The blow-up solutions for fractional heat equations on torus and Euclidean space

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Abstract. We produce a finite time blow-up solution for nonlinear fractional heat equation \((\partial_t u + (-\Delta)^{\beta/2} u = u^k)\) in modulation and Fourier amalgam spaces on the torus \(T^d\) and the Euclidean space \(\mathbb{R}^d\). This complements several known local and small data global well-posedness results in modulation spaces on \(\mathbb{R}^d\). Our method of proof rely on the formal solution of the equation. This method should be further applied to other non-linear evolution equations.

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1. Introduction

We study heat equation associated to fractional Laplacian \((-\Delta)^{\beta/2}\) of the form

\[
\begin{aligned}
\partial_t u + (-\Delta)^{\beta/2} u &= u^k, \quad (t, x) \in \mathbb{R}^+ \times \mathcal{M} \\
u(0, x) &= u_0(x),
\end{aligned}
\]

(1.1)

where \(\mathcal{M} = T^d\) (torus) or \(\mathbb{R}^d\). We denote \(\hat{\mathcal{M}}\) the Pontryagin dual of \(\mathcal{M}\), i.e.

\[
\hat{\mathcal{M}} = \begin{cases} 
\mathbb{Z}^d & \text{if } \mathcal{M} = T^d \\
\mathbb{R}^d & \text{if } \mathcal{M} = \mathbb{R}^d.
\end{cases}
\]

The fractional Laplacian \((-\Delta)^{\beta/2}\) is defined by

\[
\mathcal{F}[(-\Delta)^{\beta/2} u](\xi) = c_\beta |\xi|^\beta \mathcal{F}u(\xi) \quad (\xi \in \hat{\mathcal{M}})
\]

where \(\mathcal{F}\) denotes the Fourier transform and \(c_\beta\) is some constant. The fractional heat equation (1.1) is significantly interesting in both physics and PDEs, since it is the Poisson equation if \(\beta = 1\) and the classical heat equation if \(\beta = 2\).
In the later case (1.1) appears as a one-dimensional model for the voracity equation of incompressible and viscous fluid of three dimension.

In the 1980s, Feichtinger [10] introduced the modulation spaces $M^{p,q}(\mathcal{M})$ using short-time Fourier transform (STFT)\(^1\). The STFT of a $f \in \mathcal{S}'(\mathcal{M})$ (space of tempered distributions, see e.g. [24, Part II]) with respect to a window function $0 \neq g \in \mathcal{S}(\mathcal{M})$ (Schwartz space) is defined by

$$V_g f(x,y) = \int_{\mathcal{M}} f(t)T_x g(t)e^{-2\pi iy \cdot t}dt, \ (x,y) \in \mathcal{M} \times \hat{\mathcal{M}}$$

whenever the integral exists. Here, $T_x g(t) = g(tx^{-1})$ is the translation operator on $\mathcal{M}$. The modulation spaces $M^{p,q}_s(\mathcal{M})$ is defined as follows:

$$M^{p,q}_s(\mathcal{M}) = \left\{ f \in \mathcal{S}'(\mathcal{M}) : \|f\|_{M^{p,q}_s} = \|\|V_g f(x,y)\|_{L^p(\mathcal{M})}\langle y \rangle^s\|_{L^q(\hat{\mathcal{M}})} < \infty \right\}.$$ 

See also Remark 1.4. If $s = 0$, we write $M^{p,q}_0(\mathcal{M}) = M^{p,q}(\mathcal{M})$. It is known that

$$M^{p,q}_s(\mathcal{M}) = \left\{ H^s(\mathcal{M}) \ (\text{Sobolev space}) \quad \text{if} \ p=q=2 \right.$$ 

$$\mathcal{F}L^q(\mathcal{M}) \ (\text{Fourier-Lebesgue space}) \quad \text{if} \ \mathcal{M} = \mathbb{T}^d.$$ 

In last two decades modulation spaces have turned out to be very fruitful in the study of various non-linear PDEs on $\mathbb{R}^d$, see [1–6,25,26]. Iwabuchi [18, Theorems 1.9 and 1.13] proved local and global well-posedness of (1.1) with $\beta = 2$ for small data in some weighted modulation spaces $M^{p,q}_s(\mathbb{R}^d)$. Later Chen-Deng-Ding-Fan [8, Theorems 3.1, 1.5 and 1.6] have obtained some space-time estimates for heat semigroup $e^{-t(-\Delta)^\beta/2}$ in modulation spaces. Further, they [8, Theorems 1.5 and 1.6] proved local and global well-posedness (for small data) of (1.1) with any $\beta > 0$ in some weighted modulation spaces, see also [21, Theorem 3]. In [17,27] authors have found some critical exponent in modulation spaces and provide some local well-posedness and ill-posedness for (1.1) in weighted modulation spaces. Perhaps the best known local well-posedness for (1.1) read as follows:

**Theorem A.** ([8]) Assume that $u_0 \in M^{p,1}(\mathbb{R}^d)$ ($1 \leq p \leq \infty$). Then there exists $T_1 > 0$ such that (1.1) has a unique solution $u \in C([0,T_1), M^{p,1}(\mathbb{R}^d))$.

We note that though there is extensive literature on classical heat equation on $\mathbb{R}^d$, only a few authors have studied (1.1) on the torus $\mathbb{T}^d$ (in spite of periodic data is very important in the analysis and in applications). See e.g. [20]. On the other hand, there has been good well-posedness theory developed for non-linear Schrödinger equations on torus $\mathbb{T}^d$ but not for (1.1). We are thus also interested to study time behaviour solution of (1.1) for periodic data.

Recently in [11], Forlano and Oh have introduced **Fourier amalgam spaces** $\hat{\mathcal{W}}^{p,q}_s(\mathcal{M})$ ($1 \leq p, q \leq \infty$):

$$\hat{\mathcal{W}}^{p,q}_s(\mathcal{M}) = \left\{ f \in \mathcal{S}'(\mathcal{M}) : \|f\|_{\hat{\mathcal{W}}^{p,q}} = \left\| \chi_{n+(-\frac{1}{2}, \frac{1}{2})^d} f(\xi) \mathcal{F}f(\xi) \|_{L^p(\hat{\mathcal{M}})} \right\|_{\ell^q(\mathbb{Z}^d)} < \infty \right\}.$$ 

\(^1\)STFT is also known as windowed Fourier transform and is closely related to Fourier–Wigner and Bargmann transform. See [15, Lemma 3.1.1] and [15, Proposition 3.4.1].
Their motivation was to study well-posedness of 1D cubic NLS in these spaces, [12]. See also Remark 1.5. It is interesting to note that \( \hat{\omega}^{q,q}(\mathcal{M}) = F^L q(\mathcal{M}) \), \( \hat{\omega}^{2,q}(\mathcal{M}) = M^{2,q}(\mathcal{M}) \) and \( \hat{\omega}^{p,1}(\mathcal{M}) \) \((1 \leq p \leq \infty)\) is an algebra under point-wise multiplication. We also note that free fractional heat propagator \( e^{-t(-\Delta)^{\beta/2}} \) is uniformly bounded on \( \hat{\omega}^{p,q}(\mathcal{M}) \). Thus, by the standard fixed point argument we have the following local well-posedness theorem.

**Theorem B.** Assume that \( u_0 \in \hat{\omega}^{p,1}(\mathcal{M}) \) \((1 \leq p \leq \infty)\). Then there exists \( T_1 > 0 \) such that (1.1) has a unique solution \( u \in C([0,T_1),\hat{\omega}^{p,1}(\mathcal{M})) \).

