On one-point extensions of cluster-tilted algebras

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Abstract

We define an operation which associates to a pair \((B, M)\) where \(B\) is a cluster-tilted algebra and \(M\) is a \(B\)-module which lies in a local slice of \(B\), a new cluster-tilted algebra \(B'\). In terms of the quivers, this operation corresponds to adding one vertex (and arrows).

1 Introduction

Cluster-tilted algebras are finite dimensional associative algebras which were introduced in the context of categorifications of cluster algebras in [14,17]. Since then, these algebras have been the subject of many studies, see for example [3,6,8,11,13,15,18,20].

A striking property shown in [16, Theorem 2.13] is that deleting a vertex of a cluster-tilted algebra produces again a cluster-tilted algebra, more precisely:

If \(B\) be a cluster-tilted algebra with quiver \(Q_B\) and \(e_x\) is a primitive idempotent corresponding to a vertex \(x\) of \(Q_B\), then \(B/Be_xB\) is cluster-tilted.

Note that the quiver of \(B/Be_xB\) is obtained from \(Q_B\) be deleting the vertex \(x\) and all arrows incident to it.

The goal of this paper is to give a construction which is inverse to the deletion of vertex. Given a cluster-tilted algebra \(B\) with quiver \(Q_B\) we want to construct a cluster-tilted algebra \(B'\) whose quiver contains \(Q_B\) as a full subquiver and has one additional vertex \(x\), the extension vertex, such that \(B = B'/B'e_xB'\).

Our first construction is the one-point extension \(B[P]\) of the algebra \(B\) with respect to a projective \(B\)-module \(P\). We prove that \(B[P]\) is cluster-tilted if there exists a local slice \(\Sigma\) in \(\text{mod} \ B\) that contains \(P\), see Theorem 3.6. In this situation, the extension vertex is a source in the quiver of \(B[P]\). We also give an example which shows that the condition that \(P\) lies on a local slice is necessary, see Example 3.13.

Dually, the one-point coextensions of \(B\) with respect to an injective module is cluster-tilted if the injective module lies in a local slice. In this situation, the extension vertex is a sink.

Our second construction yields cluster-tilted algebras in which the extension vertex can have both incoming and outgoing arrows. Given any \(B\)-module \(M\) lying on a local slice \(\Sigma\) in \(\text{mod} \ B\), let \(C = B/\text{Ann}\Sigma\) be the quotient of \(B\) by the
annihilator of the local slice. It is known that $C$ is a tilted algebra, see [4]. Let $C[M]$ be its one-point extension with respect to $M$, and let $B'$ be the relation extension of $C[M]$, that is

$$B' = C[M] \ltimes \text{Ext}_{C[M]}(DC[M], C[M]).$$

Then we show that $B'$ cluster-tilted such that $B'/B'e_B = B$, see Theorem 3.9.

Along the way, we prove that for arbitrary algebras of global dimension 2, the operation of one-point extension with respect to a projective module and the operation of relation extension commute, see Theorem 3.1.

2 Preliminaries

Let $k$ be an algebraically closed field. The algebras in this paper are always finite dimensional, basic, associative $k$-algebras and the modules are always finitely generated. If $\Lambda$ is a $k$-algebra, we denote by $\text{mod}\Lambda$ the category of finitely generated right $\Lambda$-modules, by $\Gamma(\text{mod}\Lambda)$ its Auslander-Reiten quiver and by $\tau_\Lambda$ its Auslander-Reiten translation. Furthermore, $Q_\Lambda$ will denote the ordinary quiver of $\Lambda$, and $P_\Lambda(i), I_\Lambda(i)$ and $S_\Lambda(i)$ the indecomposable projective, injective and simple $\Lambda$-module at the vertex $i$ of $Q_\Lambda$, respectively. Throughout the article, we use the notation $aM$ for the direct sum of $a$ copies of the module $M$. For the representation theory of $k$-algebras, we refer to [9,10].

2.1 Slices and local slices

A path in $\text{mod}\Lambda$ with source $X$ and target $Y$ is a sequence of non-zero morphisms $X = X_0 \to X_1 \to \cdots \to X_s = Y$ where $X_i \in \text{mod}\Lambda$ for all $i$, and $s \geq 1$. A path in $\Gamma(\text{mod}\Lambda)$ with source $X$ and target $Y$ is a sequence of arrows $X = X_0 \to X_1 \to \cdots \to X_s = Y$ in the Auslander-Reiten quiver. A sectional path is a path $X = X_0 \to X_1 \to \cdots \to X_s = Y$ in $\Gamma(\text{mod}\Lambda)$ such that for each $i$ with $0 < i < s$, we have $\tau_\Lambda X_{i+1} \neq X_{i-1}$.

A slice $\Sigma$ in $\Gamma(\text{mod}\Lambda)$ is a set of indecomposable $\Lambda$-modules such that

(S1) $\Sigma$ is sincere.

(S2) Any path in $\text{mod}\Lambda$ with source and target in $\Sigma$ consists entirely of modules in $\Sigma$.

(S3) If $M$ is an indecomposable non-projective $\Lambda$-module then at most one of $M, \tau_\Lambda M$ belong to $\Sigma$.

(S4) If $M \to S$ is an irreducible morphism with $M$ and $S$ indecomposable and $S \in \Sigma$, then either $M$ belongs to $\Sigma$ or $M$ is non-injective and $\tau_\Lambda^{-1}M$ belongs to $\Sigma$.

