RIGIDITY OF TWO-DIMENSIONAL COXETER GROUPS

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Abstract. A Coxeter system \((W, S)\) is called two-dimensional if the Davis complex associated to \((W, S)\) is two-dimensional (equivalently, every spherical subgroup has rank less than or equal to 2). We prove that given a two-dimensional system \((W, S)\) and any other system \((W, S')\) which yields the same reflections, the diagrams corresponding to these systems are isomorphic, up to the operation of diagram twisting defined by Brady, McCammond, Mühlherr, and Neumann in [8]. As a step in the proof of this result, certain two-dimensional groups are shown to be reflection rigid in the sense of [8], and a result concerning the strong rigidity of two dimensional systems is given in the final section.

1. Introduction

A Coxeter system \((W, S)\) is a pair \((W, S)\) where \(W\) is a group with a presentation of the form \(\langle S | R \rangle\), \(S = \{s_i\}_{i \in I}\), and

\[R = \{(s_i s_j)^{m_{ij}} | m_{ij} \in \{1, 2, \ldots, \infty\}, m_{ij} = m_{ji}, \text{ and } m_{ij} = 1 \Leftrightarrow i = j \}\].

When \(m_{ij} = \infty\), the element \(s_i s_j\) has infinite order. A group \(W\) with such a presentation is called a Coxeter group, and \(S\) is called a fundamental generating set.

Let \(T \subseteq S\). Denote by \(W_T\) the subgroup of \(W\) generated by the elements in \(T\). Such a subgroup is called a standard parabolic subgroup of \(W\), and any conjugate of such a group is called a parabolic subgroup. If \(W_T\) is finite, \(W_T\) is called a spherical subgroup. It is well-known (see [7], for instance) that \((W_T, T)\) is a Coxeter system for any subset \(T \subseteq S\), and therefore \(W_T\) is a Coxeter group in its own right, with the obvious presentation. It is also known that any spherical subgroup \(W_T\) contains a unique longest element with respect to the set \(S\) (see [7]), which we denote by \(\Delta_T\). This element has the property that \(\Delta_T\) conjugates any element \(t \in T\) to some \(t' \in T\).

The information contained in the presentation \(\langle S | R \rangle\) above can be displayed nicely by means of a Coxeter diagram. The Coxeter diagram \(\mathcal{V}\) associated to the Coxeter system \((W, S)\) is an edge-labeled graph whose vertices are in one-to-one correspondence with the generating set \(S\) and for which there is an edge \([s_i s_j]\) labeled \(m_{ij}\) between two vertices \(s_i\) and \(s_j\) if and only if \(i \neq j\) and \(m_{ij} < \infty\).

Given a spherical subgroup \(W_T\) of \(S\), it is clear that the subgraph of \(\mathcal{V}\) induced by the generators in \(T\) is a simplex in the combinatorial sense. We call such a simplex...

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a spherical simplex, and say that it is maximal if it is not properly contained in another spherical simplex.

In the sequel, we frequently omit the word “Coxeter” when discussing groups, systems, and diagrams, as these words will be used in no other context.

It is easy to see that the diagram fully and faithfully records all of the information in the presentation $\langle S \mid R \rangle$. It is also easy to see that to a given group $W$ there may correspond more than one system (and therefore diagram). For example, the dihedral group $D_{2k}$ of order $4k$ has the presentations

$$\langle a, b \mid a^2, b^2, (ab)^{2k} \rangle$$

and

$$\langle c, d, g \mid c^2, d^2, (cd)^2, (cg)^2, (dg)^k \rangle$$

when $k$ is odd. These correspond to diagrams consisting of a single edge labeled $2k$, and a triangle with edge labels $\{2, 2, k\}$, respectively.

Therefore one may consider the question: to what extent is a given Coxeter system unique? As a first step toward answering this question, we must decide what is meant by “unique”.

We say that the group $W$ is rigid if given any two systems $(W, S)$ and $(W, S')$, there is an automorphism $\alpha \in \text{Aut}(W)$ satisfying $\alpha(S) = S'$. Equivalently, the diagrams corresponding to these two systems are isomorphic as edge-labeled graphs.

We say that $W$ is strongly rigid if such an automorphism $\alpha$ can always be chosen to lie in $\text{Inn}(W)$; i.e., any two fundamental generating sets are conjugate to one another.

We can relax these conditions slightly. We require the notion of a reflection. A reflection in the system $(W, S)$ is any conjugate $wsw^{-1}$ of a generator $s \in S$. We say that a Coxeter system $(W, S)$ is reflection rigid if given any other system $(W, S')$ which yields the same reflections, there is an automorphism $\alpha$ of $W$ satisfying $\alpha(S) = S'$. Finally, $(W, S)$ is said to be strongly reflection rigid if given any other system $(W, S')$ yielding the same reflections, such an automorphism $\alpha$ can be found in $\text{Inn}(W)$. We call $W$ reflection independent if every two systems for $W$ yield the same reflections. Clearly if $W$ is reflection independent, then (strong) rigidity and (strong) reflection rigidity are equivalent.

A number of results have been proven that characterize the groups that satisfy these rigidity conditions. Furthermore, there are other characterizations of uniqueness with which we will not concern ourselves in this paper. (See [1], [2], [3], [5], [8], [11], [16], [19], [20].)

In this paper we will generalize the method used in [3] in order to describe the extent to which two-dimensional Coxeter groups are rigid. A system $(W, S)$ is called two-dimensional (or 2-d) if no three distinct generators from $S$ generate a finite subgroup of $W$. (The term “two-dimensional” refers to the dimension of the Davis complex, a simplicial complex associated to the system $(W, S)$. See [11], [12] for more details regarding this complex and its usefulness.) The group $W$ is called two-dimensional if there exists a two-dimensional system $(W, S)$. (As a consequence of the main theorem below, we will see that this distinction is unnecessary in the presence of reflection independence.) In order to describe the results we obtain, we must introduce the important notion of diagram twisting, due to Brady, McCammond, Mühlherr, and Neumann (in [8]).
Given a Coxeter system \((W, S)\), suppose that \(T\) and \(U\) are disjoint subsets of \(S\) satisfying

1. \(W_U\) is spherical, and
2. every vertex in \(S \setminus (T \cup U)\) which is connected to a vertex of \(T\) by an edge is also connected to every vertex in \(U\), by an edge labeled 2.

Under these conditions, we may define a new diagram (and therefore new system) \(\mathcal{V}'\) for \(W\) by changing every edge \([tu]\) \((t \in T, u \in U)\) to an edge \([tu']\), where \(u' = \Delta_U^{-1}u\Delta_U\), leaving every other edge unchanged. This modification results in a generating set \(S'\) obtained from \(S\) by replacing \(t \in T\) with \(\Delta_U^{-1}t\Delta_U\).

This operation is called a diagram twist, because of the way that we “twist” around the subdiagram representing the group \(W_U\).

We require a few new terms in order to state this paper’s main results, remaining consistent with the terminology of \[19\]. If \(\mathcal{V}\) is connected and \(s\) is a vertex in \(\mathcal{V}\) such that \(\mathcal{V} \setminus \{s\}\) is disconnected, \(s\) is called a cut vertex of \(\mathcal{V}\). If \(\mathcal{V}\) has no such vertices, we say that \(\mathcal{V}\) is one-connected. If \(\mathcal{V}\) is one-connected and there exists no edge \([st]\) such that \(\mathcal{V} \setminus [st]\) is disconnected, then \(\mathcal{V}\) is called edge-connected. If \(\mathcal{V}\) is one-connected and there exists no edge \([st]\) with odd label such that \(\mathcal{V} \setminus [st]\) is disconnected, we call \(\mathcal{V}\) odd-edge-connected. (Thus \(\mathcal{V}\) is odd-edge-connected if it is edge-connected.)

**Theorem 1.1.** Let \((W, S)\) be a two-dimensional Coxeter system with diagram \(\mathcal{V}\). Then \((W, S)\) is reflection rigid, up to diagram twisting. (That is, given a system \((W, S')\) which yields the same reflections as \((W, S)\), there is a sequence of diagram twists which transforms the first system into the second.)

As a step in the proof of the main theorem, we will prove

**Theorem 1.2.** Let \((W, S)\) be a two-dimensional Coxeter system with odd-edge-connected diagram \(\mathcal{V}\). Then \((W, S)\) is reflection rigid.

Furthermore, we will prove a theorem (Theorem \[62\]) concerning the strong rigidity of 2-d Coxeter groups. Its statement will be deferred until the final section of this paper.

The above results partially generalize the similar results obtained by Mühlherr and Weidmann in \[13\]. Indeed, in this paper we will make similar use of the results of \[13\] in order to complete the proof of Theorem \[11\] (see \[13\], Section 8). However, the preliminary arguments are very different, and will be introduced in the following section. The author has also recently learned that T. Hosaka has proven independently a slightly weaker result concerning rigidity of two-dimensional Coxeter groups.

This is the third paper in a series (see \[3\], \[4\]) which makes use of similar techniques in order to establish structural properties of Coxeter groups. It is clear that these techniques can be pushed even further to prove results about yet more general Coxeter groups. This will be done in subsequent papers.

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2. Circuits and centralizers

We begin by sketching the argument that we will use to prove Theorem 2.1. Let \((W, S)\) be a 2-d system, and let \((W, S')\) be another system for \(W\). Denote the corresponding diagrams by \(\mathcal{V}\) and \(\mathcal{V}'\). Let us assume until further mention that \(W\) is reflection independent. Our goal is to show that, up to twisting, \(\mathcal{V}\) and \(\mathcal{V}'\) are identical.

The two-dimensionality of \((W, S)\) allows us to establish a matching between the edges of \(\mathcal{V}\) and the edges of \(\mathcal{V}'\), using the following result from \[10\].

**Theorem 2.1.** Let \(W\) be a Coxeter group with diagram \(\mathcal{V}\), all of whose maximal spherical simplices are of the same dimension. Then, given any other Coxeter system \((W, S')\) with diagram \(\mathcal{V}'\), there is a one-to-one correspondence \(\phi\) between the maximal spherical simplices of \(\mathcal{V}\) and those of \(\mathcal{V}'\). Moreover, for any maximal spherical simplex \(\sigma\) in \(\mathcal{V}\), there is an element \(w \in W\) such that \(wW_\sigma w^{-1} = W_{\phi(\sigma)}\).

In our case, every maximal spherical simplex is an edge, and therefore has dimension 1. We apply Theorem 2.1 to obtain a matching between the edges of \(\mathcal{V}\) and the edges of \(\mathcal{V}'\) which respects conjugacy as indicated in the theorem.

Why must each edge of \(\mathcal{V}\) be matched with an edge of \(\mathcal{V}'\)? If there were an edge \([st]\) in \(\mathcal{V}\) such that \(wW_{[st]} w^{-1} = W_\sigma\) for some \(\sigma\) of dimension \(> 1\), then \(D_n \cong W_\sigma\), where \(n\) is the order of \(st\). However, it is an easy matter (see \[1\]) to show that this can only happen if \(n = 2k\), \(k\) odd, and \(\sigma\) is a triangle with edge labels \(\{2, 2, k\}\). In this case, the central element of \(W_{[st]}\) (which is of even length with respect to \((W, S)\), and is therefore not a reflection) is a reflection in \((W, S')\), contradicting the assumption that \(W\) is reflection independent. An immediate corollary is that every system corresponding to \(W\) is 2-d, so it matters not whether we refer to the group or to the system as 2-d, provided \(W\) is reflection independent. (Alternately, one may apply Lemma 1.5 of \[11\].)

Let \(\phi\) be the matching whose existence is guaranteed by Theorem 2.1.

We now consider circuits in the diagram \(\mathcal{V}\). A **simple circuit** of length \(k\) in \(\mathcal{V}\) is a collection \(C\) of \(k\) distinct edges \(\{[s_1s_2], \ldots, [s_ks_1]\}\) for which \(s_i \neq s_j\) when \(i \neq j\). Define \(d(i, j) = \min\{|i - j|, k - |i - j|\}\) for \(1 \leq i, j \leq k\). We call a simple circuit \(C\) **achordal** if for any two vertices \(s_i \neq s_j\) in \(C\) such that \(d(i, j) > 1\), \([s_is_j]\) is not an edge in \(\mathcal{V}\).

We shall prove the following theorem.

**Theorem 2.2.** Let \((W, S), (W, S'), \mathcal{V}, \mathcal{V}', \) and \(\phi\) be as above. Let \(C\) be an achordal circuit of length \(k\) in \(\mathcal{V}\), as above. Then there is an achordal circuit \(C' = \{[\hat{s}_1\hat{s}_2], \ldots, [\hat{s}_k\hat{s}_1]\}\) in \(\mathcal{V}'\) such that \(\{\hat{s}_i, \hat{s}_{i+1}\} = \phi(\{s_i, s_{i+1}\})\) for \(1 \leq i \leq k\). Moreover, for each edge \([s_is_{i+1}]\) there is an element \(w_{i+1} \in W\) such that \(w_{i+1} s_i w_{i+1}^{-1} = \hat{s}_i\) and \(w_{i+1}s_{i+1} w_{i+1}^{-1} = \hat{s}_{i+1}\) both hold.

Therefore not only do the edges match up nicely, but the achordal circuits do as well. In fact, we can do better:

**Theorem 2.3.** Let \((W, S), (W, S'), \mathcal{V}, \mathcal{V}', \phi, C, \) and \(C'\) be as in Theorem 2.2. Let \(s_i\) and \(s_j\) be distinct vertices on \(C\), with \(\{e_i = [s_is_{i+1}], [s_{i+1}s_{i+2}], \ldots, e_j = [s_{j-1}s_j]\}\) a subpath of \(C\) between them. Let \(w_i\) and \(w_j\) be the group elements which conjugate the edges \(e_i\) and \(e_j\), respectively, to their corresponding edges in \(C'\). Then \(w_i w_j^{-1}\) can be written \(\alpha_1\alpha_2 \cdots \alpha_r\), where for every \(l\) (\(1 \leq l \leq r\)) \(\alpha_l \in S'\) and one of the following holds.
1. For every $l' \leq l' \leq k$, $\alpha_l \neq \hat{s}_{l'}$, and $\alpha_l$ commutes with at least 2 vertices which lie on $C'$. Moreover, we can find two such elements, $\hat{s}_{l_1}$ and $\hat{s}_{l_2}$, such that the path $\{[\hat{s}_{l_1}\alpha_l],[\alpha_l\hat{s}_{l_2}]\}$ separates $C'$ into two circuits, one containing $\hat{s}_{l_1}$ and the other containing $\hat{s}_{l_2}$.

