**-Ricci-Yamabe Soliton and Contact Geometry

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Abstract: It is well known that a unit sphere admits Sasakian 3-structure. Also, Sasakian manifolds are locally isometric to a unit sphere under several curvature and critical conditions. So, a natural question is: Does there exist any curvature or critical condition under which a Sasakian 3-manifold represents a geometrical object other than the unit sphere? In this regard, as an extension of the **-Ricci soliton, the notion of **-Ricci-Yamabe soliton is introduced and studied on two classes contact metric manifolds. A \((2n + 1)\)-dimensional non-Sasakian \(N(k)\)-contact metric manifold admitting **-Ricci-Yamabe soliton is completely classified. Further, it is proved that if a Sasakian 3-manifold \(M\) admits **-Ricci-Yamabe soliton \((g, V, \lambda, \alpha, \beta)\) under certain conditions on the soliton vector field \(V\), then \(M\) is **-Ricci flat, positive Sasakian and the transverse geometry of \(M\) is Fano. In addition, the Sasakian 3-metric \(g\) is homothetic to a Berger sphere and the soliton is steady. Also, the potential vector field \(V\) is an infinitesimal automorphism of the contact metric structure.

Mathematics Subject Classification 2010: 53D15, 35Q51, 53C25.

Keywords: \(N(k)\)-contact manifold, Sasakian 3-manifold, **-Ricci-Yamabe soliton, Berger sphere, Positive-Sasakian, Infinitesimal automorphism.

1. Introduction

In 2014, Kaimakamis and Panagiotidou [15] introduced the notion of **-Ricci soliton from the Ricci soliton as

\[
\frac{1}{2} \mathcal{L}_V g + S^* = \lambda g,
\]

where \(\mathcal{L}_V\) denotes the Lie derivative along the vector field \(V\), \(S^*\) is the **-Ricci tensor given by \(S^*(X, Y) = g(Q^*X, Y)\), \(Q^*\) is the **-Ricci operator, \(g\) is the Riemannian metric and \(\lambda\) is a constant. The **-Ricci tensor \(S^*\) is not symmetric in general. This definition is inconsistent if the **-Ricci tensor is not symmetric. For a consistent **-Ricci soliton on some space, the **-Ricci tensor of that space
has to be symmetric. In [11], Ghosh and Patra studied the notion of \( \ast \)-Ricci soliton on Sasakian manifolds. Further, in [10], the authors studied \( \ast \)-Ricci soliton and \( \ast \)-gradient Ricci soliton within the framework of trans-Sasakian 3-manifolds.

In 2019, Güler and Crasmareanu [14] proposed the notion of Ricci-Yamabe flow on a Riemannian manifold \((M^n,g)\) as

\[
\frac{\partial g}{\partial t}(t) + 2\alpha S(t) + \beta r(t)g(t) = 0,
\]

where \( g \) is the Riemannian metric, \( S \) is the Ricci tensor, \( r \) is the scalar curvature and \( \alpha, \beta \) are two real constants. Since \( \alpha \) and \( \beta \) are arbitrary constants, then one can freely choose the signs of \( \alpha \) and \( \beta \). This freedom of choice is very useful in differential geometry and theory of relativity. In [1, 5], the authors study space-time geometry with a bi-metric approach.

In [8], the present author proposed the idea of Ricci-Yamabe soliton from the Ricci-Yamabe flow on a Riemannian manifold \((M^n,g)\) as

\[
\mathcal{L}_V g + 2\alpha S = (2\lambda - \beta r)g,
\]

where \( \lambda, \alpha, \beta \in \mathbb{R} \). This generalizes a large class of soliton like equations such as Ricci soliton, Yamabe soliton, Einstein soliton, \( \rho \)-Einstein soliton. In [8], the present author classified two classes of almost Kenmotsu manifolds admitting Ricci-Yamabe soliton.

We now extent the notion of \( \ast \)-Ricci soliton on a contact metric manifold to a more generalized version as follows:

**Definition 1.1.** A contact metric manifold \((M,g)\) is said to admit a \( \ast \)-Ricci-Yamabe soliton (in short, \( \ast \)-RYS) \((g,V,\lambda,\alpha,\beta)\) if

\[
\mathcal{L}_V g + 2\alpha S^\ast = (2\lambda - \beta r^\ast)g,
\]

where \( \lambda, \alpha, \beta \in \mathbb{R} \) such that \( \alpha \neq 0 \) and \( r^\ast \) is the \( \ast \)-scalar curvature defined by \( r^\ast = \text{trace}(Q^\ast) \).