A local slice $\Sigma$ in $\Gamma(\text{mod}\Lambda)$ is a set of indecomposable $\Lambda$-modules inducing a connected full subquiver of $\Gamma(\text{mod}\Lambda)$ such that
(LS1) If $X \in \Sigma$ and $X \to Y$ is an arrow in $\Gamma(\text{mod } \Lambda)$ then either $Y \in \Sigma$ or $\tau Y \in \Sigma$.

(LS2) If $Y \in \Sigma$ and $X \to Y$ is an arrow in $\Gamma(\text{mod } \Lambda)$ then either $X \in \Sigma$ or $\tau^{-1} X \in \Sigma$.

(LS3) For every sectional path $X = X_0 \to X_1 \to \cdots \to X_s = Y$ in $\Gamma(\text{mod } \Lambda)$ with $X$ and $Y$ in $\Sigma$ we have $X_i \in \Sigma$, for $i = 0, 1, \ldots, s$.

(LS4) The number of indecomposables in $\Sigma$ equals the number of isoclasses of simple $\Lambda$-modules.

2.2 Sections and left sections

Let $\Lambda$ be a $k$-algebra and $\Gamma$ be a connected component of the Auslander-Reiten quiver $\Gamma(\text{mod } \Lambda)$. A full connected subquiver $\Sigma$ of $\Gamma$ is called a section if

1. $\Sigma$ has no oriented cycles.
2. $\Sigma$ intersects every $\tau_\Lambda$-orbit in $\Gamma$ exactly once.
3. Each path in $\Gamma$ with source and target in $\Sigma$ lies entirely in $\Sigma$.

A full connected subquiver $\Sigma$ of $\Gamma$ is called a left section if it satisfies condition (s1) and (s3) above as well as the following condition.

(s2') For any indecomposable $X$ in $\Gamma$ such that there exists an indecomposable $Y$ in $\Sigma$ and a path in $\Gamma$ from $X$ to $Y$, there exists a unique $m \in \mathbb{Z}_{\geq 0}$ such that $\tau_\Lambda^{-m} X \in \Sigma$.

2.3 Tilted algebras

Let $A$ be a hereditary $k$-algebra and let $n$ be the number of isoclasses of simple modules in $\text{mod } A$. An $A$-module $T$ is called a tilting module if $\text{Ext}^1_A(T, T) = 0$, and $T$ is the direct sum of $n$ non-isomorphic indecomposable $A$-modules. The corresponding endomorphism algebra $\text{End}_A T$ is called a tilted algebra.

Tilted algebras can be characterized by the existence of a slice as follows.

Theorem 2.1 [22, 4.2.3] Let $C = \text{End}_A T$ be a tilted algebra. Then the class of $C$-modules $\text{Hom}_A(T, I)$ such that $I$ is an indecomposable injective $A$-module forms a slice in $\text{mod } C$. Conversely, any slice in any module category is obtained in this way.

If $C$ is a tilted algebra and $M$ is a $C$-module lying on a slice in $\Gamma(\text{mod } C)$ then the projective dimension and the injective dimension of $M$ are at most 1. Tilted algebras have global dimension at most 2.

There is another characterization of tilted algebras in terms of sections due to Liu [21] and Skowronsky [24]. We will need the following result which has been shown by Assem.
Theorem 2.2 [2, Theorem A] Let $C$ be a $k$-algebra and let $\Sigma$ be a left section in a component $\Gamma$ of $\Gamma(\text{mod } C)$ such that

$$\text{Hom}_C(\tau_C^{-1}E', E'') = 0,$$

for all $E', E'' \in \Sigma$. Then $C/\text{Ann}_C \Sigma$ is a tilted algebra having $\Sigma$ as a slice.

For further information on tilted algebras we refer the reader to [9].

2.4 Relation extensions and cluster-tilted algebras

Following [3], we make the following definitions. If $C$ is an algebra of global dimension at most 2 then its relation extension $R(C)$ is defined as

$$R(C) = C \rtimes \text{Ext}^2_C(DC, C),$$

where $D = \text{Hom}(-, k)$ denotes the standard duality. The quiver $Q_{R(C)}$ has the same vertices as the quiver $Q_C$, and the arrows in $Q_{R(C)}$ are the arrows in $Q_C$ plus one new arrow $i \rightarrow j$ for each relation from $j$ to $i$ in a minimal system of relations for $C$, see [3]. Moreover, the dimension of $\text{Ext}^2_C(I_C(j), P_C(i))$ is equal to the number of nonzero paths from $i$ to $j$ in the quiver $Q_{R(C)}$ that use a new arrow. The paths in $Q_{R(C)}$ corresponding to $\text{Ext}^2_C(DC, C)$ are called the new paths in the relation extension $R(C)$.

An algebra is called a cluster-tilted algebra if it is the relation extension of a tilted algebra.

Remark 2.3 Originally, cluster-tilted algebras were introduced as bound quiver algebras associated to triangulations of polygons in [17], and as endomorphism algebras of cluster-tilting objects in cluster categories in [14].

Remark 2.4 Relation extensions which are not necessarily cluster-tilted have been studied in [1, 11], and a generalization of the construction is given in [7].

The relation between a tilted algebra and its cluster-tilted algebra naturally provides a strong connection between the corresponding module categories. For example, a slice in the module category of a tilted algebra embeds as a local slice in the module category of the corresponding cluster-tilted algebra.

Theorem 2.5 [3] Let $B$ be a cluster-tilted algebra. Then

1. $B$ admits a local slice.

2. For each local slice $\Sigma$ in $\Gamma(\text{mod } B)$, the algebra $C = B/\text{Ann}_B \Sigma$ is tilted with slice $\Sigma$ and $B = R(C)$.

