POLYGON OF RECOLLEMENTS AND $N$-COMPLEXES

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Abstract. We study a structure of subcategories which are called a polygon of recollements in a triangulated category. First, we study a $2n$-gon of recollements in an $(m/n)$-Calabi-Yau triangulated category. Second, we show the homotopy category $K(\text{Mor}_{N-1}(B))$ of complexes of an additive category $\text{Mor}_{N-1}(B)$ of $N-1$ sequences of split monomorphisms of an additive category $B$ has a $2N$-gon of recollements. Third, we show the homotopy category $K_N(B)$ of $N$-complexes of $B$ has also a $2N$-gon of recollements. Finally, we show there is a triangle equivalence between $K(\text{Mor}_{N-1}(B))$ and $K_N(B)$.

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0. Introduction

The notion of recollement of triangulated categories was introduced by Beilinson, Bernstein and Deligne in connection with derived categories of sheaves of topological spaces ([BBD]). One of the authors introduced the notion of stable $t$-structure in a triangulated category [M], and studied relations to recollements. Afterwards this notion was studied by many authors under a lot of names, e.g. a torsion pair, a semiorthogonal decomposition, Bousfield localization.

Definition 0.1. Let $\mathcal{D}$ be a triangulated category with the translation functor $\Sigma$. A pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of $\mathcal{D}$ is called a stable $t$-structure in $\mathcal{D}$ provided that

(a) $\mathcal{U} = \Sigma \mathcal{U}$ and $\mathcal{V} = \Sigma \mathcal{V}$.
(b) $\text{Hom}_\mathcal{D}(\mathcal{U}, \mathcal{V}) = 0$.
(c) For every $X \in \mathcal{D}$, there exists a triangle $U \rightarrow X \rightarrow V \rightarrow \Sigma U$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

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In [IKM1], we introduced the notion of polygons of recollements in a triangulated category, and studied the properties of triangle of recollements in connection with the homotopy category of unbounded complexes with bounded homologies, its quotient category by the homotopy category of bounded complexes, and the stable category of Cohen-Macaulay modules over an Iwanaga-Gorenstein ring. Moreover, we studied derived categories of ordinary complexes over an abelian category and show an angulated subcategories of the derived category of Cohen-Macaulay modules over an Iwanaga-Gorenstein ring. Moreover, we studied derived categories of ordinary complexes over an abelian category $\text{Mor}_{N-1}(A)$ of $N-1$ sequences of morphisms of $A$ ([IKM2]). In this article, we study the properties of polygons of recollements in various categories.

**Definition 0.2.** Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{U}_1, \cdots, \mathcal{U}_n$ be full triangulated subcategories of $\mathcal{D}$. An $n$-tuple $(\mathcal{U}_1, \cdots, \mathcal{U}_n)$ is called an $n$-gon of recollements in $\mathcal{D}$ if $(\mathcal{U}_i, \mathcal{U}_{i+1})$ is a stable $t$-structure in $\mathcal{D}$ $(1 \leq i \leq n)$, where $\mathcal{U}_1 = \mathcal{U}_{n+1}$.

In Section 1, we recall the notions of stable $t$-structures and recollement polygons of recollements in a triangulated category.

**Proposition 0.3** (Proposition 1.7). Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories. Let $(\mathcal{U}_1, \cdots, \mathcal{U}_n)$ and $(\mathcal{V}_1, \cdots, \mathcal{V}_n)$ be $n$-gons of recollements in $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Assume a triangle functor $F: \mathcal{D}_1 \rightarrow \mathcal{D}_2$ sends $(\mathcal{U}_1, \cdots, \mathcal{U}_n)$ to $(\mathcal{V}_1, \cdots, \mathcal{V}_n)$. If $F|_{\mathcal{U}_i}$ and $F|_{\mathcal{U}_{i+1}}$ are triangle equivalences for some $t$, so is $F$.

In Section 1, we study stable $t$-structures with relation to contravariantly finite categories and Calabi-Yau triangulated categories.

**Proposition 0.4** (Proposition 2.5). Let $\mathcal{D}$ be an $(m/n)$-Calabi-Yau triangulated category. For any functorially finite thick subcategory $\mathcal{U}_i$ of $\mathcal{D}$, we put $\mathcal{U}_{i+1} := \mathcal{U}_i^{\perp}$ for any $i$. Then we have an $l$-gon $(\mathcal{U}_1, \cdots, \mathcal{U}_l)$ of recollements in $\mathcal{D}$ for some positive divisor $l$ of $2n$.

In Section 3, we constructs polygons of recollements in the derived derived category of modules over an algebra, and them in the stable category of Cohen-Macaulay modules over an Iwanaga-Gorenstein ring.

**Theorem 0.5** (Theorem 3.6). Let $A$ be a finite dimensional $k$-algebra of finite global dimension such that $A/J_A$ is separable over a field $k$ and $\mathcal{D}^b(\text{mod } A)$ is $(m/n)$-Calabi-Yau. Let $R$ be a coherent $k$-algebra of finite self-injective dimension as both sides. For any functorially finite thick subcategory $\mathcal{U}_i$ of $\mathcal{D}^b(\text{mod } A)$, we put $\mathcal{U}_{i+1} := \mathcal{U}_i^{\perp}$ for any $i$. Then there is a positive divisor $l$ of $2n$ such that we have an $l$-gon $(\mathcal{U}_1^R, \cdots, \mathcal{U}_l^R)$ of recollements in $\mathcal{D}^b(\text{mod } R \otimes_k A)$ and an $l$-gon $(Q(\mathcal{U}_1^R), \cdots, Q(\mathcal{U}_l^R))$ of recollements in $\mathcal{C}^l(A)$.

In Section 4, we the homotopy category $\mathcal{K}(\text{Mor}_{N-1}^{	ext{sm}}(B))$ of the category $\text{Mor}_{N-1}^{	ext{sm}}(B)$ of $N - 1$ sequences of split monomorphisms in an additive category $B$.

**Theorem 0.6** (Theorem 4.8). Let $B$ be an additive category. Then there is a $2N$-gon of recollements in $\mathcal{K}(\text{Mor}_{N-1}^{	ext{sm}}(B))$:

$$\mathcal{E}^{[1,N-1]}, \mathcal{E}^{[2,N-1]}, \mathcal{E}^1, \mathcal{E}^{[1,2]}, \cdots, \mathcal{E}^{s}, \mathcal{E}^{[s+1,N-1]}, \cdots, \mathcal{E}^{N-2}, \mathcal{F}^{[N-2,N-1]}, \mathcal{E}^{N-1}, \mathcal{E}^{[1,N-2]}$$

In Section 5, we study the homotopy category $\mathcal{K}_N(B)$ of $N$-complexes of objects of an additive category $B$. 
Theorem 0.7 (Corollary 5.10). We have a recollement of $K_N(B)$:

\[
\begin{array}{ccc}
K_{N-r}(B) & \xrightarrow{i_*} & K_N(B) \\
\downarrow i_! & & \downarrow i^! \\
K_{N+1}(B) & \xleftarrow{j_*} & K_{N-r+1}(B)
\end{array}
\]

Corollary 0.8 (Corollary 5.11). There is a 2N-gon of recollements in $K_N(B)$:

\[
(F_1^{-N-2}, F_1^{-N-1}, \ldots, F_1^{-1}, F_0^1, F_1^1, \ldots, F_1^{N+2}, F_0^{N+2}, F_1^{N+1}, F_0^{N+1})
\]

In Section 6, we construct a triangle functor $F_N : K(Mor^m_{N-1}(B)) \to K_N(B)$ which sends the above 2N-gon of $K_N(B)$ to the above 2N-gon of $K_N(B)$. Therefore we have the result.

Theorem 0.9 (Theorem 6.8). Let $B$ be an additive category, then we have triangle equivalences:

\[
K^z(Mor^m_{N-1}(B)) \simeq K^z_N(B)
\]

where $z = \text{nothing}, -, +, b$.

1. Stable t-structures and recollements

We recall the notion of recollements and study their relationship with stable t-structures. This correspondence enables us to understand (co)localizations and recollements by way of subcategories instead of quotient categories. In Proposition 1.2 we see that a recollement corresponds to two consecutive stable t-structures. First we see that a (co)localization and a stable t-structure essentially describe the same phenomenon, using the methods which are similar to ones in recollements [BBD].

Next we recall the notion of a recollement which consists of a localization and a colocalization.

Definition 1.1 ([BBD]). We call a diagram

\[
\begin{array}{ccc}
\mathcal{D}' & \xrightarrow{i_*} & \mathcal{D} \\
\downarrow i_! & & \downarrow i^! \\
\mathcal{D} & \xleftarrow{j_*} & \mathcal{D}''
\end{array}
\]

of triangulated categories and functors a recollement if it satisfies the following:

1. $i_*$, $j_*$, and $j_*$ are fully faithful.
2. $(i^*, i_*)$, $(i_*, i^*)$, $(j_*, j^*)$, and $(j^*, j_*)$ are adjoint pairs.
3. There are canonical embeddings $\text{Im} j_* \hookrightarrow \text{Ker} i^*$, $\text{Im} i_* \hookrightarrow \text{Ker} j^*$, and $\text{Im} j_* \hookrightarrow \text{Ker} i^*$ which are equivalences.

We remember that a recollement corresponds to a pair of consecutive stable t-structures.

Proposition 1.2 ([Mi]).

1. Let

\[
\begin{array}{ccc}
\mathcal{D}' & \xrightarrow{i_*} & \mathcal{D} \\
\downarrow i_! & & \downarrow i^! \\
\mathcal{D} & \xleftarrow{j_*} & \mathcal{D}''
\end{array}
\]

be a recollement. Then $(U, V)$ and $(V, W)$ are stable t-structures in $\mathcal{D}$ where we put $U = \text{Im} j_*$, $V = \text{Im} i_*$ and $W = \text{Im} j_*$. 

Definition 1.5. Let \( \mathcal{D} \) be a triangulated category, and let \( \mathcal{U}_1, \cdots, \mathcal{U}_n \) be full triangulated subcategories of \( \mathcal{D} \). We call \((\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n)\) an \( n \)-gon of recollements in \( \mathcal{D} \) if \((\mathcal{U}_i, \mathcal{U}_{i+1})\) is a stable t-structure in \( \mathcal{D} \) \( (1 \leq i \leq n) \), where \( \mathcal{U}_1 = \mathcal{U}_{n+1} \).

An \( n \)-gon of recollements results in strong symmetry, and it induces three recollements as the name suggests.

Proposition 1.4 ([IKM1]). Let \( \mathcal{D} \) be a triangulated category. Then \((\mathcal{U}_1, \cdots, \mathcal{U}_n)\) is an \( n \)-gon of recollements in \( \mathcal{D} \) if and only if there is a recollement

\[
\begin{array}{c}
\mathcal{U}_i \oplus \mathcal{D} \oplus \mathcal{D}/\mathcal{U}_i \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{U}_i \oplus \mathcal{D}_i \oplus \mathcal{D}/\mathcal{U}_i \\
\end{array}
\]

such that \( \text{Im} j_1 = \mathcal{U} \) and \( \text{Im} j_n = \mathcal{W} \).

Finally we study the case that triangle functors preserve localizations, colocalizations or recollements, etc.

Definition 1.5. Let \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) be triangulated categories and let \( F : \mathcal{D}_1 \to \mathcal{D}_2 \) be a triangle functor.

1. Let \((\mathcal{U}_n, \mathcal{V}_n)\) be a stable t-structure in \( \mathcal{D}_n \) \( (n = 1, 2) \). We say that \( F\) sends \((\mathcal{U}_1, \mathcal{V}_1)\) to \((\mathcal{U}_2, \mathcal{V}_2)\) if \( F(\mathcal{U}_1) \) is contained in \( \mathcal{U}_2 \) and \( F(\mathcal{V}_1) \) is in \( \mathcal{V}_2 \).

2. Let \((\mathcal{U}_n, \cdots, \mathcal{U}_n)\) be an \( n \)-gon of recollements in \( \mathcal{D}_1 \) \( (i = 1, 2) \). We say that \( F\) sends \((\mathcal{U}_n, \cdots, \mathcal{U}_n)\) to \((\mathcal{U}_2n, \cdots, \mathcal{U}_2n)\) if \( F(\mathcal{U}_nk) \) is contained in \( \mathcal{U}_2k \) for any \( k \).

Lemma 1.6 ([IKM1]). If a triangle functor \( F : \mathcal{D}_1 \to \mathcal{D}_2 \) sends a stable t-structure \((\mathcal{U}_1, \mathcal{V}_1)\) in \( \mathcal{D}_1 \) to a stable t-structure \((\mathcal{U}_2, \mathcal{V}_2)\) in \( \mathcal{D}_2 \). Then we have the following:

1. If \( F|_{\mathcal{U}_1} \) is full (resp., faithful), then \( \text{Hom}_{\mathcal{D}_1}(U, X) \to \text{Hom}_{\mathcal{D}_2}(FU, FX) \) is surjective (resp., injective) for \( U \in \mathcal{U}_1 \) and \( X \in \mathcal{D}_1 \).

2. If \( F|_{\mathcal{V}_1} \) is full (resp., faithful), then \( \text{Hom}_{\mathcal{D}_1}(X, V) \to \text{Hom}_{\mathcal{D}_2}(FX, FV) \) is surjective (resp., injective) for \( X \in \mathcal{D}_1 \) and \( V \in \mathcal{V}_1 \).

3. If \( F\) is full and \( F|_{\mathcal{U}_1} : \mathcal{U}_1 \to \mathcal{U}_2 \) and \( F|_{\mathcal{V}_1} : \mathcal{V}_1 \to \mathcal{V}_2 \) are dense, then \( F\) is dense.
Proposition 1.7. Let $\mathcal{D}_1$, $\mathcal{D}_2$ be triangulated categories. Let $(\mathcal{U}_1, \ldots, \mathcal{U}_n)$ and $(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ be $n$-gons of recollements in $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Assume a triangle functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ sends $(\mathcal{U}_1, \ldots, \mathcal{U}_n)$ to $(\mathcal{V}_1, \ldots, \mathcal{V}_n)$. Then the following hold.

1. In the case that $n$ is odd, if $F |_{\mathcal{U}_t}$ is fully faithful (equivalent) for some $t$, so is $F$.
2. In the case that $n$ is even, if $F |_{\mathcal{U}_t}$ and $F |_{\mathcal{U}_{t+1}}$ are fully faithful (equivalent) for some $t$, so is $F$.

