Some fixed point results for the cyclic $(\alpha, \beta) - (k, \theta)$-multi-valued mappings in metric space

Haitham Qawaqneh¹, Mohd Salmi Md Noorani¹, Wasfi Shatanawi², and Habes Alsamir³

¹School of mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM, Selangor Darul Ehsan, Malaysia
E-mail: haitham.math77@gmail.com, msn@ukm.my and h.alsamer@gmail.com
²Department of Mathematics and general courses Prince Sultan University, Riyadh, Saudi Arabia
E-mail: vshatanawi@psu.edu.sa
³School of mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 UKM, Selangor Darul Ehsan, Malaysia

Abstract. In this paper, we introduce the cyclic $(\alpha, \beta) - (k, \theta)$-multi-valued mappings in metric space. Our results generalize and extend several works which are weaker as we use cyclic $(\alpha, \beta)$-contraction and multi-valued mappings in metric space. We provide an example to show the superiority of our results over corresponding fixed point results proved in metric spaces.

1. Introduction and preliminaries

One of the most important tools in fixed point is Banach contraction principle. A lot of authors have extended or generalized this contraction and proved the existence of fixed and common fixed point theorems (see [4]-[10]). The theory of multi-valued mapping has an important tool in various fields of mathematics and several authors have also extended and presented many forms of multi-valued mapping conditions on self mappings as a generalization of Banach contraction principle by using the concept of Hausdorff metric and proved the existence of fixed and common fixed point theorems in metric space and other spaces. We recite some notations, needed definitions and elementary results, for the purpose of the sequel, and $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers respectively. Let $(X, d)$ be a metric space, we denote $CL(X)$ the family of closed subsets of $X$, by $CB(X)$ the class of all nonempty closed bounded subsets of $X$, and $F(f)$ is the set of all fixed points of $f$. For $A, B \in CL(X)$, let the $H : CL(X) \times CL(X) \to \mathbb{R}^+ \cup \{\infty\}$ be defined by

$$H(A, B) = \begin{cases} \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}, & \text{if the maximum exists} \\ \infty, & \text{otherwise.} \end{cases}$$

such map $H$ is called the generalized Hausdorff metric induced by the metric $d$.

In 1969, the study of Banach-type fixed theorems of multi-valued mappings started with the work of Nadler [15], who proved that a multi-valued contractive mapping of a complete metric
space $X$ into the family of closed bounded subsets of $X$ has a fixed point.

The scope and the objectives of this paper that to generalize and extend several results for mixed cyclic $(\alpha, \beta)$–contraction and multi-valued mappings in metric space and prove fixed point of these results.

**Definition 1.1.** [15] Let $(X,d)$ be a metric space. A map $f : X \rightarrow CB(X)$ is said to be multi valued contraction if there exists $0 \leq \lambda < 1$ such that

$$H(fx,fy) \leq \lambda d(x,y),$$

for all $x,y \in X$ where $CB(X)$ denotes the family of nonempty closed subsets of $X$.

**Definition 1.2.** [15] A point of $x_0 \in X$ is said to be a fixed point of the multi-valued mapping $f$ if $x_0 \in fx_0$.

**Lemma 1.3.** [15] If $A, B \in CB(X)$ and $a \in A$, then for each $\epsilon > 0$, there exists $b \in B$ such that

$$d(a,b) \leq H(A,B) + \epsilon.$$

In 2012, Samet et al. [9] introduced the concept of $\alpha$-admissibility as follows.

**Definition 1.4.** [9] Let $f : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$. $f$ is called $\alpha$-admissible when if $x,y \in X$ such that $\alpha(x,y) \geq 1$ then we have $\alpha(fx, fy) \geq 1$.

After that, many authors using the concept of $\alpha$-admissible contractive-type mappings to study the existence of fixed point in many spaces (see [12], [13], [14] and references cited therein).

