AdS Holography in the Penrose Limit

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Abstract

We study the holographic description of string theory in a plane wave space-time by taking the Penrose limit of the usual AdS/CFT correspondence. We consider three-point functions with two BMN operators and one non-BMN operator; the latter should go over to a perturbation of the dual CFT. On the string side we take the Penrose limit of the metric perturbation produced by the non-BMN operator, and the BMN state propagates in this perturbed background. The work of Lee, Minwalla, Rangamani, and Seiberg shows that for chiral operators the AdS three-point functions agree with those calculated in the free gauge theory. However, when this is reduced to an effective plane wave amplitude by truncating the amplitude to propagate from the AdS boundary to the Penrose geodesic, we find a puzzling mismatch. We discuss possible resolutions, and future directions.
1 Introduction

Recently, the plane wave (Penrose) limit of the AdS/CFT correspondence has received much attention \[1\]. In the field theory, this amounts to selecting a single generator \(J\) of the \(SO(6)\) \(R\)-symmetry, and considering only operators that have a large charge (and small \(\Delta - J\)) with respect to this symmetry. On the supergravity side of the correspondence, one takes a limit that focuses in on a single null geodesic, yielding the plane wave metric \[2\]. There exists an exact light-cone quantization of string theory in this background \[3\] and a dictionary relating the string states to the CFT operators. Furthermore, some quantities can be expanded in terms of constants which are simultaneously small on both sides of the correspondence \[1, 4, 5, 6\]. Plane wave physics is of particular interest because of the transparent nature of the string/field theory duality; it yields a dictionary between objects which have manifestly similar structures. In particular, when analyzing the spectrum, one can see in the field theory where the “string” arises.

In the full AdS/CFT duality, there remain important problems of principle. One is the gauge theory origin of the approximate locality in the bulk spacetime. Another, related to this, is a more complete understanding of the quantum mechanics of black holes (for recent discussion see ref. \[7\]). A third is the extension of the duality to more general spacetimes, in particular those with interesting cosmologies. One might hope that the more transparent duality of the plane wave limit would provide some ideas in these directions, but this has not yet been realized. Indeed, there seems to be an important ingredient missing from the plane wave picture. Currently string theory in the plane wave background is understood as being dual to a limit of the gauge theory, but it is widely anticipated that there is some smaller CFT, not yet identified, which is precisely dual to the string side.

Our goal in this paper is to obtain some insight into this question by looking at holographic observables that are analogous to those studied in the full AdS/CFT correspondence \[8\]. These would correspond to local operators in the dual CFT, and would map to perturbations of the boundary conditions in the plane wave spacetime. An understanding of the properties of these local operators may give some insight into the nature of the CFT itself.

There has been some general discussion of holography in this context, focussing on the geometry of the plane wave spacetime \[9\], but no complete understanding has
emerged. We will approach the question in a slightly different way, by starting with a CFT matrix element and its dual in the \( AdS_5 \times S^5 \) theory, where the holographic dictionary is known, and taking the Penrose limit. It is natural to hold fixed the dimension and quantum numbers of the Hamiltonian perturbation as we take the limit, so the simplest matrix element to consider would be two BMN (large-\( J \)) operators, to prepare the initial and final string states, and one non-BMN operator corresponding to the perturbation.

The inclusion of a non-BMN operator is consistent with the arguments of refs. [10, 11] that in order to take the plane wave limit, all BMN operators must be taken to \( t = \pm \infty \), and so correspond to incoming and outgoing strings rather than perturbations of the Hamiltonian. Thus the non-BMN operators cannot naturally be regarded as limits of BMN operators, but have a different interpretation. Most of the previous work on three-point functions, beginning with refs. [12], has focused on three BMN operators, corresponding to a three-string interaction.

We begin with the three-point AdS/CFT calculation of Lee, Minwalla, Rangamani, and Seiberg (LMRS) [13] and adapt it to the plane wave limit. That is, we start with the LMRS result for the metric perturbation corresponding to the insertion of a chiral operator, and take its limit. We then calculate the amplitude for the string to propagate through this perturbed background.\(^1\) For a three-point function the position dependence is determined entirely by conformal invariance, so the normalization is the only nontrivial aspect, and we focus on this.

Of course, the results of LMRS already show that for three chiral operators the duality works, in that the normalization in the free gauge theory exactly matches that in the AdS calculation, and we reproduce this agreement in our framework. However, we argue that the agreement actually presents a puzzle. Local operators in the dual to the plane wave geometry should differ from those in the full \( AdS_5 \times S^5 \), in that the propagation from the \( AdS_5 \times S^5 \) boundary to the limiting geodesic is not part of the plane wave theory and this propagation amplitude should be truncated. When this is done the form of the amplitude changes so that it no longer has an obvious match to the dual theory. This may be related to problems found by Spradlin and Volovich in the corresponding four-point amplitude (unpublished).