Proof. By Lemma 1.6. □

2. Contravariantly finite subcategories and Stable t-structures

In this section let $k$ be a field and $D := \text{Hom}_k(-, k)$. The concept of stable $t$-structures is closely related to functorially finite subcategories [AS]. If $(\mathcal{U}, \mathcal{V})$ is a stable $t$-structure of $D$, then clearly $\mathcal{U}$ (resp., $\mathcal{V}$) is a contravariantly (resp., covariantly) finite subcategory of $D$. We shall show that a certain converse of this statement holds. For a full subcategory $\mathcal{U}$ of $D$, we put

\[ \mathcal{U}^\perp := \{ T \in D \mid \text{Hom}_D(\mathcal{U}, T) = 0 \}, \]
\[ {}^\perp \mathcal{U} := \{ T \in D \mid \text{Hom}_D(T, \mathcal{U}) = 0 \}. \]

Recall that an additive category is called Krull-Schmidt if any object is isomorphic to a finite direct sum of objects whose endomorphism rings are local.

Definition 2.1. Let $\mathcal{C}$ be an additive category, and $\mathcal{U}$ its full subcategory. For an object $X$ of $\mathcal{C}$, a morphism $U_X \xrightarrow{f} X$ with $U_X \in \mathcal{U}$ is called a right $\mathcal{U}$-approximation if $\text{Hom}_\mathcal{C}(U, f) : \text{Hom}_\mathcal{C}(U, U_X) \to \text{Hom}_\mathcal{C}(U, X)$ is surjective for any $U \in \mathcal{U}$. Moreover, a right $\mathcal{U}$-approximation $U_X \xrightarrow{f} X$ is called minimal if $g$ is an isomorphism whenever $g : U_X \to U_X$ satisfies $f \circ g = f$.

A full subcategory $\mathcal{U}$ is called a contravariantly finite subcategory if every object $X$ of $\mathcal{C}$ has a right a $\mathcal{U}$-approximation. A left a $\mathcal{U}$-approximation and a covariantly finite subcategory are defined dually. $\mathcal{U}$ is called a functorially finite subcategory if it is contravariantly finite and covariantly finite.

Proposition 2.2. Let $\mathcal{D}$ be a Krull-Schmidt triangulated category. For any contravariantly (resp., covariantly) finite thick subcategory $\mathcal{U}$ of $\mathcal{D}$, we have a stable $t$-structure $(\mathcal{U}, \mathcal{U}^\perp)$ (resp., $({}^\perp \mathcal{U}, \mathcal{U})$) in $\mathcal{D}$.

Proof. This is a consequence of Wakamatsu-type Lemma (see [IY, Prop. 3.6] for example). For the convenience of the reader, we give the proof here. Since $\mathcal{D}$ is Krull-Schmidt category and $\mathcal{U}$ is closed under direct summands, for any object $X$ of $\mathcal{D}$ there is a triangle $U_X \xrightarrow{f} X \xrightarrow{g} V \xrightarrow{h} \Sigma U_X$ such that $U_X \xrightarrow{f} X$ is a minimal right $\mathcal{U}$-approximation. For any morphism $\alpha : U \to V$ with $U \in \mathcal{U}$, by octahedral
axiom we have a commutative diagram

\[
\begin{array}{ccc}
\Sigma^{-1}U & \longrightarrow & \Sigma^{-1}U \\
\downarrow \Sigma^{-1}\alpha & & \downarrow \Sigma^{-1}\beta \\
\Sigma^{-1}V & \longrightarrow & \Sigma^{-1}V \\
\downarrow \gamma & & \downarrow \delta \\
Z & \longrightarrow & M \\
\downarrow h' & & \downarrow g' \\
U & \longrightarrow & U
\end{array}
\]

where all columns and rows of 4 terms are triangles. Then \(M\) belongs to \(U\). Since \(f\) is a right \(U\)-approximation, there is a morphism \(\delta : M \to U_X\) such that \(f \circ \delta = f'\).

Then there is a morphism \(\epsilon : Z \to \Sigma^{-1}V\) such that \(-\Sigma^{-1}h \circ \epsilon = \delta \circ h'\). Then \(\gamma\) is a split monomorphism because of the minimality of \(f\). Therefore \(\beta\) is a split monomorphism, and \(\Sigma^{-1}\alpha\) and \(\alpha\) are zero. Hence we have \(V \in U^\perp\).

**Definition 2.3.** Let \(D\) be a \(k\)-linear triangulated category such that \(\dim_k \text{Hom}_D(X,Y) < \infty\) for any \(X,Y \in D\). An autofunctor \(S : D \to D\) is called a Serre functor [BK, RV] if there exists a functorial isomorphism

\[
\text{Hom}_D(X,Y) \simeq \text{Hom}_D(Y, SX)
\]

for any \(X,Y \in D\).

We say that \(D\) is \((m/n)\)-Calabi-Yau for a positive integer \(n\) and an integer \(m\) if we have an isomorphism \(S^n \simeq \Sigma^m\) of functors

\[1\]

Immediately we have the following.

**Lemma 2.4.** Let \(D\) be a Krull-Schmidt triangulated category with a Serre functor \(S\). Then the following hold.

1. \(U^\perp = (SU)^\perp\) for any subcategory \(U\) of \(D\).
2. If \((U,V)\) is a stable \(t\)-structure and \(U\) is functorially finite in \(D\), then \((V,SU)\) is a stable \(t\)-structure and \(V\) is functorially finite in \(D\).

**Proof.**

1. Immediate from the definition of Serre functor.

2. We have \((SU)^\perp = V^\perp \) by (1). Since \(SU\) is a covariantly finite subcategory of \(D\), we have that \((V,SU)\) is a stable \(t\)-structure by the dual of Proposition 2.2. Consequently \(V\) is a functorially finite subcategory of \(D\) by Proposition 2.2.

**Proposition 2.5.** Let \(D\) be an \((m/n)\)-Calabi-Yau triangulated category. For any functorially finite thick subcategory \(U_i\) of \(D\), we put \(U_{i+1} := U_i^\perp\) for any \(i\). Then we have an \(l\)-gon \((U_1, \cdots, U_l)\) of recollements in \(D\) for some positive divisor \(l\) of \(2n\).

**Proof.** By Propositions 2.2, we have a stable \(t\)-structure \((U_1, U_2)\) in \(D\). Using Lemma 2.4(2) inductively, we have a stable \(t\)-structure \((U_i, U_{i+1})\) in \(D\) such that \(U_{i+2} = SU_i\) for any \(i\). We have the statement because of \(S^n \simeq \Sigma^m\).
3. Constructions of Stable $t$-structures

In this section, we investigate polygons of recollements in derived categories and in stable module categories.

For a ring $R$, we denote by $\text{Mod } R$ (resp., $\text{mod } R$) the category of right (resp., finitely generated right) $R$-modules, and denote by $\text{Proj } R$ (resp., $\text{Inj } R$, $\text{proj } R$) the full subcategory of $\text{Mod } R$ consisting of projective (resp., injective, finitely generated projective) modules. For right (resp., left) $R$-module $M_R$ (resp., $R N$), we denote by $\text{idim } M_R$ (resp., $\text{idim } R N$) the injective dimension of $M_R$ (resp., $R N$), and by $\text{pdim } M_R$ (resp., $\text{pdim } R N$) the projective dimension of $M_R$ (resp., $R N$). For $A$ be an abelian category and its additive subcategory $B$, we denote by $\text{D}^{b}(A)$ (resp., $K^{b}(B)$) the derived category (resp., the homotopy category) of bounded complexes of objects of $A$ (resp., $B$).

**Definition 3.1.** We call a ring $R$ Iwanaga-Gorenstein if it is Noetherian with $\text{idim}_R R < \infty$ and $\text{idim}_R R < \infty$ [Iw]. We define the category of Cohen-Macaulay $R$-modules $\text{CM } R$ and the category of large Cohen-Macaulay $R$-modules by

$$\text{CM } R := \{ X \in \text{mod } R \mid \text{Ext}^i_R(X, R) = 0 \ (i > 0) \},$$

$$\text{LCM } R := \{ X \in \text{Mod } R \mid \text{Ext}^i_R(X, \text{Proj } R) = 0 \ (i > 0) \}.$$  

Then $\text{CM } R$ forms a Frobenius category with the subcategory $\text{proj } R$ of projective-injective objects, and the stable category $\text{CM } R$ forms a triangulated category [Ha]. By [IKM1] there exist triangle equivalences

$$\text{CM } R \simeq \text{D}^{b}(\text{mod } R)/K^{b}(\text{proj } R), \quad \text{LCM } R \simeq \text{D}^{b}(\text{Mod } R)/K^{b}(\text{Proj } R).$$

For subcategories $\mathcal{U}$ and $\mathcal{V}$ of a triangulated category $\mathcal{D}$, we put

$$\mathcal{U} * \mathcal{V} := \{ X \in \mathcal{D} \mid U \rightarrow X \rightarrow V \rightarrow \Sigma U \text{ is a triangle in } \mathcal{D} (U \in \mathcal{U}, V \in \mathcal{V}) \}.\]$$

By octahedral axiom, we have $(\mathcal{U} * \mathcal{V}) * \mathcal{W} = \mathcal{U} * (\mathcal{V} * \mathcal{W})$.

**Lemma 3.2.** Let $\mathcal{D}$ be a triangulated subcategory, and $\mathcal{U}$ and $\mathcal{V}$ triangulated (resp., thick) subcategories of $\mathcal{D}$ satisfying $\text{Hom}_\mathcal{D}(\mathcal{U}, \mathcal{V}) = 0$. Then $\mathcal{U} * \mathcal{V}$ is a triangulated (resp., thick) subcategory of $\mathcal{D}$.

**Proof.** We only have to show $(\mathcal{U} * \mathcal{V}) * (\mathcal{U} * \mathcal{V}) \subset \mathcal{U} * \mathcal{V}$. Since $\text{Hom}_\mathcal{D}(\mathcal{U}, \Sigma \mathcal{V}) = 0$, we have $\mathcal{V} * \mathcal{U} = \text{add}\{\mathcal{U}, \mathcal{V}\}$. Thus we have $(\mathcal{U} * \mathcal{V}) * (\mathcal{U} * \mathcal{V}) = \mathcal{U} * (\mathcal{V} * \mathcal{U}) * \mathcal{V} = \mathcal{U} * \text{add}\{\mathcal{U}, \mathcal{V}\} * \mathcal{V} \subset (\mathcal{U} * \mathcal{U}) * (\mathcal{V} * \mathcal{V}) = \mathcal{U} * \mathcal{V}$, where $\text{add}\{\mathcal{U}, \mathcal{V}\}$ is the additive subcategory of consisting of finite direct sums of objects of $\mathcal{U}$ and $\mathcal{V}$. Thus $\mathcal{U} * \mathcal{V}$ is a triangulated subcategory of $\mathcal{D}$. If $\mathcal{U}$ and $\mathcal{V}$ are closed under direct summand, then so is $\mathcal{U} * \mathcal{V}$ (e.g. [IY, Prop. 2.1]). Thus the assertion for thick subcategories follows. \hfill $\Box$

Let $A$ and $R$ be $k$-algebras. For a subcategory $\mathcal{U}$ of $\text{D}^{b}(\text{mod } A)$, we denote by $\mathcal{U}^{R}$ the thick subcategory of $\text{D}^{b}(\text{mod } R \otimes_k A)$ generated by

$$\{ L \otimes_k X \mid L \in \text{D}^{b}(\text{mod } R), \ X \in \mathcal{U} \}.$$

The following observation gives us a lot of examples of stable $t$-structures in derived categories.

\footnotesize
\[\text{In the representation theory of orders CR, Au, Y, there is another notion of Cohen-Macaulay modules which generalizes the classical notion in commutative ring theory. These two concepts coincide for Gorenstein orders.}\]
Proposition 3.3. Let $R$ be a $k$-algebra and $A$ a finite dimensional $k$-algebra such that $A/J_A$ is a separable $k$-algebra, where $J_A$ is the Jacobson radical. For any stable t-structure $(\mathcal{U}, \mathcal{V})$ in $D^b(\text{mod } A)$, we have a stable t-structure $(\mathcal{U}^R, \mathcal{V}^R)$ in $D^b(\text{mod } R \otimes_k A)$.

Proof. Let $\mathcal{D} := D^b(\text{mod } R \otimes_k A)$. Since

$$\text{Hom}_D(L \otimes_k U, M \otimes_k V) = \text{Hom}_{D^b(\text{mod } R)}(L, M) \otimes_k \text{Hom}_{D^b(\text{mod } A)}(U, V)$$

for any $L, M \in D^b(\text{mod } R)$ and any $U, V \in D^b(\text{mod } A)$, we have $\text{Hom}_{\mathcal{D}}(\mathcal{U}^R, \mathcal{V}^R) = 0$.

Since $\mathcal{U}^R \ast \mathcal{V}^R$ is a thick subcategory of $\mathcal{D}$ by Lemma 3.2, we only have to show $\mathcal{U}^R \ast \mathcal{V}^R$ contains $\text{mod } R \otimes_k A$. Any $R \otimes_k A$-module $M$ is filtered by $R \otimes_k A$-modules $M J_A^i / M J_A^{i+1}$ which are semisimple $A$-modules. We only have to show that any $R \otimes_k A$-module $N$ which is a semisimple $A$-module belongs to $\mathcal{U}^R \ast \mathcal{V}^R$. Since the map $(A/J_A) \otimes_k (A/J_A) \to A/J_A$, $x \otimes y \mapsto xy$ is a split epimorphism of $A^{op} \otimes_k A$-modules, we have that the map $N \otimes_k (A/J_A) \to N$, $n \otimes y \mapsto ny$ is a split epimorphism of $R \otimes_k A$-modules. Since $A/J_A \in \mathcal{U} \ast \mathcal{V}$, we have that $N \otimes_k (A/J_A) \in \mathcal{U}^R \ast \mathcal{V}^R$. Thus $N \in \mathcal{U}^R \ast \mathcal{V}^R$. □

The following result gives a criterion for a stable t-structure in the derived category to give a stable t-structure in the stable category.

Lemma 3.4 ([IKM1]). Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}$ a thick subcategory of $\mathcal{D}$, and $Q : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ the canonical quotient [Ne]. For a stable t-structure $(\mathcal{U}, \mathcal{V})$ in $\mathcal{D}$, the following are equivalent, where $Q(\mathcal{U})$ is the full subcategory of $\mathcal{D}/\mathcal{C}$ consisting of objects $Q(X)$ for $X \in \mathcal{E}$.

1. $(Q(\mathcal{U}), Q(\mathcal{V}))$ is a stable t-structure in $\mathcal{D}/\mathcal{C}$.
2. $(\mathcal{U} \cap \mathcal{C}, \mathcal{V} \cap \mathcal{C})$ is a stable t-structure in $\mathcal{C}$.

The following example provides us a rich source of triangulated categories with Serre functors.

Proposition 3.5. Let $A$ be a finite dimensional $k$-algebra of finite self-injective dimension as both sides. Then the following hold.

