Half-plane diffraction problems on a triangular lattice

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Abstract
We investigate thin-slit diffraction problems for two-dimensional lattice waves. Namely, Dirichlet problems for the two-dimensional discrete Helmholtz equation on the triangular lattice in a half-plane are studied. Using the notion of the radiating solution, we prove the existence and uniqueness of a solution for the real wave number \( k \in (0, 3) \backslash \{2\sqrt{2}\} \) without passing to the complex wave number. Besides, an exact representation formula for the solution is derived. Here, we develop a numerical calculation method and demonstrate by example the effectiveness of our approach related to the propagation of wavefronts in metamaterials through two small openings.

Keywords Discrete Helmholtz equation · Dirichlet boundary value problem · Half-plane diffraction · Metamaterials · Triangular lattice model

1 Introduction

Wave propagation through discrete structures remains an active area of research today. The triangular lattice is one of the five two-dimensional Bravais lattice types and appears naturally in applications. For example, close-packed planes occur frequently in some kinds of crystals in the form of triangular lattices [1, 2]. Structures of left-handed 2D metamaterials [3], which are a host microstrip line network periodically loaded with series capacitors and shunt inductors for signal processing and filtering, can also be represented by the triangular lattices (see Fig. 1).

Motivated by applications of recent interest related to analogue circuits, crystalline materials, and metamaterials, we investigate thin-slit diffraction problems on the semi-infinite triangular lattice. For this lattice, we study Dirichlet problems for the
two-dimensional discrete Helmholtz equation in a half-plane. Analysing regular processes when waves at the level of the microstructural scales are neglected, it is possible to consider the continuum limit of corresponding equations. However, today industrial and scientific interest is growing in the study of nano- and microstructure of modern materials and composites. One of the effective ways to investigate microstructural processes in the materials is to consider their discrete structures, cf., e.g. [3–6]. Therefore, we devote our paper to the analyses of discrete Helmholtz diffraction problems mathematically formulated in Sect. 2. Although similar problem for square lattice has been studied in [7], see also [8], its extension to a triangular lattice model is not direct.

Our main interest lies in the investigation of the problems with wave number $k$ within the pass-band which is an arbitrary non-zero complex number in general. In this paper, we provide new results in more complex case when $k \in (0, 3) \{2\sqrt{2}\}$ is a real wave number. For this purpose, we use the notion of the radiating solution [9] described in Sect. 4 and the method of images to construct the Dirichlet Green’s function for the half-plane represented in Sect. 3. Unique solvability results and an exact representation formula for the solution are obtained in Sect. 4. In Sect. 5, we develop a numerical calculation method and demonstrate by examples effectiveness of our approach related to the propagation of wavefronts in metamaterials through two small openings.

2 Formulation of the problem

Let us consider a two-dimensional infinite triangular lattice $\mathfrak{T}$ defined as a periodic simple graph $\{V, E\}$, where

$$V = \{(x_1 + x_2/2, \sqrt{3}x_2/2) \subset \mathbb{R}^2 : (x_1, x_2) \in \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}\}$$

is a vertex set, and $E$ is an edge set, whose endpoints $(a, b) \in V \times V$ are adjacent points, i.e. $|a - b| = 1$, cf. Figure 1. The time-harmonic discrete waves in $\mathfrak{T}$ can be described by solutions of the following discrete Helmholtz equation:

$$(\Delta_d + k^2)u(x) = 0, \quad x = (x_1, x_2) \in \mathbb{Z}^2, \quad (2)$$

where $\Delta_d$ denotes the discrete (a 7-point) Laplacian

$$\Delta_d u(x) = u(x + e_1) + u(x - e_1) + u(x + e_2) + u(x - e_2) + u(x + e_1 - e_2) + u(x - e_1 + e_2) - 6u(x), \quad (3)$$

and $e_1 = (1, 0), e_2 = (0, 1)$ stand for the standard base of $\mathbb{Z}^2$.

