New Constants in Discrete Lieb-Thirring
Inequalities for Jacobi Matrices

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December 9, 2013

Abstract
This paper is essentially derived from the observation that some results
used for improving constants in the Lieb-Thirring inequalities for Schrödinger
operators in $L^2(-\infty, \infty)$ can be translated to the discrete Schrödinger op-
erators and more generally to Jacobi matrices. Some results were previ-
ously proved by D. Hundertmark and B. Simon and the aim of this article
is to improve the constants obtained in their article [7].

1 Introduction and main results

Let $W$ be a tridiagonal self-adjoint Jacobi matrix

$$W = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
\cdots & b_{-1} & a_{-1} & 0 & 0 & \cdots \\
\cdots & a_{-1} & b_0 & a_0 & 0 & \cdots \\
\cdots & 0 & a_0 & b_1 & a_1 & \cdots \\
\cdots & 0 & 0 & a_1 & b_2 & \cdots \\
\ddots & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix}$$

viewed as an operator in $\ell^2(\mathbb{Z})$ of complex sequences:

$$Wu(n) = a_{n-1}u(n-1) + b_nu(n) + a_nu(n+1), \quad n \in \mathbb{Z}. \quad (1)$$
In what follows we assume that \( a_n > 0, \ b \in \mathbb{R} \) and \( a_n \to 1, \ b_n \to 0, \ n \to \pm \infty \), rapidly enough. Then the essential spectrum \( \sigma_{\text{ess}} = [-2, 2] \) and \( W \) may have simple eigenvalues \( \{ E_{\pm}^j \} \) \( \forall \ j \in \mathbb{N} \cup \{ \infty \} \),
\[ E_1^+ > E_2^+ > \ldots > 2 > -2 > \ldots > E_2^- > E_1^- . \]

Applying the method of D. Hundertmark, E. Lieb and L. Thomas [6] to Jacobi matrices, D. Hundertmark and B. Simon [7] have proved the following result concerning Lieb-Thirring inequalities for Jacobi matrices:

**Theorem 1.1 (Hundertmark-Simon).** Let \( \{ b_n \}, \ \{ a_n - 1 \} \in l^1 \). Then
\[
\sum_{j=1}^{N_+} ((E_j^+)^2 - 4)^{1/2} + \sum_{j=1}^{N_-} ((E_j^-)^2 - 4)^{1/2} \leq \sum_{n=-\infty}^{\infty} |b_n| + 4 \sum_{n=-\infty}^{\infty} |a_n - 1| . \tag{2}
\]
Moreover, if \( \{ b_n \}, \ \{ a_n - 1 \} \in l^{\gamma + 1/2}, \ \gamma \geq 1/2 \), then
\[
\sum_{j=1}^{N_+} |E_j^+ - 2|^{\gamma} + |E_j^- + 2|^{\gamma} \leq c_{\gamma} \left[ \sum_{n=-\infty}^{\infty} |b_n|^{\gamma + 1/2} + 4 \sum_{n=-\infty}^{\infty} |a_n - 1|^{\gamma + 1/2} \right] \tag{3}
\]
where\[
c_{\gamma} = 2(3^{1/2} - 1)L_{\gamma,1} .
\]
where\[
L_{\gamma,1} = \frac{\Gamma(\gamma + 1)}{2\sqrt{\pi} \Gamma(\gamma + 3/2)} .
\]

**Remark.** Note that the constants in front of each sum in the right hand side of (2) are sharp and this is the only case when sharp constants in Lieb-Thirring inequalities are known for Jacobi matrices.

This paper is one of a series of papers where we would like to obtain improved and possibly sharp constants for discrete operators. In particular, the aim of this paper is to improve the constants \( c_{\gamma} \) appearing in (3) for \( \gamma \geq 1 \) by applying the method of A. Eden and C. Foias, [3] who obtained improvement of constants in Lieb-Thirring inequalities for one-dimensional Schrödinger operators acting in \( L^2(\mathbb{R}) \).
Let us first recall known results for “continuous” multi-dimensional Schrödinger operators that give us a motivation for the study of discrete problems. Let $H$ be a Schrödinger operator acting in $L^2(\mathbb{R}^d)$

$$H\psi_j := -\Delta \psi_j(x) + V\psi_j(x) = -e_j \psi_j(x). \quad (4)$$

Lieb-Thirring inequalities relate the eigenvalues $\{e_j\}$ of the operator $H$ and the potential $V \in L^{\gamma+n/2}(\mathbb{R}^d)$ via the following estimate

$$\sum_j |e_j|^\gamma \leq L_{\gamma,d} \int V_-(x)^{\gamma+d/2}dx, \quad (5)$$

where $V_- = (|V| - V)/2$ is the negative part of $V$.