2. $\alpha_l = \hat{s}_{l_1}$, in which case both $[\hat{s}_{l_1}\hat{s}_{l_2}]$ and $[\hat{s}_{l_1}\hat{s}_{l_2}]^{-1}$ are labeled 2, or $\alpha_l = \hat{s}_{l_2}$, in which case both $[\hat{s}_{l_2}\hat{s}_{l_1}]$ and $[\hat{s}_{l_2}\hat{s}_{l_1}]^{-1}$ are labeled 2.

Although Theorem 2.3 appears very technical, it addresses precisely the issues that must be faced when dealing with strong rigidity in the presence of edges labeled 2. (Compare the arguments of Section 6 in [3]; in particular, those used in Cases 1 and 2.) We note that in case no edges in $\mathcal{V}$ are labeled 2, $w_i = w_j$ must hold for all edges $w_i$ and $w_j$; thus the circuit $C$ is in this case “strongly rigid”.

Mühlherr and Weidmann also consider achordal circuits in [19], but their approach to these circuits is very different from that adopted here, where we draw upon the techniques developed in [3] and [11].

Once Theorem 2.2 and Theorem 2.3 have been established, it will be a relatively straightforward matter to reconstruct the unique (up to twisting) diagram $\mathcal{V}$ which is built up from the achordal circuits.

As will become clear, our analysis of the achordal circuits in $\mathcal{V}$ will depend upon an understanding of the centralizer $C(s)$ of an arbitrary generator $s \in S$. To that end, we recall in the next theorem the structure of $C(s)$ (first given in [9]). We also introduce notation which will remain fixed throughout the remainder of the paper.

Let $(W, S)$ be an arbitrary Coxeter system and suppose $s, t \in S$ are elements of the fundamental generating set $S$. If $m_{st} = 2k$ is even, denote by $u_{st}$ the element $(st)^k s$. We note that $u_{st}$ commutes with $t$ (in fact, $u_{st} = s$ if $st = ts$). If $m_{st} = 2k + 1$ is odd, denote by $v_{st}$ the element $(st)^k$. Note that $v_{st}^{-1} sv_{st} = t$. More generally, there is an path in the diagram $\mathcal{V}$ between two vertices $s$ and $t$ which consists entirely of odd edges if and only if $s$ and $t$ are conjugate to one another. In fact, if $\{[s], [s_1s_2], ..., [s_ks]\}$ is such a path, then $v_{s_1t}v_{s_2s_1}^{-1}s_2 \cdots v_{ss_1}$ conjugates $s$ to $t$.

Let $\bar{\mathcal{V}}$ be the graph resulting from a diagram $\mathcal{V}$ by removing all edges with even labels. As in [9], we can identify elements of the fundamental group of $\bar{\mathcal{V}}$ with paths in $\mathcal{V}$ which start and end at a fixed vertex $s \in S$ and which never backtrack. For the fixed vertex $s \in S$, let $\mathcal{B}(s)$ be a collection of simple circuits in $\bar{\mathcal{V}}$ containing $s$ such that $\mathcal{B}(s)$ generates the fundamental group of $\bar{\mathcal{V}}$.

The following was first proven by Brink in [9]. (The generators given here can be computed by arguments similar to those in [6].)

**Theorem 2.4.** Let $(W, S)$ be an arbitrary Coxeter system with diagram $\mathcal{V}$, and let $s \in S$. Then $C(s)$ is the subgroup of $W$ generated by

$$\{s\} \cup A \cup B$$

where

$$A = \{vu_{ts}v^{-1} \mid v = v_{s_1s_2} \cdots v_{sk}, t, s_i \in S, m_{ts} \text{ even}; m_{s_1s}, m_{s_2s_{i-1}} \text{ odd}\}$$

and

$$B = \{v_{s_1s}v_{s_2s_1} \cdots v_{sk} \mid \{[s], [s_1s_2], ..., [sks]\} \in \mathcal{B}(s)\}.$$
We will use this description of the centralizer $C(s)$ in the sequel.

**Remarks.** When $(W, S)$ is 2-d, it can be shown that distinct choices of $s_1, s_2, \ldots, s_k$ and $t$ in $A$ and $B$ above yield distinct generators. This may not be the case if $(W, S)$ is not 2-d.

Moreover, it is not difficult to compute a geodesic form for an element $w$ in $C(s)$. To do this, first express $w$ as a product in the given generators. Factor all occurrences of $s$ as a single letter to the end of the word and cancel, yielding either $s$ or $1$. Next, perform all “obvious” cancellation; that is, given $\alpha = v_{s_1} \cdots v_{s_{n_k}}$ and $\beta = v_{s_{i_1}}' \cdots v_{s_{i_{n_k}}'}$, in $B$, for some $i$, $0 \leq i \leq \min\{k, k'\}$ we have $s_{k-i} = s_{k-i+1} = s_{i}^e$, so that $\alpha \cdot \beta = v_{s_1} \cdots v_{s_{k-i+1}} v_{s_{k-i}}' v_{s_{k-i+1}}' \cdots v_{s_{i_{n_k}}'}$. Similar cancellation occurs in a product of two generators from $A$, and in a product of a generator from $A$ and a generator from $B$.

We claim that the word that results after such cancellation is geodesic. In order to prove this, we appeal to a result of Tits. From Section 2 of [21] we conclude that if the word resulting from the previous paragraph were not geodesic, we would be able to shorten the word by successively replacing subwords $(st)^n$ with $(ts)^n$ when $st$ has order $2n$ and subwords $(st)^n s$ with $(ts)^n t$ when $st$ has order $2n + 1$, and then canceling any adjacent occurrences of the same letter which might arise in the course of these replacements. However, thanks to two-dimensionality, no such shortening replacements can be performed, (perhaps) aside from commuting the single occurrence of $s$ that may occur at the end.

The above argument (replacing the one half of a relator with the other half) will be used again in the following sections. We refer to the process of shortening a word $w$ in the manner described above as the **Tits process** (TP).

## 3. Matching edges in a given circuit

In this section we retrace the arguments from [3], adapting them as necessary to the case of 2-d systems. In fact, many of the arguments throughout the remainder of the paper will parallel arguments from [3] (such analogous arguments will be indicated).

Let $(W, S)$ be a 2-d system with diagram $\mathcal{V}$, and let $(W, S')$ be another system, with diagram $\mathcal{V}'$, yielding the same reflections as $(W, S)$. We fix all of this notation for the remainder of the paper.

Let $C = \{[s_1 s_2], \ldots, [s_k s_1]\}$ be an anchordal circuit in $\mathcal{V}$. By Theorem 2A for every $i = 1, \ldots, k$ there exists an edge $[s_{i-1}^e s_i']$ in $\mathcal{V}'$ and an element $w_i \in W$ such that $w_i W_{s_{i-1} s_i} W_{s_{i-1} s_i}^{-1} = W_{s_i' s_{i-1} s_i'}$. By considering the possible generators for the dihedral group $W_{s_{i-1} s_i}$, we can assume that

$$w_i s_{i-1} w_i^{-1} = s_i'' s_{i-1}^{-1}$$

for some word $z_{i-1, i} \in W_{s_i'' s_{i-1} s_i'}$. Let $m = m_{i-1, i}$ be the order of $s_{i-1} s_i$. One may prove by direct computation that after suitably modifying $w_i$ we can assume $z_{i-1, i} = (s_i'' s_{i-1}^m)^j$, where $0 \leq j \leq \frac{|m|}{2}$ if $m$ is even and $0 \leq j \leq \frac{|m| - 1}{2}$ if $m$ is odd. (cf. [3], Section 4.) In particular, $z_{i-1, i} = 1$ if $m \in \{2, 3, 4\}$.

We now use the fact that each vertex $s_i$ appears in two edges in $C$. Because $s_i' = z_{i-1, i} w_i s_i w_i^{-1} z_{i-1, i}$ and $s_i'' = w_i s_i w_i^{-1}$, both $s_i'$ and $s_i''$ are conjugate to $s_i$, and therefore to each other. Let $P_i = \{[s_i' s_i], \ldots, [s_{i+r} s_i'']\}$ be a path of minimal length from $s_i'$ to $s_i''$, all of whose edges have odd labels. Then the element
Thus we obtain
\[
\tilde{v}_i = v_{s_{i,1}s'_i}v_{s_{i,2}s_{i,1}} \cdots v_{s_{i,r}s_{i,r}}
\]

conjugates \(s''_i\) to \(s'_i\).

We now compute:
\[
w_i s_i w_i^{-1} = z_{i-1,i} s'_i z_{i-1,i}^{-1} = z_{i-1,i} \tilde{v}_i s'_i \tilde{v}_i^{-1} z_{i-1,i}^{-1} = z_{i-1,i} \tilde{v}_i w_{i+1} s_i w_i^{-1} \tilde{v}_i^{-1} z_{i-1,i}^{-1}.
\]

Thus
\[
w_i^{-1} z_{i-1,i} \tilde{v}_i w_{i+1} \in C(s_i) = C(w_i^{-1} z_{i-1,i} s'_i z_{i-1,i}^{-1} w_i) = w_i^{-1} z_{i-1,i} C(s'_i) z_{i-1,i}^{-1} w_i.
\]

Finally, we obtain
\[
w_{i+1} w_i^{-1} = \tilde{v}_i^{-1} \tilde{s}_i z_{i-1,i}^{-1}
\]
for some \(\tilde{s}_i \in C(s'_i)\). Denote by \(x_i\) the word appearing on the right-hand side of (1). Then
\[
x_k x_{k-1} \cdots x_2 x_1 = w_1 w_k^{-1} w_k w_{k-1}^{-1} \cdots w_2 w_1^{-1} = 1.
\]

If we choose a geodesic representation of \(\tilde{s}_i\), each of the words \(\tilde{v}_i^{-1}, \tilde{s}_i,\) and \(z_{i-1,i}^{-1}\)
are as short as possible.

Recalling the generators of \(C(s'_i)\) given by Theorem 2.4, \(\tilde{s}_i\) may terminate with a term of the form \(u_{s''_{i-1}s'_i}\) (if \(m = m_{s'_i} s'_i\) is even) or \(v_{s'_i s''_{i-1}}\) (if \(m\) is odd). In this case, we can reduce the product appearing in (1) by multiplying this term with \(z_{i-1,i}^{-1}\) if \(z_{i-1,i} \neq 1\).

There is one other place where reduction can occur in \(x_i\). Suppose \(\tilde{s}_i\) begins with a word of the form
\[
v_{\alpha_1 s'_i s_{i,1}} v_{\alpha_2 s_{i,1}} \cdots v_{\alpha_{i-1}} v_{\alpha_{i} s''_{i}}
\]
corresponding to an odd “loop” based at \(s'_i\) in the diagram \(\mathcal{V}'\). If
\[
\tilde{v}_i = v_{s_{i,1}s'_i} v_{s_{i,2}s_{i,1}} \cdots v_{s_{i,r}s_{i,r}} \neq 1,
\]
we may have
\[
s_{i,1} = \alpha_1, s_{i,2} = \alpha_{i-1}, \ldots, s_{i,j} = \alpha_{i-j+1},
\]
where \(j \leq r\). Therefore after cancellation \(\tilde{v}_i^{-1} \tilde{s}_i\) begins with a word of the form
\[
v_{\beta_1 s'_i \beta_2 s_{i,1} \cdots v_{s_{i,r}}} (2)
\]
for some \(j \geq 0\). (When \(j = 0\), (2) has the form \(v_{s'_i s''_i}\).

After all cancellation has been performed on \(x_i\), we obtain a new product, of “even” words \(E\) (involving terms \(u_{s''_i} \)) and “odd” words \(O\) (involving terms \(v_{st}\):
\[
x_i = v_{\alpha_1 s'_i v_{\alpha_2 s_{i,1}} \cdots v_{s'_{i,\alpha_j}} E_1 O_1 E_2 O_2 \cdots E_r O_r w(s'_i, s''_{i-1})
\]
where \(w(s'_i, s''_{i-1})\) is some word in the letters \(s'_i\) and \(s''_{i-1}\). (We allow \(E_1\) and \(O_1\)
to be trivial.) If \(s'_i\) appears in \(\tilde{s}_i\) as a generator of \(C(s'_i)\), it may be absorbed by \(w(s'_i, s''_{i-1})\). The exact structure of the words \(E\) and \(O\) is governed by Theorem 2.4.

All of the terms \(v_{st}\), and all of the even terms \(u_{s''_i}\) which consist of more than a single letter \(s\) will be called long terms. In case \(u_{s''_i}\) is a long term of length \(2r + 1\).
(for \( r \geq 1 \)), we will also call the words \((ss_i')^r\) and \((s_i's)^r\) long terms by a slight abuse of terminology. Terms consisting of a single letter (either \(s_i'\) or \(s\) such that \(ss_i' = s's\)) will be called short terms.

We claim that the product given in (3) is in fact geodesic; this is shown by another application of the result of Tits. Thanks to the form of the long terms and two-dimensionality of \(W\), the only possible subwords in the right-hand side of (3) which admit replacement as in TP would come from \(s_i\). (E.g., TP may allow us to bring two instances of \(s_i'\) together, in order to cancel them.) But we have assumed \(s_i\) to be geodesic, and therefore unchanged under application of TP (such cancellations have already been performed).

We are now ready to begin the proof of Theorem 2.2, inducting upon the length \(k\) of the circuit \(C\).

4. The base cases

We continue to use the notation from the previous sections, and prove Theorem 2.2 and Theorem 2.3 for cycles of lengths 3 and 4. Some of the methods used in this section will be generalized in the following section, and so will be stated in general terms. We will use the fact that \(x_kx_{k-1}\cdots x_1 = 1\) in order to show that each word \(x_i\) must have a very specific form. The form of \(x_i\) will allow us both to identify a circuit in \(V'\) to which \(C\) corresponds as in Theorem 2.2 and to prove the statements regarding \(w_iw_j\) made in Theorem 2.3.

Hereafter we say that the words \(x_i \neq 1\) and \(x_j \neq 1\) (\(i > j\)) are adjacent in the product \(x_k\cdots x_1\) if either \(i = j + 1\) or \(x_i = 1\) for \(j < l < i\).

We first assume there is no cancellation of common short terms between two words \(x_i\) and \(x_j\). Having proven the theorems in this case, we will then indicate how to prove the theorems in general. (In fact, by two-dimensionality, the case in which \(k = 3\) yields very little such short term cancellation, as there can be no vertex \(\alpha\) not on \(C'\) such that \(\alpha\) commutes with two distinct elements \(s_i'\) and \(s_j'\).)

