Closed linear spaces consisting of strongly norm attaining Lipschitz functionals

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Abstract
Given a pointed metric space $M$, we study when there exist $n$-dimensional linear subspaces of $\text{Lip}_0(M)$ consisting of strongly norm-attaining Lipschitz functionals, for $n \in \mathbb{N}$. We show that this is always the case for infinite metric spaces, providing a definitive answer to the question. We also study the possible sizes of such infinite-dimensional closed linear subspaces $Y$, as well as the inverse question, that is, the possible sizes for a metric space $M$ in order to such a subspace $Y$ exist. We also show that if the metric space $M$ is $\sigma$-precompact, then the aforementioned subspaces $Y$ need to be always separable and isomorphically polyhedral, and we show that for spaces containing $[0, 1]$ isometrically, they can be infinite-dimensional.

Keywords Norm-attaining Lipschitz functionals · Lipschitz-free Banach space

Mathematics Subject Classification 46B04 · 46B20 · 46B87

1 Introduction

1.1 Motivation

According to Rmoutil’s result [25], there exists an infinite-dimensional Banach space $X$ (namely $c_0$ in the equivalent norm constructed by Read [24]) such that the set $\text{NA}(X) \subset X^*$.
of norm-attaining linear functionals does not contain two-dimensional linear subspaces. That is a negative answer to [15, Problem III] by Godefroy. Read’s construction was generalized in [19], where such equivalent norms with “extremely nonlinearable set of norm-attaining functionals” were constructed in a number of other Banach spaces, in particular in all separable and some non-separable Banach spaces containing $c_0$.

We address an analogous question for metric spaces $M$ and the set $\text{SNA}(M)$ of strongly norm-attaining Lipschitz functionals. Surprisingly for the authors, for Lipschitz functionals the answer happens to be just the opposite: for every infinite $M$ the corresponding set $\text{Lip}_0(M)$ always has linear subspaces of dimension at least 2 consisting of strongly norm-attaining functionals, and in fact, it contains such subspaces of arbitrarily big finite dimension. After figuring out this new fact, we study some other natural questions about possible sizes for closed linear subspaces in $\text{SNA}(M)$.

It was shown in [7, Theorem 3.2] that if $M$ is an infinite metric space, then $\text{Lip}_0(M)$ contains linear subspaces isomorphic to $\ell_\infty$, and later, an isometric version of this result was given in [8, Theorem 5]. However, the proofs cannot be adapted to the setting of strongly norm-attaining Lipschitz mappings in general and, as we show in Theorem 2, for separable $M$, the non-separable space $\ell_\infty$ cannot be embedded in $\text{SNA}(M)$, neither isometrically nor isomorphically.

### 1.2 Notation and preliminaries

Let us quickly recall some standard notation from Banach space theory (see [9] for instance). All vector spaces in this document are assumed to be real. Let $X$ be a Banach space. We denote by $X^*$, $B_X$ and $S_X$ its topological dual, its closed unit ball, and its unit sphere, respectively. The notations $\text{conv}(A)$ and $\text{aconv}(A)$ respectively mean the convex hull and the absolute convex hull of $A$. Their closures with respect to a topology $\tau$ are denoted by $\text{conv}^\tau(A)$ and $\text{aconv}^\tau(A)$, respectively, and when $\tau$ is the usual topology, it will be omitted. If $X$ and $Y$ are two Banach spaces, $\mathcal{L}(X, Y)$ is the space of bounded linear operators from $X$ to $Y$, and $\text{NA}(X, Y)$ is the subset of $\mathcal{L}(X, Y)$ consisting of all those operators that attain their norms. We will use without additional explanation the standard notation like $\ell_1^n$ for corresponding classical Banach spaces. For $n \in \mathbb{N}$, $\ell_1^n$ denotes the $n$-dimensional version of $\ell_1$, that is, the space $\mathbb{R}^n$ endowed with the norm

$$
\| (x_1, \ldots, x_n) \|_1 = \sum_{k=1}^n |x_k|,
$$

and $\ell_\infty^n$ is defined analogously with the norm $\| \cdot \|_\infty$,

$$
\| (x_1, \ldots, x_n) \|_\infty = \max_{1 \leq k \leq n} |x_k|.
$$

Recall also that $\ell_1^2$ is isometrically isomorphic to $\ell_\infty^2$.

We will also use the following concepts. Given a Banach space $X$, a set $B \subset B_{X^*}$ is a James boundary of $X$ if for every $x \in X$, there is $g \in B$ such that $g(x) = \| x \|$ (see [9, Definition 3.118]). A metric space is said to be $\sigma$-precompact if it is a countable union of precompact sets, and a metric space $M$ is said to have the small ball property if for every $\varepsilon_0 > 0$, it is possible to write $M$ as a union of a sequence $(B(x_n, r_n))_n$ of closed balls such that $r_n < \varepsilon_0$ for all $n$ and $r_n \xrightarrow{n \to \infty} 0$. It is known that $\sigma$-precompact spaces have the small ball property but the converse is not true in general (see [2, Theorem 5.2]). Finally, recall that a Banach space $X$ is polyhedral if the unit ball of every finite-dimensional subspace of
$X$ is a polytope. A space that is isomorphic to a polyhedral space is said to be isomorphically polyhedral.

