Bernstein Inequalities for Constrained Polynomial Optimization Problems.

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Abstract In this paper, we examine linear programming (LP) relaxations based on Bernstein polynomials for polynomial optimization problems (POPs). We present a progression of increasingly more precise LP relaxations based on expressing the given polynomial in its Bernstein form, as a linear combination of Bernstein polynomials. The well-known bounds on Bernstein polynomials over the unit box combined with linear inter-relationships between Bernstein polynomials help us formulate “Bernstein inequalities” which yield tighter lower bounds for POPs in bounded rectangular domains. The results can be easily extended to optimization over polyhedral and semi-algebraic domains.

Keywords Polynomial Optimization Problem · Bernstein Polynomials · Linear programming

1 Introduction

In this paper, we examine linear programming relaxations for polynomial optimization problems (POP) that seek to optimize a multivariate polynomial $p(x)$ over a compact interval domain $x \in [\ell, u]$. Our approach is based on two ideas: (a) We consider a reformulation of the problem as a linear program using Bernstein polynomials. However, doing so also increases the number of decision variables and constraints in the problem. (b) Next, we present
valid inequalities for improving the approximation. These inequalities are de-

rived from well-known properties of Bernstein polynomials that yield linear

inter-relationships between the decision variables of the linear program. Our

approach is extended to handle compact domains described by semi-algebraic

constraints. We present a branch-and-cut scheme that introduces the cutting

plane inequalities hand-in-hand with a decomposition of the feasible region.

The problem of optimizing polynomials over an interval is well-known to

be non-convex, and is in fact NP-hard. Nevertheless, well-known classes such

as linear, quadratic, or even integer linear programs can be viewed as partic-

ular cases of POPs. Also, since polynomials provide a good approximation for

non linear functions, solving POPs efficiently is a big step toward handling

more complex problems. Finally, a lot of problems arising from disparate do-

mains such as biology, robotics and engineering can be formulated as POPs.

Our interest is motivated by verification and synthesis problems for dynamical

systems such as safety, reachability and stability verification. These problems

can be reduced to POPs. In fact, the motivation of this paper comes from

our previous work, where we aim to prove stability for polynomial dynamical

systems [15]. Therein, Bernstein polynomials were used as an alternative to

the well-known sum of squares (SOS) approach in order to avoid the numerical

issues of semi-definite programming (SDP) [13,17,18]. In this regard, the

Simplex algorithm can be implemented in exact arithmetic to yield numeri-

cally validated lower bounds to the optimal value of the POP, thus formally

establishing the stability of the process. The success of the approach in a large

set of benchmarks motivates us to go further, improve the results and make

them known in an optimization context.

More precisely, we show how POPs can be relaxed to linear programs

thanks to the use of Bernstein polynomials, and a well-known reformulation-

linearization technique (RLT) described by Sherali et al [16,17]. In fact, the

properties of Bernstein polynomials inside the unit box offer us an elegant ap-

proach to improving the RLT approach. We formulate these properties as linear

inequalities to obtain guaranteed lower (upper) bounds for our minimization

(maximization) problems. This will be useful in cases where the POP does not

need to be solved exactly. In the latter case, we combine our inequalities with

a branch-and-bound decomposition process originally described by Nataraj et

al [12].

We evaluate our approach using a set of benchmarks described in Nataraj

et al [12] to characterize the effect of adding the extra Bernstein inequalities

to the RLT approach. We observe that while the addition of these inequalities

improves the lower bound, it is not sufficient for yielding tight bounds.

Next, we consider the addition of Bernstein inequalities in a “branch-and-cut”

approach that combines the addition of cutting planes “on-demand” with a

branch-and-bound decomposition of the domain. We find that all approaches

eventually yield tight bounds on the value of the global optimum. Therefore,

we compare the computational time for various approaches. Finally, we com-

pare the various approaches on benchmarks from our previous work [15] which

consists of a set of polynomial Lyapunov functions used as stability proofs for
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polynomial dynamical systems. In this particular case, our goal is to show that the functions are non-negative over a domain. We adapt the branch-and-bound scheme for this application to evaluate its effectiveness.

The results of our evaluation are mixed: we observe that adding cutting plane inequalities does result in tighter lower bounds on the optimum and therefore examining fewer cells in the branch-and-bound approach. However, this comes at the cost of obtaining larger linear programs due to the extra inequalities, and therefore, an overall larger computation time. We show that the careful consideration of inequalities to be introduced yields a “sweet spot” for better approximations using less computation time.

1.1 Organization

In Section 2, we present basic notions and properties related to Bernstein polynomials. All the results of this section are quite standard, therefore proofs are omitted. Section 3 is the core of the paper. In this section, Bernstein polynomials and their properties inside the unit box are translated into a series of inequalities yielding a corresponding set of LP relaxations of increasing precision. An iterative approach mixing these relaxations is presented, and a criterion for checking if the given lower bound meets the optimal value of the original problem are also given. In Section 4, we show how bounds can be made arbitrary tighter using some techniques such as decomposition (branch-and-bound scheme).

1.2 Related Work

Since solving a POP is generally NP-hard, existing work consists of relaxing it in order to obtain an easier problem for which efficient solvers exist. In the literature, we can distinguish two types of relaxations. The first class is called LP relaxations. These approaches approximate the POP using linear programs that can be efficiently solved using an LP solver. A popular LP relaxation is the reformulation linearization technique (RLT) given by Sherali et al [16,17]. The approach was improved by Nataraj et al [12] for solving POPs, wherein the use of the Bernstein basis was proposed as an improvement. In particular, Nataraj et al made use of the property that Bernstein polynomial coefficients over a box form a lower bound of the polynomial. In this work, we show that this property is simply the optimal value of a LP formed by a series of inequalities that relate one Bernstein polynomial to another. In doing so, we formulate numerous valid inequalities that improve substantially on this bound. Another recent approach called DSOS (Diagonally-dominant Sum of Squares) was formulated by Ali Ahmadi et al [1] by relaxing positive semi-definiteness of a matrix using the stronger condition of diagonal dominance. In fact, Ali Ahmadi’s approach can be seen as selecting a finite set of generators from the infinitely generated cone of positive polynomials in the polynomial ring $\mathbb{R}[x]$. In contrast, our approach
also adds a finite set of generators to the cone of positive polynomials over a compact interval. Naturally, both choices of finite bases involve a tradeoff that are optimal for certain classes of problems. In particular, we choose the Bernstein polynomials and utilize the set of linear inter-relationships between these. Extending our approach to possibly cover the polynomial basis used in the DSOS approach is currently under investigation.

As an alternative to LP relaxations, we can formulate SDP relaxations. In 2001, Lasserre proposed what was called a Linear matrix equality (LMI) relaxation [7]. The main idea is to map the polynomial optimization problem to an optimization problem over probability measures and then use results from moment theory. Subsequently, Parillo introduced the SOS programming approach that has become one of the most popular SDP relaxations [13]. Theoretically, following the comparison made by Lasserre [8] between SDP (LMI) and LP (RLT) relaxations, one concludes that the SDP approach is much more precise at the extra (polynomial) cost of solving an SDP. In fact the comparison points out that for the LP (RLT) relaxation, convergence results to the optimal value are not always guaranteed, in contrast to SDP relaxations. Also, the comparison shows that RLT cannot be exact whenever the global optimum belongs to the interior of the feasible set. We will show in this paper, that this claim does not remain true (see Example 2) when the Bernstein inequalities suggested here are used. Furthermore, in practice, the SDP approach suffers from numerical issues. This was pointed in our previous work [15] when using SOS programming for Lyapunov function synthesis. Other approaches like interval methods [10] and decomposition techniques exists. In this paper, we will focus on the related scheme given by Nataraj et al in [12], since it is fully based on the use of Bernstein coefficients. We will build on this approach by adding the extra Bernstein inequalities.

2 Overview of Bernstein Polynomials

Bernstein polynomials were first proposed by Bernstein as a constructive proof of Weierstrass approximation theorem [4], and are useful in many engineering design applications for approximating geometric shapes [5]. They form a basis for approximating polynomials over a compact interval, and have nice properties inside the unit box (see [11] for more details). We first examine Bernstein polynomials and their properties for the univariate case, and then extend them to multivariate polynomials (see [2,3]).

Definition 1 (Univariate Bernstein Polynomials) Given an index $i \in \{0, \ldots, m\}$, the $i^{th}$ univariate Bernstein polynomial of degree $m$ over $[0, 1]$ is given by the following expression:

$$
\beta_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}.
$$

(2.1)
Using these polynomials, monomials can be written as follows:

\[ x_i^i = \sum_{j=i}^{m} \binom{j}{i} \binom{m}{i} \beta_{j,m}(x), \text{ for all } i = 0, \ldots, m. \] (2.2)

Then, in the Bernstein polynomial basis, polynomial \( p(x) = \sum_{j=0}^{m} p_j x^j \) of degree \( m \) can be written as:

\[ p(x) = \sum_{i=0}^{m} b_{i,m} \beta_{i,m}(x) \]

where for all \( i = 0, \ldots, m \):

\[ b_{i,m} = \sum_{j=0}^{i} \binom{i}{j} \binom{m}{j} p_j. \] (2.3)

The coefficients \( b_{i,m} \) are called the Bernstein coefficients of the polynomial \( p \).

Bernstein polynomials have many interesting properties on the unit interval \([0, 1]\). We summarize the most relevant ones for our applications.

**Lemma 1** Bernstein polynomials have the following properties:

1. **Unit partition:** \( \sum_{i=0}^{m} \beta_{i,m}(x) = 1 \).
2. **Bounds:** \( 0 \leq \beta_{i,m}(x) \leq \beta_{i,m}(0) \), \( \forall i = 0, \ldots, m \).
3. **Induction:** \( \beta_{i,m-1}(x) = \frac{m-i}{m} \beta_{i,m}(x) + \frac{i+1}{m} \beta_{i+1,m}(x) \), \( \forall i = 0, \ldots, m-1 \).

