Lower bounds on the error probability of quantum channel discrimination by the Bures angle and the trace distance

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Quantum channel discrimination is a fundamental problem in quantum information science. In this study, we consider general quantum channel discrimination problems, and derive the lower bounds of the error probability. Our lower bounds are based on the triangle inequalities of the Bures angle and the trace distance. As a consequence of the lower bound based on the Bures angle, we prove the optimality of Grover’s search if the number of marked elements is fixed to some integer \( \ell \). This result generalizes Zalka’s result for \( \ell = 1 \). We also present several numerical results in which our lower bounds based on the trace distance outperform recently obtained lower bounds.

I. INTRODUCTION

The quantum channel discrimination problem is a fundamental problem in quantum information science \cite{1,2,3}. In this study, we consider a general quantum channel discrimination problem, and derive the lower bounds of the error probability.

From Helstrom’s seminar work \cite{4} and Holevo’s subsequent work \cite{5}, the error probability of the discrimination of two quantum states \( \rho \) and \( \sigma \) is characterized by the trace distance \( \frac{1}{2} \| \rho - \sigma \|_1 \), which can be computed efficiently. As a consequence, the error probability of the discrimination of two quantum channels \( \Psi \) and \( \Phi \) by algorithms invoking the quantum channel once can be characterized by the maximum of the trace distance, \( \max \frac{1}{2} \| \Phi \otimes \id(\rho) - \Psi \otimes \id(\rho) \|_1 =: \frac{1}{2} \| \Phi - \Psi \|_0 \), which is called the diamond distance. The diamond distance can be computed efficiently as well \cite{6}. However, if we consider algorithms that use the quantum channel \( n \) times, the analysis is much more complicated. If we restrict the discrimination algorithms to be non-adaptive, i.e., the given channel is called \( n \) times at once in parallel, the error probability is characterized by the diamond distance \( \frac{1}{2} \| \Phi \otimes \id - \Psi \otimes \id \|_0 \). To compute the above diamond distance, we have to solve an optimization problem on exponentially high-dimensional linear space in \( n \). Hence, it is computationally hard to compute the error probability of non-adaptive algorithms with \( n \) queries when \( n \) is large. Furthermore, for general discrimination algorithms, no simple representation of the error probability has been known even for the discrimination of two quantum channels.

The exact error probability for a fixed number of queries is known if two unitary channels are given with uniform probabilities \cite{7}. In this case, a non-adaptive algorithm achieves the minimum error probability. For general quantum channel discriminations, however, there exists some discrimination problem in which some adaptive algorithm gives a strictly smaller error probability than any non-adaptive algorithms \cite{1,2,3}.

The well-known Grover’s search problem can be regarded as an instance of a quantum channel discrimination problem if there is exactly one marked element. Grover’s search algorithm solves this problem with \( O(\sqrt{N}) \) queries, where \( N \) is the number of elements. The asymptotic optimality of Grover’s algorithm was shown in \cite{8}. Zalka showed that the error probability of Grover’s algorithm for a fixed number of queries is optimal when there is exactly one marked element \cite{9}.

Recently, lower bounds on the error probability of general quantum channel discrimination have been studied \cite{10,11,12,13}. In this study, we derive several lower bounds on the error probability of adaptive algorithms for quantum channel discrimination problems. On the lines of previous studies \cite{10,11,13}, the main technique we use to derive the lower bounds is the application of triangle inequalities for some distance measures to series of quantum states. In this work, we use two distance measures, namely, the Bures angle and the trace distance.

In this paper, we first derive lower bounds on the error probability of two quantum channel discriminations. Our technique using the Bures angle is a natural generalization of the known technique for unitary channel discrimination \cite{7}. For the discrimination of two amplitude damping channels with damping rates \( r_0 \) and \( r_1 \), we obtain a simple analytic lower bound \( \frac{1}{2} \left( 1 - \sin(n \Delta(r_0, r_1)) \right) \) if \( n \Delta(r_0, r_1) \leq \frac{\pi}{2} \), where \( \Delta(r_0, r_1) := \arccos(\sqrt{r_0 r_1} + \sqrt{(1 - r_0)(1 - r_1)}) \). This lower bound is better than the known lower bounds \cite{10,13} for some choices of the damping rates \( r_0 \) and \( r_1 \) and the number \( n \) of queries. For the lower bound based on the trace distance, we introduce “weights” for the series of quantum states to tighten the triangle inequality. We present several numerical results in which our lower bounds for two quantum channel discrimination based on the trace distance outperform recently obtained lower bounds.

We next consider a quantum channel group discrimination problem, which is a generalization of the quantum channel discrimination problem, and derive lower bounds of the error probability. Grover’s search problem is an instance of the quantum channel group discrimination problem if the number of marked elements is fixed.
to some integer $\ell$. Our lower bound based on the Bures angle shows the optimality of Grover’s algorithm in this case. This result generalizes Zalka’s result for $\ell = 1$. Our lower bound based on the trace distance yields better lower bounds for some problems considered in [5].

This paper is organized as follows. Distance measures for quantum states, including the Bures angle and the trace distance, are introduced in Section II. The quantum channel discrimination problem and the quantum channel group discrimination problem are introduced in Section III. The main results in this paper are presented in Section IV. The applications of our lower bounds are presented in Section VI. In Section VII, we show that the lower bounds derived in this paper can be computed efficiently using semidefinite programming (SDP). Finally, we add some conclude remarks in Section VIII.

II. THE TRACE DISTANCE AND THE BURES ANGLE

In this section, we introduce the notion of distance measures for quantum states, and several examples of distance measures, including the trace distance and the Bures angle. Let $L(A)$ be a set of linear operators on a quantum system $A$. Let $D(A) \subseteq L(A)$ be a set of density operators on a quantum system $A$.

Definition 1 (Distance measure for quantum states). A function

$$D : \{(\rho_A \in D(A), \sigma_A \in D(A)) \mid A: a \text{ system}\} \to [0, \infty)$$

is called a distance function if it satisfies the following conditions:

1. (Identity of indiscernibles) For any $\rho_A, \sigma_A \in D(A)$, $D(\rho_A, \sigma_A) = 0$ if and only if $\rho_A = \sigma_A$.
2. (Symmetry) For any $\rho_A, \sigma_A \in D(A)$, $D(\rho_A, \sigma_A) = D(\sigma_A, \rho_A)$.
3. (Triangle inequality) For any $\rho_A, \sigma_A, \tau_A \in D(A)$,

$$D(\rho_A, \sigma_A) \leq D(\rho_A, \tau_A) + D(\tau_A, \sigma_A).$$
4. (Monotonicity) For any $\rho_A, \sigma_A \in D(A)$ and any quantum channel $\Psi_{A\to B}$ from a quantum system $A$ to a quantum system $B$,

$$D(\Psi_{A\to B}(\rho_A), \Psi_{A\to B}(\sigma_A)) \leq D(\rho_A, \sigma_A).$$

The trace distance is a well-known distance measure for quantum states.

Definition 2 (Trace distance). For a linear operator $L_A \in L(A)$, the trace norm is defined as $\|L_A\|_1 := \text{Tr} \sqrt{L_A^* L_A}$. For density operators $\rho_A, \sigma_A \in D(A)$, the trace distance is defined as $D(\rho_A, \sigma_A) = \frac{1}{2} \|\rho_A - \sigma_A\|_1$.

The trace distance has a useful property called the joint convexity, which is $D \left( \sum p_i \rho_A^{(i)}, \sum p_i \sigma_A^{(i)} \right) \leq \sum p_i D(\rho_A^{(i)}, \sigma_A^{(i)})$ for any probability distribution $(p_i)$ and density operators $(\rho_A^{(i)})$ and $(\sigma_A^{(i)})$.

The fidelity of quantum states is defined as follows.

Definition 3 (Fidelity). For density operators $\rho_A, \sigma_A \in D(A)$, the fidelity is defined as

$$F(\rho_A, \sigma_A) := \|\sqrt{\rho_A} \sqrt{\sigma_A}\|_1.$$ 

The Bures angle is defined as follows.

Definition 4 (Bures angle). For density operators $\rho_A, \sigma_A \in D(A)$, the Bures angle is defined as

$$\mathcal{A}(\rho_A, \sigma_A) := \arccos F(\rho_A, \sigma_A).$$

The Bures angle satisfies the conditions on the distance measure for quantum states [15].

There are several distance measures induced by the fidelity. Indeed, the Bures distance $B(\rho_A, \sigma_A)$ and the sine distance $C(\rho_A, \sigma_A)$, defined as

$$B(\rho_A, \sigma_A) := \sqrt{2 - 2F(\rho_A, \sigma_A)}$$

and

$$C(\rho_A, \sigma_A) := \sqrt{1 - F(\rho_A, \sigma_A)^2},$$

satisfy the conditions on the distance measure for quantum states. These distance measures can be naturally represented in terms of the Bures angle. Indeed, for $\theta = \mathcal{A}(\rho_A, \sigma_A)$, it holds $B(\rho_A, \sigma_A) = 2 \sin(\theta/2)$ and $C(\rho_A, \sigma_A) = \sin(\theta)$. The triangle inequalities for the Bures distance and the sine distance are obtained from the triangle inequality for the Bures angle (see Appendix A). This fact means that the triangle inequality for the Bures angle gives better bounds than the triangle inequalities for the Bures distance and the sine distance. Hence, in this study, we use the Bures angle, and do not use the Bures distance or the sine distance to derive inequalities.

Although the fidelity-based distances $A$, $B$, and $C$ are not jointly convex, the fidelity has a useful property called the joint concavity, which is $F \left( \sum_i p_i \rho_A^{(i)}, \sum_i p_i \sigma_A^{(i)} \right) \geq \sum_i p_i F(\rho_A^{(i)}, \sigma_A^{(i)})$ for any probability distribution $(p_i)$ and density operators $(\rho_A^{(i)})$ and $(\sigma_A^{(i)})$.

III. QUANTUM CHANNEL DISCRIMINATION PROBLEMS

Let $A$ and $B$ be finite-dimensional quantum systems. The quantum channel discrimination problem is defined as follows.

Definition 5 (Quantum channel discrimination problem). Let $(\mathcal{O}_{A\to B}^\xi)_{\xi \in \Xi}$ be a finite family of quantum channels and $(p_\xi)_{\xi \in \Xi}$ be a probability distribution on
the quantum channels. Assume that one of the quantum channels $O^{\xi}_A \rightarrow B$ is given as an oracle with probability $p_\xi$. The quantum channel discrimination problem is the problem of determining the index of the given oracle $\xi \in \Xi$ by a quantum algorithm accessing the oracle multiple times. This problem is denoted by QCDP$(p_\xi, (O^{\xi}_A \rightarrow B))$.

We also consider the quantum channel group discrimination problem, which is a natural generalization of the quantum channel discrimination problem.