In section two we discuss the field theory amplitude. We consider the three point amplitude

\(^1\) We work to leading order in the effective string coupling \( J^2/N \), so we are considering only planar world-sheets.
function with one BMN operator mapped to negative infinity in global time and the other BMN operator to the positive infinity, thus acting as creation and destruction operators for the string state; the non-BMN operator is fixed at finite time. In section three we discuss the string amplitude. We first calculate the metric perturbation that corresponds to the non-BMN operator via the AdS/CFT dictionary and take its Penrose limit. We then look at the light-cone quantized string action in this perturbed background, and calculate the amplitude for the initial string to propagate through the perturbed background to the given final state. Our main calculational result is to reproduce the result of LMRS for three chiral operators, but in a framework that readily extends to nonchiral BMN operators and to amplitudes with additional non-BMN operators. In the final section we discuss the discrepancy that we have found, and suggest several possible resolutions. We also discuss future directions.

2 Field theory calculation

We first review the calculation of LMRS. In the $D = 4, N = 4, SU(N)$ supersymmetric Yang-Mills theory, the operators that we are studying will all be of the form

$$O^I = \kappa C_{i_1 \ldots i_k}^I \Tr(\phi^{i_1} \phi^{i_2} \cdots \phi^{i_k}) \equiv \kappa C_I^I(\phi). \tag{2.1}$$

The six scalars $\phi^i$ form the fundamental representation of the $SO(6)$ $R$-symmetry of the field theory. Here $C_I^I$ is a totally symmetric traceless rank $k$ tensor of $SO(6)$, normalized such that

$$\langle C_{i_1 \ldots i_k}^I C_{i_1' \ldots i_k'}^{I'} \rangle = \delta^I_{I'} \delta_{i_1 i_1'} \delta_{i_2 i_2'} \cdots \delta_{i_k i_k'}. \tag{2.2}$$

The constant $\kappa = (2\pi)^k/(g_{YM}^2 N)^{k/2} \sqrt{k}$ is fixed to normalize the two-point function,

$$\langle O^I(x_1) O^{I_2^*}(x_2) \rangle = \frac{\delta^I_{I_2}}{|x_{12}|^{2k}}. \tag{2.3}$$

The three-point function is then

$$\langle O^I(x_1) O^{I_2}(x_2) O^{I_3}(x_3) \rangle = \frac{1}{N} \frac{\sqrt{k_1 k_2 k_3}}{|x_{12}|^{2\alpha_3} |x_{23}|^{2\alpha_1} |x_{13}|^{2\alpha_2}} \langle C_I^I C_I^{I_2} C_I^{I_3} \rangle, \tag{2.4}$$

where $k_1, k_2, k_3$ are the ranks of $C_I^I, C_I^{I_2}, C_I^{I_3}$. The $\alpha_i$ are defined cyclically such that $\alpha_1 = (k_2 + k_3 - k_1)/2$; $\alpha_1$ is the number of contractions between $O^{I_2}$ and $O^{I_3}$, and so on. The position-dependence of the two- and three-point functions is determined by conformal symmetry.
To take the BMN limit, we decompose $SO(6) \to SO(2) \times SO(4)$. The combinations $Z = (\phi_5 + i\phi_6)\sqrt{2}$ and $\bar{Z} = (\phi_5 - i\phi_6)/\sqrt{2}$ are eigenvectors of the $SO(2)$ generator. The three operators that we will consider are

$$O^I = \tilde{k}_I \tilde{C}^I_{i_1 \ldots i_k} \text{Tr}(Z^J \phi^{i_1} \ldots \phi^{i_k} + \text{permutations}), \quad (k_1) \text{ terms},$$

$$O^J = \tilde{k}_J \tilde{C}^J_{i_1 \ldots i_k} \text{Tr}(Z^J \phi^{i_1} \ldots \phi^{i_k} + \text{permutations}), \quad (k_2) \text{ terms},$$

$$O^K = \tilde{k}_K \tilde{C}^K_{i_1 \ldots i_k} \text{Tr}(\phi^{i_1} \ldots \phi^{i_k}) \ . \quad (2.5)$$

The $\tilde{C}^I$ are totally symmetric traceless $SO(4)$ tensors of rank $\tilde{k}$, normalized $\langle C^I C^{I*} \rangle = \delta^{I_1I_2}$. We have defined

$$\tilde{k}_1 = k_1 - J \ , \quad \tilde{k}_2 = k_2 - J \ , \quad \tilde{k}_3 = k_3 \ . \quad (2.6)$$

Relating the normalizations of the $SO(6)$ and $SO(4)$ tensors gives

$$\tilde{k}_1 = \kappa_1 \left(\frac{k_1}{J}\right)^{-1/2} \ , \quad \tilde{k}_2 = \kappa_2 \left(\frac{k_2}{J}\right)^{-1/2} \ , \quad \tilde{k}_3 = \kappa_3 \ . \quad (2.7)$$

Define also $\tilde{\alpha}_1 = (\tilde{k}_2 + \tilde{k}_3 - \tilde{k}_1)/2$ and permutations of this.

