerical studies of randomized sketching have been presented during the past
decade [1,2,9,11,13,15,18], most of them linked to the Johnson-Lindenstrauss lemma [7]. To a large extent,
two perspectives have been taken. One approach starts with the least square problem directly. The authors
proposed two conditions the random matrix needs to satisfy for an accurate solution with high confidence, and
then justified that certain choices of random matrices indeed satisfy these two conditions. See the original papers [5, 14, 15] and a review [8]. The second perspective more focuses on the structure of the space spanned by $A$. The authors argued the space could be approximated by a finite number of vectors (the so-called $\gamma$-net), which further could be “embedded” using random matrices with high accuracy. See [2, 16, 17] and a review [18].

We use the latter perspective in this paper.

There are many variations of the original sketching problem. With some statistical assumptions on the perturbation in the right hand side, results could be further enhanced [13], and the sketching problem is also investigated when other constraints (such as $l_1$ constraints) are present; see for example [12].

In most previous studies, the design of $S$ varies according to the priorities of the application. For good accuracy with small $r$, random projections with sub-Gaussian variables are typically used. When the priority is to reduce the cost of computing the product $SA$, either sparse or Hadamard type matrices have been proposed, leading to “random-sampling” or FFT-type reduction in cost of the matrix-matrix multiplication. To cure “bias” in the selection process, leverage scores have been introduced; these trace their origin back to classical methods in experimental design.

In this paper, with practical inverse problems in mind, we consider the case in which $A$ and $b$ have certain tensor-type structures. For the sketched system to be formed and solved efficiently, the random sketching matrix $S$ must have a corresponding tensor structure. For these tensor-structured sketching matrices $S$, we ask: What are the requirements on $r$ to achieve a certain accuracy in the solution $x_s^*$ of the sketched system?

We consider $A$ with the following structure:

$$A_{i,:} = F_{i,1} \circ G_{i,2,:},$$

where $i = (i_1, i_2)$ is a multi-index, and $\circ$ denotes Hadamard (i.e., pointwise) multiplication. One also can refer to the matrix $A$ as the Khatri-Rao product of the matrices $F$ and $G$. Assuming $i_1 \in I_1$ and $i_2 \in I_2$, with cardinalities $n_1 = |I_1|$ and $n_2 = |I_2|$ respectively, the dimensions of these matrices are

$$F \in \mathbb{R}^{n_1 \times p}, \quad G \in \mathbb{R}^{n_2 \times p}, \quad A \in \mathbb{R}^{n \times p},$$

(5)

where $n = |I_1 \otimes I_2| = n_1 n_2$.

By defining $f_j = F_{:j} \in \mathbb{R}^{n_1}$ and $g_j = G_{:j} \in \mathbb{R}^{n_2}$, we can define $A$ alternatively as

$$a_j \overset{\text{def}}{=} A_{:,j} = f_j \otimes g_j,$$

(6)

where $a_j \in \mathbb{R}^n$ denotes the $j$th column of $A$, for $j = 1, 2, \ldots, p$. For vector $b$, we assume that it admits the same tensor structure, i.e.,

$$b = f_b \otimes g_b,$$

for some fixed $f_b \in \mathbb{R}^{n_1}$ and $g_b \in \mathbb{R}^{n_2}$.

(7)

This type of structure comes from the fact that to formulate inverse problems, one typically needs to prepare both the forward and adjoint solutions. Letting $\sigma(x)$ denote the the unknown function to be reconstructed in the inverse problem. This allows the problem to be formulated as a Fredholm integral equation of the first type:

$$\int f_{i_1}(x) g_{i_2}(x) \sigma(x) dx = \text{data}_{i_1,i_2},$$

(8)

where $f_{i_1}$ and $g_{i_2}$ solve the forward and adjoint equations respectively, equipped with boundary/initial conditions indexed by $i_1$ and $i_2$. The measured data $\text{data}_{i_1,i_2}$ typically means the data measured at $i_2$ with input source index $i_1$. To reconstruct $\sigma$, one loops over the entire list of conditions for $f_{i_1}$ ($i_1 \in I_1$) and $g_{i_2}$ ($i_2 \in I_2$). The LS formulation $\min \|Ax - b\|_2$ is the discrete version of the Fredholm integral (8).

This structure imposes requirements on the sketching matrix $S$. Since $I_1$ and $I_2$ contain conditions for different sets of equations, sketching needs to be performed within $I_1$ and $I_2$ separately. This condition is reflected by choosing the sketching matrix $S$ to have the form

$$S_{i,:} = p_i^\top \otimes q_i^\top,$$
where \( p_i \in \mathbb{R}^{n_1} \) and \( q_i \in \mathbb{R}^{n_2}, \ i = 1, \ldots, p \). The product \( SA \) then has the special form:

\[
(SA)_{i,:} = (p_i \ F) \circ (q_i \ G), \quad \text{or equivalently} \quad (SA)_{i,j} = (p_i^\top f_j)(q_i^\top g_j).
\]  

(9)

Thus, to formulate the \( i \) row in the reduced (sketched) system, we perform a linear combination of parameters in \( \mathcal{I}_1 \) according to \( p_i \) to feed in the forward solver, and a linear combination of parameters in \( \mathcal{I}_2 \) according to \( q_i \) to feed in the adjoint solver, then assemble the results in the Fredholm integral \([8]\).

With the structural requirements for \( S \) in mind, we consider the following two approaches for choosing \( S \).

Case 1: Generate two random matrices \( P \) and \( Q \), of size \( r_1 \times n_1 \) and \( r_2 \times n_2 \), respectively, and define \( S \) to be their tensor product:

\[
S = P \otimes Q \in \mathbb{R}^{r_1 r_2 \times n_1 n_2}.
\]  

(10)

Case 2: Generate two sets of \( r \) random vectors \( \{ p_i, i = 1, 2, \ldots, r \} \) and \( \{ q_i, i = 1, 2, \ldots, r \} \), with \( p_i \in \mathbb{R}^{n_1} \) and \( q_i \in \mathbb{R}^{n_2} \) for each \( i \), and define row \( i \) of \( S \) to be the tensor product of the vectors \( p_i \) and \( q_i \):

\[
S = \frac{1}{\sqrt{r}} \begin{bmatrix}
  p_1^\top \otimes q_1^\top \\
  \vdots \\
  p_r^\top \otimes q_r^\top
\end{bmatrix} \in \mathbb{R}^{r \times n_1 n_2}.
\]  

(11)

Case 2 gives greater randomness, in a sense, because the rows of \( P \) and \( Q \) are not “re-used” as in the first option.

We are not interested in designing sketching matrices of Hadamard type. In practice, \( A \) is often semi-infinite: \( F \) and \( G \) contain all possible forward and adjoint solutions, a set of infinite cardinality that cannot be prepared in advance. In practice, one can only obtain the “realizations” \( p_i^\top F \) or \( q_i^\top G \) obtained by solving the forward and adjoint equations with the parameters contained in \( p \) and \( q \). Because we use this technique to find \( SA \), rather than computing the matrix-matrix product explicitly, there is no advantage to defining \( S \) in terms of Hadamard type random matrices.

The rest of the paper is organized as follows. In Section 2, we give two examples from PDE-based inverse problem that give rise to a linear system with tensor structure. Section 3 presents classical results on sketching for general linear regression, and states our main results on sketching of inverse problem associated with a tensor structure. Sections 4 and 5 study the two different sketching strategies outlined above. Computational testing described in Section 6 validates our results.

To the best of our knowledge, there have been few theoretical results concerning tensor-structured sketching problem. In [4] the authors utilized TensorSketch as a method to embed \( A_1 \otimes A_2 \otimes \cdots \otimes A_q \) with each \( A_i \in \mathbb{R}^{n_i \times d_i} \), and in [6], the authors investigated a fast Johnson-Lindenstrauss Transform for Kronecker matrix products. Neither paper imposes the tensor structure on the random sketching matrix as we do here.

Notation. We denote the range space (column space) of a matrix \( X \) by \( \text{Range}(X) \).

2. OVERDETERMINED SYSTEMS WITH TENSOR STRUCTURE ARISING FROM PDE INVERSE PROBLEMS

Tensor structure is a common feature of PDE-based inverse problems. In the problem of reconstructing the conductivity in Electrical Impedance Tomography (EIT), we seek the solution \( \sigma(x) \) to

\[
\int \nabla \rho_1(x) \cdot \nabla \rho_2(x) \sigma(x) \, dx = \text{data}_{\phi, \psi},
\]  

(12)

where \( \rho_i \) are solutions to the following forward and adjoint diffusion equations

\[
\begin{align*}
\begin{cases}
\nabla \cdot (\sigma^*(x) \nabla \rho_1(x)) = & -\nabla \cdot (\sigma(x) \nabla \rho^*(x)), \quad x \in \Omega \\
\rho_1(x) = & 0, \quad x \in \partial \Omega
\end{cases}
& \quad x \in \Omega \\
\begin{cases}
\nabla \cdot (\sigma^*(x) \nabla \rho_2(x)) = & 0, \quad x \in \Omega \\
\rho_2(x) = & \psi(x), \quad x \in \partial \Omega
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\nabla \cdot (\sigma^*(x) \nabla \rho_1(x)) = & -\nabla \cdot (\sigma(x) \nabla \rho^*(x)), \quad x \in \Omega \\
\rho_1(x) = & 0, \quad x \in \partial \Omega
\end{cases}
& \quad x \in \Omega \\
\begin{cases}
\nabla \cdot (\sigma^*(x) \nabla \rho_2(x)) = & 0, \quad x \in \Omega \\
\rho_2(x) = & \psi(x), \quad x \in \partial \Omega
\end{cases}
\end{align*}
\]
where \( \sigma^* \) is the background media and \( \rho^* \) is the background solution defined by the following elliptic PDE:

\[
\begin{dcases}
\nabla_x \cdot (\sigma^*(x) \nabla \rho^*(x)) = 0, & x \in \Omega \\
\rho^*(x) = \phi(x), & x \in \partial \Omega.
\end{dcases}
\]

Note that the forward equation solution \( \rho_1 \) and adjoint equation solution \( \rho_2 \) depend on parameters \( \phi \) and \( \psi \), respectively. By varying the boundary condition \( \phi \) and \( \psi \), one finds infinitely many solution pairs of \( \{\rho_1(\cdot, \phi), \rho_2(\cdot, \psi)\} \), each pair giving rise to an item of data on the right-hand side of \([12]\), indicating that data is obtained with \( \psi \) as the test function at the boundary, and \( \phi \) as the input source. We can thus assemble an overdetermined Fredholm integral from \([12]\), and solve it to reconstruct \( \sigma \).