Let us first consider the case of a circuit of length 3: \(x_3x_2x_1 = 1\). Unless all three words \(x_i\) are trivial (in which case \(w_1 = w_2 = w_3\) already and \(C\) clearly corresponds to a circuit \(C'\) in \(V'\)), at least two of these words are nontrivial.

Case 1. Suppose that \(x_1 = 1\) (after renaming, if necessary). Thus \(x_3x_2 = 1\) and \(s_1 = s_1''\).

First suppose that \(x_3\) ends with the long even term \(\alpha(s_2'a)^{-1}a\) and \(x_2\) begins with the long even term \((\beta s_2')\beta^{-1}\). (If \(x_2\) were not to begin with an even term, at most one pair of letters would cancel, and an application of TP would yield a contradiction.) This implies that \(s_2' = s_2''\). Since \(s_3' \neq s_2'' = s_2',\) we can avoid the same contradiction only if \(\alpha = s_2'\) and \(\beta = s_3'\), in which case \(m = n\) and

\[
\alpha(s'_3a)^{-1} \cdot (\beta s'_2)^{-1}\beta = s'_3s'_2.
\]

In this case, easy computations (and applications of TP) show that there is no further cancellation if \(u_{s'_2s'\ 3}\) is either the first term in \(x_3\) or is preceded by another long term, and if \(u_{s'_2s'_2}\) is either the last term in \(x_2\) or is followed by another long term. Therefore, in order that \(x_3x_2 = 1\), \(u_{s'_2s'_2}\) must be preceded by \(s_3'\) in \(x_3\), and \(u_{s'_2s'_2}\) must be followed by \(s_2'\) in \(x_2\). Moreover, if there are any further terms in \(x_3\) and \(x_2\), there can be no further cancellation. Thus \(x_2 = x_3 = (s_2's_{3})\bar{w}\), and \(s_i' = s_i''\)
for \( i = 1, 2, 3 \). Moreover, \( x_1 = 1 \Rightarrow w_1 = w_2, \) and \((s'_i s'_j)^{\alpha} \) commutes with both \( s'_2 \) and \( s'_3 \). Therefore, \( w_1 s_1 w_1^{-1} = s'_1, w_1 s_2 w_1^{-1} = s'_2, \) and

\[
 w_1^{-1} s'_3 w_1 = w_3^{-1} (s'_2 s'_3)^{\overline{\alpha}} s'_3 (s'_2 s'_3)^{\overline{\alpha}} w_3 = w_3^{-1} s'_3 w_3 = s_3.
\]

Thus the same element (namely, \( w_1 \)) conjugates each \( s_i \) to \( s'_i \), proving Theorem 2.3 for \( C \).

Now suppose \( x_3 \) ends with a long odd term \((s'_3 s'_i)^{\overline{\alpha}} \) and \( x_2 \) begins with a long odd term \((\beta s''_2)^{\overline{\alpha}} \). (As before, we obtain a contradiction to \( x_3 x_2 = 1 \) if one term is odd and the other even.) In order that more than one pair of letters cancel, it must be that \( \alpha = s'_2 \) and \( \beta = s'_3 \), so that \( m = n \) and

\[
 (s'_i \alpha)^{\overline{\alpha}} \cdot (\beta s''_2)^{\overline{\alpha}} = s''_2 s'_3.
\]

As before there is no further cancellation possible if \( v s'_i s''_i \) is either the first term in \( x_3 \) or is preceded by a long term and \( v s'_i s''_i \) is either the last term in \( x_2 \) or is followed by a long term. In fact, only if \( s'_3 = s'_2 \) and \( v s'_i s''_i \) is followed in \( x_2 \) by the term \( s'_2 \) is there further cancellation. However, we are still left with a stray letter \( s''_2 \), so this product cannot occur.

However, if instead \( x_3 \) ends with \( v s'_i s''_3 \) and \( x_2 \) begins with \( v s'_i s''_2 \), we obtain the product \( s''_2 v s'_i s''_3 s'_3 s''_i s''_2 = s''_3 \) in the middle of \( x_3 x_2 \). No further cancellation is possible unless \( s'_2 = s'_3 \) and \( v s'_i s''_2 \) is followed in \( x_2 \) by the letter \( s'_2 \). In this case, it is easily seen that there are no more terms in either \( x_2 \) or \( x_3 \), so that \( s'_2 = s'_3, s''_2 = s''_3, \) and \( x_2 = x_3 = s'_3 v s'_i s''_2 \). As before, \( w_1 = w_2, \) so \( w_1 s_1 w_1^{-1} = s'_1 \) and \( w_1 s_2 w_1^{-1} = s'_2 \). Now, however, the vertex \( \hat{s}_3 \) to which \( s_3 \) is to be conjugated is not \( s'_1 \), but \( s''_3 = v s'_i s''_3 s'_3 s''_i s''_2 \). This is seen by drawing the circuit \( C' \) (compare Lemma 1.3). But note

\[
 w_1^{-1} s''_3 w_1 = w_3^{-1} v s'_i s''_3 s'_3 s'_i s''_2 s''_3 w_3 = w_3 s'_i s''_3 = s_3.
\]

Therefore a single element again conjugates the vertices appropriately. Schematically, the two possibilities above can be summarized respectively as \( x_3 = E, x_2 = E^{-1} \); and \( x_3 = O, x_2 = O^{-1} \) (as before, \( E = \text{“even”}, O = \text{“odd”} \).

Similar arguments show that \( x_{2,3} = z_{1,2} = 1 \), that \( x_3 \) cannot end with a short term, and that \( x_2 \) cannot begin with a short term. Thus the two products shown above are the only valid possibilities when \( x_1 = 1 \).

**Case 2.** Now suppose that each \( x_i \neq 1 \), and that there is no cancellation of common short terms. We give the possible forms for \( x_i \) schematically (as was done above) in the proposition below, leaving precise computations to the reader.

**Proposition 4.1.** Let \( x_i \neq 1 \) for \( i = 1, 2, 3 \). Up to renumbering, one of the following holds:

\[
 x_3 = E_1 E_2, x_2 = E_2^{-1} E_3, x_1 = E_1^{-1} E_1^{-1}; \quad x_3 = E_1 E_2, x_2 = E_2^{-1}, x_1 = E_1^{-1};
 x_3 = E_1 O_1, x_2 = O_1^{-1} O_2, x_1 = O_2^{-1} E_1 E_1^{-1}; \quad x_3 = E_1 O_1, x_2 = O_1^{-1}, x_1 = E_1^{-1};
 x_3 = E_1 O_1, x_2 = O_1^{-1} E_2, x_1 = E_2^{-1} E_1^{-1}; \quad x_3 = O_1 E_1, x_2 = E_1^{-1}, x_1 = O_1^{-1};
 x_3 = O_1 O_2, x_2 = O_2^{-1} O_3, x_1 = O_3^{-1} O_1^{-1}; \quad x_3 = O_1 O_2, x_2 = O_2^{-1}, x_1 = O_1^{-1}.
\]

Here, \( E_j \) represents either \( u_{\alpha \beta} \) or \( u_{\alpha \beta} \beta \) and \( O_j \) represents either \( v_{\gamma \delta} \) or \( v_{\gamma \delta} \). for the appropriate choices of \( \alpha, \beta, \gamma, \) and \( \delta \).
Of course, one may not be able to choose freely whether $E_j$ represents $u_{\alpha,\beta}$ or $u_{\alpha,\beta'}$, and similarly for $O_j$. That is, the exact forms of the words $E_j$ and $O_j$ are clearly interdependent. (Cf. Section 4 of [3].)

Note that in multiplying any two distinct terms $x_i$ and $x_j$, at most one long term may cancel. (That is, as we saw above, forms such as $x_1 = 1$, $x_3 = O_1 O_2$, $x_2 = O_2^{-1} O_1^{-1}$ cannot occur.) In fact, this will remain true even as we consider arbitrarily long circuits.

Remark. There are a few cases which must be handled carefully; these cases involve the affine Euclidean Coxeter groups whose diagrams are triangles with edge label multisets $\{2,3,6\}, \{2,4,4\}$, or $\{3,3,3\}$. For instance, suppose that $x_i = s_i's_i''$ and $s_i' = s_i'' + 1$ for $i = 1,2,3$. Then $x_1 x_2 x_1 = 1$, $s_i = s_i' + 1$, and the edges of $\mathcal{V}'$ which correspond to those in $C$ do indeed form a circuit of length 3. However, it can be shown (with the aid of Lemma 5.1) that no $w \in W$ satisfies $w s_i w^{-1} = s_i$. We claim that these forms of $x_i$ lead to a fundamental contradiction. Consider the parabolic subgroup $w_1 W_C w_1^{-1}$; because $w_1 s_i' w_1^{-1} = s_i', w_1 s_i w_1^{-1} = s_i''$, and $w_1 s_2 w_1^{-1} = s_2 s_3 s_1' s_3'$, $w_1 W_C w_1^{-1} \subseteq W_{C'}$. Now we apply the construction of $x_i$ given in Section 3 “in reverse”, proceeding from $C'$ to $C$. (Essentially, we compute the ratios $x_i' = w_i^{-1} w_i$. We have proven our result for $C$ (by conjugating back from the corresponding result for $C'$) provided the words $x_i' = w_i^{-1} w_i$ do not have forms similar to those of $x_i$. In particular, we may assume that $x_i' \in W_C$. Therefore, since $s_i'' = s_i''$ and $w_1 s_i' w_1 = w_1^{-1} w_3 s_3 w_3^{-1} w_1 \in W_C, W_{C'} \subseteq w_1 W_C w_1^{-1}$. Therefore $w_1 W_C w_1^{-1} = W_{C'}$. The set

$$\{w_1 s_i w_1^{-1} \mid i = 1,2,3\} = \{s_i', s_3', s_3 s_1' s_3 s_1' s_3'\}$$

must therefore generate $W_{C'}$. However, it can be shown that this set does not generate $W_{C'}$ (this is possible because the affine Euclidean group $W_C$ is not cohopfian). This gives a contradiction.

Therefore the case in which $x_i = s_i's_i''$ and $s_i'' = s_i' + 1$ for all $i$ cannot occur. Any similar case involving the affine Euclidean triangle groups can be outlawed in an analogous fashion, and we are forced to conclude that the forms for $x_1 \cdot x_2 \cdot x_3$ given above are exhaustive.

Before turning our attention to a proof of Theorem 2.3, we state the following result concerning circuits of length 4.

**Proposition 4.2.** Let $C$ be a circuit of length 4 and let $x_4 x_3 x_2 x_1 = 1$. Assume furthermore that there is no cancellation of short terms between different words $x_i$ and $x_j$. Then, up to a renumbering of the vertices, there are 27 forms for the product $x_4 \cdot x_3 \cdot x_2 \cdot x_1$ (entirely analogous to those given in Proposition 4.1). In each case, every word $x_i$ has at most two long terms, every long term of $x_i$ cancels with a long term in either $x_{i+1}$ or $x_{i-1}$, and no more than two long terms cancel in any product $x_i x_{i-1}$.

One may check that in each of the cases mentioned, the exact forms of the words $O_i$ and $E_i$ which appear are determined completely, and that the subdiagram of $\mathcal{V}'$ corresponding to these trivial products is a circuit of length 4 whose edges appear in the appropriate order. This establishes Theorem 2.3 for $k = 4$. (The form of the word which conjugates each edge $[s_i s_{i+1}]$ appropriately is easy to compute, given the forms of $x_{i+1}$ and $x_i$.)
To prove Theorem 2.3 for $C$, we must first decide to which vertex $\hat{s}_i$ in $C'$ a given vertex $s_i$ in $C$ corresponds in Theorem 2.2. To this end, we have the next lemma, which, along with the results which follow it, is stated in very general terms as it will be useful in the following section as well. We say that two long terms completely cancel provided that they comprise the same two letters (thus their product has length at most 2).

**Lemma 4.3.** Let $s_i$ be a vertex on $C$, and suppose that every word $x_i$ has one of the schematic forms $B$ (for “blank”; i.e., no long terms), $O$, $E$, $OO$, $OE$, $EO$, or $EE$, for $1 \leq i \leq k$. Suppose each long term completely cancels with a long term in an adjacent word, and no two words $x_i$ and $x_j$ allow complete cancellation of more than one long term between them. Then $\hat{s}_i = s'_i$ unless $x_i$ has one of the forms $O$, $EO$, or $OO$ and the final long term in $x_i$ completely cancels with the first long term in $x_{i-1}$. In this case, $\hat{s}_i = s''_{i-1}$.

**Proof.** The cases to consider depend on the schematic form of $x_i$. If $x_i$ has no long terms, the lemma is clearly true. We prove one of the nontrivial cases and leave the rest to the reader.

Suppose that $x_i$ has the form $OO$. The second long term in $x_i$ must cancel with the first long term in $x_{i-1}$. Therefore the long terms $v'_{i-1}$ and $v_{i-1}'$ have the same letters, and $x = s''_{i-1}$. Similarly, since the first long term in $x_i$ must completely cancel with the last in $x_{i+1}$, $\beta = s'_{i+1}$. Therefore the edges $[s'_{i} s''_{i-1}]$ and $[s'_{i+1} s''_{i}]$ meet at the vertex $s''_{i-1} = s''_{i+1}$. But these are the edges of $C'$ which match the edges $[s_{i-1} s_i]$ and $[s_i s_{i+1}]$ of $C$ as in Theorem 2.2. Therefore the vertex $s''_{i-1}$ must be matched with $s_i$.

We continue to consider arbitrary $k \geq 3$. From the results of Section 5 it will follow that provided there is no cancellation of short terms between different words $x_i$, every word $x_i$ will possess at most two long terms and have one of the seven schematic forms given above. Also, if there is no cancellation of short terms, we claim that the circuit $C$ is strongly rigid in the sense that there is a single word $w$ which conjugates every vertex of $C$ appropriately.

To construct this word $w$, we must understand the form of $x_k$. Given there is no short term cancellation, $x_k$ has either one or two long terms, perhaps followed by $s_k$. Suppose, for instance, that $x_k = v_{s''_{k}} v'_{s'} s''_{k}$. This forces $x_{k-1}$ to begin with the long term $v_{s''_{k} s'_{k-1}}$ (perhaps followed by $s''_{k} = s''_{k-1}$) and $x_{1}$ to end with the long term $v_{s''_{1} s''_{k}}$ (perhaps followed by $s''_{1} = s''_{k}$). If indeed $x_{k-1} = v_{s''_{k} s'_{k-1}}$, then

$$w_{1}w_{k-1}^{-1} = x_{k}x_{k-1}^{-1} = v_{s''_{k}} s''_{k}.$$ 

Since this product clearly does not conjugate $s_{k-1}$ to $s''_{k-1}$ as needed, $w = w_{1}$ cannot be. However, premultiplying by $\Delta_{s''_{k}}$ gives

$$\Delta_{s''_{k}} x_{k} x_{k-1} = 1,$$

and as one can check that $w = \Delta_{s''_{k}} w_{1}$ conjugates $s_{1}$ to $s''_{k}$ and $s_{k}$ to $s''_{k}$, this $w$ works for the vertices $s_{1}$, $s_{k}$, and $s_{k-1}$. We claim that such a $w$ can always be constructed in similar manner, depending on the precise form of $x_{1}$, $x_{k}$, and $x_{k-1}$, leaving the proof of this claim to the reader. (Cf. Lemma 4.2 of [2].)

