MODELS FOR THE COHOMOLOGY OF CERTAIN POLYHEDRAL PRODUCTS

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Abstract. For a commutative ring \( \mathbb{k} \) with unit, we describe and study various differential graded \( \mathbb{k} \)-modules and \( \mathbb{k} \)-algebras which are models for the cohomology of polyhedral products \( (\mathbb{C}X, X)^K \). Along the way, we prove that the integral cohomology \( H^*(D^1, S^0)^K; \mathbb{Z} \) of the real moment-angle complex is a Tor module, the one that does not come from a geometric setting. We also reveal that the apriori different cup product structures in \( H^*((DX, X)^K; \mathbb{Z}) \) and in \( H^*((D^n, S^{n-1})^K; \mathbb{Z}) \) for \( n \geq 2 \) have the same origin. As an application, this work sets the stage for studying the based loop space of \( (\mathbb{C}X, X)^K \) in terms of the bar construction applied to the differential graded \( \mathbb{Z} \)-algebras \( B(C^*(X, K)), K \) quasi-isomorphic to the singular cochain algebra \( C^*((\mathbb{C}X, X)^K; \mathbb{Z}) \).

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1. Introduction

One of the most studied algebraic invariants of topological spaces is their cohomology ring. With its rich and relatively approachable structure, cohomology is a homotopy theoretical flagship invariant harnessed by a host of mathematical disciplines.

In the realm of toric topology, polyhedral products \( (X, A)^K \) are constructed as a functorial interplay between topology and combinatorics in term of topological pairs \( (X, A) \) and a simplicial complex \( K \). More precisely, let \( K \) be a simplicial complex on the vertex set \( [m] = \{1, \ldots, m\} \) and let \( (X, A) = \{ (X_i, A_i) \}_{i=1}^m \) be an \( m \)-tuple of CW-pairs. The polyhedral product is defined by

\[
(X, A)^K = \bigcup_{\sigma \in K} (X, A)^\sigma \subseteq \prod_{i=1}^m X_i
\]

where

\[
(X, A)^\sigma = \prod_{i=1}^m Y_i, \quad Y_i = \begin{cases} X_i & \text{for } i \in \sigma \\ A_i & \text{for } i \notin \sigma. \end{cases}
\]

With \( CX_i \) being the cone on \( X_i \), we restrict our attention to polyhedral products where the topological pairs are \( (CX_i, X_i) \) for all \( i \). Of particular importance are \( X_i = S^1 \) for all \( i \) in which case the polyhedral
Buchstaber and Panov [4] showed that there is an algebra isomorphism for the real moment-angle complex, denoted by $\mathbb{R}_K$.

Before we start with the study of polyhedral products, let us lay down some basic combinatorial notations. For an abstract simplicial complex $K$ on the vertex set $[m]$, let $|K|$ denote its geometric realisation and let $K_J$ be the full subcomplex of $K$ consisting of all simplices of $K$ with vertices in $J \subseteq [m]$.

Moment-angle complexes are known as the bellwethers for the combinatorial coding of topological and geometrical properties. For example, there is a stable decomposition of a real moment-angle complex $\mathbb{R}_K = (D^1, S^0)^K$ in terms of full subcomplexes of the simplicial complex $K$ given by

$$\Sigma(D^1, S^0)^K = \bigvee_{J \subseteq [m]} |\Sigma K_J|.$$

This decomposition was generalised in [1] to a stable decomposition of polyhedral products, which specialises to the following stable decomposition of $(C \Sigma X, X)^K$,

$$\Sigma(C \Sigma X, X)^K \to \bigvee_{J \subseteq [m]} |\Sigma K_J| \wedge Y^J$$

where

$$Y^J = \bigwedge_{i \in J} Y_i$$

if $i \in J$ and $Y_i = S^0$ if $i \notin J$.

Directly from these stable decompositions, we can read off the underlying $k$-module of the cohomology of the real moment-angle complex, of the complex moment-angle complex and of the polyhedral product $(C \Sigma X, X)^K$, respectively as

$$H^*((D^1, S^0)^K; k) \cong \bigoplus_{J \subseteq [m]} H^{*-1}(|K_J|; k), \quad H^*((D^2, S^1)^K; k) \cong \bigoplus_{J \subseteq [m]} H^{*}-|J|(-1)\left(|K_J|; k\right)$$

and

$$H^*((C \Sigma X, X)^K; k) \cong \bigoplus_{J \subseteq [m]} H^*\left(|\Sigma K_J|; k\right) \otimes H^J$$

where $H^J = \bigotimes_{i \in J} H_i$ with $H_i = \tilde{H}^*(X_i; k)$ if $i \in J$ and $H_i = k$ if $i \notin J$ assuming that $\tilde{H}^*(X_i; k)$ are free $k$-modules for all $i$.

Let $K$ be a simplicial complex on the vertex set $[m]$. Denote by $k[v_1, \ldots, v_m]$ the polynomial ring on $m$ variables $v_i$ and for every set $I = \{i_1, \ldots, i_r\} \subseteq [m]$, let $v_I = v_{i_1} \cdots v_{i_r}$. Then the Stanley-Reisner ring is defined by

$$SR[K] = k[v_1, \ldots, v_m]/\mathcal{I}_K$$

where $\mathcal{I}_K = (v_I \mid I \notin K)$ is the ideal generated by those square-free monomials $v_I$ for which $I$ is not a simplex of $K$. If $v_1, \ldots, v_m$ are all of degree 2, then

$$H^*((\mathbb{C}P^\infty, *)^K; k) \cong SR[K].$$

Using the language of toric varieties Franz [7] and independently using the existence of the homotopy fibration

$$T^m = (S^1, S^1)^K \to (D^2, S^1)^K \to (\mathbb{C}P^\infty, *)^K \to (\mathbb{C}P^\infty, \mathbb{C}P^\infty)^K = BT^m$$