Let $\Omega = \hat{\Omega} \cup \Gamma$ be a discrete upper half-plane in $\mathbb{Z}^2$, where $\hat{\Omega} := \mathbb{Z} \times \mathbb{Z}^+$ and $\Gamma := \partial\Omega = \{(x_1, 0) : x_1 \in \mathbb{Z}\}$. As it was mentioned in the Introduction, our main objective is to investigate the problem of determining a unique solution $u : \Omega \to \mathbb{C}$ to the following Dirichlet problem:

$$(\Delta_d + k^2)u(x) = 0, \quad \text{in } \hat{\Omega}, \quad (4a)$$
Fig. 1 Triangular lattice. Connection between $x = (x_1, x_2) \in \mathbb{Z}^2$ and the Euclidean coordinates of the vertexes (black dots) is established via $(x_1, x_2) \rightarrow (x_1 + x_2/2, \sqrt{3}x_2/2)$.

The nearest neighbour interactions based on the triangular lattice are shown with thick blue lines. Triangular lattice can be viewed as a left-handed 2D inductor–capacitor metamaterial. A host transmission-line is loaded periodically with series capacitors and shunt inductors. (Color figure online)
\[ u(y) = f(y), \quad \text{on } \Gamma. \quad (4b) \]

Here, \( f \) is a given function supported on a finite subset of \( \Gamma \) and \( k \in (0, 3) \setminus \{2\sqrt{2}\} \).

From now on we will refer to this problem as Problem \( \mathcal{P}_D \).

### 3 Lattice Green’s function

Denote by \( G(x; y) \) the Green’s function for the discrete Helmholtz equation (2) centred at \( y \) and evaluated at \( x \). Then the function \( G(x; y) \) satisfies the equation

\[ (\Delta_d + k^2)G(x; y) = \delta_{x,y}, \tag{5} \]

where \( \delta_{x,y} \) is the Kronecker delta. For brevity, we use the notation \( G(x) \) for \( G(x; 0) \). Notice that \( G(x; y) = G(x - y) \).

Using the discrete Fourier transform and the inverse Fourier transform we get

\[ G(x) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i(x \cdot \xi)} \sigma(\xi; k) d\xi, \quad \xi = (\xi_1, \xi_2), \tag{6} \]

where

\[
\sigma(\xi; k^2) = e^{i\xi_1} + e^{-i\xi_1} + e^{i\xi_2} + e^{-i\xi_2} + e^{i\xi_1} e^{-i\xi_2} + e^{-i\xi_1} e^{i\xi_2} - 6 + k^2
\]

\[ = k^2 - 6 + 2 \cos \xi_1 + 2 \cos \xi_2 + 2 \cos(\xi_1 - \xi_2). \tag{7} \]

The lattice Green’s function \( G \) is quite well known when \( (k^2 \in \mathbb{C} \setminus [0, 9]) \) (cf., e.g. [10]). Notice that if \( (k^2 \in \mathbb{C} \setminus [0, 9]) \) then \( \sigma \neq 0 \) and, consequently, \( G \) in (6) is well defined. In this case \( G(x) \) decays exponentially when \( |x| \to \infty \). Moreover, we have

\[ G(x_1, x_2) = G(x_2, x_1) = G(-x_1, -x_2) = G(x_1 + x_2, -x_2) \tag{8} \]

for all \( x = (x_1, x_2) \in \mathbb{Z}^2 \).

Let us show \( G(x_1, x_2) = G(x_1 + x_2, -x_2) \) while other identities are trivial. According to (6), we have

\[
G(x_1 + x_2, -x_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{ix_1 \xi_1 + ix_2 \xi_2 (\xi_1 - \xi_2)}}{\sigma(\xi_1, \xi_2; k)} d\xi_1 d\xi_2
\]

\[ = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{ix_1 \eta_1 + ix_2 \eta_2}}{\sigma(\eta_1, \eta_2; k)} d\eta_2 d\eta_1, \tag{9} \]

with \( \eta_1 = \xi_1, \eta_2 = \xi_1 - \xi_2 \). The following equality can be easily obtained by changing the variable \( \eta_2 \) to \( \eta_2 = 2\pi \)

