EXISTENCE OF COROTATING AND COUNTER-ROTATING VORTEX PAIRS FOR ACTIVE SCALAR EQUATIONS

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ABSTRACT. In this paper, we study the existence of corotating and counter-rotating pairs of simply connected patches for Euler equations and the generalized SQG equations with $\alpha \in [0, 1]$. From the numerical experiments implemented for Euler equations in [12, 36, 39] it is conjectured the existence of a curve of steady vortex pairs passing through the point vortex pairs. There are some analytical proofs based on variational principle [27, 41], however they do not give enough information about the pairs such as the uniqueness or the topological structure of each single vortex. We intend in this paper to give direct proofs confirming the numerical experiments and extend these results for the gSQG equation when $\alpha \in [0, 1]$. The proofs rely on the contour dynamics equations combined with a desingularization of the point vortex pairs and the application of the implicit function theorem.

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1. Introduction

The present work deals with the dynamics of vortex pairs for some nonlinear transport equations arising in fluid dynamics. The equations that we shall consider are the generalized surface quasi-geostrophic equations gSQG which describe the evolution of the potential temperature $\theta$ through the system,

\[
\begin{cases}
\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
u = -\nabla \perp (\Delta)^{-1+\frac{\alpha}{2}} \theta, \\
\theta_{t=0} = \theta_0.
\end{cases}
\]

Here $u$ refers to the velocity field, $\nabla \perp = (-\partial_2, \partial_1)$ and $\alpha$ is a real parameter taken in $[0, 2[$. The operator $(-\Delta)^{-1+\frac{\alpha}{2}}$ is of convolution type and defined as follows

\[
(-\Delta)^{-1+\frac{\alpha}{2}} \theta(x) = \int_{\mathbb{R}^2} K_\alpha(x - y) \theta(y) dy
\]

with

\[
K_\alpha(x) = \begin{cases}
-\frac{1}{2\pi} \log |x|, & \text{if } \alpha = 0 \\
C_{\alpha} \frac{1}{2\pi |x|^\alpha}, & \text{if } 0 < \alpha < 1
\end{cases}
\]

and $C_\alpha = \frac{\Gamma(\alpha/2)}{\Gamma(1-\alpha/2)}$ where $\Gamma$ stands for the gamma function. Note that this model was proposed by Córdoba et al. in [11] as an interpolation between Euler equations and the surface quasi-geostrophic model (SQG) corresponding to $\alpha = 0$ and $\alpha = 1$, respectively. We mention that the SQG equation is used in [20, 25] to describe the atmosphere circulation near the tropopause and to track the ocean dynamics in the upper layers [30]. The mathematical analogy with the classical three-dimensional incompressible Euler equations was pointed out in [10].

In the last few years there has been a growing interest in the mathematical study of these active scalar equations. Local well-posedness of classical solutions has been discussed in various function spaces. For instance, this was implemented in the framework of Sobolev spaces [8] however, the global existence is still an open problem except for Euler equations. The second restriction with the gSQG equation concerns the construction of Yudovich solutions–known to exist globally in time for Euler equations [43]– which remains unsolved even locally in time. The main difficulty is due to the velocity which is in general singular and scales below the Lipschitz class. Nonetheless, one can say more about this issue for some special class of concentrated vortices. More precisely, when the initial data is a single vortex patch, that is, $\theta_0(x) = \chi_D$ is the characteristic function of a bounded simply connected smooth domain $D$, there is a unique local solution in the patch form $\theta(t) = \chi_{D_t}$. In this case, the boundary motion of the domain $D_t$ is described by the contour dynamics formulation. Indeed, the Lagrangian parametrization $\gamma_t : \mathbb{T} \to \partial D_t$ obeys to the following integro-differential equations

\[
\partial_t \gamma_t(w) = \int_{\mathbb{T}} K_\alpha(\gamma_t(w) - \gamma_t(\xi)) \gamma'_t(\xi) d\xi.
\]

For more details, see [9, 18, 37]. The global persistence of the boundary regularity is established for Euler equations by Chemin [9], we refer also to the paper of Bertozzi and Constantin [2] for another proof. However for $\alpha > 0$ only local persistence result is known and numerical experiments carried out in [11] reveals a singularity formation in finite time. Let us mention that the contour dynamics equation remains locally well-posed when the domain of the initial data is composed of multiple patches with different magnitudes in each component.
In this paper we shall focus on steady single and multiple patches moving without changing shape, called also V-states according to the terminology of Deem and Zabusky. Their dynamics is seemingly simple flow configurations described by rotating or translating motion but it is immensely rich and exhibits complex behaviors. In the first part we shall recall some numerical and analytical results for the isolated rotating patches and in the second one which is the central subject of this paper we shall focus on multiple vortices. The study of rotating vortex patches is an old subject and the first example goes back to Kirchhoff [28] who proved for Euler equations that an ellipse of semi-axes $a$ and $b$ rotates uniformly with the angular velocity $\Omega = ab/(a + b)^2$. About one century later, Deem and Zabusky [12] provided strong numerical evidence for the existence of rotating patches with $m$-fold symmetry for the integers $m \in \{3, 4, 5\}$. Note that a domain is said $m$-fold symmetric if it is invariant by the action of the dihedral group $D_m$. Few years later, Burbea gave in [3] an analytical proof and showed for any integer $m \geq 2$ the existence of a curve of V-states with $m$-fold symmetry bifurcating from Rankine vortex at the angular velocity $m^{-1}$. The proof relies on the use of complex analysis tools combined with the bifurcation theory. The regularity of the boundary close to Rankine vortices has been discussed very recently by the authors and Verdera in [22] and where we proved that the boundary is $C^{\infty}$ and convex. It seems that the boundary is actually analytic according to the recent result of Castro, Córdoba and Gómez-Serrano [7]. We also refer to the paper [42] where it is proved that corners with right angles is the only plausible scenario for the limiting V-states. We point out that Burbea’s approach has been successfully implemented for the gSQG equations in [7, 19] but with much more delicate computations. We found countable family of bifurcating curves at some known angular velocities related to gamma function. In the same context, it turns out that countable branches of simply connected V-states bifurcate from the ellipses, which has the advantage to get explicit parametrization, and we note that the shapes have in fact less symmetry. This was first observed numerically in [26, 31] and analytical proofs were recently discussed in [6, 23]. Another interesting configuration is to look for doubly connected V-states; that is V-states with only one hole, where the dynamics is much more rich and we capture new behaviors. In this case the V-states are governed by two coupled nonlinear equations which describe the interaction between the two connected boundaries. Their existence was first accomplished for Euler equations in [21], and roughly speaking we obtain that for higher symmetry $m$ there are two branches bifurcating from the annulus $\{b < |z| < 1\}$. Numerical experiments about the limiting V-states were also implemented in the same paper. Later, this result has been extended for the gSQG equations in [13] for $0 < \alpha < 1$ which surprisingly exhibit various completely new behaviors compared to Euler equations. For example we find rotating patches with negative and positive angular velocities for any $\alpha \in [0, 1]$. It is worthy to mention that the bifurcation in the preceding cases is obtained under the transversality assumption of Crandall-Rabinowitz corresponding to simple nonlinear eigenvalues. However the bifurcation in the degenerate case where there is crossing eigenvalues is more complicated and has been recently solved in [24].

Now we shall move to vortex pairs which is a fundamental and rich subject in vortex dynamics and serve for example to describe trailing vortices behind the wings of aircraft in steady horizontal flight or to modelize the interaction between isolated vortex and a solid wall. We shall only consider the cases of counter-rotating and corotating vortices. In the first case the most common studied configuration is two symmetric vortex pair with opposite circulations moving steady with constant speed in a fixed direction. The first classical example is a pair of point vortices with opposite circulations which translates steadily with the speed $U_{sing} = \frac{\gamma}{2\pi d}$ where $d$ is the distance separating the point vortices and $\gamma$ is the magnitude, see for instance [29]. Another nontrivial example of touching counter-rotating
vortex pair was discovered by Lamb [29] and note that the vortex is not uniform but has a smooth compactly supported profile related to Bessel functions of the first kind. On the other hand the existence of translating vortex pair of symmetric patches was revealed numerically in the papers of Deem and Zabusky [12] and Pierrehumbert [36]. Their numerical experiments suggest the existence of a curve of translating symmetric pair of patches starting from two point vortices and ending with two touching patches at right angle. We mention that Keady [27] used a variational principle in order to explore the existence part and give asymptotic estimates for some significant functionals such as the excess kinetic energy and the speed of the pairs. The basic idea is to maximize the excess kinetic energy supplemented with some additional constraints and to show the existence of a maximizer taking the form of a pair of vortex patches in the spirit of the paper of Turkington [41]. However, this approach does not give sufficient information on the pairs. For example the uniqueness of the maximizer is left open and the topological structure of the patches is not well-explored and it is not clear from the proof whether or not each single patch is simply connected as it is suggested numerically. Concerning the corotating vortex pair, which consists of two symmetric patches with the same circulations and rotating about the centroid of the system with constant angular velocity, it was investigated numerically by Saffman and Szeto [39]. They showed that when far apart, the vortices are almost circular and when the distance between them decreases they become more deformed until they touch. We remark that a pair of point vortices far away at a distance $d$ and with the same magnitude $\gamma$ rotates steadily with the angular velocity $\Omega_{\text{sing}} = \frac{\gamma}{\pi d^2}$. By using variational principle, Turkington gave in [41] an analytic proof of the existence of corotating vortex pairs but this general approach does not give enough precision on the topological structure of each vortex patch similarly to the translating case commented before. Note that Dritschel [15] calculated numerically $V$-states of vortex pairs with different shapes and discussed their linear stability. We also mention that Denisov established in [14] for a modified Euler equations the existence of corotating simply connected vortex patches and analyzed the contact point of limiting $V$-states.

To end this introductory part we point out that the subject of vortex pairs has been intensively studied during the past and it is difficult to track, know and recall here everything written about. So, we have only selected some basic results and the reader can find more details not only in this subject but also in some other connected topics in [1, 4, 5, 16, 17, 31, 32, 34, 35, 38, 40] and the references therein.

In the current paper we intend to give direct proofs for the existence of corotating and counter-rotating vortex pairs using the contour dynamics equations. We shall also extend these results to the $\text{gSQG}$ equations for $\alpha \in ]0, 1[$. Let $0 < \varepsilon < 1$, $d > 2$ and take a small simply connected domain $D_1$ containing the origin and contained in the open ball $D(0, 2)$ centered at the origin and with radius 2. Define

$$\theta_{0, \varepsilon} = \frac{1}{\varepsilon^2} \chi_{D_1^\varepsilon} + \delta \frac{1}{\varepsilon^2} \chi_{D_2^\varepsilon}, \quad D_1^\varepsilon = \varepsilon D_1, \quad D_2^\varepsilon = -D_1^\varepsilon + 2d,$$

where the number $\delta$ is taken in $\{\pm 1\}$. As we can readily observe, this initial data is composed of symmetric pair of simply connected patches with equal or opposite circulations. The main result of the paper is the following.

**Main theorem.** Let $\alpha \in [0, 1[$, there exists $\varepsilon_0 > 0$ such that the following results hold true.

(i) Case $\delta = 1$. For any $\varepsilon \in ]0, \varepsilon_0]$ there exists a strictly convex domain $D_1^\varepsilon$ at least of class $C^1$ such that $\theta_{0, \varepsilon}$ in (3) generates a corotating vortex pair for (1).

(ii) Case $\delta = -1$. For any $\varepsilon \in ]0, \varepsilon_0]$ there exists a strictly convex domain $D_1^\varepsilon$ of class $C^1$ such that $\theta_{0, \varepsilon}$ generates a counter-rotating vortex pair for (1).

Before giving the basic ideas of the proofs some remarks are in order.
Remark 1. The domain $D_1^{\varepsilon}$ is a small perturbation of the disc $D(0,\varepsilon)$, centered at zero and of radius $\varepsilon$. Moreover, it can be described by the conformal parametrization $\phi_{\varepsilon} : \mathbb{T} \to \partial D_1^{\varepsilon}$ which belongs for $0 < \alpha < 1$ to $C^{2-\alpha}(\mathbb{T})$ and for $\alpha = 0$ to $C^{1+\beta}$ for any $\beta \in [0,1]$, and satisfies 
$$
\phi_{\varepsilon}(w) = \varepsilon w + \varepsilon^{2+\alpha} f_{\varepsilon}(w) \quad \text{with} \quad \|f_{\varepsilon}\|_{C^{2-\alpha}} \leq 1.
$$
Therefore the boundary of each V-state is at least $C^1$. Note that with slight modifications we can adapt the proofs and show that the domain $D_1^{\varepsilon}$ belongs to $C^{n+\beta}$ for any fixed $n \in \mathbb{N}$. Of course, the size of $\varepsilon_0$ depends on the parameter $n$ and cannot be uniform; it shrinks to zero as $n$ grows to infinity. However, we expect the boundary to be analytic meaning that the conformal mapping possesses a holomorphic extension in $D(0,r)^\varepsilon$ for some $0 < r < 1$. The ideas developed in the recent paper [6] might be useful to confirm such expectation.

Remark 2. The proof is valid for $\alpha \in [0,1]$ but we expect that the result remains true for $\alpha \in [1,2]$. We believe that the use of the spaces introduced in [7] could be helpful for solving these cases.

Remark 3. As we shall see later, we can unify the formalism leading to the existence of corotating patches with the point vortex model. The latter one is obtained when $\varepsilon = 0$ in which case we find the classical result which says that two point vortices at distance $2d$ and with the same magnitude rotate uniformly about their center with the angular velocity $\Omega_{\text{sing}} = \frac{\alpha C_n}{\pi^2 d^{2+\alpha}}$. However when they have opposite signs they are exhibit a uniform translating motion with the speed $U_{\text{sing}} = \frac{\alpha C_n}{\pi^2 d^{2+\alpha}}$.

Next we shall sketch the basic ideas used to prove the main result. We will just restrict the discussion to the corotating pairs for Euler equations since the proofs for the remaining cases follow the same lines but with much more involved computations. The proof relies on the desingularization of point vortex pairs combined with the implicit function theorem. To be more precise, we first formulate the equations governing the corotating vortex pairs and as we shall see later in Section 3 we obtain

$$
\text{Re}\left\{\left(2\Omega(\varepsilon \bar{z} - d) + I_{\varepsilon}(z)\right) \bar{\xi}\right\} = 0, \quad \forall z \in \partial D_1
$$

with $\bar{\xi}$ being the unit tangent vector to the boundary $\partial D_1$ positively oriented and

$$
I_{\varepsilon}(z) = \frac{1}{\varepsilon} \int_{\partial D_1} \bar{\xi} - \bar{z} d\xi - \int_{\partial D_1} \frac{\bar{\xi}}{\varepsilon \xi + \varepsilon z - 2d} d\xi.
$$

The basic idea is to extend the functional defining the vortex pairs beyond $\varepsilon = 0$ corresponding to point vortex pairs and to apply after the implicit function theorem. As we can see, the first integral term in $I_{\varepsilon}(z)$ is singular and to remove the singularity we should seek for domains which are slight perturbation of the unit disc with a small amplitude of order $\varepsilon$. In other words, we look for a parametrization of $D_1$ in the form

$$
\forall w \in \mathbb{T}, \quad \phi_{\varepsilon}(w) = w + \varepsilon f(w)
$$

where the Fourier expansion of $f$ takes the form

$$
f(w) = \sum_{n \geq 1} a_n w^{-n}, \quad a_n \in \mathbb{R} \quad \text{and} \quad \|f\|_{C^{1+\beta}} \leq 1
$$

for some $\beta \in [0,1]$. Note that the singularity in $\varepsilon$ is removable owing to the symmetry of the disc. Indeed, following standard computations we get the expansion

$$
I_{\varepsilon}(\phi_{\varepsilon}(w)) = -\frac{1}{\varepsilon} w + J_{\varepsilon}(\phi_{\varepsilon}(w))
$$

(4)
where $J_\varepsilon$ belongs to the space $C^\beta$ and can be extended for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ with $\varepsilon_0 > 0$. Setting
\[ G(\varepsilon, \Omega, f(w)) \equiv \text{Im}\left\{ \left( 2\Omega[\varepsilon \phi_\varepsilon(w) - d] + I_\varepsilon(\phi_\varepsilon(w)) \right) w \phi'_\varepsilon(w) \right\}, \]
the equation of the vortex pairs is simply given by
\[ \forall w \in \mathbb{T}, \quad G(\varepsilon, \Omega, f(w)) = 0. \]
Therefore, from the expansion (4) we can get rid of the singularity in $\varepsilon$ and this is the first step towards the application of the implicit function theorem. Before giving further details we should first fix the function spaces. Let
\[ X = \left\{ f \in C^{1+\beta}(\mathbb{T}), f(w) = \sum_{n \geq 1} a_n w^{-n} \right\}, \]
\[ Y = \left\{ f \in C^\beta(\mathbb{T}), f = \sum_{n \geq 1} a_n e_n, a_n \in \mathbb{R} \right\}, \quad \hat{Y} = \left\{ f \in Y, a_1 = 0 \right\}, e_n(w) \equiv \text{Im}(w^n). \]
According to Proposition 1 the function $G : -\frac{1}{2} + \frac{1}{\varepsilon_0} [\varepsilon \mathbb{R} \times B_1] \to Y$ is well-defined and it is of class $C^1$, where $B_1$ is the open unit ball of $X$. Moreover
\[ \partial f G(0, \Omega, 0) h(w) = -\text{Im}(h'(w)). \]
However, this operator is not invertible from $X$ to $Y$ but it does from $X$ to $\hat{Y}$. The next step is to choose carefully $\Omega$ such that the image of the nonlinear functional $G$ is contained in the vector space $\hat{Y}$. This will be done in Section 4.2 and leads eventually to a new nonlinear constraint of the type $\Omega = \Omega(\varepsilon, f)$. Consequently the equation of the vortex pairs becomes
\[ F(\varepsilon, f(w)) \equiv G(\varepsilon, \Omega(\varepsilon, f), f) = 0. \]
Note that the point vortex configuration corresponds to $F(0, 0) = 0$ in which case $\Omega = \Omega_{\text{sing}} = \frac{1}{4\pi d^2}$. In addition, from the platitude of $\Omega$ we deduce that the linearized operator remains the same, that is,
\[ \partial f F(0, 0) = \partial f G(0, \Omega_{\text{sing}}, 0) \]
which is invertible from $X$ to $\hat{Y}$. Therefore and at this stage one can use the implicit function theorem which implies the local existence of a unique curve of solutions $\varepsilon \mapsto \phi_\varepsilon$ passing through $(0, 0)$ and notice that each point of this curve is a nontrivial corotating vortex pair of symmetric simply connected patches.

The remaining of the paper is organized as follows. In Section 2 we shall gather some tools dealing with function spaces and give some results on Newton and Riesz potentials. In Section 3 we shall write down the equations governing the corotating and translating vortex pairs of symmetric patches for both Euler and gSQG equations. Sections 4 and 5 are devoted to the proof of the main theorem in the case of Euler equations. In the last two sections we shall see how to extend these results to the generalized SQG equations.

**Notation.** We need to fix some notation that will be frequently used along this paper. We denote by $C$ any positive constant that may change from line to line. We denote by $\mathbb{D}$ the unit disc and its boundary, the unit circle, is denoted by $\mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function, we define its mean value by,
\[ \int_{\mathbb{T}} f(\tau) d\tau \equiv \frac{1}{2\pi} \int_{\mathbb{T}} f(\tau) d\tau, \]
where $d\tau$ stands for the complex integration. Finally, for $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we use the notation $(x)_n$ to denote the Pochhammer symbol defined by,

$$(x)_n = \begin{cases} 1 & n = 0 \\ x(x+1) \cdots (x+n-1) & n \geq 1. \end{cases}$$

2. Preliminaries and background

In this section we shall briefly recall the classical Hölder spaces on the periodic case and state some classical facts on the continuity of fractional integrals over these spaces. It is convenient to think of $2\pi$-periodic function $f : \mathbb{R} \to \mathbb{C}$ as a function of the complex variable $w = e^{in}$ rather than a function of the real variable $\eta$. To be more precise, let $f : \mathbb{T} \to \mathbb{R}^2$, be a continuous function, then it can be assimilated to a $2\pi$-periodic function $g : \mathbb{R} \to \mathbb{R}$ via the relation

$$f(w) = g(\eta), \quad w = e^{in}. $$

Hence when $f$ is smooth enough we get

$$f'(w) \equiv \frac{df}{dw} = -ie^{-in}g'(\eta).$$

Because $d/dw$ and $d/d\eta$ differ only by a smooth factor with modulus one we shall in the sequel work with $d/dw$ instead of $d/d\eta$ which appears to be more convenient in the computations. Now we shall introduce Hölder spaces on the unit circle $\mathbb{T}$.

