ORBIFOLDS AND STABLE HOMOTOPY GROUPS

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ABSTRACT. Lie groupoids generalize transformation groups, and so provide a natural language for studying orbifolds [13] and other noncommutative geometries. In this paper, we investigate a connection between orbifolds and equivariant stable homotopy theory using such groupoids. A different sort of twisted sector, along with a classical theorem of tom Dieck [17], allows for a natural definition of stable orbifold homotopy groups, and motivates defining extended unstable orbifold homotopy groups generalizing previous definitions.

1. Introduction

Homotopy groups for orbifolds have been defined in a variety of ways, going back to Thurston’s orbifold fundamental group [16] and culminating in Moerdijk’s elegant treatment, which we briefly recall here. We assume the reader is familiar with the category \( \mathbf{Gpd} \) of orbifold groupoids and their homomorphisms, as well as Morita equivalences, classifying spaces, groupoid actions, and other basic notions—see [13] or [2] for an introduction and further references. By an orbifold groupoid \( \mathcal{G} \), we mean a proper foliation Lie groupoid. We write \( G_0 \) for the manifold of objects and \( G_1 \) for the manifold of arrows. Let \( \mathcal{G} \) be an orbifold groupoid, and let \( x \in G_0 \) be a base point. The \( n^{th} \) orbifold homotopy group of \( \mathcal{G} \) based at \( x \) is

\[
\pi_n^{\text{orb}}(\mathcal{G}, x) := \pi_n(B\mathcal{G}, [x]),
\]

where \( B\mathcal{G} \) is the classifying space of \( \mathcal{G} \), and \([x]\) is the point in \( B\mathcal{G} \) corresponding to the object \( x \).

These groups have several agreeable properties. They are Morita invariant, so that they descend to the localized category \( \mathbf{Orb} \) of orbifolds. They also generalize Thurston’s fundamental group, and agree with alternative definitions, such as [4], in higher degrees. On the other hand, equivariant homotopy theory reveals that these groups are insufficient, for if \( \mathcal{G} = G \times M \) is a translation groupoid corresponding to the quotient orbifold \( \mathcal{X} = M/G \), then \( B\mathcal{G} \) is
homotopic to the Borel construction $EG \times_G M$ (see Appendix A), which fails to capture the $G$-homotopy type of $M$.

Example 1.1. Let $D$ be a disk with a smooth, fixed-point free action of the icosahedral group $\mathcal{I}$. Such an action is described in [3, pp. 55–58], and was first constructed simplicially by Floyd and Richardson [6]. Then the map $f : D \to \{\text{pt}\}$ is an equivariant map that is a nonequivariant homotopy equivalence. Accordingly, the induced map $f : E\mathcal{I} \times_\mathcal{I} D \to B\mathcal{I}$ is a homotopy equivalence. However, $\mathcal{I} \ltimes D$ is certainly not the same orbifold as $\mathcal{I} \ltimes \{\text{pt}\}$, as the former has no point with isotropy $\mathcal{I}$.

For a less complicated example, take any group $G$. Then the unique $G$-map $EG \to \{\text{pt}\}$ also induces a homotopy equivalence between the Borel constructions. However, here we must drop some smoothness or properness conditions; in other words, at least one of these translation groupoids is not a (finite dimensional) orbifold. In any case, one ought not call the translation groupoids in either example “homotopy equivalent.”

These examples suggest that it is necessary to locate additional fixed point data in topological groupoids. For in the equivariant situation, one would consider the homotopy types of various fixed point sets and easily distinguish the spaces above. Alternatively, (integer graded) stable equivariant homotopy groups also do the trick, given tom Dieck’s isomorphism [17]:

$$\varpi_n^G(X) \cong \bigoplus_{(H)} \varpi_n^{W_G H}(E W_G H_+ \wedge X^H). \quad (2)$$

Here $G$ is supposed to be a compact Lie group acting on the pointed $G$-space $X$, the sum runs over all conjugacy classes of subgroups, $W_G H = N_G H / H$, and $+$ denotes a disjoint $G$-fixed base point. This result demonstrates that these groups depend mainly on fixed point data and isotropy groups, so that one might hope that they are “Morita invariant” in some sense. We make this hope precise by defining intrinsic stable orbifold homotopy groups

$$\varpi_n^{\text{orb}}(\mathfrak{X}) := \varpi_n(B\tilde{G}_+),$$

where $\mathfrak{G}$ is an orbifold groupoid corresponding to $\mathfrak{X}$ and $\tilde{G}$ is the groupoid of fixed point sectors defined in Section 2. We can also define extended unstable orbifold homotopy groups

$$\hat{\pi}_n^{\text{orb}}(\mathfrak{X}, (x, H)) := \pi_n(B\tilde{G}, [x, H])$$

where $(x, H)$ is a base point in the fixed point sectors. The main theorem is then:

Theorem 3.1. The stable orbifold homotopy groups $\varpi_n^{\text{orb}}(\mathfrak{X})$ and extended orbifold homotopy groups $\hat{\pi}_n^{\text{orb}}(\mathfrak{X}, (x, H))$ are orbifold invariants.
We apply this result to prove the following proposition in the case where $\mathfrak{X}$ is a quotient.

**Proposition 3.3.** Let $\mathfrak{X} = M/G$ where $M$ is a smooth manifold and $G$ is a compact Lie group acting smoothly and almost freely. Then the total stable equivariant homotopy group

$$\varpi^G_{\text{tot}}(M_+) := \bigoplus_n \varpi^n(M_+)$$

is an orbifold invariant.

