Spiral magnetic field and vortex clustering in noncentrosymmetric superconductors

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We study magnetic response and vortex states in noncentrosymmetric superconductors with \( O \) or \( T \) symmetry. We microscopically derive Ginzburg-Landau free energy, which exhibits a substantial temperature dependence. For some materials the later leads to a crossover from type-1 superconductivity at elevated temperature to vortex states at lower temperature. Next we show that magnetic field can be solved in terms of complex force free fields. Using that we uncover that magnetic field of a vortex decays in spirals. Due to that intervortex and vortex-boundary interaction becomes non-monotonic with multiple minima. This implies that vortices form bound states with other vortices, antivortices and boundary.

I. INTRODUCTION

Lack of inversion symmetry in a crystal can have very profound impact on its superconducting properties. The free-energy functionals describing these, so-called noncentrosymmetric, systems can have new terms which are not allowed by symmetry in ordinary superconductors [1]. These new terms can include contributions which are linear in the gradients of the superconducting order parameter and the magnetic field \( \vec{B} \). Depending on the symmetry of the material, the free energy can feature scalar and vector products of these fields of the form \( \propto K_{ij} B_i J_j \), where \( i = x, y, z \), \( \vec{J} \propto \text{Re} [\psi^* D \psi] \), \( D \) is the gauge-invariant gradient and \( \psi \) is the order parameter, and \( K_{ij} \) are coefficients, which form depends on crystal symmetry [1]. Correspondingly, while in ordinary superconductors the externally applied field decays monotonically, in a noncentrosymmetric superconductor an externally applied field can have a spiral decay [1–5]. This raises the question of the nature of topological excitations in such materials [1–7]. The main goal of this paper is to investigate vortex physics and magnetic response of a superconductor when there is no inversion symmetry in an underlying crystal lattice.

The structure of this paper is the following: in the Section II we discuss microscopic derivation of the Ginzburg-Landau (GL) model. In the Section III, by rescaling we cast the GL model in a representation which is more convenient for calculations and analysis. In the Section IV we describe a method that solves the hydromagnetostatics of a noncentrosymmetric superconductor in the London limit in terms of complex force free fields. In Section V we define the temperature dependence of the penetration depth to show how a crossover to type-1 superconductivity appears in a class of noncentrosymmetric superconductors at elevated temperatures. In Section VI we obtain analytic and numerical vortex configuration with spiral magnetic field. In Section VII we show that system forms vortex-vortex and vortex-antivortex bound states. In Section VIII we show that vortex forms bound states with boundary.

II. MICROSCOPIC DERIVATION OF THE GINZBURG-LANDAU MODEL

The addition of the noncentrosymmetricity-induced terms \( K_{ij} B_i J_j \), to a standard GL free energy makes, in general this model unbounded from below. That is, the term is not positively-defined and is higher order in fields. While in certain cases one can still perform useful calculations in such minimally extended model, nonetheless there could be also unphysical configurations with infinitely negative energy. Therefore our starting point is a microscopic calculation of a GL model for noncentrosymmetric superconductor in the case of \( O \) or equivalently \( T \) symmetry [1], to obtain a manifestly positively defined free energy functional.

We will focus on the simplest case with the BCS type local interaction given by strength \( V > 0 \), but will include general space dependent magnetic field \( \vec{B} \). We start from Fermi-Hubbard model in path integral formulation, given by the action and partition function:

\[
S = \int_0^T d\tau d\vec{x} \sum_{\alpha,\beta=\downarrow,\uparrow} a_\alpha^\dagger(\vec{h} \cdot \vec{\sigma}) a_\beta - V a_\alpha^\dagger a_\beta^\dagger a_\gamma a_\gamma^\dagger
\]

\[
Z = \int D[a^\dagger, a] e^{-S}
\]

with Grassman fields \( a_\alpha(\tau, \vec{x}), a_\beta^\dagger(\tau, \vec{x}) \) corresponding to fermionic creation and annihilation operators and

\[
\vec{h} \equiv (\partial_\tau + E - \mu, \vec{h}), \quad \vec{\sigma} = \frac{\gamma(\vec{d}) - \gamma_0 \vec{B}(\vec{x})}{E(\vec{x})}
\]

where single electron energy is \( E(-i\nabla - e\vec{A}(\vec{x})) \) with \( E(0) = 0 \), which is \( E(k) = \frac{k^2}{2m} \) for quasi free electrons. However in our derivation we keep \( E(k) \) in general form, also suitable for band electrons. In the case of \( O \) or \( T \) symmetry we take spin-orbit coupling to be \( \gamma(\vec{d}) = \gamma_0 \vec{d} \). We consider the standard situation where the macroscopic length scale \( \lambda \), over which the quantities \( \vec{A}, \vec{B} \) change, is much bigger than Fermi length scale \( \propto 1/k_F \). We assume that the following inequalities hold:

\[
\mu \gg \omega_D \gg T_c, \quad \gamma_0 k_F \gg \omega_D \gg \mu_B B
\]
We perform Hubbard-Stratonovich transformation by introducing auxiliary bosonic field $\Delta (\tau, \vec{x})$. Hence up to a constant quartic interaction term becomes quadratic:

$$e^V \int d\tau d\vec{x} a_1^\dagger a_1 a_1^\dagger a_1 = \int D[\Delta^\dagger, \Delta] e^{-\int d\tau d\vec{x} \left( \frac{1}{2} \delta^\dagger a_1 a_1^\dagger + \Delta^\dagger \Delta + a_1^\dagger a_1 \right)}$$
(4)

