POSITIVE SCALAR CURVATURE ON MANIFOLDS WITH ODD ORDER
ABELIAN FUNDAMENTAL GROUPS

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Abstract. We introduce Riemannian metrics of positive scalar curvature on manifolds with Baas-Sullivan singularities, prove a corresponding homology invariance principle and discuss admissible products.

Using this theory we construct positive scalar curvature metrics on closed smooth manifolds of dimensions at least five which have odd order abelian fundamental groups, are non-spin and atoral. This solves the Gromov-Lawson-Rosenberg conjecture for a new class of manifolds with finite fundamental groups.

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1. Summary

In this paper we will show the following existence result for positive scalar curvature metrics.

Theorem 1.1. Let \( M \) be a closed connected smooth manifold of dimension at least 5 with odd order abelian fundamental group. Assume that \( M \) is non-spin and \( p \)-atorial for all odd primes \( p \). Then \( M \) admits a Riemannian metric of positive scalar curvature.

For the notion of \( p \)-atoriality see Definition 1.3. This condition is satisfied if \( \dim M > \text{rk} \pi_1(M) \), compare Remark 1.4 (v).

Theorem 1.1 contributes to the Gromov-Lawson-Rosenberg conjecture concerning the existence of positive scalar curvature metrics on closed manifolds, see [24, Conjecture 1.22]. It solves [6, Problem 5.11] for odd \( p \).

For finite fundamental groups of odd order the Gromov-Lawson-Rosenberg conjecture can be formulated in the following concise way, see [23, Conjecture 1.2].

Conjecture 1.2. Let \( M \) be a closed connected smooth manifold with finite fundamental group of odd order. If the universal cover of \( M \) admits a positive scalar curvature metric, then \( M \) admits a positive scalar curvature metric.

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Connected manifolds with odd order fundamental groups are orientable, and they are spin if and only if their universal covers are spin. Since simply connected closed non-spin manifolds in dimensions at least 5 admit positive scalar curvature metrics by [13] Corollary C], Conjecture 1.2 implies Theorem 1.1 without the assumption on p-atorality.

Conjecture 1.2 holds in dimensions 1 and 2, and it holds in dimension 3 by the geometrization theorem. In dimension 4 it is false [15]; hence this case must be excluded. In dimensions larger than or equal to 5 it holds for p-atoral manifolds whose fundamental groups are elementary abelian p-groups, where p is an odd prime. This result was stated in [6,7]; a gap in the argument was later pointed out in [14].

By [17, Theorem 1.8], which can be generalized to the non-spin case, Conjecture 1.2 holds for manifolds of dimensions larger than or equal to 5 whose fundamental groups have periodic cohomology.

Conjecture 1.2 is false without assuming that \( \pi_1(M) \) is of odd order, see the remarks after [23, Theorem 1.3]. In dimensions larger than 4 Conjecture 1.2 remains open in general.

**Definition 1.3.** Let \( X \) be a topological space and let \( p \) be a prime. A homology class \( h \in H_d(X; \mathbb{Z}) \) is called \( p \)-toral, if there exist \( \alpha \in \mathbb{N} \), \( \alpha \geq 1 \), and classes \( c_1, \ldots, c_d \in H^1(X; \mathbb{Z}/p^\alpha) \) such that
\[
(c_1 \cup \cdots \cup c_d)(h) \neq 0 \in \mathbb{Z}/p^\alpha.
\]
Otherwise \( h \) is called \( p \)-atoral.

A closed oriented manifold \( M \) of dimension \( d \) is called \( p \)-atoral or \( p \)-toral, respectively, if the fundamental class of \( M \), \([M] \in H_d(M; \mathbb{Z})\), has the corresponding property.

**Remark 1.4.** (i) The \( d \)-torus \( T^d = (S^1 \times \cdots \times S^1)^d \), \( d \geq 1 \), is \( p \)-toral for all \( p \), and so are all closed manifolds which are oriented bordant to \( T^d \) over the classifying space \( B(\mathbb{Z}/p)^d \), using the canonical map \( T^d = B\mathbb{Z}^d \to B(\mathbb{Z}/p)^d \).

(ii) The \( p \)-atoral homology classes form a subgroup of \( H_d(X; \mathbb{Z}) \).

(iii) A closed connected oriented manifold \( M^d \) is \( p \)-toral, if and only if \( \phi_*([M]) \in H_d(B\pi_1(M); \mathbb{Z}) \) is \( p \)-toral, where \( \phi : M \to B\pi_1(M) \) is the classifying map of the universal cover of \( M \). This uses the fact that \( \phi^* : H^1(B\pi_1(M); \mathbb{Z}/p^\alpha) \to H^1(M; \mathbb{Z}/p^\alpha) \) is an isomorphism for all \( \alpha \).

(iv) Let \( M^d \) be closed connected oriented with finite abelian fundamental group \( \pi_1(M) \). Let \( \psi : \overline{M} \to M \) be a connected cover corresponding to a Sylow \( p \)-subgroup of \( \pi_1(M) \). Then \( M \) is \( p \)-toral, if and only if \( \overline{M} \) is \( p \)-toral. This follows from the relation
\[
\left( \psi^*(c_1) \cup \cdots \cup \psi^*(c_d) \right)(\overline{M}) = \deg(\psi) \cdot (c_1 \cup \cdots \cup c_d)([M])
\]
and from the fact that \( \psi^* : H^1(M; \mathbb{Z}/p^\alpha) \to H^1(\overline{M}; \mathbb{Z}/p^\alpha) \) is an isomorphism for all \( \alpha \) since \( \pi_1(M) \) is abelian.

(v) Let \( M^d \) be closed connected oriented and let \( p \) be an odd prime with \( d > \text{rk} H^1(B\pi_1(M); \mathbb{F}_p) \).

Then \( M \) is \( p \)-atoral since for \( \alpha \geq 1 \) we have \( \text{rk} H^1(M; \mathbb{Z}/p^\alpha) = \text{rk} H^1(B\pi_1(M); \mathbb{F}_p) \), and each element in \( H^1(M; \mathbb{Z}/p^\alpha) \) has square zero for odd \( p \).

(vi) In contrast, for all \( m \geq 1 \) the orientable real projective space \( \mathbb{R}P^{2m-1} \) is \( 2 \)-toral.

(vii) One may speculate that \( p \)-toral manifolds for odd \( p \) yield counterexamples to Conjecture 1.2.

In spirit of other existence results for positive scalar curvature metrics on high dimensional manifolds the proof of Theorem 1.1 is based on the propagation of positive scalar curvature metrics along surgeries of codimension at least three [13,23]. In the paper at hand this technique is combined with the description of singular homology classes by manifolds with Baas-Sullivan singularities [1]. To this end we introduce and discuss the concept of positive scalar curvature metrics on manifolds with Baas-Sullivan singularities in Sections 3 and 4, where we also provide the construction of admissible products in this context.

The main steps of the proof of Theorem 1.1 are as follows. Let \( \Omega_*^{SO} \) denote the oriented bordism ring and fix a family \( \mathcal{Q} = \{Q_{4i}\}_{i \geq 1} \) of closed oriented manifolds of dimension \( 4i \) whose bordism classes form a set of polynomial generators of \( \Omega_*^{SO} \)/torsion, and each of which is equipped with a metric of positive scalar curvature. Such families exist by the results in [13]. Let \( \Omega_*^{SO}(\mathcal{Q}) \) denote oriented bordism with singularities in \( \mathcal{Q} \), see Section 2.
Given a topological space $X$ we define a subgroup $H^{2,+}_d(X;\mathbb{Z}) \subset H_d(X;\mathbb{Z})$, called the positive homology of $X$ with respect to $\mathcal{L}$, see Definition 5.12. This definition is based on manifolds with Baas-Sullivan singularities carrying positive scalar curvature metrics. In particular positive homology classes need not be representable by smooth manifolds. The proof of Theorem 1.1 is based on the following homology invariance principle.

**Theorem 1.5.** Let $M$ be a closed connected smooth manifold of dimension $d \geq 5$ with odd order fundamental group and which is non-spin. Let $\phi : M \to B\pi_1(M)$ be the classifying map. Then $M$ admits a metric of positive scalar curvature, if and only if $\phi_*([M]) \in H^{2,+}_d(B\pi_1(M);\mathbb{Z})$.

For proving Theorem 1.1 we can assume, using Remark 1.4 (iv) and a transfer argument similar to Proposition 1.5, that $\pi_1(M)$ is a finite abelian $p$-group, where $p$ is an odd prime. We will therefore study the positive homology $H^2_*(B\Gamma;\mathbb{Z})$ for finite abelian $p$-groups $\Gamma$.

The homology of $B\Gamma$ can inductively be computed by an exact K"unneth sequence

$$0 \to H_*(B\Gamma) \otimes H_*(B\mathbb{Z}/p^a) \xrightarrow{\otimes} H_*(B\Gamma \times B\mathbb{Z}/p^a) \to \text{Tor} (H_*(B\Gamma), H_*(B\mathbb{Z}/p^a)) \to 0.$$ 

The cross product can be realized by admissible products of manifolds with Baas-Sullivan singularities, and the same is true for the Tor-term, which is related to a homological Toda bracket. The construction of admissible products and Toda brackets for Baas-Sullivan manifolds with positive scalar curvature is non-trivial and will be developed in Sections 4 and 5 of our paper.

By a variant of the well known “shrinking one factor” argument (see Proposition 4.6) the cross product of two homology classes is positive, if one of the factors is positive. In contrast we can show positivity of Toda brackets only if both of the factors are positive, compare Corollary 5.3. This means that we cannot deal with Toda brackets involving homology classes of degree one (represented by circles), even though these Toda brackets are $p$-atorial.

In order to resolve this issue we consider homology theories $B\mathbb{P}(j)_*(-) \to H_*(-;\mathbb{Z}(p))$ for $j \geq 1$, each of which sits in a sequence of natural transformations of homology theories

$$\Omega^{SO}_*(-) \to B\mathbb{P}_*(-) \to B\mathbb{P}(j)_*(-) \to H_*(-;\mathbb{Z}(p)).$$

In this sequence $B\mathbb{P}$ denotes Brown-Peterson homology at the prime $p$ with coefficient ring $B\mathbb{P}_* = \mathbb{Z}_p[v_1, v_2, \ldots]$, $\deg(v_m) = 2p^m - 2$, and $B\mathbb{P}(j)$ is obtained by dividing out the coefficient ideal $(v_1, \ldots, v_{j-1}, v_j^2, v_{j+1}, \ldots)$, see Section 7.

**Theorem 1.6.** Let $p$ be an odd prime and let $\Gamma$ be a finite abelian $p$-group. Then $p$-atorial classes in $H_*(B\Gamma)$ which lie in the image of $B\mathbb{P}(j)_*(-) \to H_*(-;\mathbb{Z}(p))$ for all $j \geq 1$ are positive.

The proof in Section 7 is based on a detailed investigation of the (ordinary) homology of abelian $p$-groups in Section 6, a formal group law computation for the theories $B\mathbb{P}(j)$, and ideas from [14].

Theorem 1.5 and Theorem 1.6 for $\Gamma = \pi_1(M)$ imply Theorem 1.1 since $[\phi : M \to B\pi_1(M)] \in \Omega^{SO}_*(B\pi_1(M))$, and hence the class $\phi_*([M]) \in H_*(B\Gamma)$ satisfies the assumptions of Theorem 1.6.

We conjecture that Theorem 1.6 also holds for spin manifolds with vanishing $\alpha$-invariants. A proof should be based on real connective $K$-homology instead of ordinary homology, compare [25]. However the homological computations in later parts of our work do not carry over to this case in an obvious way. Hence we will restrict to the non-spin case in the present paper and leave the spin analogue of Theorem 1.1 for later investigation.

The restriction to odd order fundamental groups is inherent to our approach, since in general the construction of admissible products for manifolds with Baas-Sullivan singularities is obstructed 2-locally, see [8] for instance.

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2. Review of manifolds with Baas-Sullivan singularities

We recall some terminology, following mainly [9] Section 3.3, and fix some notation. Smooth \( d \)-dimensional manifolds with corners \( V \) are modeled on subsets \( N(k,U) = U \times [0,1]^k \subset \mathbb{R}^d \), \( U \subset \mathbb{R}^{d-k} \) open, \( 0 \leq k \leq d \), with smooth transition maps covering permutations \((t_1, \ldots, t_k) \mapsto (t_{\sigma(1)}, \ldots, t_{\sigma(k)})\) of the standard coordinates on \([0,1]^k\), cf. [9] Def. 3.14.]. In particular, manifolds with corners are equipped with preferred collar structures. The subset \( U \times \{0\} \subset N(k,U) \) defines the points of codimension \( k \) in \( N(k,U) \).

Let \( V \) be a \( d \)-dimensional manifold with corners. Every point \( x \in V^d \) has a codimension \( 0 \leq c(x) \leq d \), defined with respect to any local chart around \( x \). This leads to a decomposition of \( V \) into smooth (in general non-compact) connected submanifolds of \( V \), called strata, of various codimensions. Each stratum admits a canonical completion (by adding boundary points to local models), which is itself a manifold with corners, see [9] Def. 3.17. The union of strata of codimension at least 1 in \( V \) is denoted \( \partial V \).

We assume, as usual, that each \( x \in V \) lies in the closure of exactly \( c(x) \) codimension-1 strata of \( V \). In this case the completions of strata coincide with their respective closures in \( V \) (note that this is not true for the 1-gon, for example), which are called connected faces of \( V \).

Manifolds with Baas-Sullivan singularities were introduced in [1]. Let us recall some features of the theory which are relevant for our discussion. A decomposed manifold is a manifold with corners \( V \) together with a decomposition

\[
\partial V = \partial_0 V \cup \cdots \cup \partial_n V ,
\]

\( n \in \mathbb{N} \), where each \( \partial_i V \) is a disjoint union of connected codimension-1 faces of \( V \) and each connected codimension-1 face of \( V \) is contained in exactly one \( \partial_i V \), see [1] Def. 2.1. Each \( \partial_i V \) has an induced structure of a decomposed manifold by setting \( \partial_i(\partial_j V) = \partial_i V \cap \partial_j V \) for \( j \neq i \), and \( \partial_i(\partial_j V) = \emptyset \), \( 0 \leq i \leq j \leq n \), compare [1] p. 283.

**Definition 2.1.** We call the decomposed manifold \( \partial_0 V \) the boundary of \( V \). If \( V \) is compact and \( \partial_0 V = \emptyset \), then \( V \) is called closed.

