COMMUTATIVE \( C^* \)-SUBALGEBRAS OF SIMPLE STABLY Finte \( C^* \)-ALGEBRAS WITH REAL RANK ZERO

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Abstract. Let \( X \) be a second countable, path connected, compact metric space and let \( A \) be a unital separable simple nuclear \( \mathbb{Z} \)-stable real rank zero \( C^* \)-algebra. We classify all the unital \(*\)-embeddings (up to approximate unitary equivalence) of \( C(X) \) into \( A \). Specifically, we provide an existence and a uniqueness theorem for unital \(*\)-embeddings from \( C(X) \) into \( A \).

0. Introduction

In this paper, we study the classification problem for embeddings of a commutative \( C^* \)-algebra \( C(X) \) into a simple, real rank zero, stably finite \( C^* \)-algebra \( A \) (with other properties). This problem is closely related to, though independent of, the classification problem for the simple \( C^* \)-algebra \( A \) itself. Moreover, techniques from either subject often carry over to the other and have interesting implications (for other areas as well!).

A complete solution to the classification problem would consist of two parts: an existence theorem and a uniqueness theorem. One starting point for the existence theorem is a result of Pimsner [39] which shows that there is an interesting \(*\)-monomorphism \( \phi : C(T^2) \to A \) of \( C(T^2) \) into a unique simple AF-algebra \( A \) with \( K_0(A) \cong \mathbb{Z}[1/2] \oplus \mathbb{Z} \) with lexicographic order and order unit \((1,0)\) (here \( T^2 \) is the 2-torus). This \(*\)-monomorphism actually induces an isomorphism on the rational \( K_0 \) groups. The example of Pimsner exhibits the higher dimensional features of the AF-algebra which is also, at the same time, a noncommutative zero dimensional space (of course, every commutative \( C^* \)-algebra can be embedded in a commutative \( C^* \)-algebra with spectrum being a Cantor set - but not so with interesting induced map in K-theory). This phenomenon was subsequently intensively studied by Dadarlat, Elliott and Loring, who showed that many group homomorphisms from \( K_0(C(X)) \) to \( K_0(A) \) can be realized by \( C^* \)-homomorphisms from \( C(X) \) to \( A \). More specifically, in [13], Elliott and Loring showed that for a unital simple AF-algebra \( A \), a group homomorphism \( \mu : K_0(C(T^2)) \to K_0(A) \) can be realized by \( C^* \)-homomorphisms from \( C(T^2) \) to \( A \) if and only if \( \mu \) preserves the order unit and sends every element of the reduced \( K_0 \) group of \( C(T^2) \) to an element of \( \ker(\tau) \) for every unital trace \( \tau \in T(A) \). In [9], Dadarlat and Loring showed that given a finite CW complex \( X \), there are a unital AF-algebra \( A \) and a unital \(*\)-monomorphism...
\( \phi : C(X) \to A \) such that \( \phi \) induces an isomorphism on the rational \( K_0 \) groups. In particular, if \( X \) is connected then \( A \) can be taken to be the unique unital simple AF-algebra with \( K_0 \) group \( \mathbb{Z}[1/2] \oplus \mathbb{Z}^n \), where \( n \) is the rank of the reduced \( K_0 \) group of \( C(X) \). Here, the positive cone is \( \{ (r,m) : r \in \mathbb{Z}[1/2], \, r > 0 \text{ and } m \in \mathbb{Z}^n \} \) and the order unit is \((1,0)\). In their paper [25], Dadarlat and Loring also proved other existence results. For instance, let \( X \) be a finite CW-complex and let \( A \) be the tensor product of a unital simple AF-algebra with the UHF-algebra with dimension group being the rational numbers. If \( \mu : K_0(C(X)) \to K_0(A) \) is a group homomorphism such that (i) \( \mu([1]) = [1] \) and (ii) \( \mu \) brings the reduced \( K_0 \) group of \( C(X) \) to \( \bigcap_{\tau \in T(A)} \ker(\tau_*), \) then there is a unital \(*\)-homomorphism \( \phi : C(X) \to A \) such that \( K_0(\phi) = \mu \) (this generalizes the result of Elliott and Loring).

Li and Lin have generalized the results of Dadarlat, Elliott, Loring and Pimsner to a large class of codomain algebras \( A \). In [22], Li showed that given a finite CW complex \( X \), given a unital simple AH-algebra \( A \) with real rank zero and bounded dimension growth and given \( \alpha \in \text{KK}(C(X),A) \), \( \alpha \) can be realized by a unital \(*\)-homomorphism \( \phi : C(X) \to A \) if and only if \( \alpha \in \text{KK}(C(X),A)_{+,D(A)}(\text{KK}(C(X),A)_{+,D(A)}) \) is a subset of the KK-group \( \text{KK}(C(X),A) \); see [22] for the precise definition). In [25], Lin showed that the codomain algebra \( A \) (in Li’s result) can be replaced by an arbitrary unital simple separable nuclear \( C^* \)-algebra with real rank zero, stable rank one and weak unperforation. We note that the results of Dadarlat, Elliott, Li, Lin, Loring and Pimsner were proven after (and in the case of AF-algebras, long after!) the classification result for the corresponding simple codomain algebras. We also note that their results are incomplete in the sense that they do not fully take into account the tracial simplex (an important classification invariant). The best result in this respect was that of Li (in [22]) where for \( A \) having unique trace (in addition to other already mentioned conditions), the \(*\)-homomorphism can be constructed to realize any strictly positive probability measure on \( X \). In this paper, we complete the existence theorem by fully taking into account the tracial simplex. Our result is as follows:

0.1 Theorem: Let \( X \) be a second countable, path connected, compact metric space and let \( A \) be a unital separable simple nuclear \( C^* \)-algebra with real rank zero, stable rank one and weak unperforation. Let \( \alpha \in \text{KL}(C(X),A)_{+,1} \) and let \( \lambda : T(A) \to T(C(X)) \) be an affine continuous map such that

(a) if \( h \in \text{Aff}(T(C(X))) \) with \( h \geq 0 \) and \( h \) is not the zero function then \( \text{Aff}(\lambda)(h)(\tau) > 0 \) for all \( \tau \in T(A); \) and

(b) for every projection \( p \in C(X) \otimes K \), \( \lambda(\tau)(p) = \tau_*(\alpha([p])) \) for all \( \tau \in T(A) \).

Then there is a unital \(*\)-monomorphism \( \phi : C(X) \to A \) such that \( \text{KL}(\phi) = \alpha \) and \( T(\phi) = \lambda \).

Another part of the classification problem (for embeddings of \( C(X) \) into simple real rank zero stably finite etc. \( A \)) is the uniqueness theorem. One early result, without assuming that the codomain algebra is stably finite, is the result in [4] where Brown, Douglas and Fillmore showed that for two unital \(*\)-monomorphisms \( \phi, \psi : C(X) \to \mathbb{B}(H)/\mathcal{K}(H) \), \( \phi \) and \( \psi \) are unitarily equivalent if and only if \( \text{KK}(\phi) = \text{KK}(\psi) \). Another early result is in [15] where Gong and Lin showed that for \( X \) a compact metric space and \( A \) a unital simple separable nuclear \( C^* \)-algebra with real rank zero, stable rank one, weak unperforation and unique tracial state, two
unital \ast\text{-}monomorphisms \phi, \psi : C(X) \to A are approximately unitarily equivalent if and only if \KL(\phi) = \KL(\psi) and \tau \circ \phi = \tau \circ \psi for all \tau \in T(A). It is not hard to replace the unique tracial state condition (in Gong and Lin’s result) by the (slightly) weaker condition of countably many extreme tracial states. However, moving beyond these conditions of having small tracial simplex is difficult. This is closely related to certain recent classification problems, for simple \(C^\ast\)-algebras, where a major difficulty lies in replacing the small tracial simplex condition by arbitrary tracial simplexes (see, for example [5], [27], [35], [51] and [52]).

Recently in [29], Lin replaced the small tracial simplex condition in the codomain algebra \(A\) by the additional assumption that \(A\) had tracial rank zero (i.e., the TAF property; this is a strong property which, together with the assumptions of nuclearity and universal coefficient theorem, imply that the algebra involved is a simple unital AH-algebra with bounded dimension growth and real rank zero). We note that Lin used this result (in [29]) to generalize an interesting theorem of Kishimoto. Specifically, Lin showed that if \(A\) is a simple unital AH-algebra with real rank zero and bounded dimension growth, and if \(\alpha\) is an approximately inner \ast\text{-}automorphism of \(A\) with the tracial Rokhlin property, then the crossed product \(A \times \alpha Z\) is a unital simple AH-algebra with bounded dimension growth and real rank zero.

In this paper, we show that the restriction on the tracial simplex of \(A\) can be removed without assuming that \(A\) is TAF. However, we need to assume that \(A\) is \(Z\)-stable where \(Z\) is the Jiang-Su algebra. This condition is implied by approximate divisibility and is currently of interest in the classification program for simple \(C^\ast\)-algebras (more below). Our uniqueness result is as follows:

0.2 Theorem: Let \(X\) be a second countable, path connected, compact metric space and let \(A\) be a unital separable simple nuclear real rank zero \(Z\)-stable \(C^\ast\)-algebra. Let \(\phi, \psi : C(X) \to A\) be unital \ast\text{-}monomorphisms. Then \(\phi\) and \(\psi\) are approximately unitarily equivalent if and only if \(\KL(\phi) = \KL(\psi)\) and \(\tau \circ \phi = \tau \circ \psi\) for all \(\tau \in T(A)\).
of almost commuting self-adjoint matrices are uniformly close to pairs of exactly commuting self-adjoint matrices (see, for example, [23]). We also note that an interesting problem would be to replace the commutative algebra \(C(X)\) (in our results) by an arbitrary nuclear residually finite dimensional \(C^*\)-algebra. For instance, results of this type would lead to a proof that every simple unital separable nuclear quasidiagonal \(C^*\)-algebra is AF-embeddable (this is not even known for inductive limits of residually finite dimensional type I \(C^*\)-algebras). This in itself is interesting, but it may also in turn lead to a proof of appropriate tracial approximation properties and perhaps classification (see, for example, [5] and [27]).

Also, Theorem 0.2 is not known (except for special cases) when the codomain algebra is not real rank zero. Indeed, we do not even have a full uniqueness theorem for the case where the codomain algebra is a real rank one simple unital AH-algebra with bounded dimension growth. Hence, for this class of algebras, even though there is a complete classification using K-theory invariants, the general theory is still incomplete. One consequence of this defect is that the structure of the automorphism groups of these algebras is still not known. (See [12], [30], [36], [37], [38], [39], [41]).

Finally, the results of this paper are connected to but distinct from the existence and uniqueness theorems found in [6], [7] and [26]. The theorems in these papers require that the domain algebra be a simple TAF \(C^*\)-algebra. The first immediate result is that the tracial simplex need not be taken into account (whereas for commutative domain algebras, the tracial simplex is needed and results in much extra work). Secondly, that the domain algebra is simple TAF leads to simplication and special techniques for uniqueness, which do not work for commutative domain algebras - this is why, for example, Lin had to prove a separate uniqueness theorem for commutative domain algebras in [29]. Finally, since all the above existence theorems require uniqueness, similar remarks hold for the existence theorems.

