Strongly Intensive Cumulants: Fluctuation Measures for Systems With Incompletely Constrained Volumes

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Abstract

The cumulants of thermal variables are of general interest in physics due to their extensivity and their correspondence with susceptibilities. They become especially significant near critical points of phase transitions where they diverge along with the correlation length. Cumulant measurements have been used extensively within the field of heavy-ion physics, principally as tools in the search for a hypothetical QCD critical point along the transition between hadronic matter and QGP. The volume of individual heavy-ion collisions can be only partially constrained and, as a result, cumulant measurements are significantly biased by the limited volume resolution. We propose a class of moments called strongly intensive cumulants which can be accurately measured in the presence of unconstrained volume fluctuations. Additionally, they share the same direct relationship with susceptibilities as cumulants in many cases.
I. INTRODUCTION

The development of classes of moments that are either invariant or have clear scaling properties under various operations has been an active area of research for well over a century. The initial work on cumulants themselves predated the development of statistical partition functions and focused on their scaling properties under affine transformations [1]. They were originally called half-invariants due to these scaling properties and were later renamed cumulants due to their additivity under convolution [2]. In the field of computer vision, two-dimensional moments which are invariant under operations such as rotation, scaling, and translation have played a central role in pattern recognition [3, 4]. Within the field of particle physics, the $R$ fluctuation measures were constructed as ratios of factorial moments such that detector efficiencies would cancel resulting in moments invariant under binomial efficiency losses [5, 6].

More recently, the $\Delta$ and $\Sigma$ observables were proposed to address the issue of measurement biases resulting from the poor constraint of volume in heavy-ion collisions [7–9]. These quantities are two-dimensional second-order moments which are constructed in such a way that the volume fluctuation terms cancel. The authors coined the term strongly intensive to describe these observables because they are not only independent of the volume of the system but also of the distribution of volume within an ensemble. They have been used effectively in fluctuation analyses [10, 11], but their physical meanings are obscure relative to those of cumulants and they only measure second-order fluctuations.

In this work, we present a new set of statistical quantities which we call strongly intensive cumulants. These quantities are invariant under both convolution and mixing with distributions sharing the same strongly intensive cumulants. A direct result of this is that they can accurately be measured experimentally in situations where the volume is not well constrained. These quantities are directly related to cumulants when certain conditions are met and therefore, in physical systems, to thermodynamic susceptibilities.

The strongly intensive cumulants are of particular interest in heavy-ion collisions where cumulant measurements of conserved charges have been proposed as a signature for critical point fluctuations [12]. The experimental determination of these cumulants has been a focus of the RHIC beam energy scan [13, 14] and the published measurements have been discussed in the context of critical fluctuations as well as compared to lattice calculations in order to
determine the temperature and chemical potentials at chemical freeze-out \cite{15}. It has been shown that these measurements depend greatly on the method used to constrain collision volumes which calls into question the validity of any physics conclusions drawn from the results \cite{16}.

To demonstrate the utility and efficacy of the strongly intensive cumulants we present an analysis of net proton fluctuations in simulated heavy-ion collisions. This analysis makes clear the issues with current analysis techniques and illustrates how the strongly intensive cumulants resolve them. In light of this, we propose that the strongly intensive cumulants listed in Eq. \eqref{eq:9} be used as drop-in replacements in future heavy-ion cumulant analyses.

II. MOTIVATION

We begin by briefly reviewing the generating function formalism with respect to statistical moments, cumulants, and their relationships to volume fluctuations. Let $X = (X_1, X_2, \ldots, X_n)$ be the components of a random vector. The moment-generating function is then defined to be

$$
\phi (\xi) \equiv \langle e^{\xi_i X_i} \rangle_X = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{\mu_{r_1, r_2, \ldots, r_n}}{r_1! r_2! \cdots r_n!} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n}
$$

where the $\langle \cdots \rangle_X$ notation denotes the expectation value over the distribution of $X$ and the repeated index $i$ is implicitly summed over according to Einstein summation notation. The coefficients of the Taylor series correspond to the moments of the distribution and can be recovered from the generating function by taking derivatives and setting $\xi = 0$. Defining the operator $D_i$ to be $\partial / \partial \xi_i$, we can see that

