Disproof of a Conjecture by Rademacher on Partial Fractions

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Hans Rademacher (1892-1969)
Integer partitions

- $p(n) =$ number of ways to write $n$ as sum of positive integers (disregarding their order)
- Central object in additive number theory
- Euler: Recursion for $p(n)$
- Hardy, Ramanujan (1918):

$$p(n) \sim \frac{1}{4\sqrt{3n}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad n \to \infty$$

- Rademacher (1937): Convergent series representation
Unrestricted partition generating function:

\[ \sum_{n=1}^{\infty} p(n)x^n = \sum_{k_1 \geq 0} x^{k_1} \sum_{k_2 \geq 0} x^{2k_2} \cdots = \prod_{j=1}^{\infty} \frac{1}{1 - x^j} \]

Partitions into parts ≤ N:

\[ \sum_{n=1}^{\infty} p_N(n)x^n = \sum_{k_1 \geq 0} x^{k_1} \sum_{k_2 \geq 0} x^{2k_2} \cdots \sum_{k_N \geq 0} x^{Nk_N} = \prod_{j=1}^{N} \frac{1}{1 - x^j} \]

(Same as partitions into at most N parts)
Partial fraction decomposition

- Rademacher (*Topics in Analytic Number Theory*, 1973):

\[
\prod_{j=1}^{\infty} \frac{1}{1 - x^j} = \sum_{0 \leq h < k} \sum_{\ell=1}^{\infty} \frac{C_{hk\ell}(\infty)}{(x - e^{2\pi i h/k})^{\ell}}
\]

He gave an explicit formula for \( C_{hk\ell}(\infty) \).

- Restricted case:

\[
\prod_{j=1}^{N} \frac{1}{1 - x^j} = \sum_{0 \leq h < k \leq N} \sum_{\ell=1}^{\lfloor N/k \rfloor} \frac{C_{hk\ell}(N)}{(x - e^{2\pi i h/k})^{\ell}}
\]

- **Conjecture** (Rademacher):

\[
\lim_{N \to \infty} C_{hk\ell}(N) \overset{?}{=} C_{hk\ell}(\infty), \quad \text{for fixed } h, k, \ell.
\]
Let us write the unique algebraic partial fraction decomposition of (130.4) as

\[
\frac{1}{\prod_{m=1}^{N} (1 - x^m)} = \sum' \sum_{0 \leq h < k \leq N}^{[N/k]} \frac{C_{hkl}(N)}{(x - e^{2\pi ih}l)}.
\]

(130.5)

The \(C_{hkl}(N)\) can be obtained algebraically as expressions containing roots of unity, although the actual computation becomes soon very cumbersome with increasing \(N\). No explicit formula for \(C_{hkl}(N)\) is known, not even for the simplest case \(h = 0, k = 1, l = 1\), and variable \(N\).

I conjecture now that the partial fraction decomposition (130.5) converges termwise to the expansion (130.1). More explicitly, I propose the
Chapter 14. Analytic theory of partitions

Conjecture.

\[ \lim_{N \to \infty} C_{hkl}(N) \]

exists and is equal to

\[ C_{hkl}(\infty) = -2\pi \left( \frac{\pi}{12} \right)^{3/2} \cdot \frac{2\pi h l}{k} \cdot \frac{\omega_{hke}}{k^{5/2}} \cdot \Delta_{\alpha}^{l-1} L_{3/2} \left( -\frac{\pi^2}{6k^2} (\alpha + 1) \right) , \]

\[ \alpha = \frac{1}{24}. \]
History

- Rademacher made some computations for $N = 1, \ldots, 5$; seemed ok.
- George Andrews: Rademacher discussed the conjecture in a course 1961/62.
- Ehrenpreis, Friedmann (1993): “coefficients difficult to compute; inconclusive”.
- Davidson, Gagola (2002): $C_{011}(N)$ for $N \leq 45$. Oscillations.
- George Andrews (2003): “Rademacher’s conjecture lies at the interface of the theory of modular forms and the theory of q-series. Thus progress on this problem may require contributions from two areas that have had less contact in the past than might have been expected or hoped for.”
History

