THE STOCHASTIC PRIMITIVE EQUATIONS WITH NON–ISOTHERMAL TURBULENT PRESSURE

ANTONIO AGRESTI, MATTHIAS HIEBER, AMRU HUSSEIN, AND MARTIN SAAL

Abstract. In this paper we introduce and study the primitive equations with non–isothermal turbulent pressure and transport noise. They are derived from the Navier–Stokes equations by employing stochastic versions of the Boussinesq and the hydrostatic approximations. The temperature dependence of the turbulent pressure can be seen as a consequence of an additive noise acting on the small vertical dynamics. For such model we prove global well–posedness in $H^1$ where the noise is considered in both the Itô and Stratonovich formulations. Compared to previous variants of the primitive equations, the one considered here present a more intricate coupling between the velocity field and the temperature. The corresponding analysis is seriously more involved than in the deterministic setting. Finally, the continuous dependence on the initial data and the energy estimates proven here are new, even in the case of isothermal turbulent pressure.

Contents

1. Introduction 1
2. Physical derivations 7
3. Local and global well-posedness 12
4. Proof of Theorems 3.4, 3.6 and 3.7 20
5. Basic estimates 24
6. The main intermediate estimate 28
7. Proof of Proposition 4.2 47
8. Stratonovich formulation 49
References 52

1. Introduction

In this paper we introduce and study the stochastic primitive equation with non–isothermal turbulent pressure and transport noise. The primitive equations are one of the fundamental models for geophysical flows used to describe oceanic and atmospheric dynamics. They are derived from the Navier-Stokes equations in domains where the vertical scale is much smaller than the horizontal scale by the small aspect ratio limit. Additional information for the various versions of the deterministic primitive equations can be found, e.g. in [Ped87, Val06]. The introduction of additive and multiplicative noise into models for geophysical flows can be used on the one hand to account for numerical and empirical uncertainties and errors and on the other hand as subgrid-scale parameterizations for data assimilation, and ensemble prediction as described in the review articles [Del04, FOB+14, Pal19]. The primitive equations with non–isothermal turbulent pressure present a more intricate interplay between the velocity field and the temperature which leads to serious mathematical complications compared to the deterministic situation, see e.g. [CT07, HH20]. The

Date: September 7, 2023.

2010 Mathematics Subject Classification. Primary 35Q86; Secondary 35R60, 60H15, 76M35, 76U60.

Key words and phrases. stochastic partial differential equations, primitive equations, global well–posedness, gradient noise, stochastic maximal regularity, turbulent flows, Kraichnan’s turbulence, thermal fluctuations.

The first author has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 948819).
same difficulties also appear when comparing previous stochastic perturbations of the primitive equations (see e.g. [AHHS22, BS21, DGHT11, DGHTZ12] and the references therein) with the one considered here. A discussion of these difficulties can be found in Subsection 1.1 below.

The presence of the temperature in the balance for the turbulent pressure can be thought of as the large scale effect of thermal fluctuations acting on the small vertical dynamics. From a modeling point of view, a non–isothermal turbulent pressure may provide a new perspective on the contribution of the temperature on geophysical flows ruled by primitive equations. For instance, our hope is that the model introduced in the current paper can be used in the study on the contribution of the temperature on geophysical flows ruled by primitive equations (see e.g. [AHHS22, BS21, DGHT11, DGHTZ12] and the references therein) with the same difficulties also appear when comparing previous

The problem (1.1) is supplemented with the following boundary conditions

\begin{align}
\partial_3 v(-h, \cdot) &= \partial_3 v(\cdot, 0) = 0 \quad \text{on } \mathbb{T}^2, \\
\partial_3 \theta(-h, \cdot) &= \partial_3 \theta(\cdot, 0) + \alpha \theta(\cdot, 0) = 0 \quad \text{on } \mathbb{T}^2,
\end{align}

where \(\alpha \in \mathbb{R}\) is given and

\begin{equation}
\begin{aligned}
&v(\cdot, -h) = v(\cdot, 0) = 0 \quad \text{on } \mathbb{T}^2, \\
&\theta(\cdot, -h) = \theta(\cdot, 0) = 0 \quad \text{on } \mathbb{T}^2.
\end{aligned}
\end{equation}

Actually, in our main results we consider a generalization of the system in (1.1), see (3.1) in the main text. Moreover, our arguments also cover the case where the boundary conditions (1.2) are replaced by periodic ones. We refer to Remark 3.13 for further comments.

The aim of this paper is to show the global well–posedness in the strong setting (both analytically and probabilistically) of the system (1.1)–(1.3), see Theorems 3.6 and 3.7. In such results the noise is understood in the Itô–sense. In Section 8 we also discuss the case of Stratonovich noise. In stochastic fluid mechanics, and in particular for geophysical flows, the Stratonovich formulation of the noise is relevant, and it is seen as a more realistic model compare than the Itô one, see e.g. [BCF91, BCF92, MR05, Fla08, HLN19, AV21b] and the references therein for related mathematical results. Let us stress that the difficulties arising from the non–isothermal turbulent pressure are still present in absence of transport noise, see Subsection 1.1 for details.

The presence of the temperature in the balance for the turbulent pressure can be thought of as the large scale effect of thermal fluctuations acting on the small vertical dynamics. As in [AHHS22], we also consider dynamics driven by transport noise. Such type of noise was first introduced by R.H. Kraichanan in the study of turbulent flows [Kra68, Kra94] and it has been widely studied in the context of Navier-Stokes equations for turbulent flows, see [MR01, MR04, HLN21] for a physical justification and also [BCF91, BCF92, MR05, Fla08, HLN19, AV21b] and the references therein for related mathematical results. Let us stress that the difficulties arising from the non–isothermal turbulent pressure are still present in absence of transport noise, see Subsection 1.1 for details.

The primitive equations with non–isothermal turbulent pressure in the domain \(\Omega = \mathbb{T}^2 \times (-h, 0)\), where \(h > 0\), are given by the following system:

\begin{align}
\begin{aligned}
dv - \Delta v dt &= \left[-\nabla_H P - (v \cdot \nabla_H)v - w\partial_3 v + F_v\right] dt \\
&\quad + \sum_{n \geq 1} \left[(\phi_n \cdot \nabla) v - \nabla_H \bar{P}_n + G_{v,n}\right] d\beta^n_t, \\
d\theta - \Delta \theta dt &= \left[-(v \cdot \nabla_H) \theta - w\partial_3 \theta + F_\theta\right] dt \\
&\quad + \sum_{n \geq 1} \left[(\psi_n \cdot \nabla) \theta + G_{\theta,n}\right] d\beta^n_t,
\end{aligned}
\end{align}

(1.1a, 1.1b)

\begin{align}
\partial_3 P + \kappa \theta &= 0, \\
\partial_3 \bar{P}_n + \sigma_n \theta &= 0, \\
\div_3 v + \partial_3 w &= 0,
\end{align}

(1.1c, 1.1d, 1.1e)

Here \(\kappa, \sigma_n\) and \(\phi_n = (\phi_n^{ij})_{i,j=1}^3, \psi_n = (\psi_n^{ij})_{i,j=1}^3\) are assigned maps. Moreover \(v = (v^k)_{k=1}^3 : [0, \infty) \times \Omega \times \Omega \to \mathbb{R}^2\) denotes the horizontal component of the unknown velocity field \(u = (v, w)\) and \(w : [0, \infty) \times \Omega \times \Omega \to \mathbb{R}\) the vertical one, \(P : [0, \infty) \times \Omega \times \Omega \to \mathbb{R}\) denote the unknown pressure, \(\bar{P}_n : [0, \infty) \times \Omega \times \Omega \to \mathbb{R}\) the components of the unknown turbulent pressure and \(\theta : [0, \infty) \times \Omega \times \Omega \to \mathbb{R}\) the unknown temperature. Finally, \((\beta^n_t : t \geq 0)_{n \geq 1}\) is a sequence of standard independent Brownian motions on a given filtered probability space and \((F_v, F_\theta, G_{v,n}, G_{\theta,n})\) are given maps possibly depending on \((v, \theta, \nabla v, \nabla \theta)\) describing deterministic and stochastic forces as well as taking into account lower order effects like the Coriolis force. For the unexplained notation we refer to Subsection 1.5 below.

The problem (1.1) is supplemented with the following boundary conditions

\begin{align}
\partial_3 v(-h, \cdot) &= \partial_3 v(\cdot, 0) = 0 \quad \text{on } \mathbb{T}^2, \\
\partial_3 \theta(-h, \cdot) &= \partial_3 \theta(\cdot, 0) + \alpha \theta(\cdot, 0) = 0 \quad \text{on } \mathbb{T}^2,
\end{align}

where \(\alpha \in \mathbb{R}\) is given and

\begin{equation}
\begin{aligned}
&v(\cdot, -h) = v(\cdot, 0) = 0 \quad \text{on } \mathbb{T}^2, \\
&\theta(\cdot, -h) = \theta(\cdot, 0) = 0 \quad \text{on } \mathbb{T}^2.
\end{aligned}
\end{equation}
[BF20, HL84, FOB14, Wen14, FP22, DP22, MR01, MR04]. From an analytic point of view, the Stratonovich noise is not more difficult than the Itô one and, at least formally, one can covert the Stratonovich formulation in the Itô one up to consider some additional corrective terms. We refer to Theorem 8.2 for the global well–posedness of (1.1) with Stratonovich noise and in the strong setting.

For the readers convenience, we state here a simplified version of the Theorems 3.6 and 3.7. Below we write \( \phi_n^{\ell,m} \equiv \gamma_n^{\ell,m} \equiv (\gamma_n^{\ell,m})_{n \geq 1} \) and \( \mathbb{R}_+ \equiv (0, \infty) \).

**Theorem 1.1** (Simplified version). Let \( \kappa \) be constant, \( (\sigma_n)_{n \geq 1} \in \ell^2 \), \( G_{\nu,n}^k = G_{\theta,n} = 0 \), \( F_{\nu} = 0 \), and let \( F_{\nu} = k_0(v^2, -v^1) \) for \( k_0 \in \mathbb{R} \) be the Coriolis force. For all \( n \geq 1 \) let the maps

\[
\phi_n, \psi_n : \mathbb{R}_+ \times \Omega \times \mathcal{O} \rightarrow \mathbb{R}^3 \and \gamma_n : \mathbb{R}_+ \times \Omega \times \mathbb{T}^2 \rightarrow \mathbb{R}^{2 \times 2}
\]

be \( \mathcal{P} \otimes \mathcal{B} \)-measurable, and let for some \( \delta > 0 \) and all \( j \in \{1, 2, 3\} \), \( \ell, m \in \{1, 2\} \) be

\[
\phi^j, \psi^j \in L^{\infty}(\mathbb{R}_+ \times \Omega ; \mathcal{H}^{1+\delta}(\mathcal{O}; \ell^2)) \and \gamma^{\ell,m} \in L^{\infty}(\mathbb{R}_+ \times \Omega ; \mathcal{H}^{3+\delta}(\mathcal{T}^2; \ell^2)).
\]

Suppose that \( (\phi_n^j, \psi_n^j) \) are independent of \( x_3 \) for \( j \in \{1, 2\} \). Furthermore, assume that there exists \( \nu \in (0, 2) \) such that, a.s. for all \( t \in \mathbb{R}_+ \), \( x \in \mathcal{O} \) and \( \xi \in \mathbb{R}^3 \) the parabolicity conditions

\[
\sum_{n \geq 1} \left( \sum_{1 \leq j \leq 3} \phi_n^j(t, x)\xi_j \right)^2 \leq \nu |\xi|^2 \and \sum_{n \geq 1} \left( \sum_{1 \leq j \leq 3} \psi_n^j(t, x)\xi_j \right)^2 \leq \nu |\xi|^2
\]

hold. Then for each \( v_0 \in L^0_{\mathcal{F}_0}(\Omega; \mathcal{H}^1(\mathcal{O})) \) and \( \theta_0 \in L^0_{\mathcal{F}_0}(\Omega; \mathcal{H}^1(\mathcal{O})) \) the following hold:

1. There exists a unique global strong solution \( (v, \theta) \) to (1.1)–(1.3) satisfying

\[
(v, \theta) \in L^{2}_{loc}((0, \infty); \mathcal{H}^2(\mathcal{O}) \otimes \mathcal{H}^1(\mathcal{O})) \cap C((0, \infty); \mathcal{H}^1(\mathcal{O}) \otimes \mathcal{H}^1(\mathcal{O})) \text{ a.s.}
\]

2. For all \( T \in (0, \infty) \) and all \( \gamma > e^e \),

\[
P \left( \sup_{t \in [0, T]} \|v(t)\|_{\mathcal{H}^1} + \int_0^T \|v(t)\|_{\mathcal{H}^2} \, dt \geq \gamma \right) \lesssim_T \frac{1 + E\|v_0\|_{\mathcal{H}^1} + E\|\theta_0\|_{\mathcal{H}^1}}{\log \log \log(\gamma)},
\]

\[
P \left( \sup_{t \in [0, T]} \|\theta(t)\|_{\mathcal{H}^1} + \int_0^T \|\theta(t)\|_{\mathcal{H}^2} \, dt \geq \gamma \right) \lesssim_T \frac{1 + E\|v_0\|_{\mathcal{H}^1} + E\|\theta_0\|_{\mathcal{H}^1}}{\log \log \log(\gamma)}.
\]

3. The assignment \( (v_0, \theta_0) \mapsto (v, \theta) \) is continuous in probability in the sense of Theorem 3.7.

For the definition of \( \mathcal{P} \otimes \mathcal{B} \)-measurable, \( L^0_{\mathcal{F}_0}(\Omega; X) \), and the notation for the function spaces see Subsections 1.5 and 3.1. In the above, we have not specified the unknowns \( v \), \( P \) and \( \mathcal{D}_n \) as they are uniquely determined by \( v \) and \( \theta \) due to the divergence free condition and the hydrostatic Helmholtz projection. For comments on the relation between the regularity of the transport noise consider in this paper and Kruisch’s noise, we refer to [AHHS22, Section 1].

Physical motivations for the independence of \( (\phi_n^j, \psi_n^j) \) on \( x_3 \) for \( j \in \{1, 2\} \) are discussed in Remarks 2.2 and 2.3. In a nutshell, the small aspect ratio limit (i.e. the hydrostatic approximation [FGH+20, LT19]) shows that the primitive equations can be derived by taking the limit as \( \varepsilon \downarrow 0 \) of the Navier-Stokes equations on a thin domain \( \mathbb{T}^2 \times (-\varepsilon, 0) \) (see Figure 1), and therefore the variability in the vertical direction of the coefficients disappear in the limit and hence the independence of \( (\phi_n^j, \psi_n^j) \) on \( x_3 \) for \( j \in \{1, 2\} \) is justified. In particular, the situation for geophysical flows is different from usual turbulence models concerning Navier-Stokes equations [BE12, Tab2].

The logarithmic bounds of Theorem 1.1(2) seem rather weak. However, compared to the estimates in the deterministic setting (see e.g. [CT07]), even in absence of noise, it seems not possible to obtain in (2) more than a \( \log \log \)-decay due to three applications of Grownall’s inequality. Moreover, it seems not clear how to improve the estimates in (2) without enforcing regularity assumptions on the noise. We refer to the text below Theorem 3.6 and to Remark 3.10 for more details. The bounds in Theorem 1.1(2) remind us the estimates obtained in [GHKVZ14, Theorem 4.2], where the authors proved logarithmic moment bounds in \( \mathcal{H}^2(\mathcal{O}) \) under additional assumptions on the noise. In particular, in [GHKVZ14], it is not possible to consider gradient or transport type noises (in particular, this forces \( \sigma_n \equiv 0 \), cf. Subsection 1.1 below). However, it seems that there is no direct relation between the estimates of (2) and the above mentioned
estimate of [GHKVZ14]. In [GHKVZ14] the authors used logarithmic moment bounds to prove existence of ergodic invariant measures in $H^1(O)$. The extension of such result to the system (1.1) goes beyond the scope of this manuscript. Finally, let us mention that the continuous dependence on the initial data in (3) readily implies the Feller property for (1.1) which is a first step in the proof of existence of invariant ergodic measures, and it is based on the energy estimates in (2). We refer to Remark 3.8 for more details on the Feller property.

1.1. Novelties and description of the main difficulty. Compare to the results in [AHHS22], the major novelty of the current work is the presence of $\sigma_n \neq 0$. Here we explain the main analytic difficulty behind this fact. For simplicity, as in Theorem 1.1, in this subsection we assume that $(\sigma_n)_{n \geq 1} \in \ell^2$ is constant. Note that (1.1d) yields, for all $(x_H, x_3) \in O$ (here and below $x_H \in \mathbb{T}^2$ and $x_3 \in (h, 0)$ denote the horizontal and vertical variables, respectively) and $t \in \mathbb{R}_+$,

$$\tilde{P}_n(t, x_H, x_3) = \tilde{p}_n(t, x_H) + \sigma_n \int_{-h}^{x_3} \theta(t, x_H, \zeta) \, d\zeta$$

where $\tilde{p}_n$ depends only on $x_H \in \mathbb{T}^2$ (typically referred as turbulent surface pressure). Using the above identity in (1.1a), the following gradient noise term appears in the $v$-dynamics:

$$\sum_{n \geq 1} \sigma_n \int_{-h}^{x_3} \nabla_H \theta(t, x_H, \zeta) \, d\zeta \, d\beta^v_t,$$

where $\nabla_H = (\partial_1, \partial_3)$. In particular, as maximal $L^2$-regularity estimates show (see e.g. [AHHS22, Proposition 6.8] or Lemma 4.1), to obtain a-priori $L^\infty_t(H^1_x) \cap L^2_t(H^2_x)$-bounds for $v$ (and hence global existence for (1.1)), one needs $L^\infty_t(H^2_x)$-bounds for $\theta$. This is dramatically different from the case of isothermal turbulent pressure (i.e. $\sigma_n \equiv 0$), where it is sufficient to show $L^\infty_t(H^1_x)$-bounds for $\theta$ to obtain $L^\infty_t(H^1_x) \cap L^2_t(H^2_x)$-estimates for $v$ (see [AHHS22, Section 5]). Since $L^\infty_t(H^1_x)$-bounds for $\theta$ follow from standard energy estimates, the proof of global existence in the case $\sigma_n \equiv 0$ is essentially independent of the $\theta$-dynamics from an analytic point of view, cf. [AHHS22, Section 5]. This is not the case for (1.1) with $\sigma_n \neq 0$ where the coupling between the evolution of $v$ and the one of $\theta$ is more subtle and cannot be decoupled from $\theta$ in the $L^\infty_t(H^1_x) \cap L^2_t(H^2_x)$-estimates. Let us remark that these difficulties are also present in absence of transport noise in (1.1a)–(1.1b), i.e. $\phi_n = \psi_n \equiv 0$.

Before going further let us mention some further differences with [AHHS22]. The energy estimates and the continuous dependence on the initial data of Theorem 1.1(2)–(3) were not contained in [AHHS22] and are based on the use of a recent stochastic Gronwall’s lemma proven in [AV22a, Appendix A]. Finally, due to the presence of the term (1.4) in the $v$-dynamics (1.1a), we cannot allow for a strong–weak setting as in [AHHS22, Section 3], i.e. considering (1.1a) in the strong setting (in the sense of Sobolev spaces) and (1.1b) in the weak analytic one. Hence we only consider the strong setting, i.e. both (1.1a) and (1.1b) are understood in the strong sense.

To conclude, let us anticipate that in Theorems 3.6 and 3.7 we can even allow $(\sigma_n)_{n \geq 1}$ to depend on $(t, \omega, x_H)$, but not on $x_3$. The physical relevance of the $x_3$-independence of $\sigma_n$ is discussed in Remark 2.1. As for the $x_3$-independence of $\phi^i_n, \psi^j_n$ for $j \in \{1, 2\}$ in Theorem 1.1, the justification is via the hydrostatic approximation.

1.2. On the physical derivations. Besides the symmetry of the relations (1.1c)–(1.1d), to motivate the presence of non–isothermal balance (1.1d), in Section 2 we provide two physical derivations of (1.1). In both derivations the condition (1.1d) appears naturally. Following the strategy used in the deterministic framework we derive (1.1) by employing suitable stochastic variants of the Boussinesq and the hydrostatic approximations. In both cases the main ideas are in the Boussinesq approximation. In fluid dynamics, the Boussinesq approximation is employed in the study of buoyancy-driven flows (also referred as natural convection) and it is typical a good approximation in the context of oceanic flows. Roughly speaking, the idea behind the Boussinesq approximation is that, in a natural convection regime, the role of the compressibility is negligible in the inertial and in the convection terms, but not in the gravity term. More precisely, in the compressible
Navier–Stokes equations one assumes

\[(\rho - \rho_0)(\partial_t u - (u \cdot \nabla) u) \approx 0\]  

for some reference density \(\rho_0\). In our first approach to derive (1.1), borrowing some ideas from stochastic climate modeling (see e.g. [MTVE01]), we replace LHS(1.5) compatible with the divergence free condition of \((v, w)\), i.e. (1.1e).

(1.6) \[(\rho - \rho_0)(\partial_t u - (u \cdot \nabla) u) \approx \sum_{n \geq 1} \left[(\rho - \rho_0)k_n - \nabla P_n\right] \beta^n_t.\]

Here \(k_n \in \mathbb{R}^3\) is given and \(\nabla P_n\) is the turbulent pressure which makes the modeling assumption on the RHS(1.6) consistent with the divergence free condition of \((v, w)\).

1.3. Comments on the literature. Here we collect further references to the literature on primitive equations. Since the literature is extensive, we restrict to literature particularly relevant to this work, referring to the cited works for a more extensive and complete overview.

In the deterministic setting, the primitive equations were first studied by J.L. Lions, R. Temam and S. Wang in a series of articles [LTW92a, LTW92b, LTW93]. There, the authors proved existence of global Leray–Hopf type solutions for initial data \(v_0 \in L^2(\Omega)\). As for the Navier–Stokes equations, the uniqueness of such solutions is still open. Under additional regularity assumptions uniqueness holds, see [Ju17]. In the deterministic setting, a breakthrough result has been proven independently by C. Cao and E.S. Titi [CT07] and R.M. Kobelkov [Kob07] where they proved the global well–posedness in \(H^1\) for the primitive equations via \(L^\infty(\Omega; H^1) \cap L^2(\Omega; H^2)\) a–priori estimates. See also [KZ07] for other boundary conditions. The results of [CT07, Kob07] have been extended to the \(L^p\)–setting by the second author and T. Kashiwabara in [HK16]. Further results can be found in [GGH+20b, GGH+20a, GGH+21]. See also [HH20] for an overview.

Stochastic versions of the primitive equations have been studied by several authors. Global well–posedness for pathwise strong solutions has been established for multiplicative white noise in time by A. Debussche, N. Glatt-Holtz and R. Temam in [DGHT11] and the same authors with M. Ziane in [DGHTZ12]. There, the authors used a Galerkin approach to first show the existence of martingale solutions, and then strong existence is deduced via pathwise uniqueness and a Yamada-Watanabe type result. The global existence of solutions is then shown by energy estimates where the noise is seen as a perturbation of the linear system. The drawback of this approach is that it needs some smoothness for the noise which for instance excludes the case of gradient or transport noises. Z. Brzeźniak and J. Slavík in [BS21] employed a similar approach to show local and global existence existence for the primitive equations with small transport noise. The stochastic perturbation of the primitive equations considered in [BS21] is such that it does not act directly on the pressure when turning to the question of global existence. This allows the authors of [BS21] to overcome some of the difficulties arose in [DGHT11, DGHTZ12]. In [AHHS22], by combining energy estimates and the functional analytic setting of [AV22b, AV22c] we were able to overcome such drawbacks in presence of gradient noise and transport type noises.
1.4. **Strategy and overview.** As in [AHHS22], we take another point of view on stochastic primitive equations like (1.1) compared to [BS21, DGHT11, DGHTZ12]. More precisely, we interpret the transport and gradient noise terms as a part of the linearized system. Hence we only need to impose conditions guaranteeing that this linearization is parabolic. Such conditions are known to be optimal in the parabolic setting. With this perspective, the local existence and blow-up criterion of Theorem 3.4 follow easily from the theory of critical spaces for stochastic evolution equations developed by the M.C. Veraar and the first author in [AV22b, AV22c].

Once obtained local existence and blow-up criteria from the abstract setting of [AV22b, AV22c], we turn our attention to global well–posedness, that is the main point of the present manuscript. Here we follow the arguments of [CT07], where the main authors shows a–priori estimates for $v$ as a by–product of several concatenated estimates. In [CT07], the core of the argument is an intermediate estimate involving the \emph{barotropic} and \emph{baroclinic modes} given by

$$\bar{v} = \int_{-h}^{0} v(\cdot, \zeta) \, d\zeta$$ and $$\bar{v} = v - \bar{v},$$

respectively. Note that this is also the path also used in our previous work [AHHS22]. However, in [CT07, AHHS22], the temperature acts in the $v$–equations only as a lower order term and therefore it does \emph{not} play any role in the estimates involving $(\bar{v}, \bar{v})$, see the discussed below Theorem 1.1. The presence of $\theta$ in (1.1d) (and hence the term (1.4) in the $v$–dynamics) creates several additional terms in the estimates for $(\bar{v}, \bar{v})$ which cannot be treated as lower–order. Such terms will be described extensively at the beginning of Section 6. In particular, we need to estimate $(\bar{v}, \bar{v})$ and $\theta$ jointly exploiting some further (subtle) cancellations appearing in the energy balances. In our derivation of the energy estimates for $(\bar{v}, \bar{v})$, here and in [AHHS22], we follow the simplified approach due to T. Kashiwabara and the second author in [HK16] (also used in [HH20]). There, for instance, the $L^1$–estimates proven in [CT07] are replaced by the (apparently) weaker $L^2$–estimates.

The paper is organized as follows.

- Section 2: Physical derivations of (1.1).
- Section 3: Statements of the local and global well–posedness results of (1.1) in $H^1$.
- Section 4: Proof of the main results of Section 3 taken for granted the energy estimates of Proposition 4.2.
- Section 5: Basic energy estimates for $(v, \theta)$.
- Section 6: Proof of the crucial intermediate estimate involving $(\bar{v}, \bar{v})$ and other unknowns.
- Section 7: Proof of the energy estimates of Proposition 4.2.
- Section 8: Global well–posedness for (1.1) with noise in Stratonovich form.

1.5. **Notation.** Here we collect the main notation which will be used through the paper. $C$ denotes a constant which may change from line to line and depends only on the parameters introduced in our main assumption, namely Assumption 3.1 below.

For any integer $k \geq 1$, $s \in (0, \infty)$ and $p \in (1, \infty)$, $L^p(\mathcal{O}; \mathbb{R}^k) = (L^p(\mathcal{O}))^k$ denotes the usual Lebesgue space and $H^s(\mathcal{O}; \mathbb{R}^k)$ the corresponding Sobolev space. In the paper we also use the common abbreviation $H^s(\mathcal{O}; \mathbb{R}^k) \overset{\text{def}}{=} H^{s,2}(\mathcal{O}; \mathbb{R}^k)$. Appropriate function spaces of divergence free the velocity fields will be introduced in Subsection 3.1 and are denote by $\mathbb{H}^s(\mathcal{O})$ or $L^s(\mathcal{O})$. Function spaces which take also into accounts the boundary conditions (1.2) are defined in (3.14)–(3.15).

Since $\mathcal{O} = \mathbb{T}^2 \times (-h, 0)$, we employ the natural splitting $x \mapsto (x_1, x_3)$ where $x_1 \in \mathbb{T}^2$, $x_3 \in (-h, 0)$ and the subscript $H$ stands for horizontal. Similarly, we set $\text{div}_H = \partial_1 + \partial_2$, $\nabla_H = (\partial_1, \partial_2)$, and $\Delta_H = \text{div}_H \nabla_H$.

Similarly, for a vector $y = (y^j)_{j=1}^3 \in \mathbb{R}^3$ we write $y_H = (y^j)_{j=1}^2$ for its horizontal component. In the same spirit, we also set

$$(v \cdot \nabla_H)v \overset{\text{def}}{=} \left( \sum_{1 \leq j \leq 2} v^j \partial_j v^k \right)_{k=1}^2,$$

$$(\varphi_n \cdot \nabla)v \overset{\text{def}}{=} \left( \sum_{1 \leq j \leq 3} \varphi_n^j \partial_j v^k \right)_{k=1}^2,$$

$$(v \cdot \nabla_H)\theta \overset{\text{def}}{=} \sum_{1 \leq j \leq 2} v^j \partial_j \theta,$$

$$(\psi_n \cdot \nabla)\theta \overset{\text{def}}{=} \sum_{1 \leq j \leq 3} \psi_n^j \partial_j \theta.$$
We also employ the following usual notation for the vertical average:

\[ \int_{-h}^{0} \cdot \, d\zeta \overset{\text{def}}{=} \frac{1}{h} \int_{-h}^{0} \cdot \, d\zeta. \]

If no confusion seems likely, we write \( L^2, H^k, \mathbb{H}^k, L^2(\mathbb{F}^2) \) and \( H^k(\mathbb{F}^2) \) instead of \( L^2(\Omega; \mathbb{R}^m), H^k(\Omega; \mathbb{R}^m), \mathbb{H}^k(\Omega), L^2(\Omega; \ell^2(N; \mathbb{R}^m)) \) and \( H^k(\Omega; \ell^2(N; \mathbb{R}^m)) \) for some \( m \geq 1 \) etc.

Finally, we collect the main probabilistic notation. Through the paper we fix a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)^\infty_{t=0}, \mathbb{P})\) and we let \( \mathbb{E}[\cdot] \overset{\text{def}}{=} \int_{\Omega} \cdot \, d\mathbb{P} \). Moreover, \((\beta^n) = (\beta^n_t : t \geq 0)\) denotes a sequence of standard Brownian motion on the above mentioned probability space. We will denote by \( B_{\ell^2} \) the \( \ell^2 \)-cylindrical Brownian motion uniquely induced by the sequence \((\beta^n)_{n \geq 1}\) (see e.g. [AV22b, Example 2.12]). A stopping time \( \tau : \Omega \to [0, \infty) \) is a measurable map such that \( \{\tau \leq t\} \in \mathcal{F}_t \) for all \( t \geq 0 \). For a stopping time \( \tau \), we let \( [0, \tau] \times \Omega \overset{\text{def}}{=} \{(t, \omega) : 0 \leq \tau(\omega) \leq t\} \) and use analogous definitions for \([0, \tau) \times \Omega\) etc. By \( \mathcal{P} \) and \( \mathcal{B} \) we denote the progressive and the Borel \( \sigma \)-algebra, respectively. Moreover, we say that a map \( \Phi : \mathbb{R}_+ \times \Omega \times \mathbb{R}^m \to \mathbb{R} \) is \( \mathcal{P} \otimes \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{R}^m) \)-measurable if \( \Phi \) is \( \mathcal{P} \otimes \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{R}^m) \)-measurable.

Acknowledgements. The first author thanks Umberto Pappalettera for helpful suggestions on Section 2 and for bringing to his attention the reference [MTVE01]. The first author is grateful to Marco Romito for helpful comments related to Remarks 2.1 and 2.2.

### 2. Physical derivations

In this section we derive the primitive equations with non–isothermal turbulent pressure (1.1). In the deterministic framework, the primitive equations are derived from the compressible Navier–Stokes equations by means of the Boussinesq and hydrostatic approximations. In the current section, following the same path, we propose two derivations of (1.1) both based on suitable stochastic variants of these approximations. In the first derivation, given in Subsections 2.1–2.2, we motivate the noise leading to the non–isothermal turbulence balance (1.1e) by borrowing ideas from stochastic climate modeling (see e.g. [MTVE01, AFP21] and the reference therein). In the second one, worked out in Subsection 2.3, we derive (1.1) by looking at the Navier–Stokes equations as a two–scale system, where large and small scales are given by the horizontal and vertical ones, respectively; see Figure 1. As explained in Subsection 1.1 (see the text around (1.4)), the presence of \( \theta \) in the balance (1.1d) gives rise to a gradient type noise for the unknown \( \int_{-h}^{0} \theta \, d\zeta \) in the equations for the horizontal part of the velocity field \( v \). Hence, the two–scale viewpoint is somehow in accordance with the results obtained in [FP22, DP22] where an additive noise on the small scale dynamics give rise to a transport (or gradient) noise on the large scale dynamics. For exposition convenience, in the first derivation in Subsections 2.1–2.2, we do not consider transport noise in order to emphasize the natural appearance of the non-isothermal turbulent balance (1.1d). The former is included in the second derivation in Subsection 2.3. Let us anticipate that the derivations below also naturally lead to \( x_3 \)-independence of \((\sigma_n, \phi'_n, \psi'_n)\) for \( j \in \{1, 2\} \) used in our global well-posedness results of Theorems 1.1 and 3.6, see Remarks 2.1–2.2.

Finally, we mention that the primitive equations are often formulated by also adding an equation for the salinity. We do not consider this here, as the equation for the salinity has the same structure of the one for \( \theta \) and it does not provide any new mathematical difficulty (see e.g. [HHK16] and the reference therein).

#### 2.1. Stochastic Boussinesq approximation

In fluid dynamics, the BOUSSINESQ APPROXIMATION is employed in the study of buoyancy-driven flows (also referred as natural convection). As already mentioned in Subsection 1.2, the idea behind the Boussinesq approximation is that, in a natural convection regime, the role of the compressibility is negligible in the inertial and in the convection terms, but not in the gravity term. Next we propose an extension of such approximation in the context of stochastic Navier-Stokes equations. Let us consider the stochastic compressible anisotropic Navier-Stokes equations on the \( \varepsilon \)-dependent domain

\[ O_{\varepsilon} \overset{\text{def}}{=} \mathbb{T}^2 \times (-\varepsilon, 0) \]
where $\varepsilon > 0$ is a small parameter which measures the small vertical direction, see Figure 1; hence the velocity field $u : \mathbb{R}_+ \times \Omega \times O_\varepsilon \to \mathbb{R}^3$ and the density $\rho : \mathbb{R}_+ \times \Omega \times O_\varepsilon \to \mathbb{R}_+$ satisfy, on $O_\varepsilon$,

$$
\rho \partial_t u + \rho (u \cdot \nabla) u = -\nabla P + \mu_H \Delta_H u + \varepsilon^2 \partial_3^2 u + \mu \nabla (\text{div} \ u) + g \rho, \tag{2.1a}
$$

$$
\partial_t \rho + \text{div}(\rho u) = 0, \tag{2.1b}
$$

where $g = (0, 0, -g)$ and $\mu_H, \mu$ denotes the gravity and the dynamic viscosities, respectively. The anisotropic behavior of the viscosity in (2.1) is in accordance with physical observations of oceanic flows, see e.g. [HH20, Subsection 1.2.3].

Let $\rho_0 > 0$ be a reference density, e.g. the density of the fluid in standard conditions. The stochastic Boussinesq approximation consists in performing the following approximations:

(a) Consider $\rho \approx \rho_0$ in (2.1b), and therefore $\text{div} \ u = 0$.

(b) Approximate all the terms in (2.1a) which contains $\rho$ with a noise, expect for the buoyancy term $g \rho$. More precisely, in (2.1a), we use the following approximation

$$
(\rho - \rho_0)(\partial_t u - (u \cdot \nabla) u) = \sum_{n \geq 1} [(\rho - \rho_0)k_{n, \varepsilon} - \nabla P_n] \beta^n_t, \tag{2.2}
$$

where $\beta^n$ are standard independent Brownian motions and $k_{n, \varepsilon} \in \mathbb{R}^3$ are given for $n \geq 1$.

Recall that, in the deterministic setting, the Boussinesq approximation consists in assuming (a) and considering $(\rho - \rho_0)(\partial_t u - (u \cdot \nabla) u) \approx 0$, see e.g. [HH20, Subsection 1.2.2]. The reason not to approximate the gravity term $g \rho$ is that, experimentally, in buoyancy driven flows, such term is the most relevant in the dynamics and there is no natural approximation for it. At least formally, the RHS(2.2) has zero expected value (if we interpret the noise in the Itô formulation), cf. [MTVE01, Assumption (A.4)]. Hence, the modeling assumption on the RHS(2.2) is consistent with the usual Boussinesq approximation when one considers expectations and can be seen as a refinement of the latter. The presence of the turbulent pressure $P_n$ on the RHS(2.2) is necessary to obtain compatibility with the divergence free condition $\text{div} \ u = 0$, see (a) in the above list.

