Reduction formulas for symmetric products of spin matrices

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Abstract

We show that, for SU(2) generators of arbitrary dimension $D$, there exist identities that express the completely symmetric product of $D$ matrices in terms of completely symmetric products of fewer number of matrices. We also indicate why such identities are important in characterizing electromagnetic interactions of particles.

1 Introduction

The purpose of this article is to present several identities involving spin matrices, or equivalently SU(2) generators [1]. The identities are independent of any basis used for writing the explicit forms of the generators, and can be used to write products of a certain number of spin matrices by using products of smaller number of such matrices.

The SU(2) generators, or the spin matrices, satisfy the commutation relation [2]

$$\left[S_i, S_j\right] = i \sum_k \varepsilon_{ijk} S_k \hspace{1cm} (1.1)$$

where the summation [3] over $k$ runs from 1 to 3 since there are three generators, and $\varepsilon_{ijk}$ is the completely antisymmetric symbol with

$$\varepsilon_{123} = +1. \hspace{1cm} (1.2)$$

Eq. (1.1) shows that an antisymmetric product of two spin matrices yields terms with only one spin matrix. From this, it is easy to show that if any expression containing spin matrices is antisymmetric in two indices, the expression can be written by using smaller number of spin matrices. For example, consider the string $S_i S_j S_k - S_k S_j S_i$, which is obviously antisymmetric under the interchange of the indices $i$ and $k$. Also notice that

$$S_i S_j S_k - S_k S_j S_i = \left[S_i, S_j\right] S_k + S_j \left[S_i, S_k\right] + \left[S_j, S_k\right] S_i$$

$$= i \sum_l \left(\varepsilon_{ijl} S_k + \varepsilon_{ikl} S_j + \varepsilon_{jkl} S_l\right), \hspace{1cm} (1.3)$$
using Eq. (1.1) to write the last step. This example shows that the product of three spin matrices, antisymmetrized in two indices, can be expressed as terms which contain products of only two spin matrices. It is easy to generalize this result to products of arbitrary number of spin matrices which are antisymmetric with respect to the interchange of at least one pair of indices.

It therefore remains to be seen whether the same kind of reduction is possible for other kinds of products of spin matrices, which are not antisymmetric under the interchange of any pair of indices. It suffices to examine the possibility for completely symmetric products, because any product can be written in terms of symmetric and antisymmetric combinations.

We will show that reduction identities exist among completely symmetric combinations of spin matrices. However, unlike Eqs. (1.1) and (1.3) which are obeyed by all representations of the spin matrices, the identities that follow depend on the dimensionality of the representations. In order to present these identities, we will use curly brackets to denote symmetric combination of any number of spin matrices. Thus, for example, with two spin matrices, the symmetric product is denoted by

$$\{S_i S_j\} \equiv S_i S_j + S_j S_i ,$$

which is just the anticommutator. With three matrices, we define

$$\{S_i S_j S_k\} \equiv S_i S_j S_k + S_j S_k S_i + S_k S_i S_j + S_i S_k S_j + S_j S_i S_k$$

$$= \{S_i S_j\} S_k + \{S_j S_k\} S_i + \{S_k S_i\} S_j .$$

(1.5)

The generalization is obvious. For example, the symmetric combination with four spin matrices is given by

$$\{S_i S_j S_k S_l\} \equiv \{S_i S_j S_k\} S_l + \{S_j S_k S_l\} S_i + \{S_k S_l S_i\} S_j + \{S_l S_i S_j\} S_k .$$

(1.6)

In addition, we will also use the Kronecker delta, as well as its generalizations involving more than two indices, such as

$$\delta_{ijkl} \equiv \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} .$$

(1.7)

For the generalized Kronecker delta with $2n$ indices, each term contains a product of $n$ Kronecker deltas, and the number of such terms is

$$T_n \equiv 1 \cdot 3 \cdot 5 \cdots (2n - 1) = \frac{(2n)!}{2^n n!} .$$

(1.8)

2 Examples with various representations

We start with the examples of identities of symmetric combinations of spin matrices. As commented above, these identities depend on the dimension of the representation, so we discuss various low-dimensional representations one by one.
2.1 2-dimensional representation

The generators for the 2-dimensional representation are \( \frac{1}{2} \sigma_i \), where \( \sigma_i \) \((i = 1, 2, 3)\) are the Pauli matrices. It is well-known that the square of each Pauli matrix is the unit matrix, and different Pauli matrices anticommute. Both kinds of information can be encoded by writing

\[
2\{S_i S_j\} - \delta_{ij} I = 0. \tag{2.1}
\]

This identity reduces the symmetric combination of two spin matrices to a multiple of the unit matrix, which does not contain any spin matrix at all. Note that the symmetric combinations of larger number of spin matrices can all be built up from the symmetric combination of two spin matrices, as has been shown in Eqs. (1.5) and (1.6). Thus, Eq. (2.1) guarantees that symmetric combination of any number of 2-dimensional spin matrices can be reduced to smaller number of matrices. This general comment is true for all representations — once we obtain an identity involving a certain number of spin matrices, it guarantees that there are such identities with larger number of spin matrices. This comment will not be repeated for higher representations.

