Estimation of micromechanical strength of brittle rocks under compression

Li Xiaozhao¹, Qi Chengzhi¹, Shao Zhushan²

¹School of Civil and Transportation Engineering, Beijing University of Civil Engineering and Architecture, Beijing 100044, China
²School of Civil Engineering, Xi’an University of Architecture and Technology, Xi’an 710055, China

E-mail: lixiaozhao@bucea.edu.cn

Abstract. A unified micro-macro method is proposed to estimate short-term, long-term, and shear strength of rocks under compression. This method is formulated by combining the wing microcrack model, subcritical crack growth law, Mohr–Coulomb failure criterion, and crack–strain relation. The short-term strength is predicted by the theoretical stress–strain relationship caused by microcrack growth. The stress states for a given creep failure time and long-term strength are analyzed using the proposed theoretical three-level creep evolution relationship. Moreover, the shear strength is reviewed by the proposed predictive method for shear failure. Correlation of short-term, long-term, and shear strength is closely analyzed, and the effects of crack size, crack friction, and fracture toughness on short-term, long-term, and shear strength are discussed.

Keywords. Brittle rocks; Strengths; Micromechanics; Fracture; Damage

1. Introduction
Rock strength is of great significance to evaluations of the mechanics of surrounding rock in underground engineering. Internal microcracks in rocks play an essential role in rock strength. Furthermore, rock strength subjected to compressive loading has extensively been studied in terms of short-term, long-term, and shear strength. Hence, the study of these properties from the basis of microcrack growth in brittle rocks is highly important.

Short-term strength can usually be evaluated by stress–strain curves during stress-induced progressive failure in convenient uniaxial and triaxial compressive tests [1-3]. Based on tensile microcrack mechanisms in brittle rocks, multiple microcrack damage models have been proposed under compressive loading, with the stress–strain relationship and short-term strength widely studied [4-10]. Ashby and Sammis [11] proposed a wing microcrack model describing a relation between microcrack damage and stress, which aided the study of short-term strength under different confining pressures. Based on the wing crack model under uniaxial compression, Baud et al. [12] proposed an analytical simplification and studied the effects of porosity and crack density on the short-term strength of rocks. According to the microcrack growth mechanism, the stress–strain curves were divided into five phases: crack closure, crack initiation, stable crack growth, unstable crack growth, and short-term peak strength [13]. The stress corresponding to unstable crack growth was defined as long-term strength [14]. This generally occurs at axial stress levels between 75% and 80% of short-term strength. Inelastic strain starts to generate when axial stress is larger than the long-term strength. Lajtai et al. [15] found that the stress in unstable crack growth for unconfined samples of Lac du Bonnet granite occurred at 70% of the short-term peak strength.

Generally, the long-term strength of rock is defined by the critical stress that can transform steady-state creep into accelerating creep. Creep tests are performed under many different constant stresses [16-19]. Creep
failure time corresponding to different constant stress states have been measured. Based on the experimental creep data, an exponential fitting equation has been obtained in studies. Using the fitting function, the unconfined long-term strength or static fatigue limit is produced at 45% of short-term strength [16, 19]. Furthermore, calculation methodologies of long-term strength evaluation have been presented using the subcritical crack growth law and stress intensity factor. The Double–Torsion method test estimates the subcritical crack growth parameters. The effects of temperature and humidity on long-term strength have been studied [20-22]. Miura et al. [23] presented a micromechanics-based model considering two interacting cracks. Creep failure following tertiary (i.e., accelerating) creep was represented as an unstable extension of interacting cracks, and long-term strength was predicted. Ohnaka [24] established a law governing shear failure of rocks in the brittle-plastic transition regime under lithospheric conditions. Based on the proposed shear failure strength law, the effects of temperature, confining pressure, pore pressure and strain rate on shear failure strength were investigated by triaxial compressive tests under different conditions [25, 26]. Rock failure of brittle rock is caused by shear fracture formed from many tensile microcracks. The interaction and coalescence of microcracks play an important role in the shear fracture of brittle rocks in compression [27-29]. Thus, the shear strength is closely related to the tensile microcrack growth.

