ON THE IDEAL ASSOCIATED TO A LINEAR CODE

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Abstract. In this paper we introduce a binomial ideal derived from an arbitrary linear code. Let \( C \) be an \([n, k]\) linear code defined over the finite field \( \mathbb{F}_q \), the generators of the associated ideal are given by the rows of a generator matrix and the relations given by the additive table of the field \( \mathbb{F}_q \).

The binomials involved in the reduced Gröbner basis of such ideal w.r.t. the deglex order induce a uniquely defined test set for the code, moreover the graver basis associated to this ideal provides a universal test set which turns out to be a set containing the set of codewords of minimal support of the code. Therefore this article yields a generalization of [1, 2, 6] where this ideas were stated just for the binary case or for modular codes. In order to obtain a test set or the set of codewords of minimal support we must compute a reduced Gröbner basis of an ideal from which we known a generating set. In the last section we showed some results on the computation of the Gröbner basis.

1. REVIEW

We begin with an introduction of basic definitions and some known results from coding theory over finite fields. By \( \mathbb{Z} \) and \( \mathbb{F}_q \) we denote the ring of integers and any representation of a finite field with \( q \) elements, respectively.

Although is not obvious, first note that \( q \) must be a prime power, say \( q = p^r \), where \( p \) is prime and \( r \) is a positive integer, thus \( \mathbb{F}_q \) contains the subfield \( \mathbb{F}_p \), or equivalently \( \mathbb{F}_q \) is a vector space over \( \mathbb{F}_p \) of dimension \( r \). Moreover the set \( \mathbb{F}_q^* \) of all nonzero elements of \( \mathbb{F}_q \) is cyclic of order \( q - 1 \) under multiplication, each generator is called a primitive element of \( \mathbb{F}_q \).

Suppose \( f(x) \in \mathbb{F}_p[x] \) is an irreducible polynomial over \( \mathbb{F}_p \) of degree \( r \), then the residue class ring \( \mathbb{F}_p[x]/(f(x)) \) is actually the finite field \( \mathbb{F}_q \) with \( q = p^r \) elements. Let \( \alpha \) be a root of a polynomial \( f(x) \in \mathbb{F}_p[x] \) that is irreducible over \( \mathbb{F}_p \) and has degree \( r \), an equivalent formulation of \( \mathbb{F}_q \) is \( \mathbb{F}_p[\alpha] \). Hence although \( \mathbb{F}_q \) is unique up to

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isomorphism, it may have many different representations. We adopt the convention that $\mathbb{F}_q = \mathbb{F}_p[x]/(f(x))$ where $f(x)$ is chosen that $f(x)$ is irreducible of degree $r$ and has a root that is a primitive element of $\mathbb{F}_q = \mathbb{F}_{p^r}$. For example we express $\mathbb{F}_9$ in the form $\mathbb{F}_3[\alpha]$ where $\alpha$ is a root of the polynomial $x^2 + x + 1$, irreducible over $\mathbb{F}_3$.

An $[n,k]$ linear code $C$ over $\mathbb{F}_q$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$. We define a generator matrix of $C$ to be a $k \times n$ matrix $G$ whose row vectors span $C$, while a parity check matrix of $C$ is an $(n-k) \times n$ matrix $H$ whose null space is $C$. We will denote by $d_H(\cdot, \cdot)$ and $w_H(\cdot)$ the hamming distance and the hamming weight on $\mathbb{F}_q^n$, respectively. We write $d$ for the minimum distance of a linear code $C$, which is equal to its minimum weight. This parameter determines the error-correcting capability of $C$ which is given by $t \leq \lfloor \frac{d-1}{2} \rfloor$, where $\lfloor \cdot \rfloor$ is the greatest integer function. For a word $x \in \mathbb{F}_q^n$, its support, denoted by $\text{supp}(x)$ is defined as the set of nonzero coordinate positions, i.e., $\text{supp}(x) = \{i \mid x_i \neq 0\}$.

2. The ideal associated with a code

Let $\alpha$ be a primitive element of $\mathbb{F}_q$. We will use the following characteristic crossing functions:

$$\nabla : \mathbb{Z}^{n(q-1)} \rightarrow \mathbb{F}_q^n \quad \text{and} \quad \blacktriangle : \mathbb{F}_q^n \rightarrow \mathbb{Z}^{n(q-1)}$$

The map $\blacktriangle$ replace the class of the element $a = (a_1, \ldots, a_n)$ in $\mathbb{F}_q^n$ with

$$a_i = \alpha^{j_i} \quad \text{or} \quad a_i = 0 := j_i \quad \text{i.e.} \quad j_i \subseteq \{0, 1, \ldots, q-1\}$$

for all $i = 1, \ldots, n$, by the vector $(e_{j_1}, \ldots, e_{j_n})$ in $\mathbb{Z}^{n(q-1)}$, where $(e_1, \ldots, e_{q-1})$ denotes the canonical basis of $\mathbb{Z}^{q-1}$ and $e_0$ denotes the zero vector in $\mathbb{Z}^{q-1}$. Whereas the map $\nabla$ recover the element

$$(j_1, \alpha + \ldots + j_1 q^{-1}, \ldots, j_{n,1} \alpha + \ldots + j_{n,q-1} q^{q-1})$$

of $\mathbb{F}_q^n$ from the $n(q-1)$-tuple of integers $(j_{1,1}, \ldots, j_{1,q-1}, \ldots, j_{n,1}, \ldots, j_{n,q-1})$.

