Golden section based method of 3D modeling of icosahedron and dodecahedron in construction design

V N Vasilieva

Department of engineering and computer graphics, South Ural State University, 76, Lenin Avenue, Chelyabinsk, 454080, Russian Federation

E-mail: vasilevavn@susu.ru

Abstract. This paper provides an introduction of a three-dimensional modeling of the two most complex regular polyhedrons: icosahedron and dodecahedron. It is suggested to apply a method of the icosahedron and dodecahedron building by using rectangles with the sides ratio like in the golden section, and taking the icosahedron’s golden rectangles as a basis. The author gives information on the first mentioning of this building method by the Italian mathematician L. Pacioli in his Divine Proportion book. Based on this method, the building of the icosahedron and dodecahedron inscribed in the regular hexahedron and circumscribed about it is performed. This may be important for the design of facilities, based on the shapes of these geometrical figures, with inscribed regular hexahedrons for the purpose of a better organization of residential space of storeys. The examples of using the geometrical shapes of regular polygons (RP) in architecture are given. The building is performed using the AutoCAD package, but it also can be repeated in other known CAD software.

1. Introduction

Regular polygons (RP), except for the regular hexahedron, can be quite rarely seen, as they are, in architecture, not only because of the big faces, but also because of their complex geometry.

From the existing examples of the practical design of facilities with the shapes of the most complex RPs, the icosahedron and dodecahedron, it is known that those are related to the regular hexahedron inscribed in them in which residential space is organized.

Many methods of building the icosahedron and dodecahedron are known. One of those has become of interest to us, that is the method of building based on golden rectangles (GR) with the sides ratio like in the golden section (GS). The method becomes of special importance due to the fact that the RPs are built with the use of the regular hexahedron. This method, which is simple, elegant and is characterized with internal harmony, has become even more illustrative and multi-purpose when applied in the 3D modeling environment.

The work’s objective is to complete the method of the icosahedron and dodecahedron building using rectangles with the GS-based side ratio by means of 3D modeling.

2. Brief History

It is believed that the RPs came into being in the Pythagorean school of thought in ancient Greece in the 6th century B.C. Pythagoras of Samos and his disciples knew about tetrahedron, regular hexahedron, octahedron and the GS. Dodecahedron was first described by Hippasus of Metapontum, and icosahedron – by Theaetetus of Athens, who also gave the mathematical description of the RPs and proved that there existed only five of those [1-3]. Plato, who shared the views of the Pythagoreans
regarding the harmony of the universe, in his *Timaeus* dialogue developed the doctrine on proportions and matched the RPs against the four "elements" of nature. Thus, geometrical representatives of the numerical harmony of the Universe became known as Platonic solids. In the 3rd century B.C., the development of the ancient Greek science resulted in Euclid’s *Elements*, the final book XIII of which gives strict building of all the regular solids.

In the Renaissance era, "the talks about the GS, Platonic solids and the RPs were once again resumed since the times of Euclid" [4]. The development of the theory on the RPs during this period is related to such big names as Filippo Brunelleschi, Piero della Francesca, Luca Pacioli, Leonardo da Vinci, Albrecht Dürer, and Johannes Kepler.

The five RPs have remained popular for the whole period of the history of science. In our time, too, using the Platonic solids and the GS as a basis, “outstanding scientific discoveries of modern science [have been made], … in particular, quasicrystals and fullerenes, for which the Nobel prizes were awarded” [5].

3. Main Concepts
Regular polyhedron is a convex polyhedron every face of which is a regular $n$-gon with equal number of such faces meeting in each vertex.

Let us consider the icosahedron and dodecahedron as the most complex and interesting Platonic solids, not only due to their multiple symmetries, but also due to the presence of regular pentagons, and therefore, the GS proportions in their geometry.

They follow the Euler’s formula, which unites the number of vertices (V), faces (F) and edges (E) of any convex polyhedron into a simple ratio: $V + F = E + 2$.

4. Golden Section as a Basis for Building of the Icosahedron and Dodecahedron
An Italian mathematician and monastic scholar Luca Pacioli was a successor of the ancient Greeks in studying the Platonic solids. In his Divine Proportion book [6,7], illustrated by Leonardo da Vinci, he presented the GS and the RPs in compliance with book XIV of the Elements [4]. In the same work, for the first time, he also spoke of an “unfathomable” effect when “a side of a regular hexahedron and a
side of a triangle of a solid of 20 faces, while both limited by one and the same sphere”, correspond to the GS proportion [7]. This means that the icosahedron can be built using GRs. Such GRs had also been known before the publication by L. Pacioli, and the Parthenon in Athens is proof of that [8].

The algorithm of the GR-based building of the icosahedron is as follows: “three congruent GRs are inserted one into another, perpendicularly to each other along the middle parallel, and all that is left is to connect the vertices being nearest to each other” [9]. The method of building the dodecahedron [10] is similar to that of the icosahedron.

J. Kepler called the regular hexahedron a “parent” to all the RPs, probably because all other RPs are built most rationally, logically and illustratively when based on it. From among the many methods of the RP building [7,9-16], let us consider the AutoCAD building [17] of 3D models of the icosahedron and dodecahedron based on rectangles with the GS proportion, in which the bigger side $a$ will equal a side of a conditional regular hexahedron. Therefore, the figures may be considered as inscribed in the regular hexahedron.