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{ix_1 \eta_1 + ix_2 \eta_2}}{\sigma(\eta_1, \eta_2; k)} d\eta_2 d\eta_1 = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{ix_1 \eta_1 + ix_2 \eta_2}}{\sigma(\eta_1, \eta_2; k)} d\eta_2 d\eta_1. \tag{10} \]
The factor \( e^{i \lambda x_2 \pi} \) is equal to 1 since \( x_2 \in \mathbb{Z} \). Similar arguments give us
\[
\int_{\pi}^{0} \int_{-\pi}^{\pi} \frac{e^{i \lambda_1 \eta_1 + i \lambda_2 \eta_2}}{\sigma(\eta_1, \eta_2; k)} \, d\eta_2 \, d\eta_1 = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{i \lambda_1 \eta_1 + i \lambda_2 \eta_2}}{\sigma(\eta_1, \eta_2; k)} \, d\eta_2 \, d\eta_1. \tag{11}
\]
Taking into account the last two equalities, we get
\[
\mathcal{G}(x_1 + x_2, -x_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{i \lambda_1 \eta_1 + i \lambda_2 \eta_2}}{\sigma(\eta_1, \eta_2; k)} \, d\eta_2 \, d\eta_1 = \mathcal{G}(x_1, x_2). \tag{12}
\]
In the case \( k^2 \in (0, 9) \setminus \{8\} \) the expression (6) is understood as follows: we replace \( k^2 \) by \( k^2 + i\epsilon \) with \( 0 < \epsilon \ll 1 \) and let \( \epsilon \to 0 \) at the end of the calculation, cf. [9]. Thus, we define the lattice Green’s function for \((k \in (0, 3) \setminus \{2\sqrt{2}\})\) as a pointwise limit of
\[
(R_{x+\eta \delta x, 0}(x)) := \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{i \lambda_1 \eta_1 + i \lambda_2 \eta_2}}{\sigma(\eta; k^2 + i\epsilon)} \, d\eta \, d\xi \tag{13}
\]
as \( k^2 + i\epsilon \to k^2 + i0 \) and denote it again by \( \mathcal{G}(x) \), i.e. \( \mathcal{G}(x) = (R_{x+\eta \delta x, 0}(x)) \). Clearly, \( \mathcal{G}(x) \) is a solution to Eq. (5) and satisfies equalities (8).

### 4 Unique solvability result

In order to simplify further arguments, let us introduce the following vectors:
\[
e_1 = (1, 0), \quad e_2 = (0, 1), \quad e_3 = e_1 - e_2, \quad e_4 = -e_1, \quad e_5 = -e_2, \quad e_6 = -e_3.
\]
For any point \( x \in \mathbb{Z}^2 \) we define the neighbourhood \( F_x \) as \( \{x + e_j : j = 1, \ldots, 6\} \cup \{x\} \). We say that \( R \subset \mathbb{Z}^2 \) is a region if there exist disjoint non-empty subsets \( \hat{R} \) and \( \partial R \) of \( R \) such that

(a) \( R = \hat{R} \cup \partial R \),
(b) if \( x \in \hat{R} \) then \( F_x \subset R \),
(c) if \( x \in \partial R \) then there is at least one point \( (y \in F_x \setminus \{x\}) \) such that \( y \in \hat{R} \).

Clearly, the subsets \( \hat{R} \) and \( \partial R \) are not defined uniquely by \( R \), but henceforth, it will be always assumed that for a given region \( R \) in \( \mathbb{Z}^2 \) the sets \( \hat{R} \) and \( \partial R \) are also given and fixed.

Denote by \((\partial R)_j, j = 1, \ldots, 6\), a set of all boundary points \( y \in \partial R \) such that \( y - e_j \in \hat{R} \) and call it the sides of the boundary \( R \). Clearly, \( \partial R \) is the union of its six sides, \( \partial R = \bigcup_{j=1}^{6}(\partial R)_j \). Notice that a boundary point \( y \) can simultaneously belong to all six sides of \( R \). However, in our arguments presented below it will be always clear which side is needed to be considered. Under this condition, we define the discrete derivative in the outward normal direction \( e_j, j = 1, \ldots, 6, \)
\[
Tu(y) = u(y) - u(y - e_j), \quad y \in (\partial R)_j. \tag{14}
\]
Let us introduce the following set \( H_0 = \{(0, 0)\} \) and then define \( H_N, N \in \mathbb{N} \), with the help of recurrence formula

\[
H_N := \bigcup_{x \in H_{N-1}} F_x
\]

with \( \tilde{H}_N := H_{N-1} \) and \( (\partial H)_N := H_N \setminus \tilde{H}_N \).