To some extend, the approximation in (b) follows the philosophy of stochastic climate modeling, where there are certain unresolved variables (in our case $\rho - \rho_0$) and the main assumption is that the nonlinear interactions among unresolved variables can be represented stochastically. Such approximation has two basic advantages. Firstly, the noise keeps track of the approximations done in the balances ruling the dynamics, and secondly, the corresponding model has a reasonable complexity (both mathematically and computationally). We refer to [MTVE01, AFP21] and the references therein for more details on stochastic climate models.

**Figure 1.** A particle subject to random forces in the thin domain $O_\varepsilon = T^2 \times (-\varepsilon, 0)$.
Using the stochastic Boussinesq approximation of (a)-(b) in (2.1) we obtain, on $\Omega_\varepsilon$,

\begin{equation}
du = \left[ \nu_H \Delta_H u + \frac{\varepsilon^2}{\rho_0} \partial_3^2 u - \frac{\nabla P}{\rho_0} - (u \cdot \nabla)u + g \frac{\rho}{\rho_0} \right] dt
\end{equation}

\begin{equation}
\text{div} u = 0,
\end{equation}

where, as usual, $\nu_H \overset{\text{def}}{=} \mu_H/\rho_0$ denotes the viscosity while $k_{n,\varepsilon} = H (\varepsilon k_{n,H}, k_{n,\varepsilon}^3)$, $k_{n,H} \in \mathbb{R}^2$ and $k_{n,\varepsilon}^3 \in \mathbb{R}$ are given. The anisotropic behavior of $k_n$ reflects the anisotropic viscosity in (2.3a).

To remove the dependence on $\rho$ in (2.3), we use a state equation $\rho = \rho(\theta)$. As standard in the context of primitive equations, we assume that $\theta \mapsto \rho(\theta)$ is linear, i.e.

\begin{equation}
\rho = \rho_0 + \lambda (\theta - \theta_0)
\end{equation}

where $\lambda \in \mathbb{R}$ and $\theta_0$ denote a parameter to be determine experimentally and a reference temperature, respectively. To close the problem consisting of (2.1) and (2.4), we need an equation for $\theta$.

By using the thermal balancing with constant density, one obtains, on $\Omega_\varepsilon$,

\begin{equation}
\partial_t \theta = \kappa_H \Delta_H \theta + \varepsilon^2 \partial_3^2 \theta - (u \cdot \nabla)\theta.
\end{equation}

In the above, as in (2.1), we use anisotropic conductivity. In the sequel, to simplify the presentation, we let

\begin{equation}
\nu_H = \rho_0 = \lambda = 1 \quad \text{and} \quad \theta_0 = 0.
\end{equation}

The general case can be obtained similarly (note that (2.5) is also satisfied by $\theta - \theta_0$ for all $\theta_0 \in \mathbb{R}$).

2.2. Stochastic hydrostatic approximation. Roughly speaking, the HYDROSTATIC APPROXIMATION consists in neglecting several terms in the dynamics for the vertical component of the velocity field. From a mathematical point of view, we would like to take the limit as $\varepsilon \downarrow 0$ in (2.3)-(2.5). To this end, it is convenient to rescale the vertical variable $x_3$ in order to obtain a problem on the fixed domain $\Omega \overset{\text{def}}{=} \Omega_1 = \mathbb{T}^2 \times (-1, 0)$. Moreover, to accomodate the anisotropic behavior of viscosity and conductivity in (2.3) and (2.5), we let

\begin{equation}
u = (v, w) \quad \text{where} \quad v \in \mathbb{R}^2 \quad \text{and} \quad w \in \mathbb{R}.
\end{equation}

In other words, $v$ and $w$ are the horizontal and the vertical parts of the velocity field $u$, respectively. Let $\varepsilon > 0$ and consider the rescaled quantities: For $t \in [0, \infty)$, $x_H \in \mathbb{T}^2$ and $x_3 \in (-1, 0)$,

\begin{equation}
v_\varepsilon(t, x) \overset{\text{def}}{=} v(t, x_H, \varepsilon x_3), \quad w_\varepsilon(t, x) \overset{\text{def}}{=} \varepsilon^{-1} w(t, x_H, \varepsilon x_3),
\end{equation}

\begin{equation}
\theta_\varepsilon(t, x) \overset{\text{def}}{=} \varepsilon \theta(t, x_H, \varepsilon x_3), \quad P_\varepsilon(t, x) \overset{\text{def}}{=} \varepsilon P(t, x_H, \varepsilon x_3), \quad \bar{P}_\varepsilon(t, x) \overset{\text{def}}{=} \varepsilon \bar{P}(t, x_H, \varepsilon x_3).
\end{equation}

The choice of the rescaling is the one used in the deterministic setting and it reflects the natural size of the corresponding quantity, see e.g. [FGH+20, LT19] and [PZ22] for the rescaling of $\theta$.

Note that $(v_\varepsilon, w_\varepsilon, P_\varepsilon, \bar{P}_\varepsilon, \theta_\varepsilon)$ are defined on the fixed domain $\Omega = \mathbb{T}^2 \times (-1, 0)$. From (2.3)-(2.5), we infer that, on $\Omega$,

\begin{equation}
dv_\varepsilon = \left[ \Delta_H v_\varepsilon + \partial_3^2 v_\varepsilon - \nabla_H P_\varepsilon - (v_\varepsilon \cdot \nabla) v_\varepsilon \right] dt + \sum_{n \geq 1} \left[ \theta k_{n,H} - \nabla_H \bar{P}_\varepsilon,n \right] d\beta_n^w,
\end{equation}

\begin{equation}
d(\varepsilon^2 w_\varepsilon) = \left[ \varepsilon^2 \left( \Delta_H w_\varepsilon + \partial_3^2 w_\varepsilon - (u_\varepsilon \cdot \nabla) w_\varepsilon \right) - \partial_3 P_\varepsilon - g \theta_\varepsilon + \varepsilon g \right] dt
\end{equation}

\begin{equation}
+ \sum_{n \geq 1} \left[ - \partial_3 \bar{P}_\varepsilon,n + k_{n,\varepsilon}^3 \theta_\varepsilon \right] d\beta_n^w,
\end{equation}

\begin{equation}
d\theta_\varepsilon = \left[ \Delta \theta_\varepsilon - (u_\varepsilon \cdot \nabla) \theta_\varepsilon \right] dt,
\end{equation}

\begin{equation}
\text{div} u_\varepsilon = 0.
\end{equation}
The **stochastic hydrostatic approximation** consists in taking the formal limit as $\varepsilon \downarrow 0$ in (2.8) and assuming that the quantities in $(v_\varepsilon, P_\varepsilon, \bar{P}_\varepsilon, \theta_\varepsilon)$ converge and

\[
\lim_{\varepsilon \to 0} \varepsilon^2 w_\varepsilon = 0, \quad \text{and} \quad \lim_{\varepsilon \to 0} \varepsilon^2 (\Delta w_\varepsilon + \partial^2 w_\varepsilon - (u_\varepsilon \cdot \nabla)w_\varepsilon) = 0.
\]

We refer to [Ped87] for physical reasons for the approximation to hold. Let us remind that the limits (2.9) are justified in the deterministic setting, see e.g. [LT19, FGH+20].

Assume that the hydrostatic approximation holds and denote by $(v, P, \bar{P}, \theta)$ the limit as $\varepsilon \downarrow 0$ of $(v_\varepsilon, P_\varepsilon, \bar{P}_\varepsilon, \theta_\varepsilon)$. By (2.8a) and (2.8b), one sees that $(v, \theta)$ solve (1.1a) and (1.1b) where $G_{v,n} = \theta k_n \bar{P}$, and $F_v = F_\theta = G_{\theta,n} \equiv 0$. While, using (2.9) and (2.8b), one obtains (1.1c) and (1.1d) with $k = g$ and $\sigma_n = -k_n^3$, respectively. Therefore (1.1) follows from (2.8) by means of the stochastic hydrostatic approximation.

**Remark 2.1** ($x_3$-independence of $\sigma_n$). In our main result, i.e. Theorem 3.6, we assume that $\sigma_n$ depends on $(t, \omega, x_H) \in \mathbb{R}_+ \times \Omega \times \mathbb{T}^2$, cf. Assumption 3.5 below. Here we discuss how the $x_3$-independence arises naturally from the stochastic hydrostatic approximation. Indeed, let us assume that $k_n$ is a map on $\mathbb{R}_+ \times \Omega \times \mathcal{O}_\varepsilon$ for some $\varepsilon_0 > 0$. From a modeling point of view, it is reasonable to assume that $k_n$ is continuous in $x \in \mathcal{O}_\varepsilon$. Then repeating the argument in (2.3)-(2.7) leading to the stochastic primitive equations (1.1), one obtains in (2.8b) the stochastic perturbation is of the form $\sum_{n \geq 1} ( - \partial_t \bar{P}_n(t, x) + k_n^3(t, x_H, \varepsilon x_3) \theta(t, x) ) d\beta^3_t$. In particular, if the stochastic hydrostatic approximation (2.9) holds, then the limiting balance (1.1d) is satisfied with $\sigma_n(t, \omega, x_H) = -k_n^3(t, \omega, x_H, 0)$ (here we used the continuity of $k_n$). A similar situation arises if we also assume $\lambda$ in (2.4) is $(t, \omega, x)$-dependent instead of (2.6).

Let us conclude this remark by noticing that if one considers $k_n$ such that $k_n^3 = k_n^3(t, \omega, x_H, \varepsilon^{-1}x_3)$, then the stochastic hydrostatic approximation eventually lead to $x_3$-dependent $\sigma_n$’s. However, to the authors’ opinion, the latter choice does not seem physically relevant. Indeed, in the spirit of Boussinesq approximations, one wants to obtain a reduced model from the Navier-Stokes equations by neglecting detailed information about the vertical dynamics and this is in contrast with the rescaling of the vertical direction, which increases the order of the vertical dynamics on the limiting SPDEs as $\lim_{\varepsilon \to 0} \|k_n^3(t, x_H, \varepsilon^{-1})\|_{C^\alpha(0, \varepsilon)} = \infty$ for all $\alpha > 0$. To the authors’ knowledge, in the literature there is no derivation of the primitive equations with $x_3$-dependent coefficients and therefore we cannot compare our situation with known results.

We conclude this remark by highlighting that in Section 6 we show that the $x_3$-independence of $\sigma_n$ allows us to obtain a meaningful splitting of the stochastic primitive equations (1.1) in terms of the **barotropic** and **baroclinic modes**, whose relevance is commented in Remark 2.2 below.

### 2.3. A related derivation and the two–scale viewpoint

In this section we give another derivation of (1.1) still based on the Boussinesq and hydrostatic approximation. Here the main starting point is a two–scale interpretation of the Navier–Stokes equations for the velocity field $u = (v, w) \in \mathbb{R}^2 \times \mathbb{R}$ on the thin domain $\mathcal{O}_\varepsilon$. Indeed, as Figure 1 suggests, the primitive equations can be seen as a two–scale system where the large scale dynamics is the horizontal component of the velocity field, i.e. $v$, and the small dynamics is the vertical component, i.e. $w$. Since $w$ is somehow a small scale, from a physical point of view it is natural to consider additive noise on this component, see e.g. [AFP21, DP22, FP20, FP22, MTVE01].

To make this rigorous, as a starting point, assume that $u : \mathbb{R}_+ \times \Omega \times \mathcal{O}_\varepsilon \to \mathbb{R}^3$ and the density $\rho : \mathbb{R}_+ \times \Omega \times \mathcal{O}_\varepsilon \to \mathbb{R}_+$ satisfy, on $\mathcal{O}_\varepsilon$,

\[
\rho \, du = 
\left[ \mu_H \Delta_H u + \varepsilon^2 \partial^2 u + \mu \nabla (\text{div} u) - \nabla P - \rho (u \cdot \nabla) u + gP \right] \, dt
\]

\[
+ \sum_{n \geq 1} \left[ (g \Phi_n \cdot \nabla H) u + \varepsilon \dot{\Phi}^3_n \partial u - \nabla P + n, 3(\rho) \right] d\beta^3_n,
\]

\[
\partial_t \rho + \text{div}(\rho \, u) = 0.
\]

A derivation of (2.10) is given, for instance, in [MR01, MR04]. In the latter works, transport noise is a consequence of a stochastic dynamics at the level of fluid particles, see [MR04, eq. (2.4)]. Here, as above, $(\beta^3)_{n \geq 1}$ is a sequence of standard independent Brownian motions on some probability
space, \(g\) is the gravity vector, \(\mu_H, \mu\) are the dynamic viscosities and \(\Phi_{n,H} \in \mathbb{R}^2, \Phi_n \in \mathbb{R}\) are given.

In (2.10a) we used anisotropic viscosity as in (2.1), which is in accordance with measurements in oceanic flows. The anisotropic behavior of the transport noise reflects the different order of the leading differential operators in the deterministic and the stochastic terms. The latter fact is actually a consequence of the different (time) scaling of the Brownian noise \(d\beta^n_t\) and the time \(dt\), see e.g. [AV21b, Subsection 1.1] for a discussion. Finally, \(k_n(\rho)\) is a given function of the density \(\rho\). Results on compressible Navier–Stokes equations can be found in [BFH20, BFH21, BFHM19] and the references therein.

Next we add a structural assumption on \(k_{n,\varepsilon}(\rho) = (k_{n,\varepsilon,H}(\rho), k_{n,\varepsilon}(\rho))\), where \(k_{n,\varepsilon,H}(\rho) \in \mathbb{R}^2\) and \(k_{n,\varepsilon}(\rho) \in \mathbb{R}\). More precisely, we assume that

\[
(2.11) \quad k_{n,\varepsilon,H}(\rho) = \tilde{k}_{n,H}(\varepsilon \rho), \quad \text{where} \quad \tilde{k}_{n,H} : \mathbb{R}_+ \to \mathbb{R} \quad \text{is a given nonlinearity},
\]

\[
(2.12) \quad k_{n,\varepsilon}(\rho) = \tilde{k}_{n,\varepsilon}^3(\rho), \quad \text{where} \quad \tilde{k}_{n,\varepsilon}^3 \in \mathbb{R}.
\]

The condition (2.12) tells us that on the vertical component \(w\) is acting an additive noise per unit of mass. As mentioned above, this is in accordance with the two–scale interpretation of (2.10) in the thin domain \(\Omega_\varepsilon\). The condition (2.11) is somehow technical and it is motivated by the scaling argument as in (2.7) which will be used below. However, let us stress that, for our purpose the crucial assumption is (2.12).

Now following the scheme of Subsections 2.1–2.2, one can derive (1.1) from (2.10) and the structural assumptions (2.11)–(2.12) performing the following steps:

- **(Stochastic Boussinesq Approx. II).** Assume that the density is constant (i.e. \(\rho = \rho_0\)) in all terms in (2.10) expect in the buoyancy term \(g\rho\) and its stochastic counterpart \(k_n(\rho)\).
- **(Heat Balance and State Equation II).** The heat balance shows that the temperature \(\theta\) evolves accordingly to the equations

\[
(2.13) \quad d\theta = [\kappa_H \Delta_H \theta + \varepsilon^2 \partial_n^2 \theta - (u \cdot \nabla) \theta] \, dt + \sum_{n \geq 1} \left[ (\Psi_{n,H} \cdot \nabla) \theta + \varepsilon \psi_n^3 \partial_n \theta \right] \, d\beta^n_t,
\]

where \((\Psi_n)_{n \geq 1}\) is a sequence of vector fields. Finally, as a state equation \(\rho = \rho(\theta)\), use the linear map \(\rho = \lambda \theta\) where \(\lambda \in \mathbb{R}\).
- **(Stochastic Hydrostatic Approx. II).** The hydrostatic approximation can be performed as in Subsection 2.2, where one also needs to add in (2.9) the requirement

\[
(2.14) \quad \lim_{\varepsilon \to 0} \varepsilon^2 \left[ (\Phi_n \cdot \nabla) w_\varepsilon \right] = 0.
\]

Let us stress that, in the deterministic setting [LT19, FGH+20], one can even prove that (2.9) holds and \(\varepsilon^2 \Delta w_\varepsilon \xrightarrow{\varepsilon \to 0} 0\). Hence, it seems that (2.14) is no more demanding than the requirements in (2.9).

We conclude this subsection by admitting that there is no direct relation between the above derivation with the two–scale arguments in [FP22, DP22, MTVE01]. It would be interesting to study which contribution(s) need to be consider in the small scale equation of [DP22, Subsection 7.3] to obtain the non–isothermal balance for \(\tilde{P}_n\) of (1.1d) for the effective dynamics.

**Remark 2.2 (x₃-independence of transport noise).** Arguing as in Remark 2.1, if one assumes that \(\Phi_{n,H}\) and \(\Phi_n\) are maps on \(\mathbb{R}_+ \times \Omega \times \mathcal{O}_{t_0}\) for some \(\varepsilon_0 > 0\) which are continuous in \(x \in \mathcal{O}_{t_0}\), then the stochastic hydrostatic approximation (i.e. (2.9) and (2.14) both hold) leads to the transport noise coefficients \(\phi_n(t, \omega, x_\varepsilon) = (\Phi_{n,H}(t, \omega, x_{H,0}), \Phi_n(t, \omega, x_{H,0}))\) in (1.1a). Let us remark that the continuity of \(\Phi_n\) in \(x \in \mathcal{O}_{t_0}\) is satisfied in the physical relevant case of the Kraichnan noise (see e.g. the discussion below [MR05, eq. (1.3)]). Therefore the \(x_3\)-independence condition of Assumption 3.5 is in accordance with the physical derivation.

As in Remark 2.1, the \(x_3\)-independence of \(\phi_n\) arises if and only if one rescales also the vertical variable by \(\varepsilon^{-1}\). As in the latter remark, to the authors’ opinion, on the one hand this seems unreasonable for the horizontal part of \(\Phi_n\), i.e. \(\Phi_{n,H}\). On the other hand, rescaling of \(\Phi_n\) might be physically relevant as in (2.10a) we are weakening the strength of the contribution \(\Phi_n^3 \partial_n \nabla\) via the multiplication by \(\varepsilon\). Thus, if one assumes \(\Phi_n^3 = \Phi_n^3(t, \omega, x_{H,0}, \varepsilon^{-1} x_3)\), then this leads to an
$\phi_n^3 = \Phi_n^3$. The latter situation is also covered by our results as in Assumption 3.5 no condition on the vertical component of $\phi$ is enforced.

As it follows from Section 6, the $x_3$-independence of $\phi_{n,H}$ is necessary for the stochastic primitive equations (1.1) to behave well under the decomposition into barotropic and baroclinic modes, i.e. $v = \tau + \bar{v}$ with $\tau = \int_{-h}^0 \nu(\cdot, \zeta) \, d\zeta$. The latter is very important from in physics and in particular for the study of oceanic dynamics, see e.g. [CH19, DHC+95, Hds97, OL07, SB99, YTLR17].

We conclude by noticing that the above arguments holds with $(\Phi_n, \phi_n)$ replaced by $(\Psi_n, \psi_n)$ which appear in the temperature balance (2.13) in case $\Psi_n$ is $x$-dependent.

**Remark 2.3** (Two-dimensional turbulence). The 2d nature of the transport noise for stochastic primitive equations arose in the above introduced stochastic hydrostatic approximation (cf. Remark 2.2) is in accordance with physical measurements of turbulent flows in the ocean, which are (approximately) two dimensional, see e.g. [BE12, Car01, Rhi73, Tab02, YO88].

### 3. Local and global well-posedness

In this section we state our main results on local and global well-posedness for (1.1). Actually, we will consider the following generalization of (1.1):

\[
\begin{align*}
\partial_t v - \Delta v & = - \nabla \cdot (v \cdot \nabla) v + \nabla \cdot f_n + \nabla G_n + \dot{\gamma}_n, \\
\partial_t \theta & = - \nabla \cdot (v \cdot \nabla) \theta + \nabla G_n, \\
\partial_t \bar{p} & = \left( \sum_{k=1}^{2} \gamma_{n}^{i,j,k} [\partial_j \bar{P}_n + \partial_j \int_{-h}^{z} \sigma_n(\cdot, \zeta) \theta(\cdot, \zeta) \, d\zeta \right)^2, \\
\partial_t P & = \kappa \theta + (\nabla \cdot \nabla) \theta = 0, \\
\partial_t \bar{P}_n + \sigma_n \theta & = 0, \\
\partial_t w & = 0, \\
v(\cdot,0) & = v_0, \\
\theta(\cdot,0) & = \theta_0,
\end{align*}
\]

where the above equations hold on $\mathcal{O} = T^2 \times (-h,0)$. In (3.1c), with $\int_{-h}^{z} \sigma_n(\cdot, \zeta) \theta(\cdot, \zeta) \, d\zeta$ we understand the mapping $x \mapsto \int_{-h}^{z} \sigma_n(\cdot, \zeta) \theta(\cdot, \zeta) \, d\zeta$ where $x = (x_H, x_3) \in \mathcal{O}$ with $x_H \in T^2$ and $x_3 \in (-h,0)$. A similar notation will be also employed in the sequel if no confusion seems likely.

There are two additional terms in (3.1) compared to (1.1). Firstly, (3.1a) contains the additional term $\partial_t \bar{p}$ defined in (3.1c) which takes into account the effect of the hydrostatic turbulent pressure $\bar{P}_n$ (defined in (3.6) below) on the deterministic dynamics of $v$, i.e. (3.1a). A similar term was also considered in [AHHS22]. Secondly in (3.1e) there is an additional transport type term $(\nabla \cdot \nabla) \theta$ which is also due to the effect of the turbulent pressure. Both terms $\partial_t \bar{p}$ and $(\nabla \cdot \nabla) \theta$ are motivated by the Stratonovich formulation of (3.1) and we refer to Section 8 for further discussion. Let us mention that the term $(\nabla \cdot \nabla) \theta$ gives rise to the same mathematical difficulties of $\sigma_n \theta$ in (3.1e), and therefore the problem (3.1a) is as (analytically) difficult as (1.1). Finally let us note that, comparing (1.1) and (3.1), the terms $(F_v, F_\theta, G_{v,n}, G_{\theta,n})$ are $(v, \theta)$-dependent nonlinearities.

The system (3.1) is complemented with the following boundary conditions on $T^2$:

\[
\begin{align*}
\partial_t v(\cdot,-h) & = \partial_t v(\cdot,0) = 0, \\
\partial_t \theta(\cdot,-h) & = \partial_t \theta(\cdot,0) + \alpha \theta(\cdot,0) = 0, \\
w(\cdot,-h) & = w(\cdot,0) = 0,
\end{align*}
\]
where \( \alpha \in \mathbb{R} \) is a given constant. As mentioned in Section 1, the results below are also true in case (3.2a)–(3.2b) are replaced by periodic boundary conditions, see Remark 3.13. This section is organized as follows:

- In Subsection 3.1 we reformulate (1.1) as a stochastic evolution equations for the unknown \((v, \theta)\). To this end, we introduce the hydrostatic Helmholtz projection and appropriate function spaces of divergence free vector fields.
- In Subsection 3.2 we collect the main assumptions and definitions. In particular we provide a rigorous definition of solutions to (1.1) using Itô calculus.
- In Subsection 3.3 we state local and global well-posedness results for (1.1).

3.1. Hydrostatic Helmholtz projection and reformulation of (3.1). Let us begin by introducing the Helmholtz projection on the horizontal variables \( x_H \in \mathbb{T}^2 \) which will be denoted by \( \mathbb{P}_H \).

Let \( f \in L^2(\mathcal{O}; \mathbb{R}^2) \) and denoted by \( \mathbb{Q}_H f \triangleq \nabla_H \Psi_f \in L^2(\mathbb{T}^2; \mathbb{R}^2) \) where \( \Psi_f \in H^1(\mathbb{T}^2) \) is the unique solution to

\[ \Delta_H \Psi_f = \text{div}_H f \quad \text{on} \quad \mathbb{T}^2, \quad \text{and} \quad \int_{\mathbb{T}^2} \Psi_f \, dx = 0. \]

Then the Helmholtz projection on \( \mathbb{T}^2 \) is given by

\[ \mathbb{P}_H f \triangleq f - \mathbb{Q}_H f, \quad \text{for} \quad f \in L^2(\mathbb{T}^2; \mathbb{R}^2). \]

It is easy to see that \( \mathbb{P}_H \in \mathcal{L}(L^2(\mathbb{T}^2; \mathbb{R}^2)) \) and it is an orthogonal projection. The hydrostatic Helmholtz projection on \( \mathcal{O} \) will be denote by \( \mathbb{P} \) and it defined as, for all \( f \in L^2(\mathcal{O}; \mathbb{R}^2) \) (recall that \( \int_{-h}^0 \text{d}z = \frac{1}{h} \int_{-h}^0 \text{d}z \))

\[ \mathbb{Q}_f = \mathbb{Q}_H \left[ \int_{-h}^0 f(\cdot, \zeta) \, d\zeta \right] \quad \text{and} \quad \mathbb{P}_f \triangleq f - \mathbb{Q}_f. \]

One can check that \( \mathbb{P} \in \mathcal{L}(L^2(\mathcal{O}; \mathbb{R}^2)) \), it is an orthogonal projection and \( \text{div}_H \int_{-h}^0 (\mathbb{P} f(\cdot, \zeta)) \, d\zeta = 0 \) in \( \mathcal{D}'(\mathbb{T}^2) \) for all \( f \in L^2(\mathcal{O}; \mathbb{R}^2) \). Let

\[ L^2(\mathcal{O}) = \left\{ f \in L^2(\mathcal{O}; \mathbb{R}^2) : \text{div}_H \left( \int_{-h}^0 f(\cdot, \zeta) \, d\zeta \right) = 0 \quad \text{on} \quad \mathbb{T}^2 \right\}, \]

be endowed with the norm \( \|f\|_{L^2(\mathcal{O})} \triangleq \|f\|_{L^2(\mathcal{O}; \mathbb{R}^2)} \) and for all \( k \geq 1 \) we set

\[ H^k(\mathcal{O}) \triangleq H^k(\mathcal{O}; \mathbb{R}^2) \cap L^2(\mathcal{O}), \quad \|f\|_{H^k(\mathcal{O})} \triangleq \|f\|_{H^k(\mathcal{O}; \mathbb{R}^2)}. \]

As above, for \( \mathcal{A} \in \{L^2, H^k\} \), we write \( \mathcal{A} \) instead of \( \mathcal{A}(\mathcal{O}) \), if no confusion seems likely.

Next we reformulate (3.1) as a stochastic evolution equation on \( L^2(\mathcal{O}) \times L^2(\mathcal{O}) \) for the unknown \((v, \theta)\). As usual in the context of primitive equation we start by integrating the conditions (3.1d)-(3.1f) and we obtain, a.e. on \( \mathbb{R}_+ \times \Omega \) and for all \( (x_H, x_3) \in \mathbb{T}^2 \times (-h, 0) = \mathcal{O}, \)

\[ w(\cdot, x) = -\int_{-h}^{x_3} \text{div}_H v(\cdot, x_H, \zeta) \, d\zeta, \]

\[ p(\cdot, x_H) = p(\cdot, x_H) - \int_{-h}^{x_3} \left( \kappa(\cdot, x_H, \zeta) \theta(\cdot, x_H, \zeta) + (\pi(\cdot, x_H, \zeta) \cdot \nabla) \theta(\cdot, x_H, x_3) \right) \, d\zeta, \]

\[ \tilde{P}_n(\cdot, x_H) = \tilde{P}_n(\cdot, x_H) - \int_{-h}^{x_3} \sigma_n(\cdot, x_H, \zeta) \theta(\cdot, x_H, \zeta) \, d\zeta, \]

To obtain (3.4) we also used \( w(\cdot, -h) = 0 \) by (3.2c). Note that \( w(\cdot, 0) = 0 \) is equivalent to

\[ \int_{-h}^{0} \text{div}_H v(\cdot, \zeta) \, d\zeta = 0 \quad \text{on} \quad \mathbb{T}^2. \]

Moreover, let us stress that the pressures \((p, \tilde{P}_n)\) are independent of the vertical direction \( x_3 \in (-h, 0) \). For this reason, in the physical literature, the latter are often referred as surface pressures.
Hence, the system (3.1a)–(3.1b) can be equivalently rewritten as:
\[
dv - \Delta v \, dt = \left[ -(v \cdot \nabla_H) v - w(v) \partial_3 v - \nabla_H p + \partial_3 \tilde{p}_n \right. \\
\left. + \nabla_H \int_{-h}^h [\kappa(\cdot, \cdot) \theta(\cdot, \cdot) + (\pi(\cdot, \cdot) \cdot \nabla) \theta(\cdot, \cdot)] \, d\zeta + F_v(v, \theta, \nabla v, \nabla \theta) \right] \, dt \\
+ \sum_{n \geq 1} \left[ (\phi_n \cdot \nabla) v - \nabla_H \tilde{p}_n + \nabla_H \int_{-h}^h (\sigma_n(\cdot, \cdot) \theta(\cdot, \cdot)) \, d\zeta + G_{v,n}(v, \theta) \right] \, d\beta^n_t,
\]
(3.8a)
\[
d\theta - \Delta \theta \, dt = \left[ -(v \cdot \nabla_H) \theta - w(v) \partial_3 \theta + F_\theta(v, \theta, \nabla v, \nabla \theta) \right] \, dt \\
+ \sum_{n \geq 1} \left[ (\psi_n \cdot \nabla) \theta + G_{\theta,n}(v, \theta) \right] \, d\beta^n_t,
\]
(3.8b)
on \mathcal{O}, where
\[
w(v) \overset{\text{def}}{=} - \int_{-h}^h \text{div}_H v(\cdot, \cdot) \, d\zeta.
\]
Next applying the hydrostatic Helmholtz project \( \mathbb{P} \) on (3.8a), we obtain
\[
dv - \Delta v \, dt = \left[ \mathbb{P} \left[- (v \cdot \nabla_H) v - w(v) \partial_3 v + \partial_3 \tilde{p}_n \right] \right.
\]
\[
\left. + \mathbb{P} \left[ \nabla_H \int_{-h}^h [\kappa(\cdot, \cdot) \theta(\cdot, \cdot) + (\pi(\cdot, \cdot) \cdot \nabla) \theta(\cdot, \cdot)] \, d\zeta + F_v(v, \theta, \nabla v, \nabla \theta) \right] \right) \, dt \\
+ \sum_{n \geq 1} \mathbb{P} \left[ (\phi_n \cdot \nabla) v + \nabla_H \int_{-h}^h (\sigma_n(\cdot, \cdot) \theta(\cdot, \cdot)) \, d\zeta + G_{v,n}(v, \theta) \right] \, d\beta^n_t.
\]
(3.10)
In (3.10) we used that \( \mathbb{P} \Delta v = \Delta v \) by (3.2a) and (3.7). Note that in the stochastic part of the above, the operator \( \mathbb{P} \) cannot be (in general) removed. In particular, we have
\[
\nabla_H \tilde{p}_n = \mathbb{Q} \left[ (\phi_n \cdot \nabla) v + \nabla_H \int_{-h}^h (\sigma_n(\cdot, \cdot) \theta(\cdot, \cdot)) \, d\zeta + G_{v,n}(v, \theta) \right].
\]
A similar relation holds for \( \nabla_H p \). Using the above identity and (3.6), we get
\[
\partial_3 \tilde{p} = \left( \sum_{n \geq 1} \sum_{1 \leq j \leq 2} \sum_{k=1}^{\gamma^j_n(\mathbb{Q}(v, \theta))^j} \right)
\]
\[
\partial_{\gamma^j_n(\mathbb{Q}(v, \theta))^j},
\]
where \( (\mathbb{Q}(v, \theta))^j \) denotes the \( j \)-th coordinate of the vector \( \mathbb{Q}(v, \theta) \). Therefore, we prove that the system (3.8) is equivalent to following system of SPDEs on \( \mathcal{O} \):
\[
dv - \Delta v \, dt = \left[ \mathbb{P} \left[- (v \cdot \nabla_H) v - w(v) \partial_3 v - \mathcal{P}_\gamma(v, \theta) \right] \right.
\]
\[
\left. + \nabla_H \int_{-h}^h [\kappa(\cdot, \cdot) \theta(\cdot, \cdot) + (\pi(\cdot, \cdot) \cdot \nabla) \theta(\cdot, \cdot)] \, d\zeta + F_v(v, \theta, \nabla v, \nabla \theta) \right] \, dt \\
+ \sum_{n \geq 1} \mathbb{P} \left[ (\phi_n \cdot \nabla) v + \nabla_H \int_{-h}^h (\sigma_n(\cdot, \cdot) \theta(\cdot, \cdot)) \, d\zeta + G_{v,n}(v, \theta) \right] \, d\beta^n_t,
\]
(3.12a)
\[
d\theta - \Delta \theta \, dt = \left[ -(v \cdot \nabla_H) \theta - w(v) \partial_3 \theta + F_\theta(v, \theta, \nabla v, \nabla \theta) \right] \, dt \\
+ \sum_{n \geq 1} \left[ (\psi_n \cdot \nabla) \theta + G_{\theta,n}(v, \theta) \right] \, d\beta^n_t.
\]
(3.12b)
The above problem is complemented with the following boundary conditions on \( \mathbb{T}^2 \):
\[
\partial_3 v(\cdot, -h) = \partial_3 v(\cdot, 0) = 0,
\]
(3.13a)
\[
\partial_3 \theta(\cdot, -h) = \partial_3 \theta(\cdot, 0) + \alpha \theta(\cdot, 0) = 0,
\]
(3.13b)
where $\alpha \in \mathbb{R}$ is a given constant. Note that (3.12a) yields (3.7) in case $\int_{-h}^{0} \div_{h} v_{0}(\cdot, \zeta) \, d\zeta = 0$ where $v_{0}$ is the initial condition of $v$, see (3.1g).

3.2. Main assumptions and definitions. We begin by listing the main assumptions of this section. Below we employ the notation introduced in Subsection 1.5.

**Assumption 3.1.** There exist $M, \delta > 0$ for which the following hold.

1. For all $j \in \{1, 2, 3\}$ and $n \geq 1$, the mappings $\phi_{n}^{j}, \psi_{n}^{j}, \kappa, \rho, \sigma_{n} : \mathbb{R}_{+} \times \Omega \times \mathcal{O} \to \mathbb{R}$ are $\mathcal{P} \otimes \mathcal{B}$-measurable.

