Stability of a Three-Species Cooperative System with Time Delays and Stochastic Perturbations

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1. Introduction

In biological world, mutualism is an important interaction and many cooperative models have been proposed to describe such biological phenomena (see [1–3] and references cited therein).

In nature, time delays usually appear and bring some important influence to the dynamics of ecosystem models. Kuang [4] says that ignoring time delays means ignoring the reality. It is essential to take the influence of time delays into account in mathematical modelling [5–9]. Furthermore, in view of the complexity of natural world, single-species or two-species ecological models often cannot describe some natural phenomena accurately and many vital behaviours can only be exhibited by systems with three or more species [10–13]. Considering the effect of time delays to three-species cooperative system, we propose the following deterministic model:

\[
\begin{align*}
\frac{dN_1(t)}{dt} &= N_1(t)(r_1 - a_{11}N_1(t) + a_{12}N_2(t - \tau_1) + a_{13}N_3(t - \tau_3)), \\
\frac{dN_2(t)}{dt} &= N_2(t)(r_2 - a_{21}N_1(t - \tau_1) - a_{22}N_2(t) + a_{23}N_3(t - \tau_3)), \\
\frac{dN_3(t)}{dt} &= N_3(t)(r_3 - a_{31}N_1(t - \tau_1) + a_{32}N_2(t - \tau_2) - a_{33}N_3(t)).
\end{align*}
\]

(1)

For (1), we define

\[
\begin{align*}
\alpha_1 &= (a_{11}, -a_{12}, -a_{13})^T, \\
\alpha_2 &= (-a_{21}, a_{22}, -a_{23})^T, \\
\alpha_3 &= (-a_{31}, -a_{32}, a_{33})^T, \\
R &= (r_1, r_2, r_3)^T, \\
A &= \det(\alpha_1, \alpha_2, \alpha_3), \\
A_1 &= \det(R, \alpha_2, \alpha_3), \\
A_2 &= \det(\alpha_1, R, \alpha_3), \\
A_3 &= \det(\alpha_1, \alpha_2, R),
\end{align*}
\]

(2)

and assume that there exists a unique positive equilibrium point as follows:

\[
\begin{align*}
N_1^* &= \frac{A_1}{A}, \\
N_2^* &= \frac{A_2}{A}, \\
N_3^* &= \frac{A_3}{A}.
\end{align*}
\]

(3)
On the contrary, the growth of populations is often subject to environmental fluctuation, so it is necessary to consider stochastic perturbation in the process of mathematical modelling [14–18]. Usually, there are many kinds of stochastic perturbations. The authors in [19] proposed that the stochastic perturbation of state variable around the steady-state \( N^*_i \), \( N^*_2 \), and \( N^*_3 \) was Brownian white noise, which was proportional to the distance from the equilibrium state. Consequently, we obtain the following three-species stochastic cooperative system with time delays:

\[
\begin{align*}
\mathrm{d}N_1(t) &= N_1(t)(r_1 - a_{11}N_1(t) + a_{12}N_2(t - \tau_{12}) + a_{13}N_3(t - \tau_{13}))
+ \sigma_1 N_1(t)(N_1(t) - N^*_1)\mathrm{d}w_1(t), \\
\mathrm{d}N_2(t) &= N_2(t)(r_2 + a_{21}N_1(t - \tau_{21}) - a_{22}N_2(t)
+ a_{23}N_3(t - \tau_{23}))
+ \sigma_2 N_2(t)(N_2(t) - N^*_2)\mathrm{d}w_2(t), \\
\mathrm{d}N_3(t) &= N_3(t)(r_3 + a_{31}N_1(t - \tau_{31}) + a_{32}N_2(t - \tau_{32})
- a_{33}N_3(t))
+ \sigma_3 N_3(t)(N_3(t) - N^*_3)\mathrm{d}w_3(t),
\end{align*}
\]

with initial data

\[
N_i(\theta) = \phi_i(\theta) > 0, \quad \theta \in [-\tau, 0], \quad i = 1, 2, 3, \tag{5}
\]

where \( N_i(t) \) stands for the population size of the \( i \)-th species at time \( t \), \( r_i > 0 \) is the growth rate of \( N_i(t) \), \( a_{ij} > 0 \) represents the intraspecific competitive coefficient of \( N_i(t) \), \( a_{ij} \) is the interspecific cooperative rate, \( \tau_{ij} > 0 \) is time delay, \( \tau = \max\{\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32}\} \), \( \phi_i(\theta) \) is positive and continuous function defined on \([-\tau, 0]\), \( \sigma_i^2 \) denotes the intensity of white noise, and \( \omega_i(t)_{t \geq 0} \) is the standard independent Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{F}_t, \mathbb{P})\), \( i, j = 1, 2, 3, j \neq i \).

Furthermore, Liu [20] proposed that the stochastic perturbation of state variable was proportional to \( N_i(N_i - N^*_i) \) \( i = 1, 2, 3 \), and then, we get the following model:

\[
\begin{align*}
\mathrm{d}N_1(t) &= N_1(t)(r_1 - a_{11}N_1(t) + a_{12}N_2(t - \tau_{12}) + a_{13}N_3(t - \tau_{13}))
+ \sigma_1 N_1(t)(N_1(t) - N^*_1)\mathrm{d}w_1(t), \\
\mathrm{d}N_2(t) &= N_2(t)(r_2 + a_{21}N_1(t - \tau_{21}) - a_{22}N_2(t)
+ a_{23}N_3(t - \tau_{23}))
+ \sigma_2 N_2(t)(N_2(t) - N^*_2)\mathrm{d}w_2(t), \\
\mathrm{d}N_3(t) &= N_3(t)(r_3 + a_{31}N_1(t - \tau_{31}) + a_{32}N_2(t - \tau_{32})
- a_{33}N_3(t))
+ \sigma_3 N_3(t)(N_3(t) - N^*_3)\mathrm{d}w_3(t).
\end{align*}
\]

