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Ph.D. Thesis

A geometric approach to the study of $(\mathfrak{g}, \mathfrak{k})$-modules of finite type

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1. Notation and definitions

Throughout this thesis $\mathfrak{g}$ will be a reductive Lie algebra over the algebraically closed field $\mathbb{C}$ of characteristic 0 and $\mathfrak{k} \subset \mathfrak{g}$ will be a reductive in $\mathfrak{g}$ subalgebra. Let $\mathcal{U}(\mathfrak{k})$ be the universal enveloping algebra of $\mathfrak{k}$, $G$ be the adjoint group of $[\mathfrak{g}, \mathfrak{g}]$, $K$ be a connected subgroup of $G$ with Lie algebra $\mathfrak{k} \cap [\mathfrak{g}, \mathfrak{g}]$ and $B$ be a Borel subgroup of $K$. We work in the category of algebraic varieties over $\mathbb{C}$. All Lie algebras considered are finite-dimensional.

**Definition 1.1.** A $(\mathfrak{g}, \mathfrak{k})$-module is a $\mathfrak{g}$-module for which action of $\mathfrak{k}$ is locally finite, i.e. for which $\dim (\mathcal{U}(\mathfrak{k})m) < \infty$ for any $m \in M$, where $\mathcal{U}(\mathfrak{k})m := \{m' \in M \mid m' = um \text{ for some } u \in \mathcal{U}(\mathfrak{k}) \text{ and } m \in M\}$.

**Definition 1.2.** Let $M$ be a locally finite $\mathfrak{k}$-module, i.e a $(\mathfrak{k}, \mathfrak{k})$-module. We say that $M$ has finite type over $\mathfrak{k}$ if all isotypic components of $\mathfrak{k}$ are finite-dimensional. We say that a $(\mathfrak{g}, \mathfrak{k})$-module is of finite type if $M$ has finite type over $\mathfrak{k}$.

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

**Definition 1.3.** We say that a $\mathfrak{g}$-module $M$ affords a central character if for some homomorphism of algebras $\chi : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ we have $zm = \chi(z)m$ for all $z \in Z(\mathfrak{g})$ and $m \in M$.

**Definition 1.4.** We say that a $\mathfrak{g}$-module $M$ affords a generalized central character if for some homomorphism $\chi : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ and some $n \in \mathbb{Z}_{>0}$ we have $(z - \chi(z))^n m = 0$ for all $m \in M$ and $z \in Z(\mathfrak{g})$.

Let $M$ be a $(\mathfrak{g}, \mathfrak{k})$-module of finite type and $V$ be a simple $\mathfrak{k}$-module. We denote by $[M : V]_{\mathfrak{k}}$ the supremum over all finite-dimensional $\mathfrak{k}$-submodules $M' \subset M$ of the Jordan-Hölder multiplicities $[M' : V]_{\mathfrak{k}}$. By $[M : :]_{\mathfrak{k}}$ we denote the corresponding function from the set of simple $\mathfrak{k}$-modules to $\mathbb{Z}_{\geq 0} = \mathbb{N}$. This is nothing but the $\mathfrak{k}$-character of $M$.

**Definition 1.5.** A bounded $(\mathfrak{g}, \mathfrak{k})$-module $M$ is a $(\mathfrak{g}, \mathfrak{k})$-module which is bounded as a $\mathfrak{k}$-module, i.e. for which the function $[M : :]_{\mathfrak{k}}$ is uniformly bounded by some constant $C_M$.

**Definition 1.6.** A multiplicity-free $(\mathfrak{g}, \mathfrak{k})$-module $M$ is a $(\mathfrak{g}, \mathfrak{k})$-module which is multiplicity-free as a $\mathfrak{k}$-module, i.e. for which the function $[M : :]_{\mathfrak{k}}$ is uniformly bounded by 1.
For a finitely generated $\mathfrak{g}$-module we denote by $V(M)$ the associated variety of $M$, by $\text{Ann}M$ the annihilator of $M$ in $U(\mathfrak{g})$, by $GV(M)$ the zero set of $\text{gr}(\text{Ann}M)$ (see Subsection 3.5). We call a $G$-orbit $Gu$ in $\mathfrak{g}^*$ nilpotent if $0 \in Gu$. If $\mathfrak{g}$ is semi-simple we identify $\mathfrak{g}$ and $\mathfrak{g}^*$.

For a variety $X$ we denote by $\mathcal{O}(X)$ the structure sheaf of $X$, by $\mathbb{C}[X]$ the algebra of regular functions on $X$, by $\mathcal{D}(X)$ the sheaf of differential operators on $X$, by $\mathcal{D}(X)$ the algebra of global sections of $\mathcal{D}(X)$. We denote by $TX$ (respectively, $T^*X$) the total space of the tangent (respectively, cotangent) bundle to $X$. If $M$ is a coherent $\mathcal{D}(X)$-module, $\mathcal{V}(M) \subset T^*X$ stands for the singular support of $M$ (see also Subsection 3.4). All $\mathcal{D}(X)$-modules considered are quasicoherent.

For a finite-dimensional vector space $W$ we set $n_W := \dim W$.

**Definition 1.7.** We call an $n_W$-tuple $\bar{\lambda} := (\lambda_1, ..., \lambda_{n_W})$, $\lambda_i \in \mathbb{C}$, decreasing if $\lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0}$ for $i \geq j$. We call $\bar{\lambda}$ semi-decreasing if it is not decreasing but becomes decreasing when we remove one coordinate (cf. with O. Mathieu’s definitions of ordered/semi-ordered tuples).

Assume $n_W \geq 3$. Let $\bar{\lambda} = (\lambda_1, ..., \lambda_{n_W-1})$, $\lambda_j \in \mathbb{C}$, be a decreasing tuple and $t \in \mathbb{C}$. By adding an additional coordinate $t$ to $\bar{\lambda}$ (at any position) we obtain a semi-decreasing $n_W$-tuple $\bar{\lambda}^+$.

**Definition 1.8.** The monodromy $m(\bar{\lambda}^+)$ of $\bar{\lambda}^+$ is the number $e^{2\pi i (t-\lambda_1)}$.

**Definition 1.9.** We call an $n_W$-tuple $\bar{\lambda} := (\lambda_1, ..., \lambda_{n_W})$, $\lambda_i \in \mathbb{C}$, integral if $\lambda_i - \lambda_j \in \mathbb{Z}$ for any $i, j$. We call $\bar{\lambda}$ semi-integral if it is not integral but becomes integral when we remove one term. We call a tuple $\bar{\lambda}$ regular if $\lambda_i \neq \lambda_j$ for all pairs $i \neq j$.

Any semi-decreasing $n_W$-tuple $\bar{\lambda}$ is regular integral, singular integral, or semi-integral. If $\bar{\lambda}$ is integral then $m(\bar{\lambda}) = 1$; if $\bar{\lambda}$ is not integral then $m(\bar{\lambda}) \neq 1$.

**Definition 1.10.** We call an $n_W$-tuple a Shale-Weil tuple if $\mu_i > \mu_j$ for $i > j$, $\mu_{n_W-1} > |\mu_{n_W}|$ and $\mu_i \in \frac{1}{2} \mathbb{Z} + \mathbb{Z}$ for all $i \in \{1, ..., n_W\}$.

For a $K$-variety $X$ and a point $x \in X$ we denote by $K_x$ the stabilizer of $x$ in $K$ and by $\mathfrak{t}_x$ the Lie algebra of $K_x$. If there exists a subgroup $H \subset K$ such that $H$ is conjugate to $K_x$ for all $x$ from some open subset $\tilde{X} \subset X$, we call $H$ a generic stabilizer of $K$ on $X$; we call the Lie algebra of $H$ a generic isotropy subalgebra. By definition, $X$ is a $K$-spherical variety if $X$ is irreducible and has an open $B$-orbit.

By $\text{SL}_n$, $\text{SO}_n$, $\text{SP}_n$ we denote respectively the special linear, orthogonal and symplectic groups of $n$-dimensional vector space ($\text{SP}_n$ is defined only when $n = 2k$). By $\mathfrak{sl}_n$, $\mathfrak{so}_n$, $\mathfrak{sp}_n$ we denote the corresponding Lie algebras.
2. Introduction and brief statements of results

There are two well-known categories of \((\mathfrak{g}, \mathfrak{k})\)-modules: the category of Harish-Chandra modules and the category \(\mathcal{O}\). In the first case \(\mathfrak{k}\) is a symmetric subalgebra of \(\mathfrak{g}\) (i.e. \(\mathfrak{k}\) coincides with the fixed points of an involution of \(\mathfrak{g}\)), and in the second case \(\mathfrak{k}\) is a Cartan subalgebra \(\mathfrak{h}\) of \(\mathfrak{g}\). For both types of pairs \((\mathfrak{g}, \mathfrak{k})\) the \((\mathfrak{g}, \mathfrak{k})\)-modules in question are of finite type. I. Penkov, V. Serganova, and G. Zuckerman have proposed to study, and attempt to classify, simple \((\mathfrak{g}, \mathfrak{k})\)-modules of finite type for arbitrary reductive in \(\mathfrak{g}\) subalgebras \(\mathfrak{k}\), [PSZ], [PZ] (see also [Mi]).

Let \(X\) be the variety of all Borel subalgebras of \(\mathfrak{g}\). Let \(\lambda \in H^1(X, \Omega^{1,\text{cl}})\) be a cohomology class (\(\Omega^{1,\text{cl}}\) is the sheaf of closed holomorphic 1-forms on \(X\)) and \(\mathcal{D}^\lambda(X)\) be the corresponding sheaf of twisted differential operators on \(X\) [Be]. The cohomology class \(\lambda\) defines functors of 'localization' (Loc: \(\mathfrak{g}\)-modules to \(\mathcal{D}^\lambda(X)\)-modules) and 'global sections' (GSec: \(\mathcal{D}^\lambda(X)\)-modules to \(\mathfrak{g}\)-modules). For any central character \(\chi\) there exists a cohomology class \(\lambda\) such that GSec(Loc) is the identity functor after restriction to the category of \(\mathfrak{g}\)-modules which affords the central character \(\chi\) [BeBe2]. Any simple \(\mathfrak{g}\)-module affords a central character [Dix].

We fix \(\lambda \in H^1(X, \Omega^{1,\text{cl}})\). In the category of \(\mathcal{D}^\lambda(X)\)-modules there is a distinguished full subcategory of holonomic sheaves of modules, or simply holonomic modules. Informally, holonomic modules are \(\mathcal{D}^\lambda(X)\)-modules of minimal growth (see Definition 3.20). The simple holonomic modules are in one-to-one correspondence with the set of pairs \((L, S)\), where \(L\) is an irreducible closed subvariety of \(X\) and \(S\) is sheaf of \(\mathcal{D}^\lambda(L')\)-modules which is \(\mathcal{O}(L')\)-coherent after restriction to a suitable open subset \(L' \subset L\). Moreover, a coherent holonomic module \(S\) is locally free on \(L'\), and one could think about it as a vector bundle \(S_B\) over \(L'\) with a flat connection. Note that flat local sections of \(S_B\) are not necessarily algebraic.

Our first result is the following theorem, which we prove in Section 4 (Corollary 4.1).

**Theorem 2.1.** Let \(M\) be a simple \((\mathfrak{g}, \mathfrak{k})\)-module of finite type. Then Loc\((M)\) is a holonomic \(\mathcal{D}^\lambda(X)\)-module.

The following theorem is a 'geometric twin' of the previous one and is proved also in Section 4. The definitions needed for the statement of the theorem see in Subsection 3.1 and Subsection 3.7 or in [VP].

**Theorem 2.2.** Let \(Z \subset \mathfrak{g}^*\) be a nilpotent \(G\)-orbit, \(\mathfrak{k}^\perp\) be the annihilator of \(\mathfrak{k}\) in \(\mathfrak{g}^*\), \(N_K(\mathfrak{g}^*)\) be the \(K\)-null-cone in \(\mathfrak{g}^*\). Then the irreducible components of \(Z \cap \mathfrak{k}^\perp \cap N_K(\mathfrak{g}^*)\) are isotropic subvarieties of \(Z\).
Let $V_{g,t}$ be the set of all irreducible components of all possible intersections of $N_K(t^\perp)$ with $G$-orbits in $N_G(g^*)$. This finite set of subvarieties of $g^*$ determines a finite set $V_{g,t}$ of subvarieties of $T^*X$ and a finite set $L_{g,t}$ of subvarieties of $X$, see Section 4.

**Theorem 2.3.** Let $M$ be a finitely generated $(g,t)$-module of finite type which affords a central character, and let $(L,S)$ the pair corresponding to $\text{Loc} M$ as introduced above. Then $L$ is an element of $L_{g,t}$.

We prove this theorem in Section 4 (Corollary 4.1). Any simple Harish-Chandra module is holonomic [BeBe2] and of finite-type [HC]. For the literature on Harish-Chandra modules, see [KV] and references therein.

In the classification of simple $(g,\mathfrak{h}_g)$-modules of finite type the bounded simple modules play a crucial role. Based on this, and on the experience with Harish-Chandra modules, I. Penkov and V. Serganova have proposed to study bounded $(g,\mathfrak{t})$-modules for general reductive subalgebras $\mathfrak{t}$. A question arising in this context is, given $g$, to describe all reductive in $g$ bounded subalgebras, i.e. reductive in $g$ subalgebras $\mathfrak{t}$ for which at least one infinite-dimensional simple bounded $(g,\mathfrak{t})$-module exists. In [PS] I. Penkov and V. Serganova give a partial answer to this problem, and in particular proved an important inequality which restricts severely the class of possible $\mathfrak{t}$. They also give the complete list of bounded reductive subalgebras of $g = sl_n$ which are maximal subalgebras.

In the work [Pe] we prove the following theorem.

**Theorem 2.4.** There exists an infinite-dimensional simple bounded $(sl(V),\mathfrak{t})$-module if and only if $V$ is a $K \times C^*$-spherical variety.

We reproduce the proof of this theorem in Section 5. There are well-known pairs of algebras which admit an infinite-dimensional simple bounded module:

1. $\mathfrak{t}$ is a Cartan subalgebra of a simple Lie algebra $g$ of type A or C [F];
2. $\mathfrak{t}$ is a symmetric subalgebra of a reductive Lie algebra $g$.

Bounded $(g,\mathfrak{h}_g)$-modules are nothing but weight modules with bounded weight multiplicities, and all such simple modules have been classified by O. Mathieu [M].

As far as we know, the simple bounded Harish-Chandra modules have not been singled explicitly out within the category of all Harish-Chandra modules. For a given symmetric pair $(g,\mathfrak{t})$, Harish-Chandra

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\[1\text{In the literature on Harish-Chandra modules the condition of being of finite type is often called admissibility.}\]
modules admit only finitely many support varieties, and a Harish-Chandra module $M$ is spherical if and only if any irreducible component of the support variety $V(M)$ is spherical. There is some progress in singling out the spherical varieties among the support varieties of Harish-Chandra modules $[\text{Pan2}], [\text{Ki}].$

Let $I$ be a two-sided ideal in $U(g)$. By definition, $I$ is primitive if $I$ is the annihilator of some simple $g$-module $M$. Let $W$ be a finite-dimensional $\mathbb{C}$-vector space. To any $n_W$-tuple $\bar{\lambda} := (\lambda_1, \ldots, \lambda_{n_W})$ one assigns a weight $\lambda$ (see Subsection 3.9) and a primitive ideal $I(\lambda) \subset U(sl(W))$ (this is the annihilator of the simple $sl(W)$-module with highest weight $\lambda$), and any primitive ideal of $U(sl(W))$ arises in this way from some tuple $\bar{\lambda}$.

Let $M_1, M_2$ be simple $(g, k)$-modules. Let $AnnM_1, AnnM_2$ be the annihilators in $U(g)$ of $M_1$ and $M_2$ respectively. I. Penkov and V. Serganova $[\text{PS}]$ proved that, if $AnnM_1 = AnnM_2$ and $M_1$ is a bounded $(g, k)$-module, then $M_2$ is $k$-bounded too (see also Theorem 3.13). Moreover, a $(g, k)$-module is bounded if and only if the algebra $U(g)/AnnM$ satisfies certain relations. In Section 5 we prove the following weaker geometric version of this result.

**Theorem 2.5.** A simple $(sl(W), \mathfrak{t})$-module $M$ is bounded if and only if the associated variety $GV(M)$ is $K$-coisotropic.

This shows that 'boundedness' is not a property of a module $M$ but of the ideal $AnnM$, and moreover of the nilpotent orbit $GV(M) \subset sl(W)^*$. Theorem 2.5 also motivates our interest in the classification of $K$-coisotropic nilpotent $G$-orbits in $g^*$. In the case $g = sl(W)$, the set of $K$-coisotropic nilpotent orbits in $sl(W)^*$ is naturally identified with the set of partition equivalence classes of $K$-spherical partial $W$-flag varieties (see Subsection 3.3). We work out the classification of $K$-spherical flag varieties in Section 6.

Furthermore, we make the following conjecture.

**Conjecture 2.1.** A simple $(g, \mathfrak{t})$-module $M$ is bounded if and only if the associated variety $GV(M)$ is $K$-coisotropic.

This statement is closely related with $[\text{Pan}$, Question, p. 191] and we believe that this question will be answered soon $[\text{ZT}]$.

In the rest of the thesis we consider in greater detail four special pairs $(g, \mathfrak{t})$:

\begin{align*}
1a) \quad W &= S^2 V \\
1b) \quad W &= \Lambda^2 V \\
2a) \quad (sl(W), sl(V)) \\
2b) \quad (sp(W \oplus W^*) , gl(V))
\end{align*}

(1)
We hope that our results about these cases shed some light on how the general theory of bounded modules looks like.

In the rest of Section 2 \( V \) is a finite-dimensional vector space and \( W = S^2 V \) (in this case \( n_V \geq 3 \)) or \( W = \Lambda^2 V \) (in this case \( n_V = 2k \) and \( n_V \geq 5 \)).

For a simple \( \mathfrak{sl}(W) \)-module \( M \) we have \( \dim V(M) \geq n_W - 1 \) or \( M \) is finite-dimensional. A simple \( \mathfrak{sl}(W) \)-module \( M \) is of small growth if \( \dim V(M) \leq n_W - 1 \) (the definition of a module of small growth which is not necessarily simple is given in Section 8). The following theorem is proved in Subsection 9.3.

**Theorem 2.6.** Any simple bounded \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-module is of small growth.

We note that all bounded weight modules are also of small growth. The following result is inspired by the corresponding result of O. Mathieu for weight modules. The proof of this result is presented in Subsection 9.3.

**Theorem 2.7.** Let \( M \) be a simple bounded \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-module and \( \bar{\lambda} \) be an \( n_W \)-tuple such that \( \text{Ann} M = I(\lambda) \). Then \( \bar{\lambda} \) is semi-decreasing.

**Theorem 2.8.** Let \( M \) be a finitely generated \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-module and suppose \( I(\lambda) \subset \text{Ann} M \) for some semi-decreasing tuple \( \lambda \). Then \( M \) is \( \mathfrak{sl}(V) \)-bounded.

We prove Theorem 2.8 in Section 9.3. The primitive ideals which correspond to the semi-decreasing sequences are called Joseph ideals (see Section 7).

Fix \( t \in \mathbb{C} \). Let \( P_{\text{exist}}(W) \) be the cardinality of the set of isomorphism classes of simple perverse sheaves on \( W \) (with respect to the stratification by \( \text{GL}(V) \)-orbits) which have fixed monodromy \( e^{2\pi it} \) and are neither supported at 0 nor are smooth on \( W \). The above simple perverse sheaves on \( W \) are described, following [BR], in quiver terms in the Appendix. In particular, there are only finitely many isomorphism classes of simple perverse sheaves with a given monodromy. The following theorem 'counts' simple bounded \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-modules. We prove this theorem in Subsection 9.3.

**Theorem 2.9.** Let \( \bar{\lambda} \) be a semi-decreasing tuple. Then there exist precisely \( P_{\text{m}(\lambda)}(W) \) non-isomorphic infinite-dimensional simple \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-modules annihilated by \( I(\lambda) \).

Let \( \bar{\lambda} \) be a decreasing tuple. Let \( \mathcal{J}_\lambda \) be the set of infinite-dimensional simple bounded \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-modules annihilated by \( \text{Ker} \chi_\lambda \). Let \( \langle \mathcal{J}_\lambda \rangle \) be the free vector space generated by \( \mathcal{J}_\lambda \). The set of \( n_W \)-tuples carries
a natural action of $S_{nW}$. This action induces an action of $S_{nW}$ on $\langle J \lambda \rangle$. The $S_{nW}$-module $\langle J \lambda \rangle$ is isomorphic to a direct sum of $P_1(W)$-copies of an $(n_W - 1)$-dimensional module of $S_{nW}$.

We now turn our attention to row 2 of table (1). The space $W \oplus W^*$ has a natural symplectic form

$$\omega(\cdot, \cdot) : (x_1, l_1, x_2, l_2) \mapsto x_1(l_2) - x_2(l_1) \in \mathbb{C}.$$

There is an inclusion $gl(W) \subset sp(W \oplus W^*)$. The following theorem relates bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$-modules and bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(W))$-modules. We prove it in Subsection 9.3.

**Theorem 2.10.**

a) Let $\tilde{M}$ be a bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-module. Then any simple $\mathfrak{sl}(W)$-subquotient of $\tilde{M}$ is an $(\mathfrak{sl}(W), \mathfrak{sl}(V))$-bounded module.

b) Let $M$ be a simple bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$-module. Then there exists a simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-module $\tilde{M}$ such that $M$ is an $\mathfrak{sl}(W)$-subquotient of $\tilde{M}$.

Any tuple $\bar{\mu} = (\mu_1, ..., \mu_{nW})$ determines a two-sided ideal $I_{sp}(\bar{\mu})$ of $U(\mathfrak{sp}(W \oplus W^*))$ and any primitive ideal of $U(\mathfrak{sp}(W \oplus W^*))$ is determined by some tuple $\bar{\mu}$.

For a simple $\mathfrak{sp}(W \oplus W^*)$-module $M$, we have either $\dim V(M) \geq n_W$ or $\dim M < \infty$. A simple $\mathfrak{sp}(W \oplus W^*)$-module $M$ is of small growth if $\dim V(M) \leq n_W$ (the definition of a module of small growth which is not necessarily simple is given in Section 8). The following theorems are ‘$\mathfrak{sp}$-twins’ of Theorem 2.6, Theorem 2.7, Theorem 2.8.

**Theorem 2.11.** Any simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-module is of small growth.

**Theorem 2.12.** Let $M$ be an infinite-dimensional simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-module and $\bar{\mu}$ be an $n_W$-tuple such that $\text{Ann } M = I_{sp}(\mu)$. Then $\bar{\mu}$ is a Shale-Weil tuple.

**Theorem 2.13.** Let $M$ be a finitely generated $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-module and $I_{sp}(\mu) \subset \text{Ann } M$ for some Shale-Weil tuple $\bar{\mu}$. Then $M$ is bounded.

We prove these three theorems in Subsection 9.4. The primitive ideals which correspond to the Shale-Weil sequences are called Joseph ideals (see Section 7).

We call a Shale-Weil $n_W$-tuple $\bar{\mu}$ positive if $\mu_{nW} > 0$ and negative otherwise. Let $\mathfrak{h}_{nW}$ be the $\mathbb{C}$-affine space of $n_W$-tuples. The automorphism

$$\sigma : \mathfrak{h}_{nW}^* \rightarrow \mathfrak{h}_{nW}^*, (\mu_1, \mu_2, ..., \mu_{nW}) \mapsto (\mu_1, \mu_2, ..., -\mu_{nW})$$
is naturally identified with the set of pairs \((\bar{\mu}, M)\). Shale-Weil modules annihilated by \(I_b\). In particular, the set of simple bounded \((\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))\)-modules is naturally identified with the set of pairs \((\bar{\mu}, M)\), where \(\bar{\mu}\) is a positive Shale-Weil \(n_W\)-tuple and \(M\) is a simple bounded \((\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))\)-module annihilated by \(I_{sp}(\bar{\mu}_0)\).

Let \(\bar{\mu}\) be a positive Shale-Weil \(n_W\)-tuple. The functor \(\mathcal{H}_{\mu}^{\sigma}\) is exact and involutive. The category of modules annihilated by \(I_{sp}(\bar{\mu})\) is stable under \(\mathcal{H}_{\mu}^{\sigma}\). We denote \(\mathcal{H}_{\mu_0}^{\sigma\mu_0}\) by Inv.

The Dynkin diagram of Spin\(_{2n_W}\) has a nontrivial involution (it is unique unless \(n = 4\)) and it induces an involution \(\sigma\) on Spin\(_{2n_W}\). Furthermore, there is a unique central element \(z \in \text{Spin}_{2n_W}\) such that \(z^2 = 1\) and \(\text{Spin}_{2n_W}/\{1, z\} = \text{SO}_{2n_W}\) (for \(n = 4, z\) is uniquely determined by the additional requirement \(\sigma(z) = z\)). An odd pair of simple Spin\(_{2n_W}\)-modules is a pair \(\{L^{\bar{\mu}}, L^{\sigma\bar{\mu}}\}\) of simple Spin\(_{2n_W}\)-modules which are conjugate by \(\sigma\) and such that \(z\) acts by \(-1\) on \(L^{\bar{\mu}}\) and \(L^{\sigma\bar{\mu}}\). Obviously, \(L^{\bar{\mu}}\) and \(L^{\sigma\bar{\mu}}\) has the same dimension. Any positive Shale-Weil \(n_W\)-tuple \(\bar{\mu}\) determines an odd pair \(\{L^{\bar{\mu}}, L^{\sigma\bar{\mu}}\}\) of Spin\(_{2n_W}\)-modules (cf. [M]).

Let \(\{L, L^{\sigma}\}\) be the unique odd pair of minimal dimension. The following theorem should be understood as a mnemonic rule corresponding to the results of Subsection [9.3]. It is related to the \(\mathfrak{gl}(V)\)-characters of bounded \((\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))\)-modules.

**Theorem 2.14.** a) For any two positive Shale-Weil tuples \(\bar{\mu}_1, \bar{\mu}_2\) the categories of bounded \((\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))\)-modules annihilated by \(I_{sp}(\bar{\mu}_1)\) and \(I_{sp}(\bar{\mu}_2)\) are equivalent.

b) In particular, the set of simple bounded \((\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))\)-modules is a pair \((\bar{\mu}, M)\), where \(\bar{\mu}\) is a positive Shale-Weil \(n_W\)-tuple and \(M\) is a simple bounded \((\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))\)-module annihilated by \(I_{sp}(\bar{\mu}_0)\).

**Theorem 2.15.** Let \((\bar{\mu}, M)\) be a pair as in Theorem [2.14] and let \(\bar{\mu}M\) be the corresponding simple \((\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))\)-module. Then

\[
\frac{[\mu M : \cdot]_e + [\mu \text{Inv} M : \cdot]_e}{[M : \cdot]_e + [\text{Inv} M : \cdot]_e} = \frac{[L^{\bar{\mu}} : \cdot]_h + [L^{\sigma\bar{\mu}} : \cdot]_h}{[L^{\cdot} : \cdot]_h + [L^{\sigma : \cdot}]_h} \quad (2.1)
\]

and

\[
\frac{[\mu M : \cdot]_e - [\mu \text{Inv} M : \cdot]_e}{[M : \cdot]_e - [\text{Inv} M : \cdot]_e} = \frac{[L^{\bar{\mu}} : \cdot]_h - [L^{\sigma\bar{\mu}} : \cdot]_h}{[L^{\cdot} : \cdot]_h - [L^{\sigma : \cdot}]_h} \quad (2.2)
\]

The functions \([L^{\cdot} : \cdot]_h, [L^{\sigma : \cdot}]_h, [L^{\bar{\mu}} : \cdot]_h, [L^{\sigma\bar{\mu}} : \cdot]_h\) are known since the modules

\(L, L^{\sigma}, L^{\bar{\mu}}, L^{\sigma\bar{\mu}}\)

are finite-dimensional. Therefore the formulas (2.1), (2.2) provide two linear equations for the four unknowns.
Let $M_1, M_2$ be non-isomorphic simple $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-modules annihilated by $I_{\mathfrak{sp}}(\mu_0)$. Then both $M_1, M_2$ are multiplicity-free $\mathfrak{gl}(V)$-modules and the functions

$[M_1 : \cdot]_{\mathfrak{gl}(V)}$ and $[M_2 : \cdot]_{\mathfrak{gl}(V)}$

are pairwise disjoint, i.e. their product is the zero-function. Moreover, there are infinitely-many non-isomorphic simple $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-modules annihilated by $I_{\mathfrak{sp}}(\mu_0)$. We relate the category of bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-modules with a suitable category of perverse sheaves in Subsection 9.4.

**Theorem 2.16.** Let $\bar{\mu}$ be a positive Shale-Weil $n_W$-tuple. Then the category of $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-modules annihilated by $I_{\mathfrak{sp}}(\bar{\mu})$ is equivalent to the direct sum of two copies of the category of perverse sheaves on $W$ with respect to the stratification by $GL(V)$-orbits.

