CONFORMALLY FLAT FRW METRICS

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Abstract

We find a new family of non-separable coordinate transformations bringing the FRW metrics into the manifestly conformally flat form. Our results are simple and complete, while our derivation is quite explicit. We also calculate all the FRW curvatures, including the Weyl tensor.

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1 Introduction

The fundamental Cosmological Principle of the spatially homogeneous and isotropic (1 + 3)-dimensional Universe (at large scales) gives rise to the standard Friedman-Robertson-Walker (FRW) metrics of the form \[1, 2\]

\[ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \] (1.1)

where the function \(a(t)\) is known as the scale factor in ‘cosmic’ coordinates \((t, r, \theta, \phi)\); we use \(c = 1\) and \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\), while \(k\) is the FRW topology index taking values \((-1, 0, +1)\). Accordingly, the FRW metric (1.1) admits a 6-dimensional isometry group \(G\) that is either \(SO(1, 3)\), \(E(3)\) or \(SO(4)\), acting on the orbits \(G/\text{SO}(3)\), with the spatial 3-dimensional sections \(H^3\), \(E^3\) or \(S^3\), respectively.\(^5\)

By the coordinate change, \(dt = a(t)d\eta\), the FRW metric (1.1) can be rewritten to the form

\[ds^2 = a^2(\eta) \left[ d\eta^2 - \frac{dr^2}{1 - kr^2} - r^2 d\Omega^2 \right] \] (1.2)

which is obviously (4d) conformally flat in the case of \(k = 0\). It immediately implies that the 4d Weyl tensor of the FRW metric vanishes in the ‘flat’ case of \(k = 0\). In fact, the FRW Weyl tensor also vanishes for \(k = -1\) and \(k = +1\) (see Appendix for our explicit check). In its turn, it implies that there exist the coordinate transformations that bring the FRW metrics to the conformally flat form in the non-trivial cases of \(k = -1\) and \(k = +1\) too.

Though the fact that the FRW Weyl tensor vanishes for all topologies is known, while some special coordinate transformations bringing the FRW metric (1.2) to the conformally flat form are also known (see the end of Sec. 2), to the best of our knowledge, we are not aware of any systematic treatment of all such transformations in the literature, as well as their most general form, with an arbitrary scale factor. Because of the great importance of the FRW metrics to physics, knowing such explicit transformations is desirable, when taking advantage of the vanishing Weyl tensor. In this paper we find that the transformations in question are surprisingly simple.

Our paper is organized as follows. In Sec. 2 we consider the case of \(k = -1\) in great detail. Sec. 3 is devoted to the case of \(k = +1\). We find that

\(^5\)Our notation follows ref. [2], and it is given in Appendix.
there is the fundamental difference between those two cases, as regards the existence of separable real solutions. Possible physical applications are briefly discussed in Sec. 4. In Appendix we summarize our notation, calculate all the FRW curvatures, and verify that all the FRW Weyl tensor components vanish.

2 The open FRW case \( k = -1 \)

Let’s introduce a coordinate \( \chi \) by

\[
 r = \sinh \chi \quad (2.1)
\]

Then the FRW metric reads

\[
 -ds^2 = a^2(\eta) \left[ -d\eta^2 + d\chi^2 + \sinh^2 \chi d\Omega^2 \right] \quad (2.2)
\]

We are looking for some new local coordinates \( \xi(\eta, \chi) \) and \( R(\eta, \chi) \) in which the metric (2.2) would be manifestly conformally flat, i.e.

\[
 -ds^2 = a^2(\xi, R) A^2(\xi, R) \left[ -d\xi^2 + dR^2 + R^2 d\Omega^2 \right] \quad (2.3)
\]

where

\[
 a(\xi, R) := a(\eta(\xi, R)) \quad (2.4)
\]

and \( A \) is yet another function of \( \xi \) and \( R \). Substituting

\[
 d\xi = \xi_\eta d\eta + \xi_\chi d\chi \quad (2.5a)
\]

\[
 dR = R_\eta d\eta + R_\chi d\chi \quad (2.5b)
\]

into eq. (2.3) gives

\[
 -ds^2 = a^2(\xi, R) A^2(\xi, R) \left[ - (\xi_\eta)^2 d\eta^2 - 2 \xi_\eta \xi_\chi d\eta d\chi - (\xi_\chi)^2 d\chi^2 + (R_\eta)^2 d\eta^2 + 2 R_\eta R_\chi d\eta d\chi + (R_\chi)^2 d\chi^2 + R^2 d\Omega^2 \right] \quad (2.6)
\]

which should be the same as eq. (2.2). Hence, the functions \( A, \xi \) and \( R \) obey the following non-linear partial differential equations:

\[
 -1 = A^2[-(\xi_\eta)^2 + (R_\eta)^2] \quad (2.7a)
\]

\[
 1 = A^2[-(\xi_\chi)^2 + (R_\chi)^2] \quad (2.7b)
\]

\[
 0 = -2 \xi_\eta \xi_\chi + 2 R_\eta R_\chi \quad (2.7c)
\]

\[
 \sinh^2 \chi = A^2(\xi, R) R^2 \quad (2.7d)
\]
A substitution of eq. (2.7d) into eqs. (2.7a) and (2.7b) gives

\[(\xi,\eta)^2 = \frac{R^2}{\sinh^2 \chi} + (R,\eta)^2\] (2.8a)

\[(\xi,\chi)^2 = -\frac{R^2}{\sinh^2 \chi} + (R,\chi)^2\] (2.8b)

whereas a substitution of eqs. (2.8a) and (2.8b) into eq. (2.7c) gives

\[-\frac{R^2}{\sinh^2 \chi} + (R,\chi)^2 - (R,\eta)^2 = 0\] (2.9)