In the abovementioned studies, the short-term, long-term, and shear strength were widely studied by experiment, theory, and numerical simulation. However, a theoretical method to study these properties simultaneously has yet to be proposed. In this study, a micro-macro method is proposed by combining the microcrack model of Ashby, subcritical crack growth law, Mohr–Coulomb failure criterion and crack–strain relation. The proposed stress–strain relations are used to analyze short-term strength. The proposed three-level creep failure curve are used to calculate long-term strength. Shear strength caused by microcrack growth is studied by the proposed prediction method. The correlation of short-term, long-term, and shear strength is analyzed. The effects of crack size, friction coefficient, and fracture toughness on these strengths are also discussed.

2. Theoretical formulation

2.1. Stress–strain relation and short-term strength

Based on the wing crack model proposed by Ashby and Sammis [11] and combining it with the subcritical crack law [30] and Mohr–Coulomb failure criterion, a unified micro-macro model is proposed to analyze the shear strength, short-term strength, and long-term strength under compressive loadings of brittle rocks, as shown in Figure 1.

![Figure 1. Model of microcrack growth under compressive loadings](image)

The relation between wing crack length \( l \) and stress state \( (\sigma_1, \sigma_3) \) is expressed as follows: [31-33]:

\[
\sigma_1(l) = \sigma_1 \left[ c_3 + A_2 (c_1 + c_2) \right] - \frac{K_{\text{IC}}}{\sqrt{\pi a}}
\]

\[\frac{A_1 (c_1 + c_2)}{(1)}\]
Using Eq. (7), the peak crack length can be calculated by the peak axial strain $\varepsilon_{\text{peak}}$. The peak axial strain can be easily obtained by the stress–strain curves in a compressive test. Substituting the calculated peak crack length into Eqs. (13–16), the shear strength can be expressed as follows [32]:

$$c(l_{\text{peak}}) = \frac{K_{IC}}{2\sqrt{A_1\pi a(c_1 + A_2(c_1 + c_2))(c_1 + c_2)}}$$

where $c$ is the cohesion, $\phi$ the internal friction angle, $\sigma_{n2}$ the normal stress on shear failure plane in rocks, and $\psi$ the macroscopic angle for shear failure plane in rocks. The crack length value in parameters $c_1$, $c_2$, and $c_3$ is $l_{\text{peak}}$ for all. The correlation of shear failure plane angle and internal friction angle is $90^\circ + \phi = 2\psi$.

### 2.2. Shear strength

Based on the linear failure envelope between axial stress $\sigma_1$ and confining pressure $\sigma_3$ in the Mohr–Coulomb failure criterion and the proposed Eq. (8) at $l = l_{\text{peak}}$, the relationship between micromechanical parameters and shear strength $\tau$, cohesion, the initial friction angle can be derived as follows:[32]:

$$\tau_{\text{f}}(l_{\text{peak}}) = \sigma_{n2} \tan \phi - c$$

where

$$\sigma_{n2}(l_{\text{peak}}) = \frac{\sigma_{\text{peak}} + \sigma_3}{2} + \frac{\sigma_{\text{peak}} - \sigma_3}{2} \cos 2\psi$$

$$\phi(l_{\text{peak}}) = 90^\circ - 2\arctan \sqrt{\frac{A_1(c_1 + c_2)}{c_3 + A_2(c_1 + c_2)}}$$

$$c(l_{\text{peak}}) = \frac{K_{IC}}{2\sqrt{A_1\pi a(c_1 + A_2(c_1 + c_2))(c_1 + c_2)}}$$

where $\mu$ is the friction coefficient of the initial microcrack interfaces, $a$ the initial crack size, $\varphi$ the initial crack angle, and $\beta$ a constant. The initial damage $D_0 = 4\pi N_V(aa)/3$, where $a = \cos \varphi$, and $N_V$ is the initial microcrack density. The function $\sigma(l)$ exhibits a maximum $\sigma_{\text{peak}}$ at $l = l_{\text{peak}}$ corresponding to the short-term strength in brittle rocks. The parameter $l_{\text{peak}}$ is the crack length corresponding to the peak stress. Compressive stress is negative in theory for calculation and is positive in the plot for observation.