Using these properties, the following result holds:

**Corollary 1** On the interval \([0, 1]\), the following inequality holds [6]:

\[ \min_{i=0,\ldots,m} b_{i,m} \leq p(x) \leq \max_{i=0,\ldots,m} b_{i,m}. \] (2.4)

The equality \( \min_{i=0,\ldots,m} b_{i,m} = \min_{x \in [0,1]} p(x) \), respectively \( \max_{i=0,\ldots,m} b_{i,m} = \max_{x \in [0,1]} p(x) \), holds iff \( \min_{i=0,\ldots,m} b_{i,m} \in \{b_{0,m}, b_{m,m}\} \), respectively \( \max_{i=0,\ldots,m} b_{i,m} \in \{b_{0,m}, b_{m,m}\} \).

This is commonly called the vertex condition.

We generalize the previous notions to the case of multivariate polynomials i.e \( p(x) = p(x_1, \ldots, x_n) \) where \( x = (x_1, \ldots, x_n) \in U = [0,1]^n \). For multi indices, \( I = (i_1, \ldots, i_n) \in \mathbb{N}^n, J = (j_1, \ldots, j_n) \in \mathbb{N}^n \), we will use the following notation throughout this paper:

- \( I + J = (i_1 + j_1, \ldots, i_n + j_n) \).
- \( x^I = x_1^{i_1} \cdot x_2^{i_2} \cdots \cdot x_n^{i_n} \).
\(- I \leq J \iff i_l \leq j_l, \text{ for all } l = 1, \ldots, n.\)
\(- I = \left( \frac{i_1}{j_1}, \ldots, \frac{i_n}{j_n} \right) \text{ and } \left( \frac{I}{J} \right) = \left( \frac{i_1}{j_1} \right) \cdots \left( \frac{i_n}{j_n} \right).\)
\(- I_{r,k} = (i_1, \ldots, i_{r-1}, i_r+k, i_{r+1}, \ldots, i_n) \text{ where } r \in \{1, \ldots, n\} \text{ and } k \in \mathbb{Z}.\)

Let us fix our maximal degree \(\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n\) for a multivariate polynomial \(p (\delta_l \text{ is the maximal degree of } x_l \text{ for all } l = 1, \ldots, n).\) Then the multivariate polynomial \(p\) can be written as:

\[
p(x) = \sum_{I \leq \delta} p_I x^I \text{ where } p_I \in \mathbb{R}, \forall I \leq \delta.
\]

Multivariate Bernstein polynomials are given by products of the univariate polynomials:

\[
B_{I,\delta}(x) = \beta_{i_1,\delta_1}(x_1) \cdots \beta_{i_n,\delta_n}(x_n) \text{ where } \beta_{i_j,\delta_j}(x_j) = \binom{\delta_j}{i_j} x_j^{i_j} (1 - x_j)^{\delta_j - i_j}. \tag{2.5}
\]

Thanks to the previous notations, these polynomials can also be written as:

\[
B_{I,\delta}(x) = \binom{\delta}{I} x^I (1 - x)^{\delta - I}. \tag{2.6}
\]

The expression of monomials using these polynomials is:

\[
x^I = \sum_{J \leq I \leq \delta} \binom{J}{I} B_{J,\delta}(x), \text{ for all } I \leq \delta \tag{2.7}
\]

Now, we can give the general expression of a multivariate polynomial in the Bernstein basis:

\[
p(x) = \sum_{I \leq \delta} b_{I,\delta} B_{I,\delta}(x),
\]

where Bernstein coefficients \((b_{I,\delta})_{I \leq \delta}\) are given as follows:

\[
b_{I,\delta} = \sum_{J \leq I} \binom{J}{\delta} p_J. \tag{2.8}
\]

Therefore, the generalization of Lemma \(\text{I}\) will lead to the following properties:

**Lemma 2** For all \(x = (x_1, \ldots, x_n) \in U\) we have the following properties:

1. **Unit partition:** \(\sum_{I \leq \delta} B_{I,\delta}(x) = 1.\)

2. **Bounds:** \(0 \leq B_{I,\delta}(x) \leq B_{I,\delta}(\frac{I}{J}), \text{ for all } I \leq \delta.\)

3. **Induction:** \(B_{I,\delta_{r-1}} = \frac{\delta_r - i_r}{\delta_r} B_{I,\delta} + \frac{\delta_r - i_r + 1}{\delta_r} B_{I_{r,1},\delta}, \forall r = 1, \ldots, n, \text{ } I \leq \delta_{r-1}.\)
Also, Corollary 1 can be generalized as follows:

**Corollary 2** Let $p$ be a multivariate polynomial of degree $\delta$ over the unit box $U = [0, 1]^n$ with Bernstein coefficients $b_{I,\delta}$ where $I \leq \delta$. Then, for all $x \in U$, the following inequality holds:

$$\min_{I \leq \delta} b_{I,\delta} \leq p(x) \leq \max_{I \leq \delta} b_{I,\delta}. \quad (2.9)$$

The vertex condition holds iff the minimum value (respectively the maximum value) is reached for an index $I^* \in S_0$ where:

$$S_0 = \{ I = (i_1, \ldots, i_n) \in \mathbb{N}^n, \text{ such that } i_j \in \{0, \delta_j\}, \forall j = 1, \ldots, n \}.$$ 

Given the Bernstein coefficients $(b_{I,\delta})_{I \leq \delta}$ for a polynomial $p$, the vertex condition is quite easy to check using the steps outlined below:

1. Find $I^* := \arg\min_{I \leq \delta} (b_{I,\delta})$.
2. Check for each $j \in [1, n]$ if $I^*_j = 0$ or $I^*_j = \delta_j$.
3. If the previous step succeeds, vertex condition holds and $b_{I^*,\delta}$ is a global minimum of $p$ inside the unit box. Otherwise, vertex condition fails.

Checking the vertex condition will be an important primitive for the overall approach that will be developed in this paper.

Finally, consider an arbitrary, bounded interval $K : [x_1, \bar{x}_1] \times \cdots \times [x_n, \bar{x}_n]$, wherein $-\infty < x_j < \bar{x}_j < \infty$, for all $j = 1, \ldots, n$. It suffices to map $K$ into the unit box $U$ by applying the following change of variables from $x$ to $z$:

$$z_j = \frac{x_j - x_j^*}{\bar{x}_j - x_j^*} \text{ for all } j = 1, \ldots, n.$$ 

Doing so, the results from Lemma 2 can be transferred to arbitrary boxes $K$.

**3 Bernstein Polynomial Relaxations for Polynomial Optimization Problems**

Given a multivariate polynomial $p$ and a rectangle $K$, we consider the following optimization problem:

$$\minimize p(x) \quad \text{s.t.} \quad x \in K. \quad (3.1)$$

Whereas (3.1) is hard to solve, we will construct a linear programming (LP) relaxation, whose optimal value is guaranteed to be a lower bound on $p^*$. In this section, we will use Bernstein polynomials for the unit box ($K = [0, 1]^n$).

If $K$ is a general rectangle, we use an affine transformation to transform $p$ and $K$ back to the unit box.
3.1 Reformulation Linearization Technique (RLT)

We first recall a simple approach to relaxing polynomial optimization problems to linear programs, originally proposed by Sherali et al. \[16,17\]. We then carry out these relaxations for Bernstein polynomials, and show how the properties in Lemma 2 can be incorporated into the relaxation schemes. Recall, once again, the optimization problem (3.1) over the unit box \(K\):

\[
p^* = \min \quad p(x) \\
\text{s.t.} \quad x \in K =: [0,1]^n,
\]

where \(K\) is represented by the constraints \(K: \bigwedge_{j=1}^n x_j \geq 0 \land (1-x_j) \geq 0\). The standard RLT approach consists of writing \(p(x) = \sum_I p_I x^I\) as a linear form \(p(x) = \sum_I p_I y_I\) for fresh variables \(y_I\) that are placeholders for the monomials \(x^I\). Next, we write down as many facts about \(x^I\) over \(K\) as possible. The basic approach now considers all possible power products up to a maximal degree \(D\) i.e. of the form \(\pi_{J,\delta}: x^J (1-x)^{\delta-J}\) for all \(J \leq \delta\) where \(|\delta| = D\). Clearly if \(x \in K\) then \(\pi_{J,\delta}(x) \geq 0\). Expanding \(\pi_{J,\delta}\) in the monomial basis as \(\pi_{J,\delta}: \sum_{I \leq \delta} a_{I,J} x^I\), we write the linear inequality constraint

\[
\sum_{I \leq J} a_{I,J} y_I \geq 0. 
\]

The overall LP relaxation is obtained as

\[
\min \sum_I p_I y_I \\
\text{s.t.} \quad \sum_{I \leq J} a_{I,J} y_I \geq 0, \text{ for each } J \leq \delta. \quad (3.2)
\]

Additionally, it is possible to augment this LP by adding inequalities of the form \(\ell_I \leq y_I \leq u_I\) through the interval evaluation of \(x^I\) over the set \(K\).

Remark 1 The extra “facts” that form the constraints in Eq. 3.2 are akin to valid inequalities or cuts that incrementally refine an over-approximation of the feasible region. Unfortunately, the number of such inequalities is exponential in \(|\delta|\). Rather than adding these all at once to yield a single LP, we may add them on demand, iteratively solving a series of LPs wherein the new inequalities are introduced as cutting planes to help improve the solution.

Proposition 1 For any polynomial \(p\), the optimal value computed by the LP (3.2) is a lower bound to that of the polynomial program (3.1).

Example 1 We wish to solve the following POP (or find a lower bound for its solution):

\[
\min \quad x_1^2 + x_2^2 \\
\text{s.t.} \quad (x_1, x_2) \in [0,1]^2 
\]

(3.3)
Using the RLT technique for a degree $D = 2$ we denote by $y_{i,j}$ the fresh variables replacing the nonlinear terms $x^{(i,j)} = x_1^i x_2^j$ for all $(i,j) \in \mathbb{N}^2$ such that $i + j \leq 2$. We obtain an LP which is shown, in part, below:

$$\begin{align*}
\text{minimize} & \quad y_{2,0} + y_{0,2} \\
\text{s.t.} & \quad 0 \leq y_{2,0} \leq 1 \\
& \quad 0 \leq y_{0,2} \leq 1 \\
& \quad \ldots
\end{align*}$$

The optimal solution obtained from the LP is 0, which coincides with the optimum of the original problem.

