Insertion of a Contra-\(\alpha\)-continuous Function

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Abstract

A necessary and sufficient condition in terms of lower cut sets is given for the insertion of a contra-\(\alpha\)-continuous function between two comparable real-valued functions.

1. Introduction

The concept of a preopen set in a topological space was introduced by Corson and Michael in 1964 [4]. A subset \(A\) of a topological space \((X, \tau)\) is called preopen or locally dense or nearly open if \(A \subseteq \text{Int}(\text{Cl}(A))\). A set \(A\) is called preclosed if its complement is preopen or equivalently if \(\text{Cl}(\text{Int}(A)) \subseteq A\). The term, preopen, was used for the first time by Mashhour et al. [20], while the concept of a locally dense set was introduced by Corson and Michael [4].

The concept of a semi-open set in a topological space was introduced by Levine in 1963 [17]. A subset \(A\) of a topological space \((X, \tau)\) is called semi-open [10] if \(A \subseteq \text{Cl}(\text{Int}(A))\). A set \(A\) is called semi-closed if its complement is semi-open or equivalently if \(\text{Int}(\text{Cl}(A)) \subseteq A\).
Recall that a subset $A$ of a topological space $(X, \tau)$ is called $\alpha$-open if $A$ is the difference of an open and a nowhere dense subset of $X$. A set $A$ is called $\alpha$-closed if its complement is $\alpha$-open or equivalently if $A$ is union of a closed and a nowhere dense set.

A set is $\alpha$-open if and only if it is semi-open and preopen.

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V$-sets. Complements of $V$-sets, i.e., sets that are intersection of open sets are called $\Lambda$-sets [19].

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$-continuous [25] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subsets of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$-continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

Dontchev in [6] introduced a new class of mappings called contra-continuity. Jafari and Noiri in [12, 13] exhibited and studied among others a new weaker form of this class of mappings called contra-$\alpha$-continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 24].

Hence, a real-valued function $f$ defined on a topological space $X$ is called contra-$\alpha$-continuous (resp. contra-semi-continuous, contra-precontinuous) if the preimage of every open subset of $\mathbb{R}$ is $\alpha$-closed (resp. semi-closed, preclosed) in $X$ [6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-$\alpha$-continuous function between two comparable real-valued functions.

If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all $x$ in $X$.

The following definitions are modifications of conditions considered in [16].

A property $P$ defined relative to a real-valued function on a topological space is a $c\alpha$-property provided that any constant function has property $P$ and provided that the
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sum of a function with property \(P\) and any contra-\(\alpha\)-continuous function also has property \(P\). If \(P_1\) and \(P_2\) are \(c\alpha\)-property, the following terminology is used: (i) A space \(X\) has the weak \(c\alpha\)-insertion property for \((P_1, P_2)\) if and only if for any functions \(g\) and \(f\) on \(X\) such that \(g \leq f\), \(g\) has property \(P_1\) and \(f\) has property \(P_2\), then there exists a contra-\(\alpha\)-continuous function \(h\) such that \(g \leq h \leq f\). (ii) A space \(X\) has the \(c\alpha\)-insertion property for \((P_1, P_2)\) if and only if for any functions \(g\) and \(f\) on \(X\) such that \(g < f\), \(g\) has property \(P_1\) and \(f\) has property \(P_2\), then there exists a contra-\(\alpha\)-continuous function \(h\) such that \(g < h < f\). (iii) A space \(X\) has the weakly \(c\alpha\)-insertion property for \((P_1, P_2)\) if and only if for any functions \(g\) and \(f\) on \(X\) such that \(g < f\), \(g\) has property \(P_1\), \(f\) has property \(P_2\) and \(f - g\) has property \(P_2\), then there exists a contra-\(\alpha\)-continuous function \(h\) such that \(g < h < f\).

In this paper, it is given a sufficient condition for the weak \(c\alpha\)-insertion property. Also for a space with the weak \(c\alpha\)-insertion property, we give a necessary and sufficient condition for the space to have the \(c\alpha\)-insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a contra-\(\alpha\)-continuous function, the necessary definitions and terminology are stated.

Let \((X, \tau)\) be a topological space. Then the family of all \(\alpha\)-open, \(\alpha\)-closed, semi-open, semi-closed, preopen and preclosed will be denoted by \(\alpha O(X, \tau)\), \(\alpha C(X, \tau)\), \(sO(X, \tau)\), \(sC(X, \tau)\), \(pO(X, \tau)\) and \(pC(X, \tau)\), respectively.

