Representations of Centrally-Extended Lie Algebras over Differential Operators and Vertex Algebras¹

Xiaoping Xu²³

Institute of mathematics, Academy of Mathematics & System Sciences
Chinese Academy of Sciences, Beijing 100080, P. R. China

Abstract

We construct irreducible modules of centrally-extended classical Lie algebras over left ideals of the algebra of differential operators on the circle, through certain irreducible modules of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries. The structures of vertex algebras associated with the vacuum representations of these algebras are determined. Moreover, we prove that under certain conditions, the highest weight irreducible modules of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries naturally give rise to the irreducible modules of the simple quotients of these vertex algebras. Our results are natural generalizations of the well-known WZW models in conformal field theory associated with affine Kac-Moody algebras. They can also be viewed as quadratic generalizations of free field theory. In the very special case of $W_{1,\infty}$, our results with integral central charges are more direct and explicit than those of Kac and Radul.

¹1991 Mathematical Subject Classification. Primary 17B10, 17B69; Secondary 81Q40
²Research Supported by China NSF 10371121
³ACKNOWLEDGEMENT: Part of this work was done during the author’s visit to The University of Sydney, under the financial support from Prof. Ruibin Zhang’s ARC research grant. The author thanks Prof. Zhang for his invitation and hospitality.
1 Introduction

The well-known $W_{1+\infty}$ Lie algebra is the centrally-extended Lie algebra of differential operators on the circle. It serves as the symmetry algebra of the famous KP-hierarchy in integrable systems. In fact, it is the Lie algebra of both rank-one charged quadratic free bosonic fields and rank-one charged quadratic fermionic fields. Kac and Radul [KR1] gave a classification of quasifinite highest weight irreducible modules of the $W_{1+\infty}$ algebra. Frenkel, Kac, Radul and Wang [FKRW] proved that the categories of irreducible modules of the vertex operator algebras associated to $W_{1+\infty}$ and $W(gl_N)$ with integral central charge $N$ are equivalent. Moreover, Kac and Radul [KR2] used free quadratic bosonic fields to study the irreducible representations of the vertex operator algebras associated to $W_{1+\infty}$ with negative integral central charge. The results in [KR1] were extended to certain Lie subalgebras of $W_{1+\infty}$ by Kac, Wang and Yan [KWY].

In general, the Lie algebras of charged free quadratic bosonic fields and fermionic fields of any rank are centrally-extended general linear Lie algebras over differential operators on the circle. They are the symmetry algebras of the multi-component KP-hierarchies in integrable systems (cf. [V]). Boyallian and Liberati [BL1] studied certain classical Lie subalgebras of the Lie algebra of matrix differential operators on the circle. Ma [M1, M2] systematically investigated the conformal algebra structures and two-cocycles of centrally-extended classical Lie superalgebras over left ideals of the algebra of differential operators on the circle.

In this paper, we give constructions of irreducible modules of centrally-extended classical Lie algebras over left ideals of the algebra of differential operators on the circle, through certain irreducible modules of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries. The structures of vertex algebras associated with the vacuum representations of these algebras are determined. Moreover, we prove that under certain conditions, the highest weight irreducible modules of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries naturally give rise to the irreducible modules of the simple quotients of these vertex operator algebras. It turns out that our results resemble those for the vertex operator algebras associated with affine Kac-Moody algebras, which are called WZW models in conformal field theory. With special examples of the Hecke algebras and group algebras borne in mind, we present general constructions of representations of certain Lie subalgebras of the centrally-extended Lie algebra of the tensor algebra of any associative algebra with the algebra of differential operators on the circle. Our representation theory can be viewed as quadratic generalizations of free field theory. Below we give more detailed and technical introduction.

Throughout this paper, all the variables are formal and commute with each other. All the
vector spaces are assumed over \( \mathbb{C} \), the field of complex numbers. Denote by \( \mathbb{Z} \) the ring of integers and by \( \mathbb{N} \) the additive semigroup of nonnegative integers.

Denote \( \partial_t = d/dt \). Let

\[
\mathbb{A} = \sum_{i=0}^{\infty} \mathbb{C}[t, t^{-1}] \partial_t^i
\]

be the algebra of differential operators on the circle. Let \( M_{n \times n}(\mathbb{A}) \) be the algebra of \( n \times n \) matrices with entries in \( \mathbb{A} \). Denote by \( E_{i,j} \) the \( n \times n \) matrix with 1 as its \((i,j)\)-entry and 0 as the others. Define the vector space \( \hat{\mathfrak{gl}}(n, \mathbb{A}) \) of \( n \times n \) extended classical Lie algebras of infinite matrices with finite number of nonzero entries.

\[
\mathbb{M} = \text{the algebra of differential operators on the circle. Let } \hat{\mathfrak{g}} = \mathbb{M} \oplus \mathbb{C} \kappa \text{ and its Lie bracket:}
\]

\[
[t^{m_1} \partial_t^{r_1} E_{i_1,j_1} + \mu_1 \kappa, t^{m_2} \partial_t^{r_2} E_{i_2,j_2} + \mu_2 \kappa] = \delta_{i_1,i_2} t^{m_1} \partial_t^{r_1} E_{i_1,j_1} - \delta_{i_1,i_2} t^{m_1} \partial_t^{r_1} E_{i_1,j_1} + (-1)^{r_1} \delta_{i_1,i_2} \delta_{j_1,j_2} \delta_{r_1+r_2,m_1+m_2} r_1! r_2! \left( \frac{m_1}{r_1 + r_2 + 1} \right) \kappa
\]

for \( i, j \in \overline{1,n} \) and \( m_1, m_2, r_1, r_2 \in \mathbb{N} \). Fixed an element \( \bar{\ell} = (\ell_1, ..., \ell_n) \in \mathbb{N}^n \). The subspace

\[
\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}) = \sum_{i,j=1}^{n} \mathbb{A} \delta_\ell^{ij} E_{i,j} + \mathbb{C} \kappa
\]

of \( \hat{\mathfrak{gl}}(n, \mathbb{A}) \) forms a Lie subalgebra. The well-known Lie algebra \( W_{1+\infty} \) is the special case of \( \hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}) \) when \( n = 1 \) and \( \bar{\ell} = 0 \).

Denote \( \overline{1,n} = \{1, 2, ..., n\} \), and set \( \ell^* = n + 1 - i \) for \( i \in \overline{1,n} \). We fix \( \epsilon \in \{0, 1\} \) and take

\[
\bar{\ell} = (\ell_1, \ell_2, ..., \ell_n) \in \mathbb{N}^n \text{ such that } \{\ell_1, \ell_2, ..., \ell_n\} \subset 2\mathbb{N} + \epsilon
\]

and \( \ell_i = \ell_{i^*} \) for \( i \in \overline{1,n} \). The subspace

\[
\hat{\mathfrak{o}}(\bar{\ell}, \mathbb{A}) = \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \sum_{m \in \mathbb{Z}} \mathbb{C}(t^m \partial_t^{r+\ell_j} E_{i,j} - (-1)^r (-\partial_t)^r t^m \partial_t^{\ell_j} E_{i,j^*} + \mathbb{C} \kappa
\]

of \( \hat{\mathfrak{gl}}(n, \mathbb{A}) \) forms a Lie subalgebra. Next we suppose that \( n = 2n_0 \) is an even positive integer. Moreover, we define the parity of indices:

\[
p(i) = 0, \quad p(n_0 + i) = 1 \quad \text{for } i \in \overline{1,n_0}.
\]

The subspace

\[
\hat{\mathfrak{sp}}(\bar{\ell}, \mathbb{A}) = \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \sum_{m \in \mathbb{Z}} \mathbb{C}(t^m \partial_t^{r+\ell_j} E_{i,j} - (-1)^{p(i) + p(j)} (-\partial_t)^r t^m \partial_t^{\ell_j} E_{i,j^*} + \mathbb{C} \kappa
\]

of \( \hat{\mathfrak{gl}}(n, \mathbb{A}) \) forms a Lie subalgebra.

Note that the Lie algebras \( \hat{\mathfrak{o}}(\bar{\ell}, \mathbb{A}) \) and \( \hat{\mathfrak{sp}}(\bar{\ell}, \mathbb{A}) \) are in general not graded by conformal weights. One of the main objectives in this paper is to construct irreducible modules of \( \hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}) \), \( \hat{\mathfrak{o}}(\bar{\ell}, \mathbb{A}) \) and \( \hat{\mathfrak{sp}}(\bar{\ell}, \mathbb{A}) \) through weighted irreducible modules (may not be highest weight type) of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries.
For \( i, j \in \mathbb{T}, n \) and \( r \in \mathbb{N} \), we denote

\[
E_{i,j}(r, z) = \sum_{t \in \mathbb{Z}} t^l \partial^l_z E_{i,j} \zeta^{-l-1}.
\]

(1.8)

The \textit{vacuum module} \( \mathcal{V}_\chi \) of \( \widehat{\mathfrak{g}}l(n, \mathbb{A}) \) is a module generated by a vector \(|0\rangle\), called \textit{vacuum}, such that \( \kappa|\mathcal{V}_\chi = \chi \text{Id}_{\mathcal{V}_\chi} \),

\[
E_{i,j}(r, z)|0\rangle = \sum_{m=0}^{\infty} t^{-m-1} \partial^m z \partial^l E_{i,j}|0\rangle z^m
\]

(1.9)

for \( i, j \in \mathbb{T}, n \); \( r, \in \mathbb{N} \), and any other \( \widehat{\mathfrak{g}}l(n, \mathbb{A}) \)-module with the same property must be a quotient module of \( \mathcal{V}_\chi \). Denote by \( U(\cdot) \) the universal enveloping algebra of a Lie algebra. Suppose that \( \mathcal{G} \) is one of the Lie algebras \( \widehat{\mathfrak{g}}l(\ell, \mathbb{A}), \mathfrak{o}(\ell, \mathbb{A}) \) or \( \mathfrak{sp}(\ell, \mathbb{A}) \). The \( \mathcal{G} \)-module

\[
\mathcal{V}_\chi(\mathcal{G}) = U(\mathcal{G})|0\rangle
\]

(1.10)

is called the \textit{vacuum module} of \( \mathcal{G} \) and the corresponding representation is called the \textit{vacuum representation} of \( \mathcal{G} \). Our first main result in this paper is as follows:

\textbf{Theorem 1.1.} \textit{The module} \( \mathcal{V}_\chi(\mathcal{G}) \) \textit{is irreducible if} \( \chi \notin \mathbb{Z} \). \textit{When} \( \chi \in \mathbb{Z} \), \textit{the module} \( \mathcal{V}(\mathcal{G}) \) \textit{has a unique maximal proper submodule} \( \mathcal{V}_\chi(\mathcal{G}) \), \textit{and the quotient} \( \mathcal{V}_\chi(\mathcal{G}) = \mathcal{V}(\mathcal{G})/\mathcal{V}_\chi(\mathcal{G}) \) \textit{is an irreducible} \( \mathcal{G} \)-\textit{module}. \textit{When} \( n > 1 \) \textit{and} \( \chi \in \mathbb{N} \), \textit{the submodule}

\[
\mathcal{V}_\chi(\mathcal{G}) = U(\mathcal{G})(t^{-1} \partial^n E_{n,1})^{\chi+1}|0\rangle
\]

(1.11)

for \( \mathcal{G} = \widehat{\mathfrak{g}}l(\ell, \mathbb{A}), \mathcal{G} = \mathfrak{o}(\ell, \mathbb{A}) \) with \( \epsilon = 1 \) and \( \mathcal{G} = \mathfrak{sp}(\ell, \mathbb{A}) \) with \( \epsilon = 0 \).

If \( n > 3 \), the submodule

\[
\mathcal{V}_\chi(\mathcal{G}) = U(\mathcal{G})(t^{-1}(\partial^n E_{n-1,1} - \partial^n E_{n,2}))^{\chi+1}|0\rangle
\]

(1.12)

for \( \mathcal{G} = \mathfrak{o}(\ell, \mathbb{A}) \) with \( \epsilon = 0 \) and \( \mathcal{G} = \mathfrak{sp}(\ell, \mathbb{A}) \) with \( \epsilon = 1 \).

Suppose that \( \chi \) is a positive integer. Note \( (t^{-1} E_{n,1})^{\chi+1}|0\rangle \) is a singular vector generating the maximal proper submodules of the vacuum modules at level \( \chi \) of the affine Kac-Moody algebras \( \widehat{\mathfrak{s}l}(n, \mathbb{C}) \) and \( \mathfrak{sp}(n, \mathbb{C}) \), respectively. Moreover, \( (t^{-1}(E_{n-1,1} - E_{n,2}))^{\chi+1}|0\rangle \) is a singular vector generating the maximal proper submodules of the vacuum module at level \( \chi \) of the affine Kac-Moody algebra \( \mathfrak{o}(n, \mathbb{C}) \) when \( n \) is even. Our above results are exactly analogous in higher order differential operators to those of the affine Kac-Moody algebras.

On \( \mathcal{V}_\chi \), there exists a unique vertex algebra structure whose structure map \( Y(\cdot, z) \) satisfying \( Y(|0\rangle, z) = \text{Id}_{\mathcal{V}_\chi} \) and

\[
Y(t^{-m-1} \partial^n E_{i,j}, z) = \frac{1}{m!} \frac{d^m}{dz^m} E_{i,j}(r, z)
\]

(1.13)

for \( i, j \in \mathbb{T}, n \) and \( r, m \in \mathbb{N} \). Let \( \mathcal{G} \) be one of the Lie algebras \( \widehat{\mathfrak{g}}l(\ell, \mathbb{A}), \mathfrak{o}(\ell, \mathbb{A}) \) or \( \mathfrak{sp}(\ell, \mathbb{A}) \). The family \( (\mathcal{V}_\chi(\mathcal{G}), Y(\cdot, z)) \) forms a vertex subalgebra. If \( \chi \notin \mathbb{Z} \), the vertex algebra \( (\mathcal{V}_\chi(\mathcal{G}), Y(\cdot, z)) \)
is simple. When $\chi \in \mathbb{Z}$, the quotient space $V_\chi(G)$ forms a simple vertex algebra. If $\ell_i \in \{0, 1\}$ for $i \in \overline{1, n}$, the above vertex algebras have a Virasoro element, and thus they are vertex operator algebras.

Denote $\mathcal{Z} = \mathbb{Z} + 1/2$. Let $\mathfrak{gl}(\infty)$ be a vector space with a basis $\{\mathcal{E}_{l,k} \mid l, k \in \mathbb{Z}\}$ and multiplication:

$$\mathcal{E}_{l_1,l_2} \cdot \mathcal{E}_{k_1,k_2} = \delta_{l_2+k_1,0}\delta_{l_1,k_2} \quad \text{for } l_1, l_2, k_1, k_2 \in \mathbb{Z}.$$  \hfill(1.14)

Then $\mathfrak{gl}(\infty)$ is isomorphic to the associative algebra of infinite matrices with finite number of nonzero entries. Define the step function $H$ on $\mathcal{Z}$ by

$$H(l) = \begin{cases} 1 & \text{if } l > 0, \\ 0 & \text{if } l < 0 \end{cases} \quad \text{for } l \in \mathbb{Z}.$$  \hfill(1.15)

Set $\tilde{\mathfrak{gl}}(\infty) = \mathfrak{gl}(\infty) \oplus \mathbb{C}\kappa_0$, where $\kappa_0$ is symbol for base element. We have the following the Lie bracket on $\tilde{\mathfrak{gl}}(\infty)$:

$$[\mathcal{E}_{l_1,l_2} + \mu_1\kappa_0, \mathcal{E}_{k_1,k_2} + \mu_2\kappa_0] = \mathcal{E}_{l_1,l_2}\mathcal{E}_{k_1,k_2} - \mathcal{E}_{k_1,k_2}\mathcal{E}_{l_1,l_2}$$

$$+ \delta_{l_1+k_2,0}\delta_{l_2+k_1,0}[H(l_1)H(l_2) - H(k_1)H(k_2)]\kappa_0$$  \hfill(1.16)

for $l_1, l_2, k_1, k_2 \in \mathbb{Z}$ and $\mu_1, \mu_2 \in \mathbb{C}$. Moreover, the subspace $\mathcal{T} = \sum_{l \in \mathbb{Z}} \mathbb{C}\mathcal{E}_{l,-l} + \mathbb{C}\kappa_0$ forms a toral Cartan subalgebra of $\tilde{\mathfrak{gl}}(\infty)$. In fact, the root structures of $\mathfrak{gl}(\infty)$ and $\tilde{\mathfrak{gl}}(\infty)$ are the same. Thus they have exactly the same representation theory.

Given $\mu \in \mathbb{C}$, we set $\langle \mu \rangle_0 = 1$ and $\langle \mu \rangle_m = \mu(\mu - 1) \cdots (\mu - (m - 1))$ for $0 < m \in \mathbb{N}$. Denote by $\mathcal{T}^*$ the space of linear functions on $\mathcal{T}$. For $\lambda \in \mathcal{T}^*$, we define

$$\text{supp } \lambda = \{l \in \mathbb{Z} \mid \lambda(\mathcal{E}_{l,-l}) \neq 0\}.$$  \hfill(1.17)

Pick a weight $\lambda$ such that $\lambda(\kappa_0) = \chi$ and supp $\lambda$ is a finite set. Let $\mathcal{M}_\lambda$ be the highest weight irreducible $\mathfrak{gl}(\infty)$-module with weight $\lambda$, whose highest weight vector is annihilated by the subalgebra $\text{Span} \{\mathcal{E}_{l,k} \mid l, k \in \mathbb{Z}; \ l + k > 0\}$. Fix a constant $\iota \in \mathbb{C}$. We construct a module structure of the vertex algebra $\mathcal{V}_\chi$ whose structure map $Y_{\mathcal{M}}(\cdot, z)$ satisfying $Y_{\mathcal{M}}(\langle 0 \rangle, z) = \text{Id}_{\mathcal{M}}$ and

$$Y(t^{-1}\partial^r E_{\iota,j}, z) \equiv \sum_{l,k \in \mathbb{Z}} \langle \lambda - k \rangle_{l,n+i-1/2, k n-j+1/2} z^{-l-k-r-1} \quad (\text{mod } \mathbb{C}\kappa_0)$$  \hfill(1.18)

if $\iota \notin \mathbb{Z}$, and

$$Y(t^{-1}\partial^r E_{\iota,j}, z) \equiv \sum_{l,k=0}^n \langle -k-1 \rangle_{l,n+i-1/2, (k+i+1)n-j+1/2} z^{-l-k-\ell_i-r-2}$$

$$+ \langle k + \ell_j \rangle_{l,n+i-1/2, (l-k)n-j+1/2} z^{-l+k+\ell_j-\ell_i-r-1}$$

$$+ \langle k + \ell_j \rangle_{l,(l+i+1)n-i-1/2, (k+i+1)n-j+1/2} z^{-l+k+\ell_j-r}$$

$$+ \langle -k-1 \rangle_{l,(l+i+1)i-1/2, (k+i+1)n-j+1/2} z^{-l-k-r-1} \quad (\text{mod } \mathbb{C}\kappa_0)$$  \hfill(1.19)
if \( \iota \in \mathbb{Z} \).

Let \( m \) is a positive integer. Set \( S_m = \{\{3/2 - r, 5/2 - r, \ldots, (2m + 1)/2 - r\} \mid r \in \mathbb{Z}, m + 1\} \) the set of intervals around 0 of length \( m \) in \( \mathbb{Z} \). Define

\[
\Gamma^m = \{ \lambda \in \mathcal{T}^* \mid \lambda(\kappa_0) = -m, -s^{-1}|s|\lambda(\mathcal{E}_{\kappa_0-s}) \in \mathbb{N} \}
\]

for \( s \in \mathbb{Z} \); \( \text{supp} \lambda \subset S \) for some \( S \in S_m \).

(1.20)

**Theorem 1.2.** The family \( (\mathcal{M}, Y_{\mathcal{M}}(\mathcal{V}_\chi(\mathcal{G}), z)) \) forms an irreducible module of the vertex algebra \( (\mathcal{V}_\chi(\mathcal{G}), Y(\cdot, z)) \) for \( \mathcal{G} = \hat{\mathfrak{gl}}(\vec{\ell}, \mathbb{A}) \) and \( \mathcal{G} = \hat{o}(\vec{\ell}, \mathbb{A}), \hat{s}\mathfrak{p}(\vec{\ell}, \mathbb{A}) \) if \( \iota \notin \mathbb{Z}/2 \).

Suppose that \( \chi \) is a positive integer and

\[
\lambda(\mathcal{E}_{1/2,-1/2} - \mathcal{E}_{-1/2,1/2} + \kappa_0), \lambda(\mathcal{E}_{l+1,-l-1} - \mathcal{E}_{-l-1}) \in \mathbb{N}
\]

for \( -1/2 \neq l \in \mathbb{Z} \). The family \( (\mathcal{M}, Y_{\mathcal{M}}(\mathcal{V}_\chi(\mathcal{G}), z)) \) induces an irreducible module of the quotient simple vertex algebra \( (\mathcal{V}_\chi(\mathcal{G}), Y(\cdot, z)) \) for \( \mathcal{G} = \hat{\mathfrak{gl}}(\vec{\ell}, \mathbb{A}) \) and \( \mathcal{G} = \hat{o}(\vec{\ell}, \mathbb{A}), \hat{s}\mathfrak{p}(\vec{\ell}, \mathbb{A}) \) if \( \iota \notin \mathbb{Z}/2 \).

Assume that \( \chi = -m \) is a negative integer and \( \lambda \in \Gamma^m \). The family \( (\mathcal{M}, Y_{\mathcal{M}}(\mathcal{V}_\chi(\mathcal{G}), z)) \) induces an irreducible module of the quotient simple vertex algebra \( (\mathcal{V}_{-m}(\mathcal{G}), Y(\cdot, z)) \) for \( \mathcal{G} = \hat{\mathfrak{gl}}(\vec{\ell}, \mathbb{A}) \) and \( \mathcal{G} = \hat{o}(\vec{\ell}, \mathbb{A}), \hat{s}\mathfrak{p}(\vec{\ell}, \mathbb{A}) \) if \( \iota \notin \mathbb{Z}/2 \).

We remark that the module \( \mathcal{M} \) is an unitary module if (1.21) is satisfied or \( \lambda \in \Gamma^m \). When \( \iota \in \mathbb{Z}/2 \), we construct irreducible modules of the vertex algebras \( (\mathcal{V}_\chi(\hat{o}(\vec{\ell}, \mathbb{A})), Y(\cdot, z)) \) and \( (\mathcal{V}_\chi(\hat{s}\mathfrak{p}(\vec{\ell}, \mathbb{A})), Y(\cdot, z)) \) through highest weight modules of non-standard centrally-extended the other types of classical Lie algebras of infinite matrices with finite number of nonzero entries. Similar conclusions for their quotient simple vertex algebras hold. When \( \chi \) is a positive integer, our results on modules are natural generalizations of those for the simple vertex operators algebras associated with affine Lie algebras. When \( n > 3 \) and \( \chi \) is a positive integer, the condition (1.21) can be relaxed to obtain certain non-unitary irreducible modules of the simple vertex algebras \( \mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, \mathbb{A})), \mathcal{V}_\chi(\hat{o}(\vec{\ell}, \mathbb{A})) \) and \( \mathcal{V}_\chi(\hat{s}\mathfrak{p}(\vec{\ell}, \mathbb{A})) \). In the case \( \vec{\ell} = 0 \), our theory coincide with the charged quadratic free bosonic field theory if \( \chi = -1 \), and with the charged quadratic free fermionic field theory if \( \chi = 1 \). If \( \vec{\ell} = (1, 1, \ldots, 1) \), our irreducible modules of the vertex algebra \( (\mathcal{V}_{-1}(\hat{o}(\vec{\ell}, \mathbb{A})), Y(\cdot, z)) \) include those studied by Dong and Nagatomo [DN1,DN2].

The paper is organized as follows. In Section 2, we present the frame works of constructing representations of certain Lie subalgebras of the centrally-extended Lie algebra of the tensor algebra of any associative algebra with the algebra of differential operators on the circle. Section 3 is devoted to the constructions of irreducible modules of the Lie algebras \( \hat{\mathfrak{gl}}(\vec{\ell}, \mathbb{A}), \hat{o}(\vec{\ell}, \mathbb{A}) \) or \( \hat{s}\mathfrak{p}(\vec{\ell}, \mathbb{A}) \) from weighted irreducible modules of the centrally-extended general linear Lie algebra of infinite matrices with finite number of nonzero entries. In Section 4, we give detailed constructions of irreducible modules with the parameter \( \iota \in \mathbb{Z} + 1/2 \) of the Lie algebras \( \hat{o}(\vec{\ell}, \mathbb{A}) \).
and $\hat{sp}(\vec{\ell}, A)$ from weighted irreducible modules of certain central extensions of the Lie algebras of infinite skew matrices with finite number of nonzero entries. The cases with the parameter $\iota \in \mathbb{Z}$ are handled in Section 5. In Section 6, we study the vacuum representation of the Lie algebra $\hat{gl}(\vec{\ell}, A)$, its vertex algebra structure and vertex algebra irreducible representations. We deal with the cases for the Lie algebras $\hat{o}(\vec{\ell}, A)$ and $\hat{sp}(\vec{\ell}, A)$ in Section 7.

2 General Frames

In this section, we construct irreducible representations of certain Lie subalgebras of the centrally-extended Lie algebra of the tensor algebra of any associative algebra with the algebra of differential operators on the circle, from representations of the certain Lie subalgebras of a centrally-extended Lie algebra of the tensor algebra of the associative algebra with the algebra of infinite matrices with finite number of nonzero entries. When the associative algebra is a finite matrix algebra, the later Lie algebras are exactly centrally-extended classical Lie algebra of infinite matrices with finite number of nonzero entries.

Recall

$$\partial_t = \frac{d}{dt}$$

and the algebra of differential operators on the circle:

$$A = \sum_{i=0}^{\infty} \mathbb{C}[t, t^{-1}] \partial_t^i$$

Note

$$f(t) \partial_t^i \cdot g(t) \partial_t^j = \sum_{r=0}^{i} \binom{i}{r} f(t) \frac{d^r}{dt^r}(t) \partial_t^{i-j+r}$$

for $f(t), g(t) \in \mathbb{C}[t, t^{-1}], i, j \in \mathbb{N}.$

Let $\mathcal{A}$ be an associative algebra with an identity element $1_{\mathcal{A}}$ and a linear map $tr : \mathcal{A} \to \mathbb{C}$ such that

$$tr 1_{\mathcal{A}} \neq 0, \quad tr ab = tr ba \quad \text{for} \; a, b \in \mathcal{A}.$$  

Such a map “$tr$” is called a trace map. Set

$$\hat{\mathcal{A}} = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \oplus \mathbb{C}\kappa,$$

where $\kappa$ is a base element. According to [L] (also cf. [M1]), we have the following Lie bracket on $\hat{\mathcal{A}}$:

$$[a \otimes t^{i_1}\partial_t^{j_1} + \mu_1 \kappa, b \otimes t^{i_2}\partial_t^{j_2} + \mu_1 \kappa]$$

$$= ab \otimes t^{i_1}\partial_t^{j_1} \cdot t^{i_2}\partial_t^{j_2} - ba \otimes t^{i_2}\partial_t^{j_2} \cdot t^{i_1}\partial_t^{j_1}$$

$$+(-1)^{j_1+1} \delta_{i_1+i_2,j_1+j_2} j_1! j_2! \binom{i_1}{j_1+j_2+1} (tr ab) \kappa$$

$$\frac{d}{dt}$$

and the algebra of differential operators on the circle:

$$A = \sum_{i=0}^{\infty} \mathbb{C}[t, t^{-1}] \partial_t^i$$

2.1
for $a, b \in \mathcal{A}, i_1, i_2 \in \mathbb{Z}, j_1, j_2 \in \mathbb{N}$ and $\mu_1, \mu_2 \in \mathbb{C}$.

For two vector spaces $V_1$ and $V_2$, we denote by $LM(V_1, V_2)$ the space of linear maps from $V_1$ to $V_2$. We also use the following operator of taking residue:

$$\text{Res}_z(z^n) = \delta_{n,-1} \quad \text{for } n \in \mathbb{Z}. \quad (2.7)$$

Furthermore, all the binomials are assumed to be expanded in the nonnegative powers of the second variable.

A conformal algebra $R$ is a $\mathbb{C}[\partial]$-module with a linear map $Y^+(\cdot, z) : R \to LM(R, R[z^{-1}]z^{-1})$ satisfying:

$$Y^+(\partial u, z) = \frac{dY^+(u, z)}{dz} \quad \text{for } u \in R; \quad (2.8)$$

$$Y^+(u, z)v = \text{Res}_x e^{x\partial}Y^+(v, -x)u \quad \frac{z-x}{z}, \quad (2.9)$$

$$Y^+(u, z_1)Y^+(v, z_2) - Y^+(v, z_2)Y^+(u, z_1) = \text{Res}_z \frac{Y^+(Y^+(u, z_1-x)v, x)}{z_2-x} \quad (2.10)$$

for $u, v \in R$. We denote by $(R, \partial, Y^+(\cdot, z))$ a conformal algebra.

Define

$$\hat{R}(\mathcal{A}) = \mathcal{A}[\varsigma_1, \varsigma_2] \oplus \mathbb{C}1, \quad (2.11)$$

where $\varsigma_1, \varsigma_2$ are indeterminates and $1$ is a base element. For convenience, we denote

$$a[m_1, m_2] = a\varsigma_1^{m_1} \varsigma_2^{m_2} \quad \text{for } a \in \mathcal{A}, m_1, m_2 \in \mathbb{N}. \quad (2.12)$$

We define the action of $\mathbb{C}[\partial]$ on $\hat{R}(\mathcal{A})$ by:

$$\partial(1) = 0, \quad \partial(a[m_1, m_2]) = (m_1 + 1)a[m_1 + 1, m_2] + (m_2 + 1)a[m_1, m_2 + 1], \quad (2.13)$$

and a linear map $Y^+(\cdot, z) : \hat{R}(\mathcal{A}) \to LM(\hat{R}(\mathcal{A}), \hat{R}(\mathcal{A})[z^{-1}]z^{-1})$ by

$$Y^+(w, z)1 = Y^+(1, z)w = 0 \quad \text{for } w \in \hat{R}(\mathcal{A}) \quad (2.14)$$

and

$$Y^+(a[m_1, m_2], z)b[n_1, n_2]$$

$$= \left( -n_1 - 1 \atop m_2 \right) ^{m_1 + m_2 + n_1 \atop p} \left( m_1 \atop p \right) ab[p, n_2]z^{p-m_1-m_2-n_1-1}$$

$$- \left( -n_2 - 1 \atop m_1 \right) ^{m_1 + m_2 + n_2 \atop q} \left( m_2 \atop q \right) ba[n_1, q]z^{q-m_1-m_2-n_2-1}$$

$$+ \left( -n_1 - 1 \atop m_2 \right) \left( -n_2 - 1 \atop m_1 \right) \text{tr} ab1z^{-m_1-m_2-n_1-n_2-2} \quad (2.15)$$

for $a, b \in \mathcal{A}$ and $m_1, m_2, n_1, n_2 \in \mathbb{N}$. Then $(\hat{R}(\mathcal{A}), \partial, Y^+(\cdot, z))$ forms a conformal algebra (cf. Section 7.3 in [X1]).
Let
\[ \mathcal{A}[[z^{-1}, z]] = \left\{ \sum_{m \in \mathbb{Z}} u_m z^m \mid u_m \in \mathcal{A} \right\} \tag{2.16} \]
be the space of formal power series with coefficients in \( \mathcal{A} \). Define a linear map \( Y(\cdot, z) : \hat{R}(A) \to \mathcal{A}[[z^{-1}, z]] \) by
\[
Y(1, z) = \kappa, \quad Y(a[m_1, m_2], z) = \frac{1}{m_1! m_2!} \sum_{i \in \mathbb{Z}} a \otimes (-\partial_t)^{m_1} \partial_t^{m_2} z^{-i-1} \tag{2.17}
\]
for \( a \in A \) and \( m_1, m_2 \in \mathbb{N} \). By Lemma 3.1 in [M1], we have:

**Lemma 2.1.** The Lie bracket (2.6) on \( \mathcal{A} \) is equivalent to:
\[
[Y(u, z_1), Y(v, z_2)] = \text{Res}_{z_1 z_2}^{-1} \delta \left( \frac{z_2 + z_1}{z_1} \right) Y(Y^+(u, z_0) v, z_2) \tag{2.18}
\]
for \( u, v \in \hat{R}(A) \).