1. $K^b(\text{proj } A)$ has a Serre functor $\nu_A := - \otimes_A^L (DA)$.
2. $K^b(\text{proj } A)$ is $(m/n)$-Calabi-Yau if and only if $(DA)^{\otimes_A^L k^n} \simeq \Sigma^m A$ in $D^b(\text{mod } A^{op} \otimes_k A)$.

We have the following main result in this section.

Theorem 3.6. Let $A$ be a finite dimensional $k$-algebra of finite global dimension such that $A/J_A$ is separable over $k$ and $D^b(\text{mod } A)$ is $(m/n)$-Calabi-Yau. Let $R$ be a coherent $k$-algebra of finite self-injective dimension as both sides. For any functorially finite thick subcategory $\mathcal{U}_i$ of $D^b(\text{mod } A)$, we put $\mathcal{U}_{i+1} := \mathcal{U}_i^\perp$ for any $i$. Then there is a positive divisor $l$ of $2n$ such that we have an $l$-gon $(\mathcal{U}_1^R, \cdots, \mathcal{U}_{2n}^R)$ of recollements in $D^b(\text{mod } R \otimes_k A)$ and an $l$-gon $(Q(\mathcal{U}_1^R), \cdots, Q(\mathcal{U}_{2n}^R))$ of recollements in $\text{CM}(R \otimes_k A)$, where $Q : D^b(\text{mod } R \otimes_k A) \to \text{CM}(R \otimes_k A)$ is the canonical quotient.

Proof. According to Propositions 2.5 and 3.3, we have an $n$-gon $(\mathcal{U}_1^R, \mathcal{U}_2^R, \cdots, \mathcal{U}_n^R)$ of recollements in $D^b(\text{mod } R \otimes_k A)$. Since $A$ is of finite global dimension, there exists a triangle

$$U_i \to A \to U_{i+1} \to \Sigma U_i$$
with \( U_i \in U_i \cap K^b(\text{proj } A) \) and \( U_{i+1} \in U_{i+1} \cap K^b(\text{proj } A) \). Applying \( R \otimes_k - \), we have a triangle
\[
R \otimes_k U_i \to R \otimes_k A \to R \otimes_k U_{i+1} \to \Sigma R \otimes_k U_i
\]
with \( R \otimes_k U_i \in U^R_i \cap K^b(\text{proj } R \otimes_k A) \) and \( R \otimes_k U_{i+1} \in U^R_{i+1} \cap K^b(\text{proj } R \otimes_k A) \). By Lemma 3.4, we have a stable \( t \)-structure \( (Q(U^R_i), Q(U^R_{i+1})) \) in \( \mathbf{CM}(R \otimes_k A) \). \( \square \)

We have the following example of recollements by [Mi, Cor. 5.11].

**Proposition 3.7.** Let \( A \) be a finite dimensional \( k \)-algebra, and \( e \) an idempotent of \( A \). Assume that \( \text{Ext}_A^i(A/AeA, A/AeA) = 0 \) (\( i > 0 \)), \( \text{pdim}_A(AeA) < \infty \) and \( \text{pd}(AeA)_A < \infty \). Then we have a recollement
\[
\begin{array}{ccc}
\mathbf{D}^b(\text{mod } A/AeA) & \xrightarrow{i_*} & \mathbf{D}^b(\text{mod } A) \\
\mathbf{D}^b(\text{mod } AeA) & \xrightarrow{j_*} & \mathbf{D}^b(\text{mod } eA)
\end{array}
\]
In particular, \( \text{Im } j_* \), \( \text{Im } i_* \) and \( \text{Im } j_* \), \( \text{Im } i_* \) are stable \( t \)-structures in \( \mathbf{D}^b(\text{mod } A) \).

### 4. Recollement of \( K(\text{Mor}_{N-1}^\text{sm}(B)) \)

In this section we study the properties of complexes of the category of \( N - 1 \) sequences of morphisms in an additive category \( B \). Throughout this section \( B \) is an additive category. Then the category \( C(B) \) of complexes of objects of \( B \) is a Frobenius category such that its conflations are short exact sequences of which each term is a split exact sequence in \( B \).

**Definition 4.1.** We define the category \( \text{Mor}_{N-1}^\text{sm}(B) \) (resp., \( \text{Mor}_{N-1}(B) \)) of sequences of morphisms in \( B \) as follows.

- An object is a sequence of split monomorphisms (resp., morphisms) \( X : X^1 \xrightarrow{\alpha_1^1} \cdots \xrightarrow{\alpha_{N-2}} X^{N-1} \) in \( B \).
- A morphism from \( X \) to \( Y \) is an \( (N - 1) \)-tuple \( f = (f^1, \cdots, f^{N-1}) \) of morphisms \( f^i : X^i \to Y^i \) such that \( f^{i+1} \alpha_X^i = \alpha_Y^{i+1} f^i \) for \( 1 \leq i \leq N - 2 \).

We give technical tools to investigate the homotopy category \( K(\text{Mor}_{N-1}^\text{sm}(B)) \).

**Definition 4.2.** For an additive functor \( G : B \to B' \) between additive categories, let \( (G \downarrow 1_B) \) be a comma category, that is the category of objects \( G(X) \xrightarrow{\alpha} Y \) for \( X \in A, Y \in B' \). We denote by \( (G \downarrow 1_B)^\text{sm} \) the subcategory of \( (G \downarrow 1_B) \) consisting of objects \( G(X) \xrightarrow{\alpha} Y \), where \( \alpha \) are split monomorphisms.

**Example 4.3.** For \( 1 \leq r < N - 1 \), let \( G : \text{Mor}(B) \to \text{Mor}_{N-r-1}(B) \) (resp., \( G : \text{Mor}_r^\text{sm}(B) \to \text{Mor}_{N-r-1}^\text{sm}(B) \)) be an additive functor defined by
\[
G(X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^{r-1}} X^r) = Y^1 \xrightarrow{\beta^1} \cdots \xrightarrow{\beta^{N-r-2}} Y^{N-r-1}
\]
where \( Y^1 = \cdots = Y^{N-r-1} = X^r \) and \( \beta^1 = \cdots = \beta^{N-r-2} = 1_{X^r} \). Then the category \( (G \downarrow 1_{\text{Mor}_{N-r-1}(B)}) \) (resp., \( (G \downarrow 1_{\text{Mor}_{N-r-1}^\text{sm}(B)}) \)) is equivalent to \( \text{Mor}_{N-1}(B) \) (resp., \( \text{Mor}_{N-1}^\text{sm}(B) \)).

**Lemma 4.4.** For an additive functor \( G : B \to B' \) between additive categories, the following hold.
(1) Every complex of \((G \downarrow \text{sm} 1_{B'})\) has the following form:

\[ G(X) \xrightarrow{u_f} C(f) \]

where \(f : Y \to G(X)\) is a morphism of complexes of \(B'\), \(C(f)\) is the mapping cone of \(f\), and \(u_f\) is the canonical morphism.

(2) If a complex \(X\) of \(B\) is homotopically trivial, then a complex \(G(X) \xrightarrow{u_f} C(f)\) of \((G \downarrow \text{sm} 1_{B'})\) is isomorphic to \(0 \to \Sigma_{B'} Y\) in \(K(G \downarrow \text{sm} 1_{B'})\), where \(f : Y \to G(X)\).

(3) For a complex \(G(X) \xrightarrow{u_f} C(f)\) of \((G \downarrow \text{sm} 1_{B'})\), if \(C(f)\) is homotopically trivial, then \(G(X) \xrightarrow{u_f} C(f)\) is isomorphic to \(G(X) \xrightarrow{u_f} C(1_{G(X)})\) in in \(K(G \downarrow \text{sm} 1_{B'})\).

**Proof.** (1) It is trivial.

(2) For a morphism \(Y \xrightarrow{f} G(X)\) of complexes of \(B'\), we have a triangle in \(K(G \downarrow \text{sm} 1_{B'})\):

\[
\begin{array}{c}
0 \\
Y \\
\downarrow f \\
G(X) \\
\downarrow u_f \\
\Sigma_{B'} Y
\end{array}
\]

If \(G(X)\) is homotopically trivial, then \(G(X) \xrightarrow{1} G(X)\) is 0 in \(K(G \downarrow \text{sm} 1_{B'})\), and hence \(G(X) \xrightarrow{u_f} C(f)\) is isomorphic to \(0 \to \Sigma_{B'} Y\).

(3) For a morphism \(Y \xrightarrow{f} G(X)\) of complexes of \(B'\), we have a morphism between triangles in \(K(G \downarrow \text{sm} 1_{B'})\):

\[
\begin{array}{ccccccc}
0 & \xrightarrow{0} & G(X) & \xrightarrow{1} & G(X) & \xrightarrow{1} & 0 \\
0 & \xrightarrow{0} & G(X) & \xrightarrow{1} & G(X) & \xrightarrow{1} & 0 \\
Y & \xrightarrow{f} & G(X) & \xrightarrow{u_f} & C(f) & \xrightarrow{u_f} & \Sigma Y \\
Y & \xrightarrow{f} & G(X) & \xrightarrow{u_f} & C(f) & \xrightarrow{u_f} & \Sigma Y \\
G(X) & \xrightarrow{1} & G(X) & \xrightarrow{1} & C(1_{G(X)}) & \xrightarrow{1} & \Sigma G(X)
\end{array}
\]

If \(C(f)\) is homotopically trivial, then \(f : Y \to G(X)\) is an isomorphism in \(K(B')\), and therefore \(0, f : (0 \to Y) \to (0 \to G(X))\) is an isomorphism in \(K(G \downarrow \text{sm} 1_{B'})\). By the above morphism between triangles, \(G(X) \xrightarrow{u_f} C(f)\) is isomorphic to \(G(X) \xrightarrow{u_f} C(1_{G(X)})\) in in \(K(G \downarrow \text{sm} 1_{B'})\). \(\square\)

**Definition 4.5.** We define the following functors:

\[
D_{[s,t]} : \text{Mor}_{N-1}(B) \to \text{Mor}_{t-s+1}(B) \quad (1 \leq s \leq t \leq N - 1)
\]

\[
E^{s,N-1}_{r} : \text{Mor}_{r}(B) \to \text{Mor}_{N-1}(B) \quad (1 \leq r \leq N - 2)
\]

\[
U_{N-1} : B \to \text{Mor}_{N-1}(B)
\]

as follows. For \(X^1 \xrightarrow{\alpha^1_X} \cdots \xrightarrow{\alpha^{N-2}_X} X^{N-1} \in \text{Mor}_{N-1}(B)\),

\[
D_{[s,t]}(X^1 \xrightarrow{\alpha^1_X} \cdots \xrightarrow{\alpha^{N-2}_X} X^{N-1}) = Y^1 \xrightarrow{\alpha^1_Y} \cdots \xrightarrow{\alpha^{t-s}_Y} Y^{t-s+1}
\]
where \( Y^i = X^{i+s-1}, \alpha_Y = \alpha_X^{i+s-1} \) (\( 1 \leq i \leq t-s+1 \)). We denote \( D_{[s,t]} \) by \( D_{|s|} \).

For \( X^1 \xrightarrow{\alpha_X^1} \cdots \xrightarrow{\alpha_X^{r-1}} X^r \in \text{Mor}^{sm}_{r}(B) \),

\[
E_{r-1}^N (X^1 \xrightarrow{\alpha_X^1} \cdots \xrightarrow{\alpha_X^{r-1}} X^r) = Y^1 \xrightarrow{\alpha_Y^1} \cdots \xrightarrow{\alpha_Y^{N-2}} Y^{N-1}
\]

\( Y^i = \begin{cases} 0 (1 \leq i < N-r) \\ X^{1-N+r+1} (N-r \leq i \leq N-1) \end{cases}, \alpha_Y = \begin{cases} 0 (1 \leq i < N-r) \\ \alpha_X^{1-N+r+1} (N-r \leq i \leq N-1) \end{cases} \)

For \( X \in B \),

\[
U_{N-1}(X) = Y^1 \xrightarrow{\alpha_Y^1} \cdots \xrightarrow{\alpha_Y^{N-2}} Y^{N-1}
\]

where \( Y^i = X, \alpha_Y = 1_X \) (\( 1 \leq i \leq N-1 \)). Moreover, we use the same symbols for the corresponding functors \( D_{[s,t]} : K(\text{Mor}^{sm}_{N-1}(B)) \to K(\text{Mor}^{sm}_{s-1}(B)), E_{N-1}^s : K(\text{Mor}_{r}(B)) \to K(\text{Mor}_{N-1}(B)) \) and \( U_{N-1} : K(B) \to K(\text{Mor}_{N-1}(B)) \).

**Definition 4.6.** We define the following full triangulated subcategories of \( K(\text{Mor}^{sm}_{N-1}(B)) \):

\[
E_{[2,N-1]} = \text{Ker} D_{[1]} \quad E_{[1,N-2]} = \text{Ker} D_{[N-1]} \quad E_{[1]} = \text{Ker} D_{[2,N-1]}
\]

\[
E_\alpha = \text{Ker} D_{[1,s-1]} \cap \text{Ker} D_{[s+1,N-1]} \quad E_{N-1}^N = \text{Ker} D_{[1,N-2]}
\]

For \( 1 \leq s < t \leq N-1 \), \( \mathcal{F}_{[s,t]} \) is the full triangulated subcategory of \( K(\text{Mor}^{sm}_{N-1}(B)) \) consisting of objects \( X^1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{N-2}} X^{N-1} \) such that \( \alpha_s = \cdots = \alpha_{t-1} = 1 \).

Immediately, we have the following.

**Proposition 4.7.** The following hold.

1. A functor \( E_{N-1}^s : K(\text{Mor}_{N-2}(B)) \to K(\text{Mor}_{N-1}(B)) \) induces a triangle equivalence between \( K(\text{Mor}_{N-2}(B)) \) and \( E_{[2,N-1]} \).
2. A functor \( U_{N-1} : K(B) \to K(\text{Mor}_{N-1}(B)) \) induces a triangle equivalence between \( K(B) \) and \( \mathcal{F}_{[1,N-1]} \).

**Proof.** (1) By Lemma 4.4, every complex of \( E_{[2,N-1]} \) is isomorphic to some complex of the form

\[
0 \xrightarrow{} X^2 \xrightarrow{} \cdots \xrightarrow{} X^{N-1}
\]

Then it is easy to see that a triangle functor \( E_{N-1}^s : K(\text{Mor}_{N-2}(B)) \to E_{[2,N-1]} \) is a triangle equivalence.

(2) Every complex of \( \mathcal{F}_{[1,N-1]} \) is of the form

\[
X^1 \xrightarrow{} X^2 \xrightarrow{} \cdots \xrightarrow{} X^{N-1}
\]

Then it is easy to see that a triangle functor \( U_{N-1} : K(B) \to \mathcal{F}_{[1,N-1]} \) is a triangle equivalence.