For the deterministic system, whether there exists a positive equilibrium state is an important topic, which attracts many attentions of researchers, while stochastic model cannot tend to a positive fixed point; i.e., there exists no traditional positive equilibrium state, so it is popular to study the dynamics of stochastic system around the equilibrium state of its corresponding deterministic system. Furthermore, some kinds of delays are considered in biological systems, but many obtained results have no relation with delays and the effects of time delays are not revealed clearly [9, 11]. Time delays are the sources of instability in population dynamics, and they can cause population fluctuations, so it is interesting to study the effects of delays to the system dynamics. Motivated by these, we propose two three-species delayed cooperative systems with stochastic perturbation and aim to investigate how time delays affect the stability in probability around the equilibrium state and then by numerical examples to validate our theoretical results. To the best of our knowledge, this paper is the first attempt to investigate the influence of time delays on the stability in probability of a stochastic three-species cooperative system.

The rest work of this paper is structured as follows. Section 2 begins with definitions and some important lemmas and notations. Section 3 focuses on the asymptotical stability and stability in probability of (4) and (6), respectively. Some numerical simulations are given in Section 4 to validate our theoretical results. Finally, a brief conclusion and future direction are given in Section 5 to conclude the paper.

2. Preliminaries

Let \((\Omega, \sigma, \mathbb{P})\) be a probability space and \([f, t \geq 0]\) be a family of \( \sigma \)-algebras. We consider the following neutral stochastic differential equation:

\[
\left\{ \begin{array}{l}
d(x(t) - F(t, x_t)) = a(t, x_t)\mathrm{d}t + b(t, x_t)\mathrm{d}w(t), \\
x_0 = \phi \in H,
\end{array} \right.
\]

\]

\[(7)\]
where $H$ represents the space with all $f_0$-adapted functions $\phi(s) \in R^n$ and $x_t(s) = x(t + s), s \leq 0, \omega(t)$ denotes the m-dimensional $f_0$-adapted Brownian process, $a(t, \phi)$ and $b(t, \phi)$ are $n$-dimensional vector and $n \times m$-dimensional matrix, respectively. Define
\[
\| \phi \|_0 = \sup_{s \geq 0} |\phi(s)|,
\]
\[
\| \phi \|_1 = \sup_{s \geq 0} \mathbb{E}\{ |\phi(s)|^2 \},
\]
where $\mathbb{E}$ denotes the mathematical expectation.

**Definition 1** (see [19]). The zero solution of (7) is said to be stable in probability if for any $\epsilon, \delta > 0$, there exists a number $\delta > 0$ such that the solution $x(t) = x(t, \phi)$ satisfies $P[|x(t, \phi)| > \delta] < \epsilon$ for any initial function $\phi \in H$ such that $P[\| \phi \|_0 \leq \delta] = 1$, in which $P$ is the probability of an event.

**Lemma 1.** Systems (4) and (6) have a unique globally positive solution on $t > -\tau$ for any initial data given above, respectively.

**Remark 1.** The proof is standard. For more details, refer to [20, 21].

**Lemma 2** (see [19, 22]). If there exists a functional $V(t, x)$ such that
\[
\lambda_1 |x(t)|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|_0^2, \quad \lambda_1, \lambda_2 > 0, LV(t, x_t) \leq 0,
\]
for any function $\phi \in H$ such that $P[\| \phi \|_0 \leq \delta] = 1$, $\delta > 0$, is sufficiently small positive constant, and then, the zero solution of (7) is stable in probability.

For convenience in the later, if there is no special mention, we always denote $f(t), t \in R$, by $f$ and give the following notations to end this section:

\[
\begin{align*}
\xi_1 &= a_{21}(a_{21} + a_{23}), \\
\eta_1 &= a_{31}(a_{31} + a_{32}), \\
\xi_2 &= a_{12}(a_{12} + a_{13}), \\
\eta_2 &= a_{32}(a_{31} + a_{32}), \\
\xi_3 &= a_{13}(a_{12} + a_{13}), \\
\eta_3 &= a_{23}(a_{21} + a_{23}), \\
\Delta_1 &= 2a_{11} - a_{12} - a_{13} - \frac{a_1^2}{N_1^1}, \\
\Delta_2 &= 2a_{22} - a_{21} - a_{23} - \frac{a_2^2}{N_2^2}, \\
\Delta_3 &= 2a_{33} - a_{31} - a_{32} - \frac{a_3^2}{N_3^3}, \\
\bar{\Delta}_1 &= 2a_{11} - a_{12} - a_{13} - \frac{a_1^2}{N_1^1}, \\
\bar{\Delta}_2 &= 2a_{22} - a_{21} - a_{23} - \frac{a_2^2}{N_2^2}, \\
\bar{\Delta}_3 &= 2a_{33} - a_{31} - a_{32} - \frac{a_3^2}{N_3^3}.
\end{align*}
\]