Similar to [1] Def. 2.2] we fix a family of closed smooth manifolds \( \mathcal{P} = (P_0 = \ast, P_1, P_2, \ldots) \), called singularity types. Let \( n \in \mathbb{N} \). By definition a \( \mathcal{P}_n \)-manifold is a family of decomposed manifolds \( A = (A(\omega))_{\omega \subset \{0, \ldots, n\}} \), \( \partial A(\omega) = \partial_0 A(\omega) \cup \cdots \cup \partial_n A(\omega) \), with \( \partial_i A(\omega) = \emptyset \) for \( i \in \omega \), together with isomorphisms \( \partial_i A(\omega) \cong A(\omega,i) \times P_i \) of decomposed manifolds for \( i \in \{0, \ldots, n\} \setminus \omega \). Here we set \( \partial_i(A(\omega,i) \times P_i) := \partial_j A(\omega,i) \times P_i \) for \( 0 \leq j \leq n \) and we write \( A(\omega,i) \) instead of \( A(\omega,i) \cup \{i\} \). We also use the shorthand \( A \) for the decomposed manifold \( A(\emptyset) \).

Furthermore the following compatibility condition is required: For all \( i, j \notin \omega \), \( i \neq j \), the isomorphisms

\[
\partial_i A(\omega) \cap \partial_j A(\omega) = \partial_i(\partial_j A(\omega)) \cong \partial_j(A(\omega,i) \times P_i) = \partial_j A(\omega,i) \times P_i \cong A(\omega) \times P_j \times P_i ,
\]

\[
\partial_i A(\omega) \cap \partial_j A(\omega) = \partial_i(\partial_j A(\omega)) \cong \partial_i(A(\omega,j) \times P_j) = \partial_i A(\omega,j) \times P_j \cong A(\omega) \times P_i \times P_j
\]

coincide after composition one of them with the interchange map \( P_j \times P_i \rightarrow P_i \times P_j \).

For a \( \mathcal{P}_n \)-manifold \( A \) we define the singular part of \( A \) as

\[
\text{Sing}(A) := \bigcup_{1 \leq i \leq n} \partial_i A ,
\]

such that, obviously,

\[
\partial A = \partial_0 A \cup \text{Sing}(A) ,
\]

\[
\text{Sing}(\partial_0 A) = \partial_0 A \cap \text{Sing}(A) .
\]

There is a bordism theory \( M(\mathcal{P}_n),(-) \) based on manifolds with Baas-Sullivan singularities, see [1] p.284 ff]. Given a pair of topological spaces \((X,Y \subset X)\) elements in \( M(\mathcal{P}_n),(-)\) are represented by continuous maps \( f : A^d \rightarrow X \), where (cf. [1] Definitions 2.2. and 2.3.]

(i) \( A \) is a compact \( d \)-dimensional \( \mathcal{P}_n \)-manifold;

\footnote{Some authors use different conventions, compare, for instance. [16] Def. 2.2.]
(ii) on local models $U \times [0, 1)^k$ the map $f$ factors through the projections $U \times [0, 1)^k \xrightarrow{pr_i} U$;
(iii) for all $i \in \{1, \ldots, n\}$ the restriction $f|_{\partial_i A}$ factors as $\partial_i A \cong A(i) \times P_i \xrightarrow{pr_i} A(i) \to X$;
(iv) $f(\partial_0 A) \subset Y$.

Definition 2.2. A continuous map $f : A^d \to X$ with properties (ii) and (iii) is called compatible with the singularity structure of $A^d$.

Definition 2.3. The homology theory obtained in the limit $n \to \infty$ is called bordism with singularities in $\mathcal{P}$ and denoted $\Omega^{\mathcal{P}}_*(-)$.

There is a straightforward generalization to bordism with tangential structures. In this paper we will be working with oriented bordism with singularities $\Omega^{\mathcal{P}}_{SO,*}(-)$, where we assume that
\begin{itemize}
  \item all singularity types $P_i$ are even dimensional;
  \item all $P_i$ and $A(\omega)$ are oriented;
  \item the isomorphisms $\partial_i A(\omega) \cong A(\omega, i) \times P_i$ are orientation preserving, using the orientations of $\partial_i A(\omega) \subset A(\omega)$ determined by the outward normals.
\end{itemize}

In a similar way one may define spin bordism with singularities $\Omega^{\mathcal{P},\text{Spin}}_*(-)$, but this theory will not be considered in this paper.

Construction 2.4. We shall define a natural transformation of homology theories

$$u : \Omega^{\mathcal{P},\text{SO}}_*(-) \to H_*(-; \mathbb{Z}).$$

Let $(X, Y)$ be a pair of topological spaces, and let $f : A^d \to X$ represent an element in $\Omega^{\mathcal{P},\text{SO}}_{d}(X, Y)$ with a $\mathcal{P}_n$-manifold $A$.

Let $A'$ be the identification space with respect to the projections $\partial_i A = A(i) \times P_i \to A(i)$ for $i = 1, \ldots, n$. A straightforward computation shows $H_d(A', (\partial_0 A)'; \mathbb{Z}) \cong \mathbb{Z}$ (recall $\dim P_i = 2$ for $i \geq 1$), with a preferred generator $[A', (\partial_0 A)']$ corresponding to the given orientation of $A$. By assumption $f$ factors through a map $f' : (A', (\partial_0 A)') \to (X, Y)$, and we define

$$u([f : A \to X]) := f'([A', (\partial_0 A)']) \in H_d(X, Y; \mathbb{Z}).$$

By [22] the oriented bordism ring $\Omega^{\text{SO}}_{-}$ modulo torsion is a polynomial ring. There are closed oriented manifolds $Q_1, Q_2, \ldots, \dim Q_i = 4i$, such that

$$\Omega^{\text{SO}}_{-}/\text{torsion} \cong \mathbb{Z}[Q_1, [Q_2], \ldots],$$

where $[Q_i] \in \Omega^{\text{SO}}_{-}$ denotes the bordism class represented by $Q_i$. Since $\Omega^{\text{SO}}_{-}$ contains no odd torsion [18] the sequence $([Q_i])_{i \geq 1}$ is a regular sequence in $\Omega^{\text{SO}}_{-}\otimes \mathbb{Z}[1/2]$. Setting $\mathcal{Q} := (Q_0 = \ast, Q_1, Q_2, \ldots)$ we arrive at the following fundamental fact, proven in [1].

Proposition 2.5. The natural transformation $u$ defined in [1] induces an isomorphism

$$u : \Omega^{\mathcal{Q},\text{SO}}_{-*}(X, Y) \otimes \mathbb{Z}[1/2] \to H_*(X, Y; \mathbb{Z}[1/2])$$

for all pairs $(X, Y)$ of topological spaces.

Corollary 2.6. Let $\Gamma$ be a finite group of odd order. Then the map $u$ induces a surjective map

$$u : \Omega^{\mathcal{Q},\text{SO}}_{-*}(B\Gamma) \to H_*(B\Gamma; \mathbb{Z}).$$

Proof. This holds in degree $0$, when source and target of $u$ are equal to $\mathbb{Z}$. Let $d \geq 1$. Since $\Gamma$ is of odd order the homology group $H_d(B\Gamma; \mathbb{Z})$ is abelian of odd order. Hence for any $m_0 \geq 1$ and any $x \in H_d(B\Gamma; \mathbb{Z})$ there exists $m \geq m_0$ with $2^n \cdot x = x$. The claim is hence implied by Proposition 2.5 by clearing denominators. \qed

In other words: Homological cycles in $H_*(B\Gamma; \mathbb{Z})$ are represented by $\mathcal{Q}_n$-manifolds. Next we will introduce and study the notion of positive scalar curvature on these objects.
3. Positive scalar curvature on manifolds with Baas-Sullivan singularities

**Definition 3.1.** An **admissible** Riemannian metric on a manifold with corners $V$ is a smooth Riemannian metric $g$ on $V$ which on each local model $U \times [0,1]^k$ restricts to a product metric $g^U \oplus \eta$. Here and in the following $\eta$ denotes the standard Euclidean metric and $g^U$ is some Riemannian metric on $U \subset \mathbb{R}^{d-k}$.

**Definition 3.2.** A family of Riemannian singularity types is a family of singularity types $\mathcal{P} = (P_0, P_1, P_2, \ldots)$ together with Riemannian metrics $h_i$ on $P_i$ for $i \geq 1$.

We call a family of Riemannian singularity types **positive** if each metric $h_i$, $i \geq 1$, is of positive scalar curvature.

**Definition 3.3.** Let $\mathcal{P}$ be a family of Riemannian singularity types and let $A$ be a $\mathcal{P}_n$-manifold, possibly with boundary. An admissible metric $g$ on $A = A(\emptyset)$ is called $\mathcal{P}$-compatible, if for all $\omega \subset \{1, \ldots, n\}$ (hence for all $\omega \subset \{0, \ldots, n\}$) there is an admissible metric $g(\omega)$ on $A(\omega)$ such that $g = g(\emptyset)$ and the metric $g(\omega)$ restricts to the product metric $g(\omega, i) \oplus h_i$ on $\partial_i A(\omega) \cong A(\omega, i) \times P_i$ for all $i \in \{1, \ldots, n\} \setminus \omega$.

**Lemma 3.4.** Each $\mathcal{P}_n$-manifold admits a $\mathcal{P}$-compatible metric.

**Proof.** Use downward induction on the cardinality of $\omega \subset \{1, \ldots, n\}$, starting with $|\omega| = n$. \hfill \Box

**Construction 3.5** (Scaling $\mathcal{P}$-compatible metrics). Let $\mathcal{P}$ be a family of Riemannian singularity types and let $A$ be a $\mathcal{P}_n$-manifold together with a $\mathcal{P}$-compatible metric $g$. For $\lambda > 0$ the scaled metric $\lambda \cdot g$ is not $\mathcal{P}$-compatible unless $\lambda = 1$. The following construction will resolve this issue.

Let $\lambda > 0$ and $\delta \geq 3$. For $\omega \subset \{1, \ldots, n\}$, $\omega = (j_1, \ldots, j_k)$, $1 \leq j_1 < \cdots < j_k \leq n$, we obtain a $k$-parameter family $(g_t)_{t=(t_{j_1}, \ldots, t_{j_k})} \in [0, \delta]^k$ of Riemannian metrics on $\partial_{j_1} \times \cdots \times \partial_{j_k} A = A(\omega) \times P_{j_1} \times \cdots \times P_{j_k}$ where

$$g_t = \lambda \cdot g(\omega) \oplus \bigoplus_{j \in \omega} \left( \phi(t_j/\delta) \cdot \lambda + (1 - \phi(t_j/\delta)) \right) h_j.$$

We abbreviate $P_\omega := P_{j_1} \times \cdots \times P_{j_k}$. With the Euclidean metric $\eta$ on $[0, \delta]^k$ we obtain a smooth Riemannian metric $g_{\omega, \lambda, \delta}$ on $A(\omega) \times P_\omega \times [0, \delta]^k$ defined by

$$g_{\omega, \lambda, \delta}(a, p, t) := g_t(a, p) \oplus \eta.$$

Choose some monotonely increasing diffeomorphism $\chi : [0, 1] \rightarrow [0, \delta]$ which has derivative $\sqrt{\lambda}$ near 1 and denote the induced diffeomorphisms $[0, 1]^k \rightarrow [0, \delta]^k$ by $\chi$ as well.

For $\omega \subset \{1, \ldots, n\}$ (including $\omega = \emptyset$), $|\omega| = k$, we replace the metric $\lambda \cdot (g(\omega) \oplus \bigoplus_{j \in \omega} h_j \oplus \eta)$ on the local model $A(\omega) \times P_\omega \times [0, 1]^k \subset A$ by the metric $g_{\omega, \lambda, \delta}$ pulled back along the diffeomorphism $d\chi \times d\chi : A(\omega) \times P_\omega \times [0, 1]^k \rightarrow A(\omega) \times P_\omega \times [0, \delta]^k$. By the definition of $g_{\omega, \lambda, \delta}$ this construction is well defined and results in a smooth metric on $A$. Furthermore, by the choice of $\phi$ and since $\delta \geq 3$, there are induced local corner models on $A$ with respect to which this metric is $\mathcal{P}$-compatible.

This new metric on $A$ is denoted $g_{(\lambda, \delta)}$ and is called the $(\lambda, \delta)$-scaling of $g$. It depends on the choice of the diffeomorphism $\chi$, but this ambiguity is not relevant for our arguments.

For $n = 2$ and $\delta = 3$ the situation is illustrated in Figure 4, where the shaded region indicates the collar near $\text{Sing}(A)$ for the scaled metric $g_{(\lambda, \delta)}$.

**Definition 3.6.** Let $\mathcal{P}$ be a family of Riemannian singularity types, let $A$ be a $\mathcal{P}_n$-manifold and let $g$ be a $\mathcal{P}$-compatible metric on $A$. We say that $g$ is singularity-positive, if for all $1 \leq i \leq n$ the product metric $g(i) \oplus h_i$ on $\partial_i A = A(i) \times P_i$ is of positive scalar curvature.

**Proposition 3.7.** Let $\mathcal{P}$ be positive, let $A$ be a compact $\mathcal{P}_n$-manifold, and let $g$ be a $\mathcal{P}$-compatible metric on $A$.

Then there exists $\lambda \geq 1$ and $\delta_0 \geq 3$ such that for all $\delta \geq \delta_0$ the metric $g_{(\lambda, \delta)}$ is singularity positive.
Definition 3.8. Let $\lambda g(\varnothing)$ be a family of Riemannian singularity types, let $P$ be a $P$-compatible metric on $A$. We say that $g$ is positive, if for all $\omega \subset \{1, \ldots, n\}$ (hence for all $\omega \subset \{0, \ldots, n\}$) the metric $g(\omega)$ on $A(\omega)$ is of positive scalar curvature.

Proposition 3.9. Let $\mathcal{P}$ be a positive family of singularity types and $A$ be a compact $P_n$-manifold together with a $P$-compatible positive metric $g$. Let $\Lambda \subset (0, \infty)$ be a compact subset and let $s > 0$.

(i) There exists $\delta_0 \geq 3$ such that for all $\lambda \in \Lambda$ and $\delta \geq \delta_0$ the scaled metric $g_{(\lambda, \delta)}$ is positive.

(ii) There exists $0 < \lambda_0 \leq 1$ such that for all $0 < \lambda \leq \lambda_0$ there exists $\delta_0 \geq 3$ such that for all $\delta \geq \delta_0$ and $\omega \subset \{1, \ldots, n\}$ we have $\text{scal}_{g_{(\lambda, \delta)}(\omega)} > s$.

Proof. For $\omega \subset \{1, \ldots, n\}$ Equation (2) implies:

(i) For all $\lambda \in \Lambda$ and $t \in [0, \delta]^k$ we have $\text{scal}_{g_t} > 0$.