Unless otherwise stated we simply write $C$ for an $[n,k]$ linear code defined over the finite field $\mathbb{F}_q$. We will use the symbol $x_i$ to denote the set of variables $x_1, \ldots, x_{q-1}$; and, by abuse of notation, we write the symbol $x$ to denote the set $x_1, \ldots, x_n$. With this notation, let $a = (a_1, \ldots, a_{q-1})$ be a $(q-1)$-tuple of non-negative integers, then we set $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and, more generally, for every $b = (b_1, \ldots, b_n)$ consisting of $n$ integer vectors $b_i$ of length $q-1$, i.e. $b_i = (b_{i1}, \ldots, b_{iq-1})$ for each $i = 1, \ldots, n$, we define

$$x^b = x_1^{b_{11}} \cdots x_n^{b_{nn}} = (x_1^{b_{11}} \cdots x_{1q-1}^{b_{1q-1}}) \cdots (x_n^{b_{11}} \cdots x_{nq-1}^{b_{nq-1}}).$$

We define the ideal associated to $C$ as the binomial ideal:

$$I(C) = \langle x^a - x^b \mid a - b \in C \rangle \subseteq \mathbb{F}_2[x_1, \ldots, x_n].$$

Given the rows of a generator matrix of $C$, labelled by $w_1, \ldots, w_k$, we define the following ideal:

$$I_2 = \left\{ \left\{ x^{\blacktriangle w_i} - 1, \ldots, x^{(\alpha^{q-2} w_i)} - 1 \right\}_{i=1,\ldots,k} \cup \left\{ \mathcal{R} \left( T_+^{(i)} \right) \right\}_{i=1,\ldots,n} \right\}$$

$$= \left\{ \left\{ x^{\blacktriangle (\alpha^{q-2} w_i)} - 1 \right\}_{i=1,\ldots,k} \cup \left\{ \mathcal{R} \left( T_+^{(i)} \right) \right\}_{i=1,\ldots,n} \right\}.$$
where $\mathcal{R}\left(T_+^{(i)}\right)$ consist of all the binomials in the variable $x_i = (x_{i1}, \ldots, x_{iq-1})$ with $i = 1, \ldots, n$ associated to the relations given by the additive table of the field $\mathbb{F}_q = \langle \alpha \rangle$, i.e.,

$$\mathcal{R}\left(T_+^{(i)}\right) = \{x_{iu}x_{iv} - x_{iw} \mid \alpha^u + \alpha^v = \alpha^w\} \cup \{x_{iu}x_{iv} - 1 \mid \alpha^u + \alpha^v = 0\}.$$ 

**Theorem 2.1.** $I(\mathcal{C}) = I_2$

**Proof.** It is clear that $I_2 \subseteq I(\mathcal{C})$ since all binomials in the generating set of $I_2$ belongs to $I(\mathcal{C})$. Note that the binomials from $\mathcal{R}\left(T_+^{(i)}\right)$ poses no problem because they correspond to the binomial $x^{a0} - 1$ which fit in $I(\mathcal{C})$.

To show the converse it suffices to show that each binomial $x^{an} - x^{ab}$ of $I(\mathcal{C})$ belongs to $I_2$. By the definition of $I(\mathcal{C})$ we have that $a - b \in \mathcal{C}$. Hence

$$a - b = \lambda_1 w_1 + \ldots + \lambda_k w_k$$

with $\lambda_1, \ldots, \lambda_k \in \mathbb{F}_q$.

Note that, if the binomials $z_1 - 1$ and $z_2 - 1$ belongs to the ideal $I_2$ then, also $z_1z_2 - 1 = (z_1 - 1)z_2 + (z_2 - 1)$ belongs to $I_2$. On account of the previous line, we have:

$$x^{a-b} - 1 = \left(x^{\lambda_1 w_1} - 1\right) \prod_{i=2}^{k} x^{\lambda_i w_i} + \left(\prod_{i=2}^{k} x^{\lambda_i w_i} - 1\right)$$

$$= \left(x^{\lambda_1 w_1} - 1\right) \prod_{i=2}^{k} x^{\lambda_i w_i} + \left(\sum_{i=3}^{k} x^{\lambda_i w_i} + \ldots + x^{\lambda_k w_k} - 1\right)$$

If at least one $\lambda_i$ is nonzero with $i = 1, \ldots, k$, then the last equation forces that

$$x^{a-b} - 1 \in \left\{x^{\lambda_j w_i} - 1\right\}_{j=0, \ldots, q-1}$$

Otherwise $a - b = 0$. Therefore it is easily seen that

$$x^{a-b} - 1 \in \left\{\mathcal{R}\left(T_+^{(i)}\right)\right\}_{i=1, \ldots, n}$$

Note that we actually proved that $x^{an} - x^{ab} = (x^{a-b} - 1)x^{b} \in I_2$, which completes the proof.

