Adjoint cohomology of Heisenberg Lie superalgebras

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Abstract: Suppose the ground field \( F \) is an algebraically closed field of characteristic zero. In this paper, the adjoint cohomology of Heisenberg Lie superalgebras is studied by using the Hochschild-Serre spectral sequence. In particular, we determine the Betti numbers of the adjoint cohomology.

Keywords: Heisenberg Lie superalgebra; adjoint cohomology; spectral sequence

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0 Introduction

For a symplectic vector space, we can define its Heisenberg Lie algebra by the symplectic form, which is a two-step nilpotent Lie algebra with 1-dimensional center. Heisenberg Lie algebras have some special attention in the modern mathematics and physics because of the application of them in the commutation relations in quantum mechanics. For example, Santharoubane gave the description of the cohomology spaces for Heisenberg Lie algebras with coefficients in the trivial module over a field of characteristic zero in [1]. Sköldberg calculated the Betti numbers of Heisenberg Lie algebras in the characteristic two case in [2]. Cairns and Jambor extended Sköldberg’s result to arbitrary characteristic by directly computing the Betti numbers and showed that in characteristic two, unlike all other cases, the Betti numbers are unimodal in [3]. The adjoint cohomology of Heisenberg Lie algebras were studied in the works of Magnin [4] and Cagliero [5]. Moreover, by considering a Heisenberg Lie algebra as the nilradical of a parabolic subalgebra of a simple Lie algebra of type A, Alvarez gave a full description of the adjoint homology of the parabolic subalgebra as a module over its Levi factor in [6].

In [7], Rodríguez-Vallarte generalized Heisenberg Lie algebras by considering its supersymmetry, which are called Heisenberg Lie superalgebras, and proved that they do not admit neither supersymplectic nor superorthogonal invariant forms, however one-dimensional extensions by some appropriate homogeneous derivations do. Zhou and Chen showed the derivation algebra of a Heisenberg Lie superalgebra with an even center is simple complete, but the holomorph of it is not in [8]. Bai and Liu determined completely the Betti numbers and associative superalgebra structures for the trivial cohomology of Heisenberg Lie superalgebras by using the Hochschild-Serre spectral sequence relative to the centers in [9]. However, in the adjoint module case, less of work is done for Heisenberg Lie superalgebras.

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Throughout this paper, the ground field \( F \) is an algebraically closed field of characteristic zero and all vector spaces, algebras are over \( F \). Write \( \mathbb{Z}_2 \) for the two elements field. For a \( \mathbb{Z}_2 \)-graded vector space \( V \), denote by \( |v| \) the \( \mathbb{Z}_2 \)-degree of \( v \), where \( v \in V_0 \cup V_1 \). In this article, for a Heisenberg Lie superalgebra, we show the relation between the adjoint cohomology and trivial cohomology by using the Hochschild-Serre spectral sequence relative to its center. Moreover, we depict the Betti numbers for the adjoint cohomology.

1 Preliminaries

A Lie superalgebra is a \( \mathbb{Z}_2 \)-graded algebra whose multiplication satisfies the skew-supersymmetry and the super Jacobi identity (see [10]). For a Lie superalgebra \( \mathfrak{g} \), we inductively define the lower central series

\[
\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i], \quad i \geq 0.
\]

\( \mathfrak{g} \) is called nilpotent if \( \mathfrak{g}^n \) vanishes for some \( n \in \mathbb{N} \) (see [7]). In particular, if \( \mathfrak{g} \) is a two-step nilpotent Lie superalgebra with a one-dimensional center, \( \mathfrak{g} \) is called a Heisenberg Lie superalgebra. All finite dimensional Heisenberg Lie superalgebras are divided into two classes, according to the parities of their centers, denoted by \( h_{2m,n} \) and \( ba_n \) (see [3]).