Equations (2.8a), (2.8b) and (2.9) now imply

\[(R,\eta)^2 = (\xi,\chi)^2\] (2.10a)

\[(R,\chi)^2 = (\xi,\eta)^2\] (2.10b)

and, hence,

\[R,\eta = \sigma_1 \xi,\chi, \quad (\sigma_1 = \pm 1)\] (2.11a)

\[R,\chi = \sigma_2 \xi,\eta, \quad (\sigma_2 = \pm 1)\] (2.11b)

The original equations (2.7a) to (2.7d) are invariant under the sign flip of \(\xi\) and \(R\), so that we can remove one of those sign ambiguities. Let’s redefine \(\xi \rightarrow \sigma_1 \xi\) in order to get

\[R,\eta = \xi,\chi\] (2.12a)

\[R,\chi = \sigma_3 \xi,\eta, \quad (\sigma_3 := \sigma_1 \sigma_2 = \pm 1)\] (2.12b)

When choosing the elliptic case \(\sigma_3 = -1\), eqs. (2.12a) and (2.12b) are nothing but the Cauchy-Riemann equations, whose general solution for \(R\) and \(\xi\) is given by the real and imaginary parts, respectively, of an arbitrary complex (holomorphic) function \(F(\eta \pm i\chi)\). However, it is inconsistent with the remaining equations (2.7c) and (2.7d). Therefore, since we are interested in a real solution, we have to choose the hyperbolic case, \(\sigma_3 = +1\).

A solution must satisfy the integrability condition

\[R,\eta,\chi = R,\chi,\eta\] (2.13)
which via eqs. (2.12a), (2.12b) and (2.13) yields a linear (!) equation

\[
\left( \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \chi^2} \right) \xi = 0
\]

(2.14)

Equation (2.14) is just the two-dimensional wave equation whose general solution is given by

\[
\xi = \xi_+(\eta + \chi) + \xi_-(\eta - \chi)
\]

(2.15)

where \(\xi_+\) and \(\xi_-\) are arbitrary functions of \(\eta + \chi\) and \(\eta - \chi\), respectively. Substituting eq. (2.15) into eq. (2.12a) yields

\[
R_{,\eta} = \frac{\partial}{\partial \chi} \xi_+ + \frac{\partial}{\partial \chi} \xi_- = \frac{\partial}{\partial \eta} \xi_+ - \frac{\partial}{\partial \eta} \xi_-
\]

(2.16)

whose integration with respect to \(\eta\) gives

\[
R = \xi_+ - \xi_- + f(\chi)
\]

(2.17)

where the function \(f(\chi)\) is actually a constant because of eq. (2.12b). That constant amounts to a trivial shift of \(R\), while it can also be included into any of the functions \(\xi_+\) or \(\xi_-\), so we simply set it to zero. As a result, we get

\[
\xi = \xi_+(\eta + \chi) + \xi_-(\eta - \chi)
\]

(2.18a)

\[
R = \xi_+(\eta + \chi) - \xi_-(\eta - \chi)
\]

(2.18b)

with some yet to be determined functions \(\xi_+\) and \(\xi_-\) of a single variable.

To determine the remaining functions, we substitute eqs. (2.18a) and (2.18b) into eq. (2.7a) and get

\[
\frac{1}{\sinh^2 \chi} = \frac{4\xi_+\xi_-'}{(\xi_+ - \xi_-)^2}
\]

(2.19)

where the primes mean the derivatives with respect to \(\eta + \chi\) or \(\eta - \chi\), respectively. We now introduce the new coordinates \(x\) and \(y\) as follows:

\[
x := \eta + \chi, \quad y := \eta - \chi
\]

(2.20a)

\[
\chi = \frac{x - y}{2}, \quad \eta = \frac{x + y}{2}
\]

(2.20b)
Then (2.19) takes the form
\[
\frac{1}{\sinh^2 \frac{x-y}{2}} = \frac{4\xi_+(x)\xi_-(y)}{(\xi_+(x) - \xi_-(y))^2}
\] (2.21)

It should be satisfied for any values of \(x\) and \(y\) so, when we set \(y = 0\) in (2.21), we achieve a separation of variables,
\[
\frac{1}{\sinh^2 \frac{x}{2}}dx = \frac{4\xi_+(0)}{(\xi_+(x) - \xi_-(0))^2}d\xi_+
\] (2.22)

