Geometrical Interpretation of BRST Symmetry in Topological Yang-Mills-Higgs Theory

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Abstract

We study topological Yang-Mills-Higgs theories in two and three dimensions and topological Yang-Mills theory in four dimensions in a unified framework of superconnections. In this framework, we first show that a classical action of topological Yang-Mills type can provide all three classical actions of these theories via appropriate projections. Then we obtain the BRST and anti-BRST transformation rules encompassing these three topological theories from an extended definition of curvature and a geometrical requirement of Bianchi identity. This is an extension of Perry and Teo’s work in the topological Yang-Mills case. Finally, comparing this result with our previous treatment in which we used the “modified horizontality condition”, we provide a meaning of Bianchi identity from the BRST symmetry viewpoint and thus interpret the BRST symmetry in a geometrical setting.

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I. Introduction

Soon after Witten \cite{1} constructed topological Yang-Mills theory to generate the Donaldson invariants of smooth four-manifolds, Baulieu and Singer \cite{2} showed that Witten’s topological quantum action could be obtained by gauge fixing the classical topological action

\[ I_4 = \int_{M_4} \text{Tr} \ F \wedge F \]  

in the BRST quantization scheme. Then Perry and Teo \cite{3} argued that the asymmetry of BRST transformation rules appeared in the Baulieu and Singer’s work was caused by treating only the BRST symmetry \cite{4}, and not the anti-BRST symmetry \cite{5}. And they obtained symmetric BRST and anti-BRST transformation rules by treating them on an equal footing. They further identified the difference between the ordinary Yang-Mills theory and the topological Yang-Mills theory as follows. In ordinary Yang-Mills theory, one can impose the so-called “horizontality condition” to find BRST symmetry, and this is tantamount to requiring the vanishing Yang-Mills field strength (curvature) along the unphysical directions of ghosts. In the topological case, one can not impose this condition of vanishing field strength along the unphysical directions, and one can only impose the Bianchi identity in the ghosts-included extended space.

Parallel to this development, topological Yang-Mills-Higgs actions in two and three dimensions were also constructed. Following the Baulieu and Singer’s approach, Baulieu and Grossman \cite{6} found the topological action for magnetic monopoles by gauge fixing the following classical action in three dimensions

\[ I_3 = \int_{M_3} \text{Tr} \ F \wedge D\phi. \]  

Two dimensional case was studied by Chapline and Grossman \cite{7} by gauge fixing the following two dimensional classical action

\[ I_2 = \int_{M_2} \text{Tr} \left( F[\Phi^\dagger, \Phi] - D\Phi^\dagger \wedge D\Phi \right), \]
and the quantized theory turned out to be connected to the theory of vortices and knots.

In our previous work [8], we investigated the BRST/anti-BRST symmetry of the above topological Yang-Mills-Higgs theory in two and three dimensions by modifying the horizontality condition such that it could take care of the topological symmetry in addition to the ordinary Yang-Mills gauge symmetry. This work was done in the superconnection framework so that the scalar and vector gauge fields were treated on the same footing as a connection. Thereby we could find the BRST/anti-BRST transformation rules of the scalar and vector gauge fields at once without doing separate calculations.

In this paper, we investigate the BRST/anti-BRST symmetry of these theories through the “Bianchi identity” in the same superconnection framework. This was motivated by a question that is whether the Perry and Teo’s work could be extended to the topological Yang-Mills-Higgs case in which additional ghosts for the scalar field appears. Thus the superconnection framework became a very natural testing ground for this idea. The result is the affirmative. By comparing these two approaches, we can further provide a meaning of the Bianchi identity in the extended space from the BRST symmetry view point. This in turn allows a geometrical interpretation of the BRST/anti-BRST symmetry in the extended space.

In section II, we show that the classical actions, $I_4$, $I_3$, $I_2$, can be obtained from a classical action of topological Yang-Mills type written in superconnection language by appropriate projections depending on the dimensions of spaces to which corresponding theories belong. In section III, we find the BRST/anti-BRST transformation rules of the topological Yang-Mills-Higgs theory from the Bianchi identity in the extended space and an extended definition of curvature in the superconnection formalism. In section IV, we compare our present work with the “horizontality condition” approach which we adopted in our previous work. From
the comparison of these two approaches, we provide a geometrical meaning to the BRST symmetry in the topological Yang-Mills-Higgs theory. Section V constitutes the conclusion.