2. (Parabolicity) There exists $\nu \in (0, 2)$ such, a.s. for all $t \in \mathbb{R}_{+}$, $x \in \mathcal{O}$, $\xi \in \mathbb{R}^{d}$,

$$\sum_{n \geq 1} \left( \sum_{1 \leq j \leq 3} \phi_{n}^{j}(t, x) \xi_{j} \right)^{2} \leq \nu |\xi|^{2}, \quad \text{and} \quad \sum_{n \geq 1} \left( \sum_{1 \leq j \leq 3} \psi_{n}^{j}(t, x) \xi_{j} \right)^{2} \leq \nu |\xi|^{2}. $$

3. (Regularity) a.s. for all $t \in \mathbb{R}_{+}$, $j, k \in \{1, 2, 3\}$ and $\ell, m \in \{1, 2\}$,

$$\left\| \sum_{n \geq 1} |\phi_{n}^{j}(t, \cdot)|^{2} \right\|_{L^{3+\ell}(\Omega)}^{1/2} + \left\| \sum_{n \geq 1} |\partial_{k} \phi_{n}^{j}(t, \cdot)|^{2} \right\|_{L^{3+\ell}(\Omega)}^{1/2} \leq M,$$

$$\left\| \sum_{n \geq 1} |\psi_{n}^{j}(t, \cdot)|^{2} \right\|_{L^{3+\ell}(\Omega)}^{1/2} + \left\| \sum_{n \geq 1} |\partial_{k} \psi_{n}^{j}(t, \cdot)|^{2} \right\|_{L^{3+\ell}(\Omega)}^{1/2} \leq M,$$

$$(\gamma_{n}^{\ell, m}(t, \cdot))_{n \geq 1} \|_{L^{3+\ell}(\Omega, \ell^{2})} \leq M.$$

4. a.s. for all $t \in \mathbb{R}_{+}$, $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$,

$$\|\kappa(t, \cdot)\|_{L^{\infty}(\mathbb{T}^{2}; L^{2}(-h, 0))} + \|\partial_{i} \kappa(t, \cdot)\|_{L^{2+i}(\mathbb{T}^{2}; L^{2}(-h, 0))} \leq M,$$

$$\|\rho(t, \cdot)\|_{L^{\infty}(\mathbb{T}^{2}; L^{2}(-h, 0))} + \|\partial_{i} \rho(t, \cdot)\|_{L^{2+i}(\mathbb{T}^{2}; L^{2}(-h, 0))} \leq M.$$

5. Set $\sigma \overset{\text{def}}{=} (\sigma_{n})_{n \geq 1}$. Then a.s. for all $t \in \mathbb{R}_{+}$ and $i, j \in \{1, 2\}$,

$$\|\sigma(t, \cdot)\|_{L^{\infty}(\Omega, \ell^{2})} + \|\partial_{i} \sigma(t, \cdot)\|_{L^{2+i}(\mathbb{T}^{2}; L^{2}(-h, 0), \ell^{2})}$$

$$+ \|\partial_{i,j}^{2} \sigma(t, \cdot)\|_{L^{2+i}(\mathbb{T}^{2}; L^{2}(-h, 0), \ell^{2})} \leq M.$$

6. For all $n \geq 1$, the following mappings are $\mathcal{P} \otimes \mathcal{B}$-measurable:

$$F_{v} : \mathbb{R}_{+} \times \Omega \times \mathcal{O} \times \mathbb{R}^{6} \times \mathbb{R}^{3} \times \mathbb{R}^{2} \times \mathbb{R} \to \mathbb{R}^{2}, \quad F_{\theta} : \mathbb{R}_{+} \times \Omega \times \mathcal{O} \times \mathbb{R}^{6} \times \mathbb{R}^{3} \times \mathbb{R}^{2} \times \mathbb{R} \to \mathbb{R},$$

$$G_{v, n} : \mathbb{R}_{+} \times \Omega \times \mathcal{O} \times \mathbb{R}^{2} \times \mathbb{R} \to \mathbb{R}, \quad G_{\theta, n} : \mathbb{R}_{+} \times \Omega \times \mathcal{O} \times \mathbb{R}^{2} \times \mathbb{R} \to \mathbb{R}.$$ Set

$$G_{v} \overset{\text{def}}{=} (G_{v, n})_{n \geq 1} \quad \text{and} \quad G_{\theta} \overset{\text{def}}{=} (G_{\theta, n})_{n \geq 1}.$$

7. (Global Lipschitz nonlinearities) a.s.

$$F_{v}(\cdot, 0) \in L_{\text{loc}}^{2}(\mathbb{R}_{+} \times \mathcal{O}; \mathbb{R}^{2}), \quad F_{\theta}(\cdot, 0) \in L_{\text{loc}}^{2}(\mathbb{R}_{+} \times \mathcal{O})$$

$$G_{v}(\cdot, 0) \in L_{\text{loc}}^{2}(\mathbb{R}_{+}; H^{1}(\mathcal{O}; \ell^{2}(\mathbb{N}, \mathbb{R}^{2}))), \quad G_{\theta}(\cdot, 0) \in L_{\text{loc}}^{2}(\mathbb{R}_{+}; H^{1}(\mathcal{O}; \ell^{2})).$$

Moreover, there exists $K \geq 1$ such that, for all $u \in \{v, \theta\}$, a.e. on $\mathbb{R}_{+} \times \Omega \times \mathcal{O}$ and for all $y, y' \in \mathbb{R}^{2}$, $Y, Y' \in \mathbb{R}^{6}$, $z, z' \in \mathbb{R}$ and $Z, Z' \in \mathbb{R}^{3}$,

$$|F_{u}(\cdot, y, z, Y, Z) - F_{u}(\cdot, y', z', Y', Z')| \leq K(|y - y'| + |z - z'|$$

$$+ |Y - Y'| + |Z - Z'|),$$

$$\|G_{u}(\cdot, y, z) - G_{u}(\cdot, y', z')\|_{\ell^{2}} + \|\nabla_{y} G_{u}(\cdot, y, z) - \nabla_{y} G_{u}(\cdot, y', z')\|_{\ell^{2}} \leq K(|y - y'| + |z - z'|),$$

$$\|\nabla_{y} G_{u}(\cdot, y, z) - \nabla_{y} G_{u}(\cdot, y', z')\|_{\ell^{2}} \leq K.$$
• As in [AHHS22, Remark 3.2(c)], (2) is equivalent with the usual stochastic parabolicity and therefore (2) is optimal in the parabolic setting.

• The globally Lipschitz assumption (7) can be weaken still keeping true the results of this manuscript. We refer to Remark 3.12 for more details.

Next we define $L^2$-strong solutions to (3.1)–(3.2). Motivated by the reformulation of (3.1) performed in Subsection 3.1, we consider the equivalent system (3.12) for the unknown $(v, \theta)$ while the unknown $(P, \tilde{P}, w)$ are determinate uniquely by $(v, \theta)$. Taking into account the boundary conditions (3.13) and the divergence free condition (3.7) for the velocity $v$, we introduce the following spaces:

\begin{align}
\mathbb{H}_h^2(\Omega) & \overset{\text{def}}{=} \left\{ v \in H^2(\Omega; \mathbb{R}^2) \cap L^2(\Omega) \mid \partial_3 v(\cdot, -h) = \partial_3 v(\cdot, 0) = 0 \text{ on } T^2 \right\}, \\
H_0^2(\Omega) & \overset{\text{def}}{=} \left\{ \theta \in H^2(\Omega) \mid \partial_3 \theta(\cdot, 0) + a \partial_3 \theta(\cdot, -h) = 0 \text{ on } T^2 \right\}.
\end{align}

Note that the boundary conditions (3.13) are included in the above spaces. Hence, the spaces $\mathbb{H}^2$ and $H_0^2$ serve as regularity spaces for the unknowns $v$ and $\theta$, respectively.

Finally, we denote by $B_{\ell^2}$ the $\ell^2$-cylindrical Brownian motion induced by $(\beta^n)_{n\geq 1}$, i.e.

\begin{equation}
B_{\ell^2}(f) \overset{\text{def}}{=} \sum_{n \geq 1} \int_{\mathbb{R}^+} f_n(t) \, d\beta^n_t \quad \text{where } f = (f_n)_{n \geq 1} \in L^2(\mathbb{R}_+; \ell^2).
\end{equation}

**Definition 3.3** ($L^2$–local, maximal and global strong solutions). Let Assumption 3.1 be satisfied. Let $\tau$ be a stopping time with values in $[0, \infty]$. Consider two stochastic processes $v : [0, \tau] \times \Omega \to \mathbb{H}_h^2(\Omega)$ and $\theta : [0, \tau] \times \Omega \to H_0^2(\Omega)$.

- We say that $((v, \theta), \tau)$ is called an $L^2$–local strong solution to (3.1)–(3.2) if there exists a sequence of stopping time $(\tau_k)_{k \geq 1}$ for which the following hold:

  - $\tau_k \leq \tau$ a.s. for all $k \geq 1$ and $\lim_{k \to \infty} \tau_k = \tau$ a.s.
  - For all $k \geq 1$, the process $1_{[0, \tau_k]}(v, \theta)$ is progressively measurable.
  - a.s. we have $(v, \theta) \in L^2(0, \tau_k; \mathbb{H}_h^2(\Omega) \times H_0^2(\Omega))$ and

\begin{align}
& (v \cdot \nabla_H) v + w(v) \partial_3 v + F_v(v, \theta, \nabla v, \nabla \theta) + P_\gamma(\cdot, v, \theta) \in L^2(0, \tau_k; L^2(\mathbb{R}^2)), \\
& (v \cdot \nabla_H) \theta + w(v) \partial_3 \theta + F_\theta(v, \theta, \nabla v, \nabla \theta) \in L^2(0, \tau_k; L^2(\mathbb{R}^2)), \\
& \quad (G_{v, n}(v, \theta))_{n \geq 1} \in L^2(0, \tau_k; H^1(\mathbb{R}_+; \ell^2(\mathbb{N}; \mathbb{R}^2))), \\
& \quad (G_\theta, n(v, \theta))_{n \geq 1} \in L^2(0, \tau_k; H^1(\mathbb{R}_+; \ell^2)) .
\end{align}

- The following equality holds a.s. for all $k \geq 1$ and $t \in [0, \tau_k]$:

\begin{align*}
& v(t) - v_0 = \int_0^t \left( \Delta v(s) + \mathbb{P} \left[ \int_{-h}^h \nabla_H [\kappa(\cdot, \cdot) \theta(\cdot, \cdot) + (\pi(\cdot, \cdot) \cdot \nabla) \theta(\cdot, \cdot)] \, d\zeta \right. \right. \\
& \quad \left. \left. - (v \cdot \nabla_H) v - w(v) \partial_3 v + F_v(v, \theta, \nabla v, \nabla \theta) + P_\gamma(\cdot, v, \theta) \right) \right] \, ds \\
& \quad + \int_0^t \left( 1_{[0, \tau_k]} \mathbb{P} \left[ \phi \cdot \nabla v + \int_{-h}^h \nabla_H [\sigma(\cdot, \cdot) \theta(\cdot, \cdot)] \, d\zeta + G_{v, n}(v, \theta) \right] \right)_{n \geq 1} \, dB_{\ell^2}(s), \\
& \theta(t) - \theta_0 = \int_0^t \left[ \Delta \theta - (v \cdot \nabla_H) \theta - w(v) \partial_3 \theta + F_\theta(v, \theta, \nabla v, \nabla \theta) \right] \, ds \\
& \quad + \int_0^t \left( 1_{[0, \tau_k]} \mathbb{P} \left[ \psi \cdot \nabla \theta + G_\theta, n(v, \theta) \right] \right)_{n \geq 1} \, dB_{\ell^2}(s).
\end{align*}

In the following, we say that $(\tau_k)_{k \geq 1}$ is a localizing sequence for $(v, \tau)$.

- An $L^2$–local strong solution $((v, \theta), \tau)$ to (3.1)–(3.2) is said to be a (unique) $L^2$–maximal strong solution to (3.1)–(3.2) if for any other local solution $((v', \theta'), \tau')$ we have $\tau' \leq \tau$ a.s. and $(v, \theta) = (v', \theta')$ a.e. on $[0, \tau') \times \Omega$.

- An $L^2$–maximal strong solution $((v, \theta), \tau)$ to (3.1)–(3.2) is called an $L^2$–global strong solution if $\tau = \infty$ a.s. In such a case, we write $(v, \theta)$ instead of $((v, \theta), \tau)$.
Note that $L^2$-maximal strong solution are unique in the class of $L^2$–local strong solutions due to the above definition. By (3.17), the deterministic integrals and the stochastic integrals in the above definition are well-defined as an $L^2$–valued Bochner and $H^1$–valued Itô integrals, respectively.

3.3. Main results. To economize the notation, through this manuscript we let
\[(3.18)\quad H \overset{\text{def}}{=} \mathbb{H}^1(O) \times H^1(O) \quad \text{and} \quad V \overset{\text{def}}{=} \mathbb{H}^2(O) \times H^2(O) .\]
Below $H$ and $V$ plays the role of the trace and regularity space for (3.1)–(3.2), respectively.

We begin by stating a local existence result for (3.1)–(3.2).

**Theorem 3.4** (Local existence and blow-up criterion). Let Assumption 3.1 be satisfied. Let $(v_0, \theta_0) \in L^2_{\mathcal{P}}(\Omega; H)$. Then (3.1)–(3.2) has a (unique) $L^2$–maximal strong solution $(v, \theta, \tau)$ such that
\[(\tau > 0 \text{ a.s. and } (v, \theta) \in L^2_{\text{loc}}([0, \tau); V) \cap C([0, \tau); H) \text{ a.s.).} \]
Finally, for all $T \in (0, \infty)$,
\[(3.19)\quad \mathbb{P}\left( \tau < T, \sup_{t \in [0, \tau)} \|v(t)\|_H^2 + \int_0^\tau \|v(t)\|_V^2 \, dt < \infty \right) = 0. \]

The proof of Theorem 3.4 follows as in [AHHS22] where we checked the applicability of the abstract results of [AV22b, AV22c]. A Sketch of the proof of Theorem 3.4 will be given in Subsection 3.3. The statement (3.19) will be referred as blow-up criterion as it shows that explosion $\tau = \infty$ can only happen if either $(v, \theta) \notin C([0, \tau); H)$ or $(v, \theta) \notin L^2([0, \tau); V)$ for some $\tau < \infty$. Let us note that, since $(v, \theta) \in V$ a.e. on $[0, \tau) \times \Omega$, the blow-up criterion (3.19) is equivalent to
\[\mathbb{P}\left( \tau < T, \sup_{t \in [0, \tau]} \left[ \|v(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 \right] + \int_0^\tau \|v(t)\|_{H^2}^2 + \|	heta(t)\|_{H^2}^2 \, dt < \infty \right) = 0. \]

Let us turn now our attention to existence of global solutions to (3.1)–(3.2). In contrast to the local existence result of Theorem 3.4, the global existence is much more involved. In particular, in addition to Assumption 3.1 we will also need the following assumptions.

**Assumption 3.5.** a.s. and for all $n \geq 1$, $x = (x_H, x_3) \in \mathbb{T}^2 \times (-h, 0) = O$, $t \in \mathbb{R}^+$, $j, k \in \{1, 2\}$
\[\psi^i_n(t, x), \phi^i_n(t, x), \gamma^i_{jk}(t, x), \pi^i(t, x) \text{ and } \sigma_n(t, x) \text{ are independent of } x_3. \]

We do not know if any of the above hypotheses can be removed in general. Note that there are no additional assumptions on $(\phi^i_n, \psi^i_n)$. However, in case of isothermal turbulent pressure, the conditions on $\psi^i$ in Assumption 3.5 can be removed (see [AHHS22, Sections 3 and 6] and Remark 3.10 below). The physical relevance of the $x_3$-independence of $(\phi^i_n, \psi^i_n, \sigma_n)$ is discussed in Remarks 2.1, 2.2 and 2.3. While, for $(\gamma^i_{jk} , \pi^i)$, in the important case they are related to the Stratonovich formulation of the primitive equations (see Section 8), the $x_3$-independence comes from the one of $(\phi^i_n, \psi^i_n, \sigma_n)$ (cf. formula (8.7) in Section 8).

We are ready to state the main results of this paper. For notational convenience we set
\[(3.20)\quad \Xi(t) \overset{\text{def}}{=} \|F(t, \cdot, 0)\|_{L^2} + \|F_0(t, \cdot, 0)\|_{L^2} + \|G_v(t, \cdot, 0)\|_{H^1(E)} + \|G_\theta(t, \cdot, 0)\|_{H^1(E)} . \]
Note that $\Xi \in L^2(0, T)$ a.s. for all $T < \infty$ by Assumption 3.1(7).

**Theorem 3.6** (Global existence and energy estimates). Let Assumptions 3.1 and 3.5 be satisfied. Let $(v_0, \theta_0) \in L^2_{\mathcal{P}}(\Omega; H)$. Then there exists a (unique) $L^2$–global strong solution $(v, \theta, \tau)$ to (3.1)–(3.2) such that
\[\tau > 0 \text{ a.s. and } (v, \theta) \in L^2_{\text{loc}}([0, \infty); V) \cap C([0, \infty); H) \text{ a.s.} \]
Finally, for all $T \in (0, \infty)$ there exists $C_T > 0$, independent of $(v_0, \theta_0)$, such that, for all $\gamma > e^c$,
\[\mathbb{E}\sup_{t \in [0, T]} \|v(t)\|_{L^2}^2 + \mathbb{E}\int_0^T \|v(t)\|_{H^1}^2 \, dt \leq C_T (1 + \mathbb{E}\|v_0\|_{L^2}^2 + \mathbb{E}\|\theta_0\|_{L^2}^2 + \mathbb{E}\Xi(t)_{L^2(0, T)}), \]
\[\mathbb{E}\sup_{t \in [0, T]} \|\theta(t)\|_{L^2}^2 + \mathbb{E}\int_0^T \|\theta(t)\|_{H^1}^2 \, dt \leq C_T (1 + \mathbb{E}\|v_0\|_{L^2}^2 + \mathbb{E}\|\theta_0\|_{L^2}^2 + \mathbb{E}\Xi(t)_{L^2(0, T)}), \]
turbulent pressure, see Remark 3.10 below. The estimates of Theorem 3.6 can be (slightly) improved in case of isothermal case where one obtains estimates with exponentially increasing constants in the size of the data. Each of them costs a log-factor. The same also appears in the deterministic case where one obtains estimates with exponentially increasing constants in the size of the data. However it seems not possible to improve the estimates as they come from three applications of Theorem 3.7 in particular implies that \(H\). Therefore it seems not possible to improve the estimates as they come from three applications of the Gronwall lemma. Each of them costs a log-factor. The same also appears in the deterministic case where one obtains estimates with exponentially increasing constants in the size of the data (see e.g. [CT07]). The estimates of Theorem 3.6 can be (slightly) improved in case of isothermal turbulent pressure, see Remark 3.10 below.

Theorem 3.6 and the following result show that the problem (3.1)-(3.2) is globally well-posed. Recall that \(\zeta_n \rightarrow \xi\) in probability in \(Y\) provided \(\lim_{n \rightarrow \infty} \mathbb{P}(\|\zeta_n - \xi\| < \varepsilon) = 0\) for all \(\varepsilon > 0\).

**Theorem 3.7** (Continuous dependence on the initial data). Let Assumptions 3.1 and 3.5 be satisfied. Suppose that \((v_n, \theta_n)_{n \geq 1} \subseteq L^0_{\mathcal{G}_0}(\Omega; H)\) is a sequence of initial data converging in probability in \(H\) to some \((v_0, \theta_0)\). Let \((v_n, \theta_n)\) and \((v, \theta)\) be the \(L^2\)-global strong solutions to (3.1)-(3.2) with initial data \((v_n, \theta_n)\) and \((v_0, \theta_0)\), respectively. Then, for all \(T \in (0, \infty)\),

\[
(v_n, \theta_n) \rightarrow (v, \theta) \quad \text{as } n \rightarrow \infty \quad \text{in probability in } C([0, T]; H) \cap L^2(0, T; V).
\]

The proof of Theorems 3.6 and 3.7 will be given in Subsections 4.2 and 4.3, respectively. Both results essentially depend on the energy estimate of Proposition 4.2. The proof of the latter will be the major scope of our work and to its proof are devoted Sections 5, 6 and 7. Finally, in Section 8 we discuss the case of Stratonovich noise.

We conclude this section with several remarks related to Theorems 3.6 and 3.7.

**Remark 3.8** (Feller property). Let \((v, \theta)\) be the global strong solution to (3.1)-(3.2) provided by Theorem 3.6 with initial data \((\eta, \xi) \in H\). For all \(t \geq 0\), set

\[
[S_t \varphi](\eta, \xi) = \mathbb{E}[\varphi(v(t), \theta(t)) \mid \mathcal{F}_t] \quad \text{for all } (\eta, \xi) \in H \quad \text{and } \varphi \in C(H; \mathbb{R}).
\]

**Theorem 3.7** in particular implies that \(S_t\) maps continuously \(C(H; \mathbb{R})\) into itself. This is often referred as Feller property. In particular, our results extend [GHKV14, Theorem 1.5]. In spirit of [GHKVZ14], it would be interesting to study the existence and/or uniqueness of invariant measures. However, this goes beyond the scope of this paper.

**Remark 3.9** (\(\Omega\)–localization of energy estimates). The energy estimates in Theorem 3.6 also implies tail probability estimates for non–integrable data by using localization arguments. To see this let \((v_0, \theta_0) \in L^0_{\mathcal{G}_0}(\Omega; H)\). Fix \(\delta > 0\) and set \((v_0^{(\delta)}, \theta_0^{(\delta)}) = 1_{\{\|v_0, \theta_0\|_H \leq \delta\}}(v_0, \theta_0)\). Let \((v^{(\delta)}, \theta^{(\delta)})\) be the global strong solution to (3.1)-(3.2) with initial data \((v_0^{(\delta)}, \theta_0^{(\delta)})\) provided by Theorem 3.6. Then by [AV22b, Theorem 4.7(4)] we have \((v^{(\delta)}, \theta^{(\delta)}) = (v, \theta)\) a.e. on \(\mathbb{R}^+ \times \{\|v_0, \theta_0\|_H \leq \delta\}\). Hence, for all \(\gamma, \delta > 0\),

\[
\mathbb{P}(\sup_{t \in [0, T]} \|v(t)\|_{H^1} + \int_0^T \|v(t)\|_{H^2} dt \geq \gamma) \leq \mathbb{P}(\sup_{t \in [0, T]} \|v^{(\delta)}(t)\|_{H^1} + \int_0^T \|v^{(\delta)}(t)\|_{H^2} dt \geq \gamma, \|v_0, \theta_0\|_H \leq \delta) + \mathbb{P}(\|v_0, \theta_0\|_H > \delta)
\]

\[
\leq C_T \left(1 + 2\delta^4 + \mathbb{E}[\Xi_2^2 L^2(0, T)] + \mathbb{P}(\|v_0, \theta_0\|_H > \delta)\right) \log \log \log(\gamma),
\]

where in the last inequality we applied the third estimate of Theorem 3.6. For instance, we may choose \(\delta = \log \log \log(\gamma)\) and the above estimate shows that the tail of the r.v. \(\mathbb{P}(\sup_{t \in [0, T]} \|v(t)\|_{H^1} + \|v(t)\|_{L^2(\Omega; H)}^2 \geq \gamma)\) converges to 0 as \(\gamma \rightarrow \infty\) with an explicit rate. A similar argument also holds for the other estimates in Theorem 3.6. For the first two one also applies the Chebyshev inequality.

A similar argument also works if one only knows that \(\Xi \in L^2(0, T)\) a.s. for all \(T < \infty\).
Remark 3.10 (Improved energy estimates in case of isothermal turbulent pressure). The tail estimates of Theorem 3.6 are new even in case of isothermal turbulent pressure $\sigma_n \equiv 0$ and $\pi \equiv 0$, as considered in [AHHS22]. However, following the proofs in [AHHS22] and the one presented here, one sees that the tail estimates of Theorem 3.6 can be improved in the setting considered in [AHHS22]. Indeed, as shown in [AHHS22], the tail estimate for $\sup_1 \|\|^{2}_{H^1} + \|\|^{2}_{L^2}$ of Lemma 5.1 are not needed as a starting point. Hence, following the arguments in [AHHS22] and using the stochastic Grownall lemma of [AV22a, Lemma A.1] as in the present paper, one sees that the log-log-log($\gamma$) decay of the tail estimates in Theorem 3.6 can be improved to a log-log($\gamma$)-one.

Remark 3.11 (Non homogeneous viscosity/conductivity). Arguing as in [AHHS22, Section 7], one can check that Theorems 3.6 and 3.7 extends to the case of inhomogeneous viscosity and/or conductivity. More precisely, we may replace the terms $\Delta u$ and $\Delta \theta$ in (3.1a)-(3.1b) by

$$\mathbb{P} \left[ \sum_{1 \leq i,j \leq 3} a_{ij}^{k} \partial^{k}_{ij} v + \sum_{1 \leq k \leq 3} b_{k}^{i} \partial_{k} v \right] \quad \text{and} \quad \sum_{1 \leq i,j \leq 3} a_{ij}^{k} \partial^{k}_{ij} \theta + \sum_{1 \leq k \leq 3} b_{k}^{i} \partial_{k} \theta, \quad \text{respectively.}$$

The above situation arises in the case of noise in the Stratonovich formulation of (3.1)–(3.2), see Section 8. We may also consider 0-th order terms in (3.21). However, as they are not needed in Section 8, we do not consider such terms here. We leave the details to the interested reader.

The local existence result of Theorem 3.4 extends to such situation under suitable assumptions on $(a_{ij}, b_{ij}, a_{ij}, b_{ij})$. More precisely, in addition to Assumption 3.1(1), (3)–(7) and Assumption 3.5, one assumes that:

- (Measurability) $a_{ij}^{k}, b_{ij}, b_{ij}^{k}, \phi_{ij}^{k}, \psi_{ij}^{k}: \mathbb{R}_{+} \times \Omega \to \mathbb{R}$ are $\mathcal{P} \otimes \mathcal{B}(\Omega)$-measurable.
- (Parabolicity) There exists $\nu > 0$ such that, a.e. on $\mathbb{R}_{+} \times \Omega \times \mathbb{R}^{3}$, we have:

$$\sum_{1 \leq i,j \leq 3} \left( a_{ij}^{k} - \frac{1}{2} \sum_{n \geq 1} \phi_{ij}^{k,n} \right) \xi_{i} \xi_{j} \geq \nu |\xi|^{2} \quad \text{and} \quad \sum_{1 \leq i,j \leq 3} \left( a_{ij}^{k} - \frac{1}{2} \sum_{n \geq 1} \psi_{ij}^{k,n} \right) \xi_{i} \xi_{j} \geq \nu |\xi|^{2}.$$

- (Regularity) There exist $M, \delta > 0$ such that, a.e. on $\mathbb{R}_{+} \times \Omega$,

$$|a_{ij}^{0}|_{H^{1+\delta}(\mathcal{O};\mathbb{R}^{d})} + |a_{ij}^{0}|_{L^{1+\delta}(\mathcal{O};\mathbb{R}^{d})} + |b_{ij}^{0}|_{L^{1+\delta}(\mathcal{O};\mathbb{R}^{3})} + |b_{ij}^{0}|_{L^{1+\delta}(\mathcal{O};\mathbb{R}^{3})} \leq M.$$

- a.s. for all $t \in \mathbb{R}_{+}$, $x_{H} \in \mathbb{T}^{2}$ and $R \in \{v, \theta\}$,

$$\|a_{R}^{3,ij}(t, \cdot, 0)\|_{H^{1+\delta}(\mathbb{T}^{2})} = \|a_{R}^{3,ij}(t, \cdot, -h)\|_{H^{1+\delta}(\mathbb{T}^{2})} \leq M.$$

We refer to [AHHS22, Assumption 7.4 and Remark 7.6] for a discussion on the above conditions.

Similarly, as in [AHHS22, Section 7], the global well-posedness result of Theorems 3.6 and 3.7 also extend to the case of inhomogeneous viscosity and/or conductivity by also assuming that:

- For all $i, j \in \{1, 2\}$, the maps $(a_{ij}^{0}, b_{ij}^{0}, b_{ij}^{0})$ are independent of $x_{3}$. a.s. for all $t \in \mathbb{R}_{+}$, $x_{H} \in \mathbb{T}^{2}$ and $R \in \{v, \theta\}$,

$$a_{ij}^{3,ij}(t, x_{H}, 0) = a_{ij}^{3,ij}(t, x_{H}, -h) = 0.$$

Note that the condition on $a_{ij}^{3,ij}|_{\mathbb{T}^{2} \times (-h, 0)}$ is stronger than the one needed for local existence.

Remark 3.12 (Weakening the assumptions on the nonlinearities). Assumption 3.1(6)–(7) can be generalized still keeping true (a subset of) Theorems 3.4 and 3.6–3.7. More precisely:

(a) Theorem 3.4 holds if Assumptions 3.1(6)–(7) are replaced by $[AV22c, (HF)-(HG)]$ with $X_{0} = L^{2}(\mathcal{O}) \times L^{2}(\mathcal{O})$ and $X_{1} = H^{0}_{0}(\mathcal{O}) \times H^{0}_{0}(\mathcal{O})$. In particular, instead of the global Lipschitz condition we may require the locally Lipschitz condition (4.14) below.

(b) Theorems 3.6–3.7 still hold if Assumptions 3.1(6)–(7) are replaced by the conditions in (a) and a (sub-linear) condition: There exists $\Xi \in L^{0}(\mathcal{T} \times \mathcal{O})$ for all $T < \infty$ such that, for all $(v', \theta') \in V$ and a.e. on $\mathbb{R}_{+} \times \Omega$,

$$\|F_{u}(\cdot, v', \theta', \nabla v', \nabla \theta')\|_{L^{2}} + \|G_{u}(\cdot, v', \theta')\|_{H^{1}((T, \mathcal{O}))} \leq \Xi + \|v'\|_{H^{1}} + \|\theta'\|_{H^{1}}, \quad \text{for } u \in \{v, \theta\}.\]
Remark 3.13 (Periodic boundary conditions in all directions). The contents of Theorems 3.6 and 3.7 also hold in case the boundary conditions (3.2) are replaced by periodic ones. The proofs remain essentially unchanged up to substitute $\mathbb{T}^3$ and neglecting in the computations the contributions coming from the boundary conditions.

4. Proof of Theorems 3.4, 3.6 and 3.7

Through this section we let $H = \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega)$ and $V = \mathbb{H}_0^2(\Omega) \times \mathbb{H}_0^2(\Omega)$, cf. (3.18).

4.1. Proof of Theorem 3.4. The proof of Theorem 3.4 follows as in [AHHS22, Section 6.4] by using the results of [AV22b, AV22c] (see [AHHS22, Section 5.1] for a similar situation).

We begin by reformulating (3.1)–(3.2) as a stochastic evolution equation on the Banach space $V_0 \overset{\text{def}}{=} L^2(\Omega) \times L^2(\Omega)$ for the unknown $U \overset{\text{def}}{=} (v, \theta)$:

$$
\begin{cases}
   dU + A(\cdot)U \,dt = F(\cdot, U) \,dt + [(B_n(\cdot)U + G_n(\cdot, U))]_{n \geq 1} \,dB_\xi(t), \\
   U(0) = (v_0, \theta_0),
\end{cases}
$$

(4.1)

where $(A, B, F, G)$ are given below and $B_\xi$ is as in (3.16). Before describing $(A, B, F, G)$, we introduce some more notation. Firstly, for a weakly differentiable map $f$, we set

$$
[Jf](x) \overset{\text{def}}{=} \nabla_H \int_{-h}^{x_3} f(x_H, \zeta) \,d\zeta, \quad \text{for} \quad x = (x_H, x_3) \in \mathbb{T}^2 \times (-h, 0).
$$

Moreover, set

$$
P_{\gamma, \phi}(v, \theta) \overset{\text{def}}{=} \left( \sum_{n \geq 1} \gamma_n \phi \left[ (\phi_n \cdot \nabla) v + \nabla_H \int_{-h}^{x_3} (\sigma_n(\cdot, \zeta) \theta(\cdot, \zeta)) \,d\zeta \right] \right)^2_{k=1},
$$

$$
P_{\gamma, G}(v, \theta) \overset{\text{def}}{=} \left( \sum_{n \geq 1} \sum_{1 \leq j \leq 2} \gamma_n \phi \left[ G_{v,n}(v, \theta) ) \right] \right)^2_{k=1}.
$$

Note that $P_{\gamma, \phi}(v, \theta) + P_{\gamma, G}(v, \theta) = P_{\gamma}(v, \theta)$, where $P_{\gamma}$ is as in (3.11).

We can now make explicit the terms in (4.1):

$$
A(\cdot, U) \overset{\text{def}}{=} \left[ \Delta u - \mathbb{P} \left[ J(\kappa \theta + (\sigma \cdot \nabla) \theta) + P_{\gamma, \phi}(\cdot, v, \theta) \right] \right],
$$

(4.2)

$$
B_n(\cdot, U) \overset{\text{def}}{=} \mathbb{P} \left[ (\phi_n \cdot \nabla) v + J(\sigma_n \theta) \right], \quad B(\cdot, U) = (B_n(\cdot, U))_{n \geq 1},
$$

(4.3)

$$
F(\cdot, U) \overset{\text{def}}{=} \mathbb{P} \left[ (v \cdot \nabla_H) v + w(v) \cdot \partial_3 v + F_v(\cdot, v, \theta, \nabla v, \nabla \theta) + P_{\gamma, G}(\cdot, v, \theta) \right],
$$

(4.4)

$$
G(\cdot, U) \overset{\text{def}}{=} \mathbb{P} \left[ G_{v,n}(\cdot, v, \theta) \right], \quad G(\cdot, U) = (G_n(\cdot, U))_{n \geq 1},
$$

(4.5)

where $w(v)$ is as in (3.9).

By virtue of Definition 3.3, one can see that $((v, \theta), \tau)$ is a $L^2$–maximal (resp. $L^2$–local) strong solution to (3.1)–(3.2) if and only if $(U, \tau)$ where $U \overset{\text{def}}{=} (v, \theta)$ is an $L^2$–maximal (resp. $L^2$–local) solution to (4.1) in the sense of [AV22b, Definition 4.4] (see also [AV22c, Remark 5.6] and Lemma 4.1 below).

Now Theorem 3.4 can be prove as [AHHS22, Theorem 6.4]. To avoid repetitions, below we only give a sketch of the proof of the maximal $L^2$–regularity for the linearized system of (3.1)–(3.2) which is the main ingredient to apply the results of [AV22b, AV22c] (see [AHHS22, Proposition 6.8] for the case of isothermal turbulent pressure). Below we set $H(\ell^2) \overset{\text{def}}{=} L_2(\ell^2, H)$ where $L_2$ denotes the class of Hilbert-Schmidt operators.

Lemma 4.1 (Stochastic maximal $L^2$–regularity). Let Assumption 3.1(1)–(4) be satisfied. Fix $T \in (0, \infty)$ and let $\tau$ be a stopping time with values in $[0, T]$. Let

$$
f \in L^2_{\mathbb{P}}((0, \tau) \times \Omega; L^2 \times L^2) \quad \text{and} \quad g \in L^2_{\mathbb{P}}((0, \tau) \times \Omega; H(\ell^2)).
$$


Then there exists a unique $U \in L^2_{\mathcal{F}}((0, \tau) \times \Omega; V)$ such that, a.s. for all $t \in [0, T]$,

$$U(t) = \int_0^t (A(s)U(s) + f(s)) \, ds + \int_0^t (B_n(s)U(s) + g_n(s)) \, dB_n(s).$$

Moreover, there exists $C > 0$, independent of $(f, g)$, such that for all $U \in L^2_{\mathcal{F}}((0, \tau) \times \Omega; V)$ satisfying (4.6) one has $U \in C([0, \tau]; H)$ a.s. and

$$E[\|U\|^2_{L^2([0, \tau]; H)}} \leq C(E[\|f\|^2_{L^2([0, \tau]; L^2)}] + E[\|U\|^2_{L^2([0, \tau]; H)}]).$$

Combining the above result with [AV22b, Proposition 3.9], in (4.6) we can also allow non-trivial initial data from the space $L^2_{\mathcal{F}_0}(\Omega; H).

Proof of Lemma 4.1 – Sketch. The proof is similar to the one of [AHHS22, Proposition 6.8] and therefore we only give a sketch of the proof. As in [AHHS22] we consider only the case $\gamma \equiv 0$ as one can check that the term $P_{\gamma, \alpha}(v, \theta)$ is lower order, and therefore [AV21a, Theorem 3.2] applies.

As in [AHHS22], we used the method of continuity of [AV22c, Proposition 3.13 and Remark 3.14]. Hence, for $\lambda \in [0, 1]$, consider, on $\mathcal{O}$,

$$dv - \Delta v \, dt = \left[f_v + \lambda J(\kappa \theta + (\pi \cdot \nabla) \theta)\right] dt$$

$$+ \sum_{n \geq 1} \left[\lambda \bar{P}[(\phi_n \cdot \nabla)v] + J(\sigma_n \theta) + g_{v,n}\right] \, dB_n^v,$$

$$d\theta - \Delta \theta \, dt = d\theta + \sum_{n \geq 1} \left[(\psi_n \cdot \nabla) \theta + g_{\theta,n}\right] \, dB_n^\theta,$$

$$\int_{-\ell}^0 \text{div}_H v(\cdot, \zeta) \, d\zeta = 0,$$

$$v(\cdot, 0) = 0, \quad \text{and} \quad \theta(\cdot, 0) = 0.$$}

The above problem is complemented with the boundary conditions (3.13).