It is known that the constants $L_{\gamma,d}$ are finite if $\gamma \geq 1/2 \ (d = 1)$, $\gamma > 0 \ (d = 2)$, and $\gamma \geq 0 \ (d \geq 3)$. If $\gamma = 0 \ (d \geq 3)$ the inequality (5) is called the CLR-inequality (Cwikel-Lieb-Rozenblum). The case $\gamma = 1/2 \ (d = 1)$ was justified by T. Weidl in [11]. In all these cases we have the following Weyl asymptotic formula for the eigenvalues of the operator $H(\alpha) = -\Delta + \alpha V$

$$\sum_j |e_j|^\gamma = \alpha^{\gamma+d/2} (2\pi)^{-d} \int \int \frac{(|\xi|^2 + V(x))^\gamma}{\lambda^{\gamma+d/2}}dxd\xi + o(\lambda^{\gamma+d/2})$$

$$= \alpha^{\gamma+d/2} L_{\gamma,d}^d \int V_-(x)^{\gamma+d/2}dx + o(\lambda^{\gamma+d/2}), \quad \text{as} \ \alpha \to \infty, \quad (6)$$

where

$$L_{\gamma,d}^d = (2\pi)^{-d} \int (|\xi|^2 - 1)_- ^\gamma d\xi.$$ 

Therefore the sharpness of the constants $L_{\gamma,d}$ appearing in (5) could be compared with the values of $L_{\gamma,d}^d$. Clearly (6) implies that $L_{\gamma,d} \leq L_{\gamma,d}^d$.

In some cases the values of sharp constants $L_{\gamma,d}$ are known. However, they do not always coincide with $L_{\gamma,d}^d$. It has been proved in [10] that $L_{3/2,1} = 3/16$ and by using [11] one obtains sharp constants $L_{\gamma,1}$ for all $\gamma \geq 3/2$. Later A. Laptev and T. Weidl [8] obtained sharp constants for $L_{\gamma,d}$ for all $\gamma \geq 3/2$ in any dimension. If $\gamma = 1/2$ and $d = 1$ then $L_{1/2,1} = 1/2$ was found by D. Hundertmark, E.B. Lieb and L. Thomas in [6].
Several attempts have been made to improve estimates for the constants $L_{\gamma,d}$. For $1/2 \leq \gamma < 3/2$, Hundertmark, Laptev and Weidl [5] found the constant to be not greater than $2 L_{\gamma,d}^{cl}$. Recently this has been improved for $1 \leq \gamma \leq 3/2$ by J. Dolbeault, A. Laptev and M. Loss [2] to $c L_{\gamma,d}^{cl}$, $c = 1.8...$, using methods essentially derived from Eden and Foias [3], which we will use ourselves in quite a substantial way. Eden and Foias essentially used an interesting method to improve the constant in the Lieb-Thirring inequalities in one dimension.

Let us now introduce the operator $D$ in $l^2(\mathbb{Z})$

$$D\varphi(n) = \varphi(n+1) - \varphi(n), \quad n \in \mathbb{Z}. \quad (7)$$

We choose its adjoint to be:

$$D^*\varphi(n) = -(\varphi(n) - \varphi(n-1)).$$

Then the discrete Laplacian takes the form:

$$D^* D \varphi(n) = -\varphi(n+1) - \varphi(n-1) + 2\varphi(n).$$

The spectrum $\sigma(D^* D)$ of this operator is absolutely continuous. We have $\sigma(D^* D) = [0, 4]$ and if $b_n \geq 0$ then the discrete Schrödinger operator

$$H_D := D^* D - b_n \quad (8)$$

may have negative eigenvalues.