If \( V \) is gradient of some smooth function \( f \) on \( M \), then it is called a \( \ast \)-gradient Ricci-Yamabe soliton (in short, \( \ast \)-GRYS) and then (1.1) reduces to

\[
\nabla^2 f + \alpha S^\ast = (\lambda - \frac{1}{2}\beta r^\ast)g,
\]

where \( \nabla^2 f \) is the Hessian of \( f \) defined by \( Hess_f(X,Y) = g(\nabla_X Df,Y) \), \( D \) is the gradient operator. The \( \ast \)-RYS (or \( \ast \)-GRYS) is said to be expanding, steady
or shrinking according as $\lambda$ is negative, zero or positive respectively. Note that, the $\ast$-RYS reduces to a $\ast$-Ricci soliton if $\alpha = 1$ and $\beta = 0$.

$N(k)$-contact metric manifolds are special kind of contact metric manifolds that generalizes Sasakian manifolds. Sasakian geometry is an odd dimensional analog of the Kaehler geometry and is an interesting topic to physicists as it perceived relevance in string theory (see [7, 16]). Due to this connection with physics, we consider the notion of $\ast$-RYS and $\ast$-GRYS within the frameworks of $N(k)$-contact geometry and Sasakian 3-geometry. The paper is organized as follows: In section 2, we recall some basic relations and definitions on $N(k)$-contact and Sasakian geometry. Section 3 deals with $N(k)$-contact metric as a $\ast$-RYS. In section 4, a Sasakian 3-manifold is completely classified admitting $\ast$-RYS under certain conditions on the soliton vector field.

2. Preliminaries

An odd dimensional smooth manifold $M$ together with a structure $(\varphi, \xi, \eta, g)$ satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0, \quad (2.1)$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.2)$$

for all vector fields $X, Y$ on $M$ is called an almost contact metric manifold, where $\varphi$ is a $(1, 1)$-tensor field, $\xi$ is a unit vector field called the characteristic vector field, $\eta$ is a 1-form dual to $\xi$ and $g$ is the Riemannian metric. It is easy to see from (2.2) that $\phi$ is skew-symmetric, that is,

$$g(\varphi X, Y) = -g(X, \varphi Y). \quad (2.3)$$

A contact metric manifold is an almost contact metric manifold with $d\eta = g(X, \varphi Y)$. On a contact metric manifold, the $(1, 1)$-tensor field $h$ is defined as $h = \frac{1}{2} \mathcal{L}_{\xi} \varphi$. The tensor field $h$ is symmetric and satisfies

$$h \varphi = -\varphi h, \quad \text{trace}(h) = \text{trace}(\varphi h) = 0, \quad h\xi = 0. \quad (2.4)$$

Also, on a contact metric manifold, we have

$$\nabla_X \xi = -\varphi X - \varphi h X. \quad (2.5)$$

In [18], Tanno introduced the notion of $k$-nullity distribution on a Riemannian manifold as

$$N(k) = \{Z \in T(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$
where \( k \in \mathbb{R} \) and \( T(M) \) is the Lie algebra of all vector fields on \( M \). If the characteristic vector field \( \xi \in N(k) \), then we call a contact metric manifold as \( N(k) \)-contact metric manifold \([18]\). For a \((2n+1)\)-dimensional \( N(k) \)-contact metric manifold, we have (see \([2, 4]\))

\[
h^2 = (k - 1)\varphi^2, \quad (2.6)
\]

\[
R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \quad (2.7)
\]

\[
R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X], \quad (2.8)
\]

\[
(\nabla_X \eta)Y = g(X + hX, \varphi Y), \quad (2.9)
\]

\[
(\nabla_X \varphi)Y = g(X + hX,Y)\xi - \eta(Y)(X + hX) \quad (2.10)
\]

for all vector fields \( X, Y \) on \( M \), where \( R \) is the Riemann curvature tensor.

