AUTOMORPHISMS OF THE QUOT SCHEMES ASSOCIATED TO COMPACT RIEMANN SURFACES

INDRANIL BISWAS, AJNEET DHILLON, AND JACQUES HURTUBISE

Abstract. Let $X$ be a compact connected Riemann surface of genus at least two. Fix positive integers $r$ and $d$. Let $Q$ denote the Quot scheme that parametrizes the torsion quotients of $\mathcal{O}_X^{\oplus r}$ of degree $d$. This $Q$ is also the moduli space of vortices for the standard action of $U(r)$ on $\mathbb{C}^r$. The group $\text{PGL}(r, \mathbb{C})$ acts on $Q$ via the action of $\text{GL}(r, \mathbb{C})$ on $\mathcal{O}_X^{\oplus r}$. We prove that this subgroup $\text{PGL}(r, \mathbb{C})$ is the connected component, containing the identity element, of the holomorphic automorphism group $\text{Aut}(Q)$. As an application of it, we prove that the isomorphism class of the complex manifold $Q$ uniquely determines the isomorphism class of the Riemann surface $X$.

1. Introduction

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. Fix positive integers $r$ and $d$. Let $Q := \text{Quot}_X(\mathcal{O}_X^{\oplus r}, d)$ be the Quot scheme that parametrizes the torsion quotients $\mathcal{O}_X^{\oplus r} \rightarrow Q$ such that $\text{degree}(Q) = d$. It is a smooth complex projective manifold of dimension $rd$. Consider the standard action of $U(r)$ on $\mathbb{C}^r$. The corresponding vortices are pairs of the form $(E, \phi)$, where $E$ is a holomorphic vector bundle on $X$ of rank $r$ and

$$\phi : \mathcal{O}_X^{\oplus r} \rightarrow E$$

is a holomorphic homomorphism such that the subsheaf $\text{image}(\phi) \subset E$ is of rank $r$, equivalently, $E/\text{image}(\phi)$ is a torsion sheaf [BDW]. Therefore, the dual homomorphism

$$\phi^* : E^* \rightarrow \mathcal{O}_X^{\oplus r}$$

defines an element of $Q$ if $\text{degree}(E) = d$. Consequently, $Q$ is a moduli space of vortices on $X$.

Our aim here is to investigate the geometry of the variety $Q$. Let $\text{Aut}^0(Q)$ denote the connected component, containing the identity element, of the group of holomorphic automorphisms of $Q$. The natural action of $\text{GL}(r, \mathbb{C})$ on $\mathcal{O}_X^{\oplus r}$ produces an action of $\text{PGL}(r, \mathbb{C})$ on $Q$, let

$$F : \text{PGL}(r, \mathbb{C}) \rightarrow \text{Aut}^0(Q)$$

be the homomorphism given by this action.

We prove the following (see Theorem 3.1 and Corollary 3.2):

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Theorem 1.1. The homomorphism $F$ is an isomorphism. In particular, the homomorphism of Lie algebras
\[ \text{sl}(r, \mathbb{C}) \longrightarrow H^0(Q, TQ) \]
given by $F$ is an isomorphism.

Theorem 3.1 allows us to investigate the fixed point locus in $Q$ for the action of a maximal torus of $\text{Aut}^0(Q)$. As a consequence, we obtain the following Torelli type theorem (see Theorem 5.1):

Theorem 1.2. If $g = 2 = d$, assume that $r \geq 2$. Then the isomorphism class of the complex manifold $Q$ uniquely determines the isomorphism class of the Riemann surface $X$.

The proof of Theorem 1.2 also uses a Torelli type theorem for $\text{Sym}^d(X)$ proved in [Fa].

2. Self-product, meromorphic functions and meromorphic vector fields

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. The Cartesian product $X \times X$ will be denoted by $X^2$. Let $\Delta \subset X^2$ be the diagonal consisting of points of the form $(x, x)$ with $x \in X$. For $\ell = 1, 2$, let
\[ p_\ell : X^2 \longrightarrow X \]
be the projection to the $\ell$-th factor. The holomorphic tangent bundle of $X$ will be denoted by $TX$.

Lemma 2.1. For any $i \geq 0$,
\[ H^0(X^2, \mathcal{O}_{X^2}(i \cdot \Delta)) = H^0(X^2, \mathcal{O}_{X^2}) = \mathbb{C}. \]

For any $i \geq 0$,
\[ H^0(X^2, (p_1^*TX)^{\otimes \alpha_1} \otimes (p_2^*TX)^{\otimes \alpha_2} \otimes \mathcal{O}_{X^2}(i \cdot \Delta)) = 0, \]
where $\alpha_1$ and $\alpha_2$ are nonnegative integers with $\alpha_1 + \alpha_2 > 0$.