Let $(M, \rho)$ be a pointed metric space (that is, a metric space consisting of at least 2 points and containing a distinguished point 0). We will usually consider only one metric on $M$ which permits to write just $M$ for the metric space instead of $(M, \rho)$. We use the standard notation $\operatorname{Lip}_0(M)$ for the space of all Lipschitz mappings $f : M \to \mathbb{R}$ such that $f(0) = 0$ endowed with the Lipschitz constant as the norm, that is

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{\rho(x, y)} : x, y \in M, x \neq y \right\}.$$  

The interested reader can find a detailed study of Lipschitz spaces in the book [26].

There is a natural way to define the norm-attainment in this context. According to [20], a Lipschitz functional $f \in \operatorname{Lip}_0(M)$ is said to strongly attain its norm if there is a pair of points $x, y \in M$ with $x \neq y$ such that

$$\|f\| = \frac{|f(x) - f(y)|}{\rho(x, y)}.$$  

We will denote the set of Lipschitz functionals from $\operatorname{Lip}_0(M)$ that attain their norm strongly by $\operatorname{SNA}(M)$ (the notations $\operatorname{SA}(M)$ and $\operatorname{LipSNA}(M)$ have also been used in the literature).

The reason behind calling this natural norm-attainment strong is that this is a restrictive notion, and other weaker notions of norm-attainment, which are also natural and give interesting results, have also been introduced and studied since the initial works on the topic [17, 20] (see also [6, Section 1] for a very clean exposition of various kinds of norm-attainment for Lipschitz mappings and the relations between them).

The systematic study of norm-attaining Lipschitz mappings was started in [17, 20]. Since then, a fruitful line of research arose and continues to be very active nowadays. As we just mentioned, the notion of strong norm-attainment is a bit restrictive. This can be justified by the following facts:

- If a Lipschitz functional $f$ strongly attains its norm at some pair of points $x \neq y$, then $f$ strongly attains its norm at any pair of different points in between them (see [20, Lemma 2.2] for the details).
- If $M$ is a complete length metric space (that is, if for every $x \neq y \in M$, the distance $\rho(x, y)$ is equal to the infimum of the length of rectifiable curves joining them; note that every normed space is a metric length space), then $\operatorname{SNA}(M)$ is never dense in $\operatorname{Lip}_0(M)$ (for the details, see [3, Theorem 2.2], which improves [20, Theorem 2.3], and check also [1, 14, 18] for more background and characterizations on complete length metric spaces).

However, despite the above results, positive results have also been achieved in this direction in the recent years for some metric spaces (see for instance [3, Section 3] and [5]). Evidently, if $M$ is finite, then $\operatorname{SNA}(M) = \operatorname{Lip}_0(M)$, which means that it is a linear space, so we are mainly interested in infinite metric spaces.

An important tool in the study of Lipschitz mappings is the concept of Lipschitz-free spaces (also referred to as Arens-Eells spaces and Transportation cost spaces in the literature). Given a metric space $M$, denote $\delta : M \to (\operatorname{Lip}_0(M))^\ast$ the canonical embedding given by $\delta(x) = \delta_x$, $x \in M$, where $\delta_x$ is the evaluation functional $f \mapsto f(x)$. Then the norm-closed linear span $\mathcal{F}(M)$ of $\delta(M)$ in $\operatorname{Lip}_0(M)$ is called the Lipschitz-free space over $M$. The space $\mathcal{F}(M)$ can be seen isometrically as a predual of $\operatorname{Lip}_0(M)$ (see [3, Section 1] and the survey [16] for a solid background on Lipschitz-free spaces). The identification $(\mathcal{F}(M))^\ast = \operatorname{Lip}_0(M)$ can be explained as follows: every Lipschitz mapping $f : M \to \mathbb{R}$ can be identified with
the continuous linear mapping \( \hat{f} : \mathcal{F}(M) \rightarrow \mathbb{R} \) given by \( \hat{f}(\delta_p) \mapsto f(p) \) for \( p \in M \) and extended to the whole \( \mathcal{F}(M) \) by linearity and continuity. This identifies isometrically the spaces Lip\(_0\)(\( M \)) and (\( \mathcal{F}(M) \))^\ast. It is easy to check that SNA\((M)\) can be identified with the set of those elements of \( \mathcal{L}(\mathcal{F}(M), \mathbb{R}) \) that attain their norm at a point of the form \( \frac{\delta_x - \delta_y}{\rho(x, y)} \) for different \( x, y \in M \). This identification has been used to get many results about strongly norm-attaining Lipschitz mappings (see for instance [3, Section 3], [4, Section 2], [5], [12, Section 7] and [13, Section 4]). We refer to [8, 21, 23] for works where the possibility to embed \( \ell_1 \) into Lipschitz-free spaces was studied in depth.

Remark also, that the structure of Lip\(_0\)(\( M \)) and SNA\((M)\) does not depend on the selection of the distinguished point 0: if \( M' \) is the same metric space but with another distinguished point 0′ then the mapping \( f \mapsto f - f(0) \) is a bijective linear isometry between Lip\(_0\)(\( M \)) and Lip\(_0\)(\( M' \)), which maps SNA\((M)\) to SNA\((M')\) in fact, and it is \( \omega^* - \omega^* \) continuous, which proves that the Lipschitz-free spaces do not depend on the choice of the origin either.