5. Building the Icosahedron Inscribed in the Regular Hexahedron
The sides of rectangle $H$ (Fig. 1, a) should be in the GS ratio. Using the triangle method (Fig. 1, b), we divide intercept $a$ into parts where $a:c=c:b=\Phi$. Next, the three identical GRs are placed symmetrically and perpendicularly to each other. Points 1, 2, 3 create one of the 20 faces of the icosahedron. In this plane a triangle is created, which turns into a surface. With a Circle Packing as related to point 3 of axis O-3, 5 faces with a common vertex are built (see Fig. 1, c). For some of the faces the 3D Mirror is applied three times, where two GRs and a plane of symmetry defined by points 4, 5, 6 are used as the mirror planes. Then we transform the surfaces of the faces forming the enclosed volume into a 3D solid (see Fig. 1, d).

6. Building the Dodecahedron Inscribed in the Regular Hexahedron
Let us build the dodecahedron using the same GRs with amendments, for comparison. An interesting peculiarity is revealed in the process of building. It turns out that if the icosahedron and the dodecahedron are inscribed in one and the same regular hexahedron, their sides also follow the golden ratio $\Phi$. Therefore, the side of the dodecahedron equals $b$ – the smaller intercept of the GS (see Fig. 1, b). We introduce amendments into the geometry of the rectangles for the dodecahedron by placing intercepts $b$ in the middle of the smaller sides of the same GRs (Fig. 1, e). The ratio of the sides (of the whole to the smaller part) of the new rectangle for the dodecahedron is $a:b=2.618$, and for the icosahedron’s GR it was $a:c=1.618$.

![Figure 2. Building 3D models of the icosahedron and dodecahedron circumscribed about the regular hexahedron.](image)

The final points 1, 2, 3 of intercepts $b$ will indicate to the dodecahedron face’s plane in which to draw the Regular Pentagon / Side, having chosen p. 1 and 2. The diagonal of the pentagonal face equals side $c$ of the icosahedron. The next steps in the applying of the Circle Packing with the center in point B of arm O-B (see Fig. 1, e) and the thrice-applied 3D Mirror as related to the GR create the enclosed volume of the dodecahedron, limited with surfaces. We will use it to build the 3D solid – the dodecahedron (Fig. 1, g).
7. Building the Icosahedron and the Dodecahedron Circumscribed about the Regular Hexahedron

The dual figures were built on the corresponding edges $c$ and $b$. By turning the dodecahedron (or the icosahedron) $90^\circ$, we will create a classical dual connection of the dual RPs (Fig. 2, a). A wireframe image of this dual connection in the view following the arrow is shown in Fig. 2, b. It illustrates the structure of two GRs with sides $a$, $c$ and intercept $b$, as well as all the elements of the figures: sides – intercepts $c$ and $b$; projecting faces of the icosahedron – $i$ and of the dodecahedron – $d$; positions of vertices – points 1 and 2 – all that is required to build the icosahedron and dodecahedron circumscribed about the regular hexahedron.

In a diagram analogous to Fig. 2, b, we add the contour of the regular hexahedron with side $a$ (see Fig. 2, c), index $v$ means the figures inscribed in the regular hexahedron, $o$ – the ones circumscribed about it. First, we build the dodecahedron $d_v$, by using $p$. 1– the regular hexahedron vertex, we define the sides of its GR – $a_1$, $b_1$. The obtained $p$. 2 on the arm of the icosahedron vertices allows to build a circumscribed icosahedron $i_v$ and define the sides of its GR – $a_2$, $c_2$.

The image of the dual figures circumscribed about the regular hexahedron is given in Fig. 2, d.

8. Using the Geometrical Shapes of the Icosahedron and Dodecahedron in Architecture

Designing the RP-shaped buildings is a complex task in most cases. Studying the RPs, their geometrical properties and laws of building provides a possibility to use them in architecture and construction.

The icosahedron and dodecahedron shapes are extremely rare in architecture, that is why a project of a modular residential skyscraper (Fig. 3, a), consisting of regular hexahedrons and dodecahedrons, presented by a German architectural bureau Tammo Prinz Architects, gains attention. Inside the concrete dodecahedron, a regular-hexahedron-shaped residential space of storeys is formed, and the protruding elements transform into balconies. It is planned to construct this unusual high-rise building in Lima, the capital city of Peru.

In the Azores, architect John Shenton is working on developing and fulfilling a project on tourist natural-timber cabins shaped as the icosahedrons and dodecahedrons (see Fig. 3, b), with solar panels mounted on the rooftops [18].

Figure 3. Using the dodecahedron shape in architecture.
More often, one may see sculptures of these figures: for instance, a dodecahedron sculpture installed on a university Campus in Qatar (Fig. 3, c), or a Double Helix made of dodecahedrons at Wright State University, Ohio.