If the characteristic vector field \( \xi \) is Killing type, then a contact metric manifold is called a \( K \)-contact manifold and if the structure \((\varphi, \xi, \eta, g)\) is normal, then a contact metric manifold is called Sasakian. Also, an almost contact metric manifold is Sasakian if and only if

\[
(\nabla_X \varphi)Y = g(X + hX,Y)\xi - \eta(Y)(X + hX) \quad (2.11)
\]

holds for all vector fields \( X, Y \) on \( M \). Hence, a \( N(k) \)-contact metric manifold reduces to a Sasakian one if \( h = 0 \), that is, \( k = 1 \). On a Sasakian 3-manifold, the following relations are well known:

\[
\nabla_X \xi = -\varphi X, \quad (2.12)
\]

\[
(\nabla_X \eta)Y = g(X, \varphi Y), \quad (2.13)
\]

\[
R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.14)
\]

\[
R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X. \quad (2.15)
\]

Since a 3-dimensional Riemannian manifold is conformally flat, it’s curvature tensor can be expressed as

\[
R(X,Y)Z = \left[ S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \right] - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \quad (2.16)
\]
where $r$ is the scalar curvature and $S$ is the Ricci tensor. The Ricci tensor of a Sasakian 3-manifold can be obtained from here as

$$S(X,Y) = \frac{1}{2}[(r - 2)g(X,Y) + (6 - r)\eta(X)\eta(Y)].$$

(2.17)

Note that the scalar curvature $r$ is not constant in general. We now close this section by recalling the following definition:

**Definition 2.1.** Let $V$ be a vector field on an almost contact metric manifold $M$. If there exist a smooth function $\sigma$ on $M$ such that $\mathcal{L}_V g = 2\sigma g$, then $V$ is called a conformal vector field. In particular, if $\sigma$ is constant then $V$ is homothetic and if $\sigma = 0$, then $V$ is called a Killing vector field. Also $V$ is said to be an infinitesimal contact transformation if $\mathcal{L}_V \eta = \psi \eta$ for some smooth function $\psi$ on $M$. If $\psi = 0$, then $V$ is said to be a strict infinitesimal contact transformation. If $V$ leaves all the structure tensor fields $\varphi$, $\xi$, $\eta$ and $g$ invariant, then $V$ is called an infinitesimal automorphism of the contact metric structure.

### 3. $N(k)$-Contact Metric as a *-RYS

In this section, we study the notion of *-RYS within the framework of $N(k)$-contact metric manifolds. To prove the main theorem of this section, we need the following lemmas:

**Lemma 3.1.** ([3]) A contact metric manifold $M^{2n+1}$ satisfying the condition $R(X,Y)\xi = 0$ for all $X, Y$ is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.

**Lemma 3.2.** ([9]) A $(2n+1)$-dimensional $N(k)$-contact metric manifold is $\ast$-$\eta$-Einstein and the $\ast$-Ricci tensor is given by

$$S^\ast(X,Y) = -k[g(X,Y) - \eta(X)\eta(Y)].$$

(3.1)

**Note 3.3.** We observe that the $\ast$-Ricci tensor of a $N(k)$-contact metric manifold is symmetric. Hence, the notion of $\ast$-RYS is consistent in this setting. We are now ready to prove the main result of this section.

**Theorem 3.4.** If a $(2n+1)$-dimensional non-Sasakian $N(k)$-contact metric manifold $M$ admits $\ast$-RYS $(g, V, \lambda, \alpha, \beta)$, then

1. The manifold $M$ is $\ast$-Ricci flat.

2. The manifold $M$ is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$. 


The soliton vector field $V$ is homothetic.

Proof. We start the proof by taking $g$-trace of (3.1) that gives $r^* = -2nk$. Now, substituting (3.1) and the value of $r^*$ in (1.1) yields

$$ (L_V g)(X, Y) = [2\lambda + 2nk\beta + 2k\alpha]g(X, Y) - 2k\alpha\eta(X)\eta(Y). \quad (3.2) $$

Differentiating the foregoing equation covariantly along any vector field $Z$, we get

$$ (\nabla_Z L_V g)(X, Y) = -2k\alpha[\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y]. \quad (3.3) $$

Applying (2.9) in (3.3) yields

$$ (\nabla_Z L_V g)(X, Y) = -2k\alpha[\eta(Y)(Z + hZ, \varphi X) + \eta(X)(Z + hZ, \varphi Y)]. \quad (3.4) $$

The well known commutation formula (see [19])

$$ (L_V \nabla_X g - \nabla_X L_V g)(Y, Z) = -g((L_V \nabla)(X, Y), Z) - g((L_V \nabla)(X, Z), Y). $$

leads to

$$ g((L_V \nabla)(X, Y), Z) = \frac{1}{2}(\nabla_X L_V g)(Y, Z) + \frac{1}{2}(\nabla_Y L_V g)(X, Z) - \frac{1}{2}(\nabla_Z L_V g)(X, Y). $$