**Lemma 4.4.** There exists a word $\pi \in \{1, s''_{k}, \Delta_{s''_{k}}, \Delta_{s''_{k} s''_{k}}\}$ such that
\[ \pi w_1 s_i w_1^{-1} \pi^{-1} = \hat{s}_i \]

for \( i \in \{1, k-1, k\} \).

If \( k = 3 \), this lemma completes the proof of Theorem 2.3. We claim that in fact this word \( w \) will appropriately conjugate every vertex in \( C \), even if \( k \geq 4 \). (Cf. Proposition 4.3 of [3].)

**Lemma 4.5.** Let \( k \geq 4 \) and suppose that every word \( x_i \) has one of the forms \( B, O, E, OO, OE, EO, \) or \( EE \), for every \( 1 \leq i \leq k \). Further, suppose there is no short term cancellation between different words \( x_i \). Define \( w = \pi w_1 \) accordingly, as in Lemma 4.4. Then \( ws_i w^{-1} = \hat{s}_i \) for all \( i \), where \( \hat{s}_i \) is the vertex of \( C' \) to which \( s_i \) corresponds.

This lemma will complete the proof of Theorem 2.3 in case \( k \geq 4 \) and there is no cancellation of common short terms, assuming that every word \( x_i \) has one of the seven schematic forms shown above. (Again, this last statement can be verified by computation in case \( k = 4 \), and will be proven in Section 5 in case \( k \geq 5 \).) In order to prove Lemma 4.5, we make use of the following fact, which requires (when \( k \geq 5 \)) an argument similar to, but simpler than, that used to prove Proposition 5.4. (Essentially, Lemma 4.6 shows that once we have got over the initial “hill” by premultiplying \( w_1 \) with \( \pi \), there is sufficient cancellation between words \( x_i \) to guarantee that the ratio \( w_1 w_i^{-1} \) is very short.)

**Lemma 4.6.** Suppose that \( k \geq 4 \), that every word \( x_i \) has one of the forms \( B, O, E, OO, OE, EO, \) or \( EE \), and that there is no short term cancellation between different words \( x_i \). Define \( t_i = \pi x_k \cdots x_i \) where \( \pi \) is as above. If the final long term of \( x_i \) completely cancels with the first long term in \( x_{i-1} \), then \( t_i \in \{u, u's_i'\} \), where \( u \) is the final long term in \( x_i \); otherwise \( t_i \in \{1, s_i'\} \).

**Proof.** We prove the case in which \( x_k = v_{s_i''} s_i'' \) and this term completely cancels with the last term in \( x_1 \) (the other cases will be similar). In this case, \( \pi = \Delta s_i'' \) and \( t_i = \Delta s_i'' x_k \cdots x_i \) for all \( i \). The lemma is clearly true in case \( i = k \). Suppose that we have established the lemma for some fixed value of \( i \), and consider \( t_{i-1} \).

Suppose first that the last long term in \( x_i \) is the odd term \( v_{s_i''} \). and that this term completely cancels with the first long term in \( x_{i-1} \). First let \( x_{i-1} = v_{s'_{i-1} s''_{i-1}} \), \( \epsilon_{i-1} \). Then \( t_i = v_{s_i''} s_i'' \Rightarrow t_{i-1} = v_{s_i''} s_i'' : v_{s'_{i-1} s''_{i-1}} \epsilon_{i-1} \). Complete cancellation implies \( s_i'' = s_i'' \) and \( \alpha = s_{i-1}'' \), so \( t_{i-1} = s_{i-1}'' \epsilon_{i-1} = \{1, s_{i-1}'\} \), as needed. If instead \( x_{i-1} = v_{s'_{i-1} s''_{i-1}} u_{s''_{i-1}} \epsilon_{i-1} \), then \( t_i = s_{i-1}'' u_{s''_{i-1}} \epsilon_{i-1} = \{u_{s''_{i-1}} \}, \) as needed. Similar computations hold in case \( i = k \).

If \( t_i = v_{s_i''} \) and \( x_{i-1} = v_{s'_{i-1} s''_{i-1}} \epsilon_{i-1} \), we compute \( t_{i-1} = \{s_{i-1}'' \} \). It is at this point that a proof like that for Proposition 5.4 must be used, in order to show that \( s_{i-1}'' \) cannot appear in \( x_k \cdots x_{i-1} \). (One must show that such an occurrence of \( s_{i-1}'' \) cannot be canceled by completing the product \( t_{i-1} \cdot x_{i-2} \cdots x_1 \). This cancellation would require another occurrence of \( s_{i-1}'' \) in \( x_{i-1} \). But if there were such an occurrence, we would be able to apply Theorem 2.3 inductively in order to obtain a contradiction, as in the proof of Proposition 5.4.) Therefore this case cannot really occur. The cases in which \( t_i = v_{s_i''} \) and \( x_{i-1} \) is either \( v_{s_i''} v_{s''_{i-1}} u_{s''_{i-1}} \epsilon_{i-1} \) or \( v_{s_i''} v_{s''_{i-1}} u_{s''_{i-1}} \epsilon_{i-1} \) are outlawed by similar arguments.

We leave the proofs of the remaining cases to the reader. \( \square \)
As a consequence, \( t_is_it_i^{-1} = \hat{s}_i \) for all \( i \). Lemma 4.6 now follows almost immediately, since now
\[
ww_i^{-1} = \pi w_1 w_i^{-1} = \pi x_k \cdots x_i = t_i
\]
for all \( i \).

We now address the issue of short term cancellation. Note that not only may such cancellation occur initially (before any long terms have been canceled), it might also occur after complete cancellation of two or more long terms has been performed. For example, let \( x_4 = s_1's_1's_1's_1\alpha_1 \cdots \alpha_is'_3's'_4's'_3, x_3 = (s'_3's'_3)^2, x_2 = \alpha_1 \cdots \alpha_1, \) and \( x_1 = (s'_1's'_1)^2 \), where \( s'_1's'_4 \) and \( s'_3's'_4 \) have order 4 and for every \( i, \alpha_is'_i = s'_i\alpha_i \iff j \in \{2, 4\} \). Then after canceling the last long term in \( x_4 \) with the first (and only) term in \( x_3 \), we may cancel every \( \alpha_i \) in the product \( x_4x_3 \cdot x_2 \).

This poses no significant problems, as we have designed Theorem 2.3 to handle the possibility that such cancellation occurs. Returning to the above example, define \( w = \Delta_{s_1's_1's_1's_1}w_1 = (s_1's'_3)^2w_1 \). Then \( ws_1w_1^{-1} = s'_1 \) and \( ws_4w_1^{-1} = s'_4 \), but \( ws_3w_1^{-1} = \alpha_1 \cdots \alpha_is'_3\alpha_i \cdots \alpha_1 \). However, every \( \alpha_i \) satisfies the separation condition of Theorem 2.3 relative to \( s_3 \) and \( s'_1 \).

Let \( k = 4 \). If short term cancellation occurs at any stage in multiplying the words \( x_i \) together, arguments like those used to handle the case \( k = 3 \) may be used to prove Proposition 4.2, where short terms may now be inserted in between the schematic long terms. The same arguments show that there is no cancellation between a short term in \( x_i \) and a long term in \( x_{i-1} \) without additional cancellation of a long term from \( x_{i-1} \). (This sort of outlawed cancellation will be the focus of much of Section 5.) That is, short terms must ultimately cancel with other short terms, and long terms with long terms.

Therefore the only problem short term cancellation poses arises when completing the proof of Theorem 2.3. Yet all of the arguments from Lemma 4.3 through Lemma 4.6 still hold, with slight modification, in case there is short term cancellation. First, we give the analogue of Lemma 4.6; the proof of Lemma 4.7 is very similar.

**Lemma 4.7.** Suppose that \( k > 3 \) and that every word \( x_i \) has at most 2 long terms (we still have the same seven schematic structures). Define \( t_i \) as in Lemma 4.6 then \( t_i \) can be written geodesically as \( \alpha_1 \cdots \alpha_1u\epsilon_1 \), where
\begin{enumerate}
  \item \( u \) is the last long term in \( x_i \) if this long term completely cancels with the first long term in \( x_{i-1} \) and \( u = 1 \) otherwise,
  \item \( \epsilon \in \{1, s'_1\}, \) and
  \item for each letter \( \alpha \) either \( \alpha \in \{s'_1, s'_1\} \) or \( \alpha \not\in C' \) and \( \alpha \) commutes with some \( s'_j \), \( i \leq j \leq k \). If \( \alpha = s'_1, [s_2s_1] \) and \( [s_1s_1] \) have label 2, and if \( \alpha = s'_1, [s_{i-1}s_i] \) and \( [s_is_{i+1}] \) have label 2.
\end{enumerate}

\[
\text{Compare the letters } \alpha \text{ in this lemma with the letters } \alpha \text{ that arise in Theorem 2.3.}
\]

As when proving Lemma 4.6, a proof along the lines of Proposition 4.4 is required in order to show that \( s'_1 \) and \( s'_1 \) are the only letters of \( C' \) which can arise in as letters \( \alpha \), and only under the circumstances indicated. Consider the following example, which suggests why this should be true.

Let
where \( \alpha, \alpha_1, \alpha_2, \) and \( s_i \) commute with \( s'_i \), \( \alpha_1 \) commutes with \( s'_i \), both \( \delta \) and \( s'_i \) commute with both \( s'_i \) and \( s''_i \), and \( \alpha \) and \( s'_i \) both commute with \( s''_i \). If further \( \beta = s'_0, \gamma = s'_0, s'_5 = s''_0 \), then \( x_3 \cdots x_1 = 1 \). We compute \( w_i w_{i}^{-1} = x_9 x_8 = (\alpha s'_1)^3 \alpha \alpha_2, \) which demonstrates how \( s'_i \) may be present in \( w_i w_{i}^{-1} \). Let us also examine \( w_i w_{3}^{-1} \) and \( w_i w_{2}^{-1} \).

\[
w_i w_{3}^{-1} = x_9 \cdots x_3 = (\alpha s'_1)^3 \alpha (\delta s'_3)^5 \delta,
\]

giving us an example of a ratio \( w_i w_{j}^{-1} \) in which \( s'_j \) appears. (Notice that since \( \delta \) and \( s'_i \) do not commute, this ratio is geodesic as written.) However, in order that \( x_k \cdots x_1 = 1 \) hold, we must cancel the word \( (\delta s'_3)^5 \delta \); the only other letter which can commute with both \( \delta \) and \( s'_i \) is \( s''_i \), and we see that \( w_i w_{2}^{-1} = x_9 \cdots x_2 = 1 \), as needed.

Assume now that short terms cancel with short terms, long terms cancel with long terms, and that no word \( x_i \) has more than 2 long terms. (Again, all of these statements follow from direct computation if \( k = 4 \) and from the results of Section 5 if \( k \geq 5 \). We can now prove Theorem 2.3 in case \( k \geq 4 \).)

Reindex, as needed, so that \( i = k \). Defining \( w \) as in Lemma 2.8, \( w s_{1} w^{-1} = s_{1} \) and \( w s_{k} w^{-1} = s_{k} \) still hold. As in the proof of Lemma 2.8, \( w = t_i w_i \) for every \( i \).

Theorem 2.8 now follows from Lemma 1.9.

5. The inductive step

We have now indicated proofs of Theorem 2.3 and Theorem 2.8 for achiral circuits of length at most 4. Inductively, assume that we have established these theorems for all achiral circuits of length less than or equal to \( k - 1 \), and consider an achiral circuit \( C \) of length \( k \) in \( \mathcal{V} \). For each edge \( [s_i - s_i] \) in \( C \), Theorem 2.1 provides an edge \( [s''_{i-1} s_i] \) to which \( [s_i - s_i] \) corresponds.

Much as in the previous section, we will multiply the terms \( x_i \) together, one at a time, performing all possible cancellation and length reduction as we go. Also, as before, we begin by assuming that there is no cancellation of short terms between different words \( x_i \).

Our first lemma can be proven using the results from [11].

**Lemma 5.1.** Let \( (W, S) \) be a 2-d Coxeter system with diagram \( \mathcal{V} \), and let \( [st] \) be an edge in \( \mathcal{V} \). Let \( w \in W \) satisfy \( \{w s t^{-1}, w t^{-1}\} = \{s, t\} \).

1. If \( [st] \) has label 2, then \( w \in \{1, s, t, st\} \).

2. If \( [st] \) has label greater than 2, then \( w \in \{1, \Delta_{st} \} \), where \( \Delta_{st} \) is the longest element in \( W_{\{s, t\}} \). In case the label on \( [st] \) is odd, then \( w = 1 \iff w s t^{-1} = s \iff w t^{-1} = t \).

Now we prove a technical lemma that will often be used to reduce our problem to a case already considered.

**Lemma 5.2.** Let Theorem 2.3 and Theorem 2.8 both be proven for achiral circuits of length at most \( k - 1 \), and let \( C \) be an achiral circuit in \( \mathcal{V} \) of length \( k \). Suppose that there is a vertex \( \alpha \) adjacent to the vertices \( s_{i_1}, s_{i_2}, \ldots, s_{i_r} \) so that each of the circuits
\{[\alpha s_i], [s_is_i+1], \ldots [s_i+1\alpha] \} is achordal and of length less than k, for 1 \leq l \leq r. Then Theorem 2.2 and Theorem 2.3 both hold for C as well.

**Proof.** Reindexing, we let \( l = i \) for 1 \leq l \leq r, and let \( C_l \) denote the circuit \( \{[\alpha s_i], \ldots, [s_i+1\alpha] \} \). By hypothesis, to each \( C_l \) there is a circuit \( C'_l \) in \( \mathcal{V}' \) which corresponds, edge by edge, to \( C_l \). (As usual, we use “prime” notation to indicate the corresponding vertices.) Moreover, the ratios \( w_iw_l^{-1} \) of the elements which conjugate the edges of a given \( C_l \) are governed by Theorem 2.3.

