Multiplicity of compact group representations
and applications to Kronecker coefficients

Velleda Baldoni, Michèle Vergne

Abstract
These notes are an expanded version of a talk given by the second author. Our main interest is focused on the challenging problem of computing Kronecker coefficients. We decided, at the beginning, to take a very general approach to the problem of studying multiplicity functions, and we survey the various aspects of the theory that comes into play, giving a detailed bibliography to orient the reader. Nonetheless the main general theorems involving multiplicities functions (convexity, quasi-polynomial behavior, Jeffrey-Kirwan residues) are stated without proofs. Then, we present in detail our approach to the computational problem, giving explicit formulae, and outlining an algorithm that calculate many interesting examples, some of which appear in the literature also in connection with Hilbert series.

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1 Introduction

Let $K$ be a compact connected Lie group and let $\hat{K}$ be the set of classes of irreducible finite dimensional representations of $K$. Let $V$ be a representation of $K$, which is a direct sum (possibly infinite) of irreducible finite dimensional representations of $K$ with finite multiplicities. We write

$$V = \oplus_{\pi \in \hat{K}} m_K(\pi) V^K_{\pi}$$

where $V^K_{\pi}$ is the irreducible representation of $K$ parameterized by $\pi$. We call the function $\pi \to m_K(\pi)$ on $\hat{K}$ the multiplicity function of the representation $V$. The study of the multiplicity function $m_K(\pi)$ is important in representation theory, invariant theory, quantum information theory.
When the representation space $V$ is constructed by ” geometric quantization” of a Hamiltonian manifold $M$, the moment map $\Phi$ on $M$ gives us a geometric interpretation of the multiplicity function of the representation $V$. This is the famous $[Q,R] = 0$ theorem, obtained by Meinrenken-Sjamaar [41]. This has consequences on the qualitative properties of the function $\pi \rightarrow m_K(\pi)$ that we will recall here in two examples, which are the paradigms of geometric quantization:

- The quantization of a symplectic vector space under a linear action of $K$.
- The quantization of $T^*K$, the cotangent bundle of $K$.

In these two basic examples, we give a direct construction of the corresponding representation space $V$. We will not justify that the representation space $V$ is “the quantized space” of $M$, but we will recall results on the multiplicity function of the representation of $K$ in $V$ in terms of the moment map on $M$ which confirm the fact that $V$ is the “the quantized space” of the Hamiltonian space $M$.

**Example 1. Quantization of a symplectic vector space**

Consider $M$ a $2n$ dimensional symplectic vector space $(M = \mathbb{R}^{2n})$ with a linear symplectic action of the compact group $K$. We choose a Hermitian structure on $M$ (so we identify $M$ to $\mathbb{C}^n$) such that $K$ acts unitarily on $M$. We denote by $Sym^k(M)$ the complex space of $k$-symmetric tensors on the complex space $M$. Consider 

$$V = \bigoplus_{k=0}^{\infty} Sym^k(M),$$

the space of polynomial functions on $M^*$. 

The space $V$ may be completed as a Hilbert space, by choosing a Gaussian inner product, and the completion of $V$ is the Fock space. In the philosophy of quantization, this Fock space is the quantized space associated to the dual representation of $K$ in the symplectic space $M^*$. 

If $K$ contains the homotheties, we then have 

$$V = \bigoplus_{\pi \in \hat{K}} m_K(\pi)V^K_{\pi}$$

where $m_K(\pi)$ is finite. When $K$ is abelian, $m_K(\pi)$ is a partition function.

Remark that the knowledge of the function $\pi \rightarrow m_K(\pi)$, for $\pi$ trivial on the semi-simple part of $K$, allows to compute the Hilbert series of the ring of invariant polynomials on $M^*$ under the semi-simple group $[K,K]$. 

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**Example 2. Quantization of the cotangent space** $T^*G$

Consider a compact Lie group $G$. Let us define $V = R(G)$ to be the subspace of $C^\infty(G)$ generated by the coefficients $\langle gu_1, u_2 \rangle$ of finite dimensional representations of $G$. The space $L^2(G)$ is the Hilbert completion of $V$. In the philosophy of quantization, the space $L^2(G)$ “is” the quantized space associated to the symplectic space $T^*G$.

By Peter-Weyl theorem, under the action of $G \times G$ by left and right translations:

$$V = \oplus_{\lambda \in \hat{G}} V^G_\lambda \otimes (V^G_\lambda)^*.$$

Let $K$ be a subgroup of $G$, and consider the subgroup $G \times K$ of $G \times G$. Under the action of $G \times K$,

$$V = \oplus_{\lambda \in \hat{G}, \mu \in \hat{K}} m_{G,K}(\lambda, \mu) V^G_\lambda \otimes (V^K_\mu)^*$$

where $m_{G,K}(\lambda, \mu)$ is the multiplicity of the representation $\mu$ in the restriction of $\lambda$ to $K$. This is the so called branching coefficient.

In both of these examples, we will recall some of the qualitative properties of the multiplicity function, in particular its piecewise quasi-polynomial behavior on closed cones. These examples being particular cases of geometric quantization of Hamiltonian manifolds, the moment cone, its decomposition in cones of quasi-polynomiality, and the determination of its faces, gives us already information on the behavior of the multiplicity function.

We will then recall our method to compute the multiplicity function $m_K(\pi)$, based on computations of partition functions via multi-dimensional residues. We apply it to the challenging example of computation of Kronecker coefficients. In particular, we obtain an algorithm which enables us to compute the dilated Kronecker coefficients associated to Young diagrams with $n$ rows (but “any number” of columns and any shape) in polynomial time when $n$ is fixed. We implemented our algorithm as a simple Maple program. In particular, it computes the dilated Kronecker coefficient $g(k\alpha, k\beta, k\gamma)$ in reasonable time (reasonable=less than 20 minutes) for 3 rows. We obtain a quasi-polynomial of degree 11 for general $\alpha, \beta, \gamma$ (with coefficients periodic functions of period at most 12) see Example $\PageIndex{83}$. When $\alpha, \beta, \gamma$ are special, the degree might be much smaller. For example, the dilated Kronecker coefficient $m(k) = g(k[1, 1, 1], k[1, 1, 1], k[1, 1, 1])$ corresponds to the Hilbert series of the ring of invariants of $SL(3) \times SL(3) \times SL(3)$ in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. An efficient
A way to represent periodic functions is by using step-polynomials \((51)\). In this representation, \(m(k)\) is given by the following quasi-polynomial:

\[
1 - \frac{3}{2} \left\{ \frac{1}{3} k \right\} + \frac{3}{2} \left\{ \frac{1}{3} k \right\}^2 - \frac{3}{2} \left\{ \frac{1}{2} k \right\} - \left\{ \frac{3}{4} k \right\}^2 + \left\{ \frac{3}{4} k \right\} \left\{ \frac{1}{2} k \right\} + \left\{ \frac{1}{2} k \right\}^2 + \\
\left( \frac{1}{4} - \frac{1}{4} \left\{ \frac{1}{2} k \right\} \right) k + \frac{1}{48} k^2
\]

Here for \(s \in \mathbb{R}\), the function \(\{s\} = s - \lfloor s \rfloor \in [0, 1)\) where \(\lfloor s \rfloor\) denotes the largest integer smaller or equal to \(s\). It is easy to check that (fortunately) this result of our algorithm agrees with Kac’s determination \([26]\) of the ring of invariants \([\text{Sym}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)]^{SL(3) \times SL(3) \times SL(3)}\), which is freely generated with generators in degree 2, 3, 4 (see \([40]\)). Indeed

\[
\sum_k m(k)t^k = \frac{1}{(1 - t^2)(1 - t^3)(1 - t^4)}.
\]

Another example is

\[
m(k) = g(k[3, 3, 3], k[4, 4, 4], k[4, 4, 4]) = \text{dim}[\text{Sym}^{12k}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)]^{SL(4) \times SL(3) \times SL(3)}.
\]

Our algorithm gives the Hilbert series

\[
\sum_k m(k)t^k = \frac{1 + t^9}{(1 - t^2)^2 (1 - t^4) (1 - t) (1 - t^3)}.
\]

We will resume with the study of Kronecker coefficients in Section \([6.2]\) giving all the details. For example, we can compute easily the Hilbert series for 3, 4, 5 qubits. The result for the case of 5 qbits appears also in \([38]\) with a discrepancy due to a small misprint (see \([6.2]\)).

Our article, focused towards computational problems, is strongly inspired by the article of Christandl-Doran-Walter \([13]\) and we thank them for introducing us to the subject of quantum computing. Our computational method differ at some points from their method, and we will discuss the differences. It is based on Jeffrey-Kirwan residues. This allows us to produce a symbolic function valid inside the cone of quasi-polynomiality, in particular along a line.

We believe that this survey might be of interest. The eventual reader that just want to read a discussion of general results in Hamiltonian geometry and
some applications of these results to combinatorics, without being addicted to computing, might skip the technical parts, namely, Subsection 4.9 inside Section 4 as well as Subsection 5.2 and Subsection 5.3 inside Section 5, and might go directly to Section 6 skiping again the technical subsection 6.1.

2 Notations

Let $K$ be a compact connected Lie group and $T_K$ a maximal torus of $K$. We denote by $\mathfrak{k}$ and $t_k$ the corresponding Lie algebras. Denote by $W_k$ the Weyl group, and $\epsilon(w) = \det(t_{k}w)$ its sign representation.

The weight lattice $\Lambda_K$ of $T$ is a lattice in $t_k^*$. If $\lambda \in \Lambda_K \subset t_k^*$, it determines a one dimensional representation of $T_K$ by $t \mapsto e^{\langle \lambda, X \rangle}$, with $t = \exp(X), X \in \mathfrak{t}$. As $\lambda$ takes imaginary values on $t_k$, $e^{\langle \lambda, X \rangle}$ is of modulus 1.

We denote by $\Gamma_K \subset \mathfrak{t}_k$ the dual lattice of $\Gamma_K$: if $\lambda \in \Lambda_K$, $\gamma \in \Gamma_K$, then $\langle \lambda, \gamma \rangle$ is an integer.

Let $\Delta_k \subset \mathfrak{t}^*$ be the root system for $\mathfrak{k}$ with respect to $\mathfrak{t}_k$. If $\alpha \in \Delta_k$, its coroot $H_\alpha$ is in $\mathfrak{t}_k$, and $\langle \alpha, H_\alpha \rangle = 2$. Define $\hat{K}$ to be the set of irreducible finite dimensional classes of complex representations of $K$. Fix $\Delta_k^+$, a positive system for $\Delta_k$.

We denote by $\Lambda_{\mathfrak{k}, \geq 0}$ the "cone" of dominant weights, that is the set $\Lambda_K \cap \mathfrak{t}_k^* \geq 0$. An element $\lambda \in \Lambda_{K, \geq 0}$ is called a dominant weight.

When the group $K$ is understood, we denote $\Lambda_K$ simply by $\Lambda$, $T_K$ by $T$, $t_k$ by $t$, etc.

The following example is the only example which will be needed when discussing Kronecker coefficients.

Example 3.

We consider the case $K = U(n)$ and $T \subset K$ the torus consisting of the diagonal matrices. Then the Lie algebra $\mathfrak{k}$ consists of the $n \times n$ anti Hermitian matrices and $i\mathfrak{k}$ is the space of Hermitian matrices. If we identify $\mathfrak{k}$ and $\mathfrak{k}^*$ via the linear form $Tr(AB)$, then $t = \mathfrak{t}^*$ is the set of diagonal anti hermitian matrices. Thus the positive Weyl chamber is $i\mathfrak{t}^*_{\geq 0} = \{\xi, \langle \xi, H_\alpha \rangle \geq 0, \alpha \in \Delta_k^+\}$ with $\xi_j \in \mathbb{R}$ and $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ where $\xi$ represents the hermitian matrix.
with diagonal entries $\xi_j$. Denote by $|\xi| = \sum_i \xi_i$, the trace of the matrix corresponding to $\xi$.

An element $X \in i\mathfrak{t}^*$ is conjugated to a unique element $\xi = [\xi_1, \xi_2, \ldots, \xi_n]$ in $i\mathfrak{t}_{\geq 0}^*$, where the list $\xi_1, \xi_2, \ldots, \xi_n$ is the list of eigenvalues (with their multiplicities) of $X$ ordered decreasingly. We denote $\xi$ by $\text{Spec}(X)$.

The “cone” of dominant weights is $\Lambda_{U(n), \geq 0} = \{\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]\}$ where $\lambda_j \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

If $\lambda \in \Lambda_{U(n), \geq 0}$ is such that $\lambda_n \geq 0$, it indexes a finite dimensional irreducible polynomial representation of $GL(n, \mathbb{C})$. The corresponding subset of $\Lambda_{U(n), \geq 0}$ will be denoted by $P\Lambda_{U(n), \geq 0}$. If $\lambda \in P\Lambda_{U(n), \geq 0}$, we also identify $\lambda$ to a Young diagram with $n$ rows. The content of the corresponding diagram is the number of its boxes, that is $k = |\lambda|$. The dominant weight $[k, k, \ldots, k]$ correspond to a rectangular Young diagram with $k$ columns and $n$ rows, and index the one-dimensional representation $\text{det}(g)^k$ of $U(n)$.

Assume now $N \geq n$, then there is a natural injection from $P\Lambda_{U(n), \geq 0}$ to $P\Lambda_{U(N), \geq 0}$ obtained just by adding more zeros on the right of the sequence $\lambda$. We denote by $\hat{\lambda}$ the new sequence so obtained. □

We parameterize $\hat{K}$ by the set of elements $(\pi_\lambda, V^K_\lambda)$ where $\pi_\lambda$ denotes the irreducible finite dimensional representation of highest weight $\lambda \in \Lambda_{K, \geq 0}$ and $V^K_\lambda$ is the finite dimensional space on which $\pi_\lambda$ acts. If there is no ambiguity on the group $K$, we may write simply $V_\lambda$. If $T$ is a torus, we also write $e^\mu$ for the one dimensional representation of $T$ associated to a weight $\mu$ of $T$. The contragredient representation $V^K_\lambda^*$ of $K$ is indexed by the dominant weight $\lambda^* = -w_0(\lambda)$, where $w_0$ is the longest element in $W_k$.

Let $\pi$ be a representation of $K$ acting on a complex vector space $V$. Assume $V$ is a direct sum (possibly infinite) of irreducible finite dimensional complex representations $\pi_\lambda$ of $K$, each of them occurring with finite multiplicities, and let

$$V = \bigoplus_{\lambda \in \hat{K}} m_K(\lambda)V^K_\lambda.$$  

Assume that the restriction of the representation of $K$ to its maximal torus $T$ is also with finite multiplicities $m_T(\mu)$. We have

$$m_K(\lambda) = \sum_{w \in W_K} \epsilon(w) m_T(\lambda + \rho_T - w(\rho_T)).$$  \hspace{1cm} (1)

This is a consequence of the denominator formula:

$$\prod_{\alpha \in \Delta_T^+} (1 - e^\alpha) = \sum_{w \in W_T} \epsilon(w) e^{\rho_T - w(\rho_T)}.$$  \hspace{1cm} (2)
If $X \in \mathfrak{k}$, we denote by $K(X)$ the subgroup of $K$ stabilizing $X$. If $K$ is compact connected, then $K(X)$ is a compact connected subgroup of $K$.

If $\mathcal{H}$ is a Hermitian vector space, define $\mathcal{H}_{\text{pure}} = \{ v \in \mathcal{H}, \langle v, v \rangle = 1 \}$, the set of elements of $\mathcal{H}$ of norm 1. Such an element is called a pure state. A density matrix is a semi-positive definite Hermitian matrix with trace 1. A pure state in the space $\mathbb{C}^2$ is called a qubit, and a pure state in $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ called a $N$ qubit.

For us, a cone $C$ in a real vector space $E$ is a closed subset of $E$ containing 0, invariant by positive homotheties. We consider here only polyhedral cones, that is $\xi \in C$ if and only if $\xi$ satisfies a certain number of linear inequations $\langle X_a, \xi \rangle \geq 0$, with $X_a \in E^*$. When we say cone, it will always mean a polyhedral cone. Most of the time, $E$ will be equipped with a lattice $L$. Consider the dual lattice $L^*$ in $E^*$. We say that $C$ is a rational cone if the $X_a$ are in $L^*$. In other words, if $E = \mathbb{R}^n$, and $L = \mathbb{Z}^n$, a rational cone is a cone defined by inequations with integral coefficients.

We will say that a cone $C$ is solid if $C$ has non empty interior. If $F$ is a finite set of vectors in $E$, we denote by $\text{Cone}(F)$ the cone generated by $F$.

An affine cone is a translate $s + C$ of a cone. An open cone will mean the interior of a polyhedral cone, that is a set determined by strict linear inequations $\langle X_a, \xi \rangle > 0$. If $C$ is a polyhedral cone in a vector space, we say that $C = \bigcup C_a$ is a cone decomposition of $C$ if the $C_a$ are closed polyhedral cones of dimension equal to the dimension of $C$, and if, when $a \neq b$, the intersection $C_a \cap C_b$ is contained in the boundary of the cone $C_a$ and $C_b$.

A face $F$ of a cone $C$ generates a linear space that we call $\text{lin}(F)$. A facet of $C$ is a face of codimension 1 in $C$.

A polytope is a closed compact convex subset of $E$ determined by linear affine inequations $\langle X_a, \xi \rangle \geq c_a$, with $X_a \in E^*$, $c_a \in \mathbb{R}$. If $X_a \in L^*$, and $c_a \in \mathbb{Q}$, the polytope is rational.

### 3 Basic examples for multiplicities

**Example 4. The Littlewood-Richardson coefficients**

We consider two irreducible representations $(\pi_\lambda, V_\lambda)$ and $(\pi_\mu, V_\mu)$ of $K$. Their tensor product $V = V_\lambda \otimes V_\mu$ is a $K \times K$ representation with action defined by: $(\pi_\lambda \otimes \pi_\mu)(k_1, k_2) = \pi_\lambda(k_1) \otimes \pi_\mu(k_2)$. The group $K$ acts diagonally.
on $V$ via $\pi(k) = \pi_\lambda(k) \otimes \pi_\nu(k)$, so we may consider the representation $\pi = \pi_\lambda \otimes \pi_\nu$ restricted to $K$. We can write the classical formula

$$(V_\lambda \otimes V_\mu)|_K = \bigoplus_{\nu \in \Lambda_{\geq 0}} c_{\lambda,\mu}^\nu V_\nu$$

where $c_{\lambda,\mu}^\nu$ is the multiplicity of $\pi_\nu$ in $(\pi_\lambda \otimes \pi_\mu)|_K$.

The numbers $c_{\lambda,\mu}^\nu$ are called Littlewood-Richardson coefficients.

The function $k \to c_{k,\lambda,\mu}^{\nu}$ of $k \in \{0, 1, 2, \ldots\}$ is called the dilated Littlewood-Richardson coefficient. It follows (see Theorem 63) from the $[Q, R] = 0$ theorem that the function $k \to c_{k,\lambda,\mu}^{\nu}$ is given by a quasi-polynomial formula for $k \geq 1$ (polynomial in the case of $U(n)$, see Proposition 31).

We recall that Cochet, [14], [15], has given an algorithm to compute the dilated Littlewood-Richardson coefficient for all classical root systems. This algorithm is available on [53].

**Example 5. The Kronecker coefficients**

Let $N = n_1 n_2 \cdots n_s$ where $n_1, n_2, \ldots, n_s$ are positive integers and write $\mathbb{C}^N = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_s}$. Consider the action of the group $K = U(n_1) \times \cdots \times U(n_s)$ on the complex vector space $\mathbb{C}^N$, where $(k_1, k_2, \ldots, k_s) \in K$ acts by $k_1 \otimes k_2 \otimes \cdots \otimes k_s$ on $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_s}$. Consider the space $V = \bigoplus_{k=0}^\infty V^k$, where $V^k = Sym^k(\mathbb{C}^N)$ is the space of symmetric tensors of degree $k$. Write

$$V = \bigoplus \mathbb{C}(\lambda_1, \lambda_2, \ldots, \lambda_s) V_{\lambda_1}^{U(n_1)} \otimes \cdots \otimes V_{\lambda_s}^{U(n_s)}.$$

Here $\lambda_j$ are polynomial representations of $U(n_j)$ and are indexed by Young diagrams with $n_j$ rows.

Considering the action of the center, we see that all diagrams $\lambda_j$ occurring in $V^k$ have content $k$, so that they also index irreducible representations $\pi_{\lambda_j}$ of the symmetric group $\Sigma_k$. By Schur duality, $\mathbb{C}(\lambda_1, \lambda_2, \ldots, \lambda_s)$ is the multiplicity of the trivial representation of $\Sigma_k$ in $\pi_{\lambda_1} \otimes \cdots \otimes \pi_{\lambda_s}$. The numbers $g(\lambda_1, \lambda_2, \ldots, \lambda_s)$ are called Kronecker coefficients. The function $k \to g(k\lambda_1, k\lambda_2, \ldots, k\lambda_s)$ of $k \in \{0, 1, 2, \ldots\}$ is called the dilated Kronecker coefficient. It follows again from the $[Q, R] = 0$ theorem that the function $k \to g(k\lambda_1, k\lambda_2, \ldots, k\lambda_s)$ is given by a quasi-polynomial formula for $k \geq 1$, (Proposition 31).

Denote by $M = n_2 n_3 \cdots n_s$. In computing Kronecker coefficients, we may assume $n_1 \leq M$, and that $n_1$ is the maximum of the $n_i$. Indeed, if $n_1 \geq M$, the multiplicities $g(\lambda_1, \lambda_2, \ldots, \lambda_s)$ stabilize in the sense that $g(\lambda_1, \lambda_2, \ldots, \lambda_s)$
is non zero only if $\lambda_1$ is obtained from an element in $PA_{U(M), \geq 0}$, by adding more zeroes on its right, and multiplicities coincide. Moreover, we have

$$g(\lambda_1, \lambda_2, \ldots, \lambda_s) = g(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_s).$$

Thus it is sufficient to study Kronecker coefficients in the case where $n_1 \leq M = n_2n_3 \cdots n_s$ and where $n_i$ is the number of rows of the tableau corresponding to $\lambda_i$.

To describe an algorithm to compute the dilated Kronecker coefficient is the main computational objective of this article. We will state some of the previous results and our results in Section 6. Our algorithm uses a branching rule from $U(n_2n_3 \cdots n_s)$ to $U(n_2) \times U(n_3) \times \cdots \times U(n_s)$ in order to reduce (slightly) the size of the problem. Our Maple program is available on [53 □

## 4 Linear representation of $K$ in a Hermitian vector space $\mathcal{H}$

Let $\mathcal{H}$ be a finite dimensional Hermitian vector space provided with a representation of $K$ by unitary transformations.

Assume that $K$ contains the subgroup of homotheties $\{e^{i\theta}Id_H\}$. Consider $V = Sym(\mathcal{H})$, the space of symmetric tensors, so we have

$$Sym(\mathcal{H}) = \bigoplus_{\lambda \in \hat{K}} m^K_{\mathcal{H}}(\lambda)V^K_{\lambda}$$

where $m^K_{\mathcal{H}}(\lambda)$ is finite.

It is clear that if $m^K_{\mathcal{H}}(\alpha)$ and $m^K_{\mathcal{H}}(\beta)$ are non zero, then $m^K_{\mathcal{H}}(\alpha + \beta)$ is non zero. Indeed the product of two non zero vectors $f, g$ in $Sym(\mathcal{H})$ is non zero, and if $f, g$ are highest weight vectors of weights $\alpha, \beta$, the product is a highest weight vector of weight $\alpha + \beta$. So the support of the multiplicity function $m^K_{\mathcal{H}}(\alpha)$ is a “discrete cone” (that is a semi-group). We will relate this discrete cone to the moment map and to the Kirwan cone.

### 4.1 Examples of decomposition of $Sym(\mathcal{H})$

**Example 6.**

Let $\mathcal{H} = \mathbb{C}^n$ and let $K = S^1$ acting by the homothety

$$(z_1, z_2, \ldots, z_n) \rightarrow (e^{i\theta}z_1, e^{i\theta}z_2, \ldots, e^{i\theta}z_n).$$
Then

\[ V = \text{Sym}(\mathcal{H}) = \bigoplus_{k=0}^{\infty} \binom{n-1+k}{n-1} e^{ik\theta} \]

since \( \text{dim}(\text{Sym}^k(\mathcal{H})) = \binom{n-1+k}{n-1} \).

So the multiplicity function \( k \to \text{dim}(\text{Sym}^k(\mathcal{H})) \) is a polynomial function of \( k \).

**Example 7. The Knapsack**

Again, let \( \mathcal{H} = \mathbb{C}^n \) and let \( K = S^1 \) acting by

\[ (z_1, z_2, \ldots, z_n) \to (e^{iA_1\theta} z_1, e^{iA_2\theta} z_2, \ldots, e^{iA_n\theta} z_n), \]

where now the \( A_i \) are any positive integers.

