Fully developed isotropic turbulence: nonperturbative renormalization group formalism and fixed point solution

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We investigate the regime of fully developed homogeneous turbulence of the Navier-Stokes (NS) equation in the presence of a stochastic forcing. Using the nonperturbative (functional) version of the renormalization group (NPRG), we avoid all the difficulties that plague standard RG approaches to NS turbulence and have hindered real progress in the calculation of multi-scaling. We obtain the fixed point solution of the NPRG flow equations, which corresponds to fully developed turbulence, in $d = 2$ and $d = 3$ dimensions. The striking feature of this fixed point is that it does not entail the usual scale invariance, because of the absence of a regular limit when the integral scale (the typical length scale of energy injection) tends to infinity. We indeed show, on the basis of exact flow equations in the large wave-number limit, how violations to the Kolmogorov scaling can emerge, leaving the accurate determination of the ensuing intermittent exponents for future work.

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\section{I. INTRODUCTION}

The statistical theory of turbulence is more than seventy years old and, despite intensive efforts, it remains unsatisfactory. Two length scales play a dominant role in the phenomenology of fully developed isotropic turbulence: the microscopic (Kolmogorov) scale $\eta$ where energy is dissipated by molecular viscosity and the macroscopic integral scale $L$ where energy is injected in the system. In between these two scales and when turbulence is fully developed, the equal-time velocity correlation functions exhibit universal scaling, that is, they behave as power laws with exponents independent of the precise mechanisms of energy injection and dissipation. The energy cascade scenario (in dimensions $d > 2$) together with dimensional analysis leads to the celebrated K41 scaling law which reproduces this universal behavior although with incorrect exponents (but for the three-velocity correlator)\textsuperscript{[1,2]} Calculating these exponents beyond K41 scaling has been achieved with success in the simpler passive scalar problem\textsuperscript{[4,5]} – where a scalar field is advected by a prescribed Gaussian random field with K41-type scaling and with a very short correlation time – but it remains one of the major challenge in the Navier-Stokes case. This situation appears frustrating if compared to that of critical phenomena occurring at equilibrium, which share many common features with turbulence (e.g. scaling, chaos, universality)\textsuperscript{[3]} and where renormalization group (RG) has led in most cases to a clear understanding of the physics at play and to accurate determinations of the critical exponents\textsuperscript{[6]}. An essential difference is that, in standard equilibrium critical phenomena, a finite set of anomalous dimensions is sufficient to describe the scaling behavior of all the correlation functions, which is no longer true for turbulence. The correlation functions do follow power laws, but each with its specific exponent, which is referred to as multi-scaling and constitutes the imprint of turbulence that moreover occurs not only in space but also in time\textsuperscript{[10]}.

The most popular microscopic model to describe fully developed isotropic turbulence is based on the forced Navier-Stokes equation:

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha = -\frac{1}{\rho} \partial_\alpha p + \nu \nabla^2 v_\alpha + f_\alpha$$ \hspace{1cm} (1)

where the velocity field $\vec{v}$, the pressure field $p$, and the stochastic forcing $\vec{f}$ depend on the space-time coordinates $(t, \vec{x})$, and with $\nu$ the kinematic viscosity and $\rho$ the density of the fluid. Since we aim at studying the turbulent steady state, the stirring force $\vec{f}$ is essential to balance the dissipative nature of the (unforced) NS equation which otherwise leads to the decay of all the velocity fields. We consider in the following incompressible flows, satisfying

$$\partial_\alpha v_\alpha = 0.$$

We focus on the inertial range of wave-numbers $p$ corresponding to $L^{-1} \ll p \ll \eta^{-1}$. In this regime, the various steady-state correlation functions are expected to be universal in the sense that they do not depend on the precise form of the forcing (as long as its Fourier transform is peaked around wave-numbers of the order $L^{-1}$). This universality allows one, instead of choosing a deterministic forcing, to take averages on various forcings with an essentially arbitrary probability distribution, as long as the typical scale of the forcing remains the prescribed integral scale. One can hence conveniently choose the simplest of probability distributions: a Gaussian one with zero mean and variance

$$\langle f_\alpha(t, \vec{x}) f_\beta(t', \vec{x}') \rangle = 2\delta(t - t') N_{L^{-1},\alpha\beta}(|\vec{x} - \vec{x}'|),$$ \hspace{1cm} (3)
centered, in Fourier space, on the inverse of the integral scale $L$. As just mentioned the precise profile of $N_{\alpha\beta}(x)$ should not affect universal properties in the inertial regime and will be specified in the following.

As in critical phenomena, scaling is observed in turbulence when the microscopic scale (the Kolmogorov scale $\eta$) is sent to zero and the macroscopic one (the integral scale $L$) to infinity. In this limit, the expansion parameter, the Reynolds number, diverges. Field theoretic techniques such as renormalization group (RG) are designed to handle the large scale fluctuations developing in strongly correlated systems and we briefly review in the following some of the former attempts in turbulence (for reviews see [17, 18]).

The difficulty when applying RG in turbulence is not so much that the Reynolds number diverges when the ultra-violet (UV) scale $\eta^{-1}$ is sent to infinity. Although it could seem odd for an expansion parameter, this can be fine when RG is employed, as well known [15]. The real difficulty is to find a situation where the renormalized expansion parameter is small. For standard critical systems, this is achieved around the upper critical dimension $d_c$ and a double expansion in the coupling constant and $\epsilon = d_c - d$ renders the perturbative expansion well defined. As for turbulence, there is no upper critical dimension but a formal (second) expansion parameter $\epsilon$ can be defined through the forcing profile $N_{\alpha\beta}(p) \propto p^{1-d-2\epsilon}$ where $p$ is the wave-number [19, 22]. Typically, as explained above, $N_{\alpha\beta}(p)$ is not a power-law in Fourier space, but is instead sharply peaked around the infra-red (IR) scale $L^{-1}$, which is recovered only in a precise limit when $\epsilon \to 2$. On the other hand, for $\epsilon = 0$, the theory is exactly renormalizable and a fixed point of order $\epsilon$ is found in any dimension [23, 24] (the $d = 2$ case being particular [23]). The challenge for the perturbatively renormalized theory is therefore to extend the results obtained for $\epsilon \to 0$ to $\epsilon = 2$ which is far from trivial: The difficulties encountered are so severe that, up to now, they have hindered real progress, at least for the calculation of multi-scaling behavior [22, 24].

Let us notice that another perturbative approach, almost ignored in the subsequent literature, does not rely on an $\epsilon$-expansion but on a self-consistent determination of the quadratic part of the action around which perturbation theory is performed [26]. This approach, after elimination of what is named the “sweeping effect” (the sweeping of the smaller scales by the larger) leads to the existence of an UV attractive fixed point from which, performing an operator product expansion, can be computed the multi-scaling exponents. They turn out to be rather accurate at least for the $n < 10$ first equal-time correlation functions of the velocity differences.

A rather different field-theoretic approach, not based on RG, has been developed by L’vov, Procaccia and collaborators. To get rid of the sweeping effect that leads to severe IR singularities in renormalized perturbation theory, these authors use “quasi-Lagrangian” variables instead of the Eulerian velocities [24]. They show that the correlation functions of the differences of these variables are finite order by order in perturbation theory in both limits where the UV and IR scales are removed. As a consequence, Kolmogorov scaling holds at all finite orders of the perturbation theory. The only way out of this hindrance is to resum infinite classes of Feynman diagrams. These authors indeed show that doing so produces new singularities in terms of the integral scale which is therefore the proper renormalization scale [25–28]. They are then able to compute approximately the small order multi-scaling exponents in terms of the first one [28].

An alternative RG approach that has been extremely successful in the study of critical systems either at or out of equilibrium, is the nonperturbative renormalization group (NPRG), which is a modern version of the RG à la Wilson [40, 42]. In addition to avoiding most of the problems encountered perturbatively, such as the need of explicit resummation of IR singularities or the asymptotic nature of the renormalized series [15], this approach has led not only to very accurate [15, 52] but also fully nonperturbative [53, 58] results in many systems. A very inspiring case is the Kardar-Parisi-Zhang (KPZ) equation describing the stochastic growth of interfaces [59], which is equivalent to the Burgers equation, and which shows fully nonperturbative behavior in the rough phase [60]. Contrary to the standard perturbative RG that fails to find the relevant fixed point and the associated scaling behavior [61], the NPRG approach captures the strong coupling physics of the KPZ equation at and above one dimension [62, 65]. One of the aims of this article is to show how this method can be implemented to study Navier-Stokes turbulence. Let us mention for completeness other works using similar functional RG methods [66–69]. In particular, the settings developed in Ref. [66] and Ref. [68] are closely related to the one presented here, although the latter studied a power-law forcing, in contrast to our work and Ref. [66]. The relevant fixed point for turbulence was already found in Ref. [66] in three dimensions. However, important elements concerning the symmetries, multi-scaling and the regularization, were not identified in this early work, and we here bring them out.

Our analysis follows the general strategy of the NPRG in its modern implementation [41]. An ansatz for the scale-dependent generating functional $\Gamma_\kappa$ of the (one-particle-irreducible) correlation and response functions is proposed and its evolution is followed between the Kolmogorov scale and the macroscopic scale. The accuracy of the results obtained with this method depends on the quality of the ansatz chosen. To preserve all the symmetries of the initial problem along the RG flow is of particular importance to ensure that it takes place in the appropriate functional (in fields, momenta and frequencies) space. We indeed show how the (gauge) symmetries of the NS field theory, presented in Ref. [71], play a crucial role since they strongly constrain the choice of the ansatz. We derive the corresponding flow equations (at Next-to-
Leading Order approximation) and show that the RG flow is generically (without fine-tuning any parameter) attracted towards a fixed point, which corresponds to stationary fully developed turbulence. Within this approximation, we find that the energy spectrum and second order structure function computed at this fixed point do not deviate from K41 predictions, as already observed by Tomassini [66]. Nevertheless, eventhough the existence of the fixed point and its properties for wave-numbers smaller or comparable to the inverse integral scale is well reliable, the performed approximation is not justified at wave-numbers much larger than the inverse integral scale and its predictions in this regime should be taken with care. In fact, we prove without approximations, that the exact fixed point does not entail the usual scale invariance in the large wave-number regime, and we expound the mechanism of emergence of multi-scaling. It originates in our formalism in a property of non-decoupling of the wave-number scales (the two fundamental UV and IR scales play a role all along the flow), which is not encountered in ordinary critical behavior. We indeed explain in details how, by using the Ward identities ensuing from the gauge symmetries of the theory, a set of exact and closed equations on the two-point functions can be derived in the large (compared to the running scale of the flow) wave-number regime, and how the non-decoupling property can be proved. These equations complement the flow equations obtained from the ansatz, which are valid in the small wave-number regime. Their numerical solution, that we leave for a future publication, should allow us to obtain from first principles the deviation to the K41 scaling for the two-point functions.

II. NPRG FORMALISM FOR NAVIER-STOKES TURBULENCE

A. Navier-Stokes field theory

The NS equation in the presence of a stochastic forcing \( \tilde{f} \) stands as a Langevin equation. One can resort to the standard Martin-Siggia-Rose-Janssen-de Dominicis procedure [71, 72, 20] to derive the associated field theory. Following Ref. [71], one introduces Martin-Siggia-Rose response fields \( \tilde{v}_\alpha \) and \( \tilde{p} \) to enforce both the equation of motion (1) and the incompressibility constraint (2). Note that the pressure sector turns out to be very simple to handle since it is not renormalized [72]. Once the response fields are introduced, the stochastic forcing can be integrated out and one obtains the generating functional

\[
Z[\tilde{J}, \tilde{\bar{K}}, \bar{K}, \bar{K}] = \int D\tilde{u} D\tilde{\bar{p}} D\tilde{u} D\tilde{\bar{p}} e^{-\left(S_0[\tilde{v}, \tilde{p}, \bar{p}, \bar{p}] + 1\Delta S_{0, L^{-1}}[\tilde{v}, \tilde{\bar{u}}]\right)}
\]

\[
\times e^{\int \tilde{J}_+ \tilde{u}_+ + \tilde{\bar{u}} + K_\bar{p} + \bar{K}_\bar{p}}
\]

where \( \tilde{J}, \bar{K}, \tilde{\bar{K}} \) and \( \bar{K} \) are sources for the velocity, pressure and response fields, and where the NS action, splitted in a local and a nonlocal contribution for convenience, is given by

\[
S_0[\tilde{v}, \tilde{\bar{u}}, \tilde{p}, \bar{p}] = \int_\mathbf{x} \left\{ \tilde{p}(\mathbf{x}) \partial_t \tilde{v}_\alpha(\mathbf{x}) + \tilde{\bar{u}}_\alpha(\mathbf{x}) \left[ \partial_t \tilde{v}_\alpha(\mathbf{x}) - \nu \nabla^2 \tilde{v}_\alpha(\mathbf{x}) + v_\beta(\mathbf{x}) \partial_t \tilde{v}_\beta(\mathbf{x}) + \frac{1}{\rho} \partial_t \bar{p}(\mathbf{x}) \right] \right\}
\]

\[
\Delta S_{0, L^{-1}}[\tilde{\bar{u}}, \tilde{\bar{v}}] = -\int_{t, \tilde{\bar{x}}, \tilde{\bar{x}}'} \tilde{\bar{v}}_\alpha(t, \tilde{\bar{x}}) N_{L^{-1}, \alpha\beta}(|\tilde{\bar{x}} - \tilde{\bar{x}}'|) \tilde{\bar{v}}_\beta(t, \tilde{\bar{x}}')
\]

where \( \mathbf{x} \equiv (t, \tilde{\bar{x}}) \) and \( \int_\mathbf{x} \equiv \int d^d \tilde{\bar{x}} dt \). Let us now discuss the choice of the forcing profile. Without loss of generality, in order to preserve rotational invariance along the flow, it can be written in Fourier space as

\[
N_{\kappa, \alpha\beta}(\tilde{q}) \equiv \delta_{\alpha\beta} N_{\kappa}(\tilde{q}) + q_\alpha q_\beta \tilde{N}_\kappa(\tilde{q}),
\]

where the inverse integral scale \( L^{-1} \) is denoted \( \kappa \) in anticipation since it will be running in the following. The Fourier convention, used throughout this work, is

\[
f(q) = \int f(\mathbf{x}) e^{-i\tilde{q} \cdot \tilde{x} + i\tilde{q} \cdot \tilde{x}'}
\]

\[
f(q) = \int f(q) e^{i\tilde{q} \cdot \tilde{x} - i\tilde{q} \cdot \tilde{x}'}
\]

where \( q \equiv (\omega, \tilde{q}) \) and \( \int_\mathbf{x} \equiv \int \frac{d^d \tilde{q}}{(2\pi)^d} \frac{d\omega}{2\pi} \). In practice, due to the incompressibility condition, the term proportional to \( q_\alpha q_\beta \) plays no role and can be omitted. We further parametrize the function \( N_\kappa \) as

\[
N_\kappa(q) = D_\kappa (|\tilde{q}|/\kappa)^2 \hat{n} (|\tilde{q}|/\kappa)
\]

where \( D_\kappa \) is a scale-dependent coefficient, discussed in Sec. [14]. The stirring force profile, peaked at the inverse integral scale, can be typically shaped as

\[
\hat{n}(x) = e^{-x^2}
\]

Let us report that ten different forcing profiles have been studied in Ref. [63], which shows that the influence of the precise form of the stirring is negligible, or inexistence, as expected from universality, in the sense that the properties of the turbulent flow in the stationary regime do not change. We can hence restrict our analysis to the specific profile [9].

B. NPRG formalism

The general NPRG formalism for nonequilibrium systems in classical physics is presented in details in Ref. [74]. In the spirit of Wilson’s RG ideas, it consists in building a sequence of scale-dependent effective models such that fluctuations are smoothly averaged as the (wave-numbers) scale \( k \) is lowered from the microscopic UV scale \( k = \eta^{-1} \), where no fluctuations are yet included, to the macroscopic IR scale \( k = 0 \), where they
are all summed over \([15, 73]\). The procedure is formally the same as in equilibrium \([45]\), but with the presence of response fields, and additional requirements stemming from \(\text{It}^\text{o}\)'s discretization and causality issues \([7, 4, 73]\).