Recall
\[
\mathcal{Z} = \mathbb{Z} + \frac{1}{2} \tag{2.19}
\]
and the step function \( H \) on \( \mathcal{Z} \):
\[
H(l) = \begin{cases} 
1 & \text{if } l > 0, \\
0 & \text{if } l < 0 
\end{cases} \quad \text{for } l \in \mathbb{Z}. \tag{2.20}
\]
Set
\[
\bar{A} = \sum_{i,j \in \mathcal{Z}} \mathcal{A} t_1^i t_2^j \oplus \mathbb{C} \kappa_0, \tag{2.21}
\]
where \( t_1, t_2 \) are indeterminates and \( \kappa_0 \) is a base element. For convenience, we denote
\[
a(i, j) = at_1^i t_2^j \quad \text{for } a \in A, ~ i, j \in \mathcal{Z}. \tag{2.22}
\]
According to Section 7.3 in [X2], we have the following Lie bracket on \( \bar{A} \):
\[
[a(l_1, l_2) + \mu_1 \kappa_0, b(k_1, k_2) + \mu_2 \kappa_0] = \delta_{l_2+k_1,0} ab(l_1, k_2) - \delta_{l_1+k_2,0} ba(k_1, l_2) \\
+ \delta_{l_1+k_2,0} \delta_{l_2+k_1,0} [H(l_1) H(l_2) - H(k_1) H(k_2)](\text{tr } ab) \kappa_0 \tag{2.23}
\]
for \( a, b \in A, ~ l_1, l_2, k_1, k_2 \in \mathcal{Z} \) and \( \mu_1, \mu_2 \in \mathbb{C} \). The algebra \( \bar{A} \) is isomorphic to the corresponding central extension of the commutator Lie algebra of \( A \otimes \mathfrak{gl}(\infty) \) (cf. (1.14) and the above).

Set
\[
\bar{A}^m = \text{Span} \{ a(i, j) \mid a \in A, ~ i, j \in \mathcal{Z}; ~ m < |i|, |j|; ~ ij < 0 \} \quad \text{for } 0 < m \in \mathbb{N}. \tag{2.24}
\]
It can be verified that \( \bar{A}^m \) is a Lie subalgebra of \( \bar{A} \). Suppose that \( \mathcal{M} \) is an \( \bar{A} \)-module
\[
generated by a subspace \mathcal{M}_0 such that \( \bar{A}^m(\mathcal{M}_0) = \{0\} \) for some \( m \in \mathbb{N}. \tag{2.25}\]
Fixed a constant $\iota \in \mathbb{C}$. Define

$$
\sum_{r_1, r_2 = 0}^{\infty} \mathbb{G}_{r_1, r_2} x^{r_1} y^{r_2} z^{-r_1 - r_2} = \frac{1}{x - y} \left( \left( \frac{z + y}{z + x} \right)^{\iota} - 1 \right)
$$

Motivated from the construction of the twisted modules of spinor vertex operator algebras in [X1], we define a linear map $Y_{\hat{\mathcal{M}}}(\cdot, z) : \hat{R}(\mathcal{A}) \to LM(\mathcal{M}, \mathcal{M}[[z^{-1}, \hat{z}]])$ by

$$
Y_{\hat{\mathcal{M}}}(1, z) = \kappa_0,
$$

$$
Y_{\hat{\mathcal{M}}}(a[r_1, r_2], z) = \sum_{i, j \in \mathbb{Z}} \left( \begin{array}{c} -i - \iota - 1/2 \\ r_1 \end{array} \right) \left( \begin{array}{c} -j + \iota - 1/2 \\ r_1 \end{array} \right) a(i, j) z^{-i - j - r_1 - r_2 - 1} + \mathbb{G}_{r_1, r_2}(\text{tr } a) \kappa_0 z^{-r_1 - r_2 - 1}
$$

for $a \in \mathcal{A}$ and $r_1, r_2 \in \mathbb{N}$. The above expression make sense because of (2.25). Denote

$$
\langle \mu \rangle_0 = 1, \; \langle \mu \rangle_m = \mu(\mu - 1) \cdots (\mu - (m - 1)) \quad \text{for } \mu \in \mathbb{C}, \; 0 < m \in \mathbb{N}.
$$

**Theorem 2.2.** On the $\hat{\mathcal{A}}$-module $\mathcal{M}$, we have

$$
[Y_{\hat{\mathcal{M}}}(u, z_1), Y_{\hat{\mathcal{M}}}(v, z_2)] = \text{Res}_{z_0} z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) Y_{\hat{\mathcal{M}}}(Y^+(u, z_0)v, z_2)
$$

for $u, v \in \hat{R}(\mathcal{A})$. In particular, Lemma 2.1 and (2.30) imply that $\mathcal{M}$ provides the following representation $\sigma_{\hat{\mathcal{A}}}^\mathcal{M}$ of the Lie algebra $\hat{\mathcal{A}}$:

$$
\sigma_{\hat{\mathcal{A}}}^\mathcal{M}(\kappa) = \kappa_0,
$$

$$
\sigma_{\hat{\mathcal{A}}}^\mathcal{M}(a \otimes t^m \partial_t^l) = \sum_{l \in \mathbb{Z}} \langle -l + \iota - 1/2 \rangle_r a(m - r - l, l) + r! \delta_{r, m} \mathbb{G}_{0, r}(\text{tr } a) \kappa_0
$$

for $m \in \mathbb{Z}$, $r \in \mathbb{N}$ and $a \in \mathcal{A}$.

**Proof.** We shall use generating functions to prove the lemma. Set

$$
c[x, y] = \sum_{r_1, r_2 = 0}^{\infty} c[r_1, r_2] x^{r_1} y^{r_2} \quad \text{for } c \in \mathcal{A}.
$$

Fix $a, b \in \mathcal{A}$ and let

$$
u = a[x_1, x_2], \quad v = b[y_1, y_2].
$$

Then

$$
Y^+(u, z_0)v = \frac{1}{z_0 + x_2 - y_1} ab[z_0 + x_1, y_2] - \frac{1}{z_0 + x_1 - y_2} ba[y_1, z_0 + x_2]
$$

$$
+ \frac{1}{(z_0 + x_2 - y_1)(z_0 + x_1 - y_2)} \text{tr } ab \mathbf{1}
$$

(2.35)
by (2.15).

On the other hand, we define

\[ c(z_1, z_2) = \sum_{i,j \in \mathbb{Z}} c(i,j)z_1^{-i-\frac{1}{2}}z_2^{-j-\frac{1}{2}} \quad \text{for } c \in A. \tag{2.36} \]

Then

\[ Y_M^t(u, z_1) = a(z_1 + x_1, z_1 + x_2) + \frac{1}{x_1 - x_2} \left( \left( \frac{z_1 + x_2}{z_1 + x_1} \right)^t - 1 \right) (\text{tr } a) \kappa_0, \tag{2.37} \]

\[ Y_M^t(v, z_1) = b(z_2 + y_1, z_2 + y_2) + \frac{1}{y_1 - y_2} \left( \left( \frac{z_2 + y_2}{z_2 + y_1} \right)^t - 1 \right) (\text{tr } a) \kappa_0. \tag{2.38} \]

Hence

\[
\begin{align*}
[Y_M^t(u, z_1), Y_M^t(v, z_1)] &= [a(z_1 + x_1, z_1 + x_2), b(z_2 + y_1, z_2 + y_2)] \\
&= (z_2 + y_1)^{-1} \left( \frac{z_1 + x_2}{z_2 + y_1} \right)^t \delta \left( \frac{z_1 + x_2}{z_2 + y_1} \right) ab(z_1 + x_1, z_2 + y_2) \\
&\quad - (z_2 + y_2)^{-1} \left( \frac{z_2 + y_2}{z_1 + x_2} \right)^t \delta \left( \frac{z_1 + x_1}{z_2 + y_2} \right) ba(z_2 + y_1, z_1 + x_2) \\
&\quad + \left( \frac{(z_1 + x_2)(z_2 + y_2)}{(z_1 + x_1)(z_2 + y_1)} \right)^t \left\{ \frac{1}{(z_1 + x_1 - z_2 - y_2)(z_1 + x_2 - z_2 - y_1)} \right\} \\
&\quad - \frac{1}{(z_2 + y_1 - z_1 - x_2)(z_2 + y_2 - z_1 - x_1)} (\text{tr } ab) \kappa_0. \tag{2.39}
\end{align*}
\]

Note that

\[
\begin{align*}
\left( \frac{(z_1 + x_2)(z_2 + y_2)}{(z_1 + x_1)(z_2 + y_1)} \right)^t &\left\{ \frac{1}{(z_1 + x_1 - z_2 - y_2)(z_1 + x_2 - z_2 - y_1)} \right\} \\
&= \frac{1}{x_1 + y_1 - x_2 - y_2} \left[ \frac{(z_1 + x_2)(z_2 + y_2)}{(z_1 + x_1)(z_2 + y_1)} \right]^t \left\{ \frac{1}{z_1 + x_2 - z_2 - y_1} \right\} \\
&\quad - \frac{1}{z_1 + x_1 - z_2 - y_2} \frac{1}{z_2 + y_1 - z_1 - x_2} - \frac{1}{z_2 + y_2 - z_1 - x_1} \\
&= \frac{-z_1^{-1}}{x_1 + y_1 - x_2 - y_2} \left[ \delta \left( \frac{z_2 + y_1 - x_2}{z_1} \right) - \delta \left( \frac{z_2 + y_2 - x_1}{z_1} \right) \right] \\
&\quad - \left( \frac{z_2 + x_2 + y_2 - x_1}{z_2 + y_1} \right)^t \delta \left( \frac{z_2 + y_1 - x_2}{z_1} \right) \tag{2.40}
\end{align*}
\]

Observe that

\[
(z_2 + y_1)^{-1} \delta \left( \frac{z_1 + x_2}{z_2 + y_1} \right) = \frac{1}{z_1 + x_2 - z_2 - y_1} + \frac{1}{z_2 + y_1 - z_1 - x_2} \\
= z_1^{-1} \delta \left( \frac{z_2 + y_1 - x_2}{z_1} \right). \tag{2.41}
\]
Hence

\[
\text{Res}_{z_0} z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \frac{1}{z_0 + x_2 - y_1} ab(z_2 + z_0 + x_1, z_2 + y_2)
\]

\[
= \text{Res}_{z_0} \left( \frac{z_2 + y_1}{z_0 + x_2 - y_1} \right) z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) [(z_2 + y_1)^i ab(z_2 + z_0 + x_1, z_2 + y_2)]
\]

\[
= \text{Res}_{z_0} \left( \frac{z_2 + y_1}{z_0 + x_2 - y_1} \right) z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) [(z_1 - z_0 + y_1)^i ab(z_1 + x_1, z_2 + y_2)]
\]

\[
= \left( \frac{z_1 + x_2}{z_2 + y_1} \right)^i z_1^{-1} \delta \left( \frac{z_2 + y_1 - x_2}{z_1} \right) ab(z_1 + x_1, z_2 + y_2)
\]

\[
= (z_2 + y_1)^{-i} \left( \frac{z_1 + x_2}{z_2 + y_1} \right)^i \delta \left( \frac{z_1 + x_2}{z_2 + y_1} \right) ab(z_1 + x_1, z_2 + y_2).
\] (2.42)

Similarly, we have

\[
\text{Res}_{z_0} z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \frac{1}{z_0 + x_1 - y_2} ba(z_2 + y_1, z_2 + z_0 + x_2)
\]

\[
= (z_2 + y_2)^{-i} \left( \frac{z_2 + y_2}{z_1 + x_1} \right)^i \delta \left( \frac{z_1 + x_1}{z_2 + y_2} \right) ba(z_2 + y_1, z_1 + x_2).
\] (2.43)

Moreover,

\[
\text{Res}_{z_0} z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \frac{1}{(z_0 + x_2 - y_1)(z_0 + x_1 - y_2)}
\]

\[
\times \left\{ \left[ \frac{z_2 + y_2}{z_2 + z_0 + x_1} \right]^i + \left[ \frac{z_2 + z_0 + x_2}{z_2 + y_1} \right]^i - 1 \right\}
\]

\[
= \frac{1}{z_1 + y_1 - x_2 - y_2} \left\{ \text{Res}_{z_0} \frac{1}{z_0 + x_2 - y_1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \right. \\
\times \left[ \left( \frac{z_2 + y_2}{z_2 + z_0 + x_1} \right)^i + \left( \frac{z_2 + z_0 + x_2}{z_2 + y_1} \right)^i - 1 \right] \\
- \text{Res}_{z_2} \frac{1}{z_0 + x_2 - y_2} \delta \left( \frac{z_2 + z_0}{z_1} \right) \left[ \left( \frac{z_2 + y_2}{z_2 + z_0 + x_1} \right)^i + \left( \frac{z_2 + z_0 + x_2}{z_2 + y_1} \right)^i - 1 \right] \\
\times \left( \frac{z_2 + y_2}{z_2 + x_2 + y_2 - x_1} \right)^i \\
- \delta \left( \frac{z_2 + y_2}{z_2 + x_2 + y_2 - x_1} \right) \left( \frac{z_2 + x_2 + y_2 - x_1}{z_2 + y_1} \right)^i \right\}.
\] (2.44)

By (2.35), (2.39), (2.40) and (2.42)-(2.44), we have

\[
\text{Res}_{z_0} z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) Y_M^i (Y^+(u, z_0) v, z_2)
\]

\[
= \text{Res}_{z_0} z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \left[ \frac{1}{z_0 + x_2 - y_1} Y_M^i(ab[z_0 + x_1, y_2], z_2) \\
- \frac{1}{z_0 + x_1 - y_2} Y_M^i (ba[y_1, z_0 + x_2], z_2) \\
+ \frac{1}{(z_0 + x_2 - y_1)(z_0 + x_1 - y_2)}(\text{tr } ab) \kappa_0 \right] \\
+ \frac{1}{z_0 + x_1 - y_2} \left[ \left( \frac{z_2 + y_2}{z_2 + z_0 + x_1} \right)^i - 1 \right] (\text{tr } ab) \kappa_0
\]
\[
- \frac{1}{z_0 + x_1 - y_2} \left[ ba(z_2 + y_1, z_2 + z_0 + x_2) - \frac{1}{z_0 + x_2 - y_1} \right] (tr \ ab)\kappa_0 + \frac{1}{(z_0 + x_2 - y_1)(z_0 + x_1 - y_2)(tr \ ab)\kappa_0} \\
\times \left[ \left( \frac{z_2 + z_0 + x_2}{z_2 + y_1} \right)^\ell - 1 \right] \left( \frac{1}{z_0 + x_2 - y_1} \right) ba(z_2 + z_0 + x_1, z_2 + y_2) \\
- \text{Res}_{z_0} \left( z_0 \right)^{\ell} \left[ z_2 + z_0 \right] \frac{1}{z_0 + x_2 - y_1} \left( z_0 + x_1 - y_2 \right) \left( z_0 + x_1 - y_2 \right) \left( z_0 + x_2 - y_1 \right) \left( z_0 + x_1 - y_2 \right) \left( z_0 + x_2 - y_1 \right) \\
+ \text{Res}_{z_0} \left( z_0 \right)^{\ell} \left[ z_2 + z_0 \right] \frac{1}{z_0 + x_2 - y_1} \left( z_0 + x_1 - y_2 \right) \left( z_0 + x_1 - y_2 \right) \left( z_0 + x_2 - y_1 \right) \left( z_0 + x_1 - y_2 \right) \left( z_0 + x_2 - y_1 \right) \\
\times \left[ \left( \frac{z_2 + z_0 + x_2}{z_2 + y_1} \right)^\ell - 1 \right] \\
= [Y^t_{\mathcal{M}}(u, z_1), Y^t_{\mathcal{M}}(v, z_2)], \quad (2.45)
\]

that is (2.30) holds. Moreover, (2.32) follows from (2.17), Lemma 2.1 and (2.30). \(\square\)

Suppose that the associative algebra \(\mathcal{A}\) has \(n\) left ideals \(\{\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n\}\) such that
\[
\mathcal{A} = \bigoplus_{i=1}^n \mathcal{B}_i. \quad (2.46)
\]

Take
\[
\vec{\ell} = (\ell_1, \ell_2, ..., \ell_n) \in \mathbb{N}^n \quad (2.47)
\]

and set
\[
\hat{\mathcal{A}}_{\vec{\ell}} = \sum_{i=1}^n \mathcal{B}_i \otimes \mathbb{A} \partial_i^{\ell_i} + \mathbb{C} \kappa. \quad (2.48)
\]

Then \(\hat{\mathcal{A}}_{\vec{\ell}}\) forms a Lie subalgebra of \(\hat{\mathcal{A}}\) (cf. (2.5) and (2.6)). Let \(\mathcal{M}\) be an \(\hat{\mathcal{A}}\)-module satisfying (2.25). Consider the restricted representation \(\sigma^t_{\mathcal{M}}|\hat{\mathcal{A}}_{\vec{\ell}}\) of \(\hat{\mathcal{A}}_{\vec{\ell}}\). However, when \(\iota \in \mathbb{Z}\) and \(\vec{\ell} \neq \vec{0}\), the summation in (2.32) has redundant terms in terms of constructing irreducible modules. We want to represent the representation in reduced form. Set
\[
a(r, z) = \sum_{m \in \mathbb{Z}} a \otimes t^m \partial_i^{\ell_i} z^{-m-1} \quad \text{for} \quad a \in \mathcal{A}, \ r \in \mathbb{N}. \quad (2.49)
\]

Suppose that \(\iota \in \mathbb{Z}\) and each
\[
\mathcal{B}_j = \bigoplus_{i=1}^n \mathcal{B}_{i,j} \quad (2.50)
\]

has a subspace decomposition such that
\[
\mathcal{B}_{i,j} \cdot \mathcal{B}_{l,k} = \{0\} \quad \text{if} \quad j \neq l. \quad (2.51)
\]

By deleting the redundant terms and changing indices, we obtain:

**Theorem 2.3.** On the \(\hat{\mathcal{A}}\)-module \(\mathcal{M}\), we have the following representation \(\sigma^t_{\mathcal{M}}|\hat{\mathcal{A}}_{\vec{\ell}}\) of \(\hat{\mathcal{A}}_{\vec{\ell}}\):
\[ \sigma_{\ell,M}^*(\kappa) = \kappa_0 \quad \text{and} \quad \sigma_{\ell,M}^*(a(r, z)) \]

\[ = \sum_{0 < l_1, l_2 \in \mathbb{Z}} [(l_2 - 1/2)r + (l_1 - 1/2)z^{-l_1 - l_2} + (l_2 - 1/2)r + (l_1 - 1/2)z^{-l_1 - l_2 - 1} + (l_2 + 1/2)r + (l_1 + 1/2)z^{-l_1 - l_2 - 1} + (l_2 + 1/2)r + (l_1 + 1/2)z^{-l_1 - l_2 - 1}] + r!3_{0,r}^0(tr a)\kappa_0 z^{-r-1} \quad (2.52) \]

for \( a \in B_{i,j} \) and \( r \in \mathbb{N} \).

Suppose that \( \tau \) is a linear transformation on \( \mathcal{A} \) such that

\[ \tau^2 = \text{Id}_\mathcal{A}, \quad \tau(1_\mathcal{A}) = 1_\mathcal{A}, \quad \tau(ab) = \tau(b)\tau(a) \quad \text{for } a, b \in \mathcal{A}, \quad (2.53) \]

\[ \text{tr } \tau = \text{tr } \mathcal{A}, \quad \tau(B_{i,j}) \subset B_{\pi(i),\pi(j)} \quad \text{for } i, j \in \overline{1,n}, \quad (2.54) \]

where \( \pi \) is a permutation on \( \{1, 2, \ldots, n\} \).

Let

\[ \epsilon \in \{0, 1\} \quad (2.56) \]

and take \( \ell \in \mathbb{N}^n \) such that

\[ \{\ell_1, \ell_2, \ldots, \ell_n\} \subset 2\mathbb{Z} + \epsilon \quad (2.57) \]

and

\[ \ell_i = \ell_{\pi(i)} \quad \text{for } i \in \overline{1,n}. \quad (2.58) \]

Set

\[ \tilde{\mathcal{A}}_\ell^* = \text{Span} \{a \otimes t^m \partial_t^{\ell_i + \ell_j} - (-1)^\epsilon \tau(a) \otimes (-\partial_t)^r t^m \partial_t^{\ell_i} | i, j \in \overline{1,n}; a \in B_{i,j}; r \in \mathbb{N}, m \in \mathbb{Z}\} + \mathcal{C} \kappa. \quad (2.59) \]

It can be verified that \( \tilde{\mathcal{A}}_\ell^* \) forms a Lie subalgebra of \( \tilde{\mathcal{A}} \) (cf. (2.6), [M1]). For any \( \mathcal{A} \)-module \( \mathcal{M} \) satisfying (2.25), we have the restricted representation \( \sigma_{\mathcal{M}}^*|_{\tilde{\mathcal{A}}_\ell^*} \). For convenience, we set

\[ a_{\ell}^*(r, z) = \sum_{m \in \mathbb{Z}} (a \otimes t^m \partial_t^{\ell_i + \ell_j} - (-1)^\epsilon \tau(a) \otimes (-\partial_t)^r t^m \partial_t^{\ell_i})z^{-m-1} \quad (2.60) \]

Then for \( a \in B_{i,j} \) and \( r \in \mathbb{N} \), we have

\[ \sigma_{\mathcal{M}}^*|_{\tilde{\mathcal{A}}_\ell^*}(a_{\ell}^*(r, z)) = Y_{\mathcal{M}}^*((r + \ell_j) a[0, r + \ell_j] - (-1)^\epsilon r! \ell_i! \tau(a)[r, \ell_i], z) \]

\[ = \sum_{l_1, l_2 \in \mathbb{Z}} (-l_2 - r - 1/2)_{r+\ell_j} a(l_1, l_2)z^{-l_1 - l_2 - r} - (-1)^\epsilon r! \ell_i! \tau(a)[r, \ell_i](l_1, l_2)z^{-l_1 - l_2 - r} \]

\[ + (-1)^\epsilon \sum_{l_1, l_2 \in \mathbb{Z}} (-l_2 - r - 1/2)_{r+\ell_j} a(l_1, l_2)z^{-l_1 - l_2 - r} \]

\[ - (-1)^\epsilon r! \ell_i! \tau(a)[r, \ell_i] \quad (2.61) \]
\[
\begin{align*}
\sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} [(l + r + \ell_j + \tau - 1/2)_{r+\ell_j} a(m + l, -l - r - \ell_j) - (-1)^{\epsilon} (l + r - \tau - 1/2)_r \\
\times \{(-m - l + l_i + \tau - 1/2)_{\ell_i} \tau(a)(-l - r, m + l - \ell_i)\}] z^{-m-1} \\
+ (r + \ell_j)! \mathfrak{Z}_{0,r+\ell_j} (\text{tr } a) \kappa_0 \zeta^{-r-\ell_j-1} - (-1)^{\epsilon} r! \ell_j! \mathfrak{Z}_{r,\ell_j} (\text{tr } a) \kappa_0 z^{-r-\ell_j-1}
\end{align*}
\]

(2.61)

for \(a \in \mathcal{B}_{i,j}\) and \(r \in \mathbb{N}\) by (2.17) and (2.18). Thus

\[
\sigma_M^\tau(a \otimes t^m \partial_t^{\ell_j} \tau(a) - (-1)^{\epsilon} \tau(a) \otimes (-\partial_t)^{\ell_j} t^m \partial_t^{\ell_j})
\]

\[
\sum_{l \in \mathbb{Z}} [(l + r + \ell_j + \tau - 1/2)_{r+\ell_j} a(m + l, -l - r - \ell_j) \\
- (-1)^{\epsilon} (l + r - \tau - 1/2)_r \{(-m - l + l_i + \tau - 1/2)_{\ell_i} \tau(a)(-l - r, m + l - \ell_i)\}]
\]

\[= \sum_{l \in \mathbb{Z}} [(l + r + \ell_j + \tau - 1/2)_{r+\ell_j} a(m + l, -l - r - \ell_j) \\
- (-1)^{\epsilon} (l + r - \tau - 1/2)_r \{(-m - l + l_i - \tau - 1/2)_{\ell_i} \tau(a)(-l - r - 2i, m + l + 2i - \ell_i)\}]
\]

\[= \sum_{l \in \mathbb{Z}} (l + r + \tau - 1/2)_r \{(-l - r - \tau - 1/2)_{\ell_j} a(m + l, -l - r - \ell_j) \\
- (m + l + \tau - 1/2)_{\ell_j} \tau(a)(-r - l - 2i, m + l + 2i - \ell_i)\}. \quad (2.63)
\]

Set

\[
\alpha_{\ell}^{\tau,\ell} (l_1, l_2) = (-1)^{\epsilon} (l_2 + \tau - 1/2)_{\ell_j} a(l_1, l_2 + 2i - \ell_j) - (l_1 + \tau - 1/2)_{\ell_j} \tau(a)(l_2, l_1 + 2i - \ell_j) \quad (2.64)
\]

for \(a \in \mathcal{B}_{i,j}\) and \(l_1, l_2 \in \mathbb{Z}\). Then

\[
\alpha_{\ell}^{\tau,\ell} (l_1, l_2) = (-1)^{\epsilon} \tau(a) \alpha_{\ell}^{\tau,\ell} (l_2, l_1). \quad (2.65)
\]

Moreover, (2.61) becomes

\[
\sigma_M^\tau(a_{\ell}^{\tau} (r, z)) = \sum_{l_1, l_2 \in \mathbb{Z}} (-l_2 - \tau - 1/2)_r a_{\ell}^{\tau,\ell} (l_1, l_2) z^{-l_1 - l_2 - 2i - r - 1} + (r + \ell_j)! \mathfrak{Z}_{0,r+\ell_j} (\text{tr } a) \kappa_0 \\
\times z^{-r-\ell_j-1} - (-1)^{\epsilon} r! \ell_j! \mathfrak{Z}_{r,\ell_j} (\text{tr } a) \kappa_0 z^{-r-\ell_j-1}. \quad (2.66)
\]

For \(l, k \in \mathbb{Z}\) and \(\ell \in \mathbb{N}\), we have

\[
\langle l - 1/2 \rangle_{\ell} = (-1)^{\epsilon} \langle k - 1/2 \rangle_{\ell} \quad \text{if } l + k = \ell. \quad (2.67)
\]
Given $a \in B_{i_1,j_1}$, $b \in B_{i_2,j_2}$ and $l_1, l_2, k_1, k_2 \in \mathbb{Z}$, (2.23) and (2.64) imply

$$[a_{\ell}^{\tau}(l_1, l_2), b_{\ell}^{\tau}(k_1, k_2)]$$

$$= \left[ (-1)\hat{\ell}(l_2 + \ell_1 + 2l_2 - \ell_1) - (l_1 + \ell_1 + 2l_1 - \ell_1), \right.$$  

$$\left. (-1)\hat{\ell}(k_2 + \ell_2 + 2k_2 - \ell_2) - (k_1 + \ell_2 + 2k_1 - \ell_2) \right]$$

$$\equiv \left( l_1 + \ell_1 + 2l_1 - \ell_1 \right) \delta_{j_1,j_2} \delta_{l_1,k_1+2l_2-k_1} \left( ab \right)^{\tau}(l_1, k_2) + \delta_{j_2,j_1} \delta_{l_1+k_1+2l_2-k_1} \left( \tau(b) a \right)^{\tau}(k_2, l_2)$$

$$\left. - (l_1 + \ell_1 + 2l_1 - \ell_1) \delta_{j_2,j_1} \delta_{l_1+k_1+2l_2-k_1} \left( b a \right)^{\tau}(l_1, k_2) \right) \right.$$  

$$\left. + \delta_{j_1,j_2} \delta_{l_1+k_1+2l_2-k_1} \left( \tau(b) b \right)^{\tau}(l_1, k_2) \right) \left( \mod \mathbb{C} \kappa_0 \right).$$  

(2.68)

We define

$$\tilde{A}^{\tau}_{\ell} = \text{Span} \left\{ a_{\ell}^{\tau}(l_1, l_2) \mid i, j \in \overline{1,n}; a \in B_{i,j}, l_1, l_2 \in \mathbb{Z} \right\} + \mathbb{C} \kappa_0. \quad (2.69)$$

Then $\tilde{A}^{\tau}_{\ell}$ is a Lie subalgebra of $\tilde{A}$ by (2.68). Set

$$\tilde{\ell} = |2\ell| + \max \{ \ell_1, \ell_2, ..., \ell_n \}. \quad (2.70)$$

Define

$$\tilde{A}^{\tau}_{\ell,m} = \text{Span} \left\{ a_{\ell}^{\tau}(l_1, -l_2) \mid i, j \in \overline{1,n}; a \in B_{i,j}, m < l_1, l_2 \in \mathbb{Z} \right\} \quad \text{for} \quad \tilde{\ell} < m \in \mathbb{N}. \quad (2.71)$$

By (2.23), (2.64), (2.67) and (2.68), $\tilde{A}^{\tau}_{\ell,m}$ forms a Lie subalgebra of $\tilde{A}^{\tau}_{\ell}$.

**Theorem 2.4.** Let $\ell \in \mathbb{Z}/2$. Let $M$ be a $\tilde{A}^{\tau}_{\ell}$-module

generated by a subspace $M_0$ such that $\tilde{A}^{\tau}_{\ell,m}(M_0) = \{0\}$ for some $\tilde{\ell} < m \in \mathbb{N}$. \quad (2.72)

Then we have the following representation $\sigma_M$ of $\tilde{A}^{\tau}_{\ell}$ on $M$: $\sigma_M(\kappa) = \kappa_0$ and

$$\sigma(a^{\tau}(r,z)) = \sum_{l_1, l_2 \in \mathbb{Z}} (-l_2 - \ell - 1/2) a_{\ell}^{\tau}(l_1, l_2) z^{-l_1-l_2-2l-1} + (r + \ell) ! \Im_{0,r} (\tau a) \kappa_0$$

$$\times z^r - \ell_1 - 1 - (-1)^r \ell_1 ! \Im_{r, \ell_1} (\tau a) \kappa_0 z^{-r - \ell_1 - 1} \quad (2.73)$$

for $a \in B_{i,j}$ and $r \in \mathbb{N}$.

Next we assume $\ell \in \mathbb{Z}$. When $\tilde{\ell} \neq 0$, the summation in (2.73) has redundant terms in terms of constructing irreducible modules. We want to represent the representation in reduced form. It can be verified that the subspace

$$\mathcal{I} = \text{Span} \left\{ a_{\ell}^{\tau}(l_1 - \ell, l_2 - \ell) \mid i, j \in \overline{1,n}; a \in B_{i,j}, l_1, l_2 \in \mathbb{Z}, 0 < l_1 < \ell_i \text{ or } 0 < l_2 < \ell_j \right\} \quad (2.74)$$

forms an ideal of $\tilde{A}^{\tau}_{\ell}$.

Denote

$$Z_i = \mathbb{Z} \setminus \{-\ell + 1/2, -\ell + 3/2, ..., -\ell + \ell_1 - 1/2\} \quad \text{for} \quad i \in \overline{1,n}. \quad (2.75)$$
The subspace
\[
\mathcal{D}_\ell^{\tau,\iota} = \text{Span} \{ a^{\tau,\iota}_\ell(l_1, l_2) \mid i, j \in \overline{1,n}; \ a \in B_{i,j}, \ l_1 \in \mathbb{Z}_i, \ l_2 \in \mathbb{Z}_j \} + \mathbb{C} \kappa_0
\] (2.76)
forms a Lie subalgebra of \( \mathcal{A}_\ell^{\tau,\iota} \). Moreover,
\[
\mathcal{A}_\ell^{\tau,\iota} = \mathcal{D}_\ell^{\tau,\iota} \oplus I.
\] (2.77)

Set
\[
\mathcal{D}_{\ell,m}^{\tau,\iota} = \text{Span} \{ a^{\tau,\iota}_\ell(l_1, -l_2) \mid i, j \in \overline{1,n}; \ a \in B_{i,j}, \ m < l_1 \in \mathbb{Z}_i, \ m < l_2 \in \mathbb{Z}_j \}
\] (2.78)
for \( \tilde{\ell} < m \in \mathbb{N} \). By (2.23), (2.64) and (2.68), \( \mathcal{D}_{\ell,m}^{\tau,\iota} \) forms a Lie subalgebra of \( \mathcal{D}_\ell^{\tau,\iota} \). According to (2.73), we have:

**Theorem 2.5.** Suppose \( \iota \in \mathbb{Z} \). Let \( \mathcal{N} \) be a \( \mathcal{D}_\ell^{\tau,\iota} \)-module generated by a subspace \( \mathcal{N}_0 \) such that \( \mathcal{D}_{\ell,m}^{\tau,\iota}(\mathcal{N}_0) = \{0\} \) for some \( \tilde{\ell} < m \in \mathbb{N} \). Then we have the following representation \( \sigma_\mathcal{N} \) of \( \hat{A}_\ell^{\tau} \) on \( \mathcal{N} \):

\[
\sigma_\mathcal{N}(a^{\tau}_\ell(r, z)) = \sum_{l_1 \in \mathbb{Z}_i, l_2 \in \mathbb{Z}_j} (z^{-l_1^2 - 2l_2 - r - 1} + (r + \ell_j)! 3_{0, \ell_j}(\text{tr} a) \kappa_0 \times z^{-r - \ell_j - 1} - (-1)^{r + \ell_j} 3_{\ell_j}(\text{tr} a) \kappa_0 z^{-r - \ell_i - 1})
\] (2.80)
for \( a \in B_{i,j} \) and \( r \in \mathbb{N} \).

**Example 2.1.** Let \( k > 1 \) be integer. The Hecke algebra \( H_k \) is an associative algebra generated by \( \{ T_1, \ldots, T_{k-1} \} \) with the following defining relations

\[
T_i T_j = T_j T_i \quad \text{whenever} \ |i - j| \geq 2,
\] (2.81)
\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i^2 = (q - 1) T_i + q
\] (2.82)
for \( i, j \in \overline{1,k-1} \), where \( 0 \neq q \in \mathbb{C} \). Let \( \zeta \in \mathbb{C} \) be a fixed constant. According to Section 5 in [HKW], there exists a unique trace map “\( \text{tr} \)” of \( H_k \) such that

\[
\text{tr} (e) = \frac{1}{2}, \quad \text{tr} (a T_m b) = \zeta \text{tr} (ab) \quad \text{for} \ a, b \in H_m
\] (2.83)
with \( m \in \overline{1,k-2} \). This trace map is the key to define the well-known “Jones polynomials” of knots (e.g., cf. [HKW]).

