A Model as a Repeated Partnership Game with Discounting

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Abstract

In this paper, we present a model of Partnership Game with respect to the important role of partnership and cooperation in nowadays life. Since such interactions are repeated frequently, we study this model as a Stage Game in the structure of infinitely repeated games with a discount factor $\delta$ and Trigger strategy. We calculate and compare the payoffs of cooperation and violation and as an important result of this study, we show that each partner will adhere to the cooperation.

Keywords: game theory, Partnership Game, repeated game, Trigger strategy
JLE classification:C71, C73

1 Introduction

Since game theory examines situations in which decision-makers interact, this theory has many applications such as firms competing for business, political candidates competing for votes, bidders competing in an auction, animals fighting over prey, the arms race between countries, the relationship between parents and children, using the resources in nature, etc (see [2], [4], [7], [12], [13], and [14]). On the other hand, many of the strategic interactions in which we are involved are repeated interactions with the same people. The relationship
between the worker and the employer is an example of this type. We can use the theory of repeated games to study such behaviors. The main idea in this theory is that a player may be deterred from exploiting her short-term advantage by the threat of punishment that reduces her long-term payoff.

In repeated games with perfect information that each player can observe the strategy used by other players, considering the discount factor \( \delta \), it is possible that Nash equilibria of the repeated game (supergame) is more efficient than the Nash equilibria of the Stage Game, or one-period game. One of the important examples in this area is Cournots oligopoly game, which has been examined as an infinitely repeated game with discount factor \( \delta \) (see [1], [5], [17] and [18]).

There are many activities and projects in which people contribute and the payoffs of those activities are derived from the efforts of each of the partners. Clearly, if any of the partners makes more efforts, more success will be achieved in these activities. But since more efforts by one person are beneficial to other people, they may not have the motivation to work effectively on these projects. In fact, everyone chooses to make less effort and others to do more. With this view, another class of repeated games with imperfect information has examined models as Partnership Games with and without the discount factor \( \delta \) (see [11], [8], [9], [10], [15], [16]).

By getting the idea of Partnership Game in [11] and [16], we have presented a more complete model of participation and considering the role of collaboration in nowadays life and the fact that a collaborative activity can be repeated frequently, we study the proposed model as a Stage Game in the structure of infinitely repeated games with perfect information and the discount factor \( \delta \) between 0 and 1. The results of this research encourage individuals to adhere to collaboration and cooperation, which is one of the most important goals of a social and modern life.

2 Model formulation and basic properties

As a Complete Information Game, we assume that there is a collaborative activity with two partners. The profit of this collaborative project depends on the effort each partner spends on the project and is given by \( \alpha(x_1 + x_2 + c_1(x_1x_2)) \), where \( x_1 \) is the amount of effort spent by partner 1 and \( x_2 \) is the amount of effort spent by partner 2. Assume that \( x_1, x_2 \in [0, \alpha] \). The
value $c_1 \in [0, \frac{2}{\alpha}]$ measures how complementary the efforts of the partners are. We assume the amount of cost each player will incur for this effort is $c_2 x_i^2$, where $c_2 \in [\frac{3}{2}, 2]$. Both players choose their effort independently and simultaneously, and both want to maximize their share of the profit of the project which is equally divided between two players. So the payoff function for partner $i$ is

$$u_i(x_1, x_2) = \alpha \left( \frac{x_1 + x_2}{2} + c_1 \left( \frac{x_1 x_2}{2} \right) \right) - c_2 x_i^2.$$  