Buchstaber and Panov [4] showed that there is an algebra isomorphism

$$H^*((D^2, S^1)^K; \mathbb{Z}) \cong \text{Tor}_{k[v_1, \ldots, v_m]}(SR[K], \mathbb{Z}).$$
Similarly, Franz [6] showed that there is an additive isomorphism

\[ H^*(D^1, S^0)^K \cong \text{Tor}_{\mathbb{Z}_2[t_1, \ldots, t_m]}(SR_{\mathbb{Z}_2}[K], \mathbb{Z}_2) \]

where \( t_i \) are of degree 1 and \( SR_{\mathbb{Z}_2}[K] = \mathbb{Z}_2[t_1, \ldots, t_m]/I_K \) which cannot be extended to a multiplicative isomorphism for the canonical product on the Tor. This additive description of the cohomology of the real moment-angle complex can be related to the homotopy fibration

\[ \mathbb{Z}_2^n = (S^0, S^0)^K \longrightarrow (D^1, S^0)^K \longrightarrow (\mathbb{R}P^\infty, *)^K \longrightarrow (\mathbb{R}P^\infty, \mathbb{R}P^\infty)^K = B\mathbb{Z}_2^m. \]

In this paper we extend Franz’s result by showing, as Corollary 2.16, that \( H^*((D^1, S^0)^K; \mathbb{k}) \) is additively isomorphic to a Tor \( \mathbb{k} \)-module. However this Tor \( \mathbb{k} \)-module does not have a geometric origin and the additive isomorphism cannot be extended to a one of algebras.

 Mimicking our approach for the cohomology of the real moment-angle complex, that is, by considering the polynomial ring on suspended cohomological classes of all \( X_i \)’s and introducing a generalised Stanley-Reisner ring \( SR(X, K) \), see Definitions 2.12 and 2.13 we next try to describe \( H^*((CX, X)^K; \mathbb{k}) \) as a Tor module. This approach allows us only to identify \( H^*((CX, X)^K; \mathbb{k}) \) as a summand of such a Tor module, see Proposition 2.10. To possibly get a description of \( H^*((CX, X)^K; \mathbb{k}) \) as a Tor module, in future work we plan to replace the polynomial ring on suspended cohomology classes of all \( X_i \)’s by an algebraic object that behaves as an algebraic “delooping” of \( X_i \)’s.

In Section 3 we give two integral algebraic models \( B(X, K) \) and \( B(C^*(X), K) \) for the cohomology of \( (CX, X)^K \), see Propositions 3.2 and 3.4. The results are obtained by combining Franz’s [6] and Cai’s [5] model of the integral cohomology ring of the real moment-angle complex with the result of Bahri, Bendersky, Cohen and Gitler [2] stating that the algebraic structure in \( H^*((CX, X)^K; \mathbb{k}) \) depends only on the cup product structure in \( H^*((D^1, S^0)^K; \mathbb{k}) \) and the cup product structure in \( H^*(X_i; \mathbb{k}) \) for all \( i \). The algebraic model \( B(C^*(X); \mathbb{k}) \) is particularly interesting as it is quasi-isomorphic to the k-cochains on \( (CX, X)^K \) and therefore the bar construction can be applied to it to deduce a model over \( \mathbb{k} \) for the based loop space of \( (CX, X)^K \) when \( X_i \) have free k-cohomology.

We finish the paper by explaining how seemingly different the cup product structures in \( H^*((D^1, S^0)^K; \mathbb{Z}) \) and in \( H^*((D^n, S^{n-1})^K; \mathbb{Z}) \) for \( n \geq 2 \) have the same origin.

2. Additive models

2.1. Real moment-angle complexes. Let \( \mathbb{k} \) be a commutative ring, which in this paper will be assumed to be either the ring \( \mathbb{Z} \) of integers or a field. Let \( y_1, \ldots, y_m \) be of degree 1. Define the graded algebra \( \mathbb{k}\langle y_1, \ldots, y_m \rangle \) as

\[ \mathbb{k}\langle y_1, \ldots, y_m \rangle = T(y_1, \ldots, y_m)/(y_iy_j = y_jy_i) \]

where \( T \) is a free associative algebra. Notice that here we are assuming commutativity not graded commutativity. Considered as a non-graded object \( \mathbb{k}\langle y_1, \ldots, y_m \rangle \) is isomorphic to the polynomial algebra \( \mathbb{k}[y_1, \ldots, y_m] \).

Given a subset \( I = \{i_1, \ldots, i_r\} \subset [m] \), we denote by \( y_I \) the square-free monomial \( y_{i_1} \cdots y_{i_r} \) in \( \mathbb{k}\langle y_1, \ldots, y_m \rangle \).

Define an analogue of the Stanley-Reisner ring to be

\[ SR(K) = \mathbb{k}\langle y_1, \ldots, y_m \rangle/I_K \]

where \( I_K = \langle y_I \mid I \notin K \rangle \) is the ideal generated by those monomials \( y_I \) for which \( I \) is not a simplex of \( K \).
For $\omega_1, \ldots, \omega_m$ of degree 0, define the graded algebra $L$ as
\[
L(\omega_1, \ldots, \omega_m) = T(\omega_1, \ldots, \omega_m)/ (\omega_i^2 = 0, \ \omega_i \omega_j = -\omega_j \omega_i).
\]
As in the case of $k(y_1, \ldots, y_m)$, the multiplication is not graded commutative, and considered as a non-graded object $L(\omega_1, \ldots, \omega_m)$ is isomorphic to the exterior algebra $\Lambda(\omega_1, \ldots, \omega_m)$.

Define the bigraded differential algebra $(E, d)$ by
\[
E = T(\omega_1, \ldots, \omega_m, y_1, \ldots, y_m)/ (\omega_i^2 = 0, \ \omega_i y_j = -\omega_j y_i, \ y_i y_j = y_j y_i, \ \omega_i y_j = y_j \omega_i)
\]
such that
\[
\text{bideg } \omega_i = (-1, 1), \quad \text{bideg } y_i = (0, 1),
\]
d$\omega_i = y_i$, \quad dy_i = 0
and requiring the differential $d$ to satisfy the identity
\[
d(a \cdot b) = d(a) \cdot b + (-1)^{\deg_2(a)} a \cdot d(b)
\]
where $\text{bideg}(a) = (\deg_1(a), \deg_2(a))$.

