Quantum compiling on locally adjusted circuits of designated architecture

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We propose a method for compiling arbitrary $n$-qubit gates via circuits of fixed architecture supplemented with a sufficient number of adjustable single-qubit operations. The circuits are tested against efficiency requirements and then a method for identifying the parameters of the single-qubit operations is applied. The latter extends quantum control techniques developed by G. Harel and V. Akulin [Phys.Rev.Lett. 82, 1 (1999)] and stays computationally tractable for several qubits. We present the method of compiling using a series of $2^n$ $n$-qubit quantum Fourier transform circuits and test it for $n \leq 4$ reporting a low count of two-qubit gates.

The aim of quantum compiling$^{1,3}$ is to approximate a given unitary operations via quantum circuits composed exclusively by gates drawn from computationally$^4$ or efficiently universal sets$^5$. The demand to confine the design within specific sets of gates is due to constrains imposed by fault-tolerance and error-correction techniques. Throughout the years, a variety of effective methods have been proposed for compiling single-qubit gates$^6$–$^{13}$, nevertheless very little progress has been done towards higher dimensions because of the exponentially increasing difficulty of the problem.

A closely related problem$^2,14$ arises when to the resources of quantum compiling tunable single-qubit operations are added. This problem provides with exact results in compiling, is more tractable in higher-dimensions$^2,17,18$ and naturally results to less lengthy sequences of gates than the original problem does. A solution to this problem – even though does not provide fault-tolerant circuits, is of interest for experimental purposes and it can also form a basis for achieving circuits which can simulate many-body quantum dynamics$^{10}$. The task is mostly known in the literature as the efficient gate decomposition problem since historically this has been formulated$^{14}$ and solved by combining two-level unitary matrices decompositions$^{15}$ with Gray codes$^2,14,17,18$. More recently there are new suggestions including hybrid quantum-classical methods$^{19}$, and pure quantum control techniques$^{20}$.

In this work we occupy with the 'exact' compiling task of general unitary operations utilizing as an 'instrument' circuits of fixed but free to determine architecture and of $2^n$-folded repetitive structure. The control is exerted via the parameters of single-qubit operations added in the circuits. For instance in this work we use a series of $2^n$ $n$-qubit quantum Fourier transform (QFT) circuits$^{21–23}$. We then treat the problem of tuning the adjustable single-qubit operations employing and extending quantum control techniques developed in$^{25}$ in the context of simulating the Floquet evolution of quantum systems. In a nutshell, first the control local parameters in the circuit are adjusted to simulate a non-trivial unity and then departing from there, the target unitary is reached with successive application of gradient descent along a path in the geometric space of unitary matrices. Regarding the scalability of our proposal. We have tested the efficiency of method using QFT architecture for simulating random unitary operations acting up to 4 qubits. It is likely that by improving the sophistication of the numerical methods this number can be moderately increased.

While the proposed method of compiling is applicable on any sufficiently 'intrigued' circuit of two-qubit gates$^{24}$, in this current study we choose to exhibit the method with circuits composed by QFT circuits. This choice is justified by several arguments. QFT circuits describe a transform and that being so these operate in a balanced way on all input qubits. These circuits are built solely on the basis of single and two-qubit gates and there is no need to perform further compiling on higher order gates. On the practical level, QFT circuits are used in a good number of quantum algorithms, are considered as standard circuits and in consequence there is motivation for realizing these in practice$^{24,25}$. Finally, QFT circuits corresponding to a 4th root of unity ($U_{QFT}^4 = \hat{I}$), slightly reduce the complexity of the first step of the proposed compiling method.

Results

A generic unitary matrix acting on $n$ qubits requires $4^n – 1$ real parameters for its description—in the case where the global phase is ignored as in this work. Therefore the minimum number of local unitary single-qubit operations required in a circuit of two-qubit gates is $4^n/3$. In order these to be effective they should be interchanged by non-local gates and in consequence any proposed 'unmodulated' circuit that acts as a basis for simulation should contain at least $4^n/6$ two-qubit gates. This estimate on the scaling of two-qubit gates as $\sim 4^n$ has been conjectured before$^{14,17}$ using similar arguments.

In this work, we present the methods with circuits formed by a series of $2^n$ $n$-qubit QFT circuits. As it becomes evident in the section Methods the $2^n$-folded repetitive structure is required by the quantum control
technique. A \( n \)-qubit QFT contains \( \frac{n(n-1)}{2} 2^n \) controlled-phase gates (plus \( \sim \frac{n}{2} \) SWAP gates). Thus, and as it can be observed on the Table 1, a series of \( 2^n \) QFT circuits has a sufficient number of non-local gates if the number qubits that does not exceed 6. The fact that the number of non-local gates in the proposed circuit (or in other circuit) is sufficient does not guarantee its efficiency in simulating any given unitary. For this purpose, in the section Methods we propose and employ relevant tests.

