Distributed Economic Model Predictive Control – Addressing Non-convexity Using Social Hierarchies

Ali C. Kheirabadi$^1$ and Ryozo Nagamune$^1$

$^1$The University of British Columbia, Vancouver Campus, 2054-6250 Applied Science Lane, Vancouver, BC Canada V6T 1Z4

September 10, 2020

Abstract

This paper introduces a novel concept for addressing non-convexity in the cost functions of distributed economic model predictive control (DEMPC) systems. Specifically, the proposed algorithm enables agents to self-organize into a hierarchy which determines the order in which control decisions are made. This concept is based on the formation of social hierarchies in nature. An additional feature of the algorithm is that it does not require stationary set-points that are known a priori. Rather, agents negotiate these targets in a truly distributed and scalable manner. Upon providing a detailed description of the algorithm, guarantees of convergence, recursive feasibility, and bounded closed-loop stability are also provided. Finally, the proposed algorithm is compared against a basic parallel distributed economic model predictive controller using an academic numerical example.

1 Introduction

1.1 Background

Model predictive control (MPC) entails recursively solving an optimization problem over a finite prediction horizon to identify optimal future control input trajectories. The popularity of MPC in academic and industrial environments is primarily attributed to its capacity for handling constraints while computing control actions that minimize nonlinear performance criteria. The reader may refer to articles by Mayne et al. [1] and Mayne [2] for reviews on MPC.

In large-scale processes or multi-agent systems, implementation of MPC in a centralized manner may be impractical due to the computational complexity of global optimization and the network infrastructure required for plant-wide communication. Distributed model predictive control (DMPC) surpasses these limitations by dispersing the burden of decision-making across a multitude of independent subsystems or agents. A trade-off that arises however, is that effective algorithms governing agent coordination are required to guarantee desirable closed-loop performance. The reader may refer to review articles by Al-Gherwi et al. [3], Christofides et al. [4], and Negenborn and Maestre [5] for further details on the subject of DMPC.

MPC has traditionally been utilized as a lower-level set-point stabilizer that tracks set-points determined by upper-level stationary optimizers. Economic model predictive control (EMPC) combines these upper- and lower-level roles by employing cost functions that capture plant economics (e.g. power production or operating cost over a finite time horizon). The effect is improved economic performance; however, additional measures for ensuring stability are required since the primary control objective no longer involves regulation. The reader may refer to articles by Ellis et al. [6] and Müller and Allgöwer [7] for reviews on EMPC.

1.2 Distributed economic model predictive control

This paper addresses distributed economic model predictive control (DEMPC) of systems with non-convex objective functions and unknown stationary set-points. A typical application with such characteristics is autonomous vehicle trajectory planning [8]. DEMPC algorithms intended for such systems have been scarce in the literature as a result of challenges pertaining to stability and convergence. This subsection reviews relevant DEMPC and nonlinear DMPC algorithms as justification for the contributions of the current work.
1.2.1 Stabilizing DEMPC algorithms

Achieving stability in DEMPC requires first computing optimal stationary set-points for all agents, and then constraining state trajectories to approach these optima within the prediction horizon. If there exist of feedback control laws that are then capable of maintaining subsystems within specified bounds of their respective steady-states, stability may be guaranteed. To achieve such an outcome, theoretical studies focused on DEMPC have either treated these stationary set-points as predefined references [9], or computed their values using centralized optimization [10, 11, 12, 13, 14, 15]. The latter group of algorithms are therefore not truly distributed.

To overcome this gap, Köhler et al. [16] were the first to develop a DEMPC scheme without the requirement for centralized processing. They presumed that optimal stationary set-points were unattainable via centralized optimization, and instead had to be negotiated online between agents in a distributed manner. Consequently, in tandem with solving their local EMPC problems and obtaining optimal input trajectories, agents also performed one iterate of a distributed coordination algorithm at each sampling time to update their respective optimal steady-states. Nonetheless, this work focused on linear systems with convex cost functions and used a sequential coordination algorithm; thus suffering from lack of scalability.

1.2.2 Convergent DEMPC algorithms

If DEMPC cost functions are non-convex, agents making decisions in parallel cannot guarantee convergence of their optimal input trajectories [17]. Several alternative classes of coordination algorithms within the nonlinear DMPC literature address this convergence issue. Sequential methods first proposed by Kuwata et al. [18] and Richards and How [19] represent the simplest solution. Agents solve their local optimization problems and exchange information with their neighbors in some predetermined order. The resulting advantage is that each subsequent agent computes its input trajectory based on updated and fixed information from its predecessors; guaranteeing convergence, stability, and feasibility is thus facilitated. The major drawback is lack of scalability to large interconnected systems, since agents at the tail-end of sequence must await decisions from all other subsystems. A secondary concern involves determining the sequence order, particularly in systems with time-varying interaction topologies.

Coordination algorithms based on negotiation between agents were developed by Müller et al. [20], Maestre et al. [21], and Stewart et al. [22]. An agent receives optimal decisions from its neighbors in the form of a proposal. Then, upon computing the corresponding effects of these decisions on its local objective function, the agent may reject or approve proposals. These algorithms are capable of resolving conflict; however they face two limitations. The first is that, in order to identify the impact of a specific agent’s control trajectory on neighboring cost functions, this agent must not operate in parallel with others; thus limiting scalability. The second is that agents whose control actions are discarded at particular time-steps remain idle.

Finally, group-based DMPC methods employ the connectivity information of a plant to identify the order in which agents should solve their local MPC problems to resolve conflict. Pannek [23] proposed a covering algorithm that permitted non-interacting agent pairs to operate in parallel, while those that were coupled made decisions sequentially according to some predetermined priority rule. This algorithm eliminated the scalability issue of pure sequential DMPC; however it required a predetermined set of priority rules. Liu et al. [24] developed a clustering algorithm that assigned agents to dominant or connecting groups. Agents in dominant clusters solved their local optimization problems first, thus eliminating conflict with agents in connecting groups. The downside in this method was that a sequential algorithm was required to determine clustering. Asadi and Richards [25] employed a slot allocation algorithm wherein each agent communicated with other subsystems to randomly select an available space in the global sequential order. This method addressed the secondary drawback of sequential DMPC, which concerned determining an effective sequence order in systems with time-varying interaction topologies. Nonetheless, the fully serial nature of the algorithm still suffered from lack of scalability.

1.3 Contributions

Based on the preceding literature review, we state, to the best of our knowledge, that a DEMPC algorithm that handles non-convex cost functions and unknown stationary set-points in a scalable and truly distributed manner, with no predetermined rules, has yet to be proposed. The existing method that meets all of these criteria except for scalability and non-convexity is the algorithm of Köhler et al. [16].

The main contribution in this paper is thus a DEMPC coordination algorithm that is scalable, fully distributed, and that guarantees stability and convergence in the presence of non-convex cost functions and unknown stationary set-points. In brief, our approach borrows from the method of conflict resolution observed in nature. Namely, when it becomes apparent that agents operating in parallel generate conflicting decisions, a social hierarchy is established
to yield resolution. Additionally, proofs of convergence, recursive feasibility, and bounded closed-loop stability are provided along with validation using a numerical example.

1.4 Paper organization

The remainder of this article is organized as follows: Section 2 provides a description of the nonlinear systems and cost functions that the proposed algorithm addresses, along with an explanation of conflict and convergence issues arising from non-convex objectives; Section 3 highlights the proposed DEMPC algorithm along with proofs of convergence, feasibility, and stability; Section 4 implements the proposed method on a numerical example with non-convex cost functions; and finally, Section 5 concludes the paper with a summary of major findings, along with recommendations for future research directions.

2 Problem description

2.1 Dynamic model

We consider $N$ agents that are dynamically decoupled and uninfluenced by disturbances. The dynamics of each agent $i \in \mathcal{I} = \{1, 2, \cdots, N\}$ are represented by the following discrete-time nonlinear state-space model:

$$x_i^k = f_i(x_i, u_i),$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ denote vectors containing the $n_i$ states and $m_i$ inputs of agent $i$, and $x_i^+$ represents $x_i$ at the subsequent sampling time-step. We consider the case where $x_i$ and $u_i$ must be bounded within the convex sets $\mathcal{X}_i$ and $\mathcal{U}_i$ at all times, which results in the following state and input constraints:

$$u_i \in \mathcal{U}_i,$$  
$$x_i \in \mathcal{X}_i.$$  

With these operational bounds defined, we make the following assumptions concerning controllability and continuity.