**Remark 1.** Let $\mathcal{G}$ be the reduced Gröbner basis of the ideal $I(\mathcal{C})$ w.r.t. $\prec$. We can compute $\mathcal{G}$ using Buchberger’s algorithm, however there are some computational advantages in our case:

1. $I(\mathcal{C})$ is a binomial ideal thus the elements of $\mathcal{G}$ are also binomials.
2. The maximal length of a word in the computation is $n$ since the binomials from $\mathcal{R}\left(T_+^{(i)}\right)$ prevent the fact that two variables of the form $x_{il}$ and $x_{jm}$ with $i = j$ appear on the same word.
3. There is no coefficient growth since the binomials generating the ideal are of the form $x^{w} - x^{v}$ with $x^{w} \succ x^{v}$. Therefore we may restrict our algorithm to the ring $\mathbb{F}_2[x].$
Example 2.2. Let us consider the $[3, 2]$ linear code $C$ defined over $\mathbb{F}_9 = \frac{\mathbb{F}_3[x]}{x^2 + x + 1}$ with generator matrix:

$$G = \begin{pmatrix} 1 & 0 & \alpha + 1 \\ 0 & 1 & 2\alpha \end{pmatrix} \in \mathbb{F}_9^{2 \times 3}$$

where the primitive element $\alpha$ is a root of the irreducible polynomial $x^2 + x + 1$, i.e.

$$\mathbb{F}_9 = \{0, \alpha, \alpha^2 = \alpha + 1, \alpha^3 = 2\alpha + 1, \alpha^4 = 2, \alpha^5 = 2\alpha, \alpha^6 = 2\alpha + 2, \alpha^7 = \alpha + 2, \alpha^8 = 1 \}.$$

This representations of the field $\mathbb{F}_9$ gives the following additive table:

| $T_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|
| 1    | 5 | 3 | 8 | 7 | 0 | 4 | 6 | 2 |
| 2    | 6 | 4 | 1 | 8 | 0 | 5 | 7 |
| 3    | 7 | 5 | 2 | 1 | 0 | 6 |
| 4    | 8 | 6 | 3 | 2 | 0 |
| 5    | 1 | 7 | 4 | 3 |
| 6    | 2 | 8 | 5 |
| 7    | 3 | 1 |
| 8    | 4 |

Or equivalently,

$$\begin{cases} 
\alpha + \alpha = \alpha^5 & \alpha + \alpha^2 = \alpha^3 & \ldots & \alpha + \alpha^8 = \alpha^2 \\
\alpha^2 + \alpha^2 = \alpha^6 & \ldots & \alpha^2 + \alpha^8 = \alpha^7 \\
& \vdots \\
\alpha^8 + \alpha^8 = \alpha^4
\end{cases}$$

Therefore we obtain the following binomials associated to the previous rules:

$$\mathcal{R}\left(T_+^{(i)}\right) = \left\{ x_{ij} - x_{ij}^{(i)} : i = 1, 2, 3 \right\}$$

for $i = 1, 2, 3$.

Let us label the rows of $G$ by $w_1$ and $w_2$. By Theorem 2.1 the ideal associated to $C$ may be defined as the following binomial ideal:

$$I(C) = \left\langle \left\{ x^\alpha (\alpha^j w_1)^{-1}, x^\alpha (\alpha^j w_2)^{-1} \right\}_{j=1, \ldots, 8} \cup \left\{ \mathcal{R}\left(T_+^{(i)}\right) \right\}_{i=1, 2, 3} \right\rangle \subseteq \mathbb{F}_2[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$$

where

$$\begin{align*}
\mathbf{x}_1 &= (x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}) \\
\mathbf{x}_2 &= (x_{20}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26}, x_{27}) \\
\mathbf{x}_3 &= (x_{30}, x_{31}, x_{32}, x_{33}, x_{34}, x_{35}, x_{36}, x_{37})
\end{align*}$$
3. Computing the Gröbner basis

In this section we present an algorithm to compute a reduced Gröbner basis for the ideal associated to a given code defined over an arbitrary finite field \( F_q \). This algorithm was presented for the binary case in [1] but the extension to the general case is straightforward. For a deeper discussion in these techniques we refer the reader to [4, 5].

We denote by \( T \) the set of monomials in 
\[
\mathbb{F}_2[x] = \mathbb{F}_2[x_1, \ldots, x_n] = \mathbb{F}_2[x_{11}, \ldots, x_{1q-1}, \ldots, x_{n1}, \ldots, x_{nq-1}].
\]

Any monomial in \( \mathbb{F}_2[x] \) is a product of the form \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) with \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^{nq-1} \) and \( \alpha_i \in \mathbb{Z}^{q-1} \).

Given \( \prec \) an admissible term ordering in \( T \) it can be extended to a position over term (POT) admissible ordering \( \prec_{\text{POT}} \) in \( T^m = \{ te_l \mid t \in T \text{ and } l \in \{1, \ldots, m\} \} \), or equivalently a monomial ordering that gives priority to the position of the vector in \( (\mathbb{F}_2[x])^m \), i.e. \( x^\alpha e_l \prec_{\text{POT}} x^\beta e_j \iff i > j \) or \( i = j \) and \( x^\alpha \prec x^\beta \). Similarly, \( \prec \) can provide a term over position (TOP) admissible ordering \( \prec_{\text{TOP}} \) in \( T^m \), i.e. \( x^\alpha e_l \prec_{\text{TOP}} x^\beta e_j \iff x^\alpha \prec x^\beta \) or \( x^\alpha = x^\beta \) and \( i > j \).

Given a set of polynomials \( F = \{ f_1, \ldots, f_m \} \) in \( \mathbb{F}_2[x] \) generating the ideal \( I \). The algorithm [1] computes a Gröbner basis of \( I \) w.r.t. an admissible term ordering \( \prec \) from a Gröbner basis of the syzygy module \( M \) in \( \mathbb{F}_2[x]^{m+1} \) of the generator set \( F' = \{-1, f_1, \ldots, f_m\} \).

We give only the main ideas of the algorithm:

1. First observe that the set :
\[
(f_1, 1, 0, \ldots, 0), \quad (f_2, 0, 1, \ldots, 0), \quad \ldots, \quad (f_m, 0, 0, \ldots, 0, 1)
\]
is a basis of the syzygy module \( M \), denoted by \( G_1(M) \). Moreover it is a Gröbner basis w.r.t. a POT ordering induced from an ordering \( \prec \) in \( \mathbb{F}_2[x] \) and the weight vector \( w = (1, LT_{\prec}(f_1), \ldots, LT_{\prec}(f_m)) \).