1. \( h_{2m,n} \) has a homogeneous basis

\[
\{ z; x_1, \ldots, x_m, x_{m+1}, \ldots, x_{2m} \mid y_1, \ldots, y_n \},
\]

and the non-zero brackets are given by

\[
[x_i, x_{m+i}] = [y_j, y_j] = z, \quad 1 \leq i \leq m \text{ and } 1 \leq j \leq n.
\]

2. \( ba_n \) has a homogeneous basis

\[
\{ x_1, \ldots, x_n \mid z; y_1, \ldots, y_n \},
\]

and the non-zero brackets are given by

\[
[x_i, y_i] = z, \quad 1 \leq i \leq n.
\]

Now, we introduce the definition of the cohomology of Lie superalgebras. For more details, the reader is referred to [9–11]. Denote by \( \mathfrak{g} \) a Lie superalgebra and

\[
\{ g_1, \ldots, g_m \mid g_{m+1}, \ldots, g_{m+n} \},
\]

an ordered basis of \( \mathfrak{g} \). For 1-cochains with trivial coefficients, the differential is defined as an operation dual to the Lie-bracket:

\[
d: \mathfrak{g}^* \longrightarrow \bigwedge^2 \mathfrak{g}^*,
\]

satisfying that

\[
d(g_i^*) = \sum_{1 \leq k < l \leq m+n} (-1)^{|g_k||g_l|+1} a_{kl} g_k^* \wedge g_l^* + \frac{1}{2} \sum_{m+1 \leq k \leq m+n} a_{kk} g_k^2, \quad 1 \leq i \leq m + n,
\]
where \( a_{ij}^k \), \( 1 \leq i, k, l \leq m + n \), are the structure constants of \( \mathfrak{g} \) with respect to the basis \([1,1]\). Suppose \( k \geq 2 \). For \( k \)-cochains with trivial coefficients, \( d \) is defined by the Leibniz rule. For cochains with coefficients in any module \( M \), we set
\[
d(m) = \sum_{1 \leq i \leq m+n} (-1)^{|g_i||m|}(g_i \cdot m) \otimes g_i^*,
\]
for any \( m \in M \), \( w \in \Lambda^g \). Note that \( M \otimes \Lambda^g \) is a \( \mathfrak{g} \)-module in a natural manner:
\[
g \cdot (m \otimes w) = (g \cdot m) \otimes w + (-1)^{|g||m|} m \otimes (g \cdot w),
\]
where \( g \in \mathfrak{g} \), \( m \in M \), and \( w \in \Lambda^g \). We obtain that \( d \) is a \( \mathfrak{g} \)-module homomorphism and \( d^2 = 0 \). Denote by \( H^\bullet(g, M) \) the cohomology of \( \mathfrak{g} \) with coefficients in \( M \) defined by the cochain complex \((M \otimes \Lambda^g, d)\). From Eq. (1.2), we have
\[
H^0(\mathfrak{g}, M) = \{ m \in M \mid g \cdot m = 0 \}.
\]
In particular, we have
\[
H^0(\mathfrak{g}, \mathfrak{g}) = C(\mathfrak{g}),
\]
where \( C(\mathfrak{g}) \) is the center of \( \mathfrak{g} \). As a result, we have
\[
H^0(\mathfrak{h}_{2m,n}, \mathfrak{h}_{2m,n}) = H^0(\mathfrak{ba}_n, \mathfrak{ba}_n) = F_z.
\]
A useful tool to compute the cohomology of Lie superalgebras is spectral sequences. Suppose \( I \) is an abelian ideal of \( \mathfrak{g} \). Then there is a convergent spectral sequence \( \{E^{p,q}_r, d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r \} \) called the Hochschild-Serre spectral sequence such that
\[
E^{k,s}_2 = H^k(\mathfrak{g}/I, H^s(I, M)) \Longrightarrow H^{k+s}(\mathfrak{g}, M),
\]
(see [10, Theorem 16.6.6]).

