RICCI SOLITONS AND CURVATURE INHERITANCE ON
ROBINSON-TRAUTMAN SPACETIMES

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Abstract. The purpose of the article is to investigate the existence of Ricci solitons and the
nature of curvature inheritance as well as collineations on the Robinson-Trautman (briefly, RT)
spacetime. It is shown that under certain conditions RT spacetime admits almost Ricci soliton,
almost $\eta$-Ricci soliton, almost gradient $\eta$-Ricci soliton. As a generalization of curvature inheri-
tance \cite{20} and curvature collineation \cite{27}, in this paper, we introduce the notion of generalized
curvature inheritance and examine if RT spacetime admits such a notion. It is shown that RT
spacetime also realizes the generalized curvature (resp. Ricci, Weyl conformal, concircular, con-
harmonic, Weyl projective) inheritance. Finally, several conditions are obtained, under which
RT spacetime possesses curvature (resp. Ricci, conharmonic, Weyl projective) inheritance as
well as curvature (resp. Ricci, Weyl conformal, concircular, conharmonic, Weyl projective)
collineation.

1. Introduction

During the investigation of compact 3-dimensional manifolds with positive Ricci curvature, in
1982, Hamilton \cite{25} established the concept of Ricci flow, a process of evolving a Riemanninan
metric over time. The main idea of Hamilton was to smooth out the singularities of the metric
by establishing a new type of non-linear diffusion equation. Ricci solitons \cite{26} are the self-similar
solutions of the Ricci flow, which is a natural generalization of Einstein metrics \cite{16,18,33,61}.
There are several generalizations of Ricci solitons such as almost Ricci solitons, $\eta$-Ricci solitons,
gradient $\eta$-Ricci solitons etc. During last three decades, lots of results (see, \cite{1,13,14}) on Ricci
solitons have been appeared and now it is an active area of research in differential geometry.

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curvature collineation; Weyl conformal collineation; concircular collineation; conharmonic collineation; Weyl
projective collineation.

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If the Ricci curvature $S$ of a Riemannian manifold $(Q, g)$ satisfies
\[ \frac{1}{2} \mathcal{L}_\xi g_{ij} + S_{ij} - \mu g_{ij} = 0 \]
with the constant $\mu$ and the Lie derivative $\mathcal{L}_\xi$ in the direction of the soliton vector field $\xi$, then $Q$ is said to be a Ricci soliton. If $\xi = \nabla \zeta$ for a smooth function $\zeta$ on $Q$, then $Q$ is said to be gradient Ricci soliton with $\zeta$ as the potential function and it satisfies
\[ S_{ij} + (\nabla^2 f)_{ij} - \mu g_{ij} = 0. \]
The Ricci soliton is expanding, steady or shrinking as per the conditions $\mu < 0$, $\mu = 0$ or $\mu > 0$ respectively. If $\mu$ is non-constant, specially a smooth function, then it is called an almost Ricci soliton [30]. For a Killing soliton vector field $\xi$, the Ricci soliton becomes an Einstein manifold. If $(Q, g)$ admits a non-zero 1-form $\eta$ satisfying
\[ \frac{1}{2} \mathcal{L}_\xi g_{ij} + S_{ij} - \mu g_{ij} + \lambda (\eta \otimes \eta)_{ij} = 0, \]
$\mu, \lambda$ being constants, then it is called an $\eta$-Ricci soliton [19]. Again, the soliton is known as an almost $\eta$-Ricci soliton [17] if $\mu, \lambda$ are allowed to be smooth functions.

Again, the geometrical symmetries perform a pivotal role in the theory of general relativity as realizing the Einstein’s field equations (EFE), the establishment of gravitational potentials can be obtained by imposing the symmetries. Several kinds of geometrical symmetries in spacetimes can be achieved if the Lie derivative of certain tensor with respect to some vector field vanishes. The vanishing Lie derivative of a geometric quantity with respect to some vector field usually represents the preservation of the geometric quantity in the direction of the vector field. Such symmetries are known as collineations and they can be described if in the direction of some vector field certain geometric quantities such as metric tensor, Ricci tensor, stress-energy momentum tensor, Riemann curvature tensor, Weyl conformal curvature tensor, Weyl projective curvature tensor etc. remain invariant. The underlying vector field may be null, spacelike or timelike. The role of collineations in general relativity was demonstrated by Katzin et al. [27,28]. In the context of gravitational radiation, investigation on conservation laws consists of a fundamental symmetry property of a spacetime, known as curvature collineation as the relations between groups of motions, collineations and conservation laws of energy, momentum are crucial in understanding the geometry of a spacetime. Ahsan [4], Ahsan and
Husain [11] studied several kinds of collineations to find the relation between physical properties and geometrical symmetries of electromagnetic fields and showed that motion and Maxwell collineation are not comparable for null electromagnetic field. Using the Nijenhuis tensor, the notion of torsion collineation was introduced by Ahsan [4]. Also, several geometrical aspects of collineation have been further studied by Ahsan et al. [2, 3, 6–9], Ali and Ahsan [10, 12].

The Riemannian manifold \((Q, g)\) admits curvature (resp., Ricci, Weyl conformal, concircular, conharmonic, Weyl projective) collineation if the Lie derivative \(\mathcal{L}_\xi R\) of the Riemann curvature tensor \(R\) (resp., Ricci curvature tensor \(S\), Weyl conformal curvature tensor \(C\), concircular curvature tensor \(W\), conharmonic curvature tensor \(K\), Weyl projective curvature tensor \(P\)) vanishes, where \(\xi\) is a smooth vector field. Generalizing the notion of curvature collineation, Duggal [20] introduced the concept of curvature inheritance. In the present paper, we have introduced the notion of generalized curvature inheritance (see Definition 2.2), which includes the notion of curvature inheritance as well as curvature collineation. Then, we have showed that RT spacetime fulfills generalized curvature (resp. Ricci, Weyl conformal, concircular, conharmonic, Weyl projective) inheritance. Also, we deduce several conditions under which RT spacetime realizes the curvature (resp. Ricci, conharmonic, Weyl projective) inheritance and curvature (resp. Ricci, Weyl conformal, concircular, conharmonic, Weyl projective) collineation.

RT spacetime [32] was introduced in 1960 as a class of algebraically special solutions of EFE having a repeated principal null direction associated with a shearless, expanding and non-twisting null geodesic congruence. During the collision of two black holes the estimation of mass loss in the final phase can be derived by the RT spacetime. Also, the RT spacetime can represent the model of gravitational radiation outgoing from spatially bounded sources. Describing topological equivalent two spheres instead of strictly two spheres, the RT spacetimes generalize the notion of spherical symmetry. The RT spacetime is a crucial class of exact solutions of EFE as it is used to describe several models of gravitational waves, black holes and cosmology. The Petrov type II RT spacetime is given as follows (\([23, 31, 32, 68]\)):

\[
\begin{align*}
    ds^2 &= -2(U^0 - 2\gamma^0 r - \Psi^0_2 r^{-1})du^2 + 2dudr - \frac{r^2}{2\Omega^2}d\zeta d\bar{\zeta} \\
    &= -2(\frac{a - 2br - d}{r})dt^2 + 2dtdr - \frac{r^2}{f^2}(dx^2 + dy^2),
\end{align*}
\]  

where \(U^0, \gamma^0, \Psi^0_2\) are constants and \(\Omega\) is a everywhere non-vanishing function of \(\zeta, \bar{\zeta}\). Setting \(\zeta = x + iy\), the RT metric can be expressed as follows:

\[
    ds^2 = -2\left(\frac{a - 2br - d}{r}\right)dt^2 + 2dtdr - \frac{r^2}{f^2}(dx^2 + dy^2),
\]  

(1.1)
where \( a, b, d \) are constants and \( f \) is nowhere vanishing function of \( x, y \). Again, for the warping function \( r \), the metric tensor \( g \) of RT spacetime is the warped product \( \bar{g} \times_r \tilde{g} \) with base metric \( \bar{g} \) and fiber metric \( \tilde{g} \) given by

\[
\bar{g} = \begin{pmatrix}
-2(a - 2br - d) & 1 \\
1 & 0
\end{pmatrix} \quad \text{and} \quad \tilde{g} = \begin{pmatrix}
-\frac{1}{f^2} & 0 \\
0 & -\frac{1}{f^2}
\end{pmatrix}.
\]

In 2018, Shaikh et al. [34] explored several curvature properties of RT spacetime and showed that RT spacetime is 2-quasi-Einstein, Roter type manifold and it admits Weyl pseudosymmetric, Ricci pseudosymmetric, pseudosymmetric Weyl conformal curvature tensor, Riemann compatible Ricci tensor, recurrent Weyl conformal curvature 2-forms, pseudosymmetric energy momentum tensor. Recently, Shaikh and his co-authors commenced the investigation of curvature properties of several spacetimes such as Melvin magnetic spacetime [35], Nariai spacetime [36], Lemaitre-Tolman-Bondi spacetime [37], generalized pp-wave spacetime [40], Kantowski-Sachs spacetime [41], Vaidya-Bonner spacetime [42], Siklos spacetime [43], Som-Raychaudhuri spacetime [48], Lifshitz spacetime [60], interior black hole spacetime [44], Vaidya spacetime [53] etc. Also, it is noteworthy to mention that Sabina et al. studied various curvature properties of \((t - z)\)-type plane wave spacetime [22], Morris-Thorne wormhole spacetime [21].

The principal moto of this article is to investigate the curvature inheritance, Ricci solitons and collineations with different curvature tensors such as Ricci, conharmonic, projective curvature tensor. The different aspects of Ricci solitons has been recently studied by Ahsan et al. [1, 13, 14], Shaikh et al. [62, 67], Blaga [17]. But Ricci solitons in various spacetimes remain to be investigated yet. Therefore, a natural question arises: whether RT spacetime admits Ricci solitons or not? If so, then what would be the nature of such solitons or under which conditions it admits Ricci solitons and collineations? The answer of these questions have been given in the present paper. The technique of the present article is unconventional as we have calculated various components of different related tensors of Ricci solitons and collineations from the RT metric and then we have checked and verified the concerned governing equations with the help of a program developed by the present authors in Wolfram Mathematica.

The outline of the paper is delineated in the following ways: Section 2 is devoted to the basic rudiments of the geometric and curvature properties of RT spacetimes. Section 3 deals with the investigation of several Ricci solitons on RT spacetime and their nature. It is shown that under various conditions RT spacetime can be exhibited as an example of almost Ricci
soliton, almost $\eta$-Ricci soliton, almost gradient $\eta$-Ricci soliton. Section 4 is engaged with the investigation of the inheritance and collineations with respect to Riemann (resp. Ricci, Weyl conformal, projective, concircular and conharmonic) curvature tensor.