$$
\mu_{r_1, r_2, \ldots, r_n} = [D_1^{r_1} D_2^{r_2} \cdots D_n^{r_n} (\phi)]_{\xi=0} = \langle X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n} \rangle_X
$$

as we would expect. This approach of recovering coefficients from the generating function can be applied in the same way for both the standard cumulants and the strongly intensive cumulants that we will introduce later.
The generating function for the cumulants is then defined in terms of the moment-generating function as

\[ \psi (\xi) \equiv \ln \phi (\xi) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{\kappa_{r_1,r_2,\ldots,r_n}}{r_1!r_2!\cdots r_n!} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n} \]

where the coefficients \( \kappa_{i,j,\ldots,k} \) are the cumulants and can be expressed in terms of the moments by matching terms in the Taylor series, either combinatorially or using recursion relations [17].

The utility of the cumulants becomes clear when we consider the convolution of probability distributions. If we take two independent random vectors, \( X \) and \( Y \), then the distribution of \( Z = X + Y \) is given by the convolution of their respective distributions. The moment-generating function of \( Z \) is then given by \( \phi_Z (\xi) = \langle e^{\xi (X+Y)} \rangle \) which, due to the independence of \( X \) and \( Y \), can be factored as \( \phi_Z (\xi) = \langle e^{\xi X} \rangle_X \langle e^{\xi Y} \rangle_Y \). The cumulant-generating function is then simply \( \psi_Z (\xi) = \ln \phi_Z (\xi) = \ln \phi_X (\xi) + \ln \phi_Y (\xi) \) which is precisely the sum of the cumulant-generating functions for \( X \) and \( Y \). Thus, we see that the cumulants of two probability distributions are additive under convolution, a fact that led to their current name.

This additivity property of cumulants is closely related to their utility in physics. If we consider two volumes of matter that are each in the same thermodynamic state then the distributions of any total quantities (e.g. net charges, total energy) for the combined volume will be given by the convolution of their distributions for each of the two independent volumes. The cumulants of these distributions will necessarily be extensive and, after scaling by the volume, will give intrinsic quantities determined by the thermodynamic state of the matter. By relating the partition function to a moment-generating function we can see that the coefficients of the corresponding cumulant-generating function then encode how the mean values of total quantities change with respect to state variables like energy density or chemical potential [18].

The determination of these coefficients is of general interest in physics but there is an experimental limitation that makes it difficult. If the volume of an ensemble of systems cannot be perfectly constrained then the cumulants of the distributions depend strongly on the distribution over volume within the ensemble. To see this we define an intrinsic generating function such that
\[
\psi' (\xi) \equiv \psi (\xi)/V \tag{1}
\]
\[
= \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \kappa'_{r_1; r_2; \ldots; r_n} \xi_{r_1} \xi_{r_2} \cdots \xi_{r_n}
\]
where the primes indicate independence from the volume. We use the term volume here loosely; it really corresponds to any measure with which the cumulants scale linearly. In actuality, \( V \) can depend on the temperature, the energy density, and other quantities in addition to the volume. The important point in the context of this discussion is simply that this quantity can be factored out from the cumulant-generating function, as done in Eq. (1).

The moment-generating function can then be expressed as
\[
\phi (\xi) = e^{V \psi' (\xi)}
\]
\[
= 1 + V \psi' (\xi) + \frac{1}{2!} V^2 \psi' (\xi)^2 + \cdots
\]
with an explicit volume dependence. In any experimental context, the moments that are measured are those of a mixed distribution over an ensemble of volumes. The measured cumulants will then be described by the generating function
\[
\psi (\xi) = \ln \left\langle e^{V \psi' (\xi)} \right\rangle_V \tag{2}
\]
\[
= \ln \left( 1 + \left\langle V \right\rangle_V \psi' (\xi) + \frac{1}{2!} \left\langle V^2 \right\rangle_V \psi' (\xi)^2 + \cdots \right)
\]
which can easily be seen to not equal \( \left\langle V \right\rangle_V \psi' (\xi) \) unless \( V \) is fixed at a single value.