2.2 3-dimensional representation

For 3-dimensional representation, we cannot find any identity for the symmetric combination of two spin matrices, or the anticommutator. With three spin matrices, however, we find the identity

\[
\{S_i S_j S_k\} - 2\left(S_i \delta_{jk} + S_j \delta_{ki} + S_k \delta_{ij}\right) = 0. \tag{2.2}
\]

If \( i, j, k \) denote the same index in Eq. (2.2), the identity reduces to

\[
S^3_i = S_i, \tag{2.3}
\]

which is the obvious characteristic equation of the 3-dimensional spin matrices since the eigenvalues have to be 0 and \(\pm 1\). However, Eq. (2.2) is much more general than this characteristic equation. For example, if two of the three indices appearing in Eq. (2.2) are equal and the other is different, we obtain the relation

\[
S^2_i S_j + S_i S_j S_i + S_j S^2_i = 2S^2_j, \quad (i \neq j). \tag{2.4}
\]

And finally, if all three indices are different, Eq. (2.2) reduces to the equation

\[
S_1 S_2 S_3 + S_2 S_3 S_1 + S_3 S_1 S_2 + S_1 S_3 S_2 + S_3 S_2 S_1 + S_2 S_1 S_3 = 0. \tag{2.5}
\]

2.3 4-dimensional representation

In this case, the lowest order identity involves symmetric combination of four spin matrices, and the identity is

\[
2\{S_i S_j S_k S_l\} - 10\left(\{S_i S_j\} \delta_{kl} + (5 \text{ more similar terms})\right) + 9 \delta_{ijkl} I = 0. \tag{2.6}
\]
For $i = j = k = l$, this equation reduces to
\begin{equation}
16S_i^4 - 40S_i^2 + 9 \cdot 1 = 0,
\tag{2.7}
\end{equation}
which is easily seen to be the characteristic equation of the spin matrices whose eigenvalues are $\pm \frac{1}{2}$ and $\pm \frac{3}{2}$. In addition, Eq. (2.6) implies many other equations. For example, with $i \neq j$, we obtain the following identities:
\begin{align*}
8 \left( S_i^2 S_j^2 + S_j^2 S_i^2 + S_j S_i S_i S_j + S_j S_i S_j S_i + S_j S_i S_j S_i \right) \\
-20 \left( S_i^2 + S_j^2 \right) + 9 \cdot 1 &= 0, \tag{2.8}
\end{align*}
\begin{align*}
2 \left( S_i S_j^3 + S_j S_i S_j^2 + S_j S_i S_j S_i + S_i S_j S_j S_i \right) \\
-5 \left( S_i S_j + S_j S_i \right) &= 0. \tag{2.9}
\end{align*}
There will also be similar identities that involve three different indices $i, j, k$, of the form
\begin{align*}
2 \left( S_i S_j S_k + S_i S_k S_j + S_i S_j S_k + S_i S_k S_j S_i + S_i S_k S_j S_i \right) \\
+ S_j S_i S_k S_i + S_j S_i S_k S_i + S_j S_i S_k S_i + S_j S_i S_k S_i S_k S_j S_i \\
-5 \left( S_j S_k + S_k S_j \right) &= 0. \tag{2.10}
\end{align*}

### 2.4 5-dimensional representation

In this case, the simplest identity involves products of five spin matrices:
\begin{align*}
\{S_i S_j S_k S_l S_m \} - 10 \left( \{S_i S_j S_k \} \delta_{lm} + \text{(9 more similar terms)} \right) \\
+ 32 \left( S_i \delta_{jklm} + \text{(4 more similar terms)} \right) &= 0. \tag{2.11}
\end{align*}
As with the previous cases, this identity contains the characteristic equation:
\begin{equation}
S_i^5 - 5S_i^3 + 4S_i = 0, \tag{2.12}
\end{equation}
but it also entails many other identities for cases where all indices are not equal.