**Definition 6** (Quantum channel group discrimination problem). Let $(O^{\xi}_A \rightarrow B)_{\xi \in \Xi}$ be a finite family of quantum channels and $(p_\xi)_{\xi \in \Xi}$ be a probability distribution on the quantum channels. Let $(C_n \subseteq \Xi)_{n \in H}$ be a finite family of subsets of $\Xi$. Assume that one of the quantum channels $O^{\xi}_A \rightarrow B$ is given in the same way as QCDP$(p_\xi, (O^{\xi}_A \rightarrow B))$.

The quantum channel group discrimination problem is the problem of finding one of the indexes of a subset $\eta \in H$ that satisfies $\xi \in C_\eta$. This problem is denoted by QCGDP$(p_\xi, (O^{\xi}_A \rightarrow B), (C_n))$.

For example, let $\Xi = \{1, 2, 3, 4\}$, $H = \{A, B, C\}$, $C_A = \{1, 2\}$, $C_B = \{2, 3\}$ and $C_C = \{1, 3\}$. When $\xi = 1$, i.e., the given oracle is $O^{1}_A \rightarrow B$, the output is regarded as a correct answer if it is either of $A$ or $C$. When $\xi = 4$, any output is an incorrect answer. The quantum channel discrimination problem QCDP$(p_\xi, (O^{\xi}_A \rightarrow B))$ can be regarded as the quantum channel group discrimination problem QCGDP$(p_\xi, (O^{\xi}_A \rightarrow B), (C_n))$ where $H = \Xi$ and $C_n = \{\eta\}$ for all $\eta \in H$.

As shown in Section [13], Grover’s search problem can be regarded as an instance of quantum channel group discrimination problem. Another important example of the quantum channel group discrimination problem is the problem of evaluation of Boolean functions [16, 17].

We next consider algorithms that solve the quantum channel group discrimination problem. The general algorithm for the quantum channel group discrimination problem is formulated as follows.

**Definition 7** (Adaptive discrimination algorithm). Let $R$ be a working system. Let $(\Phi^{i}_{BR \rightarrow AR})^n_{i=1}$ be a family of quantum channels, and $(M^\eta_{BR})_{\eta \in H}$ be a positive-operator-valued measure (POVM). Then a pair $(\Phi^{i}_{BR \rightarrow AR}, (M^\eta_{BR}))$ represents the following adaptive algorithm for QCGDP$(p_\xi, (O^{\xi}_A \rightarrow B), (C_n))$.

1. Set the initial state $|0\rangle_{BR}$ in a quantum computer.
2. for $i = 1, 2, \ldots, n$ do
3. Apply the quantum channel $\Phi^{i}_{BR \rightarrow AR}$.
4. Apply the given oracle $O^{\xi}_A \rightarrow B$.
5. end for
6. Apply a measurement $(M^\eta_{BR})_{\eta \in H}$ to the state and output the observed value $\eta \in H$.

An algorithm succeeds if and only if its output $\eta \in H$ satisfies $\xi \in C_\eta$, where $\xi \in \Xi$ is the index of the given oracle $O^{\xi}_A \rightarrow B$. The minimum error probability of a QCGDP is then defined as follows.

**Definition 8** (Minimum error probability of a QCGDP). Let $(\Phi^{i}_{BR \rightarrow AR}, (M^\eta_{BR}))$ be an adaptive algorithm for QCGDP$(p_\xi, (O^{\xi}_A \rightarrow B), (C_n))$ with $n$ queries. For each $\xi \in \Xi$, a density operator $\rho^\xi_{BR}$ is defined as

$$\rho^\xi_{BR} := O^{\xi}_A \rightarrow B \circ \Phi^1_{BR \rightarrow AR} \circ O^{\xi}_A \rightarrow B \circ \Phi^2_{BR \rightarrow AR} \circ \cdots \circ O^{\xi}_A \rightarrow B \circ \Phi^n_{BR \rightarrow AR}(|0\rangle_{BR} \langle 0|_{BR}).$$

The minimum error probability of QCGDP$(p_\xi, (O^{\xi}_A \rightarrow B), (C_n))$ with $n$ adaptive queries is defined as

$$p_{err}(n) := \min_{(M^\eta_{BR})} \sum_{\xi \in \Xi} p_\xi \sum_{\eta: \xi \notin C_\eta} \text{Tr}(M^\eta_{BR} \rho^\xi_{BR}).$$

The minimum error probability of QCDP$(p_\xi, (O^{\xi}_A \rightarrow B))$ is defined in the same way.

An algorithm that achieves the minimum error probability exists for any QCGDP because the set of algorithms is compact and the error probability is continuous. The main goal of this study is to derive lower bounds of the minimum error probability $p_{err}(n)$. The error probability $p_{err}(0)$ without calling the oracle can be evaluated as follows:

$$p_{err}(0) = \min_{(M^\eta_{BR})} \sum_{\xi \in \Xi} p_\xi \sum_{\eta: \xi \notin C_\eta} \text{Tr}(M^\eta_{BR} |0\rangle_{BR} \langle 0|_{BR})$$

$$= \min_{(M^\eta_{BR})} \sum_{\eta \in H} \sum_{\xi \notin C_\eta} \text{Tr}(M^\eta_{BR} |0\rangle_{BR} \langle 0|_{BR}) \sum_{\xi \notin C_\eta} p_\xi$$

$$= \min_{\eta \in H} \sum_{\xi \notin C_\eta} p_\xi.$$

Once the quantum channels $(\Phi^{i}_{BR \rightarrow AR})^n_{i=1}$ in a discrimination algorithm are fixed, the problem is reduced to the quantum state discrimination problem. The optimal measurement and the error probability of discrimination of two quantum states were known by Helstrom [7] and Holevo [8].

**Proposition 1** (Holevo–Helstrom theorem [7, 8, 18]). Let $\rho^0_A, \rho^1_A \in D(A)$ be density operators. Let $p_0, p_1 \in [0, 1]$ be non-negative real numbers that satisfy $p_0 + p_1 = 1$. For any POVM $(M^0_A, M^1_A)$, the following holds:

$$\sum_{\xi \in (0, 1)} p_\xi \text{Tr} \left( M^\xi_A \rho^0_A \right) \leq \frac{1}{2} \left( 1 + \|p_0 \rho^0_A - p_1 \rho^1_A\|_1 \right).$$

Moreover, a POVM $(M^0_A, M^1_A)$ exists that satisfies the equality.

From the Holevo–Helstrom theorem, the error probability of discrimination of two quantum channels with a single query is given by the following proposition.
Let \( p_{\text{err}}(1) \) be the minimum error probability for QCDP\((p_0, p_1), (\mathcal{O}_A^{0} \rightarrow B, \mathcal{O}_A^{1} \rightarrow B)\) with one query. The following then holds:

\[
 p_{\text{err}}(1) = \frac{1}{2} \left( 1 - \max_{\rho_{AR} \in \mathcal{D}(AR)} \| p_0 (\mathcal{O}_A^{0} \rightarrow B \otimes \text{id}_R) (\rho_{AR}) - p_1 (\mathcal{O}_A^{1} \rightarrow B \otimes \text{id}_R) (\rho_{AR}) \|_1 \right),
\]

where \( \text{id}_R \) denotes the identity map on \( L(R) \).

If \( \dim(R) \geq \dim(A) \), the maximum of the trace norm is called the diamond norm.

Definition 9 (Diamond norm). Let \( \Psi_{A \rightarrow B} \) be a linear map from \( L(A) \) to \( L(B) \). The diamond norm of \( \Psi_{A \rightarrow B} \) is then defined as

\[
\| \Psi_{A \rightarrow B} \|_\diamond := \max_{L_A \in L(A)} \| (\Psi_{A \rightarrow B} \otimes \text{id}_R)(L_A) \|_1,
\]

for a quantum system \( R \) whose dimension is equal to \( \dim(A) \).

From the convexity of the trace norm, we can assume that \( L_A \) in the maximization problem is a rank-1 matrix. Especially, in this paper, we consider the diamond norm of Hermitian-preserving maps. In that case, we can assume that \( L_A \) is a density operator of pure state [18]. It is known that the following holds for any Hermitian preserving map \( \Lambda_{A \rightarrow B} \).

\[
\| \Lambda_{A \rightarrow B} \|_\diamond = \max_{|\phi\rangle \in S(AR)} \| (\Lambda_{A \rightarrow B} \otimes \text{id}_R)(|\phi\rangle \langle \phi|_A)_B \|_1,
\]

where \( S(A) \) denotes a set of state vectors on a quantum system \( A \) [18].

An algorithm is said to be non-adaptive if it calls the oracle in parallel. In general, the minimum error probability of non-adaptive algorithms with \( n \) queries for discriminating two quantum channels is expressed as

\[
\frac{1}{2} \left( 1 - \| p_0 (\mathcal{O}_A^{0} \rightarrow B)^\otimes_n - p_1 (\mathcal{O}_A^{1} \rightarrow B)^\otimes_n \|_1 \right),
\]

if \( \dim(R) \geq \dim(A)^n \). Although the diamond norm \( \| \Psi_{A \rightarrow B} \|_\diamond \) can be evaluated in polynomial time with respect to the dimensions of \( A \) and \( B \) via SDP [8], the dimension of \( A^\otimes_n \) is exponentially large, so that SDP does not give an efficient algorithm when \( n \) is large. For general adaptive algorithms, no simple formula expressing the minimum error probability is known even for the discrimination of two quantum channels.

IV. MAIN RESULTS

In this paper, we present four theorems on the lower bounds on the minimum error probability \( p_{\text{err}}(n) \). Let \( E \) be a finite-dimensional quantum system. A Stinespring representation \( \mathcal{O}_{A \rightarrow BE} \) of a quantum channel \( \mathcal{O}_{A \rightarrow B} \) is a linear isometry from \( A \) to \( B \otimes E \) satisfying

\[
\mathcal{O}_{A \rightarrow B}(\rho_A) = \text{Tr}_E \left( \mathcal{O}_{A \rightarrow BE} \rho_A \mathcal{O}_{A \rightarrow BE}^\dagger \right).
\]

We first show a lower bound of the error probability for the discrimination of two quantum channels by using the Bures angle.

Theorem 1. Let \( p_{\text{err}}(n) \) be the minimum error probability for QCDP\((p_0, p_1), (\mathcal{O}_A^{0} \rightarrow B, \mathcal{O}_A^{1} \rightarrow B)\) with \( n \) adaptive queries. Let \( \tau_A \) be

\[
\tau_A := \arccos \min_{|\phi\rangle \in S(AR)} \min_{\sigma \in \mathcal{D}(A)} \| \text{Tr}_B \left( \mathcal{O}_{A \rightarrow BE}^0 \sigma \mathcal{O}_{A \rightarrow BE}^1 - \rho^0 \mathcal{O}_{A \rightarrow B}^0 \rho^1 \mathcal{O}_{A \rightarrow B}^1 \right) \|_1,
\]

under the condition \( n \tau_A \leq \pi/2 \). Furthermore, if both the quantum channels are unitary channels, then there exists a discrimination algorithm that achieves the lower bound even if \( \dim(R) = 0 \).

