Singularly perturbed Neumann problem for fractional Schrödinger equations

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Abstract. This paper is concerned with a Neumann type problem for singularly perturbed fractional nonlinear Schrödinger equations with subcritical exponent. For some smooth bounded domain \( \Omega \subset \mathbb{R}^n \), our boundary condition is given by 
\[
\int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = 0 \quad \text{for } x \in \mathbb{R}^n \setminus \bar{\Omega}.
\]
We establish existence of nonnegative small energy solutions, and also investigate the integrability of the solutions on \( \mathbb{R}^n \).

1. Introduction

The main purpose of this paper is to investigate a singularly perturbed Neumann type problem for fractional Schrödinger equations. Precisely, given a smooth bounded domain \( \Omega \subset \mathbb{R}^n \), we consider the following problem
\[
\begin{aligned}
\varepsilon^2 (-\Delta)^s u + u &= |u|^{p-1} u & \text{in } \Omega, \\
\mathcal{N}_s u &= 0 & \text{on } \mathbb{R}^n \setminus \bar{\Omega}.
\end{aligned}
\tag{1.1}
\]
Here \( \varepsilon > 0, s \in (0, 1), n \geq 2, p \in (1, \frac{n+2s}{n-2s}) \), and 
\[
\mathcal{N}_s u(x) = C_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad x \in \mathbb{R}^n \setminus \bar{\Omega},
\tag{1.2}
\]
where \( C_{n,s} \) is the normalization constant in the definition of fractional Laplacian 
\[
(-\Delta)^s u(x) = C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy.
\]
This type of boundary problem for fractional Laplacian was introduced by Dipierro, Ros-Oton and Valdinoci in [18]. It corresponds to a jump process as follows: If a particle has gone to \( x \in \mathbb{R}^n \setminus \bar{\Omega} \), then it may come back to any point \( y \in \Omega \), the probability

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of jumping from \(x\) to \(y\) being proportional to \(|x - y|^{n-2s}\). From mathematical point of view, such kind of boundary conditions generalize the classical Neumann conditions for elliptic (or parabolic) differential equations. That is, if \(s \to 1\), then \(N_{s}u = 0\) becomes the classical Neumann condition. For more details, see [18]. Also in [19, 20], Du, Gunzburger, Lehoucq and Zhou introduced volume constraints for a general class of nonlocal diffusion problems on bounded domain in \(\mathbb{R}^{n}\) via a nonlocal vector calculus. If we rewrite (1.2) by a nonlocal vector calculus for fractional Laplacian (see Section 2 below), then \(N_{s}u = 0\) (with some modifications) can be considered as a special case of the volume constraints defined by [19, 20] (see Remark 2.5 below).

Other types of Neumann problems for fractional Laplacian (or other nonlocal operators) were investigated in many works [6, 11, 4, 5, 12, 13, 22]. All these conditions also have probabilistic interpretations and recover the classical Neumann problem as a limit case. A comparison between these models and ours can be found in [18, Section 7].

The singularly perturbed Neumann problem for classical nonlinear Schrödinger equations with subcritical exponent is as follows:

\[
\begin{cases}
-\varepsilon^{2} \Delta u + u = u^{p} & \text{in } \Omega, \\
\partial_{\nu}u = 0 & \text{on } \partial\Omega, \\
u > 0 & \text{in } \Omega,
\end{cases}
\] (1.3)

where \(1 < p < \frac{n+2}{n-2}\) for \(n \geq 3\) and \(p > 1\) for \(n = 2\), and \(\nu\) is the unit outward normal on \(\partial\Omega\). There is a great deal of works on this problem. We only restrict ourselves to cite a few papers, referring to the bibliography for further references. The pioneer works by Lin, Ni and Tagaki [29, 26, 30, 31] proved the existence of single-peak spike layer solution \(u_{\varepsilon}\) to (1.3). After that many interesting results concerning multi-peak spike layer solutions to (1.3) have been obtained ([23, 24, 25]). Note that a spike-layer solution has its energy or mass concentrating near isolated points (a zero-dimensional set) in \(\bar{\Omega}\). Similarly, there exist solutions to (1.3) with \(k\)-dimensional \((1 \leq k \leq n-1)\) concentration set ([27, 28, 2, 3, 16]). We refer to [32] for more other results and references.

We should also mention that concentration phenomenon for fractional Schrödinger equations has been extensively studied recently. On the total space \(\mathbb{R}^{n}\), the existence and multiplicity of spike layer solutions under various conditions were obtained by [15, 10, 9, 21]. On a bounded domain in \(\mathbb{R}^{n}\), singularly perturbed Dirichlet problem was investigated by [14]. Moreover, under classical Neumann condition, an existence result of spike solutions to Schrödinger equations involving half Laplacian (see Equation (1.7) below) was proved by [35].

We are now in a position to formulate our main results and give the idea of the proofs. Our problem (1.1) has a variational structure. More precisely, let

\[
\langle u, v \rangle_{H_{s,\Omega}^{2}} = \frac{C_{n,s}\varepsilon^{2s}}{2} \int_{\mathbb{R}^{2n}\setminus(\Omega)^{2}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy + \int_{\Omega} uv \, dx,
\] (1.4)

where \(\Omega^{c} := \mathbb{R}^{n} \setminus \Omega\). Then define the space

\[H_{s,\Omega}^{2} = \{ u : \mathbb{R}^{n} \to \mathbb{R} \text{ measurable and } \langle u, u \rangle_{H_{s,\Omega}^{2}} < \infty \}.
\]
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It is a Hilbert space with the norm \( \| \cdot \|_{H^s_\varepsilon, \Omega} = \langle \cdot, \cdot \rangle_{H^s_\varepsilon, \Omega}^{1/2} \). It follows that weak solutions to the problem (1.1) are critical points of the following functional

\[
I_\varepsilon(u) = \frac{C_{n,s,\varepsilon}^2}{4} \int_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx\,dy + \frac{1}{2} \int_{\Omega} u^2 \, dx - \frac{1}{p + 1} \int_{\Omega} |u|^{p+1} \, dx. \tag{1.5}
\]

We obtain the following existence result.

**Theorem 1.1.** If \( \varepsilon \) is sufficiently small, then there exists a nonnegative solution \( u_\varepsilon \) to (1.1). Moreover, \( u_\varepsilon \) satisfies

\[
0 < I_\varepsilon(u_\varepsilon) \leq C_1 \varepsilon^n.
\]

Consequently, \( u_\varepsilon \) is a nonconstant solution, and,

\[
\|u_\varepsilon\|_{H^s_\varepsilon, \Omega} \leq C_2 \varepsilon^{\frac{n}{2}}. \tag{1.6}
\]

Here \( C_1, C_2 \) are two positive constants depending only on \( n, s, p, \Omega \).

**Remark 1.2.** Such kind of existence results for classical Neumann problem (1.3) was obtained by Lin, Ni and Takagi [26]. Recently, in [35], Stinga and Volzone recovered the results (including concentration and regularity issues) in [26] for a fractional semilinear Neumann problem as follows:

\[
\begin{cases}
\varepsilon^2(-\Delta)^{\frac{s}{2}} u + u = u^p & \text{in } \Omega, \\
\partial_n u = 0 & \text{on } \partial \Omega, \\
u > 0 & \text{in } \Omega, 
\end{cases} \tag{1.7}
\]

where \( p \in (1, \frac{n+1}{n-1}) \), \( \nu \) is the outer unit normal to \( \partial \Omega \). Note that both of the boundary conditions in (1.3) and (1.7) are classical.

