Universal gap fluctuations in the superconductor proximity effect

M. G. Vavilov, P. W. Brouwer, and V. Ambegaokar

Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853

C. W. J. Beenakker

Instituut-Lorentz, Universiteit Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands

(March 21, 2022)

Random-matrix theory is used to study the mesoscopic fluctuations of the excitation gap in a metal grain or quantum dot induced by the proximity to a superconductor. We propose that the probability distribution of the gap is a universal function in rescaled units. Our analytical prediction for the gap distribution agrees well with exact diagonalization of a model Hamiltonian.

PACS Numbers 73.23.-b, 74.50.+r, 74.80.Fp

A normal metal in the proximity of a superconductor acquires characteristics that are typical of the superconducting state [1]. One of those characteristics is that the quasiparticle density of states vanishes at the Fermi energy. This superconductor proximity effect is most pronounced in a confined geometry, such as a thin metal film or metal grain, or a semiconductor quantum dot. In that case, provided the scattering in the normal metal is chaotic, no excitations exist within an energy gap $E_{\text{g}} \sim \hbar / \tau$, where $\tau$ is the typical time between collisions with the superconductor [2].

If the coupling to the superconductor is weak (as for the point contact coupling of Fig. 1), the functional form of the density of states becomes independent of microscopic properties of the normal metal, such as the shape, dimensionality, or mean free path. Weak coupling means that $\tau$ is much bigger than the time $\tau_{\text{erg}}$ needed for ergodic exploration of the phase space in the normal region. For a point contact with $N \gg 1$ propagating modes at the Fermi level $\varepsilon = 0$, the density of states has a square root singularity at the excitation gap [3],

$$\rho_{\text{mf}}(\varepsilon) = \frac{1}{\pi} \sqrt{\frac{\varepsilon - E_{\text{g}}}{\Delta_{\text{g}}^3}}. \quad (1)$$

For a ballistic point contact and in the absence of a magnetic field, $E_{\text{g}} = c N \delta$ and $\Delta_{\text{g}} = c' N^{1/3} \delta$, where $c = 0.048$ and $c' = 0.068$ are numerical constants and $\delta$ is the mean level spacing in the normal metal when it is decoupled from the superconductor.

Equation (1) was obtained in a self-consistent diagrammatic perturbation theory that uses $\tau \delta / \hbar \sim N^{-1}$ as a small parameter. Such a mean-field theory provides a smooth density of states for which energies can only be resolved on the scale of $\hbar / \tau \sim N \delta$, not on smaller energy scales, and is unable to deal with mesoscopic sample-to-sample fluctuations of the excitation gap. Mesoscopic fluctuations arise, e.g., upon varying the shape of a quantum dot or the impurity configuration in a metal grain. The lowest excited state $\varepsilon_1$ fluctuates from sample to sample around the mean field value $E_{\text{g}}$, with a probability distribution $P(\varepsilon_1)$. It is the purpose of this paper to go beyond mean field theory and to study the mesoscopic fluctuations of the excitation spectrum close to $E_{\text{g}}$. Our main result is that the gap distribution $P(\varepsilon_1)$ is a universal function of the rescaled energy $x = (\varepsilon_1 - E_{\text{g}}) / \Delta_{\text{g}}$, in a broad range $|x| \ll N^{2/3}$. The Fermi level itself ($\varepsilon = 0$) falls outside this range, which is why the universal gap distribution was not found in a recent related study [8]. Our main findings are illustrated in Fig. 2.

We first consider the gap distribution in the absence of a magnetic field, and then include a time-reversal symmetry breaking magnetic field. Starting point of our calculation is the effective Hamiltonian [10]

$$H = \left( \begin{array}{cc} \frac{H}{\pi W W^\dagger} & -\pi W W^\dagger \frac{H^*}{\pi W W^\dagger} \\ -\pi W W^\dagger & -H^* \end{array} \right). \quad (2)$$