3. For every tilted algebra $C$ such that $B = R(C)$ there is a local slice $\Sigma$ such that $C = B/\text{Ann}_B \Sigma$. 

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2.5 One-point extensions

Let $\Lambda$ be a $k$-algebra and $M \in \text{mod} \Lambda$. Then the one-point extension of $\Lambda$ by $M$ is the triangular matrix algebra

$$\Lambda[M] = \begin{bmatrix} \Lambda & 0 \\ M & k \end{bmatrix},$$

whose elements are of the form $$\begin{bmatrix} a & 0 \\ m & \mu \end{bmatrix},$$

with $a \in \Lambda, m \in M$ and $\mu \in k$; and its multiplication is given by

$$\begin{bmatrix} a & 0 \\ m & \mu \end{bmatrix} \begin{bmatrix} a' & 0 \\ m' & \mu' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ ma' + \mu m' & \mu \mu' \end{bmatrix}$$

using the right $\Lambda$-module structure of $M$ as well as its $k$-vector space structure.

The quiver $Q_{\Lambda}$ is a full subquiver of $Q_{\Lambda[M]}$, and $Q_{\Lambda[M]}$ has exactly one vertex more than $Q_{\Lambda}$, the extension vertex. Moreover the extension vertex is a source in $Q_{\Lambda[M]}$ and the radical of the indecomposable projective $\Lambda[M]$-module at the extension vertex is equal to $M$. The arrows in $Q_{\Lambda[M]}$ starting at the extension vertex are called extension arrows.

In this article, we shall be interested in the special case where the module $M$ is a projective $\Lambda$-module. In this case, the relation ideal of the one-point extension $\Lambda[M]$ can be identified with the relation ideal of $\Lambda$, more precisely, if $\Lambda = kQ_{\Lambda}/I$ where $I$ is an admissible ideal generated by a set of paths in $Q_{\Lambda}$ then $\Lambda[M] = kQ_{\Lambda[M]}/I'$ where $I'$ is the admissible ideal in $kQ_{\Lambda[M]}$ generated by the same set of paths. Moreover, for each indecomposable direct summand $P_{C}(j)$ of the projective module $M$, there is precisely one extension arrow from the extension vertex to the vertex $j$, counting multiplicities.

3 Main results

3.1 Relation extensions and one-point extensions

Throughout this subsection, $C$ is a $k$-algebra of global dimension at most 2, $P$ is a projective $C$-module, and we consider the one-point extension $C[P]$. We use the notation $1, 2, \ldots, n$ for the vertices of the quiver of $C$, and we use $n + 1$ to denote the extension vertex of $C[P]$. To simplify the notation, we denote the indecomposable projective (respectively injective) $C[P]$-modules simply by $P_{C}(i)$ (respectively $I_{C}(i)$). For the indecomposable projective (respectively injective) $C$-modules we continue to write $P_{C}(i)$ (respectively $I_{C}(i)$).

Let $a_{j}$ be the multiplicity of the indecomposable projective $C$-module $P_{C}(j)$ at the vertex $j$ in the direct sum decomposition of $P$, thus

$$P = \bigoplus_{j \in J} a_{j} P_{C}(j).$$

Now consider the relation extension $R(C)$ of $C$ and let $\overline{P}$ be the projective $R(C)$-module whose indecomposable summands $P_{R(C)}(j)$ have the same multiplicity
\[ \mathcal{P} = \bigoplus_{j \in J} a_j P_{R(C)}(j). \]

**Theorem 3.1** Let \( C \) be a \( k \)-algebra of global dimension at most 2, and let \( P \) be a projective \( C \)-module. Let \( \mathcal{P} \) be the corresponding projective \( R(C) \)-module defined above. Then there exists an isomorphism of \( k \)-algebras
\[ R(C[P]) \cong R(C)[\mathcal{P}]. \]

**Remark 3.2** The idea of the proof is the following. It has been shown in [3] that if \( R(C) \) is the relation extension of an algebra \( C \) of global dimension at most 2, then the set of new paths in the quiver of \( R(C) \) from a vertex \( x \) to a vertex \( y \) is given by a basis of \( \text{Ext}^2_C(I_C(y), P_C(x)) \). Since the one-point extensions in the theorem are with respect to projective modules, the extension vertex is not involved in any relations, and, consequently, the one-point extension commutes with the relation extension.

Before proving the Theorem 3.1 we need the following lemmas.

**Lemma 3.3** There is an isomorphism of right \( C \)-modules
\[ \mathcal{P} \cong P \oplus \text{Ext}^2_C(DC, P). \]

**Proof.** In \( \text{mod } R(C) \), there is a short exact sequence of the form
\[ 0 \to \text{Ext}^2_C(DC, P) \to \mathcal{P} \to P \to 0. \]
This shows that there is a \( k \) vector space isomorphism \( \mathcal{P} \cong P \oplus \text{Ext}^2_C(DC, P) \), which is also a \( C \)-module isomorphism. \( \square \)

**Lemma 3.4** 1. There is an isomorphism of \( C \)-bimodules
\[ \psi : \text{Ext}^2_{C[P]}(DC[P], C) \to \text{Ext}^2_C(DC, C) \]
acting trivially on the new paths in \( R(C[P]) \) which do not start at the extension vertex \( n + 1 \).

2. There is an isomorphism of \( C \)-bimodules
\[ \chi : \text{Ext}^2_{C[P]}(DC[P], P(n + 1)) \to \text{Ext}^2_C(DC, P) \]
acting on the new paths in \( R(C[P]) \) which start at the extension vertex \( n + 1 \) by deleting the initial arrow.

