The Non-universal behaviour of Cold Fermi Condensates with Narrow Feshbach Resonances

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In this paper we construct an effective field theory for a condensate of cold Fermi atoms whose scattering is controlled by a narrow Feshbach resonance. We show how, from first principles, it permits a hydrodynamic description of the BEC-BCS crossover from which the equation of state, intimately related to the speed of sound, can be derived. Specifically, we stress the non-universal behaviour of the equation of state at the unitary limit of infinite scattering length that arises when either, or both, of the range of the inter-atomic force and the scale of the molecular field become large.

I. INTRODUCTION: SCATTERING WITH FESHBACH RESONANCES

Cold alkali atoms whose scattering is controlled by a Feshbach resonance can form diatomic molecules with tunable binding energy on applying an external magnetic field [1]. Weak fermionic pairing gives a BCS theory of Cooper pairs, whereas strong fermionic pairing gives a BEC theory of diatomic molecules. The transition is characterised by a crossover in which, most simply, the s-wave scattering length $a_S$ diverges as it changes sign [2].

A considerable theoretical and experimental effort has been expended on understanding such macroscopic quantum systems. We wish to show that, in a formalism in which the tuning of the system by the external field is explicit because of the narrowness of the resonance, it is straightforward to derive the semiclassical attributes of the condensates (speed of sound, hydrodynamics, equation of state) analytically. Our starting point is the exemplary 'two-channel' microscopic action (in units in which $\hbar = 1$)

$$S = \int dt d^3x \left\{ \sum_{\sigma=\uparrow,\downarrow} \psi^*_\sigma(x) \left[ i \partial_t + \frac{\nabla^2}{2m} + \mu \right] \psi_\sigma(x) + U \psi^*_\uparrow(x) \psi^*_\downarrow(x) \psi_\downarrow(x) \psi_\uparrow(x) + \phi^*(x) \left[ i \partial_t + \frac{\nabla^2}{2M} + 2\mu - \nu \right] \phi(x) - g \left[ \phi^*(x) \psi_\uparrow(x) \psi_\uparrow(x) + \phi(x) \psi^*_\sigma(x) \psi^*_\sigma(x) \right] \right\}$$

(1)

for cold fermi fields $\psi_\sigma$ with spin label $\sigma = (\uparrow, \downarrow)$, which possess a narrow bound-state (Feshbach) resonance with tunable binding energy $\mu$, represented by a diatomic field $\phi$ with mass $M = 2m$ [3, 4, 5, 6]. In addition there is a simple s-wave attractive contact interaction $U > 0$, regularised at a momentum scale $\Lambda$. Standard renormalisation methods [7] allow us to remove $\Lambda$ from the formalism. In what follows we assume that such renormalisation has been made. The effective atomic coupling strength (at zero external energy-momentum) is

$$U_{eff} = U + g^2/(\nu - 2\mu) = -kFas/N_0, \quad (2)$$

comprising the contact term plus the effect of the resonance, where $N_0$ is the density of states at the Fermi surface. Tuning $\nu - 2\mu$ to zero by the application of an external magnetic field sends $|a_S| \to \infty$. This is the so-called 'unitary' regime.

This work follows on from that of an earlier paper [7], henceforth referred to as I, from which we recreate the condensate effective action (9) in the next section.

We shall show, in greater detail than in I, that the $(T = 0)$ condensate described by (11) can be understood, in the hydrodynamical approximation, as two coupled fluids, built from the fermion-paired atomic and molecular subsystems. This is sufficient to determine the equation of state (EOS) and speed of sound exactly in the mean-field approximation. There are simplifications in that a single fluid dominates in a) the deep BEC regime b) the deep BCS regime and c) the unitary regime. When a single fluid description is appropriate the EOS can show several different allometric behaviors $p \propto \rho^{1+\gamma}$ depending on the magnitudes of the effective range of the inter-atomic force and the length scale of the molecular field.

In particular, when either is large, the single fluid behaviour in the unitary regime does not show the canonical value $\gamma = 2/3$ and we do not have a conformal field theory when $|a_S| \to \infty$. There has been much activity recently on taking advantage of the dualities of Anti de-Sitter general relativity and conformal field theory (AdS/CFT) [8] to use classical black hole physics to describe cold Fermi atoms in the unitary regime in those cases when the theory is conformally invariant [9], but our models show the limitations of this analysis.