We define a linear transformation \( \tau \) on \( H_k \) by

\[
\tau(T_1 T_2 \cdots T_{r-1} T_r) = T_r T_{r-1} \cdots T_2 T_1.
\] (2.84)

17
Then $\tau$ satisfies (2.53) and (2.54) by (2.81)-(2.83) with certain choice of $\{B_{i,j}\}$, say $n = 1$ and $B_{1,1} = H_k$.

**Example 2.2.** Suppose that $G$ is a group and 1 is its identity element. Let $\mathbb{C}[G]$ be the vector space with a basis $\{\varpi(g) \mid g \in \Gamma\}$, and multiplication:

$$\varpi(g_1) \cdot \varpi(g_2) = \varpi(g_1g_2) \quad \text{for } g_1, g_2 \in \Gamma.$$  \hfill (2.85)

Then $\mathbb{C}[G]$ forms an associative algebra with the identity element $\varpi(1)$, which is called the *group algebra* of $G$. Define the “trace map” of $\mathbb{C}[G]$ by

$$\text{tr} \ \varpi(g) = \delta_{1,g} \quad \text{for } g \in \Gamma$$  \hfill (2.86)

and a linear transformation $\tau$ on $\mathbb{C}[G]$ by

$$\tau(\varpi(g)) = \varpi(g^{-1}) \quad \text{for } g \in \Gamma.$$  \hfill (2.87)

It can be verified that $\tau$ satisfies (2.53) and (2.54) by (2.85)-(2.87) with certain choice of $\{B_{i,j}\}$, say $n = 1$ and $B_{1,1} = \mathbb{C}[G]$.

The representations of $\hat{A}_\ell$ and $\hat{A}_\ell^\tau$ with $A = H_k, \mathbb{C}[G]$ and their related vertex algebras will be studied in our future works. In the rest of this paper, we will deal with the case when $A$ is a matrix algebra.

### 3 Modules Related to General Linear Algebras

In this section, we give detailed constructions of irreducible modules of the Lie algebras $\hat{A}_\ell$ in (2.48) and $\hat{A}_\ell^\tau$ in (2.59) when $A$ is the $n \times n$ matrix algebra, from weighted irreducible modules of a central extension of the general linear Lie algebra of infinite matrices with finite number of nonzero entries.

Recall that $\overline{gl}(\infty)$ be a vector space with a basis $\{\mathcal{E}_{l,k} \mid l, k \in \mathbb{Z}\}$ and the multiplication:

$$\mathcal{E}_{l_1,l_2} \cdot \mathcal{E}_{k_1,k_2} = \delta_{l_2+k_1,0} \mathcal{E}_{l_1,k_2} \quad \text{for } l_1, l_2, k_1, k_2 \in \mathbb{Z}.$$ \hfill (3.1)

Moreover, $\overline{gl}(\infty)$ is isomorphic to the associative algebra of infinite matrices with finite number of nonzero entries. Let $M_{n \times n}(\mathbb{C})$ be the algebra of $n \times n$ matrices with entries in $\mathbb{C}$. Then we have

$$M_{n \times n}(\mathbb{C}) \otimes_\mathbb{C} \overline{gl}(\infty) \cong \overline{gl}(\infty) \text{ as associative algebras.}$$  \hfill (3.2)

Take

$$A = M_{n \times n}(\mathbb{C}) \text{ in last section}$$  \hfill (3.3)
with the sum of diagonal entries as the trace map “tr”. Let \( M_{n \times n}(\mathbb{A}) \) be the algebra of \( n \times n \) matrices with entries in \( \mathbb{A} \) (cf. (2.1)-(2.3)). Again \( E_{i,j} \) is the \( n \times n \) matrix with 1 as its \((i,j)\)-entry and 0 as the others. Recall the Lie algebra

\[
\hat{\mathfrak{gl}}(n, \mathbb{A}) = M_{n \times n}(\mathbb{A}) \oplus \mathbb{C} \kappa
\]  

(3.3)

with the Lie bracket:

\[
\begin{align*}
[t^{m_1} \partial_t^{j_1} E_{i_1,j_1} + \mu_1 \kappa, \ t^{m_2} \partial_t^{j_2} E_{i_2,j_2} + \mu_2 \kappa] \\
= \delta_{j_1,i_2} t^{m_1} \partial_t^{j_1} \cdot t^{m_2} \partial_t^{j_2} E_{i_1,j_2} - \delta_{i_1,j_2} t^{m_2} \partial_t^{j_2} \cdot t^{m_1} \partial_t^{j_1} E_{i_2,j_1} \\
+ (-1)^{r_1} \delta_{i_1,j_2} \delta_{j_1,i_2} \delta_{r_1+r_2} \cdot t^{m_1} \partial_t^{j_1} \cdot t^{m_2} \partial_t^{j_2} \cdot t^{m_2} \partial_t^{j_2} E_{i_1,j_2} \\
- \delta_{i_1,j_2} t^{m_1} \partial_t^{j_1} \cdot t^{m_2} \partial_t^{j_2} \cdot t^{m_1} \partial_t^{j_1} E_{i_2,j_1}
\end{align*}
\]  

(3.4)

for \( i, j \in \mathbb{T}, \ m_1, m_2 \in \mathbb{Z}, \ r_1, r_2 \in \mathbb{N} \) and \( \mu_1, \mu_2 \in \mathbb{C} \). The Lie algebras

\[
\hat{\mathbb{A}} \cong \hat{\mathfrak{gl}}(n, \mathbb{A}).
\]  

(3.5)

Fixed an element \( \vec{l} \in \mathbb{N}^n \). We have the Lie subalgebra

\[
\hat{\mathfrak{gl}}(\vec{l}, \mathbb{A}) = \sum_{i,j=1}^{n} \mathbb{A} \partial_{\vec{l}}^{i,j} E_{i,j} + \mathbb{C} \kappa
\]  

(3.6)

of \( \hat{\mathfrak{gl}}(n, \mathbb{A}) \) of type \( \hat{\mathbb{A}}_{\vec{l}} \) (cf. (2.48)).

Let \( H \) be a toral Cartan subalgebra of a Lie algebra \( \mathcal{G} \). We always denote

\[
H^* = \text{the space of linear functions on } H.
\]  

(3.7)

A \( \mathcal{G} \)-module \( \mathcal{M} \) is called \textit{weighted} if

\[
\mathcal{M} = \bigoplus_{\nu \in T^*} \mathcal{M}_\nu, \quad \mathcal{M}_\nu = \{ u \in \mathcal{M} \mid h(u) = \nu(h)u, \ h \in H \}.
\]  

(3.8)

Recall the algebra

\[
\hat{\mathfrak{gl}}(\infty) = \mathfrak{gl}(\infty) \oplus \mathbb{C} \kappa_0
\]  

(3.9)

with the Lie bracket:

\[
\begin{align*}
[\mathcal{E}_{l_1,l_2} + \mu_1 \kappa_0, \mathcal{E}_{k_1,k_2} + \mu_2 \kappa_0] = \mathcal{E}_{l_1,l_2} \mathcal{E}_{k_1,k_2} - \mathcal{E}_{k_1,k_2} \mathcal{E}_{l_1,l_2} \\
+ \delta_{l_2+k_2} \delta_{l_1+k_1} [H(l_1)H(l_2) - H(k_1)H(k_2)] \kappa_0
\end{align*}
\]  

(3.10)

for \( l_1, l_2, k_1, k_2 \in \mathbb{Z} \) and \( \mu_1, \mu_2 \in \mathbb{C} \) (cf. (2.23)), where \( \kappa_0 \) is a base element. By (2.21)-(2.23) and (3.2), the linear map

\[
\kappa_0 \leftrightarrow \kappa_0, \quad \mathcal{E}_{i,j}(l + 1/2, k - 1/2) \leftrightarrow \mathcal{E}_{ln+i-1/2, kn-j+1/2} \quad \text{for } i, j \in \mathbb{T}, \ l, k \in \mathbb{Z},
\]  

(3.11)
gives a Lie algebra isomorphism between the Lie algebra $\mathcal{A}$ and $\tilde{\mathfrak{gl}}(\infty)$ under our assumption (3.3). Moreover, the subspace

$$\mathcal{T} = \sum_{l \in \mathbb{Z}} \mathbb{C}\mathcal{E}_{l,-l} + \mathbb{C}\kappa_0$$

(3.12)

forms a toral Cartan subalgebra of $\tilde{\mathfrak{gl}}(\infty)$. In fact, the root structures of $\tilde{\mathfrak{gl}}(\infty)$ and $\tilde{\mathfrak{gl}}(\infty)$ are the same. Thus they have exactly the same representation theory. We can take

$$\{\mathcal{E}_{l+1,-l} \mid l \in \mathbb{Z}\}$$

as positive simple root vectors (3.13) and

$$\{\mathcal{E}_{l,-l-1} \mid l \in \mathbb{Z}\}$$

as negative simple root vectors (3.14).

Furthermore,

$$[\mathcal{E}_{l+1,-l}, \mathcal{E}_{l,-l-1}] = \mathcal{E}_{l+1,-l-1} - \mathcal{E}_{l,-l}$$

for $-\frac{1}{2} \neq l \in \mathbb{Z}$ (3.15)

and

$$[\mathcal{E}_{1/2,1/2}, \mathcal{E}_{-1/2,-1/2}] = \mathcal{E}_{1/2,-1/2} - \mathcal{E}_{-1/2,1/2} + \kappa_0.$$ (3.16)

Set

$$\tilde{\mathfrak{gl}}^m(\infty) = \text{Span}\, \{\mathcal{E}_{l_1,-l_2}, \mathcal{E}_{-l_1,l_2} \mid m < l_1, l_2 \in \mathbb{Z}\}$$

for $m \in \mathbb{N}$. (3.17)

Then (3.1) and (3.10) show that $\{\tilde{\mathfrak{gl}}^m(\infty) \mid m \in \mathbb{N}\}$ are Lie subalgebras of $\tilde{\mathfrak{gl}}(\infty)$. Let $\mathcal{M}$ be a weighted $\tilde{\mathfrak{gl}}(\infty)$-module

generated by a subspace $\mathcal{M}_0$ such that $\tilde{\mathfrak{gl}}^m(\infty)(\mathcal{M}_0) = \{0\}$ for some $m \in \mathbb{N}$. (3.18)

Fixed a constant $\iota \in \mathbb{C}$. By Theorem 2.2 and (3.2), we have the following representation $\sigma^\iota_{\mathcal{M}}$ of $\tilde{\mathfrak{gl}}(\mathbf{l}, \mathbb{A})$: $\sigma^\iota_{\mathcal{M}}(\kappa) = \kappa_0$ and

$$\sigma^\iota_{\mathcal{M}}(t^m \partial_i^r E_{i,j}) = \sum_{l \in \mathbb{Z}} (l - \ell)_{r} \mathcal{E}_{(m-r-1)_{n+i-1/2,n-j+1/2} + r \delta_{i,j} \delta_{r,m} \mathfrak{S}_{0,r} \kappa_0}$$

(3.19)

for $m \in \mathbb{Z}$, $r \in \mathbb{N}$ and $i, j \in \overline{1,n}$.

**Theorem 3.1** Suppose $\iota \notin \mathbb{Z}$. Then the representation $\sigma^\iota_{\mathcal{M}}$ of $\tilde{\mathfrak{gl}}(\mathbf{l}, \mathbb{A})$ is irreducible if and only if $\mathcal{M}$ is an irreducible $\tilde{\mathfrak{gl}}(\infty)$-module.

**Proof.** Denote

$$h_{i,r} = \sigma^\iota_{\mathcal{M}}(t^r \partial_i^r E_{i,i}) = \sum_{l \in \mathbb{Z}} (l + \ell)_{r} \mathcal{E}_{ln+i-1/2,n-i+1/2} + \mathfrak{S}_{0,r} \kappa_0$$

(3.20)

for $i \in \overline{1,n}$ and $r \in \mathbb{N} + \ell_i$. Set

$$H = \sum_{i=1}^{n} \sum_{r=0}^{\infty} \mathfrak{C}h_{i,r} \subset \text{End} \mathcal{M},$$

(3.21)
the space of linear transformations on \( M \). As operators on \( M \),

\[
[h_{i,r}, \mathcal{E}_{l^n+j_1-1/2,kn-j_2+1/2}] = [\delta_{i,j_1}(l + \ell)_{r} - \delta_{i,j_2}(k + \ell)_{r}] \mathcal{E}_{l^n+j_1-1/2,kn-j_2+1/2} \quad (3.22)
\]

for \( j_1, j_2 \in \mathbb{Z} \) and \( l, k \in \mathbb{Z} \). Using generating functions, we have:

\[
\begin{align*}
&\sum_{r=0}^{\infty} \frac{1}{r!} h_{i,r+\ell_i} \mathcal{E}_{l^n+j_1-1/2,kn-j_2+1/2} \\
= &\left[ \delta_{i,j_1} \sum_{r=0}^{\infty} \frac{(l + \ell)_{r}}{r!} - \delta_{i,j_2} \sum_{r=0}^{\infty} \frac{(k + \ell)_{r}}{r!} \right] \mathcal{E}_{l^n+j_1-1/2,kn-j_2+1/2} \\
= &\left[ \delta_{i,j_1}(l + \ell)_{\ell_i} \sum_{r=0}^{\infty} \frac{(l + \ell - \ell_i)_{r}}{r!} - \delta_{i,j_2} \sum_{r=0}^{\infty} \frac{(k + \ell)_{r}}{r!} \right] \mathcal{E}_{l^n+j_1-1/2,kn-j_2+1/2} \\
= &\frac{d^{\ell_i}}{d x^{\ell_i}} [\delta_{i,j_1} (x + 1)^{l+\ell} - \delta_{i,j_2} (x + 1)^{k+\ell}] \mathcal{E}_{l^n+j_1-1/2,kn-j_2+1/2}. \quad (3.23)
\end{align*}
\]

Note that when \( \ell \notin \mathbb{Z} \),

\[
\frac{d^{\ell_i}}{d x^{\ell_i}} (x + 1)^{m_1+\ell} = \frac{d^{\ell_i}}{d x^{\ell_i}} (x + 1)^{m_2+\ell} \quad \text{for } m_1, m_2 \in \mathbb{Z} \iff m_1 = m_2 \quad (3.24)
\]

and

\[
\frac{d^{\ell_i}}{d x^{\ell_i}} [(x + 1)^{l_1+\ell} - (x + 1)^{-k_1+\ell}] = \frac{d^{\ell_i}}{d x^{\ell_i}} [(x + 1)^{l_2+\ell} - (x + 1)^{-k_2+\ell}] \quad (3.25)
\]

for \( l_1, l_2, k_1, k_2 \in \mathbb{Z} \), \( l_1 \neq -k_1 \iff l_1 = l_2, \ k_1 = k_2 \). \quad (3.26)

Denote by \( H^* \) the space of linear functions on \( H \). Given \( \rho \in H^* \), we set

\[
\tilde{g}l(\infty)(\rho) = \{ \xi \in \tilde{g}l(\infty) \mid [h, \xi] = \rho(h)\xi \text{ for } h \in H \} \quad (3.27)
\]

and

\[
\mathcal{M}(\rho) = \{ w \in \mathcal{M} \mid h(w) = \rho(h)w \text{ for } h \in H \}. \quad (3.28)
\]

Then

\[
\tilde{g}l(\infty)(0) = \mathcal{T}, \quad \dim \tilde{g}l(\infty)(\rho) = 1 \quad \text{for } 0 \neq \rho \in H^* \quad (3.29)
\]

by (3.23)-(3.26). Moreover,

\[
\tilde{g}l(\infty) = \bigoplus_{\rho \in H^*} \tilde{g}l(\infty)(\rho). \quad (3.30)
\]

Since \( \mathcal{M} \) is a weighted \( \tilde{g}l(\infty) \)-module, we have

\[
\mathcal{M} = \bigoplus_{\rho \in H^*} \mathcal{M}(\rho) \quad (3.31)
\]

by (3.12), (3.20) and (3.21). If \( V \) is a \( \tilde{g}l(\infty) \)-submodule of \( \mathcal{M} \), then \( V \) is a \( \tilde{g}l(\tilde{\ell}, \mathbb{A}) \)-submodule of \( \mathcal{M} \) by (3.19). Suppose that \( U \) is a \( \tilde{g}l(\tilde{\ell}, \mathbb{A}) \)-submodule of \( \mathcal{M} \). Then

\[
U = \bigoplus_{\rho \in H^*} U(\rho), \quad U(\rho) = U \cap \mathcal{M}(\rho). \quad (3.32)
\]
Since
\[\sigma'_{\mathcal{M}}(t^m \partial^r_i E_{i,j})(U(\rho)) = \sum_{l \in \mathbb{Z}} (t - 1)^l \mathcal{E}_{(m-r-l)n+i-1/2,ln+j+1/2}(U(\rho)) + r! \delta_{i,j} \delta_{r,m} \mathfrak{Z}_{0,r} \kappa_0(U(\rho)) \subset U,\] (3.33)
we have
\[\mathcal{E}_{(m-r-l)n+i-1/2,ln+j+1/2}(U(\rho)) \subset U\] (3.34)
for \(m \in \mathbb{Z}, r \in \mathbb{N}, i,j \in \overline{1,n}\) such that \((m,i) \neq (0,j)\) by (3.29). Observe that \(\{\mathcal{E}_{l_1,l_2} | l_1,l_2 \in Z; l_1 \neq -l_2\}\) generates the Lie algebra \(\tilde{\mathfrak{gl}}(\infty)\). Thus (3.33) implies
\[\tilde{\mathfrak{gl}}(\infty)U(\rho) \subset U \quad \text{for } \rho \in H^*.\] (3.35)
Therefore, \(U\) is a \(\tilde{\mathfrak{gl}}(\infty)\)-submodule. \(\square\)

The above theorem shows that we construct a family of irreducible representations \(\{\sigma'_{\mathcal{M}} | \iota \in \mathbb{C} \setminus \mathbb{Z}\}\) from any irreducible weighted \(\tilde{\mathfrak{gl}}(\infty)\)-module \(\mathcal{M}\) satisfying (3.18). Set
\[E_{i,j}(r,z) = \sum_{m \in \mathbb{Z}} t^m \partial^r_i E_{i,j} z^{-m-1} \quad \text{for } i,j \in \overline{1,n}, \ r \in \mathbb{N}.\] (3.36)
By a similar proof as that of Theorem 3.1, we have the following result, which was proved by Ma [M2] in more complicated form when \(\iota = 0\).

**Theorem 3.3.** Suppose \(\iota \in \mathbb{Z}\). Let \(\mathcal{M}\) be a weighted \(\tilde{\mathfrak{gl}}(\infty)\)-module satisfying (3.18). We have the following representation \(\sigma'_{\mathcal{M}}\) of \(\tilde{\mathfrak{gl}}(\vec{\iota}, \mathbb{A})\): \(\sigma'_{\mathcal{M}}(\kappa) = \kappa_0\) and
\[\sigma'_{\mathcal{M}}(E_{i,j}(r,z)) = \sum_{l,k=0}^{n} \langle -k - 1, r \rangle \mathcal{E}_{(l-i)n+i-1/2, (k+i+1)n-j+1/2} z^{-l-k-l_i-r} + \langle k + \ell_j, r \rangle \mathcal{E}_{(l-i)n+i-1/2, (k-i)n-j+1/2} z^{-l+k+l_j-l_i-r} + \langle k + \ell_j, r \rangle \mathcal{E}_{(l+i+1)n+i-1/2, (k-i)n-j+1/2} z^{-l-k+l_j-r} + \langle -k - 1, r \rangle \mathcal{E}_{(l+i+1)n+i-1/2, (k+i+1)n-j+1/2} z^{-l-k-r} \rangle + \delta_{i,j} r! \mathfrak{Z}_{0,r} \kappa_0 z^{-r-1}.\] (3.37)
for \(i,j \in \overline{1,n}\) and \(r \in \mathbb{N} + \ell_j\). Moreover, \(\sigma'_{\mathcal{M}}\) if and only if \(\mathcal{M}\) is irreducible.

Denote
\[i^* = n + 1 - i \quad \text{for } i \in \overline{1,n}.\] (3.38)
We fix \(\epsilon \in \{0,1\}\) and take
\[\vec{\iota} = (\ell_1, \ell_2, \ldots, \ell_n) \in \mathbb{N}^n \quad \text{such that } \{\ell_1, \ell_2, \ldots, \ell_n\} \subset 2\mathbb{N} + \epsilon\] (3.39)
and
\[\ell_i = \ell_{i^*} \quad \text{for } i \in \overline{1,n}.\] (3.40)
For any
\[ \sum_{i,j=1}^{n} \mu_{i,j} E_{i,j} \in A, \]
we define
\[ (\sum_{i,j=1}^{n} \mu_{i,j} E_{i,j})^* = \sum_{i,j=1}^{n} \mu_{i,j} E_{i,j}^*. \] (3.42)

Then * is an involution of A (cf. (2.53), (2.54)). Now the Lie algebra \( \bar{A}^*_\ell \) (cf. (2.59)) becomes
\[ \bar{\varnothing}(\ell, A) = \bar{A}^*_\ell = \sum_{i,j=1}^{\infty} \sum_{r=0}^{\infty} \sum_{m \in \mathbb{Z}} \mathbb{C}(m \partial_{t}^{m+r} E_{i,j} - (-1)^{r} (-\partial)^{r} m_{i}^{\ell} E_{i,j}^*) + \mathbb{C}k. \] (3.43)

Again let \( M \) be a weighted \( \bar{g}(\infty) \)-module satisfying (3.18). Then we have the restriction representation of \( \sigma_{M}^* \) on \( \bar{\varnothing}(\ell, A) \) with \( \sigma_{M}^*(\kappa) = \kappa \) and
\[ \sigma_{M}^*(t^{m+r} \partial_{t}^{m+r} E_{i,j} - (-1)^{r} (-\partial)^{r} m_{i}^{\ell} E_{i,j}^*) = \sum_{(l+\ell, j)_{l, \ell}} \sum_{r=0}^{\infty} \sum_{m \in \mathbb{Z}} \mathbb{C}(t^{l+r} \partial_{t}^{l+r} E_{i,j} - (-1)^{r} (-\partial)^{r} t^{l} E_{i,j}^*) + \mathbb{C}k. \] (3.44)

for \( i, j \in \mathbb{T}, n, m \in \mathbb{Z} \) and \( r \in \mathbb{N} \).

**Theorem 3.3** Suppose \( \ell \notin \mathbb{Z}/2 \). Then the representation \( \sigma_{M}^* \) of \( \bar{\varnothing}(\ell, A) \) is irreducible if and only if \( M \) is an irreducible \( \bar{g}(\infty) \)-module.

**Proof.** For \( i \in \mathbb{T}, n \) and \( r \in \mathbb{N} \), we define
\[ \eta_{i,r} = \sigma_{M}^*(t^{r+\ell_{i}} \partial_{t}^{r+\ell_{i}} E_{i,i} - (-1)^{r} (-\partial)^{r} t^{r+\ell_{i}} E_{i,i}^*) = \sum_{(l+\ell_{i})_{l, \ell}} \sum_{r=0}^{\infty} \sum_{m \in \mathbb{Z}} \mathbb{C}(l^{l+r} \partial_{t}^{l+r} E_{i,i} - (-1)^{r} (-\partial)^{r} l^{l} E_{i,i}^*) + \mathbb{C}k. \] (3.45)

Set
\[ H_{0} = \sum_{i=1}^{n} \sum_{r=0}^{\infty} \mathbb{C} \eta_{i,r} \subset \text{End } M, \] (3.46)
the space of linear transformations on \( M \). As operators on \( M \),
\[ [\eta_{i,r}, E_{l+n+j-1/2,kn-j+2+1/2}] = [\delta_{i,j_{1}}(l+\ell_{i})_{r+\ell_{i}} - \delta_{i,j_{1}}(l+\ell_{i})_{r+\ell_{i}} - \delta_{i,j_{2}}(k+\ell_{i})_{r+\ell_{i}} E_{l+n+j-1/2,kn-j+2+1/2} \]
\[ - \delta_{i,j_{2}}(l+\ell_{i})_{r+\ell_{i}} E_{l+n+j-1/2,kn-j+2+1/2} \] (3.47)
for \( j_{1}, j_{2} \in \mathbb{T}, n \) and \( l, k \in \mathbb{Z} \). Using generating functions, we have:
\[ \sum_{r=0}^{\infty} \sum_{l+\ell_{i}} \eta_{l,r} E_{l+n+j-1/2,kn-j+2+1/2} \]
\[ = \frac{d^{j_{1}}}{dx_{i}^{j_{1}}} [\delta_{i,j_{1}}(x+1)^{l_{i}+\ell_{i}} - \delta_{i,j_{2}}(x+1)^{l_{i}+\ell_{i}+1/l} - \delta_{i,j_{2}}(x+1)^{l_{i}+\ell_{i}} E_{l+n+j-1/2,kn-j+2+1/2} \]
\[ + \delta_{i,j_{2}}(x+1)^{k_{i}+\ell_{i}+1/l} E_{l+n+j-1/2,kn-j+2+1/2}. \] (3.48)
Since \( i \notin \mathbb{Z}/2 \), we have
\[
l + i \neq k - \ell \quad \text{for} \ l, k \in \mathbb{Z}, \ i \in \mathbb{1}, n.
\]
Thus
\[
(x+1)^{l_1+i} - (x+1)^{l_1-i} = (x+1)^{l_2+i} - (x+1)^{l_2-i} \quad \text{for} \ l_1, l_2, k_1, k_2 \in \mathbb{Z} \iff l_1 = l_2, \ k_1 = k_2 \quad (3.50)
\]
and
\[
(x+1)^{l_1+i} - (x+1)^{l_1-i} - (x+1)^{l_2-i} + (x+1)^{k_1+i} = \frac{(x+1)^{k_1+i} + k_1 - 1}{l_1} \quad (3.51)
\]
for \( l_1, k_1, l_2, k_2 \in \mathbb{Z} \) with \( l_1 \neq k_1, l_2 \neq k_2 \) if and only if \( l_1 = l_2 \) and \( k_1 = k_2 \). Denote by \( H_o^* \) the space of linear functions on \( H_o \). Given \( \rho \in H_o^* \), we set
\[
\tilde{gl}(\infty)_{[\rho]} = \{ \xi \in \tilde{gl}(\infty) \mid [h, \xi] = \rho(h)\xi \text{ for } h \in H_o \}\.
\]
Then
\[
\tilde{gl}(\infty)_{[\rho]} = \mathcal{T}, \quad \dim \tilde{gl}(\infty)_{[\rho]} = 1 \quad \text{for} \ 0 \neq \rho \in H_o^*
\]
by the above arguments. Therefore, the conclusion follows the same arguments as those in (3.29)-(3.35).

Next we suppose that
\[
n = 2n_0 \text{ is an even positive integer.} \quad (3.54)
\]
Moreover, we define the parity of indices:
\[
p(i) = 0, \quad p(n_0 + i) = 1 \quad \text{for} \ i \in \mathbb{1}, n_0.
\]
For any element in (3.41), we define
\[
(\sum_{i,j=1}^{n} \mu_{i,j} E_{i,j})^\dagger = \sum_{i,j=1}^{n} (-1)^{p(i)+p(j)} \mu_{i,j} E_{j^\ast,i^\ast}
\]
Then \( \dagger \) is an involution of \( \mathcal{A} \) (cf. (2.53), (2.54)). Now the Lie algebra \( \tilde{\mathcal{A}}_{\ell}^\dagger \) (cf. (2.59)) becomes
\[
\tilde{sp}(\ell, \mathbb{A}) = \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \sum_{m \in \mathbb{Z}} \mathbb{C}(t^m \partial_t^{r+\ell_j} E_{i,j} - (-1)^{p(i)+p(j)+r} (-\partial_t)^r t^m \partial_t^{\ell_j} E_{j^\ast,i^\ast}) + \mathbb{C} \kappa.
\]
Let \( \mathcal{M} \) be a weighted \( \tilde{gl}(\infty) \)-module satisfying (3.18). Then we have the restricted representation of \( \sigma_{\mathcal{M}}^\dagger \) on \( \tilde{sp}(\ell, \mathbb{A}) \) with \( \sigma_{\mathcal{M}}^\dagger(\kappa) = \kappa_0 \) and
\[
\sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \sum_{m \in \mathbb{Z}} \mathbb{C}(t^m \partial_t^{r+\ell_j} E_{i,j} - (-1)^{p(i)+p(j)+r} (-\partial_t)^r t^m \partial_t^{\ell_j} E_{j^\ast,i^\ast})
\]
\[
\sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \sum_{m \in \mathbb{Z}} \mathbb{C}(t^m \partial_t^{r+\ell_j} E_{i,j} - (-1)^{p(i)+p(j)+r} (-\partial_t)^r t^m \partial_t^{\ell_j} E_{j^\ast,i^\ast}) + \mathbb{C} \kappa.
\]

24
for \( i, j \in \mathbb{T}, m \in \mathbb{Z} \) and \( r \in \mathbb{N} \).

**Theorem 3.4.** Suppose \( \iota \notin \mathbb{Z}/2 \). Then the representation \( \sigma_{\mathcal{M}} \) of \( \tilde{\mathfrak{sp}}(\ell, \mathbb{A}) \) is irreducible if and \( \mathcal{M} \) is an irreducible \( \tilde{\mathfrak{gl}}(\infty) \)-module.

**Example 3.1.** Let \( \lambda \in \mathcal{T}^* \) (cf. (3.12)) such that there exists a positive integer \( m_0 \) for which

\[
\lambda(E_{l,-l}) = \lambda(E_{-l,l}) = 0 \quad \text{for} \quad m_0 < l \in \mathbb{Z}.
\]

Moreover, we set

\[
\tilde{\mathfrak{gl}}(\infty)_+ = \text{Span} \{ E_{l,k} \mid l, k \in \mathbb{Z}; l + k > 0 \}, \quad \tilde{\mathfrak{gl}}(\infty)_- = \text{Span} \{ E_{l,k} \mid l, k \in \mathbb{Z}; l + k < 0 \}
\]

and

\[
\tilde{\mathfrak{gl}}(\infty)_0 = \tilde{\mathfrak{gl}}(\infty)_+ + \mathcal{T}
\]

(cf. (3.12)). Then \( \tilde{\mathfrak{gl}}(\infty)_\pm \) and \( \tilde{\mathfrak{gl}}(\infty)_0 \) are Lie subalgebras of \( \tilde{\mathfrak{gl}}(\infty) \). Define a one-dimensional \( \tilde{\mathfrak{gl}}(\infty)_0 \)-module \( \mathbb{C}v_\lambda \) by

\[
\tilde{\mathfrak{gl}}(\infty)_+(v_\lambda) = \{0\}, \quad h(v_\lambda) = \lambda(h)v_\lambda \quad \text{for} \quad h \in \mathcal{T}.
\]

Form an induced \( \tilde{\mathfrak{gl}}(\infty) \)-module

\[
M_\lambda = U(\tilde{\mathfrak{gl}}(\infty)) \otimes_{U(\tilde{\mathfrak{gl}}(\infty)_0)} \mathbb{C}v_\lambda \cong U(\tilde{\mathfrak{gl}}(\infty)_-) \otimes_\mathbb{C} \mathbb{C}v_\lambda,
\]

that is, a Verma module. It is known that \( M_\lambda \) has a unique maximal proper submodule \( N_\lambda \).

Thus

\[
\mathcal{M} = M_\lambda/N_\lambda
\]

is a weighted irreducible \( \tilde{\mathfrak{gl}}(\infty) \)-module satisfying (3.18). Suppose

\[
\lambda_l = \lambda(E_{l+1,-l-1} - E_{l,-l}), \quad \lambda_{-1/2} = \lambda(E_{1/2,-1/2} - E_{-1/2,1/2} + \kappa_0) \in \mathbb{N}
\]

for \( -1/2 \neq l \in \mathbb{Z} \). Then

\[
N_\lambda = \sum_{r \in \mathbb{Z}} U(\tilde{\mathfrak{gl}}(\infty)_-)E_{r,-r-1}^{\lambda_r+1} \otimes v_\lambda.
\]

**Example 3.2.** Let \( m \) be a fixed positive integer. Set

\[
I = \{ i - 1/2, -i + 1/2 \mid i \in \mathbb{N} \}, \quad J = \mathbb{Z} \setminus I.
\]

The subspaces

\[
\mathcal{G} = \text{Span} \{ E_{i,k}, \kappa_0 \mid l, k \in I \} \quad \text{and} \quad \overline{\mathcal{G}} = \text{Span} \{ E_{l,k}, \kappa_0 \mid l, k \in J \}
\]

(3.68)
forms Lie subalgebras of $\tilde{gl}(\infty)$. The algebra $\mathcal{G}$ is isomorphic a one-dimensional central extension of $gl(2m, \mathbb{C})$ with

$$H = \sum_{l \in I} \mathbb{C}E_{l,-l} + \mathbb{C}\kappa_0$$

(3.69)
as a Cartan subalgebra. Moreover, the algebra $\overline{\mathcal{G}}$ is isomorphic $\tilde{gl}(\infty)$ with

$$\overline{\mathcal{H}} = \sum_{l \in J} \mathbb{C}E_{l,-l} + \mathbb{C}\kappa_0$$

(3.70)
as a Cartan subalgebra.