Furthermore, if \( \dim(R) \geq \dim(A) \), the following then holds:

\[
\tau_A = \arccos \min_{\sigma \in \mathcal{D}(A)} \| \text{Tr}_B \left( \mathcal{O}_{A \rightarrow BE}^0 \sigma \mathcal{O}_{A \rightarrow BE}^1 - \rho^0 \mathcal{O}_{A \rightarrow B}^0 \rho^1 \mathcal{O}_{A \rightarrow B}^1 \right) \|_1,
\]

where \( \mathcal{O}_{A \rightarrow BE}^0 \) and \( \mathcal{O}_{A \rightarrow BE}^1 \) are Stinespring representations of \( \mathcal{O}_{A \rightarrow B}^0 \) and \( \mathcal{O}_{A \rightarrow B}^1 \), respectively.

Note that the minimum of the fidelity

\[
\min_{|\phi\rangle \in S(AR)} \| \text{Tr}_B \left( \mathcal{O}_{A \rightarrow BE}^0 |\phi\rangle \langle \phi|_A \mathcal{O}_{A \rightarrow BE}^1 \right) \|_1
\]

is referred to as the stabilized process fidelity in [19]. We next show a lower bound based on the trace distance for the discrimination of two quantum channels.

Theorem 2. Let \( p_{\text{err}}(n) \) be the minimum error probability for QCDP\((p_0, p_1), (\mathcal{O}_A^{0} \rightarrow B, \mathcal{O}_A^{1} \rightarrow B)\) with \( n \) adaptive queries. Let \( k \in \{0, 1, \ldots, n\} \). Let \( \alpha_0, \alpha_1 \in [0, \infty) \) be non-negative real numbers that satisfy \( p_0 \alpha_0^k = p_1 \alpha_1^{n-k} \). Let \( \tau_0 \) and \( \tau_1 \) be

\[
\tau_0 := \max_{|\phi\rangle \in S(AR)} \frac{1}{2} \| (\mathcal{O}_{A \rightarrow B} - \alpha_0 \mathcal{O}_{A \rightarrow B}) |\phi\rangle \langle \phi|_A \|_1,
\]

\[
\tau_1 := \max_{|\phi\rangle \in S(AR)} \frac{1}{2} \| (\alpha_1 \mathcal{O}_{A \rightarrow B} - \mathcal{O}_{A \rightarrow B}) |\phi\rangle \langle \phi|_A \|_1.
\]

The following then holds:

\[
p_{\text{err}}(n) \geq \frac{1}{2} - p_0 \left( \sum_{i=0}^{k-1} \alpha_0^i \right) \tau_0^i - p_1 \left( \sum_{i=0}^{n-k-1} \alpha_1^i \right) \tau_1^i,
\]
for arbitrary $\alpha_0, \alpha_1$ and $k$. Furthermore, if $\dim(R) \geq \dim(A)$, the following then holds [13]:

$$
\tau_0^0 = \frac{1}{2} \| O_{A \rightarrow B}^0 - \alpha_0 O_{A \rightarrow B}^1 \|_o,
$$

$$
\tau_0^1 = \frac{1}{2} \| \alpha_1 O_{A \rightarrow B}^0 - O_{A \rightarrow B}^1 \|_o.
$$

We also show a lower bound of the error probability for QCGDP by using the Bures angle.

**Theorem 3.** Let $p_{err}(n)$ be the minimum error probability for QCGDP($\langle p_\xi \rangle, (O_{A \rightarrow B}^\xi, (C_\eta))$) with $n$ adaptive queries. For a quantum channel $\Psi_{AR \rightarrow BR}$ and $m \in \{0,1,\ldots,n\}$, $\theta_A(\Psi_{AR \rightarrow BR})$ and $\theta_m$ are defined as

$$
\theta_A(\Psi_{AR \rightarrow BR}) := \arccos \min_{\theta \in \mathbb{S}(AR) \times \Xi} \sum_{\xi \in \Xi} p_\xi F\left( O_{A \rightarrow BR}^\xi (\phi)_{AR}, \Psi_{AR \rightarrow BR}(\phi)_{AR} \right),
$$

$$
\theta_m := \arccos \sqrt{p_{err}(m)}.
$$

The following then holds:

$$
p_{err}(n) \geq \cos^2((n-m)\theta_A(\Psi_{AR \rightarrow BR}) + \theta_m),
$$

for arbitrary $\Psi_{AR \rightarrow BR}$ and $m$ satisfying $(n-m)\theta_A(\Psi_{AR \rightarrow BR}) + \theta_m \leq \pi/2$. Furthermore, if $\dim(R) \geq \dim(A)$, the following then holds:

$$
\theta_A(\Psi_{A \rightarrow B} \otimes id_R) := \arccos \min_{\sigma_A \in \mathcal{D}(A)} \sum_{\xi \in \Xi} p_\xi \left\| \text{Tr}_B \left( O_{A \rightarrow BE}^\xi \sigma_A(V_{A \rightarrow BE})^\dagger \right) \right\|_1,
$$

where $O_{A \rightarrow BE}^\xi$ is a Stinespring representation of $O_{A \rightarrow B}^\xi$ for $\xi \in \Xi$, and $V_{A \rightarrow BE}$ is a Stinespring representation of $\Psi_{A \rightarrow B}$.

Finally, we show a lower bound of the error probability for QCGDP by using the trace distance.

**Theorem 4.** Let $p_{err}(n)$ be the minimum error probability for QCGDP($\langle p_\xi \rangle, (O_{A \rightarrow B}^\xi, (C_\eta))$) with $n$ adaptive queries. Let $m$ and $k$ be integers that satisfy $0 \leq m \leq k \leq n$. Let $\alpha_0, \alpha_1 \in [0,1]$ be non-negative real numbers that satisfy $\alpha_0^{n-k} = \alpha_1^{k-m}$. For quantum channels $\Psi_{AR \rightarrow BR}^0$ and $\Psi_{AR \rightarrow BR}^1$, $\theta_0^0(\Psi_{AR \rightarrow BR}^0)$ and $\theta_0^1(\Psi_{AR \rightarrow BR}^1)$ are defined as

$$
\theta_0^0(\Psi_{AR \rightarrow BR}^0) := \max_{\phi_{AR} \in \mathbb{S}(AR)} \sum_{\xi \in \Xi} p_\xi \frac{1}{2} \left\| \left( O_{A \rightarrow BR}^\xi - \alpha_0 \Psi_{AR \rightarrow BR}^0 \right) (\phi)_{AR} \right\|_1,
$$

$$
\theta_0^1(\Psi_{AR \rightarrow BR}^1) := \max_{\phi_{AR} \in \mathbb{S}(AR)} \sum_{\xi \in \Xi} p_\xi \frac{1}{2} \left\| \left( \alpha_1 O_{A \rightarrow BR}^\xi - \Psi_{AR \rightarrow BR}^1 \right) (\phi)_{AR} \right\|_1,
$$

respectively. The following then holds:

$$
p_{err}(n) \geq p_{err}(m) - \left( \sum_{i=0}^{n-k-1} \alpha_0^i \right) \theta_0^0(\Psi_{AR \rightarrow BR}^0) - \left( \sum_{i=0}^{k-m-1} \alpha_1^i \right) \theta_0^1(\Psi_{AR \rightarrow BR}^1),
$$

for arbitrary $\Psi_{AR \rightarrow BR}^0, \Psi_{AR \rightarrow BR}^1, m, \alpha_0, \alpha_1$, and $k$.

The proofs of Theorems 1, 2, 3 and 4 are presented in Section VII. In Section VII we show that all optimization problems on quantum states $|\phi\rangle_{AR} \in \mathcal{S}(AR)$ for computing the lower bounds in Theorems 1, 2, 3 and 4 can be written as SDP when $\dim(R) \geq \dim(A)$, so that the lower bounds can be evaluated efficiently. In this case, the optimization of the lower bounds in Theorems 3 and 4 with respect to the quantum channel $\Psi_{AR \rightarrow BR}$ with a constraint $\Psi_{AR \rightarrow BR} = \Psi_{A \rightarrow B} \otimes id_R$ for some quantum channel $\Psi_{A \rightarrow B}$ can be solved efficiently as well.

**V. APPLICATIONS**

A. Application of Theorems 1 and 2

**Discrimination of two amplitude damping channels**

In this section, we derive lower bounds on the error probability of the discrimination of two amplitude damping channels by using Theorems 1 and 2.

**Definition 10.** The amplitude damping channel $E_r^\xi$ with damping rate $r \in [0,1]$ is defined by a Kraus representation

$$
E_r^\xi(\rho_A) := K_{A \rightarrow r}^0 \rho_A K_{A \rightarrow r}^0 \dagger + K_{A \rightarrow r}^1 \rho_A K_{A \rightarrow r}^1 \dagger,
$$

where $K_{A \rightarrow r}^0 := |0\rangle \langle 0|_A + \sqrt{1-r} |1\rangle \langle 1|_A$, $K_{A \rightarrow r}^1 := \sqrt{r} |0\rangle \langle 1|_A$.

Let $E$ be a two-dimensional quantum system. Let $U_{A \rightarrow AE}^1 := K_{A \rightarrow r}^0 \otimes |0\rangle \langle 0|_E + K_{A \rightarrow r}^1 \otimes |1\rangle \langle 1|_E$, then $U_{A \rightarrow AE}^1$ is a Stinespring representation of $E_r^\xi$. For QCDP($\langle p_\rho, p_{1}\rangle, (E_r^{\sigma_0}, E_r^{\sigma_1})$), we obtain the following bound of $\tau_A$ in Theorem 1 with a two-dimensional working system $R$:

$$
\tau_A = \arccos \min_{\sigma_A \in \mathcal{D}(A)} \left\| \text{Tr}_A (U_{A \rightarrow AE}^1 \sigma_A (U_{A \rightarrow AE}^1)^\dagger) \right\|_1
$$

$$
\leq \arccos \min_{\sigma_A \in \mathcal{D}(A)} \left| \text{Tr}_A (U_{A \rightarrow AE}^1 \sigma_A (U_{A \rightarrow AE}^1)^\dagger) \right|
$$

$$
= \arccos \min_{\sigma_A \in \mathcal{D}(A)} \left| \text{Tr}_A (U_{A \rightarrow AE}^1 \sigma_A (U_{A \rightarrow AE}^1)^\dagger) \right|
$$

The inequality follows from the monotonicity of the trace norm (i.e., $\| \cdot \|_1 \geq \| \text{Tr}(\cdot) \|$). Note that $\sigma_A = |1\rangle \langle 1|_A$ satisfies the equality. Let the Bhattacharyya angle of $r_0$ and


Theorem 2 where \( * \) is optimal; \( \odot \) is specific to the amplitude damping channel, the lower bounds obtained by Pirandola et al. \[4\] and Pereira and Pirandola \[14\] (alternative resource lower bound, which is specific to the amplitude damping channel), the lower bound given by Theorems 1 and 2 are better for some \( p_0 \) in Fig. 1 and better for all \( n \) in Fig. 2. In Fig. 1 the lower bound from \[4\] is identically 0. Although Theorem 1 gives the analytic lower bound, the lower bound from Theorem 2 gives better lower bound in this case.