### 3 Main results

**Nash equilibrium and the optimal amounts of effort**

Considering $\bar{x}_2$ as average effort, mathematical expectation, of the player 2 based on the belief of player 1, we calculate the Nash equilibrium by finding the best response function of each player

$$\frac{du_1(x_1, \bar{x}_2)}{dx_1} = \frac{\alpha}{2} + \frac{\alpha c_1}{2} \bar{x}_2 - 2c_2 x_1 = 0,$$

$$\frac{d^2 u_1(x_1, \bar{x}_2)}{dx_1^2} = -2c_2 < 0.$$  

Hence Equation (3.1) and second derivative test specify the best response function of player 1 as $x_1 = B_1(\bar{x}_2) = \frac{\alpha}{4c_2} (1 + c_1 \bar{x}_2)$. Similarly the best response function for player 2 is $x_2 = B_2(x_1^*) = \frac{\alpha}{4c_2} (1 + c_1 x_1^*)$. A Nash equilibrium is a pair $(x_1^*, x_2^*)$ for which $x_1^*$ is a best response to $x_2^*$ and $x_2^*$ is a best response to $x_1^*$

$$\begin{cases} 
  x_1^* = B_1(x_2^*) = \frac{\alpha}{4c_2} (1 + c_1 x_2^*) \\
  x_2^* = B_2(x_1^*) = \frac{\alpha}{4c_2} (1 + c_1 x_1^*).
\end{cases}  

(3.2)$$

Solving these two equations, we find that $x_1^* = x_2^* = \frac{\alpha}{4c_2 - \alpha c_1}$.

The payoff of each player in the Nash equilibrium is

$$u_i(x_1^*, x_2^*) = \frac{\alpha^2}{2} \left( \frac{6c_2 - \alpha c_1}{(4c_2 - \alpha c_1)^2} \right).$$  

(3.3)

We would like to calculate the optimal amount of effort as follows

$$u(x_1, x_2) = \alpha (x_1 + x_2) + \alpha c_1 (x_1 x_2) - c_2 (x_1^2 + x_2^2),$$  

3
\[
\begin{align*}
\frac{\partial u}{\partial x_1} &= \alpha + \alpha c_1 x_2 - 2c_2 x_1 = 0 \implies x_1 = \frac{\alpha(1+c_1x_2)}{2c_2}, \\
\frac{\partial u}{\partial x_2} &= \alpha + \alpha c_1 x_1 - 2c_2 x_2 = 0 \implies x_2 = \frac{\alpha(1+c_1x_1)}{2c_2}.
\end{align*}
\]

(3.4)

By solving simultaneously the two equations \(x_1 = \frac{\alpha(1+c_1x_2)}{2c_2}\) and \(x_2 = \frac{\alpha(1+c_1x_1)}{2c_2}\), the result is \(\hat{x}_1 = x_1 = \frac{\alpha}{2c_2 - \alpha c_1}\) and \(\hat{x}_2 = x_2 = \frac{\alpha}{2c_2 - \alpha c_1}\), clearly \(x^*_1 < \hat{x}_1\) and \(x^*_2 < \hat{x}_2\).

On the other hand, according to \(c_1 \in [0, \frac{2}{\alpha}]\) and \(c_2 \in [\frac{3}{2}, 2]\), we have

\[
D = \left| \begin{array}{cc}
-2c_2 & \alpha c_1 \\
\alpha c_1 & -2c_2
\end{array} \right| = 4c_2^3 - \alpha^2 c_1^2 > 0, \quad \frac{\partial^2 u}{\partial x^2_1} = -2c_2 < 0.
\]

(3.5)

So based on the second derivative test, \((\hat{x}_1, \hat{x}_2)\) is a relative maximal point for \(u(x_1, x_2)\). With a simple calculation we have \(u(\hat{x}_1, \hat{x}_2) = \frac{\alpha^2}{2c_2 - \alpha c_1}\). To determine the absolute maximum of the optimal function, we must compare values \(u(0, 0) = 0\), \(u(\alpha, \alpha) = \alpha^2 (2 - 2c_2 + \alpha c_1)\) and \(u(\hat{x}_1, \hat{x}_2) = \frac{\alpha^2}{2c_2 - \alpha c_1}\).

