Multicritical phase transitions in multiply rotating black holes

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Received 3 December 2022; revised 16 January 2023
Accepted for publication 13 February 2023
Published 22 February 2023

Abstract

We show that multi-critical points in which more than three phases coalesce are present in multiply rotating Kerr-anti de Sitter black holes in \(d\)-dimensions. We explicitly present a quadruple point for a triply rotating black hole in \(d = 8\) and a quintuple point for a quadruply rotating black hole in \(d = 10\). The maximal number of distinct phases \(n\) is one larger than the maximal number of independent rotations, and we outline a method for obtaining the associated \(n\)-tuple point. Situations also exist where more than three phases merge at sub-maximal multi-critical points. Our results show that multi-critical points in black hole thermodynamics are more common than previously thought, with systems potentially supporting many phases as long as a sufficient number of thermodynamic variables are present.

Keywords: black hole, multicritical phase transitions, Kerr-AdS

Black hole thermodynamics provides crucial guidance along the path toward a quantum theory of gravity. Asymptotically anti de Sitter (AdS) black holes have been of particular importance to this end ever since the discovery of a phase transition between thermal radiation and a large AdS black hole (known as the Hawking-Page (HP) transition [1]), which corresponds to the confinement/deconfinement of a dual quark gluon plasma [2] in the context of the AdS/Conformal Field Theory (CFT) correspondence.
Once it was understood that a cosmological constant $\Lambda$ can be understood as a thermodynamic variable (associated, for example, with a $(d-1)$-form gauge field [3]) corresponding to pressure [4–6], black holes were seen to exhibit a broad range of phase behaviour. Charged black holes can undergo Van der Waals transitions [7], the HP transition can be understood as a solid–liquid transition [8], reentrant transitions take place in rotating black holes [9], scalar couplings admit superfluid transitions [10], Lovelock black holes can have polymer-type phase transitions [11], and accelerating black holes [12] have snapping transitions in which Van der Waals behaviour suddenly disappears [13]. The resemblance of these phenomena to chemical phase transitions has prompted a molecular interpretation of the underlying constituent degrees of freedom [14], and the subject has come to be known as Black Hole Chemistry [15].

The existence of triple points, in which three black hole phases coalesce at a single pressure and temperature (analogous to ice/water/steam), were discovered some time ago in doubly rotating black holes [16], and subsequently in Lovelock gravity [17, 18]; more recently a proposal for the microstructure of black holes at such points was proposed [19]. However multi-critical points of the type seen in colloidal polymers and other heterogeneous systems [20–22], in which more than three phases merge, did not seem to be present in black hole physics. Recently multi-critical points were discovered for charged AdS black holes in non-linear electrodynamics [23].

Here we show that the family of multiply-rotating Kerr-AdS black holes [24, 25] also exhibits multi-critical behaviour. The different angular momenta introduce additional thermodynamic conjugate pairs to the system, allowing for more phases than the small/intermediate/large ones seen for doubly rotating black holes [16]. We find multiple phases separated by first order phase transitions for sufficiently high pressure and appropriate angular momenta. As the pressure is lowered, these phases merge at a single pressure and temperature; for pressures below this multi-critical point only the largest and smallest black hole phases remain, separated by a first order phase transition. We explicitly show the existence of a quadruple point for a triply-rotating black hole and a quintuple point for a black hole with four angular momenta. In general we find that a black hole with $(n+1)$ distinct angular momenta can have an $n$-tuple point, as well as lower order multi-critical points.

The importance of the Kerr-AdS class of solutions cannot be underestimated. They have been employed in advancing our understanding of gauged supergravities [26], black hole thermodynamics [15], hidden symmetries [27], geons [28], defining conserved charges [29], black rings [30], separability of wave equations [31], gravitational instabilities [32], black strings [33], quantum gravity [34], cosmic censorship [35], holographic complexity [36], ultraspinning black holes [37], and black hole superradiance [38]. Our results suggest that black hole multi-critical behaviour is common, requiring neither the introduction of unusual matter sources nor a theory of gravity different from general relativity.