In the numerical results, we optimized \( \alpha_0 \) for given \( k \) by the golden-section search because the function empirically has a single maximal point. We optimized \( k \) in an exhaustive way under the assumption that the optimum \( k^* \) is non-decreasing with respect to \( r_0 \) and \( n \).

**B. Application of Theorem 3: Grover’s search problem**

In this section, we prove the optimality of Grover’s algorithm by using Theorem 3. Let \( A \) be an \( |H| \)-dimensional quantum system, and \( \{|\eta\rangle_{\eta \in H}\} \) be its computational basis. For some fixed \( \ell \in \{1, 2, \ldots, |H|/2\} \), \( \Xi := \{\xi \subseteq H \mid |\xi| = \ell\} \). Let \( (O_A^\xi)_{\xi \in \Xi} \) be a family of unitary operators defined as \( O_A^\xi := I_A - 2 \sum_{\eta \in \xi} |\eta\rangle\langle\eta| \) and \( I_A \) is the identity operator in \( A \). Let \( (O_A^\xi)_{\xi \in \Xi} \) be a family of quantum channels defined as

\[
O_A^\xi(\rho_A) := O_A^\xi \rho_A O_A^\xi_d.
\]

Let \( (C_\eta)_{\eta \in H} \) be a family of subsets of \( \Xi \) defined as \( C_\eta := \{\xi \subseteq H \mid |\xi| = \ell\} \). The problem QCGDP((1/|\Xi|), \( (O_A^\xi), (C_\eta) \)) is then called Grover’s search problem. The following holds:

\[
\frac{1}{|\Xi|} \sum_{\xi \in \Xi} O_A^\xi = \left( \frac{|H|}{\ell} \right)^{-1} \left( \left( \frac{|H| - 1}{\ell} \right) - \left( \frac{|H| - 1}{\ell - 1} \right) \right) I_A = \left(1 - \frac{2\ell}{|H|}\right) I_A.
\]
Hence, we obtain
\[
\theta_A(\text{id}_A) = \arccos \min_{|\phi\rangle_{A'|AR} \in \mathcal{S}(A'R)} \sum_{\xi \in \Xi} \frac{1}{|\Xi|} F \left( \mathcal{O}_A^\xi (|\phi\rangle_{AR}), |\phi\rangle_{AR} \right)
\]
\[
= \arccos \min_{|\phi\rangle_{A'|AR} \in \mathcal{S}(A'R)} \sum_{\xi \in \Xi} \left| \langle \phi | \mathcal{O}_A^\xi | \phi \rangle_{AR} \right|
\]
\[
\leq \arccos \min_{|\phi\rangle_{A'|AR} \in \mathcal{S}(A'R)} \sum_{\xi \in \Xi} \frac{1}{|\Xi|} \left( \langle \phi | \mathcal{O}_A^\xi | \phi \rangle_{AR} \right)
\]
\[
= \arccos \left( 1 - \frac{2\ell}{|H|} \right)
\]
\[
= 2 \arcsin \sqrt{\frac{\ell}{|H|}}.
\]
Moreover, we obtain
\[
\theta_0 = \arccos \sqrt{1 - \frac{\max_{\eta \in H} \left( \sum_{\xi \in C_\eta} \frac{1}{|\Xi|} \right)}{H - 1}}
\]
\[
= \arcsin \sqrt{\frac{\left( \frac{H - 1}{H} \right)^{-1}}{\ell - 1}}
\]
\[
= \arcsin \sqrt{\frac{\ell}{|H|}}.
\]
Therefore, we obtain the lower bound
\[
p_{\text{err}}(n) \geq \cos^2 \left( (2n + 1) \arcsin \sqrt{\frac{\ell}{|H|}} \right),
\]
from Theorem 3 under the condition \((2n + 1) \arcsin(\sqrt{\ell}/|H|) \in [0, \pi/2]\). This lower bound is achieved by Grover’s algorithm \[20\] \[21\]. The lower bound for \(\ell = 1\) was obtained by Zalka \[11\].

### C. Application of Theorem 4 Channel position finding for amplitude damping channels

In this section, we consider the channel position finding problem introduced in \[3\] for amplitude damping channels to demonstrate the lower bound given by Theorem 4. Let \(\ell \geq 2\) be an integer. Let \(A_1, \ldots, A_\ell\) be two-dimensional quantum systems. Let \(A'\) be the composite system of \(A_1, \ldots, A_\ell\), i.e., \(A' = \bigotimes_{i=1}^{\ell} A_i\). Let \(\Xi := \{1, 2, \ldots, \ell\}\), and \(r_0, r_1 \in [0, 1]\). For each \(\xi \in \Xi\), a quantum channel \(\mathcal{O}_A^\xi\) is defined as
\[
\mathcal{O}_A^\xi := \bigotimes_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \otimes \bigotimes_{i=\xi}^{\ell} \mathcal{E}_{A_i}^{r_i}.
\]
In these quantum channels, \(\mathcal{E}_{A_i}^{r_i}\) is applied for all but one subsystem. For the one exceptional system, \(\mathcal{E}_{A_i}^{r_0}\) is applied. Our task is to find the place where \(\mathcal{E}_{A_i}^{r_0}\) is applied. This problem is formulated as the channel discrimination problem QCDP\((1/\ell, (\mathcal{O}_A^\xi))\). We consider the error probability of discrimination algorithms using a working system \(R' = R^{\otimes \ell}\) where \(R\) is a two-dimensional working system. Let \(k \in \{0, 1, \ldots, n\}\). Let \(\alpha_0, \alpha_1 \in [0, \infty)\) be non-negative real numbers that satisfy \(\alpha_0^{n-k} = \alpha_1^k\). We then obtain
\[
\left\| \mathcal{O}_{A'} - \alpha_0 \bigotimes_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \right\|_o = \left\| \bigotimes_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} - \alpha_0 \bigotimes_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \right\|_o = \left\| \prod_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} - \alpha_0 \prod_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \right\|_o = \left\| \prod_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} - \alpha_0 \prod_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \right\|_o.
\]
The second equality follows from the multiplicativity of the diamond norm \[18\]. Hence, \(\theta_0^\circ \left( \bigotimes_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \right)\) in Theorem 4 is bounded as
\[
\theta_0^\circ \left( \bigotimes_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \right) = \max_{|\phi\rangle_{A'|AR} \in \mathcal{S}(A'R')} \sum_{\xi \in \Xi} \frac{1}{2\ell} \left\| \left( \mathcal{O}_{A'} - \alpha_0 \bigotimes_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \right) |\phi\rangle_{AR} \right\|_1 \]
\[
\leq \sum_{\xi \in \Xi} \frac{1}{2\ell} \left\| \mathcal{O}_{A'} - \alpha_0 \bigotimes_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \right\|_o = \frac{1}{2} \left\| \mathcal{E}_{A'}^{r_0} - \alpha_0 \mathcal{E}_{A'}^{r_0} \right\|_o.
\]
Note that
\[
|\phi\rangle_{A'R'} = \left( \arg\max_{|\phi\rangle_{A'|AR} \in \mathcal{S}(A'R)} \left\| (\mathcal{E}_{A'}^{r_0} - \alpha_0 \mathcal{E}_{A'}^{r_0}) |\phi\rangle_{AR} \right\|_1 \right)^{\otimes \ell}
\]
satisfies the equality. Similarly, we obtain
\[
\theta_1^\circ \left( \bigotimes_{i=1}^{\ell} \mathcal{E}_{A_i}^{r_i} \right) = \frac{1}{2} \left\| \alpha_1 \mathcal{E}_{A'}^{r_0} - \mathcal{E}_{A'}^{r_1} \right\|_o.
\]
\[2\] Zalka wrote: “It seems very plausible that the proof can be extended to oracles with any known number of marked elements.” However, no formal proof is known to the best of our knowledge.
From $p_{\text{err}}(0) = 1 - 1/\ell$, we obtain

$$p_{\text{err}}(n) \geq 1 - \frac{1}{\ell} - \left( \sum_{i=0}^{n-k-1} \frac{\alpha_0}{2} \right) \| E^{(0)}_A - \alpha_0 E^{i}_{A} \|_\diamond \geq \left( \sum_{i=0}^{k-1} \frac{\alpha_1}{2} \right) \| E^{(0)}_A - E^{i}_{A} \|_\diamond.$$  

The numerical results for the channel position finding problem for amplitude damping channels are shown in Figs. 3 and 4. The distribution of the channels is uniform, i.e., $p_\xi = 1/\ell$ for $\xi \in \Xi$. The number of oracles is $\ell = 3$.

In Fig. 3 the damping rates of the channels are $r_0$ and $r_1 = r_0 + 0.01$. In Fig. 3 the number of queries is $n = 15$. The parameters $\alpha_0$ and $\alpha_1$ are optimized to minimize the lower bound. In Fig. 4 the damping rates of the channels are $r_0 = 0.10$ and $r_1 = 0.11$. Compared with a lower bound obtained by Zhuang and Pirandola [5], the lower bound given by Theorem 4 is better for all $r_0$ in Figs. 3 and for all $n$ in Fig. 4.

VI. PROOFS OF THE THEOREMS

A. Idea of the proofs

For describing the idea of the proofs, we consider a simple case QCDP((1/2, 1/2), (O^0_{AB}, O^-_{AB}, O^-_{BR})) in this section. Let ((Φ^0_{BR→AR}, M_{BR}^-)) be a discrimination algorithm that achieves the minimum error probability $p_{\text{err}}(n)$ for this problem with $n$ queries. From Proposition 4 the error probability of the algorithm is

$$\frac{1}{2} \left( 1 - \frac{1}{2} \| \rho^0_{BR} - \rho^1_{BR} \|_1 \right),$$

where

$$\rho^0_{BR} := O^0_{A→B} \circ \Phi^0_{BR→AR} \circ O^0_{A→B} \circ \Phi^{n-1}_{BR→AR} \circ \cdots \circ O^0_{A→B} \circ \Phi^{n}_0_{BR→AR} (|0\rangle\langle 0|_BR)$$

and

$$\rho^1_{BR} := O^1_{A→B} \circ \Phi^0_{BR→AR} \circ O^1_{A→B} \circ \Phi^{n-1}_{BR→AR} \circ \cdots \circ O^1_{A→B} \circ \Phi^{n}_0_{BR→AR} (|0\rangle\langle 0|_BR).$$ (3)

We use the well-known hybrid argument for upper bounding $\frac{1}{2} \| \rho^0_{BR} - \rho^1_{BR} \|_1$ [10]. Let

$$\rho^{(j)}_{BR} := O^0_{A→B} \circ \Phi^j_{BR→AR} \circ \cdots \circ O^1_{A→B} \circ \Phi^{j+1}_{BR→AR} \circ O^0_{A→B} \circ \Phi^{j}_{BR→AR} |0\rangle\langle 0|_{BR},$$

for $j = 0, 1, \ldots, n$. In the definition of $\rho^{(j)}_{BR}$, we invoke $O^0_{A→B}$ for the first $j$ oracle calls, and $O^1_{A→B}$ for the last $n - j$ oracle calls. Then, $\rho_{BR} = \rho^0_{BR}$ and $\rho^1_{BR} = \rho^0_{BR}$.