Using (3.4) in the preceding equation yields

$$ g((L_V \nabla)(X, Y), Z) = 2k\alpha[\eta(X)g(\varphi Y, Z) + \eta(Y)g(\varphi X, Z) - g(hX, \varphi Y)\eta(Z)], $$

which implies

$$ (L_V \nabla)(X, Y) = 2k\alpha[\eta(X)\varphi Y + \eta(Y)\varphi X - g(hX, \varphi Y)\xi]. \quad (3.5) $$

Substituting $Y = \xi$ in the foregoing equation, we get

$$ (L_V \nabla)(X, \xi) = 2k\alpha\varphi X. \quad (3.6) $$

Now,

$$ (\nabla_Y L_V \nabla)(X, \xi) = \nabla_Y (L_V \nabla)(X, \xi) = (L_V \nabla)(\nabla_X \xi) - (L_V \nabla)(\nabla_Y X, \xi) - (L_V \nabla)(X, \nabla_Y \xi). $$

Using (3.5),(3.6),(2.5) and (2.10) in the previous equation, we obtain

$$ (\nabla_Y L_V \nabla)(X, \xi) = 2k\alpha[g(Y + hY, X)\xi - \eta(X)(Y - \eta(Y)\xi + hY) - \eta(X)(Y + hY) + g(hX, Y - \eta(Y)\xi + hY)\xi] \quad (3.7) $$
Due to Yano [19], we have
\[(LV R)(X, Y)Z = (\nabla_X LV \nabla)(Y, Z) − (\nabla_Y LV \nabla)(X, Z),\]
Using (3.7) in the above formula, we obtain
\[(LV R)(X, \xi)\xi = (\nabla_X LV \nabla)(\xi, \xi) − (\nabla_\xi LV \nabla)(X, \xi) = -4k\alpha[X − \eta(X)\xi + hX].\] (3.8)
Now, replacing \(Y\) by \(\xi\) in (3.2) gives
\[(LV g)(X, \xi) = [2\lambda + 2nk\beta]\eta(X),\]
which leads to
\[(LV \eta)X − g(X, LV \xi) = [2\lambda + 2nk\beta]\eta(X).\] (3.9)
Putting \(X = \xi\) in the preceding equation, we can easily obtain that
\[\eta(LV \xi) = -[\lambda + nk\beta].\] (3.10)
Now, with the help of (3.9), (3.10), (2.7) and (2.8), we obtain
\[(LV R)(X, \xi)\xi = k[2\lambda + 2nk\beta](X − \eta(X)\xi).\] (3.11)
Equating (3.8) and (3.11), we have
\[k[2\lambda + 2nk\beta + 4\alpha](X − \eta(X)\xi) = -4k\alpha hX.\] (3.12)
Taking \(g\)-trace of (3.12) and using (2.4) yields
\[k[2\lambda + 2nk\beta + 4\alpha] = 0.\] (3.13)
Operating \(h\) on (3.12) and using (2.6) leads to
\[k^2[(2\lambda + 2nk\beta + 4\alpha)^2 + 16\alpha^2(k - 1)](X − \eta(X)\xi) = 0,\]
which implies
\[k^2[(2\lambda + 2nk\beta + 4\alpha)^2 + 16\alpha^2(k - 1)] = 0.\] (3.14)
Now, (3.13) and (3.14) together implies either \(k = 0\) or the following relations holds:
\[2\lambda + 2nk\beta + 4\alpha = 0\] (3.15)
and

\[(2\lambda + 2nk\beta + 4\alpha)^2 + 16\alpha^2(k - 1) = 0.\] (3.16)

**Case 1:** If \(k = 0\), then from (3.1), we have \(S^* = 0\), that is, the manifold is \(\ast\)-Ricci flat. Again from (2.7), we have \(R(X,Y)\xi = 0\) and hence, from lemma 3.1, it follows that the manifold \(M\) is locally isometric to \(E^{n+1}(0) \times S^n(4)\) for \(n > 1\) and flat for \(n = 1\). Also, equation (3.2) reduces to \(\mathcal{L}_Vg = 2\lambda g\), where \(\lambda\) is a constant. This shows that \(V\) is homothetic.

**Case 2.** If \(k \neq 0\), then (3.15) and (3.16) holds. Since \(\alpha \neq 0\), these two equations implies that \(k = 1\). This shows that \(M\) reduces to a Sasakian manifold, a contradiction to our hypothesis. This completes the proof. \(\square\)

A \(\ast\)-RYS reduces to a \(\ast\)-Ricci soliton if \(\alpha = 1\) and \(\beta = 0\). Hence, from the above theorem, we can state the following:

**Corollary 3.5.** If \((g,V,\lambda)\) is a \(\ast\)-Ricci soliton on a \((2n + 1)\)-dimensional non-Sasakian \(N(k)\)-contact metric manifold \(M\), then

1. The manifold \(M\) is \(\ast\)-Ricci flat.
2. The manifold \(M\) is locally isometric to \(E^{n+1}(0) \times S^n(4)\) for \(n > 1\) and flat for \(n = 1\).
3. The soliton vector field \(V\) is homothetic.

