FEYNMAN-KAC FORMULA FOR PERTURBATIONS OF ORDER \( \leq 1 \),
AND NONCOMMUTATIVE GEOMETRY

SEBASTIAN BOLDT AND BATU GÜNEYSU

Abstract. Let \( Q \) be a differential operator of order \( \leq 1 \) on a complex metric vector bundle \( \mathcal{E} \to \mathcal{M} \) with metric connection \( \nabla \) over a possibly noncompact Riemannian manifold \( \mathcal{M} \). Under very mild regularity assumptions on \( Q \) that guarantee that \( \nabla^\dagger \nabla / 2 + Q \) canonically induces a
holomorphic semigroup \( e^{-z\mathcal{H}_Q} \) in \( \Gamma_{L^2(\mathcal{M}, \mathcal{E})} \) (where \( z \) runs through a complex sector which contains \( [0, \infty) \)), we prove an explicit Feynman-Kac type formula for \( e^{-t\mathcal{H}_Q} \), \( t > 0 \), generalizing the standard self-adjoint theory where \( Q \) is a self-adjoint zeroth order operator. For compact \( \mathcal{M} \)'s we combine this formula with Berezin integration to derive a Feynman-Kac type formula
for an operator trace of the form

\[
\text{Tr} \left( \tilde{V} \int_0^t e^{-s\mathcal{H}_{\tilde{V}}} P e^{-t-s\mathcal{H}_{\tilde{V}}} \, ds \right),
\]

where \( V, \tilde{V} \) are of zeroth order and \( P \) is of order \( \leq 1 \). These formulae are then used to obtain a probabilistic representations of the lower order terms of the equivariant Chern character (a differential graded extension of the JLO-cocycle) of a compact even-dimensional Riemannian spin manifold, which in combination with cyclic homology play a crucial role in the context of the Duistermaat-Heckmann localization formula on the loop space of such a manifold.

1. Introduction

The classical Feynman-Kac formula states that given a real-valued (for simplicity) smooth potential \( V : \mathcal{M} \to \mathbb{R} \) on a possibly noncompact Riemannian manifold \( \mathcal{M} \) such that the symmetric Schrödinger operator \( \Delta / 2 + V \) is semibounded from below in \( L^2(\mathcal{M}) \) (defined initially on smooth compactly supported functions), one has

\[
e^{-t\mathcal{H}_V} \Psi(x) = \mathbb{E} \left[ 1_{\{t < \zeta_x^x\}} e^{-\int_0^t V(b_s^x) ds} \Psi(b_t^x) \right] \quad \text{for all } \Psi \in L^2(\mathcal{M}), \ t > 0, \ a.e. \ x \in \mathcal{M},
\]

whenever the expectation value is well-defined. Here

- \( \mathcal{H}_V \) denotes the Friedrichs realization\(^1\) of \( \Delta / 2 + V \), taking into account that in general \( \Delta / 2 + V \) need not have a unique self-adjoint realization, and \( e^{-t\mathcal{H}_V} \) is defined via spectral calculus,

- \( b^x \) is an arbitrary Brownian motion on \( \mathcal{M} \) starting from \( x \) with lifetime \( \zeta_x^x > 0 \), taking into account that \( \mathcal{M} \) need not be stochastically complete.

Vector bundle versions of this formula have played a crucial role in mathematical physics through the Feynman-Kac-Itô formula \([S05, BHL00]\) and in geometry through probabilistic proofs of the Atiyah-Singer index theorem \([B84, H02]\). In this context, one replaces \( \Delta \) with \( \nabla^\dagger \nabla \), where

\[
\nabla : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \longrightarrow \Gamma_{C^\infty}(\mathcal{M}, T^*\mathcal{M} \otimes \mathcal{E})
\]

is a metric connection on a metric vector bundle \( \mathcal{E} \to \mathcal{M} \), and the potential with a smooth pointwise self-adjoint section \( V \) of \( \text{End}(\mathcal{E}) \to \mathcal{M} \). In other words, \( V \) is a self-adjoint zeroth order

\(^1\)which corresponds to Dirichlet boundary conditions.
operator. Assuming now that the symmetric covariant Schrödinger type operator $\nabla^1 \nabla/2 + V$ in the space of square integrable sections $\Gamma_{L^2}(\mathcal{M}, \mathcal{E})$ is bounded from below, one can prove that
\begin{equation}
    e^{-tH^\nabla} \Psi(x) = \mathbb{E} \left[ 1_{\{t < \xi^x\}} V^\nabla(t)/\xi^x(t)^{-1} \Psi(b^x_t) \right]
\end{equation}
for all $\Psi \in \Gamma_{L^2}(\mathcal{M}, \mathcal{E})$, $t > 0$, a.e. $x \in \mathcal{M}$, whenever the expectation is well-defined. Here
- $H^\nabla$ is the Friedrichs realization of $\nabla^1 \nabla/2 + V$,
- $\xi^x$ denotes the stochastic parallel transport along the paths of $b^x$ (cf. section 2 below for the precise definition),
- $V^\nabla_x$ denotes the solution of the following pathwise given ordinary differential equation in $\text{End}(\mathcal{E}_x)$,
\[
    (d/dt)V^\nabla_x(t) = -V^\nabla_x(t)/\xi^x(t)^{-1} V(b^x_t)/\xi^x(t), \quad V^\nabla_x(0) = 1.
\]

These facts are well-established (cf. the appendix of [BD01]). Note that a classical assumption on the negative part $V^-$ of $V$ that guarantees that $\nabla^1 \nabla/2 + V$ is semibounded from below and that one has the uniform square-integrability
\[
    \sup_{x \in \mathcal{M}} \mathbb{E} \left[ 1_{\{t < \xi^x\}} |V^\nabla_x(t)|^2 \right] < \infty \quad \text{for all } t > 0
\]
(so that by Cauchy-Schwarz the Feynman-Kac formula holds [G12] for all $f \in \Gamma_{L^2}(\mathcal{M}, \mathcal{E})$) is given by $|V^-| \in \mathcal{K}(\mathcal{M})$, the Kato class of $\mathcal{M}$ (cf. Definition 2.4). Since bounded functions are always Kato, and since it is possible to find large (possibly weighted) $L^p + L^\infty$-type subspaces of $\mathcal{K}(\mathcal{M})$ under very weak assumptions on the geometry of $\mathcal{M}$ (cf. Proposition 2.5), the Kato class becomes very convenient in the context of Feynman-Kac formulae and their applications.

In contrast to the self-adjoint case, very little seems to be known concerning Feynman-Kac formulae in the situation where one replaces the self-adjoint zeroth order operator $V$ by an arbitrary differential operator $Q$ of order $\leq 1$, a situation that naturally leads to a non-self-adjoint theory. The aim of this paper is to provide a systematic treatment of this problem, dealing with all probabilistic and functional analytic problems that arise naturally in this context, mainly from the noncompactness of $\mathcal{M}$. Our essential insight here, which allows to detect the new probabilistic pieces of the Feynman-Kac formula explicitly and which allows to deal with some of the functional analytic problems using perturbation theory, is to decompose $Q$ canonically in the form
\[
    Q = Q^\nabla + \sigma_1(Q) \nabla,
\]
where
\[
    \sigma_1(Q) \in \Gamma_{C^\infty}(\mathcal{M}, \text{Hom}(T^* \mathcal{M} \boxtimes \mathcal{E}, \mathcal{E}))
\]
denotes the first order principal symbol of $Q$, so that $Q^\nabla := Q - \sigma_1(Q)$ is zeroth order. Since now $\nabla^1 \nabla + Q$ will typically not be symmetric in $\Gamma_{L^2}(\mathcal{M}, \mathcal{E})$, we cannot use the Friedrichs construction to get a self-adjoint operator. Instead, we use Kato’s theory of sectorial forms and operators (cf. appendix for the basics of sectorial forms/operators and holomorphic semigroups): to this end, we assume that $\nabla^1 \nabla/2 + Q$ is sectorial. It then follows from abstract results that this operator canonically induces a sectorial operator $H^\nabla_Q$ which generates a semigroup of bounded operators $e^{-zH^\nabla_Q}$ in $\Gamma_{L^2}(\mathcal{M}, \mathcal{E})$ which is holomorphic for $z$ running through some sector of the complex plane which contains $[0, \infty)$. For fixed $x \in \mathcal{M}$ let now $Q^\nabla_x$ denote the solution to the Itô equation
\[
    dQ^\nabla_x(t) = -Q^\nabla_x(t)/\xi^x(t)^{-1} (\sigma_1(Q)^\nabla(db^x_t) + Q^\nabla(b^x_t)dt) /\xi^x(t), \quad Q^\nabla_x(0) = 1,
\]
noting that one can give sense to the underlying Itô differential $\sigma_1(Q)^\nabla(db^x_t)$ using the Levi-Civita connection on $\mathcal{M}$ (cf. Section 2). With these preparations, our main result, Theorem 2.2 below, reads as follows:
Let $\nabla^4 + Q$ be sectorial and let
\begin{equation}
\sup_{x \in K} \mathbb{E} \left[ \mathbb{1}_{\{t < \xi^x\}} |Q^x_\xi(t)|^2 \right] < \infty \quad \text{for all } K \subset \mathcal{M} \text{ compact, } t > 0.
\end{equation}

Then for all $t > 0$, $\Psi \in \Gamma_{L^2}(\mathcal{M}, \mathcal{E})$, $x \in \mathcal{M}$, one has
\begin{equation}
e^{-tH_\Psi^x} \Psi(x) = \mathbb{E} \left[ \mathbb{1}_{\{t < \xi^x\}} \frac{Q^x_\xi(t)}{\xi^x(t)}^{-1}\Psi(b^x_t) \right].\end{equation}

Let us note that the locally uniform $L^2$-assumption \((1.2)\) serves two purposes: firstly, it decouples the validity of the Feynman-Kac formula from $\Psi$ (as in the above self-adjoint Kato situation). Secondly and more importantly, it allows us to conclude that the smooth representative of $e^{-tH_\Psi^x} \Psi$, which exists by local parabolic regularity, is in fact pointwise equal to the right hand side of \((1.3)\), and not only almost everywhere. This is achieved by first proving the formula on relatively compact subsets of $\mathcal{M}$ using Itô-calculus, and then letting these local formulae run through an exhaustion of $\mathcal{M}$, using a recent result for monotone convergence of nondensely defined sectorial forms (this procedure is, up to additional technical difficulties, somewhat analogous to the self-adjoint case) with a parabolic maximum principle for the heat equation (the use of which in this form being new even in the self-adjoint case). To the best of our knowledge, this pointwise identification of the smooth representative is new for stochastically incomplete $\mathcal{M}$’s even in the self-adjoint case.

Making contact with perturbation theory through Kato type assumptions, in Proposition 2.6 we prove:

Assume either
\begin{itemize}
  \item $|\Re(\sigma_1(Q))| \in L^\infty(\mathcal{M})$,
  \item $\Re(Q_\xi) \text{ is bounded from below by a constant } \kappa \in \mathbb{R}$,
  \item $|\Im(Q_\xi)| \in \mathcal{K}(\mathcal{M})$,
\end{itemize}
or
\begin{itemize}
  \item $\sigma_1(Q) \text{ is anti-selfadjoint and } |\sigma_1(Q)| \in L^\infty(\mathcal{M})$,
  \item $|\Re(Q_\xi)| \in \mathcal{K}(\mathcal{M})$,
  \item $|\Im(Q_\xi)| \in \mathcal{K}(\mathcal{M})$.
\end{itemize}

Then $\nabla^4 + Q$ is sectorial, and one has
\begin{equation}
\sup_{x \in \mathcal{M}} \mathbb{E} \left[ \mathbb{1}_{\{t < \xi^x\}} |Q^x_\xi(t)|^2 \right] < \infty \quad \text{for all } t > 0.
\end{equation}

In particular, \((1.3)\) holds true.

Note that above $\Re(A)$ and $\Im(A)$ denote, respectively, the fiberwise defined real part and imaginary part of any zeroth order operator. Since these are self-adjoint zeroth order operators, one can define their positive/negative parts using the spectral calculus fiberwise. Note that, while in the self-adjoint case one can control $|Q^x_\xi(t)|$ pathwise using Gronwall’s inequality, in the situation of Theorem 2.2 and Proposition 2.6 one has to estimate the solution of a covariant Itô-equation, which in combination with the noncompactness of $\mathcal{M}$ leads to several technical difficulties. Although the present formulation of Proposition 2.6 should cover most applications, it would be natural to replace any (lower) boundedness assumption in Proposition 2.6 with an appropriate Kato-type assumption. Although we tried hard, we have not been able to do that. It would also be very interesting to obtain non self-adjoint variants of semigroup domination \cite{BD86, BD01, O99, IS97} (also called ‘Kato-Simon inequality’ in \cite{G17}) using the Feynman-Kac formula in the above setting, keeping in mind that such estimates play a crucial role in geometric analysis (see e.g. \cite{GP13, BG20}) and in mathematical physics (where they are called...
'diamagnetic inequalities' [S77, BHL00]. In the self-adjoint case these estimates take the form
\[ |e^{-tH_\nabla} \Psi(x)| \leq e^{-tH_0} |\Psi|(x), \]
where \( v: \mathcal{M} \to \mathbb{R} \) is any scalar potential such that for all \( x \in \mathcal{M} \) every eigenvalue of \( V(x) \) is \( \geq v(x) \).