A similar problem arises in optical tomography. To reconstruct the optical property of the media, we seek the solution \( \sigma(x, v) \) to

\[
\int p_1(x, v) p_2(x, v) \sigma(x, v) \, dx \, dv = \text{data}_{\phi, \psi},
\]

where \( (x, v) \in \Omega \otimes S \) (where \( \Omega \) is the spatial domain and \( S \) is the velocity domain), and \( p_i \) are solutions to the forward and adjoint radiative transfer equations

\[
\begin{dcases}
\nabla_x \cdot p_1(x, v) = \sigma^*(x, v) L p_1(x, v) + \sigma(x, v) \rho^*(x, v), & (x, v) \in \Omega \otimes S \\
p_1(x, v) = 0, & (x, v) \in \Gamma_-
\end{dcases}
\]

and

\[
\begin{dcases}
\nabla_x \cdot p_2(x, v) = \sigma^*(x, v) L p_2(x, v), & (x, v) \in \Omega \otimes S \\
p_2(x, v) = \psi(x, v), & (x, v) \in \Gamma_+
\end{dcases}
\]

Similarly, \( \sigma^* \) is the background media and \( \rho^* \) the background equation defined from the following system:

\[
\begin{dcases}
\nabla_x \cdot \rho^* = \sigma^* L \rho^*, & (x, v) \in \Omega \otimes S \\
\rho^*(x, v) = \phi(x, v), & (x, v) \in \Gamma_-
\end{dcases}
\]

In these equations, \( L \) is an integral linear operator on \( v \), and \( \Gamma_- \) and \( \Gamma_+ \) collect incoming and outgoing coordinates (respectively) at the boundary. By varying the boundary conditions \( \phi \) and \( \psi \), one can find infinitely many solution pairs of \( \{\rho_1(\cdot, \phi), \rho_2(\cdot, \psi)\} \), and collect the corresponding data in \([13]\). The inverse Fredholm integral \([13]\) can then be solved for \( \sigma \).

When \( \sigma \) is discretized on \( n \) grid points, the reconstruction problem has the semi-infinite form \( A x \approx b \), where \( x \in \mathbb{R}^p \) is the discrete version of \( \sigma \) and \( A \) and \( b \) have infinitely many rows, corresponding to the infinitely many instances of \( \rho_1 \) and \( \rho_2 \). A fully discrete version can be obtained by considering \( n_1 \) values of \( \rho_1 \) and \( n_2 \) values of \( \rho_2 \), and setting \( n = n_1 n_2 \) to obtain a problem of the form \([1]\). In the remainder of the paper, we study the sketched form of this system \([2]\), for various choices of the sketching matrix \( S \).

3. Sketching with tensor structures

We preface our results with a definition of \((\varepsilon, \delta)\)-\(l^2\)-embedding.

**Definition 1** \(((\varepsilon, \delta)\)-\(l^2\)-embedding). Given matrix \( \tilde{A} \) and \( \varepsilon > 0 \), let \( S \) be a random matrix drawn from a matrix distribution \((\Omega, \mathcal{F}, \Pi)\). If with probability at least \( 1 - \delta \), we have

\[
||S y||^2 - ||y||^2| \leq \varepsilon ||y||^2, \quad \text{for all } y \in \text{Range}(\tilde{A}),
\]

then we say that \( S \) is an \((\varepsilon, \delta)\)-\(l^2\) embedding of \( \tilde{A} \).

Note that \([14]\) depends only on the space \( \text{Range}(\tilde{A}) \) rather than the matrix itself, so we sometimes say instead that the random matrix \( S \) is an \((\varepsilon, \delta)\)-\(l^2\) embedding of the linear vector space \( \text{Range}(\tilde{A}) \). (We use the two terms interchangeably in discussions below.)
The \((\varepsilon, \delta)\)-\(L^2\) embedding property is essentially the only property needed to bound the error resulting from sketching. It can be shown that if \(S\) is an \((\varepsilon, \delta)\)-\(L^2\) embedding for the augmented matrix \(\tilde{A} \defeq [A, b]\), then the two least-squares problems (1) and (2) are similar in the sense of (3), as the following result suggests.

**Theorem 1.** For \(\varepsilon, \delta \in (0, 1/2)\), suppose that \(S\) is an \((\varepsilon, \delta)\)-\(L^2\) embedding of the augmented matrix \(\tilde{A} \defeq [A, b] \in \mathbb{R}^{n \times (p+1)}\). Then with probability at least \(1 - \delta\), we have

\[
\|Ax^* - b\|^2 \leq (1 + 4\varepsilon)\|Ax^* - b\|^2,
\]

where \(x^*\) and \(x^*_s\) are defined in (1) and (2), respectively.

**Proof.** For any \(x \in \mathbb{R}^p\), vector \(y \defeq Ax - b\) is in \(\text{Range}(\tilde{A})\), therefore with probability greater than \(1 - \delta\), we have

\[
1 - \varepsilon \leq \frac{||Sy||^2}{\|y\|^2} = \frac{||S(Ax - b)||^2}{\|Ax - b\|^2} \leq 1 + \varepsilon \quad \text{for all } x \in \mathbb{R}^p.
\]

(15)

For \(x = x^*\), we therefore have

\[
(1 - \varepsilon)\|Ax^* - b\|^2 \leq \|S(Ax^* - b)\|^2 \leq (1 + \varepsilon)\|Ax^* - b\|^2.
\]

(16)

By the definition of \(x^*_s\) in (2), and using (15) with \(x = x^*_s\), we have

\[
(1 - \varepsilon)\|Ax^*_s - b\|^2 \leq \|S(Ax^*_s - b)\|^2 \leq \|S(Ax^* - b)\|^2 \leq (1 + \varepsilon)\|Ax^* - b\|^2.
\]

Thus, for \(0 \leq \varepsilon \leq 1/2\), we have

\[
\|Ax^*_s - b\|^2 \leq \frac{1 + \varepsilon}{1 - \varepsilon}\|Ax^* - b\|^2 \leq (1 + 4\varepsilon)\|Ax^* - b\|^2,
\]

as required. \(\square\)

Given this result, we focus henceforth on whether the various sampling strategies form an \((\varepsilon, \delta)\)-\(L^2\) embedding of the augmented matrix \(\tilde{A} = [A, b]\).

Another theorem that is crucial to our analysis, proved in [18], states that Gaussian matrices are \((\varepsilon, \delta)\)-\(L^2\) embeddings if the number of rows is sufficiently large. This result does not consider tensor structure of \(A\).

**Theorem 2 (Theorem 2.3 from [18]).** Let \(R \in \mathbb{R}^{r \times n}\) be a Gaussian matrix, meaning that each entry \(R_{ij}\) is drawn i.i.d. from a normal distribution \(\mathcal{N}(0, 1)\), and define \(S \in \mathbb{R}^{r \times n}\) to be the scaled Gaussian matrix defined by

\[
S = \frac{1}{\sqrt{r}}R.
\]

For any fixed matrix \(A \in \mathbb{R}^{n \times p}\) and \(\varepsilon, \delta \in (0, 1/2)\), this choice of \(S\) is an \((\varepsilon, \delta)\)-\(L^2\) embedding of \(A\) provided that

\[
r \geq \frac{C}{\varepsilon^2}(\|\log \delta\| + p),
\]

where \(C > 0\) is a constant independent of \(\varepsilon, \delta, n,\) and \(p\).

The lower bound of \(r\) is almost optimal: the bound is independent of the number of equations \(n\), and grows only linearly in the number of unknowns \(p\). That is, the numbers of equations and unknowns in the sketched problem (2) are of the same order. The theorem is proved by constructing a \(\gamma\)-net for the unit sphere in \(\text{Range}(A)\) and applying the Johnson-Lindenstrauss lemma.

Building on the concept of \((\varepsilon, \delta)\)-\(L^2\) embedding and the equivalence between \((\varepsilon, \delta)\)-\(L^2\) embedding and sketching (Theorem 1), we will study the lower bound for \(r\), the number of rows needed in the sketching, if the tensor structure of Case 1 or Case 2 is imposed. Our basic strategy is to decompose the tensor structure into smaller components, to which Theorem 2 can be applied.

We state the results below and present proofs in Sections 4 and 5 for the two different cases.
Recall the notation that we defined in Section 1. The matrices $F, G$ are defined in (5) and $A$ is defined in (6). Both $F$ and $G$ are assumed to have full column rank. We need to design the sketching matrix $S$ to $(\varepsilon, \delta)$-$l^2$-embed $\text{Range}(A)$, the space spanned by $\{f_b \otimes g_b\} \cup \{a_j \triangleq f_j \otimes g_j, j = 1, \ldots, p\}$. In Theorem 3 and 4, we construct the $(\varepsilon, \delta)$-$l^2$-embedding matrix of the Kronecker product $F \otimes G$, which automatically becomes a $(\varepsilon, \delta)$-$l^2$-embedding of its column submatrix $A$. Moreover, we show in Corollaries 1 and 2 that these results can be extended to construct $(\varepsilon, \delta)$-$l^2$ embeddings of the augmented matrix $A$ by constructing $(\varepsilon, \delta)$-$l^2$-embeddings of the Kronecker product of the augmented matrices $\tilde{F} \otimes \tilde{G}$, where

$$\tilde{F} = [F, f_b], \quad \tilde{G} = [G, g_b]. \quad (17)$$

For Case 1, we have the following result.

**Theorem 3.** Consider $S = P \otimes Q \in \mathbb{R}^{r_1 r_2 \times n_1 n_2}$ where $P \in \mathbb{R}^{r_1 \times n_1}, Q \in \mathbb{R}^{r_2 \times n_2}$ are independent scaled Gaussian matrices, defined by

$$P \overset{\text{def}}{=} \frac{1}{\sqrt{r_1}} R, \quad Q \overset{\text{def}}{=} \frac{1}{\sqrt{r_2}} R', \quad R_{ij}, R'_{ij} \text{ are i.i.d. normal.}$$

For any given full rank matrices $F \in \mathbb{R}^{n_1 \times p}, G \in \mathbb{R}^{n_2 \times p}$ and $A \in \mathbb{R}^{n \times p}$ as in (5) and (6), and $\varepsilon, \delta \in (0, 1/2)$, the random matrix $S$ is an $(\varepsilon, \delta)$-$l^2$ embedding of $F \otimes G$ and $A$, provided that

$$r_i \geq C \left(\frac{1}{\varepsilon^2} (|\log \delta| + p), \quad i = 1, 2, \quad (18)$$

where the constant $C > 0$ is independent of $\varepsilon, \delta, n_1, n_2,$ and $p$.

**Corollary 1.** Consider the matrices $S, F, G$, and $A$ from Theorem 3, and assume that the vector $b$ has the form (7). Then for given $\varepsilon, \delta \in (0, 1/2)$, the random matrix $S$ is an $(\varepsilon, \delta)$-$l^2$ embedding of the augmented matrix $\tilde{A} \overset{\text{def}}{=} [A, b]$, provided that

$$r_i \geq C \left(\frac{1}{\varepsilon^2} (|\log \delta| + p + 1), \quad i = 1, 2, \quad (19)$$

where the constant $C > 0$ is independent of $\varepsilon, \delta, n_1, n_2,$ and $p$.