The proof amounts to a calculation identifying the stable equivariant homotopy group with the orbifold stable homotopy group in this special case. This is analogous to the situation with orbifold $K$-theory [1]; there, one sees that $K^*_G(M) \cong K^*_{G'}(M')$ whenever $M/G$ and $M'/G'$ are the same orbifold by identifying each with the intrinsically defined orbifold $K$-theory:

$$K^*_G(M) \xrightarrow{\cong} K^*_\text{orb}(\mathfrak{X}) \xrightarrow{\cong} K^*_{G'}(M').$$

In fact, in the very special case of finite group quotients (so-called global quotients), we have:

**Corollary 3.7.** If $M/G$ and $M'/G'$ are two global quotient presentations of the same orbifold $\mathfrak{X}$, then there are isomorphisms

$$\varpi^n(M_+) \xrightarrow{\cong} \varpi^n(\mathfrak{X}) \xrightarrow{\cong} \varpi^n(M'_+),$$

for each integer $n$.

We emphasize that the stable homotopy calculation requires information about fixed points for all subgroups, whereas the $K$-theory is detected on fixed point sets for group elements (or, equivalently, on cyclic subgroups).

We make our definitions within the smooth category, since that is where our present applications lie. However, the reader will note that most of the definitions go through for well-behaved topological groupoids. The first order of business is to organize the fixed point data of an orbifold using its groupoid presentations. This is accomplished in §2. Fixed point data in hand, we define the novel unstable and stable homotopy groups in §3. After seeing how the new invariants generalize classical ones, we close with some ideas for future directions and applications in §4.

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2. Sectors

For an orbifold groupoid $\mathcal{G}$, we need to understand the fixed point data $\mathcal{G}$ encodes, and then show that it is Morita invariant. Recall that the isotropy group $G_x$ of an object $x \in G_0$ is the group $s^{-1}(x) \cap t^{-1}(x)$ of all arrows that start and end at $x$. Let

$$\tilde{S}(\mathcal{G}) := \{(x, H) \mid x \in G_0, H \subseteq G_x\}$$

be the set of all subgroups of the groupoid $\mathcal{G}$—that is, the set of all the subgroups of all the isotropy groups. The goal is to correctly topologize this set of “fixed points.” The correct topology should specialize to a disjoint union of ordinary fixed point sets in the case of a translation groupoid $\mathcal{G} \ltimes M$. In particular, one expects that for each group $H$, the subset $\tilde{S}^H(\mathcal{G}) := \{(x, K) \mid K \cong H\}$ will be open and closed, so that $\tilde{S}^H(\mathcal{G})$ is a union of connected components. It then suffices to topologize each $\tilde{S}^H(\mathcal{G})$ separately. These fixed point sectors are closely related to the twisted multisectors appearing in Chen-Ruan cohomology and orbifold $K$-theory. In fact, both constructions are rooted in Kawasaki’s earlier work [10].

2.1. Definitions. Assume that $\mathcal{G}$ is an orbifold groupoid; in particular, that $\mathcal{G}$ has all isotropy groups $G_x$ finite. Then for each finite group $H$ of order $k$, we can identify $\tilde{S}^H(\mathcal{G})$ with a subquotient space of the $k$-sectors $\tilde{S}^k(\mathcal{G})$. Recall that elements of $\tilde{S}^k(\mathcal{G})$ are ordered tuples $(g_1, \ldots, g_k)$ of arrows in $\mathcal{G}$ that all begin and end at the same place. More formally, the $k$-sectors are topologized as the fibered product

$$\tilde{S}^k(\mathcal{G}) \to (G_1)^k \to (G_0 \times G_0)^k$$

where $\Delta$ is the diagonal map. Note that $\tilde{S}^k(\mathcal{G})$ is a smooth manifold when $\mathcal{G}$ is a Lie groupoid.

Now let $\tilde{P}^H(\mathcal{G})$ be the subset of those $(g_1, \ldots, g_k) \in \tilde{S}^k(\mathcal{G})$ such that the $g_i$ are all distinct and, as a subset of $G_1$, form a group isomorphic to $H$. The symmetric group $\mathfrak{S}_k$ acts freely on $\tilde{P}^H(\mathcal{G})$ by permuting coordinates, and the quotient is in bijection with $\tilde{S}^H(\mathcal{G})$. Topologize $\tilde{S}^H(\mathcal{G})$ by declaring this to be a homeomorphism. Finally, let

$$\tilde{S}(\mathcal{G}) := \bigsqcup_H \tilde{S}^H(\mathcal{G})$$

(4)
as \( H \) runs over representatives of all isomorphism classes of subgroups of \( \mathcal{G} \). In many cases of interest (including compact orbifolds and global quotient orbifolds), \( \mathcal{G} \) will only have finitely many isomorphism types of subgroups, so that the union is taken over a finite set.

**Remark 2.1.** In fact, \( \tilde{S}(\mathcal{G}) \) has a natural smooth structure. We will see this later in the proper étale case; the extension to the proper foliation case is not difficult.

So, is this topology “correct?” Consideration of a few examples leads us to believe that it is.

**Example 2.2.** Let \( \mathcal{G} = M \) be a manifold viewed as a trivial or unit orbifold groupoid. Then \( \tilde{S}(\mathcal{G}) = \tilde{S}^{(1)}(\mathcal{G}) \) consists of only the trivial sector corresponding to the trivial group \( \langle 1 \rangle \), and \( \tilde{S}^{(1)}(\mathcal{G}) = \tilde{P}^{(1)}(\mathcal{G}) / \mathcal{G} \approx M \).