Finally, for a finite region \( R \), recall a discrete analogue of Green’s second identity

\[
\sum_{x \in \tilde{R}} (u(x) \Delta_d v(x) - v(x) \Delta_d u(x)) = \sum_{y \in \partial R} (u(y) T v(y) - v(y) T u(y)).
\]

Now let us give a definition of a radiating solution on the discrete half-plane. First, we consider the case \( k^2 \in (0, 8) \). We say that \( u : \Omega \to \mathbb{C} \) satisfies the radiation condition at infinity if

\[
\begin{cases}
 u(x) = O(|x|^{-\frac{1}{2}}), \\
u(x + e_j) = e^{i \xi_j^*(\alpha, k)} u(x) + O(|x|^{-\frac{3}{2}}), \quad j = 1, 2,
\end{cases}
\]

with the remaining term decaying uniformly in all directions \( x/|x| \), where \( x \) is characterized as \( x_1 = |x| \cos \alpha, x_2 = |x| \sin \alpha, 0 \leq \alpha \leq \pi \). Here, \( \xi_j^*(\alpha, k) \) is the \( j \)th coordinate of the point \( \xi_j^*(\alpha, k) \).

For \( k^2 \in (8, 9) \) we say that \( u : \Omega \to \mathbb{C} \) satisfies the radiation condition at infinity if it can be represented as a sum of functions \( u_1 \) and \( u_2 \) such that each of them satisfies

\[
\begin{cases}
u_i(x) = O(|x|^{-\frac{1}{2}}), \\
u_i(x + e_j) = e^{i \xi_i^*(\alpha, k)} u_i(x) + O(|x|^{-\frac{3}{2}}), \quad i, j = 1, 2,
\end{cases}
\]

with the remaining term decaying uniformly in all directions \( x/|x| \), where \( x \) is characterized as \( x_1 = |x| \cos \alpha, x_2 = |x| \sin \alpha, 0 \leq \alpha \leq \pi \). Here, \( \xi_i^*(\alpha, k) \) is the \( j \)th coordinate of the point \( \xi_i^*(\alpha, k), i = 1, 2 \).

It is shown in [9] that \( \mathcal{G}(x) \) satisfies the radiation conditions introduced above. Now we are ready to introduce a notion of a radiating solution to the discrete Helmholtz equation (2).

**Definition 1** Let \( k^2 \in (0, 8) \). A solution \( u \) to the discrete Helmholtz equation (4a) is called radiating if it satisfies the radiation condition (17).

Let \( k^2 \in (8, 9) \). A solution \( u \) to the discrete Helmholtz equation (4a) is called radiating if it satisfies the radiation condition (18).

For our purposes below we need the Dirichlet Green’s function for the half-plane, which is represented as follows:

\[
\mathcal{G}^+(x; y) = \mathcal{G}(x; y) - \mathcal{G}(\hat{x}; y), \quad y \in \mathbb{Z}^2,
\]
with $x = (x_1, x_2) \in \Omega$ and $\hat{x} = (x_1 + x_2, -x_2)$. Notice that $G^+(x; y)$ can be expressed equivalently as
\[
G^+(x; y) = G(x; y) - G(x; \hat{y}).
\] (20)

Indeed, using the property (8), we have
\[
G(\hat{x}; y) = G(x_1 + x_2 - y_1, -x_2 - y_2) = G(x_1 - y_1 - y_2, x_2 + y_2) = G(x; \hat{y}).
\] (21)

Since $(y_1, 0) = (y_1, 0)$, we immediately get $G^+(x; y) = 0$ for $y \in \Gamma$. Moreover,
\[
(\Delta_d + k^2) G^+(x; y) = (\Delta_d + k^2) G(x; y) - (\Delta_d + k^2) G(\hat{x}; y) = \delta_{x,y} - \delta_{\hat{x},\hat{y}} = \delta_{x,y}
\] (22)

for all $x \in \hat{\Omega}$ and $y \in \Omega$. From (21) we see that $G^+(x; y)$ satisfies the radiation condition. Indeed, for any fixed $y \in \Gamma$ the angle $\alpha$ from the radiation conditions (17) and (18) is the same for $G(x; y) = G(x - y)$ and $G(x; \hat{y}) = G(x - \hat{y})$ when $|x| \to \infty$.