In this paper, we will use ideas from K-theory and KK-theory. A good reference for this is [24]. We will also use the notions of finite decomposition rank and \(\mathbb{Z}\)-stability which have recently played important roles in classification theory. Good references for this are [20] and [45].

Throughout this paper, “c.p.c.” will denote “completely positive contractive”. For \(C^*\)-algebras \(A, B\), \(K(A)\) is the total K-theory of \(A\), \(\mathbb{P}(A)\) is the class of projections defined in [24] 6.1.1 (the equivalence classes of the projections in \(\mathbb{P}(A)\) generate \(K(A)\) as a group), \(KK(A, B)\) is the KK-group of \(A\) and \(B\), and (when either \(A\) or \(B\) satisfies the UCT) \(KL(A, B) = \text{Hom}_\Lambda(K(A), K(B))\) is the KL group of \(A\) and \(B\) (here, \(\Lambda\) is the collection of Bockstein operations). We also note that if \(\phi : A \to B\) is a contractive completely positive map which is “sufficiently almost multiplicative” then \(\phi\) gives a well-defined map from a finite subset of \(\mathbb{P}(A)\) into \(K(B)\); a discussion of this can be found in [28] page 8, and we will be using the notation contained there. For more details on all of the above, see [24] and [28].

1. The existence theorem

1.1 In this section, we prove an existence theorem for \(*\)-monomorphisms from \(C(X)\) into a simple real rank zero \(C^*\)-algebra \(A\) (with appropriate additional properties).
We first need a lemma concerning c.p.c. almost multiplicative maps from $C(X)$ into a matrix algebra (cf. [24] Lemma 6.2.7):

**Lemma:** Let $X$ be a compact metric space. Let $\epsilon > 0$ and a finite subset $\mathcal{F} \subseteq C(X)$ be given. Then there exists $\delta > 0$ and a finite subset $\mathcal{G} \subseteq C(X)$ satisfying the following:

For every positive integer $n \geq 1$, if $L : C(X) \to \mathbb{M}_n(C)$ is a $\mathcal{G} - \delta$ multiplicative c.p.c. map then there exists a $\ast$-homomorphism $h : C(X) \to \mathbb{M}_n$ with $h(1_{C(X)}) = r$ such that

$$\text{tr}(1 - r) < \epsilon$$

and

$$\|L(f) - (1 - p)L(f)(1 - p) + h(f)\| < \epsilon,$$

for all $f \in \mathcal{F}$.

(In the above, tr is the unique normalized trace on $\mathbb{M}_n$.)

1.2 Next, we will also need the following theorem concerning the approximation of unital positive linear maps between tracial state spaces (of commutative $C^*$-algebras) by convex combinations of maps coming from $\ast$-homomorphisms, which is due to Li [21]:

**Theorem:** Let $X$ be a path connected compact metric space. Let a finite subset $\mathcal{F} \subseteq \text{Aff}(T(C(X)))$ and $\epsilon > 0$ be given. Then there is an $N > 0$ with the following property:

For any unital positive linear map $\xi : \text{Aff}(T(C(X)))) \to \text{Aff}(T(C(Y))))$, where $Y$ is an arbitrary compact metric space, there are $N$ $\ast$-homomorphisms

$$\phi_1, \phi_2, ..., \phi_N : C(X) \to C(Y)$$

such that each $\phi_i$ is homotopy equivalent to a point evaluation (i.e., $\phi_i$ is homotopy equivalent to a $\ast$-homomorphism of the form $C(X) \to C(Y) : f \mapsto f(x_0)1_{C(Y)}$ for some point $x_0 \in X$) and such that

$$|\xi(f)(\tau) - (1/N)\sum_{i=1}^N \tau(\phi_i(f))| < \epsilon, \forall f \in \mathcal{F}, \forall \tau \in T(C(Y)).$$

1.3 To continue, we fix some notation: Let $X$ be a compact second countable metric space. Let $A$ be a unital $C^*$-algebra. We denote by $\text{KL}(C(X), A)_{+1}$ the set of all $\alpha \in \text{KL}(C(X), A)$ which satisfy the following two conditions:

i. $\alpha(1_{C(X)}) = [1_A]$ (for the induced map between $K_0$ groups).

ii. $\alpha(K_0(C(X))_{+} - \{0\}) \subseteq K_0(A) + \{0\}$.

1.4 Next, we need the following result which follows from the TAF-property and [24] Theorem 6.1.11. (The proof can be found in [24] Lemma 6.2.8. Note that the proof of this Lemma does not really require that the spectrum of the domain algebra be a finite CW complex. See also [6] Theorem 5.6 and Corollary 5.7.)

**Theorem:** Let $A$ be a unital simple non-type I AH-algebra with bounded dimension growth and real rank zero. Suppose that $X$ is a compact, second countable, path connected metric space, $\mathcal{P} \subseteq \mathbb{P}(C(X))$ is a finite subset, and

$$\alpha \in \text{KL}(C(X), A)_{+1}.$$
Then there exists a sequence of unital c.p.c. linear maps
\[ \phi_n : C(X) \to A \]
such that
\[ [\phi_n]|P = \alpha|P \]
and
\[ \|\phi_n(fg) - \phi_n(f)\phi_n(g)\| \to 0 \]
as \( n \to \infty \), for all \( f, g \in C(X) \).

(For the definition of \([\phi_n]|P\) we refer the reader to [28] page 8. As mentioned in the last paragraph of the introduction, we will be using the notation contained there.)

**Proof:** [Sketch of proof]

As already indicated, the proof can be found in [24] Lemma 6.2.8. Here, we sketch the proof for the case where \( C(X) \) has torsion-free \( K_0 \) and \( K_1 \) groups.

By [24] Theorem 6.1.11, let \( \phi_1 : C(X) \to M_n(A) \) be an almost multiplicative map and let \( \phi_2 : C(X) \to M_{n-1}(A) \) be a unital \( * \)-homomorphism with finite dimensional range such that
\[ [\phi_1]|P = [\phi_2]|P \]
(Of course, we need \( \phi_1 \) to be almost multiplicative on a sufficiently large finite subset of \( C(X) \) and for a sufficiently small positive real number.)

Since \( M_n(A) \) is tracially AF, we can find a projection \( p \in M_n(A) \) such that \( p \) is Murray-von Neumann equivalent to a subprojection of \( 1_A \), and we can find a unital c.p.c. almost multiplicative map \( \phi_3 : C(X) \to pM_n(A)p \) and a unital \( * \)-homomorphism \( \phi_4 : C(X) \to (1-p)M_n(A)(1-p) \) such that \( \phi_1 \) can be approximated by \( \phi_3 \oplus \phi_4 \) in norm on a sufficiently large finite subset of \( C(X) \), and such that
\[ [\phi_1]|P = [\phi_3]|P + [\phi_4]|P \]
(Of course, we are assuming that all the maps involved are sufficiently multiplicative so that all the above expressions are well-defined etc.)

By conjugating with a unitary if necessary, we may assume that \( p \leq 1_A \). Let \( x_0 \in X \) be an arbitrary point. Then take \( \phi : C(X) \to M \) to be the unital c.p.c. map given by \( \phi(f) = \phi_3(f) + f(x_0)(1_A - p) \), for all \( f \in C(X) \). Assuming that all the maps involved were chosen to be sufficiently multiplicative, we can check that
\[ [\phi]|P = \alpha|P \]
(This last equality is nontrivial!)

Finally, towards the existence theorem, we need the following approximate uniqueness result (for tracial rank zero codomains) of Lin (which can be found in [29] Theorem 4.6; see also [17]):

1.5 **Theorem:** Let \( X \) be a compact metric space, \( \epsilon > 0 \) and \( F \subseteq C(X) \) be a finite subset. Let \( \nu > 0 \) be such that \( |f(x) - f(y)| < \epsilon/8 \) if \( d(x,y) < \nu \) for all \( f \in F \) and all \( x, y \in X \) (\( d \) is the metric on \( X \)). Then, for any integer \( s \geq 1 \), any finite \( \nu/2 \)-dense subset \( \{x_1, x_2, ..., x_m\} \) of \( X \) for which \( O_i \cap O_j = \emptyset \) for \( i \neq j \), where \( O_i = \{x \in X : \text{dist}(x,x_i) < \nu/(2s)\} \) and any \( 1/(2s) > \sigma > 0 \), there exists \( \gamma > 0 \), a finite subset \( G \subseteq C(X) \), \( \delta > 0 \) and a finite subset \( \mathcal{P} \subseteq \mathbb{P}(C(X)) \) satisfying the following:
For any unital separable simple nuclear C*-algebra A with tracial rank zero, any \( G = \mathcal{G} - \delta \)-multiplicative unital c.p.c. linear maps \( \phi, \psi : C(X) \to A \) with \( \tau \circ \phi(g) \) within \( \gamma \) of \( \tau \circ \psi(g) \) for all \( g \in G \), if

\[
(\text{i}) \quad \mu_{\tau \circ \phi}(O_i) > \sigma \nu \text{ for all } i \text{ and for all } \tau \in T(A), \quad \text{and}
\]

\[
(\text{ii}) \quad |\phi| F = |\psi| F
\]

then there exists a unitary \( u \in A \) such that \( u \phi(f) u^* \) is within \( \epsilon \) of \( \psi(f) \) for all \( f \in F \).

Finally, if in the above, the elements of \( F \) all have norm less than or equal to one, then we can choose \( \mathcal{G} \) so that its elements all have norm less than or equal to one.

1.6 Theorem: Let \( X \) be a compact, second countable, path connected, metric space and let \( A \) be a unital separable simple AH-algebra with bounded dimension growth and real rank zero. Suppose that \( \alpha \in \text{KL}(C(X), A)_{+1} \) and suppose that \( \lambda : T(A) \to T(C(X)) \) is an affine continuous map such that

\[
(\text{i}) \quad \text{the induced map } \text{Aff}(\lambda) : \text{Aff}(T(C(X))) \to \text{Aff}(T(A)) \text{ brings nonnegative nonzero functions to functions which are strictly positive at every point; i.e., if}
\]

\[
h \in C(X, \mathbb{R}) \cong \text{Aff}(T(C(X)))
\]

is a nonnegative function which is not the zero function then

\[
\text{Aff}(\lambda)(h)(\tau) > 0
\]

for all \( \tau \in T(A) \); and

\[
(\text{ii}) \quad \text{for every } \tau \in T(A) \text{ and for every projection } p \in C(X) \otimes K,
\]

\[
\lambda(\tau)(p) = \tau_\alpha([p])
\]

Then there exists a *-momomorphism \( \phi : C(X) \to A \) such that

\[
\text{KL}(\phi) = \alpha \text{ and } T(\phi) = \lambda.
\]