Next, by introducing $b \equiv (a_1^\dagger, a_1)^T$ the partition function Eq. (1) can be written as:

$$Z = \int D[\Delta^\dagger, \Delta] D[b] e^{-\int d\tau d\vec{x} \left( \frac{1}{2} b^T H b + \Delta b^\dagger \Delta \right)}$$
(5)

where we have the matrix $H = \Delta + H_0$ with

$$\Lambda = \begin{pmatrix} 0 & -\hat{h}^T \\ \hat{h} & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} \delta^\dagger & 0 \\ 0 & \delta \end{pmatrix},$$
(6)

and by symbol with a hat we denote $2 \times 2$ matrices defined by $\hat{h} = \sigma \cdot h$ and $\hat{\delta} = \sigma \cdot (0, 0, i \Delta, 0)$. Note, that for any function of operators $f, \phi$, transposition is defined as $f^T(\partial_r, \nabla) = f(-\partial_r, -\nabla)$. Integrating out fermionic degrees of freedom $b$ in Eq. (5) we obtain:

$$Z = \int D[\Delta^\dagger, \Delta] e^{\frac{1}{2} \ln \det H - \int d\tau d\vec{x} \Delta^\dagger \Delta}$$
(7)

In the mean field approximation one assumes that $\Delta$ doesn’t depend on $\tau$ (i.e. it’s classical) and doesn’t fluctuate thermally. Hence free energy is given by:

$$F = TS = \int d\vec{x} \left[ \frac{\Delta^2}{V} - \frac{T}{2} \ln \det H \right]$$
(8)

where $\ln \det H = \ln \det (1 + H_0^{-1} \Lambda)$, and $\Lambda$ is a block matrix. To obtain the GL model we need to expand the second term in Eq. (8) in powers and derivatives of the field $\Delta$:

$$\ln \det H = \ln \det (1 + H_0^{-1} \Lambda) = \sum_{\nu=1}^{\infty} \frac{1}{\nu} \ln \det (\Lambda^\dagger \Lambda)^{\nu}$$
(9)

where the first equality is defined, up to constant in $\Delta$ and $\hat{g}$, through:

$$H_0^{-1}(\tau, \tau', \vec{x}, \vec{x'}) = \begin{pmatrix} 0 & \hat{g} \\ -\hat{g}^T & 0 \end{pmatrix} \Rightarrow \hat{h} = \delta(\vec{x} - \vec{x'}) \delta(\tau - \tau')$$
(10)

Next we define

$$\hat{g} = e^{i \phi(\vec{x}, \vec{x'}) \hat{x}}$$
(11)

so that $\hat{h}(-i \nabla - e \hat{A}(\vec{x})) = e^{i \hat{g}(-i \nabla)} \hat{f}$. Since $\hat{A}, \hat{B}$ are slowly changing functions of $\vec{x}$ we approximate $\phi(\vec{x}, \vec{x'}) \simeq i \hat{A}(\vec{x})(\vec{x} - \vec{x'})$. The Fourier transform of $g$ is given by:

$$g(\tau - \tau', \vec{x} - \vec{x'}) = e^{i \hat{A}(\vec{x})(\vec{x} - \vec{x'})} T \sum_{\nu,\gamma} \frac{1}{2(2\pi)^3}$$
(12)

$$\int d\vec{k} e^{-iw_n(\tau - \tau')} e^{i \vec{k} \cdot (\vec{x} - \vec{x'})} f(w_n, \vec{k})$$

where $w_n = 2\pi T(n + \frac{1}{2})$ is Matsubara frequency and $\hat{f}$ is a solution of the equation $\hat{h}\hat{f} = 1$. By using Fourier transformed $h(w_n, \vec{k}) = (-i w_n + E(k) - \mu, \gamma_0 \vec{k} - \mu \vec{B})$ we obtain that $f$ is given by:

$$f = \frac{\hat{h}}{\hat{h} \cdot \hat{h}}$$
(13)

where $\hat{h} = (h_0, -\vec{h})$ if $h = (h_0, \vec{h})$. We can rewrite $f$ as:

$$f = \frac{1}{2} (f_+ + f_-), \quad f_\pm = G_\pm s$$
(14)

where we use the notations $|\hat{h}| \equiv h$ and $\vec{e}_h \equiv \vec{k}/|\vec{k}|$.

\section{Second order terms}

By using Eq. (12) and substitution $\Delta^*(\vec{x'}) = e^{i (\vec{x} - \vec{x'}) \nabla} \Delta^*(\vec{x})$ we compute $\nu = 1$ term in Eq. (9) which is second order in $\Delta$:

$$\text{Tr} \left[ \hat{g} \hat{g}^T \delta \right] = 2\text{Tr} \left[ (\hat{g} \Delta) \cdot (\hat{g}^T \Delta^*) \right] =$$
$$2 \int d\vec{x} \Delta(\vec{x}) \sum_{w_n} \frac{d\vec{k}}{2(2\pi)^3} f(w_n, \vec{k}) \cdot f(-w_n, -\vec{k} + D) \Delta^*(\vec{x})$$
(15)

where $D = -i \nabla - 2 e \hat{A}(\vec{x})$ and it is acting only on $\Delta^*(\vec{x})$. The goal is to simplify $f \cdot f'$ term in Eq. (15), where $' \equiv \nu$ means dependence on $(-w_n, -\vec{k} + D)$. Hence using that $\gamma_0 k_F \gg \mu \vec{B}$ we approximate

$$|\hat{h}| \simeq \gamma_0 k - \vec{e}_k \cdot \mu \vec{B}, \quad |\hat{h}'| \simeq \gamma_0 k + \vec{e}_k \cdot (\mu \vec{B} - \gamma_0 \vec{B})$$
(16)

Then it’s easy to show that up to second order in $\frac{\mu \vec{B}}{\gamma_0 k_F}$ and $s \cdot \vec{g}' \simeq 2$ and $s \cdot \vec{g} \simeq 0$ and hence using Eq. (14) we obtain:

$$f \cdot f' \simeq \frac{1}{2} (G_- G'_- + G_+ G'_+)$$
(17)

