Classical general relativity effects to second order in mass, spin, and quadrupole moment

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Abstract

In this contribution, we calculate the light deflection, perihelion shift, time delay and gravitational redshift using an approximate metric that contains the Kerr metric and an approximation of the Erez-Rosen spacetime. The results were obtained directly using \(\text{Mathematica} \, 2018\) Wolfram Research, Inc., Version 11.3, Champaign IL. The results agree with the ones presented in the literature, but they are extended until second order terms of mass, angular momentum and mass quadrupole. The inclusion of the mass quadrupole is done by means of the metric; no expansion of the gravitational potential as in the parameterized post-Newtonian is required.

1. Introduction

Einstein’s General Theory of Relativity is a metric theory of gravity that relates the mass-energy content of the Universe with the space-time curvature through the Einstein field equations:

\[
R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = 8\pi \frac{G}{c^4} T_{ab},
\]

(1)

where the right-hand side of this equation depends on the stress-energy tensor \(T_{ab}\) which describes the mass-energy sources of gravitational fields, and the left-hand side depends on the metric elements \(g_{ab}\) which describe the space-time curvature. \(R_{ab}\) are the Ricci tensor components, \(R\) is the scalar curvature, and \(\Lambda\) is the cosmological constant. In this article, the geometrical units are employed, so that \(G = c = 1\). The cosmological constant is set \(\Lambda = 0\).

In 1916, Karl Schwarzschild discovered a solution to the Einstein field equations in vacuum, suitable for describing the spacetime in the empty space surrounding a spherical, static object [1]. Ever since then, this metric has been used to describe a wide range of phenomena, including light deflection close to a massive star, planetary precession of the perihelion, time delay and gravitational redshifts for weak fields. Erez and Rosen introduced the effects of mass quadrupole \(q\) as exact solution in 1959 [2, 3]. This derivation had some errors which were corrected by Doroshkevich et al [4], Winicour et al [5] and Young and Coulter [6]. The exact solution for a rotating black hole (BH) could only be solved as late as 1963 by Kerr [7]. There are exact solutions containing the Erez-Rosen and Kerr features, such spacetimes are cumbersome. A new approximate metric representing the spacetime of a rotating deformed body is obtained by perturbing the Kerr metric to include up to the second order of the quadrupole moment [8]. This kind of approximations is valid because the quadrupole moment is small generally for a variety of astrophysical objects. Observing the Spin of rotating BH is possible by measuring the orbital angular momentum of light propagating around it, as well as BH shadow circularity analysis [9].

In the literature, calculations that include the mass quadrupole are only done using (parameterized) post Newtonian metrics. To introduce the mass quadrupole, the gravitational potential is expressed as a multipolar expansion [10–16]. In our calculation we perform no such expansion of the gravitational potential. The quadrupole parameter is introduced from the metric.
Now, it is possible to do such calculation in a straightforward manner using software like Mathematica. In this contribution, we present the results of light deflection, perihelion shift, time delay and gravitational redshift using this software. The results were compared with the ones obtained from the Reduce software.

This paper is organized as follows. The classical tests of general relativity are described in section 2. The parameterized post-Newtonian formalism is introduced in section 3. The approximate metric with three parameters \((M, J = ma, q)\) is described in section 4. The metric potentials are expanded in a Taylor series up to second order of \(J, M\) and \(q\). The resulting metric is transformed into a Hartle-Thorne form. In section 6 we calculate the angle of the deflection of light in traveling in the equatorial plane of our metric. In section 7, we present the necessary calculations to obtain the angle of Precession of the perihelion of the orbit of a planet in the presence of a space-time described by our metric. In section 8, we calculate the time delay of light traveling between two points and in section 9, the expression for the gravitational redshift in two different positions in our space-time is obtained. The Mathematica notebook is available upon request. Our concluding remarks are presented in the last section.

2. The classical tests

In the solar system, most of the Newtonian mechanics predictions are in good agreement with observations. However, there are a few situations where general relativity (GR) is positioned as a more precise theory. Traditionally, they are Mercury’s perihelion precession, the light deflection by the Sun, the gravitational redshift of light and the time delay of light.

Mercury’s perihelion precession is the first classical test and was first noted by Le Verrier in 1859. In this phenomenon, classical contributions such as the planetary perturbations influence [17, 18], yet it remains a discrepancy of 42.77" per century. The contributions from GR reports a value of 42.95" per century. During the 1960’s and 1970’s there was a considerable controversy on the importance of the contribution of the solar oblateness mass quadrupole \(J_2\) on the perihelion precession. This discussion has relaxed as the value of the solar quadrupole has been inferred to be small, on the order of \(J_2 = (2.25 \pm 0.09) \times 10^{-7}\) [18, 19]. Using this value, it has been estimated that the contribution to the precession from the solar oblateness is of \(0.0286 \pm 0.0011"\) per century. Yet, its importance can not be specified until a reliable value of the quadrupole is known. The second test, the light deflection due to the massive body of the Sun, was famously first observed during the Eddington’s expedition in 1919 with a high degree of inaccuracy, but it was not observed with precision until the 70’s using radio wave interferometry. By this time, it was reported that the mean gravitational deflection was 1.007 \(\pm\) 0.009 times the value predicted by GR [18]. The deflection caused by the solar oblateness can be treated as a small correction. Typically, it could modify the path of ray of light in 0.2 \(\mu\) arcseconds. Other physical property that influences light deflection is the Sun’s angular momentum, as it has been calculated that the Sun’s amount of \(L \approx 2 \times 10^{41}\) kg m² s⁻¹ can be responsible for a deflection of 0.7 \(\mu\) arcseconds [20].

