Classification of Reductive Monoid Spaces Over an Arbitrary Field

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August 16, 2018

Abstract

In this semi-expository paper, we review the notion of a spherical space. In particular, we present some recent results of Wedhorn on the classification of spherical spaces over arbitrary fields. As an application, we introduce and classify reductive monoid spaces over an arbitrary field.

1 Introduction

The classification of spherical homogenous varieties, more generally of spherical actions, is an important, lively, and very interesting chapter in modern algebraic geometry. It naturally encompasses the classification theories of toric varieties, horospherical varieties, symmetric varieties, wonderful compactifications, as well as that of reductive monoids. Our goal in this paper is to give an expository account of some recent work in this field. As far as we are aware, the broadest context in which a classification of such objects is achieved, is in the theory of algebraic spaces. This accomplishment is due to Wedhorn [34]. Here, we will follow Wedhorn’s footsteps closely to derive some conclusions. At the same time, our intention is to provide enough detail to make basic definitions accessible to beginners. We will explain a straightforward application of Wedhorn’s progress to monoid schemes.

It is not completely wrong to claim that the origins of our story go back to Legendre’s work [18], where he analyzed the gravitational potential of a point surrounded by a spherical surface in 3-space. To describe the representative functions of his enterprise, he found a clever change of coordinates argument and introduced the set of orthogonal polynomials \( \{ P_n(x) \}_{n \geq 0} \) via \( \frac{1}{\sqrt{1-2hx+h^2}} = \sum_{n \geq 0} P_n(x)h^n \), which are now known as Lagrange polynomials. It has eventually been understood that the \( P_n(x) \)'s are the eigenfunctions of the operator \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) restricted to the space of \( C^\infty \) functions on the unit 2-sphere. Nowadays, bits and pieces of these elementary facts can be found in every standard calculus textbook but their generalization to higher dimensions can be explained in a conceptual way by using transformation groups.
Let $n$ be a positive integer, and let $Q_n$ denote the standard quadratic form on $\mathbb{R}^n$,

$$Q_n(x) := x_1^2 + \cdots + x_n^2, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$ 

The orthogonal group, denoted by $O(Q_n)$, consists of linear transformations $L: \mathbb{R}^n \to \mathbb{R}^n$ such that $Q_n(L(x)) = Q_n(x)$ for all $x \in \mathbb{R}^n$. It acts transitively on the $n-1$ sphere $S^{n-1} = \{ x \in \mathbb{R}^n : Q_n(x) = 1 \}$, and the isotropy subgroup in $O(Q_n)$ of a point $x \in S^{n-1}$ is isomorphic to $O(Q_{n-1})$. It is not difficult to write down a Lie group automorphism $\sigma: O(Q_n) \to O(Q_n)$ of order two such that the fixed point subgroup $O(Q_n)^\sigma$ is isomorphic to $O(Q_{n-1})$. In other words, $S^{n-1}$ has the structure of a symmetric space, that is, a quotient manifold of the form $G/K$, where $G$ is a Lie group and $K = \{ g \in G : \sigma(g) = g \}$ is the fixed subgroup of an automorphism $\sigma: G \to G$ with $\sigma^2 = id$. It is known that the Laplace-Beltrami operator $\Delta_n := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ generates the algebra of $O(Q_n)$-invariant differential operators on $S^{n-1}$. Moreover, for each $k \in \mathbb{N}$, there is one eigenspace $E_k$ of $\Delta_n$ with eigenvalue $-k(k+n-2)$; each $E_k$ defines a finite dimensional and irreducible representation of $O(Q_n)$. In addition, the Hilbert space of square integrable functions on $S^{n-1}$ has an orthogonal space decomposition, $L^2(S^{n-1}) = \bigoplus_{k=0}^\infty E_k$. The last point of this example is the most important for our purposes; the representation of $O(Q_n)$ on the polynomial functions on $S^{n-1}$ is multiplicity-free! All these facts are well known and can be found in classical textbooks such as [14] or [7].

We will give another example to indicate how often we run into such multiplicity-free phenomena in the theory of Lie groups. This time we start with an arbitrary compact Lie group, denoted by $K$, and consider $C^0(K, \mathbb{C})$, the algebra of continuous functions on $K$ with complex values. The doubled group $\bar{K} = K \times K$ acts on $K$ by translations:

$$(g, h) \cdot x = gxh^{-1} \quad \text{for all } g, h, x \in K.$$ 

In particular, we have a representation of $K \times K$ on $C^0(K, \mathbb{C})$, which is infinite dimensional unless $K$ is a finite group. Let $\mathcal{F}(K, \mathbb{C})$ denote the subalgebra of representative functions, which, by definition, are the functions $f \in C^0(K, \mathbb{C})$ such that $K \times K \cdot f$ is contained in a finite dimensional submodule of $C^0(K, \mathbb{C})$. The theorem of Peter and Weyl states that $\mathcal{F}(K, \mathbb{C})$ is dense in $C^0(K, \mathbb{C})$. Moreover as a representation of $K \times K$, $\mathcal{F}(K, \mathbb{C})$ has an orthogonal space decomposition into finite dimensional irreducible $K \times K$-representations, each irreducible occurring with multiplicity at most one.

We have a quite related, analogous statement on the multiplicity-freeness of the $G \times G$-module structure of the coordinate ring $\mathbb{C}[G]$ of a reductive complex algebraic group $G$. Indeed, it is a well known fact that on every compact Lie group $K$ there exists a unique real algebraic group structure, and furthermore, its complexification $K(\mathbb{C})$ is a complex algebraic group which is reductive. Conversely, any reductive complex algebraic group has an algebraic compact real form and this real form has the structure of a compact Lie group. Two compact Lie groups are isomorphic as Lie groups if and only if the corresponding reductive complex algebraic groups are isomorphic (see [8, 22]).

The unifying theme of these examples, as we alluded to before, is the multiplicity-freeness of the action on the function space of the underlying variety or manifold. It
turns out that the multiplicity-freeness is closely related to the size of orbits of certain subgroups. To explain this more clearly, for the time being, we confine ourselves to the setting of affine algebraic varieties that are defined over $\mathbb{C}$. But we have a disclaimer: irrespective of the underlying field of definitions, our tacit assumption throughout this paper is that all reductive groups are connected unless otherwise noted.

Now, let $G$ be a reductive complex algebraic group, and let $X$ be an affine variety on which $G$ acts algebraically. We fix a Borel subgroup $B$, that is, a maximal connected solvable subgroup of $G$. It is well known that every other Borel subgroup of $G$ is conjugate to $B$. The action of $G$ on $X$ gives rise to an action of $G$, hence of $B$, on the coordinate ring $\mathbb{C}[X]$. Let us assume that $B$ has finitely many orbits in $X$, so, in the Zariski topology, one of the $B$-orbits is open. Let $\chi$ be a character of $B$ and $x_0$ be a general point from the dense $B$-orbit. Let $f$ be a regular function that is only defined on the open orbit. Hence, we view $f$ as an element of $\mathbb{C}(X)$, the field of rational functions on $X$. If $f$ is an $\chi$-eigenfunction for the $B$-action, that is,

$$b \cdot f = \chi(b)f \quad \text{for all } b \in B,$$

then the value of $f$ on the whole orbit is uniquely determined by $\chi$ and the base point $x_0$. Indeed, any point $x$ from the open orbit has the form $x = b \cdot x_0$ for some $b \in B$, therefore,

$$f(x) = b^{-1} \cdot f(x_0) = \chi(b^{-1})f(x_0).$$

This simple argument shows that there exists at most one $\chi$-eigenvector in $\mathbb{C}[X]$ whose restriction to the open $B$-orbit equals $f$. As irreducible representations of reductive groups are parametrized by the ‘highest’ $B$-eigenvectors, now we understand that the number of occurrence of the irreducible representation corresponding to the character $\chi$ in $\mathbb{C}[X]$ cannot exceed 1. In other words, $\mathbb{C}[X]$ is a multiplicity-free $G$-module. The converse of this statement is true as well; if a linear and algebraic action $G \times \mathbb{C}[X] \to \mathbb{C}[X]$ is multiplicity-free, then a Borel subgroup of $G$ has finitely many orbits in $X$, and one of these orbits is open and dense in $X$ (see [3], as well as [32, 24]). This brings us to the (special case of a) fundamental definition that will occupy us in the rest of our paper.

**Definition 1.1.** Let $k$ be an algebraically closed field, and let $G$ be a reductive algebraic group defined over $k$. Let $X$ be a normal variety that is defined over $k$, and finally, let $\psi : G \times X \to X$ be an algebraic action of $G$ on $X$. If the restriction of the action to a Borel subgroup has finitely many orbits, then the action is called spherical, and $X$ is called a spherical $G$-variety.

Let $H \subset G$ be a closed subgroup in a reductive group $G$. The homogenous space $G/H$ is called spherical if $BH$ is an open dense subvariety in $G$ for some Borel subgroup $B \subset G$. A $G$-variety $X$ is called an equivariant embedding of $G/H$ if $X$ has an open orbit that is isomorphic to $G/H$. In particular, spherical varieties are normal $G$-equivariant embeddings of spherical homogenous $G$-varieties. To see this, let $x_0$ be a general point from the open $B$-orbit in a spherical variety $X$, and let $H$ denote the isotropy subgroup $H = \{g \in G : g \cdot x_0 = x_0\}$ in $G$. Clearly, $G/H$ is isomorphic to the open $G$-orbit, and it is a spherical subvariety of $X$. It follows that $X$ is a $G$-equivariant embedding of $G/H$. 

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It goes without saying that this theory, as we know it, owes its existence to the work of Luna and Vust [20] who classified the equivariant embeddings of spherical homogenous varieties over algebraically closed fields of characteristic 0. Their results are extended to all characteristics by Knop in [16]. What is left is the classification of spherical subgroups (over arbitrary fields) and this program is well on its way; see the recent paper [2] and the references therein. For a good and broad introduction of the field of equivariant embeddings, up to 2011, we recommend the encyclopedic treatment [31] of Timashev.