Then \( \text{Sym}(\mathcal{H}) = \sum_k m(k) e^{ik\theta} \) where \( m(k) \) is the number of solutions in non negative integers \( x_i \) of the knapsack equation

\[ A_1 x_1 + A_2 x_2 + \cdots + A_n x_n = k. \]

The computation of the function \( m(k) \) is an "intractable" problem, as illustrated in the lecture [https://www.youtube.com/watch?v=2IbJf4oX0xU&feature=youtu.be](https://www.youtube.com/watch?v=2IbJf4oX0xU&feature=youtu.be) by P. Van Hentenryck. See however [3] for results on its highest coefficients.

**Example 8. Cauchy formula**

Let \( N, n \) be positive integers, and assume \( N \geq n \). Let \( \lambda = [\lambda_1, \ldots, \lambda_n] \) be a sequence of weakly decreasing non negative integers with \( \lambda_n \geq 0 \). We consider \( \lambda \) as an element of \( \Lambda_{U(n), \geq 0} \), that is a dominant polynomial weight for \( U(n) \). To \( \lambda \in \Lambda_{U(n), \geq 0} \) is associated an irreducible representation of \( U(n) \) that we have denoted by \( V_{\lambda}^{U(n)} \). Recall that if \( N \geq n \), then there is a natural injection \( \lambda \to \tilde{\lambda} \) from \( P\Lambda_{U(n), \geq 0} \) to \( P\Lambda_{U(N), \geq 0} \) obtained just by adding more zeros on the right of the sequence \( \lambda \). The decomposition of \( \text{Sym}(\mathbb{C}^n \otimes \mathbb{C}^N) \) with respect to \( U(n) \times U(N) \) (see [33], page 63) is given by \textbf{Cauchy formula}:

\[ \text{Sym}(\mathbb{C}^n \otimes \mathbb{C}^N) = \bigoplus_{\lambda \in P\Lambda_{U(n), \geq 0}} V_{\lambda}^{U(n)} \otimes V_{\tilde{\lambda}}^{U(N)}. \] (3)

**Example 9. Clebsch-Gordan coefficients**
Consider the representation $\Pi$ of $K = U(d) \times U(d) \times U(d)$ on $\mathcal{H} = \mathfrak{gl}(d) \oplus \mathfrak{gl}(d)$ given by

$$\Pi(g, h, k)(A, B) = (gAk^{-1}, hBk^{-1}),$$

where $\mathfrak{gl}(d)$ is the Hilbert space of complex $d \times d$ matrices equipped with the trace inner product $\langle A, B \rangle := \text{Tr} AB^*$. This action factorizes through the center $Z = S^1$ of $U(d)$ embedded in $U(d) \times U(d) \times U(d)$ as a diagonal subgroup $(z \text{Id}, z \text{Id}, z \text{Id})$. We can write

$$\text{Sym}(\mathcal{H}) = \bigoplus_{\lambda, \mu, \nu} c'_{\lambda, \mu} V_{\lambda} \otimes V_{\mu} \otimes V_{\nu}^*.$$ 

Here $\lambda, \mu, \nu$ vary in $\Lambda_{U(d), \geq 0}$ and $c'_{\lambda, \mu}$ is the multiplicity of the representation $V_{\nu}$ in $V_{\lambda} \otimes V_{\mu}$. For $c'_{\lambda, \mu}$ to be non zero, we need $|\lambda| + |\mu| = |\nu|$, as seen by considering the action of the center.

### 4.2 The moment cone

In the following, the compact group $K$ will be fixed, and we denote simply by $T$ its maximal torus, $\mathfrak{t}$ its Lie algebra, etc., as we stated in the Section 2.

Consider the moment map $\mathcal{H} \to i\mathfrak{t}^*$ given by

$$\Phi_K(v)(X) = \langle Xv, v \rangle.$$ 

Here $X \in \mathfrak{t}$, and we have denoted by $v \to Xv$ the infinitesimal action of $X \in \mathfrak{t}$ in $V$ by a anti-hermitian transformation, so $\langle Xv, v \rangle$ is purely imaginary.

**Remark 10.**

We consider $\mathcal{H}$ as a symplectic manifold, with symplectic form $\frac{1}{2i} \langle dv, dv \rangle$ (if $\mathcal{H} = \mathbb{C}^n$ with coordinates $z_k$, this is the form $\frac{1}{2i} \sum_k dz_k d\overline{z}_k$), then $\frac{1}{2i} \Phi_K$ is the moment map for the action of $K$ in $\mathcal{H}$ in the sense of Hamiltonian geometry.

We consider $i\mathfrak{t}^*_\geq 0$ as a subset of $i\mathfrak{t}^*$.

**Definition 11.** Define

- $C_K(\mathcal{H}) = \Phi_K(\mathcal{H}) \cap i\mathfrak{t}^*_\geq 0$
- and
- $\Delta_K(\mathcal{H}) = \Phi_K(\mathcal{H}_{\text{pure}}) \cap i\mathfrak{t}^*_\geq 0$. 

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Assume that $K$ contains the homotheties and let $J \in \mathfrak{t}$ so that the infinitesimal action of $J$ is the identity on $\mathcal{H}$. Then a pure state $\psi$ is such that $\Phi_K(\psi)(J) = 1$.

Recall the following theorem, which is a particular case of Kirwan theorem \cite{Kirwan1984} (a proof of this theorem, following closely Mumford argument, \cite{Mumford1993}, can be found in \cite{Vergne2003}).

**Theorem 12.**

- The set $C_K(\mathcal{H}) = \Phi_K(\mathcal{H}) \cap \mathfrak{t}_\geq$ is a rational polyhedral cone. We call $C_K(\mathcal{H})$ the Kirwan cone.
- The set $\Delta_K(\mathcal{H})$ is a rational polytope. We call $\Delta_K(\mathcal{H})$ the Kirwan polytope.

Thus there exists a finite number of elements $X_a \in \Gamma$ such that

$$C_K(\mathcal{H}) = \{ \xi \in \mathfrak{t}_\geq | \langle X_a, \xi \rangle \geq 0 \}.$$

We say that the inequations $\langle X_a, \xi \rangle \geq 0$ are the inequations of the cone $C_K(\mathcal{H})$. We may normalize $X_a$ to be a primitive element in $\Gamma$, the dual lattice to $\Lambda$. The set $\Delta_K(\mathcal{H})$ is the intersection of $C_K(\mathcal{H})$ with the affine hyperplane $\langle J, \xi \rangle = 1$ and

$$C_K(\mathcal{H}) = \mathbb{R}_\geq \Delta_K(\mathcal{H})$$

is the cone over the Kirwan polytope.

It is in general quite difficult to determine the explicit inequations of the cone $C_K(\mathcal{H})$. An algorithm to describe the inequations of this cone, based on Ressayre’s notion of dominant pairs, \cite{Ressayre2002}, is given in Vergne-Walter, \cite{Vergne2003}. Let us first give the emblematic example of the Horn cone.

**Example 13. Horn problem**

Consider 3 Hermitian matrices $X, Y, Z$. Let $\text{Spec}(X), \text{Spec}(Y), \text{Spec}(Z)$ denote the list of eigenvalues of the Hermitian matrices $X, Y, Z$. The Horn inequations describe the range of the triple $(\text{Spec}(X), \text{Spec}(Y), \text{Spec}(Z))$, when the 3 matrices $X, Y, Z$ are constrained by the relation $X + Y + Z = 0$. This problem is related to the moment map for Example 9 as follows.

Consider the representation of $K = U(d) \times U(d) \times U(d)$ on $\mathcal{H} = \mathfrak{gl}(d) \oplus \mathfrak{gl}(d)$. The moment map is

$$\Phi_K(A, B) = (AA^*, BB^*, -A^*A - B^*B).$$
Since any non-negative Hermitian matrix can be written in the form $A^*A$ and since the spectra of $AA^*$ and $A^*A$ are equal, the moment cone is equal to

$$C_K(\mathcal{H}) := \{(\text{Spec}(X), \text{Spec}(Y), \text{Spec}(Z)) : X, Y \geq 0, Z \leq 0, X + Y + Z = 0\},$$

As proved in [31], [35] (see also [7]), the equations of $C_K(\mathcal{H})$ are given by the inductive system of inequalities conjectured by Horn [24]. These ineq uations are of the following form. Let $I, J, K$ be subsets of $[1, 2, \ldots, d]$, all three of them of cardinal $r < d$. Let $E_I$ be the diagonal Hermitian matrix with $r$ eigenvalues 1 in positions $I$, and others being 0. Then the triple $(E_I, E_J, E_K)$ gives rise to the inequation

$$\text{Tr}(E_ID_1) + \text{Tr}(E_JD_2) + \text{Tr}(E_KD_3) \leq 0 \quad (4)$$
on triples $(D_1, D_2, D_3)$ of Hermitian diagonal matrices.

Horn defined inductively, for every $r<d$, a set $H(r,d)$ of triples $(I,J,K)$ of subsets of cardinal $r$ of $[1,2,\ldots,d]$. Then $C_K(\mathcal{H})$ is described by the inequations (4) above, for all $(I,J,K) \in H(r,d)$ and all $r < d$, and the equation $\text{Tr}(D_1) + \text{Tr}(D_2) + \text{Tr}(D_3) = 0$.

We now consider examples related to Kronecker coefficients.

**Example 14. (Bipartite case).**

We return to Example 8. Consider $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^N$ with the action of $U(n) \times U(N)$. Using the Hermitian inner product, we identify $A \in \mathcal{H}$ to a matrix $A : \mathbb{C}^n \to \mathbb{C}^N$. Then the moment map is given by

$$\Phi_K(A) = [AA^*, A^*A]$$

with value Hermitian matrices of size $n$ and size $N$ respectively. The matrices $AA^*$ and $A^*A$ have the same non zero eigenvalues. Assume that $N \geq n$. Define

$$\text{Pit}^*_{u(n), \geq 0} = \{\xi = [\xi_1, \ldots, \xi_n] : \xi_1 \geq \xi_2 \geq \cdots \geq \xi_n \geq 0\},$$

and consider $\text{Pit}^*_{u(n), \geq 0}$ as a subset of the positive Weyl chamber $it^*_{\geq 0}$ for $U(n)$. There is a natural injection from $\text{Pit}^*_{u(n), \geq 0}$ to $\text{Pit}^*_{u(N), \geq 0}$ obtained just by adding more zeros on the right of the sequence $\xi$. We denote by $\tilde{\xi}$ the new sequence so obtained. Then we see that the Kirwan cone $C_K(\mathcal{H})$ is the ”diagonal” $(\xi, \tilde{\xi})$ with $\xi \in \text{Pit}^*_{u(n), \geq 0}$.
In the above example, we see that the cone $C_K(H)$ may have empty interior in $it^*_0$. The following general result holds.

**Lemma 15.** The cone $C_K(H)$ is a solid cone if and only if there exists $v \in H$ so that the stabilizer $K_v$ of $v$ is a finite group.

Return to the Kronecker case.

Consider $H = C^{n_1} \otimes C^{n_2} \otimes \cdots \otimes C^{n_s}$ with action of $K = U(n_1) \times \cdots \times U(n_s)$ on $H$.

We may assume $n_1 \geq n_2 \geq \cdots \geq n_s$, and let $M = n_2 n_3 \cdots n_s$. The corresponding Kirwan cone $C_K(H)$ is a subset of $\bigoplus_{i=1}^s \Pi_{t^*_0} \oplus \Pi_{t^*_0} \oplus \cdots \Pi_{t^*_0}$.

Then if $n_1 \geq M$, this cone stabilizes in the sense that $C_K(H)$ coincides with the cone associated with the sequence $(M, n_2, n_3, \ldots, n_s)$ embedded by sending $\Pi_{t^*_0}$ to $\Pi_{t^*_0}$ by the map $\xi \mapsto \tilde{\xi}$. Furthermore, considering the action of the center, we see that $C_K(H)$ is contained in the subspace $E = \{(y_1, \ldots, y_s)\}$ of $it^*_0$ defined by the $s - 1$ linear equalities $\text{Tr}(y_1) = \text{Tr}(y_2) = \cdots = \text{Tr}(y_s)$.

For later use, we recall the following result (see [52]).

**Proposition 16.** Let $H = C^{n_1} \otimes C^{n_2} \otimes \cdots \otimes C^{n_s}$ with action of $K = U(n_1) \times \cdots \times U(n_s)$ on $H$. Assume that all $n_s$ are greater or equal to 2, and that $s$ is greater or equal to 3. Assume also that $n_1 = \max(n_i)$ and $n_1 \leq n_2 \cdots n_s$. Then $C_K(H)$ is solid in the subspace $E$.

A few examples of $(n_1, n_2, \ldots, n_s)$ where the inequations of the cone $C_K(H)$ are known are:

- $(2, 2, \ldots, 2)$ (with any number $N$ of 2: the cone of $N$-qubits), (Higuchi–Sudbery-Sultz [23]).
- $(3, 3, 3)$ (Franz [20]),
- $(4, 2, 2)$ (Briand [10], Bravyi [9]),
- $(6, 3, 2)$, $(9, 3, 3)$ (Klyachko [31]),
- $(4, 4, 4)$ and $(12, 3, 2, 2)$ (Vergne-Walter, [52]) (this last case being incorrect in Klyachko).

The general case seems for the moment out of reach.

**Example 17.**
Let us write explicitly Higuchi-Sudbery-Szulc description of the cone of \( N \)-qubits (see also [9]). Consider \( \lambda_1 = [\lambda_1^1, \lambda_2^1, \ldots, \lambda_N^1], \lambda_2 = [\lambda_1^2, \lambda_2^2, \ldots, \lambda_N^2] \) a sequence of \( N \) elements of \( \text{Pit}^*_{\text{u}(2)} \geq 0 \) (that is \( \lambda_j^1 \geq \lambda_j^2 \geq 0 \)). We assume that \( \text{Tr}(\lambda_1) = \text{Tr}(\lambda_2) = \cdots = \text{Tr}(\lambda_N) \). Then \((\lambda_1, \ldots, \lambda_N) \in C_{U(2) \times \cdots \times U(2)}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2)\) if and only if

\[
\lambda_j^2 \leq \sum_{k \neq j} \lambda_k^2,
\]

for any \( j = 1, 2, \ldots, N \).

\( \square \)

**Example 18. Quantum marginals**

Let \( A, B, C \) be integers. If \( v \in \mathbb{C}^A \otimes \mathbb{C}^B \otimes \mathbb{C}^C \) is a pure state, its quantum marginals are defined as follows. Consider \( v \) as an operator \( v : \mathbb{C}^A \rightarrow \mathbb{C}^B \otimes \mathbb{C}^C \) and define \( \rho_A(v) = v^*v \). Define \( \rho_B(v), \rho_C(v) \) similarly. Then \( \rho_A(v), \rho_B(v), \rho_C(v) \) are densities matrices of size \( A, B, C \) respectively. They are called the quantum marginals of \( v \).

Consider \( \mathcal{H} = \mathbb{C}^A \otimes \mathbb{C}^B \otimes \mathbb{C}^C \), with action of \( K = U(A) \times U(B) \times U(C) \). The moment map \( \Phi_K : \mathbb{C}^A \otimes \mathbb{C}^B \otimes \mathbb{C}^C \rightarrow iu(A)^* \oplus iu(B)^* \oplus iu(C)^* \) is

\[
\Phi_K(v) = (\rho_A(v), \rho_B(v), \rho_C(v)).
\]

Given 3 densities matrices \( \rho_A, \rho_B, \rho_C \), the quantum marginal problem is to determine if there exists a pure state \( v \in \mathbb{C}^A \otimes \mathbb{C}^B \otimes \mathbb{C}^C \) with quantum marginals \( \rho_A, \rho_B, \rho_C \). We see that this pure state exists if and only if \((\rho_A, \rho_B, \rho_C)\) is in the image of the moment map \( \Phi_K \). It is thus necessary and sufficient that the triple of spectrums \((\text{Spec}(\rho_A), \text{Spec}(\rho_B), \text{Spec}(\rho_C))\) of \((\rho_A, \rho_B, \rho_C)\) satisfy a certain number of linear inequalities. Unfortunately, there is not a good understanding of what are these inequalities, except in the few low dimensional cases quoted previously.

### 4.3 Duistermaat-Heckman measure on the moment cone

For simplicity, we assume that the moment cone \( C_K(\mathcal{H}) \) intersects the interior of the Weyl chamber. Of course, this is the case when \( C_K(\mathcal{H}) \) is solid. We also assume (we can always reduce easily to this case) that the kernel of the homomorphism \( K \rightarrow U(\mathcal{H}) \) is trivial.
Definition 19. Define
\[ r = \dim_{\mathbb{C}}(\mathcal{H}) - |\Delta^+_t| - \dim C_K(\mathcal{H}). \]

Then, if the cone \( C_K(\mathcal{H}) \) is solid,
\[ r = \dim_{\mathbb{C}}(\mathcal{H}) - |\Delta^+_t| - \dim \mathfrak{t}. \]

Consider the Lebesgue measure \( dv \) on \( \mathcal{H} \) (considered as a real vector space). If \( \xi \in i\mathfrak{k}^* \), there is a natural measure \( \beta_\xi \) on the coadjoint orbit \( K\xi \) of \( \xi \) determined by the symplectic structure of \( K\xi \) (see [29]). Consider the Duistermaat-Heckman measure \( DH_K^H(\xi) \) supported on the polyhedral cone \( C_K(\mathcal{H}) \), determined by
\[ \int_V f(\Phi_K(v)) dv = \int_{\xi \in C_K(\mathcal{H})} \left( \int_{K\xi} f d\beta_\xi \right) DH_K^H(\xi). \]

Here \( f \) is a compactly supported continuous function on \( i\mathfrak{k}^* \). In short, we divide the push forward \( (\Phi_K)_*(dv) \) of the Lebesgue measure \( dv \) on \( \mathcal{H} \) by the Kirillov measure of the orbits in the image. This measure is strictly positive on the relative interior of \( C_K(\mathcal{H}) \).

Consider the Lebesgue measure \( d\xi \) on the vector space \( \text{lin}(C_K(\mathcal{H})) \) spanned by \( C_K(\mathcal{H}) \). Here we normalize \( d\xi \) so that it gives mass 1 to a fundamental domain of \( \Lambda \cap \text{lin}(C_k(\mathcal{H})) \).

The following theorem follows from Duistermaat-Heckman [19].

**Theorem 20.** There exists a cone decomposition \( C_K(\mathcal{H}) = \bigcup_{a \in F} c_a \), in closed polyhedral cones \( c_a \), and for each \( a \), there exists a homogeneous polynomial function \( d_{h_K}^H(\xi) \) of degree \( r \) on \( \text{lin}(C_K(\mathcal{H})) \) such that
\[ DH_K^H(\xi) = d_{h_K}^H(\xi) d\xi \]
if \( \xi \in c_a \).

We denote by \( d_{h_K}^H \) the function on \( C_K(\mathcal{H}) = \bigcup c_a \) such that \( d_{h_K}^H = d_{h_K}^H(\xi) \) on \( c_a \). So we obtain a piecewise polynomial function \( d_{h_K}^H \) with support \( C_K(\mathcal{H}) \) and continuous on \( C_K(\mathcal{H}) \). We call the \( c_a \) chambers of polynomiality (for the Duistermaat-Heckman measure). It is already quite difficult to determine the cone \( C_K(\mathcal{H}) \), so even more so to determine the chambers of polynomiality \( c_a \).

The function \( d_{h_K}^H(\xi) \) is related to the volume of the reduced fiber of \( \mathcal{H} \).
Definition 21. Let $\xi \in \mathfrak{t}^*$, we define the reduced fiber of $\mathcal{H}$ at $\xi$ by $\mathcal{H}_{\text{red},\xi} = \Phi_K^{-1}(K\xi)/K$.

In other words, the reduced space at $\xi$ consists in all orbits $K\nu$ of $K$ in $\mathcal{H}$ projecting on the orbit $K\xi$ under the moment map. Via the identification of $\mathcal{H}_{\text{red},\xi}$ with the GIT (geometric invariant theory) quotient of semi-stable orbits under $K_\mathbb{C}$, the reduced space can be provided with a structure of projective variety (see [43]).

If all orbits of $K$ in $\Phi_K^{-1}(K\xi)$ have the same dimension, the reduced space is an orbifold. We say that $\xi$ is a quasi-regular value. In particular, if the map $k \to k\nu$ is injective for all $\nu \in \mathcal{H}$ projecting on $\xi$, we say that the action of $K$ on $\Phi_K^{-1}(K\xi)$ is free. In this case $\xi$ is a regular value of $\Phi_K$, and $\mathcal{H}_{\text{red},\xi}$ is a smooth manifold. Furthermore, $\mathcal{H}_{\text{red},\xi}$ inherits a symplectic structure $\Omega_\xi$ from the symplectic form $\Omega$ on $\mathcal{H}$: the restriction of $\Omega$ to $\Phi_K^{-1}(K\xi)$ is the pull back of a form $\Omega_\xi$ on $\mathcal{H}_{\text{red},\xi}$. For $\xi$ in the relative interior of the Kirwan cone, and a quasi-regular value, then the reduced space $\mathcal{H}_{\text{red},\xi}$ is of dimension $r$. Duistermaat-Heckman theorem implies that $dh^\mathcal{H}_K(\xi)$ is the volume of the symplectic space $\mathcal{H}_{\text{red},\xi}$. So when we restrict the function $dh^\mathcal{H}_K$ to the line $t\xi$ ($\xi$ in the interior of $C_K(\mathcal{H})$ and quasi-regular), the function $t \to dh^\mathcal{H}_K(t\xi)$ is homogeneous of degree equal to $r$. This is also true without the quasi-regularity assumption.

Lemma 22. If $\xi$ is in the relative interior of the cone $C_K(\mathcal{H})$, then $dh^\mathcal{H}_K(t\xi) = t^r dh^\mathcal{H}_K(\xi)$.

So the homogeneous degree on a line stays constant for interior lines. But typically, the function $dh^\mathcal{H}_K$ vanishes on the boundary, except if it is constant on $C_K(\mathcal{H})$.

Example 23. (The Bloch sphere)

Let us consider a very simple example.

Let $T$ be a two dimensional torus, with Lie algebra $\mathfrak{t} = \mathbb{R}J_1 \oplus \mathbb{R}J_2$. Let $\mathfrak{t}^*$ with dual basis $J^1, J^2$. We consider the diagonal action of $T$ on $\mathcal{H} = \mathbb{C}^2$, by $\exp(\theta_1 J_1 + \theta_2 J_2)(z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$. Thus the weights $\phi_1, \phi_2$ of the action of $T$ on $\mathbb{C}^2$ are $\phi_1 = iJ^1, \phi_2 = iJ^2$.

The space of pure states divided by the action of $(e^{i\theta}, e^{i\theta})$ is the Bloch sphere. The Kirwan polytope is the segment $[\phi_1, \phi_2]$ in $\mathbb{R}\phi_1 \oplus \mathbb{R}\phi_2 = \mathfrak{t}^*$. The Kirwan cone is the cone $\mathbb{R}_{\geq 0}\phi_1 \oplus \mathbb{R}_{\geq 0}\phi_2$. The reduced fibers are points, and the Duistermaat-Heckman measure is the characteristic function on this
cone. This is just Archimedes result for which the area on the sphere between two latitudes depends only of the difference of their heights on the $z$ axes as one can see looking at the picture Figure 1.

![Figure 1: The Kirwan polytope](image)

### 4.4 Moment map and multiplicities

We continue (for simplicity) to assume that $C_K(\mathcal{H})$ intersects the interior of the Weyl chamber and that the kernel of the homomorphism $K \to U(\mathcal{H})$ is trivial.

Consider the decomposition

$$Sym(\mathcal{H}) = \bigoplus_{\lambda \in \hat{K}} m^\mathcal{H}_{K}(\lambda)V^K_{\lambda}.$$

The cone $C_K(\mathcal{H})$ is related to the multiplicities through the following basic result, which is a particular case of Mumford theorem [44](a proof of Mumford theorem, following closely Mumford argument, can be found in [8]).

**Proposition 24.** We have $m^\mathcal{H}_{K}(\lambda) = 0$ if $\lambda \notin C_K(\mathcal{H})$. Conversely, if $\lambda$ is a dominant weight belonging to $C_K(\mathcal{H})$, there exists an integer $k > 0$ such that $m^\mathcal{H}_{K}(k\lambda)$ is non zero.

Thus the support of the function $m^\mathcal{H}_{K}(\lambda)$ is contained in the Kirwan polyhedron $C_K(\mathcal{H})$ and its asymptotic support is exactly the cone $C_K(\mathcal{H})$.

**Example 25.** (The Cauchy formula, next)
Return to the example of the Cauchy formula. Let $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^N$ under the action of $U(n) \times U(N)$. The Kirwan cone has been determined in Example 14. We see that the multiplicity function determined in Example 8 is supported exactly on the set $\Lambda \cap \mathcal{C}_K(\mathcal{H})$ (and with multiplicity 1).