Assumption 1. (Weak controllability) Let the set $Z_i^s$ comprise all feasible stationary points of agent $i$ as follows:

$$Z_i^s := \{(x_i, u_i) \in \mathcal{X}_i \times \mathcal{U}_i \mid x_i = f_i(x_i, u_i)\}.$$  

All feasible stationary state vectors of agent $i$ may then be collected within the set $\mathcal{X}_i^s$, which is defined as follows:

$$\mathcal{X}_i^s := \{x_i \in \mathcal{X}_i \mid \exists u_i \in \mathcal{U}_i : (x_i, u_i) \in Z_i^s\}.$$  

Let the set $Z_i^{0 \rightarrow s}$ contain all pairings of initial state vectors $x_i^0$ and input trajectories $\overline{u}_i = (u_i^0, u_i^1, \cdots, u_i^{H-1})$ that steer agent $i$ to each feasible stationary point $x_i^s$ in $H$ time-steps, while satisfying constraints. $Z_i^{0 \rightarrow s}$ may therefore be defined as follows:

$$Z_i^{0 \rightarrow s} := \{(x_i^0, \overline{u}_i, x_i^s) \in \mathcal{X}_i \times \overline{U}_i \times \mathcal{X}_i^s \mid \exists x_i^1, x_i^2, \cdots, x_i^H : x_i^k = f_i(x_i^{k-1}, u_i^{k-1}), x_i^s \in \mathcal{X}_i, \forall k \in 1:H, x_i^H = x_i^s\},$$  

where $\overline{U}_i = \mathcal{U}_i \times \cdots \times \mathcal{U}_i = \mathcal{U}_i^H$. All possible initial state vectors $x_i^0$ that may be steered to a feasible stationary point $x_i^s$, with constraint satisfaction, are then contained within the set $\mathcal{X}_i^{0 \rightarrow s}$ defined as follows:

$$\mathcal{X}_i^{0 \rightarrow s} := \{x_i^0 \in \mathcal{X}_i \mid \exists \overline{u}_i \in \overline{U}_i, x_i^s \in \mathcal{X}_i^s : (x_i^0, \overline{u}_i, x_i^s) \in Z_i^{0 \rightarrow s}\}.$$  

For any agent $i \in \mathcal{I}$, any initial state vector $x_i^0 \in \mathcal{X}_i^{0 \rightarrow s}$, input vector trajectory $\overline{u}_i \in \overline{U}_i$, and stationary state vector $x_i^s \in \mathcal{X}_i^s$ such that $(x_i^0, \overline{u}_i, x_i^s) \in Z_i^{0 \rightarrow s}$, and any stationary input vector $u_i^s \in \mathcal{U}_i$ such that $(x_i^s, u_i^s) \in Z_i^s$, there exists a $K_\infty$ function $\gamma(\cdot)$ that satisfies the following condition:

$$\sum_{k=0}^{H-1} \|u_i^k - u_i^s\| \leq \gamma(\|x_i^0 - x_i^s\|).$$  

Assumption 2. (Lipschitz continuous dynamics) For any agent $i \in \mathcal{I}$, $f_i(\cdot)$ satisfies the following condition for Lipschitz continuity for all $(x_i^0, u_i^0), (x_i^s, u_i^s) \in \mathcal{X}_i \times \mathcal{U}_i$:

$$\|f_i(x_i^0, u_i^0) - f_i(x_i^s, u_i^s)\| \leq \Lambda_i^f \|x_i^0 - x_i^s\| + \|u_i^0 - u_i^s\|,$$  

where the scalar $\Lambda_i^f \geq 0$ is Lipschitz constant of $f_i(\cdot)$ on the set $\mathcal{X}_i \times \mathcal{U}_i$. 

3
2.2 Control objective

At each time-step, the control objective of agent $i$ is to minimize a cooperative economic stage cost function $J_i(\cdot)$ over a finite prediction horizon $H$ as follows:

$$\min_{\mathbf{x}, \mathbf{u}} \sum_{k=0}^{H-1} J_i(x^k, u^k, x^k_{-i}, u^k_{-i}),$$

(10)

where the superscript $k$ identifies the time-step number along the prediction horizon $H$, and $x_{-i}$ and $u_{-i}$ contain the state and input vectors of all agents except $i$. The objective function is influenced by the state and input vectors of all agents $j \in \mathcal{I} \setminus i$ that influence the cooperative cost function $J_j(\cdot)$ of agent $i$. We collect the indices of these agents into the set $\mathcal{N}_{-i}$. Likewise, the indices of all agents $j \in \mathcal{I} \setminus i$ whose cooperative stage cost functions $J_j(\cdot)$ are influenced by $\mathbf{x}_i$ and $\mathbf{u}_i$ are collected into the set $\mathcal{N}_{+i}$.

The objective function $J_i(\cdot)$ may be non-convex; however, it must adhere to the following assumptions concerning cooperation, boundedness, and continuity.

Assumption 3. (Neighborhood-cooperative objectives) Let each agent $i \in \mathcal{I}$ possess a stage cost function $\ell_i(\cdot)$ that represents its local economic interests. Then, let the set $\mathcal{N}_{-i}$ contain the indices of all agents $j \in \mathcal{I} \setminus i$ whose state and input vectors influence the local stage cost function $\ell_i(\cdot)$. Likewise, let the set $\mathcal{N}_{+i}$ contain the indices of all agents $j \in \mathcal{I} \setminus i$ whose local stage cost functions $\ell_j(\cdot)$ are influenced by $\mathbf{x}_i$ and $\mathbf{u}_i$.

The stage cost function $J_i(\cdot)$ for any agent $i \in \mathcal{I}$ in neighborhood-cooperative strategy that comprises the local interests of agent $i$ and each downstream neighbor $j \in \mathcal{N}_{+i}$ as follows:

$$J_i(x_i, u_i, x_{-i}, u_{-i}) := \ell_i(x_i, u_i, x_{-i}, u_{-i}) + \sum_{j \in \mathcal{N}_{+i}} \ell_j(x_j, u_j, x_{-j}, u_{-j}).$$

(11)

The vectors $x_{-i}$ and $u_{-i}$ contain the states and inputs of all agents $j \in \mathcal{N}_{-i}$. Note that $\mathcal{N}_{-i} = \mathcal{N}_{-i} \cup \mathcal{N}_{+i} \cup \mathcal{N}_{-j} \forall j \in \mathcal{N}_{+i}$.

Assumption 4. (Bounded cost function minima) Let the sets $\mathcal{X}_{-i}$ and $\mathcal{U}_{-i}$ be defined as follows:

$$\mathcal{X}_{-i} := \prod_{j \in \mathcal{N}_{-i}} \mathcal{X}_j,$$

(12)

$$\mathcal{U}_{-i} := \prod_{j \in \mathcal{N}_{-i}} \mathcal{U}_j.$$

(13)

For any agent $i \in \mathcal{I}$, there exist state and input vectors $(\mathbf{x}_{i}^*, \mathbf{u}_{i}^*, x_{-i}^*, u_{-i}^*) \in \mathcal{X}_i \times \mathcal{U}_i \times \mathcal{X}_{-i} \times \mathcal{U}_{-i}$ such that the following condition holds for all $(\mathbf{x}_{i}, \mathbf{u}_{i}, x_{-i}, u_{-i}) \in \mathcal{X}_i \times \mathcal{U}_i \times \mathcal{X}_{-i} \times \mathcal{U}_{-i}:

$$\ell_i(\mathbf{x}_{i}^*, \mathbf{u}_{i}^*, x_{-i}^*, u_{-i}^*) \leq \ell_i(\mathbf{x}_{i}, \mathbf{u}_{i}, x_{-i}, u_{-i}).$$

(14)

Assumption 5. (Lipschitz continuous objectives) For any agent $i \in \mathcal{I}$, the local cost function $\ell_i(\cdot)$ satisfies the following condition for Lipschitz continuity for all $(\mathbf{x}_{i}^a, \mathbf{u}_{i}^a, x_{-i}^a, u_{-i}^a), (\mathbf{x}_{i}^b, \mathbf{u}_{i}^b, x_{-i}^b, u_{-i}^b) \in \mathcal{X}_i \times \mathcal{U}_i \times \mathcal{X}_{-i} \times \mathcal{U}_{-i}:

$$\|\ell_i(\mathbf{x}_{i}^b, \mathbf{u}_{i}^b, x_{-i}^b, u_{-i}^b) - \ell_i(\mathbf{x}_{i}^a, \mathbf{u}_{i}^a, x_{-i}^a, u_{-i}^a)\| \leq \Lambda_i \|\mathbf{x}_{i}^b - \mathbf{x}_{i}^a\| \|x_{-i}^b - x_{-i}^a\| \|u_{i}^b - u_{i}^a\| \|u_{-i}^b - u_{-i}^a\|,$$

(15)

where the scalar $\Lambda_i \geq 0$ is Lipschitz constant of $\ell_i(\cdot)$ on the set $\mathcal{X}_i \times \mathcal{U}_i \times \mathcal{X}_{-i} \times \mathcal{U}_{-i}$.