2. Preliminaries

Let $(Q, g)$ be a smooth connected $n$-dimensional ($n \geq 3$) semi-Riemannian manifold with the Lie algebra $\chi(Q)$ of all smooth vector fields on $M$. Now, the Kulkarni-Nomizu product $\phi \wedge \psi$ of two $(0, 2)$ type symmetric tensors $\phi$ and $\psi$ is defined as ($[24, 52]$):

$$(\phi \wedge \psi)_{pqrs} = \phi_{ps} \psi_{qr} - \phi_{pr} \psi_{qs} + \phi_{qr} \psi_{ps} - \phi_{qs} \psi_{pr}.$$

Let $T^r_s(Q)$ ($r, s \geq 0$ are non-negative integers) be the space of tensor fields of type $(r, s)$ on the manifold $M$. A tensor $\Pi \in T^1_3(Q)$ is a generalized curvature tensor if $\Pi$ satisfies the following ($[29, 45]$):

(i) $\Pi(\mathcal{O}_1, \mathcal{O}_2)\mathcal{O}_3 + \Pi(\mathcal{O}_2, \mathcal{O}_1)\mathcal{O}_3 = 0$,
(ii) $\Pi(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4) = \Pi(\mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_1, \mathcal{O}_2)$ and
(iii) $\Pi(\mathcal{O}_1, \mathcal{O}_2)\mathcal{O}_3 + \Pi(\mathcal{O}_2, \mathcal{O}_3)\mathcal{O}_1 + \Pi(\mathcal{O}_3, \mathcal{O}_1)\mathcal{O}_2 = 0$ (Bianchi’s 1st identity),

where, $\Pi(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4) = g(\Pi(\mathcal{O}_1, \mathcal{O}_2)\mathcal{O}_3, \mathcal{O}_4)$ and for $i = 1, 2, 3, 4$ $\mathcal{O}_i \in \chi(Q)$. Further, if $\Pi$ satisfies Bianchi’s 2nd identity, i.e.,

$$(\nabla_{\mathcal{O}_1} \Pi)(\mathcal{O}_2, \mathcal{O}_3)\mathcal{O}_4 + (\nabla_{\mathcal{O}_2} \Pi)(\mathcal{O}_3, \mathcal{O}_1)\mathcal{O}_4 + (\nabla_{\mathcal{O}_3} \Pi)(\mathcal{O}_1, \mathcal{O}_2)\mathcal{O}_4 = 0,$$

$\nabla$ being the Levi-Civita connection on $Q$, then $\Pi$ is said to be a proper generalized curvature tensor. Some crucial examples of generalized curvature tensors are the Riemann curvature $R$, Gaussian curvature $G$, conharmonic curvature $K$, concircular curvature $W$, Weyl conformal curvature $C$, which are defined respectively as follows ($[38, 39, 40, 47, 49, 51, 54, 55]$):

$$
R_{pqrs} = g_{pa}(\partial_r \Gamma^a_{qr} - \partial_r \Gamma^a_{qs} + \Gamma^\beta_{qr} \Gamma^a_{\beta s} - \Gamma^\beta_{qs} \Gamma^a_{\beta r}) ,
$$
$$
G_{pqjs} = \frac{1}{2} (g_{ps} g_{qr} - g_{pr} g_{qs}) ,
$$
$$
K_{pqrs} = R_{pqrs} - n_3(S \wedge g)_{pqrs} ,
$$
$$
W_{pqrs} = R_{pqrs} - \kappa n_1 n_2 G_{pqrs} ,
$$
$$
C_{pqrs} = R_{pqrs} - n_3(S \wedge g)_{pqrs} + \kappa n_2 n_3 G_{pqrs} .
$$
where \( n_\lambda = \frac{1}{1+n-\lambda} \) (\( \lambda = 1, 2, 3 \)), \( \kappa \) is the scalar curvature, \( \Gamma^p_{qr} \) are connection coefficients and \( \partial_u = \frac{\partial}{\partial x_u} \). But, the Weyl projective curvature defined by

\[
P^u_{\ pqr} = R^u_{\ pqr} - 2n\delta^u_{[p}S_{qr]},
\]

is not a generalized curvature tensor. We note that \( W, C, K \) are not proper generalized curvature tensor.

**Definition 2.1.** If \( \mathcal{T} \in T^0_4(Q) \), then the manifold \( M \) is said to possess \( \mathcal{T} \) collineation for some \( \xi \in \chi(Q) \) if

\[
\mathcal{L}_\xi \mathcal{T} = 0,
\]

where \( \mathcal{L}_\xi \) denotes the Lie derivative along the vector field \( \xi \). If \( \mathcal{T} = R \) (resp. \( C, W, K, P \)), then the \( \mathcal{T} \) collineation is said to be curvature (resp. Weyl conformal, concircular, conharmonic, Weyl projective) collineation.

Generalizing the concept of curvature collineation, in 1992, Duggal [20] introduced the notion of curvature inheritance defined as follows:

\[
\mathcal{L}_\xi R = \lambda R,
\]

where \( \lambda \) is a smooth function and \( \xi \in \chi(Q) \). Now, generalizing the notion of curvature inheritance we introduce the concept of generalized \( \mathcal{T} \) inheritance, which is defined as follows:

**Definition 2.2.** If \( \mathcal{T} \in T^0_4(Q) \), then the manifold \( M \) is said to admit generalized \( \mathcal{T} \) inheritance with respect to some vector field \( \xi \) if

\[
\mathcal{L}_\xi \mathcal{T} = \lambda \mathcal{T} + \lambda_1 g \wedge g + \lambda_2 g \wedge S,
\]

where \( \lambda, \lambda_1, \lambda_2 \) are smooth functions and \( \wedge \) is the Kulkarni-Nomizu product. If \( \mathcal{T} = R \) (resp. \( C, W, K, P \)), then the generalized \( \mathcal{T} \) inheritance is said to be generalized curvature (resp. Weyl conformal, concircular, conharmonic, Weyl projective) inheritance.

If \( \lambda_1, \lambda_2 = 0 \) (resp. \( \lambda_1, \lambda_2, \lambda = 0 \)), then the generalized curvature inheritance turns into curvature inheritance (resp. curvature collineation).

**Definition 2.3.** For \( \mathcal{B} \in T^0_2(Q) \), the manifold \( M \) is said to be admitted \( \mathcal{B} \)-collineation if the Lie derivative of \( \mathcal{B} \) with respect to some vector field \( \xi \) vanishes, i.e.,

\[
\mathcal{L}_\xi \mathcal{B} = 0.
\]
If the tensor $B$ is chosen as the metric tensor $g$, Ricci curvature tensor $S$ and stress-energy momentum tensor $T$, then the $B$-collineations are also known as motion, Ricci collineation and matter collineation respectively. The underlying vector field for a motion is called Killing vector field. Generalizing the concept of Ricci collineation and Ricci (resp. Lie) inheritance, we introduced the notion of generalized Ricci (resp. Lie) inheritance, defined as follows:

**Definition 2.4.** The manifold $Q$ is said to be admitted generalized Ricci inheritance if

$$\mathcal{L}_\xi S = \lambda_S S + \lambda_g g,$$

where $\lambda_S, \lambda_g$ are smooth functions. If $\lambda_g = 0$, then it turns out to be Ricci inheritance and if $\lambda_S = 0 = \lambda_g$, then it is called Ricci collineation.

**Definition 2.5.** The manifold $Q$ is said to be admitted generalized Lie inheritance if

$$\mathcal{L}_\xi T = \lambda_T T + \lambda_g g,$$

where $\lambda_T, \lambda_g$ are smooth functions. If $\lambda_g = 0$, then it turns out to be Lie inheritance and if $\lambda_T = 0 = \lambda_g$, then it is known as Lie symmetry along $\xi$.

The components of Riemann curvature tensor and Ricci tensor of the RT spacetime are given as follows:

$$R_{1212} = -\frac{2d}{r^2}, R_{1313} = \frac{2(d-2br^2)(d+ r(-a+2br))}{r^2 f^2} = R_{1424}, R_{1323} = \frac{d-2br^2}{r f^2} = R_{1424},$$

$$R_{3434} = \frac{r(2(d+ r(-a+2br))+ r(f_x^2 + f_y^2 - f (f_{xx}+f_{yy})))}{f^4}.$$

$$S_{11} = -\frac{8b(d+r(-a+2br))}{r^2}, S_{12} = -\frac{4b}{r}, S_{33} = -\frac{2a+8br+f_x^2 + f_y^2 - f (f_{xx}+f_{yy})}{f^2} = S_{44}.$$

The scalar curvature $\kappa$ is given by

$$\kappa = \frac{4}{r^2} \left[ f_x^2 + f_y^2 - f (f_{xx} + f_{yy}) - 2a + 12br \right].$$

3. **Nature of Ricci solitons on Robinson-Trautman spacetime**

For an $n$-dimensional smooth semi-Riemannian manifold $Q$, the set $S(Q)$ of all Killing vector fields configures a Lie subalgebra of $\chi(Q)$. The maximum number of linearly independent Killing vector fields in $S(Q)$ is less than or equal to $n(n+1)/2$. The equality holds if $Q$ is of constant scalar curvature. We note that the scalar curvature $\kappa$ of RT spacetime is not a constant. In RT spacetime, the vector field $\frac{\partial}{\partial t}$ is a Killing vector field, i.e., $\mathcal{L}_{\frac{\partial}{\partial t}} g$ vanishes. In this section, we have considered several non-Killing vector fields such as $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\mu_1 \frac{\partial}{\partial r} + \mu_2 \frac{\partial}{\partial x} + \mu_3 \frac{\partial}{\partial y}$ and
\[ \nabla r^2 \text{ on the RT spacetime, where } \mu_1, \mu_2, \mu_3 \text{ are constants.} \]

For the non-Killing vector field \( X_2 = \frac{\partial}{\partial r} \), RT spacetime realizes the following relation:

\[ \mathcal{L}_{X_2} g + \frac{2r}{F_1 - 2a + 4br} S + \frac{8b}{F_1 - 2a + 4br} g - \frac{2(-d + 2br^2)}{r^2} \eta_1 \otimes \eta_1 = 0, \]

where \( \eta_1 = (1, 0, 0, 0) \) is an 1-form and \( F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \) is a smooth function such that \( F_1 - 2a + 4br \neq 0 \). Thus, we can state the following:

**Proposition 3.1.** If \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br = r \neq 0 \), then the RT spacetime admits an almost \( \eta \)-Ricci soliton given by

\[ \frac{1}{2} \mathcal{L}_\xi g + \frac{4b}{r} g - \frac{-d + 2br^2}{r^2} \eta \otimes \eta = 0, \]

where \( \eta = (1, 0, 0, 0) \) is an 1-form and \( \xi = \frac{\partial}{\partial r} \) is the soliton vector field.

