Geometrical description of quantum mechanics—transformations and dynamics

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Abstract
In this paper, we review a proposed geometrical formulation of quantum mechanics. We argue that this geometrization makes available mathematical methods from classical mechanics to the quantum framework. We apply this formulation to the study of separability and entanglement for states of composite quantum systems.

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1. Introduction
There are several sound reasons to try to formulate quantum mechanics in geometrical terms. For instance, the high degree of geometrization of classical mechanics, general relativity, gauge theories and others acts as a stimulus to geometrize quantum theories to better understand the quantum-classical transition, to formulate quantum gravity and to better understand which structures usually dealt with in quantum theories should be attributed to the ‘system’ and which to the ‘measuring apparatus’ (observer) according to the ‘Heisenberg cut’. In the spirit of Einstein’s minimal assumptions, geometrization would bring out those algebraic structures that should be ‘dynamically determined’, i.e. obtained as a solution of the Einstein equations when the distribution of energy and matter in the universe is given. Thus, by geometrization of quantum mechanics we mean to replace the used description on the Hilbert space by a description on Hilbert manifolds. In this respect, the proposal is very much similar to the transition from special relativity to general relativity: space-time is considered to be a Lorentzian manifold and the properties of the Minkowski space-time are transferred to the tangent space at each point of the space-time manifold. In particular, we go from the scalar product $\eta_{\mu\nu} X^\mu X^\nu$ to the Lorentzian metric tensor field $\eta_{\mu\nu} dx^\mu \otimes dx^\nu$, which is further generalized to non-flat space-time manifolds in the form $\eta_{\mu\nu} \theta^\mu \otimes \theta^\nu$ where $\{\theta^\mu\}$ are general 1-forms that carry the information on the non-vanishing of the curvature tensor.

Similarly, in the geometrization of quantum mechanics we go from the scalar product $\langle \psi | \psi \rangle$ on the Hilbert space to the Hermitian tensor field on the Hilbert manifold, written as $\langle d\psi | d\psi \rangle$. This would be the associated covariant (0,2)-tensor field.

If we consider as starting carrier space not $\mathcal{H}$ but its dual $\mathcal{H}^*$, say not ket-vectors but bra-vectors, in Dirac’s notations, we would obtain a (2,0)-tensor field, i.e. a contravariant tensor field. Once we consider these replacements, algebraic structures will be associated with tensorial structures and we have to take into account that there will be no more invertible linear transformations but just diffeomorphisms. The linear structure will emerge only at the level of the tangent space and will ‘reappear’ on the manifold carrier space as a choice of each observer, according to the ‘Heisenberg cut’ [1]. We must stress that ‘manifold descriptions’ are naturally appearing already in the standard approach by means of Hilbert spaces when, due to the probabilistic interpretation of quantum mechanics, we realize that pure states are not vectors in $\mathcal{H}$ but...
rather equivalence classes of vectors, i.e. rays. The set of rays, say \( R(\mathcal{H}) \), is the complex projective space associated with \( \mathcal{H} \); it is not linear and carries a manifold structure with ‘model space’ the tangent space at each point \([\psi]\). This space may be identified with the Hilbert subspace of vectors orthogonal to \( \psi \).

In geometrical terms, the transition from non-zero vectors in \( \mathcal{H} \) to the corresponding rays defines a principal \( C_0 \)-bundle on the total space \( \mathcal{H}_0 \), with base space \( R(\mathcal{H}) \). By \( C_0 \) and \( \mathcal{H}_0 \) we mean \( C \) and \( \mathcal{H} \) respectively without the zero element.

Other examples of ‘natural manifolds’ are provided by the set of density states that do not allow for linear combinations but only convex combinations. They contain submanifolds of density states with fixed rank. Of course, the group of unitary transformations provides us with another manifold (group-manifold) whose model space, the tangent submanifolds of density states with fixed rank. Of course, by the set of density states that do not allow for linear combinations, we mean \( C \).

As is well known, most operators of physical interest are unbounded; therefore their description at the manifold level will require one to consider domain problems that obscure the geometrical picture. To avoid these technical problems, we shall restrict our considerations to finite dimensional vector spaces, see [22–24].

Consider various tensor fields on these manifolds which allow one to describe observables, states along with separability and entanglement, when we deal with composite systems.

As is well known, most operators of physical interest are unbounded; therefore their description at the manifold level will require one to consider domain problems that obscure the geometrical picture. To avoid these technical problems, we shall restrict our considerations to finite dimensional vector spaces, see [22–24]. Moreover our manifolds will always be real manifolds, so they carry the usual differential calculus. In simple terms, this means that we consider the differential calculus on complex-valued functions depending on real variables. A manifold is characterized saying that each point has a neighborhood diffeomorphic to an open subset of its tangent space at that point. For instance, the simplest vector space \( \mathbb{R} \) is diffeomorphic to the open interval (–1, 1). From the manifold point of view the two sets are equivalent. For both of them the tangent space at each point is \( \mathbb{R} \).