For simplicity of presentation, we let

\[ X(\mathcal{M}) = M^{p,1}(\mathcal{M}) \quad (1 \leq p \leq 2) \quad \text{or} \quad \hat{\omega}^{p,1}(\mathcal{M}) \quad (1 \leq p \leq \infty). \]

It is now natural to ask whether the local solutions established in Theorems A and B can be extended to a global solution in time? The purpose of the present paper is to answer this question.

In fact, for \( u_0 \in X(\mathcal{M}) \), in view of Theorem A (see Proposition 3.1 below), (1.1) has a unique solution \( u \in C([0,T^*),X(\mathcal{M})) \), where \( T^* = T^*(\|u_0\|_X) \) denotes the maximal existence of time of solution. Moreover, we have either

\[ T^* = \infty \]

or

\[ T^* < \infty \quad \text{and} \quad \limsup_{t \to T^*} \|u(t)\|_X = \infty. \]

In the previous case, we say that the solution is global, while in the latter case we say that the solution blows up in the X norm in finite time and \( T^* \) is called the blow-up time. We now state our main theorem.

**Theorem 1.1.** Suppose that \( k \in \mathbb{N}, 0 < r, \gamma, \beta < \infty \) and \( r^d v_d \geq 2^d \), \( v_d \) is the volume of unit ball \( \{ x \in \mathcal{M} : |x| \leq 1 \} \). Let \( \tilde{u}_0 \geq 0, \hat{u}_0 \geq \gamma \chi_{B_0(r)}, u_0 \in X(\mathcal{M}) \quad (1 \leq p \leq 2) \) and \( \gamma^{k-1} \geq 4r^\beta (k - 1)e \). Then (1.1) has a unique blow-up solution in \( X(\mathcal{M}) \), that is, there exists a unique solution \( u(t) \) of (1.1) defined on \([0,T^*)\) such that

\[ T^* < \infty \quad \text{and} \quad \limsup_{t \to T^*} \|u(t)\|_X = \infty. \]

We have initiated the study of blow-up analysis for (1.1) on torus \( \mathbb{T}^d \) and Theorem 1.1 thus is first finite time blow-up result in \( F^L 1(\mathbb{T}^d) \) as far as we are aware. Theorem 1.1 reveals that the local solution established in Theorem A cannot be extended to a global solution in time. Our method of proof uses a formal solution of (1.1) in terms of power series expansion. This formal solution satisfies the corresponding Duhamel’s form of (1.1) (see Lemma 3.3). Finally, by performing the analysis on the Fourier transform side for component of power series expansion, we establish crucial lower bound for each component. This leads us to blow-up solution in finite time (Lemma 3.5). We note that our method of proof is inspired by the recent work of Ru and Chen in [22], where they established finite time-blow up for a classical heat equation, i.e. (1.1) with \( \beta = 2 \), in \( F^L 1(\mathbb{R}^d) \).
Remark 1.1. (1) The function \( u_0(x) = C e^{-2\pi|x|^2} \) for \( C > \frac{k-1}{4} e^{2\pi r^d}, \) \( r^d u_d \geq 1 \) satisfies the hypothesis of Theorem 1.1. (2) The function \( u_0(x) = C \sin \frac{x}{r}, x \in \mathbb{R}, C = \frac{1}{\pi} \sqrt{\frac{k-1}{4}} e^{2\pi r^d} \) satisfies the hypothesis of Theorem 1.1 for \( p = 2. \) (3) Taking Theorem A into account, the case \( p > 2 \) in Theorem 1.1 remains an interesting open question.

Remark 1.2. The cubic NLS
\[ i \partial_t u + \Delta u = |u|^2 u \]
is locally [2] well-posed in \( M^{p,1}(\mathbb{R}^d) \) but it is not yet clear whether it is globally well-posed or there exist a blow-up solution, see for instance open question raised by Ruzhansky-Sugimoto-Wang in [23, p.280]. On the other hand, Theorem 1.1 says that we can produce blow-up solution of (1.1) in \( M^{p,1}(\mathbb{R}^d) \) \((1 \leq p \leq 2). \)

We now briefly mention widely known literature. Fujita [14] proved that for (1.1) with \( \beta = 2 \) and \( d(k-1)/2 < 1 \) no non-negative global solution exists for any non-trivial initial data \( u_0 \in L^1(\mathbb{R}^d) \) (i.e. every positive solution to this initial data problem blows up in a finite time.) Later Fujita [13] proved that, for \( d(k-1)/2 < 1, \) global solutions do exist for initial data dominated by a sufficiently small Gaussian. For the critical exponent \( d(k-1)/2 = 1, \) Hayakawa [16] proved nonexistence of nonnegative and nontrivial global solutions in the case of \( d = 1, 2, \) and Kobayashi-Sirao-Tanaka [19] proved it in general dimensions (i.e. finite time blow up case). In [7,22] authors have proved blow up in finite time for (1.1) with \( \beta = 2 \) in some scale invariant Besov space and Fourier-Lebesgue spaces. We note these results deals with the classical heat equation (1.1) with \( \beta = 2 \) while in the present paper we could considered fractional heat equation (1.1) with \( \beta \neq 2 \) as well.

1.1. Further remarks

Remark 1.3. Bényi and Okoudjou in [2] have established local well-posedness for nonlinear wave and Klein-Gordon (NLKG) equations in \( M^{p,1}(\mathbb{R}^d). \) Wang and Hudzik in [25] have established small data global well-posedness for NLKG in \( M^{2,1}(\mathbb{R}^d). \) Exploiting similar ideas of Sect. 3.1, we may establish formal solution for the wave and Klein-Gordon equations. The only difference maybe at this stage is to replace fractional heat propagator by suitable free Klein-Gordon and wave propagator. It is also further expected the method of proof of the present paper should be applicable to establish finite time blow up for these equations.

Remark 1.4. Applying the frequency-uniform localization techniques, one can get an equivalent definition of modulation spaces [25] as follows. Let \( \rho \in \mathcal{S}(\mathbb{R}^d), \) \( \rho : \mathbb{R}^d \to [0, 1] \) be a smooth function satisfying \( \rho(\xi) = 1 \) if \( |\xi| \leq \frac{1}{2} \) and \( \rho(\xi) = 0 \) if \( |\xi| \geq 1. \) Let \( \rho_k \) be a translation of \( \rho, \) that is, \( \rho_k(\xi) = \rho(\xi - k) \) \((k \in \mathbb{Z}^d). \) Denote \( \sigma_k(\xi) = \frac{\rho_k(\xi)}{\sum_{l \in \mathbb{Z}^d} \rho(l)}, \) \((k \in \mathbb{Z}^d). \) The frequency-uniform decomposition operators can be defined by
\[ \square_k = F^{-1} \sigma_k F. \]
For $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, it is known [25, Proposition 2.1] that
\[
\|f\|_{M^{p,q}} = \|f\|_{L^p(\mathcal{M})}^{1+p} (1 + |k|)^s.
\]
The definition of the modulation space given above, is independent of the choice of the particular window function. See [15, Proposition 11.3.2(c)], [26].

