THE HELMHOLTZ EQUATION WITH $L^p$ DATA AND BOCHNER-RIESZ MULTIPLIERS

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Abstract. We prove the existence of $L^2$ solutions to the Helmholtz equation $(-\Delta - 1)u = f$ in $\mathbb{R}^n$ assuming the given data $f$ belongs to $L^{(2n+2)/(n+5)}(\mathbb{R}^n)$ and satisfies the “Fredholm condition” that $\hat{f}$ vanishes on the unit sphere. This problem, and similar results for the perturbed Helmholtz equation $(-\Delta - 1)u = -Vu + f$, are connected to the Limiting Absorption Principle for Schrödinger operators.

The same techniques are then used to prove that a wide range of $L^p \to L^q$ bounds for Bochner-Riesz multipliers are improved if one considers their action on the closed subspace of functions whose Fourier transform vanishes on the unit sphere.

We consider the existence of a well-defined solution map for the Helmholtz equation in Euclidean space

$$\begin{cases}
(-\Delta - 1)u = f & \text{in } \mathbb{R}^n \\
u \in L^2(\mathbb{R}^n)
\end{cases}$$

By conjugating with dilations, the same problem can be posed with an operator $(-\Delta - \lambda^2)$, $\lambda > 0$ with minimal modification. These equations are translation invariant, so it would be desirable to choose $f$ from a function space whose norm is also translation invariant. Our goal is to establish existence of solutions and a norm bound for $u$ in terms of the $L^p$-norm of the given data $f$, provided $f$ is formally orthogonal to all plane waves of unit frequency.

The Fourier dual formulation of (1) is

$$\begin{cases}
\hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 - 1} \\
\hat{u} \in L^2(\mathbb{R}^n)
\end{cases}$$

with respect to the definition $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) \, dx$. The corresponding Plancherel identity is $\|\hat{u}\|_2 = (2\pi)^n/2 \|u\|_2$.

It is immediately clear from (2) that solutions should be unique, as $|\xi|^2 - 1$ is nonzero almost everywhere and the Fourier Transform is (a scalar multiple of) a unitary map between $L^2(dx)$ and $L^2(d\xi)$. One can also infer that solutions exist only if $\hat{f}$ vanishes on the unit sphere in a suitable sense, and also the restrictions of $\hat{f}$ to the sphere of radius $r$ must be controlled as $r$ approaches 1.

It would be sufficient, for example, if the map $S(r) = \hat{f}(r \cdot)|_{S^{n-1}}$ (taking $\mathbb{R}_+$ into $L^2(S^{n-1})$) was Hölder continuous of order $\gamma > \frac{1}{2}$ at $r = 1$ and vanished there.
Then the scalar restriction function

\begin{equation}
F(r) = \|\hat{f}\|_{L^2(r, S^{n-1})}^2
\end{equation}

would be \(O(|r - 1|^{2\gamma})\), and the formula

\begin{equation}
\|\hat{u}\|_2^2 = \int_0^\infty \frac{F(r)}{(r^2 - 1)^2} \, dr
\end{equation}

would be locally integrable at \(r = 1\).

In fact the desired continuity can be achieved if \(\hat{f}\) belongs to the Sobolev space \(W^{1,2}(\mathbb{R}^n)\). While \(S(r)\) is only Hölder continuous of order exactly \(\frac{1}{2}\), the one-dimensional Hardy inequality suffices to establish integrability of (4). This argument plays a central role in Agmon’s bootstrapping method for the decay of eigenfunctions of a Schrödinger operator \([1]\). For the Helmholtz equation in particular the following result is proved there.

**Theorem 1** (Agmon). Suppose \((1 + |x|)^\beta f \in L^2(\mathbb{R}^n)\) for some \(\beta > \frac{1}{2}\), and \(\hat{f}\) vanishes on the unit sphere in the \(L^2\)-trace sense. Then there exists a unique function \(u\) such that \((-\Delta - 1)u = f\) and \((1 + |x|)^{\beta - 1}u \in L^2(\mathbb{R}^n)\).

It is not obvious that a similar result should hold for \(f \in L^p(\mathbb{R}^n)\) without weights, regardless of the exponent, as an \(L^p\) condition typically doesn’t guarantee that \(S(r)\) is Hölder continuous of any positive order. Nevertheless an \(L^2\) solution operator for the Helmholtz equation exists for data in a narrow range of \(L^p\) spaces.

**Theorem 2.** Let \(n \geq 3\) and \(\max(1, \frac{2n}{n+1}) \leq p < \frac{2n+2}{n+3}\), with \((n, p) \neq (4, 1)\). Suppose \(f \in L^p(\mathbb{R}^n)\) and \(\hat{f}\) vanishes on the unit sphere in the \(L^2\)-trace sense.

There exists a unique \(u \in L^2(\mathbb{R}^n)\) such that \((-\Delta - 1)u = f\). Furthermore, \(\|u\|_2 \leq C_{n,p} \|f\|_p\).