Integration over momenta in Eq. (15) is performed in thin shell of width $\omega_D$ near Fermi sphere at $k_F$:

$$\varepsilon_a(k_F) = 0, \quad \text{with } \varepsilon_a \equiv E(k) + a \gamma_0 k - \mu$$
(18)

where $a = \pm 1$ is the band index. Hence we can approximate $E(-\vec{k} + D) \simeq E(k) + E'(k_F) \vec{e}_k \cdot D$. Using also $\mu \gg \omega_D$ and $\gamma_0 k_F \gg \omega_D$ integral in Eq. (15) can be estimated as:

$$\int \frac{d\vec{k}}{2(2\pi)^3} \simeq N_a \int_{-\infty}^{\infty} d\varepsilon_a \int \frac{d\Omega_k}{4\pi}$$

$$N_a = \frac{1}{2\pi^2} k_F^2 v_{aF} \quad v_{aF} = E'(k_F) + a \gamma_0$$
(19)
where $N_a$ is density of states at Fermi level, $v_{aF}$ is Fermi velocity and $d\Omega_k$ is solid angle. Then we perform integration and Matsubara sum in Eq. (15) by using Eq. (17), Eq. (19) and $\omega_D \gg T$:

$$
\sum_{w_n} \int \frac{dk}{(2\pi)^3} f \cdot f' \simeq \sum_{\alpha=\pm 1} N_a \frac{1}{2T} \int d\Omega_k \frac{\omega_D}{4\pi} \left[ \ln \frac{\omega_D}{2\pi T} - \text{Re}'\Psi \left( 1 + \frac{1}{2} + i\epsilon_k - \frac{v_{aF} D - 2a\mu_B B}{4\pi T} \right) \right]
$$

(20)

where $\text{Re}'X \equiv \frac{1}{2}(X + X^\dagger)$ and $\Psi$ is digamma function.

Next we expand in $D, \vec{B}$ and average over $\epsilon_k$ in Eq. (20). Combining result with Eq. (15), Eq. (9), Eq. (8) and integrating by parts we obtain the part of the free energy which is second order in $\Delta$:

$$
F_2 = \int d\vec{x} \left[ \alpha |\Delta|^2 + \sum_{\alpha=\pm 1} K_a \left( v_{aF} D^* - 2\nu a \vec{B} \right) |\Delta|^2 \right]
$$

$$
\alpha = N \ln \frac{T}{T_c}, \quad T_c = \frac{\omega_D \hbar}{\pi \omega_D e^{-\pi T}}
$$

$$
K_a = \frac{7\zeta(3)}{6(4\pi T)^2} N_a, \quad N = N_+ + N_-
$$

(21)

Note, that kinetic term is split into two terms corresponding to different bands with covariant derivatives that apart from $\vec{A}$ have $\vec{B}$. If one opens brackets – the only noncentrosymmetric term is proportional to difference of squares of Fermi momenta of two bands:

$$
\propto \left( k_{F+}^2 - k_{F-}^2 \right) \vec{B} \cdot (\Delta D^* + D^* \Delta)
$$

(22)

### B. Fourth order term

As usual, at fourth order, it is sufficient to retain only term $\propto |\Delta|^4$ hence we neglect $\vec{A}, \vec{B}$ and difference in $\Delta$’s. To that end we consider $\nu = 2$ term in Eq. (9), which is, using Eq. (12), equal to

$$
-\frac{1}{2} \text{Tr} \left[ \left( \vec{g} \delta \vec{g} T \delta \right)^4 \right] \simeq -\frac{1}{2} \sum_{w_n} \int d\vec{x} \frac{dk}{(2\pi)^3} \text{tr} \left[ (\vec{f} \delta \vec{f} T \delta)^4 \right]
$$

$$
\simeq -\frac{1}{2} \int d\vec{x} |\Delta|^4 \sum_a N_a \sum_{w_n} \int_{-\infty}^{+\infty} |d\epsilon_a| \left( w_n^2 + \frac{\epsilon_a^2}{2} \right)^2
$$

(23)

where to go to second equality we used Eq. (14). Using Eq. (23) and Eq. (8) we obtain part of the free energy quadratic in order parameter:

$$
F_4 = \int d\vec{x} |\Delta|^4, \quad \beta = \frac{7\zeta(3)}{4(4\pi T)^2} N
$$

(24)

### III. Rescaled Form of the Microscopic GL Model

Collecting terms Eq. (21) and Eq. (24) the microscopically derived GL model for noncentrosymmetric superconductor reads:

$$
F = \int d\vec{x} \left( \frac{(\vec{B} - \vec{H})^2}{2} + F_2 + F_4 \right)
$$

(25)

Importantly, the energy of the model is clearly bounded from below i.e. the functional does not allow infinitely negative energy states. This is in contrast to the model presented in Chapter 5 of [1], which has artificial unboundedness of the energy from below [8].

We rescale the model by introducing new variables $\vec{r}, \psi, F', \vec{A}'$ by performing the following transformation:

$$
\vec{r} = \frac{1}{\sqrt{-\alpha}} \left( \frac{\beta}{2\gamma^2} \right) \vec{r}', \quad \Delta = \sqrt{-\frac{\alpha}{2\beta}} \psi
$$

$$
F = \frac{\sqrt{-\alpha}}{2(2e^2)^{\frac{3}{4}} \beta^{\frac{1}{4}}} F', \quad \vec{A}' = \frac{1}{\sqrt{2e}} \vec{A}
$$

(26)

After dropping $'$, the rescaled GL free energy can be written as:

$$
F = \int d\vec{r} \left[ \left( \vec{B} - \vec{H} \right)^2 + \sum_{\alpha=\pm 1} |\mathcal{D}_a \psi|^2 \right] - |\psi|^2 + \frac{|\psi|^4}{2}
$$

$$
\mathcal{D}_a \equiv i\nabla - \vec{A} - \kappa_a \vec{B}, \quad \kappa_a \equiv \gamma + a\nu
$$