**Example 2.3.** Let \( \mathcal{G} = G \) be a finite group. Then \( \tilde{S}(\mathcal{G}) \) is the set of all subgroups of \( G \) endowed with the discrete topology.

**Example 2.4.** Both previous examples are special cases of translation groupoids. Let \( G \) be a Lie group acting smoothly and almost freely on the manifold \( U \), so that \( \mathcal{G} = G \ltimes U \) is an orbifold groupoid. Then \( \tilde{S}(\mathcal{G}) = \bigsqcup_{H \subseteq G} U^H \) is the disjoint union of the fixed point sets. To see this, recall that in this situation

\[
\tilde{S}^k(\mathcal{G}) = \bigsqcup_{(g_1, \ldots, g_k) \subseteq G^k} U^{g_1} \cap \cdots \cap U^{g_k}.
\]

So we have

\[
\tilde{P}^H(\mathcal{G}) = \bigcup_{(g_1, \ldots, g_k) = K \subseteq G} U^K,
\]

where \( U^K \) is appearing \(|K|!\) times, and hence

\[
\tilde{S}^H(\mathcal{G}) = \tilde{P}^H(\mathcal{G}) / \mathcal{G}_k = \bigsqcup_{K \subseteq G, K \not\subseteq H} U^K.
\] (5)

Confident in this topology for \( \tilde{S}(\mathcal{G}) \), we want to add some arrows and turn it into a groupoid. After all, the fixed point sets \( U^K \) in the last example come equipped with natural actions of the normalizers \( N_G K \). Similar information may be extracted from more general groupoids. In fact, this is easy; the well-known conjugation action of \( \mathcal{G} \) on the \( k \)-sectors induces an action on \( \tilde{S}^H(\mathcal{G}) \).

**Lemma 2.5.** Let \( \mathcal{G} \) be an orbifold groupoid. Then for each \( H \), the fixed point sector \( \tilde{S}^H(\mathcal{G}) \) is naturally a smooth \( \mathcal{G} \)-space. Thus, the associated translation groupoid, \( \mathcal{G}^H := \mathcal{G} \ltimes \tilde{S}^H(\mathcal{G}) \), is again an orbifold groupoid.
Proof. The action of $G$ on $\tilde{S}^k(G)$ is given as follows: the anchor map $\pi : \tilde{S}^k(G) \to G_0$ sends $(g_1, \ldots, g_k)$ to their common source/target. The action map $\mu : G_1 \times \pi \tilde{S}^k(G) \to \tilde{S}^k(G)$ is given by conjugation: $h(g_1, \ldots, g_k) = (hg_1h^{-1}, \ldots, hg_kh^{-1})$. These maps are smooth, making the $k$-sectors into a smooth $G$-space. So all that remains is to observe that $\pi|_{\tilde{P}H(G)}$ is $\mathfrak{S}_k$-invariant and that the restriction of $\mu$ is equivariant with image in $\tilde{P}H(G)$. □

We write $\tilde{G}$ for $G \ltimes \tilde{S}(G)$, and call it the groupoid of fixed point sectors. Its subgroupoid $\tilde{G}^H$ is called the $H$-fixed sector. We can now complete our earlier translation groupoid example.

Example 2.6. Let $G = G \ltimes U$ as before. Then the identification of Example 2.4 extends to a groupoid isomorphism

$$\tilde{G} \cong G \ltimes \left( \bigsqcup_{K \subseteq G} U^K \right).$$

(6)

Further, one readily sees that the inclusion

$$\bigsqcup_{(K) \subseteq G} N_GK \ltimes U^K \hookrightarrow G \ltimes \left( \bigsqcup_{K \subseteq G} U^K \right)$$

is a weak equivalence\(^1\), where the union on the left now runs over conjugacy classes of subgroups. So, in particular, $\tilde{G} \ltimes U$ is Morita equivalent to $\bigsqcup_{(K) \subseteq G} N_GK \ltimes U^K$. This example will be quite useful when we study the fixed point sectors of proper étale groupoids. ◊

Note that the quotient space $|\tilde{G}|$ is the set of points

$$\{(x, (H)_{G_x}) \mid x \in |G|, H \subseteq G_x\},$$

(7)

where $(H)_{G_x}$ indicates the conjugacy class of $H$ in $G_x$. This recovers Kawasaki’s description.

In considering the fixed point set $M^H$ of a group action, one often disregards the trivial $H$ action and instead focuses on the action of the Weyl group $WH = NH/H$. We can also do this in the groupoid case. Let $\tilde{G}$ be an orbifold groupoid. We consider the following subset $\mathcal{K}_H$ of arrows in $\tilde{G}^H$:

$$\mathcal{K}_H := \{(l, (x, L)) \mid l \in L\}.$$

(8)

Then $\mathcal{K}_H$ forms a wide normal subgroupoid of $\tilde{G}^H$. Recall that wide simply means that a subgroupoid contains all identity arrows, and normal means that conjugates of arrows in the subgroupoid are again in the subgroupoid.

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\(^1\)Moerdijk calls such homomorphisms equivalences, c.f. [13, p. 209]. This could be misleading, since they do not form a symmetric relation.
whenever conjugation makes sense. \( \mathcal{K}_H \) is also \textit{totally intransitive}, in that it consists entirely of isotropy arrows.