Proof: Let \( \{F_l\}_{l=1}^\infty \) be an increasing sequence of finite subsets of the closed unit ball of \( C(X) \) such that the union is dense in the closed unit ball; i.e., \( C(X)_1 = \bigcup_{l=1}^\infty F_l \). Let \( \{\epsilon_l\}_{l=1}^\infty \) and \( \{\nu_l\}_{l=1}^\infty \) be strictly decreasing sequences of strictly positive real numbers such that \( \sum_{l=1}^\infty \epsilon_l < \infty \) and \( \sum_{l=1}^\infty \nu_l < \infty \), and such that \( |f(x) - f(y)| < \epsilon_l / 8 \) for all \( x, y \in X \) with \( \text{dist}(x, y) < \nu_l \) and for all \( f \in F_l \). For simplicity, we may assume that \( \nu_1 < 1 \). For each \( l \),

\[
(\text{1}) \text{ take } s = l;
\]

\[
(\text{2}) \text{ take } \{x_{1,1}, x_{1,2}, \ldots, x_{1,m_1}\} \text{ to be a } \nu_1/2 \text{-dense subset of } X \text{ for which } O_{l,j} \cap O_{l,k} = \emptyset \text{ for } j \neq k, \text{ where } O_{l,j} = \{x \in X : \text{dist}(x, x_{l,j}) < \nu_l/(2l)\}; \text{ also, put}
\]

\[
\bar{O}_{l,j} = \text{df} \{x \in X : \text{dist}(x, x_{l,j}) < \nu_l/(4l)\} \subseteq O_{l,j};
\]

\[
(\text{3}) \text{ take } \{\sigma_l\}_{l=1}^\infty \text{ to be a strictly decreasing sequence of strictly positive real numbers such that } 1/(2l) > \sigma_l > 0 \text{ and for all } j \text{ for all } l, \text{ inf}\{\mu_{\lambda(\tau)}(\bar{O}_{l,j}) : \tau \in T(A)\} > 10^n\sigma_l.
\]

For each \( l \), put the above data into Theorem 1.6 and get the following:
(a) a strictly decreasing sequence \( \{ \gamma_l \}_{l=1}^{\infty} \) of strictly positive real numbers with \( \sum_{l=1}^{\infty} \gamma_l < \infty \); 
(b) an increasing sequence \( \{ \mathcal{G}_l \}_{l=1}^{\infty} \) of finite subsets of the closed unit ball of \( C(X) \), which is dense in the closed unit ball (i.e., \( \bigcup_{l=1}^{\infty} \mathcal{G}_l = C(X) \)); we may assume that \( F_l \subseteq \mathcal{G}_l \) for all \( l \); and 
(c) a strictly decreasing sequence \( \{ \delta_l \}_{l=1}^{\infty} \) of strictly positive real numbers such that \( \sum_{l=1}^{\infty} \delta_l < \infty \).

(d) an increasing sequence \( \{ \mathcal{P}_l \}_{l=1}^{\infty} \) of finite subsets of \( \mathcal{P}(C(X)) \).

(e) for each \( l, j \), let \( f_{l,j} \) be a real-valued function such that i. \( 0 \leq f_{l,j} \leq 1 \), ii. \( f_{l,j} = 1 \) on \( O_{l,j} \) and iii. the support of \( f_{l,j} \) is contained in \( O_{l,j} \); then, expanding \( \mathcal{G}_l \) if necessary, we may assume that \( \{ f_{l,j} : 1 \leq j \leq n_l \} \subseteq \mathcal{G}_l \); 
(f) further expanding the \( \mathcal{G}_l \)s and contracting the \( \gamma_l \)s and \( \delta_l \)s if necessary, we may assume that for each \( l \) and for every unital \( C^* \)-algebra \( \mathcal{C} \), if \( \kappa_1, \kappa_2 : C(X) \to \mathcal{C} \) are \( \mathcal{G}_l - \delta_l \)-multiplicative c.p.c. maps, then they are defined on \( \mathcal{P}_l \); and if, in addition, \( \kappa_1(f) \) is within \( \gamma_l \) of \( \kappa_2(f) \) for all \( f \in \mathcal{G}_l \) then 

\[
[\kappa_1] \lvert_{\mathcal{P}_l} = [\kappa_2] \lvert_{\mathcal{P}_l}.
\]

Hence, for each \( l \), for the given data \( \mathcal{F}_l, \epsilon_l, \nu_l, s = l \), \( \{ x_{l,1}, x_{l,2}, \ldots, x_{l,n_l} \} \), \( \sigma_l \), the quantities \( \gamma_l, \mathcal{G}_l, \mathcal{P}_l \) satisfy the conclusion of Theorem 1.5.

Since \( A \) is a unital simple AH-algebra with bounded dimension growth and real rank zero, \( A \) is (the norm closure of) an increasing union \( A = \bigcup_{m=1}^{\infty} A_m \), where each \( A_m \) is a direct sum of unital homogeneous \( C^* \)-algebras with spectra being finite CW complexes with topological dimension less than or equal to three, and where (we may assume that) the inclusions are unital and injective.

Hence, we have that \( T(A) \) is the inverse limit of tracial state spaces: 

\[
T(A_1) \leftarrow T(A_2) \leftarrow T(A_3) \leftarrow \ldots \leftarrow T(A).
\]

Hence, \( \text{Aff}(T(A)) \) is the direct limit of complete order unit spaces \( \text{Aff}(T(A_m)) \):

\[
\text{Aff}(T(A_1)) \to \text{Aff}(T(A_2)) \to \text{Aff}(T(A_3)) \to \ldots \to \text{Aff}(T(A)).
\]

Note that if the spectrum of \( A_m \) is \( Y \) then \( \text{Aff}(T(A_m)) \) is isomorphic, as an order unit space, to \( C(Y, \mathbb{R}) \) (the real-valued continuous functions on \( Y \)). Note also that \( \lambda \) induces a morphism (of order unit spaces) \( \text{Aff}(\lambda) : \text{Aff}(T(C(X))) \to \text{Aff}(T(A)) \); and by the argument of [13] Theorem 25.1.1, for every \( \varepsilon > 0 \) and for every finite subset \( F^* \) of \( \text{Aff}(T(C(X))) \), there exists an integer \( N^* \geq 1 \) such that for every \( n \geq N^* \), there is a morphism (of order unit spaces) \( \text{Aff}(\lambda^*) : \text{Aff}(T(C(X))) \to \text{Aff}(T(A_n)) \) where \( \text{Aff}(\lambda^*)(f) \) is within \( \varepsilon \) of \( \text{Aff}(\lambda^*)(f) \) for every \( f \in F^* \) (viewing \( \text{Aff}(\lambda^*) \) as a map with codomain \( \text{Aff}(T(A)) \)) by taking the natural composition etc.

For each \( l \), let \( \mathcal{G}'_l \) consist of positive elements of norm less than or equal to one such that \( \mathcal{G}'_l = \{ a_1, a_2, a_3, a_4 \geq 0 : f = a_1 - a_2 + i(a_3 - a_4), f \in \mathcal{G}_l \} \). Hence, \( \{ \mathcal{G}'_l \}_{l=1}^{\infty} \) is an increasing sequence of finite subsets of \( C(X, \mathbb{R}) \) such that the union is dense in the closed unit ball of the positive elements in \( C(X, \mathbb{R}) \).

We construct sequences \( \{ \psi_l \}_{l=1}^{\infty} \) and a subsequence \( \{ N_l \}_{l=1}^{\infty} \) of the integers such that 

1. \( \psi_l : C(X_l) \to A_{N_l} \) is a unital c.p.c. \( \mathcal{G}_l - \delta_l \)-multiplicative map; 
2. \( \tau \circ \psi_l(f) \) is within \( \gamma_l/4 \) of \( \lambda(\tau)(f) \) for all \( f \in \mathcal{G}_l \); 
3. \( \mu_{\tau \circ \psi_l}(O_{l', j}) > \sigma_{l' \nu'} \) for all \( l' \leq l \), for all \( j \) and all \( \tau \in T(A) \); and
(4) $\alpha|\mathcal{P}_l = [\psi_l]|\mathcal{P}_l$.

Let us collectively denote the above conditions by “(*)”.

To simplify notation, let us denote $\rho_l = \alpha f \min\{\gamma_l/100, \sigma_l/100\}$. Fix $l$. We now construct $\psi_l$. By the same argument as that of Theorem 25.1.1, there exists an integer $M_1$ such that for all $n \geq M_1$ there is a morphism (of complete order unit spaces) $\text{Aff}(\lambda_n) : \text{Aff}(\mathcal{T}(C(X))) \to \text{Aff}(\mathcal{T}(A_n))$ such that $\text{Aff}(\lambda)(f)$ is within $\rho_l$ of $\text{Aff}(\lambda_n)(f)$ for all $f \in \mathcal{G}_l$ (of course, we mean the appropriate composition of $\text{Aff}(\lambda_n)$ with etc. to get a map with codomain $\text{Aff}(\mathcal{T}(A))$). Since $A$ is a simple unital AH-algebra with bounded dimension growth and real rank zero, and by Theorem 1.3 let $\phi_1 : C(X) \to A$ be a unital, c.p.c., almost multiplicative map and let $q$ be a projection in $A$ such that

i. there is an integer $M_2 \geq M_1$ such that $q \in A_{M_2}$ and $\tau'(q) < \rho_l$ for every $\tau' \in \mathcal{T}(A_{M_2})$;

ii. the map $C(X) \to A : f \mapsto q\phi_1(f)q$ is a c.p.c. $\mathcal{G}_l - \delta_l$-multiplicative map;

iii. there is a finite dimensional $C^*$-subalgebra $F \subseteq A_{M_2}$ with $1_F = 1_A - q$;

iv. there is a unital finite-dimensional $*$-homomorphism $h : C(X) \to F$ such that $\phi_1(f)$ is within $\rho_l$ of $q\phi_1(f)q + h(f)$ for all $f \in \mathcal{G}_l$;

v. $\alpha|\mathcal{P}_l = [\phi_1]|\mathcal{P}_l$.

Let $\phi_{l,1} : C(X) \to A$ be the unital c.p.c. $\mathcal{G}_l - \delta_l$-multiplicative map given by $\phi_{l,1}(f) = q\phi_1(f)q + h(f)$ for all $f \in C(X)$. Note that it follows, from the above conditions and by the definition of the $\mathcal{G}_l$ (and the remarks surrounding it) that $\phi_{l,1}$ is well-defined on $\mathcal{P}_l$ and

$$\alpha|\mathcal{P}_l = [\phi_1]|\mathcal{P}_l = [\phi_{l,1}]|\mathcal{P}_l.$$

Put $\mathcal{G}_l$ and $\rho_l$ into Theorem 1.2 to get the integer $L$ (which is the $N$ in the statement of Theorem 1.2). Note that $A_{M_2}$ has the form $A_{M_2} = r_1\mathbb{M}_{k_1}(C(Y_1))r_1 \oplus r_2\mathbb{M}_{k_2}(C(Y_2))r_2 \oplus \ldots \oplus r_n\mathbb{M}_{k_n}(C(Y_n))r_n$ where each $Y_j$ is a connected finite CW-complex with dimension less than or equal to three. Suppose that $F = F_1 \oplus F_2 \oplus \ldots \oplus F_n$ where for every $j$, $F_j$ is a (finite dimensional $*$-algebra) contained in $r_j\mathbb{M}_{k_j}(C(Y_j))r_j$. Since $A$ is a simple unital AH-algebra with bounded dimension growth and real rank zero, moving up building blocks if necessary (see Theorem 1.4), we may assume that for each $j$, there is a trivial projection $t_j \in r_j\mathbb{M}_{k_j}(C(Y_j))r_j$ (i.e., a projection corresponding to a trivial vector bundle over $Y_j$) such that (a) $Lt_j$ is Murray-von Neumann equivalent to a subprojection of $1_{F_j}$ in $r_j\mathbb{M}_{k_j}(C(Y_j))r_j$ and (b) $\tau'(1_{F_j} - Lt_j) < \rho_l$ for all $\tau' \in \mathcal{T}(F_j)$.

