Dynamical Chiral Symmetry Breaking in Effective Models of QCD in the Bethe–Salpeter Approach

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Abstract

Dynamical breaking of chiral symmetry in effective models of QCD is studied. Introducing a cut-off function or a non-local interaction, the Noether current must be modified and thus the Ward–Takahashi identity and the PCAC relation are modified accordingly. We point out that the pion decay constant must be defined consistently with the Noether current so that the low-energy relations are satisfied. We define the proxy of the Noether current for general effective models, which is consistent with loop expansion of the Cornwall–Jackiw–Tomboulis effective action. A general formula for the pion decay constant in terms of the Bethe–Salpeter amplitude is derived. The effective Pagels–Stokar formula is proposed which is useful to estimate the decay constant without solving the Bethe–Salpeter equation.

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1 Introduction

The program to derive the observed properties of hadrons non-perturbatively in quantum chromodynamics (QCD) has been pursued with great intensity but not accomplished yet. The concept of chiral symmetry and its spontaneous breakdown are among the most important aspects of low-energy hadron physics. The spontaneous breakdown of chiral symmetry is believed to be responsible for a large part of the low-lying hadron masses as well as for the emergence of octet pseudo-scalar mesons as Nambu–Goldstone (NG) bosons. In order to explain the observed hadron spectrum, one also needs small, explicitly chiral symmetry breaking terms, namely, the flavor dependent current quark mass terms.

Dynamical chiral symmetry breaking (DCSB) of QCD has been extensively studied in effective models in terms of light quarks. The Nambu–Jona-Lasinio (NJL) model[1], as an example, is a superb model, that presents the main idea of DCSB most concisely. It describes the spectrum and properties of NG mesons fairly well.[2] The NJL model, however, has several shortcomings. It may be too simple to represent some of the key properties of QCD. First, it does not describe color confinement and therefore the meson spectrum above $2M_Q$ threshold ($M_Q$ is the constituent quark mass) is not reproduced. Secondly, the quark interactions are not asymptotically free in NJL. This property will be crucial to describe short distance structure of mesons.

There have been various attempts to derive effective quark theories which are consistent with the asymptotic behavior and low-energy chiral symmetry breaking. One of such models is the improved ladder approximation proposed by Higashijima and Miransky.[3, 4] This model is consistent with the asymptotic behavior of the leading order renormalization group analysis of QCD.[5] It also realizes DCSB, as is seen in the non-zero quark mass function given as the solution of the Schwinger–Dyson (SD) equation. Then according to the NG theorem, the Bethe–Salpeter (BS) equation gives a massless NG boson with $J^{PC} = 0^{−+}$.[6] The numerical results for the pion decay constant $f_\pi$ and the quark condensate $\langle \bar{\psi} \psi \rangle$ are consistent with other analyses.[4] On the other hand, it is known that this approximation violates the axial-vector Ward–Takahashi identity.[8] Therefore it looks unsuitable to study the low-energy relations in QCD.
However, there exists a global chiral symmetry in the improved ladder approximation. It produces a Noether current associated with the axial transformation, namely axial-vector current $J_{5\mu}^\alpha$, which is generally modified from the QCD form $\bar{\psi}\gamma_\mu\gamma_5\frac{\lambda^\alpha}{2}\psi$. Accordingly the pion decay constant defined by

$$i p_\mu f_\pi \delta_{\alpha\beta} := \langle 0 | J_{5\mu}^\alpha (0) | \pi^\beta (p) \rangle$$

must be modified. The Ward–Takahashi identity for the modified axial-vector current is satisfied.

The improved ladder approximation is considered as a typical example of the effective model which contains a non-local interaction term, while various similar models are proposed. Among these effective models some require no modification of the axial-vector current. However, the corresponding Bethe–Salpeter equations become so complicated that practical calculation is rather difficult in such models. In Ref. [9], the truncation in the Gegenbauer polynomial expansion is employed. This procedure may violate the low-energy relations. In Ref. [10], to preserve the axial-vector Ward–Takahashi identity, the running gauge parameter is employed, which makes practical calculation hard for finite quark mass. In Ref. [13], the SD and BS equations are solved consistently. But as the wave function renormalization constant $Z_2$ deviates from 1, the vector current or the charge operator should be modified.

On the other hand, the improved ladder approximation is simple and useful to treat numerically. The aim of this paper is to study a systematic approach which repairs the low-energy relations in the effective models and makes it possible to study the low-energy relations in QCD. We propose to define the axial-vector current consistently with the approximation in the effective model, and describe the low-energy relation in terms of the solutions of the SD and BS equations. We derive a formula for the consistent pion decay constant. There have been similar studies of this problem in the literature where the NJL models with smooth cut-off regularization are studied. However our approach is more general and systematic. We study the SD and BS equations in the loop expansion using the Cornwall–Jackiw–Tomboulis(CJT) effective action formulation. We show that the BS equation has a massless bound state solution corresponding to the NG boson. We also give a formula for the decay constant which is consistent with the rainbow–ladder approximation.
In Sec.2, we present how the Noether current is modified due to the loop momentum cut-off and/or the non-local interaction. The modification of the axial Ward–Takahashi identity is also discussed.

In Sec.3, we employ the CJT effective action to study the consistency of the SD and the BS equations. The NG theorem and the pion decay constant are studied in terms of the effective action. We further consider the explicit chiral symmetry breaking due to the finite quark mass. Three cases in the patterns of the local chiral symmetry breaking of the effective models are studied in detail.

In Sec.4, we present a general formula of the pion decay constant in the loop expansion of the CJT action. The formula is given in terms of the quark full propagator as the solution of the SD equation and the pion BS amplitude. We also propose a Pagels–Stokar type formula, which gives the pion decay constant in terms only of the quark full propagator, namely, the mass function $B(q^2)$.

In Sec.5, we employ a simple numerical model to check our analytical results. We confirm that the modifications of axial-vector current and the pion decay constant are significantly large, while our new Pagels–Stokar type formula gives a good approximation.

A conclusion is given in Sec.6.

2 Noether current and Low-energy relations

Before the discussion about the chiral symmetry and low-energy relations in the approximated SD and BS equations, we consider general aspects of chiral symmetry and low-energy relations in effective models. In this section, we show how to modify the Noether current and the Ward–Takahashi identity from the original form in QCD. This gives a perspective in the following discussion for the case of the rainbow–ladder approximated SD and BS equations.

We consider a general effective model of QCD whose lagrangian density is given by

$$\mathcal{L}[\psi, \bar{\psi}] := \mathcal{L}_{\text{free}}[\psi, \bar{\psi}] + \mathcal{L}_{\text{int}}[\psi, \bar{\psi}]$$  \hspace{1cm} (2)

$$\mathcal{L}_{\text{free}}[\psi, \bar{\psi}] := \bar{\psi} f(\partial^2)(i\partial - m_0)\psi$$  \hspace{1cm} (3)

where the quark field $\psi$ is a column vector in the color, the flavor and the Dirac space. Here the function $f(\zeta)$ of $\zeta = \partial^2$ is introduced as a cut-off regularization function in order to regularize
the ultraviolet divergences coming from quark loops. The reason we introduce the cut-off function at the lagrangian level is to preserve the consistency of the SD and BS equations. If one uses the regularization that is inconsistent between the SD and BS equations, the low-energy relations based on the chiral symmetry should be violated by the regularization. The function $f(\zeta)$ should satisfy $f(\zeta = 0) = 1$. For $\zeta \gg \Lambda_{UV}^2$, $f(\zeta)$ should diverge sufficiently fast so as to regularize loop integrals. For example, the sharp cut-off is given by

$$f(\zeta) = 1 + M\theta(\Lambda_{UV}^2 - \zeta), \quad M \to \infty$$

and the Gaussian smooth cut-off is given by

$$f(\zeta) = 1 + M\exp\left(\frac{\zeta}{\Lambda_{UV}^2}\right).$$

A caveat of this procedure is that a general choice of $f(\zeta)$ may cause a difficulty in canonical quantization. Since our purpose is not to study the non-local field theory, we simply employ the path integral formulation with the action integral regularized by $f(\zeta)$. Although this treatment is not rigorous, it is sufficient in the present discussion. We thus maintain the consistency between the chiral symmetry and the regularization.