### 3.2 RLT using Bernstein Polynomials

The success of the RLT approach depends heavily on writing “facts” involving the variables $y_I$ that substitute for $x^I$. We now present the core idea of using Bernstein polynomial expansions and the richer bounds that are known for these polynomials from Lemma 2 to improve upon the basic RLT approach.

**Linear relaxations:** First, we write $p(x)$ as a weighted sum of Bernstein polynomials of degree $\delta$.

$$p(x) = \sum_{I \leq \delta} b_{I,\delta} B_{I,\delta},$$

wherein $b_{I,\delta}$ are calculated using the formula in equation (2.8). Let us introduce a fresh variable $z_{I,\delta}$ as a place holder for $B_{I,\delta}(x)$. Lemma 2 now gives us a set of linear inequalities that hold between these variables $z_{I,\delta}$. We formulate three LP relaxations, each providing a better approximation for the feasible region of the original problem (3.1).

$$\begin{align*}
\text{minimize} & \quad \sum_{I \leq \delta} b_{I,\delta} z_{I,\delta} \\
\text{s.t.} & \quad z_{I,\delta} \geq 0, \quad I \leq \delta, \\
& \quad \sum_{I \leq \delta} z_{I,\delta} = 1, \\
& \quad z_{I,\delta} \in \mathbb{R}, \quad I \leq \delta.
\end{align*}$$

**Remark 2** It is easy to see that $p_{\delta}^{(0)} = \min_{I \leq \delta} b_{I,\delta}$ (the smallest Bernstein coefficient). As a result, it can be computed quite efficiently without actually invoking an LP solver. In fact, the branch-and-bound approach of Nataraj [12] is based on this relaxation.
Using the upper bound on Bernstein polynomials from Lemma 2, we can strengthen (3.4) further, as follows:

\[ p_{\delta}^{(1)} = \text{minimize} \sum_{I \leq \delta} b_{I,\delta} z_{I,\delta} \]
\[ \text{s.t} \]
\[ z_{I,\delta} \geq 0, \quad I \leq \delta, \]
\[ z_{I,\delta} \leq B_{I,\delta} \left( \frac{1}{\delta} \right), \quad I \leq \delta, \quad \text{Upper Bounds} \] (3.5)
\[ \sum_{I \leq \delta} z_{I,\delta} = 1 \]
\[ z_{I,\delta} \in \mathbb{R}, \quad I \leq \delta. \]

Next, tighter relaxation can be obtained by adding the induction relations between Bernstein polynomials of lower degrees. More precisely, using in addition the third property of Lemma 2, we obtain the following linear program:

\[ p_{\delta}^{(2)} = \text{minimize} \sum_{I \leq \delta} b_{I,\delta} z_{I,\delta} \]
\[ \text{s.t} \]
\[ z_{I,K} \in \mathbb{R}, \quad I \leq K, \quad K \leq \delta, \]
\[ 0 \leq z_{I,K} \leq B_{I,K} \left( \frac{1}{K} \right), \quad I \leq K, \quad K \leq \delta \]
\[ \sum_{I \leq K} z_{I,K} = 1, \quad K \leq \delta \]
\[ z_{I,K_{r-1}} = \frac{K_r - i}{K_r} z_{I,K} + \frac{i + 1}{K_r} z_{I_{r-1},K}, \quad \forall r \in \{1, \ldots, n\} \text{ s.t } I_{r,1} \leq K. \] (3.6)

Remark 3 Note that Eq. (3.5) involved variables \( z_{I,\delta} \) for \( I \leq \delta \). The formulation in Eq. (3.6) involves a larger set of “lower degree” terms of the form \( z_{I,K} \) wherein \( I \leq K \) and \( K \leq \delta \). These terms are, in fact, not necessary as demonstrated in Prop. 3.

Each of these relaxations provides a lower bound on the original polynomial optimization problem.

**Proposition 2** \( p_{\delta}(0) \leq p_{\delta}(1) \leq p_{\delta}(2) \leq p^* \).

**Proof** We already know thanks to Corollary 1 that \( p_{\delta}(0) \leq p^* \).

Now, consider any feasible solution \( y \) to the problem (3.1) which is equivalent to

\[ \text{minimize} \sum_{I \leq \delta} b_{I,\delta} B_{I,\delta}(y) \]
\[ \text{s.t} \]
\[ y \in [0,1]^n. \]

We note that replacing \( z_{I,\delta} = B_{I,\delta}(y) \) the vector of all \( z_{I,\delta} \) form a feasible solution to each of the two relaxations. Therefore, \( p_{\delta}^{(j)} \leq p^* \) for all \( j \in \{1,2\} \). Also it is easy to see that these relaxations are increasing (since they are constructed by adding extra constraints). Therefore, \( p_{\delta}(0) \leq p_{\delta}(1) \leq p_{\delta}(2) \). \( \square \)
Example 2 Let’s consider \( p(x) = x^2 \) on \([-1, 1]\). For a degree \( \delta = 2 \), the minimum of Bernstein coefficient is \( p_\delta^{(0)} = -1 \). Whereas using (3.5), we found \( p_\delta^{(1)} = 0 \) which coincides with the optimum.

Now, we consider the polynomial \( p(x, y) = x^2 + y^2 \) on \([-1, 1]^2\), we found \( p_\delta^{(0)} = -2 \) and \( p_\delta^{(1)} = -0.5 \). Using (3.6), we obtain \( p_\delta^{(2)} = 0 \) which is the exact optimal value.

Now, in order to simplify relaxation (3.6), we formulate an equivalent relaxation that only uses decision variables \( z_{I,\delta} \). This is achieved by replacing lower degree variables \( z_{I,K} \) where \( K \leq \delta \) by a matrix product involving variables \( z_{I,\delta} \). More precisely, we have the following result:

Proposition 3 There exist a matrix \( A_\delta \) and a vector \( c_\delta \) such that the LP formulation in Eq. (3.6) can be written as

\[
p_\delta^{(2)} = \min_b \ b_\delta \cdot z_\delta \\
\text{s.t.} \quad 0_\delta \leq z_\delta \leq u_\delta, \\
\quad 1_\delta \cdot z_\delta = 1, \\
\quad A_\delta z_\delta \leq c_\delta,
\]

(3.7)

wherein the notation \( a_\delta \) stands for a vector \((a_{I,\delta})_{I \leq \delta}\), \( u_{I,\delta} = B_{I,\delta}(\frac{1}{2}) \), \( 0_{I,\delta} = 0 \) and \( 1_{I,\delta} = 1 \).

Proof Each Bernstein polynomial \( B_{I,K}(x) \) can be written uniquely as

\[
B_{I,K}(x) = \sum_{J \leq \delta} \hat{b}_{J}^{(I,K)} B_{J,\delta}(x)
\]

wherein \( \hat{b}_{J}^{(I,K)} \), \( J \leq \delta \) form the Bernstein coefficients for the polynomial \( B_{I,K}(x) \). Translating this, we obtain the relation

\[
z_{I,K} = \sum_{J \leq \delta} \hat{b}_{J}^{(I,K)} z_{J,\delta} = \hat{b}^{(I,K)} \cdot z_\delta
\]

wherein \( \hat{b}^{(I,K)} \) is the vector of Bernstein coefficients \((\hat{b}_{J}^{(I,K)})_{J \leq \delta}\) and \( z_\delta = (z_{J,\delta})_{J \leq \delta} \). The result can now be established by systematically replacing each variable \( z_{I,K} \) for \( K < \delta \) in (3.6) into an expression in terms of \( z_\delta \).

Remark 4 The computation of the pair \((A_\delta, c_\delta)\) in Eq. (3.7) depends only on \( \delta \), and is independent of the actual objective function. As a result, it can be computed offline, once for a given problem setup in terms of number of variables and \( \delta \).

Iterative approach: In many cases, the optimal value given by the linear program (3.7) can be obtained with fewer number of constraints i.e instead of having the constraints given by the pair \((A_\delta, c_\delta)\) only some of them are needed.
Data: POP objective $p(x)$, Constraints $g_1(x) \leq 0, \ldots, g_K(x) \leq 0$, Box $K$, Degree limit vector $\delta$.

Result: $p^{*}_d$: an underapproximation to the POP.

begin
1. Transform $p, g_1, \ldots, g_K$ over the unit box $K: [0,1]^n$
2. Compute matrices $(A_\delta, c_\delta)$
3. Initialize $(\tilde{A}_\delta, \tilde{c}_\delta)$ empty matrices
4. $\text{changeOccurred} := \text{TRUE}$
5. while $\text{changeOccurred} := \text{TRUE}$ do
6. $p_\delta, z_\delta := \text{Solution to the LP (3.8)}$
7. $\text{changeOccurred} := \text{FALSE}$
8. for each row $j$ in $(A_\delta, c_\delta)$ do
9. if $A_{\delta,j} z_\delta > c_{\delta,j}$ then
10. $\text{changeOccurred} := \text{TRUE}$
11. Add row $j$ from $A_\delta, c_\delta$ to $\tilde{A}_\delta, \tilde{c}_\delta$
12. Remove row $j$ from $(A_\delta, c_\delta)$
13. end for
14. end while

Algorithm 1: Overall algorithm for solving a POP using iterative Bernstein polynomial relaxation.

In fact, often a large number of constraints are inactive for the optimal solution. More precisely, we solve LPs of the form:

$$
p_\delta^{(2)} = \text{minimize } b_\delta \cdot z_\delta \text{ s.t. } 0_\delta \leq z_\delta \leq u_\delta, \quad l_\delta \cdot z_\delta = 1, \quad A_\delta z_\delta \leq \tilde{c}_\delta, \quad (3.8)
$$

where $(\tilde{A}_\delta, \tilde{c}_\delta)$ contains a subset of the rows in the matrix $(A_\delta, c_\delta)$. Algorithm 1 shows the overall iterative scheme.

1. Lines 2 to 4 show the initialization steps that involve computing the matrices $(A_\delta, c_\delta)$. The incremental computation involves using matrices $(\tilde{A}_\delta, \tilde{c}_\delta)$ that are initially empty.
2. Solve the linear program (3.8), which is initially the same as (3.5).
3. At each step, we obtain the current optimal value $p_\delta$ and an optimal solution $z_\delta$.
4. The for loop in line 13 iterates through all rows $j$ of the matrix $A_\delta$ such that $A_{\delta,j} z_\delta \leq c_{\delta,j}$ is violated.
5. We these violated rows to the linear program (3.5), remove them from $(A_\delta, c_\delta)$.
6. Termination happens whenever no violated rows are found in the for loop.

