A transfer operator approach to relativistic quantum wavefunction

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Abstract

The original intent of the Koopman–von Neumann formalism was to put classical and quantum mechanics on the same footing by introducing an operator formalism into classical mechanics. Here we pursue their path the opposite way and examine what transfer operators can say about quantum mechanical evolution. To that end, we introduce a physically motivated scalar wavefunction formalism for a velocity field on a 4-dimensional pseudo-Riemannian manifold, and obtain an evolution equation for the associated wavefunction, a generator for an associated weighted transfer operator. The generator of the scalar evolution is of first order in space and time. The probability interpretation of the formalism leads to recovery of the Schrödinger equation in the non-relativistic limit. In the special relativity limit, we show that the scalar wavefunction of Dirac spinors satisfies the new equation. A connection with string theoretic considerations for mass is provided.

Keywords: transfer, operator, relativistic, quantum, wavefunction

(Some figures may appear in colour only in the online journal)

1. Introduction

Dynamical systems theory can be pursued in the phase-space (Poincaré) formalism [1], or alternatively in the Koopman formalism [2–4]. The Koopman (or Koopman–von Neumann (KvN)) formalism applied in phase space leads to the probability interpretation of the associated phase-space wavefunction consistent with the Born interpretation in quantum mechanics [5]. Born’s proposal on interpretation of the square of the wavefunction as probability led to successful application of quantum mechanics to a broad swath of problems. The dichotomy between the phase-space domain of the classical wavefunction and the physical spacetime nature of the quantum wavefunction recently led to a number of efforts to reconcile the two
(see e.g. [5–13] and the rest of the articles in this volume). These works are pursued in the nonrelativistic context, although the work in [12] can be adapted to the relativistic setting by a simple reparametrization. A different approach was pursued in [14], where the spectrum of the quantum harmonic oscillator was related to the Koopman operator spectrum of the classical harmonic oscillator by a construction involving a pair of harmonic oscillators with Hamiltonians of opposite sign.

In this paper we pursue the operator-theoretic approach to derive an equation of motion—the relativistic quantum transfer equation (RQTE)—for the resulting quantum-theoretical wavefunction starting from a relativistic dynamical system on a 4-dimensional space-time. Namely, we start from the spacetime manifold, and not the phase space, and utilize Fock’s proper-time formalism [15]. The key idea is that the RQTE arises from the projection of a 4-dimensional conserved field through a complex scalar field. The resulting equation—when presented in the probabilistic interpretation—has solutions that reduce to the Schrödinger equation in the nonrelativistic limit, and the equation for the Dirac scalar in the special relativity limit.

It has been argued that the difficulty of combining quantum-theoretical and relativistic ideas stems from the fact that the formalisms are quite different: relativity theory has a geometrical ontology, whilst quantum theory has an operator-theoretic one. We show that by combining the geometrical and operator-theoretic approach within the KvN setting produces a manifestly covariant quantum theory that, besides the geometric foundation, possesses all the quantum-theoretical properties, such as superposition. Thus, the RQTE equation can enhance our understanding by providing a more direct connection between individual terms in a mathematical formula and the physical quantum and relativistic phenomena that they describe because the underlying physical objects are familiar—as familiar as those stemming from Newton’s classical mechanics when exhibited in its geometric form utilizing the theory of smooth manifolds.

The paper is organized as follows: in section 2 we introduce the relativistic setting and the notation. In section 3 we derive the RQTE under several postulates and indicate its relationship to the Schrödinger equation. We describe the class of operators—the weighted composition operators—that are generated by RQTE. In section 4 we discuss the relationship between RQTE and the Dirac equation. In section 5 we consider several examples treated within the RQTE formalism: harmonic oscillator, particle in a box and Gaussian wavepacket. We discuss the relationship of the RQTE wavefunction with mass in appendix A and relationship with notion of mass in string theory in appendix B.

2. Preliminaries

Let \( M \) denote a 4-dimensional space-time pseudo-Riemannian manifold endowed with a metric tensor \( g \). Consider the section of its tangent bundle \( TM \), the proper velocity field (the four-velocity field) \( V = dX/d\tau \) [16] where \( X(\tau) : \mathbb{R} \to M \) is the time-like world line parametrized by the proper time \( \tau \). We define the level sets of proper time \( \tau \) on \( M \) to be able to use it for evolution of the flow of \( V \). Any vector field on \( M \) can be rectified near a point \( X \) with \( V(X) \neq 0 \) [17]. Since the four-velocity field \( V \) is nonzero everywhere, there exists a neighborhood \( \mathcal{N}_X \) of any point \( X \) in which it can be rectified by a local choice of coordinates \((x_0(X), \ldots, x_3(X))\) on \( M \). Note that \((x_1, x_2, x_3)\) label points on the \( x_0 = 0 \) intersection of an individual world line. Let \( \sigma = \tau(0, x_1, x_2, x_3) \) be the proper time field over the section \( x_0 = 0 \). We can define a new parameter \( s = \tau - \sigma \) in a small neighborhood of \( X \). In this way, the 0 proper time is synchronized for all trajectories in a neighborhood. Absent topological obstructions, this can be extended to the whole of \( M \) to define a space slice \( M^t_0 \). With topological obstructions, the construction is still valid on subsets of \( M \). In this case, we redefine \( M \) to be such a subset. We keep the
notation \( \tau \) for the reparametrized proper time. The norm of \( V \) defined using the metric tensor \( g \) on \( M \) is constant, \( ||V||^2 = -c^2 \), where \( c \) is the speed of light in vacuum \(^{15}\) (we are using the \((−1, 1, 1, 1)\) metric convention). We denote by \( G^\tau : M \to M \) the flow of \( V \) on \( M \). We denote by \( D_\tau f \) the proper time derivative (i.e. the Lie derivative \(^{17}\)) of \( f \),

\[
D_\tau (\cdot) = V^\mu \frac{\partial (\cdot)}{\partial x_\mu},
\]

representing the change of a scalar physical quantity in the direction of \( V \). The manifold is equipped with the volume form with density \( \sqrt{|\det g|} \).

The flow \( G^\tau \) can be used to define the family of Koopman composition operators \(^{2}\) parametrized by \( \tau \) acting on (in general, complex) functions \( f : M \to \mathbb{C} \) by

\[
U^\tau f(X) = f \circ G^\tau (X).
\]

Note that, in contrast with Koopman’s original formulation on the phase space, \( U^\tau \) acts on functions defined on the spacetime \( M \). The operator \( D_\tau \) is the generator of the evolution \( U^\tau \).