Again, for the non-Killing vector field \( X_3 = \frac{\partial}{\partial x} \), RT spacetime fulfills the following relation:

\[ \mathcal{L}_{X_3} g - \frac{2r^2 f_x}{f(2a - 4br - F_1)} S - \frac{8br f_x}{f(2a - 4br - F_1)} g = 0, \]

where \( F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \) with \( F_1 - 2a + 4br \neq 0 \). This implies the following:

**Proposition 3.2.** If \( r^2 f_x + f \{2a - 4br - f_x^2 - f_y^2 + f(f_{xx} + f_{yy})\} = 0 \) and \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0 \), then the RT spacetime admits an almost Ricci soliton given by

\[ \frac{1}{2} \mathcal{L}_\xi g + \frac{4b}{r} g = 0, \]

where, \( \xi = \frac{\partial}{\partial x} \) is the soliton vector field. Further, if \( r \neq 0 \) is a constant \( c_1 \) and \( f \{ f(f_{xx} + f_{yy}) - f_x^2 - f_y^2 + 2a - 4bc_1 \} + c_1^2 f_x = 0 \), then the RT spacetime possesses a Ricci soliton given by

\[ \frac{1}{2} \mathcal{L}_\xi g + \frac{4b}{c_1} g = 0. \]

Again, for the non-Killing vector field \( X_4 = \frac{\partial}{\partial y} \), RT spacetime satisfies the following relation:

\[ \mathcal{L}_{X_4} g - \frac{2r^2 f_y}{f(F_1 - 2a + 4br)} S - \frac{8br f_y}{f(F_1 - 2a + 4br)} g = 0, \]

where \( F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \) and \( F_1 - 2a + 4br \neq 0 \). Thus we can state the following:
Proposition 3.3. If \( r^2 f_x + f \{ -2a + 4br + f_x^2 + f_y^2 - f (f_{xx} + f_{yy}) \} = 0 \) and \( f_x^2 + f_y^2 - f (f_{xx} + f_{yy}) - 2a + 4br \neq 0 \), then the RT spacetime admits an almost Ricci soliton given by
\[
\frac{1}{2} \mathcal{L}_\xi g + S + \frac{4b}{r} g = 0,
\]
where, \( \xi = \frac{\partial}{\partial y} \) is the soliton vector field.

Now, for the non-Killing vector field \( V_1 = \mu_1 \frac{\partial}{\partial r} + \mu_2 \frac{\partial}{\partial x} + \mu_3 \frac{\partial}{\partial y} \) (\( \mu_1, \mu_2, \mu_3 \) are constants), the following relation holds:
\[
\mathcal{L}_{V_1} g + \frac{2r \{-\mu_1 f + r (\mu_2 f_x + \mu_3 f_y)\}}{f (2a - 4br - F_1)} S + \frac{8b \{ \mu_1 f - r (\mu_2 f_x + \mu_3 f_y) \}}{f (2a - 4br - F_1)} g - \frac{2\mu_1 (-d + 2br^2)}{r^2} \eta_2 \otimes \eta_2 = 0,
\]
where \( F_1 = f_x^2 + f_y^2 - f (f_{xx} + f_{yy}) \) with \( F_1 - 2a + 4br \neq 0 \) and \( \eta_2 = (1, 0, 0, 0) \) is an 1-form. This concludes the following:

Proposition 3.4. If \( f \{ -2a + 4br - f_x^2 - f_y^2 + f (f_{xx} + f_{yy}) \} + r \{ \mu_1 f - r (\mu_2 f_x + \mu_3 f_y) \} = 0 \) and \( f_x^2 + f_y^2 - f (f_{xx} + f_{yy}) - 2a + 4br \neq 0 \), then the RT spacetime admits an almost \( \eta \)-Ricci soliton given by
\[
\frac{1}{2} \mathcal{L}_\xi g + S - 4bg - \frac{\mu_1 (-d + 2br^2)}{r^2} \eta \otimes \eta = 0,
\]
where \( \eta = (1, 0, 0, 0) \) is an 1-form and \( \xi = \mu_1 \frac{\partial}{\partial r} + \mu_2 \frac{\partial}{\partial x} + \mu_3 \frac{\partial}{\partial y} \) (\( \mu_1, \mu_2, \mu_3 \) are constants) is the soliton vector field.

Again, for the non-Killing vector field \( X = \text{grad}(r^2) = 2r \frac{\partial}{\partial r} - 4 \{ d + r (-a + 2br) \} \frac{\partial}{\partial r} \), the following relation holds:
\[
\mathcal{L}_X g - \frac{4r \{ 3d + 2r (-a + br) \}}{F_1 - 2a + 4br} S - \frac{4 \{ 2ad - 16br^2 + 4abr^2 + (-d + 2br^2) F_1 \}}{r (2a - 4br - F_1)} g - 4 \eta_3 \otimes \eta_3 = 0
\]
where \( \eta_3 = (0, 1, 0, 0) \) is an 1-form and \( F_1 = f_x^2 + f_y^2 - f (f_{xx} - f_{yy}) \) is a smooth function such that \( F_1 - 2a + 4br \neq 0 \). This leads to the following:

Proposition 3.5. If \( f_x^2 + f_y^2 - f (f_{xx} - f_{yy}) + 2 \{ 3d + 2r (-a + br) \} - 2a + 4br = 0 \) and \( f_x^2 + f_y^2 - f (f_{xx} - f_{yy}) - 2a + 4br \neq 0 \), then the RT spacetime admits an almost gradient \( \eta \)-Ricci soliton given by
\[
\frac{1}{2} \mathcal{L}_\xi g + S - \frac{2 (-6br^2 + 4abr^2 - 5bdr^2 + 3d^2 - 2adr - 4br^2 r^3 + 4abr^3 - 2b^2 r^4)}{r^2 \{ 3d + 2r (-a + br) \}} g - 2 \eta \otimes \eta = 0
\]
where $3d + 2r(-a + br) \neq 0$ with the 1-form $\eta = (0, 1, 0, 0)$ and $\xi = \nabla r^2$ is the soliton vector field.

**Remark 3.1.** If $\xi \in \chi(Q)$ can be expressed as $\xi = \alpha \frac{\partial}{\partial r} + \beta \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}$ for some constants $\alpha, \beta, \gamma$, then the RT spacetime can not be an example of an $\eta$-Yamabe soliton or a Yamabe soliton for the soliton vector field $\xi$.

**Theorem 3.1.** The RT spacetime \((1.1)\) together with $f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0$ satisfies the following properties:

(i) for the soliton vector field $\xi = \frac{\partial}{\partial r}$, the RT spacetime admits an almost $\eta$-Ricci soliton if

$$f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br - r = 0$$

with the 1-form $\eta = (1, 0, 0, 0)$, i.e.,

$$\frac{1}{2} \mathcal{L}_\xi g + S + \frac{4b}{r} g - \frac{d + 2br^2}{r^2} \eta \otimes \eta = 0,$$

(ii) for the soliton vector field $\xi = \frac{\partial}{\partial x}$, the RT spacetime possesses an almost Ricci soliton if

$$f\{f(f_{xx} + f_{yy}) - f_x^2 - f_y^2 + 2a - 4bc_1\} + r^2 f_x = 0,$$

i.e.,

$$\frac{1}{2} \mathcal{L}_\xi g + S + \frac{4b}{r} g = 0,$$

(iii) for the soliton vector field $\xi = \frac{\partial}{\partial y}$, the RT spacetime reveals an almost Ricci soliton if

$$r^2 f_x + f\{-2a + 4br + f_x^2 + f_y^2 - f(f_{xx} + f_{yy})\} = 0,$$

i.e.,

$$\frac{1}{2} \mathcal{L}_\xi g + S + \frac{4b}{r} g = 0,$$

(iv) for the soliton vector field $\xi = \mu_1 \frac{\partial}{\partial r} + \mu_2 \frac{\partial}{\partial x} + \mu_3 \frac{\partial}{\partial y}$, the RT spacetime admits an almost $\eta$-Ricci soliton if $f\{2a - 4br - f_x^2 - f_y^2 + f(f_{xx} + f_{yy})\} + r\{\mu_1 f - r(\mu_2 f_x + \mu_3 f_y)\} = 0$ with $\eta = (1, 0, 0, 0)$, i.e.,

$$\frac{1}{2} \mathcal{L}_\xi g + S - 4bg - \frac{\mu_1( - d + 2br^2) }{r^2} \eta \otimes \eta = 0,$$

(v) for the soliton vector field $\xi = \nabla r^2$, the RT spacetime possesses an almost gradient $\eta$-Ricci soliton if $f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) + 2\{3d + 2r(-a + br)\} - 2a + 4br = 0$ and $3d + 2r(-a + br) \neq 0$ with the 1-form $\eta = (0, 1, 0, 0)$, i.e.,

$$\frac{1}{2} \mathcal{L}_\xi g + S - \frac{2(-6bdr + 4abr^2 - 5bdr^2 + 3d^2 - 2adr - 4b^2 r^3 + 4abr^3 - 2b^2 r^4)}{r^2\{3d + 2r(-a + br)\}} g - 2\eta \otimes \eta = 0.$$
4. Curvature Inheritance and Collineations on Robinson-Trautman spacetime

If \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0 \), then for the vector field \( V = \mu_1 \partial_x + \mu_2 \partial_y \), the following relation holds:

\[
\mathcal{L}_V S = \lambda_S S + \lambda_g g,
\]

where

\[
\begin{align*}
\lambda_S &= -\frac{\mu_2 F_{13} + \mu_1 F_{11}}{f(2a - 4br - F_1)} \\
\lambda_g &= -\frac{4b(\mu_2 F_{13} + \mu_1 F_{11})}{rf(2a - 4br - F_1)}
\end{align*}
\] (4.1)

with \( F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \), \( F_{13} = -2f_y + f_y\{4a - 16br - 2f_x^2 + f(3f_{yy} + f_{xx})\} - f\{-2f_x f_{xy} + f(f_{yyy} + f_{xx})\} \) and \( F_{11} = -2f_x^3 + f_x\{4a - 16br - 2f_x^2 + f(f_{yy} + 3f_{xx})\} - f\{-2f_y f_{xy} + f(f_{xxx} + f_{xy})\}. \)

This leads to the following:

**Proposition 4.1.** If \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0 \), then the RT spacetime admits generalized Ricci inheritance given by

\[
\mathcal{L}_\xi S = \lambda_S S + \lambda_g g
\]

for the vector field \( \xi = \mu_1 \partial_x + \mu_2 \partial_y \), where \( \lambda_S, \lambda_g \) are given in (4.1).

