A REMARK ON THE GENERIC VANISHING OF KOSZUL COHOMOLOGY

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Abstract. We give a sufficient condition to study the vanishing of certain Koszul cohomology groups for general pairs \((X, L) \in W_{g,d}^r\) by induction. As an application, we show that to prove the Maximal Rank Conjecture (for quadrics), it suffices to check all cases with the Brill-Noether number \(\rho = 0\).

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Introduction

Let \(L\) be a base point free \(g^r_d\) on a smooth curve \(X\), the Koszul cohomology group \(K_{p,q}(X, L)\) is the cohomology of the Koszul complex at \((p, q)\)-spot

\[
\begin{array}{c}
\wedge^{p+1}H^0(L) \otimes H^0(L^{q-1}) \\
\xrightarrow{d_{p+1,q-1}} \wedge^p H^0(L) \otimes H^0(L^q) \\
\xrightarrow{d_{p,q}} \wedge^{p-1}H^0(L) \otimes H^0(L^{q+1})
\end{array}
\]

where

\[
d_{p,q}(v_1 \wedge \ldots \wedge v_p \otimes \sigma) = \sum_i (-1)^i v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_p \otimes v_i \sigma.
\]

Koszul cohomology groups \(K_{p,q}(X, L)\) completely determine the shape of a minimal free resolution of the section ring

\[
R = R(X, L) = \bigoplus_{k \geq 0} H^0(X, L^k).
\]

and therefore carry enormous amount of information of the extrinsic geometry of \(X\).

In this paper, we are interested in Green’s question [10].

Question 0.1. What do the \(K_{p,q}(X, L)\) look like for \((X, L)\) general in \(W_{g,d}^r\) (i.e. \(X\) is a general curve of genus \(g\) and \(L\) is a general \(g^r_d\) on \(X\))?

The following facts are well known (c.f. [10], [12]) for general \((X, L) \in W_{g,d}^r\).

1. We have the following picture of \(k_{p,q} = \dim_K K_{p,q}(X, L)\) (The numbers \(k_{p,q}\) in the table are undetermined.):
Table 1.

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 0 | $h^1(L)$ | 0 | ... | ... | 0 |
| 0 | $\rho$ | $k_{r-2,2}$ | ... | ... | $k_{2,2}$ |
| 0 | $k_{r-1,1}$ | $k_{r-2,1}$ | ... | ... | $k_{2,1}$ |
| 0 | 0 | 0 | ... | ... | 0 |

(2)

$$k_{p,1} - k_{p-1,2} = \chi(\text{Koszul complex})$$

$$= \binom{r+1}{p}(g-d+r) - \binom{r+1}{p+1}g + \binom{r-1}{p}d + \binom{r}{p+1}(g-1).$$

Question 0.1 seems to be too difficult to answer in its full generality. For the case $p = 1$, the Maximal Rank Conjecture (MRC) \cite{8} predicts that the multiplication map

$$\text{Sym}^2 H^0(X, L) \xrightarrow{\mu} H^0(X, L^2)$$

is either injective or surjective, or equivalently

$$\min\{k_{1,1}, k_{0,2}\} = 0.$$

Geometrically, this means that the number of quadrics in $\mathbb{P}^r := \mathbb{P}(H^0(L))$ containing $X$ is as simple as the Hilbert function of $X \subset \mathbb{P}^r$ allows.

There are many partial results about the MRC using the so-called “méthode d’Horace” originally proposed by Hirschowitz. We refer to, for instance, \cite{5}, \cite{6} for some recent results in this direction.

For higher syzygies, again there are many results (c.f. \cite{1}, \cite{2}, \cite{4}, \cite{7}, and \cite{9}). One breakthrough result is Voisin’s solution to the generic Green’s conjecture \cite{13} \cite{14}, which answers Question 0.1 for the case $L = K_X$.