2.3 Conflict under non-convexity

In this subsection, we elaborate further on the main challenge that associated with non-convex cost functions in DEMPC. Consider a simple problem with only two optimization variables $z_1$ and $z_2$, which are computed by agents 1 and 2, respectively. Further, let both agents share a common non-convex global objective function with contours plotted in Fig. [1].

Assume initial values $z_1^0$ and $z_2^0$ obtained from a previous iteration or time-step. Under parallel and fully distributed operation, each agent must assume that its neighbors optimization variable remains unchanged while locally minimizing the global objective function. As a result, agent 1 assumes that $z_2$ remains fixed at $z_2^0$ and restricts its search path to the horizontal orange line shown in Fig. [1]. Likewise, agent 2 assumes that $z_1$ is maintained at $z_1^0$, which constrains its search path to the vertical orange line.

Upon completion of its local optimization problem, agent 1 finds the local optimum located at $(\hat{z}_1^1, z_2^0)$. Agent 2 achieves the same at $(\hat{z}_1^2, \hat{z}_2^2)$. When the updated optimal variables $\hat{z}_1$ and $\hat{z}_2$ are combined however, the overall system operates at neither of the local optima identified by the individual agents. We refer to such an outcome as conflict in the current work. Specifically, we define conflict and conflict-free operation as follows.
Definition 1. (Conflict) Agent $i$ encounters conflict when its economic performance deteriorates upon considering the candidate control actions of its neighbors. More formally, consider $\hat{V}_i^s$ and $\bar{V}_i^s$ defined as follows:

\begin{align*}
\hat{V}_i^s & := J_i(\hat{x}_i^s, \hat{u}_i^s, \hat{x}^s_{-i|J}, \hat{u}^s_{-i|J}), \\
\bar{V}_i^s & := J_i(\bar{x}_i^s, \bar{u}_i^s, \bar{x}^s_{-i|J}, \bar{u}^s_{-i|J}),
\end{align*}

where $\hat{x}_i^s$ and $\hat{u}_i^s$ denote the optimal stationary state and input vectors computed by agent $i$, $\hat{x}^s_{-i|J}$ and $\hat{u}^s_{-i|J}$ contain stationary state and input vectors that agent $i$ assumes for all neighbors $j \in N_{-i|J}$, and $\bar{x}^s_{-i|J}$ and $\bar{u}^s_{-i|J}$ consist of optimal state and input vectors computed by all agents $j \in N_{-i|J}$ and communicated to agent $i$. The terms $\hat{V}_i^s$ and $\bar{V}_i^s$ represent naive and informed values of the stage cost function of agent $i$ at some stationary point. The term naive indicates that the cost function value is computed based on assumed values of neighboring agents’ state and input vectors. Contrarily, the term informed is employed when agent $i$ considers recently communicated updated optimal state and input vectors. Given these definitions, while attempting to negotiate an optimal stationary point $(\hat{x}_i^s, \hat{u}_i^s)$, agent $i$ operates in conflict with its neighbors if the following statement is true:

$$\hat{V}_i^s > \bar{V}_i^s.$$  \hfill (18)

Similarly, one may define naive and informed values of cost functions summed along the prediction horizon as follows:

\begin{align*}
\hat{V}_i & := \sum_{k=0}^{H-1} J_i(\hat{x}_i^k, \hat{u}_i^k, \hat{x}^k_{-i|J}, \hat{u}^k_{-i|J}), \\
\bar{V}_i & := \sum_{k=0}^{H-1} J_i(\bar{x}_i^k, \bar{u}_i^k, \bar{x}^k_{-i|J}, \bar{u}^k_{-i|J}),
\end{align*}

where $\bar{x}_i^k$ and $\bar{u}_i^k$ denote optimal state and input vectors computed by agent $i$ at time-step $k$ along the prediction horizon, $\bar{x}^k_{-i|J}$ and $\bar{u}^k_{-i|J}$ contain state and input vectors that agent $i$ assumes for all neighbors $j \in N_{-i|J}$ at some time-step $k$ along the prediction horizon, and $\hat{x}^k_{-i|J}$ and $\hat{u}^k_{-i|J}$ consist of optimal state and input vectors computed by all agents $j \in N_{-i|J}$ and communicated to agent $i$. Given this information, while attempting to negotiate optimal state and input trajectories $\bar{x}_i = (\bar{x}_i^0, \bar{x}_i^1, \ldots, \bar{x}_i^H)$ and $\bar{u}_i = (\bar{u}_i^0, \bar{u}_i^1, \ldots, \bar{u}_i^{H-1})$, agent $i$ operates in conflict with its neighbors if the following statement is true:

$$\hat{V}_i > \bar{V}_i.$$  \hfill (21)

Definition 2. (Conflict-free operation) Agent $i$ operates free of conflict when its economic performance improves or remains unchanged upon considering the candidate control actions of its neighbors. More formally, consider once again the values $\hat{V}_i^s$, $\bar{V}_i^s$, $\bar{V}_i$, and $\hat{V}_i$ as previously defined. While attempting to negotiate an optimal stationary point $(\hat{x}_i^s, \hat{u}_i^s)$, agent $i$ operates free of conflict with its neighbors if the following statement is true:

$$\hat{V}_i^s \leq \bar{V}_i^s.$$  \hfill (22)

While attempting to negotiate optimal trajectories $\bar{x}_i$ and $\bar{u}_i$, agent $i$ operates free of conflict with its neighbors if the following statement is true:

$$\hat{V}_i \leq \bar{V}_i.$$  \hfill (23)
In the introduction, several algorithms based on sequential operation, agent negotiation, and agent grouping that could resolve non-convex conflict were discussed. The main disadvantages of these algorithms were lack of scalability, idleness of certain agents, and the requirement of predefined rules. In the next section, we propose a fully distributed and scalable solution to the problem of conflict that addresses these drawbacks by establishing social hierarchies.

3 Social hierarchy-based DEMPC algorithm

3.1 Social hierarchy framework

In the current context, a social hierarchy consists of a finite number of levels that establish the sequence in which agents generate decisions in order to resolve conflict. The concept is based loosely on social hierarchies that appear naturally among living organisms as a means to resolve conflict and establish which individuals’ decisions take priority over those of others. These hierarchies are often determined by the evolutionary or cultural characteristics of the individuals, which are, in essence, randomly assigned. In a similar fashion, we propose a framework which permits the formation of hierarchies with elements of randomness in order to resolve conflict resulting from non-convexity.

For generality, we assume an iterative parallel coordination algorithm, the first of which was presented by Du et al. [26]. Agents synchronously solve their local optimization problems and communicate repeatedly within a single sampling time-step until some termination condition is satisfied or until a maximum number of iterations have been implemented. The socially hierarchy framework is equally applicable to non-iterative parallel methods which were first investigated by Jia and Krogh [27] and Dunbar and Murray [28].

A visual representation of the social hierarchy framework is shown in Fig. 2. Within a single iteration, there exist $N_q$ hierarchy levels which specify the order in which agents make decisions. Each agent may solve for its optimal stationary point and control trajectory only once within an iteration; however, this computation may take place within any hierarchy level. During each iteration, agents occupying hierarchy level $q = 1$ make decisions first and transfer relevant information to their neighbors. Following this step, agents allocated to hierarchy level $q = 2$ perform the same task. This trend continues until all hierarchy level computations have been performed, at which point the entire process is repeated during the next iteration. It is important to note that multiple agents may occupy the same hierarchy level, and that $N_q$ may be substantially smaller than $N$.

Two fundamental questions now arise regarding (i) how agents should sort themselves among the $N_q$ hierarchy levels in order to resolve conflict, and (ii) how should $N_q$ be determined by the control system designer. The former concern is addressed in Section 3.2 which describes and assesses a novel DEMPC coordination algorithm that utilizes the concept of a social hierarchy. The latter question is discussed in Section 3.4 which borrows elements from vertex coloring theory to establish social hierarchy properties.

3.2 DEMPC coordination algorithm

This subsection details a novel coordination algorithm for DEMPC with non-convex objectives based on the social hierarchy framework described in Section 3.1. Our approach allows agents to resolve their conflicts in a truly
distributed and scalable manner without the requirement of predefined rules or access to the full system interaction topology. Each agent achieves this outcome by occupying an appropriate level along a social hierarchy when conflict arises. In brief, our algorithm follows evolutionary principles. That is, if an agent occupying a particular hierarchy level during some iteration experiences conflict, then its current hierarchy level is detrimental to its performance and must be randomly mutated.