Consider the edge \([s_{t+1}\alpha] \), lying in the circuits \( C_l \) and \( C_{l+1} \). Applying Theorem 2.2 to these circuits yields group elements \( \tilde{w}_l \) and \( w_{l+1} \) such that

\[
\{\tilde{w}_ls_{t+1}\tilde{w}_l^{-1}, \tilde{w}_l\alpha \tilde{w}_l^{-1}\} = \{w_{l+1}s_{t+1}w_{l+1}^{-1}, w_{l+1}\alpha w_{l+1}^{-1}\}.
\]

Lemma 5.1 implies that \( w_{t+1}^{-1}\tilde{w}_l \in \{1, s_{t+1}, \alpha, s_{t+1}\alpha\} \) if \([s_{t+1}\alpha] \) has label 2 and \( \tilde{w}_l^{-1}w_{l+1} \in \{1, \Delta_{s_{t+1}}\} \) if \([s_{t+1}\alpha] \) has label greater than 2. Since \( \tilde{w}_l\alpha \tilde{w}_l^{-1} = \alpha' \) and \( \tilde{w}_ls_{t+1}\tilde{w}_l^{-1} = s'_{t+1} \), we obtain \( \tilde{w}_lw_{l+1}^{-1} \in \{1, s'_{t+1}, \alpha', s'_{t+1}\alpha'\} \) if \([s_{t+1}\alpha] \) has label 2 and \( \tilde{w}_lw_{l+1}^{-1} \in \{1, \Delta_{s'_{t+1}}\} \) if \([s_{t+1}\alpha] \) has label greater than 2.

Now, from Theorem 2.3 applied to \( C_l \), \( w_l\tilde{w}_l^{-1} \) can be written as a product \( \beta_1 \cdots \beta_p \), where for every letter \( \beta_i \), either \( \beta_i = s'_i \) (in which case \([s_0\alpha] \) has label 2), \( \beta_i = s'_{t+1} \) (in which case \([s_{t+1}\alpha] \) has label 2), or \( \beta_i \) does not lie on \( C' \). (In this last case, Theorem 2.3 shows that \( \beta_i \) does not lie on \( C'_l \); if \( \beta_i \) were to lie on \( C'_l \), \( \beta_i \) would contradict the achordality of \( C \). For later use, we note that by the separation condition of Theorem 2.3, each \( \beta_i \) must in fact commute with \( \alpha \).

We can now compute \( w_lw_{l+1}^{-1} = w_l\tilde{w}_l^{-1}\tilde{w}_lw_{l+1}^{-1} \) for each \( l \), \( 1 \leq l \leq r \). The product of these words, taken in order, is trivial in \( W \). Therefore, application of the Tits Process (TP) must yield the trivial word.

Assume that for some edge \([s_{t+1}\alpha] \) with label greater than 2, \( \tilde{w}_lw_{l+1}^{-1} = \Delta_{s'_{t+1}}\alpha' \). One easily sees that the letter \( s'_{t+1} \) appears only in this word when forming the product \( w_lw_{l+1}^{-1} \) as above, and then only in \( \Delta_{s'_{t+1}}\alpha' \). (In particular, \( s'_{t+1} \) cannot arise as a letter \( \beta_i \) in \( w_l\tilde{w}_l^{-1} \), because we have assumed \([s_{t+1}\alpha] \) has label exceeding 2.) In applying TP, no two occurrences of the letter \( s'_{t+1} \) are brought next to one another. This letter can therefore not be canceled, contradicting the product’s triviality. Therefore, \( \Delta_{s'_{t+1}}\alpha' \) cannot appear. In particular, \( \tilde{w}_ls_{t+1}\tilde{w}_l^{-1} = w_{l+1}s_{t+1}w_{l+1}^{-1} \) and \( \tilde{w}_l\alpha \tilde{w}_l^{-1} = w_{l+1}\alpha w_{l+1}^{-1} \) hold, so that there is no “twisting” at the edge \([s'_{t+1}\alpha'] \) in \( \mathcal{V}' \) when the two adjacent circuits \( C'_l \) and \( C'_{l+1} \) are met along this edge. This concludes the proof of Theorem 2.3 for \( C \).

Now for Theorem 2.3. Note first that the argument from the previous paragraph also shows that \( \tilde{w}_lw_{l+1}^{-1} \in \{1, \alpha' \} \) if the label on \([s_{t+1}\alpha] \) is 2.

Now take any two distinct vertices on \( C \) and consider the edges \( e_i = [s_is_i+1] \) and \( e_j = [s_j\ldots s_j] \) in \( C \) (as in Theorem 2.3). Suppose \( e_i \) and \( e_j \) lie on circuits \( C_i \) and \( C_m \), respectively. We suppose \( j > i \), and let \( \tilde{w}_i, \tilde{w}_j \) be the elements conjugating \( e_i \) and \( e_j \) (with respect to \( C_i \) and \( C_m \), resp.) provided by Theorem 2.3. Then

\[
\tilde{w}_i\tilde{w}_j^{-1} = \tilde{w}_i\tilde{w}_l^{-1} \cdot \tilde{w}_lw_{l+1}^{-1} \cdots \tilde{w}_m\tilde{w}_j^{-1}.
\]

(4)

From the preceding arguments we know this product can be written in the letters \( \beta \) which arise from applying Theorem 2.3 to each achordal cycle \( C_p \) in turn, as well as the letter \( \alpha' \). We will have proven Theorem 2.3 for \( C \) once we show each letter
First suppose that $\beta$ lies on $C'$ and neither $\beta = s'_p$ nor $\beta = s'_l$ holds. Then $\beta = s'_p$ for some $p$ and $[s'_p\alpha]$ is an edge labeled 2. The only two words in (4) which may contain the letter $s'_p$ are $\hat{w}_p w_{p+1}$ and $w_{p+1} \hat{w}_{p+1}$, Therefore, in applying TP to the product

$$\bar{w}_i \bar{w}_{j}^{-1} \cdot \bar{w}_j \bar{w}_{m}^{-1} \cdots \bar{w}_{l-1} \bar{w}_{l}^{-1} \cdot \bar{w}_1 \bar{w}_i^{-1} = 1$$

(5)

the instances of $s'_p$ in $\hat{w}_p w_{p+1}$ cancel with those in $w_{p+1} \hat{w}_{p+1}$. We claim that the same cancellation takes place in (4) as in (5) (that is, we do not need the context of (5) in order to use TP to cancel these occurrences of $s'_p$). This is so because none of the letters that appear in between occurrences of $s'_p$ are related to one another, except $\alpha$. (This follows from two-dimensionality and the fact that every $\beta \neq \alpha$ commutes with $\alpha$ when it appears in one of the two terms given above.) Therefore, we can perform this cancellation of letters $s'_p$ in (4), and $s'_p$ does not in fact occur.

Now suppose $\beta \notin C'$. The separating condition is satisfied unless $\beta$ is adjacent by edges labeled 2 only to $\alpha'$ and to a single letter $s'_p$ in some cycle $C'_l$, $l \leq l' \leq m$.

Yet we claim that $\beta$ must also be adjacent to some other letter $s'_q$ in $C'$. To see this, examine the product in (5) again. As this word represents the trivial element in $W$, applying TP must yield the empty word. If a letter $\beta$ occurs in $\bar{w}_i \bar{w}_j^{-1}$, it either cancels with another occurrence of $\beta$ in that same subproduct (in which case it can be removed from that subproduct altogether anyway, by an application of TP to the subproduct, as indicated above) or it cancels with an occurrence of $\beta$ lying in $\bar{w}_j^{-1} \cdots \bar{w}_i$. This other occurrence of $\beta$ comes from an application of Theorem 2.3 to yet another cycle $C''$, which lies between $C_l$ and $C_m$. “Opposite” the cycles $C_{l+1}, \ldots, C_{m-1}$. Therefore there is a vertex $s'_q$ such that $[s'_p, \beta]$ is an edge labeled 2 and such that the path $\{[s'_p, \beta], [\beta s'_q] \}$ divides $C'$ into two cycles, one containing $s'_l$, and the other containing $s'_j$, as desired.

The following proposition shows that there cannot be a great deal of cancellation between words $x_i$ and $x_j$ for which $d(i,j)$ is large. The arguments in this proof demonstrate the flavor of many of the arguments to come.

**Proposition 5.3.** Let Theorem 2.3 and Theorem 2.4 both be proven for achordal circuits of length at most $k-1$, and let $C$ be an achordal circuit in $V$ of length $k$.

Define the words $x_i$ as above. Let $x_i$, $x_j$, and $x_1$ be adjacent words in $x_k \cdots x_1$ (for $i > j > l$), and assume that there is no short term cancellation.

1. If $d(i,j) \geq 3$, then we may assume that $x_i x_j$ is a geodesic word for the group element it represents (i.e., no length reduction is possible).

2. If $d(i,j) = 2$, then at most one pair of letters cancels in forming the product $x_i x_j$.

3. Suppose $d(i,j) = 2$ and $d(j,l) = 2$. If $x_j$ consists either of the single long term $v_{s'_j}^\prime s_{j''}$ or the single long term $z_{j-1,j}^{-1} \neq 1$, then at least one of the products $x_i x_j$ or $x_j x_1$ allows no cancellation.
Now assume that entirely similar fashion.

Note that if $i = k$ in the first case, then $d(i, j) \geq 3$ forces $j \neq 1$, $j \neq 2$, and $i = k - 1$ forces $j \neq 1$. The third case will allow an easy proof of Theorem 2.2 and Theorem 2.3 in case all adjacent $x_i$ and $x_j$ words satisfy $d(i, j) \geq 2$.

**Proof.** 1. First suppose that $x_i$ and $x_j$ are adjacent and $d(i, j) \geq 3$. Assume (to derive a contradiction) that there is length reduction in the product $x_ix_j$. There are a few possibilities.

First, $x_i$ could end with a long term $(s_1'\alpha)^l$ while $x_j$ begins with a long term $(\beta s_j')^r$ (of course, $s_j' = s_j''$ if $\bar v_j = 1$). That there is reduction implies either that $\beta = s_1'$ and $\alpha = s_j''$ both hold or that $\alpha = \beta$.

In the first case, $s_1'$ and $s_j''$ are adjacent in $\mathcal{V}'$. Because $x_p = 1$ for $j < p < i$, $\{[s_i's_{i-1}'], ..., [s_j'+1s_j'']\}$ forms a circuit $D'$ of length at most $k - 1$ in $\mathcal{V}'$. If $D'$ is not achordal, we can find a smaller circuit containing a subset of these vertices which is achordal in $\mathcal{V}'$. By inductive hypothesis, there is a circuit $D$ in $\mathcal{V}$ which corresponds to $D'$ as in Theorem 2.2. However, all but one of the edges of $D'$ correspond as in Theorem 2.1 to edges of $C$; therefore $D$ is a circuit whose vertices are vertices of $C$. This implies that $C$ was not achordal, a contradiction.

In the second case, there is a vertex, $\alpha$, adjacent to both $s_1'$ and to $s_j''$. As in the previous paragraph, we find a circuit $D' \{[s_1's_{i-1}]', ..., [s_j'+1s_j''], [s_j'']\alpha, [\alpha s_1']\}$ of length at most $k - 1$. If $D'$ is not achordal, it can be subdivided into shorter achordal circuits by addition of edges $[\alpha s_p']$, $j < p < i$. By inductive hypothesis, each of these achordal circuits corresponds as in Theorem 2.2 to an achordal circuit of the same length in $\mathcal{V}$. In fact, we can “piece together” these individual circuits by pasting along the common chords which correspond as in Theorem 2.1 to the edges of $C$. This implies that $C$ was not achordal, a contradiction.

2. Now assume that $x_i$ and $x_j$ are adjacent and that $d(i, j) = 2$. As before, let us first consider the case of a long term $(s_i'\alpha)^l$ in $x_i$ canceling with a long term $(\beta s_j')^r$ of $x_j$. If there is to be more than one pair of letters canceling in $x_ix_j$, it must be that $\alpha = s_i''$ and $\beta = s_j'$. As before, this forces $s_i'$ and $s_j''$ to be adjacent, yielding a shorter achordal circuit to which the inductive hypothesis can be applied, giving a contradiction to the achordality of $C$.

As in the first case, we leave the similar arguments for the other possibilities to the reader.
3. Suppose that \( x_j = v'_i s'_j \), that \( x_i \) ends with the long term \( (s'_i\alpha)^m \), and that \( x_l \) begins with the long term \( (\beta s''_l)^n \). (The other cases for \( x_j = v'_i s'_j \) and the cases in which \( x_j = z_{j-1,j} \neq 1 \) will be left to the reader.)

Since \( s'_i = s'_j \) and \( s''_j = s''_l \) both lead to a contradiction to \( C' \)'s achordality, cancellation can only occur if \( \alpha = s'_i \) or \( \beta = s''_l \). Suppose that both of these equalities hold. Then there are circuits \( C'_1 = \{[s'_i, s'_{i-1}], [s'_{i-1} s''_j], [s''_j s'_j], [s'_j s'_i] \} \) and \( C'_2 = \{[s''_l s'_j], [s'_j s'_{j-1}], [s'_{j-1} s''_j], [s''_j s''_l] \} \) in \( V' \), sharing the edge \( [s''_l s'_j] \). First assume that these circuits are achordal. Then by Theorem 2.2, there are corresponding circuits \( C_1 \) and \( C_2 \) in \( V \). Each of \( C_1 \) and \( C_2 \) contains two consecutive edges from \( C \), and \( C_1 \) and \( C_2 \) share a common edge. Moreover, because we know the sequence of edges in \( C \), we conclude that the circuit formed by replacing the edges \( [s_i s_{i-1}], ..., [s_{i+1} s_i] \) by \( [s_i \gamma] \) and \( [\gamma s_i] \) is shorter than \( C \); if it is not achordal, then it can be subdivided by addition of chords \( [\gamma s_n] \) for some letters \( s_n \), and an application of Lemma 5.2 brings our proof to a close (by demonstrating that the “twist” apparent at the edge \( [s''_l s'_j] \) could not in fact have been, giving a contradiction).

If \( C'_1 \) and \( C'_2 \) had not been achordal, we could have shortened them by introducing the requisite edge and applying the arguments of the previous paragraphs in order to reach the same conclusion.

\[ \square \]

Suppose now that for any two adjacent words \( x_i \) and \( x_j \), \( d(i, j) \geq 2 \). (Here we may assume that there could be short term cancellation, but we suppose that all common short terms have been canceled.) Suppose also that for some \( j \), \( v_j \neq 1 \).