2. We use the FGLM [3] algorithm adapted to submodules [1], i.e. running through the terms of \( \mathbb{F}_2[x]^m \) to obtain a new basis of \( M \) w.r.t. a TOP ordering in \( \mathbb{F}_2[x]^{m+1} \).

3. The first component of each element of the new basis points to an element of the Gröbner basis of \( I \) w.r.t. \( \prec \).

4. In our particular case in the associated syzygy computation the rows corresponding to the binomials of \( \left\{ R \left( \tau_+^{(i)} \right) \right\}_{i=1,2,\ldots,n} \) are considered as implicit in the calculation.

Note that each step of the algorithm can be viewed as a Gaussian reduction if we work with tables of the following form:

- The first row is labelled with \( -1 \) and the variables \( x_1, x_2, \ldots, x_n \).
- The following rows are filled with the exponents of the normal form of each element w.r.t. the basis \( G_1(M) \) as elements of \( \mathbb{F}_q \), i.e. using the map \( \nabla \).
- Note that Gaussian elimination is performed over \( \mathbb{F}_2 \), that is the rows can only be multiplied by \( \pm 1 \).
- The algorithm in the general case is initialized with the elements 
\[
e_1, \quad e_1 + e_2, \quad \ldots, \quad e_1 + e_{m+1},
\]
where \( m \) is the number of generators of our ideal. However in our special case, the set of generators is formed by \( k \) binomials representing the rows of a generator matrix of \( \mathcal{C} \) and all its multiples in \( \mathbb{F}_q \). Thus we may initialize the table with the elements:

\[
\text{e}_1, \ \{ \ \alpha^j (\text{e}_1 + \text{e}_2), \ \ldots, \ \alpha^j (\text{e}_1 + \text{e}_k) \} \}_{j=1,\ldots,q-1}
\]

We use the following example to illustrate the described algorithm.

**Example 3.1.** Consider \( \mathcal{C} \) the [6, 3] ternary code with generator matrix

\[
G_{\mathcal{C}} = \begin{pmatrix}
1 & 0 & 0 & 2 & 2 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 2 & 1
\end{pmatrix} \in \mathbb{F}_3^{3 \times 6}.
\]

We find that

\[
I(\mathcal{C}) = \left< \left\{ f_1 = x_{12} x_{41} x_{51} - 1, \ \alpha f_1 = x_{11} x_{42} x_{52} - 1 \\
f_2 = x_{22} x_{42} x_{52} - 1, \ \alpha f_2 = x_{21} x_{41} x_{51} - 1 \\
f_3 = x_{32} x_{42} x_{51} x_{62} - 1, \ \alpha f_3 = x_{31} x_{41} x_{52} x_{61} - 1
\right\} \cup \left\{ \mathcal{R} \left( T_{+}^{(i)} \right) \right\}_{i=1,2,\ldots,6} \right>,
\]

with \( \mathcal{R} \left( T_{+}^{(i)} \right) = \{ x_{i1}^2 - x_{i2}, x_{i1} x_{i2} - 1, x_{i2} - x_{i1} \}_{i=1,2,\ldots,6} \) and where \( \alpha \) is the primitive element of \( \mathbb{F}_3 = \{ 0, \alpha = 2, \alpha^2 = 1 \} \). Take \( \prec \) to be the deglex order with

\[
\frac{x_{11} \prec x_{12}}{x_1} \prec \frac{x_{21} \prec x_{22}}{x_2} \prec \ldots \prec \frac{x_{61} \prec x_{62}}{x_6},
\]

and suppose that we want to compute the Gröbner basis of \( I(\mathcal{C}) \) w.r.t. \( \prec \), denoted by \( \mathcal{G} \). The table is first initialized to the sequence

\[
\text{e}_1, \ \{ \ \alpha^j (\text{e}_1 + \text{e}_2), \ \alpha^j (\text{e}_1 + \text{e}_3), \ \alpha^j (\text{e}_1 + \text{e}_4) \} \}_{j=1,2}
\]
Introduce $x_{21}$

|        | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|--------|-------|-------|-------|-------|-------|-------|
| $\Rightarrow (x_{21}, 0, 0, 0)$ | 0     | 0     | 2     | 0     | 0     | 0     |
| $\Rightarrow (x_{21}, x_{21}, 0, 0)$ | 0     | 1     | 2     | 0     | 2     | 0     |
| $(x_{21}, 0, x_{21}, 0)$ | 0     | 0     | 0     | 1     | 1     | 0     |
| $(x_{21}, 0, 0, x_{21})$ | 0     | 0     | 2     | 1     | 1     | 2     |
| $(x_{22}, x_{22}, 0, 0)$ | 0     | 1     | 1     | 0     | 2     | 0     |
| $(x_{22}, 0, x_{22}, 0)$ | 0     | 0     | 0     | 1     | 1     | 0     |
| $(x_{22}, 0, 0, x_{22})$ | 0     | 0     | 2     | 1     | 1     | 2     |

Hence $(x_{11} + x_{21}, x_{22}, x_{11}, 0)$ is a syzygy and therefore $x_{11} + x_{22}$ and its multiple $x_{12} + x_{21}$ belongs to the Gröbner basis $\mathcal{G}$ of $I(C)$.

