Nonclassical properties of coherent states and excited coherent states for continuous spectra

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Abstract

Based on the definition of coherent states for continuous spectra and analogous to photon-added coherent states for discrete spectra, we introduce the excited coherent states for continuous spectra. It is shown that the main axioms of Gazeau–Klauder coherent states will be satisfied, properly. Nonclassical properties and quantum statistics of coherent states, as well as the introduced excited coherent states, are discussed. In particular, through the study of quadrature squeezing and amplitude-squared squeezing, it will be observed that both classes of the above states can be classified in the intelligent states category.

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1. Introduction

Coherent states play an important role in various fields of physics, specially in quantum technologies and quantum optics [1–4]. Gazeau and Klauder introduced coherent states for systems with either discrete, continuous or both discrete and continuous spectra [5]. Later, ladder operator structure of discrete type of the latter states has been established in [6]. Recently, a theoretical scheme for the generation of Gazeau–Klauder coherent states via intensity-dependent degenerate Raman interaction and also Gazeau–Klauder squeezed states associated with solvable quantum systems with discrete spectra has been introduced and discussed by one of us [7, 8].

Recently, Ben Geloun and Klauder introduced two kinds of ladder operators for quantum systems with continuous spectra [9]. There, two distinct classes of coherent states were introduced (but only one of them, which has been constructed based on the eigenstate definition of an annihilation operator, is appropriate for the goal of this work). Then, the
authors established the axioms of the Gazeau–Klauder coherent states on states introduced by them. Moreover, the main motivation for generalizing the coherent states lies in finding the nonclassical properties (states with no classical analog). This is due to the fact that nonclassical states have received considerable attention in the fields of quantum optics, quantum cryptography, quantum communication, etc [1, 2, 10–12]. Altogether, neither of the authors who previously dealt with coherent states for continuous spectra [5, 9] pay attention to this important subject, which is of enough interest for quantum optics specialists. It is worth noting that the notions of ‘coherent states’ and ‘nonclassicality of states’ are not restricted to the radiation field. As some cases, we may refer to fermion coherent states [1], coherent spin states [13], coherent states for potentials other than harmonic oscillator [14, 15] and nonlinear coherent states of the center of mass motion of a trapped ion [16]. Atomic spin-squeezed states and the associated spin-squeezing parameter [17] are the well-known examples that squeezing as a nonclassicality sign does not necessarily relate to the radiation field. Along with these, we have generalized some of the nonclassicality criteria to the coherent states for continuous spectra.

In this paper, we will briefly review the definition of one kind of ladder operators and the associated coherent states for continuous spectra proposed by Ben Geloun and Klauder [9], which satisfies the eigenvalue problem, including a few additional remarkable points in section 2. Then, analogous to ‘photon-added coherent states’ were introduced by Agarwal and Tara [18], the ‘excited coherent states’ corresponding to continuous spectra will be introduced in section 3. Also, the axioms of Gazeau–Klauder coherent states will be examined for the introduced states. Next, some nonclassicality features of the coherent and excited coherent states for continuous spectra are investigated, respectively, in sections 4 and 5. We conclude the paper in section 6.

2. Annihilation and creation operators and the associated coherent states for continuous spectra

Let us suppose that the dynamics of a system is described by the Hamiltonian $H$, which admits a non-degenerate continuous spectrum. The corresponding eigenstates are denoted by $|E\rangle$, where they satisfy the following eigenvalue equation and orthogonality relation, respectively:

$$H|E\rangle = \omega E|E\rangle \quad \text{and} \quad \langle E|E'\rangle = \delta(E - E'),$$

in which $E > 0$ and $\hbar = 1$ is assumed. Along the study of coherent states for such systems, Ben Geloun and Klauder introduced the following annihilation operator on the continuous basis [9]:

$$a_\epsilon = \int_0^\infty C(E, \epsilon)|E - \epsilon\rangle\langle E|dE, \quad C(E, \epsilon) = e^{\alpha E - \frac{1}{2} \epsilon^2},$$

where $\epsilon > 0$ and $\alpha$ are the real parameters. The operator $a_\epsilon$ which decreases $E$ by an amount of energy $\epsilon$ may be called an ‘energy-depleting operator’. They then introduced the coherent states $|s, \gamma\rangle_\epsilon$ satisfying the eigenvalue problem