There is no statement in dimensions 1 or 2 because \(\frac{2n+2}{n+3}\) comes from Sobolev embedding. It can be disregarded if one applies any sort of cutoff to remove high frequencies. As a special case, the sharp cutoff at \(|\xi| = 1\) leaves a Bochner-Riesz multiplier of order -1. For further discussion of these operators we adopt the definition

\begin{equation}
(S^\alpha f)(\xi) = (1 - |\xi|^2)_{+}^{\alpha} f(\xi).
\end{equation}

For \(\alpha \leq -1\) we define \(S^\alpha\) by (formal) positivity of the operator rather than by analytic continuation. This preserves the multiplicative structure \(S^\alpha S^\beta = S^{\alpha + \beta}\), however it comes at the cost that \(S^\alpha\) will not have a bounded action on general Schwartz functions once \(\alpha \leq -1\).

Neither the less, \(S^\alpha\) may behave well when applied to functions whose Fourier transform vanishes on the unit sphere, as stated below.

**Theorem 3.** Let \(n \geq 2\) and \(\frac{1}{2} \leq \alpha < \frac{3}{2}\). Suppose \(f \in L^p(\mathbb{R}^n)\), \(1 \leq p \leq \frac{2n+2}{n+1+4\alpha}\) with \((\alpha, p) \neq \left(\frac{1}{2}, \frac{2n+2}{n+3}\right)\), and suppose \(\hat{f}\) vanishes on the unit sphere.

Then \(\|S^{-\alpha} f\|_2 \lesssim \|f\|_p\).

Both Theorems 2 and 3 are easily derived from the following statement, which is our main technical result.
Proposition 4. Let \( n \geq 2 \) and \( \frac{1}{2} < \alpha < \frac{3}{2} \). Suppose \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \frac{2n+2}{n+1+4\alpha} \). There is a constant \( C_\alpha \) such that

\[
\left| \int_{\frac{1}{2}<|\xi|<\frac{1}{2}} \frac{|\hat{f}(\xi)|^2}{((1-|\xi|^2)^2 + \varepsilon^2)^\alpha} \, d\xi \right| \lesssim \left\| \frac{C_\alpha}{\varepsilon^{2\alpha-1}} \| \hat{f} \|_{L^2(\mathbb{R}^{n-1})} \right\|^2 \lesssim \| f \|_{p}^2
\]

with a constant that remains bounded in the limit \( \varepsilon \to 0 \).

In both theorems, it is given that \( \hat{f} \) vanishes on the unit sphere, eliminating the \( \varepsilon^{1-2\alpha} \| f \|_{L^2(\mathbb{R}^{n-1})} \) term from the left side of (8). Assuming Proposition 4 holds, the same inequality is then true with \( \varepsilon = 0 \) by monotone convergence. The Hausdorff-Young inequality is more than sufficient to bound the left-side integral over the center region \( \{ |\xi| < \frac{1}{2} \} \) for any \( f \in L^p \), \( 1 \leq p \leq 2 \).

For Theorem 2 let \( \chi \in C_\infty^\infty(\mathbb{R}^n) \) be any smooth cutoff that is identically 1 in the ball \( \{ |\xi| \leq \frac{1}{2} \} \) and has support in the ball of radius \( \frac{1}{2} \). Theorem 2 then reduces to the \( \alpha = 1 \) case of Proposition 4 combined with a Sobolev embedding estimate for the high frequency tail \( \frac{1}{2\varepsilon} \| f \|_{\varepsilon} \). In a similar manner, all cases of Theorem 8 with \( \alpha > \frac{1}{2} \) follow from the Proposition by applying the multiplier of the unit ball, which is bounded on \( L^2(\mathbb{R}^n) \).

Finally, if \( \alpha = \frac{1}{2} \) and \( p \in [\frac{2n+2}{n+3}, \frac{2n+2}{n+1+4\alpha}] \), we have already established Theorem 8 for the pair \( (\beta, p) \) with \( \beta = \min\{(n+1)(\frac{1}{2p} - \frac{1}{4}), 1\} \). Since \( \beta > \frac{1}{2} \), it follows that \( \| S^{-\beta} f \|_2 \leq \| S^{-\beta} f \|_2 \) by Plancherel’s formula.

Sharpness of the upper exponent \( \frac{2n+2}{n+1+4\alpha} \) is verified using a Knapp-type example. Let \( \hat{f} \) be a smooth compactly supported function, suitably scaled to have support in the slab \( \{ |\xi'| \leq \delta, 1 - 2\delta^2 \leq \xi_n \leq 1 - \delta^2 \} \), where \( \xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1} \) and unit height. Then \( |f(x)| \sim \delta^{n+1} \) over the dual region \( \{|x'| \leq \delta^{-1}, |x_n| \leq \delta^{-2}\} \) and has rapid decay elsewhere. It follows that \( \| f \|_p \sim \delta^{(n+1)(1-p^{-1})} \) and \( \| S^{-\alpha} f \|_2 \sim \delta^{\frac{n+1}{2} - 2\alpha} \). If \( p > \frac{2n+2}{n+1+4\alpha} \), then \( (n+1)(1-p^{-1}) > \frac{n+1}{2} - 2\alpha \) and Theorem 8 fails by taking \( \delta \) to zero.