**Corollary 4.1.** The RT spacetime possesses Ricci inheritance given by

\[
\mathcal{L}_\xi S = \frac{\mu_2 F_{13} + \mu_1 F_{11}}{f(f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br)} S
\]

for the vector field \( \xi = \mu_1 \partial_x + \mu_2 \partial_y \) (where \( \mu_1, \mu_2 \) are constants) if the relations \( b = 0 \) and \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0 \) are satisfied, where \( F_{13} = -2f_y + f_y\{4a - 16br - 2f_x^2 + f(3f_{yy} + f_{xx})\} - f\{-2f_x f_{xy} + f(f_{yyy} + f_{xx})\} \) and \( F_{11} = -2f_x^3 + f_x\{4a - 16br - 2f_x^2 + f(f_{yy} + 3f_{xx})\} - f\{-2f_y f_{xy} + f(f_{xxx} + f_{xy})\}. \)

**Corollary 4.2.** The RT spacetime possesses Ricci collineation with respect to the vector field \( \xi = \mu_1 \partial_x + \mu_2 \partial_y \) if the relations \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0 \) and

\[
\mu_2 \left[-2f_y^3 + f_y\{4a - 16br - 2f_x^2 + f(3f_{yy} + f_{xx})\} - f\{-2f_x f_{xy} + f(f_{yyy} + f_{xx})\}\right] + \mu_1 \left[-2f_x^3 + f_x\{4a - 16br - 2f_y^2 + f(f_{yy} + 3f_{xx})\} - f\{-2f_y f_{xy} + f(f_{xxx} + f_{xy})\}\right] = 0
\] (4.2)

are satisfied, where \( \mu_1, \mu_2 \) are constants.
The components of the Kulkarni-Nomizu products $\mathcal{U} = g \wedge g$, $\mathcal{H} = g \wedge S$ and $\mathcal{D} = S \wedge S$ on RT spacetime are determined as below:

\[ \mathcal{U}_{1212} = 2 = \mathcal{U}_{2121}, \mathcal{U}_{1221} = -2 = \mathcal{U}_{2112}, \]
\[ \mathcal{U}_{1313} = \frac{4r(d + r(-a + 2br))}{t^2} = \mathcal{U}_{1414} = \mathcal{U}_{3131} = \mathcal{J}_{4141}, \]
\[ \mathcal{U}_{1313} = -\frac{4r(d + r(-a + 2br))}{t^2} = \mathcal{U}_{1414} = \mathcal{U}_{3113} = \mathcal{U}_{4114}, \]
\[ \mathcal{U}_{1323} = \frac{2r^2}{t^2} = \mathcal{U}_{1424} = \mathcal{U}_{2313} = \mathcal{U}_{2414} = \mathcal{U}_{3132} = \mathcal{U}_{3231} = \mathcal{U}_{4343} = \mathcal{U}_{4142} = \mathcal{U}_{4241} = \mathcal{U}_{4334}, \]
\[ \mathcal{U}_{1323} = -\frac{2r^2}{t^2} = \mathcal{U}_{1442} = \mathcal{U}_{2331} = \mathcal{U}_{2441} = \mathcal{U}_{3123} = \mathcal{U}_{3213} = \mathcal{U}_{3434} = \mathcal{U}_{4124} = \mathcal{U}_{4214} = \mathcal{U}_{4343}, \]

\[ \mathcal{H}_{1212} = -\frac{8b}{r} = g \wedge s_{2121}, \quad \mathcal{H}_{1221} = \frac{8b}{r} = g \wedge s_{2112}, \]
\[ \mathcal{H}_{1313} = -\frac{1}{r^2}2(d + r(-a + 2br))(-2a + 12br + F_1) = \mathcal{H}_{1414} = \mathcal{H}_{3131} = \mathcal{H}_{4141}; \]
\[ \mathcal{H}_{1323} = -\frac{2a + 12br + F_1}{t^2} = \mathcal{H}_{1424} = \mathcal{H}_{2313} = \mathcal{H}_{2414} = \mathcal{H}_{3132} = \mathcal{H}_{3231} = \mathcal{H}_{4142} = \mathcal{H}_{4241}; \]
\[ \mathcal{H}_{1331} = \frac{1}{r^2}2(d + r(-a + 2br))(-2a + 12br + F_1) = \mathcal{H}_{1414} = \mathcal{H}_{3113} = \mathcal{H}_{4114}; \]
\[ \mathcal{H}_{1332} = -\frac{2a + 12br + F_1}{t^2} = \mathcal{H}_{1442} = \mathcal{H}_{2331} = \mathcal{H}_{2441} = \mathcal{H}_{3123} = \mathcal{H}_{3213} = \mathcal{H}_{4124} = \mathcal{H}_{4214}; \]
\[ \mathcal{H}_{3434} = \frac{2r^2(-2a + 8br + F_1)}{t^4} = \mathcal{H}_{4343}; \quad \mathcal{H}_{3443} = -\frac{1}{r^2}2r^2(-2a + 8br + F_1) = \mathcal{H}_{4334}; \]

\[ \mathcal{D}_{1212} = \frac{32b^2}{r^2} = \mathcal{D}_{2121}, \quad \mathcal{D}_{1221} = -\frac{32b^2}{r^2} = \mathcal{D}_{2112}, \]
\[ \mathcal{D}_{1313} = \frac{1}{r^2}16b(d + r(-a + 2br))(-2a + 8br + F_1) = \mathcal{D}_{1414} = \mathcal{D}_{3131} = \mathcal{D}_{4141}, \]
\[ \mathcal{D}_{1323} = \frac{8b(-2a + 8br + F_1)}{r^2} = \mathcal{D}_{1424} = \mathcal{D}_{2313} = \mathcal{D}_{2414} = \mathcal{D}_{3132} = \mathcal{D}_{3231} = \mathcal{D}_{4142} = \mathcal{D}_{4241}, \]
\[ \mathcal{D}_{1331} = -\frac{1}{r^2}16b(d + r(-a + 2br))(-2a + 8br + F_1) = \mathcal{D}_{1414} = \mathcal{D}_{3113} = \mathcal{D}_{4114}; \]
\[ \mathcal{D}_{1332} = -\frac{8b(-2a + 8br + F_1)}{r^2} = \mathcal{D}_{1442} = \mathcal{D}_{2331} = \mathcal{D}_{2441} = \mathcal{D}_{3123} = \mathcal{D}_{3213} = \mathcal{D}_{4124} = \mathcal{D}_{4214}, \]
\[ \mathcal{D}_{3434} = -\frac{2(-2a + 8br + F_1)}{t^4} = \mathcal{D}_{4343}, \quad \mathcal{D}_{3443} = \frac{2(-2a + 8br + F_1)}{t^4} = \mathcal{D}_{4334}. \]
For the vector field $V = \frac{\partial}{\partial x}$ the components of $L_V R$ are

$$(\mathcal{L}_V R)_{1313} = -\frac{4(d-2br^2)(d+r(-a+2br))f_x}{r^2 f_x}, \quad (\mathcal{L}_V R)_{1414} = (\mathcal{L}_V R)_{3131} = (\mathcal{L}_V R)_{4141},$$

$$(\mathcal{L}_V R)_{1323} = -\frac{2(d-2br^2)f_x}{r^2 f_x} = (\mathcal{L}_V R)_{1424} = (\mathcal{L}_V R)_{3231} = (\mathcal{L}_V R)_{4214} = (\mathcal{L}_V R)_{4241},$$

$$(\mathcal{L}_V R)_{1333} = \frac{4(d-2br^2)(d+r(-a+2br))f_x}{r^2 f_x} = (\mathcal{L}_V R)_{1441} = (\mathcal{L}_V R)_{3113} = (\mathcal{L}_V R)_{4114},$$

$$(\mathcal{L}_V R)_{1323} = \frac{2(d-2br^2)f_x}{r^2 f_x} = (\mathcal{L}_V R)_{1442} = (\mathcal{L}_V R)_{3213} = (\mathcal{L}_V R)_{4231} = (\mathcal{L}_V R)_{4214},$$

$$(\mathcal{L}_V R)_{2313} = \frac{2(-2b\frac{d}{f^2})r f_x}{f^3} = (\mathcal{L}_V R)_{2414},$$

$$(\mathcal{L}_V R)_{3434} = -\frac{1}{f^2} r (4r f_x^3 + f_x(8(d - ar + 2br^2) + 4r f_y^2 - r f(3f_{yy} + 5f_{xx})) + r f(-2f_y f_{xy} + f(f_{xy} + f_{xxx})) = (\mathcal{L}_V R)_{4343},$$

$$(\mathcal{L}_V R)_{3443} = \frac{1}{f^2} r (4r f_x^3 + f_x(8(d - ar + 2br^2) + 4r f_y^2 - r f(3f_{yy} + 5f_{xx})) + r f(-2f_y f_{xy} + f(f_{xy} + f_{xxx})) = (\mathcal{L}_V R)_{4334},$$

and the following relation holds:

$$\mathcal{L}_V R = \lambda_R R + \lambda_1 g \wedge g + \lambda_2 g \wedge S,$$

where

$$\lambda_R = -\frac{F_4}{f(-6d + 2ar - r F_1)},$$

$$\lambda_1 = \frac{8b(-d + 2br^2)f_x}{r^2 f(F_1 - 2a + 4br)} + \frac{(2ad - 16br + 8b^2 r^3 - dF_1)F_4}{r^3 f(F_1 - 2a + 4br)(-6d + 2ar - r F_1)},$$

$$\lambda_2 = \frac{2(-d + 2br^2)f_x}{r f(F_1 - 2a + 4br)} + \frac{(-3d + 2br^2)F_4}{r f(F_1 - 2a + 4br)(-6d + 2ar - r F_1)},$$

(4.3)

with $F_1 = f_x^2 + f_y^2 - f(f_{xx} - f_{yy})$ and $F_2 = (-12d + 8ar - 8br^2)f_x - 4rf_x^2 f_x + 3rf_y f_y f_x - 4rf_x^3 + 2rf_y f_{xy} - rf^2 f_{xxy} + 5rf_x f_x f_{xx} - rf^2 f_{xxx}$ such that $F_1 - 2a + 4br \neq 0$ and $-6d + 2ar - r F_1 \neq 0$. This leads to the following:

**Proposition 4.2.** The RT spacetime possesses generalized curvature inheritance given by

$$\mathcal{L}_\xi R = \lambda_R R + \lambda_1 g \wedge g + \lambda_2 g \wedge S$$

with respect to the vector field $\xi = \frac{\partial}{\partial x}$ if $f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br \neq 0$ and $r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) + 6d - 2ar \neq 0$, where $\lambda_R, \lambda_1, \lambda_2$ are given in (4.3).
Corollary 4.3. The RT spacetime admits curvature inheritance given by

$$\mathcal{L}_\xi R = \frac{8af_x - 4f_y f_x + 3f_y f_x - 4f_x^3 + 2f_y f_{xy} - f^2 f_{xyy} + 5f_x f_{xx} - f^2 f_{xxx}}{f(f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a)} R$$

for the vector field $\xi = \frac{\partial}{\partial x}$ if $b = d = 0$, $f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a \neq 0$ and $r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) - 2ar \neq 0$.

Corollary 4.4. The RT spacetime admits curvature collineation with respect to the vector field $\frac{\partial}{\partial x}$ if $b = 0$, $f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a \neq 0$, $r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) + 6d - 2ar \neq 0$ and $(-12d + 8ar)f_x - 4r f_y^2 f_x + 3rf_y f_y f_x - 4r f_x^3 + 2r f_y f_{xy} - r f^2 f_{xyy} + 5r f_x f_{xx} - r f^2 f_{xxx} = 0$.