$$a_\epsilon|s, \gamma\rangle_\epsilon = (s e^{-i\gamma})|s, \gamma\rangle_\epsilon,$$

where $s \in [0, \infty)$ and $\gamma \in (-\infty, \infty)$. The following expression for the associated coherent states on the continuous basis has been obtained:

$$|s, \gamma\rangle_\epsilon = N_\epsilon(s) \int_0^\infty \frac{e^{sE} e^{-i\gamma E} |E\rangle}{E^2} dE,$$
where
\[ N(\epsilon) = \left[ \int_0^\infty \frac{s^2 e^{\alpha s^2}}{\cosh^2 s} \, ds \right]^{-\frac{1}{2}} \] (5)
is a normalization factor. States (4) are the same as those determined by Gazeau and Klauder for a given function \( f(E) = e^{\frac{\alpha \epsilon}{2} E^2} \) of [5]. The creation operator or ‘energy-supplying operator’ can be simply obtained by Hermitian conjugation of (2) with the result
\[ a^\dagger(\epsilon) = \int_0^\infty C(E, \epsilon) |E\rangle \langle E - \epsilon| \, dE, \] (6)
where \( C(E, \epsilon) \) is given by (2). Both annihilation and creation operators respectively introduced in (2) and (6) act to either decrease or increase the energy scale by a uniform amount \( \epsilon \) (i.e. an amount independent of the energy \( E \) itself). Although these operators are \( \epsilon \)-dependent, this effects the ‘efficiency’ with which the subtracted or added energy is received by this particular ‘energy-transforming device’.

The ladder operators \( a_\epsilon \) and \( a_\epsilon^\dagger \) obey the following commutation relation:
\[ [a_\epsilon, a_\epsilon^\dagger] = \int_0^\infty (C^2(E + \epsilon, \epsilon) - C^2(E, \epsilon)) |E\rangle \langle E| \, dE \]
\[ = 2 \sinh (\alpha \epsilon^2) \int_0^\infty e^{2\alpha \epsilon |E|} |E\rangle \langle E| \, dE. \] (7)
While the rising and lowering operators on the continuous basis are defined by Ben Geloun and Klauder in [9], we can now introduce a number-like operator on the same basis with the help of (2) and (6) as follows:
\[ N(\epsilon) = a_\epsilon^\dagger a_\epsilon = \int_0^\infty e^{2\alpha \epsilon (E - \frac{\epsilon}{2})} |E\rangle \langle E| \, dE. \] (8)
The physical meaning of this operator is not the same as that of the number operator in the discrete case, because when the usual annihilation and creation operators act on discrete states, one speaks of decreasing or increasing ‘particles’, for example, from the fact that \( |n\rangle \rightarrow |n - 1\rangle \) and the converse, but in the continuous case the ‘energy’ is decreased or increased by a uniform amount.

The following eigenvalue equation holds:
\[ N(\epsilon) |E\rangle = e^{-\alpha \epsilon^2} e^{2\alpha \epsilon E} |E\rangle. \] (9)
The number-like operator satisfies the following commutators with \( a_\epsilon \) and \( a_\epsilon^\dagger \), respectively:
\[ [N(\epsilon), a_\epsilon] = \int_0^\infty C(E, \epsilon)(C^2(E - \epsilon, \epsilon) - C^2(E, \epsilon)) |E - \epsilon\rangle \langle E| \, dE \]
\[ = -\int_0^\infty e^{2\alpha \epsilon E - \frac{\alpha \epsilon^2}{2}} (e^{2\alpha \epsilon^2} - 1) |E - \epsilon\rangle \langle E| \, dE, \] (10)
\[ [N(\epsilon), a_\epsilon^\dagger] = \int_0^\infty C(E, \epsilon)(C^2(E, \epsilon) - C^2(E - \epsilon, \epsilon)) |E - \epsilon\rangle \langle E| \, dE \]
\[ = \int_0^\infty e^{2\alpha \epsilon E - \frac{\alpha \epsilon^2}{2}} (e^{2\alpha \epsilon^2} - 1) |E - \epsilon\rangle \langle E| \, dE. \] (11)
Therefore, the operators \( a_\epsilon, a_\epsilon^\dagger \) and \( N(\epsilon) \) do not obey the standard Weyl–Heisenberg algebra. Ben Geloun and Klauder obtained the following expression at limit:
\[ \lim_{\alpha \rightarrow 0} \frac{[a_\epsilon, a_\epsilon^\dagger]}{2\alpha \epsilon^2} = \int_0^\infty |E\rangle \langle E| \, dE = I, \] (12)
where $I$ is the unity operator in the quantum Hilbert space. One may continue with calculating the commutators between $N_\epsilon$ with $a_\epsilon$ and $a_\epsilon^\dagger$, respectively, as