For both cases $W = \Lambda^2 V, S^2 V$, the category of perverse sheaves on $W$ with respect to the stratification by $GL(V)$-orbits is equivalent to a category of representations of an explicitly described quiver with relations [BG], see also the Appendix. The simple objects of this category are enumerated by pairs $(S, Y)$, where $S$ is a $GL(V)$-orbit in $W$ and $Y$ is a simple $GL(V)$-equivariant local system on $S$, i.e. a simple representation of the fundamental group $\pi_1(S)$. As $S$ has to be $GL(V)$-spherical, this group has to be finitely generated and abelian. The category contains infinitely many non-isomorphic simple objects, but only finitely many for any fixed monodromy.

Similar work for the category of bounded weight modules has been done by D. Grantcharov and V. Serganova [GrS1], [GrS2]: they have found a quiver with relations, whose category of modules is equivalent to the category of bounded weight modules (see also [SM] and [GVM]).
3. Preliminaries

3.1. Symplectic geometry. By $\mathcal{T}X$ we denote the tangent bundle of a smooth variety $X$, by $T_x X$ the tangent space to $X$ at a point $x \in X$; by $T^*_x X$ the dual to $T_x X$ space. For a smooth $G$-variety $X$ we denote by $\tau_X : \mathfrak{g} \to \mathcal{T}X$ the canonical homomorphism. Let $Y$ be a smooth subvariety of $X$. We denote by $N_{Y/X}$ and $N^*_{Y/X}$ the total spaces of the normal and conormal bundles to $Y$ in $X$ respectively.

**Definition 3.1.** Suppose that $X$ is a smooth variety which admits a closed nondegenerate two-form $\omega$. Such a pair $(X, \omega)$ is called a symplectic variety. If $X$ is a $G$-variety and $\omega$ is $G$-invariant, $(X, \omega)$ is called a symplectic $G$-variety.

**Example 3.2.** Let $X$ be a smooth $G$-variety. Then $T^*_X$ has a one-form $\alpha_X$ defined at a point $(l, x)(l \in T^*_x X)$ by the equality

$$\alpha_X(\xi) = l(\pi^* \xi)$$

for any $\xi \in T_{(l, x)}(T^*X)$, where $\pi : T^*X \to X$ is the projection. The differential $d\alpha_X$ is a nondegenerate $G$-invariant two-form on $T^*_X$ and therefore $(T^*_X, d\alpha_X)$ is a symplectic $G$-variety.

**Example 3.3.** Let $\mathfrak{g}$ be a $G$-orbit in $\mathfrak{g}^*$. Then $\mathfrak{g}$ has a Kostant-Kirillov-Souriau two-form $\omega(\cdot, \cdot)$ defined at a point $x \in \mathfrak{g}^*$ by the equality

$$\omega_x(\tau_{\mathfrak{g}} p |_x, \tau_{\mathfrak{g}} q |_x) = x([p, q])$$

for $p, q \in \mathfrak{g}$.

**Definition 3.4.** Let $(X, \omega)$ be a symplectic variety. We call a subvariety $Y \subset X$

a) isotropic if $\omega|_{T_y Y} = 0$ for the generic point $y \in Y$;

b) coisotropic if $\omega|_{(T_y Y)^\perp} = 0$ for the generic point $y \in Y$;

c) Lagrangian if $T_y Y = (T_y Y)^\perp$ for the generic point $y \in Y$ or equivalently if it is both isotropic and coisotropic.

**Example 3.5.** Let $X$ be a smooth $G$-variety and $Y \subset X$ be a smooth $G$-subvariety. Then $N^*_{Y/X}$ is Lagrangian in $T^*_X$.

**Proposition 3.6** (see for example [NG, Lemma 1.3.27]). Any closed irreducible conical (i.e. $\mathbb{C}^*$-stable) Lagrangian $G$-subvariety of $T^*_X$ is the closure of the total space of the conormal bundle $N^*_{Y/X}$ to a $G$-subvariety $Y \subset X$.

**Definition 3.7.** Let $(X, \omega)$ be a $K$-symplectic variety. We say that $(X, \omega)$ is $K$-isotropic if any $K$-orbit of some open $K$-stable set $\bar{X} \subset X$ is isotropic. We say that $(X, \omega)$ is $K$-coisotropic if any $K$-orbit of some open $K$-stable subset $\bar{X} \subset X$ is coisotropic.
Let $X$ be a $G$-variety. The map 
$$T^*X \times \mathfrak{g} \to \mathbb{C} \quad ((x, l), g) \mapsto l(\tau_X g|_x),$$
where $g \in \mathfrak{g}, x \in X, l \in T^*_xX$, induces a map
$$\phi_X : T^*X \to \mathfrak{g}^*$$
called the \textit{moment map}. This map provides the following description of a nilpotent orbit $G\mu \subset \mathfrak{g}^*$. Suppose that $P$ is a parabolic subgroup of $G$.

\textbf{Theorem 3.1} (R. W. Richardson [Rich]). The moment map $$\phi_{G/P} : T^*(G/P) \to \mathfrak{g}^*$$
is a proper morphism to the closure of some nilpotent orbit $G\mu$ and is a finite morphism over $G\mu$.

\textbf{Example 3.8.} Let $G = \text{SL}_n$ and $G/P = \mathbb{P}(\mathbb{C}^n)$. Then $\phi_{G/P}(T^*(G/P))$ (considered as a subset of $\mathfrak{sl}_n$) coincides with the set of nilpotent matrices of rank $\leq 1$. The open $\text{SL}_n$-orbit in $\phi_{G/P}(T^*(G/P))$ is a set of $\text{SL}_n$-highest weight vectors and is contained in the closure of any nonzero nilpotent $\text{SL}_n$-orbit in $\mathfrak{sl}_n$.

For $G = \text{SL}_n$, each moment map $\phi_{G/P}$ corresponding to a parabolic subgroup $P$ is a birational isomorphism of $T^*(G/P)$ with the image of $\phi_{G/P}$, and one can obtain the closure of any nilpotent orbit $G\mu$ as the image of a suitable moment map $\phi_{G/P}$ (see [CM] and references therein). Assume $X = G/P$. The following computation shows that $\phi^*_X(\omega_{G\mu}) = \text{d}\alpha_{T^*X|\phi^*_X^{-1}G\mu}$:

$$\text{d}\alpha_X|(x,t)(\tau_X p, \tau_X q) =$$

$$= (\tau_X p \cdot \alpha_X(\tau_X q))|(x,t) - (\tau_X q \cdot \alpha_X(\tau_X p))|(x,t) - \alpha_X([\tau_X p, \tau_X q])|(x,t) -$$

$$= (\tau_X q \cdot \phi^*_X p)|(x,t) - (\tau_X q \cdot \phi^*_X p) - (\phi^*_X[p, q])|(x,t) =$$

$$= (\phi^*_X[p, q])|(x,t) - (\phi^*_X[q, p])|(x,t) - (\phi^*_X[p, q])|(x,t) = x([p, q]) = \omega_x([p, q])$$

for any $p, q \in \mathfrak{g}$ and $(x, t) \in T^*X \subset \mathfrak{g}^* \times X$.

In what follows we call quotients of $\text{SL}(W)$ by parabolic subgroups 'partial $W$-flag varieties' (see Subsection 3.3).

\textbf{3.2. Spherical varieties.} Let $V$ be a finite-dimensional $K$-module. We recall that $B$ is a Borel subgroup of $K$.

\textbf{Definition 3.9.} Let $X$ be an irreducible $K$-variety. Then $X$ is called $K$-\textit{spherical} if and only if there is an open orbit of $B$ on $X$. A $K$-module $W$ is called $K$-spherical if it is $K$-spherical as a $K$-variety.

It is well known that $K$-spherical varieties have many beautiful properties. In particular, the number of $K$-orbits on such varieties is finite [VK]. A subgroup $K' \subset K$ is called \textit{spherical} if the quotient
$K/K'$ is $K$-spherical. For example, any symmetric subgroup $K' \subset K$ is spherical.

The following lemma is a reformulation in the terms of the present thesis of a result of È. Vinberg and B. Kimelfeld [VK] Thm. 2.

**Lemma 3.10.** An irreducible quasiaffine algebraic $K$-variety $X$ is $K$-spherical if and only if the space of regular functions $\mathbb{C}[X]$ is a bounded $\mathfrak{k}$-module. Moreover, if $X$ is $K$-spherical, then $\mathbb{C}[X]$ is a multiplicity-free $\mathfrak{k}$-module.

**Theorem 3.2** (D. Panyushev [Pan, Thm 2.1]). Let $X$ be a smooth irreducible $K$-variety and $M$ a smooth locally closed $K$-stable subvariety. Then the generic stabilizers of the actions of $B$ on $X,N_{M/X}$ and $N^*_{M/X}$ are isomorphic.

Let $X$ be an irreducible $K$-variety. Then there exist open subset $\tilde{X} \subset X$ such that the stabilizers $B_x$ and $B_y$ are conjugate for all $x,y \in \tilde{X}$ [Pan]. For all $x \in \tilde{X}$ there exists a unique connected reductive subgroup $L(x) \subset K_x$ such that $B_x$ is a Borel subgroup of $L(x)$ [Pan] (see also [Gr]).

Let $\text{Gr}(r;V)$ be the variety of $r$-dimensional subspaces of $V$. For $r \in \{1,\ldots,n_V-1\}$ and $x \in \text{Gr}(r;V)$ we denote by $V^r(x) \subset V$ the $r$-dimensional subspace which corresponds to $x$. We apply the construction of [Pan] to the case $X := \text{Gr}(r;V)$. Then $L(x)$ stabilizes $V^r(x)$. Therefore the datum $(K,V,r)$ determines the modules $(L(x),V^r(x))$. The type of $(L(x),V^r(x))$ does not depend on a point $x \in \tilde{X}$ and therefore the datum $(K,V,r)$ determines the pair $(L,V^r)$. One can compute the subgroup $L$ via a technique of doubled actions [Pan].

**Definition 3.11.** We denote by $c_K(X)$ the codimension in $X$ of the generic orbit of $B$.

**Remark 3.12.** The variety $X$ is $K$-spherical if and only if $c_K(X) = 0$.

**Lemma 3.13.** Let $P_1, P_2$ be parabolic subgroups of $G$ and $L_1, L_2$ be Levi subgroups of $P_1$ and $P_2$ respectively. Then

$$c_G(G/P_1 \times G/P_2) = c_{L_1}(G/P_2) = c_{L_2}(G/P_1).$$

**Proof.** Let $\tilde{B}$ be a Borel subgroup of $G$. The generic stabilizer for the action of $\tilde{B}$ on $G/L_1$ is a Borel subgroup of $L_1$. Therefore the generic stabilizer for the action of a Borel subgroup of $L_1$ on $G/P_2$ coincides with the generic stabilizer for the action of $\tilde{B}$ on $G/P_1 \times G/P_2$. We denote this stabilizer $S$. Let $r$ be the rank of $G$. Then

$$c_{L_1}(G/P_2) = \dim G/P_2 - \left(\frac{\dim L_1 + r}{2} - \dim S\right) = \dim (G/P_1 \times G/P_2) - \left(\frac{\dim G + r}{2} - \dim S\right) = c_G(G/P_1 \times G/P_2).$$
3.3. **Grassmannians.** Let $V$ be a finite-dimensional vector space. The set of flags $V_1 \subset \ldots \subset V_s \subset V$ with fixed dimensions $(n_1, \ldots, n_s)$ is a homogeneous space of the group $\text{GL}(V)$, and we denote this variety by $\text{Fl}(n_1, \ldots, n_s; V)$. We call such a variety *partial flag variety*. The varieties $\mathbb{P}(V)$ and $\text{Fl}(1; V)$ are naturally identified.

For any $r \in \{1, \ldots, n_V - 1\}$ we denote $\text{Fl}(r; V)$ by $\text{Gr}(r; V)$. For $r = 1$, the variety $\text{Gr}(r; V)$ coincides with $\mathbb{P}(V)$. We have

$$\text{Gr}(r; V) \cong \text{Gr}(n_V - r; V^*) .$$

Let $n_1, \ldots, n_s \in \{1, \ldots, n_V - 1\}$ be numbers such that

$$n_1 < \ldots < n_s$$

and $\text{Fl}(n_1, \ldots, n_s; V)$ be the corresponding partial flag variety. Let $P(x)$ be the stabilizer of a point $x \in \text{Fl}(n_1, \ldots, n_s; V)$ in $\text{SL}(V)$ and $n(x)$ be a nilpotent radical of the Lie algebra of $P(x)$. Then $n(x) \subset \mathfrak{sl}(V)$ consists of nilpotent elements, and

$$\bigcup_{x \in \text{Fl}(n_1, \ldots, n_s; V)} n(x) \subset \mathfrak{sl}(V)^*$$

coincides with the image of the moment map

$$\phi : T^* \text{Fl}(n_1, \ldots, n_s; V) \to \mathfrak{sl}(V)^* .$$

Moreover, the image of $\phi$ in $\mathfrak{sl}(V)^*$ is the closure of a unique nilpotent $\text{SL}(V)$-orbit. In this way, to any partial flag variety one assigns a unique nilpotent orbit. The condition ‘to be in closure of’ on the set of nilpotent orbits induces a partial order on the set of partial flag varieties. We will make essential use of this partial order. In what follows we refer to one partial flag variety as being higher or lower than another in terms of this partial order. Partial flag varieties which are equivalent in terms of this partial order are called *cotangent-equivalent* (see also [Kn2]).

We now describe this latter equivalence explicitly. Let $\text{Fl}_1$, $\text{Fl}_2$ be partial $W$-flag varieties and

$$(n_1, \ldots, n_s), (n'_1, \ldots, n'_s)$$

be the corresponding dimension vectors. These vectors define the following partitions

$$(n_1, n_2 - n_1, \ldots, n_V - n_s)$$

and

$$(n'_1, n'_2 - n'_1, \ldots, n_V - n'_s)$$

of $n_V$.

**Lemma 3.14** ([CM, Ch. 6.2]). Two partial flag varieties are cotangent-equivalent if and only if their corresponding partitions coincide as sets.

For example, $\text{Gr}(n_1; V)$ and $\text{Gr}(n_V - n_1; V)$ are cotangent-equivalent.
The relation between partial flag varieties and nilpotent \( \text{SL}(V) \)-orbits of \( \mathfrak{sl}(V)^* \) has been described in terms of partitions (see [CM]); the partial order on nilpotent \( \text{SL}(V) \)-orbits in \( \mathfrak{sl}(V)^* \) has been also described in terms of partitions (see [CM]). Therefore in order to check that a given partial flag variety is higher then another one it suffices to check the corresponding condition on partitions. In this way we establish in particular the following statements.

(1) Any partial flag variety is higher than or is cotangent-equivalent to \( \mathbb{P}(V) \).
(2) If \( r_1, r_2 \in \{1, \ldots, \left[ \frac{n_v}{2} \right] \} \) and \( r_1 > r_2 \), then \( \text{Gr}(r_1; V) \) is higher than \( \text{Gr}(r_2; V) \).
(3) The subset of Grassmannians \( \text{Gr}(r; V) \) is totally ordered.
(4) If \( \text{Fl} \) is a partial flag variety which is not cotangent-equivalent to \( \mathbb{P}(V) \), then \( \text{Fl} \) is higher than or is cotangent-equivalent to \( \text{Gr}(2; V) \).
(5) Any partial \( V \)-flag variety is cotangent-equivalent to one of the following

\[ \text{Gr}(r; V), \; \text{Fl}(1, 2; V), \; \text{Fl}(1, 3; V), \]

or is higher than \( \text{Fl}(1, 3; V) \).
(6) Any partial \( V \)-flag variety is cotangent-equivalent to one of the following

\[ \text{Gr}(r; V), \; \text{Fl}(1, r; V), \; \text{Fl}(1, 2, 3; V), \; \text{Fl}(2, 4; V), \]

or is higher than \( \text{Fl}(2, 4; V) \).

**Proposition 3.15.** Suppose \( \text{Fl}_1 \) is a \( K \)-spherical variety and \( \text{Fl}_2 \) is lower then \( \text{Fl}_1 \). Then \( \text{Fl}_2 \) is a \( K \)-spherical variety.

To prove the proposition we need to recall some results of I. Losev.

**Theorem 3.3** (I. Losev [Lo]). Suppose \( X \) is a strongly equidefectinal [Lo, Def. 1.2.5] normal affine irreducible Hamiltonian \( K \)-variety. Then

\[ \mathbb{C}(X)^K = \text{Quot}(\mathbb{C}[X]^K), \]

where \( \text{Quot}(\mathbb{C}[X]^K) \) is the field of fractions of \( \mathbb{C}[X]^K \).

**Corollary 3.16.** Suppose \( X \) is a strongly equidefectinal affine irreducible Hamiltonian \( K \)-variety. Then \( \mathbb{C}(X)^K = \text{Quot}(\mathbb{C}[X]^K) \).

**Proof.** Let \( X \) be the spectrum of the integral closure of \( \mathbb{C}[X] \) in \( \text{Quot}(\mathbb{C}[X]) \) and \( X \to X \) be the canonical finite map. Then \( X \) is a normal affine irreducible Hamiltonian \( K \)-variety [Kd]. A straightforward check shows that \( X \) is strongly equidefectinal. As \( \mathbb{C}[X]^K \) is a finite extension of \( \mathbb{C}[X]^K \) of degree 1 and \( \text{Quot}(\mathbb{C}[X]^K) = \mathbb{C}(X)^K = \mathbb{C}(X)^K \), we have

\[ \text{Quot}(\mathbb{C}[X]^K) = \mathbb{C}(X)^K. \]
The closure of any $G$-orbit in $\mathfrak{g}^*$ is a strongly equidefectinal affine irreducible Hamiltonian $K$-variety \cite[Corollary 3.4.1]{Lo}.

**Theorem 3.4.** Let $Z \subset \mathfrak{g}^*$ and $Z' \subset \mathfrak{g}^*$ be nilpotent $G$-orbits such that $Z' \subset \overline{Z}$. If $Z$ is $K$-coisotropic then $Z'$ is $K$-coisotropic.

**Proof.** As $K$-action on $Z$ is $K$-coisotropic, $\mathbb{C}(Z)^K$ is a Poisson-commutative subfield of $\mathbb{C}(Z)$ \cite[Ch. II, Prop. 5]{Vi}. By Theorem 3.3 this is equivalent to the Poisson-commutativity of $\mathbb{C}[Z]^K$. As $\mathbb{C}[Z']^K$ is a quotient of $\mathbb{C}[Z]^K$, $\mathbb{C}[Z']^K$ is Poisson-commutative. Then the field $\mathbb{C}(Z')^K$ is Poisson-commutative too. Therefore the $K$-action on $Z'$ is $K$-coisotropic. □

**Proof of Proposition 3.15.** The sphericity of a $K$-variety $X$ is equivalent to the $K$-coisotropicity of the $K$-variety $T^*X$. Therefore the statement follows from Theorem 3.4. □

**Corollary 3.17.** Suppose $\text{Fl}(n_1, \ldots, n_s; V)$ is a $K$-spherical variety. Then the variety $\mathbb{P}(V)$ is $K$-spherical.

The following Lemma becomes a useful tool in Section 6 and is a trivial corollary of Proposition 3.15.

**Lemma 3.18.** If some partial $W$-flag variety, which is not cotangent-equivalent to $\mathbb{P}(W)$, is $K$-spherical, then the variety $\text{Gr}(2; W)$ is $K$-spherical.

3.4. D-modules versus $\mathfrak{g}$-modules. Let $\mathfrak{h}_g \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$; $\Delta \subset \mathfrak{h}_g$ be a root system; $\Delta^+ \subset \Delta$ be a set of positive roots. Denote by $\mathfrak{h}^*$ the set of weights $\lambda$ such that $\alpha^\vee(\lambda)$ is not a strictly positive integer for any root $\alpha$ of the dual root system $\Delta^\vee \subset \mathfrak{h}_g^*$.

For a fixed $\lambda$ we denote by $D^\lambda(X)$ the sheaf of twisted differential operators on $X$ and by $D^\lambda(X)$ its space of global sections. The algebras $D^\lambda(X)$ and $D^\mu(X)$ are naturally identified if $\lambda$ and $\mu$ lie in a single shifted orbit of the Weyl group \cite{HMSW}. Moreover, any such orbit intersects $\mathfrak{h}^*$ \cite{HMSW}. If $\lambda \in \mathfrak{h}^*$, the isomorphism

$$\tau : U(\mathfrak{g})/(\text{Ker} \chi_\lambda) \xrightarrow{\sim} D^\lambda(X)$$

established in \cite{BeBe2} enables us to identify the category of $D^\lambda(X)$-modules with the category of $\mathfrak{g}$-modules affording the central character $\chi = \chi_\lambda$. We have the following functors:

$$\text{GSec} : \text{Sheaves}_{X}^{\lambda} \rightarrow \mathfrak{g}\text{-modules}^\lambda \quad \text{Loc} : \text{Sheaves}_{X}^{\lambda} \leftarrow \mathfrak{g}\text{-modules}^\lambda$$

$$\mathcal{F} \rightarrow \Gamma(X, \mathcal{F}) \quad \quad M \otimes_{(1 \otimes \tau)U(\mathfrak{g})} D^\lambda(X) \leftarrow M$$
A. Beilinson and J. Bernstein [BeBe2] have proved that if $\lambda \in \hat{h}^*$ then $GSec(Loc)$ equals the identity.

For $\lambda \in h^*$ the sheaf $D^\lambda(X)$ has a natural filtration by degree

$$0 \subset \mathcal{O}(X) \subset D_1 \subset ... \subset D(X) = \lim_{i \to \infty} D_i.$$  

The relative spectrum of the associated graded sheaf of algebras

$$\text{gr } D^\lambda(X) = \oplus_{i \in \mathbb{Z}_{\geq 0}} (D_{i+1}/D_i)$$

is isomorphic to $T^*X$.

Let $M$ be a quasicoherent $D^\lambda(X)$-module which admits $\mathcal{O}(X)$-coherent generating subsheaf $M_{\text{gen}}$ of $M$. The associated graded sheaf $\text{gr}M$ with respect to the filtration

$$0 \subset M_{\text{gen}} \subset D_1 M_{\text{gen}} \subset ... \subset M$$

is a $\text{gr}D^\lambda(W)$-module. By definition, the singular support $\mathcal{V}(M)$ of $M$ is the support of $\text{gr}M$ in $\text{Spec}_X \text{gr}D^\lambda(X) \cong T^*X$.

**Theorem 3.5** (O. Gabber [Gab]). The variety $\mathcal{V}(M)$ is coisotropic in $T^*X$. In particular

$$\dim \mathcal{V} \geq \dim X$$

for any irreducible component $\mathcal{V} \subset \mathcal{V}(M)$.

**Definition 3.19.** The $D^\lambda(X)$-module $M$ is called holonomic if

$$\dim \mathcal{V}(M) = \dim X.$$

3.5. **Associated varieties of $\mathfrak{g}$-modules.** The algebra $U(\mathfrak{g})$ has a natural filtration by degree

$$0 \subset C \subset U_1 \subset ... \subset U(\mathfrak{g}) = \bigcup_{i \in \mathbb{Z}_{\geq 0}} U_i.$$  

The associated graded algebra

$$\text{gr } U(\mathfrak{g}) = \oplus_{i \in \mathbb{Z}_{\geq 0}} (U_{i+1}/U_i)$$

is isomorphic to $S(\mathfrak{g})$. The filtration $\{U_i\}_{i \in \mathbb{Z}_{\geq 0}}$ induces the filtration on any ideal $I$ of $U(\mathfrak{g})$, namely $\{I \cap U_i\}_{i \in \mathbb{Z}_{\geq 0}}$, hence $\text{gr}I \subset S(\mathfrak{g})$ is a well defined ideal. The ideal $\text{gr}I$ of the commutative algebra $S(\mathfrak{g})$ determines the variety

$$\mathcal{V}(\text{gr}I) := \{ x \in \mathfrak{g}^* \mid f(x) = 0 \text{ for all } f \in \text{gr}I \}.$$  

In particular, if $I=\text{Ann}M$ for a $\mathfrak{g}$-module $M$, we set $GV(M) := \mathcal{V}(\text{gr}I)$.

**Theorem 3.6** ([Jo2]). For a simple $\mathfrak{g}$-module $M$ the variety $GV(M)$ is the closure of an orbit $Gu$, and furthermore $0 \in Gu$.

Suppose that $M$ is a $U(\mathfrak{g})$-module and that a filtration

$$0 \subset M_0 \subset M_1 \subset ... \subset M = \bigcup_{i \in \mathbb{Z}_{\geq 0}} M_i$$

of vector spaces is given. We say that this filtration is good if

1. $U_i M_j = M_{i+j}$;
2. $\dim M_i < \infty$ for all $i \in \mathbb{Z}_{\geq 0}$.
Such a filtration arises from any finite-dimensional space of generators $M_0$. The corresponding associated graded object $\text{gr} M = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} M_{i+1}/M_i$ is a module over $\text{gr} \mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$, and we set

$$J_M := \{ s \in S(\mathfrak{g}) \mid \text{there exists } k \in \mathbb{Z}_{\geq 0} \text{ such that } s^k m = 0 \text{ for all } m \in \text{gr} M \}.$$ 

In this way we associate to any $\mathfrak{g}$-module $M$ the variety

$$V(M) := \{ x \in \mathfrak{g}^* \mid f(x) = 0 \text{ for all } f \in J_M \}.$$ 

It is easy to check that the module $\text{gr} M$ depends on the choice of good filtration, but the ideal $J_M$ and the variety $V(M)$ does not. Indeed, let $M_0, M'_0$ be different generating spaces of $M$ and $\{ M_i \}_{i \in \mathbb{Z}_{\geq 0}}, \{ M'_i \}_{i \in \mathbb{Z}_{\geq 0}}$ be the corresponding filtrations of $M$. Then there exist $r, s \in \mathbb{Z}_{\geq 0}$ such that $M_0 \in M'_r$ and $M'_0 \in M_s$. Let $f \in S(\mathfrak{g})$ be an element of the degree $d$ such that $fM_i \subset M_{i+d-1}$ for all $i \in \mathbb{Z}_{\geq 0}$.

Then

$$f^{r+s+1}M'_{s+(d-1)(r+s+1)} \subset M'_{s+r+(d-1)(r+s+1)} = M'_{s+r+1}d-1.$$ 

Therefore $J^{r+s+1}_M \subset J'_M$. In the same way $J^{s+r+1}_M \subset J'_M$, and therefore $V(M) = V(M)'$.

**Theorem 3.7** (Bernstein’s theorem [KL, p. 118]). Let $M$ be a finitely generated $\mathfrak{g}$-module. Then $\dim V(M) \geq \frac{1}{2} \dim G V(M)$.

Let $M$ be a finitely generated $\mathfrak{g}$-module which affords a generalized central character.

**Theorem 3.8** (O. Gabber [Gab]). Let $\tilde{V}$ be an irreducible component of $V(M)$ and $Z$ be the unique open $G$-orbit of

$$G\tilde{V} := \{ x \in \mathfrak{g}^* \mid x = gv \text{ for some } g \in G \text{ and } v \in \tilde{V} \}.$$ 

Then $\tilde{V} \cap Z$ is a coisotropic subvariety of $Z$. In particular

$$\dim \tilde{V} \geq \frac{1}{2} \dim G\tilde{V}.$$ 

**Definition 3.20.** A simple $\mathfrak{g}$-module $M$ is called holonomic if

$$\dim \tilde{V} = \frac{1}{2} \dim G\tilde{V}$$ 

for any irreducible component $\tilde{V}$ of $V(M)$.

**Corollary 3.21** (S. Fernando [F]). The vector space

$$V(M)^\perp = \{ g \in \mathfrak{g} \mid v(g) = 0 \text{ for all } v \in V(M) \}$$ 

is a Lie algebra and $V(M)$ is a $V(M)^\perp$-variety.

**Theorem 3.9** (S. Fernando [F, Cor. 2.7], V. Kac [Kac2]). Set
\[ g[M] := \{ g \in g \mid \dim(\text{span}_{i \in \mathbb{Z}_{\geq 0}} \{ g^i m \}) < \infty \text{ for all } m \in M \}. \]

Then \( g[M] \) is a Lie algebra and \( g[M] \subset V(M)^\perp \).

**Corollary 3.22.** For a \((g, \mathfrak{t})\)-module \( M \), \( V(M) \subset \mathfrak{t}^\perp \) and \( V(M) \) is a \( K \)-variety.

Let \( M \) be a \((g, \mathfrak{t})\)-module and \( M_0 \) be a \( \mathfrak{t} \)-stable finite-dimensional space of generators of \( M \); \( J_M, \text{gr} M \) be the corresponding objects constructed as above. Consider the \( \text{S}(g) \)-modules

\[ J^{-i}_M \{ 0 \} := \{ m \in \text{gr} M \mid j_1 \ldots j_i m = 0 \text{ for all } j_1, \ldots, j_i \in J_M \}. \]

One can easily see that these modules form an ascending filtration

\[ 0 \subset J^{-1}_M \subset \ldots \subset \text{gr} M \]

such that

\[ \bigcup_{i=1}^{\infty} J^{-i}_M \{ 0 \} = \text{gr} M. \]

Since \( \text{S}(g) \) is a N"otherian ring, the filtration stabilizes, i.e. \( J^{-i}_M \{ 0 \} = \text{gr} M \) for some \( i \). By \( \overline{\text{gr}} M \) we denote the corresponding graded object. By definition, \( \overline{\text{gr}} M \) is an \( \text{S}(g)/J_M \)-module. Suppose that \( f \overline{\text{gr}} M = 0 \) for some \( f \in \text{S}(g) \). Then \( f \text{gr} M = 0 \) and hence \( f \in J_M \). This proves that the annihilator of \( \overline{\text{gr}} M \) in \( \text{S}(g)/J_M \) equals zero.