The components of the Weyl conformal $(0,4)$ curvature tensor on RT spacetime are

$$C_{1212} = -\frac{6d-2ar+rF_1}{3r^3}, \quad C_{1313} = \frac{1}{3r}f_x(d + r(-a + 2br))(6d - 2ar + rF_1) = C_{1414},$$

$$C_{1323} = \frac{6d-2ar+rF_1}{6rf_x^2} = C_{1424}, \quad C_{3434} = \frac{r(6d-2ar+rF_1)}{3f_x^4}.$$

Again, for the vector field $V = \frac{\partial}{\partial x}$ the components of $\mathcal{L}_V C$ are

$$\mathcal{(\mathcal{L}_V C)}_{1212} = \frac{2(6d-2ar+rF_1)}{3r^3}, \quad \mathcal{(\mathcal{L}_V C)}_{1221} = \frac{-18d+4ar-2rF_1}{3r^4}, \quad \mathcal{(\mathcal{L}_V C)}_{2112} = \mathcal{(\mathcal{L}_V C)}_{1221},$$

$$\mathcal{(\mathcal{L}_V C)}_{1313} = \frac{1}{3r}f_x(-4(3d^2 - 2adr + abr^2) + r(-d + 2br^2)F_1) = \mathcal{(\mathcal{L}_V C)}_{1331} = \mathcal{(\mathcal{L}_V C)}_{3113} = \mathcal{(\mathcal{L}_V C)}_{3131} = \mathcal{(\mathcal{L}_V C)}_{3114} = \mathcal{(\mathcal{L}_V C)}_{1414},$$

$$\mathcal{(\mathcal{L}_V C)}_{1323} = -\frac{d}{rf_x^2} = \mathcal{(\mathcal{L}_V C)}_{1323} = \mathcal{(\mathcal{L}_V C)}_{1424} = \mathcal{(\mathcal{L}_V C)}_{2313} = \mathcal{(\mathcal{L}_V C)}_{2331} = \mathcal{(\mathcal{L}_V C)}_{2414},$$

$$\mathcal{(\mathcal{L}_V C)}_{3123} = \mathcal{(\mathcal{L}_V C)}_{3213} = \mathcal{(\mathcal{L}_V C)}_{3231} = \mathcal{(\mathcal{L}_V C)}_{4123} = \mathcal{(\mathcal{L}_V C)}_{4213} = \mathcal{(\mathcal{L}_V C)}_{4231} = \mathcal{(\mathcal{L}_V C)}_{4313} = \mathcal{(\mathcal{L}_V C)}_{4331} = \mathcal{(\mathcal{L}_V C)}_{4413},$$

which leads to the following:

Proposition 4.3. If $f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br \neq 0$ and $r\{f_x^2 + f_y^2 - f(f_{xx} - f_{yy})\} - 2ar + 6d \neq 0$, then the RT spacetime possesses generalized Weyl conformal inheritance given by

$$\mathcal{L}_\xi C = \lambda_C C + \lambda_1 g \wedge g + \lambda_2 g \wedge S$$

for the vector field $\xi = \frac{\partial}{\partial x}$ with
\lambda_C = - \frac{F_5}{f(-6d + 2ar - rF_1)};
\lambda_1 = - \frac{1}{3r^3 f(F_1 - 2a + 4br)} \left[ (-12ad + 4a^2r + 72br - 24abr^2) f_x 
+ (6d - 4ar + 12br^2)(f_y^2 f_x - f f_{yy} f_x + f_x^3 - f f_x f_{xx}) + r f_x (f_x^4 + f_y^4) 
- 2 r f f_y^2 f_x (f_{xx} + f_{yy}) + r f^2 f_x (f_{xx} + f_{yy})^2 + 2 r f_x^3 (f_y^2 - f f_{xx} - a f_{yy}) \right],
\lambda_2 = \frac{2 f_x (-6d + 2ar - rF_1)}{3rf (F_1 - 2a + 4br)},

where \( F_1 = f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) \) and \( F_5 = -12df_x + 4af_x - 2rf_y^2 f_x + rff_{yy} f_x - r f_x^3 + 2 rf f_y f_{xy} - r f^2 f_{xy} + 3rf f_x f_{xx} - r f^2 f_{xxx} \) such that \( F_1 - 2a + 4br \neq 0 \) and \(-6d + 2ar - rF_1 \neq 0\).

**Corollary 4.5.** If \( f_x = 0 \) (i.e. \( f \) is a function of \( y \) only) and the relations \( f_y^2 + f f_{yy} - 2a + 4br \neq 0 \) and \( r \{ f_y^2 + f f_{yy} \} - 2ar + 6d \neq 0 \) are satisfied, then the RT spacetime possesses Weyl conformal collineation for the vector field \( \frac{\partial}{\partial x} \).

The components of the concircular \((0,4)\) curvature tensor on RT spacetime are

\[
W_{1212} = \frac{1}{6r^4} (-12d + r(-2a + 12br + F_1));
W_{1313} = \frac{1}{3r^2 f^2} (d + r(-a + 2br))(6d - 2ar + rF_1);
W_{1323} = \frac{6d - 2ar + rF_1}{6rf^2} = \tilde{C}_{1424};
W_{1414} = \frac{1}{3r^2 f^2} (d + r(-a + 2br))(6d - 2ar + rF_1);
W_{3434} = \frac{1}{6rf} r(6d + r(-5a + 6br)) + 5rF_1;
\]
Again, for the vector field $V = \frac{\partial}{\partial x}$ the components of $\mathcal{L}_V W$ are

\[
\begin{align*}
\mathcal{L}_V W_{1212} &= \frac{1}{3r^2} (2(9d + r(a - 3br)) - rF_1) = (\mathcal{L}_V W)_{2121}, \\
\mathcal{L}_V W_{1221} &= \frac{1}{3r^2} (-2(9d + r(a - 3br)) + rF_1) = (\mathcal{L}_V W)_{2112}, \\
\mathcal{L}_V W_{1313} &= \frac{1}{3r^2} (-4(3d^2 - 2adr + abr^3) + r(-d + 2b^2)rF_1) \\
&= (\mathcal{L}_V W)_{1414} = (\mathcal{L}_V W)_{1441} = (\mathcal{L}_V W)_{3131} = (\mathcal{L}_V W)_{4141}, \\
\mathcal{L}_V W_{1331} &= \frac{1}{3r^2} (4(3d^2 - 2adr + abr^3) - r(-d + 2b^2)rF_1) \\
&= (\mathcal{L}_V W)_{1441} = (\mathcal{L}_V W)_{3113} = (\mathcal{L}_V W)_{4114}, \\
\mathcal{L}_V W_{1323} &= -\frac{d}{rF_2} = (\mathcal{L}_V W)_{1424} = (\mathcal{L}_V W)_{2313} \\
&= (\mathcal{L}_V W)_{2414} = (\mathcal{L}_V W)_{3132} = (\mathcal{L}_V W)_{3231} = (\mathcal{L}_V W)_{4142} = (\mathcal{L}_V W)_{4241}, \\
\mathcal{L}_V W_{1332} &= \frac{d}{rF_2} = (\mathcal{L}_V W)_{1442} = (\mathcal{L}_V W)_{2331} \\
&= (\mathcal{L}_V W)_{2441} = (\mathcal{L}_V W)_{3123} = (\mathcal{L}_V W)_{3213} = (\mathcal{L}_V W)_{4124} = (\mathcal{L}_V W)_{4214}, \\
\mathcal{L}_V W_{3434} &= \frac{1}{3r^2} (6d + 2r(-5a + 9br) + 5rF_1) = (\mathcal{L}_V W)_{4334}, \\
\mathcal{L}_V W_{3443} &= -\frac{1}{3r^2} (6d + 2r(-5a + 9br) + 5rF_1) = (\mathcal{L}_V W)_{4334}
\end{align*}
\]

and this leads to the following:

**Proposition 4.4.** If $f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br \neq 0$ and $r\{f_x^2 + f_y^2 - f(f_{xx} - f_{yy})\} - 2ar + 6d \neq 0$, then the RT spacetime admits generalized concircular inheritance given by

\[
\mathcal{L}_\xi W = \lambda W + \lambda_1 g \wedge g + \lambda_2 g \wedge S
\]

for the vector field $\xi = \frac{\partial}{\partial x}$ with

\[
\begin{align*}
\lambda W &= -\frac{F_4}{f(-6d + 2ar - rF_1)}, \\
\lambda_1 &= \frac{\left[\frac{4b}{3r^2} (6d + 2ar - rF_1) - \frac{1}{6r^2} (-2a + 12br + F_1) \left(-12d + r(-2a + 12br + F_1)\right)\right] F_4}{2(2a - 4br - F_1)(-6d + 2ar - rF_1)} \\
&\quad - \frac{(F_1 - 2a + 12br) \{2f_y f_{xy} + f_x (f_{xx} - f_{yy}) - f(f_{yy} + f_{xx})\}}{12r^2(2a - 4br - F_1)} \\
&\quad - \frac{2b \left[2r f_x^3 + f_x \left(12d - 4ar + 2r f_y^2 - r f(3f_{xx} + f_{yy})\right) + r f \{-2f_y f_{xy} + f(f_{yy} + f_{xx})\}\right]}{3r^2 f(2a - 4br - F_1)}, \\
\lambda_2 &= \frac{f_x (-6d + 2ar - rF_1)}{3r f(-2a + 4br + F_1)} + \frac{(-3d + 2b^2)rF_4}{r f(F_1 - 2a + 4br)(-6d + 2ar - rF_1)},
\end{align*}
\] (4.5)
where $F_1 = f_x^2 + f_y^2 - f(f_{xx} - f_{yy})$ and $F_4 = (-12d + 8ar - 8br^2)f_x - 4rf_y^2f_x + 3rf_yf_yf_x - 4rf_x^3 + 2rf_yf_{xy} - rf^2f_{xy} + 5rf_xf_{xx} - rf^2f_{xxx}$.

**Corollary 4.6.** The RT spacetime possesses concircular collineation for the vector field $\frac{\partial}{\partial x}$ if the relations $b = d = 0$, $f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a \neq 0$ and $r \{f_x^2 + f_y^2 - f(f_{xx} - f_{yy})\} - 2ar \neq 0$ are satisfied.

The components of conharmonic (0,4) curvature tensor on RT spacetime are

\[
K_{1212} = -\frac{2(d - 2br^2)}{r^2}, \quad K_{1313} = \frac{1}{r^2 f_x}(d + r(-a + 2br))(2(d + r(-a + 4br)) + rF_1) = K_{1414},
\]

\[
K_{1323} = \frac{1}{2rf_x}(2(d + r(-a + 4br)) + rF_1) = K_{1424}, \quad K_{3434} = \frac{2r(d - 2br^2)}{f_x^4}.
\]

Again, for the vector field $V = \frac{\partial}{\partial x}$ the components of $\mathcal{L}_V K$ are

\[
(\mathcal{L}_V K)_{1212} = \frac{6d - 4br^2}{r^2} = (\mathcal{L}_V K)_{2121}, \quad (\mathcal{L}_V K)_{1221} = -\frac{6d + 4br^2}{r^2} = (\mathcal{L}_V K)_{2112},
\]

\[
(\mathcal{L}_V K)_{1313} = \frac{1}{r^2 f_x}(-4(d^2 - adr + br^3(3a - 8br)) + r(-d + 2br^2)F_1)
\]

\[
= (\mathcal{L}_V K)_{1414} = (\mathcal{L}_V K)_{3131} = (\mathcal{L}_V K)_{4141},
\]

\[
(\mathcal{L}_V K)_{1323} = \frac{d - 4br^2}{r^2 f_x^2} = (\mathcal{L}_V K)_{1424} = (\mathcal{L}_V K)_{2313} = (\mathcal{L}_V K)_{2414} = (\mathcal{L}_V K)_{3132}
\]

\[
= (\mathcal{L}_V K)_{3231} = (\mathcal{L}_V K)_{4142} = (\mathcal{L}_V K)_{4241},
\]

\[
(\mathcal{L}_V K)_{1331} = \frac{1}{r^2 f_x}(-4(d^2 - adr + br^3(3a - 8br)) - r(-d + 2br^2)F_1)
\]

\[
= (\mathcal{L}_V K)_{1441} = (\mathcal{L}_V K)_{3131} = (\mathcal{L}_V K)_{4141},
\]