**Proof.** Define the augmented matrices $\tilde{F}$ and $\tilde{G}$ as in (17). We have that

$$\text{Range}(\tilde{F} \otimes \tilde{G}) = \text{span} \{F \otimes G, f_1 \otimes g_b, \ldots, f_p \otimes g_b, f_b \otimes g_1, \ldots, f_b \otimes g_p, b\}.$$ 

Therefore, the linear subspace $\text{Range}(\tilde{A})$ is a subspace of $\text{Range}(\tilde{F} \otimes \tilde{G})$. By applying Theorem 3 to the augmented matrices $\tilde{F}$ and $\tilde{G}$ and using (19), we have that $S$ is an $(\varepsilon, \delta)$-$l^2$ embedding of $\text{Range}(\tilde{F} \otimes \tilde{G})$ as well as its subspace $\text{Range}(\tilde{A})$. 

The result for Case 2 is as follows.

**Theorem 4.** Let $p_i \in \mathbb{R}^{n_1}, q_i \in \mathbb{R}^{n_2}, i = 1, 2, \ldots, r$ be independent random Gaussian vectors and define the sketching matrix $S$ to have the form:

$$S = \frac{1}{\sqrt{r}} \begin{bmatrix} p_1^\top \otimes q_1^\top & \vdots & p_r^\top \otimes q_r^\top \end{bmatrix} \in \mathbb{R}^{r \times n_1 n_2}. \quad (20)$$

For any given full rank matrices $F \in \mathbb{R}^{n_1 \times p}, G \in \mathbb{R}^{n_2 \times p}$ (for $p \geq 5$) and $A \in \mathbb{R}^{n \times p}$ as in (5) and (6), and $\varepsilon, \delta \in (0, 1/2)$, the random matrix $S$ is an $(\varepsilon, \delta)$-$l^2$ embedding of $F \otimes G$ and $A$ provided that

$$r \geq C \max \left\{ \frac{1}{\varepsilon} (|\log \delta| + p^2)^3, \frac{1}{\varepsilon^{5/2}} \right\}, \quad (21)$$

where $C > 0$ is a constant independent of $\varepsilon, \delta, n_1, n_2,$ and $p$. 

Corollary 2. Consider the same matrices $S$, $F$, $G$, and $A$ as in Theorem 1 with $p \geq 5$, and assume that vector $b$ is of the form \((\bar{b})\). Then for given $\varepsilon, \delta \in (0, 1/2)$, the random matrix $S$ is an $(\varepsilon, \delta)$-$l^2$ embedding of the augmented matrix $\tilde{A} \triangleq [A, b]$ provided that

$$r \geq C \max \left\{ \frac{1}{\varepsilon} \left( |\log \delta| + (p + 1)^2 \right)^3, \frac{1}{\varepsilon^{5/2}} \right\},$$

where the constant $C > 0$ is independent of $\varepsilon$, $\delta$, $n_1$, $n_2$, and $p$.

We omit the proof since it is similar to that of Corollary 1.

Theorems 1 and 2 yield the fundamental results that, with high probability, for any fixed overdetermined linear problem, the sketched problem in which $S$ is a Gaussian matrix can achieve optimal residual up to a small multiplicative error. In particular, as will be clear in the proof later, the Case 1 tensor structured sketching $\epsilon$ bound for Case 1 depends on the relative sizes of $(\frac{\bar{b}}{\Sigma})$-

4. CASE 1: PROOF OF THEOREM 3

In this section we present the proof of Theorem 3. We start with technical results.

Lemma 1. Consider natural numbers $r_2$, $n_1$, and $n_2$, and assume that a random matrix $Q \in \mathbb{R}^{r_2 \times n_2}$ is an $(\varepsilon, \delta)$-$l^2$ embedding of $\mathbb{R}^{n_2}$, meaning that with probability at least $1 - \delta$, $Q$ preserves $l^2$ norm with $\varepsilon$ accuracy, that is,

$$\|Qx\|^2 - \|x\|^2 \leq \varepsilon \|x\|^2, \quad \forall x \in \mathbb{R}^{n_2}.$$

Then the Kronecker product $\text{Id}_{n_1} \otimes Q$ is an $(\varepsilon, \delta)$-$l^2$ embedding of $\mathbb{R}^{n_1 \times n_2}$. Similarly if $Q \in \mathbb{R}^{r_1 \times n_1}$ is an $(\varepsilon, \delta)$-$l^2$ embedding of $\mathbb{R}^{n_1}$, then $Q \otimes \text{Id}_{n_2}$ is an $(\varepsilon, \delta)$-$l^2$ embedding of $\mathbb{R}^{n_1 \times n_2}$.

Proof. The proof for the two statements are rather similar, so we prove only the first claim.

Any $x \in \mathbb{R}^{n_1 n_2}$ can be written in the following form

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n_1} \end{bmatrix}, \quad \text{where } x_i \in \mathbb{R}^{n_2}, i = 1, 2, \ldots, n_1.$$

Then

$$(\text{Id}_{n_1} \otimes Q)x = \begin{bmatrix} Q & \cdots & Q \\ \vdots & \ddots & \vdots \\ Q & \cdots & Q \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n_1} \end{bmatrix} = \begin{bmatrix} Qx_1 \\ \vdots \\ Qx_{n_1} \end{bmatrix}.$$

Thus, we have

$$\| (\text{Id}_{n_1} \otimes Q)x \|^2 = \sum_{i=1}^{n_1} \|Qx_i\|^2, \quad \|x\|^2 = \sum_{i=1}^{n_1} \|x_i\|^2. \quad (23)$$

Since $Q$ is an $(\varepsilon, \delta)$-$l^2$ embedding of $\mathbb{R}^{n_2}$, then with probability at least $1 - \delta$, for all $x_i \in \mathbb{R}^{n_2}$, we have

$$\|Qx_i\|^2 - \|x_i\|^2 \leq \varepsilon \|x_i\|^2, \quad \text{for all } i = 1, 2, \ldots, n_1. \quad (24)$$

By using this bound in (23), with probability at least $1 - \delta$, we have for all $x \in \mathbb{R}^{n_1 n_2}$ that

$$\| (\text{Id}_{n_1} \otimes Q)x \|^2 - \|x\|^2 \leq \sum_{i=1}^{n_1} \|Qx_i\|^2 - \|x_i\|^2 \leq \varepsilon \sum_{i=1}^{n_1} \|x_i\|^2 = \varepsilon \|x\|^2,$$
so that \((\text{Id}_{n_1} \otimes Q)\) is an \((\varepsilon, \delta)\)-\(l^2\) embedding of \(\mathbb{R}^{n_1 n_2}\), as claimed.

The following corollary extends the previous result and discusses the embedding property of \(P \otimes Q\).

**Corollary 3.** Assume two random matrices \(P \in \mathbb{R}^{r_1 \times n_1}\) and \(Q \in \mathbb{R}^{r_2 \times n_2}\) are \((\varepsilon, \delta)\)-\(l^2\) embeddings of \(\mathbb{R}^{n_1}\) and \(\mathbb{R}^{n_2}\) respectively. Then the Kronecker product \(P \otimes Q \in \mathbb{R}^{r_1 r_2 \times n_1 n_2}\) is an \((\varepsilon(2 + \varepsilon), 2\delta)\)-\(l^2\) embedding of \(\mathbb{R}^{n_1 n_2}\).

**Proof.** Noting that (see [64] in Appendix A),

\[
P \otimes Q = (P \otimes \text{Id}_{r_2})(\text{Id}_{n_1} \otimes Q),
\]

we have

\[
\|(P \otimes Q)x\|^2 = \|(P \otimes \text{Id}_{r_2})(\text{Id}_{n_1} \otimes Q)x\|^2 = \|(P \otimes \text{Id}_{r_2})y\|^2,
\]

where \(y \triangleq (\text{Id}_{n_1} \otimes Q)x\).

Denote by \((\Omega_1, \mathcal{F}_1, \Pi_1)\) and \((\Omega_2, \mathcal{F}_2, \Pi_2)\) the probability triplets for \(P\) and \(Q\), respectively. Since \(P\) is an \((\varepsilon, \delta)\)-\(l^2\) embedding of \(\mathbb{R}^{n_1}\), with probability at least \(1 - \delta\) in \(\Pi_1\), we have

\[
\|((P \otimes \text{Id}_{r_2})y\|^2 - \|y\|^2| \leq \varepsilon\|y\|^2.
\]

Similarly, with probability at least \(1 - \delta\) for the choice of \(Q\) in \(\Pi_2\), we have

\[
\|((\text{Id}_{n_1} \otimes Q)x\|^2 - \|x\|^2| \leq \varepsilon\|x\|^2,
\]

for all \(x \in \mathbb{R}^{n_1 n_2}\).

Combining the two inequalities, with probability at least \(1 - 2\delta\) in the joint probability space of \(\Pi_1\) and \(\Pi_2\), we have for all \(x \in \mathbb{R}^{n_1 n_2}\) that

\[
\|((P \otimes Q)x\|^2 - \|x\|^2| \leq \|(P \otimes \text{Id}_{r_2})y\|^2 - \|y\|^2| + \|((\text{Id}_{n_1} \otimes Q)x\|^2 - \|x\|^2|
\]

\[
\leq \varepsilon\|y\|^2 + \varepsilon\|x\|^2
\]

\[
= \varepsilon\|((\text{Id}_{n_1} \otimes Q)x\|^2 + \varepsilon\|x\|^2
\]

\[
\leq \varepsilon(2 + \varepsilon)\|x\|^2.
\]

This concludes the proof. \(\square\)

Now we are ready to show the proof of Theorem 3 obtained by applying Theorem 2 to Corollary 3.