Hence,

$$\frac{1}{2} \| \rho^0_{BR} - \rho^1_{BR} \|_1 = \frac{1}{2} \| \rho^{(n)}_{BR} - \rho^{(0)}_{BR} \|_1 = \sum_{j=0}^{n-1} \frac{1}{2} \| \rho^{(j+1)}_{BR} - \rho^{(j)}_{BR} \|_1.$$

Here, we used the triangle inequality of the trace distance. Hence, it is sufficient to upper bound $\frac{1}{2} \| \rho^{(j+1)}_{BR} - \rho^{(j)}_{BR} \|_1$ for each $j = 0, 1, \ldots, n - 1$. For each $j = 0, 1, \ldots, n - 1$
A. Proof of Theorem 1

In the proof of Theorem 1 we use the Bures angle in place of the trace distance. For bounding the trace distance by using the Bures angle or equivalently fidelity, we use the well-known Fuchs–van de Graaf inequality \( \|\rho_A - \rho_A^1\|_1 \leq 2\sqrt{1 - F(\rho_A, \rho_A^1)^2} \) [22]. The following lemma is a minor generalization of the Fuchs–van de Graaf inequality.

**Lemma 1.** Let \( a_0, a_1 \in [0, \infty) \) be non-negative real numbers. Let \( \rho_A^0, \rho_A^1 \in D(A) \) be density operators. The following then holds:

\[
\|a_0 \rho_A^0 - a_1 \rho_A^1\|_1 \leq \sqrt{(a_0 + a_1)^2 - 4a_0a_1 F(\rho_A^0, \rho_A^1)^2}.
\]

The equality holds if \( \rho_A^0 \) and \( \rho_A^1 \) are pure states.

The proof is given in Appendix B. From Proposition 1 the error probability of the discrimination algorithm is

\[
\frac{1}{2} \left( 1 - \|p_0 \rho_{BR}^0 - p_1 \rho_{BR}^1\|_1 \right).
\]

where \( \rho_{BR}^0 \) and \( \rho_{BR}^1 \) are defined in (3). From Lemma 1

\[
\|p_0 \rho_{BR}^0 - p_1 \rho_{BR}^1\|_1 \leq \sqrt{1 - 4p_0p_1 F(\rho_{BR}^0, \rho_{BR}^1)^2}.
\]

Hence, our goal is to lower bound \( F(\rho_{BR}^0, \rho_{BR}^1) \). Equivalently, our goal is to upper bound \( A(\rho_{BR}^0, \rho_{BR}^1) \).

From (5) for \( D = A \), we obtain

\[
A(\rho_{BR}^0, \rho_{BR}^1) \leq n \max_{\rho_{AR}} A(\rho_{A-B}(\rho_{AR}), \rho_{A-B}(\rho_{AR})).
\]

Since

\[
\max_{\rho_{AR}} A(\rho_{A-B}(\rho_{AR}), \rho_{A-B}(\rho_{AR})) = \arccos \min_{\rho_{AR}} F(\rho_{A-B}(\rho_{AR}), \rho_{A-B}(\rho_{AR})) = \tau_{A},
\]

we obtain

\[
F(\rho_{BR}^0, \rho_{BR}^1) = \cos A(\rho_{BR}^0, \rho_{BR}^1) \geq \cos n \tau_{A},
\]

if \( n \tau_{A} \leq \pi / 2 \). Hence, we obtain (1) from (6), (7) and (8).

Kawachi et al. characterized the exact minimum error probability when the two quantum channels are both unitary channels, and the distribution of the oracle is uniform (i.e., \( p_0 = p_1 = 0.5 \)) in [3]. Their lower bound can be reproduced from Theorem 1. The minimum error probability is represented by using the covering angle.

**Definition 11 (Covering angle).** Let \( \arg_{\geq 0}(z) := \min\{\varphi \geq 0 \mid z = |z|\exp(i\varphi)\} \) for a complex number \( z \). Let \( \theta_1, \theta_2, \ldots, \theta_n \) be real numbers and \( \Theta := \{\exp(i\theta_1), \exp(i\theta_2), \ldots, \exp(i\theta_n)\} \). The covering angle \( \theta_{\text{cover}} \) of \( \Theta \) is defined as

\[
\theta_{\text{cover}} := \min_{j \in \{1, 2, \ldots, n\}} \max_{i \in \{1, 2, \ldots, n\}} \arg_{\geq 0}(\exp(i\theta_i - \theta_j)).
\]

From Theorem 1 and an algorithm in [3], we obtain the following corollary.

**Corollary 1.** Let \( O_A^0 \) and \( O_A^1 \) be unitary operators. Let \( O_A^0, O_A^1 \) be quantum channels defined as \( O_A^\xi(\rho_A) := O_A^\xi \rho_A O_A^{\xi^1} \) for \( \xi \in \{0, 1\} \). Let \( p_{\text{err}}(n) \) be the minimum error probability for QCDF((\rho_0, p_1), (O_A^0, O_A^1)) with \( n \) queries. Let \( \theta_{\text{cover}} \) be the covering angle of the set of
The fidelity (the absolute value of the inner product) of the state vectors is $|\exp(i\theta_0) + \exp(i\theta_1)|/2 = \cos(n\theta_{\text{cover}}/2)$ if $n\theta_{\text{cover}} \leq \pi$. Hence, this non-adaptive algorithm achieves the minimum error probability $p_{\text{err}}(n)$ in Corollary 1. This non-adaptive algorithm requires a working system $R$ with $\dim(R) = \dim(A)^n - 1$. There also exists an optimal adaptive algorithm that does not require ancillas, i.e., $\dim(R) = 0$. In the algorithm, first prepare the initial state $(|\psi_0\rangle_A^n + |\psi_1\rangle_A^n)/\sqrt{2}$, and then apply the oracle $n$ times in serial. This algorithm achieves the minimum error probability as well.

Although $\tau_A$ is represented using the minimum of the concave function, when $\dim(R) \geq \dim(A)$, $\tau_A$ can be represented by the minimum of the convex function $\tau_A$. This is proved in Appendix C.

### C. Proof of Theorem 2

Theorem 2 is obtained similarly to Theorem 1, but using weights for quantum states. We first show the idea of the technique briefly. Let us consider a distance measure $\| \cdot \|$ under the condition $n\theta_{\text{cover}} \leq \pi$, and $p_{\text{err}}(n) = 0$ if $n\theta_{\text{cover}} > \pi$.

Theorem 2 is obtained similarly to Theorem 1, but using weights for quantum states. We first show the idea of the technique briefly. Let us consider a distance measure $\| \cdot \|$ under the condition $n\theta_{\text{cover}} \leq \pi$, and $p_{\text{err}}(n) = 0$ if $n\theta_{\text{cover}} > \pi$.

Because there are $n - 1$ internal quantum states $\rho_{BR}^{(1)}, \ldots, \rho_{BR}^{(n-1)}$ in our proof, we have to determine $n - 1$ “weights” $r_1, \ldots, r_{n-1}$, as in Fig. 5. Because the optimization of $n - 1$ parameters is computationally hard if $n$ is large, we only use three parameters $\alpha_0, \alpha_1 \in (0, 1]$, and $k \in \{1, 2, \ldots, n\}$ such that $\alpha_0^k = \alpha_1^{n-k}$, and assume that $r_{n-i} = \alpha_0^k$ if $i \leq k$ and $r_i = \alpha_1^k$ if $i \leq n-k$. We then need to optimize only two parameters $\alpha_0$ and $k$. Once the parameters are fixed, we can calculate the lower bound in a constant time with respect to the number $n$ of queries.

In the same way as for the proof of Theorem 1, we obtain the following inequality:

$$\frac{1}{2} \left\| p_0 \rho_{BR}^{(n)} - p_1 \rho_{BR}^{(0)} \right\|_1 \leq \sum_{j=0}^{k-1} \frac{1}{2} \left\| p_0 \alpha_0^{j+1} \rho_{BR}^{(n-j)} - p_0 \alpha_1^{j+1} \rho_{BR}^{(n-j-1)} \right\|_1 + \sum_{j=0}^{n-k-1} \frac{1}{2} \left\| p_1 \alpha_1^{j+1} \rho_{BR}^{(j+1)} - p_1 \alpha_1^j \rho_{BR}^{(j)} \right\|_1.$$

We used the condition $p_0 \alpha_0^k = p_1 \alpha_1^{n-k}$ in the above inequality. For the first term of the right-hand side, we
obtain
\[\begin{align*}
\sum_{j=0}^{k-1} \frac{1}{2} \left\| p_0 \alpha_j^{(n-j)} - p_0 \alpha_j^{(n-j-1)} \right\|_1 \\
\leq p_0 \sum_{j=0}^{k-1} \frac{1}{2} \left\| O^0_{A \rightarrow B} \circ \Phi^j_{BR-BR} \circ \cdots \circ \Phi^1_{BR-AR} (|0\rangle\langle 0|_B) \\
- \alpha_0 O^0_{A \rightarrow B} \circ \Phi^j_{BR-BR} \circ \cdots \circ \Phi^1_{BR-AR} (|0\rangle\langle 0|_B) \left\|_1 \\
\leq p_0 \left( \sum_{j=0}^{k-1} \alpha_j^2 \right) \max_{\rho_{AR} \in D(AR)} \frac{1}{2} \left( (O^0_{A \rightarrow B} - \alpha_0 O^0_{A \rightarrow B}) (\rho_{AR}) \right) \left\|_1 \\
= p_0 \left( \sum_{j=0}^{k-1} \alpha_j^2 \right) \tau_0^0.
\end{align*}\]

The first inequality follows from the monotonicity of the trace norm. The equality follows from the convexity of the trace distance.