Here $H$ is an $M \times M$ Hermitian matrix representing the Hamiltonian of the isolated quantum dot, and $W$ is an $M \times N$ matrix that describes the coupling to the superconductor via an $N$-mode point contact. For a ballistic point contact, $W_{mn} = \pi^{-1} \delta_{mn} (M \delta)^{1/2}$ [1]. The number $M$ is sent to infinity at the end of the calculation [2]. The effective Hamiltonian is a valid description of the low-lying excitations if the Thouless energy
According to the universality hypothesis, the spectral statistics near the edge, in rescaled variables \((\varepsilon - b)/a\) depends only on the exponent \(p\) and on the symmetry index \(\beta\) \([\beta = 1 \ (2) \text{ in the presence (absence) of time-reversal symmetry}]\). Generically, \(p\) is either \(1/2\) (soft edge) or \(-1/2\) (hard edge). For our problem, we have \(\beta = 1\), \(p = 1/2\), \(a = \pi^{2/3}\Delta_g\), \(b = E_g\), cf. Eq. (1). The corresponding gap distribution is given by

\[ P(\varepsilon) = \frac{d}{d\varepsilon} F_1 \left[ (\varepsilon - E_g)/\Delta_g \right], \]

\[ F_1(x) = \exp \left( -\frac{1}{2} \int_{-\infty}^{x} [q(x') + (x - x')q^2(x')]dx' \right). \]  

The function \(q(x)\) is the solution of

\[ q''(x) = -xq(x) + 2q^3(x), \]

with asymptotic behavior \(q(x) \to \text{Ai}(-x)\) as \(x \to -\infty\) \([\text{Ai}(x)\) being the Airy function].

The distribution (1) is shown in Fig. 3 (solid curve). It is centered at a positive value of \(x = (\varepsilon_1 - E_g)/\Delta_g\), meaning that the average gapsize \((\varepsilon_1)\) is about \(\Delta_g\) bigger than the mean-field gap \(E_g\). For small \(x\) there is a tail of the form

\[ P(x) \approx \frac{1}{4\sqrt{\pi}|x|^{3/4}} \exp \left( -\frac{2}{3}|x|^{3/2} \right), \quad x \ll -1. \]

Non-universal corrections to the distribution (1) become important for energy differences \(|\varepsilon - E_g| \gtrsim E_g\), hence for \(|x| \gtrsim N^{2/3}\). Since the width of the gap distribution is of order unity in the variable \(x\), the probability to find a sample with an excitation gap in the non-universal regime is exponentially small.

In order to verify our universality hypothesis, we compare Eq. (5) with the results of an exact diagonalization of the Hamiltonian (3). As one can see in Fig. 3, the numerical data are in good agreement with the analytical prediction. The small deviations can be attributed to the finiteness of \(N\) and \(M\) in the numerics.

Let us now consider the effect of a weak magnetic field on the gap distribution. In the effective Hamiltonian, the presence of a magnetic field is described by replacing \(H\) by

\[ H(\alpha) = H + i\alpha A. \]

Here \(A\) is an \(M \times M\) real antisymmetric matrix, whose off-diagonal elements have the same variance as those of \(H\). The parameter \(\alpha\) is proportional to the magnetic field,

\[ M\alpha^2 = \eta \left( \frac{\Phi}{\Phi_0} \right)^2 \frac{h}{\tau_{\text{erg}}}, \]

where \(\Phi\) is the magnetic flux through the quantum dot, \(\Phi_0 = h/e\) is the flux quantum, and \(\eta\) is a non-universal
numerical constant \[11\]. The case \( \alpha = 0 \) corresponds to the GOE that we considered above; the case \( \alpha = 1 \) corresponds to the Gaussian unitary ensemble (GUE) of fully broken time-reversal symmetry.

The effect of a magnetic field on the density of states in mean-field theory is known \[3\]. The square-root singularity \[4\] near the gap still holds, but the magnitude of the gap is reduced. The critical flux \( \Phi_{c} \) at which \( E_{g} = 0 \) and hence the proximity effect is fully suppressed is given by

\[
M \alpha^2 \sim N \Rightarrow \Phi_{c} \sim \Phi_{\text{0}} \sqrt{\frac{N \delta_{\text{erg}}}{\hbar}}. \tag{11}
\]

This is a much larger flux than the flux \( \Phi_{\text{bulk}} \) at which the spectral statistics in the bulk of the spectrum crosses over from GOE to GUE, which is given by \[15\]

\[
M \alpha^2 \sim 1 \Rightarrow \Phi_{\text{bulk}} \sim \Phi_{\text{0}} \sqrt{\frac{\tau_{\text{erg}} \delta}{\hbar}}. \tag{12}
\]