Introduce $x_{21}x_{31}$

|        | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|--------|-------|-------|-------|-------|-------|-------|
| $(x_{21}x_{31}, 0, 0, 0)$ | 0     | 0     | 2     | 0     | 0     | 0     |
| $(x_{21}x_{31}, x_{21}x_{31}, 0, 0)$ | 0     | 1     | 2     | 2     | 2     | 0     |
| $\Rightarrow (x_{21}x_{31}, 0, x_{21}x_{31}, 0)$ | 0     | 0     | 0     | 2     | 1     | 1     |
| $(x_{21}x_{31}, 0, 0, x_{21}x_{31})$ | 0     | 0     | 0     | 1     | 2     | 1     |
| $(x_{22}x_{32}, x_{22}x_{32}, 0, 0)$ | 0     | 1     | 1     | 1     | 2     | 0     |
| $(x_{22}x_{32}, 0, x_{22}x_{32}, 0)$ | 0     | 0     | 0     | 1     | 1     | 0     |
| $(x_{22}x_{32}, 0, 0, x_{22}x_{32})$ | 0     | 0     | 1     | 0     | 1     | 2     |

Introduce $x_{22}x_{61}$

|        | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|--------|-------|-------|-------|-------|-------|-------|
| $(x_{22}x_{61}, 0, 0, 0)$ | 0     | 0     | 1     | 0     | 0     | 0     |
| $(x_{22}x_{61}, x_{22}x_{61}, 0, 0)$ | 0     | 1     | 1     | 0     | 2     | 2     |
| $(x_{22}x_{61}, 0, x_{22}x_{61}, 0)$ | 0     | 0     | 0     | 2     | 0     | 1     |
| $\Rightarrow (x_{22}x_{61}, 0, 0, x_{22}x_{61})$ | 0     | 0     | 0     | 1     | 1     | 2     |
| $(x_{22}x_{61}, x_{22}x_{61}, x_{22}x_{61}, 0)$ | 0     | 1     | 2     | 0     | 2     | 1     |
| $(x_{22}x_{61}, 0, x_{22}x_{61}, x_{22}x_{61})$ | 0     | 0     | 1     | 0     | 1     | 1     |
| $(x_{22}x_{61}, 0, 0, x_{22}x_{61}, x_{22}x_{61})$ | 0     | 0     | 2     | 1     | 1     | 2     |

Introduce $x_{32}x_{52}$

|        | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|--------|-------|-------|-------|-------|-------|-------|
| $(x_{32}x_{52}, 0, 0, 0)$ | 0     | 0     | 0     | 1     | 1     | 0     |
| $(x_{32}x_{52}, x_{32}x_{52}, 0, 0)$ | 0     | 1     | 0     | 1     | 2     | 0     |
| $\Rightarrow (x_{32}x_{52}, 0, x_{32}x_{52}, 0)$ | 0     | 0     | 1     | 1     | 1     | 2     |
| $(x_{32}x_{52}, 0, 0, x_{32}x_{52})$ | 0     | 0     | 0     | 2     | 1     | 0     |
| $(x_{31}x_{51}, x_{31}x_{51}, 0, 0)$ | 0     | 1     | 0     | 2     | 2     | 0     |
| $(x_{31}x_{51}, 0, x_{31}x_{51}, 0)$ | 0     | 0     | 1     | 2     | 1     | 1     |
| $(x_{31}x_{51}, 0, 0, x_{31}x_{51})$ | 0     | 0     | 0     | 1     | 1     | 0     |

Similarly $(x_{32}x_{52} - x_{22}x_{61}, 0, x_{32}x_{52}, x_{22}x_{61})$ is a syzygy, so $x_{32}x_{52} - x_{22}x_{61}$ and its multiple $x_{31}x_{51} - x_{21}x_{62}$ belongs to $\mathcal{G}$. 
Continuing with Example 2.2, suppose that we want to calculate a \( \text{Gröbner} \) basis for the ideal \( I(C) \) w.r.t. the deglex order with

\[
\begin{align*}
    x_1 &< x_2 < \ldots < x_{18} < x_{21} < x_{22} < \ldots < x_{28} < x_{31} < x_{32} < \ldots < x_{38} \\
    \text{deglex order}
\end{align*}
\]

\( \mathcal{G} \) is a syzygy, so \( x_{31} x_{31} + x_{32} x_{32} + x_{33} x_{33} + x_{34} x_{34} + x_{35} x_{35} + x_{36} x_{36} \) belongs to \( \mathcal{G} \).

The remaining elements of \( \mathcal{G} \) are binomials of the set \( \{ R \left( T_i \right) \}_{i=1,2,\ldots,6} \).
We find that the ideal $I(C)$ is generated by the following binomials

$$
\left\{
\begin{align*}
    f_{11} &= x_{18}x_{32} - 1, & f_{12} &= x_{28}x_{35} - 1, \\
    \alpha f_{11} &= f_{21} = x_{11}x_{33} - 1, & \alpha f_{11} &= f_{22} = x_{21}x_{36} - 1, \\
    \alpha^2 f_{11} &= f_{31} = x_{12}x_{34} - 1, & \alpha^2 f_{11} &= f_{32} = x_{22}x_{37} - 1, \\
    \alpha^3 f_{11} &= f_{41} = x_{13}x_{35} - 1, & \alpha^3 f_{11} &= f_{42} = x_{23}x_{38} - 1, \\
    \alpha^4 f_{11} &= f_{51} = x_{14}x_{36} - 1, & \alpha^4 f_{11} &= f_{52} = x_{24}x_{31} - 1, \\
    \alpha^5 f_{11} &= f_{61} = x_{15}x_{37} - 1, & \alpha^5 f_{11} &= f_{62} = x_{25}x_{32} - 1, \\
    \alpha^6 f_{11} &= f_{71} = x_{16}x_{38} - 1, & \alpha^6 f_{11} &= f_{72} = x_{26}x_{33} - 1, \\
    \alpha^7 f_{11} &= f_{81} = x_{17}x_{31} - 1, & \alpha^7 f_{11} &= f_{82} = x_{27}x_{34} - 1, \\
\end{align*}
\right\}
$$

and the binomials of $\left\{R\left(T^{(i)}_i\right)\right\}_{i=1,2,3}$.