Similarly, we obtain
\[\begin{align*}
\sum_{j=0}^{n-k-1} \frac{1}{2} \left\| p_1 \alpha_j^{(j+1)} \rho_{BR} - p_1 \alpha_j^{(j)} \rho_{BR} \right\|_1 \\
\leq p_1 \left( \sum_{j=0}^{n-k-1} \alpha_j^2 \right) \tau_0^1.
\end{align*}\]

Therefore, we obtain
\[\begin{align*}
p_{\text{err}}(n) \\
= \frac{1}{2} - \frac{1}{2} \left\| p_0 \rho_{0,n}^{BR} - p_1 \rho_{n,n}^{BR} \right\|_1 \\
\geq \frac{1}{2} - p_0 \left( \sum_{j=0}^{k-1} \alpha_j^2 \right) \tau_0^0 - p_1 \left( \sum_{j=0}^{n-k-1} \alpha_j^2 \right) \tau_0^1.
\end{align*}\]

This proves Theorem 2

**D. Proof of Theorem 3**

Theorems 3 and 4 are also proved using the triangle inequalities for the Bures angle and the trace distance with weights, respectively. Let \((\Phi^j_{BR-AR}), (M^n_{BR})\) be a discrimination algorithm with \(n\) queries that achieves the minimum error probability \(p_{\text{err}}(n)\) of QCGDP\((\rho_{\xi}), (O^\xi_{A \rightarrow B}), (C_\eta)\)). Let \(W\) be a two-dimensional quantum system. Quantum channels \((M^\xi_{BR-AR})_{\xi \in \Xi}\) are defined as
\[\begin{align*}
M^\xi_{BR-AR}(\rho_{BR}) := \text{Tr} \left( \sum_{\eta_j \in C_\eta} M^n_{BR} \rho_{BR} \right) |0\rangle\langle 0|_W \\
+ \text{Tr} \left( \sum_{\eta_j \in C_\eta} M^n_{BR} \rho_{BR} \right) |1\rangle\langle 1|_W,
\end{align*}\]
for each \(\xi \in \Xi\). The quantum channel \(M^n_{BR-W}\) represents the following procedure: One performs POVM \((M^n_{BR})_{\eta \in H}\) to the input state \(\rho_{BR}\) and then outputs \(|0\rangle\langle 0|_W\) if the measurement outcome \(\eta \in H\) satisfies \(\xi \notin C_\eta\) and outputs \(|1\rangle\langle 1|_W\) otherwise.

Let \(\Psi_{AR-BR}\) be an arbitrary quantum channel. We define density operators as follows:
\[\begin{align*}
\rho_{BR}^{(\xi)} := O^\xi_{A \rightarrow B} \circ \Phi^j_{BR-BR} \circ \cdots \circ \Phi^1_{BR-AR} \circ \cdots \circ \Phi^1_{BR-AR} (|0\rangle\langle 0|_B) \\
\circ \Psi_{AR-BR} \circ \Phi^j_{BR-BR} \circ \cdots \circ \Phi^1_{BR-AR} (|0\rangle\langle 0|_B) \\
\sigma_{W}^{(i)} := \sum_{\xi \in \Xi} p_\xi M_{BR-W}^{\xi}(\rho_{BR}^{(n-i,\xi)}),
\end{align*}\]

\((0 \leq i \leq n)\).

The density operator \(\sigma_{W}^{(i)}\) is obtained by the following procedure: First, one picks \(\xi\) according to the probability distribution \((p_\xi)_{\xi}\). Then, one performs the discrimination algorithm in which the first \(n-1\) queries are given to \(\Psi_{AR-BR}\) in place of \(O^\xi_{A \rightarrow B}\). Finally, one performs the measurement \((M^n_{BR})_{\eta}\) and outputs \(|0\rangle\langle 0|_W\) or \(|1\rangle\langle 1|_W\) if \(\xi \notin C_\eta\) or \(\xi \in C_\eta\), respectively. Hence, the weights of \(|0\rangle\langle 0|_W\) and \(|1\rangle\langle 1|_W\) in \(\sigma_{W}^{(n)}\) are the probability of failure and success of the modified discrimination algorithm, respectively.

By applying the triangle inequality for the Bures angle to a sequence of density operators \(|1\rangle\langle 1|_W, \sigma_{W}^{(n)}, \sigma_{W}^{(n-1)}, \ldots, \sigma_{W}^{(m)}, |0\rangle\langle 0|_W\), we obtain the following inequality:
\[\begin{align*}
\frac{\pi}{2} = A \left( |1\rangle\langle 1|_W, |0\rangle\langle 0|_W \right) \\
\leq A \left( |1\rangle\langle 1|_W, \sigma_{W}^{(n)} \right) + \sum_{i=m+1}^{n} A \left( \sigma_{W}^{(i)}, \sigma_{W}^{(i-1)} \right) \\
+ A \left( \sigma_{W}^{(m)}, |0\rangle\langle 0|_W \right).
\end{align*}\]

Each term in the upper bound is evaluated as follows. For the first term of the upper bound, we obtain
\[\begin{align*}
A \left( |1\rangle\langle 1|_W, \sigma_{W}^{(n)} \right) \\
= \text{arccos} \sqrt{\left| \langle 1 | \sigma_{W}^{(n)} | 1 \rangle \right|_W} \\
= \text{arccos} \left( \sum_{\xi \in \Xi} p_\xi \sum_{\eta \in C_\eta} \text{Tr} \left( M^n_{BR} \rho_{BR}^{(0,\xi)} \right) \right) \\
= \text{arccos} \sqrt{1 - p_{\text{err}}(n)}.
\end{align*}\]

For the second term of the upper bound, we obtain
\[\begin{align*}
A \left( \sigma_{W}^{(i)}, \sigma_{W}^{(i-1)} \right) \\
= \text{arccos} F \left( \sum_{\xi \in \Xi} p_\xi M_{BR-W}^{\xi}(\rho_{BR}^{(n-i,\xi)}) \right) \\
= \text{arccos} \left( \sum_{\xi \in \Xi} p_\xi M_{BR-W}^{\xi}(\rho_{BR}^{(n-i+1,\xi)}) \right).
\end{align*}\]
\[
\begin{align*}
&\leq \arccos \left( \sum_{\xi \in \Xi} p_\xi \mathcal{F} \left( \mathcal{M}^\xi_{BR \rightarrow W} \left( \rho_{BR}^{(n-t, \xi)} \right) \right) \right) \\
&\leq \arccos \left( \sum_{\xi \in \Xi} p_\xi \mathcal{F} \left( \mathcal{S}^\xi_{A-B} \circ \Phi_{BR \rightarrow AR}^{n-t+1} \circ \rho_{BR}^{(n-t+1, \xi)} \right) \right) \\
&\leq \arccos \left( \sum_{\xi \in \Xi} p_\xi \mathcal{F} \left( \Phi_{BR \rightarrow AR}^{n-t} \circ \cdots \circ \Phi_{AR \rightarrow BR} \circ \rho_{AR} \right) \right) \\
&\leq \arccos \left( \min_{\rho_{AR} \in D(AR)} \sum_{\xi \in \Xi} p_\xi \mathcal{F} \left( \mathcal{S}^\xi_{A-B} \circ \rho_{AR} \right) \right) \\
&= \theta_A(\rho_{AR} \rightarrow BR).
\end{align*}
\]

The first inequality and the last equality follow from the joint concavity of the fidelity. The second inequality follows from the monotonicity of the fidelity. In the third inequality, we replace the quantum state

\[
\Phi_{BR \rightarrow AR}^{n-t+1} \circ \cdots \circ \Phi_{AR \rightarrow BR} \circ \rho_{AR} \rightarrow BR \circ \rho_{BR}^{(n-t, \xi)}
\]

with a general quantum state \( \rho_{AR} \).

For the last term, we obtain

\[
\begin{align*}
A \left( \sigma_W^{(m)} ; |0\rangle \langle 0|_W \right) \\
= \arccos \sqrt{\langle 0 | \sigma_W^{(m)} | 0 \rangle_W} \\
= \arccos \sqrt{\sum_{\xi \in \Xi} \sum_{\eta \in \eta_{C_n}} \text{Tr} \left( M_{BR \rightarrow BR}^\eta \rho_{BR}^{(n-m, \xi)} \right)} \\
\leq \arccos \sqrt{p_{\text{err}}(m)}.
\end{align*}
\]

From the above inequalities, we obtain

\[
\frac{\pi}{2} \leq \arccos \sqrt{1 - p_{\text{err}}(n)} + (n-m)\theta_A(\rho_{AR} \rightarrow BR) + \arccos \sqrt{p_{\text{err}}(m)}.
\]

Therefore, we obtain

\[
1 - p_{\text{err}}(n) \leq \cos^2 \left( \frac{\pi}{2} - (n-m)\theta_A(\rho_{AR} \rightarrow BR) + \arccos \sqrt{p_{\text{err}}(m)} \right) \\
= \sin^2 ((n-m)\theta_A(\rho_{AR} \rightarrow BR) + \theta_m),
\]

for any \( \rho_{AR} \rightarrow BR \) and \( m \) satisfying \( (n-m)\theta_A(\rho_{AR} \rightarrow BR) + \theta_m \in [0, \pi/2] \). Hence, we obtain the main part of Theorem 3.

Although \( \theta_A \) is represented using the minimum of the concave function, when \( \dim(R) \geq \dim(A) \) and \( \rho_{AR} \rightarrow BR = \rho_{AR} \otimes id_R \) for some quantum channel \( \rho_{AR} \rightarrow BR \), \( \theta_A \) can be represented by the minimum of a convex function [2].

**Lemma 2.** Let \( O_{A-B}^\xi \) and \( \Psi_{A-B} \) be quantum channels with Stinespring representations \( O_{A-B}^\xi \) and \( V_{A-B} \), respectively. If \( \dim(R) \geq \dim(A) \), the following holds:

\[
\begin{align*}
\min_{\phi \in \mathcal{S}(AR)} \sum_{\xi \in \Xi} p_\xi \mathcal{F} \left( O_{A-B}^\xi (|\phi\rangle \langle \phi|_{AR}) , \Psi_{A-B} \left( |\phi\rangle \langle \phi|_{AR} \right) \right) &= \min_{\sigma_A \in D(A)} \sum_{\xi \in \Xi} p_\xi \left\| \text{Tr}_B \left( O_{A-B}^\xi \sigma_A (V_{A-B} \sigma)^\dagger \right) \right\|_1 \\
The proof is the same as the proof of Lemma 3 in Appendix C.
\end{align*}
\]

**E. Proof of Theorem 4**

Theorem 4 is proved similarly to Theorem 3. The quantum states \( \rho_{BR}^{(j, \xi)} \) and \( \sigma_e^{(m)} \) are defined in the same way. We obtain the following inequality:

\[
\begin{align*}
1 &= \frac{1}{2} \left\| |1\rangle \langle 1|_W - |0\rangle \langle 0|_W \right\|_1 \\
&\leq \frac{1}{2} \left\| |1\rangle \langle 1|_W - \sigma_e^{(n)} \right\|_1 \\
&\leq \frac{1}{2} \left\| \sigma_e^{(n)} - |0\rangle \langle 0|_W \right\|_1.
\end{align*}
\]

We used the condition \( \alpha_{n-k} = \frac{\pi}{2} \) in the above inequality. Each term on the upper bound is evaluated as follows. For the first term, we obtain

\[
\begin{align*}
\frac{1}{2} \left\| |1\rangle \langle 1|_W - \sigma_e^{(n)} \right\|_1 &= \sum_{\xi \in \Xi} p_\xi \sum_{\eta \in \eta_{C_n}} \text{Tr} \left( M_{BR \rightarrow BR}^\eta (0, \xi) \right) \\
&= p_{\text{err}}(n).
\end{align*}
\]