Clearly \(u(0, 0) < u(\alpha, \alpha)\) and \(u(0, 0) < u(\hat{x}_1, \hat{x}_2)\). Considering \(l = 2c_2 - \alpha c_1\), the relation

\[
\frac{\alpha^2}{2c_2 - \alpha c_1} \geq \alpha^2 (2 - 2c_2 + \alpha c_1)
\]

is equivalent to

\[
\frac{1}{l} \geq (2 - l)
\]

that is equivalent to

\[(l - 1)^2 \geq 0,
\]

which is always true. So in this case \(u(x_1, x_2)\) has the absolute maximum in

\[
(\hat{x}_1, \hat{x}_2) = \left( \frac{\alpha}{2c_2 - \alpha c_1}, \frac{\alpha}{2c_2 - \alpha c_1} \right).
\]

Also

\[
u_i(\hat{x}_1, \hat{x}_2) = \frac{\alpha^2}{2(2c_2 - \alpha c_1)} \quad \text{for } i = 1, 2.
\]

(3.6)

As an infinitely repeated game with Crime-Trigger strategy, we consider the Partnership model as a stage game in which each of players has the same discount factor \(\delta\).
Theorem 1: If \( \delta \in \left[ \frac{(4c_2-\alpha c_1)^2}{8c_2(2c_2-\alpha c_1) + (4c_2-\alpha c_1)^2}, 1 \right) \) then the Trigger strategy is a Subgame Perfect Equilibrium, SPE.

Proof. We consider

Stage game G:
Players: Two players \( i = 1, 2 \)
Actions: \( \forall i x_i \in [0, \alpha] \)
Stage Game Payoff:

\[
u_i(x_1, x_2) = \alpha \left( \frac{x_1 + x_2}{2} + c_1 \left( \frac{x_1 x_2}{2} \right) \right) - c_2 x_i^2 \text{ for all players } i.
\]

We consider the Trigger strategy as follows

\[
\forall i S_i(h^t) = \begin{cases} \frac{\alpha}{2c_2-\alpha c_1} & \text{if } t = 1 \\ \frac{\alpha}{2c_2-\alpha c_1} & \text{if } (\hat{x}_1, \hat{x}_2), (\hat{x}_1, \hat{x}_2), (\hat{x}_1, \hat{x}_2), ... \\ \frac{\alpha}{4c_2-\alpha c_1} & \text{if otherwise}, \end{cases}
\]

where \( h^t \) is the history of game up to stage \( t \). The concept of the above strategy is that in the first step, the amount of the effort of each player is \( \frac{\alpha}{2c_2-\alpha c_1} \). If up to step \( t-1 \), each player has selected the amount \( \frac{\alpha}{2c_2-\alpha c_1} \), then the value \( \frac{\alpha}{2c_2-\alpha c_1} \) is similarly chosen in step \( t \), otherwise the value of effort of the Nash equilibrium, \( \frac{\alpha}{4c_2-\alpha c_1} \), will be selected. We assume that the first player adheres to the above strategy. In order to determine the adherence of the second player to the above strategy, we will calculate her benefits from violations and non-violations.

First we presume that both players adhere to the strategy. In this case the sequence of the players’ selective combination will be as follows

\[
\left( \frac{\alpha}{2c_2-\alpha c_1}, \frac{\alpha}{2c_2-\alpha c_1} \right), \left( \frac{\alpha}{2c_2-\alpha c_1}, \frac{\alpha}{2c_2-\alpha c_1} \right), ... .
\]

According to the above sequence, the payoff sequence for the second player is

\[
\frac{\alpha^2}{2(2c_2-\alpha c_1)}, \frac{\alpha^2}{2(2c_2-\alpha c_1)}, ... .
\]

