FACTORIZATION OF HOPF QUASIGROUOS

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Abstract. In this paper we introduce the notion of factorization in the Hopf quasigroup setting and we prove that, if \( A \) and \( H \) are Hopf quasigroups such that their antipodes are isomorphisms, a Hopf quasigroup \( X \) admits a factorization as \( X = AH \) iff \( X \) is isomorphic to a double cross product \( A \bowtie H \) as Hopf quasigroups.

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1. Introduction

Let \( \mathbb{F} \) be a field and denote by \( \otimes \) the tensor product in the category of vector spaces over \( \mathbb{F} \) denoted by \( \mathbb{F} \)-\textbf{Vect}. The double cross product of two Hopf algebras \( A \) and \( H \) in \( \mathbb{F} \)-\textbf{Vect} was introduced by Majid in [15, Proposition 3.12] (see also [16, Theorem 7.2.2]) as a new Hopf algebra structure defined in the tensor product \( A \otimes H \) and determined by a matched pair \( (A, H) \). A matched pair of Hopf algebras is a system \( (A, H) \), where \( A \) and \( H \) are Hopf algebras, \( A \) is a left \( H \)-module coalgebra with action \( \varphi_A : H \otimes A \rightarrow A \), \( H \) is a right \( A \)-module coalgebra with action \( \phi_H : H \otimes A \rightarrow H \) and some suitable compatibility conditions hold for all \( h, g \in H \) and \( a, b \in A \). Using the Heyneman-Sweedler’s convention and the notations \( \varphi_A(h \otimes a) = h \triangleright a \), \( \phi_H(h \otimes a) = h \triangleleft a \), these conditions can be written as follows:

\[
\begin{align*}
    h \triangleright 1_A &= \varepsilon(h)1_A, & h \triangleright (ab) &= (h_{(1)} \triangleright a_{(1)})((h_{(2)} \triangleleft a_{(2)}) \triangleright b), \\
    1_H \triangleleft a &= \varepsilon(a)1_H, & (hg) \triangleleft a &= (h \triangleleft (g_{(1)} \triangleright a_{(1)}))(g_{(2)} \triangleleft a_{(2)}), \\
    h_{(1)} \triangleleft a_{(1)} \otimes h_{(2)} \triangleright a_{(2)} &= h_{(2)} \triangleleft a_{(2)} \otimes h_{(1)} \triangleright a_{(1)}.
\end{align*}
\]

If \( (A, H) \) is a matched pair of Hopf algebras, the double cross product \( A \bowtie H \) of \( A \) with \( H \) is the Hopf algebra built on the vector space \( A \otimes H \) with product

\[
(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b_{(1)}) \otimes (h_{(2)} \triangleleft b_{(2)})g
\]

and tensor product unit, counit, coproduct and antipode

\[
\lambda_{A \bowtie H}(a \otimes h) = \lambda_H(h_{(2)}) \triangleright \lambda_A(a_{(2)}) \otimes \lambda_H(h_{(1)}) \triangleleft \lambda_A(a_{(1)}),
\]

where \( \lambda_H \) is the antipode of \( H \) and \( \lambda_A \) is the antipode of \( A \).

Following [16], a Hopf algebra \( X \) factorises as \( X = AH \) if there exists sub-Hopf algebras \( A \) and \( H \) with inclusion maps \( i_A \) and \( i_H \) such that the map \( \omega(a \otimes h) = i_A(a)i_H(h) \) is an isomorphism of vector spaces. As was proved by Majid in [16, Theorem 7.2.3], \( X \) factorises as \( X = AH \) iff there exists a matched pair of Hopf algebras \( (A, H) \) such that \( X \) is isomorphic to \( A \bowtie H \) as Hopf algebras.
On the other hand, the theory of distributive laws between monads was initiated by Beck \[6\] and Barr \[5\] in the seventies of the last century. A distributive law between two algebras \(A\) and \(H\) is a morphism \(\Psi : H \otimes A \rightarrow A \otimes H\) which is compatible with the algebra structures. It is well-known that a distributive law \(\Psi : H \otimes A \rightarrow A \otimes H\) induces an algebra structure on the tensor product \(A \otimes H\) commuting with the action associated to the product of \(A\) on the left and with the action associated to the product of \(H\) on the right. This algebra, denoted by \(A \otimes_{\Psi} H\), is the wreath product of \(A\) and \(H\) and its unit and product are defined by \(1_{A \otimes_{\Psi} H} = 1_A \otimes 1_H\) and

\[
\mu_{A \otimes_{\Psi} H} = (\mu_A \otimes \mu_H) \circ (id_A \otimes \Psi \otimes id_H),
\]

where \(\mu_A\), \(\mu_H\) are the corresponding products and \(id_A\), \(id_H\) are the identity morphisms for \(A\) and \(H\) respectively. It is well known that bialgebras are algebras in the category of coalgebras. From this point of view, a distributive law in the category of bialgebras is a distributive law between the underlying algebras satisfying that is a coalgebra morphism. This kind of distributive laws induce a wreath product bialgebra, where the product is the wreath product and the colagebra structure is the one associated to the tensor product coalgebra. A relevant example of these wreath products are the double crossed product quoted in the previous page where the distributive law is

\[
\Psi(h \otimes a) = h_{(1)} \triangleright b_{(1)} \otimes h_{(2)} \triangleleft b_{(2)}.
\]

In the literature we can find similar constructions of double cross products. For example, in the associative case are relevant the double cross products associated to matched pairs of groups \[23\] (i.e. Hopf algebras in the category of sets) and the double cross products associated to matched pairs of groupoids \[1\]. In \[1\] an extension of this kind of products was presented in a non-associative setting for matched pairs of Hopf quasigroups as a generalization of the results proved in \[18\] for quasigroups. The notion of Hopf quasigroup in \(\mathbb{F}\)-\text{Vect} was introduced by Klim and Majid in \[17\] and it is a particular case of unital coassociative \(H\)-bialgebra (see \[21\]) and also of a quantum quasigroup (see \[22\] and \[14\]). This non-associative generalization of Hopf algebras include as a particular cases the loop algebra for a loop \(L\) with the inverse property (see \[17\], \[9\]) and also the enveloping algebra \(U(M)\) of a Malcev algebra over \(\mathbb{F}\) (see \[17\] and \[20\]). Several articles have been published in recent years devoted to the study of some kind of products between Hopf quasigroups as for example: \[7\] and \[8\] for the theory of smash products of Hopf quasigroups; \[11\] and \[12\] for the theory of twisted smash products of Hopf quasigroups; \[12\] for Hopf quasigroups obtained by the twist double method; \[4\] for Hopf quasigroups associated to skew pairings and Hopf quasigroups obtained as double cross products of Hopf quasigroups. As was pointed in \[13\], in all these cases the product is determined by a morphism \(\Psi\) satisfying some conditions that are close to the ones involved in the classical definition of distributive law. Taking this into consideration, in \[13\] the author introduce a notion of "non-associative" distributive law, called \(a\)-comonoidal distributive law, that permits to understand the products cited in the previous lines with a general point of view. In the final section of \[13\] we can find the proof of the following fact: For two Hopf quasigropups \(A\) and \(H\), the tensor product \(A \otimes H\) with the corresponding wreath product, i.e., the product associated to an \(a\)-comonoidal distributive law, becomes a Hopf quasigroup, where the coalgebra structure is the one of the tensor product coalgebra.

Taking into account what was said in the previous paragraphs, it is natural to ask when a Hopf quasigroup \(X\) admits a factorization. To answer this question is the main motivation of this paper.

The structure of the paper is as follows. In Section 2 we recall some necessary background about Hopf (co)quasigroups in a monoidal setting and in Section 3 we present the main facts of the theory of wreath products associated to an \(a\)-comonoidal distributive law. In Section 4 we introduce the notion of factorization for Hopf quasigroups and we prove the main theorem of this paper that asserts the following: Let \(H\), \(A\), \(X\) be Hopf quasigroups such that the antipodes of \(H\) and \(A\) are
isomorphisms. Then, $X$ factorizes as $X = AH$ iff there exists a matched pair of Hopf quasigroups $(A, H)$ such that $X$ is isomorphic to the double cross product $A \triangleright H$ as Hopf quasigroups. Also, in this last section, we discuss an example of a non-commutative, non-cocommutative Hopf quasigroup constructed as a double cross product. Finally, by dualisation, in this paper we show that we can obtain similar results for Hopf coquasigroups.

2. Preliminaries

From now on $C$ denotes a strict symmetric monoidal category with tensor product $\otimes$, unit object $K$ and natural isomorphism of symmetry $c$. Recall that a monoidal category is a category $C$ equipped with a tensor product functor $\otimes : C \times C \to C$, a unit object $K$ of $C$ and a family of natural isomorphisms $a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$, $r_M : M \otimes K \to M$, $l_M : K \otimes M \to M$, in $C$ (called associativity, right unit and left unit constraints, respectively) satisfying the Pentagon Axiom and the Triangle Axiom, i.e.,

$$a_{M,N,P \otimes Q} \circ a_{M \otimes N,P,Q} = (id_M \otimes a_{N,P,Q}) \circ a_{M,N \otimes P,Q} \circ (a_{M,N,P} \otimes id_Q),$$

$$(id_M \otimes l_N) \circ a_{M,K,N} = r_M \otimes id_N,$$

where $id_X$ denotes the identity morphism for each object $X$ in $C$. A monoidal category is called strict if the associativity, right unit and left unit constraints are identities. A strict monoidal category $C$ is symmetric if it has a family of natural isomorphisms $c_{M,N} : M \otimes N \to N \otimes M$ such that the equalities

$$c_{M,N \otimes P} = (id_N \otimes c_{M,P}) \circ (c_{M,N} \otimes id_P), \quad c_{M \otimes N,P} = (t_{M,P} \otimes id_N) \circ (id_M \otimes c_{N,P}),$$

$$c_{N,M} \circ c_{M,N} = id_{M \otimes N},$$

hold for all $M, N$ in $C$.