It is interesting to understand, for a given $\lambda \in \mathcal{C}_K(\mathcal{H}) \cap \Lambda_{\geq 0}$, what is the smallest positive $k$ such that $m_{k\lambda}^H(k\lambda) \neq 0$. We call this $k$ the saturation factor. We will give one example, Example 84, where $k = 2$. We would have liked to find larger $k$, but we could not succeed.

We now describe the nature of the function $m_{k\lambda}^H(\lambda)$ on $\mathcal{C}_K(\mathcal{H})$. For this, we need to recall the definition of a periodic polynomial function.

Consider a real vector space $E$ with a lattice $L$. We can think of $E = \mathbb{R}^d$ and $L = \mathbb{Z}^d$. Let $\mathcal{C}(L)$ be the space of functions on the lattice $L$. The restriction to $L$ of a polynomial function on $E$ will be called a polynomial function on $L$. Given an integer $q$, a function on $L/qL$ will be called a periodic function on $L$ of period $q$. If $q = 1$ we just have a constant function.

**Definition 26.** A periodic polynomial function on $L$ is a function on $L$ which is a linear combination of products of polynomial functions with periodic functions.

We also say that a periodic polynomial function $p$ is a quasi polynomial function.

The space of periodic polynomial functions is graded: if $p(\lambda) = \sum_{i,j} c_i(\lambda) p_j(\lambda)$ where the polynomials $p_j$ are homogeneous of degree $k$ and the functions $c_i$ periodic, we say that $p$ is homogeneous of degree $k$. As for polynomials, we say that $p$ is of degree $k$ if $p$ is a sum of homogeneous terms of degree less or equal to $k$, and the term of degree $k$ is non zero. If all functions $c_i(\lambda)$ are of period $q$, we say that $p(\lambda)$ is of period $q$.

**Example 27.**

$$m(k) = \frac{1}{2} k^2 + k + \frac{3}{4} + \frac{1}{4} (-1)^k$$

is a periodic polynomial function of $k \in \mathbb{Z}$ and of degree 2.

In other words, if $p(\lambda)$ is of period $q$, for any $\lambda_0$, the function $\lambda \mapsto p(\lambda_0 + q\lambda)$ is a polynomial function on $L$. So we can represent a periodic
polynomial function as a family of polynomials indexed by $L/qL$. If $q$ is very large, the above description is not efficient (the numbers of cosets being quite large). Nonetheless we will give an example of this description in Section 6.3.

In practice, $p$ will be naturally obtained as a sum of quasi-polynomial functions $p_1, p_2, \ldots, p_u$ of periods $q_1, q_2, \ldots, q_u$. So $p$ is of period $q$ where $q$ is the least common multiple of $q_1, q_2, \ldots, q_u$. However, it is already more efficient to keep $p$ as represented as $\sum p_i$, the number of cosets needed to describe each $p_i$ being $q_i$, and $\sum q_i$ is usually much smaller that $q$. We thus will say that the set of periods of the quasi-polynomial function $p$ is the set \{ $q_1, q_2, \ldots, q_u$ \}.

An efficient way to represent periodic functions by step-polynomials is given in \cite{51}, \cite{4}, as in the example we gave in the introduction.

In the case of one variable, we can give the following characterization of quasi-polynomial functions $p(k)$. If the function $p(k)$ is quasi-polynomial, its generating series $\sum_{k=0}^{\infty} p(k)t^k$ is the Taylor expansion for $|t| < 1$ of a rational function $R(t) = \frac{P(t)}{\prod_{i=1}^{s} (1-t^{a_i})}$, where the $a_i$ are integers, and $P(t)$ a polynomial in $t$ of degree strictly less than $\sum a_i$. The correspondence is as follows. Consider a quasi polynomial $p(k)$ of period $q$, equal to 0 on all cosets except the coset $f + q\mathbb{Z}$, with $0 \leq f < q$. Write the polynomial function $j \rightarrow p(f + qj)$ of degree $R$ in terms of binomials: $p(f + qj) = \sum_{n=0}^{R} a(n)(j + n)$. Then

$$\sum_{j=0}^{\infty} p(f + qj)t^{f + qj} = tf \sum_{n=0}^{R} a(n) \frac{1}{(1 - t)^{n+1}}.$$

(Given a rational function $R(t) = \frac{1}{\prod_{i=1}^{s} (1-t^{a_i})}$, an algorithm (in polynomial time if the number $s$ of factors is fixed) to compute explicitly the corresponding quasipolynomial function $p(k)$ such that $\sum k p(k)t^k = R(t)$ is given in LattE integrale \cite{5}.)

As in the case of polynomial functions, to test if two quasi polynomials functions are equal, it is sufficient to test it on a sufficiently large subset.

**Lemma 28.** Let $p_1, p_2$ be two quasi polynomial functions. If there exists a open cone $\tau$, such that $p_1, p_2$ agree on a translate $s + \tau$ of $\tau$, then $p_1 = p_2$. 

Recall that $r = \dim_{\mathbb{C}}(\mathcal{H}) - |\Delta_1^+| - \dim C_K(\mathcal{H}).$
The following theorem can be considered as a quantum analogue of Duistermaat-Heckman theorem (Theorem 20).

**Theorem 29.** Consider the decomposition of the cone $C_K(\mathcal{H}) = \cup_a c_a$, in the closed cones of polynomiality $c_a$, then, for each $a$, there exists a quasi polynomial function $p^H_{K,a}$ of degree $r$ on the lattice $\Lambda$ such that if $\lambda \in c_a \cap \Lambda$

$$m^H_K(\lambda) = p^H_{K,a}(\lambda).$$

This theorem is Meinrenken-Sjamaar theorem for the particular case of the action of $K$ in the projective space associated to $\mathcal{H}$. In this work, the function $m^H_K(\lambda)$ is related to the Kawasaki-Riemann-Roch number [27] (suitably defined) of the reduced symplectic space $\mathcal{H}_{red,\lambda} = \Phi^{-1}_K(K\lambda)/K$. One may look in [45] for a different proof.

Assume the cone $C_K(\mathcal{H})$ is solid, and let $\mathcal{H}^{fin}$ be the open subset of $\mathcal{H}$ consisting of elements with finite stabilizers. Let $c_a$ be a cone of polynomiality, choose a regular value $\xi \in c_a$ of the moment map, and let $s^a_v$ be the orders of the stabilizers $K_v$, for $v \in \Phi^{-1}_K(\xi)$. Then the set of periods of the quasi-polynomial function $p^H_{K,a}$ is contained in the set $\{s^a_v\}$. In particular, if the action of $K$ on $\mathcal{H}$ has trivial generic stabilizer, the reduced spaces are smooth for regular values, and all the functions $p^H_{K,a}$ are polynomials.

Finding the set of periods is already quite non trivial. It is related to the computation of the Picard group of the reduced spaces [37]. In the examples we study here, we compute a set containing the set of periods by brutal force, as we consider here relatively small values of the rank of $K$ and the dimension of $\mathcal{H}$.

So, for any dominant weight $\lambda$ belonging to $C_K(\mathcal{H})$, the function $k \to m^H_K(k\lambda)$ is of the form: $m^H_K(k\lambda) = \sum_{i=0}^N c_i(k)k^i$ where $c_i(k)$ are periodic functions of $k$. This formula is valid for any $k \geq 0$ (so $c_0(0) = 1$). The highest degree term for which this function is non zero will be called the degree of the quasi polynomial function $m^H_K(k\lambda)$. We discuss this degree in the next subsection.

If $\lambda$ is not in $C_K(\mathcal{H})$, the function $k \to m^H_K(k\lambda)$ is just equal to 0, except for $k = 0$, and conversely, if this function is not zero, then $\lambda \in C_K(\mathcal{H})$.

An interesting particular case is when the center of $\mathfrak{k}$ is one dimensional, $\chi$ a weight of $T$ vanishing on $[\mathfrak{t}, \mathfrak{t}] \cap \mathfrak{t}$, and such that $\chi(J) = 1$. So $\chi$ indexes a one dimensional representation of $K$, and $m^H_K(k\chi) = \dim([\text{Sym}^k(\mathcal{H})][K,K])$. The corresponding generating series $\sum_{k=0}^{\infty} m^H_K(k\chi)t^k$ is the Hilbert series $HS(t)$ of the ring of invariant polynomials on $\mathcal{H}$ under $[K, K]$. So, the degree of the
function \( k \to m_K^R(k\chi) \) is the maximal number of algebraically independent invariants. There is a non-trivial invariant \( P \) (different from a constant) if and only if 0 belongs to the Kirwan polytope of the projective space \( P(\mathcal{H}) \): equivalently, if the line \( \mathbb{R}_{\geq 0}\chi \) is an edge of the cone \( C_K(\mathcal{H}) \). This is one of the first instance of the \([Q,R] = 0\) theorem, and this basic case follows from Mumford description \([13]\) of the GIT quotient. The function \( k \to m_K^R(k\chi) \) is a quasi polynomial on the full positive line \( k \geq 0 \) (not only for \( k \) sufficiently large). It is in general difficult to decide if 0 belongs to the Kirwan polytope of the projective space \( P(\mathcal{H}) \), and even more so to determine the degree of the quasi-polynomial function \( k \to m_K^R(k\chi) \).

The two following propositions are particular cases of Meinrenken-Sjamaar result.

**Proposition 30.** Let \((\lambda, \mu, \nu)\) be a triple of dominant weights for \( U(d) \), belonging to the Horn cone. The dilated Clebsh-Gordon coefficient \( k \to c_{k\lambda,k\mu}^{k\nu} \) is a polynomial function of \( k \) for \( k \geq 0 \).

(If \((\lambda, \mu, \nu)\) is not in the Horn cone, this function is identically 0, except if \( k = 0 \)).

Indeed, we have already seen the quasi-polynomial nature of \( k \to c_{k\lambda,k\mu}^{k\nu} \) for \( k \geq 0 \). Now it is easy to see that the generic stabilizer of the action of \( U(d) \times U(d) \times U(d) \) on \( \mathcal{H} \) is the center of \( U(d) \) (embedded diagonally). Here is an explicit proof. Let \( D_1, D_2 \) be two positive definite Hermitian matrices, with distinct eigenvalues. We assume \( D_1 \) diagonalizable in the basis \( e_1, e_2, \ldots, e_d \), while \( D_2 \) diagonalizable in a basis containing \( \sum e_i \). Consider \( A = D_1^{1/2}, B = D_2^{1/2} \) in \( \mathfrak{gl}(d) \) and \((g, h, k) \in U(d) \times U(d) \times U(d) \) such that \( gAk^{-1} = A, hBk^{-1} = B \). We obtain \( AA^* = D_1 = gD_1g^{-1} \), and \( BB^* = D_2 = hD_2h^{-1} \). Thus \( g \) commutes with \( D_1 \), so is diagonalizable in the standard basis. So \( g \) commutes with \( A \), and \( gAk^{-1} = A \) implies \( g = k \). Similarly \( h = k \), and \( h \) is diagonalizable in the basis diagonalizing \( D_2 \). So we have \( g = h = k \).

But \( g \) being diagonalizable in the basis \( e_i \), and in a basis containing \( \sum e_i \), the equation \( g(e_1 + e_2 + \cdots + e_d) = a(e_1 + e_2 + \cdots + e_d) \) implies that all eigenvalues of \( g \) are equal.

A proof of Proposition 30 by more combinatorial methods is given in \([18]\).

We now consider the action of \( K = U(n_1) \times U(n_2) \times \cdots \times U(n_s) \) in \( \mathcal{H} = \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_s} \).
**Proposition 31.** Let \( \lambda_i \in P_{U(n_i),\geq 0} \) be polynomial dominant weights for \( U(n_i) \). We assume that \((\lambda_1, \ldots, \lambda_s)\) belongs to the Kronecker cone \( C_K(\mathcal{H}) \). The dilated Kronecker coefficient \( k \rightarrow g(k\lambda_1, \ldots, k\lambda_s) \) is a quasi-polynomial function of \( k \) for \( k \geq 0 \).

This result is asserted in [42].

(Remark that if \((\lambda_1, \ldots, \lambda_s)\) is not in \( C_K(\mathcal{H}) \), the dilated Kronecker coefficient is identically 0, except if \( k = 0 \)).

In the Kronecker case, we do not know the set of periods of the dilated Kronecker coefficient.

**Example 32.**

For the case \( n_1 = n_2 = n_3 = 3 \), we find quasi polynomials with set of periods \( \{1, 2, 3, 4\} \) (see [6,2], leading to polynomial behavior on cosets \( f + 12\mathbb{Z} \).

For the 4 qubits case \( n_1 = n_2 = n_3 = n_4 = 2 \), we find quasi polynomials with set of periods \( \{1, 2, 3, 4\} \) (see [6,2], leading to polynomial behavior on cosets \( f + 6\mathbb{Z} \).

For the 5 qubits case \( n_1 = n_2 = n_3 = n_4 = n_5 = 2 \), we find quasi polynomials with set of periods \( \{1, 2, 3, 4, 5\} \) (see [6,2], leading to polynomial behavior on cosets \( f + 60\mathbb{Z} \).

\[\square\]

It would be desirable to describe \( m_H^K(\mu) \) as the number of integral points in a polytope. This result would imply directly the quasi polynomial behavior of \( m_H^K(k\mu) \). For Clebsh-Gordan coefficients, this is the hive polytope of Knutson-Tao [35]. The corresponding computation is implemented in [17]. For Kronecker coefficients, no general result is known.

Let us end this section by recalling the behavior of multiplicities on the faces of \( C_K(\mathcal{H}) \).

Consider a face \( F \) of the cone \( C_K(\mathcal{H}) \). If \( F \) contains elements of \( C_K(\mathcal{H}) \cap it^*_+ \), we then say that \( F \) is a regular face. If \( F \) is a regular face, let \( T_F \) be the torus with Lie algebra \( \text{lin}(F)^{\text{perp}} \). Let \( K_F \) be the centralizer of \( T_F \), with Lie algebra \( \mathfrak{t}_F \), and system of positive roots \( \Delta^+_F \). Let \( \mathcal{H}(T_F) \) be the subspace of \( \mathcal{H} \) stable by \( T_F \). This is a Hamiltonian space for \( K_F \).

The following reduction formula holds.

**Theorem 33.** Let \( F \) be a regular face of \( C_K(\mathcal{H}) \). Then for any \( \lambda \in F \), we have

\[ m_H^K(\lambda) = m_H^{K_F}(\lambda). \]
Again, this theorem is a corollary of the \([Q,R] = 0\) general theorem of Meinrenken-Sjamaar, (see also [15]).

Return to the example of the Horn cone. Let us consider a Horn triple \((I,J,K)\) such that the equation (4) determines a facet \(F_{I,J,K}\) of the Horn cone. Let \(I^c, J^c, K^c\) the complement of \(I, J, K\) in \([1, \ldots, d]\). Then the centralizer of the element \((E_I, E_J, E_K)\) is \((U(I) \times U(I^c)) \times (U(J) \times U(J^c)) \times (U(K) \times U(K^c))\). Here we denoted by \(U(I)\), the unitary group acting on \(C_I = \oplus_{i \in I} C_{e_i}\).

So when \(\lambda, \mu, \nu\) belong to the facet \(F_{I,J,K}\), the corresponding Clebsh-Gordan coefficient is the product of the Clebsh-Gordan coefficient relative to the group \(U(r)\) with the Clebsh-Gordan coefficient relative to the group \(U(d - r)\). So Theorem 33 implies the factorization theorem of [28], proved by combinatorial means.

**4.5 Comments on degrees**

We continue to assume that the moment cone \(C_K(\mathcal{H})\) intersects the interior of the Weyl chamber and that the kernel of the homomorphism \(K \to U(\mathcal{H})\) is trivial.

Recall the definition of

\[
    r = \dim_C(\mathcal{H}) - |\Delta^+_\mathfrak{i}| - \dim C_K(\mathcal{H}).
\]

Consider \(c_a\) a cone for polynomiality and the two associated functions \(d_{K,a}^H, p_{K,a}^H\). The function \(d_{K,a}^H\) is a polynomial function on \(\mathfrak{i}^\ast\), while \(p_{K,a}^H\) is a quasi polynomial function on \(\Lambda\). Then the two functions \(d_{K,a}^H, p_{K,a}^H\) are both of degree \(r\). In the case where \(K\) is abelian, (partition functions), the term of \(p_{K,a}^H\) of highest degree \(r\) is polynomial. In general, this is not usually true. The term of \(p_{K,a}^H\) of highest degree \(r\) might be a quasi polynomial and not a polynomial. For example, \(r\) might be equal to 0, however the functions \(p_{K,a}^H\) might be periodic function of period \(q > 1\).

The following example was communicated to us by M. Walter.

**Example 34.**

Let us consider \(U(2)\) and its torus \(T\). Let \(\mathfrak{i}^\ast\) with basis \(\epsilon_1 = [1, 0], \epsilon_2 = [0, 1]\) and the lattice of weights \(\Lambda = \{\lambda = [\lambda_1, \lambda_2]; \lambda_1, \lambda_2 \in \mathbb{Z}\}\).

Consider \(K = U(2)/\{\pm 1\}\) and its torus \(T_K\). Then the lattice of weights is \(\Lambda_K = \{\lambda = [\lambda_1, \lambda_2]; \lambda_1, \lambda_2 \in \mathbb{Z}; \lambda_1 + \lambda_2 \text{ even}\}\).
Consider the 3-dimensional irreducible representation of $K$ on $\mathcal{H} = \mathbb{C}^3$ (with trivial kernel), with weights $[2, 0], [1, 1], [0, 2]$.

The Kirwan cone $C_T(\mathcal{H})$ is $\mathbb{R}_{\geq 0}\epsilon_1 \oplus \mathbb{R}_{\geq 0}\epsilon_2$. Let $c_1$ be the closed cone generated by $\epsilon_1, \epsilon_1 + \epsilon_2$, and $c_2$ generated by $\epsilon_1 + \epsilon_2, \epsilon_2$.

The Duistermaat-Heckman function for $T_K$ is given by

$$dh^\mathcal{H}_{T_K}(\xi_1 \epsilon_1 + \xi_2 \epsilon_2) = \frac{1}{2}\xi_2$$
on $c_1$,

$$dh^\mathcal{H}_{T_K}(\xi_1 \epsilon_1 + \xi_2 \epsilon_2) = \frac{1}{2}\xi_1$$
on $c_2$.

The multiplicity function for $T_K$ on $\Lambda_K$ is (with $\lambda_1 + \lambda_2 \in 2\mathbb{Z}$) is

$$m^\mathcal{H}_{T_K}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) = \frac{1}{2}\lambda_2 + \frac{3}{4} + (-1)^{\lambda_2} \frac{1}{4}$$
on $c_1$,

$$m^\mathcal{H}_{T_K}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) = \frac{1}{2}\lambda_1 + \frac{3}{4} + (-1)^{\lambda_1} \frac{1}{4}$$
on $c_2$.

The (closed) Weyl chamber is $c_1$ and we have $C_K(\mathcal{H}) = c_1$.

The Duistermaat-Heckman function is

$$dh^\mathcal{H}_K(\xi_1 \epsilon_1 + \xi_2 \epsilon_2) = 1.$$

The multiplicity function is

$$m^\mathcal{H}_K(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) = \frac{1}{2}(1 + (-1)^{\lambda_1}).$$

In particular, the highest degree term (degree 0) of the multiplicity function is not a polynomial. Only when $\lambda_1$ is even, we obtain that the highest degree term of $m^\mathcal{H}_K$ (which is the constant function 1) coincide with the Duistermaat-Heckman polynomial. □

The degree $r$ is equal to 0 if and only if multiplicities are bounded. This implies that all the fibers $\Phi^{-1}(\xi)$ of the moment map are homogeneous spaces (the reduced fiber is a point). In particular, if the $K$ multiplicities of the representation $Sym(\mathcal{H})$ are bounded, they take only the values 0 or 1.

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We now consider a face $F$ of $C_K(\mathcal{H})$. The restriction to $F$ of the function $m^H_K$ is again a piecewise quasi-polynomial function. On $F \cap \mathfrak{c}_a$, the multiplicity is given by the restriction of $p^H_K$ to $F \cap \mathfrak{c}_a$. The degree of $p^H_K|_{\text{lin}(F)}$ (considered as a quasi-polynomial function $\lambda$ on the lattice $\text{lin}(F) \cap \Lambda$) might drop. If $F$ is a regular face, we can compute the degree using Theorem 33. Indeed

**Lemma 35.** If $F$ is a regular face, then the degree of the function $m^H_K$ restricted to $F$ is the same on each cone $\mathfrak{c}_a \cap F$ and is equal to $r_F = \dim_{\mathbb{C}} \mathcal{H}(T_F) - \dim(F) - |\Delta^+_F|$.

If $F$ is not a regular face, it is not easy to describe the degree of the restriction by such a simple formula. In particular if 0 is in the Kirwan polytope of $P(\mathcal{H})$, the degree on the corresponding edge of the Kirwan cone is difficult to compute.

If $F$ is a regular face, the degree of the multiplicity function restricted to any $\mathfrak{c}_a \cap F$ depends only of the linear span of $F$. Let $w$ be an element of the Weyl group. Although the Kirwan cone is not stable by the action of $W_t$, we might have two faces $F_1$ and $F_2$, so that their linear span is conjugated by $w$ (but of course not the intersection of this linear span with $C_K(\mathcal{H})$). We then obtain that the degree of the multiplicity function is the same on $F_1, F_2$, see Example 82.

Consider now a line $\{k\lambda, k = 0, 1, 2, \ldots\}$ contained in $C_K(\mathcal{H})$. It is clear that the degree of the quasi-polynomial function $k \rightarrow m^H_K(k\lambda)$ is less or equal to $r$. The reduced space $\Phi^{-1}(K\lambda)/K$, which may be singular, is provided with a structure of projective variety. Thus we can compute in principle its dimension $r_\lambda$. More precisely, let us consider the open set $P^0$ of orbits of maximal dimension in $P = \Phi^{-1}(K\lambda)$. Then $P^0/K$ is an orbifold, and then $r_\lambda = \dim P^0/K$. The following result is also a consequence of Meinrenken-Sjamaar.

**Theorem 36.** The degree of the quasi-polynomial function $k \rightarrow m^H_K(k\lambda)$ is equal to $r_\lambda$.

However, it is usually difficult to compute explicitly $r_\lambda$ using this geometric theorem. We can do it when $\lambda$ is in the interior of the Weyl chamber.

**Lemma 37.** Let $\lambda$ be in the interior of the Weyl chamber. Consider the minimal face $F$ of $C_K(\mathcal{H})$ containing $\lambda$. Then the degree of the quasi-polynomial function $k \rightarrow m^H_K(k\lambda)$ is equal to $r_F = \dim_{\mathbb{C}} \mathcal{H}(T_F) - \dim(F) - |\Delta^+_F|$.
Thus the behavior of the function $m^H_K$ is strikingly different from the behavior of the Duistermaat-Heckman measure which typically vanishes identically on the faces. So for a $\lambda$ which is not in the interior of the cone $C_K(\mathcal{H})$, the degree of the function $dh^H_K(t\lambda)$ and of the function $m^H_K(k\lambda)$ are usually different.

A point $\lambda$ is called (weakly) stable, if the degree of $k \mapsto m^H_K(k\lambda)$ is 0. In other words, the function $m^H_K(k\lambda)$ is bounded on the ray with generator $\lambda$. In this case the function $m^H_K(k\lambda)$ takes only values 0 or 1. So this degree can be equal to 0 only on the boundary of $\mathcal{H}$, except in the case where multiplicities are bounded. Meinrenken-Sjamaar result (or in fact, here GIT theory) implies the following lemma.

**Lemma 38.** A point $\lambda \in C_K(\mathcal{H})$ is (weakly) stable if and only if $\Phi^{-1}(K\lambda)$ is a $K$-orbit.

A point $\lambda$ is called a stable point, if $m^H_K(\lambda) = 1$, and if for any $\gamma \in \Lambda_{\geq 0}$, the function $k \mapsto m^H_K(\gamma + k\lambda)$ is an increasing and bounded function of $k$.

The following result, conjectured by Stembridge [49], has been obtained recently by [48].

**Proposition 39.** A weakly stable point is stable.

L. Manivel [39] has determined some faces of the Kirwan cone $C_K(\mathcal{H})$ consisting of stable points in terms of compatible imbeddings, and has described the corresponding stabilized multiplicity $\lim_{k \to \infty} m^H_K(\gamma + k\lambda)$ in geometric terms. Explicit descriptions in the case of the Kronecker cone are given in [40].

Assume that the cone $C_K(\mathcal{H})$ intersect $t^*_{>0}$. If the inequations $X_a \geq 0$ of the Kirwan cone $C_K(\mathcal{H})$ are known, the preceding discussion allows us to compute all stable points by elementary combinatorics, except on the boundary of the Weyl chamber, see Example 82.

