SOLVABILITY AND COMPLEX LIMIT BICHARACTERISTICS

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ABSTRACT. We shall study the solvability of pseudodifferential operators which are not of principal type. The operator will have complex principal symbol satisfying condition (Ψ) and we shall consider the limits of semibicharacteristics at the set where the principal symbol vanishes of at least second order. The convergence shall be as smooth curves, and we shall assume that the normalized complex Hamilton vector field of the principal symbol over the semicharacteristics converges to a real vector field. Also, we shall assume that the linearization of the real part of the normalized Hamilton vector field at the semibicharacteristic is tangent to and bounded on the tangent space of a Lagrangean submanifold at the semibicharacteristics, which we call a grazing Lagrangean space. Under these conditions one can invariantly define the imaginary part of the subprincipal symbol. If the quotient of the imaginary part of the subprincipal symbol with the norm of the Hamilton vector field switches sign from $-\to +$ on the bicharacteristics and becomes unbounded as they converge to the limit, then the operator is not solvable at the limit bicharacteristic.

1. Introduction

We shall consider the solvability for a classical pseudodifferential operator $P$ on a $C^\infty$ manifold $X$ which is not of principal type. $P$ is solvable at a compact set $K \subseteq X$ if the equation

\begin{equation}
Pu = v
\end{equation}

has a local solution $u \in \mathcal{D}'(X)$ in a neighborhood of $K$ for any $v \in C^\infty(X)$ in a set of finite codimension.

The pseudodifferential operator $P$ is classical if it has an asymptotic expansion $p_m + p_{m-1} + \ldots$ where $p_k$ is homogeneous of degree $k$ in $\xi$ and $p_m = \sigma(P)$ is the principal symbol of the operator. $P$ is of principal type if the Hamilton vector field

\begin{equation}
H_p = \sum_{j=1}^n \partial_{\xi_j} p \partial_{x_j} - \partial_{x_j} p \partial_{\xi_j}
\end{equation}

of the principal symbol $p = p_m$ does not have the radial direction $\langle \xi, \partial \xi \rangle$ at $p^{-1}(0)$, in particular $H_p \neq 0$ then. By homogeneity $H_p$ is well defined on the cosphere bundle $S^*X = \{ (x, \xi) \in T^*X : |\xi| = 1 \}$, defined by some choice of Riemannian metric, and the principal type condition means that $H_p$ is not degenerate on $S^*X$. For pseudodifferential operators of principal type, it is known from [1] and [3] that local solvability is equivalent...
to condition \((\Psi)\):

\[
\text{Im}(ap) \text{ does not change sign from } - \text{ to } +
\]

along the oriented bicharacteristics of \(\text{Re}(ap)\)

for any \(0 \neq a \in C^\infty(T^*M)\). This condition is of course trivial if the principal symbol is real valued. The oriented bicharacteristics are the positive flow-outs of the Hamilton vector field \(H_{\text{Re}(ap)} \neq 0\) on \(\text{Re}(ap) = 0\), and these are called semibicharacteristics of \(p\).

We shall consider the case when \(P\) is not of principal type, instead the complex valued principal symbol vanishes of at least second order at the double characteristics \(\Sigma_2\). We shall study necessary conditions for solvability when \(\Sigma_2\) is an involutive manifold, and since solvability is an open condition we shall assume that \(P\) satisfies condition \((\Psi)\) in the complement of \(\Sigma_2\) where it is of principal type. Naturally, condition \((\Psi)\) is empty on \(\Sigma_2\), where instead we shall have necessary conditions on the next lower term \(p_{m-1}\), called the \textit{subprincipal symbol}. The sum of the principal symbol and subprincipal symbol is called the \textit{refined principal symbol}.

Mendoza and Uhlman \[5\] studied the case when principal symbol \(p\) is a product of two real symbols having transversal Hamilton vector fields at the involutive intersection \(\Sigma_2\) of the characteristics. They proved that \(P\) is not solvable if the subprincipal symbol changes sign on the integral curves of these Hamilton vector fields on \(\Sigma_2\), which are the limits of the bicharacteristics at \(\Sigma_2\). Mendoza \[6\] generalized this to the case when the principal symbol is real and vanishes of second order at an involutive manifold \(\Sigma_2\) having an indefinite Hessian with rank equal to the codimension of the manifold. The Hessian then gives well-defined limit bicharacteristics over \(\Sigma_2\), and \(P\) is not solvable if the subprincipal symbol changes sign on any of these limit bicharacteristics. Since \(\Sigma_2\) is involutive, the limits of the bicharacteristics are tangent to the symplectic foliation of \(\Sigma_2\), see Example \[2,6\]. Thus, both \[5\] and \[6\] have constant sign of the subprincipal symbol on the limit characteristics as a necessary condition for solvability, which corresponds to condition \((P)\) on the refined principal symbol. This is natural since when the principal symbol vanishes of exactly second order one gets both directions on the limit bicharacteristics.

These results were generalized in \[2\] to pseudodifferential operators with real principal symbol for which the linearization of the Hamilton vector field is tangent to and has uniform bounds on the tangent spaces of some Lagrangean manifolds at the bicharacteristics. Then \(P\) is not solvable if condition \((\Psi)\) is not satisfied on the limit bicharacteristics, in the sense that the imaginary part of the subprincipal symbol switches sign from \(-\) to \(+\) on the semibicharacteristics when converging to the limit semibicharacteristic. The paper \[7\] studied operators of subprincipal type, where the principal symbol vanishes of at least second order at a nonradial involutive manifold \(\Sigma_2\) and the subprincipal symbol
is of principal type with Hamilton vector field tangent to $\Sigma_2$ at the characteristics, but transversal to the symplectic foliation of $\Sigma_2$. Then the operator was not solvable if the subprincipal symbol is constant on the symplectic leaves of $\Sigma_2$ after multiplication with a nonvanishing factor and does not satisfy condition ($\Psi$) on $\Sigma_2$. In fact, if the principal symbol is proportional to a real symbol, then the result of [2] gives nonsolvability generically when the subprincipal symbol is not constant on the leaves.

In this paper, we shall extend the results of [2] to pseudodifferential operators with complex principal symbols. We shall consider the limits of semibicharacteristics at the set $\Sigma_2$ where the principal symbol vanishes of at least second order. The convergence shall be as smooth curves, then the limit semibicharacteristic also is a smooth curve. We shall assume that the normalized complex Hamilton vector field of the principal symbol on the semicharacteristics converges to a real vector field on $\Sigma_2$. Then the limit semibicharacteristic are uniquely defined, and one can invariantly define the imaginary part of the subprincipal symbol. Also, we shall assume that the linearization of the real part of the normalized Hamilton vector field is tangent to and uniformly bounded on the tangent space of a Lagrangean submanifold at the semibicharacteristics, which we call a grazing Lagrangean space, see (2.8). We shall also assume uniform bounds on linearization of the imaginary part of the Hamilton vector field on the grazing Lagrangean space, see (2.11), (2.13) and Definition 2.3.

Our main result is Theorem 2.11 which essentially says that under these conditions the operator is not solvable at the limit semibicharacteristic if the quotient of the imaginary part of the subprincipal symbol with the norm of the Hamilton vector field switches sign from $-$ to $+$ on the semibicharacteristics and becomes unbounded as they converge to the limit semibicharacteristic, see (2.20). Thus a non-homogeneous version of condition ($\Psi$) on the refined principal symbol does not hold on the limit characteristics. This result implies the results of [2], [5] and [6].

2. Statement of results

Let $p$ be the principal symbol, $\Sigma = p^{-1}(0)$ be the characteristics, and $\Sigma_2$ be the set of double characteristics, i.e., the points on $\Sigma$ where $dp = 0$. Since we are going to study necessary conditions for solvability, we shall assume that $P$ satisfies condition ($\Psi$) given by (1.3) on $\Sigma_1 = \Sigma \setminus \Sigma_2$. We shall study limits at $\Sigma_2$ of semibicharacteristics, and we shall assume that the normalized limit of $H_p$ is proportional to a real vector field, in the sense that

\begin{equation}
|dp \wedge dp| \ll |dp| \quad \text{on } \Gamma_j \text{ as } j \to \infty
\end{equation}

We shall only use semibicharacteristics given by $H_{Re a p}$ such that $|Re a \nabla p| \geq c|\nabla p|$ at $\Gamma_j$ for some $c > 0$, where $\nabla p$ is the gradient of $p$. Let $\{ \Gamma_j \}_{j=1}^\infty$ be a set of semibicharacteristics
of \( p \) on \( S^*X \cap \Sigma_1 \) so that \( \Gamma_j \) are bicharacteristics of \( \text{Re} \, a_j p \) where \( 0 \neq a_j \in C^\infty \) uniformly at \( \Gamma_j \) and

\[
\text{(2.2)} \quad |\text{Re} \, a_j \nabla p| \geq c |\nabla p| \quad \text{at } \Gamma_j
\]

for some fixed \( c > 0 \), observe that \( p = 0 \) on \( \Gamma_j \). We shall assume that \( \Gamma_j \) are uniformly bounded in \( C^\infty \) when parametrized on a uniformly bounded interval (for example with respect to the arc length). The bounds are defined with respect to some choice of Riemannian metric on \( S^*X \), but different choices of metric will only change the constants. In particular, we have a uniform bound on the arc lengths:

\[
\text{(2.3)} \quad |\Gamma_j| \leq C \quad \forall \, j
\]

In fact, we have that \( \Gamma_j = \{ \gamma_j(t) : t \in I_j \} \) with \( |\gamma_j'(t)| \equiv 1 \) and \( |I_j| \leq C \), then \( |\gamma_j^{(k)}(t)| \leq C_k \) for \( t \in I_j \) and \( \forall \, j, k \geq 1 \). Let the normalized gradient \( \tilde{p} = p/|\nabla p| \) and the normalized Hamilton vector field

\[
H_{\tilde{p}} = |H_p|^{-1} H_p \quad \text{on } p^{-1}(0) \setminus \Sigma_2
\]

Then \( \Gamma_j \) is uniformly bounded in \( C^\infty \) if there exists positive constants \( c \) and \( C_k \) such that

\[
\text{(2.4)} \quad |H_{\text{Re} \, a_j \tilde{p}} \nabla \text{Re} \, a_j \tilde{p}| \leq C_k \quad \text{and} \quad |H_{\text{Re} \, a_j \tilde{p}}| \geq c \quad \text{at } \Gamma_j \quad \forall \, j, k
\]

which implies that \( |a_j| \geq c > 0 \) at \( \Gamma_j \). This means that the normalized Hamilton vector field \( H_{\text{Re} \, a_j \tilde{p}} \) is uniformly bounded in \( C^\infty \) as a non-degenerate vector field over \( \Gamma \), and this only depends on \( a_j|_{\Gamma_j} \). Observe that the semibicharacteristics have a natural orientation given by the Hamilton vector field. Now the set of semibicharacteristic curves \( \{ \Gamma_j \}_{j=1}^\infty \) is uniformly bounded in \( C^\infty \) when parametrized with respect to the arc length, and therefore it is a precompact set. Thus there exists a subsequence \( \Gamma_{j_k}, \, k \to \infty \), that converge to a smooth curve \( \Gamma \) (possibly a point), called a limit semibicharacteristic by the following definition, which generalizes the definition in [2].

**Definition 2.1.** We say that a sequence of smooth curves \( \Gamma_j \) on a smooth manifold converges to a smooth limit curve \( \Gamma \) (possibly a point) if there exist parametrizations on uniformly bounded intervals that converge in \( C^\infty \). If \( p \in C^\infty(T^*X) \), then we say that \( \{ \Gamma_j \}_{j=1}^\infty \) are a uniform family of semibicharacteristics of \( p \) if (2.3) and (2.4) hold. A smooth curve \( \Gamma \subset \Sigma_2 \cap S^*X \) is a limit semibicharacteristic of \( p \) if there exists a uniform family of semibicharacteristics of \( p \) that converge to it.

Naturally, this definition is invariant under symplectic changes of coordinates, and the set \( \{ \Gamma_j \}_{j=1}^\infty \) may have subsequences converging to several different limit semibicharacteristics, which could be points. For example, if \( \Gamma_j \) is parametrized with respect to the arc length on intervals \( I_j \) such that \( |I_j| \to 0 \), then we find that \( \Gamma_j \) converges to a limit curve which is a point. Observe that if \( \Gamma_j \) converge to a limit semibicharacteristic \( \Gamma \), then (2.3) and (2.4) must hold for \( \Gamma_j \).
Example 2.2. Let $\Gamma_j$ be the curve parametrized by

$$[0, 1] \ni t \mapsto \gamma_j(t) = (t, \cos(jt)/j, \sin(jt)/j)/\sqrt{2}$$

Since $|\gamma_j'(t)| = 1$, the curves are parametrized with respect to arc length, and we have that $\Gamma_j \rightarrow \Gamma = \{(t, 0, 0) : t \in [0, 2^{-1/2}]\}$ in $C^0$, but not in $C^\infty$ since $|\gamma_j''(t)| = j/\sqrt{2}$. If we parametrize $\Gamma_j$ with $x = jt \in [0, j]$ we find that $\Gamma_j$ converge to $\Gamma$ in $C^\infty$ but not on uniformly bounded intervals.

But we shall also need a condition on the differential of the Hamilton vector field $H_p$ at the semibicharacteristic $\Gamma$ along a Lagrangean space, which will give bounds on the curvature of the semicharacteristics in these directions. If the semicharacteristics is the bicharacteristic of $\text{Re} \ a p$ then we shall denote $\Sigma = (\text{Re} \ ap)^{-1}(0)$ and $T_w \Sigma = \ker d \text{Re} \ ap(w) \subset T(T^*X)$, where $d \text{Re} \ ap(w) \neq 0$ for $w \in \Gamma$. A section of Lagrangean spaces $L$ over a bicharacteristic $\Gamma$ is a map

$$\Gamma \ni w \mapsto L(w) \subset T_w(T^*X)$$

such that $L(w)$ is a Lagrangean space in $T_w \Sigma, \forall w \in \Gamma$. If the section $L$ is $C^1$ then it has tangent space $T L \subset T_L(T_T(T^*X))$. Observe that since $L(w) \subset T_w \Sigma$ is Lagrangean we find $d \text{Re} \ ap(w)|_{L(w)} = 0$ and $H_{\text{Re} \ ap}(w) \in L(w)$ when $w \in \Gamma$. Now we shall also have the condition that the linearization of $H_{\text{Re} \ ap}$ at $\Gamma$ is tangent to the Lagrangean space $L$.

**Definition 2.3.** Let $\Gamma$ be a semibicharacteristic of $p$, i.e., a bicharacteristic of $\text{Re}(ap)$ for some $0 \neq a \in C^\infty$. We say that a $C^1$ section of Lagrangean spaces $L$ over $\Gamma$ is a section of grazing Lagrangean spaces of $\Gamma$ if $L \subset T_\Gamma \Sigma = \ker d \text{Re} \ ap|_{\Gamma} \subset T_\Gamma(T^*X)$, and the linearization (or first order jet) of $H_{\text{Re} \ ap}$ in $T_\Gamma L$, the tangent space of $L$ at $\Gamma$.

The linearization of $H_{\text{Re} \ ap}(w)$ is given by the second order Taylor expansion of $\text{Re} \ ap$ at $w$ and since $L(w)$ is Lagrangean we find that terms in that expansion that vanish on $L(w)$ have Hamilton field parallel to $L$. Thus, the condition that the linearization of $H_{\text{Re} \ ap}(w)$ is in $T L(w)$ only depends on the restriction to $L(w)$ of the second order Taylor expansion of $\text{Re} \ ap$ at $w$. We find that Definition 2.3 is invariant under multiplication of $\text{Re} \ ap$ by non-vanishing real factors because $\text{Re} \ ap(w) = 0$ and $d \text{Re} \ ap(w)|_{L(w)} = 0$ since $L \subset T_\Gamma \Sigma$. Thus the linearization of $H_{\text{Re} \ ap}$ is determined by $\text{Hess} \text{Re} \ cap(w)|_{L(w)} - c \text{Hess} \text{Re} \ ap(w)|_{L(w)}$ when $c$ is real. Thus the linearization only depends on the argument of $a_j$ at $\Gamma_j$ so we can replace $H_{\text{Re} \ ap}(w)$ by $H_{\text{Re} \ ap}$ in the definition.