The measured cumulants instead depend on the distribution of the volume in a straightforward way [19]. This relationship can be made more clear by considering the simple example of the variance of a single variable. By expanding the logarithm in Eq. (2) and matching terms we find that the measured variance would be
\[
\kappa_2 = \left\langle V \right\rangle_V \kappa'_2 + \left( \left\langle V^2 \right\rangle_V - \left\langle V \right\rangle_V^2 \right) \left( \kappa'_1 \right)^2 \tag{3}
\]
in the presence of volume fluctuations. If the volume is constrained to a single value then the variance of \( V \) goes to zero and the relationship reduces to \( \kappa_2 = \left\langle V \right\rangle_V \kappa'_2 \), the quantity that one would truly want to measure. The measured variance will be artificially high for any other distribution of volume due to the contribution of the second term. The most extreme example of this is a situation where \( \kappa'_2 = 0 \), which would be approximately true for
the total number of atoms in a crystal lattice. In this scenario, the first term on the right hand side of Eq. (3) vanishes and the measured variance would be directly proportional to the variance of the volume in the measurement ensemble.

It may seem as though the fluctuation terms could be subtracted off from measurements such that the cumulants $\langle V \rangle_V \kappa'_i$ could be directly determined. This is true in theory, but the measurements cannot be corrected for without an exact knowledge of the volume distribution. The precise shape of this distribution is typically not known in practice and so this approach is not applicable. Instead, measurements tend to be made without any attempt at corrections and, as a result, they are biased in poorly understood ways due to the contributions from volume fluctuations.

III. DERIVATION

We introduce here a set of statistical quantities called the strongly intensive cumulants. The goal is to construct a set of non-trivial statistical quantities that can be measured in physical systems without any dependence on the volume distribution. Formally, this is equivalent to saying that the quantities are invariant under both convolution and mixing of distributions sharing the same strongly intensive cumulants. We will begin by defining the strongly intensive cumulants and then prove that they satisfy the desired properties.

Their generating function, $\psi^*$, is defined in terms of the partial differential equation

$$D_u (\psi^*) = \frac{D_u (\phi)}{D_v (\phi)}$$

(4)

where the choice of $u$ and $v$ determines the exact flavor of the generating function. We assume from here foreword that the $u$ and $v$ indices are 1 and $n$, respectively. The components of $X$ can always be rearranged such that this is the case, with the trivial exception of $u = v$. Without loss of generality, we can assume this has been done.

The choice of the first and last components of $X$ is arbitrary, but we will see in Sec. [●] that it makes no difference which component comes first with an appropriate choice of $X_n$. In this situation, $X_n$ serves as a measure of the volume and is independent of the quantities that one is primarily interested in. The general case is more subtle but thinking of a physical situation where $X_n$ is a noisy volume measurement and the choice of $X_1$ is arbitrary can be helpful in understanding how we proceed.
The strongly intensive cumulants, \(\kappa_{r_1, r_2, \ldots, r_n}\), then correspond to the Taylor series coefficients of
\[
\psi^* (\xi) = \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{\kappa^*_{r_1, r_2, \ldots, r_n}}{r_1! r_2! \cdots r_n!} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n}
\]
in the same way that cumulants are defined by the coefficients of \(\psi\). The \(\kappa^*_{0, r_2, r_3, \ldots, r_n}\) terms in this series are, as of yet, undetermined because they correspond to the integration constants obtained when integrating Eq. (4) with respect to \(\xi_1\). We choose to define these terms to be \(\kappa^*_{0, r_2, r_3, \ldots, r_n} = \kappa^*_{r_2, r_3, \ldots, r_n}\), the strongly intensive cumulant obtained when the first component of \(\xi\) is dropped such that \(\xi \rightarrow (\xi_2, \xi_3, \ldots, \xi_n)\). This process can be applied repeatedly until the first index is nonzero or, if all indices are zero, \(\kappa^*_{0,0,0,\ldots} = 0\). This definition allows us to prove properties for \(D_1 (\psi^*)\) and have them apply more generally to \(\psi^*\) because the missing \(\kappa^*_{0, r_2, r_3, \ldots, r_n}\) terms correspond to coefficients of \(D_1 (\psi^*)\) for a different choice of \(\xi\).