### 2 Main Results

Write \( \mathfrak{d}^k(r, s) \) for the dimension of \( k \)-homogeneous subspace of the super-exterior algebra generated by \( r \) even generators and \( s \) odd generators, that is,
\[
\mathfrak{d}^k(r, s) = \sum_{i=0}^{k} \binom{r}{k-i} \binom{s+i-1}{i}.
\]
Suppose that \( \mathfrak{g} \) is \( \mathfrak{h}_{2m,n} \) or \( \mathfrak{ba}_n \). For \( k \geq 0 \), set
\[
\psi^k_g : \bigwedge (\mathfrak{g}/Fz)^* \to \bigwedge (\mathfrak{g}/Fz)^*, \quad \alpha \mapsto \alpha \wedge (dz^*)_g,
\]
where
\[
(dz^*)_g = \begin{cases} 
- \sum_{i=1}^{m} x_i^* \wedge x_{m+i}^* + \frac{1}{2} \sum_{j=1}^{n} y_j^2, & \mathfrak{g} = \mathfrak{h}_{2m,n}; \\
- \sum_{i=1}^{n} x_i^* \wedge y_i^*, & \mathfrak{g} = \mathfrak{ba}_n.
\end{cases}
\]
From [9, Lemma 4.3] and the proofs of [9, Theorems 4.1 and 4.2], we obtain the following lemma:
Lemma 2.1. Suppose that $k \geq 0$ and $\mathfrak{g}$ is $\mathfrak{h}_{2m,n}$ or $\mathfrak{b}_{a,n}$. Then

$$\text{Ker } \psi_0^k = \left\{ \begin{array}{ll} 0, & \mathfrak{g} = \mathfrak{h}_{2m,n}; \\ \wedge^{k-2}(\mathfrak{b}_{a,n}/\mathbb{F} z)^* \wedge \mathbb{F}(dz^*)^\mathfrak{g} \oplus \delta_{k,n} \mathbb{F}(x_1^* \wedge \cdots \wedge x_n^*), & \mathfrak{g} = \mathfrak{b}_{a,n}. \end{array} \right.$$ 

Moreover,

$$\text{dim Ker } \psi_0^k = \left\{ \begin{array}{ll} 0, & \mathfrak{g} = \mathfrak{h}_{2m,n}; \\ \sum_{i=1}^{[\frac{k}{2}]} (-1)^{i-1}(\delta^{k-2i}(n,n) - \delta_{k-2i,n}) + \delta_{k,n}, & \mathfrak{g} = \mathfrak{b}_{a,n}. \end{array} \right.$$ 

By using the Hochschild-Serre spectral sequence to the centers, we establish the relations between the trivial cohomology and adjoint cohomology for Heisenberg Lie superalgebras. Let us first determine the Betti numbers of adjoint cohomology for Heisenberg Lie superalgebras of even center.

Theorem 1. Suppose that $k \geq 1$. Then

$$\text{dim } H^k(\mathfrak{h}_{2m,n}, \mathfrak{h}_{2m,n}) = (2m + n)(\delta^k(2m,n) - \delta^{k-2}(2m,n)) - \delta^{k+1}(2m,n) - \delta^{k-1}(2m,n).$$

Proof. From Eq. (1.5), we have

$$E_2^{k,s} = H^k(\mathfrak{h}_{2m,n}/\mathbb{F} z, H^s(\mathbb{F} z, \mathfrak{h}_{2m,n})) \Rightarrow H^{k+s}(\mathfrak{h}_{2m,n}, \mathfrak{h}_{2m,n}).$$

Note that the center $C(\mathfrak{h}_{2m,n}) = \mathbb{F} z$ and $|z| = \overline{z}$. For $k, s \geq 0$, we have

$$E_2^{k,s} = \left\{ \begin{array}{ll} H^k(\mathfrak{h}_{2m,n}/\mathbb{F} z, \mathfrak{h}_{2m,n}), & s = 0; \\ H^k(\mathfrak{h}_{2m,n}/\mathbb{F} z, \mathfrak{h}_{2m,n} \otimes \mathbb{F} z^*), & s = 1; \\ 0, & s \geq 2. \end{array} \right.$$ 