The third test, the gravitational redshift, measures the wavelength shift between two identical clocks placed at rest at different positions in a gravitational field. This was the first test to be proposed by Einstein, and was first tested by Pound, Rebka and Snider in the 1960s, as they measured the gamma radiation emitted by \(^57\)Fe, as they ascended or descended the Jefferson Physical Laboratory tower [18]. The fourth test, the gravitational time delay, was classified as such by Will and was first observed by Shapiro in 1964 when he discovered that a ray of light propagating in the gravitational field of a massive body will take more time traveling a given distance, than if the field were absent [18]. Gravitational time delay can be observed by measuring the round trip of a radio signal emitted from Earth and reflected from another body, such as another planet or a satellite. To properly measure the effect, it is necessary to do a differential measurement in the variations in the round trip as the target object moves through the Sun’s gravitational field. This task is particularly difficult as it involves taking into account the variations in the round trip as a result of the orbital motion of the target relative to Earth [19].

Another ideal probe for testing GR is the massive black hole (MBH) located in a bright and very compact astronomical radio source called Sgr A*. The fourth test to be proposed by Einstein, and was first observed during the 1919 expedition with a high degree of inaccuracy, but it was not observed with precision until the 1970’s using radio wave interferometry. By this time, it was reported that the mean gravitational deflection was 1.007 \(\pm\) 0.009 times the value predicted by GR [18]. The deflection caused by the solar oblateness can be treated as a small correction. Typically, it could modify the path of ray of light in 0.2 \(\mu\) arcseconds. Other physical property that influences light deflection is the Sun’s angular momentum, as it has been calculated that the Sun’s amount of \(L \approx 2 \times 10^{41}\) kg m² s⁻¹ can be responsible for a deflection of 0.7 \(\mu\) arcseconds [20].

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3. The parametrized post-Newtonian formalism

Although it has been very successful when compared with direct observations, GR is just one of many metric theories of gravity, and all that distinguishes one metric theory from another is the particular way in which
matter generates the metric. It is simple to perform a comparison between metric theories in the slow-motion and weak-field limit, since all of their results must agree with Newtonian physics.

The parametrized post-Newtonian (PPN) formalism is a device that allows the comparison between different theories of gravitation and experiments. It is motivated by the advent of alternative theories of gravitation other than GR during the second half of the twentieth century. It has provided a common framework to quantify deviations from GR which are small in the post-Newtonian order.

As the various theories of gravitation involve mathematical objects such as coordinates, mass variables and metric tensors, PPN formalism is provided with a set of ten parameters which describe the physical effects of these theories. The so called Eddington-Robertson-Schiff parameters $\gamma$ and $\beta$ are the only non-zero parameters in GR, hence they are significant in the study of classical tests. $\beta$ measures whether gravitational fields do interact with each other, while $\gamma$ quantifies the space-curvature produced by unit rest mass, and both their values is one in GR [18].

In this context, it is very important to mention Gaia, the ESA space astrometry mission launched in late 2013. Through its detectors, it will perform Eddington-like experiments through the comparison between the pattern of the starfield observed with and without Jupiter. For this purpose, it is vital to have a formula relevant for the monopole and quadrupole light deflection for an oblate planet. These results will provide a new independent determination of $\gamma$ and evidence of the bending effect of the mass quadrupole of a planet [23, 24]. It is currently accepted that $|1 - \gamma|$ is less than $2 \times 10^{-5}$.

It is also relevant to highlight the use of radiometric range measurements to the MESSENGER spacecraft in orbit around Mercury to estimate the precession of Mercury’s perihelion. Knowing a suitable relationship between this classical test and the quadrupole allows to decouple $\beta$ and the solar quadrupole $J_2$ to yield $(\beta - 1) = (-2.7 \pm 3.9) \times 10^{-5}$ [25]. It has been conjectured that there is another additional contribution to the perihelion advance from the relativistic cross terms in the post-Newtonian equations of motion between Mercury’s interaction with the Sun and with the other planets, as well from the interaction between Mercury’s motion and the gravitomagnetic field of the moving planets. These effects are planned to be detected by the BepiColombo mission, launched in late 2018 [26].

There have been several papers that have quantified the contributions to the classical tests from various objects in the solar system. Detection and precise measurement of the quadrupolar deflection of light by objects in the solar system, at the level of a microarcsecond positional accuracy, is important as it will allow the experimental observation of a wide range of physical phenomena that will allow to test GR in a velocity and acceleration-independent-regime. There are research lines that study the effects related to the motion of planets such as the appearance of a gravitational field due to the mass dipole and methods to properly measure the quadrupole of the planets that compensate for the effects due to their movements [27]. The values shown in Table 1 illustrate the maximal magnitudes of the various gravitational effects due to the Sun and the planets at which the gravitational light deflection from that body should still be accounted for to attain a final accuracy of 1 $\mu$as. Here, Second Order: PN is the post-Newtonian effect due to the spherically symmetric field of each body, Rotation accounts for the field caused by the rotational motion of the bodies, Fourth Order: PPN is the post-post Newtonian effect due to the mass, and Quadrupole: PN is the effect caused by the mass quadrupole [28].

Table 2 shows the values of the contributions to the gravitation delay of a radio signal as it is measured from the Earth [11].