Based on the damage definitions relating to crack length and strain, the correlation of macroscopic axial strain $\varepsilon$ and microcrack growth can be derived as follows: [32,33]:

$$\varepsilon = \varepsilon_{\text{peak}} \left( -\ln \left [ 1 - (l + aa)^{3/2} D_0 / (aa)^3 \right ] \right)^{1/m}$$

Substituting Eq. (7) into Eq. (1), the stress-strain relationship can be derived as follows:

$$\sigma_l(\varepsilon) = \frac{\sigma_1(B_1 + A_2(B_1 + B_2)) - K_{IC}/\sqrt{\pi a}}{A_1(B_1 + B_2)}$$

where

$$B_1 = \pi^{-2}(\alpha B_1 + \beta)^{3/2}$$

$$B_2 = 2\pi^{-2} \alpha^{3/2} B_1^{1/2} \left [ D_o^{3/2} - (B_1 + 1)^{3/2} \right ]$$

$$B_3 = 2(\alpha B_1)^{1/2} / \pi$$

$$B_4 = D_o^{3/2} \left [ 1 - \exp \left [ -\left(\varepsilon / \varepsilon_{\text{peak}}\right)^m \right ] \right]^{1/3} - 1$$

And the relationship between crack length and strain is

$$\varepsilon = \varepsilon_{\text{peak}} \left( -\ln \left [ 1 - (l + aa)^{3/2} D_0 / (aa)^3 \right ] \right)^{1/m}$$

where $\mu$ is the friction coefficient of the initial microcrack interfaces, $a$ the initial crack size, $\varphi$ the initial crack angle, and $\beta$ a constant. The initial damage $D_0 = 4\pi N_V(aa)/3$, where $a = \cos \varphi$, and $N_V$ is the initial microcrack density. The function $\sigma(l)$ exhibits a maximum $\sigma_{\text{peak}}$ at $l = l_{\text{peak}}$ corresponding to the short-term strength in brittle rocks. The parameter $l_{\text{peak}}$ is the crack length corresponding to the peak stress. Compressive stress is negative in theory for calculation and is positive in the plot for observation.
\[ a_f(e_{\text{peak}}) = \sigma_{\text{e_p}} \tan \phi - c \]  

(17)

where

\[ \sigma_{\text{e_p}}(e_{\text{peak}}) = \frac{\sigma_{\text{p_1}} + \sigma_{\text{p_2}}}{2} + \frac{\sigma_{\text{p_3}} - \sigma_{\text{p_1}} \cos 2\psi}{2} \]  

(18)

\[ \phi(e_{\text{peak}}) = 90^\circ - 2arc \tan \frac{A_k^\frac{B_1 + B_2}{B_1 + A_k(B_1 + B_2)}}{K_{IC}} \]  

(19)

\[ c(e_{\text{peak}}) = \frac{2\sqrt{A_k^\pi a(B_1 + A_k(B_1 + B_2))(B_1 + B_2)}}{K_{IC}} \]  

(20)

and the values of axial strain in parameters \( B_1, B_2, \) and \( B_3 \) all are \( e_{\text{peak}} \).