Due to the extension of the algorithm we only present here some steps.

|    | $-1$ | $x_1$ | $x_2$ | $x_3$ |
|----|------|------|------|------|
| $(1,0,0)$ | $1$ | $\alpha^2$ | $f_{11} = x_{18}x_{32} - 1 \equiv (1,0,\alpha + 1)$ |
| $(1,1,0)$ | $1$ | $\alpha^5$ | $f_{12} = x_{28}x_{35} - 1 \equiv (0,1,2\alpha)$ |
| $(1,0,1)$ | $\alpha$ | $\alpha^3$ | $f_{21} = x_{11}x_{33} - 1 \equiv (\alpha,0,2\alpha + 1)$ |
| $(1,0,\alpha)$ | $\alpha$ | $\alpha^6$ | $f_{22} = x_{21}x_{36} - 1 \equiv (0,\alpha,2\alpha + 2)$ |
| $(1,0,\alpha^2,0)$ | $\alpha^2$ | $\alpha^4$ | $f_{31} = x_{12}x_{34} - 1 \equiv (\alpha + 1,0,2)$ |
| $(1,0,\alpha^2)$ | $\alpha^2$ | $\alpha^7$ | $f_{32} = x_{22}x_{37} - 1 \equiv (0,\alpha + 1,\alpha + 2)$ |
| $(1,0,\alpha^3,0)$ | $\alpha^3$ | $\alpha^5$ | $f_{41} = x_{13}x_{35} - 1 \equiv (2\alpha + 1,0,2\alpha)$ |
| $(1,0,\alpha^3)$ | $\alpha^3$ | $1$ | $f_{42} = x_{23}x_{38} - 1 \equiv (0,2\alpha + 1,1)$ |
| $(1,0,\alpha^4,0)$ | $\alpha^4$ | $\alpha^6$ | $f_{51} = x_{14}x_{36} - 1 \equiv (2,0,2\alpha + 2)$ |
| $(1,0,\alpha^4)$ | $\alpha^4$ | $\alpha$ | $f_{52} = x_{24}x_{31} - 1 \equiv (0,2,\alpha)$ |
| $(1,0,\alpha^5,0)$ | $\alpha^5$ | $\alpha^7$ | $f_{61} = x_{15}x_{37} - 1 \equiv (2\alpha,0,\alpha + 2)$ |
| $(1,0,\alpha^5)$ | $\alpha^5$ | $\alpha^2$ | $f_{62} = x_{25}x_{32} - 1 \equiv (0,2\alpha,\alpha + 1)$ |
| $(1,0,\alpha^6,0)$ | $\alpha^6$ | $1$ | $f_{71} = x_{16}x_{38} - 1 \equiv (2\alpha + 2,0,1)$ |
| $(1,0,\alpha^6)$ | $\alpha^6$ | $\alpha^3$ | $f_{72} = x_{26}x_{33} - 1 \equiv (0,2\alpha + 2,2\alpha + 1)$ |
| $(1,0,\alpha^7,0)$ | $\alpha^7$ | $\alpha$ | $f_{81} = x_{17}x_{31} - 1 \equiv (\alpha + 2,0,\alpha)$ |
| $(1,0,\alpha^7)$ | $\alpha^7$ | $\alpha^4$ | $f_{82} = x_{27}x_{34} - 1 \equiv (0,\alpha + 2,2)$ |
| Introduce $x_{13}$ | $-1$ | $x_1$ | $x_2$ | $x_3$ |
|-----------------|------|-------|-------|-------|
| $\rightarrow x_{13}(1, 0, 0)$ | $\alpha^3$ | $\alpha^6$ | $\alpha^2$ | $x_{13} \equiv (2\alpha + 1, 0, 0)$ |
| $x_{13}(1, 1, 0)$ | $\alpha^6$ | $\alpha^2$ | $x_{13}f_{11} = x_{13}x_{28} - 1 \equiv (2\alpha + 2, 0, \alpha + 1)$ |
| $x_{13}(1, 0, 1)$ | $\alpha^3$ | $1$ | $\alpha^5$ | $x_{13}f_{12} = x_{13}x_{28} - 1 \equiv (2\alpha + 1, 2\alpha)$ |
| $x_{13}(1, \alpha, 0)$ | $\alpha$ | $\alpha^6$ | $x_{13}f_{21} = x_{18}x_{33} - 1 \equiv (1, 0, 2\alpha + 1)$ |
| $x_{13}(1, 0, \alpha)$ | $\alpha^3$ | $\alpha^2$ | $x_{13}f_{22} = x_{13}x_{21}x_{36} - 1 \equiv (2\alpha + 1, \alpha, 2\alpha + 2)$ |
| $x_{13}(1, \alpha^2, 0)$ | $\alpha^4$ | $\alpha^6$ | $x_{13}f_{31} = x_{14}x_{34} - 1 \equiv (2, 0, 2)$ |
| $x_{13}(1, 0, \alpha^2)$ | $\alpha^3$ | $\alpha^2$ | $x_{13}f_{32} = x_{13}x_{22}x_{37} - 1 \equiv (2\alpha + 1, \alpha + 1, 2\alpha + 2)$ |
| $x_{13}(1, \alpha^3, 0)$ | $\alpha^5$ | $\alpha^7$ | $x_{13}f_{41} = x_{17}x_{35} - 1 \equiv (\alpha + 2, 0, 2\alpha)$ |
| $x_{13}(1, 0, \alpha^3)$ | $\alpha^3$ | $1$ | $\alpha^6$ | $x_{13}f_{42} = x_{13}x_{23}x_{38} - 1 \equiv (2\alpha + 1, 2\alpha + 1, 1)$ |
| $x_{13}(1, \alpha^4, 0)$ | $\alpha^5$ | $\alpha^6$ | $x_{13}f_{51} = x_{15}x_{36} - 1 \equiv (2\alpha + 2, 0, 2\alpha)$ |
| $x_{13}(1, 0, \alpha^4)$ | $\alpha^3$ | $\alpha^4$ | $x_{13}f_{52} = x_{13}x_{24}x_{31} - 1 \equiv (2\alpha + 1, 2, \alpha)$ |
| $x_{13}(1, \alpha^5, 0)$ | $\alpha^2$ | $\alpha^7$ | $x_{13}f_{61} = x_{12}x_{37} - 1 \equiv (\alpha + 1, 0, 2\alpha)$ |
| $x_{13}(1, 0, \alpha^5)$ | $\alpha^3$ | $\alpha^5$ | $x_{13}f_{62} = x_{13}x_{25}x_{32} - 1 \equiv (2\alpha + 1, 2\alpha, \alpha + 1)$ |
| $x_{13}(1, \alpha^6, 0)$ | $\alpha$ | $1$ | $\alpha$ | $x_{13}f_{71} = x_{11}x_{38} - 1 \equiv (\alpha, 0, 1)$ |
| $x_{13}(1, 0, \alpha^6)$ | $\alpha^3$ | $\alpha^6$ | $x_{13}f_{72} = x_{13}x_{26}x_{33} - 1 \equiv (2\alpha + 1, 2\alpha + 2, 0, \alpha)$ |
| $x_{13}(1, \alpha^7, 0)$ | $0$ | $\alpha$ | $\alpha$ | $x_{13}f_{81} = x_{31} - 1 \equiv (0, 0, \alpha)$ |
| $x_{13}(1, 0, \alpha^7)$ | $\alpha^3$ | $\alpha^7$ | $\alpha^4$ | $x_{13}f_{82} = x_{13}x_{27}x_{34} - 1 \equiv (2\alpha + 1, \alpha + 2, 2)$ |