Denote

$$L_0 = \mathcal{G} + \overline{\mathcal{G}}, \quad L_- = \text{Span} \{ E_{l,r} \mid l \in J, r \in I \}, \quad L_+ = \text{Span} \{ E_{r,l} \mid r \in I, l \in J \}.$$ 

(3.71)

Then $L_0$ and $L_\pm$ are Lie subalgebras of $\tilde{gl}(\infty)$. In fact,

$$\tilde{gl}(\infty) = L_- \oplus L_0 \oplus L_+.$$ 

(3.72)

Note

$$\overline{\mathcal{G}}_\pm = \overline{\mathcal{G}} \cap \tilde{gl}(\infty)_\pm$$

(3.73)

(cf. (3.60)) are Lie subalgebras of $\overline{\mathcal{G}}$ and

$$\overline{\mathcal{G}} = \overline{\mathcal{G}}_- \oplus \overline{\mathcal{H}} \oplus \overline{\mathcal{G}}_-.$$ 

(3.74)

In particular, we have a Borel subalgebra

$$\overline{\mathcal{G}}_0 = \overline{\mathcal{H}} + \overline{\mathcal{G}}_+.$$ 

(3.75)

Take any weighted irreducible $\mathcal{G}$-module $M_0$, which may not necessarily be of highest weight type. We extend the action of $\kappa_0$ to that of $\overline{\mathcal{G}}_0$ by

$$\overline{\mathcal{G}}_+(M_0) = \mathcal{E}_{k,-k}(M_0) = \{0\} \quad \text{for} \quad k \in J.$$ 

(3.76)

Form an induced $\mathcal{G}$-module

$$M_1 = U(\overline{\mathcal{G}}) \otimes_{U(\overline{\mathcal{G}}_0)} M_0 \cong U(\overline{\mathcal{G}}_-) \otimes_{\mathbb{C}} M_0.$$ 

(3.77)

Since

$$[\mathcal{G}, \overline{\mathcal{G}}] = \{0\},$$ 

(3.78)

$M_1$ becomes an $L_0$-module by letting $\mathcal{G}$ act on the second factor. Moreover, $M_1$ has a unique maximal proper $L_0$-submodule $M_2$. Form a quotient $L_0$-module

$$\mathcal{M}_0 = M_1/M_2.$$ 

(3.79)
In fact, (3.76) yields
\[ E_{l-k}(M_0 + M_2) = E_{-l,k}(M_0 + M_2) \subset M_2 \quad \text{for} \quad m < l, k \in \mathbb{Z}. \quad (3.80) \]

Note that
\[ [\mathcal{L}_0, \mathcal{L}_\pm] \subset \mathcal{L}_\pm. \quad (3.81) \]
So
\[ \mathcal{L}' = \mathcal{L}_0 + \mathcal{L}_+ \quad (3.82) \]
form a Lie subalgebra of \( \tilde{gl}(\infty) \). We extend an action of \( \mathcal{L}' \) on \( M_0 \) from that of \( \mathcal{L}_0 \) by
\[ \mathcal{L}_+(M_0) = \{0\}. \quad (3.83) \]
The expression (3.81) implies that \( M_0 \) becomes an \( \mathcal{L}' \)-module. Form an induced \( \tilde{gl}(\infty) \)-module:
\[ \mathcal{M}_1 = U(\tilde{gl}(\infty)) \otimes_{U(\mathcal{L}')} M_0 \cong U(\mathcal{L}_-) \otimes_\mathbb{C} M_0. \quad (3.84) \]
It can be verified that \( \mathcal{M}_1 \) has a unique maximal proper \( \tilde{gl}(\infty) \)-submodule \( \mathcal{M}_2 \). The quotient
\[ \mathcal{M} = \mathcal{M}_1/\mathcal{M}_2 \quad (3.85) \]
is a weighted irreducible \( \tilde{gl}(\infty) \)-module satisfying (3.18) by (3.80).

4 Modules with \( \iota \in \mathbb{Z} + 1/2 \) Related to Skew Elements

In this section, we give detailed constructions of irreducible modules of the Lie algebras \( \tilde{A}_\iota^r \) in (2.59) with \( \iota \in \mathbb{Z} + 1/2 \) when \( A \) is the \( n \times n \) matrix algebra, from weighted irreducible modules of central extensions of the Lie algebras of infinite skew matrices with finite number of nonzero entries.

Recall the Lie algebra \( \tilde{gl}(\infty) \) defined in (3.1). The subspaces
\[ \bar{o}_d(\infty) = \sum_{k,l \in \mathbb{Z}} \mathbb{C}(E_{l,k} - E_{k,l}), \quad (4.1) \]
\[ \bar{o}_b(\infty) = \sum_{k,l \in \mathbb{Z}} \mathbb{C}(E_{l,k} - E_{k-1,l+1}), \quad (4.2) \]
and
\[ \bar{sp}(\infty) = \sum_{k,l \in \mathbb{Z}; kl<0} \mathbb{C}(E_{k,l} - E_{l,k}) + \sum_{k,l \in \mathbb{Z}; kl>0} \mathbb{C}(E_{k,l} + E_{l,k}) \quad (4.3) \]
forms Lie subalgebras skew elements in \( \tilde{gl}(\infty) \). Take
\[ \{E_{l+1,-l} - E_{-l,l+1}, E_{3/2,1/2} - E_{1/2,3/2} \mid 0 < l \in \mathbb{Z}\} \quad (4.4) \]
as positive simple root vectors of \( \bar{o}_d(\infty) \) and
\[ \{E_{l,-l-1} - E_{-l-1,l}, E_{-3/2,-3/2} - E_{-1/2,-1/2} \mid 0 < l \in \mathbb{Z}\} \quad (4.5) \]
as negative simple root vectors of $\tilde{\mathfrak{o}}_d(\infty)$. Choose

$$\{\mathcal{E}_{l,1-l} - \mathcal{E}_{-l,l+1} \mid 0 < l \in \mathbb{Z}\} \quad (4.6)$$

as positive simple root vectors of $\tilde{\mathfrak{o}}_b(\infty)$ and

$$\{\mathcal{E}_{l-1,-l} - \mathcal{E}_{-l-1,l} \mid 0 < l \in \mathbb{Z}\} \quad (4.7)$$

as negative simple root vectors of $\tilde{\mathfrak{o}}_b(\infty)$. Pick

$$\{\mathcal{E}_{l+1,-l} - \mathcal{E}_{-l,l+1}, \mathcal{E}_{1/2,1/2} \mid 0 < l \in \mathbb{Z}\} \quad (4.8)$$

as positive simple root vectors of $\mathfrak{osp}(\infty)$ and

$$\{\mathcal{E}_{l,-l-1} - \mathcal{E}_{-l-1,l}, \mathcal{E}_{-1/2,-1/2} \mid 0 < l \in \mathbb{Z}\} \quad (4.9)$$

as negative simple root vectors of $\mathfrak{osp}(\infty)$.

Again we assume $\mathcal{A} = M_{n \times n}(\mathbb{C})$ in the settings of Section 2. Recall the Lie algebra $\check{\mathfrak{A}}^{\tau,\iota}_\ell$ defined (2.69), the assumption (3.39), the involution $\ast$ defined in (3.42) and the involution $\dagger$ defined in (3.56).

**Theorem 4.1.** We have the following Lie algebra isomorphisms:

$$\check{\mathfrak{A}}^{\ast,\iota}_\ell / \mathbb{C}\kappa_0 \cong \begin{cases} 
\tilde{\mathfrak{o}}_b(\infty) & \text{if } \epsilon = 0, \ n \in 2\mathbb{N} + 1, \\
\tilde{\mathfrak{o}}_d(\infty) & \text{if } \epsilon = 0, \ n \in 2\mathbb{N}, \\
\mathfrak{osp}(\infty) & \text{if } \epsilon = 1
\end{cases} \quad (4.10)$$

and

$$\check{\mathfrak{A}}^{\dagger,\iota}_\ell / \mathbb{C}\kappa_0 \cong \begin{cases} 
\mathfrak{osp}(\infty) & \text{if } \epsilon = 0, \\
\tilde{\mathfrak{o}}_d(\infty) & \text{if } \epsilon = 1
\end{cases} \quad (4.11)$$

if $n$ is even.

**Proof.** We write

$$\iota = \iota_0 + 1/2, \quad \iota_0 \in \mathbb{Z}.$$  

(4.12)

By assumption (3.39) and (3.40),

$$\iota_i = \iota_{i^*} = 2m_i + \epsilon \text{ with } m_i \in \mathbb{N} \quad \text{for } i \in \overline{1,n}.$$  

(4.13)

Thus (2.64) and (2.69) give

$$\check{\mathfrak{A}}^{\ast,\iota}_\ell = \sum_{i,j=1}^{n} \sum_{l,k \in \mathbb{Z}} \mathbb{C}((-1)^{\epsilon}(k + m_j + \epsilon - 1)\iota_j E_{i,j}(l + m_i - \iota_0, k + \iota_0 - m_j)$$

$$- \langle l + m_i \rangle \iota_k E_{j^*,i^*}(k + m_j - \iota_0 + \epsilon - 1, l - m_i + \iota_0 + 1 - \epsilon) + \mathbb{C}\kappa_0, \quad (4.14)$$

and if $n = 2n_0$ is even,

$$\check{\mathfrak{A}}^{\dagger,\iota}_\ell = \sum_{i,j=1}^{n} \sum_{l,k \in \mathbb{Z}} \mathbb{C}((-1)^{\epsilon}(k + m_j + \epsilon - 1)\iota_j E_{i,j}(l + m_i - \iota_0, k + \iota_0 - m_j)$$

$$- (-1)^{p(i)+p(j)} \times \langle l + m_i \rangle \iota_k E_{j^*,i^*}(k + m_j - \iota_0 + \epsilon - 1, l + \iota_0 - m_i + 1 - \epsilon) + \mathbb{C}\kappa_0 \quad (4.15)$$
(cf. (3.55)). According to (3.11), we set
\[ G_0^{*,\tilde{\mu}} = \sum_{i,j=1}^n \sum_{l,k \in \mathbb{Z}} \mathbb{C}\langle -1 \rangle^{i+j} \langle k+m_j+\epsilon -3/2 \rangle \ell_i \mathcal{E}_{l+m_i-l_0} = 0 \quad \text{if} \quad \kappa \text{ is even}. \]

if \( n \) is even. Then \( G_0^{*,\tilde{\mu}} \) and \( G_0^{\dagger,*\tilde{\mu}} \) are Lie subalgebras of \( \tilde{\mathfrak{gl}}(\infty) \). The conclusion follows from (3.11). Define
\[ G_0^{*,\tilde{\mu}} = \sum_{i,j=1}^n \sum_{l,k \in \mathbb{Z}} \mathbb{C}\langle -1 \rangle^{i+j} \langle k+m_j+\epsilon -3/2 \rangle \ell_i \mathcal{E}_{l+m_i-l_0}, \]

and
\[ G_0^{\dagger,*\tilde{\mu}} = \sum_{i,j=1}^n \sum_{l,k \in \mathbb{Z}} \mathbb{C}\langle -1 \rangle^{i+j} \langle k+m_j+\epsilon -3/2 \rangle \ell_i \mathcal{E}_{l+m_i-l_0}, \]

if \( n \) is even. Then \( G_0^{*,\tilde{\mu}} \) and \( G_0^{\dagger,*\tilde{\mu}} \) are Lie subalgebras of \( \tilde{\mathfrak{gl}}(\infty) \). The conclusion follows from (3.11) and the following facts:
\[ G_0^{*,\tilde{\mu}} / \mathbb{C}\kappa_0 \cong \tilde{\mathfrak{g}}_0^* \cong \tilde{\mathfrak{b}}_0(\infty) \quad \text{if} \quad n \text{ is odd}, \]
\[ G_0^{*,\tilde{\mu}} / \mathbb{C}\kappa_0 \cong \tilde{\mathfrak{g}}_0^* \cong \tilde{\mathfrak{d}}_0(\infty) \quad \text{if} \quad n \text{ is even}, \]
\[ G_0^{*,\tilde{\mu}} / \mathbb{C}\kappa_0 \cong \tilde{\mathfrak{g}}_0^{\dagger,*\tilde{\mu}} \cong \tilde{\mathfrak{p}}_0^*(\infty), \quad G_0^{*,\tilde{\mu}} / \mathbb{C}\kappa_0 \cong \tilde{\mathfrak{g}}_0^{\dagger,*\tilde{\mu}} \cong \tilde{\mathfrak{p}}_0^*(\infty), \]
\[ G_0^{*,\tilde{\mu}} / \mathbb{C}\kappa_0 \cong \tilde{\mathfrak{g}}_0^{\dagger,*\tilde{\mu}} \cong \tilde{\mathfrak{d}}_0(\infty). \]

Next we want to study highest weight irreducible modules. Isomorphisms in (4.21)-(4.24) motivate us to adjust the definition of the Lie algebra \( \tilde{\mathfrak{gl}}(\infty) \) (cf. (3.9) and (3.10)) by modifying the coefficient of \( \kappa_0 \) in (3.10). Moreover, for any \( l \in \mathbb{Z} \), we write
\[ l = l_Q n + l_R, \quad l_Q \in \mathbb{Z}, \quad l_R \in \mathbb{Z}. \]
Let \( \tilde{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n \) and define a map \( \alpha_{\tilde{m}}^{(n)} : \mathbb{Z}^4 \to \mathbb{C} \) by:

\[
\alpha_{\tilde{m}}^{(n)}(l_1, l_2; k_1, k_2) = (H(l_1 + (m_{(l_1+1)/2})_R - n_0)H(l_2 + (n_0 - m_{(-l_2+1)/2})_R)n)
- H(k_1 + (m_{(k_1+1)/2})_R - n_0)H(k_2 + (n_0 - m_{(-k_2+1)/2})_R)n)\delta_{l_1+k_2,0}\delta_{l_2+k_1,0}
\]

(4.26)

for \( l_1, l_2, k_1, k_2 \in \mathbb{Z} \) (cf. (1.15)). Set

\[
\tilde{g}_{\tilde{m}}^{(n)}(\infty) = \tilde{g}_{\infty} + \mathbb{C} \kappa_0
\]

(4.27)

(cf. (3.1)), where \( \kappa_0 \) is a base element. We have the following Lie bracket on \( \tilde{g}_{\tilde{m}}^{(n)}(\infty) \):

\[
[\mathcal{E}_{l_1,l_2} + \mu_1 \kappa_0, \mathcal{E}_{k_1,k_2} + \mu_2 \kappa_0] = \mathcal{E}_{l_1,l_2} \mathcal{E}_{k_1,k_2} - \mathcal{E}_{k_1,k_2} \mathcal{E}_{l_1,l_2} + \alpha_{\tilde{m}}^{(n)}(l_1, l_2; k_1, k_2) \kappa_0
\]

(4.28)

for \( l_1, l_2, k_1, k_2 \in \mathbb{Z} \). In particular, the Lie algebras

\[
\tilde{g}_{\infty}^{(n)}(\infty) \cong \tilde{g}_{\infty}(\infty).
\]

Assume that (4.13) holds. Now we define

\[
\mathcal{L}_{\tilde{m}}^{(n)} = \sum_{i,j=1}^{n} \sum_{l,k \in \mathbb{Z}} \mathbb{C}((-1)^{\epsilon}(k + m_j + \epsilon - 3/2)\delta_{l+j,i-1/2,k-k-1/2} + \mathbb{C} \kappa_0,
\]

(4.30)

and

\[
\mathcal{L}_{\tilde{m}}^{(n)} = \sum_{i,j=1}^{n} \sum_{l,k \in \mathbb{Z}} \mathbb{C}((-1)^{\epsilon}(k + m_j + \epsilon - 3/2)\delta_{l+j,i-1/2,k-k-1/2} - (-1)^{i+j} + \mathbb{C} \kappa_0,
\]

(4.31)

if \( n \) is even. Then \( \mathcal{L}_{\tilde{m}}^{(n)} \) and \( \mathcal{L}_{\tilde{m}}^{(n)} \) are Lie subalgebras of \( \tilde{g}_{\tilde{m}}^{(n)}(\infty) \). Moreover,

\[
\mathcal{A}_{\tilde{m}}^{(n)} \cong \mathcal{L}_{\tilde{m}}^{(n)} \mathcal{L}_{\tilde{m}}^{(n)} \cong \mathcal{L}_{\tilde{m}}^{(n)} \mathcal{L}_{\tilde{m}}^{(n)}.
\]

Write

\[
n = 2n_0 + \epsilon, \quad n_0 \in \mathbb{N}, \quad \epsilon \in \{0, 1\}.
\]

(4.33)

We denote

\[
\epsilon^{(n)}_{\epsilon, l} = \frac{1}{\langle m_{(l+1)/2} - lQ + \epsilon - 3/2 \rangle_{lR} \langle m_{(l+1)/2} + (l+1)Q + 1/2 \rangle_{(l+1)/2}}
\]

(4.34)

and

\[
f^{(n)}_{\epsilon, l} = \langle m_{(l+1)/2} - (l+1)Q + \epsilon - 3/2 \rangle_{(l+1)/2} \mathcal{E}_{l-2, -l+1/2}
\]

(4.35)
for \( l \in \mathbb{N} + \delta_{\epsilon, 0}(\delta_{\epsilon, 0} - n_0) + \delta_{\epsilon, 1} \). When \( \epsilon = \varepsilon = 0 \), we define
\[
e_{0, -n_0}^* = \frac{1}{\langle m_{n_0} - 1/2 \rangle \ell_{n_0} \langle m_{n_0+2}^R + (n_0 + 2)Q - 1/2 \rangle \ell_{(n_0+2)^R}} \\
\times [\langle m_{n_0} - 1/2 \rangle \ell_{n_0} \mathcal{E}_{-n_0+3/2, n_0+1/2} \\
- \langle m_{(n_0+2)^R} + (n_0 + 2)Q - 1/2 \rangle \ell_{(n_0+2)^R} \mathcal{E}_{-n_0+1/2, n_0+3/2}] \quad (4.36)
\]

and
\[
f_{0, -n_0}^* = \langle m_{(n_0+2)^R} - (n_0 + 2)Q - 1/2 \rangle \ell_{(n_0+2)^R} \mathcal{E}_{-n_0-1/2, n_0-3/2} \\
- \langle m_{n_0} - 1/2 \rangle \ell_{n_0} \mathcal{E}_{-n_0-3/2, n_0-1/2} \quad (4.37)
\]

Furthermore, we let
\[
e_{1, 0}^* = \mathcal{E}_{1/2, 1/2}, \quad f_{1, 0}^* = \mathcal{E}_{-1/2, -1/2} \quad (4.38)
\]

Suppose \( \varepsilon = 0 \). We define
\[
e_{\epsilon, l}^\dagger = \frac{1}{\langle m_{lR} - lQ + \epsilon - 3/2 \rangle \ell_{lR} \langle m_{(l+1)^R} + (l + 1)Q + 1/2 \rangle \ell_{(l+1)^R}} \\
\times [(-1)^f \langle m_{lR} - lQ + \epsilon - 3/2 \rangle \ell_{lR} \mathcal{E}_{l+1, -l-1/2} - (-1)^p(l+1)^R + p(lR) \\
\times \langle m_{(l+1)^R} + (l + 1)Q + 1/2 \rangle \ell_{(l+1)^R} \mathcal{E}_{l+(\epsilon - 1)n+1/2, l+(1-\epsilon)n+1/2}] \quad (4.39)
\]

and
\[
f_{\epsilon, l}^\dagger = (-1)^f \langle m_{(l+1)^R} - (l + 1)Q + \epsilon - 3/2 \rangle \ell_{(l+1)^R} \mathcal{E}_{l-1, -l-1/2} \\
- (-1)^p(l+1)^R + p(lR) \langle m_{lR} + lQ + 1/2 \rangle \ell_{lR} \mathcal{E}_{l+(\epsilon - 1)n-1/2, l+(1-\epsilon)n-1/2} \quad (4.40)
\]

for \( l \in \mathbb{N} + 1 - n_0 \delta_{\epsilon, 0} \). Moreover, we set
\[
e_{0, -n_0}^\dagger = \mathcal{E}_{-n_0+1/2, n_0+1/2}, \quad f_{0, 0}^\dagger = \mathcal{E}_{-n_0-1/2, n_0-1/2} \quad (4.41)
\]

Furthermore, we let
\[
e_{1, 0}^\dagger = \frac{1}{\langle m_1 + 1/2 \rangle \ell_1 \langle m_2 + 2Q + 1/2 \rangle \ell_2^R} \\
\times [\langle m_1 + 1/2 \rangle \ell_1 \mathcal{E}_{3/2, 1/2} + (-1)^{p(1)+p(2)} \\
\times \langle m_2 + 2Q + 1/2 \rangle \ell_2^R \mathcal{E}_{1/2, 3/2}] \quad (4.42)
\]

and
\[
f_{1, 0}^\dagger = \langle m_2 - 2Q - 1/2 \rangle \ell_2^R \mathcal{E}_{-1/2, -3/2} \\
+ (-1)^{p(1)+p(2)} \langle m_1 + 3/2 \rangle \ell_1 \mathcal{E}_{-3/2, -1/2} \quad (4.43)
\]

For convenience, we always assume
\[
\tau \in \{*, \dagger\} \quad (4.44)
\]
Under the above settings,
\[ \{ e_{\epsilon,l}^r | l \in \mathbb{N} - \delta_{0,\epsilon} n_0 \} \text{ is set of positive simple root vectors of } \mathcal{L}^r_{\epsilon_0,\epsilon} \] (4.45)
and
\[ \{ f_{\epsilon,l}^r | l \in \mathbb{N} - \delta_{0,\epsilon} n_0 \} \text{ is set of negative simple root vectors of } \mathcal{L}^{\tau,\epsilon}_{\epsilon_0,\epsilon}. \] (4.46)

Set
\[ \vartheta_{\epsilon,l}^r = \mathcal{E}_{l-1/2,-l+1/2} - \mathcal{E}_{l+(\epsilon-1)n+1/2,l+(1-\epsilon)n-1/2} \] (4.47)
for \( l \in \mathbb{N} - n_0 \delta_{\epsilon,0} \). Define
\[ \omega_{\epsilon,l}^r = \alpha_{m}^{\epsilon}(l + 1/2, -l - 1/2; l - 1/2, -l - 1/2) + \alpha_{m}^{\epsilon}(-l + (\epsilon - 1)n + 1/2, l + (1 - \epsilon)n - 1/2) \] (4.48)
for \( l \in \mathbb{N} + \delta_{\epsilon,0}(\delta_{\epsilon,0} - n_0) + \delta_{\epsilon,1} \) if \( \tau = * \), and \( l \in \mathbb{N} + 1 - n_0 \delta_{\epsilon,0} \) when \( \tau = \dagger \). Moreover, we let
\[ \omega_{0,-n_0}^* = \alpha_{m}^{\epsilon}(-n_0 + 3/2, n_0 + 1/2; -n_0 - 1/2, n_0 - 3/2) + \alpha_{m}^{\epsilon}(-n_0 + 1/2, n_0 + 3/2; -n_0 - 3/2, n_0 - 1/2) \] (4.49)
when \( \epsilon = \varepsilon = 0 \),
\[ \omega_{1,0}^* = \alpha_{m}^{\epsilon}(1/2, 1/2; -1/2, -1/2), \] (4.50)
\[ \omega_{0,-n_0}^\dagger = \alpha_{m}^{\epsilon}(-n_0 + 1/2, n_0 + 1/2; -n_0 - 1/2, n_0 - 1/2) \] (4.51)
and
\[ \omega_{1,0}^\dagger = \alpha_{m}^{\epsilon}(3/2, 1/2; -1/2, -3/2) + \alpha_{m}^{\epsilon}(1/2, 3/2; -3/2, -1/2). \] (4.52)

Set
\[ T_{\epsilon,l}^r = \vartheta_{\epsilon,l+1}^r - \vartheta_{\epsilon,l}^r + \omega_{\epsilon,l}^r \kappa_0 \] (4.53)
for \( l \in \mathbb{N} + \delta_{\epsilon,0}(\delta_{\epsilon,0} - n_0) + \delta_{\epsilon,1} \) if \( \tau = * \), and \( l \in \mathbb{N} + 1 - n_0 \delta_{\epsilon,0} \) when \( \tau = \dagger \). Moreover, we set
\[ T_{0,-n_0}^* = \vartheta_{-n_0}^0 + \vartheta_{-n_0+2}^0 + \omega_{0,-n_0}^* \kappa_0 \] (4.54)
when \( \epsilon = \varepsilon = 0 \),
\[ T_{1,0}^* = \vartheta_{1}^1 + \omega_{1,0}^0 \kappa_0, \] \[ T_{0,-n_0}^\dagger = \vartheta_{-n_0+1}^0 + \omega_{0,-n_0}^\dagger \kappa_0 \] (4.55)
and
\[ T_{1,0}^\dagger = \vartheta_{1}^1 + \vartheta_{1}^2 + \omega_{1,0}^\dagger \kappa_0. \] (4.56)

It can be verified that
\[ [e_{\epsilon,l}^r, f_{\epsilon,l}^r] = T_{\epsilon,l}^r \quad \text{for } l \in \mathbb{N} - \delta_{0,\epsilon} n_0. \] (4.57)

Obviously
\[ [T_{\epsilon,l}^r, e_{\epsilon,l}^r] = 2e_{\epsilon,l}^r, \quad [T_{\epsilon,l}^r, f_{\epsilon,l}^r] = -2f_{\epsilon,l}^r \quad \text{for } l \in \mathbb{N} - \delta_{0,\epsilon} n_0. \] (4.58)
Set

\[ \mathcal{T}^\epsilon = \sum_{l=-n_0\delta_{\epsilon,0}}^{\infty} \mathbb{C} \vartheta_l^\epsilon + \mathbb{C} \kappa_0. \] (4.59)

Moreover, we let

\[ \mathcal{L}_{t_0,\epsilon,\pm}^{r,\bar{m}} = \sum_{i,j=1}^{n} \sum_{l,k \in \mathbb{Z}^2; |l+k|>0} \mathbb{C}((-1)^{\epsilon}(k + m_j + \epsilon - 3/2)\ell_j \mathcal{E}_{l+n-i-1/2,kn-j+1/2} - (l + m_i + 1/2)\ell_i \mathcal{E}_{k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2/} \] (4.60)

and

\[ \mathcal{L}_{t_0,\epsilon,\pm}^{\dagger,\bar{m}} = \sum_{i,j=1}^{n} \sum_{l,k \in \mathbb{Z}^2; |l+k|>0} \mathbb{C}((-1)^{\epsilon}(k + m_j + \epsilon - 3/2)\ell_j \mathcal{E}_{l+n-i-1/2,kn-j+1/2} - (l + m_i + 1/2)\ell_i \mathcal{E}_{k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2/} \times (l + m_i + 1/2)\ell_i \mathcal{E}_{k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2). \] (4.61)

Then \( \mathcal{L}_{t_0,\epsilon,\pm}^{r,\bar{m}} \) are Lie subalgebras of \( \mathcal{L}_{t_0,\epsilon}^{r,\bar{m}}, \mathcal{T}^\epsilon \) is a toral Cartan subalgebra of \( \mathcal{L}_{t_0,\epsilon}^{r,\bar{m}} \) and

\[ \mathcal{L}_{t_0,\epsilon}^{r,\bar{m}} = \mathcal{L}_{t_0,\epsilon,-}^{r,\bar{m}} \oplus \mathcal{T}^\epsilon \oplus \mathcal{L}_{t_0,\epsilon,\dagger}^{r,\bar{m}}. \] (4.62)

Furthermore, we have Borel subalgebras:

\[ \mathcal{L}_{t_0,\epsilon,0}^{r,\bar{m}} = \mathcal{T}^\epsilon + \mathcal{L}_{t_0,\epsilon,\dagger}^{r,\bar{m}} \] (4.63)

of \( \mathcal{L}_{t_0,\epsilon}^{r,\bar{m}} \).

Denote by \( (\mathcal{T}^\epsilon)^* \) the space of linear functions on \( \mathcal{T}^\epsilon \). Fix an element

\[ k_0 \in \mathbb{N} - n_0\delta_{0,\epsilon}. \] (4.64)

Take \( \lambda \in (\mathcal{T}^\epsilon)^* \) such that

\[ \lambda(\vartheta_l^\epsilon) = 0 \quad \text{for} \quad k_0 \leq l \in \mathbb{N} - n_0\delta_{0,\epsilon}. \] (4.65)

Define a one-dimensional \( \mathcal{L}_{t_0,\epsilon,0}^{r,\bar{m}} \)-module \( \mathbb{C} v_{\lambda}^{r,\epsilon} \) by

\[ \mathcal{L}_{t_0,\epsilon,\dagger}^{r,\bar{m}}(v_{\lambda}^{r,\epsilon}) = \{0\}, \quad h(v_{\lambda}^{r,\epsilon}) = \lambda(h)v_{\lambda}^{r,\epsilon} \quad \text{for} \quad h \in \mathcal{T}^\epsilon. \] (4.66)

Form an induced \( \mathcal{L}_{t,\epsilon}^{r,\bar{m}} \)-module:

\[ M_{\lambda}^{r,\epsilon} = \mathcal{U}(\mathcal{L}_{t_0,\epsilon}^{r,\bar{m}}) \otimes_{\mathcal{U}(\mathcal{L}_{t_0,\epsilon,0}^{r,\bar{m}})} \mathbb{C} v_{\lambda}^{r,\epsilon} \cong \mathcal{U}(\mathcal{L}_{t_0,\epsilon,\dagger}^{r,\bar{m}}) \otimes_{\mathbb{C}} \mathbb{C} v_{\lambda}^{r,\epsilon}. \] (4.67)

There exists a unique maximal proper submodule \( N_{\lambda}^{r,\epsilon} \) of \( M_{\lambda}^{r,\epsilon} \), and the quotient module

\[ \mathcal{M}_{\lambda}^{r,\epsilon} = M_{\lambda}^{r,\epsilon} / N_{\lambda}^{r,\epsilon} \] (4.68)

is a weighted irreducible \( \mathcal{L}_{t_0,\epsilon}^{r,\bar{m}} \)-module. Identify \( 1 \otimes v_{\lambda}^{r,\epsilon} \) with \( v_{\lambda}^{r,\epsilon} \). If

\[ \lambda_l^\epsilon = \lambda(T_{\epsilon,l}^\epsilon) \in \mathbb{N} \quad \text{for} \quad l \in \mathbb{N} - n_0\delta_{0,\epsilon}, \] (4.69)
then
\[ N^*_{\lambda} = \sum_{l=-\infty}^{\infty} U(\mathcal{L}_{\mathcal{T}_{0,\epsilon}}^{\sigma}) (f^*_{\epsilon, l})^{\lambda^*+1} v^*_{\lambda}. \] (4.70)

Set
\[ \hat{l} = |l_0| + 1 + \max \{ m_1, m_2, \ldots, m_n \}. \] (4.71)

For \( \hat{l} < s \in \mathbb{N} \), we define
\[ \mathcal{L}_{\ell, \epsilon}^{*, \sigma, s} = \sum_{i,j=1}^{N} \sum_{l, k \in \mathbb{N}(N+s)} \mathcal{C}((-1)^i (k + m_j + \epsilon - 3/2)\partial_{l_i} \mathcal{E}_{(n+1/2, k-n-j+1/2)} \]
\[ - (l + m_i + 1/2)\epsilon \mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) \] (4.72)

and
\[ \mathcal{L}_{\ell, \epsilon}^{t, \sigma, s} = \sum_{i,j=1}^{N} \sum_{l, k \in \mathbb{N}(N+s)} \mathcal{C}((-1)^i (k + m_j + \epsilon - 3/2)\partial_{l_i} \mathcal{E}_{(n+1/2, k-n-j+1/2)} \]
\[ - (-1)^{p(i)+p(j)} (l + m_i + 1/2)\epsilon \mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) \] (4.73)

Then \( \mathcal{L}_{\ell, \epsilon}^{*, \sigma, s} \) is a Lie subalgebra of \( \mathcal{L}_{\ell, \epsilon}^{*, \sigma} \) by (4.16)-(4.26).

Suppose that \( \mathcal{M} \) is an \( \mathcal{L}_{\ell, \epsilon}^{*, \sigma} \)-module

generated by a subspace \( \mathcal{M}_0 \) such that \( \mathcal{L}_{\ell, \epsilon}^{*, \sigma, s}(\mathcal{M}_0) = \{ 0 \} \) for some \( \hat{l} < s \in \mathbb{N} \). (4.74)

For instance, the above module \( \mathcal{M}_{\sigma, \epsilon}^{*, \sigma, s} \) is such an module with \( \mathcal{M}_0 = \mathbb{C} \mathcal{V}_{\lambda}^{*, \epsilon} \) and \( s = \max \{ \hat{l}, k_0 + 2 \} \).

