On the diameter of polytope

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Abstract

Bonifas et. al. [1] derived an upper bound of a polytope \( P = \{x \in \mathbb{R}^n : Ax \leq b\} \) where \( A \in \mathbb{Z}^{m \times n} \) and \( m > n \). This comment indicates that their method can be applied to the case where \( A \in \mathbb{R}^{m \times n} \), which results in an upper bound of the diameter for the general polytope \( O\left(\frac{n^3 \Delta}{\det(A^*)}\right) \), where \( \Delta \) is the largest absolute value among all \((n-1)\times(n-1)\) sub-determinants of \( A \) and \( \det(A^*) \) is the smallest absolute value among all nonzero \( n \times n \) sub-determinants of \( A \). For each given polytope, since \( \Delta \) and \( \det(A^*) \) are fixed, the diameter is bounded by \( O(n^3) \).

Keywords: diameter of polytope, linear programming.

1 Introduction

Let \( x^* \in P \) denote a vertex of \( P \) which satisfies (a) \( Ax^* \leq b \) holds and (b) there exist \( n \) linear independent rows of \( A \) where the equalities hold. Two vertices \( x^* \) and \( y^* \) are neighbors if they are connected by an edge of \( P \), which is defined by \( n-1 \) linearly independent rows of \( A \) where the equalities hold for both \( x^* \) and \( y^* \). In this way, any two vertices on \( P \) are connected by a path composed of a series of edges. The diameter of \( P \) is the integer that is the smallest number bound of the shortest path between any two vertices on \( P \).

The famous Hirsch conjecture (see [2]) states that for \( m > n \geq 2 \), diameter of \( P \) is less than \( m - n \). After extensive research for 50 years, this conjecture was disproved by Santos [3]. But the interest on the bound of the diameter of polytope is not reduced because this problem is not only hard but also has theoretical implication to the simplex method of the linear programming problem. Recently, Bonifas et. al. [1] derived an upper bound for a polytope with total unimodularity, i.e., for \( A \in \mathbb{Z}^{m \times n} \). We show in this comment that their method can be applied to general polytope where \( A \in \mathbb{R}^{m \times n} \).

Without loss of generality, we may assume that the lengths all row vectors of \( A \) are one, which can easily be achieved by normalizing the row \( A_i \), the \( i \)th row of \( A \), and dividing \( b_i \) by \( ||A_i|| \) for all \( i \). This does not change the graph of the polytope \( P \).

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2 Main results

We follow the notations and definitions of Bonifas et. al. [1]. First, assume that \( P \) is non-degenerate, i.e., each vertex has exactly \( n \) tight inequality. Let \( V \) be the set of all vertices of \( P \). The normal cone \( C_v \) of a vertex \( v \) is the set of all vectors \( c \in \mathbb{R}^n \) such that \( v \in V \) is an optimal solution of the linear programming \( \max \{ c^T x : x \in \mathbb{R}^n, Ax \leq b \} \).

Two vertices \( u \) and \( v \) are adjacent if and only if \( C_u \) and \( C_v \) share a facet. Let the unit ball \( B_n = \{ x \in \mathbb{R}^n : \|x\|_2 \leq 1 \} \).

The volume of the union of the normal cones of \( U \in V \) is defined as \( \text{vol}(S_U) = \text{vol}(\bigcup_{v \in U} C_v \cap B_n) \),

where \( S_v = C_v \cap B_n \) is defined as the sphere cone of \( C_v \).

For any two vertices \( u \) and \( v \) in \( P \), starting from \( u \) and \( v \), the breadth-first-search finds all the neighbor vertices by iteration until a common vertex is discovered. The shortest path is no more than two times the number of iterations. Let \( I_j \in V \) be the set of vertices that have been discovered in \( j \)th iteration. Clearly, if

\[
\text{vol}(S_{I_j}) \geq \frac{1}{2} \text{vol}(B_n),
\]

then, the common vertex must be found in less than \( j \) iterations, i.e., the diameter is bounded by \( 2j \). The rest effort is to estimate \( j \) such that equation (1) holds.