In this work, we identify the RG wave-numbers scale \(k\) with the inverse of the integral scale \(\kappa\), i.e. \(k \equiv \kappa\), which is therefore running and eventually sent to infinity when \(k \to 0\). For other purposes, the two scales can be kept independent, for instance to study the regime of a fixed integral scale \(L\) in the infinite volume limit (RG scale tends to zero). To achieve the separation of fluctuation modes within the NPRG procedure, a wave-number and scale dependent quadratic term is added to the original action \(S_0 + \Delta S_{0, L-1}\). We here, on the one hand, let the inverse integral scale run in the nonlocal quadratic term \(\Delta S_{0, L-1}\) - dropping the 0 index - (as in \([66]\)). On the other hand, we include an additional scale dependent quadratic term in the following way

\[
\Delta S_{\kappa}[\bar{\vartheta}, \bar{v}] = -\int_{x, \tilde{x}, \tilde{\vartheta}} \bar{v}_\alpha(t, x) N_{\kappa, \alpha\beta}(\bar{\vartheta}_\beta(t, \tilde{x})) + \int_{x, \tilde{x}, \tilde{\vartheta}} \bar{v}_\alpha(t, x) R_{\kappa, \alpha\beta}(\bar{\vartheta}_\beta(t, \tilde{x})). \tag{10}
\]

As before, by using the incompressibility of the flow, the function \(R_{\kappa}\) can be chosen diagonal. We write it in Fourier space

\[
R_{\kappa, \alpha\beta}(q) = \delta_{\alpha\beta} R_{\kappa}(q) = \delta_{\alpha\beta} \nu_{\kappa} q^2 \hat{\vartheta}(q^2 / \kappa^2) \tag{11}
\]

with \(q = |q|\) and where \(\nu_{\kappa}\) is a scale-dependent coefficient, defined in Sec. \([\sqrt{12}]\). The cutoff function \(\hat{\vartheta}(x)\) ensures the selectation of fluctuation modes: \(\hat{\vartheta}(x)\) is required to almost vanish for \(x \gtrsim 1\) such that the fluctuation modes \(v_{\alpha}(q \gtrsim \kappa)\) and \(\bar{v}_\alpha(q \gtrsim \kappa)\) are unaffected by the \(R_{\kappa}\) term in \(\Delta S_{\kappa}\), and to be large when \(x \lesssim 1\) such that the other modes \((v_{\alpha}(q \lesssim \kappa)\) and \(\bar{v}_\alpha(q \lesssim \kappa)\) are essentially frozen. One can show that this form of regulator term preserves all the symmetries and causality properties of the problem as done in a very similar case in \([63]\). We work here with the following cutoff function

\[
\hat{\vartheta}(x) = \frac{a}{e^{\vartheta} - 1} \tag{12}
\]

where \(a\) is a free parameter, which can be varied to assess the accuracy of the approximation scheme \([77]\). Let us emphasize that the introduction of the cutoff term \(R_{\kappa}\) is essential to properly implement the RG procedure and to correctly regularize the flow, both in the UV and in the IR. This constitutes a fundamental difference between the present work and the work of Ref. \([60]\), where the term \(R_{\kappa}\) is missing, and only the forcing term \(N_{\kappa}\) acts to select the fluctuation modes. Although the procedure of \([60]\) qualitatively leads to the correct behavior in \(d = 3\), it clearly prevents from studying the \(d = 2\) case because the flow equations are divergent in this dimension without the \(R_{\kappa}\) term. Conversely, the NPRG flow equations derived in the present work are valid in any dimensions.

In the presence of the regulator term \(\Delta S_{\kappa}\), the generating functional \([1]\) becomes scale dependent

\[
Z_{\kappa}[\bar{J}, \bar{\vartheta}, K, \bar{K}] = \int D\bar{\vartheta} D\bar{J} D\bar{\vartheta} D\bar{J} e^{-(S_0[\bar{\vartheta}, \bar{\vartheta}, \bar{\vartheta}, \bar{\vartheta}] + \Delta S_{\kappa}[\bar{\vartheta}, \bar{\vartheta}])}
\]

\[
\times \bar{e}^J \left(\bar{\vartheta} + \bar{\vartheta} + K + \bar{K}\right), \tag{13}
\]

Field expectation values in the presence of the external sources \(\bar{J}, \bar{\vartheta}, K,\) and \(\bar{K}\) are obtained as functional derivatives of \(W_{\kappa} = \log Z_{\kappa}\) as

\[
u_{\alpha}(x) = \langle v_{\alpha}(x) \rangle = \frac{\delta W_{\kappa}}{\delta J_{\alpha}(x)}, \quad \bar{u}_{\alpha}(x) = \langle \bar{u}_{\alpha}(x) \rangle = \frac{\delta W_{\kappa}}{\delta \bar{J}_{\alpha}(x)}
\]

and similarly for the pressure fields, for which for simplicity the same notation can be kept for the fields and their average values

\[
p(x) = \langle p(x) \rangle = \frac{\delta W_{\kappa}}{\delta K(x)}, \quad \bar{p}(x) = \langle \bar{p}(x) \rangle = \frac{\delta W_{\kappa}}{\delta \bar{K}(x)}.
\]

The effective action \(\Gamma_{\kappa}[\bar{u}, \bar{\vartheta}, p, \bar{p}]\) is defined as the Legendre transform of \(W_{\kappa}\) (up to terms proportional to \(R_{\kappa, \alpha\beta}\) or \(N_{\kappa, \alpha\beta}\)) \([45, 74]\):

\[
\Gamma_{\kappa}[\bar{u}, \bar{\vartheta}, p, \bar{p}] + W_{\kappa}[\bar{J}, \bar{\vartheta}, K, \bar{K}] = \int \bar{J} \varphi_i
\]

\[
- \int_{t, \tilde{x}, \tilde{\vartheta}} \left\{\bar{u}_\alpha R_{\kappa, \alpha\beta} \bar{u}_\beta - \bar{u}_\alpha N_{\kappa, \alpha\beta} \bar{u}_\beta\right\}
\]

\[
\tag{14}
\]

where \(\varphi_i, \ i = 1, \ldots, 4\), stand for the fields \(u_{\alpha}, \bar{u}_\alpha, p\) and \(\bar{p}\), respectively, and \(j_i\) for the sources \(J_{\alpha}, \bar{J}_{\alpha}, K\) and \(\bar{K}\), respectively. From \(\Gamma_{\kappa}\), one can derive 2-point correlation and response functions, which can be gathered in a \(4 \times 4\) matrix as

\[
[\Gamma_{\kappa}^{(2)}]_{i_1, i_2} (x_1, x_2, \{\varphi_i\}) = \frac{\delta^2 \Gamma_{\kappa}[\{\varphi_i\}]}{\delta \varphi_{i_1}(x_1) \delta \varphi_{i_2}(x_2)} \tag{15}
\]

and more generally \(n\)-point correlation functions that are also written in a \(4 \times 4\) matrix form as

\[
[\Gamma_{\kappa}^{(n)}]_{i_1, i_2, \ldots, i_n} (x_1, x_2, \ldots, \{\varphi_i\}) = \frac{\delta^{n-2} \Gamma_{\kappa}^{(2)}(x_1, x_2, \{\varphi_i\})}{\delta \varphi_{i_1}(x_3) \ldots \delta \varphi_{i_n}(x_n)}. \tag{16}
\]

The exact flow for \(\Gamma_{\kappa}[\bar{u}, \bar{\vartheta}, p, \bar{p}]\) is given by Wetterich equation, which reads in Fourier space \([43, 78]\)

\[
\partial_{\kappa} \Gamma_{\kappa} = \frac{1}{2} \text{Tr} \int_q \partial_{\kappa} R_{\kappa}(q) \cdot G_{\kappa}(q), \tag{17}
\]

where \(R_{\kappa}(q)\) represents the \(4 \times 4\) matrix with sole non-vanishing elements \([R_{\kappa}]_{22} = -2 N_{\kappa, \alpha\beta}\) and \([R_{\kappa}]_{12} = [R_{\kappa}]_{21} = R_{\kappa, \alpha\beta}\), and

\[
G_{\kappa} = \left[\Gamma_{\kappa}^{(2)} + R_{\kappa}\right]^{-1} \tag{18}
\]

is the full, that is, field-dependent, renormalized at scale \(\kappa\) propagator of the theory. When the RG scale \(\kappa\) is
lowered from the UV scale $\eta^{-1}$ to zero, $\Gamma_\kappa$ interpolates between the microscopic model $\Gamma_{\kappa=n^{-1}} = S_0$ and the full effective action $\Gamma_{\kappa=0}$ that encompasses all the macroscopic properties of the system (for a detailed discussion in nonequilibrium processes, see Ref. [74]). Differentiating Eq. (17) twice with respect to the fields and evaluating the resulting identity in a uniform and stationary field configuration $\varphi_i(x) = \varphi_i$ (since the model is analyzed in its long time and large distance regime where it is translationally invariant in space and time) one obtains the flow equation for the 2-point functions:

$$
\partial_\kappa \Gamma^{(2)}_{\kappa,i,j}(p) = \text{Tr} \int_q \partial_\kappa R_{\kappa}(q) \cdot G_\kappa(q) \cdot \left( -\frac{1}{2} \Gamma^{(4)}_{\kappa,i,j}(p, -p, q) + \Gamma^{(3)}_{\kappa,i}(p, q) \cdot G_\kappa(p + q) \cdot \Gamma^{(3)}_{\kappa,j}(-p, p + q) \right) \cdot G_\kappa(q)
$$

where the background field $\varphi_i$ dependencies have been omitted, as well as the last arguments of the $\Gamma^{(n)}_{\kappa}$ which are determined by frequency and wave-vector conservation [74].

Of course Eq. (17) cannot be solved exactly and one has to resort to an appropriate approximation scheme, adapted to the physics of the model under study, and in particular to its symmetries, see Sec. [V]

III. SYMMETRIES AND RELATED WARD IDENTITIES

In this section, we briefly review the three gauge symmetries of the NS action expounded in Ref. [70] and the ensuing non-renormalization theorems and general Ward identities derived in this Reference.

A. Symmetries

The NS action $S \equiv S_0 + \Delta S_\kappa$ given by [14] admit three gauge symmetries:

- (i) invariance under gauged shift of the pressure fields,
- (ii) time-gauged Galilean symmetry,
- (iii) invariance under time-gauged shift of the response velocity.

The symmetry (i) is the invariance of $S$ under the local shifts $p(t, \vec{x}) \rightarrow p(t, \vec{x}) + \varepsilon(t, \vec{x})$ or $\bar{p}(t, \vec{x}) \rightarrow \bar{p}(t, \vec{x}) + \bar{\varepsilon}(t, \vec{x})$. The infinitesimal time-gauged Galilean symmetry (ii) consists in the following field transformation

$$
\begin{align*}
\delta v_\alpha(x) &= -\dot{\varepsilon}_\alpha(t) + \varepsilon_\beta(t) \partial_\beta v_\alpha(x) \\
\delta \bar{v}_\alpha(x) &= \varepsilon_\beta(t) \partial_\beta \bar{v}_\alpha(x) \\
\delta p(x) &= \varepsilon_\beta(t) \partial_\beta p(x) \\
\delta \bar{p}(x) &= \varepsilon_\beta(t) \partial_\beta \bar{p}(x)
\end{align*}
$$

where $\dot{\varepsilon}_\alpha = \partial_t \varepsilon_\alpha$. When $\varepsilon(t) \equiv \varepsilon$ is an arbitrary constant vector, the transformation corresponds to a translation in space, and when $\varepsilon(t) \equiv \varepsilon \dot{t}$ it corresponds to the usual (non-gauged) Galilean transformation. Lastly, the infinitesimal time-gauged shift symmetry (iii) consists in the field transformation

$$
\begin{align*}
\delta \bar{v}_\alpha(x) &= \bar{v}_\alpha(t) \\
\delta \bar{p}(x) &= \bar{v}_\alpha(x) \varepsilon_\beta(t).
\end{align*}
$$

B. Ward identities

The Ward identities ensuing from the gauged shift symmetries (i) simply read

$$
\frac{\delta \Gamma_\kappa}{\delta p(x)} = \frac{\delta S_0}{\delta p(x)} \quad \text{and} \quad \frac{\delta \Gamma_\kappa}{\delta \bar{p}(x)} = \frac{\delta S_0}{\delta \bar{p}(x)}
$$

which means that the dependence in $p(x)$ and $\bar{p}(x)$ of both the effective action $\Gamma_\kappa$ and of the bare one $S_0$ are identical. One thus concludes that the whole pressure sector is not renormalized.

The NS action is invariant under the time-gauged Galilean transformation [20], but for the term proportional to the Lagrangian time derivative $D_t v_\alpha(x) \equiv \partial_t v_\alpha(x) + v_\beta(x) \partial_\beta v_\alpha(x)$, which variation is

$$
\delta \int_x \bar{v}_\alpha(x) D_t v_\alpha(x) \equiv - \int_x \dot{\varepsilon}_\alpha(t) \bar{v}_\alpha(x).
$$

Requiring that the change of variables [20] leaves the functional integral [13] unaltered, one obtains the following Ward identity

$$
\begin{align*}
\int_x \{ (\delta_{\alpha\beta} \partial_\beta u_\alpha(x) - \partial_\beta \partial_\alpha u_\beta(x)) \frac{\delta \Gamma_\kappa}{\delta u_\alpha(x)} + \partial_\beta \bar{u}_\alpha(x) \frac{\delta \Gamma_\kappa}{\delta \bar{u}_\beta(x)} \\
+ \partial_\beta p(x) \frac{\delta \Gamma_\kappa}{\delta p(x)} + \partial_\beta \bar{p}(x) \frac{\delta \Gamma_\kappa}{\delta \bar{p}(x)} \} = - \int_x \partial_\beta^2 \bar{u}_\beta(x),
\end{align*}
$$

which implies that both variations of the effective action and of the bare one are identical. This entails that,
apart from the term $\int \bar{u}_\alpha(x) D_\alpha u_\alpha(x)$ which is not renormalized and remains equal to its bare expression, $\Gamma_\kappa$ is invariant under time-gauged Galilean transformations.

As for the time-gauged shift symmetry (iii), the variation of the NS action under (21) is
\begin{equation}
\delta S = - \int_x \dot{v}_\beta(x), \tag{25}
\end{equation}
which implies the Ward identity
\begin{equation}
\int_x \left\{ \frac{\delta \Gamma_\kappa}{\delta \bar{u}_\alpha(x)} + u_\alpha(x) \frac{\delta \Gamma_\kappa}{\delta \bar{p}(x)} \right\} = \int_x \partial_t u_\alpha(x), \tag{26}
\end{equation}
meaning that, apart from the term $\int \dot{\bar{u}}_\alpha \partial_t u_\alpha$ which is not renormalized, the effective action $\Gamma_\kappa$ is invariant under the time-gauged shift transformations.

C. General structure of the effective action $\Gamma_\kappa$

One can infer from the previous Ward identities the general form of the effective action $\Gamma_\kappa$:
\begin{align*}
\Gamma_\kappa[\bar{u}, \bar{\bar{u}}, p, \bar{\bar{p}}] &= \int_x \left\{ \bar{u}_\alpha \left( \partial_t u_\alpha + \lambda u_\beta \partial_\beta u_\alpha + \frac{\partial_\alpha \bar{p}}{\bar{p}} \right) \right. \\
&\quad+ \bar{\bar{p}} \partial_\alpha u_\alpha \right\} + \bar{\Gamma}_\kappa[\bar{u}, \bar{\bar{u}}] \tag{27}
\end{align*}
where the explicit terms are not renormalized and thus keep their bare forms, and the functional $\Gamma_\kappa$ is invariant under time-gauged Galilean and shift transformations. The coefficient $\lambda$ is introduced in front of the nonlinear term for later power counting purposes. Of course, $\lambda$ can always be set equal to one in appropriate units. Note that including this coefficient in the original NS action induces some slight modifications of the related Ward identities by trivial factors $\lambda$. Yet, it still leads to the general form (27) of the effective action where $\lambda$ is not renormalized.

IV. ENERGY SPECTRUM AND SECOND ORDER STRUCTURE FUNCTION

In this section, we derive the general expressions for the energy spectrum and the second order structure function, that can be calculated in this formalism from the 2-point functions $F^{(2)}_{\alpha\beta}$. They are explicitly computed in Sec. 7.1 at LO approximation. These two observables are indeed related to the velocity-velocity correlation function, which is an element of the inverse matrix $G_\kappa$ of $\Gamma^{(2)}$. The general structure of the propagator matrix $G_\kappa$ is determined in Appendix A, its components in the velocity sector are reported below Eq. (30).

