Cosmology in beyond-generalized Proca theories

Shintaro Nakamura, Ryotaro Kase, and Shinji Tsujikawa

1Department of Physics, Faculty of Science, Tokyo University of Science, 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

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The beyond-generalized Proca theories are the extension of second-order massive vector-tensor theories (dubbed generalized Proca theories) with two transverse vector modes and one longitudinal scalar besides two tensor polarizations. Even with this extension, the propagating degrees of freedom remain unchanged on the isotropic cosmological background without an Ostrogradski instability. We study the cosmology in beyond-generalized Proca theories by paying particular attention to the dynamics of late-time cosmic acceleration and resulting observational consequences. We derive conditions for avoiding ghosts and instabilities of tensor, vector, and scalar perturbations and discuss viable parameter spaces in concrete models allowing the dark energy equation of state smaller than $w = -1$. The propagation speeds of those perturbations are subject to modifications beyond the domain of generalized Proca theories. There is a mixing between scalar and matter sound speeds, but such a mixing is suppressed during most of the cosmic expansion history without causing a new instability. On the other hand, we find that derivative interactions arising in beyond-generalized Proca theories give rise to important modifications to the cosmic growth history. The growth rate of matter perturbations can be compatible with the redshift-space distortion data due to the realization of gravitational interaction weaker than that in generalized Proca theories. Thus, it is possible to distinguish the dark energy model in beyond-generalized Proca theories from the counterpart in general Proca theories as well as from the ΛCDM model.

I. INTRODUCTION

After the first discovery of late-time cosmic acceleration [1], the constantly accumulating observational data of supernovae Ia [2], cosmic microwave background (CMB) [3], and baryon acoustic oscillations [4] have placed tighter bounds on the dark energy equation of state $w_{\text{DE}}$. The cosmological constant $\Lambda$ (characterized by $w_{\text{DE}} = -1$) is overall consistent with the observational data at background level, but the phantom equation of state ($w_{\text{DE}} < -1$) is also allowed from the data [5]. At the level of perturbations, the cosmic growth rate measurements of redshift-space distortions (RSD) [6, 8] and cluster counts [9] have shown tensions with the Planck CMB bound on $\sigma_8$ predicted by the Λ-cold-dark-matter (ΛCDM) model [10].

If the cosmological constant originates from the vacuum energy associated with particle physics, the vacuum energy usually acquires quantum corrections much larger than the observed dark energy scale [11, 12]. It is worth pursuing the possibility of realizing $w_{\text{DE}}$ smaller than $-1$ without theoretical pathology while modifying gravitational interactions with matter to be consistent with the cosmic growth data. Modified gravitational theories can allow for the realization of such a possibility [13, 14].

In the presence of a scalar field coupled to gravity, it is known that Horndeski theories [15] are the most general scalar-tensor theories with second-order equations of motion. There are models of the late-time cosmic acceleration in the framework of Horndeski theories—like those based on $f(R)$ gravity [16], Brans-Dicke theories [17, 18], and Galileons [19, 22]. These models can lead to $w_{\text{DE}}$ smaller than $-1$ without having ghost and instability problems [16, 18, 22]. In these models, the effective gravitational coupling $G_{\text{eff}}$ of cosmological perturbations is usually larger than the Newton constant $G$ [18, 23–26], so the growth rate of matter perturbations is enhanced compared to that in the ΛCDM model.

It is possible to perform a healthy extension of Horndeski theories in such a way that the number of propagating degrees of freedom (one scalar and two tensor modes) does not increase (see Ref. [27] for an early work). Gleyzes-Langlois-Piazza-Vernizzi (GLPV) [28] expressed the Horndeski action in terms of scalar quantities arising in the 3+1 decomposition of space-time [29] and derived new derivative interactions without imposing two conditions Horndeski theories obey. In a nutshell, there are six free functions $A_2$, $A_3$, $A_4$, $A_5$, $B_4$, and $B_5$ in GLPV theories, whereas in Horndeski theories, the functions $B_4$ and $B_5$ are related to $A_4$ and $A_5$ respectively. The beyond-Horndeski interactions of GLPV theories can give rise to several interesting effects such as the mixing of scalar and matter sound speeds [28, 30], modified growth of subhorizon perturbations with additional time derivatives [31, 32], and the appearance of solid-angle-deficit singularities [33].

For example, the covariantized Galileon model, the Lagrangian of which is derived by replacing partial derivatives of the Minkowski Galileon [19] with covariant derivatives, belongs to a class of GLPV theories. This is different from the covariant Galileon model [20] in which gravitational counterterms are added to keep the equations of motion up to second order. While the covariant Galileon is not excluded as a theoretically consistent dark energy model [22], the covariantized Galileon is plagued by the problem of a negative scalar sound speed squared induced by the scalar-matter mixing [34]. Thus, the extension outside the Horndeski domain generally leads to
nontrivial effects on the evolution of perturbations.

The scalar field is not the only possibility for driving the cosmic acceleration, but the vector field can be also the source for dark energy \[53\]. The massive vector field in Minkowski space-time (Proca theory) has one longitudinal scalar and two transverse vector modes due to the breaking of \(U(1)\) gauge invariance. If the massive vector field \(A^\mu\) is coupled to gravity, it is possible to construct second-order generalized Proca (GP) theories by keeping three propagating degrees of freedom besides two tensor polarizations \[36–38\] (see also Refs. \[33–51\]).

The existence of derivative interactions like those appearing for covariant vector Galileons in GP theories gives rise to a branch of background cosmological solutions where the temporal vector component \(\phi\) depends on the Hubble expansion rate \(H\) alone \[54, 52\]. For the covariant extended vector Galileon model in which the Lagrangians contain general powers of \(X = -\partial^2 A^\mu/2\), there exists a de Sitter attractor preceded by the dark natures of weak gravity consistent with the recent RSD and CMB measurements.

Moreover, it is of interest to study whether there are some distinct observational signatures of BGP theories as compared to GP theories and the \(\Lambda\)CDM model. To address these issues, we study the cosmology based on the covariantized extended vector Galileon model in which the temporal vector component \[59\].

Our paper is organized as follows. In Sec. \[II\] we review the action of BGP theories, and in Sec. \[III\] we discuss the background cosmology in the covariantized extended vector Galileon model. In Secs. \[IV\] and \[V\] we study no-ghost and stability conditions of tensor/vector perturbations and search for theoretically consistent parameter spaces. In Sec. \[VI\] we present scalar perturbation equations of motion and study the mixing of sound speeds for the covariantized extended vector Galileon model in detail. In Sec. \[VII\] we study the evolution of matter perturbations as well as gravitational potentials and show the possibility of observationally distinguishing dark energy models in BGP theories from those in GP theories and the \(\Lambda\)CDM model. We conclude in Sec. \[VIII\].

II. BEYOND-GENERALIZED PROCA THEORIES

We consider a massive vector field \(A^\mu\) coupled to gravity with the field tensor \(F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu\), where \(\nabla_\mu\) is a covariant derivative operator. The mass term explicitly breaks a \(U(1)\) gauge symmetry, so the longitudinal scalar mode arises in addition to two transverse vector polarizations. In GP theories with derivative couplings to gravity \[36, 38\], the equations of motion for the vector field and the metric remain of second order.

It is possible to extend GP theories in such a way that the number of propagating degrees of freedom does not increase relative to those in GP theories (one scalar, two vectors, and two tensors) \[50\]. The four-dimensional action of such extended theories (BGP theories) is given by

\[
S = \int d^4x\sqrt{-g}\left(\sum_{i=2}^{6} \mathcal{L}_i + S_M(g_{\mu\nu}, \Psi_M)\right), \tag{2.1}
\]

where \(g\) is a determinant of the metric tensor \(g_{\mu\nu}\), and

\[
\mathcal{L}_2 = G_2(X, F, Y), \tag{2.2}
\]

\[
\mathcal{L}_3 = G_3(X)\nabla_\mu A^\mu, \tag{2.3}
\]

\[
\mathcal{L}_4 = G_4(X)R + G_4,X(X)\left[(\nabla_\mu A^\mu)^2 - \nabla_\mu A_\sigma \nabla^\sigma A_\nu\right], \tag{2.4}
\]

\[
\mathcal{L}_5 = G_5(X)G_{\mu\nu} \nabla_\mu A^\nu - \frac{1}{6}G_5,X(X)\left[(\nabla_\mu A^\mu)^3 - 3\nabla_\mu A^\nu \nabla_\sigma A_\nu \nabla^\sigma A_\mu + 2\nabla_\rho A_\sigma \nabla^\sigma A^\mu A_\nu\right]
- g_5(X)\bar{F}^{\alpha\beta} \bar{F}_{\mu\nu} \nabla_\alpha A_{\beta}, \tag{2.5}
\]

\[
\mathcal{L}_6 = G_6(X)L^{\mu\nu\alpha\beta} \nabla_\mu A_\alpha \nabla_\nu A_\beta + \frac{1}{2}G_6,X(X)\bar{F}^{\alpha\beta} \bar{F}_{\mu\nu} \nabla_\alpha A_\mu \nabla_\beta A_\nu, \tag{2.6}
\]

with

\[
X = -\frac{A_\mu A^\mu}{2}, \quad F = -\frac{F_{\mu\nu} F^{\mu\nu}}{4}, \quad Y = A^\mu A_\nu F_{\mu\alpha}^{\quad \nu} F^{\alpha\beta}F_{\beta\gamma} \tag{2.7}
\]

While \(G_2\) is a function of \(X, F, Y\), the functions \(G_{3,4,5,6}\) and \(g_5\) depend on \(X\) alone. For partial derivatives with...
respect to $X$, we use the notation $G_{i,X} \equiv \partial G_i/\partial X$.

There are nonminimal derivative couplings of the vector field with the Ricci scalar $R$ and the Einstein tensor $G_{\mu\nu}$ in $\mathcal{L}_4$ and $\mathcal{L}_5$, respectively. In $\mathcal{L}_6$, there is also a derivative coupling with the double dual Riemann tensor defined by

$$L^{\mu\nu\alpha\beta} = \frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\rho\gamma\delta},$$

where $R_{\rho\gamma\delta}$ is the Riemann tensor and $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor obeying the normalization $\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\mu\nu\rho\sigma} = -4!$. For constant $G_6$, the coupling $G_6 L^{\mu\nu\alpha\beta} \nabla_\mu A_\nu \nabla_\alpha A_\beta$ is the only allowed $U(1)$ gauge-invariant interaction advocated by Horndeski\cite{39}. For $G_6$ depending on $X$, we need to introduce the second term in Eq. (2.6) to keep the equations of motion up to second order (which is also the case for the second terms in $\mathcal{L}_4$ and $\mathcal{L}_5$). The quantity $\tilde{F}^{\mu\nu}$ is the dual strength tensor defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}.$$ 

The terms $F$ and $Y$ in $G_2$ as well as the terms containing the functions $g_5$ and $G_6$ correspond to intrinsic vector modes that vanish in the scalar limit $A_\mu \to \nabla_\mu \pi$.

The Lagrangian density $\mathcal{L}^N$ in the action (2.11) arises outside the domain of GP theories. The explicit form of $\mathcal{L}^N$ is given by\cite{58}

$$\mathcal{L}^N = \mathcal{L}^N_4 + \mathcal{L}^N_5 + \mathcal{L}^N_6,$$

where

$$\mathcal{L}^N_4 = f_4 \delta^{i_1 i_2 i_3 i_4} A^{i_1} A_{i_1} \nabla^{i_2} A_{i_2} \nabla^{i_3} A_{i_3} A^{i_4},$$

$$\mathcal{L}^N_5 = f_5 \delta^{i_1 i_2 i_3 i_4} A^{i_1} A_{i_1} \nabla^{i_2} A_{i_2} \nabla^{i_3} A_{i_3} A^{i_4} \nabla^{i_4} A_{i_4},$$

$$\mathcal{L}^N_6 = f_6 \delta^{i_1 i_2 i_3 i_4} \nabla^{i_1} A_{i_1} \nabla^{i_2} A_{i_2} \nabla^{i_3} A_{i_3} A^{i_4} \nabla^{i_4} A_{i_4}.$$ 

These functions $f_4$, $f_5$, and $f_6$ depend on $X$ alone. The Lagrangian densities (2.11)–(2.14) were constructed in such a way that the relative coefficients between $G_i$ and $G_i,X$ (where $i = 4, 5, 6$) appearing in Eqs. (2.4)–(2.6) are detuned. Even with these new interactions, the propagating degrees of freedom for linear perturbations on the isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) background are the same as those in GP theories\cite{56}. On an anisotropic cosmological background, it was also shown in Ref.\cite{58} that there are no additional ghostly degrees of freedom associated with the Ostrogradski instability.