### 3. Stochastic Stability

In this section, we investigate the asymptotical stability and stable in probability of (4) and (6) around the equilibrium state of (1), respectively. Let $x_1 = N_1 - N_1^1$, $x_2 = N_2 - N_2^1$, and $x_3 = N_3 - N_3^3$, then system (4) is transformed to the following equivalent system:

\[
\begin{align*}
\frac{dx_1}{dt} &= (-a_{11}N_1^1x_1(t) + a_{12}N_1^1x_2(t - \tau_{12}) + a_{13}N_1^1x_3(t - \tau_{13}) - a_{11}x_1^2 + a_{12}x_1x_3(t - \tau_{12}) + a_{13}x_1x_3(t - \tau_{13}))dt + \sigma_1x_1dw_1(t), \\
\frac{dx_2}{dt} &= (a_{21}N_2^1x_1(t - \tau_{21}) - a_{22}N_2^1x_2(t) + a_{23}N_2^1x_3(t - \tau_{23}) - a_{22}x_2^2 + a_{21}x_2x_1(t - \tau_{21}) + a_{23}x_2x_3(t - \tau_{23}))dt + \sigma_2x_2dw_2(t), \\
\frac{dx_3}{dt} &= (-a_{33}N_3^3x_3(t) + a_{31}N_3^3x_1(t - \tau_{31}) + a_{31}N_3^3x_3(t - \tau_{32}) - a_{33}x_3^2 + a_{31}x_1x_3(t - \tau_{31}) + a_{32}x_2x_3(t - \tau_{32}))dt + \sigma_3x_3dw_3(t).
\end{align*}
\]

By the equivalent property of above transformation, the stability of (4) around the equilibrate state of (1) is equivalent to the stability of zero solution of corresponding equivalent system (11). Consequently, we only need to study the stability of zero solution of system (11).
By use of the definition of derivative, system (11) is equivalent to the following neutral system:

\[
\begin{align*}
&\frac{d}{dt}\left(x_1 + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)ds + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)ds\right) \\
&= (-a_{11}N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3 - a_{11}x_1^2 + a_{12}x_2 x_2(t - \tau_{12}) + a_{13}x_3 x_2(t - \tau_{13}))dt + \sigma_1 x_1 dw_1(t), \\
&\frac{d}{dt}\left(x_2 + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_1(s)ds + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_1(s)ds\right) \\
&= (-a_{21}N_2^* x_1 - a_{22}N_2^* x_2 + a_{23}N_2^* x_3 - a_{21}x_2^2 + a_{22}x_2 x_2(t - \tau_{21}) + a_{23}x_3 x_2(t - \tau_{23}))dt + \sigma_2 x_2 dw_2(t), \\
&\frac{d}{dt}\left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)ds + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)ds\right) \\
&= (-a_{31}N_3^* x_1 - a_{32}N_3^* x_2 - a_{33}N_3^* x_3 - a_{31}x_3^2 + a_{32}x_3 x_3(t - \tau_{31}) + a_{33}x_3 x_3(t - \tau_{32}))dt + \sigma_3 x_3 dw_3(t).
\end{align*}
\]  

(12)

The linear case of (12) is as follows:

\[
\begin{align*}
&\frac{d}{dt}\left(x_1 + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)ds + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)ds\right) \\
&= (-a_{11}N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3)dt + \sigma_1 x_1 dw_1(t), \\
&\frac{d}{dt}\left(x_2 + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_1(s)ds + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_1(s)ds\right) \\
&= (-a_{21}N_2^* x_1 - a_{22}N_2^* x_2 + a_{23}N_2^* x_3)dt + \sigma_2 x_2 dw_2(t), \\
&\frac{d}{dt}\left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)ds + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)ds\right) \\
&= (-a_{31}N_3^* x_1 - a_{32}N_3^* x_2 - a_{33}N_3^* x_3)dt + \sigma_3 x_3 dw_3(t).
\end{align*}
\]  

(13)

For the linear and neutral system (13), the following statement is true.

**Theorem 1.** If the following condition holds,

\[
(H_1)
\begin{pmatrix}
\Delta_1 & -a_{31} & -a_{31} \\
-a_{12} & \Delta_2 & -a_{32} \\
-a_{13} & -a_{32} & \Delta_3
\end{pmatrix}
\begin{pmatrix}
N_1^* \\
N_2^* \\
N_3^*
\end{pmatrix}
> 2\tau
\begin{pmatrix}
\xi_1 & \eta_1 \\
\xi_2 & \eta_2 \\
\xi_3 & \eta_3
\end{pmatrix}
\begin{pmatrix}
(N_1^*)^2 \\
(N_2^*)^2 \\
(N_3^*)^2
\end{pmatrix},
\]

(14)

then the zero solution of (13) is globally asymptotically stable, almost surely, i.e., \(\lim_{t\to\infty} x_i(t) = 0, i = 1, 2, 3\).

**Proof.** First, by the characteristics of neutral functional differential system (13), we define

\[
\begin{align*}
V_1 &= \left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)ds + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)ds\right)^2, \\
V_2 &= \left(x_2 + a_{21}N_2^* \int_{t-\tau_{21}}^{t} x_1(s)ds + a_{23}N_2^* \int_{t-\tau_{23}}^{t} x_3(s)ds\right)^2, \\
V_3 &= \left(x_3 + a_{31}N_3^* \int_{t-\tau_{31}}^{t} x_1(s)ds + a_{32}N_3^* \int_{t-\tau_{32}}^{t} x_2(s)ds\right)^2.
\end{align*}
\]  

(15)