Alizadeh et al. [2] introduced the notion of cyclic $(\alpha, \beta)$–admissible mapping which is defined as follows:

**Definition 1.5.** [2] Let $X$ be a nonempty set, $f$ be a self-mapping on $X$ and $\alpha, \beta : X \rightarrow [0, +\infty)$ be two mappings. We say that $f$ is a cyclic $(\alpha, \beta)$–admissible mapping if $x \in X$ with

$$\alpha(x) \geq 1 \Rightarrow \beta(fx) \geq 1$$

and $x \in X$ with

$$\beta(x) \geq 1 \Rightarrow \alpha(fx) \geq 1.$$

Recently, Ahmad et al. [18] presented the family $\Theta$ of functions $\theta : (0, +\infty) \rightarrow (1, +\infty)$ be a function satisfying the following conditions:

$(\theta_1)$ $\theta$ is nondecreasing,

$(\theta_2)$ $\theta$ for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n \rightarrow \infty} \theta(t_n) = 1$ if and only if $\lim_{n \rightarrow \infty} t_n = 0$,

$(\theta_3)$ $\theta$ is continuous on $(0, \infty)$.

Consistent with Ahmad et al. [18], we denote by $\Omega$ the set of all functions satisfying the conditions $(\theta_1) - (\theta_3)$.

**Example 1.6.** Define some functions as follows: for all $t > 0$, $\theta_1(t) = e^t$, $\theta_2(t) = e^{\sqrt{t}}$, $\theta_3(t) = e^{\sqrt[3]{t}}$, $\theta_4(t) = \cosh t$, $\theta_5(t) = 1 + \ln(1 + t)$, $\theta_6(t) = e^{t\sqrt{t}}$. Then $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 \in \Omega$.

In this article we motivate and inspire by ( [2], [18]) and define the $\Theta$–contraction for a new family of functions $\Omega$. Also, we present the class of an $(\alpha, k, \theta)$–contractive multi-valued mapping. After that, we establish some the existence of fixed point for this class of mappings in metric space. An example is given to support the obtained result.
2. Main result
First, we present the following lemma which will be used efficiently in the proof of our main result.

**Lemma 2.1.** Let $f : X \rightarrow X$ be a cyclic $(\alpha, \beta)-admissible$ mapping. Assume that there exist $x_0, x_1 \in X$ such that

$$\alpha(x_0) \geq 1 \Rightarrow \beta(x_1) \geq 1$$

and

$$\beta(x_0) \geq 1 \Rightarrow \alpha(x_1) \geq 1.$$  

Define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$. Then

$$\alpha(x_n) \geq 1 \Rightarrow \beta(x_m) \geq 1$$

and

$$\beta(x_n) \geq 1 \Rightarrow \alpha(x_m) \geq 1,$$

for all $m, n \in \mathbb{N}$ with $n < m$.

**Proof.** Since

$$\alpha(x_0) \geq 1 \Rightarrow \beta(fx_0) = \beta(x_1) \geq 1$$

and

$$\beta(x_0) \geq 1 \Rightarrow \alpha(fx_0) = \alpha(x_1) \geq 1.$$ 

By continuing the above process, we have

$$\alpha(x_n) \geq 1 \Rightarrow \beta(fx_n) = \beta(x_{n+1}) \geq 1$$

and

$$\beta(x_n) \geq 1 \Rightarrow \alpha(fx_n) = \alpha(x_{n+1}) \geq 1.$$ 

Since

$$\begin{cases} \alpha(x_n) \geq 1 \Rightarrow \beta(fx_n) = \beta(x_{n+1}) \geq 1, \\
\beta(x_n) \geq 1 \Rightarrow \alpha(fx_n) = \alpha(x_{n+1}) \geq 1, \end{cases}$$

for all $m, n \in \mathbb{N}$ with $n < m$. Moreover, since

$$\begin{cases} \alpha(x_n) \geq 1 \Rightarrow \beta(x_{n+2}) \geq 1, \\
\beta(x_n) \geq 1 \Rightarrow \alpha(x_{n+2}) \geq 1, \end{cases}$$

for all $m, n \in \mathbb{N}$ with $n < m$. We deduce that

$$\begin{cases} \alpha(x_n) \geq 1 \Rightarrow \beta(x_{n+3}) \geq 1, \\
\beta(x_n) \geq 1 \Rightarrow \alpha(x_{n+3}) \geq 1. \end{cases}$$

By continuing this process, we have

$$\begin{cases} \alpha(x_n) \geq 1 \Rightarrow \beta(x_m) \geq 1, \\
\beta(x_n) \geq 1 \Rightarrow \alpha(x_m) \geq 1, \end{cases}$$

for all $m, n \in \mathbb{N}$. 

\qed
Now, we present the class of a cyclic \((\alpha, \beta) - (k, \theta)\) admissible multi-valued mapping and prove some fixed point theorems on complete metric space.

