REGULARITY ON ABELIAN VARIETIES III: FURTHER APPLICATIONS

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1. INTRODUCTION

Recently we have developed a regularity theory for coherent sheaves on abelian varieties, called M-regularity (cf. [PP1], [PP2]). It is a technique geared (at the moment) mainly towards solving geometric problems related to linear series or equations for (sub-varieties of) abelian varieties. The main ingredients are the derived category theoretic context of the Fourier-Mukai functor and the systematic use of cohomological techniques. We refer the reader to the above mentioned papers for full details. The more restricted purpose of the present sequel is to describe a number of new applications of this theory in several different directions in the study of abelian varieties and irregular varieties, some of which we have announced in the previous papers.

We start in §2 by giving an overview of the general context of M-regularity. This essentially amounts to recalling some basic definitions and results from [PP1] which will be used in the subsequent sections.

In §3 we address a very familiar problem in the context of effective results for linear series, namely that of bounding the Seshadri constant measuring the local positivity of an ample line bundle. We refer the reader to the upcoming book [La2] for a comprehensive survey of the main results in this area. A considerable body of work on this problem has developed in the context of abelian varieties, where the Seshadri constants turn out to have interesting connections with metric or arithmetic invariants (cf. [La1], [Nak], [Ba1], [Ba2], [De] and also [La2] for further references). Here we explain how the Seshadri constant of a polarization L on an abelian variety is bounded below by an asymptotic version – and
in particular by the usual \( M \)-regularity index of the line bundle \( L \), as defined in [PP2]. This is the content of Theorem 3.4. The proof is a simple application of the \( M \)-regularity criterion of [PP1], via the techniques of [PP2] §3. Combining this with various bounds for Seshadri constants proved in [La1], we obtain bounds for \( M \)-regularity indices which are not apparent otherwise. A problem of more interest – at least historically – and for which we do not have an answer at this stage, is to produce uniform bounds on Seshadri constants by bounding directly the \( M \)-regularity indices and applying the result we prove here. We raise a few questions in this direction.

In §4 we shift our attention towards a cohomological study of Picard bundles. These are vector bundles on the Jacobian of a curve \( C \), whose projectivization is the symmetric product \( C_n \), for large enough \( n \) – in other words they parametrize all linear series of degree \( n \) on the curve \( C \). Picard bundles have been the focus of intensive study, especially since they are closely related to Brill-Noether theory; cf. [La2] 6.3.C and 7.2.C for a general discussion and the corresponding literature. (It seems in fact that the very definition of the Fourier functor was given by Mukai in [M1] in part with the aim of studying Picard bundles.) We combine Fourier-Mukai techniques with the use of the Eagon-Northcott resolution for special determinantal varieties in order to compute their (strong) Theta regularity. In down to earth terms this amounts to the following: it is known that all Picard bundles are negative (i.e. have ample dual bundle). However, we show that as soon as we twist them with the smallest possible polarizations, namely the theta divisor and all its translates, their higher cohomology vanishes. The same holds for all their relatively small tensor powers (cf. Theorem 4.2). This vanishing theorem has numerous practical applications, as does basically any nontrivial statement on Picard bundles. In particular we recover in a more direct fashion the main results of [PP1] §4 on the equations of the \( W_d \)'s in Jacobians, and on vanishing for pull-backs of pluritheta line bundles to symmetric products. We end the section with a new result on the equations of \( \text{Sing}(\Theta) \) on non-hyperelliptic jacobians.

In §5 we approach the problem of giving effective results for linear series on irregular varieties of maximal Albanese dimension via \( M \)-regularity for direct images of canonical (or adjoint) bundles, extending work in [PP1] §5. The main result is a theorem to the effect that, on a smooth projective minimal variety \( Y \) of general type whose Albanese map is generically finite and whose Albanese image is not ruled by subtori, the pluricanonical map given by \( |\omega_Y^3| \) is very ample outside the locus of non-finite fibers. A result of this type was also proved by Chen and Hacon [CH], under a slightly more general hypothesis, but with a less explicit conclusion. The relevance of such statements, especially the fact that the hypothesis is not too restrictive, comes from results of Green, Ein and Lazarsfeld [GL], [EL] in the context of generic vanishing theorems, explained in §5. As an example [EL], Theorem 3, says that for any variety of maximal Albanese dimension, \( \chi(\omega_Y) = 0 \) implies that the Albanese image of \( Y \) is ruled by tori. We note that at least part of this is purely conceptually explained by the notion of \( M \)-regularity. We also observe that results similar to the theorem stated above can be obtained for higher order jets, and especially also for pluri-adjoint linear series on any variety of maximal Albanese dimension.

In §6 we concentrate on the study of higher rank vector bundles on abelian varieties. By work of Mukai and others ([M3], [M4], [M1], [Um] and [Or]) it has emerged that on abelian varieties the class of vector bundles most closely resembling semistable vector bundles on curves and line bundles on abelian varieties is that of \textit{semihomogeneous} vector
bundles. A vector bundle \( E \) is semihomogeneous if every translation \( t^*_x E \) by an element in \( X \) is isomorphic to a twist \( E \otimes \alpha \) by a line bundle \( \alpha \in \text{Pic}^0(X) \). It turns out that these bundles are semistable, behave nicely under isogenies and Fourier-Mukai transforms, and have a Mumford type theta-group theory as in the case of line bundles. All of these suggest that there should exist numerical criteria for their geometric properties like global or normal generation. We show here that this is indeed the case, and the measure is precisely the Theta regularity of the bundles in question. More generally, we give a result on the surjectivity of the multiplication map on global sections for two such vector bundles (cf. Theorem 6.13). Basic examples are the projective normality of ample line bundles on any abelian variety, and the normal generation of the Verlinde bundles on the Jacobian of a curve, coming from moduli spaces of vector bundles on that curve. Note again that, although this was part of our initial motivation, we do not have to appeal to the theta-group theory of semihomogeneous bundles.

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2. M-regularity for coherent sheaves

In this section we recall the main definitions and results from [PP1]. Let \( X \) be an abelian variety of dimension \( g \) over an algebraically closed field. Given a coherent sheaf \( \mathcal{F} \) on \( X \), we denote \( S^i(\mathcal{F}) := \text{Supp}(R^i\hat{\mathcal{F}}) \). The sheaf \( \mathcal{F} \) on \( X \) is called \( M \)-regular if \( \text{codim}(S^i(\mathcal{F})) > i \) for any \( i = 1, \ldots, g \) (where, for \( i = g \), this means that \( S^g(\mathcal{F}) \) is empty). If \( \text{Supp}(R^i\hat{\mathcal{F}}) = \emptyset \), this condition is trivially verified, and the sheaf is said to satisfy the Index Theorem (I.T.) with index 0. By the base change theorem, this is equivalent to saying that \( H^i(\mathcal{F} \otimes \alpha) = 0 \) for all \( \alpha \in \text{Pic}^0(X) \) and all \( i > 0 \). Finally, an extremely useful concept in the context of irregular varieties is the following:

**Definition 2.1.** ([PP1], Definition 2.10) Let \( Y \) be an irregular variety. We define a sheaf \( \mathcal{F} \) on \( Y \) to be continuously globally generated if for any non-empty open subset \( U \subset \text{Pic}^0(Y) \) the sum of evaluation maps

\[
\bigoplus_{\alpha \in U} H^0(X, \mathcal{F} \otimes \alpha) \otimes \alpha^Y \longrightarrow \mathcal{F}
\]

is surjective.

**Theorem 2.2.** (M-regularity criterion, [PP1] Theorem 2.4 and Proposition 2.13.) Let \( \mathcal{F} \) be an \( M \)-regular sheaf on \( X \), possibly supported on a subvariety \( Y \) of \( X \). Then the following hold:

(a) \( \mathcal{F} \) is continuously globally generated.

(b) Let also \( A \) be a line bundle on \( Y \), continuously globally generated as a sheaf on \( X \). Then \( \mathcal{F} \otimes A \) is globally generated.

**Theorem 2.3.** ([PP1] Theorem 2.5) Let \( \mathcal{F} \) and \( H \) be sheaves on \( X \) such that \( \mathcal{F} \) is \( M \)-regular and \( H \) is locally free satisfying I.T. with index 0. Then, for any non-empty Zariski open set \( U \subset X \), the map

\[
\mathcal{M}_U : \bigoplus_{\xi \in U} H^0(X, \mathcal{F} \otimes P_\xi) \otimes H^0(X, H \otimes P_\xi^\vee) \longrightarrow H^0(X, \mathcal{F} \otimes H)
\]

is surjective, where $m_\xi$ denote the multiplication maps on global sections.