Now recall that since $M_2 \geq M_1$, the morphism $\text{Aff}(\lambda_{M_2}) : \text{Aff}(\mathcal{T}(C(X))) \to \text{Aff}(\mathcal{T}(A_{M_2}))$ (of order unit spaces) is such that $\text{Aff}(\lambda)(f)$ is within $\rho_l$ of $\text{Aff}(\lambda_{M_2})(f)$ for all $f \in \mathcal{G}_l$. Note that $\text{Aff}(\mathcal{T}(C(X))) \cong C(X, \mathbb{R})$ and

$$\text{Aff}(\mathcal{T}(A_{M_2})) \cong C(Y_1, \mathbb{R}) \oplus C(Y_2, \mathbb{R}) \oplus \ldots \oplus C(Y_n, \mathbb{R}),$$

where the isomorphisms are isomorphisms between order unit spaces. For each $j$, there is the natural projection map $\pi_j : C(Y_j, \mathbb{R}) \oplus C(Y_2, \mathbb{R}) \oplus \ldots \oplus C(Y_n, \mathbb{R}) \to C(Y_j, \mathbb{R})$ which is a (surjective) morphism of order unit spaces. Hence, for each $j$, we get a morphism (or, unitary positive map) $\xi_j = \pi_j \circ \text{Aff}(\lambda_{M_2}) : C(X) \to C(Y_j)$.

By the definition of $L$ and by Theorem 1.2 let $x_{j,1}, x_{j,2}, \ldots, x_{j,L} : C(X) \to C(Y_j)$
be unital $*$-homomorphisms such that each $\chi_{j,k}$ is homotopy equivalent to a point-evaluation $*$-homomorphism and such that

$$
\|\xi_j(f) - (1/L) \sum_{k=1}^{L} \text{Aff}(T(\chi_{j,k}))(f)\| < \rho_l
$$

for all $f \in \mathcal{G}_l$.

For each $j$, let $t_{j,1}, t_{j,2}, \ldots, t_{j,L}$ be the $L$ pairwise orthogonal subprojections of $1_{F_j}$, each of which is Murray-von Neumann equivalent (in $F_j$) to $t_j$; and let $h'_{j}: C(X) \to 1_{F_j} \mathbb{M}_{k_j}(C(Y_j))1_{F_j}$ be the unital $*$-homomorphism given by $h'_{j} : f \mapsto \sum_{k=1}^{L} \chi_{j,k}(f) \otimes t_{j,k} + f(z_j)(1_{F_j} - \sum_{k=1}^{L} t_{j,k})$, for some point $z_j \in X$. Let $h' : C(X) \to 1_{F_j} \mathbb{M}_{k_j}1_{F_j}$ be given by $h' = h_1' + h_2' + \cdots + h_n'$, if $h = h_1 + h_2 + \cdots + h_n$ where each $h_j : C(X) \to F_j$ is a unital $*$-homomorphism then $h_j$ is homotopy equivalent to $h'_j$, for every $j$. Hence, $h$ is homotopy equivalent to $h'$. Let $\psi : C(X) \to A$ be given by $\psi : f \mapsto q(\phi_l(f))q + h'(f)$ for all $f \in C(X)$. Note that $\psi$ is a unital c.p.c. $\mathcal{G}_l - \delta\xi$-multiplicative map. Then, by the definition of $\mathcal{G}_l$ (and the remarks surrounding it), $\psi_l$ is well-defined on $\mathcal{P}_l$ and

$$
[\psi_l]|\mathcal{P}_l = [\phi_l]|\mathcal{P}_l.
$$

Hence,

$$
[\psi_l]|\mathcal{P}_l = [\phi_l]|\mathcal{P}_l = \alpha|\mathcal{P}_l.
$$

Fix $1 \leq j \leq a$. For $\tau' \in T(r_j \mathbb{M}_{k_j}(C(Y_j))r_j)$ and $f \in \mathcal{G}_l'$,

$$
|((1/L) \sum_{k=1}^{L} \tau'_{(\chi_{j,k}(f))} - \tau'(h'_j(f))|
$$

$$
= |((1/L) \sum_{k=1}^{L} \tau'_{(\chi_{j,k}(f))} - (\sum_{k=1}^{L} \tau'_{(\chi_{j,k}(f) \otimes t_{j,k})} + \tau'(f(z_j)(1_{F_j} - \sum_{k=1}^{L} t_{j,k}))|
$$

$$
< |(1/L) \sum_{k=1}^{L} \tau'_{(\chi_{j,k}(f))} - \sum_{k=1}^{L} \tau'_{(\chi_{j,k}(f) \otimes t_{j,k})}| + \rho_l
$$

$$
< 3\rho_l.
$$

Hence, for $f \in \mathcal{G}_l'$, and for $\tau' \in T(r_j \mathbb{M}_{k_j}(C(Y_j))r_j)$,

$$
|\tau'(\xi_j(f)) - \tau'(h'_j(f))|
$$

$$
= |\tau'(\xi_j(f)) - (1/L) \sum_{k=1}^{L} \tau'_{(\chi_{j,k}(f))}| + |(1/L) \sum_{k=1}^{L} \tau'_{(\chi_{j,k}(f))} - \tau'(h'_j(f))|
$$

$$
< \rho_l + |(1/L) \sum_{k=1}^{L} (\tau'_{(\chi_{j,k}(f))} - \tau'(h'_j(f)))|
$$

$$
< 4\rho_l.
$$
Hence, for $f ∈ G_l'$ and for $τ ∈ T(A)$,
\[
|λ(τ)(f) − τ(ψ_l(f))| \\
\leq |λ(τ)(f) − τ(\text{Aff}(λ_{M^2}))(f)| + |τ(\text{Aff}(λ_{M^2})(f)) − τ(ψ_l(f))| \\
< ρ_l + |τ(\text{Aff}(λ_{M^2})(f)) − τ(ψ_l(f))| \\
= ρ_l + |τ(\text{Aff}(λ_{M^2})(f)) − τ(qψ_l(f)q) + τ(h'(f))| \\
< 2ρ_l + |τ(\text{Aff}(λ_{M^2})(f)) − τ(h'(f))| \\
< 6ρ_l.
\]

Since for $l_0 ≤ l$, $\{f_{l_0,j} : 1 ≤ j ≤ n_{l_0}\} ⊆ G_l'$ and since $σ_l < σ_{l_0}$, it follows that $τ(ψ_l(f_{l_0,j})) > σ_{l_0}$ for all $τ ∈ T(A)$ and for all $j$. Hence, $μ_{τ,ψ_l}(O_{l_0,j}) > σ_{l_0}ν_{l_0}$ for all $l_0 ≤ l$ and all $j$.

Also, from the above and definitions of $G_l$ and $G_l'$, it follows that for all $τ ∈ T(A)$ and for all $f ∈ G_l$, $|λ(τ)(f) − τ(ψ_l(f))| < 24ρ_l < γ_l/4$.

Hence, we have constructed a sequence $\{ψ_l\}_{l=1}^∞$ that satisfies the conditions in (*).

Hence, by (*) and Theorem 1.5, let $u_1$ be a unitary in $A$ such that $u_1ψ_1(f)u_1^*$ is within $ε_1$ of $ψ_1(f)$ for all $f ∈ F_1$. Again by (*) and Theorem 1.5, let $u_2$ be a unitary in $A$ such that $u_2ψ_2(f)u_2^*$ is within $ε_2$ of $u_1ψ_2(f)u_1^*$ for all $f ∈ F_2$. Repeating this process, we get a sequence $\{u_l\}_{l=0}^∞$ of unitaries in $A$ (taking $u_0 = 1_A$) such that for every $l$, $u_lψ_{l+1}(f)u_l^*$ is within $ε_l$ of $u_{l-1}ψ_l(f)u_{l-1}^*$ for all $f ∈ F_l$. Since the union of the $F_l$s is dense in the closed unit ball of $C(X)$ and since $\sum_{l=1}^∞ ε_l < ∞$, the sequence $\{u_lψ_{l+1}u_l^*\}_{l=1}^∞$ must converge pointwise to a unital $*$-homomorphism $Φ : C(X) → A$. And by our construction, $λ = T(Φ)$. Moreover, $KL(Φ) = KL(α)$.}

1.7 From Theorem 1.6 and Theorem 4.5, we get Theorem 0.1.

Finally, since a finite CW complex is the finite union of pairwise disjoint, path connected, second countable, compact metric spaces, we can replace $X$ in Theorem 1.6 by an arbitrary finite CW complex.

2. Tracially AF embeddings

2.1 It is the aim of this section to prove the lemma below. Our argument is inspired by the methods developed in [51] and [52].

**Lemma:** Let $Y$ be a compact metrizable space and $A$ a simple, separable, unital, $Z$-stable $C^*$-algebra with real rank zero. Suppose $θ : C(Y) → A$ is a unital $*$-homomorphism; let a finite subset $F ⊂ C(Y)$ and $ε > 0$ be given. Then, there is a commutative finite-dimensional $C^*$-subalgebra $B ⊂ A$ such that

(i) $\|θ(a), 1_B\| < ε \quad \forall a ∈ F$

(ii) $\text{dist}(1_Bθ(a)1_B, B) < ε \quad ∀ a ∈ F$

(iii) $τ(1_B) > 1 - ε \quad ∀ τ ∈ T(A)$

A $*$-monomorphism that satisfies the conclusion of the above lemma will be called “tracially AF-embeddings” or “TAF-embeddings” (see Definition 2.2 below).

2.2 Notation: For a $C^*$-algebra $A$, we denote by $A_∞$ the quotient $\prod_N A / \bigoplus_N A$.

2.3 Proposition: Let $Y$ be a compact metrizable space. Then, there are a zero-dimensional compact metrizable space $X$, a unital embedding $ν : C(Y) → C(X)$ and
a commutative diagram as follows:

\[
\begin{array}{ccc}
\prod_n \mathbb{C}^{k_i} & \xrightarrow{\kappa} & \prod_n \mathbb{C}^{k_i}/\bigoplus_n \mathbb{C}^{k_i} \\
\nu \downarrow & & \downarrow \\
C(X) & \xrightarrow{\nu} & C(Y)
\end{array}
\]

Here, \(C(X) = \lim_{\rightarrow} \mathbb{C}(k_i, \beta_i)\) is a representation of \(C(X)\) as a unital AF algebra, \(\kappa : C(X) \to \prod \mathbb{C}^{k_i}/\bigoplus \mathbb{C}^{k_i}\) the canonical inclusion of the inductive limit, \(q\) the quotient map and \((\psi_l)_{l \in \mathbb{N}}\) a sequence of unital \(*\)-homomorphisms.