After integration, we get
\[
\xi_+(x) - \xi_-(0) = \frac{2\xi_-(0)}{\coth \frac{\nu}{2} + c_+}
\] (2.23)

where \(c_+\) is an integration constant. Similarly, when we set \(x = 0\) in eq. (2.21), we get
\[
\xi_-(y) - \xi_+(0) = \frac{2\xi_+(0)}{\coth \frac{\nu}{2} + c_-}
\] (2.24)

and, when we set \(x = y = 0\), we get
\[
\xi_+(0) = \xi_-(0)
\] (2.25)

We can remove some integration constants, without changing a value of \(R\) by the redefinitions \(\xi_+(x) \rightarrow \xi_+(x) - \xi_-(0)\) and \(\xi_-(y) \rightarrow \xi_-(y) - \xi_+(0)\), because of eq. (2.25). Also, when differentiating eq. (2.23) with respect to \(x\) at \(x \rightarrow 0\), we find
\[
\xi'_+(0) = \xi'_-(0)
\] (2.26)

Hence, after a redefinition \(\xi_\pm \rightarrow \frac{1}{2\xi_+(0)}\xi_\pm\), we arrive at
\[
\xi_+(x) = \frac{1}{\coth \frac{\nu}{2} + c_+}
\] (2.27a)
\[
\xi_-(y) = \frac{1}{\coth \frac{\nu}{2} + c_-}
\] (2.27b)
Substituting eqs. (2.27a) and (2.27b) into eq. (2.21) for a final check, and using an identity \( \sinh(\alpha \pm \beta) = \sinh \alpha \cosh \beta \pm \cosh \alpha \sinh \beta \), just gives \( c_+ = c_- := c \). Collecting all together, we arrive at our main result

\[
\xi = \frac{1}{\coth \frac{\eta + \chi}{2} + c} + \frac{1}{\coth \frac{\eta - \chi}{2} + c} \tag{2.28a}
\]

\[
R = \frac{1}{\coth \frac{\eta + \chi}{2} + c} - \frac{1}{\coth \frac{\eta - \chi}{2} + c} \tag{2.28b}
\]

\[
A^2(\eta, \chi) = \frac{(\coth \frac{\eta + \chi}{2} + c)^2(\coth \frac{\eta - \chi}{2} + c)^2}{(\coth \frac{\eta + \chi}{2} - \coth \frac{\eta - \chi}{2})^2} \sinh^2 \chi \tag{2.28c}
\]

The inverse transformation is given by

\[
\eta = \coth^{-1} \left[ \frac{2}{\xi + R} - c \right] + \coth^{-1} \left[ \frac{2}{\xi - R} - c \right] \tag{2.29a}
\]

\[
\chi = \coth^{-1} \left[ \frac{2}{\xi + R} - c \right] - \coth^{-1} \left[ \frac{2}{\xi - R} - c \right] \tag{2.29b}
\]

We verified by a straightforward calculation that our solution (2.28) obeys the initial equations (2.7) at any value of the parameter \( c \).

It should be mentioned that we never assumed a separation of variables in solving the non-linear differential equations. However, when we choose \( c = \pm 1 \) above, eqs. (2.28a) and (2.28b) take the form

\[
\xi = \pm \left[ 1 - e^{\mp \eta} \cosh \chi \right] \tag{2.30a}
\]

\[
R = e^{\mp \eta} \sinh \chi \tag{2.30b}
\]

It is the solution that one easily gets by assuming a separation of variables, and it is precisely the one given in ref. [2] — see the footnote after eq. (113.5) overthere.

When choosing \( c = 0 \), one gets a non-separable solution

\[
\xi = \frac{2 \sinh \eta}{\cosh \eta + \cosh \chi} \tag{2.31a}
\]

\[
R = \frac{2 \sinh \chi}{\cosh \eta + \cosh \chi} \tag{2.31b}
\]

that can be found in refs. [3] [4] [5].

Thus our new solution (2.28) can also be considered as the interpolating solution between the previously known special solutions of refs. [2] and [3] [4] [5].
3  Closed FRW case, $k = +1$

A derivation of the coordinate transformation of the closed FRW metric to its manifestly conformally flat form is almost the same as that in the previous section, with \( \sinh \) and \( \cosh \) being replaced by \( \sin \) and \( \cos \), respectively. So, we skip many details here and present only our notation (in fact, very similar to that of Sec. 2) and our results.