Remark 1. When \( 0 < \alpha < \frac{1}{2} \), no vanishing condition on the unit sphere is needed in the statement of Theorem 3. The range of viable exponents is once again \( p \in [1, \frac{2n+2}{n+3}] \) including the endpoints 3. The full range of \( L^p \to L^q \) mappings in this regime is established in [2].

Remark 2. The statement of Proposition 4 is not true for \( \alpha > 1 \) if integration is limited to the inner annulus \( \{ \frac{1}{2} < |\xi| < 1 \} \). An additional remainder term of order \( \varepsilon^{2-2\alpha} \) is present in that case. The same remainder term appears with the opposite sign if one integrates over the outer annulus \( \{ 1 < |\xi| < \frac{3}{2} \} \). This illustrates a difference in behavior between “one-sided” and "two-sided" Bochner-Riesz multipliers of order below \( -1 \), with the former being modestly more singular than the latter.

Remark 3. The endpoint case \( \alpha = \frac{1}{2}, p = \frac{2n+2}{n+3} \) is quite delicate. The conclusion is certainly false if one does not assume that \( \hat{f} \) vanishes on the unit sphere. In one dimension it remains false even with the vanishing condition. Since \( S^{-\frac{1}{2}} \) in one dimension is closely related to the fractional integral operator \( I_{1/2} \), a stronger condition that \( \varepsilon^\pm i\pi f \) belongs to the Hardy space \( H^1(\mathbb{R}) \) is needed to guarantee that \( S^{-1/2} f \in L^2(\mathbb{R}) \).
The one-dimensional counterexamples do not generalize well to \( n \geq 2 \). We believe it is an open problem whether Theorem 3 is true in these endpoint cases.

It is possible to extend Theorem 3 further by interpolation with other known estimates for Bochner-Riesz operators, subject to a few technical limitations. In this paper we do not assemble a full catalog of such estimates but instead consider a family of bounds that are sharp with respect to Knapp counterexamples.

**Theorem 5.** Let \( n \geq 2 \) and \( \beta \in \left( \frac{1}{2}, \frac{3}{2} \right) \) with \( \beta \leq \frac{n+1}{4} \). Suppose \( f \in L^{\frac{2n+2}{2n+1}}(\mathbb{R}^n) \) and \( \alpha \in [\beta, 2\beta] \) with \( \alpha < 2 \). Then

\[
\|S^{-\alpha}f\|_{\frac{2n+2}{2n+1}} \lesssim \|f\|_{\frac{2n+2}{2n+1}}.
\]

**Remark 4.** The restriction \( \alpha < 2 \) may be removed if one instead considers the analytic family of operators \( \tilde{S}^{-\alpha} = \Gamma(1-\alpha)^{-1}S^{-\alpha} \). This will be evident in the proof.

Theorem 5 plays an important role in the spectral theory of Schrödinger operators \( H = -\Delta + V(x) \) with a short-range potential. Namely, it is used in a bootstrapping argument to show that any singular part of the essential spectrum of \( H \) must contain embedded eigenvalues. Thus the spectral measure on compact subsets of \( [0, \infty) \setminus \sigma_{pp}(H) \) is absolutely continuous and satisfies an assortment of uniform mapping properties. In section 2 we present a similar bootstrapping application using Theorem 2 as the primary device. These results are contained within the more general Limiting Absorption Principle of Ionescu and Schlag [4], and serve as an instructive special case.

The discussion of perturbed Schrödinger operators naturally raises the question of whether there is a similar existence theorem for the Helmholtz equation \((-\Delta + V - 1)u = f\). Using resolvent identities we are able to prove the following.

**Theorem 6.** Let \( n \geq 3 \), \( p = \frac{2n+2}{n+5} \), and suppose \( V \in L^{\frac{2n+1}{n+5}}(\mathbb{R}^n) \). There is a subspace \( X \subset L^p(\mathbb{R}^n) \), isomorphic to the subspace \( X_0 \subset L^p(\mathbb{R}^n) \) of functions whose Fourier transform vanishes on the unit sphere, with the following property: For each \( f \in X \) there exists a unique \( u \in L^2(\mathbb{R}^n) \) such that \((-\Delta + V - 1)u = f\). Furthermore, \( \|u\|_2 \leq C_n \|f\|_p \).

**Remark 5.** The integrability condition \( V \in L^{\frac{2n+1}{n+5}}(\mathbb{R}^n) \) appears as a sharp threshold for short-range potentials in both [4] and [6]. For \( 1 \leq p < \frac{2n+2}{n+5} \) a stronger set of constraints on \( V \) may be required.

## 1. Proof of Proposition 4

The proof of Proposition 4 mirrors that of the sharp Stein-Tomas restriction theorem. We follow the exposition in [7] most closely.