\[
(\mathcal{L}_V K)_{1332} = \frac{d - 4br^2}{r^2 f_x^2} = (\mathcal{L}_V K)_{1442} = (\mathcal{L}_V K)_{2331} = (\mathcal{L}_V K)_{2441} = (\mathcal{L}_V K)_{3132}
\]

\[
= (\mathcal{L}_V K)_{3231} = (\mathcal{L}_V K)_{4142} = (\mathcal{L}_V K)_{4241},
\]

\[
(\mathcal{L}_V K)_{3434} = \frac{2(d - 6br^2)}{f_x^4} = (\mathcal{L}_V K)_{4344}, \quad (\mathcal{L}_V K)_{3443} = -\frac{2(d - 6br^2)}{f_x^4} = (\mathcal{L}_V K)_{4334},
\]

and the following relation holds:

\[
\mathcal{L}_V K = \lambda_K K + \lambda_1 g \wedge g + \lambda_2 g \wedge S,
\]

where

\[
\begin{align*}
\lambda_K &= \frac{F_5}{f(-6d + 2ar - rF_1)}, \\
\lambda_1 &= \frac{16b(-d + 2br^2)f_x}{r^2 f(F_1 - 2a + 4br)} - \frac{(-d + 2br^2)F_5}{r^3 f(-6d + 2ar - rF_1)}, \\
\lambda_2 &= \frac{4(-d + 2br^2)f_x}{rf(F_1 - 2a + 4br)},
\end{align*}
\]

with $F_1 = f_x^2 + f_y^2 - f(f_{xx} - f_{yy})$ and $F_5 = (12d - 4ar)f_x + 2rf_y^2f_x - rf_yf_yf_x + 2rf_x^3 - 2rf_yf_{xy} + rf_yf_{xy} = 3rf_xf_{xx} + r^2 f_{xxx}$ such that $F_1 - 2a + 4br \neq 0$ and $-6d + 2ar - rF_1 \neq 0$. 
Proposition 4.5. If \( f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br \neq 0 \) and \( r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) - 2ar + 6d \neq 0 \), then the RT spacetime admits generalized conharmonic inheritance with respect to the vector field \( \xi = \partial \partial_x \) given by

\[
\mathcal{L}_\xi K = \lambda K + \lambda_1 g \wedge g + \lambda_2 g \wedge S,
\]

where \( \lambda, \lambda_1, \lambda_2 \) are given in (4.6).

Corollary 4.7. The RT spacetime realizes conharmonic curvature inheritance given by

\[
\mathcal{L}_\xi K = \frac{-4af_x + 2f_y^2 f_x - f f_{yy} f_x + f_{yy}^2 f_y + 2f_y^3 + 2ff_y f_{xy} + f^2 f_{xxy} - 3ff_x f_{xx} + f^2 f_{xxx}}{f(2a - (f_x^2 + f_y^2 - f(f_{xx} - f_{yy})))} K
\]

for the vector field \( \xi = \partial \partial_x \) if the relations \( b = d = 0 \), \( f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a \neq 0 \) and \( r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) - 2ar \neq 0 \) hold.

Corollary 4.8. The RT spacetime possesses conharmonic collineation with respect to the vector field \( \frac{\partial}{\partial x} \) if \( f_x = 0 \) (i.e., \( f \) is a function of \( y \) only) and the relations \( f_y^2 + f f_{yy} - 2a + 4br \neq 0 \), \( r(f_y^2 + f f_{yy}) - 2ar + 6d \neq 0 \) hold.

The components of Weyl projective (0,4) curvature tensor on RT spacetime are

\[
P_{1212} = \frac{-2d}{r} + \frac{4b}{3r}; \quad P_{1221} = \frac{2d}{r} - \frac{4b}{3r};
\]

\[
P_{1313} = \frac{2(3d-2br)(d+r(-a+2br))}{3r^2 f^2}; \quad P_{1323} = \frac{3d-2br^2}{3r f^2};
\]

\[
P_{1331} = \frac{-1}{3r^2 f^2} 2(d + r(-a+2br))(3d + 2r(-a+br) + f F_1) = P_{1441};
\]

\[
P_{1332} = \frac{-1}{3r^2 f^2} (3d + 2r(-a+br) + r F_1) = P_{1442} = P_{2331} = P_{2441};
\]

\[
P_{1414} = \frac{2(3d-2br^2)(d+r(-a+2br))}{3r^2 f^2}; \quad P_{1424} = \frac{3d-2br^2}{3r f^2} = P_{2313} = P_{2414};
\]

\[
P_{3434} = \frac{1}{3r^2 f^2} 2r(3d + 2r(-a+br) + r F_1) = -P_{3443};
\]

Again, for the vector field \( V = \frac{\partial}{\partial x} \), the components of \( \mathcal{L}_V P \) are
\[
\begin{align*}
(L_V P)_{1212} &= \frac{6d}{r^4} - \frac{4h}{3r^2} = (L_V P)_{2121}, \\
(L_V P)_{1321} &= -\frac{12d^2 + 6adr + 4br^3(a-4br)}{3r^3 f^2} = (L_V P)_{1414}, \\
(L_V P)_{1323} &= -\frac{3d^2 + 2b^2r^2}{3r^3 f^2} = (L_V P)_{1424} = (L_V P)_{2313} = (L_V P)_{2414}, \\
(L_V P)_{3231} &= (L_V P)_{3231} = (L_V P)_{4142} = (L_V P)_{4214}, \\
(L_V P)_{3131} &= \frac{1}{3r f^2} 2(6d^2 - 5adr + 2br^3(3a - 4br) - r(-d + 2br^2)F_1) = (L_V P)_{1441}, \\
(L_V P)_{3132} &= -\frac{2b + \frac{2d}{3r^2}}{3r^2 f} = (L_V P)_{1442} = (L_V P)_{2331} = (L_V P)_{2441}, \\
(L_V P)_{3123} &= \frac{-2b + \frac{2d}{3r^2}}{3r^2 f} = (L_V P)_{3213} = (L_V P)_{4124} = (L_V P)_{4214}, \\
(L_V P)_{3134} &= \frac{1}{3r f^2} 2(-6d^2 + 5adr + 2br^3(-3a + 4br) + r(-d + 2br^2)F_1) = (L_V P)_{1441}, \\
(L_V P)_{3143} &= \frac{2(6d^2 - 3adr + 2br^3(-a + 4br))}{3r^3 f^2} = (L_V P)_{1441}, \\
(L_V P)_{4334} &= \frac{1}{3r f^2} (6d + 4r(-2a + 3br) + 4r F_1) = (L_V P)_{4343}, \\
(L_V P)_{4343} &= \frac{1}{3r f^2} (-6d + 4r(-2a - 3br) - 4r F_1) = (L_V P)_{4334}
\end{align*}
\]
and the following relation holds:

\[
L_V P = \lambda_P P + \lambda_1 g \wedge g + \lambda_2 g \wedge S + \lambda_3 S \wedge S,
\]

where

\[
\begin{align*}
\lambda_P &= -\frac{F_9}{f(2a - 4br - F_1)}, \\
\lambda_1 &= \frac{F_9}{3r f(2a - 4br - F_1)} \cdot \frac{F_7}{F_8}, \\
\lambda_2 &= -\frac{F_9 \{ -108bdr + 40b^2r^2 - (9d + 2br^2)(F_1 - 2a) \}}{3rf(F_1 - 2a + 4br)^3} + \frac{F_{10}}{3rf(F_1 - 2a + 4br)^2}, \\
\lambda_3 &= -\frac{r F_9(-9d + 2ar + 2br^2 - r F_1)}{3f(F_1 - 2a + 4br)^3} - \frac{F_{12}}{3f(2a - 4br - F_1)^2}.
\end{align*}
\]
with

\[ F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \] such that \( F_1 - 2a + 4br \neq 0 \),

\[ F_7 = \left[ 2(-3d + 2br^2)(6br^2 - 2a + 8br + F_1)(-2a + 8br + F_1)(1 + 4br) \right. \]
\[ \left. + r^2(-2a + 4br + F_1)\{(-3d + 2br^2)(-2a + 8br + F_1)^2 - 16b^2(2br^2 + 3d - 2ar + rF_1)\} \right], \]

\[ F_8 = \left[ (-2a + 4br + F_1)\{16b^2r^2 + (2a - 8br - F_1)^2 \right] \]
\[ - 2(4br + 2a + F_1)(-2a + 8br + F_1)(1 - 4br) \] \neq 0,

\[ F_9 = 4af_x - 8brf_x - 2f_yf_x - 2f_y^2f_x - 2f_x^3 + 2ff_yf_{xy} - f^2f_{xyy} + 3f^2f_{xx} - f^2f_{xxx}, \]

\[ F_{10} = 2\left[ -6adf_x + 8adbf_x - 28abr^2f_x + 8b^2r^3f_x + 2br^2f_x \{7F_1 + 2f(f_{yy} - f_{xx}) \right] \]
\[ + 4br^2f^2(f_{xxx} + f_{xyy}) + 3df_xF_1 \],

\[ F_{12} = -18drf_x + 8ar^2f_x - 4br^2f_x - 3r^2f_yf_x + 3r^2f_yf_{xy} - 4r^2f_yf_x - 2r^2ff_yf_{xy} \]
\[ - r^2f^2f_{xyy} + 5r^2ff_xf_{xx} - r^2f^2f_{xxx}. \]

This leads to the following:

**Proposition 4.6.** The RT spacetime admits generalized Weyl projective inheritance given by

\[ \mathcal{L}_\xi P = \lambda_P P + \lambda_1 g \wedge g + \lambda_2 g \wedge S, \]

for the vector field \( \xi = \frac{\partial}{\partial x} \), if the relations \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0 \), \( \{(-2a + 4br + F_1)\{16b^2r^2 + (2a - 8br - F_1)^2 \right] - 2(-4br + 2a + F_1)(-2a + 8br + F_1)(1 - 4br) \] \neq 0,

\[ -9d + 2ar + 2br^2 - r(f_x^2 + f_y^2 - f(f_{xx} + f_{yy})) = 0 \] and \(-18drf_x + 8ar^2f_x - 4br^2f_x - 3r^2f_yf_x + 3r^2f_yf_{xy} + 2r^2ff_yf_{xy} - r^2f^2f_{xyy} + 5r^2ff_xf_{xx} - r^2f^2f_{xxx} = 0 \) are satisfied with \( F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \), where \( \lambda_P, \lambda_1, \lambda_2 \) are given in (3.7).

**Corollary 4.9.** The RT spacetime realizes Weyl projective inheritance given by

\[ \mathcal{L}_\xi P = \frac{f_x(4a - 8br - 2f_y^2 + ff_{yy} - 2f_x^2 + 3ff_{xx}) + 2ff_yf_{xy} - f^2f_{xyy} - f^2f_{xxx}}{f(f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br)} P \]

for the vector field \( \xi = \frac{\partial}{\partial x} \) if the conditions \( F_1 - 2a + 4br \neq 0 \), \( F_8 \neq 0 \), \(-108br^2 + 40b^2r^2 - (9d + 2br^2)(F_1 - 2a) = 0 \), \(-9d + 2ar + 2br^2 - rF_1 \) and \( F_7 = F_{10} = F_{12} = 0 \) are satisfied with \( F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \), where the functions \( F_7, F_8, F_{10}, F_{12} \) are given in (4.7).
Corollary 4.10. The RT spacetime possesses Weyl projective curvature collineation with respect to the vector field \( \xi = \frac{\partial}{\partial x} \) if the relations \( F_1 - 2a + 4br \neq 0, F_8 \neq 0 \) and \( F_9 = F_{10} = F_{12} = 0 \) hold with \( F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \), where the functions \( F_8, F_9, F_{10}, F_{12} \) are given in \((4.7)\).