$$\lim_{\alpha \to 0} \frac{[N_\epsilon, a_\epsilon^\dagger]}{2\alpha\epsilon^2} = \int_0^\infty |E - \epsilon\rangle \langle E| \ dE = \lim_{\alpha \to 0} a_\epsilon^\dagger,$$

So, we can conclude that the set $\{a_\epsilon, a_\epsilon^\dagger, N_\epsilon, I\}$ constitute the generators of a deformed version of the Heisenberg algebra.

### 3. Excited coherent states for continuous spectra

Photon-added coherent states were introduced by Agarwal and Tara [18] by the iterated actions ($m$ times) of $a_\epsilon^\dagger$ on the canonical coherent states, i.e.

$$|\alpha, m\rangle = a_\epsilon^m |\alpha\rangle \sqrt{\epsilon < a_\epsilon^m a_\epsilon^\dagger^m |\alpha\rangle},$$

where $a_\epsilon$ and $a_\epsilon^\dagger$ are the bosonic annihilation and creation operators, respectively, and $|\alpha\rangle$ is the canonical coherent state. Later, photon-added coherent states for solvable quantum systems with discrete spectra were introduced in [19]. Recently, photon-added coherent states were generated experimentally by Zavatta et al [20]. Analogously, we are now interested in the introduction of excited coherent states for continuous spectra as follows:

$$|s, \gamma, m\rangle_\epsilon = a_\epsilon^m |s, \gamma\rangle_\epsilon \sqrt{\epsilon < a_\epsilon^m a_\epsilon^\dagger^m |s, \gamma\rangle_\epsilon},$$

where $m$ is a non-negative integer. With the help of (4) and (6), these states obtain the following form:

$$|s, \gamma, m\rangle_\epsilon = N_{\epsilon, m}(s) \int_0^\infty \left( \prod_{j=0}^{m-1} e^{\alpha [sE + (m-j-\frac{1}{2})\epsilon^2]} \right) s^E e^{-i\gamma E} e^{\frac{1}{2}\alpha E^2} |E + m\epsilon\rangle \ dE,$$

in which we have set $N_{\epsilon, m}(s) = \langle s, \gamma | a_\epsilon^m a_\epsilon^\dagger^m |s, \gamma\rangle_\epsilon^{-1/2}$. Due to the relation $\prod_{j=0}^{m-1} e^{a_\epsilon [sE + (m-j-\frac{1}{2})\epsilon^2]} = e^{s\epsilon \max(2E+m\epsilon)}$, relation (17) simplifies to

$$|s, \gamma, m\rangle_\epsilon = N_{\epsilon, m}(s) \int_0^\infty e^{s\epsilon \max(2E+m\epsilon)} s^E e^{-i\gamma E} e^{\frac{1}{2}\alpha E^2} |E + m\epsilon\rangle \ dE.$$

The normalization factor, $N_{\epsilon, m}(s)$, can be obtained by requiring $\epsilon < s, \gamma, m\langle s, \gamma, m\rangle_\epsilon = 1$ and $\alpha > 0$ as

$$N_{\epsilon, m}(s) = \left[ \int_0^\infty e^{s\epsilon \max(2E+m\epsilon)} s^E e^{\frac{1}{2}\alpha E^2} \ dE \right]^{-\frac{1}{2}} = \left( \frac{4\alpha}{\pi} \right)^{\frac{1}{4}} e^{-\frac{m\epsilon^2}{2\alpha}} e^{-m^2\alpha\epsilon^2} \left[ 1 + \text{erf} \left( \frac{m\epsilon + \ln s}{\sqrt{2\alpha}} \right) \right]^{-\frac{1}{2}},$$

where erf(·) represents the Gaussian error function. Setting $m = 0$ in (18) and (19), the state $|s, \gamma, m\rangle_\epsilon$ reduces to the coherent state for continuous spectra in (4) and (5).