Proof. Let $\iota : X \longrightarrow X^2$ be the map defined by $x \longmapsto (x, x)$. It identifies $X$ with $\Delta$. From the Poincaré adjunction formula we know that $\iota^*(\mathcal{O}_{X^2}(\Delta)|_{\Delta}) = TX$; see [GH] p. 146] for Poincaré adjunction formula.

For any $i \geq 0$, consider the short exact sequence of coherent analytic sheaves on $X^2$
\[ 0 \longrightarrow \mathcal{O}_{X^2}(i \cdot \Delta) \longrightarrow \mathcal{O}_{X^2}((i + 1) \cdot \Delta) \longrightarrow \mathcal{O}_{X^2}((i + 1) \cdot \Delta)|_{\Delta} = \iota_*((TX)^{\otimes(i+1)}) \longrightarrow 0. \]

Let
\[ H^0(X^2, \mathcal{O}_{X^2}(i \cdot \Delta)) \longrightarrow H^0(X^2, \mathcal{O}_{X^2}((i + 1) \cdot \Delta)) \longrightarrow H^0(X, (TX)^{\otimes(i+1)}) \]
be the corresponding long exact sequence of cohomologies. We have
\[ H^0(X, (TX)^{\otimes (i+1)}) = 0 \]
because \( g \geq 2 \). Hence the above homomorphism
\[ H^0(X^2, \mathcal{O}_{X^2}(i \cdot \Delta)) \rightarrow H^0(X^2, \mathcal{O}_{X^2}((i + 1) \cdot \Delta)) \]
is an isomorphism. Now using downward induction on \( i \),
\[ H^0(X^2, \mathcal{O}_{X^2}(i \cdot \Delta)) = H^0(X^2, \mathcal{O}_{X^2}) = \mathbb{C}. \]
This proves the first part of the lemma.

To prove the second part of the lemma, consider the short exact sequence of coherent analytic sheaves on \( X^2 \)
\[ 0 \rightarrow (p_1^*TX)^{\otimes \alpha_1} \otimes (p_2^*TX)^{\otimes \alpha_2} \otimes \mathcal{O}_{X^2}(i \cdot \Delta) \rightarrow (p_1^*TX)^{\otimes \alpha_1} \otimes (p_2^*TX)^{\otimes \alpha_2} \otimes \mathcal{O}_{X^2}((i + 1) \cdot \Delta) \]
\[ \rightarrow t_*((TX)^{\otimes (\alpha_1 + \alpha_2 + i + 1)}) \rightarrow 0 \]
obtained by tensoring (1) with \((p_1^*TX)^{\otimes \alpha_1} \otimes (p_2^*TX)^{\otimes \alpha_2} \). Since \( H^0(X, (TX)^{\otimes m}) = 0 \) for all \( m \geq 1 \), using downward induction on \( i \), we have
\[ H^0(X^2, (p_1^*TX)^{\otimes \alpha_1} \otimes (p_2^*TX)^{\otimes \alpha_2} \otimes \mathcal{O}_{X^2}(i \cdot \Delta)) = H^0(X^2, (p_1^*TX)^{\otimes \alpha_1} \otimes (p_2^*TX)^{\otimes \alpha_2}) = 0. \]
This completes the proof. \( \square \)

Take any integer \( n \geq 1 \). Let
\[ X^n := \overbrace{X \times \cdots \times X}^{\text{n-times}} \]
be the \( n \)-fold Cartesian product. The projection of \( X^n \) to the \( \ell \)-th factor, \( 1 \leq \ell \leq n \), will be denoted by \( p_\ell \). For \( 1 \leq j < k \leq n \), let
\[ \Delta_{j,k} \subset X^n \]
be the divisor consisting of points whose of \( X^n \) \( j \)-th coordinate coincides with the \( k \)-th coordinate.

Take integers \( m_{j,k} \geq 0 \), where \( 1 \leq j < k \leq n \). Fix a pair \((j_0, k_0)\), with \( 1 \leq j_0 < k_0 \leq n \). Define \( m'_{j,k} \), \( 1 \leq j < k \leq n \), as follows:
\[ \begin{align*}
&\bullet m'_{j_0,k_0} = m_{j_0,k_0} + 1, \text{ and} \\
&\bullet m'_{j,k} = m_{j,k} \text{ if } (j, k) \neq (j_0, k_0).
\end{align*} \]

**Lemma 2.2.** For any \( i \geq 0 \), the natural inclusion
\[ H^0(X^n, \mathcal{O}_{X^n}(\sum_{1 \leq j < k \leq n} m_{j,k} \cdot \Delta_{j,k})) \rightarrow H^0(X^n, \mathcal{O}_{X^n}(\sum_{1 \leq j < k \leq n} m'_{j,k} \cdot \Delta_{j,k})) \]
is an isomorphism.