Let us first clarify notation. In the velocity sector, the elements of $G_\kappa$ are specified through two upper indices, e.g. $G_{\alpha\beta}^{uv} = \langle v_\alpha \bar{v}_\beta \rangle_c$. Because of rotational and parity invariance, any generic two-(space)index function $F_{\alpha\beta}(\omega, \vec{p})$ can be decomposed into a longitudinal and a transverse part
\begin{equation}
F_{\alpha\beta}(\omega, \vec{p}) = F_{\alpha\beta}^L(\vec{p}) F_\perp(\omega, \vec{p}^2) + F_{\alpha\beta}^T(\vec{p}) F_\parallel(\omega, \vec{p}^2) \tag{28}
\end{equation}
where the transverse and longitudinal projectors are defined by
\begin{align*}
P_{\alpha\beta}^L(\vec{p}) &= \delta_{\alpha\beta} - \frac{p_\alpha p_\beta}{p^2}, \quad \text{and} \quad P_{\alpha\beta}^T(\vec{p}) = \frac{p_\alpha p_\beta}{p^2}. \tag{29}
\end{align*}
As shown in Appendix A, the incompressibility condition entails that the components of the propagator in the velocity sector are purely transverse (that is, all the longitudinal parts vanish) and are given by
\begin{align}
G_{\alpha\beta}^{u\bar{u}}(\omega, \vec{q}) &= 0 \\
G_{\alpha\beta}^{u\bar{u}}(\omega, \vec{q}) &= \frac{1}{\Gamma_\perp^{(1,1)}(-\omega, \vec{q}) + R_\kappa(\vec{q})} \tag{30}
\end{align}

A. Energy spectrum

The energy spectrum in dimension $d$ is usually defined as
\begin{equation}
E^{(d)}(\vec{p}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \rho^{d-1} \mathcal{E}^{(d)}(\vec{p}) \tag{31}
\end{equation}
where $\mathcal{E}^{(d)}(\vec{p})$ is the Fourier transform of the equal-time velocity-velocity correlation function $\langle \bar{v}(t, \vec{x}) \cdot \bar{v}(t, \vec{0}) \rangle$.

Within this formalism, the velocity-velocity correlation function can be expressed as the inverse Fourier transform of $G_{\alpha\beta}^{u\bar{u}}(\omega, \vec{q})$, that is
\begin{align}
\langle v_\alpha(t, \vec{x}) v_\beta(0, \vec{0}) \rangle &= \int_q e^{-i(\omega t - \vec{q} \cdot \vec{x})} G_{\alpha\beta}^{u\bar{u}}(\omega, \vec{q}) \\
&= -\int_q e^{-i(\omega t - \vec{q} \cdot \vec{x})} P_{\alpha\beta}^L(\vec{q}) \frac{\Gamma^{(0,2)}_\perp(\omega, \vec{q}) - 2N_\kappa(\vec{q})}{\Gamma^{(1,1)}_\perp(\omega, \vec{q}) + R_\kappa(\vec{q})}. \tag{32}
\end{align}

It then follows that
\begin{align}
\mathcal{E}^{(d)}(\vec{p}) &= \int d^d \vec{x} e^{-i\vec{p} \cdot \vec{x}} \langle v_\alpha(t, \vec{x}) v_\beta(t, \vec{0}) \rangle \\
&= -(d - 1) \int_0^\infty \frac{d\omega}{\pi} \frac{\Gamma^{(0,2)}_\perp(\omega, \vec{p}) - 2N_\kappa(\vec{p})}{\Gamma^{(1,1)}_\perp(\omega, \vec{p}) + R_\kappa(\vec{p})}. \tag{33}
\end{align}
We focus on the inertial regime where $|\vec{p}| \gg \kappa$. In this limit the functions $N_\kappa(\vec{p})$ and $R_\kappa(\vec{p})$ tend to zero rapidly yielding
\begin{equation}
\mathcal{E}^{(d)}(\vec{p}) \approx -(d - 1) \int_0^\infty \frac{d\omega}{\pi} \frac{\Gamma^{(0,2)}_\perp(\omega, \vec{p})}{\Gamma^{(1,1)}_\perp(\omega, \vec{p})}. \tag{34}
\end{equation}
B. Second order structure function

The longitudinal structure function of order $n$ is defined as the average of the \textit{n}\textsuperscript{th} power of the equal time longitudinal velocity increment

$$S^{(n)}(\ell) = \left\langle \left[ (\vec{v}(t, \ell) - \vec{v}(t, 0)) \cdot \ell \right]^{n} \right\rangle,$$  \hspace{1cm} (35)

where $\ell = \ell/|\ell|$. Exploiting translation invariance, the second order structure function hence reads

$$S^{(2)}(\ell) = -2\ell \cdot \ell \left\langle v_{i}(t, \ell) v_{i}(t, 0) - v_{j}(t, 0) v_{i}(t, 0) \right\rangle$$  \hspace{1cm} (36)

and in this formalism

$$\left\langle v_{\alpha}(t, \ell) v_{\beta}(t, 0) - v_{\alpha}(t, 0) v_{\beta}(t, 0) \right\rangle = \int_{\mathbf{q}} G_{\alpha\beta}^{uu}(\omega, \mathbf{q}) \left[ e^{i\mathbf{q} \cdot \mathbf{\ell}} - 1 \right],$$  \hspace{1cm} (37)

and thus

$$S^{(2)}(\ell) = -2 \int_{\mathbf{q}} G_{\perp}^{uu}(\omega, \mathbf{q}) \left[ e^{i\mathbf{q} \cdot \mathbf{\ell}} - 1 \right] \left[ 1 - \frac{(\mathbf{\ell} \cdot \mathbf{q})^2}{\mathbf{q}^2} \right]$$  \hspace{1cm} (38)

with

$$\gamma_{d} \equiv \frac{4\pi^{(d-1)/2}}{(2\pi)^{d} \Gamma \left( \frac{d-1}{2} \right)}$$

$$I_{d}(v) = \int_{0}^{\pi} d\theta \sin^{d} \theta \left[ e^{iv \cos \theta} - 1 \right]$$  \hspace{1cm} (39)

$$= \int_{-1}^{1} du (1 - u^2)^{(d-1)/2} \left[ e^{ivu} - 1 \right].$$  \hspace{1cm} (40)

In dimensions $d = 2$ and $d = 3$, the integrals $I_{d}(v)$ are given by

$$I_{3}(v) = \frac{4}{v^{3}} \left[ \sin v - v \cos v - \frac{v^3}{3} \right]$$

$$I_{2}(v) = \pi \left[ -\frac{1}{2} + \frac{v J_{1}(v)}{v} \right].$$  \hspace{1cm} (41)

where $J_{1}(v)$ denotes a type $J$ Bessel function of the first kind.

Again, let us focus on the inertial regime corresponding to $\kappa \ll 1$. Since the integral in (38) is dominated by values of $q$ such that $q \ell$ is of order one, it is dominated by values of $q \gg \kappa$ in the inertial range. Accordingly, one can neglect in this regime the functions $N_{\kappa}(\mathbf{q})$ and $R_{\kappa}(\mathbf{q})$ that tend to zero rapidly and one obtains

$$S^{(2)}(\ell) \overset{\kappa \ll 1}{=} \gamma_{d} \int_{0}^{\pi} d\omega \int_{0}^{\infty} dq q^{d-1} \frac{1}{\Gamma_{\perp}^{(1,1)}(\omega, \mathbf{q})} I_{d}(q\ell).$$  \hspace{1cm} (42)
turns out to share with the NS one a very similar invariance under time-gauged Galilean transformations, and also under time-gauged shifts (although in the field itself instead of in the response field) \cite{22,24}. This similarity is easily conceivable as the KPZ equation maps onto the Burgers equation. Thus the approximation scheme devised in the context of the KPZ equation \cite{22,24} can be quite simply transposed to the NS equation.

For the KPZ equation, the solution consists in constructing an ansatz for $\Gamma_\kappa$ explicitly preserving the symmetries of the action, in particular the time-gauged Galilean symmetry. At second order (SO) of this scheme, the effective action $\Gamma_\kappa$ is truncated at quadratic order in the response field, while retaining for the 2-point functions an arbitrary dependence in wave-vectors and frequencies (and also in the field itself through arbitrary powers of the covariant time derivative). The 2-point correlation and response functions for the KPZ problem were calculated within the SO approximation in $d = 1$ \cite{22}. The existence of scaling forms for these functions could be proved analytically, and the related scaling functions in one dimension (computed numerically) turned out to reproduce with an impressive agreement the exact results established in \cite{73}, including the finest detail of their tails. The SO approximation can therefore be roughly considered as a BMW-like scheme (for the field but not for the response field) rendered compatible with the KPZ symmetries.

From a numerical viewpoint, the SO approximation is demanding in dimensions larger than one. Two simpler approximations were thus proposed in \cite{22,24}. They both consist in a simplification of the frequency sector, by either completely neglecting the frequency dependence of the 3-point vertices (leading order (LO) approximation) or approximating it (next-to-leading (NLO) approximation). The predictions obtained at NLO for some universal amplitude ratios of the KPZ problem in \cite{22,24} were very accurately confirmed in $d = 2$ in recent large-scale simulations \cite{51}. As for the LO approximation, which only retains the bare frequency dependence, but preserves the full wave-vector dependence of the 2-point functions, it clearly suffices in the KPZ problem to obtain the full phase diagram, including the strong-coupling rough phase. The related estimates for the critical exponents are in good agreement with the numerical ones in dimensions $d = 2$ and $d = 3$ \cite{22,41}. Let us underline that the values of the KPZ critical exponents are greatly improved at LO compared to those obtained within the Derivative Expansion \cite{51}, which is probably to be imputed to the derivative nature of the bare vertex.

As we merely consider, in the current work, the wave-vector dependence of the 2-point functions of the NS problem, we choose to implement the LO approximation for the NS equation, which is achieved in the next section. For the NS effective action $\tilde{\Gamma}_\kappa$ in Eq. (27), it consists in

- (i) performing a field expansion in $\tilde{u}$ at order two
- (ii) keeping only the bare wave-vector and frequency dependencies of all $n$-point functions with $n \geq 3$
- (iii) preserving an arbitrary wave-vector dependence of the 2-point functions while restricting to their bare frequency dependence

In this approach, all the symmetries of the theory are automatically encoded by writing the proposed ansatz for $\tilde{\Gamma}_\kappa$ in terms of Galilean scalars only. Notice that within the LO approximation, point (ii) implies that all $n$-point vertex functions with $n \geq 4$ vanish and point (iii) that $\tilde{\Gamma}_\kappa$ is also truncated at order two in $\tilde{u}$ since the dependence in the $D_\kappa$ covariant derivative is neglected. The LO ansatz is given by Eq. (13).

Let us comment on the validity/accuracy of the LO approximation. Point (ii) implies that this approximation is valid only for wave-vectors typically smaller than $\kappa$ and frequencies smaller than $\kappa^{1/2}$ because keeping the bare wave-vector dependence of the $n$-point functions for $n \geq 3$ is equivalent to keeping their leading terms in a wave-vector expansion. Notice that once this expansion is performed, the frequency sector is entirely fixed by the symmetries gauged in time, see Sec. VII. This expansion is certainly valid for the internal wave-vector $\vec{q}$ in Eq. (19) since it is suppressed for $|\vec{q}| > \kappa$ by the $\partial_\mu R_\kappa$ term but, a priori, is only justified for small external wave-vector, that is $|\vec{p}| < \kappa$ in Eq. (19). In fact, in most systems, including the KPZ growth, the flow of the 2-point functions actually stops when the external wave-vector becomes larger than the RG scale $\kappa$, a phenomenon called decoupling. Thus, the determination of the momentum and frequency dependencies remains accurate within the LO approximation because (i) when $\kappa > |\vec{p}|$ (and $\kappa > \nu^{1/2}$) the LO approximation is accurate and (ii) when $\kappa$ becomes smaller than $|\vec{p}|$, the flow almost stops and although the LO approximation is not valid in this region of wave-numbers, this has negligible impact on the two-point functions.

However, as stressed in the following, the NS problem is very peculiar because the RG flow of the 2-point functions does not show the decoupling property of the large wave-vector sector. Thus a complementary scheme is necessary to determine the wave-vector dependence of the $\Gamma_\kappa^{(2)}(\vec{p})$ at finite $\vec{p}$ when $\kappa \to 0$. The LO approximation analyzed below is therefore only valid in the limit $|\vec{p}| \to 0$ when $\kappa \to 0$, which is sufficient to determine the fixed point structure and critical exponents, but not to investigate multi-scaling. On the other hand, as shown in Sec. VII exact RG flow equations can be derived in the large $\vec{p}$ sector, using the very constraining gauge symmetries of the NS field theory, which can take over from the LO equations when $\kappa$ becomes smaller than $|\vec{p}|$. In fact, the multi-scaling behavior of the $n$-point functions precisely emerges in the NPRG framework from the non-decoupling of the large $\vec{p}$ sector and the associated non-trivial behavior in both $|\vec{p}|$ and $\omega$ (see Sec. VII).
As explained in the previous section, one can construct an ansatz for $\Gamma_\kappa$ in (27) which explicitly preserves the time-gauged Galilean symmetry by using as building blocks Galilean scalars (see Ref. [52] for detail). At LO, the functional $\Gamma_\kappa$ is truncated at quadratic order in the response field $\vec{u}$ and moreover, the (non-bare) frequency dependence \( i.e. a dependence in $D_\eta$ of the running functions $f^{\nu,\beta}_{\kappa,\alpha\beta}$ below) is neglected. Hence the LO ansatz simply reads

$$\tilde{\Gamma}_\kappa[\vec{u},\vec{u}] = \int_{t,\vec{x},\vec{x}'} \left\{ \tilde{u}_\alpha(t, \vec{x}) f^{\nu}_{K,\alpha\beta}(\vec{x} - \vec{x}') u_\beta(t, \vec{x}') - \tilde{u}_\alpha(t, \vec{x}) f^{\nu}_{K,\alpha\beta}(\vec{x} - \vec{x}') \tilde{u}_\beta(t, \vec{x}') \right\}. \quad (43)$$

To further specify the running functions $f^{\nu,\beta}_{\kappa,\alpha\beta}$, let us introduce the notation for the vertex functions

$$\Gamma^{(m,n)}_{\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_n}(\vec{x}_1, \ldots, \vec{x}_m, \bar{\vec{x}}_1, \ldots, \bar{\vec{x}}_n) = \frac{\delta^{n+m} \Gamma_\kappa}{\delta u_{\alpha_1}(\vec{x}_1) \ldots \delta u_{\alpha_m}(\vec{x}_m) \delta \tilde{u}_{\beta_1}(\bar{\vec{x}}_1) \ldots \delta \tilde{u}_{\beta_n}(\bar{\vec{x}}_n)} \quad (44)$$

where from now on, the explicit index $\kappa$ is dropped for the vertex functions $\Gamma^{(m,n)} \equiv \Gamma^{(m,n)}_\kappa$ and for the running functions $f^{\nu,\beta}_{\alpha\beta} \equiv f^{\nu,\beta}_{K,\alpha\beta}$. The Fourier transforms of the $\Gamma^{(m,n)}$ only depend on $n + m - 1$ wave-vectors and frequencies because of translation invariance.

The dependence of the running functions $f^{\nu}_{\alpha\beta}$ and $f^{\nu}_{\alpha\beta}$ in $\vec{x} - \vec{x}'$ is in fact through gradients as can be inferred from the Ward identities (22) and (26). The latter imply that these two functions vanish at zero wave-vector

$$f^{\nu}_{\alpha\beta}(\vec{p} = 0) = f^{\nu}_{\alpha\beta}(\vec{p} = 0) = 0. \quad (45)$$

The initial conditions of the flow at scale $\kappa = \eta^{-1}$ for the two running functions are

$$f^{\nu}_{\alpha\beta}(\vec{x} - \vec{x}')_{|\kappa = \eta^{-1}} = 0$$

$$f^{\nu}_{\alpha\beta}(\vec{x} - \vec{x}')_{|\kappa = \eta^{-1}} = -\nu \delta_{\alpha\beta} \nabla^2_x (\delta^{(d)}(\vec{x} - \vec{x}')) \quad (46)$$

to recover the original NS action at the microscopic scale.