In the process of developing new shapes of residential and public buildings in Israel in 1972-1975, an unusual residential complex called Ramot Polin (see Fig. 3, d) created of 720 dodecahedrons was built by architect Zvi Hecker, who is known for his experience in working with unique and innovative geometrical shapes.

A geodesic dome (GD) is one of the ways of practical application of the icosahedron based on a principle that the more partitions are made in the faces of the icosahedron (see Fig. 4, a), the closer the geodesic dome is to a sphere. Such constructions look like architectural facilities in the shape of a 3D polygon inscribed in a sphere or a part of the sphere.

The GD was first used in 1926 when a German engineer Walther Bauersfeld created the first planetarium. American architect Richard Burckminster Fuller developed the field of geodesic domes and obtained a patent for designing those. The most famous creation of Fuller is the U.S. Pavilion at Expo 67 in Montreal, which today holds the Biosphere museum devoted to environment protection (see Fig. 4, b) [19].

The first GD building, an architecture uncommon for Novosibirsk, was the Shar (Globe) coffee-shop (see Fig. 4, c) designed by Valery Filippov. The pavilion constructed as a sphere-shaped truss system has 4 above-ground levels and an underground one.

In Copenhagen a GD was constructed (see Fig. 4, d) as an educational project. It holds a smart house and a greenhouse for plants.

Thanks to their light weight, big bearing capability and perfect aerodynamics, geodesic domes are used for different architectural constructions: greenhouses, Expo pavilions, stadiums, swimming pools, and weather stations [20].

9. Conclusion
The algorithm for three-dimensional modeling of the icosahedron and dodecahedron is given, which is based on using of three mutually perpendicular rectangles with their sides having the GS proportions.

It has been determined that rectangles with the GS ratio of their sides can be used to build figures both inscribed in the regular hexahedron and circumscribed about it.

Based on the suggested method and due to clear orientation of the rectangles’ planes, the RPs’ building is made easier when designing facilities in the shape of these figures.
References

[1] Shal M 1883 Historical Review of the Origin and Development of Geometrical Methods vol. 2 (Moscow: Moscow Mathematical Society) p 748

[2] Kolman E 1961 Ancient History of Mathematics (Moscow: State Publishing House for Literature on Physics and Mathematics) p 235

[3] New Philosophical Encyclopedia: 4 vol. Institute of Philosophy, Russian Academy of Sciences; National Public and Scientific Foundation https://iplib.ru/greenstone3/library/collection/newphilenc/document/HASH21249142aeec8cb5b76221

[4] Martynенко G Ya 2010 The Mathematics of Harmony: Renaissance (14th–16th Centuries) (To the 500th Anniversary of Luca Paccioli’s Divine Proportion) Academy of Trinitarianism 77-657

[5] Stakhov A P 2012 The Mathematics of Harmony: Innovations in Information Technologies, in Foundations of Mathematics, in Education Naukovedenie Internet Journal (Moscow: IGUPIT) 4 p 98 https://naukovedenie.ru/PDF/33tvn412.pdf

[6] Divina proportione: opera a tutti glingegni... https://archive.org/details/divinaproportion00paci/page/n41

[7] Schetnikov A I 2007 Luca Pacioli and his Treatise “On Divine Proportion” Mathematical Education 1(41) 33–44

[8] Livio M 2015 φ – The Number of God. Golden Section as a Formula of the Universe (Moscow:ATS) p 218

[9] Science. Greatest Theories: Issue 14: Three-dimensional World. Euclid. Geometry 2015 (Moscow: De Agostini) p 168

[10] Perepelkin D I 1937 On One Building Case of the Regular Icosahedron and Regular Dodecahedron Mathematical Enlightenment 12 10–15

[11] Perepelkin D I 1949 Course on Elementary Geometry, Part 2, Solid Geometry (Moscow-Leningrad) pp 283–287

[12] Smirnova I M Regular Polygons Cascades http://www.vasmirnov.ru/Lecture/Kaskady/Kaskady.htm

[13] Gardner M 1971 Mathematical Puzzles and Games: Ch. 23 (Moscow: Mir) p 511

[14] Dolbilin N P 2001 Three Theorems on Convex Polyhedrons Kvant 5(6) 7–12

[15] Dolbilin N P, Kanel A Ya Harmony of Regular Polygons Mathematical Sketches pp 2002-2019

[16] Alsina C A 2014 Thousand Faces of Geometrical Beauty. Polyhedrons (Moscow: De Agostini) p 144

[17] Heifets A L, Loginovskiy AN, Butorina I V, Vasilieva V N 2015 Engineering 3D Computer Graphics: Study Guide and Practicum for Academic Bachelor’s Program (Moscow: Urait Publishing House) p 602

[18] https://novate.ru/blogs/261016/38571/

[19] Fuller and His Geodesic Domes 2014 Advanced Mathematics and Structural Mechanics, Engineering Equipment, Engineering Constructions: Proceedings of a Science-to-practice Conference (Moscow: MARKhI) p 35

[20] Esipova A A 2015 Using of Geodesic Domes in Construction: Advantages and Disadvantages Science and Modernity 38 8–11

Acknowledgment
The work was supported by Act 211 Government of the Russian Federation, Contract № 02.A03.21.0011.