We can also construct \( \mathcal{M} \) as Example 3.2. According to (2.60), (2.64) and (2.73), we define a representation \( \sigma_\epsilon \) of \( \mathcal{O}(\mathbb{C}, \mathbb{C}) \) and a representation \( \sigma_\mathcal{M} \) of \( \mathcal{O}(\tilde{\mathcal{M}}, \mathbb{C}) \) on \( \mathcal{M} \) as follows: \( \sigma_\epsilon(\kappa) = \kappa_0, \sigma_\mathcal{M}(\kappa) = \kappa_0 \).

\[ \sigma_\epsilon(t^{k+m_i+m_j+r+\epsilon} \partial_{l_i} E_{i,j} - (-\partial_{l_i}) t^{k+m_i+m_j+r+\epsilon} \partial_{l_i} E_{i,j}) \]
\[ = \sum_{l \in \mathbb{Z}} (l - m_j - \epsilon + 1/2)^r ((-1)^i (l + m_j + \epsilon - 3/2)\partial_{l_i} \mathcal{E}_{(n+1/2, k-n-j+1/2)} \]
\[ - (l + m_i + 1/2)\epsilon \mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) \]
\[ + ((r + l_i)! 3_{0,r+\ell_i} - r l_i! 3_{r,\ell_i}) \delta_{k+\epsilon,0} \delta_{i,j} \kappa_0 \] (4.75)

and
\[ \sigma_\mathcal{M}(t^{k+m_i+m_j+r+\epsilon} \partial_{l_i} E_{i,j} - (-1)^{p(i)+p(j)} (-\partial_{l_i}) t^{k+m_i+m_j+r+\epsilon} \partial_{l_i} E_{i,j}) \]
\[ = \sum_{l \in \mathbb{Z}} (l - m_j - \epsilon + 1/2)^r ((-1)^i (l + m_j + \epsilon - 3/2)\partial_{l_i} \mathcal{E}_{(n+1/2, k-n-j+1/2)} \]
\[ - (l + m_i + 1/2)\epsilon \mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) \]
\[ + ((r + l_i)! 3_{0,r+\ell_i} - r l_i! 3_{r,\ell_i}) \delta_{k+\epsilon,0} \delta_{i,j} \kappa_0. \] (4.76)

**Theorem 4.2.** Suppose that \( \mathcal{M} \) is a weighted \( \mathcal{L}_{\ell, \epsilon}^{*, \sigma} \)-module satisfying (4.74). Then the representation \( \sigma_\mathcal{M} \) is irreducible if and only if \( \mathcal{M} \) is irreducible.
Proof. We only prove the statement for $\sigma_s$ when $\{\ell_s = 2m_s \mid s \in \Gamma, n\}$ and $n = 2n_0$ are even. The other cases can be proved similarly. Define

$$h_{i,r} = \sigma_s(i^{-r} \partial^r \partial_i E_{i,i} - (-\partial)_i^r i^{-r} \partial^r E_{i,i}) = \sum_{l \in \mathbb{Z}} (l + m + 1/2) 2m_i + r E_{i,n+i-1/2,n-i+1/2} - E_{-(t+1)n-i+1/2,(t+1)n+i-1/2} + (r + \ell_i)! \delta_{0,r+\ell_i} - r! E_{i,r} \kappa_0$$

(4.77)

for $i \in \Gamma, n$ and $r \in \mathbb{N}$ by (4.75). Set

$$H = \sum_{i=1}^{n} \sum_{r=0}^{\infty} C h_{i,r} \subset \text{End} \mathcal{M},$$

(4.78)

the space of linear transformations on $\mathcal{M}$. As operators on $\mathcal{M}$,

$$[h_{i,r}, (k + m_j - 3/2) \ell_j \varepsilon_{n+j-1/2,n-j+1/2}$$

$$- (l + m_j + 1/2) \ell_j \varepsilon_{(k-1)n-j+1/2,(l+1)n+j-1/2}]$$

$$= \left[ \delta_{i,j} (l + m_i + 1/2) \ell_i + r - \delta_{i,j} (l + m_i - 3/2) \ell_i + r + \delta_{i,j} (k + m_j - 3/2) \ell_j + r$$

$$- \delta_{i,j} (k + m_j + 1/2) \ell_j + r \right] (k + m_j - 3/2) \ell_j \varepsilon_{(l+j-1/2,n-j+1/2}$$

$$- (l + m_j + 1/2) \ell_j \varepsilon_{(k-1)n-j+1/2,(l+1)n+j-1/2}.$$ 

(4.79)

Using generating functions, we get

$$\sum_{r=0}^{\infty} h_{i,r} x^r, (k + m_j - 3/2) \ell_j \varepsilon_{n+j-1/2,n-j+1/2}$$

$$- (l + m_j + 1/2) \ell_j \varepsilon_{(k-1)n-j+1/2,(l+1)n+j-1/2}]$$

$$= \frac{d^{\ell_i}}{dx^{\ell_i}} \left[ \delta_{i,j} (1 + x)^{-l+m_i+1/2} - \delta_{i,j} (1 + x)^{-l+m_i-3/2} + \delta_{i,j} (1 + x)^{k+m_j-3/2}$$

$$- \delta_{i,j} (1 + x)^{k+m_j+1/2} \right] (k + m_j - 3/2) \ell_j \varepsilon_{n+j-1/2,n-j+1/2}$$

$$- (l + m_j + 1/2) \ell_j \varepsilon_{(k-1)n-j+1/2,(l+1)n+j-1/2}.$$ 

(4.80)

Denote by $H^*$ the space of linear functions on $H$. Given $\rho \in H^*$, we set

$$(\mathcal{L}_{i_0,0}^{*,\tilde{m}}(\rho)) = \{ \xi \in \mathcal{L}_{i_0,0}^{*,\tilde{m}} \mid [h, \xi] = \rho(h) \xi \text{ for } h \in H \}.$$ 

(4.81)

Then

$$(\mathcal{L}_{i_0,0}^{*,\tilde{m}}) = \mathcal{T}, \quad \dim(\mathcal{L}_{i_0,0}^{*,\tilde{m}}(\rho)) = 1 \quad \text{for } 0 \neq \rho \in H^*$$

(4.82)

by (4.80). Moreover,

$$\mathcal{L}_{i_0,0}^{*,\tilde{m}} = \bigoplus_{\rho \in H^*} (\mathcal{L}^{*,\tilde{m}}_{i,0})_{(\rho)},$$ 

(4.83)

The conclusion can be proved exactly as the proof of Theorem 3.1. $\square$
5 Modules with $\ell \in \mathbb{Z}$ Related to Skew Elements

In this section, we give detailed constructions of irreducible modules of the Lie algebras $\hat{\mathcal{A}}_{\ell}^*$ in (2.59) with $\ell \in \mathbb{Z}$ when $\mathcal{A}$ is the $n \times n$ matrix algebra, from weighted irreducible modules of central extensions of the Lie algebras of infinite skew matrices with finite number of nonzero entries.

Recall the Lie algebra $\mathcal{D}_{\ell}^{*,\ell}$ defined in (2.76), and the Lie algebra $\hat{\mathfrak{gl}}(\infty)$ defined in (3.9) and (3.10). Note

$$
\mathcal{D}_{\ell}^{*,\ell} = \sum_{i,j=1}^{n} \mathcal{C}((-1)^{i}(-k-1/2)\ell_j E_{i,j}(-l-\ell_{i}, -k+\ell_{j}) - (-l-1/2)\ell_i,
\times E_{j,i}(-k-\ell_{j}, l+\ell_{i}) + \mathcal{C}((-1)^{i}(-k-1/2)\ell_j E_{i,j}(l+\ell_{i}, -k+\ell_{j})
- (-\ell_{i} + l - 1/2)\ell_j E_{j,i}(-k-\ell_{j}, l+\ell_{i}) + \mathcal{C}((-1)^{i}(-l-1)\ell_{j} E_{i,j}(l+\ell_{i}+k+\ell_{j}) + \mathcal{C}\kappa_0
$$

(5.1)

and

$$
\hat{\mathcal{D}}_{\ell}^{*,\ell} = \sum_{i,j=1}^{n} \mathcal{C}((-1)^{i}(-k-1/2)\ell_j E_{i,j}(-l-\ell_{i}, -k+\ell_{j})
- (-\ell_{i} + l - 1/2)\ell_j E_{j,i}(-k-\ell_{j}, l+\ell_{i}) + \mathcal{C}((-1)^{i}(-l-1)\ell_{j} E_{i,j}(l+\ell_{i}+k+\ell_{j}) + \mathcal{C}\kappa_0
$$

(5.2)

if $n$ is even. By (3.11), we set

$$
\mathcal{G}_{\ell}^{*,\ell} = \sum_{i,j=1}^{n} \sum_{l,k=0}^{\infty} \mathcal{C}((-1)^{i}(-k-1)\ell_j E_{(l+i+1)n+i-1/2,(-k+i-\ell_j)n-j+1/2}
- (-l-1)\ell_i E_{(l+i+1)n+i-1/2,(-k+i-\ell_j)n-j+1/2}
+ \mathcal{C}((-1)^{i}(-k-1)\ell_j E_{(l+i+1)n+i-1/2,(-k+i-\ell_j)n-j+1/2}
- (-\ell_{i} + l - 1/2)\ell_j E_{j,i}(-k-\ell_{j}, l+\ell_{i}) + \mathcal{C}\kappa_0
$$

(5.3)

and

$$
\mathcal{G}_{\ell}^{t*,\ell} = \sum_{i,j=1}^{n} \sum_{l,k=0}^{\infty} \mathcal{C}((-1)^{i}(-k-1)\ell_j E_{(l+i+1)n+i-1/2,(-k+i-\ell_j)n-j+1/2}
- (-\ell_{i} + l - 1/2)\ell_j E_{j,i}(-k-\ell_{j}, l+\ell_{i}) + \mathcal{C}\kappa_0
$$

(5.4)
Then \( G^{s,t}_H \) and \( G^{l,t}_H \) are Lie subalgebras of \( \tilde{gl}(\infty) \), and the map in (3.11) induces

\[
\tilde{D}^{s,t}_H \cong G^{s,t}_H
\]

(cf. (4.44)).

We define two functions \( \tilde{H}_1, \tilde{H}_2 : \mathcal{Z} \to \mathbb{C} \) by

\[
\tilde{H}_1(l) = \begin{cases} 
1 & \text{if } n < l < 0 \text{ or } l > 0, (l - \ell_{(l+1/2)}) n, \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\tilde{H}_2(l) = \begin{cases} 
1 & \text{if } -ln, 0 < l < (\ell_{(-l+1/2)} - \ell)n < l, \\
0 & \text{otherwise}
\end{cases}
\]

(cf. (4.25)). Moreover, we define a map \( \beta^l_{\tilde{H}} : \mathcal{Z}^4 \to \mathbb{C} \) by:

\[
\beta^l_{\tilde{H}}(l_1, l_2, k_1, k_2) = (H_1(l_1)H_2(l_2) - H_1(k_1)H_2(k_2))\delta_{l_1+k_2,0}\delta_{l_2+k_1,0}.
\]

Set

\[
\tilde{gl}^{(i)}_H(\infty) = \mathcal{Z} gl(\infty) \oplus \mathbb{C}\kappa_0
\]

(cf. (3.1), where \( \kappa_0 \) is a base element. We have the following Lie bracket on \( \tilde{gl}^{(i)}_H(\infty) \):

\[
[\mathcal{E}_{l_1, l_2} + \mu_1\kappa_0, \mathcal{E}_{k_1, k_2} + \mu_2\kappa_0] = \mathcal{E}_{l_1, l_2}\mathcal{E}_{k_1, k_2} - \mathcal{E}_{k_1, k_2}\mathcal{E}_{l_1, l_2} + \beta^l_{\tilde{H}}(l_1, l_2, k_1, k_2)\kappa_0
\]

for \( l_1, l_2, k_1, k_2 \in \mathcal{Z} \). In particular, the Lie algebras

\[
\tilde{gl}^{(0)}_H(\infty) \cong \tilde{gl}(\infty).
\]

Next we set

\[
\mathcal{L}^{s,t}_H = \sum_{i,j=1}^{n} \sum_{l,k=0}^{\infty} \left[ \mathbb{C}((-1)^{i}(-k-1)e_{(l-1)n+i-1/2,-k-1}n+j+1/2 - (-l-1)e_{i} \right.
\]

\[
\times \mathcal{E}_{-kn-j+1/2,(l-1)n+i-1/2} + \mathbb{C}((-1)^{i}(-k-1)e_{ln+i-1/2,-k-1}n+j+1/2
\]

\[
- (l+\ell_{i})e_{ln+i-1/2,l+1/2} + \mathbb{C}((-1)^{i}(k+\ell_{j})e_{ln+i-1/2,(k+1)n-j+1/2
\]

\[
- (l+\ell_{i})e_{(k+1)n-j+1/2,ln+i-1/2}] + \mathbb{C}\kappa_0
\]

and

\[
\mathcal{L}^{l,t}_H = \sum_{i,j=1}^{n} \sum_{l,k=0}^{\infty} \left[ \mathbb{C}((-1)^{i}(-k-1)e_{(l-1)n+i-1/2,-k-1}n+j+1/2 - (-1)^{p(i)+p(j)}(-l-1)e_{i}
\right.
\]

\[
\times \mathcal{E}_{-kn-j+1/2,(l-1)n+i-1/2} + \mathbb{C}((-1)^{i}(-k-1)e_{ln+i-1/2,-k-1}n+j+1/2
\]

\[
- (l+\ell_{i})e_{ln+i-1/2,l+1/2} + \mathbb{C}((-1)^{i}(k+\ell_{j})e_{ln+i-1/2,(k+1)n-j+1/2
\]

\[
- (l+\ell_{i})e_{(k+1)n-j+1/2,ln+i-1/2}] + \mathbb{C}\kappa_0
\]

(5.13)
if \( n \) is even. Then \( \mathcal{L}^\tau_{\ell,\epsilon} \) are Lie subalgebras of \( \tilde{g}_1^{(i)}(\infty) \), and

\[
\tilde{\mathcal{D}}^\tau_{\ell,\epsilon} \cong \mathcal{G}^\tau_{\ell,\epsilon} \cong \mathcal{L}^\tau_{\ell,\epsilon}
\]

(cf. (4.44)) by (5.3) and (5.4). Thus we have:

**Theorem 5.1.** The Lie algebras:

\[
\tilde{\mathcal{D}}^\tau_{\ell,\epsilon}/\mathbb{C}K_0 \cong \begin{cases} \tilde{\mathcal{D}}_{\ell,\epsilon}^\tau & \text{if } \epsilon = 0, \\ \mathcal{S}_p(\infty) & \text{if } \epsilon = 1 \end{cases}
\]

and

\[
\tilde{\mathcal{D}}_{\ell,\epsilon}^{+0}/\mathbb{C}K_0 \cong \begin{cases} \mathcal{S}_p(\infty) & \text{if } \epsilon = 0, \\ \tilde{\mathcal{D}}_{\ell,\epsilon}^\tau & \text{if } \epsilon = 1 \end{cases}
\]

(cf. (4.1), (4.3)).

Next we want to study highest weight irreducible modules of \( \mathcal{L}^\tau_{\ell,\epsilon} \). Recall the notions in (4.25). Set

\[
\mathcal{L}^\tau_{\ell,+} = \sum_{0 < k < l} \mathbb{C}\langle -kQ - 1 \rangle_{\ell_R} E_{l_1/2,-k_1/2} - \langle lQ + \ell_R \rangle_{\ell_R} E_{-k_1/2,l_1/2} - \sum_{l,k=1}^\infty \mathbb{C}\langle -1 \rangle\langle kQ + \ell_R \rangle_{\ell_R} E_{l_1/2,k_1/2} - \langle lQ + \ell_R \rangle_{\ell_R} E_{k_1/2,l_1/2},
\]

(5.17)

\[
\mathcal{L}^\tau_{\ell,-} = \sum_{l,k=1}^\infty \mathbb{C}\langle -1 \rangle\langle -kQ - 1 \rangle_{\ell_R} E_{l_1/2,-k_1/2} - \langle -lQ - 1 \rangle_{\ell_R} E_{k_1/2,-l_1/2} + \sum_{0 < l < k} \mathbb{C}\langle -1 \rangle\langle -kQ - 1 \rangle_{\ell_R} E_{l_1/2,-k_1/2} - \langle lQ + \ell_R \rangle_{\ell_R} E_{k_1/2,l_1/2},
\]

(5.18)

and

\[
\mathcal{L}^\tau_{\ell,+} = \sum_{0 < k < l} \mathbb{C}\langle -1 \rangle\langle -kQ - 1 \rangle_{\ell_R} E_{l_1/2,-k_1/2} - \langle 1 \rangle_{\ell_R}^{+} + \langle 1 \rangle_{\ell_R} \times \langle lQ + \ell_R \rangle_{\ell_R} E_{k_1/2,l_1/2} + \sum_{l,k=1}^\infty \mathbb{C}\langle -1 \rangle\langle kQ + \ell_R \rangle_{\ell_R} E_{l_1/2,k_1/2} - \langle 1 \rangle_{\ell_R}^{+} + \langle 1 \rangle_{\ell_R}^{+} \times \langle lQ + \ell_R \rangle_{\ell_R} E_{k_1/2,l_1/2},
\]

(5.19)

\[
\mathcal{L}^\tau_{\ell,-} = \sum_{l,k=1}^\infty \mathbb{C}\langle -1 \rangle\langle -kQ - 1 \rangle_{\ell_R} E_{l_1/2,-k_1/2} - \langle 1 \rangle_{\ell_R}^{+} + \langle 1 \rangle_{\ell_R} \times \langle -lQ - 1 \rangle_{\ell_R} E_{k_1/2,-l_1/2} + \sum_{0 < l < k} \mathbb{C}\langle -1 \rangle\langle -kQ - 1 \rangle_{\ell_R} E_{l_1/2,-k_1/2} - \langle 1 \rangle_{\ell_R}^{+} + \langle 1 \rangle_{\ell_R}^{+} \times \langle lQ + \ell_R \rangle_{\ell_R} E_{k_1/2,l_1/2},
\]

(5.20)

Then \( \mathcal{L}^\tau_{\ell,\pm} \) are Lie subalgebras of \( \mathcal{L}^\tau_{\ell,\tau} \) (cf. (4.44)).

Denote

\[
\vartheta_l = E_{l_1/2,-l_1/2} - E_{-l_1/2,l_1/2} \quad \text{for } l \in \mathbb{N}.
\]

(5.21)
Set
\[ T = \sum_{l=0}^{\infty} \vartheta_l + \mathbb{C}\kappa_0. \] (5.22)

Then \( T \) is a toral Cartan subalgebra of \( \mathcal{L}^\tau_{\ell, -} \). Moreover,

\[ \mathcal{L}^\tau_{\ell, -} = \mathcal{L}^\tau_{\ell, -} \oplus T \oplus \mathcal{L}^\tau_{\ell, +}. \] (5.23)

Define
\[ f_{\ell, l}^* = - (l + 1)Q - 1) \ell_{(l+1)R} \mathcal{E}_{l-1/2, -1/2} - (-1)^l (lQ + \ell_{(l+1)R}) \mathcal{E}_{l-1/2, -1/2} \] (5.24)
and
\[ f_{\ell, l}^\dagger = - (l + 1)Q - 1) \ell_{(l+1)R} \mathcal{E}_{l-1/2, -1/2} \]
\[ - (-1)^l \sum_{l}^p (l) \eta_{(l+1)R}) (lQ + \ell_{(l+1)R}) \mathcal{E}_{l-1/2, -1/2} \] (5.25)
for \( l \in \mathbb{N} + 1 \). Moreover, we define
\[ f_{0, 0}^* = - (l + 1)Q - 1) \ell_{(l+1)R} \mathcal{E}_{l-1/2, -1/2} \] (5.26)
\[ f_{1, 0}^\dagger = - (l + 1)Q - 1) \ell_{(l+1)R} \mathcal{E}_{l-1/2, -1/2} + (-1)^l \sum_{l}^p (l) \eta_{(l+1)R}) (lQ + \ell_{(l+1)R}) \mathcal{E}_{l-1/2, -1/2} \] (5.27)
\[ f_{1, 0}^* = f_{0, 0}^\dagger = \mathcal{E}_{l-1/2, -1/2}. \] (5.28)

Then \( \{ f_{\ell, l}^* | l \in \mathbb{N} \} \) is a set of negative simple root vectors of \( \mathcal{L}^\tau_{\ell, -} \).

For \( l \in \mathbb{N} + 1 \), we define
\[ \omega_l = \beta_l^\dagger (l + 1/2, -l + 1/2; -l - 1/2, -l - 1/2) + \beta_l^\dagger (-l + 1/2, l + 1/2; -l - 1/2, -l - 1/2). \] (5.29)

Moreover, we let
\[ \omega_{0, 0}^* = \beta_l^\dagger (1/2, 3/2; -3/2, -1/2) + \beta_l^\dagger (3/2, 1/2; -1/2, -3/2) \] (5.30)
and
\[ \omega_{1, 0}^* = \beta_l^\dagger (1/2, 1/2; -1/2, -1/2). \] (5.31)

Furthermore, we set
\[ T_{\ell, l}^* = \vartheta_l - \vartheta_{l+1} + \omega_l \kappa_0 \] for \( l \in \mathbb{N} + 1 \),
\[ T_{0, 0}^* = T_{1, 0}^\dagger = \vartheta_1 + \vartheta_0 + \omega_{0, 0}^* \kappa_0 \] (5.33)
and
\[ T_{1, 0}^* = T_{0, 0}^\dagger = \vartheta_0 + \omega_{1, 0}^* \kappa_0. \] (5.34)

Denote
\[ \mathcal{L}^\tau_{\ell, 0} = T + \mathcal{L}^\tau_{\ell, +}. \] (5.35)
Let $\lambda^{\tau,\epsilon}$ a linear function on $\mathcal{T}$ such that there exists $k_0 \in \mathbb{N}$ for which

$$\lambda^{\tau,\epsilon}(\partial_l) = 0 \quad \text{for } k_0 \leq l \in \mathbb{N}. \quad (5.36)$$

Define a one-dimensional $\mathcal{L}_{t,0}^{\tau,\epsilon}$-module $\mathbb{C}v^{\tau,\epsilon}$ by:

$$\mathcal{L}_{t,0}^{\tau,\epsilon}(v^{\tau,\epsilon}) = \{0\}, \quad h(v^{\tau,\epsilon}) = \lambda^{\tau,\epsilon}(h)v^{\tau,\epsilon} \quad \text{for } h \in \mathcal{T}. \quad (5.37)$$

Form an induced $\mathcal{L}_{t,0}^{\tau,\epsilon}$-module:

$$M_{\lambda^{\tau,\epsilon}} = U(\mathcal{L}_{t,0}^{\tau,\epsilon}) \otimes_{U(\mathcal{L}_{t,0}^{\tau,\epsilon})} \mathbb{C}v^{\tau,\epsilon} \equiv U(\mathcal{L}_{t,0}^{\tau,\epsilon}) \otimes_{\mathbb{C}} \mathbb{C}v^{\tau,\epsilon}. \quad (5.38)$$

There exists a unique maximal proper submodule $N_{\lambda^{\tau,\epsilon}}$ of $M_{\lambda^{\tau,\epsilon}}$, and the quotient

$$M_{\lambda^{\tau,\epsilon}} = M_{\lambda^{\tau,\epsilon}}/N_{\lambda^{\tau,\epsilon}}$$

is a weighted irreducible $\mathcal{L}_{t,0}^{\tau,\epsilon}$-module. Identify $1 \otimes v^{\tau,\epsilon}$ with $v^{\tau,\epsilon}$. When

$$\lambda_k^{\tau,\epsilon} = \lambda^{\tau,\epsilon}(T_{t,k}^\tau) \in \mathbb{N} \quad \text{for } k \in \mathbb{N}, \quad (5.40)$$

the submodule

$$N_{\lambda^{\tau,\epsilon}} = \sum_{l=0}^{\infty} U(\mathcal{L}_{t,0}^{\tau,\epsilon})(f^{\tau}_{l,t}) \lambda_{l,k}^{\tau,\epsilon} + 1_v^{\tau,\epsilon}. \quad (5.41)$$

Based on (5.3) and (5.4), we denote

$$\bar{l} = (|t| + \max \{\ell_1, \ell_2, ..., \ell_n\})n. \quad (5.42)$$

For any $\bar{l} < s \in \mathbb{N}$, we define:

$$\mathcal{L}_{t,s}^{\tau,\epsilon} = \sum_{l,k=s}^{\infty} \mathbb{C}((-1)^{\epsilon}(-kQ-1)\ell_kR \mathcal{E}_{l-1/2,-k+1/2} - (\ell_Q + \ell_{l,R})\ell_{l,R} \mathcal{E}_{-k+1/2,l-1/2}) \quad (5.43)$$

and

$$\mathcal{L}_{t,s}^{\tau,\epsilon} = \sum_{l,k=s}^{\infty} \mathbb{C}((-1)^{\epsilon}(-kQ-1)\ell_kR \mathcal{E}_{l-1/2,-k+1/2}
-(-1)^{p(l_R) + p(k_R)}(\ell_Q + \ell_{l,R})\ell_{l,R} \mathcal{E}_{-k+1/2,l-1/2}). \quad (5.44)$$

Suppose that $\mathcal{M}$ is an $\mathcal{L}_{t,0}^{\tau,\epsilon}$-module

generated by a subspace $\mathcal{M}_0$ such that $\mathcal{L}_{t,0}^{\tau,\epsilon}(\mathcal{M}_0) = \{0\}$ for some $\bar{l} < s \in \mathbb{N}. \quad (5.45)$

For instance, the above module $\mathcal{M}^{\tau,\epsilon}$ is such an module with $\mathcal{M}_0 = \mathbb{C}v^{\tau,\epsilon}$ and $s = \max\{k_0 + 2, \bar{l} + 1\}$. We can also construct $\mathcal{M}$ as Example 3.2.

Recall the notion defined in (2.60). For $i, j \in \overline{1,n}$ and $r \in \mathbb{N}$, we have

$$(E_{i,j})^{\tau,\epsilon}_{\bar{l}}(r, z) = \sum_{l \in \mathbb{Z}} (E_{i,j} \otimes \ell_l \partial_l^{\tau,\epsilon} - (-1)^{r} E_{j,i} \otimes (-\partial_l)^{r} \partial_l^{\tau,\epsilon})z^{-l-1} \quad (5.46)$$
and
\[
(E_{i,j})^t_{(r,z)} = \sum_{l \in \mathbb{Z}} (E_{i,j} \otimes t^l \partial_k^{r+l}) - (-1)^{s+p(i)+p(j)} E_{j^*,i^*} \otimes (-\partial_{\bar{\ell}})^{r} \partial_{\bar{r}}^{l} z^{-l-1}.
\]

By Theorem 2.5, we have a representation \(\sigma_*\) of \(\hat{\mathcal{O}}(\ell,\mathbb{A})\) and a representation \(\sigma^\dagger\) of \(\hat{\mathcal{H}}(\ell,\mathbb{A})\) on \(\mathcal{M}\) with \(\sigma_*(\nu) = \nu_0\), \(\sigma^\dagger(\kappa) = \kappa_0\) and
\[
\sigma_*(\nu) = \sum_{l,k=0}^\infty \langle k \rangle^r \langle -k-1 \rangle^{(l-1)n+i-1/2,-kn-j+1/2} \langle -l-1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r}
\]
\[+ \langle k \rangle^r \langle -k-1 \rangle^{(l+1)n+i-1/2,(l+1)n+i-1/2} \langle -l-1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r-1}
\]
\[+ \langle l+1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r+1}
\]
\[+ \langle l+1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r-1}
\]
\[+ \langle l+1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r+1}
\]
\[+ \langle r+\ell_i \rangle^r \mathcal{H}_{0,r+\ell_i} - (1)^{s+p(i)+p(j)} \mathcal{H}_{r+\ell_i} \delta_{\ell_i}^{0} z^{-r-\ell_i-1} \]

and
\[
\sigma^\dagger(\nu) = \sum_{l,k=0}^\infty \langle k \rangle^r \langle -k-1 \rangle^{(l-1)n+i-1/2,-kn-j+1/2} \langle -l-1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r}
\]
\[+ \langle k \rangle^r \langle -k-1 \rangle^{(l+1)n+i-1/2,(l+1)n+i-1/2} \langle -l-1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r-1}
\]
\[+ \langle l+1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r+1}
\]
\[+ \langle l+1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r-1}
\]
\[+ \langle l+1 \rangle^r_{\ell_i} \mathcal{E}_{-(l-n)+1/2,-(-l-n)+1/2} z^{l+k-r+1}
\]
\[+ \langle r+\ell_i \rangle^r \mathcal{H}_{0,r+\ell_i} - (1)^{s+p(i)+p(j)} \mathcal{H}_{r+\ell_i} \delta_{\ell_i}^{0} z^{-r-\ell_i-1} \]

By a similar proof as that of Theorem 4.2, we obtain the following theorem, which was proved by Ma [M2] in a different form when \(t = 0\).

**Theorem 5.2.** Suppose that \(\mathcal{M}\) is a weighted \(L^\infty_{\ell,\bar{\ell}}\)-module satisfying (5.45). Then the representation \(\sigma_r\) is irreducible if and only if \(\mathcal{M}\) is irreducible.
6 Vacuum Representation of $\hat{\mathfrak{gl}}(\vec{\ell},A)$

In this section, we study the vacuum representation of the Lie algebra $\hat{\mathfrak{gl}}(\vec{\ell},A)$ in (3.6) and its vertex algebra structure. Its vertex algebra irreducible representations are investigated.

Recall the algebra $A$ of differential operators given in (2.2) and (2.3). Observe that

$$ A_− = \sum_{i=0}^{\infty} \mathbb{C}[t^{-1}] t^{-1} \partial_i^i \quad \text{and} \quad A_+ = \sum_{i=0}^{\infty} \mathbb{C}[t] \partial_i^i $$

forms associative subalgebras of $A$. Moreover,

$$ A = A_- + A_+.$$

Recall the Lie algebra $\hat{\mathfrak{gl}}(n,A)$ in (3.3) and (3.4). Note that in (3.4),

$$ \delta_{r_1+r_2,m_1+m_2,r_1!r_2!}^{m_1} \left( \frac{m_1}{r_1 + r_2 + 1} \right) = 0 \quad \text{for} \quad m_1, m_2, r_1, r_2 \in \mathbb{N},$$

because if $m_1 + m_2 = r_1 + r_2$, then $m_1 < r_1 + r_2 + 1$. Thus we have the following Lie subalgebras of $\hat{\mathfrak{gl}}(n,A)$:

$$ \hat{\mathfrak{gl}}(n,A)_\pm = M_{n\times n}(A_\pm).$$

Moreover,

$$ \hat{\mathfrak{gl}}(n,A) = \hat{\mathfrak{gl}}(n,A)_- + \hat{\mathfrak{gl}}(n,A)_+ + \mathbb{C} \kappa. $$

Suppose that $\mathcal{G}$ is Lie subalgebra of $\hat{\mathfrak{gl}}(n,A)$ such that

$$ \mathcal{G} = \mathcal{G}_- + \mathcal{G}_+ + \mathbb{C} \kappa, \quad \mathcal{G}_\pm = \mathcal{G} \cap \hat{\mathfrak{gl}}(n,A)_\pm. $$

Then $\mathcal{G}_\pm$ and

$$ \mathcal{G}_0 = \mathcal{G}_+ + \mathbb{C} \kappa$$

are Lie subalgebras of $\mathcal{G}$. Take a nonzero constant $\chi \in \mathbb{C}$. Form a one-dimensional $\mathcal{G}_0$-module $\mathbb{C}|0\rangle$ by:

$$ \kappa(|0\rangle) = \chi|0\rangle, \quad \mathcal{G}_+ (|0\rangle) = \{0\}. $$

The induced $\mathcal{G}$-module

$$ \mathcal{V}_\chi(\mathcal{G}) = \mathcal{U}(\mathcal{G}) \otimes_{\mathcal{U}(\mathcal{G}_0)} \mathbb{C}|0\rangle \cong \mathcal{U}(\mathcal{G}_-) \otimes_{\mathbb{C}} \mathbb{C}|0\rangle $$

is called the vacuum module of $\mathcal{G}$ and the corresponding representation is called the vacuum representation of $\mathcal{G}$ at level $\chi$. The main objective in the rest of paper is to study $\mathcal{V}_\chi(\mathcal{G})$ and the related vertex algebra structure when $\mathcal{G}$ is one of the Lie algebras $\hat{\mathfrak{gl}}(\vec{\ell},A)$, $\hat{\mathfrak{o}}(\vec{\ell},A)$ or $\hat{\mathfrak{sp}}(\vec{\ell},A)$. For convenience, we simply denote

$$ u|0\rangle = u \otimes |0\rangle \quad \text{for} \quad u \in U(\mathcal{G}).$$
In the rest of this section, we will deal only with $\widetilde{\mathfrak{gl}}(\vec{\ell}, A)$.