From the above, we can state the following:

**Theorem 4.1.** The RT spacetime \((1.1)\) satisfies the following symmetry properties:

(i) it admits generalized Ricci inheritance given by

\[
\mathcal{L}_\xi S = \lambda_S S + \lambda_g g
\]

for the vector field \( \xi = \mu_1 \frac{\partial}{\partial x} + \mu_2 \frac{\partial}{\partial y} \) (\( \mu_1, \mu_2 \) are constants) if \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0 \), where \( \lambda_S, \lambda_g \) are given in \((4.1)\),

(ii) it reveals generalized curvature inheritance given by

\[
\mathcal{L}_\xi R = \lambda_R R + \lambda_1 g \wedge g + \lambda_2 g \wedge S
\]

for the vector field \( \xi = \frac{\partial}{\partial x} \) if \( f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br \neq 0 \) and \( r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) + 6d - 2ar \neq 0 \), where \( \lambda_R, \lambda_1, \lambda_2 \) are given in \((4.3)\),

(iii) it possesses generalized Weyl conformal inheritance given by

\[
\mathcal{L}_\xi C = \lambda_C C + \lambda_1 g \wedge g + \lambda_2 g \wedge S
\]

for the vector field \( \xi = \frac{\partial}{\partial x} \) if \( f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br \neq 0 \) and \( r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) - 2ar + 6d \neq 0 \), where \( \lambda_C, \lambda_1, \lambda_2 \) given in \((4.4)\),

(iv) it fulfills generalized concircular inheritance given by

\[
\mathcal{L}_\xi W = \lambda_W W + \lambda_1 g \wedge g + \lambda_2 g \wedge S
\]

for the vector field \( \xi = \frac{\partial}{\partial x} \) if \( f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br \neq 0 \) and \( r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) - 2ar + 6d \neq 0 \), where \( \lambda_W, \lambda_1, \lambda_2 \) are given in \((4.5)\),

(v) it realizes generalized conharmonic inheritance given by

\[
\mathcal{L}_\xi K = \lambda_K K + \lambda_1 g \wedge g + \lambda_2 g \wedge S
\]

for the vector field \( \xi = \frac{\partial}{\partial x} \) if \( f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br \neq 0 \) and \( r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) - 2ar + 6d \neq 0 \), where \( \lambda_K, \lambda_1, \lambda_2 \) are given in \((4.6)\),
(vi) it admits generalized Weyl projective inheritance given by
\[ \mathcal{L}_\xi P = \lambda_P P + \lambda_{1g} \wedge g + \lambda_{2g} \wedge S \]
for the vector field \( \xi = \frac{\partial}{\partial x} \) if the relations \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0 \), 
\[ \{(-2a+4br+F_1)(16b^2r^2+(2a-8br-F_1)^2) - 2(-4br+2a+F_1)(-2a+8br+F_1)(1-4br)\} \neq 0, \]
\(-9d+2ar+2br^2 - r(f_x^2 + f_y^2 - f(f_{xx} + f_{yy})) = 0 \) and 
\(-18drf_x + 8ar^2f_x - 4br^3f_x - 4r^2f_yf_x - 3r^2ff_yf_x - 4r^2f_x^3 + 2r^2ff_yf_{xy} - r^2f_x^2f_{xy} + 5r^2f_xf_{xx} - r^2f_xf_{xxx} = 0 \) are satisfied with 
\( F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \), where \( \lambda_P, \lambda_1, \lambda_2 \) are given in (4.7).

**Corollary 4.11.** The RT spacetime \((1.1)\) satisfies the following inheritance properties:

(i) it admits Ricci inheritance given by
\[ \mathcal{L}_\xi S = \frac{\mu_2 F_{13} + \mu_1 F_{11}}{f(f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a)} S \]
for the vector field \( \xi = \mu_1 \frac{\partial}{\partial x} + \mu_2 \frac{\partial}{\partial y} \) (\( \mu_1, \mu_2 \) are constants) if \( b = 0 \) and \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a \neq 0 \), where the functions \( F_{13}, F_{11} \) are given in (4.7),

(ii) it reveals curvature inheritance given by
\[ \mathcal{L}_\xi R = \frac{8af_x - 4f_y f_x + 3f f_y f_x - 4f_x^3 + 2f f_y f_{xy} - f^2 f_{xy} + 5f f_x f_{xx} - f^2 f_{xxx}}{f(f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a)} R \]
for the vector field \( \frac{\partial}{\partial x} \) if \( b = d = 0 \), \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a \neq 0 \) and \( r(f_x^2 + f_y^2 - f(f_{xx} + f_{yy})) - 2ar \neq 0 \),

(iii) it realizes conharmonic inheritance given by
\[ \mathcal{L}_\xi K = \frac{-4af_x + 2f_y^2 f_x - f f_y f_x + 2f_x^3 - 2f f_y f_{xy} + f^2 f_{xy} + 3f f_x f_{xx} + f^2 f_{xxx}}{f(2a - (f_x^2 + f_y^2 - f(f_{xx} + f_{yy})))} K \]
for the vector field \( \frac{\partial}{\partial x} \) if \( b = d = 0 \), \( f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a \neq 0 \) and \( r(f_x^2 + f_y^2 - f(f_{xx} + f_{yy})) - 2ar \neq 0 \),

(iv) it admits Weyl projective inheritance given by
\[ \mathcal{L}_\xi P = \frac{f_x(4a - 8br - 2f_y^2 + f f_y f_x - 2f_x^2 + 3f f_{xx}) + 2f f_y f_{xy} - f^2 f_{xy} - f^2 f_{xxx}}{f(f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br)} P \]
for the vector field \( \xi = \frac{\partial}{\partial x} \) if the conditions \( F_1 - 2a + 4br \neq 0 \), \( F_8 \neq 0 \), \(-108bdr + 40b^2r^2 - (9d + 2br^2)(F_1 - 2a) = 0 \), \(-9d + 2ar + 2br^2 - rF_1 \) and \( F_7 = F_{10} = F_{12} = 0 \) are satisfied with \( F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) \), where \( F_7, F_8, F_{10}, F_{12} \) are given in (4.7).
Corollary 4.12. The RT spacetime \((1.1)\) satisfies the following collineation properties:

(i) it admits Ricci collineation with respect to the vector field \(\mu_1 \partial_x + \mu_2 \partial_y\) (\(\mu_1, \mu_2\) are constants) if \(f_x^2 + f_y^2 - f(f_{xx} + f_{yy}) - 2a + 4br \neq 0\) and the relation \((4.2)\) hold,

(ii) it reveals curvature collineation with respect to the vector field \(\partial_x\) if \(b = 0\), \(f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a + 4br \neq 0\), \(r(f_x^2 + f_y^2 - f(f_{xx} - f_{yy})) + 6d - 2ar \neq 0\) and the condition \((4.4)\) are satisfied,

(iii) it possesses Weyl conformal collineation with respect to the vector field \(\partial_x\) if \(f_x = 0\) (i.e., \(f\) is a function of \(y\) only) and the relations \(f_y^2 + f f_{yy} - 2a + 4br \neq 0\) and \(r\{f_y^2 + f f_{yy}\} - 2ar + 6d \neq 0\) hold,

(iv) it fulfills generalized concircular collineation with respect to the vector field \(\partial_x\) if the relations \(b = d = 0\), \(f_x^2 + f_y^2 - f(f_{xx} - f_{yy}) - 2a \neq 0\) and \(r\{f_x^2 + f_y^2 - f(f_{xx} - f_{yy})\} - 2ar \neq 0\) are satisfied,

(v) it realizes conharmonic collineation with respect to the vector field \(\partial_x\) if \(f_x = 0\) (i.e., \(f\) is a function of \(y\) only) and the relations \(f_y^2 + f f_{yy} - 2a + 4br \neq 0\), \(r\{f_y^2 + f f_{yy}\} - 2ar + 6d \neq 0\) hold,

(vi) it admits Weyl projective collineation with respect to the vector field \(\xi = \frac{\partial}{\partial x}\) if the relations \(F_1 - 2a + 4br \neq 0\), \(F_8 \neq 0\) and \(F_9 = F_{10} = F_{12} = 0\) hold with \(F_1 = f_x^2 + f_y^2 - f(f_{xx} + f_{yy})\), where the functions \(F_8, F_9, F_{10}, F_{12}\) are given in \((4.7)\).

5. Conclusion

The geometrical symmetry plays a significant role in general relativity to understand the geometry of a spacetime as it is advantageous towards the solutions of EFE and beneficial in the classification of spacetimes. The curvature collineation \([27]\) is a fundamental symmetry of spacetimes and curvature inheritance \([20]\) is a generalized notion of curvature collineation. Indeed, during the study of curvature collineation and curvature inheritance in RT spacetime, we realize the necessity of a generalized notion of the curvature inheritance. In this paper, we have introduced the concept of generalized curvature inheritance, which includes the notions of curvature collineation as well as curvature inheritance. Also, we have shown that the RT spacetime admits generalized curvature (resp. Ricci, Weyl conformal, concircular, conharmonic, Weyl projective) inheritance. Finally, it is shown that under certain conditions the RT spacetime possesses curvature (resp. Ricci, conharmonic, Weyl projective) inheritance and (resp. Ricci, Weyl conformal, concircular, conharmonic, Weyl projective) collineation. Also, we have
shown that under several conditions RT spacetime turns out to be an example of almost Ricci soliton, almost $\eta$-Ricci soliton, almost gradient $\eta$-Ricci soliton with respect to certain vector fields.

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**References**

[1] Ahsan, Z., *Ricci Solitons and the Spacetime of General Relativity*, J. Tensor Soc., **12** (2018), 49–64.

[2] Ahsan, Z., *Algebraic classification of space-matter tensor in general relativity*, Indian J. Pure Appl. Math., **8**(2) (1977), 231–237.

[3] Ahsan, Z., *Algebra of space-matter tensor in general relativity*, Indian J. Pure Appl. Math., **8**(9) (1977), 1055–1061.

[4] Ahsan, Z., *Collineation in electromagnetic field in general relativity- The null field case*, Tamkang J. Maths., **9**(2) (1978), 237.

[5] Ahsan, Z., *On the Nijenhuis tensor for null electromagnetic field*, J. Math. Phys. Sci., **21**(5) (1987), 515–526.

[6] Ahsan, Z., *Symmetries of the Electromagnetic Fields in General Relativity*, Acta Phys. Sinica, **4** (1995), 337.

[7] Ahsan, Z., *A symmetry property of the space-time of general relativity in terms of the space-matter tensor*, Braz. J. Phys. **26**(3) (1996), 572-576.

[8] Ahsan, Z., *On a geometrical symmetry of the space-time of General Relativity*, Bull. Call. Math. Soc., **97**(3) (2005), 191.

[9] Ahsan, Z. and Ali, M., *On some properties of W-curvature tensor*, Palestine J. Math., **3**(1) (2014), 61–69.