While negotiating optimal trajectories $\dot{\mathbf{x}}_i^k$ and $\mathbf{u}_i^k$, agent $i$ solves the following optimization problem during each iteration:

$$\min_{\mathbf{u}, \dot{\mathbf{x}}_i} \sum_{k=0}^{H-1} J_i(\dot{\mathbf{x}}_i^k, \mathbf{u}_i^k, \dot{\mathbf{x}}_{-i|J}^k, \mathbf{u}_{-i|J}^k),$$  \hspace{1cm} (24)

subject to

$$\dot{\mathbf{x}}_i^0 = \mathbf{x}_i, \hspace{1cm} (25a)$$
$$\dot{\mathbf{x}}_i^{k+1} = f_i(\dot{\mathbf{x}}_i^k, \mathbf{u}_i^k), \hspace{1cm} (25b)$$
$$\dot{\mathbf{x}}_i^k \in \mathcal{X}_i, \hspace{1cm} (25c)$$
$$\dot{\mathbf{u}}_i^k \in \mathcal{U}_i, \hspace{1cm} (25d)$$
$$\dot{\mathbf{x}}_i^H = \mathbf{x}_i, \hspace{1cm} (25e)$$

where $\dot{\mathbf{x}}_i^k$ and $\mathbf{u}_i^k$ denote the predicted state and input vectors of agent $i$ at some time-step $k$ along the prediction horizon $H$, $\dot{\mathbf{x}}_i = (\dot{\mathbf{x}}_i^0, \dot{\mathbf{x}}_i^1, \ldots, \dot{\mathbf{x}}_i^H)$, and $\mathbf{u}_i = (\mathbf{u}_i^0, \mathbf{u}_i^1, \ldots, \mathbf{u}_i^{H-1})$ represent predicted state and input vector trajectories, and $\dot{\mathbf{x}}_{-i|J}^k$ and $\dot{\mathbf{u}}_{-i|J}^k$ contain the predicted state and input vectors that agent $i$ assumes for all neighbors $j \in \mathcal{N}_{-i|J}$ at time-step $k$. Constraint (25a) ensures that, by the end of the prediction horizon $H$, the computed optimal input trajectory $\dot{\mathbf{x}}_i^k$ leads the local state vector to a feasible and optimal steady state vector $\dot{\mathbf{x}}_i^k$.

To negotiate the vector $\dot{\mathbf{x}}_i^k$, agent $i$ solves the following optimization problem during each iteration:

$$\min_{\mathbf{u}, \dot{\mathbf{x}}_i} J_i(\dot{\mathbf{x}}_i^a, \dot{\mathbf{u}}_i^a, \dot{\mathbf{x}}_{-i|J}^a, \dot{\mathbf{u}}_{-i|J}^a),$$  \hspace{1cm} (26)

subject to

$$\dot{\mathbf{x}}_i^0 = \mathbf{x}_i, \hspace{1cm} (27a)$$
$$\dot{\mathbf{x}}_i^{k+1} = f_i(\dot{\mathbf{x}}_i^k, \mathbf{u}_i^k), \forall k \in \mathcal{I}_{0:H-1}, \hspace{1cm} (27b)$$
$$\dot{\mathbf{x}}_i^H = f_i(\dot{\mathbf{x}}_i^H, \mathbf{u}_i^k), \hspace{1cm} (27c)$$
$$\dot{\mathbf{x}}_i^a = \dot{\mathbf{x}}_i^H, \hspace{1cm} (27d)$$
$$\dot{\mathbf{x}}_i^k \in \mathcal{X}_i, \forall k \in \mathcal{I}_{0:H}, \hspace{1cm} (27e)$$
$$\dot{\mathbf{u}}_i^k \in \mathcal{U}_i, \forall k \in \mathcal{I}_{0:H-1}, \hspace{1cm} (27f)$$
$$(\dot{\mathbf{x}}_i^a, \dot{\mathbf{u}}_i^a) \in \mathcal{X}_i \times \mathcal{U}_i, \hspace{1cm} (27g)$$

where $\dot{\mathbf{x}}_i^a$ and $\dot{\mathbf{u}}_i^a$ denote the candidate stationary state and input vectors of agent $i$, and $\dot{\mathbf{x}}_{-i|J}^a$ and $\dot{\mathbf{u}}_{-i|J}^a$ contain similar information that agent $i$ assumes for all neighbors $j \in \mathcal{N}_{-i|J}$. Constraint (27a) ensures that, at the end of the prediction horizon, an input vector $\dot{\mathbf{u}}_i^k$ exists such that $(\dot{\mathbf{x}}_i^H, \mathbf{u}_i^k)$ is a stationary point. Constraint (27a) then requires that the optimal steady state vector $\dot{\mathbf{x}}_i^a$ is equal to the final state along the prediction horizon.

The difference between Problems (24) and (26) is that the latter only considers the state cost function $J_i(\cdot)$ and therefore computes a feasible and optimal steady state vector $\dot{\mathbf{x}}_i^a$ without minimizing $J_i(\cdot)$ over the prediction horizon. The reason that dynamics are considered in Problem (26) is to ensure that the computed optimal steady state vector $\dot{\mathbf{x}}_i^a$ is reachable from the initial state vector $\dot{\mathbf{x}}_i^0$. Problem (24) then computes optimal trajectories that minimize $J_i(\cdot)$ over the prediction horizon and steer the system to the reachable steady state vector $\dot{\mathbf{x}}_i^a$. A description of the proposed social hierarchy-based DEMPC algorithm now follows.

**Algorithm 1. Social hierarchy-based DEMPC coordination scheme. Implement in parallel for all agents $i \in \mathcal{I}$.**

**Initialization:**

1. Specify $N_q \geq 1$ and set $q_i = 1$.
2. Initialize $\mathbf{x}_i, \mathbf{x}_j \in \mathcal{X}_i = \mathcal{X}_i \times \ldots \times \mathcal{X}_i = \mathcal{X}_i H$, $\mathbf{u}_i \in \mathcal{U}_i$, $\dot{\mathbf{x}}_i^0 \in \mathcal{X}_i$, $\dot{\mathbf{u}}_i^0 \in \mathcal{U}_i$ such that $\dot{\mathbf{x}}_i^0 = \mathbf{x}_i, \dot{\mathbf{x}}_i^{k+1} = f_i(\dot{\mathbf{x}}_i^k, \dot{\mathbf{u}}_i^k) \forall k \in \mathcal{I}_{0:H-1}, \dot{\mathbf{x}}_i^H = \dot{\mathbf{x}}_i^a$, and $\dot{\mathbf{x}}_i^a = f_i(\dot{\mathbf{x}}_i^k, \dot{\mathbf{u}}_i^k)$. 

7
As a result, if no conflict is encountered, agent
state and input trajectories
appropriate stationary point (˘
and ˘
through all hierarchy levels sequentially and, during its allocated hier-
archy level, solves Problem (26) to obtain ˘
agent

i

stage cost function ˆ
hierarchy level may be maintained. Further, since the recently com-
puted stationary vectors ˘
and steer the system to a reachable stationary point ( ˆ
its neighbors.

Perform at each new time-step:

1. Measure x_i, and compute ˘x_i such that ˘x_i^0 = x_i and ˘x_i^{k+1} = f_i(˘x_i^k, ˘u_i^k)∀k ∈ I_0:H−1.

2. Send ˘x_i, ˘u_i, ˘x_i, and ˘u_i to all agents j ∈ N_{+i|J}, and receive ˘x_j, ˘u_j, ˘x_j, and ˘u_j from all agents j ∈ N_{−i|J}.

3. For iteration number p_s = 1, 2, · · · , N_p, do:
   (a) For sequence slot number q = 1, 2, · · · , N_q, do:
      i. If q = q_i, (i) solve Problem (26) to acquire ˘x_i and ˘u_i, (ii) compute ˘V_i according to Eq. (16), (iv) send ˘x_i and ˘u_i to all agents j ∈ N_{+i|J}, (v) receive ˘x_j and ˘u_j from all agents j ∈ N_{−i|J}, and update ˘x_j = ˘x_j^* and ˘x_j = ˘x_j^*.  
      ii. Else, receive ˘x_j and ˘u_j from all agents j ∈ N_{−i|J}, and update ˘x_j = ˘x_j^* and ˘x_j = ˘x_j^*.
   (b) Compute ˘V_i according to Eq. (17). If ˘V_i > ˘V*, randomly change q_i with uniform probability. Else, if ˘V_i ≤ ˘V*, update ˘x_i = ˘x_i^* and ˘u_i = ˘u_i^*.

4. For iteration number p = 1, 2, · · · , N_p, do:
   (a) For sequence slot number q = 1, 2, · · · , N_q, do:
      i. If q = q_i, (i) solve Problem (24) to acquire ˘x_i and ˘u_i, (ii) compute ˘V_i according to Eq. (20), (iv) send ˘x_i and ˘u_i to all agents j ∈ N_{+i|J}, (v) receive ˘x_j and ˘u_j from all agents j ∈ N_{−i|J}, and update ˘x_j = ˘x_j and ˘x_j = ˘x_j.  
      ii. Else, receive ˘x_j and ˘u_j from all agents j ∈ N_{−i|J}, and update ˘x_j = ˘x_j and ˘x_j = ˘x_j.
   (b) Compute ˘V_i according to Eq. (20). If ˘V_i > ˘V*, randomly change q_i with uniform probability. Else, if ˘V_i ≤ ˘V, update ˘x_i = ˘x_i^* and ˘u_i = ˘u_i^*.