As \( M \) is a finitely generated \( g \)-module, the \( \text{S}(g) \)-modules \( \text{gr} M \) and \( \overline{\text{gr}} M \) are finitely generated. Let \( \tilde{M}_0 \) be a \( \mathfrak{t} \)-stable finite-dimensional space of generators of \( \overline{\text{gr}} M \). Then there is a surjective homomorphism

\[ \psi : \tilde{M}_0 \mathbb{C}(\text{S}(g)/J_M) \to \overline{\text{gr}} M. \]

Set

\[ \text{Rad} M := \{ m \in \overline{\text{gr}} M \mid \text{there exists } f \in \text{S}(g)/J_M \text{ such that } fm = 0 \text{ and } f \neq 0 \}. \]

The space \( \text{Rad} M \) is a \( \mathfrak{t} \)-stable \( \text{S}(g) \)-submodule of \( \overline{\text{gr}} M \) and \( \tilde{M}_0 \not\subset \text{Rad} M \). The homomorphism \( \psi \) determines an injective homomorphism

\[ \hat{\psi} : \text{S}(g)/J_M \to \tilde{M}_0 \mathbb{C} \overline{\text{gr}} M. \]

**Proposition 3.23.** a) The module \( M \) is bounded if and only if all irreducible components of \( V(M) \) are \( K \)-spherical.

b) If the equivalent conditions of a) are satisfied, then any irreducible component \( \tilde{V} \) of \( V(M) \) is a conical Lagrangian subvariety of \( GV \).

**Proof.** a) Suppose that all irreducible components of \( V(M) \) are \( K \)-spherical. Then

\[ \tilde{M}_0 \mathbb{C}(\text{S}(g)/J_M) \]

is a bounded \( \mathfrak{t} \)-module. Therefore \( \overline{\text{gr}} M \) is bounded, which implies that \( M \) is a bounded \( \mathfrak{t} \)-module too.
Assume now that a \( g \)-module \( M \) is \( \mathfrak{t} \)-bounded. Then \( S(\mathfrak{g})/J_M \) is a \( \mathfrak{t} \)-bounded module and all irreducible components of \( V(M) \) are \( K \)-spherical. This completes the proof of a).

b) Let \( \tilde{V} \subset V(M) \) be an irreducible component and \( x \in \tilde{V} \) be a generic point. As \( x \in \mathfrak{t}^\perp \), we have

\[
x([k_1, k_2]) = \omega_x(\tau_g^* k_1|_x, \tau_g^* k_2|_x) = 0
\]

for all \( k_1, k_2 \in \mathfrak{t} \). Therefore any \( K \)-orbit in \( \mathfrak{t}^\perp \) is isotropic. As \( \tilde{V} \) is a spherical variety, \( \tilde{V} \) has an open \( K \)-orbit. Therefore \( \tilde{V} \) is Lagrangian in \( G\tilde{V} \) and this completes the proof of b). □

**Corollary 3.24** ([PS]). Let \( M \) be a finitely generated bounded \((g, \mathfrak{t})\)-module. Then \( \dim \mathfrak{h}_t \geq \frac{1}{2} \dim GV(M) \).

**Proof.** As \( M \) is bounded, \( V(M) \) is \( K \)-spherical and therefore \( \dim \mathfrak{h}_t \geq \dim V(M) \). We have \( \dim V(M) \geq \frac{1}{2} \dim GV(M) \). □

**Lemma 3.25** ([VP]). Let \( \hat{X} \) be an affine \( K \)-variety. Then \( \mathbb{C}[X] \) has finite type as a \( \mathfrak{t} \)-module if and only if \( X \) contains only finite number of the closed \( K \)-orbits. In the latter case any irreducible component of \( X \) contains precisely one closed \( K \)-orbit.

**Lemma 3.26.** A finitely generated \((g, \mathfrak{t})\)-module \( M \) has finite type over \( \mathfrak{t} \) if and only if the variety \( V(M) \) contains only finitely many closed \( K \)-orbits. In this case the unique closed orbit of \( V(M) \) is the point 0.

**Proof.** If \( V(M) \) contains the unique closed \( K \)-orbit then \( \mathbb{C}[V(M)] \) is a \( \mathfrak{t} \)-module of finite type and therefore \( \tilde{M}_0 \otimes_{\mathbb{C}} (S(\mathfrak{g})/J_M) \) and \( \overline{gr} M \) are \( \mathfrak{t} \)-modules of finite type.

Assume that \( M \) is a \( \mathfrak{t} \)-module of finite type. Then \( S(\mathfrak{g})/J_M \) is a \( \mathfrak{t} \)-module of finite type and \( V(M) \) contains only finite number of the closed \( K \)-orbits. On the other hand, \( V(M) \) is \( \mathbb{C}^* \)-stable and hence any irreducible component of \( V(M) \) contains 0. □

3.6. **Other faces of the support variety.** In this section \( X \) is a variety of Borel subalgebras of \( g \). We recall the singular support \( \mathcal{V}(M) \subset T^* X \) of any coherent \( \mathcal{D}^\lambda(X) \)-module \( M \) is defined. The correspondence between \( g \)-modules and \( \mathcal{D}^\lambda(X) \)-modules allows us to assign interesting geometric objects to a \( g \)-module.

Let \( M \) be a finitely generated \( g \)-module which affords a central character \( \chi \) and \( \lambda \in \hat{h}^* \) (see Subsection 3.4) be a weight such that \( \chi = \chi_\lambda \).

**Definition 3.27.** The **singular support of \( M \)** is the variety \(
\mathcal{V}(M) := \mathcal{V}(\text{Loc}M).
\)
Definition 3.28. The support variety $L(M)$ of $M$ is the projection of $V(M)$ to $X$.

Let $\phi_X : T^*X \to g^*$ be the moment map. D. Barlet and M. Kashiwara [BK], have proved that

$V(M) = \phi_X(V(\text{Loc}_M))$

(see also [BoBry]). Therefore we have a diagram

$\begin{align*}
V(M) \subset T^*X & \xleftarrow{\phi_X} \xrightarrow{\text{pr}} g^* \xleftarrow{\phi_X} L(M) \subset X
\end{align*}$

3.7. Hilbert-Mumford criterion. Let $X$ be an affine $K$-variety, $V$ be a $K$-module.

Theorem 3.10 (Hilbert-Mumford [VP]). The closure of any $K$-orbit $Kx \subset X$ contains a unique closed orbit $K\bar{x} \subset X$. There exists a group homomorphism $\mu : \mathbb{C}^* \to K$ such that $\lim_{t \to 0} \mu(t)x = \bar{x} \in K\bar{x}$.

The null cone $N_K(V) := \{ x \in V \mid 0 \in Kx \}$ is a closed algebraic subvariety of $V$ [VP].

Theorem 3.11 ([VP]). Fix $x \in V$. Then $0 \in K\bar{x}$ if and only if there exists a rational semisimple element $h \in \mathfrak{k}$ such that $x \in V_h^{>0}$, here $V_h^{>0}$ is the direct sum of $h$-eigenspaces in $V$ with positive eigenvalues.

Corollary 3.29 ([VP]). There exists a finite set $H \subset \mathfrak{k}$ of rational semisimple elements such that $N_K(V) := \bigcup_{h \in H} KV_h^{>0}$, where

$KV_h^{>0} := \{ v \in V \mid v = kv_h \text{ for some } k \in K \text{ and } v_h \in V_h^{>0} \}$

3.8. A monoid of projective functors. Let $\mathfrak{b}_g$ be a Borel subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}_g$ be a Cartan subalgebra of $\mathfrak{b}_g$. Let $\Delta \subset \mathfrak{h}_g^*$ be the root system of $\mathfrak{g}$, $\Delta^+ \subset \Delta$ be the set of positive roots of $\mathfrak{b}_g$, $\Pi \subset \Delta^+$ be the set of simple roots. We denote by $s_\alpha$ the reflection of $\mathfrak{h}_g^*$ with respect to $\alpha$ for $\alpha \in \Delta$, and $W^\#$ is the group generated by $\{s_\alpha\}_{\alpha \in \Delta}$, i.e. the Weyl group of $\mathfrak{g}$. Let $\alpha^\vee \in \mathfrak{h}_g$ be the coroot of $\alpha \in \Delta$, and

$\rho := \frac{1}{2} \Sigma_{\alpha \in \Delta^+} \alpha$

We introduce a partial order on $\mathfrak{h}_g^*$ (cf. [BeGu, 1.5]). If $\phi, \psi \in \mathfrak{h}_g^*$, $\gamma \in \Delta^+$ we put $\phi <^\gamma \psi$ whenever $\phi = s_\gamma \psi$ and $\gamma^\vee(\psi) \in \mathbb{Z}_{>0}$. We put $\phi < \psi$ whenever there exists sequences of weights $\phi_0, ..., \phi_n$ and of roots $\gamma_0, ..., \gamma_n$ such that

$\phi <^{\gamma_0} \phi_0 < ... <^{\gamma_n} \phi_n = \psi$. 
We write $\phi \leq \psi$ whenever $\phi = \psi \lor \phi < \psi$. A weight is called dominant if it is maximal with respect to partial order $\prec$.

We denote by $W^g_\phi$ the stabilizer in $W^g$ of $\phi \in h^*_g$. By definition, $\phi \in h^*_g$ integral if $\alpha^\vee(\phi) \in \mathbb{Z}$ for any $\alpha \in \Delta$. Furthermore, $\phi \in h^*_g$ is regular if $W^g_\phi = \{e\}$. We call a pair of weights $(\phi, \psi) \in h^*_g \times h^*_g$ correctly ordered if $\phi$ is dominant, $\phi - \psi$ is an integral weight and $\psi \leq w\psi$ for all $w \in W^g_\phi$. In addition, we define two weights $\phi_1, \phi_2$ to be equivalent if $\phi_1 - \phi_2$ is integral and $W^g_{\phi_1} = W^g_{\phi_2}$.

Let $O$ be the category of finitely generated $g$-modules such that the action of $b^g$ is locally finite and the action of $h^g$ is semisimple. Let $\lambda \in h^*_g$ be a weight. We denote by $M_\lambda$ the Verma module with the highest weight $\lambda - \rho$, and by $L_\lambda$ the unique simple quotient of $M_\lambda$. Furthermore, $\phi \in h^*_g$ is regular if $W^g_{\phi} = \{e\}$. We call a pair of weights $(\phi, \psi) \in h^*_g \times h^*_g$ correctly ordered if $\phi$ is dominant, $\phi - \psi$ is an integral weight and $\psi \leq w\psi$ for all $w \in W^g_\phi$. In addition, we define two weights $\phi_1, \phi_2$ to be equivalent if $\phi_1 - \phi_2$ is integral and $W^g_{\phi_1} = W^g_{\phi_2}$.

The modules $M_\lambda, L_\lambda, P_\lambda$ afford the same generalized central character and we denote this central character by $\chi_\lambda$. Let $I$ be a primitive ideal of $U(g)$. By a famous theorem of Duflo [Dix], $I = \text{Ann} L_\lambda$ for some weight $\lambda \in h^*_g$.

Let $E$ be a finite-dimensional $g$-module. We have the functor
\[ F_E : g\text{-mod} \to g\text{-mod}, \quad M \mapsto E \otimes M. \]

The restriction of $F_E$ to the category of $g$-modules affording a given generalized central character $\chi$ is a direct sum of a finite number of indecomposable exact functors. Following [BeG], we call a direct summand of $F_E$ a projective functor.

The set of indecomposable projective functors arising from all finite-dimensional $g$-modules $E$ is naturally identified with the set of $W^g$-orbits in $h^*_g \times h^*_g$ [BeG, 3.3]. Let $(\phi, \psi) \in h^*_g \times h^*_g$ be a correctly ordered pair and $H^\psi_\phi$ be the corresponding projective functor. Then
\[ H^\psi_\phi M_{\phi} = P_{\psi}, \]
and the functors $H^\psi_\phi$ and $H^\phi_\psi$ are adjoint one to each other. Assume that $\phi - \psi$ is integral and $W^g_\phi = W^g_\psi$. Then
\[ (H^\psi_\phi, H^\phi_\psi) \]
is a pair of mutually inverse equivalences of the respective categories of $g$-modules affording the generalized central characters $\chi_\varphi$ and $\chi_\psi$.

Let $\text{PFunc}(\chi_\lambda)$ be the set of projective functors as above which preserve the category of $g$-modules affording the generalized central character $\chi_\lambda$. Obviously $\text{PFunc}(\chi_\lambda)$ carries an additive structure and is a monoid with respect to it. We denote by $\text{PFunc}(\chi_\lambda)$ the minimal abelian group which contains $\text{PFunc}(\chi_\lambda)$. As the composition of projective functors is a projective functor, $\text{PFunc}(\chi_\lambda)$ carries a multiplicative structure. Therefore $\text{PFunc}(\chi_\lambda)$ is a unitary ring.

**Proposition 3.30.** Let $\lambda$ be a regular dominant integral weight. Then the ring $\text{PFunc}(\chi_\lambda)$ is isomorphic to the ring $\mathbb{Z}[W^g]$.

**Proof.** Let $\mathcal{O}_\lambda$ be the subcategory of $\mathcal{O}$ consisting of $g$-modules which afford the generalized central character $\chi_\lambda$. Let $[\mathcal{O}]_g$ be the free abelian group generated by the isomorphism classes of simple $g$-modules of the category $\mathcal{O}$, and let $[\mathcal{O}_\lambda]_g$ be the subgroup generated by the simple objects of $\mathcal{O}_\lambda$. Any object $M$ of $\mathcal{O}$ has finite length and therefore determines $[M] \in [\mathcal{O}]_g$.

Let $\mathcal{H}_1, \mathcal{H}_2 \in \text{PFunc}(\chi_\lambda)$. Then $\mathcal{H}_1 = \mathcal{H}_2$ if and only if $[\mathcal{H}_1(M_\lambda)] = [\mathcal{H}_2(M_\lambda)]$.

For any $w \in W^g$, we have $\mathcal{H}_1^{w\lambda} \in \text{PFunc}(\chi_\lambda)$ and $\mathcal{H}_1^{w\lambda}M_\lambda = P_{w\lambda}$. The elements $\{[M_{w\lambda}]\}_{w \in W^g}$ form a basis of the free abelian group $[\mathcal{O}_\lambda]_g$, and therefore $[\mathcal{O}_\lambda]_g$ is identified with $\mathbb{Z}^{W^g}$ as a set. We introduce an action of $W^g$ on $[\mathcal{O}_\lambda]_g$ via the formula

$$(w, [M_{w'\lambda}]) \mapsto [M_{ww'\lambda}] \text{ for any } w', w \in W^g.$$  

Any functor $\mathcal{H} \in \text{PFunc}(\chi_\lambda)$ commutes with this action. As $\{[P_{w\lambda}]\}_{w \in W^g}$ is a basis of the free abelian group $[\mathcal{O}_\lambda]_g$, the map

$$\text{PFunc}(\chi_\lambda) \to \mathbb{Z}[W^g] = \mathbb{Z}^{W^g},$$

given by $\mathcal{H} \mapsto [\mathcal{H}(M_\lambda)]$ on generators, is an isomorphism. \hfill \Box

**Theorem 3.12 (BeG).** Let $\lambda_1, \lambda_2$ be dominant weights such that the difference $\lambda_1 - \lambda_2$ is integral and $W^g_{\lambda_1} = W^g_{\lambda_2}$, i.e. $\lambda_1$ is equivalent to $\lambda_2$. Then

a) the categories of $g$-modules which afford the generalized central characters $\chi_{\lambda_1}$ and $\chi_{\lambda_2}$ are equivalent;

b) for a given $w \in W^g$, the categories of $g$-modules annihilated by $\text{Ann} L_{w\lambda_1}$ and $\text{Ann} L_{w\lambda_2}$ are equivalent.

The subcategories of locally finite $\mathfrak{g}$-modules and bounded $\mathfrak{g}$-modules are stable under this equivalence.
3.9. **Bounded weight modules.** In the joint work \[PS\] I. Penkov and V. Serganova proved that 'boundness' is not a property of a module \(M\) but of the ideal \(\text{Ann} M\). More precisely, this means the following.

**Theorem 3.13.** Let \(M\) and \(N\) be simple \((\mathfrak{g}, \mathfrak{k})\)-modules such that \(\text{Ann} M = \text{Ann} N\). Suppose that the function \([M : \cdot]_T\) is uniformly bounded by a constant \(C_M\). Then the function \([N : \cdot]_T\) is uniformly bounded by \(C_M\) (see also Theorem 2.5).

To classify bounded modules \(M\) we should first classify 'bounded ideals' \(\text{Ann} M\), i.e. ideals for which exist at least one bounded module.

We fix the following notation:
- \(\{e_i\}_{i \leq n_W} \subset W\) is a basis of \(W\), \(\{e_i^*\}_{i \leq n_W} \subset W^*\) is the dual basis;
- \(\{e_{i,j}\}_{i \leq n_W, j \leq n_W} \subset \mathfrak{gl}(W)\) stands for the elementary matrix \(e_i \otimes e_j^*\);
- \(\varepsilon_i := e_{i,i} - \frac{1}{n_W}(e_{1,1} + e_{2,2} + ... + e_{n_W,n_W})\) for \(i \leq n_W\);
- \(\mathfrak{h}_W := \text{span}(e_{i,i})_{i \leq n_W}\).

3.9.1. **The case \(\mathfrak{g} = \mathfrak{sl}(W)\).** We identify all weights of \(\mathfrak{h}^\text{sl}_W := \mathfrak{h}_W \cap \mathfrak{sl}(W)\) with the set of \(n_W\)-tuples modulo the equivalence relation

\[(\lambda_1, \lambda_2, ..., \lambda_{n_W}) \leftrightarrow (\lambda_1 + k, \lambda_2 + k, ..., \lambda_{n_W} + k)\]

We fix the set of positive roots of \(\mathfrak{sl}(W)\) in \(\mathfrak{h}^\text{sl}_W\) for which \(\rho\) is identified with \((n_W, ..., 1)\) and \(\varepsilon_i\) is identified with the \(n_W\)-tuple \((i\text{ zeros }, 1, n_W - i - 1\text{ zeros})\). Let \(\lambda\) be an \(n_W\)-tuple and \(\bar{\lambda}\) be the corresponding weight. We put \(I(\lambda) := \text{Ann} L_\lambda\). Duflo's theorem \[DIX\] claims now that, if \(I\) is any primitive ideal of \(U(\mathfrak{sl}(W))\), then

\[I = I(\lambda)\]

for some \(n_W\)-tuple \(\bar{\lambda}\).

**Theorem 3.14 (\[M]\).** The \((\mathfrak{sl}(W), \mathfrak{h}^\text{sl}_W)\)-module \(L_\lambda\) is \(\mathfrak{h}^\text{sl}_W\)-bounded if and only if \(\lambda\) is a semi-decreasing tuple.

The bounded \((\mathfrak{sl}(W), \mathfrak{h}^\text{sl}_W)\)-modules split into coherent families \[M\], and all simple modules from one coherent family share an annihilator. Let \(\bar{\lambda}\) be a semi-decreasing tuple. Assume that \(\bar{\lambda}\) is not regular integral, i.e. it is singular integral or semi-integral. Then there exists precisely one coherent family annihilated by \(\text{Ker} \chi_\lambda \[M\]. Therefore all simple bounded \((\mathfrak{sl}(W), \mathfrak{h}^\text{sl}_W)\)-modules which afford the central character \(\chi_\lambda\) have the same annihilator.
V. Serganova and D. Grantcharov [GrS1] note that the lemma below follows from the results of O. Mathieu [M]. See also Theorem 3.12.

**Lemma 3.31.** Assume $\bar{\lambda}$ is a semi-decreasing tuple and that $\bar{\lambda}$ is not regular integral. Then there exists $s \in \mathbb{C}$ such that

$$e^{2\pi is} = m(\lambda)$$

and the categories of $\mathfrak{sl}(W)$-modules annihilated by $I(\lambda)$ and $I(s, n_W - 1, ..., 1)$ are equivalent.

The subcategories of locally finite $\mathfrak{f}$-modules and bounded $\mathfrak{f}$-modules are stable under this equivalence.

Assume that $\bar{\lambda}$ is regular integral. Let $\text{ord}(\bar{\lambda})$ be a decreasing $n_W$-tuple which coincides as a set with $\bar{\lambda}$. For $k \in \{1, ..., n_W - 1\}$ we denote by $s_k$ the permutation $(k, k + 1) \in S_{n_W}$. There exist precisely $n_W - 1$ different coherent families annihilated by $\text{Ker}_\lambda$ and any such family contains $L_{s_k, \text{ord}(\lambda)}$ for some $k \in \{1, ..., n_W - 1\}$. Therefore

$$I(\lambda) = I(s_k, \text{ord}(\lambda))$$

for some $k \in \{1, ..., n_W - 1\}$ [M].

Consider the Weyl algebra of differential operators $D(W)$ on $W$. This algebra is generated by $e_i$ and $\partial e_i$ for $i \leq n_W$. Set

$$E := e_1 \partial e_1 + ... + e_{n_W} \partial e_{n_W}.$$

The operator $[E, \cdot] : D(W) \to D(W)$ is semisimple, and $D(W)$ splits into a direct sum of eigenspaces

$$\{D^i(W)\}_{i \in \mathbb{Z}}$$

with respect to this operator.

The homomorphism of Lie algebras

$$\mathfrak{gl}(W) \to D^0(W), \quad e_{i,j} \mapsto e_i \partial e_j$$

induces a surjective homomorphism

$$\phi : U(\mathfrak{gl}(W)) \to D^0(W).$$

The element $E - t$ generates a two-sided ideal in $D^0(W)$. We denote the corresponding quotient by $D^0 \mathbb{P}(W)$. Furthermore, $\phi$ induces a surjective homomorphism

$$\phi_t : U(\mathfrak{sl}(W)) \to D^t \mathbb{P}(W)$$

and

$$\text{Ker} \phi_t = I(t, n_W - 1, ..., 1),$$

see [M].
3.9.2. The case $g = \mathfrak{sp}(W \oplus W^*)$. We note that $\mathfrak{h}_W$ is a Cartan subalgebra of $\mathfrak{sp}(W \oplus W^*)$. We identify the weights $\mathfrak{h}_W$ with $n_W$-tuples. We fix the set of positive roots of $\mathfrak{sp}(W \oplus W^*)$ in $\mathfrak{h}_W$ for which $\rho$ is identified with $(n_W, \ldots, 1)$. Let $\bar{\mu}$ be an $n_W$-tuple and $\mu$ be the corresponding weight. We denote $\text{Ann} \ L_{\mu} \text{ by } I_{\mathfrak{sp}}(\mu)$. Let $I$ be a primitive ideal of $U(\mathfrak{sp}(W \oplus W^*))$. Then $I = I_{\mathfrak{sp}}(\mu)$ for some $n_W$-tuple $\bar{\mu}$.  

**Theorem 3.15 ([M]).** The $(\mathfrak{sp}(W \oplus W^*), \mathfrak{h}_W)$-module $L_{\mu}$ is $\mathfrak{h}_W$-bounded if and only if $\bar{\mu}$ is a Shale-Weil tuple. 

The bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{h}_W)$-modules split into coherent families $[M]$ and all simple modules from one coherent family share an annihilator. Let $\bar{\mu}$ be a Shale-Weil $n_W$-tuple. Then the categories of $U(\mathfrak{sp}(W \oplus W^*))$-modules annihilated by $I_{\mathfrak{sp}}(\mu_0)$ and $I_{\mathfrak{sp}}(\mu) = \text{Im} \phi_{\mathfrak{sp}}$, $D^0(W) = \text{Im} \phi_{\mathfrak{sp}}$, and $\text{I}(\mu_0) = \text{Ker} \phi_{\mathfrak{sp}}$. 

**Lemma 3.32.** Let $\bar{\mu}$ be a Shale-Weil $n_W$-tuple. Then the categories of $U(\mathfrak{sp}(W \oplus W^*))$-modules annihilated by $I_{\mathfrak{sp}}(\mu)$ and $I_{\mathfrak{sp}}(\mu_0)$ are equivalent. 

**Proof.** This is a simple corollary of Theorem 3.12 b). 

The subcategories of locally finite $\mathfrak{g}$-modules and bounded $\mathfrak{g}$-modules are stable under the equivalence of Lemma 3.32. 

The spaces $\{D^i(W)\}_{i \in \mathbb{Z}}$ define a $\mathbb{Z}$-grading of an algebra $D(W)$. Therefore the spaces 

$$\{ \oplus_{i \in \mathbb{Z}} D^{2i}(W), \oplus_{i \in \mathbb{Z}} D^{2i+1}(W) \}$$ 

define a $\mathbb{Z}_2$-grading of the algebra $D(W)$. We set 

$$D^0(W) := \oplus_{i \in \mathbb{Z}} D^{2i}(W), \quad D^1(W) := \oplus_{i \in \mathbb{Z}} D^{2i+1}(W).$$ 

The space span $\langle e_i, e_j, e_i \partial e_j, \partial e_i, \partial e_j, 1 \rangle_{i,j \leq n_W}$ is a Lie algebra with respect to commutator. This Lie algebra is isomorphic to $\mathfrak{sp}(W \oplus W^*) \oplus \mathbb{C}$. We note that span$\langle e_i, \partial e_j \rangle_{i,j \leq n_W}$ is a Lie subalgebra and is isomorphic to $\mathfrak{gl}(W)$. This defines a homomorphism of associative algebras 

$$\phi_{\mathfrak{sp}} : U(\mathfrak{sp}(W \oplus W^*)) \to D(W),$$ 

and $\text{I}(\mu_0) = \text{Ker} \phi_{\mathfrak{sp}}$, $D^0(W) = \text{Im} \phi_{\mathfrak{sp}}$. 

3.10. A lemma on $S_{n_W}$-modules. The set of isomorphism classes of simple $S_{n_W}$-modules is naturally identified with the set of partitions 

$$m_1 \geq m_2 \geq \ldots \geq m_s (m_1 + m_2 + \ldots + m_s = n_W)$$
of \( n_W \). Let \( \mathbb{C}^{n_W - 1} \) be the natural \((n_W - 1)\)-dimensional module of \( S_{n_W} \) and \( \mathbb{C} \) be the trivial one-dimensional module. Recall that \( s_i \) is the simple transposition \((i, i + 1)\). Let \( R \) be an \( S_{n_W} \)-module. For any \( i \in \{1, \ldots, n_W - 1\} \) we denote by \( R_{\bar{i}} \) the vector space
\[
\{ v \in R \mid s_j v = v \text{ for all } j \neq i \}.
\]

**Lemma 3.33.** Suppose that \( R \) is an \( S_{n_W} \)-module such that
\[
R = +i \leq n_W - 1 R_{\bar{i}}.
\]
Then \( R \) is a direct sum of several copies of \( \mathbb{C}^{n_W - 1} \) and \( \mathbb{C} \).

**Proof.** Without loss of generality we assume that \( R \) is irreducible. If \( R_{\bar{1}} \) equals zero, the sum \( +i \leq n_W - 1 R_{\bar{i}} \) is \( s_1 \)-invariant and therefore \( R \cong \mathbb{C} \). If \( R_{\bar{1}} \neq 0 \), the restriction of \( R \) to \( S_{n_W - 1} \) contains \( \mathbb{C} \) as a simple submodule. Therefore \( R \) is
\[
\mathbb{C}^{n_W - 1}, \text{ or } \mathbb{C},
\]
see [FH]. \( \square \)

### 3.11. Regular singularities.

Let \( C \) be a smooth connected affine curve. We recall that \( \mathbb{C}[C] \) is the algebra of regular functions of \( C \) and \( D(C) \) is the algebra of regular differential operators on \( C \). Let \((x) \subset \mathbb{C}[C] \) be a maximal ideal and \( x \in C \) be the corresponding point. Set
\[
D^<_{\bar{x}}(C) := \{ D \in D(C) \mid Df \subset (x) \text{ for all } f \in (x) \}.
\]
We say that a \( D(C) \)-module \( F \) has regular singularities at \( x \) if for any finite-dimensional space \( F_0 \subset F \), \( D^<_{\bar{x}}(C)F_0 \) is a finitely generated \( \mathbb{C}[C] \)-module.

Let \( C^+ \) be a unique compact connected curve which contains \( C \) as an open subset. We say that a \( D(C) \)-module \( F \) has regular singularities if for any point \( x \in C^+ \) there exists an affine open set \( C(x) \subset C^+ \) such that \( x \in C(x) \) and \( F \otimes_{\mathbb{C}[C]} \mathbb{C}[C \cap C(x)] \) considered as a \( D(C(x)) \)-module by extension from \( C \cap C(x) \) to \( C(x) \) has regular singularities at \( x \).