Considering that it is well known that every non-strict monoidal category is monoidal equivalent to a strict one, we can assume without loss of generality that the category $C$ is strict and then we omit explicitly the associativity and unit constraints. Thus, the results proved in this paper for objects and morphisms in $C$ remain valid for every non-strict symmetric monoidal category, what would include for example the category $F$- Vect of vector spaces over a field $F$, the category $R$- Mod of left modules over a commutative ring $R$, or the category $Set$ of sets. In what follows, for simplicity of notation, given objects $M, N, P$ in $C$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$.

A magma in $C$ is a pair $A = (A, \mu_A)$, where $A$ is an object in $C$ and $\mu_A : A \otimes A \to A$ (product) is a morphism in $C$. A unital magma in $C$ is a triple $A = (A, \eta_A, \mu_A)$, where $(A, \mu_A)$ is a magma in $C$ and $\eta_A : K \to A$ (unit) is a morphism in $C$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$. A monoid in $C$ is a unital magma $A = (A, \eta_A, \mu_A)$ in $C$ satisfying $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$, i.e., the product $\mu_A$ is associative. Given two unital magmas (monoids) $A$ and $B$, a morphism $f : A \to B$ in $C$ is called a morphism of unital magmas (monoids) if $f \circ \eta_A = \eta_B$ (i.e., the morphism $f$ is unitary) and $\mu_B \circ (f \otimes f) = f \circ \mu_A$ (i.e., the morphism $f$ is multiplicative).

Also, if $A, B$ are unital magmas (monoids) in $C$, the object $A \otimes B$ is a unital magma (monoid) in $C$, where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. If $A = (A, \eta_A, \mu_A)$ is a unital magma so is $A^{opp} = (A, \eta_A, \mu_A \circ c_{A,A})$.

A comagma in $C$ is a pair $D = (D, \delta_D)$, where $D$ is an object in $C$ and $\delta_D : D \to D \otimes D$ (coproduct) is a morphism in $C$. A counital comagma in $C$ is a triple $D = (D, \varepsilon_D, \delta_D)$, where $(D, \delta_D)$ is a comagma in $C$ and $\varepsilon_D : D \to K$ (counit) is a morphism in $C$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$. A comonoid in $C$ is a counital comagma in $C$ satisfying $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$, i.e., the coproduct $\delta_D$ is coassociative. If $D$ and $E$ are counital comagmas (comonoids) in $C$, a morphism
f : D → E in C is called a morphism of counital comagmas (comonoids) if \( \varepsilon_E \circ f = \varepsilon_D \) (i.e., the morphism f is counitary), and \( (f \otimes f) \circ \delta_D = \delta_E \circ f \) (i.e., the morphism f is comultiplicative).

Moreover, if D, E are counital comagmas (comonoids) in C, the object \( D \otimes E \) is a counital comagma (comonoid) in C, where \( \varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E \) and \( \delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E) \). If \( D = (D, \varepsilon_D, \delta_D) \) is a counital comagma so is \( D^{\text{cop}} = (D, \varepsilon_D, c_{D,D} \circ \delta_D) \).

Let \( f : D \to A \) and \( g : D \to A \) be morphisms between a comagma D and a magma A. We define the convolution product of f and g by \( f * g = \mu_A \circ (f \otimes g) \circ \delta_D \). If A is unital and D counital, we will say that f is convolution invertible if there exists \( f^{-1} : D \to A \) such that \( f * f^{-1} = f^{-1} * f = \varepsilon_D \otimes \eta_A \).

**Definition 2.1.** A non-associative bimonoid in the category C is a unital magma \((H, \eta_H, \mu_H)\) and a comonoid \((H, \varepsilon_H, \delta_H)\) such that \( \varepsilon_H \) and \( \delta_H \) are morphisms of unital magmas (equivalently, \( \eta_H \) and \( \mu_H \) are morphisms of counital comagmas). Then the following identities hold:

\[
\begin{align*}
(1) \quad & \varepsilon_H \circ \eta_H = \text{id}_K, \\
(2) \quad & \varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H, \\
(3) \quad & \delta_H \circ \eta_H = \eta_H \otimes \eta_H, \\
(4) \quad & \delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_H. 
\end{align*}
\]

A non-associative bimonoid is called commutative if \( \mu_H = \mu_H \circ c_{H,H} \), i.e., \( H = H^{\text{cop}} \) as unital magmas. Also, is called cocommutative if \( \delta_H = c_{H,H} \circ \delta_H \), i.e., \( H = H^{\text{cop}} \) as comonoids.

**Definition 2.2.** A non-coassociative bimonoid in the category C is a monoid \((D, \eta_D, \mu_D)\) and a comonoid \((D, \varepsilon_D, \delta_D)\) such that \( \eta_D \) and \( \mu_D \) are morphisms of unital magmas (equivalently, \( \varepsilon_D \) and \( \delta_D \) are morphisms of unital magmas). Then, as in the previous definition, the identities \((1), (2), (3)\) and \((4)\) hold.

A non-coassociative bimonoid is called cocommutative if \( \delta_D = c_{D,D} \circ \delta_D \), i.e., \( D = D^{\text{cop}} \) as counital comagmas. Also, is called commutative if \( \mu_D = \mu_D \circ c_{D,D} \), i.e., \( D = D^{\text{cop}} \) as monoids.

Now we recall the notions of Hopf quasigroup and Hopf coquasigroup in the category C.

**Definition 2.3.** A Hopf quasigroup H in C is a non-associative bimonoid such that there exists a morphism \( \lambda_H : H \to H \) in C (called the antipode of H) satisfying

\[
\begin{align*}
(5) \quad & \mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H = \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H) \\
(6) \quad & \mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) = H \otimes \varepsilon_H = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H). 
\end{align*}
\]

Note that composing with \( H \otimes \eta_H \) in \((5)\) we obtain that

\[
\lambda_H \ast \text{id}_H = \varepsilon_H \otimes \eta_H, 
\]

and composing with \( \eta_H \otimes H \) in \((6)\) we obtain

\[
\text{id}_H \ast \lambda_H = \varepsilon_H \otimes \eta_H. 
\]

Therefore, \( \lambda_H \) is convolution invertible and \( \lambda_H^{-1} = \text{id}_H \).

**Definition** \((2.3)\) is the monoidal version of the notion of Hopf quasigroup (also called non-associative Hopf algebra with the inverse property, or non-associative IP Hopf algebra) introduced in \([17]\) (in this case \( C=\mathbb{F}\text{-Vect} \)). Note that a Hopf quasigroup H is associative if and only if it is a Hopf algebra.

If H is a Hopf quasigroup in C we know that the antipode \( \lambda_H \) is unique, antimultiplicative, anticomultiplicative, i.e.,

\[
(9) \quad \lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, 
\]
\(\delta_H \circ \lambda_H = (\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H\)

and leaves the unit and the counit invariable (see \([19]\)):

\(\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.\)

Note that, by \([17, \text{Proposition 4.3}]\), if \(H\) is a commutative or cocommutative Hopf quasigroup, we have that \(\lambda^2_H = id_H\). Therefore, under \((\co)\text{commutativity conditions}, the antipode of } H \text{ is an isomorphism.}

A morphism between Hopf quasigroups \(H\) and \(A\) is a morphism \(f : H \to A\) of unital magmas and comonoids. Then (see Lemma 1.4 of \([3]\)) the equality

\(\lambda_A \circ f = f \circ \lambda_H\)

holds.

**Examples 2.4.** A quasigroup is a set \(Q\) together with a product such that for any two elements \(u, v \in Q\) the equations \(ux = v, \ xu = v\) and \(uv = x\) have unique solutions in \(Q\). A quasigroup \(L\) which contains an element \(e_L\) such that \(ue_L = u = e_Lu\) for every \(u \in L\) is called a loop. A loop \(L\) is said to be a loop with the inverse property (for brevity an I.P. loop) if to every element \(u \in L\), there corresponds an element \(u^{-1} \in L\) such that the equations \(u^{-1}(uv) = v = (vu)u^{-1}\) hold for every \(v \in L\).

If \(L\) is an I.P. loop, it is easy to show (see \([9]\)) that for all \(u \in L\) the element \(u^{-1}\) is unique and \(u^{-1}u = e_L = uu^{-1}\). Moreover, the mapping \(u \to u^{-1}\) is an anti-automorphism of the I.P. loop \(L\): \((uv)^{-1} = v^{-1}u^{-1}\). Then, \(L\) is an I.P. loop iff \(L\) is a cocommutative Hopf quasigroup in \(\text{Set}\). A concrete examples of these objects is the set of invertible el ements of the octonions \(\mathbb{O}\), the sphere \(S^7\) (or in more general way the spheres \(S^{2n-1}\)) and the 16 Moulfang loop \(\mathbb{G}_2\) associated to the octonions (see \([17, \text{Section 2}]\)).

Let \(R\) be a commutative ring and \(L\) an I.P. loop. Then, by \([17, \text{Proposition 4.7}]\), we know that

\[RL = \bigoplus_{u \in L} Ru\]

is a cocommutative Hopf quasigroup in \(R\text{-Mod}\) with product defined by the linear extension of the one defined in \(L\) and \(\delta_{RL}(u) = u \otimes u, \varepsilon_{RL}(u) = 1_R, \lambda_{RL}(u) = u^{-1}\) on the basis elements.