Possible experimental tests are considered.
II. EFFECTIVE ACTIONS

Introducing auxiliary fermion-paired bosonic fields $\Delta(x) = U\psi_1(x)\psi_1^*(x)$, $\Delta^*(x) = U\psi_1^*(x)\psi_1(x)$ renders $S$ quadratic in the fermi fields. Integrating them out [7] enables us to write $S$ in the non-local form

$$S_{NL} = -iTr \ln G^{-1} + \int dt dx \left\{ -\frac{1}{U}|\Delta|^2 + \phi^*(x) \left[ i\partial_t + \frac{\nabla^2}{2M} + 2\mu - \nu \right] \phi(x) \right\}, \quad (3)$$

in which $G^{-1}$ is the inverse Nambu Green function,

$$G^{-1} = \left( \frac{i\partial_t - \varepsilon}{\partial_{x}^2 + \varepsilon} \right) \Delta(x) \Delta^*(x) \left( \frac{i\partial_t + \varepsilon}{\partial_{x}^2 + \varepsilon} \right) \quad (4)$$

where $\Delta(x) = \Delta(x) - g\phi(x)$ represents the two-component combined condensate and phase of $\Delta(x) = \Delta(x)|e^{i\theta_2(x)}$ and $\phi(x) = -|\phi(x)|e^{i\theta_1(x)}$ the combined condensate amplitude and phase of $\Delta(x) = |\Delta(x)|e^{i\theta_2(x)}$ are then determined. The action possesses a $U(1)$ invariance under $\theta_{\Delta} \rightarrow \theta_{\Delta} + \text{const.}$, $\theta_\phi \rightarrow \theta_\phi + \text{const.}$, which is spontaneously broken: $\delta S_{NL} = 0$ permits spacetime constant gap solutions $|\Delta(x)| = |\Delta_0| \neq 0$ and $|\phi(x)| = |\phi_0| \neq 0$ (whereby $|\Delta(x)| = |\Delta_0| \neq 0$) and a Goldstone boson, the (gapless) phonon. $|\Delta_0|$ determines the density of states at the Fermi surface as $N_0 = \int d^3p/(2\pi)^3(|\Delta_0|^2/2E_p^2)$, where $E_p^2 = \frac{p^2}{2}\mu + |\Delta_0|^2$, $\varepsilon_p = p^2/(2\mu) - \mu$. If $|\Delta_0| = |\Delta_0| + |g\phi_0|$, then $|\Delta_0|/|\Delta_0| = U/|U_{\text{eff}}|$, $|\phi_0|/|\Delta_0| = g/(\nu - 2\mu)|U_{\text{eff}}|$. In addition, the system possess a gapped (Higgs') mode.

To determine the EOS it is efficient to consider just the fluctuations around the gap configurations and perturb in the small fluctuations in the scalar condensate densities $\delta|\Delta| = |\Delta| - |\Delta_0|$ and $\delta|\phi| = |\phi| - |\phi_0|$ and their derivatives. We perform a Galilean invariant long wavelength, low-frequency expansion in space and time derivatives to give

$$S_{NL} \approx S_{eff} = \int dt dx L_{eff}$$

in terms of the local effective Lagrangian density $L_{eff}$ with elliptic equations of motion. Although $\theta_{\Delta}$ and $\theta_\phi$ are not small, we assume that $\delta|\Delta|, \delta|\phi|$, and $(\theta_{\Delta} - \theta_\phi)^2$ are of the same order.