**Definition 3.34.** Let \( X \) be a smooth variety and \( \mathcal{F} \) be a coherent \( \mathcal{D}(X) \)-module. We say that \( \mathcal{F} \) has regular singularities if \( \mathcal{F} \) has regular singularities after restriction to any smooth connected affine curve.
4. Holonomicity of \((\mathfrak{g},\mathfrak{t})\)-modules of finite type

We are now going to prove Theorem 2.2 stated in Section 2.

**Proof of Theorem 2.2** Without loss of generality we assume that \(\mathfrak{g}\) is semisimple. We identify \(\mathfrak{g}\) and \(\mathfrak{g}^*\) by use of the Killing form. Let \(h \in \mathfrak{t}\) be a rational semisimple element. We denote by \(\mathfrak{g}_h\) the direct sum of \(h\)-eigenspaces of \(\mathfrak{g}\) with nonnegative eigenvalues; by \(G_h \subset G\) the parabolic subgroup with the Lie algebra \(\mathfrak{g}_h\). Set \(Z_G := G/G_h\). In a similar way we define \(\mathfrak{t}_h, K_h, Z_K\). We denote by \(\mathfrak{n}_h\) the nilpotent radical of \(\mathfrak{g}_h\). Let \(e \subset K\) be the unit element. The orbit \(Ke \subset Z_G\) is isomorphic to \(Z_K\). Put

- \(Gn_h := \{x \in \mathfrak{g} \mid x = gn\ \text{for some } n \in \mathfrak{n}_h, g \in G\}\),
- \(Kn_h := \{x \in \mathfrak{g} \mid x = kn\ \text{for some } n \in \mathfrak{n}_h, k \in K\}\),
- \(Kn_h \cap \mathfrak{t}^\perp := \{x \in \mathfrak{g} \mid x = kn\ \text{for some } n \in \mathfrak{n}_h \cap \mathfrak{t}^\perp, k \in K\}\).

Let \(\phi : T^*Z_G \to \mathfrak{g}^*\) be the moment map. It is easy to see that \(Gn_h\) coincides with \(\phi(T^*Z_G)|_{Z_K}\), \(Kn_h\) coincides with \(\phi(T^*Z_G)|_{Z_K}\), \(Kn_h \cap \mathfrak{t}^\perp\) coincides with \(\phi(N_{Z_K/Z_G}^*)\):

\[
\begin{array}{ccc}
T^*Z_G & \longrightarrow & T^*Z_G|_{Z_K} \\
\phi | & & \phi | \\
Gn_h & \longrightarrow & Kn_h \\
\end{array}
\]

As the variety \(N_{Z_K/Z_G}^*\) is isotropic in \(T^*Z_G\), the image \(\phi(N_{Z_K/Z_G}^*)\) is isotropic in \(G(n_h \cap \mathfrak{t}^\perp)\) and any subvariety \(\tilde{V} \subset Kn_h \cap \mathfrak{t}^\perp\) is isotropic in \(G\tilde{V}\). Therefore by Corollary 3.29 any subvariety \(\tilde{V} \subset \mathfrak{t}^\perp \cap N_K(\mathfrak{g}^*)\) is isotropic in \(G\tilde{V}\).

We introduce the following notation:

- \(V_{\mathfrak{g}_h}\) is the set of all irreducible components of intersections of \(N_K(\mathfrak{t}^\perp)\) with all possible \(G\)-orbits of \(N_G(\mathfrak{g}^*)\).
- \(V_{\mathfrak{g}_h}\) is the set of all possible irreducible components of the preimages of elements of \(V_{\mathfrak{g}_h}\) under the moment map \(T^*X \to \mathfrak{g}^*\).
- \(L_{\mathfrak{g}_h}\) is the set of images in \(X\) of all elements of \(V_{\mathfrak{g}_h}\).

Let \(M\) be a finitely generated \(\mathfrak{g}\)-module which affords a central character \(\chi\) and \(\lambda \in \mathfrak{h}^*\) (see Subsection 3.4) be a weight such that \(\chi = \chi_\lambda\).

**Theorem 4.1.** If \(M\) is a \((\mathfrak{g},\mathfrak{t})\)-module of finite type, then

a) the irreducible components of \(V(M)\) are contained in \(V_{\mathfrak{g}_h}\); the irreducible components of \(V(M)\) are contained in \(V_{\mathfrak{g}_h}\); the irreducible components of \(L(M)\) are contained in \(L_{\mathfrak{g}_h}\).

b) The module \(\text{Loc}M\) is holonomic. If \(M\) is simple, \(M\) is holonomic.
Proof. Let $\tilde{V} \subset V(M)$ be an irreducible component and $Z$ be the closure of $G\tilde{V}$ in $g^*$. By Theorem 3.8 the variety $\tilde{V}$ is coisotropic. On the other hand

$$\tilde{V} \subset N_K(t^\perp) \cap Z,$$

and therefore $\tilde{V}$ is isotropic. Hence $\tilde{V}$ is Lagrangian and is an irreducible component of $Z \cap N_K(g^*) \cap t^\perp$.

As intersections of $V(M)$ with any irreducible component of $M$ are isotropic,

$$\dim V(Loc\ M) \leq \dim X.$$

Therefore $Loc\ M$ is holonomic.

Assume that $M$ is simple. As all irreducible components of $V(M)$ are Lagrangian, $M$ is holonomic. □

Corollary 4.1. The statements of Theorem 2.1 and Theorem 2.3 follow from Theorem 4.1.

Corollary 4.2. Let $M$ be a simple $(g, t)$-module of finite type and $\tilde{V}$ be an irreducible component of $V(M)$. Then

$$\dim \tilde{V} = \frac{1}{2} \dim GV(M) \text{ and } G\tilde{V} = GV(M).$$

Proof. As $\tilde{V}$ is Lagrangian in $G\tilde{V}$,

$$\dim \tilde{V} = \frac{1}{2} \dim G\tilde{V}$$

(see also [GL]). On the other hand $G\tilde{V} \subset GV(M)$, and

$$\dim \tilde{V} \geq \frac{1}{2} \dim GV(M).$$

As $GV(M)$ is irreducible, $GV(M)$ coincides with $G\tilde{V}$. □
5. Bounded subalgebras of $\mathfrak{sl}(W)$

We now prove Theorem 2.4 stated in the Introduction. In this section we assume that $\mathfrak{g} = \mathfrak{sl}(W)$ for some finite-dimensional vector space $W$. First we prove Theorem 2.5.

**Proof of Theorem 2.5.** Let $Z$ be an SL($V$)-orbit open in GV($M$). Assume $M$ is a bounded ($\mathfrak{sl}(W)$, $\mathfrak{k}$)-module. Let $\tilde{V}$ be an irreducible component of $V(M)$. Then $\tilde{V} \cap Z$ is an open subset of $\tilde{V}$ and is a conical Lagrangian subvariety of $Z$. By the discussion following Example 3.8 $Z$ is $K$-birationally isomorphic to $T^*Fl$ for some partial $W$-flag variety $Fl$. As $\tilde{V}$ is a conical Lagrangian subvariety of $Z$, $\tilde{V}$ is birationally isomorphic to $N^*_Z/Fl$ for some smooth subvariety $Z \subset Fl$ (Proposition 3.6).

As $\tilde{V}$ is $K$-spherical, $N^*_Z/Fl$ is $K$-spherical. Therefore $Fl$ is $K$-spherical (Theorem 3.2) and $T^*Fl$ is $K$-coisotropic. Hence $Z$ is $K$-coisotropic.

Assume GV($M$) is $K$-coisotropic. Let $\tilde{V}$ be an irreducible component of $V(M)$ and $\tilde{Z}$ be a $G$-orbit open in $G\tilde{V}$. By the discussion following Example 3.8 $\tilde{Z}$ is $K$-birationally isomorphic to $T^*Fl$ for some partial $W$-flag variety $Fl$. As $\tilde{Z} \subset Z$, the variety $\tilde{Z}$ is $K$-coisotropic by Theorem 3.4. Therefore the variety $Fl$ is $K$-spherical and in particular has finitely many $K$-orbits. As $\tilde{V} \cap \tilde{Z} \subset \mathfrak{t}^\perp$, $\tilde{V} \cap \tilde{Z}$ is isomorphic to an irreducible subvariety of the total space of the conormal bundle to a $K$-orbit in $Fl$. Therefore

$$\dim \tilde{V} \leq \dim Fl.$$ 

On the other hand, $\tilde{V} \cap \tilde{Z}$ is coisotropic in $\tilde{Z}$ and therefore

$$\dim \tilde{V} \geq \dim Fl.$$ 

Therefore $\dim \tilde{V} = \dim Fl$ and $\tilde{V}$ is birationally isomorphic to the total space of the conormal bundle to a $K$-orbit in $Fl$. Hence $\tilde{V}$ is $K$-spherical (Theorem 3.2).

As all irreducible components of $V(M)$ are $K$-spherical, $M$ is a bounded $K$-module. $\square$

**Theorem 5.1.** If there exists a simple infinite-dimensional bounded ($\mathfrak{sl}(W)$, $\mathfrak{k}$)-module $M$, then Gr($r; W$) is a spherical $K$-variety for some $r \in \{1, ..., n_W - 1\}$.

**Proof.** By Theorem 2.5 the variety GV($M$) is $K$-coisotropic. By the discussion following Example 3.8 the variety GV($M$) is $K$-birationally isomorphic to $T^*Fl$ for some partial flag variety Fl. Hence Fl is $K$-spherical and Gr($r; W$) is a $K$-spherical variety for some $r$. $\square$
**Theorem 5.2.** Let $F_l$ be a partial $W$-flag variety. Assume that $F_l$ is $K$-spherical. Then there exists a simple infinite-dimensional multiplicity-free $(\mathfrak{sl}(W), \mathfrak{k})$-module.

*Proof.* Theorem 6.3 in [PS] proves the existence of a simple infinite-dimensional multiplicity-free $(\mathfrak{sl}(W), \mathfrak{k})$-module under the assumption that there exists a partial $W$-flag variety for which $K$ has a proper closed orbit on $F_l$ such that the total space of its conormal bundle is $K$-spherical. By Theorem 3.2 this latter condition is equivalent to the $K$-sphericity of $F_l$. It remains to consider the case when $F_l$ has no proper closed $K$-orbits on $F_l$, i.e., $K$ has only one orbit on $F_l$. If $K$ has an open orbit on $F_l$ then $[\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{sp}(W)$. However, in this last case $\text{Gr}(2; W)$ has a proper closed $K$-orbit and is $K$-spherical. □

We have thus proved the following weaker version of Theorem 2.4.

**Corollary 5.1.** A pair $(\mathfrak{sl}(W), \mathfrak{k})$ admits an infinite-dimensional bounded simple $(\mathfrak{sl}(W), \mathfrak{k})$-module if and only if $\text{Gr}(r; W)$ is a spherical $K$-variety for some $r$.

*Proof.* The statement follows directly from Theorems 5.2 and 5.1 □

**Corollary 5.2** (see also [PS], Conjecture 6.6). If there exists a bounded simple infinite-dimensional $(\mathfrak{sl}(W), \mathfrak{k})$-module, then there exists a multiplicity-free simple infinite-dimensional $(\mathfrak{sl}(W), \mathfrak{k})$-module.

*Proof.* The statement follows directly from Corollary 5.1 □

As $\mathbb{P}(W)$ is lower than or cotangent-equivalent to any other Grassmannian (see discussion following Lemma 3.14), $r$ in Corollary 5.1 can be chosen to equal 1 (see Proposition 3.15). This completes the proof of Theorem 2.4.

All finite-dimensional $\mathfrak{k}$-modules $W$ such that $\mathbb{P}(W)$ is a $K$-spherical variety are known from the work of C. Benson and G. Ratcliff [BR] (see also [Kac1] and [Le]). The list of respective pairs $(\mathfrak{k}, W)$ is reproduced in the Appendix. Theorem 2.4 implies the following.

**Corollary 5.3.** The list of pairs $(\mathfrak{sl}(W), \mathfrak{k})$ for which $\mathfrak{k}$ is reductive and bounded in $\mathfrak{sl}(W)$ coincides with the list of C. Benson and G. Ratcliff reproduced in the Appendix.
6. Spherical partial flag varieties

We recall that \( K \) is a reductive Lie group. Let \( V \) be a finite-dimensional \( K \)-module. We denote by \( \mathfrak{k} \) the image of \( \mathfrak{t} \) in \( \text{End}(V) \).

**Definition 6.1.** A \( K \)-module \( V \) is called *weakly irreducible* if it is not a proper direct sum \( V_1 \oplus V_2 \) of two \( \mathfrak{k} \)-submodules such that

\[
[\mathfrak{k} V_1, \mathfrak{k} V_1] \oplus [\mathfrak{k} V_2, \mathfrak{k} V_2] = [\mathfrak{k} V, \mathfrak{k} V].
\]

All spherical representations are classified in the work [BR] (see also [Le] and [Kac1]) and we now recall this classification. According to [BR], a \( K \)-module \( V \) is \( K \)-spherical if and only if the pair \(([\mathfrak{k} V, \mathfrak{k} V], V)\) is a direct sum of pairs \((\mathfrak{g}, V_i)\) listed in the Appendix (cf. [BR]) and in addition

\[
(\mathfrak{k} V + \mathfrak{c}_i) = N_{\mathfrak{gl}(V)}(\mathfrak{k} V + \mathfrak{c}_i)
\]

for certain abelian Lie algebras \( \mathfrak{c}_i \) attached to \((\mathfrak{g}, V_i)\) where

\[
N_{\mathfrak{gl}(V)}(\mathfrak{k} V + \mathfrak{c}_i)
\]

is the normalizer of \( \mathfrak{k} V + \mathfrak{c}_i \) inside \( \mathfrak{gl}(V) \).

Assume that \( V \) is a direct sum of two simple modules, \( V = V_1 \oplus V_2 \). For \( a, b \in \mathbb{C} \) we denote by \( h_{a,b} \) the rational semisimple element such that

\[
h_{a,b}|V_1 = a \text{Id} \quad \text{and} \quad h_{a,b}|V_2 = b \text{Id},
\]

where \( \text{Id} \) is the identity map. We use similar notation if \( V \) is semisimple of length 1 or 3.

**Theorem 6.1.** a) Assume that a partial flag variety \( \text{Fl}(n_1, ..., n_s; V) \) is not cotangent-equivalent to \( \mathbb{P}(V) \). If the partial flag variety \( \text{Fl}(n_1, ..., n_s; V) \) is \( K \)-spherical, then \( \text{Fl}(n_1, ..., n_s; V) \) is cotangent-equivalent to \( \text{Fl}(n'_1, ..., n'_s; V) \) for some datum

\[
(n'_1, ..., n'_s; [\mathfrak{k} V, \mathfrak{k} V], V)
\]

which appears in the following list.

1) Case \( s = 1 \) ("Grassmannians").
   1-1) \((r; \mathfrak{sl}_n, \mathbb{C}^n); (r; \mathfrak{so}_n, \mathbb{C}^n)(n \geq 3); (r; \mathfrak{sp}_n, \mathbb{C}^n).\)
   1-2-1) \((2; \mathfrak{sp}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(m \geq 1);\)
   1-2-2) \((2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m, \mathbb{C}^n \oplus \mathbb{C}^m);\)
   1-2-3) \((2; \mathfrak{sp}_n, \mathbb{C}^n \oplus \mathbb{C});\)
   1-2-4) \((2; \mathfrak{sp}_m \oplus \mathfrak{sp}_n, \mathbb{C}^n \oplus \mathbb{C}^m)(n \geq 1);\)
   1-2-5) \((2; \mathfrak{sp}_m \oplus \mathfrak{sp}_n, \mathbb{C}^n \oplus \mathbb{C}^m)(n, m \geq 1);\)
   1-3-1) \((2; \mathfrak{sp}_n \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(m, n, q \geq 1);\)
   1-3-1-1) \((2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m \oplus \mathfrak{sl}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(m, n, q \geq 1);\)
   1-3-1-2) \((2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(m \geq 1, n \geq 1);\)
   1-3-1-3) \((2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(n \geq 1);\)
I-3-1-4) \((2; \mathfrak{sp}_n \oplus \mathfrak{sp}_n \oplus \mathfrak{sp}_2, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)\);
I-3-2) \((r; \mathfrak{sl}_r \oplus \mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}) (n, m \geq 1)\).

II) Case \(s \geq 2\).

II-1) \((n_1, ..., n_s; \mathfrak{sl}_n, \mathbb{C}^n)\);
II-1-2) \((1, 2, 3; \mathfrak{sp}_n, \mathbb{C}^n)\);
II-1-3) \((1, r; \mathfrak{sp}_n, \mathbb{C}^n)\);
II-2) \((n_1, ..., n_s; \mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C})\);
II-2-2) \((1, r; \mathfrak{sl}_r \oplus \mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C}^m) (m, n \geq 1)\);
II-2-3) \((r_1, r_2; \mathfrak{sl}_{r_2} \oplus \mathfrak{sl}_{r_1}, \mathbb{C}^2 \oplus \mathbb{C}^m) (n \geq 1)\);
II-2-4) \((1, 2; \mathfrak{sl}_n \oplus \mathfrak{sp}_m, \mathbb{P}^n \oplus \mathbb{C}^m) (n \geq 1)\);
II-2-5) \((1, 2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m, \mathbb{C}^n \oplus \mathbb{C}^m)\).

b) For all data \((n_1, ..., n_s; \mathfrak{k}', \mathbb{V})\) from this list there exists a reductive subgroup \(K \subset \text{GL}(\mathbb{V})\) with Lie algebra \(\mathfrak{k}'\) such that \(\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]\) and \(\text{Fl}(n_1, ..., n_s; \mathbb{V})\) is a \(K\)-spherical variety.

The rest of this section is devoted to the proof of Theorem 6.1.

6.1. **Simplifications.** Let \(V_s\) and \(V_b\) be \(K\)-modules of dimensions \(n_s, n_b\) such that \(n_s, n_b \geq 1\). We start with the following remark. The varieties

\[ \text{Hom}(V_s, V_b) \text{ and } \text{Gr}(n_s; V_s \oplus V_b) \]

are birationally isomorphic. This shows that the \(K\)-module \(\text{Hom}(V_s, V_b)\) is \(K\)-spherical if and only if \(\text{Gr}(n_s; V_s \oplus V_b)\) is \(K\)-spherical.

**Theorem 6.2.** Fix \(r \in \{1, ..., n_b\}\). Let \((L, V_r^b)\) be the pair determined by the datum \((K, V_b, r)\) (see the discussion following Theorem 3.2). Then the variety \(\text{Gr}(r; V_s)\) is \(K\)-spherical if and only if the module \(\text{Hom}(V_r^b, V_s)\) is \(L\)-spherical and the variety \(\text{Gr}(r; V_r^b)\) is \(K\)-spherical.

**Proof.** Assume that \(\text{Gr}(r; V_s)\) is \(K\)-spherical. Then \(\text{Gr}(r; V_r^b)\) is \(K\)-spherical and \(\text{Gr}(r; V_r^b \oplus V_s)\) is \(L\)-spherical. The variety \(\text{Gr}(r; V_r^b \oplus V_s)\) is \(L\)-spherical if and only if \(\text{Hom}(V_r^b, V_s)\) is an \(L\)-spherical module.

Assume that the module \(\text{Hom}(V_r^b, V_s)\) is \(L\)-spherical and the variety \(\text{Gr}(r; V_r^b)\) is \(K\)-spherical. Then \(\text{Gr}(r; V_r^b \oplus V_s)\) is an \(L\)-spherical variety. Therefore the variety \(\text{Gr}(r; V_s)\) is \(K\)-spherical. \(\square\)

**Corollary 6.2.** Suppose that the variety \(\text{Gr}(r; V_s)\) is \(K\)-spherical for some \(r \in \{2, ..., n_b\}\). Then

a) \(\text{Hom}(\mathfrak{k}^r, V_s)\) is a spherical \(\text{GL}_r \times K\)-module;

b) if \(n_b \neq 2\), then the variety \(\text{Fl}(1, 2; V_r^b)\) is \(K\)-spherical.

**Proof.** The first statement is obvious. By Lemma 3.18 we can assume that \(r = 2\). Let \((L, V_r^2)\) be the pair determined by the datum \((K, V_r, 2)\). As \(\text{Gr}(2; V_r^2 \oplus V_s)\) is an \(L\)-spherical variety, \(\mathbb{P}^2(V_r^2 \oplus V_s)\) is an \(L\)-spherical variety by Proposition 3.15. Therefore \(\text{Fl}(1, 2; V_r^2)\) is a \(K\)-spherical variety. \(\square\)
The following lemma is a first approximation to Theorem 6.1.

**Lemma 6.3.** Let $W$ be a $K$-module. Suppose that $W \otimes \mathbb{C}^r$ is a spherical $K \times \text{GL}_r$-module for some $r \in \mathbb{Z}_{\geq 2}$. Then one of the following possibilities holds.

1) $r = 2$ and the datum $([\mathfrak{t}_W, \mathfrak{t}_W], W)$ appears in the following list: 
   - $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(n, m \geq 1)$,
   - $(\mathfrak{sl}_n \oplus \mathfrak{sp}_{2m}, \mathbb{C}^n \oplus \mathbb{C}^{2m})(n \geq 1)$,
   - $(\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m}, \mathbb{C}^{2n} \oplus \mathbb{C}^{2m})$, 
   - $(\mathfrak{sp}_{2n}, \mathbb{C}^{2n})$, 
   - $(\mathfrak{sl}_n, \mathbb{C}^n)$.

2) $r = 3$ and $([\mathfrak{t}_W \cap \mathfrak{sl}(W), W]) \cong (\mathfrak{sp}_{2n}, \mathbb{C}^{2n})$.

3) $r \geq 3$ and the datum $([\mathfrak{t}_W, \mathfrak{t}_W], W)$ appears in the following list: 
   - $(\mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C})(n \geq 1)$,
   - $(\mathfrak{sl}_n, \mathbb{C}^n)(n \geq 1)$,
   - $(\mathfrak{sp}_4, \mathbb{C}^4)$.

**Proof.** The result follows directly from the tables of [HR].

Let $0 < n_1 < \ldots < n_s < n_W$ and $0 < n'_1 < \ldots < n'_{s'} < n_W$ be sequences of integers. By Lemma 6.13, 

$$\text{Fl}(n_1, \ldots, n_s; W) \times \text{Fl}(n'_1, \ldots, n'_{s'}; W)$$

is $\text{GL}(W)$-spherical if and only if the variety $\text{Fl}(n'_1, \ldots, n'_{s'}; W)$ is $K$-spherical, where 

$$K = \text{GL}_{n_W - n_s} \times \ldots \times \text{GL}_{n_1}$$

is a Levi subgroup of a parabolic subgroup of $\text{GL}(W)$. Let 

$$(d_1, \ldots, d_{s+1}) \text{ and } (d'_1, \ldots, d'_{s'+1})$$

be the corresponding to $(n_1, \ldots, n_s)$ and $(n'_1, \ldots, n'_{s'})$ partitions of $n_W$.

The following theorem is a particular case of a result of E. Ponomareva [Po]; she classified $G$-spherical products of two compact $G$-homogeneous spaces for arbitrary algebraic groups $G$.

**Theorem 6.3** (see also [Pan3]). The variety 

$$\text{Fl}(n_1, \ldots, n_s; W) \times \text{Fl}(n'_1, \ldots, n'_{s'}; W)$$

is $\text{GL}(W)$-spherical if and only if the (unordered) pair of sets 

$$(\{d_1, \ldots, d_{s+1}\}, \{d'_1, \ldots, d'_{s'+1}\})$$

appears in the following list: 

- $\{\{p_1, p_2\}, \{q_1, q_2\}\}$; 
- $\{\{p_1, p_2\}, \{1, q_1, q_2\}\}$; 
- $\{\{2, p_1\}, \{q_1, q_2, q_3\}\}$; 
- $\{\{1, p_1\}, \{q_1, \ldots, q_{s'+1}\}\}$.

**Proof.** A straightforward computation shows that the pair 

$$(\{1, 1, n\}, \{1, 1, n\})$$

does not yield a spherical variety for $n \geq 1$ (see also Theorem 6.2). A straightforward computation shows that the pair 

$$(\{3, n + 1\}, \{2, 2, n\})$$

does not yield a spherical variety for $n \geq 2$ (see also Theorem 6.2). The rest of the proof is an exercise to Proposition 3.15. □
6.2. Simple spherical modules. Let $W$ be a simple $K$-module. We recall that by Lemma 3.18 if some partial $W$-flag variety, which is not cotangent-equivalent to $\mathbb{P}(W)$, is $K$-spherical, then the variety $\text{Gr}(2; W)$ is $K$-spherical too.

Lemma 6.4. Let $\text{Gr}(2; C^m \otimes C^n)$ be an $\text{SL}_n \times \text{SL}_m$-spherical variety for some $m, n \in \mathbb{Z}_{\geq 2}$. Then $m = n = 2$.

Proof. Without loss of generality we assume that $m \geq n$. Assume that $n = 2$. Then

$$\dim(b_{sl_m} \oplus b_{sl_2}) \geq \dim \text{Gr}(2; C^m \otimes C^2).$$

We have $\frac{m(m+1)}{2} + 1 \geq 2(2m - 2)$ and therefore $(m - 2)(m - 5) \geq 0$. Hence $m \in \{2, 5, 6, 7, \ldots\}$.

Assume $m \geq 5$. As $\text{Gr}(2; C^2 \otimes C^m)$ is $\text{SL}_2 \times \text{SL}_m$-spherical, it must be that $\text{Gr}(2; C^2 \otimes C^4)$ is $\text{SL}_2 \times \text{SL}_4$-spherical. This is false by dimension reasons.

Assume $n \geq 3$. Then

$$\frac{14}{18}(m^2 + n^2) \geq \frac{m^2 + m}{2} - 1 + \frac{n^2 + n}{2} - 1 + 4.$$ 

As $\dim(b_{sl_m} \oplus b_{sl_n}) \geq \dim \text{Gr}(2; C^m \otimes C^n)$, we have

$$\frac{m^2 + m}{2} - 1 + \frac{n^2 + n}{2} - 1 + 4 \geq 2mn,$$

$$\frac{14}{18}(m^2 + n^2) \geq 2mn \text{ and } m > 2n.$$ 

As $\text{Gr}(2; C^n \otimes C^m)$ is $\text{SL}_n \times \text{SL}_m$-spherical, we have $\text{Gr}(2; C^n \otimes C^{2n})$ is $\text{SL}_n \times \text{SL}_{2n}$-spherical. This implies that $\text{SL}_n \times \text{SL}_2$ is a spherical subgroup of $\text{SL}_{2n}$. Hence $n = 2$ [Kr].

Theorem 6.4. Suppose the variety $\text{Gr}(2; W)$ is $K$-spherical. Then the pair $(\mathfrak{k}_W \cap \mathfrak{sl}(W), W)$ is isomorphic to one of the following 3 pairs:

$$(\mathfrak{sl}(W), W), \ (\mathfrak{so}(W), W), \ (\mathfrak{sp}(W), W).$$

Proof. Without loss of generality we assume that $\mathfrak{t} = \mathfrak{k}_W \subset \mathfrak{gl}(W)$ and that $\mathfrak{t}$ is a semisimple Lie algebra. We recall that $B$ is a Borel subgroup of $K$.

A straightforward computation shows that $\text{Gr}(2; W)$ is $\text{SL}(W)$-spherical, and also $\text{SO}(W)$-spherical and $\text{SP}(W)$-spherical. In the rest of the proof we assume that $(\mathfrak{k}_W, W)$ is not isomorphic to

$$(\mathfrak{sl}(W), W), \ (\mathfrak{so}(W), W) \text{ and } (\mathfrak{sp}(W), W).$$

1) Assume that $\mathfrak{k}$ is a simple Lie algebra. If there is an open orbit of $B$ on $\text{Gr}(2; W)$ then there is an open orbit of $B \times \text{GL}_2$ on $W \otimes C^2$. The simple modules with an open orbit of a reductive group are classified in [SK]. In Table 1 below we reproduce, following [SK], all simple
$K \times \text{GL}_2$-modules with an open orbit of $K \times \text{GL}_2$ such that $W$ is a $K$-spherical module.