First, we introduce a coordinate \( \chi \) by

\[
    r = \sin \chi
\]

and some new coordinates \( \xi(\eta, \chi) \) and \( R(\eta, \chi) \) in which the closed FRW metric (1.2) would be manifestly conformally flat,

\[
    -ds^2 = a(\xi, R)A^2(\xi, R) \left[ -d\xi^2 + dR^2 + R^2 d\Omega^2 \right]
\]

(3.2)

It gives rise to the following set of the non-linear partial differential equations:

\[
    -1 = A^2[-(\xi, \eta)^2 + (R, \eta)^2] \quad (3.3a)
\]

\[
    1 = A^2[-(\xi, \chi)^2 + (R, \chi)^2] \quad (3.3b)
\]

\[
    0 = -2\xi, \eta \xi, \chi + 2R, \eta R, \chi \quad (3.3c)
\]

\[
    \sin^2 \chi = A^2(\xi, R)R^2. \quad (3.3d)
\]

When proceeding as in the previous section, with the integrability condition (2.13) playing the central role, we arrive at a linear wave equation again,

\[
    \left( \frac{\partial^2}{\partial \eta^2} - \frac{\partial^2}{\partial \chi^2} \right) \xi = 0 \quad (3.4)
\]

whose general solution is

\[
    \xi = \xi_+(\eta + \chi) + \xi_- (\eta - \chi) \quad (3.5)
\]

in terms of arbitrary functions \( \xi_+ \) and \( \xi_- \) of \( \eta + \chi \) and \( \eta - \chi \), respectively. Similarly, one finds that

\[
    R = \xi_+(\eta + \chi) - \xi_- (\eta - \chi) \quad (3.6)
\]
A substitution of eqs. (3.5) and (3.6) into eq. (3.3a) yields

\[
\frac{1}{\sin^2 \chi} = \frac{4\xi'_+ \xi'_-}{(\xi_+ - \xi_-)^2}
\]

(3.7)

where the primes indicate the derivatives with respect to \(\eta + \chi\) or \(\eta - \chi\), respectively. We again introduce the new coordinates \(x\) and \(y\) as follows:

\[
x := \eta + \chi, \quad y := \eta - \chi
\]

(3.8a)

\[
\chi = \frac{x - y}{2}, \quad \eta = \frac{x + y}{2}
\]

(3.8b)

and rewrite eq. (3.7) to the form

\[
\frac{1}{\sin^2 \frac{x - y}{2}} = \frac{4\xi'_+(x)\xi'_-(y)}{(\xi_+(x) - \xi_-(y))^2}
\]

(3.9)

A solution to this equation is given by

\[
\xi_+(x) = \frac{1}{\cot \frac{x}{2} + c_+}
\]

(3.10a)

\[
\xi_-(x) = \frac{1}{\cot \frac{x}{2} + c_-}
\]

(3.10b)

with the same integration constants \(c_+ = c_- = c\). Here our main new results are (cf. eqs. (2.28a) and (2.28b))

\[
\xi = \frac{1}{\cot \frac{\eta + \chi}{2} + c} + \frac{1}{\cot \frac{\eta - \chi}{2} + c}
\]

(3.11a)

\[
R = \frac{1}{\cot \frac{\eta + \chi}{2} + c} - \frac{1}{\cot \frac{\eta - \chi}{2} + c}
\]

(3.11b)

\[
A^2(\eta, \chi) = \frac{(\cot \frac{\eta + \chi}{2} + c)^2(\cot \frac{\eta - \chi}{2} + c)^2}{(\cot \frac{\eta + \chi}{2} - \cot \frac{\eta - \chi}{2})^2}\sin^2 \chi
\]

(3.11c)

The inverse transformation reads

\[
\eta = \cot^{-1} \left[ \frac{2}{\xi + R} - c \right] + \cot^{-1} \left[ \frac{2}{\xi - R} - c \right]
\]

(3.12a)

\[
\chi = \cot^{-1} \left[ \frac{2}{\xi + R} - c \right] - \cot^{-1} \left[ \frac{2}{\xi - R} - c \right]
\]

(3.12b)
The transformations (3.11a) and (3.11b) appear to be non-separable for any real value of the parameter \( c \), unlike the situation with \( k = -1 \) in the previous section. When (formally) taking \( c = \pm i \), eqs. (3.11a) and (3.11b) take the separable though complex form

\[
\xi = \pm \frac{1}{i} \left[ 1 - e^{\mp i \eta} \cos \chi \right], \quad R = e^{\pm i \eta} \sin \chi, \quad (3.13)
\]
i.e. there are no separable real solutions in the closed FRW case.

When choosing \( c = 0 \) in eqs. (3.11a) and (3.11b), we get

\[
\xi = \frac{2 \sin \eta}{\cos \eta + \cos \chi}, \quad (3.14a)
\]
\[
R = \frac{2 \sin \chi}{\cos \eta + \cos \chi}, \quad (3.14b)
\]
thus reproducing the special transformation founded earlier in refs. [3, 4, 5].

4 Conclusion

The simplicity of our results is due to the fact that we were looking for the relevant coordinate transformations in the two-dimensional space (or plane). In physical terms, the null curves of that plane are to be invariant under such transformations, while eq. (2.21) or (3.9) is nothing but the invariance condition in the null coordinates (2.20a) or (3.8a), respectively.

The real parameter \( c \) entering eqs. (2.28) and (3.11) just parametrizes the set of the coordinate transformations we found, so it does not have physical meaning. For instance, it does not appear in the standard (physically equivalent) form (1.1) or (1.2) of the FRW metric.