We will now argue that the characteristic flux \( \Phi_{\text{edge}} \) for the spectral statistics at the edge of the spectrum is intermediate between \( \Phi_{c} \) and \( \Phi_{\text{bulk}} \). We consider the effect of the magnetic field on the lowest eigenvalue \( \varepsilon_{1} \) of \( \mathcal{H} \) to second order in perturbation theory,

\[
\delta \varepsilon_{1} = \sum_{j \neq 1} \alpha^{2} \frac{|\langle j | A | j \rangle|^{2}}{\varepsilon_{1} - \varepsilon_{j}}, \quad A = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}. \tag{13}
\]

Since typically \(|\langle j | A | j \rangle|^{2} \sim M \delta^{2}/\pi^{2} \) and \( \varepsilon_{2} - \varepsilon_{1} \sim \Delta_{g} \), we see that the effect of level repulsion from the neighboring level \( \varepsilon_{2} \) on the lowest level \( \varepsilon_{1} \) becomes comparable to \( \Delta_{g} \sim N^{1/3} \delta \) if

\[
M \alpha^{2} \sim N^{2/3} \Rightarrow \Phi_{\text{edge}} \sim \Phi_{\text{0}} \sqrt{\frac{N^{2/3} \tau_{\text{erg}} \delta}{\hbar}}. \tag{14}
\]

The terms in Eq. \((13)\) with \( j \gg 1 \) give a uniform shift of all low-lying levels, and hence do not affect the fluctuations. For \( N \gg 1 \) the flux scale \[14\] for breaking time-reversal symmetry at the edge of the spectrum is much smaller than the critical flux \( \Phi_{c} \) needed to suppress the proximity effect. What is needed is \( N^{2/3} \ll N \). This condition is difficult to satisfy in a numerical calculation.

The analytical prediction for fully broken time-reversal symmetry is \[14\]

\[
P(\varepsilon) = \frac{d}{d \varepsilon} F_{2}(\varepsilon - E_{g}/\Delta_{g}), \tag{15}
\]

\[
F_{2}(x) = \exp \left( -\int_{-\infty}^{x} (x - x')^{2} q^{2}(x') dx' \right). \tag{16}
\]

This curve is shown dashed in Fig. 3. The tail for small \( x \) is now given by

\[
P(x) \approx \frac{1}{8|x|} \exp \left( -\frac{4}{3} |x|^{3/2} \right), \quad x \ll -1. \tag{17}
\]

To make contact with Ref. \[3\] we briefly discuss the implications of our results for the ensemble averaged density of states \( \langle \rho(\varepsilon) \rangle \) in the sub-gap regime. The tail of \( P(x) \) for \( x \gtrsim -1 \) is the same as the tail of \( \langle \rho \rangle \), cf. Fig. 2. We conclude that \[13\]

\[
\langle \rho(\varepsilon) \rangle \propto \exp \left( -\frac{2\beta}{3} x^{3/2} \right) \tag{18}
\]

over a broad range \( \Delta_{g} \ll E_{g} - \varepsilon \ll E_{g} \) inside the mean-field gap. A different exponential decay (with a power 2 instead of 3/2 in the exponent) was predicted recently by Beloborodov, Narozhny, and Aleiner \[3\], for the sub-gap density of states of an ensemble of superconducting grains in a weak magnetic field. Since the mean-field density of states in that problem is also of the form \[3\], the universal GUE edge statistics should apply. The reason that the universal decay \[3\] was not obtained in Ref. \[3\] is that their theory applies to the non-universal energy range \( \varepsilon \ll E_{g} \) near the Fermi level. To emphasize the significance of the universal energy range we note that the probability to have the lowest energy level in that range is larger than in the non-universal range by an exponentially large factor \( \propto \exp(\{E_{g}/\Delta_{g}\}^{3/2}) \).