**Remark 1.5.** For any given function $f$ which is locally in $B$ (Banach space) (i.e., $g f \in B, \forall g \in C^\infty_c(\mathbb{R}^d)$), we set $f_B(x) = \|fg(\cdot - x)\|_B$. In [9], Feichtinger introduced Wiener amalgam space $W(B, C)$ with local component $B$ and global component $C$ (Banach space) is defined as the space of all functions $f$ locally in $B$ such that $f_B \in C$. The space $W(B, C)$ endowed with the norm $\|f\|_{W(B, C)} = \|f_B\|_C$. Moreover, different choices of $g \in C^\infty_c(\mathbb{R}^d)$ generates the same space and yield equivalent norms. We note that Fourier amalgam spaces is a Fourier image of particular Wiener amalgam spaces, specifically, $FW(L^p, \ell^q) = \hat{w}^{p,q}$. The rest of the paper is organized as follows. In Sect. 2 recall required facts on modulation and Fourier amalgam spaces. In Sect. 3 we prove Theorem 1.1.

### 2. Preliminaries

The notation $A \lesssim B$ means $A \leq cB$ for some constant $c > 0$. The **Fourier-Lebesgue spaces** $\mathcal{F}L^p(\mathcal{M})$ is defined by
\[
\mathcal{F}L^p(\mathcal{M}) = \left\{ f \in S'(\mathcal{M}) : \|f\|_{\mathcal{F}L^p} := \|\hat{f}\|_{L^p(\mathcal{M})} < \infty \right\}.
\]

**Lemma 2.1.** (Basic Properties, see [15,23,26], see [1] and Corollary 2.7 in [2] ) Let $p, q, p_i, q_i \in [1, \infty]$ ($i = 1, 2$), $s, s_1, s_2 \in \mathbb{R}$. Then

1. $M^{p_1, q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2, q_2}(\mathbb{R}^d)$ whenever $p_1 \leq p_2$ and $q_1 \leq q_2$ and $s_2 \leq s_1$.
2. $M^{\min(p', 2), p}(\mathbb{R}^d) \hookrightarrow \mathcal{F}L^p(\mathbb{R}^d) \hookrightarrow M^{\max(p', 2), p}(\mathbb{R}^d)$, $\frac{1}{p} + \frac{1}{p'} = 1$.
3. $M^{1, 1}(\mathbb{R}^d)$ is an algebra under pointwise multiplication with norm inequality
   \[
   \|fg\|_{M^{1, 1}} \lesssim \|f\|_{M^{1, 1}} \|g\|_{M^{1, 1}}.
   \]
4. $\hat{w}^{p_1, q_1}(\mathbb{R}^d) \subset \hat{w}^{p_2, q_2}(\mathbb{R}^d)$ for $p_1 \geq p_2$ and $q_1 \leq q_2$.
5. Let $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q}$. Then we have
   \[
   \|fg\|_{\hat{w}^{p, q}} \lesssim \|\hat{f}\|_{\hat{w}^{p, q}} \|g\|_{\hat{w}^{p, q}}.
   \]
   In particular, $\hat{w}^{p, q}(\mathcal{M})$ is an $\mathcal{F}L^1$-module i.e. $\|fg\|_{\hat{w}^{p, q}(\mathcal{M})} \lesssim \|f\|_{\mathcal{F}L^1} \|g\|_{\hat{w}^{p, q}(\mathcal{M})}$.

**Proof.** We note that $FW(L^p, \ell^q) = \hat{w}^{p,q}$. Since
\[
L^{p_1} \ast L^{p_2} \subset L^p \quad \text{and} \quad \ell^{q_1} \ast \ell^{q_2} \subset \ell^q,
\]
by [9], we have $\|fg\|_{\hat{w}^{p, q}} = \|\hat{f} \ast \hat{g}\|_{W(L^p, \ell^q)} \lesssim \|\hat{f}\|_{W(L^{p_1}, \ell^{q_1})} \|\hat{g}\|_{W(L^{p_2}, \ell^{q_2})}$.
We refer to [15] for a classical foundation of these spaces and [23,26] for some recent developments for nonlinear dispersive equations and the references therein.

2.1. Linear estimates

For $f \in \mathcal{S}(\mathcal{M}), t \in [0, \infty), 0 < \beta < \infty$, we define fractional heat propagator as follows

$$U_{\beta}(t)f(x) = e^{t(-\Delta)^{\beta/2}}f(x) = \mathcal{F}^{-1}(e^{-t|\xi|^\beta}\mathcal{F}(\xi))(x) \quad (x \in \mathcal{M}, \ \xi \in \hat{\mathcal{M}}).$$

The next proposition shows that the uniform boundedness of $U_{\beta}$ on modulation spaces. Specifically, we have following

**Proposition 2.1.** Let $1 \leq p, q \leq \infty$ and $0 < \beta < \infty$. Then, for $f \in M^{p,q}(\mathbb{R}^d)$, we have

$$\|U_{\beta}(t)f\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}}.$$

**Proof.** We sketch the proof, for detail see [8, Theorem 3.1]. By Remark 1.4, we have

$$\|U_{\beta}(t)f\|_{M^{p,q}} \lesssim \left(\sum_{|k|<10} \|\Box_k U_{\beta}(t)f\|_{L^p}^q\right)^{1/q} + \left(\sum_{|k|\geq 10} \|\Box_k U_{\beta}(t)f\|_{L^p}^q\right)^{1/q}.$$

We notice almost orthogonality [1,25] relation for the frequency-uniform decomposition operators

$$\Box_k = \sum_{\|\ell\|_{\infty} \leq 1} \Box_{k+\ell} \Box_k \quad (k, \ell \in \mathbb{Z}^d) \quad (2.1)$$

where $\|\ell\|_{\infty} = \max\{|\ell_i| : \ell_i \in \mathbb{Z}, i = 1, \ldots, d\}$. By (2.1), for $|k| \geq 10$, we have

$$\Box_k U_{\beta}(t)f = \sum_{\|\ell\|_{\infty} \leq 1} \Box_{k+\ell} \Box_k U_{\beta}(t)f.$$

Using this, Young inequality and [8, Lemma 3.1] give

$$\|\Box_k U_{\beta}(t)f\|_{L^p} \lesssim \sum_{|\ell|_{\infty} \leq 1} \|\sigma_{k+\ell}e^{-t|\xi|^\beta}\|_{L^1} \|\Box_k f\|_{L^p} \lesssim \|\Box_k f\|_{L^p}.$$

Exploiting the proof of [8, Lemma 3.5], we may obtain $\|\Box_k U_{\beta}(t)f\|_{L^p} \lesssim \|\Box_k f\|_{L^p}$ for $|k| < 10$. Combining the above inequalities, we obtain the desired inequality.

3. The proof of Theorem 1.1

In this section we prove Theorem 1.1. We start with introducing the formal solution of (1.1) in the next subsection.
3.1. The formal solution of fractional heat equation

In this subsection, we introduce the formal solution of fractional heat Eq. (1.1). Assume that $k = 2$ and

\[ u_1(t) = U_\beta(t)u_0, \]
\[ u_2(t) = \int_0^t U_\beta(t-s)u_1^2 ds, \]
\[ u_3(t) = \int_0^t U_\beta(t-s)2u_1u_2 ds, \]
\[ \cdots, \]
\[ u_{2n}(t) = \int_0^t U_\beta(t-s)(2u_1u_{2n-1} + \cdots + 2u_{n-1}u_{n+1} + u_n^2) ds, \]
\[ u_{2n+1}(t) = \int_0^t U_\beta(t-s)(2u_1u_{2n} + \cdots + 2u_{n-1}u_{n+2} + 2u_nu_{n+1}) ds, \]
\[ \cdots. \]

Then $u = \sum_{i=1}^{\infty} u_i$ (formal solution) formally satisfies the integral equation

\[ u(t) = U_\beta(t)u_0 + \int_0^t U_\beta(t-s)u^2(s) ds. \]

