PALEY’S THEOREM
FOR HANKEL MATRICES
VIA THE SCHUR TEST

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Abstract. Paley’s theorem about lacunary coefficients of functions in the classical space $H^1$ on the unit circle is equivalent to the statement that certain Hankel matrices define bounded operators on $\ell^2$ of the nonnegative integers. Since that statement reduces easily to the case where the entries in the matrix are all nonnegative, it must be provable by the Schur test. We give such proofs with interesting patterns in the vectors used in the test, and we recover the best constant in the main case. We use related ideas to reprove the characterization of Paley multipliers from $H^1$ to $H^2$.

1. Introduction

Consider semi-infinite matrices with the Hankel symmetry in which the entries only depend on the sum of the indices. We use a method of Schur to prove that matrices like

$$A_v = \begin{bmatrix} v_0 & v_1 & 0 & v_2 & 0 & 0 & 0 & v_3 & \cdots \\ v_1 & 0 & v_2 & 0 & 0 & v_3 & 0 & \cdots \\ 0 & v_2 & 0 & 0 & 0 & v_3 & 0 & 0 & \cdots \\ v_2 & 0 & 0 & 0 & v_3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & v_3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & v_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & v_3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where there are relatively large gaps between most nonzero antidiagonals, act boundedly on $\ell^2$ when $v \in \ell^2$. It remains unclear how to use
the Schur test to prove boundeness, when it holds, for a general Hankel matrix with nonnegative entries.

To get \( A_v \), fix a strictly increasing sequence \( (k_j)_{j=0}^{\infty} \) of nonnegative integers, and denote its range by \( K \). Let \( (v_j)_{j=0}^{\infty} \) be a sequence of complex numbers. Let \( a_v \) be the function on the nonnegative integers with \( a_v(k_j) = v_j \) for all \( j \) and \( a_v(n) = 0 \) when \( n \notin K \). For any sequence \( (a(n))_{n=0}^{\infty} \), let \( H_a \) be the matrix with entries

\[
H_a(m, n) = a(m + n).
\]

Then let \( A_v = H_{a_v} \).

We start the indices \( m \) and \( n \) at 0 rather than 1 to simplify our discussion of Theorems 1.1 and 1.2 below. The sequence \( a_v \) and the matrix \( A_v \) also depend on the set \( K \), but we suppress that fact in the notation. Recall that \( K \) is called a Hadamard set if

\[
k_{j+1} > (1 + \varepsilon)k_j
\]

for some positive constant \( \varepsilon \) and all values of \( j \).

Most proofs of Theorem 1.1 below proceed via its counterpart Theorem 1.2, rather than working directly with the matrix formulation. It is easy, however, to convert the proof in [11, pp. 274–275] into such a direct proof. There are references to other elementary proofs in [2], but not to one via the Schur test. We discuss various forms of that test in Section 2, recall its proof in Section 3 and use it in Section 4 to prove the following statement.

**Theorem 1.1.** Let \( K \) be a Hadamard set, and let \( v \in \ell^2 \). Then the matrix \( A_v \) represents a bounded operator on \( \ell^2 \).

As noted in [11], this is equivalent to the better-known result of Paley [14] below. Given a function \( f \) in \( L^1(-\pi, \pi] \), denote its Fourier coefficients by

\[
\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int} dt.
\]

Denote the set of such functions \( f \) for which \( \hat{f}(n) = 0 \) for all \( n < 0 \) by \( H^1 \), and let \( H^2 \) be the set of functions in \( L^2 \) with the same property.

**Theorem 1.2.** Let \( K \) be a Hadamard set. If \( f \in H^1 \), then the restriction of the coefficients of \( f \) to the set \( K \) belongs to \( \ell^2(K) \).

Another way to state this is that multiplying the Fourier coefficients of \( H^1 \) functions by \( 1_K \), the indicator function of the set \( K \), gives the coefficients of functions in \( H^2 \). Sequences, like \( 1_K \), with this multiplier property are now called Paley multipliers. They played a rôle in the
resolution \cite{15} of the similarity problem of Sz.-Nagy and Halmos. In Section 5 we will reprove the following fact.

**Theorem 1.3.** Let \((a(n))_{n=0}^\infty\) be a sequence with the property that

\[
(1.2) \quad \sup_{j \geq 0} \left[ \sum_{2^j \leq n < 2^{j+1}} |a(n)|^2 \right] < \infty.
\]

Then \(a\) is a Paley multiplier.

This follows from another statement that we reprove by passing to its Hankel counterpart, and then using a variant of the Schur test.

**Lemma 1.4.** Let \((a(n))_{n=0}^\infty\) be a sequence with the property that

\[
(1.3) \quad \sum_{j \geq 0} \left[ \sum_{2^j \leq n < 2^{j+1}} |a(n)| \right]^2 < \infty.
\]

Then the sequence \((a(n)\hat{f}(n))_{n=0}^\infty\) belongs to \(\ell^1\) for every function \(f\) in \(H^1\).

