A NOTE ON THE RANK OF HEEGAARD FLOER HOMOLOGY

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ABSTRACT. We show that if K is a non-trivial knot inside the homology sphere Y, then the rank of $\hat{HFK}(Y, K)$ is strictly bigger than the rank of $\hat{HF}(Y)$.

1. INTRODUCTION

In [Ef5] the result of [Ef3] is used to show that, in a sense, the existence of an incompressible torus (almost) implies that the Heegaard Floer homology of the three manifold is non-trivial (different from that of $S^3$). More precisely, if $Y$ is a homology sphere which is not a non-trivial connected sum of two other three manifolds, then $\hat{HF}(Y; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ implies that $Y$ does not contain any incompressible torus. A combinatorial approach to Heegaard Floer homology along the lines of [SW, MOS] and [MOST] was taken in obtaining this result.

In proving this last result in [Ef5], a non-triviality result about knot Floer homology associated with a non-trivial knot in a homology sphere is used, which will be proved in this short paper. Namely, we prove:

Theorem 1.1. Let $Y$ be a homology sphere and $K \subset Y$ be a non-trivial knot. Then

$$\text{rk}(\hat{HFK}(Y, K; \mathbb{Z}/2\mathbb{Z})) > \text{rk}(\hat{HF}(Y; \mathbb{Z}/2\mathbb{Z})).$$

The non-triviality of $\hat{HFK}(Y, K; \mathbb{Z}/2\mathbb{Z})$ had been discussed before in the literature. The first step was the result of Ozsváth and Szabó (see [OS2]) which shows that Heegaard Floer homology can distinguish the genus of a knot $K$ in $S^3$. Ni (see [Ni]) extended this result to the knots in all homology spheres. If a homology sphere $Y$ is not a L-space, it is not clear at all from Ni’s argument that the rank of $\hat{HFK}(Y, K; \mathbb{Z}/2\mathbb{Z})$ is different from the rank of $\hat{HF}(Y; \mathbb{Z}/2\mathbb{Z})$. In [Hed], Hedden defines a knot $K$ in a three-manifold $Y$ to have simple Floer homology if the equality of ranks

$$\text{rk}(\hat{HFK}(Y, K; \mathbb{Z}/2\mathbb{Z})) = \text{rk}(\hat{HF}(Y; \mathbb{Z}/2\mathbb{Z}))$$

is satisfied. He relates this property to Berge conjecture on knots admitting lens space surgeries. It is interesting to try to extend the result of this paper (which is valid for homology spheres) to a more general context, at least to rational homology spheres. It seems however that the direct extension will not be true.

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2. Floer homology and surgery: A few theorems

Suppose that \( K \) is a knot inside the homology sphere \( Y \). Denote the knot Floer complex associated with \( K \) by \( \mathcal{C} = C(K) = CFK^\infty(Y, K; \mathbb{Z}/2\mathbb{Z}) \). Let \( nd(K) \) be a tubular neighborhood of \( K \) in \( Y \), which is homeomorphic to \( D^2 \times S^1 \). We may take out this solid torus and glue it back so that the resulting manifold \( Y_{p/q} = Y_{p/q}(K) \) is the three-manifold obtained by \( p/q \)-surgery on \( K \). The core \( \{0\} \times S^1 \) of the solid torus will then represent a knot \( K_{p/q} \) in \( Y_{p/q} \). If \( C(K) \) is generated by \( x_1, \ldots, x_r \), \( \mathcal{C} \) is the relative \( \text{Spin}^c \) class of \( K \). Let \( \mathcal{A}[t] \) be the free abelian group generated by those generators \( \{x_1, \ldots, x_r\} \) indexed by integer \( t \). For integers \( i, j \), let \( \mathcal{A}[i, j] \) be a copy of \( \mathcal{A}[j] \) indexed by \( i \). Let \( \mathcal{A}[i, j] = \mathbb{B} \) be a copy of the complex generated by \( \{0, j\} \) such that \( i(x) + j = 0 \) (with the differential induced from \( C(K) \)).