In finite dimensions any vector space \( V \) is isomorphic (although in a basis-dependent way, i.e. not naturally isomorphic) with its dual space, the space of scalar-valued linear maps on \( V \). In particular, for a Hilbert space we may consider a ‘starting vector space’ either the vector space of kets or the vector space of bras, to use Dirac’s notations. If we introduce an orthonormal basis \( \{|e_j\}_{j} \) for \( \mathcal{H} \), we define coordinate functions by setting

\[
\langle e_j | \psi \rangle = z^j(\psi),
\]

usually written simply as \( z^j \). By using the dual basis \( \{|e_j\}_{j} \) we find

\[
\langle \psi | e_j \rangle = \bar{z}^j(\psi^*).
\]

This means that coordinate functions \( z^j \) are defined on \( \mathcal{H} \), while coordinate functions \( \bar{z}^j \) are defined on the dual space \( \mathcal{H}^* \). By using the inner product, we can identify \( \mathcal{H} \) and \( \mathcal{H}^* \). This provides two possibilities: the scalar product \( \langle \psi | \psi \rangle \) gives rise to a covariant Hermitian (2, 0)-metric tensor on \( \mathcal{H} \),

\[
\langle d\psi | d\psi \rangle = \sum_j \langle d\psi | e_j \rangle \langle e_j | d\psi \rangle = dz^j \otimes dz^j,
\]

where we have used \( d\langle e_j | \psi \rangle = \langle e_j | d\psi \rangle \), i.e. the chosen basis is not ‘varied’, or to a contra-variant (0,2) tensor

\[
\left( \begin{array}{c} \frac{\partial}{\partial z_j} \\ \frac{\partial}{\partial z^j} \end{array} \right) = \frac{\partial}{\partial z_j} \otimes \frac{\partial}{\partial z^j}
\]

on \( \mathcal{H}^* \).

Remark 1. By considering a changing basis, a ‘moving frame’, we should deal with covariant differential calculus.

By introducing real coordinates, say

\[
z^j(\psi) = x^j(\psi) + i y^j(\psi),
\]

we find

\[
\langle d\psi | d\psi \rangle = (dx_j - idy_j) \otimes (dx^j + idy^j)
\]

\[
= (dx_j \otimes dx^j + dy_j \otimes dy^j)
\]

\[
+ i(dx_j \otimes dy^j - dy_j \otimes dx^j).
\]

This expression shows very clearly that the Hermitian tensor is equivalent to a symmetric Euclidean metric tensor (more generally a Riemannian tensor) and a skew-symmetric tensor (a symplectic 2-form).

Similarly, on \( \mathcal{H}^* \) we may consider

\[
\left( \begin{array}{c} \frac{\partial}{\partial z^j} \\ \frac{\partial}{\partial z_j} \end{array} \right) = \left( \begin{array}{c} \frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j} \\ \frac{\partial}{\partial x^j} - i \frac{\partial}{\partial y^j} \end{array} \right) \otimes \left( \begin{array}{c} \frac{\partial}{\partial x^j} - i \frac{\partial}{\partial y^j} \\ \frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j} \end{array} \right)
\]

\[
= \left( \begin{array}{c} \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^j} + \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial y^j} \\ \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial y^j} \end{array} \right) + i \left( \begin{array}{c} \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial y^j} \\ \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial y^j} - \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial x^j} \end{array} \right).
\]

This tensor field, in contravariant form, may also be considered as a bi-differential operator, i.e. we may define a binary bilinear product on real smooth functions by setting

\[
((f, g)) = \left( \frac{\partial f}{\partial x^j} + i \frac{\partial f}{\partial y^j} \right) \cdot \left( \frac{\partial g}{\partial x^j} - i \frac{\partial g}{\partial y^j} \right).
\]
which decomposes into a symmetric bracket
\[ (f, g) = \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x^j} + \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial y^j} \] (2.11)
and a skew-symmetric bracket
\[ [f, g] = \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x^j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y^j}. \] (2.12)

This last bracket defines a Poisson bracket on smooth functions defined on \( \mathcal{H} \).

Summarizing, we can replace our original Hilbert space with a Hilbert manifold, i.e. an even-dimensional real manifold on which we have tensor fields in covariant form, or tensor fields in contravariant form,

\[ g = dx_j \otimes dx^j + dy_j \otimes dy^j \] (2.13)
\[ \omega = dy_j \otimes dx^j - dx_j \otimes dy^j, \] (2.14)
along with a complex structure tensor field
\[ J = dx^j \otimes \frac{\partial}{\partial y_j} - dy^j \otimes \frac{\partial}{\partial x_j}. \] (2.17)
The contravariant tensor fields, considered as bi-differential operators, define a symmetric product and a skew-symmetric product on real smooth functions. The skew-symmetric product actually defines a Poisson bracket. Once the manifold point of view has been selected (say, \( \mathbb{R} \) has been replaced with \((-1, 1))\), we have no meaning for linear transformations; now only diffeomorphisms are available.

To recover unitary transformations, we restrict ourselves to diffeomorphisms that preserve the Poisson bracket (they are canonical transformations) and moreover preserve the symmetric product (they are isometries for the metric tensor). Their generators at the infinitesimal level are Hamiltonian vector fields, which are also Killing vector fields. It is not difficult to show that ‘Hamiltonian functions’ that define vector fields satisfying previous requirements are necessarily quadratic functions associated with Hermitian matrices. Indeed, the group of unitary diffeomorphisms emerges as the intersection of the group of canonical transformations with the group of isometries.