**Theorem 6.1.** The module $\mathcal{V}_\chi(\widetilde{\mathfrak{gl}}(\vec{\ell}, A))$ is irreducible if $\chi \notin \mathbb{Z}$. When $\chi \in \mathbb{Z}$, the module $\mathcal{V}_\chi(\widetilde{\mathfrak{gl}}(\vec{\ell}, A))$ has a unique maximal proper submodule $\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A))$, and the quotient

$$
\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A)) = \mathcal{V}(\hat{\mathfrak{gl}}(\vec{\ell}, A))/\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A))
$$

is an irreducible $\hat{\mathfrak{gl}}(\vec{\ell}, A)$-module. If $n > 1$ and $\chi \in \mathbb{N}$, the submodule

$$
\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A)) = U(\hat{\mathfrak{gl}}(\vec{\ell}, A))(t^{-1}\partial_{E_1} E_{n+1})^{\chi} | 0).
$$

**Proof.** We define

$$
\hat{\mathfrak{gl}}(\vec{\ell}, A)_{(k)} = \bigoplus_{i,j=1}^n \bigoplus_{r=0}^\infty \mathbb{C} t^{r+\gamma_j-k}\partial_{r}^{\gamma_j} E_{i,j} + \mathbb{C} \delta_{k,n} \kappa
$$

for $k \in \mathbb{Z}$. Then

$$
\hat{\mathfrak{gl}}(\vec{\ell}, A) = \bigoplus_{k \in \mathbb{Z}} \hat{\mathfrak{gl}}(\vec{\ell}, A)_{(k)}
$$

is a $\mathbb{Z}$-graded Lie algebra. Moreover, we define a $\mathbb{Z}$-grading on $\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A))$ by

$$
\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A))_{(0)} = \mathbb{C} | 0), \quad \mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A))_{(-m)} = \{0\} \quad \text{for } m \in \mathbb{N} + 1
$$

and

$$
\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A))_{(m)} = \text{Span}\{u_1 u_2 \cdots u_s | 0) \mid u_i \in \hat{\mathfrak{gl}}(\vec{\ell}, A)_- \cap \hat{\mathfrak{gl}}(\vec{\ell}, A)_{(k_i)}; \sum_{i=1}^s k_i = m\}
$$

for $m \in \mathbb{N} + 1$. Then

$$
\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A)) = \bigoplus_{k \in \mathbb{Z}} \mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A))_{(k)}
$$

is a $\mathbb{Z}$-graded $\hat{\mathfrak{gl}}(\vec{\ell}, A)$-module. Since

$$
(\mathbb{A} \partial_{r}^{\gamma_j} E_{i,j}) \cap \bigoplus_{k=0}^{k-1} \hat{\mathfrak{gl}}(\vec{\ell}, A)_{(k+\gamma_j)} \cap \hat{\mathfrak{gl}}(\vec{\ell}, A)_{-} = \bigoplus_{r=0}^{k-1} \mathbb{C} t^{r-k}\partial_{r}^{\gamma_j} E_{i,j}
$$

(cf. (6.4) and (6.6)), the character

$$
d(\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A)), q) = \sum_{k=0}^{\infty} (\text{dim } \mathcal{V}_\chi(\hat{\mathfrak{gl}}(\vec{\ell}, A))_{(k)})q^k = \prod_{i=1}^{\infty} \prod_{r=1}^{\infty} \frac{1}{(1 - q^{\gamma_i+r})^r}.
$$

Recall the Lie algebra $\widetilde{\mathfrak{gl}}(\infty)$ defined in (3.9) and (3.10). Set

$$
\widetilde{\mathfrak{gl}}(\infty)_{(-)} = \bigoplus_{0 > j, k \in \mathbb{Z}} \mathbb{C} \mathcal{E}_{j,k}, \quad \widetilde{\mathfrak{gl}}(\infty)_{(+)} = \bigoplus_{0 < j, k \in \mathbb{Z}} (\mathbb{C} \mathcal{E}_{j,k} + \mathbb{C} \mathcal{E}_{j,-k} + \mathbb{C} \mathcal{E}_{-j,k}).
$$

By (3.10), $\widetilde{\mathfrak{gl}}(\infty)_{(\pm)}$ are Lie subalgebras of $\widetilde{\mathfrak{gl}}(\infty)$ and

$$
\widetilde{\mathfrak{gl}}(\infty) = \widetilde{\mathfrak{gl}}(\infty)_{(-)} + \widetilde{\mathfrak{gl}}(\infty)_{(+)} + \mathbb{C} \mathcal{K}_0.
$$

(6.21)
Hence we have the Lie subalgebra
\[ \tilde{\mathfrak{gl}}(\infty)_{(0)} = \mathfrak{gl}(\infty)_{(+)} + \mathbb{C}\kappa_0. \] (6.22)

Define a one-dimensional \( \tilde{\mathfrak{gl}}(\infty)_{(0)} \)-module \( \mathbb{C}1 \) by
\[ \kappa_0(1) = \chi1, \quad \tilde{\mathfrak{gl}}(\infty)_{(+)}(1) = \{0\}. \] (6.23)

Form an induced \( \tilde{\mathfrak{gl}}(\infty) \)-module
\[ U_\chi = U(\tilde{\mathfrak{gl}}(\infty)) \otimes_U(\tilde{\mathfrak{gl}}(\infty)_{(-)}) \mathbb{C}1 \cong U(\mathfrak{gl}(\infty)_{(-)}) \otimes_U \mathbb{C}1, \] (6.24)
which satisfies the condition (3.18) with \( m = 0 \) and \( M_0 = \mathbb{C}1 \otimes 1 \). For convenience, we denote
\[ v1 = v \otimes 1 \quad \text{for} \quad v \in U(\mathfrak{gl}(\infty)). \] (6.25)

Note our notion
\[ E_{i,j}(r,z) = \sum_{m \in \mathbb{Z}} t^m \partial^m_z z^{-m-1}E_{i,j} \quad \text{for} \quad i,j \in \mathbb{N}, \ r \in \mathbb{N} + \ell_j. \] (6.26)

According to (3.37) (also cf. (2.52) and (3.11)), we obtain a \( \hat{\mathfrak{gl}}(\vec{l},\mathbb{A}) \)-module structure on \( U_\chi \) defined by \( \kappa = \chi \text{Id}_{U_\chi} \) and
\[ E_{i,j}(r,z) = \sum_{m \in \mathbb{Z}} (-k-1)_r (\mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2}z^{-l-k-\ell_i-r-2} \]
\[ + \mathcal{E}_{-(l+1)n+i-1/2,(k+1)n-j+1/2}z^{l-k-r-1}) \]
\[ + (-k-1)_r (\mathcal{E}_{ln+i-1/2,-kn-j+1/2}z^{-l-k+\ell_i-\ell_j-r-1} \]
\[ + \mathcal{E}_{-(l+1)n+i-1/2,-kn-j+1/2}z^{l-k+\ell_j-r})] \] (6.27)

for \( i,j \in \mathbb{N} \) and \( r \in \mathbb{N} + \ell_j \). In particular,
\[ E_{i,j}(r,z)(1) = \sum_{m \in \mathbb{Z}} (-k-1)_r (\mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2}z^{l+k+\ell_j-r} \]
\[ + \mathcal{E}_{-(l+1)n+i-1/2,(k+1)n-j+1/2}z^{l+k+\ell_j-r})] \] (6.28)

Thus we have
\[ \hat{\mathfrak{gl}}(\vec{l},\mathbb{A})_{(+)}(1) = \{0\}. \] (6.29)

By a similar proof as that of Theorem 3.1,
\[ U_\chi = U(\hat{\mathfrak{gl}}(\vec{l},\mathbb{A})_{-})1. \] (6.30)

Therefore, we have a Lie algebra module epimorphism \( \nu : V_\chi(\hat{\mathfrak{gl}}(\vec{l},\mathbb{A})) \rightarrow U_\chi \) defined by
\[ \nu(u|0)) = u1 \quad \text{for} \quad u \in U(\hat{\mathfrak{gl}}(\vec{l},\mathbb{A})_{-}). \] (6.31)
Set
\[ \ell = \min \{ \ell_1, \ell_2, \ldots, \ell_n \}. \]  
(6.32)

For \( m \in \mathbb{N} + \ell + 1 \), we let
\[ \tilde{g}(\infty)_{(-)}^{(m)} = \text{Span} \{ E_{-l+1/2,-k-j+1/2} \mid i, j \in \mathbb{N}, l, k \in \mathbb{N}; l + k + \ell_j + 1 = m \}. \]  
(6.33)

Then
\[ \tilde{g}(\infty)_{(-)} = \bigoplus_{m=\ell+1}^{\infty} \tilde{g}(\infty)_{(-)}^{(m)}. \]  
(6.34)

Moreover, we define
\[ U^{(0)}_\chi = \mathbb{C}1, \quad U^{(m)}_\chi = \{0\} \quad \text{for} \quad m \in (-N - 1) \bigcup \mathbb{T}, \ell \]  
(6.35)

and
\[ U^{(m)}_\chi = \text{Span} \{ u_1 u_2 \cdots u_s 1 \mid u_i \in \tilde{g}(\infty)_{(-)}^{(m_i)}; \sum_{i=1}^{s} m_i = m \}. \]  
(6.36)

By (6.30),
\[ U_\chi = \bigoplus_{m \in \mathbb{Z}} U^{(m)}_\chi \]  
(6.37)

is a \( \mathbb{Z} \)-graded \( \tilde{g}(\ell, \mathbb{A}) \)-module. Furthermore, (6.33) implies the character
\[ d(U_\chi, q) = \sum_{k=0}^{\infty} (\dim U^{(k)}_\chi) q^k = \prod_{i=1}^{n} \prod_{r=1}^{\infty} \frac{1}{(1 - q^{\ell_j + r})^{r_n}}. \]  
(6.38)

Therefore, (6.19) and (6.38) imply
\[ V_\chi(\tilde{g}(\ell, \mathbb{A})) \cong U_\chi. \]  
(6.39)

Let \( \lambda \) be a linear function on \( T \) (cf. (3.12)) such that
\[ \lambda(\kappa_0) = \chi, \quad \lambda(E_{l,-l}) = 0 \quad \text{for} \quad l \in \mathbb{Z}. \]  
(6.40)

Recall the Verma module \( M_\lambda \) defined in (3.63). Note
\[ U_\chi \cong M_\lambda/(\sum_{-1/2 \neq l \in \mathbb{Z}} U(\tilde{g}(\infty)_{(-)})(E_{l+1,l} \otimes v_\lambda)), \]  
(6.41)

which is irreducible if \( \chi \not\in \mathbb{Z} \) by [J1-J3]. When \( \chi \in \mathbb{N} \),
\[ \tilde{U}_\chi = U(\tilde{g}(\infty)_{(-)})(E_{l+1,l} \otimes v_\lambda)/(\sum_{-1/2 \neq l \in \mathbb{Z}} U(\tilde{g}(\infty)_{(-)})(E_{l+1,l} \otimes v_\lambda)) \]  
(6.42)

is the unique maximal proper submodule of \( U_\chi \) (cf. (3.65)). Thus
\[ U(\tilde{g}(\ell, \mathbb{A})) \nu^{-1}(E_{l+1,l} \otimes \mathbb{1}) \]  
(6.43)
is the unique maximal proper submodule of \( \mathcal{V}_\Lambda(\hat{gl}(\mathfrak{f},\mathbb{A})) \) (cf. (6.31)). When \( n > 1 \), (6.23) and (6.27) imply
\[
\nu^{-1}(E^\chi_{1/2,-1/2}^1) = (t^{-1} \partial_t^i E_n,1)^{\chi+1}|0\rangle. \tag{6.44}
\]

Next we want to present the definitions of vertex algebra and its module. For any two vector spaces \( U \) and \( W \), we denote by \( \text{LM}(U,W) \) the set of all linear maps from \( U \) to \( W \). Let \( z_1 \) and \( z_2 \) be two formal variables. We have the following convention of binomial expansions:
\[
(z_1 - z_2)^r = \sum_{l=0}^{\infty} (-1)^l \binom{r}{l} z_1^{r-l} z_2^l \quad \text{for} \quad r \in \mathbb{C}. \tag{6.45}
\]

For a vector space \( V \), we denote
\[
V[z^{-1}, z] = \{ \sum_{i=m}^{\infty} v_i z^i \mid v_i \in V, \ m \in \mathbb{Z} \}, \tag{6.46}
\]
the space of formal Laurent series with coefficients in \( V \).

A **vertex algebra** is a vector space \( V \) with a linear map \( Y(\cdot, z) : V \to \text{LM}(V,V[z^{-1}, z]) \), an element \( \partial \in \text{End} V \) and an element \( |0\rangle \in V \), satisfying the following conditions: given \( u, v \in V \),
\[
Y(|0\rangle, z) = \text{Id}_V, \tag{6.47}
\]
\[
[\partial, Y(v, z)] = \frac{d}{dz} Y(v, z), \quad Y(v, z)|0\rangle = e^{z \partial} v, \tag{6.48}
\]
\[
(z_1 - z_2)^m Y(u, z_1) Y(v, z_2) = (z_1 - z_2)^m Y(v, z_2) Y(u, z_1) \tag{6.49}
\]
for some positive integer \( m \).

An ideal \( U \) of a vertex algebra \( V \) is a vector space of \( V \) such that
\[
\partial(U) \subset U, \quad Y(v, z) U \subset U[z^{-1}, z] \quad \text{for} \quad v \in V. \tag{6.50}
\]

A vertex algebra without proper nonzero ideals is called **simple**.

A module \( W \) of a vertex algebra \((V,Y(\cdot, z),|0\rangle, \partial)\) is a vector space with a linear map \( Y_W(\cdot, z) : V \to \text{LM}(W,W[z^{-1}, z]) \) such that given \( u, v \in V \) and \( w \in W \),
\[
Y_W(|0\rangle, z) = \text{Id}_W, \quad Y_W(\partial v, z) = \frac{d}{dz} Y(v, z), \tag{6.51}
\]
\[
(z_1 - z_2)^m Y_W(u, z_1) Y_W(v, z_2) = (z_1 - z_2)^m Y_W(v, z_2) Y_W(u, z_1), \tag{6.52}
\]
\[
(z_0 + z_2)^m Y_W(u, z_0 + z_2) Y_W(v, z_2) w = (z_2 + z_0)^m Y_W(Y(u, z_0)v, z_2) w \tag{6.53}
\]
for some positive integer \( m \).

A **submodule** \( W_1 \) of \( W \) is a subspace such that
\[
Y_W(v, z) W_1 \subset W_1[z^{-1}, z] \quad \text{for} \quad v \in V. \tag{6.54}
\]
A module without proper nonzero submodule is called **irreducible**.

For a vector space $U$ and any formal power series

$$f(z) = \sum_{r \in \mathbb{Z}} u_r z^{-r-1} \text{ with } u_r \in U,$$

we define

$$f(z)^+ = \sum_{r=0}^{\infty} u_r z^{-r-1}, \quad f(z)^- = \sum_{r=0}^{\infty} u_{-r-1} z^r. \quad (6.56)$$

Now we define a linear transformation $\partial$ on $\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A})$ by

$$\partial(\kappa) = 0, \quad \partial(t^m \partial^i E_{i,j}) = -mt^{m-1} \partial^i E_{i,j} \quad (6.57)$$

for $i, j \in \overline{1,n}$, $r \in \mathbb{N}$ and $m \in \mathbb{Z}$. Moreover, we define a linear transformation $\partial$ on $\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}))$ by

$$\partial(|0\rangle) = 0, \quad \partial(u_1 u_2 \cdots u_s |0\rangle) = \sum_{i=1}^{s} u_1 \cdots u_{i-1} \partial(u_i) u_{i+1} \cdots u_s |0\rangle \quad (6.58)$$

for $u_i \in \hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A})_\cdot$. For $i, j \in \overline{1,n}$ and $r \in \mathbb{N} + \ell_j$, we denote

$$E_{i,j}(r, z) = \sum_{l \in \mathbb{Z}} t^l \partial^l E_{i,j} z^{-l-1}. \quad (6.59)$$

Furthermore, we define linear maps

$$Y^\pm(\, , z) : \hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A})_\cdot \to \text{LM}(\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A})), \mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}))[z^{-1}, z]) \quad (6.60)$$

by

$$Y^\pm(t^{-m-1} \partial^i E_{i,j}, z) = \frac{1}{m!} \frac{d^m}{dz^m} E_{i,j}(r, z)^\pm. \quad (6.61)$$

Now we define a linear map

$$Y(\, , z) : \mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A})) \to \text{LM}(\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A})), \mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}))[z^{-1}, z]) \quad (6.62)$$

by induction:

$$Y(|0\rangle, z) = \text{Id}_{\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}))}, \quad Y(u v, z) = Y^-(u, z) Y(v, z) + Y(v, z) Y^+(u, z) \quad (6.63)$$

for $u \in \hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A})_\cdot$ and $v \in \mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}))$. In particular,

$$Y((t^{-m-1} \partial^i E_{i,j})|0\rangle, z) = \frac{1}{m!} \frac{d^m}{dz^m} E_{i,j}(r, z) \quad \text{for } i, j \in \overline{1,n}, \quad r \in \mathbb{N} + \ell_j, \quad m \in \mathbb{N}. \quad (6.64)$$

By Lemma 2.1 and the general theory of conformal algebras (cf. [K] and [X2]), we have:

**Theorem 6.2.** The family $(\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A})), Y(\, , z), \partial, |0\rangle)$ forms a vertex algebra. If $\chi \notin \mathbb{Z}$, the vertex algebra $(\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A})), Y(\, , z), \partial, |0\rangle)$ is simple. When $\chi \in \mathbb{N}$, the quotient space $\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}))$ forms a simple vertex algebra. If $\ell_i \in \{0, 1\}$ for $i \in \overline{1,n}$, $\mathcal{V}_\chi(\hat{\mathfrak{gl}}(\bar{\ell}, \mathbb{A}))$ with $\chi \in \mathbb{C} \setminus \mathbb{Z}$.
and $V_\chi(\hat{gl}(\ell, A))$ with $\chi \in \mathbb{Z}$ are simple vertex operator algebras with the Virasoro element $-t^{-1}\partial_t(\sum_{i=1}^n E_{i,i})|0\rangle$.

Take constants $\chi, \iota \in \mathbb{C}$ such that $\chi \neq 0$. Let $\mathcal{M}$ be a weighted irreducible $\hat{gl}(\infty)$-module satisfying (3.18) (also cf. (3.17)) and $\kappa_0|_\mathcal{M} = \chi \text{Id}_\mathcal{M}$. For $i, j \in \mathbb{N}$ and $r \in \mathbb{N} + \ell_j$, we denote

$$E_{i,j}(r, z) = \sum_{l,k \in \mathbb{Z}} (t - k)_r \mathcal{E}_{ln+i-1/2,kn-j+1/2} z^{-l-k-r-1} + r! \mathcal{Z}_{0,r} \delta_{i,j} \kappa_0 z^{-r-1} \tag{6.65}$$

if $\iota \notin \mathbb{Z}$, and

$$E_{i,j}(r, z) = \sum_{l,k=0}^n (t - k)_r \mathcal{E}_{l-i,n+i-1/2,(k+i+1)n-j+1/2} z^{-l-k-l_i-r-2}$$

$$+ (k + \ell_j)_r \mathcal{E}_{(l-i)n+i-1/2,(k-i+1)n-j+1/2} z^{-l+k+\ell_i-\ell_j-r-1}$$

$$+ (k + \ell_j)_r \mathcal{E}_{(l-i+1)n+i-1/2,(k-i+1)n-j+1/2} z^{-l+k+\ell_i+\ell_j-r}$$

$$+ (t - k)_r \mathcal{E}_{(l-i+1)n+i-1/2,(k+i+1)n-j+1/2} z^{-l-k-r-1} + \delta_{i,j} r! \mathcal{Z}_{0,r} \kappa_0 z^{-r-1} \tag{6.66}$$

(cf. (2.26)). Furthermore, we define linear maps

$$Y_{\mathcal{M}}^{i,\pm} (\cdot, z) : \hat{gl}(\ell, A)_- \to LM(\mathcal{M}, \mathcal{M}[z^{-1}, z]) \tag{6.67}$$

by

$$Y_{\mathcal{M}}^{i,\pm} (t^{-1-m} \partial_t^m E_{i,j}, z) = \frac{1}{m!} \frac{d^m}{dz^m} E_{i,j}(r, z)^\pm. \tag{6.68}$$

Now we define a linear map

$$Y_{\mathcal{M}}^\chi (\cdot, z) : V_\chi(\hat{gl}(\ell, A)) \to LM(\mathcal{M}, \mathcal{M}[z^{-1}, z]) \tag{6.69}$$

by induction:

$$Y_{\mathcal{M}}^\chi (|0\rangle, z) = \text{Id}_\mathcal{M}, \quad Y(uv, z) = Y_{\mathcal{M}}^{i,-}(u, z) Y_{\mathcal{M}}^\chi (v, z) + Y_{\mathcal{M}}^\chi (v, z) Y_{\mathcal{M}}^{i,+}(u, z) \tag{6.70}$$

for $u \in \hat{gl}(\ell, A)_-$ and $v \in V_\chi(\hat{gl}(\ell, A))$. By Theorem 3.1, Theorem 3.2 and the general theory for vertex algebras (e.g. cf. Section 4.1 in [X2]), we have:

**Theorem 6.3.** The family $(\mathcal{M}, Y_{\mathcal{M}}^\chi (\cdot, z))$ forms an irreducible module of the vertex algebra $(V_\chi(\hat{gl}(\ell, A)), Y (\cdot, z), \partial, |0\rangle)$.

In the rest of this section, we want to show certain familiar unitary highest weight irreducible $\hat{gl}(\infty)$-modules induce irreducible modules of the quotient simple vertex algebra $(V(\hat{gl}(\ell, A)), Y (\cdot, z), \partial, |0\rangle)$ when $\chi \in \mathbb{Z}$. To this end, we need to use charged free fields.

First we use charged free fermionic field realization. Denote

$$Z_+ = \mathbb{N} + 1/2, \quad Z_- = -Z_+. \tag{6.71}$$

48
Then
\[ Z = Z_+ \cup Z_- \] (6.72)

Let \( \{ \theta_l, \bar{\theta}_l \mid l \in Z_- \} \) be a set of odd variables, that is,
\[ \theta_l \theta_k = -\theta_k \theta_l, \quad \theta_l \bar{\theta}_k = -\bar{\theta}_k \theta_l, \quad \bar{\theta}_l \bar{\theta}_k = -\bar{\theta}_k \bar{\theta}_l. \] (6.73)

Set
\[ V_f = \mathbb{C}[\theta_l, \bar{\theta}_l \mid l \in Z_-], \] (6.74)
the polynomial algebra in odd variables \( \{ \theta_l, \bar{\theta}_l \mid l \in Z_- \} \), where the subindex “f” stands for “fermionic”. Moreover, we denote
\[ \theta_l = \partial \bar{\theta}_{-l}, \quad \bar{\theta}_l = \partial \theta_{-l} \text{ for } l \in Z_+. \] (6.75)

It can be verified that we the following representation of \( \tilde{gl}(\infty) \) on \( V_f \):
\[ \kappa_0|V_f \rangle = \text{Id}_{V_f}, \quad \mathcal{E}_{l,k}|V_f \rangle = \begin{cases} \bar{\theta}_l \theta_k & \text{if } l \in Z_- \text{ or } -k \neq l \in Z_+, \\ -\theta_k \bar{\theta}_l & \text{if } -k = l \in Z_+ \end{cases} \] (6.76)
for \( l, k \in Z \). Set
\[ \partial = \sum_{l \in \mathbb{Z}} l \mathcal{E}_{-1-l,l}. \] (6.77)

Given \( i \in \overline{1,n} \), we define
\[ \theta(i, i, z) = \sum_{l \in \mathbb{Z}} \theta_{l_n-i+1/2} z^{i-l}, \quad \bar{\theta}(i, i, z) = \sum_{l \in \mathbb{Z}} \bar{\theta}_{i_n-i-1/2} \bar{z}^{-i-l-1} \] (6.78)
for \( i \in \mathbb{C} \setminus \mathbb{Z} \), and
\[ \theta(i, i, z) = \sum_{l=0}^{\infty} [\theta_{(-l+i)n-i+1/2} \bar{z}^{i-l+1} + \theta_{(l+i+1)n-i+1/2} \bar{z}^{-i-l-1}], \] (6.79)
\[ \bar{\theta}(i, i, z) = \sum_{l=0}^{\infty} [\bar{\theta}_{(-l-i)n+i-1/2} \bar{z}^{-i-l-1} + \bar{\theta}_{(-l+i+1)n+i-1/2} \bar{z}^{-i} \] (6.80)
for \( i \in \mathbb{Z} \). Then \( \{ \theta(i, i, z), \bar{\theta}(i, i, z) \mid i \in \overline{1,n} \} \) are charged free fermionic fields.

Set
\[ \Theta = \sum_{l \in \mathbb{Z}_-} \mathbb{C} \theta_l, \quad \bar{\Theta} = \sum_{l \in \mathbb{Z}_-} \mathbb{C} \bar{\theta}_l, \] (6.81)
and
\[ V_{f,0} = \mathbb{C} + \sum_{s=1}^{\infty} \bar{\Theta} \Theta^s, \quad V_{f,r} = V_{f,0} \Theta^r, \quad V_{f,-r} = \bar{\Theta} \bar{\Theta} V_{f,0} \] (6.82)
for \( r \in \mathbb{N} + 1 \). Then
\[ V_f = \bigoplus_{k \in \mathbb{Z}} V_{f,k}. \] (6.83)

We define a linear map \( Y'(\cdot, z) : V_f \to LM(V_f, V_f \{z\}) \) by induction:
\[ Y'(1, z) = \text{Id}_{V_f}, \] (6.84)
\[
Y^i(\theta_{-rn-i+1/2}u, z) = \text{Res}_{z_1} \sum_{s=0}^{\infty} (-1)^s \left( \frac{-t}{s} \right) z_1^{-s} z^s [(z_1 - z)^{s-r-\ell_i+1} \theta(t, i, z_1) Y^i(u, z)] \\
- (1)^k (-z + z_1)^{s-r-\ell_i+1} Y^i(u, z) \theta(t, i, z_1)] \quad (6.85)
\]

and

\[
Y^i(\tilde{\theta}_{-(r+1)n+i-1/2}u, z) = \text{Res}_{z_1} \sum_{s=0}^{\infty} (-1)^s \left( \frac{t}{s} \right) z_1^{-s} z^{-s} [(z_1 - z)^{s-r-\ell_i+1} \tilde{\theta}(t, i, z_1) Y^i(u, z)] \\
- (1)^k (-z + z_1)^{s-r-\ell_i+1} Y^i(u, z) \tilde{\theta}(t, i, z_1)] \quad (6.86)
\]

for \( i \in \overline{1,n} \), \( r \in \mathbb{N} \) and \( u \in V_{f,k} \). By Section 4.1 in [X2], we have

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y^i(u, z_1) Y^i(v, z_2) - (1)^{k_1 k_2} z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y^i(v, z_2) Y^i(u, z_1)
= z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{k_1^i} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y^i(V^0(u, z_0) v, z_2) \quad (6.87)
\]

for \( u \in V_{f,k_1} \) and \( v \in V_{f,k_2} \). In fact,

\[
E^i_{\ell,j}(r, z)|_{V_f} = Y^i(\tilde{\theta}_{-n+i-1/2} \theta_{-n-j+1/2}, z) \quad (6.88)
\]

(cf. (6.65), (6.66)),

\[
Y^i(\theta_{-rn+i-1/2}, z) = \frac{1}{(r + \ell_i)!} \frac{d^{r+\ell_i}}{z^{r+\ell_i}} \theta(t, i, z) \quad (6.89)
\]

and

\[
Y^i(\tilde{\theta}_{-(r+1)n-i+1/2}, z) = \frac{1}{r!} \frac{d^r}{z^r} \tilde{\theta}(t, i, z) \quad (6.90)
\]

for \( i, j \in \overline{1,n} \).

Let \( k \in \mathbb{N} + 1 \). Denote

\[
v_0 = 1, \quad v_k = \theta_{-1/2} \theta_{-3/2} \cdots \theta_{1/2-k}, \quad v_{-k} = \tilde{\theta}_{-1/2} \tilde{\theta}_{-3/2} \cdots \tilde{\theta}_{1/2-k}. \quad (6.91)
\]

Moreover, we define linear functions on \( \mathcal{T} \) in (3.10):

\[
\lambda^0(\kappa_0) = 1, \quad \lambda^0(\mathcal{E}_{i,-}) = 0 \quad \text{for} \quad l \in \mathbb{Z},
\]

\[
\lambda^{-k}(\kappa_0) = \lambda^{-k}(\mathcal{E}_{-r-1/2,-r+1/2}) = 1, \quad \lambda^{-k}(\mathcal{E}_{-l-1/2,-l+1/2}) = 0 \quad (6.93)
\]

for \( r \in \overline{0,k-1} \) and \( l \in \mathbb{Z} \setminus \{0,k-1\} \), and

\[
\lambda^k(\kappa_0) = -\lambda^k(\mathcal{E}_{r+1/2,-r-1/2}) = 1, \quad \lambda^k(\mathcal{E}_{l+1/2,-l-1/2}) = 0 \quad (6.94)
\]

for \( r \in \overline{0,k-1} \) and \( l \in \mathbb{Z} \setminus \{0,k-1\} \).

**Lemma 6.4.** Each space \( V_{f,k} \) forms a unitary irreducible highest weight module of \( \mathfrak{gl}(\infty) \)

with the highest weight \( \lambda^k \). Moreover, a linear function \( \lambda \) on \( \mathcal{T} \) satisfies (3.59), (3.65) and

\( \lambda(\kappa_0) = 1 \), if and only if \( \lambda \) is of the form \( \lambda^k \) for some \( k \in \mathbb{Z} \). The family \((V_{f,0}, Y^0(\cdot, z), 1, \partial)\)
is a simple vertex operator algebra isomorphic to \((V^1(\widehat{gl}(\vec{L},A)), Y(\cdot, z), |0\rangle, \partial)\) in Theorem 6.2. Moreover, each \((V_{f,k}, Y^i(|V_{f,0}, z)|_{V_{f,k}})\) is an irreducible \(V_{f,0}\)-module. The map \(Y^i(|V_{f,l}, z)|_{V_{f,k}}\) is an intertwining operator of type \([V_{f,l}^{V_{f,l+k}^{V_{f,k}}}, V_{f,k}]\).

Now we want to deal with higher level case. Assume \(1 < \chi \in \mathbb{N}\). Form \(\chi\)th \(\widehat{gl}(\infty)\)-module tensor:

\[
V^{(\chi)}_f = V_f \otimes V_f \otimes \cdots \otimes V_f \quad (\chi \text{ copies}).
\]

Then

\[
\kappa_0|_{V^{(\chi)}_f} = \chi \text{Id}_{V^{(\chi)}_f}.
\]

Since \(V_f\) is an unitary \(\widehat{gl}(\infty)\)-module that is a direct sum of the highest weight irreducible modules \(V_{f,k}\) with weights \(\lambda^k\) satisfying (3.59), we can apply the tensor theory of finite-dimensional general linear Lie algebras to \(V^{(\chi)}_f\). Thus \(V^{(\chi)}_f\) is a completely reducible \(\widehat{gl}(\infty)\)-module. Moreover, for each weight \(\lambda\) satisfying (3.59), (3.65) and \(\lambda(\kappa_0) = \chi\), there exists a component of \(V^{(\chi)}_f\) that is a highest weight irreducible \(\widehat{gl}(\infty)\)-module with weight \(\lambda\).

Denote

\[
1_\chi = 1 \otimes 1 \otimes \cdots \otimes 1 \quad (\chi \text{ copies}).
\]

Since \(V_f\) is also a polynomial algebra in odd variables, \(V^{(\chi)}_f\) has an extended associative tensor algebra structure. Note the space

\[
\Theta \Theta = \sum_{l,k \in \mathbb{Z}_-} \mathbb{C} \theta_l \theta_k.
\]

We define the diagonal linear map \(g : \Theta \Theta \to V^{(\chi)}_f\) by

\[
g(\theta_l \theta_k) = \sum_{i=1}^{\chi} 1 \otimes \cdots \otimes 1 \otimes \theta_l^i \theta_k \otimes 1 \otimes \cdots \otimes 1
\]

for \(l, k \in \mathbb{Z}_-\). Set

\[
V^{[\chi]}_{f,0} = \mathbb{C} 1_\chi + \sum_{r=1}^{\infty} [g(\Theta \Theta)]^r.
\]

Moreover, define the map

\[
Y^{(\chi)}(\cdot, z) = Y^i(\cdot, z) \otimes Y^i(\cdot, z) \otimes \cdots \otimes Y^i(\cdot, z) \quad (\chi \text{ copies})
\]

and

\[
\partial^{(\chi)} = \sum_{i=1}^{\chi} 1 \otimes \cdots \otimes 1 \otimes \partial \otimes 1 \otimes \cdots \otimes 1
\]

(cf. (6.77)). Then the family \((V^{[\chi]}_{f,0}, Y^{0}(\cdot, z), 1_\chi, \partial^{(\chi)})\) forms a simple vertex algebra isomorphic to \((V_\chi(\widehat{gl}((\vec{L},A))), Y(\cdot, z), |0\rangle, \partial)\) in Theorem 6.2. For each irreducible \(\widehat{gl}(\infty)\)-module component \(U\) of \(V^{(\chi)}_f\), the family \((U, Y^{\chi}(|V^{[\chi]}_{f,0}, z)|_U)\) forms an irreducible module of \((V^{[\chi]}_{f,0}, Y^{0}(\cdot, z), 1_\chi, \partial^{(\chi)})\).
Lemma 6.5. Suppose that $\chi$ is a positive integer and $\lambda$ is a weight of $\tilde{gl}(\infty)$ satisfying (3.59), (3.65) and $\lambda(\kappa_0) = \chi$. Let $\mathcal{M}$ be the irreducible highest weight $\tilde{gl}(\infty)$-module with highest weight $\lambda$. Then the family $(\mathcal{M}, Y^\lambda_{\mathcal{M}})$ defined in (6.65)-(6.70) forms an irreducible module of the simple vertex algebra $(V_{\lambda}(\tilde{gl}(\vec{\ell}, A)), Y(\cdot, z), |0\rangle, \partial)$ in Theorem 6.2, equivalently,

\[ Y^\lambda_{\mathcal{M}}((t^{-1}\partial \ell_1)E_{n,1})^{\chi+1}|0\rangle, z) = 0 \]  
(6.103)

when $n > 1$.