We mentioned in the first paragraph of this introduction that the examples of spherical embeddings include algebraic monoids. By definition, an algebraic monoid over an algebraically closed field is an algebraic variety \( M \) endowed with an associative multiplication morphism \( m : M \times M \to M \) and there is a neutral element for the multiplication. The foundations of these monoid varieties are secured mainly by the efforts of Renner and Putcha, who chiefly developed the theory for linear (affine) algebraic monoids (see [28, 25]). From another angle, Brion and Rittatore looked at the general structure of an algebraic monoid. Amplifying the importance of linear algebraic monoids, Brion and Rittatore showed that any irreducible normal algebraic monoid is a homogenous fiber bundle over an abelian variety, where the fiber over the identity element is a normal irreducible linear algebraic monoid (see Brion’s lecture notes [5]). In this regard, let us recite a result of Mumford about the possible monoid structure on a complete irreducible variety ([21, Chapter II]): if a complete irreducible variety \( X \) has a (possibly nonassociative) composition law \( m : X \times X \to X \) with a neutral element, then \( X \) is an abelian variety with group law \( m \). In other words, an irreducible and complete algebraic monoid is an abelian variety. This interesting result of Mumford is extended to families by Brion in [5].

A \( G \)-equivariant embedding is said to be simple if it has a unique closed \( G \)-orbit. A **reductive monoid** is an irreducible algebraic monoid whose unit-group is a reductive group. The role of such monoids for the theory of equivariant embeddings was understood very early; Renner recognized in [27] and [26] that the normal reductive monoids are simple \( G \times G \)-equivariant embeddings of reductive groups. Explicating this observation, Rittatore showed in [29, Theorem 1] that every irreducible algebraic monoid \( M \) is a simple \( G(M) \times G(M) \)-equivariant embedding of its unit-group \( G(M) \).

In the same paper, by using Knop’s work on colored fans, which we will describe in the sequel, Rittatore described a classification of reductive monoids in terms of colored cones. This classification is a generalization of the earlier classification of the reductive normal monoids by Renner [27] and Vinberg [33].

Let \( k \) be an algebraically closed field. We define a **toric variety over \( k \)** as a normal algebraic variety on which the torus \( (k^*)^n \) acts (faithfully) with an open orbit. Toric varieties are prevalent in the category of spherical varieties in the following sense: if \( X \) is a spherical \( G \)-variety over \( k \), then the closure in \( X \) of the \( T \)-orbit of a general point from the open \( B \)-orbit will be a toric variety. Here, \( T \) stands for a maximal torus contained in \( B \). Also, let us not forget the fact that the affine toric varieties are precisely the commutative reductive monoids, [29, Theorem 3].

Now let \( k \) be an arbitrary field, and let \( T \) be a torus that is isomorphic to \( (k^*)^n \), where both the isomorphism and the torus \( T \) are defined over \( k \). The technical term
for such a torus is \( k \)-split or split torus over \( k \). Toric varieties defined by split tori are parametrized by combinatorial objects, the so-called fans. This fact is due to Demazure in the smooth case [12] and Danilov for all toric varieties [10]. A fan \( \mathcal{F} \) in \( \mathbb{Q}^n \) is a finite collection of strictly convex cones such that 1) every face of an element \( C \) from \( \mathcal{F} \) lies in \( \mathcal{F} \); 2) the intersection of two elements of \( \mathcal{F} \) is a face of both of the cones. Here, by a cone we mean a subset of \( \mathbb{Q}^n \) that is closed under addition and the scaling action of \( \mathbb{Q}_{\geq 0} \). A face of a cone \( C \) is a subset of the form \( \{ v \in C : \alpha(v) = 0 \} \), where \( \alpha \) is a linear functional on \( \mathbb{Q}^n \) that takes nonnegative values on \( C \). Let us define two more notions that will be used in the sequel. A cone is called strictly convex, if it does not contain a line. The relative interior of \( C \), denoted by \( C^0 \), is what is left after removing all of its proper faces.

The toric varieties defined by nonsplit tori are quite interesting and their classification is significantly more intricate. Parametrizing combinatorial objects in this case, as shown by Huruguen in [15] are fans that are stable under the Galois group of a splitting field. To explain this, we extend our earlier definition of toric varieties as follows. Let \( k \) be a field, and let \( \overline{k} \) denote an algebraic closure of \( k \). Let \( T \) be a torus defined over \( k \). A normal \( T \)-variety \( Y \) is called a toric variety over \( k \) if \( Y(\overline{k}) \) is a toric variety with respect to \( T(\overline{k}) \). (This notation will be made precise in Section 2.) Let us continue with the assumption that \( Y \) is a toric variety with respect to a \( k \)-split torus \( T \), and let \( k' \subseteq k \) be a field extension with \( T \) not necessarily split over \( k' \). Let \( \Gamma \) denote the Galois group of the extension. Since all tori become split over a finite separable field extension, we will assume also that \( k' \subset k \) is a finite extension. Of course, it may happen that \( Y \) is not defined over \( k' \). If it is defined, then \( Y(k') \) is called the (!) \( k' \)-form of \( Y \). In [15, Theorem 1.22], Huruguen gives two necessary and sufficient conditions for the existence of a \( k' \)-form. The first of these two conditions is rather natural in the sense that the fan of \( Y(k) \) is stable under the action of \( \Gamma \). The second condition is also concrete, however, it is more difficult to check. Actually, it is a criterion about the quasiprojectiveness of an equivariant embedding in terms of the fan of the embedding. Its colored version, namely a quasiprojective colored fan, which is also used by Huruguen, is introduced by Brion in [4]. We postpone the precise definition of a quasiprojective colored fan to Section 4.1, but let us mention that, in [15], Huruguen shows by an example that quasiprojectiveness is an essential requirement for \( Y \) to have a \( k' \)-form.

Using the underlying idea that worked for toric varieties, in the same paper, Huruguen proves more. Let \( Y \) be a spherical homogenous variety for a reductive group \( G \) defined over a perfect field \( k \). (The perfectness assumption is a minor glitch; it will not be needed once we start to work with the algebraic spaces as Wedhorn does. Indeed, it is required here so that the homogenous varieties have rational points.) Let \( \overline{k} \) be an algebraic closure of \( k \), and let \( \Gamma \) denote the Galois group of \( k \subset \overline{k} \). The introduction of the absolute Galois group, which is often too big, is not a serious problem since in the situations that we are interested in, the absolute Galois group factors through a finite quotient. Let \( \overline{Y} \) be a \( G \)-equivariant embedding of a spherical homogenous \( G \)-variety \( Y \) that is defined over \( \overline{k} \). In [15, Theorem 2.26], Huruguen shows that a \( G \)-equivariant embedding \( \overline{Y} \) of \( Y \) has a \( k \)-form if and only if the colored fan of \( \overline{Y}(\overline{k}) \) is \( \Gamma \)-stable and it is "quasiprojective with respect to \( \Gamma \)." Once again, Huruguen shows by examples...
that the failure of the second condition implies the nonexistence of $k$-forms.

As Wedhorn shows in [34] via algebraic spaces, as soon as we get over the contrived emotional barrier set in front of us by algebraic varieties, we happily see that the existential questions (about $k$-forms) disappear. Roughly speaking, in some sense, algebraic spaces are to schemes, what schemes are to algebraic varieties. Such is the transition from spherical varieties to spherical spaces.

**Definition 1.2.** For an arbitrary base scheme $S$ and a reductive group scheme $G$ over $S$, a spherical $G$-space over $S$ is a flat separated algebraic space of finite presentation over $S$ with a $G$-action such that the geometric fibers are spherical varieties.

Notice that we have not defined reductive $S$-groups yet. But timeliness may not be the only problematic aspect of Definition 1.2. Understandably, it may look overly general at first sight. Nevertheless, the definition has many remarkable consequences. For example, according to this definition, the property for a flat finitely presented $G$-space with normal geometric fibers to be spherical is open and constructible on the base scheme. Moreover, for a flat finitely presented subgroup scheme $H$ of $G$ the property to be spherical is open and closed on the base scheme. Most importantly for our purposes, the specialization of the base scheme to the spectrum of a field in Definition 1.2 yields a general classification of spherical $G$-spaces over arbitrary fields in terms of colored fans that are stable under the Galois group. In fact, for spaces over fields, Wedhorn’s definition is much easier to use.

Since explaining the concepts associated with spherical algebraic spaces and colored fans will take a bulky portion of our paper, we postpone giving the ballistics of Wedhorn’s theory to Section 4.2. Nearing the end of our lengthy introduction, let us mention that as a rather straightforward consequence of Wedhorn’s deus ex machina we obtain a classification of “reductive monoid spaces,” the only new definition that we offer in this paper.

The organization of our paper is as follows. In Section 2, we introduce what is required to explain spherical algebraic spaces in the following order: Subsection 2.1 is on the fundamentals; we introduce the notion of scheme as a functor. Subsection 2.2 sets up the notation for algebraic spaces. In Subsection 2.3, we talk about group schemes, which is followed by Subsection 2.4 on reductive group schemes. The Subsection 2.5 is devoted to parabolic subgroups. In Subsection 2.6, we briefly discuss actions of group schemes. In Section 3, we discuss affine monoid schemes and prove a souped-up version of a result of Rittatore on the unit-dense algebraic monoids. The beginning of Section 4 is devoted to the review of colored fans. In Subsection 4.1, we review the quasiprojectiveness criterion of Brion and mention Huruguen’s theorem on the $k$-forms of spherical varieties over perfect fields. In Subsection 4.2, we present Wedhorn’s generalization of Huruguen’s results. In particular we talk about his colored fans for spherical algebraic spaces. In the subsequent Section 5, we introduce reductive monoid spaces and apply Wedhorn’s and Huruguen’s theorems. Finally, we close our paper in Subsection 5.1 by presenting an application of our observations to the lined closures of representations of reductive groups, whose geometry is investigated by De Concini in [11].