The above linear problem (4.7) coincides with (4.6) in case $\lambda = 1$ (recall that we are assuming $\gamma \equiv 0$). By the above mentioned method of continuity, it is enough to show a priori estimates for $L^2$-strong solutions of (4.7) (i.e. $(v, \theta) \in L^2((0, \tau) \times \Omega; V) \cap L^2((0, \tau); H))$) with constants independent of $\lambda$. More precisely, by [AV22c, Proposition 3.13 and Remark 3.14], it is enough to show that, for all $L^2$-strong solutions $(v, \theta)$ to (4.7),

$$E[\|v\|^2_{L^2((0, \tau); H)}] + E[\|\theta\|^2_{L^2((0, \tau); H^2)}] \leq E[f_v\|\theta\|^2_{L^2(0, \tau; L^2)}] + E[g_v\|\theta\|^2_{L^2(0, \tau; H^1(\ell^2))}$$

$$+ E[f_\theta\|v\|^2_{L^2(0, \tau; L^2)}] + E[g_\theta\|v\|^2_{L^2(0, \tau; H^1(\ell^2))}],$$

with an the implicit constant that is independent of $\lambda$. To this end, as in the proof of [AHHS22, Proposition 6.8], the idea is to use the triangular structure of the system (4.7), i.e. the velocity $v$ does not appear in the equation for the temperature (4.7b). Therefore, one can first obtain an estimate for $\theta$ and then use it in estimating $v$.

We begin by estimating $\theta$. As in [AHHS22, Proposition 6.8] (see also Step 1 in the proof of [AHHS22, Proposition 4.1]) an application of the Itô formula to $\theta \mapsto \|\nabla \theta\|_{L^2}$, an integration by part and Assumption 3.1(2) yield

$$E[\|\theta\|^2_{L^2((0, \tau); H^2)}] \leq E[f_\theta\|\theta\|^2_{L^2(0, \tau; L^2)}] + E[g_\theta\|\theta\|^2_{L^2(0, \tau; H^1(\ell^2))}]$$

where the implicit constant is independent of $\lambda \in [0, 1]$ and we set $H^1(\ell^2) \overset{\text{def}}{=} H^1(\mathcal{O}; \ell^2)$.

The same argument also applies to $v$. Since $v$ solves (4.7a), we have

$$E[\|v\|^2_{L^2((0, \tau); H^2)}] \leq E[f_v\|v\|^2_{L^2(0, \tau; L^2)}] + E[g_v\|v\|^2_{L^2(0, \tau; H^1(\ell^2))}$$

$$+ E[(\|\nabla \theta\|_{L^2})_{n \geq 1}\|\theta\|^2_{L^2(0, \tau; H^1(\ell^2))}]$$

$$+ E[\|\theta\|^2_{L^2(0, \tau; H^1(\ell^2))}],$$

where the implicit constant is independent of $\lambda \in [0, 1]$. 


By (4.9)–(4.10), to obtain (4.8), it remains to show that, for all \( \varphi \in H^2 \),
\[
(4.11) \quad \| (J(\sigma \varphi))_{n \geq 1} \|_{H^1(T^2)} + \| J(\kappa \varphi) \|_{L^2} + \| J((\pi \cdot \nabla) \varphi) \|_{L^2} \lesssim_M \| \varphi \|_{H^2},
\]
where \( M \) is as in Assumption 3.1. For brevity, we only provide some details for the estimate of \( \| (J(\sigma \varphi))_{n \geq 1} \|_{H^1(T^2)} \). The other follow similarly by using Assumption 3.1(4) instead of 3.1(5).

Let \( r \in (1, \infty) \) be such that \( \frac{1}{r} + \frac{1}{2 + \delta} = \frac{1}{2} \), where \( \delta > 0 \) is as in Assumption 3.1. To estimate \( \| (J(\sigma \varphi))_{n \geq 1} \|_{H^1(T^2)} \), firstly, note that,
\[
\| (J(\sigma \varphi))_{n \geq 1} \|_{L^2(T^2)} \lesssim \left( \left( \int_{-h}^{0} \| \sigma \|_{L^2}^2 \, d\zeta \right)^{1/2} \left( \int_{-h}^{0} \| \nabla H \sigma \|_{L^2}^2 \, d\zeta \right)^{1/2} \right)^{1/2} \| L^2(T^2) \langle \| \sigma \|_{L^\infty(T^2;L^2(-h,0))} + \| \nabla H \sigma \|_{L^2(-h,0)} \rangle \rangle \langle (i) \| \| \varphi \|_{H^1(T^2;L^2(-h,0))} \lesssim \| \varphi \|_{H^1(T^2;L^2(-h,0))}
\]
where in (i) we used \( H^1(T^2;L^2(-h,0)) \rightarrow L^r(T^2;L^2(-h,0)) \) and Assumption 3.1(5). The estimate of \( \| (\nabla J(\sigma \varphi))_{n \geq 1} \|_{L^2(T^2)} \) is similar, where one also uses that \( \partial_t J(\sigma \varphi) = \nabla H(\sigma \varphi) \),
\[
H^2 \hookrightarrow L^\infty \quad \text{and} \quad H^2 \hookrightarrow L^2(T^2;H^2(-h,0)) \rightarrow L^2(T^2;L^\infty(-h,0)),
\]
by Sobolev embeddings. This completes the proof of (4.11) and the claim of Lemma 4.1 follows. □

4.2. Proof of Theorem 3.6. As commented below the statements of Theorems 3.6 and 3.7, the following result is the key ingredient in their proofs. Recall that \( \Xi \) is defined in (3.20).

**Proposition 4.2** (Energy estimate for maximal solutions). Let Assumptions 3.1 and 3.5 be satisfied. Let \( T \in (0, \infty) \). Assume that \( (v_0, \theta_0) \in L^4_{\mathcal{L}_C}(\Omega; \mathbb{H}^1 \times H^1) \). Let \( ((v, \theta), \tau) \) be the \( L^2 \)-maximal strong solution to (3.1)–(3.2) provided by Theorem 3.4. Then
\[
(4.12) \quad \sup_{s \in [0, \tau \wedge T)} \left[ \| v(s) \|_{H^{1}}^2 + \| \theta(s) \|_{H^{1}}^2 \right] + \int_{0}^{\tau \wedge T} \left( \| v(s) \|_{H^2}^2 + \| \theta(s) \|_{H^2}^2 \right) \, ds < \infty \quad \text{a.s.}
\]
Moreover, there exists \( C_T > 0 \), independent of \((v_0, \theta_0)\), such that, for all \( \gamma > e^\tau \),
\[
\begin{align*}
\mathbb{E} \sup_{t \in [0, \tau \wedge T)} \| v(t) \|_{H^1}^2 + \mathbb{E} \int_{0}^{\tau \wedge T} \| v(t) \|_{H^2}^2 \, dt \leq & \quad C_T (1 + \mathbb{E} \Xi \| v_0 \|_{L^2(0,T)}^2 + \mathbb{E} \Xi \| \theta_0 \|_{H^2}^2), \\
\mathbb{E} \sup_{t \in [0, \tau \wedge T)} \| \theta(t) \|_{H^1}^2 + \mathbb{E} \int_{0}^{\tau \wedge T} \| \theta(t) \|_{H^2}^2 \, dt \leq & \quad C_T (1 + \mathbb{E} \Xi \| v_0 \|_{L^2(0,T)}^2 + \mathbb{E} \Xi \| \theta_0 \|_{H^2}^2), \\
\mathbb{P} \left( \sup_{s \in [0, \tau \wedge T)} \| v(t) \|_{H^1}^2 + \int_{0}^{\tau \wedge T} \| v(t) \|_{H^2}^2 \, dt \geq \gamma \right) \leq & \quad C_T \frac{(1 + \mathbb{E} \Xi \| v_0 \|_{L^2(0,T)}^2 + \mathbb{E} \Xi \| \theta_0 \|_{H^1}^2)}{\log \log (\gamma)}, \\
\mathbb{P} \left( \sup_{s \in [0, \tau \wedge T)} \| \theta(t) \|_{H^1}^2 + \int_{0}^{\tau \wedge T} \| \theta(t) \|_{H^2}^2 \, dt \geq \gamma \right) \leq & \quad C_T \frac{(1 + \mathbb{E} \Xi \| v_0 \|_{L^2(0,T)}^2 + \mathbb{E} \Xi \| \theta_0 \|_{H^1}^2)}{\log \log (\gamma)}. 
\end{align*}
\]

The proof of Proposition 4.2 is postponed to Section 7 and Sections 5–6 are preparatory to its proof. In this section we show that Theorems 3.6 and 3.7 follows from Proposition 4.2. More precisely, Theorem 3.6 follows from the blow-up criteria of Theorem 3.4 and (4.12), see e.g. the proof of [AHHS22, Theorem 3.7] for a similar situation. For the reader’s convenience, we provide some details. The estimates of Proposition 4.2 will be used to prove Theorem 3.7.

**Proof of Theorem 3.6.** By localization of solutions to stochastic evolution equations (see [AV22c, Proposition 4.13]), it is enough to consider \((v_0, \theta_0) \in L^\infty(\Omega; \mathbb{H}^1 \times H^1)\). Hence, for all \( T \in (0, \infty) \),
\[
\mathbb{P}(\tau < T) \overset{(4.12)}{=} \mathbb{P}(\tau < T, \sup_{t \in [0, \tau)} \| (v(t), \theta(t)) \|_{H^1}^2 + \int_{0}^{\tau} \| (v(t), \theta(t)) \|_{H^2}^2 \, dt) \overset{(i)}{=} 0,
\]
where in (i) we used Theorem 3.4. Since $T \in (0, \infty)$ is arbitrary, the above yields $\tau = \infty$ a.s.

The estimates in Theorem 3.6 follows from the one in Proposition 4.2 with $\tau = \infty$. \hfill $\square$

4.3. Proof of Theorem 3.7. To prove Theorem 3.7 we argue as in [AV22a]. As in the proof of Theorem 3.7 readily follows from the following result.

**Proposition 4.3.** Let Assumptions 3.1 and 3.5 be satisfied. Fix $T \in (0, \infty)$ and $(v_0, \theta_0), (v'_0, \theta'_0) \in L^3_{\mathcal{F}_0}(\Omega; H)$. Let $(v, \theta)$ and $(v', \theta')$ be the $L^2$-global strong solution to (3.1)–(3.2) provided by Theorem 3.6 with initial data $(v_0, \theta_0)$ and $(v'_0, \theta'_0)$, respectively. Then there exist mappings $\psi, N : [0, \infty) \to [0, \infty)$, independent of $(v_0, \theta_0), (v'_0, \theta'_0)$, such that, for all $R, \varepsilon > 0$,

$$
\mathbb{P}
\left(
\|(v, \theta) - (v', \theta')\|_{L^2(0,T;H)} > \varepsilon
\right)
\leq \frac{\psi(R)}{\varepsilon^2} E\|(v_0, \theta_0) - (v'_0, \theta'_0)\|^2_H
+ N(R)\left(1 + E\|\Xi\|^2_{L^2(0,T)} + E\|(v_0, \theta_0)\|^2_H + E\|(v'_0, \theta'_0)\|^2_H\right),
$$

and $\lim_{R \to \infty} N(R) = 0$.

**Proof.** To economize the notation, here we adopt the one used in Subsection 4.1 for the proof of Theorem 3.4. In particular $(A, B, F, G)$ are as in (4.2)–(4.5) and $U = (v, \theta)$. Similarly $U = (v, \theta)$, $U' = (v', \theta')$, $V_0 = L^2 \times L^2$, $H(\ell^2) \equiv \mathcal{L}_2(\ell^2, H)$ etc. Moreover, for notational convenience, we set $V_0 \equiv [0, V]_\theta$ for $\theta \in (0, 1)$ (complex interpolation).

Note that $V_{1/2} = H$. Since $V \hookrightarrow H^2 \times H^2$ and $V_0 \hookrightarrow L^2 \times L^2$, we have $V_0 \hookrightarrow H^\theta \times H^\theta$.

Next note that the difference $U_* \equiv U - U'$ solves

$$
\begin{cases}
du_* - Au_* \ dt = (F(U) - F(U')) \ dt + [Bv + (G(U) - G(U'))] \ dB_t,
\end{cases}
U_*(0) = U_0 - U'_0.
$$

Fix $T \in (0, \infty)$. By Lemma 4.1 and [AV22b, Proposition 3.9 and 3.12] there exists $C_0 > 0$, independent of $U_0, U'_0$, such that for all stopping times $(\eta, \xi)$ such that $0 \leq \eta \leq \xi \leq T$ a.s.

$$
\|U_*(\eta)\|^2_H \leq C_0 E\|U_*(\eta)\|^2_H
+ C_0 E\|F(U) - F(U')\|^2_{L^2(0,T;V_0)} + C_0 E\|G(U) - G(U')\|^2_{L^2(0,T;H(\ell^2))}.
$$

Next we estimate the nonlinearities $(F, G)$. The arguments in [AHHS22, Theorem 3.4] show the existence of $m \geq 1$, $(\rho_j)_{j=1}^m$ such that, for all $(\eta, \xi, \rho) \in X_1$,

$$
\|F(U) - F(U')\|_{V_0} + \|G(U) - G(U')\|_{H(\ell^2)} \lesssim \sum_{1 \leq j \leq m} (1 + \|U\|_{V_{3j}}^\rho \|U'\|_{V_{3j}}^{\theta_j} + \|U\|_{V_{3j}}^{\rho_j} \|U'\|_{V_{3j}}^{\theta_j})\|U - U'\|_{V_{3j}}
$$

where $\beta_j = \frac{2 + \rho_j}{2(1 + \rho_j)} \in (\frac{1}{2}, 1)$ and the implicit constant is independent of $U, U'$. Note that, by standard interpolation arguments, for $\theta_j = 2\beta_j - 1 \in (0, 1)$, we have

$$
\|x\|_{V_{3j}} \lesssim \|x\|_{H^{1-\theta_j}} \|x\|_{V_{\theta_j}}, \quad \text{for all } x \in V.
$$

Hence, for all $x, x' \in V$ and $\eta > 0$,

$$
\|x\|_{V_{\beta_j}} \|x'\|_{V_{\beta_j}} \lesssim \|x\|_{H^{1-\theta_j}} \|x\|_{V_{\beta_j}} \|x'\|_{H^{1-\theta_j}} \|x'\|_{V_{\beta_j}}
\leq C_\eta \|x\|_{H^{1-\theta}} \|x\|_{H^{\theta}} + \eta \|x\|_{V},
$$

where in (i) we used the Young’s inequality with exponents $(\frac{1}{1-\theta_j}, \frac{1}{\theta_j})$ and that $\rho_j^{\theta_j} - \rho_j^{1-\theta_j} = 1$ since $\beta_j = \frac{2 + \rho_j}{2(1 + \rho_j)}$. Combining the above estimate with (4.14) we have, for all $\eta > 0$ and $x, x' \in V$,

$$
\|F(., x) - F(., x')\|_{V_0} + \|G(., x) - G(., x')\|_{H(\ell^2)}
\leq \left(\sum_{1 \leq j \leq m} (1 + \|x\|_{H}^\rho \|x\|_{V} + \|x'\|_{H}^\rho \|x'\|_{V}) \|x - x'\|_{H} + m\eta \|x - x'\|_{V}\right).
$$
Choosing $\eta = \frac{1}{\log m}$, the above inequality and (4.13) yield, for some $c_0 > 0$ independent of $U_0, U'_0$,

$$
\tag{4.15}
E \sup_{s \in [\eta, \xi]} \|U_s(s)\|_H^2 + E \int_\eta^\xi \|U_s\|_V^2 \, ds \leq c_0 E \|U_s(\eta)\|_H^2
$$

$$
+ c_0 E \int_\eta^\xi \left( \sum_{1 \leq j \leq m} (1 + \|U\|_H^{2p_j} \|U\|_V^2 + \|U'_s\|_H^{2p_j} \|U'_s\|_V^2) \right) \|U_s\|_H^2 \, ds.
$$

Note that $M \in L^1(0, T)$ a.s. since $U, U' \in C([0, T]; H) \cap L^2(0, T; V)$ a.s. By the tail estimates of Theorem 3.6, there exists a mapping $N : [0, \infty) \to [0, \infty)$, independent of $U_0, U'_0$, such that $\lim_{R \to \infty} N(R) = 0$ and for all $R > 1$

$$
\tag{4.16}
P\left( \int_0^T M_s \, ds \geq R \right) \leq N(R) (1 + E \|\Xi\|_{L^2(0, T)}^2 + E \|(v_0, \theta_0)\|_H^2 + E \|(v'_0, \theta'_0)\|_H^2).
$$

The conclusion follows from (4.15)–(4.16) and the Gronwall lemma in [AV22a, Lemma A.1].

**Proof of Theorem 3.7.** Due to Proposition 4.3, the proof of Theorem 3.7 follows verbatim from the one of [AV22a, Theorem 3.8].

\begin{flushright}
\ding{51}
\end{flushright}

5. Basic estimates

The aim of this section is to prove the following result. Recall that $H$ is defined in (3.18).

**Lemma 5.1.** Let Assumptions 3.1 and 3.5 be satisfied. Let $T \in (0, \infty)$. Assume that $(v_0, \theta_0) \in L^1_{\omega_0}(\Omega; H)$. Let $(v, \theta, \tau)$ be the $L^2$–maximal strong solution to (3.1)–(3.2) provided by Theorem 3.4. Then, for all $\gamma > 3.4$, then, for all $\gamma > 3.4$,

$$
E \sup_{s \in [0, \tau \wedge T]} \|v(s)\|_{L^2} + E \int_0^{\tau \wedge T} \|v(s)\|_H^2 \, ds \lesssim_T 1 + E \|v_0\|_{L^2}^2 + E \|\theta_0\|_{L^2}^2 + E \|\Xi\|_{L^2(0, T)}^2,
$$

$$
E \sup_{s \in [0, \tau \wedge T]} \|\theta(s)\|_{L^2}^2 + E \int_0^{\tau \wedge T} \|\theta(s)\|_{H^2}^2 \, ds \lesssim_T 1 + E \|v_0\|_{L^2}^2 + E \|\theta_0\|_{L^2}^2 + E \|\Xi\|_{L^2(0, T)}^2,
$$

$$
P\left( \sup_{s \in [0, \tau \wedge T]} \|\theta(s)\|_{L^4}^4 \geq \gamma \right) \lesssim_T \frac{1}{\log(\gamma)} (1 + E \|v_0\|_{L^2}^2 + E \|\theta_0\|_{L^4}^4 + E \|\Xi\|_{L^2(0, T)}^2),
$$

$$
P\left( \int_0^{\tau \wedge T} \int_\Omega |\theta|^2 |\nabla \theta|^2 \, dx \, ds \geq \gamma \right) \lesssim_T \frac{1}{\log(\gamma)} (1 + E \|v_0\|_{L^2}^2 + E \|\theta_0\|_{L^4}^4 + E \|\Xi\|_{L^2(0, T)}^2),
$$

where the implicit constants in the above estimates are independent of $(v_0, \theta_0)$.

The first two inequalities are standard energy estimates and coincide with the first two estimates in Proposition 4.2. The last two estimates are rather weak and do not give any information on moments of the r.v. \( \sup_{s \in [0, \tau \wedge T]} \|\theta(s)\|_{L^4}^4 \) and \( \int_0^{\tau \wedge T} \int_\Omega |\theta|^2 |\nabla \theta|^2 \, dx \, ds \). However, it seems not possible to improve them in general. Note that \( \int_\Omega |\nabla |\theta|^2|^2 \, dx \approx \int_\Omega |\theta|^2 |\nabla \theta|^2 \, dx \). Combining the Sobolev embedding \( H^1(\Omega) \hookrightarrow L^6(\Omega) \), standard interpolation inequality and the last two estimates of Lemma 5.1 we get, for all $\eta \in [0, 1]$ and $p \equiv \frac{12}{1 + 2\eta}$, $q \equiv \frac{12}{1 + 2q}$,

$$
\tag{5.1}
P\left( \|\theta\|_{L^p(0, \tau \wedge T; L^q(\Omega))} \geq \gamma \right) \lesssim_{T, \eta} \frac{1}{\log(\gamma)} (1 + E \|v_0\|_{L^2}^2 + E \|\theta_0\|_{L^4}^4 + E \|\Xi\|_{L^2(0, T)}^2).
$$

As in Lemma 5.1, the implicit constant in (5.1) is independent of $(v_0, \theta_0)$.

The energy estimate of Lemma 5.1 and of Proposition 4.2 are based on certain cancellations of the nonlinearities in (3.1). We formulate the one needed in current work in the following.

**Lemma 5.2 (Cancellations).** Assume that $u = (u^k)_{k=1}^3 \in C^\infty(\overline\mathcal{T}; \mathbb{R}^3)$ satisfies

$$
u^3(-h) = u^3(0, 0) = 0 \quad \text{on } T^2 \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{on } \mathcal{T}.
$$
Then, for all integers \( r \geq 2 \) and all \( f \in C^\infty(\overline{\Omega}; \mathbb{R}^3) \), \( g \in C^\infty(\Omega) \),
\[
\int_\Omega |f|^r g^{r-1} [(u \cdot \nabla) g] \mathrm{d}x + \int_\Omega g^r |f|^{r-2} f \cdot [(u \cdot \nabla) f] \mathrm{d}x = 0.
\]

To prove Lemma 5.1 we use the above with \( q = 1 \). However, in the proof of Proposition 4.2 we also need the case \( f, g \neq 1 \). To check the smoothness assumptions, we will use that \( (u, f, g) \) are Sobolev maps and the density of smooth functions in Sobolev spaces.

5.1. **Proof of Lemmas 5.1–5.2.** We first prove Lemma 5.2 and afterwards Lemma 5.1.

**Proof of Lemma 5.2.** Integrating by parts, we have
\[
\int_\Omega |f|^r g^{r-1} [(u \cdot \nabla) g] \mathrm{d}x = \frac{1}{r} \int_\Omega |f|^r [(u \cdot \nabla) g] \mathrm{d}x
\]
where in \( (i) \) we used \( \text{div}\, u = 0 \) and in \( (ii) \) that \( u^3(-, -h) = u^3(\cdot, 0) = 0 \) on \( \mathbb{T}^2 \).

Before going into the proof of Lemma 5.1, let us recall the boundedness of the trace operator on the boundary \( \partial \Omega = \mathbb{T}^2 \times \{h, 0\} \) (see e.g. [Tan11, Proposition 1.6, Chapter 4]),
\[
H^{1/2+r}(\Omega) \ni f \mapsto f|_{\mathbb{T}^2 \times \{h, 0\}} \in H^r(\mathbb{T}^2 \times \{h, 0\}) \quad \text{for all} \ r > 0.
\]

**Proof of Lemma 5.1.** The first two estimates of Lemma 5.1 can be proven as in [AHHS22, Lemma 5.2] with minor modifications. To avoid repetitions we omit the details. To prove the third estimate in Lemma 5.1 we employ the stochastic Gronwall lemma [AV22a, Lemma A.1]. To this end we need a localization argument. Through this proof we fix \( T \in (0, \infty) \). For each \( j \geq 1 \), let
\[
\tau_j \overset{\text{def}}{=} \inf \{ t \in [0, \tau) : ||v(t)||_{H^1} + ||v||_{L^2(0, t; H^2)}
\]
\[+ ||\theta(t)||_{L^2} + ||\theta||_{L^2(0, t; H^2)} + ||\Xi||_{L^2(0, t; L^2)} \geq j \} \wedge T,
\]
where \( \inf \Theta \overset{\text{def}}{=} \tau \) and \( \Xi \) is as (3.20). Note that \( (\tau_j)_{j \geq 1} \) is a localizing sequence for \( (v, \tau \wedge T) \). In particular \( \lim_{j \to \infty} \tau_j = \tau \wedge T \). Moreover, by Definition 3.3 and (5.3) we have, uniformly in \( \Omega \) and for all \( j \geq 1 \) (recall that \( (H, V) \) are as in (3.18))
\[
(v, \theta) \in C([0, \tau_j]; H) \cap L^2(0, \tau_j; V) \quad \text{a.s.}
\]

Fix \( j \geq 1 \) and let \( (\eta, \xi) \) be stopping times such that \( 0 \leq \eta \leq \xi \leq \tau_j \) a.s. We claim that there exist some \( c_0 \geq 1 \) independent of \( (j, \eta, \xi, \theta_0, \theta_0) \) such that
\[
\mathbb{E} \left[ \sup_{t \in [\eta, \xi]} ||\theta(t)||_{L^2}^4 \right] + \mathbb{E} \int_\eta^\xi \int_\Omega |\theta|^2|\nabla\theta|^2 \mathrm{d}x \mathrm{d}s
\]
\[\leq c_0 (1 + \mathbb{E} ||\theta(\eta)||_{L^4}^4) + c_0 \mathbb{E} \int_\eta^\xi N(s)(1 + ||\theta(s)||_{L^4}^4) \mathrm{d}s
\]
where \( N_{\nu, \theta}(t) \overset{\text{def}}{=} 1 + ||v(t)||_{H^1}^2 + ||\theta(t)||_{H^1}^2 + ||\Xi(t)||. \) To economize the notation, for all \( j \geq 1 \), we set
\[
E_j \overset{\text{def}}{=} \sup_{t \in [0, \tau_j]} ||\theta(t)||_{L^2} + \int_0^{\tau_j} \int_\Omega |\theta|^2|\nabla\theta|^2 \mathrm{d}x.
\]

Suppose for a moment that (5.5) holds. Then, by [AV22a, Lemma A.1], we have for all \( R, \gamma > 1 \),
\[
\mathbb{P}(E_j \geq \gamma) \leq \frac{4c_0}{\gamma} e^{4c_0 R} (1 + \mathbb{E} ||\theta_0||_{L^4}^4) + \mathbb{P} \left( \int_0^{\tau_j} N(s) \mathrm{d}s \geq \frac{R}{c_0} \right)
\]
\[\leq \frac{4c_0}{\gamma} e^{4c_0 R} (1 + \mathbb{E} ||\theta_0||_{L^4}^4) + \frac{c_0}{R} (1 + \mathbb{E} ||v_0||_{L^2}^2 + \mathbb{E} ||\theta_0||_{L^2}^2 + \mathbb{E} ||\Xi||_{L^2(0, \tau_j)}^2)
\]
\[\leq \left( \frac{4c_0}{\gamma} e^{4c_0 R} + \frac{c_0}{R} \right) (1 + \mathbb{E} ||v_0||_{L^4}^4 + \mathbb{E} ||\theta_0||_{L^4}^4 + \mathbb{E} ||\Xi||_{L^2(0, \tau_j)}^4).\]
Now choosing $R = \frac{1}{x_0} \log\left(\frac{x^2}{\log(\gamma)}\right) \geq \frac{1}{x_0} \log(\gamma)$ for $\gamma$ large, we have for some $C_0 > 0$ (depending only on $c_0$),

$$P(E_j \geq \gamma) \leq C_0 \frac{1 + E\|v_0\|_{L^4}^4 + E\|\theta_0\|_{L^4}^4 + E\|\Xi\|_{L^2(0,T)}^2}{\log(\gamma)}.$$  

Since $C_0(c_0)$ is independent of $(j, v_0, \theta_0)$, the last estimate in Lemma 5.1 follows by letting $j \to \infty$ in the previous estimate.

Hence it remains to prove (5.5). To this end, we set

$$f_\theta \overset{\text{def}}{=} 1_{[0, \tau)} \times \chi F_\theta(\cdot, v, \theta, \nabla \theta, \nabla v, \nabla \theta) \quad \text{and} \quad g_{\theta, n} \overset{\text{def}}{=} 1_{[0, \tau)} \times \chi G_{\theta, n}(\cdot, v, \theta, \nabla \theta).$$

Note that, by Assumption 3.1(7) and (3.20), a.s. for all $t \in \mathbb{R}_+$,

$$\|f_\theta(t)\|_{L^2} + \|g_\theta(t)\|_{H^1(\mathbb{R}^d)} \lesssim 1 + \|v(t)\|_{H^1} + \|\theta(t)\|_{H^1}.$$  

Applying the Itô’s formula to $\theta \mapsto \|\theta\|^4_{L^4}$ (using and a standard approximation argument, see e.g. [AHHS22, Step 3 of Lemma 5.3]) we have, a.s. for all $t \in [0, T]$,  

$$\|\theta^n, \xi(t)\|_{L^4}^4 + 12 \int_0^t \int_{\mathcal{O}} 1_{[\eta, \xi]} \|\theta^n\|_{L^4}^4 \|\nabla \theta\|^2 \, dx \, ds$$

$$\|\theta(\eta)\|_{L^4}^4 + \sum_{1 \leq j \leq 3} \int_0^t \int_{\mathcal{O}} 1_{[\eta, \xi]} I_{\theta, j}(s) \, ds + \mathcal{M}(t)$$

where $I_{\theta, 1}(t) \overset{\text{def}}{=} -\alpha \int_{T^n} |\theta(\cdot, 0)|^4 \, dx$,  

$$I_{\theta, 2}(t) \overset{\text{def}}{=} 4 \int_{\mathcal{O}} \theta^n f_\theta \, dx, \quad I_{\theta, 3}(t) \overset{\text{def}}{=} 12 \sum_{n \geq 1} \int_{\mathcal{O}} \theta^2 (\psi_n \cdot \nabla) \theta + g_{\theta, n} \, dx,$$

$$\mathcal{M}(t) \overset{\text{def}}{=} 4 \sum_{n \geq 1} \int_0^t \int_{\mathcal{O}} \theta^3 (\psi_n \cdot \nabla) \theta + g_{\theta, n} \, dx \, d\beta^n$$

and we used the cancellation

$$\int_{\mathcal{O}} \theta^3 (v \cdot \nabla) \theta + w(v) \partial_3 \theta \, dx = 0,$$

which follows from Lemma 5.2 with $g = 1$, $f = \theta$, $u = (v, v(w(v)))$ and a standard density argument.  

For exposition convenience, the remain part of the proof is split into several steps.

**Step 1:** There exists $c_1$ independent of $(\eta, \xi, v_0, \theta_0)$ such that

$$E \int_\Omega |\theta|^2 |\nabla \theta|^2 \, dx \, ds \leq c_1 (1 + E\|\theta_0\|_{L^4}^4) + c_1 E \int_\Omega N(s) (1 + \|\theta(s)\|^4_{L^4}) \, dx$$

where $N$ is as below (5.5). We begin by estimating $I_{\theta, 1}$. Let $\varepsilon > 0$ be decided later. Note that, by (5.2) and interpolation, we have, a.e. on $[0, \tau) \times \Omega$,

$$I_{\theta, 1} = \|\theta(\cdot, 0)\|^2_{L^2} \leq \varepsilon \|\nabla \theta\|^2_{L^2} + C_\varepsilon \|\theta\|^4_{L^4}$$

$$\leq \varepsilon \int_{\mathcal{O}} |\theta| \|\nabla \theta\|^2 \, dx + C_\varepsilon \|\theta\|^4_{L^4}.$$  

Next we estimate $I_{\theta, 2}$:

$$|I_{\theta, 2}| \leq \|\theta^3\|_{L^2} \|f_\theta\|_{L^2} = \|\theta^2\|_{L^4}^{3/2} \|f_\theta\|_{L^2}$$

$$\overset{(i)}{\lesssim} \|\theta\|^3_{L^4} \|f_\theta\|_{L^2} + \|\nabla |\theta|^2\|_{L^2} \|\theta\|^3_{L^4} \|f_\theta\|_{L^2}$$

$$\lesssim \|\theta\|^3_{L^4} \|f_\theta\|_{L^2} + \|\theta\|^3_{L^4} \|\nabla \theta\|_{L^2} \|\theta\|^3_{L^4} \|f_\theta\|_{L^2}$$

$$\overset{(ii)}{\leq} \|\theta\|^3_{L^4} \|f_\theta\|_{L^2} + \varepsilon \|\theta\| \|\nabla \theta\|^2_{L^2} + C_\varepsilon \|\theta\|_{L^4}^{12/5} \|f_\theta\|_{L^2}^{8/5}$$

$$\leq \varepsilon \|\theta\| \|\nabla \theta\|^2_{L^2} + C_\varepsilon (1 + \|f_\theta\|^2_{L^2}) (1 + \|\theta\|^4_{L^4})$$
where in (i) we used the interpolation inequality \( \|\zeta\|_{L^3} \lesssim \|\zeta\|_{L^2}^{1/2} \|\zeta\|_{H^1}^{1/2} \) for \( \zeta \in H^1(\mathcal{O}) \) and in (ii) the Young’s inequality with exponents \( (\frac{3}{2}, \frac{3}{4}) \).

It remains to estimate \( I_{\theta,3} \). By Cauchy-Schwarz inequality we have, a.e. on \([0,\tau) \times \Omega,\)

\[
|I_{\theta,3}| \leq 6(1 + \varepsilon) \sum_{n \geq 1} \left[ \int_{\mathcal{O}} \theta^2 |(\psi_n \cdot \nabla)\theta|^2 \, dx + C \varepsilon \int_{\mathcal{O}} \theta^4 |g_{\theta,n}|^2 \, dx \right]
\]

\[
\leq 6(1 + \varepsilon) \nu \int_{\mathcal{O}} \theta^2 |\nabla\theta|^2 \, dx + C \varepsilon \sum_{n \geq 1} \int_{\mathcal{O}} \theta^2 |g_{\theta}|^2 \, dx
\]

where in the last step we used Assumption 3.1(2). We now estimate the last term appearing in the above estimate:

\[
(5.8) \quad \sum_{n \geq 1} \int_{\mathcal{O}} \theta^2 |g_{\theta,n}|^2 \, dx \leq \|\theta\|^2_{L^4} \|(g_{\theta,n})_{n \geq 1}\|^2_{L^4(\mathcal{O})} \lesssim \|\theta\|^2_{L^4} \|g_{\theta}\|^2_{H^1(\mathcal{O})},
\]

where in the last estimate we used the Sobolev embedding \( H^1(\mathcal{O}; \ell^2) \hookrightarrow L^6(\mathcal{O}; \ell^2) \).

Taking \( t = T \) in (5.7) and afterwards the expected values, the previous estimates show that

\[
12E \int_{\eta} ^{\xi} |\theta|^2 |\nabla\theta|^2 \, dx \, ds \leq (6 \nu (1 + \varepsilon) + 2 \varepsilon) E \int_{\eta} ^{\xi} |\theta|^2 |\nabla\theta|^2 \, dx \, ds + C_{\varepsilon} E \int_{\eta} ^{\xi} N(s)(1 + \|\theta\|^4_{L^4}) \, ds.
\]

Here we have also used that \( E[\mathcal{M}(T)] = E[\mathcal{M}(0)] = 0 \). Recall that \( \nu < 2 \). Thus the claim of this step follows by choosing \( \varepsilon \) so that \( (6 \nu (1 + \varepsilon) + 2 \varepsilon) < 12 \) in the above estimate.

Step 2: There exists \( c_2 > 0 \), independent of \((j, \eta, \xi, v_0, \theta_0)\), such that

\[
E \left[ \sup_{s \in [0,T]} \|\mathcal{M}(s)\| \right] \leq \frac{1}{2} E \left[ \sup_{s \in [\eta,\xi]} \|\theta(s)\|^4_{L^4} \right] + C_{\varepsilon} (1 + E\|\theta_{0}\|^4_{L^4}) + E \int_{\eta} ^{\xi} N(s)(1 + \|\theta\|^4_{L^4}) \, ds.
\]

The Burkholder-Davis-Gundy inequality yields:

\[
E \left[ \sup_{s \in [0,T]} \|\mathcal{M}(s)\| \right] \lesssim E \left[ \int_{\eta} ^{\xi} \sum_{n \geq 1} \left( \int_{\mathcal{O}} |\theta|^3 |(\psi_n \cdot \nabla)\theta + g_{\theta,n}| \, dx \right)^2 \, ds \right]^{1/2}.
\]

The Cauchy-Schwarz inequality, a.e. on \([0,\tau) \times \Omega,
\]

\[
\sum_{n \geq 1} \left( \int_{\mathcal{O}} |\theta|^3 |(\psi_n \cdot \nabla)\theta + g_{\theta,n}| \, dx \right)^2 \leq \|\theta\|^4_{L^4} \left[ \int_{\mathcal{O}} |\theta|^2 \sum_{n \geq 1} |(\psi_n \cdot \nabla)\theta + g_{\theta,n}|^2 \, dx \right]
\]

\[
\lesssim \|\theta\|^4_{L^4} \left[ \int_{\mathcal{O}} |\theta|^2 (|\nabla\theta|^2 + \|g_{\theta}\|^2_{L^2}) \, dx \right]
\]

where in the last estimate we used boundedness of \((\psi_n)_{n \geq 1}\), cf. Remark 3.2.