**Theorem 1** Let $(k \in (0,3) \setminus \{2\sqrt{2}\})$, and $u$ be a given function $\Omega \to \mathbb{C}$ that satisfies the radiation condition. The function $u|_\Gamma$ has a finite support. Then, for any point $x \in \hat{\Omega}$, we have the following representation formula:
\[
u(x) = \sum_{y \in \Gamma} (u(y)T G^+(x; y) - G^+(x; y)Tu(y)) + \sum_{y \in \Omega} G^+(x; y)(\Delta_d + k^2)u(y).
\] (23)

In particular, if $u$ is a solution to the discrete Helmholtz equation
\[
(\Delta_d + k^2)u(x) = 0 \text{ in } \hat{\Omega},
\] (24)

then
\[
u(x) = \sum_{y \in \Gamma} u(y)T G^+(x; y).
\] (25)

**Proof** Denote by $\tilde{u}$ the extension of $u$ to $\mathbb{Z}^2$ by zeros, i.e. $\tilde{u}(x) = u(x)$ if $x \in \Omega$ and $\tilde{u}(x) = 0$ if $x \in \mathbb{Z} \times \mathbb{Z}^-$. Then, for any finite region $H_N$, $N \in \mathbb{N}$, we apply Green’s second identity (16), where we take $v(y) = G^+(x; y)$. Here, note that $G^+$ is a fundamental solution for $y \in \hat{\Omega}$, and $\tilde{u}(y) = 0$ for $y \in \mathbb{Z} \times \mathbb{Z}^-$. Thus, $u(y)(\Delta_d + k^2)v(y)$ disappears in the following identity:
\[
u(y)\Delta_d v(y) - v(y)\Delta_d u(y) = u(y)(\Delta_d + k^2)v(y) - v(y)(\Delta_d + k^2)u(y)
\] (26)
as far as $y \in \mathbb{Z} \times \mathbb{Z}^-$. For $x \in \hat{H}_N$, we get

$$
\tilde{u}(x) = \sum_{y \in \hat{H}_N} \tilde{u}(y) \delta_{x,y} = \sum_{y \in \partial H_N} \left( \tilde{u}(y) T \mathcal{G}^+(x; y) - \mathcal{G}^+(x; y) T \tilde{u}(y) \right) + \sum_{y \in H_N} \left( \mathcal{G}^+(x; y) (\Delta_d + k^2) \tilde{u}(y) \right).
$$

(27)

Passing to the limit $N \to \infty$, we use exactly the same arguments as in the proof of Theorem 5.2 from [9] and, consequently, we get

$$
\tilde{u}(x) = \sum_{y \in \mathbb{Z}^2} \mathcal{G}^+(x; y) (\Delta_d + k^2) \tilde{u}(y).
$$

(28)

For the function $u$, we obtain

$$
u(x) = \sum_{y \in \Gamma} \left( u(y) T \mathcal{G}^+(x; y) - \mathcal{G}^+(x; y) T u(y) \right) + \sum_{y \in \hat{\Omega}} \mathcal{G}^+(x; y) (\Delta_d + k^2) u(y).
$$

Taking into the account that $\mathcal{G}^+(x; y) = 0$ when $y \in \Gamma$, for a solution $u$ to the discrete Helmholtz equation we get the following quality:

$$
u(x) = \sum_{y \in \Gamma} u(y) T \mathcal{G}^+(x; y), \quad x \in \hat{\Omega}.
$$

(29)

Now we are ready to prove the unique solvability result for the discrete Helmholtz equation on the semi-infinite triangular lattice.

**Theorem 2** Let $(k \in (0, 3) \setminus \{2\sqrt{2}\})$ then the Problem $\mathcal{P}_D$ has a unique radiating solution $u$ which can be represented as

$$
u(x) = \sum_{y \in \Gamma} (\delta_{x,y} - \mathcal{G}^+(x; y + e_2) - \mathcal{G}^+(x, y + e_2 - e_1)) f(y).
$$

(30)

**Proof** To prove the uniqueness result, it is sufficient to show that the corresponding homogeneous problem has only the trivial solution. Let $u$ be a radiating solution to the homogeneous problem $\mathcal{P}_D$. Then Theorem 1 immediately implies $u(x) = 0$ for all $x \in \hat{\Omega}$.