### 3 General remarks

It should be noticed that all identities given above are independent of the basis in which the spin matrices are written. If we change the basis, the matrix $S_i$ changes to
\begin{equation}
S_i' = U S_i U^\dagger, \tag{3.1}
\end{equation}
for some unitary matrix $U$. It is easily seen that such transformations have no effect on the identities. More generally, any transformation of the form
\begin{equation}
S_i' = M S_i M^{-1}, \tag{3.2}
\end{equation}
with any non-singular matrix $M$, does not change the commutation relation of Eq. \[1.1\] as well as the identities given in §[2]

In the examples given above, note that for $D$-dimensional representation of the spin matrices, the simplest identity that we obtain with the symmetric combinations contains a combination of order $D$. It is easy to see that it must be so. If we put all indices to be equal in such equations, we obtain an equation that the eigenvalues of the spin matrices must satisfy. Spin matrices in $D$-dimensional representation have $D$ distinct eigenvalues, so the equation must be of degree $D$.

It should be noted that the eigenvalues of a spin matrix of dimension $D$, with even $D$, contains all eigenvalues of a spin matrix whose dimension is also even, but smaller than $D$. The same can be said about odd $D$. As a result, Eq. \[2.6\] will also be satisfied by 2-dimensional spin matrices, and Eq. \[2.11\] by the 3-dimensional spin matrices. In fact, starting from definitions like Eq. \[1.6\] and using Eq. \[2.1\], one can easily show that Eq. \[2.6\] is satisfied. Similarly, one can build one’s way from Eq. \[2.2\] to Eq. \[2.11\].

The numerical co-efficients appearing in various identities are simple combinatorial factors. These can be worked out from the characteristic equations for the spin matrices. Let us write the characteristic equation in a form where the co-efficient of the term with the highest power is unity:

$$S_i^D + \sum_{p=1}^{[D/2]} a_p S_i^{D-2p} = 0,$$

where $[D/2]$ denotes the largest integer equal to or less than $D/2$. The generalized identity involving symmetric combinations can then be written as

$$\{S_{i_1}S_{i_2}\cdots S_{i_D}\} + \sum_{p=1}^{[D/2]} b_p \left(\{S_{i_1}\cdots S_{i_{D-2p}}\} \delta_{i_{D-2p+1}\cdots i_D} + \text{similar terms}\right) = 0,$$

where

$$b_p = 2^p p! a_p.$$

The number of “similar terms” in Eq. \[3.4\] is easily seen to be $\binom{D}{2p} - 1$, since there are $\binom{D}{2p}$ ways of choosing the indices in the symmetric product of $D-2p$ spin matrices, out of which one has already been shown explicitly.

In fact, one can do better than Eq. \[3.5\]. One can find the $b_p$’s in closed form by putting in the analytical expressions for the $a_p$’s. The latter can be found from the characteristic equations. If the largest eigenvalue $L$ equals $n$ where $n$ is a non-negative integer, the characteristic equation will be of the form

$$S_i(S_i^2 - 1^2)(S_i^2 - 2^2)\cdots(S_i^2 - n^2) = 0.$$

On the other hand, for half-integral spins, denoting the largest eigenvalue by $L = n + \frac{1}{2}$, the characteristic equation can be written as

$$\left(S_i^2 - \frac{1^2}{2^2}\right)\left(S_i^2 - \frac{3^2}{2^2}\right)\cdots\left(S_i^2 - \frac{(2n+1)^2}{2^2}\right) = 0.$$
Thus, for example, the co-efficient $a_1$ appearing in Eq. (3.3) is easily seen to be

$$a_1 = \begin{cases} -\Sigma_2(n) & \text{for } L = n, \\ -(\Sigma_2(n) + \Sigma_1(n) + \frac{1}{4}\Sigma_0(n)) & \text{for } L = n + \frac{1}{2}, \end{cases}$$  \hspace{1cm} (3.8)$$

where we have used the symbol

$$\Sigma_r(n) \equiv \sum_{q=0}^{n} q^r. \hspace{1cm} (3.9)$$

The co-efficient of the term with the smallest power of $S_i$ can also be written down easily. For odd $D$, i.e., for $L = n$, the last term is $a_n S_i$, where

$$a_n = (-1)^n (n!)^2. \hspace{1cm} (3.10)$$

For even $D$, i.e., for $L = n + \frac{1}{2}$, the last term is

$$a_{n+1} = \left( \frac{(2n + 1)!!}{2^{n+1}} \right)^2 = \left( \frac{(2n + 1)!}{2^{2n+1} n!} \right)^2. \hspace{1cm} (3.11)$$

However, as we move towards the middle of the series of terms that appear in Eq. (3.3), general expressions for the co-efficients look more and more cumbersome, although they can all be written down in closed form. For example, for $L = n$, the co-efficient $a_2$ is given by

$$a_2 = \sum_{q=0}^{n-1} \sum_{q'=q+1}^{n} q^2 q'^2 = \left( \Sigma_2(n) \right)^2 - \sum_{q=0}^{n} q^2 \Sigma_2(q). \hspace{1cm} (3.12)$$

The summation will involve the sum of three or higher powers of natural numbers. All such sums can be carried out, as shown in the Appendix.