**Table 1.** The pairs ([F, W], W) such that $W \otimes \mathbb{C}^2$ has an open orbit of $K \times \text{GL}_2$ and $W$ is a spherical $K$-module.

| No | The pair ([F, W], W) | dim(b_F \oplus \mathfrak{gl}_2) | 2n_W |
|----|----------------------|---------------------------------|------|
| 1  | (\mathfrak{sl}_2, \mathbb{C}^n)(n \geq 2) | $\frac{n(n+1)}{2} + 3$ | 2n |
| 2  | (\mathfrak{so}_n, \mathbb{C}^n)(n \geq 3) | $\frac{n(n-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor$ | 2n |
| 3  | (\mathfrak{sp}_{2n}, \mathbb{C}^{2n})(n \geq 2) | $n^2 + n$ | 4n |
| 4  | (\mathfrak{sl}_3, \mathbb{S}^2 \mathbb{C}^3) | 9 | 12 |
| 5  | (\mathfrak{sl}_{2n+1}, \mathbb{A}^2 \mathbb{C}^{2n+1}) | $2n^2 + 3n + 4$ | $4n^2 + 2n$ |
| 6  | (\mathfrak{sl}_6, \mathbb{A}^2 \mathbb{C}^6) | 24 | 30 |
| 7  | (\mathfrak{so}_7, \text{Spin}(\mathbb{C}^7)) | 16 | 16 |
| 8  | (\mathfrak{so}_{10}, \text{Spin}(\mathbb{C}^{10})) | 29 | 32 |
| 9  | (G_2, \mathbb{C}^7) | 12 | 14 |
| 10 | (E_6, \mathbb{C}^{27}) | 42 | 54 |

Here Spin(F) is any spinor module of SO(F), and $\mathbb{C}^7$ in case 9 is the unique faithful $G_2$-module of minimal dimension, $\mathbb{C}^{27}$ in case 10 is a faithful $E_6$-module of minimal dimension. If $b_\mathfrak{F} \oplus \mathfrak{gl}_2$ has an open orbit on $W \otimes \mathbb{C}^2$, then dim($b_\mathfrak{F} \oplus \mathfrak{gl}_2) \geq 2n_W$.

Under the assumption that $\mathfrak{F}$ is simple we complete the proof by the following case-by-case considerations.

Case 5. The computation $4n^2 + 2n - (2n^2 + 3n + 4) = 2n^2 - n - 4 = n^2 - 4 + n(n-1) > 0(n \geq 2)$ shows that dim($b_\mathfrak{F} \oplus \mathfrak{gl}_2) < 2n_W$.

Case 7. A generic isotropy subalgebra for the action of $\mathfrak{so}_7 \oplus \mathfrak{gl}_2$ on Spin($\mathbb{C}^7 \otimes \mathbb{C}^2$ is isomorphic to $\mathfrak{gl}_3$. All spherical subgroups of $SO_7$ are not quotients of $SL_3 \times \mathbb{C}^*$ [Kra]. Therefore Spin($\mathbb{C}^7 \otimes \mathbb{C}^2$ has no open $B(SO_7) \times \text{GL}_2$-orbit.

For the cases 4, 6, 8, 9, 10 from Table 1 we have dim($b_\mathfrak{F} \oplus \mathfrak{gl}_2) < 2n_W$ and therefore Gr(2; W) has no open B-orbit.

2) Let $\mathfrak{F}$ be a direct sum $\mathfrak{F}_1 \oplus \mathfrak{F}_2$ of noncommutative ideals. Since $W$ is a simple $K$-module, $W$ is isomorphic to $V_i \otimes V_2$, where $V_i(i = 1, 2)$ are simple $\mathfrak{F}_i$-modules such that $\mathfrak{F}_i = \mathfrak{F}_i$. As Gr(2; $V_1 \otimes V_2$) is $G$-spherical, Gr(2; $V_1 \otimes V_2$) is $SL(V_1) \times SL(V_2)$-spherical too. Since $n_{V_{i, 2}} \geq 2$, the only possibility for ($\mathfrak{F}, W$) is ($\mathfrak{so}_4, \mathbb{C}^4$) by Lemma 6.4. In this case Gr(2; W) is a spherical $K$-variety.

**Remark 6.5.** Fix $r \in \{1, \ldots, n_W - 1\}$. Then

a) the variety Gr(r; W) is SL(W)-spherical;

b) the variety Gr(r; W) is SO(W)-spherical;

c) the variety Gr(r; W) is SP(W)-spherical.
Lemma 6.6. Suppose that Fl(1, 2; W) is a K-spherical variety. Then the pair ($\mathfrak{t}_W \cap \mathfrak{sl}(W), W$) is isomorphic to one of the following pairs:

\[(\mathfrak{sl}(W), W), (\mathfrak{sp}(W), W)\].

Proof. As Fl(1, 2; W) is a K-spherical variety, the variety Gr(2; W) is K-spherical. Therefore ($\mathfrak{t}_W \cap \mathfrak{sl}(W)$, W) is isomorphic to ($\mathfrak{sl}(W)$, W), to ($\mathfrak{so}(W)$, W), or to ($\mathfrak{sp}(W)$, W).

A generic isotropy subgroup for the action of SO$_n$ on Fl(1, 2; C$^n$) is isomorphic to SO$_{n-2}$. Such a subgroup is not spherical in SO$_n$ for all $n \geq 3$ [Kr]. □

Corollary 6.7. The only partial flag varieties which are SO(W)-spherical are Gr($r; W$) for all $r \in \{1, ..., n-1\}$.

Proof. This is a simple corollary of Proposition 3.15 and of the discussion preceding this proposition. □

Remark 6.8. All partial W-flag varieties are SL(W)-spherical.

Lemma 6.9. Assume that $2|n_W$ and $n_W \geq 6$. Then the variety Fl(2, 4; W) is not SP(W)-spherical.

Proof. To the datum (SP(W), W, 4) one assigns the datum (SL$_2 \times$SL$_2$, C$^2$ $\oplus$ C$^2$). Therefore the variety Fl(2, 4; W) is SP(W)-spherical if and only if the variety Gr(2; C$^2$ $\oplus$ C$^2$) is SL$_2 \times$SL$_2$-spherical. However, Gr(2; C$^2$ $\oplus$ C$^2$) is not SL$_2 \times$SL$_2$-spherical. □

We recall that any partial W-flag variety is cotangent-equivalent to one of the following

Gr($r; W$), Fl(1, r; W), Fl(1, 2, 3; W), Fl(2, 4; W),

or is higher than Fl(2, 4; W), see subsection 3.3.

Corollary 6.10. The only SP(W)-spherical partial flag varieties, up to cotangent-equivalence, are

Gr($r; W$), Fl(1, r; W), Fl(1, 2, 3; W).

In particular, all partial C$^4$-flag varieties are SP$_4$-spherical.

6.3. Weakly irreducible spherical modules. Let W be a weakly irreducible, but not irreducible, K-spherical module. Then W is a direct sum of two nonzero simple K-modules $W_s \oplus W_b$ of dimensions $n_s, n_b$. Without loss of generality we assume that $n_b \geq n_s$.

Lemma 6.11. The variety Gr(2; W) is not K-spherical.
Proof. Assume $n_b = 2$. Then the pair $([\mathfrak{t}, \mathfrak{t}], W)$ is isomorphic to the pair $(\mathfrak{sl}_2, \mathbb{C}^2 \oplus \mathbb{C}^2)$. The variety $\text{Gr}(2; \mathbb{C}^2 \oplus \mathbb{C}^2)$ is not $\text{GL}_2$-spherical for dimension reasons.

Assume $n_b > 2$. Then the variety $\text{Fl}(1, 2; W_b)$ is $K$-spherical by Corollary 6.2. Therefore $(\mathfrak{t}_W \cap \mathfrak{sl}(W_b), W_b)$ is isomorphic to $(\mathfrak{sl}(W_b), W_b)$ or to $(\mathfrak{sp}(W_b), W_b)$. Then the pair $([\mathfrak{t}, \mathfrak{t}], W)$ is isomorphic to

$$(\mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C}^n), \text{ to } (\mathfrak{sl}_n, \mathbb{C}^n \oplus (\mathbb{C}^n)^*), \text{ or to } (\mathfrak{sp}_{2n}, \mathbb{C}^{2n} \oplus \mathbb{C}^{2n})$$

(see the tables in [BR]). Note, that $n_s = n_b$ in all three cases.

Consider the pair $(\mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C}^n)$. If the variety $\text{Gr}(2; W)$ is $K$-spherical, $\text{Gr}(2; \mathbb{C}^n \otimes \mathbb{C}^2)$ must be $\text{GL}_n \times \text{GL}_2$-spherical. This is not the case by Lemma 6.4.

Next we consider the pair $(\mathfrak{sp}_{2n}, \mathbb{C}^{2n} \oplus \mathbb{C}^{2n})$. If the variety $\text{Gr}(2; W)$ is $K$-spherical, $\text{Gr}(2; \mathbb{C}^{2n} \otimes \mathbb{C}^2)$ must be $\text{GL}_{2n} \times \text{GL}_2$-spherical. This is not the case by Lemma 6.4.

Finally consider the pair $(\mathfrak{sl}_n, \mathbb{C}^n \oplus (\mathbb{C}^n)^*)(n \geq 3)$. If the variety $\text{Gr}(2; W)$ is $\text{GL}_n$-spherical, the variety $\mathbb{C}^2 \otimes \mathbb{C}^n$ is $\text{GL}_2$-spherical by Theorem 6.2. This is not the case for dimension reasons. □

Corollary 6.12. The only $K$-spherical partial $W$-flag varieties for a weakly irreducible, but not irreducible, $K$-spherical module $W$ are $\text{Gr}(1; W)$ and $\text{Gr}(n_W - 1; W)$.

6.4. On the length of certain modules. Let $V$ be a spherical $K$-module and $W$ be a proper nonzero submodule of $V$. We recall that by Lemma 3.18, if some partial $V$-flag variety, which is not cotangent-equivalent to $\mathbb{P}(V)$, is $K$-spherical, then the variety $\text{Gr}(2; V)$ is $K$-spherical too.

Lemma 6.13. Assume that $W$ is a weakly irreducible submodule of $V$ and $\text{Gr}(2; V)$ is a $K$-spherical variety. Then the pair $(\mathfrak{sl}(W) \cap \mathfrak{t}_W, W)$ is isomorphic to $(\mathfrak{sl}(W), W)$ or to $(\mathfrak{sp}(W), W)$.

Proof. Assume $n_W = 2$. Then the pair $(\mathfrak{t}_W \cap \mathfrak{sl}(W), W)$ is isomorphic to $(\mathfrak{sl}_2, \mathbb{C}^2)$.

Assume $n_W \geq 3$. The variety $\text{Fl}(1, 2; W)$ is $K$-spherical by Corollary 6.2. Therefore the pair $(\mathfrak{t}_W \cap \mathfrak{sl}(W), W)$ is isomorphic to $(\mathfrak{sl}(W), W)$, or to $(\mathfrak{sp}(W), W)$ by Corollary 6.12 and Lemma 6.6. □

Lemma 6.14. Let $W$ be a $K$-module such that $n_W > 3$ and $\text{Gr}(2; W)$ is $K$-spherical. Then the length of $W$ as a $\mathfrak{t}$-module equals at most 3.

Proof. See Theorem 6.3. □
6.5. Concluding case-by-case considerations. The following theorem is a second approximation to Theorem 6.1.

**Theorem 6.5.** Let \( W_1, W_2 \) be \( K \)-modules of the dimensions \( n_1, n_2 \) such that \( n_1 \geq n_2 \geq 2 \). Suppose that \( \text{Hom}(W_1, W_2) \) is a \( K \)-spherical module. Then one of the following holds.

1) \([t_{W_1}, t_{W_2}] = 0, n_2 = 2 \) and \((t_{W_1}, t_{W_2}, W_1)\) is either \((\mathfrak{sl}(W_1), W_1)\), or \((\mathfrak{sp}(W_1), W_1)\);
2) \((t_{W_1}, t_{W_2}, W_2) = (\mathfrak{sl}, \mathbb{C}^2)\) and \((t_{W_1}, t_{W_2}, W_2)\) appears in the following list: \((\mathfrak{sl}(W_1), W_1), (\mathfrak{sp}(W_1), W_1), (\mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(m, n \geq 1)\), \((\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m}, \mathbb{C}^{2n} \oplus \mathbb{C}^{2m})(n \geq 1)\), \((\mathfrak{sl}_n, \mathbb{C}^n)(n \geq 3)\) and \((t_{W_1}, t_{W_2}, W_2)\) appears in the following list: \((\mathfrak{sl}(W_1), W_1), (\mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C}), (\mathfrak{sp}_4, \mathbb{C}^4)\); \(\mathbb{C}^3\) \(\mathbb{C}^2\) \(\mathbb{C}^2\)
3) \((t_{W_1}, t_{W_2}, W_2) = (\mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C})(n \geq 2)\) and \((t_{W_1}, t_{W_2}, W_2) = (\mathfrak{sl}(W_1), W_2)\);
4) \((t_{W_1}, t_{W_2}, W_2) = (\mathfrak{sl}_3, \mathbb{C}^3)\) and \((t_{W_1}, t_{W_2}, W_2) = (\mathfrak{sp}(W_1), W_2)\);
5) \((t_{W_1}, t_{W_2}, W_2) = (\mathfrak{sp}_4, \mathbb{C}^4)\) and \((t_{W_1}, t_{W_2}, W_2) = (\mathfrak{sl}(W_1), W_2)\).

**Proof.** As the variety \( \text{Hom}(W_1, W_2) \) is \( K \)-spherical, the variety \( \text{Gr}(n_1; W_1 \oplus W_2) \) is \( K \)-spherical. Therefore the variety \( \mathbb{P}(W_1 \oplus W_2) \) is \( K \)-spherical and the length of the \( K \)-module \( W_1 \oplus W_2 \) is not more then 3. Using this condition and the results of \( [BR] \) it is straightforward to derive the required list. We also use Lemma 6.13. \( \square \)

In the remainder of this section we collect the additional information needed to prove Theorem 6.1.

Let \( W \) be a nonzero \( K \)-module. Denote by \( \mathbb{C}_{a,b}^* \) the one-dimensional subgroup of \( \text{GL}(W) \) which corresponds to \( \mathbb{C} h_{a,b} \) (see Section 2). We use similar notation if \( W \) has length 1 or 3.

Fix \( r \in \{1, \ldots, n_W - 1\} \). Recall that the datum \((K, W, r)\) determines the datum \((L, W^r)\) (see the discussion following Theorem 3.2).

**Lemma 6.15.** The following data \((K, W, r)\) determine the pairs \((L, W^r)\) as follows:

1) \((\text{SL}(W), W, r) \rightarrow (\text{GL}_r, \mathbb{C}^r)\);
2) \((\text{SP}(W), W, 2) \rightarrow (\text{SL}_2, \mathbb{C}^2)\);
3) \((\text{SP}(W), W, 3) \rightarrow (\text{SL}_2 \times \mathbb{C}_{a,b}^*, \mathbb{C}^2 \oplus \mathbb{C})\);
4) \((\text{SP}(W), W, 4) \rightarrow (\text{SL}_2 \times \text{SL}_2, \mathbb{C}^2 \oplus \mathbb{C}^2)\).

**Proof.** We omit the proof. \( \square \)

**Lemma 6.16.** For any \( q \in \mathbb{Z} \geq 2 \) the following statements are equivalent.
1) The variety \( \text{Gr}(2; W \oplus \mathbb{C}^2q) \) is \( K \times \text{SP}_{2q} \)-spherical.
2) The variety \( \text{Gr}(2; W \oplus \mathbb{C}^2) \) is \( K \times \text{SL}_2 \)-spherical.
3) The variety \( \text{Hom}(\mathbb{C}^2, W) \) is \( \text{SL}_2 \times K \)-spherical.
Proof. As the datum \((\text{SP}_2, \mathbb{C}^{2q}, 2)\) determines the pair \((\text{SL}_2, \mathbb{C}^2)\), part 1) is equivalent to part 2) by Theorem 6.2. Part 3) is a reformulation of part 2). □

Lemma 6.17. For any \(q \in \mathbb{Z}_{\geq 2}\) the following statements are equivalent.
1) The variety \(\text{Gr}(3; W \oplus \mathbb{C}^{2q})\) is \(K \times \text{SP}_2\)-spherical.
2) The variety \(\text{Gr}(3; W \oplus \mathbb{C}^2 \oplus \mathbb{C})\) is \(K \times \text{SL}_2 \times C_{0,1}\)-spherical.
3) The variety \(\text{Hom}(\mathbb{C}^2 \oplus \mathbb{C}, W)\) is \(\text{SL}_2 \times C_{0,1} \times K\)-spherical.

Proof. As the datum \((\text{SP}_2, \mathbb{C}^{2q}, 3)\) determines the pair \((\text{SL}_2 \times C_{0,1}, \mathbb{C}^2 \oplus \mathbb{C})\), part 1) is equivalent to part 2) by Theorem 6.2. Part 3) is a reformulation of part 2). □

Corollary 6.18. Suppose that \(\text{Gr}(3; W \oplus \mathbb{C}^{2q})\) is a \(K \times \text{SP}_2\)-spherical variety for some \(q \in \mathbb{Z}_{\geq 2}\). Then \(\text{sl}(W) \subset \mathfrak{t}_W\).

Corollary 6.19. The variety \(\text{Fl}(1, 3; W \oplus \mathbb{C}^{2q})\) is not a \(K \times \text{SP}_2\times C_{0,1}\)-spherical variety for any \(q \in \mathbb{Z}_{\geq 2}\).

Proof. The variety \(\text{Fl}(1, 3; W \oplus \mathbb{C}^{2q})\) is \(K \times \text{SP}_2\times C_{0,1}\)-spherical if and only if the variety \(\text{Fl}(1, 3; W \oplus \mathbb{C}^2 \oplus \mathbb{C})\) is \(K \times \text{GL}_2 \times C_{0,1}\)-spherical (Lemma 6.17). The latter is false by Theorem 6.3. □

Lemma 6.20. For any \(q \in \mathbb{Z}_{\geq 3}\) the following statements are equivalent.
1) The variety \(\text{Gr}(4; W \oplus \mathbb{C}^{2q})\) is \(K \times \text{SP}_2\times C_{0,1}\)-spherical.
2) The variety \(\text{Gr}(4; W \oplus \mathbb{C}^2 \oplus \mathbb{C})\) is \(K \times \text{SL}_2 \times \text{SL}_2 \times C_{0,1,1}\)-spherical.
3) The variety \(\text{Hom}(\mathbb{C}^2 \oplus \mathbb{C}^2, W)\) is \(\text{SL}_2 \times \text{SL}_2 \times C_{0,1,1}\times K\)-spherical.

Proof. As the datum \((\text{SP}_2, \mathbb{C}^{2q}, 4)\) determines the pair \((\text{SL}_2 \times \text{SL}_2 \times C_{1,1}, \mathbb{C}^2 \oplus \mathbb{C}^2)\), part 1) is equivalent to part 2) by Theorem 6.2. Part 3) is a reformulation of part 2). □

Corollary 6.21. Suppose that \(\text{Gr}(4; W \oplus \mathbb{C}^{2q})\) is a \(K \times \text{SP}_2\)-spherical variety for some \(q \in \mathbb{Z}_{\geq 3}\). Then either \(n_W = 2\) and \(\text{sl}(W) \subset \mathfrak{t}_W\), or \(n_W = 1\).

Proposition 6.22. Let \(V\) be a \(K\)-module.
a) Suppose that \(\text{Gr}(r; V)\) is a \(K\)-spherical variety for some \(r \in \{2, \ldots, \left\lfloor \frac{n_W}{2} \right\rfloor \}\). Then the datum
\[
(r; [\mathfrak{t}_V, \mathfrak{t}_V], V)
\]

appears in the following list:
1) \((r; \mathfrak{sl}_n, \mathbb{C}^n), (r; \mathfrak{so}_n, \mathbb{C}^n)(n \geq 3), (r; \mathfrak{sp}_n, \mathbb{C}^n)\);
2) \((2; \mathfrak{sp}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(m \geq 1)\);
2-1) \((2; \mathfrak{sp}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(m \geq 1)\);
2-2) \((3; \mathfrak{sl}_n \oplus \mathfrak{sp}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(n \geq 1)\);
2-3) \((r; \mathfrak{sp}_n, \mathbb{C}^n \oplus \mathbb{C})\);
2-4) \( (r; \mathfrak{sl}_n \oplus \mathfrak{sp}_4, \mathbb{C}^n \oplus \mathbb{C}^4)(n \geq 1); \)
2-5) \( (r; \mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(n, m \geq 1); \)
3-1-1) \( (2; \mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{sl}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(m, n, q \geq 1); \)
3-1-2) \( (2; \mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(m \geq 1, n \geq 1); \)
3-1-3) \( (2; \mathfrak{sl}_n \oplus \mathfrak{sp}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(n \geq 1); \)
3-1-4) \( (2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q); \)
3-2) \( (r; \mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C})(n, m \geq 1). \)

b) For all data \((r; k', V)\) from the list there exists a reductive subgroup \(K \subset \text{GL}(V)\) with Lie algebra \(k\) such that \(\text{Gr}(r; V)\) is \(K\)-spherical and \(k' = [k, k]\).

Proof. Let \(V\) be a direct sum of nonzero simple \(\mathfrak{k}\)-modules \(V_1 \oplus V_2 \oplus ... \oplus V_l\).

By Lemma 3.18 the variety \(\text{Gr}(2; V)\) is \(K\)-spherical and therefore \(l \leq 3\) by Lemma 6.14.

If \(l = 1\), the \(K\)-module \(V\) is simple. This case is considered in Subsection 6.2 (Theorem 6.4 and Remark 6.5).

In the rest of the proof we assume that \(l \geq 2\), i.e. \(l = 2, 3\). By Lemma 6.13 for all \(i\),

\[ [\mathfrak{k}_{V_i}, \mathfrak{k}_{V_i}] = \mathfrak{sl}(V_i), \text{ or } [\mathfrak{k}_{V_i}, \mathfrak{k}_{V_i}] = \mathfrak{sp}(V_i) \text{ and } n_{V_i} \geq 4. \]

If the variety \(\text{Gr}(r; V)\) is \(K\)-spherical then the pair of sets

\[ \{r, n_V - r\}, \{n_{V_1}, ..., n_{V_l}\} \]

appears in the list of Theorem 6.3. All entries of the required list with \(r = 2\) are recovered by the list of Theorem 6.3.

In the rest of the proof we assume that \(r \geq 3\). It remains to show that the datum (3) is as in cases 2-2, 2-3, 2-4, 2-5 or 3-2. If \(\mathfrak{sl}(V_i) \subset \mathfrak{k}_{V_i}\) for all \(i\), then Theorem 6.3 implies that we are in cases 2-5 or 3-2.

In the rest of the proof we assume that

\[ [\mathfrak{k}_{V_1}, \mathfrak{k}_{V_1}] = \mathfrak{sp}(V_1) \text{ and } n_{V_1} \geq 4. \]

As \(r \geq 3\), \(\text{Gr}(3; V)\) is a \(K\)-spherical variety by Proposition 3.15.

Then

\[ \mathfrak{sl}(V_2 \oplus ... \oplus V_s) \subset \mathfrak{k}_{V_2 \oplus ... \oplus V_s}, \]

in particular, \(l = 2\). For \(r = 3\), this is case 2-2 from the list.

In the rest of the proof we assume that \(r \geq 4\). Case 2-4 corresponds to \(n_{V_1} = 4\). Assume now that \(n_{V_1} \geq 6\). As \(\text{Gr}(r; V)\) is a \(K\)-spherical variety, \(\text{Gr}(4; V)\) is a \(K\)-spherical variety. Therefore \(n_{V_1} \leq 2\) by Corollary 6.21. A straightforward computation shows that \(\text{Gr}(4; V_1 \oplus \mathbb{C}^2)\).
is not an $\text{SP}(V_1) \times \text{GL}_2$-spherical variety for all $n_{V_1}$ such that $n_{V_1} \geq 6$. Therefore $n_{V_2} = 1$ and we are in case 2-3.

This proves a). Part b) is straightforward. 

We are now ready to prove Theorem 6.1.

**Proof of theorem 6.1.** Let $V$ be a direct sum of simple nonzero $\mathfrak{k}$-modules

$$V_1 \oplus V_2 \oplus ... \oplus V_l.$$ 

If $l = 1$, the $\mathfrak{k}$-module $V$ is simple. This case is considered in Subsection 6.2 (Remark 6.8, Corollary 6.7, Corollary 6.10). In the rest of the proof we assume that $l \geq 2$. The claim of Theorem 6.1 for $s = 1$ is established in Proposition 6.22.

In the rest of the proof we also assume that $s \geq 2$. By Lemma 3.18 the variety $\text{Gr}(2; V)$ is $K$-spherical and $l \leq 3$ by Lemma 6.14. Lemma 6.13 implies, for all $i$, that

$$[\mathfrak{k}_{V_i}, \mathfrak{k}_{V_i}] = \mathfrak{sl}(V_i), \text{ or } [\mathfrak{k}_{V_i}, \mathfrak{k}_{V_i}] = \mathfrak{sp}(V_i) \text{ and } n_{V_i} \geq 4.$$ 

If the variety $\text{Fl}(n_1, ..., n_s; V)$ is $K$-spherical then the pair of sets

$$(\{n_1, ..., n_V - n_s\}, \{n_{V_1}, ..., n_{V_l}\})$$

appears in the list of Theorem 6.3. If $\mathfrak{sl}(V_i) \subset \mathfrak{k}_{V_i}$ for all $i$, then Theorem 6.3 implies that we are in cases II-1-1, II-2-1, II-2-2, II-2-3.

In the rest of the proof we assume that $[\mathfrak{k}_{V_i}, \mathfrak{k}_{V_i}] = \mathfrak{sp}(V_i)$ and $n_{V_1} \geq 4$.

It remains to show that the datum (1) is as in cases II-1-2, II-1-3, II-2-4, II-2-5.

We have now $l \leq 2$ by Theorem 6.3 and therefore $l = 2$. The variety $\text{Fl}(n_1, ..., n_s; V)$ is cotangent-equivalent to $\text{Fl}(1, 2; V)$, or to $\text{Fl}(1, 3; V)$, or is higher than $\text{Fl}(1, 3; V)$ (see Subsection 3.3). As the variety $\text{Fl}(1, 3; V)$ is not $K$-spherical by Corollary 6.19 $\text{Fl}(n_1, ..., n_s; V)$ is cotangent-equivalent to $\text{Fl}(1, 2; V)$ and we are in cases II-2-4, II-2-5.

This proves a). Part b) is straightforward.
7. Joseph ideals

In this section $\mathfrak{g}$ is semisimple. Let $\lambda \in \mathfrak{h}_0^*$ be a dominant weight. A two-sided ideal $I$ of $U(\mathfrak{g})$ determines the submodule $IM_\lambda$ of $M_\lambda$. This defines a map $\alpha$ from the lattice of ideals of $U(\mathfrak{g})$ to the lattice of submodules of $M_\lambda$.

**Theorem 7.1** (A. Joseph [Jo1, Prop. 4.3]). The map $\alpha$ is an embedding of the lattice of two-sided ideals $I \subset U(\mathfrak{g})$ such that $\text{Ker} \chi_\lambda \subset I$ to the lattice of submodules of $M_\lambda$. If $\lambda$ is regular, $\alpha$ is an isomorphism.

**Remark 7.1.** The two-sided ideals of $U(\mathfrak{g})$ have been studied extensively in the last 30 years. This section adapts some known results to the setup of the dissertation. We thank Anthony Joseph and Anna Melnikov for useful comments and references.

### 7.1. The case $\mathfrak{g} = \mathfrak{sl}(W)$

Let $\mathfrak{Q} \subset \mathfrak{sl}(W)^*$ be the non-trivial nilpotent orbit of minimal dimension. This nilpotent orbit consists of matrices of rank 1. We represent such matrices as $l \otimes v$, for some $v \in W$ and $l \in W^*$ with $l(v) = 0$.

**Definition 7.2.** Let $I$ be a primitive ideal of $U(\mathfrak{sl}(W))$. We call $I$ a *Joseph ideal* whenever $V(I) = \overline{\mathfrak{Q}}$.

**Lemma 7.3.** Let $I$ be a Joseph ideal and let $\tilde{L}$ be a finitely generated $(\mathfrak{sl}(W), \mathfrak{b}_{W}^{sl})$-module such that $I \subset \text{Ann} \tilde{L}$. Then $\tilde{L}$ is a $(\mathfrak{sl}(W), \mathfrak{b}_{W}^{sl})$-bounded module.

**Proof.** As $\tilde{L}$ is an $(\mathfrak{sl}(W), \mathfrak{b}_{W}^{sl})$-module, $V(\tilde{L}) \subset \overline{\mathfrak{Q}} \cap (\mathfrak{b}_{W}^{sl}) ^\perp$. We have

$$\overline{\mathfrak{Q}} \cap (\mathfrak{b}_{W}^{sl}) ^\perp = \{ x \in \mathfrak{sl}(W)^* \mid x = l \otimes v \text{ for some } v \in W \text{ and } l \in W^* \text{ such that } l(bv) = 0 \text{ for all } b \in \mathfrak{b}_{W}^{sl} \}.$$  

The variety $\overline{\mathfrak{Q}} \cap (\mathfrak{b}_{W}^{sl}) ^\perp$ has irreducible components $\{ \mathfrak{Q}_i \}_{i \leq n_W - 1}$, where

$$\mathfrak{Q}_i := \{ x \in \mathfrak{sl}(W)^* \mid x = l \otimes v \text{ for some } v \in \text{span} \{ e_1, ..., e_i \} \text{ and } \quad l \in \text{span} \{ e_{i+1}, ..., e_{n_W} \} \}.$$  

The dimension of $\mathfrak{Q}_i$ equals $n_W - 1$ for any $i \leq n_W - 1$. Let $\mathfrak{H}_{W}^{sl}$ be the connected subgroup of SL$(W)$ with Lie algebra $\mathfrak{b}_{W}^{sl}$. The varieties $\mathfrak{Q}_i$ are $\mathfrak{H}_{W}^{sl}$-stable and are $\mathfrak{H}_{W}^{sl}$-spherical for all $i \leq n_W - 1$. Therefore $\tilde{L}$ is a bounded $(\mathfrak{sl}(W), \mathfrak{b}_{W}^{sl})$-module by Proposition 3.23.