### 4.6 The 3-qubits example

Before explaining our method to obtain actual computations of the multiplicity function, let us give a complete simple example, [11], [10]. We followed the exposition of [13].

We consider the action of $U(2) \times U(2) \times U(2)$ on $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. 

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The representation of $U(2) \times U(2) \times U(2)$ in $\text{Sym}(\mathcal{H})$ decomposes as

$$
\text{Sym}(\mathcal{H}) = \bigoplus_{\lambda,\mu,\nu} g(\lambda, \mu, \nu) V^{U(2)}_{\lambda} \otimes V^{U(2)}_{\mu} \otimes V^{U(2)}_{\nu}
$$

over polynomial irreducible representations $\lambda, \mu, \nu$ of $U(2) \times U(2) \times U(2)$. They are indexed as $\lambda = [\lambda_1, \lambda_2], \mu = [\mu_1, \mu_2], \nu = [\nu_1, \nu_2]$ with $\lambda_i, \mu_i, \nu_i \in \mathbb{Z}, i = 1, 2, \lambda_1 \geq \lambda_2 \geq 0$, $\mu_1 \geq \mu_2 \geq 0$, $\nu_1 \geq \nu_2 \geq 0$.

Though the dimension of a Cartan subalgebra of $U(2) \times U(2) \times U(2)$ is 6, we can restrict our attention to the subset $E \subset i\mathfrak{t}^*$ defined by the equations

$$
E = \{ (\alpha, \beta, \gamma); \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \gamma_1 + \gamma_2 \}.
$$

Here $\alpha = [\alpha_1, \alpha_2]$, with $\alpha_1, \alpha_2$ real numbers, and $\alpha_1 \geq \alpha_2$, etc.

Indeed, considering the action of the center, it is easy that the Kirwan cone $C_K(\mathcal{H})$ is contained in $E$. More precisely, the two dimensional subgroup

$$
S = \{ \begin{pmatrix} t_1 & 0 \\ 0 & t_1 \end{pmatrix}, \begin{pmatrix} t_2 & 0 \\ 0 & t_2 \end{pmatrix}, \begin{pmatrix} t_3 & 0 \\ 0 & t_3 \end{pmatrix}; |t_i| = 1; t_1t_2t_3 = 1 \}
$$

of $U(2) \times U(2) \times U(2)$ is contained in its center and acts trivially on $\mathcal{H}$. So we might consider $K = (U(2) \times U(2) \times U(2))/S$ acting on $\mathcal{H}$, with Cartan subalgebra $\mathfrak{t}$. Then $i\mathfrak{t}^*$ is isomorphic to $E$.

Higuchi-Sudbery-Szulc equations [23] (see Example 17) for the Kirwan cone in $E$ are

$$
C_K(\mathcal{H}) = \left\{ (\alpha, \beta, \gamma) \in E \mid \begin{array}{l}
\alpha_1 \geq \alpha_2 \geq 0, \\
\beta_1 \geq \beta_2 \geq 0, \\
\gamma_1 \geq \gamma_2 \geq 0,
\end{array} \begin{array}{l}
\alpha_2 \leq \gamma_2 + \beta_2, \\
\beta_2 \leq \gamma_2 + \alpha_2, \\
\gamma_2 \leq \alpha_2 + \beta_2,
\end{array} \right\}.
$$

The image of pure states in $\mathcal{H}$ is contained in the three dimensional affine space:

$$
E_1 = \{ (\alpha, \beta, \gamma); \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \gamma_1 + \gamma_2 = 1 \}.
$$

The group $\Sigma_3$ of permutations of $\alpha, \beta, \gamma$ acts on $E$.

We parameterize $E$ by $\mathbb{R}^4$ by associating to $[[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\gamma_1, \gamma_2]]$ the point $(t, \alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2)$ in $\mathbb{R}^4$, with $t = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \gamma_1 + \gamma_2$. The Weyl chamber in $\mathbb{R}^4 = \{ (t, x_1, x_2, x_3) \}$ is $x_i \geq 0$, and Bravy’s equations become

$$
\{ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, t + x_1 \geq x_2 + x_3, t + x_2 \geq x_3 + x_1, t + x_3 \geq x_1 + x_2 \}.
$$
As $E_1$ is isomorphic to $\{(1, x_1, x_2, x_3)\}$, we can picture the Kirwan polytope $\Delta(H)$ in $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. It is the polytope with 5 vertices $v_1, v_2, v_3, v_4, v_5$ (see Figure 2).

For example, the point $v_1 := (0, 0, 0)$ corresponds to the triple $v_{\min}$ in $E_1$, with

$$v_{\min} = [[[1/2, 1/2], [1/2, 1/2], [1/2, 1/2]]],$$

and the point $v_2 = (1, 1, 1)$ corresponds to the triple

$$v_{\max} = [[[1, 0], [1, 0], [1, 0]].$$

The other vertices $v_3, v_4, v_5$ of $\Delta(H)$ are $v_3 = [0, 0, 1], v_4 = [0, 1, 0], v_5 = [0, 0, 1], v_4 = [0, 1, 0], v_5 = [0, 0, 1], v_4 = [0, 1, 0], v_5 = [0, 0, 1], v_4 = [0, 1, 0], v_5 = [0, 0, 1], and its permutations by $\Sigma_3$.

Thus the Kirwan polytope is an union of two tetrahedra glued over the triangle $T$ with vertices $[v_3, v_4, v_5]$. Consider the center $c = (1/3, 1/3, 1/3)$ of this triangle, corresponding to the triple $[[2/3, 1/3], [2/3, 1/3], [2/3, 1/3]], and take the subdivision of the triangle $T$ in 3 triangles, $T_1, T_2, T_3$. Then we have 6 cones of polynomiality: the 3 cones with basis the 3 tetrahedra which are the convex hulls of $(T_i, v_{\min})$ or the 3 cones with basis $(T_i, v_{\max})$.

Up to permutations by $\Sigma_3$, we obtain two cones. Writing $\kappa = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \gamma_1 + \gamma_2,$

$$C_1 = \{\alpha, \beta, \gamma ; 0 \leq \alpha_1 - \alpha_2 \leq min(\beta_1 - \beta_2, \gamma_1 - \gamma_2) \text{ and } \alpha_1 - \alpha_2 + \beta_1 - \beta_2 + \gamma_1 - \gamma_2 \leq \kappa \}$$

and

$$C_2 = \{\alpha, \beta, \gamma ; 0 \leq \alpha_1 - \alpha_2 \leq min(\beta_1 - \beta_2, \gamma_1 - \gamma_2) ; \alpha_1 - \alpha_2 + \beta_1 - \beta_2 + \gamma_1 - \gamma_2 \geq \kappa \}.$$

Figure 2: Kirwan polytope for three qubits
Figure 3 represents the cones $C_1$ or $C_2$ intersected with the hyperplane $E_1$ of $E$.

![Figure 3: Cones $C_1$ and $C_2$ inside the Kirwan polytope](image)

The multiplicities on the cone $C_K(\mathcal{H})$ have been described by Briand-Orellana-Rosas, [11], [10]. On each cone $g(\lambda, \mu, \nu)$ is a periodic polynomial of degree 1 and period 2. Set $k = \lambda_1 + \lambda_2 = \nu_1 + \nu_2 = \mu_1 + \mu_2$. Then

$$g(\lambda, \mu, \nu) =
\begin{cases}
\frac{1}{2}(\lambda_1 - \lambda_2) + \frac{1}{4} \left(-1\right)^{\lambda_1+\mu_1+\nu_1} + \frac{1}{4} \left(-1\right)^{\lambda_2+\mu_1+\nu_1} + \frac{1}{2} & \text{if } (\lambda, \mu, \nu) \in C_1 \\
\frac{1}{2} \mu_1 + \frac{1}{2} \lambda_2 - \frac{1}{2} \nu_1 + \frac{1}{4} \left(-1\right)^{\mu_1+\lambda_2+\nu_1} + \frac{3}{4} & \text{if } (\lambda, \mu, \nu) \in C_2
\end{cases}$$

In particular for example $v_{\text{min}} = [[1, 1], [1, 1], [1, 1]]$ is in $C_1$ while $v_{\text{max}} = [[1, 0], [1, 0], [1, 0]]$ is in $C_2$.

In this example, the multiplicity function takes values 0 or 1 on the boundary of $C_K(\mathcal{H})$. In particular, we see that $g([k, k], [k, k], [k, k]) = \frac{1}{2} + (-1)^k \frac{1}{2}$, so that $g([k, k], [k, k], [k, k])$ is 1, or 0, if $k$ is even or odd, and

$$\sum_{k=0}^{\infty} g([k, k], [k, k], [k, k]) t^k = \frac{1}{1 - t^2},$$

as follows from the study of the Hilbert series of the ring of invariant polynomials on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ under the action of $SL(2) \times SL(2) \times SL(2)$.

We conclude this example with Figure 4 that shows the Duistermaat-Heckman measure for three qubits. The drawing is along the line $[[1/2, 1/2], [1/2, 1/2], [1/2, 1/2]]$ to $[[1, 0], [1, 0], [1, 0]]$ between the bottom vertex $v_{\text{min}}$ and the top vertex $v_{\text{max}}$ of the Kirwan polytope up to the top vertex $c$. It grows linearly between $v_{\text{min}}$ and $c$, then decreases from $c$ to $v_{\text{max}}$. 31
Here is the discrete version of the multiplicity function. We compute $g([6k + s, 6k - s], [6k + s, 6k - s], [6k + s, 6k - s])$ for $s$ from 0 to 6. Here is the answer:

- For $k = 1$: $[1, 1, 3, 2, 2, 1, 1]$
- For $k = 2$: $[1, 1, 3, 3, 5, 4, 4, 3, 3, 2, 2, 1, 1]$
- For $k = 3$: $[1, 1, 3, 3, 5, 5, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1]$

It shows clearly the quasi polynomial behavior of the multiplicity on cones of polynomiality.

Our program computes symbolically the multiplicity function on the domains of polynomiality.

4.7 Multiplicities and Partition functions

We are now going to explicit the quasi polynomial functions determining the Duistermaat-Heckman measure for the space $\mathcal{H}$, and the multiplicity function for $\text{Sym}(\mathcal{H})$ whose existence is stated in Theorem 20 and Theorem 29.

For computing multiplicities, the connection is made through partition functions, whose computation is achieved (using techniques developed in [2]), via iterated residue of rational functions with poles on arrangement of hyperplanes. We give a simple example, explaining the philosophy of the
method, and we compare it with the algorithm of Guoce Xin [54] in Sub-section 4.9.9. The reader may want first to read this subsection. We state formulae Theorem 51 and Theorem 56 for the Duistermaat-Heckman measure and Theorem 53 and Theorem 59 for the multiplicity. Even if we could follow the same pattern, we will take a different approach for computing Kronecker coefficients. We will compute them as a byproduct of the computation of branching coefficients, as explained in detail in Section 6.

To start with, when $K = T$ is abelian, computation of multiplicities is the same problem than computing the number of integral points in a (rational) polytope, while computation of Duistermaat-Heckman function is the same problem than computing the volume of a polytope. In turn volumes and number of integral points in polytopes can be computed, as we just said, as iterated residue of rational functions with poles on arrangement of hyperplanes. The precise statements are collected in Theorem 51 and Theorem 53.

We write

$$\text{Sym}(\mathcal{H}) = \bigoplus_{\mu \in \hat{T}} m^H_T(\mu)e^\mu.$$  

Let $\Psi \subset it^*$ be the list of the weights for the action of $T$ on $\mathcal{H}$ counted with multiplicities:

$$\Psi = [\psi_1, \psi_2, \ldots, \psi_N].$$

We assume that the cone $Cone(\Psi)$ generated by $\Psi$ is a pointed cone: $Cone(\Psi) \cap -Cone(\Psi) = \{0\}$, and that the lattice of weights $\Lambda$ is generated by $\Psi$.

Let $P_\Psi$ be the function on $\Lambda$ that computes the number of ways we can write $\mu \in \Lambda$ as $\sum x_i \psi_i$ with $x_i$ nonnegative integers. The function $P_\Psi(\mu)$ is called the Kostant partition function (with respect to $\Psi$). It is thus immediate to see that

$$m^H_T(\mu) = P_\Psi(\mu).$$

The moment map $\Phi_T : \mathcal{H} \to it^*$ is given by

$$\Phi_T(\sum_{a=1}^N z_a e_a) = \sum |z_a|^2 \psi_a.$$  

Here we have used an orthonormal basis $e_1, e_2, \ldots, e_N$ of $\mathcal{H}$ where $T$ acts diagonally with weights $[\psi_1, \ldots, \psi_N]$. So the cone $C_T(\mathcal{H})$ is just the cone
Cone(Ψ) generated by the list Ψ of weights. Assume for simplicity that Ψ generates \( it^* \). It is thus a cone with non empty interior in \( it^* \).

Let \( y \in it^* \). Define the polytope

\[
\Pi_\Psi(y) = \{ [x_1, \ldots, x_N] \in \mathbb{R}^N, x_i \geq 0, \sum_{a=1}^{N} x_a \psi_a = y \}.
\]

The Duistermaat-Heckman function \( dh^H_T(y) \) is the volume of the polytope \( \Pi_\Psi(y) \) and \( m^H_T(\mu) (\mu \in \Lambda) \) is the number of integral points in the polytope \( \Pi_\Psi(\mu) \).

Define \( C_T^{reg}(H) \) to be the open subset of the cone \( C_T(H) \), where we removed from \( C_T(H) \) the union of the boundaries of the cones generated by all subsets of \( \Psi \). Let us write \( C_T^{reg}(H) = \bigcup Y_a \) where \( Y_a \) are the connected components of \( C_T^{reg}(H) \). Define \( c_a = Y_a \), so we obtain a cone decomposition

\[
C_T(H) = \bigcup_a c_a.
\]

Here all cones \( c_a \) are closed and have non empty interiors.

The following result explicit Theorem 20 in this setting and the well known behavior of the partition function.

**Proposition 40.**

- The function \( dh^H_T(y) \) is given by a homogeneous polynomial function \( d_a^\psi \) on \( c_a \). The degree of \( d_a^\psi \) is equal to \( |\Psi| - \text{dim} \ t \).

- There exists a quasi polynomial function \( p_a^\psi \) on \( \Lambda \) such that

\[
P_\Psi(\mu) = p_a^\psi(\mu)
\]

on the closed cone \( c_a \). The degree of \( p_a^\psi \) is equal to \( |\Psi| - \text{dim} \ t \) and its highest degree term is equal to \( d_a^\psi \).

- In particular \( m^H_T(\mu) = p_a^\psi(\mu) \) on the closed cone \( c_a \).

Let \( K \) be a compact Lie group acting on \( H \) by unitary transformations, and let us assume that the action of its maximal Cartan subgroup \( T \) has finite multiplicity. In other words, the weights of \( T \) in \( H \) span a pointed cone. Then

\[
\text{Sym}(H) = \bigoplus_\lambda m^H_K(\lambda)V^K_\lambda
\]
and we have, for \( \lambda \in \Lambda_{\geq 0} \),
\[
m_H^K(\lambda) = \sum_{w \in W} \epsilon(w) P_{\Psi}(\lambda + \rho_t - w\rho_t).
\]

The cone \( C_K(H) \) is contained in \( C_T(H) \cap i t_{\geq 0}^* \) but usually smaller. Let us give some simple examples.

**Example 41.**
Let us consider the standard action of \( U(2) \) on \( C^2 \) with weights \( \epsilon_1 = [1, 0], \epsilon_2 = [0, 1] \). The moment cone for \( T \) is thus \( \mathbb{R}_{\geq 0}\epsilon_1 \oplus \mathbb{R}_{\geq 0}\epsilon_2 \). The Weyl chamber is \( t_{\geq 0} = \{ \xi_1 \epsilon_1 + \xi_2 \epsilon_2; \xi_1 \geq \xi_2 \} \). We have \( \Phi_K(z) = zz^* \), if we consider \( z \in \mathbb{C}^2 \) as a map \( \mathbb{C}^2 \to \mathbb{C} \). Thus \( \Phi_K(z) \) is a Hermitian non-negative matrix of rank 1. We thus have \( \Delta_K(H) = \mathbb{R}_{\geq 0}\epsilon_1 \). In particular, the cone \( C_T(H) \) is solid in \( t^* \), while \( C_K(H) \) is not solid. □

Even when both \( C_T(H) \) and \( C_K(H) \) are solid, they can be different.

**Example 42.** \( C_T(H) \) for the 3-qubits.

Return for example to the case of 3 qubits and we keep the same notations. The weights of the action of \( T \) in \( \mathbb{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) on the weight vectors \( e_\pm \otimes e_\pm \otimes e_\pm \) with \( \mathbb{C}^2 = \mathbb{C} e_+ \oplus \mathbb{C} e_- \) are \( [1, 0], [1, 0], [1, 0] \) and its permutation by the Weyl group \( \Sigma_2 \times \Sigma_2 \times \Sigma_2 \) of \( K \). In the parameters \( \mathbb{R}^4 \), the weights are \( [1, \epsilon_1, \epsilon_2, \epsilon_3] \) with \( \epsilon_i \in \{1, -1\} \). So we see that \( C_T(H) \) is the cone over the cube \( C \) with vertices \( [\epsilon_1, \epsilon_2, \epsilon_3] \) with \( \epsilon_i \in \{-1, 1\} \). The intersection of \( C_T(H) \) with the positive Weyl chamber is the cone over \( C \) intersected with the positive quadrant. This is again a cube with vertices \( [\nu_1, \nu_2, \nu_3] \) with \( \nu_i \in \{0, 1\} \).

Figure shows the Kirwan cone \( C_K(H) \) inside \( C_T(H) \cap i t_{\geq 0}^* \). We see that the points \( (1, 1, 0) \) (and permutations) are vertices of \( C_T(H) \cap i t_{\geq 0}^* \), but they do not belong to \( C_K(H) \). The Kirwan polytope \( C_K(H) \) is obtained by removing from \( C_T(H) \) a ”neighborhood” of these vertices, namely the simplices with vertices \( (1, 1, 0), (1, 1, 1), (1, 0, 0), (0, 1, 0) \) (and its permutations). □

Assume (to simplify) that \( K \) acts on \( \mathbb{H} \) with generic finite stabilizer. Then \( C_K(H) \) is a cone in \( i t_{\geq 0}^* \) with non empty interior and it follows from the formula for Duistermaat-Heckman measure that the Kirwan cone \( C_K(H) \)
Figure 5: $K$-Kirwan polytope for three qubits inside $T$-Kirwan polytope

is the union of the solid cones $c_a \cap it^*_\geq 0$ contained in it. As seen in the preceding example, it is not easy to understand which ones have this property.

We denote by $\partial_{-\alpha} f(y) = \frac{d}{dt} f(y - t\alpha)|_{t=0}$ the derivative of a function $f$ on $i\mathfrak{t}^*$ in the direction $-\alpha$. Let $(\prod_{\alpha > 0} \partial_{-\alpha})$ the product of derivatives with respect to all negative roots in $\Delta_t$. Consider the cone decomposition $C_T(\mathcal{H}) = \cup_a c_a$.

**Theorem 43.** Assume $C_K(\mathcal{H})$ is a solid cone.

Let $C_T(\mathcal{H}) = \cup_a c_a$. Consider $c_a$ such that $c_a \cap it^*_\geq 0 \subset C_K(\mathcal{H})$.

- On $c_a \cap it^*_\geq 0$ we have
  \[ dh^K_H = (\prod_{\alpha > 0} \partial_{-\alpha}) \cdot d^\psi_a \]

- Let $p^\psi_a$ be the quasi polynomial function on $\Lambda$ such that $\mathcal{P}_\psi(\mu) = p^\psi_a(\mu)$ on $c_a$. Then, on $c_a \cap \Lambda \geq 0$, we have
  \[ m^K_H(\lambda) = \sum_{w \in W_t} \epsilon(w)p^\psi_a(\lambda + \rho_t - w\rho_t). \]

**Remark 44.**

We see that the degree of the polynomial function $dh^K_H$, and $m^K_H$ on each $c_a$ is at most equal to the degree of the corresponding functions for $T$ minus the number $|\Delta_t^+|$ of positive roots. Our algorithm will indeed be easier for $K$ than for $T$.

\[ \square \]
Proof. Consider first the open subset $\tau_Q$ of $c_a \cap it^*_0$, at distance $Q$ of its boundary. We assume $Q$ greater than the norm of $\|\rho_t - w\rho_t\|$ for all $w \in W_k$. If we use the formula

$$m^H_K(\lambda) = \sum_{w \in W_k} \epsilon(w) P_\Psi(\lambda + \rho_t - w\rho_k)$$

and $\lambda$ is in $\tau_Q$, all points $\lambda + \rho_t - w\rho_k$ are in $c_a$ and we may replace $P_\Psi$ by the function $p_\Psi^a$. We thus obtain on $\tau_Q$ the expression

$$m^H_K(\lambda) = \sum_{w \in W_k} \epsilon(w)p_\Psi^a(\lambda + \rho_t - w\rho_k).$$

However the function $\lambda \rightarrow \sum_{w \in W_k} \epsilon(w)p_\Psi^a(\lambda + \rho_t - w\rho_t)$ on the right side of the above equation is a quasi polynomial function of $\lambda$ on $c_a \cap (it^*_0 \geq 0)$. By Meinrenken-Sjamaar, Theorem 29, we know that $m^H_K(\lambda)$ coincide with a quasi polynomial function on the closed cone $c_a \cap \Lambda_{\geq 0}$, under our assumption that $c_a \cap it^*_0 \subset C_K(\mathcal{H})$.

Using Lemma 28, we obtain the equality on $c_a \cap \Lambda_{\geq 0}$. 

4.8 Back to degrees on faces

Assume that the cone $C_K(\mathcal{H})$ is solid. We rewrite the formulae for the degrees $r_F$ on faces of the multiplicities, using $\Psi$. Let $F$ be a facet of $C_K(\mathcal{H})$ determined by an equation $X = 0$ intersecting $it^*_0$ (that is $X$ is not proportional to a fundamental coroot $H_\alpha$). Then we compute $\mathcal{H}(X)$, the subspace of $\mathcal{H}$ annihilated by $X$. Its complex dimension is the cardinal of $\Psi_0$, where $\Psi_0$ is the sublist of $\Psi$ composed of the $\psi_j$ such that $\psi_j(X) = 0$. The system $\Delta^+_0$ of positive roots for $\mathfrak{t}(X)$ is the sublist of $\Delta^+_t$ composed of the roots $\alpha_j$ such that $\alpha_j(X) = 0$. So we obtain the formula

Lemma 45. On the facet $F$, the degree of the multiplicity function is $|\Psi_0| - |\Delta_0^+| - (\dim \mathfrak{t} - 1)$.

4.9 Tools for computations

From now on, we focus on the computation of $m^H_K(\lambda)$. 

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4.9.1 Residue calculations: the philosophy

Consider the case of $n$ dimensional torus $T = \{(u_1, u_2, \ldots, u_n); |u_i| = 1\}$ acting on $\mathcal{H} = \mathbb{C}^N$. We denote by $u \rightarrow u^\lambda = \prod_i u_i^{\lambda_i}$ the character of $T$ indexed by $\lambda \in \Lambda = \mathbb{Z}^n$. Assume that all weights $\psi_i$ of the action of $T$ on $\mathcal{H}$ have non negative coordinates. Almost by definition, the computation of $m_T^\mathcal{H}(\lambda)$ is the computation of the Fourier series of the function $S(u) = \prod_i \frac{1}{1-u_i^{\psi_i}}$, on $|u_i| < 1$. So we obtain, choosing $\epsilon_i$ small positive real numbers,

$$m_T^\mathcal{H}(\lambda) = \int_{|u_j|=\epsilon_j} u^{-\lambda} \prod_{j=1}^{N} \frac{1}{1-u_j^{\psi_j}} \prod_j \frac{du_j}{2i\pi u_j}. \tag{5}$$

We choose an order of integration, and the residue theorem in one variable allows to integrate the variable $u_1$, but we obtain a priori $N-1$ new integrals in $n-1$ variables, by considering all poles in $u_1$ of the functions $(1-u_1^{\psi_j})$. Not all of these poles are in the interior of the circle $|u_1| = \epsilon_1$, then we have to keep track of the branchings. At the end we obtain a very large number of products of one dimensional residues computations.

The essence of Jeffrey-Kirwan residue is to compute a priori the paths which contributes. In fact, it is easy to see that the function $\prod_{j=1}^{N} \frac{1}{1-u_1^{\psi_j}}$ can be written as a sum of functions $f_\alpha$, where each $f_\alpha$ depend of $n$ independent variables. Here $\alpha$ varies in a large set $\mathcal{T}$. For example

$$\frac{1}{(1-u_1)(1-u_1u_2)(1-u_2)} = \frac{1}{(1-u_1u_2)^2(1-u_1)} + \frac{u_2}{(1-u_1u_2)^2(1-u_2)}$$

and we can separately use the independent variables $(u_1, u_1u_2)$ or $(u_2, u_1u_2)$. In the residue computation at $\lambda$, only a certain $\mathcal{T}(\lambda)$ of these functions $f_\alpha$ will contribute. So we compute a priori what is the subset $\mathcal{T}(\lambda)$ and we are reduced to product of residues in one variables. This is the object of the computation of adapted Orlik Solomon bases. We give now the technical details of the resulting formula.