**Definition 2.4.** A coherent sheaf $\mathcal{F}$ on a polarized abelian variety $(X, \Theta)$ is called $m$-$\Theta$-regular if $\mathcal{F}((m-1)\Theta)$ is $M$-regular. We will call it strongly $m$-$\Theta$-regular if $\mathcal{F}((m-1)\Theta)$ satisfies I.T. with index 0.

We recall the "abelian" Castelnuovo-Mumford Lemma, which is in fact a consequence of the two results above.

**Theorem 2.5.** ([PP1] Theorem 6.3) Let $\mathcal{F}$ be a $0$-$\Theta$-regular coherent sheaf on $X$. Then:
(1) $\mathcal{F}$ is globally generated.
(2) $\mathcal{F}$ is $m$-$\Theta$-regular for any $m \geq 1$.
(3) The multiplication map
$$H^0(\mathcal{F}(\Theta)) \otimes H^0(O(k\Theta)) \longrightarrow H^0(\mathcal{F}((k+1)\Theta))$$
is surjective for any $k \geq 2$.

3. $M$-regularity indices and Seshadri constants

Here we express a natural relationship between Seshadri constants of ample line bundles on abelian varieties and the $M$-regularity indices of those line bundles as defined in [PP2]. This result improves the lower bound for Seshadri constants proved in [Nak], and combined with the results of [La1] provides bounds for controlling $M$-regularity. For a general overview of Seshadri constants – and in particular the statements used below – one can consult [La2] Ch.I §5.

We start by recalling the basic definition from [PP2] and by also looking at a slight variation. We will denote by $X$ an abelian variety of dimension $g$ over an algebraically closed field and by $L$ an ample line bundle on $X$.

**Definition 3.1.** The $M$-regularity index of $L$ is defined as
$$m(L) := \max\{l \mid L \otimes m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_p}^{k_p} is \ M\text{-regular for all distinct } x_1, \ldots, x_p \in X \text{ with } \Sigma k_i = l\}.$$

**Definition 3.2.** We can also define a related invariant, associated this time to just one given point $x \in X$:
$$p(L, x) := \max\{l \mid L \otimes m_x^l is \ M\text{-regular}\}.$$The definition does not depend on $x$ because of the homogeneity of $X$, and so we will denote this invariant simply by $p(L)$.

Our main interest will be in the asymptotic versions of these indices, which turn out to be related to the Seshadri constant associated to $L$.

**Definition 3.3.** The asymptotic $M$-regularity index of $L$ and its punctual counterpart are defined as
$$\rho(L) := \sup_n \frac{m(L^n)}{n}$$
and
$$\rho'(L) := \sup_n \frac{p(L^n)}{n}.$$
The main result of this section is:

**Theorem 3.4.** We have the following inequalities:

\[ \epsilon(L) = \rho'(L) \geq \rho(L) \geq 1. \]

In particular \( \epsilon(L) \geq \max\{m(L), 1\} \).

This improves a result of Nakamaye (cf. [Nak] and the references therein). Nakamaye also shows that \( \epsilon(L) = 1 \) for some line bundle \( L \) if and only if \( X \) is the product of an elliptic curve with another abelian variety, so then a similar result holds for the invariant \( \rho'(L) \). As explained in [PP2] §3, the value of \( m(L) \) is reflected in the geometry of the map to projective space given by \( L \). Here is a basic example:

**Example 3.5.** If \( L \) is very ample – or more generally gives a birational morphism outside a codimension 2 subset – then \( m(L) \geq 2 \), and so by the theorem above \( \epsilon(L) \geq 2 \). Note that on an arbitrary smooth projective variety very ampleness implies in general only that \( \epsilon(L, x) \geq 1 \) at each point.

The proof of Theorem 3.4 is a simple application of the \( M \)-regularity criterion 2.2, via the results of [PP2] §3. To understand the growth of the usual invariants we use the relationship with the notions of \( k \)-jet ampleness and separation of jets. Namely let’s denote by \( s(L, x) \) the largest number \( s \geq 0 \) such that \( L \) separates \( s \)-jets at \( x \). Recall also the following:

**Definition 3.6.** A line bundle \( L \) is called \( k \)-jet ample, \( k \geq 0 \), if the restriction map

\[ H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_X/m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_p}^{k_p}) \]

is surjective for any distinct points \( x_1, \ldots, x_p \) on \( X \) such that \( \sum k_i = k + 1 \).

Note that if \( L \) is \( k \)-jet ample, then it separates \( k \)-jets at every point. Recall from [PP2] Theorem 3.8 and Proposition 3.5 the following facts:

**Proposition 3.7.**

(i) \( L^n \) is \( (n + m(L) - 2) \)-jet ample, so in particular \( s(L^n, x) \geq n + m(L) - 2 \).

(ii) If \( L \) is \( k \)-jet ample, then \( m(L) \geq k + 1 \).

This immediately points in the direction of local positivity, since one way to interpret the Seshadri constant of \( L \) is (independently of \( x \)):

\[ \epsilon(L) = \limsup_n \frac{s(L^n, x)}{n} = \sup_n \frac{s(L^n, x)}{n}. \]

The last equality follows from the fact that jet-separation satisfies the well-known super-additivity relation \( s(L_1 \otimes L_2, x) \geq s(L_1, x) + s(L_2, x) \) for any two line bundles \( L_1 \) and \( L_2 \) on \( X \). To establish the connection with the asymptotic invariants above we also need the following

**Lemma 3.8.** For any \( n \geq 1 \) and any \( x \in X \) we have \( s(L^{n+1}, x) \geq m(L^n) \).

**Proof.** This follows immediately from the \( M \)-regularity criterion Theorem 2.2: if \( L^n \otimes m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_p}^{k_p} \) is \( M \)-regular, then \( L^{n+1} \otimes m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_p}^{k_p} \) is globally generated, and so by [PP2] Lemma 3.3, \( L^{n+1} \) is \( m(L) \)-jet ample. \( \square \)
Proof. (of Theorem 3.4.) Note first that for every $p \geq 1$ we have
\begin{equation}
m(L^n) \geq m(L) + n - 1,
\end{equation}
which follows immediately from the two parts of Proposition 3.7. In particular $m(L^n)$ is always at least $n - 1$, and so $\rho(L) \geq 1$.

Putting together the definitions, (1) and Lemma 3.8, we obtain the main inequality
\[ \epsilon(L^n) \geq \rho(L). \]
Finally, and less surprisingly, we see equally quickly that the asymptotic punctual index computes precisely the Seshadri constant. Indeed, by completely similar arguments as above, we have that for any ample line bundle $L$ and any $p \geq 1$ one has
\[ p(L^n) \geq s(L^n, x) \text{ and } s(L^{n+1}, x) \geq p(L^n, x). \]
The statement follows then from the definition. \qed

Remark 3.9. What the proof above shows is that one can give an interpretation for $\rho(L)$ similar to that for $\epsilon(L)$ in terms of separation of jets. In fact $\rho(L)$ is precisely the "asymptotic jet ampleness" of $L$ (stronger then jet separation), namely:
\[ \rho(L) = \sup_n \frac{a(L^n)}{n}, \]
where $a(M)$ is the largest integer $k$ for which the line bundle $M$ is $k$-jet ample.

In this respect, an interesting – though admittedly quite optimistic – question is whether the asymptotic $M$-regularity index computes precisely the Seshadri constant:

Question 3.10. Do we always have $\epsilon(L) = \rho(L)$?

Remark 3.11. Another, rather surprising, lower bound for the Seshadri constant of a polarization on an abelian variety has been given by Lazarsfeld in [La1]. This is expressed in terms of a metric invariant defined by Buser and Sarnak. In comparison, Lazarsfeld’s result can be made effective based on the Buser-Sarnak result giving a lower bound for that particular invariant (cf. loc. cit.). For the same reason, it would be of considerable interest to find an independent lower bound for the asymptotic invariant $\rho(L)$.

Question 3.12. Can one give independent lower bounds for $\rho(L)$ or $\rho'(L)$?