The proof of this theorem relies on critical point theory. More precisely, the functional \( I_\varepsilon \) has mountain pass structure. The key point is to construct an appropriate function \( \phi \in H^s_\varepsilon, \Omega \) such that, for some \( t_0 > 0 \), it holds that \( I_\varepsilon(t_0 \phi) \leq 0 \) and \( 0 < \sup_{t \in [0,t_0]} I_\varepsilon(t \phi) \leq C \varepsilon^n \) (\( C \) is a constant depending on \( n, s, \Omega \)). As compared with the classical case, verifying the necessary properties of \( \phi \) becomes more involved because of the fractional Laplacian. To prove the nonnegativity of the solution, we use the local realization method of Caffarelli and Silvestre [8] to show a weak maximum principle for problem (1.1) (Proposition 6.3). For more details, see Section 3, 4 and 6.

Moreover, we investigate the integrability properties of the solutions to problem (1.1). We have the following theorem.

**Theorem 1.3.** Let \( u \in H^s_\varepsilon, \Omega \). Then it holds that

1. \( u \in L^2_{\text{loc}}(\mathbb{R}^n) \),
2. if \( \mathcal{N}_u(u) = 0 \), then

\[
\int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, dx < +\infty. \tag{1.8}
\]
To prove this theorem, we need a detailed analysis of some singular integrals of the following form
\[ u[A, B] := \int_A \int_B \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx, \]
where \( A \) and \( B \) are two measurable sets in \( \mathbb{R}^n \). Such kind of integral is also important in nonlocal minimal surfaces (see e.g. \([7, 33]\)). Since \( u \in H^s_{x, \Omega}, u(\Omega, \Omega) \) and \( u(\Omega, \Omega^c) \) are finite. Then we can prove the local integrability of \( u \) by choosing appropriate balls in \( \mathbb{R}^n \). The second conclusion implies that \( u \) is \( L^s \) integrable in the sense of Silvestre \([34]\). We should note that if \( \mathcal{N}_s(u) = 0 \), then \( \lim_{|x| \to \infty} u(x) = \frac{1}{|\Omega|} \int_{\Omega} u(x) dx \) ([18, Proposition 3.13]). Therefore, in general, we can not expect to prove that \( u \) is integrable on the total space \( \mathbb{R}^n \). See Section 5.

The paper is organized as follows. In Section 2, inspired by \([19, 20]\), we give an alternative definition of \( \mathcal{N}_s(u) \) by a nonlocal vector calculus for fractional Laplacian. By this method, we recover the ingredients in the classical calculus such as integration by parts, Gauss formula, Green’s identities and so on. Then we rewrite \( \mathcal{N}_s(u) \) as the normal derivative on the boundary in the classical Neumann problems. In Section 3, we show that problem (1.1) has a variational structure. In Section 4, an existence result is proved. Section 5 gives the proof of Theorem 1.3. Finally, we prove the nonnegativity of the solution in Section 6.

2. A nonlocal vector calculus for fractional Laplacian

In this section, we establish a nonlocal vector calculus for fractional Laplacian which can be considered as a special case in \([19, 20]\). Such kind of nonlocal vector calculus enables striking analogies to be drawn between the nonlocal model and classical models for diffusion. For our applications, we define fractional divergence operator and its adjoint operator, interaction operator for fractional Laplacian. Then, formally, the ingredients in classical calculus such as integration by parts, Gauss formula, Green’s identities can be proved. Note that in what follows, the computations are formal. Strictly, we should choose appropriate function spaces to make sure that the integrals are finite, or consider the singular integrals in the principle value sense if necessary.

Fractional Laplacian of order \( s, s \in (0, 1) \), is the pseudo-differential operator with symbol \(|\xi|^{2s}\). Precisely, it can be represented by
\[ (-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(u)(\xi)), \tag{2.1} \]
where \( \mathcal{F} \) denotes the Fourier transform. Suppose that \( u \in H^{2s}(\mathbb{R}^n) \) (the 2s-th Sobolev space on \( \mathbb{R}^n \)). Then, the fractional Laplacian given by (2.1) is equivalent to the following formula (see e.g. \([17]\))
\[ (-\Delta)^s u = C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad s \in (0, 1), \]
where \( C_{n,s} \) is a normalizing constant. Let
\[ \gamma(x, y) = C_{n,s} |x - y|^{-(n+2s)}. \]
Define
\[ K(x, y) = \left( \frac{C_{n,s}}{2} \right)^{\frac{1}{2}} \left( \frac{x_1 - y_1}{|x_1 - y_1|^{\frac{n+2}{2}} + 1}, \ldots, \frac{x_n - y_n}{|x_n - y_n|^{\frac{n+2}{2}} + 1} \right). \]

Note that
\[ K(y, x) = -K(x, y) \]
and
\[ 2K(x, y) \cdot K(x, y) = \gamma(x, y). \]

In the following context of this section, \( u, v \) denote two measurable scalar functions on \( \mathbb{R}^n \), and \( v \) denotes a \( n \)-dimensional vector-valued measurable function on \( \mathbb{R}^n \times \mathbb{R}^n \).

**Definition 2.1 (Fractional Divergence).** The fractional divergence operator '\( \nabla^s \cdot \)' acting on \( v \) is defined by
\[
(\nabla^s \cdot v)(x) := \int_{\mathbb{R}^n} (v(x, y) + v(y, x)) \cdot K(x, y) dy, \quad \text{for } x \in \mathbb{R}^n.
\]

**Definition 2.2.** The operator '\( \nabla^s \)' acting on \( u \) is given by
\[
\nabla^s(u)(x, y) = (u(x) - u(y))K(x, y), \quad \text{for } x, y \in \mathbb{R}^n.
\]
\( \nabla^s \) is the adjoint operator of fractional divergence operator \( \nabla^s \cdot \). Precisely, we have

**Lemma 2.3.** It holds that
\begin{enumerate}
  \item \( \langle \nabla^s \cdot v, u \rangle = \langle v, \nabla^s u \rangle \);  
  \item \( \nabla^s \cdot \nabla^s u = (-\Delta)^s u \).
\end{enumerate}

**Proof.** (1) A direct calculation yields
\[
\langle \nabla^s \cdot v, u \rangle = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} (v(x, y) + v(y, x)) \cdot K(x, y) dy \right) u(x) dx \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)v(x, y) \cdot K(x, y)dydx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)v(y, x) \cdot K(x, y)dydx \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)v(x, y) \cdot K(x, y)dydx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(y)v(x, y) \cdot K(y, x)dydx \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)v(x, y) \cdot K(x, y)dydx - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(y)v(x, y) \cdot K(y, x)dydx \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} v(x, y) \cdot (u(x) - u(y))K(x, y)dydx = \langle v, \nabla^s u \rangle.
\]

Thus, we have (1).
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(2) Computing

\[(\nabla^s \cdot \nabla^s) u(x) = \int_{\mathbb{R}^n} ((u(x) - u(y))K(x, y) + (u(y) - u(x))K(y, x)) \cdot K(x, y) dy \]
\[= 2 \int_{\mathbb{R}^n} (u(x) - u(y))K(x, y) \cdot K(x, y) dy \]
\[= C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \]
we obtain (2). □

Corresponding to the fractional divergence operator ‘\(\nabla^s \cdot \)’, we define an interaction operator as follows.

**Definition 2.4 (Interaction Operator).** The interaction operator \(\mathcal{N}\) corresponding to fractional Laplacian is given by

\[\mathcal{N}(v)(x) = -\int_{\Omega} (v(x, y) + v(y, x)) \cdot K(x, y) dy, \quad \text{for} \ x \in \Omega^c.\]

**Remark 2.5.** We should note that in [19, 20], the interaction operator is given by

\[\tilde{\mathcal{N}}(v) := -\int_{\mathbb{R}^n} (v(x, y) + v(y, x)) \cdot K(x, y) dy\]

for \(x \in \Omega^c\). From probabilistic point of view, it corresponds to the following: If a particle is locating at \(x \in \mathbb{R}^n \setminus \bar{\Omega}\), then it may jump to any point \(y\) in \(\mathbb{R}^n\) with probability \(|x - y|^{-n-2s}\).