Define \( \overline{\mathcal{G}}^H := \widehat{\mathcal{G}}^H / \mathcal{K}_H \), and let \( \overline{\mathcal{G}} := \bigsqcup_H \overline{\mathcal{G}}^H \), where \( H \) runs over isomorphism classes of subgroups as before. These quotient groupoids have the same set of objects as the fixed point sectors, but fewer arrows. In fact, one can check that for \( G = G \ltimes M \), we have \( G \ltimes M \) Morita equivalent to \( \bigsqcup (H) W_G H \ltimes M^H \).

In general, the quotient space \( |\mathcal{G}| \) is homeomorphic to \( |\overline{\mathcal{G}}| \), but has a different orbifold structure (with less isotropy). \( \overline{\mathcal{G}} \) is called the groupoid of \textit{reduced fixed point sectors}.

The fixed point sectors \( \widehat{\mathcal{G}} \) (and the reduced sectors \( \overline{\mathcal{G}} \)) are natural in \( G \), inherit many of its properties, and respect Morita equivalence. A Lie groupoid homomorphism \( \phi : \mathcal{G} \to \mathcal{H} \) is \textit{faithful} if it is faithful as a functor (i.e., injective on Hom sets).

**Lemma 2.7.** The fixed point sector construction \( \widehat{\mathcal{G}} \) and its reduced version \( \overline{\mathcal{G}} \) are both functorial with respect to faithful orbifold groupoid homomorphisms, and both respect Morita equivalences. Moreover, \( \widehat{\mathcal{G}} \) and \( \overline{\mathcal{G}} \) are orbifold groupoids, and are étale if and only if \( \mathcal{G} \) is such a groupoid.

**Proof.** Suppose \( \phi : \mathcal{G} \to \mathcal{H} \) is a homomorphism between two orbifold groupoids. Then \( \phi \) induces a homomorphism \( \phi_* : \widetilde{S}^k(\mathcal{G}) \to \widetilde{S}^k(\mathcal{H}) \) for each \( k \), where

\[
\phi_*(g_1, \ldots, g_k) := (\phi(g_1), \ldots, \phi(g_k)).
\]

If \( \phi \) is faithful, then for each \( H \) the restriction of \( \phi_* \) to \( \widetilde{P}^H(\mathcal{G}) \) lands in \( \widetilde{P}^H(\mathcal{H}) \), and is \( \mathcal{G}_k \)-equivariant. Thus there are induced maps \( \widetilde{\phi}^H : \widetilde{S}^H(\mathcal{G}) \to \widetilde{S}^H(\mathcal{H}) \), with union \( \widetilde{\phi} : \widetilde{S}(\mathcal{G}) \to \widetilde{S}(\mathcal{H}) \). To extend \( \widetilde{\phi} \) to arrows, we just restrict the map \( \phi_1 \times \widetilde{\phi}_0 : G_1 \times \widetilde{S}(\mathcal{G}) \to H_1 \times \widetilde{S}(\mathcal{H}) \) to \( G_1 \times_{\phi} \widetilde{S}(\mathcal{G}) \). The image is automatically in \( H_1 \times_{\phi} \widetilde{S}(\mathcal{H}) \), and it is easy to check that this is a groupoid homomorphism.

The kernel \( \mathcal{K}_H \) is preserved under this homomorphism, so we also obtain an induced homomorphism between the reduced sectors.

Next suppose \( \phi \) is a weak equivalence. We show \( \widetilde{\phi} \) is as well, so that \( \widehat{\mathcal{G}} \) and \( \widehat{\mathcal{H}} \) are Morita equivalent whenver \( \mathcal{G} \) and \( \mathcal{H} \) are. This is actually a special case of a more general situation. If \( E \) is a \( \mathcal{H} \)-space, then the pullback \( \phi^* E := E \times_{\phi} G_0 \) is a \( \mathcal{G} \)-space, and \( \phi \) induces a homomorphism between the associated translation groupoids \( \mathcal{G} \times \phi^* E \) and \( \mathcal{H} \times E \). When \( \phi \) is a weak equivalence, this induced homomorphism is automatically a weak equivalence. So it suffices to show that \( \widetilde{S}(\cdot) \) is \textit{natural}, that is, \( \phi^* \widetilde{S}(\mathcal{H}) \cong \widetilde{S}(\mathcal{G}) \) as \( \mathcal{G} \)-spaces when \( \phi \) is a weak equivalence.
It is enough to check this sector by sector, so we consider the map

\[(\tilde{\phi}_H, \pi) : \tilde{S}^H(G) \to \tilde{S}^H(H) \times_\phi G_0\]

\[\left[ g_1, \ldots, g_k \right] \mapsto \left[ \phi(g_1), \ldots, \phi(g_k) \right], \pi[ g_1, \ldots, g_k ]\].

This is a smooth \(G\)-equivariant bijection, with a smooth \(G\)-equivariant inverse given by

\[\left[ h_1, \ldots, h_k, x \right] \mapsto \left[ \phi^{-1}(h_1), \ldots, \phi^{-1}(h_k) \right],\]

where \(\phi^{-1}(h_i)\) means the unique preimage in \(G_x\). Here we used the fact that \(\phi\) is fully faithful to see uniqueness, and the Cartesian square property in the definition of weak equivalences [13, p. 209] to see that the inverse is smooth. The proof for the reduced sectors is analogous.