\[ R \equiv R(\phi_x, \phi_y, \phi_z) = \exp(i\phi_x\sigma_x + i\phi_y\sigma_y + i\phi_z\sigma_z) \]  

where \( \{\sigma_j\} \) the Pauli matrices. Let us denote the totality of free local parameters in the circuit by \( \vec{\phi} \) and the subset of free parameters corresponding just to the 1/2\( n \) of the circuit by \( \{\hat{\phi}\} \). The exact placement of single-qubit gates is not crucial and just some simple rules should be respected: (a) exclude adjacent single-qubit operations on the same qubit, (b) place single-qubit operations on qubits that experience a two-qubit gate in next step, (c) keep the circuit balanced by placing approximately the same number of single-qubit operations on every qubit. By trial and error, we have concluded that the number of single-qubit adjustable gates \( R \) should be set to an approximate number \( \frac{2}{3} 4^n \), thus exceeding the required number ( \( \sim 4^n/3 \)). On the other hand, there is no need (and it is not recommended) that all local operations are independent; one can ‘squeeze’ the total number of adjustable parameters \( \vec{\phi} \) to \( \sim \frac{3}{2} 4^n \). This can be achieved by randomly attributing each of the \( \sim \frac{3}{2} 4^n \) parameters \( \vec{\phi} \) to several single-qubit operations \( R \) along the circuit. Interestingly this squeezing of parameters increases the non-linear character of the problem and strengthens the robustness of the circuit.

We have tested the circuit and methods using several random target unitary matrices and on Table 1 we present the overview of the accuracy achieved together the characteristics of the circuits in numbers. As a measure of distance between the target unitary \( \hat{U}_t \) and the unitary corresponding to the optimized circuit \( \hat{U}_t \) we use the following quantity:

\[ D(\hat{U}_t, \hat{\tilde{U}}_t) = 1 - \frac{1}{4^n} \text{ tr } \left[ \hat{U}_t \hat{\tilde{U}}_t^\dagger \right]^2. \]  

### Table 1. Characteristics of the circuits in overview

| \( n \)-qubits | \( n^2 \) of controlled-phase gates | \( 2^n \) of CNOT gates | \( \frac{n(n-1)}{2} 2^n \) of single-qubit operations | \( n^2 \) of adjustable parameters | Accuracy achieved |
|----------------|----------------------------------|-------------------------|---------------------------------|-------------------------------|-------------------|
| \( n=2 \)      | 4                                | 12                      | 32                              | 32                            | \( 10^{-10} \)     |
| \( n=3 \)      | 24                               | 24                      | 96                              | 96                            | \( 10^{-9} \)      |
| \( n=4 \)      | 96                               | 96                      | 384                             | 381                           | \( 10^{-7} \)      |
| \( n=5 \)      | 320                              | 192                     | 1024                            | 1536                          | under test        |
| \( n \)        | \( \frac{n(n-1)}{2} 2^n \)      | \( \frac{1}{2} 2^n \)   | \( \frac{1}{2} 2^n \)          | \( 4^n/2 \)                   |                   |

**Methods**

The method for tuning the local parameters \( \vec{\phi} \) of an efficient circuit consists of two steps (see Fig. 2 for its overview). We present below the method with reference...
to the specific example of QFT circuits, however this is general and can be applied to a \(2^n\) series of any circuit that has sufficient complexity in number and allocation of non-local gates.

\[ x^N + \lambda_{N-1} \left( \{ \vec{\phi} \} \right) x^{N-1} + \ldots + \lambda_1 \left( \{ \vec{\phi} \} \right) x + e^{i\chi}, \quad (3) \]

and the condition \( \sum_{j=1}^{N-1} |\lambda_j \left( \{ \vec{\phi} \} \right)| = 0 \) simply imposes the eigenvalues to be \(N\)th roots of \(-e^{i\chi}\). Therefore

\[ \sum_{j=1}^{N-1} |\lambda_j \left( \{ \vec{\phi} \} \right)| \quad (4) \]

is a convenient cost function for the gradient method at this step.

With the solution to this step being reached, we proceed by repeating the locally adjusted QFT circuit \(2^n\) times, \(\vec{\phi} = \bigcup \{ \vec{\phi} \} \). This series of locally adjusted identical circuits constitutes a non-trivial unity for the totality of the circuit (up to a phase \(-e^{i\chi}\)) and it is the starting point for the next step where all the parameters \(\vec{\phi}\) need to be adjusted.

**B. Checking the efficiency of the circuit to simulate any gate**

Thus far the total circuit has been locally engineered for reaching a non-trivial unity. One might wonder whether this step is necessary since as we have underlined \(\hat{U}_{QFT}^4 = \hat{I}\) and the setting of all local parameters \(\vec{\phi}\) in the circuit to zero, also results to unity. The reason is that at next step one needs to move towards any direction in the space of unitary operators (see Fig.2). The non-trivial unity is a necessary condition for doing so, but in addition the circuit should be efficient enough.