4.9.2 Definitions of regular elements and topes

Consider a finite set $F$ of non zero vectors in a real vector space $E$.

**Definition 46.** We say that a hyperplane $H \subset E$ is $F$-admissible if $H$ is generated by elements of $F$. We denote by $\mathcal{A}(F)$ the set of admissible hyperplanes.
Assume that $F$ generates a lattice $L$ in $E$. Let $\sigma$ be a subset of $F$, generating $E$ as a vector space. We say that $\sigma$ is a basis of $F$. We denote by $d_\sigma$ the smallest integer such that $d_\sigma L$ is contained in the lattice generated by $\sigma$.

**Definition 47.** Consider the set periods($F$) = \{d_\sigma\} where $\sigma$ runs over the basis of $F$. Let $q(F)$ is the least common multiple of the integers in periods($F$) We say that $q(F)$ is the index of the finite set $F$ (with respect to $L$).

**Definition 48.** Consider a finite set $\mathcal{F}$ of hyperplanes in a real vector space $E$. We say that $\xi \in E$ is $\mathcal{F}$ regular, if $\xi$ is not on any hyperplane belonging to $\mathcal{F}$.

Let $E_{\text{reg}}(\mathcal{F})$ be the set of $\mathcal{F}$-regular elements. A connected component $\tau$ of $E_{\text{reg}}(\mathcal{F})$ will be called a tope (with respect to $\mathcal{F}$). If $\xi$ is $\mathcal{F}$-regular, we denote by $\tau(\xi)$ the unique tope containing $\xi$.

Return to the situation where $\Psi \subset \text{it}^\ast$ is the set of weights of $T$ in $\mathcal{H}$. When $\mathcal{F}$ is the family $A(\Psi)$ of $\Psi$-admissible hyperplanes, we also say that $\xi$ is $\Psi$-regular instead of $\mathcal{F}$-regular. A tope for the family $A(\Psi)$ will be called a $\Psi$-tope.

**4.9.3 Iterated residues and Orlik-Solomon bases**

A list $\vec{\psi} = [\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_r}]$ of elements of $\Psi$ will be called a ordered basis if the elements $\psi_{i_k}$ form a basis of $\text{it}^\ast$. Let $\mathcal{B}(\Psi)$ be the set of ordered bases. An ordered base $[\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_r}]$ is an Orlik Solomon base, OS in short, if for each $1 \leq l \leq r$, there is no $j < i_l$ such that the elements $\psi_j, \psi_{i_l}, \ldots, \psi_{i_r}$ are linearly dependent. Denote by $\mathcal{OS}(\Psi)$ the set of $OS$ bases.

**Definition 49.** If $\tau$ is a $\Psi$-tope, we denote by $\mathcal{OS}(\Psi, \tau) = \{\sigma \in \mathcal{OS}(\Psi), \tau \subset \text{Cone}(\sigma)\}$. Here $\text{Cone}(\sigma) = \sum \mathbb{R}_{\geq 0} \psi_{i_j}$.

The set $\mathcal{OS}(\Psi, \tau)$ is called the set of $\text{OS}$ adapted bases to $\tau$.

We will show in Section 4.9.6 how to compute adapted bases. For an account of the theory cf. [2].

We now define the notion of iterated residue. If $f$ is a meromorphic function in one variable $z$, consider its Laurent series $\sum_n a_n z^n$ at $z = 0$. The coefficient of $z^{-1}$ is denoted by $\text{Res}_{z=0} f$. 

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Let $\vec{\sigma} = [\alpha_1, \alpha_2, \ldots, \alpha_r]$ be an ordered basis of $i\mathfrak{t}^*$. For $z \in \mathfrak{t}_C$, let $z_j = \langle z, \alpha_j \rangle$, and express a meromorphic function $f(z)$ on $\mathfrak{t}_C$ with poles on a union of hyperplanes as a function $f(z) = f(z_1, z_2, \ldots, z_r)$, (in particular $f$ may have poles on $z_j = 0$).

Consider the associated iterated residue functional defined by:

$$Res_{\vec{\sigma}}(f(z)) := \text{Res}_{z_1=0}(\text{Res}_{z_2=0} \cdot \cdots (\text{Res}_{z_r=0} f(z_1, z_2, \ldots, z_r)) \cdot \cdots).$$

(6)

4.9.4 Multiplicity and Duistermaat-Heckman function for the torus $T$

Fix a cone of polynomiality $c_a$, then $c_a$ is the union of the closures of the topes $\tau$ contained in $c_a$.

Tope were also called small chambers in [1].

We might need several topes (small chambers) so that the union of their closures is $c_a$. Figure 6 shows an example for the chambers complex versus the topes complex for the cone associated to a root system of type $A_3$.

Figure 6: 8 topes (left) versus 7 chambers (right)

Let $\xi \in i\mathfrak{t}^*$. Define the following function of $z \in \mathfrak{t}_C$.

$$s_T^\Psi(\xi, z) = e^{\langle \xi, z \rangle} \frac{1}{\prod_{\psi \in \Psi} \langle \psi, z \rangle}.$$ 

The function $z \rightarrow s_T^\Psi(\xi, z)$ has poles on the union of the hyperplanes $\langle \psi, z \rangle = 0$. An iterated residue of the function $z \rightarrow s_T^\Psi(\xi, z)$ depends of $\xi$ through the Taylor series of the function $e^{\langle \xi, z \rangle}$ at $z = 0$. So if we consider
ξ as a variable, we obtain as a result of performing an iterated residue a polynomial function of ξ.

**Definition 50.** Let τ be a Ψ-tope. Define

\[ d^Ψτ(ξ) = \sum_{σ ∈ OS(Ψ,τ)} Res_σ S^ΨT(ξ, z). \]

Thus \( d^Ψτ \) is a polynomial function of ξ.

**Theorem 51.** Let \( τ ⊂ i^* \) be a Ψ-tope, and \( ξ ∈ \overline{τ} \). If \( τ \) is contained in \( Cone(Ψ) \),

\[ dh^H_T(ξ) = d^Ψτ(ξ). \]

Let \( μ ∈ Λ \), and for \( z ∈ t_C \), define :

\[ S^ΨT(μ, z) = e^{⟨μ, z⟩} \frac{1}{\prod_{ψ ∈ Ψ}(1 - e^{-⟨ψ, z⟩})}. \]

Let \( Γ \) be the dual lattice to \( Λ \). Let \( q := q(Ψ) \) be the index of \( Ψ \). So if \( σ \) is a basis in \( i^* \) consisting of elements of \( Ψ \) then \( qΛ ⊂ \sum_{ψ ∈ Ψ} Zψ. \)

If \( γ ∈ Γ \), and we apply an iterated residue to the function \( z → S^ΨT(μ, z + \frac{2iπ}{q} γ) \), we obtain a quasi-polynomial function of \( μ \). Indeed, it depends of \( μ \) through the Taylor series at \( z = 0 \) of \( e^{⟨μ, z + \frac{2iπ}{q} γ⟩} = e^{⟨μ, \frac{2iπ}{q} γ⟩}e^{⟨μ, z⟩} \), and \( e^{⟨μ, \frac{2iπ}{q} γ⟩} \) is a periodic function of \( μ \) of period \( q \).

**Definition 52.** Let \( τ \) be a Ψ-tope. Define

\[ p^Ψτ(μ) = \sum_{σ ∈ OS(Ψ,τ)} \sum_{γ ∈ Γ/qΓ} Res_σ S^ΨT(μ, z + \frac{2iπ}{q} γ). \]

**Theorem 53.** (Szenes-Vergne) Let \( τ ⊂ i^* \) be a Ψ-tope and \( μ ∈ \overline{τ} \). If \( τ \) is contained in \( Cone(Ψ) \), then for any \( μ ∈ \overline{τ} ∩ Λ \),

\[ m^H_T(μ) = p^Ψτ(μ). \]

**Remark 54.**
We denote by $u \to u^\nu$ the character of $T_C$ associated to $\nu \in \Lambda$. Consider the function $F(u) = \frac{1}{\prod_{\psi \in \Psi}(1 - u^\psi)}$. It is clear that $m_H^T(\mu)$ is the Fourier coefficient of the function $F(u)$, expanded in the domain $|u^\psi| < 1$, for all $\psi \in \Psi$. Thus $m_H^T(\mu) = \int_{|u| = r} u^{-\mu} F(u) \frac{du}{u}$. Here $r$ is a small positive number.

Using the change of variables $u_i = e^{-z_i}$ ($z_i$ being the coordinates on $i$ associated to the basis $\sigma$), an iterated residue $\text{Res}_\sigma s_T^\Psi(\mu, z)$ can be computed as an iterated residue at $u = 1 \in T_C$ of the function $\prod_{\psi \in \Psi}(1 - u^{-\mu})$.

Similarly the other terms associated to $\gamma \in \Gamma/q\Gamma$ can be computed as iterated residues at some elements of finite order $q$ in $T_C$. Thus Szenes-Vergne theorem is a multi-dimensional residue theorem. A detailed example will be given in 4.9.9.

4.9.5 Multiplicity and Duistermaat- Heckman function for $K$

We now want to compute the functions $dh_H^K(\xi)$ and $m_H^K(\mu)$.

Let $\xi \in i\mathfrak{t}^*$. Define the following function of $z \in \mathfrak{t}_C$:

$$s_T^\Psi(\xi, z) = e^{\langle \xi, z \rangle} \prod_{\alpha \in \Delta^+} (\langle \alpha, z \rangle) \prod_{\psi \in \Psi} (\langle \psi, z \rangle).$$

**Definition 55.** Let $\tau$ be a $\Psi$-tope. Define

$$D_T^\Psi(\xi) = \sum_{\sigma \in \mathcal{O}_S(\Psi, \tau)} \text{Res}_\tau s_T^\Psi(\xi, z).$$

**Theorem 56.** Let $\tau$ be a $\Psi$-tope and let $\xi \in \mathfrak{t}^* \cap i\mathfrak{t}^*$. If $\tau \cap i\mathfrak{t}^*_< \geq 0$ is contained in the Kirwan cone $C_K(\mathcal{H})$, then for any $\xi \in \mathfrak{t}^* \cap i\mathfrak{t}^*_<$,

$$dh_H^K(\xi) = D_T^\Psi(\xi)$$

**Remark 57.**

The function $D_T^\Psi$ is easier to compute that $d_T^\Psi$ as the orders of poles in $z$ of the function $s_T^\Psi(\xi, z)$ are smaller that those of $s_T^\Psi(\xi, z)$.

In particular, if $\xi \in i\mathfrak{t}^*_<$, and is $\Psi$-regular, we can check if $\xi$ is in $C_K(\mathcal{H})$, by computing $D_T^\Psi(\xi)$ where $\tau$ is the tope containing $\xi$. 42
We can now state the result to compute $m^H_K(\lambda)$.
Let $\lambda \in \Lambda_{\geq 0}$, and for $z \in tC$ define:

$$S^\Psi_K(\lambda, z) = e^{\langle \lambda, z \rangle} \frac{\prod_{\alpha \in \Delta^+} (1 - e^{\langle \alpha, z \rangle})}{\prod_{\psi \in \Psi} (1 - e^{-\langle \psi, z \rangle})}.$$  

**Definition 58.** Let $\tau$ be a tope. Define

$$P^\Psi_\tau(\lambda) = \sum_{\sigma \in OS(\Psi, \tau)} \sum_{\gamma \in \Gamma/q\Gamma} \text{Res}_{\sigma} S^\Psi_K(\lambda, z + \frac{2i\pi}{q} \gamma).$$

**Theorem 59.** Let $\tau$ be a tope and let $\lambda \in \Lambda_{\geq 0}$. Then

1. If $\lambda \in \tau$, then $m^H_K(\lambda) = P^\Psi_\tau(\lambda)$.

2. If $\lambda \in \tau$ and $\tau \cap tC_{\geq 0} \subset C_K(H)$ then $m^H_K(\lambda) = P^\Psi_\tau(\lambda)$.

**Proof.** Consider the quasi-polynomial function, def.52:

$$p^\Psi_\tau(\mu) = \sum_{\sigma \in OS(\Psi, \tau)} \sum_{\gamma \in \Gamma/q\Gamma} \text{Res}_{\sigma} S^\Psi_T(\mu, z + \frac{2i\pi}{q} \gamma).$$

The function $P_\Psi$ is given on $\tau$ by the quasi polynomial formula, Theorem 53

$$P_\Psi(\mu) = p^\Psi_\tau(\mu).$$

We now use Theorem 43 which asserts that $m^H_K$ is given on the closure of $\tau$ by $\sum_w \epsilon(w)p^\Psi_\tau(\mu + \rho_k - w\rho_k)$. This is

$$\sum_{w \in W_K} \epsilon(w) \sum_{\sigma \in OS(\Psi, \tau)} \sum_{\gamma \in \Gamma/q\Gamma} \text{Res}_{\sigma} S^\Psi_K(\mu - w(\rho_k) + \rho_k, z + \frac{2i\pi}{q} \gamma) =$$

$$\sum_{\sigma \in OS(\Psi, \tau)} \sum_{\gamma \in \Gamma/q\Gamma} \text{Res}_{\sigma} \sum_{w} \epsilon(w) e^{-w(\rho_k) + \rho_k, z + \frac{2i\pi}{q} \gamma} \frac{e^{\langle \mu, z + \frac{2i\pi}{q} \gamma \rangle}}{\prod_{\psi \in \Psi} (1 - e^{\langle -\psi, z + \frac{2i\pi}{q} \gamma \rangle})} =$$

$$\sum_{\sigma \in OS(\Psi, \tau)} \sum_{\gamma \in \Gamma/q\Gamma} \text{Res}_{\sigma} S^\Psi_K(\mu, z + \frac{2i\pi}{q} \gamma).$$

The last equality follows from the denominator formula $\sum_w \epsilon(w)e^{\rho_k - w\rho_k} = \prod_{\alpha > 0} (1 - e^{\alpha})$. Thus $m^H_K(\mu) = P^\Psi_\tau(\mu)$ when $\mu$ is in the closure of $\tau$.  

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4.9.6 OS bases

Let \( v \) be a \( \Psi \)-regular vector. Let \( n = \dim t \). We recall here the algorithmic method based on the method of Maximal Nested Sets (MNS) of De Concini-Procesi, [16], and developed by Baldoni-Beck-Cochet-Vergne in [2] to compute \( \mathcal{O}(\Psi, \tau(v)) \). Recall that \( \mathcal{O}(\Psi, \tau(v)) \) depends only of the tope \( \tau(v) \) where \( v \) belongs. We first need to compute the set \( \mathcal{A} \) of admissible hyperplanes generated by elements of \( \Psi \) (we do not have a good method to perform this step). For each hyperplane, we choose an element \( H \in t \), so that our hyperplane is \( H^\perp \).

Anyway, assume that we have determined this set \( \mathcal{A} \) of admissible hyperplanes. Then we order \( \Psi \). We compute recursively a set of OS basis for \( \Psi \) by the following method.

- We first choose \( \psi_0 \) the lowest element of \( \Psi \), and an hyperplane \( H_1 \) where \( \psi_0 \) does not belong: \( \langle H_1, \psi_0 \rangle \neq 0 \). Let \( \Psi_1 = \Psi \cap H_1^\perp \) and \( \mathcal{A}_1 = \{ H^\perp \in \mathcal{A}, \langle H, \psi_0 \rangle = 0 \} \), that is \( \mathcal{A}_1 \) is the set of hyperplanes in \( \mathcal{A} \) that contain \( \psi_0 \).

- We choose \( \psi_1 \) the lowest element in \( \Psi_1 \) and \( H_2 \) in \( \mathcal{A}_1 \), such that \( \psi_1 \) does not belong to \( H_2 \), and such that \( H_1^\perp \cap H_2^\perp \) is spanned by \( \Psi \cap H_1^\perp \cap H_2^\perp \).

In practice, we verify this condition only at the end, but we verify that the set \( \Psi \cap H_1^\perp \cap H_2^\perp \) has at least \( n - 2 \) elements.

We can now define \( \Psi_2 = H_1^\perp \cap H_2^\perp \cap \Psi \) and \( \mathcal{A}_2 = \{ H^\perp \in \mathcal{A}_1, \langle H, \psi_1 \rangle = 0 \} \), that is \( \mathcal{A}_2 \) is the set of hyperplanes in \( \mathcal{A} \) that contain \( \psi_0, \psi_1 \).

- The inductive step is the following:
  
  suppose we have constructed a list \( [\alpha_0, \alpha_1, \ldots, \alpha_s] \subset \Psi \) and a list \( [H_1, H_2, \ldots, H_{s+1}] \) of hyperplanes with the property that
  
  - \( \alpha_i \) is the lowest element of \( \Psi \cap H_1^\perp \cap \cdots \cap H_i^\perp \).
  - \( \alpha_j \notin H_{j+1}^\perp \) and \( H_{j+1}^\perp \in \mathcal{A}_j \) where \( \mathcal{A}_j = \{ H^\perp \in \mathcal{A}, \langle H, \psi_t \rangle = 0, t = 0 \cdots j - 1 \} \).

  Thus \( \alpha_0, \alpha_1, \ldots, \alpha_{j-1} \in H_{j+1}^\perp \) and \( \alpha_{j+1} \notin H_{j+1} \).
  
  - \( \Psi \cap H_1^\perp \cap H_2^\perp \cap \cdots \cap H_{s+1}^\perp \) has \( \geq n - s - 1 \) elements.

- The next step is as follows:
  
  - we choose \( \alpha_{s+1} \) the lowest element in \( \Psi \cap H_1^\perp \cap H_2^\perp \cap \cdots \cap H_{s+1}^\perp \)
We choose an hyperplane $H_{s+2}$ containing $[\alpha_0, \alpha_1, \ldots, \alpha_s]$, i.e. $H_{s+2} \in A_{s+1}$, but not $\alpha_{s+1}$ and such that $\Psi \cap H_1^\perp \cap H_2^\perp \cap \cdots \cap H_s^\perp \cap H_{s+2}^\perp$ has $\geq n - s - 2$ elements.

- We then continue with $[H_1, H_2, \ldots, H_{s+1}, H_{s+2}]$ and $[\alpha_0, \alpha_1, \ldots, \alpha_{s+1}, \alpha_{s+2}]

- At the end, we obtain $n$ elements $[\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$.

   It may occur that we cannot go to the end. This means that at this step, the space $H_1^\perp \cap H_2^\perp \cap \cdots \cap H_k^\perp$ was not generated by its intersection with $\Psi$. However, if we arrive to the end, we obtain a basis, and by [16] we obtain a set of OS basis for $\Psi$.

We can refine this method to obtain directly the set of OS adapted basis for a regular vector $v$. The algorithm we just explained is modified easily. At each step when we look for the hyperplane $H_s$ containing $[\alpha_0, \alpha_1, \ldots, \alpha_{s-2}]$, but not $\alpha_{s-1}$, we also impose on $H_s$ the condition that the vectors $v$ and $\alpha_{s-1}$ lie on the same side of the hyperplane $H_s$. In this way, we obtain $\mathcal{OS}(\Psi, \tau(v))$. If $v$ is not in the cone $C(\Psi)$, then the algorithm returns the empty set for $\mathcal{OS}(\Psi, \tau(v))$.

Remark that if we know the equations of the cone $C(\Psi)$, then we could first check if $v$ is in the cone, before computing $\mathcal{OS}(\Psi, \tau(v))$ to shorten the procedure. However, we do not use this preliminary step.

### 4.9.7 Dilated coefficients, Hilbert series

Once we know $\mathcal{OS}(\Psi, \tau)$ the calculation is not more difficult to do with symbolic variable $\mu$, and we obtain the periodic polynomial which coincides with $m^H_K(\mu)$ on $\tau \cap \mathbb{N}_0^\infty \tau$ (and the closure if $\tau \cap \mathbb{N}_0^\infty \tau$ is contained in $C_K(H)$).

In particular, to compute the function $m^H_K(k\mu)$ on the line $\mathbb{N}_0$ as a periodic polynomial function of $k$ is not more difficult than to compute the numerical value $m^H_K(\mu)$.

A particular interesting example is the computation of Hilbert series. Assume that $\mathfrak{k} = \mathfrak{z} \oplus [\mathfrak{k}, \mathfrak{k}]$, where the center $\mathfrak{z} = \mathbb{R}J$ of $\mathfrak{k}$ acts by the homothety. Consider $\chi \in \Lambda$ such that $\chi(iJ) = 1$ and $\chi = 0$ on $i(\text{tr}[\mathfrak{k}, \mathfrak{k}])$. Then $m^H_K(k\chi) = \text{dim}[S^k(H)]^{[K,K]}$. So the series $R(t) = \sum_{k=0}^{\infty} m^H_K(k\chi)t^k$ is the Hilbert series of the ring of invariants polynomials under the action of $[K_C, K_C]$. It is of the form $\frac{P(t)}{\prod_{j=1}^{P(t)}(1-t^{\alpha_j})}$.
The degree of the quasi-polynomial function \( k \rightarrow m^K(\mathcal{H})(k\chi) \) as well as its set of periods gives some information on the number of factors and the coefficients \( a_j \) in this Hilbert series.

### 4.9.8 Advantages and Difficulties of the method

Since one of the objectives of this paper is to describe an efficient algorithm to compute multiplicities, at least for low dimension, let’s look at the weak and good points in implementing the above formulae.

- The computation of the set of admissible hyperplanes for a system \( \Psi \). For example, if we consider the space of 6 qubits, a brute force computation (taking any subset with 6 elements of \( 2^6 \) elements would lead to 7624512 computations). We do not know an efficient algorithm for computing the set \( \mathcal{A}(\Psi) \). Moreover, our system \( \Psi \) has symmetries coming from the Weyl group action. So it would be good to have an algorithm computing \( \mathcal{A}(\Psi) \) up to Weyl group action. When \( \Psi \) is a root system of rank \( r \), this set, up to Weyl group action, is just the set \( \{h_i^+, i = 1, \ldots, r\} \) where \( h_i \) are the fundamental coweights. In the Kronecker examples, we do not know the set \( \mathcal{A}(\Psi) \), although the system \( \Psi \) is quite simple, see Formula 8 inside Example 69.

- The residues calculation for the function \( S^\Psi_{\Gamma}(\mu, z) \). We do this by power series expansion, and it leads to multiply polynomials of larger and larger degree.

- The computation of the set \( \mathcal{O}S(\Psi, \tau) \). Once this set is computed, residues contributions are independent of each other. So an advantage of the iterated residue method is that each individual residue computation is easy and does not use much memory.

- The index \( q \) of the set \( \Psi \). The fact is that for arbitrary systems \( \Psi \) with \( N \)-vectors, this index can be large. So Szenes-Vergne formula does not provide a polynomial time algorithm (the dimension of \( \mathcal{H} \) being fixed).

For example, in the case of the knapsack, (Example 7), when the \( A_i \) are relatively prime, the index \( q \) is \( A_1 \cdots A_n \). Thus, if \( q \) is large, the function

\[
\sum_{\gamma \in \Gamma/q\Gamma} S^\Psi_T(\mu, z + \frac{2i\pi}{q}\gamma)
\]

should be computed in polynomial time using Barvinok determination of generating functions of cones [6]. This is the method we used in [3] to give
an efficient computation when \( n \) is fixed. However, for our computation of Kronecker examples, we just used the summation over \( L^*/qL^* \), as \( q \) was relatively small.

Let us explain at this point the method followed by Christand-Doran-Walter \([13]\). Their method (of polynomial complexity, when the dimension of \( \mathcal{H} \) is fixed) computes the numeric multiplicity by the following method. As we said, the multiplicity function \( m^H_T(\lambda) \) for \( T \) is the number of points in the polytope \( \Pi_\psi(\lambda) \). Thus \([13]\) uses the ”Barvinok algorithm”, as implemented in \([51]\), to compute \( m^H_T(\lambda) \). This computation would also be possible to do using the Latte software package \([5]\). So the above difficulty of the possibly very large index \( q \) is resolved (in polynomial time) by the power of Barvinok signed decompositions. Then they use the relation between \( m^H_K \) and \( m^H_T \) to compute the multiplicity under \( K \). Remark that this last computation can be done on parallel computing. However the number of elements \( w \) becomes very large.

We close this subsection with some questions.

**Questions**

Here are the problems that we would like to have some partial answers in interesting examples:

- Can we understand the image of the moment map, and describe the Kirwan polyhedron by inequalities?
- Can we compute the Duistermaat-Heckman measure?
- Can we compute the dilated multiplicity \( k \to m^H_K(k\lambda) = \sum_i c_i(k)k^i \), or at least say something on the periodicity of the coefficients and the degree of the function \( k \to m^H_K(k\lambda) \) ? In particular when \( \lambda \) is a one dimensional representation of \( K \), or other interesting \( \lambda \).