We remark that (6.103) can be proved easily by using the affine Lie algebra $\tilde{sl}(2, \mathbb{C})$ when $\ell_1 = \ell_n = 0$. Let $\mathcal{M}$ be the highest weight irreducible module in Example 3.1. By the locality of $Y^\lambda_{\mathcal{M}}(\cdot, z)$, (6.103) holds if and only if

\[ Y^\lambda_{\mathcal{M}}((t^{-1}\partial \ell_1)E_{n,1})^{\chi+1}|0\rangle, z)v_\lambda = 0. \]  
(6.104)

According to the above lemma with $n = 2$, (4.104) holds if

\[ \lambda(\mathcal{E}_{rn+1/2, -rn-1/2} - \mathcal{E}_{rn-1/2, -rn+1/2} + \delta_{r,0}\kappa_0) \in \mathbb{N} \quad \text{for} \quad r \in \mathbb{Z} \]  
(4.105)

and when $n > 2$

\[ \lambda(\mathcal{E}_{rn+n/2, -rn-n+1/2} - \mathcal{E}_{rn+1/2, -rn-1/2} + \delta_{r,0}) \in \mathbb{N} \quad \text{for} \quad r \in \mathbb{Z}, \]  
(4.106)

Note that the condition of (4.105) and (4.106) is weaker than (3.65) when $n > 3$. Under the condition of (4.105) and (4.106), $\mathcal{M}$ may not be a unitary $\tilde{gl}(\infty)$-module. Therefore, we have proved the following main theorem:

**Theorem 6.6.** Suppose that $\chi$ is a positive integer and $\lambda$ is a weight of $\tilde{gl}(\infty)$ satisfying (3.59), (6.105), (6.106) and $\lambda(\kappa_0) = \chi$. Let $\mathcal{M}$ be the irreducible highest weight $\tilde{gl}(\infty)$-module with highest weight $\lambda$. Then the family $(\mathcal{M}, Y^\lambda_{\mathcal{M}})$ defined in (6.65)-(6.70) forms an irreducible module of the simple vertex algebra $(V_{\lambda}(\tilde{gl}(\vec{\ell}, A)), Y(\cdot, z), |0\rangle, \partial)$ in Theorem 6.2.

Next we use charged free bosonic field realization to study the case of negative integral $\chi$. Set

\[ V_b = \mathbb{C}[x_l, \bar{x}_l \mid l \in \mathbb{Z}_-], \]  
(6.107)

the polynomial algebra in a set of ordinary commute variables \{\(x_l, \bar{x}_l \mid l \in \mathbb{Z}_-\)\} (cf. (6.71)), where the subindex “b” stands for “bosonic”. Moreover, we denote

\[ x_l = \partial \bar{x}_{-l}, \quad \bar{x}_l = -\partial x_{-l} \quad \text{for} \quad l \in \mathbb{Z}_+. \]  
(6.108)

It can be verified that we the following representation of $\tilde{gl}(\infty)$ on $V_b$:

\[ \kappa_0|_{V_b} = -\text{Id}_{V_b}, \quad \mathcal{E}_{-l,-k}|_{V_b} = -\partial x_l \partial \bar{x}_k, \quad \mathcal{E}_{l,-k}|_{V_b} = \bar{x}_l \partial \bar{x}_k, \]  
(6.109)
\[ \mathcal{E}_{-l,k}|v_b = -x_k \partial_{x_l}, \quad \mathcal{E}_{l,k}|v_b = \bar{x}_l x_k \] (6.110)

for \( l, k \in \mathbb{Z}_- \). The operator \( \partial \) acts on \( V_b \) by (6.77). Given \( i \in \overline{1, n} \), we define

\[ x(t, i, z) = \sum_{l \in \mathbb{Z}} x_{l-i+1/2} z^{l-1-t}, \quad \bar{x}(t, i, z) = \sum_{l \in \mathbb{Z}} \bar{x}_{l-i-1/2} z^{-l-1-t} \] (6.111)

for \( t \in \mathbb{C} \setminus \mathbb{Z} \), and

\[ x(t, i, z) = \sum_{l=0}^{\infty} [x_{(-l+i)n-i+1/2} z^{l+1} + x_{(l+i+1)n-i+1/2} z^{-l-1}], \] (6.112)

\[ \bar{x}(t, i, z) = \sum_{l=0}^{\infty} [\bar{x}_{(l-i)n+i-1/2} z^{-l-i-1} + \bar{x}_{-(l+i+1)+i-1} z^{2l}] \] (6.113)

for \( t \in \mathbb{Z} \). Then \( \{x(t, i, z), \bar{x}(t, i, z) \mid i \in \overline{1, n} \} \) are charged free bosonic fields.

Set

\[ X = \sum_{l \in \mathbb{Z}_-} \mathbb{C} x_l, \quad \bar{X} = \sum_{l \in \mathbb{Z}_-} \mathbb{C} \bar{x}_l, \] (6.114)

and

\[ V_{b,0} = \mathbb{C} + \sum_{s=1}^{\infty} \bar{X}^{s} X^{s}, \quad V_{b,r} = V_{b,0} X^{r}, \quad V_{b,-r} = \bar{X}^{r} V_{b,0} \] (6.115)

for \( r \in \mathbb{N} + 1 \). Then

\[ V_b = \bigoplus_{k \in \mathbb{Z}} V_{b,k}. \] (6.116)

We define a linear map \( Y^i(\cdot, z) : V_b \to LM(V_b, V_b\{z\}) \) by induction:

\[ Y^i(1, z) = \text{Id}_{V_b}, \] (6.117)

\[ Y^i(x_{-(r-n-i+1/2})u, z) = \text{Res}_{z_1} \sum_{s=0}^{\infty} (-1)^s \left( \begin{array}{c} -s \vspace{0.1cm} \hline \vspace{0.1cm} l \end{array} \right) \bar{z}_1^{-l-s} z_l^{l-1-s} [(z_1 - z)^{s-r-l-1} x(t, i, z_1) Y^i(u, z) \]

\[ -(-z + z_1)^{s-r-l-1} Y^i(u, z) x(t, i, z_1)] \] (6.118)

and

\[ Y^i(\bar{x}_{-(r+1)n+i-1/2}u, z) = \text{Res}_{z_1} \sum_{s=0}^{\infty} (-1)^s \left( \begin{array}{c} -s \vspace{0.1cm} \hline \vspace{0.1cm} l \end{array} \right) \bar{z}_1^{-l-s} \bar{z}_l^{l-1-s} [(z_1 - z)^{s-r-l-1} \bar{x}(t, i, z_1) Y^i(u, z) \]

\[ -(-z + z_1)^{s-r-l-1} Y^i(u, z) \bar{x}(t, i, z_1)] \] (6.119)

for \( i \in \overline{1, n}, \quad r \in \mathbb{N} \) and \( u \in V_{f,k} \). By Section 4.1 in [X2], we have

\[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y^i(u, z_1) Y^i(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y^i(v, z_2) Y^i(u, z_1) \]

\[ = z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{k_i} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y^i(Y^0(u, z_0) v, z_2) \] (6.120)

for \( u \in V_{b,k} \) and \( v \in V_b \). In fact,

\[ E^i_{j}(r, z)|v_b = Y^i(\bar{x}_{-n+i-1/2} x_{-n-j+1/2}, z) \] (6.121)
(cf. (6.65), (6.66))

\[ Y^i(x_{-r n + i - 1/2}, z) = \frac{1}{(r + \ell_i)!} z^{r + \ell_i} x(t, i, z) \]  

(6.122)

and

\[ Y^i(\bar{x}_{-(r+1)n - i + 1/2}, z) = \frac{d^r}{r!} \bar{x}(t, i, z) \]  

(6.123)

for \( i \in \mathbb{N} \).

Let \( k \in \mathbb{N} + 1 \). Denote

\[ v_0 = 1, \quad v_k = x_{-1/2}, \quad v_{-k} = \bar{x}_{-1/2}. \]  

(6.124)

Moreover, we define linear functions on \( T \) in (3.12):

\[ \lambda^0(\kappa_0) = -1, \quad \lambda^0(\mathcal{E}_{l,-l}) = 0 \quad \text{for} \ l \in \mathbb{Z}, \]  

(6.125)

\[ \lambda^k(\kappa_0) = -1, \quad \lambda^k(\mathcal{E}_{1/2,-1/2}) = -k, \quad \lambda^k(\mathcal{E}_{l+1/2,-l-1/2}) = 0 \]  

(6.126)

for \( 0 \neq l \in \mathbb{Z} \), and

\[ \lambda^{-k}(\kappa_0) = -1, \quad \lambda^{-k}(\mathcal{E}_{-1/2,1/2}) = k, \quad \lambda^{-k}(\mathcal{E}_{l-1/2,-l+1/2}) = 0 \]  

(6.127)

for \( 0 \neq l \in \mathbb{Z} \).

**Lemma 6.7.** Each space \( V_{b,k} \) forms a unitary irreducible highest weight module of \( \tilde{\mathfrak{gl}}(\infty) \) with the highest weight \( \lambda^k \). The family \( (V_{b,0}, Y^0(\cdot, z), 1, \partial) \) is a simple vertex operator algebra isomorphic to \( (V_{-1}(\tilde{\mathfrak{gl}}(\ell, A)), Y(\cdot, z), |0\rangle, \partial) \) in Theorem 6.2. Moreover, each \( (V_{b,k}, Y^i(|V_{b,0}, z)|V_{b,k}) \) is an irreducible \( V_{b,0} \)-module. The map \( Y^i(|V_{b,l}, z)|V_{b,k} \) is an intertwining operator of type \([V_{b,l}^{V_{b,l+k}} V_{b,k}]\).

Now we want to deal with higher level case. Assume \( 1 < \chi \in \mathbb{N} \). Form \( \chi \)th \( \tilde{\mathfrak{gl}}(\infty) \)-module tensor:

\[ V_b^{(\chi)} = V_b \otimes V_b \otimes \cdots \otimes V_b \quad (\chi \text{ copies}). \]  

(6.128)

Then

\[ \kappa_0|_{V_b^{(\chi)}} = \chi \text{Id}_{V_b^{(\chi)}}. \]  

(6.129)

Set

\[ \mathcal{L} = \sum_{l,k \in \mathbb{Z}_-} \mathbb{C} x_l \partial x_k, \quad \bar{\mathcal{L}} = \sum_{l,k \in \mathbb{Z}_-} \mathbb{C} \bar{x}_l \partial \bar{x}_k. \]  

(6.130)

Then \( \mathcal{L} \) and \( \bar{\mathcal{L}} \) are Lie subalgebras of \( \tilde{\mathfrak{gl}}(\infty)|_{V_b} \). Moreover, they are infinite-dimensional general Lie algebras. Denote
\[ H = \sum_{l \in \mathbb{Z}} x_l \partial x_l, \quad \bar{H} = \sum_{l \in \mathbb{Z}} \bar{x}_l \partial \bar{x}_l, \quad (6.131) \]

The subspace \( H \) is a toral Cartan subalgebra of \( \mathcal{L} \) and the subspace \( \bar{H} \) is a toral Cartan subalgebra of \( \bar{\mathcal{L}} \). Moreover, \( X^k \) is an irreducible highest weight \( \mathcal{L} \)-module with the highest weight \( \lambda_X^k \) determined by

\[ \lambda_X^k(x_l \partial x_l) = k \delta_{l,-1/2} \quad \text{for} \ l \in \mathbb{Z}, \quad (6.132) \]

and \( \bar{X}^k \) is an irreducible highest weight \( \bar{\mathcal{L}} \)-module with the highest weight \( \lambda_{\bar{X}}^k \) determined by

\[ \lambda_{\bar{X}}^k(\bar{x}_l \partial \bar{x}_l) = k \delta_{l,-1/2} \quad \text{for} \ l \in \mathbb{Z}. \quad (6.133) \]

For a positive integer \( s \), we define:

\[ \Gamma^s_X = \{ \lambda_X \in H^* \mid \lambda_X(x_{1/2}^{1/2} \partial x_{1/2^{1/2}}) \in \mathbb{N}, \lambda_X(x_{-1/2}^{1/2} \partial x_{-1/2^{1/2}}) = 0 \text{ for } r \in \mathbb{N}, l \in \mathbb{N} \} \]

\[ \Gamma^s_{\bar{X}} = \{ \lambda_{\bar{X}} \in \bar{H}^* \mid \lambda_{\bar{X}}(\bar{x}_{1/2}^{1/2} \partial \bar{x}_{1/2^{1/2}}) \in \mathbb{N}, \lambda_{\bar{X}}(\bar{x}_{-1/2}^{1/2} \partial \bar{x}_{-1/2^{1/2}}) = 0 \text{ for } r \in \mathbb{N}, l \in \mathbb{N} \}. \]

Set

\[ \mathcal{C} = \mathbb{C}[x_l \mid l \in \mathbb{Z}_-] = \bigoplus_{r=0}^{\infty} X^r, \quad \bar{\mathcal{C}} = \mathbb{C}[\bar{x}_l \mid l \in \mathbb{Z}_-] = \bigoplus_{r=0}^{\infty} \bar{X}^r. \quad (6.136) \]

Then

\[ V_0 = \bar{\mathcal{C}} \mathcal{C}. \quad (6.137) \]

By the tensor theory of modules of finite-dimensional general Lie algebras, the \( s \)-tensor

\[ \mathcal{C}^{(s)} = \mathcal{C} \otimes \mathcal{C} \otimes \cdots \otimes \mathcal{C} \ (s \text{ copies}) \quad (6.138) \]

can be decomposed as a direct sum of highest weight irreducible \( \mathcal{L} \)-submodules, whose set of highest weights is exactly \( \Gamma^s_X \). Similarly, the \( s \)-tensor

\[ \bar{\mathcal{C}}^{(s)} = \bar{\mathcal{C}} \otimes \bar{\mathcal{C}} \otimes \cdots \otimes \mathcal{C} \ (s \text{ copies}) \quad (6.139) \]

can be decomposed as a direct sum of highest weight irreducible \( \bar{\mathcal{L}} \)-submodules, whose set of highest weights is exactly \( \Gamma^s_{\bar{X}} \).

Since \( V_0 \) is an unitary \( \mathfrak{gl}(\infty) \)-module, \( V_0^{(\chi)} \) is a completely reducible \( \mathfrak{gl}(\infty) \)-module. Let \( s_1, s_2 \in \mathbb{N} \) such that \( s_1 + s_2 = \chi \). Suppose that \( v_1 \) is a highest weight vector of an irreducible component of the \( \mathcal{L} \)-module \( \mathcal{C}^{(s)} \) with \( \lambda_X \), and \( v_2 \) is a highest weight vector of an irreducible
component of the $L$-module $C^{(s)}$ with $\lambda_X$. Then $v_1 \otimes v_2$ is the highest weight vector of some irreducible component of the $\tilde{g}l(\infty)$-module $V_{\chi}^{(x)}$ (cf. (6.128)), whose weight $\lambda$ is given by:

$$\lambda(\kappa_0) = -\chi, \quad \lambda(\mathcal{E}_{l,-l}) = \lambda_X(x_l \partial x_l), \quad \lambda(\mathcal{E}_{-l,l}) = -\lambda_X(x_l \partial x_l), \quad \text{for } l \in \mathbb{Z}_. \quad (6.140)$$

Set

$$S_\chi = \{\{3/2 - r, 5/2 - r, ..., (2\chi + 1)/2 - r\} | r \in [0, \chi + 1]\}. \quad (6.141)$$

For $\lambda \in T^*$, we define

$$\text{supp } \lambda = \{l \in \mathbb{Z} | \lambda(\mathcal{E}_{l,-l}) \neq 0\}. \quad (6.142)$$

Set

$$\Gamma^\chi = \{\lambda \in T^* | \lambda(\kappa_0) = -\chi, -s^{-1}|s|\lambda(\mathcal{E}_{s,-s}) \in \mathbb{N} \text{ for } s \in \mathbb{Z}; \text{ supp } \lambda \subset S \text{ for some } S \in S_\chi\}. \quad (6.143)$$

Expression (6.140) implies that every element in $\Gamma^\chi$ is the highest weight of some irreducible component of the $\tilde{g}l(\infty)$-module $V_{\chi}^{(x)}$.

Denote

$$1_\chi = 1 \otimes 1 \otimes \cdots \otimes 1 \quad (\chi \text{ copies}). \quad (6.144)$$

Since $V_b$ is also a polynomial algebra, $V_{\chi}^{(x)}$ has an extended commutative associative tenor algebra structure. Note the space

$$\bar{XX} = \sum_{l,k \in \mathbb{Z}_-} \mathbb{C}\bar{x}_l x_k. \quad (6.145)$$

We define the diagonal linear map $\rho : \bar{XX} \to V_{\chi}^{(x)}$ by

$$\rho(\bar{x}_l x_k) = \sum_{i=1}^\chi 1 \otimes \cdots \otimes 1 \otimes \bar{x}_l x_k \otimes 1 \otimes \cdots \otimes 1 \quad (6.146)$$

for $l, k \in \mathbb{Z}_-$. Set

$$V_{b,0}^{[\chi]} = \mathbb{C}1_\chi + \sum_{r=1}^\infty [\rho(\bar{XX})]^r. \quad (6.147)$$

Moreover, define the map

$$Y_\chi^t(\cdot, z) = Y^t(\cdot, z) \otimes Y^t\cdots Y^t(\cdot, z) \quad (\chi \text{ copies}) \quad (6.148)$$

and

$$\partial^{(\chi)} = \sum_{i=1}^\chi 1 \otimes \cdots \otimes 1 \otimes \partial \otimes 1 \otimes \cdots \otimes 1 \quad (6.149)$$

(cf. (6.77)). Then the family $(V_{b,0}^{[\chi]}, Y_\chi^t(\cdot, z), 1_\chi, \partial^{(\chi)})$ forms a simple vertex algebra isomorphic to $(V_{-\chi}(\tilde{g}l(\hat{\ell}, A)), Y(\cdot, z), [0], \partial)$ in Theorem 6.2. For each irreducible $\tilde{g}l(\infty)$-module component
where $\chi$ and $\mathcal{V}$ and the vacuum module is an irreducible

\[ \hat{\mathfrak{gl}}(6.70) \] forms an irreducible module of the simple vertex algebra $\mathfrak{gl}$.

In this section, we study the vacuum representations of the Lie algebras $\hat{\mathfrak{gl}}(\hat{\mathfrak{g}})$ of $\hat{\mathfrak{gl}}(\hat{\mathfrak{g}}_1)$ $(\mathfrak{M}, Y_{\mathfrak{M}})$ defined in (6.1)-(6.9). Note

\[ \mathcal{V} \equiv \delta_{\ell} \mathcal{V} \] and their related vertex algebra structures. Their vertex algebra irreducible representations are investigated.

Recall the general settings in (6.1)-(6.9). Note

\[ \hat{\mathfrak{gl}}(\hat{\mathfrak{g}}_1) = \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \sum_{m=1}^{\infty} \mathbb{C}(t^{-m}\delta_{\ell}^{r+\ell} E_{i,j} - (-1)^r(-\partial)^r t^{-m}\delta_{\ell}^{r} E_{j,i}) + \mathbb{C}\kappa \] (7.1)

and

\[ \hat{\mathfrak{sl}}(\hat{\mathfrak{g}}_1) = \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \sum_{m=1}^{\infty} \mathbb{C}(t^m\delta_{\ell}^{r+\ell} E_{i,j} - (-1)^r(-\partial)^r t^m\delta_{\ell}^{r} E_{j,i}) + \mathbb{C}\kappa. \] (7.2)

The vacuum module

\[ \mathcal{V}_{\chi}(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1)) = U(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1))|0 \] (7.3)

and

\[ \hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1)|0 = \{0\}, \quad \kappa(0) = \chi|0), \] (7.4)

where $\chi \in \mathbb{C}$.

**Theorem 7.1.** The module $\mathcal{V}_{\chi}(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1))$ is irreducible if $\chi \notin \mathbb{Z}$. When $\chi \in \mathbb{Z}$, the module

\[ \mathcal{V}(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1)) \] has a unique maximal proper submodule $\mathcal{V}_{\chi}(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1))$, and the quotient

\[ \mathcal{V}_{\chi}(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1)) = \mathcal{V}(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1))/\mathcal{V}_{\chi}(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1)) \] (7.5)

is an irreducible $\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1)$-module. Assume $\chi \in \mathbb{N}$. The submodule

\[ \mathcal{V}_{\chi}(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1)) = U(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1))((t^{-1}\delta_{\ell}^{\ell} E_{n,1})^{\chi+1}|0) \] (7.6)

if $n > 1$ and $\epsilon = 1$ (cf. (4.13)). When $\epsilon = 0$ and $n > 3$, the submodule

\[ \mathcal{V}_{\chi}(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1)) = U(\hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1))((t^{-1}(\delta_{\ell}^{\ell} E_{n-1,1} - \delta_{\ell}^{\ell} E_{n,2}))^{\chi+1}|0). \] (7.7)

**Proof.** For $k \in \mathbb{Z}$, we define

\[ \hat{\mathfrak{g}}(\hat{\mathfrak{g}}_1)_{(k)} = \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \mathbb{C}(t^{-k}\delta_{\ell}^{r+\ell} E_{i,j} - (-1)^r(-\partial)^r t^{-k}\delta_{\ell}^{r} E_{j,i}) + \mathbb{C}\delta_{k,0}\kappa \] (7.8)
Thus \( \mathfrak{d}(\breve{\ell}, \mathcal{A}) \) is a \( \mathbb{Z} \)-graded Lie algebra by (2.60), (2.66) and (2.68) with \( \iota = 0 \). We remark that this grading is not conformal weight grading. Moreover, we define a \( \mathbb{Z} \)-grading on \( \mathcal{V}_\chi(\mathfrak{d}(\breve{\ell}, \mathcal{A})) \) by

\[
\mathcal{V}_\chi(\mathfrak{d}(\breve{\ell}, \mathcal{A}))(0) = \mathbb{C}|0\rangle, \quad \mathcal{V}_\chi(\mathfrak{d}(\breve{\ell}, \mathcal{A}))(-m) = \{0\} \quad \text{for } m \in \mathbb{N} + 1
\]

and

\[
\mathcal{V}_\chi(\mathfrak{g}(\breve{\ell}, \mathcal{A}))(m) = \text{Span}\{u_1 u_2 \cdots u_s \mid u_i \in \mathfrak{d}(\breve{\ell}, \mathcal{A}) - \bigcap \mathfrak{d}(\breve{\ell}, \mathcal{A})(k_i); \sum_{i=1}^{s} k_i = m\}
\]

for \( m \in \mathbb{N} + 1 \). Then

\[
\mathcal{V}_\chi(\mathfrak{d}(\breve{\ell}, \mathcal{A})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{V}_\chi(\mathfrak{d}(\breve{\ell}, \mathcal{A}))(k)
\]

is a \( \mathbb{Z} \)-graded \( \mathfrak{d}(\breve{\ell}, \mathcal{A}) \)-module. observe

\[
[ \sum_{r,s \in \mathbb{N}} \mathbb{C}(t^{-s-1} \partial_l^r E_{i,j} - (-1)^r (-\partial_l)^r t^{-s-1} \partial_l^r E_{j,*})] \bigcap \mathfrak{d}(\breve{\ell}, \mathcal{A})(k)
\]

\[
= \sum_{r=0}^{k-1} \mathbb{C}(t^{-r} \partial_l^r E_{i,j} - (-1)^r (-\partial_l)^r t^{-r} \partial_l^r E_{j,*}).
\]

By [X3],

\[
\dim(\sum_{r=0}^{k-1} \mathbb{C}(t^{-r} \partial_l^r - (-1)^r (-\partial_l)^r t^{-r})) = k \epsilon + (-1)^r \lfloor |k/2| \rfloor.
\]

Thus the character

\[
d(\mathcal{V}_\chi(\mathfrak{d}(\breve{\ell}, \mathcal{A})), q) = \sum_{k=0}^{\infty} (\dim(\mathcal{V}_\chi(\mathfrak{d}(\breve{\ell}, \mathcal{A}))(k)) q^k = \prod_{r=1}^{\infty} \frac{1}{(1-q^r)^{n(r(n-1)/2+\epsilon+(1)^r \lfloor |r/2| \rfloor)}}.
\]

Recall the Lie algebra \( \mathfrak{g}(\infty) \) defined in (3.9) and (3.10), and the Lie algebra \( \mathcal{L}_0^{*,\breve{\ell}} \) in (5.12), which is a Lie subalgebra of \( \mathfrak{g}(\infty) \) by (5.11). Next we set

\[
\mathcal{L}_0^{*,\breve{\ell}}(-) = \sum_{i,j=1}^{n} \sum_{l,k=0}^{\infty} [\mathbb{C}((-1)^r(-k-1)\ell_l E_{(-l-1)n+i-1/2,-kn-j+1/2} - (-l-1)\ell_l E_{-kn-j+1/2,-(-l-1)n+i-1/2})
\]

and

\[
\mathcal{L}_0^{*,\breve{\ell}}(+) = \sum_{i,j=1}^{n} \sum_{l,k=0}^{\infty} [\mathbb{C}((-1)^r(-k-1)\ell_l E_{l+n+i-1/2,-kn-j+1/2} - (l+\ell_l)\ell_l E_{-kn-j+1/2,l+n+i-1/2}) + \mathbb{C}((-1)^r(k+\ell_r)\ell_l E_{l+n+i-1/2,(k+1)n-j+1/2} - (l+\ell_l)\ell_l E_{(k+1)n-j+1/2,l+n+i-1/2})].
\]

By (3.10), \( \mathcal{L}_0^{*,\breve{\ell}} \) are Lie subalgebras of \( \mathcal{L}_0^{*,\breve{\ell}} \) and

\[
\mathcal{L}_0^{*,\breve{\ell}} = \mathcal{L}_0^{*,\breve{\ell}}(-) + \mathcal{L}_0^{*,\breve{\ell}}(+),
\]

where

\[
\mathcal{L}_0^{*,\breve{\ell}}(-) = \mathcal{L}_{0,-}^{*,\breve{\ell}} + \mathcal{C}_{0,0}.
\]
Recall the $\mathfrak{gl}(\infty)$-module $U_{\chi}$ defined in (6.20)-(6.24). Note

$$L_{(+)}^* \mathfrak{1} = \{0\}$$

(7.19)

because $L_{(+)}^* \mathfrak{g} \mathfrak{l}(\infty) \subset \mathfrak{g} \mathfrak{l}(\infty)_{(+)}$. Thus

$$U_{\chi}^0 = U(L_{(-)}^* \mathfrak{1})$$

(7.20)

is a $L_{0}^*$-module, which satisfies the condition (5.45) with $s = \overline{l} + 1$ (cf. (5.42)) and $\mathcal{M}_0 = \mathbb{C} \mathbb{I} \otimes \mathfrak{1}$.

Note our notion (5.46). We have the $\hat{\mathfrak{o}}(\overline{l}, \mathfrak{a})$-module structure on $U_{\chi}^0$ defined by $\kappa = \chi \text{Id}_{U_{\chi}^0}$ and

$$(E_{i,j})_r(r, z) = \sum_{l,k=0}^{\infty} |(k)_r((-1)^{\ell}(-1)^{i,j}E_{(-l-1)n+i-1/2, -kn-j+1/2}
-(-l-1)\ell_iE_{-kn-j+1/2, (-l-1)n+i-1/2})z^{l+k-r}
+|k)_r((-1)^k\ell_jE_{ln+i-1/2, -kn-j+1/2}
-(l+\ell_i)\ell_iE_{-kn-j+1/2, 2ln+i-1/2})z^{l+k+1}r-1
-(-k-\ell_j-1)r((-1)^k\ell_jE_{(k+1)n-j+1/2, -(l+1)n+i-1/2})z^{l-k}r-1
+(-k-\ell_j-1)r((-1)^k\ell_jE_{ln+i-1/2, (k+1)n+j+1/2}
-(l+\ell_i)\ell_iE_{(k+1)n-j+1/2, 2ln+i-1/2})z^{l-k}r-2
$$

(7.21)

for $i, j \in \mathbb{I} \mathbb{N}$ and $r \in \mathbb{N}$. In particular,

$$(E_{i,j})_r(r, z) \mathfrak{1} = \sum_{l,k=0}^{\infty} |(k)_r((-1)^{\ell}(-1)^{i,j}E_{(-l-1)n+i-1/2, -kn-j+1/2}
-(-l-1)\ell_iE_{-kn-j+1/2, (-l-1)n+i-1/2})z^{l+k-r} \mathfrak{1}.$$ 

(7.22)

Thus we have

$$\hat{\mathfrak{o}}(\overline{l}, \mathfrak{a})_+(\mathfrak{1}) = \{0\}.$$ 

(7.23)

By a similar proof as that of Theorem 3.1,

$$U_{\chi}^0 = U(\hat{\mathfrak{o}}(\overline{l}, \mathfrak{a})_-) \mathfrak{1}.$$ 

(7.24)

Therefore, we have a Lie algebra module epimorphism $\nu : \mathcal{V}_{\chi}(\hat{\mathfrak{o}}(\overline{l}, \mathfrak{a})) \rightarrow U_{\chi}^0$ defined by

$$\nu(u|0)) = u \mathfrak{1} \quad \text{for} \quad u \in U(\hat{\mathfrak{o}}(\overline{l}, \mathfrak{a})_-).$$ 

(7.25)

For $m \in \mathbb{N} + 1$, we let

$$L_{(-)}^* \mathfrak{g} \mathfrak{l}_{m} = \text{Span} \{(\ell_{i,j}E_{(-l-1)n+i-1/2, -kn-j+1/2}
-(-l-1)\ell_iE_{-kn-j+1/2, (-l-1)n+i-1/2}) \mid i, j \in \mathbb{I} \mathbb{N}, l, k \in \mathbb{N}; l + k + 1 = m\}.$$ 

(7.26)
Then
\[
\mathcal{L}^\star_{(-)} = \bigoplus_{m=1}^{\infty} \mathcal{L}^\star_{(-),m}.
\] (7.27)

Moreover, we define
\[
U_\chi^{\alpha,(0)} = \mathbb{C}1, \quad U_\chi^{\alpha,(m)} = \{0\} \quad \text{for} \quad m \in (-N - 1)
\] (7.28)

and
\[
U_\chi^{\alpha,(m)} = \text{Span} \left\{ u_1 u_2 \cdots u_s 1 \mid u_i \in \mathcal{L}^\star_{(-),m_i}; \sum_{i=1}^{s} m_i = m \right\}.
\] (7.29)

Expressions (7.8), (7.21) and (7.24) imply
\[
U_\chi = \bigoplus_{m \in \mathbb{Z}} U_\chi^{0,(m)}
\] (7.31)

is a \(\mathbb{Z}\)-graded \(\hat{\mathfrak{o}}(\vec{\ell}, \mathfrak{A})\)-module. Furthermore, (7.26) gives the character
\[
d(U_\chi^\alpha, q) = \sum_{m=0}^{\infty} \left( \dim U_\chi^{0,(m)} \right) z^m = \prod_{r=1}^{\infty} \frac{1}{(1 - q^r)^{n(r(n-1)/2 + r\epsilon + (-1)^r ||r/2||)}}.
\] (7.32)

Therefore, (7.15) and (7.32) yield
\[
\mathcal{V}_\chi(\hat{\mathfrak{d}}(\vec{\ell}, \mathfrak{A})) \cong U_\chi^\alpha.
\] (7.33)

Let \(\lambda\) be a linear function on \(\mathcal{T}\) defined in (5.22) such that
\[
\lambda(\kappa_0) = \chi, \quad \lambda(\partial_l) = 0 \quad \text{for} \quad l \in \mathbb{N}
\] (7.34)
(cf. (5.21)). Recall the Verma module \(M_\lambda\) defined in (5.39) with \(\iota = 0\) and \(\tau = \ast\). Note
\[
U_\chi^\alpha \cong M_\lambda / \left( \bigoplus_{l=1}^{\infty} U(\mathcal{L}^\star_{0,-} f^{*}_{c,l} \otimes v_\lambda) \right),
\] (7.35)

which is irreducible if \(\chi \notin \mathbb{Z}\) by [J1-J3]. When \(\chi \in \mathbb{N}\),
\[
\bar{U}_\chi^\alpha = U(\mathcal{L}^\star_{0,-} f^{*}_{c,0})^{\chi+1} 1
\] (7.36)
is the unique maximal proper submodule of \(U_\chi^\alpha\). Thus
\[
U(\hat{\mathfrak{d}}(\vec{\ell}, \mathfrak{A})) \nu^{-1}((f^{*}_{c,0})^{\chi+1} 1)
\] (7.37)
is the unique maximal proper submodule of \(\mathcal{V}_\chi(\hat{\mathfrak{d}}(\vec{\ell}, \mathfrak{A}))\). When \(n > 1\), (5.28), (7.21) and (7.25) imply
\[
\nu^{-1}((f^{*}_{c,0})^{\chi+1} 1) = (t^{-1}\partial^{\ell_1} E_{n,1})^{\chi+1} 1.
\] (7.38)

Moreover, if \(n > 3\), (5.26), (7.21) and (7.25) yield
\[
\nu^{-1}((f^{*}_{0,0})^{\chi+1} 1) = (t^{-1}(\partial^{\ell_1} E_{n-1,1} - \partial^{\ell_2} E_{n,2}))^{\chi+1} 1.
\] (7.39)
Since $\mathfrak{o}(\ell, \mathbb{A})$ is a Lie subalgebra of $\hat{gl}(\ell, \mathbb{A})$, we view $V_\notv(\mathfrak{o}(\ell, \mathbb{A}))$ as a subspace of $V_\notv(\hat{gl}(\ell, \mathbb{A}))$. Recall the vertex algebra $(V_\notv(\hat{gl}(\ell, \mathbb{A})), Y(\cdot, z), \partial, |0\rangle)$ defined in (6.57)-(6.63). Then

$$
(V_\notv(\mathfrak{o}(\ell, \mathbb{A})), Y(|V_\notv(\mathfrak{o}(\ell, \mathbb{A})), z)|V_\notv(\mathfrak{o}(\ell, \mathbb{A})), \partial)|V_\notv(\mathfrak{o}(\ell, \mathbb{A})), |0\rangle)
$$

(7.40)
forms a vertex subalgebra. Let $\mathcal{M}$ be a weighted irreducible $\hat{gl}(\infty)$-module satisfying (3.18) (also cf. (3.17)) and $\kappa_0|_{\mathcal{M}} = \chi\text{Id}_{\mathcal{M}}$. Recall the linear map $Y_\notv, r(\cdot, z)$ defined by (6.65)-(6.70). Now we obtain:

**Theorem 7.2.** The family (7.40) forms a vertex algebra and $(\mathcal{M}, Y_\notv, r(|V_\notv(\mathfrak{o}(\ell, \mathbb{A})), z))$ forms an irreducible vertex algebra module of the vertex algebra (7.40). If $\chi \not\in \mathbb{Z}$, the vertex algebra $(V_\notv(\mathfrak{o}(\ell, \mathbb{A})), Y(\cdot, z), \partial, |0\rangle)$ is simple. When $\chi \in \mathbb{Z}$, the quotient space $V_\notv(\mathfrak{o}(\ell, \mathbb{A}))$ forms a simple vertex algebra. If $\ell_i = \epsilon$ for $i \in \overline{1,n}$, $V_\notv(\mathfrak{o}(\ell, \mathbb{A})))$ with $\chi \in \mathbb{C} \setminus \mathbb{Z}$ and $V_\notv(\mathfrak{o}(\ell, \mathbb{A}))$ with $\chi \in \mathbb{Z}$ are simple vertex operator algebras with the Virasoro element

$$
\sum_{i=1}^{||\epsilon+1/2||} (-t^{-1}\partial \epsilon_{i,i} + (-1)^{\epsilon}(-\partial \epsilon_{i,i} t^{-1}\partial \epsilon_{i,i}^\epsilon)|0\rangle).
$$

(7.41)

Assume that $\chi$ is a positive integer. Recall the assumption (3.39) and the charged free fermionic field realization given in (6.71)-(6.86). We set

$$
R^\notv = \text{Span}\left\{\tilde{\theta}_{\ell_0} - (-1)^{\ell_0}\tilde{\theta}_{\ell_0}\theta_{\ell} \mid \ell, k \in \mathbb{Z}_-\right\}.
$$

(7.42)

Note the notion $V_f^{(\chi)}$ defined in (6.95), the notion $1_\chi$ defined in (6.97) and the map $\varrho : \hat{\Theta} \to V_f^{(\chi)}$ defined by (6.81) and (6.99). Set

$$
V_{\notv, f}^{\epsilon} = \mathbb{C}1_\chi + \sum_{t=1}^{\infty} [\varrho(R^\notv)]^t.
$$

(7.43)

In terms of (6.101) and (6.102), the family $(V_{\notv, f}^{\epsilon}, Y_{\notv, f}^{\epsilon}(\cdot, z), 1_\chi, \varrho^{(\chi)})$ forms a simple vertex algebra isomorphic to $(V_\notv(\hat{gl}(\ell, \mathbb{A})), Y(\cdot, z), |0\rangle, \partial)$ in Theorem 7.2. For each irreducible $\hat{gl}(\infty)$-module component $U$ of $V_f^{(\chi)}$, the family $(U, Y_{\notv, f}^{\epsilon}(|V_{\notv, f}^{\epsilon}, z)|U)$ forms an irreducible module of $(V_{\notv, f}^{\epsilon}, Y_{\notv, f}^{\epsilon}(\cdot, z), 1_\chi, \varrho^{(\chi)})$ when $U \not\in \mathbb{Z}/2$.