**Lemma 2.1.** The differential graded algebra $(E, d)$ with the differential given by $d(\omega_i) = y_i$ and $d(y_i) = 0$ is a free $k(y_1, \ldots, y_m)$-resolution of $k$.

**Proof.** Consider $k$ with the $k(y_1, \ldots, y_m)$-module structure given by the augmentation map sending each $y_i$ to zero. Rewrite $E$ as
\[
E = L(\omega_1, \ldots, \omega_m) \otimes_k k(y_1, \ldots, y_m)
\]
emphasising that $\omega_i y_j = y_j \omega_i$. Then $(E, d)$ together with the augmentation map $\varepsilon: E \longrightarrow k$ defines a cochain complex of $k(y_1, \ldots, y_m)$-modules
\[
0 \longrightarrow L^m(\omega_1, \ldots, \omega_m) \otimes k(y_1, \ldots, y_m) \xrightarrow{d} \cdots \xrightarrow{d} L^i(\omega_1, \ldots, \omega_m) \otimes k(y_1, \ldots, y_m) \xrightarrow{d} k(y_1, \ldots, y_m) \xrightarrow{\varepsilon} k \longrightarrow 0
\]
where $L^i(\omega_1, \ldots, \omega_m)$ is the submodule of $L(\omega_1, \ldots, \omega_m)$ generated by monomials of length $i$. We shall show that $\varepsilon: (E, d) \longrightarrow (k, 0)$ is a quasi-isomorphism. There is an obvious inclusion $\eta: k \longrightarrow E$ such that $\varepsilon \eta = \text{id}$. To finish the proof we construct a cochain homotopy between id and $\eta \varepsilon$, that is, a set of $k$-linear maps $s = \{s^{-i,j}: E^{-i,j} \longrightarrow E^{-i-1,j}\}$ satisfying the identity
\[
ds + sd = \text{id} - \eta \varepsilon.
\]
For $m = 1$, we define the map $s_1: E_1^{0,*} = k(y) \longrightarrow E_1^{-1,*}$ by
\[
s_1(a_0 + a_1 y + \cdots + a_j y^j) = \omega(a_1 + a_2 y + \cdots + a_j y^{j-1})\]
Then for $f = a_0 + a_1 y + \cdots + a_j y^j \in E_1^{0,*}$ we have $ds_1 f = f - a_0 = f - \eta \varepsilon f$ and $s_1 df = 0$. On the other hand, for $\omega f \in E_1^{-1,*}$ we have $s_1 d(\omega f) = \omega f$ and $ds_1 (\omega f) = 0$. In any case $s_1$ holds. Now we assume by induction that for $m = k - 1$ the required cochain homotopy $s_{k-1}: E_{k-1} \longrightarrow E_{k-1}$ is already constructed. Since $E_k = E_{k-1} \otimes E_1$, $\varepsilon_k = \varepsilon_{k-1} \otimes \varepsilon_1$ and $\eta_k = \eta_{k-1} \otimes \eta_1$, a direct calculation shows that the map
\[
s_k = s_{k-1} \otimes \text{id} + \eta_{k-1} \varepsilon_{k-1} \otimes s_1
\]
is a cochain homotopy between id and $\eta_k \varepsilon_k$. 
Since $L^i(\omega_1, \ldots, \omega_m) \otimes k\langle y_1, \ldots, y_m \rangle$ is a free $k\langle y_1, \ldots, y_m \rangle$-module, \([\bullet]\) is a free resolution for the $k\langle y_1, \ldots, y_m \rangle$-module $k$. This is an analog of the Koszul resolution.

Since $L(\omega_1, \ldots, \omega_m) \otimes k\langle y_1, \ldots, y_m \rangle$ is a resolution of $k$ by free $k\langle y_1, \ldots, y_m \rangle$-modules, it follows that the cohomology of the complex

$$\left(L(\omega_1, \ldots, \omega_m) \otimes_k k\langle y_1, \ldots, y_m \rangle\right) \otimes_{k\langle y_1, \ldots, y_m \rangle} SR(K) = L(\omega_1, \ldots, \omega_m) \otimes_k SR(K)$$

is isomorphic as a $k$-module to $\text{Tor}_{k\langle y_1, \ldots, y_m \rangle}(SR(K), k)$.

We aim to show that there is an additive isomorphism between the cohomology $H^*((D^0, S^1)^K; k)$ and the $k$-module $\text{Tor}_{k\langle y_1, \ldots, y_m \rangle}(SR(K), k)$. Following Buchstaber-Panov’s work \([4]\), the idea is to first reduce the differential graded algebra $(E, d)$ to a finite dimensional quotient $\bar{R}(K)$ without changing the cohomology. We then proceed by showing that as a $k$-module $\bar{R}(K)$ is quasi-isomorphic to the underlying $k$-module of a certain differential graded algebra $B(K)$ which in turn is quasi-isomorphic to the singular cochains of $(D^1, S^0)^K$.

**Definition 2.2.** Let $K$ be a simplicial complex on $[m]$. Define the differential graded algebra $(\bar{R}(K), d)$ by

$$\bar{R}(K) = L(\omega_1, \ldots, \omega_m) \otimes SR(K)/(y_i^2 = \omega_i y_i = 0)$$

with differential $d$ induced from $(E, d)$, that is, $d(w_i) = y_i$ and $d(y_i) = 0$.

**Proposition 2.3.** The quotient map

$$L(\omega_1, \ldots, \omega_m) \otimes SR(K) \longrightarrow \bar{R}(K)$$

is an algebra quasi-isomorphism.

**Proof.** There is a short exact sequence of differential graded algebras

$$0 \longrightarrow \mathcal{I} \longrightarrow \Lambda(\omega_1, \ldots, \omega_m) \otimes SR(K) \longrightarrow \Lambda(\omega_1, \ldots, \omega_m) \otimes SR(K)/\mathcal{I} \longrightarrow 0$$

where $\mathcal{I}$ is the ideal $(y_i^2 = \omega_i y_i = 0)$. Specifically, $\mathcal{I}$ is generated by monomials which are divisible by $y_i^2$ or $\omega_i y_i$ for some $i$.