**Theorem 4.8.** Let \( B \) be an additive category. Then a 2N-tuple of full subcategories

\[
(\mathcal{F}_{[1,N-1]}, E_{[2,N-1]}, E_{[1]}, \mathcal{F}_{[1,2]}, \cdots, E_{s}, \mathcal{F}_{[s,s+1]}, \cdots, E_{N-2}, \mathcal{F}_{[N-2,N-1]}, E_{N-1}, E_{[1,N-2]})
\]

is a 2N-gon of recollements in \( K(\text{Mor}^{sm}_{N-1}(B)) \).

**Proof.** First, we prove \( \text{Hom}_{K(\text{Mor}^{sm}_{N-1}(B))}(\mathcal{X}, \mathcal{Y}) = 0 \), where \( \mathcal{X}, \mathcal{Y} \) are two successive subcategories of the above 2N-tuple. By Example 4.3, Lemma 4.4 (2), any complex of \( E_{[2,N-1]} \) is isomorphic to a complex \( 0 \to X^2 \to \cdots \to X^{N-1} \) in \( K(\text{Mor}^{sm}_{N-1}(B)) \).

Then \( \text{Hom}_{K(\text{Mor}^{sm}_{N-1}(B))}(\mathcal{F}_{[1,N-1]}, E_{[2,N-1]}) = 0 \) is easy. By Example 4.3, Lemma
4.4 (3), for any object of $\mathcal{E}^1$ there is a complex $X$ such that it is isomorphic to an object $X \to C(1_X) \to \cdots \to C(1_X)$ in $K(\text{Mor}_{N-2}^\text{sm}(\mathcal{B}))$. Since it is easy to see

$$\text{Hom}_{K(\text{Mor}_{N-1}^\text{sm}(\mathcal{B}))}(\mathcal{E}^{[2,N-1]}, \mathcal{E}^1) \simeq \text{Hom}_{K(\text{Mor}_{N-1}^\text{sm}(\mathcal{B}))}(\mathcal{D}^{[2,N-1]}, \mathcal{D}^{[2,N-1]}(\mathcal{E}^1))$$

and

$$\mathcal{D}^{[2,N-1]}(X \to C(1_X) \to \cdots \to C(1_X)) = C(1_X) \to \cdots \to C(1_X)$$

is homotopically 0 in $K(\text{Mor}_{N-2}^\text{sm}(\mathcal{B}))$, $\text{Hom}_{K(\text{Mor}_{N-1}^\text{sm}(\mathcal{B}))}(\mathcal{E}^{[2,N-1]}, \mathcal{E}^1) = 0$. By Example 4.3, Lemma 4.4 (2), (3), we may assume any morphism from $\mathcal{E}^s$ to $\mathcal{F}^{[s,s+1]}$ is of the form

$$0 \to \cdots \to 0 \to X^s \to C(1_X) \to \cdots \to C(1_X)$$

$$Y^1 \to \cdots \to Y^{s-1} \to Y^s \to Y^{s+1} \to \cdots \to Y^{N-1}$$

It is easy that the above morphism is null homotopic, and then $\text{Hom}_{K(\text{Mor}_{N-1}^\text{sm}(\mathcal{B}))}(\mathcal{E}^s, \mathcal{F}^{[s,s+1]}) = 0$. Similarly, we may assume any morphism from $\mathcal{F}^{[s,s+1]}$ to $\mathcal{E}^{s+1}$ is of the form

$$X^1 \to \cdots \to X^s \to X^{s+1} \to X^{s+2} \to \cdots \to X^{N-1}$$

$$0 \to \cdots \to 0 \to Y \to C(1_Y) \to \cdots \to C(1_Y)$$

It is easy that the above morphism is null homotopic, and then $\text{Hom}_{K(\text{Mor}_{N-1}^\text{sm}(\mathcal{B}))}(\mathcal{F}^{[s,s+1]}, \mathcal{E}^{s+1}) = 0$. Since any complex of $\mathcal{E}^{N-1}$ is isomorphic to $0 \to \cdots \to 0 \to X^{N-1}$, and any complex of $\mathcal{E}^{[1,N-2]}$ is isomorphic to $Y^1 \to \cdots \to Y^{N-2} \to C(1_Y)$ in $\text{Mor}_{N-1}^\text{sm}(\mathcal{B})$, it is easy to see that $\text{Hom}_{K(\text{Mor}_{N-1}^\text{sm}(\mathcal{B}))}(\mathcal{E}^{N-1}, \mathcal{E}^{[1,N-2]}) = 0$. Since we may assume any morphism from $\mathcal{E}^{[1,N-2]}$ to $\mathcal{F}^{[1,N-1]}$ is of the form

$$X^1 \to \cdots \to X^{N-2} \to C(1_X)$$

$$Y^1 \to \cdots \to Y^{N-2} \to Y^{N-1}$$

It is easy to see it is null homotopic, and then $\text{Hom}_{K(\text{Mor}_{N-1}^\text{sm}(\mathcal{B}))}(\mathcal{E}^{[1,N-2]}, \mathcal{F}^{[1,N-1]}) = 0$.

Second, we prove $K(\text{Mor}_{N-1}^\text{sm}(\mathcal{B})) = X \ast \mathcal{Y}$, where $X, \mathcal{Y}$ are two successive subcategories of the above 2N-tuple. Let $X^1 \xrightarrow{\alpha^1} X^2 \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^{N-2}} X^{N-1}$ be a complex of $K(\text{Mor}_{N-1}^\text{sm}(\mathcal{B}))$. Since we have a short exact sequence of complexes:

$$0 \to \text{Cok} \alpha^1 \to \cdots \to \text{Cok} \alpha^{N-2} \to \text{Cok} \alpha^1$$
we have $K(\text{Mor}_{N-1}^{\rm sm}(B)) = \mathcal{F}^{[1,N-1]} * \mathcal{E}^{[2,N-1]}$. Since we have a triangle in $K(\text{Mor}_{N-1}^{\rm sm}(B))$:

\[
\begin{array}{cccccc}
0 & \rightarrow & a^1 & X^2 & a^2 & \cdots & a^{N-2} & X^{N-1} \\
\downarrow & & 1 & & & & 1 \\
X^1 & \rightarrow & a^1 & X^2 & a^2 & \cdots & a^{N-2} & X^{N-1} \\
\downarrow & & & & & & & \\
X^1 & \rightarrow & C(1_{X^2}) & \cdots & C(1_{X^{N-1}}) \\
\downarrow & & & & & & & \\
0 & \rightarrow & \Sigma a^1 & \Sigma X^2 & \Sigma a^2 & \cdots & \Sigma a^{N-2} & \Sigma X^{N-1}
\end{array}
\]

we have $K(\text{Mor}_{N-1}^{\rm sm}(B)) = \mathcal{E}^{2,N-1} * \mathcal{E}^{1}$. Since we have a triangle in $K(\text{Mor}_{N-1}^{\rm sm}(B))$:

\[
\begin{array}{cccccc}
X^1 & \rightarrow & a^1 & \cdots & a^{s-2} & X^{s-1} & a^{s-1} & X^s & a^{s+1} & \cdots & a^{N-2} & X^{N-1} \\
\downarrow & & 1 & & & & & & & & 1 \\
X^1 & \rightarrow & a^1 & \cdots & a^{s-2} & X^{s-1} & a^{s-1} & X^s & a^{s+1} & \cdots & a^{N-2} & X^{N-1} \\
\downarrow & & & & & & & & & & & \\
C(1_{X^1}) & \rightarrow & \cdots & C(1_{X^{s-1}}) & C(a^s) & a^{s+1} & C(1_{X^{s+1}}) & \cdots & C(1_{X^{N-1}}) \\
\downarrow & & & & & & & & & & & \\
\Sigma X^1 & \rightarrow & \Sigma a^1 & \Sigma X^2 & \Sigma a^2 & \cdots & \Sigma a^{s-1} & \Sigma X^s & \Sigma a^{s+1} & \cdots & \Sigma a^{N-2} & \Sigma X^{N-1}
\end{array}
\]

we have $K(\text{Mor}_{N-1}^{\rm sm}(B)) = \mathcal{E}^s * \mathcal{F}^{[s,s+1]}$. Since we have a triangle in $K(\text{Mor}_{N-1}^{\rm sm}(B))$:

\[
\begin{array}{cccccc}
X^1 & \rightarrow & \cdots & a^{s-1} & X^s & \cdots & a^{s+1} & a^{s+2} & \cdots & a^{N-2} & X^{N-1} \\
\downarrow & & 1 & & & & & & & & 1 \\
X^1 & \rightarrow & \cdots & a^{s-1} & X^s & \cdots & a^{s+1} & a^{s+2} & \cdots & a^{N-2} & X^{N-1} \\
\downarrow & & & & & & & & & & & \\
C(1_{X^1}) & \rightarrow & \cdots & C(1_{X^{s-1}}) & C(a^s) & a^{s+1} & C(1_{X^{s+1}}) & \cdots & C(1_{X^{N-1}}) \\
\downarrow & & & & & & & & & & & \\
\Sigma X^1 & \rightarrow & \Sigma a^1 & \Sigma X^2 & \Sigma a^2 & \cdots & \Sigma a^{s-1} & \Sigma X^s & \Sigma a^{s+1} & \cdots & \Sigma a^{N-2} & \Sigma X^{N-1}
\end{array}
\]
we have $K(Mor^\text{sm}_{N-1}(B)) = \mathcal{F}^{[s, s+1]} * \mathcal{E}^{s+1}$. Since we have a triangle in $K(Mor^\text{sm}_{N-1}(B))$:

$$
\begin{array}{cccccc}
0 & \rightarrow & \cdots & \rightarrow & 0 & \rightarrow & X^{N-1} \\
\downarrow & & & & \downarrow & & 1 \\
X^1 & \rightarrow & \cdots & \rightarrow & X^{N-2} & \rightarrow & X^{N-1} \\
\alpha^1 & & \alpha^{N-3} & & \alpha^{N-2} & & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^1 & \rightarrow & \cdots & \rightarrow & X^{N-2} & \rightarrow & C(1_{X^{N-1}}) \\
\downarrow & & & & \downarrow & & \Sigma X^{N-1} \\
0 & \rightarrow & \cdots & \rightarrow & 0 & \rightarrow & \Sigma X^{N-1}
\end{array}
$$

we have $K(Mor^\text{sm}_{N-1}(B)) = \mathcal{E}^{N-1} * \mathcal{E}^{[1, N-2]}$. Since we have a triangle in $K(Mor^\text{sm}_{N-1}(B))$:

$$
\begin{array}{cccccc}
X^1 & \rightarrow & \cdots & \rightarrow & X^{N-2} & \rightarrow & X^{N-1} \\
\alpha^1 & & \alpha^{N-3} & & \alpha^{N-2} & & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^{N-1} & \rightarrow & \cdots & \rightarrow & X^{N-1} & \rightarrow & X^{N-1} \\
\downarrow & & & & \downarrow & & \downarrow \\
C(\alpha^{N-2} \cdots \alpha^1) & \rightarrow & \cdots & \rightarrow & C(\alpha^{N-2}) & \rightarrow & C(1_{X^{N-1}}) \\
\downarrow & & & & \downarrow & & \downarrow \\
\Sigma X^1 & \rightarrow & \cdots & \rightarrow & \Sigma X^{N-2} & \rightarrow & \Sigma X^{N-1}
\end{array}
$$

we have $K(Mor^\text{sm}_{N-1}(B)) = \mathcal{E}^{N-1} * \mathcal{F}^{[1, N-1]}$.

5. Recollement of $K_N(B)$

We fix a positive integer $N \geq 2$. Let $B$ be an additive category. An $N$-complex is a diagram

$$
\cdots \rightarrow d^{i-1}_X X^i \rightarrow d^i_X X^{i+1} \rightarrow d^{i+1}_X \rightarrow \cdots
$$

with objects $X^i \in B$ and morphisms $d^i_X \in \text{Hom}_B(X^i, X^{i+1})$ satisfying

$$
d^{i+N-1}_X \cdots d^{i+1}_X d^i_X = 0
$$

for any $i \in \mathbb{Z}$.

A morphism between $N$-complexes is a commutative diagram

$$
\cdots \rightarrow d^{i-1}_X \rightarrow d^i_X \rightarrow d^{i+1}_X \rightarrow \cdots \\
\downarrow f^i \rightarrow \downarrow f^{i+1} \\
\cdots \rightarrow d^{i-1}_Y \rightarrow d^i_Y \rightarrow d^{i+1}_Y \rightarrow \cdots
$$

with $f^i \in \text{Hom}_B(X^i, Y^i)$ for any $i \in \mathbb{Z}$. We denote by $C_N(B)$ the category of $N$-complexes. A collection $S_N(B)$ of conflations is the collection of short exact sequences of $N$-complexes of which each term is a split short exact sequence in $B$.

**Proposition 5.1** ([IKM2]). A category $(C_N(B), S_N(B))$ is a Frobenius category.
Definition 5.2 ([IKM2]). Let \((X, d), (Y, e)\) be objects and \(f : Y \to X\) be a morphism in \(C_N(B)\). Then the mapping cone \(C(f)\) of \(f\) is given as

\[
C(f)^m = X^m \oplus \prod_{i=m+1}^{m+N-1} Y^i, d_{C(f)}^m = \begin{pmatrix}
\varepsilon & 0 & \cdots & 0 \\
0 & \varepsilon & 1 & \cdots & \cdots \\
0 & \cdot & \cdot & \cdots & \cdots \\
\cdot & \cdot & \cdot & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 1 \\
0 & -e(N-1) & \cdots & \cdots & -e \\
\end{pmatrix}
\]

Here \(d^{(N-1)}\) means the \((N-1)\)-power of \(d\).

The above mapping cone induces a morphism between conflations:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{u_X} & C(1_X) & \xrightarrow{v_X} & \Sigma X & \longrightarrow & 0 \\
\downarrow{f} & & \downarrow{\psi_f} & & \downarrow{v_f} & & \downarrow{\psi_f} & & \downarrow{0} \\
0 & \longrightarrow & Y & \xrightarrow{u_f} & C(f) & \xrightarrow{v_f} & \Sigma X & \longrightarrow & 0 \\
\end{array}
\]

Let \(I(X) = C(1_X)\), then \(I(X)\) is a projective-injective object in \(C_N(B)\). We call a sequence \(Y \xrightarrow{f} X \xrightarrow{u_f} C(1_X) \xrightarrow{v_f} \Sigma X\) a (distinguished) triangle.

A morphism \(f : X \to Y\) of \(N\)-complexes is called null-homotopic if there exists \(s^i \in \text{Hom}_B(X^i, Y^{i+N-1})\) such that

\[
f^i = \sum_{j=1}^{N-1} d_Y^{i-1} \cdots d_Y^{-j+1} s^{i+j-1} d_{X}^{i+j-2} \cdots d_X^{i}
\]

for any \(i \in \mathbb{Z}\). We denote by \(K_N(B)\) the homotopy category of \(N\)-complexes.