In fact, we have

\[ \sum_{i=1}^{\infty} \hat{u}_i = \hat{u}_1 + \int_0^t e^{-(t-s)|\xi|^\beta} \mathcal{F}(u_1^2) ds + \int_0^t e^{-(t-s)|\xi|^\beta} \mathcal{F}(2u_1u_2) ds + \cdots \]
\[ + \int_0^t e^{-(t-s)|\xi|^\beta} \mathcal{F}(2u_1u_{2n} + \cdots + 2u_nu_{n+1}) ds + \cdots \]
\[ = \hat{u}_1 + \int_0^t e^{-(t-s)|\xi|^\beta} \mathcal{F}(u_1^2 + 2u_1u_2 + \cdots) ds \]
\[ = \hat{u}_1 + \int_0^t e^{-(t-s)|\xi|^\beta} \mathcal{F}\left( \sum_{i=1}^{\infty} u_i \right)^2 ds. \]

By taking inverse Fourier transform on two sides, we can obtain that

\[ u(t) = U_\beta(t)u_0 + \int_0^t U_\beta(t-s)u^2 ds. \]

**Remark 3.1.** The formal analysis (formal solution for (1.1)) performed above will be made rigorous in the following subsections.

**Lemma 3.1.** For $f(u) = u^k$, assume

\[ u_1(t) = U_\beta(t)u_0, \]
\[ u_k(t) = \int_0^t U_\beta(t-s)u_1^k ds, \]
\[ u_{2k-1}(t) = \int_0^t U_\beta(t-s)C_k^1u_1^{k-1}u_k ds, \]
The mapping $C_c$ for some universal constant.

Consider a point in an appropriate functions space, for small $t$. By Duhamle principle, (1.1) can be rewritten as

$$u(t) = U_\beta(t)u_0 + \int_0^t U_\beta(t-s)f(u(s))ds.$$  

3.2. Local well-posedness in $X(M)$

**Proposition 3.1.** (local well-posedness) Assume that $u_0 \in X(M)$ ($1 \leq p \leq \infty$). Then there exists $T_1 > 0$ such that (1.1) has a unique solution $u \in C([0,T_1),X(M))$.

**Proof.** By Duhamle principle, (1.1) can be rewritten as

$$u(\cdot,t) = U_\beta(t)u_0 + \int_0^t U_\beta(t-s)f(u)ds := J(u),$$

where $U_\beta(t) = e^{-t(-\Delta)^{\beta/2}}$. We show that the mapping $J$ has a unique fixed point in an appropriate functions space, for small $t$. For this, we consider the Banach space $X_T = C([0,T],M^{p,1}(\mathbb{R}^d))$, with norm

$$\|u\|_{X_T} = \sup_{t \in [0,T]} \|u(\cdot,t)\|_X, \ (u \in X_T).$$

By Minkowski’s inequality for integrals and Propositions 2.1 and Lemma 2.1, we obtain

$$\left\| \int_0^t U_\beta(t-s)u^k(s)ds \right\|_X \leq T\|u^k(t)\|_X \leq T\|u^k\|_{C([0,T],X)}.$$  

By Proposition 2.1, and using above inequality, we have

$$\|Ju\|_{C([0,T],X)} \leq \|u_0\|_X + cT\|u\|_X^k,$$

for some universal constant $c$. For $M > 0$, put

$$B_{T,M} = \{u \in C([0,T],X)) : \|u\|_{C([0,T],M^{p,1})} \leq M \},$$

which is the closed ball of radius $M$, and centered at the origin in $C([0,T],X)$. Next, we show that the mapping $J$ takes $B_{T,M}$ into itself for suitable choice of
$M$ and small $T > 0$. Indeed, if we let, $M = 2\|u_0\|_X$ and $u \in B_{T,M}$, it follows that
\[
\|Ju\|_{C([0,T],X)} \leq \frac{M}{2} + cTM^3.
\]
We choose a $T$ such that $cTM^2 \leq 1/2$, that is, $T \leq \frac{1}{2cM^2}$ and as a consequence we have
\[
\|Ju\|_{C([0,T],X)} \leq \frac{M}{2} + \frac{M}{2} = M,
\]
that is, $Ju \in B_{T,M}$. By Proposition 3, and the arguments as before, we obtain
\[
\|Ju - Jv\|_{C([0,T],X)} \leq \frac{1}{2}\|u - v\|_{C([0,T],X)}.
\]
Therefore, using the Banach’s contraction mapping principle, we conclude that $J$ has a fixed point in $B_{T,M}$ which is a solution of (3.1). □

3.3. Finite time blow-up in $X(M)$

In this subsection, we prove Theorem 1.1. To this end, we start with flowing technical lemmas.

Lemma 3.2. There exists $\epsilon_0$ such that if $T_2\|u_1\|_{L^\infty_T(X)} < T_2\|u_0\|_X < \epsilon_0$ then
\[
\sum_{i=1}^{\infty} \|u_i\|_{L^\infty_T(X)} < \infty.
\]

Proof. Taking notations of Subsect. 3.1 and Proposition 2.1 and Lemma 2.1 (3) into account, we have
\[
\|u_1\|_{L^\infty_T(X)} = \|U_\beta(\cdot)u_0\|_{L^\infty_T(X)} \leq C\|u_0\|_X,
\]
\[
\|u_2\|_{L^\infty_T(X)} \leq C_0 \left\langle \left\| u_1 \right\|_X^2 \right\rangle_{L^\infty_T(X)} \leq C_0 T_2 \|u_1\|_{L^\infty_T(X)}^2,
\]
\[
\|u_3\|_{L^\infty_T(X)} \leq 2C_0 T_2 \|u_1\|_{L^\infty_T(X)} \|u_2\|_{L^\infty_T(X)},
\]
\[
\|u_4\|_{L^\infty_T(X)} \leq 2C_0 T_2 \|u_1\|_{L^\infty_T(X)} \|u_2\|_{L^\infty_T(X)} + C_0 T_2 \|u_2\|_{L^\infty_T(X)}^2,
\]
\[
\cdots.
\]

