A NON-ALGEBRAIC PATCHWORK

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Abstract

Itenberg and Shustin’s pseudoholomorphic curve patchworking is in principle more flexible than Viro’s original algebraic one. It was natural to wonder if the former method allows one to construct non-algebraic objects. In this paper we construct the first examples of patchworked real pseudoholomorphic curves in Σₙ whose position with respect to the pencil of lines cannot be realised by any homologous real algebraic curve.

1 Introduction

Viro’s patchworking has been since the seventies one of the most important and fruitful methods in topology of real algebraic varieties. It was applied in the proof of a lot of meaningful results in this field (e.g. [Vir84], [Vir89], [Shu99], [LdM], [Ite93], [Ite01], [Han95], [Bru06], [Bih], [IV], [Ber06], [Mik05], and [Shu05]). Here we will only consider the case of curves in CP² and in rational geometrically ruled surfaces Σₙ.

Viro Method allows to construct an algebraic curve A out of simpler curves Aᵢ so that the topology of A can be deduced from the topology of the initial curves Aᵢ. Namely one gets a curve with Newton polytope Δ out of curves whose Newton polytopes are the 2-simplices of a subdivision σ of Δ, and one can see the curve A as a gluing of the curves Aᵢ. Moreover, if all the curves Aᵢ are real, so is the curve A. For a detailed account on the Viro Method we refer to [Vir84] and [Vir89] for example.

One of the hypotheses of the Viro Method is that σ should be convex (i.e. the 2-simplices of σ are the domains of linearity of a piecewise linear convex function). In the original Patchworking Theorem, one also requires the curves Aᵢ to be totally nondegenerate. In particular, one can only glue nonsingular curves. Later, E. Shustin proved in [Shu98] that, under some numerical conditions depending on the types and number of the singularities, it is possible to patchwork singular curves keeping the singular points.

On the other hand, I. Itenberg and E. Shustin proved in [IS02] a pseudoholomorphic patchworking theorem: they showed that applying the Viro Patchworking with any subdivision (non necessarily convex) and with reduced curves Aᵢ with arbitrary singularities, one can glue the Aᵢ’s, keeping singular points, to obtain a real pseudoholomorphic curve. More precisely, given some (maybe singular) curves Aᵢ whose Newton polygons are the 2-simplices of a subdivision of the quadrangle with vertices (0,0), (k,0), (k,l) and (l+nk,0), Itenberg and Shustin gave a way to construct a pseudoholomorphic curve C of bidegree (k,l) in the rational geometrically ruled surface Σₙ, whose position with respect to the pencil of lines can be deduced from the initial curves Aᵢ. Isotopy types realizable by (algebraic or pseudoholomorphic) curves obtained via a patchworking procedure are called patchworked curves.
Pseudoholomorphic curves were introduced by M. Gromov in [Gro85] to study symplectic 4-manifolds. A real pseudoholomorphic curve $C$ on $\mathbb{C}P^2$ or $\Sigma_n$ is an immersed Riemann surface which is a $J$-holomorphic curve in some tame almost complex structure $J$ such that the exceptional section (in $\Sigma_n$ with $n \geq 1$) is $J$-holomorphic, $\text{conj}(C) = C$, and $\text{conj} \circ J_p = -J_p \circ \text{conj}$ (where $\text{conj}$ is the standard complex conjugation and $p$ is any point of $C$). It has been realized since then that real pseudoholomorphic curves share a lot of properties with real algebraic ones (see for example [OS02] and [OS03]). It is still unknown if there exist nonsingular real pseudoholomorphic curves in $\mathbb{C}P^2$ or in $\Sigma_n$ which are isotopic to no homologous real algebraic curves (this is the so-called real symplectic isotopy problem). Note that, not requiring the exceptional section in $\Sigma_n$, to be $J$-holomorphic, J-Y. Welschinger constructed in [Wel02] examples of real pseudoholomorphic curves on $\Sigma_n$ for $n \geq 2$ which are not isotopic to any real algebraic curve realizing the same homology class.