Applying Itô formula to \(V_1\), we have

\[
LV_1 = 2\left(x_1 + a_{12}N_1^* \int_{t-\tau_{12}}^{t} x_2(s)ds + a_{13}N_1^* \int_{t-\tau_{13}}^{t} x_3(s)ds\right) \\
\times (-a_{11}N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3) + \sigma_1^2 x_1^2 \\
= -2a_{11}N_1^* x_1^2 + 2a_{12}N_1^* x_1 x_2 + 2a_{13}N_1^* x_1 x_3 + \sigma_1^2 x_1^2 + 2a_{13}a_{23} (N_1^*)^2 \int_{t-\tau_{13}}^{t} x_1(t)x_2(s)ds \\
+ 2a_{13}^2 (N_1^*)^2 \int_{t-\tau_{12}}^{t} x_2(t)x_2(s)ds + 2a_{12}a_{13} (N_1^*)^2 \int_{t-\tau_{12}}^{t} x_3(t)x_2(s)ds - 2a_{11}a_{13} (N_1^*)^2 \\
\times \int_{t-\tau_{13}}^{t} x_1(t)x_3(s)ds + 2a_{12}a_{13} (N_1^*)^2 \int_{t-\tau_{13}}^{t} x_2(t)x_3(s)ds + 2a_{13}^2 (N_1^*)^2 \int_{t-\tau_{13}}^{t} x_3(t)x_3(s)ds \\
\leq -2a_{11}N_1^* x_1^2 + a_{13}N_1^* (x_1^2 + x_2^2) + a_{13}N_1^* (x_1^2 + x_3^2) + \sigma_1^2 x_1^2 \\
+ a_{12}a_{13} (N_1^*)^2 x_2^2 \tau + a_{13} (a_{13} + a_{12}) (N_1^*)^2 x_2^2 \tau \\
+ a_{12}a_{13} (N_1^*)^2 \int_{t-\tau_{12}}^{t} x_2(s)ds + a_{13} (a_{12} + a_{13}) (N_1^*)^2 \int_{t-\tau_{13}}^{t} x_3(s)ds.
\]

By the same way, we have
\[ LV_2 = 2 \left( x_2 + a_{21} N_2^2 \int_{t-r_{31}}^{t} x_1 (s) ds + a_{23} N_3^2 \int_{t-r_{31}}^{t} x_3 (s) ds \right) \times (a_2 N_2^* x_1 - a_{23} N_2^* x_2 + a_{23} N_3^* x_3) + \sigma^2 x_2^2 \]

\[ = -2a_{22} N_2^2 x_2^2 + 2a_{21} N_2^* x_1 x_2 + 2a_{23} N_2^* x_2 x_3 + \sigma^2 x_2^2 + 2a_{21} (N_2^*)^2 \int_{t-r_{31}}^{t} x_1 (t) x_1 (s) ds \]

\[ - 2a_{31} a_{22} (N_2^*)^2 \int_{t-r_{31}}^{t} x_2 (t) x_1 (s) ds + 2a_{21} a_{23} (N_2^*)^2 \int_{t-r_{31}}^{t} x_3 (t) x_1 (s) ds + 2a_{21} a_{23} (N_2^*)^2 \int_{t-r_{31}}^{t} x_3 (t) x_3 (s) ds \]

\[ \leq -2a_{22} N_2^2 x_2^2 + 2a_{21} N_2^* (x_1^2 + x_2^2) + 2a_{23} N_3^* (x_2^2 + x_3^2) + \sigma^2 x_2^2 \]

\[ + a_{21} (a_{21} + a_{23}) (N_2^*)^2 x_2^2 + a_{23} (a_{21} + a_{23}) (N_2^*)^2 x_3^2 \]

\[ + a_{21} (a_{21} + a_{23}) (N_2^*)^2 \int_{t-r_{31}}^{t} x_3^2 (s) ds + a_{23} (a_{21} + a_{23}) (N_2^*)^2 \int_{t-r_{31}}^{t} x_3^2 (s) ds \]

\[ \frac{LV_3}{2} = 2 \left( x_3 + a_{31} N_3^2 \int_{t-r_{31}}^{t} x_1 (s) ds + a_{32} N_3^2 \int_{t-r_{31}}^{t} x_3 (s) ds \right) \times (a_3 N_3^* x_1 + a_{32} N_3^* x_2 - a_{33} N_3^* x_3) + \sigma^2 x_3^2 \]

\[ = -2a_{33} N_3^2 x_3^2 + 2a_{31} N_3^* (x_1^2 + x_2^2) + 2a_{32} N_3^* x_2 x_3 + \sigma^2 x_3^2 + 2a_{31} (N_3^*)^2 \int_{t-r_{31}}^{t} x_1 (t) x_1 (s) ds \]

\[ + 2a_{31} a_{32} (N_3^*)^2 \int_{t-r_{31}}^{t} x_2 (t) x_1 (s) ds - 2a_{31} a_{33} (N_3^*)^2 \int_{t-r_{31}}^{t} x_3 (t) x_1 (s) ds + 2a_{32} a_{33} (N_3^*)^2 \int_{t-r_{31}}^{t} x_3 (t) x_3 (s) ds \]

\[ \leq -2a_{33} N_3^2 x_3^2 + 2a_{31} N_3^* (x_1^2 + x_2^2) + 2a_{32} N_3^* (x_2^2 + x_3^2) + \sigma^2 x_3^2 \]

\[ + a_{32} (a_{31} + a_{32}) (N_3^*)^2 x_2^2 + a_{31} (a_{31} + a_{32}) (N_3^*)^2 x_3^2 \]

\[ + a_{31} (a_{31} + a_{32}) (N_3^*)^2 \int_{t-r_{31}}^{t} x_3^2 (s) ds + a_{32} (a_{31} + a_{32}) (N_3^*)^2 \int_{t-r_{31}}^{t} x_3^2 (s) ds \]