Let \( \sigma_r \) denote the surface measure on \( r^{2n-1} \) inherited from its embedding in \( \mathbb{R}^n \). The main estimate will be a bound on \( \langle K^*_1 \ast f, f \rangle \), where

\[
K^*_1 = \int_0^\frac{1}{2} \frac{\tilde{\sigma}_r - \tilde{\sigma}_1}{(1-r^2)^2 + \varepsilon^2} dr = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\tilde{\sigma}_{1+s} - \tilde{\sigma}_1}{(s^2(2+s)^2 + \varepsilon^2)^\alpha} ds.
\]
This is almost equal to the lefthand expression in (9), with the only discrepancy arising in the coefficient of \( \langle \tilde{\sigma}_1 * f, f \rangle \). More precisely,

\[
\text{[Left side of (9)] - } \langle K^*_f * f, f \rangle = \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( (1 - r^2)^2 + \varepsilon^2 \right) - \frac{C_{\alpha}}{2^{2\alpha - 1}} \right) \langle \tilde{\sigma}_1 * f, f \rangle = O(1) \langle \tilde{\sigma}_1 * f, f \rangle.
\]

Since \( f \in L^p(\mathbb{R}^n) \) with \( p \leq \frac{2n+2}{n+3} \), the \( O(1) \) term can be absorbed into the right side of (9) by the Stein-Tomas theorem.

The integrand in (9) may become highly singular at \( s = 0 \) as \( \varepsilon \) decreases. However the denominator is approximately an even function of \( s \), while the leading order behavior of the numerator is an odd function. To be precise, let

\[
A_{\text{even}}(s) = \frac{1}{2} \left( \frac{1}{(s^2(2 + s)^2 + \varepsilon^2)^{\alpha}} + \frac{1}{(s^2(2 - s)^2 + \varepsilon^2)^{\alpha}} \right)
\]

\[
A_{\text{odd}}(s) = \frac{1}{2} \left( \frac{1}{(s^2(2 + s)^2 + \varepsilon^2)^{\alpha}} - \frac{1}{(s^2(2 - s)^2 + \varepsilon^2)^{\alpha}} \right).
\]

Then

\[
(9) \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( A_{\text{even}}(s)(\tilde{\sigma}_{1+s} - 2\tilde{\sigma}_1 + \tilde{\sigma}_{1-s}) + A_{\text{odd}}(s)(\tilde{\sigma}_{1+s} - \tilde{\sigma}_{1-s}) \right) ds
\]

The main size bounds for \( A_{\text{even}} \) and \( A_{\text{odd}} \) are:

\[
(10) \quad |A_{\text{even}}(s)| \lesssim s^{-2\alpha}, \quad |A_{\text{odd}}(s)| \lesssim s^{1-2\alpha} \quad \text{uniformly in } \varepsilon > 0.
\]

It is a common practice to estimate the inverse Fourier transform of a surface measure by decomposing the surface into smaller regions where stationary phase methods can be applied. Consider a conical decomposition \( \sum_{j=1}^{n} \eta_j(\frac{\xi}{|\xi|}) = 1 \) where each smooth cutoff \( \eta_j \) is supported in the region where \( |\xi_j| \sim |\xi| \). One may symmetrize so that each \( \eta_j \) is invariant under reflections across any one of the coordinate planes. Then (9) splits into a directional sum

\[
K^*_f = \sum_{j=1}^{n} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \eta_j \sigma_{1+s} - (\eta_j \sigma_1) \right) ds.
\]

Let \( K^*_f \) denote the \( j = n \) term of this sum and write coordinates in \( \mathbb{R}^n \) as \( (x', x_n) \) or \( (\xi', \xi_n) \). We will make further estimates on \( K^*_f \) as a representative element.

Inside the support of \( \eta_n \sigma_r \), the relationship \( \xi_n = \pm (r^2 - |\xi'|^2)^{1/2} \) expresses \( \xi_n \) as a smooth function of \( \xi' \) on each hemisphere. Then the inverse Fourier transform of \( \eta_n \sigma_r \) takes the form

\[
(\eta_n \sigma_r)(x', x_n) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \frac{r \eta_n(\frac{\xi'}{r}, \pm \sqrt{r^2 - |\xi'|^2})}{\sqrt{r^2 - |\xi'|^2}} e^{i(x' \cdot \xi' + x_n \sqrt{r^2 - |\xi'|^2})} d\xi'.
\]

For \( \frac{1}{2} < r < \frac{3}{2} \), the Hessian of the phase function is bounded below by \( x_n \) times the \( (n-1) \)-identity matrix and the initial fraction is a uniformly smooth function. This leads to the pointwise bound

\[
|\eta_n \sigma_r|^2(x', x_n) \lesssim (1 + |x_n|)^{\frac{2n}{2}}.
\]
for $r$ in this range. Furthermore one can differentiate with respect to $r$ under the integral sign to obtain bounds

$$|\partial^k_r (\eta_0 \sigma_r)(x', x_n)| \lesssim (1 + |x_n|)^{\frac{k}{2} - \frac{n}{2} + k.} \tag{12}$$

Taylor remainder estimates then imply that

$$|(\eta_0 \sigma_{1+s})(x) - (\eta_0 \sigma_{1-s})(x)| \lesssim \min(|s| (1 + |x_n|), 1)(1 + |x_n|)^{\frac{k}{2} - \frac{n}{2}},$$

and

$$|(\eta_0 \sigma_{1+s})(x) - 2(\eta_0 \sigma_1)(x) + (\eta_0 \sigma_{1-s})(x)| \lesssim \min(s^2 (1 + |x_n|)^{2}, 1)(1 + |x_n|)^{\frac{k}{2} - \frac{n}{2}}$$

while $|s| < \frac{1}{2}$. Plugging these and (10) into the appropriately modified version of (9), one concludes that

$$|K_2^s(x)| \lesssim (1 + |x_n|)^{\frac{4n - 1 - n}{2}}. \tag{13}$$

In other words, for a fixed choice of $x_n$, the restricted convolution operator

$$Tg(x') = \int_{\mathbb{R}^{n-1}} K_2^s(x' - y', x_n)g(y') dy'$$

maps $L^1(\mathbb{R}^{n-1})$ to $L^\infty(\mathbb{R}^{n-1})$ with operator norm controlled by $(1 + |x_n|)^{\frac{4n - 1 - n}{2}}$.