In Eq. (2.7), $S_M$ is the action of matter fields $\Psi_M$. We assume that the matter fields are minimally coupled to gravity. Since the vector field has a direct coupling to gravity, the matter sector feels the vector propagation through gravitational interactions. In BGP theories, it is known that the Lagrangian density $\mathcal{L}^N$ leads to a mixing between the scalar sound speed of the vector field and the matter sound speed $\frac{E_5}{2X}$\cite{56}. This is the important difference between GP theories and BGP theories, so we will estimate modifications of the sound speeds induced by $\mathcal{L}^N$ in concrete dark energy models in Sec. VI.

Moreover, the effective gravitational coupling $G_{\text{eff}}$ associated with the growth of matter perturbations should be also subject to change by new interactions of BGP theories. In particular, extra time derivatives can arise in the perturbation equations of motion, so the quasistatic approximation used in GP theories for subhorizon modes\cite{58} may lose its validity. In Sec. VII we will study how the evolution of matter perturbations and gravitational potentials is affected by the new interactions $\mathcal{L}^N$ in dark energy models within the framework of BGP theories.

Before entering the details of scalar perturbations, we will discuss the background cosmology and no-ghost and stability conditions of tensor and vector perturbations in subsequent sections to restrict the parameter space of dark energy models in BGP theories.

### III. BACKGROUND COSMOLOGY

#### A. Background equations of motion

On the flat FLRW space-time described by the line element $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$, the background equations of motion were derived in Ref.\cite{56} in the presence of a matter perfect fluid with density $\rho_M$ and pressure $P_M$. The vector-field configuration compatible with the symmetry of the FLRW background contains the temporal component $\phi(t)$ alone, i.e., $A^u = (\phi(t), 0, 0, 0)$. In Ref.\cite{58}, it was shown that the background equations depend only on four functions among ten free functions $G_{2,3,4,5,6}, g_5, f_4, f_5, f_6$ appearing in the action (2.11). It is convenient to introduce the following combinations,

$$A_2 = G_2, \quad A_3 = (2X)^{3/2} E_3, \quad A_4 = -G_4 + 2X G_{4,X} + 4X^2 f_4, \quad A_5 = -\sqrt{2} X^{3/2} \left( \frac{1}{3} G_{5,X} - 4X f_5 \right),$$

$$B_4 = G_4, \quad B_5 = (2X)^{1/2} E_5,$$

where $E_5(X)$ and $E_5(X)$ are auxiliary functions satisfying the relations

$$G_3 = E_3 + 2X E_3, \quad G_{5,X} = \frac{E_5}{2X} + E_5.$$ 

Then, the gravitational equations of motion are given by\cite{56}

$$A_2 - 6H^2 A_4 - 12H^3 A_5 = \rho_M,$$

$$\dot{A}_3 + 4(H A_4 + H A_4) + 6H (H A_5 + 2H A_5) = \rho_M + P_M,$$

where $H = \dot{a}/a$ is the Hubble expansion rate and a dot represents a derivative with respect to $t$. The temporal vector component obeys

$$\phi \left( A_{2,X} + 3H A_{3,X} + 6H^2 A_{4,X} + 6H^3 A_{5,X} \right) = 0.$$
which can be also derived from Eqs. (3.3) and (3.4).

From Eq. (3.1), it follows that

\[
A_4 + B_4 - 2XB_{4,X} = 4X^2 f_4, \tag{3.6}
\]
\[
A_5 + \frac{1}{3}XB_{5,X} = (2X)^{3/2} f_5. \tag{3.7}
\]

Since \(f_4 = 0 = f_5\) in GP theories, there are two particular relations \(A_4 + B_4 - 2XB_{4,X} = 0\) and \(A_5 + XB_{5,X}/3 = 0\).

In BGP theories, the functions \(B_4\) and \(B_5\) are not directly related to \(A_4\) and \(A_5\) due to the existence of nonvanishing functions \(f_4\) and \(f_5\). The Lagrangians containing the functions \(G_5, f_5, f_5, f_6\) do not contribute to the background equations of motion by reflecting the fact that they correspond to intrinsic vector modes.

From Eqs. (3.3)-(3.5), the background dynamics is determined by the four functions \(A_{2,3,4,5}\), but it does not depend on \(B_4\) and \(B_5\). This means that BGP theories and GP theories with the same \(A_{2,3,4,5}\) but with different \(B_{4,5}\) cannot be distinguished from each other at the background level.

**B. Covariant and covariantized extended vector Galileon models**

For concreteness, we consider the model given by the functions \([52]\)

\[
G_2 = b_2X^{p_2} + F, \quad G_3 = b_3X^{p_3}, \tag{3.8}
\]
and

\[
G_4 = \frac{M_{pl}^2}{2} + b_4X^{p_4}, \quad G_5 = b_5X^{p_5}, \tag{3.9}
\]

where \(b_{2,3,4,5}\) and \(p_{2,3,4,5}\) are constants, and \(M_{pl} = (8\pi G)^{-1}\) is the reduced Planck mass. From Eq. (3.2), we can choose the auxiliary functions \(E_3\) and \(E_5\) in the forms

\[
E_3 = b_3X^{p_3}/(1 + 2p_3) \quad \text{and} \quad E_5 = 2b_5X^{p_5}/(1 + 2p_5),
\]

respectively. Since \(f_4 = 0 = f_5\) in GP theories, the functions in Eq. (3.1) yield

\[
A_2 = b_2X^{p_2} + F, \quad A_3 = \frac{2\sqrt{2}\beta_3 p_3}{1 + 2p_3} X^{p_3+1/2},
\]
\[
A_4 = -\frac{M_{pl}^2}{2} + b_4(2p_4 - 1)X^{p_4},
\]
\[
A_5 = -\frac{\sqrt{2}}{3}b_5 p_5 X^{p_5+1/2}, \tag{3.10}
\]
and

\[
B_4 = \frac{M_{pl}^2}{2} + b_4X^{p_4}, \quad B_5 = \frac{2\sqrt{2}b_5p_5}{1 + 2p_5} X^{p_5+1/2}. \tag{3.11}
\]

The covariant vector Galileon \([53]\) corresponds to the powers \(p_2 = 1, p_3 = 1, p_4 = 2, p_5 = 2\). The model given by more general functions \(G_5\) and \(G_6\) together with couplings \(g_5\) and \(G_6\) is dubbed the covariant extended vector Galileon (covariant EVG). In this case, the functions \(B_4\) and \(B_5\) obey the relations \(A_4 + B_4 - 2XB_{4,X} = 0\) and \(A_5 + XB_{5,X}/3 = 0\) to keep the equations of motion up to second order.

In BGP theories, there are no particular constraints between \(A_4, B_4, A_5,\) and \(B_5\). Let us consider theories in which the functions \(A_{2,3,4,5}\) are the same as those in Eq. (3.10) but with the functions \(B_4\) and \(B_5\) given by

\[
B_4 = \frac{M_{pl}^2}{2}, \quad B_5 = 0. \tag{3.12}
\]

In this case we have

\[
G_4 = \frac{M_{pl}^2}{2}, \quad G_5 = 0, \tag{3.13}
\]

which are different from the functions \(G_4, G_5\) in Eq. (3.9). Note that \(G_5\) can be a nonvanishing constant, but we have set \(G_5 = 0\) without loss of generality since the constant \(G_5\) does not contribute to the dynamical equations of motion (due to the property \(\nabla^\mu G_{\mu\nu} = 0\)). The functions \(f_4\) and \(f_5\), which characterize the deviation from GP theories, are given, respectively, by

\[
f_4 = \frac{1}{4}b_4(2p_4 - 1)X^{p_4-2}, \quad f_5 = -\frac{1}{12}b_5 p_5 X^{p_5-2}. \tag{3.14}
\]

The Lagrangians of BGP theories we are considering now contain the interactions \(L_4^N\) and \(L_5^N\) besides the Einstein-Hilbert term \((M_{pl}^2/2)R\). These new terms correspond to those derived by replacing partial derivatives with covariant derivatives for the EVG model in Minkowski spacetime (analogous to the covariantized Galileon model discussed in Ref. [34]). The model given by the functions (3.8), (3.13), and (3.14) together with the other couplings \(g_5, G_6, f_5, f_6\) is dubbed the covariantized EVG.

The background cosmological dynamics in the covariantized EVG model is exactly the same as that in the covariant EVG model. Since the dynamics in the latter was studied in Ref. [52], we briefly summarize the main results. We consider the powers \(p_{3,4,5}\) satisfying

\[
p_3 = \frac{1}{2} (p + 2p_2 - 1), \quad p_4 = p + p_2, \quad p_5 = \frac{1}{2} (3p + 2p_2 - 1), \tag{3.15}
\]

where \(p\) is a positive constant. Then, the nonvanishing \(\phi\) branch of Eq. (3.5) gives rise to the following solution:

\[
\phi \propto H^{-1/p}. \tag{3.16}
\]

Since \(\phi\) grows with the decrease of \(H\), the energy density of the temporal vector component works as dark energy at late cosmological epochs. In fact, there exist de Sitter solutions characterized by constant \(\phi\) and \(H\).

For the matter action \(S_m\), we take into account the perfect fluids of nonrelativistic matter (density \(\rho_m\) and pressure \(P_m\)) and radiation (density \(\rho_r\) and pressure \(P_r = \rho_r / 3\)). We define the corresponding density parameters \(\Omega_m = \rho_m / (3M_{pl}^2 H^2)\) and \(\Omega_r = \rho_r / (3M_{pl}^2 H^2)\) as well as the dark energy density parameter

\[
\Omega_{DE} = \frac{\gamma}{p + p_2}, \tag{3.17}
\]
where
\[ y = \frac{b_2 \phi^{2p_2}}{3M_{pl}^2 H^2 2p_2} , \] (3.18)
\[ \gamma = 6p_2(2p + 2p_2 - 1)\beta_4 - (p + p_2)(1 + 4p_2\beta_5) , \] (3.19)
\[ \beta_i = \frac{p_i b_i (\phi^2 H)^{-i-2}}{2^{p_i - p_2 p_2} p_2} , \] (3.20)
with \( i = 3, 4, 5 \), from Eq. (3.5), we have the following relation for the branch \( \phi \neq 0 \):
\[ 1 + 3\beta_4 + 6(2p + 2p_2 - 1)\beta_4 - (3p + 2p_2)\beta_5 = 0 . \] (3.21)

The background equations (3.3) and (3.4) can be expressed as
\[ 3M_{pl}^2 H^2 = \rho_{DE} + \rho_m + \rho_r , \] (3.22)
\[ -2M_{pl}^2 H = \rho_{DE} + P_{DE} + \rho_m + \frac{4}{3}\rho_r , \] (3.23)
where \( \rho_{DE} \) and \( P_{DE} \) correspond to the density and the pressure associated with the vector field, respectively. Introducing the ratio \( s = p_2 / p \), the dynamical equations of motion can be expressed in the autonomous forms (5.2)
\[ \Omega'_{DE} = \frac{(1 + s) \Omega_{DE}(3 + \Omega_r - 3\Omega_{DE})}{1 + s \Omega_{DE}} , \] (3.24)
\[ \Omega'_r = -\frac{(1 + s) \Omega_{DE} - 3\Omega_{DE}}{1 + s \Omega_{DE}} , \] (3.25)
where a prime represents a derivative with respect to \( N = \ln a \). The matter density parameter is known from the relation \( \Omega_m = 1 - \Omega_{DE} - \Omega_r \). The dark energy equation of state, which is defined by \( w_{DE} = P_{DE}/\rho_{DE} \), reads
\[ w_{DE} = -\frac{3(1 + s) + s \Omega_r}{3(1 + s \Omega_{DE})} . \] (3.26)

For the dynamical system (3.24)–(3.25), there exist the three fixed points: (a) radiation: \( (\Omega_{DE}, \Omega_r) = (0, 1) \), (b) matter: \( (\Omega_{DE}, \Omega_r) = (0, 0) \), and (c) de Sitter: \( (\Omega_{DE}, \Omega_r) = (1, 0) \). During the cosmological sequence of the fixed points \( (a) \rightarrow (b) \rightarrow (c) \), the dark energy equation of state evolves as \( w_{DE} = -1 - 4s/3 \rightarrow (b) \rightarrow (c) \rightarrow -1 - s \rightarrow (c) \rightarrow (a) \). From Eqs. (3.24)–(3.25), there is the relation \( \Omega'_{DE} / \Omega_{DE} = (1 + s)(\Omega'_r / \Omega_r + 4) \), which is integrated to give \( \Omega_{DE} / \Omega_{DE}^{1+s} \propto a^{4(1+s)} \). Using this solution, the evolution of \( \Omega_{DE} \) and \( \phi \) during the radiation and the matter eras is given by
\[ \Omega_{DE} \propto t^{2(1+s)} , \quad \phi \propto t^{1/p} . \] (3.27)

For \( s > -1 \), the dark energy density parameter grows in time. Since the fixed point (c) is always stable (52), the solutions finally approach the de Sitter attractor to give rise to the late-time cosmic acceleration.

