Low energy physics of interacting bosons with a moat spectrum, and the implications for condensed matter and cold nuclear matter

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(Dated: April 9, 2021)

We discuss bosonic models with a moat spectrum, where in momentum space the minimum of the dispersion relation is on a sphere of nonzero radius. For spinless bosons with $O(N)$ symmetry, we emphasize the essential difference between $N=2$ and $N>2$. When $N=2$, there are two phase transitions: at zero temperature, a transition to a state with Bose condensation, and at nonzero temperature, a transition to a spatially inhomogeneous state. When $N>2$, previous analysis\textsuperscript{[1, 2]} suggests that a mass gap is generated dynamically at any temperature. In condensed matter, a moat spectrum is important for spin-orbit-coupled bosons. For cold nuclear or quarkyonic matter, we suggest that the transport properties, such as neutrino emission, are dominated by the phonons related to a moat spectrum; also, that at least in the quarkyonic phase the nucleons may be a non-Fermi liquid.

Several recent papers\textsuperscript{[3–5]} discuss bosonic systems with a “moat” spectrum, where the energy $\epsilon(p)$ depends upon the spatial momentum $p$ as

$$\epsilon(p)^2 = v^2(p^2 - Q^2)^2 + r,$$

where $v^2$, $r$, and especially $Q^2$ are all nonzero. The minimum of the energy is at the bottom of the moat, when $p^2 = Q^2$, and has a local maximum at zero momentum $p = 0$.

Refs.\textsuperscript{[3, 4]} suggest that such systems display certain analogies to Fermi liquids, where the gapless surface survives down to the lowest energies. In this paper we argue that this is unlikely, at least for the models considered in Refs.\textsuperscript{[3–5]}.

In two and three spatial dimensions, we argue that a system with $O(2)$ symmetry undergoes two phase transitions: at zero temperature, a transition to a state with Bose condensation, and at nonzero temperature, a transition to a spatially inhomogeneous state. At nonzero temperature the rotational symmetry in space is spontaneously broken by singling out a particular wave vector $Q$ on the moat, while at zero temperature, a Bose condensate develops at $Q$.

Even when $r = 0$, when the symmetry is non-Abelian, such as $O(N)$ with $N>2$, there is no condensate either at nonzero\textsuperscript{[1]} nor zero\textsuperscript{[2]} temperature. Instead a dynamically generated gap opens over the entire bottom of the moat, $p^2 = Q^2$. In Refs.\textsuperscript{[1, 2]} this was shown using a $O(N)$ model at large $N$, but we suggest that it occurs for all $N>2$.

Besides the question of principle, such models are of interest in both condensed matter and nuclear physics. For example, spin-orbit-coupled bosons\textsuperscript{[8]} have a moat spectrum. For Quantum ChromoDynamics (QCD)\textsuperscript{[9]}, in nuclear matter it arises for pion\textsuperscript{[10–23]} and kaon\textsuperscript{[24–27]} condensates, and in the quarkyonic regime\textsuperscript{[1, 2, 28–30]}, for chiral spirals\textsuperscript{[31–68]}. As we discuss, the moat spectrum will have important implications for both, and especially for the transport properties of nuclear matter.

Spinless bosons with a moat spectrum. The first model we consider is a model of $d$-dimensional bosons with a moat spectrum,

$$L = \int d^dx \left[ b^\dagger \partial_\tau b - \mu b^\dagger b + \frac{1}{2} g (b^\dagger b)^2 + \frac{1}{2mQ^2} b^\dagger(-\nabla^2 - Q^2)^2 b \right].$$

For a real $b$-field, this model is similar to the Landau-Brazovskii model of weak crystallization\textsuperscript{[69]}.