Finally, we address the adjectives. First, \(G\) is embedded in \(\tilde{G}\) (and in \(\bar{G}\)) as the trivial sector, so if \(\tilde{G}\) has any of the properties below, they are automatically inherited by \(G\). Conversely:

- If \(G\) is proper, then so is \(\tilde{G}\). It’s enough to show the source map of \(\tilde{G}\) is proper, and that is the map \(\text{pr}_2 : G_1 \times_\pi \tilde{S}(G) \to \tilde{S}(G)\). This is proper because the source map \(s\) of \(G\) is proper, and properness is stable under base change.
- Foliation: the natural homomorphism \(\tilde{G} \to G\) is easily seen to be faithful. Hence all isotropy groups of \(\tilde{G}\) are discrete when those of \(G\) are.
- The étale case is just like properness. Any translation groupoid \(G \rtimes E\) for an étale groupoid \(G\) is étale, since étale maps are preserved under base change.

The proof of these properties for \(\bar{G}\) is straightforward. \(\square\)

**Remark 2.8.** In fact, one expects these fixed point constructions to be functorial with respect to all groupoid homomorphisms. Up to homotopy, this certainly appears to be the case: we can replace \(\tilde{G}\) with a topological category \(\text{Gps}/G\) for which functorality is clear. Here, \(\text{Gps}\) is the category of finite groups and surjective homomorphisms, and \(\text{Gps}/G\) has objects the groupoid homomorphisms \(H \rightarrow G\) for \(H \in \text{Gps}\), with arrows the commutative diagrams

\[
\begin{array}{ccc}
H & \xrightarrow{f} & K \\
\downarrow{a} & & \downarrow{b} \\
\text{im}(a) \subseteq G & \xrightarrow{c_g} & \text{im}(b) \subseteq G,
\end{array}
\]

where \(c_g\) is conjugation by \(g \in G_1\). The natural projection \(\text{Gps}/G \to \tilde{G}\) induces a homotopy equivalence of classifying spaces.
2.2. **The Proper Étale Case.** Proper étale groupoids form a pleasant class of examples, and since every orbifold $\mathcal{X}$ may be presented as such a groupoid, they merit some elaboration. The key feature of proper étale groupoids is that they are locally isomorphic to translation groupoids $G_x \rtimes U_x$ for each $x \in G_0$. The fixed point sector construction commutes with restriction, so in the proper étale case the fixed point sectors $\widetilde{\mathcal{G}}$ are themselves locally isomorphic to $G_x \rtimes U_x$. We have seen in Example 2.4 that this is Morita equivalent to

$$\prod_{(H)} N_{G_x} H \rtimes U_x^H.$$ 

Similarly, $\mathcal{G}$ is locally modelled on

$$\prod_{(H)} W_{G_x} H \rtimes U_x^H.$$ 

These local pictures can be thought of as two atlases of orbifold charts on $|\mathcal{G}|$, endowing it with two smooth orbifold structures. Indeed, one could use the local picture to *define* the topology on the fixed point sectors. However, in nature one often encounters somewhat more general groupoids, and it is worthwhile to have fixed point sectors defined intrinsically for them as well. In fact, this will become crucial for us when we discuss equivariant stable homotopy groups.

2.3. **Finer Structure.** The fixed point sectors have a finer structure worth mentioning. Just as the collection of fixed point sets of a $G$-space come equipped with a web of maps making them into an $\mathcal{O}(G)^{\text{op}}$-space, i.e., a contravariant diagram of spaces over the *orbit category* $\mathcal{O}(G)$, so also do our sectors include into one another\(^2\). The difference in our case is that one must restrict to components over a given base point to ensure that the isotropy groups are actually subconjugate in the groupoid. Also, the inclusions we obtain are in general only defined in the localized category $\mathcal{O}\mathcal{rb}$, rather than on the level of groupoids. We will only discuss the situation for the unreduced sectors $\widetilde{\mathcal{G}}$; very similar results hold for $\mathcal{G}$ as well.

By a *connected component* or *$G$-component* of a groupoid $\mathcal{G}$, we mean the inverse image of a component of $|\mathcal{G}|$ under the quotient map. Suppose that $\mathcal{G}$ is a proper étale groupoid. We wish to understand the connected components of $\widetilde{\mathcal{G}}$. Such a component corresponds to a connected component of the classifying space, since arrows become paths under realization. Consider the following equivalence relation on $\widetilde{S}(\mathcal{G})$. First, suppose $(p, H)$ and $(q, K)$ are elements of $\widetilde{S}(\mathcal{G})$ such that $p$ and $q$ lie in the same equivariantly contractible chart $G_x \rtimes U_x$ about some $x \in G_0$. Then (up to conjugacy) we may regard $H$ and $K$

\(^2\)One might also consider the diagram over the *subgroup category* of $G$, c.f. [11, p. 206].
as subgroups of $G_x$. Write $(p, H) \sim_{\text{loc}} (q, K)$ if and only if $(H)|_{G_x} = (K)|_{G_x}$. For general points in $\tilde{S}(\mathcal{G})$, we let $(p, H) \sim (q, K)$ if there is either a finite chain of $\sim_{\text{loc}}$’s joining them, or else an arrow $g \in G_1$ with $g^{-1}Hg = K$. Equivalent elements $(p, H)$ and $(q, K)$ will be called locally conjugate.

The connected components of $\tilde{\mathcal{G}}$ are exactly these local conjugacy classes. In the case of a global quotient $G \ltimes M$, each sector corresponds to a disjoint union of fixed point sets $G \ltimes \bigsqcup_K M^K$ for $K \cong H$, and each local conjugacy class corresponds to a $G$-component of this disjoint union.