At this point, we want to check Gazeau–Klauder axioms for the introduced excited coherent states corresponding to continuous spectra. The main axioms are as follows.
(a) Continuity of labeling: it is clearly satisfied.

(b) Temporal stability: using (18) and the relevant Hamiltonian (1) readily gives
\[
e^{-iHt}|s, \gamma, m\rangle_e = e^{-im\epsilon N_{e,m}(s)} \int_0^\infty e^{i\max(2E+mc)e^{-i(\gamma+\omega t)E}} e^{i\frac{2E}{\omega_E E}} |E + m\epsilon\rangle dE
\]
\[= e^{-im\epsilon \omega t}|s, \gamma + \omega t, m\rangle_e.
\]

(c) Resolution of the identity:
\[
\int_{-\infty}^\infty dy \int_0^\infty ds \sigma_m(s)|s, \gamma, m\rangle e|s, \gamma, m\rangle = \mathbb{I}_m,
\]
where we have defined the unity operator, \(\mathbb{I}_m\), as
\[
\mathbb{I}_m = \int_0^\infty |E + m\epsilon\rangle \langle E + m\epsilon| dE,
\]
analogously to the unity operator that has been introduced for the photon-added coherent states of discrete spectra [21]. With the help of (18) and performing the integral on \(\gamma\), using \[\int_{-\infty}^\infty e^{i(\gamma - \epsilon t)} d\gamma = 2\pi \delta(E - E')\] on the left-hand side of (21), finally leads one to
the Stieltjes moment problem
\[
\int_0^\infty ds \sigma_m(s) \frac{2E}{\epsilon} = e^{\epsilon E'} \frac{e^{-m\epsilon}}{\sqrt{\alpha \pi}} \epsilon^{-2m^2 \alpha^2},
\]
where \(\epsilon = \sigma_m(s)[N_{e,m}(s)]^2\). Considering new variables \(u = \ln s\) and \(E' = E - m\epsilon\), this integral equation converted to
\[
\int_{-\infty}^\infty du \tilde{h}_m(u) e^{2E' u} = e^{\epsilon E'} \frac{e^{-m^2 \alpha E'}}{\sqrt{\alpha \pi}},
\]
with the solution \(\tilde{h}_m(u) = e^{-m^2 \alpha E'} \frac{e^{-u^2}}{\sqrt{\alpha \pi}}\). So, we have
\[
\sigma_m(s) = \frac{1}{s^{2m+1}} \frac{\epsilon^{-m\epsilon}}{\sqrt{\alpha \pi}} e^{\frac{-m^2 \alpha E'}{s}} [N_{e,m}(s)]^{-2}.
\]
Using (19), \(\sigma_m(s)\) may be finally written in the following closed form as
\[
\sigma_m(s) = \frac{1}{2\alpha} \left[ 1 + \text{erf} \left( \frac{m\epsilon + \ln s}{\sqrt{\alpha}} \right) \right].
\]
The excited coherent state will be coherent in its exact meaning, if \(\sigma_m(s) \geq 0\) exists for all ranges of \(s\). To test this requirement, in figure 1, we have displayed the function \(\sigma_m(s)\) for different values of \(m\). It is seen that \(\sigma_m(s)\) is positive for all values of \(s\) and \(\lim_{s \to \infty} \sigma_m(s) = 0\).

(d) Action identity: the action identity can be deduced from the mean value of the Hamiltonian over the excited states, which yields
\[
\epsilon(s, \gamma, m|H|s, \gamma, m\rangle_e = \omega [N_{e,m}(s)]^2 \int_0^\infty e^{i\max(2E+mc)} \frac{2E}{\omega_E E} dE \equiv \omega J(s),
\]
where the new action variable is assumed to be invertible versus \(s\). As argued in [5], if the function \(J(s)\) is invertible such that \(s(J)\) can be determined, the excited coherent states (18) satisfy the action identity axiom, appropriately.