5. Apply optimal control input ˘u_i^* to the system, update ˘u_i such that ˘u_i^k = ˘u_i^{k+1}∀k ∈ I_0:H−2 and ˘u_i^{H−1} = ˘u_i^*.

During initialization, Step 1 requires the control system designer to specify the quantity N_q of hierarchy levels and to allocate agent i to the first level. As a result, all agents initially solve their local optimization problems in parallel. Step 2 requires that feasible state and input trajectories ˘x, and ˘u, that satisfy the constraints of Eq. (25), and steer the system to a reachable stationary point (˘x_i, ˘u_i), be specified for agent i given the initial state vector x_i. Finally, Step 3 involves the exchange of candidate trajectories and stationary vectors between agent i and all of its neighbors.

Focusing on the recursive portion of the DEMPC algorithm, agent i first measures its state vector x_i and updates its local state trajectory ˘x_i, using the most up-to-date candidate input sequence ˘u. Then, Step 2 requires the exchange of candidate trajectories and stationary vectors between agent i and all of its neighbors. As a result, prior to Step 3, agent i possesses the most up-to-date assumptions on the trajectories and stationary vectors ˘x_j, ˘u_j, ˘x_j, and ˘u_j of neighbors j ∈ N_{−i|J}.

Step 3 initiates an iterative process within the current sampling time-step with the objective of identifying an appropriate stationary point (˘x_i, ˘u_i) to serve as a terminal constraint in future steps. In Step 3(a), agent i cycles through all hierarchy levels sequentially and, during its allocated hierarchy level, solves Problem (26) to obtain ˘x_i and ˘u_i. These vectors, along with the assumptions ˘x_j and ˘u_j for all j ∈ N_{−i|J}, are then used to calculate a naive stage cost function ˘V_i to be used for comparison to ˘V_i later on. Finally, agent i exchanges optimal stationary vectors with neighboring agents and updates its assumptions ˘x_i and ˘u_i for all j ∈ N_{−i|J}. Outside of its hierarchy level, agent i remains idle and only receives information from neighbors and updates its assumptions.

After all hierarchy levels have been cycled through, agent i will have received ˘x_j and ˘u_j from all neighbors j ∈ N_{−i|J}. Step 3(b) requires the computation of the informed stage cost function ˘V_i. If ˘V_i > ˘V_i, then agent i is in conflict with its neighbors while attempting to identify an optimal stationary point, and it is necessary for its hierarchy level q_i to be randomly mutated. If ˘V_i ≤ ˘V_i, then agent i is operating free of conflict, and its current hierarchy level may be maintained. Further, since the recently computed stationary vectors ˘x_i and ˘u_i did not yield conflict, they will serve as appropriate candidate information for optimization problems during the next iteration. As a result, if no conflict is encountered, agent i updates ˘x_i and ˘x_i using ˘x_i and ˘u_i.

Step 3 is terminated once the maximum number of iterations N_p has been reached. Step 4 essentially repeats Step 3, except the optimal terminal state vector ˘x_i has now been identified, and the objective is to compute optimal state and input trajectories ˘x_i and ˘u_i. The maximum number of iterations available for this step is also N_p. Step 5
requires that the control input vector corresponding to the first time-step along the prediction horizon be applied to the system. Additionally, the candidate input trajectory \( \hat{u}_i \) for the next sampling time-step is constructed by concatenating the most recent optimal input trajectory \( \bar{u}_i \) with the optimal stationary input vector \( \hat{u}_i^* \).

### 3.3 Closed-loop properties

The current subsection establishes closed-loop properties for Algorithm 1. We first demonstrate that, regardless of the interaction topology of a multi-agent system, all conflicts may be resolved in a finite number of iterations with some probability. We then address convergence, feasibility, and stability.

**Theorem 1.** *(Conflict resolution)* There exist a finite number of iterations after which, with some probability greater than zero, Inequalities \((22)\) and \((23)\) are guaranteed to be satisfied at each iteration within a single time-step.

**Proof.** This proof consists of two parts; (i) it is first necessary to prove that, for any interconnected system, at least one social hierarchy exists that will ensure system-wide conflict resolution; (ii) it is then proved that the probability of agents self-organizing according to such a social hierarchy is greater than zero during any iteration.

**Part I:** Consider a multi-agent system wherein all agents possess the same stage cost function \( J(\cdot) \) defined as follows:

\[
J(x, u) := \sum_{i \in \mathcal{I}} \ell_i(x_i, u_i, x_{-i}, u_{-i}),
\]

where \( x \in \mathbb{R}^{\sum_{i \in \mathcal{I}} n_i} \) and \( u \in \mathbb{R}^{\sum_{i \in \mathcal{I}} m_i} \) contain the \( n_i \) states and \( m_i \) inputs of all agents \( i \in \mathcal{I} \), and \( J(\cdot) \) is global-cooperative in that it considers the local interests \( \ell_i(\cdot) \) of all agents \( i \in \mathcal{I} \). The resulting dynamic optimization problem over the prediction horizon \( H \) for any agent \( i \) is therefore defined as follows:

\[
\min_{\bar{u}, x} \sum_{k=0}^{H-1} J(\hat{x}_k, \hat{u}_k),
\]

with constraints similar to those of Problem \((24)\). The vectors \( \hat{x}_k^i \) and \( \hat{u}_k^i \) denote system-wide states and inputs at time-step \( k \) along the prediction horizon. The stationary optimization problem for any agent \( i \) is defined as follows:

\[
\min_{\bar{u}_i^*, \bar{x}_i^*, \bar{x}_j^*} J(\hat{x}_k^i, \hat{u}_k^i),
\]

with constraints similar to those of Problem \((26)\). The vectors \( \hat{x}_k^i \) and \( \hat{u}_k^i \) denote system-wide steady states and inputs. We also define \( V(\cdot) \) as follows:

\[
V(x, u) := \sum_{k=0}^{H-1} J(\hat{x}_k, \hat{u}_k).
\]

As a reference case, let the agents operate in a fully sequential manner such that, at any given iteration, only one agent solves its local optimization problem. Focusing on Step 3 in Algorithm 1, only agent \( i \) updates \( \hat{x}_k^i \) and \( \hat{u}_k^i \) at some iteration \( p_i \), then transmits this information to all other agents. Furthermore, let recursive feasibility, which is established independently in Theorem 3 be presumed, and let Assumptions 2, 4, and 5 concerning continuity and bounded minima hold. Under these conditions, upon solving Problem \((30)\), each agent \( i \in \mathcal{I} \) is guaranteed to shift \( \hat{x}^i \) and \( \hat{u}^i \) along the variable space \( (\hat{x}_k^i, \hat{u}_k^i) \) until the gradient of \( J(\cdot) \) projected along \( (\hat{x}_k^i, \hat{u}_k^i) \) satisfies optimality conditions. Since this process occurs sequentially across all agents \( i \in \mathcal{I} \), \( J(\cdot) \) is guaranteed to decrease or remain unchanged after each subsequent agent’s update to \( \hat{x}_k^i \) and \( \hat{u}_k^i \), which ensures conflict-free operation as per Definition 2.

We now prove that the above result may be achieved using social hierarchies that are not necessarily fully sequential (i.e. a subset of agents may solve their local optimization problems simultaneously) and also assuming neighborhood-cooperative stage cost functions as per Assumption 3. From the perspective of any agent \( i \in \mathcal{I} \), \( J(\cdot) \) may be arranged as follows:

\[
J(x, u) = \ell_i(x_i, u_i, x_{-i}, u_{-i}) + \sum_{j \in \mathcal{N}_i} \ell_j(x_j, u_j, x_{-j}, u_{-j}) + \sum_{\kappa \in \mathcal{I} \setminus \mathcal{N}_i, \kappa \neq i} \ell_\kappa(x_\kappa, u_\kappa, x_{-\kappa}, u_{-\kappa}),
\]

where the three terms on the right-hand-side represent, from left to right, the local interests of agent \( i \), the local interests of agents whose cost functions are influenced by agent \( i \), and the local interests of all remaining agents.
whose cost functions are uninfluenced by agent $i$. Taking the gradient of $J(\cdot)$ along the variable space $(x_i, u_i)$ yields the following:

$$\nabla_{x_i, u_i} J(x, u) = \nabla_{x_i, u_i} \ell_i(x_i, u_i, x_{-i}, u_{-i}) + \nabla_{x_i, u_i} \sum_{j \in N_{-i}} \ell_j(x_j, u_j, x_{-j}, u_{-j}).$$

(33)

Note that gradient of $\ell_\kappa(\cdot)$ for all $\kappa \in I \setminus N_{+i}, \kappa \neq i$ is zero along $(x_i, u_i)$ since these expressions have no dependency upon the operation of agent $i$. The shape of $J(\cdot)$ along the variable space $(x_i, u_i)$ is therefore only influenced by $x_j$ and $u_j$ for all $j \in N_{-i,j}$, since $N_{-i,j} = N_{-i} \cup N_{+i} \cup N_{-j} \forall j \in N_{+i}$. This result yields two important consequences.