**Proof of Theorem 3** For any vector \(y\) in the span of \(F \otimes G\), we can write

\[
y = (U_F \otimes U_G)x, \quad \text{for some } x \in \mathbb{R}^p,
\]

where \(U_F \in \mathbb{R}^{n_1 \times p}\) and \(U_G \in \mathbb{R}^{n_2 \times p}\) collect the left singular vectors of matrices \(F\) and \(G\), respectively. By applying (64) from Appendix A we have

\[
(U_F \otimes U_G) = (U_F \otimes \text{Id}_{n_2})(\text{Id}_p \otimes U_G).
\]

It is easy to see that the matrix \(\text{Id}_p \otimes U_G\) has orthonormal columns, therefore it is an isometry. Similarly, the matrices \(U_F \otimes \text{Id}_{n_2}\) and \(U_F \otimes U_G\) are isometries. As a consequence, we have \(\|y\|^2 = \|x\|^2\). From (64) in Appendix A we have by defining \(\tilde{P} \triangleq PU_F \in \mathbb{R}^{r_1 \times p}\) and \(\tilde{Q} \triangleq QU_G \in \mathbb{R}^{r_2 \times p}\) that

\[
Sy = (P \otimes Q)(U_F \otimes U_G)x = (PU_F) \otimes (QU_G)x = (\tilde{P} \otimes \tilde{Q})x.
\]

Due to the orthogonality of \(U_F\) and \(U_G\), the random matrices \(\tilde{P}\) and \(\tilde{Q}\) are also independent Gaussian matrices. According to Theorem 2 for any pair \(\varepsilon, \delta \in (0, 1/2)\), by choosing \(r_i\) to satisfy

\[
r_i \geq \frac{C}{\varepsilon^2(\log \delta + p)}, \quad i = 1, 2,
\]

(25)
we have that $\tilde{P}$ and $\tilde{Q}$ are both $(\tilde{\varepsilon}, \tilde{\delta})$-$l^2$ embeddings of $\mathbb{R}^p$. Thus, from Corollary 3, the tensor product $(\tilde{P} \otimes \tilde{Q})$ is an $(\varepsilon(2 + \tilde{\varepsilon}), 2\tilde{\delta})$-$l^2$ embedding of $\mathbb{R}^{p^2}$, meaning with probability at least $1 - 2\tilde{\delta}$, we have
\[
\|((\tilde{P} \otimes \tilde{Q})x) - x\|^2 \leq \varepsilon(2 + \tilde{\varepsilon})\|x\|^2, \quad \text{for all } x \in \mathbb{R}^{p^2}.
\]
Recalling $\|x\|^2 = \|y\|^2$ and (25), we have that
\[
\|\langle S y \rangle - \|y\|\|^2 \leq \varepsilon(2 + \tilde{\varepsilon})\|y\|^2, \quad \text{for all } y \in \text{span}\{F \otimes G\}.
\]
By defining $\varepsilon = \tilde{\varepsilon}(2 + \varepsilon)$ and $\delta = 2\tilde{\delta}$, we have
\[
\tilde{\varepsilon} = \frac{\varepsilon}{\sqrt{1 + \varepsilon} + 1}, \quad \text{and} \quad \tilde{\delta} = \frac{\delta}{2}.
\]
Note that if $\varepsilon$ and $\delta$ are in $(0, 1/2)$, then $\tilde{\varepsilon}$ and $\tilde{\delta}$ are also in this interval, so (26) applies. By substituting into (26) we obtain
\[
r_i \geq \frac{C}{\varepsilon^2}(|\log \delta| + p), \quad i = 1, 2.
\]
The constant $C$ here is different from the value in (26) but can still be chosen independently of $\varepsilon$, $\delta$, $n_1$, $n_2$, and $p$. We conclude that $S = P \otimes Q$ is an $(\varepsilon, \delta)$-embedding of $F \otimes G$ and thus also an $(\varepsilon, \delta)$-embedding of $A$. □

5. Case 2: Proof of Theorem 4

In this section we investigate Case 2 sketching matrices, which have the form (20).

We prove Theorem 4 in two major steps. First, in Section 5.1, we investigate the accuracy and probability of embedding any given vector $y \in \text{Span}\{F \otimes G\}$. Second, in Section 5.2, we extend this study to deal with the whole space $\text{Span}\{F \otimes G\}$. To do so, we first build a $\gamma$-net over the unit sphere in $\text{Span}\{F \otimes G\}$ so that we can “approximate” the space using a finite set of vectors. By adjusting $\varepsilon$ and $\delta$, one not only preserves the norm, but also the angles between the vectors on the net. We then map the net back to the space to show that $S$ preserves the norm of the vectors in the whole space. This standard technique is used in [18] to prove their Theorem 2

5.1. Embedding a given vector. In this subsection, prove the following result. (The proof appears at the end of the subsection.)

**Proposition 1.** Given two full rank matrices $F$ and $G$ as in (5) and $\varepsilon \in (0, 1/2)$, let $S \in \mathbb{R}^{r \times n_1 n_2}$ have the form of (20), with $p_i$ and $q_i$, $i = 1, 2, \ldots, r$ being i.i.d. Gaussian vectors. Then for any fixed $y \in \text{Span}\{F \otimes G\}$, we have that
\[
\Pr \left( \|\langle S y \rangle - \|y\|\|^2 > \varepsilon\|y\|^2 \right) \leq 5r \exp \left( \frac{3}{4}p^{1/2} \right) \exp \left( -\frac{1}{2}r^{1/3}\varepsilon^{1/3} \right),
\]
provided that
\[
r \geq 8 \cdot 3^{3/2} \cdot \max\{\varepsilon^{-5/2}, p^{3/2}\varepsilon^{-1}\}.
\]

Essentially, this proposition says that $S$ is an $(\varepsilon, 5r \exp ((3/4)p^{1/2}) \exp (-\varepsilon^{1/3}))$-$l^2$ embedding of any $y \in \text{Span}\{F \otimes G\}$. The contribution from the factor $\exp \left( -\frac{1}{2}r^{1/3}\varepsilon^{1/3} \right)$ is small when $r$ is large.

We start with several technical lemmas. Lemma 2 identifies $\|\langle S y \rangle - \|y\|\|^2 / \|y\|$ with a particular type of random variable, and we discuss the tail bound for this random variable in Lemma 4. Lemma 3 contains some crucial estimates to be used in Lemma 4.
Lemma 2. Given two full rank matrices $F$ and $G$ as in \((5)\), consider $S$ defined as in \((20)\). Then there exists a diagonal positive semi-definite matrix $\Sigma$ with $\text{Tr}(\Sigma^2) = 1$ so that for any $y \in \text{span}\{F \otimes G\}$ with $\|y\| = 1$, we have

$$\|Sy\|^2 \sim 1 \sum_{i=1}^{r} \zeta_i^2,$$

where $\sim$ denotes equal in distribution and $\zeta_i, \eta_i \in \mathbb{R}^p$ are independent Gaussian vectors drawn from $\mathcal{N}(0, I_d)$. 

Proof. From \((20)\) we have

$$Sy = \frac{1}{\sqrt{r}} \begin{pmatrix} (p^T_1 \otimes q_1^T) y \\ \vdots \\ (p^T_r \otimes q_r^T) y \end{pmatrix} \implies \|Sy\|^2 = \frac{1}{r} \sum_{i=1}^{r} \zeta_i^2,$$

where $\zeta_i \overset{\text{def}}{=} (p_i^T \otimes q_i^T)y$. Since $p_i$ and $q_i$ are independent Gaussian vectors, all random variables $\zeta_i, i = 1, 2, \ldots, r,$ are drawn i.i.d. from the same distribution.

We consider now the behavior of $\zeta \overset{\text{def}}{=} (p^T \otimes q^T)y$ for Gaussian vectors $p$ and $q$. Notice that for any $y \in \mathbb{R}^{n_1 \times n_2} \in \text{Span}\{F \otimes G\}$, there exists $x \in \mathbb{R}^{p^2}$ such that

$$y = (U_F \otimes U_G)x, \quad \text{with } \|x\| = 1,$$

where $U_F$ and $U_G$ collect the left singular vectors of $F$ and $G$, respectively. We thus obtain from \((64)\) that

$$\zeta = (p^T \otimes q^T)y = (p^T \otimes q^T)(U_F \otimes U_G)x = ((p^T U_F) \otimes (q^T U_G))x = (\tilde{p}^T \otimes \tilde{q}^T)x,$$

where $\tilde{p} \overset{\text{def}}{=} U_F^T p \in \mathbb{R}^p$ and $\tilde{q} \overset{\text{def}}{=} U_G^T q \in \mathbb{R}^p$ are i.i.d. Gaussian vectors as well. By applying \((65)\) and \((66)\), we obtain

$$\zeta = (\tilde{p}^T \otimes \tilde{q}^T)x = \tilde{q}^T \text{Mat}(x)\tilde{p},$$

where $\text{Mat}(x) \in \mathbb{R}^{p \times p}$ is the matricization of $x$, discussed in Appendix A. By using the singular value decomposition $\text{Mat}(x) = U \Sigma V^T$, we obtain

$$\text{Tr}(\Sigma^2) = \|\text{Mat}(x)\|_F^2 = \|x\|^2 = 1,$$

where $\| \cdot \|_F$ denotes the Frobenius norm of a matrix. By substituting into \((27)\), we obtain

$$\zeta = (U^T \tilde{q}) \Sigma V^T \tilde{p} = \xi^T \Sigma \eta,$$

where

$$\xi \overset{\text{def}}{=} U^T \tilde{q} \in \mathbb{R}^p, \quad \text{and} \quad \eta \overset{\text{def}}{=} V^T \tilde{p} \in \mathbb{R}^p$$

are again i.i.d. Gaussian vectors in $\mathbb{R}^p$. This completes the proof. \hfill \Box

Lemma 3. For any fixed diagonal semi-positive definite matrix $\Sigma \overset{\text{def}}{=} \text{diag}\{\sigma_1, \ldots, \sigma_p\}$ such that $\text{Tr}(\Sigma^2) = 1$, define the random variable $\zeta$ to be $\zeta \overset{\text{def}}{=} \xi^T \Sigma \eta$, with $\xi$ and $\eta$ being i.i.d. random Gaussian vectors with $p$ components. Then $\zeta$ satisfies the following properties:

1. 

$$\Pr(\|\zeta\| > t) \leq \begin{cases} 2 \exp \left( -\frac{t^2 - (2\sqrt{t} \eta t)}{4\sqrt{t}} \right) & \text{if } \sqrt{p} \leq t \leq 2\sqrt{p} \\ 2 \exp \left( -\frac{t^2 - 3\eta t}{4\sqrt{t}} \right) & \text{if } t \geq 2\sqrt{p} \end{cases},$$

2. 

$$\mathbb{E} [\zeta^2] = 1 \quad \text{and} \quad \mathbb{E} [\zeta^4] \leq 9,$$

3. 

$$\mathbb{E} \left[ (|\zeta|^2 - \mathbb{E}[\zeta^2])^2 \right] \leq 8.$$
Proof. For any $s > 0$ and $t \geq 0$, we apply Markov’s inequality to derive

$$
\Pr(\zeta > t) = \Pr(e^{s\zeta} > e^{st}) \leq e^{-st} \mathbb{E} \left[ \exp(s\xi^\top \Sigma \eta) \right].
$$

(31)

Noting that $2\xi^\top \Sigma \eta \leq \|\Sigma^{1/2} \xi\|^2 + \|\Sigma^{1/2} \eta\|^2$, we use the independence of $\xi$ and $\eta$ to deduce that

$$
\mathbb{E} \left[ \exp(s\xi^\top \Sigma \eta) \right] \leq \mathbb{E} \left[ \exp \left(\frac{s}{2}(\|\Sigma^{1/2} \xi\|^2 + \|\Sigma^{1/2} \eta\|^2)\right) \right] = \mathbb{E} \left[ e^{(s/2)\|\Sigma^{1/2} \xi\|^2} \right] \mathbb{E} \left[ e^{(s/2)\|\Sigma^{1/2} \eta\|^2} \right].
$$