As a further bonus, the symmetric bracket, when restricted to these particular quadratic functions, defines a Jordan algebra that is compatible with the commutator bracket so that they define a Lie–Jordan algebra. By restricting our bracket \((f, g))\) to functions whose real and imaginary parts are made up of these quadratic functions, we have a new product
\[ f \ast g = ((f, g)). \] (2.18)
This product is associative and compatible with complex conjugation. By introducing a norm by means of the symmetric product, we obtain a \( \mathbb{C}^* \)-algebra. What should be stressed is that we have defined unitary diffeomorphisms and \( \mathbb{C}^* \)-algebras by using only structures available on a real smooth manifold of even dimensions. The original Hilbert space was instrumental in defining the tensor fields, but we have not used the vector space structure any more.

As is well known, due to the probabilistic interpretation of quantum mechanics, states of a quantum system are to be identified with rays of the Hilbert space according to the equivalence relation
\[ \psi_1 \sim \psi_2 : \iff \exists \lambda \in \mathbb{C} \setminus \{0\} : \psi_1 = \lambda \psi_2 \] (2.19)
on any two vectors \( \psi_1, \psi_2 \in \mathcal{H} \). If we restrict our attention to the space of rays, \( \mathcal{R}(\mathcal{H}) \), we deal with a manifold that is no more diffeomorphic to a vector space.

Before continuing with general structures and arguments, let us consider the most simple nontrivial example \( \mathcal{H} = \mathbb{C}^2 \), the Hilbert space of a two-level system. We have
\[ |e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \] (2.20)
\[ z^1 = x^1 + iy^1, \quad z^2 = x^2 + iy^2. \] (2.21)
A generic Hermitian matrix \( A \) may be decomposed by means of the Pauli matrices into
\[ A = Y_0 \sigma_0 + \bar{Y} \bar{\sigma}. \] (2.22)
A generic quadratic form whose Hamiltonian vector field is also Killing is given by
\[ f_A(\psi) = \langle \psi | A | \psi \rangle = (z_1, \bar{z}_2) A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \] (2.23)
Complex-valued functions that define the \( \mathbb{C}^* \)-algebra have the form
\[ F = f_A + i f_B = f_{A+iB}, \] (2.24)
with an associative product
\[ F \ast G = f_{A+iB} \ast g_{M+iN} = f_{(A+iB)(M+iN)}. \] (2.25)
This resulting product is not pointwise, i.e. it is a non-local product, which is an essential ingredient to take into account the quantum nature of the system we are describing.

Let us now describe the manifolds of rays. The most efficient way is to consider coordinate functions, say \((z_1, \bar{z}_2)\), and consider pure states as rank-one projectors
\[ |\psi\rangle \langle \psi| = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} (\bar{z}_1, \bar{z}_2) = \begin{pmatrix} z_1 \bar{z}_1 & z_1 \bar{z}_2 \\ z_2 \bar{z}_1 & z_2 \bar{z}_2 \end{pmatrix}. \] (2.26)
To normalize it, we set
\[ \rho_\psi = \frac{1}{z_1 \bar{z}_1 + z_2 \bar{z}_2} \begin{pmatrix} z_1 \bar{z}_1 & z_1 \bar{z}_2 \\ z_2 \bar{z}_1 & z_2 \bar{z}_2 \end{pmatrix}. \] (2.27)
We notice that
\[ \rho_\psi \cdot \rho_\psi = \rho_\psi, \quad \text{Tr}(\rho_\psi) = 1. \] (2.28)
By using the decomposition
\[ \rho_\psi = Y_0 \sigma_0 + \bar{Y} \bar{\sigma}, \] (2.29)
we find that
\[ \vec{Y} \cdot \vec{Y} = \frac{1}{4} \]  
(2.30)
and \( Y_0 = \frac{1}{2} \). Thus, the space of rays is diffeomorphic with the manifold \( S^2 \subset \mathbb{R}^3 \). This manifold is a Hilbert manifold.

Another parametrization of the ray space could be given in terms of homogeneous coordinates, say
\[ \xi = \frac{z_1}{z_2}. \tag{2.31} \]
This description, unlike the previous one, is singular when \( z_2 = 0 \); then one may use
\[ \eta = \frac{z_2}{z_1}. \tag{2.32} \]
The Schrödinger equation
\[ i\hbar \frac{d}{dt} \psi = H \psi \tag{2.33} \]
on \( \mathbb{C}^2 \), which may be written in complex Cartesian coordinates as
\[ i\hbar \frac{d}{dt} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{2.34} \]
defines on these homogeneous coordinates a Riccati-type equation \[ i\hbar \frac{d}{dt} \xi = -H_{12} + (H_{11} - H_{22}) \xi - H_{21} \xi^2. \tag{2.35} \]
This equation is nonlinear and does not define a one-parameter group of diffeomorphisms because of the singularity introduced by \( z_2 = 0 \). It is clear, however, that this behavior is an artifact of the coordinate system. It is not a singularity of the equation which describes a well-defined one-parameter group of diffeomorphisms on the sphere \( S^2 \). Moreover, this equation allows us to remark that now the superposition rule, available on \( \mathbb{C}^2 \), is here replaced by the superposition of solutions of the Riccati equation \[ 26. \] Thus, the description of interference phenomena is also possible, as it should, on the manifold of pure states if we use a generalized superposition rule \[ 33. \] We shall consider now in more general terms which tensor fields and which associated binary, bilinear brackets are available on the Hilbert manifold of rays, pure states.