It may be possible to do this for generic abelian varieties by constructing specific examples. In the other direction, there are numerous bounds on Sesahdri constants, which in turn give bounds for the $M$-regularity indices that (at least to us) are not obvious from the definition. Essentially each of the results listed in [La2] Ch.I §5 gives some sort of bound. Let’s just give a couple of examples:

Corollary 3.13. If $(J(C), \Theta)$ is a Jacobian with the usual principal polarization, then $m(n\Theta) \leq \sqrt{g} \cdot n$. On an arbitrary abelian variety, for any principal polarization $\Theta$ we have $m(n\Theta) \leq (g!)^{\frac{1}{3}} \cdot n$.

Proof. It is shown in [La1] that $\epsilon(\Theta) \leq \sqrt{g}$. We then apply Theorem 3.4. For the other bound we use the usual elementary upper bound for Seshadri constants, namely $\epsilon(\Theta) \leq (g!)^{\frac{1}{3}}$. \qed

Corollary 3.14. If $(A, \Theta)$ is a very general PPAV, then there exists at least one $n$ such that $p(n\Theta) \geq 2^\frac{1}{3}(g!)^{\frac{1}{3}} \cdot n$.
Proof. Here we use the lower bound given in [La1] via the Buser-Sarnak result.

There are of course more specific results on $\epsilon(\Theta)$ for Jacobians (cf. [De] Theorem 7), each giving a corresponding result for $m(n\Theta)$. It would be more satisfactory to have a concrete answer in this case, but note that in this generality the problem should be quite difficult, since it would also answer conjectures about the Seshadri constant (for example the fact that $\epsilon(\Theta) < 2$ characterizes hyperelliptic Jacobians).

Question 3.15. Can we understand $m(n\Theta)$ individually on Jacobians, at least for small $n$, in terms of the geometry of the curve?

As a simple example, the question above has a clear answer for elliptic curves. We know that on an elliptic curve $E$ a line bundle $L$ is $M$-regular if and only if $\deg(L) \geq 1$, i.e. if and only if $L$ is ample. From the definition of $M$-regularity we see then that if $\deg(L) = d > 0$, then $m(L) = d - 1$. This implies that on an elliptic curve $m(n\Theta) = n - 1$ for all $n \geq 1$. However, this is misleading when we look at the case of curves of higher genus. In fact in the simplest case we have the following general:

Proposition 3.16. If $(X, \Theta)$ is an irreducible principally polarized abelian variety of dimension at least 2, then $m(2\Theta) \geq 2$.

Proof. This is an immediate consequence of the existence of the Kummer map. The linear series $|2\Theta|$ induces a $2:1$ map of $X$ onto its image in $\mathbb{P}^{2g-1}$, with injective differential. Thus the cohomological support locus for $O(2\Theta) \otimes m_x \otimes m_y$ consists of two points, while the one for $O(2\Theta) \otimes m_x^2$ is empty.

4. Regularity of Picard bundles and vanishing on symmetric products

In this section we study the regularity of Picard bundles over the Jacobian of a curve, in other words we give a quantitative estimate for their positivity with respect to the natural polarization. Our study is not a direct consequence of $M$-regularity, but integrates nicely in the context of its strong version called Theta regularity (cf. §2). The point is that one can prove vanishing results by combining the Fourier-Mukai transform with classical resolution type methods for determinantal varieties, involving in particular the Eagon-Northcott complex.

Let $C$ be a smooth curve of genus $g \geq 2$, and denote by $J(C)$ the Jacobian of $C$, and by $C_n$ the $n$-th symmetric product of $C$. The objects we are interested in are the Picard bundles on $J(C)$: for a given $n \geq 2g - 1$, a Picard bundle is loosely speaking a vector bundle over $\text{Pic}^n(C)$ whose projectivization is $C_n$, so that the projectivizations of its fibers parametrize all linear series of degree $n$ on $C$ (cf. [ACGH] Ch.VII §2). We will look at such a bundle $E$ on $J(C)$, via translating by a line bundle $L \in \text{Pic}^n(C)$, so we make the convention that the vector bundle fiber of $E$ at a point $\xi \in J(C)$ is $H^0(L \otimes \xi)$. We will somewhat abusively call this the $n$-th Picard bundle of $C$.

Proposition 4.1. For any $k \geq 1$, let $\pi_k : C^k \to J(C)$ a desymmetrized Abel-Jacobi mapping and let $L$ be a line bundle on $C$ of degree $n > 0$ as above. Then $\pi_k \ast (L \boxtimes \ldots \boxtimes L)$ satisfies I.T. with index 0, and

$$(\pi_k \ast (L \boxtimes \ldots \boxtimes L)) \approx \otimes^k E,$$
where $E$ is the $n$-th Picard bundle of $C$.

Proof. This is a generalization of the well-known fact (cf. [M1] §4) that the Picard bundle $E$ is the Fourier transform of $i_*L$, where $i$ is an Abel-Jacobi embedding of $C$ in the Jacobian, and we only briefly sketch the proof. Indeed, the positivity of $L$ insures the fact that $\pi_{k*}(L \boxtimes \ldots \boxtimes L)$ satisfies I.T. with index $0$, as the fibers of the second projection to the dual Jacobian have no higher cohomology (this can be easily seen using the Künneth formula). This implies that the fibers of the Fourier transform are naturally isomorphic to $H^0(C, L \otimes \xi)^{\otimes k}$, which characterizes $\otimes^k E$. □

The following theorem is the main cohomological result we are aiming for. The point to keep in mind is that Picard bundles are known to be negative (i.e with ample dual bundle), so vanishing theorems are not automatic. We give an effective range for achieving the strongest vanishing one can hope for with the smallest possible "positive" perturbation. To be very precise, everything that follows holds if $n$ is assumed to be at least $4g - 4$. (However the value of $n$ does not affect the applications.)

**Theorem 4.2.** For every $1 \leq k \leq g - 1$, $\otimes^k E$ is strongly $2$-$\Theta$-regular.

Proof. 1 We will use loosely the notation $\Theta$ for any translate of the canonical theta divisor. The statement of the theorem becomes then equivalent to the vanishing

$$h^i(\otimes^k E \otimes \mathcal{O}(\Theta)) = 0, \ \forall \ i > 0, \ \forall \ 1 \leq k \leq g - 1.$$

To prove this vanishing we use the Fourier-Mukai transform. The first point is that Proposition 4.1 above, combined with Mukai’s main duality theorem [M1] Theorem 2.2, tells us precisely that $\otimes^k E$ satisfies W.I.T. with index $g$, and its Fourier transform is

$$\hat{\otimes^k E} = (-1)^j \pi_{k*}(L \boxtimes \ldots \boxtimes L).$$

The next point is that the cohomology groups involved can be computed on the dual Jacobian via the Fourier transform. We have the following sequence of isomorphisms:

$$H^i(\otimes^k E \otimes \mathcal{O}(\Theta)) \cong \text{Ext}^i(\mathcal{O}(\Theta), \otimes^k E) \cong \text{Ext}^i(\mathcal{O}(\Theta), \otimes^k E)$$

$$\cong \text{Ext}^i(\mathcal{O}(\Theta), (-1)^j \pi_{k*}(L \boxtimes \ldots \boxtimes L)) \cong H^i((-1)^j \pi_{k*}(L \boxtimes \ldots \boxtimes L) \otimes \mathcal{O}(\Theta)).$$

Here we are using the correspondence between the Ext groups given in [M1] Corollary 2.5, plus the fact that both $\mathcal{O}(\Theta)$ and $\otimes^k E$ satisfy W.I.T. with index $g$ and that $\hat{\mathcal{O}(\Theta)} = \hat{\mathcal{O}(\Theta)} = \mathcal{O}(\Theta)$.