First, the local optimization problem of agent $i$ is uninfluenced by the control actions of agents $j \in I \setminus N_{-i,j}, j \neq i$. Agent $i$ may therefore update its control actions in parallel with agents $j \in I \setminus N_{-i,j}, j \neq i$, and the computed optimal trajectories are guaranteed to decrease or preserve $V(\cdot)$. Second, whether agent $i$ employs the global-cooperative stage cost function $J(\cdot)$, or the neighborhood-cooperative stage cost function $J_i(\cdot)$ defined in Eq. (11), the computed optimal trajectories remain unchanged. This property is true since $\nabla_{x_i, u_i} J(\cdot) = \nabla_{x_i, u_i} J_i(\cdot)$. Therefore, if Assumption 3 concerning neighborhood-cooperative cost functions holds, and if no two agents $i$ and $j$ such that $j \in N_{-i,j} \forall i, j \in I, j \neq i$ solve their local optimization problems simultaneously, then at least one social hierarchy exists that will satisfy Inequality (22) after each iteration. The above logic may be extended to Step 4 in Algorithm 1 without modification. Part I of the proof is thus completed.

Part II: Due to its distributed nature, Algorithm 1 may lead some agents to resolve their conflicts earlier than others. However, we prove that, even in a worst-case scenario in which all agents initially encounter conflict, the probability that all conflicts will be resolved within a single iteration is greater than zero. Let $N_q$ describe the quantity of possible social hierarchies that will resolve conflict among $N$ agents. If agent must, with uniform probability, randomly choose among $N_q$ hierarchy levels, the probability $P$ that all conflicts are resolved within a single iteration is defined as follows:

$$P = N_s \left( \frac{1}{N_q} \right)^N. $$

(34)

Thus, as long as $N_s > 0, P > 0$ must also be true. Part I of this proof demonstrated that $N_s \geq 1$ for any interconnected system with neighborhood-cooperative cost functions. □

Theorem 2. (Convergence) Let $\bar{x}_i^{s,p}$ and $\bar{u}_i^{s,p}$ denote the optimal steady state vector and input trajectory $\bar{x}_i^{s}$ and $\bar{u}_i$, computed by agent $i$ at iteration $p$. Given a sufficiently large number of iterations in Steps 3 and 4 in Algorithm 1, the following inequalities are guaranteed to be satisfied for all $i \in I$:

$$\|\bar{x}_i^{s,p} - \bar{x}_i^{s,p-1}\| \leq \alpha \|\bar{x}_i^{s,p-1}\|,$$

(35)

$$\|\bar{u}_i^{s,p} - \bar{u}_i^{s,p-1}\| \leq \beta \|\bar{u}_i^{s,p-1}\|,$$

(36)

where $\alpha > 0$ and $\beta > 0$ represent fixed convergence tolerances.

Proof. Theorem 1 guarantees that, with a large enough number of iterations, system-wide conflict-resolution may be achieved with some probability greater than zero. As per Definition 2 and focusing on Step 3 from Algorithm 1 once system-wide conflict-free operation has been achieved, the inequality $\bar{V}_i^s \leq \bar{V}_i^s$ is guaranteed to be satisfied after each iteration for any agent $i \in I$. Let $\tilde{V}_i^{s,p}$ and $\tilde{V}_i^{s,p}$ denote $\bar{V}_i^s$ and $\bar{V}_i^s$ computed during iteration number $p$.

Since $\tilde{V}_i^p$ is computed using optimal state and input trajectories of agent $i$ that have been updated during iteration $p$, then $\tilde{V}_i^{s,p} \leq \tilde{V}_i^{s,p-1}$ must be true during conflict-free operation. Therefore, since $\tilde{V}_i^{s,p} \leq \tilde{V}_i^{s,p}$ and $\tilde{V}_i^{s,p} \leq \tilde{V}_i^{s,p-1}$, then $\tilde{V}_i^{s,p} \leq \tilde{V}_i^{s,p-1}$ must be true after conflicts have been resolved, which indicates that the value of $J_i(\cdot)$ computed using updated stationary points of all relevant agents is guaranteed to decrease with each subsequent iteration. If Assumption 4 concerning bounded minima holds, then $(\bar{x}_i^s, \bar{u}_i^s)$ is guaranteed to approach a local minimizer of $J_i(\cdot)$, thus satisfying Inequality (35). The above logic may be extended to Step 4 of Algorithm 1 without modification: thus concluding the proof. □

Theorem 3. (Recursive feasibility) The constraints of Problems (20) and (24) are guaranteed to be satisfied during each iteration at any sampling time $k \geq 0$ for all agents $i \in I$.

Proof. Provided that initial values of $\bar{x}_i, \bar{u}_i, \bar{x}_i^s$, and $\bar{u}_i^s$ established at the start of some time-step are feasible, then at least one viable set of values for $\bar{x}_i, \bar{u}_i, \bar{x}_i^s$, and $\bar{u}_i^s$ exists that satisfies Constraints (20) and (27), thus ensuring recursive feasibility at each subsequent iteration within the time-step. Further, since the absence of disturbances restricts state progression to $\bar{x}_i$, then feasibility is preserved at subsequent time-steps. Guaranteeing recursive feasibility thus requires that $\bar{x}_i, \bar{u}_i, \bar{x}_i^s$, and $\bar{u}_i^s$ are initially feasible. Referring to Eq. (7), the existence of $\bar{x}_i, \bar{u}_i,$
\(x^*_i\) and \(u^*_i\) such that Constraints \(25\) and \(27\) are satisfied requires that the set \(X^{0\rightarrow s}\) is not empty. Assumption \(1\) concerning weak controllability bounds the control input trajectory \(\bar{u}_i\) required to steer any initial state vector \(x^0_i \in X^{0\rightarrow s}\) to a reachable stationary point \((x_i^*, u_i^*)\) such that \((x_i^0, \bar{u}_i, x_i^*) \in Z^{0\rightarrow s}\). As a result, the set \(X^{0\rightarrow s}\) must be non-empty; thus concluding the proof.

**Theorem 4.** (Closed-loop stability) For all agents \(i \in I\), as \(k \to \infty\), the measured state vector \(x_i\) will remain bounded within a set \(X^*_i \subset X^{0\rightarrow s}\) surrounding a fixed stationary point \(x^*_i \in X^*_i\). The set \(X^*_i\) is defined as follows:

\[
X^*_i := \{x_i \in \mathcal{X}_i \mid \exists \bar{u}_i \in \mathcal{U}_i : (x_i, \bar{u}_i, x^*_i) \in Z^{0\rightarrow s}\}.
\]

**Proof.** For all agents \(i \in I\), while computing optimal state and input vector trajectories \(x_i^*\) and \(u_i^*\), Constraint \(26\) ensures that the state vector \(x_i\) always remains within a reachable set surrounding some steady state vector \(\bar{x}_i^*\). Therefore, in order to prove that this reachable set ultimately maintains a fixed value \(X^*_i\), one must prove that the terminal steady state vector \(\hat{x}_i^*\) approaches a fixed value \(x_i^*\) as \(k \to \infty\) for all agents \(i \in I\).

We assume for the time being that, after a sufficient number of iterations, agents establish a social hierarchy that permanently resolves all conflicts. That is, Inequality \(22\) is guaranteed to be satisfied at each iteration within all subsequent time-steps, and the social hierarchy therefore ceases to change. We shall refer to such a social hierarchy as a **universal social hierarchy**. Let \(\hat{V}^{s,k}_i|_A\) and \(\hat{V}^{s,k}_i|_B\) denote \(\hat{V}_i^s\) computed at the initial and final iterations, respectively, of time-step \(k\). It is clear that, after a universal social hierarchy has been established, \(\hat{V}^{s,k}_i|_B \leq \hat{V}^{s,k}_i|_A\) is satisfied at all subsequent time-steps. Since \(\hat{V}^{s,k}_i|_A\) is computed using \(\hat{x}_i^*\) and \(\hat{u}_i^*\) obtained by solving Problem \(26\) at the first iteration of time-step \(k\), then \(\hat{V}^{s,k}_i|_A \leq \hat{V}^{s,k-1}_i|_B\) must also be true. As a result, \(\hat{V}^{s,k}_i|_B \leq \hat{V}^{s,k-1}_i|_B\) must be satisfied at all time-steps following the establishment of a universal social hierarchy. If Assumption \(1\) concerning bounded minima holds, \((\hat{x}_i^*, \hat{u}_i^*)\) is guaranteed to approach some fixed point \((x_i^*, u_i^*)\) for all \(i \in I\) as \(k \to \infty\), where \((x_i^*, u_i^*)\) is some local minimizer of \(J_i(\cdot)\).