The \((n-1)\)-dimensional surface of a spherical cone \( S \) that is not on the sphere is denoted as the dockable surface \( D(S) \). Bonifas et. al. [1] showed the following:

**Lemma 2.1** Let \( S \) be a (not necessarily convex) spherical cone with \( \text{vol}(S) \leq \frac{1}{2} \text{vol}(B_n) \). Then,

\[
\frac{D(S)}{\text{vol}(S)} \geq \sqrt{\frac{2n}{\pi}}.
\]

Let \( \Delta \) denote the largest absolute value among all \((n-1)\times(n-1)\) sub-determinants of \( A \) and \( A_v \) be a \( n \times n \) matrix of \( A \) corresponding to a vertex \( v \in V \), i.e., there is a \( x \) satisfying \( Ax \leq b \) and \( A_v x = b_v \) where \( b_v \) is a sub-vector of \( b \) whose index set is the same as \( A_v \). Denote \( \det(A^*) = \min_{v \in V} \det(A_v) \), \( \det(A_v) \) is the volume of the box spanned by the (unit length) row vectors of \( A_v \). \( \det(A^*) \) can be viewed as the condition number of polytope [8]. The next lemma is a modification of Lemma 3 of Bonifas et. al. [1].

**Lemma 2.2** Let \( v \) be a vertex of \( P \). Then, one has,

\[
\frac{D(S_v)}{\text{vol}(S_v)} \leq \frac{n^{2.5} \Delta}{\det(A^*)}.
\]
Proof: The proof uses the same idea of Bonifas et. al. [1] for the case of $A \in \mathbb{R}^{m \times n}$. Let $F$ be a facet of a spherical cone $S_v$. Let $y$ be the vertex of $S_v$ not contained in the $(n-1)$ dimensional facet $F$. Let $Q$ be the convex hull of $F$ and $y$. We have $Q \subseteq S_v$ because $S_v$ is convex. Let $h_F$ be the Euclidean distance of $y$ from the hyperplane containing $F$, we have

$$\text{vol}(S_v) \geq \text{vol}(Q) = \frac{\text{area}(F) \cdot h_F}{n}.$$  

This yields

$$\frac{D(S_v)}{\text{vol}(S_v)} = \sum_{\text{facet } F} \frac{\text{area}(F)}{\text{vol}(S_v)} \leq n \sum_{\text{facet } F} \frac{1}{h_F}. \tag{4}$$

Let $a_1, a_2, \ldots, a_n$ be the row vectors of $A_v$, and $b_1, b_2, \ldots, b_n$ be the column vectors of the adjugate of $A_v$. Clearly $A_v[b_1, b_2, \ldots, b_n] = \det(A_v)I$, where $I$ is an identity matrix. This means that $a_1^T b_1 = \det(A_v)$ and $b_1 \perp \{a_2, \ldots, a_n\}$. Without loss of generality, assuming $y$ is $a_1$, clearly, $h_F$ is the projection of $a_1$ onto $b_1$. Noticing the absolute value of each component of $b_1$ is less than or equal to $\Delta$, we have

$$h_F = |a_1| \cos(\theta) = |a_1| \frac{a_1^T b_1}{|a_1| \cdot |b_1|} = \frac{a_1^T b_1}{|b_1|} = \frac{\det(A_v)}{|b_1|} \geq \frac{\det(A^*)}{\sqrt{n}\Delta}. \tag{5}$$

Substituting this into (4) completes the proof.

The aforementioned two lemmas lead to the following claim.

Lemma 2.3 Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a general polytope with $A \in \mathbb{R}^{m \times n}$ and $m > n \geq 2$. Let all $(n-1) \times (n-1)$ sub-determinants of $A$ are bounded above by $\Delta$ and $\det(A_v)$ are bounded below by $\det(A^*)$. Let $I_j \subseteq V$ be a set of vertices with $\text{vol}(I_j) \leq \frac{1}{2} \text{vol}(B_n)$. Then the volume of the neighborhood of $I_j$, denoted by $\text{vol}(S_{N(I_j)})$, satisfies

$$\text{vol}(S_{N(I_j)}) \geq \sqrt{\frac{2}{\pi n^2 \Delta}} \cdot \text{vol}(S_{I_j}). \tag{5}$$
Proof: Noticing that $D(S_{I_j})$ is part of $\sum_{v \in N(I_j)} D(S_v)$ and using Lemma 2.1 we have

$$\sum_{v \in N(I_j)} D(S_v) \geq D(S_{I_j}) \geq \sqrt{\frac{2n}{\pi}} \cdot \text{vol}(S_{I_j}).$$  \hfill (6)

Applying Lemma 2.2 we have

$$\sum_{v \in N(I_j)} D(S_v) \leq \frac{n^{2.5} \Delta}{\det(A^*)} \sum_{v \in N(I_j)} \text{vol}(S_v) = \frac{n^{2.5} \Delta}{\det(A^*)} \cdot \text{vol}(S_{N(I_j)}).$$ \hfill (7)

Combining these two inequality gives

$$\text{vol}(S_{N(I_j)}) \geq \frac{\det(A^*)}{n^{2.5} \Delta} \sqrt{\frac{2n}{\pi}} \cdot \text{vol}(S_{I_j}) = \sqrt{\frac{2}{\pi} \frac{\det(A^*)}{n^2 \Delta}} \cdot \text{vol}(S_{I_j}).$$ \hfill (8)

This completes the proof.