**Acknowledgement.** I thank the organizers of the 2017 Southern Regional Algebra Conference: Laxmi Chataut, Jörg Feldvoss, Lauren Grimley, Drew Lewis, Andrei
I am grateful to Jörg Feldvoss, Lex Renner, Soumya Dipta Banerjee, and to the anonymous referee for their very careful reading of the paper and for their suggestions, which improved the quality of the article. This work was partially supported by a grant from the Louisiana Board of Regents.

2 Notation and preliminaries

The purpose of this section is to set up our notation and provide some background on algebraic spaces, reductive \( k \)-groups, and reductive \( k \)-monoids. We tried to give most of the necessary definitions for explaining the logical dependencies. For standard algebraic geometry facts, we recommend the books [13] and [17]. (It seems to us that the Stacks Project [30] will eventually replace all standard references.) In addition, we find that Brion’s lecture notes in [6] are exceptionally valuable as a resource for the background on algebraic groups.

Notation: Throughout our paper \( k \) will stand for a field, as \( G \) does for a group. \( k \) is called a perfect field if every algebraic extension of \( k \) is separable.

2.1 Schemes.

A point that we want to make in this section is that the only way to know a group scheme is to know all (affine) group schemes related to it. In fact, this statement is a theorem.

Terminology: Let \((X, \mathcal{F})\) be a pair of a topological space \( X \) and a sheaf of rings \( \mathcal{F} \) on \( X \). If for each point \( x \in X \), the corresponding stalk \( \mathcal{F}_x \) is a local ring, then the pair \((X, \mathcal{F})\) is called a locally ringed space. An affine scheme is a locally ringed space which is of the form \((\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})\), where \( R \) is a commutative ring, and \( \text{Spec}(R) \) is its spectrum endowed with the Zariski topology. The sheaf \( \mathcal{O}_{\text{Spec}(R)} \) is the structure sheaf of \( \text{Spec}(R) \). A scheme is a locally ringed space \((X, \mathcal{O}_X)\) which has a covering by open subsets \( X = \bigcup_{i \in I} U_i \) such that each pair \((U_i, \mathcal{O}_X|_{U_i})\) is an affine scheme.

There is a tremendous advantage of using categorical language while studying a scheme in relation with others. Therefore, we proceed with the identification of schemes with their functors of points. In this regard, we will use the following standard notation throughout:

- \( \text{Obj}(\mathcal{C}) \) : the class of objects of a category \( \mathcal{C} \)
- \( \text{Mor}(\mathcal{C}) \) : the class of morphisms between the objects of \( \mathcal{C} \)
- \( \text{Mor}(X, Y) \) : the set of morphisms from \( X \) to \( Y \), where \( X, Y \in \text{Obj}(\mathcal{C}) \)
- \( \mathcal{C}^\circ \) : the category opposite to \( \mathcal{C} \)
- \( \text{(schemes)} \) : the category of schemes
- \( \text{(sets)} \) : the category of sets

The functor of points of a scheme \( X \) is the functor \( h_X : (\text{schemes})^\circ \to (\text{sets}) \) that is defined by the following assignments:
1. if $Y \in \text{Obj}((\text{schemes})^o)$, then $h_X(Y) = \text{Mor}(Y,X)$;
2. if $f \in \text{Mor}(Y,Z)$, then $h_X(f)$ is the set map

$$h_X(f) : h_X(Z) \rightarrow h_X(Y)$$

$$g \mapsto f \circ g$$

Let $h$ denote the following natural transformation:

$$h : (\text{schemes}) \rightarrow \text{functors}((\text{schemes})^o, (\text{sets}))$$

$$X \mapsto h_X$$

(2.1)

Here, $\text{functors}(-,-)$ stands for the category whose objects are functors, and its morphisms are the natural transformations between functors. It follows from Yoneda’s Lemma that (2.1) is an equivalence onto a full subcategory of the target category. In other words, a scheme $X$ is uniquely represented by its functor of points $h_X$.

**Terminology:** In the presence of a morphism $Y \rightarrow X$ between two schemes $X$ and $Y$, we will occasionally say that $Y$ is a scheme over $X$ or that $Y$ is an $X$-scheme. If there is no danger for confusion, we will write $X(Y)$ for $h_X(Y)$, which is the set of all morphisms from $Y$ to $X$. If, in addition, $Y$ is an affine scheme of the form $Y = \text{Spec}(R)$, then we often write $X(R)$ instead of $X(Y)$, and we say that $X$ is a scheme over $R$.

We want to show that the functor $h$ in (2.1) behaves well upon restriction to the category of schemes over an affine scheme. To this end, let $R$ be a commutative ring, and denote the category of $R$-schemes by $(R\text{-schemes})$. A morphism in this category is a commutative diagram of morphisms as in Figure 2.1.

![Figure 2.1: A morphism in $(R\text{-schemes})$.](image)

It is well known that the category $(R\text{-schemes})$ is equivalent to the opposite category of commutative $R$-algebras, denoted by $(R\text{-algebras})^o$. The highlight of this subsection is the following result whose proof easily follows from the definitions.

**Proposition 1.** The functor

$$h : (R\text{-schemes}) \rightarrow \text{functors}((R\text{-algebras}),(\text{sets})),$$

which is obtained from (2.1) by restriction to the subcategory $(R\text{-schemes})$, is an equivalence onto a full subcategory of the target category. In particular, a scheme over $R$ is determined by the restriction of its functor of points to the category of affine schemes over $R$. 

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For our purposes, the most important consequence of Proposition 1 is that an $R$-scheme can be thought of as a sheaf of sets on the category of $R$-algebras with respect to the Zariski topology. Closely related to this sheaf realization of schemes is the notion of an “algebraic space.” Concisely, and very roughly speaking, an algebraic space is a sheaf of sets on the category of $R$-algebras with respect to the “étale topology.” We will give a more precise definition of algebraic spaces in the next subsection. Next, we remind ourselves of some basic notions regarding the morphisms between schemes.

1. A ring homomorphism $f : R \to S$ is called flat if the associated induced functor
   
   $$f_* : (R\text{-modules}) \to (S\text{-modules})$$

   is exact, that is to say, it maps short-exact sequences to short-exact sequences. In this case, the morphism $f^* : \text{Spec}(S) \to \text{Spec}(R)$ is called flat. More generally, a map $f : X \to Y$ of schemes is called flat if the induced functor

   $$f^* : (\text{Quasicoherent sheaves on } Y) \to (\text{Quasicoherent sheaves on } X)$$

   is exact.

2. A ring homomorphism $f : R \to S$ is called of finite presentation if $S$ is isomorphic (via $f$) to a finitely generated polynomial algebra over $R$ and the ideal of relations among the generators is finitely generated. A map $f : X \to Y$ of schemes is called a map of finite presentation at $x \in X$ if there exists an affine open neighborhood $U = \text{Spec}(S)$ of $x$ in $X$ and an affine open neighborhood $V = \text{Spec}(R)$ of $Y$ with $f(U) \subseteq V$ such that the induced ring map $R \to S$ is of finite presentation. More generally, a map $f : X \to Y$ of schemes is called of locally finite presentation if it is of finite presentation at every point $x \in X$.

3. A map $f : X \to Y$ of schemes is called separated if the induced diagonal morphism $X \to X \times_Y X$ is a closed immersion.

4. A map $f : X \to Y$ of schemes is called unramified if the induced diagonal morphism $X \to X \times_Y X$ is an open immersion.

5. A map $f : X \to Y$ of schemes is called étale if it is unramified, flat, and of locally finite presentation.

Associated with these types of maps, we have two important topologies. These are

1. The topology on the category of schemes associated with the étale morphisms; this topology is called the étale topology.

2. The topology on the category of schemes associated with the set of maps, flat and of locally finite presentation; this topology is called the fppf topology.

### 2.2 Algebraic spaces.

Let $S$ be a scheme. An algebraic space $X$ over $S$ is a functor

$$X : (S\text{-schemes})^\circ \to (\text{sets})$$

satisfying the following properties:
(i) $X$ is a sheaf in the fppf topology.

(ii) The diagonal morphism $X \to X \times_S X$ is representable by a morphism of schemes.

(iii) There exists a surjective étale morphism $\tilde{X} \to X$, where $\tilde{X}$ is an $S$-scheme.

It turns out that, in this definition, specifically in part (i), replacing the fppf topology by the étale topology on the category of schemes does not cause any harm. In other words, the resulting functor describes the same algebraic space (see [30, Tag 076L]).

Let us mention in passing that the category of schemes is a full subcategory of the category of algebraic spaces. We will follow the standard assumption, as in [34] (and as in [17]), that all algebraic spaces are quasiseparated over some scheme, hence they are Zariski locally quasiseparated. This implies that our algebraic spaces are reasonable and decent in the sense of [30, Tag 03IS] and [30, Tag 03JX].

Our final note in this subsection is that if $k$ is a separably closed field, then an algebraic space over $k$ is a $k$-scheme.

2.3 Group schemes.

Let $R$ be a commutative ring. A group scheme over $R$ is an $R$-scheme whose functor of points factors through the forgetful functor from the category of groups to the category of sets. In a nutshell, an $R$-scheme $G$ is called a group scheme over $R$ if for every $R$-scheme $S$, there is a natural group structure on $G(S)$ which is functorial with respect to the morphisms $R \to S$.

Terminology: If $G$ is a (group) scheme over $R$, then the morphism $G \to \text{Spec}(R)$ is called the structure morphism.