Therefore the present value of the payoffs of the second player is

\[
\frac{\alpha^2}{2(2c_2-\alpha c_1)} + \delta \frac{\alpha^2}{2(2c_2-\alpha c_1)} + \delta^2 \frac{\alpha^2}{2(2c_2-\alpha c_1)} + ... = \frac{\alpha^2}{2(2c_2-\alpha c_1)} \frac{1}{1-\delta}.
\]
Assuming that the first player adheres to the strategy, we would like to calculate the optimal amount of the effort for the second player in case of the violation, \( \max u_2(\frac{\alpha}{2(2c_2-\alpha c_1)}, x_2) \). In this case, we have

\[
u_2\left(\frac{\alpha}{2(2c_2-\alpha c_1)}, x_2\right) = \frac{\alpha}{2} \left(\frac{\alpha}{2(2c_2-\alpha c_1)} + x_2\right) + \frac{\alpha c_1}{2} \left(\frac{\alpha x_2}{2(2c_2-\alpha c_1)}\right) - c_2 x_2^2
\]

\[
\frac{du_2}{dx_2} = \frac{\alpha}{2} + \frac{c_1 \alpha^2}{2(2c_2-\alpha c_1)} - 2c_2 x_2 = 0 \implies x_2 = \frac{\alpha}{2(2c_2-\alpha c_1)}
\]

\[
, \frac{d^2u_2}{dx_2} = -2c_2 < 0.
\]

Therefore \( x_2 = \frac{\alpha}{2(2c_2-\alpha c_1)} \) is a relative maximum for \( u_2(\frac{\alpha}{2(2c_2-\alpha c_1)}, x_2) \).

With a simple comparison between \( u_2(\frac{\alpha}{2(2c_2-\alpha c_1)}, 0) = \frac{\alpha \alpha^2}{4(2c_2-\alpha c_1)} \),

\[
u_2\left(\frac{\alpha}{2(2c_2-\alpha c_1)}, \alpha\right) = \frac{\alpha^2}{2(2c_2-\alpha c_1)} + \frac{c_2 \alpha^2}{(2c_2-\alpha c_1)}(1 - (2c_2-\alpha c_1))
\]

and

\[
u_2\left(\frac{\alpha}{2c_2-\alpha c_1}, \frac{\alpha}{2(2c_2-\alpha c_1)}\right) = \frac{\alpha^2(5c_2 - 2\alpha c_1)}{4(2c_2-\alpha c_1)^2} = \frac{\alpha^2}{2(2c_2-\alpha c_1)} + \frac{c_2 \alpha^2}{4(2c_2-\alpha c_1)^2}
\]

and considering

\[2c_2-\alpha c_1 \geq 1,
\]

it follows that \( x_2 = \frac{\alpha}{2(2c_2-\alpha c_1)} \) is an absolute maximal for \( u_2(\frac{\alpha}{2(2c_2-\alpha c_1)}, x_2) \).

The important point is

\[
u_2(\tilde{x}_1, \frac{\alpha}{2(2c_2-\alpha c_1)}) = u_2(\frac{\alpha}{2(2c_2-\alpha c_1)}, \frac{\alpha}{2(2c_2-\alpha c_1)}) = u_2(\tilde{x}_1, \tilde{x}_2) + \frac{c_2 \alpha^2}{4(2c_2-\alpha c_1)^2}
\]

so \( u_2(\tilde{x}_1, \frac{\alpha}{2(2c_2-\alpha c_1)}) > u_2(\tilde{x}_1, \tilde{x_2}) \) while \( \frac{\alpha}{2(2c_2-\alpha c_1)} < \tilde{x}_2 \).

This means that the second player can achieve more payoff with an effort less than the optimal amount of effort.