First, we’ll show explicitly that the strongly intensive cumulants are completely independent of the distribution over volume in a physical system. This can be proved very simply by substituting the volume mixed moment-generating function from earlier into Eq. (4)
\[
D_1 (\psi^*) = \frac{D_1 (\langle e^{V \psi^*(\xi)} \rangle_V)}{D_n (\langle e^{V \psi(\xi)} \rangle_V)} = \frac{\langle V e^{V \psi^*(\xi)} \rangle_V D_1 (\psi')}{\langle V e^{V \psi^*(\xi)} \rangle_V D_n (\psi')} = D_1 (\psi') \frac{D_n (\psi')} {D_n (\psi')}
\]
and canceling the volume dependent terms. This proof of their strongly intensive property is straightforward, but it is not the most clear way to demonstrate why this happens. By exploring the properties of the strongly intensive cumulants under convolution and mixing it will become more clear.

We will first consider convolutions, whereby a new random vector \(Z\) is constructed as the sum of two other random vectors \(X\) and \(Y\). For physical systems, this operation can be viewed as constructing a larger volume out of two smaller volumes. Given that \(Z = X + Y\), we find that \(\psi_Z (\xi) = \psi_X (\xi) + \psi_Y (\xi)\) or, equivalently, that \(\phi_Z (\xi) = \phi_X (\xi) \phi_Y (\xi)\). Plugging this into Eq. (4) we find that
\[
D_1 (\psi^*_Z) = \frac{D_n (\psi_X) D_1 (\psi^*_X) + D_n (\psi_Y) D_1 (\psi^*_Y)}{D_n (\psi_X) + D_n (\psi_Y)}
\]
which can be seen as a $\xi$-dependent weighted average of the differential equations defining
the strongly intensive cumulant generating functions of $X$ and $Y$.

Now let us consider the case that $X$ and $Y$ have the same set of strongly intensive
cumulants. This implies that $D_1 (\psi^*_X) = D_1 (\psi^*_Y)$ which simplifies Eq. (6) to

$$D_1 (\psi^*_Z) = \frac{D_n (\psi_X) D_1 (\psi^*_X) + D_n (\psi_Y) D_1 (\psi^*_Y)}{D_n (\psi_X) + D_n (\psi_Y)}$$

$$= \frac{D_n (\psi_X) + D_n (\psi_Y)}{D_n (\psi_X) + D_n (\psi_Y)} D_1 (\psi^*_X)$$

$$= D_1 (\psi^*_X)$$

(7)

showing that the strongly intensive cumulants are invariant under the convolution of distributions with identical strongly intensive cumulants. This demonstrates the intensive property of the strongly intensive cumulants and how it emerges from the way they combine under convolution.

The situation for mixing distributions very closely parallels that for convolving them. If convolution can be thought of as an operation for constructing new volumes then distribution mixing can be thought of as the operation of combining different volumes into an ensemble. If we imagine that half of the time we choose a random vector $Z$ according to $P_X (Z)$ and half of the time we choose it from $P_Y (Z)$ then the probability for the mixed distribution $Z$ is given by $P_Z (Z) = \frac{1}{2} (P_X (Z) + P_Y (Z))$. We could have chosen a ratio other than $\frac{1}{2}$ but the extension to a mixing parameter is trivial. The resulting moment-generating function is, like the probability distribution, the arithmetic average of those for the separate distributions: $\phi_Z (\xi) = \frac{1}{2} (\phi_X (\xi) + \phi_Y (\xi))$. This results in a strongly intensive cumulant generating function of

$$D_1 (\psi^*_Z) = \frac{D_n (\phi_X) D_1 (\psi^*_X) + D_n (\phi_Y) D_1 (\psi^*_Y)}{D_n (\phi_X) + D_n (\phi_Y)}$$

(8)

which is again a weighted average of the two independent differential equations.