In the surfaces $\mathbb{R}\Sigma_n$, there is a natural pencil of lines $L$, and one can study curves there up to fiberwise isotopy. Two curves $C_1$ and $C_2$ in $\mathbb{R}\Sigma_n$ are said to be $L$-isotopic if there exists an isotopy $\phi(t, x)$ of $\mathbb{R}\Sigma_n$ mapping $C_1$ to $C_2$ such that for any $t \in [0; 1]$, for any $p \in C_1$ and for any fiber $F$ of $\mathbb{R}\Sigma_n$, $\phi(t, F)$ is a fiber of $\mathbb{R}\Sigma_n$, and the intersection multiplicity of $C_1$ and $F$ at $p$ is the intersection multiplicity of $\phi(t, C_1)$ and $\phi(t, F)$ at $\phi(t, p)$.

A lot of examples are known of nonsingular real pseudoholomorphic curves in $\mathbb{R}\Sigma_n$ which are $L$-isotopic to no homologous real algebraic curves (see for example [OS02], [OS03], [Bru]). However, as far as we know none of those examples are constructed with the pseudoholomorphic patchworking of Itenberg and Shustin, and the question of the existence of a patchworked pseudoholomorphic curves with any kind of non-algebraic behaviour was open.

In [BB06] we proved that in the case of curves of bidegree $(3, 0)$ in $\Sigma_n$, the patchworked pseudoholomorphic curve is always isotopic to a real algebraic one in the same homology class.

In this paper we construct the first examples of patchworked real pseudoholomorphic curves in $\Sigma_n$ whose position with respect to the pencil of lines cannot be realised by any homologous real algebraic curve.

A smooth curve $C$ in $\mathbb{R}\Sigma_n$ is said to be $L$-nonsingular if $C$ intersects any fiber transversally, except for a finite number of fibers which have an ordinary tangency point with one of the branches of $C$, and intersect transversally the other branches of $C$.

A smooth curve $C$ in $\mathbb{R}\Sigma_n$ which is $L$-singular is called smooth $L$-singular.

**Theorem 1.1** For any $d \geq 3$ there exists a smooth $L$-singular real pseudoholomorphic patchworked curve of bidegree $(d, 0)$ in $\Sigma_2$ which is not $L$-isotopic to any real algebraic curve in $\Sigma_2$ of the same bidegree.

The pseudoholomorphic construction is done in Proposition 3.2 and the algebraic obstruction is proved in Proposition 3.4.

## 2 Rational geometrically ruled surfaces

In this section we fix our notations for the surfaces $\Sigma_n$. The $n$th rational geometrically ruled surface, denoted by $\Sigma_n$, is the surface obtained by taking four copies of $\mathbb{C}^2$ with coordinates $(x, y)$, $(x_2, y_2)$, $(x_3, y_3)$ and $(x_4, y_4)$, and by gluing them along $(\mathbb{C}^*)^2$ with the identifications $(x_2, y_2) = (1/x, y/x^n)$, $(x_3, y_3) = (x, 1/y)$ and $(x_4, y_4) = (1/x, x^n/y)$. Let us denote by $E$ (resp. $B$ and $F$) the algebraic curve in $\Sigma_n$ defined by the equation $\{y_3 = 0\}$ (resp. $\{y = 0\}$ and $\{x = 0\}$). The coordinate system $(x, y)$ is called standard. The projection $\pi : (x, y) \mapsto x$ on $\Sigma_n$ defines a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^1$. The
intersection numbers of $B$ and $F$ are respectively $B \circ B = n$, $F \circ F = 0$ and $B \circ F = 1$. The surface $\Sigma_n$ has a natural real structure induced by the complex conjugation in $\mathbb{C}^2$, and the real part $\mathbb{R}\Sigma_n$ of $\Sigma_n$ is a torus if $n$ is even and a Klein bottle if $n$ is odd. The restriction of $\pi$ on $\mathbb{R}\Sigma_n$ defines a pencil of lines denoted by $\mathcal{L}$.