Let’s consider the selection sequence of the players in case of the second player’s violation as follows

\[
(\frac{\alpha}{2c_2-\alpha c_1}, \frac{\alpha}{2(2c_2-\alpha c_1)}); (\frac{\alpha}{4c_2-\alpha c_1}, \frac{\alpha}{4c_2-\alpha c_1}); (\frac{\alpha}{4c_2-\alpha c_1}, \frac{\alpha}{4c_2-\alpha c_1}); \ldots
\]
So the sequence of the payoffs of player 2 is as follows
\[ \frac{\alpha^2}{2(2c_2 - \alpha c_1)} + \frac{c_2 \alpha^2}{4(2c_2 - \alpha c_1)^2} + \frac{\alpha^2}{2} \frac{6c_2 - \alpha c_1}{(4c_2 - \alpha c_1)^2}, \frac{\alpha^2}{2} \frac{6c_2 - \alpha c_1}{(4c_2 - \alpha c_1)^2}, \ldots. \]

Therefore in the violation, the present value of the payoffs of the second player is
\[ \frac{\alpha^2}{2(2c_2 - \alpha c_1)} + \frac{c_2 \alpha^2}{4(2c_2 - \alpha c_1)^2} + \delta \left( \frac{\alpha^2}{2} \frac{6c_2 - \alpha c_1}{(4c_2 - \alpha c_1)^2} \right) + \delta^2 \left( \frac{\alpha^2}{2} \frac{6c_2 - \alpha c_1}{(4c_2 - \alpha c_1)^2} \right) + \ldots \]
\[ = \frac{\alpha^2(5c_2 - 2\alpha c_1)}{4(2c_2 - \alpha c_1)^2} + \delta \left( \frac{\alpha^2(6c_2 - \alpha c_1)}{2(4c_2 - \alpha c_1)^2(1 - \delta)} \right). \]

Since player 2 will not violate if her payoff is greater than or at least equal to the non-violation, then we have to have
\[ \frac{\alpha^2}{2(2c_2 - \alpha c_1)} + \frac{1}{1 - \delta} \geq \frac{\alpha^2(5c_2 - 2\alpha c_1)}{4(2c_2 - \alpha c_1)^2} + \delta \left( \frac{\alpha^2(6c_2 - \alpha c_1)}{2(4c_2 - \alpha c_1)^2(1 - \delta)} \right). \]

It is easy to check that the above inequation is equivalent to
\[ \delta \geq \frac{(4c_2 - \alpha c_1)^2}{(4c_2 - \alpha c_1)^2 + 8c_2(2c_2 - \alpha c_1)}. \]

With the same process for player 1, one can show that if \( \delta \in \left( \frac{(4c_2 - \alpha c_1)^2}{8c_2(2c_2 - \alpha c_1) + (4c_2 - \alpha c_1)^2}, 1 \right) \) then according to the Trigger strategy, the players will continue the cooperation. Therefore the Trigger strategy is a SPE, that is, in each subgame the players choose the cooperation, and no player intends to violate because his payoffs reduce in comparison with cooperation. This completes the proof.

**Theorem 2** In the partnership game for all \( \delta \in \left( 0, \frac{(4c_2 - \alpha c_1)^2}{8c_2(2c_2 - \alpha c_1) + (4c_2 - \alpha c_1)^2} \right) \) we can define the Trigger strategy, in which each player’s level of effort, \( \bar{x} \), is greater than \( x^* \) and less than \( \hat{x} \).

Proof. We consider

Stage game G:
Players: Two players \( i = 1, 2 \)
Actions: \( \forall i \; x_i \in [0, \alpha] \)
Stage Game Payoff:
\[ u_i(x_1, x_2) = \alpha \left( \frac{x_1 + x_2}{2} + c_1 \left( \frac{x_1 x_2}{2} \right) \right) - c_2 x_i^2 \text{ for all player } i. \]
Also, we consider the Trigger strategy as follows

\[
\forall i \ S_i(h^t) = \begin{cases} 
\overline{x} & \text{if } t = 1 \\
\frac{\alpha}{(4c_2 - ac_1)} & \text{if otherwise},
\end{cases}
\]

where \(h^t\) is the history of game up stage \(t\). The concept of the above strategy is that in the first step, the amount of the effort of each player is \(\overline{x}\). If up to step \(t - 1\), each player has selected the amount \(\overline{x}\), then the value \(\overline{x}\) is similarly chosen in step \(t\) otherwise the value of effort of the Nash equilibrium, \(\frac{\alpha}{4c_2 - ac_1}\), will be selected.