If \( x_j \) contains at least two long terms, then \( x_j \) has length at least 4. Consider the product \( x_i x_j x_l \) where \( x_i \) and \( x_l \) are the adjacent words nearest to \( x_j \) on either side. At most two letters of \( x_j \) are canceled in this product. Can there be more cancellation upon multiplying more words? Suppose that \( x_l \) consists of a single letter, \( \beta \), which cancels in the product \( x_j x_l \). Then either \( \beta = s'_l \) or \( \beta s'_l = s'_l \beta \); in any case, \( s'_l = s''_l \). If \( x_m \) is the next nontrivial word after \( x_l \), the arguments used to prove part 1 of Proposition 5.3 can be applied to show that there is no cancellation in multiplying \( x_j \beta \cdot x_m \). The same can be said for the product \( x_i x_j \).

Ultimately we conclude that in the product \( x_k \cdots x_1 \), there are letters of \( x_j \) which remain uncanceled after all reduction is performed (including application of TP). Thus \( x_j \) could not have had more than one term.

If instead \( v_j \neq 1 \) and this is the only long term appearing in \( x_j \), part 3 of Proposition 5.3 allows us to reach the same conclusion: some of \( x_j \) must remain in multiplying \( x_k \cdots x_1 \).

Therefore, \( v_j = 1 \), so \( s'_i = s''_i \) for all \( i \), \( 1 \leq i \leq k \), and the circuit \( C' \) corresponds as in Theorem 2.2.

Does this matching satisfy Theorem 2.2 as well? Note that an argument similar to that in the above paragraphs shows that \( z_{i-1,i} = 1 \) must hold for all \( i \), \( 1 \leq i \leq k \) and no word \( u_{a_i a_{i-1}} \) can occur in any \( x_i \). Thus, the ratio \( u_{i+1}^{-1} w_i^{-1} \) can be written \( \alpha_1 \cdots \alpha_{i-1} \epsilon_i \), where \( \alpha_l s'_l = s'_l \alpha_l \) for \( 1 \leq l \leq i \) and \( \epsilon_i \in \{1, s'_i\} \). Suppose that \( s'_i \) were to appear. As \( x_k \cdots x_1 = 1 \), some other word, \( x_j \), must contain \( s'_i \), and therefore \( s'_i s'_j = s'_j s'_i \). This contradicts the achordality of \( C' \). Therefore \( \epsilon_i = 1 \) for all \( i \). At last, an argument similar to that used in establishing the “separation condition” of Theorem 2.2 in Lemma 5.2 shows that the same condition is satisfied in this case.
We must now turn our attention to the case in which for some adjacent words \(x_i\) and \(x_{i-1}\), \(d(i,j) = 1\). (We frequently assume that \(d(i,j) = 1\) for all adjacent words \(x_i\) and \(x_j\). Whenever this is not so, our arguments are often made simpler, as the reader is invited to verify.) The arguments we offer, though at times technical, are entirely analogous to those that have come before.

Until further notice we will assume that \(z_{i-1,i} = 1\) for all \(i\) (our arguments below will show that this must be the case anyway) and that there is no short term cancellation, as in the previous section. We distinguish between two putative types of cancellation between two adjacent words \(x_i\) and \(x_{i-1}\): \textit{complete long term cancellation} (CLTC) and \textit{incomplete long term cancellation} (ILTC).

CLTC occurs when \(x_i\) ends with a long term (followed perhaps by \(s_i'\)) and \(x_{i-1}\) begins with a long term in the same letters. In this case the resulting product has at most two letters and lies in \(\{1, s_i', s_{i-1}'', s_i''s_i'\}\). In the case that the common long term is odd, and \(\{1, s_i', s_{i-1}', s_i's_i'\}\) if the common long term is even.

ILTC is the cancellation that can conceivably occur when a long term in \(x_i\) (resp. \(x_{i-1}\)) cancels with either a short term, a single letter of a long term, or both, in \(x_{i-1}\) (resp. \(x_i\)). For instance, if \(x_i\) ends with \(uv_{\alpha}\) and \(x_{i-1}\) begins with \(\alpha u_{\beta s_{i-1}'}\) (where \(\alpha \neq s_{i-1}'\)), the letters \(\alpha\) cancel. Further, it is possible that \(\beta = s_{i}'\), so that one more pair of letters cancels. There is no more cancellation than this, leaving at least two letters of \(u_{\beta s_{i-1}'}\) intact, justifying the use of the term “incomplete”.

There will be a small number of exceptional cases of ILTC in which an entire long term is canceled; these cases will more closely resemble CLTC in some respects.

We shall describe all possible forms of CLTC and ILTC shortly.

The rough course of our remaining argument is as follows. For each \(x_i\), there is a subdiagram of \(V'\) (which we will call a \textit{piece}) whose form can be derived from \(x_i\). In case \(x_i\) has at most two long terms, the corresponding piece has one of the seven general forms examined in Section 4. When two words \(x_i\) and \(x_{i-1}\) exhibit any sort of cancellation, we are given information about how to put the corresponding pieces together, and given a chain of consecutive “short” words (in a sense to be introduced below) \(x_i, x_{i-1}, ..., x_j\), each of which cancels in some way with the previous one, a subdiagram of \(V'\) emerges which resembles a segment in the circuit \(C'\) whose existence we wish to establish.

Putting together pieces, we first show that we may assume ILTC does not occur. We then indicate how the same argument may be used to show that \(z_{i-1,i} = 1\) for all \(i\), and that no word \(x_i\) comprises more than 2 long terms. At this point we will be in a position to prove Theorem 2.22 for \(C\) much as was done for short circuits in Section 4. Then we will appeal to the final results from that section to complete the proof of Theorem 2.30 for \(C\).

We call a word \(x_i\) \textit{terse} if it has no more than 2 long terms. As we have seen, there are seven general forms for such words: \(B, O, E, OO, OE, EO, EE\), where short terms may be inserted in appropriate places. Note that in cases \(B, E,\) and \(EE, s_i' = s_i''\). The pieces which correspond to each of these terse word forms are depicted in Figure 1.

\textbf{Proposition 5.4.} We can assume that there is no incomplete long term cancellation between two words \(x_i\) and \(x_{i-1}\).

To begin our proof, let us assume that there is ILTC between \(x_i\) and \(x_{i-1}\); by reindexing, assume that \(i = k\).
There may also be ILTC between $x_{k-1}$ and $x_{k-2}$, and then too between $x_{k-2}$ and $x_{k-3}$, and so forth. We will continue to multiply terms $x_k, x_{k-1}, \ldots, x_j$ until we no longer see ILTC, keeping track of the product $x_k x_{k-1} \cdots x_j$ (and the corresponding concatenation of pieces) as we go. Unless some nontrivial word remains when all words $x_i$ are multiplied, we will either be able to apply Theorem 2.2 to a shorter circuit to obtain a contradiction, or be able to apply Lemma 5.2 to yield the desired conclusion by appeal to the inductive hypothesis.

For notational convenience, we denote the subdiagram of $\mathcal{V}'$ formed by concatenating the pieces corresponding to $x_i, x_{i-1}, \ldots, x_{i-2}$ by $\mathcal{V}'(i_1, i_2)$.

We first multiply $x_k, x_{k-1}, \cdots, x_j$ only so long as each word $x_i$ for $j \leq i \leq k - 1$ has at most one long term and there is no exceptional ILTC as defined below. (For the time being the exact form of $x_k$ is not important.) Thus at each step we multiply words of types $B, E$, and $O$ with one another. Since we have assumed that there is no short term cancellation, and because no more than one short term may cancel with a long term (otherwise we would contradict the fact that $W$ is 2-d), we are left with the following possible products for $x_i \cdot x_{i-1}$:

1. $B \cdot O$, $B \cdot E$, $O \cdot B$, $E \cdot B$,
2. $O \cdot O$, $O \cdot E$, $E \cdot O$, $E \cdot E$,
3. $Os_i' \cdot O$, $Os_i' \cdot E$, $Es_i' \cdot O$, $Es_i' \cdot E$,
4. $O\alpha \cdot O$, (*) $O\alpha \cdot E$, $E\alpha \cdot O$, (*) $E\alpha \cdot E$,
5. $O \cdot \beta E$, (*) $E \cdot \beta E$,
6. $O \cdot s_{i-1} \cdot E$, $E \cdot s_{i-1} \cdot E$.

Here, $\alpha s_i' = s_i' \alpha$ and $\beta s_{i-1}' = s_{i-1}' \beta$.

The cases marked (*) represent cases in which the ILTC may be more complicated. In each of these cases, one of the words $x_i, x_{i-1}$ can indeed be completely canceled in forming the product $x_i x_{i-1}$. For instance, if $x_i = v s_i' \gamma \alpha$ (where $\alpha s_i' = s_i' \alpha$) and $x_{i-1} = v s_{i-1}' \gamma s_{i-1}'$, then $\alpha = s_{i-1}'$ and $\gamma = s_{i-1}' \gamma$ can both hold. If $s_{i-1}' s_{i-1}$ has order $n = 3$, then $x_{i-1}$ has been completely canceled. (Incidentally, when $n = 3$, two-dimensionality implies that $s_i' \gamma$ has order at least 7. We use this
fact later.) However, one may apply Lemma 5.1 to show that further cancellation of the remainder of $x_i$ with $x_{i-2}$ would contradict the structure of $C$ (we would essentially be trying to “twist” the diagram at an edge labeled 2, which cannot be done, by Lemma 5.1). Therefore we can cancel no more. Because $x_i$ and $x_{i-2}$ do not cancel with each other, the chain of ILTC ends at this point, and may or not done, by Lemma 5.1). Therefore we can cancel no more. Because $x_i$ and $x_{i-2}$ do not cancel with each other, the chain of ILTC ends at this point, and may or not begin ILTC anew with $x_{i-2} \cdot x_{i-3}$. If a long term is completely canceled in the product $x_i \cdot x_{i-1}$, we call this case of ILTC exceptional.

As an illustration of the above products, Figure 2 displays the subdiagrams $V'(i, i-1)$ corresponding to the products $O \cdot O$, $O s'_i \cdot E = v_{s'_i} s'_i s'_i \cdot u_{s_{i-1}}$, and $E \cdot \beta E = u_{s'_i} \cdot \beta u_{s_{i-1}}$.

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
\hspace{1cm} & \hspace{1cm} & \\
$s_{i-1}$ & $s'_i$ & $s''_i$ \\
$s''_{i-1}$ & $s'_{i-1}$ & $s''_{i-1}$ \\
O \cdot O & $s'_i = \alpha$ & $s''_i = s''_i$ (\(= \gamma\)) \\
$O s'_i \cdot E$ & $s''_i = s''_i$ & \\
E \cdot \beta E & \hspace{1cm} & \hspace{1cm}
\end{tabular}

\caption{Some products of pieces}
\end{figure}

The reader may wish to draw some of the remaining subdiagrams in order to familiarize himself or herself with their appearance.

Let us now assume that $x_{k-1} \cdot x_{k-2}$ also sees ILTC, and paste to $V'(k, k-1)$ the piece corresponding to $x_{k-2}$. We observe that aside from the letters whose equality is forced in order to produce the ILTC, there can be no equality between the remaining letters in $x_k$, $x_{k-1}$, and $x_{k-2}$. For instance, in multiplying $O \cdot O s'_{k-1} \cdot E$, $s'_{k-2} = s''_{k-1}$ cannot be, for otherwise $\{[s'_k, s''_{k-1}], [s''_{k-1}, s_k], [s'_{k-1}, s_k]\}$ would be a circuit of length 3 containing two edges corresponding to edges of $C$, contradicting $C$’s achordality.

We make another observation. Consider the portion of $V'(k, k-2)$ which lies between $s'_k$ and $s''_{k-2}$; that is, the largest subdiagram of $V'(k, k-2)$ all of whose vertices lie on some simple path from $s'_k$ to $s''_{k-2}$. This subdiagram has diameter 2: any two vertices in this subdiagram can be connected by a path $P$ of length at most 2. Furthermore, the edges in $P$ can be chosen so that at most one edge in $P$ lies in $V'(k, k-1)$ and not $V'(k-1, k-2)$, and at most one edge lies in $V'(k-1, k-2)$ and not $V'(k, k-1)$.

We may generalize these observations as we continue to paste together the pieces corresponding to $x_k, x_{k-1}, \ldots, x_j$, as long as each word $x_i$ has no more than one long term and as long as $x_i \cdot x_{i-1}$ witnesses some non-exceptional case of ILTC. We have the following lemma.

**Lemma 5.5.** Suppose that each of the words $x_k, x_{k-1}, \ldots, x_j$ has at most one long term, that each word $x_i$ sees non-exceptional ILTC with the following word, and that $j > 1$. Denote by $V''$ the portion of the subdiagram $V'(k, j)$ which lies between $s'_k$ and $s''_j$ in the sense described above. Then the following all hold.

1. For every $i, j \leq i \leq k-1$, $s'_i$ and $s''_i$ lie in $V''$. 
2. Besides the letters forced to be equal by ILTC (and, of course, \( s'_i = s''_i \) in case \( x_i \) is of type E or B), there is no equality between any of the letters in \( V'' \).

3. The diameter of \( V'' \) is at most \( k - j \), and moreover any two vertices in \( V'' \) can be connected by a simple path \( P \) so that for every subdiagram \( V'(i, i - 1) \) of \( V'' \) \((j + 1 \leq i \leq k)\), at most one edge of \( P \) lies in \( V'(i, i - 1) \) and in no other such subdiagram.

Proof. We sketch a proof and leave the details to the reader.

The proof is essentially an induction on \( k - j \). In case \( k - j = 1 \), the definition of ILTC and a glance at all possible pieces yields the desired conclusions. We then assume that the result holds for all values of \( k - j \) less than a given value \( m \), and consider \( x_kx_{k-1} \cdots x_{j+1} \cdot x_j \) where \( m = k - j \).

(1) is easily proven by examining all possible pieces for products \( x_{j+1} \cdot x_j \) and appealing to the inductive hypothesis regarding \( V'(k, j + 1) \).

(3) will also follow from the corresponding fact about \( V'(k, j + 1) \) once (2) is established for \( V'(k, j) \). For purposes of illustration, let \( x_{j+1} \cdot x_j \) have the form \( O \cdot O \) (other cases can be handled in a similar fashion). Because ILTC occurs, \( s''_{j+1} = s'_j \). Assume to derive a contradiction that \( s''_j \) is equal to some generator which lies in the portion of \( V'(k, j + 1) \) between \( s'_k \) and \( s''_{j+1} \).

First assume that \( s''_j \) is not adjacent to \( s'_j \in V'(k, j + 1) \). Then by inductive hypothesis, the edge \([s'_j s''_j]\) may be concatenated with a path \( P \) of length at most \( k - (j + 1) = m - 1 \) to obtain a circuit \( D' \), whose length is at most \( m \leq k - 1 \).