### 4.9.9 A very simple example

Return to Example 4.5. We close this part in computing in two ways the multiplicity \( m^H_K(\lambda) \) in this very simple example.

- The straightforward method of computing the expansion of a Fourier series. The one dimensional residue formula is used repeatedly to replace the large poles at \( u = 0 \) in poles on \( |u| = 1 \). The branchings appearing in this straightforward becomes soon quite complicated, if the rank of \( K \) as well as
the dimension of \( \mathcal{H} \) increases. This is the method followed by \([54] \), with an efficient way to keep track of the branchings.

- The Jeffrey-Kirwan residue method. It gives a (possibly large) number of residues at \( z = 0 \), or equivalently, in exponential coordinates \( u = \exp z \), at \( u = 1 \), or some finite order elements in \( T \). The computation for each iterated residue are independent of each other.

- **The straightforward method.**

  We consider the torus \( T \) of \( U(2) \) parameterized as \( \{ (u_1, u_2); |u_1| = 1; |u_2| = 1 \} \) and the action of \( T \) on \( \mathcal{H} \) with weights \([2, 0], [1, 1], [0, 2] \). The character of the representation of \( T \) in \( \text{Sym}(\mathcal{H}) \) is obtained by computing the Fourier expansion of

  \[
  F(u_1, u_2) = \frac{1}{(1 - u_1^2)(1 - u_1 u_2)(1 - u_2^2)}
  \]

  for \( u_1, u_2 \) of modulus strictly less then than 1. Let \( \lambda = \lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 \), with \( \lambda_1, \lambda_2 \) two non negative integers. Thus, taking for example \( r_1 = r_2 = \frac{1}{2} \), we obtain the straightforward formula

  \[
  m_T^\mathcal{H}(\lambda) = \int \int_{|u_1|=r_1, |u_2|=r_2} \frac{u_1^{-\lambda_1} u_2^{-\lambda_2}}{(1 - u_1^2)(1 - u_1 u_2)(1 - u_2^2)} \frac{du_1}{2i\pi u_1} \frac{du_2}{2i\pi u_2}.
  \]

  Let us fix \( u_1 \), and integrate on \( |u_2| = r_2 \). As \( \lambda_2 \geq 0 \), the poles of the integrand on \( |u_2| \leq r_2 \) is the pole at \( u_2 = 0 \). The other poles are \( u_2 = 1, u_1 = -1, u_1 = 1/u_1 \), and there are no poles at \( \infty \). Using the one dimensional residue theorem, we obtain \( m_T^\mathcal{H}(\lambda) = A + B + C \) with

  \[
  A = -\frac{1}{2} \int_{|u_1|=r_1} u_1^{-\lambda_1} \frac{du_1}{(1 - u_1)(1 - u_1^2)} 2i\pi u_1,
  \]

  \[
  B = -\frac{(-1)^{\lambda_2}}{2} \int_{|u_1|=r_1} u_1^{-\lambda_1} \frac{du_1}{(1 - u_1)(1 - u_1^2)} 2i\pi u_1,
  \]

  \[
  C = \int_{|u_1|=r_1} u_1^{\lambda_2 - \lambda_1 + 2} \frac{du_1}{(1 - u_1^2)^2 2i\pi u_1}.
  \]

  We now integrate in \( u_1 \). However, at this step, the computation is different if \( \lambda_2 \geq \lambda_1 \) or not. Indeed in the case where \( \lambda_2 \geq \lambda_1 \), the integral \( C \) will be equal to 0, as the integrand has no poles inside \( |u_1| \leq \frac{1}{2} \). So let us assume this is the case. We then pursue in computing similarly the integrals \( A, B \) by the one dimensional residue formula, and we obtain by summing the residues
at $u_1 = 1, u_1 = -1$ of the integrands expressions in $A, B$, a sum of 4 terms adding up to

$$
\frac{1}{4}(1 + (-1)^{\lambda_1 + \lambda_2})\lambda_2 + (1 + (-1)^{\lambda_1 + \lambda_2})\frac{3}{8} + \frac{1}{8}((-1)^{\lambda_1} + (-1)^{\lambda_2}),
$$

an expression vanishing if $\lambda_1 + \lambda_2$ is odd, as it should, and coinciding with the given expression

$$
m^{H}_{TK}(\lambda) = \frac{1}{2}\lambda_1 + \frac{3}{4} + (-1)^{\lambda_1} \frac{1}{4},
$$
on the cone of polynomiality $c_2$

- Let us now employ Theorem 53 using Jeffrey Kirwan residues.
  We consider the same case $\lambda_2 \geq \lambda_1$.
  Consider the tope $\tau_2 = \{\lambda; \lambda_1 > \lambda_2 > 0\}$, the interior of $c_2$. If we compute $OS(\Psi, \tau_2)$ for the order $\Psi = [[2, 0], [1, 1], [0, 2]]$, there is only one adapted basis $\sigma = [[2, 0], [0, 2]]$. The index $q$ is equal to 2 (in the lattice $\Lambda_K$ generated by $\Psi$). The function $S_T(\lambda, z)$ is equal to

$$
e^{\lambda_1 z_1 + \lambda_2 z_2}\\(1 - e^{-2z_1})(1 - e^{-2z_2})(1 - e^{-z_1 - z_2}).
$$

A representative of $\Gamma_K/2\Gamma_K$ is $G = [-1/2, 0]$, and we obtain

$$
\sum_{\gamma \in \Gamma_K/2\Gamma_K} S_T(\lambda, z + i\pi\gamma) = S_1 + S_2
$$

with

$$
S_1 = \frac{e^{\lambda_1 z_1 + \lambda_2 z_2}}{(1 - e^{-2z_1})(1 - e^{-2z_2})(1 - e^{-z_1 - z_2})},
$$

$$
S_2 = (-1)^{\lambda_1} \frac{e^{\lambda_1 z_1 + \lambda_2 z_2}}{(1 - e^{-2z_1})(1 - e^{-2z_2})(1 + e^{-z_1 - z_2})}.
$$

The iterated residue computation $Res_{z_1=0}Res_{z_2=0}S_1$ is straightforward and we obtain $\frac{1}{2}\lambda_1 + \frac{3}{4}$ from $S_1$, and $\frac{1}{4}(-1)^{\lambda_1}$ from the second term $S_2$.

It is also worthwhile to remark in this example that the iterated residue computation depends of the order. The reverse order $Res_{z_2=0}Res_{z_1=0}S_1$ would have given the formula for the tope $\tau_1 = \{(\lambda_1, \lambda_2), \lambda_1 > \lambda_2 > 0\}$. 

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5 Branching Rules

In this section, we will write a quasi polynomial formula for the branching coefficients. This will allow, in combination with the Cauchy formula, to express the Kronecker coefficients in a different way that has at least the advantage of reducing the number of variables in the setting, Section 6.

5.1 The Branching Cone

Consider a pair $K \subset G$ of two compact connected Lie groups, with Lie algebras $k, g$ respectively. Let $\pi : g^* \to k^*$ be the projection.

We consider the following action of $G \times K$ on $G$ : $(g, k) \cdot a = gak^{-1}$. The manifold $T^*G$ (the cotangent bundle of $G$) is a $G \times K$ Hamiltonian manifold. The geometric quantization of $T^*G$ is the space $L^2(G)$. This statement can be justified, but we will not do it here.

Let us define $V = R(G)$ to be the subspace of $C^\infty(G)$ generated by the coefficients $\langle gu, u \rangle$ of finite dimensional representations of $G$ (by Peter-Weyl theorem, the space $L^2(G)$ is the Hilbert completion of $V$).

Assume $G,K$ connected, and let $T_G, T_K$ be maximal tori of $G, K$. We may assume, and we do so, that $T_K \subset T_G$. We choose Cartan subalgebras $t, t$, Weyl chambers $i t^*_G > 0, i t^*_K > 0$, and we denote the corresponding cones of dominant weights by $\Lambda G, \geq 0, \Lambda K, \geq 0$. We denote by $i t^*_{G,K} > 0$ the sum $i t^*_G \oplus i t^*_K > 0$ of the closed positive Weyl chambers relatives to $G, K$, and by $i t^*_{G,K} > 0$ its interior. We may also choose compatible root systems on $K, G$: If $\lambda$ is dominant for $G$, then the restriction of $\lambda$ to $i t$ is dominant.

For $\lambda \in \Lambda G, \geq 0$ (resp. $\mu \in \Lambda K, \geq 0$), denote by $V^G_\lambda$ (resp. $V^K_\mu$) the irreducible representation of $G$ (resp. $K$) of highest weight $\lambda$ (resp. $\mu$).

Under the action of $G \times G$,

$$V = \bigoplus_{\lambda \in \Lambda G, \geq 0} V^G_\lambda \otimes V^G_{\lambda^*}.$$ 

So, under the action of $G \times K$,

$$V = \bigoplus_{\lambda, \mu} m_{G,K}(\lambda, \mu) V^G_\lambda \otimes V^K_\mu.$$ 

($\lambda$ varies in $\Lambda G, \geq 0$, and $\mu$ in $\Lambda K, \geq 0$) where $m_{G,K}(\lambda, \mu)$ is the multiplicity of the representation $\mu$ in the restriction of $V^G_\lambda$ to $K$; it is also called the branching coefficient.
Let us write coordinates for $T^*G$. We identify the tangent bundle $TG$ to $G \times \mathfrak{g}$ through the left translations: to $(g, X) \in G \times \mathfrak{g}$ we associate $\frac{d}{dt}e^{-tX}g|_{t=0} \in T_gG$. Thus, $T^*G$ is identified to $G \times \mathfrak{g}^*$, and the moment map relative to the $G \times K$-action is the map $\Phi_G \oplus \Phi_K : T^*G \to \mathfrak{g}^* \oplus \mathfrak{k}^*$ defined by $(g, \xi) \to (\xi, -\pi(g^{-1} \cdot \xi))$. In order to relate the moment map to the branching coefficient $m_{G,K}(\lambda, \mu)$, we use the slightly modified map: $\Phi : T^*G \to i\mathfrak{g}^* \oplus i\mathfrak{k}^*$ given by

$$\Phi(g, \xi) \to (i\xi, i\pi(g^{-1} \cdot \xi)).$$

Here $g \in G$, and $\xi \in \mathfrak{g}^*$.

Let

$$C_{G,K}(T^*G) = \Phi(T^*G) \cap i\mathfrak{g}^*_+, i\mathfrak{k}^*_+, \geq 0$$

be the Kirwan cone associated to $\Phi$:

$$C_{G,K}(T^*G) = \{ (\xi, \eta) \in i\mathfrak{g}^*_+, i\mathfrak{k}^*_+; \eta \in \pi(G \cdot \xi) \}.$$

The set $C_{G,K}(T^*G)$ is a polyhedral cone in $i\mathfrak{g}^*_+, i\mathfrak{k}^*_+, \geq 0$, and is related to the branching coefficients through the following basic result.

**Proposition 60.** We have $m_{G,K}(\lambda, \mu) = 0$ if $(\lambda, \mu) \notin C_{G,K}(T^*G)$.

Conversely, if $(\lambda, \mu)$ is a pair of dominant weights contained in $C_{G,K}(T^*G)$, there exists an integer $k > 0$ such that $m_{G,K}(k\lambda, k\mu)$ is non zero.

Thus the support of the function $m_{G,K}(\lambda, \mu)$ is contained in the Kirwan polyhedron $C_{G,K}(T^*G)$ and its asymptotic support is exactly the cone $C_{G,K}(T^*G)$.

Remark that if $G = K$, the cone $C_{G,K}(T^*G)$ is just the diagonal $\{(\xi, \xi) : \xi \in \mathfrak{g}^*_+, \mathfrak{k}^*_+, \geq 0\}$ in $i\mathfrak{g}^*_+, i\mathfrak{k}^*_+, \geq 0$. However, we assume from now on that no nonzero ideal of $\mathfrak{k}$ is an ideal of $\mathfrak{g}$ (this condition excludes the preceding case). It implies the following result (Duflo, private communication).

**Lemma 61.** The polytope $C_{G,K}(T^*G)$ is solid.

Let $\pi_\lambda : G\lambda \to i\mathfrak{k}^*$ be the restriction of $\pi$ to the orbit $G\lambda$.

**Definition 62.** Define the reduced space $M_{red,\mu}^\lambda = \pi_\lambda^{-1}(K\mu)/K$.

Remark that $\Phi^{-1}(G\lambda, K\mu)/(G \times K)$ is isomorphic to $M_{red,\mu}^\lambda$ and that the reduced space $M_{red,\mu}^\lambda$ is non empty if and only if $(\lambda, \mu) \in C_{G,K}(T^*G)$.

The $[Q, R] = 0$ theorem of Meinrenken-Sjamaar relates $m_{G,K}(\lambda, \mu)$ to the Riemann-Roch number (suitably defined) of the space $M_{red,\mu}^\lambda$. As a consequence, we obtain the following theorem.
Theorem 63. There exists a decomposition of the cone \( C_{G,K}(T^*G) = \bigcup_a c_a \), in closed solid polyhedral cones \( c_a \) such that the following property holds.

For each \( a \), there exists a non zero quasi-polynomial function \( p_a \) on \( \Lambda_G \oplus \Lambda_K \) such that
\[
m_{G,K}(\lambda,\mu) = p_a(\lambda,\mu)
\]
if \( (\lambda,\mu) \in c_a \cap (\Lambda_G \oplus \Lambda_K) \).

In particular, for any pair \( (\lambda,\mu) \) of dominant weights contained in \( C_{G,K}(T^*G) \), the function \( k \to m_{G,K}(k\lambda,k\mu) \) is of the form:
\[
m_{G,K}(k\lambda,k\mu) = \sum_{i=0}^N E_i(k)k^i
\]
where \( E_i(k) \) are periodic functions of \( k \). This formula is valid for any \( k \geq 0 \), and in particular \( E_0(0) = 1 \).

An interesting example is the case of \( K \) embedded in \( G = K \times K \) by the diagonal. Recall that \( c'_{\lambda,\mu} \) is the multiplicity of \( V_\nu \) in the tensor product \( V_\lambda \otimes V_\mu \). Thus we obtain

Corollary 64. Let \( K \) embedded in \( G = K \times K \) by the diagonal. If \( (\lambda,\mu,\nu) \) is in the \( C_{G,K}(T^*G) \), the dilated Littlewood-Richardson coefficient \( k \to c'_{k\lambda,k\mu}^{k\nu} \) is a quasi-polynomial function of \( k \in \{0,1,2,\ldots\} \).

To describe the cone \( C_{G,K}(T^*G) \) is difficult, and has been the object of numerous works, notably Berenstein-Sjamaar, Belkale-Kumar, Kumar, Ressayre. We refer to the survey article, [12]. The complete description of the multiplicity function \( m_{G,K} \), in particular the decomposition of \( C_{G,K}(T^*G) \) in \( \bigcup_a c_a \) is even more so. However, we will give an algorithm where, given as input \( (\lambda,\mu) \), the output is the dilated function \( k \to m_{G,K}(k\lambda,k\mu) \). In particular, we can test if the point \( (\lambda,\mu) \) is in the cone \( C_{G,K}(T^*G) \) or not, according if the output is not zero or zero.

Consider the set \( \Psi \subset i t_\mathfrak{t}^* \) of non zero restrictions of the roots \( \Delta_\mathfrak{g}^+ \) to \( i t_\mathfrak{t} \).

We say that \( \Psi \) is the list of restricted roots.

Recall that an hyperplane in \( i t_\mathfrak{g}^* \) is \( \Psi \)-admissible if it is spanned by elements of \( \Psi \). The set \( \mathcal{A} = \mathcal{A}(\Psi) \) of admissible hyperplanes is finite. For \( H \in \mathcal{A} \), consider \( X \in \mathfrak{t}_\mathfrak{g} \) such that \( H = X^\perp \). Let \( \mathcal{W}_\mathfrak{g} \) be the Weyl group of \( G \). If \( X \in \mathfrak{t}_\mathfrak{t} \subset \mathfrak{t}_\mathfrak{g} \), consider \( wX \in \mathfrak{t}_\mathfrak{g} \) and the hyperplane
\[
H(w) = \{(\xi,\nu) \in i t_\mathfrak{g}^* \oplus i t_\mathfrak{t}^*; \langle \xi, wX \rangle - \langle \nu, X \rangle = 0 \}.
\]
We obtain a finite set of hyperplanes \( \mathcal{F} \) in \( i t_\mathfrak{g}^* \oplus i t_\mathfrak{t}^* \).

Consider a connected component \( \tau \) of the complement of the union of the hyperplanes \( H(w) \), where \( H \) varies over admissible hyperplanes in \( t_\mathfrak{t}^* \).
and \(w\) in the Weyl group of \(G\), in other words \(\tau\) is a tope for the system of hyperplanes \(\mathcal{F}\). So \(\tau\) is an open conic subset of \(i t_g^* \oplus i t_r^*\). Thus, if \((\xi, \nu) \in \tau\), for any admissible hyperplane \(H \in \mathcal{A}\) with equation \(X\), and any \(w \in \mathcal{W}_g\), we have
\[
\langle \xi, wX \rangle - \langle \nu, X \rangle \neq 0.
\] (7)

The following proposition follows from the description of the Duistermaat-Heckman measure [22].

**Proposition 65.** The facets of the cones \(c_a\) generates hyperplanes belonging to the family \(\mathcal{F}\).

Thus the following lemma follows, and will be useful.

**Lemma 66.** Fix a cone \(c_a\).

- If \(\tau\) is a tope, then \(\tau \cap it^{*}_{g,t,\geq 0}\) is either contained in \(c_a\), or is disjoint from \(c_a\).

- The closed cone \(c_a\) is the union of the closures of the sets \(\tau \cap it^{*}_{g,t,\geq 0}\) over the \(\tau\) such that \(\tau \cap c_a\) is non empty.

Remark that there might be several topes \(\tau\) needed to obtain \(c_a\).

We rephrase Theorem 63 as follows.

**Proposition 67** (“Continuity property” of \(m_{G,K}\)).

- If \(\tau\) is a tope, the function \((\lambda, \mu) \rightarrow m_{G,K}(\lambda, \mu)\) is given by a quasi polynomial function \(n_\tau\) on \(\tau \cap \Lambda_{G,K,\geq 0}\).

- If \(\tau \cap it^{*}_{g,t,\geq 0}\) is contained in the cone \(C_{G \times K}(T^*G)\), and if \((\lambda, \mu) \in \tau \cap \Lambda_{G,K,\geq 0}\), then

\[
m_{G,K}(\lambda, \mu) = n_\tau(\lambda, \mu).
\]

We finally state a result on the behavior of the function \(m_{G,K}(\lambda, \mu)\) on the boundary of the polyhedral cone \(C_{G \times K}(T^*G)\).

If \(X \in it_t\), we have an injection \(K(X) \subset G(X)\). Let \(X \in it_t\), \(H = X^\perp\), and \(w \in W\) such that \(\langle \xi, wX \rangle - \langle \nu, X \rangle \geq 0\) for all \((\xi, \nu) \in C_{G \times K}(T^*G)\). We assume that \(H(w)\) contains an element \((\lambda, \mu) \in it^{*}_{G,K,\geq 0}\), in other words that \(F\) is a regular face. Let \(F = H(w) \cap C_{G \times K}(T^*G)\) be the corresponding face of the polyhedral cone \(C_{G \times K}(T^*G)\).
As \( w \) is defined modulo the Weyl group of the stabilizer of \( X \), we may choose \( w \) such that, if \( \lambda \) is dominant for \( G \), then \( w^{-1}(\lambda) \) is dominant for \( G(X) \). If \( (\lambda, \mu) \in F \), the couple \( (w\lambda, \mu) \) is a couple of dominant weights for \((G(X), K(X))\). The following proposition follows again from the \([Q, R] = 0\) theorem of Meinrenken-Sjamaar (see also [15]).

**Proposition 68.** For any \( (\lambda, \mu) \in F \), we have \( m_{G,K}(\lambda, \mu) = m_{G(X), K(X)}(w\lambda, \mu) \).

A proof of this theorem using surjectivity of the restriction of holomorphic sections, with given invariance, is given in Ressayre [47].

**5.2 Branching theorem: a piecewise quasi-polynomial formula**

Our aim is to give an explicit quasi-polynomial formula for \( m_{G,K}(\lambda, \mu) \) on a tope \( \tau \) in terms of iterated residues.

We assume, for simplicity, that \( i_k \) contains a regular (with respect to \( \Delta_k \)) element \( X \) which is regular also for \( \Delta_g \) (this is not always the case). We use this element to define positive compatible root systems \( \Delta_g^+ \) and \( \Delta_k^+ \) as follows: \( \Delta_g^+ := \{ \alpha \in \Delta_g, \alpha(X) > 0 \} \) and \( \Delta_k^+ := \{ \alpha \in \Delta_k, \alpha(X) > 0 \} \). Given \( \alpha \in \Delta_g^+ \), denote by \( \overline{\alpha} \) the restriction \( \alpha|_{i_k} \). Then \( 0 \neq \overline{\alpha} \in \Delta_k^+ \). Thus the system \( \Psi \) in \( i_k \) consists of the restrictions of \( \Delta_g^+ \) repeated with multiplicities:

\[
\Psi = [\overline{\alpha}, \alpha \in \Delta_g^+] .
\]

The system \( \Psi \) contains \( \Delta_k^+ \). By our construction, all elements \( \psi \) in \( \Psi \) satisfy \( \langle \psi, X \rangle > 0 \).

We denote by \( \Psi \setminus \Delta_k^+ \) the list where we have removed \( \Delta_k^+ \) from \( \Psi \). More precisely, if \( \alpha \in \Delta_k^+ \) occur in \( \Psi \) with multiplicity \( m > 0 \), then \( \alpha \in \Delta_k^+ \) occur in \( \Psi \setminus \Delta_k^+ \) with multiplicity \( m - 1 \).

**Example 69.**

Let \( G = SU(n) \) and \( K = SU(n_1) \times SU(n_2) \) with \( n = n_1n_2 \). We consider \( \mathfrak{t} \) the Cartan subalgebra of \( \mathfrak{g} \) given by the diagonal matrices of trace zero and \( \mathfrak{t}_k = \mathfrak{t}_1 \times \mathfrak{t}_2 \) the Cartan subalgebras of \( \mathfrak{k} \) given by the corresponding diagonal matrices. We take the embedding from \( \mathfrak{t}_1 \times \mathfrak{t}_2 \to \mathfrak{t} \) given by

\[
diag(a_1, \ldots, a_{n_1}) \times diag(b_1, \ldots, b_{n_2}) \to
\]
diag(a_1 + b_1, a_2 + b_1, \ldots, a_n + b_1, a_1 + b_2, \ldots, a_n + b_2, \ldots, a_1 + b_{n_2}, \ldots, a_n + b_{n_2}).

We take the lexicographic order. The list of restricted roots is thus the list

$$\Psi = [(a_i - a_j + b_k - b_{\ell})].$$  \hfill (8)

There \(i, j\) varies between 1 and \(n_1\), and \(k, \ell\) and varies between 1 and \(n_2\). The couple \((i, k)\) being different from \((j, \ell)\), so all restricted roots are non zero. This lexicographic order is compatible, as one check easily, that the restrictions are not zero and we get exactly the order \(a_1 \geq a_2 \geq \cdots \geq a_{n_1}\) on \(t_1\) and similarly for \(t_2\).

Explicitly, we can take the embedding of the element

\[ X = \text{diag}(n_1, \ldots, 1) \times \text{diag}((n_2 - 1)n_1 + 1, \ldots, 1). \]

For \(n_1 = 2, n_2 = 3\) then

\[ X = \text{diag}(2, 1) \times \text{diag}(5, 3, 1) \]

and the embedded element is \(\text{diag}(7, 6, 5, 4, 3, 2)\).