II. Classical topological action with superconnections

In 1982, Thierry-Mieg and Ne’eman [9] constructed a generalized system of connections with arbitrary form degrees hinted from the old idea of Cartan’s integrable system [10], while they studied a generalized gauge theory possessing an internal supersymmetry. In mathematics, a similar concept was introduced by Quillen in 1985 [11] under a notion of superconnections, independently to Thierry-Mieg and Ne’eman’s work. Then Ne’eman and Sternberg [12] used the Quillen’s superconnection concept to study the Higgs mechanism where the Higgs field occurs as the zero-th order part of the superconnection. This work is much easier for physicists to understand the superconnection concept and also shows that the superconnection is not much different from the generalized connection of Thierry-Mieg and Ne’eman except for the existence of zero-th order connection. In this paper, we thus follow the Ne’eman and Sternberg presentation of superconnections. In general, the superconnection has all orders with odd degree forms as its even part and even degree forms as its odd part. However, in this paper we shall deal with superconnections which contain zero and one forms only since the theories we are dealing with have the scalar and vector gauge fields only. Now, we write down our superconnection as

\[ J = \begin{pmatrix} A & i\Phi \\ i\Phi^\dagger & A \end{pmatrix} \]  

(4)

where \( A, \Phi \) are Lie algebra valued one form and zero form, respectively. The multiplication rule among the elements of total \( Z_2 \)-graded “superspace” is given
\[
\begin{pmatrix}
A & C \\
D & B
\end{pmatrix}
\begin{pmatrix}
A' & C' \\
D' & B'
\end{pmatrix}
= \begin{pmatrix}
A \wedge A' + (-1)^{|D'|} C \wedge D' & A \wedge C' + (-1)^{|B'|} C \wedge B' \\
(-1)^{|A'|} D \wedge A' + B \wedge D' & (-1)^{|C'|} D \wedge C' + B \wedge B'
\end{pmatrix}
\]

(5)

where \(A, \cdots, A', \cdots\) are matrices of differential forms, and \(|A'|, |B'|, |C'|, |D'|\) denote form degrees of \(A', B', C', D', \) respectively.

The “super” curvature is defined from superconnection as
\[
\mathcal{F} = d\mathcal{J} + \mathcal{J} d
\]
where \(d\) denotes a one form differential operator given by \(d = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}\) with \(d\) denoting the ordinary one form exterior derivative times a unit matrix. From here on, we shall use the term curvature instead of “super” curvature for brevity.

Written in the component form, the curvature is given by
\[
\mathcal{F} = \begin{pmatrix}
F - \Phi \Phi^\dagger \\
iD\Phi \\
iD\Phi^\dagger \\
F - \Phi^\dagger \Phi
\end{pmatrix}
\]

(7)

where \(F = dA + A \wedge A\) and \(D\Phi = d\Phi + A\Phi - \Phi A\). Now, we claim our classical topological action as
\[
I = \int_M \text{GTr} \mathcal{F} \mathcal{F}
\]

(8)

and we explain what “GTr” means below. In general, we can write down \(\mathcal{F}\) as
\[
\mathcal{F} = \begin{pmatrix}
\mathcal{F}_{ev} \\
(\mathcal{F}_{od})_1 \\
(\mathcal{F}_{od})_2 \\
\mathcal{F}_{ev}
\end{pmatrix}
\]

thus \(\mathcal{F} \mathcal{F}\) can be written as
\[
\mathcal{F} \mathcal{F} = \begin{pmatrix}
(\mathcal{F}_{ev})^2 - (\mathcal{F}_{od})_1 (\mathcal{F}_{od})_2 & \mathcal{F}_{ev} (\mathcal{F}_{od})_1 + (\mathcal{F}_{od})_1 \mathcal{F}_{ev} \\
(\mathcal{F}_{od})_2 \mathcal{F}_{ev} + \mathcal{F}_{ev} (\mathcal{F}_{od})_2 & (\mathcal{F}_{ev})^2 - (\mathcal{F}_{od})_2 (\mathcal{F}_{od})_1
\end{pmatrix}
\]