Condition (1.2) above is also necessary for \(a\) be a Fourier multiplier from \(H^1\) to \(H^2\). Condition (1.3) is not necessary for the conclusion of Lemma 1.4. Instead, the strictly weaker condition (1.4) below is necessary if \(a \geq 0\), and sufficient in any case. That is an unpublished result of Charles Fefferman; see \[1\] page 264.

Consider the corresponding sufficiency result for Hankel matrices.

**Theorem 1.5.** Let \((a(n))_{n=0}^\infty\) be a sequence with the property that

\[
(1.4) \quad \sup_{M>0} \left\{ \sum_{j=1}^{\infty} \left[ \sum_{jM \leq n < (j+1)M} |a(n)| \right]^2 \right\} < \infty,
\]

Then the matrix \(H_a\) represents a bounded operator on \(\ell^2\).

The proofs of this in \[3, 17, 18\] all run via the equivalent statement on \((-\pi, \pi]\) and the duality between \(H^1\) and \(BMO\). In \[2\] pp. 423–424, Grahame Bennett asked for an elementary proof. There must be one using the Schur test, but we have not found it.

In Sections 6–8 we sharpen Theorem 1.1 in the following way. Denote the operator norm of \(A_v\) by \(\|A_v\|_\infty\). By uniform boundedness, that theorem is equivalent to the existence of a constant \(C_K\) for which

\[
(1.5) \quad \|A_v\|_\infty \leq C_K \|v\|_2 \quad \text{for all} \ \ell^2 \ \text{sequences} \ v.
\]

We use the Schur test and a pattern found in \[8\] to recover the fact that if \(k_{j+1} > 2k_j\) for all \(j\), then the smallest such constant \(C_K\) is \(\sqrt{2}\).
2. Schur tests

Let $F$ be the set of all finitely-supported sequences in the unit ball of $\ell^2$ of the nonnegative integers. Given a semi-infinite matrix $A$, consider the extended real number

$$\|A\|_\infty = \sup \left\{ \left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A(m,n)g(m)h(n) \right| : g, h \in F \right\} \tag{2.1}$$

This gives the usual operator norm when $A$ represents a bounded operator on $\ell^2$, and it is equal to $\infty$ otherwise.

We study conditions on the size of the entries in a Hankel matrix that make its operator norm finite. In this analysis, we may assume that the entries in the matrix are nonnegative, since the supremum above does not decrease if we replace all entries in the matrix $A$ and the sequences $g$ and $h$ by their absolute values.

**Lemma 2.1.** The following conditions on a semi-infinite matrix $A$ and its adjoint $A^*$ are equivalent when their entries are all nonnegative.

1. $A$ represents a bounded operator on $\ell^2$ with norm at most $S$.
2. For each number $T > S$ there are matrices $B$ and $C$ with nonnegative entries with the following properties.
   - (a) $A(m,n) = \sqrt{B(m,n)C(m,n)}$ for all $m$ and $n$.
   - (b) Each row of $B$ has $\ell^1$ norm at most $T$.
   - (c) Each column of $C$ has $\ell^1$ norm at most $T$.
3. For each number $T > S$ there are strictly positive vectors $u$ and $w$ for which $Au \leq Tw$ and $A^*w \leq Tu$.

Condition (3) is the frequently-used Schur test. The less-known condition (2) is easier to apply to Paley multipliers. We rewrite it using the mixed-norm notation

$$\|B\|_{(1,\infty)} = \sup_m \sum_{n=0}^{\infty} |B(m,n)|$$

and the notation $B \star C$ for the Schur (or Hadamard) product matrix with entries $B(m,n)C(m,n)$. Then condition (2) requires that

$$A \star A = B \star C, \quad \|B\|_{(1,\infty)} \leq T, \quad \|C^*\|_{(1,\infty)} \leq T, \tag{2.2}$$

and that the entries in these matrices all be nonnegative.

The fact that condition (3) implies condition (2) is implicit in proofs, like the one in [12, Section 5.2], of the Schur test. So is the fact that (2) implies (1). The fact that (1) implies (3) goes back to [10] or [9]. Since we use ideas from a proof of the lemma, we include that proof.
Remark 2.2. For some authors [6, pp. cii–cviii], the “Schur test” requires more, namely that conditions (2b) and (2c) hold with \(B\) and \(C\) replaced by \(A\). The former condition then implies that \(A\) acts boundedly on \(\ell^\infty\), and the latter that \(A\) acts boundedly on \(\ell^1\). One can then use Cauchy-Schwartz, or more sophisticated methods, to get boundedness on \(\ell^2\).