3. For all \( m \in P_c \) and all \( g \in \text{Ext}^2_{C[P]}(DC[P], C) \), we have
\[ m\psi(g) = \chi(mg). \]
Proof. First note that $I(n+1)$ is a simple module and that
\[ 0 \to P \to P(n+1) \to I(n+1) \to 0 \quad (1) \]
is a projective resolution in mod $C[P]$, and thus, $I(n+1)$ has projective dimension 1, whence $\text{Ext}^2_{C[P]}(I(n+1), C) = 0$.

Now let $i$ be a vertex different from $n+1$. Then the injective $C$-module $I_C(i)$ considered as a $C[P]$-module is a submodule of the corresponding injective $C[P]$-module $I(i)$, and the quotient is a direct sum of copies of the simple injective module $I(n+1)$. Thus we have a short exact sequence in mod $C[P]$ of the form
\[ 0 \to I_C(i) \to I(i) \to mI(n+1) \to 0 \]
for some $m \in \mathbb{Z}$.

Applying $\text{Hom}_{C[P]}(-, C)$ to this sequence and using the fact that $\text{pd } I(n+1) = 1$ yields an isomorphism
\[ \text{Ext}^2_{C[P]}(I(i), C) \to \text{Ext}^2_{C[P]}(I_C(i), C). \quad (2) \]
Moreover, given a minimal projective resolution of $I_C(i)$ in mod $C$ one obtains a minimal projective resolution of $I_C(i)$ in mod $C[P]$ simply by extending scalars, because the extension vertex $n+1$ is a source in the quiver of $C[P]$, which is not in the support of $I_C(i)$. Thus there is an isomorphism of left $C$-modules
\[ \text{Ext}^2_{C[P]}(I_C(i), C) \to \text{Ext}^2_C(I_C(i), C). \]
Composing this isomorphism with the isomorphisms in (2), we obtain an isomorphism of left $C$-modules
\[ \text{Ext}^2_{C[P]}(I(i), C) \to \text{Ext}^2_C(I_C(i), C), \]
which sends a new path in $R(C[P])$ with terminal point $i$ and starting point different from $n+1$ to the same path in $R(C)$. This isomorphism induces an isomorphism $\psi$ of the left $C$-modules
\[ \text{Ext}^2_{C[P]}(DC[P], C) \to \text{Ext}^2_C(DC, C). \]

For the right $C$-module structure, let
\[ 0 \to P_C(i) \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1 \xrightarrow{f_2} I_2 \to 0 \]
be a minimal injective resolution in mod $C$. The indecomposable components of the maps $f_i$ are given by comultiplication of paths in $Q_C$. Then, since $n+1$ is a source in $Q_C[P]$, there is a minimal injective resolution in mod $C[P]$ of the form
\[ 0 \to P_C(i) \xrightarrow{f_0} I_0 \xrightarrow{f_i} I_1 \oplus aI(n+1) \xrightarrow{f_2} I_2 \to 0 \]
where each indecomposable injective $C[P]$-module $I(j)$ appears in the direct sum decomposition of $I_h$, $h = 0, 1, 2$, with the same multiplicity as the corresponding indecomposable injective $C$-module $I_C(j)$ in $I_h$, and each indecomposable
component of \( f_h \) is given by the comultiplication with the same path as the corresponding indecomposable component of the map \( f_h \). It follows that \( \psi \) is also an isomorphism of right \( C \)-modules. This shows 1.

Since the projective dimension of \( I(n + 1) \) is 1, we have \( \text{Ext}^2_{C[P]}(I(n + 1), P(n + 1)) = 0 \). Moreover, by a similar argument as in the proof of 1, we get an isomorphism

\[
\text{Ext}^2_{C[P]}(I(i), P(n + 1)) \longrightarrow \text{Ext}^2_{C[P]}(I_C(i), P(n + 1)),
\]

for all \( i \neq n + 1 \).

Applying the functor \( \text{Hom}_{C[P]}(I_C(i), -) \) to the short exact sequence (1) and using the fact that \( I(n + 1) \) is injective yields an isomorphism

\[
\text{Ext}^2_{C[P]}(I_C(i), P) \longrightarrow \text{Ext}^2_{C[P]}(I_C(i), P(n + 1)),
\]

which sends a new path \( w \) of \( R(C[P]) \) starting at a vertex \( j \) which corresponds to an indecomposable summand \( P_C(j) \) of \( P \), to the new path \( \alpha w \) in \( R(C[P]) \) corresponding to the indecomposable summand \( P_C(j) \) of \( P \).

By the same argument as in the proof of 1, there is an isomorphism of vector spaces

\[
\text{Ext}^2_{C[P]}(I_C(i), P) \longrightarrow \text{Ext}^2_{C[P]}(I_C(i), P),
\]

and composing the three isomorphisms yields an isomorphism of vector spaces

\[
\chi : \text{Ext}^2_{C[P]}(DC[P], P(n + 1)) \longrightarrow \text{Ext}^2_{C[P]}(DC, P)
\]

which acts on the new paths in \( R(C[P]) \) that start at the extension vertex \( n + 1 \) by deleting the initial arrow. Finally, by the same argument as in the proof of 1, we see that \( \chi \) is an isomorphism of right \( C \)-modules. This shows 2.