As we are loosely writing $\Theta$ for any translate, multiplication by $-1$ does not influence the vanishing, so the result follows if we show:

$$h^i(\pi_{k*}(L \boxtimes \ldots \boxtimes L) \otimes \mathcal{O}(\Theta)) = 0, \ \forall \ i > 0.$$

Now the image $W_k$ of the Abel-Jacobi map $u_k : C_k \to J(C)$ has rational singularities (cf. [Ke2]), so we only need to prove the vanishing:

$$h^i(u_k^* (\pi_{k*}(L \boxtimes \ldots \boxtimes L) \otimes \mathcal{O}(\Theta))) = 0, \ \forall \ i > 0.$$

We are grateful to Olivier Debarre for pointing out a numerical mistake in the statement, in a previous version of this paper.
Thus we are interested in the skew-symmetric part of the cohomology group $H^i(C^k, (L \boxtimes \ldots \boxtimes L) \otimes \pi^*_k \mathcal{O}(-\Theta))$, or, by Serre duality that of

$$H^i(C^k, ((\omega_C \otimes L^{-1}) \boxtimes \ldots \boxtimes (\omega_C \otimes L^{-1})) \otimes \pi^*_k \mathcal{O}(\Theta)),$$

for $i < k$.

At this stage we can essentially invoke a Serre vanishing type argument, but it is worth noting that the computation can be in fact made very concrete. For the identifications used next we refer to [Iz] Appendix 3.1. As $k \leq g - 1$, we can write

$$\pi^*_k \mathcal{O}(\Theta) \cong (((\omega_C \otimes A^{-1}) \boxtimes \ldots \boxtimes (\omega_C \otimes A^{-1})) \otimes \mathcal{O}(-\Delta)),$$

where $\Delta$ is the union of all the diagonal divisors in $C^k$ and $A$ is a line bundle of degree $g - k - 1$. Then the skew-symmetric part of the cohomology groups we are looking at is isomorphic to

$$S^i H^1(C, \omega_C \otimes A^{-1} \otimes L^{-1}) \otimes \wedge^{k-i} H^0(C, \omega_C \otimes A^{-1} \otimes L^{-1}),$$

and since for $1 \leq k \leq g - 1$ and $n \geq 4g - 4$ the degree of the line bundle $\omega_C \otimes A^{-1} \otimes L^{-1}$ is negative, this vanishes precisely for $i < k$.  

An interesting consequence of the vanishing result for Picard bundles proved above is a new – and in some sense more classical – way to deduce Theorem 4.1 of [PP1] on the regularity of the ideal sheaves $\mathcal{I}_{W_d}$ on the Jacobian $J(C)$. This theorem has a number of nice consequences on the equations of the $W_d$’s – in particular on those of the curve $C$ – inside $J(C)$, and also to some useful vanishing results for pull-backs of theta divisors to symmetric products. For this circle of ideas we refer the reader to [PP1] §4.

For any $1 \leq d \leq g - 1$, consider $u_d : C_d \to J(C)$ to be an Abel-Jacobi mapping of the symmetric product (depending on the choice of a line bundle of degree $d$ on $C$), and denote by $W_d$ the image of $u_d$ in $J(C)$.

**Theorem 4.3.** For every $1 \leq d \leq g - 1$, the ideal sheaf $\mathcal{I}_{W_d}$ is strongly $3$-$\Theta$-regular.

**Proof.** We have to prove that:

$$h^i(\mathcal{I}_{W_d} \otimes \mathcal{O}(2\Theta) \otimes \alpha) = 0, \forall i > 0, \forall \alpha \in \text{Pic}^0(J(C)).$$

In the rest of the proof, by $\Theta$ we will understand generically any translate of the canonical theta divisor, and so $\alpha$ will disappear from the notation.

It is well known that $W_d$ has a natural determinantal structure, and its ideal is resolved by an Eagon-Northcott complex. We will chase the vanishing along this complex. This setup is precisely the one used by Fulton and Lazarsfeld in order to prove for example the existence theorem in Brill-Noether theory, and for explicit details on this circle of ideas we refer to [ACGH] Ch.VII §2.

Concretely, $W_d$ is the “highest” degeneracy locus of a map of vector bundles

$$\gamma : E \to F,$$

where $\text{rk} F = m$ and $\text{rk} E = n = m + d - g + 1$, with $m >> 0$ arbitrary. The bundles $E$ and $F$ are well understood: $E$ is the $n$-th Picard bundle of $C$, discussed above, and $F$ is a direct sum of topologically trivial line bundles. (For simplicity we are again moving the
whole construction on $J(C)$ via the choice of a line bundle of degree $n$. In other words, $W_d$ is scheme theoretically the locus where the dual map

$$\gamma^* : F^* \longrightarrow E^*$$

fails to be surjective. This locus is resolved by an Eagon-Northcott complex (cf. [Ke1]) of the form:

$$0 \rightarrow \wedge^m F^* \otimes S^{m-n} E \otimes \det E \rightarrow \ldots \rightarrow \wedge^{n+1} F^* \otimes E \otimes \det E \rightarrow \wedge^n F^* \rightarrow I_{W_d} \rightarrow 0.$$ 

As it is known that the determinant of $E$ is nothing but $O(-\Theta)$, and since $F$ is a direct sum of topologically trivial line bundles, the statement of the theorem follows by chopping this into short exact sequences, as long as we prove:

$$h^i(S^k E \otimes O(\Theta)) = 0, \quad \forall i > 0, \ \forall 1 \leq k \leq m - n = g - d - 1.$$ 

Since we are in characteristic zero, $S^k E$ is naturally a direct summand in $\otimes^k E$, and so it is sufficient to prove that:

$$h^i(\otimes^k E \otimes O(\Theta)) = 0, \quad \forall i > 0, \ \forall 1 \leq k \leq g - d - 1.$$ 

But this follows from Theorem 4.2.

**Remark 4.4.** Using [PP1] Proposition 2.9, we have a strong version of the "abelian" Castelnuovo-Mumford Lemma [PP1] Theorem 6.3, namely a strongly $m$-$\Theta$-regular sheaf is strongly $k$-$\Theta$-regular for every $k \geq m$. Thus Theorem 4.2 and Theorem 4.3 imply that $\otimes^k E$ is strongly $k$-$\Theta$-regular for every $k \geq 2$ and $I_{W_d}$ is strongly $k$-$\Theta$-regular for every $k \geq 3$.

**Question 4.5.** An interesting question, extending the result above, is the following: what is the $\Theta$-regularity of the ideal of an arbitrary Brill-Noether locus $W_d^r$?

We describe below one case in which the answer can already be given, namely that of the singular locus of the Riemann theta divisor on a non-hyperelliptic jacobian. It should be noted that in this case we do not have strong 3-$\Theta$-regularity any more (but rather only strong 4-$\Theta$-regularity, by the same [PP1] Proposition 2.9).

**Proposition 4.6.** The ideal sheaf $I_{W_{g-1}^1}$ is 3-$\Theta$-regular.

**Proof.** In fact it follows from the results of [vGI] that

$$h^i(I_{W_{g-1}^1} \otimes O(2\Theta) \otimes \alpha) = \begin{cases} 0 & \text{for } i \geq g - 2, \ \forall \alpha \in \text{Pic}^0(J(C)) \\ 0 & \text{for } 0 < i < g - 2, \ \forall \alpha \in \text{Pic}^0(J(C)) \text{ such that } \alpha \neq O_{J(C)}. \end{cases}$$

For the reader’s convenience, let us briefly recall the relevant points from Section 7 of [vGI]. We denote for simplicity, via translation, $\Theta = W_{g-1}$, (so that $W_{g-1}^1 = \text{Sing}(\Theta)$). In the first place, from the exact sequence

$$0 \rightarrow O(2\Theta) \otimes \alpha \otimes O(-\Theta) \rightarrow I_{W_{g-1}^1}(2\Theta) \otimes \alpha \rightarrow I_{W_{g-1}^1/\Theta}(2\Theta) \otimes \alpha \rightarrow 0$$

it follows that

$$h^i(J(C), I_{W_{g-1}^1}(2\Theta) \otimes \alpha) = h^i(\Theta, I_{W_{g-1}^1/\Theta}(2\Theta) \otimes \alpha) \text{ for } i > 0.$$ 

Hence one is reduced to a computation on $\Theta$. It is a standard fact (see e.g. [vGI], 7.2) that, via the Abel-Jacobi map $u = u_{g-1} : C_{g-1} \rightarrow \Theta \subset J(C)$,

$$h^i(\Theta, I_{W_{g-1}^1/\Theta}(2\Theta) \otimes \alpha) = h^i(C_{g-1}, L^{\otimes 2} \otimes \beta \otimes I_Z),$$

where $I_Z$ is the ideal sheaf of $Z$ in $J(C)$. 