3. Tensorial structures on pure states

As we have stressed in the previous section, the probabilistic interpretation requires that states are identified with rays of the Hilbert space \( \mathcal{H} \). We have identified a description in terms of normalized states, \( \langle \psi | \psi \rangle = 1 \), by setting \( \rho = | \psi \rangle \langle \psi | \). We now identify tensors, built out of the Hermitian tensor on \( \mathcal{H} \), which are defined on \( \mathcal{R}(\mathcal{H}) \), i.e. they depend on complex rays rather than on states. Thus, \( G \) and \( \Lambda \) of our previous section are modified to construct
\[ \tilde{G}(\psi) = \langle \psi | \psi \rangle G - (\Delta \otimes \Delta + \Gamma \otimes \Gamma), \tag{3.1} \]
\[ \tilde{\Lambda}(\psi) = \langle \psi | \psi \rangle \Lambda - (\Delta \otimes \Gamma - \Gamma \otimes \Delta), \tag{3.2} \]
where \( \Delta \) and \( \Gamma \) denote generating vector fields of \( \mathbb{R}^*_+ \)-dilations and \( U(1) \)-phase transformations \[ 28. \] Spelled out in coordinates we find
\[ \tilde{G}(z, \bar{z}) = \left( \sum_k \bar{z}_k z_k \right) \left( \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} + \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial y^i} \right) \\
- \sum_{l,m} \left( y^l \frac{\partial}{\partial y^m} \otimes y^m \frac{\partial}{\partial y^l} \right) \\
+ \left( y^l \frac{\partial}{\partial x^l} - x^l \frac{\partial}{\partial y^l} \right) \otimes \left( y^m \frac{\partial}{\partial x^m} - x^m \frac{\partial}{\partial y^m} \right). \tag{3.3} \]
Similarly out of \( \Lambda \), we define
\[ \tilde{\Lambda}(z, \bar{z}) = \left( \sum_k \bar{z}_k z_k \right) \left( \frac{\partial}{\partial y^j} \otimes \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial y^j} \right) \\
- \sum_{l,m} \left( y^l \frac{\partial}{\partial y^j} + x^l \frac{\partial}{\partial x^j} \right) \otimes \left( y^m \frac{\partial}{\partial x^m} - x^m \frac{\partial}{\partial y^m} \right) \\
- \left( y^m \frac{\partial}{\partial x^m} - x^m \frac{\partial}{\partial y^m} \right) \otimes \left( y^l \frac{\partial}{\partial y^j} + x^l \frac{\partial}{\partial x^j} \right). \tag{3.4} \]
By construction, the bi-differential operators have the property that
\[ ((f, g)\tilde{\Lambda}) = (\tilde{G} + i\tilde{\Lambda})(df, dg) \tag{3.5} \]
is well defined on rays whenever \( f \) and \( g \) depend only on rays and not on the representation vectors. Thus \( \tilde{\Lambda} \) induces a Poisson bracket and \( \tilde{G} \) induces a binary, bilinear symmetric bracket on pulled-back functions from the manifold \( \mathcal{R}(\mathcal{H}) \).

On functions defined on \( \mathcal{R}(\mathcal{H}) \), which enjoy the property that their Hamiltonian vector fields are also Killing vectors for \( \tilde{G} \) when they are pulled back to \( \mathcal{H} \), we are again able to define a \( \mathbb{C}^* \)-algebra. Thanks to the way in which we have defined our tensor fields, it turns out that the function \( c(\psi) = \langle \psi | \psi \rangle \) is a central element with respect to the bracket defined by \( \tilde{\Lambda} \) on \( \mathcal{F}(\mathcal{H}) \). It may be instructive to compute these brackets on expectation value functions of Hermitian operators, say
\[ e_A(\psi) = \frac{\langle \psi | A | \psi \rangle}{\langle \psi | \psi \rangle}, \quad e_B(\psi) = \frac{\langle \psi | B | \psi \rangle}{\langle \psi | \psi \rangle}. \tag{3.6} \]
We find
\[ \tilde{\Lambda}(de_A, de_B)(\psi) = e_{[(A,B)]}(\psi), \tag{3.7} \]
while
\[ \tilde{G}(de_A, de_B)(\psi) = \frac{1}{2} e_{[(A,B)]}(\psi) - e_A(\psi)e_B(\psi). \tag{3.8} \]
Clearly, when we consider \( B = \text{Id} \), we find
\[ e_{\text{Id}}(\psi) = \frac{\langle \psi | \text{Id} | \psi \rangle}{\langle \psi | \psi \rangle} = 1 \tag{3.9} \]
and we obtain
\[ \tilde{\Lambda}(de_A, de_B) = \tilde{G}(de_A, de_B) = 0. \tag{3.10} \]
By using again only functions that are infinitesimal generators of unitary diffeomorphisms on \( \mathcal{R}(\mathcal{H}) \), we may define a \( \mathbb{C}^* \)-algebra on \( \mathcal{R}(\mathcal{H}) \) by setting
\[
((e_A, e_B)) = (\tilde{G} + i\tilde{A})(de_A, de_B) + e_A \cdot e_B. \tag{3.11}
\]
By \( e_A \cdot e_B \) we mean the pointwise product of the two functions. The notation \( e_A, e_B \) is reminiscent of the expectation value functions associated with Hermitian operators, but this time they are not defined out of Hermitian operators. They are identified by simply requiring that their Hamiltonian vector fields are also Killing vector fields for \( \tilde{G} \). When \( e_A \) and \( e_B \) are extended to complex-valued functions, the previous requirement should be made separately for the real part and the imaginary part.