It is important to note that Eq. (8) is identical to Eq. (6) with $\psi$ replaced by $\phi$. Although the strongly intensive cumulants do not combine as simply as the moments under mixing or the cumulants under convolution, they have the same relationship with the moments under mixing as they do with the cumulants under convolution. Similarly, we find that

$$D_1 (\phi^*) = \frac{D_1 (\phi)}{D_n (\phi)} = \frac{1/\phi D_1 (\phi)}{1/\phi D_n (\phi)} = \frac{D_1 (\ln \phi)}{D_n (\ln \phi)} = \frac{D_1 (\psi)}{D_n (\psi)}$$
showing that the strongly intensive cumulants are directly related to the cumulants and moments of a distribution in the same way (i.e. the expressions for the strongly intensive cumulants are unchanged when all moments are replaced with cumulants or vice versa). The similarity in the relationships that the strongly intensive cumulants have with both the moments and the cumulants is at the core of why they exhibit strongly intensive behavior.

It follows from Eqs. (6) and (8) that the strongly intensive cumulants are invariant under mixing and convolution of distributions with identical strongly intensive cumulants. This was shown explicitly for convolution but the same proof employed in Eq. (7) applies to mixing when Eq. (8) is used as a starting point. The invariance under convolution implies that they will be intensive, or independent of volume, in a thermodynamic system. The combination of an ensemble with many different volumes is, in effect, distribution mixing and the strongly intensive cumulants will therefore be invariant under this operation as well. This is equivalent to saying that the strongly intensive cumulants can be measured over an ensemble of different volumes without any dependence on the volume distribution. This line of reasoning helps to illustrate why the volume terms were shown to cancel in Eq. (5).

IV. OBTAINING EXPRESSIONS FOR THE STRONGLY INTENSIVE CUMULANTS

We move now to the task of determining polynomial expressions for the strongly intensive cumulants. Let \( f(\xi) = D_n(\phi) \) and \( g(\xi) = 1/f(\xi) \) where the coefficients in the Taylor series for \( g(\xi) \) are given by \( a_{r_1,r_2,...,r_n}/(r_1!r_2!\cdots r_n!) \). We then find that

\[
0 = [D_1^{r_1}D_2^{r_2}\cdots D_n^{r_n}(f \times g)]_{\xi=0} = \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_n=0}^{r_n} \left( \begin{array}{c} r_1 \\ i_1 \end{array} \right) \left( \begin{array}{c} r_2 \\ i_2 \end{array} \right) \cdots \left( \begin{array}{c} r_n \\ i_n \end{array} \right) \times a_{i_1,i_2,...,i_n} \mu_{r_1-i_1,r_2-i_2,...,r_n-i_n-1,r_{n+1}-i_{n+1}}
\]

when at least one of \( r_1, r_2, \ldots, r_n \) are nonzero. We can rearrange this to give a recursion equation

\[
a_{r_1,r_2,...,r_n} = \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_n=0}^{r_n} \left( \begin{array}{c} r_1 \\ i_1 \end{array} \right) \left( \begin{array}{c} r_2 \\ i_2 \end{array} \right) \cdots \left( \begin{array}{c} r_n \\ i_n \end{array} \right)
\]

when \( i_1 \neq r_1 \lor i_2 \neq r_2 \lor \cdots \lor i_n \neq r_n \).
\[ \times a_{i_1,i_2,\ldots,i_n} \frac{\mu_{r_1-i_1,r_2-i_2,\ldots,r_n-i_n-1-r_{n-1}-1-r_{n+1}-i_n}}{-\mu_{0,0,\ldots,0,1}} \]

with the starting point \( a_{0,0,\ldots,0} = g(0) = 1/\mu_{0,0,\ldots,0,1} \).