We show that $H^*\mathcal{I} = 0$. For a monomial $x \in \mathcal{I}$, there is a minimal index $i(x)$ such that either $y_i^2$ or $\omega_i y_i$ divides $x$. Define $s: \mathcal{I} \longrightarrow \mathcal{I}$ on generating monomials by

$$s(x) = \omega_i(x)\frac{x}{y_i(x)}.$$

By showing that $s$ is a chain homotopy between the identity and zero, that is, $ds(x) - sd(x) = x$ the proposition statement follows.

Since $x = \frac{y_i(x)}{y_i(x)}y_i(x)$, we have

$$ds(x) - sd(x) = d(\omega_i(x)\frac{x}{y_i(x)}) - s(d(\frac{x}{y_i(x)}))y_i(x) = x + \omega_i(x)d(\frac{x}{y_i(x)}) - s(d(\frac{x}{y_i(x)}))y_i(x).$$

We first observe that $y_i^2(x)$ divides $d(x)/y_i(x)$. To see this, note that either $y_i(x)$ or $\omega_i(x)$ divide $\frac{x}{y_i(x)}$. In either case $y_i(x)$ divides $d(x)/y_i(x)$.

We claim that for $j < i(x)$, neither $y_j^2$ nor $\omega_j y_j$ divide $d(x)/y_i(x)$.
Assuming the claim for the moment, we first finish the proof. Since \( y_j^2(x) \) divides \( \frac{x}{y_i(x)} y_i(x) \) and for smaller \( j \), both \( y_j^2 \) and \( \omega_j \) do not divide \( \frac{x}{y_i(x)} y_i(x) \), it follows that \( s ( \frac{x}{y_i(x)} y_i(x) ) = \omega_i(x) d ( \frac{x}{y_i(x)} ) \).

We next prove the claim. When \( j < i \), for \( y_j^2 \) or \( \omega_j \) to divide \( \frac{x}{y_i(x)} \) either \( y_j \) or \( \omega_j \) divides \( \frac{x}{y_i(x)} \). If \( y_j \) divides \( \frac{x}{y_i(x)} \), then \( y_j^2 \) cannot divide \( x \) because \( j < i \). So if either \( y_j^2 \) or \( \omega_j \) divides \( \frac{x}{y_i(x)} \), we must have \( \omega_j \) divides \( \frac{x}{y_i(x)} \). But then neither \( \omega_j \) nor \( y_j \) can divide \( \frac{x}{y_i(x)} / \omega_j \). Thus

\[
d \left( \frac{x}{y_i(x)} \right) = d \omega_j \left[ \left( \frac{x}{y_i(x)} / \omega_j \right) \right] = y_j \left[ \left( \frac{x}{y_i(x)} / \omega_j \right) \right] + \omega_j d \left[ \left( \frac{x}{y_i(x)} / \omega_j \right) \right].
\]

Neither summand is divisible by \( y_j^2 \) or \( \omega_j y_j \). \( \square \)

**Corollary 2.4.** As \( k \)-modules,

\[
\text{Tor}_k(y_1, \ldots, y_m)(SR(K), k) \cong H^*(\bar{R}(K)).
\]

Recalling that there is the bigrading of \( \omega_i \) and \( y_i \) given by \( \text{bideg}(\omega_i) = (-1, 1) \) and \( \text{bideg}(y_i) = (0, 1) \), we can consider \( \bar{R}(K) \) as a bigraded differential algebra. Recall that for \( I = (i_1, \ldots, i_p), i_1 < \ldots < i_p, L = (y_{i_1}, \ldots, y_{i_p}), l_1 < \ldots < l_s, \) the monomial \( \omega_{i_1} \cdots \omega_{i_p} y_{i_1} \cdots y_{i_p} \), denoted by \( \omega_I y_L \).

For \( 0 \leq p \leq m \), a \( k \)-module basis for \( \bar{R}^{-p, *}(K) \) can be given as

\[
\{ \omega_I y_L | L \in K, |L| = p, I \cap L = \emptyset \}.
\]

To consider \( \bar{R}(K) \) as a differential \( k \)-module, the differential \( d: s^*(\bar{R}(K)) \to \bar{R}^{-*, *}(K) \), induced from its differential algebraic structure, is given as the \( k \)-module generators by

\[
d(\omega_I y_L) = \sum_{k=1}^{p} (-1)^{k+1} \omega_{i_1} \cdots \omega_{i_{k-1}} \omega_{i_k+1} \cdots \omega_{i_p} y_{i_1} \cdots y_{i_k} y_{i_{k+1}} \cdots y_L
\]

where \( l_r < i_k < l_{r+1} \). Notice that the sign is induced by the derivation identity (3) and if the set \( \{ l_1, \ldots, l_r, i_k, l_{r+1}, \ldots, l_s \} \notin K \), then \( \omega_{i_1} \cdots \omega_{i_{k-1}} \hat{\omega}_{i_k} \omega_{i_{k+1}} \cdots \omega_{i_p} y_{i_1} \cdots y_{i_k} y_{i_{k+1}} \cdots y_L = 0 \).

Next we relate the \( k \)-module \( \bar{R}(K) \) to the \( k \)-cohomology module of the real moment-angle complex \( (D^1, S^0)^K \) via the differential graded algebra model \( B(K) \) of \( H^*((D^1, S^0)^K; k) \) given by Cai [5] and Franz [3].