Definition 0.2. For $1 \leq p \leq r - 1$, we say property $\text{GV}(p)^r_{g,d}$ holds if for general $(X, L) \in \mathbb{W}^r_{g,d}$,

$$\min\{k_{p,1}(X, L), k_{p-1,2}(X, L)\} = 0.$$

Remark 0.3. The MRC implies that property $\text{GV}(1)^r_{g,d}$ always holds provided the Brill-Noether number $\rho := g - (r+1)(g-d+r) \geq 0$. However, property $\text{GV}(p)^r_{g,d}$ does not always hold for $p \geq 2$ (c.f Green \cite{10} (4.a.2) for more details).

In this note, we give a sufficient condition (Theorem 1.5) for $\text{GV}(p)^r_{g,d}$ to imply $\text{GV}(p)^r_{g+1,d+1}$. One could use this to set up an inductive argument for the generic vanishing of Koszul cohomology groups. In each step of the induction, $r$ is fixed and $g$, $d$ go up by 1.

In the case $p = 1$, this sufficient condition turns out to be an surprisingly simple geometric condition on the quadrics containing the first secant variety $\Sigma_1(X)$ of $X$ (Lemma 2.1). We manage to verify this geometric condition and prove

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\footnote{In this paper, we will restrict ourselves to only consider quadrics containing $X$.}
Theorem 0.4. The property $\Gamma V(1)^r_{g,d}$ implies $\Gamma V(1)^r_{g+1,d+1}$.

Based on our knowledge about the base cases of the induction, we have

Theorem 0.5. The Maximal Rank Conjecture holds for a general pair $(X, L) \in W_{g,d}$, if $h^1(L) \leq 2$.

An interesting question remaining is that for $p \geq 2$, whether the sufficient condition in Theorem 0.5 has anything to do with higher syzygies of $\Sigma_1(X)$.

1. Koszul cohomology on a singular curve

Throughout this section, let $X = Y \cup Z$ be the union of a smooth curve $Y$ of genus $g$ and $Z = \mathbb{P}^1$ meeting at two general points $u$ and $v$. Consider a line bundle $L$ (up to $\mathbb{C}^*$-action) on $X$ such that $A := L|_Y$ is a $g^r_d$ and $L|_Z = O_{\mathbb{P}^1}(1)$. Note that by construction, every section in $H^0(Y, A)$ extends uniquely to a section in $H^0(X, L)$. Thus we have an isomorphism induced by restriction to $Y$:

\[(1.1) \quad H^0(X, L) \cong H^0(Y, A).\]

Proposition 1.1. Notation as above, if $K_{p,1}(Y, A) = 0$, then $K_{p,1}(X, L) = 0$.

Proof. Consider the following commutative diagram

\[
\begin{array}{cccccc}
\wedge^{p+1} H^0(L) & \longrightarrow & \wedge^p H^0(L) \otimes H^0(L) & \longrightarrow & \wedge^{p-1} H^0(L) \otimes H^0(L^2) \\
\wedge^{p+1} H^0(A) & \longrightarrow & \wedge^p H^0(A) \otimes H^0(A) & \longrightarrow & \wedge^{p-1} H^0(A) \otimes H^0(A^2)
\end{array}
\]

where the vertical arrows are restriction maps to $Y$. The hypothesis says that the second row is exact in the middle, a simple diagram chasing gives the conclusion. \qed

Remark 1.2. The argument in Proposition 1.1 does not generalize to the case $q = 2$ because $H^0(Y, A^2)$ is not isomorphic to $H^0(X, L^2)$.