By Definition 2.3 we find that the linearization of $H_{\text{Re} \ ap}$ gives an evolution equation for the section $L$, see Example 2.4. Choosing a Lagrangean subspace of $T_{w_0} \Sigma$ at $w_0 \in \Gamma$ then determines $L$ along $\Gamma$, so $L$ must be smooth. Actually, $L$ is the tangent space at $\Gamma$ of a smooth Lagrangean submanifold of $(\text{Re} \ ap)^{-1}(0)$, see [3.30].
Example 2.4. Let \( p = \tau + ia(t, x)\xi_1 - (\langle A(t, x)x, x \rangle + 2\langle B(t, x)x, \xi \rangle + \langle C(t, x)x, \xi \rangle) / 2 \), \( (x, \xi) \in T^*\mathbb{R}^n \), where \( a(t, x) \in C^\infty \) is real valued, \( A(t, x), B(t, x) \) and \( C(t, x) \) are \( n \times n \) matrices, such that \( A(t, x) = A^t(t, x) \) and \( C(t) = C^t(t, x) \) are symmetric, and let \( \Gamma = \{ (t, 0, 0, \xi_0) : t \in I \} \). Then \( H_{Re_p} = \partial_t \) at \( \Gamma \) and

\[
(\text{Re} p)^{-1}(0) = \{ \tau = (\text{Re} A(t, x)x, x) / 2 + (\text{Re} B(t, x)x, \xi) + (\text{Re} C(t, x), \xi) / 2 \}
\]

where \( \text{Re} F \) is the given by the real part of the elements of \( F \). The linearization of the Hamilton field \( H_p \) at \( (t, 0, 0, \xi_0) \) is

\[
(2.5) \quad \partial_t + ia(t, 0)\partial_x + \langle A(t, 0)y + B^t(t, 0)\eta, \partial_\eta \rangle - \langle B(t, 0)y + C(t, 0)\eta, \partial_\eta \rangle
\]

with \( (y, \eta) \in T(T^*\mathbb{R}^n) \). Since \( d\text{Re} p = d\tau \) at \( \Gamma \), a \( C^1 \) section of Lagrangean spaces \( L(t) \subset T_\Gamma \Sigma \) must be tangent to \( \Gamma \). Thus, by choosing linear symplectic coordinates \( (y, \eta) \) we may obtain that

\[
L(t) = \{ (s, y, 0, E(t)y) : (s, y) \in \mathbb{R}^n \}
\]

where \( E(t) \in C^1 \) is real and symmetric with \( E(0) = 0 \). By applying (2.5) on \( \eta - E(t)y \), which vanishes on \( L(t) \), we obtain that \( L(t) \) is a grazing Lagrangean space if

\[
(2.6) \quad \partial_t E(t) = \text{Re} A(t, 0) + \text{Re} B(t, 0) E(t) + E(t) \text{Re} B^t(t) + E(t) \text{Re} C(t, 0) E(t)
\]

Then by uniqueness we find that \( L(t) \) is constant in \( t \) if and only if \( \text{Re} A(t, 0) \equiv 0 \), and then \( A(t, 0) = \text{Hess} p|_{L(t)} \). In general, the real part of \( \text{Hess} p|_{L(t)} \) is given by the right hand side of (2.6).

Example 2.5. If \( p \) is of principal type, then one can choose \( a \neq 0 \) and symplectic coordinates so that \( \text{Re} ap = \tau \) near \( \Gamma = \{ (t, 0, 0, \xi_0) : t \in I \} \). Then one can take any Lagrangean plane in \( \text{Ker} d\tau|_\Gamma = T_\Gamma \Sigma \) which is tangent to \( \Gamma \).

Observe that we may choose symplectic coordinates \( (t, x; \tau, \xi) \) so that \( \tau = \text{Re} ap \) and the fiber of \( L(w) \) is equal to \( \{ (s, y, 0, 0) : (s, y) \in \mathbb{R}^n \} \) at \( w \in \Gamma = \{ (t, 0, 0, \xi_0) : t \in I \} \). But it is not clear that we can do that uniformly for a family of semibicharacteristics \( \{ \Gamma_j \} \), for that we need additional conditions. We shall assume that there exists a grazing Lagrangean space \( L_j \) of \( \Gamma_j \), \( \forall j \), such that the normalized Hamilton vector field \( H_{\tilde{p}} \) satisfies

\[
(2.7) \quad \left| dH_{\tilde{p}}(w)|_{L_j(w)} \right| \leq C \quad \forall w \in \Gamma_j
\]

This is equivalent to

\[
(2.8) \quad \left| dH_p(w)|_{L_j(w)} \right| \leq C|H_p|
\]

for \( w \in \Gamma_j \) since \( L \subset T_\Gamma \Sigma \). In fact, we have that \( dH_{bp} = dbH_p + bdH_p + dpH_b \) on \( \Sigma \). Since the mapping \( \Gamma_j \ni w \mapsto L_j(w) \) is determined by the linearization of \( H_{\text{Re} a_j \tilde{p}} \) on \( L_j \), thus by \( dH_{\text{Re} a_j \tilde{p}}(w)|_{L_j(w)} \), condition (2.7) implies that \( \Gamma_j \ni w \mapsto L_j(w) \) is uniformly in \( C^1 \), see Example 2.4. Observe that condition (2.4) gives (2.7) in the direction of \( T_w \Gamma_j \subset \)
Clearly condition (2.7) is invariant under changes of symplectic coordinates and multiplications with non-vanishing real factors. In general, we only have $dH_p = \mathcal{O}(|H_p|^{-1})$ since $dH_p = \mathcal{O}(1)$, and by induction we find $\partial^\alpha H_p = \mathcal{O}(|H_p|^{-|\alpha|})$, see Proposition 3.1.

Observe that condition (2.7) gives

$$
(2.9) \quad \left| d\nabla \Re a_j \tilde{p}(w) \right|_{L_j(w)} \leq C \quad \text{for } w \in \Gamma_j \quad \forall \ j
$$

Since $\nabla \Re a_j \tilde{p}$ is uniformly proportional to the normal of the level surface $(\Re a_j p)^{-1}(0)$, condition (2.9) gives a uniform bound on the curvature of the level surface $(\Re a_j p)^{-1}(0)$ in the directions given by $L_j$ over $\Gamma_j$.

**Example 2.6.** Assume that $p(x, \xi)$ vanishes of exactly order $k \geq 2$ at the involutive submanifold $\Sigma_2 = \{ \xi' = 0 \}$, $\xi = (\xi', \xi'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, such that the localization

$$
\eta \mapsto \sum_{|\alpha| = k} \partial^\alpha_x p(x, 0, \xi'') \eta^\alpha
$$

is of principal type when $\eta \neq 0$. Then the semibicharacteristics of $p$ with $\left| \Re a_j \nabla \tilde{p} \right| \cong 1$ satisfies (2.4) and (2.7) with $L_j = \{ \xi = 0 \}$ at any point. In fact, $\left| \partial_{x'^j} p(x, \xi) \right| \cong |\xi'|^{k-1}$ and $\partial_{x'\xi''} p(x, \xi) = \mathcal{O}(|\xi'|^k)$ so $H_p = \partial_{x'\xi''} \partial_{x'} + \mathcal{O}(|\xi'|)$ and $\partial^\alpha_x \nabla \tilde{p} = \mathcal{O}(|\xi'''|^{k-1})$, $\forall \alpha$, when $|\xi'| \ll 1$ and $|\xi| \cong 1$.

Now for a uniform family of semibicharacteristics $\{ \Gamma_j \}$ we shall denote

$$
(2.10) \quad 0 < \min_{\Gamma_j} |H_p| = \kappa_j \rightarrow 0 \quad j \rightarrow \infty
$$

and we shall assume that

$$
(2.11) \quad \left| dp \wedge d\tilde{p} \right| \leq C \kappa_j^{14/3} |H_p|^2 \quad \text{at } \Gamma_j
$$

which by Leibniz’ rule means that $\left| \Re a_j \wedge d \Im \tilde{p} \right| \leq C \kappa_j^{14/3}$ on $\Gamma_j$. In fact, we have

$$
(2.12) \quad d(a p) \wedge d(\bar{a} \tilde{p}) = |a|^2 dp \wedge d\tilde{p} + 2i \Im(a \bar{p} dp \wedge d \bar{a}) + |p|^2 da \wedge d \bar{a}
$$

where the two last terms vanish on $\Sigma$. This gives a measure on the complex part of $H_p$ and gives that $H_p$ is proportional to a real vector field on $\Gamma_j$ modulo terms that are $\mathcal{O}(\kappa_j^{14/3})$.

With $L_j$ as in (2.7) we shall assume the following condition

$$
(2.13) \quad \left| d \right|_{L_j} (dp \wedge d\tilde{p})(w) \leq C \kappa_j^{4/3} |H_p|^2 \quad \text{for } w \in \Gamma_j \quad \forall \ j
$$

where the outer differential is restricted to $L_j$ on $\Gamma_j$. Observe that condition (2.13) gives an estimate on the variation of the complex part of the Hamilton vector field along $L$, whereas condition (2.7) gives an estimate on the variation of the Hamilton vector field. Using (2.8), (2.11) and (2.12) we find that (2.13) is equivalent to

$$
(2.14) \quad \left| d \right|_{L_j} (d \Re \tilde{p} \wedge d \Im \tilde{p})(w) \leq C \kappa_j^{4/3} \quad \text{for } w \in \Gamma_j \quad \forall \ j
$$
In fact, the differential of the two last terms in (2.12) vanish since $dp = 0$ on $L_j$ and if $a = |∇p|^{-1}$ then $da|_{L_j} = O(a)$ by (2.8).

If $|∇ Re \tilde{p}| \cong |∇ \tilde{p}| = 1$, then we find from (2.11) that

$$|d Im \tilde{p}(w)| \leq C\kappa_j^{14/3} \quad \text{on Ker} \ d Re \tilde{p}(w)$$

for $w \in \Gamma_j$. Since $d|_{L_j} d Re \tilde{p}(w) = O(1)$ by (2.7), we find from (2.14) that

$$d|_{L_j} d Im \tilde{p}(w) = O(\kappa_j^{4/3}) \quad \text{on Ker} \ d Re \tilde{p}(w)$$

when $w \in \Gamma_j$. The estimates (2.15) and (2.16) will be needed in order to handle the imaginary part of the principal symbol as a perturbation, see Lemmas 5.1 and 5.2.

Now, since the semibicharacteristics $\Gamma_j$ are uniform we have $|H_{Re a_j \tilde{p}}| \geq c$, which by (2.11) gives

$$Im(a_j \nabla \tilde{p}) = \beta_j Re(a_j \nabla \tilde{p}) + V_j \quad \text{at} \ \gamma_j$$

where $\beta_j = O(1)$ and $|V_j| \leq C\kappa_j^{14/3}$. The first part of the right hand side will not change the direction of $\Gamma_j$. Thus multiplying $\tilde{p}$ with the complex factor $1 - i\beta_j$ only changes the direction of the real part of the Hamilton vector field by terms that are $O(\kappa_j^{14/3})$. This only perturbs $\Gamma_j$ so that the distance to the original semibicharacteristic is $O(\kappa_j^{14/3})$. Now the derivative of the linearization of the Hamilton vector field is $O(|H_p|^{-2}) = O(\kappa_j^{-2})$, see Proposition 3.1. Thus, the linearization is changed with a bounded factor and terms that are $O(\kappa_j^{8/3})$. Thus, we find from (2.6) that the grazing Lagrangean spaces $L_j$ are only changed by terms that are $O(\kappa_j^{8/3})$. Since $\kappa_j \leq |H_p|$ on $\Gamma_j$ we find that conditions (2.8), (2.11) and (2.13) are not changed. Observe that $a_j$ is only defined on $\Gamma_j$, but since $\Gamma_j$ is a uniformly bounded smooth curve, $a_j$ can easily be uniformly extended to a neighborhood of $\Gamma_j$.

**Remark 2.7.** The family of uniform semibicharacteristics $\{ \Gamma_j \}$ satisfying condition (2.11) and the grazing Lagrangean spaces $L_j$ of $\Gamma_j$ are invariant modulo perturbations of $O(\kappa_j^{14/3})$ under different choices of $a_j$ in (2.4). Thus conditions (2.8), (2.11) and (2.13) are well defined.

Thus, the choice of $a_j$ will be irrelevant when taking the limit. Now, we shall only consider semibicharacteristics $\Gamma_j$ with tangent vectors $H_{Re a_j \tilde{p}}$ so that

$$|H_{Im a_j \tilde{p}}| \leq C\kappa_j^{14/3} \quad \text{and} \ |a_j| > 1/C \quad \text{on} \ \Gamma_j$$

which implies that $|Re \nabla a_j \tilde{p}| \geq c > 0$ when $\kappa_j \ll 1$. Then the multipliers $a_j$ are well defined on $\Gamma_j$ modulo uniformly bounded factors which have argument that are $O(\kappa_j^{14/3})$.

The invariant subprincipal symbol $p_s$ will be important for the solvability of the operator near $\Sigma_2$. For the usual Kohn-Nirenberg quantization of pseudodifferential operators, the
next lower order term is equal to

\begin{equation}
(2.19) \quad p_s = p_{m-1} - \frac{1}{2i} \sum_j \partial_x \partial_j \eta, p
\end{equation}

and for the Weyl quantization it is \(p_{m-1}\). Both of these are equal to \(p_{m-1}\) at the involutive manifold \(\Sigma_2 = \{ \xi' = 0 \}\) since then \(\partial_x p \equiv 0\) at \(\Sigma_2\).

For the subprincipal symbol \(p_s\) we shall have a condition that essentially means that condition \((\Psi)\) does not hold for the subprincipal symbol. Observe that if \((2.18)\) holds then the imaginary part of \(a_j p_s\) is well defined modulo terms that are \(O(\kappa_j^{1/3})\). Assuming \((2.18)\) we shall as in [2] assume that

\begin{equation}
(2.20) \quad \min_{\partial \Gamma_j} \int \text{Im} a_j p_s |H_p|^{-1} ds / |\log \kappa_j| \to \infty \quad j \to \infty
\end{equation}

where the integration is along the natural orientation given by \(H_{\text{Re} a_j p}\) on \(\Gamma_j\) starting at \(w_j \in \tilde{\Gamma}_j\). (Actually, it suffices that the minimum in \((2.20)\) is sufficiently large, depending on the norms of the symbol of the operator.) Since \(|H_p| \geq \kappa_j \to 0\) on \(\Gamma_j\), we find that condition \((2.20)\) is well defined independently of the choice of multiplier \(a_j\) satisfying \((2.18)\).