where $Z = u^{-1}(W_{g-1}^1)$, $L = u^*\mathcal{O}_X(\Theta)$ and $\beta = u^*\alpha$. We now use the standard exact sequence ([ACGH], p.258):

$$0 \to T_{g-1} \xrightarrow{du} H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_{C_{g-1}} \to L \otimes I_Z \to 0.$$ 

Tensoring with $L \otimes \beta$, we see that it is sufficient to prove that

$$H^i(C_{g-1}, T_{C_{g-1}} \otimes L \otimes \beta) = 0, \forall i \geq 2, \forall \beta \neq \mathcal{O}_{C_{g-1}}.$$ 

To this end we use the well known fact (cf. loc. cit.) that $T_{C_{g-1}} \cong p_*\mathcal{O}_D(D)$ where $D \subset C_{g-1} \times C$ is the universal divisor and $p$ is the projection onto the first factor. As $p_{|D}$ is finite, the degeneration of the Leray spectral sequence and the projection formula ensure that

$$h^i(C_{g-1}, T_{C_{g-1}} \otimes L \otimes \beta) = h^i(D, \mathcal{O}_D(D) \otimes p^*(L \otimes \beta)),$$

which are zero for $i \geq 2$ and $\beta$ non-trivial by [vGI], Lemma 7.24. \qed

5. **Pluricanonical maps of irregular varieties of maximal Albanese dimension**

It is well known that the minimal pluricanonical embedding working for all smooth curves of general type is the tricanonical one. It turns out that this very classical result has a generalization to arbitrary dimension. In fact Chen and Hacon ([CH], Theorem 4.4) recently proved that the tricanonical map of a smooth complex irregular variety of general type $Y$, having generically finite Albanese map and such that $\chi(\omega_Y) > 0$, is birational onto its image. The main point of this section is that the concept of M-regularity, combined with well-known results of Ein, Green and Lazarsfeld, provides – under mildly more restrictive hypotheses – a very quick and conceptually simple proof of a more explicit version of this statement, as well as of other related facts. To put things into perspective, let us recall that, by a theorem of Ein, Lazarsfeld and Green ([EL], Theorem 3), partly conjectured by Kollár, given a smooth variety of general type $Y$ with generically finite Albanese map, then $\chi(\omega_Y) \geq 0$, and if equality holds then the Albanese image of $Y$ is ruled by tori. We show the following:

**Theorem 5.1.** Let $Y$ be a smooth projective complex minimal variety of general type such that its Albanese map $a : Y \to \text{Alb}(Y)$ is generically finite. If $a(Y)$ is not ruled by tori, then $\omega_Y^{-3}$ is very ample on the open set $a^{-1}(T)$, where $T$ is the open set of points of $a(Y)$ over which the fiber of $a$ is finite.

In preparation for the proof, let us settle some preliminary results. First we introduce a slight generalization of Definition 2.1.

**Definition 5.2.** Let $a : Y \to X$ be a map from a projective variety $Y$ to an abelian variety $X$ and let $\mathcal{F}$ be a coherent sheaf on $Y$. Let also $Z \subset Y$ be a closed subset. We say that $\mathcal{F}$ is continuously globally generated away from $Z$ (with respect to the map $a$) if, for any Zariski open set $U \subset \text{Pic}^0(X)$, $Z$ does not meet the support of the cokernel of the evaluation map

$$\text{ev}_U : \bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes a^* \alpha) \otimes a^* \alpha^* \to \mathcal{F}.$$
With this terminology, we have the following trivial generalization of Theorem 2.2(b) (the proof is the same).

**Proposition 5.3.** Let $\mathcal{F}$ and $\mathcal{L}$ be respectively a coherent sheaf and an invertible sheaf on $Y$. If they are both continuously globally generated away from $Z$ (with respect to $a$) then $\mathcal{F} \otimes \mathcal{L} \otimes a^*\alpha$ is globally generated away from $Z$ for any $\alpha \in \text{Pic}^0(X)$.

From this point on, $Y$ will be a smooth projective variety, the map $a$ will be the Albanese map of $Y$, and the subset $Z$ will be the inverse image, via $a$, of the locus of points of $a(Y)$ having non-finite fiber. The key point is

**Lemma 5.4.** If $\dim Y = \dim a(Y)$ and $a(Y)$ is not ruled by tori then:

(i) $a_*\omega_Y$ is an $M$-regular sheaf on $X = \text{Alb}(Y)$;
(ii) $a_*\omega_Y$ is continuously globally generated;
(iii) $\omega_Y$ is continuously globally generated away from $Z$;
(iv) $\omega_Y^{\otimes 2} \otimes a^*\alpha$ is globally generated away from $Z$ for any $\alpha \in \text{Pic}^0(X)$.

**Proof.** (i) Given a coherent sheaf $\mathcal{F}$ on $Y$, let us denote

\[ V_i(\mathcal{F}) = \{ \alpha \in \text{Pic}^0(X) \mid h^i(\mathcal{F} \otimes a^*\alpha) > 0 \}, \]

the $i$-th cohomological support locus of $\mathcal{F}$. As we are assuming that $a$ is generically finite onto its image, by the Generic Vanishing Theorem of [GL] we have that $\text{codim} V_i(\omega_Y) \geq i$ for all $i$. Moreover, by a well-known argument of [EL] (end of the proof of Theorem 3), if $a(X)$ is not ruled by tori then $\text{codim} V_i(\omega_Y) > i$ for all $i \geq 1$. Since, by Grauert-Riemenschneider vanishing, $R^ia_*\omega_Y = 0$ for any $i \neq 0$, the projection formula gives

\[ H^i(Y, \omega_Y \otimes a^*\alpha) \cong H^i(X, (a_*\omega_Y) \otimes \alpha). \]

This implies that $\text{codim} V_i(a_*\omega_X) > i$ for all $i > 0$, which implies by base change that $a_*\omega_X$ is $M$-regular.

(ii) This follows from (i) via the $M$-regularity criterion Theorem 2.2.

(iii) It is immediate to see that, as with global generation, continuous global generation is preserved by finite maps. We then apply (ii).

(iv) This follows from (iii) and the previous Proposition. \qed

**Remark 5.5.** The concept of $M$-regularity in the case of canonical bundles on varieties of maximal Albanese dimension interprets very conceptually some of the results of [EL]. It is in fact easy to see, as noted in [PP1], that a non-zero $M$-regular sheaf has global sections. Thus by the definition, if $a_*\omega_Y$ is $M$-regular, then $\chi(\omega_Y) > 0$. So Lemma 5.4 reinterprets the result of Ein and Lazarsfeld mentioned above, using of course half of their argument, employed in the proof of 5.4 (i).

**Corollary 5.6.** ([EL], Theorem 3) If $Y$ is a smooth projective variety of maximal Albanese dimension with $\chi(\omega_Y) = 0$, then the Albanese image $a(Y)$ is ruled by tori (since $\omega_Y$ is not $M$-regular).

**Proof.** (of Theorem 5.1) The statement is equivalent to the fact that, for any $y \in Y - Z$, the sheaf $I_y \otimes \omega_Y^{\otimes 3}$ is globally generated away from $Z$. Consider such a $y$. As in deducing (iv) in the previous Lemma, via the $M$-regularity criterion the statement will follow from the fact that the sheaf $a_*(I_y \otimes \omega_Y^{\otimes 2})$ is $M$-regular, which is what we will prove.
To this end, note that by Lemma 5.4 (iv), $\omega_Y^2 \otimes a^* \alpha$ is globally generated at $y$. Since, by Kawamata-Viehweg vanishing, $H^i(Y, \omega_Y^2 \otimes a^* \alpha) = 0$ for any $i > 0$ (here we are using the hypothesis that $\omega_Y$ is big and nef), it follows easily that $H^i(I_y \otimes \omega_Y^2 \otimes a^* \alpha) = 0$, for any $\alpha \in \text{Pic}^0(X)$. On the other hand we have that $R^ia_*(I_y \otimes \omega_Y^2) = 0$ for any $i > 0$. This follows from the standard exact sequence

$$0 \to I_y \otimes \omega_Y^2 \to \omega_Y^2 \to \omega_Y^2 |_y \to 0,$$

using the fact that, by a well-known generalization of Grauert-Riemenschneider vanishing ([KM], Corollary 2.8), $R^ia_*(\omega_Y^2) = 0$ for any $i > 0$. Therefore, by the projection formula, $a_*(I_y \otimes \omega_Y^2)$ satisfies I.T. with index 0, so in particular it is M-regular. This proves the Theorem.