By this definition, we can show the following Gauss type formula.

**Lemma 2.6 (Gauss Formula).** It holds that

\[\int_{\Omega} \nabla^s \cdot v dx = \int_{\Omega^c} \mathcal{N}(v) dx. \tag{2.2}\]

**Proof.** Compute

\[\int_{\Omega} \nabla^s \cdot v dx = \int_{\Omega} \int_{\mathbb{R}^n} (v(x, y) + v(y, x)) \cdot K(x, y) dy dx \]
\[= \int_{\Omega} \int_{\mathbb{R}^n} (v(x, y) + v(y, x)) \cdot K(x, y) dy dx \]
\[+ \int_{\Omega} \int_{\Omega^c} (v(x, y) + v(y, x)) \cdot K(x, y) dy dx.\]

Since \(K(x, y) = -K(y, x)\), we have that

\[\int_{\Omega} \int_{\Omega} (v(x, y) + v(y, x)) \cdot K(x, y) dy dx = 0,\]
and
\[ \int_{\Omega} \int_{\Omega^c} (v(x, y) + v(y, x)) \cdot K(x, y) dxdy = - \int_{\Omega^c} \int_{\Omega} (v(x, y) + v(y, x)) \cdot K(x, y) dxdy. \]

Thus
\[ \int_{\Omega} \nabla^s \cdot \mathbf{v} dx = - \int_{\Omega^c} \int_{\Omega} (v(x, y) + v(y, x)) \cdot K(x, y) dxdy \]
\[ = \int_{\Omega^c} N(\mathbf{v}) dx. \]

It completes the proof. \(\square\)

Similar to the classical calculus, we also have the following integration by parts formula and Green’s identities.

**Lemma 2.7** (Integration by Parts Formula). *It holds that*
\[ \int_{\Omega} u(\nabla^s \cdot \mathbf{v}) dx - \int_{\mathbb{R}^{2\mathbb{N}\backslash\{\Omega^c\}}^2} (\nabla^s u) \cdot \mathbf{v} dxdy = \int_{\Omega^c} u N(\mathbf{v}) dx. \]

**Proof.** Calculate
\[ \int_{\Omega} u(\nabla^s \cdot \mathbf{v}) dx = \int_{\Omega} u(x) \left[ \int_{\mathbb{R}^n} (v(x, y) + v(y, x)) \cdot K(x, y) dy \right] dx \]
and
\[ \int_{\mathbb{R}^{2\mathbb{N}\backslash\{\Omega^c\}}^2} (\nabla^s u) \cdot \mathbf{v} dxdy \]
\[ = \int_{\mathbb{R}^{2\mathbb{N}\backslash\{\Omega^c\}}^2} u(x) K(x, y) \cdot v(x, y) dxdy - \int_{\mathbb{R}^{2\mathbb{N}\backslash\{\Omega^c\}}^2} u(y) K(x, y) \cdot v(x, y) dxdy \]
\[ = \int_{\Omega^c} \int_{\mathbb{R}^n} u(x)(v(x, y) + v(y, x)) \cdot K(x, y) dxdy \]
\[ + \int_{\Omega^c} \int_{\Omega^c} u(x)(v(x, y) + v(y, x)) \cdot K(x, y) dxdy. \]

Therefore,
\[ \int_{\Omega} u(\nabla^s \cdot \mathbf{v}) dx - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\nabla^s u) \cdot \mathbf{v} dxdy \]
\[ = - \int_{\Omega^c} u(x) \left[ \int_{\Omega} (v(x, y) + v(y, x)) \cdot K(x, y) dy \right] dx \]
\[ = \int_{\Omega^c} u N(\mathbf{v}) dx. \]

So, we obtain our conclusion. \(\square\)
COROLLARY 2.8 (Green’s Identities). We have ‘Green’s first identity’
\[
\int_{\Omega} v(-\Delta)^s u \, dx = \int_{\mathbb{R}^n \setminus (\Omega^c)^2} \nabla^s v \cdot \nabla^s u \, dy + \int_{\Omega^c} v \mathcal{N}(\nabla^s u) \, dx, \tag{2.3}
\]
and ‘Green’s second identity’
\[
\int_{\Omega} u(-\Delta)^s v \, dx - \int_{\Omega} v(-\Delta)^s u \, dx = \int_{\Omega^c} u \mathcal{N}(\nabla^s v) \, dx - \int_{\Omega^c} v \mathcal{N}(\nabla^s u) \, dx. \tag{2.4}
\]

PROOF. Choose the vector function \( v \) in Lemma 2.7 to be \( \nabla^s u \). Using the integration by parts formula in Lemma 2.7, we obtain the Green’s first identity (2.3). Furthermore, the Green’s second identity (2.4) is a direct conclusion from (2.3). \( \square \)

3. Variational structure

The Neumann problem (1.1) is variational. More precisely, given measurable function \( u, v : \mathbb{R}^n \to \mathbb{R} \), we define
\[
\|u\|_{H^s_{\varepsilon, \Omega}} = \varepsilon^{2s} \int_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} |\nabla^s u|^2(x, y) \, dxdy + \int_{\Omega} u^2 \, dx \tag{3.1}
\]
and
\[
\langle u, v \rangle_{H^s_{\varepsilon, \Omega}} = \varepsilon^{2s} \int_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} (\nabla^s u \cdot \nabla^s v)(x, y) \, dxdy + \int_{\Omega} uv \, dx \tag{3.2}
\]
It is easy to see that (1.4) and (3.2) are equivalent. Correspondingly, we define the space
\[ H^s_{\varepsilon, \Omega} = \{ u : \mathbb{R}^n \to \mathbb{R} \text{ measurable and } \|u\|_{H^s_{\varepsilon, \Omega}} < \infty \}. \]

REMARK 3.1. Constant functions \( v(x) \equiv c \) on \( \mathbb{R}^n \) are contained in \( H^s_{\varepsilon, \Omega} \).

REMARK 3.2. Let \( H^s(\Omega) \) be the \( s \)-th Sobolev space with the norm given by
\[
\|h\|_{H^s(\Omega)} = \int_{\Omega} \int_{\Omega^c} \frac{|h(x) - h(y)|^2}{|x - y|^{n+2s}} \, dxdy + \int_{\Omega} h^2 \, dx
\]
(See, for example, [1], [17].) Therefore, if \( u \in H^s_{\varepsilon, \Omega} \), then \( u|_{\Omega} \) is in \( H^s(\Omega) \).

LEMMA 3.3. \( H^s_{\varepsilon, \Omega} \) is a Hilbert space with inner product given by (3.2) (or, equivalently, (1.4)).