**Definition 2.2.** Let \((X, d)\) be a metric space and \(f : X \to CL(X)\) be a cyclic \((\alpha, \beta)\)-admissible mapping. We say that \(f\) is a cyclic \((\alpha, \beta) - (k, \theta)\)-admissible multi-valued mapping if there exists \(\alpha, \beta : X \times X \to [0, +\infty)\) and \(\theta \in \Theta\) such that:

\[
x, y \in X, \alpha(x)\beta(y) \geq 1 \Rightarrow \theta(H(fx, fy)) \leq \theta(M(x, y))^k,
\]

where

\[
M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\}
\]

for all \(x, y \in X\).

**Theorem 2.3.** Let \((X, d)\) be a complete metric space and \(f : X \to CL(X)\) be cyclic \((\alpha, \beta) - (k, \theta)\) admissible multi-valued mapping. Assume that the following conditions hold:

(i) there exists \(x_0 \in X\) and \(x_1 \in fx_0\) such that

\[
\alpha(x_0) \geq 1 \Rightarrow \beta(fx_0) = \beta(x_1) \geq 1
\]

\[
\beta(x_0) \geq 1 \Rightarrow \alpha(fx_0) = \alpha(x_1) \geq 1
\]

(ii) \(f\) is a cyclic \((\alpha, \beta)\)-continuous multi-valued mapping.

then \(f\) has a fixed point.

**Proof.** By starting from \(x_0\) and \(x_1 \in fx_0\) in conditions (i), we have

\[
\alpha(x_0) \geq 1 \Rightarrow \beta(x_1) = \beta(fx_0) \geq 1,
\]

\[
\beta(x_0) \geq 1 \Rightarrow \alpha(x_1) = \alpha(fx_0) \geq 1.
\]

Therefore, \(\alpha(x_0) \geq 1\) and \(\beta(x_1) \geq 1\), equivalently, \(\alpha(x_0)\beta(x_1) \geq 1\). If \(x_0 = x_1\), we derive that \(x_1 \in F(f)\) and so the proof is done. Now, we assume that \(x_0 \neq x_1\) and \(x_1 \notin fx_1\). From (1), we have

\[
1 < \theta(d(x_1, fx_1)) \leq \theta(H(fx_0, fx_1)) \leq \theta(M(x_0, x_1))^k,
\]

where

\[
M(x_0, x_1) = \max\{d(x_0, x_1), d(x_0, fx_0), d(x_1, fx_1), \frac{1}{2}[d(x_0, fx_1) + d(x_1, fx_0)]\}
\]

\[
= \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, fx_1), \frac{1}{2}d(x_0, fx_1)\}
\]

\[
\leq \max\{d(x_0, x_1), d(x_1, fx_1), \frac{1}{2}[d(x_0, x_1) + d(x_1, fx_1)]\}
\]

\[
= \max\{d(x_0, x_1), d(x_1, fx_1)\},
\]

from (2) and (3), we get

\[
1 < \theta(d(x_1, f1)) \leq \theta(H(fx_0, fx_1)) \leq \theta(M(x_0, x_1))^k.
\]
If $\max\{d(x_0, x_1), d(x_1, f x_1)\} = d(x_1, f x_1)$, then we obtain
\[
1 < \theta(d(x_1, f x_1)) \leq \theta(H(f x_0, f x_1)) \leq \theta(d(x_1, f x_1))^k,
\]
which is a contradiction. Thus $\max\{d(x_0, x_1), d(x_1, f x_1)\} = d(x_0, x_1)$. From (4), we obtain
\[
1 < \theta(d(x_1, f x_1)) \leq \theta(H(f x_0, f x_1)) \leq \theta(d(x_0, x_1))^k. \tag{5}
\]
Since $fx_1$ is compact, then there exists $x_2 \in fx_1$ such that
\[
d(x_1, x_2) = d(x_1, f x_1). \tag{6}
\]
From (5) and (6), we get
\[
1 < \theta(d(x_1, x_2)) \leq \theta(d(x_0, x_1))^k. \tag{7}
\]
If $x_1 = x_2$ or $x_2 \in fx_1$, then it follows that $x_2 \in F(f)$ and so the proof is done. Therefore, we assume that $x_2 \neq x_1$ and $x_2 \in fx_2$. Since $x_1 \in fx_0$, $x_2 \in fx_1$, $\alpha(x_0)\beta(x_1) \geq 1$ and $f$ is a cyclic $(\alpha, \beta) - (k, \theta)$–admissible multi-valued mapping, we have $\alpha(x_1)\beta(x_2) \geq 1$.