Notice that the norm $\|u_2\|_{L^\infty_T(X)}$ can be controlled by the norm $\|u_1\|_{L^\infty_T(X)}$ and so the norms of $\|u_i\|_{L^\infty_T(X)}$ also can be controlled by $\|u_1\|_{L^\infty_T(X)}$. In view of this, we have
\[
\sum_{i=1}^{\infty} \|u_i\|_{L^\infty_T(X)} \leq S
\]
where
\[
S := \|u_1\|_{L^\infty_T(X)} + C_0 T_2 \|u_1\|_{L^\infty_T(X)}^2 + C_0 T_2 \|u_1\|_{L^\infty_T(X)} \|u_2\|_{L^\infty_T(X)} + \cdots.
\]
Next we claim that there exists $\epsilon_0$ such that if the initial data $T_2\|u_0\|_X < \epsilon_0$, then $S < \infty$. To justify the claim, we let
\[
C = \frac{1}{2C_0T_2}, \quad a_1 = \|u_1(t)\|_{L_{T_2}^\infty(X)}, \quad a_2 = \sum_{j=1}^i a_j a_{2i-j} \quad \text{and}
\]
\[
a_{2i+1} = \sum_{j=1}^i a_j a_{2i+1-j}.
\]
Then we have
\[
S \leq \sum_m \frac{1}{C^m} a_m \sim \sum_m \frac{1}{C^m} a_m.
\]
Then, by induction, we can obtain that there exists $\epsilon_0$ such that if $T_2\|u_0\|_X < \epsilon_0$, then $\sum_{i=1}^\infty \frac{n_i}{m} < \sum_{i=1}^\infty \frac{1}{m+i} < \infty$ ($\epsilon$ to be decided). Indeed, if for any $m \leq 2i, a_m \leq \frac{C^m}{m+i}$, then we have
\[
a_{2i+1} \leq a_1 \frac{C^{2i}}{(2i)^{1+\epsilon}} + a_2 \frac{C^{2i-1}}{(2i-1)^{1+\epsilon}} + \cdots + \frac{C^{2i+1}}{(i(i+1))^{1+\epsilon}}
\]
\[
\leq a_1 \frac{C^{2i+1}}{(2i)^{1+\epsilon}} + a_2 \frac{C^{2i+1}}{(2i-1)^{1+\epsilon}} + \sum_{j=3}^i \frac{C^{2i+1}}{(j(2i+1-j))^{1+\epsilon}}
\]
\[
\leq \frac{a_1 (2i+1)^{1+\epsilon}}{C^{2i}} \frac{C^{2i+1}}{(2i+1)^{1+\epsilon}} + \frac{a_2 (2i+1)^{1+\epsilon}}{C^{2i-1}} \frac{C^{2i+1}}{(2i+1)^{1+\epsilon}} + \sum_{j=3}^i \frac{(2i+1)^{1+\epsilon}}{(j(2i+1-j))^{1+\epsilon}} (2i+1)^{1+\epsilon}
\]
\[
< \frac{C^{2i+1}}{(2i+1)^{1+\epsilon}}.
\]
Choose $\epsilon > 0$ so $\frac{a_1 (2i+1)^{1+\epsilon}}{C^{2i}} < 1/3, \frac{a_2 (2i+1)^{1+\epsilon}}{C^{2i-1}} < 1/3$ and $\sum_{j=3}^i \frac{(2i+1)^{1+\epsilon}}{(j(2i+1-j))^{1+\epsilon}} < 1/3$ and hence we may obtain the claim. Taking these observation into account, there exists $\epsilon_0$ such that if $T_2\|u_0\|_X < \epsilon_0$, then $u(t) = \sum_{i=1}^\infty u_i(t) \in L^\infty([0,T_2), X(M))$.

**Lemma 3.3.** Let $T_1$ and $T_2$ be as in Proposition 3.1 and Lemma 3.2. For $T = \min\{T_1, T_2\}, u(t) = \sum_{i=1}^\infty u_i(t) \in L^\infty([0,T), X(M))$ satisfies the integral equation
\[
u(t) = U_\beta(t)u + \int_0^t U_\beta(t-s)u^2 ds.
\]

**Proof.** We claim that
\[
\lim_{M \to \infty} \int_0^t U_\beta(t-s) \left( \left( \sum_{i=1}^M u_i \right) \left( \sum_{i=1}^M u_i \right) \right) ds
\]
\[
= \int_0^t U_\beta(t-s) \left( \left( \sum_{i \in \mathbb{N}} u_i \right) \left( \sum_{i \in \mathbb{N}} u_i \right) \right) ds \quad (3.2)
\]
in $X_T := L^\infty([0, T), X(M))$. By the definition of $u_i$ in the formal solution and $\bar{u}_0 \geq 0$, we have

$$D := \left\| \int_0^t U_\beta(t - s) \left( \left( \sum_{i \in \mathbb{N}} u_i \right) \left( \sum_{i \in \mathbb{N}} u_i \right) \right) \right\|_{X_T}$$

$$\leq \left\| \int_0^t U_\beta(t - s) \left( \left( \sum_{i \in \mathbb{N}} u_i \right) \left( \sum_{i \in \mathbb{N}} u_i \right) \right) \right\|_{X_T}$$

$$- \int_0^t U_\beta(t - s) \left( \left( \sum_{i = 1}^M u_i \right) \left( \sum_{i = 1}^M u_i \right) \right) \right\|_{X_T}$$

$$\leq 2C_0 \left\| \sum_{i = M+1}^{\infty} u_i \right\|_{X_T}.$$ 

In view of this and Lemma 3.2, we have that for any small $\epsilon > 0$, there exist $N$ such that for any $M > N, D < \epsilon$. This proves (3.2). Next, we show that

$$\lim_{M \to \infty} \int_0^t U_\beta(t - s) \left( \sum_{i = 1}^M u_i \right) \left( \sum_{i = 1}^M u_i \right) ds = \lim_{M \to \infty} \int_0^t U_\beta(t - s) \left( \sum_{i = M+1}^{\infty} u_i \right) \left( \sum_{i = M+1}^{\infty} u_i \right) ds$$

(3.3)

in $X_T$. Indeed, by Lemma 3.2 and the similar argument as above, we have

$$E := \left\| \int_0^t U_\beta(t - s) \left( \left( \sum_{i = 1}^M u_i \right) \left( \sum_{i = 1}^M u_i \right) \right) \right\|_{X_T}$$

$$- \int_0^t U_\beta(t - s) \left( \sum_{i = 2}^{M-1} \sum_{j = 1}^{i-1} u_j u_{(i-j)} \right) ds$$
\[\begin{align*}
\sum_{i=1}^{M} u_i = U_\beta(t_0) + \sum_{i=2}^{M} \lim_{M \to \infty} \sum_{i=1}^{M} u_i = U_\beta(t) u_0 + \lim_{M \to \infty} \sum_{i=1}^{M} \int_{0}^{t} U_\beta(t-s) \left( \sum_{q=1}^{i-1} u_q u_{i-q} \right) ds
\end{align*}\]

in \(X_T\). By the above argument, we can obtain that
\[\begin{align*}
u(t) = \sum_{i=1}^{\infty} u_i(t) \in L^\infty([0, T), X(M))
\end{align*}\]
is a solution of (1.1). \(\square\)

Let \(T\) be as in Lemma 3.3 and \(u(T)\) be the initial data, by repeating previous process (Proposition 3.1 and Lemmas 3.2 and 3.3), we can obtain \(t_2\) such that \(u(t) = \sum_{j=1}^{\infty} a_j \in L^\infty([T, t_2), X(M))\) is a unique solution of the integral equation
\[\begin{align*}
u(t) = U_\beta(t) u + \int_{0}^{t} U_\beta(t-s) u^2 ds,
\end{align*}\]
where, for \(T < t < t_2\),
\[\begin{align*}
a_1(t) &= U_\beta(t-T) w_T, \\
a_2(t) &= \int_{T}^{t} U_\beta(t-s) a_1^2 ds, \\
a_3(t) &= \int_{T}^{t} U_\beta(t-s) 2(a_1 a_2) ds, \\
a_4(t) &= \int_{T}^{t} U_\beta(t-s) (2a_1 a_3 + a_2^2) ds,
\end{align*}\]
\begin{align*}
a_5(t) &= \int_t^T U_\beta(t-s)(2a_1a_4 + 2a_2a_3)ds, \\
a_{2m}(t) &= \int_t^T U_\beta(t-s)(2a_1a_{2m-1} + \cdots + 2a_{m-1}a_{m+1} + a_m^2)ds, \\
a_{2m+1}(t) &= \int_t^T U_\beta(t-s)(2a_1a_{2m} + \cdots + 2a_{m-1}a_{m+2} + 2a_ma_{m+1})ds,
\end{align*}