The calculation of the 2-point functions from the LO ansatz is straightforward. At vanishing fields and in Fourier space, one obtains

$$\tilde{\Gamma}^{(2,0)}_{\alpha\beta}(\omega, \vec{p}) = 0$$

$$\tilde{\Gamma}^{(1,1)}_{\alpha\beta}(\omega, \vec{p}) = f^{\nu}_{\alpha\beta}(\vec{p})$$

$$\tilde{\Gamma}^{(0,2)}_{\alpha\beta}(\omega, \vec{p}) = -2 f^{\nu}_{\alpha\beta}(\vec{p}) \quad (47)$$

Within the LO approximation, all vertex functions of order $m + n \geq 4$ vanish, and the only non-zero 3-point vertex function is the bare one, which reads in Fourier space

$$\Gamma^{(2,1)}_{\alpha\beta\gamma}(\omega_1, \vec{p}1, \omega_2, \vec{p}2) = -i\lambda(p_2^0 \delta_{\beta\gamma} + p_1^0 \delta_{\alpha\gamma}). \quad (48)$$

C. Derivation of the LO flow equations

In this section, we derive the LO flow equations of the two running functions $f^{\nu}_{\alpha\beta}$ and $f^{\nu}_{\alpha\beta}$. In fact, for incompressible flows, only the transverse sector plays a role, which means that only their transverse components, denoted $f^{\nu}_{\alpha\beta}$ and $f^{\nu}_{\alpha\beta}$, are eventually needed. The flow equations of $f^{\nu}_{\alpha\beta}$ and $f^{\nu}_{\alpha\beta}$ are proportional to the projection in the transverse sector of the flow equations of the 2-point functions $\Gamma^{(1,1)}_{\alpha\beta}(0, \vec{p})$ and $\Gamma^{(0,2)}_{\alpha\beta}(0, \vec{p})$, which are given in matrix form by Eq. (19) evaluated at zero external frequency. At LO, the matrices $\Gamma^{(4)}_{\kappa,\alpha\beta}$ entering these equations are zero and only one 3-point vertex, the bare one $\Gamma^{(2,1)}_{\kappa,\alpha\beta}$, contributes in the matrices $\Gamma^{(3)}_{\kappa,\alpha\beta}$. The transverse components of the propagator in the $\vec{u}, \vec{\tilde{u}}$ sector are given within the LO approximation, \(i.e.\) with the ansatz (44), by

$$C^{u\bar{u}}_\perp(\omega, \vec{p}) = 0$$

$$C^{u\bar{u}}_\perp(\omega, \vec{p}) = \frac{1}{\omega^2 + (f^{\nu}_{\perp}(\vec{p}))^2} \quad (49)$$

where

$$\tilde{f}^{\nu}_{\perp}(\vec{p}) \equiv f^{\nu}_{\perp}(\vec{p}) + R_{\kappa}(\vec{q})$$

$$\tilde{f}^{\nu}_{\perp}(\vec{p}) \equiv f^{\nu}_{\perp}(\vec{p}) + N_{\kappa}(\vec{q}). \quad (50)$$

One has to compute the trace of the matrix product (19), and project the result onto the transverse sector. In the obtained expression, since the frequency dependence remains the bare one at LO, the integral over the internal frequency $\omega$ can be analytically carried out. These calculations are detailed in Appendix B. The resulting flow equations of the two running functions $f^{\nu}_{\perp}$ and $f^{\nu}_{\perp}$ are given by...
where \( \partial_s \equiv \kappa \partial_\kappa \).

### D. Dimensionless LO flow equations

As we seek the fixed point solution of the flow equations, we introduce renormalized and dimensionless quantities, denoted with a hat symbol. The wave-vectors and frequencies are respectively measured in units of \( \kappa \) and \( \kappa^2 \nu_\kappa \), where \( \nu_\kappa \) is the running viscosity. We thus define e.g. \( \tilde{p} = p/\kappa \) and \( \tilde{\omega} = \omega/(\kappa^2 \nu_\kappa) \). Let us consider the expression (10) of the term \( \Delta S_\kappa \). The dimension of the cutoff term \( R_{\kappa, \alpha \beta} \) is given by

\[
\left[ \int_{\vec{x}} R_{\kappa, \alpha \beta} (\vec{x} - \vec{x}') \right] = [\kappa^2 \nu_\kappa] \quad (53)
\]

and the dimension of the forcing term \( N_{\kappa, \alpha \beta} \) can be inferred from definition (6)

\[
\left[ \int_{\vec{x}} N_{\kappa, \alpha \beta} (\vec{x} - \vec{x}') \right] = [D_\kappa]. \quad (54)
\]

We associate two running anomalous dimensions \( \eta_\kappa^\rho \) and \( \eta_\kappa^\nu \) with these running coefficients as

\[
\eta_\kappa^\rho = -\partial_s \ln \nu_\kappa \quad \text{and} \quad \eta_\kappa^\nu = -\partial_s \ln D_\kappa. \quad (55)
\]

According to the previous definitions, the two functions \( \hat{r}(q^2/\kappa^2) \) and \( \hat{n}(q^2/\kappa^2) \) introduced in Eqs. (3) and (11) are dimensionless, and the flow of the regulator terms in Eqs. (51) and (52) are given by

\[
\begin{align*}
\partial_s R_\kappa (\vec{q}) &= \nu_\kappa q^2 \left( -\eta_\kappa^\rho \hat{r}(q^2) - 2q^2 \partial_q \hat{r}(q^2) \right) \\
\partial_s N_\kappa (\vec{q}) &= D_\kappa q^2 \left( -\eta_\kappa^\nu + 2\hat{n}(q^2) - 2q^2 \partial_q \hat{n}(q^2) \right). \quad (56)
\end{align*}
\]

In fact, as we now explain, the scalings of the two running coefficients are fixed by physical arguments. As stated in the introduction, energy must be permanently injected at the integral scale \( \kappa^{-1} \) to maintain a turbulent flow since the NS equation is dissipative. Hence, reaching the stationary regime of fully developed turbulence requires that the rate of power injected by unit mass of the fluid compensates the rate of power dissipated by unit mass. In any \( d > 2 \), the dissipated power is dominated by the microscopic dissipative scale, which means that it is independent of \( \kappa \). (In fact, one can safely remove the \( R_\kappa \) term (as in Ref. [68] and preserve well-behaved flow equations.) It follows that the average injected power per unit mass should be kept constant. This quantity can be expressed (see Appendix C) as

\[
\langle f_\alpha (t, \vec{x}) v_\alpha (t, \vec{x}) \rangle = \lim_{\delta t \to 0} \int_{\vec{x}'} N_{\kappa, \alpha \beta} (|\vec{x} - \vec{x}'|) G^{\alpha \bar{\alpha}} (t + \delta t, \vec{x}; t, \vec{x}') = D_\kappa \kappa^d \lim_{\delta t \to 0} \int_{\vec{q}, \vec{\omega}} \hat{N} (\vec{q}) e^{i \vec{\omega} \delta t} \hat{G}^{\alpha \bar{\alpha}} (\vec{\omega}, q).
\]

(57)

One concludes that to take the RG limit \( \kappa \to 0 \) at constant injected power requires to fix \( \eta_\kappa^\rho = d \) for all \( \kappa \), which means that \( N_\kappa (\vec{q}) \) must scale as \( \kappa^{-d} \). The case \( d = 2 \) can be deduced as a limit from \( d > 2 \). Since \( \eta_\kappa^\nu = d \) for any \( d > 2 \), this must hold also in \( d = 2 \) (except from possible logarithmic corrections, that are not considered in the present article and that do not seem to play any role in the existence of the two-dimensional fixed point, see Sec. [VI]). In the (less relevant) case \( d < 2 \), one can only deduce that \( \eta_\kappa^\nu \gtrsim d \).

Let us now determine the dimensions of the terms entering the LO ansatz (27) and (33). According to Eqs. (53) and (54), and since \( \Gamma_\kappa \) and \( \Delta S_\kappa \) have the same dimension, one deduces the dimensions of the fields: \( [u] = [\kappa^{d-2} D_\kappa \nu_\kappa^{-1}]^{1/2} \) and \( [\bar{u}] = [\kappa^{d+2} \nu_\kappa D_\kappa^{-1}]^{1/2} \). We introduce the dimensionless coupling \( \lambda_\kappa \) as

\[
\lambda = \left( \kappa^{-d+4} \nu_\kappa^3 D_\kappa^{-1} \right)^{1/2} \lambda_\kappa. \quad (58)
\]
Since $\lambda$ is not renormalized, the flow equation for $\hat{\lambda}_1$ is purely dimensional and reads

$$\partial_s \hat{\lambda}_1 = \hat{\lambda}_1 \left( \frac{d}{2} - 2 + \frac{3}{2} \eta_s^\nu - \frac{1}{2} \eta_s^D \right). \quad (59)$$

It follows that any non-Gaussian fixed point, with $\hat{\lambda}_1 \neq 0$, is characterized by a single independent exponent, e.g. $\eta_s^\nu$ with $\eta_s^\nu = 4/3 + (\eta_s^\nu - d)/3$. In particular, once $\eta_s^\nu = d$ is fixed to keep a constant energy injection rate, then at the fixed point, $\eta_s^\nu = 4/3$ independently of the dimension. Lastly, according to Eq. (59), the dimensions of the running functions $f_s^\nu$ and $f_s^D$ are the same as the ones of $R_s,\alpha\beta_s$ and $N_s,\alpha\beta_s$ in Eqs. (60) and (61) and we thus define the dimensionless functions $\hat{h}^\nu$ and $\hat{h}^D$ as

$$f_s^\nu (\hat{p}) = \nu_1 \kappa^2 \hat{p}^2 \hat{h}^\nu (\hat{p}) \quad \text{and} \quad f_s^D (\hat{p}) = D_\kappa \hat{p}^2 \hat{h}^D (\hat{p}). \quad (60)$$

Their flow equations are given by

$$\partial_s \hat{h}^\nu (\hat{p}) = \eta_s^\nu \hat{h}^\nu (\hat{p}) + \hat{p} \partial_{\hat{p}} \hat{h}^\nu (\hat{p}) + \nu_s^{-1} \partial_s f_s^\nu (\hat{p}) \quad (61)$$

$$\partial_s \hat{h}^D (\hat{p}) = (\eta_s^D + 2) \hat{h}^D (\hat{p}) + \hat{p} \partial_{\hat{p}} \hat{h}^D (\hat{p}) + D_\kappa^{-1} \partial_s f_s^D (\hat{p})$$

with the substitutions for dimensionless quantities in the flow equations (61) and (62) for $\partial_s f_s^\nu (\hat{p})$ and $\partial_s f_s^D (\hat{p})$.

VI. FIXED POINT SOLUTIONS IN $d = 2$ AND $d = 3$

In this section, we proceed to the numerical integration of the dimensionless LO flow equations (61), both in $d = 2$ and $d = 3$.

A. Fixed point functions $\hat{h}^\nu$ and $\hat{h}^D$

The flow equations (61) of $\hat{h}^\nu$ and $\hat{h}^D$ are integrated numerically – fixing $\eta_s^\nu = d$ for all $\kappa$ – from the initial conditions $\hat{h}^\nu (\hat{p}) = 0$ and $\hat{h}^D (\hat{p}) = 1$, and different values of $\lambda$. The detail of the numerical procedure is summarized in Appendix D. We observe that the flow always reaches a (fully attractive) fixed point, independently of the initial conditions, and without fine-tuning any parameter. Hence the corresponding stationary regime is universal. Along the flow, the two functions $h^{\nu,D}$ are smoothly deformed to acquire a fixed form, which is illustrated on the example of the function $\hat{h}^\nu$ in $d = 2$ and $d = 3$ in Fig. 1. The fixed point profile of the two functions in both dimensions is displayed in logarithmic scales in Fig. 1. This figure shows that both functions decay algebraically at large wave-number. The respective exponents are

$$\hat{h}^\nu (\hat{p}) \sim \hat{p}^{-4/3+\alpha} \quad \text{and} \quad \hat{h}^D (\hat{p}) \sim \hat{p}^{-(d+2)+\beta} \quad (62)$$

where $\alpha$ and $\beta$ are deviations to the expected (dimensional) scaling. We find $\alpha \approx \beta \approx 0.33$. These values are in agreement with the values estimated in $d = 3$ in Ref. [66]. It was shown in particular in this work that these exponents are independent of the choice of the stirring profile. In fact, one can prove by inspection of the regime $\hat{p} \gg \kappa$ of the LO flow equations (61) and (62) that the exponent $\alpha$ is exactly $1/3$ (see Sec. VI D). We also found numerically that $\beta = 1/3$, if not exactly at least very precisely. Notice that the dependence of this value in the regulator (via the $a$ parameter in Eq. (12)) could also be studied. However, it is meaningless within the LO approximation since this approximation is not appropriate to study with precision the large wave-number sector, as already pointed out in Sec. VA Let us now probe the effect of these deviations on physical observables.

B. Energy spectrum

Within the LO approximation, the 2-point functions are given by

$$\Gamma^{(0,2)}_r (\omega, \vec{q}) = -2D_\kappa \hat{q}^2 \hat{h}^\nu (\vec{q})$$

$$\Gamma^{(1,1)}_r (\omega, \vec{q}) = \nu_1 \kappa^2 (i\omega + \hat{q}^2 \hat{h}^\nu (\vec{q})). \quad (63)$$

Inserting these expressions into Eq. (34) and performing the integral over the frequency, one obtains for the energy
follow the asymptotics (62) and the energy spectra in which is the Kolmogorov result when based on an expansion of all wave-vectors, including the appropriate to study this regime with accuracy, since it is as explained previously, the LO approximation is not applied. However, large deviations to the Kolmogorov scaling, these tend to indicate that, although the 2-point functions do diverge (52) do not become negligible at the fixed point compared to the linear (dimensional) parts for large external wave-numbers much larger than \( \kappa \ell \). Hence the Kolmogorov scaling is again recovered (or received) \( \kappa \ell \) approximately. This seems to be a generic result. Nevertheless, this result for \( \kappa \ell \ll 1 \) is unlikely to be precise. This is further discussed in Sec. VII.

C. Second order structure function

Within the LO approximation, one obtains for the second order structure function \( S^{(2)}(\ell) \) inserting the expressions \( \hat{h}^{\nu}(\hat{q})/\hat{h}^{\nu}(\hat{p}) \).

\[
S^{(2)}(\ell) \approx -\gamma_d \kappa^{-2/3} \int_0^\infty d\hat{q} \hat{q}^{d-1} \frac{\hat{h}^{\nu}(\hat{q})}{\hat{h}^{\nu}(\hat{p})} I_d(\kappa \hat{q} \ell). \quad (67)
\]

D. Large wave-number sector and limit of the LO approximation

In this section, we analyze the large wave-number limit of the flow equations (52). This analysis reveals that the nonlinear parts of these equations given by Eqs. (52) do not become negligible at the fixed point compared to the linear (dimensional) parts for large external wave-vectors \( \hat{p} \). This means that the large wave-number sector...
does not decouple from the flow when $\kappa \ll |\vec{p}|$, which is a very unusual property. As a consequence, the existence of the fixed point does not lead to the usual scale invariance, as encountered in ordinary critical phenomena. Indeed, the non-decoupling entails that the large wave-vector sector is not determined by the small wave-vector one, that is the large $\vec{p}$ behavior – the exponents of the algebraic tails of the correlation functions – is not fixed in terms of the Kolmogorov dimensions ($\eta^{s}_{\perp}$ and $\eta^{s}_{\parallel}$) but can develop deviations.

These deviations cannot be reliably computed at LO because this approximation is fully justified only when all the wave-numbers are small. As already explained, whereas this expansion is always valid for the internal wave-vector because of the presence of the regulator term $\partial_s R_n$ which effectively cuts off the contributions $|q| \gtrsim \kappa$, it is justified only for small external wave-vectors $\vec{p}$. Hence, one has to device an alternative approximation to properly describe the large wave-number sector. This is achieved in Sec. VII. However, we prove in this section that the non-decoupling is a real feature of the NS flow equations.