For nonnegative integers \( \alpha_\ell, \ell \in \{1, \cdots, n\} \), with \( \sum_{\ell=1}^n \alpha_\ell > 0 \), the natural inclusion
\[ H^0(X^n, (\bigotimes_{\ell=1}^n (p_\ell^*TX)^{\otimes \alpha_\ell}) \otimes \mathcal{O}_{X^n}(\sum_{1 \leq j < k \leq n} m_{j,k} \cdot \Delta_{j,k})) \]
\[ \to H^0(X^n, (\bigotimes_{\ell=1}^n (p^*TX)^{\otimes \alpha_{\ell}}) \otimes O_{X^n}(\sum_{1 \leq j < k \leq n} m'_{j,k} \cdot \Delta_{j,k})) \]

is an isomorphism.

**Proof.** If \( n = 1 \), then \( \Delta_{j,k} \) are the zero divisors. Hence the lemma holds for \( n = 1 \) as \( g \geq 2 \). If \( n = 2 \), then it reduces to Lemma 221. We will prove the lemma using induction on \( n \).

Assume that the lemma is proved for all \( n \leq a - 1 \). For any \( \ell \in \{1, \cdots, a - 1\} \), let

\[
q_{\ell} : X^{a-1} \longrightarrow X
\]

be the projection to the \( \ell \)-th factor. Fix nonnegative integers \( n_{j,k} \) for every \( 1 \leq j < k \leq a - 1 \). Since the first statement of the lemma holds for \( n = a - 1 \), we conclude that

\[
H^0(X^{a-1}, O_{X^{a-1}}(\sum_{1 \leq j < k \leq a-1} n_{j,k} \cdot \Delta_{j,k})) = \mathbb{C},
\]

where \( \Delta_{j,k} \subset X^{a-1} \) is the divisor defined in (2). Since the second statement of the lemma holds for \( n = a - 1 \), we conclude that

\[
H^0(X^{a-1}, (\bigotimes_{\ell=1}^{a-1} (q_{\ell}^*TX)^{\otimes \alpha_{\ell}}) \otimes O_{X^{a-1}}(\sum_{1 \leq j < k \leq a-1} n_{j,k} \cdot \Delta_{j,k})) = 0,
\]

where \( \alpha_{\ell} \) are nonnegative integers with \( \sum_{\ell=1}^{a-1} \alpha_{\ell} > 0 \).

We will prove the lemma for \( n = a \).

Take \((j_0, k_0), \{m_{j,k}\}\) and \(\{m'_{j,k}\}\) as in the lemma. Let

\[
\iota : X^{a-1} \longrightarrow X^a = X^n
\]

be the map that sends \((x_1, \cdots, x_{a-1})\) to \((y_1, \cdots, y_a)\) such that

- \( y_c = x_c \) if \( c \leq k_0 - 1 \),
- \( y_{k_0} = x_{j_0} \), and
- \( y_c = x_{c-1} \) if \( c > k_0 \).

So \( \iota \) identifies \( X^{a-1} \) with \( \Delta_{j_0,k_0} \). The Poincaré adjunction formula says that

\[
\iota^*(O_{X^a}(\Delta_{j_0,k_0})|_{\Delta_{j_0,k_0}}) = (q_{j_0})^*TX,
\]

where \( q_{j_0} \) is defined in (3). For any \((j, k) \neq (j_0, k_0)\), let

\[
\Delta'_{j,k} := \iota^{-1}(\Delta_{j,k} \cap \Delta_{j_0,k_0}) \subset X^{a-1}
\]

be the diagonal divisor.