It should also be noted that, if one ignores functional analytic problems that arise for example from the noncompactness of \( \mathcal{M} \), it is somewhat natural that some probabilistic representation of \( e^{-tH_\nabla} \) must exist: as \( \nabla \) is metric, the operator \( \nabla^t \nabla + Q \) equals \(-\text{tr} \nabla^2 + Q\), and (see appendix, Section B), the latter nondivergence form operator can be canonically rewritten in the nondivergence form \(-\text{tr} \tilde{\nabla}^2 + \tilde{Q}\), where \( \tilde{\nabla} \) is another connection and \( \tilde{Q} \) is of zeroth order (keeping in mind that at least \( \tilde{Q} \) is somewhat implicitly given; see also Proposition 2.5 in [BGV92]). For compact \( \mathcal{M} \)’s no particular analytic problems arise, and the Feynman-Kac formula for \(-\text{tr} \tilde{\nabla}^2 + \tilde{Q} \) is formally of the type \((1.1)\), as shown in [N02] (section 8 therein). On the other hand, in our noncompact setting, the divergence form \( \nabla^t \nabla + Q \) is favourable from an analytic point of view, and (see again appendix, Section B) in this case it is in general not possible to rewrite this operator in the divergence form \( \tilde{\nabla}^t \tilde{\nabla} + \tilde{Q} \), with \( \tilde{Q} \) zeroth order. From this point of view, we believe that our formulation of the Feynman-Kac formula is optimal in the noncompact case from an analytic point of view. Moreover, our formula has even some advantages in some applications to compact \( \mathcal{M} \)’s, where the generator appears precisely in the form \( \nabla^t \nabla + \sigma_1(Q) \nabla + Q_\nabla \) (see below).

Our next main result is the following trace formula (cf. Theorem 2.3):

**Assume \( \mathcal{M} \) is compact, and let \( P \) be of order \( \leq 1 \), and let \( V, \tilde{V} \) be of zeroth order. Then for all \( t > 0 \) one has**

\[ (1.5) \]

\[
\text{Tr} \left( \tilde{V} \int_0^t e^{-sH_\nabla} P e^{-(t-s)H_\nabla} \, ds \right) = -\int \exp(-tH)(x, y) \text{Tr}_x \left( \tilde{V}(x) \mathbb{E}_t^{x,x} \left[ \mathcal{V}_\nabla^{x,y}(t) \int_0^t \mathcal{V}_\nabla^{y,y}(s) \sigma_1(P)_y^y (db_x^s) + P_{\nabla}(b_x^s) ds / \mathcal{V}_\nabla^{x,y}(t) \right] \right) \, d\mu(x),
\]

where \( \exp(-tH)(x, y) \) denotes the integral kernel of the Friedrichs realization of \( \Delta \) (in other words, the heat kernel on \( \mathcal{M} \)), and \( \mathbb{E}_t^{x,x} \) denotes the expectation with respect to the Brownian bridge starting in \( x \) and ending in \( x \) at the time \( t \).

The proof of this result is in fact reduced to \((1.3)\) using Berezin integration, a trick which has been communicated to the authors by Shu Shen. It would be very interesting to see, if at least for certain \( P \)'s it is possible to obtain \((1.5)\) using the very general Bismut derivative formulae from [BD01] in combination with the Markov property of Brownian motion. We have not worked into this direction.

Finally, we use \((1.5)\) together with a new commutation formula for spin-Dirac operators (cf. formula \((3.4)\) below) to establish a probabilistic formula for the 'first order' part of the equivariant Chern-Character \( \text{Ch}_T(\mathcal{M}) \) of a compact even-dimensional Riemannian spin manifold \( \mathcal{M} \), where \( T := S^1 \). We refer the reader to Section B for the definition of \( \text{Ch}_T(\mathcal{M}) \) and concentrate here only the probabilistic side of the formula: to this end, note that every element \( \alpha \) of the space \( \Omega_T(\mathcal{M}) \) of \( T \)-invariant differential forms on \( \mathcal{M} \times T \) can be uniquely written in the form \( \alpha = \alpha'e^\alpha dt \) with \( dt \) the volume form on \( T \). Then \( \text{Ch}_T(\mathcal{M}) \) becomes a complex linear
functional on the space

\[ C_T(\mathcal{M}) := \bigoplus_{N=0}^{\infty} \Omega_T(\mathcal{M}) \otimes (\Omega_T(\mathcal{M}) \otimes^N / (\mathbb{C} \cdot 1)). \]

In Theorem 3.1 we prove:

For all \( \alpha_0, \alpha_1 \in \Omega_T(\mathcal{M}), t > 0 \) one has

\[
\text{Ch}_T(\mathcal{M})(\alpha_0 \otimes \alpha_1) = \int_{\mathcal{M}} e^{-tH}(x,x) \text{Str}_x \left( c(\alpha'_0)(x) \overline{\mathbb{E}}_{p'}^{x,x} \left[ e^{-/(1/8) \int_{\mathbb{R}} \text{scal}(\overline{\mathcal{E}}_s^x) ds} \int_0^t f\left( \frac{\overline{\mathcal{E}}_s^x}{\overline{\mathcal{E}}_t^x} \right)^{-1} \left( 2c(*\overline{\mathcal{E}}_s^x, \alpha'_1) - c(\alpha'_1)(\overline{\mathcal{E}}_s^x) ds \right) f\left( \frac{\overline{\mathcal{E}}_s^x}{\overline{\mathcal{E}}_t^x} \right)^{-1} \right] \right) \right) \left. \right|_{t=2} d\mu(x),
\]

where

- \( \text{Str}_x \) denotes the \( \mathbb{Z}_2 \)-graded trace on \( \text{End}(\mathcal{I}_x) \), with \( \mathcal{I} \to \mathcal{M} \) the spin bundle,
- \( \overline{\mathbb{E}}_{p'}^{x,x} \) denotes the stochastic parallel transport \( \mathcal{I} \to \mathcal{M} \),
- \( c : \Omega_{C^\infty}(\mathcal{M}) \to \Gamma_{C^\infty}(\mathcal{M}, \text{End}(\mathcal{I})) \) denotes Clifford multiplication,
- \( c(*\overline{\mathcal{E}}_s^x, \alpha) \) denotes a Stratonovic differential with respect to the \( \text{End}(\mathcal{I}) \)-valued 1-form \( v \to c(v, \alpha) \),
- \( \mathbb{E}_{p'}^{x,x} \) denotes the expectation with respect to the Brownian bridge starting \( x \) and ending at the time \( t \) in \( x \).

We remark that \( \text{Ch}_T(\mathcal{M}) \) has been introduced in [GL19] in the abstract setting of \( \partial \)-summable Fredholm modules over locally convex differential graded algebras and is in fact a differential-graded refinement of the JLO-cocycle [JLO88] for ungraded algebras. When applied to a compact even dimensional Riemannian spin-manifold, this construction provides via Chen integrals an algebraic model for Duistermaat-Heckman localization on the space of smooth loops, allowing a proof of the Atiyah-Singer index theorem for twisted spin-Dirac operators in the spirit of Atiyah [AS83] and Bismut [BS85]. We refer the reader to the introduction of [GL19] for a detailed explanation of these results. Obtaining a probabilistic formula for the higher order pieces of the equivariant Chern character remains an open problem at this point.

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2. Main results

Let \( \mathcal{M} \) be a connected Riemannian manifold of dimension \( m \), where we work exclusively in the category of smooth manifolds without boundary. As such it is equipped with its Levi-Civita connection and its volume measure \( \mu \). We denote the open geodesic balls with \( B(x, r) \subset \mathcal{M} \).

Any fiberwise metric on a vector bundle will simply be denoted with \( (\bullet, \bullet) \), with \( |\bullet| := \sqrt{(\bullet, \bullet)} \). If \( \mathcal{E} \to \mathcal{M} \) is a metric vector bundle and \( p \in [1, \infty] \), then the norm on the complex Banach space of \( L^p \)-sections is denoted with

\[
\|\Psi\|_p := \left( \int |\Psi|^p d\mu \right)^{1/p}.
\]

(with the obvious replacement for \( p = \infty \)). The scalar product in the Hilbert space \( \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \) is denoted by

\[
\langle \Psi_1, \Psi_2 \rangle = \int (\Psi_1, \Psi_2) d\mu.
\]

Given another metric vector bundle \( \mathcal{F} \to \mathcal{M} \) and a differential operator

\[
P : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \longrightarrow \Gamma_{C^\infty}(\mathcal{M}, \mathcal{F})
\]
of order \( \leq k \) with smooth coefficients, its formal adjoint
\[
P^\dagger : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \longrightarrow \Gamma_{C^\infty}(\mathcal{M}, \mathcal{F})
\]
is the uniquely determined differential operator of order \( \leq k \) with smooth coefficients, which satisfies
\[
\langle P\Psi_1, \Psi_2 \rangle = \langle \Psi_1, P^\dagger \Psi_2 \rangle \quad \text{for all} \quad \Psi_1 \in \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}), \Psi_2 \in \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}).
\]

Assume from now on that \( \mathcal{E} \rightarrow \mathcal{M} \) is a metric vector bundle with a smooth metric connection
\[
\nabla : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \longrightarrow \Gamma_{C^\infty}(\mathcal{M}, T^*\mathcal{M} \otimes \mathcal{E})
\]
Given a differential operator
\[
Q : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \longrightarrow \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E})
\]
of order \( \leq 1 \), then with its first order principal symbol
\[
\sigma_1(Q) \in \Gamma_{C^\infty}(\mathcal{M}, \text{Hom}(T^*\mathcal{M}, \text{End}(\mathcal{E}))) = \Gamma_{C^\infty}(\mathcal{M}, \text{Hom}(T^*\mathcal{M} \otimes \mathcal{E}, \mathcal{E})),
\]
the operator
\[
Q_{\nabla} := Q - \sigma_1(Q)\nabla \text{ is zeroth order, thus}
\]
\[
Q_{\nabla} \in \Gamma_{C^\infty}(\mathcal{M}, \text{End}(\mathcal{E})), \quad Q = Q_{\nabla} + \sigma_1(Q)\nabla.
\]
Assume that for every \( x \in \mathcal{M} \) we are given a maximally defined Brownian motion
\[
b^x : [0, \zeta^x) \times \Omega \longrightarrow \mathcal{M}
\]
on \( \mathcal{M} \) with starting point \( x \) and explosion time \( \zeta^x > 0 \), which is defined on a fixed filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) that satisfies the usual assumptions. Let
\[
\|\xi_t^x\| : [0, \zeta^x) \times \Omega \longrightarrow \mathcal{E} \otimes \mathcal{E}^\dagger
\]
be the corresponding stochastic parallel transport with respect to the fixed metric connection, where \( \mathcal{E} \otimes \mathcal{E}^\dagger \rightarrow \mathcal{M} \times \mathcal{M} \) denotes the vector bundle whose fiber at \((a, b)\) is \( \text{Hom}(\mathcal{E}_a, \mathcal{E}_b) \). This is the uniquely determined continuous semimartingale such that \( \mathbb{N}^2 \) for all \( t \in [0, \zeta^x) \),

- one has \( \|\xi_t^x\| : \mathcal{E}_x \rightarrow \mathcal{E}_{b^x(t)} \) unitarily,
- for all \( \Psi \in \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \) one has
\[
\|\xi_t^x\|^{-1}\Psi(b^x_t) = \|\xi_t^x\|^{-1}\nabla(*d\xi^x_t)\Psi(b^x_t), \quad \|\xi_t^x\|(0) = 1.
\]

Above and in the sequel, \(*d\) stands for Stratonovic integration, while \(d\) will denote Itô integration. Note that one can integrate 1-forms in the Stratonovic sense on any manifold along any continuous semimartingale, while one can integrate 1-forms on \( \mathcal{M} \) along \( b^x \) also in the Itô sense, using the Levi-Civita connection on \( \mathcal{M} \).

Define the process
\[
Q_{\nabla}^x : [0, \zeta^x) \times \Omega \longrightarrow \text{End}(\mathcal{E}_x)
\]
as the unique solution to the Itô equation
\[
dQ_{\nabla}^x(t) = -Q_{\nabla}^x(t)\|\xi_t^x\|^{-1}(\sigma_1(Q)^\dagger(d\xi^x_t) + Q_{\nabla}(b^x_t)d\xi^x_t), \quad Q_{\nabla}^x(0) = 1.
\]
Written out explicitly, the above equation means that for all \( t \geq 0 \) one has
\[
Q_{\nabla}^x(t) = 1 - \int_0^t Q_{\nabla}^x(s)\|\xi_s^x\|^{-1}\sigma_1(Q)^\dagger(U_s^x e_j)\|\xi_s^x\|\nabla(s)dW_s^x j - \int_0^t \|\xi_s^x\|^{-1}Q_{\nabla}(b^x_s)\|\xi_s^x\|ds,
\]
a.s. on \( \{ t < \zeta^x \} \), where \( e_1, \ldots, e_m \) is the standard basis of \( \mathbb{R}^m \),
\[
U^x : [0, \zeta^x) \times \Omega \longrightarrow O(\mathcal{M}) = \bigcup_{x \in \mathcal{M}} O(\mathbb{R}^m, T_x\mathcal{M})
\]
is a horizontal lift of $\mathfrak{b}^x$ with respect to the Levi-Civita connection on $\mathcal{M}$ to the principal fiber bundle of orthonormal frames $O(\mathcal{M}) \to \mathcal{M}$, and

$$W^x := \int_0^\bullet \omega(\ast dU^x_\tau) : [0, \zeta^x) \times \Omega \longrightarrow \mathbb{R}^m$$

is the $\mathbb{R}^m$-representation of $\mathfrak{b}^x$ (in particular, $W^x$ is a Euclidean Brownian motion), with

$$\omega \in \Omega^{1,\infty}_C(O(\mathcal{M}), \mathbb{R}^m), \quad \omega_u(A) := u^{-1}(T\pi(A_u)), \quad A_u \in T_u O(\mathcal{M}), \quad u \in O(\mathcal{M}),$$

the solder 1-form of $\pi : O(\mathcal{M}) \to \mathcal{M}$. These constructions do not depend on the initial value $U^x_0 \in O(\mathbb{R}^m, T_x \mathcal{M})$.