Moreover, there is a sequence of c.p.c. maps

\[\varphi_l : \mathbb{C}^{k_i} \to C(Y)\]

such that \(q \circ ((\varphi_l)_l)_{l \in \mathbb{N}} = \iota_{C(Y)}\), where \(\iota_{C(Y)} : C(Y) \to \prod_n C(Y)\) is the canonical embedding (as constant sequences) and \(q\) also denotes the quotient map \(\prod_n C(Y) \to C(Y)_{\infty}\). If \(\dim Y = n < \infty\), we may choose the \(\varphi_l\) to be \(n\)-decomposable in the sense of [20] Definition 2.2.

**Proof:** Choose a sequence \((\mathcal{U}_l = \{U_{i,l,1}, \ldots, U_{i,l,k_i}\})_{l \in \mathbb{N}}\) of open coverings of \(Y\) with the following properties:

1. \(U_{l+1}\) refines \(U_l\) for all \(l\)
2. each \(U_{i,l,j}\) contains some element \(y_{l,j}\) of \(Y\)
3. for some fixed metric on \(Y\), \(\max_j \{\text{diam}(U_{i,l,j})\} \xrightarrow{l \to \infty} 0\).

Define \(*\)-homomorphisms \(\psi_l : C(Y) \to \mathbb{C}^{k_i}\) by \(\psi_l := \bigoplus_{j=1}^{k_i} \text{ev}_{y_{l,j}}\). Choose partitions of unity subordinate to \(\mathcal{U}_l\); interpret these as c.p.c. maps \(\varphi_l : \mathbb{C}^{k_i} \to C(Y)\). It is clear from (iii) that \(\varphi_l(\psi_l) \to \text{id}_{C(Y)}\) pointwise, whence \(q((\varphi_l(\psi_l))_{l \in \mathbb{N}}) = \iota_{C(Y)}\). Note also that, if \(\dim Y = n < \infty\), then \(\mathcal{U}_l - \) and therefore also the \(\varphi_l\) may be chosen to be \(n\)-decomposable.

Since \(U_{l+1}\) refines \(U_l\) for each \(l\), we may choose unital \(*\)-homomorphisms \(\beta_l : \mathbb{C}^{k_i} \to \mathbb{C}^{k_{i+1}}\) such that, if the \(i\)-th component of \(\beta_l(e_{i,l,j})\) is nonzero, then \(U_{l+1,i} \subset U_{l,j}\) (where \(e_{i,l,j}\) denotes the \(j\)-th canonical generator of \(\mathbb{C}^{k_i}\)). In other words, for each pair \((l+1, i)\) we choose a pair \((l, j)\) such that \(U_{l+1,i} \subset U_{l,j}\) and let this assignment represent an arrow in the Bratteli diagram \(\mathbb{C}^{k_i} \to \mathbb{C}^{k_{i+1}}\). The inductive limit \(\lim_{\rightarrow} (\mathbb{C}^{k_i}, \beta_l)\) is a commutative unital AF algebra, hence of the form \(C(X)\) for some compact zero-dimensional space \(X\); let \(\kappa : C(X) \to \prod \mathbb{C}^{k_i}/\bigoplus \mathbb{C}^{k_i}\) denote the canonical embedding.

For each \(l \in \mathbb{N}\) we have a \(*\)-homomorphism

\[\nu_l : C(Y) \xrightarrow{\psi_l} \mathbb{C}^{k_i} \xrightarrow{\beta_l} C(X)\]

using (iii) it is now straightforward to check that this sequence of maps is approximately multiplicative in the sense of [2] and induces a \(*\)-homomorphism \(\nu : C(Y) \to C(X)\) which makes our diagram commute.

2.4 The next lemma uses some technical results of [51]; it is essentially contained in the proof of Theorem 4.1 of that paper.
LEMMA: Let $A$ and $B$ be separable $C^*$-algebras, $A$ unital and $\mathcal{Z}$-stable with real rank zero. Let $n, k \in \mathbb{N}$ and $p \in A$ be a projection. Consider c.p.c. maps

$$B \xrightarrow{\phi} \mathbb{C}^k \xrightarrow{\xi} pAp$$

such that $\phi$ is n-decomposable for some $n \in \mathbb{N}$ and satisfies $\|p - \phi(1_{\mathbb{C}^k})\| \leq \eta$ for some $\eta > 0$.

Then, there is a sequence of *-homomorphisms $\varrho_m : \mathbb{C}^k \rightarrow pAp$, $m \in \mathbb{N}$, such that

$$\limsup_{m \rightarrow \infty} \|\varrho_m(1_{\mathbb{C}^k}) - \phi\xi(b)\| \leq 10(n + 1) \cdot \|\varphi\xi(b^2) - \varphi\xi(b)^2\| \geq b \in B_+,$$

$$\limsup_{m \rightarrow \infty} \|\varrho_m(1_{\mathbb{C}^k})\varphi\xi(b) - \varrho_m\varphi\xi(b)\| \leq 5(n + 1) \cdot \|\varphi\xi(b^2) - \varphi\xi(b)^2\| \geq b \in B_+$$

and

$$\tau(\varrho_m(1_{\mathbb{C}^k})) \geq \left( \frac{1}{5(n + 1)} - \eta \right) \cdot \tau(\varphi) \forall \tau \in T(A), m \in \mathbb{N}.$$

2.5 Let $(j \mapsto l_j)$ be a surjective and decreasing map $\mathbb{N} \rightarrow \mathbb{N}$. We call the sequence $(l_j)_{j \in \mathbb{N}}$ an expansion of the sequence $(l_i)_{i \in \mathbb{N}}$. For a sequence $(M_i)_{i \in \mathbb{N}}$ of maps, algebras, etc., we call the sequence $(M_i)_{j \in \mathbb{N}}$ an expansion of $(M_i)_{i \in \mathbb{N}}$. Similarly, from an inductive system $(A_i, \alpha_i)_{i \in \mathbb{N}}$ of $C^*$-algebras we obtain an expansion $(A_i, \alpha_i)_{j \in \mathbb{N}}$, where $\alpha_j = \alpha_i$ if $l_j \neq l_{j+1}$, and $\alpha_j = \text{id}_{A_{l_j}}$ if $l_j = l_{j+1}$. It is clear that $\lim_{j \rightarrow \infty} A_i \cong \lim_{j \rightarrow \infty} A_{l_j}$.

2.6 PROPOSITION: Let $A$ be a simple, separable, unital, $\mathcal{Z}$-stable $C^*$-algebra with real rank zero; let $Y$ be a compact metrizable space with $\dim Y = n < \infty$. Suppose $\theta : \mathcal{C}(Y) \rightarrow A_\infty$ is a *-homomorphism and $(p_i)_{i \in \mathbb{N}} \in \prod A$ is a sequence of projections lifting $p := \theta(1_Y)$.

Then, there is a commutative diagram as follows:

$$
\begin{array}{ccc}
\mathcal{C}(Y) & \xrightarrow{\nu} & \mathcal{C}(X) \\
\downarrow{\psi} & & \downarrow{\kappa} \\
\Pi_{N} \mathbb{C}^{k_i} & \xrightarrow{q} & \Pi_{N} \mathbb{C}^{k_i} / \bigoplus_{i} \mathbb{C}^{k_i} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\Pi_{N} p_i A p_i & \xrightarrow{q} & pA_\infty p \\
& & \subset A_\infty
\end{array}
$$

Here, the upper left triangle has the properties of diagram (1) in Proposition 2.5 (in fact, it is an expansion of (1)), the $\psi_i$ and $\varphi$ are *-homomorphisms,

$$h := \varphi \psi(1_Y) \in \theta(\mathcal{C}(Y))^0 \cap A_\infty$$

and the map $h \theta(\cdot)h$ is a *-homomorphism. Moreover, the $\varphi_i$ satisfy

$$\liminf_{l \rightarrow \infty} \left( \tau(\varphi_l \psi_l(1_Y)) - \frac{1}{5(n + 1)} \cdot \tau(p_l) \right) \geq 0 \forall \tau \in T(A).$$
Proof: Given $Y$, apply Proposition 2.3 to obtain a diagram

\[
\begin{array}{ccc}
C(Y) & \overset{\rho}{\rightarrow} & C(X) \\
(\tilde{\psi})_l & \downarrow & \\
\prod C_{k_l} & \overset{\eta}{\rightarrow} & \prod C_{k_l} / \bigoplus C_{k_l}
\end{array}
\]

and $n$-decomposable maps $\tilde{\varphi}_l : C_{k_l} \rightarrow C(Y)$. Set

\[
\bar{\varphi}_l := \theta \circ \tilde{\varphi}_l \quad \forall l \in \mathbb{N};
\]

these are $n$-decomposable c.p.c. maps $C_{k_l} \rightarrow pA_{\infty}p$ satisfying

\[
\bar{\varphi}_l \circ \tilde{\psi}_l (b) \overset{\sim}{\rightarrow} \theta(b) \quad \forall b \in C(Y).
\]

By [20], Remark 2.4 (cf. also [48], Proposition 1.2.4), each $\bar{\varphi}_l$ may be lifted to a sequence of $n$-decomposable c.p.c. maps

\[
\bar{\varphi}_{l,j} : C_{k_j} \rightarrow p_jAp_j, \quad j \in \mathbb{N}.
\]

A diagonal sequence argument now yields an expansion $(l_j)_{j \in \mathbb{N}}$ of the sequence $(l)_{l \in \mathbb{N}}$ such that

\[
\varphi_j := \tilde{\varphi}_{l_j,j} : C_{k_{l_j}} \rightarrow p_jAp_j
\]

and

\[
\psi_j := \tilde{\psi}_{l_j}
\]

satisfying

\[
q \circ ((\varphi_j \circ \psi_j)_{j \in \mathbb{N}}) = \theta.
\]

Note that the inductive system $C(X) \cong \lim_{\rightarrow} C_{k_l}$ and its expansion $\lim_{\rightarrow} C_{k_{l,j}}$ are isomorphic by [23]. Let $\nu$ denote the composition of this isomorphism with $\nu$ and write $\kappa$ for the natural inclusion of $\lim_{\rightarrow} C_{k_{l,j}}$ into $\prod C_{k_{l,j}} / \bigoplus C_{k_{l,j}}$. This yields a commutative diagram

\[
\begin{array}{ccc}
C(Y) & \overset{\nu}{\rightarrow} & C(X) \\
(\psi_l)_l & \downarrow & \\
\prod C_{k_{l,j}} & \overset{\kappa}{\rightarrow} & \prod C_{k_{l,j}} / \bigoplus C_{k_{l,j}}
\end{array}
\]

with the same properties as (1). Misusing notation, from now on we will write $C_{k_j}$ in place of $C_{k_{l,j}}$: this should not cause confusion.