[10] Ahsan, Z. and Ali, M., *Symmetries of Type D Pure Radiation Fields*. Int. J. Theo. Phys., **51** (2012), 2044-2055.

[11] Ahsan, Z. and Husain, S. I., *Null electromagnetic fields, total gravitational radiation and collineations in general relativity*, Annali di Mathematical Pura ed Applicata, **126** (1980), 379396.

[12] Ali, M. and Ahsan Z., *Ricci Solitons and Symmetries of Spacetime Manifold of General Relativity* Global J. Adv. Research Classical Mod. Geom., **1**(2) (2012), 75–84.

[13] Ali, M. and Ahsan Z., *Gravitational field of Schwarzschild soliton*, Arab J. Math. Sci., **21**(1) (2015), 15–21.

[14] Ali, M. and Ahsan, Z., *Geometry of Schwarzschild soliton*, J. Tensor Soc., **7** (2013), 49–57.

[15] Bokhari, A. H. and Qadir, A., *Collineations of the Ricci tensor*, J. Math. Phys., **34** (1993), 3543–3552.
[16] Besse, A. L., *Einstein Manifolds*, Springer-Verlag, Berlin, Heidelberg, 1987.
[17] Blaga, A. M., *eta-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat, **30**(2) (2016), 489–496.
[18] Brinkmann, H. W., *Einstein spaces which are mapped conformally on each other*, Math. Ann. **94** (1925), 119–145.
[19] Cho, J. and Kimura, M., *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J., **61**(2) (2009), 205–212.
[20] Duggal, K. L., *Curvature inheritance symmetry in Riemannian spaces with applications to fluid space times*, J. Math. Phys., **33**(9) (1992), 2989–2997.
[21] Eyasmin, S., Chakraborty, D. and Sarkar, M., *Curvature properties of Morris-Thorne wormhole metric*, J. Geom. Phys., **174** (2022), 104457.
[22] Eyasmin, S. and Chakraborty, D., *Curvature properties of (t−z)-type plane wave metric*, J. Geom. Phys., **160** (2021), 104004.
[23] Griffiths, J. B. and Podolsky, J., *Exact Space-Times in Einstein’s General Relativity*, Cambridge University Press, 2009.
[24] Głogowska, M., *Semi-Riemannian manifolds whose Weyl tensor is a Kulkarni-Nomizu square*, Publ. Inst. Math. (Beograd) (N.S.), **72**(86) (2002), 95–106.
[25] Hamilton, R. S., *Three manifolds with positive Ricci curvature*, J. Diff. Geom., **17** (1982), 255–306.
[26] Hamilton, R. S., *The Ricci flow on surfaces*, Contemp. Math., **71** (1988), 237–261.
[27] Katzin, G. H., Livine, J. and Davis, W. R., *Curvature collineations: A fundamental symmetry property of the space-times of general relativity defined by the vanishing Lie derivative of the Riemann curvature tensor*, J. Math. Phys., **10**(4) (1969), 617–629.
[28] Katzin, G. H., Livine, J. and Davis, W. R., *Groups of curvature collineations in Riemannian space-times which admit fields of parallel vectors*, J. Math. Phys., **11** (1970), 1578–1580.
[29] Nomizu, K., *On the spaces of generalized curvature tensor fields and second fundamental forms*, Osaka J. Math., **8** (1971), 21–28.
[30] Pigola, S., Rigoli, M., Rimoldi, M., Setti, A. G., *Ricci almost solitons*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **X**(5) (2011), 757–799.
[31] Robinson, I. and Trautman, A., *Some spherical gravitational waves in general relativity*, Proc. R. Soc. London, Ser. A, **265** (1962), 463–473.
[32] Robinson, I. and Trautman, A., *Spherical gravitational waves*, Phys. Rev. Lett., **4**(8) (1960), 431–432.
[33] Shaikh, A. A., *On pseudo quasi-Einstein manifolds*, Period. Math. Hungarica, **59**(2) (2009), 119–146.
[34] Shaikh, A. A., Ali, M. and Ahsan, Z., *Curvature properties of Robinson-Trautman metric*, J. Geom., **109**(38) (2018), [http://dx.doi.org/10.1007/s000022-018-0443-1](http://dx.doi.org/10.1007/s000022-018-0443-1)
[35] Shaikh, A. A., Ali, A., Alkhaldi, A. H. and Chakraborty, D., *Curvature properties of Melvin magnetic metric*, J. Geom. Phys. **150** (2020), [https://doi.org/10.1016/j.geomphys.2019.103593](https://doi.org/10.1016/j.geomphys.2019.103593).
[36] Shaikh, A. A., Ali, A., Alkhaldi, A. H. and Chakraborty, D., *Curvature properties of Nariai spacetimes*, Int. J. Geom. Methods. Mod. Phys., **17**(03) (2020), 2050034.
[37] Shaikh, A. A., Ali, A., Alkhalidi, A. H., Chakraborty, D. and Datta, B. R., *On some curvature properties of Lemaitre–Tolman–Bondi spacetime*, Gen. Relativ. Gravit., 54, 6 (2022), (21 pages), https://doi.org/10.1007/s10714-021-02890-4

[38] Shaikh, A. A., Al-Solamy, F. R. and Roy, I., *On the existence of a new class of semi-Riemannian manifolds*, Mathematical Sciences, 7:46 (2013), 1–13.

[39] Shaikh, A. A. and Binh, T. Q., *On some class of Riemannian manifolds*, Bull. Transilv. Univ. 15(50) (2008), 351–362.

[40] Shaikh, A. A., Binh, T. Q. and Kundu, H., *Curvature properties of generalized pp-wave metrics*, Kragujevac J. Math., 45(2) (2021), 237–258.

[41] Shaikh, A. A. and Chakraborty, D., *Curvature properties of Kantowski-Sachs metric*, J. Geom. Phys., 160 (2021), 103970. DOI: 10.1016/j.geomphys.2020.103970

[42] Shaikh, A. A., Datta, B. R. and Chakraborty, D., *On some curvature properties of Vaidya-Bonner metric*, Int. J. Geom. Methods Mod. Phys., 18(13) (2021), 2150205.

[43] Shaikh, A. A., Das, L., Kundu, H. and Chakraborty, D., *Curvature properties of Siklos metric*, Diff. Geom.-Dyn. Syst., 21 (2019), 167–180.

[44] Shaikh, A. A., Deszcz, R., Hasmani, A. H. and Kambholja, V. G., *Curvature Properties of Interior Black Hole Metric*, Indian J. Pure Appl. Math., 51(4) (2020), 1779–1814.

[45] Shaikh, A. A., Deszcz, R., Hotloś, M., Jelowicki, J. and Kundu, H., *On pseudosymmetric manifolds*, Publ. Math. Debrecen, 86(3-4) (2015), 433-456.

[46] Shaikh, A. A. and Kundu, H., *On equivalency of various geometric structures*, J. Geom., 105 (2014), 139–165, DOI: 10.1007/s00022-013-0200-4.

[47] Shaikh, A. A. and Kundu, H., *On warped product generalized Roter type manifolds*, Balkan J. Geom. Appl., 21(2) (2016), 82–95.

[48] Shaikh, A. A. and Kundu, H., *On curvature properties of Som-Raychaudhuri spacetime*, J. Geom. 108(2) (2016), 501–515.

[49] Shaikh, A.A. and Kundu, H., *On some curvature restricted geometric structures for projective curvature tensor*, Int. J. Geom. Meth. Mod. Phys. 15 (2018), 1850157 (38 pages).

[50] Shaikh, A. A. and Kundu, H., *On warped product manifolds satisfying some pseudosymmetric type conditions*, Diff. Geom. - Dyn. Syst., 19 (2017), 119–135.

[51] Shaikh, A. A. and Kundu, H., *On generalized Roter type manifolds*, Kragujevac J. Math 43(3) (2019), 471–493.

[52] Shaikh, A. A., Kundu, H. and Ali, Md. S., *On warped product super generalized recurrent manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N. S.) LXIV(1) (2018), 85–99.

[53] Shaikh, A. A., Kundu, H. and Sen, J., *Curvature properties of the Vaidya metric*, Indian J. Math. 61(1) (2019), 41–59.

[54] Shaikh, A. A. and Jana, S. K., *On weakly cyclic Ricci symmetric manifolds*, Ann. Pol. Math., 89(3) (2006), 139–146.
Shaikh, A. A., Kim, Y. H. and Hui, S. K.,  *On Lorentzian quasi Einstein manifolds*, J. Korean Math. Soc. **48** (2011), 669–689 and Erratum: *On Lorentzian quasi Einstein manifolds*, J. Korean Math. Soc. **48(6)** (2011), 1327–1328.

Shaikh, A. A. and Patra, A.,  *On a generalized class of recurrent manifolds*, Arch. Math. (BRNO), **46** (2010), 71–78.

Shaikh, A. A. and Roy, I.,  *On weakly generalized recurrent manifolds*, Ann. Univ. Sci. Budapest, Eötvös Sect. Math., **54** (2011) 35–45.

Shaikh, A. A., Roy, I. and Kundu, H.,  *On the existence of a generalized class of recurrent manifolds*, An. Științ. Univ. Al. I. Cuza Iași Mat. (N. S.) **LXIV(2)** (2018), 233–251.

Shaikh, A. A., Roy, I. and Kundu, H.,  *On some generalized recurrent manifolds*, Bull. Iranian Math. Soc. **43(5)** (2017), 1209–1225.

Shaikh, A. A., Srivastava, S. K. and Chakraborty, D.,  *Curvature properties of anisotropic scale invariant metrics*, Int. J. Geom. Methods. Mod. Phys., **16** (2019), 195086 (17 pages).

Shaikh, A. A., Yoon, D. W. and Hui, S. K.,  *On quasi-Einstein spacetimes*, Tsukuba J. Math., **33(2)** (2009), 305–326.

Shaikh, A. A., Mandal, P. and Mondal, C. K.,  *Diameter estimation of gradient ρ-Einstein solitons*, Journal of Geometry and Physics, **177** (2022), 104518.

Shaikh, A. A., Cunha, A. W. and Mandal, P.,  *Some characterizations of gradient Yamabe solitons*, J. Geom. Phys., **167** (2021), 104293.

Shaikh, A. A., Datta, B. R., Ali, A. and Alkhaldi, A. H.,  *LCS-manifolds and Ricci solitons*, Int. J. Geom. Methods Mod. Phys., **18(09)** (2021), 2150138.

Shaikh, A. A., Cunha, A. W. and Mandal, P.,  *Some characterizations of ρ-Einstein solitons*, J. Geom. Phys., **166** (2021), 104270.

Shaikh, A. A., Mondal, C. K. and Mandal, P.,  *Compact gradient ρ-Einstein soliton is isometric to the Euclidean sphere*, Indian J. Pure Appl. Math., **52(2)** (2021), 335–339.

Shaikh, A. A., and Mondal, C. K.,  *Isometry theorem of gradient Shrinking Ricci solitons*, J. Geom. Phys., **163** (2021), 104110.

Stephani. H., Kramer, D., Maccallum, M., Hoenselaers, C. and Herlt,  *Exact Solutions of Einstein's Field Equations*, Second Edition. Cambridge Univ. Press, UK, **2003**.