Adding both sides of \(LV_1, LV_2, \) and \(LV_3\) yields

\[ L(V_1 + V_2 + V_3) \leq \left[ -(2a_{12} - a_{13} - a_{13}) N_1^* + \sigma_1^2 + a_{21} N_2^* + a_{31} N_3^* \right] x_1^2 \]

\[ + r \left[ a_{21} (a_{21} + a_{23}) (N_2^*)^2 + a_{31} (a_{31} + a_{32}) (N_3^*)^2 \right] x_1^2 \]

\[ + a_{31} (a_{31} + a_{32}) (N_3^*)^2 \int_{t-r_{31}}^{t} x_1^2 (s) ds + a_{31} (a_{21} + a_{23}) (N_2^*)^2 \int_{t-r_{31}}^{t} x_1^2 (s) ds \]

\[ \leq \left[ -(2a_{22} - a_{23}) N_2^* + \sigma_2^2 + a_{12} N_1^* + a_{33} N_3^* \right] x_2^2 \]

\[ + r \left[ a_{12} (a_{12} + a_{13}) (N_1^*)^2 + a_{32} (a_{31} + a_{32}) (N_3^*)^2 \right] x_2^2 \]

\[ + a_{12} (a_{12} + a_{13}) (N_1^*)^2 \int_{t-r_{31}}^{t} x_2^2 (s) ds + a_{32} (a_{31} + a_{32}) (N_3^*)^2 \int_{t-r_{31}}^{t} x_2^2 (s) ds \]

\[ \leq \left[ -(2a_{33} - a_{32}) N_3^* + \sigma_3^2 + a_{13} N_1^* + a_{23} N_2^* \right] x_3^2 \]

\[ + r \left[ a_{23} (a_{21} + a_{23}) (N_2^*)^2 + a_{13} (a_{13} + a_{12}) (N_1^*)^2 \right] x_3^2 \]

\[ + a_{23} (a_{21} + a_{23}) (N_2^*)^2 \int_{t-r_{31}}^{t} x_3^2 (s) ds + a_{13} (a_{13} + a_{12}) (N_1^*)^2 \int_{t-r_{31}}^{t} x_3^2 (s) ds. \]
Define
\[ V_4 = a_{21} (a_{21} + a_{23}) (N_i^*)^2 \int_{t-t_{21}}^t \left( \theta - t + \tau_{21} \right) x_i^2 (\theta) d\theta \]
\[ + a_{31} (a_{31} + a_{32}) (N_i^*)^2 \int_{t-t_{31}}^t \left( \theta - t + \tau_{31} \right) x_i^2 (\theta) d\theta, \]
\[ V_5 = a_{12} (a_{12} + a_{13}) (N_i^*)^2 \int_{t-t_{12}}^t \left( \theta - t + \tau_{12} \right) x_i^2 (\theta) d\theta \]
\[ + a_{32} (a_{31} + a_{32}) (N_i^*)^2 \int_{t-t_{32}}^t \left( \theta - t + \tau_{32} \right) x_i^2 (\theta) d\theta, \]
\[ V_6 = a_{13} (a_{12} + a_{13}) (N_i^*)^2 \int_{t-t_{13}}^t \left( \theta - t + \tau_{13} \right) x_i^2 (\theta) d\theta \]
\[ + a_{23} (a_{21} + a_{23}) (N_i^*)^2 \int_{t-t_{23}}^t \left( \theta - t + \tau_{23} \right) x_i^2 (\theta) d\theta. \]

(19)

Let \( V = \sum_{i=1}^6 V_i \), then

\[ LV \leq \left\{ \left( -2a_{11} - a_{12} - a_{13} - \frac{\sigma_1^2}{N_i^*} \right) N_i^* + a_{21} N_i^* + a_{31} N_i^* \right\} x_i^2 (t) \]
\[ + 2a_{21} (a_{21} + a_{23}) (N_i^*)^2 + 2a_{31} (a_{31} + a_{32}) (N_i^*)^2 \right\} x_i^2 (t) \]
\[ + \left\{ \left( -2a_{22} - a_{21} - a_{23} - \frac{\sigma_2^2}{N_2^*} \right) N_2^* + a_{12} N_2^* + a_{32} N_2^* \right\} \]
\[ + 2a_{12} (a_{12} + a_{13}) (N_2^*)^2 + 2a_{32} (a_{31} + a_{32}) (N_2^*)^2 \right\} x_2^2 \]
\[ + \left\{ \left( -2a_{33} - a_{31} - a_{32} - \frac{\sigma_3^2}{N_3^*} \right) N_3^* + a_{13} N_3^* + a_{23} N_3^* \right\} \]
\[ + 2a_{23} (a_{21} + a_{23}) (N_3^*)^2 + 2a_{13} (a_{12} + a_{13}) (N_3^*)^2 \right\} x_3^2. \]

(20)

By the hypothesis \((H_1)\), it is easy to get \( LV < 0 \) along all trajectories in \( R_i^3 \) except \( N_i^* \). Hence, by the stability theory of stochastic functional differential equations \([23]\), the zero solution of (13) is globally asymptotically stable. This completes the proof. \( \square \)

Remark 2. The globally asymptotical stability of zero solution of (13) means the globally asymptotical stability of the solution of (4) around the equilibrium state of (1), that is, \( \lim_{t \to \infty} N_i (t) = N_i^*, i = 1, 2, 3. \)

Remark 3. The proof method is motivated in \([19]\). We apply the theory of neutral functional differential equation and define some suitable functionals \( V_1, V_2, \) and \( V_3 \) to obtain the sufficient conditions assuring the globally asymptotical stability of (13) around the equilibrium state \( N_i^* \), which are much different from those of \([20]\).