For each $j \in \mathbb{N}$, we may apply Lemma 2.4 with $C(Y)$ in place of $B$ and $p_j, C_{k_j}, \psi_j, \varphi_j$ and $\eta_j := \|p_j - \varphi(1_{C_{k_j}})\|$ in place of $p, C_{k_l}, \psi, \varphi$ and $\eta$, respectively, to obtain a sequence of $*$-homomorphisms

\[
\theta_{j,m} : C_{k_j} \rightarrow p_jAp_j
\]
satisfying the assertions of (2.4). Note that
\[ \| p_j - \varphi_j(1_{C^*}) \| \xrightarrow{j \to \infty} 0 \]
and
\[ \| \varphi_j \psi_j(b^2) - \varphi_j \psi_j(b)^2 \| \xrightarrow{j \to \infty} 0 \quad \forall b \in C(Y) . \]
Therefore, again by a diagonal sequence argument (and separability of \( C(Y) \)) we may choose a sequence \((m_j)_{j \in \mathbb{N}} \subseteq \mathbb{N} \) such that for the \(*\)-homomorphisms \( \varrho_j := \varrho_{j, m_j} : C^{k_j} \rightarrow p_j A p_j, j \in \mathbb{N} \)
we have
\[ \| [\varrho_j(1_{C^*}), \varphi_j(1_Y)] \| \xrightarrow{j \to \infty} 0 \quad \forall b \in C(Y) , \]
(5)
and
\[ \| \varrho_j(1_{C^*}) \varphi_j(b) - \varrho_j(b) \varphi_j(1_Y) \| \xrightarrow{j \to \infty} 0 \quad \forall b \in C(Y) \]
(6)
and
\[ \liminf_j \left( \tau(\varrho_j(1_Y)) - \frac{1}{5(n + 1)} \cdot \tau(p_j) \right) = \liminf_j \left( \tau(\varrho_j(1_{C^*})) - \frac{1}{5(n + 1)} \cdot \tau(p_j) \right) \]
\[ \geq 0 \quad \forall \tau \in T(A) . \]
(7)
The \( \varrho_j \) now yield a commutative diagram
\[ \prod C^{k_j} \xrightarrow{q} \prod C^{k_j} / \bigoplus C^{k_j} \]
\[ \prod p_j A p_j \xrightarrow{q} p A_\infty p , \]
where \( q \) denotes the map induced by the \( \varrho_j \). The above diagram will be the lower left rectangle of (2). It follows from (4) and (5) that
\[ [\varrho(1_{\prod C^{k_j} / \bigoplus C^{k_j}}), \vartheta(b)] = 0 \quad \forall b \in C(Y) . \]
Since the \( \psi_j \) are unital, this implies that
\[ h = \varrho \kappa \nu(1_Y) = q(\varrho_j(1_Y))_{j \in \mathbb{N}} \in \varrho(C(Y))' \cap A_\infty . \]
Note that (4), (5) and (6) also imply that
\[ h \vartheta(b) h = \varrho \kappa \nu(1_Y) \vartheta(b) = q(\varrho_j(b))_{j \in \mathbb{N}} \quad \forall b \in C(Y) . \]
Exchanging the indices \( j \) for \( l \), we have constructed all the ingredients for the diagram (2); above we have proved commutativity of this diagram. The proposition’s statement about traces is just (7).

2.7 PROPOSITION: Let \( Y \) be a compact metrizable space with \( \dim Y = n < \infty \). Let \( A \) be a simple, separable, \( \mathcal{Z} \)-stable \( C^* \)-algebra with real rank zero. Suppose \( \vartheta : C(Y) \rightarrow A \) is a unital \(*\)-homomorphism and let \( \varepsilon > 0 \) be given. Then, there are a commutative AF algebra \( C(X) \subseteq A_\infty \), a sequence \((p_l)_{l \in \mathbb{N}} \subseteq \prod_{\mathbb{N}} A \)
of projections lifting \( p := 1_X \in A_\infty \) and \( l_0 \in \mathbb{N} \) such that
(i) \( p \in \vartheta(C(Y))' \cap A_\infty \)
(ii) \( p \vartheta(C(Y)) \subseteq C(X) \)
(iii) \( \tau(p_l) > 1 - \varepsilon \quad \forall l \geq l_0, \tau \in T(A) . \)
Proof: For convenience, we set
\[ \mu := \frac{1}{5(n+1)}; \]
let \( \vartheta^{(0)} := \vartheta \) and \( p_l^{(0)} := 1_A \) for \( l \in \mathbb{N} \) and \( p^{(0)} := 1_{A_\infty} \). Provided that, for some \( m \in \mathbb{N} \), we have constructed a unital \(*\)-homomorphism \( \vartheta^{(m)} : \mathcal{C}(Y) \to p^{(m)}A_\infty p^{(m)} \)
together with a lift \( (p_l^{(m)})_{l \in \mathbb{N}} \in \prod A \) of \( p^{(m)} \in A_\infty \), we apply Lemma 2.6 to obtain
a diagram as in (2). Set
\[ h_l^{(m)} := \vartheta_l^{(m)}(1_Y), \quad l \in \mathbb{N}. \]
This yields a sequence of projections
\[ p_l^{(m+1)} := p_l^{(m)} - h_l^{(m)} = 1_A - \sum_{k=0}^{m} h_l^{(k)} \in A; \]
let \( p^{(m+1)} \) denote the class of \( (p_l^{(m+1)})_{l \in \mathbb{N}} \) in \( A_\infty \). By Lemma 2.6 we have
\[ q((h_l^{(m)})_{l \in \mathbb{N}}) \in \vartheta^{(m)}(\mathcal{C}(Y))' \cap p^{(m)}A_\infty p^{(m)} \]
and, since \( \vartheta^{(m)}(1_Y) = p^{(m)} \),
\[ p^{(m+1)} = p^{(m)} - q((h_l^{(m)})_{l \in \mathbb{N}}) \in \vartheta^{(m)}(\mathcal{C}(Y))' \cap p^{(m)}A_\infty p^{(m)}. \]
We may thus define a \(*\)-homomorphism
\[ \vartheta^{(m+1)} : \mathcal{C}(Y) \to p^{(m+1)}A_\infty p^{(m+1)} \subset A_\infty \]
by
\[ \vartheta^{(m+1)}(\cdot) := p^{(m+1)}\vartheta^{(m)}(\cdot). \]
Induction yields a sequence of diagrams as in (2), with \(*\)-homomorphisms
\[ \vartheta^{(m)} : \mathcal{C}(Y) \to p^{(m)}A_\infty p^{(m)}, \]
zero-dimensional spaces \( X^{(m)} \) and \(*\)-homomorphisms
\[ \vartheta^{(m)} \kappa^{(m)} : \mathcal{C}(X^{(m)}) \to p^{(m)}A_\infty p^{(m)} \]
for \( m \in \mathbb{N} \). Moreover, by (3) we have
\[ \liminf_{l \to \infty} \tau(h_l^{(m)}) - \mu \cdot \tau(p_l^{(m)}) \geq 0 \quad \forall \tau \in T(A), \quad m \in \mathbb{N}. \]
We show by induction over \( m \) that
\[ \liminf_{l \to \infty} \tau(\sum_{k=0}^{m} h_l^{(k)}) \geq \mu \cdot \sum_{k=0}^{m} (1-\mu)^k \quad \forall \tau \in T(A), \quad m \in \mathbb{N}. \]
For \( m = 0 \) this is true, since by (3),
\[ \liminf_{l \to \infty} \tau(h_l^{(0)}) - \mu \cdot \tau(p_l^{(0)}) = \liminf_{l \to \infty} \tau(h_l^{(0)}) - \mu \geq 0 \quad \forall l \in \mathbb{N}, \tau \in T(A). \]
Suppose now (8) has been verified for some \( m \in \mathbb{N} \), then
\[
\liminf_{l \to \infty} \tau(\sum_{k=0}^{m+1} h_l^{(k)}) = \lim_{l \to \infty} \left( \tau(\sum_{k=0}^{m} h_l^{(k)}) + \mu \cdot \tau(p_l^{(m+1)}) + \tau(h_l^{(m+1)}) - \mu \cdot \tau(p_l^{m+1}) \right)
\]
\[
\geq \liminf_{l \to \infty} \left( \tau(\sum_{k=0}^{m} h_l^{(k)}) + \mu \cdot \tau(p_l^{(m+1)}) \right) = \liminf_{l \to \infty} \left( \tau(\sum_{k=0}^{m} h_l^{(k)}) + \mu \cdot (1 - \mu) \cdot \tau(\sum_{k=0}^{m} h_l^{(k)}) \right) = \mu + \mu \cdot \sum_{k=1}^{m+1} (1 - \mu)^k = \mu \cdot \sum_{k=0}^{m+1} (1 - \mu)^k \forall \tau \in T(A), l \in \mathbb{N}
\]
This proves (8) for \( m + 1 \) in place of \( m \), hence for all \( m \in \mathbb{N} \) by induction.

Since \( \lim_{m \to \infty} \mu \cdot \sum_{k=0}^{m} (1 - \mu)^k = 1 \), by Dini's theorem there are \( l_0, m_0 \in \mathbb{N} \) such that
\[
(9) \quad \tau\left(\sum_{m=0}^{m_0} h_l^{(m)}\right) > 1 - \varepsilon \forall \tau \in T(A), l \geq l_0.
\]

Note that by (2),
\[
q((h_l^{(m)})_{l \in \mathbb{N}}) = q^{(m)}(1_{X^{(m)}})(1_{X^{(m)}}) \in A_\infty \forall m \in \mathbb{N}.
\]
Set \( \tilde{X} := \bigoplus_{m=0}^{m_0} X^{(m)} \), then
\[
C(\tilde{X}) \cong \bigoplus_{m=0}^{m_0} C(X^{(m)})
\]
is a commutative AF algebra, and so is its image under \( \bigoplus_{m=0}^{m_0} q^{(m)}(1_{X^{(m)}}) \in A_\infty \); this image is of the form \( C(\tilde{X}) \subset A_\infty \) for some zero-dimensional compact space \( \tilde{X} \).

Let
\[
p_l := \sum_{m=0}^{m_0} h_l^{(m)}, l \in \mathbb{N},
\]
then \((p_l)_{l \in \mathbb{N}}\) lifts \( p := 1_X \in A_\infty \). By (2) we have \( q^{(m)}(1_{X^{(m)}}) \in q(C(Y))' \cap A_\infty \forall m = 0, \ldots, m_0 \), whence \( p \in q(C(Y))' \cap A_\infty \), so assertion (i) of the proposition holds. Furthermore,
\[
p\theta(C(Y)) = \bigoplus_{m=0}^{m_0} q((h_l^{(m)})_{l \in \mathbb{N}})\theta(C(Y)) \subset \bigoplus_{m=0}^{m_0} q^{(m)}(1_{X^{(m)}})(\bigoplus_{m=0}^{m_0} q^{(m)}(1_{X^{(m)}}))(C(X^{(m)})) = C(X),
\]
whence (ii) holds; we have already verified (iii) in (9).

2.8 We are finally prepared to prove the main result of this section.
Proof: (of Lemma 2.1) We may clearly assume the elements of $F$ to be positive and normalized. Even more, we may assume that $\sum_{a \in F} a = 1_Y$. Set $n := \text{card}F - 1$ and let $\Delta^n \subset \mathbb{R}^{n+1}$ denote the full $n$-simplex. Since $C(\Delta^n)$ is the universal unital commutative $C^*$-algebra generated by $n + 1$ positive elements adding up to the unit, $C^*(F) \subset C(Y)$ is a quotient of $C(\Delta^n)$. We have $C^*(F) \cong C(Y')$ for some compact subspace $Y'$ of $\Delta^n$; by [18], Theorem III.1 (cf. also [20], Proposition 3.3 and [47], Proposition 2.19), we see that

$$\dim Y' \leq \dim \Delta^n = n.$$ 

Restricting $\theta$ to $C(Y')$ we see that it suffices to prove the assertion of the lemma for a finite-dimensional space; thus we assume that $\dim Y = n$ for some $n \in \mathbb{N}$ right away.