Because $G^{-1}$ is defined in terms of $\Delta$ it is natural to express the local Lagrangian density $L_{eff}$ in terms of the phase angles $\theta_\Delta, \theta_\phi$ and the Galilean scalar fluctuations $\delta|\Delta| = |\Delta| - |\Delta_0|$ and $\delta|\phi|$. In fact, it is convenient to rescale $\delta|\Delta|$ to $\delta|\Delta| = \kappa\epsilon$, such that $L_{eff}$ takes the form

$$L_{eff} = -\frac{1}{2}\rho \Phi G(\theta_\phi) - \frac{1}{2}\Omega^2(\theta_\Delta - \theta_\phi)^2 - \frac{1}{2}\left(\rho_{\phi}^0 + 2\alpha \epsilon\right) G(\theta_\phi, \epsilon) + \frac{N_0}{4} G^2(\theta_\phi, \epsilon) + \frac{1}{4}\gamma X^2(\theta_\phi) - \frac{1}{4}M^2\epsilon^2 + \frac{2\gamma}{U_\phi}\delta|\phi|$$

in terms of the Galilean scalar combinations $G(\theta) = \hat{\theta} + (\nabla\theta)^2/4m$. $G(\theta, \epsilon) = \hat{\theta} + (\nabla\theta)^2/4m + (\nabla\epsilon)^2/4m$, $X(\theta, \epsilon) = \hat{\epsilon} + \nabla\theta \nabla\epsilon/2m + \theta_\Delta - \theta_\phi$. We have chosen $\kappa$ so that the dimensionless $\epsilon$ has the same coefficients as $\theta_\Delta$ in its spatial derivatives. On taking $\epsilon$ identically zero in we recover the Lagrangian of [11].

The fermion number density arising from the gap equations is $\rho^0 F = \rho_{\phi}^0 + \rho_{\phi}^B$, where $\rho_{\phi}^B = \int d^3p/(2\pi)^3 [1 - \varepsilon_p/E_p]$ is the explicit fermion density, and $\rho_{\phi}^B = 2|\phi_0|^2$ is due to molecules (two fermions per molecule). We shall introduce the other coefficients, which are simple momenta integrals, as and when they are needed. It is the form of (5) rather than the detail that concerns us at the moment.

III. EQUATIONS OF STATE

The definition of a narrow resonance is that

$$\gamma_r \sim \sqrt{\Gamma/\epsilon_F} \ll 1,$$

where $\Gamma$ is the resonance width and $\epsilon_F = k_F^2/2m$ is the typical atomic kinetic energy. Unless stated otherwise, we assume narrow resonances, for which the mean field approximation can be justified [10]. Since the Fermi momentum $k_F$ increases as the density $\rho$ increases we can, in principle, make even broad resonances narrow by increasing the density. In more detail [10],

$$\gamma_r = \frac{1}{(3\pi^2)^{1/3}} \frac{m^2(2\mu - \nu)U_{eff}^2}{\rho^{1/3}g^2}$$

Note that $\gamma_r \propto g^2$ when $|\alpha_S| \rightarrow \infty$, irrespective of $U$. However, $\gamma_r$ increases as we move into the deep BEC and BCS regimes, when $|\nu|$ is large, making the narrow resonance approximation less valid. With this qualification, systems with narrow resonances include $^6Li$ with $\gamma_r \approx 0.2$. On the other hand, when $\gamma_r \gg 1$ and the narrow resonance approximation breaks down, the model effectively becomes a one-channel model in its basic properties. There are then strong similarities with the single-channel model in which $g = 0$ identically, discussed by many authors, but for which we cite [11] in particular.

The hydrodynamics of the system is encoded in $\theta_\Delta(x)$ and $\theta_\phi(x)$ and will have a natural realisation as two coupled fluids. To proceed, we ignore the density
and velocity fluctuations $\rho_e = -(\eta/4)X(\epsilon, \theta_\Delta)$ and $v_e = \nabla \epsilon/2m$ due to the condensate fluctuations $\epsilon$, in comparison to $\rho_0$ and $\tilde{v} = \nabla \theta_\Delta/2m$, the condensate velocity. The inclusion of $\rho_e$ and $v_e$ in the two-fluid model would give small fluctuating short-range sources and sinks in the fluids. All that we need for the EOS is the hydrodynamic approximation, which coarse-grains by replacing them with their (zero) averages.