PROOF. This lemma is the case \( g = 0 \) of Proposition 3.1 in [18]. We omit details of the proof here. \( \square \)

DEFINITION 3.4. We say that \( u \in H^s_{\varepsilon, \Omega} \) is a weak solution to (1.1), if, for all \( v \in H^s_{\varepsilon, \Omega} \), it holds that
\[
\varepsilon^{2s} \int_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} (\nabla^s u \cdot \nabla^s v)(x, y) \, dxdy + \int_{\Omega} uv \, dx - \int_{\Omega} |u|^{p-1} uv \, dx = 0. \tag{3.3}
\]
Remark 3.5. This definition of weak solution is the same as the classical case. In fact, by the Green’s first identity (Lemma 2.8), we have that
\[ \int_{\mathbb{R}^n \setminus \Omega^2} \nabla^s v \cdot \nabla^s u dy dx = \int_{\Omega} v(-\Delta)^s u dx - \int_{\Omega^c} v \mathcal{N}(\nabla^s u) dx. \]
Note that
\[ \mathcal{N}(\nabla^s u)(x) = C_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = \mathcal{N}_s u(x), \quad x \in \mathbb{R}^n \setminus \bar{\Omega}. \]
Then, formally, (3.3) becomes
\[ \int_{\Omega} (\varepsilon^{2s}(-\Delta)^s u + u - |u|^{p-1}u) v dx - \varepsilon^{2n} \int_{\Omega^c} v \mathcal{N}_s u dx = 0. \]
We rewrite the functional (1.5) in the nonlocal vector calculus form as follows. For all \( u \in H^s_{\varepsilon,\Omega}, \)
\[ I_{\varepsilon}(u) = \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^n \setminus \Omega^2} |\nabla^s u|^2(x, y) dy dx + \frac{1}{2} \int_{\Omega} u^2 dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx. \]
Remark 3.6. From Remark 3.2 and Sobolev embedding \( H^s(\Omega) \hookrightarrow L^q(\Omega) \) (\( q \in (1, \frac{2n}{n-2s}) \)), we have that, for all \( u \in H^s_{\varepsilon,\Omega}, \)
\[ \int_{\Omega} |u|^{p+1} dx < +\infty. \]
Proposition 3.7. Any critical point of \( I_{\varepsilon} \) is a weak solution of problem (1.1).
Proof. For any \( v \in H^s_{\varepsilon,\Omega}, \) we have that
\[ I_{\varepsilon}(u + tv) = \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^n \setminus \Omega^2} |\nabla^s (u + tv)|^2(x, y) dy dx + \int_{\Omega} (u + tv)^2 dx - \frac{1}{p+1} \int_{\Omega} |u + tv|^{p+1} dx \]
\[ = I_{\varepsilon}(t) + t \left( \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^n \setminus \Omega^2} (\nabla^s u \cdot \nabla^s v)(x, y) dy dx + \int_{\Omega} uv dx - \int_{\Omega} |u|^{p-1} uv dx \right) \]
\[ + t^2 \left( \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^n \setminus \Omega^2} |\nabla^s v|^2(x, y) dy dx + \int_{\Omega} v^2 dx + p \int_{\Omega} |u + \theta_t v|^{p-1} v^2 dx \right), \]
where \( \theta_t \in (0, 1). \) Thus
\[ I'_{\varepsilon}(u)v = \lim_{t \to 0} \frac{I_{\varepsilon}(u + tv) - I_{\varepsilon}(u)}{t} \]
\[ = \varepsilon^{2s} \int_{\mathbb{R}^n \setminus \Omega^2} (\nabla^s u \cdot \nabla^s v)(x, y) dy dx + \int_{\Omega} uv dx - \int_{\Omega} |u|^{p-1} uv dx \]
\[ = \langle u, v \rangle_{H^s_{\varepsilon,\Omega}} - \int_{\Omega} |u|^{p-1} uv dx. \]
Therefore, if \( u \) is a critical point of \( I_{\varepsilon} \), then \( u \) is a weak solution to (1.1). \( \square \)
4. An existence result

In this section, we shall investigate the following problem:

\[ \begin{cases} \varepsilon^2(-\Delta)s u + u = u_+^p & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{on } \mathbb{R}^n \setminus \bar{\Omega}, \end{cases} \tag{4.1} \]

where \( u_+ = \max\{u, 0\} \). Similar to the problem \((1.1)\), define

\[ F_\varepsilon(u) = \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} |\nabla^s u|^2(x,y) dxdy + \frac{1}{2} \int_\Omega u^2 dx - \frac{1}{p+1} \int_\Omega u_+^{p+1} dx. \]

Then, for all \( u, v \in H^s_{\varepsilon, \Omega} \),

\[ F_\varepsilon'(u) v = \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} (\nabla^s u \cdot \nabla^s v) dxdy + \int_\Omega uv dx - \int_\Omega u_+^p v dx = \langle u, v \rangle_{H^s_{\varepsilon, \Omega}} - \int_\Omega u_+^p v dx. \tag{4.2} \]

Therefore, any critical point of \( F_\varepsilon \) is a weak solution of problem \((4.1)\).

We shall prove the following existence result of problem \((4.1)\).

**Proposition 4.1.** If \( \varepsilon \) is sufficiently small, then there exists a nonconstant solution \( u_\varepsilon \) to \((1.1)\). Moreover, \( u_\varepsilon \) satisfies

\[ 0 < F_\varepsilon(u_\varepsilon) \leq C\varepsilon^n, \]

and

\[ \|u_\varepsilon\|_{H^s_{\varepsilon, \Omega}} \leq C\varepsilon^{\frac{n}{2}}, \tag{4.3} \]

where \( C \) is a positive constant depending only on \( n, s, p, \Omega \).

**Lemma 4.2.** \( F_\varepsilon \) satisfies Palais-Smale condition.

**Proof.** Let \( \{u_m\} \subset H^s_{\varepsilon, \Omega} \) be a Palais-Smale sequence such that \( |F_\varepsilon(u_m)| \leq d \), for all \( m \in \mathbb{N} \), is bounded and \( F_\varepsilon'(u_m) \to 0 \). Then

\[ d \geq \left| \frac{1}{2} \|u_m\|^2_{H^s_{\varepsilon, \Omega}} - \frac{1}{p+1} \int_\Omega (u_m)^{p+1} dx \right|. \tag{4.4} \]

Since \( F_\varepsilon'(u_m) \to 0 \), for any \( \epsilon > 0 \), there is an \( M = M(\epsilon) \) such that for all \( m \geq M \),

\[ |F_\varepsilon'(u_m)v| = \left| \left( \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} |\nabla^s u_m \cdot \nabla^s v|(x,y) dxdy + \int_\Omega u_m v dx \right) - \int_\Omega (u_m)^{p} v dx \right| \leq \epsilon \|v\|_{H^s_{\varepsilon, \Omega}}, \tag{4.5} \]

for all \( v \in H^s_{\varepsilon, \Omega} \). If we choose \( \epsilon = 1 \), \( v = u_m \), then \((4.5)\) yields

\[ \left| \int_\Omega (u_m)^{p+1} dx \right| \leq \|u_m\|^2_{H^s_{\varepsilon, \Omega}} + \|u_m\|_{H^s_{\varepsilon, \Omega}}. \tag{4.6} \]
By (4.6) and (4.4), we have that
\[
    d \geq \left( \frac{1}{2} - \frac{1}{p+1} \right) \|u_m\|_{H^s_{\epsilon, \Omega}}^2 - \frac{1}{p+1}\|u_m\|_{H^s_{\epsilon, \Omega}}.
\]
Therefore, \(\{u_m\}\) is bounded in \(H^s_{\epsilon, \Omega}\). Up to a subsequence, we assume that \(u_m \rightharpoonup u\) in \(H^s_{\epsilon, \Omega}\). By Remark 3.2 and Sobolev embedding, \(u_m \to u\) in \(L^{p+1}(\Omega)\). So \((u_m)_{p+1} \to u_{p+1}\) in \(L^{(p+1)/p}(\Omega)\). Equation (4.2) yields that
\[
    \|u_m - u\|^2_{H^s_{\epsilon, \Omega}} = \langle F'_\epsilon(u_m) - F'_\epsilon(u), u_m - u \rangle_{H^s_{\epsilon, \Omega}}
    + \int_\Omega ((u_m)^p_{p+1} - u^p_{p+1})(u_m - u).
\]
Since \(\{u_m\}\) is bounded, \(u_m \to u\) in \(H^s_{\epsilon, \Omega}\) and \(F'_\epsilon(u_m) \to 0\), we have that
\[
    \langle F'_\epsilon(u_m) - F'_\epsilon(u), u_m - u \rangle_{H^s_{\epsilon, \Omega}} \to 0, \quad \text{as } m \to \infty.
\]
By Hölder inequality, it holds that
\[
    \left| \int_\Omega ((u_m)^p_{p+1} - u^p_{p+1})(u_m - u) \right| 
    \leq \|(u_m)^p_{p+1} - u^p_{p+1}\|_{L^{(p+1)/p}(\Omega)} \|u_m - u\|_{L^{p+1}(\Omega)} \to 0,
\]
as \(m \to \infty\). Therefore, we have that \(\|u_m - u\|_{H^s_{\epsilon, \Omega}} \to 0\) as \(m \to \infty\). It completes the proof. \(\Box\)