(9)

In four dimensions, only \((\mathcal{F}_{ev})^2\) term can contribute since \(\mathcal{F}_{ev}\) is either two form or zero form. Thus we take the ordinary trace for “GTr” in order to get a meaningful result. In three dimensions, only \(\mathcal{F}_{od}\mathcal{F}_{ev}\) type terms can contribute since \(\mathcal{F}_{od}\) terms are one forms. And in this case we take “GTr” as taking the ordinary trace after the addition of the odd parts, and we denote this as “QTr” following the notation.
of the queer trace defined in Ref. [13]. In two dimensions, two types of terms can contribute, \((\mathcal{F}_{\text{ev}})^2\) and \(\mathcal{F}_{\text{od}}\mathcal{F}_{\text{od}}\). However, only \(\mathcal{F}_{\text{od}}\mathcal{F}_{\text{od}}\) type terms have second derivative terms and we take “GTr” such that these terms do not vanish. Thus we take supertrace in the two dimensional case. Given this rule, the classical topological action (8) becomes

(a) in four dimensions

\[
I = \int_{\mathcal{M}_4} \text{Tr} \mathcal{F} \mathcal{F} = 2 \int_{\mathcal{M}_4} \text{Tr} F \wedge F; \tag{10}
\]

(b) in three dimensions

\[
I = \int_{\mathcal{M}_3} \text{QTr} \mathcal{F} \mathcal{F} = 4i \int_{\mathcal{M}_3} \text{Tr} F \wedge D\phi \tag{11}
\]

where \(\phi = \frac{1}{2}(\Phi^\dagger + \Phi)\),

(c) in two dimensions

\[
I = \int_{\mathcal{M}_2} \text{STr} \mathcal{F} \mathcal{F} = 2 \int_{\mathcal{M}_2} \text{Tr} (F[\Phi^\dagger, \Phi] - D\Phi^\dagger \wedge D\Phi) \tag{12}
\]

where we used the anticommuting property of one form \(D\phi\). In this way, we retrieve all three classical actions of topological Yang-Mills-Higgs theory in Refs. [2, 6, 7].

III. Curvature, Bianchi identity, and BRST/anti-BRST symmetry

In the geometrical BRST quantization scheme, the base space is extended in such a way that the ghost/antighost sector can be constructed on the extended
space. This is done by adding a doubled fiber bundle structure to the base manifold to represent unphysical (ghost/antighost) directions. This scheme was first developed by Thierry-Mieg and Ne’eman [14, 15] with principal fiber bundle structure yielding the BRST symmetry (the ghost direction) only. Then it was further developed to yield the BRST and anti-BRST symmetry together by including the antighost direction also – a doubled fiber bundle structure [16]. In this scheme, the ghost/antighost fields are obtained from the gauge field by replacing its spacetime leg $dx^\mu$ with $dy^N$ ($d\bar{y}^N$) where $y$, $\bar{y}$ represent the fiber coordinates in a doubled fiber bundle [4, 17, 18, 19]. If one does not like the interpretation of this extended fiber bundle approach, one can take the superspace interpretation given in Refs. [20, 21], whose view was taken in Perry and Teo’s work [3]. In the superspace approach, the fiber coordinates $y$, $\bar{y}$ are replaced by a set of anticommuting variables $\theta$ and $\bar{\theta}$ which represent the coordinates of the abstract superspace extended from the spacetime basemanifold. However, the resultant BRST/anti-BRST transformation rules are exactly the same whichever approach one uses, and thus we will be careless of the subtleties among the two approaches. In the ordinary Yang-Mills theory, the BRST/anti-BRST symmetry is obtained from a condition which sets that the ordinary curvature equals to the extended curvature, which is tantamount to setting the curvature components containing vertical (fiber) directions zero, thus only the horizontal components of curvature (physical Yang-Mills field strength) in the extended space survive. For this reason, people gave the name “horizontality condition” [18] to this condition. In this paper, we denote objects in the extended space with tildes.