**Remark 2.** The emerging picture of our Hilbert manifold description is that the \( \mathbb{C}^* \)-algebra approach appears to be more general than the Hilbert space approach. Indeed, to describe the formalism on the space of rays, we have to go from the Hilbert space to the Hilbert manifold while we are not obliged to change perspective within the \( \mathbb{C}^* \)-algebra approach once the imaginary elements of the algebra are identified as infinitesimal generators of unitary diffeomorphisms.

### 4. The GNS construction

The \( \mathbb{C}^* \)-algebra we have defined on \( \mathcal{A} \subset \mathcal{F}(\mathcal{R}(\mathcal{H})) \) allows one to go from pure states to general density states by simply using the usual notion of positive, normalized linear functionals
\[
\rho \in \mathcal{D} \subset \mathcal{A}^*. \tag{4.1}
\]
As is usual with duality, from the action \( \mathcal{A} \times A \to A \) of \( \mathcal{A} \) on itself, say on the right or on the left, it is possible to induce an action on states. Starting with a state \( \rho \), the action of \( \mathcal{A} \) on \( \rho \) defines a Hilbert space \( \mathcal{H}_\rho \) with a Hermitian inner product
\[
\langle a | b \rangle_\rho = \rho(a^* b) \tag{4.2}
\]
by setting
\[
(a \cdot \rho) + (b \cdot \rho) = (a + b) \cdot \rho. \tag{4.3}
\]
The set of elements that annihilate \( \rho \) defines the bilateral Gelfand ideal \( \mathcal{J}_\rho \) and
\[
\mathcal{A} \cdot \rho \equiv \mathcal{H}_\rho \tag{4.4}
\]
becomes identified with the quotient of \( \mathcal{A} \) by the Gelfand ideal \( \mathcal{J}_\rho \). When we restrict the action of elements of \( \mathcal{A} \) on \( \rho \) only by means of invertible elements, we obtain a manifold
\[
\mathcal{G} \cdot \rho = \mathcal{G}/\mathcal{G}_\rho, \tag{4.5}
\]
which is the quotient of the group \( \mathcal{G} \) defined by the invertible elements by those which leave \( \rho \) invariant. If we restrict further the invertible elements to those which preserve the pairing \( \rho(e_A^* e_B) \), the associated group becomes the ‘unitary group’ and the corresponding manifold becomes a quotient of unitary groups.

The manifold picture emerging from (adjoint) group actions
\[
g \rho g^\dagger = \rho_g \tag{4.6}
\]
on states \( \rho \) is very interesting and clearly provides a generalization of the idea underneath coherent states. It is sufficient that out of the states generated by the group action we are able to construct a partition of unity, i.e. a completeness relation
\[
\int_{\mathcal{G}/\mathcal{G}_\rho} |\psi_g\rangle d\mathcal{G}_\rho = 1. \tag{4.7}
\]
This aspect is closely related to the notion of a tomographic set in the description of quantum mechanics by means of quantum tomography [29].

The tensorial (algebraic) structures available on the space of functions that define a \( \mathbb{C}^* \)-algebra, by duality, can be induced on the space of states. From the space of all states, one may restrict them to the manifold of states selected by the action of a group or by any other means. Therefore, in suitable conditions, we may study particular problems by considering finite-dimensional real manifolds of states
\[
\mathcal{G}/\mathcal{G}_\rho \tag{4.8}
\]
rather than the full space of states, which is usually infinite dimensional and realized by means of
\[
L^2(\mathcal{G}/\mathcal{G}_\rho). \tag{4.9}
\]
This Hilbert space may be considered as the subspace of \( L^2(\mathcal{G}) \) when \( \mathcal{G}_\rho \) is compact. We stress again that due to the completeness relation (4.7), it is in fact possible to consider finite-dimensional real submanifolds of quantum states to generate the full (infinite-dimensional) space of states.

In this setting we are going to consider manifolds of states and the induced tensor fields.

### 5. Induced tensor fields on manifolds of quantum states

On a given Hilbert manifold \( \mathcal{H} \) we consider a covariant Hermitian tensor field
\[
\frac{\langle d\psi \mid d\psi \rangle}{\langle \psi \mid \psi \rangle} - \frac{\langle d\psi \mid d\psi \rangle \langle d\psi \mid \psi \rangle}{\langle \psi \mid \psi \rangle^2}, \tag{5.1}
\]
admitting the property of having the generating vector field of \( \mathbb{C}_0 \)-transformations in its kernel. For a given embedding of a manifold \( Q \) of quantum states
\[
i_Q : Q \hookrightarrow \mathcal{H}, \tag{5.2}
\]
we find an induced covariant rank-2 tensor on \( Q \) defined by the pull-back tensor
\[
i_Q \left( \frac{\langle d\psi \mid d\psi \rangle}{\langle \psi \mid \psi \rangle} - \frac{\langle d\psi \mid d\psi \rangle \langle d\psi \mid \psi \rangle}{\langle \psi \mid \psi \rangle^2} \right). \tag{5.3}
\]
In the case that \( Q \) is a homogeneous space \( \mathcal{G}/\mathcal{G}_0 \), we may find an embedding by means of the unitary representation
\[
\mathcal{G} \to U(\mathcal{H}) \tag{5.4}
\]
of a ‘classical’ Lie group \( \mathcal{G} \) on a normalized fiducial quantum state
\[
|0\rangle \in L^2(Q) \cap C^\infty(Q). \tag{5.5}
\]
By introducing a basis $\{\theta_1\}_{j \in J}$ of left-invariant 1-forms on $G$ and a basis $\{X_j\}_{j \in J}$ on the Lie algebra of $G$, we find
\begin{equation}
\langle 0 | R(X_j) R(X_k) | 0 \rangle = \langle 0 | R(X_j) | 0 \rangle \langle 0 | R(X_k) | 0 \rangle \theta^j \otimes \Theta^k,
\end{equation}
as pull-back tensor on $G/G_0$, where $\{R(X_j)\}_{j \in J}$ define the action of the Lie algebra of $G$ by means of self-adjoint operators $[30]$. To evaluate the tensor on $T_Q \subset \mathcal{U}(\mathcal{H})$ we set $\rho_0 := |0 \rangle \langle 0 | \in u^*(\mathcal{H})$, where one finds
\begin{equation}
\langle 0 | R(X_j) R(X_k) | 0 \rangle = \langle 0 | R(X_j) | 0 \rangle \langle 0 | R(X_k) | 0 \rangle \theta^j \otimes \Theta^k
\end{equation}
which allows one to identify a symmetric and an antisymmetric tensor:
\begin{equation}
T_{(jk)}^{\rho_0} \equiv \langle [R(X_j), R(X_k)]_{\rho_0} \theta^j \otimes \Theta^k. \tag{5.8}
\end{equation}