Definition 2.2. Let $k$ be a field and let $G$ be a group scheme over $k$. We will call $G$ a $k$-group (or, an algebraic group over $k$) if $G$ is of finite type as a scheme over $k$. If $k'$ is a subfield of $k$, then $G$ is said to be defined over $k'$ if $G$, as a scheme, and all of its group operations as morphisms are defined over $k'$.

Let $S$ be a scheme, and let $G$ be a group scheme over $S$. $G$ is called affine, smooth, flat, or separated, respectively, if the structure morphism $G \to S$ is affine, smooth, flat, or separated, respectively. We have some remarks regarding these properties:

1. Any $k$-group $G$ is separated as an algebraic scheme. Indeed, the diagonal in $G \times G$ is closed as being the inverse image of the “identity” point of $G(k)$ under the morphism $t : G \times G \to G$ defined by $t(g, h) = gh^{-1}$.

2. Let $k$ be a perfect field, and let $G$ be a $k$-group scheme. If the underlying scheme of $G$ is reduced, that is to say, the structure ring has no nilpotents, then $G$ is smooth (see [13, Pg 287]).

3. The Cartier’s theorem states that if the characteristic of the underlying field $k$ is 0, then a $k$-group scheme is reduced (see [21, Pg 101]). It follows that, in characteristic 0, all $k$-groups are smooth.
Next, we will define a particular \( k \)-group that is of fundamental importance for the whole development of algebraic group theory.

**Example 2.3.** Let \( V \) be a finite dimensional vector space over \( k \). The *general linear group*, denoted by \( GL(V) \), is the group functor such that

\[
GL(V)(S) = \text{the automorphism group of the sheaf of } \mathcal{O}_S \text{-modules } \mathcal{O}_S \otimes_k V,
\]

where \( S \) is a \( k \)-scheme. By choosing a basis for \( V \), we see that \( GL(V)(S) \) is isomorphic to the group of invertible \( n \times n \) matrices with coefficients in the algebra \( \mathcal{O}_S(S) \), hence, \( GL(V) \) is represented by the open affine scheme

\[
GL_n = \{ P \in \mathbb{A}^{n^2} : \det P \neq 0 \}.
\]

It follows that \( GL(V) \) is smooth and connected.

**Definition 2.4.** A group scheme \( G \) is called *linear* if it is isomorphic to a closed subgroup scheme of \( GL_n \) for some positive integer \( n \). If \( V \) is a vector space, then a homomorphism \( \rho : G \to GL(V) \), which is a morphism of schemes, is called a *linear representation* of \( G \) on \( V \). In this case, \( V \) is called a \( G \)-module.

Note that if a \( k \)-group scheme is linear, then it is an affine scheme. Note conversely that, every \( k \)-group scheme which is affine and of finite type is a linear group scheme (see [6, Proposition 3.1.1]).

In some sense, atomic pieces of \( k \)-groups are given by the following two very special \( k \)-groups:

**Example 2.5.** Let \( + \) and \( \cdot \) denote the addition and the multiplication operations on the field \( k \).

- The additive 1-dimensional \( k \)-group, denoted by \( \mathbf{G}_a \), is the affine line \( \mathbb{A}^1_k \) considered with the group structure \((k,+)\).
- The multiplicative 1-dimensional \( k \)-group, denoted by \( \mathbf{G}_m \), is \( \mathbb{A}^1_k - \{0\} \) considered with the group structure \((k^*,\cdot)\).

**Definition 2.6.** A \( k \)-group \( G \) is called a torus if there exists an isomorphism \( \zeta : G \to \mathbf{G}_m \times \cdots \times \mathbf{G}_m \) (\( n \) copies, for some \( n \geq 1 \)) over some field containing \( k \). Assuming that \( G \) is a torus defined over \( k' \), and that \( k' \) is a subfield of \( k \), \( G \) is called \( k' \)-split if both of \( G \) and \( \zeta \) are defined over \( k \).

A modern proof of the following basic result can be found in [9, Appendix A].

**Theorem 1** (Grothendieck). Let \( k' \) and \( k \) be two fields such that \( k' \subset k \). Let \( G \) be a smooth connected affine \( k \)-group. If \( G \) is defined over \( k' \), then \( G \) contains a maximal \( k' \)-split torus \( T \) such that \( T(\overline{k}) \) is a maximal torus of \( G(\overline{k}) \).
2.4 Reductive group schemes.

Almost any fact about reductive group schemes can be found in Conrad’s SGA3 replacement [9].

Let $G$ be a $k$-group, and let $\rho : G \hookrightarrow GL(V)$ be a finite dimensional faithful linear representation. An element $g$ from $G$ is called semisimple (respectively unipotent) if the linear operator $\rho(g)$ on $V$ is diagonalizable (respectively, unipotent). It is not difficult to check that these notions (semisimplicity and unipotency) are independent of the faithful representation, and they are preserved under $k$-homomorphisms. Therefore, the following definition is unambiguous:

**Definition 2.7.** Let $G$ be a linear algebraic $k$-group. A $k$-subgroup $U$ of $G$ is called unipotent if every element of $U$ is unipotent.

Next, we give the definition of a reductive group. However, we will do this in the opposite of the chronological development of the subject to emphasize the differences. So, let us start with the definition of the relative reductive group schemes. This is most useful for studying properties that are preserved in families over a commutative ring.

Let $S$ be a scheme, and let $G$ be a smooth $S$-affine group scheme over $S$. Let $s$ be a point from $S$, and denote by $G_{\overline{s}}$ the geometric fiber $G \times_S \text{Spec}(\overline{k(s)})$ of $G \to S$. Here, $k(s)$ is the residue field of $s$. If for each $s \in S$ the fiber $G_{\overline{s}}$ is a connected reductive group, then $G$ is called a reductive $S$-group.

So, what is a reductive $k$-group over an algebraically closed field? We take this opportunity to define the ‘unipotent radical’ and the ‘radical’ of an algebraic group. Let $G$ be a $k$-group (affine or not). There is a maximal connected solvable normal linear algebraic subgroup, denoted by $\mathcal{R}(G(\overline{k}))$, and it is called the radical of $G$ (see [6, Lemma 3.1.4]). If $\mathcal{R}(G(\overline{k}))$ is trivial, then $G(\overline{k})$ is called semisimple. The unipotent radical of $G(\overline{k})$, denoted by $\mathcal{R}_u(G(\overline{k}))$, is the maximal connected normal unipotent subgroup of $G(\overline{k})$. If $\mathcal{R}_u(G(\overline{k}))$ is trivial, then $G(\overline{k})$ is called reductive. Clearly, semisimplicity implies reductivity. Notice also that we have no connectedness assumption here. If the characteristic of $k$ is 0, then the property of reductiveness of the identity component of $G$ is equivalent to the semisimplicity of all linear representations of $G$. This equivalence fails in positive characteristics (see [9, Remark 1.1.13.]).

In the rest of our paper we will focus mainly on the “relative” reductive group schemes over fields.

**Definition 2.8.** Let $k$ be a field. A $k$-group $G$ is called reductive if the geometric fiber $G_{\overline{k}}$ (which we take to be equal to $G(\overline{k})$) is a connected reductive group in the sense of the previous paragraph.

2.5 Parabolic subgroups.

Let $G$ be a reductive $k$-group. A subgroup $P(\overline{k})$ of $G(\overline{k})$ is called parabolic if $G(\overline{k})/P(\overline{k})$ has the structure of a projective variety. More generally, a smooth affine $k$-subgroup
$P$ of $G$ is called a parabolic subgroup if $P(\bar{k})$ is a parabolic subgroup of $G(\bar{k})$.

A Borel subgroup in $G(\bar{k})$ is a maximal connected solvable subgroup. More generally, a parabolic $k$-subgroup $B$ of $G$ is called a Borel subgroup if $B(\bar{k})$ is a Borel subgroup in $G(\bar{k})$.

A fundamentally important result that is due to Borel (see [1, Theorem 11.1]) states that any two Borel subgroups are conjugate in $G(k)$, and furthermore, for any Borel $k$-subgroup $B(k)$ the quotient $G(\bar{k})/B(\bar{k})$ is projective. Of course, according to the above definition of parabolic $k$-subgroups, a Borel subgroup in $G(k)$ is a (minimal) parabolic $k$-subgroup.

**Example 2.9.** The upper triangular subgroup $T_n$ of $GL_n$ is a Borel subgroup. As a consequence of the Lie-Kolchin Theorem (see [1, Corollary 10.2]) any connected solvable group admits a faithful representation with image in $T_n$.

It may happen that $G(k)$ is defined over a field but has no nontrivial Borel subgroup. This holds true, even for some classical groups, as we will demonstrate in the next classic example (from [1]).

**Example 2.10.** Let $k$ be a field whose characteristic is not 2, and let $V$ be $k$-vector space. Let $Q$ be a nondegenerate quadratic form on $V$, and let $F$ denote its symmetric bilinear form. We assume that $Q$ is isotropic, that is to say, there exists a nonzero vector $v \in V$ such that $Q(v) = 0$. A subspace is called isotropic if it contains an isotropic vector; a subspace is called anisotropic if it contains no nonzero isotropic vector; a subspace is called totally isotropic if it consists of isotropic vectors only. A hyperbolic plane is a two-dimensional subspace $E$ of $V$ with a basis $\{e, f\}$ with respect to which the restriction of $F$ has the form $F(x_1e + x_2f, y_1e + y_2f) = x_1y_2 + x_2y_1$.

By the Witt’s Decomposition Theorem we know that the dimension $q$ of a maximal totally isotropic subspace is an invariant of $Q$. More precisely, it states that $V$ contains $q$ linearly independent hyperbolic planes $H_1, \ldots, H_q$, and $V$ is an orthogonal direct sum of the form

$$V \cong V_0 \oplus \bigoplus_{i=1}^{q} H_i,$$

where $V_0$ is an anisotropic subspace. Let $Q_0$ denote the restriction of $Q$ to $V_0$.