Once the FRW metric is transformed into the conformally flat form, there exist the 15-parametric group of four-dimensional conformal transformations (see e.g., ref. [6]) that keeps the conformally flat form of the metric. Therefore, we can combine our one-parametric transformations in eqs. (2.28) or (3.11) with the conformal transformations in four space-time dimensions, in order to get a much larger non-trivial 16-parametric family of the coordinate transformations bringing the standard \( k \neq 0 \) FRW metrics to the manifestly conformally-flat form.

Though our results are rather technical, we believe that they may have interesting physical applications in cosmology and early Universe (see e.g.,
ref. [7]). The reason is that the FRW metrics are fully determined (up to a scale factor) by the symmetry, being independent upon equations of motion.

All modern theories of quantum gravity, and especially string theory imply modifications of Einstein equations [8, 9]. They are believed to be crucial for any deeper understanding of inflation and Big Bang. Whatever those modifications are, they are going to include more fields and higher-curvature (or higher-derivative) terms in the effective gravitational equations of motion, so that our considerations in this paper could be quite useful for any such analysis at the level of the effective field equations. Those equations include the full curvature, not just the Ricci tensor, so that all the FRW curvature components are needed in their explicit form (see Appendix).

The ‘non-flat’ FRW metrics in their manifestly conformally flat form may be particularly useful for applications of superstrings/M-theory to cosmology, because there are higher powers of curvature in the superstrings/M-theory effective field equations to all orders in the string slope parameter and the string coupling constant [10, 11], while the FRW curvatures take their simplest form just in such ‘conformally flat’ coordinates. At the same time, it comes with the price: in the ‘conformally flat’ coordinates the matter is not static anymore and is not even homogeneous in general, though it still appears to be centrally-symmetric with respect to an arbitrary point in space (at the origin of the coordinate system).

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Appendix: FRW curvatures

In this Appendix we calculate all the FRW curvatures, as well as the FRW Weyl tensor. To calculate the Riemann tensor $R_{\mu\nu ab}$, we choose to work with the spin connection $\omega_{\mu ab}$ and the vierbein $e_{\mu}^a$. We use lower case Greek letters for curved (spacetime) indices, and either lower case latin letters or lower case Greek letters with bars for flat (target space) indices. The definitions are (see e.g., ref. [12])

\begin{align*}
\omega_{\mu ab} &= \frac{1}{2} e_{\mu}^c (\Omega_{cab} + \Omega_{bac} + \Omega_{bca}) \\
\Omega_{abc} &= e_{\mu}^a e_{\nu}^b (\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c}) \\
\Omega_{bac} &= -\Omega_{abc} \quad \omega_{\mu ba} = -\omega_{\mu ab} \\
R_{\mu\nu ab} &= S_{\mu\nu ab} + K_{\mu\nu ab} \\
S_{\mu\nu ab} &= \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} \\
K_{\mu\nu ab} &= \omega_{\mu a c} \omega_{\nu cb} - \omega_{\nu a c} \omega_{\mu cb}
\end{align*}

To check their equivalence to the standard definition of Riemann curvature in terms of Christoffel symbols,

\begin{align*}
R_{\nu \rho \sigma} &= \partial_\rho \Gamma_{\nu \sigma}^\mu - \partial_\sigma \Gamma_{\nu \rho}^\mu + \Gamma_{\rho \alpha}^\mu \Gamma_{\nu \sigma}^\alpha - \Gamma_{\sigma \alpha}^\mu \Gamma_{\nu \rho}^\alpha \\
\Gamma_{\mu \nu} &= \frac{1}{2} g^{\lambda \mu} (g_{\alpha \mu, \nu} + g_{\alpha \nu, \mu} - g_{\mu \nu, \alpha}) \\
\partial_\mu e_{\nu}^a + \omega_{\mu b}^a e_{\nu}^b - \Gamma_{\mu \nu}^\lambda e_{\lambda}^a = 0 \\
R_{\mu \nu}^a b &= \partial_\mu \omega_{\nu}^a b - \partial_\nu \omega_{\mu}^a b + \omega_{\mu c}^a \omega_{\nu}^c b - \omega_{\nu c}^a \omega_{\mu}^c b \\
[D_\mu, D_\nu] V^a &= R_{\mu \nu}^a b V^b
\end{align*}
where $D_\mu$ stands for the covariant derivative acting on local Lorentz (flat) indices only (in the vierbein formalism),

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega^a_{\mu b} e^b_\nu$$  \hspace{1cm} (4.7)