**Definition 1.** Let $0 < \beta < 1$. We denote by $C^\beta(\mathbb{T})$ the space of continuous functions $f$ such that

$$\|f\|_{C^\beta(\mathbb{T})} \equiv \|f\|_{L^\infty(\mathbb{T})} + \sup_{x \neq y \in \mathbb{T}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty. $$

For any integer $n$ the space $C^{n+\beta}(\mathbb{T})$ stands for the set of functions $f$ of class $C^n$ whose $n$-th order derivatives are Hölder continuous with exponent $\beta$. This space is equipped with the usual norm,

$$\|f\|_{C^{n+\beta}(\mathbb{T})} \equiv \|f\|_{L^\infty(\mathbb{T})} + \left\| \frac{d^n f}{dw^n} \right\|_{C^\beta(\mathbb{T})}. $$

Recall that the Lipschitz (semi)-norm is defined as follows.

$$\|f\|_{\text{Lip}(\mathbb{T})} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}. $$

Now we list some classical properties that will be used later in several sections.

(i) For $n \in \mathbb{N}, \beta \in [0, 1]$ the space $C^{n+\beta}(\mathbb{T})$ is an algebra.

(ii) For $K \in L^1(\mathbb{T})$ and $f \in C^{n+\beta}(\mathbb{T})$ we have the convolution law,

$$\|K * f\|_{C^{n+\beta}(\mathbb{T})} \leq \|K\|_{L^1(\mathbb{T})}\|f\|_{C^{n+\beta}(\mathbb{T})}. $$

The next result is used frequently. It deals with fractional integrals of the following type,

$$\mathcal{T}(f)(w) = \int_{\mathbb{T}} K(w, \tau) f(\tau) d\tau,$$

with $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ being a singular kernel satisfying some properties. The problem on the smoothness of this operator will appear naturally when we shall deal with the regularity of the nonlinear functional defining steady vortex pairs. The result that we shall discuss with respect to this subject is classical and whose proof can be found for instance in [19, 33].

**Lemma 1.** Let $0 \leq \alpha < 1$ and consider a function $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ with the following properties. There exits $C_0 > 0$ such that,
(i) $K$ is measurable on $\mathbb{T} \times \mathbb{T}\setminus\{(w, w), w \in \mathbb{T}\}$ and
\[ |K(w, \tau)| \leq \frac{C_0}{|w - \tau|^\alpha}, \quad \forall w \neq \tau \in \mathbb{T}. \]

(ii) For each $\tau \in \mathbb{T}$, $w \mapsto K(w, \tau)$ is differentiable in $\mathbb{T}\setminus\{\tau\}$ and
\[ |\partial_w K(w, \tau)| \leq \frac{C_0}{|w - \tau|^{1+\alpha}}, \quad \forall w \neq \tau \in \mathbb{T}. \]

Then

A) The operator $\mathcal{T}$ defined by (5) is continuous from $L^\infty(\mathbb{T})$ to $C^{1-\alpha}(\mathbb{T})$. More precisely, there exists a constant $C_\alpha$ depending only on $\alpha$ such that
\[ \|\mathcal{T}(f)\|_{1-\alpha} \leq C_\alpha C_0 \|f\|_{L^\infty}. \]

B) For $\alpha = 0$ the operator $\mathcal{T}$ is continuous from $L^\infty(\mathbb{T})$ to $C^\beta(\mathbb{T})$ for any $0 < \beta < 1$. That is, there exists a constant $C_\beta$ depending only on $\beta$ such that
\[ \|\mathcal{T}(f)\|_{\beta} \leq C_\beta C_0 \|f\|_{L^\infty}. \]

As a by-product we obtain a result that will be frequently used through this paper.

**Corollary 1.** Let $0 < \alpha < 1$, $\phi : \mathbb{T} \to \phi(\mathbb{T})$ be a bi-Lipschitz function with real Fourier coefficients and define the operator
\[ \mathcal{T}_\phi : f \mapsto \int_{\mathbb{T}} \frac{f(\tau)}{\phi(w) - \phi(\tau)} d\tau, \quad w \in \mathbb{T}. \]

Then $\mathcal{T}_\phi : L^\infty(\mathbb{T}) \to C^{1-\alpha}(\mathbb{T})$ is continuous with the estimation,
\[ \|\mathcal{T}_\phi(f)\|_{C^{1-\alpha}(\mathbb{T})} \leq C \left( \|\phi^{-1}\|_{Lip(\mathbb{T})}^\alpha \|\phi\|_{Lip(\mathbb{T})}^2 + \|\phi\|_{Lip(\mathbb{T})}^{-1} \|\phi^{-1}\|_{Lip(\mathbb{T})}^{1+\alpha} \right) \|f\|_{L^\infty(\mathbb{T})}, \]

where $C$ is a positive constant depending only on $\alpha$.

### 3. Steady vortex pairs models

The aim of this section is to derive the equations governing co-rotating and translating symmetric pairs of patches. In the first step, we shall write down the equations for the rotating pairs for Euler and $gSQG$ equations. In the second step, we shall be concerned with counter-rotating vortex pairs sometimes called translating pairs. Notice that we prefer using the conformal parametrization because it is more convenient in the computations.

#### 3.1. Corotating vortex pair

Let $D_1$ be a bounded simply connected domain containing the origin and contained in the ball $B(0, 2)$. For $\varepsilon \in [0, 1]$ and $d > 2$ we define the domains
\[ D_1^\varepsilon = \varepsilon D_1 \quad \text{and} \quad D_2^\varepsilon = -D_1^\varepsilon + 2d. \]

Set
\[ \theta_{0, \varepsilon} = \frac{1}{\varepsilon^2} \chi_{D_1^\varepsilon} + \frac{1}{\varepsilon^2} \chi_{D_2^\varepsilon} \]

and assume that this gives rise to a rotating pairs of patches about the centroid of the system $(d, 0)$ and with an angular velocity $\Omega$. According to [21], this condition holds true if and only if
\[ \text{Re}(-i \Omega (z - d) \vec{n}) = \text{Re}(\overline{v(z)} \vec{n}), \quad \forall z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon, \]

where $\vec{n}$ is the exterior unit normal vector to the boundary of $D_1^\varepsilon \cup D_2^\varepsilon$ at the point $z$. Next we shall discuss separately Euler equations and the case $0 < \alpha < 1$. 

3.1.1. *Euler equations*. It is well-known that the velocity can be recovered for the vorticity according to Biot-Savart law,

\[
\overline{v}(z) = -\frac{i}{2\pi \varepsilon^2} \int_{D_1^\varepsilon} \frac{dA(\zeta)}{z - \zeta} - \frac{i}{2\pi \varepsilon^2} \int_{D_2^\varepsilon} \frac{dA(\zeta)}{z - \zeta}, \quad \forall z \in \mathbb{C}.
\]

From Green-Stokes formula we record that

\[
-\frac{1}{\pi} \int_D \frac{dA(\zeta)}{z - \zeta} = \oint_{\partial D} \frac{\xi - z}{z - \xi} d\xi, \quad \forall z \in \mathbb{C}.
\]

Therefore

(7) \( \text{Re}\{2\Omega(\overline{z} - d) + I(z)\} \overrightarrow{\tau} = 0, \quad \forall z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon, \)

with \( \overrightarrow{\tau} \) being the unit tangent vector to \( \partial D_1^\varepsilon \cup \partial D_2^\varepsilon \) positively oriented and

\[
I(z) = \frac{1}{\varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi + \frac{1}{\varepsilon^2} \int_{\partial D_2^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi.
\]

Changing in the last integral \( \xi \to -\xi + 2d \), which sends \( \partial D_2^\varepsilon \) to \( \partial D_1^\varepsilon \), we get

\[
I(z) = \frac{1}{\varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi - \frac{1}{\varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{\overline{\xi} + 2d}{\xi - z} d\xi.
\]

We can check that if the equation (7) is satisfied for all \( z \in \partial D_1^\varepsilon \), then it will be surely satisfied for all \( z \in \partial D_2^\varepsilon \). This follows easily from the identity

\[
I(-z + 2d) = -I(z).
\]

Now observe that when \( z \in \partial D_1^\varepsilon \) then \( -z + 2d \notin \overline{D_1} \) and thus residue theorem allows to get

\[
I(z) = \frac{1}{\varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi - \frac{1}{\varepsilon^2} \int_{\partial D_1^\varepsilon} \overline{\xi} + z - 2d d\xi.
\]

Denote \( \Gamma_1 = \partial D_1 \) then by the change of variables \( \xi \mapsto \varepsilon\xi \) and \( z \mapsto \varepsilon z \) the equation (7) becomes

\[
\text{Re}\{2\Omega(\varepsilon\overline{z} - d) + I_\varepsilon(z)\} \overrightarrow{\tau} = 0, \quad \forall z \in \Gamma_1,
\]

with

\[
I_\varepsilon(z) \equiv I(\varepsilon z)
\]

\[
= \frac{1}{\varepsilon} \int_{\Gamma_1} \frac{\overline{\xi} - \overline{z}}{\xi - z} d\xi - \int_{\Gamma_1} \frac{\overline{\xi}}{\varepsilon\xi + \varepsilon z - 2d} d\xi
\]

\[
\equiv I_1^\varepsilon(z) - I_2^\varepsilon(z).
\]

We shall search for domains \( D_1 \) which are small perturbations of the unit disc with an amplitude of order \( \varepsilon \). More precisely, we shall in the conformal parametrization \( \phi : \mathbb{T} \to \partial D_1 \) look for a solution in the form

\[
\phi(w) = w + \varepsilon f(w), \quad \text{with} \quad f(w) = \sum_{n \geq 1} \frac{a_n}{w^n}, \quad a_n \in \mathbb{R}.
\]

We remark that the assumption \( a_n \in \mathbb{R} \) means that the domain \( D_1 \) is symmetric with respect to the real axis. Setting \( z = \phi(w) \), then for \( w \in \mathbb{T} \) a tangent vector to the boundary at the point \( z \) is given by

\[
\overrightarrow{\tau} = i w \phi'(w) = i w (1 + \varepsilon f'(w)).
\]
Thus the steady vortex pairs equation becomes

\begin{equation}
\text{Im}\left\{ \left( 2\Omega\left[ \varepsilon w + \varepsilon^2 f(w) - d \right] + I_\varepsilon(\phi(w)) \right) w(1 + \varepsilon f'(w)) \right\} = 0, \quad \forall w \in \mathbb{T}.
\end{equation}

Notice that we have used that $f$ has real Fourier coefficients and thus $\overline{f(w)} = f(\overline{w})$. By using the notation $A = \tau - w$ and $B = f(\tau) - f(w)$ we can write for all $w \in \mathbb{T}$

\[
I_\varepsilon^1(\phi(w)) = \frac{1}{\varepsilon} \int_{\mathbb{T}} \tau - w + \varepsilon(f(\tau) - f(w)) (1 + \varepsilon f'(\tau)) d\tau
\]

\[
= \int_{\mathbb{T}} \frac{A + \varepsilon B}{A + \varepsilon B} f'(\tau) d\tau + \int_{\mathbb{T}} \frac{AB - \overline{AB}}{A + \varepsilon B} d\tau + \frac{1}{\varepsilon} \int_{\mathbb{T}} \frac{A}{A} d\tau
\]

\[
= \int_{\mathbb{T}} \frac{A + \varepsilon B}{A + \varepsilon B} f'(\tau) d\tau + \int_{\mathbb{T}} \frac{AB - \overline{AB}}{A + \varepsilon B} d\tau - \frac{1}{\varepsilon} w
\]

where we have used the obvious formula

\[
\int_{\mathbb{T}} \frac{A}{A} d\tau = -\overline{w} \int_{\mathbb{T}} \frac{d\tau}{\tau} = -\overline{w}.
\]

This leads to a significant cancellation and the singular term will disappear from the full nonlinearity due in particular to the symmetry of the disc,

\[
\text{Im}\left\{ I_\varepsilon^1(\phi(w)) w(1 + \varepsilon f'(w)) \right\} = \text{Im}\left\{ \left( \int_{\mathbb{T}} \frac{A + \varepsilon B}{A + \varepsilon B} f'(\tau) d\tau + \int_{\mathbb{T}} \frac{AB - \overline{AB}}{A + \varepsilon B} d\tau \right) w(1 + \varepsilon f'(w)) \right\}
\]

\[
- \text{Im}(f'(w)), \quad \forall w \in \mathbb{T}.
\]

For the second term $I_\varepsilon^2(\phi(w)$ it takes the form

\[
I_\varepsilon^2(\phi(w) = \int_{\mathbb{T}} (\tau + \varepsilon f(\tau))(1 + \varepsilon f'(\tau)) d\tau.
\]

Hence the steady vortex pairs equation is equivalent to

\begin{equation}
G(\varepsilon, \Omega, f) \equiv \text{Im}(F(\varepsilon, \Omega, f)) = 0
\end{equation}

with

\[
F(\varepsilon, \Omega, f(w)) = 2\Omega \left( \varepsilon w + \varepsilon^2 f(w) - d \right) w(1 + \varepsilon f'(w)) - f'(w)
\]

\[
+ \left( \int_{\mathbb{T}} \frac{A + \varepsilon B}{A + \varepsilon B} f'(\tau) d\tau + \int_{\mathbb{T}} \frac{AB - \overline{AB}}{A + \varepsilon B} d\tau \right) w(1 + \varepsilon f'(w))
\]

\[
- \left( \int_{\mathbb{T}} \frac{(\tau + \varepsilon f(\tau))(1 + \varepsilon f'(\tau))}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} d\tau \right) w(1 + \varepsilon f'(w))
\]

\[
\equiv F_1(\varepsilon, \Omega, f(w)) + F_2(\varepsilon, f(w)) + F_3(\varepsilon, f(w)).
\]

3.1.2. gSQG equations. First we remark that the equation (6) can be written in the form,

\begin{equation}
\Omega \text{Re} \left\{ (z - d) \overline{\tau} \right\} = \text{Im}\left\{ v(z) \overline{\tau} \right\}, \quad \forall z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon,
\end{equation}

where as before $\overline{\tau}$ denotes a tangent vector to the boundary at the point $z$. This equation is equivalent to

\[
\text{Re} \left\{ \left( \Omega(z - d) + iv(z) \right) \overline{\tau} \right\} = 0, \quad \forall z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon.
\]
The velocity can be recovered from the boundary as follows, see for instance [19],

\[ v(z) = \frac{C_0}{2\pi \varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{1}{|z - \xi|^\alpha} d\xi + \frac{C_0}{2\pi \varepsilon^2} \int_{\partial D_2^\varepsilon} \frac{1}{|z + \xi - 2d|^\alpha} d\xi, \quad \forall z \in \mathbb{C}. \]

Using in the last integral the change of variables \( \xi \mapsto -\xi + 2d \), we deduce that

\[ v(z) = \frac{C_0}{2\pi \varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{1}{|z - \xi|^\alpha} d\xi - \frac{C_0}{2\pi \varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{1}{|z + \xi - 2d|^\alpha} d\xi, \quad \forall z \in \mathbb{C}. \]

We point out that by a symmetry argument if the equation (10) is satisfied for all \( z \in \partial D_1^\varepsilon \) then it will be also satisfied for all \( z \in \partial D_2^\varepsilon \). This follows from the identity

\[ v(-z + 2d) = -v(z). \]

As \( D_1^\varepsilon = \varepsilon D_1 \) then using a change of variable the equation becomes

\[ \text{Re}\left\{ \left( \Omega(\varepsilon z - d) + I_\varepsilon(z) \right) \overline{\tau} \right\} = 0, \quad \forall z \in \partial D_1, \]

with

\[ I_\varepsilon(z) = -\frac{C_0}{\varepsilon^{1+\alpha}} \int_{\partial D_1^\varepsilon} \frac{1}{|z - \xi|^\alpha} d\xi + \frac{C_0}{\varepsilon} \int_{\partial D_1^\varepsilon} \frac{1}{|\varepsilon z + \varepsilon \xi - 2d|^\alpha} d\xi. \]

We shall look for the domains \( D_1 \) which are small perturbation of the unit disc with an amplitude of order \( \varepsilon^{1+\alpha} \). More precisely, we shall in the conformal parametrization \( \phi : \mathbb{T} \to \partial D_1 \) look for a solution in the form

\[ \phi(w) = w + \varepsilon^{1+\alpha} f(w) = w + \varepsilon^{1+\alpha} \sum_{n \geq 1} \frac{a_n}{w^n}, \quad a_n \in \mathbb{R}. \]

For \( w \in \mathbb{T} \) the conjugate of a tangent vector is given by \( \overline{\tau} = -i \overline{w} \overline{\phi'(w)} \) and therefore for any \( w \in \mathbb{T} \),

\[ G(\varepsilon, \Omega, f(w)) \equiv \text{Im}\left\{ \Omega \left[ \varepsilon w + \varepsilon^{2+\alpha} f(w) - d \right] + I(\varepsilon, f(w)) \right\} \overline{w} \left( 1 + \varepsilon^{1+\alpha} \overline{f'(w)} \right) = 0, \]

with

\[ I(\varepsilon, f(w)) = -\frac{C_0}{\varepsilon^{1+\alpha}} \int_{\mathbb{T}} \frac{\phi'(\tau)d\tau}{|\phi(w) - \phi'(\tau)|^\alpha} + \frac{C_0}{\varepsilon} \int_{\mathbb{T}} \frac{\phi'(\tau)d\tau}{|\varepsilon \phi(w) + \varepsilon \phi'(\tau) - 2d|^\alpha}, \]

\[ \equiv -I_1(\varepsilon, f(w)) + I_2(\varepsilon, f(w)). \]

We shall split \( G \) into three terms

\[ G = G_0 - G_1 + G_2 \]

with

\[ G_0(\varepsilon, \Omega, f(w)) = \text{Im}\left\{ \Omega \left[ \varepsilon w + \varepsilon^{2+\alpha} f(w) - d \right] \overline{w} \left( 1 + \varepsilon^{1+\alpha} \overline{f'(w)} \right) \right\}, \]

\[ G_1(\varepsilon, f(w)) = \text{Im}\left\{ I_1(\varepsilon, f(w)) \overline{w} \left( 1 + \varepsilon^{1+\alpha} \overline{f'(w)} \right) \right\}, \]

and

\[ G_2(\varepsilon, f(w)) = \text{Im}\left\{ I_2(\varepsilon, f(w)) \overline{w} \left( 1 + \varepsilon^{1+\alpha} \overline{f'(w)} \right) \right\}. \]
3.2. Counter-rotating vortex pair. As for the corotating pairs we shall distinguish between Euler equations and the case $0 < \alpha < 1$. Let $D_1$ be a bounded domain containing the origin and contained in the ball $B(0, 2)$. For $\varepsilon \in [0, 1]$ and $d > 2$ we define as before

$$D_1^\varepsilon = \varepsilon D_1 \quad \text{and} \quad D_2^\varepsilon = -D_1^\varepsilon + 2d.$$ 

Set

$$\theta_0 = \frac{1}{\varepsilon^2} \chi_{D_1^\varepsilon} - \frac{1}{\varepsilon^2} \chi_{D_2^\varepsilon}$$

and assume that $\theta_0$ travels steadily in the $(Oy)$ direction with uniform velocity $U$. Then in the moving frame the pair of the patches is stationary and consequently the analogous of the equation (6) is

(16) \quad \text{Re}\left\{ (v(z) + iU) \mathbf{n} \right\} = 0, \quad \forall z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon.