Let $p : \tilde{\mathcal{G}} \to \mathcal{G}$ be the natural projection. For each $x \in G_0$, the fiber $p^{-1}(x) \subseteq \tilde{\mathcal{G}}$ is the discrete groupoid with objects the subgroups of $G_x$, and morphisms given by the conjugation action of $G_x$. For each subgroup $H \subseteq G_x$, let

$$T_H \subseteq \tilde{S}(\mathcal{G})$$

be the connected component of $\tilde{\mathcal{G}}$ containing $(x, H)$; so $T_H$ consists of the points locally conjugate to $(x, H)$.

Now suppose $f : G_x/H \to G_x/K$ is an arrow in $\mathcal{O}(G_x)$. Then there is $g \in G_x$, determined up to conjugation by elements of $K$ on the right, such that $gHg^{-1} \subseteq K$ and $f(sH) = sgK$ for each $s \in G_x$. We define an orbifold morphism $f^# : \mathcal{G}|_{T_K} \to \mathcal{G}|_{T_H}$. For $(x', K') \in T_K$, take a local chart $G_{x'} \ltimes U_{x'}$ for $\mathcal{G}$ at $x'$. Then $\mathcal{G}|_{T_K}$ is locally isomorphic to $G_{x'} \ltimes \bigsqcup_{L \in (K')G_{x'}} U_{x'}^L$ at $(x', K')$. Similarly, $\mathcal{G}|_{T_H}$ is locally isomorphic to $G_{x'} \ltimes \bigsqcup_{N \in (H')G_{x'}} U_{x'}^N$ at $(x', H')$. Moreover, because $(x', K')$ is locally conjugate to $(x, K)$ and $(x', H')$ is locally conjugate to $(x, H)$, we may identify $H'$ with a subgroup of $K'$ up to conjugacy in $G_{x'}$. Fixing such an identification determines a faithful homomorphism

$$G_{x'} \ltimes \bigsqcup_{L \in (K')G_{x'}} U_{x'}^L \to G_{x'} \ltimes \bigsqcup_{N \in (H')G_{x'}} U_{x'}^N.$$ 

Together, these local homomorphisms (defined up to conjugation by an element of $G_{x'}$) determine the desired orbifold morphism $f^#$. As $f$ varies over $\mathcal{O}(G_x)$, one obtains a contravariant functor from $\mathcal{O}(G_x)$ to $\mathsf{Orb}$. Note, however, that all the morphisms $G_x/H \to G_x/K$ in $\mathcal{O}(G_x)$ go to the same orbifold morphism, since morphisms that differ by conjugations in the local groups are identified in the localized category $\mathsf{Orb}$. So the functor factors through the category with objects the conjugacy classes of subgroups of $G_x$, and a single morphism $(H) \to (K)$ if and only if $H$ is subconjugate to $K$.

If $\mathcal{G}$ is a general orbifold groupoid, we can arrange to have a weak equivalence $\epsilon : \mathcal{E} \to \mathcal{G}$ where $\mathcal{E}$ is proper étale. In this case, the subgroupoids generated
by the images of the components $T_H$ give rise to a Morita equivalent $\mathcal{O}(G_x)^{\text{op}}$-orbifold involving components of $\tilde{G}$.

3. Homotopy Groups

Now that we know something about the fixed point data of an orbifold, it is easy to define interesting new invariants. Here, we stick to homotopy groups, but any other (weak) homotopy functor could also be applied.

3.1. Definitions. The $n^{\text{th}}$ stable orbifold homotopy group of an orbifold groupoid $\mathcal{G}$ is

$$\omega_n^{\text{orb}}(\mathcal{G}) := \omega_n(B\tilde{\mathcal{G}}_{+}),$$

where on the right we take the ordinary stable homotopy of the classifying space $B\tilde{\mathcal{G}}$ with a disjoint base point $+$ added.

If $(x, H)$ is a base point in $\tilde{\mathcal{G}}$, the $n^{\text{th}}$ (unstable) extended orbifold homotopy group of $\mathcal{G}$ is

$$\hat{\pi}_n^{\text{orb}}(\mathcal{G}, (x, H)) := \pi_n(B\tilde{\mathcal{G}}, [x, H]),$$

where $[x, H]$ is the point in the classifying space corresponding to $(x, H) \in \tilde{S}(\mathcal{G})$.

When $\mathcal{G}$ is a groupoid presentation of an orbifold $\mathfrak{X}$, we sometimes write $\omega_n^{\text{orb}}(\mathfrak{X})$ for $\omega_n^{\text{orb}}(\mathcal{G})$ and $\hat{\pi}_n^{\text{orb}}(\mathfrak{X}, (x, H))$ for $\hat{\pi}_n^{\text{orb}}(\mathcal{G}, (x, H))$. This is justified because both constructions are Morita invariant: the weak homotopy types of $B\tilde{\mathcal{G}}$ and $B\mathcal{G}$ depend only on the Morita class of $\tilde{\mathcal{G}}$ and $\mathcal{G}$, which in turn are determined by the Morita class of $\mathcal{G}$ by Lemma 2.7. It follows that both invariants descend to the localized category $\text{Orb}$ of orbifolds. We state this important fact as a theorem.

Theorem 3.1. The stable orbifold homotopy groups $\omega_n^{\text{orb}}(\mathfrak{X})$ and extended orbifold homotopy groups $\hat{\pi}_n^{\text{orb}}(\mathfrak{X}, (x, H))$ are orbifold invariants.