We do not know how to compute the set \(A(\Psi)\) of admissible hyperplanes for the system \(\Psi\), for any \((n_1, n_2)\). We computed it for a few examples (see also \([52]\)), but we do not see a general pattern. Furthermore, the cardinal of the set \(A(\Psi)\) (up to Weyl group action) seems to grow quickly. □

Let \(\lambda \in \Lambda_{G, \geq 0}\) be the highest weight of an irreducible representation of \(G\). The character \(\chi_\lambda\) of \(V^G_\lambda\) is given by the Hermann Weyl character formula:

\[ \chi_\lambda|_{T_G} = \sum_{w \in W_0} \frac{e^{w(\lambda)}}{\prod_{\alpha \in \Delta^+_G} (1 - e^{-w(\alpha)})}. \]

Restricting on \(T_K\), we obtain:

\[ \chi_\lambda|_{T_K} = \sum_{w \in W_0} \frac{e^{w(\lambda)}}{\prod_{\alpha \in \Delta^+_K} (1 - e^{-w(\alpha)})}. \]

Let \(w \in W_0\). Let \(z \in (t_\ell)_\mathbb{C}\). Consider the meromorphic function of \(z\) given by

\[ s^{w}_{\lambda, \mu}(z) = \frac{e^{(w(\lambda) - \mu, z)}}{\prod_{\alpha \in \Delta^+_K} (1 - e^{-(w(\alpha), z)})}. \]
Define

\[
S^w_{\lambda, \mu}(z) = \prod_{\beta \in \Delta^+_t} (1 - e^{-\langle \beta, z \rangle}) s^w_{\lambda, \mu}(z) = \frac{\prod_{\beta \in \Delta^+_t} (1 - e^{-\langle \beta, z \rangle}) e^{i\langle w(\lambda) - \mu, z \rangle}}{\prod_{\alpha \in \Delta^+_g} (1 - e^{-\langle w(\alpha), z \rangle})}.
\]

Consider a tope \( \tau \subset i^*_g \oplus i^*_t \) for the system of hyperplanes \( \mathcal{F} \). Let \( (\xi, \nu) \in \tau \). Then for each \( w \in W_g \), the element \( w(\xi) - \nu \) is \( \Psi \)-regular. Given \( w \in W_g \), we note \( a(w(\xi) - \nu) \) the tope for \( \Psi \), which contains the element \( w(\xi) - \nu \). The tope \( a(w(\xi) - \nu) \) depends only from \( w \) and \( \tau \), so we denote it by \( a_w^\tau \). We have defined the set \( \mathcal{OS}(\Psi, a^\tau_w) \) of adapted basis to the tope \( a^\tau_w \) (Def. 49). Let \( \Gamma_K \) be the dual lattice to \( \Lambda_K \), and let \( q = q(\Psi) \) be the index of \( \Psi \).

**Proposition 70.** Let \( \tau \) be a tope in \( i^*_g \oplus i^*_t \) for the system \( \mathcal{F} \). Define for \( (\lambda, \mu) \in \Lambda_G \oplus \Lambda_K \),

\[
p_{\tau}(\lambda, \mu) = \sum_{w \in W_g} \sum_{\gamma \in \Gamma_K/q\Gamma_K} \sum_{\sigma \in \mathcal{OS}(\Psi, a^\tau_w)} \text{Res}_{\rightarrow} S^w_{\lambda, \mu}(z + \frac{2i\pi}{q} \gamma).
\]

Then, the function \( p_{\tau} \) is a quasi-polynomial function on \( \Lambda_G \oplus \Lambda_K \).

**Proof.** Explicitly,

\[
p_{\tau}(\lambda, \mu) = \sum_{w \in W_g} \sum_{\gamma \in \Gamma_K/q\Gamma_K} \sum_{\sigma \in \mathcal{OS}(\Psi, a^\tau_w)} \text{Res}_{\rightarrow} \frac{e^{i\langle w(\lambda) - \mu, z + \frac{2i\pi}{q} \gamma \rangle}}{\prod_{\alpha \in \Delta^+_g} (1 - e^{-\langle w(\alpha), z + \frac{2i\pi}{q} \gamma \rangle})} \prod_{\beta \in \Delta^+_t} (1 - e^{-\langle \beta, z + \frac{2i\pi}{q} \gamma \rangle}).
\]

\( \square \)

**Theorem 71.** Let \( \tau \) be a tope, and let \( (\lambda, \mu) \in \tau \cap \Lambda_{G,K,\geq 0} \). Then

1. if \( (\lambda, \mu) \notin C_{G,K}(T^*G) \),

\[
m_{G,K}(\lambda, \mu) = p_{\tau}(\lambda, \mu).
\]

2. if \( (\lambda, \mu) \in C_{G,K}(T^*G) \), and the tope \( \tau \) intersect \( C_{G,K}(T^*G) \), then

\[
m_{G,K}(\lambda, \mu) = p_{\tau}(\lambda, \mu).
\]
To test if a regular element \((\lambda, \mu)\) is in the cone \(C_{G,K}(T^*G)\). Take the tope \(\tau\) containing \((\lambda, \mu)\). The point \((\lambda, \mu)\) is in the cone if and only if the quasi polynomial function \(k \to p_\tau(k\lambda, k\mu)\) is non zero.

Before going into the proof of this theorem, let us make some remarks of how to somewhat reduce the complexity of this formula.

**Remark 72.**

Since the system \(\Psi\) contains the system \(\Delta_{G_0}^+\), the function \(S_{\lambda,\mu}^w(z)\) could be written as function with the smaller denominator \(\prod_{\psi \in \Psi/\Delta_{G_0}^+} (1 - e^{-\langle \psi, z \rangle})\) and the residues could be taken over adapted OS basis of the system \(\Psi \setminus \Delta_{G_0}^+\). Not so many \(w\) giving a non zero contribution to the formula. Indeed, at least \(w\) has to be such that \(w\lambda - \mu\) is in the cone generated by the restricted roots. This is the so called valid permutations for \((\lambda, \mu)\) defined by Cochet in [15], and his algorithm construct them recursively. For example, as computed by Pamela Harris, [21], when \(\Psi\) is the system \(f\) positive roots of \(A_r\), \(\lambda\) the highest root and \(\mu = 0\), the number of \(w\) such that \(w\lambda\) is in the cone generated by the positive roots is the Fibonacci number \(f(r)\), much smaller that \((r + 1)!\), the order of the Weyl group.

\(\square\)

**Proof.** Consider 

\[
\chi_{\lambda|_{T_K}} = \sum_{w \in W_0} \frac{e^{w(\lambda)}}{\prod_{\alpha \in \Delta_0^+} (1 - e^{-w(\alpha)})}.
\]

Using our regular element \(X\), we rewrite this formula polarizing the linear form \(w(\alpha)\): if \(\langle w(\alpha), X \rangle < 0\), we replace \(w(\alpha)\) by its opposite; we then make use of the identity \(\frac{1}{1 - e^{-\beta}} = -e^{-\beta}/1 - e^{-\beta}\). Precisely, write \(\Psi = \Psi^1 \cup \Psi^2\) as the disjoint union of the two sets:

\[
\Psi^1_w = \{w\alpha, \alpha \in \Delta_0^+, \langle w(\alpha), X \rangle > 0\}
\]

\[
\Psi^2_w = \{-w\alpha, \alpha \in \Delta_0^+, \langle w(\alpha), X \rangle < 0\}.
\]

If \(s_w = |\Psi^2_w|\) and \(e^{g_w} = \prod_{w(\alpha), \langle w(\alpha), X \rangle < 0} e^{w(\alpha)}\), then we obtain that \(\chi_{\lambda|_{T_K}}\) is equal to

\[
\sum_{w \in W_0} \left(\frac{e^{w(\lambda)}}{\prod_{\psi \in \Psi^1_w} (1 - e^{-\psi})} \frac{(-1)^{s_w} e^{g_w}}{\prod_{\psi \in \Psi^2_w} (1 - e^{-\psi})}\right) = \sum_{w \in W_0} \left(\frac{e^{w(\lambda)}}{\prod_{\psi \in \Psi} (1 - e^{-\psi})}\right).
\]
Lemma 73. The following holds on $T_K$:

$$
\chi_\lambda = \sum_{w \in W} \sum_{\mu \in \Lambda^K} (-1)^{s_w} \mathcal{P}_\Psi(w(\lambda) + g_w - \mu) e^\mu
$$

where $\mathcal{P}_\Psi$ is the partition function determined by the restricted roots $\Psi$. So

$$
m_{G,T}(\lambda, \mu) = \sum_{w \in W} (-1)^{s_w} \mathcal{P}_\Psi(w(\lambda) + g_w - \mu). \tag{9}
$$

When $K$ is the maximal torus $T_G$, the formula above is Kostant multiplicity formula for a weight [33]. The formula (9) is obtained by the same method.

Let us now use the formula

$$
m_{G,K}(\lambda, \mu) = \sum_{\tilde{w} \in W_t} \epsilon(\tilde{w}) m_{G,T}(\lambda, \mu - \tilde{w}(\rho_t) + \rho_t).
$$

We obtain for $(\lambda, \mu) \in \Lambda_{G,K,\geq 0}$

$$
m_{G,K}(\lambda, \mu) = \sum_{\tilde{w} \in W_t} \epsilon(\tilde{w}) \sum_{w \in W} (-1)^{s_w} \mathcal{P}_\Psi(w(\lambda) + g_w - (\mu - \tilde{w}(\rho_t) + \rho_t))
$$

(we may rewrite this expression as a sum of partition functions for $\Psi \backslash \Delta^+_t$, obtaining Heckman formula, [22], [34]).

Suppose first that $(\lambda, \mu) \in \tau \cap \Lambda_{G,K,\geq 0}$ is regular and ”very” far away from all the walls $H(w)$. So for all $w, \tilde{w}$, the element $w(\lambda) - \mu + g_w + \tilde{w}(\rho_t) - \rho_t$ is still in the tope $a^+_w$ for $\Psi$ containing $w(\lambda) - \mu$. We thus can employ the iterated residue formula (Formula 6) for $\mathcal{P}_\Psi(w(\lambda) + g_w - (\mu - \tilde{w}(\rho_t) + \rho_t))$ on $a^+_w$. We obtain that $m_{G,K}(\lambda, \mu)$ is equal to

$$
\sum_{w \in W_t} \epsilon(\tilde{w}) (-1)^{s_w} \sum_{\tilde{w} \in W} \sum_{\gamma \in \Gamma^K/\Gamma^K} \sum_{\sigma \in \text{OS}(\Psi, a^+_w)} \text{Res}_z \frac{e^{(w(\lambda) + g_w - (\mu - \tilde{w}(\rho_t) + \rho_t), z + \frac{2\pi z}{q} \gamma)}}{\prod_{\psi \in \Psi} (1 - e^{-\langle \psi, z + \frac{2\pi z}{q} \gamma \rangle})}.
$$

Inverting the polarization process, we rewrite

$$
(-1)^{s_w} \frac{e^{(w(\lambda) + g_w, z + \frac{2\pi z}{q} \gamma)}}{\prod_{\psi \in \Psi} (1 - e^{-\langle \psi, z + \frac{2\pi z}{q} \gamma \rangle})} = \frac{e^{(w(\lambda), z + \frac{2\pi z}{q} \gamma)}}{\prod_{\alpha \in \Delta^+_t} (1 - e^{-\langle \alpha, z + \frac{2\pi z}{q} \gamma \rangle})}.
$$
So, remembering that \(\prod_{\beta \in \Delta^+_t}(1 - e^{-\beta}) = \sum_we^{-\rho_t + \bar{w}(\rho_t)}\), we obtain

\[ m_{G,K}(\lambda, \mu) = p_{\tau}(\lambda, \mu) \]

when \((\lambda, \mu) \in \tau \cap \Lambda_{G,K,\geq 0}\) is regular and “very” far away from all the walls \(H_w\). Theorem 63 asserts that \(m_{G,K}(\lambda, \mu)\) is given by a quasi polynomial formula on \(\tau \cap \Lambda_{G,K,\geq 0}\) or even on \(\tau \cap i_{g,\Sigma}^*\). Theorem 70 asserts that \(m_{G,K}(\lambda, \mu)\) is given by a quasi polynomial formula on \(\tau \cap \Lambda_{G,K,\geq 0}\) or even on \(\tau \cap i_{g,\Sigma}^*\) if \(\tau \cap i_{g,\Sigma}^*\) is contained in \(C_{G,K}(T^*G)\).

Using Lemma 28 we obtain Theorem 70. \(\square\)

5.3 Singular case

When the stabilizer of \(\lambda\) is large, that is \(\langle \lambda, H_\alpha \rangle\) is equal to 0 for a large number of \(\alpha\), then we can rewrite the Hermann Weyl formula for the character in a way that takes advantage of this.

Fix a subset \(\Sigma\) of the simple roots of \(\Delta^+_g\), and let \(l\) be the Levi subalgebra of \(g\), with simple root system \(\Sigma\). Let \(\Delta^+_l\) be its positive root system.

Let \(i_{g,\Sigma}^*\) be the set of the elements \(\xi \in i_{g,\Sigma}^*\) such that \(\langle \xi, H_\alpha \rangle = 0\) for all \(\alpha \in \Sigma\). We define consistently \(\Lambda_{G,\Sigma}^0 = \Lambda_G \cap i_{g,\Sigma}^*,\) a lattice in \(i_{g,\Sigma}^*,\) \(\Lambda_{G,K,\geq 0} = \Lambda_{G,K,\geq 0} \cap i_{g,\Sigma}^*,\) and similarly.

We also define

\[ C_{G,K}^\Sigma(T^*G) = \{ (\xi, \nu) \in C_{G,K}(T^*G); \xi \in i_{g,\Sigma}^* \}. \]

In other words, \(\nu\) must belong to the projection on \(i_{g,\Sigma}^*\) of the singular orbit \(G\xi\).

The cone \(C_{G,K}^\Sigma(T^*G)\) is solid in \(i_{g,\Sigma}^*,\) if and only if there exists \(\xi \in i_{g,\Sigma}^*\) such that the projection on \(i_{g,\Sigma}^*\) of the singular orbit \(G\xi\) has a non zero interior in \(i_{g,\Sigma}^*\). In other words the Kirwan polytope \(\pi(G\xi) \cap i_{g,\Sigma}^*\) is solid.

Example 74.

Consider the embedding of \(K = U(n_2) \times U(n_3)/Z\) in \(G = U(n_2n_3)\). Here \(n_2, n_3 \geq 2\) and \(Z\) is the subgroup \(\{t_2Id, t_3Id\}\) of the center of \(U(n_2) \times U(n_3)\) with \(t_2t_3 = 1\). We take \(\lambda\) a weight of \(G\) with more than two non zero coordinates. Then \(\pi(G\lambda)\) has interior in \(i_{g,\Sigma}^*\).

\(\square\)

Let

\[ \Delta_u = \Delta^+_g \backslash \Delta^+_l \]

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and denote by $W_l$ the Weyl group of $l$.

For any $\lambda \in \Lambda_{G,\geq 0}\cap it^*_\Sigma$, we can write the character formula on $T_G$ and the restriction on $T_K$ as:

$$
\chi_{\lambda|T_G} = \sum_{w \in W_l/W_l} \frac{e^{w(\lambda)}}{e^{w(\lambda)/W_l}} \prod_{\alpha \in \Delta_u} (1 - e^{-\langle w(\alpha), z \rangle}).
$$

$$
\chi_{\lambda|T_K} = \sum_{w \in W_l/W_l} \frac{e^{w(\lambda)}}{e^{w(\lambda)/W_l}} \prod_{\alpha \in \Delta_u} (1 - e^{-\langle w(\alpha), z \rangle}).
$$

In the regular case, the Levi component is just the Cartan subalgebra, the parabolic is the Borel subalgebra, and $\Delta_u = \Delta^+_l$.

To compute $m_{G,K}(\lambda, \mu)$ for $\Lambda_{G,\geq 0}\cap it^*_\Sigma$ by iterated residues, it is tempting to replace the function $S^w_{\lambda, \mu}$ by the function

$$
S^\Sigma_{\lambda, \mu}(z) = \prod_{\beta \in \Delta^+_l} (1 - e^{-\langle \beta, z \rangle}) \frac{e^{\langle w(\lambda)/W_l, \mu, z \rangle}}{\prod_{\alpha \in \Delta_u} (1 - e^{-\langle w(\alpha), z \rangle})}.
$$

We consider the system of hyperplanes $F_\Sigma$ in $it^*_\Psi \oplus it^*_l$ defined by the equations $\langle \xi, wX \rangle - \langle \xi, X \rangle = 0$, where $X$ is an equation for a $\Psi$-admissible hyperplane, and $w \in W_l$. If $(\xi, \nu) \in it^*_\Psi \oplus it^*_l$ is in a tope $\tau_\Sigma$ for $F_\Sigma$, then $(\xi, \nu)$ is in a unique tope $\tau$ for $F$, and determines a tope $a^w_{\tau}$ in $it^*_l$.

The following proposition is clear.

**Proposition 75.** Let

$$
p^\Sigma_{\tau}(\lambda, \mu) = \sum_{w \in W_l/W_l} \sum_{\gamma \in \Gamma_K/\Gamma_K} \sum_{\sigma \in \text{OS}(\Psi_a^\Sigma)} \text{Res}_{\sigma} S^\Sigma_{\lambda, \mu}(z + 2i\pi \gamma).q
$$

Then $p^\Sigma_{\tau}(\lambda, \mu)$ is a quasi polynomial function on $\Lambda^\Sigma_G \oplus \Lambda_K$.

Remark that if $\tau_\Sigma$ is a tope for $F_\Sigma$, then $\tau_\Sigma \cap C^\Sigma_{G,K}(T^*G)$ is empty if $C^\Sigma_{G,K}(T^*G)$ is not solid. If the cone $C^\Sigma_{G,K}(T^*G)$ is solid, it is the union of the closures of the open sets $\tau_\Sigma \cap t^*_\Psi_{l, t \geq 0}$ contained in $C^\Sigma_{G,K}(T^*G)$.

**Theorem 76.** Let $\tau_\Sigma$ a tope in $it^*_\Psi \oplus it^*_l$ for $F_\Sigma$, and let $(\lambda, \mu) \in \tau_\Sigma \cap \Lambda^\Sigma_{G,K, \geq 0}$. Then
1. If \((\lambda, \mu) \notin C_{G,K}^\Sigma(T^*G)\),

\[ m_{G,K}(\lambda, \mu) = p_{\tau}^\Sigma(\lambda, \mu). \]

2. If \((\lambda, \mu) \in C_{G,K}^\Sigma(T^*G)\), and the tope \(\tau_\Sigma\) intersects \(C_{G,K}^\Sigma(T^*G)\), then

\[ m_{G,K}(\lambda, \mu) = p_{\tau}^\Sigma(\lambda, \mu). \]

**Proof.** If the set \(\tau_\Sigma \cap C_{G,K}^\Sigma(T^*G)\), is contained in \(\tau \cap C_{G,K}(T^*G)\), so we know (by before) that on \(\tau \cap \Lambda_{G,K,>0}\), the function \(m_{G,K}\) is given by a quasi polynomial formula, so a fortiori its restriction to \(\tau_\Sigma \cap \Lambda_{G,K,>0}\). So it is sufficient to prove that when \(\tau_\Sigma \cap \Lambda_{G,K,>0}\) is sufficiently far away from all walls belonging to \(F_\Sigma\), then \(m_{G,K}(\lambda, \mu)\) coincide with \(p_{\tau}^\Sigma\).

The proof is very similar to the preceding proof, so we skip details.

Let \(w \in W_g\). Because of our assumption on compatible systems, namely the existence of a regular element \(X\), we can define \(\Psi_{w,u} = \Psi_{w,u}^1 \cup \Psi_{w,u}^2\) with \(\Psi_{w,u}^1 = \{\overline{w(\alpha)}, \alpha \in \Delta_u, \langle w(\alpha), X \rangle > 0\}\), \(\Psi_{w,u}^2 = \{-\overline{w(\alpha)}, \alpha \in \Delta_u, \langle w(\alpha), X \rangle < 0\}\). Elements in \(\Psi_{w,u}\) are positive on \(X\), so \(\Psi_{w,u}\) is contained in \(\Psi\). In contrast to the regular case, \(\Psi_{w,u}\) depends of \(w\) and may not contain \(\Delta_u^+\).

Let \(s_{w}^\Sigma = |\Psi_{w,u}^2|\) and \(e_{w}^\Sigma = \prod_{w(\alpha), \langle w(\alpha), X \rangle < 0} e^{\overline{w(\alpha)}}\) then we obtain

\[ \chi_{\lambda|\tau_\Sigma} = \sum_{w \in W_g/W_\ell} \left( \frac{e^{\overline{w(\lambda)}}(-1)^{s_{w}^\Sigma} e_{w}^\Sigma}{\prod_{\psi \in \Psi_{w,u}} (1 - e^{-\psi})} \right) \]

and

\[ m_{G,K}(\lambda, \mu) = \sum_{w \in W_\ell} e(\overline{w}) \sum_{w \in W_g/W_\ell} (-1)^{s_{w}^\Sigma} \mathcal{P}_{\Psi_{w,u}}(\overline{w(\lambda)} + q_{w}^\Sigma - (\mu - \overline{w(\rho_\ell)} + \rho_\ell)). \]

The point \((\lambda, \mu)\) being in \(\tau_\Sigma\), the point \(\overline{w(\lambda)} - \mu\) is in \(a_{w}^\tau\). We can assume that \((\lambda, \mu)\) is sufficiently far away from all walls, so that \(\overline{w(\lambda)} + q_{w}^\Sigma - (\mu - \overline{w(\rho_\ell)} + \rho_\ell)\) is also in \(a_{w}^\tau\), so that we can apply the iterated residue formula for

\[ \mathcal{P}_{\Psi_{w,u}}(\overline{w(\lambda)} + q_{w}^\Sigma - (\mu - \overline{w(\rho_\ell)} + \rho_\ell)) \]

as a sum of iterated residue with respect to adapted OS basis for \(a_{w}^\tau\). Then we proceed as in the preceding case, reversing the polarization process, and obtain

\[ m_{G,K}(\lambda, \mu) = p_{\tau}^\Sigma(\lambda, \mu) \]
provided \((\lambda, \mu)\) is in \(\tau\) and sufficiently for away from the walls.

If \(\tau \cap C^G_{G, K}(T^* G) \subset \tau \cap C_{G, K}(T^* G)\) the formula is quasi polynomial, so we obtain our theorem.

\[\square\]

# 6 Kronecker coefficients and examples

We describe now our approach to compute Kronecker coefficients, the result is summarized in Corollary \ref{cor:kronecker}.

Consider \(N = n_1 \cdots n_s\) and assume that \(n_1\) is the maximum of the \(n_i\). Write \(\mathcal{H} = \mathbb{C}^N = \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_s}\) and consider the action of \(U(n_1) \times \cdots \times U(n_s)\) in \(\text{Sym}(\mathcal{H})\).

Thus

\[
\text{Sym}(\mathcal{H}) = \sum g(\mu_1 \otimes \cdots \otimes \mu_s) V_{\mu_1}^{U(n_1)} \otimes \cdots \otimes V_{\mu_s}^{U(n_s)}.
\]

We want to compute \(g(\mu_1, \ldots, \mu_s)\) and the dilated coefficients.

Let \(M = n_2 \cdots n_s\) and consider the embedding of \(K = U(n_2) \times \cdots \times U(n_s)\) in \(G = U(M)\) determined by: \((k_2, \ldots, k_s)(v_2 \otimes \cdots \otimes v_s) = k_2 v_2 \otimes \cdots \otimes k_s v_s\), as we explained in Example \ref{ex:kronecker}

Using the Cauchy formula \ref{eq:cauchy} we can write

\[
\text{Sym}(\mathbb{C}^N) = \text{Sym}(\mathbb{C}^{n_1} \otimes \mathbb{C}^M) = \sum_{\mu_1 \in \mathfrak{P}(U(n_1))} V_{\mu_1}^{U(n_1)} \otimes V_{\mu_1}^{U(M)}.
\]

Write the decomposition of \(V_{\mu_1}^{U(M)}\) restricted to \(K\):

\[
V_{\mu_1}^{U(M)} = \bigoplus_{\mu_2 \in U(n_2), \ldots, \mu_s \in U(n_s)} m_{G, K}(\bar{\mu}_1, \mu_2 \otimes \cdots \otimes \mu_s) V_{\mu_2}^{U(n_2)} \otimes \cdots \otimes V_{\mu_s}^{U(n_s)}.
\]

Remember that the polynomial irreducible representation of \(U(n_k)\) are parameterized by the highest weight \(\gamma = [\gamma_1, \ldots, \gamma_{n_k}]\) with \(\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n_k} \geq 0\) and that \(|\gamma| = \sum \gamma_i\). Taking care of the fact that if \(\mu_1 \otimes \cdots \otimes \mu_s\) occur in \(\text{Sym}(\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_s})\) we must have that the actions on the centers must be the same, we obtain:

**Corollary 77.**

\[
g(\mu_1 \otimes \cdots \otimes \mu_s) = \begin{cases} m_{G, K}(\bar{\mu}_1, \mu_2 \otimes \cdots \otimes \mu_s) & \text{if } |\mu_1| = |\mu_2| = \cdots = |\mu_s| \\ 0 & \text{otherwise} \end{cases}
\]
We take advantage of the formula to somewhat reduce the computation of a tensor product with \( s \) factors to an analogous computation with \( s - 1 \) factors. So we will compute some of the Kronecker coefficients as a corollary of the branching theorem.

The case \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4 \) corresponds to give explicit formulae for the decomposition of the representation of the symmetric group associated to partitions with at most two rows. Complete expressions for these functions have already been obtained by Briand et al.

We list the examples we can compute in Subsection 6.2. For example, we can compute the dilated Kronecker coefficients for \( \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \), as well as some examples for \( \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4 \).

