EXACT ANALYTICAL SOLUTION FOR ONE NONLINEAR VARIATIONAL PROBLEM OF THE CAVITATION THEORY

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In this paper we investigate the limiting values of the lift and drag coefficients of profiles in the Helmholtz–Kirchhoff (infinite cavity) flow. The coefficients are based on the wetted arc length of profile surfaces. Namely, for a given value of the lift coefficient we find minimum and maximum values of the drag coefficient. Thereby, we determine maximum and minimum values of the lift-to-drag ratios.

In the theory of aero and hydrofoils there are known two classical models for studying flows past a profile. For the first model the flow is continuous (fig. 1a)) and for the second one the flow is separated with formation of an infinite cavity (fig. 1b)). If we assume that the flow is steady, irrotational and incompressible, then for the first model the drag force \( D = 0 \) (d’Alembert’s paradox) and the lift force \( L \) is defined by the well-known Kutta–Joukowskii theorem:

\[
L = -\rho v_0 \Gamma, \quad \Gamma = \int_0^l (\mathbf{v} \cdot \mathbf{\tau}) ds. \tag{1}
\]

Here \( \rho \) is the density of the fluid, \( v_0 \) is the velocity at infinity, \( \Gamma \) is the circulation around the profile, \( l \) is the perimeter of the profile surface, \( s \) is the arc abscissa of the profile contour, reckoned from the trailing edge point \( A \), \( (\mathbf{v} \cdot \mathbf{\tau}) \) is the dot product of the velocity vector \( \mathbf{v} \) at the point on the profile surface and the tangential unit vector \( \mathbf{\tau} \), directed toward increase of \( s \). For the continuous model the point \( B \) with the arc abscissa \( s = l \) coincides with the point \( A \) for which \( s = 0 \). If \( l_1 \) is the arc abscissa of the stagnation point \( O \) and \( v = |\mathbf{v}| \), then

\[
(\mathbf{v} \cdot \mathbf{\tau}) = -v(s) \text{ for } 0 \leq s \leq l_1, \quad (\mathbf{v} \cdot \mathbf{\tau}) = v(s) \text{ for } l_1 \leq s \leq l. \tag{2}
\]

Figure 1: a) — continuous flow over an aerofoil, b) — Helmholtz–Kirchhoff flow with an infinite cavity past a profile.

As one can see from (1), to compute the lift force for the continuous model one needs only to know the velocity distribution \( v(s) \) along the profile surface. Moreover, if \( v(s) \) is known, the contour of the profile can be restored by means of solving the so-called inverse boundary-value problem of aerodynamics [1]. The Kutta–Joukowskii theorem played an outstanding role in the theory of aerofoils and was used many times for aerodynamic shape optimization (see, for example, [1, 2]).
Consider now the second classical model — the Helmholtz–Kirchhoff (infinite cavity) flow (fig. 1b)). According to this model the flow detaches from the profile surface at the points $A$ and $B$, and an infinite cavity with a constant pressure, equal to the incident pressure, forms behind the profile. The velocity on the free streamlines $AI$ and $BI$ is constant and equals the incident velocity $v_0$. As previously the stagnation point is denoted by $O$ and the arc abscissa $s$ is reckoned from the point $A$.

For the Helmholtz–Kirchhoff flow formulas analogous to (1) have been recently obtained in the works [3], [4]:

$$L = \rho v_0 \int_0^l (v \cdot \tau) \log \frac{v_0}{v} \, ds, \quad D = \frac{\rho v_0}{4\pi} \left( \int_0^l \frac{v}{\sqrt{\varphi}} \log \frac{v_0}{v} \, ds \right)^2,$$

where $l$ is the length of the wetted arc $AOB$ of the profile, $\varphi = \varphi(s)$ is the distribution of potential along $AOB$:

$$\varphi = \int_s^{l_1} v(s) \, ds \quad \text{for} \quad 0 \leq s \leq l_1, \quad \varphi = \int_{l_1}^s v(s) \, ds \quad \text{for} \quad l_1 \leq s \leq l,$$

$l_1$ is the arc abscissa of the critical point $O$.

Nowadays the Helmholtz–Kirchhoff model is treated as a limiting case of cavity flows ([5], p. 23), when the pressure in the cavity tends to the incident pressure, and the sizes of the cavity become infinitely large. In the theory of cavity flows the first Brillouin condition [5] plays an important role: the pressure in the cavity is minimal. Then the velocity on the free streamlines $AI$ and $BI$ is maximal, and therefore

$$v(s) \leq v_0, \quad 0 \leq s \leq l.$$

This implies that in formulas [3] the factor $\log \frac{v_0}{v} \geq 0$.

