On Cartan Spaces with the \( m \)-th Root Metric

\[
K(x, p) = m \sqrt{a_{i_{1}i_{2}...i_{m}}(x)p_{i_{1}}p_{i_{2}}...p_{i_{m}}}
\]

Gheorghe Atanasiu and Mircea Neagu
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Abstract

The aim of this paper is to expose some geometrical properties of the locally Minkowski-Cartan space with the Berwald-Moór metric of momenta \( L(p) = \sqrt{p_{1}p_{2}...p_{n}} \). This space is regarded as a particular case of the \( m \)-th root Cartan space. Thus, Section 2 studies the \( v \)-covariant derivation components of the \( m \)-th root Cartan space. Section 3 computes the \( v \)-curvature d-tensor \( S^{hijk} \) of the \( m \)-th root Cartan space and studies conditions for \( S^{3} \)-likeness. Section 4 computes the \( T \)-tensor \( T^{hijk} \) of the \( m \)-th root Cartan space. Section 5 particularizes the preceding geometrical results for the Berwald-Moór metric of momenta.

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1 Introduction

Owing to the studies of E. Cartan, R. Miron [6, 7], Gh. Atanasiu [2] and many others, the geometry of Cartan spaces is today an important chapter of differential geometry, regarded as a particular case of the Hamilton geometry. By the Legendre duality of the Cartan spaces with the Finsler spaces studied by R. Miron, D. Hrimiuc, H. Shimada and S. V. Sabău [8], it was shown that the theory of Cartan spaces has the same symmetry like the Finsler geometry, giving in this way a geometrical framework for the Hamiltonian theory of Mechanics or Physical fields. In a such geometrical context we recall that a Cartan space is a pair \( C^{n} = (M^{n}, K(x, p)) \) such that the following axioms hold good:

1. \( K \) is a real positive function on the cotangent bundle \( T^{*}M \), differentiable on \( T^{*}M \backslash \{0\} \) and continuous on the null section of the canonical projection

\[
\pi^{*} : T^{*}M \rightarrow M;
\]

2. \( K \) is positively 1-homogenous with respect to the momenta \( p_{i} \);
3. The Hessian of \( K^2 \), with the elements 
\[
g^{ij}(x, p) = \frac{1}{2} \frac{\partial^2 K^2}{\partial p_i \partial p_j},
\]
is positive-defined on \( T^*M \setminus \{0\} \).

On the other hand, in the last two decades, physical studies due to G. S. Asanov [1], D. G. Pavlov [9], [10] and their co-workers emphasize the important role played by the Berwald-Moór metric \( L : TM \to \mathbb{R} \),
\[
L(y) = (y^1 y^2 ... y^n)^{\frac{1}{m}},
\]
in the theory of space-time structure and gravitation as well as in unified gauge field theories.

For such geometrical-physics reasons, following the geometrical ideas exposed by M. Matsumoto and H. Shimada in [4], [5] and [11] or by ourselves in [3], in this paper we investigate some geometrical properties of the \( m \)-th root Cartan space which is a natural generalization of the locally Minkowski-Cartan space with the Berwald-Moór metric of momenta.

2 The \( m \)-th root metric and \( v \)-derivation components

Let \( C^n = (M^n, K(x, p)) \), \( n \geq 4 \), be an \( n \)-dimensional Cartan space with the metric
\[
K(x, p) = \sqrt[\!m\!]\left( a^{i_1 i_2 ... i_m}(x) p_{i_1} p_{i_2} ... p_{i_m} \right),
\]
where \( a^{i_1 i_2 ... i_m}(x) \), depending on the position alone, is symmetric in all the indices \( i_1, i_2, ..., i_m \) and \( m \geq 3 \).

**Definition 2.1** The Cartan space with the metric (2.1) is called the \( m \)-th root Cartan space.

Let us consider the following notations:
\[
\begin{align*}
a^i &= \left[ a^{i_1 i_2 ... i_m}(x) p_{i_2} p_{i_3} ... p_{i_m} \right] / K^{m-1}, \\
a^{ij} &= \left[ a^{i_1 i_2 ... i_m}(x) p_{i_3} p_{i_4} ... p_{i_m} \right] / K^{m-2}, \\
a^{ijk} &= \left[ a^{i_1 i_2 i_3 ... i_m}(x) p_{i_4} p_{i_5} ... p_{i_m} \right] / K^{m-3}.
\end{align*}
\]