$f(\zeta)$ also determines a scale $\Lambda_{UV}$ at which the bare quark mass $m_0$ is evaluated. To compare the bare mass $m_0$ the renormalized quark mass $m_R$ in QCD, one must impose a suitable renormalization condition. In general $m_0$ is a diagonal flavor matrix, i.e., $m_0 = \text{diag}(m_u, m_d, m_s)$ for $N_f = 3$. But in this paper we deal only with a flavor independent mass or the $SU(3)$ limit. The generalization to the flavor dependent masses can be also done. Although there is a difficulty in choosing the center of mass coordinate for the quark–antiquark bound states physically, the low-energy relations hold for any choice of the center of mass coordinate.

A general non-local 4-quark interaction is written by

$$\mathcal{L}_{\text{int}}[\psi, \bar{\psi}](x) := -\frac{1}{2}\int pp'qq' \mathcal{K}_{mm',nn'}(p, p'; q, q') \times \bar{\psi}_m(p)\psi_{n'}(p')\bar{\psi}_n(q)\psi_{n'}(q')e^{-i(p+p'+q+q')x}$$

where $\int_p$ denotes $\int d^4p (2\pi)^4$ and the Fourier transformation of the quark field is defined by

$$\bar{\psi}(p) := \int d^4x e^{ipx} \bar{\psi}(x),$$

$$\psi(p) := \int d^4x e^{ipx} \psi(x).$$
The indices \( m, n, \ldots \) are combined indices \( m := (a, i, f), n := (b, j, g), \ldots \) with Dirac indices \( a, b, \ldots \) and color indices \( i, j, \ldots \) and flavor indices \( f, g, \ldots \). We do not consider the quantum correction and renormalization for the interaction itself, because we assume that effective models have already contained such effects. \( \mathcal{K}^{mm',nn'}(p, p'; q, q') \) is an interaction kernel which we can assume without loss of generality

\[
\mathcal{K}^{mm',nn'}(p, p'; q, q') = \mathcal{K}^{nn',mm'}(q, q'; p, p').
\]  

We further assume that the interaction is invariant under the global \( SU(3)_L \times SU(3)_R \) transformation

\[
\mathcal{K}^{mm',nl}(p, p'; q, q')(i\frac{\lambda^\alpha}{2})_{ln'} = (i\frac{\lambda^\alpha}{2})_{nl}\mathcal{K}^{mm',ln'}(p, p'; q, q')
\]  

\[
\mathcal{K}^{mm',nl}(p, p'; q, q')(i\gamma_5\frac{\lambda^\alpha}{2})_{ln'} = -(i\gamma_5\frac{\lambda^\alpha}{2})_{nl}\mathcal{K}^{mm',ln'}(p, p'; q, q')
\]

where \( \lambda^\alpha \) is the Gell-Mann matrix \( (\alpha = 1, \ldots, 8) \) for \( N_F = 3 \) with normalization condition \( \text{tr}(F)[\lambda^\alpha\lambda^\beta] = 2\delta^{\alpha\beta} \). Thus the violation of the global chiral symmetry comes from the quark mass term only. We here concentrate only on the chiral symmetric 4-quark interaction for simplicity.

The generalization to the multi-quark interaction is straightforward.\[23\] Some effective models in Ref.\[24, 16, 23\] contain the interaction which violates the global \( U(1)_A \) symmetry, and our approach can be generalized to such models as well.

Under the infinitesimal axial transformation

\[
\psi(x) \rightarrow \psi'(x) := (1 + i\gamma_5\frac{\lambda^\alpha}{2}\theta^\alpha(x))\psi(x),
\]

the action

\[
S[\psi, \bar{\psi}] := \int d^4x \mathcal{L}[\psi, \bar{\psi}](x)
\]

changes by

\[
\Delta_5 S[\psi, \bar{\psi}] := S[\psi', \bar{\psi}'] - S[\psi, \bar{\psi}]
\]

\[
\equiv \int d^4x \theta^\alpha(x) \left\{ \overline{\psi}(x)(\bar{\psi} f(\overline{\partial^2})\gamma_5\frac{\lambda^\alpha}{2} + f(\partial^2)\bar{\theta}\gamma_5\frac{\lambda^\alpha}{2})\psi(x) 
\right.
\]

\[
- 2m_0\overline{\psi}(x)f(\overline{\partial^2}) + f(\partial^2)\bar{\psi}\gamma_5\frac{\lambda^\alpha}{2}\psi(x)
\]

\[
- \int_{pp'qq'} \left( \mathcal{K}^{mm',nn'}(p, p'; -p - p' - q, q') - \mathcal{K}^{mm',nn'}(p, p'; q, -p - p' - q) \right)
\]

\[
\times \overline{\psi}_m(p)\psi_m(p')\overline{\psi}_n(q)(i\gamma_5\frac{\lambda^\alpha}{2}\psi)_n'(q')e^{-i(p + p' + q + q')x} \right\}.
\]
One finds that the third term in the right hand side (RHS) of Eq. (14) vanishes if the momentum dependence of the interaction kernel is such that

$$\mathcal{K}^{nn',nn'}(p, p'; q, q') = \mathcal{K}^{nn',nn'}(p + p'; q + q').$$

(15)

This relation is satisfied when the kernel is generated by one gluon exchange whose coupling depends only on the transfer momentum.\[^{9, 10, 12, 13}\] Furthermore if the cut-off function $f(\zeta)$ is identically 1, we obtain the operator identity

$$\partial^\mu J_5^\mu(x) = 2m_0 J_5^\alpha(x)$$

(16)

for

$$J_5^\mu(x) := \overline{\psi}i\gamma_5 \frac{\lambda^\alpha}{2} \psi(x),$$

(17)

$$J_5^\alpha(x) := \overline{\psi}i\gamma_5 \frac{\lambda^\alpha}{2} \psi(x).$$

(18)

For general $\mathcal{K}^{nn',nn'}(p, p'; q, q')$ and $f(\zeta)$, Eq. (14) is written as

$$\Delta_5 S[\psi, \overline{\psi}] = \int d^4x \theta^\alpha(x) \left\{ \partial^\mu J_5^\alpha - 2m_0 I_5^\alpha \right\}(x)$$

(19)

with

$$J_5^\mu(x) := I_5^\mu(x) - K_\mu^\alpha(x),$$

$$I_5^\alpha(x) := \int_{p,q} \overline{\psi}(p) \left( \gamma_\mu f_1(-(p^2, -q^2) + (p - q)_\mu(\theta - \theta') f_2(-p'^2, -q'^2) \right) \gamma_5 \frac{\lambda^\alpha}{2} \psi(q) e^{-i(p+q)x},$$