2.3. Creep behavior and long-term strength

For a mode-I crack in the rock, the subcritical crack growth rate is expressed as [33]:

\[ \frac{dl}{dt} = v(\pi a)^{n/2} \left( \frac{(A_k^\sigma - A_k^\nu)(c_1 + c_2) + \sigma c_3}{K_{IC}} \right) \]  

(21)

where the exponent \( n \) is the stress corrosion index. \( v \) is the characteristic crack velocity. For solving the time-dependent wing crack length (i.e., \( l(t) \)) in the numerical integration of Eq. (21)), the initial crack length \( l_0 \) can be calculated by Eq. (1) at a given axial stress state. Substituting the solution for the evolution for wing crack length into Eq. (7), the time-dependent axial strain is [33]:

\[ e(t) = e_o \left( -\ln \left[ 1 - (l(t) + a \alpha)^{D_o} / (a \alpha)^{D_o} \right] \right)^{1/m} \]

(22)

Based on the proposed expression of creep evolution, the long-term strength can be calculated. In other words, the accelerated creep stage cannot be found from the numerical results of Eq. (22) when the applied stress is smaller than the long-term strength.

3. Results

Based on the experimental data of Jinping marble in China, the critical stress intensity factor \( K_{IC} = 1.61 \text{MPa} \cdot \text{m}^{1/2} \), index of stress corrosion \( n = 57 \), characteristic crack velocity \( v = 0.16 \text{m/s} \), initial damage \( D_o = 0.048 \), crack angle \( \phi = 45^\circ \), friction coefficient \( \mu = 0.51 \), initial crack size \( a = 3.1 \text{mm} \), material constant \( m = 1 \), \( \epsilon_o = 0.0147 \), and \( \beta = 0.32 \). The particular selected method for model parameters can be found in Refs. [32,33].

The stress–strain constitutive relationships at different confining pressures are shown in Fig. 2(a). The axial stress increases and approaches peak stress, reaching failure with increasing axial strain. The short-term strength (i.e., peak stress) and peak axial strain both increase with increase in confining pressure. Furthermore, the evolution of axial strain is shown in Fig. 2(b). The axial strain exhibits three phases: decelerating, steady-state, and accelerating deformation. The results verify the basis of the theoretical model [33].

![Figure 2. Verification of theoretical model](image-url)
3.1 Correlation of short-term, long-term and shear strength
The stress state for a given CFT (creep failure time), crack initiation stress, and long-term strength are shown in Fig. 3. The stress state for a given CFT linearly increases with increase in confining pressure. Furthermore, incrementing axial stress (i.e., \( \Delta \sigma_1 = 3 \) MPa at \( \sigma_3 = 10 \) MPa) results in varying magnitude of creep failure time. This shows the strong dependence of creep failure on the stress state. In the case of smaller confining pressure (0 and 10 MPa), long-term strength equals crack initiation stress. For higher confining pressure (20, 30, and 40 MPa), long-term strength is larger than crack initiation stress.

The short-term, long-term, and shear strength under different confining pressures are wholly given in Fig. 4. The correlation of short-term, long-term, and shear strength for brittle rocks can be intuitively seen. The percentage between long-term and short-term strength is in the range of 34%–69% for confining pressure of 0–40 MPa. This percentage is predicted to be \( \sim 45\% \) by Schmidtke and Lajtai [16] and Damjanac and Fairhurst [19] based on the uniaxial compressive creep test of Lac du Bonnet Granite. Our calculated results of long-term strength agree well with the published prediction results. The difference may be due to difficulty in determining the long-term strength of rock by the experimental method.

![Figure 3. Relationship between long-term strength, axial stress for given CFT (creep failure time) and confining pressure](image)

![Figure 4. Relationship between short-term, long-term, shear strength and confining pressure](image)

4. Discussions
4.1. Effects of model parameters on short-term strength
The effects of crack size, friction coefficient and fracture toughness on short-term strength are shown in Fig. 5. Increments in initial crack size weaken the short-term strength, as seen in Fig. 5(a). The initial crack size corresponds to the initial damage for a given number of cracks \( N_c \). In Fig. 5(b), increase in friction coefficient between crack interfaces leads to increase in short-term strength. This phenomenon is in good agreement with the experimental results of Park and Bobet [34]. The friction between closed flaw interfaces under compression is considered in their test. In Fig. 5(c), increase in fracture toughness leads to increase in short-term strength.
4.2. Effects of model parameters on shear strength
The effects of crack size, friction coefficient, and fracture toughness on shear strength are shown in Fig. 6. Increase in friction coefficient and fracture toughness, and decrease in crack size lead to increase in shear strength.