The angular Euler-Lagrange (EL) equations are then

$$\frac{\partial}{\partial t} \rho_F + \nabla (\rho_F \tilde{v}) - 2\Omega^2 (\theta_\Delta - \theta_\phi) = 0,$$

$$\frac{\partial}{\partial t} \rho_B + \nabla (\rho_B u) + 2\Omega^2 (\theta_\Delta - \theta_\phi) = 0,$$  \hspace{1cm} (9)

where $u = \nabla \theta_\phi/2m$ and $\rho_F = \rho_0^F - N_0 G(\theta_\Delta, \epsilon) + 2\alpha \epsilon$, $\rho_B = 2|\phi|^2 \approx \rho_0^B + 4|\phi_0| \delta \phi$. Putting these together gives

$$\frac{\partial}{\partial t} (\rho_B + \rho_F) + \nabla (\rho_B u + \rho_F \tilde{v}) = 0,$$  \hspace{1cm} (10)

the continuity equation for two coupled fluids, as given in $I$.

Equation (10) can be written in a more transparent form. Whereas the explicit fermion density in molecules $\rho_0^F$ in (10) appears in conjunction with the velocity $u$ of the molecular component of the fluid, the generalised fermion pair density $\rho_F$ is coupled to the combined condensate velocity $\tilde{v}$, rather than $v = \nabla \theta_\Delta/2m$. However, from the definition of $\Delta$ it follows that $\theta_\Delta = b\theta_\phi + (1 - b)\theta_\phi$, where $b = |\Delta_0|/|\Delta_0| = U/U_{eff}$. In consequence, $\tilde{v} = b v + (1 - b) u$, whereby (10) can be written as

$$\frac{\partial}{\partial t} (\tilde{v} \rho_B + \tilde{v} \rho_F) + \nabla (\tilde{v} \rho_B u + \tilde{v} \rho_F \tilde{v}) = 0,$$  \hspace{1cm} (11)

where $\tilde{v} \rho_F = b \rho_F$ and $\tilde{v} \rho_B = b \rho_B + (1 - b) \rho_B$. That is, the effective molecular density $\tilde{v} \rho_B$ describes point-particle bosons together with a cloud of fermionic Cooper pairs, which deplete the effective fermion pair density $\tilde{v} \rho_F$. Because the fluids are coupled, the condensate moves as a single entity with velocity $\tilde{v}$.

The Bernoulli equations from which the EOS follows are derived from these EL equations on substituting for the densities. Again neglecting $\rho_e$ and $v_e$, the EL equation for $\theta_\Delta$ can be written as the simple Bernoulli equation

$$m \dot{\tilde{v}} + \nabla \left[ \delta h_F + \frac{1}{2} m \tilde{v}^2 \right] = 0,$$  \hspace{1cm} (12)

where $\delta h_F = (\rho_F - \rho_0^F - 2\alpha \epsilon)/2N_0$ is the specific enthalpy. After substituting $\epsilon$ from its EL equation

$$\dot{\delta \epsilon} + 2\alpha G(\theta_\Delta) - (4\epsilon g/U)\delta \phi \approx 0$$  \hspace{1cm} (13)

in $\delta h_F$, the enthalpy can be expressed in terms of the density fluctuations $\delta \rho_F = \rho_F - \rho_0^F$, and $\delta \rho_B = 4|\phi_0| \delta \phi$ as

$$\delta h_F = \frac{\delta \rho}{\rho_0^F} F F + \delta \rho_B K F B$$  \hspace{1cm} (14)

where

$$K_{FF} = \frac{M^2}{2(N_0 M^2 + 4\alpha^2)}, \quad K_{FB} = \frac{-g\delta}{U|\phi_0|[(N_0 M^2 + 4\alpha^2)}/(N_0 M^2 + 4\alpha^2)].$$  \hspace{1cm} (15)

Complementarily, the EL equation for $\theta_\phi$ has the form

$$m \dot{\tilde{u}} + \nabla \left[ \delta h_B + \frac{1}{2} m u^2 - \frac{1}{16m \rho_0^B} \nabla^2 \delta \rho_B \right] = 0,$$  \hspace{1cm} (16)

where $h_B$ permits the decomposition

$$\delta h_B = \frac{\delta \rho}{\rho_0^B} K_{BF} \delta \rho_F + K_{BB} \delta \rho_B.$$  \hspace{1cm} (17)