Then we have

$$D_\mu D_\nu V^a = D_\mu D_\nu (e^a_\alpha V^\alpha)$$

$$= D_\mu [(D_\nu e^a_\alpha) V^\alpha + e^a_\alpha D_\nu V^\alpha]$$

$$= D_\mu D_\nu [\Gamma^\beta_{\nu \alpha} e^a_\beta V^\alpha + e^a_\alpha \partial_\nu V^\alpha]$$

$$= \partial_\mu \Gamma^\beta_{\nu \alpha} e^a_\beta V^\alpha + \Gamma^\beta_{\nu \alpha} \Gamma^\gamma_{\mu \beta} e^a_\gamma V^\alpha + \Gamma^\beta_{\nu \alpha} e^a_\beta \partial_\mu V^\alpha$$

$$+ \Gamma^\beta_{\mu \alpha} e^a_\beta \partial_\nu V^\alpha + e^a_\alpha \partial_\mu \partial_\nu V^\alpha$$  \hspace{1cm} (4.8)

which gives rise to the commutator (4.6) in the form

$$R^b_{\mu \nu a} V^b = [D_\mu, D_\nu] V^a$$

$$= e^a_\gamma V^\alpha [\partial_\mu \Gamma^\gamma_{\nu \alpha} + \Gamma^\gamma_{\nu \alpha} \Gamma^\beta_{\mu \beta} - \partial_\nu \Gamma^\gamma_{\mu \alpha} - \Gamma^\gamma_{\nu \alpha} \Gamma^\beta_{\mu \beta}]$$

$$= e^a_\gamma V^\alpha R^\gamma_{\alpha \mu \nu}$$  \hspace{1cm} (4.9)

Now the equivalence follows

$$R_{\mu \nu a b} = e^a_\alpha e^b_\beta R_{\alpha \beta \mu \nu} = e^a_\alpha e^b_\beta R_{\mu \nu \alpha \beta}$$  \hspace{1cm} (4.10)

This equivalence is of course well-known, see e.g., ref. ([6]), so we consider our proof here as merely a consistency check of our notation.

When a metric is of the ‘diagonal’ type

$$ds^2 = (A_0)^2 (dz^0)^2 - \sum_{\mu=1}^3 (A_\mu)^2 (dz^\mu)^2$$  \hspace{1cm} (4.11)

it is convenient to use a diagonal vierbein

$$e^\mu_\mu = A_\mu$$  \hspace{1cm} (4.12)

The FRW metric (1.2) is diagonal, with the components

$$g_\eta := g_{\eta \eta} = +a^2$$  \hspace{1cm} (4.13a)

$$g_r := g_{rr} = -\frac{a^2}{1-kr^2}$$  \hspace{1cm} (4.13b)

$$g_\theta := g_{\theta \theta} = -a^2 r^2$$  \hspace{1cm} (4.13c)

$$g_\phi := g_{\phi \phi} = -a^2 r^2 \sin^2 \theta$$  \hspace{1cm} (4.13d)

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so that we have

\[ A_\eta = a, \quad A_r = \frac{a}{\sqrt{1 - kr^2}}, \quad A_\theta = ar, \quad A_\phi = ar \sin \theta \]  

(4.14)

and

\[ g_\mu = \eta_{\mu\nu}(A_\mu)^2 \]  

(4.15)

with the almost minus signature of our choice (as in ref. [2])

\[ \eta = \text{diag}(+ - - -) \]  

(4.16)

The non-vanishing spin connection components of a diagonal metric are (no summation over \( \mu \) and \( \nu \))

\[ \omega_{\mu\nu\rho} = \eta_{\mu\nu} \frac{1}{A_\nu} \partial_\rho A_\mu \quad (\mu \neq \nu) \]  

(4.17)

In the FRW case we have

\[
\begin{align*}
\omega_{\tau\tau\eta} &= -\frac{1}{a} \frac{\dot{a}}{\sqrt{1 - kr^2}} \\
\omega_{\theta\theta\eta} &= -\frac{1}{a} \dot{a}r \\
\omega_{\phi\phi\eta} &= -\sqrt{1 - kr^2} \\
\omega_{\phi\theta\eta} &= -\frac{1}{a} \dot{a}r \sin \theta \\
\omega_{\phi\phi\theta} &= -\sqrt{1 - kr^2} \sin \theta \\
\omega_{\phi\theta\theta} &= - \cos \theta
\end{align*}
\]

where the dots denote the derivatives with respect to \( \eta \),

\[ \dot{a} := a_{\cdot\eta} \]  

(4.18)

The Riemann curvature (4.1d) is a sum of eqs. (4.1c) and (4.11). The \( S_{\mu\nu\rho\sigma} \) does not vanish when at least one of its indices \( \rho \) and \( \sigma \) is either \( \mu \) or \( \nu \). First, we take \( \rho = \mu \) and \( \sigma \neq \nu \). Then we find

\[
\begin{align*}
S_{\mu\nu\sigma} &= \left[ (\partial_\nu \ln A_\sigma)(\partial_\sigma \ln A_\mu) - (\partial_\nu \ln A_\mu)(\partial_\sigma \ln A_\sigma) - (\partial_\nu \partial_\sigma \ln A_\mu) \right] \\
S_{\mu\nu\sigma} &= \left[ (\partial_\nu \ln A_\nu)(\partial_\mu \ln A_\mu) - (\partial_\nu \ln A_\mu)(\partial_\mu \ln A_\nu) - (\partial_\nu \partial_\mu \ln A_\mu) \right] \\
&\quad + \eta_{\mu\nu}\eta_{\rho\sigma} \frac{A_\nu^2}{A_\mu^2} \left[ (\partial_\mu \ln A_\mu)(\partial_\rho \ln A_\nu) - (\partial_\mu \ln A_\nu)(\partial_\rho \ln A_\mu) - (\partial_\mu \partial_\rho \ln A_\nu) \right] \quad (4.19a)
\end{align*}
\]