Remark 2.3. Sometimes, one can exhibit strictly positive vectors \(u\) and \(v\) for which condition (3) holds in the stronger form where \(Au = Sw\) and \(A^*w = Sv\). Then \(u\) is an eigenvector of \(A^*A\) with eigenvalue \(S^2\), so that \(\|A\|_\infty \geq S\). Since the Schur test makes \(S\) an upper bound for that norm, \(\|A\|_\infty = S\) in such a case. Moreover, if \(A\) is symmetric then \(u + w\) is an eigenvector of \(A\) with eigenvalue \(S\). We exploit this possibility in Section 8.

3. A proof of equivalence

When condition (3) in Lemma 2.1 holds, let

\[
B = A \star \left( \frac{u(n)}{w(m)} \right)_{m,n=0}^\infty, \quad \text{and} \quad C = A \star \left( \frac{w(m)}{u(n)} \right)_{m,n=0}^\infty
\]

respectively. Then \(B \star C = A \star A\). For this choice of \(B\) and \(C\), conditions (2b) and (2c) are equivalent to requiring that \(Au \leq Tw\) and \(A^*w \leq Tu\).

Suppose now that condition (2) holds in any way. Apply the Cauchy-Schwarz inequality to the sum

\[
\sum_{m=0}^\infty \sum_{n=0}^\infty A(m,n)g(m)h(n) = \sum_{m=0}^\infty \sum_{n=0}^\infty \sqrt{B(m,n)g(m)}\sqrt{C(m,n)h(n)}
\]

to get the upper bound

\[
\left[ \sum_{m=0}^\infty \sum_{n=0}^\infty B(m,n)g(m)^2 \right]^{1/2} \left[ \sum_{m=0}^\infty \sum_{n=0}^\infty C(m,n)h(n)^2 \right]^{1/2}.
\]

By condition (2b), the inner sum in the first factor above is no larger than \(\{Tg\}(m)\). Since \(\|g\|_2 \leq 1\), the first factor is at most \(\sqrt{T}\). Reversing the order of summation in the second factor and using condition (2c) gives the same upper bound for that factor when \(\|h\|_2 \leq 1\). The matrix \(A\) therefore represents a bounded operator on \(\ell^2\) with norm at most \(T\). Use this for all numbers \(T > S\) to get that \(\|A\|_\infty \leq S\), and that (2) \(\implies\) (1).

Finally, assume that condition (1) holds, and fix \(T > S\). Suppose initially that \(A\) is symmetric. Given any strictly positive sequence \(d\)
in $\ell^2$, let $u$ be the sum of the convergent series
\begin{equation}
(3.2) \quad d + \frac{Ad}{T} + \frac{A^2d}{T^2} + \ldots
\end{equation}
Then $Au \leq Tu$, and condition (3) holds with $w = u$.

When $A$ is not symmetric, apply the reasoning above to $A^*A$ and to $AA^*$, both with norms $S^2$, to get strictly positive vectors $\hat{u}$ and $\check{u}$ for which
\[ A^*A\hat{u} \leq T^2\hat{u}, \quad \text{and} \quad AA^*\check{u} \leq T^2\check{u}. \]
Then let
\begin{equation}
(3.3) \quad u = T\hat{u} + A^*\check{u}, \quad \text{and} \quad w = A\hat{u} + T\check{u},
\end{equation}
and check that $Au \leq Tw$ and $A^*w \leq Tu$. So (1) $\Rightarrow$ (3). \hfill $\square$

**Remark 3.1.** There are several dichotomies here. First, the proof above shows that if condition (2) holds, then it must hold with factors $B$ and $C$ equal to Schur products of $A$ with matrices of rank one in which all entries are positive.

Next, when $A$ is symmetric and acts boundedely on $\ell^2$, the proof yields conditions (3) and (2) with $u = w$ and $C = B^*$. But if condition (3) or (2) is satisfied in any way for a symmetric matrix with nonnegative entries, then boundedness follows, and both conditions can be satisfied symmetrically. This also follows by simply replacing the strictly positive vectors $u$ and $w$ that satisfy condition (3) with $u + w$.

Finally, the proof that (1) $\Rightarrow$ (3) yields strictly positive vectors in $\ell^2$ that satisfy condition (3), while the proof that (3) $\Rightarrow$ (1) only requires that $u$ and $w$ be strictly positive, without necessarily belonging to $\ell^2$. This occurs in [13] and [4], where the sequence $(1/\sqrt{n+1})$ is used to prove one of the Hilbert inequalities with a best constant. Some of the sequences that we use in Section 8 also do not belong to $\ell^2$.

## 4. Proving Paley

Since sums of bounded operators are bounded, the conclusion of Theorem 1.1 also holds for sets $K$ that are unions of finitely many Hadamard sets. Similarly, it suffices to prove the theorem for sets $K$ that are *strongly lacunary* in the sense that
\begin{equation}
(4.1) \quad k_{j+1} > 2k_j \quad \text{for all } j,
\end{equation}
because each Hadamard set is a union of finitely many strongly lacunary sets.