The functions in the eigenspace at 0 of \( D_\tau \) are conserved quantities, since

\[
D_\tau f = 0
\]

implies \( f \) is conserved on the world line \( X(\tau) \). In terms of the Koopman operator evolution, for such \( f \) we get

\[
U^\tau f(X) = f(X).
\]

In line with the terminology used in Koopman operator theory \(^{3, 18}\) we call functions \( g : M \to \mathbb{C} \) observables. By identification with the associated, position-dependent operators, the terminology is consistent with that of quantum mechanics.

### 3. Wavefunction evolution

Consider a field \( \rho \) conserved under trajectories of \( V \) on \( M \). We denote the covariant 4-gradient of a function on \( M \) by \( D \). The restriction of \( \rho \) onto level sets of the complex field \( e^{iY} \) of modulus 1, with phase \( Y \) reads

\[
\rho \frac{D}{De^{iY}} = \frac{\rho}{i[DY]e^{iY}}.
\]

We assume that the density \( \rho \) is not observed directly, but is projected via a complex scalar field \( e^{iY} \), as indicated by equation (5) and shown in figure 1. The geometry can be described as that of a fiber bundle over \( M \), while \( \rho e^{iY} \) is a horizontal lift of the spacetime propagation of a string (see appendix B). It is interesting to note that fiber bundle geometry has found use in many different physical contexts \(^{19–21}\). Here it emerges from the idea that we are observing a conserved field \( \rho \)—that has the physical interpretation of the oscillation wavenumber of a string—through the field \( Y \).

This construction renders the appearance of complex numbers in quantum mechanics a natural consequence of geometry.

Given this geometric formulation, we use the following postulates:

#### 3.1. Postulates

(a) There is a function \( \rho : M \to \mathbb{R} \) that is constant on trajectories of \( V \) satisfying

\[
D_\tau \rho = 0.
\]
We argue in the appendix A that the oscillation wavenumber and is related to mass (and thus energy).

(b) The observable wavefunction \( \psi \) is the pushforward of \( \rho \) by an observable \( e^{iY} \) given by (see figure 1 for the geometrical representation of this postulate)

\[
\psi = \frac{\rho}{i|DY|e^{i\gamma}} = \frac{\rho}{iK} e^{-i\gamma} = \frac{\rho}{K} e^{-(Y + \pi/2)},
\]

where \( Y \) is a phase and \( K = |DY| \). This, in turn, implies

\[
\rho = iK e^{iY} \psi.
\]

(c) \( \rho/|DY| \) is an invariant density for \( V \).

**Remark 1.** The last postulate is natural in view of the fact that—when extending the classical action—velocity relationship relativistic, and identifying \( Y \) with relativistic action, \( DY \) is proportional to the space-time velocity \( V \). Thus, the density \( \rho/|DY| \) is inversely proportional to the velocity magnitude and thus is invariant.

Note that the factor \( i \) is used in the wavefunction definition just for convenience of the calculations below since the constant phase of the wavefunction is irrelevant. In the following, we identify \( S \) with the action integral defined by

\[
S = \int_0^T \mathcal{L} \, dt,
\]

where \( \mathcal{L} \) is the Lagrangian of the motion. The specific expressions for \( S \) in the relativistic and nonrelativistic case are given below in formulas (42) and (43), respectively. Under the above assumptions, we prove the following:

**Theorem 1.** Let \( Y = -S/\hbar \), where \( \hbar \) is the reduced Planck constant, and \( D_x S = \mathcal{L} \), analogous to the standard notions of the action \( S \) and the Lagrangian \( \mathcal{L} \). The wavefunction \( \psi \) satisfies the RQTE equation

\[
\frac{\rho}{i|DY|e^{i\gamma}} = \rho
\]
\[ i\hbar D_\tau \psi = -\mathcal{L}\psi - i\hbar \text{div} V \psi, \quad (10) \]

where, in coordinates,

\[ \text{div} V = \frac{1}{\sqrt{|g|}} \sum_j \frac{\partial \sqrt{|g|} V^j}{\partial X_j} \quad (11) \]

is divergence with respect to volume element, where \(|g| = |\det g_{ij}|\) is the absolute value of the determinant of the metric tensor.

**Proof.** By assumption 1, \( D_\tau \rho = 0 \), and we have

\[ i\hbar D_\tau \psi = -\hbar \frac{D_\tau K}{K^2} e^{-iY} - i\hbar \frac{\rho}{K} D_\tau Y e^{-iY} = -i\hbar \frac{D_\tau K}{K} \psi + \hbar D_\tau Y \psi \]

\[ = -i\hbar \frac{D_\tau K}{K} \psi - \mathcal{L}\psi. \quad (12) \]

Now we show that \( K = |DY| \) must satisfy

\[ D_\tau |DY| = |DY| \text{div} V. \quad (13) \]

Since \( \rho/|DY| \) is an invariant density,

\[ D_\tau (\rho/|DY|) = -\rho |DY|^{-2} D_\tau |DY| = -\rho/|DY| \text{div} V \quad (14) \]

implying

\[ D_\tau |DY| = |DY| \text{div} V. \quad (15) \]

Now (12) yields

\[ i\hbar D_\tau \psi = \left[-\mathcal{L} - i\hbar \text{div} V\right] \psi, \quad (16) \]

where \( D_\tau \) is the proper time derivative, \( \mathcal{L} \) is the Lagrangian, and \( \text{div} \) is the divergence of the vector field \( V \) with respect to \( \sqrt{|g|} \).