**Corollary 7.4.** a) Let $I$ be a Joseph ideal and let $\lambda \in \mathfrak{h}_W^{sl*}$ be a weight such that $I = \text{Ann} L_\lambda$. Then $L_\lambda$ is a bounded $\mathfrak{b}_{W}^{sl}$-module. In particular, $\lambda$ is a semi-decreasing tuple.

b) If $\bar{\lambda}$ is a semi-decreasing $n_W$-tuple then $I(\bar{\lambda})$ is a Joseph ideal.
Proof. Part a) is trivial. We have
\[
\dim \text{GV}(M) \leq 2 \dim V(M)
\]
by Theorem 3.7. As \( \bar{\lambda} \) is semi-decreasing, \( L_\lambda \) is bounded and therefore \( V(L_\lambda) \) is \( H^0_{\bar{\lambda}} \)-spherical. Hence \( \dim \text{GV}(M) \leq 2(n_W - 1) \). The only nilpotent orbit which satisfies this inequality is \( Q \). \( \square \)

Corollary 7.5 ([Jo4 Table 3]). Let \( I \subset U(\mathfrak{sl}(W)) \) be a two-sided ideal. Then \( I \) is a Joseph ideal if and only if one of the following conditions holds:
\begin{itemize}
  \item[a)] there exists a semi-integral semi-decreasing tuple \( \bar{\lambda} \) such that \( I = I(\bar{\lambda}) \);
  \item[b)] there exists a singular integral semi-decreasing tuple \( \bar{\lambda} \) such that \( I = I(\lambda) \);
  \item[c)] there exists a regular integral semi-decreasing tuple \( \bar{\lambda} \) and a number \( k \in \{1, ..., n_W - 1\} \) such that \( I = I(s_k \lambda) \).
\end{itemize}

Proof. Follows from Corollary 7.4 and the discussion about coherent families in Subsection 3.9. \( \square \)

Lemma 7.6. Let \( k \in \{1, 2, ..., n_W - 1\} \) and \( \lambda \) be a regular integral dominant weight. Then \( \text{Ann} L_{s_k \lambda} \) is a submaximal ideal of \( U(\mathfrak{sl}(W)) \), i.e. for any two ideals \( I_1, I_2 \) such that \( \text{Ann} L_{s_k \lambda} \subseteq I_1, I_2 \) we have \( I_1 = I_2 \).

Proof. The Verma module \( M_\lambda \) has a unique maximal submodule \( \text{rad}^1 M_\lambda \) such that \( M_\lambda/\text{rad}^1 M_\lambda \) is a simple module \( L_\lambda \). Let \( \text{rad}^2 M_\lambda \) be the minimal submodule of \( \text{rad}^1 M_\lambda \) such that \( \text{rad}^1 M_\lambda/\text{rad}^2 M_\lambda \) is a semisimple \( \mathfrak{sl}(W) \)-module. The module \( M_\lambda/\text{rad}^1 M_\lambda \) is finite-dimensional and \( \text{rad}^1 M_\lambda/\text{rad}^2 M_\lambda \) has a unique submodule isomorphic to \( L_{s_k \lambda} \). Let \( \text{rad}^{1+s_k} M_\lambda \) be a submodule of \( \text{rad}^1 M_\lambda \) such that \( \text{rad}^1 M_\lambda/\text{rad}^{1+s_k} M_\lambda = L_{s_k \lambda} \). Let \( I \) be the corresponding to \( \text{rad}^{1+s_k} M_\lambda \) two-sided ideal. Then \( I \) is submaximal and \( I \subset \text{Ann} L_{s_k \lambda} \). As the unique ideal which is larger then \( I \) is of finite codimension, we have \( \text{Ann} L_{s_k \lambda} = I \). Therefore the ideal \( \text{Ann} L_{s_k \lambda} \) is submaximal. \( \square \)

Let \( \bar{\lambda} \) be an \( n_W \)-tuple. We can consider the algebra \( U(\mathfrak{sl}(W))/I(\lambda) \) as an \( (\mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r) \)-module, where both \( \mathfrak{sl}(W)_l \) and \( \mathfrak{sl}(W)_r \) are isomorphic to \( \mathfrak{sl}(W) \) and the action is given by
\[
(g_1, g_2)m = g_1 m - m g_2
\]
for \( g_1, g_2 \in \mathfrak{sl}(W) \) and \( m \in U(\mathfrak{sl}(W))/I(\lambda) \). Fix \( i \in \{1, ..., n_W - 1\} \). The action of
\[
\mathfrak{sl}(W)_d := \{(g, 0) \mid g \in \mathfrak{sl}(W) \} \subset \mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r
\]
on \( U(\mathfrak{sl}(W))/I(s_i \rho) \) is locally finite. Therefore \( U(\mathfrak{sl}(W))/I(s_i \rho) \) is an \( (\mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r, \mathfrak{sl}(W)_d) \)-module. This module affords the central character \( (\chi_\rho, \chi_\rho) \). The category of \( (\mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r, \mathfrak{sl}(W)_d) \)-modules
which afford the central character $(\chi_\rho, \chi_\rho)$ is equivalent to the category of $(\mathfrak{sl}(W), b^\ell_W)$-modules which afford the central character $\chi_\rho$. The equivalence is given by the functor
\[ H_{M_\rho} : \mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r \mod \rightarrow \mathfrak{sl}(W) \mod, \]
\[ M \mapsto M \otimes_{\mathfrak{sl}(W)} M_\rho, \]
see [BeG, Prop. 5.9]. The action of the $\mathfrak{sl}(W)_l$-simple modules and $\mathfrak{sl}(W)_r$-modules commutes with the functor $H_{M_\rho}$.

Fix $i, j \in \{1, ..., n_W - 1\}$. Let $P_j$ be the maximal parabolic subgroup of $\text{SL}(V)$ for which the space span$(e_r)_{r \leq j}$ is stable and $p_j$ be the Lie algebra of $P_j$. We have
\[ F_W H_\rho = \Sigma_{i \leq n_W} H_\rho^{\rho + \varepsilon_i}, \]
and
\[ H_\rho^{\rho + \varepsilon_i} = H_\rho^{\rho + \varepsilon_{i+1}} H_\rho^{\rho + \varepsilon_{i+1}} = H_\rho^{\rho + \varepsilon_{i+1}} H_\rho^{\rho + \varepsilon_i} \]
for all $i \leq n_W - 1$.

**Lemma 7.7.** Assume $i \neq j$. Then
\[ H_\rho^{\rho + \varepsilon_i+1}(M_\rho/\text{rad}^{1+\varepsilon_j} M_\rho) = 0 \text{ and } H_\rho^{\rho + \varepsilon_j}(M_\rho/\text{rad}^{1+\varepsilon_i} M_\rho) = 0. \]

**Proof.** The module $M_\rho/\text{rad}^{1+\varepsilon_j} M_\rho$ is an $(\mathfrak{sl}(W), p_j)$-module. Therefore
\[ H_\rho^{\rho + \varepsilon_i+1}(M_\rho/\text{rad}^{1+\varepsilon_j} M_\rho) \]
is an $(\mathfrak{sl}(W), p_j)$-module. If $H_\rho^{\rho + \varepsilon_i+1}(M_\rho/\text{rad}^{1+\varepsilon_j} M_\rho) \neq 0$, then some simple quotient of $H_\rho^{\rho + \varepsilon_i+1} M_\rho$ is an $(\mathfrak{sl}(W), p_j)$-module. If $i \neq j$, a unique simple quotient of
\[ H_\rho^{\rho + \varepsilon_i+1} M_\rho = M_{\rho + \varepsilon_{i+1}} \]
is not an $(\mathfrak{sl}(W), p_j)$-module, i.e. the action of $p_j$ on $L_{\rho + \varepsilon_{i+1}}$ is not locally finite. Therefore
\[ H_\rho^{\rho + \varepsilon_i+1}(M_\rho/\text{rad}^{1+\varepsilon_j} M_\rho) = 0 \text{ and } H_\rho^{\rho + \varepsilon_j}(M_\rho/\text{rad}^{1+\varepsilon_i} M_\rho) = 0. \]

\[ \square \]

As $[P_{s, \rho}] = [M_\rho] + [M_{s, \rho}]$, the projective functor $H_\rho^{s_i \rho}$ is identified with $s_i+1$ under the identification of $\text{Pf}\mathfrak{sl}(\chi_\rho)$ with $\mathbb{Z}[S_{n_W}]$. As the set $\{s_i + 1\}_{i \leq n_W - 1}$ generates $\mathbb{Z}[S_{n_W}]$, the set $\{H_\rho^{s_i \rho}\}_{i \leq n_W - 1}$ generates the algebra $\text{Pf}\mathfrak{sl}(\chi_\rho)$.

**Corollary 7.8.** Let $M$ be an $\mathfrak{sl}(W)$-module. Suppose that $M$ is annihilated by a submaximal ideal $I(s_j \rho)$ for some $j \leq n_W - 1$. Then
\[ H_\rho^{\rho + \varepsilon_i+1} M = 0 \text{ for all } i \neq j. \]
Remark 7.9. The notion of left cell [Jo6] (see also [Vo2]) relates nilpotent orbits with Weyl group modules. Moreover, the ordering of primitive ideals is related to left cells via tensoring by finite-dimensional representations. As the projective functors are building blocks of the functors of tensoring with a finite-dimensional representation, the action of the projective functors on a primitive ideal $I$ is closely related [Jo5] with the nilpotent orbit $V(I)$ (see also [Jo3]). Corollary 7.8 is an explicit example of this relationship.

Proposition 7.10. Let $M$ be an infinite-dimensional finitely generated $\mathfrak{sl}(W)$-module which affords the generalized central character $\chi_\rho$. Then there exists $i \leq n_W - 1$ such that $\mathcal{H}_\rho^{i+1} M \neq 0$.

Proof. Without loss of generality we assume that $M = \mathfrak{sl}(W)M$. In [BeBe] the authors mention that there exists a Borel subalgebra $b \subset \mathfrak{sl}(W)$ with nilpotent radical $n$ such that $(M/nM)^b \neq 0$. We have

$$(M/nM)^b = (M_\rho \otimes C M)/(\mathfrak{sl}(W)(M_\rho \otimes C M)) \neq 0 \quad (4).$$

As the sequence

$$\bigoplus_{i \leq n_W - 1} P_{s_i, \rho} \to M_\rho \to L_\rho \to 0$$

is exact, the sequence

$$\frac{\bigoplus_{i \leq n_W - 1} P_{s_i, \rho} \otimes C M}{\mathfrak{sl}(W) \bigoplus_{i \leq n_W - 1} P_{s_i, \rho} \otimes C M} \to \frac{M_\rho \otimes C M}{\mathfrak{sl}(W)(M_\rho \otimes C M)} \to \frac{L_\rho \otimes C M}{\mathfrak{sl}(W)(L_\rho \otimes C M)} \to 0$$

is exact. As

$$(L_\rho \otimes C M)/\mathfrak{sl}(W)(L_\rho \otimes C M) = M/\mathfrak{sl}(W)M = 0,$$

formula (4) implies

$$(P_{s_i, \rho} \otimes C M)/\mathfrak{sl}(W)(P_{s_i, \rho} \otimes C M) \neq 0$$

for some $i \leq n_W - 1$. A straightforward computation shows that

$$\frac{P_{s_i, \rho} \otimes C M}{\mathfrak{sl}(W)(P_{s_i, \rho} \otimes C M)} = \frac{\mathcal{H}_\rho^{i+1} M_\rho \otimes C \mathcal{H}_\rho^{n_W - i - 1} M}{\mathfrak{sl}(W)(\mathcal{H}_\rho^{i+1} M_\rho \otimes C \mathcal{H}_\rho^{n_W - i - 1} M)} = \frac{M_\rho \otimes C \mathcal{H}_\rho^{n_W - i - 1} M}{\mathfrak{sl}(W)(M_\rho \otimes C \mathcal{H}_\rho^{n_W - i - 1} M)}.$$

Therefore $\mathcal{H}_\rho^{n_W - i - 1} M \neq 0$. □
7.2. The case $g = \mathfrak{sp}(W \oplus W^*)$. Let $\mathcal{Q}_\mathfrak{sp} \subset \mathfrak{sp}(W \oplus W^*)^*$ be the non-trivial nilpotent orbit of minimal dimension. This nilpotent orbit consists of matrices of rank 1. We represent such matrices as $v \otimes v$ with $v \in W \oplus W^*$.

**Definition 7.11.** Let $I$ be a primitive ideal of $U(\mathfrak{sp}(W \oplus W^*))$. We call $I$ a Joseph ideal whenever $V(I) = \mathcal{Q}_\mathfrak{sp}$.

**Lemma 7.12.** Let $I$ be a Joseph ideal and let $\tilde{L}$ be a simple $(\mathfrak{sp}(W \oplus W^*), \mathfrak{b}_W^{\mathfrak{sp}})$-module. Then $\tilde{L}$ is an $(\mathfrak{sp}(W \oplus W^*), \mathfrak{h}_W)$-bounded module.

**Proof.** As $\mathfrak{g}$ is a bounded Lie algebra, we have $V(\tilde{L}) \subset \tilde{Q}_\mathfrak{sp} \cap \mathfrak{b}_W^{\mathfrak{sp}}$. Furthermore,

$$(\mathfrak{b}_W^{\mathfrak{sp}})^{\perp} \cap \tilde{Q}_\mathfrak{sp} = \{ x \in \mathfrak{sp}(W \oplus W^*)^* | x = v \otimes v \text{ for some } v \in W \oplus W^* \}$$

such that $(v, bv) = 0$ for all $b \in \mathfrak{b}_W^{\mathfrak{sp}}$.

The variety $(\mathfrak{b}_W^{\mathfrak{sp}})^{\perp} \cap \tilde{Q}_\mathfrak{sp}$ has the unique irreducible component $\mathcal{Q}_1$, where $\mathcal{Q}_1 := \{ x \in \mathfrak{sp}(W \oplus W^*)^* | x = v \otimes v \text{ for some } v \in \text{span}\{e_1, ..., e_n\} \} \subset \mathfrak{b}_W^{\mathfrak{sp}}$.

The dimension of $\mathcal{Q}_1$ equals to $n_W$. Let $H_W$ be an irreducible subgroup of $\text{Sp}(W \oplus W^*)$ with Lie algebra $\mathfrak{h}_W$. The variety $\mathcal{Q}_1$ is $H_W$-stable and is $H_W$-spherical. Therefore $\tilde{L}$ is a bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{h}_W)$-module by Proposition 3.23.

**Corollary 7.13.** a) Let $I$ be a Joseph ideal and let $\mu$ be an $n_W$-tuple such that $I = \text{Ann} L_\mu$. Then $L_\mu$ is a bounded $\mathfrak{h}_W$-module. In particular, $\mu$ is a Shale-Weil tuple.

b) Let $\mu$ be a Shale-Weil $n_W$-tuple then $I_{\mathfrak{sp}}(\mu)$ is a Joseph ideal.

**Proof.** Part a) is trivial. We have $\dim GV(M) \leq 2\dim V(M)$ by Theorem 3.4. As $\mu$ is Shale-Weil, $L_\mu$ is bounded and therefore $V(L_\mu)$ is $H_W$-spherical. Hence $\dim GV(M) \leq 2n_W$. The only nilpotent orbit which satisfies this inequality is $\mathcal{Q}_\mathfrak{sp}$.

We recall that $\sigma$ stands to the reflection with respect to the simple long root of $\mathfrak{sp}(W \oplus W^*)$ for our fixed choice of a Borel subalgebra, see Subsection 3.9.

**Lemma 7.14.** Let $\mu$ be a Shale-Weil tuple. Then $\text{Ann} L_\mu$ is a maximal ideal of $U(\mathfrak{sp}(W \oplus W^*))$.

**Proof.** Without loss of generality we assume that $\mu$ is positive, in particular $\mu$ is dominant. Let $I$ be an ideal of $U(\mathfrak{sp}(W \oplus W^*))$ such that $I(\mu) \subsetneq I$. Then $I(\mu)M_\mu \subsetneq IM_\mu$. As $I$ annihilates $M_\mu/IM_\mu$, we have $I \subset \text{Ann} L_\mu'$ for some simple $\mathfrak{h}_W$-bounded module $L_\mu'$. Therefore either $\mu' = \mu$ or $\mu' = \sigma \mu$. Hence $I \subset I(\mu)$ and $I = I(\mu)$.
Let $\bar{\mu}$ be a Shale-Weil $n_W$-tuple. Let $\bar{\mu}'$ be an $n_W$-tuple and let $\mu' \in h^*_W$ be the corresponding weight.

**Lemma 7.15.** Suppose $\mathcal{H}_\mu L_\mu \neq 0$. Then $\mu'$ is a Shale-Weil tuple.

**Proof.** The $\mathfrak{sp}(W \oplus W^*)$-module $L_\mu$ is $h_W$-bounded and therefore $\mathcal{H}_\mu L_\mu$ is $h_W$-bounded. As $L_\mu$ is a quotient of $M_\mu$, $\mathcal{H}_\mu L_\mu$ is a quotient of $\mathcal{H}_\mu M_\mu$. As $L_\mu'$ is the unique simple quotient of $\mathcal{H}_\mu M_\mu$, $L_\mu'$ is an $h_W$-bounded $\mathfrak{sp}(W \oplus W^*)$-module. Therefore $\mu'$ is a Shale-Weil tuple by Theorem 3.15. □

**Corollary 7.16.** Let $M$ be an $\mathfrak{sp}(W \oplus W^*)$-module such that $I_{\mathfrak{sp}(\mu)} = \text{Ann} M$. Suppose $\mathcal{H}_\mu' M \neq 0$. Then $\mu'$ is a Shale-Weil $n_W$-tuple.
8. Modules of small growth

Definition 8.1. Let $M$ be a $\mathfrak{g}$-module. We say that $M$ is $Z(\mathfrak{g})$-finite if $\dim Z(\mathfrak{g})m < \infty$ for all $m \in M$, where $Z(\mathfrak{g}) := \{m' \in M \mid m' = zm \text{ for some } m \in M \text{ and } z \in Z(\mathfrak{g})\}$.

8.1. The case $\mathfrak{g} = \mathfrak{sl}(W)$.

Proposition 8.2. If $M$ is an infinite-dimensional finitely generated $\mathfrak{sl}(W)$-module then $\dim V(M) \geq n_W - 1$.

Proof. The variety $GV(M)$ is a union of several nilpotent $SL(W)$-orbits in $\mathfrak{sl}(W)^*$. There is a unique nonzero nilpotent orbit of minimal dimension equal to $2(n_W - 1)$. As $M$ is infinite-dimensional, $\dim V(M) \geq \frac{1}{2} \dim GV(M) \geq n_W - 1$ by Theorem 3.7.

Definition 8.3. Let $M$ be a finitely generated $\mathfrak{sl}(W)$-module which is $Z(\mathfrak{sl}(W))$-finite. We say that $M$ is of small growth if $\dim V(M) \leq n_W - 1$, i.e. if $\dim V(M)$ equals either $n_W - 1$ or 0.

The $\mathfrak{sl}(W)$-modules of small growth form a full subcategory of the category of $\mathfrak{sl}(W)$-modules. This subcategory is stable under the projective functors.

We recall that a finitely generated $\mathfrak{g}$-module $M$ has a graded version $\text{gr}M$ which is a finitely generated $S(\mathfrak{g})$-module. The sum of ranks\(^2\) of $\text{gr}M$ over the function rings of all irreducible components of $V(M)$ of maximal dimension is called the Bernstein number of $M$ (see also [KL, p. 78]), and we denote this number $b(M)$.

Proposition 8.4. Any $\mathfrak{sl}(W)$-module $M$ of small growth is of finite length.

Proof. As $M$ is finitely generated, it is enough to check the descending chain condition for $M$. Let $\ldots \subset M_{-i} \subset \ldots \subset M_{-0} = M$ be a strictly descending chain of $\mathfrak{sl}(W)$-submodules. These $\mathfrak{sl}(W)$-submodules $M_{-i}$ are finitely generated. We have $\dim V(M_{-i}) = n_W - 1$ for all $i \in \mathbb{Z}_{\geq 0}$ or $\dim V(M_{-i}) = 0$ for some $i \in \mathbb{Z}_{\geq 0}$. In the second case $\dim M_{-i} < \infty$ and therefore the chain $\ldots \subset M_{-i}$ stabilizes.

Assume that $\dim V(M_{-i}) = n_W - 1$ for all $i \in \mathbb{Z}_{\geq 0}$. If $i \geq j$ then $b(M_{-i}) \leq b(M_{-j})$, i.e. the sequence $\{b(M_{-i})\}_{i \in \mathbb{Z}_{\geq 0}}$ is decreasing. Therefore there exists $i \in \mathbb{Z}_{\geq 0}$ such that $b(M_{-i}) = b(M_{-j})$ for all $j \geq i$.

\(^2\)The rank of finitely generated module over a commutative ring is the rank of a maximal free submodule.
i. Therefore \( \dim V(M_{-i}/M_{-j}) < n_W - 1 \) and \( M_{-i}/M_{-j} \) is a finite-dimensional \( \mathfrak{sl}(W) \)-module for all \( j \geq i \). We have an inclusion

\[
M_{-i}/ \cap_{j \geq i} M_{-j} \hookrightarrow \bigoplus_{j \geq i} M_{-i}/M_{-j}.
\]

The right-hand side is a direct sum of a finite-dimensional \( \mathfrak{sl}(W) \)-modules. As \( M_{-i} \) is \( Z(\mathfrak{sl}(W)) \)-finite and finitely generated, only a finite number of simple \( \mathfrak{sl}(W) \)-modules appear in this direct sum. Therefore \( M_{-i}/ \cap_{j \geq i} M_{-j} \) is a finite-dimensional \( \mathfrak{sl}(W) \)-module. \( \square \)

**Proposition 8.5.** Let \( I \) be the annihilator of a simple infinite-dimensional \( \mathfrak{sl}(W) \)-module \( M \) of small growth. Then \( I \) is a Joseph ideal.

**Proof.** As \( \dim GV(M) \leq 2\dim V(M) \) (Theorem 3.7),

\[
\dim GV(M) \leq 2(n_W - 1).
\]

As \( M \) is infinite-dimensional, \( I \) is a Joseph ideal. \( \square \)

Let \( J \) be the set of simple \( \mathfrak{sl}(W) \)-modules of small growth, and \( \langle J \rangle \) be the vector space generated by the set \( J \). The action of the projective functors on the category of modules of small growth defines an action of the projective functors on \( \langle J \rangle \) by linear operators. As any \( \mathfrak{sl}(W) \)-module \( M \) of small growth has finite length, \( M \) defines the vector \( [M] \in \langle J \rangle \) which is a sum of the simple \( M \)-subquotients with their multiplicities in \( M \).

Let \( \lambda \) be a decreasing \( n_W \)-tuple and \( J_{\lambda} \) be the set of simple \( \mathfrak{sl}(W) \)-modules of small growth annihilated by \( \operatorname{Ker} \chi_\lambda \), and \( \langle J_{\lambda} \rangle \) be the free vector space generated by \( J_{\lambda} \). The action of \( \operatorname{PFunc}(\chi_\lambda) \) defines an action of \( S_n^{\chi_{\lambda}(\rho)} \) on \( \langle J_{\lambda} \rangle \), see Proposition 3.30. We recall that \( C^{\operatorname{sgn}} \) is the sign representation of \( S_n^{\chi_{\lambda}(\rho)} \). Set \( \langle J_{\lambda} \rangle^{\operatorname{sgn}} := \langle J_{\lambda} \rangle \otimes C^{\operatorname{sgn}} \). As vector spaces \( \langle J_{\lambda} \rangle \) and \( \langle J_{\lambda} \rangle^{\operatorname{sgn}} \) are identical.

**Lemma 8.6.** An \( S_{n_W} \)-module \( \langle J_{\lambda} \rangle^{\operatorname{sgn}} \) is a direct sum of copies of \( \mathbb{C}^{n_W-1} \) and \( \mathbb{C} \).

**Proof.** Without loss of generality we assume that \( \lambda = \rho \) (\( \rho \) is defined in Subsection 3.8). The action of \( H_s^{\chi_{\rho}(\rho)} \) on \( J_{\lambda} \) corresponds to multiplication by \( (s_k - 1) \) on \( \langle J_{\lambda} \rangle^{\operatorname{sgn}} \) (cf. Corollary 7.8). Let \( M \) be a simple \( \mathfrak{sl}(W), \mathfrak{sl}(V) \)-module of small growth and \( k \) be a number such that \( \operatorname{Ann} M = I(s_k \rho) \). Then \( H_s^{\chi_{\rho}(\rho)} M = 0 \) for all \( k \neq l \) (Corollary 7.8). As the simple modules generate \( \langle J_{\lambda} \rangle^{\operatorname{sgn}} \),

\[
\langle J_{\lambda} \rangle^{\operatorname{sgn}} = \bigoplus_{1 \leq i \leq n_W - 1} \langle J_{\lambda} \rangle_i^{\operatorname{sgn}}.
\]

Therefore as an \( S_{n_W} \)-module \( \langle J_{\lambda} \rangle^{\operatorname{sgn}} \) is a direct sum of copies of \( \mathbb{C}^{n_W-1} \) and \( \mathbb{C} \) (Lemma 3.33). \( \square \)
Lemma 8.7. Let $M$ be a simple $\mathfrak{sl}(W)$-module of small growth annihilated by $\text{Ker} \chi_{\rho}$ and $k \in \{1, \ldots, n_W - 1\}$ be a number. Then, for some $j \in \{1, 2\}$,

$$[\mathcal{H}_{\rho}^{\rho + \varepsilon_k} M] = j[M']$$

for a simple $\mathfrak{sl}(W)$-module $M'$ of small growth.