(ii) There exists $0 < \lambda_0 \leq 1$ such that for all $0 < \lambda \leq \lambda_0$ and $t \in [0, \delta]^k$ we have $\text{scal}_{g_t} > s$.

Using the O'Neill formula and the compactness of $\Lambda$ this implies:

(i) There exists $\delta_0 \geq 3$ such that for $\lambda \in \Lambda$ and $\delta \geq \delta_0$ we have $\text{scal}_{g_{(\lambda, \delta)}} > 0$.

(ii) There exists $0 < \lambda_0 \leq 1$ such that for all $0 < \lambda \leq \lambda_0$ there exists $\delta_0 \geq 3$ such that for all $\delta \geq \delta_0$ we have $\text{scal}_{g_{(\lambda, \delta)}} > s$. 

Proof. Since the metrics $h_1, \ldots, h_n$ are of positive scalar curvature and $A$ is compact we find some (large) $\lambda \geq 1$ such that the metric $\lambda \cdot g(i) \oplus h_i$ on $\partial_i A = A(i) \times P_i$ is of positive scalar curvature for all $1 \leq i \leq n$.

By the additivity of scalar curvature in Riemannian products and since $\lambda \geq 1$ the metric $g_t$ in (2) is of positive scalar curvature whenever $t_i \leq \delta/3$ for some $1 \leq i \leq k$.

For $\omega \subset \{1, \ldots, n\}$, $|\omega| = k$, we obtain Riemannian submersions

$$
(A(\omega) \times P_\omega \times [0, \delta]^k, g_{\omega, \lambda, \delta}) \longrightarrow ([0, \delta]^k, \eta),
$$

whose fibres are equipped with the metrics $g_t$.

By the O’Neill formula for the scalar curvature in Riemannian submersions [2, Formula (9.37)] we find $\delta_0 \geq 3$ such that for all $\delta \geq \delta_0$ the metric $g_{\omega, \lambda, \delta}$ is of positive scalar curvature on the subset

$$
\{(a, p, t) \in A(\omega) \times P_\omega \times [0, \delta]^k \mid 0 \leq t_i \leq \delta/3 \text{ for some } 1 \leq i \leq k\}.
$$

This implies the assertion of Proposition 3.7. 

We need the following variation of Definition 3.6.

Figures 1. $\mathcal{P}_n$-manifold $A$ with scaled metric $g_{(\lambda, \delta)}$
The same argument can be applied to all \( A(\omega) \) instead of \( A \) so that we can pass to the maximum of the resulting constants \( \delta_0 \) in (i), respectively to the minimum of the resulting constants \( \lambda_0 \) and the maximum of the resulting constants \( \delta_0 \) in (ii).

**Corollary 3.10.** In the situation of Proposition 3.7 let \( C \subset \partial_0 A \) be a union of components of \( \partial_0 A \) and assume that the restriction of \( g \) to \( C \) is positive (see Definition 3.8).

Then there exists \( \lambda \geq 1 \) and \( \delta_0 \geq 3 \) such that for all \( \delta \geq \delta_0 \) the scaled metric \( g(\lambda, \delta) \) is singularity positive and restricts to a positive metric on \( C \).

**Proof.** Let \( \lambda \) and \( \delta_0 \) be chosen as in Proposition 3.7. The claim follows from part (i) of Proposition 3.9 applied to \( A := C \) and \( \Lambda := \{ \lambda \} \), possibly after passing to some larger \( \delta_0 \).

We can now show the following bordism principle.

**Proposition 3.11.** Let \( \mathcal{P} \) be a family of positive singularity types and let \( V \) be a compact \( \mathcal{P}_n \)-manifold with \( \dim V \geq 6 \). Assume that the boundary \( \partial_0 V \) decomposes as a disjoint union \( \partial_0 V = A \sqcup M \), where \( A \) is a closed \( \mathcal{P}_n \)-manifold equipped with a \( \mathcal{P} \)-compatible positive metric, and \( M \) is a closed smooth manifold. Furthermore assume that the inclusion \( M \hookrightarrow V \) is a 2-equivalence.

Then \( M \) carries a Riemannian metric of positive scalar curvature.

**Proof.** By Corollary 3.10 we find a \( \mathcal{P} \)-compatible singular-positive metric \( g \) on \( V \) which restricts to a positive metric on \( A \subset \partial_0 V \).

For \( 1 \leq \ell \leq k \leq n \) we consider the \( \ell \)th face

\[
\partial(0,1)^k = \{(x_1, \ldots, x_k) \in (0,1]^k \mid x_\ell = 1\} \subset (0,1]^k.
\]

Each \( \partial(0,1)^k \) can be identified with \( (0,1)^{k-1} \) in a canonical way, and \( \partial(0,1)^k \) is equipped with a collar of width 0.1 equal to

\[
\partial((0,1)^k) \times (0,9,1) = \{(x_1, \ldots, x_k) \in (0,1)^k \mid 0.9 < x_\ell \leq 1\} \subset (0,1)^k.
\]

For \( 1 \leq k \leq n \) we fix hypersurfaces \( \mathcal{H}^{k-1} \subset (0,1]^k \) subject to the following conditions:

\( \mathcal{H}^0 = \{1/2\} \subset (0,1] \);

\( \mathcal{H}^{k-1} \) is invariant under permutations \( (0,1)^k \rightarrow (0,1)^k, (t_1, \ldots, t_k) \mapsto (t_{\sigma(1)}, \ldots, t_{\sigma(k)}) \);

if \( \mathcal{H}^{k-2} \) has been constructed, then \( \mathcal{H}^{k-1} \) is of product form in the collar neighborhood of width 0.1 of each codimension 1 face \( \partial((0,1)^k) \subset (0,1)^k, 1 \leq \ell \leq k \), and meets this face in \( \mathcal{H}^{k-2} \);

the metric \( \gamma_{k-1} \) on \( \mathcal{H}^{k-1} \) induced from the Euclidean metric \( \eta \) on \( (0,1)^k \) is of non-negative scalar curvature.

One explicit construction of \( \mathcal{H}^{k-1} \) is by smoothly attaching a collar neighborhood of width 1/5 to the shifted spherical segment \( (1/5, \ldots, 1/5) + \{x \in (0,1]^k \mid \|x\| = 3/10\} \subset (0,1]^k \).

Replacing \( U \times [0,1]^k \) by \( U \times \mathcal{H}^{k-1} \) in all local corner models of \( V \) we obtain a smooth hypersurface \( \partial W \) contained in the collar neighborhood of \( \partial V \), where we recall that \( \partial V \) is the set of points of codimension at least 1 in \( V \). The hypersurface \( \partial W \) is the boundary of a smooth embedded codimension zero submanifold \( W \) of \( V \), which we may think of \( V \) with “smoothened corners”.

We obtain a decomposition \( \partial W = C_0 \sqcup C_1 \) where \( C_0 \) and \( C_1 \) are disjoint smooth submanifolds of \( \partial W \) with \( C_1 = M \). Furthermore \( C_1 \hookrightarrow W \) is a 2-equivalence.

We claim that the smooth manifold \( C_0 \) carries a Riemannian metric of positive scalar curvature, such that Theorem 3.11 follows from the usual bordism principle for positive scalar curvature metrics, see [25, Extension Theorem 3.3].

By assumption the induced metrics on the local models \( V(\omega) \times \prod_{i \in \omega} P_i \times [0,1)^k \) of \( V \), \( \omega \subset \{1, \ldots, n\}, \omega \neq \emptyset, |\omega| = k \), are of product form \( g(\omega) \oplus \bigoplus_{i \in \omega} h_i \oplus \eta \) and of positive scalar curvature, as \( g \) is singularity-positive. Furthermore the metric \( g \) is of positive scalar curvature in the collar neighborhood \( A \times P_0 \times [0,1) \) for \( g \) restricts to a positive metric on \( A \).

Since the metrics \( \gamma_{k-1} \) on \( \mathcal{H}^{k-1} \) have non-negative scalar curvature this implies that the restricted metrics \( g(\omega) \oplus \bigoplus_{i \in \omega} h_i \oplus \gamma_{k-1} \) for \( \omega \subset \{1, \ldots, n\} \) as well as on \( A \times \mathcal{H}^0 = A \times \{1/2\} \).

Altogether we obtain a positive scalar curvature metric on \( C_0 \) as required. \( \square \)
Let $\mathcal{Q} := (Q_0 = \ast, Q_1, Q_2, \ldots)$ be a family of singularity types as in Proposition 2.4. For $i \geq 1$, we can assume that $Q_i$ is equipped with a positive scalar curvature metric $h_i$, compare [13], such that $\mathcal{Q}$ is a family of positive singularity types in the sense of Definition 3.2.

**Definition 3.12.** Let $X$ be a topological space. A homology class $h \in H_d(X; \mathbb{Z})$ is called positive with respect to $\mathcal{Q}$, if there is a bordism class $[f : A^d \to X] \in \Omega_d^{SO,\mathcal{Q}}(X)$ with the following properties:

- $A$ carries a $\mathcal{Q}$-compatible positive metric (see Definition 3.8);
- $u([f : A^d \to X]) = h$, where $u : \Omega_d^{SO,\mathcal{Q}}(X) \to H_d(X; \mathbb{Z})$ was defined in Construction 2.4.

The subgroup of all positive homology classes with respect to $\mathcal{Q}$ is denoted $H_d^{\mathbb{Z}}(X; \mathbb{Z})$.

**Proof of Theorem 1.5.** For the nontrivial implication assume $\phi_*([M]) \in H_d^{\mathbb{Z}}(B\pi_1(M); \mathbb{Z})$. We write $\phi_*([M]) = u([f : A^d \to B\pi_1(M)])$ where $A$ is equipped with a $\mathcal{Q}$-compatible positive metric.

Using an inclusion $\ast \to B\pi_1(M)$ the manifold $M$ represents a class $[M] \in \Omega_d^{SO}(B\pi_1(M))$. Then $\beta := [\phi : M \to B\pi_1(M)] - [M] \in \Omega_d^{SO}(B\pi_1(M))$, the reduced oriented bordism group of $B\pi_1(M)$. Since $\Omega_d^{SO}(B\pi_1(M))$ is a finite abelian group of odd order by assumption on $\pi_1(M)$ and by the Atiyah-Hirzebruch spectral sequence, we find, for each $m_0 > 0$, an $m \geq m_0$ with $2^m \cdot \beta = \beta$.

Each element in the kernel of the map $u : \Omega_d^{SO,\mathcal{Q}}(B\pi_1(M)) \to H_d(B\pi_1(M); \mathbb{Z})$ from Corollary 2.6 is 2-power torsion by Proposition 2.5, and hence, using $d > 0$, there is some $m_0 \in \mathbb{N}$ with

$$2^{m_0} \cdot ([f : A^d \to B\pi_1(M)] - \beta) = 0 \in \Omega_d^{SO,\mathcal{Q}}(B\pi_1(M)).$$

Hence there is an $m \geq m_0$ with

$$2^m \cdot [f : A^d \to B\pi_1(M)] = \beta = [\phi : M \to B\pi_1(M)] - [M] \in \Omega_d^{SO,\mathcal{Q}}(B\pi_1(M)).$$

Since $d \geq 5$ we can represent $[M] \in \Omega_d^{SO}$ by a closed oriented smooth $d$-manifold $N$ with a positive scalar curvature metric by [13] Corollary C. By [14] there exists a compact oriented $\mathcal{Q}$-bordism $V$ between $M \sqcup \overline{N} \to B\pi_1(M)$ and $\bigsqcup_{2^m} (f : A \to B\pi_1(M))$ over $B\pi_1(M)$. Here $\overline{N}$ denotes $N$ with the reversed orientation.

We can assume that the inclusion $M \hookrightarrow V$ is a 2-equivalence by applying surgeries to the interior of $V$ and using the fact that the universal cover of $M$ is non-spin (since $M$ is non-spin and $\pi_1(M)$ is of odd order). Hence the claim of Theorem 1.5 is implied by Proposition 3.11.

**Remark 3.13.** The language developed in this section allows an alternative approach to the results in [12].

4. Admissible products

The cartesian product of two manifolds $A$ and $B$ with corners carries an induced structure of a manifold with corners. However, the construction of the product of $\mathcal{P}_n$-manifolds as $\mathcal{P}_n$-manifold is more involved.

In order to illustrate the issue let $A$ and $B$ be smooth manifolds with boundaries diffeomorphic to the closed manifold $P_1$. This induces the structure of $\mathcal{P}_1$-manifolds on $A$ and $B$ where $A(1) = B(1) = \{\ast\}$. We obtain $\partial(A \times B) = (P_1 \times B) \cup (A \times P_1)$, but this does not induce the structure of a $\mathcal{P}_1$-manifold on $A \times B$ (even after straightening the $\pi/2$-angle at $\partial A \times \partial B$), since the $P_1$-factors on the two pieces of $\partial(A \times B)$ correspond to different $P_1$-factors in the intersection $(P_1 \times B) \cap (A \times P_1) = P_1 \times P_1$. Therefore an additional construction is required, which, roughly speaking, interchanges these two factors at the glueing region.

An obstruction theory for this problem was developed in [3, 19, 21, 27], showing in particular that the problem can be solved after inverting 2. Here we will present a somewhat different, explicit geometric construction, which is well adapted to our purpose. In the following we will work in an oriented setting and in particular assume that all singularity types $P_i$, $i \geq 1$, are even dimensional.
Let $A$ and $B$ be $\mathcal{P}_n$-manifolds with decompositions

$$\partial A = \partial_0 A \cup \cdots \cup \partial_n A, \quad \partial B = \partial_0 B \cup \cdots \cup \partial_n B.$$ 

In the remainder of the construction we fix a two dimensional compact hexagonal manifold $\bar{X}$ with corners, see the dark grey region in Figure 2.

For $\omega \subset \{1, \ldots, n\}$ we will construct a manifold with corners $A \times_\omega B$, which, intuitively, is the cartesian product $A \times B$ with all codimension 2-singularities $\partial_i A \times \partial_i B = (A(i) \times B(i)) \times P_i \times P_i$, $i \in \omega$, resolved. The construction is by induction on the cardinality of $\omega$.

For $\omega = \emptyset$ we set $A \times_\omega B := A \times B$, the cartesian product of $A$ and $B$ with its induced structure of a manifold with corners. In addition we smoothen the $\pi/2$-angle appearing at $\partial_0 A \times \partial_0 B$.

Assume that $1 \leq \ell \leq n$ and $A \times_\omega B$ has been constructed whenever $|\omega| = \ell - 1$. Let $\omega \subset \{1, \ldots, n\}$ with $|\omega| = \ell$.