The covariant and the covariantized EVG models can be distinguished from each other at the level of linear cosmological perturbations. Since the perturbations can be decomposed into tensor, vector, and scalar modes, we will separately study the behavior of each mode in subsequent sections.

IV. TENSOR PERTURBATIONS

A. Stability conditions

We begin with tensor perturbations \( h_{ij} \) given by the line element
\[ ds_t^2 = -dt^2 + a^2(t) (\delta_{ij} + h_{ij}) \, dx^i \, dx^j . \] (4.1)

Due to the transverse and traceless conditions \( \partial_i h_{ij} = 0 \) and \( \partial^i h_i = 0 \), there are two polarization modes \( h_+ \) and \( h_\times \) for \( h_{ij} \). In terms of the unit vectors \( e^+_i \) and \( e^\times_i \), the dynamical equations of motion can be expressed in the autonomous forms (2.1) is given by (56)
\[ \delta_T^{(2)} = \sum_{\lambda=+\times} \int dt \, dt' \, dx^3 \, \frac{q_T}{8} \left[ \frac{\dot{h}^2}{\lambda^2} - \frac{c_s^2}{a^2} (\partial h_{ij})^2 \right] , \] (4.2)
where \( q_T = -2(A_4 + 3H A_5) \), (4.3)
\[ c_s^2 = \frac{2B_4 + B_5}{q_T} . \] (4.4)

The conditions for avoiding ghosts and Laplacian instabilities correspond to \( q_T > 0 \) and \( c_s^2 > 0 \), respectively. The ghost condition is determined by the quantities \( A_4 \) and \( A_5 \) appearing in the background equations of motion. Since \( c_s^2 \) contains \( B_4 \) and \( B_5 \), the theories with same values of \( A_4, A_5 \) but with different values of \( B_4, B_5 \) can be distinguished from the tensor propagation speed. The intrinsic vector modes associated with the functions \( g_5, G_6, f_5, f_6 \) affect neither \( q_T \) nor \( c_s^2 \).

We write the function \( A_4 \) in the form
\[ A_4(X) = -\frac{M_{pl}^2}{2} + \hat{A}_4(X) , \] (4.5)
where \( \hat{A}_4(X) \) is a function of \( X \). For covariant and covariantized EVG models, \( \hat{A}_4(X) = b_4(2p_4 - 1) X^{p_4} \). From the background equations (3.3)–(3.4), the density \( \rho_{DE} \) and the pressure \( P_{DE} \) in Eqs. (3.22) and (3.23) are given, respectively, by
\[ \rho_{DE} = -A_2 + 6H^2 \hat{A}_4 + 12H^3 A_5 , \] (4.6)
\[ P_{DE} = A_2 - \hat{A}_3 - 3(3H^2 + 2H) \hat{A}_4 - 4H \hat{A}_4 - 12H^2 + \hat{H} \right) A_5 - 6H^2 \hat{A}_5 . \] (4.7)

We are now considering the case in which \( \rho_{DE} \) and \( P_{DE} \) are responsible for the late-time cosmic acceleration. During the radiation and matter eras, both \( \rho_{DE} \) and \( P_{DE} \) are suppressed relative to the background density \( \rho_M \approx M_{pl}^2 H^2 \), so the conditions
\[ \{ |A_2|, |H|A_3|, H^2|A_4|, H^3|A_5| \} \ll M_{pl}^2 H^2 \] (4.8)
are satisfied. Under these conditions, the quantity \( q_T = M_{pl}^2 - 2A_4 - 6HA_5 \) is approximately given by

\[
(q_T)_{\text{early}} \simeq M_{pl}^2,
\]

(4.9)

which means that the tensor ghost is absent in the early cosmological epoch. In the late Universe, there are contributions from the terms \( \tilde{A}_4 \) and \( A_5 \) to \( q_T \), but as long as the condition

\[
2\tilde{A}_4 + 6HA_5 < M_{pl}^2
\]

(4.10)

is satisfied, there is no tensor ghost.

For the estimation of \( c_T^2 \), we express \( B_4 \) in the form

\[
B_4(X) = \frac{M_{pl}^2}{2} + \tilde{B}_4(X),
\]

(4.11)

where \( \tilde{B}_4(X) \) is a function of \( X \). In the early cosmological epoch, the functions \( A_4 \) and \( A_5 \) should satisfy Eq. (4.8) to realize the consistent background dynamics, in which regime Eq. (4.3) reduces to

\[
(c_T^2)_{\text{early}} \simeq 1 + \frac{2\tilde{B}_4 + \tilde{B}_5}{M_{pl}^2}.
\]

(4.12)

In GP theories, the functions \( \tilde{B}_4 \) and \( B_5 \) are subject to the constraints \( \tilde{A}_4 + \tilde{B}_4 - 2XB_4, X = 0 \) and \( 3A_5 + XB_5, X = 0 \), so they also satisfy the conditions similar to those of \( A_4 \) and \( A_5 \), i.e., \( |\tilde{B}_4| \ll M_{pl}^2 \) and \( |H|B_5| \ll M_{pl}^2 \) for power-law functions of \( \tilde{B}_4 \) and \( B_5 \). Provided that \( |B_5| \) is at most of the order of \( |H|B_5| \), we have that \( c_T^2 \simeq 1 \) in the early cosmological epoch in GP theories.

In BGP theories the functions \( \tilde{B}_4 \) and \( B_5 \) are independent of \( \tilde{A}_4 \) and \( A_5 \), respectively, so \( c_T^2 \) is not necessarily close to 1 at high redshifts. If the quantities \( |\tilde{B}_4| \) and \( |H|B_5| \) are not much smaller than \( M_{pl}^2 \) in the early cosmological epoch, the deviation of \( c_T^2 \) from 1 is significant at low redshifts. This leads to either the Laplacian instability \( (c_T^2 < 0) \) or the highly superluminal propagation speed squared \( (c_T^2 \gg 1) \) being possibly incompatible with observational bounds of \( c_T^2 \). Then, it is safe to consider the situation in which the two conditions

\[
|\tilde{B}_4| \ll M_{pl}^2, \quad |H|B_5| \ll M_{pl}^2
\]

(4.13)

are satisfied. In this case, \( c_T^2 \simeq 1 \) at high redshifts.

On the de Sitter solution, we have \( \tilde{B}_5 = 0 \), so Eq. (4.4) reduces to

\[
(c_T^2)_{\text{dS}} = \left(1 + \frac{2\tilde{B}_4}{M_{pl}^2}\right) \left(1 - \frac{2\tilde{A}_4}{M_{pl}^2} \frac{6HA_5}{M_{pl}^2}\right)^{-1}.
\]

(4.14)

Since the inequality (15) does not generally hold on the de Sitter solution, there is the deviation of \( (c_T^2)_{\text{dS}} \) from 1. Moreover, we also have the contribution to \( (c_T^2)_{\text{dS}} \) from the term \( \tilde{B}_4 \), which is different between GP theories and BGP theories.

B. Covariant and covariantized EVG models

For concreteness, we consider the covariant and covariantized EVG models in which the functions \( A_{2,3,4,5} \) are given by Eq. (3.10) with the powers \( \frac{1}{2} \). The difference between the two models arises from the functions \( B_4 \) and \( B_5 \). Since the parameter \( \Omega_{DE} \) satisfies the relation \( y = (p + p_2)\Omega_{DE}/\gamma \), the quantity \( \Omega_{DE} \) reduces to

\[
q_T = M_{pl}^2 \left[ 1 - \frac{6p_2}{\gamma} \{(2p + 2p_2 - 1)\beta_4 - (p + p_2)\beta_5\} \Omega_{DE} \right].
\]

(4.15)

Since \( \Omega_{DE} \to 0 \) in the asymptotic past, we recover the property (4.9). On the de Sitter solution \( (\Omega_{DE} = 1) \), the no-ghost condition \( (q_T > 0) \) is satisfied for \( |\beta_4| \) and \( |\beta_5| \) much smaller than 1.

In the covariant EVG model, the background dynamics restricts \( c_T^2 \) to be close to 1 in the early cosmological epoch. In the covariantized EVG model \( (\tilde{B}_4 = 0, B_5 = 0) \), the conditions (4.13) are automatically satisfied, so \( c_T^2 \simeq 1 \) at high redshifts.

On the de Sitter solution \( (\Omega_{DE} = 1) \), the tensor propagation speed squared (4.14) in the covariant EVG model \( (\tilde{B}_4 = b_4 X) \) reads

\[
(c_T^2)_{\text{dS1}} = 1 - \frac{6p_2(2\beta_4 - \beta_5)}{1 - 2p_2\beta_5}.
\]

(4.16)

On the other hand, the covariantized EVG model corresponds to \( \tilde{B}_4 = 0 \), so Eq. (4.14) yields

\[
(c_T^2)_{\text{dS2}} = 1 - \frac{6p_2[(2p + 2p_2 - 1)\beta_4 - (p + p_2)\beta_5]}{(p + p_2)(1 - 2p_2\beta_5)}.
\]

(4.17)

If \( |\beta_4| \) and \( |\beta_5| \), both \( (c_T^2)_{\text{dS1}} \) and \( (c_T^2)_{\text{dS2}} \) tend to be away from 1. However, as long as the conditions

\[
|\beta_4| \ll 1, \quad |\beta_5| \ll 1
\]

(4.18)

are satisfied, they do not deviate much from 1. From Eqs. (1.10) and (1.11), we have \( (c_T^2)_{\text{dS2}} - (c_T^2)_{\text{dS1}} = 6p_2\beta_4/[(p + p_2)(1 - 2p_2\beta_5)] \), the difference of which gets more significant for larger \( |\beta_4| \). In summary, under the conditions (4.13), there are neither ghosts nor Laplacian instabilities in both covariant and covariantized EVG models.

From the CMB observations, the tensor propagation speed squared is constrained to be \( c_T^2 < 1.30 \pm 0.79 \) at 95% confidence level by assuming that \( c_T^2 \) is constant [61]. From the gravitational Cherenkov radiation, there exists the tight bound \( 1 - c_T^2 < 2 \times 10^{-15} \) for the subluminal propagation [64], but the corresponding energy,

\[1 \] The superluminal case is not subject to the Cherenkov-radiation constraint. The superluminal propagation does not necessarily imply the violation of causality; since, in many theories including k-essence [62] and Galileons [63], the appearance of closed causal curves can be avoided.
$\sim 10^{10}$ GeV, is much higher than any reasonable cutoff associated with the late-time cosmic acceleration. From binary pulsars timing data, the deviation of $c_T$ from 1 is constrained to the level of $10^{-2}$ [65]. Under the conditions [4.18], $c_T^2$ is very close to 1 during the cosmic expansion history, so the above-mentioned observational bound of $c_T^2$ can be satisfied.