(27)

with new parameters:

$$
\kappa_c = \sqrt{\frac{\beta}{2e^2}} \sum_{\alpha=\pm 1} K_a v_{aF}, \quad \vec{H} = \frac{\sqrt{2\beta}}{-\alpha} \vec{H}
$$

$$
\gamma = \sqrt{-\alpha} \left( \sum_{\alpha=\pm 1} aK_a v_{aF} \right) 2\mu_B \kappa_c \left( \frac{2e^2}{\beta} \right)^{\frac{3}{4}}
$$

$$
\nu = \sqrt{-\alpha K_c} \left( \sum_{\alpha=\pm 1} v_{aF} \right) 2\mu_B \kappa_c \left( \frac{2e^2}{\beta} \right)^{\frac{3}{4}}
$$

(28)

Two conclusions can be drawn here: (i) the noncentrosymmetric term Eq. (22) has the prefactor $\gamma$. It means that sign of $\gamma$ determines weather left or right handed states are preferable. It is proportional to microscopic spin-orbit coupling $\gamma \propto \gamma_0$ if $\gamma_0 k_F \ll \mu$. On the other hand, the parameter $\nu$ appears due to the coupling to Zeeman magnetic field. (ii) Note, that $\gamma, \nu \propto \sqrt{-\alpha}$ and hence for $T \rightarrow T_c$ we get $\gamma, \nu \rightarrow 0$. It means that close to critical temperature noncentrosymmetric superconductor will behave as usual superconductor with GL parameter $\kappa_c$.

Varying Eq. (27) with respect to $\psi^*$ and $\vec{A}$ we obtain...
the following Euler-Lagrange (EL) equations:

$$\sum_a \frac{D_a^2 \psi}{2\kappa_c} - \psi + |\psi|^2 \psi = 0, \quad c.c. = 0$$

$$\nabla \times \left[ \vec{B} - \vec{H} - \sum_a \kappa_a \vec{j}_a \right] = \sum_a \vec{j}_a$$

with $\vec{j}_a = \frac{\text{Re}(\psi^* D_a \psi)}{\kappa_c}$ and boundary conditions for unitary vector $\vec{n}$ orthogonal to the boundary:

$$\vec{n} \cdot \sum_a D_a \psi = 0, \quad c.c. = 0$$

$$\vec{n} \times \left[ \vec{B} - \vec{H} - \sum_a \kappa_a \vec{j}_a \right] = 0$$

IV. MAGNETIC FIELD CONFIGURATION AS THE SOLUTION TO COMPLEX FORCE-FREE EQUATION

At temperatures lower than critical one, the magnetic field behavior is more complex. To analyze that, we start with considering the London limit which we define as a regime where the length scales of density variations are much smaller than the characteristic length scale of the magnetic field decay length scales. Therefore in the London limit one neglects the density variations. We show that configuration of the magnetic field is represented by superposition of fields satisfying complex force free equation.

We make the London approximation where the order parameter is set to $\psi = 0$ at $r < \xi$ to model a core of a vortex positioned at $r = 0$. Away from the core it recovers to bulk value $\psi = e^{i\phi}$, where $\phi(\vec{r})$ is the phase of the order parameter. Using this ansatz and $\kappa_c \gg 1$ we see that first of the GL equations in Eq. (29) is of the order parameter. Using this ansatz and $\psi$ recovers to bulk value should be zero. By introducing a differential operator $S$ the equation, that determines $\vec{B}$, can then be written as:

$$SS^* \vec{B} = 0, \quad S = -\eta + \nabla \times, \quad SS^* = S^* S$$

$$\eta \equiv \eta_1 + i\eta_2 = \frac{-\gamma + i\gamma_2}{\gamma^2 + \gamma_2^2}, \quad \eta_2 = \sqrt{\frac{\kappa_c}{2} + \nu^2}$$

$$S\vec{W} = 0$$

where we defined complex force free field $\vec{W}$, equation for which can be rewritten in usual form $\nabla \times \vec{W} = \eta \vec{W}$, but with complex $\eta$.

By using Eq. (32) we obtain that

$$S^* \vec{B} = c \vec{W}$$

where $c$ is arbitrary complex valued constant. Subtracting complex conjugated from Eq. (33) we obtain solution for magnetic field $\vec{B}$ in terms of complex force free field $\vec{W}$:

$$\vec{B} = \text{Re} \vec{W}$$

Note, that we absorbed multiplicative complex constant into definition of $\vec{W}$ in the last step.

Below we will need to calculate the energy of the system. By using the Eq. (27), up to a constant it can be written as:

$$F = \int d\vec{r} \left[ \frac{\eta_2^2 B^2}{\kappa_c} - \vec{B} \cdot \vec{H} + \frac{j^2}{\kappa_c} \right]$$

where $\vec{j} \equiv \nabla \phi + \vec{A} + \gamma \vec{B}$, which should be found from the second equation in Eq. (29):

$$\eta_2^2 \nabla \times \vec{B} + \gamma \nabla \times \vec{j} + \vec{j} = 0$$

Note, that taking curl of this equation gives Eq. (31). This gives the solution:

$$\vec{j} = \eta_2 \text{Im} \vec{W}$$

Hence energy of any configuration can be written simply as:

$$F = \int d\vec{r} \left[ \frac{\eta_2^2 |\vec{W}|^2}{\kappa_c} - \text{Re} \vec{W} \cdot \vec{H} \right]$$

To obtain $\vec{W}$ one can just straightforwardly solve Eq. (32) $S\vec{W} = 0$, but we believe that it is more elegant to employ the trick used by Chandrasekhar and Kendall [9]. Namely, solution for $\vec{W}$ is made of auxiliary functions:

$$\vec{W} = \vec{T} + \frac{1}{\eta} \nabla \times \vec{T}, \quad \vec{T} = \nabla \times (\vec{v} f(\vec{r}))$$

$$\nabla^2 f + \eta^2 f = 0, \quad \vec{v} = \text{const} \quad \text{or} \quad \vec{v} \propto \vec{r}$$