| Introduce $x_{28}$ | $-1$ | $x_1$ | $x_2$ | $x_3$ |
|-----------------|------|-------|-------|-------|
| $\rightarrow x_{28}(1, 0, 0)$ | $1$ | $1$ | $\alpha^2$ | $x_{28} \equiv (0, 1, 0)$ |
| $x_{28}(1, 1, 0)$ | $\alpha$ | $\alpha^2$ | $x_{28}f_{11} = x_{18}x_{28}x_{32} - 1 \equiv (1, 1, \alpha + 1)$ |
| $x_{28}(1, 0, 1)$ | $\alpha^4$ | $\alpha^6$ | $x_{28}f_{12} = x_{24}x_{35} - 1 \equiv (0, 2, 2\alpha)$ |
| $x_{28}(1, \alpha, 0)$ | $\alpha^2$ | $\alpha^6$ | $x_{28}f_{21} = x_{11}x_{28}x_{33} - 1 \equiv (\alpha, 1, 2\alpha + 1)$ |
| $x_{28}(1, 0, \alpha)$ | $\alpha^6$ | $\alpha^7$ | $x_{28}f_{32} = x_{27}x_{37} - 1 \equiv (0, \alpha + 2, \alpha + 2)$ |
| $x_{28}(1, \alpha^2, 0)$ | $\alpha^4$ | $\alpha^7$ | $x_{28}f_{41} = x_{13}x_{28}x_{35} - 1 \equiv (2\alpha + 1, 1, 2\alpha)$ |
| $x_{28}(1, 0, \alpha^2)$ | $\alpha^5$ | $1$ | $\alpha^6$ | $x_{28}f_{42} = x_{26}x_{38} - 1 \equiv (0, 2\alpha + 2, 1)$ |
| $x_{28}(1, \alpha^3, 0)$ | $\alpha^4$ | $1$ | $\alpha^6$ | $x_{28}f_{51} = x_{14}x_{28}x_{36} - 1 \equiv (2, 1, 2\alpha + 2)$ |
| $x_{28}(1, 0, \alpha^3)$ | $\alpha^5$ | $0$ | $\alpha$ | $x_{28}f_{52} = x_{31} - 1 \equiv (0, 0, \alpha)$ |
| $x_{28}(1, \alpha^4, 0)$ | $\alpha^5$ | $1$ | $\alpha^7$ | $x_{28}f_{61} = x_{15}x_{28}x_{37} - 1 \equiv (2\alpha, 1, \alpha + 2)$ |
| $x_{28}(1, 0, \alpha^4)$ | $\alpha^6$ | $1$ | $\alpha^7$ | $x_{28}f_{62} = x_{23}x_{32} - 1 \equiv (0, 2\alpha + 1, \alpha + 1)$ |
| $x_{28}(1, \alpha^5, 0)$ | $\alpha^6$ | $1$ | $\alpha$ | $x_{28}f_{71} = x_{18}x_{28}x_{38} - 1 \equiv (2\alpha + 2, 1, 1)$ |
| $x_{28}(1, 0, \alpha^5)$ | $\alpha^5$ | $\alpha^3$ | $x_{28}f_{72} = x_{25}x_{33} - 1 \equiv (0, 2\alpha, 2\alpha + 1)$ |
| $x_{28}(1, \alpha^6, 0)$ | $\alpha^5$ | $\alpha$ | $\alpha$ | $x_{28}f_{81} = x_{17}x_{28}x_{31} - 1 \equiv (\alpha + 2, 1, \alpha)$ |
| $x_{28}(1, 0, \alpha^6)$ | $\alpha^5$ | $\alpha^3$ | $x_{28}f_{82} = x_{21}x_{34} - 1 \equiv (0, \alpha, 2)$ |
| Introduce $x_{31}$ | $-1$ | $x_1$ | $x_2$ | $x_3$ |
|------------------|------|-------|-------|-------|
| $x_{31}(1,0,0)$  |      |       |       | $\alpha$ |
| $x_{31}(1,1,0)$  | 1    | $\alpha^3$ | $x_{31}f_{11} = x_{18}x_{33} - 1 \equiv (1,0,2\alpha+1)$ |
| $\rightarrow x_{31}(1,0,1)$ | 1 | 0 | $x_{31}f_{12} = x_{28} - 1 \equiv (0,1,0)$ |