For $i = 1, \ldots, q$ we choose a basis $\{e_i, e_{n-q+i}\}$ for $H_i$ in such a way that the following identities are satisfied:

$$F(e_i, e_i) = F(e_{n-q+i}, e_{n-q+i}) = 0 \quad \text{and} \quad F(e_i, e_{n-q+i}) = 1.$$

Here $n$ is the dimension of $V$. Let $\{e_j : j = q+1, \ldots, n-q\}$ denote a basis for $V_0$. For each pair $e_i, e_{n-q+i}$ ($i = 1, \ldots, q$) of basis elements and $x \in \overline{k}$ we have a linear map $s_i(x) : V(\overline{k}) \to V(\overline{k})$ defined by

$$s_i(x)e_i = xe_i, \quad s_i(x)e_{n-q+i} = x^{-1}e_{n-q+i}, \quad \text{and} \quad s_i(x)f = f \quad \text{for} \quad f \in H_i^\perp.$$

Clearly, the $s_i(x)$'s are semisimple and generate a diagonal torus, denoted by $T$, whose elements expressed in the basis $\{e_1, \ldots, e_n\}$ of $V$ are of the form

$$\text{diag}(a_1, \ldots, a_q, 1, \ldots, 1, a_1^{-1}, \ldots, a_q^{-1}), \quad \text{where} \ a_i \in \overline{k}.$$
If $G$ denotes $\text{SO}(Q)$, the $k$-group consisting of linear automorphisms of $V$ that preserves the quadratic form $Q$ and of determinant 1, then $S := T \cap G$ is in fact a maximal torus of $G$. Note that the centralizer of $S$ in $G$, denoted by $\mathcal{Z}_G(S)$, is isomorphic to the product $S \times \text{SO}(Q_o)$.

The group $G$ is a reductive $k$-group, and since $S$ is a maximal torus, by definition, $\mathcal{Z}_G(S)$ is a Cartan subgroup of $G$. Therefore $\mathcal{Z}_G(S)$ is contained in a minimal parabolic $k$-subgroup $P$ of $G$. In fact, $\mathcal{Z}_G(S)$ is equal to the Levi component of $P$. Observe that, if $n > 2q + 2$, then $\text{SO}(Q_o)$ is not commutative, hence $\mathcal{Z}_G(S)$ is not contained by a Borel subgroup. We conclude that if $n > 2q + 2$, then $G$ does not have any Borel subgroups.

For connected semisimple $k$-groups, where $k$ is a perfect field, the question of the existence of Borel subgroups has a nice answer.

**Theorem 2** (Ono). Let $k'$ and $k$ be two fields such that $k' \subset k$ and $k'$ is perfect. If $G$ is a connected semisimple $k$-group, then $G$ posses a Borel subgroup $B$ defined over $k'$ if and only if a maximal torus $T$ of $G$ is $k'$-split. In this case, all Borel subgroups containing $T(k')$ are conjugate by the elements of the group $N_G(T)(k')$, where $N_G(T)$ denotes the normalizer group of $T$ in $G$.

**Proof.** See [23].

Finally, we finish this section by mentioning another important related result.

**Theorem 3** (Chevalley). The parabolic subgroups of any connected linear algebraic $k$-group are connected, and moreover, the normalizer of a parabolic subgroup in $G(k)$ is equal to $P(k)$.

**Proof.** See [1, Section 11].

### 2.6 Actions of group schemes.

Let $G$ be a group scheme over $S$. An action of $G$ on an $S$-scheme $X$ is an $S$-morphism

$$a : G \times_S X \to X$$

such that for any $S$-scheme $T$ the morphism $a(T) : G(T) \times X(T) \to X(T)$ is an action of $G(T)$ on $X(T)$. In this case, $X$ is called a $G$-scheme. If the action map $a$ is clear from the context we will denote $a(g, x)$ by $g \cdot x$, where $g \in G$, $x \in G(S)$.

Let $X$ be a $G$-scheme with respect to an action $a : G \times X \to X$. If $X$ is an affine scheme, then for any scheme $S$ and $g \in G(S)$ we have an $O_S$-algebra automorphism defined as follows:

$$\rho(g) : O_S \otimes_k \Gamma(X, k) \to O_S \otimes_k \Gamma(X, k)$$

$$1 \otimes f(-) \mapsto 1 \otimes (f \circ a(g^{-1}, -)),$$

where $\Gamma(X, k)$ is the global section functor applied to $X$. These automorphisms patch up to give a linear representation

$$\rho : G \to GL(\Gamma(X, k)).$$

(2.11)

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Remark 2.12. If $X$ is a $G$-scheme of finite type, then the representation (2.11) decomposes $\Gamma(X, k)$ into a union of finite dimensional $G$-submodules ([6, Proposition 2.3.4]). Furthermore, if $X$ is affine, then there exists a closed $G$-equivariant immersion of $X$ into a finite dimensional representation of $G$ ([6, Proposition 2.3.5]).

We close this section by mentioning an important theorem of Brion on projective equivariant embeddings of algebraic groups. Let $G$ be a $k$-group, and let $H \subseteq G$ be a subgroup scheme. An equivariant compactification of the homogenous space $G/H$ is a proper $G$-scheme $X$ equipped with an open equivariant immersion $G/H \to X$ with schematically dense image.

Theorem 4 (Brion). Let $G$ be a $k$-group, and let $H \subseteq G$ be a subgroup scheme. Then $G/H$ has an equivariant compactification by a projective scheme.

Proof. See [6, Section 5.2].

Some remarks are in order:

1. The homogenous space $G/H$ of Theorem 4 is quasiprojective; this is a well-known theorem for algebraic groups in the classical sense. But notice here that Brion proves the result for not necessarily reduced algebraic groups; of course, if the ground field is of characteristic zero, then Cartier’s theorem implies that $G$ is reduced.

2. If in addition $G$ is smooth, then $G/H$ has an equivariant compactification by a normal projective scheme.

3. If the characteristic of $k$ is 0, then every homogenous space has a smooth projective equivariant compactification.

4. Over any imperfect field, there exists smooth connected algebraic groups having no smooth compactification (see [6, Remark 5.2.3] for an example).

3 Algebraic monoids

For an algebro-geometric introduction to the theory of (not necessarily affine) monoid and semigroup schemes we recommend Brion’s lecture notes [5] (see also [13, Chapter II] \footnote{This book is one of the few if not the only book in algebraic geometry that acknowledges monoid schemes as part of the theory of group schemes.}).

Let us define a monoid scheme by relaxing the condition of invertibility in the definition of group schemes. More precisely, let $R$ be a commutative ring. A monoid scheme over $R$ is an $R$-scheme whose functor of points factors through the forgetful functor from the category of monoids to the category of sets.

Definition 3.1. Let $k$ be an algebraically closed field. An algebraic monoid over $k$, also called a $k$-monoid, is a monoid scheme over $k$ whose underlying scheme is separated and of finite type.
Remark 3.2. Our definition of $k$-monoids is somewhat more general than the one that is used by Brion in [5] since we do not assume reducedness of the underlying scheme structure.

It is not difficult to see that the category of $k$-groups forms a full subcategory of the category of $k$-monoids. We are mainly interested in the affine $k$-monoids but let us first mention a general result.

Let $M$ be a $k$-monoid, and let $G = G(M)$ denote its unit-group. Then $M$ is called unit-dense if $G$ is dense in $M$. A (weaker) form of the following theorem was first proven by Rittatore. Here, we will make use of Brion’s proof from [5, Section 3].

Theorem 5. Let $M$ be a unit-dense irreducible $k$-monoid, and let $G$ denote the unit-group of $M$. If $G$ is affine, then so is $M$.

Proof. If $M$ is reduced, then the result follows from [5, Theorem 2]. If $M$ is not reduced, then we pass to a normalization $\bar{M}$ which factors through the reduction $M_{\text{red}}$. This follows from the fact that $M$ is irreducible (hence Noetherian). Therefore, we can apply [30, Lemma 28.52.2]. But a normalization is a finite morphism, therefore $M_{\text{red}} \to M$ is a finite morphism. Since $M_{\text{red}}$ is affine (once again by [5, Theorem 2]), and since the image of an affine scheme under a finite morphism is affine, $M$ is an affine scheme as well. \hfill □

4 From spherical varieties to spherical spaces

We start with reviewing the classification schematics of the spherical embeddings over an algebraically closed field. Throughout this section $k$ will denote an algebraically closed field and we assume that all varieties are defined over $k$. As usual, let $G$ be a reductive group, let $B$ be a Borel subgroup, and let $T$ be a maximal torus contained in $B$. If $K$ is an algebraic group, we will denote by $X^*(K)$ the group of characters of $K$. Note that $X^*(B) = X^*(T)$. This is because the commutator subgroup of $B$ coincides with the unipotent radical $R_u(B)$, and $B = R_u(B) \rtimes T$.

Let $Y$ denote $G/H$, where $H$ is a spherical subgroup of $G$. Quotients of affine groups, in particular $Y$, have the structure of a quasiprojective variety. Recall that $Y$ is a spherical $G$-variety if and only if $B$ has only finitely many orbits with respect to the left multiplication action on $Y$. By a theorem of Brion, this is equivalent to the statement that $B$ has an open orbit in $Y$. Thus, it should come as no surprise that the $B$-invariant rational functions on $Y$ are among the main players in this game.

