Generalized quadrature for solving singular integral equations of Abel type in application to infrared tomography

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Abstract

We propose the generalized quadrature methods for numerical solution of singular integral equation of Abel type. We overcome the singularity using the analytic computation of the singular integral. The problem of solution of singular integral equation is reduced to nonsingular system of linear algebraic equations without shift meshes techniques employment. We also propose generalized quadrature method for solution of Abel equation using the singular integral. Relaxed errors bounds are derived. In order to improve the accuracy we use Tikhonov regularization method. We demonstrate the efficiency of proposed techniques on infrared tomography problem. Numerical experiments show that it make sense to apply regularization in case of highly noisy (about 10\%) sources only. That is due to the fact that singular integral equations enjoy selfregularization property.

Keywords: integral equations, singular kernels, quadrature, regularization, Abel equation, infrared tomography, midpoint quadrature.

1. Introduction

Numerical methods for solving a variety of singular integral equations (SIE) are offered in many publications, here readers may refer to [2–5, 13, 14, 18, 21, 22, 24, 27, 31, 34, 37, 40, 41] and others. A one-dimensional SIE of the 1st and 2nd kind with Cauchy kernels Hilbert kernels, logarithmic et al., as well as two-dimensional, nonlinear SIE have been addressed. In present article we concentrate on Abel singular integral equation [1, 2, 6, 7, 8, 14, 22, 24, 26, 35, 37, 39, 40]

\begin{equation}
2 \int_{x}^{R} \frac{r}{\sqrt{r^{2} - x^{2}}} k(r) \, dr = q(x), \quad 0 \leq x \leq R,
\end{equation}

where $k(r)$ is desired function, $q(x)$ is the source function. Equations (1) are widely used in practical models including plasma diagnostics, thermal tomography, X-ray CT, spectroscopy, galaxy clusters astrophysics, etc. In all these problems the object of interest enjoy the axial (or spherical) symmetry. Abel equation is also can be written as

\begin{equation}
\int_{0}^{x} \frac{k(r)}{\sqrt{x - r}} \, dr = q(x), \quad 0 \leq x \leq R.
\end{equation}

It has been studied in this form is [2, 5, 13, 14, 18, 21, 22, 27, 29, 31, 34, 39, 41]. Equation (2) describes various problems in mechanics (such as tautohron problem), scattering and other problems. Of course, one may transfer SIE (1) into SIE (2) and vice versa, but it makes it more complicated to analyse their physical meaning.
Let us below outline the main algorithms for numerical solution of SIE and singular integrals computation. For more details readers may refer to [36, 37].

1. Algorithms based on relevant mesh shift. In papers [3, 4] the discrete meshes of knots with respect to variables \( r \) and \( x \) are introduced, i.e. \( r_j = jh, x_i = r_i + \Delta, j = 0, 1, \ldots, n, r_n = R \), where step \( h = R/n \), \( \Delta \) is mesh shift which is \( h/2 \) [3] or \( \Delta \in (0, h/2) \) [4]. Introduction of the shift \( \Delta \) enables singularity overcoming when it comes to quadrature rules application. But such algorithms need this shift selection.

2. Quadrature type methods. One of the popular methods (here readers may refer to work [3]) is Discrete Vortices Method where the integral with Cauchy kernel

\[
\frac{1}{2\pi} \int_{-1}^{1} \frac{\gamma(x)}{x-x_0} \, dx = f(x_0), -1 < x_0 < 1,
\]

is approximated with lift rectangles quadrature rule and using meshes on \( x \) and \( x_0 \) with shift \( \Delta = h/2 \). This gives the system of linear algebraic equations (SLAE) with non zero main diagonal.