Another important tool is the well-known McShane’s extension theorem [26, Theorem 1.33] that allows us to extend \( f \in \text{Lip}_0(M_1) \) to \( \bar{f} \in \text{Lip}_0(M_2) \), with \( M_1 \subset M_2 \), in such a way that \( \|f\| = \|\bar{f}\| \).

Given a metric space \( M \), in this text, the expression linear subspaces of SNA\((M)\) should be understood as linear subspaces of Lip\(_0\)(\( M \)) consisting of strongly norm-attaining Lipschitz functionals. Also, we use below the following slang. Let \( Y \) be a Banach space and \( M \) be a pointed metric space. We say that \( Y \) embeds in SNA\((M)\) (or equivalently SNA\((M)\) contains a copy of \( Y \)), if there is a linear isometric embedding \( U : Y \rightarrow \text{Lip}_0(M) \) such that \( U(Y) \subset \text{SNA}(M) \).

### 1.3 The structure of the article

The rest of the paper is structured in three sections as follows.

In Sect. 2, we ask for what metric spaces \( M \) there exist \( n \)-dimensional linear subspaces inside of SNA\((M)\), for \( n \in \mathbb{N} \). The main result of the section, Theorem 1, asserts that if \( M \) contains at least \( 2^n \) points, then SNA\((M)\) contains a subspace isometric to \( \ell_1^n \), and so, Corollary 2 characterizes that SNA\((M)\) contains \( n \)-dimensional linear subspaces if and only if \( M \) has more than \( n \) points, providing a definitive answer to our question.

In Sect. 3 we study possible sizes for linear subspaces of SNA\((M)\) for metric spaces \( M \), and in particular, if there can be infinite-dimensional or even non-separable subspaces. We show in Proposition 1 that, actually, any Banach space \( Y \) is a subspace of SNA\((M)\) for a suitable metric space \( M \). We also tackle some kind of an inverse problem: how “small” a metric space \( M \) can be so that a given Banach space \( Y \) is a subspace of SNA\((M)\). We give a characterization for this in Theorem 2. We also study similar questions for some particular classes of metric spaces. In particular, we show in Theorem 3 that if \( M \) is a \( \sigma \)-precompact pointed metric space, then any closed linear subspace \( Y \) of SNA\((M)\) is separable and isomorphically polyhedral. We also show in Proposition 4 that in any metric space that contains \([0, 1]\) isometrically, such closed linear subspaces can be chosen to be infinite-dimensional.

Finally, in Sect. 4, we exhibit some remarks and questions that remain open for now despite our attempts to solve them.
2 Finite-dimensional subspaces

In this section, we will study the existence of $n$-dimensional linear subspaces in $\text{SNA}(M)$, where $M$ is a pointed metric space and $n \in \mathbb{N}$. Our main result from the section states that if $M$ contains at least $2^n$ points (in particular, if $M$ is infinite), then $\text{SNA}(M)$ contains an isometric copy of $\ell_1^n$ (see Theorem 1). This provides a shocking contrast when compared to the classical theory of norm-attaining functionals, where Rmoutil showed that an infinite-dimensional Banach space $X$ introduced by Read satisfies that $\text{NA}(X, \mathbb{R})$ has no 2-dimensional subspaces (see [25]). In order to prove our main result in this direction, we need a bit of preparatory work.

First of all, recall that if a finite pointed metric space $M$ has exactly $n > 1$ distinct points, for some $n \in \mathbb{N}$, then $\text{Lip}_0(M) = \text{SNA}(M)$, and it is an $(n-1)$-dimensional Banach space.

**Remark 1** Note that, in general, we cannot claim that if a Banach space $Y$ is a linear subspace of $\text{SNA}(K)$ for some metric space $K$, then $Y$ is also linearly isometric to a subspace of $\text{SNA}(M)$ for metric spaces $M$ containing $K$ as a subspace. One may be tempted to use McShane’s extension theorem in order to try to get such a result, but the extensions do not behave well like a linear subspace in general. However, the well-behaving norm $\| \cdot \|_1$ will allow us to get a result in this direction, as Lemma 1 below shows.

**Lemma 1** Let $M$ be a pointed metric space such that for some subspace $K$ of $M$, $\text{SNA}(K)$ contains a linear subspace isometrically isomorphic to $\ell_1^n$ for some $n \in \mathbb{N}$. Then, $\text{SNA}(M)$ also contains a linear subspace isometrically isomorphic to $\ell_1^n$.

**Proof** Let $E \subset \text{Lip}_0(K)$ be a linear isometric copy of $\ell_1^n$ consisting of strongly norm-attaining functionals. Then, there are $f_1, \ldots, f_n \in S_E \subset S_{\text{Lip}_0(K)}$ such that for all $a_1, \ldots, a_n \in \mathbb{R}$,

$$\left\| \sum_{k=1}^n a_k f_k \right\| = \sum_{k=1}^n |a_k|.$$ 

Let $g_1, \ldots, g_n \in S_{\text{Lip}_0(M)}$ be norm-preserving extensions of $f_1, \ldots, f_n$ respectively. Then, by the triangle inequality, for all $a_1, \ldots, a_n \in \mathbb{R}$,

$$\left\| \sum_{k=1}^n a_k g_k \right\| \leq \sum_{k=1}^n |a_k|.$$ 

On the other hand, there is a pair of different points $t_1, t_2 \in K$ at which $\sum_{k=1}^n a_k f_k$ strongly attains its norm. This gives us