V. VECTOR PERTURBATIONS

A. Stability conditions

The vector perturbation arises from the spatial component $A^i$ of the vector field. We express the intrinsic vector mode $E_j$ in $A^i$, as $(A^i)_V = E_j \delta^{ij} / a^2(t)$, where $E_j$ satisfies the transverse condition $\partial_j E_j = 0$. We also consider the metric perturbation $V_i$ described by the line element

$$ds_V^2 = -dt^2 + 2V_i dt dx^i + a^2(t) \delta_{ij} dx^i dx^j ,$$

(5.1)

where we have chosen the flat gauge. The vector perturbation also obeys the transverse condition $\partial^i V_i = 0$. The combination

$$Z_i = E_i + \phi(t) V_i$$

(5.2)

corresponds to a dynamical degree of freedom with two transverse polarizations. The matter perfect fluid can be associated with the late-time cosmic acceleration. From the small-scale limit, the resulting second-order action for the element $V_i$ can be expressed as

$$S_V^{(2)} \sim \int dt dx \sum_{i=1}^2 q_{VV} \left[ \dot{Z}_i^2 - \frac{c_V^2}{a^2} (\partial Z_i)^2 \right],$$

(5.3)

where

$$q_V = G_2 F + 2G_2 Y \phi^2 - 4g_5 \phi \phi + 2G_6 H^2$$

$$+ 2G_6 X H^2 \phi^2 + 4\tilde{f}_6 H^2 \phi^2,$$

(5.4)

$$c_V^2 = 1 + \frac{2(A_4 + B_4 + 3H A_5)^2}{\phi^2 q_V q_{VV}}$$

$$+ \frac{2(G_6 H - G_2 Y \phi^2) - 2(H \phi - \dot{\phi})(G_6 X H \phi - g_5)}{q_V}$$

$$- \frac{2}{q_{VV}} \left[ \tilde{f}_5 H \phi^3 + 2\tilde{f}_6 \phi (H \phi - \dot{\phi}) \right].$$

(5.5)

The functions $F$ and $Y$ in $G_2$ as well as the functions $g_5, G_5$, and $\tilde{f}_6$ affect the quantity $q_V$ (which characterizes the vector no-ghost condition). Besides these intrinsic vector modes, the function $\tilde{f}_5$ also leads to a modification to the vector propagation speed $c_V$. The difference of $c_V^2$ between GP theories and BGP theories arises through the functions $B_4$ (or $f_4$) and $\tilde{f}_5, \tilde{f}_6$.

To understand the effect of the terms beyond the domain of GP theories, we consider the theories of nonvanishing functions $\tilde{f}_5, \tilde{f}_6$, and

$$G_2 = F + g_2(X), \quad g_5 = 0, \quad G_6 = 0,$$

(5.6)

where $g_2(X)$ is a function of $X$. Then, Eqs. (5.3) and (5.5) reduce, respectively, to

$$q_V = 1 + 4\tilde{f}_6 H^2 \phi^2,$$

(5.7)

$$c_V^2 = \frac{1}{q_V} \left[ 1 + \frac{(g_T - 2B_4)^2}{2\phi^2 q_T} - 2\tilde{f}_5 H \phi^3 + \frac{\dot{\phi}}{H \phi} (q_V - 1) \right].$$

(5.8)

For positive $\tilde{f}_6$, the vector ghost is absent. If $\tilde{f}_6$ is negative and the function $|\tilde{f}_6| H^2 \phi^2$ grows in time, there is a possibility for the appearance of ghosts. To avoid this, we require the condition

$$|\tilde{f}_6| H^2 \phi^2 \ll 1$$

(5.9)
in the early cosmological epoch. Moreover, the condition $\tilde{f}_6 H^2 \phi^2 > -1/4$ needs to be satisfied on the de Sitter solution.

For the theories with $\tilde{f}_5 = 0$, the vector propagation speed squared (5.8) on the de Sitter solution ($\phi = 0$) reduces to

$$(c_V^2)_{\text{dS}} = \frac{1}{q_V} \left[ 1 + \frac{(g_T - 2B_4)^2}{2\phi^2 q_T} \right],$$

(5.10)

which is positive under the no-ghost conditions $g_T > 0$ and $q_V > 0$. Provided that $q_V$ is close to 1 in the early cosmological epoch, the last term in the square brackets of Eq. (5.8) is also suppressed relative to the first term. Hence, the Laplacian instability can be avoided for the theories with $\tilde{f}_5 = 0$ and $\tilde{f}_6 \neq 0$.

In the presence of the coupling $\tilde{f}_5$, the term $-2\tilde{f}_5 H^3$ in Eq. (5.8) modifies the value of $c_V^2$. If $\tilde{f}_5$ is positive and the function $\tilde{f}_5 H^3$ grows in time, $c_V^2$ can be negative. To avoid this Laplacian instability, we require the condition

$$\tilde{f}_5 H^3 \ll 1$$

(5.11)
in the early cosmological epoch. On the de Sitter solution, the term $\tilde{f}_5 H^3$ should not be large either to satisfy the stability condition $c_V^2 > 0$.

B. Covariantized EVG model

For concreteness, let us consider the covariantized EVG model with the functions (5.6) and the power-law couplings

$$\tilde{f}_5(X) = c_5 X^{g_5}, \quad \tilde{f}_6(X) = c_6 X^{g_6},$$

(5.12)
where $c_5, c_6, q_6$, and $q_6$ are constants. Since $\dot{\phi}^2 \propto H^{-1}$ for the background, we introduce the dimensionless constant

$$\lambda = \left(\frac{\phi}{M_{\text{pl}}}\right)^p \frac{H}{m},$$

(5.13)

where $m$ ($>0$) is a mass scale related to the function $g_2(X)$ in Eq. (5.6) as $g_2(X) = b_2 X^{p_2}$ with $b_2 = -m^2 M_{\text{pl}}^{2(1-p_2)}$. The negative value of $b_2$ is chosen to avoid the appearance of tensor ghosts in the limit that $G_5 \to 0$ [52]. Then, the quantity $q_V$ reads

$$q_V = 1 + 2^{2-q_6} \tilde{c}_6 \lambda^2 u^2(1+q_6-p),$$

(5.14)

where $\tilde{c}_6 = c_6 m^2 M_{\text{pl}}^{2(1+q_6)}$, and

$$u \equiv \frac{\phi}{M_{\text{pl}}} = \left[-2p_2 3\lambda^2 (p + p_2) \Omega_{\text{DE}} \right]^{\frac{1}{2(p+2)}}.$$  

(5.15)

For $q_6 > p - 1$, the function $u^{2(1+q_6-p)}$ in Eq. (5.14) increases with the growth of $\phi$.

In Fig. 1 we plot the evolution of $q_V$ for $q_6 = 5, p_2 = 1, p = 5, \lambda = 1$ with three different values of $\tilde{c}_6$. For $\tilde{c}_6 > 0$, $q_V$ starts to grow from the value close to 1, and then it approaches a constant larger than 1 on the de Sitter attractor. For negative $\tilde{c}_6$, $q_V$ decreases toward the value smaller than 1. Numerically, we find that the condition $\tilde{c}_6 \gtrsim -6$ is required for avoiding the vector ghost.

For the theories with $\tilde{f}_5 = 0 = \tilde{f}_6$, Eq. (5.13) reduces to $c_V^2 = 1 + (q_T - 2B_4)^2/(2\phi^2 q_T)$, which is larger than 1 under the no-ghost condition $q_T > 0$ of tensor perturbations. In case (a) of Fig. 2 we plot the evolution of $c_V^2$ for the covariant EVG model with $B_4$ given by Eq. (5.14). The vector propagation speed squared starts to evolve from the value close to 1, and then it finally approaches a superluminal value on the de Sitter attractor. For the covariantized EVG model, the quantity $B_4$ is different, so $c_V^2$ exhibits some difference at late times compared to the covariantized EVG model [see case (b) of Fig. 2].

The case (c) of Fig. 2 corresponds to the covariantized EVG model for $\tilde{f}_5 = 0$ and $\tilde{f}_6 = c_6 X^5$ with $c_6 = 1$. For $c_6 > 0$ and $q_6 > p - 1$, the quantity $q_V$ grows toward a constant larger than 1 [see case (a) of Fig. 1]. This leads to the suppression of $c_V^2$ in Eq. (5.10) on the de Sitter solution. In case (c) of Fig. 2 $c_V^2$ is in fact smaller than 1 at low redshifts. As we already mentioned, the theories with $\tilde{f}_5 = 0$ generally lead to the values of $(c_V^2)_{\text{AS}}$ larger than zero.

In the covariantized EVG model, we have

$$\tilde{f}_5 H \phi^3 = 2^{-q_6} \tilde{c}_5 \lambda u^{2q_5-p+3},$$

(5.16)

where $\tilde{c}_5 = m M_{\text{pl}}^{1+2q_6} c_5$. In case (d) of Fig. 2 we show the evolution of $c_V^2$ for $q_5 = 2, \tilde{c}_5 = 1, q_6 = 5, \tilde{c}_6 = 1, p = 5$, and $\lambda = 1$, in which case the quantity $\tilde{f}_5 H \phi^3$ grows in proportion to $\phi^2$. Since $c_V^2 \simeq 0.29$ on the de Sitter solution, the Laplacian instability is absent.
the covariantized EVG model studied in the numerical simulation of Fig. 2 (c_6 = 1), the condition c_F > 0 is satisfied for c_6 \lesssim 1.5.

In summary, provided that the conditions \[56\text{ and } 5.11\text{ are satisfied for } f_6 < 0 \text{ and } f_5 > 0, \text{ the ghosts and Laplacian instabilities of vector perturbations do not generally arise from the BGP interactions } \mathcal{L}^N.

VI. SCALAR PERTURBATIONS

A. Stability conditions

Let us proceed to the discussion of no-ghost and stability conditions of scalar perturbations. The temporal and spatial components of the vector field contain the scalar perturbations \(\delta \phi\) and \(\chi_V\), respectively, as

\[ A^0 = \phi(t) + \delta \phi, \quad A^i = \frac{1}{a^2(t)} \delta \phi \partial_j \chi_V. \]  

We also consider the perturbed line element with scalar metric perturbations \(\alpha\) and \(\chi\) in the flat gauge, as

\[ ds^2 = -(1 + 2\alpha) dt^2 + 2\partial_i x^i dx + (a^2(t)) \delta_{ij} dx^i dx^j. \]  

If we consider two scalar fields \(\sigma_r\) and \(\sigma_m\) with kinetic terms \(Z_r = -g^{\mu\nu} \partial_\mu \sigma_r \partial_\nu \sigma_r/2\) and \(Z_m = -g^{\mu\nu} \partial_\mu \sigma_m \partial_\nu \sigma_m/2\) for the matter sector of scalar perturbations, then the k-essence action

\[ S_M = \int d^4 x \sqrt{-g} [P_r(Z_r) + P_m(Z_m)] \]  

can describe the perfect fluids of radiation and non-relativistic matter (labeled by \(r\) and \(m\), respectively) \[56\text{ and } 6.6\]. At the background level, the fluid densities are \(\rho_i = 2Z_i P_{i,iz} - P_i\), where \(i = r, m\). The density perturbation \(\delta \rho_i\), the pressure perturbation \(\delta P_i\), and the velocity potential \(v_i\) are given, respectively, by \[56\]

\[ \delta \rho_i = (P_{i,iz} + 2Z_i P_{i,iz}z_i) \delta Z_i, \]  

\[ \delta P_i = P_{i,iz} \delta Z_i, \]  

\[ v_i = \frac{\delta \sigma_i}{\delta_i}, \]  

where \(\delta Z_i = \delta \sigma_i - \delta \sigma_i^2 \alpha\).