The space of $B$-semiinvariant rational functions on $Y$ is denoted by $k(Y)^{(B)}$. In other words,

$$k(Y)^{(B)} = \{ f \in k(Y) : b \cdot f = \chi(b)f \text{ for all } b \in B \text{ and for some character } \chi \text{ of } B \}.$$  

(4.1)

The gist of the classification schematics for spherical varieties will take place inside the vector space $\text{Hom}_{\mathbb{Z}}(X^*(B), \mathbb{Q})$. Indeed, it is easy to check that $k(Y)^{(B)}$ is a subgroup
of $k(Y)$ and that the assignment
\[ k(Y)^{(B)} \rightarrow X^*(B) \]
\[ f \mapsto \chi_f, \]
where $\chi_f$ is a character as in (4.1) is an injective group homomorphism. We will denote the image of (4.2) by $\Omega_Y$, and we will denote the $\mathbb{Q}$-vector space associated with the dual of $\Omega_Y$ by $\mathbb{Q}_Y$. In other words,
- $\Omega_Y := \{ \chi \in X^*(B) : b \cdot f = \chi(b)f \text{ for all } b \in B \text{ and for some } f \in k(Y)^{(B)} \}$;
- $\mathbb{Q}_Y := \text{Hom}_\mathbb{Z}(\Omega_Y, \mathbb{Q})$.

We occasionally refer to $\Omega_Y$ as the character group of the homogenous variety $Y$ since in the special case, where $Y = G \times G/\text{diag}(G) \cong G$ viewed as a spherical $G \times G$-variety, the character group of $Y$ is isomorphic to the ‘ordinary’ character group, $\Omega_Y \cong X^*(B)$.

We look closely at the divisors and their invariants on $Y$. A function $\nu : k(Y) \rightarrow \mathbb{Q}$ is called a $\mathbb{Q}$-valued discrete valuation on $k(Y)$ if for every $a, b \in k(Y)^*$, we have:
1. $\nu(k(Y)^*) \cong \mathbb{Z}$ and $\nu(k) = \{0\}$;
2. $\nu(ab) = \nu(a) + \nu(b)$;
3. $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ provided $a + b \neq 0$.

We notice here that every $\mathbb{Q}$-valued discrete valuation $\nu$ on $k(Y)$ defines a function on $\Omega_Y$. More precisely, there is a map
\[ \rho : \{ \text{Q-valued discrete valuations on } k(Y) \} \rightarrow \mathbb{Q}_Y \]
\[ \nu \mapsto \rho_\nu \]
such that $\rho_\nu$ is the map that sends an element $\chi = \chi_f$ of $\Omega_Y$ to $\nu(f)$, where $f$ is the $B$-semiinvariant that specifies $\chi$ (so we wrote $\chi = \chi_f$). Indeed, since $\chi$ is defined uniquely by $f$ (up to a scalar), $\rho_\nu$ is well-defined. Moreover, it is easy to check that $\rho_\nu(\chi_f \chi_g) = \rho_\nu(\chi_f) + \rho_\nu(\chi_g)$, hence $\rho_\nu \in \mathbb{Q}_Y$.

- A valuation $\nu$ is called $G$-invariant if $\nu(g \cdot a) = \nu(a)$ for every $g \in G$ and $a \in k(Y)$.

We will denote the set of all $\mathbb{Q}$-valued $G$-invariant discrete valuations on $Y$ by $\mathcal{V}_Y$.

It is not completely obvious, but nevertheless true, that the restriction
\[ \rho|_{\mathcal{V}_Y} : \mathcal{V}_Y \rightarrow \mathbb{Q}_Y \]
is an injective homomorphism of abelian groups.

The reason for which we started looking at discrete valuations defined on $Y$ in the first place is because many important discrete valuations come from irreducible hypersurfaces. From now on, we will refer to irreducible hypersurfaces as prime divisors. For any spherical action of $G$ on $Y$, we will consider the set of all $B$-stable prime divisors in $Y$. More generally, if $Y'$ is a normal spherical $G$-variety, then a color of $Y'$ is defined as a $B$-stable, but not $G$-stable, prime divisor. In our case, since $G$ acts transitively on $Y = G/H$, any $B$-stable prime divisor in $Y$ is non-$G$-stable.
The set of all $B$-stable, but not $G$-stable, prime divisors in $Y$ is denoted by $\mathcal{D}_Y$. The elements of $\mathcal{D}_Y$ are called the colors of $Y$.

Let us point out that on a noetherian integral separated scheme, which is regular in codimension one, the local ring associated with each prime divisor is a discrete valuation ring (DVR). In particular, since our $Y$ is a smooth variety, the local ring of a color of $Y$ at its generic point is a DVR, and we have the map

$$\tilde{\rho} : \mathcal{D}_Y \to \mathcal{Q}_Y,$$

which is defined as the composition of $\rho$ with the map that assigns a color to the corresponding discrete valuation.

**Definition 4.4.** The colored cone of $Y = G/H$ is the pair $(C_Y, \mathcal{D}_Y)$ where $C_Y$ is the cone in $\mathcal{Q}_Y$ that is generated by $\rho(\mathcal{V}_Y)$ and $\tilde{\rho}(\mathcal{D}_Y)$.

So far what we have are some ‘birational invariants’ that are defined solely for $Y = G/H$, and we have not given any indication of how they are related to its embeddings. To see how all these basic ingredients come together to play a role, next, we introduce the notion of a colored fan. This will give us a generalization of the combinatorial classification of toric varieties.

Let $\overline{Y}$ be a $G$-equivariant embedding of $Y$. Let $\mathcal{D}_{\overline{Y}}$ denote the set of $B$-stable, but not $G$-stable, prime divisors of $\overline{Y}$. Clearly, $\mathcal{D}_Y$ is a subset of $\mathcal{D}_{\overline{Y}}$. Since $Y$ is the open $G$-orbit in $\overline{Y}$, we have $k(Y) = k(\overline{Y})$. In particular, there is an extension of (4.3) to a map $\tilde{\rho} : \mathcal{D}_{\overline{Y}} \to \mathcal{Q}_Y$. Let

$$\pi : \mathcal{D}_{\overline{Y}} \to \mathcal{D}_Y$$

denote the partially defined map $\pi(S) = S \cap Y$, whenever $S \cap Y$ is an element of $\mathcal{D}_Y$.

Let us mention in passing that both of the sets $\mathcal{D}_Y$ and $\mathcal{D}_{\overline{Y}}$ contain a finite number of elements since $\overline{Y}$ is spherical. Note also that the set of $G$-invariant discrete valuations on $\overline{Y}$ is equal to $\mathcal{V}_Y$.

- The pair $(\mathcal{C}_{Y' \to \overline{Y}}, \mathcal{D}_{Y' \to \overline{Y}})$ is called the colored cone of the $G$-orbit $Y'$. A face of $(\mathcal{C}_{Y' \to \overline{Y}}, \mathcal{D}_{Y' \to \overline{Y}})$ is a pair of the form $(\mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ is a face of $\mathcal{C}_{Y' \to \overline{Y}}$ such that
  (i) $\mathcal{C} \cap (\mathcal{C}_{Y' \to \overline{Y}})^0 \neq \emptyset$;
  (ii) $\tilde{\rho}^{-1}(\mathcal{C}) \cap \mathcal{D}_{Y' \to \overline{Y}} \neq \emptyset$.

Any colored cone satisfies the following defining properties:
The cone $\mathcal{C}_{Y',\rightarrow \overline{Y}}$ is generated by $\overline{\rho}(D_{Y',\rightarrow \overline{Y}})$ and a finite number of vectors of the form $\rho(S)$, where $S \in \mathcal{V}_{Y',\rightarrow \overline{Y}}$.

C2. The relative interior of $\mathcal{C}_{Y',\rightarrow \overline{Y}}$ has a nonempty intersection with the set $\rho(\mathcal{V}_{Y',\rightarrow \overline{Y}})$.

C3. $\mathcal{C}_{Y',\rightarrow \overline{Y}}$ is strictly-convex, that is to say, $\mathcal{C}_{Y',\rightarrow \overline{Y}} \cap (-\mathcal{C}_{Y',\rightarrow \overline{Y}}) = \{0\}$.

C4. 0 is not an element of $\overline{\rho}(D_{Y',\rightarrow \overline{Y}})$.

Now we are ready to introduce the combinatorial objects which parametrize the $G$-equivariant embeddings of $Y$.

**Definition 4.5.** The following (finite) set is called the colored fan of $\overline{Y}$:

$$\mathcal{F}_{\overline{Y}} := \{(\mathcal{C}_{Y',\rightarrow \overline{Y}}, D_{Y',\rightarrow \overline{Y}}) : Y' \text{ is a } G\text{-orbit in } \overline{Y}\}.$$

The colored fans satisfy the following defining properties:

F1. Every face of a colored cone in $\mathcal{F}_{\overline{Y}}$ is an element of $\mathcal{F}_{\overline{Y}}$.

F2. For every $G$-invariant valuation $\nu$ in $\mathcal{V}_{Y}$, there exists at most one colored cone $(\mathcal{C},D)$ in $\mathcal{F}_{\overline{Y}}$ such that $v \in \mathcal{C}$.

It is easy to make abstract versions of colored fans. Let $V$ be a finite dimensional vector space over $\mathbb{Q}$. Starting with a subset $\mathcal{V}$ of $V$ and a finite set $\mathcal{D}$ together with a set map $\overline{\rho} : \mathcal{D} \rightarrow V$, we define a colored fan associated with $(V, \mathcal{V}, \mathcal{D}, \overline{\rho})$ as a finite collection of pairs $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is a cone in $V$ and $\mathcal{E}$ is a subset of $\mathcal{D}$ satisfying the properties F1–F2. Of course, $\mathcal{V}$ plays the role of $\rho(\mathcal{V}_{Y})$ in $\mathcal{Q}_{Y}$ and $\overline{\rho} : \mathcal{D} \rightarrow V$ plays the role of $\overline{\rho} : \mathcal{D}_{Y} \rightarrow \mathcal{Q}_{Y}$.