Our first result is:

**Theorem 1.2.** Let $b_n \geq 0$, $\{b_n\}_{n=-\infty}^{\infty} \in l^{3/2}(\mathbb{Z})$. Then the negative eigenvalues $\{e_j\}$ of the operator in (8) are discrete and they satisfy the inequality

$$\sum_j |e_j| \leq \frac{\pi}{\sqrt{3}} L_{1,1}^{cl} \sum_{n \in \mathbb{Z}} b_n^{3/2}. \quad (9)$$

The standard argument due to Aizenman and Lieb [1] implies that we obtain the following spectral inequalities for moments $\gamma \geq 1$. 

4
Theorem 1.3. Let $b_n \geq 0$, $\{b_n\}_{n=-\infty}^{\infty} \in l^{\gamma+1/2}(\mathbb{Z})$, $\gamma \geq 1$. Then the negative eigenvalues $\{e_j\}$ of the operator in (9) satisfy the inequality

$$\sum_j |e_j|^\gamma \leq C \sum_{n \in \mathbb{Z}} b_n^{\gamma+1/2},$$

(10)

where as in the continuous case

$$C = \frac{\pi}{\sqrt{3}} L_{\gamma,1}^d.$$

By changing sign of the operator we immediately obtain a version of Theorem 1.3 for positive eigenvalues of the operator $-D^*D + b_n$, $b_n \geq 0$.

Corollary 1.4. Let $b_n \geq 0$, $\{b_n\}_{n=-\infty}^{\infty} \in l^{\gamma+1/2}(\mathbb{Z})$, $\gamma \geq 1$. Then the positive eigenvalues $\{e_j\}$ of the operator $-D^*D + b_n$ satisfy the inequality

$$\sum_j |e_j|^\gamma \leq C \sum_{n \in \mathbb{Z}} b_n^{\gamma+1/2},$$

(11)

where as in the continuous case

$$C = \frac{\pi}{\sqrt{3}} L_{\gamma,1}^d.$$

Our main result for Jacobi matrices is the following statement:

Theorem 1.5. Let $\gamma \geq 1$, $\{b_n\}_{n=-\infty}^{\infty}, \{a_n - 1\}_{n=-\infty}^{\infty} \in l^{\gamma+1/2}$. Then for the eigenvalues of the operator (11) we have

$$\sum_j |E_j^- + 2|^{\gamma} + |E_j^+ - 2|^{\gamma} \leq 3^{\gamma-1/2} \frac{\pi}{\sqrt{3}} L_{\gamma,1}^d \left( \sum_n |b_n|^{\gamma+1/2} + 4 \sum_n |a_n - 1|^{\gamma+1/2} \right).$$

(12)

Remark. Comparing the constants in right hand sides of (3) and (12) we note that the latter constant is around 1.1 times better.

2 Some auxiliary results

In order to prove our main results we consider some auxiliary statements.
Lemma 2.1. Let $\varphi, D\varphi \in l^2(\mathbb{Z})$. Then for any $n \in \mathbb{Z}$

$$| \varphi(n) |^2 \leq \|\varphi\|_{l^2} \|D\varphi\|_{l^2}$$

i.e.

$$| \varphi(n) |^2 \leq \left( \sum_{k=-\infty}^{\infty} |\varphi(k)|^2 \right)^{1/2} \left( \sum_{k=-\infty}^{\infty} |D\varphi(k)|^2 \right)^{1/2}.$$

Proof. For any $n \in \mathbb{Z}$ we have

$$|\varphi(n)|^2 = \frac{1}{2} \left| \sum_{k=-\infty}^{n} D(\varphi^2(k)) - \sum_{k=n+1}^{\infty} D(\varphi^2(k)) \right|$$

$$\leq \frac{1}{2} \left( \sum_{k=-\infty}^{\infty} |D(\varphi^2(k))| + \sum_{k=n+1}^{\infty} |D(\varphi^2(k))| \right)$$

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} |\varphi^2(k + 1) - \varphi^2(k)|$$

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} |D\varphi(k)| \left( |\varphi(k + 1)| + |\varphi(k)| \right).$$