Setting $\hbar = c = G = 1$, the general metric for multiply rotating Kerr-AdS black holes is [24, 25]

$$\text{d}s^2 = -W\left(1 + \frac{r^2}{F}\right)\text{d}r^2 + \frac{2m}{U}\left(W\text{d}\tau - \sum_{i=1}^{N} \frac{a_i \mu_i^2 \text{d}\phi_i}{E_i}\right)^2 + \sum_{i=1}^{N} \frac{r^2 + a_i^2}{E_i} \mu_i^2 \text{d}\phi_i^2 + \frac{U d\tau^2}{F - 2m} + \sum_{i=1}^{N+i} \frac{r^2 + a_i^2}{E_i} d\mu_i^2$$

$$- \frac{l^{-2}}{W(1 + r^2/P)} \left(\sum_{i=1}^{N+i} \frac{r^2 + a_i^2}{E_i} \mu_i \text{d}\mu_i\right)^2$$

(1)
in $d$ spacetime dimensions, with metric functions

$$W = \sum_{i=1}^{N+\epsilon} \mu^2_i / \Xi_i, \quad U = r^d \sum_{i=1}^{N+\epsilon} \mu^2_i / r^2 + a^2_i \prod_{j=1}^{N}(r^2 + a^2_j),$$

$$F = r^{d-2}(1 + \frac{\mu^2}{F^2}) \prod_{i=1}^{N}(r^2 + a^2_i), \quad \Xi_i = 1 - a^2_i / F^2$$  \hspace{1cm} (2)

and where $N = \frac{1}{2}(d - 1 - \epsilon)$, is the maximal number of independent rotations, with $\epsilon = 0/1$ for odd/even spacetime dimensions. The coordinates $\mu_i$ obey the constraint $\sum_{i=1}^{N+\epsilon} \mu^2_i = 1$. $I$ is the AdS radius, $m$ is the mass parameter, and $a_i$ are the rotation parameters. The horizon radius $r_+$ can be determined by finding the largest root of the equation $F - 2m = 0$. For constant $J_i$, the $a_i$ parameters are functions of $r_+$ and the thermodynamic pressure $P$, where

$$P = -\frac{\Lambda}{8\pi}, \quad \Lambda = -\frac{(d-1)(d-2)}{2F^2}$$ \hspace{1cm} (3)

in Planckian units $\ell_P^2 = \frac{\hbar}{mc^2}$ [15]. The mass $M$, angular momenta $J_i$, and the thermodynamically conjugate angular velocities $\Omega_i$ are [39]

$$M = \frac{m\Sigma_{d-2}}{4\pi(\prod_i \Xi_i)} \left( \sum_{i=1}^{N} \frac{1}{\Xi_i} - \frac{1-\epsilon}{2} \right),$$

$$J_i = \frac{a_i m \Sigma_{d-2}}{4\pi \Xi_i (\prod_i \Xi_i)}, \quad \Omega_i = \frac{a_i (1 + \frac{\mu^2}{F^2})}{r^2 + a^2_i},$$ \hspace{1cm} (4)

where $\Sigma_{d-2} = \frac{2\pi \cdot \kappa_{d-1}}{\Gamma(\frac{d}{2})}$. The temperature $T = \frac{\kappa}{2\pi}$ and entropy $S$ are determined in terms of $r_+$

$$T = \frac{1}{2\pi} \left[ r_+ (\frac{\mu^2}{F^2} + 1) \sum_{i=1}^{N} \frac{1}{a^2_i + r_+^2} - \frac{1}{r_+} \left( \frac{1}{2} - \frac{r_+^2}{2F^2} \right) \right],$$

$$S = \frac{\Sigma_{d-2}}{4\pi r_+^2} \prod_{i=1}^{N} \frac{a^2_i + r^2_+}{\Xi_i}.$$ \hspace{1cm} (5)

The thermodynamically stable state of the system is given by the global minimum of the Gibbs free energy $G = M - TS$ [15].

The first law is [39, 40]

$$dM = TdS + \sum_{i=1}^{N} \Omega_i dJ_i + VdP$$ \hspace{1cm} (6)

with the thermodynamic volume

$$V = \frac{r_+ A}{d-1} + \frac{8\pi}{(d-1)(d-2)} \sum_{i=1}^{N} a_i J_i$$ \hspace{1cm} (7)

and $A$ the area of the outermost horizon. Taking the variation of $G$ and substituting (6) yields

$$dG = \sum_{i=1}^{N} \Omega_i dJ_i - SdT + VdP$$ \hspace{1cm} (8)

and so $dG = -SdT$ for constant $P$ and $J_i$, in which case the extrema of $G(r_+)$ and $T(r_+)$ occur at the same $r_+$ values. Consequently $T$ alone determines the existence and distribution
Figure 1. T and $T'$ for different values of $P$. $d = 8, J_1 = 7.967, J_2 = 1.24, J_3 = 0.12798$.