Theorem 5.3 ([IKM2]). A category \(K_N(B)\) is a triangulated category.

Definition 5.4. Let \(N\) be an integer greater than 2. For any integer \(s\), we define functions \(\iota^{(N-1)}_s : \mathbb{Z} \rightarrow \mathbb{Z}, \rho^{(N)}_s : \mathbb{Z} \rightarrow \mathbb{Z}\) as follows.

\[
\iota^{(N-1)}_s(s + i + kN) = \begin{cases} 
  s + k(N - 1) & (i = 0) \\
  s + i - 1 + k(N - 1) & (0 < i < N) 
\end{cases}
\]

\[
\rho^{(N)}_s(s + i + k(N - 1)) = \begin{cases} 
  s + kN & (i = 0) \\
  s + i + 1 + kN & (0 < i < N - 1) 
\end{cases}
\]

For an \((N-1)\)-complex \(X = (X^i, d_X^i)\), we define a complex \(I^{(N-1)}_s(X)\) by

\[
I^{(N-1)}_s(X)^i = X^{\iota^{(N-1)}_s(i)}, \\
d^{(N-1)}_{I_s}(X) = \begin{cases} 
  d^{(N-1)}_{s^{(i)}_s(i)} & (\iota^{(N-1)}_s(i) < \iota^{(N-1)}_s(i + 1)) \\
  1 & (\iota^{(N-1)}_s(i) = \iota^{(N-1)}_s(i + 1)).
\end{cases}
\]
Lemma 5.5. A functor $I_s^{(N-1)} : \mathcal{C}_{N-1}(\mathcal{B}) \to \mathcal{C}_N(\mathcal{B})$ induces the triangle functor $I_s^{(N-1)} : \mathcal{K}_{N-1}(\mathcal{B}) \to \mathcal{K}_N(\mathcal{B})$.

Proof. Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{C}_{N-1}(\mathcal{B})$. Consider the mapping cone $C(I_0^{(N-1)}(f))$ of $I_0^{(N-1)}(X) \xrightarrow{I_0^{(N-1)}(f)} I_0^{(N-1)}(Y)$, then we have an exact sequence $0 \to I_0^{(N-1)}(Y) \to C(I_0^{(N-1)}(f)) \to \Sigma(I_0^{(N-1)}(X)) \to 0$ in $\mathcal{C}_{N-1}(\mathcal{B})$. Let $Z$ be an $N$-complex defined by

$$Z^k = \begin{cases} Y_0^{(N-1)}(k) \oplus \bigoplus_{i=k+1}^{k+N-2} X_0^{(N-1)}(i) \oplus X_0^{(N-1)}(k) & (k \equiv 0 \text{ mod } N) \\ Y_0^{(N-1)}(k) \oplus \bigoplus_{i=k+1}^{k+N-1} X_0^{(N-1)}(i) & (k \not\equiv 0 \text{ mod } N) \end{cases}$$

and $W$ an $N$-complex defined by

$$W^k = \begin{cases} \bigoplus_{i=k+1}^{k+N-2} X_0^{(N-1)}(i) \oplus X_0^{(N-1)}(k) & (k \equiv 0 \text{ mod } N) \\ \bigoplus_{i=k+1}^{k+N-1} X_0^{(N-1)}(i) & (k \not\equiv 0 \text{ mod } N) \end{cases}$$

For an $N$-complex $Y = (Y^i, d^i_Y)$, we define a complex $J_{s}^{(N)}(Y)$ by

$$J_{s}^{(N)}(X)^i = X_{s}^{(N)}(i),$$

$$d^j_{J_{s}^{(N)}}(X) = d^j_{I_{s}^{(N)}}(i+1) \ldots d^j_{I_{s}^{(N)}}(i).$$

Then $I_{s}^{(N-1)} : \mathcal{C}_{N-1}(\mathcal{B}) \to \mathcal{C}_N(\mathcal{B})$ and $J_{s}^{(N)} : \mathcal{C}_N(\mathcal{B}) \to \mathcal{C}_{N-1}(\mathcal{B})$ are functors.
Let \( g : C(I_0^{(N-1)}(f)) \to Z \) be an isomorphism defined by

\[
g^k = \begin{pmatrix}
1 & f & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-1 & -1 & -1 & \cdots & -1
\end{pmatrix} \quad (k \equiv 0 \, \text{mod} \, N),
\]

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix} \quad (k \equiv 1 \, \text{mod} \, N),
\]

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix} \quad (k \equiv 2 \, \text{mod} \, N),
\]

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix} \quad (k \equiv 3 \, \text{mod} \, N),
\]

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix} \quad (k \equiv N-1 \, \text{mod} \, N)
\]

Then we have the following isomorphism between short exact sequences:

\[
0 \longrightarrow I_0^{(N-1)}(Y) \longrightarrow C(I_0^{(N-1)}(f)) \longrightarrow \Sigma(I_0^{(N-1)}(X)) \longrightarrow 0
\]

\[
0 \longrightarrow I_0^{(N-1)}(Y) \xrightarrow{u} Z \xrightarrow{\Sigma} W \longrightarrow 0
\]

where \( u' = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v' = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \). Therefore, \( C(I_0^{(N-1)}(f)) \simeq I_0^{(N-1)}(C(f)) \)
and \( \Sigma(I_0^{(N-1)}(X)) \simeq I_0^{(N-1)}(\Sigma(X)) \) in \( K_N(B) \).

\[\Box\]

**Proposition 5.6.** For a functor \( J_s^{(N)} : C_N(B) \to C_{N-1}(B) \), the following hold.

1. \( J_s^{(N)} \) induces the triangle functor \( J_s^{(N)} : K_N(B) \to K_{N-1}(B) \).
2. \( J_s^{(N)} \) is a right adjoint of \( I_s^{(N-1)} \).
3. \( J_s^{(N-1)} \) is a right adjoint of \( J_s^{(N)} \).
4. The adjunction arrow \( \mathbf{1} \to J_s^{(N)} I_s^{(N-1)} \) is an isomorphism.
5. The adjunction arrow \( J_s^{(N)} I_s^{(N-1)} \to \mathbf{1} \) is an isomorphism.
Definition 5.7. We denote \( L^{(N-r)}_N = L^{(N-1)}_N \ldots L^{(N-r)}_N : K_{N-r}(B) \to K_N(B) \) and \( L^{\geq N-r} = L^{(N-r+1)}_N \ldots L^{(N)}_N : K_N(B) \to K_{N-r}(B) \). For \( 1 \leq r < N \), we define the full subcategory of \( K_N(B) \)

\[
\mathcal{F}_r = \{ (X, d) \in K_N(B) \mid d^{i+k} = 1_X, \forall i \forall k (0 \leq k < r) \text{ for } i \equiv s \mod N \}
\]

Corollary 5.8. \( L^{\leq N-r}_r, L^{\geq N-r}_r \) are triangle functors such that \( L^{\geq N-r}_r \) is a right adjoint of \( L^{\leq N-r}_r \) (resp., \( L^{\geq N-r}_N \)), and that \( \mathcal{F}_r = \text{im} L^{\geq N-r}_r \) is a full triangulated subcategory of \( K_N(B) \).

Proof. By Lemma 5.5, Proposition 5.6. \( \square \)

Theorem 5.9. For \( 1 \leq r < N \), \( (\mathcal{F}_r, \mathcal{F}_r^{N-r+1}) \) is a stable t-structure in \( K_N(B) \).

Proof. We may assume \( s = 0 \). It is easy to see that \( \text{Hom}_{K_N(B)}(\mathcal{F}_r, \mathcal{F}_r^{N-r+1}) = 0 \).

Let \( X \) be an \( N \)-complex. By Proposition 5.6, the adjunction arrow induce a triangle

\[
L^{\geq N-r}_r X \xrightarrow{\varepsilon_X} X \xrightarrow{u} C(\varepsilon_X) \xrightarrow{v} \Sigma L^{\leq N-r}_r X \xrightarrow{d} L^{\leq N-r}_r X.
\]

\( \square \)
Let \( \sigma := \tau_0^{(N-1)} \cdots \tau_0^{(N-r)} \tau_0^{(N-r+1)} \cdots \tau_0^{(N)} \), and \( V \) be an \( N \)-complex defined by

\[
V^k = \begin{cases}
\bigoplus_{i=k+1}^{k+N-1} X^\sigma(i) & (k \equiv 0, \cdots, r \mod N) \\
X^k \oplus \bigoplus_{i=k+1}^{k+N-1} X^\sigma(i) & (k \equiv r+1, \cdots, N-1 \mod N)
\end{cases}
\]

Let \( h : V \to C(\epsilon_X) \) be a monomorphism defined by

\[
h^k = \begin{pmatrix}
0 & \cdots & 0 & 0 \\
1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & d & 1 \\
0 & \cdots & -d & 1
\end{pmatrix}
\]

(\( k \equiv r \mod N \)),

\[
N-r \begin{pmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
-d & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & -d & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -1 & 1 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & -1 & 1 \\
0 & \cdots & 0 & 0 & \cdots & 0 & -1
\end{pmatrix}
\]

(\( k \equiv r+1 \mod N \)),

\[
N-r \begin{pmatrix}
1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
-d & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & -d & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & -1 & 1 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & -1 & 1 \\
0 & \cdots & 0 & 0 & \cdots & 0 & -1
\end{pmatrix}
\]

(\( k \equiv r+2 \mod N \)),

\[
r+1 \begin{pmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
-1 & 1 & \cdots & 0 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 1 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & -1 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & \cdots & -1 & 1 \\
0 & \cdots & 0 & 0 & \cdots & 0 & -1
\end{pmatrix}
\]

(\( k \equiv 0 \mod N \)),
We have a recollement of Corollary 5.10. Therefore, we have a triangle \( \triangledown \) in
\[ \begin{array}{c|c|c}
\text{} & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 0 \\
\hline
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \]
\((k \equiv 1 \mod N), \cdots \)

Then we have the following conflations.
\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
-1 & 1 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -1 \\
0 & \cdots & 0 & 0
\end{pmatrix}
\]
\((k \equiv r - 1 \mod N)\)

Let \( p : C(\xi_X) \to \bigoplus_{r+1}^N \bigoplus_{r+1}^N \) be an epimorphism defined by
\[
p^k = \begin{cases}
1 & 0 & \cdots & 0 \\
dN-r & dN-r-2 & \cdots & d \\
dN-r-2 & dN-r-3 & \cdots & d \\
& \vdots & \ddots & \vdots \\
& 1 & \cdots & 0 \\
& 0 & \cdots & -d
\end{cases}
\]
\((k \equiv r \mod N), \cdots \)

Then we have the following conflations.
\[
0 \to \bigoplus_{r+1}^N \bigoplus_{r+1}^N(X) \to 0
\]

Since \( V \) is a projective object in \( C_N(B) \), we have an isomorphism in \( K_N(B) \)
\[
C(\xi_X) \simeq \bigoplus_{r+1}^N \bigoplus_{r+1}^N(X)
\]

Therefore, we have a triangle \( U \to X \to V \to \Sigma X \) such that \( U \in \mathcal{F}^r_s, V \in \mathcal{F}^r_{s+1} \).

**Corollary 5.10.** We have a recollement of \( K_N(B) \):
\[
\begin{array}{ccc}
K_{N-r}(B) & \xrightarrow{i_{rs}} & K_N(B) \\
\xleftarrow{j_{rs}} & & \xrightarrow{j_{rs}} K_{r+1}(B)
\end{array}
\]

where \( i_s = \bigoplus_{r+1}^N, j_s = \bigoplus_{r+1}^N \), \( i_s = \bigoplus_{r+1}^N, j_s = \bigoplus_{r+1}^N \), \( j_s = \bigoplus_{r+1}^N \) and \( j_s = \bigoplus_{r+1}^N \).

**Proof.** By Theorem 5.9, \( \mathcal{F}^{N-r+1}_s \) and \( \mathcal{F}^{N-r+1}_s \) are stable t-structures in \( K_N(B) \). By the proof of Theorem 5.9, the adjunction arrows induce triangles in \( K(B) \). Hence we have the statement. \( \square \)
Corollary 5.11. For any integer \( s \),
\[
(F^{N-2}_{s+1}, F^1_s, F^1_{s+2}, F^1_{s+3}, \ldots, F^1_{s+r}, \ldots, F^1_{s+N-1}, F^{N-2}_{s+r}, \ldots, F^{N-2}_{s+2N-2}, F^1_{s+2N-1})
\]
is a 2N-gon of recollements in \( K_N(B) \).

6. Triangle equivalence between homotopy categories

\[
\mu^{s}_{r} C : X^{s+r} \longrightarrow X^{s+r+1} \longrightarrow \cdots \longrightarrow X^{s-1} \longrightarrow X^{s}
\]
be an \( N \)-complex satisfying that \( X^{s-i} = C \) (\( 0 \leq i \leq r - 1 \)), \( d^{s-i} = 1_C \) (\( 0 < i \leq r - 1 \)).

Lemma 6.1. Let \( B \) be an additive category. Consider \( \text{Mor}_{N-1}^\text{sm}(B) \) as a subcategory of \( C_N(B) \), then the following hold.

1. For every object \( X \) of \( \text{Mor}_{N-1}^\text{sm}(B) \), there are objects \( C_1, \ldots, C_{N-1} \) of \( B \) such that \( X \cong \prod_{i=1}^{N-1} \mu_i^{N-1} C_i \).
2. For any \( X, Y \in \text{Mor}_{N-1}^\text{sm}(B) \), we have isomorphisms
\[
\text{Hom}_{\text{Mor}_{N-1}^\text{sm}(B)}(X, Y) = \text{Hom}_{C_N(B)}(X, Y) = \text{Hom}_{K_N(B)}(X, Y).
\]
3. For any \( X, Y \in \text{Mor}_{N-1}^\text{sm}(B) \), we have \( \text{Hom}_{K_N(B)}(X, \Sigma^j Y) = 0 \) (\( i \neq 0 \)).