For the second term, we obtain

\[
\begin{align*}
\sum_{j=0}^{n-k-1} \frac{1}{2} \left\| \sigma_e^{(n-j)} - \sigma_e^{(n-j+1)} \right\|_1 &= \sum_{j=0}^{n-k-1} \frac{1}{2} \left\| \sigma_e^{(j+1)} - \sigma_e^{(j)} \right\|_1 \\
&= \sum_{j=0}^{n-k-1} \frac{1}{2} \left\| M_{BR \rightarrow BR}^\xi (\rho_{BR}^{(j, \xi)}) \right\|_1
\end{align*}
\]
\[ \Psi \leq \sum_{j=0}^{n-k-1} \frac{\alpha_0^j}{2} \sum_{\xi \in \Xi} p_\xi \left\| O_\xi^{j+1} \circ \Phi^{j+1}_{BR} \right\|_1 \]
\[ \circ \Psi_{AR-BR} \circ \Phi^{j}_{BR} \circ \cdots \circ \Psi_{AR-BR} \]
\[ \circ \Phi^{j}_{BR} \circ \Phi^{j}_{BR} \circ \cdots \circ \Psi_{AR-BR} \]
\[ \circ \Psi_{AR-BR} \circ \Phi^{j}_{BR} \circ \cdots \circ \Psi_{AR-BR} \]
\[ \circ \Phi^{j}_{BR} \circ \Psi_{AR-BR} \circ \cdots \circ \Psi_{AR-BR} \]
\[ \circ \Phi^{j}_{BR} \circ \Psi_{AR-BR} \circ \cdots \circ \Psi_{AR-BR} \]
\[ \circ \Phi^{j}_{BR} \circ \Psi_{AR-BR} \circ \cdots \circ \Psi_{AR-BR} \]
\[ \circ \Phi^{j}_{BR} \circ \Psi_{AR-BR} \circ \cdots \circ \Psi_{AR-BR} \]
\[ \circ \Phi^{j}_{BR} \circ \Psi_{AR-BR} \circ \cdots \circ \Psi_{AR-BR} \]

The first inequality and the second equality follows from the monotonicity of the trace norm.

From the above inequalities, we obtain

\[ \sum_{i=0}^{k-m-1} \frac{1}{2} \left\| \sigma_i^{m+j+1} - \sigma_i^m \right\|_1 \]
\[ \leq \sum_{i=0}^{k-m-1} \left\| \sigma_i^{m+j} \right\|_1 \]
\[ = \sum_{i=0}^{k-m-1} \alpha_i \left\| \sigma_i^{m+j} \right\|_1 \]
\[ = \sum_{i=0}^{k-m-1} \alpha_i \theta_i^{m+j} \left( \Psi_{AR-BR} \right) \]

For the last term, we obtain

\[ \sum_{\xi \in \Xi} p_\xi \sum_{\eta \in \Xi} \sum_{\xi \in \Xi} \| M_{BR}^{\eta} \|_1 \| \rho_{BR}^{\eta} \|_1 \]
\[ \leq 1 - p_{err}(m) \]

From the above inequalities, we obtain

\[ 1 \leq p_{err}(n) + \sum_{i=0}^{k-m-1} \alpha_i \left( \theta_i^{m+j} \left( \Psi_{AR-BR} \right) \right) \]
\[ + \sum_{i=0}^{k-m-1} \alpha_i \left( \theta_i^{m+j} \left( \Psi_{AR-BR} \right) \right) + 1 - p_{err}(m) \]

By replacing last \( n - k \) applications of \( \Psi_{AR-BR} \) with \( \Psi_{AR-BR}^j \) and first \( k \) applications of \( \Psi_{AR-BR} \) with \( \Psi_{AR-BR}^j \), we obtain Theorem 4.
can be represented as a solution of SDP. Similarly to the above case, \( \min_{\Psi_{A\rightarrow B}} \theta^0(\Psi_{A\rightarrow B} \otimes \text{id}_R) \) is a solution of the following SDP:

\[
\min_{\lambda, Y_{BA}, Z_{BA}^\xi, P_{BA}} \lambda
\]

subject to:

\[
J(O^\xi_{A\rightarrow B}) - \alpha_0 P_{BA} = Y_{BA} - Z_{BA}^\xi \quad (\forall \xi \in \Xi)
\]

\[
\lambda I_A \geq \frac{1}{2} \sum_{\xi \in \Xi} p_\xi \left( \text{Tr}_B(Y_{BA}^\xi + \text{Tr}_B(Z_{BA}^\xi)) \right)
\]

\[
P_{BA} \geq 0, \quad \text{Tr}_B(P_{BA}) = I_A.
\]

Hence, all lower bounds obtained in this paper can be evaluated and optimized on the condition \( \Psi_{A\rightarrow B} = \Psi_{A\rightarrow \mathbb{C}} \otimes \text{id}_R \) efficiently.

**VIII. CONCLUDING REMARKS**

In this study, we derive lower bounds on the error probability of quantum channel group discrimination problem on the basis of the triangle inequalities for the Bures angle and the trace distance. It would be interesting to investigate if other distance measures exist that yield tighter and efficiently computable lower bounds.

The function \( \cos \theta \leftrightarrow \cos n\theta \) is known as the Chebyshev polynomial. The Chebyshev polynomial has some extremal properties. The polynomial method for lower bounding the query complexity is relevant to the Chebyshev polynomial \([16, 24]\). Hence, it is interesting to study the connection between the lower bounds in Theorems\([1]\) and \([2]\) using the Bures angle and the polynomial method.

Our technique can be used together with the technique based on the port-based teleportation \([4]\). The combination of two techniques may improve the lower bound.

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**Appendix A: Triangle inequalities for the Bures distance and the sine distance**

In this appendix, we show the triangle inequalities for the Bures distance and the sine distance from the triangle inequality for the Bures angle. For arbitrary quantum states \( \rho_A, \sigma_A, \) and \( \tau_A \in \mathcal{D}(A) \), the Bures angle satisfies the triangle inequality

\[
\mathcal{A}^0(\rho_A, \sigma_A) \leq \mathcal{A}^0(\rho_A, \tau_A) + \mathcal{A}^0(\tau_A, \sigma_A).
\]

Then,

\[
\mathcal{B}(\rho_A, \sigma_A) = 2 \sin \left( \frac{\mathcal{A}^0(\rho_A, \sigma_A)}{2} \right)
\]

\[
\leq 2 \sin \left( \frac{\mathcal{A}^0(\rho_A, \tau_A) + \mathcal{A}^0(\tau_A, \sigma_A)}{2} \right)
\]

because the sine function is monotonically increasing in \([0, \pi/2]\). From the trigonometric addition formula,

\[
2 \sin \left( \frac{\mathcal{A}^0(\rho_A, \tau_A) + \mathcal{A}^0(\tau_A, \sigma_A)}{2} \right)
\]

\[
= 2 \sin \left( \frac{\mathcal{A}^0(\rho_A, \tau_A)}{2} \right) \cos \left( \frac{\mathcal{A}^0(\tau_A, \sigma_A)}{2} \right)
\]

\[
+ 2 \sin \left( \frac{\mathcal{A}^0(\tau_A, \sigma_A)}{2} \right) \cos \left( \frac{\mathcal{A}^0(\rho_A, \tau_A)}{2} \right)
\]

\[
\leq 2 \sin \left( \frac{\mathcal{A}^0(\rho_A, \tau_A)}{2} \right) + 2 \sin \left( \frac{\mathcal{A}^0(\tau_A, \sigma_A)}{2} \right).
\]

This proves the triangle inequality for the Bures distance.

The triangle inequality for the sine distance

\[
\sin \left( \mathcal{A}^0(\rho_A, \sigma_A) \right) \leq \sin \left( \mathcal{A}^0(\rho_A, \tau_A) + \mathcal{A}^0(\tau_A, \sigma_A) \right)
\]

can be obtained in the same way. If \( \mathcal{A}^0(\rho_A, \tau_A) + \mathcal{A}^0(\tau_A, \sigma_A) \leq \pi/2 \), the triangle inequality is obtained from

\[
\sin \left( \mathcal{A}^0(\rho_A, \tau_A) + \mathcal{A}^0(\tau_A, \sigma_A) \right) \geq 2 \pi \left( \mathcal{A}^0(\rho_A, \tau_A) + \mathcal{A}^0(\tau_A, \sigma_A) \right) > 1.
\]

**Appendix B: Proof of Lemma 1**

The following proof is a generalization of the proof of Fuchs–van de Graaf inequality in \([18]\). From Uhlmann’s theorem, there exist pure states \( |\phi^0\rangle_{AR}; |\phi^1\rangle_{AR} \in \mathcal{S}(AR) \) satisfying the equalities \( \text{Tr}_R (|\phi^0\rangle \langle \phi^0|_{AR}) = \rho_A^0 \), \( \text{Tr}_R (|\phi^1\rangle \langle \phi^1|_{AR}) = \rho_A^1 \), and \( |\langle \phi^0 | \phi^1\rangle_{AR} = F(\rho_A^0, \rho_A^1) \).

Therefore, we obtain

\[
\|a_0 \rho_A^0 - a_1 \rho_A^1\|_1 \leq \|a_0 \text{Tr}_R (|\phi^0\rangle \langle \phi^0|_{AR}) - a_1 \text{Tr}_R (|\phi^1\rangle \langle \phi^1|_{AR})\|_1
\]

\[
\leq \|a_0 |\phi^0\rangle \langle \phi^0|_{AR} - a_1 |\phi^1\rangle \langle \phi^1|_{AR}\|_1 = \sqrt{(a_0 + a_1)^2 - 4a_0a_1 |\langle \phi^0 | \phi^1\rangle_{AR}|^2}
\]

\[
= \sqrt{(a_0 + a_1)^2 - 4a_0a_1 F(\rho_A^0, \rho_A^1)^2}.
\]

The inequality follows from the monotonicity of the trace norm.

**Appendix C: Minimum of the fidelity**

**Lemma 3.** Let \( \mathcal{O}_{A\rightarrow B}^0 \) and \( \mathcal{O}_{A\rightarrow B}^1 \) be quantum channels with Stinespring representations \( \mathcal{O}_{A\rightarrow BE}^0 \) and \( \mathcal{O}_{A\rightarrow BE}^1 \),
respectively. If \( \dim(R) \geq \dim(A) \), the following holds:

\[
\min_{|\psi|_{AR} \in \mathcal{S}(AR)} F(\mathcal{O}^{0}_{A-B}(|\psi\rangle_{AR}), \mathcal{O}^{1}_{A-B}(|\psi\rangle_{AR})) = \min_{\sigma_A \in \mathcal{D}(A)} \| \text{Tr}_B \left( \mathcal{O}^{0}_{A-B} \sigma_A \mathcal{O}^{1\dagger}_{A-B} \right) \|_1.
\]

Proof.

\[
F(\text{Tr}_E \left( |\psi^0\rangle\langle\psi^0|_{CE} \right), \text{Tr}_E \left( |\psi^1\rangle\langle\psi^1|_{CE} \right)) = \max_{U_E} \left\langle |\psi^0\rangle |U_E \left| \psi^0\right\rangle_{CE} \right\rangle = \max_{U_E} \text{Tr}_E \left( U_E |\psi^0\rangle \langle\psi^0|_{CE} \right) \|_{\text{Tr}} = \max_{U_E} \text{Tr}_E \left( U_E \text{Tr}_C \left( |\psi^0\rangle\langle\psi^0|_{CE} \right) \right) = \| \text{Tr}_C \left( |\psi^0\rangle\langle\psi^0|_{CE} \right) \|_1,
\]

where \( U_E \) is a unitary operator. The first equality follows from Uhlmann’s theorem. The last equality follows from the duality of the Schatten \( p \)-norm.