3.2.1. Euler equations. One has from (16)

(17) \quad \text{Re}\left\{ (2U + I(z)) \mathbf{\tau} \right\} = 0, \quad \forall z \in \partial D_1^\varepsilon \cup \partial D_2^\varepsilon,

with

$$I(z) = \frac{1}{\varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{\xi - z}{\xi - \zeta} d\xi - \frac{1}{\varepsilon^2} \int_{\partial D_2^\varepsilon} \frac{\xi - z}{\xi - \zeta} d\xi.$$

In the last integral changing $\xi$ to $-\xi + 2d$ which sends $\partial D_2^\varepsilon$ to $\partial D_1^\varepsilon$ we get

$$I(z) = \frac{1}{\varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{\xi - z}{\xi - \zeta} d\xi + \frac{1}{\varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{\xi + z - 2d}{\xi + z - 2d} d\xi.$$

We can check that if the equation (17) is satisfied for all $z \in \partial D_1^\varepsilon$ then it is also satisfied for all $z \in \partial D_2^\varepsilon$. This follows from the identity

$$I(-z + 2d) = I(z).$$

Now observe that when $z \in \partial D_1^\varepsilon$ then $-z + 2d \notin \overline{D_1^\varepsilon}$ and using residue theorem we obtain

$$I(z) = \frac{1}{\varepsilon^2} \int_{\partial D_1^\varepsilon} \frac{\xi - z}{\xi - \zeta} d\xi + \int_{\Gamma_1} \frac{\xi}{\xi + \varepsilon z - 2d} d\xi.$$

Let $\Gamma_1 = \partial D_1$ then by change of variables $\xi \to \varepsilon \xi$ and $z \to \varepsilon z$. The equation (17) becomes

$$\text{Re}\left\{ (2U + I_\varepsilon(z)) \mathbf{\tau} \right\} = 0, \quad \forall z \in \Gamma_1,$$

with

$$I_\varepsilon(z) = I(\varepsilon z)$$

$$= \frac{1}{\varepsilon} \int_{\Gamma_1} \frac{\xi - \zeta}{\xi - z} d\xi + \int_{\Gamma_1} \frac{\xi}{\xi + \varepsilon z - 2d} d\xi$$

$$\equiv I_1^\varepsilon(z) + I_2^\varepsilon(z).$$

we shall now use the conformal parametrization of the boundary $\Gamma_1$,

$$\phi(w) = w + \varepsilon f(w), \quad \text{with} \quad f(w) = \sum_{n \geq 1} \frac{a_n}{w^n}, a_n \in \mathbb{R}.$$

Setting $z = \phi(w)$ and $\xi = \phi(\tau)$, then for $w \in \mathbb{T}$ a tangent vector at the point $\phi(w)$ is given by

$$\mathbf{\tau} = iw \phi'(w) = iw(1 + \varepsilon f'(w)).$$
The V-states equation becomes
\[ \text{Im}\left\{(2U + I_\epsilon(\phi(w)))w(1 + \epsilon f'(w))\right\} = 0, \quad \forall w \in \mathbb{T}. \]

As in the rotating case, with the notation \( A = \tau - w \) and \( B = f(\tau) - f(w) \) we get for \( w \in \mathbb{T} \)
\[ I_\epsilon^1(\phi(w)) = \int_\mathbb{T} \frac{A + \epsilon B}{A + \epsilon B} f'(\tau)d\tau + \int_\mathbb{T} \frac{A B - A B}{A(A + \epsilon B)} d\tau - \frac{1}{\epsilon}. \]

This yields
\[ \text{Im}\left\{I_\epsilon^1(\phi(w))w(1 + \epsilon f'(w))\right\} = \text{Im}\left\{(\int_\mathbb{T} \frac{A + \epsilon B}{A + \epsilon B} f'(\tau)d\tau + \int_\mathbb{T} \frac{A B - A B}{A(A + \epsilon B)} d\tau\}w(1 + \epsilon f'(w))\right\} - \text{Im}(f'(w)), \quad \forall w \in \mathbb{T}. \]

For the second term \( I_\epsilon^2(\phi(w)) \) it takes the form
\[ I_\epsilon^2(\phi(w)) = \int_\mathbb{T} \frac{(\tau + \epsilon \overline{f(\tau)})(1 + \epsilon f'(\tau))}{\epsilon(\tau + w) + \epsilon^2(f(\tau) + f(w)) - 2d} d\tau. \]

Hence the V-states equation becomes
\[ G(U, \epsilon, f) \equiv \text{Im}(F(U, \epsilon, f)) = 0 \]

with
\[ F(U, \epsilon, f(w)) = 2Uw(1 + \epsilon f'(w)) - f'(w) \]
\[ + \left( \int_\mathbb{T} \frac{A + \epsilon B}{A + \epsilon B} f'(\tau)d\tau + \int_\mathbb{T} \frac{A B - A B}{A(A + \epsilon B)} d\tau\right)w(1 + \epsilon f'(w)) \]
\[ + \left( \int_\mathbb{T} \frac{\tau + \epsilon f(\tau)}{\epsilon(\tau + w) + \epsilon^2(f(\tau) + f(w)) - 2d} (1 + \epsilon f'(\tau))d\tau\right)w(1 + \epsilon f'(w)) \]
\[ \equiv F_1(U, \epsilon, f(w)) + F_2(\epsilon, f(w)) + F_3(\epsilon, f(w)). \]

3.2.2. Case 0 < \( \alpha < 1 \). The equation (16) can be written in the form
\[ \text{Re}\left\{(v(z) - iU)\overline{n}\right\} = 0, \quad \forall z \in \partial D_1^1 \cup \partial D_2^2. \]

The velocity associated to this model is
\[ v(z) = \frac{C_\alpha}{2\pi \epsilon^2} \int_{\partial D_1^1} \frac{1}{|z - \xi|^\alpha} d\xi - \frac{C_\alpha}{2\pi \epsilon^2} \int_{\partial D_2^2} \frac{1}{|z - \xi|^\alpha} d\xi, \quad \forall z \in \mathbb{C}. \]

In the last integral changing \( \xi \) to \( -\xi + 2d \) which sends \( \partial D_1^1 \) to \( \partial D_2^2 \) we get
\[ v(z) = \frac{C_\alpha}{2\pi \epsilon^2} \int_{\partial D_1^1} \frac{1}{|z - \xi|^\alpha} d\xi + \frac{C_\alpha}{2\pi \epsilon^2} \int_{\partial D_2^2} \frac{1}{|z + \xi - 2d|^\alpha} d\xi, \quad \forall z \in \mathbb{C}. \]

Therefore the V-states equation be can be written in the form
\[ \text{Re}\left\{(-U + I_\epsilon(z))\overline{n}\right\} = 0, \quad \forall z \in \partial D_1 \]

with
\[ I_\epsilon(z) \equiv \frac{C_\alpha}{\epsilon^{1+\alpha}} \int_{\partial D_1} \frac{1}{|z - \xi|^\alpha} d\xi + \frac{C_\alpha}{\epsilon} \int_{\partial D_1} \frac{1}{|\epsilon z + \epsilon \xi - 2d|^\alpha} d\xi. \]

Using the conformal parametrization,
\[ \phi(w) = w + \epsilon^{1+\alpha} f(w) \]
\[ \equiv w + \epsilon^{1+\alpha} \sum_{n \geq 1} a_n w^n. \]
For \( w \in \mathbb{T} \) the conjugate of a tangent vector is given by
\[
\overline{z}' = -i \overline{w} \phi'(w).
\]
Therefore for any \( w \in \mathbb{T} \),
\[
G(\varepsilon, \Omega, f(w)) \equiv \text{Im} \left\{ -U + I(\varepsilon, f(w)) \frac{1 + \varepsilon^{1+\alpha} f'(w)}{1 + \varepsilon^{1+\alpha} f'(w)} \right\} = 0,
\]
with
\[
I(\varepsilon, f(w)) = \frac{C_\alpha}{\varepsilon^{1+\alpha}} \int_\mathbb{T} \frac{\phi' (\tau) d\tau}{|\phi(w) - \phi(\tau)|^\alpha} + \frac{C_\alpha}{\varepsilon} \int_\mathbb{T} \frac{\phi' (\tau) d\tau}{|\varepsilon \phi(w) + \varepsilon \phi(\tau) - 2d|^\alpha}
\]
(21)
\[
\equiv I_1(\varepsilon, f(w)) + I_2(\varepsilon, f(w)).
\]
We shall split, as before, \( G \) into three terms
(22)
\[
G = G_0 + G_1 + G_2
\]
with
\[
G_0(\varepsilon, \Omega, f(w)) = -U \text{Im} \left\{ \frac{1 + \varepsilon^{1+\alpha} f'(w)}{1 + \varepsilon^{1+\alpha} f'(w)} \right\},
\]
\[
G_1(\varepsilon, f(w)) = \text{Im} \left\{ I_1(\varepsilon, f(w)) \frac{1 + \varepsilon^{1+\alpha} f'(w)}{1 + \varepsilon^{1+\alpha} f'(w)} \right\}
\]
and
\[
G_2(\varepsilon, f(w)) = \text{Im} \left\{ I_2(\varepsilon, f(w)) \frac{1 + \varepsilon^{1+\alpha} f'(w)}{1 + \varepsilon^{1+\alpha} f'(w)} \right\}.
\]

4. Corotating vortex pair for Euler equations

In this section we will prove the existence of rotating pairs of patches for the planar Euler equations. Recall that the equations governing the boundaries of the vortices were formulated in the subsection 3.1.1. The first goal is to discuss the regularity of the functionals defining the V-states. In the subsection 4.2 we shall see how the angular velocity is uniquely determined through the geometry of the domain. Finally, in the subsection 4.3 we will get the existence of the vortex pairs as a consequence of the implicit function theorem on Banach spaces and discuss the arguments leading to the convexity of the each single vorticity.

4.1. Extension and regularity of the functional \( G \). The main idea to prove the existence of rotating vortex pairs is to apply the implicit function theorem to the equation (9). For this purpose we have to check that the function \( G \) defined there satisfies some regularity conditions. First we need to fix some function spaces. Let \( \beta \in ]0,1[ \) and consider the spaces
\[
X = \left\{ f \in C^{1+\beta}(\mathbb{T}), f(w) = \sum_{n \geq 1} a_n w^{-n} \right\},
\]
\[
Y = \left\{ f \in C^\beta(\mathbb{T}), f = \sum_{n \geq 1} a_n e_n, a_n \in \mathbb{R} \right\}, \quad \tilde{Y} = \left\{ f \in Y, a_1 = 0 \right\}, \quad e_n(w) = \text{Im}(w^n).
\]
For \( r > 0 \) we denote by \( B_r \) the open ball of \( X \) centered at zero and of radius \( r \). The next result deals with some properties of the function \( G \).

**Proposition 1.** The following assertions hold true.

(i) The function \( G \) can be extended from \( \] -\frac{1}{2}, \frac{1}{2}[ \times \mathbb{R} \times B_1 \rightarrow Y \) as a \( C^1 \) function.

(ii) Two initial point vortex \( \pi \delta_0 \) and \( \pi \delta_{(2d,0)} \) rotate uniformly about \( (d,0) \) with the angular velocity
\[
\Omega_{\text{sing}} \equiv \frac{1}{4d^2}.
\]
(iii) For $\Omega \in \mathbb{R}$ and $h \in X$, we have
\[ \partial_f G(0, \Omega, 0) h(w) = -\text{Im}\{h'(w)\}. \]

(iv) For any $\Omega \in \mathbb{R}$, the operator $\partial_f G(0, \Omega, 0) : X \to \hat{Y}$ is an isomorphism.

**Remark 4.** By adapting the proof below we can check that the preceding proposition remains true if we change in the definition of $X$ and $Y$ the parameter $\beta$ by $n + \beta$ for any $n \in \mathbb{N}$.

**Proof.** (i) We will start with the regularity of the functional
\[ G_1(\varepsilon, \Omega, f) = \text{Im}\left\{ 2\Omega (\varepsilon \bar{w} + \varepsilon^2 \overline{f(w)} - d) w(1 + \varepsilon f'(w)) - f'(w) \right\}. \]
Clearly this function can be defined from the set $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1$ to $Y$ because the function in the brackets is in $C^\beta(\mathbb{T})$, and is obtained as sums and products of functions with real coefficients. In order to prove its differentiability we have to compute the partial derivatives of $G_1$.

\[ \partial_\varepsilon G_1(\varepsilon, \Omega, f) = \text{Im}\left\{ 2\Omega (\varepsilon\bar{w} + \varepsilon^2 \overline{f(w)}) w(1 + \varepsilon f'(w)) + 2\Omega (\varepsilon \bar{w} + \varepsilon^2 \overline{f(w)} - d) w f'(w) \right\}, \]
and clearly this is a continuous function from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1$ to $Y$. Taking now the derivative in $\Omega$ we get
\[ \partial_\Omega G_1(\varepsilon, \Omega, f) = \text{Im}\left\{ 2\left( \varepsilon \bar{w} + \varepsilon^2 \overline{f(w)} - d \right) w (1 + \varepsilon f'(w)) \right\}, \]
which is continuous from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1$ to $Y$. Let’s note that $G_1$ is a polynomial also in $f$ and $f'$ and consequently the derivative is also polynomial in $f$ and $f'$. Thus, it is a continuous function from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1$ to $Y$. It is an easy computation to check that
\[ \partial_f G_1(0, \Omega, 0)(h) = -\text{Im}\{h'(w)\}. \]
Let’s take now
\[ G_2(\varepsilon, f) = \text{Im}\left\{ \int_T \frac{\bar{A} + \varepsilon B}{A + \varepsilon B} f' d\tau + \int_T \frac{A \bar{B} - \bar{A} B}{A(A + \varepsilon B)} d\tau \right\} w(1 + \varepsilon f'(w)) \]
\[ = \text{Im}\left\{ (G_{21} + G_{22}) w(1 + \varepsilon f'(w)) \right\}. \]
To prove that $G_2(\varepsilon, f)$ is a function from $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1$ to $Y$ it is enough to verify that the functions $G_{21}(\varepsilon, f)$ and $G_{22}(\varepsilon, f)$ satisfies the same property. The function
\[ G_{21}(\varepsilon, f) = \int_T \frac{\tau - \bar{w} + \varepsilon(f(\tau) - f(w))}{\tau - w + \varepsilon (f(\tau) - f(w))} f'(\tau) d\tau \]
is given by an integral operator. Since $f$ is in $C^{1+\beta}(\mathbb{T})$, we will have that $G_{21}$ is in the space $C^\beta(\mathbb{T})$ if the kernel
\[ K(\tau, w) = \frac{\tau - \bar{w} + \varepsilon(f(\tau) - f(w))}{\tau - w + \varepsilon (f(\tau) - f(w))} \]
satisfies the hypothesis of Lemma 1 for $\alpha = 0$. For $\tau \neq w$
\[ |K(\tau, w)| \leq 1, \]
and moreover
\[ |\partial_w K(\tau, w)| \leq \frac{|1 + \varepsilon f'(w)(\tau - \bar{w}) + \varepsilon(f(\tau) - f(w))|}{|\tau - w| + \varepsilon (f(\tau) - f(w))^2} + \frac{1}{w^2} + \frac{1 + \varepsilon f'(w)}{|\tau - w| + \varepsilon (f(\tau) - f(w))}, \]
\[ \leq \frac{M^2 + M}{|\tau - w|}, \]
where \( M = \frac{1 + \varepsilon\|f\|_{C^{1+\alpha}(\mathbb{T})}}{1 - \varepsilon\|f\|_{C^{1+\alpha}(\mathbb{T})}} \). Now to check that this function has real coefficients we have to show that \( G_{21}(\varepsilon, f)(w) = G_{21}(\varepsilon, f)(\overline{w}) \). Using the change of variable \( \eta = \tau \), it is an easy computation to see that
\[
G_{21}(\varepsilon, f)(w) = \int_T \frac{\tau - w + \varepsilon(f(\tau) - f(w))}{\tau - w + \varepsilon(f(\tau) - f(w))} f'(\tau) d\tau = \int_T \frac{\eta - w + \varepsilon(f(\eta) - f(w))}{\eta - w + \varepsilon(f(\eta) - f(w))} f'(\eta) d\eta = G_{21}(\varepsilon, f)(\overline{w}).
\]
On the other hand the function
\[
G_{22}(\varepsilon, f) = \int_T \frac{(\tau - w)(f(\tau) - f(w)) - (\tau - w)(f(\tau) - f(w))}{(\tau - w)((\tau - w) + \varepsilon(f(\tau) - f(w))} d\tau
\]
will be in the space \( C^\beta(\mathbb{T}) \) if the kernel
\[
K(\tau, w) = \frac{(\tau - w)(f(\tau) - f(w)) - (\tau - w)(f(\tau) - f(w))}{(\tau - w)((\tau - w) + \varepsilon(f(\tau) - f(w))}
\]
satisfies the hypothesis of Lemma 1 for \( \alpha = 0 \). For \( \tau \neq w \),
\[
|K(\tau, w)| \leq \frac{2\|f\|_{C^{1+\alpha}}}{1 - \varepsilon\|f\|_{C^{1+\alpha}(\mathbb{T})}}.
\]
Therefore
\[
|\partial_w K(\tau, w) \leq \frac{C}{|\tau - w|},
\]
where the constant \( C \) depends on \( \varepsilon \) and \( \|f\|_{C^{1+\beta}(\mathbb{T})} \). To check that the function \( G_{22} \) has real coefficients one can repeat the same computations used for the function \( G_{21} \).

Now we will verify that the function \( G_2 \) is of class \( C^1 \) from \((-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1 \) to \( Y \). To do so, we will check the continuity of the partial derivatives of \( G_{21} \) and \( G_{22} \). Simple computations prove that
\[
\partial_w G_{21} = \int_T \frac{f(\tau) - f(\overline{w})}{\tau - w + \varepsilon(f(\tau) - f(w))} f'(\tau) d\tau - \int_T \frac{\tau - w + \varepsilon(f(\tau) - f(\overline{w}))}{(\tau - w + \varepsilon(f(\tau) - f(w))} (f(\tau) - f(w)) f'(\tau) d\tau
\]
and
\[
\partial_w G_{22} = -2i \int_T \frac{\Im\{(\tau - w)(f(\tau) - f(\overline{w}))\}}{(\tau - w)((\tau - w) + \varepsilon(f(\tau) - f(w))} (f(\tau) - f(w)) d\tau.
\]
The existence and the continuity of this partial derivative can be obtained proving that the kernels that appear in the integral operators satisfy the conditions of Lemma 1. For \( h \in X \) we will compute the Gâteaux derivative in the direction \( h \) of the function \( G_2 \). For it we only need to calculate the Gâteaux derivatives of the functions \( G_{21} \) and \( G_{22} \).
\[
\partial_f G_{21}(\varepsilon, f)h(w) = \int_T \frac{\varepsilon(h(\tau) - h(\overline{w}))}{\tau - w + \varepsilon(f(\tau) - f(w))} f'(\tau) d\tau + \int_T \frac{\tau - w + \varepsilon(f(\tau) - f(\overline{w}))}{\tau - w + \varepsilon(f(\tau) - f(w))} h'(\tau) d\tau
\]
and
\[
\partial_f G_{22}(\varepsilon, f)h(w) = 2i \int_T \frac{\Im\{(\tau - w)(h(\tau) - h(\overline{w}))\}}{(\tau - w)((\tau - w) + \varepsilon(f(\tau) - f(w))} (h(\tau) - h(w)) d\tau
\]

Moreover the Gâteaux derivative of the \( G_{22} \) in the direction \( h \) is given by
\[
\partial_f G_{22}(\varepsilon, f)h(w) = 2i \int_T \frac{\Im\{(\tau - w)(f(\tau) - f(\overline{w}))\}}{(\tau - w)((\tau - w) + \varepsilon(f(\tau) - f(w))} (f(\tau) - f(w)) d\tau
\]

Moreover the Gâteaux derivative of the \( G_{22} \) in the direction \( h \) is given by
Again Lemma 1 applied to the kernels that appear in the Gâteaux derivatives of the functions $G_{21}$ and $G_{22}$ will give the existence and the continuity of the functions $\partial_f G_{21}$ and $\partial_f G_{22}$. On the other hand,

$$\partial_f G_2(0, 0)(h) = \text{Im} \left\{ \left( \partial_f G_{21}(0, 0)(h) - \partial_f G_{22}(0, 0)(h) \right) w \right\}.$$ 

Moreover, by the residue theorem, we can compute explicitly the partial derivatives at $(0, 0)$,

$$\partial_f G_{21}(0, 0)(h) = \int_T \frac{\tau - w}{\tau - w} h'(\tau) d\tau = 0$$

and

$$\partial_f G_{22}(0, 0)(h) = 2i \int_T \frac{\text{Im} \{(\tau - w)(h(\tau) - h(w))\}}{(\tau - w)^2} d\tau = 0.$$ 