(32)

For the first term on the RHS of (32), using independence of the $\xi_i$ and the concave Jensen’s inequality, we have that

$$
\mathbb{E} \left[ e^{(s/2)\|\Sigma^{1/2} \xi\|^2} \right] \leq \prod_{i=1}^p \left( \mathbb{E} \left[ e^{s\xi_i^2/2} \right] \right)^{\sigma_i} \leq \prod_{i=1}^p \left( \mathbb{E} \left[ e^{s\xi_i^2/2} \right] \right)^{\sigma_i},
$$

where we used $0 \leq \sigma_i \leq 1$, $i = 1, 2, \ldots, r$ to apply the concave Jensen’s inequality, and $\xi_i \sim \mathcal{N}(0, 1)$. According to Proposition 2 (see Appendix A.2), $\xi_i^2 - 1$ is a sub-exponential random variable with parameters $(2, 4)$. Thus from (67), with $\lambda = 2, b = 4$, and $s$ replaced by $s/2$, we have

$$
\mathbb{E} \left[ e^{(s/2)\|\Sigma^{1/2} \xi\|^2} \right] \leq \left( \mathbb{E} \left[ e^{s\xi_i^2/2} \right] \right)^{\sigma_i} = \left( \mathbb{E} \left[ e^{s\xi_i^2/2} \right] \right)^{\sigma_i} \leq e^{(s^2/2)\text{Tr}(\Sigma)},
$$

for $0 < s < 1/2$.

Since, by Hölder’s inequality, we have

$$
\text{Tr}(\Sigma) = \sum_{i=1}^p \sigma_i \leq \left( \sum_{i=1}^p \sigma_i \right)^{1/2} \sqrt{p} = \sqrt{p},
$$

it follows that

$$
\mathbb{E} \left[ e^{(s/2)\|\Sigma^{1/2} \xi\|^2} \right] \leq e^{(s^2/2)\sqrt{p}},
$$

for $s \in (0, 1/2)$.

The same bound holds for second term on the right-hand side of (32). When we substitute these bounds into (31) and (32), we obtain

$$
\Pr(\zeta > t) \leq \exp \left( \sqrt{p} s^2 - (t - \sqrt{p}) s \right).
$$

By minimizing the right-hand side over $s \in [0, 1/2]$, we obtain

$$
\Pr(\zeta > t) \leq \begin{cases} 
    e^{-\frac{(t-\sqrt{p})^2}{2\sqrt{p}}} & \text{if } \sqrt{p} \leq t \leq 2\sqrt{p} \\
    e^{-\frac{(2t-3\sqrt{p})^2}{4\sqrt{p}}} & \text{if } t \geq 2\sqrt{p}
\end{cases}
$$

Due to symmetry, we have the same bound for $\Pr(\zeta < -t)$, so (28) follows.

To show the second statement, we notice that

$$
\mathbb{E} \left[ (\zeta^2 - \mathbb{E}[\zeta^2])^2 \right] = \mathbb{E}[\zeta^4] - (\mathbb{E}[\zeta^2])^2.
$$

(33)

By considering $\zeta = \sum_{i=1}^p \sigma_i \xi_i \eta_i$, the second moment can be calculated directly:

$$
\mathbb{E}[\zeta^2] = \mathbb{E} \left[ \sum_{i,j=1}^p \sigma_i \sigma_j \xi_i \xi_j \eta_i \eta_j \right] = \mathbb{E} \left[ \sum_{i=1}^p \sigma_i^2 \xi_i^2 \eta_i^2 \right] = \sum_{i=1}^p \sigma_i^2 = 1,
$$

(34)

where we used the independence of $\xi_i$ and $\eta_i$, the fact that $\mathbb{E} \xi_i = \mathbb{E} \eta_i = 0$ and $\mathbb{E} \xi_i^2 = \mathbb{E} \eta_i^2 = 1$. 

To control the fourth moment, we notice that

$$\mathbb{E}[\zeta^4] = \mathbb{E}\left[\sum_{i,j,k,l} \sigma_i \sigma_j \sigma_k \sigma_l \xi_i \xi_j \xi_k \xi_l \eta_i \eta_j \eta_k \eta_l\right].$$

Due to the independence and the fact that all odd moments vanish for Gaussian random variables, the only terms in the summation that survive either have all indices equal ($i = j = l = k$) or two indices equal to one value while the other two indices equal a different value, for example $i = j$ and $k = l$ but $i \neq k$. Altogether, we obtain

$$\mathbb{E}[\zeta^4] = 3\mathbb{E}\left[\sum_{i \neq k} \sigma_i^2 \sigma_k^2 \xi_i^2 \xi_k^2 \eta_i^2 \eta_k^2\right] + \mathbb{E}\left[\sum_i \sigma_i^4 \xi_i^4 \eta_i^4\right],$$

where the coefficient in front of the first term comes from $(\binom{4}{1}/\binom{2}{1}) = 3$. Considering $\mathbb{E} \xi^2 = 1$ and $\mathbb{E} \xi^4 = 3$, we have

$$\mathbb{E}[\zeta^4] = 3\sum_{i \neq k} \sigma_i^2 \sigma_k^2 + 9 \sum_i \sigma_i^4 = 3 \sum_{i,k=1}^p \sigma_i^2 \sigma_k^2 + 6 \sum_i \sigma_i^4 \leq 3 \sum_{i,k=1}^p \sigma_i^2 \sigma_k^2 + 6 \sum_i \sigma_i^2 = 3 \left(\sum_{i=1}^p \sigma_i^2\right) \left(\sum_{k=1}^p \sigma_k^2\right) + 6 \sum_i \sigma_i^2 = 9,$$

where we used $\sigma_i^4 \leq \sigma_i^2$. By substituting (34) and (35) into (33), we have

$$\mathbb{E}\left[(|\zeta|^2 - \mathbb{E}[\zeta^2])^2\right] = \mathbb{E}[\zeta^4] - (\mathbb{E}[\zeta^2])^2 \leq 9 - 1^2 = 8,$$

which concludes the proof. \(\square\)

**Lemma 4.** Let $\zeta_i, i = 1, 2, \ldots, r$ be i.i.d. copies of the random variable $\zeta$ defined in Lemma 3. Then if

$$r \geq 8 \cdot 3^{3/2} \cdot \max\{t^{-5/2} \cdot p^{3/2} t^{-1}\},$$

we have

$$\Pr\left(\left|\frac{1}{r} \sum_{i=1}^r (\zeta_i^2 - \mathbb{E}[\zeta_i^2])\right| > t\right) \leq 5r \exp\left(\frac{3}{4} p^{1/2}\right) \exp\left(-\frac{1}{2} r^{1/3} t^{1/3}\right), \quad \text{for } t \in [0, 1].$$

**Proof.** Let $E^t$ be the event defined as follows:

$$E^t \overset{\text{def}}{=} \left\{\frac{1}{r} \sum_{i=1}^r (\zeta_i^2 - \mathbb{E}[\zeta_i^2]) > t\right\}.$$

Due to the symmetry of $\sum_{i=1}^r \zeta_i^2 - \mathbb{E}[\zeta_i^2]$, the probability in (37) is $2 \Pr(E^t)$. We now estimate $\Pr(E^t)$. For any fixed large number $M$, we define the following event, for $i = 1, 2, \ldots, r$:

$$E_i^M \overset{\text{def}}{=} \{\zeta_i^2 \leq M\} = \{\zeta_i^2 - 1 \leq M - 1\}.$$

Clearly, we have

$$\Pr\left(E^t\right) = \Pr\left(E^t \cap \bigcap_{i=1}^r E_i^M\right) + \Pr\left(E^t \cap \left(\bigcap_{i=1}^r E_i^M\right)^C\right).$$

We now estimate the two terms.
By combining (40) and (41) in (38), we have

\[ \Pr (E^t \cap (\cap_{i=1}^r E_i^M)) = \Pr (E^t \mid (\cap_{i=1}^r E_i^M)) \cdot \Pr ((\cap_{i=1}^r E_i^M)) \leq \Pr (E^t \mid (\cap_{i=1}^r E_i^M)). \]  

(39)

Denoting \( X_i \stackrel{d}{=} \zeta_i^2 - \mathbb{E} [\zeta_i^2] \), and realizing that \( \mathbb{E} [\zeta_i^2] = 1 \) according to (29) of Lemma 3, then \( E_i^M = \{ X_i \leq M - 1 \} \). Estimating (39) now amounts to controlling the probability of \( \sum_{i=1}^r X_i > rt \) assuming that \( X_i \leq M - 1 \) for all \( i = 1, 2, \ldots, r \). By applying Bernstein's inequality (68), we have

\[ \Pr (E^t \mid (\cap_{i=1}^r E_i^M)) = \Pr \left( \sum_{i=1}^r X_i > rt \mid X_i \leq M - 1, \; i = 1, 2, \ldots, r \right) \leq \exp \left( - \frac{rt^2}{\mathbb{E} [X_i^2]} - (M - 1)rt/3 \right). \]

From (30), we have \( \mathbb{E} [X_i^2] \leq 8 \) from Lemma 3; we further have:

\[ \Pr (E^t \mid (\cap_{i=1}^r E_i^M)) \leq \exp \left( - \frac{3rt^2}{48 + 2(M - 1)t} \right), \]

(40)

which gives the upper bound of the first term in (38).