The following generalization of Theorem 5.1 to higher order jets is proved the same way (cf. also [PP2] §3).

**Theorem 5.7.** Let $Y$ be a smooth projective complex minimal variety of general type such that its Albanese map $a : Y \to \text{Alb}(Y)$ is generically finite. If $a(Y)$ is not ruled by tori, then, for any $k \geq 0$, $\omega_Y^{\otimes k+2}$ is $k$-jet ample on the open set $a^{-1}(T)$, where $T$ is the open set of points of $a(Y)$ over which the fiber of $a$ is finite.

To complete the picture, we recall the following general result, indicated in [PP1], Remark 5.3, and partly obtained independently by Chen and Hacon (compare e.g. [CH], Corollaries 3.3 and 4.3). We omit the proof, which is very similar to the previous one.

**Theorem 5.8.** Let $Y$ be a smooth projective complex variety such that its Albanese map $a : Y \to \text{Alb}(Y)$ is generically finite, and let $L$ be a big and nef line bundle on $Y$. Then, for any $k \geq 0$, $(\omega_Y \otimes L)^{\otimes k+2}$ is $k$-jet ample on the open set $a^{-1}(T)$, where $T$ is the open set of points of $a(Y)$ over which the fiber of $a$ is finite. For instance, if $Y$ is minimal of general type, then $\omega_Y^{\otimes 6}$ is very ample on the open set $a^{-1}(T)$.

6. Numerical study of semihomogeneous vector bundles

An idea that originated in work of Mukai is that on abelian varieties the class of vector bundles to which the theory of line bundles should generalize naturally is that of semihomogeneous bundles (cf. [M1], [M3], [M4]). These vector bundles are semistable, behave nicely under isogenies and Fourier transforms, and have a Mumford type theta group theory as in the case of line bundles (cf. [Um]). The purpose of this section is to show that this analogy can be extended to include effective global generation and normal generation statements dictated by specific numerical invariants measuring positivity.

The guiding problem is the following: given a semihomogeneous bundle $E$ on an abelian variety $X$, such that $h^0(E) \neq 0$, find sufficient "positivity" conditions on $E$ which ensure that $E$ is globally or normally generated. Recall that the later is Mumford’s terminology for the surjectivity of the multiplication map $H^0(E) \otimes H^0(E) \to H^0(E^{\otimes 2})$ – the model to keep in mind is the global generation of $\mathcal{O}(2\Theta)$ and the projective normality of $\mathcal{O}_X(3\Theta)$ for any ample divisor $\Theta$ on $X$. We would like to integrate this into the general regularity theory of [PP1], since $\Theta$-regularity as described in §2 is precisely a measure of
the (cohomological) positivity of $E$. As a minimal requirement we have to ask that $E$ be 0-$\Theta$-regular (recall that this means that $E(-\Theta)$ is $M$-regular). Note that this automatically implies $h^0(E) \neq 0$.

**Basics on semihomogeneous bundles.** Let $X$ be an abelian variety of dimension $g$ over an algebraically closed field. As a general convention, for a numerical class $\alpha$ we will use the notation $\alpha > 0$ to express the fact that $\alpha$ is ample. If the class is represented by an effective divisor, then the condition of being ample is equivalent to $\alpha^g > 0$.

Let $L$ be a line bundle on $X$. We denote by $\phi_L$ the isogeny defined by $L$: 

$$
\phi_L : X \rightarrow \operatorname{Pic}^0(X) \cong \hat{X},
$$

$$
x \mapsto t_x^* L \otimes L^{-1}.
$$

Note that if $L$ is a principal polarization, $\phi_L$ is just a self-duality of $X$.

**Definition 6.1.** ([M3]) A vector bundle $E$ on $X$ is called *semihomogeneous* if for every $x \in X$, $t_x^* E \cong E \otimes \alpha$, for some $\alpha \in \operatorname{Pic}^0(X)$.

It is a general principle, described later in this subsection, that the study of arbitrary semihomogeneous bundles can be reduced to that of *simple* semihomogeneous ones, i.e. those with no nontrivial endomorphisms. We will use a few basic properties of simple semihomogeneous bundles which can be found in [M3]. In the next lemmas $E$ is a simple semihomogeneous of rank $r$ on $X$.

**Lemma 6.2.** ([M3] Proposition 7.3) There exists an isogeny $\pi : Y \rightarrow X$ and a line bundle $M$ on $Y$ such that

$$
\pi^* E \cong \bigoplus_r M.
$$

**Lemma 6.3.** ([M3] Theorem 5.8(iv)) There exists an isogeny $\phi : Z \rightarrow X$ and a line bundle $L$ on $Z$ such that

$$
\phi_* L = E.
$$

The second lemma implies that any simple semihomogeneous bundle satisfies an Index Theorem analogous to the line bundle one (cf. [Mu1] §16).

**Lemma 6.4.** Let $E$ be a nondegenerate (i.e. $\chi(E) \neq 0$) simple semihomogeneous bundle on $X$. Then exactly one cohomology group $H^i(E)$ is nonzero.

**Proof.** This follows immediately from the similar property of the line bundle $L$ in Lemma 6.3. \hfill \Box

We will be concerned with semihomogeneous bundles which have some sort of positivity, so in particular are nondegenerate and have global sections. A first property is that they have well-defined Fourier-Mukai transforms.

**Lemma 6.5.** Assume $E$ is nondegenerate simple semihomogeneous, such that $h^0(E) \neq 0$. Then $E$ satisfies I.T. with index 0.

**Proof.** For every $\alpha \in \operatorname{Pic}^0(X)$, there exists an $x \in X$ such that $E \otimes \alpha \cong t_x^* E$. This implies that

$$
H^0(E \otimes \alpha) \cong H^0(t_x^* E) \cong H^0(E) \neq 0.
$$
By Lemma 6.4 this means that $h^i(E \otimes \alpha) = 0$ for all $i > 0$ and all $\alpha \in \text{Pic}^0(X)$, that is $E$ satisfies I.T. with index 0.

The numerical measure of positivity used here is $\Theta$-regularity. Recall from §2 that $E$ is $m$-$\Theta$-regular if $E(-(m-1)\Theta)$ is $M$-regular. It is easy to see that in the case of semihomogeneous bundles this coincides with strong $m$-$\Theta$-regularity.

**Lemma 6.6.** A semihomogeneous bundle $E$ is $m$-$\Theta$-regular if and only if $E(-(m-1)\Theta)$ satisfies I.T. with index 0.

**Proof.** The more general fact that an $M$-regular semihomogeneous bundle satisfies I.T. with index 0 follows quickly from Lemma 6.2 above. More precisely the line bundle $M$ in its statement is forced to be ample since it has a twist with global sections and positive Euler characteristic. □

Mukai shows in [M3] §6 that the semihomogeneous bundles are Gieseker semistable (while the simple ones are in fact stable). Moreover, any semihomogeneous bundle has a Jordan-Hölder filtration in a strong sense:

**Proposition 6.7.** ([M3] Proposition 6.18) Let $E$ be a semihomogeneous bundle on $X$, and let $\delta$ be the equivalence class of $\frac{\det(E)}{\text{rk}(E)}$ in $NS(X) \otimes \mathbb{Z} \mathbb{Q}$. Then there exist simple semihomogeneous bundles $F_1, \ldots, F_n$ whose corresponding class is the same $\delta$, and semihomogeneous bundles $E_1, \ldots, E_n$, satisfying:

- $E \cong \bigoplus_{i=1}^n E_i$.
- Each $E_i$ has a filtration whose factors are all isomorphic to $F_i$.

Since the positivity of $E$ is carried through to the factors of a Jordan-Hölder filtration as in the Proposition above, standard inductive arguments allow us to immediately reduce the study below to the case of simple semihomogeneous bundles, which we do freely in what follows.

**A numerical criterion for normal generation.** The main result of this section is that the normal generation of a semihomogeneous vector bundle is dictated by an explicit numerical criterion. We assume all throughout that all the semihomogeneous vector bundles involved satisfy the minimal positivity condition, namely that they are $0$-$\Theta$-regular (which in particular is a numerical criterion for global generation, by Theorem 2.2).