Free $b$ bosons do not condense, but interacting bosons can. We begin by integrating out fluctuations in the density. When the average density is large, fluctuations in the density are massive and can be integrated out by a change in variables, $b = \sqrt{\rho} e^{i\phi}$, so that

$$L = \int d^Dx \left\{ i\delta\phi, \phi + \frac{g}{2} (\rho - \mu/g)^2 + \frac{1}{2mQ^2} \left[ \rho^{1/2} e^{-i\phi}(\nabla^2 + Q^2)^2 e^{i\phi}\rho^{1/2} \right] \right\}.$$}

We assume that the interaction is weak, replacing $\rho$ by its average value, $\rho \to \rho_0$. The conventional Hartree approximation yields $\mu = g\rho_0$, and sets an upper bound on the interaction strength, $g\rho_0 \ll Q^2/m$. A lower bound...
follows by comparing \( \mu \), evaluated in Hartree approximation for bosons, with an alternative “fermionized” state in two dimensions specific for the moat spectrum \( \mathcal{S} \), \( g\rho_0 \gg \rho_0^2/mQ^2 \). These two constraints are compatible at low density, \( \rho_0 \ll Q^2 \).

We now show that \( \phi = Qr + \alpha \) is a stable ansatz, where \( Q^2 = Q^2 \) and the direction of \( Q \) is arbitrary. This choice breaks the rotational symmetry, where the order parameter is the current \( J = i b^\dagger \nabla b \). Notice that introduction of several wave vectors decreases the interaction energy. There is a second order parameter, namely \( \alpha \); for \( \rho = \bar{\rho} \) the current breaks the rotational symmetry, where the order \( Q \) is arbitrary. This form of the correlator lies along the \( x \) axis. This form of the correlator is

\[
\mathcal{L} = \frac{(\partial_\alpha)^2}{2g} + \frac{\rho_0}{2mQ^2}
\]

It is convenient to rescale \( \alpha = M^{1/2} \bar{\alpha} \) and \( \nu^2 = M/g \), where \( M = mQ^2/\rho_0 \), so that Eq. (1) becomes

\[
\mathcal{L} = \nu^2(\partial_\alpha)^2 + 2Q(\partial^2Q + \frac{M^{1/2}(\nabla^2})^2 + (\nabla^2 \bar{\alpha})^2, \\
\text{for which the bare inverse propagator is}
\]

\[
(\bar{\alpha})^{-1} = \nu^2 \omega^2 + \gamma p^2 + (p^2)^2, \\
\text{where } \gamma = 4Q^2. \text{ Here and in what follows we assume that } Q \text{ lies along the } x \text{ axis. This form of the correlator is preserved at } T = 0 \text{ because of rotational invariance and since Eq. (1) is infrared finite at zero temperature. Indeed, an infinitesimal change } \bar{\alpha} \to \bar{\alpha} + B \cdot r, \text{ where } B \cdot Q = 0 \text{ does not change the action. This implies the absence of a term } \sim p^2.
\]

To find corrections to the propagator we rewrite the last term in \( \mathcal{S} \) as \( (2Q(\partial_\alpha) + (\nabla^2))^2/(2M) \), so to remove spurious \( p^2 \) corrections to the self energy, which are removed by shifting \( Q \to Q_0 - (\nabla^2)/(2Q_0) \). We also distinguish between \( Q \) in \( M \) and the coefficient in front of \( \partial_\alpha \) since these two quantities renormalize differently.

The crucial difference between our analysis and that of Ref. \( \mathcal{S} \) is their neglect of higher order terms in the inverse propagator, \( \sim (p^2)^2 \), Eq. (1), while we include them. We show that this term ensures that the Bose condensate is stable at zero temperature.

In \( D \) spatial dimensions the first correction to the self energy is

\[
\Sigma^{(1)} = \rho_0^2(2Q)^2MT \sum_n \int \frac{d^Dp}{(2\pi)^D} \frac{(2p^2 + p^2)^2}{(\nu^2 \omega^2 + 4Q^2 p^2 + (p^2)^2)}, \\
\text{This integral converges in the infrared at zero temperature, } T = 0 \text{ for } D > 1 \text{ and diverges for } D < 3 \text{ at } T = 0. \text{ Since } G^{-1} = G_0^{-1} - \Sigma, \text{ this singular diagram leads to reduction of the longitudinal stiffness. At zero temperature in } D = 2, \text{ the correction to the stiffness is } \delta\gamma/\gamma \sim -M/\nu = (M Q_0^2/\rho_0)^{1/2}, \text{ which is the small parameter of the expansion. We show that the stiffness is a nonanalytic function of } T. \text{ At zero temperature, the single particle correlation function is}
\]