(21)

$$K_\mu^\alpha(x) := \int_{pp'qq'} \frac{i(p + p' + q + q')_\mu}{(p + p' + q + q')^2} \times \left( \mathcal{K}^{nn',nn'}(p, p'; -p - p' - q', q') - \mathcal{K}^{nn',nn'}(p, p'; q, -p - p' - q) \right) \times \overline{\psi}_m(p) \psi_{m'}(p') \psi_n(q) (i\gamma_5 \frac{\lambda^\alpha}{2} \psi)(q') e^{-i(p+p'+q+q')x},$$

(22)

$$I_5^\alpha(x) := \overline{\psi}(x) \frac{f(\overline{\partial}^2) + f(\partial^2)}{2},$$

(23)

$$f_1(-p^2, -q^2) := \frac{f(-p^2) + f(-q^2)}{2}, \quad f_2(-p^2, -q^2) := \frac{f(-p^2) - f(-q^2)}{2(p^2 - q^2)}.$$
Because the effective lagrangian contains non-standard momentum dependencies, the Noether axial-vector current is non-local and cannot be calculated in the standard procedure. To avoid confusion, we call Eq.(17) the *naive* axial-vector current and call Eq.(20) the *true* axial-vector current.

When chiral symmetry is broken dynamically, the Schwinger–Dyson equation has a solution with non-zero quark mass function and the Bethe–Salpeter equation gives a massless pion state which must appear as a pole of the Noether current according to the NG theorem. We define the *effective* decay constant $\tilde{f}_\pi$ associated with the Noether current by

$$iP_{B\mu} \tilde{f}_\pi := \langle 0 | \tilde{J}^\alpha_{5\mu} (0) | P \rangle$$

which is compared to the *naive* decay constant $f_\pi$ defined by

$$iP_{B\mu} f_\pi := \langle 0 | J^\alpha_{5\mu} (0) | P \rangle$$

where $| P \rangle$ denotes a pion state with normalization condition $\langle P | P \rangle = (2\pi)^3 2m_0 \delta^3 (P - P')$ and $P_{B\mu} := (\sqrt{M^2_\pi + P^2}, P)$ denotes the on-shell momentum. The matrix element of Eq.(25) between a pion state $\langle P |$ and the vacuum $| 0 \rangle$ gives

$$M^2_\pi \tilde{f}_\pi = -2m_0 i \int _q \frac{f(q^2 + P^2_B) + f(q^2 - P^2_B)}{2} \text{tr} \left[ \chi(q; P_B) \gamma^5 \frac{\lambda^\alpha}{2} \right]$$

where

$$q_B := q - \frac{P_B}{2}, \quad q_B := q + \frac{P_B}{2}$$

and the BS amplitude $\chi(q; P_B)$ and its conjugate are defined by

$$\chi(q; P_B) := e^{iP_B x} \int d^4 (x - y) e^{iq(x - y)} \langle 0 | T \psi (x) \overline{\psi} (y) | P \rangle,$$

$$\chi(q; P_B) := e^{-iP_B x} \int d^4 (y - x) e^{iq(y - x)} \langle P | T \psi (y) \overline{\psi} (x) | 0 \rangle,$$

with $X := \frac{x + y}{2}$. It should be noted that Eq.(28) is an exact relation representing Eq.(25), where exact means it holds for any $m_0$.

The Ward–Takahashi identity can be derived by the standard path integral procedure

$$-i\partial^\mu \langle 0 | T \psi(x) \overline{\psi} (y) \tilde{J}^\alpha_{5\mu} (0) | 0 \rangle = -2im_0 \langle 0 | T \psi(x) \overline{\psi} (y) I^\alpha_5 (0) | 0 \rangle$$

$$+ i \gamma_5 \frac{\lambda^\alpha}{2} \langle 0 | T \psi(x) \overline{\psi} (y) | 0 \rangle \delta^4 (x) + \langle 0 | T \psi(x) \overline{\psi} (y) | 0 \rangle i \gamma_5 \frac{\lambda^\alpha}{2} \delta^4 (y).$$
Note that in Eq.(32) we replace the $T^*$ product to the $T$ product. (This procedure is not rigorous but is justified when $\tilde{J}_{\mu}^\alpha(0)$ and $I_5^\alpha(0)$ are local currents.) From this identity, one obtains the NG solution of the BS amplitude in the limit of $m_0 \to 0$,

$$\chi(q;0) = \frac{1}{f_\pi}\{i\frac{\lambda^\alpha}{2}\gamma_5, S_F(q)\}$$

(33)

where $S_F(q)$ is the full propagator of the quark. For a small quark mass $m_0$, one expects that Eq.(33) gives an approximate solution of the BS equation. An approximation of the exact relation Eq.(28) is obtained by substituting Eq.(33) into Eq.(28)

$$M_\pi^2 f_\pi^2 \simeq -2 m_0 \langle \bar{\psi}\psi \rangle_0, \quad \langle \bar{\psi}\psi \rangle_0 := -\int_q f(-q^2)\text{tr}[S_F(q)_{m_0=0}].$$

(34)

The chiral condensate $\langle \bar{\psi}\psi \rangle_0$ is considered to be evaluated at the scale $\Lambda_{\text{UV}}$. To compare it the renormalized one in QCD, one must impose a suitable renormalization condition. This is the Gell-Mann–Oakes–Renner mass formula.

In conclusion, we have formally proved that the low-energy relations are satisfied if we redefine the axial-vector current and the decay constant $f_\pi$ according to the global chiral symmetry of the effective lagrangian. In the next section, we show how the above relations are described by the quark propagator and the BS amplitude in the framework of the rainbow–ladder approximation. Later, we will see that the difference between $f_\pi$ and $f_\pi$ is significantly large in the improved ladder model.

### 3 Analysis with CJT Action

In this section, we consider the chiral symmetry property of the SD and BS equations when they are truncated. The similar discussion has been already done in the case of QED or for a specific interaction kernel which satisfies Eq.(15).\cite{20, 21} However, it is non-trivial in the case of the general non-local interactions. Especially it is difficult to find the formula of the decay constant $\bar{f}_\pi$ because the Noether current is complicated. For the systematic discussion, we use the Cornwall, Jackiw and Tomboulis (CJT) effective action formulation.

The Cornwall, Jackiw and Tomboulis effective action formulation is one of the most powerful and useful methods to derive the SD and BS equations consistently with the chiral symmetry.\cite{22}
We study the chiral property with this formulation following Munczek. The CJT action is given by

$$\Gamma[S_F] := i\text{Tr}\ln[S_F] - i\text{Tr}[S_0^{-1}S_F] + \Gamma_{\text{loop}}[S_F].$$  \hspace{1cm} (35)$$

The last term of Eq. (35), $i\Gamma_{\text{loop}}[S_F]$ is given by the sum of all Feynman amplitudes of 2-particle irreducible vacuum diagrams with two or more loops in which every bare quark propagator $S_0$ is replaced by the full one $S_F$. We will show that the truncation in the loop expansion preserves properties of the chiral symmetry.