To this aim, we now study the large $\vec{p}$ sector of the flow equations (61). We observed that the dimensionless running functions $\hat{h}^{s,0}$ reach a fixed point, which means by definition $\partial_s \hat{h}^{s}(\vec{p}) = \partial_s \hat{h}^{0}(\vec{p}) = 0$ and $\eta^{s}_{\parallel}$ acquires its fixed point value $\eta^{s}_{\parallel} = 4/3$. Let us assume that the nonlinear terms of the flow equations (61), denoted $NL^{s} \equiv \partial_s f^{s}(\vec{p})/(\nu, \kappa p^2)$ and $NL^{0} \equiv \partial_s f^{0}(\vec{p})/(D_\perp \kappa p^2)$ and which explicit expressions are given by (62) and (63), become negligible in the large wave-number limit $|\vec{p}| \gg \kappa$ compared to the linear terms, that is, they decouple. One then deduces that the general solutions at the fixed point of the remaining homogeneous (linear) parts of the flow equations are the scaling forms

$$
\hat{h}^{s}_{\perp}(\vec{p}) = \hat{p}^{-4/3} \xi^{s}_{\perp} \quad \text{and} \quad \hat{h}^{0}(\vec{p}) = \hat{p}^{-(d+2)/3} \xi^{0} \quad (69)
$$

where $\xi^{s,0}$ are constants. Had we considered wave-vector and frequency dependent running functions, we would have obtained, e.g. $\hat{h}^{s}(\hat{\omega}, \hat{p}) = \hat{p}^{-4/3} \xi^{s}(\hat{\omega}/\hat{p}^{2/3})$. We now show that this leads to a contradiction, that is, these scaling solutions are incompatible with the assumed decoupling of the large wave-number sector. For this, the leading contribution in $\hat{p}$ of each nonlinear term $NL^{s}$ and $NL^{0}$ can be analytically determined substituting the two functions $\hat{h}^{s,0}$ with the scaling solutions (69). One obtains that the dominant contributions of $NL^{s}$ and $NL^{0}$ are respectively $\hat{p}^{-4/3}$ and $\hat{p}^{-(d+2)}$. Hence, they are not negligible (sub-dominant) compared to the linear terms, but of the same order. One concludes that in LO approximation the nonlinear parts of the flow equations do not decouple at the fixed point, which invalidates the general scaling solutions (69).

This non-decoupling property entails that the existence of the fixed point does not generate the usual scale invariance. As a matter of fact, we found in the previous section that the two functions $\hat{h}^{s,0}$ do decay algebraically, but not with the Kolmogorov exponents (69). Instead, they exhibit the deviations $\alpha$ and $\beta$ to these scalings following the asymptotics Eqs. (62). Of course, the existence of the non-decoupling property for the exact NS flow equations cannot be asserted at the LO level and the obtained values of these deviations are not to be fully trusted. However, we now specifically address the large wave-number regime, and prove that non-decoupling indeed occurs in the exact NS flow.

VII. EXACT NPRG FLOW EQUATIONS IN THE LARGE WAVE-NUMBER REGIME

The LO approximation is justified and expected to be accurate in the small wave-number regime $|\vec{p}| \lesssim \kappa$. In this section, we devise an alternative approximation for the large wave-number regime, $|\vec{p}| \gtrsim \kappa$ which becomes exact in the limit $|\vec{p}| \gg \kappa$, or equivalently in the limit $\kappa \to 0$ for any fixed external wave-vector and frequency. The flow equations for the 2-point functions can indeed be exactly closed in this limit using Ward identities [83].

A. Derivation of the flow equations in the large wave-number limit

Let us first show that many of the diagrams contributing to the exact flow of the 2-point functions (19), represented diagrammatically in Figs. 4 and 5, are zero:

- diagrams involving $\Gamma^{(n,0)}_{\kappa}$ vertices vanish, as a consequence of the general properties of NPRG within the Janssen-de Dominicis formalism and Ito’s discretisation (74, 76) (the corresponding diagrams are not depicted in Figs. 4 and 5).
- causality implies that the diagrams (b) of Figs. 4 and 5 vanish in Ito’s discretisation (74, 76). More
precisely, response functions are causal (the ending point is at strictly larger times than the starting point). Moreover, in all vertices, the latest time corresponds to an outgoing leg and the earliest one to an incoming leg, which implies that the corresponding diagrams cannot be ordered in a causal way (or, equivalently, their integral on internal frequency is exactly zero).

We then show that the remaining diagrams in Figs. 4 and 5 are either negligible, or closed (expressed in terms of 2-point functions) in the large $|\vec{p}|$ limit. Indeed, the presence of the regulator term $\partial_s R_\kappa$ (implicit in the $\tilde{\partial}_s$ operator) effectively cuts off the internal wave-vector $\vec{q}$ integral to values of order $|\vec{q}| \ll \kappa$ (and similar conclusions can be drawn when $\vec{p} + \vec{q}$ is cut off instead of $\vec{q}$, see Appendix F). In the limit of large external wave-number $|\vec{p}| \gg \kappa$, the internal wave-vector $\vec{q}$ is hence negligible in all vertices compared to $\vec{p}$ and can be set to zero. This implies that, in this limit, the 3- and 4-point vertex functions are to be evaluated at one, respectively two, vanishing wave-vectors, and this implies

- if the zero wave-vector is carried by a $\vec{u}$-leg, then the corresponding vertex is vanishing as a consequence of the time-gauged shift symmetry (encoded in the general Ward identity \([23]\)) and thus such diagrams are negligible in the large $|\vec{p}|$ limit (diagrams (a) and (d) in Fig. 4 and (a) in Fig. 5).

- if the zero wave-vector is carried by a $\vec{u}$-leg, then the corresponding vertex is exactly related to lower order vertices by time-gauged Galilean identities and thus such diagrams are closed in the large $|\vec{p}|$ limit.

Let us make this last assertion explicit. There are two 3-point vertex functions, $\Gamma^{(2,1)}$ and $\Gamma^{(1,2)}$, involved in the (non-zero) diagrams (c) of Fig. 4 and (c) and (d) of Fig. 5 which are to be evaluated at a zero wave-vector on a $\vec{u}$-leg. As derived in Appendix E, they are related by a time-gauged Galilean Ward identity to 2-point vertex functions as

$$\Gamma^{(2,1)}_{\alpha\beta\gamma}(\nu, \vec{q}; \omega, \vec{p}) = -\frac{q^\alpha}{p^\gamma} \left(\Gamma^{(1,1)}_{\beta\gamma}(\nu + \omega, \vec{q}) - \Gamma^{(1,1)}_{\gamma\beta}(\omega, \vec{q})\right)$$

(70)

$$\Gamma^{(1,2)}_{\alpha\beta\gamma}(\nu, \vec{q}; \omega, \vec{p}) = -\frac{q^\alpha}{p^\gamma} \left(\Gamma^{(0,2)}_{\beta\gamma}(\nu + \omega, \vec{q}) - \Gamma^{(0,2)}_{\gamma\beta}(\omega, \vec{q})\right).$$

(71)

Similarly, one can deduce from the Ward identities \([13], \[51], \[70]\) and \([71]\) that the two 4-point vertex functions, $\Gamma^{(2,2)}$ and $\Gamma^{(0,4)}$, involved in the remaining diagrams (e) of Figs. 4 and 5 to be evaluated at two vanishing wave-vectors, are related to 2-point vertex functions as

$$\Gamma^{(2,2)}_{\alpha\beta\gamma\delta}(\omega,\vec{q};\omega,\vec{p}) = \frac{p^\alpha p^\beta}{\omega^2} \left[\Gamma^{(0,2)}_{\gamma\delta}(\omega + \nu, \vec{p}) - 2\Gamma^{(0,2)}_{\gamma\delta}(\nu, \vec{p}) + \Gamma^{(0,2)}_{\gamma\delta}(-\omega + \nu, \vec{p})\right]$$

(72)

$$\Gamma^{(3,1)}_{\alpha\beta\gamma\delta}(\omega,\vec{q};\omega,\vec{p}) = \frac{p^\alpha p^\beta}{\omega^2} \left[\Gamma^{(1,1)}_{\gamma\delta}(\omega + \nu, \vec{p}) - 2\Gamma^{(1,1)}_{\gamma\delta}(\nu, \vec{p}) + \Gamma^{(1,1)}_{\gamma\delta}(-\omega + \nu, \vec{p})\right].$$

(73)

Thus, the expressions of all the non-vanishing diagrams contributing to the flows of the 2-point functions can be exactly closed, i.e. expressed in terms of 2-point functions only, in the limit of large external wave-number.

Let us emphasize that, since the Galilean symmetry is gauged in time, no approximation is needed on the internal frequency once the internal wave-vector is neglected. This is a great advantage since an expansion on the internal frequency would not be justified as it is not cut off. Indeed, the regulator functions $N_\kappa$ and $R_\kappa$ only depend on momenta, but not on frequencies (which is required in order to maintain the various symmetries of the model along the flow). Hence, the internal momentum can safely be neglected, and once this is done, the internal frequency dependence is entirely fixed by the symmetries and no approximation is necessary in the frequency sector.

\[ \text{FIG. 5. Diagrammatic representation of the exact flow equation of } \Gamma^{(n,0)}(\nu, \vec{p}), \text{ with } \partial_s \equiv \partial_s R_\kappa \frac{\partial}{\partial R_\kappa} + \partial_s N_\kappa \frac{\partial}{\partial N_\kappa}. \text{ The combinatorial factors are not explicitly written, and diagrams involving } \Gamma^{(n,0)}(\nu, \vec{p}) \text{ vertices are omitted since they are vanishing.} \]
The non-zero diagrams (c), (e) of Fig. 3 and (c), (d), (e) of Fig. 4 are explicitly calculated in Appendix F. Gathering

\[ \partial_s \Gamma^{(1,1)}(\nu, p) = \frac{(d-1)}{d} \nu^2 \int \frac{d^d q}{(2\pi)^d} \left\{ \frac{\Gamma^{(1,1)}(\nu, \bar{p}) - \Gamma^{(1,1)}(\nu, \bar{p})}{\omega} \right\}^2 G^{nn}(\omega, q) \]

\[ \partial_s \Gamma^{(0,2)}(\nu, p) = \frac{1}{\nu^2} \left\{ \frac{\Gamma^{(0,2)}(\nu, \bar{p}) - \Gamma^{(0,2)}(\nu, \bar{p})}{\omega} \right\} \times \frac{\partial_s}{\partial q} \int G^{nn}(\omega, q). \] 

We study below the decoupling property of these equations and its consequences.

B. Study of the (non-)decoupling

In this section, we prove that the large wave-number sector does not decouple in the NS flow equations Eqs. (74) and (75). Our strategy is as previously to assume that such a decoupling does take place and then to show that this leads to a contradiction.

We consider the inertial regime of wave-numbers \(|\bar{p}|\) much larger than the running inverse integral scale \(\kappa\) and much smaller than the microscopic Kolmogorov scale. The effective action \(\Gamma_\kappa\) has thus already approached the IR attractive fixed point, and it is convenient to rewrite Eqs. (74) and (75) in terms of dimensionless quantities:

\[ \partial_s \Gamma^{(1,1)}(\nu, \bar{p}) = \nu^2 \kappa^2 \Gamma^{(1,1)}(\nu, \bar{p}) - \bar{p} \partial \bar{p} \Gamma^{(1,1)}(\nu, \bar{p}) \]

\[ \partial_s \Gamma^{(0,2)}(\nu, \bar{p}) = \partial_q \left\{ \frac{\Gamma^{(0,2)}(\nu, \bar{p}) - \Gamma^{(0,2)}(\nu, \bar{p})}{\omega} \right\} \times \frac{\partial_s}{\partial q} \int G^{nn}(\omega, q). \]

We substitute the obtained solutions (79) in the right hand sides of Eqs. (74) and (75) and show that they are negligible (decoupled), that is, \(\partial_s \Gamma^{(0,2)}(\nu, \bar{p}) \approx 0\) in Eqs. (76). The general solutions of the remnant homogeneous linear equations are the scaling forms

\[ \hat{\Gamma}^{(1,1)}(\nu, \bar{p}) = \nu^{2} 3 \chi^{(1,1)}(\nu, \bar{p}) \]

\[ \hat{\Gamma}^{(0,2)}(\nu, \bar{p}) = \nu^{-d} \chi^{(0,2)}(\nu, \bar{p}) \]

or, equivalently,

\[ \Gamma^{(1,1)}(\nu, p) = \nu^{2} 3 \chi^{(1,1)}(\nu, \bar{p}) \]

\[ \Gamma^{(0,2)}(\nu, p) = \nu^{-d} \chi^{(0,2)}(\nu, \bar{p}) \]

Both functions \(\chi^{(i,j)}(z)\) and \(\hat{\chi}^{(i,j)}(z)\) are equal up to some (non-universal) normalisations (of the functions and of their arguments).

Let us now prove that this is inconsistent. For this, we substitute the obtained solutions (79) in the right hand sides of Eqs. (74) and (75) and show that they are not negligible compared to the other terms of Eqs. (76). To determine the behavior of the right hand sides of Eqs. (74) and (75), we assign to each quantity appropriate powers of \(p\):

- the internal wave-vector \(q\) is tailored to values \(q \sim \kappa\) by the presence of the term \(\partial \bar{p} R_\kappa\), that is, it is of order one as \(p \gg \kappa\).
- the external frequency \(\nu\) scales as \(p^{2/3}\).
- the internal frequency \(\omega\) satisfies \(\omega \ll \nu \sim p^{2/3}\), as shown below.
As a matter of fact, the internal frequency is not cut off by the regulator and it is not clear a priori which region of integration on $\omega$ dominates. There are essentially two scales, $\kappa$ and $p$. If one assumes that $\omega \sim \nu \sim p^{2/3}$ then the resulting integral on $\omega$ behaves for small $\omega$ as $\int d\omega/\omega^2$ which is IR divergent (see the note at the end of this subsection). This means that the dominating internal frequencies are $\omega \ll \nu \sim p^{2/3}$. In this limit, the flow equations (80) and (81) acquire a simpler form

$$
\partial_s \Gamma^{(1)}_\perp (\nu, \vec{p}) = \frac{(d-1)}{d} p^2 I_0 \left\{ \frac{1}{2} \partial_s^2 \Gamma^{(1)}_\perp (\nu, \vec{p}) \right\} - \left( \partial_s \Gamma^{(1)}_\perp (\nu, \vec{p}) \right)^2 G^{uu}_\perp (-\nu, \vec{p}) \right\} \quad (80)
$$

$$
\partial_s \Gamma^{(2)}_\perp (\nu, \vec{p}) = \frac{(d-1)}{d} p^2 I_0 \left\{ \frac{1}{2} \partial_s^2 \Gamma^{(2)}_\perp (\nu, \vec{p}) \right\} - 2 \partial_s \Gamma^{(1)}_\perp (\nu, \vec{p}) \times \Re \left[ \partial_s \Gamma^{(1)}_\perp (\nu, \vec{p}) G^{uu}_\perp (-\nu, \vec{p}) \right] + \left[ \partial_s \Gamma^{(1)}_\perp (\nu, \vec{p}) \right]^2 G^{uu}_\perp (\nu, \vec{p}) \right\} \quad (81)
$$

where

$$
I_0 = \hat{\partial}_s \int_{\omega, \vec{q}} G^{uu}_\perp (\omega, \vec{q}).
$$

Under this form, it is manifest that the relevant scale for the internal frequency $\omega$ is $\kappa$ since the integral $I_0$ does not depend on the external scales $p$ or $\nu$.

Substituting the scaling solutions (72) in the right hand sides of Eqs. (80) and (81) one obtains that the equations for $\Gamma^{(1)}_\perp (\nu, \vec{p})$ and $\Gamma^{(0,2)}_\perp (\nu, \vec{p})$ behave as $p^{4/3}$ and $p^{-d+2/3}$, respectively. This yields violations of the scaling which are not marginal, but quite substantial: the right hand sides are not sub-leading compared to the left hand ones but dominating by a factor $p^{2/3}$. This clearly is not consistent and proves that there is no decoupling of the large wave-number sector $|\vec{p}| \gg \kappa$.

Note:

Let us assume that in the regime $|\vec{p}| \gg \kappa$, the dominating region of integration is $\omega \sim \nu \sim p^{2/3}$ and determine the behavior of the integral:

$$
\hat{\partial}_s \int_{\vec{q}} G^{uu}_\perp (\omega, \vec{q}) = \int_q \left( \frac{\Gamma^{(0,2)}_\perp (\omega, \vec{q}) + N_k(q)}{\Gamma^{(1)}_\perp (\omega, \vec{q})^2} \right) \partial_q R_k(q) - \int_q \left( \frac{\partial_q N_k(q)}{\Gamma^{(1)}_\perp (\omega, \vec{q})^2} \right) + c.c. \quad (82)
$$

The first term is of order $p^{-4/3}$ and the second one, dominated by the contribution proportional to $N_k(q)$, is of order $p^{-2}$. One concludes from this power-counting that the right hand sides of Eqs. (72) and (73) for $\Gamma^{(1,1)}_\perp (\nu, \vec{p})$ and $\Gamma^{(0,2)}_\perp (\nu, \vec{p})$ are of order $p^{2/3}$ and $p^{-d}$, respectively. Thus, if one assumes the decoupling of the large $|\vec{p}|$ sector and that the dominant internal frequencies are of order $\omega \sim \nu \sim p^{2/3}$, the right hand sides of the flow equations at the fixed point are not negligible compared to the left hand sides, but of the same order. One finds as well that this is inconsistent. However, this estimate is not correct because the resulting integral on $\omega$ behaves for small $\omega$ as $\int d\omega/\omega^2$. In fact, the right hand sides were found to be even dominant, behaving as $p^{4/3}$ and $p^{-d+2/3}$, respectively, which confirms that the contribution of frequencies $\omega \ll \nu$ is much larger than that of frequencies of order $\omega \sim \nu$.