**Definition 2.1.** Let \(A\) be a subset of a topological space \((X, \tau)\). We define the subsets \(A^\Lambda\) and \(A^V\) as follows:

\[
A^\Lambda = \bigcap \{O : O \supseteq A, O \in (X, \tau)\} \quad \text{and} \quad A^V = \bigcup \{F : F \subseteq A, F^c \in (X, \tau)\}.
\]

In [7, 18, 23], \(A^\Lambda\) is called the kernel of \(A\).

We define the subsets \(\alpha(A^\Lambda), \alpha(A^V), \rho(A^\Lambda), \rho(A^V), s(A^\Lambda)\) and \(s(A^V)\) as follows:

\[\text{Earthline J. Math. Sci. Vol. 2 No. 2 (2019), 383-393}\]
\[ \alpha(A^\Lambda) = \bigcap\{O : O \supseteq A, O \in \alpha(O(X, \tau))\}, \]
\[ \alpha(A^V) = \bigcup\{F : F \subseteq A, F \in \alpha(C(X, \tau))\}, \]
\[ p(A^\Lambda) = \bigcap\{O : O \supseteq A, O \in p(O(X, \tau))\}, \]
\[ p(A^V) = \bigcup\{F : F \subseteq A, F \in p(C(X, \tau))\}, \]
\[ s(A^\Lambda) = \bigcap\{O : O \supseteq A, O \in s(O(X, \tau))\} \]
and
\[ s(A^V) = \bigcup\{F : F \subseteq A, F \in s(C(X, \tau))\}. \]

\(\alpha(A^\Lambda)\) (resp. \(p(A^\Lambda), s(A^\Lambda)\)) is called the \(\alpha\)-kernel (resp. prekernel, semi-kernel) of \(A\).

The following first two definitions are modifications of conditions considered in [14, 15].

**Definition 2.2.** If \(\rho\) is a binary relation in a set \(S\), then \(\overline{\rho}\) is defined as follows: \(x \overline{\rho} y\) if and only if \(y \rho v \) implies \(x \rho v\) and \(u \rho x \) implies \(u \rho y\) for any \(u\) and \(v\) in \(S\).

**Definition 2.3.** A binary relation \(\rho\) in the power set \(P(X)\) of a topological space \(X\) is called a strong binary relation in \(P(X)\) in case \(\rho\) satisfies each of the following conditions:

1. If \(A_i \rho B_j\) for any \(i \in \{1, \ldots, m\}\) and for any \(j \in \{1, \ldots, n\}\), then there exists a set \(C\) in \(P(X)\) such that \(A_i \rho C\) and \(C \rho B_j\) for any \(i \in \{1, \ldots, m\}\) and any \(j \in \{1, \ldots, n\}\).

2. If \(A \subseteq B\), then \(A \overline{\rho} B\).

3. If \(A \rho B\), then \(\alpha(A^\Lambda) \subseteq B\) and \(A \subseteq \alpha(B^\Lambda)\).

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4.** If \(f\) is a real-valued function defined on a space \(X\) and if \(\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}\) for a real number \(\ell\), then \(A(f, \ell)\) is called a lower indefinite cut set in the domain of \(f\) at the level \(\ell\).
We now give the following main result:

**Theorem 2.1.** Let \( g \) and \( f \) be real-valued functions on the topological space \( X \), in which \( \alpha \)-kernel sets are \( \alpha \)-open, with \( g \leq f \). If there exists a strong binary relation \( \rho \) on the power set of \( X \) and if there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \), then \( A(f, t_1) \rho A(g, t_2) \), then there exists a contra-\( \alpha \)-continuous function \( h \) defined on \( X \) such that \( g \leq h \leq f \).

**Proof.** Theorem 2.1 in [22].

**Theorem 2.2.** Let \( P_1 \) and \( P_2 \) be \( \alpha \)-property and \( X \) be a space that satisfies the weak \( \alpha \)-insertion property for \( (P_1, P_2) \). Also assume that \( g \) and \( f \) are functions on \( X \) such that \( g < f \), \( g \) has property \( P_1 \) and \( f \) has property \( P_2 \). The space \( X \) has the \( \alpha \)-insertion property for \( (P_1, P_2) \) if and only if there exist lower cut sets \( A(f - g, 3^{-n+1}) \) and there exists a decreasing sequence \( \{D_n\} \) of subsets of \( X \) with empty intersection and such that for each \( n \), \( X \setminus D_n \) and \( A(f - g, 3^{-n+1}) \) are completely separated by contra-\( \alpha \)-continuous functions.