To study the relation between $K_{p-1,2}(X, L)$ and $K_{p-1,2}(Y, A)$, we use the duality relation [3 p. 21]

\[K_{p-1,2}(Y, A)^{\vee} \cong K_{r-p,0}(Y, A; K_Y)\]

and compare $K_{r-p,0}(Y, A; K_Y)$ with $K_{r-p,0}(X, L; \omega_X)$. Here $\omega_X$ is the dualizing sheaf of $X$. Its restriction $\omega_X|_Y \cong K_Y(p+q)$ and $\omega_X|_Z \cong O_{\mathbb{P}^1}$. One checks easily that restriction to $Y$ induces the following isomorphisms:

\[H^0(X, \omega_X) \cong H^0(Y, K_Y(u + v)),\]
\[H^0(X, \omega_X \otimes L^{-1}) \cong H^0(Y, K_Y \otimes A^{-1}),\]
\[H^0(X, \omega_X \otimes L) \cong H^0(Y, K_Y \otimes A(u + v)).\]

Denote $M_A$ the kernel bundle associated to a globally generated line bundle $A$, defined by the exact sequence

\[0 \to M_A \to H^0(Y, A) \otimes O_Y \overset{ev}{\longrightarrow} A \to 0.\]
Taking \((r - p)\)-th wedge product, we obtain
\[
0 \to \bigwedge^{r-p}M_A \to \bigwedge^{r-p}H^0(M) \otimes \mathcal{O}_Y \to \bigwedge^{r-p-1}M_A \otimes A \to 0.
\]
Tensoring the above sequence with \(K_Y\), we obtain an isomorphism \cite[Section 2.1]{[3]}
\[
H^0(\bigwedge^{r-p}M_A \otimes K_Y) \cong \text{Ker}(\delta_0 : \bigwedge^{r-p}H^0(A) \otimes H^0(K_Y) \to \bigwedge^{r-p-1}H^0(A) \otimes H^0(K_Y \otimes A)),
\]
and therefore,
\[
(1.5) \quad K_{r-p,0}(Y, A; K_Y) \cong \frac{H^0(\bigwedge^{r-p}M_A \otimes K_Y)}{\bigwedge^{r-p+1}H^0(A) \otimes H^0(K_Y \otimes A^{-1})}.
\]

**Proposition 1.3.** We have an isomorphism
\[
K_{r-p,0}(X, L; \omega_X) \cong \frac{H^0(\bigwedge^{r-p}M_A \otimes K_Y(u + v))}{\bigwedge^{r-p+1}H^0(A) \otimes H^0(K_Y \otimes A^{-1})}.
\]

**Proof.** Consider the following diagram
\[
\begin{array}{c}
\bigwedge^{r-p+1}H^0(L) \otimes H^0(\omega_X \otimes L^{-1}) \\
\xrightarrow{d_-} \bigwedge^{r-p}H^0(L) \otimes H^0(\omega_X) \\
\xrightarrow{d_0} \bigwedge^{r-p-1}H^0(L) \otimes H^0(\omega_X \otimes L) \\
\end{array}
\]
\[
\xrightarrow{\cong} \\
\xrightarrow{\cong}
\begin{array}{c}
\bigwedge^{r-p+1}H^0(A) \otimes H^0(K_Y \otimes A^{-1}) \\
\xrightarrow{\delta_0} \bigwedge^{r-p}H^0(A) \otimes H^0(K_Y(u + v)) \\
\xrightarrow{\delta_0} \bigwedge^{r-p-1}H^0(A) \otimes H^0(K_Y \otimes A(u + v)),
\end{array}
\]
where the vertical arrows are induced by restriction to \(Y\). By definition, \(K_{r-p,0}(X, L; \omega_X)\) is the cohomology in the middle of the first row. By Equations (1.4) to (1.3), all three vertical arrows are isomorphisms, thus
\[
\text{Ker}(\delta_0) \cong \text{Ker}(\delta_0) \cong H^0(\bigwedge^{r-p}M_A \otimes K_Y(u + v)).
\]
and the statement follows immediately.

**Corollary 1.4.** Notation as above, if
\[
h^0(\bigwedge^{r-p}M_A \otimes K_Y) = h^0(\bigwedge^{r-p}M_A \otimes K_Y(u + v)),
\]
or equivalently,
\[
h^0((\bigwedge^pM_A \otimes A)(-u - v)) = h^0(\bigwedge^pM_A \otimes A) - 2 \binom{r}{p},
\]
then
\[
K_{r-p,0}(X, L; \omega_X) \cong K_{r-p,0}(Y, A; K_Y).
\]

**Proof.** Immediate. The equivalence of the two assumptions followed from Riemann-Roch and the fact that \(\bigwedge^{r-p}M_A' \cong \bigwedge^pM_A \otimes A\).