In the three-point function there are $\left(\alpha_J^3\right)$ permutations that give nonzero contractions between the $Z$ fields, and so

$$\langle C^{I_1} C^{I_2} C^{I_3} \rangle = \left(\frac{\alpha^3}{J}\right) \left(\frac{k_1}{J}\right)^{-1/2} \left(\frac{k_2}{J}\right)^{-1/2} \langle \tilde{C}^{I_1} \tilde{C}^{I_2} \tilde{C}^{I_3} \rangle \ . \quad (2.8)$$

Then

$$\langle O^{I_1}(x_1) O^{I_2}(x_2) O^{I_3}(x_3) \rangle = \frac{1}{N} \frac{(\tilde{\alpha}_3 + J)! \sqrt{\tilde{k}_1! \tilde{k}_2! \tilde{k}_3}}{\tilde{\alpha}_3! \sqrt{(k_1 + J - 1)!(k_2 + J - 1)!}} \frac{(\tilde{C}^{I_1} \tilde{C}^{I_2} \tilde{C}^{I_3})}{|x_{12}|^{2(J+\tilde{\alpha}_3)} |x_{23}|^{2\tilde{\alpha}_1} |x_{13}|^{2\tilde{\alpha}_2}} \ . \quad (2.9)$$

Taking the large-$J$ limit with the tilded quantities held fixed gives

$$\langle O^{I_1}(x_1) O^{I_2}(x_2) O^{I_3}(x_3) \rangle \approx J^{-\frac{1}{2}+\frac{1}{2} \tilde{k}_3} \sqrt{\tilde{k}_1! \tilde{k}_2! \tilde{k}_3} \frac{(\tilde{C}^{I_1} \tilde{C}^{I_2} \tilde{C}^{I_3})}{|x_{12}|^{2(J+\tilde{\alpha}_3)} |x_{23}|^{2\tilde{\alpha}_1} |x_{13}|^{2\tilde{\alpha}_2}} \ . \quad (2.10)$$

To facilitate the comparison to string theory we rewrite this vacuum three-point function as an operator matrix element. We first convert from the $R^4$ description of the
boundary to $S^3 \times R$ via $x_i = e^{\tau_i} \hat{x}_i$. Here $\tau_i$ is Euclidean global time, and $\hat{x}_i$ is a point on the unit $S^3$. The chiral primaries $O^I$ each pick up a conformal factor, $O^I = e^{k\tau}O'^I$, so that for $-\tau_1 \gg 1$ and $\tau_2 \gg 1$ the position dependent factors in eq. (2.10) become $e^{-k_1(\tau_3 - \tau_1) - k_2(\tau_2 - \tau_3)}$. Similarly the two-point function becomes

$$\langle O'^I_2(\tau_2, \hat{x}_2)O'^I_1(\tau_1, \hat{x}_1) \rangle = \delta^{I_1I_2}e^{-k(\tau_2 - \tau_1)}$$

so that $O^I(\tau, \hat{x})|0\rangle \rightarrow e^{k\tau}|I\rangle$. Then

$$\langle I_2|O^I_3(\tau_3, \hat{x}_3)|I_1\rangle \approx \frac{J^{1-\frac{1}{2}\tilde{k}_3}}{N}\sqrt{\tilde{k}_1\tilde{k}_2\tilde{k}_3}\frac{1}{\alpha_3!}e^{-(k_1-k_2)\tau_3}\langle \tilde{C}I_1\tilde{C}I_2\tilde{C}I_3\rangle.$$ (2.12)

Note that this can be Wick-rotated readily from Euclidean to Minkowski global time.

Since the field theory amplitude does not involve the 't Hooft parameter (it has been scaled out), the BMN limit is simply large $J$ with fixed $\tilde{C}$ and $J^2/N$. In the final form (2.12), $J$ enters only as a power, with the exponent depending on $\tilde{k}_3$. Thus to obtain a finite limit we must rescale $O^I_3$ by $J^{\frac{1}{2}\tilde{k}_3-1}$. Note that for the mass operator, with $\tilde{k}_3 = 2$, no rescaling is needed: this can couple to the $SO(4) \phi^i$ without suppression (aside from the $1/N$ from the normalization of single-trace operators). For each increase of $\tilde{k}_3$ by two, one additional $SO(4)$ propagator must appear in an adjacent cyclic position, costing a factor of $J$. The factor $\frac{1}{N}$ is the suppression expected from an interaction of three strings, the tensor contraction follows from $SO(4)$ invariance, and the $\tau$-dependence from the Heisenberg equation of motion.

### 3 String theory calculation

According to the AdS/CFT dictionary, chiral operators on the boundary correspond to perturbations of the supergravity fields in the bulk. Thus the matrix element (2.12) represents the amplitude for a single string to make a transition while propagating through the perturbed background produced by the non-BMN operator $O^I_3$. In section 3.1 we obtain the form of the perturbed background, and in section 3.2 we use light-cone string theory to calculate the transition amplitude.
3.1 The metric perturbation