Observe that if \((2.20)\) holds then there must be a change of sign of \(\text{Im} a_j p_s\) from \(-\) to \(+\) on \(\Gamma_j\), and

\begin{equation}
(2.21) \quad \max_{\Gamma_j} (1)^{\pm 1} \text{Im} a_j p_s / |H_p| / |\log \kappa_j| \to \infty \quad j \to \infty
\end{equation}

for both signs. Observe that condition \((2.20)\) for \(a_j\) satisfying \((2.18)\) is invariant under symplectic changes of coordinates and multiplication with elliptic pseudodifferential operators, thus under conjugation with elliptic Fourier integral operators. In fact, multiplication only changes the subprincipal symbol with uniform non-vanishing factors and terms proportional to \(|\nabla p| = |H_p|\). By multiplying with \(a_j\) we may for simplicity assume that \(a_j \equiv 1\). Then by choosing symplectic coordinates \((t, x; \tau, \xi)\) near a given point \(w_0 \in \Gamma_j\) so that \(\text{Re} p = \alpha \tau\) near \(w_0\) with \(\alpha = |\text{Re} \nabla p| \neq 0\), we obtain that \(\partial_{x_k} \partial_{x_h} \text{Re} p = 0\) at \(\Gamma_j\), \(\forall k\), and \(\partial_{t_k} \partial_{\tau} \text{Re} p = \partial_{t_k} \alpha = \partial_{\tau} |\text{Re} \nabla p|\) at \(\Gamma_j\) near \(w_0\). Thus, the second term in \((2.19)\) only gives terms which are either real or gives terms in condition \((2.20)\) which are bounded by

\begin{equation}
(2.22) \quad \left| \int \partial_t |\text{Re} \nabla p| / |\nabla \text{Re} p| ds / |\log (\kappa_j)| \right| = O\left( |\log (|\nabla \text{Re} p|)| / |\log (\kappa_j)| \right) = O(1)
\end{equation}

when \(j \gg 1\) since \(|\text{Re} \nabla p| \approx |\nabla p| \geq \kappa_j \to 0\) on \(\Gamma_j\) by \((2.18)\). Thus we obtain the following result.

Remark 2.8. We may replace the subprincipal symbol \(p_s\) by \(p_{m-1}\) in \((2.20)\), since the difference is bounded as \(j \to \infty\).
One can define the reduced principal symbol as \( p + p_s \), see Definition 18.1.33 in [4]. Then (2.20) means that a non-homogeneous version of condition (\( \Psi \)) does not hold for the reduced principal symbol.

**Example 2.9.** If \( p \) is real and vanishes of exactly order \( k \geq 2 \) at an involutive manifold \( \Sigma_2 \), then we find that \( |H_p| \equiv d^{k-1} \) on \( S^*X \) where \( d \) is the homogeneous distance to \( \Sigma_2 \). If \( \text{Im} p_s \) changes sign from \( - \) to \( + \) on the semibicharacteristics and vanishes of order \( \ell \) at \( \Sigma_2 \), then (2.20) holds if and only if \( \ell < k - 1 \). When \( k = 2 \) this means that \( \text{Im} p_s \) changes sign from \( - \) to \( + \) on the limit bicharacteristic, as in the results of [5] and [6].

We shall study the microlocal solvability, which is given by the following definition. Recall that \( H^{loc}_{(s)}(X) \) is the set of distributions that are locally in the \( L^2 \) Sobolev space \( H_{(s)}(X) \).

**Definition 2.10.** If \( K \subset S^*X \) is a compact set, then we say that \( P \) is microlocally solvable at \( K \) if there exists an integer \( N \) so that for every \( f \in H^{loc}_{(N)}(X) \) there exists \( u \in \mathcal{D}'(X) \) such that \( K \cap \text{WF}(Pu - f) = \emptyset \).

Observe that solvability at a compact set \( M \subset X \) is equivalent to solvability at \( S^*X \mid M \) by [4, Theorem 26.4.2], and that solvability at a set implies solvability at a subset. Also, by Proposition 26.4.4 in [4] the microlocal solvability is invariant under conjugation by elliptic Fourier integral operators and multiplication by elliptic pseudodifferential operators. The following is the main result of the paper.

**Theorem 2.11.** Let \( P \in \Psi^m_{cl}(X) \) have principal symbol \( \sigma(P) = p \) satisfying condition (\( \Psi \)), and subprincipal symbol \( p_s \). Let \( \Gamma_j \subset S^*X, j = 1, \ldots \) be a uniform family of semibicharacteristics of \( p \) so that (2.8), (2.11), (2.13) and (2.20) hold for some \( a_j \) satisfying (2.18) and grazing Lagrangean spaces \( L_j \) of \( \Gamma \). Then \( P \) is not microlocally solvable at any limit semibicharacteristics of \( \{ \Gamma_j \}_j \).

In fact, if there exists a limit semibicharacteristic, then we can choose a subsequence of semibicharacteristics \( \Gamma_j \) converging to it, which gives conditions (2.13) and (2.23) for these \( \Gamma_j, \forall j \). Observe that if the principal symbol is real, then conditions (\( \Psi \)), (2.11) and (2.13) are trivially satified, and we obtain Theorem 2.9 in [2].

To prove Theorem 2.11 we shall use the following result. Let \( \|u\|_{(\nu)} \) be the \( L^2 \) Sobolev norm of order \( k \) for \( u \in C_0^\infty \) and \( P^* \) the \( L^2 \) adjoint of \( P \).

**Remark 2.12.** If \( P \) is microlocally solvable at \( \Gamma \subset S^*X \), then Lemma 26.4.5 in [4] gives that for any \( Y \subset X \) such that \( \Gamma \subset S^*Y \) there exists an integer \( \nu \) and a pseudodifferential operator \( A \) so that \( \text{WF}(A) \cap \Gamma = \emptyset \) and

\[
(2.23) \quad \|u\|_{(-N)} \leq C(\|P^*u\|_{(\nu)} + \|u\|_{(-N-\nu)} + \|Au\|_{(0)}) \quad u \in C_0^\infty(Y)
\]
We shall use Remark 2.12 to prove Theorem 2.11 in Section 6 by constructing approximate local solutions to \( P^* u = 0 \). We shall first prepare and get a microlocal normal form for the adjoint operator, which will be done in Section 3. We shall then apply \( P^* \) to an oscillatory solution, for which we shall solve the eikonal equation in Section 4 and the transport equations in Section 5.

3. The normal form

In the following we assume that the conditions in Theorem 2.11 holds with some limit semibicharacteristic, observe that then (2.3) and (2.4) hold for \( \Gamma_j \). We shall prepare the operator to a normal form as in [2], but since the principal symbol now is complex valued the preparation will be slightly different. First we shall put the adjoint operator \( P^* \) on a normal form uniformly and microlocally near the semibicharacteristics \( \Gamma_j \subset \Sigma \cap S^*X \) converging in \( C^\infty \) to \( \Gamma \subset \Sigma_2 \). This will present some difficulties since we only have conditions at the semibicharacteristics. By the invariance, we may multiply with an elliptic operator so that the order of \( P^* \) is \( m = 1 \) and \( P^* \) has the symbol expansion \( p + p_0 + \ldots \), where \( p \) is the principal symbol. By Remark 2.8 we may assume that \( p_0 \) is the subprincipal symbol, and as before we shall assume (2.18) so that \( |\text{Re} \nabla p| \cong |\nabla p| \).

Observe that \( p = 0 \) on \( \Gamma_j \) and for the adjoint the signs in (2.20) are reversed, changing it to

\[
\max_{\partial \Gamma_j} \int \text{Im} a_j p_0 |H_p|^{-1} ds / |\log \kappa_j| \to -\infty \quad j \to \infty
\]

where \( \kappa_j \) given by (2.10). Changing the starting point \( w_j \) of the integration to the maximum of the integral in (3.1) only improves the estimate so we may assume that

\[
\int \text{Im} a_j p_0 / |H_p| ds \leq 0 \quad \text{on } \Gamma_j
\]

with equality at \( w_j \in \Gamma_j \). Since \( \nabla p_0 \) and \( \nabla H_p \) are bounded on \( S^*X \) and \( |H_p| \geq \kappa_j \) on \( \Gamma_j \), we find that \( |H_p| \) and \( p_0 / |H_p| \) only change with a fixed factor and a bounded term on an interval of length \( \lesssim \kappa_j \) on \( \Gamma_j \). Thus, we find that integrating \( \text{Im} a_j p_0 / |H_p| \) over such intervals only gives bounded terms. Therefore, by (2.21) we may assume that

\[
|\Gamma_j| \gg \kappa_j
\]

and that condition (3.1) holds on some intervals of length \( \cong \kappa_j \) at the endpoints of \( \Gamma_j \).

Now we choose

\[
1 \leq \lambda_j = \kappa_j^{-1/\varepsilon} \leftrightarrow \kappa_j = \lambda_j^{-\varepsilon}
\]
for some $0 < \varepsilon \leq 1$ to be determined later. Then we may replace $|\log \kappa_j|$ with $\log \lambda_j$ in (3.1). By choosing a subsequence and renumbering, we may assume by (2.20) that

$$\max_{\partial \Gamma_j} \int \Im a_j p_0/|H_p| \, ds \leq -j \log \lambda_j$$

and that this also holds on some intervals of length $\cong \kappa_j$ at the endpoints of $\Gamma_j$. Next, we introduce the normalized principal and subprincipal symbols

$$\tilde{\p} = p/|H_p| \quad \text{and} \quad \tilde{p}_0 = p_0/|H_p|$$

Then we have that $H_{\tilde{p}}|_{\Gamma_j} \in C^\infty$ uniformly for the grazing Lagrangean space $L_j$ of $\Gamma_j$, $|H_{\tilde{p}}| = 1$ on $\Gamma_j$ and $dH_{\tilde{p}}|_{L_j}$ is uniformly bounded at $\Gamma_j$ by (2.4) and (2.7). We find that condition (3.5) becomes

$$\max_{\partial \Gamma_j} \int \Im a_j \tilde{p}_0 \, ds \leq -j \log \lambda_j$$

Observe that because of condition (2.21) we have that $\partial \Gamma_j$ has two components since $\Im a_j \tilde{p}_0$ has opposite sign there, thus $\Gamma_j$ is a uniformly embedded curve.

In the following we shall consider a fixed semibicharacteristic $\Gamma_j \subset \Sigma \cap S^*X$ and suppress the index $j$, so that $a = a_j$, $\Gamma = \Gamma_j$, $L = L_j$ and $\kappa = \lambda^{-\varepsilon} = \kappa_j$ for some $\varepsilon > 0$ to be determined later. Observe that the preparation will be uniform in $j$ with $\lambda$ as parameter, assuming the conditions in Theorem 2.11. Now $H_{\Re a\tilde{p}} \in C^\infty$ uniformly on $\Gamma$ but not in a neighborhood. By (2.4) we may define the first order Taylor expansion of $\Re a\tilde{p}$ at $\Gamma$ uniformly. Since $\Gamma \in C^\infty$ uniformly, we can choose local uniform coordinates so that $\Gamma = \{ (t, 0) ) : t \in I \subset \mathbb{R} \}$ locally. In fact, we can take a local parametrization $\gamma(t)$ of $\Gamma$ with respect to the arc length and choose the orthogonal space $M \subset \mathbb{R}^{n-1}$ to the tangent vector of $\Gamma$ at a point $w_0$ with respect to some local Riemannian metric. Then $\mathbb{R} \times M \ni (t, w) \mapsto \gamma(t) + w$ is uniformly bounded in $C^\infty$ with a uniformly bounded inverse near $(t_0, 0)$ giving local coordinates near $\Gamma = \{ (t, 0) : t \in I \}$. We may then complete $t$ to a uniform symplectic coordinate system. Multiplying with the uniformly bounded function $a(t, 0)$ we may assume that $a(t, 0) \equiv 1$. We can define the first order Taylor term of $\Re \tilde{p}$ at $\Gamma$ by

$$\rho(t, w) = \partial_w \Re \tilde{p}(t, 0) \cdot w \quad w = (x, \tau, \xi)$$

which is uniformly bounded. This can be done locally, and by using a uniformly bounded partition of unity we obtain this in a fixed neighborhood of $\Gamma$. Going back to the original coordinates, we find that $\rho \in C^\infty$ uniformly near $\Gamma$ and $\Re \tilde{p} - \rho = O(d^2)$, but the error is not uniformly bounded. Here $d$ is the homogeneous distance to $\Gamma$, i.e., the distance with respect to the homogeneous metric

$$dt^2 + |dx|^2 + (d\tau^2 + |d\xi|^2)/((\tau, \xi))^2$$
But by condition \((2.7)\) we find that the second order derivatives of \(\tilde{p}\) along the Lagrangean space \(L\) at \(\Gamma\) are uniformly bounded. We shall use homogeneous coordinates, i.e., local coordinates which are normalized with respect to the homogeneous metric \((3.9)\).

By completing \(\tau = q\) in \((3.8)\) to a uniformly bounded homogeneous symplectic coordinate system \((\tau, w) = (\tau, x, \tau, \xi)\) near \(\Gamma\) and conjugating with the corresponding uniformly bounded Fourier integral operator we may assume that
\[
\Gamma = \{ (t, 0; 0, \xi_0) : t \in I \} \subset S^* \mathbb{R}^n
\]
for \(|\xi_0| = 1\) and some bounded interval \(I \ni 0\), and that \(\text{Re} \tilde{p} \cong \tau\) modulo second order terms at \(\Gamma\). The second order terms are not uniformly bounded, but \(d\nabla \tilde{p}|_L\) is uniformly bounded at \(\Gamma\) by \((2.7)\). Since \(d \text{Re} \tilde{p} = d\tau\) on \(\Gamma\) we find that \(H_{\text{Re} \tilde{p}}|\Gamma = D_t\) and since \(L \subset (dp)^{-1}(0)\) we may obtain that \(L = \{ (t, x; 0, 0) \} \) at any given point at \(\Gamma\) by choosing suitable linear symplectic coordinates \((x, \xi)\). We find from \((2.7)\) that
\[
|d\nabla \tilde{p}(t, 0; 0, \xi_0)| \lesssim 1 \quad t \in I
\]
Condition \((2.15)\) gives
\[
|\partial_{t,x,\xi} \text{Im} \tilde{p}(t, 0; 0, \xi_0)| \lesssim \kappa^{14/3} = \lambda^{-14\varepsilon/3} \quad t \in I
\]
and condition \((2.16)\) gives
\[
|d \partial_{t,x,\xi} \text{Im} \tilde{p}(t, 0; 0, \xi_0)| \lesssim \lambda^{-4\varepsilon/3} \quad t \in I
\]
Here \(a \lesssim b\) (and \(b \gtrsim a\)) means that \(a \leq Cb\) for some \(C > 0\).

Let
\[
q(t, w) = |\nabla p(t, w)| \geq \lambda^{-\varepsilon} \quad \text{at } \Gamma
\]
and extend \(q\) so that it is homogeneous of degree 0, then \(q\) is the norm of the homogeneous gradient of \(p\). Recall that \(\lambda \gg 1\) is a parameter that depends on the bicharacteristic \(\Gamma\). Since the symbols are homogeneous, we shall restrict them to \(S^* \mathbb{R}^n\). There we shall choose coordinates \((t, w)\) so that \(w = 0\) on \(\Gamma\), and then localize in conical neighborhoods depending on the parameter \(\lambda\). We have \(|\nabla \tilde{p}| \equiv 1\) at \(\Gamma\), higher derivatives are not uniformly bounded but can be handled by the using the metric
\[
g_\varepsilon = (dt^2 + |dw|^2)\lambda^{2\varepsilon} \quad w = (x, \tau, \xi)
\]
and the symbol classes \(f \in S(m, g_\varepsilon)\) defined by \(\partial^\alpha f = \mathcal{O}(m\lambda^{|\alpha|\varepsilon}), \forall \alpha\).

**Proposition 3.1.** If \((3.10)\) and \((3.14)\) hold then \(q\) is a weight for \(g_\varepsilon, q \in S(q, g_\varepsilon)\) and \(\tilde{p}(t, w) \in S(\lambda^{-\varepsilon}, g_\varepsilon)\) when \(|w| \leq c\lambda^{-\varepsilon}\) for some \(c > 0\) on \(S^* \mathbb{R}^n\) when \(t \in I\).