\[
\begin{align*}
S_{\mu\nu\rho} &= \left[ (\partial_\nu \ln A_\rho)(\partial_\rho \ln A_\mu) - (\partial_\nu \ln A_\mu)(\partial_\rho \ln A_\rho) - (\partial_\nu \partial_\rho \ln A_\mu) \right] \\
&\quad + \eta_{\mu\nu}\eta_{\rho\sigma} \frac{A_\nu^2}{A_\mu^2} \left[ (\partial_\mu \ln A_\mu)(\partial_\rho \ln A_\nu) - (\partial_\mu \ln A_\nu)(\partial_\rho \ln A_\mu) - (\partial_\mu \partial_\rho \ln A_\nu) \right] \quad (4.19b)
\end{align*}
\]
Next we calculate $\partial_{\nu}(\ln A_\mu)$:

$$
(\ln A_{\eta})_{,\eta} = (\ln a)_{,\eta}
$$

$$
(\ln A_{r})_{,\eta} = (\ln a)_{,\eta}
$$

$$
(\ln A_{r})_{,r} = \frac{kr}{1 - kr^2}
$$

then

$$
(\ln A_{\theta})_{,\eta} = (\ln a)_{,\eta}
$$

$$
(\ln A_{\theta})_{,r} = \frac{1}{r}
$$

and

$$
(\ln A_{\phi})_{,\eta} = (\ln a)_{,\eta}
$$

$$
(\ln A_{\phi})_{,r} = \frac{1}{r}
$$

$$
(\ln A_{\phi})_{,\theta} = \frac{1}{\tan \theta}
$$

All the other components of $\partial_{\nu}(\ln A_\mu)$ vanish.

Similarly, the components $S_{\mu\nu}^{\mu\sigma}$ (no sums!) are given by

$$
S_{\theta r\theta r} = -\frac{1}{r} (\ln a)_{,\eta}
$$

and

$$
S_{\phi r\phi r} = -\frac{1}{r} (\ln a)_{,\eta}
$$

$$
S_{\phi \theta\phi \theta} = -\frac{1}{\tan \theta} (\ln a)_{,\eta}
$$

while otherwise zero.

To derive the remaining components $S_{\mu\nu}^{\mu\nu}$ (no sums!), we define

$$
B_{\mu\nu} = (\ln A_{\nu})_{,\nu} (\ln A_{\mu})_{,\nu} - [(\ln A_{\mu})_{,\nu}]^2 - (\ln A_{\mu})_{,\nu\nu}
$$

(4.20)
The $S_{\mu\nu}^{\mu\nu}$ can now be rewritten to

$$S_{\mu\nu}^{\mu\nu} = B_{\mu\nu} + \eta^{\mu\nu} \eta_{\rho\sigma} \frac{A^2_{\rho}}{A^2_{\mu}} B_{\nu\mu}$$  \hspace{1cm} (4.21)

The non-vanishing components of $B_{\mu\nu}$ are

$$B_{r\eta} = -(\ln a)_{\eta,\eta}$$
$$B_{\theta\eta} = -(\ln a)_{\eta,\eta}$$
$$B_{\theta r} = \frac{k}{1 - kr^2}$$
$$B_{\phi\eta} = -(\ln a)_{\eta,\eta}$$
$$B_{\phi r} = \frac{k}{1 - kr^2}$$
$$B_{\phi\theta} = 1$$

It is now easy to get

$$S_{\eta\eta}^{r\eta} = \frac{1}{1 - kr^2} (\ln a)_{\eta,\eta}$$
$$S_{\eta\theta}^{\eta\theta} = r^2 (\ln a)_{\eta,\eta}$$
$$S_{\eta\phi}^{\eta\phi} = r^2 \sin^2 \theta (\ln a)_{\eta,\eta}$$
$$S_{r\theta}^{r\theta} = r^2 k$$
$$S_{r\phi}^{r\phi} = r^2 \sin^2 k$$
$$S_{\theta\phi}^{\theta\phi} = \sin^2 \theta$$

As regards a derivation of the $K_{\mu\nu}^{\rho\sigma}$ components from eq. (4.1f), they can be only non-zero when at least one of the indices $\rho$ or $\sigma$ is either $\mu$ or $\nu$.