The differential graded algebra \( B(K) \) is presented with generators \( s_i \) and \( t_i \) such that \( \deg(s_i) = 0 \) and \( \deg(t_i) = 1 \) and the relations

\[
s_i s_i = s_i, \quad t_i s_i = t_i, \quad s_i t_i = 0, \quad t_i t_i = 0, \quad \prod_{j \in \sigma, \sigma \notin K} t_j = 0
\]

for all \( 1 \leq i \leq m \) and all variables with different indices are graded commutative, that is,

\[
s_i s_j = s_j s_i, \quad t_i t_j = -t_j t_i, \quad s_i t_j = t_j s_i \quad \text{for} \ i \neq j.
\]

The differential \( d: B^*(K) \to B^*(K) \) is given by

\[
ds_i = -t_i, \quad dt_i = 0
\]

and extended using the Leibniz rule \( d(a \cdot b) = d(a) \cdot b + (-1)^{\deg(a)} a \cdot d(b) \). Franz [6] proved that \( B(K) \) is quasi-isomorphic to \( C^*((D^1, S^0)^K, k) \), the singular \( k \)-cochains of the real moment-angle complex \( (D^1, S^0)^K \).
We treat $B(K)$ as a bigraded differential algebra by setting that $\text{bideg}(s_i) = (-1, 1)$ and $\text{bideg}(t_i) = (0, 1)$ for all $1 \leq i \leq m$. Then the $k$-module basis for $B^{-p,*}(K), 0 \leq p \leq m$ is similar to the one for $\bar{R}^{-p,*}(K)$ and is given by

$$\{ s_I t_L | L \in K, I \cap L = \emptyset \}$$

where $s_I = s_{i_1} \cdots s_{i_p}$ for $I = (i_1, \ldots, i_p)$ and $t_L = t_{l_1} \cdots t_{l_r}$ for $L = (l_1, \ldots, l_s)$. The $k$-module differential $d: B^{*,*}(K) \to B^{*,*}(K)$ is given by

$$d(s_I t_L) = \sum_{k=1}^{p} (-1)^{r+1} s_{i_1} \cdots s_{i_{k-1}} s_{i_k+1} \cdots s_{i_p} t_{l_1} \cdots t_{l_r} t_{l_{k+1}} \cdots t_{l_s}$$

where $l_r < i_k < l_{r+1}$. Notice that in $B(K)$ the summand $s_{i_1} \cdots s_{i_{k-1}} s_{i_k+1} \cdots s_{i_p} t_{l_1} \cdots t_{l_r} t_{l_{k+1}} t_{l_{k+1}} \cdots t_{l_s} = 0$ if $(l_1, \ldots, l_r, i_k, l_{r+1}, \ldots, l_s) \notin K$. The graded-commutative Leibniz contributes $-1$ as $\text{deg}(s_i) = 0$ and $d(s_{i_k}) = -t_{i_k}$. The additional $(-1)^r$ in the differential formula comes about from $t_{i_k}$ passing $r$ many $t_j$’s.

**Definition 2.5.** Let an additive isomorphism $f: \bar{R}^{-p,*}(K) \to B^{-p,*}(K)$ be defined by

$$f(\omega_I y_L) = \epsilon(I, L)s_I t_L$$

where $\epsilon(I, L)$ is the sign of the permutation that converts $IL$, the concatenation of $I$ followed by $L$, into an increasing sequence.

**Proposition 2.6.** The $k$-isomorphism $f$ commutes up to sign with the differentials. Specifically, if $\alpha \in \bar{R}^{-p,*}(K)$ then

$$fd_R(\alpha) = (-1)^p dB f(\alpha).$$

**Proof.** For $\alpha = \omega_I y_L$,

$$d(\alpha) - \sum (-1)^{k+1} \omega_{i_1} \cdots \omega_{i_{k-1}} \omega_{i_k+1} \cdots \omega_{i_p} y_{l_1} \cdots y_{l_r} y_{l_{k+1}} \cdots y_{l_s}.$$ 

We compute the sign of the permutation that converts

$$i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p, l_1, \ldots, l_r, i_k, l_{r+1}, \ldots, l_s$$

into an increasing sequence.

To start, note that the sequence of $i$’s and the sequence of $l$’s are increasing. Therefore it suffices to permute the $l_j$’s to the left starting with $l_1$ followed by increasing $l_j$’s.

The sign of the permutation that converts (7) into an increasing sequences differs from $\epsilon(I, L)$ in two ways.

First, we move the indices $l_1, \ldots, l_r$ to the left. Note that $l_j < i_k$ for $j \leq r$. These $l_j$’s are meshed into $i_1, \ldots, i_{k-1}$. The number of permutations needed to place $l_j$ into its final position differs by $-1$ from the number needed to order $IL$ because $i_k$ is missing. So moving $l_1, \ldots, l_r$ differs from $\epsilon(I, L)$ by $(-1)^r$.

Second, we have to move $i_k$. Moving this index contributes $(-1)^{p-k}$.

Thus the sign of the permutation that converts (7) into an increasing sequence differs from $\epsilon(I, L)$ by $(-1)^{r+k+p}$. Therefore the map $f$ multiplies the $k$th term in $d_R(\alpha)$ by $(-1)^{r+k+p}\epsilon(I, L)$ which is $(-1)^p \epsilon(I, L)$ times the coefficient of the $k$-th term in $dB$. 

$\square$
Corollary 2.7. The chain map $f$ induces a $\mathbb{k}$-module isomorphism
\[
\text{Tor}_{\mathbb{k}}(\pi_1, \ldots, \pi_n)(SR(K), \mathbb{k}) \to H^*(B(K)) \cong H^*((D^1, S^0)^K; \mathbb{k}).
\]
\[
\square
\]

2.2. A generalisation to polyhedral products. In many ways, a combinatorial contribution of the simplicial complex $K$ to homotopy theoretical properties of the real and complex moment-angle complexes is comparable and usually suggests a formulation of those properties for polyhedral products $(CX, X)^K$, see for example [33]. Having that both cohomology $\mathbb{k}$-modules $H^*((D^2, S^1); \mathbb{k})$ and $H^*((D^1, S^0); \mathbb{k})$ can be expressed in terms of appropriate Tor $\mathbb{k}$-modules, we generalise the previous discussion to give an additive description of $H^*((CX, X)^K; \mathbb{k})$ as a direct summand of $\text{Tor}_A(M, \mathbb{k})$, where $A$ and $M$ are defined below.

Throughout the remainder of the paper, when the cohomology $H^*((CX, X)^K; \mathbb{k})$ of the polyhedral product $(CX, X)^K$ is considered we assume that $H^*(X_i; \mathbb{k})$ are free $\mathbb{k}$-modules for all $i$. If not explicitly stated all cohomology groups of topological spaces are taken with $\mathbb{k}$ coefficients.