Consequently $\partial_f G_2(0, 0)(h) = 0$. Let’s now study the function

$$G_3(\varepsilon, f) = -\text{Im} \left\{ \left( \int_T \frac{\tau + \varepsilon f'(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} (1 + \varepsilon f'(\tau)) d\tau \right) w(1 + \varepsilon f'(w)) \right\}$$

$$= -\text{Im} \left\{ G_{31}(\varepsilon, f) w(1 + \varepsilon f'(w)) \right\}.$$ 

So, the regularity of the function $G_3$ is equivalent to the regularity of the function $G_{31}$. Now, this function is given by an integral operator with kernel

$$K(\tau, w) = \frac{\tau + \varepsilon f'(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d}.$$ 

It is clear that $|K(\tau, w)| \leq C$ and moreover

$$|\partial_w K(\tau, w)| = \left| \frac{(\tau + \varepsilon f'(\tau))(\varepsilon + \varepsilon^2 f'(w))}{(\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d)^2} \right| \leq C.$$ 

Since $1 + \varepsilon f'(\tau)$ is in $C^3(\mathbb{T})$ and applying Lemma 1 to the above kernel we get that $G_{31}$ is a function in $C^3(\mathbb{T})$. To prove that $G_{31}$ has real coefficients one only has to repeat the arguments given in the case of the function $G_{21}$. Now, to check that the function $G_{31}$ is in $C^1$ we compute its partial derivatives

$$\partial_\varepsilon G_{31} = \int_T \frac{f(\tau)(1 + \varepsilon f'(\tau))}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} d\tau + \int_T \frac{(\tau + \varepsilon f'(\tau)) f'(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} d\tau$$

$$- \int_T \frac{(\tau + \varepsilon f'(\tau))(\tau + w + 2\varepsilon(f(\tau) + f(w)))}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d^2} (1 + \varepsilon f'(\tau)) d\tau.$$ 

Easy computations, using Lemma 1, prove that these operators are continuous from $\left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{R} \times B_1$ to $C^3(\mathbb{T})$. Since they are functions with real coefficients we can conclude that $\partial_\varepsilon G_3$ is continuous from $\left(-\frac{1}{2}, \frac{1}{2}\right) \times \mathbb{R} \times B_1$ to $Y$. On the other hand, we can compute the Gâteaux derivative of $G_{31}$ in a given direction $h \in X$

$$\partial_f G_{31}(\varepsilon, f)(h) = \varepsilon \int_T \frac{h(\tau)(1 + \varepsilon f'(\tau))}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - d} d\tau$$

$$+ \varepsilon \int_T \frac{(\tau + \varepsilon f'(\tau)) h'()}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - d} d\tau$$

$$- \varepsilon^2 \int_T \frac{(\tau + \varepsilon f'(\tau))(h(\tau) + h(w))}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - d^2} (1 + \varepsilon f'(\tau)) d\tau.$$
Again it is an easy computation to verify that the integral operators defined by these partial
derivatives are continuous and so we obtain that $\partial f G_3$ is continuous from \((-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R} \times B_1\)
to $Y$. Moreover we have that $\partial f G_{31}(0, 0)(h) = 0$, and consequently
\[
\partial f G_{31}(0, 0)(h) = 0.
\]

Therefore (i) and (iii) are proved. Note that when $\varepsilon = 0$ one should get the two point
vortices. Indeed, we can easily check that
\[
G(0, \Omega, 0) = \text{Im}\left\{ \left( -2\Omega d + \frac{1}{2d} \right) w \right\}
\]
and therefore $G(0, \Omega, 0) = 0$ if and only if
\[
\Omega = \Omega_{\text{sing}} = \frac{1}{4d^2},
\]
and so (ii) is obtained.
To prove (iv) we use that $\partial f G(0, \Omega, 0)(h) = -\text{Im}\{h'\}$, therefore we can conclude that
the linear mapping $\partial f G(0, \Omega, 0) : X \rightarrow \hat{Y}$ is an isomorphism. \hfill $\square$

4.2. **Relationship between the angular velocity and the boundary.** As we have seen
in Proposition 1 the linear operator $\partial f G(0, \Omega, 0)$ is an isomorphism from $X$ to $\hat{Y}$ and not
to the space $Y$. However the functional $G$ has its range in $Y$ which contains strictly $\hat{Y}$.
The strategy will be to choose $\Omega$ in such a way that the range of $G$ is contained in $\hat{Y}$.
This condition is enough strong to uniquely determine $\Omega$ and it means that the first Fourier
coefficients in the expansion of $G$ should be zero. This latter condition is equivalent to
\[
(23) \quad \int_{T} F(\Omega, \varepsilon, f(w)) w^2 dw - \int_{T} F(\Omega, \varepsilon, f(w)) dw = 0.
\]

We recall that $F$ was defined in (9). Then using residue theorem we get
\[
\int_{T} F_1(\Omega, \varepsilon, f(w)) w^2 dw = 2\Omega \left( -d + \varepsilon^3 \int_{T} f(w) w f'(w) dw \right)
\]
and
\[
\int_{T} F_1(\Omega, \varepsilon, f(w)) dw = 2\Omega \left( -d \varepsilon \int_{T} w f'(w) dw + \varepsilon^3 \int_{T} f(w) w f'(w) dw \right).
\]
This last identity can be written in the form
\[
\int_{T} F_1(\Omega, \varepsilon, f(w)) dw = 2\Omega \left( \varepsilon \int_{T} f(w) dw + \varepsilon^3 \int_{T} f(w) w f'(w) dw \right).
\]
Consequently
\[
\int_{T} F_1(\Omega, \varepsilon, f(w)) w^2 dw - \int_{T} F_1(\Omega, \varepsilon, f(w)) dw = 2\Omega \left( -d \left[ 1 + \varepsilon \int_{T} f(w) dw \right] \right.
\]
\[
+ \left. \varepsilon^3 \int_{T} f(w) f'(w)(\overline{w} - w) dw \right).
\]

Now we shall look for the contribution of $F_3$. First
\[
F_3(\varepsilon, f(w)) = -\bar{F}_3(w)(1 + \varepsilon f'(w)),
\]
with
\[
\bar{F}_3(\varepsilon, f(w)) \equiv \int_{T} \frac{\tau + \varepsilon f(\tau)}{\varepsilon(\tau + w) + \varepsilon^2 \left( f(\tau) + f(w) \right) - 2d(1 + \varepsilon f'(\tau))} d\tau.
\]
We write
\[
\frac{\tau + \varepsilon f(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w))} - 2d = -\frac{\tau}{2d} + \varepsilon \frac{f(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} + \varepsilon \frac{\tau + w + \varepsilon(f(\tau) + f(w))}{2d \varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d \tau}
\]
\[
= -\frac{\tau}{2d} + \varepsilon g_3(\varepsilon, \tau, w).
\]
Thus
\[
\tilde{F}_3(\varepsilon, f(w)) = -\frac{1}{2d} + \varepsilon \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau)) d\tau.
\]
Hence
\[
\int_T F_3(\Omega, \varepsilon, f(w)) w^2 dw = \frac{1}{2d} - \varepsilon \int_T \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau)) w(1 + \varepsilon f'(w)) d\tau dw
\]
\[
\int_T F_3(\Omega, \varepsilon, f(w)) d\omega = -\frac{\varepsilon}{2d} \int_T f(\tau) d\tau - \varepsilon \int_T \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau)) w(1 + \varepsilon f'(w)) d\tau dw.
\]
Consequently
\[
\int_T F_3(\Omega, \varepsilon, f(w)) w^2 dw - \int_T F_3(\Omega, \varepsilon, f(w)) d\omega = \frac{1}{2d} + \frac{\varepsilon}{2d} \int_T f(\tau) d\tau
\]
\[
- \varepsilon \int_T \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau))(w - w)(1 + \varepsilon f'(w)) d\tau dw.
\]
On the other hand using residue theorem we get
\[
F_2(\varepsilon, f(w)) = \varepsilon \int_T \frac{AB - AB}{A(A + \varepsilon B)} f'(\tau) d\tau w(1 + \varepsilon f'(w))
\]
\[
+ \varepsilon \int_T \frac{(AB - AB)B}{A^2(A + \varepsilon B)} d\tau w(1 + \varepsilon f'(w))
\]
\[
\equiv \varepsilon g_2(\varepsilon, w) w(1 + \varepsilon f'(w)).
\]
Thus
\[
\int_T F_2(\Omega, \varepsilon, f(w)) w^2 dw - \int_T F_2(\Omega, \varepsilon, f(w)) d\omega = \varepsilon \int_T g_2(\varepsilon, w)(w - w)(1 + \varepsilon f'(w)) d\omega.
\]
The equation (23) becomes
\[
2\Omega \left( d \left[ 1 + \varepsilon \int_T f(w) d\omega \right] - \varepsilon^3 \int_T f(w) f'(w)(w - w) d\omega \right) = \frac{1}{2d} + \frac{\varepsilon}{2d} \int_T f(\tau) d\tau
\]
\[
+ \varepsilon \int_T g_2(\varepsilon, w)(w - w)(1 + \varepsilon f'(w)) d\omega
\]
\[
+ \varepsilon \int_T \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau))(w - w)(1 + \varepsilon f'(w)) d\tau dw
\]
\[
\equiv \frac{1}{2d} + \frac{\varepsilon}{2d} T_1(\varepsilon, f).
\]
which can be written in the form

\[
\Omega = \Omega(\varepsilon, f)
\]

\[
= \frac{1}{4d^2} \frac{1 + \varepsilon T_1(\varepsilon, f)}{1 - \varepsilon T_2(\varepsilon, f)}
\]

\[
(24)
\]

with

\[
T_2(\varepsilon, f) = - \int_T f(w)dw + \frac{\varepsilon^2}{d} \int_T f(w) f'(w)(w - w)dw.
\]

Now we intend to discuss the regularity of \(\Omega\).

**Proposition 2.** The function \(\Omega : (\frac{1}{2}, \frac{1}{2}) \times B_1 \rightarrow \mathbb{R}\) defined in (24) is a \(C^1\) function.

**Proof.** It is enough to check that the functions \(T_1(\varepsilon, f)\) and \(T_2(\varepsilon, f)\) are \(C^1\) functions and moreover \(|T_2(\varepsilon, f)| < 2\). Since \(f\) has real coefficients it is clear that \(T_2(\varepsilon, f) \in \mathbb{R}\) and

\[
|T_2(\varepsilon, f)| \leq \|f\|_{C^{1+\beta}(\mathbb{T})} + \frac{\varepsilon^2}{d} \|f\|_{C^{1+\beta}(\mathbb{T})}^2 < 2.
\]

On the other hand, \(T_2\) is polynomial in \(\varepsilon, f\) and \(f'\) and so its derivatives. Thus, we can conclude that \(T_2\) is a \(C^1\) function from \((\frac{1}{2}, \frac{1}{2}) \times B_1\) to \(\mathbb{R}\). Let's take now the functional

\[
T_1(\varepsilon, f) = \int_T f(\tau) d\tau + 2d \int_T g_2(\varepsilon, w)(w - w)(1 + \varepsilon f'(w))dw
\]

\[
+ 2d \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau))(w - w)(1 + \varepsilon f'(\tau))d\tau dw,
\]

where

\[
g_2(\varepsilon, f) = \int_T A\overline{B} - \overline{A}B f'(\tau) d\tau + \int_T \frac{(A\overline{B} - \overline{A}B)B}{A^2(A + \varepsilon B)} d\tau,
\]

with \(A = \tau - w, B = f(\tau) - f(w)\) and

\[
g_3(\varepsilon, f) = \int_T \frac{f(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(\tau + w + f(\tau) + f(w))} - 2d + 2d \frac{\varepsilon(\tau + w + \varepsilon f(\tau + f(w)))}{\varepsilon(\tau + w) + \varepsilon^2 f(\tau + f(w))} - 2d\tau.
\]

Since \(|\varepsilon| < \frac{1}{2}\) and \(\|f\|_{C^{1+\beta}} < 1\) we get that \(g_3\) is a bounded function. Moreover

\[
|g_2(\varepsilon, f)(w)| \leq 2 \int_T \left| \frac{\text{Im}\{(\tau - w)(f(\tau) - f(w))\}}{(\tau - w)(\tau - w + \varepsilon(f(\tau) - f(w)))} f'(\tau) \right| |d\tau|
\]

\[
+ 2 \int_T \left| \frac{\text{Im}\{(\tau - w)(f(\tau) - f(w))\}(f(\tau) - f(w))}{(\tau - w)^2(\tau - w + \varepsilon(f(\tau) - f(w)))} \right| |d\tau| \leq C,
\]

where in the last inequality we use again that \(|\varepsilon| < \frac{1}{2}\) and \(\|f\|_{C^{1+\beta}} < 1\). To prove that \(T_1\) is a \(C^1\) function it is enough to check that the partial derivatives of \(g_2(\varepsilon, f)\) and \(g_3(\varepsilon, f)\) are continuous functions on \((-\frac{1}{2}, \frac{1}{2}) \times B_1\). Observe that,

\[
\partial_\varepsilon g_2(\varepsilon, f) = -2i \int_T \frac{\text{Im}\{(\tau - w)(f(\tau) - f(w))\}}{(\tau - w)(\tau - w + \varepsilon(f(\tau) - f(w)))^2} (f(\tau) - f(w)) f'(\tau) d\tau
\]

\[
- 2i \int_T \frac{\text{Im}\{(\tau - w)(f(\tau) - f(w))\}}{(\tau - w)^2(\tau - w + \varepsilon(f(\tau) - f(w)))^2} (f(\tau) - f(w))^2 d\tau.
\]
It is easy to verify that the kernels involved in the above integral operators satisfy the conditions of Lemma 1 and so we can conclude that $\partial_\varepsilon g_2(\varepsilon, f)$ is a continuous function from $(-\frac{1}{2}, \frac{1}{2}) \times B_1$ to $\mathbb{R}$. For any direction $h \in X$ straightforward computations yield

$$
\partial_f g_2(\varepsilon, f)(h) = 2i \int_T \frac{\text{Im}\{(\tau - w)(h(\overline{\tau}) - h(\overline{w}))\}}{(\tau - w)(\tau - w + \varepsilon(f(\tau) - f(w)))} f'(\tau)d\tau
+ 2i \int_T \frac{\text{Im}\{(\tau - w)(f(\overline{\tau}) - f(\overline{w}))\}}{(\tau - w)(\tau - w + \varepsilon(f(\tau) - f(w)))} h'(\tau)d\tau
- 2i\varepsilon \int_T \frac{\text{Im}\{(\tau - w)(f(\tau) - f(w))\}}{(\tau - w)(\tau - w + \varepsilon(f(\tau) - f(w)))} (h(\tau) - h(w)) f'(\tau)d\tau
+ 2i \int_T \frac{\text{Im}\{(\tau - w)^2(h(\tau) - h(w))\}}{(\tau - w)(\tau - w + \varepsilon(f(\tau) - f(w)))} (f(\tau) - f(w))d\tau
+ 2i \int_T \frac{\text{Im}\{(\tau - w)^2(f(\tau) - f(w))\}}{(\tau - w)(\tau - w + \varepsilon(f(\tau) - f(w)))} (h(\tau) - h(w))d\tau.
$$

Again the kernels involved in the integral operators satisfy the conditions in Lemma 1 and so $\partial_f g_2(\varepsilon, f)(h)$ defines a continuous function from $(-\frac{1}{2}, \frac{1}{2}) \times B_1$ to $\mathbb{R}$. Reproducing similar computations one can prove that $g_3(\varepsilon, f)$ is a $C^1$ function from $(-\frac{1}{2}, \frac{1}{2}) \times B_1$ to $\mathbb{R}$.

\[ \Box \]

4.3. \textbf{Existence and convexity.} In this section we will finish the proof of the existence of corotating vortex pairs and show the convexity of each single vortex forming the vortex pair. Recall that the equation of the $V$-states is given by

$$
\hat{G}(\varepsilon, f(w)) \equiv \text{Im} \left\{ F(\Omega(\varepsilon, f), \varepsilon, f(w)) \right\} = 0, \ \forall w \in \mathbb{T}.
$$

Our goal is to prove the following result.

\begin{proposition}
The following holds true.

(i) The linear operator $\partial_f \hat{G}(0, 0) : X \rightarrow \hat{Y}$ is an isomorphism and

$$
\partial_f \hat{G}(0, 0) h(w) = -\text{Im}\{h'(w)\}.
$$

(ii) There exists $\varepsilon_0 > 0$ such that the set

$$
\left\{ (\varepsilon, f) \in [-\varepsilon_0, \varepsilon_0] \times B_1, \text{ s.t. } \hat{G}(\varepsilon, f) = 0 \right\}
$$

is parametrized by one-dimensional curve $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \mapsto (\varepsilon, f_\varepsilon)$ and

$$
\forall \varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}, f_\varepsilon \neq 0.
$$

(iii) If $(\varepsilon, f)$ is a solution then $(-\varepsilon, \tilde{f})$ is also a solution, where

$$
\forall w \in \mathbb{T}, \ \tilde{f}(w) = f(-w)
$$

(iv) For all $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$, the domain $D_\varepsilon^\ast$ is strictly convex.

\end{proposition}

\begin{proof}
(i) From the composition rule

$$
\partial_f \hat{G}(0, 0) h(w) = \partial_\Omega G(0, \Omega_{\text{sing}}, 0) \partial_f \Omega(0, 0) h + \partial_f G(0, \Omega_{\text{sing}}, 0) h(w).
$$

\end{proof}
From the formula of $\Omega(\varepsilon, f)$ in Proposition 2 we deduce that
\[ \frac{\partial f}{\partial t} \Omega(0, 0) = \frac{d}{dt} \Omega(0, th(w)) \big|_{t=0} = 0 \]
and therefore
\[ \partial f \hat{G}(0, 0) h(w) = \partial f G(0, \Omega_{\text{sing}}, 0) h(w). \]
Combining this identity with Proposition 1 we obtain the desired result.

(ii) As we have seen before $\hat{G} : [0, \frac{1}{2}, \frac{1}{2}] \times B_1 \to \hat{Y}$ is $C^1$ and $\partial f \hat{G}(0, 0) : X \to \hat{Y}$ is an isomorphism. Thus we can apply the implicit function theorem. More precisely, there exist $\varepsilon_0 > 0$ and a $C^1$ function $f : (-\varepsilon_0, \varepsilon_0) \to B_1$, such that for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the function $f_\varepsilon$ satisfies
\[ \text{Im} \left\{ F' (\Omega(\varepsilon, f_\varepsilon), \varepsilon, f_\varepsilon(w)) \right\} = 0, \quad \forall w \in T, \]
and so we can assert that $f$ defines a rotating vortex pair. It remains to check that $f_\varepsilon \neq 0$ for $\varepsilon \neq 0$. To this end, we will prove that for any $\varepsilon$ small enough and any $\Omega$ we can not get a vortex pair with $f = 0$. So, it means that
\[ G(\varepsilon, \Omega, 0) \neq 0. \]
It is easy to check from (9) that
\[ F_1(\varepsilon, \Omega, 0) = 2\Omega(\varepsilon - d w) \quad \text{and} \quad F_2(\varepsilon, 0) = 0. \]
However to compute $F_3$ we proceed by Taylor expansion as follows,
\[ F_3(\varepsilon, 0) = -w \int_{\mathbb{T}} \frac{\tau}{\varepsilon(\tau + w)} - \frac{2d}{2d} d\tau = w \sum_{n \in \mathbb{N}} \frac{\varepsilon^n}{(2d)^n+1} \int_{\mathbb{T}} (\tau + w)^n d\tau = \sum_{n \in \mathbb{N}} \frac{\varepsilon^n}{(2d)^n+1} w^{n+1}, \]
which gives in turn
\[ F_3(\varepsilon, 0) = \frac{w}{2d - \varepsilon w}. \]
Consequently
\[ G(\varepsilon, \Omega, 0) = \text{Im} \left\{ -2d\Omega w + \frac{w}{2d - \varepsilon w} \right\}; \]
and this quantity is not zero if $\varepsilon \neq 0$ is small enough.