Choose some $i \in \omega$ and consider the collar neighborhood

$$\left(\partial_i A \times [0, 1]\right) \times_{\omega \setminus \{i\}} \left(\partial_i B \times [0, 1]\right) = \left(\partial_i A \times_{\omega \setminus \{i\}} \partial_i B\right) \times [0, 1)^2 \subset A \times_{\omega \setminus \{i\}} B$$

of the codimension-2 face $\partial_i A \times_{\omega \setminus \{i\}} \partial_i B \subset A \times_{\omega \setminus \{i\}} B$. The manifold $A \times_\omega B$ is obtained by removing this collar neighborhood from two disjoint copies of $A \times_{\omega \setminus \{i\}} B$ and gluing in the handle $(\partial_i A \times_{\omega \setminus \{i\}} \partial_i B) \times \bar{X}$ as indicated in Figure 2 where $\bar{X}$ is drawn in dark grey color. The factor $P_i \times P_i$ appearing in

$$\partial_i A \times_{\omega \setminus \{i\}} \partial_i B = (A(i) \times_{\omega \setminus \{i\}} B(i)) \times \partial_i A \times_{\omega \setminus \{i\}} \partial_i B$$

is glued to the left hand copy of $(A \times_{\omega \setminus \{i\}} B) \setminus \left(\partial_i A \times_{\omega \setminus \{i\}} \partial_i B \times [0, 1)^2\right)$ by the identity map, and to the right hand copy by the interchange map $(p_1, p_2) \mapsto (p_2, p_1)$.

The interchange map $P_i \times P_i \to P_i \times P_i$ is orientation preserving, since $P_i$ is even dimensional, and hence the manifold $A \times_\omega B$ carries an induced orientation.

**Remark 4.1.** The manifold $A \times_\omega B$ does not depend on the choice of $i \in \omega$, up to canonical diffeomorphism.

For $i \in \omega$ we set

$$\partial_i (A \times_\omega B) := 2 \cdot \left(\left(\partial_i A \times_{\omega \setminus \{i\}} B\right) \cup_{\partial_i A \times_{\omega \setminus \{i\}} \partial_i B} (A \times_{\omega \setminus \{i\}} \partial_i B)\right),$$

where the two copies on the right hand side correspond to the upper and lower blue boundary piece in Figure 2. Notice that

$$(\partial_i A \times_{\omega \setminus \{i\}} B) \cap (A \times_{\omega \setminus \{i\}} \partial_i B) = \partial_i A \times_{\omega \setminus \{i\}} \partial_i B = (A(i) \times_{\omega \setminus \{i\}} B(i)) \times P_i \times P_i$$

and that the identification along this subspace interchanges the two factors in $P_i \times P_i$, hence realizing our initial goal.

In particular we get an induced isomorphism

$$\partial_i (A \times_\omega B) \cong (A \times_\omega B)(i) \times P_i$$

where

$$\begin{equation}
(A \times_\omega B)(i) := 2 \cdot \left(\left(A \times_{\omega \setminus \{i\}} B(i)\right) \cup_{(A(i) \times_{\omega \setminus \{i\}} B(i)) \times P_i} (A(i) \times_{\omega \setminus \{i\}} B(i))\right).
\end{equation}$$

This concludes the induction step.

**Definition 4.2.** The manifold $A \tilde{\times} B := A \times_{\{1, \ldots, n\}} B$ is called the *admissible product* of $A$ and $B$.

**Proposition 4.3.** The admissible product $A \tilde{\times} B$ carries an induced structure of a $\mathcal{P}_n$-manifold.

**Proof.** By construction $A \tilde{\times} B$ carries the structure of a manifold with corners (with respect to appropriate local models) and is a decomposed manifold with decomposition $\partial (A \tilde{\times} B) = \partial_0 (A \tilde{\times} B) \cup \cdots \cup \partial_n (A \tilde{\times} B)$, where we set

$$\partial_0 (A \tilde{\times} B) := \left(\partial_0 A \tilde{\times} B\right) \cup_{\partial_0 A \tilde{\times} \partial_0 B} (A \tilde{\times} \partial_0 B).$$

(Recall the smoothening of the $\pi/2$-angle at $\partial_0 A \times \partial_0 B$ at the initial stage of the inductive construction).
It remains to define the decomposed manifolds $(A \times B)(\omega)$ for $\omega \subset \{1, \ldots, n\}$ such that the compatibility condition for decomposed manifolds is fulfilled (see page 4). At first we study the case when $\omega$ has two elements.

Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. By (5) we have

$$\partial_j \partial_i (A \times B) = 2 \cdot \left( \partial_j (A \times B(i)) \cup (\partial_j (A(i) \times B(i))) \times P_j \right) \times P_i,$$

and likewise

$$\partial_i \partial_j (A \times B) = 2 \cdot \left( \partial_i (A \times B(j)) \cup (\partial_i (A(j) \times B(j))) \times P_i \right) \times P_j.$$

Furthermore

$$\left( A \times B \right)(i) = 2 \cdot \left( A \times B(i) \right) \cup \left( A(i) \times B(i) \right) \times P_i,$$

and likewise

$$\left( A \times B \right)(j) = 2 \cdot \left( A \times B(j) \right) \cup \left( A(j) \times B(j) \right) \times P_j.$$

Similar computations apply to $(A \times B)(i,j)$ and $(A \times B)(j,i)$. Setting

$$\left( A \times B \right)(i,j) := 4 \cdot \left( A(i) \times B(i,j) \right) \cup \left( A(i) \times B(i,j) \right) \cup \left( A(j) \times B(i,j) \right) \cup \left( A(j) \times B(i,j) \right) \times P_i \times P_j,$$

where we glue

- $A \times B \times i, j$ and $A(i) \times B(i, j)$ along $A(i) \times B(i, j) \times P_i$,
- $A \times B \times i, j$ and $A(j) \times B(i, j)$ along $A(j) \times B(i, j) \times P_j$,
- $A(i, j) \times B$ and $A(j) \times B(i, j)$ along $A(i, j) \times B(i, j) \times P_i$.  

Figure 2. Construction of admissible products
Assume that working with collar factors \([0, \lambda, \mu > 0] \) the local models \(U \) be canonically extended to local models \(k \). Let \(\tilde{\omega} \) be a compact \(\widetilde{\mathcal{P}_n} \)-manifold and let \(\omega \) be a compact \(\widetilde{\mathcal{P}_n} \)-manifold. This results in a bilinear map of bordism theories

\[
\tilde{\omega} : M(\tilde{\mathcal{P}_n})(X) \times M(\tilde{\mathcal{P}_n})(S) \to M(\tilde{\mathcal{P}_n})(X \times S),
\]

and this construction extends to relative bordism groups. With the natural transformation \(u : M(\tilde{\mathcal{P}_n})_*(\cdot) \to \Omega^\infty SO_\mathcal{P}(\cdot) \to \mathcal{H}_*(\cdot) \) from Construction \(2.4 \) and our construction of admissible products we hence obtain:

**Proposition 4.4.** Let \(\times\) denote the cross product in singular homology. Then for all pairs of topological spaces \((X, Y)\) and \((S, T)\) we have a commutative diagram

\[
\begin{array}{ccc}
M(\tilde{\mathcal{P}_n})_d(X, Y) \times M(\tilde{\mathcal{P}_n})_e(S, T) & \xrightarrow{\tilde{x}} & M(\tilde{\mathcal{P}_n})_{d+e}(X \times S, X \times T \cup S \times Y) \\
\downarrow{u \times u} & & \downarrow{u} \\
H_d(X, Y) \times H_e(S, T) & \xrightarrow{(a, b) \rightarrow 2^n(a \times b)} & H_{d+e}(X \times S, X \times T \cup S \times Y).
\end{array}
\]

Let \(A\) and \(B\) be \(\mathcal{P}_n\)-manifolds and let \(g\) and \(h\) be \(\mathcal{P}\)-compatible metrics on \(A\) and \(B\). Let \(\lambda, \mu > 0\) and \(\delta, \epsilon \geq 9\). With this choice of \(\delta, \epsilon\) the local models \(U \times [0, 1]^k\) on \(A\) and \(B\) can be canonically extended to local models \(U \times [0, 3]^k\) on which the scaled metrics \(g(\lambda, \delta)\) and \(h(\mu, \epsilon)\) restrict to product metrics \(g^U \oplus \eta\), respectively \(h^U \oplus \eta\), with the Euclidean metric \(\eta\) on \([0, 3]^k\).

We equip the hexagonal manifold \(X\) with some admissible Riemannian metric \(\sigma\) with respect to which each side has length 3. With these data we construct a metric \(g(\lambda, \delta)\tilde{\oplus}h(\mu, \epsilon)\) on \(A \times B\) along the inductive construction of \(A \times B\) in the proof of Proposition \(4.3\), starting with the product metric \(g \oplus h\) on \(A \times B\) and working with collar factors \([0, 3]^2\) and \([0, 3]\) instead of \([0, 1]^2\) and \([0, 1]\) in Figure \(2\). Here we recall that according to Definition \(5.3\) each singularity type \(P_i\), \(i \geq 1\), is equipped with the fixed metric \(h_i\), such that the interchange map on \(P_i\) is an isometry with respect to \(h_i \oplus h_i\).

By the choice of \(\delta, \epsilon\) and the metric \(\sigma\) on \(X\) we hence obtain a \(\mathcal{P}\)-compatible metric \(g(\lambda, \delta)\tilde{\oplus}h(\mu, \epsilon)\) on \(A \times B\).

**Definition 4.5.** We call \(g(\lambda, \delta)\tilde{\oplus}h(\mu, \epsilon)\) the admissible product metric of \(g(\lambda, \delta)\) and \(h(\mu, \epsilon)\).

We obtain the following version of the well known “shrinking one factor” principle.

**Proposition 4.6.** Assume that \(A\) and \(B\) are compact and \(g\) is positive (see Definition \(7.8\)). Then for any \(\mu \geq 1\) and \(\epsilon \geq 9\) there exists \(0 < \lambda \leq 1\) and \(\delta \geq 9\) such that for all \(\delta' \geq \delta\) the following holds:

(i) The metric \(g(\lambda, \delta')\tilde{\oplus}h(\mu, \epsilon)\) on \(A \times B\) is positive (see Definition \(7.8\)).

(ii) The metric \(g(\lambda, \delta')\) is positive.

(iii) Let \(C\) be a compact \(\mathcal{P}_n\)-manifold, let \(k\) be a \(\mathcal{P}\)-compatible metric on \(C\), and let \(\nu > 0\) and \(\theta \geq 9\) such that the scaled metric \(k(\mu, \theta)\) is positive. Then \(g(\lambda, \delta')\tilde{\oplus}k(\mu, \theta)\) is positive.
Proposition 5.1. \textit{Let } $X, S$ \textit{be topological spaces and consider the Künneth sequence of singular homology groups}

$$0 \to H_*(X) \otimes H_*(S) \xrightarrow{\cdot \cdot} H_*(X \times S) \xrightarrow{\cdot \cdot} \text{Tor}(H_*(X), H_*(S))_{*-1} \to 0.$$ 

\textit{In this section we study positivity of homology classes (see Definition 5.12) related to the homological cross product }$\times$\textit{ and the Tor term in this sequence. Propositions 2.5, 4.4 and 4.6 (i) imply the following result.}

**Proposition 5.1.** \textit{Let } $a \in H_d(X)$, $b \in H_*(S)$, \textit{and assume that at least one of these classes is positive. Then there exists } $m \in \mathbb{N}$ \textit{such that } $2^m \cdot (a \times b) \in H_{d+e}(X \times S)$ \textit{is positive.}

Next we discuss the Tor-term in the Künneth sequence. \textit{Let } $a \in H_d(X)$, $b \in H_*(S)$, \textit{and } $r \geq 2$ \textit{be an integer with } $ra = 0 \neq rb$. \textit{Let } $(C_*(X), \partial)$ \textit{and } $(C_*(S), \bar{\partial})$ \textit{be the integral chain complexes of } $X$ \textit{and } $S$. \textit{We pick chains } $\overline{\partial} \in C_{d+1}(X)$ \textit{and } $\overline{b} \in C_{e+1}(S)$ \textit{whose boundaries represent } $ra$ \textit{and } $rb$ \textit{respectively. The cycle}

$$\tag{7} \frac{1}{r} \cdot \partial(\overline{\partial} \otimes \overline{b}) \in (C_*(X) \otimes C_*(S))_{d+e+1}$$

\textit{represents a Toda bracket coset}

$$\langle a, r, b \rangle \subset H_{d+e+1}(X \times S)$$

\textit{for the submodule } $(a \times H_{e+1}(S)) \oplus (H_{d+1}(X) \times b) \subset H_{d+e+1}(X \times S)$, \textit{which is independent from the choice of } $\overline{\partial}$ \textit{and } $\overline{b}$. \textit{It is well known [10] that Toda brackets generate a submodule of } $H_*(X \times S)$ \textit{which maps surjectively onto } $\text{Tor}(H_*(X), H_*(S))_{*-1}$. \textit{Now let}

$$[\alpha : A \to X] \in \Omega_d^{SO, g}(X) \text{ and } [\beta : B \to S] \in \Omega_e^{SO, \bar{g}}(Y)$$

\textit{map to } $a \in H_d(X)$ \textit{and } $b \in H_*(S)$ \textit{under the natural transformation } $u$ \textit{from Construction 2.3. By Proposition 2.5 there exists } $m \geq 1$ \textit{such that}

$$2^m \cdot r \cdot [\alpha : A \to X] = 0 \text{ and } 2^m \cdot r \cdot [\beta : B \to Y] = 0 .$$

\textit{Hence, up to a factor } $2^m$ \textit{and possibly after passing to some larger } $n$, \textit{we can assume that } $\bigsqcup_{r} (A \to X)$ \textit{and } $\bigsqcup_{r} (B \to S)$ \textit{can be extended to maps } $\overline{\alpha} : V \to X$ \textit{and } $\overline{\beta} : W \to S$ \textit{for some compact oriented } $\mathcal{O}_n$\textit{-manifolds } $V$ \textit{and } $W$ \textit{with boundaries } $\partial V = \bigsqcup_{r} A$ \textit{and } $\partial W = \bigsqcup_{r} B$, \textit{where } $\overline{\alpha}$ \textit{and } $\overline{\beta}$ \textit{are compatible with the singularity structures of } $V$ \textit{and } $W$.