4 Utility

Are these identities useful? In other words, can they be used to tackle some physics problem? We can think of one such utility.

If we put a spin-$\frac{1}{2}$ fermion like the electron in a magnetic field $\vec{B}$, the electron experiences an interaction proportional to the field strength. The fact is summarized by saying that the electron has a magnetic dipole moment. The dipole moment is proportional to spin, the only inherent property of a particle that transforms like a vector, so that the interaction can be written as some constant times $\vec{S} \cdot \vec{B}$. One might ask, why doesn’t the electron have any quadrupole moment? If it did, there would have been interactions of the form

$$H_{\text{int}} = (\text{constant}) \times \sum_{i,j} \{S_i S_j\} \frac{\partial F_i}{\partial x_j}, \hspace{1cm} (4.1)$$

where $F_i$ could be either the electric field or the magnetic field. It is clear why the symmetric product of the two spin vectors is used: the antisymmetric combination
could have been reduced to only one spin matrix through Eq. (1.1), and the resulting interaction would be none other than the dipole interaction. However, the symmetric product can be reduced by Eq. (2.1), so this interaction is not a quadrupole interaction at all. By the same argument, a spin-1 particle can have a quadrupole moment in addition to a dipole moment, and therefore can have an interaction term with the electromagnetic field of the form shown in Eq. (4.1). Such interactions were identified in a quantum field theoretical investigation of the electromagnetic vertex of spin-1 particles [4]. However, a spin-1 particle cannot have any higher moment, because the symmetric product with three spin matrices can be reduced to products of smaller number of matrices, as shown in Eq. (2.2). Spin-\(\frac{3}{2}\) particles can also have octupole moments, and so on.

It is true that the utility mentioned above depends not on the explicit form of any identity but on the fact that such identities exist. It would be of interest to see whether the explicit forms of various identities shown in §2 can be important in some other physical context.

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**Appendix: Finding the sums \(\Sigma_r(n)\)**

For \(r = 0\), we trivially obtain

\[
\Sigma_0(n) = n + 1. \tag{A.1}
\]

In order to find \(\Sigma_r(n)\) for an integer \(r \geq 1\), we start from the identity

\[
\sum_{q=0}^{n} q^{r+1} = \sum_{q=0}^{n} (q+1)^{r+1} - (n+1)^{r+1}. \tag{A.2}
\]

Writing the binomial series for the power of \(q+1\), we obtain [5]

\[
\Sigma_r(n) = \frac{1}{r+1} \left[ (n+1)^{r+1} - \sum_{p=0}^{r-1} \binom{r+1}{p} \Sigma_p(n) \right]. \tag{A.3}
\]

Starting from Eq. (A.1), one can now climb the ladder to the sums of higher powers [6]:

\[
\begin{align*}
\Sigma_1(n) &= \frac{1}{2} n(n+1), \tag{A.4} \\
\Sigma_2(n) &= \frac{1}{6} n(n+1)(2n+1), \tag{A.5} \\
\Sigma_3(n) &= \frac{1}{4} n^2(n+1)^2, \tag{A.6} \\
\Sigma_4(n) &= \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1), \tag{A.7}
\end{align*}
\]

and so on.
References

[1] For an introduction to the group SU(2) and its representations, see textbooks on group theory. Textbooks on quantum mechanics also discuss SU(2) representations.

[2] Strictly speaking, the spin matrices are $\hbar$ times the SU(2) generators, and therefore a factor of $\hbar$ should appear on the right side of Eq. (1.1) if we are talking of the spin matrices. We use the phrases “SU(2) generators” and “spin matrices” interchangeably, thus assuming a system of units in which $\hbar = 1$.

[3] Note that nowhere in this article we assume automatic summation over repeated indices. All summations are indicated explicitly.

[4] J. F. Nieves and P. B. Pal, Phys. Rev. D 55, 3118 (1997) [hep-ph/9611431].

[5] This is a trivial generalization of one of the methods used for evaluating $\Sigma_2(n)$ given in R. L. Graham, D. E. Knuth and O. Patashnik, Concrete Mathematics: A Foundation for Computer Science, (Addison-Wesley; 2nd edition, 1994).

[6] Sums up to the 10th power appear on the internet site http://mathworld.wolfram.com/PowerSum.html