On the other hand, consider a commutative ring \(R\) with \(1/2\) and \(1/3\) in \(R\). A Malcev algebra \((M, [\ , \ ]\) over \(R\) is a free module over \(R\) with a bilinear anticommutative operation \([\ , \ ]\) on \(M\) satisfying that:

\[[J(a,b,c),a] = J(a,b,[a,c]),\]

where \(J(a,b,c) = [[a,b],c] - [[a,c],b] - [a,[b,c]]\) is the Jacobian in \(a, b, c\) (see \([20]\)). By the construction given in \([20]\), there exists a cocommutative Hopf quasigroup structure in \(R\text{-Mod}\) associated to \(M\). Indeed, consider the not necessarily associative algebra \(U(M)\) defined as the quotient of \(R\{M\}\), the free non-associative algebra on a basis of \(M\), by the ideal \(I(M)\) generated by the set

\[\{ab - ba - [a,b], (a,x,y) + (x,a,y), (x,a,y) + (x,y,a) / a, b \in M, x, y \in R\{M\}\},\]

where \((x,y,z) = (xy)z - x(yz)\) is the usual additive associator. By \([20, \text{Proposition 4.1}]\) and \([17, \text{Proposition 4.8}]\), the diagonal map \(\delta_{U(M)} : U(M) \to U(M) \otimes U(M)\) defined by \(\delta_{U(M)}(x) = 1 \otimes x + x \otimes 1\) for all \(x \in M\), and the map \(\varepsilon_{U(M)} : U(M) \to R\) defined by \(\varepsilon_{U(M)}(x) = 0\) for all \(x \in M\), both extended to \(U(M)\) as algebra morphisms; together with the map \(\lambda_{U(M)} : U(M) \to U(M)\), defined by \(\lambda_{U(M)}(x) = -x\) for all \(x \in M\) and extended to \(U(M)\) as an antialgebra morphism, provide a cocommutative Hopf quasigroup structure on \(U(M)\).
**Definition 2.5.** A Hopf coquasigroup $D$ in $C$ is a non-coassociative bimonoid such that there exists a morphism $\lambda_D : D \rightarrow D$ in $C$ (called the antipode of $D$) satisfying

\[(D \otimes \mu_D) \circ (\delta_D \circ \lambda_D) \circ \delta_D = D \otimes \eta_D = (D \otimes \mu_D) \circ (D \otimes \lambda_D \otimes D) \circ (\delta_D \otimes D) \circ \delta_D\]

and

\[(\mu_D \otimes D) \circ (\lambda_D \circ \delta_D) \circ \delta_D = \eta_D \otimes D = (\mu_D \otimes D) \circ (D \otimes \lambda_D \otimes D) \circ (D \otimes \delta_D) \circ \delta_D.\]

Note that composing with $\varepsilon_D \otimes D$ in (13) we obtain (9) and composing with $D \otimes \varepsilon_D$ in (14) we obtain (10). Then, as in the quasigroup case, $\lambda_D$ is convolution invertible and $\lambda_D^{-1} = id_D$.

It is obvious that, a Hopf coquasigroup $D$ is coassociative, i.e., $D$ is a comonoid, if and only if $D$ is a Hopf algebra. Moreover, as in the Hopf quasigroup case, if $D$ is a Hopf coquasigroup, the antipode $\lambda_D$ is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariant. In this setting a morphism between two Hopf coquasigroups $D$ and $B$ is a morphism $g : D \rightarrow B$ of monoids and counital comagmas. Therefore, (12) holds, i.e., $\lambda_B \circ g = g \circ \lambda_D$.

**Example 2.6.** Let $F$ be a field. By [17] the algebraic variety $F[S^7]$ is an example of Hopf coquasigroup. Also, there is a natural action of $\mathbb{Z}_2^n$ on $F[S^7]$ which leads to a cross coproduct $F[S^7] \times \mathbb{Z}_2^n$ as the first example of noncommutative noncocommutative Hopf coquasigroup (see [17, Proposition 5.10, Example 5.11]).

On the other hand, we can obtain examples of Hopf coquasigroups as duals of finite Hopf quasigroups. Following [2], in the next definition we recall the notion of finite object in the category $C$.

**Definition 2.7.** An object $P$ in $C$ is finite if there exists $P^*$ in $C$, called the dual object of $P$, such that $(P \otimes -, P^* \otimes -, \alpha_P, \beta_P)$ is an adjoint pair.

If $f : P \rightarrow Q$ is a morphism between finite objects, we define the dual morphism of $f$ as $f^* : Q^* \rightarrow P^*$ where $f^* = (P^* \otimes (\beta_Q(K) \circ (f \otimes Q^*)) \circ (\alpha_P(K) \otimes Q^*)$.

If $P$ and $Q$ are finite objects, $P \otimes Q$ is a finite object where $(P \otimes Q)^* = Q^* \otimes P^*$ because, if $(P \otimes -, P^* \otimes -, \alpha_P, \beta_P)$ and $(Q \otimes -, Q^* \otimes -, \alpha_Q, \beta_Q)$ are adjoint pairs, then

$$(P \otimes Q \otimes -, Q^* \otimes P^* \otimes -, \alpha_{P \otimes Q}, \beta_{P \otimes Q})$$

with

$$\alpha_{P \otimes Q} = (Q^* \otimes \alpha_P(K) \otimes Q \otimes -) \circ (\alpha_Q(K) \otimes -)$$

and

$$\beta_{P \otimes Q} = (\beta_Q(K) \otimes -) \circ (P \otimes \beta_Q(K) \otimes P^* \otimes -),$$

is an adjoint pair. Also, for morphisms we have that $(f \otimes g)^* = g^* \otimes f^*$. On the other hand, if $P$ is a finite object with adjoint pair $(P \otimes -, P^* \otimes -, \alpha_P, \beta_P)$, $P^*$ is finite object where $P^{**} = P$ because $(P^* \otimes -, P \otimes -, \alpha_{P^*}, \beta_{P^*})$ with $\alpha_{P^*} = (c_{P^*} \otimes P \otimes -) \circ \alpha_P$ and $\beta_{P^*} = \beta_P \circ (c_{P^*} \otimes P \otimes -)$ is an adjoint pair.

Then, by the properties of quoted in the previous paragraph, if $H$ is a finite Hopf quasigroup, it is easy to prove that the dual object $H^*$ is a Hopf coquasigroup with $\eta_{H^*} = \varepsilon_H^*, \mu_{H^*} = \delta_H^*, \varepsilon_{H^*} = \eta_H^*, \delta_{H^*} = \mu_H^* \text{ and } \lambda_{H^*} = \lambda_H^*$ as antipode. Therefore, if $H$ is finite object in $C$, $H$ is a Hopf quasigroup if $H^*$ is a Hopf coquasigroup. Similarly, $H$ is a Hopf coquasigroup iff $H^*$ is a Hopf quasigroup.

Finally, by [19] Corollary 1, we know that, if $H$ is a finite Hopf (co)quasigroup, the antipode of $H$ is an isomorphism.
3. Wreath (co)products for Hopf (co)quasigroups

In this section we recall the main notions and results introduced and proved in [13] about the wreath product of Hopf quasigroups. Following [13], this kind of products are the ones associated to \(a\)-comonoidal distributive laws.

**Definition 3.1.** Let \(H, A\) be Hopf quasigroups. A morphism \(\Psi : H \otimes A \to A \otimes H\) is said to be a distributive law of \(H\) over \(A\) if the following identities hold.

\[
\begin{align*}
\Psi \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) &= (\mu_A \otimes H) \circ (A \otimes \Psi) \circ (\Psi \otimes A) \circ (\lambda_H \otimes \lambda_A \otimes A), \\
\Psi \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes \lambda_A) &= (A \otimes \mu_H) \circ (\Psi \otimes H) \circ (H \otimes \Psi) \circ (H \otimes \lambda_H \otimes \lambda_A), \\
\Psi \circ (\eta_H \otimes A) &= A \otimes \eta_H,
\end{align*}
\]

respectively. Then, in this case, the conditions of the definition of distributive law for Hopf quasigroups are the ones that we can find in the classical definition of distributive law between monoids, i.e., \(\Psi\) is compatible with the unit and the product of \(A\) and \(H\).

**Definition 3.2.** Let \(H, A\) be Hopf quasigroups and let \(\Psi : H \otimes A \to A \otimes H\) be a distributive law of \(H\) over \(A\). The distributive law \(\Psi\) is said to be comonoidal if it is a comonoid morphism, i.e., the following identities hold.

\[
\begin{align*}
\delta_{A \otimes H} \circ \Psi &= (\Psi \otimes \Psi) \circ \delta_{H \otimes A}, \\
(\varepsilon_A \otimes \varepsilon_H) \circ \Psi &= \varepsilon_H \otimes \varepsilon_A,
\end{align*}
\]

**Definition 3.3.** Let \(H, A\) be Hopf quasigroups and let \(\Psi : H \otimes A \to A \otimes H\) be a comonoidal distributive law of \(H\) over \(A\). We will say that \(\Psi\) is an \(a\)-comonoidal distributive law of \(H\) over \(A\) if the following identities hold.

\[
\begin{align*}
(A \otimes \mu_H) \circ (\Psi \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \circ ((\lambda_H \otimes H) \circ \delta_H) \circ A \otimes H) &= \varepsilon_H \otimes A \otimes H, \\
(A \otimes \mu_H) \circ (\Psi \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \circ ((H \otimes \lambda_H) \circ \delta_H) \circ A \otimes H) &= \varepsilon_H \otimes A \otimes H, \\
(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) &= A \otimes H \otimes \varepsilon_A, \\
(\mu_A \otimes H) \circ (\mu_A \otimes \Psi) \circ (A \otimes \Psi \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) \circ (\lambda_A \otimes A) &= A \otimes H \otimes \varepsilon_A,
\end{align*}
\]
In [13] Theorem 3.1 the author prove that, for an a-comonoidal distributive law $\Psi$ of $H$ over $A$, the wreath product $A \otimes H$ built on $A \otimes H$ with the wreath product magma

$$(27) \quad \mu_{A \otimes \Psi H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H)$$

unit $\eta_{A \otimes \Psi H} = \eta_A \otimes \eta_H$, counit $\varepsilon_{A \otimes \Psi H} = \varepsilon_A \otimes H$, coproduct $\delta_{A \otimes \Psi H} = \delta_{A \otimes H}$ and antipode

$$(28) \quad \lambda_{A \otimes \Psi H} = \Psi \circ (\lambda_A \otimes \lambda_H) \circ c_{H,A},$$

is a Hopf quasigroup.