\[
G(\tau, r) = \langle b(\tau, r)b(0, 0) \rangle \approx \rho_0 \delta(Qr) e^{-\xi(0, 0)}, \\
\text{where } \xi \text{ is the condensate.}
\]

\[
\text{Thus the bosons spontaneously choose a wave vector on the circle } |Q| = Q \text{ and condense. However, at } T \neq 0 \text{ the integral diverges in the infrared for } D < 4, \text{ so there is no condensation.}
\]

To determine what happens at nonzero temperatures we concentrate on classical fluctuations, corresponding to zero Matsubara frequency. For \( D = 3 \) the first correction to the stiffness diverges logarithmically. The renormalization group equations are \( (\gamma = 4Q^2): \)

\[
\frac{d\gamma}{d\xi} = -\gamma^{1/2} M; \quad \frac{dM}{d\xi} = 10^{-\gamma^{1/2} M^2}, \\
\text{where } \xi = \ln(Q_0/\rho)/(8\pi), \text{ } \gamma = 4Q_0^2(M_0/M)^{1/10} \text{ and } M = M_0/[1 - (19M_0/16\pi Q_0) \ln(Q_0/\rho)]^{20/19}. \text{ This implies that for } D = 3 \text{ at } T \neq 0 \text{ the longitudinal stiffness disappears at the moment } p_0 \sim Q_0 \exp[-16\pi Q_0/19M_0]. \text{ At this scale the fluctuations of } \nabla \alpha \text{ become of the order of } Q_0 \text{ and the spectrum becomes effectively isotropic around } Q. \text{ It is interesting that at nonzero frequency the infrared divergence in (1) is cut by the frequency itself. Therefore there is also a frequency scale above which the stiffness remains finite.}
\]

As far as the broken rotational symmetry associated with the finite current \( J \), the average order parameter remains finite, at least until some critical temperature, \( Q(T) = Q_0 - (\nabla \alpha) = Q_0 - \text{const} T \). Moats in cold nuclear/quarkyonic matter. In this section we would like to comment on the role of a moat spectrum for cold nuclear or quarkyonic matter [1, 2, 30–32]. Consider \( N_f \) flavors of massless quarks coupled to a \( SU(N_c) \) gauge theory. From the left- and right-handed quarks \( q^a_{L,R} \), one can form the gauge invariant quantity,

\[
\Phi^{ab}(x) = \bar{q}^a_{L}(x) g^{ab} \bar{q}^b_R(x),
\]

\[
\text{where } g^{ab} \text{ is the \( SU(N_c) \) structure constant.}
\]
where \(i, j, \ldots, N_c\) are indices for the fundamental representation of the SU\((N_c)\) gauge group, and \(a, b, \ldots, N_f\) for the flavor symmetry of SU\((N_f)\). In vacuum breaks spontaneously to SU\((N_f)\). There is also an axial \(U(1)_A\) symmetry which is broken dynamically by topologically nontrivial fluctuations, but this probably remains strongly broken until extremely high densities [71].

With dynamical quarks there is no precise measure of confinement, but at asymptotically high temperature or baryon density the pressure approaches that of a nearly ideal gas of quarks and gluons. Our interest here is what happens at low temperature as the quark chemical potential decreases, and one enters a quarkyonic phase [1, 2, 28, 68]. While the free energy is approximately that of free quarks, the excitations near the edge of the Fermi surface are confined. As the chemical potential decreases further, quarkyonic matter becomes hadronic, with a free energy which far from quarkish, and again excitations near the Fermi surface which are confined. This illustrates the basic continuity between hadronic and quarkyonic matter.