- Munagi (2008): Writes in favor of the conjecture.
- Sills, Zeilberger (2013): Recurrence for $C_{01\ell}(N)$. Computations up to $N = 1000$. No convergence, but oscillations and exponential growth, apparently.
- Drmota, SG (2013): This is true, and Rademacher’s conjecture is incorrect. Details to follow.
- O’Sullivan (2013): Disproof by another approach.
$C_{011}(N)$ for $N = 1, \ldots, 100$ and Rademacher's conjectured limit
Disproof of the conjecture

**Theorem** (Drmota, SG 2013):

For any integer \( \ell \geq 1 \), we have the asymptotics

\[
C_{0,1,\ell}(N) = b^N N^{-\ell-1} H_\ell(N) + O(b^N N^{-\ell-117/112}), \quad N \to \infty,
\]

where \( b \approx 1.07 \), and \( H_\ell \) is a bounded periodic function with period \( p \approx 31.96 \).

- Exponential growth + oscillations
- No convergence
- \( b \) and \( p \) defined by transcendental equation (involving dilogarithm)
- \( b \) and \( p \) independent of \( \ell \)
Proof idea

- Contour integral representation of $C_{0,1,\ell}(N)$
- Split integration contour
- Left part dominates
- **Main Step 1 (left part):** Approximate the integrand (Mellin transform asymptotics)
- **Main Step 2 (left part):** Saddle point method
- **Main Step 3 (right part):** Direct estimates.
Integral representation by Cauchy’s formula

\[ \prod_{j=1}^{N} \frac{1}{1 - x^j} = \cdots + \frac{C_{0,1,\ell}(N)}{(x-1)^\ell} + \cdots \]

For small \( r > 0 \):

\[ C_{0,1,\ell}(N) = \frac{1}{2i\pi} \int_{|x-1|=r} (x-1)^{\ell-1} \prod_{j=1}^{N} \frac{1}{1 - x^j} \, dx \]

\[ = \frac{1}{2i\pi} \int_{|x|=r} x^{\ell-1} \prod_{j=1}^{N} \frac{1}{1 - (x+1)^j} \, dx \]

From now on \( \ell = 1 \), for better readability.
Integral representation by Cauchy’s formula

Substitute $x + 1 = e^{z/N}$

$$C_{0,1,\ell}(N) = \frac{1}{2i\pi} \int_{|x|=r} \prod_{j=1}^{N} \frac{1}{1 - (x + 1)^j} dx$$

$$= \frac{1}{2i\pi} \frac{1}{N} \int_{|z|=r} e^{z/N} \prod_{j=1}^{N} \frac{1}{1 - e^{zj/N}} dz$$

Goal: Asymptotics for $N \to \infty$
Easy observation: Reflection formula

\[ \prod_{j=1}^{N} \frac{1}{1 - e^{zj/N}} = (-1)^N \prod_{j=1}^{N} \frac{e^{-zj/N}}{1 - e^{-zj/N}} \]

\[ = (-1)^N e^{-z(N+1)/2} \prod_{j=1}^{N} \frac{1}{1 - e^{-zj/N}} \]

- \( \Re z > 0 \implies \) exponential decay of \( e^{-z(N+1)/2} \)
- Left half-circle (\( \Re z < 0 \)) dominates
Plan

- Define
  \[ g(z, N) := \log \prod_{j=1}^{N} \frac{1}{1 - e^{zj/N}} \]

- Recall:
  \[ C_{0,1,\ell}(N) = \frac{1}{2i\pi} \frac{1}{N} \int_{|z|=r} e^{z/N + g(z, N)} dz \]

- Split the integration contour

- \( \Re z < -N^{-7/8} \): Approximate \( g \), saddle point method (Steps 1, 2)

- \( \Re z \geq -N^{-7/8} \): Estimate \( g \) directly \( \Rightarrow \) negligible (Step 3)

- We need two estimates for \( g \) with sufficient accuracy and sufficient validity region.
Approximate the integrand