**Lemma 4.3.** There exists a \(\rho > 0\) such that \(F_\epsilon(u) > 0\) if \(0 < \|u\|_{H^s_{\epsilon, \Omega}} < \rho\) and \(F_\epsilon(u) \geq \beta > 0\) if \(\|u\|_{H^s_{\epsilon, \Omega}} = \rho\).

**Proof.** By Remark 3.2 and Sobolev embedding,
\[
    \int_\Omega u^p_{p+1} \, dx \leq \int_\Omega |u|^{p+1} \, dx \leq \|u\|^p_{H^s_{\epsilon, \Omega}}.
\]
Since \(p > 1\), the conclusion of this lemma holds. \(\Box\)

**Lemma 4.4.** For sufficiently small \(\epsilon > 0\), there exists a nonconstant function \(\phi \in H^s_{\epsilon, \Omega}\) and positive constants \(t_0\) such that \(F_\epsilon(t_0 \phi) = 0\) and \(F_\epsilon(t \phi) \leq C\epsilon^n\) if \(t \in [0, t_0]\). Here \(C\) is a constant depending on \(n, s, \Omega\).

To prove this lemma, we construct a special function. Without loss of generality, we assume that \(0 \in \Omega\). Let \(\epsilon \in (0, 1)\) is small enough so that \(B_{2\epsilon} \subset \Omega\). Define
\[
    \phi(x) = \begin{cases} 
        \epsilon^{-n}(1 - \epsilon^{-1}|x|) & \text{if } |x| < \epsilon, \\
        0 & \text{if } |x| \geq \epsilon.
    \end{cases}
\]

**Lemma 4.5.** For sufficiently small \(\epsilon\), \(\phi \in H^s_{\epsilon, \Omega}\). Precisely, we have that
\[
    \|\phi\|_{H^s_{\epsilon, \Omega}} \leq \frac{C}{\epsilon^n},
\]
where \(C\) is a positive constant depending on \(n, s, p\) and \(\Omega\).
Proof. A direct calculus yields

\[
\int_{\Omega} \phi^2(x) dx = \int_{B_1} \frac{1}{\varepsilon^{2n}} \left( 1 - \frac{|x|}{\varepsilon} \right)^2 dx
\]

\[
= \frac{\omega_{n-1}}{\varepsilon^{2n}} \int_0^\varepsilon \left( 1 - \frac{r}{\varepsilon} \right)^2 r^{n-1} dr = \left( \frac{2\omega_{n-1}}{(n+1)(n+2)} \right) \frac{1}{\varepsilon^n}. \tag{4.8}
\]

Here \(\omega_{n-1}\) is the area of unit sphere in \(\mathbb{R}^n\). Thus it remains us to estimate

\[
\varepsilon^{2s} \int_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} |\nabla^s \phi|^2(x, y) dxdy = \frac{C_{n,s} \varepsilon^{2s}}{2} \int_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dxdy. \tag{4.9}
\]

Compute

\[
\int_{\mathbb{R}^{2n} \setminus (\Omega^c)^2} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dxdy
\]

\[
= \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dxdy + \int_{\Omega^c} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dxdy
\]

\[
+ \int_{\Omega} \int_{\Omega^c} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dxdy = \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dxdy + 2 \int_{\Omega^c} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dxdy. \tag{4.10}
\]

Calculate

\[
\int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} dxdy
\]

\[
= \int_{B_1} \int_{B_1} \frac{1}{\varepsilon^n} \left( 1 - \frac{|x|}{\varepsilon} \right) - \frac{1}{\varepsilon^{n}} \left( 1 - \frac{|y|}{\varepsilon} \right)^2 dxdy + \int_{\Omega \setminus B_1} \int_{B_1} \frac{|1}{\varepsilon^n} \left( 1 - \frac{|x|}{\varepsilon} \right)^2 dxdy
\]

\[
+ \int_{B_1} \int_{\Omega \setminus B_1} \frac{|1}{\varepsilon^n} \left( 1 - \frac{|y|}{\varepsilon} \right)^2 dxdy = \int_{B_1} \int_{B_1} \frac{1}{\varepsilon^n} \left( 1 - \frac{|x|}{\varepsilon} \right) - \frac{1}{\varepsilon^n} \left( 1 - \frac{|y|}{\varepsilon} \right)^2 dxdy + 2 \int_{\Omega \setminus B_1} \int_{B_1} \frac{1}{\varepsilon^n} \left( 1 - \frac{|x|}{\varepsilon} \right)^2 dxdy.
\]

\[
:= T_1 + T_2. \tag{4.11}
\]
Estimating $T_1$, we have that

$$T_1 = \frac{1}{\varepsilon^{2n+2}} \int_{B_1} \int_{B_1} \frac{|x| - |y|^2}{|x - y|^{n+2s}} \, dxdy$$

$$\leq \frac{1}{\varepsilon^{2n+2}} \int_{B_1} \int_{B_1} \frac{|x - y|^2}{|x - y|^{n+2s}} \, dxdy = \frac{1}{\varepsilon^{2n+2}} \int_{B_1} \int_{B_1} \frac{1}{|x - y|^{n+2s}} \, dxdy$$

$$\leq \frac{1}{\varepsilon^{2n+2}} \int_{B_2(\varepsilon)} \int_{B_2(\varepsilon)} \frac{1}{|x - y|^{n+2s}} \, dxdy = \frac{\omega_{n-1}}{\varepsilon^{2n+2}} \int_{B_1} \left\{ \int_0^{2\varepsilon} \frac{1}{r^{n+2s}} r^{n-1} \, dr \right\} dy$$

$$\leq \frac{C}{\varepsilon^{n+2s}}.$$ 

And calculate $T_2$:

$$T_2 = \frac{2}{\varepsilon^{2n+2}} \int_{\Omega \setminus B_2} \int_{B_1} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} \, dxdy$$

$$= \frac{2}{\varepsilon^{2n+2}} \int_{\Omega \setminus B_2} \int_{B_1} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} \, dxdy + \frac{2}{\varepsilon^{2n+2}} \int_{B_2 \setminus B_1} \int_{B_1} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} \, dxdy.$$ 