**Example 3.4.** As was proved in [13], the Hopf quasigroups defined by the twisted double method in [12] are examples of wreath product Hopf quasigroups. Also, the smash products of Hopf quasigroups in the sense of [7] are examples of wreath product Hopf quasigroups as well as the twisted smash products defined in [11].

**Example 3.5.** In [13] Example 2.8 the author proved that the theory of double cross products of Hopf quasigroups, introduced in [4], provides an interesting family of a-comonoidal distributive laws. In the following lines we recall the main details.

Let $H$ be a Hopf quasigroup. The pair $(M, \varphi_M)$ is said to be a left $H$-quasimodule if $M$ is an object in $\mathcal{C}$ and $\varphi_M : H \otimes M \rightarrow M$ is a morphism in $\mathcal{C}$ (called the action) satisfying

$$(29) \quad \varphi_M \circ (\eta_H \otimes M) = id_M$$

and

$$(30) \quad \varphi_M \circ (H \otimes \varphi_M) \circ (((H \otimes \lambda_H) \circ \delta_H) \otimes M) = \varepsilon_H \otimes M = \varphi_M \circ (\lambda_H \otimes \varphi_M) \circ (\delta_H \otimes M).$$

Given two left $H$-quasimodules $(M, \varphi_M)$, $(N, \varphi_N)$ and a morphism $f : M \rightarrow N$ in $\mathcal{C}$, we will say that $f$ is a morphism of left $H$-quasimodules if

$$(31) \quad \varphi_N \circ (H \otimes f) = f \circ \varphi_M.$$ 

We will say that a unital magma $A$ is a left $H$-quasimodule magma if it is a left $H$-quasimodule with action $\varphi_A : H \otimes A \rightarrow A$ and the following equalities

$$(32) \quad \varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A,$$

$$(33) \quad \mu_A \circ \varphi_{A \otimes A} = \varphi_A \circ (H \otimes \mu_A),$$

hold, i.e., $\eta_A$ and $\mu_A$ are quasimodule morphisms where the action on $A \otimes A$ is defined by $\varphi_{A \otimes A} = (\varphi_A \otimes \varphi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$

A comonoid $A$ is a left $H$-quasimodule comonoid if it is a left $H$-quasimodule with action $\varphi_A$ and

$$(34) \quad \varepsilon_A \circ \varphi_A = \varepsilon_H \circ \varepsilon_A,$$

$$(35) \quad \delta_A \circ \varphi_A = \varphi_{A \otimes A} \circ (H \otimes \delta_A),$$

hold, i.e., $\varepsilon_A$ and $\delta_A$ are quasimodule morphisms.

Replacing (30) by the equality

$$(36) \quad \varphi_M \circ (H \otimes \varphi_M) = \varphi_M \circ (\mu_H \otimes M),$$

we have the definition of left $H$-module and the ones of left $H$-module magma and comonoid. Note that the pair $(H, \mu_H)$ is not an $H$-module but it is an $H$-quasimodule. Morphisms between left $H$-modules are defined as for $H$-quasimodules and we denote the category of left $H$-modules by $H\text{Mod}$. Obviously we have similar definitions for the right side.

In [4] Corollary 5.4 the authors prove that, if $A$, $H$ are Hopf quasigroups, $(A, \varphi_A)$ is a left $H$-module comonoid, $(H, \phi_H)$ is a right $A$-module comonoid and

$$\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A},$$
the following assertions are equivalent:

(i) The double cross product $A \bowtie H$ built on the object $A \otimes H$ with product
\[ \mu_{A \bowtie H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H) \]
and tensor product unit, counit and coproduct, is a Hopf quasigroup with antipode
\[ \lambda_{A \bowtie H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H}. \]

(ii) The equalities
\[ \varphi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A, \]
\[ \phi_H \circ (\eta_H \otimes A) = \eta_H \otimes \varepsilon_A, \]
\[ (\phi_H \otimes \varphi_A) \circ \delta_{H \otimes A} = c_{A,H} \circ \Psi, \]
\[ \varphi_A \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A \otimes A) = \mu_A \circ (A \otimes \varphi_A) \circ ((\Psi \circ (\lambda_H \otimes \lambda_A)) \otimes A), \]
\[ \mu_H \circ (\phi_H \otimes \mu_H) \circ (\lambda_H \otimes \Psi \otimes H) \circ (\delta_H \otimes A \otimes H) = \varepsilon_H \otimes \varepsilon_A \otimes H, \]
\[ \phi_H \circ (\mu_H \otimes A) \circ (H \otimes \lambda_H \otimes \lambda_A) = \mu_H \circ (\phi_H \otimes H) \circ (H \otimes (\Psi \circ (\lambda_H \otimes \lambda_A))), \]
\[ \mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes \lambda_A) \circ (A \otimes H \otimes \delta_A) = A \otimes \varepsilon_H \otimes \varepsilon_A, \]
\[ \mu_A \circ (\mu_A \otimes \varphi_A) \circ (A \otimes \Psi \otimes A) \circ (A \otimes H \otimes ((\lambda_A \otimes A) \otimes \delta_A)) = A \otimes \varepsilon_H \otimes \varepsilon_A, \]

hold.

Under the conditions (37)-(45), we can prove that $\Psi$ is an example of a-cocommutative distributive law of $H$ over $A$ (see [13, Example 2.8]) and then $A \bowtie H$ an example of wreath product associated to $\Psi$.

Using the terminology that can be found in the literature on double cross products of Hopf algebras, we will say that, if $A$ and $H$ are in the conditions of this example, $(A, H)$ is a matched pair of Hopf quasigroups.

Following, [17, Definition 4.12], we will say that a Hopf quasigroup $A$ is a left $H$-module Hopf quasigroup if it is a a left $H$-module monoid and comonoid. By, [17, Proposition 4.14], we know that if we denote the action of $H$ over $A$ by $\varphi_A$ and we assume that $H$ is cocommutative, there is a left cross product Hopf quasigroup $A \rtimes H$ built on $A \otimes H$ with tensor coproduct and unit and
\[ \mu_{A \rtimes H} = (\mu_A \otimes \mu_H) \circ (A \otimes \Psi \otimes H), \quad \mu_{A \rtimes H} = \Psi \circ (\lambda_H \otimes \lambda_A) \circ c_{A,H} \]
where $\Psi = (\varphi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)$. Then, the Hopf quasigroup $A \rtimes H$ is an example of double cross product of Hopf quasigroups with $\phi_H = H \otimes \varepsilon_A$.

The previous definitions and results can be extended to the wreath coproduct setting of Hopf coquasigroups by dualization as follows.

**Definition 3.6.** Let $D, B$ be Hopf coquasigroups. A morphism $\Omega : D \otimes B \to B \otimes D$ is said to be a codistributive law of $D$ over $B$ if the following identities
\[ (B \otimes \lambda_B \otimes \lambda_D) \circ (\delta_B \otimes D) \circ \Omega = (B \otimes \lambda_B \otimes \lambda_D) \circ (B \otimes \Omega) \circ (D \otimes \delta_B), \]
\[ (\lambda_B \otimes \lambda_D \otimes D) \circ (B \otimes D) \circ \Omega = (\lambda_B \otimes \lambda_D \otimes D) \circ (\Omega \otimes D) \circ (D \otimes \Omega) \circ (\delta_D \otimes B), \]
\[ (\varepsilon_B \otimes D) \circ \Omega = D \otimes \varepsilon_B, \]
(49) \[(B \otimes \varepsilon_D) \circ \Omega = \varepsilon_D \otimes B,\]
hold.

If the antipodes of \(D\) and \(B\) are isomorphisms (for example, if \(D\) and \(B\) are finite objects), the identities (46) and (47) are equivalent to

(50) \[(\delta_B \otimes D) \circ \Omega = (B \otimes \Omega) \circ (\Omega \otimes B) \circ (D \otimes \delta_B),\]

(51) \[(B \otimes \delta_D) \circ \Omega = (\lambda_B \otimes \lambda_D \otimes D) \circ (\Omega \otimes D) \circ (D \otimes \Omega) \circ (\delta_D \otimes B),\]

respectively. Then, in this case, the conditions of the definition of codistributive law for Hopf quasigroups are the ones that we can find for codistributive laws between comonoids, i.e., \(\Omega\) is compatible with the counit and the coproduct of \(D\) and \(B\).

**Definition 3.7.** Let \(D\), \(B\) be Hopf coquasigroups and let \(\Omega : D \otimes B \to B \otimes D\) be a codistributive law of \(D\) over \(B\). The codistributive law \(\Omega\) is said to be monoidal if it is a monoid morphism, i.e., the following identities

(52) \[\Omega \circ \mu_{D \otimes B} = \mu_{B \otimes D} \circ (\Omega \otimes \Omega),\]

(53) \[\Omega \circ (\eta_D \otimes \eta_B) = \eta_B \otimes \eta_D,\]

hold.