(iii) Using the definition of $\tilde{f}$ one can check that $T_i(-\varepsilon, \tilde{f}) = -T_i(\varepsilon, f)$, for $i = 1, 2$ and so by (24) we obtain that
\[ \Omega(\varepsilon, f) = \Omega(-\varepsilon, \tilde{f}). \]
Taking the decomposition of $F = F_1 + F_2 + F_3$ given in (9) we only need to check that $F_i(\varepsilon, \Omega, f)(-w) = -F_i(-\varepsilon, \Omega, \tilde{f})(w)$, for $i = 1, 2, 3$. Since $\tilde{f}'(w) = -f'(w)$ we have
\[ F_1(-\varepsilon, \Omega, \tilde{f})(w) = 2\Omega \left( -\varepsilon \bar{w} + \varepsilon^2 \tilde{f}'(\bar{w}) - d \right) w (1 - \varepsilon \tilde{f}'(w)) - \tilde{f}'(w) = -2\Omega \left( \varepsilon \bar{w} + \varepsilon^2 f(-\bar{w}) - d \right) w (1 + \varepsilon f'(-w)) - f'(-w) \]
\[ = -F_1(\varepsilon, \Omega, f)(-w). \]
Straightforward computations will lead to the same properties for the functions $F_2$ and $F_3$. Consequently,
\[ F(\varepsilon, \Omega, f)(w) - F(-\varepsilon, \Omega, \tilde{f})(-w) \]
and therefore $(-\varepsilon, \tilde{f})$ defines a curve of solutions for $0 < \varepsilon < \varepsilon_0$.

**(iv)** First we shall make the following comment. As it was mentioned in Remark 4 one can reproduce the preceding proofs when we replace $\beta$ by $n + \beta$ with $n \in \mathbb{N}$. Therefore the implicit function theorem gives that the function $\tilde{f}_\varepsilon$ belongs to $C^{n+1+\beta}$ for any fixed $n$. Of course, the size of $\varepsilon_0$ is not uniform with respect to $n$ and it shrinks to zero as $n$ grows to infinity. Now to prove the convexity of the domain $D_\varepsilon$ we shall reproduce the same arguments of [22]. Recall that the outside conformal mapping associated to this domain is given by

$$\phi(w) = \varepsilon w + \varepsilon^2 \tilde{f}_\varepsilon(w)$$

and the curvature can be expressed by the formula

$$\kappa(\theta) = \frac{1}{|\phi'(w)|} \Re \left( 1 + w \frac{\phi''(w)}{\phi'(w)} \right).$$

It is plain that

$$1 + w \frac{\phi''(w)}{\phi'(w)} = 1 + \varepsilon w \frac{f''(w)}{1 + \varepsilon f'(w)}$$

and so

$$\Re \left( 1 + w \frac{\phi''(w)}{\phi'(w)} \right) \geq 1 - |\varepsilon| \frac{|f''(w)|}{1 - |\varepsilon| f'(w)} \geq 1 - \frac{|\varepsilon|}{1 - |\varepsilon|},$$

which is non-negative if $|\varepsilon| < 1/2$. Thus the curvature is strictly positive and therefore the domain is strictly convex.

□

5. **Counter-rotating vortex pair for Euler equations**

In this section we will prove the existence of planar translating pairs of vortex patches with velocity $U$ in the direction $OY$. The proof is similar to that of the corotating pairs.

5.1. **Extension and regularity of the functional $G$.** To prove the existence of translating pairs we will apply the implicit function theorem to the equation (8). To do this we have to check that the function $\text{Im} F$ satisfies the hypothesis of the theorem.

**Proposition 4.** The following holds true.

(i) The function $G$ defined in (8) can be extended to $C^1$ function from $]-\frac{1}{2}, \frac{1}{2}] \times \mathbb{R} \times B_1 \rightarrow Y$.

(ii) Two initial point vortex $\pi \delta_0$ and $-\pi \delta(2d, 0)$ move uniformly in the direction $(Oy)$ with velocity $U_{\text{sing}} \equiv \frac{1}{4d}$.

(iii) For $U \in \mathbb{R}$ and $h \in X$, we have

$$\partial_f G(0, U, 0) h(w) = -\text{Im}\{h'(w)\}.$$ 

(iv) For any $U \in \mathbb{R}$, the operator $\partial_f G(0, U, 0) : X \rightarrow \hat{Y}$ is an isomorphism.

**Proof.** (i) The function $G$ as defined in (8) is $G = \text{Im}(F_1 + F_2 + F_3) = G_1 + G_2 + G_3$. Notice that the functions $G_2$ and $G_3$ coincide with the functions $G_2$ and $G_3$ appeared in the corotating pairs discussed in the foregoing sections and so they are $C^1$ from $]-\frac{1}{2}, \frac{1}{2}] \times B_1$ to $Y$. Note also that it is obvious that $G_1$ enjoys the required regularity.

(ii) For $\varepsilon = 0$ one has

$$F(U, 0, 0) = (2U - \frac{1}{2d}) w$$
and therefore $\text{Im} F(U, 0, 0) = 0$ if and only if

$$U = U_{\text{sing}} \equiv \frac{1}{4d}.$$  

(iii)-(iv) For any given direction $h \in X$ we have

$$\partial_f G_2(0, 0)(h) = \partial_f G_3(0, 0)(h) = 0.$$  

So, it remains to check the properties for the function $G_1 = \text{Im}\{2Uw(1 + \varepsilon f'(w)) - f'(w)\}$. This a polynomial function in $U, \varepsilon$ and $f'$ and thus we conclude that $G_1$ is $C^1$ from $]-\frac{1}{2}, \frac{1}{2}[ \times \mathbb{R} \times B_1$ to $Y$. Moreover simple calculations give

$$\partial_f G(0, 0)h(w) = -\text{Im}\{h'(w)\}.$$  

From this expression we get that the map $\partial_f G(0, U, 0) : X \to \hat{Y}$ is an isomorphism.

□

5.2. Relationship between the speed and the boundary. We shall follow the same strategy of the Subsection 4.2. According to Proposition 4 the linear operator $\partial_f G(0, U, 0)$ is an isomorphism from $X$ to $\hat{Y}$ which is strictly contained in $Y$. Note that he functional $G$ has its range in $Y$ and we shall choose $U$ in such a way that the range of $G$ is contained in $\hat{Y}$. To guarantee $G(U, \varepsilon, \cdot) : X \to \hat{Y}$ we should put a nonlinear constraint on $U$ such that the Fourier coefficient of $e_1$ disappears in the expansion of $G(U, \varepsilon, f)$. This constraint is

$$\int_T F(U, \varepsilon, f(w))\overline{w}^2 dw - \int_T F(U, \varepsilon, f(w))dw = 0. \quad (26)$$

Note that by residue theorem

$$\int_T F_1(U, \varepsilon, f(w))\overline{w}^2 dw = 2U$$

and

$$\int_T F_1(U, \varepsilon, f(w))dw = 2U \varepsilon \int_T w f'(w)dw = -2U \varepsilon \int_T f(w)dw.$$  

Consequently

$$\int_T F_1(\Omega, \varepsilon, f(w))\overline{w}^2 dw - \int_T F_1(\Omega, \varepsilon, f(w))dw = 2U \left(1 + \varepsilon \int_T f(w)dw\right).$$  

Now we shall look for the contribution of $F_3$. First

$$F_3(\varepsilon, f(w)) = \widetilde{F}_3(w)w(1 + \varepsilon f'(w)),$$

with

$$\widetilde{F}_3(\varepsilon, f(w)) \equiv \int_T \frac{\tau + \varepsilon f'(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d(1 + \varepsilon f'((\tau)))d\tau.}$$
We write

\[
\begin{align*}
\frac{\tau + \varepsilon f(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} &= \frac{-\tau}{2d} + \varepsilon \frac{f(\tau)}{\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} \\
&\quad + \varepsilon \frac{\tau + w + \varepsilon(f(\tau) + f(w))}{2d\varepsilon(\tau + w) + \varepsilon^2(f(\tau) + f(w)) - 2d} \\
&\equiv -\frac{\tau}{2d} + \varepsilon g_3(\varepsilon, \tau, w).
\end{align*}
\]

Thus

\[
\widetilde{F}_3(\varepsilon, f(w)) = -\frac{1}{2d} + \varepsilon \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau))d\tau.
\]

Hence

\[
\int_T F_3(\Omega, \varepsilon, f(w))w^2dw = -\frac{1}{2d} + \varepsilon \int_T \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau))w(1 + \varepsilon f'(w))d\tau dw \\
\int_T F_3(\Omega, \varepsilon, f(w))dw = \frac{\varepsilon}{2d} \int_T f(\tau)d\tau + \varepsilon \int_T \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau))w(1 + \varepsilon f'(w))d\tau dw.
\]

Consequently

\[
\int_T F_3(\Omega, \varepsilon, f(w))w^2dw - \int_T F_3(\Omega, \varepsilon, f(w))dw = -\frac{1}{2d} - \frac{\varepsilon}{2d} \int_T f(\tau)d\tau \\
+ \varepsilon \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau))(w - w)(1 + \varepsilon f'(w))d\tau dw.
\]

On the other hand using residue theorem we get

\[
F_2(\varepsilon, f(w)) = \varepsilon \int_T \frac{AB - \overline{AB}}{A(A + \varepsilon B)}f'(\tau)dw(1 + \varepsilon f'(w)) \\
+ \varepsilon \int_T \frac{(\overline{AB} - AB)B}{A^2(A + \varepsilon B)}d\tau (1 + \varepsilon f'(w)) \\
\equiv \varepsilon g_2(\varepsilon, w)(1 + \varepsilon f'(w)).
\]

Thus

\[
\int_T F_2(\Omega, \varepsilon, f(w))w^2dw - \int_T F_2(\Omega, \varepsilon, f(w))dw = \varepsilon \int_T g_2(\varepsilon, w)(w - w)(1 + \varepsilon f'(w))dw.
\]

The equation (26) becomes

\[
2U\left(1 + \varepsilon \int_T f(w)dw\right) = \frac{1}{2d} + \frac{\varepsilon}{2d} \int_T f(\tau)d\tau \\
+ \varepsilon \int_T g_2(\varepsilon, w)(w - w)(1 + \varepsilon f'(w))dw \\
+ \varepsilon \int_T g_3(\varepsilon, \tau, w)(1 + \varepsilon f'(\tau))(w - w)(1 + \varepsilon f'(w))d\tau dw \\
\equiv \frac{1}{2d} + \frac{\varepsilon}{2d} T_1(\varepsilon, f).
\]
which may be written in the form
\[
U = U(\varepsilon, f)
\]
\[
= \frac{1}{4d} \frac{1 + \varepsilon T_1(\varepsilon, f)}{1 + \varepsilon T_2(f)}
\]
\[
= U_{sing} + \frac{\varepsilon}{4d} \frac{T_1(\varepsilon, f) - T_2(f)}{1 + \varepsilon T_2(f)}
\]
(27)
with
\[
T_2(f) = \int_T f(w)dw.
\]

In the next section we will apply the implicit function theorem to the composition of the functions \( G \) and \( U \) and for this reason we need to check the differentiability properties of the function \( U \).

**Proposition 5.** The function \( U : (\frac{1}{2}, \frac{1}{2}) \times B_1 \rightarrow \mathbb{R} \) defined in (27) is a \( C^1 \) function.

**Proof.** To verify that the function \( U \) satisfies the desired conditions it is enough to check it for the operators \( T_1 \) and \( T_2 \). The function \( T_1 \) coincides with the functional \( T_1 \) in Proposition 2 and so we get its regularity. On the other hand it is clear that \( T_2 \) defines a \( C^1 \) function from \([-\frac{1}{2}, \frac{1}{2}] \times B_1 \) to \( \mathbb{R} \). \( \square \)

### 5.3. Existence of translating patches.

This section will be devoted to prove the existence of translating pairs. The equation of this pairs of patches is given by
\[
\hat{G}(\varepsilon, f(w)) \equiv \text{Im} \left\{ F(U(\varepsilon, f), \varepsilon, f(w)) \right\} = 0, \quad \forall w \in T.
\]

**Proposition 6.** The following holds true.

(i) The linear operator \( \partial_f \hat{G}(0, 0) : X \rightarrow \hat{Y} \) is an isomorphism and
\[
\partial_f \hat{G}(0, 0) = -\text{Im}\{h'(w)\}
\]

(ii) There exists \( \varepsilon_0 > 0 \) such that the set
\[
\left\{ (\varepsilon, f) \in [-\varepsilon_0, \varepsilon_0] \times B_1, \text{ s.t. } \hat{G}(\varepsilon, f) = 0 \right\}
\]
is parametrized by one-dimensional curve \( \varepsilon \in [-\varepsilon_0, \varepsilon_0] \mapsto (\varepsilon, f_\varepsilon) \) and
\[
\forall \varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}, f_\varepsilon \neq 0.
\]

(iii) If \((\varepsilon, f)\) is a solution then \((-\varepsilon, \tilde{f})\) is also a solution, where
\[
\forall w \in T, \quad \tilde{f}(w) = f(-w)
\]

(iv) For all \( \varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\} \), the domain \( D_\varepsilon^x \) is strictly convex.

**Proof.** By Proposition 4 and Proposition 5 we know that
\[
\hat{G} : [-\frac{1}{2}, \frac{1}{2}] \times B_1 \rightarrow \hat{Y}
\]
defines a \( C^1 \) function. On the other hand it is a simple computation to see that
\[
\partial_f U(0, 0) = 0.
\]
Therefore we have
\[
\partial_f \hat{G}(0, 0) h(w) = -\text{Im} \{h'(w)\}, \quad \forall h \in X.
\]
Thus \( \partial_f \hat{G}(0, 0) : X \rightarrow \hat{Y} \) is an isomorphism.

To prove (ii), using the conditions of (i), we apply the implicit function theorem and so we
have that there exist \( \varepsilon_0 \) and a \( C^1 \) function \( f : ( -\varepsilon_0, \varepsilon_0 ) \to B_1 \), such that for any \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \) the function \( f_\varepsilon \) satisfies

\[
\text{Im} \left\{ F(U(\varepsilon, f_\varepsilon), \varepsilon, f_\varepsilon(w)) \right\} = 0, \quad \forall w \in \mathbb{T},
\]

and so we can assert that \( f \) defines a translating vortex pair. Now it remains to prove that \( f_\varepsilon \neq 0 \). It is easy to check that \( F_1(\varepsilon, \Omega, 0) = 2Uw \). On the other hand, the functionals \( F_2 \) coincides with the same functional in Proposition 3 and for \( F_3 \) we only have to change the sign. Then \( F_2(\varepsilon, 0) = 0 \) and

\[
F_3(\varepsilon, 0) = \frac{w}{\varepsilon w - 2d}.
\]

Consequently

\[
\forall w \in \mathbb{T}, \quad G(\varepsilon, \Omega, 0) = \text{Im} \left\{ 2Uw + \frac{w}{\varepsilon w - 2d} \right\}
\]

and this quantity is not zero if \( \varepsilon \neq 0 \).

Now, straightforward computations like in (iii) of Proposition 3 give the claim in (iii).

To prove (iv) one only has to follow the same arguments that in Proposition 3 since the parametrization of the boundary is the same that in this case.

\[ \square \]

6. Corotating vortex pair for the gSQG equation

This section is devoted to the construction of pairs of symmetric rotating patches. The equations are detailed in Section 3 and the strategy follows the same lines of Euler equations. Thus we shall start with extending the functionals and studying their regularity. In Section 6.2 we will establish the relation between the angular velocity and the boundary of the single patch. In the last section we will prove the existence of the corotating pairs of patches mentioned in (ii) of the Main theorem.

6.1. Extension and regularity of \( G \). This section is devoted to the extension of the functional \( G \) in a neighborhood of 0 in the variable \( \varepsilon \) in order to be able to apply the implicit function theorem. Once the extension is given we shall prove the \( C^1 \) regularity in all the parameters. Now we shall fix the function spaces. For the exponent \( \alpha \in ]0, 1[ \) given in the fractional Laplacian of the SQG equation we set

\[
X = \left\{ f \in C^{2-\alpha}(\mathbb{T}), f(w) = \sum_{n \geq 1} a_n w^{-n} \right\},
\]

\[
Y = \left\{ f \in C^{1-\alpha}(\mathbb{T}), f = \sum_{n \geq 1} a_n e_n, a_n \in \mathbb{R} \right\}, \quad \hat{Y} = \left\{ f \in Y, a_1 = 0 \right\},
\]

\[
e_n(w) \equiv \text{Im}(w^n).
\]

For \( r > 0 \) we define \( B_r \) as the open ball of \( X \) centered at zero and of radius \( r \). The main result of this section is described as follows.

**Proposition 7.** The following assertions hold true.

(i) The function \( G \) can be extended from \( ]-\frac{1}{2}, \frac{1}{2}[ \times \mathbb{R} \times B_1 \) to \( Y \) as a \( C^1 \) function.

(ii) Two initial point vortex \( \pi \delta_0 \) and \( \pi \delta(2d,0) \) rotate uniformly about \((d,0)\) with the angular velocity

\[
\Omega_{\text{sing}} \equiv \frac{\alpha C_\alpha}{(2d)^{2+\alpha}}.
\]
(iii) For any $\Omega \in \mathbb{R}$ and $h = \sum_{n \geq 1} a_n w^n \in X$, we have
\[
\partial_f G(0, \Omega, 0) h(w) = \sum_{n \geq 1} a_n \hat{\gamma}_n c_{n+1}
\]
with
\[
\hat{\gamma}_n = \frac{\alpha C\Gamma(1-\alpha)}{4\Gamma^2(1-\frac{\alpha}{2})} \left( \frac{2(1+n)}{1-\frac{\alpha}{2}} \right) - \frac{(1+\frac{\alpha}{2})_n}{(1-\frac{\alpha}{2})_n} - \frac{(1+\frac{\alpha}{2})_{n+1}}{(1-\frac{\alpha}{2})_{n+1}}
\]

(iv) For any $\Omega \in \mathbb{R}$, the operator $\partial_f G(0, \Omega, 0) : X \rightarrow \hat{Y}$ is an isomorphism.