It is often useful to know for estimates that the processes of the form $\mathcal{Q}_{\nu}^x$ factor as follows:

**Remark 2.1.** a) Let $\alpha \in \Omega^{1,\infty}_C(\mathcal{M}, \text{End}(\mathcal{E}))$, $V, W \in \Gamma^{\infty}_C(\mathcal{M}, \text{End}(\mathcal{E}))$ and let

$$C : [0, \zeta^x) \times \Omega \longrightarrow \text{End}(\mathcal{E}_x)$$

be the solution to

$$dC(t) = -C(t)\| \nu \|_t^{-1}(V(b^x_t) + \alpha(db^x_t) + W(b^x_t)dt)\| \nu \|_t^{-1}, \quad C(0) = 1.$$ 

Such a $C$ factors as follows: let

$$A : [0, \zeta^x) \times \Omega \longrightarrow \text{End}(\mathcal{E}_x)$$

be the solution to

$$dA(t) = -A(t)\| \nu \|_t^{-1}(\alpha(db^x_t) + W(b^x_t)dt)\| \nu \|_t^{-1}, \quad A(0) = 1.$$ 

Then $A$ is invertible and

$$A^{-1} : [0, \zeta^x) \times \Omega \longrightarrow \text{End}(\mathcal{E}_x)$$

is the solution to

$$dA(t)^{-1} = \| \nu \|_t^{-1}(\alpha(db^x_t) + W(b^x_t)dt)\| \nu \|_t^{-1}A(t)^{-1}, \quad A(0)^{-1} = 1.$$ 

Let $B$ be the solution to

$$dB(t) = -B(t)A(t)\| \nu \|_t^{-1}V(b^x_t)\| \nu \|_t^{-1}A(t)^{-1}dt, \quad B(0) = 1.$$ 

Then by the Itô product rule we have

$$d(B(t)A(t)) = (dB(t)A(t) + B(t)dA(t) + dB(t)dA(t)$$

$$= -B(t)A(t)\| \nu \|_t^{-1}V(b^x_t)\| \nu \|_t^{-1}A(t)^{-1}dtA(t)$$

$$- B(t)A(t)\| \nu \|_t^{-1}(\alpha(db^x_t) + W(b^x_t)dt)\| \nu \|_t^{-1}$$

$$+ \text{summands containing } dt \text{ and } db^x_t, \text{ or } dt \text{ and } dt,$$

so that by uniqueness $C = AB$.

b) As a particular case of the above situation, Let

$$\mathcal{Q}_{1, \nu}^x : [0, \zeta^x) \times \Omega \longrightarrow \text{End}(\mathcal{E}_x)$$

be the solution to

$$d\mathcal{Q}_{1, \nu}^x(t) = -\mathcal{Q}_{1, \nu}^x(t)\| \nu \|_t^{-1} \sigma_1(Q)^3(db^x_t)\| \nu \|_t^{-1}, \quad \mathcal{Q}_{1, \nu}^x(0) = 1,$$

and let $\mathcal{Q}_{2, \nu}^x$ be the solution to

$$d\mathcal{Q}_{2, \nu}^x(t) = -\mathcal{Q}_{2, \nu}^x(t)\mathcal{Q}_{1, \nu}^x(t)\| \nu \|_t^{-1}Q_{\nu}(b^x_t)\| \nu \|_t^{-1}Q_{1, \nu}^x(t)^{-1}dt, \quad \mathcal{Q}_{2, \nu}^x(t) = 1.$$ 

Then we have

$$\mathcal{Q}_{\nu}^x(t) = \mathcal{Q}_{2, \nu}^x(t)\mathcal{Q}_{1, \nu}^x(t).$$
Any differential operator
\[ Q : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \rightarrow \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \]
induces a densely defined sesqui-linear form
\[ \langle \nabla^\dagger \nabla/2 + Q \rangle \psi_1, \psi_2 \rangle \in \mathbb{C} \]
in \( \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \). In case this form is sectorial it is automatically closable (stemming from a sectorial operator), and we denote the closed operator in \( \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \) induced by the closure of \( h_Q^\nabla \) with \( H_Q^\nabla \) in the sense of Theorem \( \text{A.2} \) from the appendix. It follows that \( H_Q^\nabla \) generates a holomorphic semigroup (cf. appendix)
\[ \left( e^{-zh_Q^\nabla} \right)_{z \in \Sigma_{\alpha,\beta}} \subset \mathcal{L}(\Gamma_{L^2}(\mathcal{M}, \mathcal{E})) , \]
which is defined on some sector of the form
\[ \Sigma_{\alpha,\beta} = \{ r e^{\sqrt{-1} \alpha} : r > 0, \alpha \in (-\beta, \beta) \} \quad \text{for some } \beta \in (0, \pi/2] . \]

In the situation of a trivial complex line bundle with its trivial connection (identifying sections with functions) we will omit the dependence on the connection in the notation. In particular, \( H \geq 0 \) stands for the Friedrichs realization of the scalar Laplace-Beltrami operator \( \Delta/2 \) in \( L^2(\mathcal{M}) \).

**Theorem 2.2.** Let
\[ Q : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \rightarrow \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \]
be a differential operator of order \( \leq 1 \). Assume that \( h_Q^\nabla \) is sectorial and that
\[ \sup_{x \in K} \mathbb{E} \left[ 1_{\{ t < \xi^x \}} |Q_{\mathcal{V}}^x(t)|^2 \right] < \infty \quad \text{for all } K \subset \mathcal{M} \text{ compact}, t > 0 . \]
Then for all \( t > 0, \Psi \in \Gamma_{L^2}(\mathcal{M}, \mathcal{E}), x \in \mathcal{M}, \) one has
\[ e^{-tH_Q^\nabla} \psi(x) = \mathbb{E} \left[ 1_{\{ t < \xi^x \}} Q_{\mathcal{V}}^x(t)/\xi^x(t)^{-1} \psi(b_y^x) \right] . \]

**Remark 2.3.** By local parabolic regularity, the time dependent section \( (t, x) \mapsto e^{-tH_Q^\nabla} \psi(x) \)
has a representative which is smooth on \( (0, \infty) \times \mathcal{M}, \) and \( (2.5) \) means that the RHS of this equation is precisely this smooth representative. This pointwise identification, which is based on the locally uniform integrability assumption \( (2.4) \), is highly nontrivial in the stochastically incomplete case and even slightly improves the existing results in the 'usual' Feynman-Kac setting \( (\sigma_1(Q) = 0 \text{ and } Q_{\mathcal{V}} \text{ self-adjoint}) \), where so far only an \( \mu \)-almost everywhere equality has been established.

**Proof of Theorem 2.2.** We omit the dependence on \( \nabla \) of several data in the notation, whenever there is no danger of confusion. Fix \( x \in \mathcal{M}, t > 0 \) and pick an exhaustion \( (U_l)_{l \in \mathbb{N}} \) of \( \mathcal{M} \) with open connected relatively compact subsets having a smooth boundary. Let \( H_{Q,l} \) be defined with \( \mathcal{M} \) replaced by \( U_l \) (note that this corresponds to Dirichlet boundary conditions). It suffices to show that (with an obvious notation) for all \( \Psi \in \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \) and all \( l \) large enough such that \( \Psi \) is supported in \( U_l \) one has
\[ e^{-tH_{Q,l}} \psi(x) = \mathbb{E} \left[ 1_{\{ t < \xi^x \}} Q_{\mathcal{V}}^x(t)/\xi^x(t)^{-1} \psi(b_y^x) \right] . \]
Indeed, we have
\[ \lim_{t \rightarrow \infty} \left\| e^{-tH_{Q,l}} \psi - e^{-tH_Q} \psi \right\|_2 = 0 . \]
by an abstract monotone convergence theorem for nondensely defined sectorial forms (Theorem 3.7 in [CtE18]), and furthermore for every compact set $K \subset \mathcal{M}$ with $x \in K$ we have
\[
\sup_{y \in K} \left| \mathbb{E} \left[ (1_{t < \zeta^y} - 1_{t < \zeta^y}) \mathcal{Q}^y(t)/\mathcal{Q}^y(t)^{-1} \Psi(b^y) \right] \right| \\
\leq \sup_{y \in K} \| \Psi \|_{\infty} \mathbb{E} \left[ (1_{t < \zeta^y} - 1_{t < \zeta^y}) \right]^{1/2} \mathbb{E} \left[ (1_{t < \zeta^y}) \mathcal{Q}^y(t)^2 \right]^{1/2} \\
\leq \sup_{y \in K} \mathbb{E} \left[ (1_{t < \zeta^y}) \mathcal{Q}^y(t)^2 \right]^{1/2} \| \Psi \|_{\infty} (e^{-tH}1(y) - e^{-tH}1(y))^{1/2}.
\]

The latter expression converges to zero as $l \to \infty$ by a maximum principle for the heat equation of Dodziuk [D83], which shows that the RHS of (2.5) is continuous in $x$, and that in view of (2.7) one has (2.5) for $\mu$-a.e $x \in \mathcal{M}$. A posteriori this equality holds for all $x$, as both sides are continuous in $x$. If $\Psi$ is only square integrable, we can pick a sequence of smooth compactly supported sections $(\Psi_n)_{n \in \mathbb{N}}$ with $\|\Psi_n - \Psi\|_2 \to 0$. Given an open relatively compact subset $U \subset \mathcal{M}$ with $x \in U$, we have
\[
e^{-tH} : \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \to \Gamma_{C_b}(U, \mathcal{E})
\]
algebraically by elliptic regularity (where $\Gamma_{C_b}(U, \mathcal{E})$ denotes the Banach space of continuous bounded sections of $\mathcal{E}|_U \to U$ equipped with the uniform norm), and a posteriori continuously by the closed graph theorem, we then have
\[
\lim_{n \to \infty} e^{-tH} \Psi_n(x) = e^{-tH} \Psi(x),
\]
and
\[
\mathbb{E} \left[ (1_{t < \zeta^y}) \mathcal{Q}^y(t)^2 (\Psi_n(b^y) - \Psi(b^y)) \right] \\
\leq \mathbb{E} \left[ (1_{t < \zeta^y}) \mathcal{Q}^y(t)^2 \right]^{1/2} \mathbb{E} \left[ (1_{t < \zeta^y}) \left| \Psi_n(b^y) - \Psi(b^y) \right|^2 \right]^{1/2} \\
= \mathbb{E} \left[ (1_{t < \zeta^y}) \mathcal{Q}^y(t)^2 \right]^{1/2} \left( \int e^{-tH}(x,y) |\Psi_n(y) - \Psi(y)|^2 d\mu(y) \right)^{1/2} \\
\leq \mathbb{E} \left[ (1_{t < \zeta^y}) \mathcal{Q}^y(t)^2 \right]^{1/2} \left( \sup_{y \in \mathcal{M}} e^{-tH}(x,y) \right)^{1/2} \|\Psi_n - \Psi\|_2,
\]
which tends to 0 as $n \to \infty$ and proves (2.5) again.