- We want to approximate

\[ g(z, N) = \log \prod_{j=1}^{N} \frac{1}{1 - e^{zj/N}} \]

- For Mellin transform, \( N \) has to be real

- Taylor series:

\[ g(z, N) = -\sum_{j=1}^{N} \log(1 - e^{zj/N}) \]

\[ = \sum_{j=1}^{N} \sum_{k=1}^{\infty} \frac{1}{k} e^{zjk/N} = \sum_{k=1}^{\infty} \frac{1}{k} \frac{1 - e^{kz}}{e^{-kz/N} - 1} \]
Mellin transform, for fixed $z$ with $\Re(z) < 0$, and $\Re(s) < -1$

\[
\mathcal{M}g(z, \cdot)(s) = \int_0^\infty g(z, x)x^{s-1}dx
\]

\[
= \sum_{k=1}^\infty \frac{1 - e^{kz}}{k} \int_0^\infty \frac{x^{s-1}}{e^{-kz/x} - 1}dx
\]

\[
= \sum_{k=1}^\infty \frac{1 - e^{kz}}{k} (-kz)^s \Gamma(-s) \zeta(-s)
\]

\[
= (-z)^s \Gamma(-s) \zeta(-s) \left( \sum_{k=1}^\infty k^{s-1} - \sum_{k=1}^\infty k^{s-1} e^{kz} \right)
\]

\[
= (-z)^s \Gamma(-s) \zeta(-s) \left( \zeta(1 - s) - \text{Li}_{1-s}(e^z) \right)
\]

- Polylogarithm $\text{Li}_\nu(w) = \sum_{k \geq 1} w^k / k^\nu$, for $|w| < 1$ and $\nu \in \mathbb{C}$
Mellin inversion: Poles map to asymptotic elements

- Mellin inversion formula:

\[ g(z, N) = \frac{1}{2i\pi} \int_{-3/2-i\infty}^{-3/2+i\infty} \mathcal{M}g(z, \cdot)(s) N^{-s} ds \]

- Recall:

\[ \mathcal{M}g(z, \cdot)(s) = (-z)^s \Gamma(-s) \zeta(-s) \left( \zeta(1-s) - \text{Li}_{1-s}(e^z) \right) \]

- Shift integration path to the right ($\Re s = 8/7$)
- Collect residues
- $\Gamma(-s)$ has simple poles at $s = 0, 1, 2, \ldots$
- $\zeta(-s)$ has a simple pole at $s = -1$
- $\zeta(1-s)$ has a simple pole at $s = 0 \implies$ double pole
- $\text{Li}_{1-s}(e^z)$ is an entire function of $s$
Mellin inversion: Poles map to asymptotic elements

- Calculate residues:

\[
\text{res}_{s=-1} \mathcal{M} g(z, \cdot)(s) N^{-s} = \frac{N}{z} (\zeta(2) - \text{Li}_2(e^z))
\]

\[
\text{res}_{s=0} \mathcal{M} g(z, \cdot)(s) N^{-s} = \frac{1}{2} \log N + \frac{1}{2} \left( \log 2\pi + \log(1 - e^z) - \log(-z) \right)
\]

- Integrand expansion for fixed \( z \) with \( \Re z < 0 \), as \( N \to \infty \):

\[
g(z, N) = \frac{1}{z} \left( \text{Li}_2(e^z) - \frac{\pi^2}{6} \right) N - \frac{1}{2} \log N
\]

\[
- \frac{1}{2} \left( \log 2\pi + \log(1 - e^z) - \log(-z) \right) + O(1/N)
\]
Mellin inversion: Refined estimate

- We need uniformity w.r.t. $z$
- Refined estimate:

$$g(z, N) = \frac{1}{z} \left( \text{Li}_2(e^z) - \frac{\pi^2}{6} \right) N - \frac{1}{2} \log N$$

$$- \frac{1}{2} \left( \log 2\pi + \log(1 - e^z) - \log(-z) \right) + h(N)$$

The function $h$ is

(i) uniformly $O(N^{-1/2})$ if $|\arg z| \geq \pi/2 + \varepsilon$, $z$ is bounded away from 0, and $z = O(N^{1/2})$,