### 4. Sasakian 3-Metric as a \(\ast\)-RYS

In the preceding section, the Sasakian case is omitted. In this section, we consider an extended notion of \(\ast\)-RYS by considering the soliton vector field \(V\) as a gradient vector field and an infinitesimal contact transformation on Sasakian 3-manifold. We start this section with the following discussion:

A contact metric manifold \(M\) is said to be \(\eta\)-Einstein if there exist two smooth functions \(a\) and \(b\) such that

\[S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)\]

for all vector fields \(X, Y\) on \(M\). A \((2n + 1)\)-dimensional \(\eta\)-Einstein Sasakian manifold with \(a = -2\) and \(b = 2n + 2\) is known as null-Sasakian. An example of such a manifold is a Sasakian space form \(\mathbb{R}^{2n+1}\) with constant \(\varphi\)-sectional curvature \(-3\), which is identifiable with a Heisenberg group. Also, \(\eta\)-Einstein Sasakian manifold with \(a > -2\) is called positive-Sasakian. In this case, the
transverse geometry of $M$ is Fano. For more details, we refer the reader to go through [6]. From (2.17), we see that a Sasakian 3-manifold is $\eta$-Einstein. Thus a Sasakian 3-manifold is null-Sasakian if $r = -2$ and positive-Sasakian if $r > -2$. For $r > -2$, the transverse geometry of the Sasakian 3-manifold is Fano.

In [11], Ghosh and Patra obtained the expression of the $*$-Ricci tensor for a $(2n + 1)$-dimensional Sasakian manifold which involves the Ricci tensor. In [10], the present author together with Majhi presents a expression of the $*$-Ricci tensor for a trans-Sasakian 3-manifold, which is a generalization of the Sasakian 3-manifold. From lemma 3.1 of [10] or using (2.17) in lemma 3.1 of [11], we obtain the expression of the $*$-Ricci tensor for Sasakian 3-manifold $M$ as

$$S^*(X,Y) = \frac{1}{2}(r - 4)[g(X,Y) - \eta(X)\eta(Y)]$$

for all vector fields $X$, $Y$ on $M$. Note that, $S^*$ is symmetric. Taking $g$-trace of (4.1), we have $r^* = r - 4$. We now consider $V$ as gradient of some smooth function $f : M \rightarrow \mathbb{R}$ in the definition of $*$-RYS and prove the following theorem.

**Theorem 4.1.** If $(g, V, \lambda, \alpha, \beta)$ be a $*$-RYS on a Sasakian 3-manifold $M$ such that either (i) $V$ is a gradient vector field or (ii) $V$ is an infinitesimal contact transformation, then

1. The manifold $M$ is $*$-Ric flat.
2. The Sasakian 3-manifold $M$ is positive-Sasakian and the transverse geometry of $M$ is Fano.
3. The Sasakian 3-metric $g$ is homothetic to a Berger sphere.
4. The soliton vector field $V$ is an infinitesimal automorphism of the contact metric structure.
5. The $*$-RYS is steady.

**Proof.** (i) If the soliton vector field $V$ is gradient of some smooth function $f$ on $M$, then the equation (1.2) can be exhibited as

$$\nabla_X Df = [\lambda - \frac{1}{2}(r - 4)\beta]X - \alpha Q^*X$$

(4.2)

for any vector field $X$ on $M$, where $V = Df$ and $r^* = r - 4$ is used. Differentiating (4.2) covariantly along any vector field $Y$ yields

$$\nabla_Y \nabla_X Df = [\lambda - \frac{1}{2}(r - 4)\beta]\nabla_Y X - \frac{1}{2}\beta(Yr)X - \alpha \nabla_Y Q^*X.$$  

(4.3)
Interchanging $X$ and $Y$ in the preceding equation, we have

$$\nabla_X \nabla_Y Df = [\lambda - \frac{1}{2}(r - 4)\beta] \nabla_X Y - \frac{1}{2} \beta(Xr)Y - \alpha \nabla_X Q^* Y.$$  \hspace{1cm} (4.4)

From (4.2), we get

$$\nabla_{[X,Y]} Df = [\lambda - \frac{1}{2}(r - 4)\beta](\nabla_X Y - \nabla_Y X) - \alpha Q^*(\nabla_X Y - \nabla_Y X).$$  \hspace{1cm} (4.5)