Consider the short exact sequence of coherent analytic sheaves on \( X^a \)

\[
0 \longrightarrow O_{X^a}(\sum_{1 \leq j < k \leq a} m_{j,k} \cdot \Delta_{j,k}) \longrightarrow O_{X^a}(\sum_{1 \leq j < k \leq a} m'_{j,k} \cdot \Delta_{j,k}) \longrightarrow (O_{X^a}(\sum_{1 \leq j < k \leq a} m'_{j,k} \cdot \Delta_{j,k}))|_{\Delta_{j_0,k_0}}
\]

\[
= \iota_*(q_{j_0})^*(TX)^{\otimes m'_{j_0,k_0}} \otimes O_{X^{a-1}}(\sum_{(j,k) \neq (j_0,k_0)} m'_{j,k} \cdot \Delta'_{j,k})) \longrightarrow 0,
\]
where $\Delta'_{j,k}$ is defined in (7) and $q_{j0}$ is defined in (3); the identification

$$O_{X^a}(\sum_{1 \leq j < k \leq a} m'_{j,k} \cdot \Delta_{j,k}) \rightarrow (O_{X^a}(\sum_{1 \leq j < k \leq a} m'_{j,k} \cdot \Delta_{j,k}))|_{\Delta_{j0,k0}}$$

$$= \iota_*((q_{j0})^* (TX)^{\otimes m'_{j0,k0}} \otimes O_{X^{a-1}}(\sum_{(j,k) \neq (j0,k0)} m'_{j,k} \cdot \Delta'_{j,k}))$$

in (8) is constructed using the isomorphism in (6). From (5) we know that

$$H^0(X^{a-1} \otimes (p'_{j0})^* (TX)^{\otimes m'_{j0,k0}} \otimes O_{X^{a-1}}(\sum_{(j,k) \neq (j0,k0)} m'_{j,k} \cdot \Delta'_{j,k})) = 0.$$  

Therefore, from the long exact sequence of cohomologies associated to the short exact sequence in (8) we conclude that

$$H^0(X^a, O_{X^a}(\sum_{1 \leq j < k \leq a} m_{j,k} \cdot \Delta_{j,k})) = H^0(X^a, O_{X^a}(\sum_{1 \leq j < k \leq a} m'_{j,k} \cdot \Delta_{j,k})) .$$

This proves the first statement of the lemma for $n = 1$. Therefore, the proof of the first statement of the lemma is complete by induction on $n$.

We will now prove the second statement. Take $\{\alpha_\ell\}$ as in the second statement of the lemma. For $\ell \in \{1, \cdots, a-1\}$, define $\alpha'_\ell$ as follows:

- $\alpha'_{\ell} = \alpha_\ell$ if $\ell < j0$,
- $\alpha'_{j0} = \alpha_{j0} + \alpha_{k0}$,
- $\alpha'_{\ell} = \alpha_\ell$ if $j0 < \ell < k0$, and
- $\alpha'_{\ell} = \alpha_{\ell+1}$ if $\ell \geq k0$.

Note that $\sum_{\ell=1}^{a} \alpha_\ell = \sum_{\ell=1}^{a-1} \alpha'_\ell$.

Let

$$0 \rightarrow \left(\bigotimes_{\ell=1}^{a} (p^*_\ell TX)^{\otimes \alpha'_\ell}\right) \otimes O_X^a(\sum_{1 \leq j < k \leq a} m_{j,k} \cdot \Delta_{j,k}) \rightarrow \left(\bigotimes_{\ell=1}^{a} (p^*_\ell TX)^{\otimes \alpha_\ell}\right) \otimes O_X^a(\sum_{1 \leq j < k \leq a} m'_{j,k} \cdot \Delta_{j,k})$$

$$\rightarrow \left(\bigotimes_{\ell=1}^{a-1} (p^*_\ell TX)^{\otimes \alpha'_\ell}\right) \otimes (q_{j0})^* (TX)^{\otimes m'_{j0,k0}} \otimes O_{X^{a-1}}(\sum_{(j,k) \neq (j0,k0)} m'_{j,k} \cdot \Delta'_{j,k}) \rightarrow 0,$$

be the short exact sequence of coherent analytic sheaves on $X^a$ obtained by tensoring (8) with $\bigotimes_{\ell=1}^{a} (p^*_\ell TX)^{\otimes \alpha'_\ell}$. From (5) we know that

$$H^0(X^{a-1}, \left(\bigotimes_{\ell=1}^{a-1} (p^*_\ell TX)^{\otimes \alpha'_\ell}\right) \otimes (q_{j0})^* (TX)^{\otimes m'_{j0,k0}} \otimes O_{X^{a-1}}(\sum_{(j,k) \neq (j0,k0)} m'_{j,k} \cdot \Delta'_{j,k})) = 0.$$  

Therefore, from the long exact sequence of cohomologies associated to the above short exact sequence of sheaves we conclude that the second statement of the lemma holds for $n = a$. This completes the proof by induction on $n$. 

\[\square\]

**Proposition 2.3.** For any $n \geq 1$ and nonnegative integers $m_{j,k}, 1 \leq j < k \leq n$,

$$H^0(X^n, TX^n \otimes O_{X^n}(\sum_{1 \leq j < k \leq n} m_{j,k} \cdot \Delta_{j,k})) = 0,$$
where $TX^n$ is the holomorphic tangent bundle of $X^n$.