\textit{By (7) and Proposition 4.4 the coset } $2^m \cdot \langle a, r, b \rangle \subset H_*(X \times S)$ \textit{is represented by}

$$\tag{8} (\overline{\alpha} \times \overline{\beta}) \cup (a \times \overline{b}) : (V \times B) \cup_{\partial V \times \partial B = A \times B} (A \times W) \to X \times S .$$

\textit{Let } $A$ \textit{and } $B$ \textit{be equipped with } $\mathcal{O}$\textit{-compatible positive metrics } $g$ \textit{and } $h$. \textit{The metrics } $\bigsqcup_{r} g$ \textit{on } $\bigsqcup_{r} A$ \textit{and } $\bigsqcup_{r} h$ \textit{on } $\bigsqcup_{r} B$, \textit{can be extended to (not necessarily positive) } $\mathcal{O}$\textit{-compatible metrics } $\overline{g}$ \textit{and } $\overline{h}$ \textit{on } $V$ \textit{and } $W$ \textit{(compare the proof of Lemma 4.1).}

\textit{By Proposition 3.9 (i) we find } $\delta_0, \epsilon_0 \geq 9$ \textit{such that for all } $\delta \geq \delta_0$ \textit{and } $\epsilon \geq \epsilon_0$ \textit{the scaled metrics } $g_{(1, \delta)}$ \textit{and } $h_{(1, \epsilon)}$ \textit{are positive.}

\textit{Choose } $(\lambda, \delta)$ \textit{for } $A$ \textit{according to Proposition 4.6 for the scaled metric } $\overline{h}_{(1, \delta_0)}$ \textit{on } $W$, \textit{and in an analogous fashion choose } $(\mu, \epsilon)$ \textit{for } $B$ \textit{for the scaled metric } $\overline{g}_{(1, \epsilon_0)}$ \textit{on } $V$. \textit{With these choices}
the admissible product metrics $\tilde{g}_{(1,\delta_0)} \oplus h_{(\mu, \epsilon)}$ on $V \times B$ and $g_{(\lambda, \delta)} \oplus h_{(1,\delta_0)}$ on $A \times W$ are positive. In order to glue the induced metrics on the common boundary $\bigcup A \times B$ we need the following result.

**Lemma 5.2.** The metrics $g_{(1,\delta_0)} \oplus h_{(\mu, \epsilon)}$ and $g_{(\lambda, \delta)} \oplus h_{(1,\delta_0)}$ on $A \times B$ are isotopic, hence concordant, through positive $\mathcal{L}$-compatible metrics.

*Proof.* Set $\Lambda = [\lambda, 1] \subset \mathbb{R}$ and choose $\delta_0^* \geq \delta_0, \delta$ according to Proposition 5.3 (i) for this $\Lambda$. We find isotopies through positive $\mathcal{L}$-compatible metrics on $A$:

$\triangleright$ from $g_{(1,\delta_0)}$ to $g_{(1,\delta_0^*)}$, by the choice of $\delta_0^*$;

$\triangleright$ from $g_{(1,\delta_0^*)}$ to $g_{(\lambda, \delta)}$, by the choice of $\delta_0^*$;

$\triangleright$ from $g_{(\lambda, \delta)}$ to $g_{(\lambda, \delta)}$, by the choice of $(\lambda, \delta)$.

Hence, by the choice of $(\mu, \epsilon)$, we obtain a smooth isotopy from $g_{(1,\delta_0)} \oplus h_{(\mu, \epsilon)}$ to $g_{(\lambda, \delta)} \oplus h_{(\mu, \epsilon)}$ through positive $\mathcal{L}$-compatible metrics, see Proposition 5.3 (ii).

In an analogous fashion we find a smooth isotopy from $g_{(\lambda, \delta)} \oplus h_{(1,\delta_0)}$ to $g_{(\lambda, \delta)} \oplus h_{(\mu, \epsilon)}$ through positive $\mathcal{L}$-compatible metrics, thus finishing the proof of Lemma 5.2.

We obtain the following counterpart of Proposition 5.1.

**Proposition 5.3.** Let $a \in H_d(X)$ and $b \in H_e(Y)$ be positive homology classes and let $r \geq 2$ with $ra = 0 = rb$. Then for each element $x \in \langle a, r; b \rangle \subset H_{d+e+1}(X \times S)$ there exists $m \geq 1$ such that $2^m \cdot x$ is positive.

*Proof.* The $\mathcal{L}_n$-manifold $(V \times B) \cup_{\partial(V \times B)} = B \times X \cup_{\partial(B \times X)} A \times W$ in $\mathbb{R}$ carries a $\mathcal{L}$-compatible positive metric by Lemma 5.2. Hence the claim holds for the specific element $x$ represented by $(V \times B) \cup (A \times W) \to X \times S$.

The claim also holds for each element $x \in (a \times H_{e+1}(S)) \oplus (H_{e+1}(X) \times b) \subset H_{d}(X \times S)$, since $a$ and $b$ are positive, see Proposition 5.1.

Let $\Gamma_1$ and $\Gamma_2$ be finite groups of odd order and set $X = B\Gamma_1$ and $S = B\Gamma_2$. Let $d \geq 1$. Then $H_d(X \times S)$ is a finite abelian groups of odd order and hence for all $x \in H_d(X \times S)$ and $m_0 \geq 1$ there exists $m \geq m_0$ with $2^m \cdot x = x$. By Propositions 5.1 and 5.3 we conclude:

**Corollary 5.4.** Let $a \in H_d(B\Gamma_1)$ and $b \in H_e(B\Gamma_2)$ where $d, e \geq 1$. Then:

(i) If either $a$ or $b$ is positive, then $a \times b$ is positive.

(ii) Let $r \geq 2$ with $ra = 0 = rb$ and let $a$ and $b$ be positive. Then $(a, r; b) \subset H_{d+e+1}(B\Gamma_1 \times B\Gamma_2)$ only contains positive classes.

This result will be crucial for the computations in the next sections.

6. Homology of abelian $p$-groups

Let $p$ be a prime, where $p = 2$ is allowed unless stated otherwise. Given an integer $\alpha \geq 1$ we denote by $G_{\alpha}$ the cyclic group of order $p^\alpha$ with generator $g_{\alpha}$ and neutral element $1_{\alpha}$. The group operation in $G_{\alpha}$ is written multiplicatively. We denote by $\mathbb{Z}G_{\alpha}$ the integral group ring of $G_{\alpha}$.

Let $\alpha \geq 1$ and let $(C(\alpha)_+, \partial_\cdot)$ denote the $\mathbb{Z}$-graded $\mathbb{Z}$-free chain complex with one generator $c_\alpha$ in each degree $d \geq 1$ and differential

$$\partial(c_\alpha) = \begin{cases} p^\alpha \cdot c_{\alpha-1} & \text{for even } \alpha, \\ 0 & \text{for odd } \alpha. \end{cases}$$

This is the reduced cellular chain complex with integer coefficients of the standard CW-model of the classifying space $BG_{\alpha}$ with one cell in each non-negative dimension. We hence recover the well known computation (see $\mathbb{R}$ (II.3.1)])

$$H_d(C(\alpha)_+, \partial_\cdot) \cong \tilde{H}_d(BG_{\alpha}) = \begin{cases} \mathbb{Z}/p^\alpha & \text{for odd } \alpha, \\ 0 & \text{for even } \alpha. \end{cases}$$
For $\alpha_1 \leq \cdots \leq \alpha_n$ we consider the abelian $p$-group
\[ \Gamma = G_{\alpha_1} \times \cdots \times G_{\alpha_n}, \]
and the smash product of classifying spaces
\[ \widetilde{B}\Gamma = BG_{\alpha_1} \wedge \cdots \wedge BG_{\alpha_n}. \]

We obtain
\[ (9) \quad \tilde{H}_*(\widetilde{B}\Gamma) \cong H_*(C_*(1) \otimes \cdots \otimes C_*(n)), \]
where $C_*^{(i)} = C(\alpha_{i})$, $i = 1, \ldots, n$, refers to the $i$-th cyclic factor in the group $\Gamma$.

We will construct explicit cycles generating $H_*(C_*(1) \otimes \cdots \otimes C_*(n))$.

**Definition 6.1.** For $1 \leq j \leq n$, a family $(i_1, \ldots, i_j)$ with $1 \leq i_1 < \cdots < i_j \leq n$, and a family of positive integers $(m_{i_1}, \ldots, m_{i_j})$ we define a cycle of degree $2m_{i_1} + \cdots + 2m_{i_j} - 1$ by
\[ \text{Tor}(c_{2m_{i_1} - 1}^{(i_1)}, \ldots, c_{2m_{i_j} - 1}^{(i_j)}) := \frac{1}{p^{m_j}} \partial(c_{2m_{i_1}}^{(i_1)} \otimes \cdots \otimes c_{2m_{i_j}}^{(i_j)}) \in C^{(i_1)} \otimes \cdots \otimes C^{(i_j)}. \]

This cycle represents a homology class of order $p^{m_j}$, which, in the terminology of Section 5, represents an iterated Toda bracket.

Let $(s_1, \ldots, s_{n-j})$, $1 \leq s_1 < \cdots < s_{n-j} \leq n$, be the unique family which is complementary to $(i_1, \ldots, i_j)$; this family is empty if $j = n$. Suppressing a signed permutation of tensor factors we obtain cycles
\[ \text{Tor}(c_{2m_{i_1} - 1}^{(i_1)}, \ldots, c_{2m_{i_j} - 1}^{(i_j)}) \otimes c_{2m_{s_1} - 1}^{(s_1)} \otimes \cdots \otimes c_{2m_{s_{n-j}} - 1}^{(s_{n-j})} \in C_*(1) \otimes \cdots \otimes C_*(n), \]
which in the following will be called special.

**Proposition 6.2.** The reduced integral homology $\tilde{H}_*(\widetilde{B}\Gamma)$ is generated by special cycles with $i_1 = 1$.

**Proof.** We apply induction on $n$. In the induction step we set $\mathcal{E}_n := C_*(1) \otimes \cdots \otimes C_*(n)$ and consider the exact K"unneth sequence
\[ 0 \longrightarrow H_*(\mathcal{E}_n) \otimes H_*(C_*^{(n+1)}) \longrightarrow H_*(\mathcal{E}_n \otimes C_*^{(n+1)}) \longrightarrow \text{Tor}(H_*(\mathcal{E}_n), H_*(C_*^{(n+1)}))_{s-1} \longrightarrow 0. \]
The induction assumption and the construction of $\text{Tor}(H_*(\mathcal{E}_n), H_*(C_*^{(n+1)}))$ together with $\alpha_1 \leq \cdots \leq \alpha_n$ show that homology classes represented by special cycles
\[ \text{Tor}(c_1^{(1)}, \ldots, c_1^{(i)}, c^{(n+1)}_1 \otimes \cdots \otimes c^{(s_{n-j})}_1) \]
map to a generating set of $\text{Tor}(H_*(\mathcal{E}_n), H_*(C_*^{(n+1)}))$.

The image of left hand map in the K"unneth sequence satisfies the claim by the induction assumption. \qed

**Example 6.3.** Let $n = 3$ and $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$. Then
\[ 0 \neq \{ \text{Tor}(c_1^{(2)}, c_1^{(3)}) \otimes c_1^{(1)} \} \in \tilde{H}_4(\widetilde{B}\Gamma). \]

Proposition 6.2 can be verified in this case by computing $\text{Tor}(c_1^{(2)}, c_1^{(3)}) \otimes c_1^{(1)}$ as
\[ - \text{Tor}(c_1^{(1)}, c_1^{(3)}) \otimes c_1^{(2)} - p \cdot \text{Tor}(c_1^{(1)}, c_1^{(2)}) \otimes c_1^{(3)} = - \text{Tor}(c_1^{(1)}, c_1^{(3)}) \otimes c_1^{(2)} - \partial(c_1^{(1)} \otimes c_1^{(2)} \otimes c_1^{(3)}). \]

Next we will derive some explicit formulas for maps in group homology induced by group homomorphisms. We consider the homological chain complex in non-negative degrees
\[ (F(\alpha), \partial_{\alpha}) := \cdots \longrightarrow ZG_{\alpha} \overset{\nu_{\alpha}}\longrightarrow ZG_{\alpha} \overset{\tau_{\alpha}}\longrightarrow ZG_{\alpha} \overset{\nu_{\alpha}}\longrightarrow ZG_{\alpha} \overset{\tau_{\alpha}}\longrightarrow ZG_{\alpha} \overset{\nu_{\alpha}}\longrightarrow ZG_{\alpha} \overset{\tau_{\alpha}}\longrightarrow ZG_{\alpha} \overset{\nu_{\alpha}}\longrightarrow \cdots \longrightarrow \Z \longrightarrow 0. \]
where the differentials are given by multiplication with $\tau_{\alpha} := g_{\alpha} - 1_{\alpha}$ and $\nu_{\alpha} := \sum_{i=0}^{\alpha-1}(g_{\alpha})^i$, respectively. With the augmentation map $\varepsilon_{\alpha} : ZG_{\alpha} \rightarrow \Z$ induced by the group homomorphism $G_{\alpha} \rightarrow \{1\}$ we obtain an exact sequence
In other words \((F(\alpha)_*, \partial_*)\) is a \(ZG_\alpha\)-free resolution of the \(ZG_\alpha\)-module \(Z\), cf. [S] (I.6.3)]. Note the canonical isomorphism of reduced chain complexes
\[
C(\alpha)_* = \left( \ker : F(\alpha)_* \otimes_{ZG_\alpha} Z \xrightarrow{\epsilon_{\alpha}} Z \right).
\]
Let \(\alpha, \beta \geq 1\), let \(\lambda \geq 0\) with \(p^{\beta} | (\lambda \cdot p^\alpha)\), and consider the group homomorphism
\[
\phi : G_\alpha \to G_\beta, \quad g_\alpha \mapsto (g_\beta)^\lambda.
\]
Then each \(ZG_\beta\)-module can be regarded as a \(ZG_\alpha\)-module via the ring map \(Z\phi : ZG_\alpha \to ZG_\beta\).