Theorem 2. If \((H_1)\) holds, then the zero solution of system (12) is stable in probability; that is, system (4) around the equilibrium state of (1) is stable in probability.
Proof. For system (12), define $V_1$, $V_2$, and $V_3$ as before and assume that there exists a number $\delta > 0$ such that $\sup_{t \geq 0} |x_i(s)| < \delta, i = 1, 2, 3$. By the Itô formula, we calculate $LV_1, LV_2$, and $LV_3$ along system (12), respectively, then

\[
LV_1 = 2 \left( x_1 + a_{12}N_1^* \int_{t - \tau_{12}}^{t} x_2(s)ds + a_{13}N_1^* \int_{t - \tau_{13}}^{t} x_3(s)ds \right)
\]

\[
\cdot \left( -a_1N_1^* x_1 + a_{12}N_1^* x_2 + a_{13}N_1^* x_3 - a_{11}x_1^2 + a_{12}x_1x_2(t - \tau_{12}) + a_{13}x_1x_3(t - \tau_{13}) \right) + \sigma_1^2 x_1^2
\]

\[
\leq -2a_{11}N_1^* x_1^2 + a_{12}N_1^* \left( x_1^2 + x_2^2 \right) + a_{13}N_1^* \left( x_1^2 + x_3^2 \right)
\]

\[
+ \sigma_1^2 x_1^2 + a_{12}(a_{12} + a_{13})(N_1^*)^2 x_2^2 \tau + a_{13}(a_{13} + a_{12})(N_1^*)^2 x_3^2 \tau
\]

\[
+ a_{12}(a_{12} + a_{13})(N_1^*)^2 \int_{t - \tau_{12}}^{t} x_2^2(s)ds + a_{13}(a_{12} + a_{13})(N_1^*)^2 \int_{t - \tau_{13}}^{t} x_3^2(s)ds
\]

\[
+ 2\delta(a_{12} + a_{13})x_1^2 + (a_{12} + a_{13})N_1^* \hat{\delta} x_1^2 + \delta a_{12}^2 N_1^* \int_{t - \tau_{12}}^{t} x_2^2(s)ds + \delta a_{13}^2 N_1^* \int_{t - \tau_{13}}^{t} x_3^2(s)ds,
\]

\[
LV_2 = 2 \left( x_2 + a_{21}N_2^* \int_{t - \tau_{21}}^{t} x_1(s)ds + a_{23}N_2^* \int_{t - \tau_{23}}^{t} x_3(s)ds \right)
\]

\[
\cdot \left( -a_{22}N_2^* x_2 + a_{21}N_2^* x_1 + a_{23}N_2^* x_3 + a_{21}x_2x_1(t - \tau_{21}) - a_{22}x_2^2 + a_{23}x_2x_3(t - \tau_{23}) \right) + \sigma_2^2 x_2^2
\]

\[
\leq -2a_{22}N_2^* x_2^2 + a_{21}N_2^* \left( x_1^2 + x_2^2 \right) + a_{23}N_2^* \left( x_2^2 + x_3^2 \right) + \sigma_2^2 x_2^2
\]

\[
+ a_{21}(a_{21} + a_{23})(N_2^*)^2 x_1^2 \tau + a_{23}(a_{21} + a_{23})(N_2^*)^2 x_3^2 \tau
\]

\[
+ a_{21}(a_{21} + a_{23})(N_2^*)^2 \int_{t - \tau_{21}}^{t} x_1^2(s)ds + a_{23}(a_{21} + a_{23})(N_2^*)^2 \int_{t - \tau_{23}}^{t} x_3^2(s)ds + 2\delta(a_{21} + a_{23})x_2^2
\]

\[
+ (a_{21} + a_{23})^2 N_2^* \hat{\delta} x_2^2 + \delta a_{21}^2 N_2^* \int_{t - \tau_{21}}^{t} x_1^2(s)ds + \delta a_{23}^2 N_2^* \int_{t - \tau_{23}}^{t} x_3^2(s)ds,
\]

\[
LV_3 = 2 \left( x_3 + a_{31}N_3^* \int_{t - \tau_{31}}^{t} x_1(s)ds + a_{32}N_3^* \int_{t - \tau_{32}}^{t} x_2(s)ds \right)
\]

\[
\cdot \left( a_{31}N_3^* x_1 + a_{32}N_3^* x_2 - a_{33}N_3^* x_3 + a_{31}x_3x_1(t - \tau_{31}) + a_{32}x_3x_2(t - \tau_{32}) - a_{33}x_3^2 \right) + \sigma_3^2 x_3^2
\]

\[
\leq -2a_{33}N_3^* x_3^2 + a_{31}N_3^* \left( x_1^2 + x_3^2 \right) + a_{32}N_3^* \left( x_2^2 + x_3^2 \right) + \sigma_3^2 x_3^2
\]

\[
+ a_{32}(a_{31} + a_{32})(N_3^*)^2 x_2^2 \tau + a_{31}(a_{31} + a_{32})(N_3^*)^2 x_1^2 \tau
\]

\[
+ a_{31}(a_{31} + a_{32})(N_3^*)^2 \int_{t - \tau_{31}}^{t} x_1^2(s)ds
\]

\[
+ a_{32}(a_{31} + a_{32})(N_3^*)^2 \int_{t - \tau_{32}}^{t} x_2^2(s)ds + 2\delta(a_{31} + a_{32})x_3^2 + (a_{31} + a_{32})^2 N_3^* \hat{\delta} x_3^2
\]

\[
+ \delta a_{31}^2 N_3^* \int_{t - \tau_{31}}^{t} x_1^2(s)ds + \delta a_{32}^2 N_3^* \int_{t - \tau_{32}}^{t} x_2^2(s)ds.
\]