(ii) uniformly $O(N^{33/112})$ if $z$ is bounded, bounded away from 0 and $\pm 2i\pi$, $|\Im z| < 8$, and $\Re z < -N^{-7/8}$. 
Mellin inversion: Refined estimate

Proof is based on the estimates ($\Re s = 8/7, \Im s \to +\infty$)

$$|N^{-s}| = N^{-\Re s},$$

$$|(-z)^s| = |z|^{\Re s} e^{-\Im(s) \arg(-z)},$$

$$|\Gamma(-s)| \sim \sqrt{2\pi} e^{-\frac{1}{2} \pi \Im(s) (\Im s)}^{\Re s-1/2},$$

$$\zeta(-s) = O((\Im s)^{\Re s+1/2}),$$

$$\zeta(1 - s) = O((\Im s)^{\Re s-1/2}),$$

$$\text{Li}_{1-s}(e^z) = O((\Im s)^{\Re s-1/2})$$
Step 1 done (approximate integrand in left half-plane)

\[ C_{0,1,1}(N) \approx \frac{1}{(2\pi N)^{3/2} i} \int_{\Re z \leq 0} \sqrt{-\frac{z}{1 - e^z}} \exp \left( \frac{z}{N} + \frac{N}{z} \left( \text{Li}_2(e^z) - \frac{\pi^2}{6} \right) \right) dz \]
Recall the proof idea

- Contour integral representation of $C_{0,1,\ell}(N)$
- Split integration contour
- Left part dominates
- ✓ **Main Step 1 (left part):** Approximate the integrand (Mellin transform asymptotics)
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- **Main Step 3 (right part):** Direct estimates.
Step 2: Saddle point asymptotics

- Method goes back to Riemann (1863) and Debye (1909)
- Move integration contour through saddle points
- Then: Laplace method
- Here: Two conjugate saddle points
- Location of saddle points yields exponential growth
- Local behavior of integrand yields subexponential factors
Saddle point asymptotics

- Dominating factor of integrand:
  \[ \exp \left( \frac{N}{z} \left( \text{Li}_2(e^z) - \frac{\pi^2}{6} \right) \right) \]

- Saddle point equation:
  \[ \log(1 - e^z) + \frac{1}{z}(\text{Li}_2(e^z) - \frac{\pi^2}{6}) = 0 \]

- Saddle points: \( z_0 \approx -1.61 + 7.42i \), and \( \bar{z}_0 \)

- Independent of \( N \)
Axis of the saddle point

- **Argument of the axis:**

\[
a = \frac{\pi}{2} - \frac{1}{2} \arg \left. \frac{d^2}{dz^2} \left( \frac{1}{z} \left( \text{Li}_2(e^z) - \frac{\pi^2}{6} \right) \right) \right|_{z=z_0}
\]

\[
= \frac{\pi}{2} - \frac{1}{2} \arg \left( \frac{e^{z_0}}{z_0(1 - e^{z_0})} \right)
\]

\[
\approx 1.79
\]

- **Direction of steepest descent:**

\[
\rho = \exp(i a)
\]
New integration contour

\[ z_1 = 5i \]

\[ z_5(N) = -\sqrt{N} \]

Width of central part is \( O(N^{-39/112}) \)

\[ z_2(N) = z_0 - \rho N^{-39/112} \]

\[ z_3(N) = z_0 + \rho N^{-39/112} \]
Local expansion near the saddle point

- Saddle point segment:
  \[ z = z_0 + t\rho, \quad -N^{-39/112} \leq t \leq N^{-39/112} \]

- Expansion of integrand:
  \[ g(z, N) = -N \log(1 - e^{z_0}) - \frac{1}{2} \alpha N t^2 - \frac{1}{2} \log N + \text{const} + O(N^{-5/112}), \]

where

\[ \alpha := -\frac{\rho^2 e^{z_0}}{z_0 (1 - e^{z_0})} \approx 0.028 \]
Gaussian integral