From the following two sequences

$$E_2^{k-2,1} \xrightarrow{d_2^{k-2,1}} E_2^{k,0} \xrightarrow{d_2^{k,0}} E_2^{k+2,-1},$$

and

$$E_2^{k-2,2} \xrightarrow{d_2^{k-2,2}} E_2^{k,1} \xrightarrow{d_2^{k,1}} E_2^{k+2,0},$$

we have

$$E_\infty^{k,s} = E_3^{k,s} = \left\{ \begin{array}{ll} H^k(\mathfrak{h}_{2m,n}/\mathbb{F} z, \mathfrak{h}_{2m,n})/\text{Im } d_2^{k-2,1}, & s = 0; \\ \text{Ker } d_2^{k,1}, & s = 1; \\ 0, & s \geq 2. \end{array} \right.$$ 

Moreover, we have

$$H^k(\mathfrak{h}_{2m,n}, \mathfrak{h}_{2m,n}) \cong H^k(\mathfrak{h}_{2m,n}/\mathbb{F} z, \mathfrak{h}_{2m,n})/\text{Im } d_2^{k-2,1} \bigoplus \text{Ker } d_2^{k-1,1}. \quad (2.1)$$
Set
\[ i_X = \begin{cases} i + m, & 1 \leq i \leq m; \\ i - m, & m + 1 \leq i \leq 2m, \end{cases} \quad \sigma(i) = \begin{cases} -1, & 1 \leq i \leq m; \\ 1, & m + 1 \leq i \leq 2m, \end{cases} \]
and \( \mathcal{V}_{h_{2m,n}} = \text{span}_F \{ x_1, \ldots, x_{2m} \mid y_1, \ldots, y_n \} \). Note that \( h_{2m,n}/Fz \) is abelian. From Eqs. (1.2)–(1.4), we have
\[ d((h_{2m,n}/Fz)^*) = d(z) = 0, \quad d(x_i) = \sigma(i) z \otimes x_i^*, \quad d(y_j) = -z \otimes y_j^*, \]
and
\[ (dz)^{h_{2m,n}} = -\sum_{i=1}^m x_i^* \wedge x_{m+i}^* + \frac{1}{2} \sum_{j=1}^n y_j^2, \]
and
\[ H^0(h_{2m,n}/Fz, h_{2m,n}) = Fz, \quad (2.2) \]
where \( 1 \leq i \leq 2m, 1 \leq j \leq n \). Moreover, for \( k \geq 1 \), we have
\[ d \left( \mathcal{V}_{2m,n} \bigotimes \bigwedge^{k-1} (h_{2m,n}/Fz)^* \right) = Fz \bigotimes \bigwedge^k (h_{2m,n}/Fz)^*, \quad (2.3) \]
and
\[ H^k(h_{2m,n}/Fz, h_{2m,n}) = \text{Ker} \, d \bigcap \left( \mathcal{V}_{h_{2m,n}} \bigotimes \bigwedge^k (h_{2m,n}/Fz)^* \right), \quad (2.4) \]
Now, we consider the linear mappings:
\[ d^{0,1}_2 : H^0(h_{2m,n}/Fz, h_{2m,n}) \bigotimes Fz^* \rightarrow H^2(h_{2m,n}/Fz, h_{2m,n}), \]
such that \( d^{0,1}_2 (z \otimes z^*) = z \otimes (dz)^{h_{2m,n}} \), and
\[ d^{k,1}_2 : H^k(h_{2m,n}/Fz, h_{2m,n}) \bigotimes Fz^* \rightarrow H^{k+2}(h_{2m,n}/Fz, h_{2m,n}), \]
\[ x \otimes z^* \mapsto x \wedge (dz)^{h_{2m,n}}, \]
where \( k \geq 1 \), and \( x \in H^k(h_{2m,n}/Fz, h_{2m,n}) \). From Eqs. (2.2) and (2.3), we have
\[ \text{Ker} \, d^{0,1}_2 = F(z \otimes z^*). \quad (2.5) \]
For \( k \geq 1 \), define the linear mapping
\[ \bar{d}^{k,1}_2 : \left( \mathcal{V}_{h_{2m,n}} \bigotimes \bigwedge^k (h_{2m,n}/Fz)^* \right) \bigotimes Fz^* \rightarrow \mathcal{V}_{h_{2m,n}} \bigotimes \bigwedge^{k+2} (h_{2m,n}/Fz)^*, \]
\[ x \otimes z^* \mapsto x \wedge (dz)^{h_{2m,n}}, \]
where \( x \in \mathcal{V}_{h_{2m,n}} \bigotimes \bigwedge^k (h_{2m,n}/Fz)^* \). Then, from Lemma 2.1 and Eq. (2.4),
\[ \text{Ker} \, d^{k,1}_2 \subseteq \text{Ker} \, \bar{d}^{k,1}_2 = 0. \quad (2.6) \]
Moreover, from Eqs. (2.1), (2.4)–(2.6), we have