Proof. (1) It is trivial.
(2) It is easy because the term-length of objects of \( \text{Mor}_{N-1}^\text{sm}(B) \) is less than \( N \).
(3) For every \( C \in B \), \( 1 \leq r \leq N - 1 \) and any integer \( i \), the canonical injection and projection induce an conflation in \( C(B) \)
\[
0 \to \mu_r^{iN+N-1} C \to \mu_r^{iN+N-1} C \to \mu_{N-r}^{(i+1)N-r-1} C \to 0.
\]
Since \( \mu_r^{iN+N-1} C \) is a projective-injective object in \( C_N(B) \), we have isomorphisms in \( K_N(B) \):
\[
\Sigma^j \mu_r^{N-1} C = \begin{cases} \mu_r^{N-r} (1-j)N/2 - r - 1 C & (j \equiv 1 \text{ mod } 2) \\ \mu_r^{N-r} (2-j)N/2 - 1 C & (j \equiv 0 \text{ mod } 2) \end{cases}
\]
For every \( C, C' \in B \) and any \( 1 \leq r, r' \leq N - 1 \), we have
\[
\text{Hom}_{K_N(B)}(\mu_r^{N-1} C, \Sigma^j \mu_r^{N-1} C') = 0 \ (j \neq 0).
\]
By (1), we have the statement. \( \square \)

Definition 6.2. For every \( C \in B \), let
\[
\Xi^j \mu_r^{N-1} C = \begin{cases} \mu_r^{N-r} (1-j)N/2 + N - r - 1 C & (j \equiv 1 \text{ mod } 2) \\ \mu_r^{N-r} (2-j)N/2 - 1 C & (j \equiv 0 \text{ mod } 2) \end{cases}
\]
By the proof of Lemma 6.1, for every \( M \in \text{Mor}_{N-1}^\text{sm}(B) \) and any \( i \in \mathbb{Z} \), there exist the projective-injective object \( I'(\Xi^i M) \) and a functorial conflation in \( C_N(B) \)
\[
0 \to \Xi^i M \overset{u^i_{M \Xi^i M}}{\longrightarrow} I'(\Xi^i M) \overset{v^i_{M \Xi^i M}}{\longrightarrow} \Xi^{i+1} M \to 0
\]
that is, every morphism \( f : M \to N \) in \( \text{Mor}_{N-1}(\mathcal{B}) \) uniquely determines morphisms \( I'(f) : I'(M) \to I'(N) \) and \( \Xi f : \Xi M \to \Xi N \) which have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Xi I' M & \longrightarrow & \Xi I'(\Xi I M) & \longrightarrow & \Xi^{i+1} M & \longrightarrow & 0 \\
| & & | & & | & & | & & |
\Xi f & \quad & I'(\Xi f) & \quad & \Xi^{i+1} f & \quad & \Xi^{i+1} f & \quad & 0 \\
0 & \longrightarrow & \Xi I' N & \longrightarrow & \Xi I'(\Xi I N) & \longrightarrow & \Xi^{i+1} N & \longrightarrow & 0
\end{array}
\]

For any (ordinary and \( N \)) complex \( X = (X^i, d^i) \), we define the following truncations:

\[
\begin{align*}
\tau_{\leq n} X : & \cdots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0 \longrightarrow \cdots , \\
\tau_{\geq n} X : & \cdots \longrightarrow 0 \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow X^{n+2} \longrightarrow \cdots , \\
\tau_{[m, n]} X : & \cdots \longrightarrow X^m \longrightarrow \cdots \longrightarrow X^n \longrightarrow 0 \longrightarrow \cdots , \\
\tau_n X : & \cdots \longrightarrow 0 \longrightarrow X^n \longrightarrow 0 \longrightarrow \cdots .
\end{align*}
\]

**Lemma 6.3.** Let \( \mathcal{B} \) be an additive category. There exists an exact functor \( F_N : C(\text{Mor}_{N-1}(\mathcal{B})) \to C_N(\mathcal{B}) \) which sends \( \text{Mor}_{N-1}(\mathcal{B}) \) to \( \text{Mor}_{N-1}(\mathcal{B}) \) as a subcategory of \( K_N(\mathcal{B}) \) such that \( F_N \) induces a triangle functor \( F_N : K(\text{Mor}_{N-1}(\mathcal{B})) \to K_N(\mathcal{B}) \).

**Proof.** Let \( T : C(\text{Mor}_{N-1}(\mathcal{B})) \to C(\text{Mor}_{N-1}(\mathcal{B})) \) be a translation functor. We will prove the statement by the following steps:

Step 1. We have a functor \( F_N : \prod_i T^i \mathcal{B} \to C_N(\mathcal{B}) \) which preserves split exact sequences.

For \( T^i X \in T^i \text{Mor}_{N-1}(\mathcal{B}) \), let \( F_N(T^i X) = \Xi X \), then \( F_N : \prod_i T^i \text{Mor}_{N-1}(\mathcal{B}) \to K_N(\mathcal{B}) \) which preserves split exact sequences.

Step 2. We have a functor \( F_N : C^b(\text{Mor}_{N-1}(\mathcal{B})) \to C_N(\mathcal{B}) \) which preserves conflations.

For \( i < j \), let \( X : X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots \longrightarrow X^j \in C^b(\text{Mor}_{N-1}(\mathcal{B})) \). By induction on \( j - i \), we construct a functor \( F_N \). For \( X \), we have a commutative diagram in \( C^b(\mathcal{B}) \):

\[
\begin{array}{cccccc}
0 & \longrightarrow & T^{-i} X^i & \longrightarrow & I(T^{-i} X^i) & \longrightarrow & T^{-i+1} X^i & \longrightarrow & 0 \\
| & & | & & | & & | & & |
| & & | & & | & & | & & |
\tau_{\geq i+1} X & \longrightarrow & \tau_{\geq i} X & \longrightarrow & T^{-i+1} X^i & \longrightarrow & 0 \\
0 & \longrightarrow & T^{-i} X^i & \longrightarrow & I(T^{-i} X^i) & \longrightarrow & T^{-i+1} X^i & \longrightarrow & 0
\end{array}
\]

where all rows are exact. Then we have a commutative diagram in \( C_N(\mathcal{B}) \):

\[
(6.4)
\begin{array}{cccccc}
0 & \longrightarrow & F_N(T^{-i} X^i) & \longrightarrow & I(F_N(T^{-i} X^i)) & \longrightarrow & F_N(T^{-i+1} X^i) & \longrightarrow & 0 \\
| & & | & & | & & | & & |
| & & | & & | & & | & & |
F_N(d^i) & (A) & \gamma_i & & & & & & \\
0 & \longrightarrow & F_N(\tau_{\geq i+1} X) & \longrightarrow & C(F_N(d^i)) & \longrightarrow & F_N(T^{-i+1} X^i) & \longrightarrow & 0
\end{array}
\]

where all rows are conflations. We take \( F_N(\tau_{\geq i} X) = C(F_N(d^i)) \). By successive step, we have a functor \( F_N : C^b(\text{Mor}_{N-1}(\mathcal{B})) \to C_N(\mathcal{B}) \). For any inflation (resp., deflation) \( f : X \to Y \) in \( C^b(\mathcal{B}) \), \( F_N(T^i f) : F_N(T^i X) \to F_N(T^i Y) \) and \( I'(F_N(T^i f)) : I'(F_N(T^i X)) \to I'(F_N(T^i Y)) \) are split monomorphism (resp., epimorphism) by Definition 6.2. Then

\[
0 \to F_N(T^i X) \to F_N(\tau_{\geq i+1} X) \oplus I'(F_N(T^{-i} X^i)) \to F_N(\tau_{\geq i} X) \to 0
\]
is an inflation. Let $0 \to X \to Y \to Z \to 0$ be an conflation in $C^1(M_{N - 1}(B))$. Consider a small Frobenius subcategory $C$ of $C_{N}(M_{N - 1}(B))$ which contains $F_{N}(T^{i}X)$, $F_{N}(T^{i}Y)$, $F_{N}(T^{i}Z)$, $\iota^{i}F_{N}(\tau \geq i X)$, $\iota^{i}F_{N}(T^{i - 1}Y)$, $\iota^{i}F_{N}(T^{i - 1}Z)$, $F_{N}(\tau \geq i X)$, $F_{N}(\tau \geq i Y)$ and $F_{N}(\tau \geq i Z)$ ($i \in \mathbb{Z}$). Then by the induction on $j-i$, 9-lemma implies that

$$0 \to F_{N}(\tau \geq i X) \to F_{N}(\tau \geq i Y) \to F_{N}(\tau \geq i Z) \to 0$$

is a short exact sequence in some abelian category. According to Proposition 7.2, it is a conflation in $C$. Therefore, $F_{N}$ preserves conflations.

Step 3. We have a functor $F_{N} : C^{-}(M_{N - 1}(B)) \to C_{N}(B)$ which preserves conflations. For $X \in C^{-}(B)$, we have $X = \lim_{j \to \infty} \tau \geq -1 X$. By the diagram 6.4, we have

$$\tau \geq \frac{j}{2 + 1} F_{N}(\tau \geq i X) = \tau \geq \frac{j}{2 + 1} F_{N}(\tau \geq i - 1 X)$$

Then there exists $\lim_{i \to \infty} F_{N}(\tau \geq -1 X)$, and we take $F_{N}(X) = \lim_{i \to \infty} F_{N}(\tau \geq -1 X)$. It is not hard to see that $F_{N}$ becomes a functor and preserves conflations.

Step 4. We have a functor $F_{N} : C(M_{N - 1}(B)) \to C_{N}(B)$ which preserves conflations. Let $X : \cdots \to X^{i} \xrightarrow{d^i} X^{i+1} \to \cdots \in C(B)$. By induction on $i$, we construct a functor $F_{N}$. For $X$, we have a commutative diagram in $C^{-}(M_{N - 1}(B))$:

$$0 \to T^{i - 1} X^{i+1} \to \tau \leq i X \to \tau \leq i X \to 0$$

where all rows are exact. Then we have a commutative diagram in $C_{N}(B)$:

$$(6.5)$$

$$0 \to F_{N}(T^{i - 1} X^{i+1}) \to \Sigma^{-1} C(F_{N}(d^{i})) \to F_{N}(\tau \leq i X) \to 0$$

where all rows are conflations. We take $F_{N}(\tau \leq i X) = C(F_{N}(d^{i}))$. By the diagram 6.5, we have

$$\tau \leq \frac{j}{2 + 1} F_{N}(\tau \leq i X) = \tau \leq \frac{j}{2 + 1} F_{N}(\tau \leq i + 1 X)$$

Then there exists $\lim_{i \to \infty} F_{N}(\tau \leq i X)$, and we take $F_{N}(X) = \lim_{i \to \infty} F_{N}(\tau \leq i X)$. Since the commutative diagram (B) is an exact square, it is not hard to see that $F_{N}$ becomes a functor and preserves conflations.

Step 5. $F_{N}$ sends projective-injective objects in $C(M_{N - 1}(B))$ to projective-injective objects in $C_{N}(B)$. Every projective-injective object in $C(M_{N - 1}(B))$ is a direct summand of some biproduct $\oplus_{i \in \mathbb{Z}} T^{i} C(1_{M_{i}})$ with $M_{i} \in M_{N - 1}(B)$. Since $F_{N}(C(1_{M_{i}})) \simeq C(1_{F_{N}(M_{i})})$ is projective-injective in $C_{N}(B)$, we have the statement.

According to Proposition 7.3, $F_{N} : C(M_{N - 1}(B)) \to C_{N}(B)$ induces a triangle functor $F_{N} : K(M_{N - 1}(B)) \to K_{N}(B)$. $\square$

**Lemma 6.6.** The following hold.
(1) Every complex $X$ of $\mathcal{C}(\text{Mor}^\text{sm}_{N-1}(B))$ is of the form

$$X_1 \xrightarrow{\alpha^1} \oplus_{i=1}^{N-1} X_i \xrightarrow{\alpha^2} \cdots \xrightarrow{\alpha^{N-2}} \oplus_{i=1}^{N-1} X_i$$

where $\alpha^i = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$ and $(\oplus_{i=1}^{N-1} X_i, d^i_X = \begin{bmatrix} d_{11}^i & d_{12}^i & \cdots & d_{1N-1}^i \\ 0 & d_{22}^i & \cdots & d_{2N-1}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{N-1,N-1}^i \end{bmatrix})$ is a complex of $B$. 

(2) For a complex $X$ of $\mathcal{C}(\text{Mor}^\text{sm}_{N-1}(B))$, the $N$-complex $F_N(X)$ is equal to $(Y^j, d^j_Y)$ where

$$Y^j = \left( \oplus_{r=1}^{k} X_r^{2j} \right) \oplus \left( \oplus_{r=k}^{N-1} X_r^{2j-1} \right)$$

and

$$d^j_Y = \begin{cases} 
\begin{bmatrix} 
1 & 0 & \cdots & 0 & d_{1k}^{j-1} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & d_{2k}^{j-1} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \cdots \\
\vdots & \vdots & \cdots & 1 & d_{k-1,k}^{j-1} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & 0 & \ddots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 
\end{bmatrix} & (j \equiv -1 \mod N), \\
\begin{bmatrix} 
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 
\end{bmatrix} & \text{otherwise} 
\end{cases}$$

where $i = 2 \left\lfloor \frac{r}{N} \right\rfloor$, $0 \leq k \leq N - 1$ and $k \equiv j \mod N$.

(3) $\mathcal{E}^1_{(E^{[1,N-2]}_N)} \subset \mathcal{F}_{N-1}$

**Proof.** (1) It is trivial.

(2) For a complex $X \in \mathcal{C}(\text{Mor}^\text{sm}_{N-1}(B))$, we assume $F_N(\tau_{\geq 2j}X)$ satisfies the statement. For $2j = i + 1$, we have the following equations in the diagram 6.4

$$F_N(T^{-i}X^i) = \oplus_{r=1}^{N-1} \mu_r^{N-1+jN} X_r^{2j-1} \quad I'(F_N(T^{-i}X^i)) = \oplus_{r=1}^{N-1} \mu_r^{N-1+jN} X_r^{2j-1}$$

By easy calculations, $\gamma_{k+jN}^i : I'(F_N(T^{-i}X^i))^k \rightarrow Y^{k+jN}$ is equal to $d_{Y^j}^{k+N}$ in the statement for $0 \leq k < N$. Therefore, we have the statement for $F_N(\tau_{\geq i}X)$. For $2j = i$, we have the following equations in the Diagram 6.4

$$F_N(T^{-i}X^i) = \oplus_{r=1}^{N-1} \mu_r^{-1+(j+1)N} X_r^{2j} \quad I'(F_N(T^{-i}X^i)) = \oplus_{r=1}^{N-1} \mu_r^{-1+(j+1)N} X_r^{2j}$$

$$F_N(T^{-i}X^i) = \oplus_{r=1}^{N-1} \mu_r^{-1+jN} X_r^{2j}$$
By easy calculations, \( \gamma_j^{k+jN} : I'(F_N(T^{-i}X)) \to Y^{k+jN} \) is a \((k+1) \times k\) matrix \[
\begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix}
\] for \(1 \leq k < N-1\), and \( \gamma_j^{N-1+jN} : I'(F_N(T^{-i}X)) \to Y^{k+jN} \) is equal to \( d_Y^{N-1+jN} \) in the statement. Therefore, we have the statement for \( F_N(\tau_{\geq i}X) \).