It remains to show (2.6): By parabolic regularity, the time dependent section
\[
\Psi_s(y) := e^{-(t-s)H} \Psi(y)
\]
of $\mathcal{E}|_{U_t} \to U_t$ extends smoothly to $[0, t] \times U_t$ and $\Psi_s$ vanishes in $\partial U_t$ for all $s \in [0, t)$. Define a continuous semimartingale by
\[
N : [0, t \wedge \zeta^x] \times \Omega \to \mathcal{E}_x, \quad N_s := \mathcal{Q}(s)/\mathcal{Q}(s)^{-1} \Psi_s(b^x).
\]
Then we have
\[
\begin{aligned}
dN_s &= (dQ^x(s))/\|x\|^2(s)^{-1}\Psi_s(b_s^x) + Q^x(s)d/\|x\|^2(s)^{-1}\Psi_s(b_s^x) + dQ^x(s)d/\|x\|^2(s)^{-1}\Psi_s(b_s^x) \\
&= -Q^x(s)/\|x\|^2(s)^{-1}(\sigma_1(Q)^{\hat{\Psi}}(db_s^x)) + Q\nabla(b_s^x)ds\Psi_s(b_s^x) \\
&+ Q^x(s)/\|x\|^2(s)^{-1}\nabla\Psi_s(*db_s^x)(b_s^x) + /\|x\|^2(s)^{-1}\partial_s\Psi_s(b_s^x)ds \\
&- Q^x(s)/\|x\|^2(s)^{-1}(\sigma_1(Q)^{\hat{\Psi}}(db_s^x)) + Q\nabla(b_s^x)ds/\|x\| \\
&\times (\|x\|^2(s)^{-1}\nabla\Psi_s(*db_s^x)(b_s^x) + /\|x\|^2(s)^{-1}\partial_s\Psi_s(b_s^x)ds) \\
\equiv & -Q^x(s)/\|x\|^2(s)^{-1}(Q\nabla(b_s^x)ds\Psi_s(b_s^x)) \\
&+ Q^x(s)/\|x\|^2(s)^{-1}\nabla\Psi_s(*db_s^x)(b_s^x) - 1/2\|x\|^2(s)^{-1}\nabla^1\nabla\Psi_s(b_s^x)ds + /\|x\|^2(s)^{-1}\partial_s\Psi_s(b_s^x)ds \\
&- Q^x(s)/\|x\|^2(s)^{-1}(\sigma_1(Q)^{\hat{\Psi}}(db_s^x)) + Q\nabla(b_s^x)ds/\|x\| \\
&\times (\|x\|^2(s)^{-1}\nabla\Psi_s(*db_s^x)(b_s^x) + 1/2\|x\|^2(s)^{-1}\nabla^1\nabla\Psi_s(b_s^x)ds + /\|x\|^2(s)^{-1}\partial_s\Psi_s(b_s^x)ds) \\
\equiv & -Q^x(s)/\|x\|^2(s)^{-1}Q\nabla(b_s^x)ds\Psi_s(b_s^x) + Q^x(s)\left(-1/2\|x\|^2(s)^{-1}\nabla^1\nabla\Psi_s(b_s^x)ds + /\|x\|^2(s)^{-1}\partial_s\Psi_s(b_s^x)ds\right) \\
&- Q^x(s)/\|x\|^2(s)^{-1}(\sigma_1(Q)^{\hat{\Psi}}(db_s^x)) + Q\nabla(b_s^x)ds/\|x\| \\
&\times (\|x\|^2(s)^{-1}\nabla\Psi_s(*db_s^x)(b_s^x) + 1/2\|x\|^2(s)^{-1}\nabla^1\nabla\Psi_s(b_s^x)ds + /\|x\|^2(s)^{-1}\partial_s\Psi_s(b_s^x)ds) \\
&= -Q^x(s)/\|x\|^2(s)^{-1}Q\nabla(b_s^x)ds\Psi_s(b_s^x) + Q^x(s)\left(-1/2\|x\|^2(s)^{-1}\nabla^1\nabla\Psi_s(b_s^x) + /\|x\|^2(s)^{-1}\partial_s\Psi_s(b_s^x)ds\right) \\
&- Q^x(s)/\|x\|^2(s)^{-1}\sigma_1(Q)^{\hat{\Psi}}(db_s^x)\nabla\Psi_s(*db_s^x)(b_s^x) \\
&= -Q^x(s)/\|x\|^2(s)^{-1}Q\nabla(b_s^x)ds\Psi_s(b_s^x) + Q^x(s)\left(-1/2\|x\|^2(s)^{-1}\nabla^1\nabla\Psi_s(b_s^x) + /\|x\|^2(s)^{-1}\partial_s\Psi_s(b_s^x)ds\right) \\
&- Q^x(s)/\|x\|^2(s)^{-1}\sigma_1(Q)^{\hat{\Psi}}(db_s^x)\nabla\Psi_s(*db_s^x)(b_s^x) \\
&= 0,
\end{aligned}
\]
where \(\equiv\) stands for equality up to continuous local martingales. In the above calculation, we have used the Itô product rule, the differential equation for \(Q^x\), the formula
\[
d/\|x\|^2(s)^{-1}\Psi_s(b_s^x) = /\|x\|^2(s)^{-1}\nabla\Psi_s(*db_s^x)(b_s^x) + /\|x\|^2(s)^{-1}\partial_s\Psi_s(b_s^x),
\]
which follows from applying (2.11) to the metric connection \(\pi^*\nabla\) on the metric vector bundle \(\pi^*E \to \mathcal{M} \times [0, \infty)\) with the projection \(\pi : \mathcal{M} \times [0, \infty) \to \mathcal{M}\), the covariant Stratonvic-to-Itô formula
\[
/\|x\|^2(s)^{-1}\nabla\Psi_s(*db_s^x)(b_s^x) = /\|x\|^2(s)^{-1}\nabla\Psi_s(*db_s^x)(b_s^x) + 1/2/\|x\|^2(s)^{-1}\nabla^1\nabla\Psi_s(b_s^x)ds,
\]
and
\[
\partial_s\Psi_s = ((1/2)\nabla^1\nabla + \sigma_1(Q)\nabla + Q\nabla)\Psi_s.
\]
This shows that \(N\) is a continuous local martingale. Since \(U_1\) is relatively compact, \(N\) is in fact a martingale: indeed, a.s., for all \(s > 0\) we have in \(\{s < \zeta^2\}\) from the differential equation for \(Q^x\) and Jensen’s inequality
\[
|Q^x(s)|^2 \leq C + C\left|\int_0^s Q^x(r)/\|x\|^2(r)^{-1}\sigma_1(Q)^{\hat{\Psi}}(db_s^x)/\|x\|^2(r)\right|^2 + Cs\int_0^s |Q^x(r)|^2|Q(b_s^x)|^2dr,
\]
so that by the Burkholder-Davis-Gundy inequality, with
\[
\vartheta_n := \inf\{r \geq 0 : |Q^x(r)| > n\}, \quad n \in \mathbb{N},
\]
one has
\[
\mathbb{E}\left[ \sup_{s \leq t \wedge \zeta_t^x} |Q^x(s \wedge \vartheta_n)|^2 \right] \\
\leq C' + C' \mathbb{E} \left[ \int_0^{t \wedge \zeta_t^x} |Q^x(r \wedge \vartheta_n)|^2 \sigma_1(Q)^2(b_r^x)^2 \, dr \right] + t \mathbb{E} \left[ \int_0^{t \wedge \zeta_t^x} |Q^x(r \wedge \vartheta_n)|^2 |Q \sigma(b_r^x)|^2 \, dr \right] \\
\leq C' + C' \left( \sup_{y \in U_l} |\sigma_1(Q)(y)|^2 \right) \mathbb{E} \left[ \int_0^{t \wedge \zeta_t^x} |Q^x(r \wedge \vartheta_n)|^2 \, dr \right] \\
+ t \left( \sup_{y \in U_l} |Q \sigma(y)|^2 \right) \mathbb{E} \left[ \int_0^{t \wedge \zeta_t^x} |Q^x(r \wedge \vartheta_n)|^2 \, dr \right] \\
\leq C_{Q,l} + (C_{Q,l} + tC_{Q,l}) \mathbb{E} \left[ \int_0^{t \wedge \zeta_t^x} |Q^x(r \wedge \vartheta_n)|^2 \, dr \right] \\
\leq C_{Q,l} + (C_{Q,l} + tC_{Q,l}) \mathbb{E} \left[ \int_0^{t \wedge \zeta_t^x} \sup_{s \leq t \wedge \zeta_t^x} |Q^x(s \wedge \vartheta_n)|^2 \, ds \right],
\]
where $C, C'$ are universal constants, and $C_{Q,l}$ depends only on $\|Q \sigma\|_\infty$ and $\|\sigma_1(Q)\|_\infty$. As a consequence, for all $T > 0$ with $t \leq T$, Gronwall’s inequality gives
\[
\mathbb{E}\left[ \sup_{s \leq t \wedge \zeta_t^x} |Q^x(s \wedge \vartheta_n)|^2 \right] \leq C_{Q,l} e^{C_{Q,l,T} t},
\]
where $C_{Q,l,T}$ only depends on $Q, l, T$, and so
\[
(2.8) \quad \mathbb{E}\left[ \sup_{s \leq t \wedge \zeta_t^x} |Q^x(s)|^2 \right] = \mathbb{E}\left[ \max_{s \leq t \wedge \zeta_t^x} |Q^x(s)|^2 \right] = \mathbb{E}\left[ \lim_{n} \max_{s \leq t \wedge \zeta_t^x} |Q^x(s \wedge \vartheta_n)|^2 \right] \\
\leq \lim_{n} \inf \mathbb{E}\left[ \sup_{s \leq t \wedge \zeta_t^x} |Q^x(s \wedge \vartheta_n)|^2 \right] \leq C_{Q,l} e^{C_{Q,l,T} t} < \infty
\]
by Fatou’s lemma. We arrive at
\[
\mathbb{E}\left[ \sup_{s \leq t \wedge \zeta_t^x} |N_s|^2 \right] \leq \left( \sup_{s \in [0,t], y \in U_l} |\Psi(y)|^2 \right) \mathbb{E}\left[ \sup_{s \leq t \wedge \zeta_t^x} |Q^x(s)|^2 \right] < \infty,
\]
so that
\[
\mathbb{E}\left[ \sup_{s \leq t \wedge \zeta_t^x} |N_s| \right] \leq \mathbb{E}\left[ \sup_{s \leq t \wedge \zeta_t^x} |N_s|^2 \right]^{1/2} < \infty.
\]
which shows that $N$ is a martingale, as claimed.
We thus have
\[
e^{-tH \varphi} \Psi(x) = \mathbb{E}[N_0] = \mathbb{E}[N_{t \wedge \zeta^x_t}]
\]
\[
= \mathbb{E} \left[ Q^x(t \wedge \zeta^x_t)\|^2 (t \wedge \zeta^x_t)^{-1} \Psi_{t \wedge \zeta^x_t} (b^x_{t \wedge \zeta^x_t}) \right]
\]
\[
= \mathbb{E} \left[ (1_{t < \zeta^x_t} + 1_{t \geq \zeta^x_t}) Q^x(t \wedge \zeta^x_t)\|^2 (t \wedge \zeta^x_t)^{-1} \Psi_{t \wedge \zeta^x_t} (b^x_{t \wedge \zeta^x_t}) \right]
\]
\[
= \mathbb{E} \left[ 1_{t < \zeta^x_t} Q^x(t)\|^2 (t)^{-1} \Psi_{t \wedge \zeta^x_t} (b^x_{t \wedge \zeta^x_t}) + \mathbb{E} \left[ 1_{t \geq \zeta^x_t} Q^x(\zeta^x_t)\|^2 (\zeta^x_t)^{-1} \Psi_{t \wedge \zeta^x_t} (b^x_{t \wedge \zeta^x_t}) \right] \right].
\]

This completes the proof. 

In order to evaluate the somewhat abstract assumptions from Theorem 2.2 we recall the definition of the Kato class (referring the reader to [G17 SV96 AS S2 GI12 S93] and the references therein for some fundamental results concerning this class):

**Definition 2.4.** A Borel function $w : \mathcal{M} \to \mathbb{R}$ is said to be in the Kato class $\mathcal{K}(\mathcal{M})$ of $\mathcal{M}$, if

\[
\lim_{t \to 0^+} \sup_{x \in \mathcal{M}} \int_0^t \mathbb{E} \left[ 1_{t < \zeta^x_t} |w(b^x_{t \wedge \zeta^x_t})| \right] ds = 0.
\]

By Khashminskii’s lemma [G17], $w \in \mathcal{K}(\mathcal{M})$ implies

\[
\sup_{x \in \mathcal{M}} \mathbb{E} \left[ 1_{t < \zeta^x_t} |w(b^x_{t \wedge \zeta^x_t})| \right] < \infty \quad \text{for all } t > 0, \ p \in [1, \infty).
\]

One trivially always has $L^\infty(\mathcal{M}) \subset \mathcal{K}(\mathcal{M})$, and under a mild control on the geometry one has $L^p + L^\infty$-type subspaces of the Kato class. For example, one has (cf. Chapter VI in [G17] and the appendix of [BrG]):

**Proposition 2.5.** a) Assume there exists a Borel function $\theta : \mathcal{M} \to (0, \infty)$ with

\[
\sup_{x \in \mathcal{M}} e^{-tH} (x, y) \leq \theta(y) t^{-m/2} \quad \text{for all } 0 < t < 1, \ y \in \mathcal{M}.
\]

Then one has

\[
L^p_0(\mathcal{M}) + L^\infty(\mathcal{M}) \subset \mathcal{K}(\mathcal{M}), \quad \text{for all } p \geq 1 \text{ if } m = 1, \text{ and all } p > m/2 \text{ if } m \geq 2,
\]

where $L^p_0(\mathcal{M})$ denotes the weighted $L^p$-space of all equivalence classes of Borel functions $f$ on $\mathcal{M}$ such that $\int |f|^p \theta d\mu < \infty$.

b) If $\mathcal{M}$ is geodesically complete and quasi-isometric to a Riemannian manifold with Ricci curvature bounded from below by a constant, then one has

\[
L^p_{1, \mu(B(\mathcal{M}))}(\mathcal{M}) + L^\infty(\mathcal{M}) \subset \mathcal{K}(\mathcal{M}), \quad \text{for all } p \geq 1 \text{ if } m = 1, \text{ and all } p > m/2 \text{ if } m \geq 2.
\]

Given an endomorphism $A$ on a metric vector bundle, we denote with

\[
\Re(A) := (1/2)(A + A^\dagger)
\]

its real part and with

\[
\Im(A) := -\sqrt{-1}(A - \Re(A))
\]

its imaginary part, so that $A = \Re(A) + \sqrt{-1} \Im(A)$, where $\Re(A)$ and $\Im(A)$ are self-adjoint (and then, for example, the positive and negative parts $\Re(A) \pm \geq 0$ are defined via the fiberwise spectral calculus, giving $\Re(A) = \Re(A)^+ - \Re(A)^-$). Note also that $\Re(A) = \Re(A^\dagger)$, and that $\Re(A) = U \Re(B) U^\dagger$ if $A = U B U^\dagger$ for some unitary $U$. 

Proposition 2.6. Let
\[ Q : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \rightarrow \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \]
be a differential operator of order \( \leq 1 \).

a) Assume
- \( |\Re(\sigma_1(Q))| \in L^\infty(\mathcal{M}) \),
- \( \Re(Q_{\nabla}) \) is bounded from below by a constant \( \kappa \in \mathbb{R} \),
- \( |\Im(Q_{\nabla})| \in \mathcal{K}(\mathcal{M}) \).

Then \( h_{Q}^\nabla \) is sectorial and
\[
\sup_{x \in \mathcal{M}} \mathbb{E} \left[ \left| Q_{\nabla}^{x}(t) \right|^2 \right] < \infty \quad \text{for all } t > 0,
\]
in particular, (2.5) holds true.

b) Assume
- \( \sigma_1(Q) \) is anti-selfadjoint and \( |\sigma_1(Q)| \in L^\infty(\mathcal{M}) \),
- \( |\Re(Q_{\nabla})| \in \mathcal{K}(\mathcal{M}) \),
- \( |\Im(Q_{\nabla})| \in \mathcal{K}(\mathcal{M}) \).