In general $\vec{v}$ can be chosen differently than in Eq. (39). But for our purposes it’s sufficient to restrict $\vec{v}$ to be:

$$\vec{v} = \text{const}, \quad \vec{v} \in \text{Re}, \quad |\vec{v}| = 1$$

Namely, we will use the following expression for $\vec{W}$ for vortices and surface currents:

$$\vec{W} = \eta f \vec{e}_z - \vec{e}_z \times \nabla f$$

It’s crucial to note that representation of $\vec{B}$ in terms of complex force free fields Eq. (34) is general: i.e. it holds for usual centrosymmetric superconductor as well as for noncentrosymmetric one. Using Eq. (40) energy Eq. (38) is simplified to

$$F = \int d\vec{r} \left[ \frac{\eta_2^2}{\kappa_c} \left( |\nabla f|^2 + |\eta f|^2 \right) - \vec{B} \cdot \vec{H} \right]$$
In order to consider vortices with windings \( n_i \) placed at \( \vec{r}_i \), the equation for \( f \) Eq. (39) should be modified as (discussed in more detail below):

\[
\nabla^2 f + \eta^2 f = -2\pi \eta \sum_i n_i \delta(x-x_i,y-y_i) \equiv \eta \delta
\]

(42)

V. CROSSOVER TO TYPE-1 SUPERCONDUCTIVITY AT ELEVATED TEMPERATURES

The coherence length \( \xi \) characterizes recovery of the order parameter from a small perturbation. Hence setting \( \vec{A} = 0 \) in Eq. (27) we immediately obtain that \( \psi \propto e^{-x/\xi} \) with:

\[
\xi = \frac{1}{\sqrt{2}\kappa_c}
\]

(43)

To obtain penetration depth one needs to solve for Meissner state in London limit. The Meissner state in the non-centrosymmetric superconductors was discussed before in [1, 5, 10] for similar models. Here we rederive it for our model Eq. (27) using method that we outlined in the previous section.

Consider superconductor positioned at \( x > 0 \) and external magnetic field \( \vec{H} \) is parallel to the boundary. As usual we assume that fields depend only on \( x \). Then Eq. (39) is easily solved resulting in \( f(x) = ce^{i\gamma x} \), since we demand \( f(x \to \infty) \to 0 \). Where \( c \) is complex multiplicative constant. To determine \( c \) we use boundary condition Eq. (30), which now becomes:

\[
\vec{n} \times \text{Re} \left[ \frac{i\vec{W}}{\eta} - \frac{\kappa_c}{2\eta_2} \vec{H} \right] = 0 \quad \Rightarrow \quad c = -\frac{i\kappa_c}{2\eta_2} \vec{H}
\]

(44)

where \( \vec{H} = H_3 + iH_2 \) and we used that \( \vec{A} = \text{Re} \frac{\vec{W}}{\eta} \) from Eq. (34). From Eq. (40) we obtain magnetic field, which can be represented by linear combination of components of \( \vec{B} \) parallel to the boundary \( \vec{B} = B_3 + iB_2 \):

\[
\vec{B} = -\frac{i\eta\kappa_c}{2\eta_2} \vec{H} e^{i\eta x} \propto e^{-\eta_2 x + i\eta_1 x}
\]

(45)

While the magnetic field has a spiral decay, its modulus has an exponential decay. That allows to define the penetration depth for magnetic field as the inverse of imaginary part of \( \eta \):

\[
\lambda = \frac{1}{\eta_2}
\]

(46)

Importantly, when going inside superconductor vector of magnetic field rotates with period \( \frac{2\pi}{\eta} \) forming right handed spiral (helical) structure, see Fig. 1. Note, that handness of the state is set by the sign of \( \eta_1 \). Also observe that the operator SS* that determines the configuration of \( \vec{B} \) is invariant under inversion (parity) transformation \( \mathbb{P} : \vec{r} \to -\vec{r} \) and the model is centrosymmetric only if \( \eta_1 = 0 \). It’s also apparent from the fact that \( \eta_1 \propto \gamma \), where \( \gamma \) is, as was shown above, the parameter that determines the degree of noncentrosymmetry of the material.

The ratio of the magnetic field penetration length and coherence length for the noncentrosymmetric superconductor reads

\[
\frac{\lambda}{\xi} = \frac{1 + \frac{\kappa_c}{\pi c} (\gamma^2 + \nu^2)}{\sqrt{1 + \frac{2}{\pi c} \nu^2}}
\]

(47)

Note, that \( \gamma, \nu \) strongly depend on \( T \) and go to zero for \( T \to T_c \), see Eq. (28). Since \( \gamma/\nu \approx \text{const} \) the ratio \( \lambda/\xi \) increases when temperature is decreased. Assume that, for example, for \( T \to T_c \) we have type-1 superconductivity with \( \kappa_c < 1 \). Then for lower temperature superconductor can allow vortex states, if noncentrosymmetry is strong enough to achieve \( \lambda \gtrsim \xi \). This type of behavior was reported for noncentrosymmetric superconductor AuBe [11].

We obtained the Eq. (47) by considering noncentrosymmetric superconductor with \( O \) or \( T \) symmetry. Noncentrosymmetric systems with different symmetry, have terms of different structure but with the same scaling, corresponding to spin-orbit and Zeeman coupling terms. It means that for any symmetry it is expected to have strong dependence of these noncentrosymmetric terms on temperature. Consequently, if \( \kappa_c < 1 \) and \( \gamma, \nu \) terms are large enough, one should expect the crossover between different types in noncentrosymmetric superconductors.