With this convention the assignments (using \(\lambda \cdot p^{\beta-\beta} \in \mathbb{N}_0\))
\[
\phi_{2m}(1_\alpha) := (\lambda \cdot p^{\beta-\beta})^m \cdot 1_\beta,
\]
\[
\phi_{2m-1}(1_\alpha) := (\lambda \cdot p^{\beta-\beta})^{m-1} \cdot \sum_{i=0}^{\lambda-1} (g_\beta)^{i}.
\]
uniquely extend to \(ZG_\alpha\)-linear maps \(ZG_\alpha \to ZG_\beta\) and an explicit computation shows that we obtain an augmentation preserving map of \(ZG_\alpha\)-linear chain complexes
\[
\cdots \xrightarrow{\nu} ZG_\alpha \xrightarrow{\tau} ZG_\alpha \xrightarrow{\nu} ZG_\alpha \xrightarrow{\tau} ZG_\alpha \xrightarrow{\nu} ZG_\alpha \xrightarrow{\tau} ZG_\alpha \cdots
\]
\[
\cdots \xrightarrow{\nu} ZG_\beta \xrightarrow{\tau} ZG_\beta \xrightarrow{\nu} ZG_\beta \xrightarrow{\tau} ZG_\beta \xrightarrow{\nu} ZG_\beta \xrightarrow{\tau} ZG_\beta \cdots
\]
After applying the functor \(\otimes_{ZG_\alpha} Z\) and passing to reduced chain complexes we obtain an induced chain map \(\phi_* : C(\alpha)_* \to C(\beta)_*\) given by
\[
\phi_{2m}(c_{2m}) = (\lambda \cdot p^{\beta-\beta})^m \cdot c_{2m},
\]
\[
\phi_{2m-1}(c_{2m-1}) = \lambda \cdot (\lambda \cdot p^{\beta-\beta})^{m-1} \cdot c_{2m-1}.
\]
The map induced in homology can be identified with the map
\[
(B\phi)_* : \tilde{H}_*(BG_\alpha) \to \tilde{H}_*(BG_\beta),
\]
compare [S] (II.6.1)]. The following lemma evaluates the previous formula in some specific cases.

**Lemma 6.4.** (i) Let \(\alpha \leq \beta\) and \(\lambda = p^{\beta-\alpha}\). Then
\[
\phi_*(c_{2m}) = c_{2m} \quad \text{and} \quad \phi_*(c_{2m-1}) = p^{\beta-\alpha} \cdot c_{2m-1}.
\]
(ii) Let \(\alpha \geq \beta\) and \(\lambda = 1\). Then
\[
\phi_*(c_{2m}) = (p^{\beta-\beta})^m \cdot c_{2m} \quad \text{and} \quad \phi_*(c_{2m-1}) = (p^{\beta-\beta})^{m-1} \cdot c_{2m-1}.
\]
(iii) Let \(\alpha = \beta\). Then
\[
\phi_*(c_{2m}) = \lambda^m \cdot c_{2m} \quad \text{and} \quad \phi_*(c_{2m-1}) = \lambda^m \cdot c_{2m-1}.
\]

**Lemma 6.5.** Consider the diagonal map \(\Delta : G_\alpha \to G_\alpha \times G_\alpha, \quad g \mapsto (g, g)\). Then the induced map in homology \(\Delta_* : \tilde{H}_*(BG_\alpha) \to \tilde{H}_*(BG_\alpha \wedge BG_\alpha)\) is represented on the chain level by
\[
\Delta_*(c_{2m-1}) = \sum_{i=1}^{2m-2} c_i \otimes c_{2m-1-i}.
\]

**Proof.** It is enough to carry out this computation after passing to coefficients \(Z/p^\alpha\). Using the universal coefficient theorem \(H_*(BG_\alpha; \mathbb{Z}/p^\alpha) = \text{Hom}(H^*(BG_\alpha; \mathbb{Z}/p^\alpha), \mathbb{Z}/p^\alpha)\) the claim follows from the well known ring structure of \(H^*(BG_\alpha; \mathbb{Z}/p^\alpha)\).

**Definition 6.6.** A cycle \(c \in C_*(G_\alpha) \otimes \cdots \otimes C_*(G_\alpha)\) is called **positive**, if the homology class \([c] \in \tilde{H}_*(BG_\alpha)\) is positive with respect to \(\mathcal{P}\) in the sense of Definition 5.12.

\[2\phi_{2m-1}(\nu_\alpha \cdot 1_\alpha) = (\lambda p^{\beta-\beta})^{m-1} \cdot \sum_{i=0}^{\lambda-1} (g_\beta)^i \cdot \sum_{i=0}^{\lambda-1} (g_\beta)^i = (\lambda p^{\beta-\beta})^{m} \cdot \sum_{i=0}^{\lambda-1} (g_\beta)^i = \nu_\beta \cdot \phi_{2m}(1_\alpha).\] A similar computation shows \(\phi_{2m}(\tau_\alpha \cdot 1_\alpha) = \tau_\beta \cdot \phi_{2m+1}(1_\alpha)\)
Obviously the cycles $c_{2m-1} \in C(\alpha)$ are positive for $m \geq 2$ since these can be represented by classifying maps of lens spaces $S^{2m-1}/(\mathbb{Z}/p^m) \to BG_\alpha$. Moreover, if one of the cycles $c'$ or $c''$ is positive, then the tensor product $c' \otimes c''$ is positive by Corollary 5.4 (i). In the remaining part of this section we investigate the cycles appearing in Definition 6.1.

**Proposition 6.7.** For $n \geq 2$ the cycle $Tor(c_1^{(1)}, \ldots, c_1^{(n)}) \in C_*^{(1)} \otimes \cdots \otimes C_*^{(n)}$ is positive.

**Proof.** Note that for $n \geq 2$ the cycle $c_{2m-1}$ is positive. Let us first deal with the case $\alpha_1 = \cdots = \alpha_n =: \alpha$. We consider the diagonal homomorphism
\[
\phi : G_\alpha \to G_\alpha \times \cdots \times G_\alpha, \quad g_\alpha \mapsto (g_\alpha, \ldots, g_\alpha).
\]
For $n = 2$ Lemma 6.5 shows $\phi_\ast(c_3) = Tor(c_1, c_1)$ and $\phi_\ast(c_5) = Tor(c_1, c_1) + Tor(c_1, c_1)$. For $n = 3$ the second computation implies $\phi_\ast(c_3) = Tor(c_1, c_1, c_1)$ by applying Lemma 6.5 to one factor of $G_\alpha \times G_\alpha$. By a similar argument we obtain $\phi_\ast(c_{2m-1}) = Tor(c_1, \ldots, c_1)$ for arbitrary $n$, finishing the proof of Proposition 6.7.

The case of arbitrary $\Gamma = G_{\alpha_1} \times \cdots \times G_{\alpha_n}$ follows by applying the chain map
\[
C(\alpha_1)_* \otimes \cdots \otimes C(\alpha_n)_* \to C(\alpha_1)_* \otimes \cdots \otimes C(\alpha_n)_*
\]
induced by the group homomorphism $G_{\alpha_1} \to G_{\alpha_1}$, $g_{\alpha_1} \mapsto (g_{\alpha_1})^{\alpha_1^{-1}}$, on the $i$-th tensor factor, and referring to Lemma 6.3 (i) and Definition 6.1.

For the remainder of this section we assume that $p$ is odd.

**Proposition 6.8.** The following cycles in $C(\alpha_1)_* \otimes C(\alpha_2)_*$ are positive for $m, n \geq 2$:

(i) $Tor(c_{2m-1}, c_{2m-1});$

(ii) $Tor(c_1, c_{2m-1}) + Tor(c_{2m-1}, c_1);$

(iii) $p \cdot Tor(c_{2m-1}, c_1)$ and $p \cdot Tor(c_1, c_{2m-1});$

(iv) $Tor(c_{2m-1}, c_1)$ and $Tor(c_1, c_{2m-1})$ in case $\alpha_1 < \alpha_2$.

**Proof.** For $m, n \geq 2$ the cycle $Tor(c_{2m-1}, c_{2m-1})$ is positive by Corollary 5.4 (ii).

For parts (ii) and (iii) let $\lambda \in \mathbb{N}_0$ and consider the group homomorphism
\[
f_\lambda : G_{\alpha_1} \to G_{\alpha_1} \times G_{\alpha_2}, \quad g_{\alpha_1} \mapsto (g_{\alpha_1}, (g_{\alpha_2})^{\lambda (p^{m-1} - 1)}).
\]
Then (i) and (iii) of Lemma 6.4 together with Lemma 6.5 and Definition 6.1 show
\[
(f_\lambda)_\ast(c_{2m+1}) \lambda \cdot Tor(c_{2m-1}, c_1) + \cdots + \lambda^m \cdot Tor(c_1, c_{2m-1}).
\]
Together with (i) this implies (ii) by setting $\lambda = 1$.

For $m \geq 2$ we have $p \mid (p^m - p)$, but $p^2 \nmid (p^m - p)$. Together with (iii) this implies that linear combinations of $(f_1)_\ast(c_{2m+1})$ and $(f_1)_\ast(c_{2m+1})$ realize $p \cdot Tor(c_{2m-1}, c_1)$ and $p \cdot Tor(c_1, c_{2m-1})$ as positive cycles, finishing the proof of (iii).

Let $\alpha_1 < \alpha_2$ and consider the group homomorphism
\[
f : G_{\alpha_2} \to G_{\alpha_1} \times G_{\alpha_2}, \quad g_{\alpha_2} \mapsto (g_{\alpha_1}, g_{\alpha_2}).
\]
Lemma 6.4 (ii), together with Lemma 6.5 and Definition 6.1 show
\[
f_\ast(c_{2m+1}) = Tor(c_1, c_{2m-1}) + \cdots + p^{(m-1)\alpha_1 - \alpha_1} Tor(c_{2m-1}, c_1).
\]
Together with (iii) this implies that $Tor(c_{2m-1}, c_1)$ is positive. Combining this with part (ii) we conclude that also $Tor(c_{2m-1}, c_1)$ is positive. This finishes the proof of part (iv).
7. Almost BP-representable homology

The proof of Theorem 1.6 requires some more preparation. Let $p$ be an odd prime. Recall that the coefficient ring for Brown-Peterson theory $BP$ at $p$ is isomorphic to a polynomial ring

$$BP_* = \mathbb{Z}(p)[v_1, v_2, \ldots],$$

where $v_n \in BP_{2p^n-2}$. As usual we set $v_0 = p$.

Let $j \geq 1$. Relying on the theory of MU-ring spectra [11 V.4.1] we obtain a multiplicative homology theory $BP(j)$ by dividing out the regular coefficient ideal $(v_1, \ldots, v_{j-1}, v_j^2, v_{j+1}, \ldots)$ from the Brown-Peterson spectrum $BP$. It carries a complex orientation induced by the complex orientation of $BP$. We have a factorization of multiplicative complex oriented homology theories

$$BP \to BP(j) \to \mathbb{H} \mathbb{Z}(p)$$

which on the level of coefficients is given by the canonical projections

$$\mathbb{Z}(p)[v_1, v_2, \ldots] \to \mathbb{Z}(p)(1, v_j) \to \mathbb{Z}(p),$$

where $\mathbb{Z}(p)(1, v_j)$ is the free $\mathbb{Z}(p)$-module with generators 1, $v_j$ and ring structure satisfying $v_j^2 = 0$.

**Definition 7.1.** Let $X$ be a topological space. A class $H_*(X; \mathbb{Z}(p))$ is called *almost BP-representable*, if it lies in the image of the canonical maps $BP(j)_*(X) \to H_*(X; \mathbb{Z}(p))$ for all $j \geq 1$.

The subgroup of almost BP-representable classes is denoted $RH_*(X; \mathbb{Z}(p))$ and its intersection with the reduced homology $\tilde{H}_*(X; \mathbb{Z}(p))$ is denoted $RH_*(X; \mathbb{Z}(p))$.

We have

$$RH_*(X; \mathbb{Z}(p)) = \bigcap_{j \geq 1} E_2^{0,j}(j) \subset H_*(X; \mathbb{Z}(p)),$$

where $E_2^{0,j}(j) \subset E_{2,0}^{j}(j)$ is the subgroup of permanent cycles in the $E^2$-term of the Atiyah-Hirzebruch spectral sequence

$$E_2^{a,t} = H_* \mathbb{Z}(X;BP(j)_t) \Longrightarrow BP(j)_{a+t}(X).$$

Since $BP(j)_*$ is a free $\mathbb{Z}(p)$-module in every degree the universal coefficient theorem implies

$$H_*(X;BP(j)_*) = H_*(X;\mathbb{Z}) \otimes BP(j)_* = H_*(X;\mathbb{Z}) \otimes \mathbb{Z}(p)(1, v_j),$$

and hence $E_2^{a,t}(j)$ is non-zero precisely for $t = 0$ and $t = 2p^j - 2$.

Let us apply this to $X = B\Gamma$, where $\Gamma = G_{\alpha_1} \times \cdots \times G_{\alpha_n}$, $\alpha_1 \leq \cdots \leq \alpha_n$. We will work with the chain model description $\tilde{H}_*(B\Gamma) \cong H_*(C^{(1)}_\ast \otimes \cdots \otimes C^{(n)}_*; \mathbb{Z}/p^{\alpha_1})$ from Section 6. By the universal coefficient and Künneth theorems we have an injective map

$$H_* \mathbb{Z}(C^{(1)}_\ast \otimes \cdots \otimes C^{(n)}_*; \mathbb{Z}/p^{\alpha_1}) \to H_* \mathbb{Z}(C^{(1)}_\ast \otimes \cdots \otimes C^{(n)}_*; \mathbb{Z}/p^{\alpha_1}) \cong H_*(C^{(1)}_*; \mathbb{Z}/p^{\alpha_1}) \otimes \cdots \otimes H_*(C^{(n)}_*; \mathbb{Z}/p^{\alpha_1}),$$

which is compatible with differentials in the Atiyah-Hirzebruch spectral sequence. Since these differentials act as derivations it is enough to analyse the differentials

$$\partial_2^{p^j-1} \colon \tilde{H}_*(BG_{\alpha}; \mathbb{Z}/p^j) \to \tilde{H}_{*-2p^j+1}(BG_{\alpha}; \mathbb{Z}/p^j) \otimes v_j$$

for $1 \leq t \leq \alpha$ in the Atiyah-Hirzebruch spectral sequence for the homology theories $BP(j)_*(\cdot; \mathbb{Z}/p^j)$, $j \geq 1$ obtained by dividing out the coefficient ideal $(p^j)$ from $BP(j)$ (which amounts to introducing coefficients $\mathbb{Z}/p^j$ for the theory $BP(j)$).

The differential $\partial_2^{p^j-1}$ in (11) is determined by the following formal group law computation.

**Lemma 7.2.** Let $\alpha \geq 1$ and let $p^{\alpha} : \mathbb{C}^{\infty} \to \mathbb{C}^{\infty}$ be the map induced by $S^1 \to S^1$, $t \mapsto t^{p^{\alpha}}$, using the identification $BS^1 = \mathbb{C}^{\infty}$. Then for the complex orientation $x \in BP(j)^2(\mathbb{C}^{\infty})$ we have

$$(p^{\alpha})^*(x) = p^{\alpha} \cdot x + p^{\alpha-1}v_j \cdot x^{p^j} + R_\alpha$$

where $R_\alpha \in p^{\alpha} \cdot BP(j)^*(\mathbb{C}^{\infty})$. 