Then estimate

$$\frac{2}{\varepsilon^{2n+2}} \int_{\Omega \setminus B_2} \int_{B_1} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} \, dxdy$$

$$= \frac{2}{\varepsilon^{2n+2}} \int_{B_1} \int_{\Omega \setminus B_2} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} \, dxdy \leq \frac{2}{\varepsilon^{2n+2}} \int_{B_1} \int_{\Omega \setminus B_2} \frac{\varepsilon^2}{|y|^{n+2s}} \, dxdy$$

$$\leq \frac{2\omega_{n-1}}{\varepsilon^{2n}} \int_{B_1} \left\{ \int_0^{\infty} \frac{1}{r^{n+2s}} r^{n-1} \, dr \right\} dx = \frac{\omega_{n-1}}{s \varepsilon^{2n+2}} \int_{B_1} dx$$

$$\leq \frac{C}{\varepsilon^{n+2s}}.$$ 

and

$$\frac{2}{\varepsilon^{2n+2}} \int_{B_2 \setminus B_1} \int_{B_1} \frac{(\varepsilon - |x|)^2}{|x - y|^{n+2s}} \, dxdy$$

$$\leq \frac{2}{\varepsilon^{2n+2}} \int_{B_2 \setminus B_1} \int_{B_1} \frac{|y|}{|x - y|^{n+2s}} \, dxdy \leq \frac{2}{\varepsilon^{2n+2}} \int_{B_2 \setminus B_1} \int_{B_1} \frac{1}{|x - y|^{n+2s}} \, dxdy$$

$$\leq \frac{2}{\varepsilon^{2n+2}} \int_{B_2 \setminus B_1} \int_{B_2(\varepsilon)} \frac{1}{|y|^{n+2s}} \, dxdy = \frac{2\omega_{n-1}}{\varepsilon^{2n+2}} \int_{B_2 \setminus B_1} \left\{ \int_0^{3\varepsilon} \frac{1}{r^{n+2s}} r^{n-1} \, dr \right\} dy$$

$$= \frac{2 \cdot 3^{2-2s} \omega_{n-1}}{\varepsilon^{2n+2}} \int_{B_2 \setminus B_1} \int_{B_2 \setminus B_1} dx = \frac{2 \cdot 3^{2-2s}(2^n - 1)\omega_{n-1}}{n \varepsilon^{2n+2}}.$$ 

Therefore, we obtain that

$$\int_{\Omega} \int_{\Omega} \frac{(|\phi(x) - \phi(y)|^2)}{|x - y|^{n+2s}} \, dxdy = T_1 + T_2 \leq \frac{C}{\varepsilon^{n+2s}},$$

(4.12)
where $C$ is a positive constant depending on $n, s, \Omega$.

Finally, from $B_{2\varepsilon} \subset \Omega$, we have
\[
2 \int_{\Omega} \int_{\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 2 \int_{\Omega} \int_{B_{2\varepsilon}} \frac{1}{|x - y|^{n+2s}} \, dx \, dy \leq \frac{C}{\varepsilon^{n+2s}} \int_{B_{2\varepsilon}} \frac{1}{|y|^{n+2s}} \, dx \leq \frac{C}{\varepsilon^{n+2s}}.
\]

Summarizing the estimates (4.9), (4.10), (4.11), (4.12) and (4.13), we have that
\[
\varepsilon^{2s} \int_{\Omega} (\nabla^s \phi(x, y))^2 \, dx \, dy \leq \frac{C}{\varepsilon^n},
\]
where $C$ is a positive constant depending on $n, s, \Omega$. From (4.8) and (4.14), we obtain (4.7). This completes the proof. \qed

Define
\[
g(t) = F_\varepsilon(t\phi), \quad t \geq 0.
\]

**Lemma 4.6.** Assume that $\varepsilon > 0$ sufficiently small, there exist $t_1$ and $t_2$ with $0 < t_1 < t_2$ such that

1. for $t > t_1$, $g'(t) < 0$;
2. for $t > t_2$, $g(t) < 0$.

**Proof.** By a similar argument as in the proof of Lemma 4.5, we obtain that, for $t > 0$,
\[
\frac{C_{n,s}}{4} \int_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|t\phi(x) - t\phi(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + \frac{1}{2} \int_{\Omega} (t\phi)^2 \, dx \leq C_0 t^2 \varepsilon^{-n},
\]
where $C_0$ is a positive constant depending on $n, s, \Omega$. Moreover,
\[
\int_{\Omega} (t\phi)^{p+1} \, dx = \frac{t^{p+1}}{\varepsilon^{n(p+1)}} \int_{B_{2\varepsilon}} \left(1 - \frac{|x|}{\varepsilon}\right)^{p+1} \, dx = \frac{\omega_{n-1} t^{p+1}}{\varepsilon^{n(p+1)}} \int_0^\varepsilon \left(1 - \frac{r}{\varepsilon}\right)^{p+1} r^{n-1} \, dr = \frac{\omega_{n-1} t^{p+1}}{\varepsilon^{np}} \int_0^1 (1 - \rho)^{p+1} \rho^{n-1} \, d\rho = \frac{\alpha \omega_{n-1} t^{p+1}}{\varepsilon^{np}},
\]
where $\alpha := \int_0^1 (1 - \rho)^{p+1} \rho^{n-1} \, d\rho$. Let $t_2 = \left(\frac{C_{n(p+1)}}{\alpha \omega_{n-1}}\right)^{1/(p+1)} \varepsilon^n$. Then for all $t > t_2$, it holds that $g(t) = F_\varepsilon(t\phi) < 0$. 


Next compute
\[
g'(t) = \frac{C_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (\Omega)^2} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} \, dx \, dy + t \int_{\Omega} \phi^2 \, dx - t^p \int_{\Omega} \phi^{p+1} \, dx
\]
\[
\leq 2C_0 t \varepsilon^{-n} - \frac{\alpha \omega_{n-1} t^p}{\varepsilon^{np}}.
\]
Let \( t_1 = \left( \frac{2C_0}{\alpha \omega_{n-1}} \right)^{\frac{1}{p-1}} \varepsilon^n \). Thus if choosing \( t > t_1 \), we have that \( g'(t) < 0 \). Since \( p > 1 \), it holds that \( t_1 < t_2 \). This completes the proof. \( \square \)

**Proof of Lemma 4.4.** By Lemma 4.3, it holds that \( g(t) > 0 \) if \( t \) is positive and sufficiently small. Then from Lemma 4.6, we have that there exists a \( t_0 > 0 \) such that \( g(t_0) = 0 \). Moreover, estimate
\[
\max_{t \geq 0} g(t) = \max_{0 \leq t \leq t_1} g(t)
\]
\[
\leq \max_{0 \leq t \leq t_1} \left\{ C_0 t^2 \varepsilon^{-n} - \frac{1}{p+1} \int_{\Omega} (t\phi)^{p+1} \, dx \right\}
\]
\[
\leq \max_{0 \leq t \leq t_1} C_0 t^2 \varepsilon^{-n}
\]
\[
= C_0 t_1^2 \varepsilon^{-n} = C_1 \varepsilon^n,
\]
where \( C_1 = C_0 \left( \frac{2C_0}{\alpha \omega_{n-1}} \right)^{\frac{2}{p-1}} \). It completes the proof. \( \square \)

**Proof of Proposition 4.1.** (1) Let \( e = t_0 \phi \) and \( \Gamma = \{ \gamma \in C([0,1], H^{s \epsilon, \Omega}) \mid \gamma(0) = 0 \text{ and } \gamma(1) = e \} \). Then by Mountain Pass Theorem, we have that
\[
c := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} F_\varepsilon(\gamma(s)) > 0
\]
is a critical value of \( F_\varepsilon \). Then there exists a critical point \( u_\varepsilon \) such that
\[
F_\varepsilon(u_\varepsilon) = c \leq C_1 \varepsilon^n.
\]
Note that the unique constant solution to (4.1) is \( u \equiv 1 \) on \( \mathbb{R}^n \). A direct calculate yields
\[
F_\varepsilon(1) = \left( \frac{1}{2} - \frac{1}{p+1} \right) |\Omega| > 0,
\]
where \( |\Omega| \) denotes the volume of \( \Omega \). Thus for \( \varepsilon \) small enough, we have that
\[
F_\varepsilon(u_\varepsilon) < F_\varepsilon(1).
\]
Therefore, \( u_\varepsilon \) is a nonconstant solution to (1.1).