VI. STRUCTURE OF A SINGLE VORTEX

In contrast to the Meissner state, vortex solutions are much less studied in the noncentrosymmetric case. Earlier, solutions were obtained as a series expansion [1, 5], which didn’t exhibit any spiral structure of magnetic field. In this section we show how the method that we developed in Eq. (32), Eq. (34) and Eq. (39) allows to obtain exact solution which turns out to be structurally different.

Consider a vortex positioned at \( x, y = 0 \) and that it’s translationally invariant along \( z \) direction. As a check we obtain a numerical solution of the three dimensional system. Then using polar coordinates \( \rho \) and \( \theta \) for auxiliary function \( f \) Eq. (39) becomes Bessel equation with complex parameter:

\[
\rho^2 f_{\rho\rho} + \rho f_{\rho} + \eta^2 \rho^2 f + f_{\theta\theta} = 0
\]

(48)

Then the general solution is:

\[
f = \sum_{j=-\infty}^{+\infty} c_j e^{ij\theta} H_j^{(1)}(\eta \rho)
\]

(49)
Figure 1. Magnetic field vector \( \vec{B} \) in the model Eq. (27) with \( \kappa_c = 20, \gamma = 20, \nu = 1 \) obtained in the London approximation for the (a) Right handed Meissner state: \( \vec{B} \) on a line going inside the superconductor; (b) Right handed Vortex: \( \vec{B} \) on a line going radially along \( \rho \) away from the vortex core. Color is changing from red to blue when going from boundary or vortex core into the bulk of superconductor. Note, that handness of states is determined by the sign of \( \gamma \).

where we chose \( \mathcal{H}_j^{(1)} \) – Hankel function of the first kind to obtain appropriate asymptotic \( f \to 0 \) for \( \rho \to \infty \). Note that this asymptotic is the same as for the surface currents in the Meissner state Eq. (45), namely \( f \propto e^{i\eta \rho} \).

Next in order to determine complex constants \( c_n \) we have to take into account r.h.s of Eq. (31), i.e. make sure that magnetic field is consistent in the vortex core. It’s possible to show that this is equivalent to the following equation for \( f \):

\[
\nabla^2 f + \eta^2 f = -2\pi \eta \delta(x,y) \quad (50)
\]

where we used that phase winds \( n \) times around the core, i.e. \( \phi = n \theta \) and hence:

\[
\nabla \times \nabla \phi = 2\pi n e^z \delta(x,y) = n e_z \nabla^2 \ln \rho \quad (51)
\]

Hence we set \( c_j = 0 \) for \( j \neq 0 \) in Eq. (49) since only \( \mathcal{H}_0^{(1)} (\eta \rho) \to \frac{2\pi}{\eta} \ln \rho \) for \( \rho \to 0 \) as desired. Using Eq. (50) we obtain that \( c_0 = \frac{\pi}{\eta} n \). Finally magnetic field of a vortex, see Fig. 1, with winding \( n \) is given by:

\[
\vec{B} = \text{Re} \left[ \frac{i\pi}{2} n \eta (\eta e^z - e^z \times \nabla) \mathcal{H}_0^{(1)} (\eta \rho) \right] \quad (52)
\]

For \( \nu, \gamma \to 0 \) this expression, as expected, gives the usual result \( \vec{B} = -e_z \frac{\kappa_0 (\pi / \lambda_c)}{\lambda_c^2} \). In polar coordinates Eq. (52) can be written as:

\[
\vec{B} = \text{Re} \left[ \frac{i\pi}{2} n \eta^2 \left( 0, \mathcal{H}_1^{(1)} (\eta \rho), \mathcal{H}_0^{(1)} (\eta \rho) \right) \right] \quad (53)
\]

Then for \( \rho \to \infty \) since \( \mathcal{H}_1^{(1)} \to -i \mathcal{H}_0^{(1)} \propto e^{i\eta \rho} / \sqrt{\rho} \) magnetic field forms the right handed spirals as in the case of the Meissner state Eq. (45) but instead in a radial direction:

\[
\vec{B} = B_z + i B_\theta \propto \frac{e^{i\eta \rho}}{\sqrt{\rho}} \quad (54)
\]

Note, that this is general observation that decaying magnetic field forms spirals with handness determined by the sign of \( \gamma \).

For the vortex obtained as numerical solution of the full GL model Eq. (27) see Fig. 2. Solutions were obtained using nonlinear conjugate gradient algorithm, parallized on CUDA enabled GPU [12]. Discretisized grid had 512×512×32 points. To verify results we used grids of different sizes like 128³.

A. First critical magnetic field

By using the previously obtained solution \( f = \frac{i\pi}{2} n \eta \mathcal{H}_0^{(1)} (\eta \rho) \) and energy given by Eq. (41), we obtain energy of a vortex \( \mathcal{F}_v \) with winding \( n \). We can express it in terms of first critical magnetic field \( H_{c1} \):

\[
\mathcal{F}_v = 2\pi n (n H_{c1} + H)
\]

\[
H_{c1} = \frac{\eta_2}{\kappa_c} \eta_1 \frac{1}{\eta_1} \frac{\eta_2}{\eta_1} \ln \left[ \frac{2 e^{-\gamma\text{Euler}}}{|\eta| \xi} \right] \quad (55)
\]

where \( \gamma\text{Euler} \simeq 0.577 \) is Euler Gamma, magnetic field \( H_z = H > 0 \) and for single vortex \( n = -1 \).
Uz energy per unit length in the Eq. (56) and its complex conjugate we obtain the
which is a generalized form of Eq. (50). Then by using
it. Now lets consider only two vortices
δ by subtracting from Eq. (57) energies of single vortices Eq.
δ vortex combination since
there is as usual current going around the vortex core, there is a part of current going along the vortex core, alternating the
direction.