Proof. (i) We shall start with the proof of the regularity for the function $G_0$ which is the easiest one. The suitable extension of this function, still denoted $G_0$, is given by
\[
G_0(\varepsilon, \Omega, f(w)) = \Omega \text{Im} \left\{ (\varepsilon w + \varepsilon^2 \varepsilon |\alpha f(w) - d) \overline{w} \left( 1 + \varepsilon |\alpha f'(w) \right) \right\}
\]
First $G_0$ is well-defined from $]-\frac{1}{2}, \frac{1}{2}[ \times \mathbb{R} \times B_1$ to $Y$ because, using the algebra structure of Hölder spaces with positive regularities, the function between the brackets belongs to $C^{1-\alpha}(\mathbb{T})$ and has real Fourier coefficients. To prove that this functional is $C^1$ it suffices to check that the partial derivatives exist and are continuous. It is clear that
\[
\partial_{\varepsilon} G_0(\varepsilon, \Omega, f(w)) = \Omega \text{Im} \left\{ \left( w + (2\varepsilon \varepsilon |\alpha + \alpha \text{sign}(\varepsilon) |\varepsilon |^{1+\alpha}) f(w) \right) \overline{w} \left( 1 + \varepsilon |\alpha f'(w) \right) \right\}
\]
\[
+ \Omega(|\varepsilon |^{\alpha + \alpha \text{sign}(\varepsilon) |\varepsilon |^{\alpha-1}) \text{Im} \left\{ (\varepsilon w + \varepsilon^2 |\alpha f(w) - d) \overline{w} f'(w) \right\}
\]
This function is polynomial in $f$ and $f'$ and therefore it is continuous from $]-\frac{1}{2}, \frac{1}{2}[ \times \mathbb{R} \times B_1$ to $Y$. On the other hand, the partial derivative with respect to $\Omega$ it is given by
\[
\partial_{\Omega} G_0(\varepsilon, \Omega, f(w)) = \text{Im} \left\{ (\varepsilon w + \varepsilon^2 |\alpha f(w) - d) \overline{w} \left( 1 + \varepsilon |\alpha f'(w) \right) \right\}
\]
and this is obviously continuous from $]-\frac{1}{2}, \frac{1}{2}[ \times \mathbb{R} \times B_1$ to $Y$. Note also that $G_0$ is polynomial with respect to $f$ and $f'$ and consequently $\partial_f G_0$ exists and is continuous. This concludes the fact that $G_0$ is $C^1$. It is easy to check that for any direction $h \in X$
\[
(29) \quad \partial_f G_0(0, \Omega, 0)(h) = 0.
\]
For the remaining functionals the situation is much more complicated. As we shall see the reasoning is very classical and we will give just some significant details. We first start with the term $G_2$. To find the suitable extension note that the ansatz of the solution is very crucial and allows to get rid of the singularity in $\varepsilon$. Recall that
\[
G_2(\varepsilon, f(w)) = \text{Im} \left\{ I_2(\varepsilon, f(w)) L(\varepsilon, f(w)) \right\}
\]
with $L(\varepsilon, f(w)) = \overline{w} \left( 1 + \varepsilon |\alpha f'(w) \right)$
where we have extended the tangent vector to $L$ and as previously,
\[
L : ] -\frac{1}{2}, \frac{1}{2}[ \times B_1 \rightarrow Y
\]
is well-defined and is of class $C^1$. Therefore it suffices to prove that $I_2$ can be extended from $]-\frac{1}{2}, \frac{1}{2}[ \times \mathbb{R} \times B_1$ to $Y$ as a $C^1$ function. The key point is Taylor formula:

\[
\frac{1}{|A + B|^\alpha} = \frac{1}{|A|^\alpha} - \alpha \int_0^1 \frac{\text{Re}(A B) + t |B|^2}{|A + t B|^{2+\alpha}} dt
\]

which is true for any complex numbers $A, B$ such that $|B| < |A|$. As an application we get

\[
\frac{1}{|\varepsilon \phi(w) + \varepsilon \phi(\tau) - 2d|^\alpha} = \frac{1}{(2d)^\alpha} - \alpha \int_0^1 \frac{-2d \varepsilon \text{Re}[\phi(\tau) + \phi(w)] + t \varepsilon^2 |\phi(\tau) + \phi(w)|^2}{|t \varepsilon \phi(w) + t \varepsilon \phi(\tau) - 2d|^{2+\alpha}} dt.
\]

We mention that the condition $|B| < |A|$ is satisfied because

\[
|\varepsilon \phi(w) + \varepsilon \phi(\tau)| \leq 2\varepsilon \|\phi\|_{L^\infty} \leq 4\varepsilon < d.
\]

Consequently

\[
I_2(\varepsilon, f(w)) = -\alpha C_\alpha \int_T \int_0^1 \frac{-2d \text{Re}[\phi(\tau) + \phi(w)] + t \varepsilon |\phi(\tau) + \phi(w)|^2}{|t \varepsilon \phi(w) + t \varepsilon \phi(\tau) - 2d|^{2+\alpha}} \phi'(\tau) d\tau dt,
\]

where we have used the fact

\[
\int_T \phi'(\tau) d\tau = 0.
\]

Thus the suitable extension of this functional is

\[
I_2(\varepsilon, f(w)) = -\alpha C_\alpha \int_T \int_0^1 \frac{-2d \text{Re}[\phi(\tau) + \phi(w)] + t \varepsilon |\phi(\tau) + \phi(w)|^2}{|t \varepsilon \phi(w) + t \varepsilon \phi(\tau) - 2d|^{2+\alpha}} \phi'(\tau) d\tau dt
\]

\[
\equiv -\alpha C_\alpha \int_T \int_0^1 \mathcal{K}_2(\tau, w) \phi'(\tau) d\tau,
\]

with $\phi(w) = w + \varepsilon |\varepsilon|^\alpha f(w)$. We shall now check that this extension defines a $C^1$ function from $] - \frac{1}{2}, \frac{1}{2} [ \times B_1$ to $Y$. First, the integral operator is well-defined since the kernel $\mathcal{K}_2$ is not singular and satisfies the hypothesis of Lemma 1 for any $f \in B_1$, $|\mathcal{K}_2(\tau, w)| \leq C$ and $|\partial_w \mathcal{K}_2(\tau, w)| \leq C$ for some constant $C$ and thus

\[
\|I_2(\varepsilon, f)\|_{C^{1-\alpha}(T)} \leq C \|\phi\|_{L^\infty} \leq C.
\]

For any $f \in X$ we have that

\[
I_2(0, 0) = \frac{\alpha C_\alpha}{(2d)^{1+\alpha}} \int_T \int_0^1 \tau d\tau dt
\]

\[
= \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}.
\]

In addition, for $f \in B_1$ and $h \in X$ one has

\[
\partial_f I_2(0, f)h(w) = \frac{d}{ds} I_2(0, f(w) + s h(w))|_{s=0}
\]

\[= 0.
\]

As $\partial_f L(0, f) = 0$ then we deduce

\[
\partial_f G_2(0, f) = 0.
\]
For a future use, we shall apply once again (31) to $I_2(\varepsilon, f)$ in order to get
\[
I_2(\varepsilon, f(w)) = \frac{\alpha C_\alpha}{(2d)_{1+\alpha}} \int_T \text{Re}(\phi(\tau))\phi'(\tau)d\tau - \frac{\alpha C_\alpha}{(2d)_{2+\alpha}} \frac{\varepsilon}{2} \int_T |\phi(\tau)|^2 \phi'(\tau)d\tau
\]
\[+ \alpha C_\alpha (2+\alpha) \varepsilon \int_T K_2(\tau, w)\phi'(\tau)d\tau,
\]
(35)
\[
K_2(\tau, w) = \int_0^1 \int_0^1 \left( -2d \text{Re}\{\Phi(\tau, w)\} + t \varepsilon|\Phi(\tau, w)|^2 \right) \left( -2d t \text{Re}\{\Phi(\tau, w)\} + st \varepsilon|\Phi(\tau, w)|^2 \right) dt ds,
\]
with
\[
\Phi(\tau, w) = \phi(\tau) + \phi(w).
\]
On the other hand, since $\phi(w) = w + \varepsilon|\alpha|^a f(w)
\[
\int_T \int_0^1 \text{Re}(\phi(\tau))\phi'(\tau)d\tau dt = \frac{1}{2} + \varepsilon|\alpha|^a \int_T \text{Re}(\tau)f'(\tau)d\tau + \varepsilon|\alpha|^a \int_T \text{Re}(f(\tau))\phi'(\tau)d\tau.
\]
Therefore one gets
\[
I_2(\varepsilon, f(w)) = \frac{\alpha C_\alpha}{2(2d)_{1+\alpha}} + \varepsilon I_2(\varepsilon, f(w)).
\]
Using that the kernel $K_2$ satisfies the conditions of Lemma 1, that is,
\[
|K_2(\tau, w)| \leq C, \quad |\partial_w K_2(\tau, w)| \leq C,
\]
one can verify that the function $I_2 : -\frac{1}{2}, \frac{1}{2}[\times B_1 \to C^{1-\alpha}](\mathbb{T})$ is well-defined. To prove that it is indeed of class $C^1$ we shall look for its derivatives and study their continuity. The computations are straightforward and resemble to those done for Euler equations and thus we will skip the details. Inserting the formula (36) into the expression of $G_2$ allows to get the decomposition
\[
G_2(\varepsilon, f) \equiv -\frac{\alpha C_\alpha}{2(2d)_{1+\alpha}} e_1(w) - \varepsilon R_2(\varepsilon, f)
\]
with $R_2 : -\frac{1}{2}, \frac{1}{2}[\times B_1 \to C^{1-\alpha}](\mathbb{T})$ being a $C^1$ function. Let us now move to the extension of the function $G_1$ defined in (22). We can split an extension of $I_1(\varepsilon, f)$ into three parts as follows,
\[
I_1(\varepsilon, f(w)) = C_\alpha \int_T \frac{f'(\tau)d\tau}{|\phi(w) - \phi(\tau)|^\alpha} + C_\alpha \frac{1}{\varepsilon|\alpha|^\alpha} \int_T \frac{d\tau}{|\tau - w|^\alpha}
\]
\[+ \frac{C_\alpha}{\varepsilon|\alpha|^\alpha} \int_T \left( \frac{1}{|\phi(w) - \phi(\tau)|^\alpha} - \frac{1}{|\tau - w|^\alpha} \right) d\tau
\]
\[= C_\alpha \int_T \frac{f'(\tau)d\tau}{|\phi(w) - \phi(\tau)|^\alpha} + \frac{C_\alpha}{\varepsilon|\alpha|^\alpha} w + C_\alpha \int_T K(\varepsilon, \tau, w)d\tau
\]
(38)
\[= I_{11}(\varepsilon, f(w)) + \frac{C_\alpha}{\varepsilon|\alpha|^\alpha} w + I_{12}(\varepsilon, f(w))
\]
with
\[
K(\varepsilon, \tau, w) = \frac{1}{\varepsilon|\alpha|^\alpha} \left( \frac{1}{30} \frac{1}{|\phi(w) - \phi(\tau)|^\alpha} - \frac{1}{|\tau - w|^\alpha} \right)
\]
which implies that
\[
\hat{C}_\alpha = \frac{\alpha \Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})} C_\alpha = \frac{\Gamma(1-\alpha)\Gamma(1+\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(2-\frac{\alpha}{2})\Gamma^2(1-\frac{\alpha}{2})}.
\]

Note that we have used the identity, see \[19, \text{Lemma 2}\]
\[
\int_T \frac{d\tau}{|\tau-w|^{\alpha}} = \frac{\alpha \Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})} w.
\]

Consequently,
\[
G_1(\varepsilon, f(w)) = \text{Im}\left\{ I_{11}(\varepsilon, f(w))\overline{\varepsilon(1+\varepsilon|\alpha f'(w)|)} \right\}
\]
\[
+ \text{Im}\left\{ I_{12}(\varepsilon, f(w))\overline{\varepsilon(1+\varepsilon|\alpha f'(w)|)} \right\} - \hat{C}_\alpha \text{Im}(f'(w)).
\]

The last term defines a linear operator from $X$ to $Y$ and therefore it is smooth. It remains to study the first and second terms. This amounts to studying the terms $I_{11}$ and $I_{12}$. The first part is extended as usual through the formula
\[
I_{11}(\varepsilon, f(w)) = C_\alpha \int_T \frac{f'(\tau)}{|\phi(\tau) - \phi(w)|^{\alpha}}, \quad \text{with} \quad \phi(w) = w + \varepsilon|\alpha f(w)|.
\]

First to check that $I_{11}$ is well-defined we use Corollary 1 which implies that
\[
\|I_{11}(\varepsilon, f)\|_{C^{1-\alpha}(T)} \leq C\|f'\|_{L^\infty}
\]
\[
\leq C\|f\|_{C^{2-\alpha}(T)}.
\]

Now we shall prove that $I_{11}$ is $C^1$ and for this purpose it suffices to check the existence of the partial derivatives and their continuity in strong topology. The partial derivative with respect to $\varepsilon$ can be easily computed and we find
\[
\partial_\varepsilon I_{11}(\varepsilon, f(w)) = -\alpha C_\alpha (|\varepsilon|^{\alpha} + \alpha \varepsilon \text{sign}(\varepsilon)|\varepsilon|^{\alpha-1}) \int_T \frac{\text{Re}[\overline{(\tau-w)}(f(\tau) - f(w))]}{|\phi(\tau) - \phi(w)|^{2+\alpha}} f'(\tau) d\tau
\]
\[
- \alpha C_\alpha \varepsilon|\varepsilon|^{\alpha} (|\varepsilon|^{\alpha} + \alpha \varepsilon \text{sign}(\varepsilon)|\varepsilon|^{\alpha-1}) \int_T \frac{|f(\tau) - f(w)|^2}{|\phi(\tau) - \phi(w)|^{2+\alpha}} f'(\tau) d\tau.
\]

Introduce the kernels
\[
K_1(\tau, w) = \frac{\text{Re}[\overline{(\tau-w)}(f(\tau) - f(w))]}{|\phi(\tau) - \phi(w)|^{2+\alpha}}, \quad K_2(\tau, w) = \frac{|f(\tau) - f(w)|^2}{|\phi(\tau) - \phi(w)|^{2+\alpha}}
\]

Then for $\tau \neq w$
\[
|K_1(\tau, w)| \leq \frac{\|f\|_{Lip}}{\|\phi^{-1}\|_{2+\alpha}} |\tau - w|^{-\alpha}
\]
\[
\leq C|\tau - w|^{-\alpha}.
\]

and in a similar way
\[
|K_2(\tau, w)| \leq C|\tau - w|^{-\alpha}.
\]

Moreover
\[
|\partial_w K_1(\tau, w)| + |\partial_w K_2(\tau, w)| \leq C|\tau - w|^{-1-\alpha}.
\]
Therefore using Lemma 1 we deduce that $\partial_t I_{11}(\varepsilon, f) \in C^{1-\alpha}(\mathbb{T})$ and the dependence on $(\varepsilon, f) \in [-\frac{1}{2}, \frac{1}{2}] \times B_1$ is continuous. More details in a similar context can be found in [19]. The partial derivative with respect to $f$ in the direction $h \in X$ is given by

$$
\partial_f I_{11}(\varepsilon, f)h = C_{\alpha} \int_\mathbb{T} \frac{h'(\tau)d\tau}{|\phi(\tau) - \phi(w)|^\alpha} - \alpha C_{\alpha} \varepsilon |\varepsilon|^\alpha \int_\mathbb{T} \frac{\text{Re}[\phi(\tau) - \phi(w)](\partial_t f(\tau) - h(w))|f'(\tau)|}{|\phi(\tau) - \phi(w)|^{2+\alpha}} d\tau,
$$

which is continuous from $[-\frac{1}{2}, \frac{1}{2}] \times B_1$ to $C^{1-\alpha}(\mathbb{T})$. In particular we get for any $h \in X$

$$
\partial_f I_{11}(0, 0)h = C_{\alpha} \int_\mathbb{T} \frac{h'(\tau)}{|\tau - w|^{\alpha}} d\tau.
$$

We shall now move to the extension and the regularity of $I_{12}$ defined in (38). It can be extended through its kernel as follows,

$$
K(\varepsilon, \tau, w) = \frac{1}{|\varepsilon|^\alpha} \left( \frac{1}{|\phi(w) - \phi(\tau)|^\alpha} - \frac{1}{|\tau - w|^\alpha} \right), \quad \phi(w) = w + \varepsilon |\varepsilon|^\alpha f(w).
$$

Now using (31) we find that

$$
K(\varepsilon, \tau, w) = -\alpha \int_0^{\frac{1}{2}} \text{Re}\left( \frac{(\tau - w)h(\tau)}{|\tau - w + t\varepsilon|^{2+\alpha}} \right) dt - \alpha \varepsilon |\varepsilon|^\alpha \int_0^{\frac{1}{2}} \frac{t|f(\tau) - f(w)|^2}{|\tau - w + t\varepsilon|^{2+\alpha}} dt.
$$

By straightforward computations we can check that there exists an absolute constant $C$ such that for $(\varepsilon, f) \in [-\frac{1}{2}, \frac{1}{2}] \times B_1$ and for $\tau \neq w$

$$
|K(\varepsilon, \tau, w)| \leq \frac{C}{|\tau - w|^{1+\alpha}} \quad \text{and} \quad |\partial_w K(\varepsilon, \tau, w)| \leq \frac{C}{|\tau - w|^{1+\alpha}}.
$$

Therefore using Lemma 1 again we can conclude that $I_{12}(\varepsilon, f)$ is well-defined and belongs to $C^{1-\alpha}(\mathbb{T})$. The regularity with respect to $\varepsilon$ is straightforward since $\varepsilon \mapsto K(\varepsilon, \tau, w)$ is $C^1$ and

$$
|\partial_{\varepsilon} K(\varepsilon, \tau, w)| \leq \frac{C}{|\tau - w|^{1+\alpha}} \quad \text{and} \quad |\partial_{\varepsilon} \partial_{\varepsilon} K(\varepsilon, \tau, w)| \leq \frac{C}{|\tau - w|^{1+\alpha}}
$$

which implies that $\partial_{\varepsilon} I_{12}(\varepsilon, f)$ is well-defined for $(\varepsilon, f) \in [-\frac{1}{2}, \frac{1}{2}] \times B_1$ and belongs to $C^{1-\alpha}(\mathbb{T})$. The continuity can be done in a similar way. Moreover by (42) we have that

$$
I_{12}(0, 0) = 0.
$$

The existence of partial derivative with respect to $f$ can be done without difficulty and we can check that this derivative is continuous. Thus we establish that $I_{12}$ is $C^1$ and in particular we deduce that

$$
\partial_f I_{12}(0, 0)h(w) = -\alpha C_{\alpha} \int_\mathbb{T} \frac{\text{Re}\left( (\tau - w)(h(\tau) - h(w)) \right)}{|\tau - w|^{2+\alpha}} d\tau.
$$
Putting together (40), (41) and (44) we find for $h \in X$,
\[
\partial_f G_1(0, 0) h(w) = C_\alpha \text{Im} \left\{ \bar{w} \int_T \frac{h'(\tau) d\tau}{|\tau - w|^\alpha} \right\} - \tilde{C}_\alpha \text{Im} \{h'(w)\}
\]
(45) 
\[- \alpha C_\alpha \text{Im} \left\{ \bar{w} \int_T \frac{\text{Re} \left( (\bar{\tau} - \bar{w})(h(\tau) - h(w)) \right)}{|\tau - w|^{2+\alpha}} d\tau \right\}.\]
According to (29), (34) and (45) we get for any $\Omega \in \mathbb{R}$ and $h \in X$,
\[
\partial_f G(0, \Omega, 0) h(w) = \partial_f G_0(0, \Omega, 0) h(w) - \partial_f G_1(0, 0) h(w) + \partial_f G_2(0, 0) h(w)
\]
\[= -C_\alpha \text{Im} \left\{ \bar{w} \int_T \frac{h'(\tau) d\tau}{|\tau - w|^\alpha} \right\} + \tilde{C}_\alpha \text{Im} \{h'(w)\}
\]
\[+ \alpha C_\alpha \text{Im} \left\{ \bar{w} \int_T \frac{\text{Re} \left( (\bar{\tau} - \bar{w})(h(\tau) - h(w)) \right)}{|\tau - w|^{2+\alpha}} d\tau \right\}
\]
\[\equiv \mathcal{L}_1 h + \mathcal{L}_2 h + \mathcal{L}_3 h.
\]
Finally note that the extension for $G$ is obtained by putting together (15), (28), (30), (32), (38), (40) and (42).
(ii) When $D_{1, \varepsilon} = \varepsilon \mathbb{D}$ then from (3) we get in a weak sense
\[
\lim_{\varepsilon \to 0} \theta_{0, \varepsilon} = \pi \delta_0 + \pi \delta_{(2d, 0)}.
\]
Now from the extension of $G$ we can check that for $\varepsilon = 0$
\[
G(0, \Omega, 0) = \left( \Omega d - \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} \right) e_1
\]
which implies that this a solution if and only if
\[
\Omega = \Omega_{\text{sing}} \equiv \frac{\alpha C_\alpha}{(2d)^{2+\alpha}}.
\]
This means that two point vortices $\pi \delta_0$ and $\pi \delta_{2d}$ rotate uniformly about their center $(d, 0)$ with the angular velocity $= \Omega_{\text{sing}}$.
(iii) We will start with computing $\partial_f G_1(0, 0)$ whose expression is given in (45). It is easy to see that
\[
\alpha C_\alpha \bar{w} \int_T \frac{\text{Re} \left( (\bar{\tau} - \bar{w})(h(\tau) - h(w)) \right)}{|\tau - w|^{2+\alpha}} d\tau = \frac{1}{2} \alpha C_\alpha \bar{w} \int_T \frac{(\tau - w)(h(\tau) - h(w)) d\tau}{|\tau - w|^{2+\alpha}}
\]
\[+ \frac{1}{2} \alpha C_\alpha \bar{w} \int_T \frac{(\bar{\tau} - \bar{w})(h(\tau) - h(w)) d\tau}{|\tau - w|^{2+\alpha}}
\]
\[\equiv I_4(h(w)) + I_5(h(w)).
\]
According [19, p.360] these terms were computed and take the form
\[
I_4(h(w)) = \frac{\alpha \left( 1 + \frac{\alpha}{2} \right) C_\alpha \Gamma(1 - \alpha)}{2(2 - \alpha) \Gamma^2(1 - \alpha/2)} \sum_{n \geq 1} a_n \left( 1 - \left( \frac{2 + \alpha}{2 - \alpha} \right)^n \right) w^{n+1}
\]
(46)
and

$$I_5(h(w)) = -\frac{\alpha C_\alpha \Gamma(1 - \alpha)}{4\Gamma^2(1 - \alpha/2)} \sum_{n \geq 1} a_n \left( 1 - \frac{(\frac{\alpha}{2})_n}{(-\frac{\alpha}{2})_n} \right) w^{n+1}.$$  