The group $H_2(\Sigma_n, \mathbb{Z})$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and is generated by the classes of $B$ and $F$. Moreover, one has $E = B - nF$. An algebraic or pseudoholomorphic curve on $\Sigma_n$ is said to be of bidegree $(k, l)$ if it realizes the homology class $kB + lF$ in $H_2(\Sigma_n, \mathbb{Z})$. Its equation in $\Sigma_n \setminus E$ is

$$\sum_{i=0}^{k} a_{k-i}(X, Z) Y^i$$

where $a_j(X, Z)$ is a homogeneous polynomial of degree $nj + l$.

3 Proof of the main Theorem

Our constructions use the Pseudoholomorphic Patchworking Theorem (see [IS02]). The fact that the curves constructed have a position with respect to $\mathcal{L}$ which is not realizable by an algebraic curve comes from a condition on the degree of a certain univariate polynomial.

**Proposition 3.1** Let $d \geq 3$ be a natural number and note $k = \left\lfloor \frac{d}{2} \right\rfloor$. Choose $k$ real numbers $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$ and define the curve $C_{\text{sing}}$ by the equation

$$\prod_{i=1}^{k} (Y^2 - \alpha_i X) \quad \text{if } d \text{ is even}$$

$$Y \prod_{i=1}^{k} (Y^2 - \alpha_i X) \quad \text{if } d \text{ is odd}.$$

Then, for $\epsilon > 0$ small enough, the real algebraic curve $C = C_{\text{sing}} + \epsilon(\nu^d + X - X^{d-1})$ in $\mathbb{CP}^2$ satisfies

- $C$ is nonsingular,
- at the point $[0 : 0 : 1]$ (resp. $[1 : 0 : 0]$), the curve $C$ is locally given by the equation $Y^d + X = 0$ (resp. $Y^d - X^{d-1} = 0$),
- there exists a real number $a > 0$ such that the line of equation $X - aZ = 0$ intersects the curve $C$ in $d$ distinct real points.

**Proof.** Straightforward.

Let us denote by $\tilde{C}$ the curve defined by the equation $X^d C(X, \frac{Y}{X})$. Then, the Newton polygon of $\tilde{C}$ is the triangle with vertices $(0, 0), (d + 1, 0)$ and $(2d - 1, 0)$ and the curve $\tilde{C}$ is locally given by the equation $Y^d + X^{d+1} = 0$ (resp. $Y^d - X^{2d-1} = 0$) at the point $[0 : 0 : 1]$ (resp. $[1 : 0 : 0]$). Moreover, there exists a real number $b > 0$ such that the line of equation $X - bZ = 0$ intersects the curve $\tilde{C}$ in $d$ distinct real points.

Define the polynomials $P_1 = Y^d + X + 1$, $P_2 = Y^d - X^{d-1} + X^d$, $P_3 = Y^d + X^d + X^{d+1}$ and $P_4 = Y^d - X^{2d-1} + X^{2d}$. The curves $P_1$ and $P_3$ (resp. $P_2$ and $P_4$) have a maximal tangency point with the line $X + Z = 0$ (resp. $X - Z = 0$).

One can now patchwork the polynomials $C$, $\tilde{C}$, $P_1$, $P_2$, $P_3$, and $P_4$ (see Figure 1). According to
Itenberg and Shustin’s theorem (see [IS02]), one can glue pseudoholomorphically all the pieces of the patchwork keeping the tangency conditions with respect to \( L \). This proves the following proposition.

**Proposition 3.2** There exists a patchworked nonsingular real pseudoholomorphic curve \( \Gamma \) of bidegree \((d, 0)\) in \( \Sigma_2 \) such that there exist real numbers \( x_1 < x_2 < 0 < z_1 < x_4 < z_2 < x_6 \) satisfying

- \( \Gamma \) has a maximal tangency point with the fibers \( X - x_iZ = 0 \) with \( i = 1, \ldots, 4 \),
- \( \Gamma \) has \( d \) distinct real intersection points with the fibers \( X - z_iZ = 0 \) with \( i = 1, 2 \).