Figure 2. Vortex obtained numerically in the three dimensional model Eq. (27) with \( \kappa_c = 0.01 \), \( \gamma = 2.5 \), \( \nu = 0 \). (a) White streamlines show the force lines of the Magnetic field starting from the middle cross section, which is colored according to the \( B \) modulus. Note periodical structure in radial direction, which corresponds to spirals as in analytic solution Fig. 1. (b) Streamline plot for current \( \vec{J} \equiv \nabla \times \vec{B} \). Observe that current configuration is very similar to that of magnetic field. While there is as usual current going around the vortex core, there is a part of current going along the vortex core, alternating the
direction.

Thermodynamic critical magnetic field \( H_c \) is defined as \( H \) when energy of the uniform superconducting state \( \psi = 1 \) and \( \vec{A} = 0 \) is zero. It is easy to show that in our rescaled units it is \( H_c = 1 \). Note, that vortices will be present only if \( H_{c1} < H_c \). This inequality is satisfied since \( H_{c1} \to 0 \) in the limit \( \frac{T}{T_c} \to \infty \) if \( \gamma, \nu \ll \frac{2}{\sqrt{\kappa c}} \). Note, that this is true also for \( \kappa_c < 1 \), when for \( T \to T_c \) system crossovers to type-1 superconductivity.

VII. INTERVORTEX INTERACTION AND VORTEX BOUND STATES

Here we compute interaction energy of vortices by using the Eq. (41). Consider a set of vortices with windings \( n_i \) placed at \( \vec{r}_i \) with cores parallel to \( \vec{e}_z \). Then \( f \) satisfies the following equation:

\[
\nabla^2 f + \eta^2 f = -2\pi \eta \sum_i n_i \delta(x-x_i, y-y_i) \equiv \eta \delta
\]

\[
f = \sum_i \frac{i\pi}{2} n_i \eta \mathcal{H}_0^{(1)} (\eta|\vec{r} - \vec{r}_i|)
\]

which is a generalized form of Eq. (50). Then by using the Eq. (56) and its complex conjugate we obtain the energy per unit length in \( z \) direction:

\[
\mathcal{F} = \int dx dy \left[-\frac{\eta^2}{\kappa c} \text{Im}(f) - H_z \right] \delta
\]

(57)

where we also used that the flux of the vortices is fixed by \( \delta \). The integral in Eq. (57) is easily performed for any vortex combination since \( \delta \) contains the Dirac delta’s in it. Now lets consider only two vortices \( i = 1, 2 \). Hence subtracting from Eq. (57) energies of single vortices Eq.

(55) we obtain the interaction energy \( U \) as a function of distance \( R \) between them:

\[
U(R) = 2\pi^2 n_1 n_2 \frac{\eta^2}{\kappa c} \text{Re} \left[ \eta \mathcal{H}_0^{(1)} (\eta R) \right]
\]

(58)

Importantly, the intervortex interaction energy \( U \), see Fig. 3, changes sign. Analytically asymptotics for big \( R \) is given by:

\[
U(R) \propto n_1 n_2 e^{-\eta R} \cos (\eta_1 R + \phi_0)
\]

(59)

where \( \phi_0 \) is some phase. Hence the system forms vortex-vortex and vortex-antivortex pairs. Those will form stable states at distances \( R \) corresponding to local minima in \( U \) appearing with period \( \frac{2\pi}{\eta_1} \). This behavior
is due to the fact that in noncentrosymmetric superconductor vortices are represented by “circularly polarized” cylindrical magnetic field Eq. (52) with period \( \frac{2\pi}{n} \), see Fig. 2 and Fig. 4. Two or more of them brought together will form interference pattern of two point sources which, when moving them apart, will alternate between in phase and out of phase with the same period.

We obtained the bound states numerically in the full nonlinear GL model given by Eq. (27). The Fig. 4 shows two example of such bound states.

VIII. VORTEX-BOUNDARY INTERACTION

Another unconventional effect in noncentrosymmetric superconductors transpires in the physics of vortex-boundary interaction. Consider a half infinite superconductor positioned at \( x > 0 \) and right handed vortex with winding \( n \) placed at \( x = R \) and \( y = 0 \). Here we consider the London limit, and neglect the effects associated with the gap variations near the surface [13], and the nonlinear effects appearing at the scale of the vortex core [14].

External magnetic field is set to be \( \vec{H} = (0, 0, H) \). Then auxiliary field \( f \) should satisfy the following equation inside the superconductor:

\[
\nabla^2 f + \eta^2 f = -2\pi\eta\delta(x - R, y) \equiv \eta\delta
\]  

supplemented by the boundary conditions that \( f \) is zero at \( x \to \infty \). From Eq. (30) we obtain the following boundary conditions at \( x = 0 \):

\[
\text{Im}[\eta^* \partial_x f] = 0, \quad \text{Im} [f] = -\frac{\kappa_c}{2\eta_2} H
\]  

Since Eq. (60) is linear in \( f \) it is convenient to write solution as superposition of Meissner state, vortex and image of a vortex as:

\[
f = f_m + f_v + \tilde{f}_i
\]

\[
f_m = -\frac{i\kappa_c}{2\eta_2} H e^{i\eta x}
\]

\[
f_v = \frac{i\pi}{2} \tilde{\eta} \gamma H_0^{(1)} \left( \eta \sqrt{(x - R)^2 + y^2} \right)
\]

where \( f_m \) and \( f_v \) were found in the previous sections. Note, that since Meissner state \( f_m \) satisfies boundary conditions Eq. (61), the vortex and image \( f_v + \tilde{f}_i \) should satisfy Eq. (61) with zero right hand side.

Remember that with the London model, for usual superconductor image of the vortex is just its mirror reflection in the boundary, which is modeled by antivortex positioned outside the superconductor, see [15]. This configuration then satisfies both equation Eq. (60) and boundary conditions Eq. (61). By contrast in our case for noncentrosymmetric superconductor unfortunately it’s not possible to use this approach. Namely, mirror reflection of right handed vortex inside the superconductor is left handed antivortex outside, which indeed satisfies boundary conditions Eq. (61), but equation for \( \tilde{B} \) Eq. (31) (more complicated version of Eq. (60)) is not satisfied. This is simply because antivortex is left handed but the equation is right handed, or vice versa for \( \gamma < 0 \).