It remains to show 3. For the action of \( m \) on \( g \), we must consider \( m \in C[P] \) as an element of \( P = \text{rad} P(n + 1) \), thus \( m \) is a linear combination of non-constant paths starting at \( n + 1 \). Then \( mg \) is given by composing the paths in \( m \) and \( g \). Applying \( \chi \) to \( mg \) means deleting the initial arrow from each of these paths, so that the new starting point is in the top of \( P \).

On the other hand, for the action of \( m \) on \( \psi(g) \), we must consider \( m \in C \) as an element of the projective \( C \)-module \( P \), thus \( m \) is given by a linear combination of paths in \( Q_C \), which are obtained by deleting the initial arrow from each of the paths in the expression for \( m \in \text{rad} P(n + 1) \) discussed above. The product \( m\psi(g) \) is then given by the composition of the paths in \( m \) and \( \psi(g) \). This shows 3. \( \square \)

Proof of Theorem 3.1. With the notation of Lemmas 3.3 and 3.4, we define a map

\[
\phi : R(C[P]) \longrightarrow R(C)[\overline{\mathcal{P}}_{R(C)}]
\]

by

\[
\phi \left( \begin{bmatrix} c & 0 \\ m & \mu \end{bmatrix}, \begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix} \right) = \begin{bmatrix} (c, \psi(g)) & 0 \\ (m, \chi(h)) & \mu \end{bmatrix}
\]

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where \( c \in C, m \in P, \mu \in k \), so \( \begin{bmatrix} c & 0 \\ m & \mu \end{bmatrix} \) is an element of \( C[P] \), and \( g \in \text{Ext}^2_{C[P]}(DC[P], C), \ h \in \text{Ext}^2_{C[P]}(DC[P], P(n+1)) \), and the two zeros in the matrix \( \begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix} \) correspond to \( \text{Ext}^2_{C[P]}(I(n+1), C[P]) = 0 \). The map \( \phi \) is clearly bijective and the following computation shows that \( \phi \) is a homomorphism of \( k \)-algebras.

First note that

\[
\begin{bmatrix} c & 0 \\ m & \mu \end{bmatrix} \begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix} = \begin{bmatrix} cg' + gc' & 0 \\ mg' + \mu h' + hc' & 0 \end{bmatrix}.
\]

Applying \( \phi \) to this expression yields

\[
\begin{bmatrix} cc' & 0 \\ mc' + \mu m' & \mu \mu' \end{bmatrix} \begin{bmatrix} cg' + gc' & 0 \\ mg' + \mu h' + hc' & 0 \end{bmatrix}.
\]

On the other hand,

\[
\phi \left( \begin{bmatrix} c & 0 \\ m & \mu \end{bmatrix} \begin{bmatrix} g & 0 \\ h & 0 \end{bmatrix} \right) \phi \left( \begin{bmatrix} c' & 0 \\ m' & \mu' \end{bmatrix} \begin{bmatrix} g' & 0 \\ h' & 0 \end{bmatrix} \right)
\]

is equal to

\[
\begin{bmatrix} (cc', \psi(g) + \psi(g')) & 0 \\ (mc' + \mu m', \chi(mg' + \mu h' + hc')) & \mu \mu' \end{bmatrix}
\]

and the result follows from Lemma 3.4.

\[\Box\]

### 3.2 One-point extensions of cluster-tilted algebras

We now study the question when the one-point extension of a cluster-tilted algebra is again cluster-tilted. The following lemma seems to be well-known; we provide a proof for the convenience of the reader.

**Lemma 3.5** Let \( C \) be a tilted algebra with slice \( \Sigma \) and let \( M \) be a module in \( \Sigma \). Then the one-point extension \( C[M] \) is tilted with slice \( \Sigma' = \Sigma \cup P(n+1) \).

**Proof.** Let \( \{1, 2, \ldots, n\} \) be the set of vertices in \( Q_C \) and let \( n+1 \) denote the extension vertex in \( Q_{C[M]} \). The full subquivers in both Auslander-Reiten quivers \( \Gamma(\text{mod } C) \) and \( \Gamma(\text{mod } C[M]) \), whose points are the predecessors of \( \Sigma \), are equal. In \( \Gamma(\text{mod } C[M]) \), the new indecomposable projective \( C[M] \)-module \( P(n+1) \) lies on a new \( \tau_{C[M]} \)-orbit with one arrow \( M' \to P(n+1) \) for each indecomposable summand \( M' \) of \( M \), counting multiplicities. Therefore, the full subquiver given
by \( \Sigma' = \Sigma \cup \{ P(n+1) \} \) satisfies the condition (s1) and (s2') in the definition of a left section.

To show that \( \Sigma' \) satisfies condition (s3), observe that any non-constant path in \( \Gamma(\text{mod } C[P]) \) that ends in \( P(n+1) \) must pass through one of its immediate predecessors, which are given by the indecomposable summands of \( M \), because \( M \) is the radical of \( P(n+1) \). Since \( M \) lies in \( \Sigma \), it follows that every path in \( \Gamma(\text{mod } C[M]) \) that ends in \( P(n+1) \) restricts to a path that ends in a point in \( \Sigma \). On the other hand, every path in \( \Gamma(\text{mod } C[M]) \) that ends in a point in \( \Sigma \) is actually a path in \( \Gamma(\text{mod } C) \). Hence the condition (s3) for \( \Sigma' \) in \( \Gamma(\text{mod } C[M]) \) follows from the condition (s3) for \( \Sigma \) in \( \Gamma(\text{mod } C) \). This shows that \( \Sigma' \) is a left section.