In this formalism, the gravitational potential of an axially symmetric body can be written in the following form [29]

---

### Table 1. Order of magnitude of the contributions PN, PPN, PNQ and PNR to the deviation angle of a light ray grazing the solar limb as predicted by GR [28].

| Stellar object | Second order: PN ($\mu$as) | Rotation ($\mu$as) | Fourth order: PPN ($\mu$as) | Quadrupole: PN ($\mu$as) |
|---------------|---------------------------|-------------------|----------------------------|------------------------|
| Sun           | $1.75 \times 10^{-6}$     | 0.7               | 11                         | $\sim 1$               |
| Mercury       | 83                        | —                 | —                          | —                      |
| Venus         | 493                       | —                 | —                          | —                      |
| Earth         | 574                       | —                 | 0.6                        | —                      |
| Mars          | 116                       | —                 | —                          | 0.2                    |
| Jupiter       | 16270                     | 0.2               | —                          | 240                    |
| Saturn        | 5780                      | —                 | —                          | 95                     |
| Uranus        | 2080                      | —                 | —                          | 8                      |
| Neptune       | 2533                      | —                 | —                          | 10                     |
The metric, we will employ to do the calculations was generated in a perturbative form using the Kerr spacetime as seed metric. This approximate rotating spacetime with quadrupole moment written in standard form is as follows [8, 30]:

$$ds^2 = -\frac{\Delta}{\rho^2}[e^{-\psi}dt - ae^{\psi}\sin^2\theta d\phi]^2 + \frac{\sin^2\theta}{\rho^2}(\bar{t}^2 + a^2) e^{\psi}d\phi - ae^{-\psi}dt]^2 + \rho^2 e^{2\psi}\left(\frac{dr^2}{\Delta} + d\theta^2\right),$$

where

$$\Delta = \bar{r}^2 - 2Mr + a^2,$$

$$\rho^2 = \bar{r}^2 + a^2\cos^2\theta,$$

$$\psi = \frac{\bar{q}}{\bar{r}}P_2 + \frac{3}{\bar{r}^4} \bar{q}P_2,$$

$$\chi = \frac{\bar{q}}{\bar{r}^3}P_2 + \frac{1}{3} \frac{Mq}{\bar{r}^5}(5P_2^2 + 5P_2 - 1) + \frac{1}{9} \frac{q^2}{\bar{r}^6}(25P_2^2 - 21P_2^2 - 6P_2^2 + 2),$$

$$P_2 = \frac{1}{2}(3\cos^2\theta - 1).$$

This spacetime has three parameters, namely mass $M$, spin, $J = Ma$ (a as the Kerr rotation parameter) and $q$, the mass quadrupole. It contains the Kerr and the Schwarzschild metrics. This metric is an approximation to the Erez-Rosen metric ($q^3 \approx 0$).

According to [8], the Taylor series up to second order of $a, J, M$ and $q$ gives

$$g_{tt} = -\left(1 - 2\frac{M}{\bar{r}} + 2\frac{Ma^2}{\bar{r}^3}\cos^2\theta - 2\frac{\bar{q}}{\bar{r}^3}P_2 - 2\frac{Mq}{\bar{r}^4}P_2 + 2\frac{q^2}{\bar{r}^6}P_2^2\right),$$

$$g_{t\theta} = -2\frac{J}{\bar{r}}\sin^2\theta,$$

$$g_{rr} = 1 + 2\frac{M}{\bar{r}} + 4\frac{M^2}{\bar{r}^3} - \frac{a^2}{\bar{r}^2}\sin^2\theta - 2\frac{Ma^2}{\bar{r}^3}(1 + \sin^2\theta) - 4\frac{Ma^2}{\bar{r}^4}(2 + \sin^2\theta) + 2\frac{\bar{q}}{\bar{r}^3}P_2 + \frac{2}{3}\frac{Mq}{\bar{r}^5}(5P_2^2 + 11P_2 - 1) + \frac{2}{9}\frac{q^2}{\bar{r}^6}(25P_2^2 - 21P_2^2 - 6P_2^2 + 2),$$

$$g_{\theta\theta} = \bar{r}^2\left(1 + \frac{a^2}{\bar{r}^2}\cos^2\theta + \frac{2}{3}\frac{q}{\bar{r}^3}P_2 + \frac{2}{3}\frac{Mq}{\bar{r}^4}(5P_2^2 + 5P_2 - 1) + \frac{2}{9}\frac{q^2}{\bar{r}^6}(25P_2^2 - 12P_2^2 - 6P_2^2 + 2)\right),$$

$$g_{\phi\phi} = \bar{r}^2\sin^2\theta\left(1 + \frac{a^2}{\bar{r}^2} + 2\frac{Ma^2}{\bar{r}^3}\sin^2\theta + \frac{2}{3}\frac{q}{\bar{r}^3}P_2 + \frac{6}{3}\frac{Mq}{\bar{r}^4}P_2 + 2\frac{q^2}{\bar{r}^6}P_2^2\right).$$