$$\left\| \sum_{k=1}^n a_k g_k \right\| = \frac{\left| \left( \sum_{k=1}^n a_k g_k \right)(t_1) - \left( \sum_{k=1}^n a_k g_k \right)(t_2) \right|}{\rho(t_1, t_2)} = \frac{\left| \left( \sum_{k=1}^n a_k f_k \right)(t_1) - \left( \sum_{k=1}^n a_k f_k \right)(t_2) \right|}{\rho(t_1, t_2)} = \sum_{k=1}^n |a_k|,$$

so $\sum_{k=1}^n a_k g_k$ strongly attains its norm. \qed

In particular, if we were able to embed $\ell_1^n$ spaces isometrically in $\text{SNA}(M)$ for a finite pointed metric space $M$, we could use the previous lemma to obtain the same result for all metric spaces containing $M$. 

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In the recent works [21, 23], the existence of $\ell^n_1$ and $\ell_1$ subspaces of Lipschitz-free spaces was studied in depth, providing an answer to [8, Question 2]. This has proven to be an important tool in our case, and we will use the cited below first half of [21, Theorem 14.5] in the proof of our main result. For the sake of completeness and easy reference, we include below the formal statement.

**Lemma 2** ([21, Theorem 14.5]) *For every $n \in \mathbb{N}$, if a pointed metric space $M$ contains $2n$ elements, then $\mathcal{F}(M)$ contains a 1-complemented subspace isometric to $\ell^n_1$.***

Recall that it is not true in general that if $Y$ is a subspace of a Banach space $X$, then $Y^*$ is isometric to a subspace of $X^*$ (for instance, recall that $\ell_1$ embeds isometrically in $C([0, 1])$, but $\ell_\infty$ does not embed isometrically in $C([0, 1])^*$); however, the scenario is different if $Y$ is 1-complemented.

**Lemma 3** Let $X$ be a Banach space that contains a 1-complemented subspace $Y$. Then $Y^*$ embeds isometrically as a subspace of $X^*$.

**Proof** Let $P : X \to Y$ be a norm-one projection. Consider the mapping $U : Y^* \to X^*$ such that for all $y^* \in Y^*$, $U(y^*) := y^* \circ P$, that is, for all $x \in X$, $U(y^*)(x) := y^*(P(x))$. Then $U$ is an isometric embedding, as desired. Indeed, just note that for each $y^* \in Y^*$, we have

$$
\|U(y^*)\| = \sup_{x \in B_X} \|U(y^*(x))\| = \sup_{x \in B_X} |y^*(P(x))| = \sup_{y \in B_Y} |y^*(y)| = \|y^*\|.
$$

□

Finally, recall the following well-known result, for which it is sufficient to consider the span of $n$ vectors in $\ell^{2n-1}_\infty$ with $\pm 1$ coordinates, built analogously to the Rademacher functions on $[0, 1]$.

**Lemma 4** If $n \in \mathbb{N}$, then $\ell^n_1$ is isometric to a subspace of $\ell^{2n-1}_\infty$.***

We now have all the necessary tools for the proof of the main result of the section.

**Theorem 1** *Let $n > 1$ be a natural number, and let $M$ be a pointed metric space with at least $2^n$ distinct points. Then, there exists a linear subspace of $\text{SNA}(M)$ which is isometrically isomorphic to $\ell^n_1$.***

**Proof** First of all, consider a metric subspace $K$ of $M$ containing exactly $2^n$ distinct points. By Lemma 2, $\mathcal{F}(K)$ contains a 1-complemented subspace isometric to $\ell^{2n-1}_1$. Recall that $\mathcal{F}(K)^*$ is isometric to $\text{Lip}_0(K)$, and that $(\ell^{2n-1}_1)^*$ is isometric to $\ell^{2n-1}_\infty$, so by Lemma 3, $\text{Lip}_0(K) = \text{SNA}(K)$, and it contains a subspace isometric to $\ell^{2n-1}_\infty$. Applying Lemma 4 we deduce that $\text{SNA}(K)$ contains a subspace isometric to $\ell^n_1$ as well. Finally, by Lemma 1, $\text{SNA}(M)$ also contains a subspace isometric to $\ell^n_1$. □

**Corollary 1** If $M$ is an infinite pointed metric space, then for all $n \in \mathbb{N}$, $\text{SNA}(M)$ contains an $n$-dimensional subspace isometric to $\ell^n_1$.

**Corollary 2** Let $n \in \mathbb{N}$. For a pointed metric space $M$, the following statements are equivalent:

1. $\text{SNA}(M)$ contains $n$-dimensional linear subspaces.
2. $M$ contains at least $n + 1$ points.
Remark 2 In the first preprint version of the document, we provided a completely constructive proof of Theorem 1 for \( n = 2 \). In order to do this, for any given metric space \( M \) with exactly 4 points, we provided two Lipschitz functionals \( f_1 \) and \( f_2 \) such that span \((f_1, f_2)\) is a subspace of \( \text{SNA}(M) \) isometric to \( \ell^2_1 \), and then we applied Lemma 1 for bigger metric spaces. At that preprint we left the case \( n > 2 \) as an open question despite conjecturing it to be true after our attempts to solve it. From that result, using the relation between \( \text{SNA}(M) \) and \( \text{NA}(\mathcal{F}(M), \mathbb{R}) \) and the fact that if \( X^* \) contains an isometric copy of \( \ell^2_1 \) consisting of norm-attaining functionals, then \( X \) contains an isometric copy of \( \ell^2_1 \), we deduced as a corollary that if \( M \) contains at least 4 points, then \( \mathcal{F}(M) \) contains an isometric copy of \( \ell^2_1 \), a result which we believed to be new. However, Mikhail Ostrovskii kindly pointed us out the existence of a result which is clearly stronger than the corollary we obtained in the preprint version [21, Theorem 14.5], and this has allowed us to significantly improve our original results by providing a definitive answer to our question. We are deeply indebted to Mikhail Ostrovskii for this.