Proof. Since

$$TX^n = \bigoplus_{\ell=1}^n p_\ell^*TX,$$

where $p_\ell$ is the projection of $X^n$ to the $\ell$-th factor, the proposition follows from the second statement in Lemma 2.2. $\square$

3. Vector fields on the Quot scheme

Let $E^0 := \mathcal{O}^{\oplus r}_X$ be the trivial holomorphic vector bundle over $X$ of rank $r$. Fix a positive integer $d$. Let

$$Q := \text{Quot}_X(E^0, d)$$

be the Quot scheme that parametrizes the torsion quotients of $E^0$ of dimension $d$. So each point of $Q$ represents a quotient

$$(9) \quad \varphi : E^0 \to Q$$

of the $\mathcal{O}_X$-module $E^0$ such that

- $Q$ is a torsion $\mathcal{O}_X$-module, and
- $\dim H^0(X, Q) = d$.

The obstruction to the smoothness of the variety $Q$ at the point $Q \in Q$ is given by $\text{Ext}^1_{\mathcal{O}_X}(\ker(\varphi), Q)$ [BC, p. 1, Theorem 2]. For any $\varphi$ as in (9), since $\ker(\varphi)$ is locally free, and the dimension of the support of $Q$ is zero, we have

$$\text{Ext}^1_{\mathcal{O}_X}(\ker(\varphi), Q) = 0.$$

Therefore, $Q$ is an irreducible smooth complex projective variety. Its dimension is $rd$.

Consider the tautological action of the group $\text{Aut}(E^0) = \text{GL}(r, \mathbb{C})$ on $E^0$. It produces an effective action of $\text{PGL}(r, \mathbb{C})$ on $Q$

$$(10) \quad \text{PGL}(r, \mathbb{C}) \times Q \longrightarrow Q.$$  

Consider the Lie algebra $sl(r, \mathbb{C})$ (the space of $r \times r$ complex matrices of trace zero) of $\text{PGL}(r, \mathbb{C})$. Let

$$(11) \quad \gamma : sl(r, \mathbb{C}) \longrightarrow H^0(Q, TQ)$$

be the homomorphism of Lie algebras given by the action of $\text{PGL}(r, \mathbb{C})$ on $Q$ in (10). (The Lie algebra structure of $H^0(Q, TQ)$ is given by Lie bracket of vector fields.)

Theorem 3.1. The homomorphism $\gamma$ in (11) is an isomorphism.

Proof. The homomorphism $\gamma$ is injective because the homomorphism from $sl(r, \mathbb{C})$ to the space of holomorphic vector fields on $\mathbb{CP}^{r-1}$, given by the standard action of $\text{PGL}(r, \mathbb{C})$ on $\mathbb{CP}^{r-1}$, is injective.
Let $\text{Sym}^d(X)$ be the $d$-fold symmetric product of $X$. It parametrizes the effective divisors on $X$ of degree $d$. Let

$$U \subset \text{Sym}^d(X)$$

be the Zariski open subset parametrizing distinct $d$ points; so $U$ parametrizes the reduced effective divisors of degree $d$. The group of permutations of $\{1, \cdots, d\}$ will be denoted by $S_d$. Let

$$(12) \quad f : \tilde{U} := X^d \setminus \bigcup_{1 \leq j < k \leq d} \Delta_{j,k} \rightarrow U$$

be natural projection, where $\Delta_{j,k}$ is defined in (2). We note that $f$ is an étale Galois covering with Galois group $S_d$.

Sending any quotient $Q \in \mathcal{Q}$ of $E^0$ to the scheme-theoretic support of $Q$, we obtain a morphism

$$(13) \quad \bar{\beta} : \mathcal{Q} \rightarrow \text{Sym}^d(X).$$

Define

$$\mathcal{U} := \bar{\beta}^{-1}(U) \subset \mathcal{Q}.$$ 

The restriction of $\bar{\beta}$ to $\mathcal{U}$ will be denoted by $\beta$. We note that

$$(14) \quad \beta : \mathcal{U} \rightarrow U$$

is a smooth proper surjective morphism. The fiber of $\beta$ over any $z \in U$ is the Cartesian product

$$(15) \quad \mathcal{U}_z := \beta^{-1}(z) = \prod_{x \in z} P(E^0_x).$$

So $\mathcal{U}_z$ is isomorphic to $(\mathbb{CP}^{r-1})^d$.