Exact relaxation: The decision variables $z_{I,\delta}$ introduced during the RLT technique are fresh variables that substitute the nonlinear polynomials $B_{I,\delta}(x)$. A sufficient condition for an exact relaxation to hold is that optimal solutions $z_{I,\delta}^{*} = B_{I,\delta}(x^{*})$ for all $I \leq \delta$ where $x^{*} \in K$. It is easy to see that when this happens, $x^{*}$ is in fact a global optimum for our problem.
Proposition 4. Let \( z^*_\delta \) be the optimal value given by our relaxation. If there exist \( x^* \in \mathcal{K}(= [0,1]^n) \) such that
\[
x^*I = \sum_{0 \leq J \leq \delta} \binom{\delta}{J} z_{J,\delta}^*, \quad \text{for all } I \leq \delta,
\] then the relaxation is exact i.e. \( p^* = p_\delta^* \) and \( x^* \) is the global minimum.

Proof. The conditions (3.9) can be written as a constraint matrix \( B_\delta z^*_\delta = (x^*I)_{I \leq \delta} \) where \( B_\delta \) is the matrix given by (3.9). If this condition holds then we have \( z_{I,\delta}^* = B_{I,\delta}(x^*) \) for all \( I \leq \delta \) which implies that \( p_\delta^* = \sum_{I \leq \delta} b_{I,\delta} B_{I,\delta}(x^*) = p(x^*) \). This shows that \( p_\delta^* \) is also an upper bound and prove that the condition is sufficient.

The converse is not necessarily true: it is easy to construct examples wherein \( z^*_\delta \) is optimal and \( p_\delta^* \) coincides with a global optimum, but \( z^*_\delta \neq B_\delta(x^*) \) for any \( x^* \) in the domain. In fact, since \( z^*_\delta \) is not unique, then we can have \( z_{I,\delta}^* \neq B_{I,\delta}(x^*) \) whereas \( \sum_{I \leq \delta} b_{I,\delta} z_{I,\delta}^* = \sum_{I \leq \delta} b_{I,\delta} B_{I,\delta}(x^*) \).

Given an optimal solution \( z^*_\delta \), we now provide a procedure that attempts to possibly find a \( x^* \in \mathcal{K} \) such that \( B_\delta(x^*) = z^*_\delta \):

1. Each variable \( x_j \) is itself a polynomial and thus can be written uniquely in the Bernstein form as \( x_j = \sum_{I \leq \delta} a_{I,j}^x \binom{\delta}{I} B_{I,\delta}(x) \), wherein \( a_{I,j}^x \) are the Bernstein coefficients of \( x_j \).
2. Therefore, compute a nominal vector \( \hat{x} \) as \( \hat{x}_j = \sum_{I \leq \delta} a_{I,j}^x \binom{\delta}{I} z^*_I \).
3. Use \( \hat{x} \) to check if \( z^*_\delta = B_\delta(\hat{x}) \). If yes, we conclude exactness of our relaxation with \( x^* = \hat{x} \), and stop. Otherwise, we conclude that no such \( x^* \) exists.

Example 3. Consider the problem in Example 2. For the univariate case, one can check that relaxation (3.5) is exact. In fact, the optimal solution is \( z^*_\delta = (0.25, 0.5, 0.25) \). Using the previous remark, we found \( \hat{x} = 0.5 \) and we check that \( z_{I,\delta}^* = B_{I,\delta}(\hat{x}) \) for all \( I \leq \delta \). Then the relaxation is exact and \( x^* = 0.5 \) which corresponds to zero after a linear transformation to \([-1,1]\). For the bivariate case, relaxation (3.6) is exact but the condition (3.9) does not hold. This is due to the fact that \( z^*_\delta \) is not unique.

3.3 Numerical examples

Thus far, we have presented three LP relaxations using Bernstein polynomials. For the formulation in Eq. (3.6), we provide a technique to reduce the number of variables by computing matrices \((A_\delta, c_\delta)\) that substitute constraints over variables \( z_{I,K} \) for \( K < \delta \) in terms of variables \( z_\delta \) (Eq. (3.7)). Next, we provide an iterative approach that avoids an upfront solution to Eq. (3.7), considering an iterative and incremental addition of constraints as in Eq. (3.8). Also, our approach thus far is monolithic: we translate a single instance of a POP into a LP formulation without considering subdivisions of the feasible region \( \mathcal{K} \).
For the “monolithic” Bernstein relaxation, we consider benchmark problems proposed by Nataraj et al. [12]. In Table 1, we report the optimal values of the proposed relaxations, the size of matrices $A_\delta$, $\tilde{A}_\delta$, the number of iterations needed, and the computation time for the matrix $A_\delta$ (if the computation time exceeds 30 minutes, we print ‘TO’). We find that considerable reduction is made by considering $\tilde{A}_\delta$ instead of $A_\delta$ and also a considerable improvement in the lower bound is obtained when transitioning from the simple formulation in (3.6) to the larger formulation in (3.7). However, we find that, in many cases, a monolithic LP relaxation by itself is not able to provide tight bounds on the optimal value.

Example 4 Let’s consider the Himmelbeau function taken from [12], shown as example ID 1 in Table 1. The POP is given by

$$p(x_1, x_2) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2 \text{ on } [-5, 5]^2. \quad (3.10)$$

Solving the LP formulation (3.6) yields $p_\delta^{(2)} = -856.42$. If one used the relaxation (3.6), then we have a linear program with 324 variables and 341 constraints, without counting the roughly 628 bounds constraints on our variables. Instead, we can solve the linear program given by (3.7). In that case, we only have 25 decision variables. The matrix $A_\delta$ will contain 200 rows and 25 columns. Using the iterative approach, however, we just need 3 iterations to obtain $p_\delta^{(2)}$ where the matrix $A_\delta$ contains 6 rows, in all. Thus, we achieve a significant reduction in the size of the LP and hence the cost of solving it.
However, in spite of these improvements, the objective value when using (3.8) is $-856.416$. This is a very coarse lower bound on the actual optimal value which is $p^* = 0$. One reason for getting a poor bound is that the considered box is relatively big and that the optimal solution $x^*$ is located quite far from the edges.

This motivates the Branch and Bound algorithms we are going to present in the next section. Before doing that, we will briefly show how one can extend the previous relaxations in the case of non rectangular domains.

3.4 Extension to polyhedral and semi algebraic sets

If $K$ is a bounded polyhedral set, our POP can be formulated as follows:

$$\begin{align*}
\min p(x) \\
s.t. & \quad x \in [0,1]^n, \\
& \quad A_0 x \leq b_0,
\end{align*}$$

(3.11)

where $A_0 \in \mathbb{R}^{m \times n}$ and $b_0 \in \mathbb{R}^m$. In fact, it suffices to compute a bounding box for the polyhedral set $K$ and then map the problem to the unit box.

**Proposition 5** Using the same notation, we build the following LP:

$$\begin{align*}
\min p^* & = \min b_\delta \cdot z_\delta \\
\text{s.t.} & \quad 0_\delta \leq z_\delta \leq u_\delta, \\
& \quad 1_\delta \cdot z_\delta = 1, \\
& \quad \tilde{A}_\delta z_\delta \leq \tilde{c}_\delta, \\
& \quad \sum_{I \leq \delta} (A_{I,\delta} z_{I,\delta}) \leq b_0.
\end{align*}$$

(3.12)

Then $p^*_\delta \leq p^*$, where $p^*$ is the optimal value of (3.11).

**Proof** The proof follows directly from the following property:

$$\forall x \in [0,1]^n, \quad \sum_{I \leq \delta} \frac{1}{\delta} B_{\delta,1}(x) = x.$$

Now, If $K$ is a bounded semi-algebraic set, our POP can be formulated as follows:

$$\begin{align*}
\min p(x) \\
\text{s.t.} & \quad x \in [0,1]^n, \\
& \quad g_i(x) \leq 0, \quad \forall i = 1, \ldots, m.
\end{align*}$$

(3.13)

where $p$ and $g_i$ are multivariate polynomials of degree less than $\delta$ for all $i = 1, \ldots, m$. Then, we have the following result:
Proposition 6 Recall LP (3.12) below:

\[
p_\delta^* = \minimize_{b_\delta} b_\delta(p) \cdot z_\delta \tag{3.14}
\]
\[
\text{s.t.} \quad 0 \leq z_\delta \leq u_\delta,
\]
\[
1_\delta \cdot z_\delta = 1,
\]
\[
\tilde{A}_\delta z_\delta \leq \tilde{c}_\delta.
\]
\[
\hat{b}_\delta(g_i) \cdot z_\delta \leq 0, \forall i = 1, \ldots, m.
\]

where \(b_\delta(p) = (b_{I,\delta}(p))_{I \leq \delta}\) and \(b_\delta(g_i) = (b_{I,\delta}(g_i))_{I \leq \delta}\) are Bernstein coefficients of respectively \(p\) and \(g_i\) for all \(i = 1, \ldots, m\).

Then \(p_\delta^* \leq p^*\), where \(p^*\) is the optimal value of (3.13).

Proof It suffices to write polynomials \(g_i\), for all \(i = 1, \ldots, m\), in the Bernstein basis up to the degree \(\delta\) and replace Bernstein polynomials using fresh variables \(z_{I,\delta}\) for all \(I \leq \delta\).

4 Precision Improvements

We will now consider three different approaches to improving our relaxation using the improved LP formulations proposed in this section:

(a) We will show how further properties of Bernstein polynomials can result in multifacaffine constraint system that can be converted back into a LP through dualization. However, we will see that doing so yields impractically large LPs. Therefore, this approach is of theoretical interest.

(b) Next, we will consider using higher degrees \(\delta\) in our LP formulations beyond the degrees of the original POP. However, we observe that the convergence is linear in \(\frac{1}{\delta}\), and thus quite poor when compared to the growth in running times.

(c) Finally, we will use a branch-and-bound scheme that decomposes our problem domain into multiple smaller boxes, using many pruning ideas to limit the number of branches needed. In this context, we examine whether the improved LP relaxations can translate into fewer decompositions of the feasible region.