Let
\[
\mathcal{A}_q = \bigoplus_{s \in \mathbb{Z}} (\mathbb{A}, [\frac{s}{q}] ), \quad \mathbb{B}_q = \bigoplus_{s \in \mathbb{Z}} (\mathbb{B}, [\frac{s}{q}] ),
\]
and define \( h^p, v : \mathcal{A}_q \to \mathbb{B}_q \) as the sum of the respective maps
\[
h^p : (s, \mathbb{A}, [\frac{s}{q}]) \mapsto (s + p, \mathbb{B}), \quad v_s : (s, \mathbb{A}, [\frac{s}{q}]) \mapsto (s, \mathbb{B}).
\]

If \( t = [s/q] \), the map \( v_s \) is defined by projecting \( \mathcal{A}[t] = C\{i, j - t = 0\} \) on \( \mathbb{B} = C\{i = 0\} \), while \( h^p \) is defined by first projecting \( \mathcal{A}[t] \) on \( C\{j - t = 0\} \), and then using the chain homotopy equivalence of this last complex with \( \mathbb{B} \).

Ozsváth and Szabó proved the following theorem in [OS1]:

**Theorem 2.1.** The homology of the mapping cone \( \mathcal{M}(D^p_q) \) of \( D_q = h^p + v : \mathcal{A}_q \to \mathbb{B}_q \) is \( \widehat{HF}(Y_{p/q}; \mathbb{Z}/2\mathbb{Z}) \).

In [EF1] we extended this result to compute \( \widehat{HF}^\bullet(Y_n(K), K_n) \). The simplest version of the surgery formula proved in that paper is the following:

**Theorem 2.2.** With the above settings, the homology group \( \widehat{HF}(Y_n(K), K_n; s; \mathbb{Z}/2\mathbb{Z}) \) associated with \( K_n \) and the \( \text{Spin}^c \)-class \( s \in \mathbb{Z} \) is isomorphic to the homology of the following mapping cone
\[
C_n(s) = \left( \mathbb{B} \{\geq s\} \xrightarrow{i} \mathbb{B} \xrightarrow{\iota} \mathbb{B} \{\geq n - s\} \right),
\]
where \( i \) denotes the inclusion map, or equivalently the homology of the mapping cone
\[
\mathbb{B} \{j < s\} \xleftarrow{\iota} \mathbb{B} \{j > s - n\}.
\]

For a vector space \( E \), let \( |E| \) denote the rank of \( E \). The following is a quick corollary of this theorem:

**Corollary 2.3.** For \( K \subset Y \) as above, if \( |\widehat{HF}(Y)| = |\widehat{HF}(Y, K)| \), then for \( n = 1, 2, 3, \ldots \) we would have \( |\widehat{HF}(Y_n)| = |\widehat{HF}(Y_n, K_n)| \).

**Proof.** In [EF1] we showed that all the surgery computations depend only on the quasi-isomorphism type of the chain complex \( C(K) \). The differential \( d : C \to C \) of this complex may be written as \( d = \sum_{i,j \geq 0} d^{i,j} \), where \( d^{i,j} \) takes \( C(k,l) \) to \( C(k-i,l-j) \). Since the coefficient ring is a field, \( C(K) \) may be replaced with a
new chain complex, which is quasi-isomorphic to \( C(K) \) and will still be denoted by \( C(K) \), such that \( d^{0,0} = 0 \). If \( |HF(Y)| = |HF(K, K)| \), the above assumptions imply that all \( d^{0,j}, j = 1, 2, \ldots \) are trivial. Since changing the role of the two marked points in the Heegaard diagram for \((Y, K)\) does not change the chain homotopy type of the complex, it is implied that \( d^{0,j}, j = 1, 2, \ldots \) are also zero. As a result, the differential of each \( A_d \) complex, and the differential of \( B \) would be trivial, and we may use the map \( h^n \) in the statement of the theorem for cancelation. This implies that the homology group \( HF(Y_n) \) has the same rank as the homology of the following mapping cone

\[
\bigoplus_{s \in \mathbb{Z}} M(B\{j < s\} \to B\{j > n - s\}).
\]

Together with the above theorem, this completes the proof of corollary.