This gives \(p = q\tilde{p} \in S(q\lambda^{-\varepsilon}, g_\varepsilon)\) when \(|w| \leq c\lambda^{-\varepsilon}\). Observe that \(b \in S^{1}_{1-\varepsilon,\varepsilon}\) if and only if \(b \in S(\lambda^\varepsilon, g_\varepsilon)\) in homogeneous coordinates when \(|\xi| \approx \lambda \gtrsim 1\). In fact, in homogeneous
coordinates $z$ this means that $\partial^\nu_b = O(\|\xi\|^{\nu+|\alpha|\varepsilon})$. Therefore, we obtain by homogeneity that $\tilde{p} \in S_{1-\varepsilon}^1$ and $q^{-1} \in S_{1-\varepsilon}^1$ when $|w| \lesssim \lambda^{-\varepsilon} \equiv |\xi|^{-\varepsilon} \lesssim 1$.

**Proof.** We are going to use the previously chosen coordinates $(t, w)$ on $S^*\mathbb{R}^n$ so that $\Gamma = \{ (t, 0) : t \in I \}$. Now $\partial^2 p = O(1)$, $q \geq \lambda^{-\varepsilon}$ at $\Gamma$ by (3.14) and

$$\partial q = \text{Re} \nabla \mathfrak{p} \cdot (\partial \nabla p)/q \quad \text{when } q \neq 0$$

which is uniformly bounded. We find that $q(s, w) \equiv q(t, 0)$ when $|s - t| + |w| \leq c\lambda^{-\varepsilon}$ for small enough $c > 0$, so $q$ is a weight for $g_\varepsilon$ there. This gives that $|p(t, w)| \lesssim q(t, w)\lambda^{-\varepsilon}$, $|\nabla p(t, w)| = q(t, w)$ and $|\partial^\alpha p| \lesssim 1 \lesssim q^{\varepsilon} \lesssim q^{\lambda(\varepsilon-1)}$ for $|\alpha| \geq 2$, which gives $p \in S(q\lambda^{-\varepsilon}, g_\varepsilon)$ when $|w| \leq c\lambda^{-\varepsilon}$ and $t \in I$.

We find from (3.16) that $\partial q = \alpha/q$ where $\alpha \in S(q^2\lambda^\varepsilon, g_\varepsilon)$ when $|w| \leq c\lambda^{-\varepsilon}$ since $\nabla p \in S(q, g_\varepsilon)$ in this domain. By induction over the order of differentiation of $q$ we obtain from (3.16) that $q \in S(q, g_\varepsilon)$ when $|w| \leq c\lambda^{-\varepsilon}$, which gives the result.

As before, we take the restriction of $\tilde{p}$ to $|\xi| = 1$, use local coordinates $(t, w)$ on $S^*\mathbb{R}^n$ so that (3.10) holds with $\xi_0 = 0$ and put $Q(t, w) = \lambda^{\varepsilon}\tilde{p}(t\lambda^{-\varepsilon}, w\lambda^{-\varepsilon})$ when $t \in I_\varepsilon = \{ t\lambda^\varepsilon : t \in I \}$. Recall that $\lambda \gg 1$ is fixed, depending on $\Gamma$. Then by Proposition 3.1 we find that $Q \in C^\infty$ uniformly when $|w| \lesssim 1$ and $t \in I_\varepsilon$, $\partial_{t, \xi} \text{Re} Q \equiv 0$ and $|\partial_{t, x, \xi} \text{Re} Q| \equiv 0$ when $w = 0$ and $t \in I_\varepsilon$. Thus we find $|\partial_{t, x} Q| \neq 0$ for $|w| \lesssim 1$ and $t \in I_\varepsilon$. By using Taylor’s formula at $\Gamma$ we can write $Q(t, x; \tau, \xi) = \tau + h(t, x; \tau, \xi)$ when $|w| \lesssim 1$ and $t \in I_\varepsilon$, where $h = |\nabla \text{Re} h| = 0$ at $w = 0$. By using the Malgrange preparation theorem, we obtain

$$\tau = a(t, w)(\tau + h(t, w)) + s(t, x, \xi) \quad |w| \lesssim 1 \quad t \in I_\varepsilon$$

where $a$ and $s \in C^\infty$ uniformly, $a \neq 0$, and on $\Gamma$ we have $a = 1$ and $s = |\nabla \text{Re} s| = 0$. In fact, this can be done uniformly, first locally in $t$ and then by a uniform partition of unity for $t \in I_\varepsilon$. This gives

$$a(t, w)Q(t, w) = \tau - s(t, x, \xi) \quad |w| \lesssim 1 \quad t \in I_\varepsilon$$

In the original coordinates, we find that

$$\lambda^{\varepsilon}\tilde{p}(t, w) = a^{-1}(t\lambda^{\varepsilon}, w\lambda^{\varepsilon})(\tau\lambda^{\varepsilon} - s(t\lambda^{\varepsilon}, x\lambda^{\varepsilon}, \xi\lambda^{\varepsilon}))$$

and thus

$$\tilde{p}(t, w) = b(t, w)(\tau - r(t, x, \xi)) \quad |w| \lesssim \lambda^{-\varepsilon} \quad t \in I$$

where $0 \neq b \in S(1, g_\varepsilon)$, $r(t, x, \xi) = \lambda^{-\varepsilon}s(t\lambda^{\varepsilon}, x\lambda^{\varepsilon}, \xi\lambda^{\varepsilon}) \in S(\lambda^{-\varepsilon}, g_\varepsilon)$ when $|w| \lesssim \lambda^{-\varepsilon}$, and $t \in I$, $b = 1$ and $r = |\nabla \text{Re} r| = 0$ on $\Gamma$. By condition (3.11) we find that

$$|d\nabla r|_{L_1} \leq C \quad \text{at } \Gamma$$

since $r$ is constant in $\tau$. Similarly, by conditions (3.12) and (3.13) we find that

$$|\nabla \text{Im} r| \lesssim \lambda^{-14\varepsilon/3} \quad \text{at } \Gamma$$
and

\begin{equation}
(3.21) \quad |d\nabla \text{Im } r| \leq C\lambda^{-4\varepsilon/3} \quad \text{at } \Gamma
\end{equation}

Extending by homogeneity, we obtain this preparation where the homogeneous distance in \((x, \xi)\) to \(\Gamma\) is \(\lesssim \lambda^{-\varepsilon}\), then \((3.19)\)–\((3.21)\) hold with the homogeneous gradient. Now, the symbol \(b\) is homogeneous but it is not in \(S^0_{0,0}\) uniformly, instead it will have uniform bounds in a larger symbol class. In the following, we shall denote by \(\Gamma\) the rays in \(T^*\mathbb{R}\) that goes through the semibicharacteristic. Recall that \(\tilde{\rho} = p/q\), where \(q \in S(q, g_\varepsilon)\) when \(|w| \lesssim \lambda^{-\varepsilon}\) and is homogeneous of degree 0. By homogeneity we obtain from \((3.18)\) that

\[ b^{-1}q^{-1}p(t, x; \tau, \xi) = \tau - r(t, x, \xi) \]

where \(b^{-1} \in S^0_{1-\varepsilon, \varepsilon}\), \(q^{-1} \in S^0_{1-\varepsilon, \varepsilon}\) and \(\tau - r \in S^0_{1-\varepsilon, \varepsilon}\) when \(|\xi| \gtrsim \lambda\) and the homogeneous distance \(d(x, \xi)\) to \((0, \xi_0)\) is less than \(c|\xi|^{-\varepsilon} \lesssim \lambda^{-\varepsilon}, c > 0\). In fact, in homogeneous coordinates this means that \(b^{-1} \in S(1, g_\varepsilon), q^{-1} \in S(\lambda^\varepsilon, g_\varepsilon)\) and \(r \in S(\lambda^{1-\varepsilon}, g_\varepsilon)\) when \(|\xi| \gtrsim \lambda\).

Take a homogeneous cut-off function \(\chi(x, \xi) \in S^0_{0,0}\) supported where \(d(x, \xi) \lesssim \lambda^{-\varepsilon}\) so that \(b \geq c_0 > 0\) in \(\text{supp } \chi\) and \(\chi = 1\) when \(d \leq c\lambda^{-\varepsilon}\) for some \(c > 0\), then we have \(\chi \in S^0_{1-\varepsilon, \varepsilon}\) uniformly when \(|\xi| \gtrsim \lambda\). We take the homogeneous symbol \(B = \chi b^{-1}q^{-1} \in S^0_{1-\varepsilon, \varepsilon}\) uniformly when \(|\xi| \gtrsim \lambda\) and we compose the corresponding pseudodifferential operator \(B \in \Psi^0_{1,0}\) with \(P^\ast\). Since \(P^\ast \in \Psi^1_{1,0}\) we obtain an asymptotic expansion of \(BP^\ast\) in \(S^1_{1-\varepsilon, \varepsilon}\) for \(j = 0, 1, 2, \ldots\) when \(|\xi| \gtrsim \lambda\). But actually the symbol is in a better class. The principal symbol is

\[(\tau - r(t, x, \xi))\chi \in S^1_{1-\varepsilon, \varepsilon} \quad \text{for } |\xi| \gtrsim \lambda\]

and the calculus gives that the homogeneous term is equal to

\begin{equation}
(3.22) \quad \frac{i}{2} H_p(\chi b^{-1}q^{-1}) + \chi b^{-1}q^{-1}p_0
\end{equation}

where \(p_0\) is the homogeneous term of the expansion of \(P^\ast\). As before, we shall use homogeneous coordinates. Then Proposition \(3.1\) gives \(p = q\tilde{\rho} \in S(q\lambda^{-\varepsilon}, g_\varepsilon)\) when \(|\xi| \cong \lambda\) and since \(\chi b^{-1}q^{-1} \in S(q^{-1}, g_\varepsilon) \subset S(\lambda^\varepsilon, g_\varepsilon)\) when \(|\xi| \cong \lambda\), we find that the terms in \((3.22)\) are in \(S(\lambda^\varepsilon, g_\varepsilon)\) when \(d \lesssim \lambda^{-\varepsilon}\) and by homogeneity in \(S^1_{1-\varepsilon, \varepsilon}\) when \(d \lesssim |\xi|^{-\varepsilon} \lesssim \lambda^{-\varepsilon}\).

The value of \(H_p\) at \(\Gamma\) is equal to \(qd\xi\) modulo terms with coefficients that are \(O(\lambda^{-14\varepsilon/3})\) by \((3.20)\) so the value of \((3.22)\) is equal to

\begin{equation}
(3.23) \quad \frac{1}{2\i} \partial_t q/q + p_0/q = \frac{D_t|\nabla p|}{2|\nabla p|} + \frac{p_0}{|\nabla p|} \quad \text{at } \Gamma
\end{equation}

modulo \(O(\lambda^{-8\varepsilon/3})\). Here \(|\nabla p| = \sqrt{(|\partial_s p|^2/|\xi|^2 + |\partial_t p|^2}|\) is the homogeneous gradient, and the error of this approximation is bounded by \(\lambda^{2\varepsilon}\) times the homogeneous distance \(d\) to \(\Gamma\), since \((3.22)\) is in \(S(\lambda^\varepsilon, g_\varepsilon)\). Observe that \(p_0/|\nabla p|\) is equal to the normalized subprincipal
symbol of $P^*$ on $S^*\mathbb{R}^n$ given by (3.6). But we have to estimate the error terms in this preparation.

**Definition 3.2.** For $0 < \varepsilon < 1/2$ and $R \in S^\mu_{\theta, \delta}$ where $\theta + \delta \geq 1$, $\theta > \varepsilon$ and $\delta < 1 - \varepsilon$, we say that $S^*X \ni (x_0, \xi_0) \notin \operatorname{WF}_\varepsilon(R)$ if for any $N$ there exists $c_N > 0$ so that $R \in S_{\theta, \delta}^{-N}$ when the homogeneous distance to the ray $\{(x_0, \theta \xi_0) : \theta \in \mathbb{R}_+\}$ is less than $c_N|\xi|^{-\varepsilon}$.

For a family of operators $R_j \in \Psi^\mu_{\theta, \delta}$, $j = 1, \ldots$, we say that $S^*X \ni (x_j, \xi_j) \notin \operatorname{WF}_\varepsilon(R_j)$ uniformly with respect to $\lambda_j \geq 1$, if for any $N$ there exists $C_N > 0$ so that $R_j \in S_{\theta, \delta}^{-N}$ uniformly in $j$ when the homogeneous distance to the ray $\{(x_j, \theta \xi_j) : \theta \in \mathbb{R}_+\}$ is less than $C_N|\xi|^{-\varepsilon} \leq CC_N\lambda_j^{-\varepsilon}$ for some $C > 0$.

By the calculus, this means that there exist $A_j \in \Psi^0_{1-\varepsilon, \varepsilon}$ so that $A_j \geq c > 0$ when the distance to the ray through $(x_j, \xi_j)$ is less than $C_N|\xi|^{-\varepsilon} \lesssim \lambda_j^{-\varepsilon}$ such that $A_j R_j \in \Psi^{-N}$ uniformly. This neighborhood is in fact the points with fixed $g_\varepsilon$ distance to the ray through $(x_j, \xi_j)$ when $|\xi| \gtrsim \lambda_j$. For example, if the homogeneous cut-off functions $\chi_j$ is equal to 1 where the homogeneous distance to the ray $\{(x_j, \theta \xi_j) : \theta \in \mathbb{R}_+\}$ is less than $C_N\lambda_j^{-\varepsilon}$ then $(x_j, \xi_j) \notin \operatorname{WF}_\varepsilon(1 - \chi_j)$ uniformly with respect to $\lambda_j$. It follows from the calculus that Definition 3.2 is invariant under composition with classical elliptic pseudodifferential operators and under conjugation with elliptic homogeneous Fourier integral operators preserving the fiber, by the conditions on $\theta$ and $\delta$. We also have that $\operatorname{WF}_\varepsilon(R)$ grows when $\varepsilon$ shrinks and $\operatorname{WF}_\varepsilon(R) \subset \operatorname{WF}(R)$.

Now we can use the Malgrange division theorem in order to make the lower order terms independent on $\tau$ when $d \lesssim \lambda^{-\varepsilon}$, starting with the subprincipal symbol $\tilde{p}_0 \in S^\varepsilon_{1-\varepsilon, \varepsilon}$ of $BP^*$ given by (3.22). Then restricting to $|\xi| = 1$ and rescaling as before so that $Q_0(t, w) = \lambda^{-\varepsilon}\tilde{p}_0(t\lambda^{-\varepsilon}, w\lambda^{-\varepsilon}) \in C^\infty$ uniformly, we obtain that

$$Q_0(t, w) = \tilde{c}(t, w)(\tau - s(t, x, \xi)) + q_0(t, x, \xi) \quad |w| \lesssim 1 \quad t \in I_\varepsilon$$

where $s$ is given by (3.17), and $\tilde{c}$ and $q_0$ are uniformly in $C^\infty$. This can be done uniformly, first locally and then by a partition of unity for $t \in I_\varepsilon$. We find in the original coordinates that

$$\tilde{p}_0(t, w) = c(t, w)(\tau - r(t, x, \xi)) + q_0(t, x, \xi) \quad d \lesssim \lambda^{-\varepsilon} \quad t \in I$$

where $q_0(t, w) = \lambda^\varepsilon q_0(t\lambda^\varepsilon, w\lambda^\varepsilon) \in S(\lambda^\varepsilon, g_\varepsilon)$ and $c(t, w) = \lambda^\varepsilon \tilde{c}(t\lambda^\varepsilon, w\lambda^\varepsilon) \in S(\lambda^{2\varepsilon}, g_\varepsilon)$. By using a partition of unity, we obtain (3.24) uniformly when the homogeneous distance to $\Gamma$ is $\lesssim \lambda^{-\varepsilon}$. By homogeneity we find as before that $c$ is homogeneous of degree $-1$ and $q_0$ is homogeneous of degree 0, which gives $c \in S^\varepsilon_{1-\varepsilon, \varepsilon}$ and $q_0 \in S^\varepsilon_{1-\varepsilon, \varepsilon}$ when $|\xi| \gtrsim \lambda$. Now the composition of the operators having symbols $c$ and $\tau - r$ gives error terms that are homogeneous of degree $-1$ and are uniformly in $S^\varepsilon_{1-\varepsilon, \varepsilon}$ when $|\xi| \gtrsim \lambda$. Thus if $\varepsilon < 1/3$ then by multiplication with an pseudodifferential operator with symbol $1 - c$ we can make the
subprincipal symbol independent of $\tau$. By iterating this procedure we can successively make any lower order terms independent of $\tau$ when the homogeneous distance $d$ to $\Gamma$ is less than $c\lambda^{-\epsilon}$. By applying a homogeneous cut-off function $\chi$ as before we obtain the following result.