4.1 Further Valid Inequalities

We now consider techniques for adding further valid inequality constraints to the overall problem. As before, our goal is to ensure that the added constraints are affine, or can somehow be converted to an affine system of constraints.

Adding Known Positive Polynomials: One simple approach, following recent developments in so-called diagonally dominant sum-of-squares is to add polynomials that are easy to show nonnegative such as \(D_{ij}(x) : (x_i - x_j)^{2d} \geq 0\) and \(E_{ij}(x) : (x_i + x_j)^{2d} \geq 0\), for pairs \(x_i, x_j\) to the system of constraints for degrees \(d \leq \frac{1}{2} \min(\bar{\delta}_i, \bar{\delta}_j)\) [1]. To add such polynomials, we convert \(D_{ij}\) and \(E_{ij}\) to
the Bernstein basis, perform RLT by replacing Bernstein polynomials $B_{I,\delta}(x)$ with a fresh variable $z_{I,\delta}$. The resulting constraints will also be added to the matrix $(A_\delta, c_\delta)$ and possibly included in the matrix $(\tilde{A}_\delta, \tilde{c}_\delta)$. However, the cone of positive polynomials over $K$ is not finitely generated cone (even when we consider positive polynomials of bounded degrees). Therefore, an addition of finitely many generators cannot be useful for all problems, in general.

4.1.1 Adding Multiaffine Constraints

In this section, we briefly sketch a further approach to LP relaxations that involves adding multiaffine constraints and relaxing the resulting set of constraints back to a linear program. The multiaffine constraints are given by product of Bernstein polynomials. Consider Bernstein polynomials $p_1(x) := B_{I_1,\delta_1}(x)$, $p_2 := B_{I_2,\delta_2}$, $\ldots$ $p_j := B_{I_j,\delta_j}$.

Claim The product $p_1 \times p_2 \times \cdots \times p_j$ is of the form $c(I_1, \ldots, I_j, \delta_1, \ldots, \delta_j) \times B_{I_1+\cdots+I_j,\delta_1+\cdots+\delta_j}$, where $c(I_1, \ldots, I_j, \delta_1, \ldots, \delta_j)$ is a constant coefficient given by the ratio of the binomial coefficients.

This allows us to provide additional constraints in the formulation (3.6) of the form:

$$z_{I_1,K_1}z_{I_2,K_2}\cdots z_{I_j,K_j} = c(I_1, \ldots, I_j, \delta_1, \ldots, \delta_j)z_{I_1+\cdots+I_j,K_1+\cdots+K_j}$$

The addition of these constraints yields a system of linear multiaffine constraints of the following form:

$$\begin{align*}
\min & \quad c\hat{z}_\delta \\
\text{s.t.} & \quad \tilde{A}_\delta \hat{z}_\delta \leq \tilde{c}_\delta & \text{Linear relationships between Bernstein polynomials} \\
& \quad \hat{z}_\delta \in [L_\delta, U_\delta] & \text{bounds constraints} \\
& \quad z_{i_1} \cdots z_{i_j} = c_i z_i & \text{multiaffine equality constraints}
\end{align*}$$

(4.1)

As such, the multi-affine system above is, in fact, a nonlinear system of constraints. However, the following result by Ben Sassi and Girard [14], shows that any such system can be relaxed to yield a linear programs.

Claim (Ben Sassi + Girard [14]). The multi-affine formulation in Eq. (4.1) can be relaxed to yield a linear program whose optimal value lower bounds that of the multi-affine system (4.1).

The central idea behind Ben Sassi and Girard’s result involves writing down the Lagrangian $L(z, \mu, \lambda)$ involving the primal variables $z$ and multipliers $\mu, \lambda$ for the equalities and inequalities in the optimization problem (4.1). It is noted that the function $L$ is multi-affine in $z$, and also that the optimal value of a multi-affine function in a box $[L_\delta, U_\delta]$ is achieved at its vertices. Therefore, the dual is obtained as $\min_{\nu} L(\mu, \lambda) = \min_{v \in V} L(v, \mu, \lambda)$, where $V$ represents the vertices of the box $[L_\delta, U_\delta]$. As a result of this, the resulting LP is exponential in the number of variables in $z_\delta$, which is already $O(n^|\delta|)$.

As a result, even the addition of additional multi-affine facts involving Bernstein polynomials can cause an unacceptable blowup in the problem size.
Table 2 Improvement of the lower bound by considering higher dimension relaxations

| δ  | (5,4)  | (5,5)  | (6,6)  | (10,10) | (20,19) | (20,20) |
|----|--------|--------|--------|---------|---------|---------|
| $p_\delta^{(2)}$ | -738.918 | -582.783 | -436.57 | -165.89 | -63.89 | -62.23 |

4.2 Higher Degree Relaxations

To improve the precision of the computed lower bound one can increase the degree of the relaxation $\delta$. However, if we use the simpler formulations in Eq. (3.5), then increasing $\delta$ alone does not necessarily yield a better optimal value.

**Example 5** In the Example 2, we saw that for $\delta = (2, 2)$, the optimal value $p_\delta^{(1)} = -0.5$ using formulation in Eq. (3.5). Now increasing the degree to $\delta' = (3, 2)$, one can verify that $p_{\delta'}^{(1)} = -0.59$ which is a worse bound.

However, if we used the formulation in Eq. (3.6) or the equivalent formulations in (3.7) and (3.8), then it is easy to see that increasing the degree $\delta$ will result in the addition of more constraints to the LP and thus, cannot make the lower bound worse. Increasing the degree of the approximation eventually results in tighter bounds that asymptotically converge to the globally optimal bound. This is motivated by the following result by Lin and Rokne [9]:

**Proposition 7** For a degree $K \in \mathbb{N}^n$, let $b_{I,K}$ denote Bernstein coefficients for a polynomial $p$. Then:

$$\|b_{I,K} - p \left( \frac{I}{K} \right) \| = O \left( \frac{1}{k_1} + \cdots + \frac{1}{k_n} \right).$$

Nevertheless, this convergence can be quite slow in practice.

**Example 6** Let’s consider again the POP (3.3), by increasing the degree, we obtain the results reported in Table 2. The results show initially large improvements upon increasing the degree. However, it is clear that large degrees are needed to approach the optimal value of $p^* = 0$.

This motivates us to consider the approaches developed in the previous section inside a branch-and-bound solver that recursively partitions the feasible region into smaller region, while lower bounding the optimal value inside each region using the approach considered here. In this setting, a better lower bound can potentially lead to fewer branches, and therefore a better performance.

4.3 Branch-and-bound scheme

In this section, we consider the branch-and-bound approach for solving POPs and integrate the improved LP formulation in Algorithm 1 into our overall branch-and-bound scheme. Our branch-and-bound scheme is built on top of
Bernstein Inequalities for Constrained Polynomial Optimization Problems.

Data: Objective: \( p(x) \), constraints \( g_1(x) \geq 0, \ldots, g_k(x) \geq 0 \) and domain \( x \in \mathcal{K} \).

Result: Lower bound \( p \) to the optimal value of POP.

```plaintext
begin
worklistOfBoxes := \{ \mathcal{K} \}
glbMin := +\infty
while worklistOfBoxes ≠ \emptyset do
    B := pop (worklistOfBoxes )
    /* Call Algorithm 1 as a subroutine */
    (pB, zB) := Compute Bernstein lower bound for \( p(x) \) on \( B \n
    /* Check if we have to branch further */
    exact := Check if (pB, zB) is an exact solution to \( B \n
    /* 1. Is the value computed exact for \( B \) */
    monotone := Check if \( \partial p/\partial x \) is sign invariant over \( B \) for each \( x \).

    /* 2. Monotonicity check. */
    terminal := Check if we can terminate the branch-and-bound for \( B \n
    if monotone then
        /* 4. Create edge subproblem \( \hat{p}, \hat{B} \). */
        pB := branchAndBound(\( \hat{p}, \hat{B} \) )
        glbMin := min(glbMin, pB)
    else if terminal or exact then
        /* Branch into multiple subboxes */
        (B_1, \ldots, B_k) := splitBox (B)
        add B_1, \ldots, B_k to worklistOfBoxes
    else
        /* Branch into multiple subboxes */
        B_1, \ldots, B_k := splitBox (B)
        add B_1, \ldots, B_k to worklistOfBoxes
```

Algorithm 2: Basic Branch-And-Bound scheme for solving POPs.

previous work by Nataraj et al. \[12\] that is based on a simple formulation that involves finding the minimum Bernstein coefficient inside each box decomposition considered by the algorithm. Additionally, their approach uses properties such as the vertex condition and a monotonicity condition (described below) to detect leaf nodes. We augment our approach directly inside their framework by iteratively solving LPs as described in Algorithm 1. While solving a LP is more expensive than finding the minimum Bernstein coefficient, we show that the extra overhead is offset by our ability to consider fewer boxes.

4.3.1 Overview of Branch and Bound Algorithm

The main idea of the branch-and-bound (BB) algorithm is to keep subdividing the rectangular domain into sub-boxes until a termination condition can be obtained. Algorithm 2 shows the basic branch and bound scheme. It involves repeated decompositions of the original box \( \mathcal{K} \) to construct a `worklistOfBoxes` that should become empty (ideally) in order to ensure termination.

The algorithm’s behavior and performance depends critically on three key operations: (a) The precise relaxation used to compute the bound for \( p(x) \) in
line 6. (b) The exactness test in lines 7 and termination check in line 9 and (c) The branching step in line 16.

4.3.2 Exactness Test

The exactness test is performed to infer if the current lower bound \(p_B\) for \(p(x)\) over a given box \(B\) is in fact the optimal value. This is achieved by testing for the vertex condition and a monotonicity condition. The vertex condition is described in Corollary 2 (page 7). This is quite easy to test once we transform the problem from the current box \(B\) to \([0,1]^n\) using the mapping \(x' = T(x)\), and compute the Bernstein coefficients of \(p(T(x))\).