Writing (10) as

\[ D_\tau \psi = \frac{d\psi}{d\tau} = \left[ \frac{i}{\hbar} \mathcal{L} - \text{div} V \right] \psi \quad (17) \]

and directly integrating with respect to proper time, (10) has the solution

\[ \psi(Y, \tau) = \psi_0(G^{-\tau}(Y)) e^{-\int_0^\tau \text{div} V(G^{-\tau}(Y)) ds} e^{iS(\tau, Y)/\hbar}, \]

\[ = \psi_0(X_0) e^{-\int_0^\tau \text{div} V(G^{-\tau}(Y)) ds} e^{iS(\tau, Y)/\hbar}, \quad (18) \]

where \( \psi(Z, 0) = \psi_0(Z) \), and \( X_0 = X(G^{-\tau}(Y)) \) is the initial position at \( \tau = 0 \) of trajectory landing at \( Y \) at \( \tau \).
3.2. Relationship with the Schrödinger equation

Note that for any power $\left(\rho/|DY|\right)^{\alpha}$
\[
D_{\tau}\left(\rho/|DY|\right)^{\alpha} = -\alpha\rho^{\alpha}|DY|^{-(\alpha+1)}D_{\tau}|DY|
\] (19)
and thus, for $\alpha = 1/2$
\[
D_{\tau}\left(\rho/|DY|\right)^{1/2} = -\frac{1}{2}\rho^{1/2}|DY|^{-3/2}D_{\tau}|DY| = -\frac{1}{2}\left(\rho/|DY|\right)^{1/2}\text{div}V,
\] (20)
where the last equation is obtained using (15). Thus, the evolution equation for
\[
\varphi = \left(\rho/|DY|\right)^{1/2}e^{-iY}
\] (21)
is
\[
\imath\hbar D_{\tau}\varphi = -\mathcal{L}\varphi - \frac{\hbar}{2}\text{div}V\varphi.
\] (22)
Integrating equation (22), like in (17) yields the solution
\[
\varphi(Y,\tau) = \varphi_{0}(G^{-1}(Y)e^{-\frac{\hbar}{2}\int_{\mathbb{R}}^{\tau}\text{div}V(G^{-1}(Y))ds}e^{\mathcal{S}(\tau,Y)/\hbar},
\] = \varphi_{0}(X_{0})e^{-\frac{\hbar}{2}\int_{\mathbb{R}}^{\tau}\text{div}V(G^{-1}(X_{0}))ds}e^{\mathcal{S}(\tau,Y)/\hbar}.
\] (23)
Replacing $\tau$ in (55) with the classical coordinate time and assuming a flat geometry of space-time, this solution for the wavefunction reduces to the one derived from the Schrödinger equation by Holland ([22], equation (7.3), where the Lagrangian incorporates the so-called ‘quantum potential’), thus providing the proof that equation (22) in the nonrelativistic setting is equivalent to the Schrödinger equation. Thus, utilizing the square root $\left(\rho/|DY|\right)^{1/2}$ of the invariant measure of $V$, we obtain the unitary evolution equation 22 that admits a probabilistic interpretation\(^1\) for the square of its solution $|\phi\rangle = \phi|\phi\rangle$.

Remark 2. It is well known [23] that if $M$ is totally finite, separable measure space, with $\sigma$-algebra $B$, and measure $m$, and $T$ an invertible measurable non-singular transformation on this space (i.e. one for which $m(E) = 0$ if and only if $m(T^{-1}E) = 0$), then there is a measure $\mu$ such that
\[
\mu(T^{-1}E) = \int_{E}\mu(x)dm(x).
\] (24)
As a consequence, there is a unitary weighted composition operator $\mathcal{U}_{\mu}$ associated with $T$, defined on measurable functions $f$ by
\[
\mathcal{U}_{\mu}f(x) = \mu^{1/2}f(T^{-1}(x)).
\] (25)
Since unitary evolution is $L^{2}$-norm preserving, physically it expresses the probability conservation constraint. Thus the choice of $\alpha = 1/2$ in (19) amounts to choosing the probabilistic interpretation of the underlying dynamics.

Remark 3. The equation (18) can serve as a template for the path integral formulation of the current theory.

\(^1\) While the RQTE equation (10), its proof and ontology stated in the Postulates—that I have first presented at the MIT physics seminar in 2019—are new, an equation similar to the probability amplitude evolution equation (22), albeit in non-relativistic setting, and derived by different methods and with different ontology, has appeared in the work [12], appendix C, in 2020, as pointed out privately to me by the author, Dr Ilon Joseph. The resulting equation can be easily adapted to the relativistic setting by using the methodology described here, that involves synchronization of proper time, and reparametrization using it.
3.3. Relationship to weighted composition operators

Based on (18) a group of evolution operators \( \mathcal{W}^\tau \) parametrized by proper time can be defined:

\[
\mathcal{W}^\tau \psi = \pi \cdot \psi \circ G^{-\tau}
\]

(26)

where \( \pi : M_t \to C \)

\[
\pi(y) = e^{-\int_0^\tau \text{div}(G(G^{-\tau}(Y)))dt} \frac{e^{i S(\tau,Y)}}{\hbar}.
\]

(27)

The operators \( \mathcal{W}^\tau \) belong to the class of the so-called weighted composition operators [24].

4. Special relativity case: Dirac equation

In this section we show that the scalar wave amplitude \( \psi \) of a solution to Dirac equation\(^2\) satisfies equation (10). Note here that it is only in the divergence part that the equation (10) differs from the probability amplitude equation (16), and we will see below that for Dirac equation the divergence is 0. We start with the Dirac equation in the form

\[
i \hbar \frac{\partial \psi}{\partial t} = -i \hbar c \alpha_j \cdot \nabla_j \psi + \beta mc^2 \psi
\]

(28)

where \( \psi = s \psi \), \( s \) is a 4-component spinor, \( \psi \) is a scalar function, \( \alpha_j \)’s and \( \beta \) are matrices defined by

\[
\beta = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix},
\alpha_1 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

(29)

\[
\alpha_2 = \begin{bmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & -i & 0 & 0 \\
i & 0 & 0 & 0
\end{bmatrix},
\alpha_3 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}.
\]

(30)

Let

\[
n = \sqrt{\frac{E + mc^2}{2mc^2}},
\]

(31)

be the normalization constant. The Dirac spinors for the frame moving with velocity \( v \) are [25]

\[
u_1 = n \begin{bmatrix}
1 \\
0 \\
\frac{p_x}{E + mc^2} \\
\frac{p_y}{E + mc^2} \\
\frac{p_z}{E + mc^2}
\end{bmatrix}, \nu_2 = n \begin{bmatrix}
0 \\
1 \\
\frac{p_x - p_y}{E + mc^2} \\
\frac{p_x + p_y}{E + mc^2} \\
\frac{p_z}{E + mc^2}
\end{bmatrix},
\]

(32)

\[
u_1 = n \begin{bmatrix}
\frac{p_x}{E + mc^2} \\
\frac{p_y}{E + mc^2} \\
0 \\
1 \\
\frac{p_z}{E + mc^2}
\end{bmatrix}, \nu_2 = n \begin{bmatrix}
\frac{(p_x - p_y)c}{E + mc^2} \\
\frac{(p_x + p_y)c}{E + mc^2} \\
0 \\
1
\end{bmatrix},
\]

where \( E \) is the energy and \( p_j \) are components of momentum. Let

\[
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad p^2 = p_x^2 + p_y^2 + p_z^2.
\]