**Lemma 3.3** Let \( P(X, Y) \) be a real polynomial of the form

\[
Y^d + a_2(X)Y^{d-2} + a_3(X)Y^{d-3} + \ldots + a_d(X),
\]

and \( x \) be a real number. If the univariate polynomial \( Q_x(Y) = P(x, Y) \) has only real roots then \( a_2(x) \leq 0 \) and \( a_2(x) = 0 \) if and only if \( 0 \) is a root of order \( d \).

**Proof.** Let \( y_i \) be the \( d \) roots of \( Q_x \). The sum \( \sum_{i=1}^{d} y_i \) is 0 and the second coefficient satisfies \( a_2 = \sum_{i<j} y_i y_j \). Thus, one has \( \sum_{i<j} y_i y_j = -1/2 \sum y_i^2 \) which proves the lemma.

Using Lemma 3.3, we easily prove that there are no algebraic curves of the same bidegrees as curves in Proposition 3.2 having the same positions with respect to the line pencil \( L \).

**Proposition 3.4** There does not exist a nonsingular real algebraic curve \( \Gamma \) of bidegree \((d, 0)\) in \( \Sigma_2 \) such that there exist real numbers \( x_1 < x_2 < 0 < z_1 < x_4 < z_2 < x_6 \) satisfying

- \( \Gamma \) has a maximal tangency point with the fibers \( X - x_iZ = 0 \) with \( i = 1, \ldots, 4 \),
- \( \Gamma \) has \( d \) distinct real intersection points with the fibers \( X - z_iZ = 0 \) with \( i = 1, 2 \).

**Proof.** Suppose such a real algebraic curve \( \Gamma \) exists. Then, in an appropriate standard system of coordinates on \( \Sigma_2 \), the curve \( \Gamma \) has the following equation

\[
Y^d + a_2(X)Y^{d-2} + a_3(X)Y^{d-3} + \ldots + a_d(X)
\]

where \( a_i(X) \) is real a polynomial of degree \( 2i \). According to Lemma 3.3, one has \( a_2(z_i) < 0 \), so the polynomial \( a_2 \) is not identically zero. According to Lemma 3.3, one has \( a_2(x_i) = 0 \), so the \( x_i \)'s are exactly the simple roots of the polynomial \( a_2 \). However, as \( z_1 < x_4 < z_2 \) and \( a_2(z_1) \) and \( a_2(z_2) \) have the same sign, there should exist an extra root \( x_5 \) of \( a_2 \) in the interval \( \]z_1; z_2[ \). Hence, \( a_2 \) would be a non-null polynomial of degree 4 with at least 5 roots, which is impossible.
4 Concluding remarks

1. The patchwork in Section 3 can be realized algebraically as soon as one keeps only any three out of the four maximal tangency points. Patchworking a curve with Newton polygon the triangle with vertices \((1,0), (0,d)\) and \((2d-1,0)\), one can keep two of them. Applying Shustin’s theorems (see [Shu05], [Shu98]), one can keep the maximal tangency point coming either from the curve \(P_2\) or from the curve \(P_3\).

2. It is fairly easy to generalise the main theorem to other rationally ruled surfaces and to construct a lot of other examples. For instance in the same way one proves that there exists pseudoholomorphic patchworked curves of bidegree \((d,0)\) in \(\Sigma_n\) which are not \(\mathcal{L}\)-isotopic to any such real algebraic curve as soon as \(n \geq 2\) and \(d \geq 3\). One can also construct examples in \(\Sigma_1\) of bidegree \((d,0)\) for \(d \geq 5\).

3. This paper is part of a work in progress in which we investigate tangencies of curves with respect to a pencil of lines. For real algebraic curves, one can obtain restrictions valid in any degree by means of certain subresultants. We point out that most of these restrictions do not hold for real pseudoholomorphic curves. Our proof here relies on the simplest example of algebraic prohibitions obtained in this way.

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