We offer three answers to the question “What do Schur-test proofs of Paley’s theorem look like?” In this section, we specify a simple factorization that satisfies condition (2) in Lemma 2.1 whenever $K$ is
a union of finitely-many Hadamard sets. In Section 6, we use that factorization when $K$ is strongly lacunary to get a strictly positive vector $u$ that satisfies the symmetric version

\begin{equation}
A_v u \leq Tu
\end{equation}

of condition (3) in the lemma. We discuss the pattern in this example in Section 7. In Section 8, we modify the example to recover the best constant in the inequality $\|A_v\|_\infty \leq C_K \|v\|_2$ when $K$ is strongly lacunary.

A set is a union of finitely many Hadamard sets if and only there is a constant $M$ so that there are at most $M$ members of that set in each interval $[m, 2m)$. As in [16], this property also characterizes sets $K$ for which the conclusion of Theorem 1.2 holds. Similar reasoning applies to Theorem 1.1.

For any such set $K$, get $B$ and $C$ from $A_v$ as follows. Let $B$ and $C$ both match $A_v$ on the main diagonal. Let $B$ match $A_v \star A_v$ above that diagonal. In that region, let $C(m, n) = 1$ on the special antidiagonals where $m + n = k_j$ for some $j$, and let $C(m, n) = 0$ otherwise. Reverse the roles of $B$ and $C$ below the main diagonal. Then $B \star C = A_v \star A_v$, and $C = B^*$.

Since $\|C^*\|_{1,\infty} = \|B\|_{1,\infty}$, it suffices to show that either mixed norm is finite. In each row sum in $B$, the diagonal term is at most $\|v\|_2$, while the sum of the terms to the right of it is at most $(\|v\|_2)^2$. The sum of the terms to the left of the diagonal is the number of 1’s below the diagonal in that row of the matrix $B$. In the $m$-th row, this is the number of indices $k_j$ in the interval $[m, 2m)$. Therefore

\begin{equation}
\sum_{n=0}^{\infty} B(m, n) \leq M + \|v\|_2 + (\|v\|_2)^2.
\end{equation}

\section{Paley multipliers}

\textit{Proof of Lemma 1.4} The same arguments giving the equivalence of Theorems 1.2 and 1.1 show that Lemma 1.4 is equivalent to the statement that if a sequence $a$ satisfies condition (1.3), then the Hankel matrix $H_a$ acts boundedly on $\ell^2$. It suffices to prove that boundedness for strictly positive sequences $a$.

In that case, apply condition (2) in Lemma 2.1 as follows. Let $B$ match $H_a$ along the main diagonal. Above that diagonal, get the entries in $B$ by multiplying the entries in each column of $H_a$ by the sum along that column above the diagonal. In the $n$-th column, for instance,
multiply above the diagonal by
\[ \sum_{0 \leq m < n} H_a(m, n) = \sum_{n \leq r < 2n} a(r). \]

To the left of the diagonal, divide the entries in the \( n \)-th row of \( H_a \) by the same positive sum. Let \( C = B^* \), and check that \( H_a \star H_a = B \star C \).

Split the \( \ell^1 \) norm of the \( n \)-th row of \( B \) into the part before the diagonal, the diagonal term, and the part after the diagonal. The division above makes the the norm of the first part equal to 1. The diagonal term is no larger than \( \|a\|_2 \). The sum beyond the diagonal is equal to
\[ \sum_{s > n} H_a(n, s) \left[ \sum_{s \leq r < 2s} a(r) \right] = \sum_{n \leq r < 2n} a(n + s) \left[ \sum_{s \leq r < 2s} a(r) \right]. \]

In the inner sum above, the index \( r \) is always greater than \((n + s)/2\) and smaller than \(2(n + s)\). So the double sum is majorized by
\[ \sum_{i=1}^{\infty} a(i) \left[ \sum_{i/2 < r < 2i} a(r) \right], \]
which does not depend on \( n \). To bound this expression, write it dyadically as
\[ (5.1) \sum_{j \geq 0} \left\{ \sum_{2^j \leq i < 2^{j+1}} a(i) \left[ \sum_{i/2 < r < 2i} a(r) \right] \right\}. \]

Here the index \( r \) in the innermost sum is always greater than \( 2^{j-1} \) and smaller than \( 2^{j+2} \). So the quantity \( (5.1) \) is majorized by
\[ \sum_{j \geq 0} \left( \sum_{2^j-1 < i < 2^{j+2}} a(i) \right)^2, \]
which is finite if condition (1.3) holds. That gives the desired uniform bound on the \( \ell^1 \) norms of the rows of \( B \). \( \square \)

**Proof of Theorem 1.3.** Multiplying coefficients of \( H^1 \) functions by a sequence \( b \) always gives an \( \ell^2 \) sequence if and only if the product sequences \( bc \) with \( \ell^2 \) sequences \( c \) are always multipliers from \( \hat{H}^1 \) to \( \ell^1 \).