4.3.3 Monotonicity Test

The monotonicity test (originally proposed by Nataraj et al. [12]) checks whether \(0 \notin \partial p_r / \partial x_r\) for \(r = 1, \ldots, n\) where \(x \in B\). If the partial derivative \(p_r\) w.r.t some \(x_r\) is sign invariant over \(B\), then the global minimum of \(p\) in \(B\) can be obtained at one of the bounds: \(x_r = \ell_r\) or \(x_r = u_r\), depending on the sign of \(p_r\). The derivative \(p_r\) is also expressed using Bernstein polynomials, where the coefficients are computed directly from the Bernstein coefficients of \(p\). The monotonicity test is computed along each dimension \(x_r\) by computing the Bernstein coefficients of \(\partial p / \partial x_r\). If the polynomial is deemed monotone along \(x_r\), then depending on the sign of the partial derivative, \(x_r\) is substituted by its lower (partial derivative is positive) or upper (partial derivative is negative) bound in \(B\). In particular, further decomposition of \(B\) is unnecessary in this case. However, since the global minimum may lie along a facet, we create an “edge” subproblem \(\hat{p}\) by substituting \(x_i = \ell_i\) for each monotonically increasing variable \(x_i\) and \(x_i = u_i\) for each monotonically decreasing \(x_i\). The resulting subproblem has strictly fewer variables than the original problem, and is solved recursively using the same branch-and-bound procedure.

4.3.4 Termination Test

The main termination test compares the current lower bound for the box \(B\) against the best upper bound \(\hat{p}\) obtained by sampling feasible points in the original feasible region \(\mathcal{K}\). If the lower bound \(p_B \leq (1 - \epsilon)\hat{p}\) (alternatively \(p_B \leq -\epsilon\) when \(\hat{p} = 0\)), we do not subdivide the box further. Another approach to cutting off the branch-and-bound imposes a bound on the volumes of boxes that can be subdivided.

4.3.5 Computing Lower Bounds

Next, we consider the computation of lower bounds to a polynomial \(p(x)\) over a box \(B\). This is a key step in our branch-and-bound scheme. We consider the three relaxations defined in Eqs. (3.4), (3.5), and (3.6). As mentioned earlier, using (3.4) is equivalent to computing the minimal Bernstein coefficient as
originally suggested by Nataraj et al. \cite{12}. However, the relaxtions in \cite{3.5} and \cite{3.6} involve solving linear programming problems that are more expensive when compared to finding the smallest Bernstein coefficient. On the other hand, the advantage is that we obtain tighter bounds that may allow us to use fewer decompositions.

As a further optimization, we build a function called “First-LP” that attempts to provide a lower bound for \cite{4.2} directly without using a LP solver by finding a dual feasible solution for it. We rewrite \cite{4.2} as follows:

$$\begin{align*}
\min & \quad b^T z \\
\text{s.t.} & \quad z \geq 0 \\
& \quad -z \geq -u \\
& \quad l^T z = 1
\end{align*}$$

\noindent wherein $b$ is the vector of Bernstein coefficients and $u$ represents the vector of upper bounds. Let us sort the Bernstein coefficients in $b$ and without loss of generality we write:

$$b_1 \leq b_2 \leq \cdots \leq b_N.$$

Next, let $b_l$ be an index such that $b_l \leq 0$ and $b_{l+1} > 0$. If $b_1 \geq 0$ then $z^* = 0$ is an optimal solution to \cite{4.2}. On the other hand, if $b_N \leq 0$, then we take $l = N$. Note that $b_1$ is the optimal value for the relaxation \cite{3.4}. Next, we choose the index $q = \max\{i \in [1, l-1] \mid \sum_{j=1}^i u_j \leq 1\}$.

**Lemma 3** The optimal value of \cite{4.2} is lower bounded by $\max(b_1, b_{q+1} + \sum_{j=1}^q b_j u_j)$.

**Proof** We first formulate the dual to \cite{4.2}. Let us use the multiplier $\lambda$ corresponding to the upper bound constraints $-z \geq u$ and $\mu$ corresponding to the equality constraint $l^T z = 1$. The (simplified) dual LP is given as

$$\begin{align*}
The set the dual solutions as $\lambda_j = -b_j$ for $j \in [1, q]$ and $\lambda_j = 0$ for $j > q$. Finally we set $\mu = b_{q+1}$. We can verify that all the dual constraints are satisfied. Thus our solution is dual feasible. We also note that it yields a dual objective value of $b_{q+1} + \sum_{j=1}^q b_j u_j$ as required. In contrast, setting $\mu = b_1$ and $\lambda = 0$ yields another dual feasible solution. The rest follows by applying the standard weak duality theorem for linear programs.

It is possible to provide precise conditions under which the dual feasible solution is in fact dual optimal, and obtain a corresponding primal optimal solution. The advantage of using a dual lower bound in a branch-and-bound scheme is that it provides an improved bound over \cite{3.4} but at a reduced computational cost that involves sorting the Bernstein coefficients and performing a linear time scan over them to identify the indices $l, q$ which is less expensive than solving \cite{4.2}. For \cite{3.6}, a lower bound is obtained by considering the optimal value given by First-LP, construct an associate feasible solution to it, then perform an iterative approach to improve this optimal value.
4.3.6 Numerical Results

The algorithms described thus far were implemented inside the MATLAB™ environment using the inbuilt `linprog` function for solving linear programs. We compare our three algorithms using a set of 18 benchmarks to evaluate whether the additional inequalities lead to (a) fewer boxes being examined by our branch-and-bound scheme and (b) overall improvement in the computation time. The first 16 benchmarks are collected from Nataraj et al. [12] (taken in the same order). In addition to those, we consider two further challenging examples:

- The 3-dimensional Motzkin example (ID=17):
  \[ p(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6, \quad R = [-0.5, 0.5]^3. \]

- The 4-dimensional algebraic example (ID=18):
  \[ p(x_1, x_2, x_3, x_4) = x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4x_1 x_2 x_3 x_4 - 1, \quad R = [-0.1, 0.1]^4. \]

A termination test threshold \( \epsilon = 10^{-9} \) is fixed for computing the global minimum for the first 16 benchmarks. For the Motzkin example ID 17, we fix \( \epsilon = 10^{-5} \) and \( \epsilon = 10^{-3} \) for example ID 18 to deal with numerical issues in using the MATLAB’s LP solver. We expect commercial LP solvers such as CPLEX to provide us with more robustness.

Table 3 shows the results obtained for the various benchmarks using the LP relaxations labeled 0, 1 and 2, respectively in column Ineq. These correspond to the LPs in (3.4), (3.5) and (3.6) while (3.4) is computed exactly (since it is only given by the smallest Bernstein coefficients) whereas only lower bounds are computed for (3.5) and (3.6) using the results of the previous section.

For completeness, we also report, separately, the results over the subproblems generated by the monotonicity tests.

**Comparing number of subdivisions:** Did the use of a larger LP at each step yield fewer cells? From Table 3, we observe that indeed the use of a larger LP formulation with more inequalities did lead to roughly a 10% reduction in the number of cells examined, especially for the larger instances.

**Comparing total time:** Despite the reduction in the number of cells, the overall computation time for LP relaxation 2 was slightly larger. This is clearly due to the cost of the iterative approach (since some LPs need to be performed). However, for relaxation 1, the lower bound given by the First-LP avoid us solving LPs, which turns out to be advantageous. Indeed, the advantage vanishes as soon as we use an LP solver for relaxation 1, as demonstrated by a single example in Table 3.

**Accuracy of Results:** Because of the adaptive nature of our branch-and-bound scheme, we obtain solutions that are consistently close to the actual global optima.
Table 3 Performance of the Cuts for benchmark problems taken from Nataraj et al. [12] + two more examples. Legend: ID: problem ID as given in [12] + two more examples, Ineq.: the LP used for lower bounding 0; LP [5,5], 1:lower bound on [5,5], 2:Lp [3,3]. Sub. the number of subdivisions, Time: time taken in seconds, Cutoff: number of boxes removed using the cut-off test, Mono: number of box removed using the monotonicity test, Opt: the optimal value, Sub*, Cutoff*, Time*: Total number of subdivisions, number cutoff and time spent solving recursive subproblems.