Example 7.4.

Proposition 7.3. For $R$ where $C$ holds for $(11)$

Lemma 7.5.

Definition 7.6.

Proof. Induction on $\alpha$. For $\alpha = 1$ we use the $p$-typical formal group law of BP to obtain $(p)^*(x) = px + v_j \cdot x^{p^j}$, possibly after multiplying the generator $v_j$ with some unit in $\mathbb{Z}(p)$. Hence the assertion holds for $\alpha = 1$.

Using $v_j^2 = 0 \in \text{BP}(j)_*$ we inductively obtain for $\alpha \geq 1$

$$(p^{\alpha+1})^*(x) = (p)^*((p^\alpha)^*)(x) =$$

$$= p \cdot (p^\alpha \cdot x + p^{\alpha-1}v_j \cdot x^{p^j} + R_\alpha) + v_j \cdot (p^{\alpha} \cdot x + p^{\alpha-1}v_j \cdot x^{p^j} + R_\alpha)^{p^j}$$

$$= p^{\alpha+1} \cdot x + p^\alpha \cdot v_j \cdot x^{p^j} + R_{\alpha+1},$$

where $R_{\alpha+1} = pR_\alpha + v_j \cdot (p^{\alpha} \cdot x + p^{\alpha-1}v_j \cdot x^{p^j} + R_\alpha)^{p^j}$. $\square$

For $j \geq 1$ and the induced orientation $x \in \text{BP}(j)^2(\mathbb{CP}^\infty; \mathbb{Z}/p^j)$ we therefore have

$$(p^\alpha)^*(x) = \begin{cases} p^{\alpha-1} \cdot v_j \cdot x^{p^j} \text{ for } \ell = \alpha, \\ 0 \text{ for } 1 \leq \ell < \alpha. \end{cases}$$

Observe that $(p^\alpha)^*(x) \in \text{BP}(j)^2(\mathbb{CP}^\infty; \mathbb{Z}/p^j)$ is equal to the BP$(j)(-; \mathbb{Z}/p^j)$-theoretic Euler class of the fibre bundle $S^1 \hookrightarrow B\mathbb{Z}/p^\alpha \to \mathbb{CP}^\infty$. This implies the following fact where we set $C_*^{BP}(\mathbb{Z}/p^\ell) := C_*(\mathbb{Z}/p^\ell)$.

Proposition 7.3. For $\alpha, j \geq 1$ and $1 \leq \ell \leq \alpha$ the differential

$$\partial_{c^{p^j-1}}^{2p^j-1} : \tilde{H}_* (BG_\alpha; \mathbb{Z}/p^j) \to \tilde{H}_{*-2p^j+1} (BG_\alpha; \mathbb{Z}/p^j) \otimes v_j$$

introduced in (11) is given by

$$\partial_{c}^{2p^j-1} : C_*(\mathbb{Z}/p^\ell) \to C_{*-2p^j+1}(\mathbb{Z}/p^\ell) \otimes v_j,$$

$$c_d \mapsto \begin{cases} p^{\alpha-1} \cdot c_{d-2p^j+1} \otimes v_j \text{ for even } d > 2p^j - 1 \text{ and } \ell = \alpha, \\ 0 \text{ otherwise} \end{cases}.$$}

Example 7.4. (i) For $\alpha_1 \neq \alpha_2$ all cycles in $C_*(\alpha_1) \otimes C_*(\alpha_2)$ are almost BP-representable. This is clear for cycles of the form $c_{2m-1} \otimes c_{2n-1}$ and follows for the cycles $\text{Tor}(c_{2m-1}, c_{2n-1})$ from Definition 7.1 and Proposition 7.3.

(ii) For $\alpha, j \geq 1$ the cycle $\text{Tor}(c_1, c_{2p^j-1}) \in C_*(\alpha) \otimes C_*(\alpha)$ is not almost BP-representable, since it does not survive to $E_2^{2p^j-1}(\mathbb{Z}/p^\alpha \times BG_\alpha)$.

We will now concentrate on the almost BP-representable homology of $B(G_\alpha)^\alpha$. Let

- $c_d$ be the free $\mathbb{Z}$-graded $\mathbb{F}_p$-module with one generator $c_d$ in each positive degree $d \geq 1$;
- $(C_*)^n$ be its $n$-fold tensor product;
- $\partial(j)_*$ be the differential on $(C_*)^n$ of degree $2p^j - 1$, $j \geq 0$, which acts as a derivation and satisfies

$$\partial(j)_*(c_d) := \begin{cases} c_{d-2p^j+1} \text{ for even } d > 2p^j - 1, \\ 0 \text{ otherwise} \end{cases};$$

- $E_{p^r}^n := \ker \partial(j)_* \subset (C_*)^n$ for $0 \leq r \leq \infty$;
- $D_{p^r}^n := \{c_{d_1} \otimes \cdots \otimes c_{d_n} \in E_{p^r}^n | \text{ exactly one } d_i \text{ even} \} \subset E_{p^r}^n$ for $0 \leq r \leq \infty$.

We denote by $C(\alpha)_*^n$ the $n$-fold tensor power of the chain complex $C(\alpha)_*$. The following fact is implied by Corollary 7.3.

Lemma 7.5. The projection $C(\alpha)_*^n \to (C_*)^n$ induces an isomorphism $\overline{\text{RH}}_*(B\Gamma) \otimes \mathbb{F}_p \cong E_{\bullet}^{\infty, n}.

Definition 7.6. We define the following $\mathbb{Z}$-graded $\mathbb{F}_p$-modules:

- $N_* := \text{span}\{c_{2m-1} | m \geq 1\} \subset C_*;
- L_* := \text{span}\{y_{2m} | m \geq 1\}$, where $y_{2m}$ are free generators, considered as a chain complex with zero differential;
- $L_{<p^k} := \text{span}\{y_{2m} | 1 \leq m < p^k\} \subset L_*$ for $k \geq 0$.
Note that the canonical projection $C_* \to L_*, c_{2m} \mapsto y_{2m}, c_{2m-1} \mapsto 0$, is a map of chain complexes. Let $(N_*)^n$ be the $n$-fold tensor product of $N_*$ for $n \geq 1$.

Let $\phi : (G_\alpha)^k \to (G_\alpha)^n$ be a group homomorphism. We obtain an induced map

$$\phi_* : \tilde{H}_*(\Lambda^k B\alpha; \mathbb{Z}/p^n) \to \tilde{H}_*(\Lambda^k B\alpha; \mathbb{Z}/p^n)$$

in group homology. Using the identifications $\tilde{H}_*(\Lambda^k B\alpha; \mathbb{Z}/p^n) \otimes \mathbb{F}_p \cong (\alpha)_*^\ell \otimes \mathbb{F}_p \cong (\alpha_*)^\ell$ for $\ell = k, n$ this induces a map $\phi_* : (\alpha_*)^k \to (\alpha_*)^n$ of graded $\mathbb{F}_p$-modules.

**Definition 7.7.** For $n \geq 1$ we set

$$\mathcal{L}^n_* := \text{span}\{\phi_*((N_*)^k) | \phi : (G_\alpha)^k \to (G_\alpha)^n \text{ injective homomorphism, } 1 \leq k \leq n\} \subset (G_\alpha)^n.$$

**Lemma 7.8.** $\mathcal{L}^n_* \subset \mathcal{G}^{n,\infty}$.

**Proof.** This follows from the functoriality of the Atiyah-Hirzebruch spectral sequence for BP$(j)$-theory and Corollary 7.8.

We will show in Corollary 7.13 that equality occurs in Lemma 7.8.

**Proposition 7.9.** For $n \geq 1$ the canonical projection $\mathcal{L}^{n+1}_* \to (N_*)^n \otimes \mathbb{L}_{<p}^n$ is surjective.

**Proof.** The proof given for $\alpha = 1$ in [13 Prop. 5.3.] generalizes to larger $\alpha$. For notational reasons we work with the additive group $\mathbb{Z}/p^n$ instead of $G_\alpha$.

For $0 \leq \lambda_1, \ldots, \lambda_n \leq p - 1$ we consider the injective group homomorphism

$$\phi_{(\lambda_1, \ldots, \lambda_n)} : (\mathbb{Z}/p^n)^n \to (\mathbb{Z}/p^n)^{n+1}$$

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \lambda_1 x_1 + \cdots + \lambda_n x_n).$$

For all $\ell \geq 1$ we have an $\mathbb{F}_p$-algebra isomorphism

$$H^*(B(\mathbb{Z}/p^n)\mathbb{L}; \mathbb{F}_p) \cong \mathbb{F}_p[t_1, \ldots, t_\ell] \otimes \Lambda(s_1, \ldots, s_\ell),$$

where $t_1, \ldots, t_\ell$ are indeterminates of degree 2 and $s_1, \ldots, s_\ell$ are indeterminates of degree 1.

The map induced in $F_p$-cohomology by $B\phi_{(\lambda_1, \ldots, \lambda_n)} : B(\mathbb{Z}/p^n)^n \to B(\mathbb{Z}/p^n)^{n+1}$ satisfies

$$\phi_{(\lambda_1, \ldots, \lambda_n)} : (\mathbb{Z}/p^n)^n \to (\mathbb{Z}/p^n)^{n+1} (t_1^{n+1} s_1 \cdot \cdots \cdot t_n^{n+1} s_n) \cdot (\lambda_1 t_1 + \cdots + \lambda_n t_n)^\nu$$

for $\nu \geq 0$. This computation uses the ring structures of $H^*(B(\mathbb{Z}/p^n)\mathbb{L}; \mathbb{F}_p)$ for $\ell = n, n + 1$.

The $(p^n \times p^n)$-Vandermonde-matrix

$$X := \begin{pmatrix} 1 & (\lambda_1 t_1 + \cdots + \lambda_n t_n) & \cdots & (\lambda_1 t_1 + \cdots + \lambda_n t_n)^{p^n-1} \end{pmatrix}_{0 \leq \lambda_1, \ldots, \lambda_n < p}$$

(where the subscript parameterizes the rows) with entries in $\mathbb{F}_p[t_1, \ldots, t_n]$ has determinant

$$\prod_{0 \leq \lambda_1, \ldots, \lambda_n < \mu_1, \ldots, \mu_n} ((\lambda_1 - \mu_1) t_1 + \cdots + (\lambda_n - \mu_n) t_n) \neq 0,$$

applying the lexicographic order to the index set. Hence the column vectors of $X$ are linearly independent over $\mathbb{F}_p[t_1, \ldots, t_n]$.

Setting $N_* := H^{odd}(B\mathbb{Z}/p^n; \mathbb{F}_p)$ this means, in view of (12), that the map

$$\bigoplus_{0 \leq \lambda_1, \ldots, \lambda_n < p} \phi_{(\lambda_1, \ldots, \lambda_n)} : (N_*)^n \otimes H^{0 \leq 2m < 2p^n}(B\mathbb{Z}/p^n; \mathbb{F}_p) \to \bigoplus_{0 \leq \lambda_1, \ldots, \lambda_n < p} (N_*)^n$$

is injective. Dualizing this statement over $\mathbb{F}_p$ we conclude that the map

$$\sum_{0 \leq \lambda_1, \ldots, \lambda_n < p} \phi_{(\lambda_1, \ldots, \lambda_n)} : \bigoplus_{0 \leq \lambda_1, \ldots, \lambda_n < p} (N_*|^n) \to (N_*)^n \otimes \text{span}_{\mathbb{F}_p}(c_0, \ldots, c_{2(p^n-1)})$$

is surjective.

**Proposition 7.10.** For $n \geq 0$ the following holds.
(i) We have a commutative diagram of surjective maps

\[
\begin{array}{ccc}
\mathcal{G}^{n+1,n-1} & \xrightarrow{\pi} & (\mathcal{N}_n)^{n} \otimes L_* \\
\partial(n) & \downarrow & \downarrow \partial(n)_* \\
(\mathcal{N}_n)^{n+1} & & \\
\end{array}
\]

where \(\pi\) is the canonical projection.

(ii) The projection \(\ker(\partial(n)_*) \to (\mathcal{N}_n)^{n} \otimes L_{<p^n}\) is an isomorphism.

(iii) \(\mathcal{D}^{n+1,n}(= \ker(\partial(n)_*)) = \mathcal{D}^{n+1,n}_{<p^n}\).

Proof. We apply induction on \(n\), the case \(n=0\) being clear. Let \(n > 0\), \(c \in (\mathcal{N}_n)^n\) and \(m > 0\). We will show \(c \otimes y_{2m} \in \text{im}(\pi)\). Let \(0 \leq \ell < n-1\). Using the inductive assumption (i) we find \(c(\ell) \in \mathcal{D}^{n-\ell,1}_{\deg(c)+2p^{\ell-1}}, 0 \leq \ell \leq n-1\), with \(\partial(\ell)_*(c(\ell)) = c\) and using the induction assumption again for \(\ell < j < n-1\) we can arrange in addition that \(\partial(j)_*(c(\ell)) = 0\) for \(\ell < j \leq n-1\).

Then

\[
(13) \quad c \otimes c_{2m} = \sum_{\ell=0}^{n-1} c(\ell) \otimes c_{2m-(2p^{\ell-1})} \in \mathcal{D}^{n+1,n-1}_* \\
\]

and \(\pi\) sends this element to \(c \otimes y_{2m} \in (\mathcal{N}_n)^n \otimes L_*\).

If \(c \in \mathcal{D}^{n+1,n-1}_* \cap \ker(\pi)\), then \(c \in \mathcal{D}^{n,n-1}_* \otimes \mathcal{N}_*\) by the definition of \(\mathcal{D}^{n,n-1}_*\) and hence \(c \in \mathcal{D}^{n,\infty}_* \otimes N_*\), using the inductive assumption (iii). We conclude \(\partial(n)_*(c) = 0\) such that \(\overline{\partial(n)}_*\) is well defined.

Next let \(c \in (\mathcal{N}_n)^n\). We claim \(c \otimes c_{2m-1} \in \text{im}(\partial(n)_*)\) for all \(m > 0\). The proof is by induction on \(\deg(c)\). As in (13) we find \(c(\ell) \in \mathcal{D}^{n-\ell,1}_{\deg(c)+2p^{\ell-1}}, 0 \leq \ell \leq n-1\), with

\[
\sum_{\ell=0}^{n-1} c(\ell) \otimes c_{2m+(2p^{\ell-1})-2} \in \mathcal{D}^{n+1,n-1}_* \\
\]

Then

\[
\partial(n)_*(c \otimes c_{2m+2p^{\ell-2}} - \sum_{\ell=0}^{n-1} c(\ell) \otimes c_{2m+(2p^{\ell-1})-2}) = c \otimes c_{2m-1} - \sum_{\ell=0}^{n-1} \partial(n)_*(c(\ell)) \otimes c_{2m+2(p^{\ell-1})-1} \\
\]

and for \(0 \leq \ell < n-1\) we have

\[
de(\partial(n)_*(c(\ell)) = \deg(c(\ell)) - (2p^n - 1) = (\deg(c) + 2p^{\ell-1}) - (2p^n - 1) < \deg(c).\]

Hence we can apply induction on \(\deg(c)\), finishing the proof of (i).