\[
\dim H^k(h_{2m,n}, h_{2m,n}) = \begin{cases} 
\dim \operatorname{Ker} d \cap (V_{h_{2m,n}} \otimes (h_{2m,n}/Fz)^*) + 1, & k = 1; \\
\dim \operatorname{Ker} d \cap (V_{h_{2m,n}} \otimes \Lambda^2(h_{2m,n}/Fz)^*), & k = 2; \\
\dim \operatorname{Ker} d \cap (V_{h_{2m,n}} \otimes \Lambda^k(h_{2m,n}/Fz)^*) - \dim \operatorname{Ker} d \cap (V_{h_{2m,n}} \otimes \Lambda^{k-2}(h_{2m,n}/Fz)^*), & k \geq 3.
\end{cases}
\] (2.7)

Thus, for \(k \geq 1\), it is sufficient to consider the dimension of \(\operatorname{Ker} d \cap (V_{h_{2m,n}} \otimes \Lambda^k(h_{2m,n}/Fz)^*)\). For \(k \geq 1\), set

\[
\overline{d}_k : V_{h_{2m,n}} \otimes \Lambda^k(h_{2m,n}/Fz)^* \rightarrow Fz \otimes \Lambda^{k+1}(h_{2m,n}/Fz)^*,
\]

such that \(\overline{d}_k(x) = d(x), x \in V_{h_{2m,n}} \otimes \Lambda^k(h_{2m,n}/Fz)^*\). Then, from Eq. (2.8), \(\overline{d}_k\) is surjective. Moreover, we have

\[
\dim \operatorname{Ker} d \cap (V_{h_{2m,n}} \otimes \Lambda^k(h_{2m,n}/Fz)^*) = \dim \operatorname{Ker} \overline{d}_k = \dim V_{h_{2m,n}} \otimes \Lambda^k(h_{2m,n}/Fz)^*
\]

\[
- \dim Fz \otimes \Lambda^{k+1}(h_{2m,n}/Fz)^* = (2m + n)b^k(2m, n) - b^{k+1}(2m, n).
\]

From Eq. (2.7), the proof is complete. \(\square\)

For \(k, n \in \mathbb{Z}\), set

\[
a^k_n = \sum_{i=1}^{[\frac{k}{2}]} (-1)^{i-1}(\delta^{k-2i}(n, n) - \delta_{k-2i,n}) + \delta_{k,n}
\]

and

\[
b^k_n = \delta^k(n, n) - a^k_n.
\]

We are in position to determine the Betti numbers of adjoint cohomology for Heisenberg Lie superalgebras of odd center.

**Theorem 2.** Suppose that \(k \geq 1\). Then

\[
\dim H^k(ba_n, ba_n) = 2n\delta^k(n, n) - \delta^{k+1}(n, n) + 1 + \sum_{i=1}^{k-1}(2n\alpha^{k-i}_n - \beta^{k-i-1}_n - \delta_{k,n})
\]

\[
+ \sum_{i=0}^{k-3}(\delta^{k-i-1}(n, n) - \beta^{k-i-3}_n - 2n\beta^{k-i-2}_n).
\]
Proof. Set $\mathcal{V}_{ba_n} = \text{span}_{F}\{x_1, \ldots, x_n \mid y_1, \ldots, y_n\}$. From Eq. (1.5), we have