Our next goal is to show that \( \text{Hom}_{C[M]}(\tau_{C[M]}^{-1}E', E'') = 0 \) for all \( E', E'' \in \Sigma' \). Clearly it suffices to check this property for indecomposable modules \( E', E'' \). Let us suppose first that \( \text{Hom}_{C}(M, E') = 0 \). In that case it follows from [23, Corollary XV 1.7] that \( \tau_{C[M]}^{-1}E' = \tau_{C}^{-1}E' \). Thus if \( E'' \in \Sigma \) then \( \text{Hom}_{C[M]}(\tau_{C[M]}^{-1}E', E'') = 0 \) because \( \Sigma \) is a slice in \( \text{mod } C \). On the other hand, if \( E'' = P(n+1) \) then any non-zero morphism in \( \text{Hom}_{C[M]}(\tau_{C[M]}^{-1}E', E'') \) would factor through \( M = \text{rad } P(n+1) \in \Sigma \) and, again, we conclude that \( \text{Hom}_{C[M]}(\tau_{C[M]}^{-1}E', E'') = 0 \).

Now suppose that \( \text{Hom}_{C}(M, E') \neq 0 \). It follows from the construction of \( \Sigma' \) that \( E' \) is either (i) a direct summand of \( M \) or (ii) \( E' = P(n+1) \). Suppose first that \( E' \) is a summand of \( M \). Without loss of generality, we may assume that \( M \) has no injective summands. Consider the following almost split sequence in \( \text{mod } C[M] \)

\[
0 \rightarrow M \rightarrow P(n+1) \oplus X \rightarrow \tau_{C[M]}^{-1}M \rightarrow 0.
\]

Then \( X \) is a \( C \)-module, and we have the following almost split sequence in \( \text{mod } C \)

\[
0 \rightarrow M \rightarrow X \rightarrow \tau_{C}^{-1}M \rightarrow 0.
\]

If \( E'' \in \Sigma \), then applying \( \text{Hom}_{C}(-, E'') \) to the exact sequence \( \langle 3 \rangle \) shows that \( \text{Hom}_{C}(X, E'') \cong \text{Hom}_{C}(M, E'') \). Moreover, \( \text{Hom}(P(n+1), E'') = 0 \), since \( E'' \) is not supported on the extension vertex \( n+1 \). Therefore, applying \( \text{Hom}(-, E'') \) to the exact sequence \( \langle 3 \rangle \) yields an exact sequence

\[
0 \rightarrow \text{Hom}(\tau_{C[M]}^{-1}M, E'') \rightarrow \text{Hom}(X, E'') \xrightarrow{g} \text{Hom}(M, E''),
\]

with \( g \) an isomorphism. Consequently, \( \text{Hom}(\tau_{C[M]}^{-1}M, E'') = 0 \) if \( E'' \in \Sigma \).

On the other hand, if \( E'' = P(n+1) \) then \( \text{Hom}(X, E'') = 0 \). Therefore applying \( \text{Hom}(-, E'') \) to the exact sequence \( \langle 3 \rangle \) yields an exact sequence

\[
0 \rightarrow \text{Hom}(\tau_{C[M]}^{-1}M, E'') \rightarrow \text{Hom}(P(n+1), E'') \xrightarrow{g'} \text{Hom}(M, E''),
\]

where the morphism \( g' \) is injective, since \( M \) is the radical of \( E'' = P(n+1) \), and, again, we conclude that \( \text{Hom}(\tau_{C[M]}^{-1}M, E'') = 0 \).

It remains the case where \( E' = P(n+1) \). Since \( M \) lies in the slice \( \Sigma \) in \( \text{mod } C \), we have \( \text{id}_{C}M \leq 1 \). Let

\[
0 \rightarrow M \rightarrow I_{C}^{0}(M) \rightarrow I_{C}^{1}(M) \rightarrow 0
\]
be a minimal injective resolution of $M$ in $\text{mod} \ C$. Then we have a minimal injective resolution in $\text{mod} \ C[M]$ of the form

$$0 \to M \to I^0(M) \to I^1(M) \oplus mI(n+1) \to 0$$

where $m \geq 0$ and the multiplicity of each indecomposable injective $C[M]$-module $I(j)$ at $j \neq n+1$ in $I^0(M)$ (respectively $I^1(M)$) is equal to the multiplicity of the indecomposable injective $C$-module $I_C(j)$ in $I^0_C(M)$ (respectively $I^1_C(M)$).

In particular, $\text{id}_{C[M]} M \leq 1$. Using the horseshoe lemma on the short exact sequence

$$0 \to M \to P(n+1) \to I(n+1) \to 0$$

we conclude that $\text{id}_{C[M]} P(n+1) \leq 1$, and a standard Lemma from representation theory, see for example [9, Lemma IV 2.7], implies

$$\text{Hom}(\tau^{-1}_{C[M]} P(n+1), C[M]) = 0.$$ 

Now the Auslander-Reiten formula yields an isomorphism

$$\text{Hom}(\tau^{-1}_{C[M]} P(n+1), E'') \cong D\text{Ext}^1(E'', P(n+1)).$$

If $E'' = P(n+1)$ then the right hand side is zero. To compute the right hand side in the case where $E'' \in \Sigma$, we apply the functor $\text{Hom}_{C[M]}(E'', -)$ to the short exact sequence

$$0 \to M \to P(n+1) \to I(n+1) \to 0$$

to conclude that there exists a surjective map

$$\text{Ext}^1_{C[M]}(E'', M) \to \text{Ext}^1_{C[M]}(E'', P(n+1)).$$

But $\text{Ext}^1_{C[M]}(E'', M) = 0$, because both $E''$ and $M$ lie in the slice $\Sigma$ of $\text{mod} \ C$. This completes the proof that $\text{Ext}^1_{C[M]}(E'', E') = 0$, for all $E', E'' \in \Sigma'$.