(2) Since \( u_\varepsilon \) is a solution to (4.1), we have that
\[
\varepsilon^{2s} \int_{\mathbb{R}^{2n} \setminus (\Omega)^2} |\nabla s u_\varepsilon|^2(x,y) \, dx \, dy + \int_{\Omega} u_\varepsilon^2 \, dx = \int_{\Omega} (u_\varepsilon)^{p+1} \, dx.
\]
Then by the definition of $F_{\varepsilon}$, it holds that

$$F_{\varepsilon}(u_\varepsilon) = \frac{1}{2} \left( \varepsilon^{2s} \int_{\mathbb{R}^{2n}\backslash(\Omega^c)^2} |\nabla^s u_\varepsilon|^2(x, y) dx dy + \int_{\Omega} u_\varepsilon^2 dx \right) - \frac{1}{p+1} \int_{\Omega} (u_\varepsilon)^{p+1} dx$$

$$= \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \varepsilon^{2s} \int_{\mathbb{R}^{2n}\backslash(\Omega^c)^2} |\nabla^s u_\varepsilon|^2(x, y) dx dy + \int_{\Omega} u_\varepsilon^2 dx \right)$$

$$= \left( \frac{1}{2} - \frac{1}{p+1} \right) \|u_\varepsilon\|_{H^s_{\varepsilon, \Omega}}^2.$$ 

Thus, choosing $C_2 = 2C_1 \left( \frac{p+1}{p-1} \right)$, we obtain (4.3). This completes the proof. \(\square\)

5. Integrability of $u_\varepsilon$

In this section, we prove Theorem 1.3.

Proof of Theorem 1.3. (1) Without loss of generality, we assume that $0 \in \Omega$. Then there exists $R_0 > 0$ such that $B_{2R_0} \subset \Omega$. Since $u \in H^s_{\varepsilon, \Omega}$, it holds that

$$I := \int_{\Omega} \int_{\Omega^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy dx < +\infty.$$ 

Particularly,

$$\int_{B_{R_0}} \int_{\Omega^c \cap \Omega^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy dx \leq I < +\infty.$$ 

Here $\rho > 0$ is constant and $\Omega_\rho = \{ y \in \mathbb{R}^n \mid d(y, \Omega) < \rho \}$. A direct computation yields

$$\int_{B_{R_0}} \int_{\Omega^c \cap \Omega^c} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dy dx \geq \int_{B_{R_0}} \int_{\Omega_\rho \cap \Omega^c} \frac{|u(x)|^2}{d(x) + \rho} dy dx + \int_{B_{R_0}} \int_{\Omega_\rho \cap \Omega^c} \frac{|u(y)|^2}{|x-y|^{n+2s}} dy dx - 2\int_{B_{R_0}} \int_{\Omega_\rho \cap \Omega^c} \frac{|u(x)u(y)|}{|x-y|^{n+2s}} dy dx := T_1 + T_2 - T_3.$$ 

We now estimate these three terms. Firstly,

$$T_1 = \int_{B_{R_0}} |u(x)|^2 \left\{ \int_{\Omega_\rho \cap \Omega^c} \frac{1}{d(x) + \rho} dy \right\} dx \geq \frac{|\Omega_\rho \cap \Omega^c|}{(d(\Omega) + \rho)^{n+2s}} \int_{B_{R_0}} |u(x)|^2 dx := a.$$ 

where $d(\Omega)$ denotes the diameter of $\Omega$ and $|\Omega_\rho \cap \Omega^c|$ is the volume of $\Omega_\rho \cap \Omega^c$. Note that $\int_{B_{R_0}} |u(x)|^2 dx < \infty$ since $\|u\|_{H^s(\Omega)} \leq \|u\|_{H^s_{\varepsilon, \Omega}}$. Thus $a$ is a nonnegative constant.
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depending on \( \Omega, R_0, \rho, n, s \). Secondly,

\[
T_2 \geq \frac{1}{(d(\Omega) + \rho)^{n+2s}} \int_{B_{R_0}} \int_{\Omega \cap \Omega^c} |u(y)|^2 dy dx
\]

\[
= \frac{|B_{R_0}|}{(d(\Omega) + \rho)^{n+2s}} \int_{\Omega \cap \Omega^c} |u(y)|^2 dy
\]

\[
:= b \int_{\Omega \cap \Omega^c} |u(y)|^2 dy.
\]

Here \( b \) is a positive constant depending on \( \Omega, R_0, \rho, n, s \). Finally,

\[
T_3 = 2 \int_{B_{R_0}} |u(x)| \int_{\Omega \cap \Omega^c} \frac{|u(y)|}{|x - y|^{n+2s}} dy dx
\]

\[
\leq \frac{2}{d(B_{R_0}, \partial \Omega)^{n+2s}} \int_{B_{R_0}} |u(x)| \left\{ \int_{\Omega \cap \Omega^c} |u(y)| dy \right\} dx
\]

\[
= \frac{2}{d(B_{R_0}, \partial \Omega)^{n+2s}} \int_{\Omega \cap \Omega^c} |u(y)| dy
\]

\[
:= c \int_{\Omega \cap \Omega^c} |u(y)| dy,
\]

where \( c \) is also a nonnegative constant depending on \( \Omega, R_0, \rho, n, s \). Therefore, we obtain that

\[
I \geq a + b \int_{\Omega \cap \Omega^c} |u(y)|^2 dy - c \int_{\Omega \cap \Omega^c} |u(y)| dy. \tag{5.1}
\]

Note that by the proof of (5.1), it holds that, for any \( X \subset \Omega \cap \Omega^c \),

\[
I \geq a + b \int_X |u(y)|^2 dy - c \int_X |u(y)| dy.
\]

We then argue by contradiction. Assume that \( u \notin L^2(\Omega \cap \Omega^c) \), that is

\[
\int_{\Omega \cap \Omega^c} |u(y)|^2 dy = +\infty.
\]

From (5.1), we have that \( u \notin L^1(\Omega \cap \Omega^c) \). Let

\[
A_k := \{ y \in \Omega \cap \Omega^c \mid |u(y)| > 2^k \}
\]

and

\[
D_k := A_k \setminus A_{k+1} = \{ y \in \Omega \cap \Omega^c \mid 2^k < |u(y)| \leq 2^{k+1} \}.
\]

Set \( d_k \) to be the measure of \( D_k \). Let \( N_1 \) be a positive integer such that \( 2^{N_1 - 1} > \frac{\varepsilon}{b} \). Then, for all \( N_2 > N_1 \),

\[
\int_{A_{N_1} \setminus A_{N_2}} |u(y)| dy = \sum_{k=N_1}^{N_2} \int_{A_k \setminus A_{k+1}} |u(y)| dy \leq \sum_{k=N_1}^{N_2} 2^{k+1} d_k \to +\infty, \quad \text{as } N_2 \to \infty, \tag{5.2}
\]
and
\[ \int_{A_{N_1} \setminus A_{N_2}} |u(y)|^2 dy = \sum_{k=N_1}^{N_2} \int_{A_k \setminus A_{k+1}} |u(y)|^2 dy \geq \sum_{k=N_1}^{N_2} 2^{2k} d_k \to +\infty, \quad \text{as } N_2 \to \infty. \]

Since
\[ \sum_{k=N_1}^{N_2} 2^{2k} d_k > 2^{N_1-1} \sum_{k=N_1}^{N_2} 2^{k+1} d_k, \]
we have that
\[ I \geq a + b \int_{A_{N_1} \setminus A_{N_2}} |u(y)|^2 dy - c \int_{A_{N_1} \setminus A_{N_2}} |u(y)| dy \]
\[ > a + 2^{N_1-1} b \sum_{k=N_1}^{N_2} 2^{k+1} d_k - c \sum_{k=N_1}^{N_2} 2^{k+1} d_k. \]

It is a contradiction to (5.2). Therefore, \( u \in L^2(\Omega_\rho \cap \Omega^c) \). Note that \( u \in L^2(\Omega) \), we obtain that \( u \in L^2(\Omega_\rho) \). Since \( \rho \) is arbitrary, it follows that \( u \in L^2_{\text{loc}}(\mathbb{R}^n) \).