**Proposition 3.3.** Assume that (2.3), (2.4), (2.8), (2.11), (2.13) and (2.20) hold uniformly for $\Gamma_j$, $L_j$ and $\lambda_j$ satisfying (3.1) for some $\epsilon > 0$. By conjugating with uniformly bounded elliptic homogeneous Fourier integral operators and multiplying with uniformly bounded homogeneous elliptic operators we may assume that $m = 1$, $a_j \equiv 1$ and $\Gamma_j$ is given by (3.10). If $0 < \epsilon < 1/3$ then for any $c > 0$ we can obtain that $B_j P^* = Q_j + R_j \in \Psi^{1-\epsilon}_{1-\epsilon,\epsilon}$ where $B_j \in \Psi^0_{1-\epsilon,\epsilon}$ uniformly, $\Gamma_j \cap \text{WF}_R(R_j) = \emptyset$ uniformly, and the symbol of $Q_j$ is equal to

\begin{equation}
\tau - r(t, x, \xi) + q_0(t, x, \xi) + r_0(t, x, \xi) \quad \text{when } d_j(x, \xi) \leq c|\xi|^{-\epsilon} \lesssim \lambda_j^{-\epsilon} \quad \text{and } t \in I
\end{equation}

where $d_j$ is the homogeneous distance to $\Gamma_j$. Here $r$ is homogeneous of degree 1 and $q_0$ is homogenous of degree 0, $r \in C^1_{1-\epsilon,\epsilon}$, $q_0 \in C^1_{1-\epsilon,\epsilon}$ and $r_0 \in C^3_{1-\epsilon,\epsilon}$ uniformly. We also have $r = |\nabla \text{Re } r| = 0$, $\nabla \text{Im } r = O(\lambda^{-14\epsilon/3})$, $d\nabla \text{Re } r|_L = O(1)$ and $d\nabla \text{Im } r|_L = O(\lambda^{-4\epsilon/3})$ on $\Gamma_j$. We find that $q_0$ is equal to

\begin{equation}
\frac{D_t|\nabla p(t, 0)|}{2|\nabla p(t, 0)|} + \frac{p_0(t, 0)}{|\nabla p(t, 0)|} \quad \text{when } d_j(x, \xi) \leq c|\xi|^{-\epsilon} \lesssim \lambda_j^{-\epsilon} \quad \text{and } t \in I
\end{equation}

modulo terms that are $O(\lambda^{-8\epsilon/3} + \lambda^{2\epsilon}d_j)$ where $|\nabla p| = \sqrt{|\partial_x p|^2/|\xi|^2 + |\partial_\xi p|^2}$ is the homogeneous gradient of $p$.

We shall apply the operator in Proposition 3.3 on oscillatory solutions having frequencies $\xi$ of size $\lambda$, see Proposition 3.5. Observe also that the integration of the term $D_t|\nabla p(t, 0)|/2|\nabla p(t, 0)|$ in (3.26) will give terms that are

$$O(\log(|\nabla p(t, 0)|)) = O(|\log(\lambda)| + 1)$$

which do not affect condition (2.20).

Recall that $L$ is a smooth section of Lagrangean spaces $L(w) \subset T_w \Sigma \subset T_w(T^*\mathbb{R}^n)$, $w \in \Gamma$, such that the linearization of the Hamilton vector field $H_{\text{Re } p}$ is in $TL$ at $\Gamma$. Here $\Sigma = (\text{Re } p)^{-1}(0)$ and $T_w \Sigma = \text{Ker } d\text{Re } p(w)$ where $d\text{Re } p(w) \neq 0$ for $w \in \Gamma$. By Proposition 3.3 we may assume that $\Gamma = \{(t, 0; 0, \xi_0) : t \in I\}$, $0 \in I$, and we may parametrize $L(t) = L(w)$ where $w = (t, 0, \xi_0)$ for $t \in I$. Now since $T^*\mathbb{R}^n$ is a linear space, we may identify the fiber of $T_w(T^*\mathbb{R}^n)$ with $T^*\mathbb{R}^n$. Since $L(w) \subset T_w \Sigma$ and $w \in \Gamma$ we find that $d\tau = 0$ in $L(w)$. Since $L(w)$ is Lagrangean, we find that $t$ lines are parallel to $L(w)$. By choosing linear symplectic coordinates in $(x, \xi)$ we obtain that $L(0) = \{(s, y; 0, 0) : (s, y) \in \mathbb{R}^n\}$, then by condition (3.19) we find that $\partial_x \nabla r(0, 0, \xi_0)$ is
uniformly bounded. Since \( d\tau = 0 \) on \( L(t) \) and \( L(t) \) is Lagrangean we find by continuity for small \( t \) that
\[
L(t) = \{ (s, y; 0, A(t)y) : (s, y) \in \mathbb{R}^n \}
\]
where \( A(t) \) is real, continuous and symmetric for \( t \in I \) and \( A(0) = 0 \). Since the linearization of the Hamilton vector field \( H_{\text{rep}} \) at \( \Gamma \) is tangent to \( L \), we find that \( L \) is parallel under the flow of that linearization. Since \( L(t) \) is Lagrangean, the evolution of \( t \mapsto L(t) \) is determined by the restriction of the second order Taylor expansion of \( r(t, w) \) to \( L(t) \). For \( (3.27) \) this restriction is given by the second order Taylor expansion of
\[
R(t, x) = \text{Re} r(t, x, \xi_0 + A(t)x)
\]
thus \( \partial^2_x R(t, 0) \) is uniformly bounded by condition \( (2.7) \). The linearized Hamilton vector field is
\[
\partial_t + \langle \partial^2_x R(t, 0)x, \partial_\xi \rangle = \partial_t + \langle (\partial^2_x \text{Re} r(t, 0, \xi_0) + \partial_\xi \partial_x \text{Re} r(t, 0, \xi_0)A \nonumber \\
+ A \partial_x \partial_\xi \text{Re} r(t, 0, \xi_0) + A \partial^2_\xi \text{Re} r(t, 0, \xi_0)A)x, \partial_\xi \rangle
\]
Applying this on \( \xi - A(t)x \), which vanishes identically on \( L(t) \) for \( t \in I \), we obtain that the evolution of \( L(t) \) is given by
\[
(3.28) \quad A'(t) = \partial^2_x \text{Re} r(t, 0, \xi_0) + \partial_\xi \partial_x \text{Re} r(t, 0, \xi_0)A(t) \nonumber \\
+ A(t)\partial_\xi \partial_x \text{Re} r(t, 0, \xi_0) + A(t)\partial^2_\xi \text{Re} r(t, 0, \xi_0)A(t)
\]
with \( A(0) = 0 \). This is locally uniquely solvable and the right-hand side is uniformly bounded as long as \( A \) is bounded. Observe that by uniqueness, \( A(t) \equiv 0 \) if and only if \( \partial^2_x \text{Re} r(0, \xi_0) \equiv 0 \), \( \forall t \). But since \( (3.28) \) is non-linear, the solution could become unbounded if \( \partial^2_x \text{Re} r \neq 0 \) and \( \partial^2_\xi \text{Re} r \neq 0 \) so that \( \|A(s)\| \to \infty \) as \( s \to t_1 \in I \). This means that the angle between \( L(t) = \{ (s, y; 0, A(t)y) : (s, y) \in \mathbb{R}^n \} \) and the vertical space \( \{ (s, 0; 0, \eta) : (s, \eta) \in \mathbb{R}^n \} \) goes to zero, but that is only a coordinate singularity.

In general, since we identify the fiber of \( T_w(T^*\mathbb{R}^n) \) with \( T^*\mathbb{R}^n \) we may define \( R(t, x, \xi) \) for each \( t \) so that
\[
(3.29) \quad R(t, x, \xi) = \text{Re} r(t, x, \xi_0 + \xi) \quad \text{when} \ (0, x; 0, \xi) \in L(t) \nonumber \]
Then \( R = \text{Re} r \) on \( L \) and we find that
\[
(3.30) \quad \tau - \langle R(t)z, z \rangle / 2 \in C^\infty \nonumber \]
if \( z = (x, \xi) \) and \( R(t) = \partial^2_x R(t, 0, 0)|_t(t) \). Observe that we find from \( (3.19) \) that \( (3.30) \) is uniformly in \( C^\infty \) in \( z \) and uniformly continuous in \( t \). We find that \( R(0) = \partial^2_x \text{Re} r(t, 0, \xi_0) \) and in general \( R(t) \) is given by the right hand side of \( (3.28) \). Now we can complete \( t, \tau - \langle R(t)z, z \rangle / 2 \) and \( (x, \xi)|_{t=0} \) to a uniform homogeneous symplectic coordinates system so that \( \Gamma = \{ (t, 0, \xi_0) : t \in I \} \) and \( L(0) = \{ (s, y; 0, 0) : (s, y) \in \mathbb{R}^n \} \). In fact, \( (x, \xi) \)
satisfies a linear evolution equation \( H_t(x, \xi) = 0 \) and has the same value when \( t = 0 \), so \( (x, \xi) = 0 \) and \( H_t = \partial_t \) on \( \Gamma \). Since this is done by integration in \( t \), it gives a uniformly bounded linear symplectic transformation in \( (x, \xi) \) which is uniformly \( C^1 \) in \( t \). It is given by a uniformly bounded elliptic Fourier integral operator \( F(t) \) on \( \mathbb{R}^{n-1} \) which is uniformly \( C^1 \) in \( t \). We will call this type of Fourier integral operator a \( C^1 \) section of Fourier integral operators on \( \mathbb{R}^{n-1} \). This will give uniformly bounded terms when we conjugate \( F(t) \) with a first order differential operator in \( t \), for example the normal form of \( P^* \) given by (3.25). For \( t \) close to 0 the section \( F(t) \) is given by multiplication with \( e^{i(A(t)x,x)} \), where \( A(t) \) solves (3.28). For general \( t \) we can put \( F(t) \) on this form after a linear symplectic transformation in \( (x, \xi) \). Observe that \( F(t) \) is continuous on local \( L^2 \) Sobolev spaces in \( x \), uniformly in \( t \), since it is continuous with respect to the norm \( \| (1 + |x|^2 + |D_x|^2)^k u \| \), \( \forall k \). In fact, it suffices to check this for the generators of the group of Fourier integral operators corresponding to linear symplectic transformations of \( (x, \xi) \), which are given by the partial Fourier transforms, linear transformations in \( x \) and multiplication with \( e^{i(Ax,x)} \) where \( A \) is real and symmetric.

We find in the new coordinates that \( p = \tau - r_1 \), where \( r_1(t, x, \xi) \) is independent of \( \tau \) and satisfies \( \partial_z^2 \text{Re} r_1(t, 0, 0) \big|_L(t) \equiv 0 \). This follows since
\[
p(t, x; \tau, \xi) = \tau - \langle R(t)z, z \rangle / 2 - r_1(t, x, \tau, \xi) \quad z = (x, \xi)
\]
where \( \partial_z^2 \text{Re} r_1(t, 0, 0) \big|_L(t) \equiv 0 \). We also have that \( \partial_{t,r_1} = -\{ t, r_1 \} = -\{ t, r \} \equiv 0 \), which is invariant under the change of symplectic coordinates. Similarly we find that the lower order terms \( p_j(t, x, \xi) \) remain independent of \( \tau \) for \( j \leq 0 \). Since the evolution of \( L \) is determined by the second order derivatives of the principal symbol along \( L \) by Example 2.1, we find that \( L(t) \equiv \{(t, x; 0, 0) : (t, x) \in \mathbb{R}^n \} \) after the change of coordinates. Since \( L \) is a grazing Lagrangean space, the linearization of \( H_{Re p} \) at \( \Gamma \) is tangent to \( L \). Thus \( \partial_x \text{Re} r_1 = \partial_z^2 \text{Re} r_1 = 0 \), \( \nabla \text{Im} r_1 = \mathcal{O}(\lambda_j^{-14/3}) \) and condition (3.21) gives that \( \partial_{t,x} \nabla \text{Im} r_1 = \mathcal{O}(\lambda_j^{-14/3}) \) at \( \Gamma_j \). Changing notation so that \( r = r_1 \) and \( p(t, x; \tau, \xi) = \tau - r(t, x, \xi) \) we obtain the following result.

**Proposition 3.4.** By conjugating with a uniformly bounded \( C^1 \) section of Fourier integral operators on \( \mathbb{R}^{n-1} \), we may assume that the symplectic coordinates in Proposition 3.3 are chosen so that the grazing Lagrangean space \( L(w) \equiv \{(t, x; 0, 0) : (t, x) \in \mathbb{R}^n \} \), \( \forall w \in \Gamma \), which gives that \( \partial_x \text{Re} r = \partial_z^2 \text{Re} r = 0 \), \( \partial_{t,x} \nabla \text{Re} r = \mathcal{O}(1) \), \( \nabla \text{Im} r_1 = \mathcal{O}(\lambda_j^{-14/3}) \) and \( \partial_{t,x} \nabla \text{Im} r = \mathcal{O}(\lambda_j^{-14/3}) \) at \( \Gamma_j \).