In work [5] the “onion peeling” method for solution of SIE [1] is suggested. Here region \( r \in [0, R] \) is approximated with rings \( \Delta r \) wide of constant values \( k \in (r_j - \Delta r/2, r_j + \Delta r/2) \) for each \( r_j \). Here meshes are assumed to be uniform (\( \Delta = 0 \)). The main idea in this method is that integral \( \int_{r_j - \Delta r/2}^{r_j + \Delta r/2} \sqrt{r^2 - x^2} \, dr \) can be computed analytically and its finite. Further midpoint quadrature is used resulting systems of linear algebraic equations with upper triangular matrix with respect to \( k_j = k(x_j) \). The similar method is suggested in [37].

3. Solution approximation. In works [2, 24, 30, 31, 34, 40] et al., the desired solution \( k(r) \) (as well as the right-hand side \( q(x) \)) is approximated with an orthogonal polynomial, shifted Legendre polynomials, normalized Bernstein polynomials, algebraic or trigonometric polynomial or polynomial spline with coefficients determined with minimum of discrepancy between the left-hand side and right-hand side of [1]. This leads to a projection method (the Galerkin method, the collocation method, the method of splines, the quadrature method, the least squares method, etc.) and to the solution of a SLAE wrt the corresponding polynomial coefficients.

In these algorithms, there is a self-regularization, and in the case of using the relative shift of meshes, the shift \( \Delta \) plays the role of the regularization parameter. Namely if \( \Delta \) is closer to \( h/2 \) then solution \( k(r) \) is more stable, but it makes reduction of resolving capability of the method. If \( \Delta \) is closer to zero, then solution is less stable but resolving capability of the method is higher. In all these algorithms, a SLAE is with prevailing (but not infinite) matrix diagonal.

We also note a number of algorithms. Equation [1], as is known, has an analytical solution \([1, 2, 6, 8, 24, 30, 37]\)

\[
k(r) = -\frac{1}{\pi} \int_{r}^{R} \frac{q'(x)}{\sqrt{x^2 - r^2}} \, dx, \quad 0 \leq r \leq R.
\]

However, solution [3] contains derivative \( q'(x) \) of experimental (noisy) function \( q(x) \) and the problem of differentiation is ill-posed [38]. Moreover, integral in [3] is improper (singular). Nevertheless, a number of the following algorithms is proposed to compute the solution according to [3].

4. Interpolation and quadrature method. In [6], derivative \( q'(x) \) was computed using interpolation on three (and two) neighboring points (discrete values of \( x \)). Integral \( \int_{r}^{R} \frac{dx}{\sqrt{x^2 - r^2}} \) (cf. [3]) is computed analytically (without singularity). The similar algorithm was suggested in [37] using generalized left rectangles formula.

5. Approximation of the right-hand side \( q(x) \) is used in works [18, 24, 40]. Function \( q(x) \) is suggested to be approximated by a linear combination of smoothing polynomials (or splines) uniform for the whole interval \( x \in [0, R] \). Derivative \( q'(x) \) is computed using polynomial (or spline) differentiation. Solution \( k(r) \) in accordance with [3] is computed by summing the integral in [3] along segments that performed analytically (see [40, pp. 188–189]).

6. Algorithm without using derivative \( q'(x) \). In [9], formula [3] is converted (by means of integration by parts) into the following expression that does not contain derivative \( q'(x) \) (cf. [8, 40]):

\[
k(r) = -\frac{1}{\pi} \left[ \frac{q(R) - q(r)}{\sqrt{R^2 - r^2}} + \int_{r}^{R} \frac{x[q(x) - q(r)]}{\sqrt{(x^2 - r^2)^3}} \, dx \right], \quad 0 \leq r \leq R.
\]
This algorithm is implemented, e.g., in paper [40, pp. 217–220] using the cubic spline (see [23, p. 273]) for $q(x)$.

7. Use of regularization. Abel’s equation (1) enjoys self-regularizing property due to the singularity, as a result the problem of its solving is moderately ill-posed [8]. This means that above mentioned algorithms are moderately stable. Nevertheless, in papers [1, 7, 8, 14] et al., the Tikhonov regularization method [10, 16, 38, 39] was used to enhance the stability of algorithms.