(33)

\(^2\) \( \psi \) is known to be a solution to the Klein–Gordon equation.
The classical approach to deriving the Schrödinger equation from the Dirac equation can be interpreted as the fact that when we fix the spin vector we showed that the solution of the RQTE equation (37)

\[ u'_1 u_1 = u'_2 u_2 = v'_1 v_1 = v'_2 v_2 = \gamma, \]
\[ u'_1 \beta u_1 = u'_2 \beta u_2 = 1, \]
\[ v'_1 \beta v_1 = v'_2 \beta v_2 = -1, \]
\[ u'_1 \alpha_1 u_1 = v'_1 \alpha_1 v_1 = u'_2 \alpha_1 u_2 = v'_2 \alpha_1 v_2 = \nu \gamma / c, \]
\[ u'_1 \alpha_2 u_1 = v'_1 \alpha_2 v_1 = u'_2 \alpha_2 u_2 = v'_2 \alpha_2 v_2 = \nu \gamma / c, \]
\[ u'_1 \alpha_3 u_1 = v'_1 \alpha_3 v_1 = u'_2 \alpha_3 u_2 = v'_2 \alpha_3 v_2 = \nu \gamma / c. \]

(34)

Letting \( \psi = u_1 \psi^+_u \), and premultiplying (28) by \( u'_1 \), leads to

\[ i \hbar \frac{\partial \psi^+_u}{\partial t} = -i \hbar v \cdot \nabla \psi^+_u + \frac{mc^2}{\gamma} \psi^+_u. \]

(35)

Using proper time

\[ \tau = \frac{t}{\gamma}, \]

(36)

we obtain

\[ i \hbar D_\tau \psi^+_u = mc^2 \psi^+_u, \]

(37)

which is the equation (10) for the case of constant velocity with the relativity Lagrangian \( \mathcal{L} = -mc^2 \). Calculating similarly, for \( \psi = u_2 \psi^-_v, v_2 \psi^+_v, v_2 \psi^-_v \) we obtain

\[ i \hbar D_\tau \psi^+_u = mc^2 \psi^+_u \]
\[ i \hbar D_\tau \psi^-_u = mc^2 \psi^-_u \]
\[ i \hbar D_\tau \psi^+_v = -mc^2 \psi^+_v \]
\[ i \hbar D_\tau \psi^-_v = -mc^2 \psi^-_v \]

(38)

where the subscript \( u \) denotes the positive energy, \( v \) the negative energy solutions, and superscripts \( \pm \) refer to spin up and spin down solutions. These resemble equations governing the Dirac particle in rest frame, where \( \tau = t, v = 0 \), and reduce to those in the limit.

**Remark 4.** This can be interpreted as the fact that when we fix the spin vector \( u_0 \), the Dirac equation reduces to the equation we derived. It is of interest that the velocity can be interpreted as positive or negative, depending on the sign of the Lagrangian \( \pm mc^2 \), just like the Feynman–Stückelberg interpretation of positrons moving backwards in time.

**Remark 5.** The classical approach to deriving the Schrödinger equation from the Dirac equation relies to splitting of the spinor into large and small components, the small components being of order \( \nu / c \), which leads to decoupling of these and derivation of the Pauli equation, or by the means of the Foldy–Wouthuysen transformation [26]. The Schrödinger equation can then be obtained e.g. in the limit of the weak magnetic field. The developments here show that no such decoupling is necessary to establish the Schrödinger equation as the classical limit of the Dirac equation. Namely, in section 3.2 we showed that the solution of the RQTE equation satisfies the Schrödinger equation in the limit \( \tau \to t \) (i.e. \( \nu / c \to 0 \)). This provides for an additional physical understanding of the Dirac equation since the dynamics of the scalar part of the Dirac spinor is governed by the generator of a transfer operator, and this fact does not depend on the representation (e.g. choice of matrices). In fact, the Dirac equation itself can be derived from considerations similar to the ones advanced here, in representation-invariant form, as we show in an upcoming publication.
5. Examples

5.1. The non-relativistic case of flat 1-dimensional configuration space

Consider a 1-dimensional space and proper time, depicted in figure 2. The proper time is denoted by \( \tau \). We denote \( v = \dot{x} = dx/dt \), and assume—for simplicity of notation—that \( v \) is positive. Setting \( ds = c d\tau \), we have \( c^2 d\tau^2 = c^2 dt^2 - dx^2 \) or alternatively \( c^2 dt^2 = c^2 d\tau^2 + dx^2 \).

We have

\[
v = \dot{x} = \frac{c}{J},
\]

(39)

(see figure 2).

More generally, let \( v \) be the norm of the configuration space velocity. \( K \) is exactly the cosine of the angle between the normal to the surface spanned by trajectories in space-proper time and the line of sight to the space slice, \( 1/J = K = v/c \). The ‘velocity’ in the proper time direction is \( v_\tau = c \sin \theta \).

We set the observable phase \( Y \) to satisfy, in any dimension

\[
-\frac{\hbar D_x Y}{m} = V,
\]

(40)

where \( D_x Y \) is the reduction of the differential \( DY \) to the space slice of \( M \). Then, in the non-relativistic case

\[
-\frac{\hbar |D_x Y|}{m} = v = \frac{c}{J},
\]

(41)
where $v = |v|$, and $m$ is a constant$^3$. Thus the velocity measures the spatial change in the phase of the observable. Recall $Y = -S/\hbar$. The relativistic action with no external potentials, with spatial coordinates is $(x,y,z)$ is given by

$$S = \alpha \int \sqrt{-c^2(\frac{dt}{d\tau})^2 + (\frac{dx}{d\tau})^2 + (\frac{dy}{d\tau})^2 + (\frac{dz}{d\tau})^2} \, d\tau,$$

(42)

where $\alpha = imc$, the classical action is

$$S_c = \frac{m}{2} \int [\dot{x}^2 + \dot{y}^2 + \dot{z}^2] \, dt,$$

(43)

where $t$ is the classical time and $\tau = t$. Note that with identification $m = \rho \frac{\hbar}{c}$, that has dimensions of mass (see appendix A), we get

$$\frac{|DY_x|}{\rho} = \frac{1}{J} \Rightarrow |DY_x| = \frac{\rho}{J}. \quad (45)$$