Expanding the action \[2.1\] up to quadratic order in scalar perturbations, the second-order action reads \[56\]

\[ S^{(2)}_S = \int dt d^3 x a^3 \left\{ \left( w_1 \alpha + w_2 \delta \phi \right) \frac{\partial^2 \chi}{a^2} - w_3 \left( \frac{\partial \alpha}{\alpha} \right)^2 
+ w_4 \alpha^2 - \frac{w_3}{4} \frac{(\partial \delta \phi)^2}{a^2} + w_5 \frac{(\delta \phi)^2}{a^2} - \frac{w_3}{4} \frac{(\partial \psi)^2}{a^2} 
+ \frac{w_7}{2} \frac{(\partial \psi)^2}{a^2} - (3H w_1 - 2w_4) \frac{\delta \phi}{a^2} 
+ \alpha \left[ \frac{w_3}{2} \frac{\partial^2 (\delta \phi)}{a^2} + w_3 \frac{\partial^2 \psi}{a^2} - w_6 \frac{\partial^2 \psi}{a^2} \right] 
- \left( w_8 \psi - w_3 \psi \right) \frac{\partial^2 (\delta \phi)}{2a^2 \phi^2} \right\} + (S_M)_S^{(2)}, \]  

where \(\psi \equiv \chi_V + \phi(t) \chi\), and

\[ w_1 = -A_{3.4} M^2 + 4H(A_{4} - A_{4.4} \phi^2) + 6H^2(2A_{5} - A_{5.5} \phi^2), \]  

\[ w_2 = w_1 + 2H q_T, \]  

\[ w_3 = -2\phi^2 q_V, \]  

\[ w_4 = 3H(w_2 - H q_T) + w_5, \]  

\[ w_5 = \frac{1}{2} \phi^4 \left( A_{2.2} + 3H A_{3.3} \right) + 6H^2 A_{4.4} + 6H^3 A_{5.5}, \]  

\[ w_6 = -\frac{1}{2\phi} \left[ 4H(q_T - 2B_{4}) - w_8 \right], \]  

\[ w_7 = \frac{2(q_T - 2B_{4})}{\phi^2} H + \frac{w_8}{2\phi^3}, \]  

\[ w_8 = 2w_2 + 4H \phi^2 (2B_{4} - HB_{5}X). \]  

The last term in Eq. \[6.7\] corresponds to the second-order matter action \((S_M)_S^{(2)} = \int dt d^3 x \mathcal{L}_M\), with the Lagrangian

\[ L_M = \sum_{i=r,m} a^3 \left[ \frac{1}{2} (P_{i,iz} + \sigma_i^2 P_{i,iz} z_i) \left( \dot{\sigma_i}^2 - 2\sigma_i \alpha \dot{\sigma_i} \right) 
- \frac{1}{2\sigma_i^2} P_{i,iz} \left\{ (\dot{\sigma_i})^2 + 2\sigma_i \partial \chi \dot{\sigma_i} \alpha \right\} 
+ \frac{1}{2} \sigma_i^2 (P_{i,iz} + \sigma_i^2 P_{i,iz} z_i) \alpha^2 \right]. \]  

Varying the action \[6.7\] with respect to \(\alpha, \chi, \delta \phi,\) and \(\partial \psi\), respectively, we obtain the perturbation equations of mo-
tion in Fourier space as

\[
\sum_{i=r,m} \delta \rho_i - 2w_4\alpha + (3Hw_1 - 2w_4) \frac{\delta \phi}{\phi} + \frac{k^2}{a^2} (\dot{\gamma} + w_1 \chi - w_6 \psi) = 0 ,
\]
\[
\sum_{i=r,m} (\rho_i + P_i) v_i + w_1 \alpha + \frac{w_2}{\phi} \delta \phi = 0 ,
\]
\[
(3Hw_1 - 2w_4) \alpha - 2w_5 \frac{\delta \phi}{\phi} + \frac{k^2}{a^2} \left( \frac{1}{2} \dot{\gamma} + w_2 \chi - \frac{w_5}{2\phi} \psi \right) = 0 ,
\]
\[
\dot{\gamma} + \left( H - \frac{\phi}{\rho} \right) \dot{\gamma} + 2\phi (w_6\alpha + \psi) + \frac{w_8}{\phi} \delta \phi = 0 ,
\]

where

\[
\gamma = \frac{w_8}{\phi} \left( \dot{\psi} + \delta \phi + 2\phi \delta \right) .
\]

The matter perturbation equations of motion, which follow from the continuity equations \( \delta T^{\mu\nu}_{\text{matter}} = 0 \) and \( \delta T^{\mu}_{\text{matter}} = 0 \) for perturbations of the energy-momentum tensor \( T^{\mu}_{\nu} = P_{i,Z} \partial^{\mu} Z_i \delta_{\nu} Z_i + \delta^{\nu}_{\nu} P_i \), are given by

\[
\dot{\rho}_i + 3H (1 + c_i^2) \rho_i + \frac{k^2}{a^2} (\rho_i + P_i) (\chi + v_i) = 0 ,
\]
\[
\dot{v}_i - 3H c_i^2 v_i - c_i^2 \frac{\delta \rho_i}{\rho_i + P_i} - \alpha = 0 ,
\]

where \( i = r, m \), and

\[
c_i^2 = \frac{P_{i,Z}}{\rho_i + P_i} = \frac{P_{i,Z} + 2Z_i P_{i,Z}}{P_{i,Z} + 2Z_i P_{i,Z}} .
\]

By using Eqs. (6.17)–(6.19) with Eqs. (6.14)–(6.16), one can express \( \alpha, \chi, \delta \phi \) in terms of \( \psi, \delta \sigma_r, \delta \sigma_m \) and their derivatives. Then, the second-order action (6.18) can be expressed in the form \( S^{(2)} = \int dt d^3x L \), with the Lagrangian

\[
L = a^3 \left( \dot{\psi}^2 K \dot{\chi}^2 + \frac{k^2}{a^2} \dot{\chi}^2 G \chi^2 - \dot{\chi}^2 M \dot{\chi}^2 - \dot{\chi}^2 B \dot{\chi}^2 \right) ,
\]

where \( K, G, M, B \) are \( 3 \times 3 \) matrices and \( \dot{\chi}^2 = (\psi, \delta \sigma_r, \delta \sigma_m) \). In the small-scale limit, the nonvanishing components of the matrices \( K \) and \( G \) are given by

\[
K_{11} = Q_S + \xi_r^2 K_{11} + \xi_m^2 K_{33} ,
K_{22} = \frac{1}{2} (P_{r,Z} + \delta^2_{m} P_{r,Z} ,
K_{33} = \frac{1}{2} (P_{m,Z} + \delta^2_{m} P_{m,Z} ,
K_{12} = K_{21} = \xi_r K_{22} ,
K_{13} = K_{31} = \xi_m K_{33} ,
\]

and

\[
G_{11} = \mathcal{G} + \dot{\mu} + H \mu ,
G_{22} = \frac{1}{2} P_{r,Z} ,
G_{33} = \frac{1}{2} P_{m,Z} ,
G_{12} = G_{21} = \xi_r G_{22} ,
G_{13} = G_{31} = \xi_m G_{33} ,
\]

with

\[
Q_S = \frac{H^2 q_T (3w_7^2 + 4q_T w_4)}{(w_1 - 2w_2)^2 \phi^2} ,
\]
\[
\xi_r = -\frac{w_2 \delta_{r}}{(w_1 - 2w_2) \phi} , \quad \xi_m = -\frac{w_2 \delta_{m}}{(w_1 - 2w_2) \phi} ,
\]
\[
\xi_r = -\frac{(w_8 - w_6 \phi) \delta_{r}}{(w_1 - 2w_2) \phi} , \quad \xi_m = -\frac{(w_8 - w_6 \phi) \delta_{m}}{(w_1 - 2w_2) \phi} ,
\]
\[
G = \frac{w_1 w_8 (4w_2 w_6 \phi - w_1 w_8) - 4w_2 w_6^2 \phi^2}{4w_3 (w_1 - 2w_2)^2 \phi^2} - \frac{w_7}{4} ,
\]
\[
\mu = \frac{2w_2 w_6 \phi - w_1 w_8}{4(w_1 - 2w_2)^2 \phi^2} .
\]

Under the no-ghost conditions \( K_{22} > 0 \) and \( K_{33} > 0 \) of the matter fields, the positivity of \( K \) is ensured for \( Q_S > 0 \).

The scalar propagation speeds \( c_S \) are the solutions to the dispersion relation given by \( \det (c_S^2 K - G) = 0 \), i.e.,

\[
(c_S^2 K_{11} - G_{11}) (c_S^2 K_{22} - G_{22}) (c_S^2 K_{33} - G_{33})
- (c_S^2 K_{12} - G_{12})^2 (c_S^2 K_{33} - G_{33})
- (c_S^2 K_{13} - G_{13})^2 (c_S^2 K_{22} - G_{22}) = 0 .
\]

It is useful to notice the following relation,

\[
w_8 - (w_6 \phi + w_2) = -4H \phi^4 (f_4 + 3H \phi f_5) ,
\]

where we used Eqs. (6.20) and (6.27). Since \( f_4 = 0 = f_5 \) in GP theories, we have that \( w_8 = w_6 \phi + w_2 \). Provided that \( f_4 = 0 = f_5 \), the same relation holds even in BGP theories with nonvanishing functions \( f_4, f_5 \). In such cases, we have \( \xi_r = \xi_2, \xi_m = \xi_2 \), so that \( K_{12}/K_{22} = G_{12}/G_{22} \) and \( K_{13}/K_{33} = G_{13}/G_{33} \). Then, Eq. (6.29) gives the three decoupled solutions

\[
c_r^2 = \frac{G_{22}}{K_{22}} ,
\]
\[
c_r^2 = G_{33} / K_{33} ,
\]
\[
c_r^2 = \frac{1}{Q_S} (G_{11} - \xi_r^2 G_{22} - \xi_m^2 G_{33}) ,
\]

where \( c_r \) corresponds to the scalar propagation speed arising from the longitudinal mode of the vector field.

In BGP theories with nonvanishing functions \( f_4 \) and \( f_5 \), the three propagation speeds are mixed with each
other. To quantify the deviation from GP theories in the scalar sector, we define the following quantities:

\[
\alpha_p = \frac{\xi c^2 - 1}{\xi r_1} = \frac{w_S - (w_0 \phi + w_2)}{w_2}, \quad (6.34)
\]

\[
\beta_{\rho r} = \frac{2\xi^2 G_{22} \alpha_p}{Q_S} = \frac{w_2(w_0 - w_0 \phi - w_2)(\rho_r + P_r)}{[3w^2 + 4qT w_4]qTH^2}, \quad (6.35)
\]

\[
\beta_{\rho m} = \frac{2\xi^2 G_{33} \alpha_p}{Q_S} = \frac{w_2(w_0 - w_0 \phi - w_2)(\rho_m + P_m)}{[3w^2 + 4qT w_4]qTH^2}. \quad (6.36)
\]

In the limit that \(c^2_m \to 0\), one of the solutions to Eq. (6.29) is given by \(c^2_S = 0\), whereas the other two solutions are

\[
c^2_S = \frac{1}{2} \left[ c^2 + c^2 - \beta \pm \sqrt{(c^2 - c^2 + \beta)^2 + 2c^2 \beta \beta \rho r} \right], \quad (6.37)
\]

where \(c^2_p\) is of the same form as Eq. (6.33), i.e.,

\[
c^2_p = \frac{1}{Q_S} \left[ G + \mu + H \mu - \frac{w^2_2(\rho_r + P_r + \rho_m + P_m)}{2(w_1 - 2w_2)} \right], \quad (6.38)
\]

and

\[
\beta_p = \beta_{\rho r} + \beta_{\rho m}. \quad (6.39)
\]

If the deviation from GP theories is small, then the contribution \(2c^2 \alpha_p \beta \rho_r\), to \(c^2_S\) should be subdominant to the term \((c^2 - c^2 + \beta)^2\) in Eq. (6.37). In this case, one of the solutions to Eq. (6.37) reduces to \(c^2_S \simeq c^2_r\), while another solution reads

\[
c^2_S \simeq c^2_p - \beta. \quad (6.40)
\]

Thus, the deviation from GP theories (\(\beta_0 \neq 0\)) in the scalar sector leads to the value of \(c^2_S\) different from \(c^2_p\). The Laplacian instability can be avoided for \(c^2_S > 0\). The sound speeds derived above are the generalizations of the single-fluid case discussed in Ref. 52.