Let $C(X) \subset A_\infty$ be a commutative AF algebra and $(p_l)_{l \in \mathbb{N}} \subset A$ a sequence of projections satisfying the assertions of Proposition 2.7. By 2.7(i) we then have $1_X \in \theta(C(Y))' \cap A_\infty$; by (ii), $1_X \theta(C(Y)) \subset C(X)$. But then there is a unital finite-dimensional $C^*$-subalgebra $B$ of $C(X)$ such that dist$(1_X \theta(a) 1_X, B) < \varepsilon/2 \forall a \in F$; note that $B$ is commutative and that we may assume $1_B = 1_X$. Since $(p_l)_{l \in \mathbb{N}}$ lifts $1_X$, $1_X A_\infty 1_X$ is a quotient of $\prod_{l \in \mathbb{N}} p_l A p_l$. Finite-dimensional $C^*$-algebras are semiprojective ([32], Chapter 14), so there is a *-homomorphism

$$\sigma : B \rightarrow \prod_{l \in \mathbb{N}} p_l A p_l$$

lifting id$B$. It is clear that all but finitely many of the $\sigma_l$ (the components of $\sigma$) are unital. Now if $l_0 \in \mathbb{N}$ is large enough, then dist$(\theta(a), \sigma_{l_0}(B)) < \varepsilon \forall a \in F$, $\sigma_{l_0}(1_B) = p_{l_0}$ and $||\theta(a), p_{l_0}|| < \varepsilon \forall a \in F$. Since $\tau(p_{l_0}) > 1 - \varepsilon \forall \tau \in T(A)$ (for large enough $l_0$) by Proposition 2.7(iii), $B := \sigma_{l_0}(B)$ satisfies the assertion of the lemma.

3. The uniqueness theorem

Towards the uniqueness theorem, we need the following stable uniqueness result which can be found in [16] Theorem 3.1 (see also [16] Remark 1.1, [15] and [29]):

3.1 Theorem: Let $X$ be a compact metric space. For any $\varepsilon > 0$ and any finite subset $F \subseteq C(X)$, there exist $\delta > 0$, $\eta > 0$, an integer $N > 0$, a finite subset $G \subseteq C(X)$ and a finite subset $P \subseteq P(C(X))$ satisfying the following:

For any unital simple separable nuclear $C^*$-algebra $A$ with real rank zero, stable rank one and weakly unperforated $K_0$ group, for any $\eta$-dense subset $\{x_1, x_2, \ldots, x_k\}$ in $X$, and any $G - \delta$-multiplicative c.p.c. linear maps $\phi, \psi : C(X) \rightarrow A$, if

$$[\phi]P = [\psi]P,$$

then there exists a unitary $u \in M_{Nk+1}(A)$ such that

$$u(\phi(f) \oplus f(x_1) 1_N \oplus f(x_2) 1_N \oplus \ldots f(x_k) 1_N) u^* \approx \psi(f) \oplus f(x_1) 1_N \oplus f(x_2) 1_N \oplus \ldots f(x_k) 1_N$$

for all $f \in F$.

3.2 Definition: Let $X$ be a compact metric space, and let $A$ be a unital simple separable $C^*$-algebra. A *-monomorphism $\phi : C(X) \rightarrow A$ is said to be a TAF-embedding (i.e., a tracially AF-embedding) if for every $\varepsilon > 0$, for every finite
subset $F \subseteq A$ and for every nonzero positive element $a \in A$, there is a projection $p \in A$ and there is a finite dimensional $C^*$-subalgebra $F \subseteq A$ such that

(i) $p$ is Murray-von Neumann equivalent to a subprojection of her($a$) (the hereditary subalgebra of $A$ generated by $a$),

(ii) $1_F = 1 - p$,

(iii) $\|pa - ap\| < \epsilon$ for all $a \in F$, and

(iv) $pap$ is within $\epsilon$ of an element of $F$ for all $a \in F$.

From Lemma 2.1 it follows that if $F$ is a unital separable simple real rank zero $\mathcal{Z}$-stable $C^*$-algebra, and if $X$ is a compact second countable metric space, then any unital $*$-monomorphism $\phi : C(X) \to A$ is a TAF-embedding.

3.3 Proposition: Let $X$ be a compact metric space, and let $A$ be a unital separable simple nuclear $C^*$-algebra with real rank zero, stable rank one and weakly unperforated $K_0$ group. Let $\phi : C(X) \to A$ be a unital TAF-embedding.

Then for every $\epsilon > 0$, for every finite subset $F \subseteq A$, for every integer $N \geq 1$ and for every nonzero positive element $a \in A$, there exists a real number $\delta$ with $0 < \delta < \epsilon$, there exists a finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq X$ which is $\delta$-dense in $X$, and there exists a projection $p \in A$ and a $F - \epsilon$-multiplicative c.p.c. $L_1 : C(X) \to pAp$ such that

(i) $p$ is Murray-von Neumann equivalent to a subprojection of her($a$) (there hereditary subalgebra of $A$ generated by $a$),

(ii) $\|\phi(f) - \phi(f)p\| < \epsilon$ for all $f \in F$, and

(iii) there exists pairwise orthogonal projections $p_0, p_1, p_2, \ldots, p_n, t$ with

$$p_0 = p, \sum_{i=0}^{n} p_i + t = 1_A, \text{ and } Np \preceq p_i$$

for $i \neq 0$ (here $\preceq$ is the relation of being Murray-von Neumann equivalent to a subprojection) and there exists a finite dimensional $*$-homomorphism

$$h_1 : C(X) \to tAt$$

such that

$$\phi(f) = \text{within } \epsilon \text{ of } L_1(f) + \sum_{i=1}^{n} f(x_i)p_i + h_1(f), \text{ for all } f \in F.$$

Proof: For simplicity, let us assume that the elements of $F$ all have norm less than or equal to one. Let $d$ be the metric on the space $X$. Let $\delta$ be a real number such that $0 < \delta < \epsilon/10$ and such that for every $x, y \in X$, if $d(x, y) < \delta$ then $|f(x) - f(y)| < \epsilon/10$. Now let $\{x_1, x_2, \ldots, x_n\}$ be a $\delta$-dense subset of $X$. We may assume that $d(x_i, x_j) > 0$ for $i \neq j$. For each $i$, let $0 < \delta_i < \delta$ be such that $B(x_i, \delta_i) \cap B(x_j, \delta_j) = \emptyset$ for $i \neq j$. (Here, $B(x_i, \delta_i)$ is the closure of the open ball $B(x_i, \delta_i)$ with radius $\delta_i$ about $x_i$.) For each $i$, let $f_i$ be a positive real-valued function with $0 \leq f_i \leq 1$, $f_i(x_i) = 1$ and $\text{supp}(f_i) \subseteq B(x_i, \delta_i)$, where $\text{supp}(f_i)$ is the (compact subset of $X$ which is the) support of $f_i$. Since $\phi$ is a $*$-monomorphism and since $A$ is simple, for each $i$, $\inf\{\tau(\phi(f_i)) : \tau \in T(A)\} > 0$ (strictly greater than zero). Hence, let $s$ be the strictly positive real number given by $s = \inf\{\tau(\phi(f_i)) : \tau \in T(A), 1 \leq i \leq n\}$. Let $F' = \phi F \cup \{f_i : 1 \leq i \leq n\}$.

Now apply Lemma 1.1 on $X$, $F'$ and $\epsilon_1 = \min\{\epsilon/10, s/(10(N + 10))\}$ to get $G$ and $\rho$ ($\rho$ is the $\delta$ in Lemma 1.1). We may assume that $\rho < \epsilon_1$. Making $\rho$ smaller if necessary, we may assume that the elements of $G$ all have norm less than or equal
to one. Since the \( * \)-monomorphism \( \phi : C(X) \to A \) is a TAF-embedding and since finite dimensional \( C^* \)-algebras are injective von Neumann algebras, let \( p \in A \) be a projection and let \( F \subseteq A \) be a finite dimensional \( C^* \)-subalgebra with \( 1_F = 1 - p \) such that

i. \( p \) is Murray-von Neumann equivalent to a subprojection of her(a), the hereditary subalgebra of \( A \) generated by \( a \),

ii. \( \tau(p) < \epsilon_1 \) for all \( \tau \in T(A) \),

iii. \( \|p\phi(f) - \phi(f)p\| < \epsilon_1 \) for all \( f \in G \), and

iv. there exists a unital c.p.c. \( \rho \)-multiplicative map \( L : C(X) \to F \) such that \( \phi(f) \) is within \( \epsilon_1 \) of \( p\phi(f)p + L(f) \) for all \( f \in G \).

Applying Lemma 1.1 to \( L \), we get a \( * \)-homomorphism \( h : C(X) \to F \) with a projection \( q = h(1_{C(X)}) \) such that \( \tau'(1 - p - q) < \epsilon_1 \) for all \( \tau' \in T(F) \) and such that \( L(f) \) is within \( \epsilon_1 \) of \( (1 - p - q)L(f)(1 - p - q) + h(f) \) for all \( f \in F' \). Note that we must have that \( \tau(1 - p - q) < \epsilon_1 \) for all \( \tau \in T(A) \).

Hence, \( \phi(f) \) is within \( 2\epsilon_1 \) of \( L'(f) + h(f) \) for all \( f \in F' \), where

\[
L'(f) = df \cdot p\phi(f)p + (1 - p - q)L(f)(1 - p - q)
\]

(so \( L'(1_{C(X)}) = 1 - q \)). Note that (after some computation) \( L' \) is \( \mathcal{F} \)-\( 3\epsilon_1 \)-multiplicative. Also, \( \tau(q) > (1 - \epsilon_1)^2 \) (which in turn is greater than \( 1 - 2\epsilon_1 \)) for all \( \tau \in T(A) \).

