Binary Polar Codes are Optimized Codes for Bitwise Multistage Decoding

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Polar codes are considered the latest major breakthrough in coding theory. Polar codes were introduced by Arıkan in 2008. In this letter, we show that the binary polar codes are the same as the optimized codes for bitwise multistage decoding (OCBM), which have been discovered before by Stolte in 2002. The equivalence between the techniques used for the constructions and decodings of both codes is established.

Introduction: Polar codes were introduced by Arıkan in 2008, who proved that they can achieve the symmetric capacity of binary-input discrete memoryless channels [1]. About seven years earlier, a class of codes called optimized codes for bitwise multistage decoding were constructed by Stolte [2]. We will refer to the codes of [2] as OCBM codes. Although they were numerically shown to have near-capacity performance [2], they have not received much attention as there was no explicit proof of their capacity achieving property. Polar codes have received a lot of attention since Arıkan showed that they have an explicit construction and are provably capacity achieving. The channel polarization phenomenon, introduced by Arıkan [1], has since been extended to other channels, such as the Gaussian channel with signal to noise ratio (SNR) SNR, and channel capacity C0 = C0(SNR), which can be calculated by numerical integration, cf. [15]. The equivalent SNR (EqSNR) of the equivalent channel observed by u is SNR, = 2SNR0, and of that observed by v is SNR, = C0 − 1(2C0(SNR0) − c(SNR0))). Assuming these two equivalent channels are still Gaussian with the corresponding calculated equivalent SNRs, then the EqSNR’s of all N equivalent bit-channels can be calculated by log2(N) recursions. It is clear that C0 ≥ C0 ≥ C0 by the EqSNR method, which is the channel polarization phenomenon later observed by Arıkan. However, Arıkan [1] was the first to explicitly prove that the ratio of the number of good bit-channels (whose error probability approaches zero) to the length of the code approaches the channel capacity C0. It is worth noting that the sum-capacity observation can also be used to construct OCBM codes for other channels.

Polar codes [1] were constructed using the Bhattacharyya parameters (BPs) [16] as the reliability measure, where the BPs for the binary erasure channel can be recursively calculated to choose the k information bit-channels and freeze the other N − k bit-channels. Different approaches were later developed to construct polar codes for other channels, notably the BIAWGNC, such as the estimation of bit-channel reliability by lower and upper bounds using degrading and upgrading channel quantizations [17], and the density evolution with Gaussian approximation method (DE-GA) [18]. Similar to the EqSNR method, DE-GA assumes that the equivalent channels after each polarization step are Gaussian. Consequently, their corresponding detection log-likelihood ratios (LLRs) are assumed to have symmetric Gaussian distributions whose variance is twice their mean, i.e. SNR = |L0|/2 for expected LLR L0. The DE-GA method [18] uses the well-known density evolution to track the means of the distributions and has been generalized to the case of two non-identical input channels with expected LLRs L1 and L2 where the expected LLRs of the degraded and upgraded channels can be respectively calculated by

\[ |L_{\text{d}}| = \phi^{-1}(1 - (1 - \phi(L_1)) (1 - \phi(L_2))) \quad \text{and} \quad L_{\text{u}} = L_1 + L_2, \]

such that

\[ \phi(x) = \begin{cases} 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, & \text{if } x > 0, \\ 1, & \text{if } x = 0. \end{cases} \]

Similar to the EqSNR method, the equivalent LLRs given by the DE-GA method can be calculated recursively.

We implemented both the EqSNR and GA-DE methods using numerical integrations and lookup tables. To construct a code with rate k/N, using either the EqSNR or GA-DE methods, the k bit-channels with the lowest error probability, or equivalently with the highest EqSNRs or largest expected LLRs, are chosen to carry the information bits, and the other bit-channels are frozen to zero. It is clear that in case of identical input LLRs, the LLRs of the upgraded channels obtained by both methods are twice the input LLRs. We calculated the relationship between the input LLR L0 and the estimated LLR of the degraded channel (Ld) for the BIAWGNC. We numerically verify in Fig. 1 that both methods also result in an almost same degrading channel effect, except that the GA-DE method has better numerical accuracy than the EqSNR method at the extreme ranges due to the capping effect of the capacity functions used to calculate the EqSNR.