Proof. Assume that $\mathcal{H}_{\rho}^{\rho + \varepsilon_k} M$ is nonzero. The identity homomorphism

$$\text{Id}: \mathcal{H}_{\rho}^{\rho + \varepsilon_k} M \to \mathcal{H}_{\rho}^{\rho + \varepsilon_k} M$$

gives rise to a nonzero homomorphism

$$M \to \mathcal{H}_{\rho}^{\rho + \varepsilon_k} \mathcal{H}_{\rho}^{\rho + \varepsilon_k} M = \mathcal{H}_{\rho}^{\rho + \rho} M.$$

Let $M'$ be a simple subquotient of $\mathcal{H}_{\rho}^{\rho + \varepsilon_k} M$ such that $M$ is a subquotient of $\mathcal{H}_{\rho}^{\rho + \varepsilon_k} M'$. As $\mathcal{H}_{\rho}^{\rho + \varepsilon_k} \mathcal{H}_{\rho}^{\rho + \varepsilon_k} = 2\text{Id}$, we have $\mathcal{H}_{\rho}^{\rho + \varepsilon_k} \mathcal{H}_{\rho}^{\rho + \varepsilon_k} M' = M' \oplus M'$. Therefore $\mathcal{H}_{\rho}^{\rho + \varepsilon_k} M$ is semisimple of length 2 or less and $M'$ is, up to isomorphism, the only simple constituent of $M$. □

Lemma 8.8. Let $M$ be a simple $\mathfrak{sl}(W)$-module of small growth annihilated by $I(\rho + \varepsilon_k)$. Then, for some $j \in \{1, 2\}$,

$$[\mathcal{H}_{\rho}^{\rho + \varepsilon_k} M] = j[M'] + [T],$$

where $M'$ is a simple $\mathfrak{sl}(W)$-module of small growth and $T$ is some $\mathfrak{sl}(W)$-module of small growth such that

$$\mathcal{H}_{\rho}^{\rho + \varepsilon_k} T = 0, \mathcal{H}_{\rho}^{\rho + \varepsilon_k} M' = \frac{2}{j} M.$$

Proof. Let $0 \subset M_0 \subset \ldots \subset M_i = M$ be a composition series of $\mathcal{H}_{\rho}^{\rho + \varepsilon_k} M$. As $\mathcal{H}_{\rho}^{\rho + \varepsilon_k} \mathcal{H}_{\rho}^{\rho + \varepsilon_k} M = 2\text{Id}$, the number of simple subquotients $M_{i+1}/M_i$ such that $\mathcal{H}_{\rho}^{\rho + \varepsilon_k} (M_{i+1}/M_i)$ is nonzero does not exceed 2. Assume that there is a unique such subquotient $M'$. Then there exists a semisimple $\mathfrak{sl}(W)$-module $T$ such that

$$\mathcal{H}_{\rho}^{\rho + \varepsilon_k} [M] = [M'] + [T],$$

and

$$[\mathcal{H}_{\rho}^{\rho + \varepsilon_k} M'] = 2[M], \mathcal{H}_{\rho}^{\rho + \varepsilon_k} [T] = 0.$$

Suppose that such a subquotient is not unique. Then there exist two simple $\mathfrak{sl}(W)$-modules $\hat{M}_1$ and $\hat{M}_2$ and a semisimple $\mathfrak{sl}(W)$-module $T$ such that

$$[\mathcal{H}_{\rho}^{\rho + \varepsilon_k} M] = [\hat{M}_1] + [\hat{M}_2] + [T]$$

and

$$[\mathcal{H}_{\rho}^{\rho + \varepsilon_k} \hat{M}_1] = [\mathcal{H}_{\rho}^{\rho + \varepsilon_k} \hat{M}_2] = [M].$$

We have $\text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho}^{\rho + \varepsilon_k} \hat{M}_1, \mathcal{H}_{\rho}^{\rho + \varepsilon_k} \hat{M}_1) = \text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho}^{\rho + \varepsilon_k} \mathcal{H}_{\rho}^{\rho + \varepsilon_k} \hat{M}_1, \hat{M}_1)$.
Then there exists a sequence of nonzero $\mathfrak{sl}(W)$-homomorphisms

$$\mathcal{H}_\rho^\rho \hat{M}_1 \to \hat{M}_1 \to \mathcal{H}_\rho^\rho \hat{M}_1.$$ 

Therefore there exists an $\mathfrak{sl}(W)$-homomorphism $\phi_1 : \mathcal{H}_{\rho+\varepsilon_k}^\rho M \to \mathcal{H}_{\rho+\varepsilon_k}^\rho M$ whose image is $\hat{M}_1$. In the same way there exists an $\mathfrak{sl}(W)$-homomorphism

$$\phi_2 : \mathcal{H}_\rho^\rho M \to \mathcal{H}_\rho^\rho M$$

with image $\hat{M}_2$. Assume that $\hat{M}_1 \neq \hat{M}_2$. Then both compositions $\phi_1 \circ \phi_2$ and $\phi_2 \circ \phi_1$ equal zero. We have

$$\text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_\rho^\rho M, \mathcal{H}_\rho^\rho M) = \text{Hom}_{\mathfrak{sl}(W)}(M, \mathcal{H}_{\rho+\varepsilon_k}^\rho M) = \text{Hom}_{\mathfrak{sl}(W)}(M, \mathcal{H}_{\rho+\varepsilon_k}^\rho M).$$

As $M$ is simple, $\text{Hom}_{\mathfrak{sl}(W)}(M, \mathcal{H}_{\rho+\varepsilon_k}^\rho M) = \text{Hom}_{\mathfrak{sl}(W)}(\text{Id}, \mathcal{H}_{\rho+\varepsilon_k}^\rho)$, where the right-hand side refers to homomorphisms of functors. Therefore

$$\text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho+\varepsilon_k}^\rho M, \mathcal{H}_{\rho+\varepsilon_k}^\rho M) = \text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho+\varepsilon_k}^\rho, \mathcal{H}_{\rho+\varepsilon_k}^\rho).$$

However,

$$\text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho+\varepsilon_k}^\rho, \mathcal{H}_{\rho+\varepsilon_k}^\rho) = \text{Hom}_{\mathfrak{sl}(W)}(P_{s_k\rho}, P_{s_k\rho}) = \mathbb{C}[x]/(x^2).$$

As $\phi_1 \circ \phi_2 = 0$, $\phi_1$ and $\phi_2$ are identified with $\alpha_1 x$ and $\alpha_2 x$ for some constants $\alpha_1, \alpha_2 \in \mathbb{C}$. Then $\alpha_1$ is proportional to $\alpha_2$ and therefore

$$\hat{M}_1 \cong \hat{M}_2.$$ 

□

**Corollary 8.9.** There is a natural bijection between the set of simple infinite-dimensional $\mathfrak{sl}(W)$-modules of small growth annihilated by $I(s_k\rho)$ and the set of simple $\mathfrak{sl}(W)$-modules of small growth annihilated by $I(\rho + \varepsilon_k)$.

**Lemma 8.10.** The subspace of $S_{nW}$-invariants in $\langle \partial \lambda \rangle$ is one-dimensional and is generated by the class of the simple finite-dimensional module.

**Proof.** Without loss of generality we assume that $\lambda = \rho$. Let $\{\alpha_i\}$ be numbers such that $\Sigma \alpha_i [M_i]$ is non-zero and $S_{nW}$-invariant. Then $\mathcal{H}_{\rho+\varepsilon_k}^\rho \alpha_i [M_i] = 0$, i.e. $\Sigma \alpha_i \mathcal{H}_{\rho+\varepsilon_k}^\rho M_i = 0$. Therefore $\alpha_i = 0$ for all infinite-dimensional simple modules annihilated by $I(s_k\rho)$. Hence $\alpha_i \neq 0$ implies that $M_i$ is finite-dimensional. □

**Corollary 8.11.** For any $i, j \in \{1, ..., n_W - 1\}$ there is a natural bijection between the set of simple infinite-dimensional $\mathfrak{sl}(W)$-modules of small growth annihilated by $I(s_i\rho)$ and the set of simple infinite-dimensional $\mathfrak{sl}(W)$-modules of small growth annihilated by $I(s_j\rho)$. 

The following statement can be considered as a weak analogue of Theorem 3.12.

**Corollary 8.12.** Let $\overline{\lambda}_1, \overline{\lambda}_2$ be semi-decreasing $n_W$-tuples such that $m(\lambda_1) = m(\lambda_2)$.

There is a natural bijection between the set of simple infinite-dimensional $\mathfrak{sl}(W)$-modules of small growth annihilated by $I(\lambda_1)$ and the set of simple infinite-dimensional $\mathfrak{sl}(W)$-modules of small growth annihilated by $I(\lambda_1)$ and $I(\lambda_2)$.

**Proof.** This follows from Lemma 3.31, Corollary 8.9 and Theorem 3.12. □

8.2. **The case $\mathfrak{g} = \mathfrak{sp}(W \oplus W^*)**. All proofs for this subsection are similar to their counterparts in the previous subsection. We leave them to the reader.

**Proposition 8.13.** If $M$ is an infinite-dimensional finitely generated $\mathfrak{sp}(W \oplus W^*)$-module, then $\dim V(M) \geq n_W$.

**Definition 8.14.** Let $M$ be a finitely generated $\mathfrak{sp}(W \oplus W^*)$-module which is $Z(\mathfrak{sp}(W \oplus W^*))$-finite. We say that $M$ is of small growth if $\dim V(M) \leq n_W$, i.e. if $\dim V(M)$ equals either $n_W$ or 0.

The $\mathfrak{sl}(W)$-modules of small growth form a full subcategory of the category of $\mathfrak{sp}(W \oplus W^*)$-modules. This subcategory is stable under the projective functors.

**Proposition 8.15.** Any $\mathfrak{sp}(W \oplus W^*)$-module $M$ of small growth is of finite length.

**Proposition 8.16.** Let $I$ be an annihilator of a simple infinite-dimensional $\mathfrak{sl}(W)$-module $M$ of small growth. Then $I$ is a Joseph ideal.

Let $\mu$ be a Shale-Weil tuple. The set PFuc($\chi_{\mu}$) is generated by one involutive element $K_{\alpha}^{\mu}$ which preserve the set of simple modules. We recall that $\overline{\mu}_0$ is an $n_W$-tuple $(n_W - \frac{1}{2}, n_W - \frac{3}{2}, \ldots, \frac{1}{2})$.

**Proposition 8.17.** The categories of $\mathfrak{sp}(W \oplus W^*)$-modules of small growth annihilated by $I(\mu_0)$ and $I(\mu)$ are equivalent.
9. Categories of \((\mathfrak{sl}(S^2V), \mathfrak{sl}(V))\) and \((\mathfrak{sl}(\Lambda^2V), \mathfrak{sl}(V))\)-modules

9.1. Construction of \((D(W), \mathfrak{t})\)-modules. We recall that \(K\) is a connected reductive Lie group with Lie algebra \(\mathfrak{k}\) and a Borel subgroup \(B\). Let \(\mathfrak{g}\) be an associative algebra and \(\psi: U(\mathfrak{g}) \rightarrow \mathfrak{g}\) be a homomorphism of associative algebras injective on \(\mathfrak{g}\). We identify \(\mathfrak{g}\) with its image. Any \(\mathfrak{g}\)-module can be considered as a \(\mathfrak{k}\)-module.

**Definition 9.1.** (cf. Def. 1.1) A \((\mathfrak{g}, \mathfrak{k})\)-module is a \(\mathfrak{g}\)-module which is locally finite as a \(\mathfrak{k}\)-module.

**Definition 9.2.** (cf. Def. 1.5) A bounded \((\mathfrak{g}, \mathfrak{k})\)-module is a \((\mathfrak{g}, \mathfrak{k})\)-module bounded as a \(\mathfrak{k}\)-module.

Let \(W\) be a spherical \(K\)-module. There is an obvious homomorphism \(U(\mathfrak{g}) \rightarrow D(W)\) (see the discussion at the end of Subsection 3.9.1).

The algebra of differential operators \(D(W)\) has a natural filtration by degree

\[ 0 \subset \mathbb{C}[W] \subset D_1 \subset \ldots \subset D(W) = \bigcup_{i \in \mathbb{Z}_{\geq 0}} D_i. \]

The associated graded algebra

\[ \text{gr} D(W) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (D_{i+1}/D_i) \]

is isomorphic to \(\mathbb{C}[T^*W]\). Let \(M\) be \(D(W)\)-module with a finite-dimensional space of generators \(M_{\text{gen}}\). This defines a filtration

\[ \{D_i M_{\text{gen}}\}_{i \in \mathbb{Z}_{\geq 0}} \]

of \(M\). The associated graded space

\[ \text{gr} M := \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (D_{i+1}M_{\text{gen}}/D_iM_{\text{gen}}) \]

is a finitely generated \(\mathbb{C}[T^*W]\)-module. We denote the support of this \(\mathbb{C}[T^*W]\)-module as \(\mathcal{V}(\text{Loc} M)\). As finitely generated \(D(W)\)-modules are in a natural one-to-one correspondence with coherent \(D(W)\)-modules, \(M\) corresponds to the \(D(W)\)-module \(\text{Loc} M\). The variety \(\mathcal{V}(\text{Loc} M)\) coincides with the singular support of \(\text{Loc} M\) (see Section 11) and is conical and coisotropic in \(T^*W\). In what follows we assume that \(M\) is a finitely generated \((D(W), \mathfrak{t})\)-module.

**Proposition 9.3.** a) The module \(M\) is bounded if and only if all irreducible components of \(\mathfrak{t}\)-module \(Y\) are \(K\)-spherical.

b) If the equivalent conditions of a) are satisfied, any irreducible component \(\tilde{V}\) of \(\mathcal{V}(\text{Loc} M)\) is a conical Lagrangian subvariety of \(T^*W\).

**Proof.** The proof repeats the proof of Proposition 3.23 verbatim. \(\square\)

As explained in Section 2, to a holonomic \(D(W)\)-module \(M\) one assigns a pair \((S, Y)\), where \(S \subset W\) is an irreducible subvariety and \(Y\)
is an $\mathcal{O}(S)$-coherent $\mathcal{D}(S)$-module. Let $K_{ss}$ be the connected simply-connected semisimple group with Lie algebra $[\mathfrak{k}, \mathfrak{k}]$, $A(K)$ be the center of $K$ and $\mathfrak{a} \subset \mathfrak{k}$ be the Lie algebra of $A(K)$.

**Lemma 9.4.** The module $M$ is $\mathfrak{k}$-bounded. If $M$ is simple then $M$ is $\mathfrak{k}$-multiplicity free. The $\mathcal{D}(W)$-module $\text{Loc}_M$ is holonomic and has regular singularities with respect to the stratification by $K$-orbits on $W$.

**Proof.** Let $\phi_{\mathfrak{k}}^{-1}(0) \subset T^*W$ be the union of conormal bundles to all $K$-orbits in $W$. As $\mathfrak{t}$ acts locally finitely on $M$, $\mathcal{V}(M) \subset \phi_{\mathfrak{k}}^{-1}(0)$. Since the dimension of any irreducible component of $\phi_{\mathfrak{t}}^{-1}(0)$ is $n_W$ and $\dim \mathcal{V}(M) \geq n_W$ (Theorem 3.5), $\mathcal{V}(M)$ is the union of irreducible components of $\phi_{\mathfrak{t}}^{-1}(0)$. Therefore $\text{Loc}_M$ is holonomic.

Let $\tilde{V}$ be an irreducible component of $\mathcal{V}(M)$. It is the total space of the conormal bundle to a $K$-orbit $S \subset W$. As $W$ is a $K$-spherical variety, $\tilde{V}$ is a spherical variety [Pan]. Therefore $M$ is a $\mathfrak{k}$-bounded module by Proposition 9.3. Assume $M$ is simple. Since the algebra $D(W)^{\mathfrak{k}}$ is commutative [Kn1], $M$ is multiplicity free by [PS2 Corollary 5.8].

We have to prove that $\text{Loc}_M$ has regular singularities. Without loss of generality we assume that $M$ is simple. Let $(S, Y)$ be the pair corresponding to $\text{Loc}_M$. Then $S$ is $K$-stable. As $W$ is $K$-spherical, $S$ is a $K$-orbit. Since $\text{Loc}_M$ is $K_{ss}$-equivariant, $Y$ is also $K_{ss}$-equivariant. As $\mathfrak{a}$ acts locally finitely on $M$, $\mathfrak{a}$ acts locally finitely on $\Gamma(A(K))s, M|_{A(K)\lambda_s}$ for any point $s \in S$. Since $K$ is a quotient of $K_{ss} \times A(K)$, $Y$ has regular singularities. Therefore $\text{Loc}_M$ has regular singularities (cf. [Bo 12.11]).

**Lemma 9.5.** If a holonomic $D(W)$-module $\text{Loc}_M$ has regular singularities with respect to the stratification of $W$ by $K$-orbits, then $M$ is $\mathfrak{k}$-locally finite.

**Proof.** Without loss of generality assume that $M$ is simple. Let $(S, Y)$ be the pair corresponding to $\text{Loc}_M$. Then $S$ is a $K$-orbit. As $S$ is a homogeneous $K_{ss} \times A(K)$-space, the sheaf $Y$ is $K_{ss}$-equivariant [Bo 12.11]. Hence $\text{Loc}_M$ is $K_{ss}$-equivariant and therefore $[\mathfrak{k}, \mathfrak{k}]$ acts locally finitely on $M$. As $Y|_{\Lambda(K)s}$ has regular singularities for any point $s \in S$, $\mathfrak{a}$ acts locally finitely on $\Gamma(A(K)s, M|_{A(K)s})$ for any point $s \in S$. Therefore $\mathfrak{a}$ acts locally finitely on $M$. As $\mathfrak{t} = [\mathfrak{k}, \mathfrak{k}] \oplus \mathfrak{a}$, $\mathfrak{t}$ acts locally finitely on $M$. □

By a famous theorem of M. Kashiwara [Ka], the category of holonomic $D(W)$-modules with regular singularities with respect to a given stratification is equivalent to the category of perverse sheaves on $W$ with respect to the same stratification.
Corollary 9.6. The category of bounded \((D(W), \mathfrak{t})\)-modules is equivalent to the category of perverse sheaves on \(W\) with respect to the stratification of \(W\) by \(K\)-orbits.

The categories of such perverse sheaves has been described explicitly for the stratifications of \(S^2V\) and \(\Lambda^2V\) by the \(GL(V)\)-orbits \([BG]\). In particular, in \([BG]\) the authors have described the simple objects of this categories.

The following theorem provides a checkable condition under which the functions \([M_1, M_2]: \cdot \mapsto \mathfrak{t}\) are pairwise disjoint (see Section 2) for non-isomorphic simple \((D(W), \mathfrak{t})\) modules \(M_1\) and \(M_2\). The proof, which we present below, follows essentially the proof of \([PS2, Corollary 5.8]\).

Theorem 9.1. Assume \(D(W)^\mathfrak{t}\) is the image of \(Z(\mathfrak{t})\). Let \(E\) be a simple \(\mathfrak{t}\)-module. Then there exists at most one simple \((D(W), \mathfrak{t})\)-module \(M\) such that \(\text{Hom}_{\mathfrak{t}}(E, M) \neq 0\).

To prove Theorem 9.1 we need the following preparatory lemma.

Lemma 9.7. Let \(E\) be a simple finite-dimensional \(\mathfrak{t}\)-module. Then for any \(n \in \mathbb{Z}_{\geq 1}\) and \(\phi \in \text{Hom}(E, E \otimes \mathbb{C}^n)\) there exist \(\phi_1, \ldots, \phi_s \in \text{Hom}_{\mathfrak{t}}(E, E \otimes \mathbb{C}^n)\) such that \(\phi_i \in U(\mathfrak{t}) \phi U(\mathfrak{t})\) and \(\phi \in +_i U(\mathfrak{t}) \phi_i U(\mathfrak{t})\).

Proof. We omit the proof. \(\square\)

Proof of Theorem 9.1. Let \(R\) be the isotypic component of \(E\) in \(D(W) \otimes U(\mathfrak{t})\) and \(pr\) denote the projection \(D(W) \otimes U(\mathfrak{t}) \rightarrow R\). For any \(g \in D(W)\) there exists \(g' \in D(W)\) such that \(pr(gm) = g'm\) for any \(m \in E\). Therefore \(g'\) induces a homomorphism \(\phi\) from \(E\) to \(R\). Let \(\phi_i\) be as in Lemma 9.7. Then for any \(i\) there exist \(g_i \in D(W)\) such that \(g_im = \phi_im\) for any \(m \in E\). We have \([g_i, k]m = 0\) for any \(k \in \mathfrak{t}\) and \(m \in E\). As the adjoint action of \(\mathfrak{t}\) on \(D(W)\) is semisimple, \(g_i = c_i + a_i\) for some \(c_i, a_i \in D(W)\) such that \(c_i \in D(W)^\mathfrak{t}\) and \(a_im = 0\) for all \(m \in E\). Hence \(g_iE \subset U(\mathfrak{t})E\) and therefore \(R = E\).

Let \(M\) be the unique maximal submodule of \(D(W) \otimes U(\mathfrak{t})\) which does not contain \(R = E\). Then the quotient \(D(W) \otimes U(\mathfrak{t})E/\hat{M}\) is simple and, up to isomorphism, is the unique simple \((D(W), \mathfrak{t})\)-module for which \(\dim \text{Hom}_{\mathfrak{t}}(E, \cdot) \neq 0\). \(\square\)

Let \(\mathfrak{t} = \mathfrak{gl}(V)\) and let \(W\) equal \(\Lambda^2V\) or \(S^2V\). Then \(D(W)^\mathfrak{t}\) is the image of \(Z(\mathfrak{t})\) \([HU]\). Therefore, for non-isomorphic simple \((D(W), \mathfrak{t})\)-modules \(M_{1, 2}\), the functions \([M_{1, 2}: \cdot]_\mathfrak{t}\) are pairwise disjoint, i.e. the product of these functions is the zero function (see Section 2).
9.2. A useful algebraic trick. We recall that $E$ is the Euler operator and that $\{D^i(W)\}_{i \in \mathbb{Z}}$ are the $E$-eigenspaces of $D(W)$ (see the discussion at the end of Subsection 3.9.1). In this section we relate the category of $(D(W), E)$-modules to the categories of $D^0(W)$ and $D^0(W)$-modules (see also [PS2]). Let $M$ be a locally finite $E$-module. Fix $t \in \mathbb{C}$. Set

$$M_t := \{m \in M \mid (E - t)^n m = 0 \text{ for some } n \in \mathbb{Z}_{\geq 0}\}.$$ 

If $M$ is a $(D(W), E)$-module which is semisimple an an $E$-module, then $M_t$ is a $D^t\mathbb{P}(W)$-module.

**Definition 9.8.** We say that a $(D(W), E)$-module $M$ is a $(D(W), E)$-module with monodromy $e^{2\pi it}$ if $M = \oplus_{j \in \mathbb{Z}} M_{t+j}$.

Any simple $(D(W), E)$-module with a locally finite action of $E$ is a module with a monodromy. Let $\delta_0$ be the delta function at $0 \in W$. The functions $1$ and $\delta_0$ generate $(D(W), E)$-modules which we denote $D(W)1$ and $D(W)\delta_0$.

**Lemma 9.9.** Let $M$ be a simple $(D(W), E)$-module with monodromy $e^{2\pi it}$. Assume that

$$\dim M_t < \infty \text{ and } n_W \geq 2.$$ 

Then $M$ is isomorphic to one of the following modules:

$$0, D(W)1, D(W)\delta_0.$$

**Proof.** Let $\dim M_t \neq 0$. Then the action of $\mathfrak{sl}(W)$ on $M_t$ is locally finite and therefore $M$ is a locally finite $\mathfrak{sl}(W)$-module. As $W$ is a spherical $\text{SL}(W)$-module, $M$ is a holonomic $(D(W), E)$-module with regular singularities by Lemma 9.3. Let $\text{Sh}$ be the corresponding to $M$ simple perverse sheaf. Then $\text{Sh}$ is constructible with respect to the stratification $\{\{0\}, W - \{0\}\}$ of $W$. As

$$\pi_1(\{0\}) = \pi_1(W - \{0\}) = 0,$$

$\text{Sh}$ is the simple perverse sheaf corresponding to $\{0\}$ or $W - \{0\}$, in both cases the local system being trivial. The latter two perverse sheaves correspond to the simple $(D(W), E)$-modules $D(W)\delta_0$ and $D(W)1$, respectively. Therefore $M$ is isomorphic to $0$, to $D(W)1$, or to $D(W)\delta_0$.

Assume that $M_t = 0$. Then, for all $i \leq n_W$, the action of $e_i$ on $M$ is locally finite, or the action of $\partial_{e_i}$ is locally finite for all $i$. In the first case $M$ is isomorphic to $D(W)\delta_0$ by Kashiwara’s theorem. In the second case $D(W)$ is isomorphic to $D(W)1$ by Kashiwara’s theorem (we can interchange the roles of $e_i$ and $\partial_{e_i}$).

**Lemma 9.10.** a) Let $M$ be a nonzero simple $(D(W), E)$-module with monodromy $e^{2\pi it}$ which is not isomorphic to $D(W)1$ and $D(W)\delta_0$. Then $M_t$ is a simple infinite-dimensional $D^t\mathbb{P}(W)$-module.
b) Let $M_{res}$ be a simple infinite-dimensional $D^t\mathbb{P}(W)$-module. Then the $D(W)$-module

$$D(W) \otimes_{D^0(W)} M_{res}$$

has a unique simple quotient $\tilde{M}$. This quotient is not isomorphic to $D(W)1$ or to $D(W)\delta_0$ and has monodromy $e^{2\pi it}$. Moreover, there is an isomorphism $\tilde{M}_t = M_{res}$ of $D^t\mathbb{P}(W)$-modules.

**Proof.** a) By Lemma 9.9 we have $\dim M_t = \infty$. Let $\tilde{M}_t \subset M_t$ be a $D^t\mathbb{P}(W)$-submodule of $M_t$. The inclusion $\tilde{M}_t \to M_t$ induces a homomorphism

$$\text{inc}: D(W) \otimes_{D^0(W)} \tilde{M}_t \to M_t$$

of $\mathbb{Z}$-graded $D(W)$-modules. We have

$$(D(W) \otimes_{D^0(W)} \tilde{M}_t)_t = \tilde{M}_t,$$

and the map

$$\text{inc}_t: (D(W) \otimes_{D^0(W)} \tilde{M}_t)_t \to M_t$$

is the inclusion map. As $M$ is simple, $\text{inc}$ equals 0 or is surjective and therefore $\tilde{M}_t$ is isomorphic to 0 or to $M_t$. Therefore a) follows.

Any submodule $\tilde{M}$ of $D(W) \otimes_{D^0(W)} M_{res}$ either intersects $M_{res}$ trivially, or coincides with $D(W) \otimes_{D^0(W)} M_{res}$. Therefore $D(W) \otimes_{D^0(W)} M_{res}$ has a unique maximal submodule $\tilde{M}$ and this submodule is isomorphic to a direct sum of copies of $D(W)1$ and $D(W)\delta_0$. The quotient

$$\tilde{M} := (D(W) \otimes_{D^0(W)} M_{res})/\tilde{M}$$

is simple and $\tilde{M}_t = M_{res}$. As $M_{res}$ is infinite-dimensional, $\tilde{M}$ is not isomorphic to $D(W)1$ or to $D(W)\delta_0$. \(\square\)

This reduces the study of (simple) $D^t\mathbb{P}(W)$-modules to a study of some (simple) modules over the Weyl algebra. Sometimes the latter is much easier.

We recall that $D(W) = D^0(W) \oplus D^1(W)$ is a $\mathbb{Z}_2$-graded algebra. We say that a $D^0(W)$-module has half-monodromy $e^{\pi it}$ if $M$ is a direct sum $\oplus_{j \in \mathbb{Z}} M_{t+2j}$. We denote by

$$(D(W), E)\text{-mod}$$

the category of $D(W)$-modules with a locally finite action of $E$ and by

$$(D(W), E)\text{-mod}^{e^{2\pi it}}$$

the subcategory of $(D(W), E)\text{-mod}$ of modules with monodromy $e^{2\pi it}$ for any $t \in \mathbb{C}$. Similarly we define

$$(D^0(W), E)\text{-mod}$$

and

$$(D^0(W), E)\text{-mod}^{e^{\pi it}}.$$
The sign \( \cong \) stands for equivalence of categories. For any \( t \in \mathbb{C} \) we have two functors:

\[
\begin{align*}
\text{Res}_{e^{2\pi it}} & : \text{D}(W)\text{-mod}^{e^{2\pi it}} \to \text{D}^{0}(W)\text{-mod}^{e^{\pi it}}, \\
M & \mapsto \oplus_{j \in \mathbb{Z}} M_{t+2j}; \\
\text{Ind}_{e^{\pi it}} & : \text{D}^{0}(W)\text{-mod}^{e^{\pi it}} \to \text{D}(W)\text{-mod}^{e^{2\pi it}}, \\
M & \mapsto \text{D}(W) \otimes_{\text{D}^{0}(W)} M.
\end{align*}
\]

**Theorem 9.2.** The functors \( \text{Ind} \) and \( \text{Res} \) are mutually inverse equivalences of the categories of \( (\text{D}(W), E)\text{-mod}^{e^{2\pi it}} \) and \( (\text{D}^{0}(W), E)\text{-mod}^{e^{\pi it}} \).

**Proof.** For a \( \text{D}(W) \)-module \( M \) with monodromy \( e^{2\pi it} \), set

\[
M_{\bar{t}} := \oplus_{j \in \mathbb{Z}} M_{t+2j+1}.
\]

Then \( M = M_{\bar{t}} \oplus M_{\bar{t}-1} \), and this is a \( \mathbb{Z}_{2} \)-grading on \( M \) considered as a \( \text{D}(W) \)-module.

Let \( M_{\bar{t}} \) be a \( \text{D}^{0}(W) \)-module with half-monodromy \( e^{\pi it} \). The \( \text{D}(W) \)-module

\[
\text{D}(W) \otimes_{\text{D}^{0}(W)} M_{\bar{t}}
\]

is \( \mathbb{Z}_{2} \)-graded and has monodromy \( e^{2\pi it} \). Moreover,

\[
(D(W) \otimes_{\text{D}^{0}(W)} M_{\bar{t}})_{\bar{t}} = \text{D}^{0}(W) \otimes_{\text{D}^{0}(W)} M_{\bar{t}} = M_{\bar{t}}.
\]

Therefore \( \text{Res}_{e^{2\pi it}} \circ \text{Ind}_{e^{\pi it}} \) is the identity functor.

Let \( M \) be a \( \text{D}(W) \)-module with monodromy \( e^{2\pi it} \). If \( M_{\bar{t}} = 0 \), then \( M \) has finite length and, up to isomorphism, the only simple constituents of it are \( \text{D}(W)1 \) and \( \text{D}(W)\delta_{0} \). In both cases \( M_{\bar{t}} \neq 0 \). There is a natural homomorphism

\[
\psi : \text{D}(W) \otimes_{\text{D}^{0}(W)} M_{\bar{t}} \to M.
\]

We have

\[
(D(W) \otimes_{\text{D}^{0}(W)} M_{\bar{t}})_{\bar{t}} = (\text{D}(W)^{0} \otimes_{\text{D}^{0}(W)} M_{\bar{t}} = M_{\bar{t}}.
\]

Moreover,

\[
\psi_{|t} : (D(W) \otimes_{\text{D}^{0}(W)} M_{\bar{t}})_{\bar{t}} \to M_{\bar{t}}
\]

is an isomorphism. Therefore \( (M/\text{Im}\psi)_{\bar{t}} = 0 \) and hence \( M/\text{Im}\psi = 0 \). This shows that \( \text{Ind}_{e^{\pi it}} \circ \text{Res}_{e^{2\pi it}} \) is the identity functor. \( \Box \)

**Corollary 9.11.** We have

\[
(\text{D}^{0}(W), E)\text{-mod} \cong (\text{D}(W), E)\text{-mod} \oplus (\text{D}(W), E)\text{-mod}.
\]

**Proof.** We have
\[ (D^0(W), E)\text{-mod} \cong \bigoplus_{t \in \mathbb{C}/\mathbb{Z}} (D^0(W), E)\text{-mod}_{e^{2\pi it}}; \]
\[ (D(W), E)\text{-mod} \cong \bigoplus_{t \in \mathbb{C}/\mathbb{Z}} (D(W), E)\text{-mod}_{e^{2\pi it}}. \]

On the other hand \( (D^0(W), E)\text{-mod} \cong (D(W), E)\text{-mod}_{e^{2\pi it}} \). The two half-monodromies \( e^{\pi it}, -e^{\pi it} = e^{\pi i(t+1)} \) combine into monodromy \( e^{2\pi it} \), and
\[ (D^0(W), E)\text{-mod} \cong (D(W), E)\text{-mod} \oplus (D(W), E)\text{-mod}. \]

9.3. A description of \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-modules. In this section
\[ K := \text{SL}(V), \mathfrak{g} := \mathfrak{sl}(W) \]
where \( W := \Lambda^2 V(n_V = 2k, n_V \geq 5) \) or \( W = S^2 V(n_V \geq 3) \).