First we presume that both players adhere to the strategy in which case the sequence of the players’ selective combination will be as follows

\((\overline{x}, \overline{x}), (\overline{x}, \overline{x}), (\overline{x}, \overline{x}), \ldots\).

According to the above sequence, the payoff sequence for player 2 is

\[u_2(\overline{x}, \overline{x}), u_2(\overline{x}, \overline{x}), u_2(\overline{x}, \overline{x}), \ldots\]

where \(u_2(\overline{x}, \overline{x}) = \alpha \overline{x} + \overline{x}^2(\frac{ac_1}{2} - c_2)\).

Therefore the present value of the payoffs of player 2 in case of non-violation is

\[(\alpha \overline{x} + \overline{x}^2(\frac{ac_1}{2} - c_2)) + \delta(\alpha \overline{x} + \overline{x}^2(\frac{ac_1}{2} - c_2)) + \delta^2(\alpha \overline{x} + \overline{x}^2(\frac{ac_1}{2} - c_2)) + \ldots\]

\[= (\alpha \overline{x} + \overline{x}^2(\frac{ac_1}{2} - c_2)) \frac{1}{1 - \delta}.

Assuming that player 1 selects the level of effort \(\overline{x}\) and player 2 intends to violate from \(\overline{x}\), we calculate the optimal amount of effort, \(x_\ast\), that maximizes her payoff

\[u_2(\overline{x}, x_\ast) = \frac{\alpha}{2}(\overline{x} + x_\ast) + \frac{\alpha c_1}{2}(\overline{x}x_\ast) - c_2 x_\ast^2\]

\[
\frac{du_2}{dx_\ast} = \frac{\alpha}{2} + \frac{\alpha c_1}{2} \overline{x} - 2c_2 x_\ast = 0 \implies x_\ast = \frac{\alpha}{4c_2} (1 + c_1 \overline{x})
\]

\[
\frac{d^2u_2}{dx_\ast^2} = -2c_2 < 0.
\]
Therefore, according to the second derivative test, \( x_* = \frac{\alpha}{4c_2}(1 + c_1\overline{x}) \) is a relative maximal point for \( u_2(\overline{x}, x_*) \).

Also since \( c_1 \in [0, \frac{2}{\alpha}] \) and \( c_2 \in [\frac{3}{2}, 2] \) it is easy to get \( x_* = \frac{\alpha}{4c_2}(1 + c_1\overline{x}) \leq \frac{\alpha}{2} \).

To determine the absolute maximum of \( u_2(\overline{x}, x_*) \), we need to compare the values \( u_2(\overline{x}, 0) = \frac{\alpha}{2}(\overline{x} + \alpha(1 + c_1\overline{x}) - 2c_2\alpha) \) and

\[
u_2(\overline{x}, x_*) = \frac{\alpha}{2}(\overline{x} + \alpha(1 + c_1\overline{x}))\]

Clearly always \( u_2(\overline{x}, x_*) > u_2(\overline{x}, 0) \).

On the other hand,

\[
u_2(\overline{x}, x_*) \geq u_2(\overline{x}, \alpha) \iff \frac{\alpha}{2}(\overline{x} + \frac{\alpha}{8c_2}(1 + c_1\overline{x}))^2 \geq \frac{\alpha}{2}(\overline{x} + \alpha(1 + c_1\overline{x}) - 2c_2\alpha) \iff \frac{(1 + c_1\overline{x})}{8c_2} + \frac{2c_2}{(1 + c_1\overline{x})} \geq 1 \iff (4c_2 - (1 + c_1\overline{x}))^2 \geq 0.

Because \( 4c_2 - (1 + c_1\overline{x})^2 \geq 0 \) is always true, then always

\[
u_2(\overline{x}, x_*) \geq u_2(\overline{x}, \alpha).