In this work, we develop the following variant for numerical solving some SIE. We make the meshes of nodes in $r$ and $x$ coincide (i.e. $\Delta = 0$) and eliminate the singularities using the generalized quadrature formula (cf. [19, 36, 37]). However, such a technique can be applied to not all SIE. For example, it is not applicable to SIE with the Cauchy kernel, but it is applicable to some SIE with logarithmic and other (weakly singular) kernels. In this paper, we consider the solution of equation (1) wrt the desired function $k(r)$, as well as numerical computation of $k(r)$ according to (3) by the generalized quadrature method with use of Tikhonov regularization.

2. The generalized quadrature method

Let us describe method using generalized left rectangles formula in application to numerical solving SIE (1) (the first method) and to computation of $k(r)$ according to (3) (the second method).

It is to be noted here that in [6, 8], the numerical method “onion-peeling” is suggested for computation of integrals in (1) [8] and in [3, 6]. Here, the uniform coinciding node meshes in $r$ and $x$ and the middle rectangles quadrature formula have been employed. In [37], also uniform coinciding meshes have been employed combined with more usable the left rectangles formula.

In present paper, we use nonuniform meshes and left rectangles resulting more generic and convenient algorithm. The solution error estimates for equation (1) by the generalized quadrature method are also derived. This method is described below in two variants (the first and second methods).

2.1. First quadrature method

First quadrature method employs generalized left rectangle formula. Let us introduce nonuniform (but coinciding) meshes on $x$ and $r$ as follows

$$0 = x_1 = r_1 < x_2 = r_2 < \ldots < x_i = r_i < \ldots < x_n = r_n = R.$$  

Here, $R = r_{\max}$ is boundary value such as $k(R + 0) = 0$. On each interval $[r_j, r_{j+1})$, $j = 1, 2, \ldots, n - 1$ we suppose approximately $k(r) = k(r_j) \equiv k_j = \text{const}$.  

We have the following

**Lemma 2.1.** Under condition (5), one has the equality

$$\int_{r_j}^{r_{j+1}} \frac{r}{\sqrt{r^2 - x^2}} k(r) \, dr = \left( \sqrt{r_j^2 - x^2} - \sqrt{r_{j+1}^2 - x^2} \right) k_j, \quad j \in [1, n - 1], \quad x \leq r_j < r_{j+1} \leq R.$$  

**Proof.** Integral (table)

$$\int_{r_j}^{r} \frac{r}{\sqrt{r^2 - x^2}} \, dr = \sqrt{r^2 - x^2} \quad \text{for} \quad x \leq r,$$

whence, taking into account (5), we obtain (6). \qed

**Definition 2.1.** We call formula (5) as generalized quadrature formula of left rectangles (cf. (19)) for the specific singularity $r/\sqrt{r^2 - x^2}$, and multipliers $\sqrt{r_{j+1}^2 - x^2} - \sqrt{r_j^2 - x^2}$ are the quadrature coefficients of this singularity.
The specifics of the formula (6) is that the singular integral \( \int_{r_{j-1}}^{r_j} \frac{r}{\sqrt{r^2 - x_i^2}} dr \) is calculated analytically accurate and without peculiarity. If it is calculated numerically by the usual left rectangles quadrature formula, then at \( x = r_j \) there will be a division by zero.