We have shown that $\Sigma'$ satisfies the hypotheses of Theorem 2.2. Moreover, $\text{Ann}_C \Sigma = 0$, because $\Sigma$ is a slice in $\text{mod} \ C$, hence a faithful $C$-module. This implies that $\text{Ann}_{C[M]} \Sigma' = 0$, because the only paths in the quiver $Q_{C[M]}$ of $C[M]$, which are not already in the subquiver $Q_C$, must start at the extension vertex $n+1$ and therefore do not annihilate the direct summand $P(n+1)$ of $\Sigma'$. Now Theorem 2.2 implies that $C[M]$ is a tilted algebra with slice $\Sigma'$. \qed

**Theorem 3.6** Let $B$ be a cluster-tilted algebra and $P$ a projective $B$-module whose indecomposable summands lie in a local slice of $\Gamma(\text{mod} \ B)$. Then $B[P]$ is cluster-tilted.

**Remark 3.7** Note that the extension vertex in $B[P]$ is a source.
Proof. Let $\Sigma$ be a local slice of $B$ containing $P$. By Theorem 2.5, the algebra $C = B/\text{Ann}\Sigma$ is tilted with slice $\Sigma$, and $B = R(C)$ is the relation extension of $C$. Now it follows from Lemma 3.5 that $C[P]$ is a tilted algebra with slice $\Sigma' = \Sigma \cup \{P(n+1)\}$, and Theorem 3.1 implies that the one-point extension $B[P]$ is isomorphic to the relation extension $R(C[P])$ of the tilted algebra $C[P]$. By definition, this means that $B[P]$ is a cluster-tilted algebra. \hfill \square

Remark 3.8 Dually, if $I$ is an injective $B$-module whose indecomposable summands lie in a local slice of $\Gamma(\text{mod} B)$, then the one-point coextension of $B$ with respect to $I$ is cluster-tilted. In this situation, the extension vertex is a sink.

3.3 Further extensions of cluster-tilted algebras

We consider now a different construction of an extension of a cluster-tilted algebra, which produces a cluster-tilted algebra whose extension vertex can have incoming and outgoing arrows.

Theorem 3.9 Let $B$ be a cluster-tilted algebra with local slice $\Sigma$, and let $C = B/\text{Ann}\Sigma$ be the corresponding tilted algebra. Let $M$ be any module in $\Sigma$. Then $B' = R(C[M])$ is a cluster-tilted algebra such that

(a) $Q_{B'}$ contains $Q_B$ as a full subquiver and $Q_{B'}$ has one new vertex $n+1$;
(b) $\Sigma' = \Sigma \cup \{P(n+1)\}$ is a local slice;
(c) $B'/B'e_{n+1}B' = B$;
(d) $\text{rad} P(n+1) = M$ and $I(n+1)/S(n+1) = \tau_B M = \tau_{B'} M$.

Moreover, the quiver of $B'$ is obtained from the quiver of $B$ by adding one vertex $x$ and one arrow $n+1 \to i$ for each indecomposable summand $S(i)$ of top $M$, and one arrow $j \to n+1$ for each indecomposable summand $S(j)$ of $I(n+1)/S(n+1)$, counting multiplicities.

Proof. Lemma 3.5 implies that $C[M]$ is a tilted algebra with slice $\Sigma' = \Sigma \cup \{P(n+1)\}$ whose quiver has one more vertex than the quiver of $C$, and therefore $B'$ is a cluster-tilted algebra with local slice $\Sigma'$ and one new vertex $n+1$, which proves (a) and (b).

(c) The algebra $B'/B'e_{n+1}B'$ is cluster-tilted by [10, Theorem 2.13], and since cluster-tilted algebras are determined by their quivers, it suffices to show that the quiver of $B$ is the same as the quiver of $B'/B'e_{n+1}B'$. The quivers of the relation extensions $B = R(C)$ and $B' = R(C[M])$ are obtained from the quivers of $C$ and $C[M]$ respectively by adding $e_{ij}$ arrows $i \to j$ where $e_{ij}$ is the $k$-dimension of the vector space $\text{Ext}_C^2(S(i), S(j))$ and $\text{Ext}_{C[M]}^2(S(i), S(j))$, respectively. Now if $i$ and $j$ are different from $n+1$ then the dimensions of $\text{Ext}_C^2(S(i), S(j))$ and $\text{Ext}_{C[M]}^2(S(i), S(j))$ are equal. Moreover the quivers of $C$ and $C[M]$ differ only by arrows starting at the extension vertex $n+1$. Since the point $n+1$ is deleted when taking the quotient $B'/B'e_{n+1}B'$, we conclude that the quivers of $B$ and $B'/B'e_{n+1}B'$ are equal.

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(d) Since $P(n+1)$ lies in the local slice $\Sigma'$ it follows that its radical in $\text{mod } B'$ is the same as its radical in $\text{mod } C[M]$, thus $\text{rad } P(n+1) = M$.