It seems to us that the original direct construction of \( f_1 \) and \( f_2 \) mentioned above may be of independent interest, so we permit ourselves to give it below without proof.

Let \( M \) be a pointed metric space consisting of exactly 4 points. We enumerate the elements of \( M \) as \( x_1, x_2, x_3, x_4 \) in such a way that \( x_1 = 0 \) and

\[
\begin{align*}
\rho(x_1, x_4) + \rho(x_2, x_3) &= \min \{ \rho(x, \beta) + \rho(y, \delta) : M = \{x, \beta, y, \delta\} \}.
\end{align*}
\]

Then the requested \( f_1, f_2 \in \text{Lip}_0(M) \) whose span is isometric to \( \ell^2_1 \) can be chosen as follows:

\[
f_1 : \begin{cases} 
f_1(x_1) = 0, \\
f_1(x_2) = \rho(x_1, x_4) - \rho(x_2, x_4), \\
f_1(x_3) = \rho(x_1, x_4) - \rho(x_2, x_4) + \rho(x_2, x_3), \\
f_1(x_4) = \rho(x_1, x_4),
\end{cases}
\]

\[
f_2 : \begin{cases} 
f_2(x_1) = 0, \\
f_2(x_2) = \rho(x_1, x_2), \\
f_2(x_3) = \rho(x_1, x_2) - \rho(x_2, x_3), \\
f_2(x_4) = \rho(x_1, x_4).
\end{cases}
\]

3 Size of subspaces

We start this section by showing that there exist metric spaces \( M \) for which \( \text{SNA}(M) \) contains “big” Banach subspaces. Actually, any Banach space \( Y \) can be a subspace of \( \text{SNA}(M) \) for a suitable metric space \( M \).

Proposition 1 If \( Y \) is a Banach space, then it is a subspace of \( \text{SNA}(BY^*) \).

Proof Let \( Y \) be any Banach space. Consider the metric space \( BY^* \). For each \( y \in Y \), let \( \delta_y : BY^* \rightarrow \mathbb{R} \) be the evaluation map \( \delta_y(y^*) := y^*(y) \), for all \( y^* \in BY^* \), which is clearly a linear mapping. For each \( y \in Y \), there exists some \( y^* \in BY^* \) such that \( y^*(y) = \|y\| \). It is immediate to check that \( \delta_y \) is in \( \text{Lip}_0(BY^*) \) with Lipschitz constant \( \|y\| \), and that it attains its norm strongly at the pair \((0, y^*)\). Therefore, \( Y \) is a subspace of \( \text{SNA}(BY^*) \). □

A natural question arises now: given a Banach space \( Y \), how small can a metric space \( M \) be so that \( Y \) is a linear subspace of \( \text{SNA}(M) \)? From the previous proposition, it is clear that if \( Y \) has separable dual, then \( M \) can be chosen to be separable.
What if $Y^*$ is not separable? For instance, we have seen in Theorem 1 that if $M$ is an infinite pointed metric space, then $\text{SNA}(M)$ contains isometrically all the $\ell_1^n$ spaces as linear subspaces, so it is natural to wonder if it also contains, say, $\ell_1$. However, this is not the case in general, as we are about to see. Theorem 2 below shows that separability of $Y^*$ actually characterizes the possibility of $M$ being separable. In order to prove it, we rely on the concept of James boundary introduced in Sect. 1 (see [9, Definition 3.118]) and also on the following result by Gilles Godefroy.

**Proposition 2** (Godefroy, [9, Corollary 3.125]) *Let $X$ be a Banach space. If $X$ has a separable James boundary, then $X^*$ is separable.*

**Theorem 2** *For a Banach space $Y$, the following assertions are equivalent.

1. There is a separable pointed metric space $M$ and a closed linear subspace $Z \subset \text{Lip}_0(M)$ such that $Z$ is isometric to $Y$ and $Z \subset \text{SNA}(M)$.

2. There is a separable Banach space $X$ and a closed linear subspace $Z_1 \subset X^*$ such that $Z_1$ is isometric to $Y$ and $Z_1 \subset \text{NA}(X, \mathbb{R})$.

3. $Y^*$ is separable.

**Proof**

(1) implies (2): It is sufficient to consider $X = \mathcal{F}(M)$ and use the identification of $\text{Lip}_0(M)$ with $X^*$. With this identification $Z \subset \text{Lip}_0(M)$ identifies with a subspace of $Z_1 \subset X^*$ and all elements of $Z_1$ remain to be norm-attaining as elements of $X^*$.