We first review the AdS/CFT dictionary \[8\]. We begin with coordinates in which the Euclidean AdS\(_5\) metric is
\[
\text{d}s^2 = R^2 \frac{\text{d}x_0^2 + \text{d}x \cdot \text{d}x}{x_0^2} .
\] (3.1)
Each scalar chiral operator \(O^I\) gives rise to a scalar field \(\phi^I\) on AdS\(_5\), with the dictionary \[14\]
\[
O^I(x) = c_I \lim_{x_0 \to 0} x_0^{-\Delta_I} \phi^I(x, x_0) .
\] (3.2)
That is, the fluctuating part of the field \(\phi^I\) has the normalizable behavior \(x_0^{\Delta_I}\); after extracting this \(x_0\)-dependence, the boundary limit of the bulk field operator is the boundary operator. We take the fields \(\phi^I\) to be normalized canonically,
\[
S = \frac{1}{2} \sum_I \int_{\text{AdS}_5} \text{d}^4x \text{d}x_0 \sqrt{-g} (\partial_\mu \phi^I \partial^\mu \phi^I + m^2_I \phi^I) ,
\] (3.3)
where \(m^2_I = \Delta_I (\Delta_I - 4)/R^2\). We take the operators \(O^I\) to have normalized two-point functions (2.3); for the present discussion we take a real basis for the fields and operators. These two normalizations determine the coefficient \(c_I\). The canonical scalar propagator satisfies
\[
\lim_{x'_0 \to 0} x'_0^{-\Delta} \langle 0|\phi(x, x_0)\phi(x', x_0')|0 \rangle = A(\Delta) \left[ \frac{x_0}{x_0^2 + (x - x')^2} \right]^\Delta ,
\] (3.4)
where
\[
A(\Delta) = \frac{\Delta - 1}{2\pi^2 R^3} .
\] (3.5)
(For \(\text{AdS}_{d+1}\), \(A(\Delta) = \Gamma(\Delta)/2\pi^{d/2}\Gamma(\Delta + 1 - d/2)R^{d-1}\).) It follows that \(c_I^{-2} = A(\Delta_I)\), and that
\[
\langle 0|\phi^I(x, x_0)O^I(x')|0 \rangle = A(\Delta_I)^{1/2} \left[ \frac{x_0}{x_0^2 + (x - x')^2} \right]^{\Delta_I} .
\] (3.6)
To take the Penrose limit it will be convenient to put the \(\text{AdS}_5 \times S^5\) metric in the coordinates
\[
\text{d}s^2 = R^2 \left[-(1 + u^2)\text{d}t^2 + d\vec{u} \cdot d\vec{u} - \frac{u^2 \text{d}u^2}{1 + u^2} \right] + R^2 \left[(1 - v^2)\text{d}\psi^2 + d\vec{v} \cdot d\vec{v} + \frac{v^2 \text{d}v^2}{1 - v^2} \right] .
\] (3.7)
The $AdS_5$ part is related to the earlier metric \[^3.1\] by a change of coordinates followed by a Wick rotation. The coordinate change is

$$e^\tau = (\bar{x}_0^2 + x^2)^{1/2}, \quad \bar{u} = \bar{x}/x_0 .$$ \hfill (3.8)

It is then straightforward to Wick rotate $\tau \to (1 - i\epsilon)it$ on both sides of the duality. The duality dictionary \[^3.6\] becomes

$$\langle 0 | T \phi^I(t, \bar{u}) O^J(t', \hat{x}') | 0 \rangle = \frac{A(\Delta_I)^{1/2}}{2^{\Delta_I} \left( \sqrt{1 + \bar{u}^2 \cos[(1 - i\epsilon)(t - t')] - \bar{u} \cdot \hat{x}'} \right)^{\Delta_I}}.$$ \hfill (3.9)

We have included a factor $e^{i\Delta_I t'}$ from the conformal transformation of $O^I$ in going from $R^4$ to $R \times S^3$.

LMRS express the metric perturbation associated with a chiral operator in terms of scalar fields $s^I$ on $AdS_5$. Following the conventions in LMRS, these are related to canonically normalized fields by

$$s^I = \frac{2\left( k_I - 3 \right)/2}{N\sqrt{k_I(k_I - 1)}} \phi^I ;$$ \hfill (3.10)

note that $k_I = \Delta_I$. The perturbation takes the form $h_{\mu\nu} = Y^I h^I_{\mu\nu}$ and $h_{\alpha\beta} = Y^I h^I_{\alpha\beta}$, where the indices $\mu, \nu, \ldots$ correspond to the $AdS_5$ space, and the indices $\alpha, \beta$ correspond to the $S_5$ space, and the $S^5$ harmonics $Y^I$ are defined in the appendix. The tensor perturbations are related to the scalars $s^I$ by

$$h^I_{\mu\nu} = -\frac{6k_I}{5} s^I g_{\mu\nu} + \frac{4R^2}{5(k_I + 1)} (5\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\sigma \nabla^\sigma) s^I$$

$$= -\frac{2k_I(k_I - 1)}{(k_I + 1)} g_{\mu\nu} s^I + \frac{4R^2}{(k_I + 1)} \nabla_\mu \nabla_\nu s^I ,$$

$$h^I_{\alpha\beta} = 2k_I s^I g_{\alpha\beta} .$$ \hfill (3.11)

Combining eqs. \(^3.5\), \(^3.9\), \(^3.10\), and \(^3.11\) gives the metric perturbation.