Bahri, Bendersky, Cohen and Gitler [1] gave a stable homotopy decomposition of $(CX, X)^K$ by establishing the homotopy equivalence
\[
\Sigma(CX, X)^K \to \Sigma \bigvee_{J \subseteq [m]} |\Sigma K_J| \wedge Y^J
\]
where
\[
Y^J = \bigwedge_{i \in [m]} Y_i \text{ with } Y_i = X_i \text{ if } i \in J \text{ and } Y_i = S^0 \text{ if } i \notin J.
\]
They further showed [2] that the map
\[
H^*((CX, X)^K; \mathbb{k}) \to \bigoplus_{J \subseteq [m]} H^*(|\Sigma K_J|; \mathbb{k}) \otimes H^J
\]
where $H^J = \bigotimes_{i \in [m]} H_i$ with $H_i = \tilde{H}^*(X_i; \mathbb{k})$ if $i \in J$ and $H_i = \mathbb{k}$ if $i \notin J$ is an isomorphism of rings. This specialises to an isomorphism
\[
H^*((D^1, S^0)^K; \mathbb{k}) \cong \bigoplus_{J \subseteq [m]} H^*(|\Sigma K_J|; \mathbb{k}).
\]
It is proven in [2] that the cup product on $H^*((D^1, S^0)^K; \mathbb{k})$ restricts to a pairing
\[
H^*(|\Sigma K_J|; \mathbb{k}) \otimes H^*(|\Sigma K_L|; \mathbb{k}) \to H^*(|\Sigma K_{J \cup L}|; \mathbb{k}).
\]
This in turn induces a product on
\[
\bigoplus_{J \subseteq [m]} H^*(|\Sigma K_J|; \mathbb{k}) \otimes H^J
\]
which is called the $*$-product in [2].

To give a $\mathbb{k}$-module model for $H^*((CX, X)^K; \mathbb{k})$, we shall only use the underlying additive isomorphism in (9). In Section 2.3 we shall invoke the multiplicative structure.

By Corollaries [2,4] and [2,7] the cohomology $\mathbb{k}$-module structure of $(D^1, S^0)^K$ is also modeled by the differential graded algebra $R(K)$. We next identify the summands in (10) with sub dga’s of $R(K)$.
Definition 2.8. For an element $\omega_I y_L \in \check{R}(K)$, define its support as

$$\text{supp}(\omega_I y_L) = I \cup L.$$ 

The differential sub-module of $\check{R}(K)$ generated by monomials with support $J$ is denoted by $\check{R}_J(K)$.

The following lemma is a direct consequence of [6] and [4, Lemma 4.5.1].

Lemma 2.9. There is an algebra isomorphism

$$H^*(\check{R}_J(K); \mathbb{k}) \cong H^*(\Sigma|K_J|); \mathbb{k}).$$

□

Definition 2.10. Let $(C(X, K), d)$ be a differential graded $\mathbb{k}$-module defined by

$$C(X, K) = \bigoplus_{J \subseteq [m]} \check{R}_J(K) \otimes H^J$$

where $H^J = \bigotimes_{i \in [m]} H_i$ and $H_i = \check{H}^*(X_i; \mathbb{k})$ if $i \in J$ and $H_i = \mathbb{k}$ if $i \notin J$.

The differential $d: C^*(X, K) \to C^*(X, K)$ is given as the tensor product of the differential on $\check{R}_J(K)$, induced from the dga $\check{R}(K)$, and the trivial differential on $H^J$, induced by the trivial differential on $H^*(X_i)$.

Combining Lemma 2.9 with (9), the following statement holds.

Lemma 2.11. There is a $\mathbb{k}$-module isomorphism

$$H^*((C(X, X^K); \mathbb{k}) \to H^*(C(X, K)).$$

□

To integrate the topological structure of the polyhedral product $(C X, X^K$ into our cohomology model, we start by generalising the polynomial algebra $\mathbb{k}[y_1, \ldots, y_m]$ to a free commutative algebra on the cohomology of $\Sigma X$.

Definition 2.12. For $CW$-complexes $X_i, 1 \leq i \leq m$, let

$$\mathbb{k}(\bigoplus_{1 \leq i \leq m} H^*(\Sigma X_i)) = T(\bigoplus_{1 \leq i \leq m} H^*(\Sigma X_i))/\langle \alpha \beta = \beta \alpha | \alpha, \beta \in \bigoplus_{1 \leq i \leq m} \check{H}^*(\Sigma X_i) \rangle$$

where $T(M)$ denotes the free associative algebra generated by a free $\mathbb{k}$-module $M$.

To algebra (12) we associate a generalised Stanley-Reisner ring.

Definition 2.13. For $CW$-complexes $X_i, 1 \leq i \leq m$, define the generalised Stanley-Reisner ring $SR(X, K)$ as

$$SR(X, K) = \mathbb{k}(\bigoplus_{1 \leq i \leq m} \check{H}^*(\Sigma X_i))/\mathcal{I}_K$$

where $\mathcal{I}_K$ is the ideal generated by square free monomial

$$\alpha_{i_1} \cdots \alpha_{i_t}, \text{ where } \alpha_{i_j} \in \check{H}^*(X_{i_j}) \text{ and } \{i_1, \cdots, i_t\} \notin K.$$
We prove that $H^*(C(X, K))$ additively splits off $\text{Tor}_{k(\bigoplus \tilde{H}^*(X_i))} (SR(X, K), k)$. To this end we choose an ordered bases $B_i$ for $\tilde{H}^*(X_i)$,

$$B_i = \{b_{i,1}, \ldots, b_{i,k_i}\}$$

which induces an ordering on $\bigoplus \tilde{H}^*(X_i)$ by saying that $b < b'$ if $b \in \tilde{H}^*(X_i), b' \in \tilde{H}^*(X_j)$ and $i < j$.

To make $\bigoplus \tilde{H}^*(X_i)$ into a bigraded object, let $\text{bideg}(b) = (0, |b|)$.