For the wavefunction, we have

$$\psi = \frac{\rho}{i|D_x Y(e^{iY})} = -iJ e^{i\gamma}. \quad (46)$$

We consider the flat 1 + 1 space-time. As above, we assume that the wavefunction $\rho$ satisfies

$$\frac{\partial \rho}{\partial \tau} + v_x \frac{\partial \rho}{\partial x} = 0. \quad (47)$$

i.e. $\rho$ is invariant on space-time trajectories. Here $v_x = dx/d\tau$. We let the observation field $f: \mathbb{R}^2 \to \mathbb{C}$ be given by

$$f = e^{i\gamma}. \quad (48)$$

The 'observable wavefunction’ $\psi$ on the $x$ axis is defined by

$$\psi = \frac{\rho}{i|D_x Y(e^{iY})} = \frac{\rho}{i|\frac{\partial}{\partial x}|e^{i\gamma}} = \frac{\rho}{iKe^{i\gamma}}. \quad (49)$$

We proceed to derive an equation of evolution for $\psi$. We set

$$D_\tau Y = Y_x + v_x Y_x = \tilde{L}, \quad (50)$$

and obtain

$$\psi_\tau = -v_x \psi_x - v_x \psi - i\tilde{L} \psi, \quad (51)$$

or, more compactly

$$i\hbar \frac{\partial \psi}{\partial \tau} = \mathcal{H} \psi, \quad (52)$$

where

$$\mathcal{H} = \left(-i\hbar v_x \frac{\partial}{\partial x} + \hbar \tilde{L}\right) - i\hbar \epsilon. \quad (53)$$

$^3$ In classical optics, $J = c/v$ is the index of refraction.
The equation (52), extended to $d$-dimensional configuration space reads
\[ \psi_{\tau} = -v_{\tau} \nabla \cdot \psi - \nabla \cdot v_{\tau} \psi - i\tilde{L}\psi, \] (54)
and has the solution
\[ \psi(y, \tau) = \psi_0(x(X^{-\tau}(y)))e^{-\int_0^\tau \text{div}v_{\tau}(x(X^{-\tau}(y)))dx}e^{iS(\tau y)/\hbar}, \] (55)
where $\psi(z,0) = \psi_0(z)$, and $X^{-\tau}(y)$ is the initial position at $\tau = 0$ of trajectory landing at $y$ at $\tau$.

In next sections we treat the non-relativistic case that makes use of these relationships, where we assume the quantum part of the potential is small compared with the classical potential.

Remark 6. If we set $Y = -S/\hbar$, the first term on the right side of (51) is just the quantization of the classical Hamiltonian
\[ H = v_{\tau} p + \hbar \tilde{L} = v_{\tau} p - \mathcal{L}, \] (56)
where $p$ gets replaced by $-i\hbar \partial/\partial x$ and $\mathcal{L} = -\hbar \tilde{L}$ is the Lagrangian.

It is thus clear that in the current theory velocity and momentum are treated separately, like in the context of Dirac equation in Heisenberg representation [27, 28], or Schwinger’s variational principle [29].

5.2. The Lagrangian

The relativistic Lagrangian for a particle with no charge is usually stated as
\[ \mathcal{L}_0 = -\frac{mc^2}{\gamma}. \] (57)

In [15] Fock developed the so-called proper-time formalism, that utilizes proper time as an independent variable and derived the relativistic Lagrangian for a particle with no charge as
\[ \mathcal{L} = \frac{m}{2} \|\mathbf{V}\|^2 - \frac{mc^2}{2}. \] (58)

Since then, the proper time formalism has proved useful in relativistic physics in a variety of contexts [30]. In the examples below, we utilize the Fock Lagrangian in equation (10). For the non-relativistic limit of the harmonic oscillator and particle-in-a-box, we utilize a recent formulation that relates the classical potential $U$ to the metric tensor component $g_{00}$ in mechanics on classical static curved spaces utilizing Gibbons, formulation [31] (note the change of metric convention in [31] that does not change the result):
\[ ds^2 = g_{00}c^2 dt^2 - |dx|^2. \] (59)

Let $U$ be a scalar potential source. As [32] shows, the condition $mv^2/2 - U << mc^2$ leads to
\[ g_{00} \approx 1 + \frac{2U}{mc^2} \] (60)
\[ \mathcal{L}_c \approx \frac{mv^2}{2} - \frac{mc^2}{2} g_{00} = \frac{mv^2}{2} - U - \frac{mc^2}{2}. \] (61)

It is interesting to note that the constant $mc^2/2$ stems from the time-component of the metric tensor. This is of consequence for the zero-point energy calculation in the examples that follow.

---

4 These papers show a classical, geometry-based theory is possible that exhibits features of the electron—such as spin, present in quantum theory, but not superposition. The effort here shows that a theory can be found in which all these properties are present, and yet the theory has a combination of geometry-based and operator-theoretic ontology.
Example 1 (Free Particle). Consider the free particle moving in flat 4-dimensional space-time with constant four-velocity \( V \). The divergence \( \nabla \cdot V = 0 \). Recall that
\[
t = \frac{\tau}{\sqrt{1 - \nu^2/c^2}} = \tau \gamma, \quad \gamma = 1/\sqrt{1 - \nu^2/c^2}.
\] (62)
Denote the space components of \( V \) by \( U \). Since the velocity is constant, the Lagrangian reads
\[
L = -mc^2,
\] (63)
and thus from RQTE we get
\[
{i\hbar\gamma}{\partial\psi}{\partial t} + i\hbar U \cdot \nabla \psi = mc^2 \psi.
\] (64)
Let \( u = U/\gamma = (U_1/\gamma, U_2/\gamma, U_3/\gamma) \). Then
\[
{i\hbar\gamma}{\partial\psi}{\partial t} + i\hbar \gamma u \cdot \nabla \psi = -L \psi,
\] (65)
the expression obtained from the Schrödinger equation.

5.3. Dispersion relationship
We next derive the dispersion relationship for the wave
\[
\psi(x, \tau) = A e^{i(k \cdot x - \omega \tau)}.
\] (66)
From (65) we get
\[
h \omega - \hbar k \cdot u = \frac{mc^2}{\gamma} = -L_0.
\] (67)
Also, with \( E = h \omega \) and \( p = k \hbar \) we obtain
\[
E = -L_0 + p \cdot u = mc^2 \gamma,
\] (68)
and thus we get the correct relativistic expression for energy.