2. For the second term in (38), we note that

\[ \Pr \left( E^t \cap (\cap_{i=1}^r E_i^M)^c \right) \leq \Pr ((\cap_{i=1}^r E_i^M)^c) = \Pr (\cup_{i=1}^r (E_i^M)^c) \leq r \Pr ((E_i^M)^c). \]

By applying (28) from Lemma 3 with \( t = \sqrt{M} \), we have

\[ \Pr ((E_i^M)^c) = \Pr (\zeta_i^2 > M) = \Pr \left( |\zeta_i| > \sqrt{M} \right) \leq \begin{cases} 2e^{-\frac{\sqrt{\pi} - \sqrt{\pi}^2}{4\sqrt{\pi}}} & \text{if } p \leq M \leq 4p, \\ 2e^{-\frac{(2\pi - \sqrt{\pi} - 3\pi)}{4\sqrt{\pi}}} & \text{if } M \geq 4p, \end{cases} \]

and thus

\[ \Pr (E^t \cap (\cap_{i=1}^r E_i^M)^c) \leq \begin{cases} 2re^{-\frac{(\sqrt{\pi} - \sqrt{\pi}^2)}{4\sqrt{\pi}}} & \text{if } p \leq M \leq 4p, \\ 2re^{-\frac{(2\pi - \sqrt{\pi} - 3\pi)}{4\sqrt{\pi}}} & \text{if } M \geq 4p. \end{cases} \]

(41)

By combining (40) and (41) in (38), we have

\[ \Pr (E^t) \leq \exp \left( - \frac{3rt^2}{48 + 2(M - 1)t} \right) + \begin{cases} 2re^{-\frac{(\sqrt{\pi} - \sqrt{\pi}^2)}{4\sqrt{\pi}}} & \text{if } p \leq M \leq 4p, \\ 2re^{-\frac{(2\pi - \sqrt{\pi} - 3\pi)}{4\sqrt{\pi}}} & \text{if } M \geq 4p. \end{cases} \]

(42)

To find a sharp bound of \( \Pr (E^t) \), we choose a suitable value of \( M \). We set

\[ M = r^{2/3}t^{2/3}, \]

(43)

where \( r \) satisfies the lower bound (45). Since \( r \geq 8 \cdot 3^{1/2} \cdot p^{3/2}t^{-1} \), we have \( r^{2/3} \geq 12pt^{-2/3} \), so that

\[ M = r^{2/3}t^{2/3} \geq 12p > 4p, \]

(44)

so the second case applies in (42). Since \( r \geq 3^{1/2} \cdot 2^3 \cdot t^{-5/2} \), we have \( r^{2/3} \geq 12t^{-5/3} \), so that

\[ Mt = r^{2/3}t^{5/3} \geq 12, \]

so that, for the denominator of the first term in (42), we have

\[ 48 + 2(M - 1)t = 6Mt + 48 - 2t - 4Mt \leq 6Mt. \]

(45)
we obtain
\[
\Pr(E^t) \leq \exp\left(\frac{1}{2} \frac{rt}{M}\right) + 2r \exp\left(\frac{3}{4} p^{1/2}\right) \exp\left(-\frac{1}{2} M^{1/2}\right).
\] (46)

With our chosen $M$ from (43), we see that the two exponential terms involving $M$ in this expression are both equal to $\exp(-r^{1/3}t^{1/3}/2)$. Additionally, since $p \geq 1$ and $r \geq 1$, we have $2r \exp(3p^{1/2}/4) > 4$. Thus, from (46), we obtain

\[
\Pr(E^t) \leq (5/2) r \exp\left(\frac{3}{4} p^{1/2}\right) \exp\left(-\frac{1}{2} r^{1/3} t^{1/3}\right).
\] (47)

We obtain the result by multiplying the right-hand side by 2, as discussed at the start of the proof. \hfill \Box

Proposition 1 is a direct consequence of Lemmas 2 and 3.

Proof of Proposition 1. For any $y \in \text{Span}\{F \otimes G\}$, denote $\hat{y} = \frac{y}{\|y\|}$, then $\|\hat{y}\| = 1$. From Lemma 2, we have

\[
\|Sy\|^2 - \|\hat{y}\|^2 = \frac{1}{r} \sum_{i=1}^{r} \zeta_i^2 ,
\]

where $\xi_i, \eta_i \in \mathbb{R}^p$ are independent Gaussian vectors drawn from $\mathcal{N}(0, \Sigma_\eta)$, and thus

\[
\frac{\|Sy\|^2 - \|\hat{y}\|^2}{\|\hat{y}\|^2} = \frac{1}{r} \sum_{i=1}^{r} \zeta_i^2 - 1.
\]

By setting $t = \varepsilon$ in (47) from Lemma 4, we have

\[
\Pr\left(\left|\frac{\|Sy\|^2 - \|\hat{y}\|^2}{\|\hat{y}\|^2}\right| > \varepsilon\right) = \Pr\left(\frac{1}{r} \sum_{i=1}^{r} (\zeta_i^2 - 1) > \varepsilon\right) \leq 5r \exp\left(\frac{3}{4} p^{1/2}\right) \exp\left(-\frac{1}{2} r^{1/3} t^{1/3}\right),
\]

conditioned on $r \geq 8 \cdot 3^{3/2} \cdot \max\{\varepsilon^{-5/2}, p^{3/2} \varepsilon^{-1}\}$, as required. \hfill \Box

5.2. Proof of Theorem 4. Proposition 1 shows that the probability of the sketching matrix $S$ of the form (20) preserving the norm of a fixed given vector chosen from the range space $\text{Range}(F \otimes G)$. To show the preservation of norm holds true over the entire column space, we follow the technique in the proof in [18]. We construct a $\gamma$-net over the unit sphere in $\text{Range}(F \otimes G)$ and show that for $r$ sufficiently large, with high probability, the angles between any vectors in the net will be preserved with high accuracy. Preservation of angles on the $\gamma$-net can be translated to the norm preservation over the entire space.

We show in Lemma 5 that angles can be preserved with the sampling matrix $S$ of the form (20). In Lemma 7, we calculate the cardinality of the $\gamma$-net. The fact that preservation of angle leads to the preservation of norms on the space is justified in Lemma 6. The three results can be combined into a proof for Theorem 4, which we complete at the end of the section.

Lemma 5. Let $V$ be a collection of vectors in $\mathbb{R}^n$ with cardinality $|V| = f$ and let

\[
\tilde{V} \defeq \{ u \pm \nu : u, \nu \in V \}.
\]

Suppose a random matrix $S$ $(\varepsilon, \delta)$-$t^2$ preserves norm of all $\tilde{v} \in \tilde{V}$. That is, for each $\tilde{v} \in \tilde{V}$, with probability at least $1 - \delta$, if we have

\[
\|S\tilde{v}\|^2 - \|\tilde{v}\|^2 < \varepsilon \|\tilde{v}\|^2,
\]

then $S$ preserves the angle between all elements in $V$ with probability at least $1 - 4f^2 \delta$, namely:

\[
\Pr\left(\|S\tilde{v}\| - \langle u, \nu \rangle \leq \varepsilon \|u\| \|\nu\|\right) > 1 - 4f^2 \delta, \quad \text{for all } u, \nu \in V.
\]
Proof. Without loss of generality, we assume all vectors in \( V \) are unit vectors. It is straightforward to see that 
\[ |\hat{V}| \leq f^2. \]
Since \( S \) is embeds all vectors in \( \hat{V} \), we have
\[ \Pr \left( |\|Sv\| - \|v\| | < \varepsilon \|v\|^2 \right) \leq 1 - f^2\delta, \quad \forall v \in \hat{V}. \tag{48} \]

Considering \( u, v \in V \), we denote \( s \triangleq u + v \in \hat{V} \) and \( t \triangleq u - v \in \hat{V} \) and use the parallelogram equality:
\[ \langle u, v \rangle = \frac{1}{4} \left( |s|^2 - |t|^2 \right), \quad \langle Su, Sv \rangle = \frac{1}{4} \left( |Ss|^2 - |St|^2 \right), \]
so that
\[ \langle Su, Sv \rangle - \langle u, v \rangle = \frac{1}{4} \left( |Ss|^2 - |s|^2 - (|St|^2 - |t|^2) \right). \]

From (48), we have, with probability at least \( 1 - f^2\delta \), that
\[ |\langle Su, Sv \rangle - \langle u, v \rangle| \leq \frac{1}{4} \left( |Ss|^2 - |s|^2 + |St|^2 - |t|^2 \right) \]
\[ \leq \frac{\varepsilon}{4} \left( |s|^2 + |t|^2 \right) = \frac{\varepsilon}{4} \left( |u + v|^2 + |u - v|^2 \right) = \frac{\varepsilon}{4} (2|u|^2 + 2|v|^2) = \varepsilon, \]
which completes the proof. \( \square \)

We now define the \( \gamma \)-net, and show that preservation of angles on this net leads to preservation of norms.

**Definition 2.** Denote the unit sphere in space \( \text{Range}(F \otimes G) \) by \( S \), that is,
\[ S \triangleq \left\{ y \in \mathbb{R}^{n_1n_2} : \ y = (F \otimes G)x \text{ for some } x \in \mathbb{R}^p \text{ and } \|y\| = 1 \right\}. \tag{49} \]

For fixed \( \gamma \in (0, 1) \), we call \( G \) a \( \gamma \)-net of \( S \) if \( G \) is a finite subset of \( S \) such that for any \( y \in S \), there exists \( w \in G \) such that \( \|w - y\| \leq \gamma \).

**Lemma 6.** Let \( S \) and \( G \) be as in Definition 2, for some \( \gamma \in (0, 1) \). Then preservation of angle on \( G \) leads to the preservation of norm in \( S \). That is, if
\[ |\langle Sw, Sw' \rangle - \langle w, w' \rangle| \leq \varepsilon, \quad \text{for all } w, w' \in G, \tag{50} \]
then
\[ \|Sy\|^2 - \|y\|^2 \leq \frac{\varepsilon}{(1 - \gamma)^2}, \quad \text{for all } y \in S. \]

**Proof.** We claim first that any \( y \in \text{Range}(F \otimes G) \) can be expressed as an infinite series (not necessarily unique):
\[ y = y^0 + y^1 + y^2 + \ldots, \tag{51} \]
where
\[ y^i = c_i w_i, \quad |c_i| \leq \gamma^i \quad \text{and} \quad w_i \in G. \]

We use the definition of \( G \) to construct this representation. First, according to the definition, there exists \( w_0 \in G \) such that
\[ \|y - c_0 w_0\| \leq \gamma, \quad \text{with} \quad c_0 = 1. \]

Let \( y^0 = w_0 \), then it is immediate that \( \frac{y - y^0}{\|y - y^0\|} \in S \). Using the definition of \( G \) again, there exists \( w_1 \in G \) such that
\[ \left\| \frac{y - y^0}{\|y - y^0\|} - w_1 \right\| \leq \gamma, \]
or equivalently
\[ \|y - y^0 - c_1 w_1\| \leq \gamma \|y - y^0\| \leq \gamma^2, \quad \text{where} \quad c_1 \triangleq \|y - y^0\|. \]

The rest of the construction follows inductively.
To prove the result, we multiply this representation of \( y \) by \( S \) and take norms to obtain
\[
\|Sy\|^2 = \sum_{i,j=0}^{\infty} \langle Sy^i, Sy^j \rangle = \sum_{i,j=0}^{\infty} c_ic_j \langle Sw_i, Sw_j \rangle, \quad \text{and} \quad \|y\|^2 = \sum_{i,j=0}^{\infty} \langle y_i, y_j \rangle = \sum_{i,j=0}^{\infty} c_ic_j \langle w_i, w_j \rangle. \tag{52}
\]
According to \([50]\), we have
\[
\|\|Sy\|^2 - \|y\|^2\| \leq \varepsilon \sum_{i,j=0}^{\infty} |c_ic_j| |\langle Sw_i, Sw_j \rangle - \langle w_i, w_j \rangle| \leq \varepsilon \sum_{i,j=0}^{\infty} |c_ic_j| \leq \frac{\varepsilon}{(1-\gamma)^2},
\]
as required. \( \square \)

The size of the net can also be controlled, as shown below.