To show the existence results, let us first check that $u(x)$ satisfies the boundary condition (4b). Since $\mathcal{G}^+(x; y) = 0$ for all $x \in \Gamma$ and any $y$ we get

$$
u(x_1, 0) = \sum_{y_1 \in \mathbb{R}} (\delta_{x_1,y_1} - \delta_{0,0}) f(y) = f(x_1).
$$

(31)
Now let us check that \( u(x) \) satisfies the Helmholtz equation \((4a)\). For \( x_2 > 1 \) all terms \((\Delta_d + k^2)\delta_{x,y} , (\Delta_d + k^2)\mathcal{G}^+(x, y + e_2) = \delta_{x,y+e_2} \) and \((\Delta_d + k^2)\mathcal{G}^+(x, y + e_2 - e_1) = \delta_{x,y+e_2-e_1} \) are equal to zero. Consequently, \((\Delta_d + k^2)u(x) = 0\) for points \( x = (x_1, x_2) \) with \( x_2 > 1 \). It remains to consider the case \( x_2 = 1 \). By the direct calculation we have

\[
(\Delta_d + k^2)\delta_{x,y} f(y) = f(x_1) + f(x_1 + 1),
\]
(32)

\[
(\Delta_d + k^2)\mathcal{G}^+(x, y + e_2) f(y) = f(x_1),
\]
(33)

\[
(\Delta_d + k^2)\mathcal{G}^+(x, y + e_2 - e_1) f(y) = f(x_1 + 1),
\]
(34)

and as a result \((\Delta_d + k^2)u(x) = 0\). Thus, \( u(x) \) is a solution to \((4a)\). Since \( \mathcal{G}^+ \) satisfies the radiation condition and \( f \) is supported on the finite subset of \( \Gamma \) it follows that \( u(x) \) is the unique radiating solution to the problem \((4a), (4b)\).

\[\Box\]

5 Numerical results

The main difficulty for numerical evaluation of the solution \((30)\) is to compute the lattice Green’s function. For this purpose, we apply the method developed in [11]. Using 8-fold symmetry, we need only to compute the lattice Green’s function \( \mathcal{G}(i, j) \) with \( i \geq j \geq 0 \). Following to [11], let us introduce the vectors \( \mathcal{V}_{2p} = (\mathcal{G}(2p, 0), \mathcal{G}(2p - 1, 1), \ldots, \mathcal{G}(p, p))^\top \) and \( \mathcal{V}_{2p+1} = (\mathcal{G}(2p + 1, 0), \mathcal{G}(2p, 1), \ldots, \mathcal{G}(p + 1, p))^\top \) that collect all distinct Green’s functions \( \mathcal{G}(i, j) \) with “Manhattan distances” \(| i | + | j |\) of \( 2p \) and \( 2p + 1 \), respectively. For any Manhattan distance larger than 1, equation

\[
(\Delta_d + k^2)\mathcal{G}(x) = \delta_{x,0},
\]
(35)

can be written in the matrix form \( \gamma_n(k)\mathcal{V}_n = \alpha_n(k)\mathcal{V}_{n-1} + \beta_n(k)\mathcal{V}_{n+1} \), where \( \alpha_n(k), \beta_n(k) \), and \( \gamma_n(k) \) are sparse matrices (cf., Appendix A). Notice that only the dimensions of these matrices depend on \( n \). It is shown in [11] that, for any \( n \geq 1 \), we have

\[
\mathcal{V}_n = A_n(k)\mathcal{V}_{n-1},
\]
(36)

where the matrices \( A_n(k) \) are defined by the following recurrence formula:

\[
A_n(k) = [\gamma_n(k) - \beta_n(k)A_{n+1}]^{-1}\alpha_n(k).
\]
(37)

They can be computed starting from a sufficiently large \( N \) with \( A_{N+1}(k) = 0 \). Here, it is worth mentioning that for \( k = 2 \) we need to choose a better “initial guess” than \( A_{N+1}(k) = 0 \), since in this case \( \det \gamma_n(k) = 0 \) and the matrix \( \gamma_n(k) - \beta_n(k)A_{n+1} \) is not invertible.

Once \( A_n(k) \) are known, we have \( \mathcal{V}_n = A_n(k) \ldots A_1(k)\mathcal{V}_0 \), where \( \mathcal{V}_0 = \mathcal{G}(0, 0) \). In particular, \( \mathcal{V}_1 = \mathcal{G}(1, 0) = A_1(k)\mathcal{G}(0, 0) \) which, together with \( 6\mathcal{G}(1, 0) - (6 - k^2)\mathcal{G}(0, 0) = 1 \), gives \( \mathcal{G}(0, 0) = 1/[6A_1(k) - 6 + k^2] \). This completes the calculation of the Green’s function using elementary operations and no integrals. Notice also one more important advantage of this method. The \( A_n(k) \) matrices are calculated coming
down from asymptotically large Manhattan distances. As they are propagated towards smaller Manhattan distances, it definitely gives us the physical solution.