By Lemma 1.4, the product sequences have this property if
\[ \sum_{j \geq 0} \left[ \sum_{2^j \leq n < 2^{j+1}} \|b(n)\||c(n)| \right]^2 < \infty. \]
If condition (1.2) holds with the sequence $a$ replaced by $b$, then the Cauchy-Schwartz inequality yields the condition above for every $\ell^2$ sequence $c$. □

6. Folding patterns

When $K$ is strongly lacunary, and $v$ is strictly positive, ask that the matrix $B$ in Section 4 satisfy the version of equation (3.1) with $w = u$, that is
\begin{equation}
B = A_v \ast \left( \frac{u(n)}{u(m)} \right)_{m,n=0}^\infty.
\end{equation}
Unless $m + n = k_j$ for some $j$, the entries $B(m, n)$ and $A_v(m, n)$ both vanish, and condition (6.1) then places no restriction on the nonzero numbers $u(n)$ and $u(m)$. There is also no restriction if $m = n = k_j/2$, when that is an integer, since $B$ and $A_v$ agree along the main diagonal.

In the remaining cases, $A(m, n) = v_j$, while $B(m, n) = 1$ if $(m, n)$ lies below the main diagonal, and $B(m, n) = v_j^2$ above that diagonal. In the former case, equation (6.1) yields that $u(m) = v_j u(n)$, that is
\begin{equation}
(6.1) \quad u(k_j - n) = v_j u(n) \quad \text{when } 0 \leq n < k_j/2.
\end{equation}
Applying condition (6.1) above the main diagonal gives the equivalent conclusion with $m$ and $n$ interchanged.

Rescale the unknown positive vector $u$ to make $u(0) = 1$. Formula (6.2) remains valid. Apply it when $n = 0$ to get that
\[ u(k_j) = v_j u(0) = v_j \quad \text{for all } j. \]
By strong lacunarity, $k_i < k_j/2$ when $i < j$. By formula (6.2),
\[ u(k_j - k_i) = v_j u(k_i) = v_j v_i \quad \text{for all } i < j. \]
The fact that $k_i - k_h < k_j/2$ when $h < i < j$ then gives that
\[ u(k_j - k_i + k_h) = v_j v_i v_h \quad \text{when } h < i < j. \]
Continue in this way.

Strong lacunarity implies that each nonnegative integer, $k$ say, has at most one representation as a, possibly empty, sum of the form
\begin{equation}
(6.3) \quad k = k_{j_1} - k_{j_2} + k_{j_3} - \cdots \pm k_j, \quad \text{where } j_1 > j_2 > j_3 > \cdots.
\end{equation}
When $k$ has that form, the analysis above shows that
\begin{equation}
(6.4) \quad u(k) = v_{j_1} v_{j_2} v_{j_3} \cdots v_{j_r}.
\end{equation}
Denote the set of integers with an alternating representation (6.3) by $\text{Fold}(K)$. If $k_j = 2^j - 1$ for all $j$, then that set consists of all nonnegative integers. In that case, formula (6.4), with the usual convention
that the empty product is 1, completely determines the sequence $u$. Moreover, $u$ is strictly positive since $v$ is. It suffices to prove the theorem for such $v$’s.

When $k_0 > 0$, or $k_{j+1} > 2k_j + 1$ for some value(s) of $j$, proceed as follows. If $0 \notin K$, augment $K$ with 0, and then reindex $K$. For each nonnegative integer $j$, let $L_j$ and $R_j$ be the sets of integers in the intervals $[0, k_j]$ and $[k_{j+1} - k_j, k_{j+1}]$ respectively. Strong lacunarity makes $L_j$ and $R_j$ disjoint. They cover $L_{j+1}$ if and only if $k_{j+1} = 2k_j + 1$.

Algorithm 6.1. Start with $u(0) = 1$, and define $u$ iteratively by imposing the following conditions in the set $L_{j+1}$.

1. The values of $u$ on $R_j$ are just those on $L_j$ listed in reverse order and multiplied by $v_{j+1}$. 
2. At integers, if any, in the first half $(k_j, k_{j+1}/2]$ of the gap between $L_j$ and $R_j$ let $u$ take any positive values.
3. In the part of that gap strictly the right of the midpoint $k_{j+1}/2$, use the values from the part of the gap strictly the left of $k_{j+1}/2$ listed in reverse order and multiplied by $v_{j+1}$.

Then $u$ satisfies condition (6.1), and equation (6.4) holds on the set $\text{Fold}(K)$.

Remark 6.2. A simpler way to satisfy condition (3) in Lemma 2.1 is to allow $u$ to take any fixed positive constant value in each gap, rather than satisfying condition (3) above. The corresponding matrix $B$ then differs from the one used in Section 4.

7. Related Patterns

Patterns with folding properties similar to those in Algorithm 6.1 have occurred in settings where the sequence $u$ does not have to be nonzero or even real-valued. In those cases, let $u$ vanish in any gaps between the sets $L_j$ and $R_j$, instead of being positive there as in the algorithm. Formula (6.4) still gives the values of $u$ on $\text{Fold}(K)$, and $u$ vanishes off that set.