Since players are always looking for less effort and more payoff, even if \( u_2(\overline{x}, x_*) = u_2(\overline{x}, \alpha) \) then player 2 always chooses less effort, \( x_* = \frac{\alpha}{4c_2}(1 + c_1\overline{x}) \).

In the above argument, \( x_* \) is the optimal amount of effort for player 2 when player 1 selects \( \overline{x} \).

This way, the selection sequence of the players in case of the second player’s violation is

\[(\overline{x}, \frac{\alpha}{4c_2}(1 + c_1\overline{x})), (\alpha \frac{\alpha}{4c_2 - \alpha c_1} \frac{\alpha}{4c_2 - \alpha c_1}), (\alpha \frac{\alpha}{4c_2 - \alpha c_1} \frac{\alpha}{4c_2 - \alpha c_1}), \ldots \]

According to the above sequence, the sequence of the payoffs of player 2 is as follows

\[
\frac{\alpha}{2}(\overline{x} + \frac{\alpha}{8c_2}(1 + c_1\overline{x}))^2, \ \frac{\alpha^2}{2}(\frac{6c_2 - \alpha c_1}{4c_2 - \alpha c_1})^2, \ \frac{\alpha^2}{2}(\frac{6c_2 - \alpha c_1}{4c_2 - \alpha c_1})^2, \ldots.
\]

So in case of violation, the present value of the payoffs of the second player is

\[
\left(\frac{\alpha}{2}(\overline{x} + \frac{\alpha}{8c_2}(1 + c_1\overline{x})), \delta \frac{\alpha^2}{2}(\frac{6c_2 - \alpha c_1}{4c_2 - \alpha c_1})^2, \delta^2 \frac{\alpha^2}{2}(\frac{6c_2 - \alpha c_1}{4c_2 - \alpha c_1})^2, \ldots \right).\]
\[ = \left( \frac{\alpha}{2} (\overline{x} + \frac{\alpha}{8c_2} (1 + c_1\overline{x})) \right) + \frac{\alpha^2}{2} \left( \frac{6c_2 - \alpha c_1}{(4c_2 - \alpha c_1)^2} \right) \left( \frac{\delta}{1 - \delta} \right). \]

Obviously, player 2 will adhere to Trigger strategy if

\[ (\alpha \overline{x} + \overline{x}^2 (\alpha c_1 - c_2)) \frac{1}{1 - \delta} \geq \left( \frac{\alpha}{2} (\overline{x} + \frac{\alpha}{8c_2} (1 + c_1\overline{x})) \right) + \frac{\alpha^2}{2} \left( \frac{6c_2 - \alpha c_1}{(4c_2 - \alpha c_1)^2} \right) \left( \frac{\delta}{1 - \delta} \right). \]

By calculating, it is determined that the above inequality is equivalent to

\[ A = \frac{-1}{16c_2} ((4c_2 - \alpha c_1)^2 - \alpha^2 c_1^2 \delta) = -(8c_2(2c_2 - \alpha c_1) + \alpha^2 c_1^2 (1 - \delta)) < 0, \]

\[ B = \frac{\alpha}{8c_2} ((4c_2 - \alpha c_1) + \delta(4c_2 + \alpha c_1)) > 0 \]

\[ C = -\frac{\alpha^2}{16c_2} (\delta(32c_2^2 - \alpha^2 c_1^2) + 1) < 0. \]

Put \( p(\overline{x}) = A\overline{x}^2 + B\overline{x} + C \), then this equation has \( \sqrt{\Delta} = \frac{2c_2\alpha\delta}{4c_2 - \alpha c_1} \) and its roots are

\[ \overline{x}_1 = \frac{\alpha}{4c_2 - \alpha c_1}, \quad \overline{x}_2 = \frac{\alpha}{4c_2 - \alpha c_1} \frac{(4c_2 - \alpha c_1)^2 - \delta \alpha^2 c_1^2 + 32c_2 \delta c_1^2}{(4c_2 - \alpha c_1)^2 - \delta \alpha^2 c_1^2}. \]