One can also determine the size of $T$ as an operator on $L^2(\mathbb{R}^{n-1})$. This bound is given by the essential supremum of the $x'$-Fourier transform of the convolution kernel $K_2^s$. Since $K_2^s$ is a superposition of the inverse Fourier transforms of $(\eta_0 \sigma_s)$ as in (11), the $x'$-Fourier transform reverses the procedure in all except the $x_n$ variable. More precisely,

$$\int_{\mathbb{R}^{n-1}} e^{-ix' \cdot x'} K_2^s(x', x_n) dx' = \frac{1}{2\pi} \int_{\{\xi'\} \times \mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{e^{ix_n \xi_n (\sigma_{1+s} - \sigma_1) \eta_n}}{(s^2 (2 + s)^2 + \epsilon^2)^n} ds d\xi_n.$$

If the integral over $s$ is split into even and odd contributions as in (9), the result is

$$\int_{\mathbb{R}^{n-1}} e^{-ix' \cdot x'} K_2^s(x', x_n) dx'$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^{n-1}} \left( A_{\text{even}}(s) \int_{\{\xi'\} \times \mathbb{R}} e^{ix_n \xi_n (\sigma_{1+s} - 2 \sigma_1 + \sigma_{1-s}) \eta_n} d\xi' + A_{\text{odd}}(s) \int_{\{\xi'\} \times \mathbb{R}} e^{ix_n \xi_n (\sigma_{1+s} - \sigma_{1-s}) \eta_n} d\xi' \right) ds$$

For a fixed choice of $\xi' \in \mathbb{R}^{n-1}$ and radius $r > 0$, the line $\{\xi'\} \times \mathbb{R}$ intersects the support of $\sigma_r$ only when $\xi_n = \pm \sqrt{r^2 - |\xi'|^2}$. Thus for $|\xi'| < r$

$$\int_{\{\xi'\} \times \mathbb{R}} e^{ix_n \xi_n \sigma_r \eta_n} d\xi' = \frac{2 \cos \left( x_n \sqrt{r^2 - |\xi'|^2} \right) \eta_n \left( \xi_n, \sqrt{1 - (|\xi'|/r)^2} \right)}{\sqrt{1 - (|\xi'|/r)^2}}$$

and is zero otherwise. The denominator accounts for the angle of intersection between the line and surface. It is bounded away from zero within the support of $\eta_n$, so the integral expression is a smooth bounded function of $\xi'$ and $r$. Within the range $\frac{1}{2} < r < \frac{3}{2}$, its first two derivatives with respect to $r$ are bounded by
provided 2

(1 + |x_n|) and (1 + |x_n|)^2 respectively. It follows that

\[ \left| \int_{\mathbb{R}^{n-1}} e^{-ik' \cdot x'} K_2(x', x_n) \, dx' \right| \lesssim \int_{-\frac{i}{2}}^{\frac{i}{2}} A_{\text{even}}(s) \max(s^2(1 + |x_n|)^2, 1) + A_{\text{odd}}(s) \max(|s|(1 + |x_n|), 1) \, ds \]

\[ \lesssim (1 + |x_n|)^{2\alpha - 1} \]

and therefore T is a bounded operator on \( L^2(\mathbb{R}^{n-1}) \) with norm comparable to \(|x_n|^{2\alpha - 1}\). Interpolating with the previous \( L^1 \to L^\infty \) bound shows that

\[ \| Tg \|_{L^p(\mathbb{R}^{n-1})} \lesssim (1 + |x_n|)^{2\alpha + \frac{n-3}{p} + \frac{1-n}{p}} \| g \|_{L^p(\mathbb{R}^{n-1})}, \quad 1 \leq p \leq 2. \]

Returning to the action of \( K_2^\gamma \) on functions in \( \mathbb{R}^n \), these estimates imply that

\[ \| K_2^\gamma \ast f \|_p \lesssim \left\| \int_{-\infty}^{\infty} (1 + |x_n - y_n|)^{2\alpha + \frac{n-3}{p} + \frac{1-n}{p}} \| f(\cdot, y_n) \|_{L^p(\mathbb{R}^{n-1})} \, dy_n \right\|_{L^p(\mathbb{R})} \]

(14)

provided \( 2\alpha + \frac{n-3}{p} + \frac{1-n}{p} \leq \frac{2}{p} - 2 \), or more simply \( 1 \leq p \leq \frac{2n+2}{n+1} \). The last step is a restatement of the Hardy-Littlewood-Sobolev inequality in one dimension.