Hence, the Young inequality yields

\[
E \left[ \sup_{s \in [0,T]} \|\mathcal{M}(s)\| \right] \lesssim E \left[ \left( \sup_{s \in [\eta,\xi]} \|\theta(s)\|^4_{L^4} \right)^{1/2} \left( \int_{\eta} ^{\xi} \int_{\mathcal{O}} |\theta|^2 (|\nabla\theta|^2 + \|g_{\theta}\|^2_{L^2}) \, dx \, ds \right)^{1/2} \right]
\]

\[
\leq \frac{1}{2} E \left[ \sup_{s \in [\eta,\xi]} \|\theta(s)\|^4_{L^4} \right] + C E \int_{\eta} ^{\xi} \int_{\mathcal{O}} |\theta|^2 (|\nabla\theta|^2 + \|g_{\theta}\|^2_{L^2}) \, dx \, ds.
\]

The claim of Step 2 follows by combining the previous estimate with Step 1 and (5.8).

Step 3: Proof of (5.5). Taking \( E \left[ \sup_{t \in [0,T]} |\cdot| \right] \) on both sides of (5.7) the claim follows by repeating the estimates for \((I_{\theta,j})_{j=1}^3\) performed in Step 1 and using the estimate for \( \mathcal{M} \) of Step 2. Note that the term \( \frac{1}{2} E \left[ \sup_{s \in [\eta,\xi]} \|\theta(s)\|^4_{L^4} \right] \) can be absorbed on the LHS of the corresponding estimate since \( \theta^{n,\xi} = \theta((\cdot \vee \eta) \land \xi) \). \( \square \)
6. The main intermediate estimate

The aim of this section is to obtain the following key estimate for the $L^2$–maximal strong solution to (3.1)–(3.2), which is the main ingredient in the proof of Proposition 4.2. As in [AHHS22, Subsection 5.2], inspired by the seminal work of C. Cao and E.S. Titi [CT07], the main estimate involves the barotropic and baroclinic modes, i.e.

\[(6.1)\]

\[
\bar{\mathbf{v}} \equiv \int_{-h}^{0} v(\cdot, \zeta) \, d\zeta \quad \text{and} \quad \tilde{v} \equiv v - \bar{\mathbf{v}}.
\]

**Lemma 6.1** (Main intermediate estimate). Let Assumptions 3.1 and 3.5 be satisfied. Fix $T \in (0, \infty)$. Assume that $(v_0, \theta_0) \in L^4_{\beta_0}(\Omega; \mathbb{H}^1 \times H^1)$. Let $((v, \theta), \tau)$ be the $L^2$–maximal strong solution to (3.1)–(3.2) provided by Theorem 3.4. For all $s \in [0, \tau)$ set

\[
X_s \equiv \left\| \bar{v}(s) \right\|_{L^2(\Omega)}^4 + \left\| \mathbf{\bar{v}}(s) \right\|_{H^1(T^2)}^2 + \left\| \partial_3 v(s) \right\|_{L^2(\Omega)}^2 + \left\| \partial_3 \theta(s) \right\|_{L^2(\Omega)}^2,
\]

\[
Y_s \equiv \left\| \bar{v}(s) \left\| \nabla \bar{v}(s) \right\|_{L^2(\Omega)}^2 + \left\| \mathbf{\bar{v}}(s) \right\|_{H^2(T^2)}^2 + \left\| \partial_3 v(s) \right\|_{H^1(\Omega)}^2 \right. + \left. \left\| \partial_3 \theta(s) \right\|_{H^1(\Omega)}^2 \right\|_{L^2(\Omega)}^2
\]

Then there exists $C_T > 0$, independent of $(v_0, \theta_0)$, such that, for all $\gamma > e$,

\[(6.2)\]

\[
\mathbb{P}\left( \sup_{s \in [0, \tau \wedge T]} X_s + \int_0^{\tau \wedge T} Y_s \, ds \geq \gamma \right) \lesssim_T \frac{1 + E\|\mathbf{\bar{v}}\|^2_{L^2(0,T)} + E\|v_0\|^2_{H^1} + E\|\theta_0\|^2_{H^1}}{\log \log(\gamma)}.
\]

The proof of the above result requires several steps which are spread over this section. The proof of Lemma 6.1 will be given in Subsection 6.12.

Lemma 6.1 can be seen as an extension of [AHHS22, Lemma 5.3] to the case of non-isothermal turbulent pressure. Note that the estimate of the tail probability (6.2) was not given in [AHHS22]. In case of isothermal turbulent pressure (i.e. $\sigma \equiv 0$ and $\pi \equiv 0$), the decay factor $(\log \log(\gamma))^{-1}$ on RHS(6.2) can be replaced by $(\log(\gamma))^{-1}$, cf. Remark 3.10 for a similar situation.

As in [AHHS22], to prove the above main estimate we follow the approach of second named author and T. Kashiwabara in [HK16]. There the main idea was to prove three estimates separately for the variables $\bar{v}$, $\tilde{v}$ and $\partial_3 v$. Afterwards, one multiplies these estimates with suitable constants and summing them up, one obtains a closed estimate (cf. [AHHS22, Lemma 5.3]). Since in [AHHS22] we were concerned with the case of isothermal turbulent pressure, we were able to follow the strategy of [HK16] as the temperature $\theta$ played only a minor role in the estimates (see the discussion in Subsection 1.1). Indeed, in case of isothermal turbulent pressure, the energy bound for $\sup_p \|\theta\|^2_{L^2} + \|\theta\|^2_{L^2(T^2)}$ in Lemma 5.1 already gives enough information on $\theta$ to obtain global well-posedness, see the proof of [AHHS22, Theorem 3.7]. However, this is not true in case of non–isothermal turbulent pressure. Indeed, if $\sigma_\alpha \neq 0$, then the term

\[(6.3)\]

\[
\mathbb{P}\left( \int_{-h}^{0} \nabla H(\sigma_\alpha(\cdot, \zeta) \theta(\cdot, \zeta)) \, d\zeta \right) \, d\beta_0^1
\]

appearing in (3.8a) cannot be controlled via Lemma 5.1 in the strong setting, cf. Lemma 4.1. In other words, the action of $\theta$ through the term (6.3) in the $v$-equation is not lower order. Hence, in contrast to [AHHS22], we need to consider the equations for $v$ and $\theta$ jointly. This gives rise to some new terms in the equations for $\bar{v}$ and $\tilde{v}$ which we are going to describe. To explain the new quantities arising in the estimates, let us follow the argument in [AHHS22, Lemma 5.3] and therefore we first look at the estimate for $\bar{v}$. Taking the averaging operator $\tau = \int_{-h}^{0} \cdot \, d\zeta$ in (3.8a), one sees that the following term appears

\[(6.4)\]

\[
\int_{-h}^{0} \nabla H(\sigma_\alpha(\cdot) \theta(\cdot, \zeta)) \, d\zeta = \nabla H(\sigma_\alpha(\cdot) \int_{-h}^{0} \theta(\cdot, \zeta) \, d\zeta) = -\nabla H(\sigma_\alpha(\cdot) \bar{\theta}(\cdot))
\]

where we used that $\sigma_\alpha$'s are $x_3$–independent by Assumption 3.5 and we set

\[(6.5)\]

\[
\bar{\theta} \equiv \int_{-h}^{0} \theta(\cdot, \zeta) \zeta \, d\zeta.
\]
Remark 6.2 (Physical interpretation of $\hat{\theta}$). Recall that $\theta$ is proportional to $\rho$, cf. (2.4) and (2.6). Hence the ratio $\hat{\theta}/\theta$ is equal to the center of gravity in the vertical direction.

To repeat the argument of [AHHS22, Step 1 of Lemma 5.3], by stochastic maximal $L^2$-regularity (cf. Lemma 4.1), to obtain $L^2_t(H^1_0) \cap L^2_t(H^2_0)$--estimates for $\tau$, we need $L^2_t(H^1_0)$--estimates for $\hat{\theta}$. To obtain the latter estimate we take weighted average operator $\hat{\gamma} = \int_{-h}^0 \cdot \zeta \, d\xi$ of (3.8b) and the following term appears
\begin{align}
(6.6) \quad w(v)\partial_3\theta &= \int_{-h}^0 w(v)\partial_3\theta \, d\zeta \\
&= \left[\int_{-h}^0 (w(v)\zeta)\theta(\cdot, \zeta) - \text{div}_H v(\cdot, \zeta)\theta(\cdot, \zeta)\zeta\right] \, d\zeta \\
&= \left[\int_{-h}^0 \left(\int_0^\infty \text{div}_H v(\cdot, \xi) \, d\xi\right)\theta(\cdot, \zeta) - \text{div}_H v(\cdot, \zeta)\theta(\cdot, \zeta)\zeta\right] \, d\zeta \\
&= \int_{-h}^0 \left[-\text{div}_H v(\cdot, \zeta)\left(\int_{-h}^\infty \theta(\cdot, \xi) \, d\xi\right) + \text{div}_H v(\cdot, \zeta)\theta(\cdot, \zeta)\zeta\right] \, d\zeta.
\end{align}

where in (i) we use an integration by parts and $[w(v)](-h, t) = [w(v)](0, t) = 0$, in (ii) (3.9) and $\int_{-h}^0 \text{div}_H v \, d\zeta = 0$. Therefore, to obtain $L^2_t(H^1)$--estimates for $\hat{\theta}$ we need to bound the products
\begin{align}
(6.7) \quad \|\theta\|_{L^2((0, t) \times \mathcal{O})} \quad \text{and} \quad \left\|\int_{-h}^0 \theta(\cdot, \zeta) \, d\zeta\right\|_{L^2((0, t) \times \mathcal{O})}^2,
\end{align}

such quantities can be estimated by applying the Itô formula to the functionals
\begin{align}
(\bar{v}, \theta) \rightarrow \|\theta\|_{L^2(\mathcal{O})} \quad \text{and} \quad (\bar{v}, \theta) \rightarrow \left\|\int_{-h}^0 \theta(\cdot, \zeta) \, d\zeta\right\|_{L^2(\mathcal{O})}^2,
\end{align}

respectively. For details, we refer to Subsections 6.8 and 6.9. The first and second quantity in (6.7) also arises in the estimate for $\bar{v}$. Fortunately, compared to [AHHS22, Lemma 5.3], no further terms appears in the estimate for $\partial_3\tau$ see Subsection 6.5. Finally, as it will turn out, to bound the quantities in (6.7), we need an estimate also for the quantities
\begin{align}
\left\|\theta\right\|_{L^2((0, t) \times \mathcal{O})} \quad \text{and} \quad \left\|\int_{-h}^0 \theta(\cdot, \zeta) \, d\zeta\right\|_{L^2((0, t) \times \mathcal{O})}^2
\end{align}

respectively. To bound the above terms we apply the Itô formula to the functionals
\begin{align}
\theta \mapsto \left\|\theta\right\|_{L^2(\mathcal{O})} \quad \text{and} \quad \theta \mapsto \left\|\int_{-h}^0 \theta(\cdot, \zeta) \, d\zeta\right\|_{L^2(\mathcal{O})}^2
\end{align}

After that, Lemma 6.1 follows by multiplying each estimate with a suitable constant and summing them up, see [AHHS22, Step 4 of Lemma 5.3] for a similar situation.

To economize the notation, below, for $(t, \omega) \in \mathbb{R}_+ \times \Omega, x_H \in \mathbb{T}^2$ and $x_3 \in (-h, 0)$, we let
\begin{align}
(6.8) \quad \Theta(t, \omega, x_H, x_3) \overset{\text{def}}{=} \int_{-h}^{x_3} \theta(t, \omega, x_H, \zeta) \, d\zeta.
\end{align}

Next we give overview of this section.

- Subsection 6.1: Equations for the new quantities $(\bar{v}, \tilde{v}, \hat{\theta})$.
- Subsection 6.2: Set-up of the proof of Lemma 6.1.
- Subsection 6.3: Estimate for $\sup_t \|\bar{v}\|_{L^2_0 \cap L^2_t H^1_0}$.
- Subsection 6.4: Estimate for $\sup_t \|\hat{\theta}\|_{H^1_0}$ and $\|\hat{\theta}\|_{L^2_t H^2_0}$.
- Subsection 6.5: Estimate for $\sup_t \|\partial_3 \tau\|_{L^2_0}$ and $\|\partial_3 \tau\|_{L^2_t H^1_0}$.
- Subsection 6.6: Estimate for $\sup_t \|\partial_3 \theta\|_{L^2_0}$ and $\|\partial_3 \theta\|_{L^2_t H^1_0}$.
- Subsection 6.7: Estimate for $\sup_t \|\tilde{v}\|_{L^2_0}$ and $\|\tilde{v}\|_{L^2_t L^2_0}$.
- Subsection 6.8: Estimate for $\|\bar{v}\|_{L^2_0}$ and $\|\hat{\theta}\|_{L^2_t H^1_0}$.
- Subsection 6.9: Estimate for $\|\Theta\|_{L^2_0 L^2_0}$ and $\|\tilde{v}\|_{L^2_t L^2_0}$. 

Subsection 6.10: Estimate for \( \| \theta \| \| \nabla \Theta \|_{L^2_t L^2_x} \).

Subsection 6.11: Estimate for \( \| \Theta \| \| \nabla \Theta \|_{L^2_t L^2_x} \).

Subsection 6.12: Lemma 6.1 obtained by multiplying with suitable constants the estimates of Subsections 6.3-6.11 and then summing them up.

In following subsections, the assumptions of Lemma 6.1 are in force. In particular, \( (v, \theta) \) is the \( L^2 \)-maximal strong solution to (3.1)-(3.2) provided by Theorem 3.4, see Definition 3.3.

6.1. System of SPDEs for the unknown \((\pi, \tilde{v}, \tilde{\theta})\). By Definition 3.3 and Assumptions 3.1 and (3.5), \((v, \theta, \tau)\) is an \( L^2 \)-local strong solution to (cf. Definition 3.3)

\[
\begin{align*}
\dot{v} &= \left( \Delta v + \mathbb{P} \left[ -(v \cdot \nabla v - w(v) \partial_3 v + L_{\pi, \gamma} \theta + \mathcal{P}_{\gamma, \phi}(v, \theta) + f_v) \right] \right) dt \\
& \quad + \sum_{n \geq 1} \mathbb{P} \left[ (\phi_n \cdot \nabla v) + \sigma_n \int_{-h} \nabla_{\H} \theta(\cdot, \zeta) d\zeta + g_{v, n} \right] d\beta^n_t, \\
\dot{\theta} &= \left[ \Delta \theta - (v \cdot \nabla \H) \theta - w(v) \partial_3 \theta + f_\theta \right] dt + \sum_{n \geq 1} \left[ (\psi_n \cdot \nabla \theta + g_{\theta, n}) \right] d\beta^n_t,
\end{align*}
\]

(6.9a) (6.9b) (6.9c)

where, for \( n \geq 1 \), \( Q \) as in Subsection 1.5 and on \([0, \tau) \times \Omega \), we set

\[
\begin{align*}
\mathcal{L}_{\pi, \gamma} &\overset{\text{def}}{=} (\pi \cdot \nabla \H) \int_{-h} \nabla_{\H} \theta(\cdot, \zeta) d\zeta + \int_{-h} \pi^3(\cdot, \zeta) \partial_3 \nabla_{\H} \theta(\cdot, \zeta) d\zeta, \\
\mathcal{P}_{\gamma, \phi}(v, \theta) &\overset{\text{def}}{=} \sum_{n \geq 1} \sum_{1 \leq j \leq 2} \gamma_n^j \left( Q \left[ (\phi_n \cdot \nabla) v + \sigma_n \int_{-h} \nabla_{\H} \theta(\cdot, \zeta) d\zeta \right] \right)^j, \\
g_{v, n} &\overset{\text{def}}{=} G_{v, n}(\cdot, v, \theta), \\
g_{\theta, n} &\overset{\text{def}}{=} G_{\theta, n}(\cdot, v, \theta), \\
f_v &\overset{\text{def}}{=} F_v(\cdot, v, \nabla v, \nabla \theta) + \int_{-h} \nabla(\kappa(\cdot, \zeta) \theta(\cdot, \zeta)) d\zeta \\
& \quad + \sum_{n \geq 1} \sum_{1 \leq j \leq 2} \gamma_n^j (Q [g_{v, n}])^j, \\
f_\theta &\overset{\text{def}}{=} F_\theta(\cdot, v, \nabla v, \nabla \theta).
\end{align*}
\]

(6.10a) (6.10b) (6.10c) (6.10d) (6.10e) (6.10f)

Finally, let us recall that (6.9) is complemented with the following boundary conditions:

\[
\begin{align*}
\partial_3 v(\cdot, -h) &= \partial_3 v(\cdot, 0) = 0 \quad \text{on } T^2, \\
\partial_3 \theta(\cdot, -h) &= \partial_3 \theta(\cdot, 0) + \alpha \theta(\cdot, 0) = 0 \quad \text{on } T^2.
\end{align*}
\]

(6.11a) (6.11b)

Note that to derive (6.9) we used that \( \sigma_n \) and \( \pi = (\pi^1, \pi^2) \) are \( x_3 \)-independent by Assumption 3.5 and \( \int_{-h} \partial_3 \pi^3(\cdot, \zeta) \partial_3 \theta(\cdot, \zeta) d\zeta = \partial_3 \pi^3 \theta \) by (6.11b). As above, here \( (\cdot)^j \) denotes the \( j \)-th coordinate of the corresponding vector. \( L^2 \)-local strong solutions to (6.9) can be defined as in Definition 3.3, we omit the details for brevity.

The logic behind the definition (6.10) is that the quantities in (6.10c)-(6.10d) are lower-order in the sense that they can be estimated (in strong \( L^2 \)-norms) due to the standard energy estimates of Lemma 5.1 and Assumption 3.1(7) (cf. (6.17)-(6.18) below). This is not the case for the linear operators in \((v, \theta)\) appearing (6.10b), due to our (relatively) weak regularity assumptions on \((\gamma, \pi)\) in Assumption 3.1. It is easy to see that, under additional assumption on \((\gamma, \pi)\), also the quantities in (6.10a)-(6.10b) can by the energy estimates in Lemma 5.1. However, it would be unnatural to enforce the regularity assumptions on \((\gamma, \pi)\) as they will appear naturally when dealing with Stratonovich formulation of (3.1), see Section 8.

Next we derive SPDEs for the unknown \((\pi, \tilde{v}, \tilde{\theta})\). We begin by considering \( \pi = f v(\cdot, \zeta) d\zeta \). To this end, let us recall that \( \piH \) denotes the Helmholtz projection acting on the horizontal variable \( x_H \in T^2 \) where \( x = (x_H, x_3) \in \Omega \), see Subsection 1.5. Since \( \piH \pi = \piH \pi \), applying the vertical
average $\tau = \int_{-h}^{0} \cdot d\zeta$ in (6.9a) and using Assumption 3.5, $(\tau, \tau)$ is a $L^2$–local strong solution the following problem on $\mathbb{T}^2$:

$$
\begin{align*}
d\tau &= \left( \Delta_H \tau + \mathbb{P}_H \left[- (\tau \cdot \nabla_H) \tau - \mathbb{F}(\bar{v}) \right)
\right) dt \\
&\quad + \sum_{n \geq 1} \mathbb{P}_H \left[ (\phi_n, \nabla) \tau + \hat{\phi}_n \hat{\tau} \cdot \nabla \tau - \sigma_n \nabla \hat{\theta} + \bar{g}_{\tau, n} \right] d\beta^n_t,
\end{align*}
$$

(6.12a)

$$
\mathcal{F}(\bar{v}) \overset{\text{def}}{=} \left( \bar{v} \cdot \nabla_H \bar{v} + \bar{v}(\text{div}_H \bar{v}) \right),
$$

(6.12b)

$$
\tau(\cdot, 0) = \tau_0 = \int_{-h}^{0} v_0(\cdot, \zeta) d\zeta,
$$

(6.12c)

where $\phi_{n, H} \overset{\text{def}}{=} (\phi_n^1, \phi_n^2)$. To obtain (6.12a) we also used (6.4),

$$(\bar{v} \cdot \nabla_H \bar{v}) + w(v) \partial_3 \bar{v} = (\bar{v} \cdot \nabla_H) \bar{v} + (\bar{v} \cdot \nabla_H) \bar{v} + (\text{div}_H \bar{v}) \bar{v}$$

which follows from $\bar{v} = 0$ and an integration by parts, and $\mathbb{P}_{\gamma, \phi}(v, \theta) = \mathbb{P}_{\gamma, \phi}(v, \theta)$ which follows from the $x_3$–independent of $\gamma_{\ell}^{a, k}$ (see Assumption 3.5) and the fact that $Q[\gamma] = Q_{H}[\gamma]$ is $x_3$–independent as well (see (3.3)). Here, as above, by $L^2$–local strong solution to (6.12) we understand that $(\bar{v}, \tau)$ solves (6.12) in its natural integral form, cf. Definition 3.3. Note that $\text{div}_H \bar{v} = 0$ since $v_0 \in \mathbb{H}$. Hence, by (6.12a),

$$
\text{div}_H \tau = 0 \quad \text{a.e. on} \ [0, \tau) \times \Omega \times \mathbb{T}^2.
$$

Next, we derive a system of SPDEs for $\bar{v}$. To this end, we apply the deviation from the vertical average operator $\bar{v} = \cdot - \tau$ in (6.9). Note that $\mathbb{F}(\mathbb{F}) = \mathbb{F} - \bar{f}$ for all $f \in L^2(\Omega; \mathbb{R})$ by (3.3). Using (6.4), one sees that $(\bar{v}, \tau)$ is a $L^2$–local strong solution to

$$
\begin{align*}
d\bar{v} &= \left[ \Delta_{\bar{v}} - (\bar{v} \cdot \nabla_H) \bar{v} - w(v) \partial_3 \bar{v} + \mathcal{E}(\bar{v}, \tau) \\
&\quad + L_{\pi, \gamma} + (\pi_H \cdot \nabla_H) \nabla_H \hat{\theta} + \pi^3 \partial_3 \nabla_H \theta + f \right] dt \\
&\quad + \sum_{n \geq 1} \left[ (\phi_n \cdot \nabla) \bar{v} - \hat{\phi}_n \hat{\tau} \cdot \nabla \bar{v} + T_n \theta + \bar{g}_{\tau, n} \right] d\beta^n_t,
\end{align*}
$$

(6.14a)

$$
\mathcal{E}(\bar{v}, \tau) \overset{\text{def}}{=} - (\bar{v} \cdot \nabla_H) \bar{v} - (\bar{v} \cdot \nabla_H) \bar{v} + \mathcal{F}(\bar{v}),
$$

(6.14b)

$$
T_n \theta \overset{\text{def}}{=} \sigma_n \left( \int_{-h}^{0} \nabla_H \theta d\zeta - \nabla_H \hat{\theta} \right),
$$

(6.14c)

$$
\bar{v}(\cdot, 0) = \bar{v}_0 \overset{\text{def}}{=} v_0 - \tau_0.
$$

(6.14d)

where $\mathcal{F}$ is as in (6.12b) and we used that $\partial_3 v = \partial_3 \bar{v}$. By (6.11a) we also have

$$
\partial_3^2 v(\cdot, -h) = \partial_3 v(\cdot, 0) = 0 \quad \text{on} \ \mathbb{T}^2.
$$

Before going further, let us note that, by (6.13) we have

$$
w(v) = w(\bar{v}) \quad \text{a.e. on} \ [0, \tau) \times \Omega \times \mathcal{O}.
$$

The previous identity will be often used in the following without further mentioning it.

Finally, we consider $\hat{\theta}$. By taking the weighted average operator $\hat{\theta} = \int_{-h}^{0} \cdot d\zeta$ in the second equation of (6.9), we have that $(\hat{\theta}, \tau)$ is an $L^2$–local strong solution to

$$
\begin{align*}
d\hat{\theta} &= \left[ \Delta_H \hat{\theta} - (\bar{v} \cdot \nabla_H) \hat{\theta} - (\bar{v} \cdot \nabla_H) \hat{\theta} - R(v, \theta) + f_{\hat{\theta}} \right] dt \\
&\quad + \sum_{n \geq 1} \left[ (\psi_{n, H} \cdot \nabla_H) \hat{\theta} + \hat{\psi}_n \hat{\tau} \cdot \nabla \hat{\theta} + \bar{g}_{\hat{\theta}, n} \right] d\beta^n_t,
\end{align*}
$$

(6.16a)

$$
R(v, \theta) \overset{\text{def}}{=} \int_{-h}^{0} \left[ - \Theta \text{div}_H \bar{v} + \theta \text{div}_H \bar{v} \zeta \right] d\zeta,
$$

(6.16b)
Lemma 6.1. To this end, let \((f, \psi)\) where we used that (6.16c)\[\frac{\partial}{\partial t} \theta(\cdot, 0) = \hat{\theta}(\cdot) \equiv \int_0^\theta \theta_0(\cdot, \zeta) \zeta d\zeta,\]
where we used that \(\psi_1, \psi_2\) are \(x_3\)-independent by Assumption 3.5, the identity (6.6) and \(\frac{\partial}{\partial \zeta} \theta(\cdot, 0) \equiv \theta_0(\cdot, \zeta)\) on \(\mathbb{T}^2\), since \(\frac{\partial}{\partial \zeta} \theta(\cdot, -h) = 0\) on \(\mathbb{T}^2\).

6.2. Preparation of the proof of Lemma 6.1. In this subsection we prepare the proof of Lemma 6.1. To this end, let \((f_v, f_0, g_v, g_0)\) be as in (6.10c)-(6.10f). As remarked below (6.10), such terms can be estimated by using Lemma 5.1. More precisely, let

\[
L_t \equiv [1 + \|v(t)\|^2_{L^2} + \|\theta(t)\|^2_{L^2}]
\]

By (3.20), Assumptions 3.1(3)–(7) and (3.5) (see also (6.20)–(6.21) below), there exists \(K \geq 1\) independent of \((v_0, \theta_0)\) such that, a.s. for all \(t \in [0, \tau)\),

\[
L_t \leq K(1 + \|v(t)\|^2_{L^2} + \|\theta(t)\|^2_{L^2})(1 + \|v(t)\|^2_{H^1} + \|\theta(t)\|^2_{L^2} + \|1 + \|v(t)\|\|\theta(t)\|\|^2_{L^2}).
\]

Hence, by the Chebyshev inequality, Lemma 5.1 and (5.1) with \(\eta = 1/2\), we have

\[
P\left(\int_0^\tau L_s ds \geq \gamma\right) \lesssim \frac{1 + E\|v\|^2_{L^2(0,T)} + E\|v_0\|^2_{H^1} + E\|\theta_0\|^2_{H^1}}{\log(\gamma)}
\]

for \(\gamma > 1\), where the implicit constant on the RHS(6.18) is independent of \((v_0, \theta_0)\).

We are ready to set up the proof of Lemma 6.1. Fix \(T \in (0, \infty)\) and let \((\tau_j)_{j \geq 1}\) be as in (5.3). Recall that \(\lim_{j \to \infty} \tau_j = \tau \land T\) a.s. and (5.4) holds. Let \((X_t, Y_t)\) and \(L_t\) be as in Lemma 6.1 and (6.17), respectively. Finally fix two stopping times \((\eta, \xi)\) such that \(0 < \eta < \xi \leq \tau\) a.s. for some \(j \geq 1\). The aim of this section is to prove the existence of \(c_0 \geq 1\) independent of \((j, \eta, \xi, v_0, \theta_0)\) such that

\[
E\left[\sup_{t \in [\eta, \xi]} (X_t + \|\hat{\theta}(s)\|^2_{H^1(\mathbb{T}^2)}) + E\int_{\eta}^{\xi} (Y_s + \|\hat{\theta}(s)\|^2_{H^2(\mathbb{T}^2)}) ds\right]
\]

\[
\leq c_0 (1 + E|X_\eta| + E\|\theta(\eta)\|^4_{L^4} + E\|\hat{\theta}(\eta)\|^2_{H^1(\mathbb{T}^2)})
\]

\[
+ c_0 E\int_{\eta}^{\xi} L_s (1 + X_s + \|\theta(s)\|^4_{L^4}) ds.
\]

The presence of \(\|\theta\|^4_{L^4}\) on the RHS(6.19) will prove convenient later, cf. the last comments in Subsection 6.8.8. However, this terms do not create additional problems as they have been already estimated in Lemma 5.1.

Next we show the sufficiency of (6.19) for Lemma 6.1 to hold. Let \(X_t' \equiv X_t + \|\theta(t)\|^4_{L^4}\). By adding the estimates (5.5) and (6.19), we can apply the stochastic Grownall lemma of [AV22a, Lemma A.1] with \((X, Y, f, c_0)\) replaced by \((X', Y, L, 2c_0)\). Since \(X_t \leq X_t'\), the previous mentioned Grownall lemma implies, for all \(R, \gamma > 1\),

\[
P\left(\sup_{t \in [0, \tau \land \tau_j]} X_s + \int_0^{\tau \land \tau_j} Y_s ds \geq \gamma\right) \leq \frac{8c_0}{\gamma} e^{8c_0 R} (1 + E|X_0| + E\|\theta_0\|^4_{L^4}) + P\left(\int_0^{\tau \land T} L_s ds \geq \frac{R}{c_0}\right)
\]

\[
\leq \left(\frac{8c_0}{\gamma} e^{8c_0 R} + \frac{C}{\log R}\right) (1 + E\|v\|^2_{L^2(0,T)} + E\|v_0\|^2_{H^1} + E\|\theta_0\|^2_{H^1})
\]

where in the last step we used (6.18) and \(E|X_0| + E\|\theta_0\|^4_{L^4} \lesssim 1 + E\|v_0\|^2_{H^1} + E\|\theta_0\|^2_{H^1}\). Choosing \(R = \frac{1}{8c_0} \log\left(\frac{\gamma}{\log(\gamma)}\right)\) for \(\gamma > 1\) large and letting \(j \to \infty\), one can readily check that the above estimate yields (6.2).

The remaining part of this section is devoted to the proof of (6.19) where \((\eta, \xi)\) are two stopping times such that \(0 \leq \eta < \xi \leq \tau\) a.s. for some \(j \geq 1\) and \(T \in (0, \infty)\) is also fixed. The proof of
(6.19) requires a long preparation which will be the scope of Subsections 6.3-6.11. The proof of (6.19) is postponed to Subsection 6.12. Before starting into the proof of the estimates, we collect some facts which will be used frequently. Firstly, by Assumption 3.1(5) and 3.5 as well as the Sobolev embedding \( H^{1,2+\delta}(\mathbb{T}^d, \ell^2) \hookrightarrow L^\infty(\mathbb{T}^d, \ell^2) \) we have, a.s. for all \( t \in \mathbb{R}_+ \),

\[
\| \partial_t^j \sigma(t, \cdot) \|_{L^\infty(\mathbb{T}^d, \ell^2)} \lesssim_{M, \delta} 1 \quad \text{for all } j \in \{1, 2\} \text{ and } k \in \{0, 1\},
\]

\[
\| \pi^j(t, \cdot) \|_{L^\infty(\mathbb{T}^2)} \lesssim_{M, \delta} 1 \quad \text{for all } j \in \{1, 2\}.
\]

Secondly, we recall the following standard interpolation inequalities:

\[
\| f \|_{L^4(\mathbb{T}^2)} \lesssim \| f \|_{L^2(\mathbb{T}^2)}^{1/2} \| f \|_{H^1(\mathbb{T}^2)}^{1/2}, \quad \text{for } f \in H^1(\mathbb{T}^2),
\]

\[
\| f \|_{L^4(\mathcal{O})} \lesssim \| f \|_{L^2(\mathcal{O})} \| f \|_{H^1(\mathcal{O})}, \quad \text{for } f \in H^1(\mathcal{O}).
\]

To prove (6.19) we also use (small) parameters \( \varepsilon_i, \delta_i \in (0, \infty) \), where \( i \in \{1, \ldots, 9\} \), which will be used to absorb energy terms on the LHS of the corresponding estimate. The parameter \( \delta_i \) is chosen in the \( i \)-th subsection among Subsections 6.3-6.11 and the \( \varepsilon_i \)'s are chosen in Subsection 6.12.

Finally, to economize the notation, we do not display the dependence of the constants on \( T \).

### 6.3. Estimate for \( \sup_t \| \overline{\pi} \|_{H^s_x} \) and \( \| \overline{\pi} \|_{L^2_T H^s_x} \)

In this subsection we prove that

\[
\mathbf{E} \left[ \sup_{t \in [0, \xi]} \| \overline{\pi}(t) \|_{H^1(\mathbb{T}^2)} \right] + \mathbf{E} \int_0^\xi \| \overline{\pi} \|_{H^2(\mathbb{T}^2)}^2 \, ds \leq C_1 \left( 1 + \mathbf{E} \| \overline{\pi}(\eta) \|_{H^1(\mathbb{T}^2)}^2 \right)
\]

\[
+ \mathbf{E} \int_0^\xi L_s \| \overline{\pi} \|_{H^1(\mathbb{T}^2)}^2 \, ds + \mathbf{E} \int_0^\xi \| \nabla \overline{\pi} \|_{L^2}^2 \, ds + \mathbf{E} \int_0^\xi \| \nabla \overline{\pi} \|_{L^2}^2 \, ds + \mathbf{E} \int_0^\xi \| \nabla \overline{\pi} \|_{L^2}^2 \, ds
\]

where \( C_1 \) is a constant independent of \( j, \eta, \xi, \nu_0, \theta_0 \).

The estimate (6.24) follows as the one in [AHHS22, Lemma 5.3, Step 1] with minor modifications. The only additional term comes from the presence of \( \sigma_n \nabla \hat{\theta} \) in the stochastic part of (6.12a). To estimate the latter, note that (recall that \( (M, \delta) \) are as in Assumption 3.1),

\[
\mathbf{E} \int_0^\xi \left\| \partial_t \sigma_n \nabla \hat{\theta} \right\|_{L^2(\mathbb{T}^2)}^2 \, ds \lesssim_{M, \delta} \mathbf{E} \int_0^\xi \| \nabla \theta \|_{H^2(\mathbb{T}^2)}^2 \, ds
\]

\[
\mathbf{E} \int_0^\xi \left\| \pi^j_\delta \nabla \theta \right\|_{L^2(\mathbb{T}^2)}^2 \, ds \lesssim_{M, \delta} \mathbf{E} \int_0^\xi \| \nabla \theta \|_{L^2}^2 \, ds,
\]

\[
\mathbf{E} \int_0^\xi \| (\sigma_n \nabla \hat{\theta})_{n \geq 1} \|_{H^1(\mathbb{T}^2, \ell^2)}^2 \, ds \lesssim_{M, \delta} \mathbf{E} \int_0^\xi \| \hat{\theta} \|_{H^2(\mathbb{T}^2)}^2 \, ds.
\]

Using the above, the estimate (6.24) follows as from the one in [AHHS22, Lemma 5.3, Step 1] adding also the term \( \mathbf{E} \int_0^\xi (\| \hat{\theta} \|_{H^2(\mathbb{T}^2)}^2 + \| \nabla \theta \|_{L^2}^2) \, ds \) on the RHS of the corresponding estimate.

### 6.4. Estimate for \( \sup_t \| \hat{\theta} \|_{H^s_x} \) and \( \| \hat{\theta} \|_{L^2_T H^s_x} \)

The aim of this subsection is to prove the following estimate:

\[
\mathbf{E} \left[ \sup_{t \in [0, \xi]} \| \hat{\theta}(t) \|_{H^1(\mathbb{T}^2)}^2 \right] + \mathbf{E} \int_0^\xi \| \hat{\theta} \|_{H^2(\mathbb{T}^2)}^2 \, ds
\]

\[
\leq C_2 \left( 1 + \mathbf{E} \| \hat{\theta}(\eta) \|_{H^1(\mathbb{T}^2)}^2 \right) + \mathbf{E} \int_0^\xi L_s (1 + \| \hat{\theta} \|_{H^1(\mathbb{T}^2)}^2) \, ds + \mathbf{E} \int_0^\xi \| \nabla \hat{\theta} \|_{L^2}^2 \, ds + \mathbf{E} \int_0^\xi \| \nabla \hat{\theta} \|_{L^2}^2 \, ds + \mathbf{E} \int_0^\xi \| \hat{\theta} \|_{L^2}^2 \, ds,
\]

where \( C_2 \) is a constant independent of \( (j, \eta, \xi, \nu_0, \theta_0) \) and \( L_s \) is as in (6.17).