5.4. deBroglie relationships
We observe that one of our postulates is conservation of the wavenumber \( \rho \) along the spacetime trajectory. Because of the discussion in appendix B, leading to equation (B.4) we assume the natural frequency and wavenumber are related by
\[
\omega = \rho c \gamma \tag{69}
\]
which is just the ‘coordinate time’ version of the relationship (B.3). The identification \( m = \rho \hbar / c \) leads to
\[
h \omega = mc^2 \gamma,
\] (70)
the first deBroglie wave-particle relationship. The dispersion equation (67) now yields
\[
mc^2 \gamma - \hbar k \cdot u = \frac{mc^2}{\gamma},
\]
\[
mc^2 - \frac{\hbar k \cdot u}{\gamma} = \frac{mc^2}{\gamma^2} = mc^2 - mu \cdot u,
\]
\[
\frac{\hbar k \cdot u}{\gamma} = mu \cdot u,
\]
which in turn gives

\[ u = \frac{\hbar k}{m\gamma}, \]  

which is the second deBroglie relationship as we set \( p = \hbar u \gamma \).

5.5. The relativistic wavepacket

The general solution to (65) reads

\[ \psi(x, t) = \psi_0(x - ut) e^{-i mc^2 t/\hbar \gamma}. \]  

This solution is physical as long as \( |\psi_0| dx \) is finite and thus can be normalized to 1. For simplicity, we restrict to 1 spatial dimension. Assume an initial Gaussian wavepacket of variance \( \sigma^2 \):

\[ \psi_0(x) = A e^{-x^2/2\sigma^2} = 2\pi \sigma A \hat{\mathcal{R}} \phi(k) e^{ikx} dk \]  

where

\[ \phi(k) = e^{-\sigma^2 k^2/2}. \]  

By the second deBroglie relationship derived above, the velocity \( u \) is

\[ u = k\hbar /m\gamma. \]  

Integrating over the possible wavenumbers in a wavepacket, we get

\[ \psi(x, t) = \frac{A}{2\pi \sigma} \hat{\mathcal{R}} \phi(k) e^{ikx} e^{-it E(k) / \hbar} dk. \]  

It is notable that the wavepacket width is suppressed (over the Schrödinger wavepacket derived below) due to the \( \hbar/m\gamma \) term. In fact, \( \sigma^2 + i \frac{\hbar^2}{2m} \approx \sigma^2 \) as \( v \approx c \). Such a suppression was observed in numerical simulations of electrons accelerated in intense laser fields [33, 34] using the Dirac equation considered on section 4.

Remark 7. The solution (77) can be interpreted in terms of energy as follows:

\[ \psi(x, t) = A \left( \frac{2\pi \sigma}{\sigma^2 + i \frac{\hbar^2}{m\gamma}} \right) e^{-\frac{E}{\hbar} x} e^{i \frac{\sigma^2 k^2}{2\hbar}}. \]  

For wavepacket of small width, where \( \gamma(k) \approx \text{const.} \), we get

\[ \psi(x, t) = A \left( \frac{2\pi \sigma}{\sigma^2 + i \frac{\hbar^2}{m\gamma}} \right) e^{-\frac{E}{\hbar} x} e^{i \frac{\sigma^2 k^2}{2\hbar}}. \]  

Remark 7. The solution (77) can be interpreted in terms of energy as follows:
And we see that the wavepacket is the combination of waves with positive energy. This is in contrast with the Dirac equation, where the combination of positive and negative energy states is used \[35\]. In fact, one can interpret QRTE as an equation valid for the case when spin is fixed \[36\].

5.6. Non-relativistic dispersion relationship

The non-relativistic case is obtained by approximating the Lagrangian with

\[
L_0 = -\frac{mc^2}{\gamma} + \frac{mu^2}{2} = -\frac{mc^2}{\gamma} + \frac{mk^2h^2}{2m^2\gamma^2} = -\frac{mc^2}{\gamma} + \frac{k^2h^2}{2m\gamma^2}.
\]

(80)

Applying the Dirac equation [35], we set \(\gamma \approx 1\) for a single particle in 1 spatial dimension, we obtain

\[
\psi(x, t) = \psi_0(x - ut) e^{\frac{imc^2 t}{\hbar}}.
\]

(81)

Integrating over the possible velocities in a 1 + 1 wavepacket, we get

\[
\psi(x - ut) = a e^{-imc^2 t/\hbar} \int_R e^{-k^2(\sigma^2 + 2i\sigma t/m)/2} e^{ikx} dk
\]

\[
= a e^{-imc^2 t/\hbar} \left( \frac{\sigma^2}{\sigma^2 + i\hbar/m} \right)^{1/2} e^{\frac{-x^2}{2(\sigma^2 + i\hbar/m)}},
\]

(82)

which is also the result obtained from the Schrödinger equation.

The dispersion relationship is

\[
h\omega - hku = mc^2 - \frac{mu^2}{2}
\]

\[
E = \hbar^2 \frac{k^2}{m} - \frac{mk^2h^2}{2m^2} + mc^2
\]

\[
= \frac{k^2h^2}{2m} + mc^2
\]

(83)

which, apart from the constant \(mc^2\) term is the expression obtained from the Schrödinger equation.

**Example 2.** We consider the example of the classical one-dimensional harmonic oscillator. The velocity field of the harmonic oscillator in classical space-time (\(\tau = t\)) is given by (see figure 3)

\[
\dot{x} = A\cos \omega t
\]

\[
\dot{\omega} \approx 1.
\]

(84)

Thus, the divergence of the vector field is 0. The Lagrangian reads [32]

\[
L = L_c - \frac{1}{2}mc^2 = \frac{1}{2}mv^2 - \frac{1}{2}kx^2 - \frac{1}{2}mc^2.
\]

(85)

The trajectory of the harmonic oscillator in space-time, taken for simplicity with the initial conditions \((x_0, \dot{x}_0 = 0)\) satisfies

\[
x(t) = x_0\cos \omega t
\]

\[
\dot{x}(t) = x_0\omega \sin \omega t
\]

(86)
where $\omega^2 = k/m$. From (53), the eigenvalue equation reads

$$i\hbar \frac{d\psi}{dt} + \mathcal{L}\psi = -\beta \psi$$

where $\beta$ is an eigenvalue. Integrating over the period of the trajectory, we have

$$i\hbar \int_{\psi_0}^{\psi} \frac{d\psi}{\psi} = \int_0^T (-\beta - \mathcal{L}) dt.$$  \hspace{1cm} (88)

and thus

$$\psi(T) = \psi(0)e^{\frac{i}{\hbar} \int_0^T (\beta + \mathcal{L}) dt}.$$ \hspace{1cm} (89)