Define $\tilde{V}_4, \tilde{V}_5$, and $\tilde{V}_6$ as follows:
where \( V_j \) denotes the left-hand side of (22), and the right-hand side is given by:

\[
LV \leq \left\{ \begin{array}{l}
\left( -2a_{11} - a_{12} - a_{13} - \frac{\sigma^2}{N^*_1} \right) N^*_1 + a_{21} N^*_2 + a_{31} N^*_3 \right. & + 2\tau a_{21} (a_{21} + a_{23}) (N^*_2)^2 \\
+ 2\tau a_{31} (a_{31} + a_{32}) (N^*_3)^2 & + \tau \theta \left( a_{12}^2 N^*_1 + a_{32}^2 N^*_3 \right) x^*_1(t) \\
+ \left[ \left( -2a_{22} - a_{23} - \frac{\sigma^2}{N^*_2} \right) N^*_2 + a_{12} N^*_1 + a_{32} N^*_3 \right] & + 2\tau a_{12} (a_{12} + a_{13}) (N^*_1)^2 \\
+ 2\tau a_{32} (a_{31} + a_{32}) (N^*_3)^2 & + \tau \theta \left( a_{13}^2 N^*_1 + a_{23}^2 N^*_2 \right) x^*_2(t) \\
+ \left[ \left( -2a_{33} - a_{32} - \frac{\sigma^2}{N^*_3} \right) N^*_3 + a_{13} N^*_1 + a_{23} N^*_2 \right] & + 2\tau a_{31} (a_{12} + a_{13}) (N^*_1)^2 \\
+ 2\tau a_{13} (a_{12} + a_{13}) (N^*_1)^2 & + \tau \theta \left( a_{21}^2 N^*_2 + a_{31}^2 N^*_3 \right) x^*_3(t).
\end{array} \right.
\]
Similarly, the stability of (6) around the equilibrate state of (1) is equivalent to the stability of zero solution of (24). Consequently, we only need to study the stability of zero solution of (24). By use of transformation, system (24) is equivalent to the following system:

\[
\begin{aligned}
\frac{d}{dt} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
&= \begin{bmatrix}
    a_{12} N_1^* + a_{13} N_2^* \\
    a_{12} N_1^* \\
    a_{13} N_2^*
\end{bmatrix} \int_{t-	au_{11}}^{t} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} ds \\
&\quad + \begin{bmatrix}
    a_{11} N_1^* \\
    a_{12} N_1^* \\
    a_{13} N_2^*
\end{bmatrix} \int_{t-	au_{13}}^{t} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} ds \\
&\quad + \begin{bmatrix}
    a_{21} N_2^* \\
    a_{22} N_2^* \\
    a_{23} N_2^*
\end{bmatrix} \int_{t-	au_{21}}^{t} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} ds \\
&\quad + \begin{bmatrix}
    a_{31} N_3^* \\
    a_{32} N_3^* \\
    a_{33} N_3^*
\end{bmatrix} \int_{t-	au_{31}}^{t} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} ds \\
&\quad + \sigma_1 x_1 (x_1 + N_1^*) dt + \sigma_2 x_2 (x_2 + N_2^*) dt + \sigma_3 x_3 (x_3 + N_3^*) dt
\end{aligned}
\]

Consider the following linear case of (25):

\[
\begin{aligned}
\frac{d}{dt} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
&= \begin{bmatrix}
    a_{12} N_1^* + a_{13} N_2^* \\
    a_{12} N_1^* \\
    a_{13} N_2^*
\end{bmatrix} \int_{t-	au_{11}}^{t} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} ds \\
&\quad + \begin{bmatrix}
    a_{11} N_1^* \\
    a_{12} N_1^* \\
    a_{13} N_2^*
\end{bmatrix} \int_{t-	au_{13}}^{t} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} ds \\
&\quad + \begin{bmatrix}
    a_{21} N_2^* \\
    a_{22} N_2^* \\
    a_{23} N_2^*
\end{bmatrix} \int_{t-	au_{21}}^{t} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} ds \\
&\quad + \begin{bmatrix}
    a_{31} N_3^* \\
    a_{32} N_3^* \\
    a_{33} N_3^*
\end{bmatrix} \int_{t-	au_{31}}^{t} \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} ds \\
&\quad + \sigma_1 x_1 (x_1 + N_1^*) dt + \sigma_2 x_2 (x_2 + N_2^*) dt + \sigma_3 x_3 (x_3 + N_3^*) dt
\end{aligned}
\]

For system (25), the following result holds.

**Theorem 3.** If the following (H2) holds,

\[
(H_2) \begin{bmatrix}
    \tilde{\lambda}_1 & -a_{21} & -a_{31} \\
    -a_{12} & \tilde{\lambda}_2 & -a_{32} \\
    -a_{13} & -a_{23} & \tilde{\lambda}_3
\end{bmatrix}
\begin{bmatrix}
    N_1^* \\
    N_2^* \\
    N_3^*
\end{bmatrix}
> 2 \begin{bmatrix}
    0 & \xi_1 & \xi_3 \\
    \xi_2 & 0 & \eta_1 \\
    \xi_3 & \eta_3 & 0
\end{bmatrix}
\begin{bmatrix}
    (N_1^*)^T \\
    (N_2^*)^T \\
    (N_3^*)^T
\end{bmatrix},
\]

then system (25) around the equilibrium state \( N_1^* \) is stable in probability; i.e., system (6) around the equilibrium state \( N_1^* \) is stable in probability.

**Proof.** Applying the same manner as before, consider the linear system (26) and define \( V_1, V_2, \) and \( V_3 \) as before. By using the Itô formula and computing the derivatives of \( V_1, V_2, \) and \( V_3 \) along solutions of (26), then
Remark 5. Theorems 1–3 show that both time delays $(\tau_{ij}, i, j = 1, 2, 3)$ and stochastic disturbances $(\sigma_i, i = 1, 2, 3)$...
have important impacts on the globally asymptotical stability and stability in probability around the positive equilibrium state. Conditions \((H_1)\) and \((H_2)\) imply that too large time delays or white noises may destroy the stability of above systems, which are also verified by numerical simulations (see Figures 1 and 2 in the next section).