Now we can express the strongly intensive cumulants in terms of these coefficients in a similar manner. We find that

\[
\kappa_{r_1,r_2,\ldots,r_n}^* = \left[ D_1^{r_1-1} D_2^{r_2} \cdots D_n^{r_n} (D_1 (\phi (\xi)) g (\xi)) \right] \\
= \sum_{i_1=0}^{r_1-1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_n=0}^{r_n} \binom{r_1-1}{i_1} \binom{r_2}{i_2} \cdots \binom{r_n}{i_n} \\
\times a_{i_1,i_2,\ldots,i_n} \mu_{r_1-i_1,r_2-i_2,\ldots,r_n-i_n}
\]

which provides a full prescription for finding the strongly intensive cumulants for \( r_1 > 0 \). If \( r_1 = 0 \) then the first component can be removed, as described earlier. For the special case of \( n = 2 \) and \( r_2 = 0 \) this can be simplified to

\[
\kappa_r^* \equiv \kappa_{r,0}^* = \frac{1}{\mu_{0,1}} \left( \mu_{r,0} - \sum_{i=1}^{r-1} \binom{r - 1}{i - 1} \mu_{r-i,i} \kappa_i^* \right)
\]

where the second zero index has been dropped. This can be done without ambiguity because the strongly intensive cumulants are trivial in one dimension. This recursive equation can be used to find the explicit expressions

\[
\begin{align*}
\kappa_1^* &= \frac{\mu_{1,0}}{\mu_{0,1}} \\
\kappa_2^* &= \frac{\mu_{2,0}}{\mu_{0,1}} - \frac{\mu_{1,0} \mu_{1,1}}{\mu_{0,1}^2} \\
\kappa_3^* &= \frac{\mu_{3,0}}{\mu_{0,1}} - \frac{2 \mu_{2,0} \mu_{1,1} + \mu_{1,0} \mu_{2,1}}{\mu_{0,1}^2} + \frac{2 \mu_{1,0} \mu_{1,1}^2}{\mu_{0,1}^3} \\
\kappa_4^* &= \frac{\mu_{4,0}}{\mu_{0,1}} - \frac{3 \mu_{3,0} \mu_{1,1} + \mu_{1,0} \mu_{3,1}}{\mu_{0,1}^2} - \frac{3 \mu_{2,0} \mu_{2,1}}{\mu_{0,1}^2} \\
&\quad + \frac{6 \mu_{2,0} \mu_{1,1}^2 + 6 \mu_{1,0} \mu_{2,1} \mu_{1,1}}{\mu_{0,1}^3} - \frac{6 \mu_{1,0} \mu_{1,1}^3}{\mu_{0,1}^4}
\end{align*}
\]

which we expect will be most practical in common usage.

V. RELATION TO OTHER QUANTITIES

The name “strongly intensive cumulants” may seem contradictory, as they are not cumulative like the standard cumulants. This name was chosen due to their close relationship
with cumulants. We’ll work with a fixed volume system so that there are no volume fluctuation terms anywhere. Now consider the case where $X_n$ is completely uncorrelated from $X_1, X_2, \ldots, X_{n-1}$ at fixed volume such that $\langle X_1^{r_1} X_2^{r_2} \cdots X_{n-1}^{r_{n-1}} X_n^{r_n} \rangle = \langle X_1^{r_1} X_2^{r_2} \cdots X_n^{r_n} \rangle$ for any choice of $r_1, r_2, \ldots, r_n$. It then follows that

$$D_1 (\psi^*) = \frac{D_1 (\phi)}{D_n (\phi)} = \frac{D_1 (\phi)}{D_n ((e^{\xi_i X_i})_X)} = \frac{D_1 (\phi)}{\langle X_n e^{\xi_i X_i} \rangle_X}$$

which directly implies

$$k_{r_1, r_2, \ldots, r_n}^* = \frac{k_{r_1, r_2, \ldots, r_n}}{k_{0, 0, \ldots, 0, 1}}$$  \hspace{1cm} (10)$$

for $r_1 > 0$ and $n > 1$. The definition of the integration constant was, however, chosen such that this equality holds for $r_1 = 0$ as well. Thus, we find that the strongly intensive cumulants are equal to their corresponding cumulants normalized by a volume term given that $X_n$ is independent from the other components. For the case of cumulant ratios, which are used frequently in experimental contexts, this normalization cancels.

When we extend this to a variable volume, we find that the left hand side of Eq. (10) does not change because it is strongly intensive. This means that it corresponds to what the cumulant ratio on the right hand side would be in the absence of volume fluctuations. In a very real sense, this quantity corresponds to what one would want to measure in the absence of volume fluctuations.