Let us now formulate the main result as following

**Theorem 2.1.** Numerical solution of equation (1) according to the first quadrature method is defined as following recursion

\[
\begin{align*}
  k_{n-1} &= \frac{q_{n-1}}{p_{n-1,n-1}}, \\
  k_i &= \frac{q_i/2 - \sum_{j=1}^{n-1} p_{ij}k_j}{p_{ij}}, \quad i = n - 2, n - 3, \ldots, 1, \\
  k_n &= k_{n-2} + \left( \frac{r_n - r_{n-2}}{r_{n-1} - r_{n-2}} \right) (k_{n-1} - k_{n-2}),
\end{align*}
\]

where \( k_i \equiv k(r_i), q_i \equiv q(x_i) \).

\[
p_{ij} = \sqrt{r_{j+1}^2 - x_i^2} - \sqrt{r_j^2 - x_i^2},
\]

Proof. Integral in (1) is sum of integrals (6), i.e.

\[
\int_{x_i}^{R} \frac{r}{\sqrt{r^2 - x_i^2}} k(r) \, dr = \sum_{j=1}^{n-1} \left( \sqrt{r_{j+1}^2 - x_i^2} - \sqrt{r_j^2 - x_i^2} \right) k_j = q_i/2,
\]

\( i = 1, 2, \ldots, n - 1 \).

This is the SLAE wrt \( \{k_j\}_{j=1}^n \). SLAE (9) is upper triangular and its solution can be recursively constructed. From (9) for \( i = n - 1, n - 2, \ldots, 1 \) we eventually obtain \( k_{n-1}, k_{n-2}, \ldots, k_1 \) according to (7). As to the value of \( k_n \equiv k(R) \), it can not be found by this scheme, but can be additionally determined as \( k_n = 0 \) from physical concepts or \( k_n = k_{n-1} \) or can be derived using linear extrapolation (35), as was done in (7).

In (8), formulae of type (6) are also given, but for the case of uniform (and coinciding) meshes in \( r \) and \( x \) and using the middle rectangles quadrature formula. Furthermore, important formulae of type (7) is not given.

Formulæ (4)–(9) are more common and more convenient than in (8) and formulæ (7) give a solution in the explicit form. The method according to (4)–(9) for solving the equation (1) is called in (37) the generalized quadrature method for solving SIE (1). This paper presents a more general formulæ (4) and (7) than in (37). In Sec. 3, estimates of the errors for this method are given.

2.2. Second quadrature method

The second method is generalized quadrature method for computation of singular integral (3) giving the solution \( k(r) \). Let us assume \( k'(x) \) to be computed with some stable method. As in the first method we introduce node meshes (4). On each \( [x_i, x_{i+1}], i = 1, 2, \ldots, n - 1 \) we assume

\[
q'(x) = q'(x_i) \equiv q_i' = \text{const}.
\]

Proof. Integral \( \int_{x_i}^{x_{i+1}} \frac{q'(x)}{\sqrt{x^2 - r^2}} \, dx \) is equal to

\[
\int_{x_i}^{x_{i+1}} \frac{q'(x)}{\sqrt{x^2 - r^2}} \, dx = \ln \frac{x_{i+1} + \sqrt{x_{i+1}^2 - r^2}}{x_i + \sqrt{x_i^2 - r^2}} q_i',
\]

\( i = 1, 2, \ldots, n - 1, \quad r \leq x_i < x_{i+1} \leq R \).

Proof. Integral \( \int \frac{dx}{\sqrt{x^2 - r^2}} = \ln \left( x + \sqrt{x^2 - r^2} \right) \) for \( r \leq x \) is the table integral. Taking into account the condition (10) we obtain (11).
Lemma 3.1. Integral 
\[ \int_{r_i}^{r_j} \frac{k(r) dr}{\sqrt{r^2 - x_i^2}} \] is a result of the following recurrence formula:

\[ k_j = \frac{-1}{\pi} \sum_{i=1}^{n-1} g_{ij} q_i', \quad j = 2, 3, \ldots, n - 1, \]

where

\[ g_{ij} = \frac{x_{i+1} + \sqrt{x_{i+1}^2 - r_j^2}}{x_i + \sqrt{x_i^2 - r_j^2}}. \]

Proof. Integral in (5) is sum of integrals (11) over the separate intervals \([x_i, x_{i+1})\), i.e.