Due to the simplicity of formulas [3] one can formulate different optimization problems in which one needs to determine a velocity distribution that satisfies the Brillouin condition [1] and has some optimal property. One of such problems has been solved in the works [3], [4]. Namely, it has been found the velocity distribution that under the Brillouin condition [1] provides a global maximum of the lift force. It has been established that for the profile of maximum lift the length $l_1 = 0$ (the points $A$ and $O$ coincide) and the optimal velocity distribution $v(s) = e^{-1}v_0 = \text{const}$, where $e$ is the base of natural logarithms. It follows from [3] that $L_{\text{max}} = \rho v_0^2 le^{-1}$ and this is the global maximum of the lift force. But such a formulation does not take into account at all the cavitation drag, which is defined by the second equation in [3]. If $v(s) = e^{-1}v_0$, then according to [3] the drag force $D = \rho v_0^2 l/(\pi e)$, and the lift-to-drag ratio of the profile of maximum lift is $\kappa = L/D = \pi$. To obtain profiles with a greater lift-to-drag ratio it seems to be natural to introduce the drag $D$ in the optimization process.

Let us introduce the lift and drag coefficients $C_L$ and $C_D$:

$$C_L = \frac{2L}{\rho v_0^2 l}, \quad C_D = \frac{2D}{\rho v_0^2 l},$$

based on the wetted arc length $l$.

At the end of the paper [3] as a variant of a further perspective direction of investigations it has been formulated the problem of finding velocity distributions that provide
maximum of the lift-to-drag ratio \( \kappa = \frac{L}{D} = \frac{C_L}{C_D} \) under the given lift force \( L \) (given \( C_L \)). In this paper we present an exact analytical solution to this problem. Besides, for sake of completeness, we find velocity distributions that provide minimum of the lift-to-drag ratio \( \kappa \). Thereby, for fixed values of the lift coefficient \( C_L \) we determine exact upper and lower bounds of this important for applications hydrodynamic characteristic.

Let \( l_2 = l - l_1 \) be the length of the arc \( OB \). We introduce two dimensionless functions \( u_1(\sigma) \) and \( u_2(\sigma), 0 \leq \sigma \leq 1 \), such that

\[
\frac{v}{v_0} = \begin{cases} u_1\left(\frac{l_1 - s}{l_1}\right) & \text{on } OA; \\ u_2\left(\frac{s - l_1}{l_2}\right) & \text{on } OB. \end{cases}
\]

(5)

Since the velocity \( v \geq 0 \), the functions \( u_1(\sigma) \) and \( u_2(\sigma) \) are nonnegative. Under the Brillouin condition (4) they satisfy the inequalities

\[
u_1(\sigma) \leq 1, \quad u_2(\sigma) \leq 1.
\]

(6)

By means of (3) we express the lift and drag coefficients in terms of \( u_1(\sigma) \) and \( u_2(\sigma) \):

\[
C_L = 2\left\{ (1 - \varepsilon)I[u_2] - \varepsilon I[u_1] \right\}, \quad C_D = \frac{1}{2\pi} \left\{ \sqrt{1 - \varepsilon J[u_2]} + \sqrt{\varepsilon J[u_1]} \right\}^2,
\]

(7)

where \( \varepsilon = l_1/l \), \( I[u] \) and \( J[u] \) are nonlinear functionals of \( u(\sigma), 0 \leq \sigma \leq 1 \):

\[
I[u] = -\int_0^1 u(\sigma) \log u(\sigma) \, d\sigma, \quad J[u] = -\int_0^1 \frac{u(\sigma) \log u(\sigma) \, d\sigma}{\sqrt{\int_0^\sigma u(\sigma_1) \, d\sigma_1}}.
\]

(8)

As one can see from (3), under the Brillouin condition (6) the values of the functionals \( I[u] \) and \( J[u] \) at \( u = u_1(\sigma) \) and \( u = u_2(\sigma) \) are nonnegative.