Studies in lower dimensional models show that at low temperature and nonzero density a spatially inhomogeneous solution arises in 1 + 1 [51, 57, 63, 72, 81] and 2 + 1 dimensions [68, 51, 31]. For spatially inhomogeneous states pairing occurs between a particle at one edge of the Fermi surface, with momentum \(k_F\), and a hole on the other edge, with momentum \(-k_F\). Fig. (3) of Ref. 31]. Because the pairing is between a particle-hole pair, the condensate carries a net momentum \(2k_F\).

In a gauge theory a gauge invariant order parameter can be constructed in terms of the quark fields. Since the other edge, with momentum \(k\), the condensate carries a net momentum \(2k\). Because the pairing is between a particle-hole pair and a hole on the other edge, with momentum \(-k\), Fig. (3) of Ref. 31]. Because the pairing is between a particle-hole pair, the condensate carries a net momentum \(2k\).

The global symmetry can be enlarged by the spin degrees of freedom. In quarkyonic matter, the flavor SU\((N_f)\) symmetry increases to a SU\((2N_f)\) symmetry of spin and flavor when magnetic interactions can be ignored [30]. Similarly, in nuclear matter an increased spin-flavor symmetry is exact at infinite \(N_c\) [83], where it is related to the supermultiplet symmetry of Wigner [84, 85]. Our analysis, which here is entirely qualitative, is very similar in either case.

In greater than one spatial dimension, the direction of the density wave is chosen spontaneously, and the Fermi surface is covered by patches of kink condensates [1, 2, 30, 32]. Because kink crystals are spatially periodic, they spontaneously break translational symmetry along the condensate axis, and generate phonons as the associated Goldstone modes. There are non-Abelian phonons, associated with flavor rotations of matrix field \(G^{ab}\), and Abelian, associated with the overall \(U(1)\) phase of \(\text{det}(G)\) [30, 92]. In greater than one spatial dimension, the stiffness of the phonons vanishes in the transverse direction, as in Eq. 9. At nonzero temperature, the absence of the transverse stiffness leads to strong fluctuations which generate a finite correlation length for the non-Abelian phonons, while the Abelian phonon remain massless [1, 30, 32]. At zero temperature, the correlation length for the non-Abelian phonons diverges exponentially at zero temperature, Eq. (21) of Ref. [30].

In contrast, in a phenomenological \(O(N)\) sigma model with a moat spectrum, the correlation length for the non-Abelian phonons remains finite even at zero temperature [80]. It is not clear if the difference is relevant, as the mass gap at zero temperature is much smaller than at nonzero temperature. Further the chiral symmetry is only approximate in QCD, which thus generates a small mass gap for the flavored phonons in any case. The Abelian phonon is related only to fermion number, and so is always massless.

The phonons can play an essential role in transport properties in neutron/quarkyonic stars. Consider, for example, cooling through the emission of neutrinos [87–92]. In that case, the flavored phonon can decay into a virtual nucleon pair, and thereby through the weak interaction into a lepton neutrino pair [93]:

\[
\mathcal{L}_W \sim ig_W [\bar{\nu}_e (Q \cdot \gamma)(1 - \gamma_5) \nu_e + \bar{\nu}_\mu (Q \cdot \gamma)(1 - \gamma_5) \nu_\mu] \times \left( \text{Tr}(G^+ Q \cdot \nabla G^+ \tau^+) \right),
\]

where \(\tau^+\) is a Pauli matrix acting on flavor indices and \(G\) is the field for the non-Abelian phonon [30]. The Abelian phonon only has diagonal couplings to nucleons, and so only produces neutrinos (and leptons) through processes of second order in the weak interactions. This decay process is analogous to neutrino emission by pion condensates [87, 92].