The perturbation becomes much simpler in the Penrose limit. Setting $t = x^+ + x^-/R^2$, $\psi = x^+ - x^-/R^2$, $\bar{u} = \bar{w}/R$, $\bar{v} = \bar{y}/R$, and sending $R \to \infty$, the general scalar perturbation \(^3.9\) becomes

$$\langle 0 | T \phi^I(t, \bar{x}) O^J(t', \hat{x}') | 0 \rangle \to \frac{A(\Delta_I)^{1/2}}{2^{\Delta_I} \cos^\Delta_I [(1 - i\epsilon)(x^+ - t')] .} \hfill (3.12)$$
This leading behavior depends on the bulk position only through $x^+$. Also,

$$h_{MN}dx^Mdx^N \rightarrow (h_{tt} + h_{\psi\psi})dx^+dx^+ .$$

(3.13)

For the non-BMN operator, the $SO(6)$ tensor $C^I$ depends only on the $SO(4)$ directions, and so $Y^I \rightarrow R^{-k_3}C^I(y)$. Assembling all factors,

$$h_{++} = h_{tt} + h_{\psi\psi} = \frac{4R^2}{k_1 + 1}(k_1^2 + \partial_t^2)s^I,$$

$$= \frac{R^{2-k_3}(k_1 + 1)\sqrt{k_1}}{N2^{k_3/2}\cos^{k_1/2}[(1+i)(x^++t)]}C^I(y) .$$

(3.14)

Notice that the perturbation, for which $k_1 = \tilde{k}_3$, has the same $R^{2-\tilde{k}_3}$ scaling as the gauge theory amplitude (2.12), since $J \propto R^2$. Thus to obtain a finite limit on either side we must scale the non-BMN operator by $R^{-k_3}$, as well as a factor of $N$ to offset the string coupling. The case $\tilde{k}_3 = 2$, whose limit is $1/N \times$ finite, corresponds to a perturbation quadratic in the $y$, and so it is simply a time- and direction-dependent modulation of the quadratic terms in the plane wave metric. Higher $\tilde{k}_3$ correspond to nonlinearities which scale away in the plane wave limit. Notice that the metric perturbation has poles at infinitely many values of $x^+$. This strange dependence is due to the propagation of the bulk field from the $AdS_5$ boundary to the plane wave geodesic and will be discussed further later.

This is the only metric perturbation that we will need for our calculation. However, it is instructive to consider the form that a metric perturbation derived from a string-like operator would take. Consider, for example, the perturbation that would correspond to $O^{I_1}$ at the boundary point $(z, \vec{x}) = (0, \vec{0})$, which is the negative infinity of the global time. In this case,

$$\langle 0| s^{I_1}(t, \vec{x}) | I_1 \rangle = \frac{2^{k_1/2}(k_1 + 1)}{4N\sqrt{k_1}}e^{-ik_1 t}(1 + u^2)^{-k_1/2} .$$

(3.15)

The spherical harmonic is

$$Y^{I_1}(\psi, \vec{v}) = \left(\frac{k_1}{J}\right)2^{-J/2}e^{i\psi}J/2C^{I_1}(i\vec{v})$$

(3.16)

Now taking the Penrose limit with $\tilde{k}_1 = k_1 - J$ and $p_- = -2J/R^2$ fixed, the scalar perturbation becomes

$$\langle s^{I_1} \rangle Y^{I_1} \rightarrow \frac{|p_-|^{\tilde{k}_1/2}\sqrt{J}}{4N\sqrt{k_1}}e^{-ip_-x^- - ik_1 x^+ - |p_-|(w^2+y^2)/4}C^{I_1}(y) .$$

(3.17)
Note the gaussian term which restricts the perturbation to the plane \( \vec{w} = \vec{y} = 0 \). Actually, to restrict to the neighborhood of the limiting geodesic \( x^- = 0 \), we would need to superpose different \( J \)-states to form a wavepacket in \( x^- \); it will not be necessary to do this explicitly.

This scalar perturbation scales as \( R^{-2} \sim J^{-1} \). There is the explicit \( \sqrt{J} \), an implicit \( \sqrt{J} \) from the plane wave behavior in \( x^- \), and a further \( R^{-4} \) since \( N^{-1} \propto \sqrt{G_N} R^{-4} \).

The metric perturbation (3.11) contains an additional factor or \( R^2 \sim J \) and so is finite in the Penrose limit (there are terms in \( h_{++} \) which scale as \( J \), but these cancel). This is expected because this is the field corresponding to a normalized string state. We could complete the calculation of the three-point function directly as in LMRS, by evaluating the trilinear supergravity amplitude (see also ref. [15]). However, this would only apply to chiral operators \( |I_1 \rangle \). We will instead adopt a first-quantized description which applies to general BMN operators.

Note the dependence of the perturbation (3.17) on \( x_- \). If we were to try to describe string propagation in this supergravity background (which is not necessary to the calculation), \( p_- \) would no longer be a conserved quantity, and in the light-cone description the three-string interaction would change the string length.