Then \( h_{Q}^\nabla \) is sectorial and one has (2.10), in particular, (2.5) holds true.

Proof. We have
\[
h_{Q}^\nabla = h_a + h_b + h_c + h_d + h_e,
\]
where
\[
\begin{align*}
h_a(\Psi_1, \Psi_2) &:= (1/2) \langle \nabla \Psi_1, \nabla \Psi_2 \rangle, \quad h_b(\Psi_1, \Psi_2) := \langle \sigma_1(Q) \nabla \Psi_1, \Psi_2 \rangle, \\
h_c(\Psi_1, \Psi_2) &:= \langle \Re(Q_{\nabla})^{+} \Psi_1, \Psi_2 \rangle, \quad h_d(\Psi_1, \Psi_2) := \langle \Re(Q_{\nabla})^{-} \Psi_1, \Psi_2 \rangle, \\
h_e(\Psi_1, \Psi_2) &:= \langle \Im(Q_{\nabla}) \Psi_1, \Psi_2 \rangle.
\end{align*}
\]

a) We have
\[
|h_b(\Psi, \Psi)| \leq \|\sigma_1(Q)\|_{\infty} \|\nabla \Psi\| \|\Psi\| \leq \|\sigma_1(Q)\|_{\infty} \left( C_e \|\Psi\|^2 + \epsilon h_a(\Psi, \Psi) \right),
\]
and (as Kato perturbations of Bochner-Laplacians are infinitesimally form small; cf. Lemma VII.4 in [G17])
\[
|h_e(\Psi, \Psi)| \leq \left( C_e \|\Psi\|^2 + \epsilon h_a(\Psi, \Psi) \right),
\]
which shows that \( h_a + h_b + h_c \) is sectorial, as \( h_a \) is so (cf. Theorem [A.1] in the appendix). Moreover,
\[
h_c(\Psi, \Psi) + h_d(\Psi, \Psi) = \langle \Re(Q_{\nabla}) \Psi, \Psi \rangle
\]
is bounded from below, so that the sum
\[
h = h_a + h_b + h_c + h_e + h_d
\]
of sectorial forms is sectorial, too.
Let \( v \in E_x \). Almost surely, for all \( s > 0 \), we have in \( \{ s < \zeta^x \} \) by the Itô product rule,
\[
\begin{align*}
&d \left( Q^x_v(s)^t v \right)^2 = 2 \Re \left( d Q^x_v(s)^t v, Q^x_v(s)^t v \right) + (d Q^x_v(s)^t v, d Q^x_v(s)^t v) \\
&\leq -2 \left( \frac{d}{d t} \Re (\sigma_1(Q)^b(b^x)) \right) \left( Q^x_v(s)^t v, Q^x_v(s)^t v \right) \\
&\quad - 2 \left( \frac{d}{d t} \Re (Q_V(s)^b) \right) \left( Q^x_v(s)^t v, Q^x_v(s)^t v \right) ds \\
&\quad + |\sigma_1(Q)^b(b^x)|^2 |Q^x_v(s)^t v|^2 ds \\
&\leq -2 \left( \frac{d}{d t} \Re (\sigma_1(Q)^b(b^x)) \right) \left( Q^x_v(s)^t v, Q^x_v(s)^t v \right) \\
&\quad - 2 \Re \left( \sigma_1(Q)^b \right) \left( Q^x_v(s)^t v, |Q^x_v(s)^t v|^2 \right) ds \\
&\quad + \| \Re (\sigma_1(Q)) \| \| Q^x_v(s)^t v \|^2 ds.
\end{align*}
\]

With the sequences of stopping times \( \vartheta_n \) and \( \zeta^x \) as in the proof of Theorem 2.2, the Itô isometry and Jenßen’s inequality imply that for all \( t > 0 \),
\[
\begin{align*}
&\mathbb{E} \left[ \left( Q^x_v(t \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right]^2 \\
&\leq 1 + 2 \mathbb{E} \left[ \int_0^t \left( \frac{d}{d t} \Re (\sigma_1(Q)^b(b^x)) \right) \left( Q^x_v(r)^t v, Q^x_v(r)^t v \right) \right]_{r=s \wedge \vartheta_n \wedge \zeta^x}^{2 \Re} \\
&\quad - 2 \Re \left[ \int_0^t \mathbb{E} \left[ \left( Q^x_v(s \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right] ds + \| \Re (\sigma_1(Q)) \| \mathbb{E} \left[ \left( Q^x_v(s \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right] ds \\
&\quad \leq 1 + 2 \| \Re (\sigma_1(Q)) \| \mathbb{E} \left[ \int_0^t \left( Q^x_v(s \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right] ds + \| \Re (\sigma_1(Q)) \| \mathbb{E} \left[ \left( Q^x_v(s \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right] ds \\
&\quad - 2 \Re \left[ \int_0^t \mathbb{E} \left[ \left( Q^x_v(s \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right] ds + \| \Re (\sigma_1(Q)) \| \mathbb{E} \left[ \left( Q^x_v(s \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right] ds \\
&\quad \leq 1 + 2 \| \Re (\sigma_1(Q)) \| + 2 \| \Re (\sigma_1(Q)) \| \mathbb{E} \left[ \int_0^t \left( Q^x_v(s \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right] ds \\
&\quad - 2 \Re \left[ \int_0^t \mathbb{E} \left[ \left( Q^x_v(s \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right] ds + \| \Re (\sigma_1(Q)) \| \mathbb{E} \left[ \left( Q^x_v(s \wedge \vartheta_n \wedge \zeta^x) \right)^t v \right] ds.
\end{align*}
\]

By Gronwall’s lemma and Fatou’s lemma, this estimate implies
\[
\mathbb{E} \left[ 1_{\{ t < \zeta^x \} \} \left| Q^x_v(t)^t v \right|^2 \right] \leq \lim_i \mathbb{E} \left[ 1_{\{ t < \zeta^x_i \} \} \left| Q^x_v(t)^t v \right|^2 \right] \\
= \lim_i \mathbb{E} \left[ 1_{\{ t < \zeta^x_i \} \} \left| Q^x_v(t \wedge \zeta^x_i)^t v \right|^2 \right] \leq \lim_i \mathbb{E} \left[ 1_{\{ t < \zeta^x_i \} \} \left| Q^x_v(t \wedge \vartheta_n \wedge \zeta^x_i)^t v \right|^2 \right] \\
\leq C Q e^{C Q} \zeta^x < \infty,
\]
uniformly in \( x \in M \).

b) As in the proof of part a),
\[
|h_b(\Psi, \Psi)| \leq \| \sigma_1(Q) \| \left( C_\epsilon \| \Psi \|^2 + \epsilon h_a(\Psi, \Psi) \right),
\]
and

\[ |h_d(\Psi, \Psi)| \leq (C \|\Psi\|^2 + ch_a(\Psi, \Psi)), \]
\[ |h_e(\Psi, \Psi)| \leq (C \|\Psi\|^2 + ch_a(\Psi, \Psi)), \]

which shows that \( h_a + h_b + h_d + h_e \) is sectorial, and \( h_e \) is nonnegative so that \( h \) is sectorial. In the notation of Remark \[2.1\] a.s., for all \( s > 0 \) we have in \( \{s < \zeta^x\} \),

\[ dQ^x_{1, v}(s)^{-1} = /\xi(s)^{-1}\sigma_1(Q)^\dagger(d\mathbf{b}_s^x)/\xi(s) \cdot Q^x_{1, v}(s)^{-1}, \quad Q^x_{1, v}(0)^{-1} = 1, \]

and

\[ dQ^x_{1, v}(s)^* = -/\xi(s)^{-1}\sigma_1(Q)^\dagger/\xi(s) \cdot Q^x_{1, v}(s)^*, \quad Q^x_{1, v}(0)^* = 1, \]

which shows that \( Q^x_{1, v}(s) \) is unitary, if \( \sigma_1(Q) \) is anti-selfadjoint. Thus we have

\[ |Q^x_{1, v}(s)| = |Q^x_{2, v}(s)| Q^x_{1, v}(s) |Q^x_{2, v}(s)| \leq |Q^x_{2, v}(s)|. \]

For all \( v \in \mathcal{E}_x \) (as both \( Q^x_{1, v}(s) \) and the parallel transport are unitary),

\[ (d/ds) |Q^x_{2, v}(s)\rangle v|^2 = 2\Re( (d/ds) Q^x_{2, v}(s) \rangle v, Q^x_{2, v}(s) \rangle v) \]
\[ = -2\Re( Q^x_{1, v}(s) /\xi(s)^{-1} Q^x \mathbf{b}_s^x /\xi(s) \cdot Q^x_{1, v}(s)^{-1} Q^x_{2, v}(s) \rangle v, Q^x_{2, v}(s) \rangle v) \]
\[ = -2\Re( Q^x_{1, v}(s) /\xi(s)^{-1} Q^x \mathbf{b}_s^x /\xi(s) \cdot Q^x_{1, v}(s)^{-1} Q^x_{2, v}(s) \rangle v, Q^x_{2, v}(s) \rangle v) \]
\[ \leq 2\Re( Q^x \mathbf{b}_s^x ) - |Q^x_{2, v}(s)\rangle v|^2 \]

and so by Gronwall, a.s., for all \( t > 0 \) we have in \( \{t < \zeta^x\} \),

\[ |Q^x_{2, v}(t)|^2 = |Q^x_{2, v}(t)^\dagger|^2 \leq e^{2\Re( Q^x \mathbf{b}_s^x ) - |d\mu| } \]

and finally

\[ \sup_{x \in \mathcal{M}} \EE \left[ 1_{\{t < \zeta^x\}} e^{2\Re( Q^x \mathbf{b}_s^x ) - |d\mu| } \right] < \infty. \]

by Khashiminskii’s lemma. \( \blacksquare \)

Given \( x \in \mathcal{M} \), let \( (\mathbb{P}^{x, y}_{t})_{t > 0, y \in \mathcal{M}} \) be the bridge measures associated with \( \mathbf{b}(x) \): for all \( t > 0, y \in \mathcal{M} \), the measure \( \mathbb{P}^{x, y}_{t} \) is the uniquely determined probability measure (cf. [P03], p. 36) on \( \{t < \zeta^x\} \) equipped with the sigma-algebra \( \mathcal{F}_t^{b^x_{t-s}} \) such that

\[ \mathbb{P}^{x, y}_{t}(A) = \EE \left[ 1_{A} \frac{p(t-s, \mathbf{b}_s^x, y)}{p(t, x, y)} \right] \quad \text{for all } 0 < s < t, A \in \mathcal{F}_t^{b^x_{t-s}}. \]

This provides a pointwise disintegration of Brownian motion, in the sense that for all \( t > 0, x, y \in \mathcal{M} \) one has

\[ \mathbb{P}(A) = \int e^{-tH}(x, y) \mathbb{P}^{x, y}_{t}(A) d\mu(y) \quad \text{for all } A \in \mathcal{F}_t^{b^x} \cap \{t < \zeta^x\}, \]
\[ \mathbb{P}^{x, y}_{t}(\mathbf{b}_t^x = y) = 1. \]

We remark that one has to locally complete these probability spaces so that \( Q^x_v(t) \) and \( /\xi_v(t) \) become \( \mathcal{F}_t^{b^x_{t-s}} \)-measurable (cf. p. 250 in [HT94] for a precise treatment of this issue.)

We immediately get the following consequence of Theorem \[2.2\].

**Corollary 2.7.** In the situation of Theorem \[2.2\] for all \( t > 0, x, y \in \mathcal{M} \) one has

\[ e^{-tH/\xi}(x, y) = \int_M e^{-tH}(x, y) \mathbb{P}^{x, y}_{t}(Q^x_v(t)//\xi_v(t)^{-1}). \]


Remark 2.8. The precise meaning of this result is as follows: there exists a unique jointly smooth map
\[ (0, \infty) \times \mathcal{M} \times \mathcal{M} \ni (t, x, y) \mapsto e^{-tH^\Sigma_y}(x, y) \in \text{Hom}(\mathcal{E}_y, \mathcal{E}_x) \in \mathcal{E} \otimes \mathcal{E}^\dagger \]
such that for all \( t > 0, x \in \mathcal{M}, \Psi \in \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \) one has
\[ \int |e^{-tH^\Sigma_y}(x, y)|^2 d\mu(y) < \infty, \quad e^{-tH^\Sigma_y}(x, y) = \int e^{-tH^\Sigma_y}(x, y)\Psi(y)d\mu(y), \]
(this follows from the proof of Theorem II.1 in [G17], where the required self-adjointness and semiboundedness of the operator \( \hat{P} \) is only used to get a semigroup which is holomorphic in a sector of the complex plane which contains \((0, \infty)\), and Corollary 2.7 states this map is pointwise equal to the RHS of (2.12).