First, we set $\rho = \mu$ and $\sigma \neq \nu$. Then we have

$$K_{\mu\nu}^{\mu\sigma} = (\partial_{\nu} \ln A_{\mu})(\partial_{\sigma} \ln A_{\nu})$$  \hspace{1cm} (4.22)

whose non-vanishing components are

$$K_{\theta r}^{\theta \eta} = \frac{1}{r} (\ln a)_{\eta,\eta}$$
and

\[ K_{\phi r}^{\phi} = \frac{1}{r} (\ln a)_{,\eta} \]
\[ K_{\phi \theta}^{\phi} = \frac{1}{\tan \theta} (\ln a)_{,\eta} \]
\[ K_{\phi \phi}^{\phi} = \frac{1}{\tan \theta} \frac{1}{r} \]

Next, we set \( \rho = \mu \) and \( \sigma = \nu \) in order to calculate \( K_{\mu\nu}^{\mu\nu} \) (no sums!)

\[ K_{\mu\nu}^{\mu\nu} = - \sum_{\kappa \neq \mu, \nu} \eta_{\kappa}^{\mu} \eta_{\nu}^{\eta} \frac{A_{k}^{2}}{A_{k}^{2}} (\partial_{\kappa} \ln A_{\mu}) (\partial_{\kappa} \ln A_{\nu}) \quad (4.23) \]

The non-vanishing components are given by

\[ K_{r \theta}^{r \theta} = r^{2} [(\ln a)_{,\eta}]^{2} \]
\[ K_{r \phi}^{r \phi} = r^{2} \sin^{2} \theta [(\ln a)_{,\eta}]^{2} \]
\[ K_{\theta \phi}^{\theta \phi} = r^{2} \sin^{2} \theta [(\ln a)_{,\eta}]^{2} - \sin^{2} \theta (1 - kr^{2}) \]

We are now in a position to get all the non-vanishing components of the Riemann curvature from eq. (4.1d). We find

\[ R_{\eta\eta}^{\eta} = \frac{1}{1 - kr^{2}} (\ln a)_{,\eta} \quad (4.24a) \]
\[ R_{\eta\theta}^{\eta} = r^{2} ((\ln a)_{,\eta})_{,\eta} \quad (4.24b) \]
\[ R_{\eta\phi}^{\eta} = r^{2} \sin^{2} \theta ((\ln a)_{,\eta})_{,\eta} \quad (4.24c) \]
\[ R_{\theta\theta}^{\theta} = r^{2} \left[ k + ((\ln a)_{,\eta})^{2} \right] \quad (4.24d) \]
\[ R_{\phi\phi}^{\phi} = r^{2} \sin^{2} \theta \left[ k + ((\ln a)_{,\eta})^{2} \right] \quad (4.24e) \]

Accordingly, all the non-vanishing components of the Ricci tensor are given by

\[ R_{\eta\eta} = -3(\ln a)_{,\eta} \quad (4.25a) \]
\[ R_{rr} = \frac{1}{1 - kr^{2}} \left[ (\ln a)_{,\eta} + 2k + 2((\ln a)_{,\eta})^{2} \right] \quad (4.25b) \]
\[ R_{\theta\theta} = r^{2} \left[ (\ln a)_{,\eta} + 2k + 2((\ln a)_{,\eta})^{2} \right] \quad (4.25c) \]
\[ R_{\phi\phi} = r^{2} \sin^{2} \theta \left[ (\ln a)_{,\eta} + 2k + 2((\ln a)_{,\eta})^{2} \right] \quad (4.25d) \]
Finally, the scalar curvature is

\[ R = -\frac{6}{a^2} \left[ (\ln a),, \eta + k + ((\ln a),\eta)^2 \right] \]  

(4.26)

The Weyl curvature tensor is the traceless part of the Riemann curvature tensor, and it is defined by

\[ C^\rho_{\mu \nu \sigma} = R^\rho_{\mu \nu \sigma} - \frac{1}{2} \left( \delta^\rho_\mu R_{\nu \sigma} - \delta^\rho_\nu R_{\mu \sigma} - g_{\sigma \mu} R^\rho_{\nu} + g_{\sigma \nu} R^\rho_{\mu} \right) + \frac{1}{6} R \left( \delta^\rho_\mu g_{\nu \sigma} - \delta^\rho_\nu g_{\mu \sigma} \right). \]  

(4.27)

When using eqs. (5.24), (5.25) and (5.26), it is easy to verify that all the Weyl tensor components do vanish in the case of the FRW curvature,

\[ C^\rho_{\mu \nu \sigma} = 0 \]  

(4.28)

To conclude the Appendix, a few comments are in order.

Of course, there are other ways to describe the vierbein formalism much shorter (e.g., when using the differential forms). Also, when using some deeper known results about physics and geometry, it may be possible to derive the Ricci tensor faster (e.g., when using its perfect fluid type, the rotational invariance, together with the Friedman and Raychaudhuri equations from the textbooks). However, all that would not help in a calculation of the FRW curvature components, which was one of our main objectives in this paper.

The fact of the vanishing Weyl curvature also follows from the standard (Petrov) classification [13] of curved spacetimes: the local isotropy is allowed by the Petrov types D and N only, however the symmetries of those types do not include spacial rotations. So, our explicit check of the vanishing Weyl tensor may be considered as a non-trivial check of our calculations of the FRW curvatures.
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