(2) implies (3): Assume that such a separable Banach space $X$ exists, denote $J : X \to X^{**}$ the canonical embedding of $X$ into its bidual and $R : X^{**} \to Z_1^*$ the natural restriction operator. The condition $Z_1 \subset \text{NA}(X, \mathbb{R})$ means that for every $f \in Z_1$ there is $x \in B_X$ such that $f(x) = \|f\|$, so in other words $((R \circ J)(x))(f) = \|f\|$. Consequently, $(R \circ J)(B_X)$ is a separable James boundary of $Z_1$, so $Z_1^*$ is separable by Proposition 2. Hence, we have that $Y^*$ is separable.

(3) implies (1): If $Y^*$ is separable, take $M = B_{Y^*}$ and apply Proposition 1.

As a consequence of the above result, there exist infinite metric spaces $M$ such that $\text{SNA}(M)$ does not contain linear subspaces isometrically isomorphic to $\ell_1$.

**Remark 3** Note that a direct proof that (2) implies (1) in Theorem 2 can be achieved by considering $M = B_X$. In this case, the operator $U$ that maps each $f \in X^*$ to its restriction on $M$ is an isometric embedding with the property that if $f$ is norm-attaining then $U(f)$ is strongly norm-attaining on $M$. So the subspace $Z := U(Z_1)$ is what we are looking for.

The next Theorem 3 shows in a similar way that if $M$ is “small” then the restrictions on Banach subspaces in $\text{SNA}(M)$ happen to be much stronger. In the proof we will use [11, Corollary 2.2].

**Proposition 3** ([11, Corollary 2.2]) *If $X$ has a boundary that can be covered by a set of the form $\bigcup_{j=1}^\infty \text{conv}^{w^*}(K_j)$, where each $K_j$ is countably infinite and $w^*$-compact, then $X$ is isomorphically polyhedral.*

**Corollary 3** *If $X$ has a boundary that can be covered by a countable number of compact sets, then $X$ is separable and isomorphically polyhedral.*

**Proof** Let the boundary $W$ of $X$ be covered by $\bigcup_{j=1}^\infty W_j$ for compact sets $W_j$. Then, the boundary is separable, so by Godefroy’s result (Proposition 2), $X^*$ is separable, and then $X$ is separable as well.
According to [22, Proposition 1.e.2], every compact subset $W_j$ is included in a subset of the form $\overline{\text{conv}}\{x_{j,k}^*\}_{k \in \mathbb{N}} \subset X^*$ where $\|x_{j,k}^*\| \xrightarrow{k \to \infty} 0$. Thus, the boundary $W^*$ has the property from Proposition 3, so $X$ is isomorphically polyhedral. □

Remark that the same result follows from an “internal” characterization from [10].

**Theorem 3** Let $M$ be a $\sigma$-precompact pointed metric space, then all Banach subspaces in $\text{SNA}(M)$ are separable and isomorphic to polyhedral spaces.

**Proof** Let $\{M_n\}_{n=1}^\infty$ be a sequence of precompact sets satisfying that $M \subset \bigcup_{j=1}^\infty M_n$. For each $n \in \mathbb{N}$, denote $\Delta_n = \{\delta_x : x \in M_n\} \subset \text{Lip}_0(M)^*$. By our assumption, each $\Delta_n$ is precompact in $\text{Lip}_0(M)^*$. Then $\overline{\text{conv}}(\Delta_n - \Delta_m)$ is compact for every $n, m \in \mathbb{N}$. The set

$$\text{Mol} = \left\{ \frac{\delta_t - \delta_{\tau}}{\rho(t, \tau)} : t \neq \tau \in M \right\} \subset \bigcup_{m,n,k \in \mathbb{N}} k \cdot \overline{\text{conv}}(\Delta_n - \Delta_m)$$

is covered by a countable number of norm-compact sets.

Let $Y \subset \text{SNA}(M)$ be a Banach space. Denote $R : \text{Lip}_0(M)^* \to Y^*$ the natural restriction operator. Then by the continuity of $R$, $R(\text{Mol})$ is covered by a countable number of norm-compact sets as well. The set $R(\text{Mol}) \cap S_{Y^*}$ forms a boundary for $Y$ (by the definition of strong norm-attainment), so the statement follows from Corollary 3. □

Note that every compact space and every $\mathbb{R}^n$, with $n \in \mathbb{N}$, are $\sigma$-compact. In particular, every linear subspace in $\text{SNA}([0, 1])$ is separable and isomorphically polyhedral. It is worth noting that such subspaces can be infinite-dimensional, as we will see in Example 1.

Since all Lipschitz functions are absolutely continuous, one can identify (see for instance [26, Example 1.6.5]) the space $\text{Lip}_0([0, 1])$ isometrically with the space $L_\infty([0, 1])$, where the isometric isomorphism between them is just the differentiation operator (which exists almost everywhere):

$$U : \text{Lip}_0([0, 1]) \to L_\infty([0, 1])$$

$$f \mapsto U(f) = f'.$$

It is clear from this and [20, Lemma 2.2] that $U(\text{SNA}([0, 1]))$ is the subset of $L_\infty([0, 1])$ consisting of functions that attain their norm $\| \cdot \|_\infty$ throughout an interval with non-empty interior. We get the following result.