(2) Let \( R \) be a positive constant such that \( \Omega \subset B_R(0) \). By the proof of Proposition 3.13 in [18], we know that if \( \mathcal{N}_s(u) = 0 \), then \( u \) is bounded in \( B_R(0) \) with
\[ \lim_{|x| \to \infty} u(x) = \frac{1}{|\Omega|} \int_{\Omega} u(x) dx \quad \text{uniformly in } x. \]

Therefore,
\[ \int_{B_R(0)} \frac{|u(x)|}{1 + |x|^{n+2s}} dx \leq C \int_{B_R(0)} \frac{1}{1 + |x|^{n+2s}} dx < +\infty, \]
where \( C \) is a constant such that \( \sup_{B_R(0)} |u| \leq C \). So, we only need to consider \( u \) on \( B_R(0) \). From the conclusion (1), it follows that
\[ u|_{B_R(0)} \in L^2(B_R(0)). \]

Thus, we have that
\[ \int_{B_R(0)} \frac{|u(x)|}{1 + |x|^{n+2s}} dx < +\infty. \]

This completes the proof. \( \square \)

**Remark 5.1.** Theorem 1.3 yields that the solution \( u_\varepsilon \) obtained in Section 4 is regular enough to use the local realization method for fractional Laplacian in Caffarelli and Silvestre [8].
6. Nonnegativity of $u_\varepsilon$

In this section, we prove that $u_\varepsilon$ is nonnegative by a weak maximum principle. Let $\mathbb{R}^{n+1}_+ = \{(x_1, \ldots, x_n, y) \in \mathbb{R}^{n+1} : y > 0\}$. Then $\partial \mathbb{R}^{n+1}_+ = \mathbb{R}^n$. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the extension function $U : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ that satisfies the following problem:

$$\begin{cases}
\text{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\
U(x, 0) = f(x) & \text{on } \partial \mathbb{R}^{n+1}_+,
\end{cases}$$

where $a = 1 - 2s$. By Poisson formula ([8, Section 2.4]), we have that

$$U(x, y) = \int_{\mathbb{R}^n} P(x - \xi, y) f(\xi) d\xi,$$

where $P$ is the Poisson kernel

$$P(x, y) = C_{n,a} \frac{y^{1-a}}{(|x| + |y|)^{n+1-a}}.$$

Furthermore, it holds that

$$- \lim_{y \to 0^+} y^a \partial_y U(x, y) = (-\Delta)^s U(x, 0) = (-\Delta)^s f(x).$$

For the details of proof, see [8, Section 3.1].

We now choose $f = u_\varepsilon$ and

$$U_\varepsilon(x, y) = \int_{\mathbb{R}^n} P(x - \xi, y) u_\varepsilon(\xi) d\xi.$$

Then we obtain that $U_\varepsilon$ satisfies (6.1) and

$$- \lim_{y \to 0^+} y^a \partial_y U(x, y) = (-\Delta)^s u_\varepsilon(x).$$

Let $\mathcal{C}_{\Omega,1} = \Omega \times (0, 1) \subset \mathbb{R}^{n+1}_+$. Note that $\partial \mathcal{C}_{\Omega,1} \cap \partial \mathbb{R}^{n+1}_+ = \Omega$. We consider the problem

$$\begin{cases}
\text{div}(y^a \nabla v) = 0 & \text{in } \mathcal{C}_{\Omega,1}, \\
- \lim_{y \to 0^+} y^a \partial_y v = h & \text{on } \Omega.
\end{cases}$$

**Definition 6.1.** Given a function $h \in L^1(\Omega)$, we say that $v$ is a weak solution to (6.2) if

$$y^a |\nabla v|^2 \in L^1(\mathcal{C}_{\Omega,1})$$

and

$$\int_{\mathcal{C}_{\Omega,1}} y^a \nabla v \cdot \nabla w - \int_{\Omega} hw = 0$$

for all $w \in C^\infty(\mathcal{C}_{\Omega,1})$ such that $w \equiv 0$ on $\partial \mathcal{C}_{\Omega,1} \setminus \Omega$.

**Lemma 6.2.** If $h(x) = (u_\varepsilon)^p_+(x) - u_\varepsilon(x)$, then $U_\varepsilon$ is a weak solution to (6.2).
Proof. Since \((-\Delta)^s u_\varepsilon + u_\varepsilon = (u_\varepsilon)^p\) in \(\Omega\), we have that
\[
-\lim_{y \to 0^+} y^a \partial_y U(x, y) = h(x) \quad \text{on } \Omega.
\]
Therefore, it holds that, for all \(w \in C^\infty(\mathcal{C}_{\Omega,1})\) such that \(w \equiv 0\) on \(\partial \mathcal{C}_{\Omega,1}\setminus \Omega\),
\[
\int_{\mathcal{C}_{\Omega,1}} y^a \nabla U_\varepsilon \cdot \nabla w = -\int_{\mathcal{C}_{\Omega,1}} \text{div}(y^a \nabla U_\varepsilon)w + \int_{\partial \mathcal{C}_{\Omega,1}} [(y^a \nabla U_\varepsilon) \cdot \mathbf{n}]w
\]
\[
= -\int_{\Omega} \lim_{y \to 0^+} y^a \partial_y U_\varepsilon(x, y) w.
\]
Here \(\mathbf{n}\) denotes the unit out normal vectors on \(\partial \mathcal{C}_{\Omega,1}\). It follows that \(U_\varepsilon\) is a weak solution to (6.2). This completes the proof. \(\square\)

We now prove the following

Proposition 6.3 (weak maximum principle). It holds that \(u_\varepsilon \geq 0\) on \(\Omega\).

Proof. By Lemma 6.2, \(U_\varepsilon\) is a weak solution to (6.2). Then, choosing \((U_\varepsilon)_- = \min\{U_\varepsilon, 0\}\) as a test function, we have that
\[
0 = \int_{\mathcal{C}_{\Omega,1}} y^a \nabla U_\varepsilon \cdot \nabla (U_\varepsilon)_- - \int_{\Omega} ((U_\varepsilon)^p_+(x, 0) - U_\varepsilon(x, 0)) (U_\varepsilon)_-(x, 0)
\]
\[
= \int_{\mathcal{C}_{\Omega,1}} y^a |\nabla (U_\varepsilon)_-|^2 + \int_{\Omega} (U_\varepsilon)^2_+(x, 0).
\]
It yields that \(U_\varepsilon(x, 0) = u_\varepsilon(x) \geq 0\) on \(\Omega\). This completes the proof. \(\square\)

Proof of Theorem 1.1. From Proposition 6.3, \(u_\varepsilon\) is nonnegative. Therefore, \(u_\varepsilon\) is a solution to the problem (1.1). Then it holds that
\[
I_\varepsilon(u_\varepsilon) = F_\varepsilon(u_\varepsilon) \leq C \varepsilon^n
\]
and
\[
\|u_\varepsilon\|_{\mathcal{H}_s^{\varepsilon, \Omega}} \leq C \varepsilon^{n/2},
\]
where \(C\) is a positive constant depending only on \(n, s, p, \Omega\). This completes the proof. \(\square\)

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