In order for $\psi$ to be periodic, we have the condition

$$\int_0^T (\beta + \mathcal{L}) dt = n2\pi\hbar$$

where $n \in \mathbb{Z}$. Now,

$$\int_0^T \mathcal{L} dt = \int_0^T \left( \frac{1}{2} mv^2 - \frac{1}{2} kx^2 \right) dt = \int_0^T \left( \frac{1}{2} m(x_0\omega \sin\omega t)^2 - \frac{1}{2} k(x_0\cos\omega t)^2 \right) dt$$

$$= \frac{x_0^2}{2} (m\omega^2 - k) \int_0^T (\sin\omega t)^2 dt = 0,$$ \hspace{1cm} (91)

since

$$\int_0^T (\sin\omega t)^2 dt = \int_0^T (\cos\omega t)^2 dt.$$ \hspace{1cm} (92)

Figure 3. The geometry of motion of the harmonic oscillator in classical space-time, with $A = 1, \omega = 2$. 

where $\omega^2 = k/m$. From (53), the eigenvalue equation reads

$$i\hbar \frac{d\psi}{dt} + \mathcal{L}\psi = -\beta \psi$$

where $\beta$ is an eigenvalue. Integrating over the period of the trajectory, we have

$$i\hbar \int_{\psi_0}^{\psi} \frac{d\psi}{\psi} = \int_0^T (-\beta - \mathcal{L}) dt.$$  \hspace{1cm} (88)

and thus

$$\psi(T) = \psi(0)e^{\frac{i}{\hbar} \int_0^T (\beta + \mathcal{L}) dt}.$$ \hspace{1cm} (89)

In order for $\psi$ to be periodic, we have the condition

$$\int_0^T (\beta + \mathcal{L}) dt = n2\pi\hbar$$

where $n \in \mathbb{Z}$. Now,

$$\int_0^T \mathcal{L} dt = \int_0^T \left( \frac{1}{2} mv^2 - \frac{1}{2} kx^2 \right) dt = \int_0^T \left( \frac{1}{2} m(x_0\omega \sin\omega t)^2 - \frac{1}{2} k(x_0\cos\omega t)^2 \right) dt$$

$$= \frac{x_0^2}{2} (m\omega^2 - k) \int_0^T (\sin\omega t)^2 dt = 0,$$ \hspace{1cm} (91)

since

$$\int_0^T (\sin\omega t)^2 dt = \int_0^T (\cos\omega t)^2 dt.$$ \hspace{1cm} (92)
and thus, from (90)
\[ \beta T - \frac{1}{2}mc^2T = n2\pi\hbar. \] (93)

From our previous consideration, using \( m = \rho \hbar / c \), we have
\[ \frac{1}{2}mc^2T = \frac{1}{2}\rho\hbar T = \frac{1}{2}\omega T. \] (94)

where \( \omega = \rho c \) is de Broglie wave frequency. Finally, we get
\[ \beta = \frac{1}{2}\omega + n\hbar = \omega \left( n + \frac{1}{2} \right). \] (95)

which is exactly the standard result on the spectrum of the harmonic oscillator. The zero point energy \( \omega \hbar / 2 \) arises from the oscillation of the observational field, since it comes from the Lagrangian term.

Note that the nature of the spectrum is typical of weighted composition operators \[37\], and \( n\hbar \omega \) is the spectrum of the underlying composition operator generated by setting \( L = 0 \).

**Example 3.** Consider the example of particle in a box of length \( l \). Since between impacts with the walls the particle has constant velocity \( v \), the classical limit of the Fock Lagrangian\[32\]
\[ L = \frac{1}{2}mv^2. \] (96)

The particle moves with constant velocity \( v \) between the walls. The eigenvalue problem reads
\[ -i\hbar \frac{\partial \psi}{\partial x} = (\beta - L)\psi. \] (97)

By integrating from 0 to \( l \)
\[ \frac{\beta - L}{\hbar} \frac{l}{v} = n\pi \] (98)
i.e.
\[ \lambda - L = \hbar \frac{n\pi v}{T} \] (99)

for \( n \neq 0 \) as \( n = 0 \) leads to a trivial eigenfunction 0. De Broglie momentum relationship
\[ p = mv = \frac{\hbar}{\lambda_p} = \frac{2\pi \hbar}{\lambda_p} \Rightarrow v = \frac{2\pi \hbar}{m\lambda_p}, \] (100)

where \( \lambda_p \) is the particle wavelength, leads to
\[ L = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{4\pi^2\hbar^2}{m^2\lambda_p^2} = \frac{2\pi^2\hbar^2}{m\lambda_p^2} \] (101)
\[ \lambda = \hbar^2 \frac{n\pi^2}{2ml\lambda_p} - L. \] (102)

Now we ask for ‘spatial’ resonance, namely that the wavelength of string vibration is a subharmonic of the wavelength of the trajectory:
\[ \lambda_p = \frac{2l}{n} \] (103)

\(^5\) Omitting the constant \( mc^2 \) term, which however has to be included to obtain the correct limit of the relativistic setting \[38\]; here we are interested in recovering the result classically deduced from the Schrödinger equation.)
we get

\[ L = \frac{\pi^2 \hbar^2 n^2}{2mL^2} \]  

(104)

\[ \lambda = \frac{\pi^2 \hbar^2 n^2}{mL^2} - L = \frac{\pi^2 \hbar^2 n^2}{2mL^2}. \]  

(105)

Note that the relationship (103) indicates the nonlinearity of the dynamics: in the case of the harmonic oscillator treated in example 2, the frequency of oscillation of the trajectory was matched to the frequency of oscillation of the string. Here, the trajectory motion contains all of the harmonics of the base frequency, and the oscillation of the string can excite any of these.

6. Discussion and conclusions

Starting from several postulates, in this paper we present a RQTE governing the evolution of a wavefunction transported by a four-velocity field over a spacetime manifold. The key physical assumption is the existence of a complex scalar field (the horizontal lift) of the dynamics. When a probabilistic interpretation is sought, the solution of the equation reduces in the non-relativistic limit to the solution of the Schrödinger equation. In the special relativity limit, the equation is satisfied by the scalar part of the Dirac spinor. We obtained the classically known spectra from the RQTE formalism in the specific non-relativistic physical cases—the harmonic oscillator and the particle in the box. We additionally considered the problem of the Gaussian wavepacket. The solution of RQTE in this case yields a prediction that indicates reduction of wavepacket spreading in the limit when velocity approaches the speed of light in vacuum.