6.1 The algorithm to compute Kronecker coefficients

We refer to Section 5.3 and Section 6 for the notation.

We are given a sequence of \( s \) strictly positive integers \([n_1, \ldots, n_s]\) and for each integer \( n_i \) a sequence \( \nu_i \) of integers: \( \nu_i = [\nu_{i1}, \ldots, \nu_{in_i}] \) with \( \nu_{i1} \geq \nu_{i2} \geq \cdots \geq \nu_{in_i} \geq 0 \). Each \( \nu_i \) parametrizes an irreducible polynomial representation of \( U(n_i) \) of highest weight \( \nu_i \).

Write \( N = \prod_{i=1}^s n_i \) and \( M = \prod_{i=2}^s n_i \). We want to compute the Kronecker coefficients \( g(k\nu_1, \cdots, k\nu_s) \) dilated by an integer \( k \) that is the multiplicity of the tensor product representation \( k\nu_1 \otimes \cdots \otimes k\nu_s \) in \( \text{Sym}(\mathbb{C}^N) \). Our approach uses Cauchy formula together with the computation of the branching coefficients to reduce the number of parameters. We may assume that \( n_1 \leq M \) and that \( |\nu_1| = |\nu_2| = \cdots = |\nu_s| \).

We set \( G = U(M) \) and \( K = U(n_2) \times \cdots \times U(n_s) \).

The first reduction step is:

- If \( \nu_1 = |\nu_2| = \cdots = |\nu_s| \) then \( g(\nu_1 \otimes \cdots \otimes \nu_s) = m_{G,K}(\hat{\nu}_1, \nu_2 \otimes \cdots \otimes \nu_s) \) where \( \hat{\nu}_1 \) is the highest weight representation of \( U(M) \) obtained by \( \nu \) adding \( M - n_1 \) zeros and the branching coefficient \( m_{G,K} \) is computed in Theorem 76 via the function defined in Proposition 75.

Let us write \( \lambda = \hat{\nu}_1, \mu = \nu_2 \otimes \cdots \otimes \nu_s \). If \( n_1 < M \), \( \lambda \) is a singular weight for the group \( U(M) \). Denote by \( \Sigma \) the set of simple roots \( \{e_{n_1+2} - e_{n_1+1}, \ldots, e_{M-1} - e_{M-1}\} \) of \( U(M) \). Let \( I = \mathfrak{u}(M - n_1) \) the Lie algebra with this simple root system. We have \( \langle \lambda, H_\alpha \rangle = 0 \) for all \( \alpha \in \Sigma \).

Then \( (\lambda, \mu) \in \Lambda^\Sigma_{\geq 0} \oplus \Lambda^K_{\geq 0} \) with the notations as in 5.3. Let us review the key steps of the algorithm to compute \( m_{G,K} \). See the discussion in 4.9.8 outlining the limits of the implementation.
Given as input \((\lambda,\mu)\), we wish to compute the branching coefficients. Recall that:

\[
m_{G,K}(\lambda,\mu) = \sum_{w \in W} \sum_{\gamma \in \Gamma_K/q\Gamma_K} \sum_{\sigma \in \mathcal{OS}(\Psi,a^w_\sigma)} Res_{\tau\rightarrow} S^w_{\lambda,\mu}(z + \frac{2i\pi\gamma}{q}). \tag{10}
\]

One of the tricky point in computing the right hand side of equation (10) is to find a \(\mathcal{F}_\Sigma\)-tope \(\tau_\Sigma\) such that \((\lambda,\mu) \in \tau_\Sigma\). We do this by computing a regular point inside the Kirwan cone sufficiently small and deform \((\lambda,\mu)\) along the line from \((\lambda,\mu)\) to this interior point.

1. We list all the equations \(X\) of the \(\mathcal{F}\)-admissible hyperplanes.

2. For each such equation given by \(X\) and for \(w \in W\), we compute \(H(X,w) = \langle \lambda, wX \rangle - \langle \mu, X \rangle\). As \((\lambda,\mu)\) is in the lattice of weights, and \(X\) in the dual lattice, \(H(X,w)\) is an integer.

   Remember \(\lambda,\mu\) are our fixed input.

3. If \(H(X,w) \neq 0\) \(\forall w, X\), then \((\lambda,\mu)\) is \(\mathcal{F}_\Sigma\)-regular and then it is in a tope \(\tau_\Sigma\). A fortiori it is in a unique \(\mathcal{F}\) tope \(\tau\) and therefore \(w(\lambda) - \mu\) is in a unique tope \(a^\tau_w \in \mathfrak{i}_\mathfrak{k}\).

   In conclusion we can compute \(\mathcal{OS}(\Psi,a^\tau_w)\).

4. Else if \(H(w,X) = 0\) for some \(X\) and \(w\), then we deform as follows:
   
   (a) We find \(\epsilon = (\epsilon_1,\epsilon_2)\) in the interior of \(C_{G,K}^\Sigma\). We can find this point, in the cases we treat, because either we have the equations of the Kirwan cone, either we know some points in the Kirwan cone by directly computing projections.

   (b) We rescale \(\epsilon\) so that \(|\langle wX,\epsilon_1 \rangle - \langle X,\epsilon_2 \rangle| < 1/2\), so \((\lambda,\mu) + t\epsilon\) stays in the same tope \(\tau_\Sigma\) for all \(0 < t < 1\).

   (c) We define \((\lambda_{def},\mu_{def})\) as \((\lambda + \epsilon_1,\nu + \epsilon_2)\).

5. We can now pick the tope \(\tau\) defined by \((\lambda_{def},\mu_{def})\) and compute \(\mathcal{OS}(\Psi,a^\tau_w)\) as in step 3.
6. Now we compute

\[ S_{\lambda,\mu}^{\Sigma,w}(z) = \prod_{\beta \in \Delta^+_t} (1 - e^{-\langle \beta, z \rangle}) \frac{e^{\langle w(\lambda) - \mu, z \rangle}}{\prod_{\alpha \in \Delta_u} (1 - e^{-\langle w(\alpha), z \rangle})} \]

and the residue along an \(\mathcal{OS}\) basis with an appropriate series expansion.

7. Finally to compute \(m_{G,K}(\lambda,\mu)\), we have to sum the contribution from \(w \in \mathcal{W}_g/\mathcal{W}_l\) over the set \(\gamma \in \Gamma_K/q\Gamma_K\) and \(\sigma \in \mathcal{OS}(\Psi,a^{\sigma}_w)\). Each individual term of these two sums, that is if we fix a \(\gamma\) and a \(\sigma\), is easy to compute, but there can be really many of these terms, making possibly the computation very long.

Observe once again that in particular, we can test if the point \((\lambda,\mu)\) is in the cone \(C_{G,K}(T^*G)\) or not, according if the output is not zero or zero.

It is not more difficult to compute with symbolic variables \((\lambda,\mu)\) belonging to the closure of a tope. However to describe all possible topes (the chamber decomposition of \(C_K(\mathcal{H})\)) seems very difficult. So our input is \((\lambda,\mu)\), the output is either the numeric value, either the dilated coefficient \(k \rightarrow m_{G,K}(k\lambda,k\mu)\), or (in low dimensions), a tope \(\tau\) containing \(\lambda,\mu\) in its closure and the quasi-polynomial function in both variables \(\lambda,\mu\) coinciding with \(m_{G,K}(\lambda,\mu)\) on the tope \(\tau\).

**Remark 78.** Rectangular tableaux

Remark that if \(\mu_i\) is a rectangular tableau, then \(\lambda\) is even more singular. This enable us to compute more easily using a larger set \(\Sigma\) (reducing then the number of parabolic roots). When all \(\mu_i\) are rectangular tableaux, this corresponds to the case of Hilbert series. We list the corresponding results in the last subsection 6.3.

### 6.2 Examples

The first two examples have already been treated in the literature.

**Example 79.** The case of 3-qbits : \(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2\)

This case has been treated in complete details in [4.6] and is due to [11].

**Example 80.** The case of : \(\mathbb{C}^4 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2\)
This example has been studied in complete details by [11]. The number of chambers of polynomiality is 74 and on each chamber the quasipolynomial is of degree 2.

The multiplicity function \( k \rightarrow m(k\lambda, k\mu, k\nu) \) is a quasi polynomial function of the form

\[
f(k) + (-1)^c k
\]

where \( f(k) \) is a polynomial of at most degree two and \( c \) is a constant. Here is an example. For \( \lambda = [5, 3, 2, 1] \), \( \mu = \nu = [6, 5] \), then we obtain

\[
1/4 k^2 + 1/2 k + 5/8 + 3/8 (-1)^k
\]

Remark that all points \((\alpha, \beta, \gamma)\) in the boundary of the Kirwan cone are stable, thus \( g(k\alpha, k\beta, k\gamma) \) is 0 or 1.

**Example 81. The case of 4-qbits : \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \)**

We consider the action of of \( K = U(2) \times U(2) \times U(2) \times U(2) \) on \( \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) with \( (k_1, k_2, k_3, k_4) \) acting as \( k_1 \otimes k_2 \otimes k_3 \otimes k_4 \). The Kirwan polytope has been described by Higuchi-Sudbery-Szulc, [23].

We have no idea of the number chambers for polynomiality. Nevertheless, given highest weights \( \alpha, \beta, \gamma, \delta \), we can compute \( g(k\alpha, k\beta, k\gamma, k\delta) \) as a periodic polynomial in \( k \).

It is a polynomial of degree at most 7 and period 6. Here is an example. When \( \alpha = \beta = \gamma = \delta = [2, 1] \) then we compute:

\[
m(k\alpha, k\beta, k\gamma, k\delta) =
\]

\[
\frac{23}{241920} k^7 + \frac{13}{5760} k^6 + \frac{155}{6912} k^5 + \frac{139}{1152} k^4 + \left( \frac{81601}{207360} + \frac{1}{1536} (-1)^k \right) k^3 +
\]

\[
\left( \frac{9799}{11520} + (-1)^k \frac{5}{256} \right) k^2 + \left( \frac{38545}{32256} + (-1)^k \frac{179}{1536} \right) k + P(k)
\]

where

\[
P(k) = \left( \frac{5}{243} + \frac{1}{243} \theta \right) (\theta^2)^k + \left( \frac{1}{243} - \frac{1}{243} \theta \right) \theta^k + \frac{5279}{6912} + \frac{51}{256} (-1)^k
\]

is of period 6 and \( \theta \) is a primitive root \( \theta^3 = 1 \). The values of \( P(k) \) on \( 0, 1, 2, 3, 4, 5 \) are

\[
[1, 5725, 76, 77, 77, 5597, 10368, 81, 128, 81, 10368]
\]

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Figure 7 shows the Duistermaat-Heckman measure for four qbits. The
drawing is along the line from $v_{\text{min}} = [\{1/2, 1/2\}, \{1/2, 1/2\}, \{1/2, 1/2\}, \{1/2, 1/2\}]$
to $v_{\text{top}} = [[0, 1], [1, 0], [1, 0], [1, 0]]$. The function $t \to DH_K^H(tv_{\text{min}} + (1-t)v_{\text{top}})$
is a spline of degree 7 with singularities at $t = \{0, 1/5, 1/3, 1/2, 1\}$.

![Figure 7: Duistermaat-Heckman measure for four qbits](image)

**Example 82. The case of : $\mathbb{C}^6 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$**

When $n_2 = 3$, $n_3 = 2$, it is sufficient to consider the case when $n_1 = 6$.

Then the multiplicity function $k \to m(k\lambda, k\mu, k\nu)$ is a quasi polynomial
function of the form

$$f(k) + (-1)^k g(k) + h(k)$$

where $f(k)$ is a polynomial of $k$ generally of degree 8, $g(k)$ of degree 2 and
$h(k)$ is a periodic function of $k \mod 6$.

Here is an example where the degree of the polynomial is the expected one.

We fix $\lambda = [1500, 1052, 940, 492, 268, 156]$, $\mu = [2110, 1438, 860]$, $\nu = [2748, 1660]$ and compute:

$$m(k\lambda, k\mu, k\nu) =$$

\[
\frac{20160143036868818273}{540} k^8 + \frac{2401100429169038668}{945} k^7 + \frac{236265968398572733}{3240} k^6 + \\
\frac{311654396584249}{270} k^5 + \frac{11934644414969}{1080} k^4 + \frac{53468894201}{810} k^3 + \frac{44636639}{180} k^2 + \frac{378739}{630} k + \\
\frac{232}{243} + \theta^k \left( \frac{2}{81} + \frac{1}{243} \theta \right) + (\theta^2)^k \left( \frac{5}{243} - \frac{1}{243} \theta \right)
\]
where $\theta$ is a third primitive root of 1. The term of the degree zero in $m(k\lambda, k\mu, k\nu)$ is a periodic function of $k$ whose values are

$$\left[ 1, \frac{25}{27}, \frac{76}{81}, 1, \frac{25}{27}, \frac{76}{81} \right]$$

We finish by noticing that for $k = 1 \ldots 6$, the values of $m(k\lambda, k\mu, k\nu)$ are given by the following list

$$[1, 39948532219001323, 9887333657571493818, 250556011548476811713, 2488623801870416780185, 14783083490287618355455].$$

We now compute examples of multiplicity on the walls.

The walls of the Kirwan cone have been described by Klyachko in [32].

Given $\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6]$, $\mu = [\mu_1, \mu_2, \mu_3]$ and $\nu = [\nu_1, \nu_2]$, the walls are by the following 5 types of inequalities in $\lambda, \mu, \nu$.

More precisely, for each of the inequality below, there is a particular non empty subset $S$ of $\Sigma_6 \times \Sigma_3 \times \Sigma_2$ (where $\Sigma_k$ is the set of permutations of $k$ elements) computed by Klyachko such that the permuted inequality is a wall of the corresponding Kirwan cone.

| type | equation |
|------|----------|
| I $\nu_1 - \nu_2 - \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 \leq 0$ | $\nu_1 - \nu_2 - \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 \leq 0$ |
| II $\mu_1 + \mu_2 - 2 \mu_3 - \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + 2 \lambda_5 + 2 \lambda_6 \leq 0$ | $\mu_1 - 2 \mu_3 - \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 + 3 \lambda_6 \leq 0$ |
| III $2 \mu_1 - 2 \mu_3 + \nu_1 - \nu_2 - 3 \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 + 3 \lambda_6 \leq 0$ | $2 \mu_1 + 2 \mu_2 - 4 \mu_3 + 3 \nu_1 - 3 \nu_2 - 5 \lambda_1 - 5 \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + 7 \lambda_6 \leq 0$ |
| IV $4 \mu_1 - 2 \mu_2 - 2 \mu_3 + 3 \nu_1 - 3 \nu_2 - 7 \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + 5 \lambda_5 + 5 \lambda_6 \leq 0$ | $4 \mu_1 - 2 \mu_2 - 2 \mu_3 + 3 \nu_1 - 3 \nu_2 - 7 \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + 5 \lambda_5 + 5 \lambda_6 \leq 0$ |

The following table list elements $v = [\lambda, \mu, \nu]$ in the relative interior of the corresponding wall type facet, together with the value of $m(k\lambda, k\mu, k\nu)$. As asserted in Lemma 37 for the listed value $m(k\lambda, k\mu, k\nu)$ has the maximum possible degree.

| vector | $[\lambda, \mu, \nu]$ | $m(k\lambda, k\mu, k\nu)$ |
|--------|---------------------|--------------------------|
| $v_I$  | $[288, 192, 174, 120, 30, 6], [343, 270, 197], [654, 156]$ | $1 + 17k$ |
| $v_{II}$ | $[600, 372, 300, 156, 96, 12], [876, 552, 108], [930, 606]$ | $1 + \frac{311}{2} k + \frac{4101}{2} k^2 + 242154 k^3$ |
| $v_{III}$ | $[188, 140, 92, 52, 20, 4], [304, 152, 40], [340, 156]$ | $1$ |
| $v_{IV}$ | $[276, 204, 120, 66, 30, 6], [351, 273, 78], [552, 150]$ | $1 + 36k$ |
| $v_{V}$ | $[276, 198, 126, 66, 48, 6], [406, 201, 113], [536, 184]$ | $1 + 41k$ |
In particular, we see that all facets of type $v_{III}$ consists of stable elements. There are 20 such facets (obtained by considering the special permutations computed by Klyachko).

As we noticed just after Theorem 36 we don’t know how to compute the degree when $\mu$ is singular, see Subsection 6.3 for the case of three rectangular tableaux. Here is an example for which the degree is smaller. Consider $\lambda = [9, 7, 5, 3, 2, 1]$, $\mu = [9, 9, 9]$, $\nu = [14, 13]$, then

$$m(k\lambda, k\mu, k\nu) = \frac{13}{64} (-1)^k + \frac{67}{64} k + \frac{617}{64} k^2 + \frac{19}{24} k^3 + \frac{55}{288} k^4 + \frac{\theta^k (-2 \theta + 8) + \theta^2 k (2 \theta + 10) + 85}{144} + \frac{3}{16} (-1)^k$$

Here $\theta$ is again a primitive root $\theta^3 = 1$. Thus the term of degree zero is a periodic function $r$ of $k$ such that

$$[r(0), r(1), r(2), r(3), r(4), r(5)] = [1, 71/216, 17/27, 5/8, 19/27, 55/216]$$

Of course, the value of $m(0, 0, 0)$ is equal to 1. Here $m(\lambda, \mu, \nu) = 5$ and for instance the value $m(17\lambda, 17\mu, 17\nu) = 344715$.

**Example 83. The case of 3-quthrits.** : $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

Here $\mathcal{H} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. The multiplicity function $k \rightarrow m(k\lambda, k\mu, k\nu)$ is a quasi polynomial function of the form

$$f(k) + (i)^k g(k) + h(k)$$

where $f(k)$ is a polynomial of $k$ of degree at most 11 and $h(k)$ is a periodic function of $k$ (mod 12). The actual numerical values are computed in a rather quick time.

Let us give an example of the dilated Kronecker coefficient. The periodic term for the coefficient of degree 0 $m(k\lambda, k\mu, k\nu)$ with $\lambda = \mu = \nu = [4, 3, 2]$ is given by

$$\begin{bmatrix}
1166651 & 13403 & 29899 & 59 & 1166651 & 235 & 980027 & 59 & 32203 & 13403 & 980027 \\
5308416 & 20736 & 65536 & 81 & 5308416 & 256 & 5308416 & 81 & 65536 & 20736 & 5308416
\end{bmatrix}$$

In this case $m(k\lambda, k\mu, k\nu)$ has precisely degree 11.

Here is a numerical example. For $\lambda = [5, 2, 1]$, $\mu = [4, 2, 2]$, $\nu = [3, 3, 2]$ we compute $m(\lambda, \mu, \nu) = 4$.

We close with one last example that we had promised.

**Example 84.**
Here $\mathcal{H} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$.

\[
g([k[1, 1, 1], k[1, 1, 1], k[2, 1]]) = \frac{1}{4} (-1)^k + \frac{5}{12} + (1 - \theta) \frac{1}{9} \theta^{2k} + (2 + \theta) \frac{1}{9} \theta^k + \frac{1}{6} k
\]

and for instance the list of the values for $0 \leq k \leq 30$ are

\[
[1, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 6]
\]

So the saturation factor is 2.

### 6.3 Rectangular tableaux and Hilbert series

We give a list of the Kronecker coefficients for the following situation of rectangular tableaux. We use the following notations: $(\mathbb{C}^2)^3 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, $[[1, 1]]^3 = [[1, 1], [1, 1], [1, 1]]$, $\mathbb{C}^{[4, 3, 3]} = \mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and similarly. In the following Table the second column refers to the choice of the parameters $[\lambda, \mu, \nu]$ and the third column to the value of the Hilbert series $\sum_k m(k)t^k$ where $m(k)$ is the Kronecker coefficient $(k\lambda, k\mu, k\nu)$.

| type   | parameters                          | value               |
|--------|-------------------------------------|---------------------|
| $(\mathbb{C}^2)^3$ | $[[1, 1]]^3$                             | $\frac{1}{1-t^2}$ |
| $(\mathbb{C}^2)^4$ | $[[1, 1]]^4$                             | $\frac{1}{(1-t)(1-t^2)^2(1-t^3)}$. |
| $(\mathbb{C}^3)^3$ | $[[1, 1]]^3$                             | $HS_{22222}$ |
| $\mathbb{C}^{[4, 3, 3]}$ | $[[3, 3, 3, 3], [4, 4, 4], [4, 4, 4]]$ | $\frac{1}{(1-t^2)^2(1-t^3)(1-t^4)}$. |

where

\[
HS_{22222} = \sum g(k[1, 1], k[1, 1], k[1, 1], k[1, 1], k[1, 1])t^k = \frac{P(t)}{Q(t)}
\]

where

\[
P(t) = t^{52} + 16t^{48} + 9t^{47} + 82t^{46} + 145t^{45} + 383t^{44} + 770t^{43} + 1659t^{42} + 3024t^{41} + 5604t^{40} + 9664t^{39} + 15594t^{38} + 24659t^{37} + 36611t^{36} + 52409t^{35} + 71847t^{34} + 95014t^{33} + 119947t^{32} + 146849t^{31} + 172742t^{30} + 195358t^{29} + 214238t^{28} + 225699t^{27} + 229752t^{26} + 225699t^{25} + 214238t^{24} + 195358t^{23} + 172742t^{22} + 146849t^{21} + 119947t^{20} + 95014t^{19} + 71847t^{18} + 52409t^{17} + 36611t^{16} + 24659t^{15} + 15594t^{14} + 9664t^{13} + 5604t^{12} + 3024t^{11} + 1659t^{10} + 770t^{9} + 383t^{8} + 145t^{7} + 82t^{6} + 9t^{5} + 16t^{4} + 1
\]
and
\[ Q(t) = (1 - t^2)^5(1 - t^3)(1 - t^4)^5(1 - t^5)(1 - t^6)^5. \]

We remark that for the 5-qubits the result in [38] correspond to the series \( \sum_k m(k)t^{2k} \) and has a misprint on the value of the coefficient \( a_n \) for \( n = 42 \) (corresponding to the coefficient of \( t^{21} \) in our formula), as the numerator is not palindromic. So the value \( a_n \) for \( n = 42 \) in [38] has to be replaced by 146849.

We report for completeness the value of the Kronecker coefficients in the examples considered, we omit the actual expression for the Kronecker coefficients in the 5-qubits case and the one for \( g(k[3,3,3],k[4,4,4],k[4,4,4]) \) because the formula is too long to be reproduced here.

\[
\begin{align*}
g(k[1,1],k[1,1]) &= \frac{1}{2} + \frac{1}{2}(-1)^k \\
g(k[1,1],k[1,1],k[1,1],k[1,1]) &= \frac{23}{36} + \frac{1}{4}(-1)^k + \frac{1}{27}\theta^k(2+\theta) + \frac{1}{27}\theta^2k(1-\theta) + \left(\frac{29}{48} + \frac{1}{16}(-1)^k\right)k + \frac{1}{16}k^2 + \frac{k^3}{72} \\
g(k[1,1,1],k[1,1,1],k[1,1,1]) &= \frac{107}{288} + \frac{9}{32}(-1)^k + \left(1 + (-1)^k\right)\frac{1}{16}k^k + \left(1 + (-1)^{k+1}\right)\frac{1}{16}k^{k+1} + \frac{1}{9}\theta^2k + \frac{1}{9}g^k + \left(\frac{1}{16}(-1)^k + \frac{3}{16}\right)k + \frac{1}{48}k^2
\end{align*}
\]

where \( \theta \) is a third root of unity. For \( g(k[1,1,1],k[1,1,1],k[1,1,1]) \) we report, as an example, the expressions on cosets. We have twelve cosets and thus a sequence of 12 polynomials given by the following list

\[
\begin{align*}
[1 + \frac{1}{4}k + \frac{1}{48}k^2, \frac{7}{48} + \frac{1}{8}k + \frac{1}{48}k^2, \frac{5}{12} + \frac{1}{4}k + \frac{1}{48}k^2, \frac{7}{16} + \frac{1}{8}k + \frac{1}{48}k^2, \\
\frac{2}{3} + \frac{1}{4}k + \frac{1}{48}k^2, \frac{7}{48} + \frac{1}{8}k + \frac{1}{48}k^2, \frac{3}{4} + \frac{1}{4}k + \frac{1}{48}k^2, \frac{5}{48} + \frac{1}{8}k + \frac{1}{48}k^2, \\
\frac{2}{3} + \frac{1}{4}k + \frac{1}{48}k^2, \frac{3}{16} + \frac{1}{8}k + \frac{1}{48}k^2, \frac{5}{12} + \frac{1}{4}k + \frac{1}{48}k^2, \frac{5}{48} + \frac{1}{8}k + \frac{1}{48}k^2]
\end{align*}
\]

The following is the list of values of the Kronecker coefficients computed by the above formula for \( 0 \leq k \leq 20 \):

\[ \{1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 8, 7, 10, 8, 12, 10, 14\}. \]

Once again the saturation factor is 2.
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