Finally, we demonstrate our theoretical and numerical approaches for the thin-slit diffraction problem on triangular lattice. For this purpose we take \( k = \sqrt{2} \) and consider the wave \( \mathcal{U}(x_1, x_2, t) = \exp(\im x_2 \pi / 3 - \im \omega t) \) on the semi-infinite lattice \( \mathbb{Z} \times \mathbb{Z}^- \) which encounters an obstacle at \( x_2 = 0 \) with two small openings formed by four nodes \( \{(-11, 0), (-10, 0), (10, 0), (11, 0)\} \). It is easy to check that \( \exp(\im x_2 \pi / 3) \) satisfies the discrete Helmholtz equation. Thus, our goal is to evaluate numerically a radiating solution of Problem \( \mathcal{P}_D \), where for the given data, we take \( f(\pm 10) = f(\pm 11) = 1 \) and \( f(m) = 0, (m \in \mathbb{Z} \setminus \{-11, -10, 10, 11\}) \).

The results of numerical evaluations are plotted in Fig. 2 in original coordinates of \( \mathbb{R} \), where (a) shows the density plots of the Green’s function \( \text{Re} \ G \), (b) shows the density plots of \( \cos(\im x_2 \pi / 3) \) on the lower half-plane and \( \text{Re} \ u \) for the case \( k = \sqrt{2} \) on the upper half-plane, and (c) presents the graph of \( \text{Re} \ u(\cdot, 8 \sqrt{3}) \). Some key features of the numerical solution can be immediately observed, namely, as expected, the symmetry of \( \text{Re} \ u \) and contributions from the wavefront that create a variable intensity.

In our second example, we compare solutions of the classical single-slit diffraction problem and of its discrete version for particular cases. For this purpose, in Problem \( \mathcal{P}_D \) we take \( f(m) = 1, m \in \Gamma_1 \), and \( f(m) = 0, m \in \Gamma_0 \), where \( \Gamma_1 = \{0, \pm 1, \cdots \pm 10\} \) and \( \Gamma_0 = \mathbb{Z} \setminus \Gamma_1 \). The results of numerical evaluations of the diffraction pattern and intensity graph al level \( 10 \sqrt{3} \) are shown in Fig. 3, where the graphs of \( |u(\cdot, 10 \sqrt{3})| \) are plotted when \( k = 0.01 \), and \( k = 1/\sqrt{2} \). We also plot in blue (dashed line) the diffraction pattern predicted by standard Rayleigh–Sommerfeld diffraction theory [12, 13] for a two-dimensional continuum:

\[
\mathcal{U}(x_1, x_2) = -\frac{k x_1}{2t} \int_{-10}^{10} H_1(k \sqrt{(x_1 - y_1)^2 + x_2^2}) \frac{1}{\sqrt{(x_1 - y_1)^2 + x_2^2}} dy_1,
\]

where \( H_1 \) is the Hankel function of the first kind. As expected, the continuum theory diverges from the numerical experiment as \( k \) increases.

In Fig. 4, we present the density plots of solutions for discrete and continuous frameworks near the right edge point \((10, 0)\). Since the gap is relatively wide we observe small diffraction effects. However, we clearly see that the diffraction effects are larger for the discrete framework.

6 Discussion

In this paper, we have constructed the discrete scattering theory for the two-dimensional discrete Helmholtz equation with a real wave number \( k \in (0, 3) \setminus \{2 \sqrt{2}\} \) for the semi-infinite triangular lattices. The main objective was to prove the unique solvability result and derive an exact formula for the solution to a Dirichlet problem. For simplicity, we restricted ourselves to compact boundary data. For non-compact boundary data, we should require some extra conditions at infinity, cf. [8].
Fig. 2 Plots are represented in original coordinates of the triangular lattice $\mathbb{T}$, $k = \sqrt{2}$.
Fig. 3 The graphs of $|u(\cdot, 10\sqrt{3})|$. Results from the numerical experiment for Problem $P_D$ are plotted in red. In blue (dashed line) we plot the results of continuum Rayleigh–Sommerfeld diffraction theory. (Color figure online)
Fig. 4. The density plots of Re $u$ are presented near the edge point $(10,0)$ for discrete and continuous frameworks.