Similarly, we have the same result for the diagram 6.5.

(3) By Lemma 4.4, any complex of \( \mathcal{E}^{[1,N-2]} \) is isomorphic to the mapping cone of the following morphism between complexes:

\[
\begin{array}{ccccccc}
X' : & 0 & 0 & \cdots & 0 & \oplus_{i=1}^{N-1} X_i \\
\downarrow f & & & & & \\
X : & X_1 & \alpha_1^1 & \oplus_{i=1}^2 X_i & \alpha_2^2 & \cdots & \alpha_{N-2}^N X_i & \oplus_{i=1}^{N-1} X_i \\
\end{array}
\]

Consider a morphism between the Diagram 6.4 for \( F_N(\tau_{\geq i}X') \) and the Diagram 6.4 for \( F_N(\tau_{\geq i}X) \). Then \( F_N(X') = (Y^j, d_Y^j) \), where

\[
Y^j = \begin{cases}
\oplus_{r=1}^{N-1} X_r & (j \equiv -1 \mod N) \\
\oplus_{r=1}^{N-1} X_r^{2i-1} & \text{otherwise}
\end{cases}
\]

\[
d_Y^j = \begin{cases}
\begin{bmatrix}
d_{i1} & d_{i2} & \cdots & d_{iN-1} \\
0 & d_{i2} & \cdots & d_{iN-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & d_{iN-1}
\end{bmatrix} & (j \equiv -2 \mod N), \\
\text{identity} & \text{otherwise}
\end{cases}
\]

where \( i = 2 \left\lfloor \frac{j}{N} \right\rfloor \), \(0 \leq k < N-1\) and \( k \equiv j \mod N\). Moreover, \( F_N(f) : F_N(X') \to F_N(X) \) is equal to \( g : Y' \to Y \) where

\[
g^j = \begin{cases}
\begin{bmatrix}
d_{i1} & \cdots & d_{ik} & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & 1 & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix} & (j \equiv 0,-1 \mod N) \\
\text{identity} & \text{otherwise}
\end{cases}
\]

where \( i = 2 \left\lfloor \frac{j}{N} \right\rfloor \), \(0 < k < N-1\) and \( k \equiv j \mod N\). By the proof of Theorem 5.9, we have \( Y' = \bigoplus_{i=0}^N F_0^i(Y) \) and \( \epsilon_Y = g \). Therefore \( C(F_N(f)) \simeq C(g) \in F_{N-1}^1 \).

\begin{lemma}
The following hold for the above triangle functor \( E_N : \mathcal{K}(\text{mor}_N^m(\mathcal{B})) \to \mathcal{K}_N(\mathcal{B}) \).
\begin{enumerate}
\item \( E_N(\mathcal{E}^{[s,N-1]} \cap \mathcal{F}^{[s,N-1]}) \subset \mathcal{F}_{s+1}^{s-1} \cap \mathcal{F}_{s+1}^{N-1} \).
\item \( E_N(\mathcal{E}^s) \subset \mathcal{F}_{s+1}^{N-2} \).
\end{enumerate}
\end{lemma}

\textbf{Proof.} (1) According to Lemma 4.4, we may assume any complex \( Y \) of \( \mathcal{E}^{[s,N-1]} \cap \mathcal{F}^{[s,N-1]} \) is of the form

\[
0 \to \cdots \to 0 \to Y^s \xrightarrow{\alpha^s} \cdots \xrightarrow{\alpha^{N-2}} Y^{N-1}
\]
Step 1. Therefore, according to Lemma 6.6 (2), it is easy to see that

\[ X^i = \begin{cases} 
X^{2[i]} - 1 & (i \equiv 0, 1, \cdots, s - 1 \mod N) \\
X^{2[i]} & (i \equiv s, s + 1, \cdots, N - 1 \mod N)
\end{cases} \]

\[ d^i = \begin{cases} 
1_X^{2[i]} & (i \equiv 0, 1, \cdots, s - 2 \mod N) \\
d^{2[i]} - 1 & (i \equiv s - 1 \mod N) \\
1_X^{2[i]} & (i \equiv s, s + 1, \cdots, N - 2 \mod N) \\
d^{2[i]} & (i \equiv N - 1 \mod N)
\end{cases} \]

Therefore \( \mathcal{E}_N(\mathcal{E}^{[s,N-1]} \cap \mathcal{F}^{[s,N-1]}) \subset \mathcal{F}^{[0,s-1]}_0 \cap \mathcal{F}^{[0,N-s-1]}_0 \).

(2) Any complex \( Y \) of \( \mathcal{E}^s \) is isomorphic to the mapping cone of a morphism \( \iota : Y_1 \to Y_2 \) in \( C(\text{Mor}_{N-1}^{\text{sm}}(\mathcal{B})) \):

\[ Y_1 : \quad 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow Y^s \alpha^s+1 \longrightarrow \cdots \longrightarrow Y^{N-1} \]

\[ Y_2 : \quad 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow Y^s \alpha^s \longrightarrow Y^s+1 \alpha^{s+1} \longrightarrow \cdots \longrightarrow Y^{N-1} \]

where \( X = (X^i, d^i) = Y^s = \cdots = Y^{N-1} \) is a complex of \( \mathcal{B} \) and \( \alpha^s = \cdots = \alpha^{N-2} = 1_X \). By the construction of \( F_N \) in Theorem 6.3, \( \mathcal{E}_N(Y_2) = \mathcal{L}_0^N \mathcal{E}_N(Y_1) \in \mathcal{F}^{[0,N-1]}_0 \) and \( \mathcal{E}_N(\iota) = \mathcal{L}_0^N \mathcal{E}_N(Y_1) \in \mathcal{F}^{[0,N-1]}_0 \). By the Proof of Theorem 5.9, the mapping cone \( C(\mathcal{E}_N(\iota)) \) is isomorphic to a complex \( (X^i, d^i) \) where

\[ X^i = \begin{cases} 
X^{2[i]} - 1 & (i \equiv 0, 1, \cdots, s - 1 \mod N) \\
X^{2[i]} & (i \equiv s, s + 1, \cdots, N - 1 \mod N)
\end{cases} \]

\[ d^i = \begin{cases} 
1_X^{2[i]} & (i \equiv 0, 1, \cdots, s - 2 \mod N) \\
d^{2[i]} - 1 & (i \equiv s - 1 \mod N) \\
1_X^{2[i]} & (i \equiv s, s + 1, \cdots, N - 1 \mod N) \\
d^{2[i]} & (i \equiv N - 1 \mod N)
\end{cases} \]

Therefore \( \mathcal{E}_N(\mathcal{E}^s) \subset \mathcal{F}^{N-2}_{s+1} \).

\[ \square \]

**Theorem 6.8.** Let \( \mathcal{B} \) be an additive category, then we have triangle equivalences:

\[ K^*(\text{Mor}_{N-1}^{\text{sm}}(\mathcal{B})) \simeq K^1_N(\mathcal{B}) \]

where \( \mathcal{I} = \text{nothing}, -, +, b \).

**Proof.** We prove the statement by the following steps:

Step 1. \( \mathcal{E}_N : K(\text{Mor}_{N-1}^{\text{sm}}(\mathcal{B})) \to K_N(\mathcal{B}) \) sends

\((\mathcal{F}^{[1,N-1]}, \mathcal{E}^{[2,N-1]}, \mathcal{F}^{[1,2]}, \cdots, \mathcal{E}^{s}, \mathcal{F}^{[s,s+1]}, \cdots, \mathcal{E}^{N-2}, \mathcal{F}^{[N-2,N-1]}, \mathcal{E}^{N-1}, \mathcal{E}^{[1,N-2]})\) to

\((\mathcal{F}_1^{N-2}, \mathcal{F}_0^{N-2}, \mathcal{F}_1^{N-2}, \cdots, \mathcal{F}_1^{s+1}, \cdots, \mathcal{F}_1^{N-2}, \mathcal{F}_1^{N-2}, \mathcal{F}_0^{N-2}, \mathcal{F}_1^{[1,N-1]}).\)

According to Lemma 6.6 (2), it is easy to see that

\[
\mathcal{E}_N(\mathcal{F}^{[1,N-1]}_0) \subset \mathcal{F}^{[1,N-1]}_0, \quad \mathcal{E}_N(\mathcal{F}^{[s,s+1]}_0) \subset \mathcal{F}^{[s,s+1]}_0, \quad (1 \leq s < N - 1)
\]
By Lemma 6.7, we have $F_N(\mathcal{E}^s) \subset \mathcal{F}^N_{s+1}$ ($1 \leq s < N - 1$). By Lemma 6.6 (3), we have $F_N(\mathcal{E}^{[1,N-2]}) \subset \mathcal{F}_N^{1,N-2}$.

Step 2. $E_N$ induces a triangle equivalence between $\mathcal{F}^{1,N-1}$ and $\mathcal{F}_N^{1,N-2}$.

Consider the following diagram:

\begin{equation}
\begin{array}{ccc}
K(B) & \xrightarrow{id} & K(B) \\
\downarrow U_{N-1} & & \downarrow L_{N-1}^N \\
K(Mor_{N-1}(B)) & \xrightarrow{E_N} & K_N(B)
\end{array}
\end{equation}

By Proposition 4.7, Lemma 6.6, it is not hard to see that we have a functorial isomorphism $F_N \circ U_{N-1} \simeq L_{N-1}^N$. By Proposition 4.7, $U_{N-1}$ induces a triangle equivalence between $K(B)$ and $\mathcal{F}^{1,N-1}$. On the other hand, by Corollary 5.10, $L_{N-1}^N$ induces a triangle equivalence between $K_{N-1}(B)$ and $\mathcal{F}_N^{1,N-2}$. Therefore $E_N$ induces a triangle equivalence between $\mathcal{F}^{1,N-1}$ and $\mathcal{F}_N^{1,N-2}$.

Step 3. $E_N$ induces a triangle equivalence between $\mathcal{E}^{[2,N-1]}$ and $\mathcal{F}_0^1$.

Consider the following diagram:

\begin{equation}
\begin{array}{ccc}
K(Mor_{N-2}(B)) & \xrightarrow{E_{N-1}} & K_{N-1}(B) \\
\downarrow E_{N-1}^{N-2} & & \downarrow L_{N-1}^N \\
K(Mor_{N-1}(B)) & \xrightarrow{E_N} & K_N(B)
\end{array}
\end{equation}

Similarly, we have a functorial isomorphism $E_N \circ E_{N-1}^{N-1} \simeq L_{N-1}^{N-1} \circ E_{N-1}$. By Proposition 4.7, $E_{N-1}^{N-1}$ induces a triangle equivalence between $K(B)$ and $\mathcal{F}^{1,N-1}$. On the other hand, by Corollary 5.10, $L_{N-1}^{N-1}$ induces a triangle equivalence between $K_{N-1}(B)$ and $\mathcal{F}_0^1$. By the assumption of induction on $N$, $E_{N-1}$ is a triangle equivalence. Therefore $E_N$ induces a triangle equivalence between $\mathcal{E}^{[2,N-1]}$ and $\mathcal{F}_0^1$.

According to Proposition 1.7, $F_N : K(Mor_{N-1}^{sm}(B)) \to K_N(B)$ is a triangle equivalence. Moreover, it is easy to see the above proof is available for the case $E_{N-1}^d : K^d(Mor_{N-1}^{sm}(B)) \to K_{N}(B)$, where $d = -, +, b$. \hfill $\square$

In [IKM2], we studied the derived category of $N$-complexes. We have results of [IKM2] Corollaries 4.11, 4.12 under the weaker condition.

**Lemma 6.9.** Let $\mathcal{D}$ be a triangulated category, $\mathcal{C}, \mathcal{U}$ full triangulated subcategories of $\mathcal{C}$. We assume that for any $X \in \mathcal{D}$ there is a triangle $C_X \to X \to U_X \to \Sigma C_X$ such that $C_X \in \mathcal{C}$ and $U_X \in \mathcal{U}$. Then we have a triangle equivalence

$$\mathcal{C}/(\mathcal{C} \cap \mathcal{U}) \simeq \mathcal{D}/\mathcal{U}.$$  

**Proof.** Let $E : \mathcal{C} \to \mathcal{D}$ be the canonical embedding, $Q' : \mathcal{C} \to \mathcal{C}/(\mathcal{C} \cap \mathcal{U})$, $Q : \mathcal{D} \to \mathcal{D}/\mathcal{U}$ the canonical quotients. Then there is a triangle functor $F : \mathcal{C}/(\mathcal{C} \cap \mathcal{U}) \to \mathcal{D}/\mathcal{U}$ such that $Q \circ E = F \circ Q'$. By the assumption, $F$ is obviously dense. Given
\(X_1, X_2 \in \mathcal{C}\), any morphism in \(\mathcal{D}/\mathcal{U}\) is represented by the following in \(\mathcal{D}\):

\[(6.10)\]

\[
\begin{array}{c}
X_1 \\
\downarrow s_1 \\
F(C_1) \\
\downarrow g \\
F(C_2)
\end{array}
\]

where \(X_1 \xrightarrow{s_1} F(C_1) \to U_1 \to \Sigma X_1\) is a triangle in \(\mathcal{D}\) with \(U_1 \in \mathcal{U}\). By the assumption, there is a triangle \(C_2 \xrightarrow{s_2} X_1 \to U_2 \to \Sigma C_2\) with \(U_2 \in \mathcal{U}\). Therefore, \(F\) is a full dense functor. Let \(f : C_1 \to C_2\) be a morphism in \(\mathcal{C}/(\mathcal{C} \cap \mathcal{U})\) such that \(F(f) = 0\) in \(\mathcal{D}\). Then \(F(f)\) is represented by the diagram \(6.10\) where \(g = 0\). In the above, \(g \circ s_2 = 0\), and then \(f = 0\) in \(\mathcal{C}/(\mathcal{C} \cap \mathcal{U})\). Hence \(F\) is an equivalence. \(\square\)

We say that \(\mathcal{A}\) is an Ab3 category provided that it has any coproduct of objects. Moreover, \(\mathcal{A}\) is an Ab4 category provided that it has any coproduct of objects, and that the coproduct of monics is monic.

**Proposition 6.11.** Let \(\mathcal{A}\) be an Ab4 category with enough projectives, and \(\mathcal{P}\) the category of projective objects, \(\mathcal{P}\) the full subcategory of \(\mathcal{A}\) consisting of projective objects. Then the following hold.