So, we have established that the excited coherent states for continuous spectra (18) maintain in the family of Gazeau–Klauder type.
4. Nonclassical properties of ‘coherent states’ for continuous spectra

Before investigating the nonclassicality features of our introduced excited states, in this section, some of the nonclassical properties of ‘coherent states’ for continuous spectra (4) will be studied. For this purpose, we will discuss the Mandel parameter, second-order correlation function, quadrature squeezing and amplitude-squared squeezing.

4.1. Mandel parameter

Several parameters were introduced to characterize the statistical properties. The most popular one among them is the Mandel parameter $Q$, which was frequently used to measure the deviation from Poissonian distribution [22]. Calculating the mean value of $N_\epsilon$ with respect to the states (4) yields

$$\langle N_\epsilon \rangle = \epsilon \langle s, \gamma | N_\epsilon | s, \gamma \rangle_\epsilon = s^{2\epsilon}. \tag{28}$$

Also, noting that

$$N_\epsilon^2 = \int_0^\infty \int_0^\infty e^{2\epsilon(\epsilon E - \frac{1}{2}\epsilon^2)} |E\rangle \langle E| dE, \tag{29}$$

we obtain

$$\langle N_\epsilon^2 \rangle = \epsilon \langle s, \gamma | N_\epsilon^2 | s, \gamma \rangle_\epsilon = e^{2\alpha^2} s^{4\epsilon}. \tag{30}$$

Therefore, extending the Mandel parameter definition to continuous spectra seems to be possible [22], i.e.

$$Q_\epsilon = \frac{\langle N_\epsilon^2 \rangle - \langle N_\epsilon \rangle^2}{\langle N_\epsilon \rangle} = 1. \tag{31}$$
Figure 2. The plots of the Mandel parameter of coherent states for continuous spectra against $s$ with $\alpha = 10$ for $\epsilon = 0.01$ (dot-dashed curve), $\epsilon = 0.07$ (dashed curve) and $\epsilon = 0.15$ (solid curve).

The $Q_\epsilon$ quantity determines the quantum statistics of the radiation-field states for continuous spectra, i.e. it is super-Poissonian (if $Q_\epsilon > 0$), sub-Poissonian (if $Q_\epsilon < 0$) or Poissonian (if $Q_\epsilon = 0$). Using (28) and (30) in (31), the Mandel parameter for $|s, \gamma\rangle_\epsilon$ states will be obtained in the closed form as follows:

$$Q_\epsilon = s^2 \left( e^{2s\epsilon^2} - 1 \right) - 1.$$  (32)

It is noticeable that all of the quantities $\langle N_\epsilon \rangle$, $\langle N_\epsilon^2 \rangle$ and last $Q_\epsilon$, obtained in the closed form, are independent of $\gamma$. In figure 2, we have plotted the Mandel parameter of coherent states for continuous spectra versus $s$ for different values of $\epsilon$, keeping $\alpha$ fixed at 10. The figure shows that the coherent states for continuous spectra obey sub-Poissonian statistics ($Q_\epsilon < 0$) for some intervals of $s$. From the figure, it is seen that as $\epsilon$ decreases the interval of $s$, for which sub-Poissonian statistics reveals will be increased. For small values of $\epsilon$, the Mandel parameter is almost equal to $-1$. This is the value of the Mandel parameter corresponding to the number states as the most nonclassical states. Noting that in the presented approach $\epsilon > 0$ is the continuity parameter, one can be sure that for really continuous states the sub-Poissonian behavior of the states may not become weaker than what we have shown in figures. In figure 3, we have displayed the Mandel parameter of coherent states for continuous spectra versus $s$ for different values of $\alpha$, keeping $\epsilon$ fixed at 0.07. From the figure, we find that smaller $\alpha$ shows more nonclassical aspects.

4.2. Second-order correlation function

By developing the second-order correlation function definition [23] to quantum systems with continuous spectra, we define

$$g_\epsilon^2(0) = \frac{\langle a_\epsilon^2 a_\epsilon^2 \rangle}{\langle a_\epsilon^2 \rangle^2}.$$  (33)
Figure 3. The plots of the Mandel parameter of coherent states for continuous spectra against $s$ with $\epsilon = 0.07$ for $\alpha = 10$ (dot-dashed curve), $\alpha = 15$ (dashed curve) and $\alpha = 20$ (solid curve).