**Fig. 1. BIAWGNC Degrading Channel Transfer Function**

Systematic encoding is known to minimize the bit error rate (BER) of binary linear codes under maximum likelihood decoding [20]. Systematic encoding of OCBM codes was done by identifying the set of k independent output code coordinates \( O \) given the set of input indices I, setting \( O \) to carry the desired information bits, erasing the remaining coordinates, and then decoding to recover the erased bits [2]. Systematic encoding has also been considered for polar codes by choosing the k output coordinates depending on I, setting them to the

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desired information, and recursively solving for the unknown coordinates \([21]\). It was also noted in \([21]\) that systematic encoding of polar codes can alternatively be done by successive cancellation decoding (SCD) after erasing the unknown coordinates, which is the same as that was adopted for OCBM codes. Whereas OCBM codes were constructed as generalized concatenated codes, generalized concatenations of polar codes were later developed, cf. \([18,22,23]\).

**Decoding algorithms:** The recursive Plotkin construction of RM codes was used to devise low complexity recursive decoding algorithms. Schnabl and Bossert devised a recursive multistage decoder (MSD) for RM codes based on their recursive Plotkin construction \([6]\). Consequently, an MSD decoder was proposed for OCBM codes \([2]\), by deploying the difference of probabilities metric, where for an BIAWGNC \(h_k = P(c_k = +1|y_k) - P(c_k = -1|y_k) = \tanh(L_k/2)\), for code bit \(c_k\), channel output \(y_k\), and corresponding channel LLL \(L_k\). For code length \(N\), the reliability of the equivalent bit-channels of the inner code \(v\) are first calculated using the channel observations \(h_k^{(v)} = h_k - h_{k-2}^{(v)}\), from which the information bits \(v\) are estimated as \(\hat{v}_k\). The estimated bits of the outer code \(v\) are then used together with the channel observations to calculate the reliability of bit-channels of the outer code \(u\) as \(h_k^{(u)} = h_k - h_{k-2}^{(u)}\), from which the information bits of the outer code \(u\) are estimated. This decoder will be referred to as MSD(\(h\)) and can be implemented recursively, where the reliability values calculated at a certain stage are then passed to the next stage till the outer codes are of unit length. Hence, its computational complexity is of \(O(N \log_2(N))\) \([8]\). The successive cancellation decoder proposed for decoding polar codes is similarly recursive with complexity \(O(N \log_2(N))\), but uses the likelihood ratios as reliability values, and will be referred to as SCD(\(L\)) \([1]\). Based on the previous works on recursive decoding of RM codes \([18,21]\), Stolte \([2]\) also considered sequential stack decoding, list decoding, and list decoding with different permutations of the OCBM codes to improve their finite length decoding performance. Similarly, stack decoding and list decoding have been later considered for decoding polar codes \([24,25]\). It has been observed that list decoding of OCBM codes has better performance than their stack decoding and approaches their maximum likelihood decoding performance \([2]\).

**Numerical Comparisons:** In Fig. 2 we compared the simulated block (BLER) and bit error rate (BER) performances of polar codes and OCBM codes with rate 0.5 on the BIAWGNC. We show the error-rates at different code lengths \(N\) \(\in\{2^{10}, 2^{14}, 2^{17}\}\). The codes are reconstructed at each SNR point to illustrate the effectiveness of the construction method. We refer to the codes constructed by the GA-DE method as polar codes, and those constructed by the EqSNR method as OCBM codes. For the code length with \(N = 2^{10}\), we also show the code constructed by estimating the bit-channel reliability with Monte-Carlo simulations of genie-aided SCD (Genie). It is observed that all three codes have almost the same performance, except at high SNRs due to numerical inaccuracies. We also compare the performances of the SCD(\(L\)) and MSD(\(h\)) decoders, developed for polar and OCBM codes, respectively. It is observed that their performances are very close, except at higher SNRs and longer codes, where the MSD(\(h\)) decoder gives a better performance by using the difference of probabilities metric.

We conclude that the binary polar codes are the same as the OCBM codes, and the methods developed for decoding both codes are equivalent. M. El-Khamy (Samsung Modem R&D, San Diego, USA, and Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt)
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**Numerical Comparisons:**

![Fig. 2. Polar vs OCBM codes of rate 0.5, and lengths up to 131,072 bits](image_url)

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