**Lemma 7.** Let \( S \) be the unit sphere of \( F \otimes G \), defined in \([49]\). Then for any \( \gamma \in (0,1) \), there exists a \( \gamma \)-net \( G \) of \( S \) such that
\[
|G| \leq \left(1 + \frac{4}{\gamma}\right)^{p^2}.
\]
**Proof.** First, let \( S^{p^2-1} \subset \mathbb{R}^{p^2} \) be the unit sphere. In this sphere, one can construct a \( \frac{1}{4} \)-net \( G' = \{x_i, i = 1, 2, \ldots \} \). Furthermore, we set each \( x_i \) on \( S^{p^2-1} \) at least \( \gamma/2 \) away from each other, meaning that the balls \( B(x_i, \gamma/4) \) centered at these points with radius \( \gamma/4 \) are disjoint. We thus obtain
\[
\cup_{x \in \mathcal{G}} B(x, \gamma/4) \subset B(0, 1 + \gamma/4),
\]
meaning that
\[
\sum_{x \in \mathcal{G}'} \text{Vol}(B(x, \gamma/4)) \leq \text{Vol}(B(0, 1 + \gamma/4)) \implies |\mathcal{G}'| \text{Vol}(B(0, 1)) \left(\frac{\gamma}{4}\right)^{p^2} \leq \text{Vol}(B(0, 1)) \left(\frac{\gamma + 4}{4}\right)^{p^2}.
\]
As a consequence, we have
\[
|\mathcal{G}'| \leq \left(\frac{\gamma + 4}{\gamma}\right)^{p^2} = \left(1 + \frac{4}{\gamma}\right)^{p^2}.
\]
Denote by \( U_F \) and \( U_G \) the left singular vectors of \( F \) and \( G \), respectively, and define \( \mathcal{G} \) to be the image of \( U_F \otimes U_G \) applied to \( \mathcal{G}' \), that is,
\[
\mathcal{G} \overset{\text{def}}{=} \{y = (U_F \otimes U_G)x : x \in \mathcal{G}'\}.
\]
Since \( S \) can be rewritten as
\[
S = \{y \in \mathbb{R}^{n_1 n_2} : y = (U_F \otimes U_G)x \quad \text{for some} \ x \in \mathbb{R}^{p^2} \quad \text{and} \quad \|x\| = 1\},
\]
we see that \( U_F \otimes U_G \) is indeed an isometry between \( S^{p^2-1} \) and \( S \). Notice that \( \mathcal{G}' \) is a \( \gamma \)-net of \( S^{p^2-1} \), its image \( \mathcal{G} \) under the isometry \( U_F \otimes U_G \) is therefore a \( \gamma \)-net of \( \mathcal{G} \). The lemma follows since \( |\mathcal{G}| = |\mathcal{G}'| \). \( \square \)

**Proof of Theorem 4** Without loss of generality, it suffices to show \( S \) preserves norm with high accuracy and high probability over the unit sphere in \( \text{Range}(F \otimes G) \), defined by
\[
S \overset{\text{def}}{=} \{y \in \mathbb{R}^{n_1 n_2} : y = (F \otimes G)x \quad \text{for some} \ x \in \mathbb{R}^{p^2} \quad \text{and} \quad \|y\| = 1\}.
\]
Note from Lemma 7 that for given \( \gamma \in (0,1) \), one can construct a \( \gamma \)-net \( \mathcal{G} \) of \( S \) of size \( f = (1 + \frac{4}{\gamma})^{p^2} \). Given \( \varepsilon_1 \in (0,1/2) \), then on this \( \mathcal{G} \), according to Proposition 1 and Lemma 7 if we assume
\[
r \geq 8 \cdot 3^{3/2} \cdot \max\{\varepsilon_1^{5/2}, \varepsilon_1^{3/2} \varepsilon_1^{-1}\} \tag{53}
\]
then with probability at least $1 - \delta_2$ with
\[ \delta_2 \leq 20r^2 \exp \left( \frac{3}{4} p^{1/2} \right) \exp \left( -\frac{1}{2} r^{1/3} \varepsilon^{1/3} \right) = 20r \left( 1 + \frac{4}{\gamma} \right)^{2p^2} \exp \left( \frac{3}{4} p^{1/2} \right) \exp \left( -\frac{1}{2} r^{1/3} \varepsilon^{1/3} \right), \] 
we have that $S$ preserves angles, that is,
\[ |\langle Sw, Sw' \rangle - \langle w, w' \rangle| \leq \varepsilon_1, \quad \text{for all } w, w' \in G, \]
According to Lemma 6, $S$ embeds $S$, that is,
\[ \|Sy\|^2 - \|y\|^2 \leq \varepsilon, \quad \text{for all } y \in S, \quad \text{where } \varepsilon \equiv \varepsilon_1 \left( \frac{1}{1 - \gamma} \right)^2. \]
First, we need to convert the condition \((53)\) into one involving $\varepsilon$. We obtain
\[ r \geq 8 \cdot 3^{3/2} \cdot \max \{ \varepsilon^{-5/2} (1 - \gamma)^{-5}, p^{3/2} \varepsilon^{-1} (1 - \gamma)^{-2} \}. \] 
Second, we must alter the lower bound on $r$ to ensure that the right-hand side of \((54)\) is smaller than the given value of $\delta$, that is,
\[ \delta \geq 20r \left( 1 + \frac{4}{\gamma} \right)^{2p^2} \exp \left( \frac{3}{4} p^{1/2} \right) \exp \left( -\frac{1}{2} r^{1/3} \varepsilon^{1/3} (1 - \gamma)^{2/3} \right), \] 
or equivalently,
\[ \log \delta \geq \log 20 + \log r + 2p^2 \log (1 + 4/\gamma) + \frac{3}{4} p^{1/2} - \frac{1}{2} r^{1/3} \varepsilon^{1/3} (1 - \gamma)^{2/3}. \] 
Note that for $p \geq 5$ and $\gamma \in (0, 1)$, we have $20 < 3 < .1p^2 \log (1 + 4/\gamma)$ and $.75p^{1/2} < .1p^2 \log (1 + 4/\gamma)$. Thus a sufficient condition for \((57)\) is that
\[ \log \delta \geq \log r + 2.2p^2 \log (1 + 4/\gamma) - \frac{1}{2} r^{1/3} \varepsilon^{1/3} (1 - \gamma)^{2/3}. \] 
Denoting
\[ \alpha \equiv \varepsilon^{1/3} (1 - \gamma)^{2/3} \quad \text{and} \quad \beta \equiv \frac{1}{3} \left( 2.2p^2 \log (1 + 4/\gamma) + |\log \delta| \right), \]
we have $\alpha \in (0, 1)$ for any $\varepsilon, \gamma \in (0, 1)$. By using these definitions, we see that \((58)\) is equivalent to
\[ \frac{\alpha}{6} r^{1/3} - \log r^{1/3} \geq \beta, \] 
for which the combination of the following two conditions is sufficient:
\[ \frac{\alpha}{12} r^{1/3} - \log r^{1/3} \geq 0, \quad (60a) \]
\[ \frac{\alpha}{12} r^{1/3} \geq \beta. \quad (60b) \]
Condition \((60a)\) can be rewritten to
\[ r \geq \frac{12^2 \beta^3}{\alpha^3} = \frac{4^3}{\varepsilon (1 - \gamma)^2} \left( 2.2p^2 \log (1 + 4/\gamma) + |\log \delta| \right)^3, \]
for which a sufficient condition is
\[ r \geq \frac{8.8^3}{\varepsilon (1 - \gamma)^2} \log^3 (1 + 4/\gamma) \left( p^2 + |\log \delta| \right)^3. \] 
The condition \((60a)\) requires $h(r^{1/3}) \geq 0$, where $h(x) \equiv \frac{\alpha}{12} x - \log x$. Since
\[ h'(x) = \frac{\alpha}{12} - \frac{1}{x} \geq 0, \]
we see that \( h \) is an increasing function for \( x > 12/\alpha \). By noting that
\[
h \left( \frac{12}{\alpha^{5/2}} \right) = \alpha^{-3/2} - \log(12) + \frac{5}{2} \log \alpha \geq 0, \quad \text{for} \ \alpha \in (0,0.33),
\]
and
\[
\frac{12}{\alpha^{5/2}} > \frac{12}{\alpha}, \quad \alpha \in (0,1),
\]
we have for \( \alpha \in (0,0.33) \) that
\[
h(r^{1/3}) \geq 0, \quad \text{if} \ \ r^{1/3} \geq \frac{12}{\alpha^{5/2}},
\]
which leads to
\[
r \geq \frac{12^3}{\epsilon^{5/2}(1-\gamma)^5}. \quad (62)
\]
We are free to choose \( \gamma \in (0,1) \) in a way that ensures that \( \alpha \in (0,0.33) \). In fact, by setting \( \gamma = 3/4 \), we have
\[
\alpha = \epsilon^{1/3}(1/4)^{3/2} < 0.33, \quad \text{for} \ \epsilon \in (0,0.5).
\]

By combining the conditions \(62\) and \(61\), and setting \( \gamma = 3/4 \), we have
\[
r \geq \max \left\{ \frac{\bar{C}_1}{\epsilon^{5/2}}, \frac{\bar{C}_2}{\epsilon} \left( p^2 + |\log \delta| \right)^3 \right\},
\]
with \( \bar{C}_2 = 8.8^3 \cdot 4^2 \log^3(19/3) \approx 6.8e4 \), and \( \bar{C}_1 = 12^3 \cdot 4^5 \). \(\square\)

We could change the weight in the separation of \(59\) into \(60a\) and \(60b\), one could arrive at different (possibly better) constants \(\bar{C}_1\) and \(\bar{C}_2\) in the final expression. However, our priority is to show dependence of \( r \) on \( \epsilon, \delta, \) and \( p \) (and not \( n \)), and optimization of the constants is less important.

6. Numerical Tests

This section presents some numerical evidence of the effectiveness of our sketching strategies. To set up the experiment, we generate two matrices \( F = [f_1, \ldots, f_p] \in \mathbb{R}^{n_1 \times p} \) and \( G = [g_1, \ldots, g_p] \in \mathbb{R}^{n_2 \times p} \) using:
\[
F = U_F \Sigma_F V_F^T, \quad \text{and} \quad G = U_G \Sigma_G V_G,
\]
where \( U_F \in \mathbb{R}^{n_1 \times p}, \ U_G \in \mathbb{R}^{n_2 \times p}, \ V_F \in \mathbb{R}^{p \times p} \) and \( V_G \in \mathbb{R}^{p \times p} \) are generated by taking the QR-decomposition of random matrices with i.i.d Gaussian entries. The diagonal entries of \( \Sigma_F \) and \( \Sigma_G \) are independently drawn from \( \mathcal{N}(1,0.04) \). Matrix \( A \in \mathbb{R}^{n \times p} \) is then defined by setting \( a_j = f_j \otimes g_j \), with \( n = n_1 n_2 \). We further generate the reference solution \( x_{\text{ref}} \in \mathbb{R}^p \) whose entries are drawn from \( \mathcal{N}(1,0.25) \). The right-hand-side vector \( b \in \mathbb{R}^n \) encodes a small amount of noise; we set
\[
b = Ax_{\text{ref}} + 10^{-6} \xi.
\]
where each entry of \( \xi \) is drawn from \( \mathcal{N}(0,1) \). We compute \( x^* \) using \(\square\).