It is important to note here that this correspondence with the cumulants depends only on the independence of $X_n$ at fixed volume and not at all on the shape of its distribution. This means that any estimate of the volume can be used for $X_n$, regardless of its noise profile. Any independent quantity that has some dependence on the volume can be used as $X_n$ if it is adjusted such that the dependence is linearly proportional. This opens up the possibility of using volume estimates for $\xi_n$ that have been previously considered too noisy or irregular for volume determination.

It is also worth mentioning that $k_{r_1, r_2, \ldots, r_n}^*$ should equal zero when $X_n$ is chosen to be independent at fixed volume, given that at least one of $r_1, r_2, \ldots, r_{n-1}$ are nonzero. This follows trivially from the cumulant correspondence demonstrated in Eq. (10) because all joint cumulants involving $X_n$ must be zero in order for it to be entirely independent. This means
that in experimental contexts the strongly intensive cumulants with $r_n = 0$ will likely be the most interesting. It also suggests that evaluating how far the strongly intensive cumulants with $r_n > 0$ are from zero might be useful for evaluating systematic errors involving the independence of $X_n$.

The second-order strongly intensive cumulants are also closely related to the $\Sigma$ and $\Delta$ fluctuation measures that were mentioned earlier [7–9]. Namely,

$$\Sigma [X_1, X_2] = \kappa^*_2 [X_1, X_2] + \kappa^*_2 [X_2, X_1]$$
$$\Delta [X_1, X_2] = \kappa^*_2 [X_1, X_2] - \kappa^*_2 [X_2, X_1]$$

when the normalization factor on $\Sigma$ and $\Delta$ is chosen to be $\mu_1^0/\mu_2^0$, and the $[X_1, X_2]$ denotes the order of the random vector components. The strongly intensive cumulants have the same properties as these quantities but make the relationship to the underlying physics more explicit.

VI. A PRACTICAL EXAMPLE

We consider here a practical example from heavy-ion physics. Higher order fluctuations of conserved quantities in nuclear collisions has been a topic of considerable interest in recent years. The most discussed of these quantities has been the ratio of $\kappa_4/\kappa_2$ for net proton fluctuations as measured by the STAR experiment. With respect to these measurements, it has frequently been reported that “there are interesting trends, including e.g. the drop in the kurtosis of the net-proton distribution at $\sqrt{s_{NN}} = 27$ and 19.6 GeV” [20]. We will study simulated nuclear collisions to quantify the magnitude of bias introduced by volume fluctuations and to demonstrate the utility of the strongly intensive cumulants for addressing this issue.

Our analysis was run over 35 million Au+Au events at $\sqrt{s_{NN}} = 7.7$ GeV that were generated using the UrQMD model [21, 22]. The procedure followed was designed to closely mirror those used by STAR [13, 14]. The net proton number in each event was quantified as the number of protons minus the number of antiprotons with transverse momenta in the range $0.4 \text{ GeV} < p_T < 0.8 \text{ GeV}$ and pseudorapidities between $-0.5 < \eta < 0.5$. The detector efficiency for these particles was assumed to be unity in order to disentangle volume fluctuations from other effects. The approximate volume of the system was measured using
the multiplicity of charges particles with $0.5 < |\eta| < 1.0$ and $p_T > 0.15$ GeV, a quantity called Ref$_{\text{mult}:2}$ within STAR. This is the centrality quantity used in the net charge, rather than net proton, analysis at STAR but it was chosen to allow for a measurable $X_n$ variable.

Centrality was defined as the integrated percentile of Ref$_{\text{mult}:2}$, from the largest values to the smallest. The cumulant ratios were computed for each individual value of Ref$_{\text{mult}:2}$ and then averaged across each centrality bin. This was done to minimize the impact of binning on the results. This procedure was repeated three times with different binomial efficiencies for Ref$_{\text{mult}:2}$ each time: $p = 1/3$, $p = 2/3$, and $p = 1$. The efficiency of $p = 2/3$ is the most realistic but the other two were included to demonstrate the effect of better and worse volume resolutions. Additionally, the analysis was repeated with a multiplicity variable that counted all pions and kaons produced in each event. The results obtained using this multiplicity variable represent an ideal case of resolution where the biases induced by volume fluctuations should be largely eliminated.