Step by step we can obtain a sequence \( \{t_j\}_{j=0}^\infty, t_0 = 0, t_1 = T \) and \( t_j \to T^* \) such that for every \( 0 \leq i < \infty \) and \( t_i < t < t_{i+1}, u(t) = \sum_{j=1} a_j(t) \in L^\infty([t_i, t_{i+1}), X(\mathcal{M})) \) is the unique solution of integral equation

\[ u(t) = U_\beta(t-t_i)u(t) + \int_{t_i}^{t_{i+1}} U_\beta(t-s)u^2ds, \]

where, for \( t_i < t < t_{i+1}, \)
\begin{align*}
a_1(t) &= U_\beta(t-t_i)u(t_i), \\
a_2(t) &= \int_{t_i}^t U_\beta(t-s)a_1^2ds, \\
a_3(t) &= \int_{t_i}^t U_\beta(t-s)2(a_1a_2)ds, \\
a_4(t) &= \int_{t_i}^t U_\beta(t-s)(2a_1a_3 + a_2^2)ds, \\
a_5(t) &= \int_{t_i}^t U_\beta(t-s)(2a_1a_4 + 2a_2a_3)ds, \\
\cdots, \\
a_{2m}(t) &= \int_{t_i}^t U_\beta(t-s)(2a_1a_{2m-1} + \cdots + 2a_{m-1}a_{m+1} + a_m^2)ds, \\
a_{2m+1}(t) &= \int_{t_i}^t U_\beta(t-s)(2a_1a_{2m} + \cdots + 2a_{m-1}a_{m+2} + 2a_ma_{m+1})ds, \\
\cdots.
\end{align*}

In this way, we can obtain \( u(t) = \sum_{j=1} a_j(t) \in L^\infty([0, T^*), X(\mathcal{M})) \) is the unique solution of integral equation

\[ u(t) = U_\beta(t)u + \int_0^t U_\beta(t-s)u^2ds. \quad (3.4) \]

Moreover, if \( T^* < \infty, \) then

\[ \|u(t)\|_{L^\infty([0,T^*),X)} = \infty. \]

By the similar argument as above we can obtain the corresponding results for \( f(u) = u^k. \)
Lemma 3.4. Let $u_0 \in X(\mathcal{M})$ and $\hat{u}_0 \geq 0$. Then
\[ \sum_{j=1}^{\infty} \hat{a}_j(t) \geq \sum_{j=1}^{\infty} \hat{u}_j(t), \]
where
\[ u_1(t) = U_\beta(t)u_0, \quad u_2(t) = \int_0^t U_\beta(t-s)u_1^2(ds), \]
\[ u_3(t) = \int_0^t U_\beta(t-s)2u_1u_2 ds = \int_0^t U_\beta(t-s) \sum_{\Lambda_3} u_{i_1} u_{i_2} ds, \cdots, \]
\[ u_{2n}(t) = \int_0^t U_\beta(t-s)(2u_1^2 u_{2n-1} + \cdots + 2u_{n-1} u_{n+1} + u_n^2) ds \]
\[ = \int_0^t U_\beta(t-s) \sum_{\Lambda_{2n}} u_{i_1} u_{i_2} ds, \]
\[ u_{2n+1}(t) = \int_0^t U_\beta(t-s)(2u_1 u_{2n} + \cdots + 2u_{n-1} u_n + 2u_n u_{n+1}) ds \]
\[ = \int_0^t U_\beta(t-s) \sum_{\Lambda_{2n+1}} u_{i_1} u_{i_2} ds, \cdots, \]
and $\Lambda_j = \{(i_1, i_2) : i_1 + i_2 = j, i_m \in \mathbb{N}, m = 1, 2\}$. Actually, by the fact that for $0 \leq t < T^*$, $\sum_{j=1}^{\infty} \hat{a}_j \in L^1(\mathcal{M})$, we have $\sum_{j=1}^{\infty} a_j = \mathcal{F}\left(\sum_{j=1}^{\infty} a_j(t)\right)$. So, $\sum_{j=1}^{\infty} a_j(t)$ and $\sum_{j=1}^{\infty} \hat{a}_j(t)$ are non negative term series. Moreover, by $\{u_j(t)\}_{j=1}^{\infty}$ is the rearrangement of $\{a_j(t)\}_{j=1}^{\infty}$.

Proof. We just prove that $\sum_{j=1}^{\infty} \hat{a}_j \leq \sum_{j=1}^{\infty} \hat{u}_j$. For $t_1 = T < t < t_2$, by $\sum_{j=1}^{\infty} a_j(t) \in L^\infty([0,T), X(\mathcal{M}))$ satisfies the integral heat Eq. (3.4), we have
\[
\hat{a}_1(t) = e^{-(t-T)\|\xi\|^\beta} \hat{u}(T) = e^{-(t-T)\|\xi\|^\beta} \mathcal{F}\left(U_\beta(T)u_0 + \int_0^T U_\beta(T-s)u_1^2(s)ds\right) \\
= e^{-t\|\xi\|^\beta} \hat{u}_0 + \int_0^T e^{-(t-s)\|\xi\|^\beta} \left(\sum_{j=1}^{\infty} \hat{a}_j(s)\right) \left(\sum_{j=1}^{\infty} \hat{a}_j(s)\right) ds \\
= e^{-t\|\xi\|^\beta} \hat{u}_0 + \sum_{j=1}^{\infty} \int_0^T e^{-(t-s)\|\xi\|^\beta} \sum_{\Lambda_j} \hat{a}_{i_1} \hat{a}_{i_2} ds,
\]
\[
\hat{a}_2(t) = \int_0^t e^{-(t-s)\|\xi\|^\beta} (\hat{a}_1 * \hat{a}_1) ds, \cdots, \hat{a}_i(t) = \int_0^t e^{-(t-s)\|\xi\|^\beta} \sum_{\Lambda_i} \hat{a}_{i_1} \hat{a}_{i_2} ds, \cdots.
\]
Moreover, by $\hat{a}_i(t) = \hat{u}_i(t)$ for $0 \leq t \leq T$, we have
\[ \hat{u}_1(t) = e^{-t|\xi|^\beta} \hat{u}_0, \]
\[ \hat{u}_2(t) = \int_0^t e^{-(t-s)||\xi|^\beta} (\hat{u}_1 * \hat{u}_1)ds = \int_0^T e^{-(t-s)||\xi|^\beta} (\hat{u}_1 * \hat{u}_1)ds \]
\[ + \int_T^t e^{-(t-s)||\xi|^\beta} (\hat{u}_1 * \hat{u}_1)ds \]
\[ = \int_0^t e^{-(t-s)||\xi|^\beta} (\hat{u}_1 * \hat{u}_1)ds + \int_T^t e^{-(t-s)||\xi|^\beta} (\hat{u}_1 * \hat{u}_1)ds, \]
\[ \hat{u}_3(t) = \int_0^t e^{-(t-s)||\xi|^\beta} 2(\hat{u}_1 * \hat{u}_2)ds = \int_0^T e^{-(t-s)||\xi|^\beta} 2(\hat{u}_1 * \hat{u}_2)ds \]
\[ + \int_T^t e^{-(t-s)||\xi|^\beta} 2(\hat{u}_1 * \hat{u}_2)ds \]
\[ = \int_0^T e^{-(t-s)||\xi|^\beta} 2(\hat{a}_1 * \hat{a}_2)ds \]
\[ + \int_T^t e^{-(t-s)||\xi|^\beta} 2(\hat{u}_1 * \hat{u}_1)ds, \]
\[ \ldots \]