Let us rewrite \( I[u] \) and \( J[u] \) in terms of classical functionals of calculus of variations. To do so we transform the function \( u(\sigma) \) to \( \lambda(\sigma) \):

\[
\lambda(\sigma) = \sqrt{2 \int_0^\sigma u(\sigma_1) \, d\sigma_1}.
\]

Then

\[
I[\lambda] = -\int_0^1 \lambda \lambda' \log(\lambda \lambda') \, d\sigma, \quad J[\lambda] = -\sqrt{2} \int_0^1 \lambda \log(\lambda \lambda') \, d\sigma.
\]

(9)

It is clear that \( \lambda(\sigma) \geq 0 \). Besides, \( u(\sigma) = \lambda(\sigma) \lambda'(\sigma) \), whence \( \lambda'(\sigma) \geq 0 \).

The solution of the basic problem of finding absolute extrema of the lift-to-drag ratio \( \kappa \) under the given value of \( C_L \) is based on solving the following auxiliary problem.

**Auxiliary problem.** Find the function \( \lambda(\sigma), \sigma \in [0, 1] \):

\[
\lambda(0) = 0, \quad \lambda'(\sigma) \geq 0,
\]

(10)

which delivers a global minimum (maximum) to the functional \( J[\lambda] \) under the constraint \( I[\lambda] = q \) (\( q \) is given), and the complementary condition

\[
\lambda(\sigma) \lambda'(\sigma) \leq 1.
\]

(11)
Without the complementary condition (11) the auxiliary problem is a constrained problem of calculus of variations with a free right endpoint. Let us find a solution by the Lagrange multiplier rule without regard for the nonstandard condition (11). To do so we construct the augmented cost functional

\[
P[\lambda] = - \int_0^1 \lambda'(\lambda + k) \log(\lambda \lambda') d\sigma = - \int_0^1 E(\lambda, \lambda') d\sigma,
\]

where \( k \) is a real constant. We write the Euler equation [6]

\[
E_{\lambda} - \frac{d}{d\sigma} E_{\lambda'} = 0,
\]

which for the functional \( P[\lambda] \) takes the form

\[
\frac{\lambda''(\lambda + k)}{\lambda'} + \lambda' = 0.
\]

Integrating this equation yields

\[
\lambda'(\sigma)[\lambda(\sigma) + k] = c,
\]

where \( c \) is a constant. Because the right endpoint of the desired function \( \lambda(\sigma) \) is free, there holds the relation \([E_{\lambda'}]_{\sigma=1} = 0 \) (see [6]), which can be reduced to

\[
\lambda(1)\lambda'(1) = 1/e.
\]

Equation (13) subject to the condition \( \lambda(0) = 0 \) can be easily integrated and has two solutions:

\[
\lambda(\sigma) = -k + \sqrt{2c\sigma + k^2} \quad \text{for} \quad k > 0,
\]

\[
\lambda(\sigma) = -k - \sqrt{2c\sigma + k^2} \quad \text{for} \quad k < 0.
\]

We demonstrated that the functions of the form (15), (16) define the maximum of the functional \( J[\lambda] \), and as a result, do not define the minimum of the lift-to-drag ratio \( \kappa \), but its maximum, which we determine in the paper only for sake of completeness. Thus, application of the classical approach does not lead to finding extrema that are of most practical interest. To solve the auxiliary problems we use the technique developed earlier in papers [7]–[11] for investigation of extremal problems of the jet and cavity theory. This technique is based on Jensen’s inequality (12), theorem 204).

We denote by \( J_{\text{min}}(q) \) and \( J_{\text{max}}(q) \), correspondingly, the global minimum and maximum of the functional \( J[\lambda] \) for a given value of \( I[\lambda] = q \). A full solution to the auxiliary problem is given by

**Theorem 1** 1) The function \( J_{\text{min}}(q) \) is defined by the parametric equations

\[
q = q(b) = \frac{1}{2} \left[ b^2 - k^2 + k(b - a) - k^2 \log \frac{b}{a} \right],
\]

\[
J_{\text{min}} = \sqrt{2} \left( k \log \frac{b}{a} + b + k \right),
\]

where \( b \in (\sqrt{2}/e, \sqrt{2}) \),

\[
k = K(b) = \frac{(e - 1)b(b^2e - 2)}{2 + (e - 2)b^2e}, \quad a = \frac{b^2e - 2}{(e - 1)b}.
\]
The global minimum is achieved by the function

\[
\lambda(\sigma) = \begin{cases} 
\sqrt{2\sigma} & \text{for } 0 \leq \sigma < \gamma, \\
-k + \sqrt{2c(\sigma - \gamma) + (a + k)^2} & \text{for } \gamma \leq \sigma \leq 1,
\end{cases}
\]

where \( c = \frac{2 - b^2}{2 + (e - 2)b^2} \).