Remark 3.2. Regarding the “finer structure” discussed above, we see that $\hat{\pi}_n^{\text{orb}}(\mathcal{G}, (x, -))$ may be viewed as an $\mathcal{O}(G_x)^{\text{op}}$-diagram of groups and (conjugacy classes of) group homomorphisms as $H$ runs though the subgroups of $G_x$.

3.2. Relation to Classical Theories. To give some idea of what these new homotopy groups measure, we can compare them with existing theories for orbifolds and equivariant spaces. The main result is that our new stable homotopy groups include equivariant stable homotopy groups as a special case.

3.2.1. Classical Orbifold Homotopy Theory. The extended orbifold homotopy groups are generalizations of the classical orbifold homotopy groups. The latter appear as the contribution of the untwisted sector $\tilde{\mathcal{G}}^{(1)}$ in the extended groups:

$$\pi_n^{\text{orb}}(\mathcal{G}, x) \cong \hat{\pi}_n^{\text{orb}}(\mathcal{G}, (x, (1))).$$

(11)
As we vary the group $H$ in the base point $(x, H)$, we have seen that we obtain an $O(G_x)^{op}$-orbifold. For a given $H$, the extended homotopy group corresponds to the classical orbifold homotopy group $\pi^\text{orb}_n(\tilde{G}^H, (x, H))$, which will depend up to isomorphism only on the component $T_H$ of the base point $(x, H)$. So in the equivariant situation $\mathcal{G} = G \rtimes M$, we are calculating the ordinary homotopy groups of the Borel construction $ENH \times^N_M M^H$. What’s more, Moerdijk has shown [12] that classes in $\pi^\text{orb}_n(\mathcal{G}, x)$ can be represented by based generalized morphisms $S^n \to \mathcal{G}$, where the sphere $S^n$ is viewed as a unit groupoid. Consequently, elements in $\check{\pi}^\text{orb}_n(\mathcal{G}, (x, H))$ can be represented by pointed generalized morphisms $S^n \to \tilde{G}^H$, which themselves correspond to faithful pointed generalized morphisms $H \times S^n \to \mathcal{G}$. Here, $H$ is acting trivially on $S^n$, so the translation groupoid is really the same thing as the product $H \times S^n$.

It is a remarkable fact in equivariant homotopy theory that a $G$-map is an equivariant homotopy equivalence if and only if it induces homotopy equivalences on all fixed point sets. It would be interesting to see if any similar statement holds for orbifold groupoids.

3.2.2. Equivariant Stable Homotopy Theory. It turns out that in good situations, the stable homotopy groups we have defined for orbifolds really calculate a classical invariant.

**Proposition 3.3.** Let $X = M/G$ where $M$ is a smooth manifold and $G$ is a compact Lie group acting smoothly and almost freely. Then the total stable equivariant homotopy group

$$\varpi^G_{\text{tot}}(M_+) := \bigoplus_n \varpi^G_n(M_+)$$

is an orbifold invariant.

**Proof.** We need a calculational lemma.

**Lemma 3.4.** Let $\mathcal{G}$ be an orbifold groupoid Morita equivalent to $G \rtimes M$, where $G$ is a compact Lie group acting smoothly and almost freely on the manifold $M$. Then

$$\varpi^\text{orb}_n(\mathcal{G}) \cong \bigoplus_{(H)} \varpi^{W_G H}_{n + d(H)}(EW_G H_+ \wedge M^H_+),$$

where $W_G H = N_G H / H$ and $d(H) = \dim_{\mathbb{R}}(W_G H)$. 

Proof of Lemma: Since $\mathcal{G}$ is Morita equivalent to $G \ltimes M$, their fixed point sectors are also Morita equivalent by the Lemma 2.7. Also,

$$B\mathcal{G} \sim B\left(\coprod_{(H)} W_G H \ltimes M^H\right)$$

$$\sim \coprod_{(H)} EW_G H \times_{W_G H} M^H,$$

where $\sim$ denotes weak homotopy equivalence. Thus, we calculate:

$$\varpi_n^{\text{orb}}(\mathcal{G}) \cong \varpi_n\left(\coprod_{(H)} EW_G H \times_{W_G H} M^H\right)$$

$$\cong \bigoplus_{(H)} \varpi_n(EW_G H_+ \wedge_{W_G H} M^H_+)$$

$$\cong \bigoplus_{(H)} \varpi_{n+d(H)}(S^{d(H)} \wedge EW_G H_+ \wedge_{W_G H} M^H_+)$$

$$\cong \bigoplus_{(H)} \varpi_{n+d(H)}^{W_G H}(EW_G H_+ \wedge M^H).$$

The last isomorphism is a Wirthmüller isomorphism, obtained by noting that $EW_G H_+ \wedge M^H_+$ is a free $W_G H$-space. Here, we have regarded $S^{d(H)}$ as the representation sphere corresponding to the adjoint action of the trivial subgroup of $W_G H$ on $T_e W_G H$. \qed

The proposition now follows, since the lemma shows that the total stable equivariant homotopy of $M_+$ is isomorphic to the total orbifold stable homotopy of $\mathcal{G}$, which is an orbifold invariant by Theorem 3.1. \qed

Remark 3.5. The above is the strongest result that could be hoped for while maintaining Morita invariance. The numbers $d(H)$ are manifestly not Morita invariant, for given $G \ltimes M$, the groupoid $(G \times S^1) \ltimes (M \times S^1)$ is Morita equivalent but has all $d(H)$ increased by one.