Using the CJT action, the SD equation and the inhomogeneous BS equation are derived by

$$\frac{\delta \Gamma[S_F]}{\delta S_F(x, y)} = 0,$$ \hspace{1cm} (36)

$$\frac{1}{i} \frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(x, y) \delta S_{Fm'n'}(y', x')} G_{C;m'm'\nu\nu'}^{(2)}(y'x'; x''y'') = \delta_{m'n'}\delta_{\nu\nu'}\delta(x'' - x)\delta(y - y'')$$ \hspace{1cm} (37)

respectively, where the repeated indices are summed and the repeated arguments are integrated. $G_{C;m'm'\nu\nu'}^{(2)}(y'x'; x''y'')$ is the two-body connected Green function defined by

$$G_{C:mmm'n'n'}^{(2)}(yx; x'y') := \langle 0|T\psi_n(y)\overline{\psi}_m(x)\psi_{m'}(x')\overline{\psi}_{n'}(y')|0 \rangle$$

$$- \langle 0|T\psi_n(y)\overline{\psi}_m(x)|0 \rangle \langle 0|T\psi_{m'}(x')\overline{\psi}_{n'}(y')|0 \rangle.$$  \hspace{1cm} (38)

By representing $G_{C;m'm'\nu\nu'}^{(2)}(y'x'; x''y'')$ in the spectral form and taking the pion pole term, we express $G_{C}^{(2)}$ in terms of the BS amplitude of the pion as

$$G_{C:mmm'n'n'}^{(2)}(yx; x'y') = \int_P e^{-i(P - P_B)X + i(P + P_B)X'}$$

$$\times i\chi_{mm}(y, x; P_B)\chi_{m'm'}(x', y'; P_B) \frac{X}{P^2 - M^2 + i\epsilon} + R_{nnm'n'}^{(2)}(yx; x'y')$$ \hspace{1cm} (39)

with

$$X := \frac{x + y}{2}, \hspace{0.5cm} X' := \frac{x' + y'}{2},$$ \hspace{1cm} (40)

$$P = (P_0, P), \hspace{0.5cm} P_B = (\sqrt{M^2 + P^2}, P_B).$$ \hspace{1cm} (41)

The regular term $R_{nnm'n'}^{(2)}(yx; x'y')$ denotes the contributions from excited states. The BS amplitude $\chi_{nm}(y, x; P_B)$ in Eq. (39) is a solution of the homogeneous BS equation, given by

$$\frac{\delta^2 \Gamma[S_F]}{\delta S_{Fmn}(x, y) \delta S_{Fm'n'}(y', x')} \chi_{n'm'}(y', x'; P_B) = 0.$$ \hspace{1cm} (42)
In order to obtain the normalized BS amplitude, we go back to Eq.(37) and determine the normalization.

The question is whether the chiral properties of the effective model are preserved when various approximations are taken into account. To answer this let us consider a local infinitesimal axial transformation of $S_F(x, y)$

$$S_F(x, y) \rightarrow S'_F(x, y) := (1 + i \gamma_5 \frac{\lambda}{2} \theta^a(x)) S_F(x, y)(1 + i \gamma_5 \frac{\lambda}{2} \theta^a(y))$$ (43)

Under this local transformation, the change of the CJT action is

$$\Delta_5 \Gamma[S_F] \equiv \frac{\delta \Gamma[S_F]}{\delta S_{F n'm'}(y', x')} \{ i \gamma_5 \frac{\lambda}{2} \theta^a, S_F \}_{n'm'}(y', x')$$ (44)

with

$$\{ i \gamma_5 \frac{\lambda}{2} \theta^a, S_F \}(y, x) := i \gamma_5 \frac{\lambda}{2} \theta^a(y) S_F(y, x) + S_F(y, x) i \gamma_5 \frac{\lambda}{2} \theta^a(x).$$ (45)

Then the following equation holds:

$$G^{(2)}_{C;m''n''mm}(x''y''; yx) \frac{\delta \left( \Delta_5 \Gamma[S_F] \right)}{\delta S_{Fnm}(y, x)}$$

$$\equiv G^{(2)}_{C;m''n''nm}(x''y''; yx) \frac{\delta^2 \Gamma[S_F]}{\delta S_{Fnm}(y, x) \delta S_{Fm''n''}(x', y') \{ i \gamma_5 \frac{\lambda}{2} \theta^a, S_F \}_{m'n'}(x', y')$$

$$+ \frac{\delta \Gamma[S_F]}{\delta S_{Fm'n'}(x', y') \{ G^{(2)}_{C;m''n''m'l}(x''y''; x'y')(i \gamma_5 \frac{\lambda}{2} \theta^a(y'))_{lm} + (i \gamma_5 \frac{\lambda}{2} \theta^a(x'))_{ml} G^{(2)}_{C;m''n''ln'}(x''y''; x'y') \}}$$

$$= i \{ i \gamma_5 \frac{\lambda}{2} \theta^a, S_F \}_{m''n''}(x'', y'').$$ (46)

Here for the last equality we use the SD equation (36) and the inhomogeneous BS equation (37). Eq.(46) is a key equation in our study of the system of the SD and BS equations.

In the following, we consider three cases in which the local chiral invariance is broken in different ways. First, we discuss the naive case where neither cut-off nor momentum dependent interactions are included. Second, we take into account the loop cut-off function. Finally we consider the case where the interaction kernel is not locally chiral invariant. In each case, we will show that the "pion" becomes massless in the chiral limit, and present the formula for the pion decay constant. We further show the PCAC relation in each case.
3.1 Naive Case

First we consider the case that the cut-off function is not introduced and the interaction does not modify the axial-vector current. In this case, truncation to an arbitrary subset of diagrams in $\Gamma_{\text{loop}}[S_F]$ does not violate the local chiral invariance. Therefore, the following results are valid in each order of the loop expansion of $\Gamma_{\text{loop}}[S_F]$. The first term $i\text{TrLn}[S_F]$ in Eq.(35) is also invariant because

$$\Delta_5 i\text{TrLn}[S_F] \equiv i\text{Tr}[S_F^{-1}\{i\gamma_5 \lambda^\alpha/2, S_F\}] \equiv 2\text{itr}[i\gamma_5 \lambda^\alpha/2] \int d^4x \theta^\alpha(x) \equiv 0. \quad (47)$$

Then only the second term $-i\text{Tr}[S_0^{-1}S_F]$ in Eq.(35) contributes in the left hand side (LHS) of Eq.(46), and gives in momentum space

$$\int_q G^{(2)}_{\xi,m''m'n'm'n}(p, q; P) \left(i\gamma_5 \lambda^\alpha/2(2m_0 + P)\right)_{mn} = i\left(i\gamma_5 \lambda^\alpha/2S_F(p - P/2)\right)_{m''n''} + i\left(S_F(p + P/2)i\gamma_5 \lambda^\alpha/2\right)_{m''n''}. \quad (48)$$

Here the Fourier transformation of $G^{(2)}_{\xi,m''m'n'm'n}(x'y'; yx)$ is defined by

$$G^{(2)}_{\xi,m''m'n'm'n}(x'y'; yx) = \int_{pqP} e^{-i(p(x'-y') + q(y-x) + P(x'-x))} G^{(2)}_{\xi,m''m'n'm'n}(p, q; P). \quad (49)$$