The third surgery formula is the combinatorial rational surgery formula from \([3, 4]\). Let \( \mathbb{H}_\bullet(K) \) denote the group \( HFK(Y_\bullet, K_\bullet) \) for \( \bullet \in \{\infty, 1, 0\} \). The holomorphic triangle construction gives two maps \( \phi, \varphi : \mathbb{H}_\infty(K) \to \mathbb{H}_1(K) \) and two other maps \( \psi, \psi : \mathbb{H}_1(K) \to \mathbb{H}_0(K) \) so that the following two sequences are exact:

\[
\mathbb{H}_\infty(K) \xrightarrow{\phi} \mathbb{H}_1(K) \xrightarrow{\varphi} \mathbb{H}_0(K), \quad \text{and}
\]

\[
\mathbb{H}_\infty(K) \xrightarrow{\psi} \mathbb{H}_1(K) \xrightarrow{\psi} \mathbb{H}_0(K).
\]

The homology of the mapping cones of \( \phi \) (or \( \varphi \)) and \( \psi \) (or \( \psi \)) are \( \mathbb{H}_0(K) \) and \( \mathbb{H}_\infty(K) \) respectively (see \([3, 4]\)). With the above notation fixed, we proved the following surgery formula in \([3, 4]\):

**Theorem 2.4.** Let \( K \) be a knot in a homology sphere \( Y \) and let the complexes \( \mathbb{H}_\bullet = \mathbb{H}_\bullet(K), \ \bullet \in \{\infty, 1, 0\} \) and the maps \( \phi, \varphi, \psi, \psi \) between them be as above. The homology of \( Y_{p/q}(K) \), the manifold obtained by \( \frac{p}{q} \)-surgery on \( K \) (for positive integers \( p, q \) with \( (p, q) = 1 \)), may be obtained as the homology of the complex

\[
\mathbb{H} = \left( \bigoplus_{i=1}^q \mathbb{H}_\infty(i) \right) \oplus \left( \bigoplus_{i=1}^{p+q} \mathbb{H}_1(i) \right) \oplus \left( \bigoplus_{i=1}^p \mathbb{H}_0(i) \right),
\]

where each \( \mathbb{H}_\bullet(i) \) is a copy of \( \mathbb{H}_\bullet \). The differential \( D \) of this complex is the sum of the following maps

\[
\phi^i = \phi : \mathbb{H}_\infty(i) \to \mathbb{H}_1(i), \quad \phi = \varphi : \mathbb{H}_\infty(i) \to \mathbb{H}_1(i + p), \quad i = 1, 2, \ldots, q
\]

\[
\psi^j = \psi : \mathbb{H}_1(j + q) \to \mathbb{H}_0(j), \quad \psi = \psi : \mathbb{H}_1(j) \to \mathbb{H}_0(j), \quad j = 1, 2, \ldots, p.
\]

Moreover, we have shown the following link between theorem 2.4 and theorem 2.2 which is proved in \([3, 4]\):

**Theorem 2.5.** Under the identification of \( \mathbb{H}_\infty(K, s) \) with \( B\{s\} \), \( \mathbb{H}_1(K, s) \) with the homology of the complex

\[
C_1(s) = \left( B\{\geq s\} \xrightarrow{u} B \xleftarrow{u} B\{\geq -s\} \right),
\]

the map \( \phi : \mathbb{H}_1(K) \to \mathbb{H}_\infty(K) \) the transpose of the map induced by the sum of maps \( \phi_s \) which take the quotient complex \( B\{s\} \) of the complex \( C_1(s) \) to its quotient \( B\{s\} \rightleftarrows \mathbb{H}_\infty(K, s) \). The map \( \varphi \) is the transpose of the sum of maps \( \varphi_s \) which are induced using the map \( \phi \), followed by the isomorphism \( B\{s\} \rightleftarrows B\{-s\} \).
3. Proof of the main result

In this section, using the surgery formulas quoted in the previous section we prove:

Theorem 3.1. Suppose that $K$ is a knot inside a homology sphere $Y$. If the groups \( \tilde{\text{HF}}(Y, K; \mathbb{Z}/2\mathbb{Z}) \) and \( \tilde{\text{HF}}(Y; \mathbb{Z}/2\mathbb{Z}) \) are isomorphic (or equivalently, if they have the same rank), then $K$ is the trivial knot in $Y$.