**Example 1** If $M = [0, 1]$, then $\text{SNA}(M)$ contains linear subspaces isometrically isomorphic to $c_0$.

**Proof** Consider the set $A$ of functions $g : [0, 1] \to \mathbb{R}$ such that the following holds for some $a = (a_1, a_2, \ldots) \in c_0$:

$$g(x) = a_k, \quad \text{if } x \in \left[1 - \frac{1}{k}, 1 - \frac{1}{k+1}\right], \quad k = 1, 2, \ldots$$

$$g(1) = 0$$

Then $(A, \| \cdot \|_\infty)$ is a linear subspace of $U(\text{SNA}([0, 1]))$ which is isometrically isomorphic to $c_0$, and this finishes the proof. □

Naturally, the previous example remains true if one changes $[0, 1]$ with $[a, b] \subset \mathbb{R}$.

Finally, the next result shows that one can extend the existence of $c_0$ in $\text{SNA}([0, 1])$ to $\text{SNA}(M)$ for any pointed metric space $M$ that contains $[0, 1]$ isometrically (for instance, any normed space).
Proposition 4 If $M$ is any pointed metric space containing $[0, 1]$ isometrically, then $\text{SNA}(M)$ contains linear subspaces isometrically isomorphic to $c_0$.

Proof Let $M$ and $Z$ be metric spaces such that $Z \subset M$. Assume that there exists some retraction $F : M \to Z$ with Lipschitz constant 1, that is:

$$\begin{align*}
F(z) &= z, & \text{for all } z \in Z, \\
\rho(F(a), F(b)) &\leq \rho(a, b), & \text{for all } a, b \in M.
\end{align*}$$

Let $T : \text{Lip}_0(Z) \to \text{Lip}_0(M)$ be such that for all $f \in \text{Lip}_0(Z)$, $T(f) := f \circ F$. Thus, for all $x \in M$, we have $(T(f))(x) = f(F(x))$. It is clear that $T$ is linear. Moreover, $T(\text{Lip}_0(Z))$ is a subspace of $\text{Lip}_0(M)$ (and $T(\text{SNA}(Z)$ is a subset of $\text{SNA}(M)$). Hence any linear subspace of $\text{Lip}_0(Z)$ yields a subspace of $\text{Lip}_0(M)$.

All that remains is to note that if $X$ is any metric space containing $[0, 1]$ isometrically, then the mapping $F$ exists. Indeed, the identity operator $\text{Id}$ on $[0, 1]$ is a Lipschitz function with constant 1, and by McShane’s extension theorem, it can be extended to the whole $X$ preserving its Lipschitz constant.

Observe that Proposition 4 applies to all normed spaces. This should be once more compared with the classical theory of norm-attaining functionals, where there exist Banach spaces $X$ such that $\text{NA}(X, \mathbb{R})$ does not have 2-dimensional subspaces (see [25]).

4 Open Questions and remarks

We have seen in Theorem 1 that if $M$ is any infinite pointed metric space, then $\text{SNA}(M)$ contains linear subspaces of every possible finite dimension. However, we do not know if every infinite pointed metric space $M$ satisfies that $\text{SNA}(M)$ contains infinite-dimensional linear subspaces, and Theorem 2 gives us some restrictions.

Question 1 Is it true that for every infinite complete pointed metric space $M$ the corresponding $\text{SNA}(M)$ contains infinite-dimensional closed (or at least non-closed) linear subspaces?

Note that for Question 1, there is no chance to have isometric copies of $\ell_1$ as a tool for its solution in general, since if the space $M$ is separable, the candidates for linear subspaces of $\text{SNA}(M)$ need to have separable dual (see Theorem 2). Actually, in all the examples of infinite $M$ that we were able to analyze in depth, $\text{SNA}(M)$ contains isomorphic copies of $c_0$, but we do not know if this is true in general.

Question 2 Is it true that for every infinite complete pointed metric space $M$ the corresponding $\text{SNA}(M)$ contains an isomorphic copy of $c_0$?

Let us comment that, as it was mentioned in Sect. 1.1, in [7, Theorem 3.2] it was shown that if $M$ is an infinite metric space, then $\text{Lip}_0(M)$ contains a linear subspace isomorphic to $\ell_\infty$, and an isometric version was achieved later in [8, Theorem 5]. However, if one examines the proofs, there is no chance for them to be applicable to $\text{SNA}(M)$ in general (we refer once more to Theorem 2).

It has been shown in Theorem 3 that if $M$ is a $\sigma$-precompact pointed metric space, then all subspaces of $\text{SNA}(M)$ are separable and isomorphically polyhedral. However, the authors do not know if the same holds for the weaker small ball property.

Question 3 Let $M$ be a pointed metric space with the small ball property. Is it true that all subspaces of $\text{SNA}(M)$ are separable and isomorphically polyhedral?
Note after review: A. Avilés, G. Martínez-Cervantes, A. Rueda Zoca, and P. Tradacete recently communicated to us that they were able to solve our Questions 1 and 2 affirmatively (see the preprint arXiv:2204.12529).