In which \( \overline{x}_1 < \overline{x}_2 \) and \( \overline{x}_1 \) is the Nash equilibrium. Also,

\[ \delta < \frac{(4c_2 - \alpha c_1)^2}{32c_2^3 - 16\alpha c_1 c_2 + \alpha^2 c_1^2} \]

\[ \iff \frac{32\delta c_2^2}{32\delta c_2^2} < \frac{2c_2}{2c_2 - \alpha c_1} \]

\[ \iff 1 + \left( \frac{32\delta c_2^2}{32\delta c_2^2} \right) < 1 + \left( \frac{2c_2}{2c_2 - \alpha c_1} \right) = \frac{4c_2 - \alpha c_1}{2c_2 - \alpha c_1} \]

\[ \iff \frac{\alpha}{4c_2 - \alpha c_1} \frac{(4c_2 - \alpha c_1)^2 - \delta \alpha^2 c_1^2 + 32c_2 \delta c_1^2}{(4c_2 - \alpha c_1)^2 - \delta \alpha^2 c_1^2} < \frac{2c_2}{2c_2 - \alpha c_1} \]

So, according to the above calculations \( \frac{\alpha}{4c_2 - \alpha c_1} < \overline{x}_2 < \frac{\alpha}{2c_2 - \alpha c_1} \).

By specifying the sign \( p(\overline{x}) \), it follows that \( p(\overline{x}) \) is always nonnegative between two roots. On the other hand, if \( x_1 = x_2 = \overline{x} \) then the calculations indicate that \( u_i(\overline{x}, \overline{x}) = \alpha \overline{x} + (\overline{x})^2 (\alpha c_1 - c_2) \) has two roots of 0 and \( \frac{2\alpha}{2c_2 - \alpha c_1} \) and it is maximal in \( \frac{2\alpha}{2c_2 - \alpha c_1} \). So, if \( \frac{4c_2 - \alpha c_1}{2c_2 - \alpha c_1} < \overline{x} < \frac{\alpha}{2c_2 - \alpha c_1} \) then \( u_i(\overline{x}, \overline{x}) \) is an increasing function. So, if players choose the level of more effort, their payoffs will be greater. Therefore, the purpose of solving \( p(\overline{x}) \geq 0 \) is the largest \( \overline{x} \) for which \( p(\overline{x}) \geq 0 \). Hence the highest value of \( \overline{x} \) is \( \overline{x}_2 \).
Corollary 1. We have \( \lim_{\delta \to 0} x = \lim_{\delta \to 0} \frac{\alpha}{4c_2 - \alpha c_1} \frac{(4c_2 - \alpha c_1)^2 - 4\alpha c_1^2 + 32\delta c_1^2}{(4c_2 - \alpha c_1)^2 - \delta c_1^2} = \frac{\alpha}{4c_2 - \alpha c_1} \cdot \frac{(4c_2 - \alpha c_1)^2}{8c_2(4c_2 - \alpha c_1) + (4c_2 - \alpha c_1)^2} \cdot \frac{32\delta c_1^2}{(4c_2 - \alpha c_1)^2} \cdot \frac{\alpha}{4c_2 - \alpha c_1} = \frac{\alpha}{4c_2 - \alpha c_1}. \)

Then above calculations and Theorem 2 imply that \( x \) is the Nash equilibrium level for each player whenever \( \delta \to 0 \), and each player will choose \( \hat{x} \) for the level of effort if \( \delta \to \frac{8c_2(4c_2 - \alpha c_1) + (4c_2 - \alpha c_1)^2}{32c_2^2 - 16\alpha c_1 c_2} \).

Corollary 2 In the partnership game, considering the Trigger strategy, for each \( \delta \in (0, 1) \) the level of effort in an infinitely repeated game is determined.

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