(a) Discrete framework, $k = 0.01$

(b) Continuous framework, $k = 0.01$

(c) Discrete framework, $k = 1/\sqrt{2}$

(d) Continuous framework, $k = 1/\sqrt{2}$
Similarly to the continuum theory, we used the notion of the radiating solution for the continuous Helmholtz equation. In the present paper, the problems under consideration have an infinite boundary. Within the continuum framework it is well known that, in general, when the surface is unbounded, we cannot neglect the contribution of that surface waves at infinity. In this case, the Sommerfeld radiation condition is no longer appropriate and a proper modification is needed, cf. [14–19]. In the case of the square lattice, we have proposed sufficient conditions for the given boundary data at infinity, which ensures to have an unique radiation solution to the corresponding problem, cf. [8]. Here, in order to avoid further technical difficulties, we restricted ourselves with compactly supported data on the boundary.

Finally, let us note that comparing the results of problems in the continuous framework and results obtained in the continuum limit of discrete problems deserves the high interest of scientists, however, it is beyond the scope of this paper.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A: Sparse matrices

The sparse matrices $\alpha_n(k)$, $\beta_n(k)$, and $\gamma_n(k)$ are defined as follows: if $n = 2p$ then $\alpha_{2p}(k)$ is a $(p + 1) \times p$ matrix such that $\alpha_{2p}(k) \mid_{i, i} = 1$, $i = \overline{1, p}$, $\alpha_{2p}(k) \mid_{i, i+1} = 1$, $i = \overline{2, p}$, while $\alpha_{2p}(k) \mid_{p+1, p} = 2$, and all other matrix elements are zero. The $\beta_{2p}(k)$ is a $(p + 1) \times (p + 1)$ matrix such that $\beta_{2p}(k) \mid_{i, i} = 1$, $i = \overline{1, p}$, $\beta_{2p}(k) \mid_{i, i+1} = 1$, $i = \overline{2, p}$, while $\beta_{2p}(k) \mid_{p+1, p+1} = \beta_{2p}(k) \mid_{i, 2} = 2$, and all other matrix elements are zero. The $\gamma_{2p}(k)$ is a $(p+1) \times (p+1)$ matrix such that $\gamma_{2p}(k) \mid_{i, i} = 6 - k^2$, $i = \overline{1, p}$, $\gamma_{2p}(k) \mid_{i, i+1} = \gamma_{2p}(k) \mid_{i, i-1} = 1$, $i = \overline{2, p}$, and $\gamma_{2p}(k) \mid_{1, 2} = \gamma_{2p}(k) \mid_{p+1, p} = -2$.

If $n = 2p + 1$ then $\alpha_{2p+1}(k)$ is a $(p + 1) \times (p + 1)$ matrix such that $\alpha_{2p+1}(k) \mid_{i, i} = 1$, $i = \overline{1, p + 1}$, $\alpha_{2p+1}(k) \mid_{i, i+1} = 1$, $i = \overline{2, p + 1}$, and all other matrix elements are zero. The $\beta_{2p+1}(k)$ is a $(p + 1) \times (p + 2)$ matrix such that $\beta_{2p+1}(k) \mid_{i, i} = 1$, $i = \overline{1, p + 1}$, $\beta_{2p+1}(k) \mid_{i, i+1} = 1$, $i = \overline{2, p + 1}$, while $\beta_{2p+1}(k) \mid_{i, 2} = 2$, and all other matrix elements are zero. The $\gamma_{2p+1}(k)$ is a $(p + 1) \times (p + 1)$ matrix such that $\gamma_{2p+1}(k) \mid_{i, i} = 6 - k^2$, $i = \overline{1, p}$, $\gamma_{2p+1}(k) \mid_{i, i+1} = 5 - k^2$, while $\gamma_{2p+1}(k) \mid_{i, i+1} = -1$, $i = \overline{2, p}$, $\gamma_{2p+1}(k) \mid_{i, i-1} = 1$, $i = \overline{2, p + 1}$, and $(\gamma_{2p+1}(k) \mid_{1, 2} = -2)$. Finally, $\gamma_{1}(k)$ is a $1 \times 1$ matrix with an element $4 - k^2$.

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