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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**References**

1. Avilés, A., Martínez-Cervantes, G.: Complete metric spaces with property (Z) are length spaces. J. Math. Anal. Appl. 473(1), 334–344 (2019). https://doi.org/10.1016/j.jmaa.2018.12.051
2. Behrends, E., Kadets, V.: Metric spaces with the small ball property. Stud. Math. 148(3), 275–287 (2001). https://doi.org/10.4064/sm148-3-6
3. Cascales, B., Chiclana, R., García-Lirola, L.C., Martín, M., Rueda-Zoca, A.: On strongly norm attaining Lipschitz maps. J. Funct. Anal. 277(6), 1677–1717 (2019). https://doi.org/10.1016/j.jfa.2018.12.006
4. Chiclana, R., García-Lirola, L.C., Martín, M., Rueda-Zoca, A.: Examples and applications of the density of strongly norm attaining Lipschitz maps. Rev. Mat. Iberoam. 37(5), 1917–1951 (2021). https://doi.org/10.4171/rmi/1253
5. Chiclana, R., Martín, M.: The Bishop-Phelps-Bollobás property for Lipschitz maps. Nonlinear Anal. 188, 158–178 (2019). https://doi.org/10.1016/j.na.2019.06.002
6. Choi, G., Choi, Y.S., Martín, M.: Emerging notions of norm attainment for Lipschitz maps between Banach spaces. J. Math. Anal. Appl. 483(1), Paper No. 123600 (2020). https://doi.org/10.1016/j.jmaa.2019.123600
7. Cúth, M., Doucha, M., Wojtaszczyk, P.: On the structure of Lipschitz-free spaces. Proc. Am. Math. Soc. 144(9), 3833–3846 (2016). https://doi.org/10.1090/proc/13019
8. Cúth, M., Johans, M.: Isometric embedding of $\ell_1$ into Lipschitz-free spaces and $\ell_\infty$ into their duals. Proc. Am. Math. Soc. 145(8), 3409–3421 (2017). https://doi.org/10.1090/proc/13590
9. Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V.: Banach Space Theory. The Basis for Linear and Nonlinear Analysis, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York (2011). https://doi.org/10.1007/978-1-4419-7515-7
10. Fonf, V.P.: Three characterizations of polyhedral Banach spaces. Ukrainian Math. J. 42(9), 1145–1148 (1990). https://doi.org/10.1007/BF01056615
11. Fonf, V.P., Smith, R.J., Troyanski, S.: Boundaries and polyhedral Banach spaces. Proc. Am. Math. Soc. 143(11), 4845–4849 (2015). https://doi.org/10.1090/proc/12644
12. García-Lirola, L.C., Petitjean, C., Procházka, A., Rueda Zoca, A.: Extremal structure and duality of Lipschitz free spaces. Mediterr. J. Math. 15(2), Paper No. 69 (2018). https://doi.org/10.1007/s00009-018-1113-0
13. García-Lirola, L.C., Petitjean, C., Rueda Zoca, A.: On the structure of spaces of vector-valued Lipschitz functions. Stud. Math. 239(3), 249–271 (2017). https://doi.org/10.4064/sm8694-1-2017
14. García-Lirola, L.C., Procházka, A., Rueda Zoca, A.: A characterisation of the Daugavet property in spaces of Lipschitz functions. J. Math. Anal. Appl. 464(1), 473–492 (2018). https://doi.org/10.1016/j.jmaa.2018.04.017
15. Godefroy, G.: The Banach space $c_0$. Extracta Math. 16(1), 1–25 (2001)
16. Godefroy, G.: A survey on Lipschitz-free Banach spaces. Comment. Math. 55(2), 89–118 (2015). https://doi.org/10.14708/cm.v55i2.1104
17. Godefroy, G.: On norm attaining Lipschitz maps between Banach spaces. Pure Appl. Funct. Anal. 1(1), 39–46 (2016)
18. Ivakhno, Y., Kadets, V., Werner, D.: The Daugavet property for spaces of Lipschitz functions. Math. Scand. 101(2), 261–279 (2007). https://doi.org/10.7146/math.scand.a-15044
19. Kadets, V., López, G., Martín, M., Werner, D.: Equivalent norms with an extremely nonlineable set of norm attaining functionals. J. Inst. Math. Jussieu 19(1), 259–279 (2020). https://doi.org/10.1017/S1474748018000087
20. Kadets, V., Martín, M., Soloviova, M.: Norm-attaining Lipschitz functionals. Banach J. Math. Anal. 10(3), 621–637 (2016). https://doi.org/10.1215/17358787-3639646
21. Khan, S.S., Mim, M., Ostrovskii, M.I.: Isometric copies of $\ell_\infty^n$ and $\ell_1^n$ in transportation cost spaces on finite metric spaces. In: The mathematical legacy of Victor Lomonosov. Adv. Anal. Geom. 2, 189–203. (2020). https://doi.org/10.1515/9783110656756-014
22. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces I and II. Springer, Berlin (1996)
23. Ostrovka, S., Ostrovskii, M.I.: On relations between transportation cost spaces and $\ell_1$. J. Math. Anal. Appl. 491(2), Paper No. 124338 (2020). https://doi.org/10.1016/j.jmaa.2020.124338
24. Read, C.J.: Banach spaces with no proximinal subspaces of codimension 2. Israel J. Math. 223(1), 493–504 (2018). https://doi.org/10.1007/s11856-017-1627-3
25. Rmoutil, M.: Norm-attaining functionals need not contain 2-dimensional subspaces. J. Funct. Anal. 272(3), 918–928 (2017). https://doi.org/10.1016/j.jfa.2016.10.028
26. Weaver, M.: Lipschitz Algebras. World Scientific, River Edge (1999). https://doi.org/10.1142/4100

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