The main result of the comment follows from Lemma 2.3.

Theorem 2.1 Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a general polytope with $A \in \mathbb{R}^{m \times n}$ and $m > n \geq 2$. Let all $(n-1) \times (n-1)$ sub-determinants of $A$ are bounded above by $\Delta$ and $\det(A_v)$ are bounded below by $\det(A^*)$. Then, the diameter of the polytope $P$ is bounded by $O\left(\frac{n^3 \Delta}{\det(A^*)}\right)$.

Proof: We assume that the breadth-first-method starts from vertex $v$. For $j \geq 1$ and $\text{vol}(S_{I_j-1}) \leq \frac{1}{2} \cdot \text{vol}(B_n)$, using Lemma 2.3 we have

$$\text{vol}(S_{I_j}) \geq \left(1 + \sqrt{\frac{2}{\pi} \frac{\det(A^*)}{n^2 \Delta}}\right)^j \cdot \text{vol}(S_{I_{j-1}})$$

$$\geq \left(1 + \sqrt{\frac{2}{\pi} \frac{\det(A^*)}{n^2 \Delta}}\right)^j \cdot \text{vol}(S_{I_0}),$$ \hfill (9)

where $S_{I_0} = S_v$ includes a simplex $J_n$ spanned by $n + 1$ vertices composed of 0 and $n$ row vectors of $A_v$ (see Figure 1). Since the volume of $J_n$ is given by

$$\text{vol}(J_n) = \frac{\det(A_v)}{n!} \geq \frac{\det(A^*)}{n!},$$

we have

$$\text{vol}(S_{I_0}) \geq \text{vol}(J_n) \geq \frac{\det(A^*)}{n!}.$$ \hfill (10)

Assuming $n$ is even (which is easy to derive the result but the order of the estimation remains the same for odd $n$), we have

$$\text{vol}(B_n) = \frac{\pi^n}{n!}.$$ \hfill (11)
The condition $\text{vol}(S_I) \leq \frac{1}{2} \cdot \text{vol}(B_n)$ implies

$$\frac{1}{2} \cdot \text{vol}(B_n) = \frac{1}{2n!} \pi^n \geq \text{vol}(S_I) \geq \left(1 + \sqrt{\frac{2}{\pi}} \frac{\det(A^*)}{n^2 \Delta}\right)^j \frac{\det(A^*)}{n!},$$

or

$$\pi^n \geq 2 \det(A^*) \left(1 + \sqrt{\frac{2}{\pi}} \frac{\det(A^*)}{n^2 \Delta}\right)^j, \quad (12)$$

For $0 \leq c \leq 1$, it has $\ln(1 + c) \geq c/2$. Therefore, we can rewrite (12) as

$$n \ln \pi \geq \ln(2 \det(A^*)) + j \ln \left(1 + \sqrt{\frac{2}{\pi}} \frac{\det(A^*)}{n^2 \Delta}\right) \geq \ln(2 \det(A^*)) + j \sqrt{\frac{1}{2\pi}} \frac{\det(A^*)}{n^2 \Delta}. \quad (13)$$

This shows $j = O \left(\frac{n^2 \Delta}{\det(A^*)}\right)$.

**Remark 2.1** This upper bound $O \left(\frac{n^2 \Delta}{\det(A^*)}\right)$ is not just related to $n$ and $m$ like the ones of Kalai-Kleitman, Sukegawa, and Todd [3, 6, 7] but also to the condition numbers of the vertices of $A_v$. If the rays of all $S_v$ are almost perpendicular, then $\det(A^*)$ will be close to one. Otherwise, if for some $v$ the rays of $S_v$ are almost linear dependent, then $\det(A^*)$ will be close to zero, and the diameter of the polytope may increase significantly. Therefore, $\det(A^*)$ can be viewed as the condition number of the polytope.

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