Now for all \( i \), \( \phi(f_i) \) is within \( 2\epsilon_1 \) of \( L'(f_i) + h(f_i) \). Hence, for all \( i \) and for all \( \tau \in T(A) \), \( \tau(\phi(f_i)) \) is within \( 2\epsilon_1 \) of \( \tau(L'(f_i)) + \tau(h(f_i)) \). Also, note that \( \tau(L'(f_i)) < 1 - (1 - \epsilon_1)^2 = 2\epsilon_1 - \epsilon_1^2 \). Hence, for all \( i \) and for all \( \tau \in T(A) \),

\[
\tau(h(f_i)) = (\tau(h(f_i)) + \tau(L'(f_i))) - \tau(L'(f_i))
\geq (\tau(h(f_i)) + \tau(L'(f_i)) - 2\epsilon_1 + \epsilon_1^2
\geq (\tau(\phi(f_i)) - 2\epsilon_1) - 2\epsilon_1 + \epsilon_1^2
= \tau(\phi(f_i)) - 4\epsilon_1 + \epsilon_1^2
\geq s - 4\epsilon_1 + \epsilon_1^2
\]

Recall that \( \epsilon_1 = \min\{\epsilon/10, s/(10(N + 10))\} \). Hence, for all \( \tau \in T(A) \) and for all \( i \),

\[
\tau(h(f_i)) \geq s - 4s/(5(N + 10))
\geq (5Ns + 50s - 4s)/(5(N + 10))
= (5N + 46)s/(5N + 50)
\]

Now suppose that \( r_1, r_2, \ldots, r_l \) are pairwise orthogonal projections in \( h(1_{C(X)})F h(1_{C(X)}) \) and \( y_1, y_2, \ldots, y_l \) points in \( X \) such that for all \( f \in C(X) \), \( h(f) = \sum_{j=1}^l f(y_j)r_j \). For each \( i \), let \( S_i = \{ j : d(y_j, x_i) < \delta_i \} \). Then for each \( i \) and for each \( \tau \in T(A) \), \( \tau(\sum_{j \in S_i} r_j) \geq (5N + 46)s/(5N + 50) \). Replace \( h \) by a finite-dimensional \( * \)-homomorphism \( h' : C(X) \to F \) where for all \( f \in C(X) \),

\[
h'(f) = df \cdot \sum_{i=1}^n f(x_i) \sum_{j \in S_i} r_j + \sum_{j \not\in S_i} f(y_j)r_j
\]

Hence, for each \( f \in F \), \( h(f) \) is within \( \epsilon/10 \) of \( h'(f) \). Recall that \( L'(1) = 1 - q \) and \( \tau(q) > 1 - 2\epsilon_1 \geq 1 - s/(5N + 50) \) for all \( \tau \in T(A) \). Hence, \( \tau(L'(1)) < s/(5N + 50) \) for all \( \tau \in T(A) \). Hence, \( N\tau(L'(1)) < \tau(\sum_{j \in S_i} r_j) \) for all \( \tau \in T(A) \). Hence, for
For each $i$ may assume that $\tilde{\eta}$. Taking the above data and putting them into Theorem 1.5, we get quantities the elements of $F$ that $\epsilon > 0$.

3.4 Proof: (of Theorem 0.2) The “only if” direction follows from [40] 5.4. Hence, we need only prove the “if” direction.

Firstly, by [23] Theorem 4.5, there is a unital simple AH-algebra $A_0$ with bounded dimension growth and real rank zero and there is a unital $*$-homomorphism $\Phi : A_0 \to A$ such that $\Phi$ induces an order isomorphism (which, of course, respects the Bockstein operations) between the full $K$-groups $K(A_0)$ and $K(A)$; and since $A_0$ and $A$ are both real rank zero, this induces an isomorphism between the tracial simplexxes $T(A)$ and $T(A_0)$. Henceforth, to simplify notation, we may identify $K(A_0)$, $T(A_0)$ with $K(A)$, $T(A)$ respectively and take the induced maps $K(\Phi)$, $T(\Phi)$ to be identity maps. Hence, by Theorem 1.6 let $\theta : C(X) \to A_0 \subseteq A$ be a unital $*$-monomorphism such that $KL(\theta) = KL(\phi)$ ($= KL(\psi)$) and $\tau \circ \theta = \tau \circ \phi$ for all $\tau \in T(A)$. To prove that $\phi$ and $\psi$ are approximately unitarily equivalent, it suffices to show that $\phi$ and $\theta$ are approximately unitarily equivalent.

Let $\epsilon > 0$ and a finite subset $F \subseteq C(X)$ be given. Let us assume that the elements of $F$ all have norm less than or equal to one. Let $\nu > 0$ be such that $|f(x) - f(y)| < \epsilon/8$ if $\text{dist}(x, y) < \nu$ for all $x, y \in X$ and all $f \in F$. We may assume that $\nu < 1$ Now

i. take $s = \nu$;

ii. take a finite $\nu/2$-dense set $\{x_1, x_2, \ldots, x_m\}$ in $X$ for which $O_i \cap O_j = \emptyset$ whenever $i \neq j$, where $O_i = \{x \in X : \text{dist}(x, x_i) < \nu/2\}$; and also, put $\tilde{O}_i = \{x \in X : \text{dist}(x, x_i) < \nu/4\}$.

iii. and take a real number $\sigma$ where $0 < \sigma < 1/2$ and where $\mu_{\tau \circ \phi}(< \nu/4)$.

Taking the above data and putting them into Theorem 1.5, we get quantities $\gamma, F_1$, $\epsilon_1$ and $P_1$ ($F_1, \epsilon_1, P_1$ are the $G$, $\delta$, $P$ respectively in the conclusion of Theorem 1.5).

For each $i$, let $f_i$ be the real-valued function on $X$ with (a) $0 \leq f_i \leq 1$, (b) $f_i = 1$ on $\tilde{O}_i$ and (c) the support of $f_i$ is contained in $O_i$. Expanding $F_1$ if necessary, we may assume that $F_1$ contains $\{f_i : 1 \leq i \leq m\} \cup F$. We may also assume that all the elements of $F_1$ have norm less than or equal to one.

Now put $\epsilon$ and $F_1$ into Theorem 1.5 to get quantities $\epsilon_2, \eta, N > 0$, $F_2$ and $P_2$ ($\epsilon_2$, $F_2$, $P_2$ are the $\delta$, $\gamma$, $\mathcal{P}$ in the conclusion of Theorem 1.5). Contracting $\epsilon_2$ if necessary, we may assume that $\epsilon_2 < \epsilon_1$. Expanding $F_2$ if necessary, we may assume that $F_1 \subseteq F_2$ and the elements of $F_2$ all have norm less than or equal to one. Finally, expanding $P_2$ if necessary, we may assume that $P_1 \subseteq P_2$.

Let $\epsilon_3 > 0$ and let $F_3$ be a finite subset of $C(X)$ such that

1. $\epsilon_3 < \epsilon_2$,
2. $\epsilon_3 < \min\{\gamma/100, \sigma/100, \eta/100\}$,
3. $F_2 \subseteq F_3$, and
4. if $\rho_1, \rho_2 : C(X) \to A$ are two $F_3 - \epsilon_3$-multiplicative c.p.c. maps then $\rho_1$ and $\rho_2$ are both well-defined on $P_2$. Moreover, if $\rho_1(f)$ is within $\epsilon_3$ of $\rho_2(f)$ for all $f \in F_3$ then

$$[\rho_1]_{P_2} = [\rho_2]_{P_2}.$$
Now by Lemma 2.1 φ is a TAF-embedding. Therefore, put φ, ε₃, F₃, N, and a
projection whose value at every unital trace is strictly less than ε₃ into Proposition
Then as a consequence, we get a real number δ with 0 < δ < ε₃, a δ-dense subset
{x₁, x₂, ..., xₙ} of X, and we get a projection p ∈ A and a F₃ - ε₃-multiplicative
c.p.c. map L₁ : C(X) → pAp such that
(a) τ(p) < ε₃ for all τ ∈ T(A),
(b) there exists pairwise orthogonal projections p₀, p₁, p₂, ..., pₙ, t with p₀ = p,
∑ᵢ₌₀ⁿ pᵢ + t = 1, and Np ≤ pᵢ for i ≠ 0 (here ≤ is the relation of being
Murray-von Neumann equivalent to a subprojection) and there exists a
unital finite dimensional *-homomorphism h₁ : C(X) → tAₜ such that
φ(f) ≃ δ L₁(f) + \sumᵢ₌₁ⁿ f(xᵢ)pᵢ + h₁(f)
for all f ∈ F₃.

Let φ₁ : C(X) → A be the c.p.c. F₃ - ε₃-multiplicative map given by φ₁ : f ↦ L₁(f) + \sumᵢ₌₁ⁿ f(xᵢ)pᵢ + h₁(f). By the definition of ε₃, φ₁ (and also L₁) is
well-defined on P₃. Moreover,
[φ]P₃ = [φ₁]P₃.

Also, since τ ∘ φ(fᵢ) > 10σ for all τ ∈ T(A) and for all i, we must have that
τ ∘ φ₁(fᵢ) > 9σ for all τ ∈ T(A) and for all i.

Now let q₀, q₁, q₂, ..., qₙ, t' be pairwise orthogonal projections in A₀ such that (a)
1ₐ₀ = ∑ᵢ₌₁ⁿ qᵢ + t', (b) qᵢ is Murray-von Neumann equivalent (in A) to pᵢ, and (c)
t' is Murray-von Neumann equivalent (in A) to t.

Since X is a path connected, KL(φ) - KL(h₁) ∈ KL(C(X), q₀ A₀ q₀)₁₋₁. Hence,
by Theorem 1.6, let L₂ : C(X) → q₀ A₀ q₀ be a unital *-homomorphism such that
KL(L₂) = KL(φ) - KL(h₁). Let h₂ : C(X) → t'Aₜ be a unital finite
dimensional *-homomorphism which is unitarily equivalent (with unitary in A) to
h₁. Now consider the *-homomorphism φ₂ : C(X) → A given by φ₂ : f ↦
L₂(f) + \sumᵢ₌₁ⁿ f(xᵢ)qᵢ + h₂(f). Clearly,
[φ₁]P₃ = [φ₂]P₃.

Hence, by Theorem 3.1 let u ∈ A be a unitary such that
uφ₁(f)u* ≈ ε₂ φ₂(f)
for all f ∈ F₃. Hence, since τ ∘ φ₁(f) > 9σ for all f ∈ F₃ and for all τ ∈ T(A),
φ₂(f) > 8σ for all f ∈ F₃ and for all τ ∈ T(A). Hence, μₜₗφ₂(Ωₜ) > σν for all
f ∈ F₃, all τ ∈ T(A) and all i.

Also,
\|τ ∘ θ(f) - τ ∘ φ₂(f)\|
= \|τ ∘ φ(f) - τ ∘ φ₂(f)\|
≤ \|τ ∘ φ(f) - τ ∘ φ₁(f)\| + \|τ ∘ φ₁(f) - τ ∘ φ₂(f)\|
< ε₃ + \|τ ∘ φ₁(f) - τ ∘ φ₂(f)\|
< ε₃ + ε₂
≤ γ,
for all \( f \in \mathcal{F}_3 \).

Finally,
\[
[\theta]|P_3 = [\phi]|P_3 = [\phi_1]|P_3 = [\phi_2]|P_3.
\]

Hence, it follows, by Theorem 1.5, that there is a unitary \( v \in A_0 \) such that
\[
v\phi_2 (f)v^* \approx_\epsilon \theta (f),
\]
for all \( f \in \mathcal{F} \). Hence,
\[
v u \phi (f) u^* v^* \approx_\epsilon \theta (f),
\]
for all \( f \in \mathcal{F} \). Since \( \epsilon \) and \( \mathcal{F} \) were arbitrary, \( \phi \) and \( \theta \) are approximately unitarily equivalent. Similar for \( \psi \) and \( \theta \). Hence \( \phi \) and \( \psi \) are approximately unitarily equivalent as required.

3.5 Finally, note that since a finite CW complex is the finite union of pairwise disjoint, path connected, second countable, compact metric spaces, we can replace \( X \) in Theorem 0.2 by an arbitrary finite CW complex.

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