As in Subsection 6.3, the proof of (6.25) follows the line of [AHHS22, Lemma 5.3, Step 1]. Recall that \( \hat{\theta} \) satisfies (6.16). Next, let us denote by \( \mathcal{SMR}^\circ \) the set of couple of operators
having maximal $L^2$–regularity on a time interval $(0, T)$ on given spaces $(X_0, X_1)$, see Lemma 4.1 and [AV22b, Section 3] for the notation and examples. By repeating the arguments in Lemma 4.1, one sees that $(-\Delta_H, (\psi_{nH} \cdot \nabla)_{n \geq 1}) \in \mathcal{SMR}_c^\infty(0, T)$ with $X_0 = L^2(\mathbb{T}^2)$ and $X_1 = H^2(\mathbb{T}^2)$ (see also [AV21a] for the $L^p$–setting). Thus, by [AV22b, Proposition 3.10] and (6.16), there exists $\tilde{C}$ independent of $(j, \eta, \xi, v_0, \theta_0)$ such that

$$E \left[ \sup_{t \in [\eta, \xi]} \left\| \tilde{\theta}(t) \right\|^2_{H^1(\mathbb{T}^2)} \right] + E \int_\eta^\xi \left\| \tilde{\theta}(t) \right\|^2_{H^2(\mathbb{T}^2)} ds \leq \tilde{C} \left[ E \left\| \tilde{\theta}(\eta) \right\|^2_{H^1(\mathbb{T}^2)} + \sum_{1 \leq j \leq 5} \tilde{I}_j \right]$$

where

$$\tilde{I}_1 \overset{\text{def}}{=} E \int_\eta^\xi \left\| (\nabla \cdot \nabla H) \tilde{\theta} \right\|^2_{L^2(\mathbb{T}^2)} ds, \quad \tilde{I}_2 \overset{\text{def}}{=} E \int_\eta^\xi \left\| (\nabla \cdot \nabla H) \tilde{\theta} \right\|^2_{L^2(\mathbb{T}^2)} ds,$$

$$\tilde{I}_3 \overset{\text{def}}{=} E \int_\eta^\xi \left\| R(v, \theta) \right\|^2_{L^2(\mathbb{T}^2)} ds, \quad \tilde{I}_4 \overset{\text{def}}{=} E \int_\eta^\xi \left\| \tilde{\theta} \right\|^2_{H^2(\mathbb{T}^2)} ds,$$

$$\tilde{I}_5 \overset{\text{def}}{=} E \int_\eta^\xi \left\| \left( \frac{\nabla^2}{\eta} \tilde{\theta} \right)_{n \geq 1} \right\|^2_{H^1(\mathbb{T}^2)} ds.$$

Let us estimate each term separately. Note that

$$\tilde{I}_1 + \tilde{I}_3 \lesssim_h E \int_\eta^\xi \left( \left\| \nabla \right\|_{L^2(\mathbb{T}^2)}^2 + \left\| \theta \right\|_{L^2(\mathbb{T}^2)}^2 + \left\| \nabla \theta \right\|_{L^2(\mathbb{T}^2)}^2 \right) ds.$$

Moreover, applying (6.22) twice,

$$\tilde{I}_2 \leq E \int_\eta^\xi \left\| \nabla \theta \right\|_{L^2(\mathbb{T}^2)}^2 \left\| \nabla \tilde{\theta} \right\|_{L^2(\mathbb{T}^2)}^2 ds \leq E \int_\eta^\xi \left\| \nabla \theta \right\|_{L^2(\mathbb{T}^2)} \left\| \nabla \tilde{\theta} \right\|_{L^2(\mathbb{T}^2)} \left\| \nabla \tilde{\theta} \right\|_{H^1(\mathbb{T}^2)} \left\| \theta \right\|_{H^1(\mathbb{T}^2)} \left\| \nabla \theta \right\|_{H^2(\mathbb{T}^2)} ds \leq \tilde{C}_0 E \int_\eta^\xi \left\| \nabla \theta \right\|_{L^2(\mathbb{T}^2)} \left\| \nabla \tilde{\theta} \right\|_{H^1(\mathbb{T}^2)} ds + \frac{1}{2\tilde{C}} E \int_\eta^\xi \left\| \tilde{\theta} \right\|^2_{H^2(\mathbb{T}^2)} ds$$

where $\tilde{C}$ is as in (6.26), and $\tilde{C}_0$ is a constant independent of $(v_0, \theta_0, \eta, \xi, j)$. Finally, from (5.2) and Remark 3.2, we have

$$\tilde{I}_4 \lesssim E \int_\eta^\xi L_s ds \quad \text{and} \quad \tilde{I}_5 \lesssim_{\tilde{M}} E \int_\eta^\xi L_s ds + E \int_\eta^\xi \left\| \nabla \theta \right\|_{L^2(\mathbb{T}^2)} ds.$$

Putting together the previous estimate, one sees that there exists a constant $C_2$ independent of $(v_0, \theta_0, \eta, \xi, j)$ for which (6.25) holds.

6.5. **Estimate for** $\sup_{t \in [\eta, \xi]} \left\| \tilde{\theta}(t) \right\|_{L^2}$ **and** $\left\| \partial_3 v \right\|_{L^2(\mathbb{T}^2)}$. The aim of this subsection is to prove the following estimate: For all $\varepsilon_3 \in (0, \infty)$,

$$E \left[ \sup_{t \in [\eta, \xi]} \left\| \partial_3 v(t) \right\|^2_{L^2} \right] + E \int_\eta^\xi \left\| \nabla \partial_3 v(t) \right\|^2_{L^2} ds$$

$$\leq C_3 \left( 1 + E \left\| \partial_3 v(\eta) \right\|^2_{L^2} + E \int_\eta^\xi \left\| \nabla \right\|_{L^2}^2 ds \right)$$

$$+ C_{3, \varepsilon_3} E \int_\eta^\xi L_s \left( 1 + \left\| \partial_3 v \right\|^2_{L^2} \right) ds + \varepsilon_3 E \int_\eta^\xi \left\| \partial_3 \nabla \theta \right\|^2_{L^2} ds,$$

where $C_3, C_{3, \varepsilon_3}$ are constants independent of $(j, \eta, \xi, v_0, \theta_0)$ and $C_3$ is also independent of $\varepsilon_3$. Finally, $L_s$ is as in (6.17).

As before, here we follow the arguments Step 2 of [AHHS22, Lemma 5.3] with minor modifications. For notational convenience, as in the previous mentioned reference, we set $v_3 \overset{\text{def}}{=} \partial_3 v$. The estimate (6.27) follows almost verbatim as in [AHHS22, Lemma 5.3, Step 2] up to considering the additional term coming from $L_{\pi, \gamma, \theta} \, dt$ and $\sum_{n \geq 1} \sigma_n \int_{-\infty}^t \nabla \theta(\cdot, \zeta) \, d\zeta \, d\beta_t^n$ in (6.9a) in the Itô
formula for $v \rightarrow \| \partial_3 v \|_{L^2_T}$.

Let us begin by noticing that, the $\sigma_n$-contribution does not provide any additional problem as (recall that $\sigma_n$ is $x_3$-independent by Assumption 3.5)

$$
\sum_{n \geq 1} \mathbb{E} \int_{\eta}^{\xi} \left| \partial_3 \sigma_n \int_{-h}^{\eta} \nabla_H \theta(\cdot, \zeta) \, d\zeta \right|^2 \, dx ds \lesssim \mathbb{E} \int_{\eta}^{\xi} \| \nabla \theta \|_{L^2_x}^2 \, ds \lesssim \mathbb{E} \int_{\eta}^{\xi} L_s \, ds.
$$

To estimate the contribution of $L_{\pi, \gamma} \theta \, dt$, note that, in the Itô formula for $v \rightarrow \| \partial_3 v \|_{L^2_T}$ it gives rise to the term $\mathbb{E} \int_{\eta}^{\xi} R \, ds$ where

$$
R \overset{\text{def}}{=} \int_{\Omega} \mathbb{P}[L_{\pi, \gamma} \theta] \partial_3 v_3 \, dx.
$$

Recall that $\partial_3 \mathbb{P} f = \partial_3 f$ by (3.3). Integrating by parts and using (6.11a), we have

$$
R = - \int_{\Omega} \nabla \theta \left[ (\pi_H \cdot \nabla_H) \theta \right] \cdot v_3 \, dx - \int_{\Omega} \pi^3 H \nabla \theta \cdot v_3 \, dx.
$$

Note that, integrating by parts in the horizontal variables, for all $\varepsilon_0 > 0$,

$$
|R_1| = \left| \int_{\Omega} \left[ (\pi_H \cdot \nabla_H) \theta \right] \text{div}_H v_3 \, dx \right| (6.21) \leq \varepsilon_0 \| \nabla v_3 \|_{L^2_T} + C_{\varepsilon_0} \| \nabla \theta \|_{L^2_T}.
$$

To estimate $R_2$ note that $\pi^3 \in H^{1, 2+\delta}(T^2; L^2(-h, 0)) \hookrightarrow L^\infty(T^2; L^2(-h, 0))$ uniformly in $\mathbb{R}_+ \times \Omega$ by Assumption 3.1(4). Since $H^r(\Omega) \hookrightarrow L^2(T^2; H^r(-h, 0)) \hookrightarrow L^2(T^2; L^\infty(-h, 0))$ for all $r \in (\frac{1}{2}, 1)$, by interpolation, one sees that

$$
|R_2| \leq \varepsilon_3 \| \nabla \theta \|_{L^2_T} + \delta_3 \| \nabla v_3 \|_{L^2_T} + C_{\delta_3, \varepsilon_3} \| v_3 \|_{L^2_T}.
$$

By using the above estimates for $R$ and choosing $\delta_3$ small enough (independently on $(j, \eta, \xi, v_0, \theta_0, \varepsilon_3)$), one can check that the arguments in Step 2 of [AHHS22, Lemma 5.3] yield the estimate (6.27).

### 6.6. Estimate for $\sup_{t \in [\eta, \xi]} \| \partial_3 \theta \|_{L^2_T}$ and $\| \partial_3 \theta \|_{L^2_T \mu^T}$. In this subsection we prove that:

$$
\mathbb{E} \left[ \sup_{t \in [\eta, \xi]} \| \partial_3 \theta(t) \|_{L^2_T}^2 \right] + \mathbb{E} \int_{\eta}^{\xi} \| \partial_3 \theta \|_{H^1}^3 \, ds \leq C_4 (1 + \mathbb{E} \| \partial_3 \theta(\eta) \|_{L^2_T}^2 + \mathbb{E} \| \theta(\eta) \|_{L^2_T}^2)
$$

(6.28)

$$
+ C_4 \left( \mathbb{E} \int_{\eta}^{\xi} L_s (1 + \| \partial_3 \theta \|_{L^2_T}^2) \, ds + \mathbb{E} \int_{\eta}^{\xi} \| \theta \|_{L^2_T}^2 \, ds \right),
$$

where $C_4^{(4)}$ are constants independent of $(j, \eta, \xi, v_0, \theta_0)$.

Here the idea is to apply the Itô formula to (see the proof of [AHHS22, Proposition 6.8] for a similar situation)

$$
\theta \mapsto \mathcal{F}_\alpha(\theta) \overset{\text{def}}{=} \| \partial_3 \theta \|_{L^2_T}^2 + \alpha \| \theta(\cdot, 0) \|_{L^2_T}^2.
$$

For notational convenience, we set $\theta_3^{(1)} \overset{\text{def}}{=} \partial_3 \theta$ and $\theta_3^{(\eta, \xi)} \overset{\text{def}}{=} \theta_3((\cdot, \eta) \wedge \xi)$. Combining a standard approximation argument (cf. the proof of [AHHS22, Proposition 6.8]), the Itô formula, the boundary conditions (6.11b) and integrating by parts, one can check that, a.s. for all $t \in [0, T]$,

$$
\| \theta_3^{(\eta, \xi)}(t) \|_{L^2_T}^2 + \alpha \| \theta_3^{(\eta, \xi)}(t, \cdot, 0) \|_{L^2_T}^2 = \| \theta_3(\eta) \|_{L^2_T}^2 + \alpha \| \theta(\eta, \cdot, 0) \|_{L^2_T}^2
$$

(6.29)

$$
+ 2 \int_0^t 1_{[\eta, \xi]} E(s) \, ds + \sum_{1 \leq j \leq 3} \int_0^t 1_{[\eta, \xi]} I_j(s) \, ds + M(t)
$$

where $E \overset{\text{def}}{=} - \int_{\Omega} \Delta \theta \partial_3 \theta_3 \, dx$ gives the energy contribution and

$$
I_1 \overset{\text{def}}{=} 2 \int_{\Omega} f_\theta \partial_3 \theta_3 \, dx, \quad I_2 \overset{\text{def}}{=} -2 \int_{\Omega} [(v \cdot \nabla_H) \theta + w(\theta) \partial_3 \theta_3] \, dx,
$$

$$
I_3 \overset{\text{def}}{=} \sum_{n \geq 1} \left( \int_{\Omega} \left| \partial_3 ([\psi_n \cdot \nabla] \theta) + \partial_3 g_{\theta, n} \right|^2 \right) \, dx
$$
integrating by parts and using (6.11b), we have, a.e. on $\Omega$ \(×\)

\[
E(\cdot,0) \lesssim \|f(x_H,\cdot)\|_{H^{1/2+r}(-\infty,0)} \quad \text{for all } x_H \in \mathbb{T}^2 \text{ for all } r > 0, \text{ with implicit constant independent of } x_H.
\]

Hence, by integrating over $x_H \in \mathbb{T}^2$, we have

\[
\|f(\cdot,0)\|_{L^2(\mathbb{T}^2)} \lesssim \|f(x_H,\cdot)\|_{H^{1/2+r}(-\infty,0)} \quad \text{on } \mathbb{T}^2.
\]

In particular the second term on the LHS(6.29) is lower order compared to $\|\theta_3\|_{L^2}^2$ and we do not need to estimate it further. The same also applies for the second term on the RHS(6.29) for which we can use that (6.31) implies $\|\theta(\eta,\cdot,0)\|_{L^2(\mathbb{T}^2)} \lesssim \|\theta(\eta)\|_{L^2} + \|\partial_3\theta(\eta)\|_{L^2}$.

The estimates of the remaining terms are worked out in the following subsections. The proof of (6.28) is given in Subsection 6.6.4 below. In the following $\varepsilon, \delta_4 \in (0, \infty)$ are positive parameters which will be chosen in Subsections 6.12 and 6.6.4, respectively.

6.6.1. Estimate of $E$. Since $\theta \in H^3_H$, by standard approximation argument we may assume that $\theta \in C^3(\Omega)$ and satisfies (6.11b). Note that, integrating by parts,

\[
E = \alpha \int_{\mathbb{T}^2} (\Delta \theta(\cdot,0)) \theta(\cdot,0) \, dx_H + \int_{\mathbb{O}} \Delta \theta_3 \theta_3 \, dx
\]

\[
= \alpha \int_{\mathbb{T}^2} (\Delta \theta(\cdot,0)) \theta(\cdot,0) \, dx_H + \int_{\mathbb{T}^2} \partial_3 \theta_3(\cdot,0) \theta_3(\cdot,0) \, dx_H - \int_{\mathbb{O}} |\nabla \theta_3|^2 \, dx
\]

\[
(6.11b) \quad \alpha \int_{\mathbb{T}^2} (\Delta \theta(\cdot,0)) \theta(\cdot,0) \, dx_H - \alpha^3 \int_{\mathbb{T}^2} |\theta(\cdot,0)|^2 \, dx_H - \int_{\mathbb{O}} |\nabla \theta_3|^2 \, dx.
\]

The last term on the RHS of the previous equality gives rise to the second term on the LHS(6.28).

To conclude, we show that $(\epsilon_0, \epsilon_1)$ are lower-order compared to such term, i.e. for all $\varepsilon > 0$

\[
|\epsilon_0| + |\epsilon_1| \leq \varepsilon \|\nabla \theta_3\|_{L^2}^2 + C_{\varepsilon} \|\theta_3\|_{L^2}^2.
\]

Note that (5.2) already implies that $\epsilon_1$ is lower order. To estimate $\epsilon_1$, note that, by (6.11b) and integrating by parts,

\[
\epsilon_0 = -\int_{\mathbb{T}^2} |\nabla \theta(\cdot,0)|^2 \, dx_H - \alpha^2 \int_{\mathbb{T}^2} |\theta(\cdot,0)|^2 \, dx_H.
\]

Due to (5.2), it is clear that the second term on RHS(6.33) is lower order. The same also holds for the first term as one can readily check by applying (6.31) and a standard interpolation argument.

6.6.2. Estimate of $I_2$. For notational convenience, as above, we set $u \defeq (v, w(v))$. Note that, integrating by parts and using (6.11b), we have, a.e. on $\Omega \times [\eta, \xi]$,

\[
I_2 = \int_{\mathbb{T}^2} (v(\cdot,0) \cdot \nabla_H) \theta(\cdot,0) \theta(\cdot,0) \, dx_H - \int_{\mathbb{O}} [(u_3 \cdot \nabla) \theta] \theta_3 \, dx - \int_{\mathbb{O}} [(u \cdot \nabla) \theta] \theta_3 \, dx
\]

\[
= \int_{\mathbb{T}^2} (v(\cdot,0) \cdot \nabla_H) \theta(\cdot,0) \theta(\cdot,0) \, dx_H - \int_{\mathbb{O}} [(u_3 \cdot \nabla) \theta] \theta_3 \, dx
\]

\[
(6.28) \quad I_2 \defeq \int_{\mathbb{T}^2} (v(\cdot,0) \cdot \nabla_H) \theta(\cdot,0) \theta(\cdot,0) \, dx_H - \int_{\mathbb{O}} [(u_3 \cdot \nabla) \theta] \theta_3 \, dx.
\]
where the last equality follows from Lemma 5.2 and an approximation argument. Next we rewrite $I'_2$. To this end, note that $u_3 = (v_3, -\text{div}Hv)$. Hence, using an integration by parts and $\text{div}u_3 = 0$, we have, a.e. on $[\eta, \xi] \times \Omega$,

$$I'_2 = -\int_{\mathbb{T}^2} \text{div}_Hv(\cdot, 0)\theta(\cdot, 0)\theta_3(\cdot, 0) \, dx_H - \int_{\mathcal{O}} \theta[(u_3 \cdot \nabla)\theta_3] \, dx \tag{6.11b}$$

Finally, since $\text{div}_H\tilde{v} = \text{div}_Hv$ and $\tilde{v}_3 = v_3$, we have, a.e. on $[\eta, \xi] \times \Omega$,

$$|I''_2| = -\sum_{1 \leq i,j \leq 2} \int_{\mathcal{O}} \tilde{v}_i(\nabla)\theta(\partial_i\theta_3) \, dx + \int_{\mathcal{O}} (\text{div}_H\tilde{v}) \theta(\partial_i\theta_3) \, dx.$$  

Therefore, by Cauchy-Schwartz inequality we have, for all $\delta_4 > 0$ and a.e. on $[\eta, \xi] \times \Omega$,

$$|I''_2| \lesssim \int_{\mathcal{O}} \|
abla \tilde{v}\| \|\nabla \theta_3\| \, dx \leq \delta_4 \|\nabla \theta_3\|^2_{L^2} + C_{\delta_4} \|\nabla \tilde{v}\| \|\theta\|^2_{L^2}.$$  

It remains to estimate the boundary terms $(b_0, b_1)$. Recall that $L$ is as in (6.17). We claim that, a.e. on $[\eta, \xi] \times \Omega$,

$$|b_0| + |b_1| \lesssim L. \tag{6.34}$$

We prove the latter fact for $b_0$, for the $b_1$-term the proof is analogue. To this end, note that

$$|b_0| \approx \left|\int_{\mathbb{T}^2} (v(\cdot, 0) \cdot \nabla_v)[\theta(\cdot, 0)] \, dx_H\right| \leq \||v(\cdot, 0)\|_{H^{\frac{1}{2}}(\mathbb{T}^2, \mathbb{R}^2)} \|
abla_v \theta(\cdot, 0)\|_{H^{\frac{1}{2}}(\mathbb{T}^2, \mathbb{R}^2)} \approx \|v(\cdot, 0)\|_{H^1} \|\theta\|_{H^1} \lesssim \|v\|_{H^1} \|\theta\|_{H^1} \lesssim L.$$  

Since $\|\theta^2\|_{H^1} \lesssim \|\theta\|^4_{L^4} + ||\theta||^2\|\nabla \theta\|^2_{L^2}$, we have $|b_0| \lesssim L$ as desired. Thus (6.34) is proved.

6.6.3. Estimate of $I_3$. The Cauchy–Schwarz inequality, (5.2) and standard interpolation arguments show that, a.e. on $[\eta, \xi] \times \Omega$,

$$|I_3| \leq (1 + \delta_1) \sum_{n \geq 1} \int_{\mathcal{O}} \|\psi_n \cdot \nabla\theta_3\|^2 \, dx + C_{\delta_4} \sum_{n \geq 1} \left(\|\nabla \psi_n\|^2 \|\theta_3\|^2 + \|\text{div}g_n, n\|^2\right) \, dx$$

$$\leq \nu(1 + \delta_4) \int_{\mathcal{O}} \|
abla \theta_3\|^2 \, dx + C_{\delta_4}(\|\psi_n\|_{H^{1 + \frac{1}{2}r}(\mathcal{O})})^2 \|\theta_3\|^2_{L^2} + \|\nabla g\|_{H^{1 + \frac{1}{2}r}(\mathcal{O})},$$

where in the last inequality we used Assumption 3.1(2) and $r \in (1, 6)$ satisfies $\frac{1}{r+3} + \frac{1}{r} = \frac{1}{2}$.  

Recall that $\|\psi_n\|_{H^{1 + \frac{1}{2}r}(\mathcal{O})} \leq M$, by Assumption 3.1(3). Since $H^\theta(\mathcal{O}) \hookrightarrow L^r(\mathcal{O})$ for some $\theta \in (0, 1)$, by standard interpolation theory, we have a.e. on $[\eta, \xi] \times \Omega$,

$$|I_3| \leq \nu(1 + 2\delta_4) \int_{\mathcal{O}} \|
abla \theta_3\|^2 \, dx + C_{\delta_4}(\|\theta_3\|^2_{L^2} + \|\nabla g\|^2_{H^{1 + \frac{1}{2}r}(\mathcal{O})}).$$

6.6.4. Estimate of the martingale $M$ and proof of (6.28). Taking expectations in (6.29) with $t = T$, choosing $\delta_4 > 0$ sufficiently small (independently of $(j, \eta, \xi, v_0, \theta_0)$), and using that $E[M(T)] = 0$, one has

$$E \int_{\eta}^{\xi} \|
abla \theta_3\|^2_{L^2} \, ds \leq c_4 \left(1 + E\|\theta_3(\eta)\|^2_{L^2}\right)$$

$$+ c_4 \left(E \int_{\eta}^{\xi} \|
abla \tilde{v}\|^2_{L^2} \, ds + E \int_{\eta}^{\xi} L_s(1 + \|\theta_3\|^2_{L^2}) \, ds \right),$$

where $c_4$ is a constant independent of $(j, \eta, \xi, v_0, \theta_0)$.  

Arguing as in Step 2 of Lemma 5.1, the Burkholder-Davis-Gundy inequality and Assumption 3.1(3) readily yield, for some $C > 0$ independent of $(j, \eta, \xi, v_0, \theta_0)$,

$$
E \left[ \sup_{t \in [\eta, \xi]} |M_t| \right] \leq \frac{1}{2} E \left[ \sup_{s \in [\eta, \xi]} \| \theta_3(s) \|_{L^2}^2 \right] + CE \int_{\eta}^{\xi} \left( \| \nabla \theta_3 \|_{L^2}^2 + \| \theta \|_{H^1}^2 + \| g_0 \|_{H^2(T_2)}^2 \right) ds
$$

(6.35)

\[
\leq \frac{1}{2} E \left[ \sup_{s \in [\eta, \xi]} \| \theta_3(s) \|_{L^2}^2 \right] + C(1 + E \| \theta_3(\eta) \|_{L^2}^2)
\]

\[+ CE \int_{\eta}^{\xi} \left( \| \nabla \theta \|_{L^2}^2 + L_s(1 + \| \theta_3 \|_{L^2}^2) \right) ds
\]

Now (6.28) follows by taking $E[\sup_{t \in [\eta, \xi]} | \cdot |]$ in (6.29) and using the above estimates.

6.7. Estimate for $\sup_t \| \bar{\nu} \|_{L^4}^4$ and $\| \bar{\nu} \|_{L^7 L^2}$. In this subsection we prove the following estimate: For all $\varepsilon_5 \in (0, \infty)$,

$$
E \left[ \sup_{t \in [\eta, \xi]} \| \bar{\nu}(t) \|_{L^4}^4 \right] + E \int_{\eta}^{\xi} \left( \| \nabla \bar{\nu} \|_{L^2}^2 \right) ds \leq C_{5, \varepsilon_5} \left( 1 + E \| \bar{\nu}(\eta) \|_{L^4}^4 \right)
$$

(6.36)

$$
+ C_5 E \int_{\eta}^{\xi} \left( \| \nabla H \theta \|_{L^2}^2 \right) ds + C_{5, \varepsilon_5} E \int_{\eta}^{\xi} L_s(1 + \| \bar{\nu} \|_{L^4}^4) ds
$$

$$
+ \varepsilon_5 E \int_{\eta}^{\xi} \left( \| \partial_3 v \|_{H^1}^2 + \| \theta \|_{H^2(T_2)}^2 \right) ds,
$$

where $C_5, C_{5, \varepsilon_5}$ are constants independent of $(j, \eta, \xi, v_0, \theta_0)$ and $C_5$ is also independent of $\varepsilon_5$. Finally, $L_s$ is as in (6.17).

As in Subsections 6.3 and 6.5, here we can follow the proof of [AHHS22, Lemma 5.3, Step 4]. More precisely, following [AHHS22] we apply the Itô formula to $\bar{\nu} \mapsto \| \bar{\nu} \|_{L^4}^4$. Comparing (6.12) with [AHHS22, eq. (5.23)], we have the following additional terms $(\mathcal{L}_{\pi, \gamma} \theta + \mathcal{L}_{\pi, \gamma} \bar{\theta} - \partial_3 \nabla H \theta) ds$ and $\sum_{n \geq 1} \mathcal{T}_n \theta d \beta_n^\eta$. Here, we content ourself to provide a suitable estimate for the Itô corrections related to the $\mathcal{T}_n$-term when applying the Itô formula to $v \mapsto \| \bar{\nu} \|_{L^4}^4$, i.e., the term

$$
E \int_{\eta}^{\xi} \left( \int_\mathcal{O} \sum_{n \geq 1} \| \bar{\nu} \|_{L^2}^2 \| \mathcal{T}_n \theta \|_{L^2}^2 dx \right) ds.
$$

The contributions related to the terms in the deterministic part can be estimated similarly, noticing that, by (6.10a), $\mathcal{L}_{\pi, \gamma} \theta = (\pi H \cdot \nabla H) \theta + R_0$ where

$$
R_0 \leq \int_{-h}^{0} \| \pi_3(\cdot, \cdot) \partial_3 \nabla H(\cdot, \cdot) \theta(\cdot, \cdot) | d\zeta.
$$

To estimate the quantity in (6.37), note that, a.e. on $\Omega \times [0, \tau)$,

$$
\sum_{n \geq 1} \int_\mathcal{O} \| \bar{\nu} \|_{L^2}^2 \| \mathcal{T}_n(\theta) \|_{L^2}^2 dx \lesssim_M \left( \int_\mathcal{O} \| \bar{\nu}(\cdot, x) \|_{L^2}^2 \theta(\cdot, x_H, \zeta) d\zeta \right)^2 dx + \int_\mathcal{O} \| \bar{\nu} \|_{L^2}^2 \| \theta \|_{H^2(T_2)}^2 dx.
$$

The second term on the RHS of the previous can be further estimated as follows:

$$
\int_\mathcal{O} \| \bar{\nu} \|_{L^2}^2 \| H \theta \|_{L^2}^2 dx \leq \| \| \bar{\nu} \|_{L^2}^2 \| H \theta \|_{L^2}^2 \|
\leq \| \| \bar{\nu} \|_{L^2}^2 \| \theta \|_{H^1(T_2)} \| \theta \|_{H^2(T_2)}
\lesssim \varepsilon_5 \| \bar{\nu} \|_{L^2}^2 \| \theta \|_{H^1(T_2)} \| \theta \|_{H^2(T_2)} + C_{\varepsilon_5} L \| \bar{\nu} \|_{L^4}^4.
$$

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]
where $L_s$ is as in (6.17). With the above estimates available, one can check that the estimate [AHHS22, eq. (5.54)] extends to (6.14) and one gets (6.36).

6.8. **Estimate for $||\overline{v}||\nabla \theta ||_{L^2_t L^2_x}$ and $||\theta ||\nabla \overline{v} ||_{L^2_t L^2_x}$**. The aim of this subsection is to prove the following estimate: For all $\varepsilon_6 \in (0, \infty)$,

\begin{equation}
E \int_\eta^\xi \left[ ||\overline{v}||\nabla \theta ||_{L^2}^2 \right] ds + E \int_\eta^\xi \left[ ||\theta ||\nabla \overline{v} ||_{L^2}^2 \right] ds \leq C_6, \varepsilon_6 \left( 1 + E ||\theta(\eta)||_{L^4}^4 + E ||\overline{v}(\eta)||_{L^4}^4 \right)
\end{equation}

\begin{equation}
\leq C_{6, \varepsilon_6} \int_\eta^\xi L_t \left( 1 + ||\overline{v}||_{L^4} + ||\theta||_{L^4}^4 \right) ds
+ \varepsilon_6 \int_\eta^\xi \left[ ||\overline{v}||^2 + ||\nabla \overline{v}||_{H^2(\Omega)}^2 + ||\theta||_{H^2(\Omega)}^2 + ||\partial_3 \nabla v||_{L^2}^2 + ||\partial_3 \nabla \theta||_{L^2}^2 \right] ds
\end{equation}

\begin{equation}
+ C_6 \int_\eta^\xi \left[ ||\theta ||\nabla \overline{v} \Theta ||_{L^2}^2 \right] ds
\end{equation}

where $C_6, C_{6, \varepsilon_6}$ are constants independent of $(j, \eta, \xi, v_0, \theta_0)$ and $C_6$ is also independent of $\varepsilon_6$. Finally, $L_s$ is as in (6.17).

To prove (6.38), we apply the Itô formula to the functional $(\overline{v}, \theta) \mapsto ||\overline{v}||\theta ||_{L^2}^2$. To this end, recall that $\overline{v}$ and $\theta$ satisfy the SPDEs (6.14) and (6.9b), respectively. Moreover, we let

\[ \theta^{\eta, \xi} \overset{\text{def}}{=} \theta((\cdot \lor \eta) \land \xi) \quad \text{and} \quad \overline{v}^{\eta, \xi} \overset{\text{def}}{=} \overline{v}((\cdot \lor \eta) \land \xi). \]

Applying the Itô formula to $(\overline{v}, \theta) \mapsto ||\theta^{\eta, \xi}(t)||_{L^2}^2$ we have, a.s. for all $t \in \mathbb{R}_+$,

\begin{equation}
||\theta^{\eta, \xi}(t)||_{L^2}^2 = ||\theta(\eta)||_{L^2}^2 ||\overline{v}(\eta)||_{L^2}^2 + \sum_{1 \leq j \leq 4} \int_0^t 1_{[\eta, \xi]} I_{2, j}(s) ds + N_t
\end{equation}

where $N$ is a $L^1(\Omega)$-martingale, such that $E[N_t] = 0$ for all $0 \leq t \leq T$, and

\begin{align*}
J_1 & = 2 \int_\Omega (\theta^2 \overline{v} \cdot \Delta \overline{v} + |\overline{v}|^2 \theta \Delta \theta) \, dx, \\
J_2 & = 2 \int_\Omega (\theta^2 \overline{v} \cdot (f v + F(\overline{v})) + |\overline{v}|^2 \theta f_\theta) \, dx, \\
J_3 & = 2 \int_\Omega \theta^2 \overline{v} \cdot \left( \mathcal{L}_{\pi, \gamma, \theta} + (\pi_H \cdot \nabla H) \hat{\theta} + \pi^3 \partial_3 \nabla H \theta \right) \, dx ds, \\
J_4 & = -2 \int_\Omega \theta^2 \overline{v} \cdot ([\overline{v} \cdot \nabla H] v) \, dx, \\
J_5 & = \sum_{n \geq 1} \int_\Omega |\overline{v}|^2 [(\psi_n \cdot \nabla) \theta + g_{\theta, n}]^2 \, dx, \\
J_6 & = \int_\Omega |\theta|^2 [\phi_n \cdot \nabla \overline{v} - \phi^2_n \partial_3 \overline{v} + T_n(\theta) + \tilde{g}_{n, \theta}]^2 \, dx, \\
J_7 & = 2 \sum_{n \geq 1} \int_\Omega \theta [(\psi_n \cdot \nabla) \theta + g_{\theta, n} |\overline{v}| [\phi_n \cdot \nabla \overline{v} - \phi^2_n \partial_3 \overline{v} + T_n(\theta) + \tilde{g}_{n, \theta}] \, dx.
\end{align*}

and we used that, a.e. on $[0, \tau] \times \Omega$,

\[ \int_\Omega \left( |\overline{v}|^2 \theta [(\nabla \cdot \nabla H) \theta] + |\theta|^2 \overline{v} \cdot [(\nabla \cdot \nabla H) \overline{v}] \right) \, dx = 0, \]

\[ \int_\Omega \left( |\overline{v}|^2 \theta [(\overline{v} \cdot \nabla) \theta] + |\theta|^2 \overline{v} \cdot [(\overline{v} \cdot \nabla) \overline{v}] \right) \, dx = 0, \]

where $\overline{u} = (\overline{v}, w(\overline{v}))$ and $w(\overline{v})$ is as in (3.9). The above follows from Lemma 5.2, (6.13) and a standard approximation argument. Let us remark that the application of the Itô formula in (6.39)
requires an approximation argument similar to the one used in Step 3 of [AHHS22, Lemma 5.3]. To avoid repetitions, we omit the details.

For the reader’s convenience, we collect the estimates of $(J_j)_{j=1}^7$ in the following subsections. The proof of (6.38) will be given in Subsection 6.8.8. Below $\varepsilon_6, \delta_6 \in (0, \infty)$ are positive parameters which will be chosen in Subsections 6.12 and 6.8.8, respectively.