Now we apply the Cauchy-Schwarz inequality and obtain

$$|\varphi(n)|^2 \leq \frac{1}{2} \sum_{k=-\infty}^{\infty} |D\varphi(k)| \left( |\varphi(k + 1)| + |\varphi(k)| \right)$$

$$\leq \frac{1}{2} \left( \sum_{k=-\infty}^{\infty} |D\varphi(k)|^2 \right)^{1/2} \left[ \left( \sum_{k=-\infty}^{\infty} |\varphi(k + 1)|^2 \right)^{1/2} + \left( \sum_{k=-\infty}^{\infty} |\varphi(k)|^2 \right)^{1/2} \right]$$

$$= \left( \sum_{k=-\infty}^{\infty} |D\varphi(k)|^2 \right)^{1/2} \left( \sum_{k=-\infty}^{\infty} |\varphi(k)|^2 \right)^{1/2}.$$

The proof is complete. \(\square\)

The next result is a discrete version of the result obtained by Eden and Foias in [3].

Lemma 2.2 (Discrete Generalised Sobolev Inequality). Let \(\{\psi_j\}_{j=1}^{N}\) be an orthonormal system of function in \(l^2(\mathbb{Z})\) and let \(\rho(n) = \sum_{j=1}^{N} \psi_j^2(n)\). Then

$$\sum_{n \in \mathbb{Z}} \rho^3(n) = \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^{N} |\psi_j(n)|^2 \right)^3 \leq \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} |D\psi_j(n)|^2.$$
Proof. Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{C}^N \). Then by Lemma 2.1 we obtain that for every \( n \in \mathbb{Z} \):

\[
\left| \sum_{j=1}^N \xi_j \psi_j(n) \right| \leq \left( \sum_{j,k=1}^N \xi_j \xi_k \langle \psi_j, \psi_k \rangle \right)^{1/4} \left( \sum_{j,k=1}^N \xi_j \xi_k \langle D\psi_j, D\psi_k \rangle \right)^{1/4}
\]

\[
\leq \left( \sum_{j=1}^N \xi_j^2 \right)^{1/4} \left( \sum_{j,k=1}^N \xi_j \xi_k \langle D\psi_j, D\psi_k \rangle \right)^{1/4}.
\]

If we set \( \xi_j = \psi_j(n) \) then the latter inequality becomes

\[
\rho(n) = \sum_{j=1}^N |\psi_j(n)|^2 \leq \rho^{1/4}(n) \left( \sum_{j,k=1}^N \psi_j(n) \overline{\psi_k(n)} \langle D\psi_j, D\psi_k \rangle \right)^{1/4}.
\]

Thus

\[
\rho^3(n) \leq \sum_{j,k=1}^N \psi_j(n) \overline{\psi_k(n)} \langle D\psi_j, D\psi_k \rangle.
\]

If we sum both sides also taking into account that \( \{ \psi_j(n) \} \) is an orthonormal system, then we arrive at

\[
\sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^N |\psi_j(n)|^2 \right)^3 \leq \sum_{j=1}^N \left( \sum_{n \in \mathbb{Z}} |D\psi_j(n)|^2 \right).
\]

Lemma 2.3. Discrete Schrödinger operators

\[-D^*D + b_n \quad \text{and} \quad D^*D - 4 + b_n\]

are unitary equivalent.

Proof. Indeed, let \( \mathcal{F} \) be the Fourier transform

\[
\mathcal{F}\varphi(\theta) = \hat{\varphi}(\theta) = \sum_{n=-\infty}^{\infty} \varphi(n)e^{in\theta}, \quad \theta \in (0, 2\pi).
\]