Using Eq.(39), Eq.(48) becomes

$$\int_q \left[\frac{i\chi_{m''m'}(p; P_B)\chi_{mn}(q; P_B)}{P^2 - M_x^2 + i\epsilon} + R_{m''m'n'n}(p, q; P)\right] \left(i\gamma_5 \lambda^\alpha/2(2m_0 + P)\right)_{mn} = i\left(i\gamma_5 \lambda^\alpha/2S_F(p - P/2)\right)_{m''n''} + i\left(S_F(p + P/2)i\gamma_5 \lambda^\alpha/2\right)_{m''n''}. \quad (50)$$

with

$$\chi_{mn}(x, y; P_B) = e^{-iP_B\lambda^\alpha/2} \int_q e^{-iq(x-y)}\chi_{mn}(q; P_B); \quad (51)$$

$$\overline{\chi}_{mn}(y, x; P_B) = e^{iP_B\lambda^\alpha/2} \int_q e^{-iq(y-x)}\overline{\chi}_{mn}(q; P_B). \quad (52)$$

Chiral Limit

In the chiral limit $m_0 \to 0$, the CJT action is chiral invariant. We confirm that the "pion" becomes massless in this limit because if $M_x \neq 0$ LHS of Eq.(46) diverges in the limit $P^2 \to M_x^2$, while RHS converges. In this case LHS of Eq.(50) becomes

$$\text{LHS of Eq.}(50) = \int_q \frac{i}{P^2} \text{tr} \left[\overline{\chi}(q; P_B)i\gamma_5 \lambda^\alpha/2(P)\chi_{m''n''}(p; P_B)\right] + \int_q R_{m''m'n'n}(p, q; P)(i\gamma_5 \lambda^\alpha/2(P)_{mn}. \quad (53)$$
If we take the soft limit \( P \to 0 \) after taking the on-shell limit \( P \to P_B \) (i.e. \( P_0 \to \sqrt{P^2} \)), the second term of Eq.(53) vanishes and the first term becomes

\[
\text{1st term of Eq.(53)} \rightarrow i f_\pi \chi_{m'\nu'}(p; 0), \quad (54)
\]

with

\[
f_\pi = \lim_{P \to P_B} \frac{1}{P^2} \int_q \text{tr} [\chi(q; P_B) i \gamma_5 \frac{\lambda_\alpha}{2} P]. \quad (55)
\]

On the other hand RHS of Eq.(50) becomes

\[
\text{RHS of Eq.(50)} \rightarrow i \{ i \gamma_5 \frac{\lambda_\alpha}{2}, S_F(p) \}_{m'\nu'}. \quad (56)
\]

in the same limit. Then we obtain the equation

\[
\chi_{mn}(q; 0) = \frac{1}{f_\pi} \{ i \gamma_5 \frac{\lambda_\alpha}{2}, S_F(q) \}_{mn}. \quad (57)
\]

This is the same equation as Eq.(33) which was derived from the axial-vector Ward–Takahashi identity. It should be noted that Eq.(57) is valid in the truncated SD and BS equations in which \( \Gamma_{\text{loop}}[S_F] \) is expanded to a finite number of loops.

**Finite Quark Mass**

When \( m_0 > 0 \), the first term in the brackets of Eq.(50) will diverge in the on-shell limit \( P \to P_B \). Then the equation

\[
\int_q \text{tr} [\chi(q; P_B) i \gamma_5 \frac{\lambda_\alpha}{2} (2m_0 + P_B)] = 0 \quad (58)
\]

must be satisfied and we obtain

\[
M_\pi^2 f_\pi = -2m_0 i \int_q \text{tr} [\chi(q; P_B) \gamma_5 \frac{\lambda_\alpha}{2}] \quad (59)
\]

where \( f_\pi \) is given by Eq.(55). This equation is again same as Eq.(28). Using the approximation Eq.(57), we obtain the GMOR relation

\[
M_\pi^2 f_\pi^2 \simeq -2m_0 \langle \bar{\psi} \psi \rangle_0, \quad \langle \bar{\psi} \psi \rangle_0 := -\int_q \text{tr} [S_F(q) m_0=0]. \quad (60)
\]

Note again that any truncation of \( \Gamma_{\text{loop}}[S_F] \) is guaranteed to satisfy all the above equations.
3.2 The Case of the Cut-off Regularization

In order to regularize loop integrals, we introduce the cut-off function in the kinetic term of the lagrangian as in Eq.(3). As far as this regularization is taken, the interaction term of the effective action is not modified and therefore the chiral invariance of $\Gamma_{\text{loop}}[S_F]$ is not violated. When the cut-off function is introduced, the same procedure can be applied with the replacement

$$i\gamma_5 \lambda^2 \frac{1}{2} (2m_0 + P) \quad \rightarrow \quad i\gamma_5 \lambda^2 \left( 2m_0 \frac{f(-q_-^2) + f(-q_+^2)}{2} - f(-q_-^2)q_- - f(-q_+^2)q_+ \right),$$

$$q_- := q - \frac{P}{2}, \quad q_+ := q + \frac{P}{2}.$$

As a result the decay constant $\tilde{f}_\pi$ is given by

$$\tilde{f}_\pi = \lim_{P \to P_B} \frac{1}{P^2} \int_q \text{tr} \left[ X(q; P_B) i\gamma_5 \lambda^2 \left\{ \frac{f(-q_-^2) + f(-q_+^2)}{2} P + (f(-q_-^2) - f(-q_+^2))q \right\} \right].$$

This $\tilde{f}_\pi$ coincides with the definition (20). Therefore the exact relation Eq.(59) is modified to

$$M^2_\pi \tilde{f}_\pi = -2m_0 i \int_q \frac{f(-q_-^2) + f(-q_+^2)}{2} \text{tr} \left[ X(q; P_B) \gamma_5 \lambda^2 \right]$$

and the definition of quark condensate is modified to

$$\langle \bar{\psi} \psi \rangle_0 = - \int_q f(-q^2) \text{tr} [S_F(q)_{m_0=0}].$$

Under these modifications, Eqs.(57) and (60) hold.

3.3 The Case of the Non-local Interaction

Here we consider the case that the interaction modifies the axial-vector current. To simplify the argument, we omit the cut-off function $f(\zeta)$ in this subsection. We employ the two-loop approximation of $\Gamma_{\text{loop}}[S_F]$ such that

$$\Gamma_{\text{loop}}[S_F] = -\frac{1}{2} \int d^4x K^{m_1 m_2, n_1 n_2} (i\partial_{x_1}, i\partial_{x_2}; i\partial_{y_1}, i\partial_{y_2})$$

$$\times \left[ S_{Fm_2n_1}(x_2, y_1) S_{Fn_2n_1}(y_2, y_1) - S_{Fc_2n_1}(x_2, y_1) S_{Fc_2n_1}(y_2, x_1) \right]_*. $$

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where the symbol $\ast$ means to take $x_1, x_2, y_1, y_2 \to x$ after all the derivatives are operated. Note that this approximation corresponds to the rainbow approximation in the SD equation and the ladder approximation in the BS equation. Then Eq. (46) reduces in momentum space to

$$
\int_q G_{C,m'n'm}^{(2)}(p, q; P) \left( i\gamma_5 \frac{\lambda^\alpha}{2} (2m_0 + \not{P}) + E^\alpha(q; P) \right)_{mn} \\
= i \left( i\gamma_5 \frac{\lambda^\alpha}{2} S_F(p - \frac{P}{2}) \right)_{m'n''} + i \left( S_F(p + \frac{P}{2}) i\gamma_5 \frac{\lambda^\alpha}{2} \right)_{m'n''},
$$