The normalized supporting element
\[
l^i = \dot{\partial}^i K, \quad \text{where} \quad \dot{\partial}^i = \frac{\partial}{\partial p_i},
\]
the fundamental metrical d-tensor
\[
g^{ij} = \frac{1}{2} \dot{\partial}^i \dot{\partial}^j K^2
\]
and the angular metrical d-tensor

\[ h^{ij} = K \dot{\theta}^i \dot{\theta}^j K \]

are given by

\[ l^i = a^i, \]

\[ g^{ij} = (m - 1)a^{ij} - (m - 2)a^i a^j, \]  \hspace{1cm} (2.3)

\[ h^{ij} = (m - 1)(a^{ij} - a^i a^j). \]

**Remark 2.2** From the positively 1-homogeneity of the \( m \)-th root Cartan metrical function \((2.7)\) it follows that we have

\[ K^2(x, p) = g^{ij}(x, p)p_i p_j = a^{ij}(x, p)p_i p_j. \]

Let us suppose now that the d-tensor \( a^{ij} \) is regular, that is there exists the inverse matrix \((a^{ij})^{-1} = (a_{ij})\). Obviously, we have

\[ a_i \cdot a^i = 1, \]

where

\[ a_i = a_{is} a^s = \frac{p_i}{K}. \]

Under these assumptions, we obtain the inverse components \( g_{ij}(x, p) \) of the fundamental metrical d-tensor \( g^{ij}(x, p) \), which are given by

\[ g_{ij} = \frac{1}{m - 1} a_{ij} + \frac{m - 2}{m - 1} a_i a_j. \]  \hspace{1cm} (2.4)

The relations (2.2) and (2.3) imply that the components of the \( v \)-torsion d-tensor

\[ C^{ijk} = -\frac{1}{2} \dot{\theta}^k g^{ij} \]

are given in the form

\[ C^{ijk} = -\frac{(m - 1)(m - 2)}{2K} \left( a^{ijk} - a^{ij} a_k - a^{ik} a^j - a^{kij} + 2a^i a^j a^k \right). \]  \hspace{1cm} (2.5)

Consequently, using the relations (2.4) and (2.5), together with the formula

\[ a_s a^{sjk} = a^{jk}, \]

we find the components of the \( v \)-derivation

\[ C^{jk}_i = g_{is} C^{sjk} \]

in the following form:

\[ C^{jk}_i = -\frac{(m - 2)}{2K} \left[ a^{ijk} - \left( \delta_i^j a^k + \delta_i^k a^j \right) + a_s (2a^i a^k - a^{jk}) \right], \]  \hspace{1cm} (2.6)

where

\[ a^{ijk} = a_{is} a^{sjk}. \]

From (2.6) we easily find the following geometrical result:
Proposition 2.3 The torsion covector

\[ C^i = C^i_r \]

is given by the formula

\[ C^i = -(m-2) \left( a^r_i - n a^i \right), \]

where \( n = \dim M \).

3 The \( v\)–curvature d-tensor \( S^{hijk} \)

Taking into account the relations (2.5) and (2.6), by calculation, we obtain

Theorem 3.1 The \( v\)–curvature d-tensor

\[ S^{hijk} = C^i_r C^{r hk} - C^i_r C^{r hj} \]

can be written in the form

\[ S^{hijk} = \frac{(m-1)(m-2)^2}{4K^2} A_{(j,k)} \{ a^{ij} a^{r hk} - a^{ij} (a^{hk} - a^b a^k) + a^i a^j a^{bhk} \}, \]

where \( A_{(j,k)} \) means an alternate sum.

Remark 3.2 Using the relations (2.5), we underline that the \( v\)–curvature d-tensor \( S^{hijk} \) can be written as

\[ K^2 S^{hijk} = \frac{(m-2)^2}{4} \left[ (h^{ij} h^{ik} - h^{jk} h^{ij})/(m-1) + (m-1) U^{hijk} \right], \quad (3.1) \]

where

\[ U^{hijk} = a^{ij} a^{r hk} - a^{ik} a^{r hj}. \quad (3.2) \]

In the sequel, let us recall the following important geometrical concept [4]:

Definition 3.3 A Cartan space \( C^n = (M^n, K(x,p)) \), \( n \geq 4 \), is called \( S3\)-like if there exists a positively 0-homogenous scalar function \( S = S(x,p) \) such that the \( v\)–curvature d-tensor \( S^{hijk} \) to have the form

\[ K^2 S^{hijk} = S \{ h^{ij} h^{ik} - h^{jk} h^{ij} \}. \quad (3.3) \]

Let \( C^n = (M^n, K(x,p)) \), \( n \geq 4 \), be the \( m\)-th root Cartan space. As an immediate consequence of the above definition we have the following important result:

Theorem 3.4 The \( m\)-th root Cartan space \( C^n \) is an \( S3\)-like Cartan space if and only if the d-tensor \( U^{hijk} \) is of the form

\[ U^{hijk} = \lambda \{ h^{ij} h^{ik} - h^{jk} h^{ij} \}, \quad (3.4) \]

where \( \lambda = \lambda(x,p) \) is a positively 0-homogenous scalar function.
Proof. Taking into account the formula (3.1) and the condition (3.4), we find the scalar function (see (3.3))

\[ S = \frac{(m - 2)^2}{4} \left[ (m - 1)\lambda + \frac{1}{m - 1} \right]. \tag{3.5} \]

\[ \]

4 The \( T \)-tensor \( T^{hij} \)

Let \( N = (N_{ij}) \) be the canonical nonlinear connection of the \( m \)-th root Cartan space with the metric (2.1), whose local coefficients are given by [8]

\[ N_{ij} = \gamma_{ij}^0 - \frac{1}{2}\gamma_{h0}^0 \partial^h g_{ij}, \]

where

\[ \partial_k = \frac{\partial}{\partial x^k}, \]
\[ \gamma_{jk}^i = \frac{g^{ir}}{2}(\partial_k g_{rj} + \partial_j g_{rk} - \partial_r g_{jk}), \]
\[ \gamma_{ij}^0 = -\gamma_{ij}^0 p_s, \]
\[ \gamma_{h0}^0 = \gamma_{hr}^0 f^s p_t p_s. \]

Let

\[ CT(N) = (H_{jk}^i, C_{ijk}^j) \]

be the Cartan canonical connection of the \( m \)-th root Cartan space with the metric (2.1). The local components of the Cartan canonical connection \( CT(N) \) have the expressions [8]

\[ H_{jk}^i = \frac{g^{ir}}{2}(\delta_k g_{rj} + \delta_j g_{rk} - \delta_r g_{jk}), \]
\[ C_{ijk}^j = g_{is} C_{sjk}^i = -\frac{g_{is}}{2} \partial^k g_{js}, \]

where

\[ \delta_j = \partial_j + N_{js} \partial^s. \]

In the sequel, let us compute the \( T \)-tensor \( T^{hij} \) of the \( m \)-th root Cartan space, which is defined as [11]

\[ T^{hij} \overset{\text{def}}{=} KC_{hij}^k + l^h C_{ij}^k + l^i C_{jk}^k + l^j C_{ki}^k + l^k C_{hij}, \]

where \( \overset{\text{def}}{=} \) denotes the local \( v \)-covariant derivation with respect to \( CT(N) \), that is we have

\[ C_{hij}^{|k} = \partial^k C_{hij} + C_{rij}^{|k} C_{r}^{|k} + C_{hjr}^{|k} C_{r}^{|k} + C_{hri}^{|k} C_{r}^{|k}. \]
Using the definition of the local $v$–covariant derivation [8], together with the relations (2.6) and (2.3), by direct computations, we find the relations:

\[ K^k_l = a_k^l, \]

\[ a^i_l = \frac{(m-1)}{K} (a^i_k - a^k_i), \] \hspace{1cm} (4.1)

\[ a^{ij}_l = \frac{(m-2)}{K} (a^{ik}_j + a^{jk}_i - 2a^i_j a^k_l) = \frac{(m-2)}{(m-1)K} (h^{ik}_j l^j + h^{jk}_i l^i). \]

Suppose that we have $m \geq 4$. Then, the notation

\[ a^{hijk} = \left[ a^{hij}_{...m}(x) p_i^{...m} \right] / K^{m-4} \]

is very useful. In this context, we can give the next geometrical results:

**Lemma 4.1** The $v$–covariant derivation of the tensor $a^{hij}$ is given by the following formula:

\[ a^{hij}_l = \frac{(m-3)}{K} a^{hij}_l + \frac{m}{2K} a^{hij}_l a^k - \frac{(m-2)}{2K} \left( a_k^k a^i_l + a_k^l a^i_k - a^i_k a^k_l \right) + (m-1) (m-2) K \left( h^{ik}_j l^j + h^{jk}_i l^i \right). \] \hspace{1cm} (4.2)

**Proof.** Note that, by a direct computation, we obtain the relation

\[ \frac{\partial a^{hij}}{\partial p_k} = \frac{(m-3)}{K} (a^{hij} - a^{hij} a^k). \] \hspace{1cm} (4.3)

Finally, using the definition of the local $v$–covariant derivation, together with the formulas (4.3) and (2.6), we find the equality (4.2).