### B. Covariantized EVG model

We compute the quantities \(Q_S\) and \(c^2_S\) for the covariantized EVG model to discuss theoretically viable parameter spaces. Under the no-ghost condition \(q_T > 0\) of tensor perturbations, we require that the quantity \(q_S = 3w^2 + 4qT w_4\) in \(Q_S\) is positive. This amounts to the condition

\[
q_S = -2^{2-p_2} b_2 p_2 (p + p\Omega_{DE}) M^2_{pl} (1 + p^2) u^{2p_2} \times [1 - 6(2p + 2p_2 - 1)\beta_4 + 2(3p + 2p_2)\beta_5] > 0. \quad (6.41)
\]

In the limit that \(|\beta_4| \ll 1\) and \(|\beta_5| \ll 1\), the condition (6.41) is satisfied for \(b_2 < 0\) with positive values of \(p_2, p\), and \(u = \phi/M_{pl}\). Even for \(q_S > 0\), there are cases in which the term \(w_1 - 2w_2\) in the denominator of \(Q_S\) crosses zero.

The quantity \(w_1 - 2w_2\) can be expressed as

\[
w_1 - 2w_2 = -2HM^2_{pl} (1 - \Omega_{DE} w_c) \quad (6.42)
\]

where \(w_c \equiv 1 + p + p_2 - (p + p_2)(2\beta_5 p_2 - 1)/\gamma\). Provided that the dark energy density parameter is in the range \(0 < \Omega_{DE} < 1\), the rhs of Eq. (6.42) remains negative for \(w_c < 1\), i.e.,

\[
(p + 1)(2\beta_5 p_2 - 1)/\gamma > 1. \quad (6.43)
\]

For the theories with \(\beta_5 = 0\) and \(p > -1\), the condition \(\gamma < 0\) is necessary to satisfy Eq. (6.43).

In the covariantized EVG model, the quantity \(\beta_p\) arising from the deviation from GP theories yields

\[
\beta_p = \frac{2\Omega_{DE}[3(1 - \Omega_{DE}) + \Omega_r]}{p + p\Omega_{DE}} A_1 / A_2. \quad (6.44)
\]

where

\[
A_1 = p_2[\beta_4(1 - 2p - 2p_2) + \beta_5(p + p_2)], \quad (6.45)
\]

\[
A_2 = p + 2pp_2[\beta_4(2 - 3\Omega_{DE}) - 6\beta_4(1 - \Omega_{DE})] + p_2[1 + 2p\beta_5(2 - 3\Omega_{DE}) + 6\beta_4(1 - \Omega_{DE})]. \quad (6.46)
\]

The quantity \(\beta_p\) vanishes in the limit that \(\Omega_{DE} \to 0\). Moreover, we also have \(\beta_p \to 0\) in the de Sitter limit \((\Omega_{DE} \to 1\) and \(\Omega_r \to 0\)). Hence, the quantity \(\beta_p\) can deviate from zero only during the transition from the matter era to the de Sitter epoch.

During the radiation, deep matter, and de Sitter epochs, we can employ the approximation \(c^2_S \simeq c^2_p\) in Eq. (6.40), so the corresponding value of \(c^2_S\) in each cosmological epoch reads
In the limit that $|\beta_4| \ll 1$ and $|\beta_5| \ll 1$, Eqs. (6.47)–(6.49) reduce to

$$(c_s)_{n}^2 = \frac{1}{3p^2} \left[ 4p^2 - 2 + \frac{3p(2p^2 + 2p - 1)(2 - 3p - 3p^2)\beta_4 + (p + p^2)\{1 - (4 - 6p - 4p^2)\beta_5}\}}{(p + p^2)\{1 - 6(2p + 2p^2 - 1)\beta_4 + (6p + 4p^2)\beta_5}\} \right],$$

$$(c_s)_{m}^2 = \frac{1}{6p^2} \left[ 6p^2 - 3 + \frac{p^2(6p^2 + 2p - 1)(3 - 5p - 5p^2)\beta_4 + (p + p^2)(5 - 2(9 - 15p - 10p^2)\beta_5)}{(p + p^2)\{1 - 6(2p + 2p^2 - 1)\beta_4 + (6p + 4p^2)\beta_5}\} \right],$$

$$(c_s)_{as}^2 = \frac{2p^2(1 - 6(p + 2p^2 - 1)\beta_4 + (6p + 4p^2)\beta_5) + \{1 - p^2\{1 - 6(2p + 2p^2 - 1)\beta_4 + 2(1 + 3p + 2p^2)\beta_5}\}}{3(p + p^2)(1 - 2p^2)\beta_4}(q_v u^2)_{as},$$

(6.47)

(6.48)

(6.49)

The two stability conditions $(c_s)_{n}^2 > 0$ and $(c_s)_{m}^2 > 0$ are ensured for

$$3p + 4p^2 - 2 > 0 \quad \text{(if } p_2 < 1/2\),}$$

$$5p + 6p^2 - 3 > 0 \quad \text{(if } p_2 > 1/2\).}$$

(6.50)

(6.51)

For positive integers $p$ and $p_2$, these conditions are trivially satisfied. If we demand the absence of Laplacian instabilities on the de Sitter attractor, we require that $(c_s)_{as}^2 > 0$. For $p + p_2 > 0$, this condition translates to

$$2p^2 + (1 - p^2)(q_v u^2)_{as} > 0,$$

(6.52)

where we used the no-ghost condition of vector perturbations. In the limit that $q_v \to \infty$, the de Sitter stability is ensured for $p_2 \leq 1$, whereas, in another limit $q_v \to 0$, the de Sitter solution is stable for any positive value of $p_2$.

In the covariant EVG model, the quantity $\beta_P$ vanishes, so $c_P^2$ is exactly equivalent to $c_P^2$. Since $c_P^2$ contains the functions that depend on $B_4, B_5$ (like $u_6, u_7, u_8$), the value of $c_P^2$ in the covariant EVG model is different from that in the covariantized EVG model. In the Appendix we show the values of $c_P^2$ in the covariant EVG model during the radiation and early matter eras ($\Omega_{DE} \to 0$) as well as during the de Sitter epoch ($\Omega_{DE} \to 1, \Omega_r \to 0$); see Eqs. (A1)–(A3). They are indeed different from Eqs. (6.47)–(6.49), but in the limit that $|\beta_4| \ll 1$ and $|\beta_5| \ll 1$, they reduce to the values (6.50)–(6.52), respectively. This means that the difference of $c_P^2$ between the two models mostly arises from the different choices of the functions $B_4$ and $B_5$ in $c_P^2$. Apart from the transient period from the matter era to the de Sitter epoch, the quantity $\beta_P$ is close to zero in the covariantized EVG model, so the contribution of $\beta_P$ to Eq. (6.30) should be small relative to $c_P^2$.

To confirm the above analytic estimation, we numerically compute $c_P^2$ for $|\beta_4|$ and $|\beta_5|$ smaller than the order of 1. In the top panel of Fig. 3 the evolution of $c_P^2$ for $p = 1, p = 5, \beta_4 = 0.01, \beta_5 = 0.03$, and $\lambda = 1$ is plotted in both covariantized and covariant EVG models with vanishing functions $g_5, G_6, f_5, f_6$. In this case, the stability conditions (6.47)–(6.49) are automatically satisfied. The numerical values of $c_P^2$ exhibit excellent agreement with Eqs. (6.47)–(6.49) in the covariant EVG model and Eqs. (A1)–(A3) in the covariant EVG model.

In the top panel of Fig. 3 we also show the evolution of $c_P^2$ in the covariantized EVG model as a bold dotted line. The value of $c_P^2$ in this model is almost identical to $c_P^2$ apart from the tiny deviation around today. As we see in the bottom panel of Fig. 3 the quantity $\beta_P$ has a peak around $z = 0$ with the asymptotic behavior $\beta_P \to 0$ in the past and the future. Since the condition $c_P^2 \gg \beta_P$ always holds during the cosmic expansion history, $c_P^2$ is practically identical to $c_P^2$ in the covariant EVG model. As we see in the top panel of Fig. 3 the value of $c_P^2$ in the covariantized EVG model differs from that in the covariant EVG model. This is mostly attributed to the difference of $c_P^2$ between the two models.

## VII. MATTER DENSITY PERTURBATIONS AND GRAVITATIONAL POTENTIALS

To confront BGP theories with the observations of large-scale structures and weak lensing, we need to study the evolution of matter density perturbations and gravitational potentials. For this purpose, we define the gauge-invariant density contrast $\delta$ of nonrelativistic matter (satisfying $w = 0$ and $c_{m}^2 = 0$) as

$$\delta \equiv \frac{\delta \rho_m}{\rho_m} + 3Hv .$$

(7.1)

We also introduce the gauge-invariant gravitational potentials

$$\Psi \equiv \alpha + \chi, \quad \Phi \equiv H\chi$$

and the gravitational slip parameter

$$\eta \equiv - \frac{\Phi}{\Psi} .$$

(7.2)

(7.3)

Taking the time derivative of Eq. (6.22) and using Eq. (6.23), the density contrast of nonrelativistic matter obeys

$$\dot{\delta} + 2H\delta + \frac{k^2}{a^2} \Psi = 3\dot{\beta} + 6H\dot{\beta} .$$

(7.4)
The effective gravitational coupling $G_{\text{eff}}$, which is a key quantity that determines the growth rate of matter perturbations according to Eq. (7.3), is known by solving the other perturbation equations of motion.

Another important quantity associated with the deviation of light rays in weak lensing observations is given by

$$\Phi_{\text{eff}} \equiv \Phi - \Psi = -(\eta + 1)\Psi.$$  

In General Relativity, the gravitational slip parameter (7.3) is equivalent to 1 in the absence of the anisotropic stress, so that $\Phi_{\text{eff}} = -2\Psi = 2\eta$. In BGP theories, the quantity $\eta$ generally varies in time at low redshifts, so it affects the evolution of $\Phi_{\text{eff}}$.

### A. Quasistatic approximation for subhorizon perturbations

To test for BGP theories with the observations of large-scale structures and weak lensing, we are primarily interested in the evolution of nonrelativistic matter perturbations for the modes deep inside the Hubble radius. As long as the oscillating mode of a scalar degree of freedom is negligible relative to the matter-induced mode, it is known that the so-called quasistatic approximation is sufficiently accurate for perturbations deep inside the sound horizon ($c_{\text{S}}^2 k^2 / a^2 \gg H^2$) in Horndeski theories and generalized Proca theories. Under this approximation scheme, the dominant contributions to the perturbation equations of motion are those containing the matter perturbation $\delta\rho_m$ and the term $k^2 / a^2$. We assume that $c_{\text{S}}^2$ is not very close to zero, so that the condition $c_{\text{S}}^2 k^2 / a^2 \gg H^2$ holds for perturbations associated with observed large-scale structures.