C. Consequences of the non-decoupling and intermittency

This non-decoupling property is extremely peculiar. It means that correlation functions remain sensitive to the integral scale, which is completely different from what occurs in critical phenomena, where correlation functions have a well-behaved infinite-volume limit. The origin of this difference can be intuitively understood. A dissipative system such as a fluid cannot sustain well-defined stationary correlation functions without injection of energy, and thus it remains in some way sensitive to the corresponding scale, the integral scale, even at much larger wave-length scales. It is therefore reasonable to infer that this violation of scaling is general in fully developed turbulence, and not restricted to 2-point correlation functions. This could explain the origin of intermittency: correlation functions are dominated by the existence of a IR fixed point, leading to power-law behavior, but the absence of decoupling prevents the existence of usual scaling (determined by a finite set of critical exponents), and opens the door to multi-scaling (or multi-fractality).

Let us make one further step and try to explain why the lowest order structure functions display only very small corrections to the Kolmogorov scaling exponents. Obviously the fourth theorem forbids anomalous corrections for the $S^{(3)}$ structure function. As for $S^{(2)}$, one can justify very small but non-zero corrections in the following way. In the regime of large wave-numbers, the equations (80) and (81) become exact. These equations read in terms of the connected 2-point correlation functions

$$
\partial_s G^{uu}_\perp (\nu, \vec{p}) = \frac{(d-1)}{2d} p^2 I_0 \partial_p^2 G^{uu}_\perp (\nu, \vec{p}) \quad (83)
$$

$$
\partial_s G^{uu}_\perp (\nu, \vec{p}) = \frac{(d-1)}{2d} p^2 I_0 \partial_p^2 G^{uu}_\perp (\nu, \vec{p}) \quad (84)
$$

Again, if the dimensionless associated functions approach a fixed point and decoupling is assumed, one can show that right hand sides are enhanced with respect to the other terms by a factor $p^{2/3}$. However, the functions that are usually measured experimentally are not directly the functions $G^{uu}_\perp (\nu, \vec{p})$ or $G^{uu}_\perp (\nu, \vec{p})$, but correlators at equal times such as

$$
\int d\omega/2\pi G^{uu}_\perp (\omega, \vec{p}) \quad (85)
$$

see e.g. Eq. (43). As the function $G^{uu}_\perp (\omega, \vec{p})$ is expected to remain regular in frequency when $\kappa \to 0$, one can
integrate Eq. (A2) over the frequency, which yields
\[
\partial_s \int \frac{d\omega}{2\pi} G^{uu}_{\perp}(\omega, \vec{p}) \ll \text{leading terms.} \tag{86}
\]

This means that the leading term that violates the decoupling is zero when integrated over frequencies. Accordingly, the possible leading intermittency correction to this quantity comes from a sub-leading contribution. This could explain the smallness of the deviation for the second order structure function. On the contrary, higher order \(n\)-point functions bear a more complicated frequency structure and such compensations are very unlikely to occur, and thus intermittency effects could be much larger in higher order structure functions, as observed in experiments.

VIII. CONCLUSION

In this paper, we expounded the NPRG formalism to investigate the regime of fully developed isotropic turbulence of the NS equation in the presence of a stochastic forcing. We then implemented a simple approximation based on the NS symmetries, called the LO approximation, to solve the NPRG flow equations. By numerically integrating these equations, we found a fully attractive fixed point in dimension \(d = 2\) and \(d = 3\), governing the stationary regime of fully developed turbulence. The striking feature of this fixed point is the emergence of the stationary regime of fully developed turbulence. The fixed point entails power-law behavior for the correlation functions (and the energy spectrum), and may be larger for higher order ones. The value of the corresponding intermittency exponents can be computed by integrating the exact flow equations obtained in the large wave-number regime, which will be investigated in a future work. More generally, we are confident that the present work constitutes a solid basis for future investigation of NS turbulence using NPRG methods.

IX. ACKNOWLEDGMENTS

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APPENDIX A: GENERAL STRUCTURE OF THE NPRG PROPAGATOR

In this Appendix, we establish the general structure of the \(4 \times 4\) propagator matrix \(G_\kappa\) defined as the inverse of \([\Gamma_\kappa^{(2)} + R_\kappa]\). The matrix elements of \([\Gamma_\kappa^{(2)} + R_\kappa]\) are obtained by taking functional derivatives of \(\mathcal{L}\) and \(\Pi\) with respect to two of the fields \(u_\alpha\), \(\bar{u}_\alpha\), \(p\) and \(\bar{p}\). They are given in Fourier space and at zero fields by, (omitting the \(\kappa\) indices to alleviate notation)

\[
\Gamma_\kappa^{(2)}(\omega, \vec{p}) + R_\kappa(\vec{p}) = \frac{u_\alpha}{p} \Gamma^{(1,1)}_{\alpha\beta}(\omega, \vec{p}) + R_{\kappa, \alpha\beta}(\vec{p}) \frac{\bar{u}_\beta}{\bar{p}} - \beta = 1,2
\]

Using rotational invariance and parity, one may infer that the propagator matrix is endowed with the following generic structure

\[
G_\kappa(\omega, \vec{p}) = \begin{pmatrix}
G_{u_\alpha}^{u_\beta}(\omega, \vec{p}) & G_{u_\alpha}^{\bar{u}_\beta}(\omega, \vec{p}) & G_{u_\alpha}^{p}(\omega, \vec{p}) & G_{u_\alpha}^{\bar{p}}(\omega, \vec{p}) \\
G_{\bar{u}_\alpha}^{u_\beta}(\omega, \vec{p}) & G_{\bar{u}_\alpha}^{\bar{u}_\beta}(\omega, \vec{p}) & G_{\bar{u}_\alpha}^{p}(\omega, \vec{p}) & G_{\bar{u}_\alpha}^{\bar{p}}(\omega, \vec{p}) \\
p & -ip_\beta G_{u_\alpha}^{u_\beta}(\omega, \vec{p}) & ip_\alpha G_{u_\alpha}^{p}(\omega, \vec{p}) & ip_\alpha G_{u_\alpha}^{\bar{p}}(\omega, \vec{p}) \\
\bar{p} & -ip_\beta G_{\bar{u}_\alpha}^{\bar{u}_\beta}(\omega, \vec{p}) & ip_\alpha G_{\bar{u}_\alpha}^{p}(\omega, \vec{p}) & ip_\alpha G_{\bar{u}_\alpha}^{\bar{p}}(\omega, \vec{p})
\end{pmatrix}
\tag{A2}
\]

in obvious notation for the two upper indices of the different matrix elements of \(G_\kappa\). The latters are obtained by requiring that the product \([\Gamma_\kappa^{(2)} + R_\kappa](\omega, \vec{p})G_\kappa(\omega, \vec{p})\)
is the identity matrix. This yields in the pressure sector

\begin{align*}
G^{pp}(\omega, \vec{p}) &= \frac{2\rho}{p^2} f_0^\nu(\vec{p}) \\
G^{\vec{p}}\vec{p}(\omega, \vec{p}) &= \frac{\rho}{p^2} (-i\omega + f_0^\nu(\vec{p})) \\
G^{\vec{p}}\vec{p}(\omega, \vec{p}) &= 0.
\end{align*}

(A3)

In the mixed sector, only two elements are non-vanishing, which are

\begin{align*}
G^{u\vec{p}}(\omega, \vec{p}) &= -\frac{1}{p^2}, \quad G^{\vec{p}u}(\omega, \vec{p}) = \frac{\rho}{p^2}.
\end{align*}

\textbf{(A4)}

\section*{APPENDIX B: DERIVATION OF THE NPRG FLOW EQUATIONS AT LO}

In this Appendix, we derive the flow equations for the transverse components of the two running functions \(f_0^\nu(\vec{p})\) and \(f_0^\nu(\vec{p})\) of the LO ansatz. They are related to the flows of \(\Gamma^{(1,1)}_{\alpha\beta}(\nu = 0, \vec{p})\) and \(\Gamma^{(0,2)}_{\alpha\beta}(\nu = 0, \vec{p})\), respectively, which we now calculate.

\textbf{Flow equations of} \(\Gamma^{(1,1)}_{\alpha\beta}\) \textbf{and} \(\Gamma^{(0,2)}_{\alpha\beta}\) \textbf{at LO}

According to Eq. (110), the flow equation of \(\Gamma^{(1,1)}_{\alpha\beta}(\nu = 0, \vec{p})\) is given by

\begin{equation}
\partial_\nu \Gamma^{(1,1)}_{\alpha\beta}(\nu = 0, \vec{p}) = \text{Tr} \int_{\omega, \vec{q}} \partial_\omega R_\kappa(\vec{q}) \cdot G_\kappa(\vec{q}) \cdot \left\{ \Gamma^{(3)}_{\kappa,\nu\alpha}(\vec{p}, \vec{q}) \cdot G_\kappa(\vec{p} + \vec{q}) \cdot \Gamma^{(3)}_{\kappa,\nu\beta}(\vec{p} + \vec{q}, -\vec{p}) \right\} \cdot G_\kappa(\vec{q}),
\end{equation}

(A1)

omitting the contributions of the 4-point vertices which are vanishing at LO. As apparent in (A1), the regulator matrix \(R_\kappa\) has only three non-vanishing sectors: \([R_\kappa]_{12} = [R_\kappa]_{21} = R_\kappa(\vec{p})\) and \([R_\kappa]_{12} = -2N_\kappa(\vec{p})\). Moreover, at LO, only the vertex function \(\Gamma^{(2,1)}_{\kappa,\nu}\) is non-zero and contributes to the \(\Gamma^{(3)}_{\kappa,\nu}\) matrices. Performing the matrix product (B1) and taking the trace, one is left with only four terms, which are

\begin{equation}
\partial_\nu \Gamma^{(1,1)}_{\alpha\beta}(\nu = 0, \vec{p}) = \lambda^2 \int_{\omega, \vec{q}} \partial_\omega R_\kappa(\vec{q}) \left\{ i(q_i \delta_{i\beta} - (q + p)_i \delta_{i\beta}) i((p + q)_\alpha \delta_{kj} - p_k \delta_{\alpha j}) P_{ij}^{(\nu)}(\vec{q}) P_{k\nu}^{(\nu)}(\vec{\nu} + \vec{q}) \right\} \left[ \frac{2\tilde{f}_0^\nu(\vec{p} + \vec{q})}{(\omega^2 + (f_0^\nu(\vec{q}))^2)^2} \right] + \lambda^2 \int_{\omega, \vec{q}} \partial_\omega R_\kappa(\vec{q}) \left\{ \frac{4\tilde{f}_0^\nu(\vec{q}) f_0^\nu(\vec{q})}{(\omega^2 + (f_0^\nu(\vec{q}))^2)^2} \right\} \left( -i\omega + f_0^\nu(\vec{q}) \right)^2 \left( \tilde{f}_0^\nu(\vec{p} + \vec{q}) \right)^2
\end{equation}

(A2)

with the notation

\begin{equation}
\tilde{f}_0^\nu(\vec{q}) = f_0^\nu(\vec{q}) + R_\kappa(\vec{q}) \quad \text{and} \quad \tilde{f}_0^\nu(\vec{q}) = f_0^\nu(\vec{q}) + N_\kappa(\vec{q}).
\end{equation}

(B3)

This flow equation is to be projected onto the transverse sector. Hence, all the terms proportional to \(p_\alpha\) or \(p_\beta\) can be discarded as \(P_{\alpha\beta}^{(\nu)}(\vec{p})p_\alpha = P_{\alpha\beta}^{(\nu)}(\vec{p})p_\beta = 0\). The two tensor structures in Eq. (112) can be then simplified in the following way

\begin{equation}
T^{(1,1)}_\alpha \equiv -P_{ij}^{(\nu)}(\vec{q})(p + q)_\alpha \delta_{kj} - p_k \delta_{\alpha j}) P_{k\nu}^{(\nu)}(\vec{p} + \vec{q})(q_i \delta_{i\beta} - (p + q)_i \delta_{i\beta}) + \text{ longitudinal parts}
\end{equation}

\begin{equation}
= \left[ -\vec{p}^2 + \frac{(\vec{p} \cdot (\vec{p} + \vec{q}))^2}{(\vec{p} + \vec{q})^2} \right] \delta_{\alpha\beta} - 2\frac{q_\alpha q_\beta}{(\vec{p} + \vec{q})^2} \vec{p} \cdot (\vec{p} + \vec{q}) + \text{ longitudinal parts}
\end{equation}

(B4)
where only the transverse contributions are explicitly specified.

Similarly, the flow equation of $\Gamma_{\alpha\beta}^{(0,2)}(\nu = 0, \vec{p})$ is given by Eq. (B5)

$$\partial_\nu \Gamma_{\alpha\beta}^{(0,2)}(\nu = 0, \vec{p}) = \text{Tr} \int_{\omega, \vec{q}} \partial_\nu \mathcal{R}_\omega(\vec{q}) \cdot G_\omega(q) \cdot \left\{ \Gamma_{\alpha\beta}^{(3)}(p, q) \cdot G_\omega(p + q) \cdot \Gamma_{\alpha\beta}^{(3)}(p + q, -\vec{p}) \right\} \cdot G_\omega(q)$$

omitting the vanishing 4-point vertices (at LO). Only three terms are left in the trace of the matrix product (B5), which are

$$\partial_\nu \Gamma_{\alpha\beta}^{(0,2)}(\nu = 0, \vec{p}) = \lambda^2 \int_{\omega, \vec{q}} i((q + p)_\mu \delta_{\mu\alpha} - q_\mu \delta_{\alpha\mu})i((q + p)_\mu \delta_{\mu\beta} - (p + q)_\mu \delta_{\beta\mu})P_{\alpha\beta}(\vec{q})P_{\alpha\beta}(\vec{p} + \vec{q})$$

$$\times \left\{ \partial_\nu \mathcal{R}_\omega(\vec{q}) \frac{4f_{\nu}^{\perp}(\vec{q})f_{\nu}^{\perp}(\vec{q})}{(\omega^2 + (f_{\nu}^{\perp}(\vec{q}))^2)^2} - 2 \frac{\partial_\nu \nabla_\omega(q)}{\omega^2 + (f_{\nu}^{\perp}(\vec{q}))^2} \right\} \frac{2f_{\nu}^{\perp}(\vec{q} + \vec{p})}{\omega^2 + (f_{\nu}^{\perp}(\vec{q} + \vec{p}))^2}.$$  

As previously, ignoring the terms proportional to $p_\alpha$ and $p_\beta$, the tensor structure in this equation simplifies to

$$T^{(0,2)} = -P_{\alpha\beta}(p)\delta_{\alpha\beta}p_{\perp}(\vec{q}) + (p_{\perp} - \vec{q})_\mu (-p_\mu \delta_{\alpha\beta} - p_k \delta_{\beta\mu}) + \text{longitudinal parts}$$

$$= \left[ 2p_{\perp}^2 + \frac{(\vec{p} \cdot \vec{q})^2}{(\vec{p} + \vec{q})^2} - \frac{(\vec{p} \cdot \vec{q})^2}{\vec{q}^2} \right] \delta_{\alpha\beta} + 2 \frac{q_\alpha q_\beta}{\vec{q}^2(\vec{p} + \vec{q})^2} \left( \vec{p}^2 \vec{q} \cdot \vec{q} + 2(\vec{p} \cdot \vec{q})^2 - \vec{p}^2 \vec{q}^2 \right) + \text{longitudinal parts.}$$