| $x_{31}(1,\alpha,0)$ | $\alpha$ | 1 | $x_{31}f_{21} = x_{11}x_{38} - 1 \equiv (\alpha,0,1)$ |
| $x_{31}(1,0,\alpha)$ | $\alpha^2$ | $\alpha^4$ | $x_{31}f_{22} = x_{21}x_{34} - 1 \equiv (0,\alpha,2)$ |
| $x_{31}(1,\alpha^2,0)$ | $\alpha^7$ | $\alpha^6$ | $x_{31}f_{31} = x_{12}x_{37} - 1 \equiv (\alpha+1,0,\alpha+2)$ |
| $\rightarrow x_{31}(1,\alpha^3,0)$ | $\alpha^3$ | 0 | $x_{31}f_{41} = x_{13} - 1 \equiv (2\alpha+1,0,0)$ |
| $x_{31}(1,\alpha^4,0)$ | $\alpha^3$ | $\alpha^2$ | $x_{31}f_{42} = x_{23}x_{32} - 1 \equiv (0,2\alpha+1,0+1)$ |
| $x_{31}(1,\alpha^5,0)$ | $\alpha^4$ | $\alpha^4$ | $x_{31}f_{51} = x_{14}x_{34} - 1 \equiv (2,0,2)$ |
| $x_{31}(1,\alpha^6,0)$ | $\alpha^5$ | $\alpha^5$ | $x_{31}f_{52} = x_{24}x_{35} - 1 \equiv (0,2,2\alpha)$ |
| $x_{31}(1,\alpha^7,0)$ | $\alpha^6$ | $\alpha^2$ | $x_{31}f_{61} = x_{15}x_{36} - 1 \equiv (2\alpha,0,2\alpha+2)$ |
| $x_{31}(1,0,\alpha^6)$ | $\alpha^5$ | $\alpha^3$ | $x_{31}f_{62} = x_{25}x_{33} - 1 \equiv (0,2\alpha,2\alpha+1)$ |
| $x_{31}(1,0,\alpha^7)$ | $\alpha^6$ | 1 | $x_{31}f_{71} = x_{18}x_{32} - 1 \equiv (2\alpha+2,0,\alpha+1)$ |
| $x_{31}(1,0,\alpha^8)$ | $\alpha^7$ | $\alpha^7$ | $x_{31}f_{72} = x_{26}x_{38} - 1 \equiv (0,2\alpha+2,1)$ |
| $x_{31}(1,\alpha^8,0)$ | $\alpha^7$ | $\alpha^7$ | $x_{31}f_{81} = x_{17}x_{35} - 1 \equiv (\alpha+2,0,2\alpha)$ |
| $x_{31}(1,\alpha^9,0)$ | $\alpha^7$ | $\alpha^7$ | $x_{31}f_{82} = x_{27}x_{37} - 1 \equiv (0,\alpha+2,2\alpha+2)$ |

Thus $x_{31}e_1 + x_{31}(e_1 + e_2)$ is a syzygy and therefore $x_{13} + x_{31}$ belongs to the Gröbner basis of $I(C)$, denoted by $\mathcal{G}$. Consequently we can add all the multiples of $(\alpha, 0, 2\alpha + 1)$ to $\mathcal{G}$. We also have the syzygy $x_{28}e_1 + x_{31}(e_1 + e_2)$. Similar to the previous case $x_{28} + x_{31}$ and all its multiples belongs to $\mathcal{G}$.

That is to say,

$$\left\{ x_{11} - x_{37}, x_{12} - x_{38}, x_{13} - x_{31}, x_{14} - x_{32}, x_{15} - x_{33}, x_{16} - x_{34}, x_{17} - x_{35}, x_{18} - x_{36}, x_{21} - x_{32}, x_{22} - x_{33}, x_{23} - x_{34}, x_{24} - x_{35}, x_{25} - x_{36}, x_{26} - x_{37}, x_{27} - x_{38}, x_{28} - x_{31} \right\} \subseteq \mathcal{G}.$$

The remaining elements of $\mathcal{G}$ are binomials of the set $\mathcal{R}(T_i^+)$. 

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