Then

\[
\mathcal{F}(D^*D - 4 + b_n)\mathcal{F}\varphi(\theta) = 2(1 - \cos \theta - 4)\hat{\varphi}(\theta) + \int_0^{2\pi} \hat{b}(\theta - \tau)\hat{\varphi}(\tau) \, d\tau.
\]
Therefore using periodicity of \( \hat{b} \) and denoting \( \hat{\psi}(\theta) = \hat{\phi}(\theta + \pi) \) we find
\[
\int_0^{2\pi} (-2 - \cos \theta)|\hat{\phi}(\theta)|^2 + \int_0^{2\pi} \int_0^{2\pi} \hat{b}(\theta - \tau) \hat{\phi}(\tau) \overline{\hat{\phi}(\theta)} \, d\tau d\theta
\]
\[
= \int_0^{2\pi} (-2 + \cos \theta)|\hat{\psi}(\theta)|^2 + \int_0^{2\pi} \int_0^{2\pi} \hat{b}(\theta - \tau) \hat{\psi}(\tau) \overline{\hat{\psi}(\theta)} \, d\tau d\theta.
\]
Since \( \mathcal{F}(-D^*D)\mathcal{F}^* \) is unitary equivalent to \(-2 + \cos \theta\) and this proves the lemma.

\( \square \)

### 3 Proof of Theorems \([1.2]\) and \([1.3]\)

We begin with the proof of Theorem \([1.2]\).

**Proof.** Let \( \{\psi_j\}_{j=1}^N \) be the orthonormal system of eigenvectors corresponding to the negative eigenvalues \( \{-e_j\} \) of the discrete Schrödinger operator:

\[
D^*D\psi_j - b_n\psi_j = -e_j\psi_j,
\]

where we assume that \( b_n \geq 0 \) \( \forall \ n \in \mathbb{N} \).

Then by using the latter result and Hölder’s inequality we obtain just like before

\[
\sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^N |\psi_j|^2 \right)^3 - \left( \sum_{n \in \mathbb{Z}} b_n^{3/2} \right)^{2/3} \left( \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^N |\psi_j|^2 \right)^3 \right)^{1/3}
\]
\[
\leq \sum_{j=1}^N \left( \sum_{n \in \mathbb{Z}} \left( |D\psi_j|^2 - b_n|\psi_j|^2 \right) \right) = - \sum_{j=1}^N e_j.
\]

Denote

\[
X = \left( \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^N |\psi_j|^2 \right)^3 \right)^{1/3},
\]

then the latter inequality can be written as

\[
X^3 - \left( \sum_{n \in \mathbb{Z}} b_n^{3/2} \right)^{2/3} X \leq - \sum_{j=1}^N e_j.
\]
Maximising the left hand side we find

$$X = \frac{1}{\sqrt{3}} \left( \sum b_{n/2}^{3/2} \right)^{1/3}.$$ 

We substitute this back into (3):

$$- \sum_{j=1}^{N} e_j \geq \frac{1}{3\sqrt{3}} \sum_{n \in \mathbb{Z}} b_{n}^{3/2} - \frac{1}{\sqrt{3}} \sum_{n \in \mathbb{Z}} b_{n}^{3/2}$$

$$= - \frac{2}{3\sqrt{3}} \sum_{n \in \mathbb{Z}} b_{n}^{3/2}$$

and we finally obtain the discrete version of the Lieb-Thirring Inequality:

$$\sum_{j=1}^{N} |e_j| \leq \frac{2}{3\sqrt{3}} \sum_{n \in \mathbb{Z}} b_{n}^{3/2} \quad (13)$$

The proof is complete.

Let $B$ be a Beta-function

$$B(x, y) = \int_{0}^{1} t^{x-1}(1-t)^{y-1} dt.$$ 

**Proof of Theorem 1.3**

Let $B$ be a Beta-function

$$B(x, y) = \int_{0}^{1} \tau^{x-1}(1-\tau)^{y-1} d\tau.$$ 

Then by scaling we obtain that for any $\gamma > 1$ and $\mu \in \mathbb{R}$

$$\mu_{+}^{\gamma} = B^{-1}(\gamma - 1, 2) \int_{0}^{\infty} \tau^{\gamma-2}(\mu - \tau)_{+} d\tau.$$ 