$$E^{k,s}_2 = H^k(ba_n/Fz, H^s(Fz, ba_n)) \Rightarrow H^{k+s}(ba_n, ba_n).$$

For $k \geq 0$, and $0 \leq i \leq k$ we have

$$E^{k-i,i}_2 = H^{k-i}(ba_n/Fz, ba_n) \bigotimes Fz^{s'i}.$$

From the following sequence

$$E^{k-i-2,i+1}_2 \xrightarrow{d^{k-i-2,i+1}_2} E^{k-i,i}_2 \xrightarrow{d^{k-i,i}_2} E^{k-i+2,i-1}_2,$$

we have

$$E^{k-i,i}_\infty = E^{k-i}_3 = \text{Ker } d^{k-i,i}_2 / \text{Im } d^{k-i-2,i+1}_2.$$

In particular,

$$\dim H^k(ba_n, ba_n) = \sum_{i=0}^{k} \left( \dim \text{Ker } d^{k-i,i}_2 + \dim \text{Ker } d^{k-i-2,i+1}_2 - \dim H^{k-i-2}(ba_n/Fz, ba_n) \right). \quad (2.8)$$

Thus, it is sufficient to consider the linear mapping:

$$d^{k-i,i}_2 : H^{k-i}(ba_n/Fz, ba_n) \bigotimes Fz^{s'i} \rightarrow H^{k-i+2}(ba_n/Fz, ba_n) \bigotimes Fz^{s'i-1},$$

$$x \otimes z^{s'i} \mapsto ix \wedge (dz^{s})_{ba_n} \otimes z^{s'i-1},$$

where $x \in H^{k-i}(ba_n/Fz, ba_n)$. Note that $ba_n/Fz$ is abelian. From Eqs. (1.2)–(1.4), we have

$$d((ba_n/Fz)^s) = d(z) = 0, \quad d(x_i) = -z \otimes y_i^s, \quad d(y_i) = z \otimes x_i^s, \quad (dz^s)_{ba_n} = -\sum_{i=1}^{n} x_i^s \wedge y_i^s,$$

and

$$H^0(ba_n/Fz, ba_n) = Fz, \quad (2.9)$$

where $1 \leq i \leq n$. Moreover, for $k \geq 1$, we have

$$d \left( \mathcal{V}_{ba_n} \bigotimes \bigwedge^{k-1}(ba_n/Fz)^s \right) = Fz \bigotimes \bigwedge^{k}(ba_n/Fz)^s, \quad (2.10)$$

and

$$H^k(ba_n/Fz, ba_n) = \text{Ker } d \bigcap \left( \mathcal{V}_{ba_n} \bigotimes \bigwedge^{k}(ba_n/Fz)^s \right). \quad (2.11)$$