4.3. Effects of model parameters on the long-term strength
The effects of crack size, friction coefficient, and fracture toughness on long-term strength are shown in Fig. 7. Increase in friction coefficient and fracture toughness, and decrease in crack size lead to increase in
long-term strength. Furthermore, the variation of long-term strength along with fracture toughness is very small under the given larger confining pressures (e.g., for $\sigma_3 = 20, 30, \text{ and } 40 \text{ MPa in Fig. 7c}). Fracture toughness is predicted to have little effect on long-term strength at higher confining pressure.

Figure 7. Effects of (a) crack size, (b) friction coefficient and (c) fracture toughness on long-term strength

5. Conclusions
A unified micro-macro method is proposed to predict short-term, long-term, and shear strength using the wing microcrack model, subcritical crack growth law, Mohr–Coulomb failure criterion, and a crack–strain relation. The following conclusions are drawn from this study:
(1) Crack initiation stress, short-term strength, and shear strength linearly increase with increments in confining pressure. Axial stress line arly increases with increasing confining pressure under a given creep failure time. The creep failure time strongly depends on the stress state. The long-term strength is equal to crack initiation stress under low confining pressure and greater than crack initiation stress under high confining pressure.
(2) The effects of initial crack size, friction coefficient, and fracture toughness on short-term, long-term, and shear strength are discussed. Increase in initial crack size and decrease in friction coefficient and fracture toughness lead to decrease in short-term, long-term, and shear strength. Note that fracture toughness has little effect on the long-term strength under high confining pressure.