We shall apply the adjoint \( P^* \) of the operator on the form in Proposition 3.3 on approximate solutions on the form
\[
(3.31) \quad u_\lambda(t, x) = \exp(i\lambda(\langle x, \xi_0 \rangle + \omega(t, x))) \sum_{j=0}^{M} \varphi_j(t, x) \lambda^{-je}
\]
where $|\xi_0| = 1$, the phase function $\omega(t, x) \in S(\lambda^{-7\varepsilon}, g_0)$ is real valued and the amplitudes $\varphi_j(t, x) \in S(1, g_0)$ have support where $|x| \leq \lambda^{-\delta}$. Here $\delta \geq \varepsilon$ and $g$ are positive constants to be determined later. The phase function $\omega(t, x)$ will be constructed in Section 4 see Proposition 4.2. Observe that we have assumed that $\varepsilon < 1/3$ in Proposition 3.3 but we shall impose further restrictions on $\varepsilon$ later on. We shall assume that $\varepsilon + \delta < 1$, then if $p(t, x, \xi) \in \Psi_{1-\varepsilon, \varepsilon}^1$ when $|\xi| \cong \lambda$ we obtain the asymptotic expansion

\[ p(t, x, D_x)(\exp(i\lambda(\langle x, \xi_0 \rangle + \omega(t, x)))\varphi(t, x)) \]

\[ \sim \exp(i\lambda(\langle x, \xi_0 \rangle + \omega(t, x))) \sum_\alpha \partial^\alpha p(t, x, \lambda(\xi_0 + \partial_x \omega(t, x))) R_\alpha(\omega, \lambda, D) \varphi(t, x) / \alpha! \]

where $R_\alpha(\omega, \lambda, D) \varphi(t, x) = D_y^\alpha(\exp(i\lambda \omega(t, x, y)) \varphi(t, y))|_{y=x}$ with

\[ \omega(t, x, y) = \omega(t, y) - \omega(t, x) + (x-y)\partial_x \omega(t, x) \]

and the error term is of the same size as the next term in the expansion. See for example Theorem 3.1 in [7, Chapter VI], which is for classical pseudo differential operators, phase functions and amplitudes, but the proof is easily adapted to the case when these depend uniformly on parameters. Observe that since $|\partial_x \omega| \cong \lambda^{-4\varepsilon} \ll 1$ the expansion only involves the values of $p(t, x, \xi)$ where $|\xi| \cong \lambda \gg 1$. Using this expansion we find that if $p$ is given by (3.25) then

\[ e^{-i\lambda(\langle x, \xi_0 \rangle + \omega(t, x))} p(t, x, D_{t,x}) e^{i\lambda(\langle x, \xi_0 \rangle + \omega(t, x))} \varphi(t, x) \]

\[ \sim \lambda(\partial_t \omega(t, x) - r(t, x, \xi_0 + \partial_x \omega)) \varphi(t, x) \]

\[ + D_t \varphi(t, x) - \sum_j \partial_{\xi_j} r(t, x, \xi_0 + \partial_x \omega) D_{x_j} \varphi(t, x) + q_0(t, x, \xi_0 + \partial_x \omega) \varphi(t, x) \]

\[ + \sum_{jk} \partial_{\xi_j} \partial_{\xi_k} r(t, x, \xi_0 + \partial_x \omega)(\lambda^{-1} D_{x_j} D_{x_k} \varphi(t, x) + i \varphi(t, x) D_{x_j} D_{x_k} \omega(t, x)) / 2 + \ldots \]

which gives an expansion in $S(\lambda^{1-\varepsilon-j(1-\delta-\varepsilon)}, g_0)$, $j \geq 0$, if $\delta + \varepsilon < 1$ and $\varepsilon < 1/4$. In fact, since $|\xi| \cong \lambda$ every $\xi$ derivative on terms in $S_{1-\varepsilon, \varepsilon}^1$ gives a factor that is $O(\lambda^{\varepsilon-1})$ and every $x$ derivative of $\varphi$ gives a factor that is $O(\lambda^\delta)$. A factor $\lambda D_x^\alpha \omega$ requires $|\alpha| \geq 2$ number of $\xi$ derivatives of a term in the expansion of $P^*$, which gives a factor that is $O(\lambda^{1+(-7+3|\alpha|)\varepsilon-|\alpha|([1-\varepsilon])}) = O(\lambda^{1-\varepsilon-|\alpha|([1-4\varepsilon])}) = O(\lambda^{1+\varepsilon})$. Similarly, the expansion coming from terms in $P^*$ that have symbols in $S_{1-\varepsilon, \varepsilon}^1$ gives an expansion in $S_{1-\varepsilon, \varepsilon}^0$, $j \geq 0$. Thus, if $\delta + \varepsilon < 2/3$ and $\varepsilon < 1/4$ then the terms in the expansion are $O(\lambda^{\delta+2\varepsilon-1})$ except the terms in (3.33), and for the last ones we find that

\[ \sum_{jk} \partial_{\xi_j} \partial_{\xi_k} r(t, x, \xi_0 + \partial_x \omega)(\lambda^{-1} D_{x_j} D_{x_k} \varphi + i \varphi D_{x_j} D_{x_k} \omega) = O(\lambda^{2\delta+\varepsilon-1} + \lambda^{3\varepsilon-\delta}) \]

In fact, $\partial_{\xi_j} \partial_{\xi_k} r(t, x, \xi_0 + \partial_x \omega) = O(\lambda^\varepsilon)$ and $D_{x_j} D_{x_k} \omega = O(\lambda^{2\varepsilon} d)$ when $\varphi \neq 0$, since we have $D_{x_j} D_{x_k} \omega = 0$ when $x = 0$, and $d = O(\lambda^{-\delta})$ in supp $\varphi$. 
The error terms in (3.34) are of equal size if \(2\delta + \varepsilon - 1 = 3\varepsilon - \delta\), thus \(\delta = (1+2\varepsilon)/3 \geq \varepsilon\) since \(\varepsilon \leq 1\). Since \(\delta + \varepsilon < 1\) we obtain that \(4\varepsilon - 1 < 3\varepsilon - \delta = (7\varepsilon - 1)/3 < 0\) if \(\varepsilon < 1/7\) and \(1 - \delta - \varepsilon = (2 - 5\varepsilon)/3 > 1/3\) if \(\varepsilon < 1/5\). Thus we obtain the following result.

**Proposition 3.5.** Assume that \(p\) is given by (3.25), \(\omega(t,x) \in S(\lambda^{-7\varepsilon},g_{3\varepsilon})\) is real valued with \(\partial_x\omega(t,0) \equiv \partial^2_x\omega(t,0) \equiv 0\), and \(\varphi_j(t,x) \in S(1,g_3)\) has support where \(|x| \lesssim \lambda^{-\delta}\) with positive \(\delta\) and \(\varepsilon\). If \(\delta = (1 + 2\varepsilon)/3\) and \(\varepsilon < 1/7\), then (3.33) has an expansion in \(S(\lambda^{1-\varepsilon-j(2-5\varepsilon)/3},g_\delta), \ j \geq 0\), and is equal to

\[
(3.35) \quad \lambda(\partial_t \omega(t,x) - r(t,x,\xi_0 + \partial_x \omega)) \varphi(t,x) + D_t \varphi(t,x) - \sum_j \partial_{\xi_j} r(t,x,\xi_0 + \partial_x \omega) D_{\xi_j} \varphi(t,x) + q_0(t,x,\xi_0 + \partial_x \omega) \varphi(t,x)
\]

modulo terms that are \(O(\lambda^{(7\varepsilon - 1)/3}) = O(\lambda^{2\varepsilon + \varepsilon - 1})\).

In Section 5 we shall choose \(\varepsilon = 1/8\) which gives \(\delta = 5/12\), \((2 - 5\varepsilon)/3 = 11/24\) and \((7\varepsilon - 1)/3 = -1/24\), so we may take \(\rho = 1/24\) in (3.31).

4. The eikonal equation

Making the real part of the first term in the expansion (3.33) equal to zero gives the eikonal equation

\[
(4.1) \quad \partial_t \omega - \text{Re} s(t,x,\partial_x \omega) = 0 \quad \omega(0,x) \equiv 0
\]

where \(s(t,x,\xi) = r(t,x,\xi_0 + \xi)\). The imaginary part of the first term will be treated as a perturbation. We shall solve the eikonal equation approximatively after scaling, since we solve the real part it will be similar to the argument in [2]. We choose coordinates \((t,x,\xi)\) on \(S^*\mathbb{R}^n\) so that \(\Gamma\) is given by (3.10). We find that \(s \in S(\lambda^{-\varepsilon},g_{\varepsilon})\) when \(|x| + |\xi| \lesssim \lambda^{-\varepsilon}\) by Proposition 3.3 and we may assume that \(L(t) \equiv \{ (t,x,0,0) \}, \ \forall t\), by Proposition 3.4. But \(s\) is also in another symbol class by the following refinement of Proposition 3.3.

**Proposition 4.1.** Assuming Propositions 3.3 and 3.4 we have

\(s \in S(\lambda^{-7\varepsilon},\lambda^{6\varepsilon}(dt^2 + |dx|^2) + \lambda^{8\varepsilon}|d\xi|^2)\)

when \(|x| \lesssim \lambda^{-3\varepsilon}, |\xi| \lesssim \lambda^{-4\varepsilon}\) and \(t \in I\).

**Proof.** Since \(s \in S(\lambda^{-\varepsilon},g_{\varepsilon})\) when \(|x| + |\xi| \lesssim \lambda^{-\varepsilon}\) by Proposition 3.3, we find that

\[
(4.2) \quad |\partial^\alpha_x \partial_{\xi}^\beta s| \lesssim \lambda^{(\alpha + |\beta| - 1)\varepsilon} \lesssim \lambda^{(3|\alpha| + 4|\beta| - 7)\varepsilon}
\]

when \(|x| + |\xi| \lesssim \lambda^{-\varepsilon}\), if and only \(|\alpha| + |\beta| - 1 \leq 3|\alpha| + 4|\beta| - 7\), i.e.,

\[2|\alpha| + 3|\beta| > 5\]

Thus, we only have to check the cases \(|\alpha| + |\beta| \leq 2\) and \(|\beta| \leq 1\). Since the Lagrange remainder term is in the symbol class, we only have to check the derivatives at \(x = \)
\( \xi = 0 \). Then we obtain (4.2) since \( s(t,0,0) = 0, \partial s(t,0,0) = O(\lambda^{-14\varepsilon/3}) \) by (3.20), \( \partial_{t,\alpha}\partial_{\xi}s(t,0,\xi_0) = O(1) \) and \( \partial^2_{t,\alpha}s(t,0,0) = O(\lambda^{-4\varepsilon/3}) \) by (3.19) and (3.21).

Observe that the estimates for \( \partial \text{Im} s \) and \( \partial_{t,\alpha}\text{Im} s \) at \( \Gamma \) are better than the symbol estimates, which will be important in the proof of Lemma 5.2. Next, we scale and put \((x,\xi) = (\lambda^{-3\varepsilon}y, \lambda^{-4\varepsilon}\eta)\). When \(|y| + |\eta| \leq c \) we find

(4.3) \( (y,\eta) \mapsto f(t,y,\eta) = \lambda^{7\varepsilon} s(t,\lambda^{-3\varepsilon}y, \lambda^{-4\varepsilon}\eta) \in C^\infty \)

and \( y \mapsto \omega_0(t,y) = \lambda^{7\varepsilon}\omega(t,\lambda^{-3\varepsilon}y) \in C^\infty \) uniformly. Then the eikonal equation (4.1) is

(4.4) \( \partial_t\omega_0 - \text{Re} f(t,y,\partial_y\omega_0) \equiv 0 \quad \omega_0(0,y) = 0 \)

when \(|y| \leq c\). We can solve (4.4) by solving the Hamilton-Jacobi equations:

(4.5) \[
\begin{align*}
\partial_t y &= -\partial_\eta \text{Re} f(t,y,\eta) \\
\partial_t \eta &= \partial_y \text{Re} f(t,y,\eta)
\end{align*}
\]

with initial values \((y(0),\eta(0)) = (z,0)\). Since we have uniform bounds on \((y,\eta) \mapsto f(t,y,\eta)\), we find that (4.5) has a uniformly bounded \( C^\infty \) solution \((y(t),\eta(t))\) if \((z,0)\) is uniformly bounded. By taking \( z \) derivatives of the equations, we find that \( z \mapsto (y(t,z),\eta(t,z)) \in C^\infty \) uniformly. By (4.3) we find that \( (\partial_t y,\partial_t \eta) \) is uniformly bounded, and by taking repeated \( t \), \( z \) derivatives of (4.5) we find that \( (\partial^k_t\partial^\alpha_y y,\partial^k_t\partial^\alpha_z \eta) = O(\lambda^{3(k-1)\varepsilon}) \).

Letting \( \partial_{y0}\omega_0(t,y(t,z)) = \eta(t,z) \) and \( \partial_{\eta0}\omega_0(t,y(t,z)) = \text{Re} f(t,y(t,z),\eta(t,z)) = O(1) \) when \(|y| \leq c\), we obtain the solution \( \omega_0(t,y) \in S(1,\lambda^{6\varepsilon} dt^2 + |dy|^2) \) to (4.4). (Actually, we have \( \partial_t\omega_0 \in S(1,\lambda^{6\varepsilon} dt^2 + |dy|^2))\.) Since \( \nabla \text{Re} f = 0 \) on \( \Gamma \) we find by uniqueness that \( y = \eta = 0 \) when \( z = 0 \) which gives \( \omega_0(t,0) \equiv \partial_t\omega_0(t,0) \equiv \partial_{y0}\omega_0(t,0) \equiv 0 \). Since \( \partial_{y0}\text{Re} f(t,0,0) = \partial_y^2 \text{Re} f(t,0,0) = 0 \) we find by differentiating (4.4) twice that

\[
\partial_t \partial^2_{y0}\omega_0(t,0) = \partial_y\partial_\eta \text{Re} f(t,0,0)\partial^2_{y0}\omega_0(t,0) + \partial^2_{y0}\omega_0(t,0)\partial_\eta\partial_y \text{Re} f(t,0,0)
\]

Since \( \partial^2_{x0}\omega(0,x) \equiv 0 \) we find by uniqueness that \( \partial^2_{x0}\omega(t,0) \equiv 0 \).

In the original coordinates we find that that if \( x(0) = O(\lambda^{-3\varepsilon}) \) and \( \xi(0) = 0 \) then \( x(t,x_0) = O(\lambda^{-3\varepsilon}) \) and \( \xi(t,x_0) = O(\lambda^{-4\varepsilon}) \) for any \( t \in I \). The scaling also gives that

(4.6) \( \omega(t,x) = \lambda^{-7\varepsilon}\omega_0(t,\lambda^{3\varepsilon} x) \in S(\lambda^{-7\varepsilon},g_{3\varepsilon}) \quad |x| \lesssim \lambda^{-3\varepsilon} \)

and we have \( \omega(t,0) \equiv \partial_\varepsilon\omega(t,0) \equiv \partial^2_x\omega(t,0) \equiv 0 \). (Actually, \( \partial_\varepsilon\omega(t,x) \in S(\lambda^{-7\varepsilon},g_{3\varepsilon}) \) when \(|x| \lesssim \lambda^{-3\varepsilon}\). By the symbol estimates, we find \( \partial_\varepsilon\omega(t,x) = O(\lambda^{-4\varepsilon}) \) when \(|x| \lesssim \lambda^{-3\varepsilon}\). Thus, we obtain the following result.

**Proposition 4.2.** Let \( 0 < \varepsilon < 1/3 \), and assume that Propositions 3.3 and 3.4 hold.
Then there exists a real \( \omega(t,x) \in S(\lambda^{-7\varepsilon},g_{3\varepsilon}) \) satisfying \( \partial_\varepsilon\omega = \text{Re} r(t,x,\xi_0 + \partial_\varepsilon\omega) \) when \(|x| \lesssim \lambda^{-3\varepsilon} \) and \( t \in I \) such that \( \omega(t,0) \equiv \partial_\varepsilon\omega(t,0) \equiv \partial^2_x\omega(t,0) \equiv 0 \). If \( 3\varepsilon \leq \delta \leq 4\varepsilon \) we find
that the values of \((t, x; \lambda \partial_t \omega(t, x), \lambda(\xi_0 + \partial_x \omega(t, x)))\) have homogeneous distance \(\lesssim \lambda^{-\delta}\) to the rays through \(\Gamma\) when \(|x| \lesssim \lambda^{-\delta}\) and \(t \in I\).

5. THE TRANSPORT EQUATIONS

The next term in (3.33) is the transport equation, which by homogeneity is equal to

\[
D_p \varphi + q_0 \varphi + i r_0 \varphi = 0 \quad \text{at} \quad \Gamma = \{(t, 0; 0, \xi_0) : t \in I\}.
\]

where \(D_p = D_t - \sum_j \partial_{x_j} r(t, x, \xi_0 + \partial_x \omega(t, x)) \partial_{x_j}\)

and

\[
q_0(t) \equiv D_t |\nabla p(t, 0, \xi_0)|/|\nabla p(t, 0, \xi_0)| + p_0(t, 0, \xi_0)/|\nabla p(t, 0, \xi_0)| = \mathcal{O}(\lambda^\varepsilon)
\]

modulo \(\mathcal{O}(\lambda^{-8\varepsilon/3} + \lambda^{2\varepsilon}|x|)\) when \(|x| \lesssim \lambda^{-\varepsilon}\) by (3.26). Here the real valued \(\omega(t, x) \in S(\lambda^{-7\varepsilon}, g_{3\varepsilon})\) is given by Proposition 4.2. Since the transport equation is given by a complex vector field, the treatment is different to the one in [2]. But essentially we shall treat the complex part of the transport equation as a perturbation.

**Lemma 5.1.** If \(3\varepsilon \leq \delta \leq 7\varepsilon/2\) then we have that

\[
D_p = D_t + \sum_j \langle a_j(t) \cdot x \rangle D_j + R(t, x, D)
\]

where \(a_j(t) \in C^\infty(\mathbb{R}, \mathbb{R}^{n-1})\) uniformly, \(\forall j\), and \(R(t, x, D)\) is a first order differential operator in \(x\) with coefficients that are \(\mathcal{O}(\lambda^{3\varepsilon-2\delta})\) when \(|x| \lesssim \lambda^{-\delta}\).