**Definition 3.8.** Let \(D\), \(B\) be Hopf coquasigroups and let \(\Omega : D \otimes B \to B \otimes D\) be a monoidal codistributive law of \(D\) over \(B\). We will say that \(\Omega\) is an \(a\)-monoidal codistributive law of \(D\) over \(B\) if the following identities

(54) \[(D \otimes B \otimes (\mu_D \circ (D \otimes \lambda_D))) \circ (D \otimes \Omega \otimes D) \circ (\delta_D \otimes \Omega) \circ (\delta_D \otimes B) = D \otimes B \otimes \eta_D,\]

(55) \[(D \otimes B \otimes (\mu_D \circ (\lambda_D \otimes D))) \circ (D \otimes \Omega \otimes D) \circ (\delta_D \otimes \Omega) \circ (\delta_D \otimes B) = D \otimes B \otimes \eta_D,\]

(56) \[((\mu_B \circ (B \otimes \lambda_B)) \otimes D \otimes B) \circ (B \otimes \Omega \otimes B) \circ (\Omega \otimes \delta_B) \circ (D \otimes \delta_B) = \eta_B \otimes D \otimes B,\]

(57) \[((\mu_B \circ (\lambda_B \otimes B)) \otimes D \otimes B) \circ (B \otimes \Omega \otimes B) \circ (\Omega \otimes \delta_B) \circ (D \otimes \delta_B) = \eta_B \otimes D \otimes B,\]

hold.

Note that, if \(D\) and \(B\) are Hopf algebras and \(\Omega : D \otimes B \to B \otimes D\) is a codistributive law between the comonoids \(D\) and \(B\), the equalities (54), (55), (56) and (57) always hold. On the other hand, it is obvious that, if \(H\), \(A\) are finite Hopf quasigroups, \(\Psi\) is an \(a\)-comonoidal distributive law of \(H\) over \(A\) iff \(\Psi^*\) is an \(a\)-monoidal codistributive law of \(H^*\) over \(A^*\).

If we dualize [13, Theorem 3.1] we have the following: For an \(a\)-monoidal codistributive law \(\Omega : D \otimes B \to B \otimes D\) of \(D\) over \(B\), the wreath coproduct \(D \otimes_\Omega B\) built on \(D \otimes B\) with the wreath coproduct comagma

(58) \[\delta_{D \otimes_\Omega B} = (D \otimes \Omega \otimes B) \circ (\delta_D \otimes \delta_B)\]
counit \(\varepsilon_{D \otimes_\Omega B} = \varepsilon_D \otimes \varepsilon_B\), unit \(\eta_{D \otimes_\Omega B} = \eta_{D \otimes B}\), product \(\mu_{D \otimes_\Omega B} = \mu_{D \otimes B}\) and antipode

(59) \[\lambda_{D \otimes_\Omega B} = \varepsilon_{B,D} \circ (\lambda_B \otimes \lambda_D) \circ \Omega,\]
is a Hopf coquasigroup.
Example 3.9. The dual of Example 3.5 provides a family of \( a \)-monoidal codistributive laws. In this case we work with the theory of double cross coproducts of Hopf coquasigroups.

Let \( D \) be a Hopf coquasigroup. The pair \( (P, \rho_P) \) is said to be a right \( D \)-quasicomodule if \( P \) is an object in \( C \) and \( \rho_P : P \to P \otimes D \) is a morphism in \( C \) (called the coaction) satisfying

\[
(P \otimes \varepsilon_D) \circ \rho_P = \text{id}_P
\]

and

\[
(P \otimes (\mu_D \circ (\lambda_D \otimes D))) \circ (\rho_P \otimes D) \circ \rho_P = P \otimes \eta_D = (P \otimes \mu_D) \circ (\rho_P \otimes \lambda_D) \circ \rho_P.
\]

Given two right \( D \)-quasicomodules \( (P, \rho_P) \), \( (Q, \rho_Q) \) and one morphism \( g : P \to Q \) in \( C \), we will say that \( f \) is a morphism of right \( H \)-quasicomodules if

\[
(g \otimes D) \circ \rho_P = \rho_Q \circ g.
\]

We will say that a counital comagma \( B \) is a right \( D \)-quasicomodule comagma if it is a right \( D \)-quasicomodule with coaction \( \rho_B : B \to B \otimes D \) and the following equalities

\[
(\varepsilon_B \otimes D) \circ \rho_B = \varepsilon_B \otimes \eta_D,
\]

\[
\rho_B \otimes B \circ \delta_B = (\delta_B \otimes D) \circ \rho_B,
\]

hold, i.e., \( \varepsilon_B \) and \( \delta_B \) are morphisms of right quasicomodules where the coaction on \( B \otimes B \) is defined by \( \rho_B \otimes B = (B \otimes B \otimes \mu_D) \circ (B \otimes \varepsilon_{B,D} \otimes D) \circ (\rho_B \otimes \rho_B) \).

A monoid \( B \) is a right \( D \)-quasicomodule monoid if it is a right \( D \)-quasicomodule with coaction \( \rho_B \) and

\[
\rho_B \circ \eta_B = \eta_B \otimes \eta_D,
\]

\[
\rho_B \circ \mu_B = (\mu_B \otimes D) \circ \rho_{B \otimes B},
\]

hold, i.e., \( \eta_B \) and \( \mu_B \) are right quasicomodule morphisms.

Replacing (61) by the equality

\[
(\rho_P \otimes D) \circ \rho_P = (P \otimes \delta_D) \circ \rho_P,
\]

we have the definition of right \( D \)-comodule and the ones of right \( D \)-comodule comagma and monoid. Note that the pair \( (D, \delta_D) \) is not an \( D \)-comodule but it is an \( D \)-quasicomodule. Morphisms between right \( D \)-comodules are defined as for \( D \)-quasicomodules and we denote the category of right \( D \)-comodules by \( \text{Mod}^D \). Obviously we have similar definitions for the left side.

Then by the dual proof of [4, Corollary 5.4] we can prove the following. Let \( B, D \) be Hopf coquasigroups. Let \( (B, \rho_B) \) be a right \( D \)-comodule monoid and let \( (D, r_D) \) be a left \( B \)-comodule monoid and write

\[
\Omega = \mu_{B \otimes D} \circ (r_D \otimes \rho_B).
\]

Then, the following assertions are equivalent:

(i) The double cross coproduct \( D \otimes B \) built on the object \( D \otimes B \) with coproduct

\[
\delta_{D \otimes B} = (D \otimes \Omega \otimes B) \circ (\delta_D \otimes \delta_B)
\]

and tensor product counit, unit and product, is a Hopf coquasigroup with antipode

\[
\lambda_{D \otimes B} = \varepsilon_{B,D} \circ (\lambda_B \otimes \lambda_D) \circ \Omega.
\]
The equalities
\begin{align}
(\varepsilon_B \otimes D) \circ \rho_B &= \varepsilon_B \otimes \eta_D,
(B \otimes \varepsilon_D) \circ \tau_D &= \eta_B \otimes \varepsilon_D,
\mu_{B \otimes D} \circ (\rho_B \otimes r_D) &= \Omega \circ c_{B,D},
(B \otimes \lambda_B \otimes \lambda_D) \circ (\delta_B \otimes D) \circ \rho_B &= (B \otimes ((\lambda_B \otimes \lambda_D) \circ \Omega) \circ (\rho_B \otimes B) \circ \delta_B,
(D \otimes B \otimes \mu_D) \circ (D \otimes \Omega \otimes \lambda_D) \circ (\delta_D \otimes r_D) \circ \delta_D &= D \otimes \eta_B \otimes \eta_D,
(D \otimes B \otimes (\mu_D \circ (\lambda_D \otimes D))) \circ (D \otimes \Omega \otimes D) \circ (\delta_D \otimes r_D) \circ \delta_D &= D \otimes \eta_B \otimes \eta_D,
(\lambda_B \otimes \lambda_D \otimes D) \circ (B \otimes \delta_D) \circ r_D &= (((\lambda_B \otimes \lambda_D) \circ \Omega) \circ D) \circ (D \otimes r_D) \circ \delta_D,
(\mu_B \otimes D \otimes B) \circ (\lambda_B \otimes \Omega \otimes B) \circ (\rho_B \otimes \delta_B) \circ \delta_B &= \eta_B \otimes \eta_D \otimes B,
((\mu_B \circ (B \otimes \lambda_B)) \otimes D \otimes B) \circ (B \otimes \Omega \otimes B) \circ (\rho_B \otimes \delta_B) \circ \delta_B &= \eta_B \otimes \eta_D \otimes B,
\end{align}
hold.

Under the conditions (68)-(76), we can prove that $\Omega$ is an example of a-monoidal codistributive law of $D$ over $B$ and then $D \bowtie B$ an example of wreath coproduct associated to $\Omega$.

As in the Hopf quasigroup setting, if $B$ and $D$ are in the conditions of this example, we will say that $(B, D)$ is a matched pair of Hopf coquasigroups.

Recall that cross coproducts Hopf coquasigroups (see [17, Definition 5.12, Proposition 5.14]) are examples of double cross coproducts of Hopf coquasigroups because it is the dual of cross products Hopf quasigroups (see the last paragraph of Example 3.5).

### 4. Factorizations

In the classical theory of Hopf algebras over a field $F$ it is said that a Hopf algebra $X$ factorises as $X = AH$ if there exists Hopf subalgebras $A$ and $H$, with inclusions morphisms $i_A : A \to X$, $i_H : H \to X$, such that the morphism $\omega_X = \mu_X \circ (i_A \otimes i_H) : A \otimes H \to X$ is an isomorphism. Dually, if $X$ is finite dimensional, $\omega_X$ is an isomorphism iff the dual morphism $\omega_X^* = (i_H^* \otimes i_A^*) \circ \delta_B^*$ is an isomorphism. By [10, Theorem 7.2.3], $X = AH$ iff $X = A \bowtie H$, i.e., $X$ is obtained as a double cross product of Hopf algebras (or $(A, H)$ is a matched pair of Hopf algebras), equivalently, $X^* = H^* \bowtie A^*$ ($((A^*, H^*)$ is a comatched pair of Hopf algebras).