Now, in [8] a transformation was found that converts this expanded metric (5) into the expanded Hartle-Thorne (HT) metric changing $q \rightarrow Ma^2 - q$ that included the second order in $q$, it is
where
\[
\begin{align*}
f_1 &= -\frac{1}{9} (4 + 4P_2 - 5P_2^2) \\
f_2 &= -\frac{1}{72} (43 + 24P_2^2 - 40P_2^3) \\
g_1 &= \frac{1}{6} (2 - 5P_2) \cos \theta \sin \theta \\
g_2 &= \frac{1}{6} (2 - 5P_2) P_2 \cos \theta \sin \theta \\
h_1 &= -\frac{1}{2} \sin^2 \theta \\
h_2 &= -\frac{1}{2} \sin^2 \theta \\
h_3 &= -3 \cos^2 \theta \\
h_4 &= -\frac{1}{2} \cos \theta \sin \theta \\
h_5 &= -\cos \theta \sin \theta.
\end{align*}
\]

the transformed metric components take the following form [8]
\[
\begin{align*}
g_{tt} &= \left(1 - 2U + 2 \frac{Q}{r^3} P_2 - \frac{2}{3} \frac{J^2}{r^4} (2P_2 + 1) + 2 \frac{MQ}{r^4} P_2 + 2 \frac{Q^2}{r^6} P_2^2\right) \\
g_{rr} &= -\frac{2}{r} \sin^2 \theta \\
g_{\theta \theta} &= 1 + 2U + 4U^2 - 2 \frac{Q}{r^3} P_2 + \frac{2J_1}{r^4} (8P_2 - 1) - 10 \frac{MQ}{r^4} P_2 + \frac{1}{12} \frac{Q^2}{r^6} (8P_2^2 - 16P_2 + 77) \\
g_{\phi \phi} &= r^2 \left(1 - 2 \frac{Q}{r^3} P_2 + \frac{J^2}{r^4} P_2 - 5 \frac{MQ}{r^4} P_2 + \frac{1}{36} \frac{Q^2}{r^6} (44P_2^2 + 8P_2 - 43)\right) \\
g_{\theta \phi} &= r^2 \sin^2 \theta \left(1 - 2 \frac{Q}{r^3} P_2 + \frac{J^2}{r^4} P_2 - 5 \frac{MQ}{r^4} P_2 + \frac{1}{36} \frac{Q^2}{r^6} (44P_2^2 + 8P_2 - 43)\right).
\end{align*}
\]

where $U = M/r$ and $P_2 = (3 \cos^2 \theta - 1)/2$. This new expanded HT form with second order quadrupole moment is a more convenient way to calculate the quantities we are going to obtain, because it is in Schwarzschild spherical coordinates.

5. The geodesic equation

The space-time interval between two events is defined as,
\[
ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta.
\]

We can equate the interval with a proper time $d\tau$ and so write down the following equation,
\[
\mu = g_{\alpha \beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau},
\]

where $\mu$ is a parameter to be defined. For massive particles moving across spacetime its trajectories are described by time-like intervals ($ds^2 < 0$), so we set $\mu = +1$, while light trajectories are described by light-like intervals ($ds^2 = 0$) and so we set $\mu = 0$. The former case is suitable for describing planetary motion, as its the case for planetary perihelion, while light deflection and time delay, which are light related, are described by the latter. The geodesic equations help to calculate the path with the shortest proper time between two points,
\[
\frac{d}{d\tau} \left( g_{\alpha \beta} \frac{dx^\beta}{d\tau} \right) - \frac{1}{2} \partial_\gamma g_{\alpha \beta} \frac{dx^\alpha}{d\tau} \frac{dx^\gamma}{d\tau} = 0.
\]
The geodesic equation is related to conserved quantities, as in our case when we set \( \alpha = t \),

\[
\frac{d}{d\tau} \left( g_{\alpha\alpha} \frac{dx^\alpha}{d\tau} + g_{\alpha\beta} \frac{dx^\beta}{d\tau} \right) = 0. 
\] (12)

We can set the conserved quantity related with the energy \( E \),

\[
g_{\alpha\alpha} \frac{dx^\alpha}{d\tau} + g_{\alpha\beta} \frac{dx^\beta}{d\tau} = -E. 
\] (13)

When we set \( \alpha = \phi \) we obtain a conserved quantity related to the density of angular momentum along the \( z \)-axis, \( L_z \),

\[
g_{\phi\phi} \frac{dx^\phi}{d\tau} + g_{\phi\alpha} \frac{dx^\alpha}{d\tau} = L_z. 
\] (14)

These relations can be reversed to obtain:

\[
\frac{dt}{d\tau} = -\frac{1}{\rho^2} \left[ -Eg_{\phi\phi} - g_{\phi\alpha} L_z \right], 
\] (15)

\[
\frac{d\phi}{d\tau} = -\frac{1}{\rho^2} \left[ g_{\alpha\alpha} L_z + E g_{\alpha\phi} \right], 
\] (16)

where \( \rho^2 = g_{\phi\phi}^2 - g_{\phi\alpha} g_{\alpha\phi} \). Equations (15) and (16) can be combined to,

\[
\frac{d\phi}{dt} = \frac{d\phi}{d\tau} \frac{d\tau}{dt} = \frac{g_{\alpha\alpha} L_z + E g_{\phi\alpha}}{-E g_{\phi\phi} - g_{\phi\alpha} L_z}. 
\] (17)

### 6. Light deflection

The effect is represented in figure 1. We set \( \mu = 0 \) in (10) and rearranging provides an equation for \( dt/d\tau \). We can use the substitution \( u = 1/r \) to obtain up to order \( O(M^2, Q^2, J^2) \):

\[
\frac{d^2u}{d\phi^2} = -2J \frac{E^3}{L_z} + \left( 12J^2 \frac{E^4}{L_z^2} - 8JM \frac{E^3}{L_z} - 1 \right) u + \left( 3Q \frac{E^2}{L_z^2} + 3M \right) u^3 \\
+ \left( -24JQ \frac{E^3}{L_z^2} + 34J^2 \frac{E^2}{L_z} + 10MQ \frac{E^2}{L_z^2} \right) u^3 + \left( -81J^2 \frac{2}{2} + 3MQ \frac{93Q^2}{4} \right) u^5 + 33Q^2 u^7 
\] (18)
This equation can only be solved by perturbation theory. For this purpose, we propose a solution of the form