Proof. Let $K$ be a knot of genus $g$ inside the homology sphere $Y$. By the result of Ni ([Ni]) we have \( \tilde{\text{HF}}(Y, K, i) = 0 \) for $|i| > g$ and that this group is non-trivial for $i = \pm g$. With the assumption of the previous section on the complex $C(K)$, we conclude that \( h_s \) is an isomorphism for $s \leq -g$ and \( v_s \) is an isomorphism for $s \geq g$.

Let $p/q < 1$ be a positive rational number and use theorem 2.1 for computing \( Y_{p/q} = \tilde{\text{HF}}(Y_{p/q}) \). We may use the maps \( h_p \) to kill some of the generators of \( (s, A[[s/q]]) \) against the appropriate generators of \( (s + p, B) \). If we do so, what remains from the first complex would be isomorphic to \( B\{j < t\} \) and what remains from the second complex will be isomorphic to \( B\{j > t\} \), where \( t = [s/q] \). The remainder of the complex will look like a complex \( E = \oplus_{s \in \mathbb{Z}} E[s] \), where

\[
E[s] = M(B\{j < \lfloor s/q \rfloor\} \to B\{j > -\lfloor (s-p)/q \rfloor\}).
\]

Since $p/q < 1$, this sub-complex of $E$ is the same as \( M(B\{j < t\} \to B\{j > t\}) \) if $s$ is congruent to either of $p, p + 1, \ldots, q - 1$ modulo $q$, and is the same as \( M(B\{j < t\} \to B\{j > 1 - t\}) \) otherwise. This is implied that

\[
Y_{p/q} \cong pY_1 \oplus (q-p)\mathbb{H}_0 = p\mathbb{H}_1 \oplus (q-p)\mathbb{H}_0.
\]

Similarly, for arbitrary $p/q > 0$ we can compute the rank $y_{p/q}$ of $Y_{p/q}$ in terms of the ranks $h_n$ of $\mathbb{H}_n$:

\[
y_{p/q} = (p - q[p/q])h_{[p/q]} + (q[p/q] - p)h_{[p/q]}.
\]

Next we examine the same computation using theorem 2.4. The complex in the statement of theorem may be written as a vector space

\[
\mathbb{H} = \bigoplus_{i=1}^{p} \mathbb{H}_0 \oplus \bigoplus_{i=1}^{q} \mathbb{H}_\infty \oplus \bigoplus_{i=1}^{p+q} \mathbb{H}_1.
\]

The differential of this complex is a map $D : \mathbb{H} \to \mathbb{H}$ which has the following matrix form according to this decomposition

\[
D = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \Phi_{p,q} \\
\Psi_{p,q} & 0 & 0
\end{pmatrix},
\]

where $\Psi_{p,q} : \oplus^{p+q}\mathbb{H}_1 \to \oplus^p\mathbb{H}_0$ and $\Phi_{p,q} : \oplus^{p+q}\mathbb{H}_1 \to \oplus^q\mathbb{H}_\infty$ are given by the following matrix presentations (here $A^t$ is the transpose of the matrix $A$):

\[
\Psi_{p,q} = \begin{pmatrix}
\psi & 0 & \cdots & \psi & 0 & \cdots & 0 & \cdots & 0 \\
0 & \psi & \cdots & 0 & \psi & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \psi & 0 & \cdots & 0 & \psi & \cdots & 0
\end{pmatrix},
\]

Proof of the main result
The distance between the maps $\psi$ and $\bar{\psi}$ at each row is $q$ and the distance between the maps $\phi^t$ and $\bar{\phi}^t$ at each row is $p$. The rank of $D$ will be equal to the sum of the ranks of $\Phi_{p,q}$ and $\Psi_{p,q}$, which will be denoted by $x$ and $z$ respectively. Then $y_{p/q}$ will be equal to $\text{rnk}(D) = 2(x + y)z$.