The main target of this section is to prove that for Hopf quasigroups we have similar results. In other words, the exact factorization problem for Hopf quasigroups is equivalent to the existence of a double cross product of Hopf quasigroups.

**Definition 4.1.** Let $X$ be a Hopf quasigroup in $C$. Let $H$, $A$ be Hopf subquasigroups of $X$ with inclusion morphisms $i_H : H \to X$, $i_A : A \to X$ respectively. Let $\omega_X$ and $\theta_X$ be the morphisms defined by

$$
\omega_X = \mu_X \circ (i_A \otimes i_H) : A \otimes H \to X, \quad \theta_X = \mu_X \circ (i_H \otimes i_A) : H \otimes A \to X.
$$

We will say that $X$ factorizes as $X = AH$ if $\omega_X$ is an isomorphism and the following identities

\begin{align}
\mu_X \circ (\omega_X \otimes X) &= \mu_X \circ (i_A \otimes (\mu_X \circ (i_H \otimes X))),
\mu_X \circ (X \otimes \omega_X) &= \mu_X \circ ((\mu_X \circ (X \otimes i_A)) \otimes i_H),
\mu_X \circ ((\theta_X \circ (\lambda_H \otimes \lambda_A)) \otimes X) &= \mu_X \circ ((i_H \circ \lambda_H) \otimes (\mu_X \circ ((i_A \circ \lambda_A) \otimes X)),
\end{align}

hold.
(80) \[ \mu_X \circ (X \otimes (\theta_X \circ (\lambda_H \otimes \lambda_A))) = \mu_X \circ ((\mu_X \circ (X \otimes (i_H \circ \lambda_H))) \otimes (i_A \circ \lambda_A)) \]
hold.

Note that, if \( H \) and \( A \) are finite we can remove the antipodes in (79) and (80). Then these identities become in

(81) \[ \mu_X \circ (\theta_X \otimes X) = \mu_X \circ (i_H \otimes (\mu_X \circ (i_A \otimes X))), \]

(82) \[ \mu_X \circ (X \otimes \theta_X) = \mu_X \circ ((\mu_X \circ (X \otimes i_H)) \otimes i_A). \]

Finally, note that \( \omega_X \) and \( \theta_X \) are comonoidal morphisms because \( i_A, i_H \) are comonoidal morphisms and (2) and (3) holds for the Hopf quasigroup \( X \). Obviously, if \( \omega_X \) is a comonoidal isomorphism, its dual is a monoidal isomorphism.

**Example 4.2.** Suppose that \((A,H)\) is a matched pair of Hopf quasigroups. By Example 3.5, the double cross product \(A \bowtie H\) is a Hopf quasigroup. The morphisms \(i_A = A \otimes \eta_H : A \to A \bowtie H\) and \(i_H = \eta_A \otimes H : H \to A \bowtie H\) are morphisms of Hopf quasigroups because (17), (18) holds and (1) and (2) also hold for \( H \) and \( A \). By the properties of the units and (18), we obtain that

\[ \omega_{A \bowtie H} = id_{A \bowtie H}, \quad \theta_{A \bowtie H} = \Psi. \]

Then, by (18),

\[ \mu_{A \bowtie H} \circ (i_A \otimes (\mu_{A \bowtie H} \circ (i_H \circ A \bowtie H))) = \mu_{A \bowtie H} = \mu_{A \bowtie H} \circ (\omega_{A \bowtie H} \otimes A \bowtie H) \]

and (77) holds. Similarly, by (17), we obtain (18) because

\[ \mu_{A \bowtie H} \circ ((\mu_{A \bowtie H} \circ (A \bowtie H \circ i_A)) \otimes i_H) = \mu_{A \bowtie H} = \mu_{A \bowtie H} \circ (A \bowtie H \circ \omega_{A \bowtie H}). \]

On the other hand,

\[ \mu_{A \bowtie H} \circ ((\theta_{A \bowtie H} \circ (\lambda_H \otimes \lambda_A)) \otimes A \bowtie H) = (\mu_A \otimes \mu_H) \circ (A \otimes (\Psi \otimes H) \circ (\Psi \circ (\lambda_H \otimes \lambda_A)) \otimes A \otimes H) \quad \text{(by the unit properties)} \]

\[ = (A \otimes \mu_H) \circ ((\Psi \circ (H \otimes \mu_A) \circ (\lambda_H \otimes \lambda_A)) \otimes H) \quad \text{(by 15)} \]

\[ = \mu_{A \bowtie H} \circ ((i_H \circ \lambda_H) \otimes (\mu_{A \bowtie H} \circ (i_A \circ \lambda_A) \otimes A \bowtie H)) \quad \text{(13)}, \]

and (79) holds. Similarly,

\[ \mu_{A \bowtie H} \circ (A \bowtie H \circ (\theta_{A \bowtie H} \circ (\lambda_H \otimes \lambda_A))) = (\mu_A \otimes \mu_H) \circ (A \otimes (\Psi \otimes H) \circ (\Psi \circ (\lambda_H \otimes \lambda_A))) \quad \text{(by the unit properties)} \]

\[ = (\mu_A \otimes H) \circ (A \otimes (\Psi \circ (\mu_H \circ A) \circ (H \otimes \lambda_H \otimes \lambda_A))) \quad \text{(by 16)} \]

\[ = \mu_{A \bowtie H} \circ ((\mu_{A \bowtie H} \circ (A \bowtie H \circ (i_H \circ \lambda_H))) \otimes (i_A \circ \lambda_A)) \quad \text{(by 14)}, \]

and, as a consequence, (80) also holds.

Therefore, any double cross product of Hopf quasigroups induces an example of factorization.

**Theorem 4.3.** Let \( H, A \) be Hopf subquasigroups of a Hopf quasigroup \( X \) with inclusion morphisms \( i_H : H \to X, i_A : A \to X \) respectively. If \( X \) factorises as \( X = AH \), the morphism

\[ \Psi = \omega^{-1}_X \circ \theta_X : H \otimes A \to A \otimes H \]
is a comonoidal distributive law of \( H \) over \( A \). Moreover, if the antipodes of \( H \) and \( A \) are isomorphisms \( \Psi \) is an a-comonoidal distributive law of \( H \) over \( A \).

**Proof.** The condition (15) of Definition 3.1 holds because:

\[ \omega_X \circ (\mu_A \otimes H) \circ (A \otimes (\Psi \otimes A) \circ (\Psi \circ A) \circ (\lambda_H \otimes \lambda_A \otimes A) = \mu_X \circ ((\mu_X \circ (i_A \otimes i_H)) \otimes i_A) \circ (A \otimes (\omega^{-1}_X \circ \theta_X) \circ ((\omega^{-1}_X \circ \theta_X \circ (\lambda_H \otimes \lambda_A)) \otimes A) \quad \text{(by definition of \( \omega_X \) and the condition of monoid morphism for \( i_A \))} \]

\[ = \mu_X \circ ((i_A \otimes (\omega^{-1}_X \circ \theta_X)) \circ ((\omega^{-1}_X \circ \theta_X \circ (\lambda_H \otimes \lambda_A)) \otimes A) \quad \text{(by 79)} \]

\[ = \mu_X \circ ((i_A \otimes (\mu_X \circ (i_H \otimes i_A))) \circ ((\omega^{-1}_X \circ \theta_X \circ (\lambda_H \otimes \lambda_A)) \otimes A) \quad \text{(by definition of \( \theta_X \))} \]
and this implies that (17) of Definition 3.1 holds. In the same way, by the unit properties and the condition of monoid morphism for \( i_A \), we obtain (26).

By a similar proof, using that \( H \) is a monoidal morphism, (77) instead (78), (78) instead (77) and (80) instead (79), we obtain that \( \Psi \) satisfies (16) of Definition 3.1. Also, by the unit properties and the condition of monoid morphism for \( i_A \) we have that

\[
\Psi \circ (H \otimes \eta_A) = \omega_X^{-1} \circ i_H = \omega_X^{-1} \circ \omega_X \circ (\eta_A \otimes H) = \eta_A \otimes H
\]

and this implies that (17) of Definition 3.1 holds. In the same way, by the unit properties and the condition of monoid morphism for \( i_H \), we prove that (18) of Definition 3.1 holds. Therefore, \( \Psi \) is distributive law of \( H \) over \( A \).

On the other hand, \( \Psi \) is comonoidal because it is a composition of comonoid morphisms.

Finally, if the antipodes of \( H \) and \( A \) are isomorphisms, \( \Psi \) is \( a \)-comonoidal distributive law because, as in the previous case, if we compose with \( \omega_X \), we have

\[
\omega_X \circ (A \otimes \mu_H) \circ (\Psi \otimes \mu_H) \circ (H \otimes \Psi \otimes H) \circ ((\lambda_H \otimes H) \otimes \delta_H) \otimes \lambda_H \otimes A \otimes H)
\]

and then, (23) of Definition 3.3 holds. By a similar proof we can show that (24) of Definition 3.3 also holds. On the other hand, (26) follows from

\[
\omega_X \circ (A \otimes \mu_H) \circ (\Psi \otimes \mu_H) \circ (A \otimes H \otimes ((\lambda_H \otimes A) \otimes \delta_A))
\]

and similarly, we obtain (26).

As a consequence of the previous theorem we obtain the following result.