Therefore we may inductively apply Theorem 2.2 and find a circuit \( D \) in \( V \) to which \( D' \) corresponds, edge for edge. However, because \( P \) can be chosen as in (3), either the edges of \( V \) which correspond to those of \( D' \) do not form a circuit at all, or they contradict the achordality of \( C \).

The case in which \( s''_j \) and \( s'_j \) are adjacent in \( V'(k, j + 1) \) requires a different argument. However, using the facts that the edges of \( C \) must correspond as in Theorem 2.2 to distinct edges of \( C'' \) and that \( C \) is achordal, one may complete the proof in this case as well. The reader is encouraged to experiment with a few different cases in order to verify this claim. (Compare the example for \( j = k - 2 \) given before the statement of Lemma 5.4)

Notice that we have assumed \( j > 1 \) so that we can appeal to inductive hypothesis regarding the length of \( C \). If \( j = 1 \) the argument in Lemma 5.5 will still go through as long as the equality that occurs is not \( s''_k = s''_1 \) or \( s'_k = s'_1 \). Unfortunately, we must consider this case: suppose that \( s''_i = s''_{i-1} \) for \( i = 1, \ldots, k \). Then if \( x_i = s'_i s''_i \) for every \( i \), \( x_kx_{k-1} \cdots x_1 = 1 \) clearly holds, and there is ILTC between any two adjacent terms in this product! (This case is analogous to the case of the affine Euclidean groups that arose when \( k = 3 \).) However, we claim that in this case, we can either appeal to Lemma 5.2 or derive a contradiction.

The key here is that \( x_kx_{k-1} \cdots x_j = s'_k s''_j \) for every \( j = 1, \ldots, k \). That is, in the ILTC that occurs in this product, every letter of each term \( x_{k-1}, x_{k-2}, \ldots, x_{j+1} \) is canceled, because every product \( x_i \cdot x_{i-1} \) is of the form \( O \cdot O \) and each term \( x_i \) has length 2. The number of letters in this reduced product which follow \( s'_k \) never increases.

Let every product \( x_i \cdot x_{i-1} \) be of a type which permits at most one pair of letters to cancel (i.e., not of types 4 or 5). Assuming that the number of letters following the last occurrence of \( s'_k \) never increases (as above), none of the words \( x_i \) can contain
an even long term. Moreover, we see also that there can be no products of the type $O\alpha \cdot O$, and $s_i s_i''$ cannot have order greater than 3.

Even if we allow products $x_i \cdot x_{i-1}$ which admit more than one pair of letters to cancel, the length of the product $x_k \cdots x_j$ will increase. Indeed, even in cases 4 and 5, cancellation of more than one pair of letters implies (by the two-dimensionality of $W$) that one of the long terms involved in the cancellation is indeed quite long. For example, suppose that $x_1 \cdot x_{i-1}$ is of type $O\alpha \cdot E$, so $x_i \cdot x_{i-1} = v s_i' s_i'' \alpha \cdot u \beta s_{i-1}'$, where $s_i' s_i''$ has order $m$ and $\beta s_{i-1}'$ has order $n$. If more than a pair of letters cancels, $\alpha = \beta$ and $s_i' = s_{i-1}'$. Then $n = 4 \Rightarrow m \geq 7$ and $m = 3 \Rightarrow n \geq 6$. Therefore none of these terms can occur if $x_k \cdots x_1 = 1$ holds.

We are therefore left with only a few types of product that $x_i \cdot x_{i-1}$ can be, all involving only terms of types $O$ and $B$. First assume that there are no blank terms. Arguing as in the proof of Lemma 5.2, we can show that each subdiagram $V'(k, j)$ appears as a sequence of triangles, each sharing an edge with the last. As in that proof, once we have three such triangles in a row, we can apply Lemma 5.2 and conclude.

Note that there cannot be two non-trivial blank terms in a row, for if there were, they would not cancel with each other, since we are assuming that there is no short term cancellation. Each blank term $x_i$ which occurs between two odd terms $x_{i+1}$ and $x_{i-1}$ must admit cancellation with both $x_{i+1}$ and $x_{i-1}$. Moreover, as above, one can rule out products $O s_i' \cdot B$ and $O \alpha \cdot B$ (where $\alpha s_i' = s_i''$). Thus the above blank term $x_i$ must be $s''_{i+1} s'_{i-1}$. If $x_i$ is a non-trivial blank term following a string of trivial blank terms, it must be followed by an odd term $x_{i-1}$, so that $x_i = s_{i-1}'$; similarly, if $x_i$ is a non-trivial blank term which precedes a string of trivial blank terms, $x_i = s''_{i+1}$. Finally, there can be no more than two terms of type $O$ in a row, as otherwise, we would obtain three consecutive triangles to which Lemma 5.2 could be applied, as in the previous paragraph.

Using this information one can piece together the subdiagram corresponding to the product $x_k \cdots x_1$: it has a rather regular form, consisting of a sequence of edges, triangles, and “diamonds” (pairs of triangles sharing an edge), each such component sharing a single vertex with the last. In any such configuration, the assumption that $s''_1 = s'_k$ will yield a shorter achordal circuit $D'$ in $V'$ to which Theorem 2.2 can be applied. This circuit must correspond with a circuit $D$ in $V$. However, as in the proof of Lemma 5.3, the edges of $V$ which correspond to those of $D'$ either do not form a circuit at all or contradict the achordality of $D$. If we assume that $s''_1 = s'_k$ instead, one can see that $s''_1$ must hold as well, for given the above form of each $x_i$, $x_k \cdots x_1 = 1$ would not hold if $s''_1 \neq s''_k$. In this case, we may again construct a shorter achordal circuit $D'$ to which Theorem 2.2 can be applied inductively to obtain a contradiction.

Thus if $j = 1$ in the above chain of ILTC, we have proven Proposition 5.4. If $j \neq 1$, we have computed a reduced word which forms a “prefix” for $x_k \cdots x_1$. We will now argue that in completing the product $x_k \cdots x_j \cdot x_{j-1} \cdots x_1$, there can be almost no cancellation of this prefix. Essentially, we show how we can continue to multiply words $x_i$ (possibly with a great deal of cancellation) until a new ILTC chain as above is encountered, and then repeat the process until $x_k \cdots x_1$ is obtained.

If the ILTC chain above ends at $x_j$ ($j \geq 2$), it does so for one of the following reasons.

1. $x_j \cdot x_{j-1}$ is already reduced (there is no cancellation),
2. $x_j \cdot x_{j-1}$ sees exceptional ILTC and $x_{j-1}$ is completely canceled in this product,

3. $x_j \cdot x_{j-1}$ sees ILTC and $x_{j-1}$ has at least two long terms.

Consider for a moment the second case. In this case, as mentioned immediately following the definition of ILTC, it is easily shown that there is no further cancel-

lation between $x_k \cdots x_{j-1}$ and $x_{j-2}$ once $x_{j-1}$ is completely canceled. Therefore, this case can be argued in much the same way as the first, with $x_{j-2}$ in place of $x_{j-1}$.

Suppose first that $x_k \cdots x_j \cdot x_{j-1}$ admits no further cancellation. If $j = 2$ we are done. Otherwise, we consider the product $x_{j-1} \cdot x_{j-2}$. If there is no cancellation here either, we may continue by considering $x_{j-2} \cdot x_{j-3}$, and now we have the additional advantage of $x_{j-1}$ serving as a “buffer” between $x_j$ and $x_{j-2}$ which effectively forbids further cancellation of letters in $x_k \cdots x_j$. If $x_{j-1} \cdot x_{j-2}$ admits ILTC instead, either we begin a new chain of ILTC between words $x_i$ with at most one long term each or $x_{j-2}$ has at least two long terms. In the latter case, the first long term in $x_{j-2}$ again serves as a buffer preventing further cancellation with $x_k \cdots x_j$ when further terms $x_{j-3}, x_{j-4}, \ldots$ are multiplied.

Finally, it is possible that $x_{j-1} \cdot x_{j-2}$ admits CLTC. If $x_{j-1}$ has more than 2 terms, the first terms serve as a buffer, as above, preventing further cancellation with $x_k \cdots x_j$. Otherwise, we must be more careful.

The following lemmas are useful when considering CLTC between terse words.

**Lemma 5.6.** Suppose that $x_i \cdot x_{i-1}$ admits CLTC. Then in multiplying $(x_k \cdots x_i) \cdot (x_{i-1} \cdots x_1)$, at most one one pair of long terms (those admiting the CLTC) and one additional pair of letters cancels. (The letters from this additional pair may be in either $x_{i+1}$ or $x_{i-2}$.)

Lemma 5.6 essentially tells us that not much more than the completely canceled long terms cancels. It is proven by straightforward arguments similar to those used to prove Proposition 5.5.

**Lemma 5.7.** Suppose that each of the words $x_i, x_{i-1}, \ldots, x_r$ is terse, that each word $x_i$ ($1 \leq i \leq r$) sees CLTC with the following word, and $r > 1$. Then the vertices $s'_i, s'_{i-1}, \ldots, s'_{r}$ and $s'_1, s'_{r-1}, \ldots, s'_1$ all lie on a simple path $P$ of length at most $r - t + 2$, and equality between any of these letters only occurs when forced by CLTC. Moreover, the edges $[s'_1 s'_{l-1}], [s'_{l-1} s'_{l-2}], \ldots, [s'_{r-1} s'_{1}]$ lie in this order on $P$, with no intervening edges.

Lemma 5.7 is proven in much the same way as was Lemma 5.5. Analogously, it allows us to multiply successive terse words $x_i, x_{i-1}, \ldots$ as long as each such word admits CLTC with the next. As was the case with Lemma 5.5, the subdiagram of $\mathcal{V}'$ that emerges from Lemma 5.7 closely resembles the circuit $C'$ we wish to construct.

The essence of the remaining proof is as follows. Having finished with the initial chain $x_k \cdots x_j$ of words exhibiting ILTC, either we begin a new chain of ILTC or we begin a chain of CLTC, perhaps with a buffer in between (provided by non-terse words or words $x_i$ and $x_{i-1}$ which do not cancel). After completing the next chain of ILTC or CLTC, the same occurs, and we repeat the process.

Arguments similar to those used to prove Lemmas 5.5 and 5.6 show that in transitioning from one chain of cancellation to another without a buffer in between,
we maintain the “diametric” property described by those lemmas. That is, the
diameter of the portion of the subdiagram $V'(k, j)$ lying between $s'_k$ and $s''_j$ is small
enough to allow an inductive appeal to Theorem 2.2, which forbids any backtracking
produced by equality of vertices of $V'(k, j)$ other than that forced by ILTC or CLTC.
On the other hand, if a buffer does appear between two chains, this buffer prevents us
from canceling every letter of the preceding chain, so that ultimately the product
$x_k \cdots x_1$ cannot be trivial.

This discussion has focused upon the first case mentioned above (in which
$x_j \cdot x_{j-1}$ admits no cancellation). Clearly the third case can be handled in a
similar fashion.

There is one difficulty which must be overcome, and it concerns the single pair
of letters that could be canceled in addition to the CLTC in Lemma 5.6. If this
additional cancellation occurs at the end of the initial chain of ILTC, such cancel-
ation may “expose” the letter $s'_k$ at the end of $x_k \cdots x_{j-1}$, and this letter could
then be canceled when $x_1$ is at last multiplied with $x_k \cdots x_2$.

In most such cases, we may solve the problem by an appeal to the inductive
hypothesis of either Theorem 2.2 or Lemma 5.2. For example, consider the following
case, in which $k = 6$: $x_6 = s'_6 s''_5 = s'_6 s'_5$, $x_5 = s'_5 s''_5 (\alpha s''_5)^3$, $x_4 = (\beta s'_4)^3 s''_5$, $x_3 = x_2 = 1$, $x_1 = s''_5$.

Here $x_6 \cdots x_1 = 1$, given that $\alpha = s'_4$, $\beta = s''_5$, and $s'_6 s'_1 = s'_1 s''_6$. However, one can
draw the subdiagram of $V'$ determined by the above equalities and see that there
is a circuit of length 5 to which we may apply Theorem 2.2 inductively in order to
contradict the achordality of $C$. (Alternatively, one could apply Lemma 5.2 in this
case as well.)

Schematically, the example above begins with the short ILTC chain $x_6 \cdot x_5$ of
type $O \cdot OEs''_5$. There are a number of other cases where the initial ILTC chain
$x_k \cdots x_j$ yields a prefix ending with $s'_k$, and in almost all of these cases one can
apply at least one of the two arguments mentioned above. As when $x_k \cdots x_1$ was a
single ILTC chain, a problem arises when there are not enough adjacent triangles
to apply Lemma 5.2 inductively, and when there is no place to apply Theorem 2.2
to a shorter circuit. This situation is illustrated by the following example.

Let $x_6 = s'_6 s''_5$, $x_5 = s''_5$, $x_4 = (s'_4 s''_4)^3 s'_4$, $x_3 = (s'_3 s''_3)^3 s''_3$, $x_2 = 1$, and $x_1 = s''_6$,
where $s'_4 \neq s''_4$ and $s'_3 \neq s''_3$ are commuting pairs, $s''_3 = s'_4$, and $s''_4 = s'_3$. Then the
subdiagram of $V'$ corresponding to the product $x_6 \cdots x_1$ contains only a single pair
of triangles which meet at an edge (ruling out use of Lemma 5.2) and no circuit of
length less than 6 to which an application of Theorem 2.2 is helpful.

We assert that in any case to which we can apply neither Lemma 5.2 nor Theorem 2.2
a similar configuration arises in $V$ and in $V'$. Namely, there exists a vertex
$\gamma$ in $V$ such that for some $i$, the vertices $s_i$, $s_{i-1}$, $s_{i-2}$, and $\gamma$ appear as in Figure 3.a, where both $[s_i s_{i-1}]$ and $[s_{i-1} s_{i-2}]$ have labels greater than 2, and $[s_{i-1} \gamma]$ has an odd label. Finally, the two triangles shown in Figure 3.a correspond as in
Theorem 2.2 to the triangles in $V'$ shown in Figure 3.b. We can argue much as we
did when $x_k \cdots x_1$ was a single ILTC chain to prove that this configuration must
arise.

Now let $\tilde{w}_i$ conjugate each vertex in $\{s_i, s_{i-1}, \gamma\}$ to the appropriate vertex in
$\{s'_i, s''_{i-1}, s''_{i-2}\}$ and $\tilde{w}_{i-1}$ conjugate each vertex in $\{s_{i-1}, s_{i-2}, \gamma\}$ to the appropriate
vertex in $\{s''_{i-1}, s''_{i-2}, s''_{i-3}\}$. (These are the words whose existence is guaranteed by

Theorem 2.3 in the case $k = 3$.) Let $w_i$ and $w_{i-1}$, as always, denote the elements conjugating the edges $[s_is_{i-1}]$ and $[s_{i-1}s_{i-2}]$, so that $x_{i-1} = w_iw_{i-1}^{-1}$.