Let $H$ and $H'$ be two closed subgroups in $G$ such that the homogenous varieties

$$Y := G/H \text{ and } Z := G/H'$$

are spherical. Let

$$\varphi : Y \rightarrow Z$$

be a morphism of varieties. If $\varphi$ is $G$-equivariant, then the resulting map on the character groups $\varphi^* : X^*(Z) \rightarrow X^*(Y)$ is injective, hence, the ‘dual’ linear map $\varphi_* : \mathcal{Q}_{Y} \rightarrow \mathcal{Q}_{Z}$ is surjective. Furthermore, we have $\varphi_*(\mathcal{V}_{Y}) = \mathcal{V}_{Z}$. Let $\mathcal{D}_{\varphi}$ denote the set of colors of $Y$ that are mapped into $Z$ dominantly,

$$\mathcal{D}_{\varphi} := \{D \in \mathcal{D}_{Y} : \varphi(D) \text{ is dense in } Z\}.$$

In other words, $\mathcal{D}_{\varphi}$ is the set of colors of $Y$ which are too big, so, we may ignore(!) them in the combinatorial setup. We set

$$\mathcal{D}_{\varphi} := \mathcal{D}_{Y} - \mathcal{D}_{\varphi}.$$

**Definition 4.6.** Let $\varphi : Y \rightarrow Z$ be a $G$-equivariant morphism between two spherical homogenous $G$-varieties. Let $\overline{Y}$ and $\overline{Z}$ denote two equivariant embeddings of $Y$ and $Z$, respectively. Let $Y'$ and $Z'$ be two $G$-orbits in $\overline{Y}$ and $\overline{Z}$, respectively. The map $\varphi$ is said to be a morphism between the colored cones $(\mathcal{C}_{Y',\rightarrow \overline{Y}}, D_{Y',\rightarrow \overline{Y}})$ and $(\mathcal{C}_{Z',\rightarrow \overline{Z}}, D_{Z',\rightarrow \overline{Z}})$ if we have
1. $\varphi_* (\mathcal{C}_Y \hookrightarrow Y) \subseteq \mathcal{C}_Z \hookrightarrow Z$, and
2. $\varphi (\mathcal{D}_Y \cap \mathcal{D}_Y' \hookrightarrow Y) \subseteq \mathcal{D}_Z \hookrightarrow Z$.

The map $\varphi$ is said to be a morphism between the colored fans $\mathcal{F}_Y$ and $\mathcal{F}_Z$ if for every cone $(C, D)$ in $\mathcal{F}_Y$ there exists a cone $(C', D')$ in $\mathcal{F}_Z$ such that $\varphi : (C, D) \to (C', D')$ is a morphism of cones.

The following result, which is proven by Knop in [16], is a generalization of the classification result of Luna and Vust for simple embeddings.

**Theorem 6** (Knop). Let $Y$ be a spherical homogenous $G$-variety, and let $B$ be a Borel subgroup in $G$. The assignment $Y \mapsto \mathcal{F}_Y$ is a bijective correspondence between the isomorphism classes of $G$-equivariant embeddings of $Y$ and the isomorphism classes of colored fans associated with $(Q_Y, V_Y, D_Y, \tilde{\rho})$. In fact, this assignment is an equivalence between the category of equivariant embeddings of $Y$ and the category of colored fans associated with $(Q_Y, V_Y, D_Y, \tilde{\rho})$.

**Remark 4.7.** As we mentioned before the theorem of Knop, the role of colored fans for simple embeddings was already known. In fact, Luna and Vust had shown in [20] that the colored cone $(\mathcal{C}_Z \hookrightarrow Y, \mathcal{D}_Z \hookrightarrow Y)$, where $Z \hookrightarrow Y$ is the closed orbit of $Y$, uniquely determines $Y$.

**Remark 4.8.** It is not difficult to check that all definitions pertaining to the colored cones make sense (definable) if we use a separably closed field instead of an algebraically closed field.

### 4.1 Quasiprojective colored fans.

It is useful to know when an equivariant embedding of a spherical homogenous variety is affine, projective, or more generally quasiprojective. Such criteria are found by Brion in [4]. Here we only give Brion’s criterion for quasiprojectiveness.

**Theorem 7** (Brion). Let $\mathcal{F}_Y$ be the colored fan of an equivariant embedding $\overline{Y}$ of a spherical homogenous $G$-variety $Y$. In this case, $\overline{Y}$ is quasiprojective if and only if for each colored cone $C_Z := (\mathcal{C}_Z \hookrightarrow \overline{Y}, \mathcal{D}_Z \hookrightarrow \overline{Y})$ in $\mathcal{F}_Y$ there exists a linear form, denoted by $\ell_Z$, on $Q_Y$ such that the following two conditions are satisfied:

1. If $C_Z = (\mathcal{C}_Z \hookrightarrow \overline{Y}, \mathcal{D}_Z \hookrightarrow \overline{Y})$ and $C'_Z = (\mathcal{C}'_Z \hookrightarrow \overline{Y}, \mathcal{D}'_Z \hookrightarrow \overline{Y})$ are two elements from $\mathcal{F}_Y$, then the restrictions of the corresponding linear forms onto $\mathcal{C}_Z \hookrightarrow \overline{Y} \cap \mathcal{C}'_Z \hookrightarrow \overline{Y}$ are the same.
2. If $C_Z = (\mathcal{C}_Z \hookrightarrow \overline{Y}, \mathcal{D}_Z \hookrightarrow \overline{Y})$ and $C'_Z = (\mathcal{C}'_Z \hookrightarrow \overline{Y}, \mathcal{D}'_Z \hookrightarrow \overline{Y})$ are two distinct elements from $\mathcal{F}_Y$, and if a vector $\chi \in Q_Y$ lies in the intersection of the interior of $\mathcal{C}_Z \hookrightarrow \overline{Y}$ with the image $\rho(V_Y)$, then $\ell_Z(\chi) > \ell_{Z'}(\chi)$.

Following Huruguen, we call a colored fan whose cones satisfy the requirements of Theorem 7 a quasiprojective colored fan. We know from (the remarks following) Theorem 4 that there are plenty of quasiprojective colored fans, especially over perfect fields.

With this precise definition of quasiprojectiveness at hand, now we are able to state Huruguen’s result.
Theorem 8 (Huruguen). Let $k$ be a perfect field, let $G$ be a connected reductive group that is defined over $k$, and let $\overline{Y}(k)$ be an embedding of a spherical homogenous spherical $G$-variety $Y$ defined over $\overline{k}$. We assume that the fan of $\overline{Y}$ is $\Gamma$-stable. In this case, $\overline{Y}$ admits a $k$-form if and only if for every cone $C_Z := (C_{Z \to \overline{Y}}, D_{Z \to \overline{Y}})$ in $F_{\overline{Y}}$, the colored fan consisting of the cones $(\sigma(C_{Z \to \overline{Y}}), \sigma(D_{Z \to \overline{Y}})), \sigma \in \Gamma$ as well as all of its faces are quasiprojective.

Proof. See [15, Theorem 2.26].

4.2 Spherical spaces over arbitrary fields.

We will start with giving a brief summary of Wedhorn’s work on the classification of spherical spaces. For all unjustified claims (and for some definitions) we refer the reader to [34] and to the references therein.

Definition 4.9. Let $k$ be a field, and let $G$ be a reductive $k$-group. Recall that this amounts to the requirement that $G_{\overline{k}}$ is a connected reductive group. According to [34, Remark 2.2], an algebraic space $X$ over $k$ with an action of $G$ is $G$-spherical if $X_{\overline{k}}$ is a spherical $G_{\overline{k}}$-variety.

Let $\overline{k}$ denote a fixed algebraic closure of $k$, let $k_s$ denote the separable closure of $k$, and let us denote by $\Gamma$ the Galois group of the extension $k_s/k$. (Here, we are intentionally vague about our choices because it does not matter which separable closure we choose.) In the sequel we will look at continuous and linear actions of $\Gamma$ on some structures. When we speak of a continuous action of $\Gamma$ on a set $X$, we will treat $X$ with the discrete topology. The important point here is that if $X$ is a finite set, or, if the action of $\Gamma$ is linear on some finite dimensional vector space $X$, then the action is continuous if and only if it factors through some finite discrete quotient of $\Gamma$. This fact should alleviate a possible pain of confronting a large absolute Galois group such as $\Gamma$ of $\mathbb{Q}_s/\mathbb{Q}$.

If $\overline{Y}$ is a spherical $G$-space, then there exists a homogenous spherical $G$-space $Y$ such that $\overline{Y}$ is a spherical embedding of $Y$. This actually amounts to the statement that $Y$ is the unique open minimal $G$-invariant subspace of $\overline{Y}$. By definition, a $G$-invariant subspace in an algebraic space $\overline{Y}$ is minimal if there exists no proper non-empty $G$-invariant subspace of $\overline{Y}$.

Theorem 9 (Wedhorn). Let $G$ be a reductive $k$-group, and let $\overline{Y}$ be a spherical $G$-scheme viewed as an equivariant embedding of the spherical homogenous scheme $G/H$. If $k$ is separably closed, then the assignment $\overline{Y} \mapsto \overline{Y}_{\overline{k}}$ induces a bijection between the isomorphism classes of spherical embeddings of $G/H$ over $k$ and the isomorphism classes of spherical embeddings of $G/H$ over $\overline{k}$.

Notice that the bijection between isomorphism classes that is mentioned in Theorem 9 is essentially the application of the base change functor from $k$ to $\overline{k}$. In general, this does not give an equivalence of categories. A straightforward example is produced by the left translation action of $G = G_m$ on $Y = G_m$. Luckily, since the definition of colored fans works over separably closed fields, and since we have faithfully flat descent.
upon restriction, the classification reduces to the classification over algebraically closed fields. The caveat is that one needs to consider all $G$-invariant minimal subschemes of the spherical space.

**Corollary 1.** Let $G$ be a reductive $k$-group, and let $\overline{Y}$ be a spherical $G$-space viewed as an equivariant embedding of the spherical homogenous space $Y := G/H$. If $k$ is separably closed, then the assignment $\overline{Y} \mapsto (C_{Y \hookrightarrow \overline{Y}^\prime}, D_{Y \hookrightarrow \overline{Y}^\prime})_{Y^\prime}$, where $Y^\prime$ runs over all minimal $G$-invariant subschemes of $\overline{Y}$, is an equivalence between the category of equivariant embeddings of $Y$ (over $k$) and the category of colored fans associated with $(Q_Y, V_Y, D_Y, \tilde{\rho})$.