\[ \int_{r_i}^{r_j} \frac{q_i'(x)}{\sqrt{x^2 - r_j^2}} dx = \sum_{i=1}^{n-1} \ln \frac{x_{i+1} + \sqrt{x_{i+1}^2 - r_j^2}}{x_i + \sqrt{x_i^2 - r_j^2}} q_i', \quad j = 1, 2, \ldots, n - 1. \]

As a result, solution (3) in the discrete form is \( \{k_j\}_{j=2}^{n-1} \) according to (12), (13). For \( j = 1 \) this (second) method due to (14) gives uncertainty \( \infty \cdot 0 \) for \( i = j = 1 \) since \( x_1 = r_1 = q_1' = 0 \). In this case let’s use the first method to determine \( k_1 \). Using (7) and (8) for \( i = 1 \) we find \( k_1 \), ref. (12). As to \( k_n \) it can be calculated using linear extrapolation (see (7)).

The advantage of the above two methods is that they do not require the relative shift of meshes and their integrals with singularities \( r/\sqrt{r^2 - x^2} \) and \( 1/\sqrt{r^2 - x^2} \) are calculated analytically and without divergences.

However, these methods are not suitable for all singularities, e.g., for numerical computation of hypersingular integral with the Cauchy kernel \( \int_{-1}^{1} \frac{dx}{r^2 - x^2} \) for \( i = 1 \).

3. Error estimates

Let us derive errors estimates for solution of SIE (1) using first quadrature method (cf. [37, 39, 20]).

3.1. Quadrature error on small interval

Let us estimate quadrature error for computing integral (6) on separate small interval \([r_j, r_{j+1})\) due to approximation (5) (while without the measurement error for \( q(x) \)).

Lemma 3.1. Integral

\[ \int_{r_i}^{r_j} \frac{r^{i+1}}{\sqrt{r^2 - x_i^2}} k(r) dr, \quad x_i \leq r_j < r_{j+1} \leq R, \]

\[ i = 1, 2, \ldots, n - 1, \quad j = i, \ldots, n - 1, \]

when using the generalized left rectangles formula (6) and taking account of quadrature error caused by the approximation (5) is equal to (refinement of formula (6))

\[ \int_{r_i}^{r_{j+1}} \frac{r^{i+1}}{\sqrt{r^2 - x_i^2}} k(r) dr = p_{ij} k_j + \Delta e_{ij}, \]

where

\[ p_{ij} = \sum_{i=1}^{n-1} g_{ij} q_i'. \]
where \( p_{ij} \) are quadrature coefficients \(^{[8]}\) and \( \Delta \varepsilon_{ij} \) is quadrature error of computation of integral \(^{[15]}\) approximately equal

\[
\Delta \varepsilon_{ij} = \frac{k'(\xi_j)}{2} \left( r_{j+1} - 2r_j \right) \left[ \frac{r_j^2 - x_j^2}{r_{j+1}^2 - x_j^2} + \frac{r_{j+1}^2 - x_j^2}{r_j^2 - x_j^2} \right] + \frac{1}{2} x_j^2 \ln \frac{r_{j+1} + \sqrt{r_{j+1}^2 - x_j^2}}{r_j + \sqrt{r_j^2 - x_j^2}}, \quad \xi_j \in [r_j, r_{j+1}).
\]  \( ^{(17)} \)

**Proof.** Using the first method we assume \( \bar{k}(r) = k_j, \ r \in [r_j, r_{j+1}) \) (see \(^{[5]}\)), i.e. we represent function \( k(r) \) by the interpolation Lagrange zero degree polynomial \(^{[17]}\). Error of such interpolation is \( \Delta k_j(r) = k(r) - k_j = k'(\xi_j)(r - r_j) \), where \( \xi = \xi_j(r) \in [r_j, r_{j+1}) \). Then

\[
k(r) = k_j + k'(\xi)(r - r_j). \]