\[ u = u_0 \cos \phi + c_0 u_m + J(u^3_m u_{Q1} + u^3_m u_{Q2} + u^3_m u_{Q3}) + M(u^3_0 u_{M1} + u^3_0 u_{M2} + u^3_0 u_{M3}) \]

or,

\[ J = \frac{4 E^3}{L^3} \cos \phi, \]

and so on. For this part, we stuck to the general solutions to the differential equation as in [31],

\[ \frac{d^2 y}{dx^2} + y = \cos(nx) \]

to be

\[ y = \frac{1}{n^2 - 1} \cos(nx) \]

for \( n = 1 \) and

\[ y = \frac{1}{2} \sin \phi \]

for \( n = 1 \). The approximate solution is:

\[ u = u_0 \cos \phi - 2Ju^3_m + \frac{1}{2} Qu^3_m (3 - \cos 2\phi) + \frac{1}{2} Qu^3_m (3 - \cos 2\phi) + \frac{1}{2} Qu^3_m (3 - \cos 2\phi) + \frac{1}{2} Qu^3_m (3 - \cos 2\phi) \]

The closest approach \( u_m \) occurs when \( \phi = 0 \), so:

\[ u_m = u_0 - 2Ju^3_m + Ju^3_m + Qu^3_m - J^2 u^3_m \left( \frac{27u^2_0}{16} + \frac{17}{16} u^2_m \right) + \frac{3}{4} Qu^3_m \left( \frac{1551u^4_0}{1024} - \frac{31}{32} u^4_m - \frac{3}{4} u^4_m \right) \]

This method brings up a number of equations of the form:

\[ \frac{d^2 u_{04}}{d\phi^2} = -u_{04} + 4 E^3 \cos \phi, \]

or,

\[ \frac{d^2 u_{14}}{d\phi^2} = -u_{14} + 3 \cos^2 \phi, \]

This equation can only be solved by perturbation theory. For this purpose, we propose a solution of the form

\[ u = u_0 \cos \phi + c_0 u_m + \frac{1}{2} Qu^3_m (3 - \cos 2\phi) + \frac{1}{2} Qu^3_m (3 - \cos 2\phi) + \frac{1}{2} Qu^3_m (3 - \cos 2\phi) + \frac{1}{2} Qu^3_m (3 - \cos 2\phi) \]

The closest approach \( u_m \) occurs when \( \phi = 0 \), so:

\[ u_m = u_0 - 2Ju^3_m + Ju^3_m + Qu^3_m - J^2 u^3_m \left( \frac{27u^2_0}{16} + \frac{17}{16} u^2_m \right) + \frac{3}{4} Qu^3_m \left( \frac{1551u^4_0}{1024} - \frac{31}{32} u^4_m - \frac{3}{4} u^4_m \right) \]

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The closest approach \( u_m \) occurs when \( \phi = 0 \), so:

\[ u_m = u_0 - 2Ju^3_m + Ju^3_m + Qu^3_m - J^2 u^3_m \left( \frac{27u^2_0}{16} + \frac{17}{16} u^2_m \right) + \frac{3}{4} Qu^3_m \left( \frac{1551u^4_0}{1024} - \frac{31}{32} u^4_m - \frac{3}{4} u^4_m \right) \]
Table 3. Order of magnitude of some of the contributions to the deviation angle of a light ray grazing the solar limb as predicted by our model.

| Stellar object | First order: Mass ($\mu$as) | Rotation ($\mu$as) | Second order: Mass ($\mu$as) | Quadrupole ($\mu$as) |
|----------------|----------------------------|------------------|----------------------------|---------------------|
| Sun            | $1.751 \times 10^9$         | $6.991 \times 10^{-1}$ | $7.224 \times 10^{-8}$     | $9.627 \times 10^9$  |
| Mercury        | $8.292 \times 10^3$         | $3.287 \times 10^{-7}$ | $1.621 \times 10^{-8}$     | —                   |
| Venus          | $4.929 \times 10^2$         | $1.011 \times 10^{-6}$ | $5.728 \times 10^{-7}$     | —                   |
| Earth          | $5.736 \times 10^2$         | $2.960 \times 10^{-2}$ | $7.759 \times 10^{-7}$     | $6.210 \times 10^{-1}$ |
| Mars           | $1.158 \times 10^2$         | $3.490 \times 10^{-2}$ | $3.163 \times 10^{-8}$     | $2.275 \times 10^{-3}$ |
| Jupiter        | $1.641 \times 10^1$         | $1.705 \times 10^{-1}$ | $6.353 \times 10^{-4}$     | $2.421 \times 10^{2}$ |
| Saturn         | $5.802 \times 10^1$         | $4.320 \times 10^{-2}$ | $7.937 \times 10^{-5}$     | $9.549 \times 10^{-4}$ |
| Uranus         | $2.172 \times 10^1$         | $3.336 \times 10^{-3}$ | $1.112 \times 10^{-5}$     | $2.607 \times 10^{-4}$ |
| Neptune        | $2.508 \times 10^3$         | $8.357 \times 10^{-3}$ | $1.483 \times 10^{-5}$     | $1.003 \times 10^{-1}$ |