For (ii) we observe on the one hand \(\dim \ker(\overline{\partial(n)}_*) = \dim ((\mathcal{N}_n)^n \otimes L_{<p^n})\) for \(d \geq 0\), since, by an elementary dimension count,

\[
\dim ((\mathcal{N}_n)^n \otimes L_*) = \dim ((\mathcal{N}_n)^n \otimes L_{<p^n}) + \dim ((\mathcal{N}_n)^{n+1} \otimes L_{<p^{n-1}}) \\
\]

and \(\overline{\partial(n)}_*\) is surjective by (i). On the other hand the map

\[
\mathcal{G}^{n+1,n}_{<p^n} \subset \mathcal{G}^{n+1,n}_{<p^n} \xrightarrow{\pi} (\mathcal{N}_n)^n \otimes L_* \longrightarrow (\mathcal{N}_n)^n \otimes L_{<p^n} \\
\]

is surjective by Lemma 7.9, such that the projection \(\ker(\overline{\partial(n)}_*) \to (\mathcal{N}_n)^n \otimes L_{<p^n}\) is also surjective. This implies assumption (ii).

For assertion (iii) let \(c \in \mathcal{D}^{n+1,n}_*\). By Lemmas 7.8 and 7.9 we can add an element in \(\mathcal{G}^{n+1,n}_{<p^n}\) such that the projection to \((\mathcal{N}_n)^n \otimes L_{<p^n}\) vanishes. Since \(\partial(n)_*(c) = 0\) and \(\ker(\overline{\partial(n)}_*)\) maps isomorphically to \((\mathcal{N}_n)^n \otimes L_{<p^n}\), we obtain \(\pi(c) = 0\) in \((\mathcal{N}_n)^n \otimes L_*\). We conclude \(c \in \mathcal{D}^{n,n}_{<p^n} \otimes N_* = \mathcal{D}^{n,\infty}_* \otimes N_*\) by the induction assumption (iii). Since \(\mathcal{D}^{n,\infty}_* \otimes N_* \subset \mathcal{D}^{n+1,\infty}_*\) assertion (iii) follows.

We notice the following implication of Proposition 7.10 (ii).

**Corollary 7.11.** \((\ker : \mathcal{G}^{n+1,\infty}_* \to (\mathcal{N}_n)^n \otimes L_{<p^n}) \subset (\ker : \mathcal{G}^{n+1,\infty}_* \to (\mathcal{N}_n)^n \otimes L_*)\).
Proposition 7.12. Let $\mathcal{J}_n$ denote the set of families $(J_1, \ldots, J_n)$, where $J_1 = N_\alpha$ or $J_i = L_{<p^k}$ and $k$ is the number of $J_j$, $j < i$, with $J_j = N_\alpha$. Then the following statements hold:

(i) The canonical map $\Psi^n : \mathcal{C}_{*}^{n,\infty} \to \bigoplus \mathcal{J}_n J_1 \otimes \cdots \otimes J_n$ is injective.

(ii) The restriction (cf. Lemma 7.8) $\Psi^n|_{\mathcal{Z}^n}$ is surjective.

Corollary 7.13. $\mathcal{L}_{*}^{n} = \mathcal{C}_{*}^{n,\infty}$.

Proof of Proposition 7.12. We work by induction on $n$ and assume (i) and (ii) have been shown for $n$. For $J = (J_1, \ldots, J_n) \in \mathcal{J}_n$ let $k(J)$ denote the number of factors $J_i = N_\alpha$. Furthermore we set $D_{J,i} = C_\alpha$ for $J_i = N_\alpha$ and $D_{J,i} = L_{<p^k}$ for $J_i = L_{<p^k}$. Consider the composition of projections

\[
\mathcal{C}_{*}^{n+1,\infty} \xrightarrow{\pi_1} \left( \bigoplus_{\mathcal{J}_n} D_{J,1} \otimes \cdots \otimes D_{J,n} \right) \otimes C_\alpha \xrightarrow{\pi_2} \left( \bigoplus_{\mathcal{J}_n} J_1 \otimes \cdots \otimes J_n \right) \otimes L_\alpha.
\]

Note that $\pi_1$ is a chain map (using the zero differential on $L_{<p^k}$). For the induction step of (i) let $c \in \ker \Psi^{n+1}$ and $c' \in D_{J,1} \otimes \cdots \otimes D_{J,n} \otimes C_\alpha$ be one component of $\pi_1(c)$ where $J \in \mathcal{J}_n$. By assumption the image of $\pi_2(c')$ under the map

\[
D_{J,1} \otimes \cdots \otimes D_{J,n} \otimes C_\alpha \to J_1 \otimes \cdots \otimes J_n \otimes L_{<p^k} \tag{7}
\]

is zero. By Lemma 7.11 we have

\[
\left( \ker : \mathcal{C}_{*}^{n+1,\infty} \to (N_\alpha)^{k(J)} \otimes L_{<p^k} \right) \subset \left( \ker : \mathcal{C}_{*}^{n+1,\infty} \to (N_\alpha)^{k(J)} \otimes L_\alpha \right)
\]

and hence $\pi_2(c') = 0$. Applying this to all components $c'$ of $\pi_1(c)$ we conclude $\pi_2(\pi_1(c)) = 0$.

Let $\pi : \mathcal{L}_{*}^{n+1} \to \mathcal{C}_{*}^{n,\infty} \otimes L_\alpha$ be the projection. Since $(\Psi^n \otimes \mathrm{id}) \circ \pi = \pi_2 \circ \pi_1$ our induction assumption (i) implies $\pi(c) = 0$. Hence $c \in \ker \Psi^n \otimes N_\alpha$, which is equal to 0, again by induction assumption (i). This completes the proof of the assertion $\ker \Psi^{n+1} = 0$.

For the induction step of (ii) let $J = (J_1, \ldots, J_{n+1}) \in \mathcal{J}_{n+1}$. We have to show $J_1 \otimes \cdots \otimes J_{n+1} \subset \Psi^{n+1}(\mathcal{L}_{*}^{n+1})$. By induction we have $J_1 \otimes \cdots \otimes J_n \subset \Psi^n(\mathcal{L}_{*}^{n})$. In particular $J_1 \otimes \cdots \otimes J_n \otimes N_\alpha \subset \Psi^{n+1}(\mathcal{L}_{*}^{n} \otimes N_\alpha)$. Since $\mathcal{L}_{*}^{n} \otimes N_\alpha \subset \mathcal{L}_{*}^{n+1}$ we can hence restrict to the case $J_{n+1} = L_{<p^k}$.

For each injective group homomorphism $\phi : (G_\alpha)^{k(J)} \to (G_\alpha)^{k(J+1)}$ we obtain an induced map $(N_\alpha)^{k(J)} \to (N_\alpha)^{k(J)} \otimes L_{<p^k}$ and hence an induced map $J_1 \otimes \cdots \otimes J_n \to J_1 \otimes \cdots \otimes J_n \otimes L_{<p^k}$. By Lemma 7.3 the images of these maps for different $\phi$ span $J_1 \otimes \cdots \otimes J_n \otimes L_{<p^k}$. Since $J_1 \otimes \cdots \otimes J_n \subset \Psi^n(\mathcal{L}_{*}^{n})$ we conclude $J_1 \otimes \cdots \otimes J_n \otimes L_{<p^k} \subset \Psi^{n+1}(\mathcal{L}_{*}^{n+1})$ by the functoriality of group homology and the fact that the composition of injective group homomorphisms is injective. \hfill $\square$

Remark 7.14. (i) Corollary 7.13 for $\alpha = 1$ implies that almost BP-representable classes in $H_1(B(Z/p)^n)$ already lie in the image of $\Omega_{*}^{SO}(B(Z/p)^n) \to H_*(B(Z/p)^n)$.

(ii) Corollary 7.13 also holds for $p = 2$.

(iii) We believe that Proposition 7.10 may be a first step towards an algebraic proof of the Conner-Floyd conjecture (see [20] for a topological proof).

We will now complete the proof of Theorem 1.6.

Lemma 7.15. The $p$-toral homology classes in $H_*(C_{*}^{(1)} \otimes \cdots \otimes C_{*}^{(n)})$ are concentrated in degree $n$.

Proof. We have $H_{d}(C_{*}^{(1)} \otimes \cdots \otimes C_{*}^{(n)}) = 0$ for $d < n$. Furthermore a $p$-toral class can have degree at most $d$, since for odd $p$ and $\alpha \geq 1$ each element in $H^1(BG; Z/p)$ has square zero. \hfill $\square$

Lemma 7.16. Let $j \geq 2$, $1 \leq i_1 < \cdots < i_j \leq n$ and $m_{i_1}, \ldots, m_{i_j} \geq 1$. Then the following cycles in $C_{*}^{(i_1)} \otimes \cdots \otimes C_{*}^{(i_j)}$ are almost BP-representable, positive and $p$-toral:

(i) $\mathrm{Tor}(c_{2m_{i_1}-1}^{(i_1)}, \ldots, c_{2m_{i_j}-1}^{(i_j)})$ if $\alpha_{i_1} < \alpha_{i_j}$;

(ii) $p \cdot \mathrm{Tor}(c_{2m_{i_1}-1}^{(i_1)}, \ldots, c_{2m_{i_j}-1}^{(i_j)})$ without any further restriction.
Proof. The almost BP-representability follows from Corollary 4.3 and Definition 6.1.

For positivity let $r$ denote the number of $m_{i_1}, \ldots, m_{i_r}$ which are equal to 1. By Proposition 6.7 and (iterated use of) Corollary 5.4 positivity holds for $r = 0$ or $r \geq 2$. We will therefore assume $r = 1$. Let $1 \leq \ell \leq j$ be the unique index with $m_{i_\ell} = 1$. If $a_{i_\ell} < a_{i_j}$ we find $1 \leq k \leq j$ with $a_{i_\ell} = a_{i_k}$. Then $\text{Tor}(c_{m_{i_\ell}-1}, c_{m_{i_j}-1})$ is positive by Proposition 6.8 (iv). If $a_{i_\ell} = a_{i_j}$, then $p \cdot \text{Tor}(c_{m_{i_\ell}-1}, c_{m_{i_j}-1})$ is positive for any $1 \leq k \leq j$ with $k \neq \ell$ by Proposition 6.8 (iii). Hence for $r = 1$ the claim is again implied by Corollary 5.4.

The $p$-atorality follows from Lemma 7.15.

Proof of Theorem 7.6. We use induction on $n = \text{rk} \Gamma$, where the case $n = 1$ is clear. By induction on $n$ it is enough to deal with $p$-atoral classes in $\text{RH}_p(\hat{B}\Gamma)$.

First assume $a_1 < a_n$. By Proposition 6.2 and Lemma 7.16 (i) we can restrict to $p$-atoral almost BP-representable cycles of the form

$$c = \sum_{m_1 \geq 1} c(m) \otimes c_{2m_1-1} \in (C^{(1)}(\alpha) \otimes \cdots \otimes C^{(n-1)}(\alpha)) \otimes C^{(n)}(\alpha).$$

Since $c$ is almost BP-representable if follows from Corollary 7.3 combined with $\partial^{2n-1}(c_{2m_1-1}) = 0$ for $m_1 \geq 1$ and $j \geq 1$ that $c(m) \in C^{(1)}(\alpha) \otimes \cdots \otimes C^{(n-1)}(\alpha)$ is an almost BP-representable cycle for $m_1 \geq 1$. Furthermore for $m_1 \geq 2$ the cycle $c(m) \otimes c_{2m_1-1}$ is positive and $p$-atoral by Lemma 7.13.

We can therefore restrict to almost BP-representable $p$-atoral cycles of the form

$$c = c(1) \otimes c_1 \in (C^{(1)}(\alpha) \otimes \cdots \otimes C^{(n-1)}(\alpha)) \otimes C^{(n)}(\alpha).$$

Then the cycle $c(1)$ is $C^{(1)}(\alpha) \otimes \cdots \otimes C^{(n-1)}(\alpha)$ is almost BP-representable and $p$-atoral by the corresponding assumption on $c$.

By induction on $n$ it hence remains to deal with the case $a_1 = \cdots = a_n$. Let $\Gamma = (G_\alpha)^n$ and let $c \in (C(\alpha)_*)^n$ be almost BP-representable and $p$-atoral. Consider the composition $\omega : (C(\alpha)_*)^n \to (C_*)^n \cong \text{RH}_p(\hat{B}\Gamma) \otimes \mathbb{F}_p$, where the last isomorphism is from Lemma 7.5. Proposition 7.12 implies that $\omega(c)$ is a sum of elements in $\text{RH}_p(\hat{B}\Gamma)$ each of which is represented by a map

$$\psi : L^{2m_1-1} \times \cdots \times L^{2m_k-1} \to B(G_\alpha)^k \xrightarrow{B\phi} B(G_\alpha)^n \to \hat{B}\Gamma,$$

where $1 \leq k \leq n$ and $\phi : (G_\alpha)^k \to (G_\alpha)^n$ is some injective group homomorphism. Now

- if at least one $m_i \geq 2$ then $L^{2m_1-1} \times \cdots \times L^{2m_k-1}$ is $p$-atoral by Lemma 7.15 and carries a positive scalar curvature metric, and hence $\psi$ represents a $p$-atoral positive class in $\text{RH}_p(\hat{B}\Gamma)$;
- if $m_1 = \cdots = m_k = 1$ and $\psi$ represents a non-zero homology class, then this class is $p$-atorial, since $\phi$ is injective.

Since $c$ is assumed to be $p$-atorial, we can hence assume that the case $m_1 = \cdots = m_k = 1$ does not occur and therefore we can add a $p$-atorial almost BP-representable positive cycle to $c$ in order to attain $\omega(c) = 0$. This implies $c = p \cdot c'$ for a cycle $c' \in (C(\alpha)_*)^n$.

By Proposition 6.2 applied to $c'$ and Lemma 7.16 (ii) applied to $c = pc'$ we can assume that $c$ is a linear combination of cycles of the form

$$c_{2m_1-1} \otimes \cdots \otimes c_{2m_n-1}.$$
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