By applying cyclic $(\alpha, \beta) - (k, \theta)$–admissible multi-valued condition, we have
\[
\theta(H(f x_1, f x_2)) \leq \alpha(x_1)\beta(x_2)\theta(H(f x_1, f x_2)) \leq \theta(M(x_1, x_2))^k. \tag{8}
\]
where
\[
M(x_1, x_2) = \max\{d(x_1, x_2), d(x_1, f x_1), d(x_2, f x_2), \frac{1}{2}[d(x_1, f x_2) + d(x_2, f x_1)]\}
\]
\[
= \max\{d(x_1, x_2), d(x_1, x_2), d(x_2, f x_2), \frac{1}{2}d(x_1, f x_2)\}
\]
\[
= \max\{d(x_1, x_2), d(x_2, f x_2), \frac{1}{2}[d(x_1, x_2) + d(x_2, f x_2)]\}
\]
\[
= \max\{d(x_1, x_2), d(x_2, f x_2)\}.
\]
If $M(x_1, x_2) = d(x_2, f x_2)$, we have:
\[
1 < \theta(d(x_2, f x_2)) \leq \theta(H(f x_1, f x_2)) \leq \theta(d(x_2, f x_2))^k,
\]
which is a contradiction. Thus, if $M(x_1, x_2) = d(x_1, x_2)$, we get
\[
1 < \theta(d(x_2, f x_2)) \leq \theta(H(f x_1, f x_2)) \leq \theta(d(x_1, x_2))^k. \tag{9}
\]
Since $fx_2$ is compact, then there exists $x_3 \in fx_2$ such that
\[
d(x_2, x_3) = d(x_2, f x_2). \tag{10}
\]
From (9) and (10), we obtain
\[
1 < \theta(d(x_2, x_3)) \leq \theta(d(x_1, x_2))^k \leq \theta(d(x_0, x_1))^{k^2}.
\] (11)

By continuing this procedure, we construct the sequence \(\{x_n\}\) in \(X\) such that \(x_{n+1} \neq x_n \in fx_n\), again, since \(f\) is a cyclic \((\alpha, \beta)\)-admissible mapping and by Lemma (2.1), we have
\[
\alpha(x_n) \geq 1 \quad \text{and} \quad \beta(x_n) \geq 1
\]
for all \(n \in \mathbb{N}\). This implies that
\[
\alpha(x_n)\beta(x_{n+1}) \geq 1,
\]
and
\[
1 < \theta(d(x_{n+1}, x_{n+2})) \leq \theta(H(f x_n, f x_{n+1})) \leq \theta(d(x_0, x_1))^{k^n},
\]
for all \(n \geq n_1\). This shows that \(\lim_{n \to \infty} \theta(d(x_n, x_{n+1})) = 1\) and so \(\lim_{n \to \infty} (d(x_n, x_{n+1})) = 0\). By our assumptions about \(\theta\), it follows that there exist \(n_1 \in \mathbb{N}\) and \(r \in (0, 1)\) such that
\[
(d(x_n, x_{n+1})) \leq \frac{1}{n^r},\]
for all \(n \geq n_1\).

Now, for \(m > n > n_1\) we have
\[
d(x_n, x_m) = \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \left(\frac{1}{i^r}\right).
\]
Since \(0 < r < 1\), \(\sum_{i=n}^{m-1} \left(\frac{1}{i^r}\right)\) converges. This implies that \(\lim_{n, m \to \infty} d(x_m, x_n) = 0\) and by the continuity of \(\theta\), we have \(\lim_{n \to \infty} d(x_m, x_n) = 0\). Thus \(\{x_n\}\) is Cauchy sequence in \((X, d)\) and there exists \(z \in X\) such that \(x_n \to z\) as \(n \to \infty\) for all \(n \in \mathbb{N}\).

We suppose that condition (i) hold. Hence \(\alpha(x_n)\beta(z) \geq 1\). From (1) and by the cyclic \((\alpha, \beta)\)-continuity of multi-valued mapping \(f\), we have
\[
\lim_{n \to \infty} H(f x_n, f z) = 0
\] (12)
for all \(n \in \mathbb{N}\), which implies that
\[
\theta(d(z, f z)) = \lim_{n \to \infty} d(x_{n+1}, f z) \leq \lim_{n \to \infty} H(f x_n, f z) = 0.
\]
Therefore, \(z \in f z\) and hence \(f\) has a fixed point.