The deflection angle $\Delta \phi = 2\delta$ can be found using the condition $u(\pi/2 + \delta) = 0$, that is:

$$
\Delta \phi = 4M\mu_m - 4\mu\mu_n^2 + 4Q\mu_n^6 + \left( \frac{8 + \frac{195}{32}\pi}{2(2 + \pi)} \right) M^2\mu_m^4 + 2(2 + \pi)JM\mu_m^3
\]

$$
+ (4 - 15\pi)JQ\mu_n^5 - \left( \frac{4}{4} - \frac{15}{4}\pi \right) M^2\mu_m^2 - \left( \frac{8}{4} - \frac{75}{4}\pi \right) MQ\mu_m^4 + \left( \frac{705}{128}\pi - 4 \right) Q^2\mu_n^6.
$$

(24)

This result agrees with the result expected from the Schwarzschild metric,

$$
\Delta \phi \approx 4M\mu_m - \left( \frac{4}{4} - \frac{15}{4}\pi \right) M^2\mu_m^2,
$$

up to second order in mass [31]. The evaluation of some these terms for a ray of light grazing the solar limb is presented in table 3.

7. Precession of the perihelion

The effect is represented in figure 2. First, we use the geodesic equation (11) to find the conserved quantities, and the equations (15) and (16). Using these new identities, it is possible to calculate $d r / d \tau$ setting $u = 1$ in (10) and imposing a planar orbit ($\theta = \pi/2$). After this, the well known variable change $u = u(\phi)$ by means of:

$$
\frac{du}{d\phi} = \frac{du}{d\tau} \frac{d\tau}{d\phi}.
$$

(25)

After taking the second derivative with respect to $\phi$, we found up to order $O(M^2, Q^2, f^2)$, the result is:

$$
\frac{d^2u}{d\phi^2} = 2J^2 \frac{E}{L_z} - 2J^2 \frac{E}{L_z} + M \frac{F}{L_z} + u \left( 12J^4 \frac{E^4}{L_z^4} - 8JM \frac{E^3}{L_z^4} - 12J^2 \frac{E^2}{L_z^2} - 1 \right)
\]

$$
+ u^2 \left( 3M + 3Q^2 \frac{E^2}{L_z^2} - \frac{3}{2} \frac{Q}{L_z} \right) + u^3 \left( -24JQ \frac{E^3}{L_z^2} + 34J^2 \frac{E^2}{L_z^2} + 10MQ \frac{E^2}{L_z^2} \right)
\]

$$
+ 16J^2 \frac{E}{L_z} - 34 \frac{J^2}{L_z} + u^5 \left( -93Q^4 \frac{E^2}{L_z^2} - 81J^2 + \frac{111Q^2}{4} \frac{E^2}{L_z^2} + \frac{3}{2} \frac{MQ}{L_z} \right) + 33Q^2 u^7.
$$

(26)

We can consider a perturbation $u = u_c + u_m w(\phi)$, where $w$ is the wobble function we want to find. As such, given that $w << 1$, it satisfies the harmonic equation:
Table 4. Order of magnitude of the contributions to the gravitational periastron and perihelion precessions in the orbits of the star S2 and Mercury, respectively.

| Body     | First order: mass (as/cent) | Second order: mass (as/cent) | Quadrupole (as/cent) |
|----------|----------------------------|-----------------------------|----------------------|
| S2       | 73.807 5                   | 0.001 012 07                | —                    |
| Mercury  | 41.162                      | 4.723 01 × 10⁻⁶             | 4.723 01 × 10⁻⁶      |

\[
\frac{d^2 w}{d\theta^2} + w = \left( \frac{M}{r_c} + 3 \frac{Q}{L_z^2 r_c} (2E^2 - 1) + 3 \frac{J^2}{r_c} \left( \frac{4E^4 - 4E^2}{L_z^4} + \frac{34E^2 - 34}{2r_c^2} - \frac{135}{2r_c^4} \right) \right) \\
- 8EJ \frac{M}{L_z^2} + 24E \frac{JQ}{L_z^3 r_c} (2 - 3E^2) + \frac{15}{2} \frac{MQ}{r_c^4} \left( \frac{4E^2}{L_z^2} + \frac{1}{r_c^2} \right) \\
+ \frac{3}{4} \frac{Q^2}{r_c^4} \left( 185 - 155E^2 + 308 \frac{r_c^4}{L_z^2} \right) w. 
\]

(27)

It provides an angular frequency \( \omega \) value for which \( w = A \cos(\omega \phi + \phi_0) \). The orbit perihelion \( \Delta \phi \) occurs when \( w(\phi) \) is a minimum, i.e. when the argument of the cosine function is \( \pi + 2\pi n \). \( \Delta \phi \) can be found using the condition \( \omega \Delta \phi = 2\pi \). Although other methods can be used [32], by using the common substitution

\[
\hat{\omega} = \frac{E^2 - 1}{2},
\]

along the Schwarzschild circular orbit approximation

\[
\hat{\omega} \approx -\frac{M}{r_c} + \frac{L_z^2}{2r_c^2} - \frac{ML_z^2}{2r_c^4}
\]

this implies:

\[
\Delta \phi = 6\pi \frac{M}{r_c} + 3\pi \frac{Q}{r_c} \left( \frac{1}{L_z^2} + \frac{2}{r_c^2} \right) - 3\pi \frac{J^2}{r_c} \left( \frac{4}{L_z^4} + \frac{59}{2r_c^2} \right) - 8\pi \frac{JQ}{L_z^3 r_c} \sqrt{L_z^2 + r_c^2} \left( \frac{1}{L_z^2} + \frac{1}{r_c^2} \right) \\
+ 24\pi \frac{JQ}{L_z^3 r_c} \sqrt{L_z^2 + r_c^2} \left( \frac{1}{L_z^2} + \frac{3}{r_c^2} \right) + 27\pi \frac{M^2}{r_c^4} + 3\pi \frac{MQ}{2r_c^4} \left( \frac{30}{L_z^2} + \frac{53}{r_c^2} \right) \\
+ \frac{9}{4} \pi \frac{Q^2}{r_c^4} \left( \frac{3}{L_z^4} + \frac{22}{L_z^2 r_c^2} + \frac{63}{r_c^4} \right)
\]