We define $(L(\tilde{H}^*(X)), d)$ and $(E(\tilde{H}^*(X)), d)$, the natural generalisation of (1) and (2), as

$$L(\tilde{H}^*(X)) = L(\bigoplus_i u B_i) = T(\bigoplus_i u B_i)/(ub_i^2 = 0, \quad ub_iub_j = -ub_jub_i)$$

$$E((\tilde{H}^*(X)) = L(\tilde{H}^*(X)) \otimes k( \bigoplus_{1 \leq i \leq m} \tilde{H}^*(X_i))$$

where $uB_i$ is generated by classes $ub$ of bidegree $(-1, |b|)$ corresponding to $b \in B_i$. Define the differential $d$ on $E(\tilde{H}^*(X))$ by $d(u b) = b$, $d(b) = 0$ and by requiring that $d$ satisfies the Leibniz identity $d(a \cdot b) = d(a) \cdot b + (-1)^{\text{deg}_E(a)} a \cdot d(b)$, where $\text{bideg}(a) = (\text{deg}_E(a), \text{deg}_{SR}(a))$.

As in Lemma 2.1, $E(\tilde{H}^*(X))$ is a resolution of $k$. Consequently the cohomology of the Koszul complex

$$L(\bigoplus_i u B_i) \otimes SR(X, K)$$

is $\text{Tor}_{k(\bigoplus \tilde{H}^*(X_i))} (SR(X, K), k)$.

**Definition 2.14.** Let $K$ be a simplicial complex on $[m]$ and let $X = \{X_i\}_{i=1}^m$ be CW-complexes. Define the differential graded algebra $(R(X, K), d)$ by

$$R(X, K) = L(\bigoplus_i u B_i) \otimes SR(X, K)/(b_i^2 = (ub_i)b_i = 0)$$

with differential $d$ induced from the differential graded algebra $(E(\tilde{H}^*(X)), d)$.

Following the lines of the proof of Proposition 2.3, we established that the finite differential graded algebra $R(X, K)$ is a model for the $\text{Tor}$ $k$-module $\text{Tor}_{k(\bigoplus \tilde{H}^*(X_i))} (SR(X, K), k)$.

**Lemma 2.15.** The quotient map

$$L(\bigoplus_i u B_i) \otimes SR(X, K) \rightarrow R(X, K)$$

is an algebra quasi-isomorphism.

To describe an additive basis of $R(X, K)$ we note that $\bigoplus uB_i$ inherits an ordering from the ordering of $\bigoplus B_i$. A basis for $R(X, K)$ is given by

$$(13) \quad \{ub_{i_1,k_1} \cdots ub_{i_s,k_s}b_{i_1,j_1} \cdots b_{i_t,j_t}\}$$

where

(i) if $ub$ is a factor, then $b$ is not a factor;
(ii) $ub_{i_1,k_1} < \cdots < ub_{i_s,k_s}$, $b_{i_1,j_1} < \cdots < b_{i_t,j_t}$;
(iii) $\{l_1, \cdots, l_t\} \in K$. 


There is a similar basis for $C(X, K)$,

$$\{ \omega_{i_1} \cdots \omega_{i_s} y_{i_1} \cdots y_{i_t} \otimes \left[ s^{-1} b_{i_1, k_1} \cdots s^{-1} b_{i_s, k_s} \right] \otimes \left[ s^{-1} b_{i_1, j_1} \cdots s^{-1} b_{i_t, j_t} \right] \}.$$  

The difference between the basis [13] and basis [14] is that the integers $\{i_1, \cdots, i_s, l_1, \cdots, l_t\}$ are all distinct in [14].

We now compare the differential graded algebras $R(X, K)$ and $C(X, K)$.

**Proposition 2.16.** The cohomology $H^*((C^k, X)^K; k)$, seen as a $k$-module, is a direct summand of the $k$-module $\mathrm{Tor}^{(\bigoplus \tilde{H}^*(\Sigma X_i))}_{\bigoplus}(SR(X, K), k)$.

**Proof.** There are maps of differential $k$-modules:

$$h: C(X, K) \rightarrow R(X, K)$$

and

$$g: R(X, K) \rightarrow C(X, K)$$

given by

$$h(\omega_{i_1} \cdots \omega_{i_s} y_{i_1} \cdots y_{i_t} \otimes \left[ s^{-1} b_{i_1, k_1} \cdots s^{-1} b_{i_s, k_s} \right] \otimes \left[ s^{-1} b_{i_1, j_1} \cdots s^{-1} b_{i_t, j_t} \right]) = (u b_{i_1, k_1} \cdots u b_{i_s, k_s})(b_{i_1, j_1} \cdots b_{i_t, j_t})$$

and

$$g((u b_{i_1, k_1} \cdots u b_{i_s, k_s})(b_{i_1, j_1} \cdots b_{i_t, j_t})) = \omega_{i_1} \cdots \omega_{i_s} y_{i_1} \cdots y_{i_t} \otimes \left[ s^{-1} b_{i_1, k_1} \cdots s^{-1} b_{i_s, k_s} \right] \otimes \left[ s^{-1} b_{i_1, j_1} \cdots s^{-1} b_{i_t, j_t} \right].$$

The homomorphisms $h$ and $g$ commute with differentials and $g \circ h = id$. \hfill $\square$

We note that $H^*((C^k, X)^K; k) \cong \mathrm{Tor}^{(\bigoplus \tilde{H}^*(\Sigma X_i))}_{\bigoplus}(SR(X, K), k)$ if $X_i = S^n$ for each $i$. For $\tilde{H}^*(X_i)$ with more than one generator, the map $g$ is zero on generators, $(u b_{i_1, k_1} \cdots u b_{i_s, k_s})(b_{i_1, j_1} \cdots b_{i_t, j_t})$ whenever a repetition occurs in the list

$$i_1, \ldots, i_s, l_1, \ldots, l_t$$

obtained by dropping the second subscripts in the bi-indexing.

### 3. Algebra Models

In this section we coalesce the Bahri-Bendersky-Cohen-Gitler $s$-product [11] on $H^*((C^k, X)^K)$ with the Cai [5] and Franz [6] differential algebra $B(K)$ to give a natural differential algebra $B(X, K)$ whose cohomology is isomorphic the cohomology of $(C^k, X)^K$.