(47)

It follows that

$$\mathcal{L}_3 h(w) = \text{Im}\{I_4(h(w)) + I_5(h(w))\} = \frac{\alpha C_\alpha \Gamma(1 - \alpha)}{4\Gamma^2(1 - \alpha/2)} \sum_{n \geq 1} a_n \beta_n e_{n+1}.$$  

(48)

with

$$\beta_n = \left( 1 - \frac{(\frac{\alpha}{2})_n}{(-\frac{\alpha}{2})_n} \right) + \frac{1 + \frac{\alpha}{2}}{1 - \frac{\alpha}{2}} \left( 1 - \frac{(2 + \frac{\alpha}{2})_n}{(2 - \frac{\alpha}{2})_n} \right)$$

$$= \left( 1 - \frac{(\frac{\alpha}{2})_n}{(-\frac{\alpha}{2})_n} \right) + \frac{1 + \frac{\alpha}{2}}{1 - \frac{\alpha}{2}} - \frac{(1 + \frac{\alpha}{2})_{n+1}}{(1 - \frac{\alpha}{2})_{n+1}}$$

$$= \frac{2}{1 - \frac{\alpha}{2}} + \frac{(1 + \frac{\alpha}{2})_{n-1}}{(1 - \frac{\alpha}{2})_{n-1}} - \frac{(1 + \frac{\alpha}{2})_{n+1}}{(1 - \frac{\alpha}{2})_{n+1}}.$$  

(49)

Regarding the first term $\mathcal{L}_1(h(w))$ it may be rewritten in the form

$$\mathcal{L}_1(h(w)) = \text{Im}\{I_3(h(w))\}$$

with

$$I_3(h(w)) = -C_\alpha \overline{w} \mathcal{F} h'(\tau) \frac{\Gamma(1 - \alpha)}{\Gamma^2(1 - \alpha/2)} \sum_{n \geq 1} n a_n \overline{w}^{n+1} \frac{(\frac{\alpha}{2})_n}{(1 - \frac{\alpha}{2})_n}.$$  

$$= C_\alpha \sum_{n \geq 1} n a_n \overline{w} \mathcal{F} h'(\tau) \frac{(\frac{\alpha}{2})_n}{(1 - \frac{\alpha}{2})_n}.$$  

(50)

Once again we get in view of [19, p.360]

$$I_3(h(w)) = \frac{C_\alpha \Gamma(1 - \alpha)}{\Gamma^2(1 - \alpha/2)} \sum_{n \geq 1} n a_n \overline{w}^{n+1} \frac{(\frac{\alpha}{2})_n}{(1 - \frac{\alpha}{2})_n}.$$  

(51)

Therefore

$$\mathcal{L}_1(h(w)) = -\frac{C_\alpha \Gamma(1 - \alpha)}{\Gamma^2(1 - \alpha/2)} \sum_{n \geq 1} n a_n \overline{w}^{n+1} \frac{(\frac{\alpha}{2})_n}{(1 - \frac{\alpha}{2})_n} e_{n+1}.$$  

For $\mathcal{L}_2$ we readily get by (39),

$$\mathcal{L}_2 h(w) = \tilde{C}_\alpha \sum_{n \geq 1} n a_n e_{n+1}$$

$$= \frac{\alpha C_\alpha \Gamma(1 - \alpha)}{\Gamma^2(1 - \alpha/2)} \sum_{n \geq 1} n a_n e_{n+1}.$$  

(52)

Putting together the preceding identities yields to

$$\partial_f G(0,0) h(w) = \mathcal{L}_1 h(w) + \mathcal{L}_2 h(w) + \mathcal{L}_3 h(w)$$

$$= \frac{\alpha C_\alpha \Gamma(1 - \alpha)}{4\Gamma^2(1 - \alpha/2)} \sum_{n \geq 1} a_n \gamma_n e_{n+1},$$  

(52)
with
\[ \gamma_n = \beta_n - 4 \frac{\left(\frac{\alpha}{2}\right)_n}{\alpha (1 - \frac{\alpha}{2})_n} n + 4 \frac{\alpha}{2} n \]
\[ = \beta_n - 2n \frac{(1 + \frac{\alpha}{2})_{n-1}}{(1 - \frac{\alpha}{2})_n} + 2n \frac{1}{1 - \frac{\alpha}{2}} \]
\[ = \frac{2(1 + n)}{1 - \frac{\alpha}{2}} + \frac{(1 + \frac{\alpha}{2})_{n-1}}{(1 - \frac{\alpha}{2})_n} - \frac{(1 + \frac{\alpha}{2})_{n+1}}{(1 - \frac{\alpha}{2})_{n+1}} - \frac{2n}{n - \frac{\alpha}{2}} \frac{(1 + \frac{\alpha}{2})_{n-1}}{(1 - \frac{\alpha}{2})_{n-1}} \]
\[ = \frac{2(1 + n)}{1 - \frac{\alpha}{2}} - \frac{(1 + \frac{\alpha}{2})_n}{(1 - \frac{\alpha}{2})_n} - \frac{(1 + \frac{\alpha}{2})_{n+1}}{(1 - \frac{\alpha}{2})_{n+1}}. \]

(iv) Now we shall prove that \( \partial_f G(0, \Omega, 0) : X \to \hat{Y} \) is an isomorphism. To check that this operator is one-to-one it is enough to prove the following: There exist two constants \( C_1 > 0 \) and \( C_2 > 0 \) such that for any \( n \geq 1 \)
\[ C_1 n \leq \gamma_n \leq C_2 n. \]

It easy to check that
\[ \frac{(1 + \frac{\alpha}{2})_n}{(1 - \frac{\alpha}{2})_n} < \frac{n + \frac{\alpha}{2}}{1 - \frac{\alpha}{2}} \beta, \]
where \( \beta = \frac{1 + \frac{\alpha}{2}}{2 - \frac{\alpha}{2}} < 1 \). Therefore we deduce by simple computations that for \( \alpha \in [0, 1] \)
\[ \gamma_n > \frac{2(1 + n)}{1 - \frac{\alpha}{2}} - \beta \frac{n + \frac{\alpha}{2}}{1 - \frac{\alpha}{2}} - \beta \frac{n + 1 + \frac{\alpha}{2}}{1 - \frac{\alpha}{2}} \geq C_1(\alpha)n. \]

On the other hand, we readily get
\[ \gamma_n \leq C_2(\alpha)n, \]
and hence the proof of (53) is achieved. It remains to prove that \( \partial_f G(0, 0) \) is onto. Let \( g \in \hat{Y} \), we shall prove that the equation \( \partial_f G(0, 0)h = g \) admits a solution \( h \in X \). The functions \( g \) and \( h \) take the form
\[ g(w) = \frac{\alpha C_\alpha \Gamma(1 - \alpha)}{4 \Gamma^2(1 - \frac{\alpha}{2})} \sum_{n \geq 1} b_n e_{n+1}(w) \quad \text{and} \quad h(w) = \sum_{n \geq 1} a_n w^n. \]

Therefore using (52) and (53) the equation \( \partial_f G(0, 0)h = g \) is equivalent to
\[ a_n = \frac{b_n}{\gamma_n}, \quad n \geq 1. \]

The only point to check is \( h \in C^{2-\alpha}(\mathbb{T}) \), that is
\[ w \in \mathbb{T} \mapsto \sum_{n \geq 1} \frac{b_n}{\gamma_n} w^n \in C^{2-\alpha}(\mathbb{T}). \]

From [19, p.358], there exists a constant \( C > 0 \) such that
\[ \frac{(1 + \frac{\alpha}{2})_n}{(1 - \frac{\alpha}{2})_n} = C n^\alpha + O\left( \frac{1}{n^{1-\alpha}} \right) \]
which implies in turn that
\[ \gamma_n = n \left( \frac{2}{1 - \frac{\alpha}{2}} - 2C \frac{1}{n^{1-\alpha}} + O\left( \frac{1}{n^{2-\alpha}} \right) \right). \]
It is not difficult to show that $h \in L^\infty(\mathbb{T})$ and thus it remains to check that $h' \in C^{1-\alpha}(\mathbb{T})$. Note that

$$-wh'(w) = \sum_{n \geq 1} \frac{nb_n}{\gamma_n} \bar{w}^n$$

and

$$\frac{nb_n}{\gamma_n} = \frac{1}{2 - \alpha} \left( 2C \frac{1}{n^{1-\alpha}} + O \left( \frac{1}{n^{2-\alpha}} \right) \right)$$

$$= \left( \frac{1}{2} - \frac{\alpha}{2} \right) b_n + \frac{C(1 - \frac{\alpha}{2})}{n^{1-\alpha} \left( \frac{1}{2} - \frac{\alpha}{2} \right) - 2C} b_n + O \left( \frac{1}{n^{2-\alpha}} \right) b_n$$

$$\equiv \frac{1}{2} \left( \frac{1}{2} - \frac{\alpha}{2} \right) b_n + \alpha_n b_n + O \left( \frac{1}{n^{2-\alpha}} \right) b_n.$$ 

Using the continuity of Szegö projector $\Pi$ in $C^{1-\alpha}(\mathbb{T})$ we obtain easily that

$$\tilde{h} : w \mapsto \sum_{n \geq 1} b_n \bar{w}^n \in C^{1-\alpha}(\mathbb{T}).$$

Define the kernels

$$K_1(w) = \sum_{n \geq 1} \alpha_n \bar{w}^n \quad \text{and} \quad K_2(w) = \sum_{n \geq 1} O \left( \frac{1}{n^{2-\alpha}} \right) \bar{w}^n.$$ 

The remainder term of $-wh'$ is given by

$$K_1 \ast \tilde{h} + K_2 \ast \tilde{h}.$$ 

As the kernel $K_2 \in L^\infty(\mathbb{T}) \subset L^1(\mathbb{T})$ then $K_2 \ast \tilde{h} \in C^{1-\alpha}(\mathbb{T})$. In [19, p.363-366] we established that $K_1 \in L^1(\mathbb{T})$ and therefore we obtain $K_1 \ast \tilde{h} \in C^{1-\alpha}(\mathbb{T})$ and this gives finally $h' \in C^{1-\alpha}(\mathbb{T})$ which concludes the proof.

\[\Box\]

6.2. Relationship between the angular velocity and the boundary. As we have seen in Proposition 7, the operator $\partial_f G(0, \Omega, 0)$ is an isomorphism from $X$ to $\hat{Y}$ and not $Y$. However the range of the nonlinear functional $G$ is contained in $\hat{Y}$ which is bigger than $Y$. The strategy is to choose carefully $\Omega$ in such a way that the range of $G$ is contained in $\hat{Y}$ in order to be able to apply the implicit function theorem. The main goal of this section is to establish the following result.

**Proposition 8.** There exists a $C^1$ function $\mathcal{R} : [-\frac{1}{2}, \frac{1}{2}] \times B_1 \rightarrow \mathbb{R}$ such that, for

$$\Omega = \Omega(\varepsilon, f) = \frac{\alpha C_\alpha}{(2d)^{2+\alpha}} + \mathcal{R}(\varepsilon, f),$$

the function $\hat{G} : [-\frac{1}{2}, \frac{1}{2}] \times B_1 \rightarrow \hat{Y}$ given by

$$\hat{G}(\varepsilon, f) = G(\varepsilon, \Omega(\varepsilon, f), f)$$

is well-defined and is of class $C^1$. Moreover,

$$\forall f \in B_1, \mathcal{R}(0, f) = 0 \quad \text{and} \quad \hat{G}(0, 0) = 0.$$ 

**Proof.** It means that $\Omega$ is chosen such that the first Fourier coefficient of $G$ (the component of $e_1$) is zero. From (13) we can deduce that the constraint is given by

$$\int_{\mathbb{T}} G(\varepsilon, \Omega, f(w))dw = 0.$$
From the decomposition (15) this assumption is equivalent to

\begin{equation}
A_0 = A_1 - A_2,
\end{equation}

with

\[ A_j = -2i \int_T G_j(\varepsilon, \Omega, f(w)) \, dw. \]

Note that \( A_j \) is the Fourier coefficient of \( e_1 = \text{Im}(w) \) in \( G_j \) and when \( G_j = \text{Im}(F_j) \) then

\[ A_j = \int_T F_j(\varepsilon, \Omega, f(w)) (\overline{\omega}^2 - 1) \, dw. \]

Looking for the Fourier expansion of \( \text{Im} \left\{ \Omega(\varepsilon w - d) \overline{w} \left( 1 + \varepsilon |\varepsilon|^{\alpha} f'(w) \right) \right\} \) we deduce that the coefficient of \( e_1 = \text{Im}(w) \) is

\[ \Omega d \left( 1 + \varepsilon |\varepsilon|^{\alpha} \int_T f(\tau) d\tau \right). \]

In a similar way the Fourier coefficient of \( e_1 \) in \( \Omega \varepsilon^2 |\varepsilon|^{\alpha} \text{Im} \left\{ f(w) \overline{w} (1 + \varepsilon |\varepsilon|^{\alpha} f'(w)) \right\} \) is

\[ \Omega \varepsilon^3 |\varepsilon|^{2\alpha} \int_T f(w) \overline{f'(w)} (\overline{w}^3 - \overline{w}) \, dw. \]

Consequently,

\begin{align*}
A_0 &= \Omega d \left( 1 + \varepsilon |\varepsilon|^{\alpha} \int_T f(\tau) d\tau \right) \\
&\quad + \Omega \varepsilon^3 |\varepsilon|^{2\alpha} \int_T f(w) \overline{f'(w)} (\overline{w}^3 - \overline{w}) \, dw \\
&\equiv \Omega d + \Omega \varepsilon |\varepsilon|^{\alpha} R_0(\varepsilon, f)
\end{align*}

with \( R_0 : [-\frac{1}{2}, \frac{1}{2}] \times B_1 \rightarrow \mathbb{R} \). We point out that for any \( (\varepsilon, f) \in [-\frac{1}{2}, \frac{1}{2}] \times B_1 \)

\[ |R_0(\varepsilon, f)| \leq d\|f\|_{L^\infty} + 2\varepsilon |\varepsilon|^{2+\alpha} \|f\|_{L^\infty} \|f'\|_{L^\infty} \leq d + 2\varepsilon |\varepsilon|^{2+\alpha}. \]

So, it means that the function \( R_0 \) is well-defined. It is clear that it is differentiable with continuity in the \( \varepsilon \)-variable and moreover the function is polynomial in \( f \) and \( f' \) and so its derivative satisfies the required assumption. Therefore we conclude that \( R_0 \) is a \( C^1 \) function from \( [-\frac{1}{2}, \frac{1}{2}] \times B_1 \) to \( \mathbb{R} \). Now we shall compute \( A_1 \) associated to \( G_1 \) which is described by (40) and (38). Thus

\[ -A_1 = \int_T I_{11}(\varepsilon, f(w)) (\overline{w} - \overline{w}^3) \, dw + \int_T I_{12}(\varepsilon, f(w)) (\overline{w} - \overline{w}^3) \, dw \]

\[ + \varepsilon |\varepsilon|^{\alpha} \int_T \left[ I_{11}(\varepsilon, f(w)) + I_{12}(\varepsilon, f(w)) \right] \overline{f'(w)} (\overline{w} - \overline{w}^3) \, dw \]

\[ \equiv A_{11} + A_{12} + \varepsilon |\varepsilon|^{\alpha} A_{13}. \]

To calculate \( A_{11} \) we use (38) which yields

\[ A_{11} = C_{\alpha} \int_T \int_T \frac{f'(\tau)(\overline{w} - \overline{w}^3)}{|\tau - w|^{3\alpha}} \, dw \, d\tau + C_{\alpha} \int_T \int_T f'(\tau)(\overline{w} - \overline{w}^3) K(\tau, w) \, dw \, d\tau \]

with

\[ K(\tau, w) = \frac{1}{|\phi(\tau) - \phi(w)|^{\alpha}} - \frac{1}{|\tau - w|^{3\alpha}}, \quad \phi(w) = w + \varepsilon |\varepsilon|^{\alpha} f(w). \]
From the Fourier expansion (50) we conclude that
\[ \int_{-T}^{T} \int_{-T}^{T} \frac{f'(\tau)(\overline{w} - \overline{w}^3)}{|\tau - w|^\alpha} dw d\tau = 0 \]
and therefore
\[ A_{11} = C_\alpha \int_{-T}^{T} \int_{-T}^{T} f'(\tau)(\overline{w} - \overline{w}^3) K(\tau, w) dw d\tau. \]

According to (42) we get
\[ (57) \quad A_{11} = C_\alpha \varepsilon |\varepsilon|^\alpha \int_{-T}^{T} \int_{-T}^{T} f'(\tau)(\overline{w} - \overline{w}^3) K(\varepsilon, \tau, w) dw d\tau. \]

For the term \( A_{12} \) recall from (38) that
\[ I_{12}(\varepsilon, f(w)) = C_\alpha \int_{-T}^{T} K(\varepsilon, \tau, w) d\tau. \]

Combining (42) with (31) we get
\[ K(\varepsilon, \tau, w) = -\alpha \frac{\text{Re} \left( (\overline{\tau} - \overline{w})(f(\tau) - f(w)) \right)}{|\tau - w|^{2+\alpha}} \]
\[ + \alpha(2 + \alpha) \varepsilon |\varepsilon|^\alpha \int_{0}^{1} \int_{0}^{1} \frac{\hat{K}(t, \varepsilon, \tau, w)}{|\tau - w + st\varepsilon|^{\alpha}(f(\tau) - f(w))|^{4+\alpha}} dt ds \]
\[ - \alpha \varepsilon |\varepsilon|^\alpha \int_{0}^{1} \frac{t|f(\tau) - f(w)|^2}{|\tau - w + t\varepsilon|^{\alpha}(f(\tau) - f(w))|^{2+\alpha}} dt, \]

with
\[ \hat{K}(t, \varepsilon, \tau, w) \equiv \text{Re} \left( (\overline{\tau} - \overline{w})(f(\tau) - f(w)) \right) \left[ t \text{Re} \left( (\overline{\tau} - \overline{w})(f(\tau) - f(w)) \right) + st\varepsilon |\varepsilon|^\alpha |f(\tau) - f(w)|^2 \right]. \]

Thus
\[ I_{12}(\varepsilon, f(w)) = -\alpha C_\alpha \int_{-T}^{T} \frac{\text{Re} \left( (\overline{\tau} - \overline{w})(f(\tau) - f(w)) \right)}{|\tau - w|^{2+\alpha}} d\tau + \varepsilon |\varepsilon|^\alpha I_{12}(\varepsilon, f(w)). \]

Using that the function \( I_{12} \) is a sum of terms defined by integral operators and those operators have kernels satisfying the conditions of Lemma 1, we can conclude that \( I_{12} : [-\frac{1}{2}, \frac{1}{2}] \times B_1 \to Y \) is a \( C^1 \) function. Now using (48) we deduce that
\[ \int_{-T}^{T} \int_{-T}^{T} \frac{\text{Re} \left( (\overline{\tau} - \overline{w})(f(\tau) - f(w)) \right)}{|\tau - w|^{2+\alpha}} (\overline{w} - \overline{w}^3) d\tau dw = 0 \]
which implies that
\[ A_{12} = \int_{-T}^{T} I_{12}(\varepsilon, f(w))(\overline{w} - \overline{w}^3) dw \]
\[ = \varepsilon |\varepsilon|^\alpha \int_{-T}^{T} I_{12}(\varepsilon, f(w))(\overline{w} - \overline{w}^3) dw. \]
Finally we get

\[-A_1 = A_{11} + A_{12} + \varepsilon |\varepsilon|^{\alpha} A_{13} \]

\[= C_\alpha \varepsilon |\varepsilon|^{\alpha} \int_{\mathbb{T}} \int_{\mathbb{T}} f'(\tau)(\overline{w} - \overline{w}^3)K(\varepsilon, \tau, w)dw d\tau \]

\[+ \varepsilon |\varepsilon|^{\alpha} \int_{\mathbb{T}} \mathcal{I}_{12}(\varepsilon, f(w))(\overline{w} - \overline{w}^3)dw + \varepsilon |\varepsilon|^{\alpha} A_{13} \]

\[\equiv - \varepsilon |\varepsilon|^{\alpha} \mathcal{R}_1(\varepsilon, f). \tag{58} \]