As required, $K_{BB} = K_{FB}$ and

$$K_{BB} = (\nu - 2\mu) U_{eff}/[8|\phi_0|^2 - U - g^2/2|\phi_0|^2 4\alpha^2 + N_0 M^2].$$  \hspace{1cm} (18)

The $\nabla^2 \delta \rho_B \propto \nabla^2 \delta \phi$ term in (10) is just as we would expect from a theory of a pure bosonic gas. In the hydrodynamic approximation such derivatives of $\delta \phi$ are also ignored comparatively, and the resulting equation

$$m \dot{\tilde{u}} + \nabla \left[ \delta h_B + \frac{1}{2} m u^2 \right] = 0,$$  \hspace{1cm} (19)

is taken in conjunction with (12) in determining the EOS. It follows from (14) and (17) that

$$\delta \rho_F (\rho_0^F K_{FF} - \rho_0^B K_{FB}) = \delta \rho_B (\rho_0^B K_{BB} - \rho_0^F K_{FB}).$$  \hspace{1cm} (20)

We have learned from (10) and (11) that, however it may be partitioned, the total fermion density is $\rho_F + \rho_B$, whose fluctuation is $\delta \rho = \delta \rho_F + \delta \rho_B$. Eqs. (14) and (17) then collapse to give the EOS of the condensate as

$$d p = \frac{\rho_0^F \rho_0^B (K_{FF} K_{BB} - K_{FB} K_{BB})}{(\rho_0^F K_{FF} + \rho_0^B K_{BB} - \rho_0^B K_{BB} - \rho_0^F K_{FB})},$$  \hspace{1cm} (21)

where we remember that $K_{FB} = K_{BB} < 0$. Eq. (21) is the key equation of this paper.

As a very good check on our calculations, the time derivatives of the EL equations for $\theta_\phi$ and $\theta_\phi$ determine the dispersion relations of the modes. It follows directly that the speed of sound $v$ is given as

$$v^2 = \frac{\rho_0^F}{m} \left( \frac{K_{FF} K_{BB} - K_{FB} K_{BB}}{K_{FF} + K_{BB} - 2K_{FB}} \right).$$  \hspace{1cm} (22)

Neither $dp/d\rho$ nor $v^2$ depend on the coefficients $\eta$, $\Omega^2$ in (10) or the scaling parameter $\kappa$ and we use the results of [2] to identify the coefficients in the $K$s as
\( \alpha = \int d^3p/(2\pi)^3(\tilde{\Delta}_0|\epsilon_p|/2E_p^3) \) and \( \tilde{M}^2 = 2(2/U - \beta) > 0 \), where

\[
\beta = \int \frac{d^3p}{(2\pi)^3} \left( \frac{\epsilon_p^2}{E_p^3} - \frac{1}{(p^2/2m)} \right) < 0. \tag{23}
\]

We assume that all the parameters in \( dp/d\rho \) and \( v^2 \) have been renormalised as in I. On substituting this and the \( K_s \)s above into \( v^2 \) of (22) we do, indeed, recover the results of I. Unfortunately, although \( dp/d\rho \) is given in terms of straightforward momenta integrals, in general it bears no simple relationship to \( v^2 \). This is not surprising since the system comprises two coupled fluids.

We conclude this section with a further comment on the role of the order parameter fluctuations \( \epsilon \) and \( \delta|\phi| \) (deviations from homogeneous gap solutions). The coarse-graining described above is not only sufficient to obtain the EOS, but obligatory within the approximation [5], that corresponds to taking only the lowest relevant field derivatives, in that the fluctuations are too short-range or too fast [11] to be properly accounted for by it.

To go beyond the EOS requires an understanding of the fluctuations of the gapped ‘Higgs’ mode mentioned earlier, built from order parameter fluctuations. (This is trivially so in the limiting case of \( g = 0 \) for which the Higgs field is just \( \epsilon \) [11].) Equally, the dynamics of the Higgs mode cannot be obtained reliably from the approximation [5] and we need to incorporate higher-order field derivatives non-perturbatively to describe it (e.g. see [10]).

For our purposes we can ignore the Higgs mode and we revert to our results of (21) and (22).