Let us now substitute \(^{(18)}\) into \(^{(15)}\), we get \(^{(16)}\), where

\[
\Delta \varepsilon_{ij} = k'(\xi_j) \int_{r_j}^{r_{j+1}} \frac{r(r - r_j)}{\sqrt{r^2 - x_j^2}} dr.
\]  \( ^{(19)} \)

Integral in \(^{(19)}\) can be analytically computed giving us an estimate \(^{(17)}\). □

It is to be noted that derivative \( k'(\xi) \) in \(^{(17)}\) can be approximated with

\[
k'(\xi_j) = \frac{k_{j+1} - k_j}{r_{j+1} - r_j}, \quad j = 1, 2, \ldots, n - 1,
\]  \( ^{(20)} \)

or by other way \(^{[37]}\).

### 3.2. Quadrature error

Let us estimate quadrature error of the solution of equation \(^{(1)}\) due to the approach \(^{[5]}\) (without error \( q(x) \)). We formulate this as a theorem.

**Theorem 3.1.** Errors of numerical solution of equation \(^{(1)}\) by the first generalized quadrature method according to \(^{[7]}\) are computed with the following recurrence

\[
\begin{cases}
\Delta k_n = \Delta k_{n-1} = \frac{\Delta \varepsilon_{n-1, n-1}}{P_{n-1, n-1}}, \\
\Delta k_i = \varepsilon_i - \sum_{j=1}^{n-1} p_{ij} \Delta k_j, & i = n - 2, n - 3, \ldots, 1,
\end{cases}
\]

\( ^{(21)} \)

where

\[
\varepsilon_i = \sum_{j=1}^{n-1} \Delta \varepsilon_{ij}.
\]

\( ^{(22)} \)

Here, \( p_{ij}, \Delta \varepsilon_{ij} \) and \( k'(\xi_j) \) are computed based on \(^{[8]}\), \(^{[17]}\) and \(^{[20]}\) respectively.

**Proof.** Let us write integral in \(^{(1)}\) as sum of integrals wrt intervals

\[
\int_{r_i}^{r_{i+1}} \frac{r}{\sqrt{r^2 - x_i^2}} k(r) \, dr = \sum_{j=1}^{n-1} \int_{r_j}^{r_{j+1}} \frac{r}{\sqrt{r^2 - x_j^2}} (k_j + \Delta k_j(r)) \, dr
\]

\[
= \sum_{j=1}^{n-1} \int_{r_j}^{r_{j+1}} \frac{r}{\sqrt{r^2 - x_j^2}} (k_j + k'(\xi)(r - r_j)) \, dr, \quad i = 1, 2, \ldots, n - 1.
\]
Then
\[
\sum_{j=1}^{n-1} \int_{r_j}^{r_{j+1}} \frac{r}{\sqrt{r^2 - x_i^2}} \Delta k_j(r) \, dr = \sum_{j=1}^{n-1} \int_{r_j}^{r_{j+1}} \frac{r (r - r_j)}{\sqrt{r^2 - x_i^2}} k'(\xi) \, dr, \quad i = 1, 2, \ldots, n - 1.
\] (23)

In order to compute the integral in the left-hand side of (23), we employ the formula of left rectangles, i.e., we assume \( \Delta k_j(r) = \Delta k(r_j) \equiv \Delta k_j = \text{const} \), \( r \in [r_j, r_{j+1}) \) and compute the integral using the generalized formula in similar way with (6). Integral in the right-hand side of (23) is equal to \( \Delta e_i \), due to (19). Then
\[
\sum_{j=1}^{n-1} p_{ij} \Delta k_j = e_i, \quad i = 1, 2, \ldots, n - 1,
\] (24)

where \( e_i \) denote sum (22). Here, (24) is a SLAE wrt \( \{\Delta k_i\}_{i=1}^{n} \). It is also assumed that \( \{k_i\}_{i=1}^{n} \) are computed using (7) in advance. Then values \( \{\varepsilon_i\}_{i=1}^{n-1} \) are computed using (22), (17) and (20). We solve SLAE (24) with upper triangular matrix and obtain solution (21) in the recurrent form, adding the condition \( \Delta k_n = \Delta k_{n-1} \).