Note that \( \int_0^T e^{-(t-s)||\xi|^\beta} 2(\hat{a}_1 * \hat{a}_2)ds \) and \( \int_0^t e^{-(t-s)||\xi|^\beta} 2(\hat{a}_1 * \hat{a}_2)ds \) are expansion term of \( \hat{a}_1 \), \( \int_T^t e^{-(t-s)||\xi|^\beta} (\hat{u}_1 * \hat{u}_1)ds \) and \( \int_T^t e^{-(t-s)||\xi|^\beta} (\hat{u}_1 * \hat{u}_1)ds \) are the expansion term of \( \hat{a}_2 \),... Then by induction

\[ u_{2n}(t) = \int_0^t U_\beta(t-s)(2u_1u_{2n-1} + \cdots + 2u_{n-1}u_{n+1} + u_n^2)ds \]
\[ = \int_0^t U_\beta(t-s) \sum_{\Lambda_{2n}} u_{i_1}u_{i_2}ds, \]
\[ u_{2n+1}(t) = \int_0^t U_\beta(t-s)(2u_1u_{2n} + \cdots + 2u_{n-1}u_n + u_{n+1}u_{n+1})ds \]
\[ = \int_0^t U_\beta(t-s) \sum_{\Lambda_{2n+1}} u_{i_1}u_{i_2}ds, \]
\[ \ldots \]

..., and comparing the expansion term of \( \hat{a}_i \) and \( \hat{u}_i \) (by splitting the integral \( \int_0^t \) into \( \int_0^T \) and \( \int_T^t \)), it is easy to see that every expansion term of \( \hat{u}_i \) is a expansion term of \( \hat{a}_i \). (Here, the every expansion term is non negative.) Then, by induction, we have \( \sum_{j=1}^{\infty} \hat{a}_j \leq \sum_{j=1}^{\infty} \hat{u}_j \leq L^1(\hat{\mathcal{M}}) \) for \( 0 < t < T^* \). We have
\[ \mathcal{F} \left( \sum_{j=1}^{\infty} u_j \right) = \sum_{j=1}^{\infty} \hat{u}_j \]
Moreover, by the dominated convergence theorem, we have
\[
\left\| \sum_{j=1}^{\infty} u_j \right\|_{L^\infty_{t,x}([0,T^*],X)} \leq \left\| \sum_{j=1}^{\infty} \|u_j\| \right\|_{L^\infty_{t,x}([0,T^*])}.
\]

The following lemma will play crucial role to prove Theorem 1.1.

**Lemma 3.5.** There exists \(0 < T < \infty\) such that
\[
\left\| \sum_{j=1}^{\infty} u_j \right\|_{L^\infty([0,T],X)} = \infty.
\]

**Proof.** Let
\[
E_{x,y} = \{ x, y \in \mathbb{R}^d : |x|^\beta \leq |x-y|^\beta + |y|^\beta \},
\]
\[
F_{x,y} = \{ x, y \in \mathbb{R}^d : |x|^\beta \geq |x-y|^\beta + |y|^\beta \}.
\]
By \(\hat{u}_0 \geq 0, \hat{u}_0 \geq \gamma \chi_{B_0(r)}\) and \(r^d v_d \geq 2^d\) (which implies that \(\chi_{B_0(r)} \ast \chi_{B_0(r)}(\xi) \geq \chi_{B_0(r)}(\xi)\)), we have
\[
\hat{u}_1(t, \xi) = e^{-t|\xi|^\beta} \hat{u}_0(\xi) \geq e^{-t|\xi|^\beta} \gamma \chi_{B_0(r)}(\xi) = a_1(t, \xi),
\]
\[
\hat{u}_2(t, \xi) = \int_0^t e^{-(t-s)|\xi|^\beta} F(u_1^2)(s, \xi)ds
\]
\[
= \int_0^t e^{-(t-s)|\xi|^\beta} \left( (e^{-s|\xi|^\beta} \hat{u}_0) * (e^{-s|\xi|^\beta} \hat{u}_0) \right) (\xi)ds
\]
\[
= \int_0^t e^{-(t-s)|\xi|^\beta} \int_{\mathbb{R}^d} e^{-s|\xi-y|^\beta} e^{-s|y|^\beta} \hat{u}_0(\xi-y) \hat{u}_0(y)dyds
\]
\[
\geq \gamma^2 \int_0^t e^{-(t-s)|\xi|^\beta} \int_{\mathbb{R}^d} e^{-s|\xi-y|^\beta} e^{-s|y|^\beta} \chi_{B_0(r)}(\xi-y) \chi_{B_0(r)}(y)dyds
\]
\[
= \gamma^2 \int_0^t e^{-(t-s)|\xi|^\beta} \int_{\mathbb{R}^d} e^{-s|\xi-y|^\beta} e^{-s|y|^\beta} \chi_{B_0(r)}(\xi-y) \chi_{B_0(r)}(y)dyds
\]
\[
\leq \chi_{B_0(r)}(\xi-y) \chi_{B_0(r)}(y) |E_{\xi,y} + \chi_{B_0(r)}(\xi-y) \chi_{B_0(r)}(y) |F_{\xi,y} dyds.
\]
Notice that for \(\xi, y \in E_{x,y}\) and \(0 < s < t\), we have \(e^{-(t-s)|\xi|^\beta} e^{-s|\xi-y|^\beta} e^{-s|y|^\beta} \leq e^{-t|\xi|^\beta} e^{-t|\xi-y|^\beta} e^{-t|y|^\beta}\). Similarly, for \(\xi, y \in F_{x,y}\), we have \(e^{-(t-s)|\xi|^\beta} e^{-s|\xi-y|^\beta} e^{-s|y|^\beta} \leq e^{-t|\xi|^\beta} e^{-t|\xi-y|^\beta} e^{-t|y|^\beta}\).
Dividing the integral into two parts and arguing as before, we obtain

\[
\hat{u}_3(t, \xi) = 2 \int_0^t e^{-(t-s)|\xi|^\beta} (\hat{u}_1 + \hat{u}_2) (\xi) ds
\]

By (3.5) and (3.6), we have

\[
\hat{u}_3(t, \xi) \geq 2\gamma \int_0^t e^{-(t-s)|\xi|^\beta} \left( (e^{-s|\xi|^\beta} \chi_{B_0(r)}) * (\hat{u}_2)(s, \xi) \right) ds
\]

\[
= 2\gamma \int_0^t e^{-(t-s)|\xi|^\beta} \int_{\mathbb{R}^d} e^{-s|\xi-\xi_1|^\beta} \chi_{B_0(r)}(\xi-\xi_1) \hat{u}_2(s, \xi_1) d\xi_1 ds
\]

\[
\geq \gamma^3 e^{-2r^\beta} t \int_0^t e^{-(t-s)|\xi|^\beta} s \int_{\mathbb{R}^d} e^{-s|\xi-\xi_1|^\beta} \chi_{B_0(r)}(\xi-\xi_1) d\xi_1 ds
\]