The remaining task is to prove that a universal social hierarchy is in fact attainable after a sufficient number of iterations. The existence of at least one such social hierarchy therefore already has been established in the proof for Theorem \(1\). Namely, if Assumption \(3\) concerning neighborhood-cooperative cost functions holds, and if no two agents \(i\) and \(j\) such that \(j \in N_{-i,j}\) for all \(i, j \in I, j \neq i\) make decisions simultaneously, then Inequality \(22\) is guaranteed to be satisfied at every iteration within any sampling time. If conflict persists, then agents will, after a sufficient number of iterations, self-organize according to a universal social hierarchy with some probability greater than zero; thus concluding the proof.

### 3.4 Determining social hierarchy properties

One final issue that must be addressed concerns the selection of \(N_q\). In order to guarantee that a universal social hierarchy is attainable, \(N_q\) must be large enough such that a social hierarchy wherein no neighboring pairs of agents operate in parallel is permissible. This goal invokes the vertex coloring problem from graph theory \([29]\). In brief, vertex coloring of a graph requires assigning colors to all nodes such that no two interconnected nodes share the same color. Returning to the context of the current article, each node signifies an agent, each color represents a specific level along a social hierarchy, and interconnection symbolizes cost function coupling. In graph theory, the chromatic number refers to the minimum number of colors required to complete the vertex coloring problem. Therefore, in the current context, \(N_q\) should be equal to or greater than the chromatic number of the system graph.

### 4 Numerical example

#### 4.1 Problem description

Consider the mechanical system described in Fig. \(3\). This setup consists of \(N\) square plates that are supported by spring-damper systems and perfectly aligned at equilibrium. Each square plate has a mass \(m = 1.0\) kg, side length \(L = 0.25\) m, and is supported by stiffness and damping coefficients \(k = 1.0\) N/m and \(c = 1.0\) kg/s. Further, the vertical position of each plate is controlled via an input force \(u_i\) that is regulated by agent \(i\). The resulting continuous-time dynamics of each plate are expressed in state-space form as follows:

\[
\begin{bmatrix}
\dot{x}_i \\
\dot{v}_i
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\frac{k}{m} & -\frac{c}{m}
\end{bmatrix} \begin{bmatrix}
x_i \\
v_i
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{u_i}{m}
\end{bmatrix},
\]

where \(x_i\) and \(v_i\) denote the vertical position and velocity of plate \(i\). For the remainder of the current section, we work with the discrete-time form of Eq. \((38)\).
The economic control objective of each agent $i$ is to, without excessive actuation, minimize the overlap area between its respective plate and those of neighboring agents $i - 1$ and $i + 1$. The resulting stage cost function of agent $i$ is therefore expressed as follows:

$$J_i(x_i, u_i, x_{-i}) = \frac{1}{|N_{-i}|} \sum_{j \in N_{-i}} A_{i,j} + u_i^T u_i,$$

where the set $N_{-i}$ contains the indices of all neighboring plates $j$ that are physically adjacent to plate $i$, $|N_{-i}|$ is the cardinality of the set $N_{-i}$, and $A_{i,j}$ defines the overlap area between agents $i$ and $j$ as follows:

$$A_{i,j} = \begin{cases} 0, & |x_i - x_j| \geq L, \\ L(L - |x_i - x_j|), & |x_i - x_j| < L. \end{cases}$$

Note that, although simple in terms of system dynamics, the numerical problem described above entails nonlinear and non-convex cost functions. Physically, non-convexity stems from the property that any two adjacent plates may be relocated in multiple ways to minimize their respective overlap areas.

### 4.2 Social hierarchy-based DEMPC properties

The interaction graph for an example problem with ten plates is shown in Fig. 4. The solution to the vertex coloring problem for this example involves simply alternating the color of each subsequent node, which yields a chromatic number of two regardless of the quantity of vertices. An appropriate choice for the number of hierarchy levels is thus $N_q = 2$.

We implement prediction and control horizons of $H = 5$ time-steps and a sampling period of 1.0 sec. Within a single sampling period, $N_p = 5$ iterations are permitted for the negotiation of optimal stationary points (i.e. Step 3 in Algorithm 1), followed by five more iterations for trajectory optimization (i.e. Step 4 in Algorithm 1). Finally, the actuation force of any agent is bounded as $-0.25 N \leq u_i \leq 0.25 N$. These values were selected to limit the steady-state displacement of each plate to a maximum of $L = 0.25 m$.

Finally, all plates are initially at rest with zero displacement from equilibrium (i.e. $x_i = 0$ and $u_i = 0$) and are therefore perfectly aligned with their neighbors. The state and input vector trajectories of all agents are initialized as $x_i = (0, \cdots, 0)$ and $u_i = (0, \cdots, 0)$. 
4.3 Simulation results

4.3.1 Social hierarchy-based DEMPC – ten plates
The first results we present pertain to five simulations involving ten plates. The outcomes in these simulations will differ due to the element of randomness in the proposed coordination algorithm. In Fig. 5, we show the variation in social hierarchy levels over the iteration number for all five simulations. Rather than displaying the actual hierarchy level of any particular agent, which would yield a cluttered image, we instead present the cumulative number of social hierarchy level changes. That is, let some cumulative counter start at zero for each simulation, then, each time an agent alters its social hierarchy level, the cumulative count increases by one.

What is observed in Fig. 5 is that, in all five simulations, the cumulative count of social hierarchy variations ultimately reaches a fixed value. This outcome indicates that, with a sufficient number of iterations, the agents sort themselves along a social hierarchy that guarantees conflict-free operation according to Definition 2 in all future iterations, thus validating Theorem 1 concerning conflict resolution.

Next, we plot the evolution of the global cost function $V$ over the iteration number in Fig. 6 for all five simulations, with $V$ computed as follows:

$$V = \sum_{i \in I} \dot{V}_i.$$  \hspace{1cm} (41)

Note that the evolution of $V$ differs in each case due to the element of stochasticity inherent to the proposed algorithm. However, in all five simulations, a reduction in $V$ to some locally-optimal value is evident after conflicts have been resolved. Further, within each time-step (i.e. within each 20 iteration interval), the reduction or preservation of $V$ is apparent after agents have settled on an appropriate social hierarchy. This latter outcome validates Theorem 2 on convergence.

As a reference in Fig. 6, we plot (using a black dotted line) the globally-optimal evolution of $V$, which is obtained by initializing the agents’ hierarchy levels according to the universal social hierarchy from Fig. 4. In this case, the agents do not alter their social hierarchy levels as conflict-free operation is guaranteed from the beginning of the simulation. As a result, $V$ is reduced or preserved within each time-step immediately from the start of the simulation. This result validates the existence of a universal social hierarchy.

In Fig. 7, we plot the evolution of stationary target positions over the iteration number for five simulations. Rather than displaying the stationary targets of individual agents, which would yield a cluttered image, we plot the mean stationary target $x_s$ of the entire plant, which is computed as follows:

$$x_s = \frac{1}{N} \sum_{i \in I} \bar{x}_s^i.$$  \hspace{1cm} (42)

It is evident from Fig. 7 that values of $x_s^i$ for all agents $i \in I$ ultimately converge to fixed values in all simulations. This outcome validates Theorem 4 concerning bounded closed-loop stability, which requires fixed stationary targets.

Finally, we have plotted the locations of all plates at the final sampling time in each simulation in Fig. 8. The color of each plate indicates its social hierarchy level (i.e. blue denotes $q_i = 1$, red denotes $q_i = 2$). The globally-optimal
Figure 6: Evolution of the global cost function for five simulations of 10 plates.

Figure 7: Evolution of the mean system-wide stationary target position for five different simulations involving 10 plates.
configuration would involve all adjacent plates being relocated in opposite directions; however, the social hierarchy-based DEMPC algorithm is only capable of finding a locally-optimal layout wherein some plates (e.g. plate 8 in simulation 1) must remain at the origin to minimize overlap with their neighbors.

4.3.2 Parallel vs. social hierarchy-based DEMPC

We compare the performance of the proposed social hierarchy-based DEMPC to a basic parallel DEMPC algorithm wherein all agents solve their local EMPC problems simultaneously and exchange stationary vectors and trajectories with their neighbors. In essence, a parallel DEMPC algorithm is identical to Algorithm 1, except with $N_q = 1$.