**Proof.** Theorem 2.1 in [21].

3. Applications

The abbreviations \( c\alpha c, \quad cpc \) and \( csc \) are used for contra-\( \alpha \)-continuous, contra-precontinuous and contra-semi-continuous, respectively.

Before stating the consequences of Theorems 2.1, 2.2, we suppose that \( X \) is a topological space whose \( \alpha \)-kernel sets are \( \alpha \)-open.

**Corollary 3.1.** If for each pair of disjoint preopen (resp. semi-open) sets \( G_1, G_2 \) of \( X \), there exist \( \alpha \)-closed sets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1 \), \( G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \), then \( X \) has the weak \( \alpha \)-insertion property for \( (cpc, cpc) \) (resp. \( (csc, csc) \)).

**Proof.** Corollary 3.1 in [22].
Corollary 3.2. If for each pair of disjoint preopen (resp. semi-open) sets $G_1, G_2$, there exist $\alpha$-closed sets $F_1$ and $F_2$ such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then every contra-precontinuous (resp. contra-semi-continuous) function is contra-$\alpha$-continuous.

Proof. Corollary 3.2 in [22].

Corollary 3.3. If for each pair of disjoint preopen (resp. semi-open) sets $G_1, G_2$ of $X$, there exist $\alpha$-closed sets $F_1$ and $F_2$ of $X$ such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then $X$ has the $\alpha$-insertion property for (cpc, cpc) (resp. (csc, csc)).

Proof. Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$ are cpc (resp. csc), and $g < f$. Set $h = (f + g)/2$, thus $g < h < f$, and by Corollary 3.2, since $g$ and $f$ are contra-$\alpha$-continuous functions hence $h$ is a contra-$\alpha$-continuous function.

Corollary 3.4. If for each pair of disjoint subsets $G_1, G_2$ of $X$, such that $G_1$ is preopen and $G_2$ is semi-open, there exist $\alpha$-closed subsets $F_1$ and $F_2$ of $X$ such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then $X$ have the weak $\alpha$-insertion property for (cpc, cpc) and (csc, cpc).

Proof. Corollary 3.4 in [22].

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.1. The following conditions on the space $X$ are equivalent:

(i) For each pair of disjoint subsets $G_1, G_2$ of $X$, such that $G_1$ is preopen and $G_2$ is semi-open, there exist $\alpha$-closed subsets $F_1, F_2$ of $X$ such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$.

(ii) If $G$ is a semi-open (resp. preopen) subset of $X$ which is contained in a preclosed (resp. semi-closed) subset $F$ of $X$, then there exists an $\alpha$-closed subset $H$ of $X$ such that $G \subseteq H \subseteq \alpha(H^\lambda) \subseteq F$.

Proof. Lemma 3.1 in [22].
Lemma 3.2. Suppose that $X$ is a topological space. If each pair of disjoint subsets $G_1$, $G_2$ of $X$, where $G_1$ is preopen and $G_2$ is semi-open, can be separated by $\alpha$-closed subsets of $X$, then there exists a contra-$\alpha$-continuous function $h : X \to [0, 1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.

Proof. Lemma 3.2 in [22].

Lemma 3.3. Suppose that $X$ is a topological space such that every two disjoint semi-open and preopen subsets of $X$ can be separated by $\alpha$-closed subsets of $X$. The following conditions are equivalent:

(i) Every countable covering of semi-closed (resp. preclosed) subsets of $X$ has a refinement consisting of preclosed (resp. semi-closed) subsets of $X$ such that for every $x \in X$, there exists an $\alpha$-closed subset of $X$ containing $x$ such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{G_n\}$ of semi-open (resp. preopen) subsets of $X$ with empty intersection there exists a decreasing sequence $\{F_n\}$ of preclosed (resp. semi-closed) subsets of $X$ such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$ and for every $n \in \mathbb{N}$, $G_n \subseteq F_n$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\{G_n\}$ is a decreasing sequence of semi-open (resp. preopen) subsets of $X$ with empty intersection. Then $\{G_n^c : n \in \mathbb{N}\}$ is a countable covering of semi-closed (resp. preclosed) subsets of $X$. By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every $V_n$ is an $\alpha$-closed subset of $X$ and $\alpha(V_n^c) \subseteq G_n^c$. By setting $F_n = \alpha((V_n^c)^c)$, we obtain a decreasing sequence of $\alpha$-closed subsets of $X$ with the required properties.