Top. $T$ at three different pressures, attaining local extrema at the roots of $T'$. A maximum of three pairs of maxima and minima is displayed for $P = 0.121$, indicating three swallowtails. Bottom. The derivative $T'$ at these three pressures, showing different numbers of roots.

of swallowtails in the $G$–$T$ plot, with the cusps of the swallowtails corresponding to the zeros of $T' = \frac{\partial T}{\partial r_+}$. It is possible to choose constant $J_i$ so that $T'$ has a new set of local maxima and minima for each new rotation. Adjusting the pressure changes the locations of these extrema; the maximum number of coexistent states is attained when $T'(r_+)$ has a root between every local extremum. Unlike the doubly rotating case [16] where the system only depends on the ratio between the two angular momenta, in general the $a_i$ rotational parameters are not invariant under constant scaling of $J_i$, and so we will directly fix the $J_i$.

We begin by illustrating the existence of a quadruple point for a triply rotating black hole in $d = 8$, the minimal value needed to support four distinct phases. The temperature is

$$ T = \frac{1}{2\pi} \left[ r_+ \left( 1 + \frac{8r_+^2 \pi P}{21} \right) \sum_{i=1}^{3} \frac{1}{a_i^2 + r_+^2} - \frac{1}{r_+} \left( \frac{1}{2} - \frac{4r_+^2 \pi P}{21} \right) \right] \tag{9} $$

where the $a_i$ are calculated numerically as functions of $r_+$ using (4), as an analytic solution is not possible. All of our results satisfy the constraint $|a_i| < l$, so that the metric (1) remains well-defined.

For large $P$, no phase transitions are present. As $P$ decreases, $T(r_+)$ develops extrema (shown in figure 1) and new thermodynamic phenomena arise illustrated in figure 2. At the critical pressure $P_{c_1} \approx 0.24335$, a first order phase transition between the smallest black hole and a larger black hole appears and the $G$–$T$ plot has a single swallowtail. Further lowering the
Figure 2. $G$–$T$ plot with three rotations. $d = 8, J_1 = 7.967, J_2 = 1.24, J_3 = 0.12798$. At $P = 0.141$ (black curve), only two swallowtails exist indicating two stable phase transitions between three distinct phases. As the pressure is lowered a third swallowtail appears (red curve) signifying four distinct phases. At $P_q = 0.121$, these three swallowtails merge at a quadruple point (green curve). For pressures lower than $P_q$, only one stable first order phase transition is seen (blue curve). Dashed lines indicate negative specific heat.

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While there is a finite range of values of $J_i$ that admit quadruple points, in general these will not be present for most choices of $J_i$. Instead for sufficiently low pressures the four distinct phases will each be separated by first-order phase transitions, and the four branches of the coexistence plot merge in two places at two distinct triple points. Similarly, with appropriate choices of $J_i$, a quadruply rotating black hole in $d = 10$ can support five distinct phases at a fixed pressure. With the added $a_4$ parameter in $T(r_+, P, J_1, J_2, J_3, J_4)$, $T'(r_+)$ can admit another pair of extrema, which can then be adjusted to line up with the other three pairs seen for $d = 8$. $P$ is chosen in the same way so that a root appears between each extremum of $T$, yielding five distinct black hole phases and the maximal number of four swallowtails, indicating four first order phase transitions, which appear in the $G$–$T$ diagram for a range of fixed $P$. These merge at sufficiently small $P$, shown in figure 4. The $P$–$T$ behaviour is analogous to that of the quadruple point: one stable phase transition is present for small pressures, above which the coexistence curve splits into four branches at the quintuple point, each branch terminating at a distinct critical point as $P$ is increased.
Figure 3. $P-T$ phase diagram for quadruple point with three rotations. $d = 8$, $J_1 = 7.967$, $J_2 = 1.24$, $J_3 = 0.12798$. For low pressures only one stable phase transition between the smallest and largest black holes exists (red curve). At $P = P_q$, four phases (smallest, small, large, largest) coexist at $T_q \approx 0.3606$. For $P_q < P < P_c$, three stable first order phase transitions are observed between four phases. All three coexistence curves terminate at their respective critical points.

Figure 4. $G-T$ plot of quintuple point with four rotations. $d = 10$, $J_1 = 24.48$, $J_2 = 4.33$, $J_3 = 1.2$, $J_4 = 0.1435$. Four swallowtails in the Gibbs free energy merge at one point for $P = 0.231$. Dashed lines indicate negative specific heat.