**IV. SINGLE FLUID REGIMES**

With the qualifications above, there are circumstances in which we have a description in terms of a single fluid. Most simply, in the deep BCS regime (where \( \rho_B \approx 0 \)) and the deep BEC regime (where \( \rho_F \approx 0 \)), the system of [10] behaves as a single fluid, of Cooper pairs or molecules respectively. A characteristic of these extremes is that the terms that are negligible in \( dp/d\rho \) are also negligible in \( v^2 \), whereby we get the simple result \( dp/d\rho \propto mv^2 \), as we shall now show.

**A. The BCS regime**

In the deep BCS regime \( \rho_B^0 = 2|\phi_0|^2 \) vanishes, as does \( \alpha \) because of particle-hole symmetry. As a result, \( K_{BB} \) becomes very large and

\[
\frac{dp}{d\rho} \approx \rho_F^0 K_{FF} \approx \frac{\rho}{2N_0} \approx m v^2. \tag{24}
\]

The EOS follows directly. At the Fermi surface \( (\mu \approx \epsilon_F) \) the density of states \( N_0 \propto m \rho^{1/3} \)

\[
\frac{dp}{d\rho} \propto \mu \propto \rho^{2/3} \tag{25}
\]

or, equivalently, \( p \propto \rho^{1+\gamma} \) where \( \gamma = 2/3 \). There is no pressure from molecules in this limit. This is in agreement with the work of other authors (e.g. [17, 18]).

**B. The BEC regime**

The deep BEC regime (large negative \( \nu \) ) is characterised by small \( N_0 \), large \( \alpha \), and \( M^2 \approx 4/U \). Unlike the BCS regime, \( K_{FF} \) dominates the \( K \)s, so that \( v^2 \approx \rho_B K_{BB}/m \). Inspection shows that

\[
\frac{dp}{d\rho} \approx \frac{1}{2m} v^2, \tag{26}
\]

since \( \rho_B^0 K_{FF} \approx \rho_B^0 |K_{BF}| \). We have shown in I that in the BEC regime \( v^2 \propto \rho \), giving us the EOS

\[
p \propto \rho^{1+\gamma} \quad \text{with} \quad \gamma = 1. \tag{27}
\]

This is the expected value of \( \gamma \), corresponding to a molecular condensate with repulsive interaction [19].

**C. The Unitary limit**

The extremes above permit relatively trivial single-fluid descriptions. More important is the ‘unitarity’ limit describing the central region \( \nu = 2\mu \) where \( |a_S| \rightarrow \infty \). This is described by a single fluid, not because \( \rho_B \) or \( \rho_F \) vanishes, but because \( b = U/U_{eff} \rightarrow 0 \). This is essentially a one-channel system [20], represented by the single fluid \( \rho_F = 0 \), \( \tilde{\rho}_F = \rho \) in terms of the \( \tilde{\rho} \)s of [11]. We stress that the divergence of \( |a_S| \) is not a signal of singular behaviour. The \( K \)’s that define the behaviour of the system vary continuously as we pass from the BCS to BEC regimes through the unitarity regime, and versa (e.g. see [13]), for which perturbative methods such as ours are valid for narrow enough resonances (e.g. see Fig. 4 of [10]).

In addition to the scattering length \( a_S \), the general model possesses two other important length scales, the effective range of the force \( r_0 \) where \( \gamma_r \sim \hbar/k_F|r_0| \) as well as the length scale \( \xi = (4m(2\mu - \nu)U_{eff}/U)^{-1/2} \) for the molecular field. In general both \( r_0 = O(g^{-2}) \) and \( \xi = O(g^2/U)^{-1/2} \) are small near the unitary limit for broad resonances with large resonance coupling \( g \), and the only effective dimensionless parameter is \( k_F|a_S| \).

On it diverging, the theory shows universal behaviour, with conformal invariance. In principle, AdS/CFT duality then enables us to convert the difficult strong coupling calculations needed to describe the system into more tractable weak coupling boundary calculations for
classical black holes [8]. However, for narrow resonances $|r_0|$ becomes large and this universal behaviour is broken. Further, for small $g^2/U$ we see that $\xi$ becomes large, again leading to the breakdown of universal behaviour.