For any sequence $(v_j)_{j \geq 0}$ of complex numbers, define functions $U^{(j)}_e$ and $U^{(j)}_o$ on the interval $(-\pi, \pi]$ as follows.

Algorithm 7.1. Let $U^{(0)}_e = 1$ and $U^{(0)}_o = 0$. Given $U^{(j-1)}_e$ and $U^{(j-1)}_o$, let

\begin{align*}
U^{(j)}_e(t) &= U^{(j-1)}_e(t) + v_j \exp(ik_j t) U^{(j-1)}_o(t), \\
U^{(j)}_o(t) &= U^{(j-1)}_o(t) + v_j \exp(ik_j t) U^{(j-1)}_e(t).
\end{align*}
These trigonometric polynomials connect with Theorems 1.1 and 1.2 in two ways. First, rational functions of the form \( \frac{U_o^{(j)}}{U_o^{(j)}} \), with \( v \) replaced by a related sequence obtained via the Schur algorithm rather than the Schur test, were used in [8] to prove Theorem 1.2 and to discover a useful extension of it. Second, for a nonnegative sequence \( v \), the coefficients of the function \( t \mapsto U_e^{(j)}(-t) + U_o^{(j)}(t) \)
match the sequence \( u \) on \( \text{Fold}(K) \cap L_j \), and they vanish otherwise. As noted in [8], a very similar folding pattern for coefficients occurs in the modification of the Rudin-Shapiro polynomials used in [5] and [7]; the only difference is that the plus sign in equation (7.1) is replaced with a minus sign.

Remark 7.2. If \( k_j = 2^j - 1 \) for all \( j \), then \( u \) can be represented by a product that takes the place of the series (3.2) that was used in the proof of Lemma 2.1. Split the matrix \( A_v \) as a formal sum

\[
A_v = \sum_{j=1}^{\infty} A_v^{(j)},
\]

where \( A_v^{(j)} \) matches \( A_v \) on the antidiagonal where \( m + n = k_j \), and vanishes elsewhere. Let \( u^{(J)} \) be the sequence obtained by stopping Algorithm 6.1 when \( j = J \). Let \( e^{(0)} \) be the transpose of \( (1, 0, 0, 0, \cdots) \). Then

\[
u^{(J)} = [I + A_v^{(J)}] [I + A_v^{(J-1)}] \cdots [I + A_v^{(1)}] e^{(0)}.
\]

Expand this as

\[
I + \sum_{j=1}^{J} \left\{ A_v^{(j)} [I + A_v^{(j-1)}] \cdots [I + A_v^{(1)}] \right\} e^{(0)}.
\]

The \( j \)-th matrix summand above times \( e^{(0)} \) vanishes off the set \( R_j \). Since these sets are disjoint, \( u \) is equal to the infinite product

\[
\cdots [I + A_v^{(J)}] [I + A_v^{(J-1)}] \cdots [I + A_v^{(1)}] e^{(0)}.
\]

8. Recovering the best constant

Continue to assume that \( K \) is strictly lacunary and contains 0. In proving that

\[
\|A_v\|_{\infty} \leq \sqrt{2}\|v\|_2
\]
in that case, it is enough to consider nontrivial nonnegative sequences \( v \) with finite support, rescale them so that \( \|v\|_2 = 1/\sqrt{2} \), and show that \( \|A_v\|_\infty \leq 1 \).

We will apply Algorithm 6.1 to a sequence \( c \) obtained from such a sequence \( v \), and use the Schur test to confirm that \( \|A_v\|_\infty \leq 1 \). We will use the inverse process going from \( c \) to \( v \) to exhibit cases where \( \|A_v\|_\infty = 1 \) while \( v \) exceeds \( 1/\sqrt{2} \) by as little as we like. So the constant \( \sqrt{2} \) in inequality (8.1) is best possible.

The inverse process is easier to describe. As in [8], given \( (c_j)_{j=0}^\infty \), form sequences \( v^{(j)}(c) \) as follows. Let \( v^{(0)}(c) \) be the transpose of \( (c_0, 0, 0, \cdots) \). Given \( v^{(j-1)}(c) \) pass to \( v^{(j)}(c) \) by setting the \( J \)-th component of \( v^{(j)}(c) \) equal to \( c_j \), and multiplying all earlier entries in \( v^{(j-1)}(c) \) by \( (1 - |c_j|^2) \).

Inequality (8.1) and the fact that \( \sqrt{2} \) is the best constant there follow easily from the two lemmas below. The first one is equivalent, in the usual way, to previous results [8] about functions on \((−\pi, \pi)\). Here, we prove it more directly using the Schur test. For completeness, we also include a modified proof of the second lemma, which is essentially in [8].

**Lemma 8.1.** If \( |c_j| \leq 1 \) for all \( j \leq J \), then \( \|A_{v^{(j)}(c)}\|_\infty \leq 1 \). If \( c_j \) is real for all \( j \leq J \), and \( c_0 = 1 \), then \( \|A_{v^{(j)}(c)}\|_\infty = 1 \).