(28)

This result agrees with the result expected from the Schwarzschild metric, \( \Delta \phi \approx 6\pi M/r_c \), up to first order in mass. For the perihelion precession of Mercury some of the contributions can be computed as is shown in table 4. The gravitational periastron precession in the orbit of the star S2 are also included, and they agree with the reported value in literature of 12 arcmin per orbit (≈75 arcsec per century) near the pericentre [21].

8. Time delay

The effect is represented in figure 3, as the path of rays of light are turned away from their classical trajectories. The curvature induced in the spacetime surrounding a massive body increases the travel time of light rays relative to what would be the case in flat space. Let \( b \) be the maximum approach distance of a ray of light traveling near a massive body. If the beam traveled in a straight line, then \( r \cos \phi = b \). This means that

\[
d\phi = \frac{bdr}{\sqrt{r^2 - b^2}}.
\]

By using \( d\theta = 0 \), it is possible to extract \( dt \) from \( g_{\mu\nu} dx^\mu dx^\nu = 0 \), so we obtain:

\[
dt = \frac{dr}{\sqrt{r^2 - b^2}} \left[ 2M + r - 2\frac{M}{r} + \frac{Mb^2}{r^2} + \frac{Q}{r^3} + \frac{5Q}{r^3} + 27\frac{J^2b^2}{r^2} - 4\frac{JMb}{r^3} - 2\frac{JQb}{r^3} \right] \\
+ 4\frac{M^2}{r} - 2\frac{M^2b^2}{r^2} - \frac{1}{2} \frac{M^2}{r^3} + 5\frac{MQ}{r^3} + 5\frac{MQb^2}{r^3} + \frac{31}{8} \frac{Q^2b^2}{r^3} - \frac{33}{8} \frac{QQ^2b^2}{r^3}.
\]

(29)
Performing an integration to go from a planet at position $r_0$ to another planet at $r_p$, to find the time delay:

$$\Delta t = d_c + d_p + 2J \left( \frac{b}{r_c d_e} + \frac{b}{r_p d_p} - \frac{r_c}{bd_e} - \frac{r_p}{bd_p} \right) + 2M \log \left( \frac{(r_c + d_c)(r_p + d_p)}{b^2} \right)$$

$$+ M \left( \frac{b^2}{r_c d_e} + \frac{b^2}{r_p d_p} - \frac{r_c}{d_e} - \frac{r_p}{d_p} \right) + Q \left( \frac{d_e}{b^2} + \frac{d_p}{b^2} \right) - 27J^2 \left( \frac{b^2}{r_c^3 d_e} - \frac{b^2}{r_p^3 d_p} \right)$$

$$+ \frac{53}{32} J^2 \left( \frac{1}{r_c^3 d_e} + \frac{1}{r_p^3 d_p} \right) + \frac{1}{32} J \left( \frac{\pi}{b^3} - \theta_e - \theta_p + \frac{1}{b^2 d_e} + \frac{1}{b^2 d_p} \right)$$

$$+ 2JM \left( \frac{b}{r_c^3 d_e} + \frac{b}{r_p^3 d_p} - \frac{1}{bd_e} - \frac{1}{bd_p} \right) + 2JM \left( \frac{\theta_e}{b^2} + \frac{\theta_p}{b^2} - \frac{\pi}{b^4} \right)$$

$$+ \frac{1}{2} JQ \left( \frac{b}{r_c^3 d_e} + \frac{b}{r_p^3 d_p} \right) + \frac{3}{4} JQ \left( \frac{\theta_e}{b^4} + \frac{\theta_p}{b^4} \right) - \frac{3}{4} JQ \left( \frac{1}{b^3 d_e} + \frac{1}{b^3 d_p} + \frac{\pi}{b^4} \right)$$

$$+ \frac{1}{4} JQ \left( \frac{1}{br_c^4 d_e} + \frac{1}{br_p^4 d_p} \right) + \frac{5}{8} JQ \left( \frac{b^4}{r_c^6 d_e} + \frac{b^4}{r_p^6 d_p} \right) + \frac{9}{16} MQ \left( \frac{b^2}{r_c^4 d_e} + \frac{b^2}{r_p^4 d_p} \right)$$

$$+ \frac{37}{16} MQ \left( \frac{\pi}{b^3} - \theta_e - \theta_p + \frac{1}{b^2 d_e} + \frac{1}{b^2 d_p} \right) + \frac{1}{16} MQ \left( \frac{1}{d_e} + \frac{1}{d_p} \right) - \frac{75}{32} MQ \left( \frac{1}{r_c^2 d_e} + \frac{1}{r_p^2 d_p} \right)$$

$$+ \frac{11}{16} Q^2 \left( \frac{b^2}{r_c^6 d_e} + \frac{b^2}{r_p^6 d_p} \right) + \frac{21}{128} Q^2 \left( \frac{\pi}{b^5} - \theta_e - \theta_p + \frac{1}{b^4 d_e} + \frac{1}{b^4 d_p} \right)$$

$$- \frac{7}{128} Q^2 \left( \frac{1}{b^6 d_e} + \frac{1}{b^6 d_p} \right) - \frac{51}{64} Q^2 \left( \frac{1}{r_c^4 d_e} + \frac{1}{r_p^4 d_p} \right)$$

where $d_c = \sqrt{r_c^2 - b^2}$, $d_p = \sqrt{r_p^2 - b^2}$, $\theta_e = \sin^{-1}(b/r_c)$, and $\theta_p = \sin^{-1}(b/r_p)$.