Corollary 2.4. Let \((X, d)\) be a metric space and \(f : X \to CL(X)\) be a cyclic \((\alpha, \beta)\)-admissible mapping. We say that \(f\) is a cyclic \((\alpha, \beta) - (k, \theta)\)-admissible multi-valued mapping if there exists \(\alpha, \beta : X \times X \to [0, +\infty)\) and \(\theta \in \Theta\) such that:
\[
\alpha(x)\beta(y)\theta(H(f x, f y)) \leq \theta(\max\{d(x, y), d(x, f x), d(y, f y), \frac{1}{2}\left|d(x, f y) + d(y, f x)\right|\}^k),
\]
Assume that the following conditions hold:
(i) there exists $x_0 \in X$ and $x_1 \in fx_0$ such that

$$\alpha(x_0) \geq 1 \Rightarrow \beta(fx_0) = \beta(x_1) \geq 1$$

and

$$\beta(x_0) \geq 1 \Rightarrow \alpha(fx_0) = \alpha(x_1) \geq 1$$

(ii) $f$ is a cyclic($\alpha, \beta$)-continuous multi-valued mapping.

then $f$ has a fixed point.

Proof. Let $\alpha(x)\beta(y) \geq 1$ for every $x, y \in X$. Then by 2.4, we have:

$$\theta(H(fx, fy)) \leq \alpha(x)\beta(y)\theta(H(fx, fy))$$

$$\leq \theta(\max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\})^k,$$

this guides that $f$ cyclic ($\alpha, \beta$) - ($k, \theta$)-admissible multi-valued mapping. So, by following the proof Theorem (2.3) we obtain the desired outcome.

Example 2.5. Let $x = [0, \infty)$ and $d : X \times X \rightarrow [0, \infty)$ be defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $f : X \rightarrow X$ and $\alpha, \beta : X \rightarrow [0, \infty)$ by

$$fx = \begin{cases} \frac{x}{4} & \text{if } x \in [0, 1], \\ 2x & \text{if } x \in (1, \infty) \end{cases}$$

$$\alpha(x) = \begin{cases} \frac{x+3}{2} & \text{if } x \in [0, 1], \\ 0 & \text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} \frac{x+5}{3} & \text{if } x \in [0, 1], \\ 0 & \text{otherwise}. \end{cases}$$

It’s easy to see that $(X, d)$ is a metric space.

Now, we want to show that the Theorem 2.3 can be guarantee the existence of fixed point of $f$. Firstly, we will show that $f$ is a cyclic ($\alpha, \beta$)-admissible mapping

For $x, y \in X$, we have

$$\alpha(x) \geq 1 \Rightarrow x \in [0, 1] \Rightarrow \beta(fx) = \beta(f\frac{x}{4}) = \frac{x+20}{12} \geq 1$$

and

$$\beta(x) \geq 1 \Rightarrow x \in [0, 1] \Rightarrow \alpha(fx) = \alpha(f\frac{x}{4}) = \frac{x+12}{8} \geq 1.$$
Let $\alpha(x)\beta(x) \geq 1$. Then for $x, y \in [0, 1]$, we have

$$\theta(H(fx, fy)) = \theta(\frac{1}{4}|x - y|)$$

$$= e^{\frac{1}{2}|x-y|e^{\frac{1}{4}|x-y|}}$$

$$\leq \frac{1}{2} e^{\sqrt{|x-y|e^{\frac{1}{4}|x-y|}}}$$

$$= \frac{1}{2} e^{\frac{1}{2}d(x,y)e^{d(x,y)}}$$

$$\leq \frac{1}{2} e^{M(x,y)e^{M(x,y)}}$$

$$= \theta(M(x, y))^k.$$

So, all conditions of Theorem (2.3) hold with $k = \frac{1}{2}$, which implying that $f$ has fixed point.

3. Conclusion and future works

We have introduced the cyclic $(\alpha, \beta) - (k, \theta)$--multi-valued mappings in metric space and we have generalized several works which were weaker as we used cyclic $(\alpha, \beta)$--contraction and multi-valued mappings in metric space. We have provided an example to show the superiority of our results over corresponding fixed point results proved in metric spaces. Otherwise we suggest to generalize more results in other spaces like $b$--metric space, metric like space and others.

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