Let $e_{j}(\tau)$ be the eigenvalues of the operator $D^{*}D - (b_{n} - \tau)_{+}$. Then by variational principle for the negative eigenvalues $-(e_{j} - \tau)_{+}$ of the operator $D^{*}D - b_{n} + \tau$ we have

$$(e_{j} - \tau)_{+} \leq e_{j}(\tau).$$
Therefore for any $\gamma > 1$ applying Theorem 1.2 to the operator $D^*D - (b_n - \tau)_+$ we find
\[
\sum_j e_j^\gamma = B^{-1}(\gamma - 1, 2) \int_0^\infty \sum_j (e_j - \tau)_+ \tau^{\gamma - 2} d\tau
\leq B^{-1}(\gamma - 1, 2) \int_0^\infty \sum_j e_j(\tau)_+ \tau^{\gamma - 2} d\tau
\leq \frac{2}{3\sqrt{3}} B^{-1}(\gamma - 1, 2) \int_0^\infty \sum_n \tau^{\gamma - 2}(b_n - \tau)^{3/2} d\tau
= \frac{2}{3\sqrt{3}} B^{-1}(\gamma - 1, 2) B(\gamma - 1, 5/2) \sum_n b_n^{\gamma + 1/2}
= \frac{\pi}{\sqrt{3}} L_{\gamma, 1} \sum_n b_n^{\gamma + 1/2}
\]
which proves the required statement.

4 Proof of Theorem 1.5

Let us introduce notations
\[
(b_n)_+ = \max(b_n, 0), \quad (b_n)_- = -\min(b_n, 0)
\]
and let
\[
W(\{a_n\}, \{b_n\})u(n) = Wu(n) = a_{n-1}u(n-1) + b_nu(n) + a_nu(n+1).
\]

Note that using this notation and Lemma 2.3 we have
\[
D^*D + b_n = W(\{a_n \equiv 1\}, \{b_n + 2\}), \quad (14)
\]
\[
-D^*D + b_n = W(\{a_n \equiv -1\}, \{b_n - 2\}). \quad (15)
\]

Hundertmark and Simon [7] made an observation, which allows us to use bounds for $a_n \equiv 1$ for the general case. Namely for any $a_n \in \mathbb{R}$
\[
\begin{pmatrix}
-|a_n - 1| & 1 \\
1 & -|a_n - 1|
\end{pmatrix}
\leq
\begin{pmatrix}
0 & a_n \\
a_n & 0
\end{pmatrix}
\leq
\begin{pmatrix}
|a_n - 1| & 1 \\
1 & |a_n - 1|
\end{pmatrix}
\quad (16)
\]
Thus the above bound implies by repeated use at each point of indices:

\[ W(\{a_n \equiv 1\}, \{b^{-}_n\}) \leq W(\{a_n\}, \{b_n\}) \leq W(\{a_n \equiv 1\}, \{b^{+}_n\}) \] (17)

where \( b^{\pm}_n \) is given by

\[ b^{\pm}_n = b_n \pm (|a_{n-1} - 1| + |a_n - 1|). \]

Thus using (14) and Theorem 1.3 and the first inequality in (17) we have

\[ \sum_{j=1}^{N_-} |E_j^- + 2| \gamma \leq \frac{\pi}{\sqrt{3}} L_{\gamma,1} \sum_n \left(b_n - + |a_{n-1}| + |a_n| \right)^{\gamma+1/2}. \] (18)

Similarly using (15) and Corollary 1.4 and the second inequality in (17) we find

\[ \sum_{j=1}^{N_+} |E_j^+ - 2| \gamma \leq \frac{\pi}{\sqrt{3}} L_{\gamma,1} \sum_n \left(b_n + + |a_{n-1}| + |a_n| \right)^{\gamma+1/2}. \] (19)

Note that for any \( q \geq 1, \)

\[ (\alpha + \beta + \gamma)^q = 3^q \left(\frac{\alpha}{3} + \frac{\beta}{3} + \frac{\gamma}{3}\right)^q \leq 3^q(\alpha^q + \beta^q + \gamma^q). \] (20)

Applying (20) to each of (18) and (19) and summing them up we finally arrive at

\[ \sum_{j=1}^{N_-} |E_j^- + 2| \gamma + \sum_{j=1}^{N_+} |E_j^+ - 2| \gamma \leq 3^{\gamma-1/2} \frac{\pi}{\sqrt{3}} L_{\gamma,1} \left(\sum_n |b_n|^{\gamma+1/2} + 4 \sum_n |a_n - 1|^{|\gamma+1/2}}\right). \]

The proof of Theorem 1.5 is complete.

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