We employ the quasistatic approximation explained above without taking into account the radiation. Then, Eqs. (6.17) and (6.19) reduce, respectively, to

$$\delta\rho_m + \frac{k^2}{a^2} (\psi + w_1 \chi - w_6 \psi) \simeq 0,$$

$$\chi \simeq -2w_2 \chi + \frac{w_8}{\phi} \psi,$$

so we obtain

$$\delta\rho_m \simeq -\frac{k^2}{a^2} \frac{w_1 - 2w_2 \Phi + w_2}{H} \frac{\Phi - \omega_\rho}{\phi} (1 + \alpha_p) \psi,$$

where $\alpha_p$ is defined by Eq. (6.31). Eliminating the velocity potential $v_m$ from Eqs. (6.18) and (6.22), it follows that

$$\delta\rho_m + 3H\delta\rho_m + \frac{k^2}{a^2} \left( \frac{\rho_m}{H} \Phi - w_1 \alpha - \frac{w_2}{\phi} \delta\phi \right) = 0.$$
where the terms we use the definition of dimensionless quantities as first-order differential equations. Let us introduce the for BGP theories with a nonvanishing value of $\mu = \phi \mu_w + \psi \Phi$, and
\[
\delta \rho_m = \frac{1}{H}(w_1 - 2w_2 + w_1 - 2w_2) - \phi w_3 \]
(7.12)
\[
\mu_2 = \phi^2 w_2 w_6 + (1 + \alpha_P)\phi(w_2^2 + H w_2 w_3 + w_2 w_3)
- \phi w_2 w_3 + \alpha_P \phi w_2 w_3.
(7.13)
\]
Differentiating Eq. (7.3) with respect to $t$ and eliminating the terms $\dot{Y}$ and $\dot{\phi}$ from Eq. (6.20), it follows that
\[
2\phi^2 w_2(1 + \alpha_P)\Psi + \mu_3 \Phi + \mu_4 \psi - \frac{2\alpha_P \phi^2 w_2}{H} \Phi \simeq 0,
(7.14)
\]
where
\[
\mu_3 = \frac{2\phi}{H w_3} \left[ \phi^2 w_2 w_6 + \phi(w_2^2 + H w_2 w_3 + w_2 w_3) - \phi w_2 w_3 
+ \frac{\alpha_P \phi^2 w_2}{H}(H w_2 + \dot{w}_3) \right],
(7.15)
\]
\[
\mu_4 = -\frac{1}{w_3} \left[ \phi^2(2w_3 w_7 + w_5^2) + \phi^2\{H w_3 w_6 + w_3 \dot{w}_6 + 2(1 + \alpha_P)w_6 \} + \phi\{(1 + \alpha_P)(H w_2 w_3 + w_2 \dot{w}_3) + \alpha_P w_2 \dot{w}_3 - \phi w_2 \dot{w}_3 + (1 + \alpha_P)^2 w_3 \}
- 2(1 + \alpha_P)\phi \dot{w}_2 w_3 \right].
(7.16)
\]
In GP theories, we have $\alpha_P = 0$, in which case the two terms containing the time derivatives $\dot{w}$ and $\dot{\phi}$ vanish in Eqs. (7.11) and (7.12). Then, the three equations (7.9), (7.11), and (7.12) are closed, so they can be explicitly solved for $\Psi$, $\Phi$, and $\psi$ in a way similar to that in BP theories [54]. This property does not hold for BGP theories with a nonvanishing value of $\alpha_P$. In this case, we need to deal with Eqs. (7.11) and (7.12) as first-order differential equations. Let us introduce the dimensionless quantities
\[
\epsilon_\Psi = \frac{\psi}{H \Psi}, \quad \epsilon_\Phi = \frac{\dot{\phi}}{H \Phi}.
(7.17)
\]
On using Eqs. (7.9), (7.11), and (7.12), we can express $\Psi$, $\Phi$, and $\psi$ in the following forms:
\[
\Psi \simeq -\frac{F_1}{\phi \mu_5} \frac{a^2}{k^2} \delta \rho_m,
(7.18)
\]
\[
\Phi \simeq \frac{F_2}{\mu_5} \frac{a^2}{k^2} \delta \rho_m,
(7.19)
\]
\[
\psi \simeq \frac{F_3}{\mu_5} \frac{a^2}{k^2} \delta \rho_m,
(7.20)
\]
where
\[
\frac{\delta \rho_m}{\phi \mu_5} \simeq -(1 + \alpha_P)H w_2[(w_1 - 2w_2) w_3 \mu_3 - 2(1 + \alpha_P)w_2 \mu_4] + \phi(w_1 - 2w_2) [(w_1 - 2w_2) w_3 \mu_4 - 2(1 + \alpha_P)w_2 \mu_2]
+ 2\alpha_P(1 + \alpha_P)\phi H w_2^2 w_3 (w_1 - 2w_2) (\epsilon_\phi - \epsilon_\psi),
(7.21)
\]
\[
F_1 = H\mu_2 \mu_3 - \mu_1 \mu_4 + \alpha_P \phi w_2 \{H \mu_3 \mu_5 \psi
- 2\phi \epsilon_\phi (\mu_2 + \alpha_P \phi H w_2 w_3 \psi) \}
(7.22)
\]
\[
F_2 = \phi H [2(1 + \alpha_P)w_2 \mu_2 + \alpha_P \phi H w_2 w_3 \epsilon_\psi
- (w_1 - 2w_2) \mu_3 \}
(7.23)
\]
\[
F_3 = \phi H [(w_1 - 2w_2) w_3 \mu_3 - 2\alpha_P \phi \epsilon_\psi]
- 2(1 + \alpha_P) \mu_2 \mu_1.
(7.24)
\]
For $\alpha_P \neq 0$, Eqs. (7.18)–(7.20) are not closed, so we need to solve the other perturbation equations of motion to find the evolution of $\epsilon_\phi$ and $\epsilon_\psi$ for a given model.

In BGP theories with nonvanishing $f_5$, $f_6$ but with vanishing $f_4$, $f_5$, the quantity $\alpha_P$ vanishes, so Eqs. (7.18)–(7.20) are closed. In such cases, the effect beyond GP theories arises only through the quantity $w_3 = -2\phi^2 \psi$. The Lagrangian densities $L_3^N$ and $L_6^N$, which are associated with the intrinsic vector modes, modify the quantity $\psi$. This modification affects the evolution of $\Psi$, $\Phi$, and $\psi$ in a way similar to that in BP theories [54]. For the modes deep inside the Hubble radius, the rhs of Eq. (7.14) can be neglected relative to its lhs, such that
\[
\dot{\psi} + 2H \dot{\psi} - 4\pi G_{\text{eff}} \rho_m \delta \simeq 0.
(7.25)
\]
Using the approximation $\delta \rho_m \simeq \rho_m \delta$ in Eq. (7.18), the effective gravitational coupling can be estimated as
\[
G_{\text{eff}} = \frac{F_1}{4\pi \phi \mu_5}.
(7.26)
\]
It is possible to rewrite $G_{\text{eff}}$ by using physical quantities like $\delta_S$ and $\delta^2_S$. In BGP theories with $\alpha_P = 0$, the form of $G_{\text{eff}}$ is exactly the same as Eq. (5.29) of Ref. [54]. Analogous to what happens in GP theories [54], there is a tendency that $G_{\text{eff}}$ gets smaller for $q_r$ approaching $0$.

From Eqs. (7.18) and (7.19), the effective gravitational potential $\Phi_{\text{eff}}$ under the quasistatic approximation reads
\[
\Phi_{\text{eff}} = \frac{F_1 + F_2 \phi}{\phi \mu_5} \frac{a^2}{k^2} \rho_m \delta.
(7.27)
\]
In General Relativity, the quantity $(F_1 + F_2 \phi)/(\phi \mu_5)$ is equivalent to $8\pi G$, but the same quantity generally varies in time in BGP theories with $\alpha_P = 0$. Moreover, the different growth of $\delta$ affects the evolution of $\Phi_{\text{eff}}$. In BGP theories with $f_4 \neq 0$ and $f_5 \neq 0$, we have $\alpha_P \neq 0$, so the terms containing $\alpha_P$ in Eqs. (7.21)–(7.24) lead to the modifications to $\Psi$, $\Phi$, $\psi$. In such cases, Eqs. (7.20) and (7.21) contain the time derivatives $\dot{\psi}$ and $\dot{\phi}$. Solving the full perturbation equations to compute $\dot{\psi}$, $\dot{\phi}$ and substituting them into Eqs. (7.18)–(7.20), we can check whether the resulting values of $\Psi$, $\Phi$, and $\psi$ reproduce those derived by the full numerical integration. In Sec. VII B, we will do so in the covariantized EVG model.
the terms containing \( \alpha \) in scalar perturbation equations of motion are also suppressed in the early Universe relative to those associated with the background. Since \( F_1/(\phi\mu_5) \approx F_2/\mu_5 \approx 4\pi G \) in this regime, the gravitational potentials in Eqs. (7.18)–(7.19) behave as

\[
-\Psi \simeq \Phi \simeq 4\pi G(a^2/k^2)\rho_m \delta \quad \text{for } z \gg 1.
\]

Since the matter density contrast evolves as \( \delta \propto a \) during the deep matter era, we have that \(-\Psi \simeq \Phi \simeq \text{constant} \) in this regime.

In the late Universe, the dynamics of \( \Psi, \Phi, \) and \( \psi \) is modified by the growth of the density of vector derivative interactions. In Fig. 4, we observe that the gravitational potentials \(-\Psi \) and \( \Phi \) start to vary at low redshifts with the parameter \( \eta = -\Phi/\Psi \) deviating from 1. By solving the full perturbation equations numerically, we compute the time derivatives \( \psi \) and \( \Phi \) and substitute them into Eqs. (7.18)–(7.19). As we see in Fig. 4, the solutions derived under this approximation scheme exhibit good agreement with the full numerical results. We confirm that this is also the case for the matter perturbation equation (7.25) with the effective gravitational coupling (7.26). If the terms \( \epsilon_\psi \) and \( \epsilon_\Phi \) are ignored in Eqs. (7.18), (7.19), and (7.20), there are some deviations from the full numerical solutions at late times. Hence, the derivative terms \( \psi \) and \( \Phi \) should be included for deriving the solutions to the subhorizon perturbations accurately. This means that the “quasistatic” approximation does not hold in the usual sense for the theories with \( \alpha \neq 0 \).

Let us proceed to the discussion of the effective gravitational coupling \( G_{\text{eff}} \) and the matter density contrast \( \delta \). In BGP theories with \( f_4 = 0 = f_5 \), the BGP modifications to scalar perturbations arise only from \( f_6 \) through the quantity \( w_3 = -2f^2q_\nu \). In the covariantized EVG model with Eq. (5.6), the quantity \( q_\nu \) is given by \( q_\nu = 1 + 4f_6H^2\phi^2 \). On using the expression of \( G_{\text{eff}} \) given in Eq. (5.29) of Ref. [54], it is possible to realize \( G_{\text{eff}} < G \) for \( 0 < q_\nu \ll 1 \) in BGP theories with \( \alpha \neq 0 \).

For the function \( \tilde{q} \), the parameter \( q_\nu \) reduces to Eq. (5.14), so \( q_\nu \) can be a positive constant for \( q_6 = p - 1 \). As shown in Ref. [54], however, the realization of \( G_{\text{eff}} \) smaller than \( G \) requires that \( q_\nu \) is quite close to zero. Moreover, the deviation of \( G_{\text{eff}} \) from \( G \) is not so significant that it is still difficult for it to be compatible with the RSD data (see the left panel of Fig. 2 of Ref. [54]). We can also consider the time-varying functions \( q_\nu \) (say, \( q_6 > p - 1 \) and \( \epsilon_6 < 0 \)), but in such cases, we require further tunings to ensure the stability condition \( q_\nu > 0 \).

In BGP theories with nonvanishing functions \( f_4 \) and \( f_5 \), the additional terms arising from \( \alpha \) to scalar perturbation equations of motion can modify the evolution of \( G_{\text{eff}} \) at low redshifts. To understand the effect of the \( \alpha \) term, we first consider the covariant EVG model and then discuss the covariantized EVG model later. In GP theories, the value of \( G_{\text{eff}} \) on the de Sitter solution is generally given by [54]

\[
(G_{\text{eff}})_{ds} = \frac{H(2H\dot{\phi}^2q_\nu - w_6\dot{\phi} - w_2)}{4\pi((2H\dot{\phi}^2q_\nu - w_6\dot{\phi})(w_2 + 2Hq_\nu) + w_1w_2)}.
\] (7.29)

In the covariant EVG model with \( p_2 = 1, \ p = 5 \) and...
ally approach zero toward the de Sitter solution, but the

The time derivatives $\dot{\psi}$ and $\dot{\Phi}$ generally approach zero toward the de Sitter solution, but the

\[ f_{\sigma} = G_6 (\text{i.e., } q_V = 1) \], for example, Eq. (7.29) reduces to

\[
(G_{\text{eff}})_{\text{as}} = \frac{(1 - 11\beta_4 + 4\beta_5)[2 - 108\beta_4 + 56\beta_5 + (1 - 11\beta_4 + 4\beta_5)(u_{\text{as}})^2]}{2 + 1584\beta_4^2 + 8\beta_5(8 + 3\beta_5/2) - 12\beta_4(11 + 116\beta_5/5) + 6(11\beta_4 - 6\beta_5)(1 - 11\beta_4 + 4\beta_5)(u_{\text{as}})^2}.
\]

Case (b) shown in Fig. 5 corresponds to the covariant EVG model with $\beta_4 = 5.00 \times 10^{-2}$, $\beta_5 = 6.78 \times 10^{-2}$, and $u_{\text{as}} = 1.193$, so that $(G_{\text{eff}})_{\text{as}} = 0.839G$ from Eq. (7.30). Although $G_{\text{eff}} < G$ on the de Sitter attractor, $G_{\text{eff}}$ temporally grows from the value close to $G$ after the end of the matter era, and then it starts to decrease toward the value smaller than $G$. Since $G_{\text{eff}} > G$ during most of the epoch by today, the growth rate of $G_{\text{eff}}$ is larger than that in the $\Lambda$CDM model for $z > 0$. This property can be confirmed by the numerical integration of $f_{\sigma_8}$ plotted in Fig. 5, where $f \equiv \delta/(H\delta)$ and $\sigma_8$ is the amplitude of $\delta$ at the comoving $8\,h^{-1}$ Mpc scale ($h$ is the normalized Hubble constant $H_0 = 100h$ km sec$^{-1}$ Mpc$^{-1}$). The values of $f_{\sigma_8}$ in case (b) are larger than those of the $\Lambda$CDM model in the redshift range $0 \leq z \lesssim 1$.