In the following result we assume for simplicity that \( \mathcal{M} \) is compact, in order to not obscure the algebraic machinery behind its proof, and to guarantee the required trace class property:

Theorem 2.9. Assume \( \mathcal{M} \) is compact. Let \( V \in \Gamma_{C^\infty}(\mathcal{M}, \text{End}(\mathcal{E})) \) (considered as a differential operator of order \( \leq 1 \) in \( \mathcal{E} \rightarrow \mathcal{M} \)) and let
\[ P : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \rightarrow \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \]
be a differential operator of order \( \leq 1 \) and denote its closure in \( \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \), defined a priori on \( \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E}) \), with \( P \) again. Then for all \( t > 0 \) the operator
\[ \int_0^t e^{-sH^\Sigma_y}P e^{-(t-s)H^\Sigma_y} ds \in \mathcal{L}(\Gamma_{L^2}(\mathcal{M}, \mathcal{E})), \]
is given for all \( x, y \in \mathcal{M} \) by
\[ \int_0^t e^{-sH^\Sigma_y}P e^{-(t-s)H^\Sigma_y} ds (x, y) = -e^{-tH}(x, y)E^{x,y} \left[ \mathcal{V}_\Sigma(t) \int_0^t \| \mathcal{V}(s(an+1)) \|_\Sigma(s) (db^\Sigma(s) + P\mathcal{V}(b^\Sigma(s))ds) \right]^{-1}, \]
in particular, for every \( \tilde{V} \in \Gamma_{C^\infty}(\mathcal{M}, \text{End}(\mathcal{E})) \) one has
\[ \text{Tr} \left( \tilde{V} \int_0^t e^{-sH^\Sigma_y}P e^{-(t-s)H^\Sigma_y} ds \right) = -\int e^{-tH}(x, x) \text{Tr}_x \left( \tilde{V}(x)E^{x,x}_\Sigma(x) \left[ \mathcal{V}_\Sigma(t) \int_0^t \| \mathcal{V}(s(an+1)) \|_\Sigma(s) (db^\Sigma(s) + P\mathcal{V}(b^\Sigma(s))ds) \right]^{-1} \right) d\mu(x). \]

This result has to be read as follows: by elliptic regularity, for all \( t > 0 \), the function
\[ [0, t] \ni s \mapsto e^{-sH^\Sigma_y}P e^{-(t-s)H^\Sigma_y} \Psi \in \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \]
is well-defined and continuous, so
\[ \int_0^t e^{-sH^\Sigma_y}P e^{-(t-s)H^\Sigma_y} \Psi ds \]
is well-defined in the sense of \( \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \)-valued Riemann integrals. Furthermore,
\[ \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \ni \Psi \mapsto \int_0^t e^{-sH^\Sigma_y}P e^{-(t-s)H^\Sigma_y} \Psi ds \in \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \]
is bounded, and our proof shows that \( \int_0^t e^{-sH^\Sigma_y}P e^{-(t-s)H^\Sigma_y} ds \) has a jointly smooth integral kernel in the sense of Remark 2.8 and that this smooth representative is pointwise equal to the RHS of (2.14). The asserted trace formula then follows from the fact that if an operator \( A_1 \) in
\( \Gamma L^2(\mathcal{M}, \mathcal{E}) \) has a smooth integral kernel and \( A_2 \) is zeroth order, then \( A_2A_1 \) has the smooth integral kernel \( [A_2A_1](x, y) = A_2(x)A_1(x, y) \) and \( A_2A_1 \) is trace class (as \( \mathcal{M} \) is compact) with

\[
\text{Tr} (A_2A_1) = \int_{\mathcal{M}} \text{Tr}_x (A_2(x)A_1(x, x)) d\mu(x),
\]

where \( \text{Tr}_x \) denotes the finite dimensional trace on \( \text{End}(\mathcal{E}_x) \).

**Proof of Theorem 2.4** Denote with \( \Lambda(\mathbb{R}) = \mathbb{R} \oplus \Lambda^1(\mathbb{R}) \) the Grassmann algebra over \( \mathbb{R} \), which is generated by \( 1 \in \mathbb{R} \) and \( \theta \in \Lambda^1(\mathbb{R}) \). In particular, we have \( \theta^2 = 0 \). Given a linear space \( \mathcal{A} \), the Berezin integral is the linear map

\[
\int_{\Lambda(\mathbb{R})} : \mathcal{A} \otimes \Lambda(\mathbb{R}) \to \mathcal{A}, \quad a + b\theta \longmapsto \int_{\Lambda(\mathbb{R})} (a + b\theta) d\theta := b, \quad a, b \in \mathcal{A},
\]

which picks the \( \theta \)-coefficient. Note that if \( \mathcal{A} \) is an associative algebra, then so is \( \mathcal{A} \otimes \Lambda(\mathbb{R}) \). With the differential operator

\[
V + P^\theta := V + \theta P : \Gamma C^\infty(\mathcal{M}, \mathcal{E} \otimes \Lambda(\mathbb{R})) = \Gamma C^\infty(\mathcal{M}, \mathcal{E}) \otimes \Lambda(\mathbb{R}) \to \Gamma C^\infty(\mathcal{M}, \mathcal{E} \otimes \Lambda(\mathbb{R})),
\]

of order \( \leq 1 \), the operator \( H^V_{\Gamma} + P^\theta \) in

\[
\Gamma L^2(\mathcal{M}, \mathcal{E} \otimes \Lambda(\mathbb{R})) = \Gamma L^2(\mathcal{M}, \mathcal{E}) \otimes \Lambda(\mathbb{R})
\]

is well-defined and in fact equal to the operator sum \( H^V_{\Gamma} + P^\theta \) (as \( \mathcal{M} \) is compact). The perturbation series

\[
e^{-tH^V_{\Gamma} + P^\theta} = 1 + \sum_{j=1}^\infty \int_{0 < t_1 < \ldots < t_j < t} e^{-t_1 H^V_{\Gamma} + P^\theta} e^{-(t_2 - t_1) H^V_{\Gamma} + P^\theta} \cdots e^{-(t_j - t_{j-1}) H^V_{\Gamma} + P^\theta} dt_1 \cdots dt_n
\]

cancels after \( j \geq 2 \) because of \( \theta^2 = 0 \), and we have

\[
(2.15) \quad \int_{\Lambda(\mathbb{R})} e^{tH^V_{\Gamma} + P^\theta} d\theta = \int_0^t e^{-sH^V_{\Gamma} + P^\theta} e^{-(t-s)H^V_{\Gamma}} ds,
\]

in particular, \( \int_0^t e^{-sH^V_{\Gamma} + P^\theta} e^{-(t-s)H^V_{\Gamma}} ds \) has a jointly smooth integral kernel in the sense of Remark 2.8. By Corollary 2.7 and Remark 2.1 we have

\[
e^{-tH^V_{\Gamma} + P^\theta}(x, y) = e^{-tH}(x, y) \mathbb{E}^{x,y}_{\Gamma} \left[ V^x_{\Gamma}(t) P^x_{\theta, \Gamma}(t) / V^x_{\Gamma}(t) \right],
\]

where

\[
P^x_{\theta, \Gamma} : [0, \zeta^x) \times \Omega \to \text{End}(\mathcal{E}_x \otimes \Lambda(\mathbb{R}))
\]

denotes the unique solution of

\[
dP^x_{\theta, \Gamma}(t) = -\partial \mathcal{P}^x_{\theta, \Gamma}(t) / \mathcal{V}^x_{\Gamma}(t)^{-1} (\sigma_1(P^\theta)^b db_t^x + P^\theta db_t^x) / \mathcal{V}^x_{\Gamma}(t), \quad \mathcal{P}^x_{\theta, \Gamma}(0) = 1.
\]

Because of \( \theta^2 = 0 \) the time ordered exponential series

\[
\mathcal{P}^x_{\theta, \Gamma}(t) = 1 + \sum_{j=1}^\infty \int_{\{0 \leq t_1 \leq \ldots \leq t_j \leq t\}} \prod_{i=1}^j \theta \left( -\mathcal{V}^x_{\Gamma}(t_1)^{-1} (\sigma_1(P)^b db_t^x + P db_t^x) dt_1 \right) / \mathcal{V}^x_{\Gamma}(t_1)
\]

has only two summands, giving

\[
\int_{\Lambda(\mathbb{R})} e^{tH^V_{\Gamma} + P^\theta}(x, y) d\theta
\]

\[
= -e^{-tH}(x, y) \mathbb{E}^{x,y}_{\Gamma} \left[ V^x_{\Gamma}(t) \int_0^t / \mathcal{V}^x_{\Gamma}(s)^{-1} (\sigma_1(P)^b db_t^x + P db_t^x) ds / \mathcal{V}^x_{\Gamma}(s) \right] / \mathcal{V}^x_{\Gamma}(t)^{-1},
\]

which in view of (2.15) is the claimed formula.
3. Applications to noncommutative geometry

In this section we present an application of Theorem 2.9 to recent results concerning an algebraic model given in [GL19] for Duistermaat-Heckman localization on the space of smooth loops in a compact Riemannian spin manifold. We refer the reader to [LM89] for details on spin geometry (noting that a brief introduction can also be found in [H02]).

Assume \( \mathcal{M} \) is a compact Riemannian spin manifold of even dimension, with \( \mathcal{I} \to \mathcal{M} \) its spin bundle, which is naturally \( \mathbb{Z}_2 \)-graded by an endomorphism \( \gamma \in \Gamma_{C^\infty}(\mathcal{M}, \text{End}(\mathcal{S})) \). The vector bundle \( \mathcal{I} \to \mathcal{M} \) inherits a metric and a metric connection \( \nabla \) from the Riemannian metric and the Levi-Civita connection on \( \mathcal{M} \). Let

\[
D : \Gamma_{C^\infty}(\mathcal{M}, \mathcal{I}) \to \Gamma_{C^\infty}(\mathcal{M}, \mathcal{I})
\]

denote the induced Dirac operator and let

\[
c : \Omega_{C^\infty}(M) \to \Gamma_{C^\infty}(\mathcal{M}, \text{End}(\mathcal{I})), \quad c(\alpha_1 \wedge \cdots \wedge \alpha_p) \Psi := \frac{1}{p!} \alpha_1 \cdots \alpha_p \cdot \Psi,
\]

\( \alpha_1, \ldots, \alpha_p \in \Omega^1_{C^\infty}(M), \quad \Psi \in \Gamma_{C^\infty}(\mathcal{M}, \mathcal{I}), \)

denote the natural extension of the (dual) Clifford multiplication

\[
\Omega^1_{C^\infty}(M) \to \Gamma_{C^\infty}(\mathcal{M}, \text{End}(\mathcal{I})), \quad \alpha \mapsto (\Psi \mapsto \alpha \cdot \Psi)
\]

from 1-forms to arbitrary differential forms. The operator \( D \) (defined a priori on \( \Gamma_{C^\infty}(\mathcal{M}, \mathcal{I}) \)) is essentially self-adjoint in \( \Gamma_{L^2}(\mathcal{M}, \mathcal{I}) \), and its unique self-adjoint realization will be denoted with the same symbol again. With \( \mathbb{T} := S^1 \) let

\[
\Omega^1_{\mathbb{T}}(\mathcal{M}) := \Omega_{C^\infty}(\mathcal{M} \times \mathbb{T})^\mathbb{Z}
\]

denote the space of \( \mathbb{T} \)-invariant differential forms on \( \mathcal{M} \times \mathbb{T} \). Each element \( \alpha \in \Omega^1_{\mathbb{T}}(\mathcal{M}) \) can be uniquely written in the form \( \alpha = \alpha' + \alpha''dt \) with \( dt \) the volume form on \( \mathbb{T} \). Define a complex linear space by

\[
\mathcal{C}_{\mathbb{T}}(\mathcal{M}) := \bigoplus_{N=0}^{\infty} \Omega^1_{\mathbb{T}}(\mathcal{M}) \otimes \left( \Omega^1_{\mathbb{T}}(\mathcal{M})^{\otimes N}/(\mathbb{C} \cdot 1) \right).
\]

Since, \( \mathcal{M} \) is compact, \( e^{-tD^2} \) is trace class for all \( t > 0 \). In this situation, the Chern Character \( \text{Ch}_{\mathbb{T}}(\mathcal{M}) \) is a linear functional\(^2\)

\[
\text{Ch}_{\mathbb{T}}(\mathcal{M}) : \mathcal{C}_{\mathbb{T}}(\mathcal{M}) \to \mathbb{C},
\]

that has been introduced in [GL19]. The formula for \( \text{Ch}_{\mathbb{T}}(\mathcal{M}) \) is given as follows: define \( F_T : \mathcal{C}_{\mathbb{T}}(\mathcal{M}) \to \{ \text{differential operators of order } \leq 2 \text{ in } \mathcal{I} \to \mathcal{M} \} \) by

\[
F_T(\alpha_0) = c(d\alpha'_0) - [D, c(\alpha'_0)] - c(\alpha''_0)
\]

\[
F_T(\alpha_0 \otimes \alpha_1) = (-1)^{|\alpha'_0|} \left( c(\alpha'_0 \wedge \alpha'_1) - c(\alpha'_0)c(\alpha'_1) \right),
\]

\[
F_T(\alpha_0 \otimes \cdots \otimes \alpha_N) = 0 \quad \text{for all } N \geq 3.
\]

Above, \( [D, c(\alpha)] \) denotes a \( \mathbb{Z}_2 \)-graded commutator (where differential forms are \( \mathbb{Z}_2 \)-graded through even/odd form degrees). Explicitly, one has

\[
[D, c(\alpha)] = Dc(\alpha) - (-1)^p c(\alpha)D, \quad \text{if } \alpha \in \Omega^p_{C^\infty}(\mathcal{M}).
\]