6.8.1. Estimate of $J_1$. Integrating by parts and using the boundary conditions (3.2), we have

$$\int_{\Omega} \theta^2 \bar{v} \cdot \Delta \bar{v} \, dx = - \int_{\Omega} \theta^2 |\nabla \bar{v}|^2 \, dx - 2 \sum_{1 \leq i,j \leq 3} \int_{\Omega} \theta \bar{v}^i \partial_i \bar{v}^j \partial_j \theta \, dx$$

and

$$\int_{\Omega} |\bar{v}|^2 \Delta \theta \, dx = - \alpha \int_{\Omega} |\bar{v}(\cdot, x, 0)|^2 \theta(\cdot, x, 0)^2 \, dx \quad \text{and} \quad \int_{\Omega} |\bar{v}|^2 |\nabla \theta|^2 \, dx - 2 \sum_{1 \leq i,j \leq 3} \int_{\Omega} \theta \bar{v}^i \partial_i \bar{v}^j \partial_j \theta \, dx.$$ 

By the boundedness of the trace operator (5.2), for any $r \in (\frac{1}{2}, 1)$,

$$\int_{\mathbb{R}^d} |\bar{v}(\cdot, x, 0)|^2 |\theta(\cdot, x, 0)|^2 \, dx \lesssim_{r} \|\theta\|_{L^r}^2 \lesssim_r \|\bar{v}\|^2 \|\nabla \theta\|^2,$$

$$\lesssim_r \|\bar{v}\|^{2(1-r)}_{L^2} \left( \|\bar{v}\|^2_{L^2} + \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} |\bar{v}|^2 \, dx \right)^r$$

$$\leq C_{r, \delta_6} \|\bar{v}\|^2_{L^2} + \delta_6 \left( \int_{\Omega} |\bar{v}|^2 \, dx + \int_{\Omega} |\nabla \theta|^2 \, dx \right)$$

$$\leq C_{r, \delta_6} \|\bar{v}\|^4_{L^4} + \|\theta\|^4_{L^4} + \delta_6 \left( \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} |\bar{v}|^2 \, dx \right).$$

By Cauchy-Schwartz inequality, we have

$$\sum_{1 \leq i,j \leq 3} \left| \int_{\Omega} \theta \bar{v}^i \partial_i \bar{v}^j \partial_j \theta \, dx \right| \leq \varepsilon_6 \int_{\Omega} |\bar{v}|^2 |\nabla \theta|^2 \, dx + C_{\varepsilon_6} \int_{\Omega} |\theta|^2 |\nabla \theta|^2 \, dx.$$ 

Summarizing the previous estimates, we have, a.e. on $[0, \tau) \times \Omega$,

$$J_1 \leq - (2 - \delta_6) \left( \int_{\Omega} \theta^2 |\nabla \bar{v}|^2 \, dx + \int_{\Omega} |\bar{v}|^2 |\nabla \theta|^2 \, dx \right)$$

$$+ \varepsilon_6 \int_{\Omega} |\bar{v}|^2 |\nabla \theta|^2 \, dx + C_{\varepsilon_6, \delta_6} (L + \|\bar{v}\|^4_{L^4} + \|\theta\|^4_{L^4}).$$

where we have also used that $\int_{\Omega} |\theta|^2 |\nabla \theta|^2 \, dx \leq L$ by (6.17).

6.8.2. Estimate of $J_2$. Let us write $J_2 = J_{2,1} + J_{2,2}$ where

$$J_{2,1} \overset{\text{def}}{=} 2 \int_{\Omega} \theta^2 \bar{v} \cdot (f_v + \mathcal{F}(\bar{v})) \, dx \quad \text{and} \quad J_{2,2} \overset{\text{def}}{=} 2 \int_{\Omega} |\bar{v}|^2 \theta \rho \, dx.$$ 

To estimate such terms, observe that

$$||\bar{v}||_{L^6}^2 = ||\bar{v}||_{L^3}^6 \overset{(6.23)}{\lesssim} \left( ||\bar{v}||_{L^2}^2 \right)^{1/2} \left( ||\nabla \bar{v}||_{L^2}^2 \right)$$

$$= ||\bar{v}||_{L^4}^2 + ||\nabla v||_{L^2} \left( ||\bar{v}||_{L^2} ||\nabla \bar{v}||_{L^2} \right)^{1/2}.$$ 

Thus, since $||\mathcal{F}(\bar{v})||_{L^2} \lesssim ||\bar{v}|| |\nabla \bar{v}||_{L^2}$ due to (6.12b), we have, a.e. on $[\eta, \xi] \times \Omega$,

$$|J_{2,1}| \lesssim \|\bar{v}\|_{L^3}^3 ||\bar{v}||_{L^2} \left( ||f_v||_{L^2} + ||\mathcal{F}(\bar{v})||_{L^2} \right)$$

$$\lesssim \|\theta\|_{L^6} \left( ||\bar{v}||_{L^4} + ||\nabla \bar{v}||_{L^4} \left( ||f_v||_{L^2} + ||\nabla \bar{v}||_{L^2} \right) \right)$$

$$\lesssim \varepsilon_6 ||\bar{v}||_{L^4}^2 + C_{\varepsilon_6} ||f_v||_{L^2}^2 + C_{\varepsilon_6} \|\theta\|^2_{L^6} (1 + ||\bar{v}||_{L^4}^4)$$

where in the last step we applied the Young inequality twice.
Similarly, we can estimate $J_{2, 2}$. Indeed, a.e. on $[\eta, \xi] \times \Omega$,

$$
|J_{2, 2}| \lesssim \|f_0\|_{L^2} \|\bar{v}\|_{L^3} \|\theta\|_{L^6}
$$

$$
= \|f_0\|_{L^2} \|\bar{v}\|_{L^6} \|\theta\|_{L^6}
$$

$$
\leq \varepsilon_6 \|\bar{v}\|_{\Delta \nabla \bar{v}} \|\bar{v}\|^2_{L^2} + C_{\varepsilon_6} \|f_0\|^2_{L^2} + C_{\varepsilon_6} (1 + \|\theta\|^8_{L^6})(1 + \|\bar{v}\|_{L^s}^4),
$$

where in the last step we applied the Young’s inequality twice again.

6.8.3. **Estimate of $J_3$.** Let us decompose $J_3$ as $J_3 = J_{3, 1} + J_{3, 2} + J_{3, 3}$ where

$$
J_{3, 1} \overset{\text{def}}{=} \int_{\Omega} \theta^2 \bar{v} \cdot [ (\pi_H \cdot \nabla H) \nabla H \Theta ] \, dx,
$$

$$
J_{3, 2} \overset{\text{def}}{=} \int_{\Omega} \theta^2 \bar{v} \cdot \left( \int_{-h} \pi^3 (\cdot, \zeta) \partial_3 \nabla H \theta (\cdot, \zeta) \, d\zeta \right) \, dx,
$$

$$
J_{3, 3} \overset{\text{def}}{=} \int_{\Omega} \theta^2 \bar{v} \cdot (\pi_H \cdot \nabla H) \nabla H \Theta \, dx,
$$

$$
J_{3, 4} \overset{\text{def}}{=} \int_{\Omega} \theta^2 \bar{v} \cdot \pi^3 \partial_3 \nabla H \Theta \, dx.
$$

We begin by looking at $J_{3, 1}$. Integrating by parts, we have, a.e. on $[\eta, \xi] \times \Omega$,

$$
|J_{3, 1}| \leq \int_{\Omega} |\nabla H \pi_H| \theta^2 |\bar{v}| |\nabla H \Theta| \, dx
$$

$$
+ \int_{\Omega} |\theta| |\nabla H \Theta| |\bar{v}| |\nabla H \Theta| \, dx + \int_{\Omega} \theta^2 |\nabla H \bar{v}| |\nabla H \Theta| \, dx.
$$

By Cauchy–Schwartz inequality and Assumption 3.1(4),

$$
|J_{3, 1}| \leq \delta_6 \left( \int_{\Omega} |\bar{v}|^2 |\nabla \theta|^2 \, dx + \int_{\Omega} |\bar{v}|^2 |\nabla \bar{v}|^2 \, dx \right)
$$

$$
+ C_{\delta_6} \int_{\Omega} |\theta|^2 |\nabla H \Theta|^2 \, dx + C_{\delta_6} (1 + \|\theta\|^8_{L^6})(1 + \|\bar{v}\|_{L^s}^4).
$$

Similarly, one can readily check that, a.e. on $[\eta, \xi] \times \Omega$,

$$
\sum_{2 \leq j \leq 4} |J_{3, j}| \leq \delta_6 \left( \int_{\Omega} |\bar{v}|^2 |\nabla \theta|^2 \, dx + \int_{\Omega} |\bar{v}|^2 |\nabla \bar{v}|^2 \, dx \right)
$$

$$
+ \varepsilon_6 \left( \|\partial_3 \nabla \theta\|^2_{L^2} + \|\partial_3 \nabla \theta\|^2_{L^2} + \|\bar{\theta}\|_{H^2(\mathcal{T}_2)}^2 \right) + C_{\varepsilon_6} (1 + \|\theta\|^8_{L^6})(1 + \|\bar{v}\|_{L^s}^4).
$$

6.8.4. **Estimate of $J_4$.** The Hölder inequality and the embedding $H^1 \hookrightarrow L^6$ yield, a.e. on $[\eta, \xi] \times \Omega$,

$$
|J_4| \lesssim \|\theta^2\|_{L^2} \|\bar{v}\|_{L^2} \|\nabla H \bar{v}\|_{L^6(\mathcal{T}_2)}
$$

$$
\lesssim \|\theta\|^2_{L^6} \|\bar{v}\|^2_{L^6} \|\nabla H \bar{v}\|_{L^6(\mathcal{T}_2)} \leq \varepsilon_6 \|\bar{v}\|^2_{H^2(\mathcal{T}_2)} + C_{\delta_6} \|\theta\|^4_{L^6} \|\bar{v}\|^4_{L^6}.
$$

6.8.5. **Estimate of $J_5$.** We begin by noticing that, for all $\varepsilon_0 \in (0, \infty)$ and a.e. on $[0, \tau] \times \Omega$,

$$
|J_5| \overset{(i)}{\leq} (\nu + \delta_6) \int_{\Omega} |\bar{v}|^2 |\nabla \theta|^2 \, dx + C_{\delta_6} \int_{\Omega} |\theta|^2 \|g_0\|_{L^2}^2 \, dx
$$

$$
\overset{(ii)}{\leq} (\nu + \delta_6) \int_{\Omega} |\bar{v}|^2 |\nabla \theta|^2 \, dx + C_{\delta_6} \|\theta\|^2_{L^4} \|g_0\|^2_{H^2(\mathcal{T}_2)}
$$

where in (i) we used Assumption 3.1(2) and in (ii) that $H^1(\ell^2) \hookrightarrow L^4(\ell^2)$. 

6.8.6. Estimate of $J_6$. To begin, note that, a.e. on $[0, \tau) \times \Omega$,

$$
|J_6| \leq (\nu + \delta_6) \int_{\Omega} |\theta|^2 |\nabla \tilde{v}|^2 \, dx
$$

$$
+ C_{\delta_6} \left( \|\partial_y^3 \tilde{v} \|_{H^1(\ell^2)}^2 + \left\| |\nabla \Theta| \right\|_{L^2}^2 + \sum_{n \geq 1} \int_{\Omega} |\theta|^2 |\partial_x \tilde{v}|^2 \, dx \right).
$$

Next we estimate the last term on the RHS of the previous inequality. To this end, note that

$$
\text{by} \ |\partial_x^3 \tilde{v}|^2 \text{ is } x_3 \text{-independent. Therefore,}
$$

$$
\sum_{n \geq 1} \int_{\Omega} |\theta|^2 |\partial_x^3 \tilde{v}|^2 \, dx \lesssim_n \|\theta\|_{L^2(-h, 0; L^4)}^2 \sum_{n \geq 1} \|\partial_x \tilde{v}\|_{L^4(\Omega^2)}^2
$$

$$
\lesssim (\|\theta\|_{L^2(-h, 0; L^4)}^2 \|\partial_x \tilde{v}\|_{L^4(\Omega^2)}) \sum_{n \geq 1} \|\partial_x \tilde{v}\|_{L^4(\Omega^2)} \lesssim M \|\theta\|_{L^2(\ell^2)} \|\tilde{v}\|_{H^1(\ell^2)}
$$

$$
\leq C_{\delta_6} (1 + \|\theta\|_{L^2(\ell^2)}^2) \|\tilde{v}\|_{H^1(\ell^2)} + \epsilon_6 \|\nabla \tilde{v}\|_{L^2}^2,
$$

where in (i) we used (6.22), the Cauchy-Schwartz inequality and $\ell^2(L^2) = L^2(\ell^2)$. Finally, (ii) follows from $\|\partial_x \tilde{v}\|_{L^4(\Omega^2)} \lesssim M 1$ as commented in Remark 3.2.

6.8.7. Estimate of $J_7$. By Cauchy–Schwarz inequality, we have

$$
|J_7| \leq \epsilon_6 \sum_{n \geq 1} \int_{\Omega} |\tilde{v}|^2 |(\phi_n \cdot \nabla) \tilde{v} - \phi_n \nabla \tilde{v} + T_n(\theta) + g_{n, \theta}|^2 \, dx
$$

$$
+ C_{\epsilon_6} \sum_{n \geq 1} \int_{\Omega} |\theta| |(\psi_n \cdot \nabla) \theta + g_{n, \theta}|^2 \, dx
$$

$$
\leq \epsilon_6 \int_{\Omega} \left( |\tilde{v}|^2 |\nabla \tilde{v}|^2 + |\nabla \tilde{v}|^2 |\tilde{v}|^2 |\nabla \Theta|^2 \right) \, dx
$$

$$
+ C_{\epsilon_6} \left( \|\theta\|_{H^1(\ell^2)}^2 + \left\| |\nabla \Theta| \right\|_{L^2}^2 + \|\theta\|_{L^2(\ell^2)}^2 \right)
$$

where in the last inequality we used that $\|\phi_n\|_{L^\infty(\ell^2)} \lesssim M 1$.

6.8.8. Proof of (6.38). Recall that $\nu < 2$ by Assumption 3.1(2). Due to the estimates of Subsection 6.8.1–6.8.7 with $\epsilon_0$ sufficiently small and independent of $(\eta, \xi, \nu, \theta_0)$, the claimed estimate follows by taking $t = T$ and the expected value on both sides of (6.39) as well as by using $E[N_T] = 0$.

Note that, in contrast to the previous subsections, we do not take $E[\sup_{t \in [0, T]} |\cdot|]$ on both sides of (6.39). This would eventually give us an estimate for $E[\sup_{t \in [0, T]} \|\tilde{v}(t)\|_{H^1(\ell^2)}]$. However, this already follows from the $L^\infty(L^2)$–estimates for $\tilde{v}$ and $\theta$ proven in Subsection 6.7 and Lemma 5.1, respectively.

6.9. Estimate for $\|\Theta\|_{L^2(\ell^2)}$ and $\|\nabla \Theta\|_{L^2(\ell^2)}$. The aim of this subsection is to prove the following estimate: For all $\epsilon_7 \in (0, \infty)$,

$$
E \int_{\eta}^{\xi} \left( \|\Theta\|_{L^2(\ell^2)}^2 + E \int_{\eta}^{\xi} \|\nabla \Theta\|_{L^2(\ell^2)}^2 \right) \, ds \leq C_{7, \epsilon_7} (1 + E \|\theta(\eta)\|_{L^2(\ell^2)}^2 + E \|\tilde{v}(\eta)\|_{L^2(\ell^2)}^2)
$$

$$
+ \epsilon_7 \left( E \int_{\eta}^{\xi} \|\theta\|_{L^2(\ell^2)}^2 \, ds + E \int_{\eta}^{\xi} \left\| |\nabla \Theta| \right\|_{L^2(\ell^2)}^2 \, ds + E \int_{\eta}^{\xi} \left\| |\nabla \tilde{v}| \right\|_{L^2(\ell^2)}^2 \, ds \right)
$$

$$
+ \epsilon_7 \left( E \int_{\eta}^{\xi} \left\| |\nabla \tilde{v}| \right\|_{L^2(\ell^2)}^2 \, ds + E \int_{\eta}^{\xi} \left\| |\nabla \tilde{v}| \right\|_{L^2(\ell^2)}^2 \, ds + E \int_{\eta}^{\xi} \|\nabla \tilde{v}\|_{L^2(\ell^2)}^2 \, ds \right)
$$

$$
+ C_{7, \epsilon_7} E \int_{\eta}^{\xi} L_\eta(1 + \|\tilde{v}\|_{L^2(\ell^2)} + \|\theta\|_{L^2(\ell^2)}^2) \, ds
$$

where $\epsilon_7$ and $\epsilon_6$ are positive constants.
Lemma 6.3 involving convective term, which will be useful in the application of such formula. Recall that \( \Theta = \int_{-h}^t \theta(\cdot, \zeta) \, d\zeta \), see (6.8). In the following result we show cancellation properties involving convective term, which will be useful in the application of such formula.

**Lemma 6.3** (Cancellation). Let \( v \in C^\infty(\Omega; \mathbb{R}^2) \) and set \( w(v) = -\int_{-h}^t \text{div}_H v(\cdot, \zeta) \, d\zeta, \ u = (v, w(v)) \). Then, for all \( \theta \in C^\infty(\Omega) \),

\[
\int_\Omega \Theta^2 v \cdot [(u \cdot \nabla)v] \, dx + \int_\Omega |v|^2 \Theta \left( \int_{-h}^t (u \cdot \nabla) \theta \, d\zeta \right) \, dx = \int_\Omega |v|^2 \Theta \left[ \int_{-h}^t (v \cdot \nabla_H) \theta \, d\zeta - (v \cdot \nabla_H) \Theta + \int_{-h}^t \text{div}_H v \theta \, d\zeta \right] \, dx
\]

where \( \Theta = \int_{-h}^t \theta(\cdot, \zeta) \, d\zeta \), see (6.8).

The key point is that in the RHS(6.43) the vertical component \( w(v) \) of \( u \) does not appear.

**Proof of Lemma 6.3.** Since \( |w(v)|(\cdot, -h) = 0 \) on \( T^2 \),

\[
\int_{-h}^t w(v) \partial_3 \theta \, d\zeta = w(v) \theta + \int_{-h}^t \text{div}_H v \theta \, d\zeta
\]

\[
= w(v) \partial_3 \Theta - \int_{-h}^t \text{div}_H v \theta \, d\zeta
\]

\[
= (u \cdot \nabla) \Theta - (v \cdot \nabla_H) \Theta + \int_{-h}^t \text{div}_H v \theta \, d\zeta.
\]

Hence (6.43) follows by using that \( \int_\Omega |v|^2 \Theta (u \cdot \nabla ) \Theta + |\Theta|^2 v \cdot (u \cdot \nabla ) v \, dx = 0 \), cf. Lemma 5.2.

Next we apply the Itô's formula to the functional in (6.42). As in Subsection 6.8, a standard approximation argument shows that

\[
\| \Theta^{\eta, \xi}(t) \|^2 \cdot \| \tilde{\omega}^{\eta, \xi}(t) \|^2 \|_{L^2} = \| \Theta(\eta) \|^2 \cdot \| \tilde{\omega}(\eta) \|^2 \|_{L^2}
\]

\[
+ \sum_{1 \leq j \leq 7} \int_0^t 1_{[\eta, \xi]} K_{2,j}(s) \, ds + \mathcal{N}_t
\]

where \( \mathcal{N}_t \) is a \( L^1(\Omega) \)-martingale, such that \( E[\mathcal{N}_t] = 0 \) for all \( 0 \leq t \leq T \), and

\[
K_1 \overset{\text{def}}{=} 2 \int_\Omega \left( \Theta^2 \tilde{v} \cdot \Delta \tilde{v} + |\tilde{v}|^2 \Theta \int_{-h}^t \Delta \theta(\cdot, \zeta) \, d\zeta \right) \, dx,
\]

\[
K_2 \overset{\text{def}}{=} \int_\Omega |\tilde{v}|^2 \Theta \left[ \int_{-h}^t (\tilde{v} \cdot \nabla_H) \theta \, d\zeta - (\tilde{v} \cdot \nabla_H) \Theta + \int_{-h}^t \text{div}_H \tilde{v} \theta \, d\zeta \right] \, dx,
\]

\[
K_3 \overset{\text{def}}{=} 2 \int_\Omega \left( \Theta^2 \tilde{v} \cdot (f_v + \mathcal{F}(\tilde{v})) + |\tilde{v}|^2 \Theta \int_{-h}^t f_\theta(\cdot, \zeta) \, d\zeta \right) \, dx,
\]

\[
K_4 \overset{\text{def}}{=} 2 \int_\Omega \Theta^2 \tilde{v} \cdot \left( \mathcal{L}_{\sigma \xi} \theta + (\pi_H \cdot \nabla_H) \Theta + \pi^2 \partial_3 \nabla_H \theta \right) \, dx ds,
\]

\[
K_5 \overset{\text{def}}{=} -2 \int_\Omega \Theta^2 \tilde{v} \cdot [\tilde{v} \cdot \nabla_H \pi] \, dx,
\]

\[
K_6 \overset{\text{def}}{=} \sum_{n \geq 1} \int_\Omega |\tilde{v}|^2 \left( \int_{-h}^t \left[ (f_n \cdot \zeta) \cdot \nabla \theta(\cdot, \zeta) + g_n \theta(\cdot, \zeta) \right] \, d\zeta \right)^2 \, dx,
\]
44 AGRESTI, HIEBER, HUSSEIN, AND SAAL

\[ K_7 \overset{\text{def}}{=} \int_{\Omega} |\Theta|^2 (\phi_n \cdot \nabla) \tilde{v} - \phi_n^3 \partial_3 \tilde{v} + T_n(\theta) + \tilde{g}_{n,v}^5|^2 \, dx, \]

\[ K_8 \overset{\text{def}}{=} 2 \sum_{n \geq 1} \int_{\Omega} \Theta \left( \int_{-h} \left[ (\psi_n(\cdot, \zeta) \cdot \nabla (\cdot, \zeta) + g_{\theta,n}(\cdot, \zeta) \right] d\zeta \right) \]

\[ \cdot \left[ (\phi_n \cdot \nabla) \tilde{v} - \phi_n^3 \partial_3 \tilde{v} + T_n(\theta) + \tilde{g}_{n,v}^5 \right] d\zeta, \]

where we used Lemma 6.3 with \((v, \theta)\) replaced by \((\tilde{v}, \Theta)\), and by Lemma 5.2,

\[ \int_{\Omega} \left[ \Theta^2 \tilde{v} \cdot [(\nabla \cdot \nabla_H) \tilde{v}] + |\tilde{v}|^2 \Theta \left( \int_{-h} (\nabla \cdot \nabla_H) \theta d\zeta \right) \right] \]

\[ = \int_{\Omega} \Theta^2 \tilde{v} \cdot [(\nabla \cdot \nabla_H) \tilde{v}] + |\tilde{v}|^2 \Theta (\nabla \cdot \nabla_H) \theta \, dx = 0. \]

As before, we collect the estimates of \((K_j)_{j=1}^7\) in the following subsections. The estimates of Below \(\varepsilon, \varepsilon_0 \in (0, \infty)\) are positive parameters which will be chosen in Subsections 6.12 and 6.8.8, respectively.

6.9.1. Estimate of \(K_1\). Integrating by parts, we have, a.e. on \([\eta, \xi] \times \Omega,\)

\[ \int_{\Omega} \Theta^2 \Delta \tilde{v} \cdot \tilde{v} \, dx = -2 \int_{\Omega} \Theta^2 |\nabla \tilde{v}|^2 \, dx - 2 \sum_{1 \leq i,j \leq 3} \int_{\Omega} \Theta \partial_i \Theta \partial^j \tilde{v} \partial^j \tilde{v} \, dx, \]

and by (6.11b),

\[ \int_{\Omega} |\tilde{v}|^2 \left( \int_{-h} \Delta \theta(\cdot, \zeta) d\zeta \right) \, dx = \int_{\Omega} |\tilde{v}|^2 \Theta \Delta_H \theta \, dx + \int_{\Omega} |\tilde{v}|^2 \Theta \partial_3 \theta \, dx \]

\[ = -2 \int_{\Omega} |\tilde{v}|^2 |\nabla_H \theta|^2 \, dx - 2 \sum_{1 \leq i,j \leq 3} \int_{\Omega} \Theta \nabla_H \theta \cdot \nabla_H \tilde{v} \partial^j \tilde{v} \, dx + \int_{\Omega} |\tilde{v}|^2 \Theta \partial_3 \theta \, dx. \]

Hence, a.e. on \([\eta, \xi] \times \Omega,\)

\[ K_1 \leq -2 \int_{\Omega} (\Theta^2 |\nabla \tilde{v}|^2 + |\tilde{v}|^2 |\nabla_H \Theta|^2) \, dx \]

\[ + \varepsilon_7 \int_{\Omega} (|\tilde{v}|^2 |\nabla \tilde{v}|^2 + |\partial_3 \nabla \theta|^2) \, dx \]

\[ + C_{\varepsilon_7} \left( \int_{\Omega} \Theta^2 |\nabla_H \theta|^2 \, dx + (1 + \|\theta\|_{L^6}^6)(1 + \|\tilde{v}\|_{L^4}^4) \right). \]

6.9.2. Estimate of \(K_2\). We write \(K_2 = K_{2,1} + K_{2,2} + K_{2,3}\) where

\[ K_{2,1} \overset{\text{def}}{=} \int_{\Omega} |\tilde{v}|^2 \Theta \left( \int_{-h} (\tilde{v} \cdot \nabla_H) \theta d\zeta \right) \, dx, \]

\[ K_{2,2} \overset{\text{def}}{=} - \int_{\Omega} |\tilde{v}|^2 \Theta \left[ (\tilde{v} \cdot \nabla_H) \Theta \right] \, dx, \]

\[ K_{2,3} \overset{\text{def}}{=} \int_{\Omega} |\tilde{v}|^2 \Theta \left( \int_{-h} \text{div}_H \tilde{v} \theta d\zeta \right) \, dx. \]

Note that, a.e. on \([\eta, \xi] \times \Omega,\)

\[ \left| \int_{\Omega} |\tilde{v}|^2 \Theta \left( \int_{-h} (\tilde{v} \cdot \nabla_H) \theta d\zeta \right) \, dx \right| \lesssim_h \|\tilde{v}\|_{L^2} \|\Theta\|_{L^6} \|\tilde{v} \cdot \nabla_H \theta\|_{L^2} \]

\[ \leq \varepsilon_7 \|\tilde{v}\|_{L^6} \|\nabla \theta\|_{L^2}^2 + C_{\varepsilon_7} \|\tilde{v}\|_{L^6}^2 \|\theta\|_{L^6}^2 \]

\[ \overset{(i)}{\leq} \varepsilon_7 \|\tilde{v}\|_{L^6} \|\nabla \theta\|_{L^2}^2 + \varepsilon_7 \|\tilde{v}\|_{L^6} \|\nabla \tilde{v}\|_{L^2}^2 + C_{\varepsilon_7} \left( 1 + \|\theta\|_{L^6}^6 \right) \|\tilde{v}\|_{L^4}^4 \]

where in (i) we used (6.40). With similar arguments, we have

\[ K_{2,2} \leq \varepsilon_7 \|\tilde{v}\|_{L^6} \|\nabla H \theta\|_{L^2}^2 + \varepsilon_7 \|\tilde{v}\|_{L^6} \|\nabla \tilde{v}\|_{L^2}^2 + C_{\varepsilon_7} \left( 1 + \|\theta\|_{L^6}^6 \right) \|\tilde{v}\|_{L^4}^4, \]
\[ K_{2,3} \leq \varepsilon T \|\nabla \tilde{v}\|_{L^2}^2 + \varepsilon T \|\nabla \tilde{v}\|_{L^2}^2 + C_{\varepsilon \eta} (1 + \|\theta\|_{L^6}^4) \|\tilde{v}\|_{L^4}^2. \]

Putting together the estimates of (6.47), one sees that \( \mathbf{E} \int_\eta^\xi K_2 \, ds \) is bounded by the RHS (6.41).

6.9.3. **Proof of (6.41).** One can readily check that the terms \( \mathbf{E} \int_\eta^\xi K_i \, ds \) appearing in (6.41) can be estimated by the RHS (6.41) by slightly modifying the arguments of Subsection 6.8. Now (6.41) follows the estimates of \( \mathbf{E} \int_\eta^\xi K_i \, ds \) by taking the expected value in (6.44).

6.10. **Estimate for \( \|\theta\|_{L^2}^2 \).** The aim of this subsection is to prove the following estimate: For all \( \varepsilon_8 \in (0, \infty) \),

\[
\mathbf{E} \int_\eta^\xi \|\theta\|_{L^2}^2 \, ds \leq C_{8,\varepsilon} (1 + \mathbf{E} \|\theta(\eta)\|_{L^4}^4)
+ C_{8,\varepsilon} \mathbf{E} \int_\eta^\xi L_s (1 + \|\theta\|_{L^4}^4 + \|\tilde{v}\|_{L^4}^4) \, ds
+ \varepsilon_8 \left( \mathbf{E} \int_\eta^\xi \|\nabla \tilde{v}\|_{L^2}^2 \, ds + \mathbf{E} \int_\eta^\xi \|\nabla \tilde{v}\|_{L^2}^2 \, ds + \mathbf{E} \int_\eta^\xi \|\nabla \tilde{v}\|_{L^2}^2 \, ds \right),
\]

where \( C_8, C_{8,\varepsilon} \) are constants independent of \( (j, \eta, \xi, \nu, \theta_0) \) and \( C_8 \) is also independent of \( \varepsilon_7 \). As above, \( L_s \) is as in (6.17).

As before, to prove the main estimate, the idea is to apply the Itô formula to a particular function. Here we employ the following function. Here we employ the following

\[
\theta \mapsto \int_{-h}^0 |\theta|^2 |\theta(\cdot, \xi) d\zeta|^2 dx.
\]

The proof of (6.45) essentially follows the line of Subsections 6.8 and 6.9 with the exception of using the functional (6.46) instead of the one used there. Here we content ourselves in estimating the term appearing in the corresponding Itô formula involving the convection term, i.e.,

\[
Q \stackrel{\text{def}}{=} \int_{-h}^0 \left( \Theta^2 \theta (\tilde{u} \cdot \nabla) \theta + \theta^2 \Theta \left( \int_{-h}^0 [(\tilde{u} \cdot \nabla) \theta] d\zeta \right) \right) dx.
\]

Here, as above, \( \tilde{u} = (\tilde{u}, \tilde{w}(\tilde{v})) \). Note that the analogue term with \( \tilde{u} \) replaced by \( \tilde{u} = (\tilde{u}, 0) \) vanishes due to Lemma 5.2. Repeating the argument in Lemma 6.3, we have

\[
Q = \int_{-h}^0 |\theta|^2 \Theta \left( \int_{-h}^0 (\tilde{u} \cdot \nabla) \theta d\zeta - (\tilde{v} \cdot \nabla) \Theta + \int_{-h}^0 \text{div}_H \tilde{v} \Theta d\zeta \right) dx.
\]

Hence

\[
Q \leq \varepsilon_8 \left( \|\nabla \tilde{v}\|_{L^2}^2 \|\Theta\|_{L^2}^2 + \|\nabla \tilde{v}\|_{L^2}^2 \|\nabla \Theta\|_{L^2}^2 + \|\tilde{v}\|_{L^2}^2 \|\Theta\|_{L^2}^2 \right) + C_{9,\varepsilon} \|\theta\|_{L^6}^6 \|\Theta\|_{L^6}^6.
\]

It remains to estimate the last term in the previous inequality. Note that, using the Hölder inequality with exponents (3, 6), we have

\[
\|\theta\|_{L^6}^6 \leq \|\Theta\|_{L^6}^6 \|\Theta\|_{L^6}^6 \lesssim_h \|\theta\|_{L^6}^6.
\]

Since \( \|\theta\|_{L^6}^6 \leq L \) by (6.17), one sees that \( \mathbf{E} \int_\eta^\xi Q \, ds \) can be estimated by the RHS (6.45).

6.11. **Estimate for \( \|\Theta\|_{L^2}^2 \).** The aim of this subsection is to prove the following estimate: For all \( \varepsilon_9 \in (0, \infty) \),

\[
\mathbf{E} \int_\eta^\xi \|\theta\|_{L^2}^2 \, ds \leq C_{9,\varepsilon} (1 + \mathbf{E} \|\theta(\eta)\|_{L^4}^4)
+ C_{9,\varepsilon} \mathbf{E} \int_\eta^\xi L_s (1 + \|\theta\|_{L^4}^4 + \|\tilde{v}\|_{L^4}^4) \, ds
+ \varepsilon_9 \left( \mathbf{E} \int_\eta^\xi \|\nabla \tilde{v}\|_{L^2}^2 \, ds + \mathbf{E} \int_\eta^\xi \|\nabla \tilde{v}\|_{L^2}^2 \, ds + \mathbf{E} \int_\eta^\xi \|\nabla \tilde{v}\|_{L^2}^2 \, ds \right),
\]
where $C_9, C_{9, \varepsilon_0}$ are constants independent of $(j, \eta, \xi, v_0, \theta_0)$ and $C_9$ is also independent of $\varepsilon_0$. Finally, $L_s$ is as in (6.17).

Here we apply the Itô formula to the functional $\theta \to \| \int_{-h}^t (\tilde{u} \cdot \nabla) \theta \, d\zeta \|_{L^4}^4$. As in Subsection 6.10, we content ourselves to estimate the term coming from the convective term:

$$Q_0 \overset{\text{def}}{=} \int_0^T \Theta^3 \left( \int_{-h}^t (\tilde{v} \cdot \nabla) \theta \, d\zeta \right) \, dx \times dx$$

$$= \int_0^T \Theta^3 \left[ \left( \int_{-h}^t (\tilde{v} \cdot \nabla) \theta \, d\zeta \right) - (\bar{v} \cdot \nabla) \Theta + \int_{-h}^t \operatorname{div} H \tilde{v} \theta \, d\zeta \right] \, dx,$$

where the last equality follows from the argument of Lemma 6.3. Hence, as in the previous subsection, one can readily check that $\mathbb{E} \int_0^T Q_0 \, ds$ can be estimated by the RHS of (6.47).

### 6.12. Proof of Lemma 6.1

Here we conclude the proof of Lemma 6.1 using the estimates proven in Subsections 6.3–6.11. As explained in Subsection 6.2, it remains to prove (6.19) with $c_0$ is independent of $(j, \eta, \xi, v_0, \theta_0)$. Now the idea is to multiply the estimates of Subsections 6.3–6.11 by suitable positive constants $(\alpha_k)_{k=1}^9$ and then sum up the resulting estimates. Then we choose $\alpha_k$'s such that the latter estimate is equivalent to (6.19).

To highlight the core of the argument we denote the quantities appearing in the estimates of Subsections 6.3–6.11 as follows:

$$* \overset{\text{def}}{=} \sum_{1 \leq i \leq 9} \tilde{I}^i$$

$$1 \overset{\text{def}}{=} \mathbb{E} \int_0^T \| \tilde{\theta} \|_{L^2}^2 \, ds,$$

$$2 \overset{\text{def}}{=} \mathbb{E} \int_0^T \| \tilde{\theta} \|_{L^2}^2 \, ds,$$

$$3 \overset{\text{def}}{=} \mathbb{E} \int_0^T \| \nabla \tilde{\theta} \|_{L^2}^2 \, ds,$$

$$4 \overset{\text{def}}{=} \mathbb{E} \int_0^T \| \nabla \tilde{\theta} \|_{L^2}^2 \, ds,$$

$$6 \overset{\text{def}}{=} \mathbb{E} \int_0^T \left( \| \tilde{v} \|_{L^2}^2 + \left\| \tilde{\theta} \right\|_{L^2}^2 \right) \, ds,$$

$$7 \overset{\text{def}}{=} \mathbb{E} \int_0^T \left( \left\| \tilde{v} \right\|_{L^2}^2 \right) \, ds,$$

$$8 \overset{\text{def}}{=} \mathbb{E} \int_0^T \left( \left\| \tilde{\theta} \right\|_{L^2}^2 \right) \, ds,$$

$$9 \overset{\text{def}}{=} \mathbb{E} \int_0^T \left( \left\| \tilde{\theta} \right\|_{L^2}^2 \right) \, ds,$$

$$L \overset{\text{def}}{=} \mathbb{E} \int_0^T L_1 (1 + X_t + \| \tilde{\theta}(t) \|_{H^1(T^2)}^2) \, ds,$$

where $X_t$ is as in Lemma 6.1. Comparing the estimates of Subsections 6.3–6.11 and (6.19), one sees that the energy terms $E_i$ for $i \in \{1, \ldots, 9\}$ are the one we would like to absorb.

It will be proved conveniently later to derive a consequence of (6.41) and (6.47). Indeed, we would like to have a constant in front of the last term on RHS (6.41) which does not blow-up as $\varepsilon_0 \downarrow 0$. To this end, using the estimate (6.47) with $\varepsilon_0 = \varepsilon \eta / (2(C_{7, \varepsilon_0} \vee 1))$ in (6.41) with $\varepsilon_0$ replaced by $\varepsilon_0/2$, we get

$$7 \leq \varepsilon \left( 2 + \frac{3}{4} + \frac{4}{5} + \frac{6}{7} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} \right) + C_{7, \varepsilon_0} \left( 1 + L \right),$$

where $C_{7, \varepsilon_0}$ is a constant independent of $(j, \eta, \xi, v_0, \theta_0)$.