How do the nucleons near the Fermi surface contribute to the transport properties? In the quarkyonic phase, a model with a confining potential reduces to QCD in 1 + 1 dimensions at nonzero density. The phase diagram of this model is not known, as there are only results for a single, heavy quark by Bringoltz [74]. For a Nambu-Jona-Lasinio model in 1 + 1 dimensions, using conformal symmetry and the truncated spectrum approach, it has
been shown that the theory could be a non-Fermi liquid with gapless but incoherent nucleons \[46, 51\]. Nucleon operators are expressible in terms of the bosonic fields following the rules of bosonization. Existence of this non-Fermi liquid regime depends upon a value of a parameter, \(K\), which is the coefficient of the kinetic term for the Abelian phonon. Making the simplest assumption that \(K = 1\), corresponding to weak interactions the NJL model at nonzero density is a non-Fermi liquid, a type of “strange” metal familiar from high-\(T_c\) superconductivity \[94\] and holography \[95\].

The saddle point condition is
\[
\frac{1}{g} = \frac{T}{\sum_n} \int \frac{d^dp}{(2\pi)^D} \frac{1}{\omega_n^2 + \frac{1}{m^2}(p^2 - Q^2)^2 + M^2}. 
\]

At \(T = 0\) we have
\[
\frac{2}{g} = \int \frac{d^dp}{(2\pi)^D} \frac{1}{\mid \frac{1}{m^2}(p^2 - Q^2)^2 + M^2 \mid^{1/2}}. 
\]

The gap \(M\) is finite for any \(D\), in particular for \(D = 2\) we have
\[
M = \left( \frac{p_{\text{max}}^2 - Q^2}{m} \right)^{1/2} Q \exp \left( -\frac{4\pi}{mg} \right). 
\]

Hence the spectrum is gapped but retains its moat-like form, unlike the \(U(1)\) model.

A.M.T. was supported by the U.S. Department of Energy, Office of Science, Materials Sciences and Engineering Division under contract DE-SC0012704. R.D.P. was supported by the U.S. Department of Energy under contract DE-SC001270 and by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Co-design Center for Quantum Advantage (C²QA) under contract DE-SC001270. A.M.T. and R.D.P. were also supported by B.N.L. under the Lab Directed Research and Development program 18-036. A.M.T. is grateful to L. Glazman for early discussions; R.D.P. thanks J. Schaffner-Bielich and J. Lattimer for discussions. We thank E. Lake, S. Sur, X.-G. Zhang and G. Chen for discussions of their work.

**SUPPLEMENTARY MATERIAL**

In this Section for the sake of completeness we repeat the calculations for the O(N) nonlinear sigma model from \[2\].

We consider the Lagrangian density in \(D + 1\) dimensions
\[
\mathcal{L} = \frac{N}{2g} \left\{ \left( \partial_t n \right)^2 + \frac{1}{m^2} \left[ (\nabla^2 + Q^2)n \right]^2 \right\}, \quad \sum_{a=1}^{N} n_a^2 = 1. 
\]

We will treat this model in large \(N\) approximation. The Green’s function is
\[
\langle \langle n^a(\omega, -p)n^n(\omega, p) \rangle \rangle = \frac{1}{gN} \frac{1}{\omega^2 + \frac{1}{m^2}(p^2 - Q^2)^2 + M^2}. 
\]

The saddle point condition is
\[
\frac{1}{g} = \frac{T}{\sum_n} \int \frac{d^dp}{(2\pi)^D} \frac{1}{\omega_n^2 + \frac{1}{m^2}(p^2 - Q^2)^2 + M^2}. 
\]

At \(T = 0\) we have
\[
\frac{2}{g} = \int \frac{d^dp}{(2\pi)^D} \frac{1}{\mid \frac{1}{m^2}(p^2 - Q^2)^2 + M^2 \mid^{1/2}}. 
\]
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[96] The Lagrangian coupling the nucleons to the phonons is $\mathcal{L}_{NN\chi} = \int d^4x \left( N \cdot \partial \chi + \frac{1}{2} (\partial \chi)^2 \right) N$. The anisotropic coupling of the phonon is dictated by the spontaneous breaking of the rotational symmetry, as for the phonon Lagrangian. Since the kink crystal does not need to respect reflection symmetry along the axis of the kink crystal, a coupling linear in $N \cdot \partial \chi$ is allowed.