**Lemma 5.8.** Let the configuration pictured in Figure 3 appear. Then, using the notation from the preceding paragraph, $x_{i-1} \neq v_{s_i's_{i-1}''}$.

In the above example, $i = 1$, and Lemma 5.8 finishes the proof of Proposition 5.4 by contradicting the fact that $x_{k} = v_{s_6's_6''}$.

**Proof.** Assume that $s_i''$ and $s_{i-1}''$ both have odd order. Using Lemma 5.1, it is easy to prove that $w_iw_{i-1}^{-1} \in \{1, v_{s_i's_{i-1}''} \}$, $w_{i-1}w_{i-1}^{-1} \in \{1, v_{s_{i-1}'s_{i-2}''} \}$, and $\hat{w}_i\hat{w}_{i-1}^{-1} = v_{s_{i-1}'s_{i-2}''}$.

Therefore

$$x_{i-1} = w_iw_{i-1}^{-1} = w_i\hat{w}_i^{-1}w_{i-1}\hat{w}_{i-1}^{-1}w_{i-1}^{-1},$$

which from the above computations can be one of four different products, none of which is $v_{s_{i-1}'s_{i-1}''}$.

This completes the proof of Proposition 5.4 by taking care of the last remaining cases.

We are now close to a proof of Theorem 2.2. Recall we have assumed throughout $z_{i-1,i} = 1$ for all $i$. We note now that the arguments we have developed above prove that $z_{i-1,i} = 1$ must hold. Suppose that $x_i = x_i'z_{i-1,i}$, where $z_{i-1,i} \neq 1$. Then the product $x_i \cdot x_{i-1}$ can be treated much like a case of ILTC (as indeed $x_i \cdot x_{i-1}$ can never be trivial if $z_{i-1,i} \neq 1$), and we can modify all of the arguments above to take this possibility into account.

We may also use the arguments above to show that every word $x_i$ is terse; otherwise some $x_i$ would serve as a “buffer” which would prohibit the product $x_k \cdots x_2 \cdot x_1$ from being trivial.

The remaining words must all exhibit CLTC with one another. In order to avoid contradicting Lemma 5.7, there can be no equality (except that forced by CLTC) between any elements $s_i', s_{i-1}', \ldots, s_2', s_1'$, at the last step, in multiplying $x_k \cdots x_2 \cdot x_1$, we must complete the circuit $C'$ of length $k$, corresponding to $C$. Therefore Theorem 2.2 is proven in case there is no cancellation of short terms.
However, we can modify all of our arguments to take care of such cancellation as well. If \( x_i = x'_i \alpha_1 \cdots \alpha_l \) and \( x_{i-1} = \alpha_1 \cdots \alpha_{l'} x'_{l-1} \) where each \( \alpha_j \) is a short term, then we can cancel all of the letters \( \alpha_j \) and apply our ILTC and CLTC arguments to \( x'_i \) and \( x'_{i-1} \) instead. Intermediate short term cancellation that arises after long terms have been canceled is met in a similar fashion.

To prove Theorem 2.3, we observe that in performing complete cancellation of words \( x_i \) and \( x_{i-1} \), the precise form of the long terms in each of these words is forced, as in Section 4 when all cases in which \( k \in \{3, 4\} \) were considered. Also, Lemmas 4.5 and 4.6 were proven in complete generality, and as indicated in that section, Lemmas 4.5, 4.6, and 4.7 follow from arguments similar to the proof of Proposition 5.4 in case \( k \geq 5 \). Applying these results concludes our proof.

6. PIECING CIRCUITS TOGETHER

We have now established Theorem 2.2 and Theorem 2.3. Assume now that \( V \) is odd-edge-connected, and let \( C_1 \) and \( C_2 \) be two circuits in \( V \) which share at least one edge. Let \( C'_1 \) and \( C'_2 \) be the circuits in \( V' \) to which \( C_1 \) and \( C_2 \) correspond as in Theorem 2.2 respectively. Then \( C'_1 \) and \( C'_2 \) share edges corresponding to the common edges of \( C_1 \) and \( C_2 \). If there is more than one common edge, there is no “twisting” at any edge, so \( C_1 \cup C_2 \) and \( C'_1 \cup C'_2 \) are isomorphic as edge-labeled graphs.

Suppose there is a single common edge, \( \{st\} \), with an odd label (if the label of this edge is even, there can be no twisting at this edge). Because \( V \) is odd-edge-connected, the removal of this edge does not disconnect the diagram \( V \), and we can find a simple path \( P \) in \( V \) leading from a vertex in \( x_1 \in C_1 \setminus \{s, t\} \) to a vertex \( x_2 \in C_2 \setminus \{s, t\} \). Moreover, we can choose this path so that \( x_1 \) and \( x_2 \) are as close to \( s \) as possible, and so that \( P \) is of minimal length among paths satisfying this first condition. (Both of these conditions can be met by replacing subpaths of \( P \) with shorter paths as needed.) Denote by \( P_1 \) the path from \( x_1 \) to \( s \), and by \( P_2 \) the path from \( s \) to \( x_2 \). Then the path \( P_1 P_2 P^{-1} \) is a circuit, \( D \), and \( D \) is achordal, except perhaps for edges \( \{sy_i\} \), where \( y_i \in P \). Subdivide \( D \) by adding these edges, as necessary, into circuits \( D_1, \ldots, D_r \).

We have obtained a picture very similar to that considered in Lemma 5.2. An argument almost identical to the proof of that lemma now shows that twisting can occur neither at \( \{sy_i\} \) for any \( i \), nor at \( \{st\} \). Therefore, \( C_1 \cup C_2 \) and \( C'_1 \cup C'_2 \) are isomorphic as edge-labeled graphs.

If \( V \) is odd-edge-connected, it is easy to see that every vertex lies on some achordal circuit. Thus by piecing together achordal circuits which share at least one edge in the manner described above, we prove Theorem 1.2. If \( V \) is still one-connected but the removal of some odd edge \( e \) disconnects \( V \), we can induct on the number of “odd-edge-indecomposable” pieces into which \( V \) may be divided by removing such edges in order to prove Theorem 1.1.

Now suppose that \( V \) is connected but not one-connected. In this case, we can apply the same technique as used by M"uhlherr and Weidmann in \[19\] to prove their Main Theorem. (See Section 8 of \[19\]. Theorem 1.2 here serves as the base case for the inductive proof.) This technique draws heavily upon the results of \[13\] and \[18\]. The latter paper details a canonical decomposition for a given Coxeter group, arrived at through an application of Bass-Serre theory. As was done in \[19\], we
may prove Theorem 7.2 by inducting upon the number of “vertex-indecomposable” pieces into which the diagram \( V \) can be broken by removing separating vertices.

Appealing to \( 14 \) and \( 15 \) (as was done in \( 19 \)), we complete the proof in case \( V \) is not connected. This concludes the verification of Theorem 1.1.

There is an immediate corollary of Theorem 1.1 regarding the structure of Artin groups. Recall that the Artin group \( A(S) \) corresponding to a given Coxeter presentation \( W \cong (S \mid R) \) is found merely by deleting from \( R \) the relators \( s^2, s \in S \). (Therefore there is an epimorphism from \( A(S) \) to \( W \) which maps each element \( s^2 \) to the identity, for \( s \in S \).) Clearly the diagram \( V \) for \((W, S) \) completely determines the group \( A(S) \) as well as the group \( W \). We may define reflections, rigidity, and reflection rigidity, in Artin groups in exactly the same way we have defined them for Coxeter groups.

From Theorem 7.2 of \( 8 \), we derive the following result.

**Theorem 6.1.** Let \( A(S) \) be the Artin group corresponding to the two-dimensional Coxeter system \((W, S)\), with diagram \( V \). Then \( A(S) \) is reflection rigid, up to diagram twisting. (That is, given any other Coxeter system \((W', S')\) such that \( A(S) \) and \( A(S') \) yield the same reflections, the diagram for \( A(S') \) can be derived from \( V \) by a sequence of diagram twists.)

What else can be said? As we have seen, achordal circuits in \( V \) are nearly strongly rigid; conjugating words for the various vertices in such a circuit \( C \) differ only by products \( \alpha_1 \cdots \alpha_l \), for appropriately chosen \( \alpha_i \). If \( V \) has no edges labeled 2, every achordal circuit is strongly rigid, and arguing as in \( 14 \), we recover another fact proven in \( 19 \): if \( W \) is a skew-angled reflection independent Coxeter group and the diagram \( V \) for the system \((W, S)\) is edge-connected, then \( W \) is strongly rigid.

We can still say something when \( V \) contains edges labeled 2. Let \((W, S)\) be an arbitrary Coxeter system, with diagram \( V \), and let \( s \) be a vertex in \( V \). As in \( 2 \), \( 3 \), and \( 5 \) we define the 2-star, \( st_2(s) \), of \( s \) to be the set of vertices

\[ \{s\} \cup \{t \in V \mid [st] \text{ is an edge labeled 2} \} \subseteq V. \]

We have the following theorem. (Compare this with the main theorem of \( 9 \)).

**Theorem 6.2.** Suppose that \((W, S)\) is a reflection independent two-dimensional Coxeter system whose diagram \( V \) has at least 3 vertices. Then \( V \) is strongly rigid if \( V \) is edge-connected and there are no vertices \( s, t_1, t_2 \in V \) such that the removal of \( st_2(s) \) separates \( V \) into at least 2 components, \( t_1, t_2 \notin st_2(s) \), and \( t_1 \) and \( t_2 \) lie in different components of the full subdiagram of \( V \) induced by the vertices \( V \setminus st_2(s) \).

**Proof.** We have already seen that if \( V \) has three 3 vertices, \( W \) is strongly rigid. Therefore we may assume that \( V \) has at least 4 vertices.

Let \( V \) satisfy both of the conditions put forth in the statement of the theorem, and let \((W', S')\) be another Coxeter system for \( W \), with diagram \( V' \). Note that because the diagram is edge-connected (and therefore odd-edge-connected), Theorem 7.2 shows that it is reflection rigid, and therefore rigid, because \( W \) is assumed to be reflection independent. Therefore \( V \) and \( V' \) are isomorphic, and the achordal circuits in these diagrams match up as in Theorem 7.2.

We first claim that every edge (and therefore every vertex) of \( V \) must lie on an achordal circuit. In fact, it is easy to show that if \([st]\) did not lie on any circuit, then removing \([st]\) from \( V \) would separate the diagram, contradicting our hypotheses. Thus every edge lies on a circuit, which can be shortened, if needed,
to make it simple and achordal. We claim that to a given achordal circuit \( C \) there is an element \( w_C \in W \) which conjugates each vertex of \( C \) to the appropriate vertex of \( V' \).

From Section 4, this is so if the circuit has length 3. Thus, we may assume that

\[
C = \{[s_1s_2], ..., [s_ks_1]\}, \ k \geq 4.
\]

Let \( s_i \neq s_j \) be vertices on \( C \). Let \( w_i \) and \( w_j \) be elements which conjugate \( s_i \) to \( \hat{s}_i \) and \( s_j \) to \( \hat{s}_j \), respectively. We assume that \( i > j \). As in Theorem 2.3, \( w_i w_j^{-1} \) is a product of elements \( \alpha_i \in V' \) which either satisfy that theorem’s separation condition or lie in \( \{\hat{s}_i, \hat{s}_j\} \).

In case \( V = C \), the result follows from [1], so we may suppose that \( V \neq C \). Consider a letter \( \alpha \) appearing in the ratio \( w_i^{-1} w_j \) which does not lie on \( C' \). The removal of \( st_2(\alpha) \) separates \( C' \) into various subarcs, but because the second condition on \( V \) in the statement of the theorem does not obtain, removing \( st_2(\alpha) \) does not disconnect \( V' \). By two-dimensionality, \( \alpha \) is not adjacent to at least two letters in \( C' \) by edges labeled 2, so \( C' \setminus st_2(\alpha) \) is not empty. Therefore given any two subarcs \( P_1 \) and \( P_2 \) into which \( C' \) is divided by removing \( st_2(\alpha) \), there is a path \( P \) lying completely in \( V' \setminus st_2(\alpha) \) which connects \( P_1 \) and \( P_2 \). Let \( P_1 \) and \( P_2 \) be “adjacent” subarcs of \( C' \), lying on either side of the vertex \( \hat{s}_i \) (where \( [\hat{s}_i, \alpha] \) is an edge labeled 2). By replacing portions of \( P \) with the appropriate paths, we may assume that the endpoints of \( P \) are as close as possible to \( \hat{s}_i \) and that \( P \) is as short as possible among all paths lying in \( V' \setminus st_2(\alpha) \) connecting these endpoints. Fix such a path \( P \), with endpoints \( x_1 \in P_1 \) and \( x_2 \in P_2 \). Denote by \( Q_l \) the subpath of \( P_l \) from \( x_l \) to \( \hat{s}_i \), for \( l = 1, 2 \). Then by our choice of \( P \), the circuit \( D = PQ_2Q_1^{-1} \) is achordal, except perhaps for edges \([\hat{s}_i y] \), for some \( y \in P \). Add such edges as needed to subdivide \( D \) into achordal circuits \( D_1, ..., D_r \). An argument we have now seen twice before implies that since \( \hat{s}_i \) is the only vertex in any of these circuits which is adjacent to \( \alpha \) by an edge labeled 2, \( \alpha \) cannot appear in any of the ratios of conjugating elements for the edges in each circuit \( D_l \). Therefore if \( \hat{s}_i \) lies in \( P_1 \) and \( \hat{s}_j \) lies in \( P_2 \), the ratio of the conjugating elements associated to these vertices cannot contain \( \alpha \).

Repeating this procedure for every pair of adjacent subarcs \( P_1 \) and \( P_2 \), and then for every element \( \alpha \) whose 2-star separates \( C' \), we see that no such \( \alpha \) can occur. Therefore the only letters \( \alpha \) that appear in \( w_i w_j^{-1} \) are \( \hat{s}_i \) and \( \hat{s}_j \). Thus we have effectively reduced the problem to the case in which \( V = C \).

Now consider two achordal circuits \( C_1 \) and \( C_2 \) which share at least one edge. Using arguments almost entirely like those just applied, one can show that \( C_1 \) and \( C_2 \) share a common conjugating element. Therefore, since every vertex in \( V \) lies on some achordal circuit, we have proven the theorem.

\[\square\]

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