If $g^{2}_{\epsilon}(0) < 1$ ($g^{2}_{\epsilon}(0) \geq 1$), the state exhibits nonclassical (classical) behavior. With respect to the coherent states associated with continuous spectra (4), the following mean value is analytically obtained:

$$\langle \alpha^2 \rangle = \epsilon \langle s, \gamma | \alpha^2 | s, \gamma \rangle_{\epsilon} = s^{4\epsilon}. \quad (34)$$

Using (28) and (34), the second-order correlation function for coherent states of continuous spectra will be exactly 1, just as in the canonical coherent states. Therefore, coherent states for continuous spectra do not show this nonclassical behavior.

4.3. Quadrature squeezing

As another criteria for nonclassicality of states, we study quadrature squeezing [24] of the coherent states in (4). For this purpose, let us consider the following Hermitian quadrature operators:

$$X_1 = \frac{\alpha^\dagger + \alpha}{\sqrt{2}}, \quad Y_1 = \frac{\alpha^\dagger - \alpha}{\sqrt{2}i}. \quad (35)$$

The expectation values of $X_1$ and $X_1^2$ with respect to the states (4) are analytically given by

$$\langle X_1 \rangle = \epsilon \langle s, \gamma | X_1 | s, \gamma \rangle_{\epsilon} = s^{\epsilon} \cos(\gamma \epsilon) \quad (36)$$

and

$$\langle X_1^2 \rangle = \epsilon \langle s, \gamma | X_1^2 | s, \gamma \rangle_{\epsilon} = \frac{1}{2}s^{2\epsilon} \cos(2\gamma \epsilon) + \frac{1}{2}s^{2\epsilon}(e^{2\epsilon} + 1), \quad (37)$$

respectively. Thus, the dispersion of $X_1$ may be written in the closed form as

$$\langle X_1 \rangle^2 = \langle X_1^2 \rangle - \langle X_1 \rangle^2 = \frac{1}{2}s^{2\epsilon}(e^{2\epsilon} - 1). \quad (38)$$
The same result is obtained for the dispersion of operator $Y_1$:

$$\langle \Delta Y_1 \rangle^2 = \langle Y_1^2 \rangle - \langle Y_1 \rangle^2 = \frac{1}{4} s^2 \alpha (e^{2\alpha \epsilon^2} - 1).$$  \hfill (39)

Also, due to the relation

$$[X_1, Y_1] = \frac{i}{2} [a_\epsilon, a_\epsilon^\dagger] = i \sinh(\alpha \epsilon^2) \int_0^\infty e^{2\alpha \epsilon E} |E\rangle \langle E| dE,$$

one readily may arrive at

$$\langle [X_1, Y_1] \rangle = \epsilon \langle s, \gamma | [X_1, Y_1] | s, \gamma \rangle_{\epsilon} = \frac{i}{2} s^2 \alpha (e^{2\alpha \epsilon^2} - 1).$$  \hfill (40)

Therefore, it is clear that $X_1$ and $Y_1$ minimize the Heisenberg uncertainty relation. Indeed, we have equal fluctuations in $X_1$ and $Y_1$, and

$$(\Delta X_1)^2 (\Delta Y_1)^2 = \frac{1}{4} |\langle [X_1, Y_1] \rangle|^2.$$  \hfill (41)

Clearly, no squeezing may be expected. It is worth mentioning that, in a sense, with respect to the quadratures in (35), the states defined in (4) are of ‘intelligent-type’ states [25, 26].