Moreover, from Eqs. (2.9) and (2.11),

$$\dim H^k(ba_n/Fz, ba_n) = \left\{ \begin{array}{ll} 1, & k = 0; \\ \dim \text{Ker } d \bigcap \left( \mathcal{V}_{ba_n} \bigotimes \bigwedge^{k}(ba_n/Fz)^s \right), & k \geq 1. \end{array} \right. \quad (2.12)$$
In order to compute the dimension of $\text{Ker } d^{k-i,i}_2$, we define the linear mapping
\[
d_{2}^{k-i,i}: \bigotimes \left( \bigwedge^{k-i} (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* \right) \otimes \mathbb{F}z^{i} \rightarrow \bigotimes \left( \bigwedge^{k-i+2} (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* \right) \otimes \mathbb{F}z^{i-1},
\]
where $k \geq 1$, $1 \leq i \leq k - 1$ and $x \in \mathcal{V}_{\mathfrak{b}_{\mathfrak{n}}} \otimes \bigwedge^{k-i} (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^*$. Set $X_n^* = x_1^* \wedge \cdots \wedge x_n^*$. From Lemma 2.1, we have
\[
\dim \text{Ker } d^{k-i,i}_2 = \mathcal{V}_{\mathfrak{b}_{\mathfrak{n}}} \bigotimes \left( \bigwedge^{k-i-2} (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* \bigwedge \mathbb{F}(dz^*)_{\mathfrak{b}_{\mathfrak{n}}} \bigoplus \delta_{k-i,n} X_n^* \right) \bigotimes \mathbb{F}z^i.
\]
Then, for $k \geq 1$,
\[
\begin{align*}
\text{Ker } d^{k-i,i}_2 &= \begin{cases} 
\dim \text{Ker } d \bigcap \left( \mathcal{V}_{\mathfrak{b}_{\mathfrak{n}}} \bigotimes \bigwedge^k (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* \right), &i = 0; \\
\dim \text{Ker } d \bigcap \tilde{\mathcal{V}}_{k,i} \bigotimes \mathbb{F}z^i, &1 \leq i \leq k - 1; \\
\mathbb{F}(z \otimes z^i), &i = k.
\end{cases}
\end{align*}
\]
In particular,
\[
\dim \text{Ker } d^{k-i,i}_2 = \begin{cases} 
\dim \text{Ker } d \bigcap \left( \mathcal{V}_{\mathfrak{b}_{\mathfrak{n}}} \bigotimes \bigwedge^k (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* \right), &i = 0; \\
\dim \text{Ker } d \bigcap \tilde{\mathcal{V}}_{k,i}, &1 \leq i \leq k - 1; \\
1, &i = k.
\end{cases}
\tag{2.13}
\]
To compute the dimensions of $\text{Ker } d \bigcap \left( \mathcal{V}_{\mathfrak{b}_{\mathfrak{n}}} \bigotimes \bigwedge^k (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* \right)$ and $\left( \text{Ker } d \bigcap \tilde{\mathcal{V}}_{k,i} \right)$, $k \geq 1$, $1 \leq i \leq k - 1$, we set
\[
\tilde{d}_k : \mathcal{V}_{\mathfrak{b}_{\mathfrak{n}}} \bigotimes \bigwedge^k (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* \rightarrow \mathbb{F}z \bigotimes \bigwedge^{k+1} (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^*,
\]
such that $\tilde{d}_k(x) = d(x)$, $x \in \mathcal{V}_{\mathfrak{b}_{\mathfrak{n}}} \bigotimes \bigwedge^k (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^*$, and
\[
\tilde{d}_{k,i} : \tilde{\mathcal{V}}_{k,i} \rightarrow \mathbb{F}z \bigotimes \left( \bigwedge^{k-i-1} (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* \bigwedge \mathbb{F}(dz^*)_{\mathfrak{b}_{\mathfrak{n}}} \right),
\]
such that $\tilde{d}_{k,i}(x) = d(x)$, $x \in \tilde{\mathcal{V}}_{k,i}$. Then, from Eq. (2.10), $\tilde{d}_k$ and $\tilde{d}_{k,i}$ are surjective. Moreover, from Lemma 2.1, we have
\[
\dim \text{Ker } d \bigcap \left( \mathcal{V}_{\mathfrak{b}_{\mathfrak{n}}} \bigotimes \bigwedge^k (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* \right) = \dim \text{Ker } \tilde{d}_k
\]
\[
= \dim \mathcal{V}_{\mathfrak{b}_{\mathfrak{n}}} \bigotimes \bigwedge^k (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^* - \dim \mathbb{F}z \bigotimes \bigwedge^{k+1} (\mathfrak{b}_{\mathfrak{n}}/\mathbb{F}z)^*
\]
\[
= 2n \delta^k(n, n) - \delta^{k+1}(n, n),
\]
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\[ \dim \ker d \cap \tilde{V}_{k,i} = \dim \ker \tilde{d}_{k,i} \]
\[ = \dim \tilde{V}_{k,i} - \dim \bigwedge^{k-i-1}(\mathfrak{b}_n/Fz)^* \bigwedge \mathbb{F}(dz^*)_{\mathfrak{b}_n} \]
\[ = 2n \dim \ker \psi_{\mathfrak{b}_n}^{k-i} - \mathfrak{d}^{k-i-1}(n, n) + \dim \ker \psi_{\mathfrak{b}_n}^{k-i-1}. \]

Moreover, from Lemma 2.1, Eqs. (2.8), (2.12) and (2.13), the proof is complete. \( \square \)

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