**Theorem 4.2** The $T$–tensor $T^{hijk}$ of the $m$-th root Cartan space is given by the expression

\[ T^{hijk} = -\frac{(m-1)(m-2)(m-3)}{2K} a^{hijk} + \frac{(m-1)(m-2)^2}{4K} \cdot \left( a_k^k a^i_l + a_k^l a^i_k - a^i_k a^k_l \right) - \frac{m(m-1)(m-2)}{4K} \cdot \left( a^{hij}_l a^k + a^{hjk}_l a^i + a^{ijk}_l a^k - a^{hk}_l a^i - a^{hj}_l a^k + a^{hj}_l a^k + a^{hj}_l a^k \right). \] \hspace{1cm} (4.4)

**Proof.** It is obvious that we have the equality

\[ T^{hijk} = (KC^{hij})^k_l + l^i C^{ijk} + l^j C^{ikh} + l^h C^{khi}. \]

Consequently, differentiating $v$–covariantly the relation (2.5) multiplied by $K$ and using the formulas (4.1), together with the Lemma 4.1, by laborious computations, it follows the required result.
5 The particular case of Berwald-Moór metric of momenta

Let us consider now the particular case when \( m = n \geq 4 \) and

\[
a^{i_1 i_2 \ldots i_n}(x) = \begin{cases} 
1/n!, & i_1 \neq i_2 \neq \ldots \neq i_n \\
0, & \text{otherwise}.
\end{cases}
\]

In this special case, the \( m \)-th root metric (2.1) becomes the Berwald-Moór metric of momenta [8]

\[
K(p) = \sqrt[n]{p_1 p_2 \ldots p_n}. \quad (5.1)
\]

By direct computations, we deduce that the \( n \)-dimensional locally Minkowski-Cartan space \( C^n = (M^n, K(p)) \) endowed with the Berwald-Moór metric of momenta (5.1) is characterized by the following geometrical entities and relations denoted by (E-R):

\[
a^i = K \cdot \frac{1}{p_i}, \quad a_i = \frac{p_i}{K}, \quad a_i \cdot a^i = \frac{1}{n} \text{ (no sum by } i),
\]

\[
a^{ij} = \begin{cases} 
\frac{n}{n-1} \cdot a^i a^j, & i \neq j \\
0, & i = j
\end{cases}
\]

\[
a_{ij} = \begin{cases} 
n \cdot a_i a_j, & i \neq j \\
-n(n-2) \cdot (a_i)^2, & i = j
\end{cases}
\]

\[
a^{ijk} = \begin{cases} 
\frac{n^2}{(n-1)(n-2)} \cdot a^i a^j a^k, & i \neq j \neq k \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
a^{hijk} = \begin{cases} 
\frac{n^3}{(n-1)(n-2)(n-3)} \cdot a^h a^i a^j a^k, & h \neq i \neq j \neq k \\
0, & \text{otherwise}.
\end{cases}
\]

Moreover, the equalities (E-R) imply that the components \( a^{ijk} \) are given by the formulas:

\[
a^{ijk} = -\frac{n^2}{(n-1)(n-2)} \cdot a_i a_j a_k, \quad i \neq j \neq k
\]

\[
a^{ik} = a^{ki} = \frac{n}{n-1} \cdot a^k, \quad i \neq k \text{ (no sum by } i)
\]

\[
a^{kk} = 0, \quad \forall i = 1, n, \text{ (no sum by } k).
\]

In this context, we obtain the following important geometrical result:
Theorem 5.1  The locally Minkowski-Cartan space $C^n = (M^n, K(p))$, $n \geq 4$, endowed with the Berwald-Moór metric of momenta (5.1) is characterized by the following geometrical properties:

1. The torsion covector $C^i$ vanish;
2. $S3$-likeness with the scalar function $S = -1$;
3. The $T$–tensor $T^{hijk}$ vanish.

Proof. 1. It is easy to see that we have
$$\sum_r a^r_i = \sum_{r,s} a^{rs} a^{sir} = n \sum_{r,s} a^r_s a^{sir} = n \sum_r a^r_i = na^i.$$  

2. It is obvious that we have
$$h^{ij} = \begin{cases} a^ia^j, & i \neq j \\ -(n-1) \cdot (a^i)^2, & i = j. \end{cases}$$

Consequently, by computations, we obtain
$$U^{hijk} = -\frac{n^2}{(n-1)^2(n-2)^2} \left\{ h^{hij}h^{hik} - h^{hjk}h^{hij} \right\},$$
where $U^{hijk}$ is given by the relation (3.2). It follows what we were looking for (see the equalities (3.4) and (3.5)).

3. Using the relation (4.4) and the formulas (E-R) and (5.2), by laborious computations, we deduce that $T^{hijk} = 0$. ■

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Gheorghe ATANASIU and Mircea NEAGU
University "Transilvania" of Brașov
Faculty of Mathematics and Informatics
Department of Algebra, Geometry and Differential Equations
Bd. Eroilor, Nr. 29, Brașov, BV 500019, Romania.

gh_atanasiu@yahoo.com
mirceaneagu73@yahoo.com