### 3.2 The string amplitude

The three-point function (2.12) is the amplitude for a string in some initial state corresponding to \( O^I_1 \) to propagate through the perturbed metric produced by \( O^I_3 \) and come out as a string in the state produced by \( O^I_2 \). This is particularly simple in light-cone gauge, since the perturbation just affects \( g_{++} \):

\[
g_{++} = -\frac{1}{2} z^2 + h_{++} , \quad (3.18)
\]

where \( h_{++} \) is given by eq. (3.14) with \( k \to \tilde{k}_3 \) and \( t' \to t_3 \). Also, \( (z^1, \ldots, z^8) = (y^1, \ldots, y^4, w^1, \ldots, w^4) \). The action can be written down immediately,

\[
S = \frac{1}{4\pi\alpha'} \int_0^{\pi\alpha'|p_-|} d\sigma \left[ \dot{z}^2 - \dot{z}'^2 - z^2 + h_{++}(x^+, \vec{y}) \right] + \text{fermionic} . \quad (3.19)
\]

We will only consider bosonic excitations, so we do not need the fermionic terms.

Expanding in the perturbation we have

\[
\langle I_2 | O^{I_3}(x_3) | I_1 \rangle = \frac{i}{4\pi\alpha'} \int_{-\infty}^{\infty} dx^+ \int_{0}^{2\pi\alpha'|p_-|} d\sigma \langle I_2 | h_{++}(x^+, \vec{y}) | I_1 \rangle , \quad (3.20)
\]

\( ^2 \)Roughly speaking, the range of \( x^- \) is \( O(R^2) \) so we must divide by \( R \) to obtain a normalized state.

10
where the matrix element on the right-hand side is in the light-cone one-string Hilbert space. Expanding the fields in creation and annihilation operators, the perturbation changes the oscillator state of the incoming string. Using

\[ z^i = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{\omega_n |p_-|}} \left[ a_n \left( e^{-i\omega_n x^+ + im/|p_-|} + a_n^\dagger e^{i\omega_n x^+ - im/|p_-|} \right) \right], \quad (3.21) \]

the normalized initial and final states are

\[ |I_a\rangle = \frac{1}{\sqrt{k_a!}} \tilde{\alpha}_1^{i_1} \cdots \tilde{\alpha}_a^{i_a} a_{00}^{i_0} \cdots a_{0a}^{i_a} |0; p_-\rangle, \quad a = 1, 2. \quad (3.22) \]

Because we are considering strings dual to chiral primaries in the CFT, only zero modes are excited. Thus in the perturbation we need also keep only the zero modes of the \( z \) fields.

In all,

\[ \langle I_2|h_{++}(x^+, \vec{y})|I_1\rangle = \frac{\tilde{k}_3 + 1}{N^{k_3/2} |p_-|^{k_3/2} \sqrt{k_1!k_2!}} \langle 0|\tilde{C}^{I_2}(\tilde{a}_0)\tilde{C}^{I_3}(\tilde{a}_0 e^{-ix^+} + \tilde{a}_0^\dagger e^{ix^+})\tilde{C}^{I_1}(\tilde{a}_0^\dagger)|0\rangle \]

\[ \cos(k_3 + 2[(1 - i\epsilon)(x^+ - t_3)]) \]

\[ e^{-i(k_1 - k_2)x^+} \langle \tilde{C}^{I_1} \tilde{C}^{I_2} \tilde{C}^{I_3} \rangle. \quad (3.23) \]

The matrix element is

\[ \frac{k_1!k_2!k_3!}{\tilde{\alpha}_1!\tilde{\alpha}_2!\tilde{\alpha}_3!} \]

\[ e^{-i(k_1 - k_2)x^+} \langle \tilde{C}^{I_1} \tilde{C}^{I_2} \tilde{C}^{I_3} \rangle. \quad (3.24) \]

The time integral is then

\[ i \int_{-\infty}^{\infty} dx^+ \frac{e^{-i(k_1 - k_2)x^+}}{\cos(k_3 + 2[(1 - i\epsilon)(x^+ - t_3)]} = \frac{2^{k_3 + 1} \tilde{\alpha}_1!\tilde{\alpha}_2!}{(k_3 + 1)!} e^{-i(k_1 - k_2)x^+} \]

\[ (3.25) \]

Collecting all factors, we have

\[ \langle I_2|O^{I_3}(t_3, \vec{x}_3)|I_1\rangle = \frac{(|p_-| R^2/2)^{1-k_3/2} \sqrt{k_1!k_2!k_3}}{\tilde{\alpha}_3!} \langle \tilde{C}^{I_1} \tilde{C}^{I_2} \tilde{C}^{I_3} \rangle e^{-i(k_1 - k_2)t_3} \]

\[ (3.26) \]

This agrees with the field theory result \((2.12)\).

## 4 Discussion

We should emphasize that the agreement between the CFT amplitude \((2.12)\) and the string amplitude \((3.26)\) was a foregone conclusion. LMRS \([13]\) have already shown that,
due to supersymmetric nonrenormalization, the free CFT and AdS three-point functions match exactly. We are simply repeating their calculation in a different framework that is adapted to the plane wave limit and so can be applies also arbitrary nonchiral BMN state, but for external supergravity states our calculation must be equivalent to theirs.³

However, if we attempt to interpret this result in terms of a holographic dual to the plane wave theory, things are not so simple. We would like to identify operators that are local in time: a local operator at time $t'$ should correspond to a perturbation of the plane wave spacetime that goes to $\delta(x^+ - t')$ as we approach the boundary. Even without being too specific about what we mean by the boundary of the plane wave spacetime, it is clear that this is not the case for the perturbation (3.14) because the time dependence

$$\cos^{-k_1-2}(1 - i\epsilon)(x^+ - t')$$

factors out, and this does not become a delta function no matter how we define and approach the boundary.