Of course, the theorem and its corollary that we just presented here give us something new (compared to Knop’s theorem) only when the characteristic of $k$ is nonzero.

Now we proceed with the general case and assume that $G$ is a reductive $k$-group. Let $Y$ be a spherical homogenous $G$-variety, and let $Y$ be a spherical embedding of $Y$. Both of $Y$ and $\overline{Y}$ are assumed to be defined over $k$. Note that Borel subgroups always exist over separably closed fields, whence we fix a Borel subgroup $B$ in $G$ despite the fact that $B$ may not have any $k$-rational points. This is where we start to notice a departure from Huruguen’s work.

There is a natural action of the Galois group $\Gamma$ on the space of $B$-semiinvariants $k_s(Y)(B)$. In particular, $\Gamma$ acts on the $k_s$-vector space $\Omega_{Y_{k_s}}$ continuously and linearly. Moreover, it acts continuously on the valuation cone $V_{Y_{k_s}}$ as well as on the set of colors $D_{Y_{k_s}}$, and the maps $\rho : V_{Y_{k_s}} \rightarrow Q_{Y_{k_s}}$ and $\tilde{\rho} : D_{Y_{k_s}} \rightarrow Q_{Y_{k_s}}$ are $\Gamma$-equivariant.

- A colored fan $\mathcal{F}_{\overline{Y}_{k_s}}$ is said to be $\Gamma$-invariant if its colored cones are permuted by the action of $\Gamma$.

**Theorem 10** (Wedhorn). Let $G$ be a reductive $k$-group, and let $\overline{Y}$ be a spherical $G$-space viewed as an equivariant embedding of the spherical homogenous $G$-space $Y := G/H$, which is defined over $k$. Then the assignment $\overline{Y}_{k_s} \mapsto (C_{Y \hookrightarrow \overline{Y}_{k_s}^\prime}, D_{Y \hookrightarrow \overline{Y}_{k_s}^\prime})_{Y^\prime}$, where $Y^\prime$ runs over all minimal $G$-invariant subschemes of $\overline{Y}_{k_s}$, induces an equivalence between the category of equivariant embeddings of $Y$ over $k$ and the category of $\Gamma$-invariant colored fans associated with $(Q_{Y_{k_s}}, V_{Y_{k_s}}, D_{Y_{k_s}}, \tilde{\rho})$.

## 5 Reductive monoids over arbitrary fields

In this section we will consider the reductive monoids that are defined over arbitrary fields. We will show how Wedhorn’s theorems are applicable to the algebraic monoid setting.

**Definition 5.1.** A reductive $k$-monoid is a $k$-monoid whose unit-group is a reductive $k$-group in the sense of Definition 2.8.

In particular, according to our Definition 5.1, the unit-group of a reductive $\overline{T}$-monoid is a connected reductive monoid, conforming with our tacit assumption from the introduction as well as with that of [29].
Remark 5.2. Let $M$ be a reductive monoid defined over an algebraically closed field, and let $G$ denote its unit-group. The following results are recorded in [29]:

1. $G$ is dense in $M$;
2. $M$ is affine;
3. the reductive monoids are exactly the affine $G \times G$-embeddings of reductive groups;
4. the commutative reductive monoids are exactly the affine embeddings of tori;
5. the isomorphism classes of reductive monoids with unit-group $G$ are in bijection with the strictly convex polyhedral cones of $Q_G$ generated by all of the colors and a finite set of elements from $V_G$.

Now we propose a definition for ‘monoid algebraic spaces.’ Probably this definition exists in the literature, however, we could not locate it. For our monoid space definition, once again, we will relax the definition of a group algebraic space (as given in Stacks Project Tag 043G).

Definition 5.3. Let $S$ be a scheme, and let $B$ be an algebraic space that is separated over $S$.

- A monoid algebraic space over $B$ is a pair $(M, m)$, where $M$ is a separated algebraic space over $B$ and $m : M \times_B M \to M$ is a morphism of algebraic spaces over $B$ with the property that, for every scheme $T$ over $B$, the pair $(M(T), m)$ is a monoid.
- A morphism $\psi : (M, m) \to (M', m')$ of monoid algebraic spaces over $B$ is a morphism $\psi : M \to M'$ of algebraic spaces such that, for every $T/B$, the induced map $\psi : M(T) \to M'(T)$ is a homomorphism of monoids.

Definition 5.4. A reductive monoid space over a scheme $S$ is a monoid algebraic space $M$ over $S$ such that $M \to S$ is flat, of finite presentation over $S$, and for all $s \in S$ the geometric fiber $M_s$ is a reductive $k$-monoid.

Clearly, if a monoid algebraic space $M$ over a field $k$ is a scheme, then $M$ is a $k$-monoid in the sense of Definition 3.1 but the converse is not true. Indeed, in [15] Huruguen has found an example of a smooth toric variety of dimension 3 that is split over a quadratic extension of $k$, having no $k$-forms. This pathological example shows that even for the purposes of classifying reductive monoids over an arbitrary field one needs to venture into the category of algebraic spaces.

Extending Rittatore’s classification to reductive monoid spaces, we record the following observations which are simple corollaries of Wedhorn’s theorems combined with Rittatore’s results.

Recall that the reductive monoids with unit-group $G$ are $G \times G$-equivariant embeddings of $G$. When we speak of ‘colors’ in this context, we always mean the colors of $G$ as a $G \times G$-spherical $k$-group.
Theorem 11. Let $k$ be a field, and let $G$ denote a reductive $k$-group. Let $M$ be a reductive monoid space with $G$ as the group of invertible elements.

1. If $k$ is separably closed, then the assignment $M \leadsto (C_{Y'} \hookrightarrow M, D_G)_{Y'}$, where $Y'$ runs over all minimal $G \times G$-invariant subschemes of $M$, is an equivalence between the category of reductive monoid spaces over $k$ and the category of strictly convex colored polyhedral cones of $Q_G$ generated by all of the colors of $G$ and a finite set of elements from $V_G$.

2. Let $k_s$ be a separable closure of $k$. If $k$ is properly contained in $k_s$, then the following categories are equivalent:
   
   (a) the category of reductive monoid spaces over $k$ with unit-group $G$;
   
   (b) the category of $\Gamma$-invariant strictly convex colored polyhedral cones of $Q_{G_k}$ generated by all of the colors of $G_k$ and a finite set of elements of $V_{G_k}$.

   Here, $\Gamma$ is the Galois group of the extension $k_s/k$.

It is now desirable to know exactly which reductive monoid schemes over a field have a $k$-form.

Theorem 12. Let $k$ be a perfect field, let $M$ be a reductive monoid defined over $\overline{k}$ with unit-group $G$, and assume that $G$ is defined over $k$. In this case, $M$ has a $k$-form if and only if its colored fan, which is a strictly convex polyhedral cone, is stable under the action of absolute Galois group of $k \subset \overline{k}$.

**Proof.** By Theorem 5, we know that $M$ is affine, therefore, its colored fan is automatically quasiprojective. Now our result follows from Theorem 8. \qed

5.1 $k$-forms of lined closures.

For the next application we restrict our attention to the field of complex numbers, and we assume that the reader is familiar with the highest weight theory.

It is well known that any complex irreducible affine monoid $M$ admits a faithful finite dimensional rational monoid representation. In other words, there exists a finite dimensional vector space $V$ and an injective monoid homomorphism

$$\rho : M \rightarrow End(V),$$

which is a morphism of varieties (see [25]). We notice, in the light of Remark 2.12, that this fact holds true more generally for all irreducible affine $k$-monoids, where $k$ is an arbitrary field. In particular, $(V, \rho|_G)$ is a faithful rational representation of the unit-group $G$, and $M \cong \rho(G)$ in $End(V)$. In this case, we will write

$$M = M_V.$$ \hfill (5.5)

Now, let $V_\lambda$ denote the irreducible representation of $G$ with highest weight $\lambda$. The saturation of $\lambda$, denoted by $\Sigma_\lambda$, is the set of all dominant weights that are less than or equal to $\lambda$,

$$\Sigma_\lambda := \{\mu : \mu \text{ is dominant and } \mu \leq \lambda\}.$$
Let $V_{\Sigma, \lambda}$ denote the representation $\oplus_{\mu \in \Sigma} V_{\mu}$, and let $M_{\lambda}$ denote the reductive monoid defined by $V_{\Sigma, \lambda}$ as in (5.5). In a similar manner, we will denote $M_{V_{\lambda}}$ by $M_{\lambda}$. (These are special cases of the “multi-lined closure” construction of Li and Putcha in [19].) Clearly, both of the monoids $M_{\lambda}$ and $M_{\lambda}$ are reductive.

In [11], De Concini analyzed the geometric properties of $M_{\lambda}$ in relation with that of $M_{\lambda}$, and he proved the following theorem.

**Theorem 13** (DeConcini). 1. $M_{\lambda}$ is a normal variety with rational singularities.
2. $M_{\lambda}$ is the normalization of $M_{\lambda}$.
3. $M_{\lambda}$ and $M_{\lambda}$ are equal if and only if $\lambda$ is minuscule, that is to say, $\Sigma_{\lambda} = \{\lambda\}$.

We finish our paper with a theorem whose proof will be given somewhere else.

**Theorem 14.** The reductive monoid $M_{\lambda}$, hence its normalization $M_{\lambda}$ have an $\mathbb{R}$-form if and only if there exists an involutory automorphism $\theta$ of the reductive unit-group $G$ of $M_{\lambda}$ such that $\theta^*\lambda = -\lambda$.

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