**Definition 3.1.** Let $K$ be a simplicial complex on $[m]$ and let $X = \{X_i\}_{i=1}^m$ be CW-complexes. Define a differential bigraded non-commutative algebra $(B(X, K), d)$ as

$$B(X, K) = \bigoplus_{J \subset [m]} B_J(K) \otimes H^J$$

where $B_J(K)$ is a subalgebra of $B(K)$ consisting of elements with support $J$ and $H^J = \bigotimes_{i \in [m]} H_i$ with $H_i = \tilde{H}^*(X_i; k)$ if $i \in J$ and $H_i = k$ if $i \notin J$.

The differential $d$ on $B(X, K)$ is given as the tensor product of the differential on $B_J(K)$, induced from the dga $B(K)$, and the trivial differential on $H^J$, induced by the trivial differential on $H^*(X_i)$.
We now recognise $B(X, K)$ as a dga model for the cohomology of $(CX, X)^K$.

**Proposition 3.2.** Let $K$ be a simplicial complex on $[m]$ and let $X = \{X_i\}_{i=1}^m$ be CW-complexes with $H^*(X_i)$ being free $k$-modules. Then the dga $B(X, K)$ is quasi-isomorphic to $H^*((CX, X)^K; k)$.

**Proof.** The statement follows as a direct consequence of the Bhari-Bendersky-Cohen-Gitler description of the cup product structure on $H^*((CX, X)^K; k)$ given by the $s$–product [11] the Künneth theorem and the result of Franz [6] stating that the dga $B(K)$ is quasi-isomorphic to $C^*((D^1, S^0)^K)$. 

We next enhance the construction to give a dga which is quasi-isomorphic to $C^*((CX, X)^K, k)$, the singular cochains of $(CX, X)^K$.

**Definition 3.3.** Let $K$ be a simplicial complex on vertex set $[m]$ and let $X = \{X_i\}_{i=1}^m$ be CW-complexes. Define a differential bigraded non-commutative algebra $(B(C^*(X), K), d)$ as

$$B(C^*(X), K) = \bigoplus_{J \subseteq [m]} B_J(K) \otimes C^J$$

where $B_J(K)$ is a subalgebra of $B(K)$ consisting of elements with support $J$ and $C^J = \bigotimes_{i \in [m]} C_i$ with $C_i = C^*(X_i, k)$ if $i \in J$ and $C_i = k$ if $i \notin J$.

The differential $d$ on $B(C^*(X), K)$ is given as the tensor product of the differential on $B_J(K)$, induced from the dga $B(K)$, and the differential on $C^J$, induced by the differential on $C^*(X_i, k)$.

**Proposition 3.4.** Let $K$ be a simplicial complex on $[m]$ and let $X = \{X_i\}_{i=1}^m$ be CW-complexes with $H^*(X_i)$ being free $k$-modules. Then the dga $C^*((CX, X)^K)$ is quasi-isomorphic to the dga $B(C^*(X), K)$.

**Proof.** The statement is a straightforward consequence of Proposition 3.2 and the Künneth formula. 

The bar construction applied to $B(C^*(X), K)$ gives a model for the loops space of $(CX, X)^K$ when spaces $X_i$ have torsion free $k$-cohomology. We will return to this point in a future paper.

We finish the paper by comparing the dga $C(X, K)$ with the dga $B(X, K)$. Restricting to the $k$-module structures, we observe that $B(X, K)$ is a differential bigraded $k$-module model for $H^*((CX, X)^K; k)$.

By extending the map $f: \bar{H}(K) \to B(K)$ in Definition 2.5, we define an isomorphism of differential $k$-modules $f_X: C(X, K) \to B(X, K)$ by

$$f_X(\omega_1 y L \otimes h) = \epsilon(I, L)s_I t_L \otimes h$$

where $\epsilon(I, L)$ is the sign of the permutation that converts $IL$, the concatenation of $I$ followed by $L$, into an increasing sequence.

Straightforwardly, following the proof of Proposition 2.6, we have the following statement.

**Lemma 3.5.** The additive isomorphism $f_X$ commutes up to sign with the differentials. Specifically,

$$f_X d_{C(X, K)}(\omega_1 y L \otimes h) = (-1)^{|I|} d_{B(X, K)} f_X(\omega_1 y L \otimes h).$$
There is a natural algebra structure on $C(X, K) = \bigoplus_{J \subset [m]} \bar{R}_J(K) \otimes H^J$ induced by the algebra structures on $\bar{R}(K)$ and $H^*(X_i)$ for all $i$. Notice that $\bar{R}(K)$ is a commutative algebra while $B(K)$ is not commutative. Therefore, although $B(X, K)$ and $C(X, K)$ are isomorphic as $k$-modules, the isomorphism cannot be extended to the one of algebras. However, in the case when all $X_i$'s are suspension spaces the algebra structure of $B(X, K)$ reduces to the one of $C(X, K)$.

**Proposition 3.6.** The algebra structures on $B(X, K)$ and $C(X, K)$ coincide up to sign on the product of classes with disjoint support. In particular, the algebra structures are isomorphic up to sign if $X_i$ is a suspension space for all $i$.

**Proof.** It is enough to see that dgas $\bar{R}(K)$ and $B(K)$ differ only on elements with repeated indices. Notice that in Definition 2.2 of $\bar{R}(K)$, we quotient out $y_i^2$ and $\omega_i y_i$. Therefore elements in which indices are repeated are trivial. On the other hand, the defining relations of $B(K)$ state that the products with repeated indices, such as $s_i s_i = s_i$ and $t_i s_i = t_i$, are not trivial. Moreover, together with $s_i t_i = 0$, these relations imply that $B(K)$ is a non-commutative dga.

If we assume that $X_i$'s are suspension spaces, then since the diagonal map on suspension spaces is null-homotopic, the cup product on $H^J$ is trivial if indices are repeated. This trivialises products with repeated indices in $B(X, K)$ as well despite the product being non-trivial in $B(K)$. \qed

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