Following this geometrical BRST scheme, we first extend the superconnection as

$$\mathcal{J} = J + C + \bar{C}$$  \hspace{1cm} (13)

where $C$ and $\bar{C}$ are the first generation ghost and antighost for $J$, which are given
by

\[ C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} \bar{c} & 0 \\ 0 & \bar{c} \end{pmatrix}. \]  

(14)

Here \( c, \bar{c} \) denote \( c = A_Ndy^N, \bar{c} = \bar{A}_Ndy^N \), and represent the ghost and antighost fields, respectively. In this extended space, the curvature is given by

\[ \tilde{\mathcal{F}} = \tilde{d} \tilde{\mathcal{J}} + \tilde{\mathcal{J}} \tilde{\mathcal{F}} \]  

(15)

where

\[ \tilde{d} = d + s + \bar{s}. \]  

(16)

Here, \( s \) and \( \bar{s} \) denote one form exterior derivative operators acting on ghost and antighost directions expressed in superconnection language, as \( d \) does in spacetime directions. Now, following the spirit of Refs. [2, 3], we identify the curvature components in unphysical directions with new fields. One main difference is that here we have the first generation ghost and antighost fields which are one forms in the extended space (having only \( dy \) or \( d\bar{y} \)), because in the superconnection formalism we have one form curvature components due to the scalar field.

\[ \tilde{\mathcal{F}} = \begin{pmatrix} F - \Phi \Phi^\dagger + \psi + \bar{\psi} + m + \lambda + \bar{m} & i(D\Phi + \xi + \bar{\xi}) \\ i(D\Phi^\dagger + \xi^\dagger + \bar{\xi}^\dagger) & F - \Phi^\dagger \Phi + \bar{\psi} + \psi + m + \lambda + \bar{m} \end{pmatrix} \]  

(17)

where \( \psi, \bar{\psi}, m, \lambda, \bar{m} \) are the first and second generation ghost and antighost fields for the two form curvature \( F \), and \( \xi, \bar{\xi} \) are the first generation ghost and antighost fields for the one form curvature \( D\Phi \):

\[ \begin{align*}
\psi &= \mathcal{F}^1_{\mu N}dx^\mu dy^N \\
\bar{\psi} &= \mathcal{F}^{-1}_{\mu N}dx^\mu d\bar{y}^N \\
m &= \mathcal{F}^2_{MN}dy^M dy^N \\
\lambda &= \mathcal{F}^0_{MN}dy^M d\bar{y}^N \\
\bar{m} &= \mathcal{F}^{-2}_{MN}d\bar{y}^M dy^N \\
\xi &= \mathcal{F}^1_Ndy^N \\
\bar{\xi} &= \mathcal{F}^{-1}_Nd\bar{y}^N
\end{align*} \]  

(18)
where upper indices 1, −1, 2, etc., represent ghost numbers. For instance, \( \psi \) has ghost number 1 and \( \bar{\psi} \) has ghost number −1.

The curvature in the extended space should also satisfy the Bianchi identity,

\[
\dd d\tilde{F} + [\tilde{J}, \tilde{F}] = 0.
\]  

Thus we have two conditions: (a) we have to equate Eq. (15) with Eq. (17), and (b) the Bianchi identity Eq. (19). The rules of BRST/anti-BRST symmetry are obtained from these two conditions. The BRST/anti-BRST transformation rules for the components of the extended superconnection \( \tilde{J} \) are given by the first condition.

\[
\text{even part : } & \quad sA + dc + cA + Ac = \psi, \\
& \quad \bar{s}A + d\bar{c} + \bar{c}A + A\bar{c} = \bar{\psi}, \\
& \quad sc + cc = m, \\
& \quad \bar{s}\bar{c} + \bar{c}\bar{c} = \bar{m}, \\
& \quad s\bar{c} + \bar{s}c + c\bar{c} = \lambda, \\
\text{odd part : } & \quad s\Phi + c\Phi - \Phi c = \xi, \\
& \quad \bar{s}\Phi + \bar{c}\Phi - \Phi\bar{c} = \bar{\xi}.
\]  

The second condition, the Bianchi identity, gives the BRST/anti-BRST transformation rules for the components of the extended curvature \( \tilde{F} \).