\(^2\)In fact, \( \text{Ch}_{\mathbb{T}}(\mathcal{M}) \) extends continuously to a certain completion of \( \mathcal{C}_{\mathbb{T}}(\mathcal{M}) \), but we shall not be concerned with this fact here.
For natural numbers $L \leq N$ denote with $P_{L,N}$ all tuples $I = (I_1, \ldots, I_L)$ of subsets of \{1, \ldots, N\} with
\[ I_1 \cup \cdots \cup I_L = \{1, \ldots, N\} \]
and with each element of $I_a$ smaller than each element of $I_b$ whenever $a < b$. Given
- $\alpha_1 \otimes \cdots \otimes \alpha_N \in \Omega_T(\mathcal{M})^\otimes N$,
- $I = (I_1, \ldots, I_L) \in P_{L,N}$,
- $1 \leq a \leq L$,
we set
\[ \alpha_{I_a} := \alpha_{i+1} \otimes \cdots \otimes \alpha_{i+l}, \quad \text{if } I_a = \{j \mid i < j \leq i+l\} \text{ for some } i, l. \]
Then with $\text{Str}(\bullet) := \text{Tr}(\gamma \bullet)$ the $\mathbb{Z}_2$-graded trace on $\mathcal{L}(\Gamma_{L^2}(\mathcal{M}, \mathcal{F}))$, one has
\[
\text{Ch}_T(\mathcal{M})(\alpha_0 \otimes \cdots \otimes \alpha_N) := \sum_{L=1}^N (-1)^L \sum_{I \in P_{L,N}} \int_{\{0 \leq s_1 \leq \cdots \leq s_L \leq 1\}} \text{Str} \left( c(\alpha_0) e^{-s_1 D^2} F_T(\alpha_{I_1}) \times \
\times e^{-(s_2-s_1) D^2} F_T(\alpha_{I_2}) \cdots e^{-(s_L-s_{L-1}) D^2} F_T(\alpha_{I_L}) e^{-(1-s_L) D^2} \right) \, ds_1 \cdots ds_L.
\]
By definition the $N = 0$ part of the Chern character is given explicitly by
\[
(3.1) \quad \text{Ch}_T(\mathcal{M})(\alpha_0) = \text{Str} \left( c(\alpha_0') e^{-D^2} \right),
\]
and the $N = 1$ part is given explicitly by
\[
(3.2) \quad \text{Ch}_T(\mathcal{M})(\alpha_0 \otimes \alpha_1) = -\text{Str} \left( \int_0^1 c(\alpha_0') e^{-s D^2} F_T(\alpha_1) e^{-(1-s) D^2} \, ds \right).
\]
By the Lichnerowicz formula we have
\[
(3.3) \quad D^2 = \nabla^\dagger \nabla + (1/4) \text{scal},
\]
so that the $N = 0$ piece of $\text{Ch}_T(\mathcal{M})$ is given by the probabilistic expression
\[
\text{Ch}_T(\mathcal{M})(\alpha_0) = \int_M e^{-tH(x, x)} \text{Str}_x \left( c(\alpha_0')(x) e^{x^x} \left[ e^{-(1/8) \int_0^1 \text{scal}(b_x^x) \, ds} / \sqrt{t} \left( \int_0^1 \frac{(2c(s \text{scal}(b_x^x) - c(\alpha_0')(b_x^x)) \, ds)}{\sqrt{t}} \right)^{1/2} \right] \right) \, d\mu(x),
\]
with $\text{Str}_x$ the $\mathbb{Z}_2$-graded trace on $\text{End}(\mathcal{F}_x)$. We are going to use Theorem 2.9 to deduce a probabilistic representation for the $N = 1$ piece of $\text{Ch}_T(\mathcal{M})$:

**Theorem 3.1.** Let $\mathcal{M}$ be a compact even dimensional Riemannian spin manifold. Then for all $\alpha_0, \alpha_1 \in \Omega_T(\mathcal{M})$ one has
\[
\text{Ch}_T(\mathcal{M})(\alpha_0 \otimes \alpha_1) = \int_M e^{-tH(x, x)} \text{Str}_x \left( c(\alpha_0')(x) e^{x^x} \left[ e^{-(1/8) \int_0^1 \text{scal}(b_x^x) \, ds} / \sqrt{t} \left( \int_0^1 \frac{(2c(s \text{scal}(b_x^x) - c(\alpha_0')(b_x^x)) \, ds)}{\sqrt{t}} \right)^{1/2} \right] \right) \, d\mu(x).
\]

**Proof.** Applying Theorem 2.9 with $V := (1/8) \text{scal}$, $\bar{V} := \gamma$ and $P := F_T(\alpha_1)$, and noting that by (3.3) one has $H_V = D^2$, for all $x, y \in \mathcal{M}$, we immediately get
\[
\text{Str} \left( \int_0^1 e^{-s D^2} F_T(\alpha_1) e^{-(1-s) D^2} \, ds \right) = \int_M e^{-tH(x, x)} \text{Str}_x \left( c(\alpha_0')(x) e^{x^x} \left[ e^{-(1/8) \int_0^1 \text{scal}(b_x^x) \, ds} / \sqrt{t} \left( \int_0^1 \frac{(2c(s \text{scal}(b_x^x) - c(\alpha_0')(b_x^x)) \, ds)}{\sqrt{t}} \right)^{1/2} \right] \right) \, d\mu(x).
\]

With the product
\[
\star : \Gamma_{C^\infty}(\mathcal{M}, T, \mathcal{M} \otimes \mathcal{F}) \otimes \Omega_{C^\infty}(\mathcal{M}) \to \Gamma_{C^\infty}(\mathcal{M}, \mathcal{F}), \quad (X \otimes \varphi) \star \alpha := c(X, \alpha) \varphi,
\]
where $X, \alpha$ denotes the contraction of the form $\alpha$ by the vector field $X$, we are going to prove in a moment the formula
\[
(3.4) \quad [D, c(\alpha)] \varphi = c \left( (d + d^\dagger) \alpha \right) \varphi - 2(\nabla \varphi)^{\otimes d} \star \alpha, \quad \alpha \in \Omega_{C^\infty}(\mathcal{M}), \quad \varphi \in \Gamma_{C^\infty}(\mathcal{M}, \mathcal{F}).
\]
Given this identity, we find
\[ \sigma_1(F_T(\alpha_1))^\flat(X) = 2c(X_\alpha_1') \]
for all vector fields \( X \) on \( \mathcal{M} \),
and furthermore
\[ F_T(\alpha_1)_\nabla = -c(d^\dagger \alpha_1') - c(\alpha_1''), \]
so that the above is
\[ (3.6) \]
Using the Itô-to-Stratonovich rule
\[ c(db_{x_\alpha}^x \alpha') = c(*db_{x_\alpha}^x \alpha') + \frac{1}{2} c(d^\dagger \alpha')(b_s^x) ds, \]
we arrive at
\[ \text{Str} \left( \int_0^1 e^{-D^2 F_T(\alpha_1)c-(1-s)D^2} ds \right) \]
\[ = \int_\mathcal{M} e^{-tH(x,x)}e^{(1/2)\int_0^1 \text{scal}(b_s^x) ds} \int_0^1 f_{\mathcal{M}}(s)^{-1} \left( 2c(*db_{x_\alpha}^x \alpha_1') - c(d^\dagger \alpha_1')(b_s^x) ds \right) f_{\mathcal{M}}(s)/f_{\mathcal{M}}(t)^{-1} \biggr|_{t=-2d\mu(x)}, \]
which is the claimed formula.
It remains to prove \((3.4)\). To this end, denote with \( \text{Cl}(\mathcal{M}) \to \mathcal{M} \) the Clifford bundle and with
\[ \gamma : \Omega^{\infty}(\mathcal{M}) \to \Gamma^{\infty}(\mathcal{M}, \text{Cl}(\mathcal{M})) \]
the natural isomorphism. Then we have
\[ (d + \tilde{d}) \alpha = D^{\text{Cl}(M)} \alpha \]
(cf. \cite{LM89}, Chapter II, Thm. 5.12), with \( D^{\text{Cl}(M)} \) the natural Dirac operator on \( \text{Cl}(\mathcal{M}) \to \mathcal{M} \).
Assume now \( \alpha \in \Omega^p(\mathcal{M}) \) and pick a local orthonormal frame \((e_1, \ldots, e_m)\). Write \( \alpha = \sum_I \alpha_I e_{i_1}^s \wedge \ldots \wedge e_{i_p}^s \) with some increasingly ordered multi-index \( I = (i_1, \ldots, i_p) \). One has
\[ (3.5) \]
\[ [D, c(\alpha)] \varphi = Dc(\alpha) \varphi - (-1)^p c(\alpha) \varphi \]
\[ = \sum_{j=1}^n \sum_{I} \left( e_j \cdot \nabla_{e_j} (\alpha_I e_{i_1} \cdots e_{i_p} \cdot \varphi) + (-1)^{p+1} \alpha_I e_{i_1} \cdots e_{i_p} \cdot e_j \cdot \nabla_{e_j} \varphi \right) \]
\[ = \sum_{j=1}^n \sum_{I} \left( e_j \cdot \nabla_{e_j}^{\text{Cl}(\mathcal{M})} (\alpha_I e_{i_1} \cdots e_{i_p}) \cdot \varphi + e_j \cdot \alpha_I e_{i_1} \cdots e_{i_p} \nabla_{e_j} \varphi + (-1)^{p+1} \alpha_I e_{i_1} \cdots e_{i_p} \cdot e_j \cdot \nabla_{e_j} \varphi \right) \]
\[ = \sum_{j=1}^n \sum_{I} \left( e_j \cdot \nabla_{e_j}^{\text{Cl}(\mathcal{M})} (\alpha_I e_{i_1} \cdots e_{i_p}) \cdot \varphi + \alpha_I (e_j \cdot e_{i_1} \cdots e_{i_p} + (-1)^{p+1} e_{i_1} \cdots e_{i_p} \cdot e_j) \cdot \nabla_{e_j} \varphi \right) \]
\[ (3.6) \]
\[ = (D^{\text{Cl}(\mathcal{M})} \alpha) \cdot \varphi + \sum_{j=1}^n \sum_{I} \left( \alpha_I (e_j \cdot e_{i_1} \cdots e_{i_p} + (-1)^{p+1} e_{i_1} \cdots e_{i_p} \cdot e_j) \cdot \nabla_{e_j} \varphi \right). \]
Fix now \( I \) and \( j \). In case \( j \neq i_k \) for all \( k = 1, \ldots, p \), one has
\[ e_j \cdot e_{i_1} \cdots e_{i_p} = (-1)^p e_{i_1} \cdots e_{i_p} \cdot e_j. \]
In case \( j = i_k \) for some \( 1 \leq k \leq p \), one has
\[ e_j \cdot e_{i_1} \cdots e_{i_p} = e_{i_k} \cdot e_{i_1} \cdots e_{i_p} = (-1)^{k-1} e_{i_1} \cdots e_{i_k} \cdot e_{i_k} \cdots e_{i_p} = (-1)^{k} e_{i_1} \cdots \hat{e}_{i_k} \cdots e_{i_p} \]
and
\[(1)^{p+1}e_{i_1} \cdots e_{i_p} \cdot e_j = (1)^{p+1}e_{i_1} \cdots e_{i_p} \cdot e_{i_k} = (1)^{p+1+p-k}e_{i_1} \cdots e_{i_k} \cdots e_{i_p} \]
\[= (1)^{2p+2-k}e_{i_1} \cdots \widehat{e}_{i_k} \cdots e_{i_p} = (-1)^ke_{i_1} \cdots \widehat{e}_{i_k} \cdots e_{i_p}.\]

So the RHS of (3.6) equals
\[
\sum_{I} \sum_{j=1}^{p} (-1)^{k-1} \alpha_I e_{i_1} \cdots \widehat{e}_{i_k} \cdots e_{i_p} \cdot \nabla e_{i_k} \varphi.
\]

Assume again \(I\) and \(j\) are fixed and that \(j = i_k\) for some \(k\). Then by the definition of the product \(*\),
\[
(e_j \otimes \nabla e_j \varphi) \ast \alpha_I e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* = c(e_j, \alpha_I e_{i_1}^* \wedge \cdots \wedge e_{i_p}^*) \nabla e_j \varphi
\]
\[= c((-1)^k \alpha_I e_{i_1}^* \wedge \cdots \widehat{e}_{i_k} \wedge \cdots e_{i_p}^*) \nabla e_{i_k} \varphi
\]
\[= (-1)^k \alpha_I e_{i_1} \cdots \widehat{e}_{i_k} \cdots e_{i_p} \cdot \nabla e_{i_k} \varphi.
\]

As one has \((\nabla \varphi)_{\otimes d} = \sum_{j=1}^{n} e_j \otimes \nabla e_j \varphi\), (3.7) equals
\[
c((d + d^*) \alpha) \varphi - 2 \sum_{I, j=1}^{n} (e_j \otimes \nabla e_j \varphi) \ast \alpha_I e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* = c((d + d^*) \alpha) \varphi - 2(\nabla \varphi)_{\otimes d} \ast \alpha,
\]

completing the proof. 

**Appendix A. Facts on sectorial forms and operators**

In this appendix, we have collected some definitions and facts on sectorial forms and operators, following the presentation from section VI in [K].

A densely defined sesquilinear form \(h\) in a Hilbert space \(\mathcal{H}\) is called *sectorial*, if there exist numbers \(\beta \in [0, \pi/2), \gamma \in \mathbb{R}\) such that
\[
\{ h(\Psi, \Psi) : \Psi \in \text{Dom}(h), \|\Psi\| = 1 \} \subset \{ z \in \mathbb{C} : |\arg(z - \gamma)| \leq \beta \}.
\]

Above, \(\gamma\) is called a vertex of \(h\) and \(\beta\) an angle of \(h\).