**Lemma 8.2.** If \( \|v\|_2 \leq 1/\sqrt{2} \) and \( v_j = 0 \) for all \( j > J \), then \( v = v^{(J)}(c) \) for a sequence \( c \) with the property that \( 0 \leq c_j \leq 1/\sqrt{2} \) for all \( j \). On the other hand, \( \|v^{(J)}(c)\|_2 = \sqrt{(J + 2)/(2J + 2)} \) if \( c_j = 1/\sqrt{J + 1} \) for all \( j \).

**Proof of Lemma 8.2.** Both parts are clear when \( J = 0 \). Let \( J > 0 \), and assume that both hold when \( J \) is replaced by \( J-1 \). Given a sequence \( v \) for which \( |v_j| < 1 \) for all \( j > 0 \) and \( v_j = 0 \) for all \( j > J \), replace \( v_J \) by 0 and divide all earlier entries in \( v \) by \( (1 - |v_J|^2) \) to get a sequence \( v' \).

Let \( \varepsilon = \|v\|^2 - 1/2 \) and \( \varepsilon' = \|v'\|^2 - 1/2 \). Then

\[
\varepsilon = \left(1 - |v_J|^2\right)^2 \left[\frac{1}{2} + \varepsilon'\right] + |v_J|^2 - \frac{1}{2}.
\]

Expand the product \( (1 - |v_J|^2)^2 \) \( (1/2) \) and simplify to get that

(8.2) \[
\varepsilon = \frac{1}{2} |v_J|^4 + (1 - |v_J|^2)^2 \varepsilon'.
\]

It follows that if \( \varepsilon' > 0 \), then \( \varepsilon > 0 \), contrary to the assumption in the first part of the lemma that \( \|v\|_2 \leq 1/\sqrt{2} \). Therefore, \( \varepsilon' \leq 0 \), and \( \|v'\|_2 \leq 1/\sqrt{2} \) in that part. By the inductive assumption \( v' \) is equal to \( v^{(J-1)}(c) \) for a sequence with the property that \( |c_j| \leq 1/\sqrt{2} \) for all \( j \). Replacing \( c_J \) by \( v_J \) then makes \( v = v^{(J)}(c) \).
In the first part of the lemma, replacing $v$ replaces $c$ enough changes in $(c)$ and that norm is not affected by changes in $< c$ assume that $0$ holds when some positive value $J$ magonal where the sequence $v$ A since the matrix entries $v$ vanish when $m$ $(8.3)$ $A$ | since $P$ \[ J \] \[ 2(J+1)^2 + \left( 1 - \frac{1}{J+1} \right)^2 \frac{1}{2(J+1)} \] \[ = \frac{1}{2(J+1)^2} + \left( \frac{J}{J+1} \right)^2 \frac{1}{(2J)} = \frac{1}{2(J+1)} \] □

Proof of Lemma \[ 8.4 \]

In the first part of the lemma, replacing $c$ by $|c|$ replaces $v^J(c)$ by $|v^J(c)|$, and does not decrease $\|A_{v^J(c)}\|_\infty$. Small enough changes in $(c_0, \cdots c_J)$ change $\|A_{v^J(c)}\|_\infty$ by as little as we like, and that norm is not affected by changes in $c_j$ when $j > J$. So we may assume that $0 < c_j < 1$ for all $j$.

Build a strictly positive sequence $u$ by using Algorithm 6.1 with the sequence $v$ replaced by $c$. Since the matrix entries $A_{v^J(c)}(m,n)$ vanish when $m+n > k_J$, each product vector $A_{v^J(c)}u$ is finite, moreover $[A_{v^J(c)}u](m) = 0$ for all $m > k_J$.

The desired upper bound on $\|A_{v^J(c)}\|_\infty$ follows by the Schur test if

\[(8.3)\]

$A_{v^J(c)}u \leq u.$

Since $|v_0| \leq 1$, inequality \[(8.3)\] is clear when $J = 0$. Assume that it holds when some positive value $J$ is replaced by $J - 1$.

Let $P_j$ be the Hankel matrix with entries equal to $1$ on the antidiagonal where $m+n = k_J$ and to $0$ otherwise. Then

$A_{v^J(c)} = (1 - c_j^2)A_{v^{J-1}(c)} + c_j P_j.$

Since the matrix entries $A_{v^{J-1}(c)}(m,n)$ vanish when $m+n > k_{J-1}$, and $A_{v^{J-1}(c)}u \leq u$, matters reduce to showing that

\[(8.4)\]

$[c_j P_j u](m) \leq \begin{cases} c_j^2 u(m) & \text{when } m \leq k_{J-1}; \\ u(m) & \text{otherwise}. \end{cases}$

It will turn out that equality holds in the first case above and also when $k_J/2 < m \leq k_J$, while strict inequality holds otherwise. Multiplying the vector $u$ by the matrix $P_j$ lists the entries $(u_0, u_1, \cdots u_J)$ in reverse order, and annihilates all other entries. So the inequality for the second case above is strict when $m > k_J$, because $[c_j P_j u](m)$ vanishes in that subcase, but $u(m) > 0$.