Now, the curvature tensor $R$ is given by

$$R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df.$$  \hspace{1cm} (4.6)

Using (4.3)-(4.5) in the foregoing equation, we get

$$R(X,Y)Df = \frac{1}{2} \beta(Yr)X - \frac{1}{2} \beta(Xr)Y - \alpha[(\nabla_X Q^*)Y - (\nabla_Y Q^*)X].$$  \hspace{1cm} (4.6)

From (4.1), we have

$$Q^* X = \frac{1}{2}(r - 4)[X - \eta(X)\xi].$$  \hspace{1cm} (4.7)

With the help of (2.12) and (2.13), from (4.7), we obtain

$$(\nabla Y Q^*)X = \frac{1}{2}(Yr)[X - \eta(X)\xi] - \frac{1}{2}(r - 4)[g(\varphi X,Y)\xi - \eta(X)\varphi Y].$$  \hspace{1cm} (4.8)

Applying (4.8) in (4.6), we obtain

$$R(X,Y)Df = \frac{1}{2} \beta(Yr)X - \frac{1}{2} \beta(Xr)Y - \alpha[(\nabla_X Q^*)Y - (\nabla_Y Q^*)X].$$  \hspace{1cm} (4.6)

Substituting $X = \xi$ in (4.9) and noting $(\xi r) = 0$ (as $\xi$ is Killing), we infer that

$$R(\xi,Y)Df = \frac{1}{2} \beta(Yr)\xi + \frac{1}{2} \alpha(r - 4)\varphi Y.$$  \hspace{1cm} (4.10)

Taking inner product of the preceding equation with $X$ yields

$$g(R(\xi, Y)Df, X) = \frac{1}{2} \beta(Yr)\eta(X) + \frac{1}{2} \alpha(r - 4)g(\varphi Y, X).$$  \hspace{1cm} (4.10)

Since $g(R(\xi, Y)Df, X) = -g(R(\xi, Y)X, Df)$, then using (2.15), we get

$$g(R(\xi, Y)Df, X) = -g(X, Y)(\xi f) + \eta(X)(Y f).$$  \hspace{1cm} (4.11)
Equating (4.10) and (4.11) and then antisymmetrizing yields
\[ \frac{1}{2} \beta(Yr)\eta(X) - \frac{1}{2} \beta(Xr)\eta(Y) + \alpha(r - 4)g(\varphi Y, X) = \eta(X)(Yf) - \eta(Y)(Xf). \] (4.12)

Replacing \( X \) by \( \phi X \) and \( Y \) by \( \phi Y \) in (4.12) yields
\[ \alpha(r - 4)g(\phi X, Y) = 0, \]
which implies \( r = 4 \) as \( \alpha \neq 0 \) by definition of *-RYS. Hence, from (4.1), we get \( S^* = 0 \). This proves (1).

Since \( r = 4 > -2 \), then \( M \) is positive-Sasakian and the transverse geometry of \( M \) is Fano proving (2).

The Tanaka-Webster curvature (see [12]) of a Sasakian 3-manifold is given by \( W = \frac{1}{4}(r + 2) \). Since \( r = 4 \), then \( W = \frac{3}{2} \). Following the classification given by Guilfoyle [13] for \( 0 < W < 2 \), we conclude that \( g \) is homothetic to a Berger sphere. This proves (3).

Now, equation (4.12) reduces to
\[ \eta(X)(Yf) - \eta(Y)(Xf) = 0. \]
Putting \( X = \xi \) in the preceding equation yields
\[ (Yf) - \eta(Y)(\xi f) = 0, \]
which implies \( Df = (\xi f)\xi \). Therefore, we get \( V = Df = (\xi f)\xi \), that is, \( V \) is pointwise collinear with \( \xi \). For simplicity, we write \( c = (\xi f) \). Then
\[ (\mathcal{L}_Y g)(X,Y) = (\mathcal{L}_{\xi f} g)(X,Y) = (Xc)\eta(Y) + (Yc)\eta(X). \] (4.13)

Using (4.13), \( S^* = 0 \) and \( r^* = 0 \) in (1.1), we have
\[ (Xc)\eta(Y) + (Yc)\eta(X) = 2\lambda g(X,Y). \] (4.14)
Substituting \( X = Y = \xi \) in (4.14) yields
\[ 2(\xi c) = 2\lambda. \] (4.15)
Let \( \{e_i\} \) be any orthonormal frame in \( M \). Now, substituting \( X = Y = e_i \) in (4.14) and summing over \( i \), we obtain
\[ 2(\xi c) = 6\lambda. \] (4.16)
The equations (4.15) and (4.16) together implies \( \lambda = 0 \) and hence, the \( * \)-GRYS is steady proving (5).