2) The function \( J_{\max}(q) \) is defined by the parametric equations

\[
q = q(b) = \frac{1}{2} \left[ b^2 + kb - k^2 \log \frac{b + k}{k} \right],
\]

\[
J_{\max} = \sqrt{2} \left( k \log \frac{b + k}{k} + b \right),
\]

where \( k = K_1(b) = -\frac{b(b^2e - 2)}{2(b^2e - 1)} \), \( b \in (0, \sqrt{2/e}] \).

The global maximum is achieved by the functions

\[
\lambda(\sigma) = -k - \sqrt{2c\sigma + k^2} \quad \text{for } q \in (0, q_*), \quad b \in (0, \sqrt{1/e}),
\]

\[
\lambda(\sigma) = \sigma / \sqrt{c} \quad \text{for } q = q_*,
\]

\[
\lambda(\sigma) = -k + \sqrt{2c\sigma + k^2} \quad \text{for } q \in (q_*, q_{\max}), \quad b \in (\sqrt{1/e}, \sqrt{2/e}],
\]

where \( c = \frac{b^2}{2(b^2e - 1)} \).

A full solution to the problem of finding the absolute maximum of the lift-to-drag ratio \( \kappa \) is given by

**Theorem 2** The maximum value of the lift coefficient \( C_L = 2/e \). At a given value of the lift coefficient \( C_L \in (0, 2/e] \) the global minimum of the drag coefficient is \( C_{D_{\min}} = \frac{1}{2\pi} J_{\min}^2(C_L/2) \), and the global maximum of the lift-to-drag ratio \( \kappa_{\max} = 2\pi C_L / J_{\min}^2(C_L/2) \).

The problem of finding the minimum of the lift-to-drag ratio (the maximum of the drag) at a given \( C_L \) turns out to be more complex. We established that

\[
C_{D_{\max}} = \max_{\varepsilon, q_1, q_2} \frac{1}{2\pi} \left\{ \sqrt{1 - \varepsilon J_{\max}(q_2)} + \sqrt{\varepsilon J_{\max}(q_1)} \right\}^2,
\]

where \( \varepsilon, q_1 \) and \( q_2 \) satisfy the constraints

\[
\varepsilon \in [0, 1), \quad 0 \leq q_1, q_2 \leq q_{\max}, \quad (1 - \varepsilon)q_2 - \varepsilon q_1 = C_L/2 \in (0, q_{\max}], \quad q_{\max} = 1/e.
\]

The solution to the problem \( (20), (21) \) was found numerically by means of the standard function Maximize of the package Mathematica 8.0.

In fig. 2 we demonstrate the dependencies of the minimal drag coefficient \( C_{D_{\min}} \) and maximal drag coefficient \( C_{D_{\max}} \) on the lift coefficient \( C_L \), and in table 1 we show the maximal and minimal lift-to-drag ratios \( \kappa_{\max}, \kappa_{\min} \) for different \( C_L \). In fig. 2 the dash-and-dot line demonstrates the dependence \( C_D \) on \( C_L \) for the flat plate. The coefficients \( C_D \) and \( C_L \) for the flat plate are defined by Rayleigh’s well-known formulas ([5], p. 83):

\[
C_D = \frac{2\pi \sin^2 \alpha}{4 + \pi \sin \alpha}, \quad C_L = \frac{\pi \sin 2\alpha}{4 + \pi \sin \alpha}, \quad \kappa = \cot \alpha,
\]

where \( \alpha \) is the angle of attack.

As one can see from fig. 2 the dash-and-dot line lies entirely between the curves \( C_{D_{\min}}(C_L) \) and \( C_{D_{\max}}(C_L) \). It is worthy of note that for any profile in the Helmholtz–Kirchhoff flow the point \( (C_L, C_D) \) always lies between the curves \( C_{D_{\min}}(C_L) \) and \( C_{D_{\max}}(C_L) \).
Figure 2: Dependencies $C_D_{\text{min}}$ and $C_D_{\text{max}}$ on $C_L$ (the dash-and-dot line is the dependence $C_D$ on $C_L$ for a flat plate).

| $C_L$ | 0   | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | $2/e$ |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-------|
| $\kappa_{\text{max}}$ | $\infty$ | 224.88 | 99.1015 | 57.0649 | 35.9197 | 23.0608 | 14.1997 | 7.0821 | $\pi$ |
| $\kappa_{\text{min}}$ | 0 | 0.107495 | 0.219695 | 0.342541 | 0.48536 | 0.666406 | 0.933793 | 1.53824 | $\pi$ |

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