| ID | Ineq. | Sub | Time | Cutoff | Mono | Sub* | Cutoff* | Time* | Opt |
|----|-------|-----|------|--------|------|------|--------|-------|-----|
| 1  | 0     | 164 | 1.2  | 55     | 62   | 5    | 7      | 0.02  | 0   |
| 1  | 1     | 155 | 1.1  | 67     | 46   | 5    | 7      | 0.02  | 0   |
| 1  | 2     | 147 | 2.5  | 47     | 61   | 5    | 7      | 0.02  | 0   |
| 2  | 0     | 100 | 1.1  | 14     | 61   | 2    | 6      | 0.01  | -1.032 |
| 2  | 1     | 97  | 1.1  | 15     | 59   | 2    | 6      | 0.01  | -1.032 |
| 2  | 2     | 97  | 3.0  | 13     | 61   | 2    | 6      | 0.01  | -1.032 |
| 3  | 0     | 194 | 0.4  | 86     | 11   | 3    | 2      | 0.00  | 0   |
| 3  | 1     | 176 | 0.4  | 85     | 6    | 3    | 2      | 0.00  | 0   |
| 3  | 2     | 173 | 0.8  | 78     | 11   | 3    | 2      | 0.00  | 0   |
| 4  | 0     | 1319| 4.8  | 751    | 4    | 9    | 3      | 0.02  | 0   |
| 4  | 1     | 1199| 4.5  | 684    | 4    | 9    | 3      | 0.02  | 0   |
| 4  | 2     | 1098| 23.2 | 625    | 4    | 9    | 3      | 0.03  | 0   |
| 5  | 0     | 388 | 3    | 126    | 146  | 20   | 22     | 0.07  | -7  |
| 5  | 1     | 371 | 2.9  | 134    | 133  | 17   | 19     | 0.06  | -7  |
| 5  | 2     | 371 | 3.4  | 122    | 145  | 17   | 19     | 0.07  | -7  |
| 6  | 0     | 784 | 19.2 | 371    | 266  | 36   | 33     | 0.50  | 0   |
| 6  | 1     | 763 | 18.8 | 371    | 254  | 34   | 31     | 0.47  | 0   |
| 6  | 2     | 707 | 33.1 | 318    | 263  | 32   | 29     | 0.66  | 0   |
| 7  | 0     | 0   | 0.1  | 0      | 0    | 0    | 0      | -36.713 | 0 |
| 7  | 1     | 0   | 0.1  | 0      | 0    | 0    | 0      | -36.713 | 0 |
| 7  | 2     | 0   | 0.1  | 0      | 0    | 0    | 0      | -36.713 | 0 |
| 8  | 0     | 637 | 15.6 | 307    | 190  | 43   | 40     | 0.58  | 0   |
| 8  | 1     | 615 | 15.2 | 300    | 181  | 40   | 37     | 0.54  | 0   |
| 8  | 2     | 580 | 27.3 | 267    | 185  | 40   | 37     | 0.61  | 0   |
| 9  | 0     | 3   | 0.1  | 2      | 503  | 307  | 4.25   | -3.18 | 0 |
| 9  | 1     | 3   | 0.1  | 0      | 2    | 498  | 304    | 4.25  | -3.18 |
| 9  | 2     | 3   | 0.1  | 0      | 2    | 498  | 304    | 4.88  | -3.18 |
| 10 | 0     | 0   | 0.1  | 0      | 0    | 0    | 0      | -20.8 | 0   |
| 10 | 1     | 0   | 0.1  | 0      | 0    | 0    | 0      | -20.8 | 0   |
| 10 | 2     | 0   | 0.1  | 0      | 0    | 0    | 0      | -20.8 | 0   |
| 11 | 0     | 1794| 51.5 | 1165   | 484  | 67   | 73     | 0.69  | -16 |
| 11 | 1     | 1542| 44.4 | 1050   | 371  | 66   | 72     | 0.69  | -16 |
| 11 | 2     | 1525| 51.6 | 989    | 415  | 66   | 72     | 0.80  | -16 |
| 12 | 0     | 18  | 0.1  | 0      | 1    | 0    | 0      | -30.25 | 0 |
| 12 | 1     | 18  | 0.1  | 0      | 1    | 0    | 0      | -30.25 | 0 |
| 12 | 2     | 18  | 0.1  | 0      | 1    | 0    | 0      | -30.25 | 0 |
| 13 | 0     | 101 | 0.1  | 6      | 0    | 0    | 0      | -0.25 | 0   |
| 13 | 1     | 101 | 0.1  | 6      | 0    | 0    | 0      | -0.25 | 0   |
| 13 | 2     | 101 | 0.1  | 6      | 0    | 0    | 0      | -0.25 | 0   |
| 14 | 0     | 0   | 0.1  | 0      | 0    | 0    | 0      | -1.44 | 0   |
| 14 | 1     | 0   | 0.1  | 0      | 0    | 0    | 0      | -1.44 | 0   |
| 14 | 2     | 0   | 0.1  | 0      | 0    | 0    | 0      | -1.44 | 0   |
| 15 | 0     | 118 | 0.1  | 7      | 0    | 0    | 0      | -0.25 | 0   |
| 15 | 1     | 118 | 0.1  | 7      | 0    | 0    | 0      | -0.25 | 0   |
| 15 | 2     | 118 | 0.1  | 7      | 0    | 0    | 0      | -0.25 | 0   |
| 16 | 0     | 18  | 0.4  | 3      | 0    | 0    | 0      | -1.74 | 0   |
| 16 | 1     | 18  | 0.4  | 3      | 0    | 0    | 0      | -1.74 | 0   |
| 16 | 2     | 18  | 0.4  | 3      | 0    | 0    | 0      | -1.74 | 0   |
| 17 | 0     | 17874| 1161 | 5860   | 452  | 600  | 404    | 22.92 | 0   |
| 17 | 1     | 16775| 1081 | 5772   | 290  | 600  | 404    | 23.23 | 0   |
| 17 | 2     | 16641| 1431 | 5626   | 452  | 600  | 404    | 24.22 | 0   |
| 18 | 0     | 12684| 36496| 48080  | 2122  | 2616 | 2240   | 330   | -1  |
| 18 | 1     | 12033| 3424 | 4949   | 1447  | 2480 | 2120   | 314   | -1  |
| 18 | 2     | 11983| 3967 | 3941   | 2414  | 2416 | 2062   | 333   | -1  |
Table 4 Comparison between 'First-LP' and Linprog performances

| ID | Cut | LP | Sub | Time | Cutoff | Mono | Sub* | Cutoff* | Time* | Opt |
|----|-----|----|-----|------|--------|------|------|--------|-------|-----|
| 1  | 1   | 'First-LP' | 155 | 1.11 | 67 | 46 | 5 | 7 | 0.02 | 0 |
| 1  | 1   | Linprog | 140 | 3.57 | 62 | 42 | 5 | 7 | 0.07 | 0 |

4.3.7 Lyapunov Stability Proofs

A standard approach to prove stability for polynomial dynamical systems is to find a polynomial Lyapunov certificate which consists on a positive definite function decreasing along the trajectories inside a region of interest. More precisely, let $V$ be a polynomial candidate Lyapunov function, $\dot{V}$ its derivative and $R$ the region of interest taken as a rectangle containing zero (the equilibrium point). To verify the asymptotic stability of the equilibrium, we should verify that: $\min_{x \in R} V(x) \geq 0$ and $\min_{x \in R} -\dot{V}(x) \geq 0$.

The advantage while solving POPs arising from Lyapunov function synthesis problems is that a global minimum is known in advance. In fact since usually $V(0) = \dot{V}(0) = 0$, then we already know that zero is the global minimum of a true Lyapunov function. Therefore, a good branch-and-bound decomposition scheme for this problem decomposes around the equilibrium to maximize the opportunity for exact relaxations [15].

To show the efficiency of the zero decomposition, we consider 9 Benchmarks given in our earlier work [15], taken in order. The goal is to verify that the candidate functions are indeed Lyapunov functions. In all these examples, the region of interest is $R = [-1, 1]^n$. We propose to check the validity of these results by computing $p_V^*$ and $\dot{V}^*$ which are lower bounds on $V$ and $-\dot{V}$ inside $R$ using the smallest Bernstein coefficient $(p_V^*(0), p_V^*(0))$ and relaxation (3.3) $(pV^*(1), p\dot{V}^*(1))$. We report in Table 5 the results we obtained where stability is said verified once a precision of $10^{-9}$ is reached. In the appendix we give a detailed description of the Benchmarks, the Lyapunov function and their associated Lie derivatives.

Table 5 Proving bounds on Lyapunov functions and their derivatives. Legend: EX - ID of the example taken from Ben Sassi et al. [15]. $p_V^*(j)$: Lower bounds to optimal value obtained by using LP relaxation id $j$, $p_{\dot{V}^*}(j)$: Lower bounds on optimal value of Lyapunov derivative.

| EX | $p_V^*(0)$ | $p_V^*(1)$ | $p_{\dot{V}^*}(0)$ | $p_{\dot{V}^*}(1)$ | Stability |
|----|-------------|-------------|-----------------|-----------------|-----------|
| 1  | $-9.2 \times 10^{-14}$ | $-0.0825$ | $-5.3 \times 10^{-14}$ | $-3.5 \times 10^{-12}$ | ✓ |
| 2  | $-5.4 \times 10^{-10}$ | 0 | $-2.2 \times 10^{-10}$ | 0 | ✓ |
| 3  | $-2.7 \times 10^{-9}$ | 0 | $-8.7 \times 10^{-10}$ | $-1 \times 10^{-14}$ | ✓ |
| 4  | $-1.3 \times 10^{-10}$ | 0 | $-3.7 \times 10^{-11}$ | $-1.4 \times 10^{-14}$ | ✓ |
| 5  | $-3.4 \times 10^{-14}$ | $-3.5 \times 10^{-12}$ | $-6.9 \times 10^{-13}$ | $-4.1 \times 10^{-13}$ | ✓ |
| 6  | $-1.5 \times 10^{-14}$ | $-2.1 \times 10^{-14}$ | $-9.5 \times 10^{-14}$ | $-3.8 \times 10^{-14}$ | ✓ |
| 7  | $-10.9788$ | $-10.9788$ | $-7.9 \times 10^{-11}$ | $-3.3 \times 10^{-14}$ | X |
| 8  | $-9.9 \times 10^{-10}$ | $-1.7 \times 10^{-14}$ | $-3.8 \times 10^{-11}$ | $-3.3 \times 10^{-13}$ | ✓ |
5 Conclusions

We present a novel approach to deal with polynomial optimization problems (POPs) by relaxing them to bigger size linear programs. The key idea is to use Bernstein polynomials in order to build LPs that can handle many of the relations between non-linear terms missed because of the linearization process. Contrarily to the standard RLT approach, the given LPs are easily implementable since only a Bernstein framework is needed (coefficients, bounds and change of variable). Thanks to the properties of Bernstein polynomials, tighter bounds than RLT are obtained and various techniques to improve the precision of these bounds are given. We show that our relaxations can be used to improve the Branch and Bound scheme given by Nataraj [12]. The main drawback faced in the latter case was the extra cost of solving LPs. We already find a way to avoid this for our first linear relaxation but not for the more precise one. This is definitely a first goal future work. Also, we manage to extend our Brand and Bound algorithms in the case of semi algebraic constraints.