Thus we get \( (\xi c) = 0 \). Then using \( \lambda = 0 \) and putting \( Y = \xi \) in (4.14), we get \( (Xc) = 0 \) for any vector field \( X \), which implies \( c = (\xi f) \) is a constant. Therefore, \( V \) is a constant multiple of \( \xi \). Since \( c \) is a constant, then (4.13) gives \( \mathcal{L}_V g = 0 \). It is easy to see that \( \mathcal{L}_V \xi = \mathcal{L}_c \xi = 0 \). Now, \( \mathcal{L}_V g = 0 \) and \( \mathcal{L}_V \xi = 0 \) together implies \( \mathcal{L}_V \eta = 0 \). With the help of (2.11) and (2.12), it can be easily proved that \( \mathcal{L}_V \varphi = 0 \). Therefore, \( V \) leaves all the structure tensors \( \varphi, \xi, \eta \) and \( g \) invariant, that is, \( V \) is an infinitesimal automorphism of the contact metric structure proving (4) and this completes the proof of (i).

(ii) If \( V \) is an infinitesimal contact transformation, then there exist a smooth function \( f \) on \( M \) such that

\[
\mathcal{L}_V \eta = f \eta. \tag{4.17}
\]

Since \( d\eta(X,Y) = g(X,\varphi Y) \), then

\[
(\mathcal{L}_V d\eta)(X,Y) = \mathcal{L}_V d\eta(X,Y) - d\eta(\mathcal{L}_V X,Y) - d\eta(X,\mathcal{L}_V Y) \\
= \mathcal{L}_V g(X,\varphi Y) - g(\mathcal{L}_V X,\varphi Y) - g(X,\varphi \mathcal{L}_V Y) \\
= (\mathcal{L}_V g)(X,\varphi Y) + g(X, (\mathcal{L}_V \varphi)Y). \tag{4.18}
\]

Substituting the value of \( (\mathcal{L}_V g)(X,\varphi Y) \) from (1.1) in (4.18), we get

\[
(\mathcal{L}_V d\eta)(X,Y) = -2\alpha S^*(X,\varphi Y) + (2\lambda - \beta r^*)g(X,\varphi Y) + g(X, (\mathcal{L}_V \varphi)Y). \tag{4.19}
\]

Since (4.17) holds, then we have

\[
\mathcal{L}_V d\eta = d\mathcal{L}_V \eta = df \wedge \eta + f d\eta, \tag{4.20}
\]

which implies

\[
(\mathcal{L}_V d\eta)(X,Y) = \frac{1}{2}[(Xf)\eta(Y) - (Yf)\eta(X)] + fg(X,\varphi Y). \tag{4.21}
\]

Equating (4.19) and (4.21), we get

\[
g(X, (\mathcal{L}_V \varphi)Y) = \frac{1}{2}[(Xf)\eta(Y) - (Yf)\eta(X)] \\
+ [f + \alpha(r - 4) - 2\lambda + \beta(r - 4)]g(X,\varphi Y),
\]

which implies

\[
(\mathcal{L}_V \varphi)Y = \frac{1}{2}[\eta(Y)Df - (Yf)\xi] + [f + \alpha(r - 4) - 2\lambda + \beta(r - 4)]\varphi Y. \tag{4.22}
\]
Replacing $Y$ by $\xi$ in the preceding equation, we have
\[
(L_V \varphi)\xi = \frac{1}{2} [Df - (\xi f)\xi].
\] (4.23)

Taking $g$-trace of the equation (1.1) yields
\[
\text{Div } V = \frac{3}{2} [2\lambda - \beta(r - 4)] - \alpha(r - 4),
\] (4.24)

where ‘Div’ stands for divergence. Let $\Sigma$ be the volume form of $M$, that is, $\Sigma = \eta \wedge (d\eta)^n \neq 0$. Lie differentiating this along the vector field $V$ and applying the formula $L_V \Sigma = (\text{Div } V)\Sigma$, (4.17) and (4.20), we obtain $\text{Div } V \Sigma = (n + 1)f\Sigma$, which implies
\[
\text{Div } V = (n + 1)f.
\] (4.25)

Equating (4.24) and (4.25), we obtain
\[
(n + 1)f = \frac{3}{2} [2\lambda - \beta(r - 4)] - \alpha(r - 4).
\] (4.26)