\[
\text{even part : } & \quad s\psi + d\psi + Am + c\psi - mA - mc = 0, \\
& \quad \bar{s}\bar{\psi} + d\bar{\psi} + A\bar{m} + \bar{c}\bar{\psi} - \bar{m}A - \bar{m}\bar{c} = 0, \\
& \quad s\bar{\psi} + \bar{s}\psi + d\lambda + A\lambda + c\bar{\psi} + \bar{c}\psi - \lambda A - \psi\bar{c} - \bar{\psi}c = 0, \\
& \quad sm + cm - mc = 0, \\
& \quad \bar{s}m + \bar{c}m - \bar{m}\bar{c} = 0, \\
& \quad s\lambda + \bar{s}m + c\lambda + \bar{c}m - m\bar{c} - \lambda c = 0, \\
& \quad s\bar{m} + \bar{s}\lambda + c\bar{m} + \bar{c}\lambda - \lambda\bar{c} - \bar{m}c = 0.
\]
odd part : \[ s\xi + c\xi + \Phi m - m\Phi + \xi c = 0, \]
\[ \bar{s}\xi + \bar{c}\xi + \Phi \bar{m} - \bar{m}\Phi + \bar{\xi}\bar{c} = 0, \]
\[ s\bar{\xi} + s\xi + c\bar{\xi} + c\xi + \Phi \lambda - \lambda\Phi + \xi\bar{c} + \bar{\xi}c = 0. \]

As usual, we have to introduce auxiliary fields to completely fix the BRST/anti-BRST transformation rules. We first define auxiliary fields

\[
\begin{align*}
sb &= b, \\
s\bar{b} &= [b, \bar{c}] + \lambda, \\
s\bar{c} &= [c, \bar{c}] - [\bar{c}, \psi], \\
s\bar{m} &= [\bar{c}, \lambda] - [\bar{c}, m], \\
s\bar{\lambda} &= -\eta - [c, \lambda] - [\bar{c}, \bar{m}], \\
s\bar{\xi} &= -\zeta - [\Phi, \lambda] - [c, \bar{\xi}] - [\bar{c}, \xi].
\end{align*}
\]

then we get from Eqs.\((20,21)\)

\[
\begin{align*}
\bar{s}c &= -b - [c, \bar{c}] + \lambda, \\
\bar{s}\psi &= \kappa - D\lambda - [c, \bar{\psi}] - [\bar{c}, \psi], \\
\bar{s}m &= -\eta - [c, \lambda] - [\bar{c}, m], \\
\bar{s}\lambda &= -\bar{\eta} - [c, \bar{m}] - [\bar{c}, \lambda], \\
\bar{s}\xi &= -\zeta - [\Phi, \lambda] - [c, \bar{\xi}] - [\bar{c}, \xi].
\end{align*}
\]

The nilpotency of BRST/anti-BRST transformation operators, \(s^2 = \bar{s}^2 = 0\), determines all the rest

\[
\begin{align*}
\bar{s}b &= 0, \\
\bar{s}b &= [b, \bar{c}] - \bar{\eta}, \\
\bar{s}\kappa &= 0, \\
\bar{s}\kappa &= -[b, \bar{\psi}] + D\bar{\eta} - [\bar{c}, \kappa] + [\bar{m}, sA], \\
\bar{s}\eta &= 0, \\
\bar{s}\eta &= [b, \lambda] - [c, \bar{\eta}] - [\bar{c}, \eta] - [\bar{m}, sc], \\
\bar{s}\bar{\eta} &= 0, \\
\bar{s}\bar{\eta} &= [b, \bar{m}] - [\bar{c}, \bar{\eta}], \\
\bar{s}\zeta &= 0, \\
\bar{s}\zeta &= [b, \bar{\xi}] - [\Phi, \bar{\eta}] - [\bar{c}, \zeta] - [\bar{m}, s\Phi].
\end{align*}
\]
where $sA$, $sc$, $s\Phi$ are given in Eq. (20). In Eqs. (23, 24), $[,]$ denotes a graded commutator. For instance, $[c, \bar{c}] = c\bar{c} + \bar{c}c$, and $[b, \bar{c}] = b\bar{c} - \bar{c}b$ since $c, \bar{c}$ are anticommuting fields and $b$ is a commuting field. In this way, we obtain all the transformation rules of the BRST/anti-BRST symmetry in topological Yang-Mills-Higgs theory obtained in Refs. [3, 6, 7].