(68)

$E^\alpha_{mn}(q; P) := \int_k \left[ \left\{ K_{in,n'm'}(-q - \frac{P}{2}, q + \frac{P}{2}; -k, k) \right. \right.$

$- K_{in,n'm'}(-q + \frac{P}{2}, q - \frac{P}{2}; -k, k) \left. \right\} (i\gamma_5 \frac{\lambda^\alpha}{2})_{mi} S_F m'n'(k) \\
+ \left\{ K_{n'n',mn}(-k + P, k; -q - \frac{P}{2}, q - \frac{P}{2}) \right. \\
- K_{n'n',mn}(-k, k + P; -q - \frac{P}{2}, q - \frac{P}{2}) \left. \right\} (i\gamma_5 \frac{\lambda^\alpha}{2} S_F)_{m'n'}(k) \\
+ \left\{ K_{n'n',mn'}(-k, q - \frac{P}{2}; -q - \frac{P}{2}, k) \right. \\
- K_{n'n',mn'}(-k, q - \frac{P}{2}; -q + \frac{P}{2}, k) \left. \right\} (i\gamma_5 \frac{\lambda^\alpha}{2} S_F)_{m'n'}(k) \\
+ \left\{ K_{n'n',mn'}(-q - \frac{P}{2}, q - \frac{P}{2}; -q - \frac{P}{2}, k) \right. \\
- K_{n'n',mn'}(-q - \frac{P}{2}, q - \frac{P}{2}; -q + \frac{P}{2}, k) \left. \right\} (i\gamma_5 \frac{\lambda^\alpha}{2} S_F)_{m'n'}(k)$

(69)

Substituting Eq. (39), Eq. (68) becomes

$$
\int_q \left[ i \frac{\chi_{m'n'}(p; P_B)\chi_{nm}(q; P_B)}{P^2 - M^2 + i\epsilon} + R_{m'n'nm}(p, q; P) \right] \left( i\gamma_5 \frac{\lambda^\alpha}{2} (2m_0 + \not{P}) + E^\alpha(q; P) \right)_{mn} \\
= i \left( i\gamma_5 \frac{\lambda^\alpha}{2} S_F(p - \frac{P}{2}) \right)_{m'n''} + i \left( S_F(p + \frac{P}{2}) i\gamma_5 \frac{\lambda^\alpha}{2} \right)_{m'n''}.
$$

(70)

It is easy to see that $E^\alpha(q; P)$ becomes zero in the limit $P \to 0$. This is expected because LHS of Eq. (68) or Eq. (70) should be finite in the chiral limit as RHS is. It is also easy to check that $E^\alpha(q; P)$ vanishes if $K_{m'n',mn'}(p, p'; q, q')$ satisfies Eq. (13).

**Chiral Limit**

In the chiral limit, the global chiral symmetry breaking implies the existence of the NG boson again. Then we obtain

$$
\text{LHS of Eq. (70)} = \int_q \frac{i}{P^2} \text{tr} [\chi(q; P_B)(i\gamma_5 \frac{\lambda^\alpha}{2} \not{P} + E^\alpha(q; P))] \chi_{m'n''}(p; P_B)
$$

15
\[ + \int_q R_{m'n'mn}(p, q; P)(i\gamma_5 \frac{\lambda^\alpha}{2} P)_{mn}. \] (71)

Taking the soft limit \( P \to 0 \) after the on-shell limit \( P \to P_B \) (i.e. \( P_0 \to \sqrt{P^2} \)) the second term of Eq.(71) vanishes and the first term becomes

\[ \text{1st term of Eq.(71)} \to i \tilde{f}_\pi \chi_{m'n'n'}(p; 0) \] (72)

where we use a (new) definition of \( \tilde{f}_\pi \),

\[ \tilde{f}_\pi := \lim_{P \to P_B} \frac{1}{P^2} \int_q \text{tr}[\chi(q; P_B)(i\gamma_5 \frac{\lambda^\alpha}{2} P + E^\alpha(q; P))]. \] (73)

On the other hand RHS of Eq.(70) becomes

\[ \text{RHS of Eq.(70)} \to i \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(p)\}_{m'n'n'} \] (74)

in the same limit. Thus we obtain the relation

\[ \chi_{mn}(q; 0) = \frac{1}{\tilde{f}_\pi} \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(q)\}_{mn}. \] (75)

This is the same equation as Eq.(33) which was derived from the axial-vector Ward–Takahashi identity. Therefore we conclude that the definition of \( \tilde{f}_\pi \) in Eq.(73) is equivalent to the definition Eq.(26).

**Finite Quark Mass**

When \( m_0 > 0 \), the first term in the brackets of Eq.(70) diverges in the on-shell limit \( P \to P_B \). Then the equation

\[ \int_q \text{tr}[\chi(q; P_B)(i\gamma_5 \frac{\lambda^\alpha}{2}(2m_0 + P_B) + E^\alpha(q; P_B))] = 0 \] (76)

must hold. From Eq.(73) we obtain

\[ M^2_\pi \tilde{f}_\pi = -2m_0i \int_q \text{tr}[\chi(q; P_B)\gamma_5 \frac{\lambda^\alpha}{2}]. \] (77)

This equation again coincides with Eq.(28) when \( f(\zeta) \equiv 1 \) and implies that our choice of \( \tilde{f}_\pi \) in Eq.(73) is consistent with Eq.(26). The GMOR relation is written as

\[ M^2_\pi \tilde{f}_\pi^2 \simeq -2m_0 \langle \bar{\psi}\psi \rangle_0, \quad \langle \bar{\psi}\psi \rangle_0 := -\int_q \text{tr}[S_F(q)]_{m_0=0}. \] (78)
Thus we have proved that the truncation of $\Gamma_{\text{loop}}[S_F]$ preserves the low energy property of the effective model if one uses the appropriate formula of the decay constant given by Eq.(73).

Up to now, we neglect the cut-off function. Introducing both the effects of the cut-off function and the non-local interaction, the formula for the decay constant in the approximation Eq.(77) is given by

$$\tilde{f}_\pi = \lim_{P \to P_B} \frac{1}{P^2} \int_q \text{tr} \left[ \overline{\chi}(q; P_B) \left\{ i\gamma_5 \frac{\lambda^\alpha}{2} \left( \frac{f(-q_-^2) + f(-q_+^2)}{2} P + (f(-q_-^2) - f(-q_+^2))q \right) + E^\alpha(q; P) \right\} \right].$$

(79)

It should be noted here that we have defined $\tilde{f}_\pi$ in Eq.(73) so as to reproduce the low-energy relations Eqs.(75) and (77). The definition Eq.(73) is obtained from Eq.(26) directly by taking the following approximations

$$\langle 0 | T \psi \bar{\psi} | P \rangle \mapsto \langle 0 | T \psi \bar{\psi} | P \rangle_{\text{ladder}},$$

(80)

$$\langle 0 | T \psi \bar{\psi} \psi \bar{\psi} | P \rangle \mapsto \sum \langle 0 | T \psi \bar{\psi} | 0 \rangle_{\text{rainbow}} \langle 0 | T \psi \bar{\psi} | P \rangle_{\text{ladder}}.$$  

(81)