By degenerating to the pair \((X, L)\), we obtain

**Theorem 1.5.** Suppose a general pair \((Y, A)\) in \(\mathcal{W}_{g,d}^r\) satisfies one of the two conditions:
\[
(1) \quad K_{p,1}(Y, A) = 0;
(2) \quad K_{p-1,2}(Y, A) = 0 \text{ and the vector bundle } \bigwedge^pM_A \otimes A \text{ satisfies } (1.6) \text{ for some } u, v \in Y.
\]
Then the property \(\text{GV}(p)_{g+1, d+1}^r\) holds.
2. The case $p = 1$

In the case $p = 1$, Equation (1.6) has a very geometric interpretation.

**Lemma 2.1.** For a pair $Y \xrightarrow{\phi} \mathbb{P}$ in $\mathcal{W}^r_{g,d}$, the vector bundle $M_A \otimes A$ satisfies equation (1.6) for some $u, v \in Y$ if and only if there exists a quadric hypersurface $Q \subset \mathbb{P}^r$ containing $Y$ but not containing its first secant variety $\Sigma_1(Y)$.

**Proof.** ($\Leftarrow$) The "$\geq"$ direction of (1.6) is automatically satisfied. For the other direction, consider the following diagram with exact rows

\[
\begin{array}{ccccccc}
0 & \to & H^0(M_A \otimes A(-u - v)) & \to & H^0(A) \otimes H^0(A(-u - v)) & \to & H^0(A^2(-u - v)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(M_A \otimes A) & \to & H^0(A) \otimes H^0(A) & \to & H^0(A^2).
\end{array}
\]

We need to show

$$\dim_\mathbb{C} \operatorname{Ker}(\mu') \leq \dim_\mathbb{C} \operatorname{Ker}(\mu) - 2r.$$  

Denote $H_{u,v} := H^0(A) \otimes H^0(A(-u - v))$ and $\overline{H}_{u,v}$ be its image in \[\frac{H^0(A) \otimes H^0(A)}{\wedge^2 H^0(A)} \cong S^2 H^0(A).\]

Note that

$$\overline{H}_{u,v} \cong \frac{H_{u,v}}{H_{u,v} \cap \wedge^2 H^0(A)},$$

is the space of quadrics which contain the secant line $\overline{uv}$. Thus

$$\dim_\mathbb{C} \overline{H}_{u,v} = \binom{r + 2}{2} - 3.$$

We claim that

$$H_{u,v} \cap \wedge^2 H^0(A) = \wedge^2 H^0(A(-u - v)).$$

This is because

\[
\begin{align*}
\dim_\mathbb{C} H_{u,v} \cap \wedge^2 H^0(A) &= \dim_\mathbb{C} H_{u,v} - \dim_\mathbb{C} \overline{H}_{u,v} \\
&= (r + 1)(r - 1) - \left[ \binom{r + 2}{2} - 3 \right] = \binom{r - 1}{2} \\
&= \dim_\mathbb{C} \wedge^2 H^0(A(-u - v)).
\end{align*}
\]

The claim is proved.