**IV. Comparison with the “horizontality condition” approach**

The BRST symmetry of the ordinary Yang-Mills theory can be obtained from the so-called horizontality condition [18]. On the other hand, the BRST symmetry of topological Yang-Mills theory is not obtained through a strict application of the horizontality condition, rather it was obtained thorough a modified definition of curvature and the Bianchi identity in the extended space [3]. In Ref. [8], we modified the horizontality condition such that it could yield the complete BRST symmetry of topological Yang-Mills-Higgs theory. The rationale of this modified horizontality condition is the following. In the ordinary Yang-Mills case, the curvature in the extended space has vanishing components along the vertical directions which represent gauge fiber orbits of classical gauge symmetry, and this fact is expressed as the horizontality condition

$$\tilde{F} = \tilde{d}\tilde{A} + \tilde{A}\tilde{A} = dA + AA = F \quad (25)$$

where $\tilde{d} = d + s + \bar{s}$ and $\tilde{A} = A + c + \bar{c}$. In the topological case, we have larger symmetry than the gauge symmetry and this extra symmetry also has to be gauge fixed. That means we need extra ghosts besides the ordinary ones $(c, \bar{c})$ orginated from the gauge symmetry. Hence we modify the horizontality condition by adding “permissible” ghosts to the extended curvature

$$\tilde{F}_T = F, \quad \text{where} \quad \tilde{F}_T = \tilde{F} + \tilde{F}' \quad (26)$$

such that $\tilde{F}'$ consists of ghosts (antighosts) only and satisfies the nilpotency of

11
BRST symmetry, \( s^2 \tilde{F}'(= s^2 \tilde{F}') = 0 \). Also this \( \tilde{F}' \) has to be chosen in such a way that it respects \( s^2 \tilde{A}(= s^2 \tilde{A}) = 0 \). Through this way we can obtain the correct BRST/anti-BRST symmetry of topological Yang-Mills theory of Ref. [3]. What we explained so far is for the topological Yang-Mills case, not including the Higgs field. In order to encompass the topological Yang-Mills-Higgs case [6, 7], we carried out the same procedure in the superconnection framework in our previous work [8]:

\[
\tilde{F}'_T = \tilde{F}, \quad \text{where} \quad \tilde{F}'_T = \tilde{F} + \tilde{F}'
\]

with \( \tilde{F} \), \( \tilde{F} \) given by Eqs.(7,15), respectively, and

\[
\tilde{F}' = \begin{pmatrix}
\psi + \bar{\psi} + m + \lambda + \bar{m} & i(\xi + \bar{\xi}) \\
i(\xi^\dagger + \bar{\xi}^\dagger) & \psi + \bar{\psi} + m + \lambda + \bar{m}
\end{pmatrix}.
\]

Now, comparing the above approach with the Bianchi identity approach that we carried out in this paper, we note two things. First, the newly defined components of the extended curvature in the Bianchi identity approach correspond to the additional curvature \( \tilde{F}' \) (or \( \tilde{F}' \)) in the modified horizontality condition approach with a negative sign. Second, the requirement of the Bianchi identity for the newly defined curvature in Eq.(17) is replaced with the BRST/anti-BRST nilpotency condition on the extra curvature \( \tilde{F}' \) (or \( \tilde{F}' \)) in the modified horizontality condition approach. Now the first observation tells us that the newly defined curvature components \( (\psi, \bar{\psi}, m, \lambda, \bar{m}, \xi, \bar{\xi}) \) in the extended space given in Eq.(17) represent the existence of topological symmetry other than the ordinary gauge symmetry which is taken care of by the ghost sector of the extended connection \( \tilde{A} \) (or \( \tilde{\cal J} \)). The second observation tells us that the Bianchi identity in the extended space is simply another expression of the BRST/anti-BRST nilpotency condition for the extra ghost/antighost fields which appear as the new curvature components. In fact, in the Bianchi identity approach, the Bianchi identity in the extended space also implies this point:

\[
\tilde{d}\tilde{F} + [\tilde{A}, \tilde{F}] = 0, \quad \text{where} \quad \tilde{F} = \tilde{d}\tilde{A} + \tilde{A}\tilde{A}
\]