4 General Formula of the Decay Constant

So far we have treated the lowest order (rainbow-ladder) approximation in the CJT action. When we proceed to the higher order terms, the two-loop formula Eq.(79) should be modified to an appropriate form. Then we can prove that the truncation of $\Gamma_{\text{loop}}[S_F]$ preserves the low energy property of the effective model. We note that the consistency is guaranteed in the loop expansion of the CJT action formulation. Further approximations inconsistent with the loop expansion will violate the low energy properties. An example is to take the leading terms of the Chebychev polynomial expansion of the BS amplitude, although the numerical result shows that the violation of the PCAC relation is generally small.[12, 13]

In order to derive a general formula of the decay constant, we consider the transformation property of the CJT action. Under the infinitesimal local axial transformation, the change of the classical action can be written as

$$\Delta_5 S[\psi, \bar{\psi}] = \int d^4 x \theta^\alpha(x) \left( \partial^\mu \tilde{J}_5^\alpha \bar{\psi}(x) - M^\alpha[\psi, \bar{\psi}](x) \right).$$

(82)
Here $M^\alpha[\psi, \bar{\psi}](x)$ comes from the globally variant terms, such as the quark mass term and $\tilde{J}^\alpha_{5\mu}[\psi, \bar{\psi}](x)$ is an effective axial-vector current.

The change of the CJT action corresponding to the classical action must be written as

$$\Delta_5 \Gamma[S_F] = \int d^4x \theta^\alpha(x) \left( \partial^\mu \tilde{J}^\alpha_{5\mu}[S_F](x) - M^\alpha[S_F](x) \right).$$

Again $M^\alpha[S_F](x)$ comes from the globally variant terms and $\tilde{J}^\alpha_{5\mu}[S_F](x)$ is a proxy of the effective axial-vector current in the CJT action.

When the effective axial-vector current $\tilde{J}^\alpha_{5\mu}[\psi, \bar{\psi}](x)$ has a non-local interaction term, the exact proxy $\tilde{J}^\alpha_{5\mu}[S_F](x)$ is an infinite sum of the Feynman amplitudes which come from the expansion of $\Gamma_{\text{loop}}[S_F]$. If one truncates the expansion of $\Gamma_{\text{loop}}[S_F]$, the approximated proxy $\tilde{J}^\alpha_{5\mu}[S_F](x)$ is built of a finite sum of the Feynman amplitudes.

For example, in the lowest loop approximation Eq. (84) and with the cut-off function $f(\zeta)$ we obtain

$$\partial^\mu \tilde{J}^\alpha_{5\mu}[S_F](z) = \int d^4x \int_p e^{-ip(z-x)} f(-p^2) \left\{ \text{tr}[\psi_i \gamma_5 \dfrac{\lambda^\alpha}{2} S_F(z,x)] - \text{tr}[\psi_i S_F(x,z) i \gamma_5 \dfrac{\lambda^\alpha}{2}] \right\}$$

$$- \int e^{ikz} \int d^4x K^{n_1 n_2 n_1 n_2} \left( i \partial_{x_1} i \partial_{x_2} ; i \partial_{y_1} i \partial_{y_2} \right) \left[ S_{Fm_1 n_1 n_2 m_1 n_2} (x_2, y_1) (S_F i \gamma_5 \dfrac{\lambda^\alpha}{2})_{n_2 n_1} (y_2, y_1) e^{-iky_1} + (i \gamma_5 \dfrac{\lambda^\alpha}{2} S_F)_{n_2 n_1} (y_2, y_1) e^{-iky_2} \right] \right|,$$

$$M^\alpha[S_F](z) = -m_0 \int d^4x \int_p e^{-ip(z-x)} f(-p^2) \left\{ \text{tr}[i \gamma_5 \dfrac{\lambda^\alpha}{2} S_F(z,x)] + \text{tr}[S_F(x,z) i \gamma_5 \dfrac{\lambda^\alpha}{2}] \right\}.$$  

To calculate the decay constant, we define the formula

$$\tilde{f}_\pi = \lim_{P \rightarrow P_B} \dfrac{i P^\mu}{P^2} \chi_{nm}(x,y; P_B) \dfrac{\delta \tilde{J}^\alpha_{5\mu}[S_F](0)}{\delta S_{Fnm}(x,y)}$$

$$= \lim_{P \rightarrow P_B} \dfrac{1}{P^2} \chi_{nm}(x,y; P_B) \dfrac{\delta \partial^\mu \tilde{J}^\alpha_{5\mu}[S_F](0)}{\delta S_{Fnm}(x,y)}.$$  

(86)

(87)

corresponding to the exact definition (28), or

$$\tilde{f}_\pi := \lim_{P \rightarrow P_B} \dfrac{i P^\mu}{P^2} \langle P | \tilde{J}^\alpha_{5\mu}[\psi, \bar{\psi}](0) | 0 \rangle.$$  

(88)

Therefore systematically approximated $\tilde{f}_\pi$ can be obtained for any truncation of $\Gamma_{\text{loop}}[S_F]$.

The previous formula Eq. (73) coincides with Eq. (87) if the rainbow-ladder approximation is employed.
4.1 Pagels–Stokar Formula

Pagels and Stokar proposed a useful approximation for the decay constant $f_\pi$ in the chiral limit in terms of the constituent quark mass function $B(q^2)$ of the SD equation.\textsuperscript{25, 28} The Pagels–Stokar formula is

$$f_\pi^2 = \frac{N_C}{8\pi^2} \int_0^\infty dq_E^2 q_E^2 \frac{B(-q_E^2) \left[ 2B(-q_E^2) + q_E^2 B'(-q_E^2) \right]}{(q_E^2 + B^2(-q_E^2))^2} \quad (89)$$

where $B(-q_E^2)$ is defined by the ansatz

$$S_F(q) = \frac{i}{\bar{q} - B(q^2)} \quad (90)$$

instead of the general form of the SD solution

$$S_F(q) = \frac{i}{A(q^2)\bar{q} - B(q^2)} \quad (91)$$

$B'(x)$ denotes the derivative of $B(x)$ and $q_E$ denotes the Euclidean momentum i.e. $q_E^2 = -q^2$.

This formula Eq.(89) can be derived from an approximated BS amplitude with the Ward–Takahashi identity for the axial-vector current. Therefore when the axial-vector current is modified, it should also be modified. In this section, we propose a new formula similar to the Pagels–Stokar formula in the effective model.

The BS amplitude is given by

$$\chi_{nm}(q; 0) = \frac{1}{f_\pi} \{i\gamma_5 \frac{\lambda^\alpha}{2}, S_F(q)\}_{nm} \quad (92)$$

from the Ward–Takahashi identity for the axial-vector current in the chiral limit. But this solution is not sufficient to calculate $\tilde{f}_\pi$, because $\tilde{f}_\pi$ is related to the derivative of the BS amplitude with respect to the total momentum $P_B := (\sqrt{P^2}, P)$. The amputated BS amplitude(or BS vertex) $\hat{\chi}_{nm}(q; P_B)$ is defined by

$$\hat{\chi}_{nm}(q; P_B) := S^{-1}_{Fnm'}(q + \frac{P_B}{2})\chi_{n'm'}(q; P_B)S^{-1}_{Fm'm}(q - \frac{P_B}{2}), \quad (93)$$

and in the chiral limit, we obtain from Eq.(92)

$$\hat{\chi}_{nm}(q; 0) = \frac{1}{f_\pi} \{i\gamma_5 \frac{\lambda^\alpha}{2}, S^{-1}_F(q)\}_{nm}. \quad (94)$$
Consider an approximation for the BS amplitude

\[
\chi_{nm}(q; P_B) = S_{F_{nn'}}(q + \frac{P_B}{2})\tilde{\chi}_{n'm'}(q; 0)S_{F_{m'm}}(q - \frac{P_B}{2})
\]