Nonetheless, in many regards there is still de facto universality for realistic systems [see Figs. 30 of Chen et al. [14]) for many observables. To see deviations from non-universality the best way is to look for the deviation of the EOS from its canonical behaviour.

After UV renormalisation, the gap equation at this limit becomes

\[ 0 = \frac{1}{U_{\text{eff}}} = \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2E_p} - \frac{1}{2(p^2/2m)} \right]. \]  

(28)

This fixes

\[ |\tilde{\Delta}_0| = c\mu, \]  

(29)

where $c \approx 1.16$. In turn, this relates the density $\rho$ to $\mu$ as

\[ \rho = \rho_B + \rho_F = \frac{2c^2}{g^2}\mu^2 + \frac{c_1}{2\pi^2}(2m\mu)^{3/2} \]  

(30)

where the first and second terms are $\rho_B$ and $\rho_F$ respectively and $c_1 \approx 1.47$.

Although we have constructed the model with narrow resonances in mind the formalism permits extension to broad resonances [11] and we consider all possibilities. In the first instance, for broad resonances ($\gamma_r \gg 1$), for which $k_F|r_0| \ll 1$, $\rho \approx \rho_F \gg \rho_B$, whereas for narrow resonances ($\gamma_r \ll 1$), for which $k_F|r_0| \gg 1$, $\rho \approx \rho_B \gg \rho_F$.

In determining the EOS we stress that, in the unitary limit, the resonance width is independent of $U$.

1. Broad resonances

The canonical conformal symmetry in the unitarity regime arises when both $k_F|r_0| \ll 1$ and $k_F\xi \ll 1$ or, equivalently, we have large resonance coupling $g$, with $g^2 \gg U\mu, \mu^2/(2m\mu)^{3/2}$. For such values $\rho_FK_{FF} \ll \rho_BK_{BB}, \rho|K_{BF}|$, and we recover the ‘universal’ behavior

\[ \frac{d\rho}{d\rho} \propto \mu^{1/2} \propto \rho^{1/4}. \]  

(31)

or $p \propto \rho^{1+\gamma}$ with $\gamma = 2/3$, as in the BCS regime.

Although our approximation is not wholly reliable in that, for broad resonances, the renormalization of the molecular boson is expected to contribute sizable corrections to the EOS in such a strongly coupled regime [19], this is the correct value [17], observed experimentally (see Fig. 15 of [17]).

For broad resonances, $\mu \sim k_F^2/2m$, whereby $k_F\xi \sim (U\mu/g^2)^{1/2}$. For relatively smaller $g$ ($U\mu \gg g^2 \gg \mu^2/(2m\mu)^{3/2}$), even though $|r_0|$ remains small, conformal invariance is broken by $k_F\xi \gg 1$. In these circumstances $\rho_FK_{FF} \gg \rho_BK_{BB}, \rho|K_{BF}|$, whereby

\[ \frac{d\rho}{d\rho} \propto \mu^0 \propto \rho^0. \]  

(32)

giving the EOS $p \propto \rho^{1+\gamma}$ with $\gamma = 0$.

2. Narrow resonances

When the resonance is narrow the mean field approximation is robust [10]. There is no contradiction in having strong self-interaction $U$ ($U \gg \mu/(2m\mu)^{3/2} \gg g^2/\mu$), when the conformal symmetry, already broken by $k_F|r_0| \gg 1$, is further broken with

\[ k_F\xi = \left( \frac{g^2}{U\mu} \right)^{-1/2} \left( \frac{\mu}{(2m\mu)^{3/2}} \right)^{1/3} \left( \frac{g^2}{\mu} \right)^{-1/3} \gg 1. \]  

(33)

In this case we find $\rho_FK_{FF} \gg \rho|K_{BF}| \gg \rho_BK_{BB}$, giving

\[ \frac{d\rho}{d\rho} \propto \mu^0 \propto \rho^0. \]  

(34)

That is, $p \propto \rho^{1+\gamma}$ where $\gamma = 0$. This is the behaviour of a free Bose gas, expected in this limit [21].