Solutions of RQTE lead to evolutions governed by specific type of transfer operators—the weighted composition operators. We believe this observation can be useful in further development of the theory and connections between the mathematical literature on such operators (see e.g. [24]).

It is of interest to note that from our postulates an interesting relationship between RQTE wavefunction and mass (or equivalently energy) emerges. In fact, the concept of mass that arises coincides with the concept of mass stemming from string theory considerations.

Extension of this theory for different spin particles is possible. We hope the developed theory might be useful in numerical methods needed for quantum computing problems.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Mass and the wavefunction

Recall that, from our postulates, \( \psi = (\rho/i|DY|)e^{iT} \). We note that \( \psi \), being a complex number, is nondimensional. For \( \psi \) to be nondimensional, \( \rho \) must have dimensions of a spatial wavenumber, \([1/L]\) where \( L \) is the unit of length.

Thus, one can think of the trajectories as carrying waves propagating in time with a certain frequency \( \nu \), where \( \rho = \nu/c \). Since \( \rho h/c \) has dimensions of mass, this is to indicate that spacetime trajectories oscillating at higher frequencies have higher ‘mass’. Vibrations theory teaches that higher frequencies of oscillation indicate higher stiffness, leading to the idea that mass reflects the ‘stiffness’ of the underlying trajectory. Interestingly, this is in line with the concept of mass in string theory that is related to tension of the string \([39, 40]\), see appendix B.

Our postulates thus consistent with classical ideas on mass: (a) two objects with a different mass fall at the same speed, which is the consequence of our assumption that \( \rho \) does not affect the velocity of objects in spacetime, and (b) it is harder to change the speed of an object (i.e. bend its trajectory in spacetime) if it has larger mass. In addition, the relationship \( m = \rho h/c \) introduces both \( h \) and \( c \) into classical mechanics, since the classical momentum can be written as \( p = m\nu/c \), where \( p \) is the linear momentum and \( \nu \) is the velocity of the particle.

In fact, the definition of \( Y = -S/\hbar \) is necessary precisely to offset the fact that in classical mechanics \( m \) and not \( \rho \) is used. The meaning of the constant \( \hbar \) emerges as that of a conversion factor between the wavelength of oscillation of a particular space-proper time trajectory, and the associated, classically observable, mass. Mass, as defined here, is conceptually the rest mass of special relativity.

An analogy offers itself to lend physical intuition about the postulates: the situation is similar to that of observing objects moving at the bottom of a swimming pool through a wavefield on the surface. If the size of the object is much larger than the typical wavelength of the wavefield, their can be seen without an uncertainty proportional to that wavelength—small compared with the size of the object. However, if the object size is comparable to the wavelength, then the uncertainty in observation is large. In our case, the wavelength is \( 1/\rho = \hbar/mc \), or precisely the reduced Compton wavelength.

Appendix B. Relationship to string theory

The ideas in this paper are consistent with deBroglie’s wave theory of matter, as we saw in section 5.4. But they are also supported by a mechanical model: the existence of the conserved wavenumber \( \rho \) indicates that the nature of the underlying object is a string, traveling through space-time at speed \( c \). Consider the case of

\[
S(\tau, y) = -mc^2\tau
\]

arising from the relativistic Lagrangian \( \mathcal{L} = -mc^2 \) (see the section 5.2 on the Lagrangian). The frequency of oscillation of the observation field is

\[
\Omega = \frac{mc^2}{\hbar}.
\]

Since the string has wavenumber \( \rho \), the associated natural frequency \( \omega_s \) of oscillation of the string is

\[
\omega_s = \rho c.
\]
In the case of resonance

\[ I = \frac{\omega_s}{\Omega} = \frac{\rho c \hbar}{mc^2}, \tag{B.4} \]

which implies the wavenumber-mass relationship \( m = \rho \hbar / c \) that we postulated based on dimensional grounds in appendix A. This implies the existence of a matter object of mass \( m \) provided there is a resonance between the internal frequency of oscillation of the string and the frequency of oscillation of the observation field.

Now we show that this analysis is consistent with the basic ideas in string theory. Consider an open string of length \( l_s \) \[41\]. Let the rest mass per unit length of the string be \( \mu_0 \). Then the resonance condition reads

\[ mc^2 = \mu_0 l_s c^2 = \rho \hbar c. \tag{B.5} \]

Let \( T_0 \) be the string tension. From \[41\], equation (7.26)

\[ \sigma_1 T_0 = E = mc^2, \tag{B.6} \]

we have

\[ \sigma_1 T_0 = \rho \hbar c \Rightarrow T_0 = \frac{\rho \hbar c}{\sigma_1}. \tag{B.7} \]

Now we identify

\[ \rho = \frac{1}{l_s}. \tag{B.8} \]

This implies that conservation of \( \rho \) on space-time trajectory is equal to the assumption of conservation of open string length. We get

\[ T_0 = \frac{\hbar c}{\sigma_1 l_s}. \tag{B.9} \]

From (7.64) in \[41\] we have for the string rotational velocity \( \omega_s \)

\[ \frac{\omega_s}{c} = \frac{\pi}{\sigma_1} \Rightarrow \sigma_1 = \frac{\pi c}{\omega_s} = \frac{\pi l_s}{2}, \tag{B.10} \]

where \( \sigma_1 \) is the boundary value of the string parameter \( \sigma \in [0, \sigma_1] \), and the last part emerges from using \( \omega_s l_s / 2 = c \) from string theory (string ends have speed of light velocity if it has rotational velocity \( \omega_s \)), or alternatively by noting that

\[ \omega_s = 2\omega = 2\rho c = \frac{2c}{l_s}. \tag{B.11} \]

from the current theory, since in string theory the frequency of repeat of string shape \( \Phi \) for rotating string in spacetime is half the frequency (twice the period) based on string length. Thus,

\[ T_0 = \frac{2\hbar c}{\pi l_s^2}. \tag{B.12} \]

This is precisely what emerges from the formula (9.101) in \[42\].

An interesting aspect of this relationship is the nature of the energy term \( mc^2 \)—this is associated with the observable field oscillation, not with the string oscillation! In contrast, in string theory, the energy \( E \) is the assumed property of the string.
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