We see that the tensor coefficients $T_{(jk)}^{\rho_0}$ are only dependent on the fiducial state $\rho_0$ and the chosen Lie algebra representation, but not on the individual points $q \in Q$. These coefficients decompose into a symmetric and an antisymmetric part:
\begin{equation}
T_{(jk)}^{\rho_0} := \langle [R(X_j), R(X_k)]_{\rho_0} \theta^j \otimes \Theta^k, \tag{5.9}
\end{equation}
which allows one to identify a symmetric and an antisymmetric tensor
\begin{equation}
G_\rho \equiv \langle [R(X_j), R(X_k)]_{\rho_0} \theta^j \otimes \Theta^k, \tag{5.10}
\end{equation}
on $G$. We point out that these coefficients may be expressed by means of the expectation values
\begin{equation}
\epsilon_{R(X_j)} := \frac{\langle 0 | R(X_j) | 0 \rangle}{\langle 0 | 0 \rangle} \tag{5.13}
\end{equation}
associated with the operators $R(X_j)$, i.e. $T_{(jk)}^{\rho_0}$ is $\tilde{G}(\text{det}_{R(X_j)}, \text{det}_{R(X_k)})$ evaluated at $\rho_0$, and similarly, $T_{(jk)}^{\rho_0}$ is $\Lambda_{\rho_0}$ evaluated at $\rho_0$. In conclusion, the symmetric and the anti-symmetric pull-back structures define functions
\begin{equation}
g \times g \times \mathcal{R}(\mathcal{H}) \to \mathbb{R}, \tag{5.14}
\end{equation}
which become bilinear on the Lie algebra $g$ once the fiducial state $\rho_0$ is fixed.

**Remark 3.** By pulling back the expectation value functions from $H_0$ to the group, we could try to define contravariant tensors on $G$ by using the coefficients $T_{(jk)}^{\rho_0}$; however, these tensors would not be defined in the directions of the isotropy group.

In general, we may use any state, say a positive normalized functional $\rho \in u^*(\mathcal{H})$, and consider, in analogy with (5.11) and (5.12), the classical tensors on the group manifold associated with the quantum density state $\rho$ in terms of a symmetric and an imaginary skew-symmetric (0,2)-tensor:
\begin{equation}
L_{(jk)}^{\rho} := \rho([R(X_j), R(X_k)]) \theta^j \otimes \Theta^k, \tag{5.15}
\end{equation}
\begin{equation}
L_{(jk)}^{\rho} := \rho([R(X_j), R(X_k)]) \theta^j \wedge \Theta^k, \tag{5.16}
\end{equation}
respectively $[31]$. In the following section we will see that in particular the symmetric tensor (5.15) will admit a direct application on the characterization of entanglement of mixed bipartite systems.

**Remark 4.** One may ask whether there exists an ‘isometrical’ embedding into a ‘surrounding’ Hilbert space such that the later tensors (5.15) and (5.16) can be identified as pulled-back tensors like in the case of (5.11) and (5.12). Here indeed, by proceeding towards the more general case of manifolds of quantum operations (e.g. unitarily related Hermitian matrices $[32]$), we may take the intrinsic rather than the extrinsic geometric point of view. For instance, by starting from the operator-valued (0,2)-tensor field $dU^+(q) \otimes dU(q)$, we could define directly
\begin{equation}
\rho(dU^+(q) \otimes dU(q)), \tag{5.17}
\end{equation}
associating a left invariant tensor field on the group manifold without a dependence of an embedding. Similarly to what happens in the GNS construction, this tensor will not be non-degenerate. It will be degenerate along the intersection of $R(T_Q)$ with the Gelfand ideal associated with $\rho$. Therefore, the tensor is not degenerate on the quotient space $G/G_0$, $G_0$ being the group associated with the sub-algebra of the Gelfand ideal.