Of course, we began with a local operator in the AdS picture. The time dependence (4.1) arises from propagation from the AdS boundary to the limiting geodesic. In particular there are poles when $x^+ - t'$ is an odd multiple of $\pi/2$. This represents the minimum propagation time from the AdS boundary to the geodesic, plus multiple reflections from the AdS boundary. This propagation is not part of the plane wave theory: the factor (4.1) should be replaced with a delta-function to obtain a local operator in the plane wave theory. In other words, the plane wave spacetime is essentially the tangent space to the limiting geodesic, so that even its boundary is in an infinitesimal neighborhood of the geodesic, and contributions arising outside this neighborhood should be renormalized away.

Replacing the factor (4.1) by $\delta(x^+ - t)$ has a puzzling effect on the amplitude: the factor

$$2^{k_3+1}\tilde{\alpha}_1!\tilde{\alpha}_2!/(\tilde{k}_3 + 1)!$$

from the time integral (3.25) is gone, and the result no longer agrees with the free field theory calculation. Further, noting that the $\tilde{\alpha}_{1,2}$ depend jointly on the quantum

³To be precise the supergravity description is strictly valid only for $gN/J^2 \ll 1$, otherwise there is a large invariant built from the string momentum and spacetime curvature. However, our string calculation shows that there are no corrections involving this parameter.
numbers of all three operators, agreement cannot be restored simply by renormalization of the operators.

It is interesting to examine in more detail the role that this factor plays in the calculation. Figure 1 shows the diagrams that contribute to the CFT and world-sheet calculations. We see that they are the same diagram except that the CFT calculation has $Z$ contractions and counts only planar diagrams, whereas the world sheet calculation counts both planar and non-planar graphs but has no $Z$ contractions. The similarities follow the usual plane-wave intuition: the insertion of $\phi_i$ fields into the $\text{Tr} Z^J$ correspond to $a_0^{ij}$ excitations of the string.

The following is then a summary of how the two sides of the calculation come to be equal:

$$\left(\sqrt{(J + k_2)(J + k_1)}k_3\right) \left(\frac{J^{-k_3/2}\sqrt{k_1!k_2!}}{\tilde{k}_3!}\right) \left(\frac{\tilde{k}_1!\tilde{k}_2!\tilde{k}_3!}{\tilde{\alpha}_1!\tilde{\alpha}_2!\tilde{\alpha}_3!}\right) \left(\frac{\tilde{\alpha}_1!\tilde{\alpha}_2!(k_3 + 1)!}{\tilde{\alpha}_3!}\right) \left(\frac{1}{\sqrt{k_1!k_2!}}\right) \left(\frac{1}{\sqrt{\tilde{k}_1!\tilde{k}_2!}}\right).$$

If we focus on factors of $\tilde{\alpha}_i!$, which are the only ones that cannot be absorbed into the
normalizations of the states, then the field theory has only a factor of $\alpha_3^{-1}$, which arises from the $\binom{\tilde{\alpha}_3 + J}{J}$ permutations of the $Z$ fields among the $\tilde{\alpha}_3 + J$ contractions between $O_1$ and $O_2$. The world-sheet calculation has a factor of $\tilde{\alpha}_1!\tilde{\alpha}_2!\tilde{\alpha}_3!$ from the zero-mode contraction factor. This factor accounts for the planar and non-planar contractions of the zero mode operators, and there is no corresponding factor from the planar contractions in the field theory. Rather, the time integral brings in an offsetting factor of $\tilde{\alpha}_1!\tilde{\alpha}_2!$. For holographic plane-wave operators defined as above, this factor is absent.

Now let us try to understand the disagreement between the free CFT calculation and the amplitudes obtained with the natural holographic operators:

1) The most sweeping possibility is that there is a problem with our basic premise that there is a holographic dual to the plane wave string theory. We see no reason to come to such a strong conclusion limiting the generality of the holographic principle, given the more conservative possibilities below.

2) It may be that the correct holographic observables are not the local operators that we have attempted to define. Indeed, there are other contexts where a holographic dual exists but does not have local operators [16]. Further, it has been argued that the geometry of the plane wave spacetime is such that the natural observable is a two-dimensional $S$-matrix in the $x^\pm$ directions [17], rather than the AdS-like observables that we consider. However, one of the authors of ref. [17] has informed us that there are difficulties with this proposal. Moreover, we note that the local operators that we consider have a natural definition on the supergravity side, in terms of metric perturbations proportional to delta functions in $x^+$, so this suggests that they are naturally defined in the dual theory as well; in the cases [16] without local operators, the problem was already visible on the supergravity side. In a sense our amplitudes are the $S$-matrix: although the perturbations are instantaneous in the light-cone time $x^+$, they are propagating on null lines in spacetime — this is the usual pathology of the massless $S$-matrix in the light-cone in $1 + 1$ dimensions.

3) The above discussion of combinatorics suggests that the dual theory might not be a planar theory. This is intriguing, but seems difficult to implement given that the construction of strings as BMN states seems to depend in an essential way on the planarity.