More generally the five phases can exist at fixed pressure without the presence of a quintuple point. In this case, the $P-T$ phase diagram can exhibit three triple points or a quadruple point and a triple point for particular choices of $J_i$. Figure 5 shows a system with one quadruple point and a triple point at a lower pressure.

Multi-critical behaviour will be present for any multiply rotating AdS black hole for an appropriate choice of angular momenta. In general, a $d = (2n + \epsilon)$ dimensional spacetimes supports up to $n - 1$ independent rotations, yielding $n = \lfloor \frac{d}{2} \rfloor$ potential distinct phases in $d$ dimensions, where $\epsilon = 0, 1$ in even/odd dimensions. New phases (and thus higher degrees of multicriticality) will appear with increasing even dimension (increasing $n$), since one of the exponents in the temperature function vanishes in odd dimensions. The necessary conditions to obtain the maximum amount of phases are unclear, but we have found numerically that the magnitudes of $J_i$ must be sufficiently spaced out. Assuming all phases are separated by first order phase transitions, the coexistence curve has at most $n - 1$ branches in regions where all $n$
Fig. 5. $P - T$ phase diagram with four rotations. $d = 10$, $J_1 = 24.48$, $J_2 = 4.331$, $J_3 = 1.1973$, $J_4 = 0.155$. For $P < P_{tr} \approx 0.22695$, the only stable phase transition observed is one of first order between the largest and smallest black hole phases (blue). At $P = P_{tr}$, a new phase emerges at a triple point, above which three phases are present. Two additional coexistence curves appear on the right branch at a quadruple point $(P = P_{q} = 0.231)$. Five distinct black hole phases exist for $P \in (P_{q}, P_{c_4} \approx 0.23883)$. All coexistence curves terminate at critical points as the pressure is increased.

phases remain, and only one branch as $P \to 0$. All $n - 1$ branches of the curve eventually merge to a single branch as pressure is decreased, which can be in the form of an $n$-tuple point, or many other lower order multi-critical points at different pressures. Scaling the $J_i$ by a constant factor does not change the phase behaviour, but rather shifts the locations of critical points in the $P-T$ phase diagram. With sufficiently large scaling, multi-critical points can be pushed arbitrarily close to $P = T = 0$.

Multi-critical points in multiply rotating black holes are unlike those found in the context of non-linear electrodynamics \cite{23} with regards to the Gibbs phase rule, which relates the degrees of freedom $F$ in a simple thermodynamic system to the number of coexistence phases $P$ and the number of chemical constituents. The generalized Gibbs phase rule \cite{22}:

$$F = W - P + 1.$$  \hspace{1cm} (10)  

replaces the notion of chemical constituents with the number of thermodynamic conjugate pairs $W$, which is directly applicable in the context of black hole thermodynamics. The $n$-tuple points in non-linear electrodynamics were discovered to have at minimum $n$ degrees of freedom and required two additional conjugate pairs for each new phase \cite{23}, whereas in the Kerr-AdS case the $n$-tuple points always have a lower bound of $F = 2$, and only one added rotation is needed for a new phase. This disparity is likely due to the fact that $T$ depends nonlinearly on the angular momenta $J_i$, in contrast to its linear dependence on the coupling constants in non-linear electrodynamics.

The presence of multi-critical behaviour of vacuum black holes in $d$-dimensional Einstein gravity raises a number of interesting questions. It would be of great interest to know the necessary and sufficient conditions for multi-criticality to occur, and to find the conditions (if any) under which it is possible for multiple phases to coalesce as pressure is increased. It would likewise be interesting to understand the implications of multi-criticality for the microstructure of black holes. An analysis of the de Sitter case would likewise be interesting, though the
challenge there is dealing with the distinct temperatures of the black hole and cosmological horizons. This can be dealt with by placing the black hole in a cavity \[41–45\], though in the multi-rotating case the structure of the cavity will be difficult to describe geometrically. Finally, recent work has shown that the results of black hole chemistry are amenable to a holographic interpretation \[36, 46–57\]. It would be most interesting to understand what the holographic duals are of multicritical points.

Data availability statement

No new data were created or analysed in this study.

Acknowledgment

This work supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC), Perimeter Institute and the University of Waterloo are situated on the Haldimand Tract, land that was promised to the Haudenosaunee of the Six Nations of the Grand River, and is within the territory of the Neutral, Anishnawbe, and Haudenosaunee peoples.

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