**Theorem 6.8.** Let $E$ be a rank $r$ semihomogeneous bundle on the polarized abelian variety $(X, \Theta)$, and assume that $E$ is $0$-$\Theta$-regular. Then, for any $x \in X$, the multiplication map

$$H^0(E) \otimes H^0(t_x^*E) \longrightarrow H^0(E \otimes t_x^*E)$$

is surjective provided that

$$\frac{1}{r} \cdot c_1(E(-\Theta)) + \frac{1}{r'} \cdot \phi_{\Theta}^*c_1(\hat{E}(-\Theta)) > 0,$$

where $r' = \text{rk}(\hat{E}(-\Theta))$ (recall that $\phi_{\Theta}$ is the isogeny $X \to \hat{X}$ induced by $\Theta$).
Remark 6.9. Although most conveniently written in terms of the Fourier-Mukai transform, the statement of the theorem is indeed a numerical condition intrinsic to the vector bundle $E$, since by [M2] Cor. 1.18 one has:

$$c_1(\hat{E}(-\Theta)) = -PD_{2g-2}(ch_{g-1}(E(-\Theta))),$$

where $PD$ denotes the Poincaré duality map

$$PD_{2g-2} : H^{2g-2}(J(X), \mathbb{Z}) \to H^2(J(X), \mathbb{Z}),$$

and $ch_{g-1}$ the $(g-1)$-st component of the Chern character. Note also that

$$rk(\hat{E}(-\Theta)) = h^0(E(-\Theta)) = \frac{1}{r^{g-1}} \cdot \frac{c_1(E(-\Theta))^g}{g!}$$

by [M1] Cor. 2.8.

This implies in a particular case the following explicit statement, one which will be generalized though at the end of the section.

Corollary 6.10. A $(-1)$-$\Theta$-regular semihomogeneous bundle is normally generated.

Proof. The hypothesis means that $E(-2\Theta)$ satisfies I.T. with index 0. Consider an isogeny $f : Y \to X$ as in Lemma 6.3, so that there exists a line bundle $L$ on $Y$ with $f_*L = E(-\Theta)$. The assumption on $E$ implies that $L$ can be written as the tensor product of two ample line bundles. Since $f$ has 0-dimensional fibers, the Grothendieck-Riemann-Roch theorem implies that

$$c_1(E(-\Theta)) = f_*c_1(L) \text{ and } ch_{g-1}(E(-\Theta)) = f_*\left(\frac{c_1(L)^{g-1}}{(g-1)!}\right).$$

By Remark 6.9 we have then that $c_1(\hat{E}(-\Theta)) = -PD_{2g-2}(f_*\left(\frac{c_1(L)^{g-1}}{(g-1)!}\right))$. We are then reduced to doing a line bundle computation on $Y$, which follows by standard methods, as for example in [Be].

Examples. There are two basic classes of examples of (strongly) $(-1)$-$\Theta$-regular bundles, and both turn out to be semihomogeneous. They correspond to the properties of linear series on abelian varieties and on moduli spaces of vector bundles on curves, respectively.

Example 6.11. (Projective normality of line bundles.) For every ample divisor $\Theta$ on $X$, the line bundle $L = O_X(m\Theta)$ is $(-1)$-$\Theta$-regular for $m \geq 3$. Thus we recover the classical fact that $O_X(m\Theta)$ is projectively normal for $m \geq 3$.

Example 6.12. (Verlinde bundles.) Let $U_C(r,0)$ be the moduli space of rank $r$ and degree 0 semistable vector bundles on a smooth projective curve $C$ of genus $g \geq 2$. This comes with a natural determinant map $det : U_C(r,0) \to J(C)$, where $J(C)$ is the Jacobian of $C$. To a generalized theta divisor $\Theta_N$ on $U_C(r,0)$ (depending on the choice of a line bundle $N \in \text{Pic}^{g-1}(C)$) one associates for any $k \geq 1$ the $(r,k)$-Verlinde bundle on $J(C)$, defined by $E_{r,k} := det_*O(k\Theta_N)$ (cf. [Po]). It is shown in loc. cit. that the numerical properties of $E_{r,k}$ are essential in understanding the linear series $|k\Theta_N|$ on $U_C(r,0)$. It is noted there that $E_{r,k}$ are polystable and semihomogeneous.
A basic property of these vector bundles is the fact that
\[ r_j^* E_{r,k} \cong \oplus O_J(kr \Theta_N), \]
where \( r_j \) denotes multiplication by \( r \) on \( J(C) \) (cf. [Pa] Lemma 2.3). Noting that the pull-back \( r_j^* O_J(\Theta_N) \) is numerically equivalent to \( O(r^2 \Theta_N) \), we obtain that \( E_{r,k} \) is 0-\( \Theta \)-regular iff \( r \geq r + 1 \), and \(( -1) \)-\( \Theta \)-regular iff \( r \geq 2r + 1 \). This implies by the statements above that \( E_{r,k} \) is globally generated for \( r \geq r + 1 \) and normally generated for \( r \geq 2r + 1 \). These are precisely the results [Po] Proposition 5.2 and Theorem 5.9(a), the second obtained there by ad-hoc (though related) methods.

**Proof of the numerical criterion.** We will prove a natural generalization of Theorem 6.8, which guarantees the surjectivity of multiplication maps for two arbitrary semihomogeneous bundles. This could be seen as an analogue of Butler’s theorem [Bu] for semistable vector bundles on curves.

**Theorem 6.13.** Let \( E \) and \( F \) be semihomogeneous bundles on \((X, \Theta)\), both satisfying I.T. with index 0. Then the multiplication maps
\[ H^0(E) \otimes H^0(t_x^* F) \rightarrow H^0(E \otimes t_x^* F) \]
are surjective for all \( x \in X \) if the following holds:
\[ \frac{1}{r_F} \cdot c_1(F(-\Theta)) + \frac{1}{r_E} \cdot \phi \cdot c_1(E(-\Theta)) > 0. \]

We can assume \( E \) and \( F \) to be simple by the considerations in §2, and we will do so in what follows. We begin with a few technical lemmas. Let us recall first that, given two sheaves \( \mathcal{E} \) and \( \mathcal{G} \) on \( X \), their skew Pontrjagin product (see [Pa] §1) is defined as
\[ \mathcal{E} \star \mathcal{G} := p_1^*((p_1 + p_2)^* (\mathcal{E} \otimes p_2^* (\mathcal{G}))), \]
where \( p_1 \) and \( p_2 \) are the projections from \( X \times X \) to the two factors.

**Lemma 6.14.** ([Pa] Theorem 3.1) The multiplication map
\[ H^0(E) \otimes H^0(t_x^* F) \rightarrow H^0(E \otimes t_x^* F) \]
is surjective for any \( x \in X \) if the skew-Pontrjagin product \( E \star F \) is globally generated, so in particular if \( (E \star F) \) is 0-\( \Theta \)-regular.

**Lemma 6.15.** For all \( i \geq 0 \) we have:
\[ h^i((E \star F) \otimes \mathcal{O}_X(-\Theta)) = h^i((E \star \mathcal{O}_X(-\Theta)) \otimes F). \]

**Proof.** This follows from Lemma 3.2 in [Pa] if we prove the following vanishings:

1. \( h^i(t_x^* E \otimes F) = 0, \forall i > 0, \forall x \in X. \)
2. \( h^i(t_x^* E \otimes \mathcal{O}_X(-\Theta)) = 0, \forall i > 0, \forall x \in X. \)

We treat them separately:

1. By Lemma 6.2 we know that there exist isogenies \( \pi_E : Y_E \rightarrow X \) and \( \pi_F : Y_F \rightarrow X \), and line bundles \( M \) on \( Y_E \) and \( N \) on \( Y_F \), such that \( \pi_E^* E \cong \oplus M \) and \( \pi_F^* F \cong \oplus N \).

Now on the fiber product \( Y_E \times_X Y_F \), the pull-back of \( t_x^* E \otimes F \) is a direct sum of line bundles numerically equivalent to \( p_1^* M \otimes p_2^* N \). This line bundle is ample and
has sections, and so no higher cohomology by the Index Theorem. Consequently
the same must be true for $t^*_x E \otimes F$.