Example 3.6. Let $\mathcal{X}$ be an effective $n$-dimensional orbifold. Then $\mathcal{X}$ may be presented as a quotient $F/O(n)$, where $F$, the frame bundle of $\mathcal{X}$, is a smooth manifold with an almost free $O(n)$-action. Then

$$\varpi_{n+d(H)}^{O(n)}(F_+) \cong \varpi_{n+d(H)}^{\text{orb}}(\mathcal{X})$$

is an orbifold invariant. \diamond

We can draw another interesting conclusion from the lemma.
**Corollary 3.7.** If $M/G$ and $M'/G'$ are two global quotient presentations of the same orbifold $\mathfrak{X}$, then there are isomorphisms

$$\varpi^G_n(M_+) \xrightarrow{\cong} \varpi^{\text{orb}}_n(\mathfrak{X}) \xrightarrow{\cong} \varpi^{G'}_n(M'_+)$$

for each integer $n$.

**Proof.** In this case, $d(H)$ is zero for every subgroup of $G$ or $G'$. The result follows immediately from tom Dieck's isomorphism (Equation (2)). $\square$

**Example 3.8.** Suppose $\mathfrak{X}$ is a connected global quotient orbifold. Then the orbifold universal cover (c.f. [16]) of $\mathfrak{X}$ is a manifold $Y$. The orbifold fundamental group $\Gamma := \pi_1^{\text{orb}}(\mathfrak{X})$ acts on $Y$ with quotient $\mathfrak{X}$. If $H_1$ and $H_2$ are any two normal subgroups of $\Gamma$ acting freely on $Y$, then $M_1/G_1$ and $M_2/G_2$ are two different global quotient presentations of $\mathfrak{X}$, where $M_i = Y/H_i$ and $G_i = \Gamma/H_i$. In fact, any two such presentations where the $M_i$ are connected arise in this way. The corollary says that $\varpi^{G_1}_n(M_1+) \cong \varpi^{G_2}_n(M_2+)$, which may be confirmed via explicit calculation. $\diamondsuit$

### 4. Further Questions

We have called our new invariants “homotopy groups,” but it is still unclear exactly what is meant by homotopy equivalent orbifold groupoids or orbifolds. Natural transformations between groupoid homomorphisms realize to homotopies of maps between classifying spaces; so our groups have at least one sort of homotopy invariance. However, the interval $I$ can be viewed as a unit groupoid, and so one could also study homotopies of the form $G \times I \rightarrow H$. These realize into maps $B G \times (\Delta^\infty \times I) \rightarrow B H$, whose significance is less clear.

A more comprehensive treatment would involve setting up a model structure on the category $\text{Gpd}$ (or $\text{Orb}$) and studying its relationship to our homotopy group functors. In a future paper, we hope to construct such a model category using techniques related to the homotopy theory of schemes and the universal homotopy theory of Dugger [5], as suggested by Dan Isaksen [8]. The motivation for this approach is as follows: our fixed point sectors can be identified with mapping spaces of (faithful) homomorphisms from finite groups into $\mathfrak{G}$. In broad strokes, looking at morphisms from various groups into the groupoid $\mathfrak{G}$ is analogous to studying various sorts of points in a scheme or stack. Indeed, this should already be a familiar notion for equivariant topologists, given that one often identifies $M^H$ with the mapping space $\text{map}_G(G/H, M)$. The hope is that this abstract approach might shed light on the question at the end of Section 3.2.1: namely, to what extent the classical orbifold homotopy types of the fixed point sectors determine the “homotopy type” of the orbifold in some reasonable model structure? Progress along similar lines appears in the very
recent preprint of Noohi [14] regarding homotopy groups of topological stacks (see also [9], [7]).

**Appendix A. Classifying Spaces of Translation Groupoids**

Let $G$ be a compact Lie group acting on $M$, and let $G = G \ltimes M$ be the translation groupoid. The homotopy equivalence $B\!G \simeq EG \times_G M$ seems to be a folk theorem. We give a quick proof generalizing Segal’s arguments for the group case in [15].

Let $\underline{G}$ be the groupoid with objects $G$ and arrows $G \times G$, so that there is a unique isomorphism $(g_1, g_2) : g_1 \to g_2$ between every two objects. Hence, this category is equivalent to the trivial category with one object and one morphism. So $B\!G$ is a model for $EG$, since the free simplicial $G$-action on the nerve realizes to a free $G$-action on the contractible space $B\!G$. Now consider the product groupoid $G \times M$, where $M$ denotes the unit groupoid on $M$. Since $B$ respects products, $B(G \times M) = B\underline{G} \times BM \simeq EG \times M$.

$G$ acts on the groupoid $G \times M$ by automorphisms as follows. For each object $(g_1, m)$, let $g(g_1, m) = (g_1 g^{-1}, gm)$; for each arrow $(g_1, g_2, m)$, let $g(g_1, g_2, m) = (g_1 g^{-1}, g_2 g^{-1}, gm)$. This action induces a simplicial $G$-action on the nerve, and one obtains the diagonal action on $EG \times M$ upon realization. Moreover, the quotient groupoid under the action is isomorphic to $G \ltimes M$ via the homomorphism sending the object orbit $G(g_1, m)$ to $g_1 m$ and the arrow orbit $G(g_1, g_2, m)$ to $(g_2 g_1^{-1}, g_1 m)$. Consequently, we may identify $EG \times_G M \simeq B(G \times M)/G$ with $B(G \ltimes M)$, up to homotopy.

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