To confirm equality in the subcase where $k_J - k_{J-1} \leq m \leq k_J$, that is when $m \in R_{J-1}$, recall that $u(m)$ is defined there by listing the values of $u$ on $L_{J-1}$ in reverse order and multiplying them by $c_J$. Forming $[c_j P_j u](m)$ does the same things.
The definition \( u(m) \) also makes \( [c_J P_J u](m) = u(m) \) inside the second half of the gap where \( k_{J-1} < m < k_J - k_{J-1} \). In the first half of that gap, where \( k_{J-1} < m \leq k_J / 2 \), the values \( [c_J P_J u](m) \) are equal to \( c_J u(k_J - m) \). This is equal to \( c_J u(m) \) if \( m = k_J / 2 \), and to \( c_J^2 u(m) \) if \( m < k_J / 2 \), making \( [c_J P_J u](m) < u(m) \) in either case. The analysis in the gap \( (k_{J-1}, k_J) \) is even simpler for the choice of \( u \) proposed in Remark 6.2 when there is strict inequality in the whole gap.

Finally, suppose that \( m \leq k_{J-1} \), that is \( m \in L_{J-1} \). The values of \( c_J P_J u \) on \( L_{J-1} \) are those of \( u \) on \( R_{J-1} \) listed in reverse order and multiplied by \( c_J \). On the other hand, the values of \( u \) on \( R_{J-1} \) come from those on \( L_{J-1} \) by another reversal of order and another multiplication by \( c_J \). So \( [c_J P_J u](m) = c_J^2 u(m) \) again, and inequality (8.4) holds.

The conclusion that \( \| A_{u(J)}(c) \|_\infty = 1 \) in the second part of the lemma now follows if 1 is an eigenvalue \( A_{u(J)}(c) \). For that purpose, let \( u^{(J)} \) be the vector obtained from \( u \) by replacing all values of \( u \) off \( \text{Fold}(K) \cap L_J \) by 0. Now \( A_{u(J)}(c) u^{(J)} = u^{(J)} \) if \( J = 0 \), because \( v_0 = 1 \).

Suppose that \( A_{u(J-1)}(c) u^{(J-1)} = u^{(J-1)} \). Then

\[
[A_{u(J-1)}(c) u^{(J)}](m) = \begin{cases} u^{(J)}(m) & \text{when } m \leq k_{J-1} \\ 0 & \text{otherwise.} \end{cases}
\]

Proving that \( A_{u(J)}(c) u^{(J)} = u^{(J)} \) therefore reduces to checking that

\[
[c_J P_J u^{(J)}](m) = \begin{cases} c_J^2 u^{(J)}(m) & \text{when } m \leq k_{J-1}; \\ u^{(J)}(m) & \text{otherwise.} \end{cases}
\]

Both sides of the equation above vanish when \( m > k_J \) and also in the gap where \( k_{J-1} < m < k_J - k_{J-1} \). For the same reasons as before, the two sides agree when \( k_J - k_{J-1} \leq m \leq k_J \) and when \( m \leq k_{J-1} \).

\[ \square \]

Remark 8.3. Inequality (8.1) also holds for strongly lacunary sets that do not contain 0, because it does when 0 \( \in K \) and \( v_0 = 0 \). To see that the constant \( \sqrt{2} \) is still best possible in those cases, again use the sequences \( v^{(J)}(c) \) specified in the second part of Lemma 8.2. The 0-th component of \( v^{(J)}(c) \) is

\[
\prod_{j=1}^{J} (1 - |c_j|^2) = \prod_{j=1}^{J} \frac{j}{j+1} = \frac{1}{J+1},
\]

which does not vanish, but tends to 0 as \( J \to \infty \).
Remark 8.4. When $K$ is strictly lacunary, the argument in Section 6 shows that condition (3) in the Schur test is also satisfied by the sequence $u$ arising from $v$ itself rather than from $c$. This yields inequality (1.5) with $C_K = 2$ rather than $\sqrt{2}$. Indeed, strict lacunarity makes the number of indices $k_i$ in the interval $[m, 2m)$ at most 1; moreover, if there is such an index, then $k_{i+1} > 2m$, and the diagonal term in the $m$-th row of $A_v$ vanishes. The outcome is the improvement

$$\sum_{n=0}^{\infty} B(m, n) \leq \max\{1, \|v\|_2\} + (\|v\|_2)^2$$

on the estimate (1.3). By the Schur test, the right-hand side above is an upper bound for $\|A_v\|_\infty$. Rescaling $v$ so that $\|v\|_2 = 1$ makes that upper bound equal to $2\|v\|_2$.

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