We may assume that in an appropriate basis for $\mathbb{Z}$, the maps $\phi, \bar{\phi}, \psi$ and $\bar{\psi}$ have the following block forms

$$
\phi = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},
$$

$$
\bar{\phi} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \psi = \begin{pmatrix} m & n \\ l & k \end{pmatrix}.
$$

We may write $x$ as $q \text{rnk}(\phi) + x_{p,q}$. If $q = ip + r$ with $i \geq 0$ and $0 \leq r < p$, then for $i = 0$ we have $x_{p,q} = qx_0$ and for $i > 0$ we have $x_{p,q} = rx_i + (p - r)x_{i-1}$, where $x_i$ is the rank of the following matrix

$$
A_i(a, b, c, d) = \begin{pmatrix} d & 0 & \ldots & 0 \\ cb & d & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ ca^{i-1}b & ca^{-2}b & \ldots & d \\ a^ib & a^{i-1}b & \ldots & b. \end{pmatrix}
$$

A similar expression exists for $z$ as $p \text{rnk}(\bar{\psi}) + z_{p,q}$. If $p = jq + s$ with $j \geq 0$ and $0 \leq s < q$, for $j = 0$ we have $z_{p,q} = p\alpha_0$ and for $j > 0$ we have $z_{p,q} = sz_j + (q - s)z_{j-1}$, where $z_j$ is the rank of $A_j(a^i, m^i, k^i, l^i)$. Since the rank of $\phi$ is equal to $1/2(h_\infty + h_1 - h_0)$ (and the rank of $\bar{\psi}$ is equal to $1/2(h_0 + h_1 - h_\infty)$) we obtain that $y_{p/q}$ may be written as $y_{p/q} = ph_\infty + qh_0 - 2(x_{p,q} + z_{p,q})$.

First, let us assume that $q = ip + r$ with $i > 0$. The two computations above imply that

$$ph_1 + (q - p)h_0 = ph_\infty + qh_0 - 2(rx_i + (p - r)x_{i-1} + p\alpha_0).$$

We may rewrite this equation as

$$p(h_1 - h_0 - h_\infty + 2(x_{i-1} + \alpha_0)) = 2r(x_i - x_{i-1}),$$

and conclude that $x_i = x_0$ for $i > 0$, and that $h_1 = h_0 + h_\infty - 2(x_0 + \alpha_0)$. This is true since the values of $r$ and $p$ may be changed without changing $i$. The first equality implies that $d = ca^ib = 0$ for $i \geq 0$.

Similarly, if we assume that $p = jq + s$ with $j > 0$, it is implied that

$$sh_{j+1} + (q - s)h_j = ph_\infty + qh_0 - 2(qx_0 + sz_j + (q - s)z_{j-1}),$$

which may be reformulated as

$$q(h_j - jh_\infty - h_0 + 2(x_0 + z_{j-1})) = s(h_j + h_\infty - h_{j+1} - 2(z_j - z_{j-1})).$$

Thus, for $j > 0$ we have $h_j = jh_\infty + h_0 - 2(x_0 + z_{j-1})$. Since $h_j = y_j$ should be asymptotic to $jy_\infty = jh_\infty$, we conclude that $z_j$ should be bounded and consequently,
\( l = mn^i k = 0, \) for \( i \geq 0. \) This means that \( z_j \) is the rank of \( (m^i, (mn)^i, \ldots, (mn^j)^i). \)

Note that \( h_2 = 2h_1 - h_0, \) which implies that \( z_1 = 2z_0 + x_0, \) i.e. the rank of \( (m^i, (mn)^i) \) is at least two times the rank of \( m^i, \) plus \( x_0. \) Automatically, we conclude that \( x_0 = 0, \) and that \( h_j = jh_\infty + h_0 - 2z_{j-1}. \) Note that the first equality implies that \( b = 0, \) i.e. \( \text{Ker}(\phi) = \text{Ker}(\overline{\phi}). \) Since both \( \phi \) and \( \overline{\phi} \) preserve the relative Spin\(^c\) class, this means that the sizes of the images of \( \phi^j \) and \( \overline{\phi} \) in each relative Spin\(^c\) class are equal.

By theorem 2.5, if we note that the complex \( \mathcal{B} \) has trivial differential, it is implies that for \( s > 0 \) the map \( \overline{\phi} \) is trivial, and the map \( \phi^j \) is surjective, while for \( s < 0 \) the converse is true. This can happen if and only if the target complex \( \mathcal{B}\{s\} \) is trivial for \( s \neq 0, \) i.e. if \( K \) is trivial by the result of Ni \([\text{Ni}]\). \( \square \)

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