Recall the charged free bosonic field realization given in (6.107)-(6.119). We set

$$
R_\notv = \text{Span}\left\{\tilde{x}_l x_{k} - (-1)^{\ell_0}\tilde{x}_k x_{l} \mid \ell, k \in \mathbb{Z}_-\right\}.
$$

(7.44)

Note the notion $V_b^{(\chi)}$ defined in (6.128), the notion $1_\chi$ defined in (6.144) and the map $\varrho : \hat{\Theta} \to V_b^{(\chi)}$ defined by (6.114) and (6.146). Set

$$
V_{\notv, b}^{\epsilon} = \mathbb{C}1_\chi + \sum_{t=1}^{\infty} [\varrho(R_\notv)]^t.
$$

(7.45)
According to (6.148) and (6.149), the family \((V^{\circ,b}_\chi, Y^0_\chi(\cdot, z), 1_\chi, \partial^{(\chi)})\) forms a simple vertex algebra isomorphic to \((V_{-\chi}(\hat{\ell}, A)), Y(\cdot, z), |0\rangle, \partial)\) in Theorem 7.2. For each irreducible \(\tilde{gl}(\infty)\)-module component \(U\) of \(V^\chi_b\), the family \((U, Y^\chi(\cdot, z)|_U)\) forms an irreducible module of \((V^{\circ,b}_\chi, Y^0_\chi(\cdot, z), 1_\chi, \partial^{(\chi)})\) when \(\iota \notin \mathbb{Z}/2\). Thus we have:

**Theorem 7.3.** Suppose that \(\chi\) is a positive integer. Assume \(\lambda\) is a weight of \(\tilde{gl}(\infty)\) satisfying (3.59), (6.105), (6.106) and \(\lambda(\kappa_0) = \chi\). Let \(\mathcal{M}\) be the highest weight irreducible \(\tilde{gl}(\infty)\)-module with highest weight \(\lambda\). Then the family \((\mathcal{M}, Y^\chi(\cdot, z))\) defined in (6.65)-(6.70) forms an irreducible module of the simple vertex algebra \((V_{-\chi}(\hat{\ell}, A)), Y(\cdot, z), |0\rangle, \partial)\) in Theorem 7.2 when \(\iota \notin \mathbb{Z}/2\).

If \(\lambda \in \Gamma^\chi\) (cf. (6.143)) and \(\mathcal{M}\) is the irreducible highest weight \(\tilde{gl}(\infty)\)-module with highest weight \(\lambda\), then the family \((\mathcal{M}, Y^\chi_{\kappa}(\cdot, z))\) defined in (6.65)-(6.70) forms an irreducible module of the simple vertex algebra \((V_{-\chi}(\hat{\ell}, A)), Y(\cdot, z), |0\rangle, \partial)\) in Theorem 7.2 when \(\iota \notin \mathbb{Z}/2\).

Assume \(\iota \in \mathbb{Z} + 1/2\). Recall the Lie algebra \(\mathcal{L}^\chi_{\iota, \kappa}\) defined in (4.30) and the highest weight irreducible \(\mathcal{L}^\chi_{\iota, \kappa}^{e}\)-module \(\mathcal{M}^\chi_{\iota, \kappa}\) defined in (4.68) with \(\tau = \ast\) and \(\lambda(\kappa_0) = \chi\). By (4.75), we define operators

\[
E^{(i,j)}_{\iota, \kappa}(r, z) = \sum_{l, k \in \mathbb{Z}} (k - m_j - \epsilon + 1/2)_{\iota, \kappa} \left((-1)^i (k + m_j + \epsilon - 3/2) + \frac{1}{\epsilon} \delta_{l, 0} \delta_{k, 0} z^{-l - k - m_j - r - \epsilon - 1}ight)
\]

on \(\mathcal{M}^\chi_{\iota, \kappa}\) for \(i, j \in \Gamma, n \in \mathbb{N}\) and \(r \in \mathbb{N}\). For \(l \in \mathbb{Z}\), we define

\[
\varphi(l) = \begin{cases} 
-1 & \text{if } l - 1/2 + (m_{1_R} - \iota_0)n > 0, \\
1 & \text{if } l - 1/2 + (m_{1_R} - \iota_0)n < 0
\end{cases}
\]

(cf. (4.13) and (4.25)) by (4.16). Moreover, for \(\lambda \in (T^\chi)^\ast\) (cf. (4.47) and (4.59)), we define

\[
\text{supp } \lambda = \{l - 1/2 + (m_{1_R} - \iota_0)n \mid l \in \mathbb{N} - n_0 \delta_{\epsilon, 0}, \lambda(\varphi(l)) \neq 0\}.
\]

Now we let

\[
\Gamma^\chi_{\iota, \kappa} = \{\lambda \in (T^\chi)^\ast \mid \lambda(\kappa_0) = -\chi, \varphi(l) \lambda(\varphi(l)) \in \mathbb{N} \text{ for } l \in \mathbb{N} - n_0 \delta_{\epsilon, 0}; \text{ supp } \lambda \subset S \text{ for some } S \in \mathcal{S}_{\chi}\}
\]

(cf. (6.141)).

Let \(\iota \in \mathbb{Z}\). Recall the Lie algebra \(\mathcal{L}^\chi_{\iota, \kappa}\) defined in (5.12) and the highest weight irreducible \(\mathcal{L}^\chi_{\iota, \kappa}^{e}\)-module \(\mathcal{M}^\chi_{\iota, \kappa}\) defined in (5.39) with \(\tau = \ast\) and \(\lambda^\chi_{\iota, \kappa}(\kappa_0) = \chi\). By (5.48), we define operators
We define linear maps \((\text{cf. (6.141))}\).

\[
E_{i,j}^{t,\epsilon}(r, z) = \sum_{l, k=0}^{\infty} \left[ (k)_r ((-1)^\epsilon (-k - 1)_j E_{-(l-1)n+i-1/2, -kn-j+1/2}^{l+k-r} - (-l - 1)_{l\epsilon} E_{-kn-j+1/2, -(l-1)n+i-1/2}^{l+k-r} + (k)_r ((-1)^\epsilon (-k - 1)_j E_{ln+i-1/2, -kn-j+1/2}^{l+k-r} - (l + \ell_i)_i E_{-kn-j+1/2, ln+i-1/2}^{l+k-r-1} + (-k - \ell_2 - 1)_r ((-1)^\epsilon (k + \ell_j)_j E_{-(l+1)n+i-1/2, (k+1)n-j+1/2}^{l+k-r} - (-l - 1)_{l\epsilon} E_{(k+1)n-j+1/2, -(l+1)n+i-1/2}^{l+k-r-1} + (-k - \ell_j - 1)_r ((-1)^\epsilon (k + \ell_j)_j E_{ln+i-1/2, (k+1)n-j+1/2}^{l+k-r} - (l + \ell_i)_i E_{(k+1)n-j+1/2, ln+i-1/2}^{l+k-r-1} + ((r + \ell_i)! \delta_{i,j} - (-1)^\epsilon r! \delta_{i,j} \delta_{i,j} z^{r-\ell_i-1} \right)
\]

(7.50)
on \mathcal{M}_{\lambda^{\epsilon,\epsilon}} for \(i, j \in \overline{1, n}\) and \(r \in \mathbb{N}\). For \(l \in \mathbb{N}\), we define

\[
\tilde{\varphi}(l) = \begin{cases} 
-1 & \text{if } l - 1/2 - \epsilon > 0, \\
1 & \text{if } l - 1/2 - \epsilon < 0
\end{cases}
\]

(7.51)by (5.3). Moreover, for \(\lambda \in \mathcal{T}^{\star}\) (cf. (5.21) and (5.22)), we define

\[
\text{supp } \lambda = \{ l - 1/2 - \epsilon \mid l \in \mathbb{N}, \lambda(\partial_l) \neq 0 \}.
\]

(7.52)
Now we let

\[
\Gamma_{\epsilon,\ell}^X = \{ \lambda \in \mathcal{T}^{\star} \mid \lambda(\kappa_0) = -\chi, \tilde{\varphi}(l) \lambda(\partial_l^\epsilon) \in \mathbb{N} \text{ for } l \in \mathbb{N};
\text{supp } \lambda \subset S \text{ for some } S \in \mathcal{S}_X \}
\]

(7.53)(cf. (6.141)).For convenience, we denote

\[
\mathcal{M} = \begin{cases} 
\mathcal{M}_{\lambda^{\epsilon,\epsilon}} \text{ in (4.68)} & \text{if } \epsilon \in \mathbb{Z} + 1/2, \\
\mathcal{M}_{\lambda^{\epsilon,\epsilon}} \text{ in (5.39)} & \text{if } \epsilon \in \mathbb{Z}.
\end{cases}
\]

(7.54)We define linear maps

\[
Y^{t,\pm}_{\mathcal{M}}(\cdot, z) : \partial(\partial_l, \mathbb{A})_- \rightarrow LM(\mathcal{M}, \mathcal{M}[z^{-1}, z])
\]

(7.55)by

\[
Y^{t,\pm}_{\mathcal{M}}(t^{-m-1}\partial_{\ell_i}^{\epsilon+\ell_j} E_{i,j} - (-1)^\epsilon (-\partial_{\ell_i}) t^{-m-1}\partial_{\ell_i}^{\epsilon} E_{j+\epsilon,i}, z) = \frac{1}{m! z^m} E^{t,\pm}_{\epsilon}(r, z)
\]

(7.56)for \(i, j \in \overline{1, n}\) and \(r, m \in \mathbb{N}\). Now we define a linear map

\[
Y^{t}_{\mathcal{M}}(\cdot, z) : V_X(\partial(\partial_l, \mathbb{A})) \rightarrow LM(\mathcal{M}, \mathcal{M}[z^{-1}, z])
\]

(7.57)by induction:

\[
Y^{t}_{\mathcal{M}}([0], z) = \text{Id}_{\mathcal{M}}, \quad Y(uv, z) = Y^{t,\pm}_{\mathcal{M}}(u, z)Y^{t}_{\mathcal{M}}(v, z) + Y^{t}_{\mathcal{M}}(v, z)Y^{t,\pm}_{\mathcal{M}}(u, z)
\]

(7.58)for \(u \in \partial(\partial_l, \mathbb{A})_-\) and \(v \in V_X(\partial(\partial_l, \mathbb{A}))\).
By Theorem 4.2, Theorem 5.2, the general theory for vertex algebras (e.g. cf. Section 4.1 in [X2]), the charged free fermionic field realization and the charged free bosonic field realization, we obtain:

**Theorem 7.4.** Assume \( \iota \in \mathbb{Z}/2 \). The family \((M, Y_M(\cdot, z))\) forms an irreducible module of the vertex algebra \((V_\chi(\tilde{\ell}, A)), Y(\cdot, z), \partial, |0\rangle)\). Suppose that \( \chi \) is a positive integer. If (4.69) and (5.40) hold, then the family \((M, Y_M(\cdot, z))\) induces an irreducible module of the quotient simple vertex algebra \((V_\chi(\tilde{\ell}, A)), Y(\cdot, z), |0\rangle, \partial)\) in Theorem 7.2. When \( \lambda \in \Gamma_{\iota}^X \) with \( \iota \in \mathbb{Z} + 1/2 \) in (7.49) and \( \lambda^{*, \epsilon} \in \Gamma_{\iota}^X \) with \( \iota \in \mathbb{Z} \) in (7.53), the family \((M, Y_M(\cdot, z))\) induces an irreducible module of the quotient simple vertex algebra \((V_\chi(\tilde{\ell}, A)), Y(\cdot, z), |0\rangle, \partial)\) in Theorem 7.2.

Recall the general settings in (6.1)-(6.8). Observe

\[
\text{Theorem 7.5.} \quad \text{Assume } \iota \in \mathbb{Z}/2. \text{ The module } V_\chi(\tilde{p}(\tilde{\ell}, A)) \text{ is irreducible if } \chi \notin \mathbb{Z}. \text{ When } \chi \in \mathbb{Z}, \text{ the module } V(\tilde{p}(\tilde{\ell}, A)) \text{ has a unique maximal proper submodule } \tilde{V}_\chi(\tilde{p}(\tilde{\ell}, A)), \text{ and the quotient }
\]

\[
V_\chi(\tilde{p}(\tilde{\ell}, A)) = V(\tilde{p}(\tilde{\ell}, A))/\tilde{V}_\chi(\tilde{p}(\tilde{\ell}, A)) \quad (7.63)
\]

is an irreducible \( \tilde{p}(\tilde{\ell}, A) \)-module. Assume \( \chi \in \mathbb{N} \). The submodule

\[
\tilde{V}_\chi(\tilde{p}(\tilde{\ell}, A)) = U(\tilde{p}(\tilde{\ell}, A))(t^{-1}\partial_{E_{n,1}})^{\chi+1}|0\rangle \quad (7.64)
\]

if \( n > 1 \) and \( \epsilon = 0 \) (cf. (4.13)). When if \( \epsilon = 1 \) and \( n > 3 \),

\[
\tilde{V}_\chi(\tilde{p}(\tilde{\ell}, A)) = U(\tilde{p}(\tilde{\ell}, A))(t^{-1}(\partial_{E_{n-1,1}} - \partial_{E_{n,2}}))^{\chi+1}|0\rangle. \quad (7.65)
\]

Since \( \tilde{p}(\tilde{\ell}, A) \) is a Lie subalgebra of \( \tilde{gl}(\tilde{\ell}, A) \), we view \( V_\chi(\tilde{p}(\tilde{\ell}, A)) \) as a subspace of \( V_\chi(\tilde{gl}(\tilde{\ell}, A)) \). Recall the vertex algebra \((V_\chi(\tilde{gl}(\tilde{\ell}, A)), Y(\cdot, z), \partial, |0\rangle)\) defined in (6.57)-(6.63). The family

\[
(V_\chi(\tilde{p}(\tilde{\ell}, A)), Y(\tilde{p}(\tilde{\ell}, A)), z)|V_\chi(\tilde{p}(\tilde{\ell}, A)), \partial|V_\chi(\tilde{p}(\tilde{\ell}, A)), |0\rangle) \quad (7.66)
\]
forms a vertex subalgebra. Let $\mathcal{M}$ be a weighted irreducible $\tilde{gl}(\infty)$-module satisfying (3.18) (also cf. (3.17)) and $\kappa_0|_{\mathcal{M}} = \chi \text{Id}_{\mathcal{M}}$. Recall the operator the linear map $Y_{\mathcal{M}}(\cdot, z)$ defined by (6.65)-(6.70). Now we obtain

**Theorem 7.6.** The family (7.66) forms a vertex algebra and $(\mathcal{M}, Y_{\mathcal{M}}(\cdot, z))$ with $i \not\in \mathbb{Z}/2$ forms an irreducible vertex algebra module of the vertex algebra (7.66). If $\chi \not\in \mathbb{Z}$, the vertex algebra $(V_\chi(\tilde{sp}(\ell, A)), Y(\cdot, z), |0\rangle)$ is simple. When $\chi \in \mathbb{Z}$, the quotient space $\tilde{V}_\chi(\tilde{sp}(\ell, A)))$ forms a simple vertex algebra. If $\ell_i = \epsilon$ for $i \in \mathbb{Z}/n$, $V_\chi(\tilde{sp}(\ell, A))$ with $\chi \in \mathbb{C} \setminus \mathbb{Z}$ and $\tilde{V}_\chi(\tilde{sp}(\ell, A))$ with $\chi \in \mathbb{Z}$ are simple vertex operator algebras with the Virasoro element

$$
\left\langle (\alpha + 1)/2 \right| (-t^{-1}\partial E_{i,i} + (-1)^i(-\partial)^{k_0}t^{-1}\partial E_{t_i,i}) \right| 0\rangle.
$$

Suppose that $\chi$ is a positive integer. Assume $\lambda$ is a weight of $\tilde{gl}(\infty)$ satisfying (3.59), (6.105), (6.106) and $\lambda(\kappa_0) = \chi$. Let $\mathcal{M}$ be the irreducible highest weight $\tilde{gl}(\infty)$-module with highest weight $\lambda$. Then the family $(\mathcal{M}, Y_{\mathcal{M}}(\cdot, z))$ induces an irreducible module of the simple vertex algebra $(V_\chi(\tilde{sp}(\ell, A)), Y(\cdot, z), |0\rangle, \partial)$ when $i \not\in \mathbb{Z}/2$.

If $\lambda \in \Gamma^\text{e}$ (cf. (6.143)) and $\mathcal{M}$ is the highest weight irreducible $\tilde{gl}(\infty)$-module with highest weight $\lambda$, then the family $(\mathcal{M}, Y_{\mathcal{M}}(\cdot, z))$ induces an irreducible module of the simple vertex algebra $(V_{-\chi}(\tilde{sp}(\ell, A)), Y(\cdot, z), |0\rangle, \partial)$ when $i \not\in \mathbb{Z}/2$.

Assume $\epsilon \in \mathbb{Z} + 1/2$. Recall the Lie algebra $L_{\epsilon}^{\dagger, \tilde{m}}$ defined in (4.31) and the highest weight irreducible $L_{\epsilon}^{\dagger, \tilde{m}}$-module $M_{\lambda}^{\dagger, \epsilon}$ defined in (4.68) with $\tau = \dagger$ and $\lambda(\kappa_0) = \chi$. By (4.76), we define operators

$$
E_{i,j}^{\dagger, \epsilon}(r, z) = \sum_{l, k \in \mathbb{Z}} \langle l - m_j - \epsilon + 1/2 \rangle r((\lambda)^i(k + m_j + \epsilon - 3/2)_{\ell_j}E_{l+n-i-1/2, kn-j+1/2}

-(-1)^{p(i)+p(j)}\langle l + m_i + 1/2 \rangle _{\ell_i}E_{(k-\epsilon)_{n-i+1/2, (l-\epsilon)n+i+1/2)}z^{-l-k-m_i-m_j-r-\epsilon-1}

+\langle r + \ell_i \rangle \langle 3_{r, \ell_i} \rangle \delta_{i,j}k_0z^{-\ell_i-r-1}
$$

(6.68)
on $M_{\lambda}^{\dagger, \epsilon}$ for $i, j \in \mathbb{Z}/n$ and $r \in \mathbb{N}$.

Let $\epsilon \in \mathbb{Z}$. Recall the Lie algebra $L_{\epsilon}^{\dagger, \tilde{e}}$ defined in (5.13) and the highest weight irreducible $L_{\epsilon}^{\dagger, \tilde{e}}$-module $M_{\lambda_{\dagger, \epsilon}}$ defined in (5.39) with $\tau = \dagger$ and $\lambda_{\dagger, \epsilon}(\kappa_0) = \chi$. By (5.49), we define operators

$$
E_{i,j}^{\dagger, \epsilon}(r, z) = \sum_{l, k = 0} \langle l \rangle r(\langle -1 \rangle ^\epsilon(\langle -k - 1 \rangle \ell_jE_{(l-1)n+i-1/2, -kn-j+1/2} z^{l+k-r}

-(-1)^{p(i)+p(j)}\langle -l - 1 \rangle _{\ell_i}E_{-kn-j+1/2, (l-1)n+i-1/2)}z^{l+k-r}

+\langle k \rangle _r(\langle -1 \rangle ^\epsilon(\langle -k - 1 \rangle \ell_jE_{l+n-i-1/2, -kn-j+1/2} z^{-l-k-\ell_i-r-1})

-\langle l \rangle _{\ell_i}E_{-kn-j+1/2, n+i-1/2)}z^{-l-k-\ell_i-r-1}
$$

(6.69)
\[
+(-k - \ell_2 - 1)_r((-1)'^r(k + \ell_j)t_j^E_{-(l+1)n+i-1/2,(k+1)n-j+1/2} \\
-(-l - 1)\ell_i^E_{(k+1)n-j+1/2,-(l+1)n+i-1/2})z^{l-k-\ell_j-r-1} \\
+(-k - \ell_j - l)_r((-1)'^r(k + \ell_j)t_j^E_{ln+i-1/2,(k+1)n-j+1/2} \\
-\ell_i^E_{(k+1)n-j+1/2,ln+i-1/2})z^{l-k-\ell_j-r-1} \\
+(r + \ell_i)!3_{0,r+\ell_i} - (-1)^r!\ell_i!3_{r,\ell_i})\delta_{i,j}n_{0z}^{-r-\ell_i-1}
\]

on \(\mathcal{M}_{\lambda^+,\epsilon}\) for \(i, j \in \mathbb{Z}\) and \(r \in \mathbb{N}\).

For convenience, we denote

\[
\mathcal{M} = \begin{cases} 
\mathcal{M}_{\lambda^+,\epsilon} \text{ in (4.68)} & \text{if } \epsilon \in \mathbb{Z} + 1/2, \\
\mathcal{M}_{\lambda^+,\epsilon} \text{ in (5.39)} & \text{if } \epsilon \in \mathbb{Z}.
\end{cases}
\]

We define linear maps

\[
Y_{\mathcal{M}}^{i,\pm}(\cdot, z) : \hat{sp}(\bar{\ell}, \bar{\lambda}) \rightarrow LM(\mathcal{M}, \mathcal{M}[z^{-1}, z])
\]

by

\[
Y_{\mathcal{M}}^{i,\pm}(t^{-m-1}\partial_t^{r+\ell_j}E_{i,j} - (-1)^{p(i) + p(j)}\epsilon(-\partial_t)^{r}t^{-m-1}\partial_t^{\ell_j}E_{j,i}, z) = \frac{1}{m!dz^m}E_{i,j}^{\pm}(r, z)
\]

for \(i, j \in \mathbb{Z}\) and \(r, m \in \mathbb{N}\). Now we define a linear map

\[
Y_{\mathcal{M}}^{i}(\cdot, z) : V_\lambda (\hat{sp}(\bar{\ell}, \bar{\lambda})) \rightarrow LM(\mathcal{M}, \mathcal{M}[z^{-1}, z])
\]

by induction:

\[
Y_{\mathcal{M}}^{i}(0, z) = \text{Id}_{\mathcal{M}}, \quad Y(uv, z) = Y_{\mathcal{M}}^{i,-}(u, z)Y_{\mathcal{M}}^{i}(v, z) + Y_{\mathcal{M}}^{i}(v, z)Y_{\mathcal{M}}^{i,+}(u, z)
\]

By for \(u \in \hat{sp}(\bar{\ell}, \bar{\lambda})_-\) and \(v \in V_\lambda (\hat{sp}(\bar{\ell}, \bar{\lambda}))\).

By Theorem 4.2, Theorem 5.2, the general theory for vertex algebras (e.g. cf. Section 4.1 in [X2]), the charged free fermion field realization and the charged free bosonic field realization, we obtain:

**Theorem 7.7.** Assume \(\epsilon \in \mathbb{Z}/2\). The family \((\mathcal{M}, Y_{\mathcal{M}}^{i}(\cdot, z))\) forms an irreducible module of the vertex algebra \((V_\lambda (\hat{sp}(\bar{\ell}, \bar{\lambda})), Y(\cdot, z), \partial, |0\rangle)\). Suppose that \(\lambda \) is a positive integer. If (4.69) and (5.40) hold, then the family \((\mathcal{M}, Y_{\mathcal{M}}^{i}(\cdot, z))\) induces an irreducible module of the quotient simple vertex algebra \((V_\lambda (\hat{sp}(\bar{\ell}, \bar{\lambda})), Y(\cdot, z), |0\rangle, \partial)\). When \(\lambda \in \Gamma_{\lambda^+,\epsilon}\) with \(\epsilon \in \mathbb{Z} + 1/2\) in (7.49) and \(\lambda^+ \in \Gamma_{\lambda^+,\epsilon}\) with \(\epsilon \in \mathbb{Z}\) in (7.53), the family \((\mathcal{M}, Y_{\mathcal{M}}^{i}(\cdot, z))\) induces an irreducible module of the quotient simple vertex algebra \((V_{-\lambda} (\hat{sp}(\bar{\ell}, \bar{\lambda})), Y(\cdot, z), |0\rangle, \partial)\).

**References**
[A] D. Adamović, Representations of the vertex algebra $W_{1+\infty}$ with a negative integer central charge, *Commun. Algebra* 29 (2001), no.7, 3153-3166.

[AFMO1] H. Awata, M. Fukuma, Y. Matsu and S. Odake, Quasifinite highest weight modules over the super $W_{1+\infty}$ algebra, *Commun. Math. Phys.* 170 (1995), 151-179.

[AFMO2] H. Awata, M. Fukuma, Y. Matsu and S. Odake, Character and determinant formulae of Quasifinite representation of the $W_{1+\infty}$ algebra, *Commun. Math. Phys.* 172 (1995), 377-400.

[Bo] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* 83 (1986), 3068-3071.

[BL1] C. Boyallian and J. Liberati, Classical Lie subalgebras of the Lie algebra of matrix differential operators on the circle, *J. Math. Phys.* 42 (2001), no.8, 3735-3753.

[BL2] C. Boyallian and J. Liberati, On modules over matrix quantum pseudo-differential operators, *Lett. Math. Phys.* 60 (2002), no.1, 73-85.

[BL3] C. Boyallian and J. Liberati, Representations of a symplectic type subalgebra of $W_{1+\infty}$, *J. Math. Phys.* 44 (2003), no.5, 2192-2205.

[BKL] C. Boyallian, V. Kac and J. Liberati, Finite growth representations of infinite Lie conformal algebras, *J. Math. Phys.* 44 (2003), no.2, 754-770.

[DN1] C. Dong and K. Nagatomo, Classification of irreducible modules for the vertex operator algebra $M(1)^+$, *J. Algebra* 216 (1999), no. 1, 384-404.

[DN2] C. Dong and K. Nagatomo, Classification of irreducible modules for the vertex operator algebra $M(1)^+$ II, higher rank, *J. Algebra* 240 (2001), no. 1, 289-325.

[FKR] E. Frenkel, V. Kac, A. Radul and W. Wang, $W_{1+\infty}$ and $W(gl_N)$ with central charge $N$, *Commun. Math. Phys.* 170 (1995), 337-357.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Mathematics 134, Academic Press Inc., Boston, 1988.

[HKW] P. Harpe, M. kervaire and C. Weber, On the Jones polynomial, *L’Enseig. Math.* 32 (1986), 271-235.

[J1] J. C. Jantzen, Zur charakterformel gewisser darstellungen halbeinfacher gruppen und Lie-algebra, *Math. Z.* 140 (1974), 127-149.
[J2] J. C. Jantzen, Kontravariante formen auf induzierten Darstellungen habeinacher Lie-algebren, *Math. Ann.* **226** (1977), no. 1, 53-65.

[J3] J. C. Jantzen, Moduln mit einem höchsten gewicht, *Lecture Note in Math.* **750**, Springer, Berlin, 1979.

[K] V. G. Kac, *Vertex algebras for beginners*, University lectures series, Vol **10**, AMS. Providence RI, 1996.

[KR1] V. Kac and A. Radul, Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, *Commun. Math. Phys.* **157** (1993), 429-457.

[KR2] V. Kac and A. Radul, Representation theory of the vertex algebra $\mathcal{W}_{1+\infty}$, *Transf. Groups* **1** (1996), 41-70.

[KWY] V. Kac, W. Wang and C. Yan, Quasifinite representations of classical Lie subalgebras of $\mathcal{W}_{1+\infty}$, *Adv. Math.* **139** (1998), 56-140.

[L] W. Li, 2-cocycles on the algebra of differential operators, *J. Algebra* **122** (1989), 64-80.

[M1] S. Ma, Conformal and Lie superalgebras motivated from free fermionic fields, *J. Phys. A* **36** (2003), no.6, 1759-1787.

[M2] S. Ma, Conformal and Lie superalgebras related to the differential operators on the circle, *Ph.D. Thesis, The Hong University of Science and Technology*, 2003.

[V] J. van de Leur, The $W_{1+\infty}(gl_s)$-symmetries of the s-component KP hierarchy, *J. Math. Phys.* **37** (1996), no.5, 2315-2337.

[X1] X. Xu, On spinor vertex operator algebras and their modules, *J. Algebra* **191** (1997), 427-460.

[X2] X. Xu, *Introduction to Vertex Operator Superalgebras and Their Modules*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1998.

[X3] X. Xu, Skew-symmetric differential operators and combinatorial identities, *Mh. Math.* **127** (1999), 143-258.

[X4] X. Xu, Simple conformal superalgebras of finite growth, *Algebra Colloq.* **7**(2000), 205-240.

[X5] X. Xu, Equivalence of conformal superalgebras to Hamiltonian operators, *Algebra Colloq.* **8** (2001), 63-92.