(28)
The above Bianchi identity is valid due to the nilpotency of the extended exterior derivative \( \tilde{d} = d + s + \bar{s} \), and the nilpotency of \( \tilde{d} \) implies the nilpotency of the BRST symmetry, \( s^2 = \bar{s}^2 = 0 \). Also in the Bianchi identity approach, the extended curvature is defined to contain the extra curvature \( \tilde{F}' \) (or \( \tilde{F}' \)) of the modified horizontality condition approach. Thus in the Bianchi identity approach one does not require the horizontality condition and instead the new degrees of freedom in the unphysical directions are allowed. For instance for the topological Yang-Mills case, we define the extended curvature \( \tilde{F} \) on which we do not impose the horizontality condition as follows:

\[
\tilde{F} = \tilde{d}\bar{A} + \bar{A}\tilde{A} = F + \psi + \bar{\psi} + m + \lambda + \bar{m}.
\]  

(29)

Since \( \tilde{A} = A + c + \bar{c} \) and \( \tilde{d} = d + s + \bar{s} \), the above definition of \( \tilde{F} \) allows the gauge field \( A \) to carry extra symmetry represented by the ghost \( \psi \) as we have seen in Eq.(20). And the symmetry property of the additional ghosts (\( \psi, \bar{\psi}, etc. \)) is constrained by the Bianchi identity that any acceptable curvature should satisfy. And this constraint on the symmetry property of the additional ghosts is nothing but the nilpotence property of the BRST symmetry.

In other words, the ordinary curvature is not a meaningful geometrical object by itself in topological field theory, rather it has to contain its own ghosts representing the extra topological symmetry and this new curvature has to satisfy the Bianchi identity in the extended space. Thus in the geometrical setting, the BRST symmetry of topological Yang-Mills-Higgs theory is represented by the extended curvature containing all “permissible” ghosts, and the BRST symmetry of these new ghosts are restricted by the Bianchi identity in the extended space. I.e., from the BRST symmetry view point only the extended curvature containing all “permissible” ghosts is a geometrically meaningful object in topological Yang-Mills-Higgs theory.

V. Conclusion
In this paper, we found the rules for the BRST/anti-BRST symmetry encompassing topological Yang-Mills-Higgs theory in two, three and four dimensions. This was done in the superconnection framework so that the scalar field is regarded as a part of a connection as is the vector gauge field. Using the superconnection language, we obtain the classical topological actions in two, three and four dimensions from a classical action of topological Yang-Mills type through appropriate projections depending on the dimensions of spacetimes to which corresponding theories belong. In this framework, the BRST/anti-BRST rules for the scalar and vector gauge fields are obtained together rather than separately. This also tells us that the usefulness of the superconnection language when one deals with the scalar and vector gauge fields, since in the ordinary treatment the BRST/anti-BRST rules for these two fields are obtained separately. As a result, we extend the work of Perry and Teo [3] in the topological Yang-Mills case to the topological Yang-Mills-Higgs case in two [7] and three [6] dimensions using the Bianchi identity. And comparing this work with our previous work of the modified horizontality condition approach, we conclude the following in topological Yang-Mills-Higgs theory: First, the newly defined ghost components of the extended curvature in the Perry and Teo’s work can be identified as the objects representing the extra topological symmetry which the theory possesses. Second, the symmetry property that these new ghosts should obey is constrained by the Bianchi identity in the extended space, and this requirement is nothing but the nilpotency condition of the BRST symmetry in another guise. Thus in theories with topological symmetry, it can be said that if things are expressed in the extended space which contains the ghost directions, then one can treat the BRST symmetry in a geometrical setting in which the curvature contains all the “permissible” ghosts, and the BRST symmetry due to topological symmetry is constrained by the Bianchi identity in this extended space.
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