(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of semi-closed (resp. preclosed) subsets of $X$, we set for $n \in \mathbb{N}$, $G_n = \left(\bigcup_{i=1}^{n} H_i\right)^c$. Then $\{G_n\}$ is a decreasing sequence of semi-open (resp. preopen) subsets of $X$ with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of preclosed (resp. semi-closed) subsets of $X$ such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$ and for every $n \in \mathbb{N}$, $G_n \subseteq F_n$. Now we define the subsets
$W_n$ of $X$ in the following manner:

$W_1$ is an $\alpha$-closed subset of $X$ such that $F_1^c \subseteq W_1$ and $\alpha(W_1^\alpha) \cap G_1 = \emptyset$.

$W_2$ is an $\alpha$-closed subset of $X$ such that $\alpha(W_1^\alpha) \cup F_2^c \subseteq W_2$ and $\alpha(W_2^\alpha) \cap G_2 = \emptyset$, and so on. (By Lemma 3.1, $W_n$ exists).

Then since $\{F_n^c : n \in \mathbb{N}\}$ is a covering for $X$, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for $X$ consisting of $\alpha$-closed sets. Moreover, we have

(i) $\alpha(W_n^\alpha) \subseteq W_{n+1}$.

(ii) $F_n^c \subseteq W_n$.

(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.

Now setting $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus \alpha(W_{n-1}^\alpha)$.

Then since $\alpha(W_{n-1}^\alpha) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of $\alpha$-closed sets and covers $X$. Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Finally, consider the following sets:

$$
\begin{align*}
S_1 \cap H_1, & \quad S_1 \cap H_2, \\
S_2 \cap H_1, & \quad S_2 \cap H_2, \quad S_2 \cap H_3, \\
S_3 \cap H_1, & \quad S_3 \cap H_2, \quad S_3 \cap H_3, \quad S_3 \cap H_4, \\
& \vdots \\
S_i \cap H_1, & \quad S_i \cap H_2, \quad S_i \cap H_3, \quad S_i \cap H_4, \quad \ldots, \quad S_i \cap H_{i+1} \\
& \vdots
\end{align*}
$$

These sets are $\alpha$-closed sets, cover $X$ and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is an $\alpha$-closed set containing $x$ that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}\}$
Corollary 3.5. If every two disjoint semi-open and preopen subsets of \( X \) can be separated by \( \alpha \)-closed subsets of \( X \), and in addition, every countable covering of semi-closed (resp. preclosed) subsets of \( X \) has a refinement that consists of preclosed (resp. semi-closed) subsets of \( X \) such that for every point of \( X \) we can find an \( \alpha \)-closed subset containing that point such that it intersects only a finite number of refining members, then \( X \) has the weakly \( c\alpha \)-insertion property for (cpc, csc) (resp. (csc, cpc)).

Proof. Since every two disjoint semi-open and preopen sets can be separated by \( \alpha \)-closed subsets of \( X \), therefore by Corollary 3.4, \( X \) has the weak \( c\alpha \)-insertion property for (cpc, csc) and (csc, cpc). Now suppose that \( f \) and \( g \) are real-valued functions on \( X \) with \( g < f \), such that \( g \) is cpc (resp. csc), \( f \) is csc (resp. cpc) and \( f - g \) is csc (resp. cpc). For every \( n \in \mathbb{N} \), set

\[
A(f - g, 3^{-n+1}) = \{ x \in X : (f - g)(x) \leq 3^{-n+1} \}.
\]

Since \( f - g \) is csc (resp. cpc), hence \( A(f - g, 3^{-n+1}) \) is a semi-open (resp. preopen) subset of \( X \). Consequently, \( \{ A(f - g, 3^{-n+1}) \} \) is a decreasing sequence of semi-open (resp. preopen) subsets of \( X \) and furthermore since \( 0 < f - g \), it follows that

\[
\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \emptyset.
\]

Now by Lemma 3.3, there exists a decreasing sequence \( \{ D_n \} \) of preclosed (resp. semi-closed) subsets of \( X \) such that \( A(f - g, 3^{-n+1}) \subseteq D_n \) and

\[
\bigcap_{n=1}^{\infty} D_n = \emptyset.
\]

But by Lemma 3.2, the pair \( A(f - g, 3^{-n+1}) \) and \( X \setminus D_n \) of semi-open (resp. preopen) and preopen (resp. semi-open) subsets of \( X \) can be completely separated by contra-\( \alpha \)-continuous functions. Hence by Theorem 2.2, there exists a contra-\( \alpha \)-continuous function \( h \) defined on \( X \) such that \( g < h < f \), i.e., \( X \) has the weakly \( c\alpha \)-insertion property for (cpc, csc) (resp. (csc, cpc)).

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