(2) Since $E$ is semihomogeneous, we have $t^*_x E \cong E \otimes \alpha$ for some $\alpha \in \Pic^0(X)$, and so:

$$h^i(t^*_x E \otimes O_X(-\Theta)) = h^i(E \otimes O_X(-\Theta) \otimes \alpha) = 0,$$

since $E(-\Theta)$ satisfies I.T. with index 0.

\[\square\]

Let us assume from now on for simplicity that the polarization $\Theta$ is symmetric. This
makes the proofs less technical, but the general case is completely similar since everything
depends (via suitable isogenies) only on numerical equivalence classes.

Lemma 6.16. We have

$$E \circ O_X(-\Theta) \cong \phi^*_G((-1_X)^* E \otimes O_X(-\Theta)) \otimes O(-\Theta).$$

Proof. This follows from Mukai’s general Lemma 3.10 in [M1]. \[\square\]

Putting together Lemmas 6.14, 6.15 and 6.16 we get the following cohomological
criterion for surjectivity of multiplication maps in our given situation.

Proposition 6.17. Under the hypotheses above, the multiplication maps

$$H^0(E) \otimes H^0(t^*_x F) \to H^0(E \otimes t^*_x F)$$

are surjective if we have the following vanishing:

$$h^i(\phi^*_G((-1_X)^* E \otimes O_X(-\Theta)) \otimes F(-\Theta)) = 0, \forall i > 0.$$

We are now in a position to give the proof of Theorem 6.13. To this end we only need
to understand the best numerical assumptions under which the cohomological requirement
in Proposition 6.17 is satisfied.

Proof. (of Theorem 6.13.) We first apply Lemma 6.2 to $G := \phi^*_G((-1_X)^* E(-\Theta)$ and $H := F(-\Theta)$: there exist isogenies $\pi_G : Y_G \to X$ and $\pi_H : Y_H \to X$, and line bundles $M$ on $Y_G$ and $N$ on $Y_H$, such that $\pi^*_G G \cong \oplus M$ and $\pi^*_H H \cong \oplus N$. Consider the fiber product $Z := Y_G \times_X Y_H$, with projections $p_G$ and $p_H$. Denote by $p : Z \to X$ the natural composition.

By pulling everything back to $Z$, we see that

$$p^*(G \otimes H) \cong \bigoplus_{r_G, r_H} (p^*_G M \otimes p^*_H N).$$

This implies that our desired vanishing $H^i(G \otimes H) = 0$ (cf. Proposition 6.17) holds as long as

$$H^i(p^*_G M \otimes p^*_H N) = 0, \forall i > 0.$$
Now $c_1(p_G^*M) = p_G^*c_1(M) = \frac{1}{r_G} p^* c_1(G)$ and similarly $c_1(p_H^*N) = p_H^* c_1(N) = \frac{1}{r_H} p^* c_1(G)$. Finally we get

$$c_1(p_G^*M \otimes p_H^*N) = p^* \left( \frac{1}{r_G} \cdot c_1(G) + \frac{1}{r_H} \cdot c_1(H) \right).$$

Thus all we need to have is that the class

$$\frac{1}{r_G} \cdot c_1(G) + \frac{1}{r_H} \cdot c_1(H)$$

be ample, and this is clearly equivalent to the statement of the theorem. \qed

We conclude by noting that the reason we only sketched the proof of Corollary 6.10 is that in fact under that particular hypothesis we have a much more general statement which works for every vector bundle on a polarized abelian variety, via substantially subtler methods.

**Theorem 6.18.** Let $E$ and $F$ be $(-1)$-regular vector bundles on $X$ (i.e. such that $E(-2\Theta)$ and $F(-2\Theta)$ are $M$-regular). Then the multiplication map

$$H^0(E) \otimes H^0(F) \to H^0(E \otimes F)$$

is surjective.

**Proof.** We use an argument inspired by techniques first introduced by Kempf, and rendered easy by the results of [PP1]. Let us consider the diagram

$$\bigoplus_{\xi \in U} H^0(E(-2\Theta) \otimes P_\xi) \otimes H^0(2\Theta \otimes P_\xi^\vee) \otimes H^0(F) \longrightarrow H^0(E) \otimes H^0(F)$$

Under the given hypotheses, the bottom horizontal arrow is onto by the general Theorem 2.3. On the other hand, the abelian Castelnuovo-Mumford Lemma Theorem 2.5 insures that each one of the components of the vertical map on the left is surjective. Thus the composition is surjective, which gives the surjectivity of the vertical map on the right. \qed

**Corollary 6.19.** Every $(-1)$-regular vector bundle is normally generated.

**References**

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths and J. Harris, *Geometry of algebraic curves*, Grundlehren 267, Springer, (1985).

[Ba1] Th. Bauer, Seshadri constants and periods of polarized abelian varieties, Math. Ann. 312 (1998), 607–623. With an appendix by the author and T. Szemberg.

[Ba2] Th. Bauer, Seshadri constants on algebraic surfaces, Math. Ann. 313 (1999), 547–583.

[Be] A. Beauville, Quelques remarques sur le transformation de Fourier dans l’anneau de Chow d’une variété abélienne, in *Algebraic Geometry*, Tokyo/Kyoto 1982, LNM 1016 (1983), 238–260.

[Bu] D. Butler, Normal generation of vector bundles over a curve, J. Diff. Geom. 39 (1994), 1–34.

[CH] J. A. Chen and C. Hacon, Linear series of irregular varieties, Proceedings of the symposium on *Algebraic Geometry in East Asia*, World Scientific (2002)

[De] O. Debarre, Seshadri constants of abelian varieties, to appear.

[EL] L. Ein and R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular varieties, J. Amer. Math. Soc. 10 (1997), 243–258.
B. van Geemen and E. Izadi, The tangent space to the moduli space of vector bundles on a curve and the singular locus of the theta divisor of the Jacobian, Journal of Alg. Geom. 10 (2001), 133–177.

M. Green and R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, Invent. Math. 90 (1987), 389–407.

E. Izadi, Deforming curves representing multiples of the minimal class in Jacobians to non-Jacobians I, preprint mathAG/0103204.

G. Kempf, Complex abelian varieties and theta functions, Springer-Verlag 1991.

G. Kempf, On the geometry of a theorem of Riemann, Ann. of Math. 98 (1973), 178–185.

J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge University Press 1998.

H. Lange and Ch. Birkenhake, Complex abelian varieties, Springer-Verlag 1992.

R. Lazarsfeld, Lengths of periods and Seshadri constants on abelian varieties, Math. Res. Lett. 3 (1996), 439–447.

R. Lazarsfeld, Positivity in algebraic geometry, book in preparation.

S. Mukai, Duality between $D(X)$ and $D(\mathcal{X})$ with its application to Picard sheaves, Nagoya Math. J. 81 (1981), 153–175.

S. Mukai, Fourier functor and its application to the moduli of bundles on an abelian variety, In: Algebraic Geometry, Sendai 1985, Advanced studies in pure mathematics 10 (1987), 515–550.

S. Mukai, Semi-homogeneous vector bundles on an abelian variety, J. Math. Kyoto Univ. 18 (1978), 239–272.

S. Mukai, Abelian variety and spin representation, preprint.

D. Mumford, Abelian varieties, Second edition, Oxford Univ. Press 1974.

D. Mumford, On the equations defining abelian varieties, Invent. Math. 1 (1966), 287–354.

M. Nakamaye, Seshadri constants on abelian varieties, Amer. J. Math. 118 (1996), 621–635.

D. Orlov, On equivalences of derived categories of coherent sheaves on abelian varieties, preprint math.AG/9712017.

W. Oxbury and C. Pauly, Heisenberg invariant quartics and $SU_C(2)$ for a curve of genus four, Math. Proc. Camb. Phil. Soc. 125 (1999), 295–319.

G. Pareschi, Syzygies of abelian varieties, J. Amer. Math. Soc. 13 (2000), 651–664.

G. Pareschi and M. Popa, Regularity on abelian varieties I, J. Amer. Math. Soc. 16 (2003), 285–302.

G. Pareschi and M. Popa, Regularity on abelian varieties II: basic results on linear series and defining equations, J. Alg. Geom. 13 (2004), 167–193.

M. Popa, Verlinde bundles and generalized theta linear series, Trans. Amer. Math. Soc. 354 (2002), 1869–1898.

H. Umemura, On a certain type of vector bundles over an abelian variety, Nagoya Math. J. 64 (1976), 31–45.

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