Inserted as an image right handed anti-vortex satisfies Eq. (60), but not boundary conditions Eq. (61).

So to obtain an “image” configuration \( f_i \) we have to solve explicitly the Eq. (60). We did that by performing Fourier transform in \( y \) direction and solving corresponding equations Eq. (60) for \( f_v + f_i \) together subjected to boundary condition Eq. (61) with zero r.h.s., which gives:

\[
f_i(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_i(x, k) e^{iky} dk
\]

\[
\tilde{f}_i(x, k) = -\frac{\pi n \eta}{s} e^{-sx} \left( e^{-sR} - 2\frac{\text{Re}(s\eta^2)}{\text{Im}(s\eta^2)} \text{Im} (e^{-sR}) \right)
\]

with \( s = \sqrt{k^2 - \eta^2} \)  

To obtain energy we integrate by parts Eq. (41) and use Eq. (60), which gives:

\[
\mathcal{F} = \int_0^\infty dx \int_{-\infty}^{\infty} dy \left[ \frac{\tilde{\eta}_2}{\kappa_c} \text{Im} (f) - H \right] \delta - \int_{-\infty}^{\infty} dy \frac{H}{2} \frac{\partial_x f}{\eta} \bigg|_{x=0}
\]

where we obtain, compared to Eq. (57), the last term which is boundary integral. Now inserting solutions Eq. (62) and Eq. (63) up to constant terms we obtain interaction energy of vortex and boundary, see Fig. 5:

\[
U_b(R) = -2\pi n H \text{Re} \left[ e^{i\eta R} \right] + 2\pi n \frac{\tilde{\eta}_2}{\kappa_c} \text{Im} [f_v(R, 0)] + \mathcal{F}_v
\]

For a large distance away from the boundary \( R \), the main contribution to the interaction energy comes from first term in Eq. (65) and hence it has the same asymptotics as for vortex vortex interaction, namely we obtain \( U_b \propto \text{Re} \left[ e^{i\eta R} \right] \), which has minimums with period \( \frac{2\pi}{\eta} \), see Fig. 5. Physically it means that when increasing magnetic field vortices will tend to stick near the boundary and only when there will be considerable amount of them occupying these minima vortices will start going into the bulk of superconductor.

For \( \gamma \to 0 \) \( f_i \) in the second term in Eq. (65) becomes antivortex as in [15]. But we believe physical interpretation in [15] of the first term in Eq. (65) as Meissner-vortex and second term as vortex-image interactions is not fully justified. Firstly, when integrating by parts energy Eq. (41) these terms obtained from combining energy and flux from the field configuration of vortex and image. Secondly, half of the first term in Eq. (65) actually comes from boundary integral in Eq. (64) due to vortex-image interaction.
Figure 4. (a) Vortex-vortex and (b) vortex-antivortex bound states obtained numerically in the three dimensional model Eq. (27) with \( \kappa_c = 0.01 \), \( \gamma = 2.5 \), \( \nu = 0 \). White streamlines show the force lines of the Magnetic field starting from the middle cross section, which is colored according to the \( B \) modulus.

Figure 5. Vortex boundary interaction energy \( U_b \) Eq. (65) for \( \kappa_c = 20 \), \( \gamma = 20 \), \( \nu = 1 \) as function of distance from vortex to boundary \( R \) for several values of external magnetic field \( H \). Plot is cut off for small distances and presented for \( R > \xi \). Interaction clearly has minimums with period \( 2\pi |\eta_1| \) for any nonzero \( H \), which lead to bound states of vortex and boundary.

IX. CONCLUSIONS

We considered the physics of magnetic field behavior and vortex states in a non-centrosymmetric superconductor. The main conclusion of the microscopic part of the paper is that type of magnetic response in a non-centrosymmetric superconductor has significant temperature dependence and one can expect materials which are type-1 close to critical temperature to exhibit vortex states at lower temperatures.

The vortex states in this systems are unconventional. The demonstrated spiral-like decay of magnetic field away from a vortex leads to multiple minima in the intervortex interaction potentials and thus formation of vortex bound state. While in multicomponent and multiband systems a formation of strong vortex bound states appears due to disparity of multiple coherence and magnetic field penetration lengths [16–21], the result is strikingly different from the vortex interaction in single-component systems. For the single-component centrosymmetric GL model, the intervortex interaction is monotonic and it purely repulsive for GL parameter \( \kappa > 1 \) (in different convention for \( \kappa > 1 / \sqrt{2} \)) and purely attractive for \( \kappa < 1 \). At a critical coupling \( \kappa = 1 \) vortices do not interact at any distance [22–26]. The lack of interaction is a finite-tuning associated with a saturation of Bogomolnyi bound where the repulsive magnetic and current-current interaction exactly cancels the attractive core-core interaction at all distances. It was noticed that if one goes beyond the GL model the cancellation will not hold and a tiny attractive interaction appears [27]. This small attraction exists in an extremely narrow region near the Bogomolnyi point (\( \kappa = 1 \)) because it requires fine tuning to cancel the dominant GL forces [28, 29]. By contrast we see that the attractive intervortex force appears in the noncentrosymmetric superconductor at the level of dominant contribution, already in the GL and the London models and is not related to neither proximity to nor existence of Bogomolny point.

We find that vortices have a similar oscillating sign of interaction with Meissner current close to the boundaries. The properties may potentially be utilized for new type of control of vortex matter for fluxonics and vortex-based cryocomputing applications.

Note added: In the process of completion of this work we received a preprint by Garaud, Chernodub and Kharzeev [30] where similar results are obtained. The submission of this work and [30] was coordinated.
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