Take any

$$\theta \in H^0(\mathcal{Q}, T\mathcal{Q}).$$

Let $\theta_0$ be the restriction of $\theta$ to $\mathcal{U}$. Let

$$d\beta : T\mathcal{U} \rightarrow \beta^*TU$$

be the differential of the projection $\beta$ in (14). Since the fibers of $\beta$ are connected and projective, we conclude that $d\beta(\theta_0)$ is the pullback of a holomorphic vector field on $U$. Let $\theta'$ be the holomorphic vector field on $U$ such that

$$d\beta(\theta_0) = \beta^*\theta'.$$

Let

$$\theta'' := f^*\theta' \in H^0(\tilde{U}, T\tilde{U})$$

be the pullback, where $f$ is the projection in (12). Since $\theta_0$ is the restriction of a holomorphic vector field on $\mathcal{Q}$, it follows that $\theta''$ is meromorphic on $X^d$, or in other words,

$$\theta'' \in H^0(X^d, TX^d \otimes \mathcal{O}_{X^d}( \sum_{1 \leq j < k \leq d} m_{j,k} \cdot \Delta_{j,k})).$$
for sufficiently large integers $m_{j,k}$. Therefore, from Proposition 2.3 we conclude that $\theta'' = 0$. Hence
\[ d\beta(\theta_0) = 0. \]
In other words, $\theta_0$ is vertical for the projection $\beta$.

Let
\[ \text{ad}(E^0) \subset \text{End}(E^0) = E^0 \otimes (E^0)^* \]
be the subbundle of rank $r^2 - 1$ defined by the sheaf of endomorphisms of trace zero. For any point $x \in X$, we have
\[ H^0(P(E^0_x), TP(E^0_x)) = \text{ad}(E^0)_x = sl(r, \mathbb{C}). \]
In view of (15),
\[ H^0(U_z, TU_z) = \bigoplus_{x \in z} \text{ad}(E^0)_x = \bigoplus_{x \in z} sl(r, \mathbb{C}) \]
for all $z \in U$. We will show that the restriction to $U_z$ of any holomorphic vector field on $Q$ is a constant diagonal element of $\bigoplus_{x \in z} sl(r, \mathbb{C})$ which is independent of $z$.

Let
\[ Z := f^*Q = \tilde{U} \times_U U \longrightarrow \tilde{U} \]
be the pullback to $\tilde{U}$ of the fiber bundle $\beta : U \longrightarrow U$. The natural projection
\[ \phi : Z := f^*Q \longrightarrow U \]
is an étale Galois covering with Galois group $S_d$.

As before, let $\theta_0$ be the restriction to $U$ of a holomorphic vector field on $Q$. Consider the vector field
\[ \theta_1 := \phi^*\theta_0 \in H^0(Z, T\Sigma) \]
on $Z$, where $\phi$ is the projection in (10). We know that $\theta_1$ is vertical for the projection $Z \longrightarrow \tilde{U}$.

Note that the fibers of the projection $Z \longrightarrow \tilde{U}$ are canonically identified with $(\mathbb{C}P^{r-1})^d$. Therefore, the vector field $\theta_1$ is a holomorphic function on $\tilde{U}$ with values in $sl(r, \mathbb{C})^\oplus d$. This holomorphic function is meromorphic on $X^d$ because $\theta_0$ is the restriction to $U$ of a holomorphic vector field on $Q$. From the first part of Lemma 2.2 we know that there are no nonconstant meromorphic functions on $X^d$ that are holomorphic on $\tilde{U}$. Hence $\theta_1$ is a constant function from $\tilde{U}$ to $sl(r, \mathbb{C})^\oplus d$. This function $\tilde{U} \longrightarrow sl(r, \mathbb{C})^\oplus d$ has to be invariant under the action of the Galois group $S_d$ on $\tilde{U}$ because $\theta_1$ is the pullback of a vector field on $U$. Therefore, there is an element
\[ v \in sl(r, \mathbb{C}) \]
such that $\theta_1$ is the constant function $(v, \cdots, v)$. Since $U$ is dense in $Q$, this immediately implies that the injective homomorphism $\gamma$ in (11) is surjective. □

Let $\text{Aut}(Q)$ be the group of holomorphic automorphisms of $Q$. It is a complex Lie group with Lie algebra $H^0(Q, TQ)$; as before, the Lie algebra operation on $H^0(Q, TQ)$ is given by the Lie bracket of vector fields. The connected component of $\text{Aut}(Q)$ containing the
identity element will be denoted by $\text{Aut}^0(\mathcal{Q})$. The following is an immediate consequence of Theorem 3.1.