In conclusion, we have argued that the proximity effect in a mesoscopic system has a gap distribution which is universal once energy is measured in units of the energy scale \( \Delta_{g} \propto (E_{g} \delta^{2})^{1/3} \) defined from the mean-field density
TABLE I. Characteristic energy and magnetic flux scales for the spectral statistics in the bulk and at the edge of the spectrum and for the size of the gap.

|                      | Energy scale | Flux scale |
|----------------------|--------------|------------|
| Bulk statistics      | $\delta$    | $\Phi_0\tau_{\text{erg}}^{1/2}/h^{1/2}$ |
| Edge statistics      | $E_g^{3/2}/\delta^{3/3}$ | $\Phi_0\tau_{\text{erg}}^{1/2}E_g^{3/3}/h^{1/2}$ |
| Gap size             | $E_g^{3/2}/\delta^{3/3}$ | $\Phi_0\tau_{\text{erg}}^{1/2}E_g^{3/3}/h^{1/2}$ |

of states $\rho(\varepsilon) = [(\varepsilon - E_g)^{31/2}/\pi]$. This universal distribution is the same as the distribution of the smallest eigenvalue of the Gaussian orthogonal or unitary ensembles from random-matrix theory, depending on whether time-reversal symmetry is broken or not. We have identified the magnetic field scale for breaking time-reversal symmetry and verified our results by exact diagonalization of an effective Hamiltonian. Characteristic energy and magnetic field scales are summarized in Table 1. The universality of our prediction should offer ample opportunities for experimental observation.

We thank I. Aleiner, I. Beloborodov, E. Mishchenko, and B. Narozhny for useful discussions. This work was supported by the Cornell Center for Materials Research under NSF grant No. DMR–9632275 and by the Dutch Science Foundation NWO/FOM.

[1] M. Tinkham, *Introduction to Superconductivity* (McGraw Hill, New York, 1995).
[2] A. A. Golubov and M. Yu. Kupriyanov, Sov. Phys. JETP 69, 805 (1989).
[3] W. Belzig, C. Bruder, and G. Schön, Phys. Rev. B 54, 9443 (1996).
[4] J. A. Melsen, P. W. Brouwer, K. M. Frahm, and C. W. J. Beenakker, Europhys. Lett. 35, 7 (1997); Physica Scripta 69, 223 (1997).
[5] A. Lodder and Yu. V. Nazarov, Phys. Rev. B 58, 5783 (1998).
[6] F. Zhou, P. Charlat, B. Spivak, and B. Pannetier, J. Low Temp. Phys. 110, 841 (1998).
[7] W. Ihra, M. Leadbeater, J. L. Vega, and K. Richter, cond-mat/9909101.
[8] In a quantum dot or metal grain of size $R$, with Fermi velocity $v_F$ and mean free path $\ell$, one has $1/\tau_{\text{erg}} \sim v_F R^{-2} \min(\ell, R)$.
[9] I. S. Beloborodov, B. N. Narozhny, and I. L. Aleiner, cond-mat/9912217.
[10] K. M. Frahm, P. W. Brouwer, J. A. Melsen, and C. W. J. Beenakker, Phys. Rev. Lett. 76, 2981 (1996).
[11] C. W. J. Beenakker, Rev. Mod. Phys. 69, 731 (1997).
[12] In our numerical computations it was necessary to choose the ratio $M/N$ relatively small in order to achieve the limit $N \gg 1$ needed for universality of the gap distribution. For finite $M/N$, the mean-field result still holds, but now with coefficients $c$ and $c'$ for the energy scales $E_g$ and $\Delta_g$ that weakly depend on $M/N$. For the comparison of the numerical data with the prediction of random-matrix theory we calculated $E_g$ and $\Delta_g$ from the mean-field theory (Eq. (8) of Ref. 1) for the values of $M/N$ used in the exact diagonalizations. No fit parameters are involved in this procedure.
[13] M. L. Mehta, *Random Matrices* (Academic, New York, 1991).
[14] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. 299, 189 (1998).
[15] E. Kanzieper and V. Freilikher, Phys. Rev. Lett. 78, 3806 (1997).
[16] C. A. Tracy and H. Widom, Commun. Math. Phys. 159, 151 (1994); 177, 727 (1996).
[17] A. Pandey and M. L. Mehta, Commun. Math. Phys. 87, 449 (1983).
[18] The complete random matrix-theory prediction is $\langle \rho(x) \rangle = -x Ai^2(x) + [Ai(x)]^2 + \frac{1}{2} \delta_{\beta, 1} Ai(1 - x Ai(1)) dy$. The $\beta = 1$ result is plotted in Fig. 2.