In the following we apply the estimates of Subsections 6.3–6.7 and 6.10-6.11 as well as (6.48) with

$$\varepsilon_i \equiv \varepsilon \in (0, \infty) \quad \text{for all } i \in \{1, \ldots, 9\}$$

where $\varepsilon$ is chosen below. Let $(C_{i, \varepsilon}, C_i)_{i=1}^9$ be the constants introduced in Subsections 6.3–6.7 and 6.10-6.11 and set

$$C_{0, \varepsilon} \overset{\text{def}}{=} \max_{1 \leq i \leq 9, i \neq 7} \frac{C_{i, \varepsilon} \vee C_{7, \varepsilon}^2}{C_{7, \varepsilon}} \quad \text{and} \quad C_0 \overset{\text{def}}{=} \max_{1 \leq i \leq 9, i \neq 7} C_i.$$
as (6.48) imply:

\[
\begin{align*}
\mathbb{E} \left[ \sup_{s \in [\eta, \xi]} \| \tau(s) \|_{H^1}^2 \right] + 1 & \leq C_0 (3 + 5) + C_{0, \varepsilon} (I + L) + \varepsilon \star, \\
\mathbb{E} \left[ \sup_{s \in [\eta, \xi]} \| \tilde{\theta}(s) \|_{H^1}^2 \right] + 2 & \leq C_0 (1 + 6 + 7) + C_{0, \varepsilon} (I + L) + \varepsilon \star, \\
\mathbb{E} \left[ \sup_{s \in [\eta, \xi]} \| \partial_s v(s) \|_{L^2}^2 \right] + 2 & \leq C_0 (5 + C_{0, \varepsilon} (I + L) + \varepsilon \star, \\
\mathbb{E} \left[ \sup_{s \in [\eta, \xi]} \| \partial_s \theta(s) \|_{L^2}^2 \right] + 4 & \leq C_0 (6 + C_{0, \varepsilon} (I + L) + \varepsilon \star, \\
\mathbb{E} \left[ \sup_{s \in [\eta, \xi]} \| \hat{v}(s) \|_{L^4}^4 \right] + 5 & \leq C_0 (7 + C_{0, \varepsilon} (I + L) + \varepsilon \star, \\
6 & \leq C_0 (8 + C_{0, \varepsilon} (I + L) + \varepsilon \star, \\
7 & \leq C_{0, \varepsilon} (I + L) + \varepsilon \star, \\
8 & \leq C_{0, \varepsilon} (I + L) + \varepsilon \star, \\
9 & \leq C_{0, \varepsilon} (I + L) + \varepsilon \star,
\end{align*}
\]

consequence of (6.26)

consequence of (6.27)

consequence of (6.28)

consequence of (6.36)

consequence of (6.38)

consequence of (6.45)

consequence of (6.47)

In the above estimates, \( \varepsilon \in (0, \infty) \) is a free parameter which will be fixed later. Multiplying the above estimates by \( \alpha_i \in [1, \infty) \) and then summing them up, we have:

\[
(6.49) \quad \mathbb{E} \left[ \sup_{s \in [\eta, \xi]} X_s \right] + \sum_{1 \leq i \leq 9} c_i \leq C_0, \varepsilon \alpha (I + L) + \alpha \varepsilon \star,
\]

where we used the definition of \( X_i \) in Lemma 6.1 and we set \( \alpha \overset{\text{def}}{=} \sum_{1 \leq i \leq 9} \alpha_i \),

\[
\begin{align*}
\alpha_1 & = \alpha_2, \\
\alpha_3 & = \alpha_3 - C_0 \alpha_1, \\
\alpha_5 & = \alpha_5 - C_0 (\alpha_1 + \alpha_3), \\
\alpha_7 & = \alpha_7 - C_0 (\alpha_2 + \alpha_5), \\
\alpha_9 & = \alpha_9,
\end{align*}
\]

Note that the terms \( \sum \) can be absorbed on the RHS(6.49) as \( \sum_{1 \leq i \leq 9} \alpha_i \geq 1 \).

Next we show that there exists a choice of \( \alpha_i \) such that \( \alpha_i \equiv 1 \). To see this, one can split the argument in several steps as follow:

\begin{itemize}
  \item Choose \( \alpha_1 = \alpha_2 = 1 \) and \( \alpha_9 = 1 \).
  \item Choose \( \alpha_3 = \alpha_4 = C_0 + 1 \).
  \item Choose \( \alpha_5 = \alpha_6 = C_0 (C_0 + 2) + 1 \).
  \item Choose \( \alpha_7 = C_0 (\alpha_2 + \alpha_5) + 1 \) and \( \alpha_8 = C_0 \alpha_6 + 1 \).
\end{itemize}

With the above choice we have \( \alpha_i \equiv 1 \) and \( \min_{1 \leq i \leq 9} \alpha_i \geq 1 \). Thus (6.49) yields (6.19) choosing \( \varepsilon = (2C_0 \alpha)^{-1} \) and recalling that \( C_0 \) is independent of \( (J, \eta, \xi, v_0, \theta_0) \).

7. PROOF OF PROPOSITION 4.2

To prove Proposition 4.2 we collect some useful facts. Let \( X_t, Y_t \) be as in Lemma 6.1. For notational convenience, we set

\[
\begin{align*}
X_s & \overset{\text{def}}{=} 1 + ||v(s)||_{L^2}^2 + ||\theta(s)||_{L^2}^2 + X_s, \\
Y_s & \overset{\text{def}}{=} 1 + ||v(s)||_{H^1}^2 + ||\theta(s)||_{H^1}^2 + Y_s.
\end{align*}
\]

By Lemmas 5.1 and 6.1, we have, for some constant \( c_{0, T} \) independent of \( (v_0, \theta_0) \), for all \( \gamma > 1 \),

\[
(7.1) \quad \mathbb{P} \left( \sup_{s \in [0, \tau^\wedge T]} X_s + \int_0^{\tau^\wedge T} Y_s \, ds \geq \gamma \right) \leq c_{0, T} \frac{(1 + \mathbb{E} \|X\|_{L^2[0,T]}^2) + \mathbb{E} \|v_0\|_{H^1}^4 + \mathbb{E} \|\theta_0\|_{H^1}^4}{\log \log(\gamma)}.
\]
Proof of Proposition 4.2. Let \((\tau_j)_{j \geq 1}\) be as in (5.3). As above the estimate is reduced to an application of the stochastic Gronwall lemma [AV22a, Lemma A.1]. To simplify the notation we write \(U = (v, \theta)\) and \((A, B, F, G)\) are as in (4.2)–(4.5). Recall that \(H = \mathbb{R}^1 \times H^1\) and \(V = \mathbb{R}^2 \times H^2\).

We claim that there exists \(C_0 > 0\) independent of \((v_0, \theta_0)\) such that, for all \(j \geq 1\) and all stopping times \((\eta, \xi)\) satisfying \(0 \leq \eta \leq \xi \leq \tau_j\),

\[
\mathbb{E} \sup_{s \in [\eta, \xi]} \|U(s)\|_{H^1}^2 + \mathbb{E} \int_{\eta}^{\xi} \|U(s)\|_{V}^2 \, ds \leq C_0 \left[ 1 + \mathbb{E} \|U(\eta)\|_{H^1}^2 + \mathbb{E} \int_{\eta}^{\xi} (1 + X_{\theta}^2) \mathcal{Y}_s (1 + \|U\|_{H^1}^2) \, ds \right].
\]

**Step 1: Sufficiency of (7.2).** Recall that \(\lim_{j \to \infty} \tau_j = \tau \wedge T\) a.s. by (5.3) and \(U \in C([0, \tau); H) \cap L^2_{\text{loc}}([0, \tau); V)\) a.s. (recall that \((H, V)\) are as in (3.18)). The stochastic Gronwall lemma [AV22a, Lemma A.1], (7.1) and the fact that \(c_0\) is independent of \((j, U_0)\) ensure that, for all \(R, \gamma > 1\),

\[
\mathbb{P} \left( \sup_{s \in [0, \tau \wedge T]} \|U(s)\|_{H}^2 + \int_0^{\tau \wedge T} \|U(s)\|_{V}^2 \, ds \geq \gamma \right) \leq \left( \frac{c_F}{\gamma} e^{c_F R} + \frac{c_F}{\log(\log(R))} (1 + \mathbb{E} \|\Xi\|_{L^2_{\mathbb{R}^1 \times \mathbb{R}^2}}^2 + \mathbb{E} \|U_0\|_{H}^2) \right).
\]

Here \(c_F\) is a constant which depends only on \((c_0, c_0, T)\). Choosing \(R = R(\gamma) = \frac{1}{c_F} \log(\frac{\gamma}{\log(\log(\gamma))})\) for \(\gamma\) large, one obtains the estimates claimed in Proposition 4.2.

**Step 2: Proof of (7.2).** Reasoning as in the proof of Proposition 4.3, by Lemma 4.1 and [AV22b, Proposition 3.9] there exists \(C_0 > 0\), independent of \((U_0, U_0')\), such that for all stopping times \((\eta, \xi)\) satisfying \(0 \leq \eta \leq \xi \leq T\) a.s. one has

\[
\mathbb{E} \sup_{s \in [\eta, \xi]} \|U(s)\|_{H}^2 + \mathbb{E} \int_{\eta}^{\xi} \|U(s)\|_{V}^2 \, ds \leq C_0 \left[ \mathbb{E} \|U(\eta)\|_{H}^2 + \sum_{j=1}^{6} I_j \right],
\]

where

\[
I_1 \overset{\text{def}}{=} \mathbb{E} \int_{\eta}^{\xi} \|v - \nabla H\|_{L^2}^2 \, ds, \quad I_2 \overset{\text{def}}{=} \mathbb{E} \int_{\eta}^{\xi} \|w(v) \partial_3 v\|_{L^2}^2 \, ds,
\]
\[
I_3 \overset{\text{def}}{=} \mathbb{E} \int_{\eta}^{\xi} \|v - \nabla H\|_{H^1}^2 \, ds, \quad I_4 \overset{\text{def}}{=} \mathbb{E} \int_{\eta}^{\xi} \|w(v) \partial_3 \theta\|_{L^2}^2 \, ds,
\]
\[
I_5 \overset{\text{def}}{=} \mathbb{E} \int_{\eta}^{\xi} \|F(v, \theta)\|_{L^2}^2 \, ds, \quad I_6 \overset{\text{def}}{=} \mathbb{E} \int_{\eta}^{\xi} \|F_\theta(v, \theta)\|_{L^2}^2 \, ds,
\]
\[
I_7 \overset{\text{def}}{=} \mathbb{E} \int_{\eta}^{\xi} \|G(v, \theta)\|_{H^1(\mathbb{R}^2)}^2 \, ds, \quad I_8 \overset{\text{def}}{=} \mathbb{E} \int_{\eta}^{\xi} \|G_\theta(v, \theta)\|_{H^1(\mathbb{R}^2)}^2 \, ds.
\]

By Assumption 3.1(7) and (3.20) (or, more generally, the condition in Remark 3.12(b)), we have

\[
\sum_{5 \leq j \leq 8} I_j \lesssim (1 + \mathbb{E} \|\Xi\|_{L^2(0, T)}^2 + \mathbb{E} \|v\|_{L^2(0, T; L^2)}^2 + \mathbb{E} \|\theta\|_{L^2(0, T; L^2)}^2) \lesssim 1 + \mathbb{E} \|\Xi\|_{L^2(0, T)}^2 + \mathbb{E} \int_{\eta}^{\xi} \mathcal{Y}_s \, ds.
\]

To estimate the remaining terms, let us recall the following useful estimate:

\[
\|w(v)\|_{L^{\infty}([0, T])} \lesssim \|v\|_{L^2([0, T]; H^1)} \lesssim \|v\|_{H^1}^{1/2} \|v\|_{H^2}^{1/2},
\]

where the last inequality follows from (6.22). The terms \(I_1\) and \(I_2\) can be estimated as in the proof of [AHHS22, Proposition 5.1]. The arguments given there shows:

\[
I_1 + I_2 \leq \frac{1}{4C_0} \mathbb{E} \int_{\eta}^{\xi} \|v(s)\|_{H^2}^2 \, ds + C_1 \mathbb{E} \int_{\eta}^{\xi} (1 + X_{\theta}^2) \mathcal{Y}_s (1 + \|v(s)\|_{H^1}^2) \, ds
\]
where $C_0 \geq 1$ is as in (7.3) and $C_1$ is independent of $(j, \eta, \xi, v_0, \theta_0)$. However, the above estimate can also be obtained by (slightly) modifying the argument below where we estimate $I_3$ and $I_4$.

To estimate $I_3$, note that, $I_3 \leq 2(I_{3,1} + I_{3,2})$ where

$$I_{3,1} \overset{\text{def}}{=} \mathbb{E} \int_\eta^\xi \| \nabla \cdot \nabla_H \theta \|^2_{L^2} \, ds$$

and

$$I_{3,2} \overset{\text{def}}{=} \mathbb{E} \int_\eta^\xi \| (\nabla \cdot \nabla_H \theta) \|^2_{L^2} \, ds,$$

since $v = \nabla + \bar{v}$. Note that $I_{3,2} \leq \mathbb{E} \int_\eta^\xi \| \nabla \theta \|^2_{L^2} \, ds \leq \mathbb{E} \int_\eta^\xi \mathcal{Y}_s \, ds$ and

$$I_{3,2} \lesssim \mathbb{E} \int_\eta^\xi \| \nabla \theta \|_{H^1} \, ds \lesssim \mathbb{E} \int_\eta^\xi \| \nabla \theta \|_{H^1} \, ds$$

where in (i) we used the Sobolev embedding $H^1 \hookrightarrow L^6$ and in (ii) (6.23). Here $C$ depends only on $C_0$. In particular $C$ is independent of $(j, \eta, \xi, v_0, \theta_0)$.

Finally we estimate $I_4$. The H"older inequality, (6.22) and (7.4) yield

$$I_4 \leq \mathbb{E} \int_\eta^\xi \| v \|^2_{L^\infty(-h,0; L^4(T^2))} \| \partial_3 \theta \|^2_{L^2(-h,0; L^4(T^2))} \, ds$$

$$\lesssim \mathbb{E} \int_\eta^\xi \| v \|_{H^1} \| v \|_{H^2} \| \partial_3 \theta \|_{L^2} \| \partial_3 \theta \|_{H^1} \, ds$$

$$\leq \frac{1}{8C_0} \mathbb{E} \int_\eta^\xi \| v \|^2_{H^2} \, ds + C_2 \mathbb{E} \int_\eta^\xi \mathcal{X}_s \mathcal{Y}_s \| v(s) \|^2_{H^1} \, ds.$$

Hence, for some $C_2$ is independent of $(j, \eta, \xi, v_0, \theta_0)$,

$$I_3 + I_4 \leq \frac{1}{4C_0} \mathbb{E} \int_\eta^\xi (\| v(s) \|^2_{H^2} + \| \theta(s) \|^2_{H^1}) \, ds$$

$$+ C_2 \mathbb{E} \int_\eta^\xi (1 + \mathcal{X}_s)^2 \mathcal{Y}_s (1 + \| v(s) \|^2_{H^1} + \| \theta(s) \|^2_{H^1}) \, ds.$$

Using the estimates (7.5)-(7.6) in (7.3), one gets (7.2) as desired.

8. **Stratonovich Formulation**

In this section we analyze the case of primitive equations (1.1) where the noise in understood in the Stratonovich formulation. More precisely, following the reformulation of (1.1) as (3.8) with $\gamma = \pi = 0$ (see also (3.10)), here we consider

$$dv - \Delta v \, dt = \mathbb{P} \left[ - (v \cdot \nabla_H) v - w(v) \partial_3 v \right.$$

$$- \nabla_H \int_{-h} \left( \kappa(\cdot, \zeta) \theta(\cdot, \zeta) \right) \, d\zeta + F_v(v, \theta, \nabla v, \nabla \theta) \right] dt$$

$$+ \sum_{n \geq 1} \mathbb{P} \left[ (\phi_n \cdot \nabla) v + \int_{-h} \nabla_H (\sigma_n(\cdot, \zeta) \theta(\cdot, \zeta)) \, d\zeta \right] \circ d\beta_t^n$$

$$\theta - \Delta \theta \, dt = \left[ (v \cdot \nabla_H) \theta - w(v) \partial_3 \theta + F_\theta(v, \theta, \nabla v, \nabla \theta) \right] dt + \sum_{n \geq 1} (\psi_n \cdot \nabla) \theta \circ d\beta_t^n$$

$$v(\cdot, 0) = v_0, \quad \theta(\cdot, 0) = \theta_0,$$

on $\mathcal{O} \overset{\text{def}}{=} T^2 \times (-h,0)$, where $\circ$ and $\mathbb{P}$ denote the Stratonovich integration and the hydrostatic Helmholtz projection see e.g. [Gar09] and Subsection 3.1, respectively. As in the previous sections,
There exist Assumption 8.1. consequence of the Stratonovich formulation and the temperature dependent turbulent pressures, viscosity and/or conductivity discussed in Remark 3.11. As we will see below, the term \( \partial_t \hat{\rho} + (\pi \cdot \nabla)\theta \) in (3.1c)-(3.1d) will appear naturally. The same also applies for inhomogeneous \( \sigma_n \)–term, respectively. Last but not least, lower order terms are mathematically easier to deal with. We the details to the interested reader.

In applications the Stratonovich formulation of the noise is often preferred as it is more close to numerical simulations due to Wong–Zakai type results [Fla11] and to two scale type arguments [BF20, MR01, MR04] and Section 2 for physical motivations of the transport noise terms and for the \( \sigma_n \)–term, respectively. Last but not least, lower order terms are mathematically easier to deal with. We the details to the interested reader.

To study (8.1)–(8.2) we need the following assumptions.

\[(8.1) \quad \phi^j_n(x), \psi^j_n(x) \text{ and } \sigma_n(x) \text{ are independent of } x_3.\]

(3) (Regularity) a.s. for all \( j, k \in \{1, 2, 3\} \) and \( i \in \{1, 2\} \),

\[
\left\| \left( \sum_{n \geq 1} |\phi^j_n|^2 \right)^{1/2} \right\|_{L^{3+\delta}(\Omega)} + \left\| \left( \sum_{n \geq 1} |\sigma_n|^2 \right)^{1/2} \right\|_{L^{3+\delta}(\Omega)} \leq M,
\]

\[
\left\| \left( \sum_{n \geq 1} |\psi^j_n|^2 \right)^{1/2} \right\|_{L^{3+\delta}(\Omega)} + \left\| \left( \sum_{n \geq 1} |\sigma_n|^2 \right)^{1/2} \right\|_{L^{3+\delta}(\Omega)} \leq M,
\]

\[
\|\kappa(t, \cdot\|_{L^\infty([T^2; L^2(-h,0)])} + \|\partial_t \kappa(t, \cdot\|_{L^{2+\delta}([\mathbb{T}^2; L^2(-h,0)])} \leq M,
\]

\[
\|\sigma_n(t, \cdot\|_{H^{2+\delta}([\mathbb{T}^2; \mathbb{F}])} \leq M.
\]

(4) a.s. for all \( n \geq 1 \) and \( x_H \in \mathbb{T}^2 \),

\[
\hat{\rho}^3(x_H, 0) = \hat{\rho}^3(x_H, -h) = 0.
\]

Next under Assumption 8.1 we (formally) rewrite (8.1)–(8.2) in the form (3.8) with suitable \((\pi, \gamma)\). To this end, let us recall that, for two stochastic processes \((X_t, Y_t)\), their joint quadratic variation at time \( t \) is denoted by \([X,Y]_t\). By (8.1b), at least formally we have \([\theta, \beta^n] = \int_0^t (\psi_n \cdot \nabla)\theta \, ds\). Moreover, formally from [Kun97, Thereom 2.3.5, p. 60],

\[
\int_0^t (\psi_n \cdot \nabla)\theta \, d\beta^n_s = \int_0^t (\psi_n \cdot \nabla)\theta \, d\beta^n_s + \frac{1}{2} \int_0^t (\psi_n' \cdot \nabla)\theta \, d\beta^n_s \big| t
\]

\[
= \int_0^t (\psi_n' \cdot \nabla)\theta \, d\beta^n_s + \int_0^t L^* \theta \, ds.
\]

where

\[
L^* \theta \overset{\text{def}}{=} \frac{1}{2} \sum_{n \geq 1} (\psi_n' \cdot \nabla) \left[ (\psi_n' \cdot \nabla)\theta \right]
\]
\[ = \frac{1}{2} \sum_{n \geq 1} \sum_{i,j \leq 3} \left( \psi_n^i \psi_n^j \partial_{ij} \theta + \psi_n^i (\partial_i \phi_n^j) \partial_j \theta \right). \]

The reformulation of the Stratonovich noise in (8.1a) is computationally more involved. To shorten the notation, similar to Subsection 4.1, we set

\[ \mathcal{J} f(x) \overset{\text{def}}{=} \nabla_H \int_{-h}^{x_3} \theta(x_H, \zeta) \, d\zeta \]

where \( x = (x_H, x_3) \in \mathbb{T}^2 \times (-h, 0) = \mathcal{O} \).

Note that by Assumption 8.1(2) and the linearity of \( \mathcal{J} \), at least formally,

\[ \int_0^t \mathbb{P} \left[ (\phi_n \cdot \nabla) v + \mathcal{J}(\sigma_n \theta) \right] \circ d\beta^n_s = \int_0^t \mathbb{P} \left[ (\phi_n \cdot \nabla) v + \mathcal{J}(\sigma_n \theta) \right] d\beta^n_s \]

\[ + \frac{1}{2} \mathbb{P} \left[ (\phi_n \cdot \nabla) \mathcal{J}(\sigma_n \theta) \right]_{C_1} + \frac{1}{2} \mathbb{P} \left( \mathcal{J}(\sigma_n [\theta, \beta^n]) \right)_{C_2}. \tag{8.4} \]

Next we formally compute the corrective terms \((C_v, C_\theta)\). We begin by taking a look at \( C_\theta \). Recall that \([\theta, \beta^n] = \int_0^t (\psi_n \cdot \nabla) \theta \, ds \) by (8.1b). Hence, formally,

\[ C_\theta = \int_0^t \mathbb{P} \left[ \mathcal{J}(\sigma_n [(\psi_n \cdot \nabla) \theta]) \right] \, ds. \]

To compute \( C_v \) we begin by looking at \([v, \beta^n]_t \). To this end, note that, by (8.1a), we formally have

\[ [v, \beta^n]_t = \int_0^t \mathbb{P} \left[ (\phi_n \cdot \nabla) v + \mathcal{J}(\sigma_n \theta) \right] \, ds. \]

To economize the notation, set \( \nabla_H \tilde{p}_n = Q[(\phi_n \cdot \nabla) v + \mathcal{J}(\sigma_n \theta)] \). Thus

\[ C_v = \int_0^t \mathbb{P} \left[ (\phi_n \cdot \nabla)(\phi_n \cdot \nabla) v \right] \, ds \]

\[ + \int_0^t \mathbb{P} \left[ (\phi_n \cdot \nabla) \mathcal{J}(\sigma_n \theta) \right] \, ds - \int_0^t \mathbb{P} \left[ (\phi_n \cdot \nabla) \nabla_H \tilde{p}_n \right] \, ds. \tag{8.5} \]

Next we rewrite the terms \((C_v,1, C_v,2)\) conveniently. Recall that, due to our notation, \( \phi_{n,H} = (\phi_n^j)_{j=1}^3 \) and that \( (\phi_{n,H}, \sigma_n) \) are \( x_3 \)-independent by Assumption 8.1(2). Hence

\[ (\phi_n \cdot \nabla) \mathcal{J}(\sigma_n \theta) = (\phi_{n,H} \cdot \nabla_H) \mathcal{J}(\sigma_n \theta) + \phi_n^3 \nabla_H \mathcal{J}(\sigma_n \theta) \]

\[ = \mathcal{J}(\sigma_n (\phi_{n,H} \cdot \nabla_H) \theta) + \phi_n^3 \nabla_H \mathcal{J}(\sigma_n \theta). \]

Finally, we consider \( C_v,2 \). By (3.3) and the fact that \( \tilde{p}_n \)'s are \( x_3 \)-independent,

\[ \mathbb{P}[(\phi_n \cdot \nabla) \nabla_H \tilde{p}_n] = \mathbb{P}[(\phi_{n,H} \cdot \nabla_H) \nabla_H \tilde{p}_n] = - \sum_{1 \leq i \leq 2} \mathbb{P}[(\nabla_H \phi_n^i) \partial_i \tilde{p}_n]. \]

Hence, putting together the previous identity, we have

\[ \mathbb{P}\left[ (\phi_n \cdot \nabla) v + \mathcal{J}(\sigma_n \theta) \right] \circ d\beta^n_s = \mathbb{P}\left[ (\phi_n \cdot \nabla) v + \int_{-h}^{x_3} \nabla_H (\sigma_n (\cdot, \zeta) \bar{\theta}(\cdot, \zeta)) \, d\zeta \right] d\beta^n_s \]

\[ + \mathbb{P}\left[ \mathcal{L}_x v + \mathcal{P}_x (v, \theta) \right] \, dt \]

\[ + \mathbb{P}\left[ \int_{-h}^{x_3} (\pi(\cdot, \zeta) \cdot \nabla) \theta(\cdot, \zeta) \, d\zeta + \frac{1}{2} \sum_{n \geq 1} \phi_n^3 \nabla_H (\sigma_n \theta) \right] \, dt. \tag{8.6} \]
where \( \mathcal{L}_\theta v \equiv \frac{1}{2} \sum_{n \geq 1} (\phi_n \cdot \nabla)[(\phi_n \cdot \nabla)v], \mathcal{P}_\gamma(v, \theta) \) as in (3.11) with \( G_{v,n} = 0 \) and \((\pi, \gamma)\) given by

\[
(8.7) \quad \pi^j \equiv \begin{cases} 
\frac{1}{2} \sum_{n \geq 1} \sigma_n (\psi^j_n + \phi^j_n) & \text{for } j \in \{1, 2\}, \\
\frac{1}{2} \sum_{n \geq 1} \sigma_n \psi^j_n & \text{otherwise},
\end{cases} \quad \text{and} \quad \gamma_n = \frac{1}{2} (\partial_i \phi^j_n)_{i,j=1}.
\]

Therefore (8.3) and (8.6)–(8.7) show that (8.1) can be (formally) rephrased as (3.12) (in the reformulation of (3.12)) by choosing \((\pi, \gamma)\) as in (8.7), \(F_v = \frac{1}{2} \sum_{n \geq 1} \phi^3_n \nabla_H (\sigma_n \theta), F_\theta = 0, G_v = G_\theta = 0\) and the differential operators \((\Delta v, \Delta \theta)\) replaced by \((\Delta v + \mathcal{L}_\theta v, \Delta \theta + \mathcal{L}_\theta \theta)\). As we commented in Remark 3.11 the case of inhomogeneous viscosity and/or diffusivity fits in our framework. In particular, the definition of (global) \(L^2\)-solution to (3.1)–(3.2) given in Definition 3.3 carries over to (8.1)–(8.2).

Before we formulate the main result of this section. As in (3.18), we let \(H = H^1(\Omega) \times H^1(\Omega)\) and \(V = H^2_N(\Omega) \times H^2_H(\Omega)\), where \(H^2_N(\Omega)\) and \(H^2_H(\Omega)\) are defined in (3.14) and (3.15), respectively.

**Theorem 8.2** (Global well-posedness – Stratonovich formulation). Let Assumption 8.1 be satisfied. Let \((v_0, \theta_0) \in L^0_{\mathcal{F}_0}(\Omega; H)\). Then (8.1)–(8.2) has a unique global \(L^2\)-strong solution \((v, \theta)\) such that

\[(v, \theta) \in C([0, \infty); H) \cap L^2_{\text{loc}}([0, \infty); V) \text{ a.s.}\]

Moreover, the following hold:

- The estimates of Theorem 3.6 hold for the global \(L^2\)-strong solution \((v, \theta)\) to (8.1)–(8.2).
- The assignment \((v_0, \theta_0) \mapsto (v, \theta)\) is continuous in the sense of Theorem 3.7.

**Proof.** One can readily check that Assumption 8.1 are stronger than Assumptions 3.1 and 3.5. For instance the parabolicity assumption of Assumption 3.1(2) (see Remark 3.11 for the case of inhomogeneous viscosity and/or conductivity) are automatically satisfied. Moreover, Assumption 3.5 follows from Assumption 8.1(2) and (8.7). Thus Theorem 8.2 follows from Theorems 3.6–3.7 and Remark 3.11.

**Remark 8.3** (Weakening Assumption 8.1(4) – Local existence for (8.1)). Theorem 3.4 also applies to the Stratonovich formulation (8.1). In particular, the local existence result of Theorem 3.4 holds for (8.1) provided Assumption 8.1(1) and (3)–(4). By the first part of Remark 3.11, to extend the local existence result of Theorem 3.4, the condition in Assumption 8.1(4) can be weakened to the following: There exist \(K, \eta > 0\) such that, a.s. for all \(j \in \{1, 2\},

\[
\left\| \sum_{n \geq 1} \phi^3_n (\cdot, 0) \phi^j_n (\cdot, 0) \right\|_{H^{\frac{1}{2}+\eta}(\mathbb{T}^2)} + \left\| \sum_{n \geq 1} \phi^3_n (\cdot, -h) \phi^j_n (\cdot, -h) \right\|_{H^{\frac{1}{2}+\eta}(\mathbb{T}^2)} \leq K.
\]

**References**

[AHHS22] A. Agresti, M. Hieber, A. Hussein, and M. Saal. The stochastic primitive equations with transport noise and turbulent pressure. Stochastics and Partial Differential Equations: Analysis and Computations, pages 1–81, 2022.

[AV21a] A. Agresti and M.C. Veraar. Stochastic maximal \(L^p(L^q)\)-regularity for second order systems with periodic boundary conditions. arXiv preprint arXiv:2106.01274, 2021. To appear in Annales de l’institut Henri Poincaré (B) Probability and Statistics.

[AV21b] A. Agresti and M.C. Veraar. Stochastic Navier-Stokes equations for turbulent flows in critical spaces. arXiv preprint arXiv:2107.03953, 2021.

[AV22a] A. Agresti and M.C. Veraar. The critical variational setting for stochastic evolution equations. arXiv preprint arXiv:2206.00520, 2022.

[AV22b] A. Agresti and M.C. Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces Part I. Stochastic maximal regularity and local existence. Nonlinearity, 35(8):4100, 2022.

[AV22c] A. Agresti and M.C. Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces part II. Journal of Evolution Equations, 22(2):1–96, 2022.

[AFP21] S. Assing, F. Flandoli, and U. Pappalettera. Stochastic model reduction: convergence and applications to climate equations. J. Evol. Equ., 21(4):3813–3848, 2021.

[AG01] P. Azérad and F. Guillén. Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics. SIAM journal on mathematical analysis, 33(4):847–859, 2001.
[BFH20] D. Breit, E. Feireisl, and M. Hofmanová. On the long time behavior of compressible fluid flows excited by random forcing. *arXiv preprint arXiv:2012.07476*, 2020.

[BFHM19] D. Breit, E. Feireisl, M. Hofmanová, and B. Maslowski. Stationary solutions to the compressible Navier-Stokes system driven by stochastic forces. *Probab. Theory Related Fields*, 174(3-4):981–1032, 2019.

[BCF91] Z. Brzeźniak, M. Capiński, and F. Flandoli. Stochastic partial differential equations and turbulence. *Math. Models Methods Appl. Sci.*, 1(1):41–59, 1991.

[BCF92] Z. Brzeźniak, M. Capiński, and F. Flandoli. Stochastic Navier-Stokes equations with multiplicative noise. *Stochastic Anal. Appl.*, 10(5):523–532, 1992.

[BS21] Z. Brzeźniak and J. Slavík. Well-posedness of the 3D stochastic primitive equations with multiplicative and transport noise. *Journal of Differential Equations*, 296:617–676, 09 2021.

[CT07] C. Cao and E.S. Titi. Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics. *Ann. of Math. (2)*, 166(1):245–267, 2007.

[Car01] N.C. Carton. Hydrodynamical modeling of oceanic vortices. *Surveys in Geophysics*, 22(3):179–263, 2001.

[CH19] N.C. Constantinou and A.M. Hogg. Eddy saturation of the southern ocean: A baroclinic versus barotropic perspective. *Geophysical Research Letters*, 46(21):12202–12212, 2019.

[DGHT11] A. Debussche, N. Glatt-Holtz, and R. Temam. Local martingale and pathwise solutions for an abstract fluids model. *Physica D: Nonlinear Phenomena*, 240(14):1123–1144, 2011.

[DGHTZ12] A. Debussche, N. Glatt-Holtz, R. Temam, and M. Ziane. Global existence and regularity for the 3D stochastic primitive equations of the ocean and atmosphere with multiplicative white noise. *Nonlinearity*, 25(7):2093–2118, jun 2012.

[DP22] A. Debussche and U. Pappalettera. Second order perturbation theory of two-scale systems in fluid dynamics. *arXiv preprint arXiv:2206.07775*, 2022.

[Del04] T. Delsole. Stochastic models of quasigeostrophic turbulence. *Surveys in Geophysics*, 25:107–149, 2004.

[DHC+95] B.D. Dushaw, B.M. Howe, B.D. Cornuelle, P.F. Worcester, and D.S. Luther. Barotropic and baroclinic tides in the central north pacific ocean determined from long-range reciprocal acoustic transmissions. *Journal of Physical Oceanography*, 25(4):631–647, 1995.

[Fla08] F. Flandoli. An introduction to 3D stochastic fluid dynamics. In *SPDE in hydrodynamic: recent progress and prospects*, pages 51–103. Springer, 2008.

[Fla11] F. Flandoli. Random perturbation of PDEs and fluid dynamic models, volume 2015 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010, École d’Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[FP20] F. Flandoli and U. Pappalettera. Stochastic modelling of small-scale perturbation. *Water*, 12(10):2950, 2020.

[FP22] F. Flandoli and U. Pappalettera. From additive to transport noise in 2D fluid dynamics. *Stochastics and Partial Differential Equations: Analysis and Computations*, pages 1–41, 2022.

[FOB+14] C.L.E. Franzke, T.J. O’Kane, J. Berner, P.D. Williams, and V. Lucarini. Stochastic climate theory and modeling. *WIREs Climate Change*, 6(1):63–78, 2014.

[FGH+20] K. Furukawa, Y. Giga, M. Hieber, A. Hussein, T. Kashiwabara, and M. Wrona. Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier-Stokes equations. *Nonlinearity*, 33(12):6502–6516, 2020.

[Gar09] C. Gardiner. *Stochastic Methods: A Handbook for the Natural and Social Sciences*. Springer, 2009.

[GGH+20a] Y. Giga, M. Gries, M. Hieber, A. Hussein, and T. Kashiwabara. Analyticity of solutions to the primitive equations. *Math. Nachr.*, 293(2):284–304, 2020.

[GGH+20b] Y. Giga, M. Gries, M. Hieber, A. Hussein, and T. Kashiwabara. The hydrostatic Stokes semi-group and well-posedness of the primitive equations on spaces of bounded functions. *J. Funct. Anal.*, 279(3):108561, 46, 2020.

[GHH+21] Y. Giga, M. Gries, M. Hieber, Amru Hussein, and Takahito Kashiwabara. The primitive equations in the scaling-invariant space $L^p(L^1)$. *J. Evol. Equ.*, 21(4):4145–4169, 2021.

[GHKVZ14] N. Glatt-Holtz, I. Kukavica, V. Vicol, and M. Ziane. Existence and regularity of invariant measures for the three dimensional stochastic primitive equations. *Journal of Mathematical Physics*, 55(5):051504, 2014.

[HH20] M. Hieber and A. Hussein. *An Approach to the Primitive Equations for Oceanic and Atmospheric Dynamics by Evolution Equations*, pages 1–109. Springer International Publishing, Cham, 2020.

[HHK16] M. Hieber, A. Hussein, and T. Kashiwabara. Global strong $L^p$ well-posedness of the 3D primitive equations with heat and salinity diffusion. *Journal of Differential Equations*, 261(12):6950–6981, 2016.