Summing over the \( n \) pieces of the conical decomposition concludes the proof.

2. Application to Embedded Resonances of \(-\Delta + V\)

Statements like Theorem 2 are useful for constraining the spectral measure of Schrödinger operators \(-\Delta + V\) with a scalar perturbation \(V \in L^r(\mathbb{R}^n)\). We present a tidy application here; one can find a more extensive set of tools and results in [4].

Suppose \( n \geq 3 \) and \( V \in L^{\frac{2n+2}{n+2}}(\mathbb{R}^n) \) is real-valued. It is known that the free resolvent \( R_0^\uparrow(\lambda) = \lim_{\epsilon \to 0^+} (\Delta - \lambda + i\epsilon)^{-1} \) maps \( L^{\frac{2n+2}{n+2}}(\mathbb{R}^n) \) to \( L^{\frac{2n+2}{n+2}}(\mathbb{R}^n) \) for each \( \lambda > 0 \) [2]. One may present the resolvent of the perturbed operator \( H = -\Delta + V \) using identities such as

\[ R_0^\uparrow(\lambda) := \lim_{\epsilon \to 0^+} (H - (\lambda + i\epsilon))^{-1} = (I + R_0^\uparrow(\lambda)V)^{-1}R_0^\uparrow(\lambda). \]

The mapping bounds for \( R_0^\uparrow(\lambda) \) extend naturally to the perturbed resolvent \( R_V^\uparrow(\lambda) \) provided there exists a suitable operator inverse for \((I + R_0^\uparrow(\lambda)V)\). Under the given condition \( V \in L^{\frac{2n+2}{n+2}}(\mathbb{R}^n) \), this is a compact perturbation of the identity on \( L^{\frac{2n+2}{n+2}}(\mathbb{R}^n) \). By the Fredholm alternative, it only fails to be invertible if there exists a function \( g \in L^{\frac{2n+2}{n+2}}(\mathbb{R}^n) \) such that \( g = -R_0^\uparrow(\lambda)Vg \).

Such a function also has the property \((R_0^\uparrow(\lambda)Vg, Vg) = -(g, Vg) \in \mathbb{R}\), where \((\cdot, \cdot)\) is the sesquilinear pairing between \( L^{\frac{2n+2}{n+2}} \) and its dual. The imaginary part of the left-hand pairing is equal to \( c\lambda^{-1/2}||Vg||_{L^2(\mathbb{R}^n)}^2 \), hence the Fourier transform of \( Vg \) vanishes on the sphere radius \( \sqrt{\lambda} \). Furthermore, \( g \) is a solution of the Helmholtz equation \((\Delta - \lambda)g = -Vg\).

The statement of Theorem 2 can be modified to accommodate any operator \(-\Delta - \lambda, \lambda > 0\), by conjugating with dilations of order \( \sqrt{\lambda} \). Write \( V = V_1 + V_2 \), where \( V_1 \) is bounded and compactly supported, and \( ||V_2||_{\frac{2n+2}{n+1}} < \delta \) for a quantity \( \delta > 0 \) to be chosen in a moment. We have

\[ ||g||_2 \leq C_{n, \lambda} ||Vg||_{\frac{2n+2}{n+1}} \leq C_{n, \lambda} (||V_1||_{\frac{2n+2}{n+1}} ||g||_{\frac{2n+2}{n+1}} + \delta ||g||_2). \]
If \( C_{n,\lambda} \delta < \frac{1}{2} \), the last term can be moved to the left side of the inequality so that 
\[ \|g\|_2 \lesssim \|g\|_2 \frac{\delta}{2}. \]

The conclusion is that resonances cannot be embedded into the continuous spectrum of \( H \); only true eigenfunctions in \( L^2 \) are possible. However it is also known that embedded eigenvalues do not exist if the potential is real and belongs to \( L^{\frac{n+1}{n+5}}(\mathbb{R}^n) \) \([6]\), so in fact the spectrum of \( H \) is purely absolutely continuous.

### 3. Perturbed Helmholtz equation

Theorem 2 admits a relatively easy extension to the equation \((-\Delta + V - 1)u = f\).

Factorize the perturbed Helmholtz operator as
\[ -\Delta + V - 1 = (I + VR^+_0(1))(-\Delta - 1) \]
where \( R^+_0(1) = \lim_{\epsilon \to 0^+} (-\Delta - (1 + i\epsilon))^{-1} \). In this case the choice of resolvent continuations is unimportant, as both \( R^+_0(1) \) and \( R^-_0(1) \) act the same when applied to functions in the range of \(-\Delta - 1\). Then there should exist \( L^2 \) solutions of \((-\Delta + V - 1)u = f\) whenever \( f = g + VR^+_0(1)g \) and the unperturbed equation \((-\Delta - 1)u = g\) has solutions in \( L^2(\mathbb{R}^n) \).