- From the second order term of the expansion:

\[
\int_{-N^{-39/112}}^{N^{-39/112}} \exp\left(-\frac{1}{2} \alpha N t^2\right) dt \sim \frac{1}{\sqrt{\alpha N}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{\frac{2\pi}{\alpha N}},
\]

- Saddle point integral:

\[
\int_{z_2(N)}^{z_3(N)} e^{f(z,N)} dz = \frac{\rho(-z_0)^{1/2}}{\sqrt{\alpha(1 - e^{z_0})}} \frac{1}{N} (1 - e^{z_0})^{-N}(1 + O(N^{-5/112}))
\]
There is also a lower saddle point

- Lower saddle point integral:

\[
\int_{\bar{z}_3(N)}^{\bar{z}_2(N)} e^{g(z,N)} \, dz = -\int_{z_2(N)}^{z_3(N)} e^{g(z,N)} \, dz
\]

- Recall:

\[
C_{0,1,1}(N) = \frac{1}{2i\pi} \frac{1}{N} \int_{|z|=r} e^{z/N} \prod_{j=1}^{N} \frac{1}{1 - e^{z_j/N}} \, dz
\]

- Contribution of both saddle points:

\[
\frac{1}{\pi N} \Im \left( \int_{z_2(N)}^{z_3(N)} e^{g(z,N)} \, dz \right) \sim \frac{1}{\sqrt{\alpha \pi} N^2} \Im \left( \frac{\rho (-z_0)^{1/2}}{\sqrt{1 - e^{z_0}}} \right) (1 - e^{z_0})^{-N}
\]
Tail estimates

- Integrand $F(z, N) := e^{z/N} \prod_{j=1}^{N} \frac{1}{1-e^{z_j/N}}$
- Estimates:
  \[
  \int_{z_1}^{z_2} 1_{\{\Re z \leq -N^{-7/8}\}} F(z, N) dz = O\left( b^N \exp \left( -\frac{1}{3} \alpha N^{17/56} \right) \right)
  \]
  \[
  \int_{z_3}^{z_4} F(z, N) dz = O\left( b^N \exp \left( -\frac{1}{3} \alpha N^{17/56} \right) \right)
  \]
  \[
  \int_{z_4}^{z_5} F(z, N) dz = \exp(O(N^{1/2}))
  \]

- All are $\ll b^N N^{-2}$
- Note: $\Re z \leq -N^{-7/8}$ is validity region of Mellin estimate
Recall the proof idea

- Contour integral representation of $C_{0,1,\ell}(N)$
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Step 3: Estimate close to imaginary axis

Lemma:

$$\int_{z_1}^{z_2} 1_{\{\Re z \geq -N^{-7/8}\}} F(z, N) \, dz = O(0.85^N).$$

Proof:

- Directly from $F(z, N) = e^{z/N} \prod_{j=1}^{N} \frac{1}{1 - e^{z_j/N}}$
- Euler’s summation formula
- Some (tedious) estimates, one of them proved by CAD (Cylindrical Algebraic Decomposition)
Step 3: Estimate in the right half-plane

Lemma:

\[
\frac{1}{N} \frac{1}{2i\pi} \int_{|z|=5} 1_{\{\Re z > 0\}} F(z, N) dz = O(0.95^N).
\]

Proof:
Use reflection formula, recycle parts of proof from left half-plane.
Conclusion

- Rademacher’s conjecture: “partial fraction decompositon” and “$\lim_{N \to \infty}$” commute
- Disproved by us
- (Interesting question: Why didn’t Rademacher do it?!)
- Is there some relation of the p.f. coefficients of $\prod_{j \geq 1}(1 - x^j)^{-1}$ and $\prod_{j=1}^{N}(1 - x^j)^{-1}$?
- Numerical observation by O’Sullivan: Maybe convergence of Cesàro means

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} C_{h,k,\ell}(n) \overset{?}{=} C_{h,k,\ell}(\infty)$$
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