Analyzing carefully all the terms in $A_1$, as in the foregoing cases, one may conclude that $\mathcal{R}_1 : [\mathbb{T}] \to \mathbb{R}$ is well-defined and $C^1$. So, it remains to compute $A_2$ which is described by (54) and (37),

\[A_2 = \int_{\mathbb{T}} F_2(\varepsilon, f(w))(\overline{w}^2 - 1)dw \]

\[= \int_{\mathbb{T}} \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}(\overline{w}^3 - \overline{w})dw - \varepsilon \mathcal{R}_2(\varepsilon, f) \]

\[\equiv - \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} - \varepsilon \mathcal{R}_2(\varepsilon, f) \tag{59} \]

and we can check that $\mathcal{R}_2 : [\mathbb{T}] \to \mathbb{R}$ is well-defined and $C^1$. Combining (54) with (55), (58) and (59) we deduce that,

\[\Omega \left( d + \varepsilon |\varepsilon|^{\alpha} \mathcal{R}_0(\varepsilon, f) \right) = \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \varepsilon |\varepsilon|^{\alpha} \mathcal{R}_1(\varepsilon, f) + \varepsilon \mathcal{R}_2(\varepsilon, f). \]

According to (56), since $d > 2$ then for any $(\varepsilon, f) \in \left[ -\frac{1}{2}, \frac{1}{2} \right][$B_1] we obtain

\[d + \varepsilon |\varepsilon|^{\alpha} \mathcal{R}_0(\varepsilon, f) \geq d - d |\varepsilon|^{1+\alpha} - 2|\varepsilon|^{3+2\alpha} \geq \frac{d}{4}. \]

Therefore

\[\Omega = \Omega(\varepsilon, f) \]

\[= \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \varepsilon |\varepsilon|^{\alpha} \mathcal{R}_1(\varepsilon, f) + \varepsilon \mathcal{R}_2(\varepsilon, f) \]

\[\equiv \Omega_{\text{sing}} + \mathcal{R}(\varepsilon, f), \quad \Omega_{\text{sing}} = \frac{\alpha C_\alpha}{(2d)^{2+\alpha}}, \]

where $\mathcal{R} : [\mathbb{T}] \to \mathbb{R}$ is $C^1$ because it is obtained as an algebraic combination of $C^1$ functions without zeros in the denominator.

\[\square \]

6.3. Existence of corotating vortex pair. Recall from Proposition 8 that the existence of solutions to the V-states equation can be transformed into the resolution of

\[\tilde{G}(\varepsilon, f) = 0, \quad (\varepsilon, f) \in \left[ -1/2, 1/2 \right][B_1] \]

with $\tilde{G}$ being the functional defined by

\[\tilde{G}(\varepsilon, f(w)) = G(\varepsilon, \Omega(\varepsilon, f), f) \]

and $\Omega(\varepsilon, f)$ was introduced in Proposition 8. The main result is the following.

Proposition 9. The following holds true.

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(i) The linear operator $\partial_f \widehat{G}(0, 0) : X \to \widehat{Y}$ is an isomorphism and
\[
\partial_f \widehat{G}(0, 0) h(w) = \sum_{n \geq 1} a_n \hat{\gamma}_n e_{n+1}
\]
with
\[
\hat{\gamma}_n = \frac{\alpha C_a \Gamma(1 - \alpha)}{4 \Gamma^2(1 - \frac{3}{2})} \left( \frac{2(1 + n)}{1 - \frac{3}{2}} - \frac{(1 + \frac{3}{2})}{n} - \frac{(1 + \frac{3}{2})_{n+1}}{(1 - \frac{3}{2})_{n+1}} \right).
\]

(ii) There exists $\varepsilon_0 > 0$ such that the set
\[
\{ (\varepsilon, f) \in [-\varepsilon_0, \varepsilon_0] \times B_1, \text{s.t. } \widehat{G}(\varepsilon, f) = 0 \}
\]
is parametrized by one-dimensional curve $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \mapsto (\varepsilon, f_\varepsilon)$ and
\[
\forall \varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}, f_\varepsilon \neq 0.
\]

(iii) If $(\varepsilon, f)$ is a solution then $(-\varepsilon, \tilde{f})$ is also a solution, where
\[
\tilde{f}(w) = f(-w), \quad \forall w \in \mathbb{T}.
\]

(iv) For all $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$, the domain $D^*_\varepsilon$ is strictly convex.

Proof. (i) From the composition rule
\[
\partial_f \widehat{G}(0, 0) h(w) = \partial_\Omega G(0, \Omega_{\text{sing}}, 0) \partial_f \Omega(0, 0) h(w) + \partial_f G(0, \Omega_{\text{sing}}, 0) h(w).
\]
From the formula of $\Omega(\varepsilon, f)$ in Proposition 8 we deduce that
\[
\partial_f \Omega(0, 0) = \frac{d}{d\varepsilon} \Omega(0, \text{th}(w))|_{\varepsilon=0}
\]
and therefore
\[
\partial_f \widehat{G}(0, 0) h(w) = \partial_f G(0, \Omega_{\text{sing}}, 0) h(w).
\]
Combining this identity with Proposition 7 we deduce the desired result.

(ii) As we have seen before $\widehat{G} : -\frac{1}{2}, \frac{1}{2}[\times B_1 \to \widehat{Y}$ is $C^1$ and combining the point (i) with Proposition 7-(iv) we deduce that $\partial_f \widehat{G}(0, 0) : X \to \widehat{Y}$ is an isomorphism. Therefore we can conclude using the Implicit Function Theorem. It remains to check that $f_\varepsilon \neq 0$ for $\varepsilon \neq 0$. For this purpose, we will prove that for any $\varepsilon$ small enough and any $\Omega$ we cannot get a vortex pair with $f = 0$. So, it means that for $\varepsilon \neq 0$ we should get
\[
G(\varepsilon, \Omega, 0) \neq 0.
\]
According to (15), (28), (40) and (43), one finds
\[
G_0(\varepsilon, \Omega, 0) = -\Omega d \text{Im}(\bar{w}) \quad \text{and} \quad G_1(\varepsilon, \Omega, 0) = 0.
\]
To compute $G_2(\varepsilon, 0)$ it is enough to calculate $I_2(\varepsilon, 0)$ because $I_1(\varepsilon, 0) = 0$. The exact computations turns to be much more involved. Thus we shall give the expansion of $I_2(\varepsilon, 0)$ at the order one in $\varepsilon$. Applying (35) one gets
\[
I_2(\varepsilon, 0) = \frac{\alpha C_a}{2(2d)^{1+\alpha}} - \frac{\alpha C_a \varepsilon}{2(2d)^{2+\alpha}} \int_\mathbb{T} |\tau + w|^2 d\tau + \frac{\alpha C_a}{2(2d)^{2+\alpha}} \int_\mathbb{T} \left( \text{Re}[\tau + w] \right)^2 d\tau + \varepsilon^2 O(1)
\]
\[
= \Omega_{\text{sing}} d - \frac{\alpha C_a \varepsilon}{2(2d)^{2+\alpha}} w + \frac{\alpha C_a (2 + \alpha) \varepsilon}{4(2d)^{2+\alpha}} (w + \bar{w}) + \varepsilon^2 O(1),
\]
and so
\[ G_2(\varepsilon, 0) = \text{Im}\{I_2(\varepsilon, 0)\bar{w}\}. \]
Therefore the V-states equations becomes
\[
\text{Im}\left\{ (I_2(\varepsilon, 0) - \Omega d)\bar{w} \right\} = d(\Omega_{\text{sing}} - \Omega)\text{Im}(\bar{w}) + \varepsilon \frac{\alpha C\varepsilon}{4(2d)^{2+\alpha}} \text{Im}(\bar{w}^2) + \varepsilon^2 O(1)
\]
and this equation is impossible for \( 0 < \varepsilon \leq \varepsilon_0 \) with \( \varepsilon_0 \) small enough depending on \( d \) and \( \alpha \).

(iii) By the definition of \( \hat{f} \) one can check that \( R_i(-\varepsilon, \hat{f}) = -R_i(\varepsilon, f) \), for \( i = 0, 1, 2 \). Consequently we obtain that
\[ \Omega(\varepsilon, f) = \Omega(-\varepsilon, \hat{f}). \]
Taking the decomposition of \( G = G_0 - G_1 + G_2 \) given in (15) we only need to check that \( G_i(\varepsilon, \Omega, f)(w) = -G_i(-\varepsilon, \hat{f})(w) \), for \( i = 0, 1, 2 \). Since \( \hat{f}'(w) = -f'(-w) \) we have
\[
G_0(-\varepsilon, \Omega, \hat{f})(w) = \text{Im}\left\{ \Omega(-\varepsilon w + \varepsilon^2 |\varepsilon|^\alpha \hat{f}(w) - d)\bar{w}(1 - \varepsilon |\varepsilon|^\alpha \hat{f}'(w)) \right\}
\]
\[ = -\text{Im}\left\{ \Omega(\varepsilon w - \varepsilon^2 |\varepsilon|^\alpha f(-w) - d)(-\bar{w})(1 + \varepsilon |\varepsilon|^\alpha f'(-w)) \right\}
\]
\[ = -G_0(\varepsilon, \Omega, f)(-w). \]
Straightforward computations will give us the same properties for the functions \( G_1 \) and \( G_2 \). Consequently,
\[ G(\varepsilon, \Omega, f)(w) = -G(-\varepsilon, \Omega, \hat{f})(-w) \]
and therefore \((-\varepsilon, \hat{f})\) defines a curve of solutions for \( 0 < \varepsilon < \varepsilon_0 \). By uniqueness of the one-dimensional curve we deduce that the negative values of \( \varepsilon \) does not lead to new shapes: we get just those obtained for \( \varepsilon > 0 \).

(iv) First we remark that the results and the arguments in Proposition 7 also hold replacing the space \( C^{2-\alpha}(\mathbb{T}) \) by \( C^{n+2-\alpha}(\mathbb{T}) \) for any \( n \in \mathbb{N} \). Therefore the implicit function theorem gives that the function \( f_\varepsilon \) belongs to \( C^{n+2-\alpha} \) for any fixed \( n \). Of course, the size of \( \varepsilon_0 \) is not uniform with respect to \( n \) and it shrinks to zero as \( n \) grows to infinity. To prove the convexity of the domain \( D_1^\varepsilon \), recall that the outside conformal mapping associated to this domain is given by
\[ \phi(w) = w + \varepsilon |\varepsilon|^\alpha f_\varepsilon(w). \]
The domain is strictly convex if and only if
\[
\text{Re}\left( 1 + w \frac{\phi''(w)}{\phi'(w)} \right) = \text{Re}\left( 1 + \varepsilon |\varepsilon|^\alpha w \frac{f''(w)}{1 + \varepsilon |\varepsilon|^\alpha f'(w)} \right)
\]
\[ \geq 1 - |\varepsilon|^{1+\alpha} \frac{|f''(w)|}{1 - |\varepsilon|^{1+\alpha}|f'(w)|} \geq 1 - \frac{|\varepsilon|^{1+\alpha}}{1 - |\varepsilon|^{1+\alpha}} > 0, \]
and the last inequality is satisfied because \( |\varepsilon| < 1/2 \). Thus, the curvature is strictly positive and the domain is strictly convex.

\[ \square \]

7. Counter-rotating vortex pair for the GSQG equations

In this section we shall prove the existence of translating vortex pair for the SQG equations. The proofs will be highly shortened because all the arguments were seen in the preceding sections. Note that the equations defining these V-states are described in the subsection 3.2.
7.1. **Extension and regularity of** $G$. The section is devoted to prove the regularity of the functionals involved in the existence of counter-rotating vortex pairs for the gSQG equations.

**Proposition 10.** The following holds true.

(i) The function $G$ can be extended from $\left[-\frac{1}{2}, \frac{1}{2}\right] \times \mathbb{R} \times B_1 \to Y$ as a $C^1$ function.

(ii) Two initial point vortex $\pi \delta_0$ and $-\pi \delta_{(2d,0)}$ move uniformly in the direction $(Oy)$ with velocity

$$U_{\text{sing}} \equiv \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}.$$

(iii) For $\Omega \in \mathbb{R}$ and $h = \sum_{n \geq 1} a_n w^n \in X$, we have

$$\partial_f G(0, \Omega, 0) h(w) = -\sum_{n \geq 1} a_n \hat{\gamma}_n e_{n+1},$$

with

$$\hat{\gamma}_n = \frac{\alpha C_\alpha \Gamma(1-\alpha)}{4\Gamma^2(1-\frac{n}{2})} \left( \frac{2(1+n)}{1 - \frac{n}{2}} - \frac{(1 + \frac{\alpha}{2})_n}{(1 - \frac{n}{2})_n} - \frac{(1 + \frac{n}{2})_{n+1}}{(1 - \frac{n}{2})_{n+1}} \right).$$

(iv) For any $U \in \mathbb{R}$, the operator $\partial_f G(0, U, 0) : X \to \hat{Y}$ is an isomorphism.

**Proof.** (i) The proof is quite similar to (i) of Proposition 7. The only slight difference is in the treatment of $G_0$ which is clearly differentiable with continuity in $\varepsilon$ and moreover is a polynomial in $\Omega, f, f'$, and so its derivatives in this variables are also continuous. Note that $G_1$ and $G_2$ are the same functions that in the rotating case.

(ii) Observe that

$G_0(0, U, 0) = U e_1$.

According to (37) one finds

$G_2(0, 0) = -\frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} e_1$.

Using (38), (40) and (43) we obtain

$G_1(0, 0) = 0$.

Therefore we get

$G(0, U, 0) = \left(U - \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}\right) e_1$

and consequently $G(0, U, 0) = 0$ if and only if

$U = U_{\text{sing}} = \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}}$.

(iii) and(iv). It is easy to see that

$\partial_f G_0(0, U, 0) = 0$,

and so

$$\partial_f G(0, U, 0) = \partial_f G_0(0, U, 0) + \partial_f G_1(0, 0) + \partial_f G_2(0, 0)$$

$$= \partial_f G_1(0, 0) + \partial_f G_2(0, 0).$$

On the other hand by (34) $\partial_f G_2(0, 0) = 0$, and so this operator coincides, after a change of sign, with the linearized operator in the rotating case and whose formula was stated in Proposition 7. \qed
7.2. Relationship between the speed and the boundary. As for the rotating case the image of $X$ by $G(\varepsilon, U, \cdot)$ is contained in $Y$ and not necessary in $\hat{Y}$. Therefore to apply implicit function theorem we should impose a constraint between $U, \varepsilon$ and $f$. The main goal of this section is to establish the following result

**Proposition 11.** There exists a $C^1$ function $\mathcal{R} : \mathbb{R} \rightarrow \mathbb{R}$ such that with the choice

$$U = U(\varepsilon, f) = \frac{\alpha C_\alpha}{(2d)^{2+\alpha}} + \mathcal{R}(\varepsilon, f),$$

the function $\hat{G} : \mathbb{R} \rightarrow \hat{Y}$ given by

$$\hat{G}(\varepsilon, f) = G(\varepsilon, U(\varepsilon, f), f)$$

is well-defined and is $C^1$. Moreover,

$$\forall f \in B_1, \mathcal{R}(0, f) = 0 \quad \text{and} \quad \hat{G}(0, 0) = 0.$$

**Proof.** It will happen if the first Fourier coefficient of $G$ (the component of $e_1$) is zero. From (22) we can deduce that the constraint is given by

$$\int T G(\varepsilon, U, f(w))dw = 0.$$ 

From the decomposition (22) this assumption is equivalent to

$$A_0 = -A_1 - A_2,$$

with

$$A_j = -2i \int T G_j(\varepsilon, \Omega, f(w))dw.$$

Note that $A_j$ is the Fourier coefficient of $e_1 = \text{Im}(w)$ in $G_j$ and when $G_j = \text{Im}(F_j)$ then

$$A_j = \int T F_j(\varepsilon, \Omega, f(w))((\overline{w}^2 - 1)dw.$$

The computation of $A_0$ is easy,

$$A_0 = -U \int T (1 + \varepsilon|\varepsilon|^\alpha f'(\overline{w}))(\overline{w}^3 - \overline{w})dw.$$

Since

$$f(w) = \sum_{n \geq 1} a_n \overline{w}^n \quad \text{and} \quad f'(w) = -\sum_{n \geq 1} na_n \overline{w}^{n+1},$$

then

$$\int T f'(\overline{w})(\overline{w}^3 - \overline{w})dw = a_1$$

$$= -\int T f(\tau)d\tau.$$ 

Consequently

$$A_0 = U \left(1 + \varepsilon|\varepsilon|^\alpha \int T f(\tau)d\tau\right).$$

Combining (61) with (62), (58) and (59) one has

$$U \left(1 + \varepsilon|\varepsilon|^\alpha \int T f(\tau)d\tau\right) = \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \varepsilon R_2(\varepsilon, f) - \varepsilon|\varepsilon|^\alpha R_1(\varepsilon, f).$$
Thus
\[
U = \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + \varepsilon R_2(\varepsilon, f) - \varepsilon|\varepsilon|^\alpha R_1(\varepsilon, f)
\]
\[
\equiv \frac{\alpha C_\alpha}{2(2d)^{1+\alpha}} + R(\varepsilon, f).
\]

Similarly to the rotating case, \( R : ]-\frac{1}{2}, \frac{1}{2}[ \times B_1 \rightarrow \mathbb{R} \) is well-defined and \( C^1 \). On the other hand, by construction of the function one can see that \( R(0, f) = 0 \). Moreover
\[
G_0(0, U, 0) = -Ue_1, \quad G_1(0, 0) = 0 \quad \text{and} \quad G_2(0, 0) = \frac{\alpha C_\alpha}{2(2d)^{2+\alpha}}e_1,
\]
and consequently \( \hat{G}(0, 0) = 0 \).

\[ \Box \]

7.3. **Existence of counter-rotating vortex pairs.** Recall that the V-states equation can be written in the form
\[
\hat{G}(\varepsilon, f) = 0, \quad (\varepsilon, f) \in ]-1/2, 1/2[ \times B_1,
\]
with \( \hat{G} \) being the functional defined by
\[
\hat{G}(\varepsilon, f(w)) = G(\varepsilon, U(\varepsilon, f), f).
\]
The main result whose proof is quite similar to that of Proposition 9 and left to the reader is the following.

**Proposition 12.** The following holds true.

(i) The linear operator \( \partial_f \hat{G}(0, 0) : X \rightarrow \hat{Y} \) is an isomorphism and
\[
\partial_f \hat{G}(0, 0)h(w) = -\sum_{n \geq 1} a_n \gamma_n e_{n+1}
\]
with
\[
\gamma_n = \frac{\alpha C_\alpha \Gamma(1-\alpha)}{4 \Gamma^2(1-\alpha/2)} \left( \frac{2(1+n)}{1-\alpha/2} - \frac{(1+\frac{n}{2})}{(1-\frac{n}{2})} - \frac{(1+\frac{n}{2})}{(1-\frac{n}{2})} \right).
\]

(ii) There exists \( \varepsilon_0 > 0 \) such that the set
\[
\{ (\varepsilon, f) \in [-\varepsilon_0, \varepsilon_0] \times B_1 \text{ s.t. } \hat{G}(\varepsilon, f) = 0 \}
\]
is parametrized by one-dimensional curve \( \varepsilon \in [-\varepsilon_0, \varepsilon_0] \mapsto (\varepsilon, f_\varepsilon) \) and
\[
\forall \varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}, f_\varepsilon \neq 0.
\]

(iii) If \( (\varepsilon, f) \) is a solution then \( (-\varepsilon, f_\varepsilon) \) is also a solution, where
\[
\tilde{f}(w) = f(-w), \quad \forall w \in \mathbb{T}.
\]

(iv) For all \( \varepsilon \in [-\varepsilon_0, \varepsilon_0] \), the domain \( D_1^\varepsilon \) is strictly convex.

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