A sectorial form \(h\) in \(\mathcal{H}\) is called *closed* if for all \(\Psi \in \mathcal{H}\) which admit a sequence \((\Psi_n) \subset \text{Dom}(h)\) with
\[
\|\Psi_n - \Psi\| \to 0, \quad h(\psi_n - \psi_l, \psi_n - \psi_l) \to 0 \quad \text{as} \quad n, l \to \infty,
\]
and \(h\) is called *closable* if it has a closed extension; in this case \(h\) has a smallest closed extension \(\overline{h}\), called the closure of \(h\). Sums of sectorial forms are sectorial, and sums of closed forms are closed (on their natural domain of definition; Theorem 131 p. 319 in [K]).

A densely defined operator \(S\) in \(\mathcal{H}\) is called *sectorial*, if the form \(h_S\) given by \(\text{Dom}(h_S) = \text{Dom}(S)\) and \(h_S(\Psi_1, \Psi_2) = \langle \Psi_1, S \Psi_2 \rangle\) is sectorial. If a form \(h\) in \(\mathcal{H}\) is induced by a sectorial operator \(S\) in \(\mathcal{H}\), in the sense that \(h = h_S\), then \(h\) is closable (Theorem 1.27 p. 318 in [K]).

**Theorem A.1.** If \(h\) is sectorial and the form \(h'\) satisfies \(\text{Dom}(h) \subset \text{Dom}(h')\) and admits constants \(a \in [0, \infty), b \in [0, 1]\) such that
\[
|h'(\Psi, \Psi)| \leq a \|\Psi\|^2 + b|h(\Psi, \Psi)| \quad \text{for all} \quad \Psi \in \text{Dom}(h),
\]
then the form \(h + h'\) is
- sectorial,
- closed if and only if \(h\) is closed,
- closable if and only if \(h\) is closable; and then \(\text{Dom}(h + h') = \overline{\text{Dom}(h)}\).

**Proof.** This is Theorem 1.33 p. 320 in [K].
Given \( \beta \in (0, \pi/2) \) set
\[
\Sigma_{\beta} = \{ re^{\sqrt{-1} \alpha} : r > 0, \alpha \in (-\beta, \beta) \}
\]
and
\[
\Sigma_{\alpha, \beta} := \Sigma_{\beta} \cup \{0\} = \{ re^{\sqrt{-1} \alpha} : r \geq 0, \alpha \in (-\beta, \beta) \}.
\]

A family of bounded operators \( (T_z)_{z \in \Sigma_{\alpha, \beta}} \) in \( \mathcal{H} \), with some \( \beta \in (0, \pi/2) \), is called a holomorphic semigroup, if
- \( z \mapsto T_z \) is holomorphic in \( z \in \Sigma_{\beta} \),
- \( T_{z+z'} = T_z T_{z'} \) for all \( z, z' \in \Sigma_{\alpha, \beta} \),
- \( z \mapsto T_z \) is strongly continuous in \( z = 0 \) and \( T(0) = 1 \).

It follows that the restriction of \( T \) to \( [0, \infty) \) is a strongly continuous semigroup, and if \( S \) is the generator of this semigroup, then for every \( \Psi_0 \in \mathcal{H} \), the function
\[
[0, \infty) \ni t \mapsto T(t)\Psi_0 \in \mathcal{H}
\]
is the uniquely determined strongly continuous function \( \Psi : [0, \infty) \to \Gamma_{L^2}(\mathcal{M}, \mathcal{E}) \) which is strongly differentiable on \( (0, \infty) \) taking values in \( \text{Dom}(S) \) thereon, such that
\[
(\frac{d}{dt})\Psi(t) = S\Psi(t), \quad t > 0, \quad \Psi(0) = \Psi_0.
\]

Thus, one essential property of holomorphic semigroups is that the above initial value problem has a unique solution for every initial value in \( \mathcal{H} \), rather than just for initial values in the domain of the generator (cf. Remark 1.22 on p. 492 in [K]).

Finally, there is the following representation theorem:

**Theorem A.2.** For every closed sectorial form \( h \) in \( \mathcal{H} \) there exists a uniquely densely defined, closed, and sectorial operator \( S \) in \( \mathcal{H} \) such that \( \text{Dom}(S) \subset \text{Dom}(h) \) and
\[
(A.1) \quad h(\Psi_1, \Psi_2) = \langle S\Psi_1, \Psi_2 \rangle \quad \text{for all } \Psi_1 \in \text{Dom}(S), \Psi_2 \in \text{Dom}(h).
\]

Moreover, \( -S \) generates a holomorphic semigroup in \( \mathcal{H} \), to be denoted with \( z \mapsto e^{-zS} \).

**Proof.** The existence of a densely defined, closed, and sectorial \( S \) satisfying \((A.1)\) is the statement of Theorem 2.1 on p. 322 in [K]. In fact, it is stated there that \( S \) is actually \( m \)-sectorial, which by Theorem 1.24 on p. 492 in [K] implies that \( -S \) generates a holomorphic semigroup, as for some \( r \in \mathbb{R} \), the form induced by \( S + r \) has a vertex 0 (see also Theorem 1.14 in ([AS6])).

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**Appendix B. Some Formulae for Second Order Differential Operators**

Given a Riemannian manifold \( \mathcal{M} \) with its volume measure \( \mu \), let \( \mathcal{E} \to \mathcal{M} \) be a vector bundle which is equipped with a covariant derivative \( \nabla \).

**B.1. Nondivergence form.** Suppose we are given a second order linear differential operator \( L \) acting on the sections of \( \mathcal{E} \to \mathcal{M} \) via
\[
L = -\text{tr} \nabla^2 + A \nabla + V
\]
with \( A \in \Gamma_{C^\infty}(\mathcal{M}, \text{Hom}(T^* \mathcal{M} \otimes \mathcal{E}, \mathcal{E})) \) and \( V \in \Gamma_{C^\infty}(\mathcal{M}, \text{End}(\mathcal{E})) \).

We introduce the notation
\[
A_X := A(X_j \otimes (\cdot))
\]
for all smooth vector fields \( X \) on \( \mathcal{M} \), and we define a new covariant derivative on \( \mathcal{E} \to \mathcal{M} \) by
\[
\nabla^A_X := \nabla_X - \frac{1}{2} A_X.
\]

Here, weak/strong/norm holomorphy are equivalent by the uniform boundedness principle.
Then, with a local orthonormal frame \((e_1, \ldots, e_n)\) for \(T^*\mathcal{M} \to \mathcal{M}\), we calculate straightforwardly for every \(\varphi \in \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E})\),

\[
-\text{tr}(\nabla^A)^2 \varphi = -\sum_{i=1}^{n} \left( \nabla_{e_i}^A \nabla_{e_i}^A \varphi - \nabla_{\nabla_{e_i}^A e_i} \varphi \right)
= -\sum_{i=1}^{n} \left( \nabla_{e_i} - \frac{1}{2} A_{e_i} \right) \left( \nabla_{e_i} - \frac{1}{2} A_{e_i} \right) \varphi + \sum_{i=1}^{n} \left( \nabla_{\nabla_{e_i}^A e_i} - \frac{1}{2} A \nabla_{e_i} e_i \right) \varphi
= -\text{tr} \nabla^2 \varphi + \sum_{i=1}^{n} \left( \frac{1}{2} A_{e_i} \nabla_{e_i} \varphi + \frac{1}{2} \nabla_{e_i} \left( A_{e_i} \varphi \right) - \frac{1}{4} A_{e_i}^2 \varphi - \frac{1}{2} A \nabla_{e_i} e_i \varphi \right)
= -\text{tr} \nabla^2 \varphi + \sum_{i=1}^{n} \left( \frac{1}{2} A_{e_i} \nabla_{e_i} \varphi + \frac{1}{2} \nabla_{e_i} \left( A_{e_i} \varphi \right) - \frac{1}{4} A_{e_i}^2 \varphi - \frac{1}{2} A \nabla_{e_i} e_i \varphi \right)
= -\text{tr} \nabla^2 \varphi + A \nabla \varphi + \sum_{i=1}^{n} \left( \frac{1}{2} \nabla_{e_i} \left( A_{e_i} \varphi \right) - \frac{1}{4} A_{e_i}^2 \varphi - \frac{1}{2} A \nabla_{e_i} e_i \varphi \right),
\]

so that

\[
L = -\text{tr} \ (\nabla^A)^2 + V^\nabla A,
\]

where \(V^\nabla A \in \Gamma_{C^\infty}(\mathcal{M}, \text{End}(\mathcal{E}))\) is given by

\[
V^\nabla A := V - \sum_{i=1}^{n} \left( \frac{1}{2} \nabla_{e_i} (A_{e_i}) - \frac{1}{4} A_{e_i}^2 - \frac{1}{2} A \nabla_{e_i} e_i \right).
\]

**B.2. Divergence form.** We now assume additionally that our vector bundle \(\mathcal{E} \rightarrow \mathcal{M}\) is endowed with a bundle metric \(h(\cdot, \cdot)\). We do not, however, assume any compatibility between \(\nabla\) and \(h\), i.e. in general we do not have

\[
Xh(\varphi, \psi) = h(\nabla_X \varphi, \psi) + h(\varphi, \nabla_X \psi),
\]

for all \(\varphi, \psi \in \Gamma_{C^\infty}(\mathcal{M}, \mathcal{E})\) and all smooth vector fields \(X\) on \(\mathcal{M}\).

Let us first prove that the formal adjoint \(\nabla^\dagger\) of \(\nabla\) is given by

\[
\nabla^\dagger(X^\otimes \varphi) = -\nabla_X \varphi - \text{div} X \cdot \varphi - ((\nabla_X h)(\cdot, \varphi))^\ast,\]

for all \(\varphi\) and \(X\) as above. Indeed, using \(X(f) + f \text{div} X = \text{div}(fX)\) for all smooth functions \(f\) on \(\mathcal{M}\), we have for all smooth compactly supported sections \(\varphi\) of \(\mathcal{E} \rightarrow \mathcal{M}\),

\[
\int_{\mathcal{M}} h(\varphi, \nabla^\dagger (X^\otimes \psi)) d\mu = \int_{\mathcal{M}} h(\varphi, \nabla^\dagger (X^\otimes \psi)) d\mu
= \int_{\mathcal{M}} (h(\varphi, \nabla_X \psi) - h(\varphi, \psi) \cdot \text{div} X - (\nabla_X h)(\varphi, \psi)) d\mu
= \int_{\mathcal{M}} (-h(\varphi, \nabla_X \psi) - h(\varphi, \psi) \cdot \text{div} X - \nabla_X h(\varphi, \psi) + h(\nabla_X \varphi, \psi)
+ h(\varphi, \nabla_X \psi)) d\mu
= \int_{\mathcal{M}} (h(\nabla_X \varphi, \psi) - \text{div} (h(\varphi, \psi)X)) d\mu
= \int_{\mathcal{M}} h(\nabla_X \varphi, \psi) d\mu
= \int_{\mathcal{M}} h(\nabla \varphi, X^\otimes \psi) d\mu.
\]
Let now a second order linear differential operator $L$ be given by 
\[ L = \nabla^\dagger \nabla + A \nabla + V \]
with $A, V$ as in the previous section. Note that now the second order part of $L$ is in divergence form, in contrast to the previous section. We are going to carry out a calculation similar to the one in the previous section to know exactly for which $A$’s the operator 
\[ (\nabla^A)^\dagger \]
is of zeroth order.

First of all, one has 
\[
(\nabla^A_h)(\varphi, \psi) = X_h(\varphi, \psi) - h(\nabla^A_X \varphi, \psi) - h(\varphi, \nabla^A_X \psi) = X_h(\varphi, \psi) - h(\nabla_X \varphi, \psi) - h(\varphi, \nabla_X \psi) + \frac{1}{2}h(\varphi, A_X \psi) + \frac{1}{2}h(\varphi, A_X \psi) = (\nabla_X h)(\varphi, \psi) + h(\varphi, \frac{1}{2}(A_X + A^\dagger_X)\psi),
\]
which, in turn, implies 
\[
(\nabla^A)^\dagger (X^\varphi \otimes \varphi) = \nabla^\dagger (X^\varphi \otimes \varphi) - \frac{1}{2}A^\dagger_X \varphi.
\]

A calculation analogous to the one in the previous section then yields 
\[
(\nabla^A)^\dagger \nabla^A \varphi = \nabla^\dagger \varphi + \sum_{i=1}^{n} \frac{1}{2}(A_{e_i} - A^\dagger_{e_i})\nabla_{e_i} \varphi + \frac{1}{2} \sum_{i=1}^{n} (\nabla_{e_i}(A_{e_i})\varphi + \text{div } e_i \cdot A_{e_i} \varphi + (\nabla_{e_i} h)(\cdot, A_{e_i} \varphi)^2)
\]
\[+ \sum_{i=1}^{n} \frac{1}{2}A^\dagger_{e_i} A_{e_i} \varphi \]
\[= \nabla^\dagger \varphi + \frac{1}{2}(A - A^\dagger) \nabla \varphi + \frac{1}{2} \sum_{i=1}^{n} (\nabla_{e_i}(A_{e_i})\varphi + \text{div } e_i \cdot A_{e_i} \varphi + (\nabla_{e_i} h)(\cdot, A_{e_i} \varphi)^2 + \frac{1}{2}A^\dagger_{e_i} A_{e_i} \varphi),\]
where we suggestively wrote $A^\dagger$ for the section defined by $A^\dagger(X^\varphi) = A^\dagger_X \varphi$. Note that $A - A^\dagger$ is skewsymmetric.

The above calculation shows that, if $A \neq 0$, then one has $\Box 13.1$ if and only if $A_X$ is skewsymmetric for all $X$.

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Mathematisches Institut, Universität Leipzig, Leipzig, Germany
Email address: boldt@math.uni-leipzig.de

Fakultät für Mathematik, Technische Universität Chemnitz, Chemnitz, Germany
Email address: batu.guenysu@math.tu-chemnitz.de