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6 Appendix

**Benchmark #1:** Consider the two variable polynomial ODE:

\[
\frac{dx}{dt} = -12.5x + 2.5x^2 + 2.5y^2 + 10x^2y + 2.5y^3.
\]

\[
\frac{dy}{dt} = -y - y^2.
\]

**Lyapunov function :**

\[2x^2 5y^2\]

**Lyapunov derivative function :**

\[40x^3y + 10x^3 - 50x^2 + 10xy^3 + 10xy^2 - 10y^3 - 10y^2\]

**Benchmark #2:** Consider the two variable polynomial ODE:

\[
\frac{dx}{dt} = 6.933333x^3 + 4.566667x^2 - 21.5x.
\]

\[
\frac{dy}{dt} = 6.933333x^3 + 0.4x^2y + 2.066667x^2 + xy^2 + 0.6xy - 9x - y^2 - y.
\]

**Lyapunov function :**

\[5x^2 - 4xy + 5y^2\]

**Lyapunov derivative function :**

\[41.6x^4 + 40x^3y + 37.4000x^3 - 179x^2 + 10xy^3 + 10xy^2 - 10y^3 - 10y^2\]

**Benchmark #3:** Consider the two variable polynomial ODE:

\[
\frac{dx}{dt} = -1.5x - x^2 + 0.5xy + 0.5y^2 - 2x^3 + x^2y.
\]

\[
\frac{dy}{dt} = -0.5y.
\]
Lyapunov function:
\[ 5x^2 + 5y^2. \]

Lyapunov derivative function:
\[ -20x^4 + 10x^3y - 10x^3 + 5x^2y - 15x^2 + 5xy^2 - 5y^2. \]

**Benchmark #4:** Consider the two variable polynomial ODE:
\[ \begin{align*}
\frac{dx}{dt} &= -2x^3 - 0.5xy - 0.5x, \\
\frac{dy}{dt} &= 0.25xy^2 - 0.125xy + 0.25y^2 - 0.4125y.
\end{align*} \]

Lyapunov function:
\[ 5x^2 + 5y^2. \]

Lyapunov derivative function:
\[ -20x^4 - 5x^2y - 5x^2 + 2.5xy^3 - 1.25xy^2 + 2.5y^3 - 4.125y^2. \]

**Benchmark #5:** Consider the three variable polynomial ODE:
\[ \begin{align*}
\frac{dx}{dt} &= -2x^3 - 0.5xy - 0.5x - z^3 - z^2, \\
\frac{dy}{dt} &= 0.25xy^2 - 0.125xy + 0.25y^2 - 0.4125y, \\
\frac{dz}{dt} &= -z^2 - z.
\end{align*} \]

Lyapunov function:
\[ 5x^2 + 5y^2 + 5z^2. \]

Lyapunov derivative function:
\[ -20x^4 - 5x^2y - 5x^2 + 2.5xy^3 - 1.25xy^2 + 2.5y^3 - 4.125y^2 - 10z^3 - 10z^2. \]

**Benchmark #6:** Consider the three variable polynomial ODE:
\[ \begin{align*}
\frac{dx}{dt} &= -0.5x^3y + 0.5x^3z^2 - 3x^3 + y^5 - y^4 + yz^4 - z^4, \\
\frac{dy}{dt} &= 0.25y^2 - 0.25y, \\
\frac{dz}{dt} &= yz^4 + z^4 - 2z^3.
\end{align*} \]
Consider the three variable polynomial ODE:

\[ \begin{align*}
&\text{Lyapunov function :} \\
&1.9150x^4 + 5\cdot x^3 + 3\cdot x^2 y^2 + 5\cdot x^2 z^2 + 3\cdot 3.9396x^2 - 2.5409xy^3 + 2.5409xy^2 + 5y^3 + 5y^2 z^2 + 5z^3 + 5z^2.
\end{align*} \]

\[ \begin{align*}
&\text{Lyapunov derivative function :} \\
&-3.8300x^6 y + 3.8300x^6 z^2 - 22.98x^6 - 7.5x^5 y + 7.5x^5 z^2 - 45x^5 - 5x^4 y^3 + 5x^4 y^2 z^2 - 30x^4 y^2 - 5x^4 y z^2 - 3.8956x^4 y + 4.5x^4 z^4 - 4 - 26.0604x^4 z^2 - 23.6375x^3 y^3 + 7.66x^3 y^5 + 5x^3 y^3 z^2 + 6.3524x^3 y^2 z^2 + 7.6228x^3 y z^2 + 10x^3 y z^2 - 3.8956x^3 y + 4.5x^3 z^4 - 4 - 26.0604x^3 z^2 - 23.6375x^2 y^3 + 7.66x^2 y^5 - 5x^2 y^3 z^2 + 6.3524x^2 y^2 z^2 + 7.6228x^2 y z^2 + 10x^2 y z^2 - 3.8956x^2 y + 4.5x^2 z^4 - 4 - 26.0604x^2 z^2 - 23.6375x y^3 + 7.66x y^5 + 5x y^3 z^2 + 6.3524x y^2 z^2 + 7.6228x y z^2 + 10x y z^2 - 3.8956x y + 4.5x z^4 - 4 - 26.0604x z^2 - 23.6375y^3 + 7.66y^5 + 5y^3 z^2 + 6.3524y^2 z^2 + 7.6228y z^2 + 10y z^2 - 3.8956y + 4.5z^4 - 4 - 26.0604z^2 - 23.6375z y^3 + 7.66z y^5 + 5z y^3 z^2 + 6.3524z y^2 z^2 + 7.6228z y z^2 + 10z y z^2 - 3.8956z + 4.5z^3 - 4 - 26.0604z^2.
\end{align*} \]

\[ \begin{align*}
&\text{Benchmark #7:} \quad \text{Consider the three variable polynomial ODE:} \\
&dx \over dt = -0.5x^3 y + 0.5x^3 z^2 - x^3 + y^4 z + y^4 - y^3 + y^3 z + y^2 - z^2.
\end{align*} \]

\[ \begin{align*}
&dy \over dt = 0.5y^2 z - 0.5y^2 - 2y.
\end{align*} \]

\[ \begin{align*}
&dz \over dt = -y z^2 + y z + z^2 - z.
\end{align*} \]
**Benchmark #8:** Consider the three variable polynomial ODE:

\[
\begin{align*}
\frac{dx}{dt} &= -0.5x^3y + 0.5x^3z^2 - x^3 + y^4z + y^4 - yz^3 + 3yz^2 + z^3 - 3z^2 \\
\frac{dy}{dt} &= y^4z - 2y^3 - z^3 + 3z^2 \\
\frac{dz}{dt} &= z^2 - 3z \\
\end{align*}
\]

Lyapunov function:

\[2.3519x^5 - 1.0449x^4y + 0.3429x^4z + 5x^4 + 4.4496x^3y - 2.5x^3y + 5x^3z^2 + 5x^3z + 5x^2y^3 + 5x^2y^2 + 2x^2 + 3.6461xy^4 + 5xy^3 + 5xy^2z^2 + 5xyz^3 + 5xyz^2 + 5xz^4 + 5xz^3 + 2.5863xz^2 - 0.4325y^5 + 4.9487y^4z + 5y^4 + 5y^3z^2 + 4.0594y^3 + 5y^2z^3 + 5y^2z^2 + 5yz^4 + 5yz^3 + 4.5809yz^2 + 1.8568xz^5 + 5z^4 - 0.7475z^3 + 5z^2.\]

Lyapunov derivative function:

\[-5.8798x^7y + 5.8798x^7z - 2 - 11.7597x^7 + 2.0899x^6y - 2 - 0.6857x^6yz - 0.58203x^6y + 0.6857x^6z^3 + 10x^6z^2 - 1.3715x^6z^2 + 6.6743x^5y^3 + 3 + 6.6743x^5y^2z^2 - 2 + 9.5986x^5y^2z^2 - 11.25x^5y^2z^2 + 7.5x^5y^2z - 4 - 7.5x^5y^2z - 5 - 15x^5y^2z + 10 + 7.1148x^4y^4z^5 + 7.8046x^4y^4z^5 + 5x^4y^4z^3 + 2 - 12.9101x^4y^4z - 2 + 2x^4y^4z^5 + 16.7597x^4y^4z^3 + 3 + 20.729x^4y^4z^2 - 5x^4y^4z^5 + 14.9019x^3y^4z^2 + 3 + 18.8712x^3y^3z^2 - 4 + 2.5x^3y^3z^2 - 4 + 1.6797x^3y^3z^2 - 3 + 20.0392x^3y^3z^2 - 2 + 2.5x^3y^3z^2 - 4 + 7.5x^3y^3z^2 - 5 + 125x^3y^3z^2 - 3 + 125x^3y^3z^2 - 2 + 49.2436x^3y^3z^2 - 3 + 22.3365x^3y^3z^2 - 3 - 100.0863x^3y^3z^2 - 3 + 23.4847x^3y^3z^2 - 6 + 125x^3y^3z^2 - 2 + 7.5x^3y^3z^2 - 5 + 5x^3y^3z^2 - 10x^3y^3z^2 - 3 + 23.4847x^3y^3z^2 - 2 + 7.5x^3y^3z^2 - 5 + 125x^3y^3z^2 - 2 + 50.0182x^3y^3z^2 - 3 + 67.75x^3y^3z^2 - 2 + 10x^3y^3z^2 - 5 - 15x^3y^3z^2 - 4 - 10x^3y^3z^2 - 5 + 30x^3y^3z^2 - 4 + 5xy^3z^2 - 3 - 15xy^3z^2 - 2 - 10xy^3z^2 - 6 + 20xy^3z^2 - 5 + 45xy^3z^2 - 4 - 45xy^3z^2 + 5xz^2 + 10xz^2 - 5 - 60xz^2 - 4 - 29.8273xz^2 - 3 - 45.5180xz^2 + 2 + 1.4833y^3 + 8x + 5.8088y + 19.7947yz^2 + 2 + 5.205y^5z^7 - 10.6745y^7 + 20y^6z^3 - 10y^6z^2 - 51.7682y^6z - 27.8218y^6z + 15yz^5 + 4x + 6.3539y^5z^3 - 24.0617y^5z^2 + 14.3566y^5z^2 + 10y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2 + 14y^5z^2
\]

**Benchmark #9:** Consider the three variable polynomial ODE:

\[
\begin{align*}
\frac{dx}{dt} &= 0.05x^2y + 0.05x^2z - 0.05x^2y - 0.05x^2z + 0.05xy + 0.05yz - 0.05x + 0.125y^3z - 0.125y^3 \\
\frac{dy}{dt} &= 0.125y^2z - 0.125y^2 + 0.2yz^2 + 0.2yz^2 + 0.2yz^2 - 0.2z^2 \\
\frac{dz}{dt} &= -0.1z^2 + 0.1z \\
\end{align*}
\]
Lyapunov function:
\[ 2.5x^2 + 2.5y^2 + 5z^2 \]

Lyapunov derivative function:
\[
0.25x^3yz + 0.25x^3y - 0.25x^3z - 0.25x^2yz + 0.25x^2y - 0.25x^2z - 0.25x^2 + 0.25x^2yz + 0.25x^2y - 0.25x^2z - 0.25x^2 + 0.625xy^3z \\
-0.6250xy^3 + 0.6250xy^2z - 0.6250xy^2 + xyz^5 + xyz^4 - xz^5 - xz^4 + 0.625y^3z - 0.625y^3 + 0.625y^2z \\
-0.625y^2 + yz^5 + yz^4 - xz^3 - z^3 - 2.
\]