### Flow equations of $f_{\nu}^{\perp}$ and $f_{\nu}^{\perp}$

According to the expressions (17) of the 2-point functions at LO in the velocity sector, the flow equations of the transverse functions $f_{\nu}^{\perp}$ and $f_{\nu}^{\perp}$ may be defined as

$$\partial_\nu f_{\nu}^{\perp}(\vec{p}) = \frac{1}{(d - 1)} P_{\alpha\beta}(\vec{p}) \delta_{\alpha\beta} \Gamma_{\alpha\beta}^{(1,1)}(\nu = 0, \vec{p})$$

$$\partial_\nu f_{\nu}^{\perp}(\vec{p}) = -\frac{1}{2(d - 1)} P_{\alpha\beta}(\vec{p}) \partial_\nu \Gamma_{\alpha\beta}^{(0,2)}(\nu = 0, \vec{p}).$$

The flow equations of $\partial_\nu \Gamma_{\alpha\beta}^{(1,1)}$ and $\partial_\nu \Gamma_{\alpha\beta}^{(0,2)}$ are proportional to the tensor structures (14) and (17), respectively, which projections onto the transverse sector are straightforward, using

$$P_{\alpha\beta}(\vec{p}) \delta_{\alpha\beta} = d - 1 \quad \text{and} \quad P_{\alpha\beta}(\vec{p}) q_\alpha q_\beta = q^2 - \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2}.$$ 

One obtains

$$\partial_\nu f_{\nu}^{\perp}(\vec{p}) = \frac{\lambda^2}{(d - 1)} \int_{\omega, \vec{q}} \left\{ \frac{2f_{\nu}^{\perp}(\vec{q}) f_{\nu}^{\perp}(\vec{q})}{\omega^2 + (f_{\nu}^{\perp}(\vec{q}))^2} \left[ (-\vec{p}^2 + \frac{(\vec{p} \cdot \vec{q})^2}{(\vec{p} + \vec{q})^2}) (d - 1) - 2\vec{p} \cdot \vec{q} \left( 1 - \frac{(\vec{p} \cdot \vec{q})^2}{\vec{q}^2\vec{p}^2} \right) \right] + \frac{2\partial_\nu \nabla_\omega(q)}{\omega^2 + (f_{\nu}^{\perp}(\vec{q}))^2} \right\}$$

$$\times \left\{ \left[ -\vec{p}^2 + \frac{(\vec{p} \cdot \vec{q})^2}{\vec{q}^2} \right] (d - 1) + 2\vec{p} \cdot (\vec{p} + \vec{q}) \left( \vec{q}^2 - \frac{(\vec{p} \cdot \vec{q})^2}{\vec{p}^2} \right) \right\} \right\}$$

$$\partial_\nu f_{\nu}^{\perp}(\vec{p}) = -\frac{\lambda^2}{2(d - 1)} \int_{\omega, \vec{q}} \left\{ \left[ \partial_\nu \mathcal{R}_\omega(\vec{q}) \frac{4f_{\nu}^{\perp}(\vec{q}) f_{\nu}^{\perp}(\vec{q})}{\omega^2 + (f_{\nu}^{\perp}(\vec{q}))^2} \right] \left[ -\frac{\partial_\nu \nabla_\omega(q)}{\omega^2 + (f_{\nu}^{\perp}(\vec{q}))^2} \right] \frac{2f_{\nu}^{\perp}(\vec{q} + \vec{p})}{\omega^2 + (f_{\nu}^{\perp}(\vec{q} + \vec{p}))^2}$$

$$\times \left\{ \left[ 2\vec{p}^2 + \frac{(\vec{p} \cdot \vec{q})^2}{(\vec{p} + \vec{q})^2} - \frac{(\vec{p} \cdot \vec{q})^2}{\vec{q}^2} \right] (d - 1) + 2\frac{1}{\vec{q}^2(\vec{p} + \vec{q})^2} \left( \vec{p}^2 \vec{q} \cdot \vec{q} + 2(\vec{p} \cdot \vec{q})^2 - \vec{p}^2 \vec{q}^2 \right) \right\}.$$ 

(B10)
Within the LO approximation, the frequency dependence remains the bare one, and the integration over the internal frequency $\omega$ can be carried out analytically in the above expressions. Denoting $A = f_1'(q)^2$ and $B = \tilde{f}_2'(\mathbf{p} + \tilde{q})^2$, the different frequency integrals are given by

\[
\begin{align*}
I_1 &= \int \frac{d\omega}{2\pi} \frac{1}{(-i\omega + A)^2} \frac{1}{\omega^2 + B^2} = \frac{1}{2B(A + B)^2} \\
I_2 &= \int \frac{d\omega}{2\pi} \frac{1}{i\omega + B} \frac{1}{\omega^2 + A^2} = \frac{1}{2A(B + A)} \\
I_3 &= \int \frac{d\omega}{2\pi} \frac{1}{i\omega + B} \frac{1}{\omega^2 + A^2} = \frac{2A}{4A^3(B + A)^2} \\
I_4 &= \int \frac{\omega}{\omega^2 + B^2} \frac{1}{\omega^2 + A^2} = \frac{1}{2AB(A + B)} \\
I_5 &= \int \frac{\omega}{\omega^2 + B^2} \frac{1}{\omega^2 + A^2} = \frac{2A}{4A^3(B + A)^2}
\end{align*}
\]  
(B11)

which yields the two flow equations (51) and (52).

**APPENDIX C: AVERAGE INJECTED POWER PER UNIT MASS**

The injected power per unit mass is $f_\alpha(x)v_\alpha(x)$. The average of a quantity linearly depending on the stochastic forcing $\mathbf{f}$ can be calculated using the Janssen-de Dominici procedure (see Ref. [7] for detail), which yields

\[
\text{E}
\]

\[
\langle f_\alpha(t, x)\mathcal{O}[\mathbf{v}] \rangle = 2\int_{\mathbf{v}'} N_{L-1,\alpha\beta}(|x - x'|) \langle \mathbf{v}_\beta(t, x')\mathcal{O}[\mathbf{v}] \rangle. 
\]  
(C1)

Averages of quantities linear in $\mathbf{f}$ are hence related to response functions. In the particular case where $f_\alpha$ and $\mathcal{O}$ are defined at equal times, one must carefully consider the Itô’s prescription. As a matter of fact, the average of the injected power per unit mass would naively be given, according to Eq. (C1), by

\[
\langle f_\alpha(t, x)\mathcal{O}[\mathbf{v}] \rangle \approx \langle f_\alpha(t, x)v_\alpha(t, x) \rangle = 2\int_{\mathbf{v}'} N_{L-1,\alpha\beta}(|x - x'|) \langle v_\alpha(t, x)\mathcal{O}[\mathbf{v}] \rangle. 
\]  
(C2)

but the response function at equal times is zero because of Itô’s prescription. However, the precise meaning of equal time must be carefully specified in discrete time. In particular, for the injected power, one should examine the kinetic energy theorem and properly discretize it. The energy at a given space point comes from both direct injection by the external force and transfer from its neighbouring points. We here only seek the variation of the velocity induced by the external force and thus omit transferred power from one point to another. Accordingly, in discrete time, Itô forward discretization is

\[
\tilde{f}_\alpha(t, \tilde{x}) = \partial_t v_\alpha(t, \tilde{x}) = \frac{1}{\delta t} (v_\alpha(t + \delta t, \tilde{x}) - v_\alpha(t, \tilde{x})). 
\]  
(C3)

Thus, the discretized kinetic energy theorem is

\[
\partial_t \left( \frac{1}{2} f_\alpha(t, x) v_\alpha(t, x) \right) = \left[ (v_\alpha(t + \delta t, x) - v_\alpha(t, x)) \right]/(2\delta t) \\
= \frac{1}{2} \left( \tilde{f}_\alpha(t, \tilde{x}) v_\alpha(t + \delta t, \tilde{x}) + v_\alpha(t, \tilde{x}) \right)
\]  
(C4)

which indicates that half of the quantity to be averaged is not defined at coinciding but at successive times and the associated response function has a non-zero contribution. In conclusion, the average injected power is precisely defined as

\[
\langle f_\alpha(t, \tilde{x}) v_\alpha(t, \tilde{x}) \rangle = \lim_{\delta t \to 0^+} \int_{\mathbf{v}'} N_{L-1,\alpha\beta}(|x - x'|) \times \langle v_\alpha(t + \delta t, \tilde{x})\mathcal{O}[\mathbf{v}] \rangle. 
\]  
(C5)

which, apart from the correct limit process and the one half factor, coincides with the expression (C2).

**APPENDIX D: NUMERICAL INTEGRATION OF THE LO FLOW EQUATIONS**

In this Appendix, we expound the detail of the numerical procedure implemented to integrate the LO flow equations (61). The running functions $h^{\nu,\alpha}(\hat{p})$ only depend on the modulus $\hat{p}$ of the wave-vector $\hat{p}$. The dimensionless wave-numbers are discretized on a $\sqrt{\hat{p}_{\text{max}}}$ grid of typical size $\sqrt{\hat{p}_{\text{max}}} \approx 30$ and spacing $\Delta \equiv \Delta \sqrt{\hat{p}} \approx 1/20$. The integrals over the internal wave-vector $\hat{q}$ are calculated numerically using Simpson’s rule, in cartesian coordinates, chosen with the $q_1$ axis along the external wave-vector $\hat{p}$ and the $(d - 1)$ other axes $\hat{q}_i$ spanning the hyperplane perpendicular to $\hat{p}$. With this choice, one simply has

\[
\hat{p} \cdot \hat{q} = \hat{p} \hat{q}_1 \quad \text{and} \quad (\hat{p} + \hat{q})^2 = (\hat{p} + \hat{q}_1)^2 + \sum_{i=2}^d \hat{q}_i^2.
\]

As a square root grid is used for the wave-numbers, the values of the functions $h^{\nu,\alpha}(\hat{p} + \hat{q})$ for arguments $|\hat{p} + \hat{q}|^{1/2}$
not falling onto mesh points are interpolated using cubic splines.

The presence of the $\partial_s R_\kappa$ terms in (51) and (52) ensures that the integrands decrease exponentially with $\hat{q}$, such that the internal wave-number integral can be safely cut at an upper finite bound $\hat{p}_{\text{up}} \leq \hat{p}_{\text{max}}$. For wave-numbers such that $|\hat{\vec{p}} + \hat{\vec{q}}| > \hat{p}_{\text{max}}$, the functions $\hat{h}_\nu, D(\hat{\vec{p}} + \hat{\vec{q}})$ are extended outside the grid using power law extrapolations. This corresponds to the expected asymptotics of the flowing functions, at least close to the fixed point. The derivative terms $\hat{p} \partial_\hat{p}$ are computed using 5-point differences. For the propagation in renormalization time $s$, explicit Euler time stepping is used with a typical time step $\Delta s = -1 \times 10^{-4}$. Starting at $s = 0$ from the bare action ($\hat{h}_\nu(\hat{\vec{p}}) = 1$ and $\hat{h}_\nu D(\hat{\vec{p}}) = 0$), we observe that the two functions $\hat{h}_\nu, D$ are smoothly deformed from their flat initial shapes to acquire their fixed point profiles, typically after $|s| \gtrsim 8$. The fixed point profiles are recorded at $s = -30$ (e.g. in Fig. 2).

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**APPENDIX E: WARD IDENTITIES FOR THE 3- AND 4-POINT VERTEX FUNCTIONS**

In this Appendix, we derive the Ward identities for the 3- and 4-point vertex functions which originate in the time-gauged shift and Galilean symmetries.

1. Ward identities for the time-gauged shift symmetry

Let us consider the Ward identity (26) associated with the time-gauged shift symmetry. Differentiating this equation with respect to $\bar{\vec{u}}_\alpha(t_y, \vec{y})$ and evaluating it at vanishing fields, one obtains

$$\int \hat{\vec{x}} \Gamma^{(0,2)}_{\alpha \beta}(t, \vec{x}, t_y, \vec{y}) = 0 \quad (E1)$$

or equivalently in Fourier space

$$\Gamma^{(0,2)}_{\alpha \beta}(\omega, \vec{p} = \vec{0}) = 0. \quad (E2)$$

By taking additional derivatives of (26) with respect to either fields $\vec{u}$ or $\bar{\vec{u}}$ and evaluating the resulting identity at zero external fields, one can infer the general property

$$\Gamma^{(m,n)}_{\alpha_1, \cdots, \alpha_{n+m}}(\omega_1, \vec{p}_1, \cdots, \omega_m, \vec{p}_m, \omega_{m+1}, \vec{p}_{m+1} = \vec{0}, \cdots) = \delta_{n1} \delta_{m1} \delta_{\alpha_1 \alpha_{m+1}} i\omega, \quad (E3)$$

which means that any $(m, n)$-point vertex function with a zero wave-vector on a $\bar{\vec{u}}$-leg vanishes, except the function $n = m = 1$ which keeps its bare form.

2. Ward identities for the time-gauged Galilean symmetry

Let us derive the Ward identities for the 3- and 4-point vertex functions ensuing from the time-gauged Galilean symmetry. Retaining only the terms that give a non-vanishing contribution to the 3- and 4-point vertex functions in the velocity sector, the Ward identity (24) reads

$$\int \hat{\vec{x}} \left\{ \partial_\alpha \bar{u}_\beta(t_x, t_x) \frac{\delta \Gamma_\kappa}{\delta u_\beta(t_x, t_x)} + \partial_t \frac{\delta \Gamma_\kappa}{\delta u_\alpha(t_x, t_x)} + \partial_\alpha \bar{u}_\beta(t_x, t_x) \frac{\delta \Gamma_\kappa}{\delta \bar{u}_\beta(t_x, t_x)} \right\} = - \int \hat{\vec{x}} \partial_t^2 \bar{u}_\alpha(t_x, t_x). \quad (E4)$$

Differentiating this equation with respect to $\bar{u}_\beta(t_y, \vec{y})$, and evaluating the resulting identity at vanishing fields, one obtains

$$\delta_{\alpha \beta} \partial_t^2 \delta(t - t') + \int \hat{\vec{x}} \partial_t \Gamma^{(1,1)}_{\alpha \beta}(t, \vec{x}, t_y, \vec{y}) = 0 \quad (E5)$$

which leads in Fourier space to

$$\Gamma^{(1,1)}_{\alpha \beta}(\omega, \vec{p} = \vec{0}) = i\omega \delta_{\alpha \beta}. \quad (E6)$$
Then, taking two derivatives of Eq. (E7) with respect to \( u_\mu(t_y, \vec{y}) \) and \( \bar{u}_\nu(t_z, \vec{z}) \) yields at vanishing fields

\[
\int \mathcal{F} \left\{ \partial_\tau \Gamma^{(2,1)}_{\alpha\mu\nu}(t_x, \vec{x}, t_y, \vec{y}, t_z, \vec{z}) - \delta(t_x-t_y)\delta(\vec{x}-\vec{y})\partial_\alpha \Gamma^{(1,1)}_{\mu\nu}(t_y, \vec{y}, t_z, \vec{z}) - \delta(t_x-t_z)\delta(\vec{x}-\vec{z})\partial_\alpha \Gamma^{(1,1)}_{\mu\nu}(t_y, \vec{y}, t_z, \vec{z}) \right\} = 0. \tag{E7}
\]

This provides in Fourier space an exact identity relating the 3-point vertex \( \Gamma^{(2,1)} \) with a zero wave-vector on a \( \vec{u} \)-leg to the 2-point function \( \Gamma^{(1,1)} \), which reads

\[
\Gamma^{(2,1)}_{\alpha\beta\gamma}(\omega_1, \vec{p}_1) = \bar{0}; \omega_2, \vec{p}_2 = -\frac{p_2^0}{\omega_1} \left( \Gamma^{(1,1)}_{\beta\gamma}(\omega_1 + \omega_2, \vec{p}_2) - \Gamma^{(1,1)}_{\beta\gamma}(\omega_2, \vec{p}_2) \right). \tag{E8}
\]

Similarly, taking two derivatives of Eq. (E7) with respect to \( \bar{u} \), one obtains at vanishing fields

\[
\int \mathcal{F} \left\{ \partial_\tau \Gamma^{(1,2)}_{\alpha\mu\nu}(t_x, \vec{x}, t_y, \vec{y}, t_z, \vec{z}) - \delta(t_x-t_y)\delta(\vec{x}-\vec{y})\partial_\alpha \Gamma^{(0,2)}_{\mu\nu}(t_y, \vec{y}, t_z, \vec{z}) - \delta(t_x-t_z)\delta(\vec{x}-\vec{z})\partial_\alpha \Gamma^{(0,2)}_{\mu\nu}(t_y, \vec{y}, t_z, \vec{z}) \right\} = 0. \tag{E9}
\]