We first plot the evolution of $V$ over the iteration number using a parallel DEMPC algorithm in Fig. 9 for five simulations, each consisting of a different number of plates. The parallel DEMPC algorithm is in fact able to naturally resolve conflicts and ultimately decrease $V$ to some locally minimum value. However, as the quantity of agents $N$ increases, a greater number of iterations is required to decrease $V$ to a locally-minimum value. For instance, with $N = 10$, $V$ reaches a local minimum in 20 iterations. If $N = 80$ however, over 100 iterations are required.

In Fig. 10, we plot the same information as Fig. 9 except using the proposed social hierarchy-based DEMPC algorithm. The improvement is clear. The value of $N$ has no discernible effect on the number of iterations necessary for reducing $V$ to a locally-minimum value. In each simulation, approximately 30 iterations are required for $V$ to settle at some minimum value. This outcome results from the fact that the likelihood of any agent resolving conflict locally is dependent solely on its neighborhood interaction topology. Thus, raising $N$ should not impact the number of iterations required for system-wide conflict resolution. This outcome is further validation of the scalability of the social hierarchy-based method.

5 Conclusion

We have presented a novel concept for addressing non-convexity in cost functions of distributed economic model predictive control systems with unknown terminal stationary targets. This concept involves agents self-organizing into a finite hierarchy using evolutionary principles, and ultimately enables agents to make decisions that are mutually beneficial with those of their neighbors. Theorems guaranteeing convergence, recursive feasibility, and bounded closed-loop stability have also been provided for the proposed social hierarchy-based algorithm.
Figure 9: Evolution of the global cost function $V$ using a parallel DEMPC algorithm for five simulations involving different quantities of plates.

Figure 10: Evolution of the global cost function $V$ using the proposed social hierarchy-based DEMPC algorithm for five simulations involving different quantities of plates.
These theorems were validated using a numerical example involving a series of suspended square plates wherein each agent attempted to minimize the overlap area between its respective plate and those of its neighbors. Results showed that, across five simulations, the proposed algorithm was capable of establishing a social hierarchy that reduced the system-wide cost function to some locally-minimum value. Another observation from numerical results was that increasing the size of the distributed system (i.e. the number of plates and agents) had no discernible effect on the number of iterations required to minimize cost function values to local minima. This behavior was not observed when using a parallel DEMPC algorithm with no mechanism for addressing non-convexity.

We recommend the following items as future topics of research for further developing the proposed social hierarchy-based DEMPC algorithm:

- Developing non-iterative algorithms using compatibility constraints as first proposed by Dunbar and Murray [28].
- Employing Lyapunov constraints to guarantee asymptotic stability rather than bounded stability.
- Guaranteeing convergence, feasibility, and stability under the effects of coupled dynamics and constraints.
- Ensuring robustness in the presence of bounded disturbances in the system dynamics and cost functions.
- Application of the proposed algorithm to distributed system with weak dynamics coupling such as autonomous vehicle trajectory planning and wind farm control.

References

[1] D.Q. Mayne, J.B. Rawlings, C.V. Rao, and P.O.M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[2] David Q. Mayne. Model predictive control: Recent developments and future promise. *Automatica*, 50(12):2967–2986, 2014.

[3] Walid Al-Gherwi, Hector Budman, and Ali Elkamel. Robust distributed model predictive control: A review and recent developments. *Canadian Journal of Chemical Engineering*, 89(5):1176–1190, 2011.

[4] Panagiotis D. Christofides, Riccardo Scattolini, David Muñoz de la Peña, and Jinfeng Liu. Distributed model predictive control: A tutorial review and future research directions. *Computers and Chemical Engineering*, 51:21–41, 2013.

[5] R. R. Negenborn and J. M. Maestre. Distributed model predictive control: An overview and roadmap of future research opportunities. *IEEE Control Systems Magazine*, 34(4):87–97, 2014.

[6] Matthew Ellis, Helen Durand, and Panagiotis D. Christofides. A tutorial review of economic model predictive control methods. *Journal of Process Control*, 24(8):1156–1178, 2014.

[7] Matthias A. Muller and Frank Allgower. Economic and Distributed Model Predictive Control: Recent Developments in Optimization-Based Control. *SICE Journal of Control, Measurement, and System Integration*, 10(2):39–52, 2017.

[8] Jan Eilbrecht and Olaf Stursberg. Hierarchical solution of non-convex optimal control problems with application to autonomous driving. *European Journal of Control*, 50:188–197, 2019.

[9] Ruigang Wang, Ian R. Manchester, and Jie Bao. Distributed economic MPC with separable control contraction metrics. *IEEE Control Systems Letters*, 1(1):104–109, 2017.

[10] Jaehwa Lee and David Angeli. Cooperative distributed model predictive control for linear plants subject to convex economic objectives. In *Proceedings of the IEEE Conference on Decision and Control*, number 6, pages 3434–3439. IEEE, 2011.

[11] Jaehwa Lee and David Angeli. Distributed cooperative nonlinear economic MPC. In *Proceedings of the 20th International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, 2012.

[12] P. A.A. Driessen, R. M. Hermans, and P. P.J. Van Den Bosch. Distributed economic model predictive control of networks in competitive environments. In *Proceedings of the IEEE Conference on Decision and Control*, pages 266–271, 2012.
[13] Xianzhong Chen, Mohsen Heidarinejad, Jinfeng Liu, and Panagiotis D. Christofides. Distributed economic MPC: Application to a nonlinear chemical process network. *Journal of Process Control*, 22(4):689–699, 2012.

[14] Fahad Albalawi, Helen Durand, and Panagiotis D. Christofides. Distributed economic model predictive control with Safeness-Index based constraints for nonlinear systems. *Systems and Control Letters*, 110:21–28, 2017.

[15] Inga J. Wolf, Holger Scheu, and Wolfgang Marquardt. A hierarchical distributed economic NMPC architecture based on neighboring-extremal updates. In *Proceedings of the American Control Conference*, pages 4155–4160. IEEE, 2012.

[16] Philipp N. Köhler, Matthias A. Müller, and Frank Allgöwer. A distributed economic MPC framework for cooperative control under conflicting objectives. *Automatica*, 96:368–379, 2018.

[17] Jinfeng Liu, Xianzhong Chen, David Muñoz De La Peña, and Panagiotis D. Christofides. Iterative distributed model predictive control of nonlinear systems: Handling asynchronous, delayed measurements. *IEEE Transactions on Automatic Control*, 57(2):528–534, 2012.

[18] Yoshiaki Kuwata, Arthur Richards, Tom Schouwenaars, and Jonathan P. How. Distributed robust receding horizon control for multivehicle guidance. *IEEE Transactions on Control Systems Technology*, 15(4):627–641, 2007.

[19] A Richards and J P How. Robust distributed model predictive control. *International Journal of Control*, 80(9):1517–1531, 2007.

[20] Matthias A. Müller, Marcus Reble, and Frank Allgöwer. Cooperative control of dynamically decoupled systems via distributed model predictive control. *International Journal of Robust and Nonlinear Control*, 22:1376–1397, 2012.

[21] J. M. Maestre, D. Muñoz De La Peña, E. F. Camacho, and T. Alamo. Distributed model predictive control based on agent negotiation. *Journal of Process Control*, 21(5):685–697, 2011.

[22] Brett T. Stewart, Stephen J. Wright, and James B. Rawlings. Cooperative distributed model predictive control for nonlinear systems. *Journal of Process Control*, 21(5):698–704, 2011.

[23] Jürgen Pannek. Parallelizing a state exchange strategy for noncooperative distributed NMPC. *Systems and Control Letters*, 62(1):29–36, 2013.

[24] Peng Liu, Arda Kurt, and Umit Ozguner. Distributed Model Predictive Control for Cooperative and Flexible Vehicle Platooning. *IEEE Transactions on Control Systems Technology*, 27(3):1115–1128, 2019.

[25] Elham (Fatemeh) Asadi and Arthur Richards. Scalable distributed model predictive control for constrained systems. *Automatica*, 93:407–414, 2018.

[26] Xiaoning Du, Ugeng Xi, and Shaoyuan Li. Distributed model predictive control for large-scale systems. In *Proceedings of the American Control Conference*, pages 3142–3143, 2001.

[27] Dong Jia and Bruce H. Krogh. Distributed model predictive control. In *Proceedings of the American Control Conference*, pages 2767–2772, 2001.

[28] William B. Dunbar and Richard M. Murray. Distributed receding horizon control for multi-vehicle formation stabilization. *Automatica*, 42(4):549–558, 2006.

[29] Gary Chartrand and Ping Zhang. *Chromatic Graph Theory*. CRC Press, Taylor and Francis Group, New York, NY, 2008.