4.4. Amplitude-squared squeezing

In order to examine whether the coherent states (4) exhibit higher-order squeezing, particularly amplitude-squared squeezing, or not, we introduce the following Hermitian operators:

$$X_2 = \frac{a_\epsilon^2 + a_\epsilon^{\dagger 2}}{2}, \quad Y_2 = \frac{a_\epsilon^2 - a_\epsilon^{\dagger 2}}{2i}.$$  \hfill (43)

In this case, the dispersions are straightforwardly obtained in the closed form as

$$(\Delta X_2)^2 = (\Delta Y_2)^2 = \frac{1}{4} s^4 \alpha (e^{8\alpha \epsilon^2} - 1).$$  \hfill (44)

In general, for the commutator of $X_2$ and $Y_2$, one obtains

$$[X_2, Y_2] = \frac{i}{2} [a_\epsilon^2, a_\epsilon^{\dagger 2}] = i \sinh(4\alpha \epsilon^2) \int_0^\infty e^{4\alpha \epsilon E} |E\rangle \langle E| dE,$$

from which we obtain

$$\langle [X_2, Y_2] \rangle = \epsilon \langle s, \gamma | [X_2, Y_2] | s, \gamma \rangle_{\epsilon} = \frac{i}{2} s^4 \alpha (e^{8\alpha \epsilon^2} - 1).$$  \hfill (45)

Hence, one has

$$(\Delta X_2)^2 (\Delta Y_2)^2 = \frac{1}{4} |\langle [X_2, Y_2] \rangle|^2,$$

and therefore, no squeezing occurs even in amplitude-squared operators and they are also minimum uncertainty or intelligent states. It is noticeable that neither of the expectation values calculated in the first- and second-order squeezing depends on $\gamma$.

5. Nonclassical properties of ‘excited coherent states’ for continuous spectra

In this section, the nonclassicality of the introduced ‘excited coherent states’ for continuous spectra (18) will be investigated, using the same criteria that have been used in the previous section.
5.1. Mandel parameter

By the definition of the number-like operator (8), the following mean values for excited coherent states are analytically obtained:

\[
\langle N_{\epsilon,m} \rangle = \epsilon \langle s, \gamma, m | N_{\epsilon} | s, \gamma, m \rangle_{\epsilon} = s^2 e^{4m \epsilon^2},
\]
and

\[
\langle N_{\epsilon,m}^2 \rangle = \epsilon \langle s, \gamma, m | N_{\epsilon}^2 | s, \gamma, m \rangle_{\epsilon} = e^{2\epsilon^2 (1+4m)} s^4 e.
\]

Finally, the Mandel parameter obtains the following closed form:

\[
Q_{\epsilon,m} = s^2 e^{4m \epsilon^2} \left( e^{2\epsilon^2} - 1 \right) - 1,
\]

where the subscript \( m \) indicates the order of excited coherent states. It is seen that \( Q_{\epsilon,m} \) is also \( \gamma \)-independent. Note that setting \( m = 0 \) in (50) recovers (32). Figure 4 shows the plots of the Mandel parameter of excited coherent states for continuous spectra versus \( s \) for different values of \( \epsilon \), fixing \( \alpha = 3 \) and \( m = 2 \). Also, we have plotted the Mandel parameter of excited coherent states versus \( s \) for different values of \( \alpha \), while keeping \( \epsilon \) fixed at 0.07 in figure 5. The figures show sub-Poissonian statistics for some intervals of \( s \). Our further calculations show that these intervals will be wider as \( \epsilon \) or \( \alpha \) decreases. In figure 6, the Mandel parameter of excited coherent states for continuous spectra has been plotted versus \( s \) for various values of \( m \). From the figure we find that, unlike the photon-added coherent states in [18], nonclassicality depth and range will decrease with increasing \( m \).

5.2. Second-order correlation function

Now, we can also study the second-order correlation function for excited coherent states associated with continuous spectra. (18). For these states, one analytically obtains

\[
\langle a_{\epsilon}^\dagger a_{\epsilon}^\dagger | a_{\epsilon}^\dagger a_{\epsilon}^\dagger | s, \gamma, m \rangle_{\epsilon} = s^4 e^{8m \epsilon^2}.
\]

Figure 4. The plots of the Mandel parameter of excited coherent states for continuous spectra against \( s \) with \( \alpha = 3 \) and \( m = 2 \) for \( \epsilon = 0.01 \) (dot-dashed curve), \( \epsilon = 0.07 \) (dashed curve) and \( \epsilon = 0.15 \) (solid curve).
Figure 5. The plots of the Mandel parameter of excited coherent states for continuous spectra against $s$ with $\epsilon = 0.07$ and $m = 2$ for $\alpha = 3$ (dot-dashed curve), $\alpha = 5$ (dashed curve) and $\alpha = 7$ (solid curve).