6. Tensor characterization of quantum entanglement

6.1. Pure states

As a particular application, we shall consider the problem of separability and entanglement for states of composite systems $H_A \otimes H_B \cong \mathbb{C}^N \otimes \mathbb{C}^N$. [6.1]

By identifying $G = U(H_A) \times U(H_B)$ with the subgroup of transformations, which leave entanglement, respectively, the Schmidt coefficients of a state, invariant, we find a pull-back tensor (5.10) on
\begin{equation}
U(H_A) \times U(H_B)/G_{\rho_0}, \tag{6.2}
\end{equation}
which admits in the real part a Riemannian coefficient matrix
\begin{equation}
(G_{(jk)}) = \begin{pmatrix} G^A & G^{AB} \\ G^{AB} & G^B \end{pmatrix}, \tag{6.3}
\end{equation}
with the sub-block matrices
\begin{equation}
G_{(jk)}^A = \begin{bmatrix} \langle \sigma_j | \sigma_k \rangle_{\rho_0} - \langle \sigma_j \otimes 1 \rangle_{\rho_0} \langle \sigma_k \otimes 1 \rangle_{\rho_0} \\
\langle 1 \otimes \sigma_j | \sigma_k \rangle_{\rho_0} - \langle 1 \otimes \sigma_j \rangle_{\rho_0} \langle 1 \otimes \sigma_k \rangle_{\rho_0} \end{bmatrix},
\end{equation}
\begin{equation}
G_{(jk)}^B = \begin{bmatrix} \langle \sigma_j | \sigma_k \rangle_{\rho_0} - \langle \sigma_j \otimes 1 \rangle_{\rho_0} \langle \sigma_k \otimes 1 \rangle_{\rho_0} \\
\langle 1 \otimes \sigma_j | \sigma_k \rangle_{\rho_0} - \langle 1 \otimes \sigma_j \rangle_{\rho_0} \langle 1 \otimes \sigma_k \rangle_{\rho_0} \end{bmatrix},
\end{equation}
\begin{equation}
G_{(jk)}^{AB} = \begin{bmatrix} \langle \sigma_j | \sigma_k \rangle_{\rho_0} - \langle \sigma_j \otimes 1 \rangle_{\rho_0} \langle \sigma_k \otimes 1 \rangle_{\rho_0} \\
\langle 1 \otimes \sigma_j | \sigma_k \rangle_{\rho_0} - \langle 1 \otimes \sigma_j \rangle_{\rho_0} \langle 1 \otimes \sigma_k \rangle_{\rho_0} \end{bmatrix},
\end{equation}
whenever one chooses a tensor-product representation of $U(H_A) \times U(H_B)$ and its Lie algebra $[31]$. With this
Riemannian pull-back tensor we are able to characterize entanglement without the need of performing the computational effort of a singular value decomposition into Schmidt coefficients: In particular, one finds

$$\rho_0 \text{ is separable } \Leftrightarrow G^{AB} = 0 \Leftrightarrow G = G^{A} \otimes G^{B}. \quad (6.5)$$

Moreover, it turns out that the sub-block matrix $G^{AB}$ is useful to compute the distance to separable states, which has been identified in [33] by the quantity

$$\text{Tr}( (R)^\dagger R ) \quad (6.6)$$

with

$$R := \rho_0 - \rho_0^A \otimes \rho_0^B. \quad (6.7)$$

Here we find

$$\text{Tr}( (G^{AB})^\dagger G^{AB} ) \sim \text{Tr}( (R)^\dagger R ). \quad (6.8)$$

On the other hand, we recall that the pull-back tensor offers in its imaginary part also an antisymmetric tensor. In particular on the same homogeneous space of entangled quantum states we have therefore not only Riemannian, but also pre-symplectic tensor coefficients to evaluate. Here we encounter the coefficient matrix

$$(\Lambda_{ijk}) = \begin{pmatrix} \Lambda^A & 0 \\ 0 & \Lambda^B \end{pmatrix} \quad (6.9)$$

with

$$\Lambda_{ijk} \equiv \begin{cases} \Lambda^A_{ijk} = [\sigma_j, \sigma_k]_\perp \otimes 1_{\rho_0} \\
\Lambda^B_{ijk} = 1 \otimes [\sigma_j, \sigma_k]_\perp \otimes 1_{\rho_0} \end{cases} \quad (6.10)$$

For what concerns the entanglement of the fiducial state, it turns out that the pre-symplectic part of the pull-back tensor behaves in an opposite way to the Riemannian part [31]:

$$\rho_0 \text{ is maximal entangled } \Leftrightarrow \Lambda = \Lambda^A = \Lambda^B = 0. \quad (6.11)$$

### 6.2. Mixed states

We may end up with an identification of an applicable tensor construction for the entanglement characterization of mixed states. The latter are identified in the Schrödinger picture as elements of the convex hull

$$D(\mathcal{H}_A \otimes \mathcal{H}_B) \equiv D(\mathbb{C}^N \otimes \mathbb{C}^N) \quad (6.12)$$

of pure states associated to the bi-partite system $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$ discussed in the previous section. In particular, by evaluating the coefficients $L_{ijk}$ of the intrinsic geometrically defined rank-2 tensor field (5.15) on the orbits generated by the local unitary group $U(N) \times U(N)$ in $u^*(\mathcal{H})$, we find a neat connection to a separability criterion proposed by de Vicente [34]. By analogy to the case of pure states orbits, we consider $\rho \in D(\mathbb{C}^N \otimes \mathbb{C}^N)$, a density state on an orbit $Q/\mathcal{Q}_\rho$ generated by $Q = U(N) \times U(N)$. The tensor coefficients

$$L_{ijk} = \text{Tr}(\rho[R(X_j), R(X_k)]_\rho), \quad (6.13)$$

defined on this orbit may then become identified according to the coefficient matrix

$$(L_{ijk}) = \begin{pmatrix} L^A & L^{AB} \\ L^{AB} & L^B \end{pmatrix} \quad (6.14)$$

with the sub-block matrices

$$L_{ijk} \equiv \begin{cases} \Lambda^A_{ijk} = [\sigma_j, \sigma_k]_\perp \otimes 1_{\rho_0} \\
\Lambda^B_{ijk} = 1 \otimes [\sigma_j, \sigma_k]_\perp \otimes 1_{\rho_0} \end{cases} \quad (6.15)$$