Dividing the integral into two parts and arguing as before, we obtain

\[
\hat{u}_3(t, \xi) \geq \gamma^3 e^{-4r^\beta} t \int_0^t s \int_{\mathbb{R}^d} \chi_{B_0(r)}(\xi-\xi_1) \chi_{B_0(r)}(\xi_1) d\xi_1 ds
\]

\[
+ \int_{\mathbb{R}^d \setminus E_{\xi, \xi_1}} e^{-t|\xi-\xi_1|^\beta} \chi_{B_0(r)}(\xi-\xi_1) e^{-t|\xi|^\beta} \chi_{B_0(r)}(\xi_1) d\xi_1 ds
\]

\[
= \gamma^3 e^{-4r^\beta} t^2 \int_0^t \chi_{B_0(r)}(\xi-\xi_1) \chi_{B_0(r)}(\xi_1) d\xi_1 ds
\]

\[
\geq \gamma^3 e^{-4r^\beta} t^2 e^{-t|\xi|^\beta} \int_{\mathbb{R}^d \setminus E_{\xi, \xi_1}} \chi_{B_0(r)}(\xi-\xi_1) \chi_{B_0(r)}(\xi_1) d\xi_1 ds
\]

\[
\hat{u}_3(t, \xi) \geq \gamma^3 e^{-4r^\beta} t^2 e^{-t|\xi|^\beta} \int_{\mathbb{R}^d \setminus E_{\xi, \xi_1}} \chi_{B_0(r)}(\xi-\xi_1) \chi_{B_0(r)}(\xi_1) d\xi_1 ds
\]

\[
+ \gamma^3 e^{-8r^\beta} t^2 e^{-t|\xi|^\beta} \int_{\mathbb{R}^d \setminus E_{\xi, \xi_1}} \chi_{B_0(r)}(\xi-\xi_1) \chi_{B_0(r)}(\xi_1) d\xi_1 ds
\]

\[
\geq \gamma^3 e^{-8r^\beta} t^2 e^{-t|\xi|^\beta} [\chi_{B_0(r)} * \chi_{B_0(r)}](\xi)
\]

\[
\hat{u}_3(t, \xi) = a_3(t, \xi),
\]

\[
\hat{u}_{2n}(t, \xi) = \int_0^t e^{-(t-s)|\xi|^\beta} F(2u_1 u_{2n-1} + \cdots + 2u_{n-1} u_{n+1} + u_n^2)(\xi) ds
\]
\[ \geq \int_0^t e^{-(t-s)|\xi|^\beta} (2a_1(s, \xi) * a_{2n-1}(s, \xi) + \cdots + 2a_{n-1}(s, \xi) * a_n(s, \xi) + a_n(s, \xi) * a_n(s, \xi)) ds 
\]

\[ = a_2n(t, \xi), \]

\[ \overline{u}_{2n+1}(t, \xi) = \int_0^t e^{-(t-s)|\xi|^\beta} F(2u_1u_{2n} + \cdots + 2u_{n-1}u_{n+2} + 2u_nu_{n+1})(\xi) ds \]

\[ \geq \int_0^t e^{-(t-s)|\xi|^\beta} (2a_1(s, \xi) * a_{2n}(s, \xi) + \cdots + 2a_{n-1}(s, \xi) + a_{n+1}(s, \xi) * a_{n+1}(s, \xi)) ds \]

\[ = a_{2n+1}(t, \xi), \]

\[ \cdots. \]

Firstly, we have \( a_1(t, \xi) = \gamma e^{-t|\xi|^\beta} \chi_{B_0(r)}(\xi) \). Secondly, if

\[ a_i(t, \xi) \geq \gamma^i e^{-4r^\beta(i-1)t} e^{-t|\xi|^\beta} \chi_{B_0(r)}(\xi) \]

for \( i < 2n \), then we have

\[ a_{2n}(t, \xi) = \int_0^t e^{-(t-s)|\xi|^\beta} (2 \int_{\mathbb{R}^d} a_1(s, \xi - y) a_{2n-1}(s, y) dy + \cdots + 2 \int_{\mathbb{R}^d} a_{n-1}(s, \xi - y) a_n(s, y) dy + \int_{\mathbb{R}^d} a_n(s, \xi - y) a_n(s, y) dy) ds \]

\[ \geq (2n-1)\gamma^{2n} \int_0^t e^{-(t-s)|\xi|^\beta} \]

\[ \int_{\mathbb{R}^d} s^{2n-2} e^{-4r^\beta(2n-2)s} e^{-s|\xi-y|^\beta} e^{-s|y|^\beta} \chi_{B_0(r)}(\xi - y) \chi_{B_0(r)}(y) dy ds \]

\[ \geq (2n-1)\gamma^{2n} e^{-4r^\beta(2n-2)t} \int_0^t s^{2n-2} e^{-(t-s)|\xi|^\beta} \]

\[ \int_{\mathbb{R}^d} e^{-s|\xi-y|^\beta} e^{-s|y|^\beta} \chi_{B_0(r)}(\xi - y) \chi_{B_0(r)}(y) dy ds \]

Dividing integral into two parts as before, we have

\[ a_{2n}(t, \xi) \geq (2n-1)\gamma^{2n} e^{-4r^\beta(2n-2)t} \int_0^t s^{2n-2} \]

\[ \left( \int_{\mathbb{R}^d \cap E_{t, y}} e^{-(t-s)|\xi|^\beta} e^{-s|y|^\beta} \chi_{B_0(r)}(\xi - y) \chi_{B_0(r)}(y) dy \right. \]

\[ \left. + e^{-t|\xi|^\beta} \int_{\mathbb{R}^d \cap F_{t, y}} \chi_{B_0(r)}(\xi - y) \chi_{B_0(r)}(y) dy \right) ds \]

\[ \geq \gamma^{2n} e^{-4r^\beta(2n-2)t} 2^{2n-1} \left( \int_{\mathbb{R}^d \cap E_{t, y}} e^{-(t-s)|\xi|^\beta} e^{-s|y|^\beta} \chi_{B_0(r)}(\xi - y) \chi_{B_0(r)}(y) dy ds \right. \]

\[ \left. + e^{-t|\xi|^\beta} \int_{\mathbb{R}^d \cap F_{t, y}} \chi_{B_0(r)}(\xi - y) \chi_{B_0(r)}(y) dy \right) \]
By similar argument as above, we can obtain that for general $k$,

$$
\|u_{i}\|_{L^\infty_T(X)} \geq \|u_{i}\|_{F_{L_1}} \geq \left\| \gamma^i e^{-4r^\beta(i-1)t} t^{i-1} e^{-t|\xi|^\beta} \chi_{B_0(r)}(\xi) \right\|_{L^1_T} = \infty.
$$

By similar argument as above, we can obtain that for general $k$,

$$
a_{i k - (i-1)}(t, \xi) \geq \gamma^{i k - (i-1)} e^{-4r^\beta(k-1)t} t^{i-1} e^{-t|\xi|^\beta} \chi_{B_0(r)}(\xi), i = 1, 2, \ldots
$$

Then, by Lemma 2.1, for $\gamma \geq 4r^\beta e$, if $T \geq \frac{1}{4r^\beta}$, we have

$$
\sum_{i=1}^{\infty} \|u_{i}\|_{L^\infty_T(X)} \geq \sum_{i=1}^{\infty} \left\| \gamma^i e^{-4r^\beta(i-1)t} t^{i-1} e^{-t|\xi|^\beta} \chi_{B_0(r)}(\xi) \right\|_{L^1_T} = \infty.
$$

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 3.1 and Lemmas 3.2, 3.3, we have, for $0 < T < T^*$,

$$
u(t) = \sum_{i=1}^{\infty} a_i(t) \in L^{\infty}_{loc}([0, T^*), X(M))
$$

is a solution of (1.1). By Lemma 3.4, $\|\sum_{i=1}^{\infty} u_{i}(t)\|_{L^{\infty}_{loc}((0, T^*), X)} \leq \|\sum_{i=1}^{\infty} a_i(t)\|_{L^{\infty}_{loc}((0, T^*), X)}$. By Lemma 3.5, we have, for $T^* \geq \frac{1}{4r^\beta(k-1)}$, $\|\sum_{i=1}^{\infty} u_{i}(t)\|_{L^{\infty}((0, T^*), X)} = \infty$. This completes the proof. \qed
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Declarations

Conflict of interest There is no conflict of interest for this article.

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