4) The simplest explanation is that the calculations do not match because there are corrections from interactions on the CFT side. These would be the analog of $gN$
corrections in the gauge theory, but now in whatever is the smaller theory corresponding to the plane wave spacetime. The field theory calculations are valid for small $gN$ and the string calculations are valid for large $gN$. The absence of $gN$ corrections in other parts of the plane wave duality is somewhat miraculous [5], and may be due in part to the large symmetry of the problem [18]. There is no strong reason to expect this to extend to the less BMN-like observables that we consider. On the other hand, it is not clear how to reconcile this with the nonrenormalization of the three point function that was found in the full gauge theory [13].

Spradlin and Volovich have independently considered the four-point analog of our calculation (unpublished), comparing correlation functions of two BMN operators and two non-BMN operators and compared them with second-order perturbations of the world-sheet theory. Here the position-dependence is not fully determined by conformal invariance. They also find a discrepancy between the field theory and string theory results, suggesting the need for $gN$ corrections.

Finally, we note again that metric perturbations with a delta-function dependence in $x^+$ do seem to be natural observables. These would correspond to the operators of a quantum mechanical theory, consistent with the conformal geometry of the plane wave spacetime [9]. Also, while we have considered only operators with fixed $J$, it may be possible by appropriate renormalization to define also local operators (at finite times) with large $J$.

There are various interesting extensions of our calculation. The limiting procedure that we have considered gives operators that depend on only four of the eight $z$-fields. By taking instead spherical harmonics on $S^3$ we can obtain operators with arbitrary $z$-dependence. The extension to nonzero modes of the string states is straightforward, and may give some additional hint as to the origin of the discrepancy that we have found. It would be interesting, but more challenging, to consider also perturbations corresponding to excited string states. Finally, we emphasize again the problem of determining the nongravitational dual to string theory in the plane wave spacetime.

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A Spherical harmonics

Here we define the spherical harmonics that appeared in section 3. We also repeat the calculation of the matrix element (2.8) in a slightly different way. We continue the convention that the reduced (3-sphere) objects are denoted with a tilde. They are normalized such that

\[ C_{i_1...i_k} x_{i_1} \cdots x_{i_k} = x^k Y^I(x_1, ..., x_6) \]  

(A.1)

and

\[ \tilde{C}_{i_1...i_k} x_{i_1} \cdots x_{i_k} = x^k \tilde{Y}^I(x_1, ..., x_4) \]  

(A.2)

With this normalization,

\[ \frac{1}{\pi^3} \int_{S^5} Y^{I_1} Y^{I_2} = \frac{\delta^{I_1 I_2}}{2^{k-1}(k+1)(k+2)}, \]  

(A.3)

\[ \frac{1}{2\pi^2} \int_{S^3} \tilde{Y}^{I_1} \tilde{Y}^{I_2} = \frac{\delta^{I_1 I_2}}{2^k(k+1)}. \]  

(A.4)

Defining

\[ x_5 = x \cos \theta \cos \psi, \quad x_6 = x \cos \theta \sin \psi, \]  

(A.5)

the spherical harmonics are related

\[ Y^{I_1} = 2^{-J/2} \sqrt{\frac{(J + \tilde{k}_1)!}{J! \tilde{k}_1!}} e^{i J \psi} \cos J \theta \sin^{\tilde{k}_1 \theta} \tilde{Y}^{I_1}, \]

\[ Y^{I_2} = 2^{-J/2} \sqrt{\frac{(J + \tilde{k}_2)!}{J! \tilde{k}_2!}} e^{-i J \psi} \cos J \theta \sin^{\tilde{k}_2 \theta} \tilde{Y}^{I_2}, \]

\[ Y^{I_3} = \sin^{\tilde{k}_3 \theta} \tilde{Y}^{I_3}. \]  

(A.6)

Then

\[ \langle C^{I_1} C^{I_2} C^{I_3} \rangle = (\Sigma/2 + 2)!2^{\Sigma/2-1} \alpha_1! \alpha_2! \alpha_3! k_1!k_2!k_3! \times \frac{1}{\pi^3} \int_{S^5} Y^{I_1} Y^{I_2} Y^{I_3} \]

\[ = \frac{2^\Sigma/2(J + \tilde{\Sigma}/2 + 2)!(J + \tilde{\alpha}_3)!(J + \tilde{\alpha}_1 \tilde{\alpha}_2)!}{\pi^2 J! \tilde{k}_3! \sqrt{k_1!k_2!(J + \tilde{k}_1)!(J + \tilde{k}_2)!}} \]

16
Here $\Sigma = k_1 + k_2 + k_2$ and $\tilde{\Sigma} = \tilde{k}_1 + \tilde{k}_2 + \tilde{k}_3$. The factor of $\cos^{2J+1}\theta$ becomes a gaussian which in narrowly peaked on the geodesic $\theta = 0$. Performing the $y$ integral, this reproduces the right-hand side of eq. (2.8), in the large-$J$ limit. The integral corresponds to a zero-dimensional field theory calculation of the contractions of four scalar fields $y_i$. In the world-sheet calculation of section 3, the zero mode contractions have this same form.

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