This result agrees with the result expected from the Schwarzschild metric, up to first order in mass [33]. Some of the contributions of the gravitational delay of light grazing the solar limb and the planets as predicted by our model are presented in table 5.

9. Gravitational redshift

The effect is represented in figure 4. It is possible to calculate a redshift factor by comparing the proper time for observers located at two different values of $r$, assuming a planar orbit, $\theta = \pi/2$. 

![Figure 3. Time delay of light signals.](image-url)
This result agrees with the result expected from the Schwarzschild metric, up to first order in mass \[34\]. The gravitational redshift in the orbit of the star S2 agrees with the reported value in literature of \[10^3 \text{km s}^{-1}/c\] near the pericentre \[21, 22\], as it is shown in table 6.

### 10. Conclusions

We reviewed the calculations of the classical experiments in GR with an approximative metric and taking into account all second order terms of mass, angular momentum and mass quadrupole. If we neglect these terms our results agree with the ones in the literature. By using our results, it could now be possible to estimate the value of second order terms of mass, quadrupole and angular momentum and determine how well they adapt to the predicted phenomena in the classical tests.

In PPN theory these results were obtained, but in this theory the quadrupole moment is introduced in the expansion of the mass potential. Here, this effect is introduced by the metric in a straightforward way. Our calculations were done in a simple manner using Mathematica. Moreover, we developed a Mathematica notebook, which is available upon request. The notebook is divided in sections, each one corresponding to a classical test. These calculations in the PPN method are rather complicated, but it would be interesting to expand them using the PPN methods.

### Table 5.

| Stellar object | Second order: PN (ns) | Rotation (ns) | Fourth order: PPN (ns) | Quadrupole: PN (ns) |
|----------------|-----------------------|---------------|------------------------|---------------------|
| Sun            | \[1.096 \times 10^6\] | \[7.869 \times 10^{-3}\] | \[3.033 \times 10^{-1}\] | \[5.417 \times 10^{-2}\] |
| Mercury        | \[3.508 \times 10^{-2}\] | \[1.296 \times 10^{-11}\] | \[2.388 \times 10^{-12}\] | \[---\] |
| Venus          | \[4.352 \times 10^{-1}\] | \[9.896 \times 10^{-11}\] | \[2.093 \times 10^{-10}\] | \[---\] |
| Mars           | \[6.510 \times 10^{-2}\] | \[1.916 \times 10^{-9}\] | \[6.485 \times 10^{-12}\] | \[6.243 \times 10^{-6}\] |
| Jupiter        | \[1.746 \times 10^2\]  | \[1.954 \times 10^{-4}\] | \[2.718 \times 10^{-6}\] | \[1.387 \times 10^{-1}\] |
| Saturn         | \[5.722 \times 10^2\]  | \[4.192 \times 10^{-5}\] | \[2.876 \times 10^{-7}\] | \[4.630 \times 10^{-2}\] |
| Uranus         | \[1.016 \times 10^3\] | \[1.322 \times 10^{-6}\] | \[1.646 \times 10^{-8}\] | \[5.164 \times 10^{-3}\] |
| Neptune        | \[1.248 \times 10^3\] | \[3.392 \times 10^{-6}\] | \[2.249 \times 10^{-8}\] | \[2.036 \times 10^{-3}\] |

### Table 6.

| Stellar object | First Order: Mass (\text{km s}^{-1}/c) | Second Order: Mass (\text{km s}^{-1}/c) |
|----------------|----------------------------------------|----------------------------------------|
| Sun            | \[103.24\]                             | \[0.053\]                              |
| Mercury        | \[0.053\]                              | \[292.3\]                              |

\[
\frac{\lambda_r}{\lambda_e} \approx 1 + M \left( \frac{1}{r} - \frac{1}{r_e} \right) + \frac{Q}{2} \left( \frac{1}{r^2} - \frac{1}{r_e^2} \right) + \frac{3}{2} M \left( \frac{1}{r^2} - \frac{2}{r_e} - \frac{1}{r_e^2} \right) + \frac{M Q}{r_e^4} - \frac{1}{2 r_e^2} - \frac{2}{2 r_e^2} - \frac{1}{r_e^4} + \frac{Q^2}{2} \left( \frac{1}{r_e^6} - \frac{1}{8 r_e^6} + \frac{1}{8 r_e^6} \right) \]

(31)

This result agrees with the result expected from the Schwarzschild metric, up to first order in mass \[34\]. The gravitational redshift in the orbit of the star S2 agrees with the reported value in literature of \[10^3 \text{km s}^{-1}/c\] near the pericentre \[21, 22\], as it is shown in table 6.

### Figure 4. Gravitational redshift.
As future work, it is planned to include the spin octupole and the mass hexadecapole, because now, these relativistic multipoles are currently considered in neutron stars calculations. For instance, to determine the innermost stable circular orbit or the precession frequencies, these relativistic multipole moment play an important role [35, 36]. Moreover, it would be interesting to investigate the effect of the quadrupole moment in the gravitational lens effect. To do it, one has to employ the PPN formalism. The results of this research can also serve as a basis for predicting the effects of rotation when better MBH spin measurements have been made.

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