**Corollary 3.2.** The subgroup $\text{PGL}(r, \mathbb{C}) \subset \text{Aut}(\mathcal{Q})$ in (10) coincides with $\text{Aut}^0(\mathcal{Q})$.

**4. Torus action on the Quot scheme**

Let $T \subset \text{PGL}(r, \mathbb{C})$ be the maximal torus consisting of diagonal matrices. Restrict the action of $\text{PGL}(r, \mathbb{C})$ on $\mathcal{Q}$ in (10) to the subgroup $T$. Let $\mathcal{Q}^T \subset \mathcal{Q}$ be the subset fixed pointwise by this action of $T$; it is a disjoint union of complex submanifolds of $\mathcal{Q}$. We will recall the description of the connected components of $\mathcal{Q}^T$ given in [Bi].

Consider a point of $\mathcal{Q}$ given by an exact sequence

$$0 \longrightarrow \mathcal{F} \overset{\iota}{\longrightarrow} \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{Q} \longrightarrow 0.$$  

This exact sequence corresponds to a fixed point of the torus action on $\mathcal{Q}$ if and only if the homomorphism $\iota$ decomposes as

$$\iota = \bigoplus_{j=1}^{r} \iota_j : \bigoplus_{j=1}^{r} \mathcal{L}_j \hookrightarrow \mathcal{O}_X^{\oplus r},$$

where each $\iota_j$ is the inclusion of some ideal sheaf

$$\iota_j : \mathcal{L}_j \hookrightarrow \mathcal{O}_X$$

(see [Bi, p. 610]). We may write $\mathcal{L}_j = \mathcal{O}_X(-D_j)$, where $D_j$ is the divisor for $\iota_j$. We have

$$\sum_{j=1}^{r} \deg(D_j) = d$$

by the definition of $\mathcal{Q}$.

Denote by $\text{Part}_r^d$ the set of partitions of $d$ of length $r$. So $(d_1, d_2, \cdots, d_r) \in \text{Part}_r^d$ if and only if $d_j$ are nonnegative integers with

$$\sum_{j=1}^{r} d_j = d.$$  

By $\text{Sym}^0(X)$ we will mean a point.

**Proposition 4.1.** The fixed point locus is a disjoint union

$$\mathcal{Q}^T = \coprod_{(d_1, \cdots, d_r) \in \text{Part}_r^d} \text{Sym}^{d_1}(X) \times \cdots \times \text{Sym}^{d_r}(X).$$
Proof. If \( r = 1 \), then \( \text{Quot}(O_X, e) \) is the symmetric product \( \text{Sym}^e(X) \), as the map in \([13]\) is an isomorphism. If for each \( 1 \leq i \leq m \),
\[
f_i : O_X \rightarrow Q_i
\]
is a torsion quotient of \( O_X \) of degree \( e_i \), then
\[
\bigoplus_{i=1}^m f_i : O_X^{\oplus m} \rightarrow \bigoplus_{i=1}^m Q_i
\]
is a torsion quotient of \( O_X^{\oplus m} \) of degree \( \sum_{i=1}^m e_i \). Therefore, for each partition
\[
(d_1, \cdots, d_r) \in \text{Part}^d_r
\]
we have an inclusion map
\[
\text{Sym}^{d_1}(X) \times \cdots \times \text{Sym}^{d_r}(X) \hookrightarrow O_X^{\oplus r}.
\]

It is clear that these subschemes together map onto the fixed point locus of the torus action. Further, the union is clearly disjoint. \( \square \)

The cohomology algebra of \( \text{Sym}^e(X) \) was computed by Macdonald [Ma, p. 325, (6.3)]. In particular, he showed that
\[
b_1(\text{Sym}^e(X)) := \dim H^1(\text{Sym}^e(X), \mathbb{Q}) = 2g.
\]
Consequently,
\[
b_1(\text{Sym}^{d_1}(X) \times \cdots \times \text{Sym}^{d_r}(X)) = 2g(\sum_{d_i \neq 0} 1).
\]
Therefore, for elements \((d_1, \cdots, d_r) \in \text{Part}^d_r\), the first Betti number \( b_1(\text{Sym}^{d_1}(X) \times \cdots \times \text{Sym}^{d_r}(X)) \) attains the minimum value if and only if some \( d_i \) is \( d \) and the rest are zero.