This intrinsic geometrically defined tensor now admits the following application:

$$\rho_0 \text{ is separable } \quad (6.16)$$

$$\Rightarrow \frac{2}{N} \text{Tr}(\sqrt{L^{AB}} L^{AB}) \leq \frac{1}{2} (N^2 - N) \quad (6.17)$$

$$\Rightarrow \rho_0 \text{ is separable (for max. mixed subsystems in } N=2). \quad (6.18)$$

One notes that the required coefficients are given in the off-diagonal block elements $L^{AB}_{ijk} := C_{ijk}$ of the coefficient matrix $(L_{ijk})$, defined by

$$C_{ijk} = \text{Tr}(\rho[\sigma_j \otimes 1, 1 \otimes \sigma_k]_{-N^2}), \quad (6.19)$$

yielding

$$C_{ijk} = \text{Tr}(\rho[\sigma_j \otimes \sigma_k]_{-N^2}). \quad (6.20)$$

One then directly completes the proof by means of theorem 1 and the following discussion leading to corollary 1 in [34], by identifying the Ky Fan Norm

$$\|C\|_{KF} := \text{Tr}(\sqrt{C^\dagger C}), \quad (6.21)$$

within inequality (6.17). Let us apply the above criterion on an explicit example:

### Example. Werner states for the case $N = 2$

Consider a density state in $D(\mathbb{C}^2 \otimes \mathbb{C}^2)$, defined as a convex combination of a maximal entangled pure state associated to a vector

$$|\phi^+\rangle := \frac{1}{\sqrt{2}} \left( |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \right). \quad (6.22)$$

and a maximal mixed state

$$\rho^* := \frac{1}{2} I, \quad (6.23)$$

according to

$$\rho_W := x |\phi^+\rangle \langle \phi^+| + (1-x) \rho^*, \quad (6.24)$$

with $x \in [0,1]$. The latter state is referred in the literature to the class of Werner states [35]. By identifying

$$\rho_W \equiv \rho \in A^* \cong M_4(\mathbb{C}), \quad (6.25)$$

we find a symmetric tensor on a six-dimensional real submanifold

$$\text{SU}(2) \times \text{SU}(2) \subset U(4) \subset A^* \quad (6.26)$$
whose coefficient matrix reads

\[
(L_{ijk}) = \begin{pmatrix}
1 & 0 & 0 & x & 0 & 0 \\
0 & 1 & 0 & 0 & -x & 0 \\
0 & 0 & 1 & 0 & 0 & x \\
x & 0 & 0 & 1 & 0 & 0 \\
0 & -x & 0 & 0 & 1 & 0 \\
0 & 0 & x & 0 & 0 & 1
\end{pmatrix}, \quad (6.27)
\]

where one identifies within a decomposition

\[
(L_{ijk}) := \begin{pmatrix}
A & C \\
C & B
\end{pmatrix}
\]

the block of diagonal elements

\[
C = \begin{pmatrix}
x & 0 & 0 \\
0 & -x & 0 \\
0 & 0 & x
\end{pmatrix}. \quad (6.29)
\]

The latter is identically related to the symmetric tensor coefficients \( L_{ijk} \) for \( 1 \leq j \leq 3 \) and \( 5 \leq k \leq 6 \). By computing the Ky Fan norm of \( C \), one finds

\[
\text{Tr}(\sqrt{C^*C}) = 3x, \quad (6.30)
\]

where we conclude according to the criterion (6.18) that \( \rho_W \) is separable iff

\[
x \leq \frac{1}{3}. \quad (6.31)
\]

7. Conclusions and outlook

We have shown that a geometrical description of quantum mechanics is possible and that many concepts and constructions available in classical mechanics are also available in the quantum framework. The richer structure emerging in the quantum setting allows us to introduce not only Poisson brackets but also Jordan brackets and Lie–Jordan algebras, a description based just on observables not only Poisson brackets but also Jordan brackets and Lie–Jordan algebras, a description based just on observables available in the quantum framework. The richer structure available in classical mechanics are also emerging in the quantum setting allows us to introduce antisymmetrized products of the symplectic structure and of the symplectic potential (absolute Poincaré invariants), one may introduce higher order tensors by means of similar constructions in terms of

\[
(U^\dagger \otimes dU \otimes U^\dagger \otimes dU)(g) \otimes \cdots \otimes (U^\dagger \otimes dU \otimes U^\dagger \otimes dU)(g).
\]

As we will show elsewhere, this approach allows us to deal with \( n \)-fold expectation values and correlation functions. We believe that a worked-out geometrization of quantum theories may provide very useful suggestions for a unification of quantum mechanics and general relativity.

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