On the other hand, since the vertex $n+1$ is a source in the quiver of $C[M]$, the socle factor $I(n+1)/S(n+1)$ of the corresponding injective in the relation extension $R(C[M])$ is given by $D\text{Ext}^2_{C[M]}(I(n+1), C[M])$. In order to compute the latter, we apply $\text{Hom}(\cdot, C[M])$ to the short exact sequence

$$0 \rightarrow M \rightarrow P(n+1) \rightarrow I(n+1) \rightarrow 0$$

to get an isomorphism

$$\text{Ext}^1_{C[M]}(M, C[M]) \cong \text{Ext}^2_{C[M]}(I(n+1), C[M]).$$

Moreover, since the projective dimension of the $C[M]$-module $M$ is at most 1, the Auslander-Reiten formula implies that

$$D\text{Hom}_{C[M]}(C[M], \tau_{C[M]}M) \cong \text{Ext}^1_{C[M]}(M, C[M]),$$

and finally there is an isomorphism $\text{Hom}_{C[M]}(C[M], \tau_{C[M]}M) \cong \tau_{C[M]}M$. Combining these three isomorphisms shows that $I(n+1)/S(n+1) = \tau_{C[M]}M$, and (d) follows because $\tau_{C[M]}M = \tau_CM = \tau_BM = \tau_{B'}M$, since $M$ lies in $\Sigma$, see [6, Proposition 3].

The statement about the arrows in the quiver of $B'$ follows from (d).

\[\square\]

\textbf{Remark 3.10} If $M$ is projective then the cluster-tilted algebra $B'$ of the theorem is the one-point extension $B[P]$ of $B$ in $P$.

\textbf{Remark 3.11} In any cluster-tilted algebra, $\tau \text{ rad } P(i)$ is always equal to $I(i)/S(i)$, see for example [2, Lemma 5].

\section{3.4 Examples}

\textbf{Example 3.12} Let $B$ be the cluster-tilted algebra of type $D_4$ given by the quiver

\begin{center}
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (1,1) {2};
\node (3) at (0,-1) {3};
\node (4) at (1,-1) {4};
\draw[->] (1) -- (2) node[anchor=west] {$\alpha$};
\draw[->] (1) -- (3) node[anchor=west] {$\gamma$};
\draw[->] (2) -- (3) node[anchor=south] {$\beta$};
\draw[->] (2) -- (4) node[anchor=west] {$\epsilon$};
\draw[->] (3) -- (4) node[anchor=west] {$\delta$};
\end{tikzpicture}
\end{center}
bound by the relations \( \alpha \beta + \gamma \delta, \epsilon \alpha, \epsilon \gamma, \beta \epsilon, \delta \epsilon \). It’s Auslander-Reiten quiver is the following

\[
\begin{array}{c}
\cdots \rightarrow 4 \\
3 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \\
4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \\
\end{array}
\]

where the two points with label 4 must be identified.

To illustrate Theorem 3.6, let \( P = P(1) \oplus P(2) \oplus P(3) \). Then \( P \) lies on the local slice \( \Sigma \) which is illustrated by the boldface modules in the Auslander-Reiten quiver above, and the one-point extension \( B[P] \) is the cluster-tilted given by the following quiver bound by the same relations.

\[
\begin{array}{c}
5 \rightarrow 1 \\
\alpha \rightarrow 2 \rightarrow \beta \\
\gamma \rightarrow 3 \rightarrow \delta \\
\end{array}
\]

To illustrate Theorem 3.9, let \( M = 2 \) be the simple module \( S(2) \). Then \( \tau M = P(3) \). The module \( M \) lies on a local slice which is obtained from the local slice \( \Sigma \) above by replacing \( P(3) \) with \( M \). The corresponding tilted algebra \( C \) is given by the quiver

\[
\begin{array}{c}
1 \rightarrow 3 \\
\alpha \rightarrow 2 \rightarrow \beta \\
\gamma \rightarrow 3 \rightarrow \delta \\
\end{array}
\]

bound by the relation \( \alpha \beta + \gamma \delta \). The one-point extension \( C[M] \) is given by the quiver

\[
\begin{array}{c}
1 \rightarrow 3 \\
\alpha \rightarrow 2 \rightarrow \beta \\
\gamma \rightarrow 3 \rightarrow \delta \\
5 \rightarrow \rho \\
\end{array}
\]
bound by the relations $\alpha\beta + \gamma\delta$, $\rho\beta$. Finally the relation extension $B' = R(C[M])$ is given by the quiver

![Quiver Diagram](image)

bound by the relations $\beta\epsilon$, $\epsilon\alpha + \sigma\rho$, $\delta\epsilon$, $\epsilon\gamma$, $\beta\sigma$, $\alpha\beta + \gamma\delta$, $\rho\beta$.

**Example 3.13** This example shows that the condition that the projective module $P$ lies on a local slice in Theorem 3.6 is necessary, even if $P$ is indecomposable. Let $B$ be the cluster-tilted algebra of type $\tilde{A}_3, 1$ given by the following quiver bound by the relations $\alpha\beta, \beta\gamma, \gamma\alpha$.

![Quiver Diagram](image)

To see that $B$ is cluster-tilted, one can apply the mutations in vertex $3$ and then in vertex $4$ to obtain an acyclic $\tilde{A}_3, 1$ quiver.

The projective $B$-module $P(3)$ lies in a regular component of the Auslander-Reiten quiver, and therefore there is no local slice containing $P(3)$. The one-point extension $B[P(3)]$ has the following quiver.

![Quiver Diagram](image)

This algebra is not cluster-tilted. In fact, this algebra corresponds to the triangulation of a punctured annulus shown in Figure 4, which means that its quiver is not mutation equivalent to an acyclic quiver, see [19], and therefore the algebra is not cluster-tilted.
Figure 1: A triangulation of an annulus with one puncture which corresponds to the one-point extension in Example 3.13.

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