From (1.1), we write
\[
(L_V g)(X, \xi) = [2\lambda - \beta(r - 4)]\eta(X),
\]
which implies
\[
(L_V \eta)X - g(X, L_V \xi) = [2\lambda - \beta(r - 4)]\eta(X).
\] (4.27)

Using (4.17) in (4.27) gives
\[
f \eta(X) - g(X, L_V \xi) = [2\lambda - \beta(r - 4)]\eta(X).
\] (4.28)

Substituting $X = \xi$ in the foregoing equation gives
\[
\eta(L_V \xi) = [f - 2\lambda + \beta(r - 4)].
\] (4.29)

Now, substitution of $X = \xi$ in (4.27) yields
\[
\eta(L_V \xi) = -\frac{1}{2} [2\lambda - \beta(r - 4)].
\] (4.30)

Equating (4.29) and (4.30), we get
\[
f = \frac{1}{2} [2\lambda - \beta(r - 4)].
\] (4.31)
We now use the equation (4.31) in (4.28) to obtain
\[ \mathcal{L}_V \xi = -f \xi. \] (4.32)

With the help of (4.32), it can be easily seen that \((\mathcal{L}_V \varphi) \xi = 0\) and hence, equation (4.23) reduces to \(Df = (\xi f) \xi\). Taking inner product of this with \(Y\), we have
\[ df(Y) = (\xi f) \eta(Y), \] (4.33)

which implies \(df = (\xi f) \eta\). Now, taking exterior derivative of this, we have \(d^2 f = d(\xi f) \wedge \eta + (\xi f) d\eta\). Applying \(d^2 = 0\) on it and then taking wedge product with \(\eta\) yields \((\xi f) \eta \wedge d\eta = 0\). Since \(\eta \wedge d\eta \neq 0\) on \(M\), then \((\xi f) = 0\) and therefore, equation (4.33) implies \(f\) is constant. Integrating both sides of (4.25) over \(M\) and applying the divergence theorem we get \(f = 0\). Thus, equations (4.17) and (4.32) implies \(\mathcal{L}_V \eta = \mathcal{L}_V \xi = 0\). From (4.31), we have \([2\lambda - \beta(r - 4)] = 0\) and hence, from (4.26), we get \(r = 4\) as \(\alpha\) is a non-zero constant, which implies \(\lambda = 0\) proving (5). Therefore, equation (4.22) gives \(\mathcal{L}_V \varphi = 0\) and equation (4.1) gives \(S^* = 0\). Now, from (1.1) yields \(\mathcal{L}_V g = 0\). Thus \(V\) is an infinitesimal automorphism of the contact metric structure. Since \(r = 4\), then by similar arguments as in part (i), the statements (1), (2) and (3) holds. This completes the theorem. \(\square\)

**Remark 4.2.** If \(M\) is complete and since \(r = 4 > 0\), then by Myers theorem [17], \(M\) is necessarily compact.

**Remark 4.3.** The \(*\)-RYS reduces to a \(*\)-Ricci soliton if \(\alpha = 1\) and \(\beta = 0\). Thus the results of the above theorem holds for a Sasakian 3-manifold admitting \(*\)-Ricci soliton.

It is proved that a \(N(k)\)-contact metric or a Sasakian 3-metric satisfying the \(*\)-RYS is \(*\)-Ricci flat. Thus a natural question is

**Question 4.4.** Does there exist an almost contact metric \(g\) such that \(g\) admits \(*\)-RYS or \(*\)-GRYS whose \(*\)-Ricci tensor is not identically zero or in other words, the manifold is not \(*\)-Ricci flat ?

**Further Scope of Study**

The notion of \(*\)-RYS is a generalized version of \(*\)-Ricci soliton. For instance, one can see that the equation (1.1) provides several soliton like equations such as

- \(*\)-Yamabe soliton for \(\alpha = 0\) and \(\beta = 2\).
- \(*\)-Einstein soliton for \(\alpha = 1\) and \(\beta = -1\).
**-Ricci-Yamabe Soliton

- **-ρ-Einstein soliton for α = 1 and β = −2ρ.

The notion of **-Yamabe soliton is not introduced yet. So, this can be a platform for introducing the notion of **-Yamabe soliton. In this article, the author studied the notion of **-RYS within the framework of N(k)-contact and Sasakian geometry. There is a large class of almost contact metric manifolds to study this notion. Further, one can consider this new notion on para-contact manifolds or Lorentzian manifolds.

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