Figure 6. The plots of the Mandel parameter of excited coherent states for continuous spectra against $s$ with $\epsilon = 0.07$ and $\alpha = 3$ for $m = 2$ (dot-dashed curve), $m = 5$ (dashed curve) and $m = 7$ (solid curve).
With the help of (48) for \( \langle \alpha | \alpha \rangle \), one clearly has, just as the coherent states for continuous spectra and canonical coherent states, \( g_{\epsilon,m}^2(0) = 1 \). This implies that these states do not exhibit nonclassical behavior (independent of the values of \( m \) and \( \epsilon \)), while they possess sub-Poissonian statistics.

5.3. Quadrature squeezing

With the help of the definitions of the Hermitian operators \( X \) and \( Y \) introduced in (35), the dispersions with respect to the excited coherent states in (18) are given in the closed form by

\[
(\Delta X_1)^2_m = (\Delta Y_1)^2_m = \frac{1}{2} \frac{s}{\epsilon} e^{4m\epsilon} (e^{2\epsilon^2} - 1). \tag{52}
\]

It also follows that

\[
\langle [X_1, Y_1] \rangle_m = \epsilon \langle s, \gamma, m | [X_1, Y_1] | s, \gamma, m \rangle = \frac{i}{2} \frac{s}{\epsilon} e^{4m\epsilon} (e^{2\epsilon^2} - 1), \tag{53}
\]

and again one has

\[
(\Delta X_1)^2_m = (\Delta Y_1)^2_m = \frac{1}{2} |\langle [X_1, Y_1] \rangle_m|. \tag{54}
\]

Therefore, no quadrature squeezing occurs for our introduced excited states and they saturate the Heisenberg uncertainty relation. These states may also be called as intelligent states, regarding \( X_1 \) and \( Y_1 \) quadratures. Again, putting \( m = 0 \), (52) and (53) reduce to (38) and (41), respectively.

5.4. Amplitude-squared squeezing

By using (43) and (18), the following expressions for dispersions of \( X_2 \) and \( Y_2 \) with respect to excited coherent states will be analytically obtained:

\[
(\Delta X_2)^2_m = (\Delta Y_2)^2_m = \frac{1}{4} s^4 e^{8m\epsilon} (e^{8\epsilon^2} - 1). \tag{55}
\]

Also, one can calculate the commutator of \( X_2 \) and \( Y_2 \) as follows:

\[
\langle [X_2, Y_2] \rangle_m = \epsilon \langle s, \gamma, m | [X_2, Y_2] | s, \gamma, m \rangle = \frac{i}{2} s^4 e^{8m\epsilon} (e^{8\epsilon^2} - 1). \tag{56}
\]

Therefore,

\[
(\Delta X_2)^2_m = (\Delta Y_2)^2_m = \frac{1}{2} |\langle [X_2, Y_2] \rangle_m|. \tag{57}
\]

and obviously no squeezing occurs in amplitude-squared operators, too. Setting \( m = 0 \) in (55) and (56) recovers (44) and (46), respectively.

6. Summary and conclusion

In summary, after presenting a brief review on the 'coherent state' for 'quantum systems with continuous spectra' recently introduced by Ben Geloun and Klauder in [9], and at the same time giving a few additional remarkable points, we have introduced the 'excited coherent states' for such systems. The four axioms of Gazeau–Klauder coherent states and among them the resolution of the identity requirement, needed for the over-completeness of the states, are established regarding the introduced exited states. Then, we studied the nonclassicality features of 'coherent' and 'excited coherent states' by discussing the quantum statistical properties and nonclassicality features of the two mentioned classes of states by evaluating the Mandel parameter, second-order correlation function, and first- and second-order squeezing. Interestingly, all of these quantities have been deduced analytically. Through this, it is found that both the coherent and excited coherent states obey sub-Poissonian statistics for some
values of their variables. We observed that, unlike the photon-added coherent states of discrete spectra [18], increasing the order of excitation of excited coherent states for continuous spectra reduces the regions and depths of nonclassicality of the states. Even though the two classes of states exhibit neither quadrature squeezing nor amplitude-squared squeezing for all values of their variables, we also found that they can be considered as ‘intelligent states’, in the context of quantum states associated with continuous spectra.

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