**Theorem 4.4.** Let \( H, A \) be Hopf subquasigroups of a Hopf quasigroup \( X \) such that the antipodes of \( H \) and \( A \) are isomorphisms. Assume that \( X \) factorises as \( X = AH \). Then, \( \omega_X \) is an isomorphism of Hopf quasigroups between the wreath product \( A \odot_X H \) and \( X \), where \( \Psi \) is the \( a \)-comonoidal distributive law defined in the previous theorem.

**Proof.** To complete the proof we only need to show that \( \omega_X \) is an isomorphism of unital magmas. Indeed, trivially \( \omega_X \circ \eta_{A \odot_X H} = \eta_X \). On the other hand,

\[
\mu_X \circ (\omega_X \otimes \omega_X)
\]
In the next theorem we will prove that, in the same way as in the case of Hopf algebras, every factorization of Hopf quasigroups comes from a matched pair of Hopf quasigroups.

**Theorem 4.5.** Let $H$, $A$, $X$ be Hopf quasigroups such that the antipodes of $H$ and $A$ are isomorphisms. If $X$ factorizes as $X = AH$, there exists a matched pair of Hopf quasigroups $(A, H)$ such that $X$ is isomorphic to $A \bowtie H$ as Hopf quasigroups.

**Proof.** Let $\Psi$ be the morphism defined in Theorem 4.3. Define the actions by

$$\varphi_A = (A \otimes \varepsilon_H) \circ \Psi, \quad \phi_H = (\varepsilon_A \otimes H) \circ \Psi.$$  

Then, $(A, H)$ is a matched pair of Hopf quasigroups. Indeed, $(A, \varphi_A)$ is a left $H$-module because, by (18) and (11) for $H$, we have

$$\varphi_A \circ (\eta_H \otimes A) = (A \otimes \varepsilon_H) \circ \Psi \circ (\eta_H \otimes A) = A \otimes (\varepsilon_H \circ \eta_H) = \text{id}_A,$$

and, by (2) for $H$ and (21),

$$\varphi_A \circ (H \otimes \varphi_A) = (A \otimes (\varepsilon_H \otimes \mu_H)) \circ (\Psi \otimes H) \circ (H \otimes \Psi) = \varphi_A \circ (\mu_H \otimes A).$$

Also, using that $\psi$ is a comonoid morphism, the naturality of $c$ and the properties of the counits, we have that $(A, \varphi_A)$ is a left $H$-module comonoid. Similarly we can prove that $(H, \phi_H)$ is a right $H$-module comonoid.

On the other hand, (37) follows from (17) and (38) from (18). The identity (39) is a consequence of the condition of comonoid morphism of $\Psi$ and the properties of the counits. On the other hand, (40) follows by (13) (or (19) because the antipodes are isomorphisms). The equality (41) follows by (23) and (12) can be proved thanks to (21). Similarly to (40), (13) follows by (16) (or (20) because the antipodes are isomorphisms). Finally, (42) is a consequence of (26) and (43) follows by (25).

Note that, by the condition of comonoid morphism of $\Psi$, the naturality of $c$ and the properties of the units, we have that

$$\Psi = (\varphi_A \otimes \phi_H) \circ \delta_{H \otimes A}.$$

Therefore, $A \otimes \Psi H = A \bowtie H$ and, by Theorem 4.4, $X$ is isomorphic to $A \bowtie H$ as Hopf quasigroups.

**Theorem 4.6.** Let $H$, $A$, $X$ be Hopf quasigroups such that the antipodes of $H$ and $A$ are isomorphisms. Then, $X$ factorizes as $X = AH$ iff there exists a matched pair of Hopf quasigroups $(A, H)$ such that $X$ is isomorphic to $A \bowtie H$ as Hopf quasigroups.

**Proof.** The proof follows by Theorem 4.5 and Example 4.2.

Of course, the definitions and results of this section admit a dual version in the Hopf coquasigroup setting. We will state everything below, omitting the proofs, since these follow by duality from those done in the context of Hopf quasigroups.

**Definition 4.7.** Let $Y$ be a Hopf coquasigroup in $\mathcal{C}$. Let $D$, $B$ be Hopf coquasigroups such that there exist Hopf coquasigroup epimorphisms $p_D : Y \to D$ and $p_B : Y \to B$. Let $u_Y$ and $v_Y$ be the morphisms defined by

$$u_Y = (p_D \otimes p_B) \circ \delta_Y : Y \to D \otimes B, \quad v_Y = (p_B \otimes p_D) \circ \delta_Y : Y \to B \otimes D.$$
We will say that $Y$ cofactorizes as $Y = D \bullet B$ if $u_Y$ is an isomorphism and the following identities hold.

\begin{align}
(83) \quad (Y \otimes u_Y) \circ \delta_Y &= (((Y \otimes p_D) \circ \delta_Y) \otimes p_B) \circ \delta_Y,
(84) \quad (u_Y \otimes Y) \circ \delta_Y &= (p_D \otimes ((p_B \otimes Y) \circ \delta_Y)) \circ \delta_Y,
(85) \quad (Y \otimes ((\lambda_B \otimes \lambda_D) \circ v_Y)) \circ \delta_Y &= (((Y \otimes (\lambda_B \circ p_B)) \circ \delta_Y) \otimes (\lambda_D \circ p_D)) \circ \delta_Y,
(86) \quad (((\lambda_B \otimes \lambda_D) \circ v_Y) \otimes Y) \circ \delta_Y &= ((\lambda_B \circ p_B) \otimes (((\lambda_D \circ p_D) \otimes Y) \circ \delta_Y)) \circ \delta_Y,
\end{align}

hold.

Note that, if $D$ and $B$ are finite we can remove the antipodes in (85) and (86). Then these identities became in

\begin{align}
(87) \quad (Y \otimes v_Y) \circ \delta_Y &= (((Y \otimes p_B) \circ \delta_Y) \otimes p_D) \circ \delta_Y,
(88) \quad (v_Y \otimes Y) \circ \delta_Y &= (p_B \otimes ((p_D \otimes Y) \circ \delta_Y)) \circ \delta_Y.
\end{align}

Finally, note that $u_Y$ and $v_Y$ are monoid morphisms because $p_D$, $p_B$ are monoid morphisms and (2) and (4) holds for the Hopf coquasigroup $Y$. Obviously if $u_Y$ is a monoid isomorphism, its dual is a comonoid isomorphism.

**Example 4.8.** Suppose that $(B, D)$ is a comatched pair of Hopf coquasigroups. By Example 3.9 the double cross coproduct $D \otimes B$ is a Hopf coquasigroup. The morphisms $p_B = \varepsilon_D \otimes B : D \otimes B \to B$ and $p_D = D \otimes \varepsilon_B : D \otimes B \to D$ are morphisms of Hopf coquasigroups such that

\[ u_{D \otimes B} = id_{D \otimes B}, \quad v_{D \otimes B} = \Omega \]

and the equalities (33)-(36) hold. Therefore, any double cross coproduct of Hopf coquasigroups induces an example of cofactorization.

**Theorem 4.9.** Let $D$, $B$ and $Y$ be Hopf coquasigroups such that there exist Hopf coquasigroup epimorphisms $p_D : Y \to D$ and $p_B : Y \to B$. If $Y$ cofactorises as $Y = D \bullet B$, the morphism

\[ \Omega = v_Y \circ u_Y^{-1} : D \otimes B \to B \otimes D \]

is a monoidal codistributive law of $D$ over $B$. Moreover, if the antipodes of $D$ and $B$ are isomorphisms $\Omega$ is an $a$-monoidal codistributive law of $D$ over $B$.

**Theorem 4.10.** Let $D$, $B$ and $Y$ be Hopf coquasigroups such that there exist Hopf coquasigroup epimorphisms $p_D : Y \to D$, $p_B : Y \to B$ and suppose that the antipodes of $D$ and $B$ are isomorphisms. Assume that $Y$ cofactorises as $Y = D \bullet B$. Then, $u_Y$ is an isomorphism of Hopf coquasigroup between $Y$ and the wreath coproduct $D \otimes_\Omega B$, where $\Omega$ is the $a$-monoidal codistributive law defined in the previous theorem.

**Theorem 4.11.** Let $D$, $B$, $Y$ be Hopf coquasigroups such that the antipodes of $D$ and $B$ are isomorphisms. If $Y$ cofactorises as $Y = D \bullet B$, there exists a comatched pair of Hopf coquasigroups $(B, D)$ such that $Y$ is isomorphic to $D \otimes B$ as Hopf coquasigroups.

**Theorem 4.12.** Let $D$, $B$, $Y$ be Hopf coquasigroups such that the antipodes of $D$ and $B$ are isomorphisms. Then, $Y$ cofactorises as $Y = D \bullet B$ iff there exists a comatched pair of Hopf coquasigroups $(B, D)$ such that $Y$ is isomorphic to $D \otimes B$ as Hopf coquasigroups.

**Example 4.13.** Let $S_3 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ be the nonabelian group where $\sigma_0$ is the identity, $o(\sigma_1) = o(\sigma_2) = o(\sigma_3) = 2$ and $o(\sigma_4) = o(\sigma_5) = 3$. Let $u$ be an additional element such that $u^2 = 1$. Then, by [10] Theorem 1], the set

\[ L = M(S_3, 2) = \{\sigma_i u^\alpha ; \alpha = 0, 1\} \]
is an I.P. loop where the product is defined by
\[ \sigma_i u^\alpha \bullet \sigma_j u^\beta = (\sigma_i^\nu \sigma_j^\mu) u^{\alpha + \beta}, \quad \nu = (-1)^\beta, \ \mu = (-1)^{\alpha + \beta} \]
and the inverse by
\[ (\sigma_i u^\alpha)^{-1} = \sigma_i^{(-1)^{\alpha + 1}} u^\alpha. \]

Let \( \mathbb{F} \) be a field such that \( \text{Char}(\mathbb{F}) \neq 2 \) and denote the tensor product over \( \mathbb{F} \) as \( \otimes \). By Examples \[2.3\] we have that \( A = \mathbb{F}L \) is a cocommutative Hopf quasigroup.