In BGP theories with $\alpha_F \neq 0$, the estimation (7.20) loses its validity. The time derivatives $\dot{\psi}$ and $\dot{\Phi}$ generally approach zero toward the de Sitter solution, but the

The values of $f_{\sigma}$ in case (b) are larger than those of the $\Lambda$CDM model in the redshift range $0 \leq z \lesssim 1$.

As we see in case (a) of Fig. 6, the values of $f_{\sigma_8}$ at low redshifts are smaller than those of the $\Lambda$CDM model. By
FIG. 6. Evolution of $f_{\sigma_8}$ vs the redshift $z$ (in the regime $0 \leq z \leq 2$) for the four cases (a), (b), (c), and (d) corresponding to the models in Fig. 5. The initial conditions of perturbations are chosen to satisfy Eqs. (7.19) and (7.20) and $\psi = 0$, $\Phi = 0$ with the comoving wave number $k = 230aH_0$ and $\sigma_8(z = 0) = 0.82$. The evolution of $f_{\sigma_8}$ in the $\Lambda$CDM model is plotted as a dashed bold line. We also show the bounds of $f_{\sigma_8}$ with error bars constrained from the RSD measurements.

Using the best-fit value of $\sigma_8(z = 0)$ constrained by the Planck CMB measurement [10], case (a) can be compatible with most of the recent RSD data. This behavior arises from the existence of nonvanishing terms $\alpha_P$ beyond the domain of GP theories. Thus, the BGP theories offer an interesting possibility of realizing weak gravitational interactions consistent with the RSD measurements.

The evolution of $G_{\text{eff}}$ is subject to modifications for different choices of $\beta_4$ and $\beta_5$. In the right panel of Fig. 5, we plot the evolution of $G_{\text{eff}}/G$ in the covariantized EVG model [case (c)] and in the covariant EVG model [case (d)] for $\beta_4 = 1.00 \times 10^{-2}$ and $\beta_5 = 0$ with the other model parameters the same as those used in the left panel. Since $\eta_{45} = 1.172$ in these cases, the estimation (7.30) gives $(G_{\text{eff}})_{\text{dis}} = 1.159G$ on the de Sitter solution in the covariant EVG model. In case (d) of Fig. 5, $G_{\text{eff}}$ starts to evolve from the value close to $G$ and then it continuously grows to the asymptotic value $1.159G$. In case (c), the existence of nonvanishing terms $\alpha_P$ leads to a different value of $G_{\text{eff}} (\approx 1.2G)$ on the de Sitter solution. As we see in Fig. 5, the effective gravitational coupling in case (c) is also larger than $G$ during the cosmic expansion history. Since the growths of $G_{\text{eff}}/G$ in cases (c) and (d) are similar to each other for $z \geq 0$, the values of $f_{\sigma_8}$ are also degenerate. In Fig. 6, cases (c) and (d) do not fit the RSD data very well due to the property $G_{\text{eff}} > G$.

In Fig. 7, we plot the evolution of $\Phi_{\text{eff}}$ defined by (7.6) in the covariantized EVG model for cases (a) and (c) in Fig. 5. The weak lensing gravitational potential in case (a) decreases faster than that in the $\Lambda$CDM model for $z \geq 0$, whereas in case (c), $\Phi_{\text{eff}}$ initially exhibits tiny growth and starts to decrease by today. This difference arises from the different evolution of $\Psi$ as well as $\eta$. We expect that future observations of weak lensing offer the possibility of distinguishing between the covariantized EVG model and the $\Lambda$CDM model.

VIII. CONCLUSIONS

We have studied the cosmology in BGP theories with five propagating degrees of freedom (one scalar, two vectors, and two tensors) on the flat FLRW background. Compared to second-order GP theories with the Lagrangian densities (2.2)–(2.6), there are four additional derivative interactions given by Eqs. (2.11)–(2.14). The latter interactions are detuned to keep the equations of motion up to second order, but they still do not cause the Ostrogradski instability with the Hamiltonian unbounded from below.

At the background level, the equations of motion (3.3)–(3.5) contain four functions $A_{2,3,4,5}$ defined by Eq. (3.4). In GP theories, they are associated with the four functions $G_{2,3,4,5}$ in $L_{2,3,4,5}$. In BGP theories, the additional functions $f_4$ and $f_5$, which are related to the intrinsic scalar mode, also arise from $L_{3}^{N}$ and $L_{5}^{N}$. Introducing
the functions $B_4$ and $B_5$ as Eq. (3.1), there are two relations (3.3) and (3.7) between $A_4, A_5, f_4,$ and $f_5$. Since $f_4 = 0 = f_5$ in GP theories, the functions $B_4$ and $B_5$ are directly related to $A_4$ and $A_5$. In BGP theories, the existence of two free functions $B_4$ and $B_5$ leads to modifications to the evolution of cosmological perturbations. Moreover, the additional two functions $f_5$ and $f_6$ in $L_5^N$ and $L_6^N$, which are associated with intrinsic vector modes, also affect the dynamics of vector and scalar perturbations.

Since our interest is the application of BGP theories to the late-time cosmic acceleration, we have explored the cosmological dynamics for a concrete dark energy scenario called the covariantized EVG model. In GP theories, there is a counterpart dubbed the covariant EVG model. In these two models, the functions $A_{2,3,4,5}$ are the same, but the functions $B_{4,5}$ are different, i.e., Eq. (3.1) for the covariant EVG and Eq. (3.12) for the covariantized EVG. Hence, the background expansion history is the same in both cases with the dark energy equation of state given by Eq. (3.26). Since the background so-called quasistatic approximation, we showed that the existence of BGP Lagrangian densities $L_4^N$ and $L_5^N$ gives rise to time derivatives $\psi$ and $\Phi$, while they do not appear in GP theories. Hence, the perturbation equations for the scalar degree of freedom $\psi$ and gravitational potentials $\Psi$ and $\Phi$ are not closed even under this approximation scheme. In BGP theories, we need to solve the full perturbation equations of motion in order to know the evolution of perturbations accurately. Computing the time derivatives $\dot{\psi}$ and $\dot{\Phi}$ by the full numerical integration and substituting them into Eqs. (7.18) and (7.19), they can reproduce the full numerical solutions to $\Psi$ and $\Phi$; see Fig. 4.

In both covariantized and covariant EVG models, we studied the evolution of the effective gravitational coupling $G_{\text{eff}}$ and the growth rate of matter perturbations. Even when the values of $G_{\text{eff}}$ on the de Sitter attractor are similar to each other between the two models, the behavior of $G_{\text{eff}}$ during the transition from the matter era to the de Sitter epoch is generally different (e.g., the left panel of Fig. 5). In the covariantized EVG model, it is possible to realize the situation in which $G_{\text{eff}}$ decreases to the value like $G_{\text{eff}} \approx 0.8 G$ by today. In this case, the growth rate of matter perturbations is smaller than that in the LCDM model, so the covariantized EVG model can be compatible with the recent RSD data of $f\sigma_8$ even by using the Planck best fit of $\sigma_8(z = 0)$; see Fig. 6. This behavior of weak gravity occurs by the existence of the BGP derivative interactions $L_4^N$ and $L_5^N$.

In the covariant EVG model, the existence of intrinsic vector modes allows the possibility of $G_{\text{eff}} < G$, but this requires that the quantity $q_\nu$ is quite close to zero. Moreover, the values of $G_{\text{eff}}$ in the redshift range $0 \leq z < 1$ are not significantly smaller than $G$ in general, so the realization of weak gravity in the covariant EVG model is limited compared to the covariantized EVG model. Hence, it is possible to distinguish between the two models from the $f\sigma_8$ data of RSD measurements. Depending on the model parameters, the covariantized EVG model can also lead to $G_{\text{eff}}$ larger than $G$ (like the right panel of Fig. 5), so it may be possible to exclude some parameter spaces from the RSD data. The weak lensing gravitational potential $\Phi_{\text{eff}}$ also exhibits the difference from that in the LCDM (see Fig. 7), so this information can be used to place constraints on the covariantized EVG model further.

We have thus shown that BGP theories allow the construction of a concrete dark energy model with the equation of state $w_{\text{DE}}$ smaller than $-1$, while the growth rate of matter perturbations can be compatible with the RSD data by reflecting the property $G_{\text{eff}} < G$. A similar at-
tempt was carried out in GLPV scalar-tensor theories [32], but it was later found that the model proposed for realizing $G_{\text{eff}} < G$ is plagued by the problem of solid-angle-deficit singularities at the center of a spherically symmetric body [33]. In BGP theories, solid-angle-deficit singularities do not generally arise due to the existence of a temporal vector component [30]. It remains to be seen whether future high-precision observations including RSD and weak lensing show some evidence that the covariantized EVG model is favored over the $\Lambda$CDM model.

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**Appendix A: SCALAR PROPAGATION SPEED FOR THE COVARIANT EVG MODEL**

In this Appendix, we compute the scalar propagation speed squared $c_s^2$ for the covariant EVG model in the limits $\Omega_{\text{DE}} \to 0$ (radiation and early matter eras) and $\Omega_{\text{DE}} \to 1, \Omega_r \to 0$ (de Sitter era). Since $\beta_p$ vanishes in this case, $c_s^2$ is exactly equivalent to $c_p^2$. During the radiation, early matter, and de Sitter eras, we obtain the following values of $c_s^2$, respectively:

\[
(c_s)_r^2 = \frac{2 - 3p - 4p_2 - 2\beta_3(3p + 2p_2 - 1)(3p + 4p_2 - 6) + 6\beta_4[6 + 6p^2 + 8(p_2 - 2)p_2 + p(14p_2 - 17)]}{3p^2(6(2p + 2p_2 - 1)\beta_4 - (4p_2 + 6p)\beta_5 - 1)} \\
(c_s)_m^2 = \frac{3 - 5p - 6p_2 - 2\beta_5[9 + 3p(5p - 11) + 4p_2(3p_2 + 7p - 6)] + 6\beta_4[9 + p(10p - 27) + 2p_2(6p_2 + 11p - 12)]}{6p^2(6(2p + 2p_2 - 1)\beta_4 - (4p_2 + 6p)\beta_5 - 1)} \\
(c_s)_{\text{deS}}^2 = \frac{\xi((p + p_2)\xi - \gamma\{(\gamma + (1 + p)(1 - 2p_2)\beta_5)\}(q + u)d\gamma)}{6\gamma^2(2p_2\beta_5 - 1)(\gamma + p(1 - 2p_2)\beta_5)(q + u)d\gamma},
\]

where

\[
\xi = \gamma (1 + p)(1 - 2p_2)\beta_5[1 + 6(5 - 2p_2 - 2p_2)\beta_4 - 2(6 - 3p_2 - 2p_2)\beta_5] + [\gamma + p(1 - 2p_2)\beta_5][\gamma + (p - 1)(1 - 2p_2)\beta_5].
\]

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