Let \( X_0 \) be the subspace of functions in \( L^p(\mathbb{R}^n) \) whose Fourier transform vanishes on the unit sphere, as defined in the statement of Theorem 6. For \( p = \frac{2n+2}{n+5} \), Theorem 2 indicates that the latter problem admits solutions precisely when \( g \in L^p(\mathbb{R}^n) \) also belongs to \( X_0 \subset L^p(\mathbb{R}^n) \). The substance of Theorem 6 is that the correspondence between \( g \) and \( f \) is an isomorphism of subspaces of \( L^p(\mathbb{R}^n) \). This statement is proved below.

**Proposition 7.** **Assume the conditions of Theorem 6** namely that \( p = \frac{2n+2}{n+5} \) and 
\[ V \in L^{\frac{n+1}{n+2}}(\mathbb{R}^n) \]. Let \( J : X_0 \to L^p(\mathbb{R}^n) \) be the inclusion map. The linear operator 
\( J + VR^+_0(1) : X_0 \to L^p(\mathbb{R}^n) \) is an isomorphism onto its range.

**Proof.** The fact that is a bounded operator is a direct consequence of Theorem 2 which effectively states that \( R^+_0(1) \) is a bounded map from \( X_0 \) to \( L^2(\mathbb{R}^n) \). It is injective by the result in \([6]\), as \( R^+_0(1)g \) would be an \( L^2 \) eigenfunction of \(-\Delta + V - 1\) for any \( g \) in the nullspace of \( J + VR^+_0(1) \).

In fact \( VR^+_0(1) \) is a compact operator from \( X_0 \) into \( L^p(\mathbb{R}^n) \). For smooth compactly supported \( V \) it acts compactly on the larger domain \( L^p(\mathbb{R}^n) \). Approximating \( V \in L^{\frac{n+1}{n+2}}(\mathbb{R}^n) \) preserves compactness of \( VR^+_0(1) \) over the restricted domain \( X_0 \).

The argument that \( (J + VR^+_0(1))X_0 \subset L^p(\mathbb{R}^n) \) is closed is nearly identical to the analogous statement in the Fredholm Alternative. Let \( f_n = (J + VR^+_0(1))g_n \) be a sequence converging to \( f \in L^p \). If \( g_n \) has a bounded subsequence, then by compactness \( VR^+_0(1)g_n \) has a convergent subsequence and so does \( g_n = f_n - VR^+_0(1)g_n \). The limit point \( g \in X_0 \) satisfies \((J + VR^+_0(1))g = f\).

If \( \lim_{n \to \infty} \|g_n\|_p = +\infty \), consider the normalized functions \( \tilde{g}_n = g_n/\|g_n\|_p \). This sequence satisfies \((J + VR^+_0(1))\tilde{g}_n \to 0\), and by compactness there is a convergent subsequence of \( VR^+_0(1)\tilde{g}_n \) with limit \(-g\). Then the same subsequence of \( \tilde{g}_n \) converges to \( g \), which has unit norm and belongs to the nullspace of \( J + VR^+_0(1) \). That would violate the injectivity property of the map.

Having ruled out unbounded (subsequences of) \( g_n \), it follows that \( f \in (J + VR^+_0(1))X_0 \) as in the first case, making the range closed. By the closed graph theorem, \( J + VR^+_0(1) \) is then an isomorphism onto its range. \( \square \)
4. Extensions via Interpolation

The subspace of $L^p$ consisting of functions whose Fourier transform vanishes on the unit sphere is not particularly well suited to interpolation. The Fourier-vanishing condition not preserved by lattice operations or by the complex-analytic families used in the Riesz-Thorin theorem. As a further impediment, it is not obvious that one can approximate each element by a sequence of simple functions (or compactly supported functions, or Schwartz functions) whose Fourier transforms also vanish on the sphere.

We able to prove Theorem 5 via complex interpolation of operators and some careful avoidance of the above obstacles. Suppose $f$ and $g$ are simple functions with compact support. Let $\tilde{S}^z$ be the “analytic” Bochner-Riesz operators defined by

$$\tilde{S}^z = \frac{1}{\Gamma(z+1)}S^z$$

for real-valued $z > -1$, and by analytic continuation to $z \in \mathbb{C}$.

The key observation is that for $\text{Re } z > -2$, and for functions whose Fourier transform vanishes on the unit sphere, $\Gamma(z+1)\tilde{S}^z f = S^z f$ (The singularity at $z = -1$ is removable in this case). Proposition 4 establishes the same observation about “two-sided” Bochner-Riesz operators over the larger range $\text{Re } z > -3$.

It is true by construction that the function $\langle \tilde{S}^z f, g \rangle$ is holomorphic in $z$ for any pair of simple functions $f$ and $g$. This remains true by uniform convergence in the halfplane $\text{Re } z > -\frac{1}{2}$ if we take limits to a generic element $f \in L^2_\mathbb{R}^{n+2}$.

Then

$$G(z) := \Gamma(z+1)\langle \tilde{S}^z f, g \rangle$$

is meromorphic over the same domain, with residues at the negative integers determined by $\langle \tilde{S}^{-k} f, g \rangle$. Since $\tilde{S}^{-1}$ agrees (up to a scalar multiple) with convolution against $\tilde{\sigma}_1$, if we further assume that $\hat{f}$ vanishes on the unit sphere then in fact the singularity of $G(z)$ at $z = -1$ is removable. The slightly modified function

$$\tilde{G}(z) := (z + 2)\Gamma(z+1)\langle \tilde{S}^z f, g \rangle = (z + 2)G(z)$$

is meromorphic with poles at the negative integers $k \leq -3$.