Lemma 9.12. Let \( M \) be a simple bounded \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-module. Then \( \text{Ann} M \) is a Joseph ideal.

Proof. By Theorem 2.5 the variety \( GV(M) \) is \( \text{SL}(V) \)-coisotropic. According to the discussion following Example 3.8, \( GV(M) \) is birationally isomorphic to \( T^* \text{Fl} \) for some partial \( W \)-flag variety \( \text{Fl} \). Hence \( \text{Fl} \) is an \( \text{SL}(V) \)-spherical variety. As the only partial \( W \)-flag varieties which are \( \text{SL}(V) \)-spherical are \( \mathbb{P}(W) \) and \( \mathbb{P}(W^*) \), \( GV(M) \) coincides with \( \Omega \) (cf. Example 3.8).

We are now ready to prove Theorems 2.6–2.8.

Proof of Theorem 2.6. As \( \text{Ann} M \) is a Joseph ideal, \( V(M) \subset \Omega \). As \( M \) has finite type over \( K \), we have \( \dim V(M) = \frac{1}{2} \dim \Omega = n_W - 1 \) and \( M \) is of small growth.

Proof of Theorem 2.7. By Lemma 9.12 the ideal \( \text{Ann} M \) is a Joseph ideal. Therefore \( \text{Ann} M = I(\lambda) \) for some semi-decreasing \( n_W \)-tuple \( \lambda \) by Corollary 7.5.

Proof of Theorem 2.8. As \( \text{Ann} M = I(\lambda) \) for some semi-decreasing tuple \( \lambda \), then \( V(\text{Ann} M) = GV(M) = \Omega \) (Corollary 7.5). The variety \( \mathbb{P}(W) \) is \( \text{SL}(V) \)-spherical, and hence the variety \( T^* \mathbb{P}(W) \) is \( \text{SL}(V) \)-coisotropic. As \( \Omega \) is birationally isomorphic to \( T^* \mathbb{P}(W) \) (Example 3.8), \( \Omega \) is \( \text{SL}(V) \)-coisotropic. Therefore \( M \) is a bounded \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-module by Theorem 2.5.

Let \( B_t(W) \) denote the cardinality of the set of isomorphism classes of simple infinite-dimensional bounded \((\mathfrak{sl}(W), \mathfrak{sl}(V))\)-modules annihilated by \( I(t, n_W - 1, ..., 1) \). Let \( P_{2\pi it}(W) \) be the cardinality of the set of isomorphism classes of simple perverse sheaves on \( W \) (with respect
to the stratification by GL(V)-orbits) such that they have fixed monodromy $e^{2\pi it}$ and are neither supported at 0 nor are smooth on $W$. The monodromies of the simple perverse sheaves on $W$ with respect to the stratification by GL(V)-orbits are described in quiver terms in the Appendix.

**Lemma 9.13.** We have $B_t(W) = \mathcal{P}_{e^{2\pi it}}(W) < \infty$ for all $t \in \mathbb{C}$.

**Proof.** Let $\lambda_t \in \mathfrak{h}_W^{*}$ be the weight corresponding to $(t, n_W - 1, ..., 1)$. The quotient $U(\mathfrak{sl}(W))/\text{Ann } L_{\lambda_t}$ is isomorphic to $D(\mathbb{P}(W))$ (see the discussion at the end of Subsection 3.9.1). Therefore the isomorphism classes of simple infinite-dimensional $(\mathfrak{sl}(W), \mathfrak{sl}(V))$-modules annihilated by $I(\lambda_t)$ are in one-to-one correspondence with the simple $(D(W), \mathfrak{gl}(V))$-modules with monodromy $e^{2\pi it}$, which are not isomorphic to $D(W)1$ or $D(W)\delta_0$ (see Lemma 9.10).

For a semi-decreasing $n_W$-tuple $\lambda$ we denote by $B_{\lambda}(W)$ the cardinality of the set of isomorphism classes of simple infinite-dimensional bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$-modules annihilated by $I(\lambda)$. Then for some $t \in \mathbb{C}$ such that $e^{2\pi it} = m(\lambda)$, we have

$$B_{\lambda}(W) = B_t(W) = \mathcal{P}_{m(\lambda)}(W) < \infty,$$

by Lemma 3.31 and Corollary 8.12. This proves Theorem 2.9.

We are now ready to prove Theorem 2.10

**Proof of Theorem 2.10.** a) Let $M$ be a bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-module. We can consider $M$ as an $(\mathfrak{gl}(W), \mathfrak{gl}(V))$-module. Let $E$ be a generator of the center of $\mathfrak{gl}(W)$. Then $M$ is a direct sum of $E$-eigenspaces $\oplus_{\zeta \in \mathbb{C}} M_{\zeta}$. As $M$ is a bounded $(\mathfrak{gl}(W), \mathfrak{gl}(V))$-module, $M_{\zeta}$ is a bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$-module.

b) Any simple bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$-module annihilated by $I(s_1 \rho)$ is an $(\mathfrak{sl}(W), \mathfrak{sl}(V))$-submodule of an $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-module. Any other simple bounded module is isomorphic to a subquotient of a tensor product $F \otimes M$ where $F$ is a finite-dimensional SL(W)-module and $M$ is a simple $(\mathfrak{sl}(W), \mathfrak{sl}(V))$-module annihilated by $I(\rho + \varepsilon_k)$. Let $F_{\mathfrak{sp}}$ be an SP$(W \oplus W^*)$-module which contains $F$ as an SL(W)-submodule (such a module always exists, see [Gr]). Then $F_{\mathfrak{sp}} \otimes M$ is a bounded $\mathfrak{gl}(V)$-module and contains $M$ as an $\mathfrak{gl}(W)$-submodule.

**9.4. Categories of $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-modules.** In this section

$$K := \text{GL}(V), \ g := \mathfrak{sp}(W \oplus W^*)$$

where $W := \Lambda^2 V(n_V = 2k, n_V \geq 5)$ or $W = S^2 V(n_V \geq 3)$.

**Lemma 9.14.** Let $M$ be a simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-module. Then $\text{Ann } M$ is a Joseph ideal.
Proof. Let $\tilde{Q}$ be the image of the moment map $\phi : T^* P \to \mathfrak{sp}(W \oplus W^*)$. Assume that $GV(M)$ does not equal to $0$ or to $\bar{Q}_{sp}$. Then $[\text{CM} 6.2]$ $\tilde{Q} \subset GV(M)$ and $\dim GV(M) \geq 2(2n_W - 1)$. Therefore $\dim \mathfrak{b}_{gl(V)} \geq 2n_W - 1$ by Corollary 3.24. This inequality is false. □

We are now ready to prove Theorems 2.11—2.14 and Theorem 2.16.

Proof of Theorem 2.11. As $\text{Ann} M$ is a Joseph ideal, we have $V(M) \subset \bar{Q}_{sp} \cap \mathfrak{k}_{sp}$. As $\dim \bar{Q}_{sp} \cap \mathfrak{k}_{sp} = n_W$, $M$ is of small growth. □

Proof of Theorem 2.12. As $M$ is a bounded module, $\text{Ann} M$ is a Joseph ideal. Therefore $\text{Ann} M = I(\bar{\mu})$ by Corollary 7.13. □

Proof of Theorem 2.13. As $M$ is an $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-module, we have $V(M) \subset \mathfrak{gl}(V)^{\perp} \cap Q_{sp}$. Moreover, $Q_{sp}$ is a quotient of $W \oplus W^*$ by $\mathbb{Z}_2$ (see Subsection 7.2), hence $\mathfrak{gl}(V)^{\perp} \cap Q_{sp}$ is a quotient by $\mathbb{Z}_2$ of the union of conormal bundles to $GL(V)$-orbits. Since $W$ is $GL(V)$-spherical, these conormal bundles are $GL(V)$-spherical. Therefore any irreducible component of $V(M)$ is $GL(V)$-spherical, and $M$ is $\mathfrak{gl}(V)$-bounded by Proposition 3.23. □

We recall that $\bar{\mu}_0$ is an $n_W$-tuple $(n_W - \frac{1}{2}, n_W - \frac{3}{2}, \ldots, \frac{1}{2})$ and $\mu_0 \subset \mathfrak{h}^*_W$ is the corresponding weight.

Proof of Theorem 2.14. Let $\bar{\mu}_1, \bar{\mu}_2$ be Shale-Weil $n_W$-tuples. The categories of bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-modules annihilated by $I(\mu_1)$ and $I(\mu_2)$ are equivalent by Theorem 3.12. □

Proof of Theorem 2.16. The category of $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-modules annihilated by $I(\mu)$ is equivalent, via the functors $H_{\mu_0}^\mu$ and $H_{\sigma_0}^\mu$, to the category of $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-modules annihilated by $I(\mu_0)$. The second category is equivalent to the direct sum of two copies of the category of perverse sheaves on $W$ with respect to the stratification by $GL(V)$-orbits (see Subsection 9.2). □

Let $\langle \mathfrak{z}_{\mu_0} \rangle$ be the free vector space generated by the isomorphism classes of simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$-modules annihilated by $\text{Ker} \chi_{\mu_0}$. We recall that $\text{PFrem}(\chi_{\mu_0})$ acts on $\langle \mathfrak{z}_{\mu_0} \rangle$ by linear operators (Section 3.8). Let $\bar{\mu}'$ be an $n_W$-tuple such that $H_{\mu_0}'(\mathfrak{z}_{\mu_0}) \neq 0$. Then $\bar{\mu}'$ is a Shale-Weil tuple by Corollary 7.16. Therefore the action of $\text{PFrem}(\chi_{\mu_0})$ collapses to an action of $H_{\mu_0}'$ and $H_{\sigma_0}^{\mu_0}$. The functor $H_{\mu_0}'$ is involutive and we call it $\text{Inv}$ (see also Section 2).
Let $M$ be a simple $U(\mathfrak{sp}(W \oplus W^*))$-module of small growth annihilated by $I(\mu_0)$. Let $E$ be a finite-dimensional simple $\mathfrak{sp}(W \oplus W^*)$-module. We recall that $F_E := E \otimes \cdot$. Then $F_E \mathcal{H}_{\mu_0}^\mu$ is a direct sum $\oplus_i \mathcal{H}_{\mu_0}^{\mu_i}$ of indecomposable projective functors. Therefore

$$E \otimes M = \oplus_i \mathcal{H}_{\mu_0}^{\mu_i} M.$$ 

If $\mu_i$ is not a Shale-Weil tuple for some $i$, then $\mathcal{H}_{\mu_0}^{\mu_i} M = 0$ (Corollary 7.16). If $\mu_i$ is a negative Shale Weil tuple for some $i$, then $\mathcal{H}_{\mu_0}^{\mu_i} = \mathcal{H}_{\mu_0}^{\sigma_i} \text{Inv}$. Therefore

$$F_E M = \oplus_{i \leq s} \mathcal{H}_{\mu_0}^{\mu_i} M + (\oplus_{i \leq s} \mathcal{H}_{\mu_0}^{\mu_i} \text{Inv} M)$$

for some positive Shale-Weil tuples $\mu_1', ..., \mu_s'$ and $\mu_1'', ..., \mu_s''$.

Let $\mathcal{J}$ be the free vector space generated by the simple $\mathfrak{sp}(W \oplus W^*)$-modules of small growth and $\langle \mathcal{J} \rangle$ be the free vector space generated by $\mathcal{J}$ (cf. Section 8). One can express the projective functors $\mathcal{H}_{\mu_0}^{\mu_i}$ in terms of the functors $F_E$. More precisely, for any Shale-Weil tuple $\bar{\mu}$ we have

$$\mathcal{H}_{\mu_0}^{\mu_i} = (\oplus_{i \leq s} c_i F_{E_i} \mathcal{H}_{\mu_0}^{\mu_i}) + (\oplus_{i \leq s'} c_i' F_{E_i'} \text{Inv})$$

for some finite-dimensional $\mathfrak{sp}(W \oplus W^*)$-modules $E_1, ..., E_s, E_1', ..., E_{s'}$ and some integers $c_1, ..., c_s, c_1', ..., c_{s'}$; this is an equality for linear operators on $\langle \mathcal{J} \rangle$. Therefore

$$\mathcal{H}_{\mu_0}^{\mu_i} ([M] + \text{Inv} M) = ([M] + \text{Inv} M) \otimes \mathbb{C} (\oplus_i c_i [E_i] + \oplus_i c_i' [E_i'])$$

and

$$\mathcal{H}_{\mu_0}^{\mu_i} ([M] - \text{Inv} M) = ([M] - \text{Inv} M) \otimes \mathbb{C} (\oplus_i c_i [E_i] - \oplus_i c_i' [E_i']).$$

This shows that if we are given the functions $[M : \cdot]_t$ and $[\text{Inv} M : \cdot]_t$, the coefficients $c_i, c_i'$, and the functions $[E_i : \cdot]_t, [E_i' : \cdot]_t$, we can find the functions $[\mathcal{H}_{\mu_0}^{\mu_i} M : \cdot]_t$ and $[\mathcal{H}_{\mu_0}^{\mu_i} \text{Inv} M : \cdot]_t$ as formal power series.

Moreover, it is enough to determine the coefficients $c_i, c_i'$ and the modules $E_i, E_i'$ for one simple module $M$ of small growth annihilated by $I(\mu_0)$. This work has been done by O. Mathieu [M] for a simple module $L_{\mu_0}$, and we now explain the result.

Let $\bar{\mu}$ be a positive Shale-Weil tuple and be the coefficients $c_i, c_i'$, the modules $E_i, E_i'$ be as in formula (5). Let $L$ and $L^\sigma$, $L^{\bar{\mu}}$ and $L^{\sigma \bar{\mu}}$ be finite-dimensional $\text{Spin}_{2nW}$-modules as in Section 2. Then

$$([L^{\bar{\mu}} : \cdot]_{bW} + [L^{\sigma \bar{\mu}} : \cdot]_{bW}) =$$

$$\oplus_i c_i [E_i : \cdot]_{bW} + \oplus_i c_i' [E_i' : \cdot]_{bW} \otimes ([L : \cdot]_{bW} + [L^{\sigma} : \cdot]_{bW}),$$

$$([L^{\bar{\mu}} : \cdot]_{bW} - [L^{\sigma \bar{\mu}} : \cdot]_{bW}) =$$

$$\oplus_i c_i [E_i : \cdot]_{bW} - \oplus_i c_i' [E_i' : \cdot]_{bW} \otimes ([L : \cdot]_{bW} - [L^{\sigma} : \cdot]_{bW}).$$

As the functions

$$[L^{\bar{\mu}} : \cdot]_{bW}, [L^{\sigma \bar{\mu}} : \cdot]_{bW}, [L : \cdot]_{bW}, [L^{\sigma} : \cdot]_{bW}$$
are computed by the Weyl character formula, we consider the above mentioned formula as an explicit description of the coefficients $c_i, c'_i$ and the $\mathfrak{sp}(W \oplus W^*)$-modules $E_i, E'_i$.

The mystery of this formula is that the function $[L : \cdot]_{h_W}$ comes from the universe of $\text{Spin}_{2n_W}$-modules and the functions $[E_i : \cdot]_{h_W}$ come from the universe of $\text{SP}_{2n_W}$-modules.
10. APPENDIX

10.1. Results of C. Benson, G. Ratcliff, V. Kac. The classification of spherical modules has been worked out in several steps. V. Kac has classified simple spherical modules in [Kac1], C. Benson and G. Ratcliff have classified all spherical modules in [BR]. The classification is contained also in paper [Le] of A. Leahy. Below we reproduce their list.

Let $W$ be a $K$-module. Then $W$ is a spherical $\mathfrak{k}$-variety if and only if the pair $(\mathfrak{t}, \mathfrak{k}, W)$ is a direct sum of pairs $(\mathfrak{t}_i, \mathfrak{k}_i, W_i)$ listed below and in addition $\mathfrak{k} = \mathfrak{gl}(W) \oplus \mathfrak{c}_i$ for certain abelian Lie algebras $\mathfrak{c}_i$ attached to $(\mathfrak{t}_i, \mathfrak{k}_i, W_i)$. Here $N_{\mathfrak{gl}(W)} \mathfrak{k}$ stands for the normalizer of $\mathfrak{k}$ in $\mathfrak{gl}(W)$ and $\mathfrak{c}_i$ is a 0-, 1- or 2-dimensional Lie algebra listed in square brackets after the pair $(\mathfrak{t}_i, \mathfrak{k}_i, W_i)$. This subalgebra is generated by linear operators $h_1$ and $h_{m,n}$. By definition, $h_1 = \text{Id}$. The notation $h_{m,n}$ is used only when $W = W_1 \oplus W_2$; in this case $h_{m,n}|W_1 = m \cdot \text{Id}$ and $h_{m,n}|W_2 = n \cdot \text{Id}$. The notation $(\mathfrak{t}_i, \{W_i, W_i\})'$ is shorthand for $(\mathfrak{t}_i, \{W_i, W_i\})$. Finally, $\omega_i$ stands for the $i$-th fundamental weight and the corresponding fundamental representation. We follow the enumeration convention for fundamental weights of $\mathfrak{vo}$.

Table [10]: Weakly irreducible spherical pairs.

0) $(0, \mathbb{C})[0]$.

i) Irreducible representations of simple Lie algebras:
1) $(\mathfrak{sl}_n, \{\omega_1, \omega_{n-1}\})[\mathcal{C}h_1](n \geq 2)$;
2) $(\mathfrak{so}_n, \omega_1)[0](n \geq 3)$;
3) $(\mathfrak{sp}_{2n}, \omega_1)[\mathcal{C}h_1](n \geq 2)$;
4) $(\mathfrak{sl}_n, \{2 \omega_1, 2 \omega_{n-2}\})[0](n \geq 3)$;
5) $(\mathfrak{sl}_{2n+1}, \{\omega_2, \omega_{n+2}\})[\mathcal{C}h_1](n \geq 2)$;
6) $(\mathfrak{sl}_{2n}, \{\omega_2, \omega_{n-2}\})[0](n \geq 3)$;
7) $(\mathfrak{so}_7, \omega_3)[0]$;
8) $(\mathfrak{so}_8, \{\omega_3, \omega_4\})[0]$;
9) $(\mathfrak{so}_9, \omega_4)[0]$;
10) $(\mathfrak{so}_{10}, \{\omega_4, \omega_5\})[\mathcal{C}h_1]$;
11) $(\mathfrak{e}_6, \omega_1)[0]$;
12) $(\mathfrak{g}_2, \omega_1)[0]$.

ii) Irreducible representations of nonsimple Lie algebras:
1) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m, \{\omega_1, \omega_{n-1}\} \otimes \{\omega_1, \omega_{m-1}\})[\mathcal{C}h_1](m > n \geq 2)$;
2) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m, \{\omega_1, \omega_{n-1}\} \otimes \{\omega_1, \omega_{m-1}\})[0](n \geq 2)$;
3) $(\mathfrak{sl}_2 \oplus \mathfrak{sp}_{2n}, \omega_1 \otimes \omega_1)[0](n \geq 2)$;
4) $(\mathfrak{sl}_3 \oplus \mathfrak{sp}_{2n}, \{\omega_1, \omega_2\} \otimes \omega_1)[0](n \geq 2)$;
5) $(\mathfrak{sl}_n \oplus \mathfrak{sp}_4, \{\omega_1, \omega_{n-1}\} \otimes \omega_1)[\mathcal{C}h_1](n \geq 5)$;
6) $(\mathfrak{sl}_4 \oplus \mathfrak{sp}_4, \{\omega_1, \omega_3\} \otimes \omega_1)[0]$.
iii) Reducible representations of Lie algebras:
1) \((\mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{sl}_2; (\{\omega_1, \omega_{n-1}\} \oplus \{\omega_1, \omega_{n-1}\}) \otimes \omega_1)[C_{h,0} \oplus C_{h,1}](n, m \geq 3)\); 
2) \((\mathfrak{sl}_n; \{\omega_1 \oplus \omega_1, \omega_{n-1} \oplus \omega_{n-1}\})[C_{h,1}](n \geq 3)\); 
3) \((\mathfrak{sl}_n; \omega_1 \oplus \omega_{n-1})[C_{h,1}](n \geq 3)\); 
4) \((\mathfrak{sl}_2n; \{\omega_1, \omega_{n-1}\} \oplus \{\omega_2, \omega_{n-2}\})[C_{h,0}](n \geq 2)\); 
5) \((\mathfrak{sl}_2n+1; \omega_1 \oplus \omega_2)[C_{h,1}](n \geq 2)\); 
6) \((\mathfrak{sl}_2n+1; \omega_{n-1} \oplus \omega_2)[C_{h,1}](n \geq 2)\); 
7) \((\mathfrak{sl}_n \oplus \mathfrak{sl}_m; \{\omega_1, \omega_{n-1}\} \otimes (\mathbb{C} \oplus \{\omega_1, \omega_{m-1}\})[C_{h,0}](2 \leq n < m)\); 
8) \((\mathfrak{sl}_n \oplus \mathfrak{sl}_m; \omega_1 \otimes \omega_{n-1} \oplus (\mathbb{C} \oplus \{\omega_1, \omega_{m-1}\})[C_{h,1}](m \geq 2, n \geq m+2)\); 
9) \((\mathfrak{sl}_n \oplus \mathfrak{sl}_m; \{\omega_1, \omega_{n-1}\} \oplus \{\omega^{*} (= \omega_{n-1}) \oplus \omega^{*} (= \omega_1)\}[C_{h,1}](2 \leq n < m)\); 
10) \((\mathfrak{sl}_n \oplus \mathfrak{sl}_m; \{\omega_1, \omega_{n-1}\} \oplus \{\omega^{*} (= \omega_{n-1}) \oplus \omega^{*} (= \omega_1)\}[C_{h,1}](m \geq 2, n \geq m+2)\); 
11) \((\mathfrak{sl}_n \oplus \mathfrak{sp}_{2m} \oplus \mathfrak{sl}_2; (\{\omega_1, \omega_{n-1}\} \oplus \omega_1) \otimes \omega_1)[C_{h,0}](n \geq 3, m \geq 1)\); 
12) \((\mathfrak{sl}_2; \omega_1 \otimes \omega_1)[0]\); 
13) \((\mathfrak{sl}_n \oplus \mathfrak{sl}_n; \{\omega_1, \omega_{n-1}\} \oplus \{\omega^{*} (= \omega_{n-1}) \oplus \omega^{*} (= \omega_1)\}[C_{h,1}](n \geq 2)\); 
14) \((\mathfrak{sl}_n+1 \oplus \mathfrak{sl}_n; \{\omega_1, \omega_{n-1}\} \oplus \{\omega^{*} (= \omega_{n-1}) \oplus \omega^{*} (= \omega_1)\}[C_{h,1}](n \geq 2)\); 
15) \((\mathfrak{sl}_2 \oplus \mathfrak{sp}_{2n}; \omega_1 \otimes (\mathbb{C} \oplus \omega_1)[0](n \geq 2)\); 
16) \((\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2; (\omega_1 \otimes \omega_1) \otimes \omega_1)[0](m, n \geq 2)\); 
17) \((\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2; (\omega_1 \otimes \omega_1) \otimes \omega_1)[0]\); 
18) \((\mathfrak{so}_8; \{\omega_1 \oplus \omega_3, \omega_1 \oplus \omega_4, \omega_3 \oplus \omega_4\})[0]\).

10.2. Results of T. Braden and M. Grinberg. Let \(n\) be a positive integer and \(V\) be a \(\mathbb{C}\)-vector space of dimension \(2n\). Then the category of perverse sheaves on \(\Lambda^2 V\) with respect to the stratification by \(GL(V)\)-orbits is equivalent to the category of representations of the following quiver \(A\) with relations:

\[
A_0 \xleftarrow{q_{01}} A_1 \xleftarrow{q_{12}} \cdots \xleftarrow{q_{n-2,n-1}} A_{n-1} \xleftarrow{q_{n-1,n}} A_n,
\]

\(\xi_i\) and \(\nu_i\) are invertible for all \(i, \xi_i = \nu_i\) for \(i \in \{1, \ldots, n-1\},\)

where \(\xi_i := 1 + q_{i-1}p_{i-1}\) for \(i \in \{1, \ldots, n\}\), and \(\nu_i := 1 + p_iq_i\) for \(i \in \{0, \ldots, n-1\}\). Let \(R\) be a representation of the quiver \(A\). If \(R\) is simple, the invertible operators \(\{\nu_0^n, \nu_1^{n-1}\xi_1, \ldots, \xi_n^n\}\) are proportional to the identity map with a fixed constant \(c \in \mathbb{C}^*\). We call \(c\) the monodromy of \(R\).

The set of eigenvalues of \(1 + p_iq_i\) is independent from \(i\), and we call them eigenvalues of \(R\). If \(R\) is simple, this set consists of one element \(\lambda\). For a given eigenvalue \(\lambda \neq 1\), there exists precisely one simple representation of \(A\) with eigenvalue \(\lambda\). The simple representations of \(A\) with eigenvalue \(1\) are enumerated by the vertices of the quiver.
By definition, the *support of* \( R \) is the set of vertices corresponding to non-zero vector spaces. The corresponding to \( R \) perverse sheaf is supported at 0 if and only if \( R \) is supported at \( A_0 \). The corresponding to \( R \) perverse sheaf is smooth along \( W \) if and only if \( R \) is supported at \( A_n \).

Let now \( V \) be a \( C \)-vector space of dimension \( n \). Then the category of perverse sheaves on \( S^2 V \) with respect to the stratification by \( \text{GL}(V) \)-orbits is equivalent to the category of representations of the following quiver \( B \) with relations:

\[
B_0 \xrightarrow{p_{1}} B_1 \xrightarrow{\xi_1} \cdots \xrightarrow{\xi_{n-2}} B_{n-1} \xrightarrow{\xi_{n-1}} B_n,
\]

\( \xi_i \) and \( \nu_i \) are invertible for all \( i \), \( \xi_i^2 = \nu_i^2 \) for \( i \in \{1, \ldots, n-1\} \),

\[
p_j \nu_{j+1} = -\nu_j p_j, \quad q_j \nu_j = -\nu_{j+1} q_j, \quad p_j \xi_{j+1} = -\xi_j p_j, \quad q_j \xi_j = -\xi_{j+1} q_j
\]

whenever both sides of equalities are well-defined. Here \( \xi_i := 1 + q_{i-1} p_{i-1} \) for \( i \in \{1, \ldots, n\} \), \( \nu_i := 1 + p_i q_i \) for \( i \in \{0, \ldots, n-1\} \).

Let \( R \) be a representation of the quiver \( B \). Assume \( R \) is simple. Then the invertible operators \( \{\nu_0^n, \nu_1^{n-1} \xi^1, \ldots, \xi^n\} \) are proportional to the identity map with a fixed constant \( c \in C^* \). We call \( c \) the *monodromy* of \( R \). The set of eigenvalues of \((-1)^i \xi_i \) consists of one element \( \bar{\xi} \); the set of eigenvalues of \((-1)^i \nu_i \) consists of one element \( \bar{\nu} \). We call the pair \((\bar{\xi}, \bar{\nu})\) the *spectrum* of \( R \). We have \( \bar{\xi} = \bar{\nu} = 1 \) or \( \bar{\xi} = -\bar{\nu} \). The simple representations of \( B \) with spectrum \((1,1)\) are enumerated by the inner vertices of the \( B \)-quiver. For \( \lambda \neq \pm 1 \) there exists precisely one simple representation of \( B \) with the spectrum \((\lambda, -\lambda)\). The simple representations of \( B \) with spectra \((1, -1)\) and \((-1, 1)\) are enumerated by the vertices of the quiver.

By definition, the *support of* \( R \) is the set of vertices corresponding to non-zero vector spaces. The corresponding to \( R \) perverse sheaf is supported at 0 if and only if \( R \) is supported at \( B_0 \). The corresponding to \( R \) perverse sheaf is smooth along \( W \) if and only if \( R \) is supported at \( B_n \).
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