Remark 2. It is to be noted here that formulae (21) give errors of the solution \( \Delta k_i \) with their signs (cf. [37, 39]) in contrast with other works where absolute values \( |\Delta k_i| \) or upper bounds \( |\Delta k_i| \leq \ldots \) or upper bounds by the norm \( \|\Delta k_i\| \leq \ldots \) etc. are given. This enable us to obtain the refined solution
\[
\tilde{k}_i = k_i + \Delta k_i, \quad i = 1, 2, \ldots, n,
\] (25)

using \( \{k_i\}_{i=1}^{n} \) from (7) and errors \( \{\Delta k_i\}_{i=1}^{n} \) from (21).

Remark 3. Errors of numerical solution given in (21) are obtained with regard to only the quadrature errors and the error of the right-hand side \( q(x) \) of equation (1) is set equal to zero. Let us take into account the measurement errors \( \{\delta_i\}_{i=1}^{n} \) of the source function \( q(x) \). In [20, 39], the error estimates for numerical solution of the V olterra integral equations of the first and second kind are derived taking into account both quadrature and source function errors. In similar way we can generalize recurrence formula (21) to the case of errors \( \{\delta_i\}_{i=1}^{n} \) as follows
\[
\begin{cases}
|\Delta k_n| = |\Delta k_{n-1}| = \frac{|\Delta e_{n-1} - \varepsilon_{n-1}| + |\delta_{n-1}|}{p_{n-1,n-1}}, \\
|\Delta k_i| = |\varepsilon_i| + |\delta_i| + \sum_{j=i+1}^{n-1} p_{ij} |\Delta k_j|, \quad i = n - 2, n - 3, \ldots, 1.
\end{cases}
\] (26)

But estimates (26) give overstated estimates of \( |\Delta k_i|_{i=1}^{n} \) due to using the operation \( | \cdot | \) (absolute value).

4. Numerical illustration

4.1. Software implementation

Proposed two generalized quadrature methods have been implemented in MatLab 7.10 (R2010a). Following the first method we search \( \{k_i\}_{i=1}^{n} \) using (7), (8). Errors \( \{\Delta k_i\}_{i=1}^{n} \) are calculated using (21), as well as (8), (17), (20) and (22). Refined solution \( \{\tilde{k}_i\}_{i=1}^{n} \) is calculated with (25). Tikhonov regularization [10, 16, 38, 39]
\[
k_\alpha = (\alpha E + A^T A)^{-1} A^T f
\] (27)
is employed, where the discrepancy principle [25] is used for choosing the regularization parameter \( \alpha > 0 \):
\[
||\Delta k_\alpha - f|| = \delta.
\] (28)

Here, \( f = q/2 \), \( E \) is identity matrix, \( A \) is matrix of the SLAE (9) represented as
\[
Ak = f,
\] (29)

where
\[
A_{ij} = \begin{cases} 
p_{ij}, & j \geq i, \\
0, & \text{otherwise},
\end{cases} \quad i, j = 1, 2, \ldots, n - 1.
\] (30)

Following the second method, solution \( k(r) \) has been computed using singular integral in (3) based on generalized left rectangles formula according to (12) and (13).

The first method was developed and implemented in software in more detail than the second method.
4.2. Infrared tomography example

The first method has been applied to axially symmetric flame diagnostics using infrared tomography [1, 5, 7, 8, 11, 12, 15, 29, 36]. Fig. 1 shows measured output intensity $I_m(x)$ of rays (m from measurement) which go through gas, undergo absorption and emission and are accepted by detectors. Measurements have been performed in Technical University of Denmark, Department of Chemical and Biochemical Engineering (before 1 January 2012 Risø DTU) within a joint project [12, 11].