(95)

\[
= \frac{1}{f_\pi}S_{F_{nn'}}(q + \frac{P_B}{2})\{i\gamma_5 \frac{\lambda^\alpha}{2}, S_{F}^{-1}(q)\}n'm'S_{F_{m'm}}(q - \frac{P_B}{2}).
\]

(96)

In this paper we call this the Pagels–Stokar ansatz. This is a very useful formula because it gives the approximated BS amplitude for the "pion" in terms only of the solution of the SD equation. As this specifies the dependence on the total momentum \(P_B\), one can estimate the decay constant without solving the BS equation, for instance using our formula Eq.(79) or Eq.(87). The result is given by

\[
(f_{PS}^\pi)^2 = \lim_{P \to P_B} \frac{1}{P^2} \int q \text{tr} \left[ S_{F}(q - \frac{P_B}{2})\{i\gamma_5 \frac{\lambda^\alpha}{2}, S_{F}^{-1}(q)\}S_{F}(q + \frac{P_B}{2}) \right.
\]

\[
\times \left\{ i\gamma_5 \frac{\lambda^\alpha}{2} \left( \frac{f(-q_+^2) + f(-q_-^2)}{2} P + (f(-q_+^2) - f(-q_-^2))q \right) + E^\alpha(q; P) \right\}
\]

(97)

where no summation with respect to \(\alpha\) is taken. This formula Eq.(97) is useful to estimate \(f_{\pi}\) while the validity of the Pagels–Stokar ansatz Eq.(96) is not established. Eq.(97) reduces to the original Pagels–Stokar formula (89) after the Wick rotation when (i) one neglects the effects of the cut-off function \(f(\zeta)\) and the local 4-quark interaction to the axial-vector current, and (ii) if one assumes the ansatz (90) of the SD solution.

5 Numerical Results for a Concrete Example

In this section we show some numerical results as an example of our approach. We here employ the effective model introduced by Aoki et al.\cite{6}. The interaction is given by the one gluon exchange with the running coupling constant which depends on the special set of momenta according to the Higashijima–Miransky approximation. In the infrared region the running coupling constant is assumed to be constant. We do not elucidate the detail of this model here. (See Ref.\cite{29}.) Our choice of the parameters is \(\Lambda_{QCD} = 500\text{MeV}, t_{IF} = -0.5, t_0 = -3.0\) and the ultraviolet cut-off parameter \(\Lambda_{UV} = 2.0\text{GeV}\). We have solved the SD equation and the BS equation for the pion in the Euclidean momentum in the chiral limit. As is pointed out in Ref.\cite{4}, the consistency of the SD and BS equations in the CJT action formulation guarantees
that the pion becomes massless in the chiral limit. We have confirmed this in our numerical calculation.

In the rainbow–ladder approximation, the exact value of the pion decay constant in the chiral limit is given by Eq. (79) with $P_B^2 = 0$, which gives

$$f_\pi = 72 \text{MeV}.\quad (98)$$

This can be first compared with the naive value $f_\pi$ in which $E^\alpha(q,p)$ term coming from the non-local interaction is omitted from Eq. (79) (See Eq. (57)). The result* is

$$f_\pi = 122 \text{MeV}.\quad (99)$$

This is more than 70% deviated from the true value. Thus we see that the correction term $E^\alpha(q,p)$ is essential. In Ref. [6], the authors normalize the BS amplitude by the true decay constant $\tilde{f}_\pi$ as in Eq. (92). To calculate the decay constant, however, the naive definition Eq. (27) is employed. As a result, their definition of the decay constant gives

$$\sqrt{f_\pi \tilde{f}_\pi} = 94 \text{MeV}.\quad (100)$$

Next we examine the Pagel–Stokar formula. The original Pagels–Stokar formula Eq. (89) gives

$$f_{\pi}^{PS} = 96 \text{MeV} \quad (101)$$

which is close to Eq. (100) as is pointed out* in Ref. [6]. Our new formula Eq. (97) gives

$$f_{\pi}^{PS} = 73 \text{MeV} \quad (102)$$

which almost coincides with Eq. (98). Thus we find that the new formula gives rather good prediction, although only the SD equation is to be solved in order to calculate Eq. (97). Thus this is the most economical way to estimate the pion decay constant.

*2 The cut-off regularization is applied for the naive value Eq. (99).

*3 The value of $f_{\pi}^{PS}$ is different from Ref. [6], because of the different choice of the parameters $t_{IF}, \Lambda_{QCD}$ and $\Lambda_{UV}$. 
6 Conclusion

In this paper, we have discussed dynamical chiral symmetry breaking in effective chiral quark models of QCD. Because the effective models may contain loop momentum cut-off as well as non-local interactions, the conserved axial-vector current is modified accordingly. Then the low-energy constants and relations, such as the pion decay constant, the Gell-Mann–Oakes–Renner relation, become very complicated. Nevertheless, using the CJT action formulation, we have proved that the combination of the SD and BS equations preserve the chiral symmetry and that the BS equation bears the NG pion solution. It is also shown that the pion decay constant \( f_\pi \) must be defined according to the modified axial-vector current and that such \( f_\pi \) satisfies the PCAC relation.

We have derived a general formula of the pion decay constant in terms of the quark full propagator and the pion BS amplitude. The formula is consistent with the loop expansion of the CJT effective action. A numerical analysis given in Sec.5 shows that the consistency of the SD and BS equations with the chiral symmetry is essential for the low-energy relations. We have proposed a Pagels–Stokar type formula which gives the pion decay constant in terms only of the mass function \( B(q^2) \) of the SD equation. We have found that this formula gives a very good approximation and therefore saves computation time.

Our intention is to apply the general formulation given in the present paper to the study of the light \( q\bar{q} \) mesons from the chiral symmetry viewpoint. As we employ realistic effective models, beyond the NJL model, the chiral symmetry is not trivially conserved. Thus we need a consistent approach of the system of the SD and BS equations. We believe that the present formulation gives a consistent view of the dynamical symmetry breaking in the effective model analyses of low-energy hadrons.

In a separate paper[29], we study the pion in the improved ladder model with finite quark mass. The introduction of finite quark mass breaks the chiral symmetry explicitly. It is important and interesting to investigate how far the chiral symmetry can be applied. For instance, the Gell-Mann–Oakes–Renner relation is proved from the chiral symmetry in the \( m_0 \rightarrow 0 \) limit. Its deviation at finite \( m_0 \) should be studied. Such a study will give an indication to the applicability of the chiral perturbation theory.[30]
The axial $U(1)$ symmetry is known to be broken by the anomaly, which may be caused by the instanton configuration of QCD. In effective models for quarks, the $U(1)_{A}$ breaking may be represented by the instanton mediated interaction. In our study of a realistic model in the flavor $SU(3)_{C}$, we employ the Kobayashi–Maskawa–’t Hooft interaction, which consists of a six-quark vertex. The present general formulation can be easily extended to such six-quark interaction with an appropriate momentum dependence. Our formula for the pion decay constant is applicable while the proxy of the axial-vector current requires additional terms from the new interaction.

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