On the other hand, let \( H_4 \) be the 4-dimensional Taft Hopf algebra. This Hopf algebra is the smallest non-commutative, non-cocommutative Hopf algebra. The basis of \( H_4 \) is \( \{1, x, y, w = xy\} \) and the multiplication table is defined by

|   | x | y | w |
|---|---|---|---|
| x | 1 | w | y |
| y | -w | 0 | 0 |
| w | -y | 0 | 0 |

The coproduct of \( H_4 \) is given by
\[ \delta_{H_4}(x) = x \otimes x, \ \delta_{H_4}(y) = y \otimes x + 1 \otimes y, \ \delta_{H_4}(w) = w \otimes 1 + x \otimes w, \]
\[ \varepsilon_{H_4}(x) = 1_x, \ \varepsilon_{H_4}(y) = \varepsilon_{H_4}(w) = 0, \]
and the antipode \( \lambda_{H_4} \) is described by
\[ \lambda_{H_4}(x) = x, \ \lambda_{H_4}(y) = w, \ \lambda_{H_4}(w) = -y. \]

Following \[3\] Example 4.12 the morphism \( \tau : A \otimes H_4 \to \mathbb{F} \), defined by
\[ \tau(\sigma_i u^\alpha \otimes z) = \begin{cases} 1 & \text{if } z = 1 \\ (-1)^\alpha & \text{if } z = x \\ 0 & \text{if } z = y, w \end{cases} \]
is a skew pairing such that \( \tau = \tau^{-1} \). Then, by \[3\] Proposition 5.2, \( (A, \varphi_A) \), where
\[ \varphi_A = (\tau \otimes A \otimes \tau) \circ (A \otimes H_4 \otimes \delta_A \otimes H_4) \circ \delta_{A \otimes H_4} \circ c_{H_4, A}, \]
is a left \( H_4 \)-module comonoid and \( (H_4, \phi_{H_4}) \), where
\[ \phi_{H_4} = (\tau \otimes H_4 \otimes \tau) \circ (A \otimes H_4 \otimes c_{A, H_4} \otimes H_4) \circ (A \otimes H_4 \otimes A \otimes \delta_{H_4}) \circ \delta_{A \otimes H_4} \circ c_{H_4, A}, \]
is a right \( H \)-module comonoid. Moreover, by \[3\] Corollary 5.6, the pair \( (A, H_4) \) is a matched pair of Hopf quasigroups. More concretely,
\[ \varphi_A(1 \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha, \ \varphi_A(x \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha, \ \varphi_A(y \otimes \sigma_i u^\alpha) = \varphi_A(w \otimes \sigma_i u^\alpha) = 0, \]
and
\[ \phi_{H_4}(1 \otimes \sigma_i u^\alpha) = 1, \ \phi_{H_4}(x \otimes \sigma_i u^\alpha) = x, \ \phi_{H_4}(y \otimes \sigma_i u^\alpha) = (-1)^\alpha y, \ \phi_{H_4}(w \otimes \sigma_i u^\alpha) = (-1)^\alpha w. \]

Then,
\[ \Psi(1 \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha \otimes 1, \ \Psi(x \otimes \sigma_i u^\alpha) = \sigma_i u^\alpha \otimes x, \]
\[ \Psi(y \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes y, \ \Psi(w \otimes \sigma_i u^\alpha) = (-1)^\alpha \sigma_i u^\alpha \otimes w. \]

Therefore, the product table of \( A \otimes H_4 \) is

|   | \sigma_j u^\beta \otimes 1 | \sigma_j u^\beta \otimes x | \sigma_j u^\beta \otimes y | \sigma_j u^\beta \otimes w |
|---|---------------------|---------------------|---------------------|---------------------|
| \sigma_i u^\alpha \otimes 1 | \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes 1 | \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes x | \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes y | \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes w |
| \sigma_i u^\alpha \otimes x | \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes x | \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes x | \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes x | \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes x |
| \sigma_i u^\alpha \otimes y | (-1)^\beta \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes y | (-1)^{\beta + 1} \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes w | 0 | 0 |
| \sigma_i u^\alpha \otimes w | (-1)^\beta \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes w | (-1)^{\beta + 1} \sigma_i u^\alpha \otimes \sigma_j u^\beta \otimes y | 0 | 0 |
and the antipode is given by:
\[
\lambda_{A\bowtie H_4}(\sigma_i u^\alpha \otimes 1) = \sigma_i^{(-1)^{\alpha+1}} u^\alpha \otimes 1, \quad \lambda_{A\bowtie H_4}(\sigma_i u^\alpha \otimes x) = \sigma_i^{(-1)^{\alpha+1}} u^\alpha \otimes x,
\]
\[
\lambda_{A\bowtie H_4}(\sigma_i u^\alpha \otimes y) = (-1)^\alpha \sigma_i^{(-1)^{\alpha+1}} u^\alpha \otimes w, \quad \lambda_{A\bowtie H_4}(\sigma_i u^\alpha \otimes x) = (-1)^{\alpha+1} \sigma_i^{(-1)^{\alpha+1}} u^\alpha \otimes y.
\]

Then, \( A \bowtie H_4 \) is an example of non-commutative, non-cocommutative Hopf quasigroup and its dual \( (A \bowtie H_4)^* = (H_4)^{\infty} A^\ast \) is an example of non-commutative, non-cocommutative Hopf coquasigroup.

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References

[1] Aguiar, M., Andruskiewitsch, N., Representations of matched pairs of groupoids and applications to weak Hopf algebras, de la Peña, José A. (ed.) et al., Algebraic structures and their representations. Proceedings of "XV Coloquio Latinoamericano de Álgebra", Cocoyoc, Morelos, México, July 20-26, 2003. Providence, RI: American Mathematical Society (AMS), Contemporary Mathematics 376, 127-173 (2005).
[2] Alonso Álvarez, J.N., Fernández Vilaboa, J.M., López López, M.P., Villanueva Novoa, E., y González Rodríguez, R.: Caenepeel, Stefaan (ed.) et al., Rings, Hopf algebras, and Brauer groups. Proceedings of "The Fourth Week on Algebra and Algebraic Geometry", SAGA-4, Antwerp and Brussels, Belgium, September 12-17, 1996. New York, NY: Marcel Dekker. Lect. Notes Pure Appl. Math. 197, 11-41 (1998).
[3] Alonso Álvarez J. N., Fernández Vilaboa J. M., González Rodríguez R., Soneira Calvo, C., Projections and Yetter-Drinfel’d modules over Hopf (co)quasigroups, J. Algebra 443 (2015), 153-199.
[4] Alonso Álvarez J. N., Fernández Vilaboa J. M., González Rodríguez R., Multiplication alteration by two-cocycles. The nonassociative version, Bull. Malays. Math. Sci. Soc. 43 (2020), 3557-3615.
[5] Barr, M., Composite cotriples and derived functors, in: Seminar on Triples and Categorical Homology Theory, B. Eckmann (ed.), Springer LNM 80 (1969), 119-140.
[6] Beck, J., Distributive laws, in: Seminar on Triples and Categorical Homology Theory, B. Eckmann (ed.), Springer LNM 80 (1969), 119-140.
[7] Brzeziński T., Jiao Z. M., Actions of Hopf quasigroups, Comm. Algebra 40 (2012), 681-696.
[8] Brzeziński T., Jiao Z. M., R-smash products of Hopf quasigroups, Arab. J. Math. 1 (2012), 39-46.
[9] Bruck R. H., Contributions to the theory of loops, Trans. Amer. Math. Soc. 60 (1946), 245-354.
[10] Chein O., Moufang loops of small order I, Trans. Amer. Math. Soc. 188 (1974), 31-51.
[11] Fang X., Wang, S. H., Twisted smash products for Hopf quasigroups, J. Southeast Univ. 27 (2011), 343-346.
[12] Fang X., Torrecillas B., Twisted smash products and L-R smash products for biquasimodule Hopf quasigroups, Comm. Algebra 42 (2014), 4204-4234.
[13] González Rodríguez, R., Distributive laws and Hopf quasigroups, preprint (2021).
[14] Im, B., Nowak, A. W., Smith, J. D. H., Algebraic properties of quantum quasigroups, J. Pure Appl. Algebra 225 (2021), 106539.
[15] Majid, S., Physics for algebraist: non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction, J. Algebra 130 (1990), 17-64.
[16] Majid, S., Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.
[17] Klim J., Majid S.; Hopf quasigroups and the algebraic 7-sphere, J. Algebra 323 (2010), 3067-3110.
[18] Klim J., Majid S.; Bicrossproduct Hopf quasigroups, Comment. Math. Univ. Carolin. 51 (2010), 287-304.
[19] López López M. P., Villanueva Novoa E., The antipode and the (co)invariants of a finite Hopf (co)quasigroup, Appl. Cat. Struct. 21 (2013), 237-247.
[20] Pérez-Izquierdo J. M., Shestakov I. P., An envelope for Malcev algebras, J. Algebra 272 (2004), 379-393.
[21] Pérez-Izquierdo J. M., Algebras, hyperalgebras, non-associative bialgebras and loops, Adv. Math. 208 (2007), 834-876.
[22] Smith, J. D. H., Quantum quasigroups and loops, J. Algebra 456 (2016) 46-75.
[23] Takeuchi, M., Matched pairs of groups and bismash products of Hopf algebras, Comm. Algebra 9 (1981), 841-882.