Figure 1. Measured noisy intensity $I_m(x)$ (difference of intensity in active and passive regimes). The mesh is nonuniform, number of nodes $n = 11$.

Intensity $I_m(x)$ is recalculated into $q_m(x) = -\ln[I_m(x)/B(T_0)]$ (the right-hand side of equation (1)), where $B(T_0)$ is the Planck function of rays source with its temperature $T_0 = 894.4^\circ C$. Fig. 2 shows function $q_m(x)$.

Figure 2. Dimensionless right-hand side $q_m(x)$ of SIE (1), $n = 11$.

Fig. 3 shows results of solution of SIE (1) wrt absorption coefficient $k_m(r)$ by the first method of generalized quadratures according to (7) and (8).

As we see, the solution $k_m(r)$ suffer from significant artificial perturbations. This is due to too great step of node mesh (in other words, the smallness of $n$), as well as measurement errors in function $I_m(x)$. Here we also demonstrate behavior of solution $k_m(r)$ derived with Tikhonov regularization using (27), (29), (30) and (2). Regularization parameter $\alpha$ is chosen using discrepancy principle (28) where $\delta = 0.037$, as a result $\alpha = 10^{-0.09} = 0.813$. Fig. 3 demonstrates that solution has been smoothed by regularization method.

To reduce the grid step in $x$ as well as to moderately smooth the fluctuations in the function $I_m(x)$, a spline approximation was used [23, 35, 40]. Fig. 4 shows an approximation of the function $I_m(x)$ by cubic smoothing spline using the m-function csaps.m.

The smoothed values of $I(x)$ were generated with splines (Fig. 4) and then SIE (1) was resolved with generalized quadrature (7). Fig. 5 shows obtained solution $k(r)$. We also applied solution using Tikhonov regularization (27) for $\alpha = 10^{-2}$. Fig. 5 shows regularized solution $k_\alpha(r)$.
Figure 3. Absorption coefficient $k_m(r)$ computed by the first generalized quadrature method and $k_{m0}(r)$ computed by Tikhonov regularization, cm$^{-1}$.

Figure 4. Measured $I_m(x)$ values are marked with ◦ ($n=11$), values $I_m(x)$ for $n=20$ marked with ● and spline interpolated values are marked with solid line.

Figure 5. Absorption coefficient $k(r)$ (without regularization) and $k_\alpha(r)$ (with regularization) after spline smoothing of $I_m(x)$, $n=20$, cm$^{-1}$.
Fig. 4 and 5 demonstrate that application of spline based smoothing enable mesh step reduction for $x$ (causing increase of $n$). This allows (moderate) smoothing $k(r)$ and $k_n(r)$. Moreover, in case of noisy $I(x)$ and big step of the mesh, regularization slightly improves solution as shown in Fig. 3. In case of $< 1\%$ errors and small step (Fig. 4) solutions $k(r)$ (without regularization) and $k_n(r)$ (with regularization) are obtained practically the same (Fig. 5). It confirms that the problem of solving the singular integral equations is moderately ill-posed and has the property of self-regularization.

5. Conclusion

In this paper we outlined two new methods of numerical solution of singular integral equation (SIE) of Abel type. The methods are based on the use of generalized quadrature formula of left rectangles. Specificity of methods is that singular integrals are computed analytically and without peculiarities. We derive recurrence formulæ for solution, generally speaking, on a nonuniform node mesh. Estimates of quadrature errors of solution with regard to their sign in the absence and in the presence of errors in the right-hand side are found. In order to enhance the stability of the solution we used Tikhonov regularization. However, SIE enjoy self-regularization, therefore it is advisable to apply the Tikhonov regularization method only if there is a significant error ($\sim 10\%$) in the right-hand side and rough mesh step (when the number of nodes is small: $n \sim 10$). The method has been applied for solution of infrared tomography problem.

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