Finally, the analysis was repeated using strongly intensive cumulants rather than standard cumulants. The $X_1$ component corresponded again to the net proton number while $X_2$ was chosen to be the number of charged pions and kaons with $0.4 \text{ GeV} < p_T < 0.8 \text{ GeV}$ and $-0.5 < \eta < 0.5$. The results can be seen in Fig. 1.

The most striking feature of the results is how dramatically the cumulant measurements are shifted by volume fluctuations, in some cases by well over a factor of two. The features observed in the STAR data are on order of 20-40% which can easily fall within the unquantified systematic errors caused by volume fluctuations. This is particularly true considering that $\sqrt{s_{NN}} = 27$ and 19.6 GeV were run with different detector configurations than the other Beam Energy Scan (BES) energies and therefore have significantly different systematics.

Another interesting thing to note is the drastic shift in the cumulant ratios in the most central events. This same trend can be seen in the data and it can be explained by a suppression of volume fluctuations when selecting on the highest multiplicity events. For mid-central events, a single multiplicity value corresponds to high multiplicity fluctuations from smaller volumes and low multiplicity fluctuations from larger volumes. This results in a wide range of possible volumes. For the highest multiplicity values there are only upward fluctuations due to there being roughly a maximum volume attainable in a collision. This leads to a tighter constraint of the volume near this maximum value for the most central events. This, at first glance, might suggest that the most central collisions are the most
Figure 1. Net Proton Cumulant Ratios

Standard and strongly intensive cumulant ratios of the net proton distribution in UrQMD generated Au+Au events at $\sqrt{s_{NN}} = 7.7$ GeV as a function of centrality. The square markers indicate standard cumulant ratios and the circle markers indicate the strongly intensive cumulant ratios for various centrality resolutions. The points are offset slightly along the $x$–axis for clarity.

The blue band indicates what the standard cumulant ratios would be with an ideal centrality resolution. We see that the strongly intensive cumulants correspond very closely with the ideal centrality resolution scenario regardless of the actual centrality resolution. The standard cumulant ratios, on the other hand, are very significantly biased and depend quite strongly on the centrality resolution. This shows very clearly that the strongly intensive cumulants more accurately measure the desired cumulant ratios in the inevitable presence of volume fluctuations.

Moving on to the strongly intensive cumulants we find that they are nearly identical for each of the resolution settings. Some small differences can be seen due to physics correlations between the protons in $X_1$ and the pions and kaons in $X_2$, but, even still, the effects of volume fluctuations are suppressed by well over an order of magnitude compared to the standard cumulant ratios. The remaining differences could be largely eliminated by a more careful choice of $X_2$. These quantities aren’t only invariant, they can also be seen to match almost exactly with what the cumulant ratios would be in the case of ideal centrality resolution. This very clearly illustrates both the invariance under volume fluctuations of the strongly intensive cumulants as well as their correspondence to the cumulants in the absence of

trustworthy. Unfortunately, this is not the case in practice because these same bins are the most sensitive to pileup events and secondary collision background.
VII. CONCLUSIONS

A new set of statistical moments called the strongly intensive cumulants have been presented. These quantities have been shown to be invariant under both convolution and mixing with distributions sharing identical strongly intensive cumulants. This property allows them to be experimentally determined in ensembles of physical systems where the distribution over volume is unknown. We have studied a practical example from heavy-ion physics where the measurement of cumulant ratios has been shown to be extremely biased due to the uncertainties in constraining the system volume. In this same example, the strongly intensive cumulant ratios have been shown to be almost entirely independent of how well constrained the volume is. Furthermore, they have been shown to correspond to the cumulant values that would be measured if the volume could be almost perfectly constrained. For these reasons, it is clear that the strongly intensive cumulants are better suited for determining underlying physics in systems with imperfectly constrained volumes.

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