**Proof.** As before we shall use the translation \(s(t, x, \xi) = r(t, x, \xi_0 + \xi)\), then

\[
s(t, x, \xi) \in S(\lambda^{-\varepsilon}, g_{\varepsilon}) \bigcap S(\lambda^{-7\varepsilon}, \lambda^{6\varepsilon}(d^2 + |dx|^2) + \lambda^{8\varepsilon}|d\xi|^2)
\]

when \(|x| \lesssim \lambda^{-3\varepsilon}, |\xi| \lesssim \lambda^{-4\varepsilon}\) and \(t \in I\) by Proposition 4.2. Since \(\partial_x^2 \omega(t, 0) = 0\) we find from Taylor’s formula that \(a_j(t) = -\partial_x \partial_{x_j} \text{Re} s(t, 0, 0)\) which is uniformly bounded by (5.19). The coefficients of the error term \(R\) are given by \(\partial_x \text{Im} s\) and the second order Lagrange remainder term of the coefficients of \(\partial_x \text{Re} s\). By Propositions 3.3, 3.4 and 4.2 we find from Taylor’s formula that

\[
\partial_x \text{Im} s(t, x, \partial_x \omega(t, x)) = \partial_x \text{Im} s(t, 0, 0) + \partial_x \partial_x \text{Im} s(t, 0, 0) x
\]

\[
+ \partial_x^2 \text{Im} s(t, 0, 0) \partial_x \omega(t, x) + \mathcal{O}(\lambda^{2\varepsilon}(|x|^2 + \lambda^{4\varepsilon}|x|^4))
\]

\[
= \mathcal{O}(\lambda^{-4\varepsilon} + \lambda^{-\varepsilon}|x| + \lambda^{3\varepsilon}|x|^2 + \lambda^{-6\varepsilon}) = \mathcal{O}(\lambda^{3\varepsilon-2\delta})
\]

when \(|x| \lesssim \lambda^{-\delta}\) since \(3\varepsilon \leq \delta \leq 7\varepsilon/2\). In fact, \(\partial_x \text{Im} s = \mathcal{O}(\lambda^{-14\varepsilon/3})\) and \(\partial_x \partial_x \text{Im} s = \mathcal{O}(\lambda^{-4\varepsilon/3})\) at \(\Gamma\), \(\partial_x^2 s = \mathcal{O}(\lambda^\varepsilon)\), \(\partial^3 s = \mathcal{O}(\lambda^{2\varepsilon})\) and \(\partial_x \omega(t, x) = \mathcal{O}(\lambda^{2\varepsilon}|x|^2) = \mathcal{O}(\lambda^{-4\varepsilon})\) when \(|x| \lesssim \lambda^{-\delta}\) since \(\delta \geq 3\varepsilon\). Similarly we find that the second order Lagrange remainder term
of the coefficients of $\partial_\xi \operatorname{Re} s$ are $O(\lambda^{2\varepsilon}(|x|^2 + \lambda^{4\varepsilon}|x|^4)) = O(\lambda^{2\varepsilon - 2\delta})$ when $|x| \lesssim \lambda^{-\delta} \ll \lambda^{-\varepsilon}$, which proves the result. \hfill $\square$

We also have to estimate the term $r_0(t, x) = \lambda \operatorname{Im} r(t, x, \partial_x \omega(t, x))$ which in fact is bounded according to the following lemma.

**Lemma 5.2.** If $\varepsilon = 1/8$ and $\delta = (1 + 2\varepsilon)/3 = 5/12$ then $r_0(t, x) \in S(1, g_\delta)$ for $|x| \lesssim \lambda^{-\delta}$ and $t \in I$.

Observe that we need that $\varepsilon < 1/7$ and $\delta = (1 + 2\varepsilon)/3$ in order to use the expansion of Proposition 3.5 and when $\varepsilon = 1/8$ we get $\delta = 5/12 = 10\varepsilon/3 < 7\varepsilon/2$.

**Proof.** As before we shall use scaling $(t, x, \xi) = (\lambda^{-3\varepsilon}s, \lambda^{-3\varepsilon}y, \lambda^{-4\varepsilon}\eta)$, and write $f(s, y, \eta) = \lambda^{7\varepsilon}r(t, x, \xi)$ where $\omega_0(s, y) = \lambda^{7\varepsilon}\omega(t, x) \in C^\infty$ uniformly so that $\partial_y \omega_0(s, y) = \lambda^{4\varepsilon}\partial_y \omega(t, x)$ when $|x| \leq c\lambda^{-3\varepsilon}$ and $t \in I$, which we shall assume in the following.

This gives

$$r_0(t, x) = \lambda^{1 - 7\varepsilon} \operatorname{Im} f(s, y, \partial_y \omega_0(t, y))$$

and we shall show that

$$F(s, y) = \operatorname{Im} f(s, y, \partial_y \omega_0(t, y)) \in S(\lambda^{-\varepsilon}, g_\delta) \quad \text{when } |y| \leq c\lambda^{-\varepsilon}$$

where $\varrho = \delta - 3\varepsilon = \varepsilon/3$. Since $\varepsilon = 1/8$, this will give the result. Taylor’s formula gives

$$F(s, y) = \partial_y \operatorname{Im} f(s, 0, 0)y + \langle \partial_y^2 \operatorname{Im} f(s, 0, 0)y, y \rangle / 2$$

$$+ \partial_y \operatorname{Im} f(s, 0, 0)\langle \partial_y^3 \omega_0(s, 0)y, y \rangle / 2 + R(s, y) \quad |y| \leq c\lambda^{-\varepsilon}$$

where $R(s, y) \in C^\infty$ uniformly and vanishes of order 3 at $y = 0$ since $f(s, 0, 0) = \partial_y \omega_0(s, 0) = \partial_y^2 \omega_0(s, 0) = 0$ when $t \in I$ by Propositions 3.4 and 1.2. Thus

$$R(s, y) = O(|y|^3) = O(\lambda^{-3\varepsilon}) = O(\lambda^{-\varepsilon}) \quad \text{when } |y| \leq c\lambda^{-\varepsilon}$$

since $\varrho = \varepsilon/3$. Now one loses at most a factor $y = O(\lambda^{-\varepsilon}) = O(\lambda^{-\varepsilon/3})$ when taking a derivative of $R(s, y)$, giving a factor $O(\lambda^{\varrho})$, so $R(s, y) \in S(\lambda^{-\varepsilon}, g_\delta)$.

It remains to consider the first three terms in (5.6) and as before it suffices to consider derivatives of order less than 3 at $y = 0$. Since $\partial_y^3 \omega_0(s, 0) \in C^\infty$ uniformly we only have to estimate $\partial_y \operatorname{Im} f(s, 0, 0)$ and $\partial_{s,y}^k \operatorname{Im} f(s, 0, 0)$ when $k \leq 2$. We obtain from (3.20) that

$$\partial_y \operatorname{Im} f(s, 0, 0) = \lambda^{3\varepsilon} \partial_\xi \operatorname{Im} r(t, 0, \xi_0) = O(\lambda^{-5\varepsilon/3}) = O(\lambda^{-\varepsilon + 2\varepsilon})$$

Similarly, (3.20) gives

$$\partial_{s,y} \operatorname{Im} f(s, 0, 0) = \lambda^{4\varepsilon} \partial_{t,x} \operatorname{Im} r(t, 0, \xi_0) = O(\lambda^{-2\varepsilon/3}) = O(\lambda^{-\varepsilon + \varepsilon})$$

and (3.21) gives that $\partial_{s,y}^2 \operatorname{Im} f(s, 0, 0) = \lambda^{\varepsilon} \partial_{t,x}^2 \operatorname{Im} r(t, 0, \xi_0) = O(\lambda^{-\varepsilon/3}) = O(\lambda^{-\varepsilon + 2\varepsilon})$. \hfill $\square$
By a change of $t$ variable we may assume that (3.2) and (3.5) hold with the integration starting at $t = 0$. We obtain new variables $z$ in $\mathbb{R}^{n-1}$ by solving

$$\partial_t z_j = \langle a_j(t), z \rangle, \quad z_j(0) = x_j \quad \forall j$$

Then $D_t + \sum_j \langle a_j(t), x \rangle D_{x_j}$ is transformed into $D_t$ but $D_{x_j} = D_{z_j}$ is unchanged, and we will for simplicity keep the notation $(t, x)$. The linear change of variables is uniformly bounded since $a_j \in C^\infty$, so it preserves the neighborhoods $|x| \lesssim \lambda^{-\nu}$ and the symbol classes $S(\lambda^\mu, g_\nu), \forall \mu, \nu$. We shall then solve the approximate transport equation

$$D_t \varphi + (q_0(t) + i r_0(t, x)) \varphi = 0 \quad (5.7)$$

where $\varphi(0, x) \in S(1, g_\delta)$ is supported where $|x| \lesssim \lambda^{-\delta}$, $q_0(t)$ is given by (5.3) and $r_0$ by (5.4). If we assume $3\varepsilon \leq \delta \leq 7\varepsilon/2$ then by Lemma 5.1 the approximation errors $R \varphi$ will be in $S(\lambda^{3\varepsilon-\delta}, g_\delta)$. In fact, since $\partial_x$ maps $S(1, g_\delta)$ into $S(\lambda^\delta, g_\delta)$ we find $R(t, x, D_x) \varphi_0 \in S(\lambda^{3\varepsilon-\delta}, g_\delta)$ when $|x| \lesssim \lambda^{-\delta}$. We find from Proposition 3.1 that $q_0 \in S(\lambda^{\varepsilon}, g_\varepsilon)$, and if $\varepsilon = 1/8$ and $\delta = (1 + 2\varepsilon)/3$ then we find from Lemma 5.2 that $r_0 \in S(1, g_\delta)$ when $|x| \lesssim \lambda^{-\delta}$ and $t \in I$.

If we choose the initial data $\varphi(0, x) = \phi_0(x) = \phi(\lambda^\delta x)$, where $\phi \in C_0^\infty$ satisfies $\phi(0) = 1$, we obtain the solution

$$\varphi(t, x) = \phi_0(x) \exp(-iB(t, x)) \quad (5.8)$$

where $\partial_t B(t, x) = q_0(t) + i r_0(t, x)$ and $B(0, x) = 0$. We find that $\exp(-iB(t, x)) \in S(1, g_\delta)$ uniformly since condition (3.2) holds with $a_j \equiv 1$, $\partial_t B(t, x) = q_0(t) + i r_0(t, x) \in S(\lambda^\varepsilon, g_\varepsilon) + S(1, g_\delta) \subset S(\lambda^\delta, g_\delta)$ and

$$\partial_x B(t, x) = i \int_0^t \partial_x r_0(s, x) \, ds \in S(\lambda^\delta, g_\delta)$$

by Proposition 3.1 and Lemma 5.2. Thus $\varphi \in S(1, g_\delta)$ uniformly and we find by (5.8) that $|\varphi(t, x)| \leq C|\phi(\lambda^\delta x)|$ so $|x| \lesssim \lambda^{-\delta}$ in supp $\varphi$, which also holds in the original $x$ coordinates.

After solving the eikonal equation and the approximate transport equation, we find from Proposition 3.5 that the terms in the expansion (3.33) are $O(\lambda^{3\varepsilon-\delta})$ if $\varepsilon < 1/7$ and $\delta = (1 + 2\varepsilon)/3$, and all the terms contain the factor $\exp(-iB(t, x))$. We take $\varepsilon = 1/8$ and $\delta = 5/12$ which gives $3\varepsilon - \delta = -1/24 > -\varepsilon/2$ so $3\varepsilon < \delta < 7\varepsilon/2$. Then the expansion in Proposition 3.5 is in multiples of $\lambda^{-1/24}$, and since the error terms of (3.33) are $O(\lambda^{-1/24})$ we will take $\varepsilon = 1/24$ and $\varphi_0 = \varphi$ in the definition of $u_\lambda$ given by (3.31).

The approximate transport equation for $\varphi_k$ in (3.31), $k > 0$, is

$$D_t \varphi_k + (q_0(t) + i r_0(t, x)) \varphi_k = \lambda^{k/24} R_k \exp(iB(t, x)) \quad k \geq 1 \quad (5.9)$$

with $R_k$ is uniformly bounded in the symbol class $S(\lambda^{-k/24}, g_{5/12})$ and is supported where $|x| \lesssim \lambda^{-5/12}$. In fact, $R_k$ contains the error terms from the transport equation (5.1).
and also the terms that are \( O(\lambda^{-k/24}) \) in (3.33) depending on \( \varphi_j \) for \( j < k \). Taking
\[ \varphi_k = \exp(-iB(t,x))\phi_k \]
we obtain the equation
\begin{equation}
D_t \phi_k = \lambda^{k/24}R_k \in S(1,g_{5/12})
\end{equation}
with initial values \( \phi_k(0,x) = 0 \), which can be solved with \( \phi_k \in S(1,g_{5/12}) \) uniformly having support where \( |x| \lesssim \lambda^{-5/12} \). Since \( \exp(-iB(t,x)) \in S(1,g_{5/12}) \) uniformly we find that \( \varphi_k \in S(1,g_{5/12}) \) uniformly having support where \( |x| \lesssim \lambda^{-5/12} \). Proceeding by induction we obtain a solution to (3.33) modulo \( O(\lambda^{-N/24}) \) for any \( N \).

**Proposition 5.3.** Assuming Propositions 3.3 and 3.4 and choosing \( \varepsilon = 1/8, \delta = 5/12 \) and \( \rho = 1/24 \) we can solve the transport equations (5.9) and (5.10) with \( \varphi_k \in S(1,g_{5/12}) \) having support where \( |x| \lesssim \lambda^{-5/12} \), such that \( \varphi_0(0,0) = 1 \) and \( \varphi_k(0,x) \equiv 0, k \geq 1 \).

Now, we get localization in \( x \) from the initial values and the transport equation. To get localization in \( t \) we use that \( \Im B(t) \leq C \) so that \( \Re(-iB) \leq C \). Near \( \partial \Omega \) we may assume that \( \Re(-iB(t)) \ll -\log \lambda \) in an interval of length \( O(\lambda^{-\varepsilon}) = O(\lambda^{-1/8}) \) by (3.5). Thus by applying a cut-off function \( \chi(t) \in S(1,\lambda^{1/4}dt^2) \subset S(1,g_{5/12}) \) such that \( \chi(0) = 1 \) and \( \chi'(t) \) is supported where (3.5) holds, i.e., where \( \varphi_k = O(\lambda^{-N}), \forall k \), we obtain a solution modulo \( O(\lambda^{-N}) \) for any \( N \). In fact, if \( u_\lambda \) is defined by (3.31) and \( Q \) by Proposition 3.3 then \( Q\chi u_\lambda = \chi Q u_\lambda + [Q,\chi]u_\lambda \) where \( [Q,\chi] = D_t \chi \) is supported where \( u_\lambda = O(\lambda^{-N}) \) which gives terms that are \( O(\lambda^{-N}), \forall N \). Thus, by solving the eikonal equation (4.1) for \( \omega \) and the transport equations (5.9) for \( \varphi_k \) for \( k \leq 24N \), we obtain that \( Q\chi u_\lambda = O(\lambda^{-N}) \) for any \( N \) and we get the following remark.

**Remark 5.4.** In Proposition 5.3 we may assume that \( \varphi_k(t,x) = \phi_k(\lambda^{5/12}t,\lambda^{5/12}x) \in S(1,g_{5/12}), k \geq 0 \), with \( \phi_k \in C_0^\infty \) having support where \( |x| \lesssim 1 \) and \( |t| \lesssim \lambda^{5/12}, k \geq 0 \).