1. We have a triangle equivalence
   \[K(\text{Mor}_{N-1}^{sm}(\mathcal{P}))/K^b(\text{Mor}_{N-1}^{sm}(\mathcal{P})) \simeq D(\text{Mor}_{N-1}(\mathcal{A})).\]
2. We have a triangle equivalence
   \[K_N(\mathcal{P})/K^b_N(\mathcal{P}) \simeq D_N(\mathcal{A}).\]

Here \(K^b(\text{Mor}_{N-1}^{sm}(\mathcal{P}))\) (resp., \(K^b_N(\mathcal{P})\)) is the full subcategory of \(K(\text{Mor}_{N-1}^{sm}(\mathcal{P}))\) (resp., \(K_N(\mathcal{P})\)) consisting of complexes (resp., \(N\)-complexes) of which all homologies are null.

**Proof.** (1) It is easy to see that \(\text{Mor}_{N-1}^{sm}(\mathcal{P})\) is the full subcategory of \(\text{Mor}_{N-1}(\mathcal{A})\) consisting of projective objects, and that \(\text{Mor}_{N-1}(\mathcal{A})\) is an Ab4 category with enough projectives. According to [BN], for any complex \(X \in K(\text{Mor}_{N-1}(\mathcal{A}))\), there is a quasi-isomorphism \(P \to X\) with \(P \in K(\text{Mor}_{N-1}^{sm}(\mathcal{P}))\). By Lemma 6.9, we have
   \[K(\text{Mor}_{N-1}^{sm}(\mathcal{P}))/K^b(\text{Mor}_{N-1}^{sm}(\mathcal{P})) \simeq D(\text{Mor}_{N-1}(\mathcal{A})).\]

(2) According to [IKM2] Theorem 2.23, for any complex \(X \in K_N(\mathcal{A})\), there is a quasi-isomorphism \(P \to X\) with \(P \in K_N(\mathcal{P})\). By Lemma 6.9, we have
   \[K_N(\mathcal{P})/K^b_N(\mathcal{P}) \simeq D_N(\mathcal{A}).\]

**Corollary 6.12.** Let \(\mathcal{A}\) be an abelian category with enough projectives, and \(\mathcal{P}\) the category of projective objects. Then we have triangle equivalences

\[K^{-}(\text{Mor}_{N-1}^{sm}(\mathcal{P})) \simeq K^{-}(\mathcal{P}), K^b(\text{Mor}_{N-1}^{sm}(\mathcal{P})) \simeq K^b_N(\mathcal{P}), \]
\[D^{-}(\text{Mor}_{N-1}(\mathcal{A})) \simeq D^{-}(\mathcal{A}).\]

Moreover if \(\mathcal{A}\) is an Ab4 category, then

\[D(\text{Mor}_{N-1}(\mathcal{A})) \simeq D_N(\mathcal{A}).\]

**Proof.** According to [IKM2] Lemma 4.8, we have a triangle equivalence
\[K^{-}(\text{Mor}_{N-1}^{sm}(\mathcal{P})) \simeq D^{-}(\text{Mor}_{N-1}(\mathcal{A})).\] By [IKM2] Theorem 2.18, we have a triangle equivalence \(K_N^{-}(\mathcal{P}) \simeq D_N^{-}(\mathcal{A})\). By Theorem 6.8, we have the statement. \(\square\)
7. Appendix

In this section, we give results concerning Frobenius categories. Let \( \mathcal{C} \) be an exact category with a collection \( \mathcal{E} \) of exact sequences in the sense of Quillen [Qu]. An exact sequence \( 0 \to X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \to 0 \) in \( \mathcal{E} \) is called a conflation, and \( f \) and \( g \) are called an inflation and a deflation, respectively. An additive functor \( F : \mathcal{C} \to \mathcal{C}' \) is called exact if it sends conflations in \( \mathcal{C} \) to conflations in \( \mathcal{C}' \). An exact category \( \mathcal{C} \) is called a Frobenius category provided that it has enough projectives and enough injectives, and that any object of \( \mathcal{C} \) is projective if and only if it is injective. In this case, the stable category \( \underline{\mathcal{C}} \) of \( \mathcal{C} \) by projective objects is a triangulated category (see [Ha]).

Remark 7.1. For a Frobenius category \( \mathcal{C} \), we treat the only case that for any object \( X \) of \( \mathcal{C} \) we can choose conflations
\[
0 \to X \xrightarrow{u_X} I_C(X) \xrightarrow{v_X} \Sigma C X \to 0
\]
\[
0 \to \Sigma C^{-1} X \xrightarrow{u_{Y^{-1} X}} P_C(X) \xrightarrow{v_{Y^{-1} X}} X \to 0
\]
where \( I_C(X) \) and \( P_C(X) \) are projective-injective objects in \( \mathcal{C} \). For a morphism \( f : X \to Y \) in \( \mathcal{C} \), we have a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{u_X} & X & \xrightarrow{I_C(X)} & \Sigma C X & \xrightarrow{\Sigma f} & \Sigma C Y & \xrightarrow{v_Y} & 0 \\
\downarrow f & & \downarrow I_f & & \downarrow \Sigma f & & \downarrow v_f & & \\
0 & \xrightarrow{u_Y} & Y & \xrightarrow{I_C(Y)} & \Sigma C Y & \xrightarrow{\Sigma f} & \Sigma C Y & \xrightarrow{v_Y} & 0
\end{array}
\]

It is easy to see that \( \Sigma f \) is uniquely determined in the stable category \( \underline{\mathcal{C}} \). Therefore \( \underline{\mathcal{C}} \) has a suspension functor \( \Sigma : \underline{\mathcal{C}} \to \underline{\mathcal{C}} \).

x

Proposition 7.2 ([Ke] A.2 Proposition). If \((\mathcal{C}, \mathcal{E})\) is a small exact category, there is an equivalence \( G : \mathcal{C} \to \mathcal{M} \) onto a full subcategory \( \mathcal{M} \) of an abelian category \( \mathcal{A} \) such that \( \mathcal{M} \) is closed under extensions and that \( \mathcal{E} \) is formed by the collection of sequences \( 0 \to X \xrightarrow{\xi} Y \xrightarrow{\eta} Z \to 0 \) inducing exact sequences in \( \mathcal{A} \):
\[
0 \to G(X) \xrightarrow{G(f)} G(Y) \xrightarrow{G(g)} G(Z) \to 0
\]

Proposition 7.3. Let \( \mathcal{C}, \mathcal{C}' \) be Frobenius categories, \( F : \mathcal{C} \to \mathcal{C}' \) an exact functor. If \( F \) sends projective objects in \( \mathcal{C} \) to projective objects in \( \mathcal{T}' \), then it induces the triangle functor \( F^\prime : \underline{\mathcal{C}} \to \underline{\mathcal{C}'} \).

Proof. For \( X \in \mathcal{C} \), since \( F(I_C(X)), I_{C'}(F(X)) \) are projective-injective objects in \( \mathcal{C}' \), we have a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{F(X)} & F(I_C(X)) & \xrightarrow{F(\Sigma C X)} & F(\Sigma C X) & \xrightarrow{v_X} & 0 \\
\downarrow \theta_X & & \downarrow \eta_X & & \downarrow \eta_X & & \\
0 & \xrightarrow{F(X)} & I_{C'}(F(X)) & \xrightarrow{\Sigma C' F(X)} & 0 \\
\downarrow \theta_X' & & \downarrow \eta_X' & & \\
0 & \xrightarrow{F(X)} & F(I_C(X)) & \xrightarrow{F(\Sigma C X)} & F(\Sigma C X) & \xrightarrow{v_X} & 0
\end{array}
\]
There are $\gamma_X : F(\Sigma C X) \to F(I C(X))$ and $\gamma'_X : \Sigma C F(X) \to I C(F(X))$ such that $F(v_X) \gamma_X = 1_{F(\Sigma C X)} - \eta'_X \eta_X$ and $v_F(X) \gamma'_X = 1_{\Sigma C F(X)} - \eta_X \eta'_X$ in $C'$. Then $\eta_X$ is an isomorphism in $C'$. For a morphism $f : X \to Y$ in $C$, we have the following diagram by the diagrams of the above and Definition 7.1:

$$
\begin{array}{c}
0 & \longrightarrow & F(X) & \xrightarrow{F(u_X)} & F(I(X)) & \xrightarrow{F(v_X)} & F(\Sigma X) & \longrightarrow & 0 \\
| & & | & & | & & | & & |
0 & \longrightarrow & F(Y) & \xrightarrow{F(I_f)} & F(I(Y)) & \xrightarrow{F(\Sigma_f)} & F(\Sigma Y) & \longrightarrow & 0 \\
| & & | & & | & & | & & |
0 & \longrightarrow & F(X) & \xrightarrow{F(f)} & I(F(X)) & \xrightarrow{\Sigma_f} & F(\Sigma X) & \longrightarrow & 0 \\
| & & | & & | & & | & & |
0 & \longrightarrow & F(Y) & \xrightarrow{u_{F(Y)}} & I(F(Y)) & \xrightarrow{v_{F(Y)}} & F(\Sigma Y) & \longrightarrow & 0
\end{array}
$$

where the diagrams on the top, the bottom, the front and the back surfaces are commutative. Since $(\theta_Y F(I_f) - I_{F(f)} \theta_X) F(u_X) = u_{F(Y)} F(f) - I_{F(f)} u_{F(X)} = 0$ there is a morphism $\delta : F(\Sigma X) \to F(I(Y))$ such that $\theta_Y F(I_f) - I_{F(f)} \theta_X = \delta F(v_X)$. Then we have $\eta_Y F(\Sigma_f) - \Sigma_{F(f)} \eta_X = v_{F(Y)} \delta$, and then $\eta_Y F(\Sigma(f)) = F(\Sigma(f)) \eta_X$ in $C'$. Therefore we have a functorial isomorphism $\eta : F \circ \Sigma C \xrightarrow{\sim} \Sigma C' \circ F$. For a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in $C$, we may have a morphism between conflations:

$$
\begin{array}{c}
0 & \longrightarrow & X & \xrightarrow{\gamma_X} & I(X) & \xrightarrow{v_X} & \Sigma X & \longrightarrow & 0 \\
| & & | & & | & & | & & |
0 & \longrightarrow & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X & \longrightarrow & 0
\end{array}
$$

There are morphisms $\psi_{F(f)} : I C(F(X)) \to Z'$ and $z : F(Z) \to \Sigma X$ such that we have a commutative diagram

$$
\begin{array}{c}
0 & \longrightarrow & F(X) & \xrightarrow{F(u_X)} & F(I(X)) & \xrightarrow{F(v_X)} & F(\Sigma X) & \longrightarrow & 0 \\
| & & | & & | & & | & & |
0 & \longrightarrow & F(Y) & \xrightarrow{F(I_f)} & F(I(Y)) & \xrightarrow{F(\Sigma_f)} & F(\Sigma Y) & \longrightarrow & 0 \\
| & & | & & | & & | & & |
0 & \longrightarrow & F(X) & \xrightarrow{F(f)} & I(F(X)) & \xrightarrow{\Sigma_f} & F(\Sigma X) & \longrightarrow & 0 \\
| & & | & & | & & | & & |
0 & \longrightarrow & F(Y) & \xrightarrow{u_{F(Y)}} & I(F(Y)) & \xrightarrow{v_{F(Y)}} & F(\Sigma Y) & \longrightarrow & 0
\end{array}
$$

Similarly, there is a morphism $z' : Z' \to F(Z)$ such that we have the above commutative diagram of which all vertical arrows are reversed. Put $\zeta = z' z + F(\psi_f) \gamma_X F(h)$, we have a commutative diagram

$$
\begin{array}{c}
0 & \longrightarrow & F(Y) & \xrightarrow{F(g)} & F(Z) & \xrightarrow{F(h)} & F(\Sigma X) & \longrightarrow & 0 \\
| & & | & & | & & | & & |
0 & \longrightarrow & F(Y) & \xrightarrow{F(g)} & F(Z) & \xrightarrow{F(h)} & F(\Sigma X) & \longrightarrow & 0
\end{array}
$$
Then \( \bar{z}' \bar{z} \) is an isomorphism in \( C' \). Similarly \( \bar{z}' \bar{z} \) is also an isomorphism in \( C' \), and then \( \bar{z} \) is an isomorphism in \( C' \). Therefore we have a commutative diagram in \( C' \):

\[
\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
\downarrow & & \downarrow \\
F(X) & \xrightarrow{F(f)} & F(Y)
\end{array}
\begin{array}{ccc}
F(Z) & \xrightarrow{F(g)} & F(\Sigma X) \\
\downarrow & & \downarrow \\
F(\Sigma X) & \xrightarrow{\Sigma h} & \Sigma F(X)
\end{array}
\]

where all vertical arrows are isomorphisms. Hence \( F \) induces a triangle functor \( F : C \to C' \). \( \square \)

**Proposition 7.4.** Let \( C, C' \) be Frobenius categories, \( F : C \to C' \), \( G : C' \to C \) exact functors such that \( F \) is a right adjoint of \( G \). Then \( F \) and \( G \) induce triangle functors \( F : C \to C' \), \( G : C' \to C \) such that \( F \) is a right adjoint of \( G \).

**Proof.** Let \( 0 \to A \to B \to C \to 0 \) be a conflation in \( C \), then \( 0 \to F(A) \to F(B) \to F(C) \to 0 \) is a conflation in \( C' \). For a projective-injective object \( P \) of \( C' \), we have an isomorphism between exact sequences:

\[
\begin{array}{cccc}
0 & \to & \text{Hom}_C(P, F(A)) & \to \text{Hom}_C(P, F(B)) & \to \text{Hom}_C(P, F(C)) & \to 0 \\
\uparrow & & \uparrow & & \uparrow & \\
0 & \to & \text{Hom}_C(G(P), A) & \to \text{Hom}_C(G(P), B) & \to \text{Hom}_C(G(P), C)
\end{array}
\]

Then \( G(P) \) is a projective-injective object in \( C \). Similarly, if \( Q \) is a projective-injective object of \( C \), then \( F(Q) \) is also a projective-injective object of \( C' \). By Proposition 7.3, \( F \) and \( G \) induce the triangle functors \( F : C \to C' \), \( G : C' \to C \). Given \( X \in C' \), consider a conflation

\[ 0 \to X \to I(X) \to \Sigma X \to 0 \]

For any \( Y \in C \), we have an isomorphism between exact sequences

\[
\begin{array}{cccc}
\text{Hom}_C(I(X), F(Y)) & \to \text{Hom}_C(X, F(Y)) & \to \text{Hom}_C(X, F(Y)) & \to 0 \\
\uparrow & & \uparrow & & \uparrow & \\
\text{Hom}_C(G(I(X)), Y) & \to \text{Hom}_C(G(X), Y) & \to \text{Hom}_C(G(X), Y) & \to 0
\end{array}
\]

Then we have an isomorphism \( \text{Hom}_C(X, F(-)) \simeq \text{Hom}_C(G(X), -) \). Similarly we have an isomorphism \( \text{Hom}_C(-, F(Y)) \simeq \text{Hom}_C(G(-), Y) \). \( \square \)

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