Assuming once again that $\hat{f}$ vanishes on the unit sphere, Theorem 3 provides a bound on the line $z = -\beta + i\mu$,

$$\|\Gamma(1 - \beta + i\mu)\tilde{S}^{-\beta+i\mu} f\|_2 \lesssim \|f\|_2 \frac{2n+1}{\beta+1}.$$  

The constant does not depend on $\mu$ because any one of the Fourier multipliers $(1 - |\xi|^2)^{i\beta}$ is an isometry on $L^2$. Therefore

$$|\tilde{G}(-\beta + i\mu)| \lesssim (1 + |\mu|)\|f\|_2 \frac{2n+1}{\beta+1} \|g\|_2.$$  

On the line $z = -2\beta + i\mu$, we need the following estimates.

**Proposition 8.** Let $\frac{1}{2} < \beta < \frac{3}{2}$ and $\beta \leq \frac{n}{n+1}$. The inequality

$$\|(2 - 2\beta + i\mu)\Gamma(1 - 2\beta + i\mu)\tilde{S}^{-2\beta+i\mu} f\|_2 \frac{2n+1}{\beta+1} \lesssim (1 + |\mu|)\|f\|_2 \frac{2n+1}{\beta+1}$$

holds uniformly for all $\mu \in \mathbb{R}$ and all $f \in L^2_\mathbb{R}^{n+2}(\mathbb{R}^n)$. 


Sketch of Proof. Proposition 8 follows from the same argument as the endpoint Stein-Tomas theorem, using the fact that the convolution kernel of \( \tilde{S}(z) \) has an asymptotic description

\[
\tilde{S}^z(|x|) \sim \frac{C_n}{|x|^{\frac{n-1}{2} + z}} \cos \left( \left| x \right| - \frac{(n-3)\pi}{4} + \frac{\pi}{2} \right)
\]

for large \(|x|\). Note that for complex \(z\), oscillations of the cosine function in this formula have amplitude approximately \(e^{(\pi/2)|\text{Im}z|}\).

For \(z = -2\beta + i\mu\), the prefactor \((z+2)\Gamma(z+1)\) is dominated by \((1+|z|)e^{-(\pi/2)|\text{Im}z|}\), using Stirling’s approximation when \(\mu\) is large, and the absence of poles for \(\text{Re}z > -3\) when \(\mu\) is small. Hence the product \((z+2)\Gamma(1+z)\tilde{S}^z\) enjoys mapping estimates that are uniform in \(\mu\) along this line.

Consequently \(|\hat{G}(-2\beta + i\mu)| \leq (1 + |\mu|)\|f\|_2 \|g\|_{\frac{2n+2}{n+1+4\beta}} \|. If one constructs \(g_z\) to be a holomorphic family of simple functions (as in Riesz-Thorin interpolation) that belong isometrically to \(L^2(\mathbb{R}^n)\) along the line \(\text{Re}z = -\beta\), and to \(L^{\frac{2n+2}{n+1+4\beta}}(\mathbb{R}^n)\) along the line \(\text{Re}z = -2\beta\), it follows from the Three-Lines Theorem that

\[
\| (z+2)\Gamma(z+1)\langle \tilde{S}^z f, g_z \rangle \| \lesssim (1 + |\text{Im}z|)\|f\|_2 \|g_z\|_{\frac{2n+2}{n+1+4\beta}}.
\]

For a fixed real value \(\beta \leq \alpha \leq 2\beta\), one can arrange for \(g_{-\alpha}\) to be any simple function. It follows by duality and density of simple functions that

\[
\|\Gamma(z+1)\tilde{S}^{-\alpha}f\|_{\frac{2n+2}{n+1+\alpha+4\beta}} \lesssim \frac{1}{|2-\alpha|}\|f\|_{\frac{2n+2}{n+1+4\beta}}.
\]

For \(\alpha < 2\) and with the assumption that \(\hat{f}\) vanishes on the unit sphere, the left hand function is exactly \(S^{-\alpha}f\). This is the norm bound claimed in Theorem 4.

As a final note, we observe that estimates can also be made for \(\alpha < \beta\) by interpolating between Theorem 2 and other known bounds for Bochner-Riesz operators. To give a simple example, \(S^{-\alpha}\) maps \(L^p(\mathbb{R}^2)\) to itself for each \(\alpha < -\frac{1}{2}\), and if \(\hat{f}\) vanishes on the unit circle for a function \(f \in L^1(\mathbb{R}^2)\) it is also true that \(S^{-\frac{3}{2}}f \in L^2(\mathbb{R}^2)\). Complex interpolation then suggests that \(S^{0}f \in L^q(\mathbb{R}^2)\) for all \(q > \frac{3}{2}\). This is a modest improvement over the generic \(L^1 \hookrightarrow L^{4/3+}\) bound for the ball multiplier. We do not claim that the exponent \(q = \frac{3}{2}\) is sharp, and suspect that the range of exponents can be extended further toward 1 by other methods.

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