6. The Proof of Theorem 2.11

For the proof we will need the following modification of [1] Lemma 26.4.14 which is Lemma 7.1 in [2]. Recall that \( \mathcal{D}_t = \{ u \in \mathcal{D}' : \text{WF}(u) \subset \Gamma \} \) for \( \Gamma \subset T^*\mathbb{R}^n \), and that \( \|u\|_k \) is the \( L^2 \) Sobolev norm of order \( k \) of \( u \in C_0^\infty \).

**Lemma 6.1.** Let
\begin{equation}
\|u_\lambda\|_{(-N)} \leq C\lambda^{-N(\varepsilon + \rho)}
\end{equation}
If $\varphi_0(x_0) \neq 0$ and $\text{Im } \omega(x_0) = 0$ for some $x_0$ then there exists $c > 0$ and $\lambda_0 \geq 1$ so that
\begin{equation}
\|u_\lambda\|_{(-N)} \geq c\lambda^{-(N+\frac{\delta}{2})+(\varepsilon+\varrho)} \lambda \geq \lambda_0
\end{equation}
Let $\Sigma = \bigcap_{\lambda \geq 1} \bigcup_j \text{supp } \varphi_j(\lambda \cdot)$ and let $\Gamma$ be the cone generated by
\begin{equation}
\{(x, \partial \omega(x)), \ x \in \Sigma, \ \text{Im } \omega(x) = 0\}
\end{equation}
then for any real $m$ we find $\lambda^m u_\lambda \to 0$ in $\mathcal{D}'_\Gamma$ so $\lambda^m A u_\lambda \to 0$ in $C^\infty$ if $A$ is a pseudodifferential operator such that $\text{WF}(A) \cap \Gamma = \emptyset$. The estimates are uniform if $\omega \in C^\infty$ uniformly with fixed lower bound on $|d \text{Re } \omega|$, and $\varphi_j \in C^\infty$ uniformly.

We shall use Lemma 6.1 for $u_\lambda$ in (3.31), then $\omega$ will be real valued and $\Gamma$ in (6.4) will be the bicharacteristic $\Gamma_j$ converging to a limit bicharacteristic.

**Proof of Lemma 6.1.** We shall adapt the proof of [4, Lemma 26.4.14] to this case. By making the change of variables $y = \lambda^\varepsilon x$ we find that
\begin{equation}
\hat{u}_\lambda(\xi) = \lambda^{(n-1)\delta/2-n\varepsilon} \sum_{j=0}^{M} \lambda^{-j\kappa} \int e^{i\lambda^\varrho \omega(y) - (y,\xi/\lambda^\varepsilon)} \varphi_j(\lambda^{\delta-\varepsilon} y) dy
\end{equation}
Let $U$ be a neighborhood of the projection on the second component of the set in (6.4). When $\xi/\lambda^{\varrho} \notin U$ then for $\lambda \gg 1$ we have that
\begin{align*}
\bigcup_j \text{supp } \varphi_j(\lambda^{\delta-\varepsilon} \cdot) \ni y \mapsto (\lambda^\varrho \omega(y) - (y,\xi/\lambda^\varepsilon)) / (\lambda^\varrho + |\xi|/\lambda^\varepsilon) \\
= (\omega(y) - (y,\xi/\lambda^{\varrho}))/ (1 + |\xi|/\lambda^{\varrho})
\end{align*}
is in a compact set of functions with non-negative imaginary part with a fixed lower bound on the gradient of the real part. Thus, by integrating by part in (6.5) we find for any positive integer $m$ that
\begin{equation}
|\hat{u}_\lambda(\xi)| \leq C_m \lambda^{-(n-1)\delta/2+m(\delta-\varepsilon)} (\lambda^\varrho + |\xi|/\lambda^\varepsilon)^{-m} \xi/\lambda^{\varrho} \notin U \quad \lambda \gg 1
\end{equation}
This gives any negative power of $\lambda$ for $m$ large enough since $\delta < \varepsilon + \varrho$. If $V$ is bounded and $0 \notin \overline{V}$ then since $u_\lambda$ is uniformly bounded in $L^2$ we find
\begin{equation}
\int_{\tau V} |\hat{u}_\lambda(\xi)|^2 (1 + |\xi|^2)^{-N} d\xi \leq C_V \tau^{-2N} \quad \tau \geq 1
\end{equation}
Using this estimate with $\tau = \lambda^{\varrho}$ together with the estimate (6.6) we obtain (6.2). If $\chi \in C^\infty_0$ then we may apply (6.6) to $\chi u_\lambda$, thus we find for any positive integer $j$ that
\begin{equation}
|\chi \hat{u}_\lambda(\xi)| \leq C_j \lambda^{-(n-1)\delta/2+j(\delta-\varepsilon)} (\lambda^\varrho + |\xi|/\lambda^\varepsilon)^{-j} \xi \in W \quad \lambda \gg 1
\end{equation}
if $W$ is any closed cone with $\Gamma \cap (\text{supp } \chi \times W) = \emptyset$. Thus we find that $\lambda^m u_\lambda \to 0$ in $\mathcal{D}'_\Gamma$ for every $m$. To prove (6.3) we assume $x_0 = 0$ and take $\psi \in C^\infty_0$. If $\text{Im } \omega(0) = 0$ and
\[ \varphi(0) \neq 0 \] we find
\[ \chi^{n(\varepsilon + \varphi) - (n-1)\delta/2} e^{-i\lambda^e \operatorname{Re} \omega(0)} \langle u_{\lambda}, \psi(\lambda^{\varepsilon + \varphi}) \rangle = \int e^{i\lambda^e (\omega(x/\lambda^e) - \omega(0))} \varphi_j(x/\lambda^e + \varphi) \lambda^{-j\kappa} \, dx \]
\[ \rightarrow \int e^{i(\operatorname{Re} \partial_z \omega(0), x)} \varphi_j(x) \varphi_0(0) \, dx \quad \lambda \to +\infty \]
which is not equal to zero for some suitable \( \psi \in C_0^\infty \). In fact, we have \( \varphi_j(x/\lambda^e + \varphi) = \varphi_j(0) + O(\lambda^{\delta - \varepsilon - \varphi}) \to \varphi_j(0) \) when \( \lambda \to \infty \), because \( \delta < \varepsilon + \varphi \). Since
\[ \| \varphi(\lambda^{\varepsilon + \varphi}) \|_{(N)} \leq C \lambda^{(N-n/2)(\varepsilon + \varphi)} \]
we obtain that \( 0 < c \leq \lambda^{(N+\frac{n}{2})(\varepsilon + \varphi) - (n-1)\delta/2} \| u \|_{(-N)} \) which gives (6.3) and the lemma. □

**Proof of Theorem 2.71** Assume that \( \Gamma \) is a limit bicharacteristic of \( P \). We are going to show that (2.23) does not hold for any \( \nu \), \( N \) and any pseudodifferential operator \( A \) such that \( \Gamma \cap \operatorname{WF}(A) = \emptyset \). This means that there exists \( 0 \neq u_j \in C_0^\infty \) such that
\[\| u_j \|_{(-N)}/(\| P^* u_j \|_{(\nu)} + \| u_j \|_{(-N-n)} + \| Au_j \|_{(0)}) \to \infty \quad \text{when} \ j \to \infty \]
which will contradict the local solvability of \( P \) at \( \Gamma \) by Remark 2.12.

Let \( \Gamma_j \subset \Sigma \cap S^* X \) be a sequence of semibicharacteristics of \( p \) that converges to the limit bicharacteristic \( \Gamma \subset \Sigma_2 \) and let \( \lambda_j \) be given by (2.10) and (3.4) with \( \varepsilon > 0 \) which will be chosen later. Now the conditions and conclusions are invariant under symplectic changes of homogeneous coordinates and multiplication by elliptic pseudodifferential operators. By Proposition 3.3, we may assume that the coordinates are chosen so that \( \Gamma_j = I \times (0, 0, \xi_j) \) with \( |\xi_j| = 1 \), and for any \( 0 < \varepsilon < 1/3 \) and \( c > 0 \) we can write \( B_j P^* = Q_j + R_j \in \Psi^{1-\varepsilon}_{1-\varepsilon} \) where \( B_j \in \Psi^{\varepsilon}_{1-\varepsilon} \) uniformly, \( \Gamma_j \cap \operatorname{WF}_x(R) = \emptyset \) uniformly and \( Q_j \) has symbol
\[ \tau - r(t, x, \xi_j) + q_0(t, x, \xi_j) + r_0(t, x, \xi_j) \]
when the homogeneous distance to \( \Gamma_j \) is less than \( c|\xi|^{-\varepsilon} \sim \lambda_j^{-\varepsilon} \). We have that \( r_0 \in S^{3\varepsilon-1}_{1-\varepsilon} \), \( q_0 \in S^\varepsilon_{1-\varepsilon} \) is given by (3.26), and \( r \in S^{1-\varepsilon}_{1-\varepsilon} \) with real part vanishing of second order at \( \Gamma_j \), and the bounds are uniform in the symbol classes.

Now, we may replace the norms \( \| u \|_{(s)} \) in (6.7) by the norms
\[ \| u \|^2_s = \| (D_x)^s u \|^2 = \int \langle \xi \rangle^{2s} |\hat{\varphi}(\tau, \xi)|^2 \, d\tau d\xi \]
and the corresponding spaces \( H_s \). In fact, the quotient \( \langle \xi \rangle / \langle (\tau, \xi) \rangle \approx 1 \) when \( |\tau| \lesssim |\xi| \), thus in a conical neighborhood of \( \Gamma \). So replacing the norms in the estimate (6.7) only changes the constant and the operator \( A \) in the estimate (2.23). By using Proposition 3.3, we may assume that the grazing Lagrangean space \( L_j(w) \equiv \{ (s, y; 0, 0) : (s, y) \in \mathbb{R}^n \} \), \( \forall w \in \Gamma_j \), after conjugation with a uniformly bounded \( C^1 \) section \( F(t) \) of homogeneous Fourier integral operators, then \( \partial_y^2 \operatorname{Re} r = 0 \) at \( \Gamma_j \). Observe that for each \( t \) we find that \( F(t) \)
is uniformly continuous in local $H_s$ spaces, which we may use in (6.7) after changing $A$. Also the conjugation of $F(t)$ with the operator with symbol (6.8) has a uniformly bounded expansion. In fact, this follows since $t \mapsto F(t) \in C^1$ are homogeneous Fourier integral operators in the $x$ variables and these preserve the symbol classes. By changing $A$ again, we may then replace the local $\|u\|_s$ norms by the norms $\|u\|_{(s)}$ in (6.7) so that we can use Lemma 6.1.

Now, by choosing $\delta = 5/12$, $\varepsilon = 1/8$ and $\varrho = 1/24$ and using Propositions 3.5, 4.2, 5.3 and Remark 5.4 we can for each $\Gamma_j$ construct approximate solution $u_{\lambda_j}$ on the form (3.31) so that $Qu_{\lambda_j} = O(\lambda_j^{-k})$, for any $k$. The real valued phase function is equal to $\langle x, \xi_j \rangle + \omega_j(t, x)$ where $|\xi_j| = 1$ and $\omega_j(t, x) \in S(\lambda_j^{-7/8}, g_{3/8})$ and the values of

$$(t, x; \lambda_j \partial_t \omega_j(t, x), \lambda_j(\xi_j + \partial_x \omega_j(t, x)))$$

have homogeneous distance $\lesssim \lambda_j^{-5/12}$ to the rays through $\Gamma_j$ when $|x| \lesssim \lambda_j^{-5/12}$, thus on supp $u_{\lambda_j}$. Observe that if $\lambda_j \gg 1$ then we have that $|\xi_0 + \partial_x \omega_j(t, x)| \approx 1$ in supp $u_{\lambda_j}$. In fact, we have

$$\omega_j(t, x) = \lambda_j^{-7/8} \tilde{\omega}_j(\lambda_j^{3/8} t, \lambda_j^{3/8} x)$$

where $\tilde{\omega}_j \in C^\infty$ uniformly so $\partial_x \omega_j = O(\lambda_j^{-1/2})$. Now

$$\lambda_j(\langle x, \xi_j \rangle + \omega_j(t, x)) = \lambda_j^{5/8}(\lambda_j^{3/8} x, \xi_j) + \lambda_j^{1/8} \tilde{\omega}_j(\lambda_j^{3/8} t, \lambda_j^{3/8} x) \quad \text{when } |x| \lesssim \lambda_j^{-5/12}$$

thus $\delta = 5/12$, $\varrho = 5/8$, $\varepsilon = 3/8$ and $\kappa = 1/24$ in (6.1) so $\varepsilon + \varrho = 1 > \delta > \varepsilon$.

The amplitude functions for $u_{\lambda_j}$ are $\varphi_{k,j}(t, x) = \phi_{k,j}(\lambda_j^{5/12} t, \lambda_j^{5/12} x)$ where $\phi_{k,j} \in C^\infty$ uniformly in $j$ with fixed compact support in $x$, but in $t$ the support is bounded by $C\lambda_j^{-5/12}$. Thus $u_{\lambda_j}$ will satisfy the conditions in Lemma 6.1 uniformly. Clearly differentiation of $Qu_{\lambda_j}$ can at most give a factor $\lambda_j$ since $\delta < \varepsilon + \varrho = 1$. Because of the bound on the support of $u_{\lambda_j}$ we may obtain that

$$(6.9) \quad \|Qu_{\lambda_j}\|_{(\nu)} = O(\lambda_j^{-N-n})$$

for any given $\nu$.

If $\text{WF}(A) \cap \Gamma = \emptyset$, then we find $\text{WF}(A) \cap \Gamma_j = \emptyset$ for large $j$, so Lemma 6.1 gives $\|Au_{\lambda_j}\|_{(0)} = O(\lambda_j^{-N-n})$ when $j \to \infty$. On supp $u_{\lambda_j}$ we have $x = O(\lambda_j^{-5/12})$ so the values of $(t, x; \lambda_j \partial_t \omega_j(t, x), \lambda_j(\xi_j + \partial_x \omega_j(t, x)))$ have homogeneous distance $\lesssim \lambda_j^{-5/12}$ to the rays through $\Gamma_j$. Thus, if $R_j \in S_{7/8,1/8}^9$ such that $WF_{1/8}(R_j) \cup \Gamma_j = \emptyset$ uniformly then we find from the expansion (3.32) that all the terms of $R_j u_{\lambda_j}$ vanish for large enough $\lambda_j$. In fact, since $\lambda_j^{-5/12} \ll \lambda_j^{-1/8}$ for $j \gg 1$, we find for any $\alpha$ and $K$ that

$$\partial^\alpha R_j(t, x; \lambda_j((0, \xi_j) + \partial_t \omega_j(t, x))) = O(\lambda_j^{-K})$$
in $\bigcup_k \text{supp} \varphi_{k,j}$. As before, we find that $\|R_j u_{\lambda_j}\|_{(\nu)} = \mathcal{O}(\lambda_j^{-N-n})$ by the bound on the support of $u_{\lambda_j}$, so we obtain from (6.9) that

$$\|P^* u_{\lambda_j}\|_{(\nu)} = \mathcal{O}(\lambda_j^{-N-n})$$

for any given $\nu$.

Since $\varepsilon + \varrho = 1$ and $\delta > 0$ we also find from Lemma 6.1 that

$$\lambda_j^{-N} = \lambda_j^{-N(\varepsilon + \varrho)} \gtrsim \|u_{\lambda_j}\|_{(-N)} \gtrsim \lambda_j^{-(N + \frac{n}{2})(\varepsilon + \varrho) + (n-1)\delta/2} \geq \lambda_j^{-N-n/2}$$

when $\lambda_j \geq 1$. We obtain that (6.7) holds for $u_j = u_{\lambda_j}$ when $j \to \infty$, so Remark 2.12 gives that $P$ is not solvable at the limit bicharacteristic $\Gamma$. 

\[ \square \]

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