Remark 6. Theorems 1–3 highlight the effects of time delays and stochastic disturbances on the stability around the positive equilibrium state, where conditions \((H_1)\) and \((H_2)\) seem complexity, but they are not difficult to be verified by computation software like Matlab.

4. Numerical Simulations

In this section, we present some numerical simulations to verify the theoretical result. After giving values of all parameters, by applying the method [24] and writing some suitable Matlab code, we can obtain the numerical results as follows. Let \(a_{11} = 1, a_{12} = 0.2, a_{13} = 0.3, a_{21} = 0.3, a_{22} = 1, a_{23} = 0.2, a_{31} = 0.2, a_{32} = 0.3, a_{33} = 1, r_1 = 1, r_2 = 1.2, \) and \(r_3 = 1.5.\) Then, an easy computation yields that \(A = 0.785, A_1 = 1.783, A_2 = 1.928, A_3 = 2.098 \) and \(N_1^* = 2.2713, N_2^* = 2.4561, N_3^* = 2.6726.\)

If \(\sigma_1 = 0.2, \sigma_2 = 0.1, \sigma_3 = 0.3, \) and \(\tau = 0.5,\) with the help of Matlab, by computation, we have

\[
\begin{align*}
\Delta_1 &= 1.4824, \\
\Delta_2 &= 1.4959, \\
\Delta_3 &= 1.4663, \\
\bar{\Delta}_1 &= 1.4091, \\
\bar{\Delta}_2 &= 1.4754, \\
\bar{\Delta}_3 &= 1.2592, \\
\end{align*}
\]

and hence,

\[
\begin{pmatrix}
\Delta_1 & -a_{21} & -a_{31} & N_1^* \\
-a_{12} & \Delta_2 & -a_{32} & N_2^* \\
-a_{13} & -a_{23} & \Delta_3 & N_3^*
\end{pmatrix} = \begin{pmatrix}
2.0956 \\
2.4180 \\
2.7462
\end{pmatrix},
\]

\[
\begin{pmatrix}
\bar{\Delta}_1 & -a_{21} & -a_{31} & N_1^* \\
-a_{12} & \bar{\Delta}_2 & -a_{32} & N_2^* \\
-a_{13} & -a_{23} & \bar{\Delta}_3 & N_3^*
\end{pmatrix} = \begin{pmatrix}
1.9291 \\
2.3677 \\
2.1927
\end{pmatrix},
\]

By verification, both \((H_1)\) and \((H_2)\) hold, and then, Theorems 2 and 3 hold. Theorem 2 implies system (4) is stable around the equilibrium of (1). The state graphs of each population are illustrated in Figure 3. Furthermore,
Figure 3: Stable case of system (4) with $\sigma_1 = 0.2, \sigma_2 = 0.1, \sigma_3 = 0.3$, and $\tau = 0.5$: (a) equilibrium of (1) with $N_1^* = 2.2713$, $N_2^* = 2.4561$, and $N_3^* = 2.6726$; (b) the state graph of three species of (4); (c), (d), and (e) are state graphs of populations $N_1, N_2$, and $N_3$ of (4) around their corresponding equilibrium points $N_1^*, N_2^*$, and $N_3^*$, respectively.
Theorem 3 indicates system (6) around the equilibrium $N_i^*$ is also stable. The state graphs of each population are illustrated in Figure 4.

If $\tau = 0.8$, other parameters are same as before, then

$$2r \begin{pmatrix} 0 & \xi_1 & \eta_1 \\ \xi_2 & 0 & \eta_2 \\ \xi_3 & \eta_1 & 0 \end{pmatrix} \begin{pmatrix} (N_1^*)^2 \\ (N_2^*)^2 \\ (N_3^*)^2 \end{pmatrix} = \begin{pmatrix} 2.5906 \\ 5.397 \\ 2.945 \end{pmatrix}.$$ (31)

An easy computation yields that neither $(H_1)$ nor $(H_2)$ holds; hence, (4) and (6) around the equilibrium point may be unstable (see Figures 1(a) and 1(b), respectively). The state graphs of each population in Figure 1 show that too large time delays destroy the stability of (4) and (6).

Finally, if $\sigma_1 = 0.2, \sigma_2 = 0.1, \sigma_3 = 0.3$, and $\tau = 0.5$, then $\Delta_1 = -0.2611, \Delta_2 = -0.1286, \Delta_3 = 0.0434$. Hence, $(H_1)$ does not hold and system (4) may be unstable, which is shown in Figure 2(a). If $\sigma_1 = 0.9, \sigma_2 = 0.9, \sigma_3 = 0.9, \tau = 0.5$, then $\Delta_1 = -0.3398, \Delta_2 = -0.4894, \Delta_3 = -0.6648$. Hence, $(H_2)$ does not hold and system (6) may be unstable too (see Figure 2(b)). The state graphs of each population in Figure 2 show that too large stochastic disturbances destroy the stability of above systems.
5. Conclusion and Future Direction

In this paper, we investigate the stability of two three-species cooperative systems with time delays and stochastic disturbances. Theorem 1 gives the sufficient conditions assuring the globally asymptotical stability around the equilibrium state of the corresponding deterministic system. Theorems 2 and 3 imply that systems (4) and (6) are stable in probability around the equilibrium states under some conditions, respectively. By giving Remarks 3–5, we clarify the difference of our proof from some existing methods and, particularly, show that time delays and stochastic perturbations bring some significant impacts on the globally asymptotical stability and stability in probability.

In this paper, we assume time delays are constants, whereas in practice, time delays may be time-varying (see [25–27]), so it is necessary to study the dynamics with time-varying delays. Furthermore, for multispecies predator-prey system or multispecies competitive system, whether some similar results can be obtained? These are interesting and left for our future work.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors have read and approved the final manuscript.

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