This yields in Fourier space an exact identity relating the 3-point vertex \( \Gamma^{(1,2)} \) with a zero wave-vector on its \( \vec{u} \)-leg to the 2-point function \( \Gamma^{(0,2)} \) as

\[
\Gamma^{(1,2)}_{\alpha\beta\gamma}(\omega_1, \vec{p}_1) = \bar{0}; \omega_2, \vec{p}_2 = -\frac{p_2^0}{\omega_1} \left( \Gamma^{(0,2)}_{\beta\gamma}(\omega_1 + \omega_2, \vec{p}_2) - \Gamma^{(0,2)}_{\beta\gamma}(\omega_2, \vec{p}_2) \right). \tag{E10}
\]

As for the 4-point vertices, taking one additional derivative of Eq. (E7) with respect to \( \bar{u}_\rho(t_s, \vec{s}) \) and evaluating the obtained identity at vanishing fields yields

\[
\int \mathcal{F} \left\{ \partial_\tau \Gamma^{(2,2)}_{\alpha\mu\rho\nu}(t_x, \vec{x}, t_y, \vec{y}, t_z, \vec{z}) - \delta(t_x-t_y)\delta(\vec{x}-\vec{y})\partial_\alpha \Gamma^{(1,2)}_{\mu\rho\nu}(t_y, \vec{y}, t_s, \vec{s}, t_z, \vec{z}) - \delta(t_x-t_z)\delta(\vec{x}-\vec{z})\partial_\alpha \Gamma^{(1,2)}_{\mu\rho\nu}(t_y, \vec{y}, t_s, \vec{s}, t_z, \vec{z}) \right\} = 0. \tag{E11}
\]

Similarly, taking one additional derivative of Eq. (E7) with respect to \( u_\rho(t_s, \vec{s}) \) and evaluating the ensuing identity at zero fields leads to

\[
\int \mathcal{F} \left\{ \partial_\tau \Gamma^{(3,1)}_{\alpha\mu\rho\nu}(t_x, \vec{x}, t_y, \vec{y}, t_s, \vec{s}, t_z, \vec{z}) - \delta(t_x-t_y)\delta(\vec{x}-\vec{y})\partial_\alpha \Gamma^{(2,1)}_{\mu\rho\nu}(t_y, \vec{y}, t_s, \vec{s}, t_z, \vec{z}) - \delta(t_x-t_z)\delta(\vec{x}-\vec{z})\partial_\alpha \Gamma^{(2,1)}_{\mu\rho\nu}(t_y, \vec{y}, t_s, \vec{s}, t_z, \vec{z}) \right\} = 0. \tag{E12}
\]

Fouquier transforming the two previous relations, one deduces two exact identities relating the 4-point vertices \( \Gamma^{(2,2)} \) and \( \Gamma^{(3,1)} \) with one zero wave-vector on a \( \vec{u} \)-leg to 3-point functions

\[
\Gamma^{(2,2)}_{\alpha\beta\gamma\delta}(\omega_1, \vec{p}_1 = \bar{0}; \omega_2, \vec{p}_2, \omega_3, \vec{p}_3) = -\frac{1}{\omega_1} \left[ p_2^0 \Gamma^{(1,2)}_{\beta\gamma\delta}(\omega_1 + \omega_2, \vec{p}_2, \omega_3, \vec{p}_3) + p_3^0 \Gamma^{(1,2)}_{\beta\gamma\delta}(\omega_2, \vec{p}_2, \omega_1 + \omega_3, \vec{p}_3) \right] + (p_2 - p_3)^0 \Gamma^{(1,2)}_{\beta\gamma\delta}(\omega_2, \vec{p}_2, \omega_1 + \omega_3, \vec{p}_3) \tag{E13}
\]

\[
\Gamma^{(3,1)}_{\alpha\beta\gamma\delta}(\omega_1, \vec{p}_1 = \bar{0}; \omega_2, \vec{p}_2, \omega_3, \vec{p}_3) = -\frac{1}{\omega_1} \left[ p_2^0 \Gamma^{(2,1)}_{\beta\gamma\delta}(\omega_1 + \omega_2, \vec{p}_2, \omega_3, \vec{p}_3) + p_3^0 \Gamma^{(2,1)}_{\beta\gamma\delta}(\omega_2, \vec{p}_2, \omega_1 + \omega_3, \vec{p}_3) \right] + (p_2 - p_3)^0 \Gamma^{(2,1)}_{\beta\gamma\delta}(\omega_2, \vec{p}_2, \omega_1 + \omega_3, \vec{p}_3) \tag{E14}
\]

**APPENDIX F: EXACT FLOW EQUATIONS FOR THE 2-POINT VERTEX FUNCTIONS IN THE LARGE EXTERNAL WAVE-NUMBER LIMIT**

In this Appendix, we derive an expression for the flow equations of \( \Gamma^{(0,2)}_{\mu, \bar{p}} \) and \( \Gamma^{(1,1)}_{\mu, \bar{p}} \) which becomes exact in the limit of large external wave-number \( |\bar{p}| \gg \kappa \). The diagrams entering these flow equations are schematically depicted in Figs. 4 and 5. Some of them, diagrams (a), (b), (d) of Fig. 4 and diagrams (a), (b) of Fig. 5 are vanishing (see Sec. VII). We calculate below the contributions of the remaining non-zero diagrams.
Flow equation of $\Gamma^{(0,2)}(\nu, \vec{p})$ in the large $|\vec{p}|$ limit

We separately analyze the three diagrams (c), (d) and (e) of Fig. 5 which give non-vanishing contributions to the flow of $\Gamma^{(0,2)}$. We begin with determining the expression of diagram (d) in the limit of large external wave-number $|\vec{p}| \gg \kappa$. Introducing the operator $\tilde{\partial}_s \equiv \partial_s R_\kappa \frac{\partial}{\partial R_\kappa} + \partial_s N_\kappa \frac{\partial}{\partial N_\kappa}$, this contribution may be written as

$$\left[ \partial_s \Gamma^{(0,2)}_{\alpha \beta} (\nu, \vec{p}) \right] (d) = -\frac{1}{2} \tilde{\partial}_s \int_{\omega, q} \Gamma^{(2,1)}_{ij\alpha} (\omega, q, -\omega - \nu, -\vec{p} - \vec{q}) G_{jk}^{uu} (\omega + \nu, \vec{p} + \vec{q}) \Gamma^{(2,1)}_{kl\beta} (\omega + \nu, \vec{p} + \vec{q}, -\omega, -\vec{q}) G_{li}^{uu} (\omega, q). \quad \text{(F1)}$$

Either the operator $\tilde{\partial}_s$ acts on $G_{li}^{uu} (\omega, q)$ and the internal wave-vector $\vec{q}$ is cut off to $|\vec{q}| \lesssim \kappa$ so that it is negligible compared to $|\vec{p}|$ and can be set to zero. Or it acts on $G_{jk}^{uu} (\omega + \nu, \vec{p} + \vec{q})$, in which case the combination $\vec{p} + \vec{q}$ is cut off. Changing variables, this last contribution identifies with the first one. Hence, in the large $|\vec{p}|$ limit, the flow equation (F1) becomes

$$\left[ \partial_s \Gamma^{(0,2)}_{\alpha \beta} (\nu, \vec{p}) \right] (d) = -\int_{\omega} \Gamma^{(2,1)}_{ij\alpha} (\omega, \vec{0}, -\omega - \nu, -\vec{p}) G_{jk}^{uu} (\omega + \nu, \vec{p}) \Gamma^{(2,1)}_{kl\beta} (\omega + \nu, \vec{p}, -\omega, \vec{0}) \tilde{\partial}_s \int_{\vec{q}} G_{li}^{uu} (\omega, \vec{q}). \quad \text{(F2)}$$

Then, using the Ward identity (70) and projecting onto the transverse sector, one deduces

$$P^{(1)}_{\alpha \beta} (\vec{p}) \left[ \partial_s \Gamma^{(0,2)}_{\alpha \beta} (\nu, \vec{p}) \right] (d) = -(d - 1) \left( 1 - \frac{1}{d} \right) p^2 \int_{\omega} \frac{1}{\omega^2} \left[ \Gamma^{(1,1)}_\perp (-\nu, \vec{p}) - \Gamma^{(1,1)}_\perp (-\nu - \omega, \vec{p}) \right] \times \left[ \Gamma^{(1,1)}_\perp (\nu + \omega, \vec{p}) - \Gamma^{(1,1)}_\perp (\nu + \omega, \vec{p}) \right] \tilde{\partial}_s \int_{\vec{q}} G_{li}^{uu} (\omega, \vec{q}) + \text{c.c.} \quad \text{(F3)}$$

where parity in $\vec{p}$ and the identity $\int (\vec{p} \cdot \vec{q})^2 f(q^2) = \frac{p^2}{d} \int q^2 f(q^2)$ were used. The contribution of diagram (c) of Fig. 5 can be written as

$$\left[ \partial_s \Gamma^{(0,2)}_{\alpha \beta} (\nu, \vec{p}) \right] (c) = -\int_{\omega, q} \Gamma^{(1,2)}_{ia\beta} (\omega, q, \nu, \vec{p}) G_{jk}^{uu} (-\omega - \nu, \vec{p} + \vec{q}) \Gamma^{(2,1)}_{kl\beta} (\omega + \nu, \vec{p} + \vec{q}, -\omega, -\vec{q}) \tilde{\partial}_s G_{li}^{uu} (\omega, q) + \text{c.c.} \quad \text{(F4)}$$

where the second equality holds in the large $|\vec{p}|$ limit, where the internal wave-vector $\vec{q}$ is negligible compared to $\vec{p}$. Inserting the Ward identities (70) and (71) for the 3-point vertices and projecting onto the transverse sector, one obtains

$$P^{(1)}_{\alpha \beta} (\vec{p}) \left[ \partial_s \Gamma^{(0,2)}_{\alpha \beta} (\nu, \vec{p}) \right] (c) = -(d - 1) \frac{2}{d} p^2 \int_{\omega} \frac{1}{\omega^2} \left[ \Gamma^{(0,2)}_\perp (\omega + \nu, \vec{p}) - \Gamma^{(0,2)}_\perp (\nu, \vec{p}) \right] \times \left[ \Gamma^{(1,1)}_\perp (\omega + \nu, \vec{p}) - \Gamma^{(1,1)}_\perp (\nu, \vec{p}) \right] \times G_{li}^{uu} (-\omega - \nu, \vec{p}) \tilde{\partial}_s \int_{\vec{q}} G_{li}^{uu} (\omega, \vec{q}) + \text{c.c.} \quad \text{(F5)}$$

Similarly, the contribution of diagram (e) of Fig. 5 simplifies to

$$\left[ \partial_s \Gamma^{(0,2)}_{\alpha \beta} (\nu, \vec{p}) \right] (e) = \frac{1}{2} \tilde{\partial}_s \int_{\omega, q} \Gamma^{(2,2)}_{ij\alpha\beta} (\omega, q, -\omega, -\vec{q}, \nu, \vec{p}) G_{ij\alpha\beta}^{uu} (\omega, q) = \frac{1}{2} \tilde{\partial}_s \int_{\omega, q} \Gamma^{(2,2)}_{ij\alpha\beta} (\omega, q, -\omega, \vec{0}, \nu, \vec{p}) \int_{\vec{q}} G_{ij\alpha\beta}^{uu} (\omega, \vec{q}) \quad \text{(F6)}$$

which transverse projection reads, using the Ward identity (72) for the 4-point vertex function with two vanishing wave-vectors on its $\vec{u}$-legs,

$$P^{(1)}_{\alpha \beta} (\vec{p}) \left[ \partial_s \Gamma^{(0,2)}_{\alpha \beta} (\nu, \vec{p}) \right] (e) = \frac{1}{2} (d - 1) - \frac{2}{d} p^2 \int_{\omega} \frac{1}{\omega^2} \left[ \Gamma^{(0,2)}_\perp (\omega + \nu, \vec{p}) - 2 \Gamma^{(0,2)}_\perp (\nu, \vec{p}) + \Gamma^{(0,2)}_\perp (-\omega + \nu, \vec{p}) \right] \tilde{\partial}_s \int_{\vec{q}} G_{li}^{uu} (\omega, \vec{q}). \quad \text{(F7)}$$

The exact flow equation of $\Gamma^{(0,2)}(\nu, \vec{p})$ in the large external wave-number limit is the sum of the three contributions (F3), (F5) and (F7), which yields Eq. (75).
Flow equation of $\Gamma^{(1,1)}_{\perp}(\nu, \vec{p})$ in the large $\vec{p}$ limit

Only the two diagrams (c) and (e) of Fig. 4 give a non-vanishing contribution to the flow of $\Gamma^{(1,1)}_{\perp}$ in the large external wave-number limit. The contribution of diagram (c) can be written as

$$
\left[ \partial_s \Gamma^{(1,1)}_{\perp \alpha \beta}(\nu, \vec{p}) \right]_{(c)} = -\partial_s \int_{\omega, \vec{q}} \Gamma^{(2,1)}_{\perp \alpha \beta}(\omega, \vec{p}, \nu, \vec{p}) G^{\perp}_{jk}(\omega - \nu, \vec{p} + \vec{q}) G^{\perp}_{ij}(\omega, \vec{p}, \vec{q})
$$

$$
= -\int_{\omega, \vec{q}} \Gamma^{(2,1)}_{\perp \alpha \beta}(\omega, \vec{p}, \nu, \vec{p}) G^{\perp}_{jk}(\omega - \nu, \vec{p}) G^{\perp}_{ij}(\omega, \vec{p}, \vec{q})
$$

$$
-\int_{\omega, \vec{q}} \Gamma^{(2,1)}_{\perp \alpha \beta}(\omega - \nu, \vec{p}, \nu, \vec{p}) G^{\perp}_{jk}(\omega - \nu, \vec{p}, \vec{q}) G^{\perp}_{ij}(\omega, \vec{p}, \vec{q})
$$

where again, the second equality holds in the large $\vec{p}$ transverse projection of this expression, inserting the Ward identity (70), is given by

$$
(F8)
$$

and its transverse projection, using the Ward identity (73), is given by

$$
P^{\perp \alpha \beta}_{\perp \alpha \beta}(\vec{p}) \left[ \partial_s \Gamma^{(1,1)}_{\perp \alpha \beta}(\nu, \vec{p}) \right]_{(c)} = \frac{(d - 1)^2}{d} p^2 \int_{\omega} \left[ \Gamma^{(1,1)}_{\perp \alpha \beta}(\omega, \vec{p}, \nu, \vec{p}) \right]^2 G^{\perp}_{\perp \alpha \beta}(\omega - \nu, \vec{p}, \vec{q})
$$

Lastly, the contribution of diagram (e) of Fig. 4 is very similar to the one of diagram (e) in the flow of $\Gamma^{(0,2)}_{\perp}$ and its transverse projection, using the Ward identity (73), is given by

$$
P^{\perp \alpha \beta}_{\perp \alpha \beta}(\vec{p}) \left[ \partial_s \Gamma^{(1,1)}_{\perp \alpha \beta}(\nu, \vec{p}) \right]_{(e)} = \frac{1}{2} P^{\perp \alpha \beta}_{\perp \alpha \beta}(\vec{p}) \partial_s \int_{\omega, \vec{q}} \Gamma^{(3,1)}_{\perp \alpha \beta}(\omega, \vec{p}, \nu, \vec{p}) G^{\perp}_{\perp \alpha \beta}(\omega, \vec{q})
$$

$$
= \frac{1}{2} \int_{\omega, \vec{q}} \Gamma^{(3,1)}_{\perp \alpha \beta}(\omega, \vec{p}, \nu, \vec{p}) \partial_s \int_{\omega} G^{\perp}_{\perp \alpha \beta}(\omega, \vec{q})
$$

$$
= \frac{1}{2} \frac{(d - 1)^2}{d} p^2 \int_{\omega} \left[ \Gamma^{(1,1)}_{\perp \alpha \beta}(\omega, \vec{p} - \vec{q}) - 2 \Gamma^{(1,1)}_{\perp \alpha \beta}(\omega, \vec{p}) + \Gamma^{(1,1)}_{\perp \alpha \beta}(\omega + \vec{p}) \right] \partial_s \int_{\vec{q}} G^{\perp}_{\perp \alpha \beta}(\omega, \vec{q})
$$

$$
(F10)
$$

The exact flow equation of $\Gamma^{(1,1)}_{\perp}(\nu, \vec{p})$ in the large external wave-number limit is the sum of the two contributions (F9) and (F10), which yields Eq. (74).

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