References
[1] Brace WF, Paulding B, Scholz C. Dilatancy in the fracture of crystalline rocks. J Geophys Res 1966; 71: 3939–3953.
[2] Chang SH, Lee CI. Estimation of cracking and damage mechanisms in rock under triaxial compression by moment tensor analysis of acoustic emission. Int J Rock Mech Min Sci 2004; 41: 1069–1086.
[3] Lajtai EZ. Microscopic fracture processes in a granite. Rock Mech Rock Eng 1998; 31(4): 237–250.
[4] Ouyang Z, Elsworth D. A phenomenological failure criterion for brittle rock. Rock Mech Rock Eng 1991; 24: 133–153.
[5] Gupta V, Bergstrom J. Compressive failure of rocks by shear faulting. J Geophys Res 1998; 103(B10): 23875−23895.
[6] Yamashita T. Generation of microcracks by dynamic shear rupture and its effects on rupture growth and elastic wave radiation. Geophys J Int 2000; 143: 395−406.
[7] Zhang P, Li N, Li XB, Nordlund E. Compressive failure model for brittle rocks by shear faulting and its evolution of strength components. Int J Rock Mech Min Sci, 2009; 46: 830−841.
[8] Graham-Brady L. Statistical characterization of meso-scale uniaxial compressive strength in brittle materials with randomly occurring flaws. Int J Solids Struct 2010; 47(18): 2398−2413.
[9] Zhou JW, Xu WY, Yang XG. A microcrack damage model for brittle rocks under uniaxial compression. Mech Res Commun, 2010; 37(4): 399−405.
[10] Vergara MR, Jan MVS, Lorig L. Numerical model for the study of the strength and failure modes of rock containing non-persistent joints. Rock Mech Rock Eng 2016; 49: 1211−1226.
[11] Ashby MF, Sammis CG. The damage mechanics of brittle solids in compression. Pure Appl Geophys 1990; 133(3): 489−521.
[12] Baud P, Wong TF, Zhu W. Effects of porosity and crack density on the compressive strength of rocks. Int J Rock Mech Min Sci 2014; 67(4): 202−211.
[13] Martin CD, Chandler NA. The progressive fracture of Lac du Bonnet granite. Int J Rock Mech Min Sci Geomech Abstr 1994; 31(6): 643−659.
[14] Bieniawski ZT. Mechanism of brittle fracture of rock, Parts I, II and III. Int J Rock Mech Min Sci Geomech Abstr 1967; 4(4): 395−430.
[15] Lajtai EZ, Carter BJ, Duncan EJS. Mapping the state of fracture around cavities. Eng Geol 1991; 31: 277−289.
[16] Schmidtke RH, Lajtai EZ. The long-term strength of lac du bonnet granite. Int J Rock Mech Min Sci Geomech Abstr 1985; 22(6): 461−465.
[17] Masuda K. Effects of water on rock strength in a brittle regime. J Struct Geol 2001; 23(11): 1653−1657.
[18] Shin K, Okubo S, Fukui K, Hashiba K. Variation in strength and creep life of six japanese rocks. Int J Rock Mech Min Sci 2005; 42(2): 251−260.
[19] Damjanac B, Fairhurst C. Evidence for a long-term strength threshold in crystalline rock. Rock Mech Rock Eng 2010; 43(5): 513−531.
[20] Nara Y, Takada M, Mori D, Owada H, Yoneda T, Kaneko K. Subcritical crack growth and long-term strength in rock and cementitious material. Int J Fract 2010; 164(1): 57−71.
[21] Nara Y, Yamanaka H, Oe Y, Kaneko K. Influence of temperature and water on subcritical crack growth parameters and long-term strength for igneous rocks. Geophys J Int 2013; 193(1): 47−60.
[22] Nara Y. Effect of anisotropy on the long-term strength of granite. Rock Mech Rock Eng 2015; 48(3): 959−969.
[23] Miura K, Okui Y, Horii H. Micromechanics-based prediction of creep failure of hard rock for long-term safety of high-level radioactive waste disposal system. Mech Mater 2003; 35(02): 587−601.
[24] Ohnaka M. A shear failure strength law of rock in the brittle-plastic transition regime. Geophys Res Lett 1995; 22(1): 25−28.
[25] Odedra A, Ohnaka M, Mochizuki H, Sammonds P. Temperature and pore pressure effects on the shear strength of granite in the brittle-plastic transition regime. Geophys Res Lett 2001; 28(15): 3011−3014.
[26] Kato A, Ohnaka M, Mochizuki H. Constitutive properties for the shear failure of intact granite in seismogenic environments. J Geophys Res 2003; 108(B1): 2060.
[27] Moore DE, Lockner DA. The role of microcracking in shear-fracture propagation in granite. J Struct Geol 1995; 17(1), 95–114.

[28] Healy D, Jones RR, Holdsworth RE. Three-dimensional brittle shear fracturing by tensile crack interaction. Nature 2006; 439: 64–67.

[29] Misra S. Deformation localization at the tips of shear fractures: An analytical approach. Tectonophysics 2011; 503, 182–187.

[30] Atkinson BK. Subcritical crack growth in geological materials. J Geophys Res 1984; 89(B6): 4077–4114.

[31] Brantut N, Baud P, Heap MJ, Meredith PG. Micromechanics of brittle creep in rocks. J Geophys Res 2012; 117(B08412): 1–12.

[32] Li XZ, Shao ZS. Fan LF. A micro–macro method for predicting the shear strength of brittle rock under compressive loading. Mech Res Commun 2016; 75: 13-19.

[33] Li XZ, Shao ZS. Investigation of macroscopic brittle creep failure caused by microcrack growth under step loading and unloading in rocks. Rock Mech Rock Eng 2016;49(7): 2581-2593.

[34] Park CH, Bobet A. Crack initiation, propagation and coalescence from frictional flaws in uniaxial compression. Eng Fract Mech 2010; 77: 2727–2748.