Three sketching strategies will be considered, the first two cases from \(10\) and \(11\), and a third standard strategy that does not take account of the tensor structure in \( A \).

Case 1: Set \( S = P \otimes Q \) (normalized), as defined in \(10\) with entries in \( P \in \mathbb{R}^{r_1 \times n_1} \) and \( Q \in \mathbb{R}^{r_2 \times n_2} \) drawn i.i.d. from \( \mathcal{N}(0,1) \). Notice here that \( r = r_1 r_2 \).

Case 2: Set \( S_{i,:} = p_i^1 \otimes q_i^r \) (normalized), as defined in \(11\), with entries in vectors \( \{p_i\} \) and \( \{q_i\} \) drawn i.i.d. from \( \mathcal{N}(0,1) \) for all \( i = 1, \ldots, r \).

Random Gaussian: \( S = R \in \mathbb{R}^{r \times n} \) (normalized), with entries in \( R \) drawn i.i.d. from \( \mathcal{N}(0,1) \).
The random Gaussian choice is not practical in this context, but is widely used, with well understood properties. We include it here as a reference.

For these three choices of $S$, we compute the solution $x^*$ of the sketched LS problem \[ (2) \], and compare the sketching solution with the standard least-squares solution. In particular, we evaluate the following relative error
\[
\text{Error} = \frac{f(x^*) - f(x^*_{\text{standard}})}{f(x^*)}, \quad \text{with} \quad f(x) = \|Ax - b\|^2_2.
\]
(63)

For each strategy, we draw 10 independent samples of $S$ and compute the median relative error. We discuss how this relative error depend on $r$, $\varepsilon$, and $n$.

6.1. Dependence on $r$. We set $\varepsilon = 0.5$, $\delta = 10^{-3}$, $p = 10$ and $n_1 = n_2 = 10^2$, and choose the following values for $r$: 256, 1024, 4096, 16384 and 65536. As shown in Figure 1, the relative error for all three strategies decreases with $r$; all are of the order of $r^{-1}$. Case 1 is slightly worse than Case 2, which is very similar to the random Gaussian reference case, but much more economical.

![Figure 1. Dependence of relative error on $r$ for the three sketching strategies.](image)

6.2. Dependence on $\varepsilon$. We set $p$, $n$, and $\delta$ as above, and choose $r$ as in (18) in Theorem 3, that is,
\[
r = r_1 r_2 \quad \text{with} \quad r_1 = r_2 = \frac{1}{\varepsilon^2} \left( |\log \delta| + p \right).
\]

Table 1 shows the dependence of $r$ on $\varepsilon$, for the values of $\varepsilon$ of interest to us.

| $\varepsilon$ | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 |
|---------------|-----|-----|-----|-----|-----|
| $r$           | 441 | 729 | 1225| 2209| 4624|

Table 1. Dependence of the number of rows $r$ in the sketching matrix $S$ on the parameter $\varepsilon$. 

In Figure 2 we plot the dependence of the relative error on $\varepsilon$, averaging over five tests as above. The random Gaussian strategy generally gives the best relative error, but the Case 2 strategy, which is much more practical, is nearly optimal, particularly for larger values of $\varepsilon$. The Case 1 strategy is also competitive, for values of $\varepsilon$ closer to 1.

6.3. Dependence on $n$. We show that experimentally, as predicted by our theory, the error does not depend on the dimension $n$ of the ambient space. We fix $\varepsilon = 0.5$, $\delta = 10^{-3}$, and $r = 2209$. We try the following values of $n_1$ and $n_2$ (with $n_1 = n_2$): 50, 100, 150, 200, 250. Results are plotted in Figure 3.

Appendix A. Key Identities and Inequalities

Some identities and inequalities used repeatedly in the text are collected here.

A.1. Identities of the Kronecker product. Let $A = (a_{ij}) \in \mathbb{R}^{r_1 \times n_1}$, $B = (b_{ij}) \in \mathbb{R}^{r_2 \times n_2}$. Then the Kronecker product of $A$ and $B$ forms a matrix of size $r_1 r_2 \times n_1 n_2$ defined by:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n_1}B \\ a_{21}B & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{r_11}B & a_{r_12}B & \cdots & a_{r_1n_1}B \end{bmatrix}.$$

The following properties hold.

1. Let $A \in \mathbb{R}^{r_1 \times n_1}$, $B \in \mathbb{R}^{r_2 \times n_2}$, $C \in \mathbb{R}^{n_1 \times p_1}$ and $D \in \mathbb{R}^{n_2 \times p_2}$, then we have the mixed-product property:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD). \quad (64)$$

2. Let $A \in \mathbb{R}^{r_1 \times n_1}$, $B \in \mathbb{R}^{r_2 \times n_2}$, and $X \in \mathbb{R}^{n_1 \times n_2}$. Further denote by $\text{vec}(X)$ the vectorization of $X$ formed by stacking the columns of $X$ into a single column vector, then

$$(B \otimes A)\text{vec}(X) = \text{vec}(AXB^T). \quad (65)$$
Equivalently, given the same $A, B$ and $x \in \mathbb{R}^{n_1 \times n_2}$, denote $\text{Mat}(x) \in \mathbb{R}^{n_1 \times n_2}$ the matricization of the vector $x$ by aligning subvectors of $x$ that are of length $n_1$ into a matrix with $n_2$ columns, then

$$(B \otimes A)x = \text{vec}(\text{AMat}(x)B^T).$$

(66)

A.2. Sub-exponential random variables and Bernstein inequality. Properties of sub-exponential random variables used in the proofs are defined here.

**Definition 3. Sub-Exponential random variable** A random variable $X \in \mathbb{R}$ is said to be sub-exponential with parameters $(\lambda, b)$ (denoted as $X \sim \text{subE}(\lambda, b)$) if $\mathbb{E}X = 0$ and its moment generating function satisfies

$$\mathbb{E}e^{sX} \leq \exp\left(\frac{s^2\lambda^2}{2}\right), \text{ for all } |s| \leq \frac{1}{b}.$$  

(67)

We have the following.

**Proposition 2.** Let $Z \sim \mathcal{N}(0,1)$, then $X \overset{\text{def}}{=} Z^2 - 1$ is sub-exponential with parameters $(2, 4)$.

We conclude with the well known Bernstein inequality.

**Proposition 3** (Bernstein inequality). Let $X_1, \ldots, X_n$ be i.i.d. mean zero random variables. Suppose that $|X_i| \leq M$ for all $i = 1, \ldots, n$, then for any $t > 0$,

$$\Pr\left(\sum_{i=1}^{n} X_i \geq t\right) \leq \exp\left(\frac{-t^2/2}{\sum_{i=1}^{n} \mathbb{E}[X_i^2] + Mt/3}\right).$$

(68)

**References**

[1] Haim Avron, Huy Nguyen, and David Woodruff, Subspace embeddings for the polynomial kernel, Advances in neural information processing systems, 2014, pp. 2258–2266.
[2] K. Clarkson, P. Drineas, M. Magdon-Ismail, M. Mahoney, X. Meng, and D. Woodruff, *The fast cauchy transform and faster robust linear regression*, SIAM Journal on Computing 45 (2016), no. 3, 763–810.

[3] Kenneth L Clarkson and David P Woodruff, *Low-rank approximation and regression in input sparsity time*, Journal of the ACM (JACM) 63 (2017), no. 6, 54.

[4] Huaian Diao, Zhao Song, Wen Sun, and David Woodruff, *Sketching for kronecker product regression and p-splines*, Proceedings of the twenty-first international conference on artificial intelligence and statistics, 201809, pp. 1299–1308.

[5] Petros Drineas, Michael W. Mahoney, S. Muthukrishnan, and Tamás Sarlós, *Faster least squares approximation*, Numerische Mathematik 117 (2011 Feb), no. 2, 219–249.

[6] Ruhui Jin, Tamara G. Kolda, and Rachel Ward, *Faster johnson-lindenstrauss transforms via kronecker products*, 2019.

[7] W. B. Johnson and J. Lindenstrauss, *Extensions of Lipschitz mappings into a Hilbert space*, Contemporary Mathematics 26 (1984), 189–206.

[8] Michael W. Mahoney, *Randomized algorithms for matrices and data*, Foundations and Trends® in Theoretical Computer Science 3 (2011), no. 2, 123–224.

[9] Xiangrui Meng and Michael W Mahoney, *Low-distortion subspace embeddings in input-sparsity time and applications to robust linear regression*, Proceedings of the forty-fifth annual acm symposium on theory of computing, 2013, pp. 91–100.

[10] Jelani Nelson and Huy L Nguyên, *Osnap: Faster numerical linear algebra algorithms via sparser subspace embeddings*, 2013 ieee 54th annual symposium on foundations of computer science, 2013, pp. 117–126.

[11] Rasmus Pagh, *Compressed matrix multiplication*, ACM Trans. Comput. Theory 5 (August 2013), no. 3, 9:1–9:17.

[12] M. Pilanci and M. J. Wainwright, *Randomized sketches of convex programs with sharp guarantees*, IEEE Transactions on Information Theory 61 (2015 Sep.), no. 9, 5096–5115.

[13] Garvesh Raskutti and Michael W. Mahoney, *A statistical perspective on randomized sketching for ordinary least-squares*, Journal of Machine Learning Research 17 (2016), no. 213, 1–31.

[14] Vladimir Rokhlin and Mark Tygert, *A fast randomized algorithm for overdetermined linear least-squares regression*, Proceedings of the National Academy of Sciences 105 (2008), no. 36, 13212–13217.

[15] Tamas Sarlós, *Improved approximation algorithms for large matrices via random projections*, 2006 47th annual ieee symposium on foundations of computer science (focs’06), 2006, pp. 143–152.

[16] Christian Sohler and David P. Woodruff, *Subspace embeddings for the l1-norm with applications*, Proceedings of the forty-third annual acm symposium on theory of computing, 2011, pp. 755–764.

[17] David Woodruff and Qin Zhang, *Subspace embeddings and lp-regression using exponential random variables*, Proceedings of the 26th annual conference on learning theory, 201312, pp. 546–567.

[18] David P Woodruff, *Sketching as a tool for numerical linear algebra*, Foundations and Trends® in Theoretical Computer Science 10 (2014), no. 1–2, 1–157.