However, for relatively weak self-interaction ($U, g^2/\mu \ll \mu/(2m\mu)^{3/2}$), with smaller $k_F\xi$ than previously, we find $\rho_BK_{BB} \ll \rho_FK_{FF}, \rho|K_{BF}|$, leading to

\[ \frac{d\rho}{d\rho} \propto \mu^{1/2} \propto \rho^{1/4}. \]  

(35)

That is, $p \propto \rho^{1+\gamma}$ where $\gamma = 1/4$.

As we anticipated earlier, in all of the cases listed above it happens that $dp/d\rho \propto m\nu^2$, and we can equally well read off the behaviour from that of $\nu^2$. Away from these extremes we see, from [30], that an allometric representation is not justified, although the effective exponent $\gamma_{eff} = d(\ln dp/d\rho)/d\ln \rho$ will interpolate between these values.

V. EXPERIMENTAL TESTS

The best experimental test of our non-canonical equations of state is to determine the EOS exponent $\gamma$ directly by observing the expansion of the condensate in elongated traps on removing the potential [15, 17]. This is governed by the hydrodynamical equations and to see how this carries over to our model we need to examine the single fluid nature of the unitary regime in more detail.

Consider an elongated axially symmetric harmonic
narrow resonances should be the easiest to identify. For narrow resonances in the unitary limit \( \gamma = 2/3 \) corresponds to strongly coupled broad resonances. For instance, when \( \omega_\perp / \omega_z = 0.1 \), the 'universal' behaviour \( \gamma = 2/3 \) corresponds to strongly coupled broad resonances. For narrow resonances in the unitary limit \( \gamma \) interpolates between \( \gamma = 0 \) and \( \gamma = 1/4 \) and \( \gamma = 1 \) arises in the BEC regime.

The aspect ratio as a function of the time for the expansion of the condensates with different allometric behaviours for its EOS when \( \lambda = \omega_\perp / \omega_z = 0.1 \). The 'universal' behaviour \( \gamma = 2/3 \) corresponds to strongly coupled broad resonances. For narrow resonances in the unitary limit \( \gamma \) interpolates between \( \gamma = 0 \) and \( \gamma = 1/4 \) and \( \gamma = 1 \) arises in the BEC regime.

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VI. CONCLUSIONS

We have seen that an explicitly tunable two-channel model for cold Fermi gases with a narrow Feshbach resonance has a two-component order parameter. With each complex component having its own phase it is not surprising that, for the purposes of its EOS, the system can be described in a very transparent way by two coupled fluids.

The system can be driven from one of Cooper pairs to one of tightly bound diatomic molecules by applying an external magnetic field. The single fluid limits of the deep BEC and BCS regimes, in which either the densities of Cooper pairs or molecules are zero, are familiar. Our emphasis here has been on the very different single-fluid limit of the unitary regime, at which the scattering length diverges, in which neither density vanishes. For this we have shown that the EOS, with an exponent interpolating between \( \gamma = 0 \) and \( \gamma = 1/4 \), will differ strongly from the canonical behaviour \( \gamma = 2/3 \) of a conformal field theory, according as one, or both, of the effective range of the inter-atomic force and the length scale of the molecular field become large. This can be confirmed by measurements of the aspect ratio for elongated condensates.

We conclude with an observation on formalism. Although the two-fluid description is very natural, the hydrodynamic equations can be reformulated as Gross-Pitaevskii (GP) equations for coupled complex GP fields \( \Psi_F = \sqrt{\rho_F}e^{i\phi_F}/\sqrt{2} \) and \( \Psi_B = \phi = \sqrt{\rho_B}e^{i\phi_B}/\sqrt{2} \), one field for each density and phase. However, since phases are field logarithms, this coupling is unmanageably logarithmic in the fields, in general. It is only when a single-fluid description is possible that the GP formalism is useful e.g. in the BEC regime, when \( \Psi_F \propto \Delta \).

Focusing on the EOS, it is evident that the hydrodynamical effect is greater in the direction of larger density gradients, as anticipated. For a larger value of the exponent \( \gamma \) on EOS, it is found that the shape of the condensates at asymptotical times changes more dramatically. The non-canonical behaviour interpolating between \( \gamma = 1/4 \) and \( \gamma = 0 \) for narrow resonances should be the easiest to identify.

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