Hence Proposition [4.1] has the following corollary:

**Corollary 4.2.** Consider the first Betti number of the connected components of \( Q_T \). If the first Betti number of a connected component attains the minimum value, then this component is isomorphic to \( \text{Sym}^d(X) \).

Since any maximal torus of \( \text{PGL}(r, \mathbb{C}) \) is conjugate to \( T \), Proposition [4.1] and Corollary [4.2] remain valid if \( T \) is replaced by any other maximal torus of \( \text{PGL}(r, \mathbb{C}) \).

5. The Torelli Theorem

As before, \( Q := \text{Quot}_X(O_X^{\oplus r}, d) \) is the Quot scheme with \( d \geq 1 \). If \( g = 2 = d \), then we assume that \( r > 1 \).

Let \( X' \) be a compact connected Riemann surface of genus \( g' \), with \( g' \geq 2 \). Fix positive integers \( r' \) and \( d' \). If \( g' = 2 = d' \), then we assume that \( r' > 1 \). Let
\[
Q' := \text{Quot}_{X'}(O_{X'}^{\oplus r'}, d')
\]
be the Quot scheme parametrizing the torsion quotients of \( O_{X'}^{\oplus r'} \) of degree \( d' \).
Theorem 5.1. The complex manifolds \( Q \) and \( Q' \) are biholomorphic if and only if the following conditions hold:

- \( X \) is biholomorphic to \( X' \),
- \( r = r' \), and
- \( d = d' \).

Proof. Assume that \( Q \) is biholomorphic to \( Q' \). We will show that the three conditions in the theorem hold.

Fix a maximal torus

\[
T_0 \subset \text{Aut}^0(Q),
\]

where \( \text{Aut}^0(Q) \), as before, is the connected component the automorphism group of \( Q \) containing the identity element. Consider the action of \( T_0 \) on \( Q \). Let \( \beta \) be the minimum value of the first Betti number of the connected components of the fixed point locus \( Q^{T_0} \). Take a connected component \( M \subset Q^{T_0} \) such that

\[
b_1(M) = \beta.
\]

From Corollary 4.2 we know that \( M = \text{Sym}^d(X) \).

First assume that at least one of the following two conditions holds:

1. \( \dim M \neq 2 \)
2. \( b_1(M) \neq 4 \).

These conditions imply that \( g > 2 \) if \( d = 2 \). Fakhruddin proved that for any compact connected Riemann surface \( Y \), and for any positive integer \( d \) such that \( d \neq 2 \) if \( \text{genus}(Y) = 2 \), the isomorphism class of \( \text{Sym}^d(Y) \) uniquely determines the isomorphism class of \( Y \) \cite{Fakhruddin} Theorem 1]. From this we conclude that \( X \) is isomorphic to \( X' \).

Considering the dimension of \( T_0 \) we conclude that \( r = r' \). Considering the dimension of \( Q \) we conclude that \( d = d' \).

Now consider the remaining case where

\[
\dim M = 2 = \frac{b_1(M)}{2}.
\]

Note that these imply that \( g = 2 = d \). Hence \( r \geq 2 \) by the assumption.

Let \( \tilde{\beta} \) is the maximum value of the first Betti number of the connected components of the fixed point locus \( Q^{T_0} \). Let \( \tilde{M} \subset Q^{T_0} \) be a connected component with

\[
b_1(\tilde{M}) = \tilde{\beta}.
\]

From Proposition 4.1 and (17) we know that \( \tilde{M} = X \times X \).

Let \( X \) and \( Y \) be compact connected Riemann surfaces of genus two such that \( X \times X \) is isomorphic to \( Y \times Y \). Fixing an isomorphism \( F : X \times X \to Y \times Y \), consider the two maps

\[
X \to Y, \quad x \mapsto f_i \circ F(x, x_0),
\]
where $f_i$ is the projection of $Y \times Y$ to the $i$-th factor. One of them is a nonconstant map, hence it is an isomorphism. Therefore, the isomorphism class of $X$ is uniquely determined by the isomorphism class of $X \times X$. This completes the proof. □

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

Department of Mathematics, Middlesex College, University of Western Ontario, London, ON N6A 5B7, Canada

E-mail address: adhill3@uwo.ca

Department of Mathematics, McGill University, Burnside Hall, 805 Sherbrooke St. W., Montreal, Que. H3A 0B9, Canada

E-mail address: jacques.hurtubise@mcgill.ca