We then obtain

\[
F(\mathcal{O}^{0}_{A-B}(|\psi\rangle_{AR}), \mathcal{O}^{1}_{A-B}(|\psi\rangle_{AR})) = F(\text{Tr}_E \left( \mathcal{O}^{0}_{A-B} |\psi\rangle_{AR} \mathcal{O}^{0\dagger}_{A-B} \right), \text{Tr}_E \left( \mathcal{O}^{1}_{A-B} |\psi\rangle_{AR} \mathcal{O}^{1\dagger}_{A-B} \right)) = \| \text{Tr}_{BR} \left( \mathcal{O}^{0}_{A-B} |\psi\rangle_{AR} \mathcal{O}^{0\dagger}_{A-B} \right) \|_1 = \| \text{Tr}_B \left( \mathcal{O}^{0}_{A-B} \text{Tr}_R \left( |\psi\rangle_{AR} \mathcal{O}^{1\dagger}_{A-B} \right) \right) \|_1.
\]

When \( \dim(R) \geq \dim(A) \), any \( \sigma_A \in \mathcal{D}(A) \) can be represented by \( \text{Tr}_R \left( |\psi\rangle\langle\psi|_{AR} \right) \) for some \( |\psi\rangle_{AR} \in \mathcal{S}(AR) \). Hence, Lemma is proved.

\[ \square \]

Appendix D: SDP representations

1. SDP for the fidelity

The fidelity of \( \rho_A \) and \( \sigma_A \) is represented as an solution of the following SDP [25].

\[
\begin{align*}
\min_{Y_Z, Z_A} & : \frac{1}{2} \left( \langle \rho_A, Y_A \rangle + \langle \sigma_A, Z_A \rangle \right) \\
\text{subject to} & : \begin{bmatrix} Y_A & I_A \\ I_A & Z_A \end{bmatrix} \succeq 0.
\end{align*}
\]

Hence, \( \min_{|\psi\rangle_{AR}, Y_{BR}, Z_{BR}} F(\Phi_{A-B}(|\psi\rangle_{AR}), \Psi_{A-B}(|\psi\rangle_{AR})) \) is the optimal value of

\[
\begin{align*}
\min_{|\psi\rangle_{AR}, Y_{BR}, Z_{BR}} & : \frac{1}{2} \left( \langle \Phi_{A-B}(|\psi\rangle_{AR}), Y_{BR} \rangle \\
& + \langle \Psi_{A-B}(|\psi\rangle_{AR}), Z_{BR} \rangle \right) \\
\text{subject to} & : \begin{bmatrix} Y_{BR} & I_{BR} \\ I_{BR} & Z_{BR} \end{bmatrix} \succeq 0
\end{align*}
\]

For \( |\psi\rangle_{AR} = \sum_{a,c} \Psi_{a,c} |a\rangle_{A} |c\rangle_{R} \), we obtain

\[
\Phi_{A-B}(|\psi\rangle_{AR}) = \sum_{a,b,c,d} \Psi_{a,c} \psi_{b,d}^{*} \Phi_{A-B}(|a\rangle_{B} |b\rangle_{A} \otimes |c\rangle_{R} \langle d|_{R})
\]

\[
= \sum_{a,b} \Phi_{A-B}(|b\rangle_{A} \otimes \sum_{c,d} \Psi_{a,c} \psi_{b,d}^{*} |c\rangle_{R} \langle d|_{R})
\]

\[
= \sum_{a,b} \Phi_{A-B}(|b\rangle_{A} \otimes L_{A-R} |b\rangle_{A} \otimes L_{A-R}^{\dagger}
\]

\[
\iff (I_{B} \otimes L_{A-R}) J(\Phi_{A-B})(I_{B} \otimes L_{A-R})
\]

where \( L_{A-R} := \sum_{a,c} \Psi_{a,c} |c\rangle_{R} \langle a|_{A} \). Let

\[
Y_{BA} : = (I_{B} \otimes L_{A-R}^{\dagger}) Y_{BR} (I_{B} \otimes L_{A-R})
\]

\[
Z_{BA} : = (I_{B} \otimes L_{A-R}^{\dagger}) Z_{BR} (I_{B} \otimes L_{A-R}).
\]

Then, we obtain

\[
\langle \Phi_{A-B}(|\psi\rangle_{AR}), Y_{BR} \rangle = \langle J(\Phi_{A-B}), Y_{BA}^{*} \rangle
\]

\[
\langle \Psi_{A-B}(|\psi\rangle_{AR}), Z_{BR} \rangle = \langle J(\Phi_{A-B}), Z_{BA}^{*} \rangle
\]

and

\[
\begin{bmatrix} Y_{BR} & I_{BR} \\ I_{BR} & Z_{BR} \end{bmatrix} \succeq 0 \implies L \begin{bmatrix} Y_{BR} & I_{BR} \\ I_{BR} & Z_{BR} \end{bmatrix} L^{\dagger} \succeq 0
\]

where

\[
L := \begin{bmatrix} I_{B} \otimes L_{A-R}^{\dagger} & 0 \\ 0 & I_{B} \otimes L_{A-R} \end{bmatrix}
\]

Since

\[
L \begin{bmatrix} Y_{BA} & I_{B} \otimes \sigma_{A} \\ I_{B} \otimes \sigma_{A} & Z_{BA} \end{bmatrix} L^{\dagger} \succeq 0
\]

\[
\sigma_{A} \succeq 0, \text{Tr}(\sigma_{A}) = 1.
\]

Next, we will show that the optimal value of \( \text{D3} \) is equal to the optimal value of \( \text{D1} \) if \( \dim(R) \geq \dim(A) \). Even if we replace \( \sigma_{A} \geq 0 \) with \( \sigma_{A} > 0 \), and replace min with inf in \( \text{D3} \), the optimal value of \( \text{D3} \) is unchanged from the continuity. Let \( Y_{BA}^{*}, Z_{BA} \in \mathcal{S}(\mathcal{A}) \) be an arbitrary feasible solution of \( \text{D3} \) with \( \sigma_{A} > 0 \). Let \( \sigma_{A} = \sum_{a} \lambda_{a} |\varphi_{a}\rangle \langle \varphi_{a}|_{A} \) be the spectral decomposition of \( \sigma_{A} \). From \( \dim(R) \geq \dim(A) \), \( \sigma_{A} = L_{A-R}^{\dagger} L_{A-R} \) for \( L_{A-R} := \sum_{a} \lambda_{a}^{1/2} |\varphi_{a}\rangle |\varphi_{a}|_{A} \). Let \( J_{A-R} := \sum_{a} \lambda_{a}^{1/2} |\varphi_{a}\rangle |\varphi_{a}|_{A} \). Then, \( L_{A-R}^{\dagger} J_{A-R} = I_{A} \). Then, it is easy to verify that

\[
Y_{BR} : = (I_{B} \otimes J_{A-R}) Y_{BA} (I_{B} \otimes J_{A-R}^{\dagger})
\]

\[
Z_{BR} : = (I_{B} \otimes J_{A-R}) Z_{BA} (I_{B} \otimes J_{A-R}^{\dagger})
\]

\[
|\psi\rangle : = \sum_{a} \sqrt{\lambda_{a}} |\varphi_{a}\rangle \langle \varphi_{a}|_{A} |a\rangle_{R}
\]
is a feasible solution of $\text{[D1]}$, and have the same objective value. This implies that the optimal value of $\text{[D3]}$ is equal to the optimal value of $\text{[D1]}$ if $\dim(R) \geq \dim(A)$.

The dual of $\text{[D3]}$ is

$$
\begin{array}{c}
\max_{\lambda} \quad \lambda \left[ U_{BA} W_{BA} \right] \\
\text{subject to:} \quad \left[ U_{BA} W_{BA} \right] \succeq 0,
\end{array}
$$

where

$$
\begin{align*}
\min & \quad \lambda \langle \Phi_{A-B}, Z^t_{BA} \rangle \\
\text{subject to:} & \quad \langle \Phi_{A-B}, Y^t_{BA} \rangle \geq 0,
\end{align*}
$$

where $\lambda_{\Phi_{A-B}} = \lambda_{\Phi_{A-B}}(\lambda_{\Phi_{A-B}})$. Hence, the optimal value of the following SDP is at least

$$
\max \langle \Phi_{A-B}, Y^t_{BA} \rangle.
$$

From $\text{[D2]}$, we obtain

$$
\Lambda_{A\rightarrow B}(|\psi\rangle_{AR}) = (I_B \otimes L_{A\rightarrow R})J(\Lambda_{A\rightarrow B})(I_B \otimes L_{A\rightarrow R}).
$$

Hence, the optimal value of the following SDP is at least the optimal value of $\text{[D1]}$.

$$
\max \langle \Phi_{A-B}, Y^t_{BA} \rangle.
$$

When $\dim(R) \geq \dim(A)$, we can show that the optimal value of $\text{[D5]}$ is equal to the optimal value of $\text{[D4]}$ in the same way as the previous section.

The dual of $\text{[D5]}$ is

$$
\min \lambda_{Y_{BA} \circ Z_{BA}} \max \lambda_{\Phi_{A-B}} \langle \Phi_{A-B}, X^t_{BA} \rangle - \lambda_{\Phi_{A-B}}(\lambda_{\Phi_{A-B}}) \geq 0
$$

where $\lambda_{\Phi_{A-B}} = \lambda_{\Phi_{A-B}}(\lambda_{\Phi_{A-B}})$. Hence, the dual SDP of $\text{[D5]}$ is

$$
\max \lambda_{\Phi_{A-B}} \langle \Phi_{A-B}, Y^t_{BA} \rangle.$$

Note that $\|\text{Tr}_B(|\Lambda_{A-B}|)\|_{\infty}$ is an upper bound of $\|\Lambda_{A-B}\|_\infty$, but not necessarily equal to $\|\Lambda_{A-B}\|_\infty$. 

2. SDP for the trace norm

The trace norm of an Hermitian matrix $H_A$ is represented as an solution of the following SDP.

$$
\max \langle H_A, X_A \rangle
$$

subject to: $-I_A \preceq X_A \preceq I_A$. 

Hence, for an Hermitian-preserving map $\Lambda_{A\rightarrow B}$, $\max_{|\psi\rangle_{AR}} \|\Lambda_{A\rightarrow B}(|\psi\rangle_{AR})\|_1$ is equal to

$$
\max_{|\psi\rangle_{AR}, X_{BR}} \langle \Lambda_{A\rightarrow B}(|\psi\rangle_{AR}), X_{BR} \rangle
$$

subject to: $-I_B \preceq X_{BR} \preceq I_B$.
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