A simple method for checking the efficiency of the circuit around a given point or in other words its ability to reach with local controls any point at the neighborhood, is the following. Employ the \(4^n\) elements of the unitary matrix \(\hat{U}\) at the given point (here the non-trivial unity) to construct a \(4^n \times K\) Jacobian matrix \(J\) for the \(K\) local parameters \(\phi_k\):

\[ \frac{\partial \left( Re \left[ \hat{U}_{ij} \right] + Im \left[ \hat{U}_{ij} \right] \right)}{\partial \phi_k}. \quad (5) \]

Then calculate the eigenvalues of \(JJ^T\). If the spectrum has only one vanishing eigenvalue (that comes as result
of $\text{det} \hat{U} = 1$) then the system is controllable in its neighborhood by the variation of the local parameters $\phi$. If more than one vanishing or very small eigenvalues exist in the spectrum, then the system is of reduced controllability and one in principle cannot expect to reach the next point with good accuracy.

The controllability around unity is important because the accuracy of the first steps is crucial for a final successful outcome. Experience also shows that a system controllable around unity is controllable everywhere, while the inverse statement naturally does not hold true. For the circuits we propose in this work, controllability has been checked around non-trivial unity and random points.

C. Second step: reaching the target via small steps starting from unity

Every unitary matrix $\hat{U}$ is generated by a Hermitian matrix $\hat{H}$ as

$$\hat{U} = e^{i\hat{H}}. \quad (6)$$

Given the target unitary, $\hat{U}_t$, at this step of the method we build a series of unitary operators

$$\hat{U}_t^{(j,M)} = e^{i\sqrt{M} \hat{H}}, \quad j = 1, \ldots, M, \quad (7)$$

where $\hat{U}_t^{(M,M)} = \hat{U}_t$. A randomly picked unitary $\hat{U}_t$ usually is at large distance from $\hat{I}$, i.e., $D(\hat{U}_t, \hat{I}) \approx 1$, while $D(\hat{U}_t^{(j,M)}, \hat{I})$ is increasing approximately linearly with $j$ for $j < M/2$.

The technique to reach $\hat{U}_t$ starting from the non-trivial unity is now straightforward. One applies gradient descent starting from $\hat{I}$ to $\hat{U}_t^{(1,M)}$ using as input the solution found in step 1 of the method but employing now all $2^n$ local parameters. The output is then used as input to the second step $\hat{U}_t^{(1,M)} \rightarrow \hat{U}_t^{(2,M)}$ and so on. The parameter $M$ should be chosen such as the distance $D(\hat{U}_t^{(j,M)}, \hat{U}_t^{(j+1,M)})$ remains constant or decreases along the path.

Let us note here that a quasi-Newton optimization method can be used as an alternative to the gradient descent method. For the purpose, one may use a square $(4^n - 1) \times (4^n - 1)$ Jacobian ‘ distilled’ from $\hat{U}_t$. Though, in order to treat the saddle-points which appear along the path, one should keep ‘active’ all the $K$ parameters and interchange these in the Jacobian matrix—so that this remains invertible at every step.

D. Further technical details on the method

On the page www.qubit.kz the programs for the second step of compiling can be downloaded and their technical details can be explored. On the other hand these are not complicated programs, can be easily reproduced and very likely improved. For the purpose we list here some key technical details.

The first step of the method involves a gradient descent method using the cost function with $2^n$ local parameters. In average, the gradient method is successful after trying out $\sim 50$ randomly selected initial sets of local parameters as starting point. The step of the method can be initialized to 0.05 and then its size be decreased in the case of increase of the cost function. One may consider as successful any outcome with of the order $10^{-2}$ or less.

The second step is more time consuming and the crucial point is to identify an optimum number of $M$ divisions. The answer of course depends on the accuracy one wants to achieve; higher $M$ results to higher final accuracy. For our tests we have used $M = 20$ for achieving the final precision reported in Table 1. Concerning the gradient descent, it is important to combine it with a simple line search on the step at every application of the method.

Finally, we note here that for compiling 2-qubit gates where the number of parameters is low, the application of the methods is unnecessary; a standard simulated annealing method or random search for minimizing the cost function can implement the task with very high accuracy ($10^{-15}$).

Discussion

We have proposed and tested a method for exact compiling of quantum gates that stays efficient up to at least four qubits. The main advantage of the method lies on its applicability to any convenient circuit architecture of provable efficiency. In addition, the structure of the circuit being repetitive—composed by $2^n$ identical unmodulated circuits, makes it appealing for experimental implementations.

On the other hand, the method is exact and as consequence the length of the resulting circuits clearly exceeds the limits of current quantum technologies. It would be interesting thus to identify a way to trade systematically the accuracy of the final result for a lower number of control parameters and for circuits of lower depth. Such a trade is also expected to push the applicability of the method into higher dimensional Hilbert spaces.

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