By hypothesis, $\overline{\operatorname{Ker}(\mu)} \notin \overline{H}_{u,v}$ for some $u, v \in Y$ (since $Q \notin \overline{H}_{u,v}$), then it follows that

$$\dim_\mathbb{C} (\operatorname{Ker}(\mu')) = \dim_\mathbb{C} (\operatorname{Ker}(\mu) \cap \overline{H}_{u,v}) \leq \dim_\mathbb{C} (\operatorname{Ker}(\mu) \cap \overline{H}_{u,v}) \leq \dim_\mathbb{C} (\operatorname{Ker}(\mu)) - 1 =: m - 1.$$
Thus
\[\dim_{\mathbb{C}}(\text{Ker}(\mu')) \leq m - 1 + \dim_{\mathbb{C}}(\wedge^2 H^0(A) \cap H_{u,v})\]
\[= m - 1 + \dim_{\mathbb{C}}(\wedge^2 H^0(A(-u-v)))\]
\[= m - 1 + \binom{r-1}{2} = m + \binom{r+1}{2} - 2r\]
\[= \dim_{\mathbb{C}}(\text{Ker}(\mu)) - 2r.\]
(\(\Rightarrow\)) Reverse the above argument we get the “only if” part.

\(\Box\)

**Lemma 2.2.** Suppose \(Y \hookrightarrow \mathbb{P}^r\) is a nondegenerate curve in \(\mathbb{P}^r\), then there does not exist any quadric hypersurface \(Q\) containing \(\Sigma_1(Y)\).

**Proof.** Suppose \(Y \subset \Sigma_1(Y) \subset Q\) for some quadric \(Q\). Fix a point \(u \in Y\), since \(Q\) contains \(\Sigma_1(Y)\), \(Q\) must contain the variety \(J(u,Y)\) of lines joining \(u\) and \(Y\). Thus the quadric \(Q\) is singular at \(u\). (If \(Q\) is smooth at \(u\), a secant line \(uw\) \(\subset Q\) if and only if \(uw\) \(\subset T_u Q\). Choose \(w \in Y \setminus T_u Q\), we have \(uw \subset J(u,Y) \subset \Sigma_1(Y)\) but \(uw \notin Q\).) Since \(u\) is chosen arbitrarily, we conclude that \(Q\) is singular along \(Y\). This is absurd since the singular locus of a quadric is a linear subspace in \(\mathbb{P}^r\) which cannot contain the nondegenerate curve \(Y\). \(\Box\)

**Proof. of Theorem 0.4** Follows immediately from Theorem 1.5 and Lemmas 2.1, 2.2

### 3. Applications to the Maximal Rank Conjecture

As an application of Theorem 0.4 we obtain a proof of Theorem 0.5.

We say a triple \((g, r, d)\), or equivalently \((g, r, h^1 = g - d + r)\) is a base case for the MRC if the Brill-Noether number \(\rho := g - (r + 1)h^1 = 0\).

**Theorem 3.1.** If the MRC holds for all \(\rho = 0\) cases, then it holds for arbitrary \(\rho \geq 0\) case.

**Proof.** Apply Theorem 0.4 and induction. Start with any base case, for which we assume property \(\Gamma V(1)_{g,d}^r\) holds. In each step of the induction, \(r\) and \(h^1\) is fixed and \(g\) (equivalently \(\rho\) or \(d\)) goes up by 1. \(\Box\)

The MRC for the base cases are known to be true when \(h^1 \leq 2\). According to the value of \(h^1\), we have the following.

1. \(h^1 = 0\). We have \(g = 0\) and \(d = r \geq 1\), i.e. \((Y, A) = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))\). The rational normal curves are projectively normal.
2. \(h^1 = 1\). In this case, \(g = r + 1, d = 2r\), i.e \((Y, A) = (Y, K_Y)\). By Nother’s theorem, canonical curves are projectively normal \((r \geq 2)\).
3. \(h^1 = 2\). Then \(g = 2r + 2, d = 3r\). Such pairs \((Y, A)\) are projectively normal for \(r \geq 4\) is the main result of [11] (The MRC is easy to check when \(r = 2\) or 3).

Farkas [9] also proved that \(\Gamma V(1)_{s(2s+1),2s(s+1)}^{2s}\) holds for any \(s \geq 1\). This covers the base cases when \(h^1 = s\) and \(r = 2s\).
Proof. of Theorem 0.5  Follows immediately from the base cases with $h^1 \leq 2$ and Theorem 3.1

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