MULTIPLIERS AND LACUNARY SETS IN NON-AMENABLE GROUPS
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§ 0. Introduction.

Let $G$ be a discrete group.

Let $\lambda : G \to B(\ell_2(G), \ell_2(G))$ be the left regular representation. A function $\varphi : G \to \mathbb{C}$ is called a completely bounded multiplier (= Herz-Schur multiplier) if the transformation defined on the linear span $K(G)$ of $\{\lambda(x), x \in G\}$ by

$$\sum_{x \in G} f(x)\lambda(x) \to \sum_{x \in G} f(x)\varphi(x)\lambda(x)$$

is completely bounded (in short c.b.) on the $C^*$-algebra $C^*_\lambda(G)$ which is generated by $\lambda$ ($C^*_\lambda(G)$ is the closure of $K(G)$ in $B(\ell_2(G), \ell_2(G))$).

One of our main results (stated below as Theorem 0.1) gives a simple characterization of the functions $\varphi$ such that $\varepsilon\varphi$ is a c.b. multiplier on $C^*_\lambda(G)$ for any bounded function $\varepsilon$, or equivalently for any choice of signs $\varepsilon(x) = \pm 1$. We wish to consider also the case when this holds for “almost all” choices of signs. To make this precise, equip $\{-1,1\}^G$ with the usual uniform probability measure. We will say that $\varepsilon\varphi$ is a c.b. multiplier of $C^*_\lambda(G)$ for almost all choice of signs $\varepsilon$ if there is a measurable subset $\Omega \subset \{-1,1\}^G$ of full measure (note that $\Omega$ depends only on countably many coordinates) such that for any $\varepsilon$ in $\Omega$ $\varepsilon\varphi$ is a c.b. multiplier of $C^*_\lambda(G)$. (Note that $\varphi$ is necessarily countably supported when this holds, so the measurability issues are irrelevant.)

Theorem 0.1. The following properties of a function $\varphi : G \to \mathbb{C}$ are equivalent

(i) For all bounded functions $\varepsilon : G \to \mathbb{C}$ the pointwise product $\varepsilon\varphi$ is a c.b. multiplier.

(ii) For almost all choices of signs $\varepsilon \in \{-1,1\}^G$, the product $\varepsilon\varphi$ is a c.b. multiplier.

(iii) There is a constant $C$ and a partition of $G \times G$ say $G \times G = \Gamma_1 \cup \Gamma_2$ such that

$$\sup_{s \in G} \sum_{t \in G} |\varphi(st)|^2 1_{\{(s,t) \in \Gamma_1\}} \leq C^2 \text{ and } \sup_{t \in G} \sum_{s \in G} |\varphi(st)|^2 1_{\{(s,t) \in \Gamma_2\}} \leq C^2.$$
(iv) There is a constant $C$ such that for all finite subsets $E, F \subset G$ with $|E| = |F| = N$ we have

$$\sum_{(s,t) \in E \times F} |\varphi(st)|^2 \leq C^2 N.$$ 

(v) There is a constant $C$ such that for any Hilbert space $H$ and for any finitely supported function $a : G \to B(H)$ we have

$$\left\| \sum_{x \in G} \varphi(x) \lambda(x) \otimes a(x) \right\|_{B(\ell_2(G,H))} \leq C \max \left\{ \left\| \left( \sum a(x)^* a(x) \right)^{1/2} \right\|, \left\| \left( \sum a(x) a(x)^* \right)^{1/2} \right\| \right\}.$$ 

**Note:** The properties (iii) and (iv) could have been stated equivalently with the function $(s, t) \to \varphi(st^{-1})$ (which would have been perhaps more natural) or $(s, t) \to \varphi(s^{-1}t)$ instead of $(s, t) \to \varphi(st)$. We chose the simplest notation.

This theorem is proved in section 2 below.

**Remark:** The papers [W] and [B3] show that amenable groups are characterized by the property that all multipliers $\varphi$ satisfying (iii) in Theorem 0.1 are necessarily in $\ell_2(G)$. Hence the preceding statement is of interest only in the non-amenable case. Moreover, on the free group with finitely many generators, the radial functions which satisfy the above property (iii) are characterized in [W].

The equivalence of (iii) and (iv) is already known. It was proved by Varopoulos [V1] in his study of the projective tensor product $\ell_\infty \hat{\otimes} \ell_\infty$ and the Schur multipliers of $B(\ell_2, \ell_2)$. Let $(e_s)$ (resp. $(e_t)$) denote the canonical basis of $\ell_\infty(S)$ (resp. $\ell_\infty(T)$). We will denote by $\tilde{V}(S, T)$ the set of functions $\psi : S \times T \to \mathbb{C}$ such that

$$\sup_{E \subset S, F \subset T, \|E\| < \infty, \|F\| < \infty} \left\| \sum_{s \in E, t \in F} \psi(s, t) e_s \otimes e_t \right\|_{\ell_\infty(S) \hat{\otimes} \ell_\infty(T)} < \infty.$$ 

More precisely, Varopoulos proved

**Theorem 0.2.** ([V1]) Let $S, T$ be arbitrary sets. The following properties of a function $\psi : S \times T \to \mathbb{C}$ are equivalent.

(i) For all bounded functions $\varepsilon : S \times T \to \mathbb{C}$ the pointwise product $\varepsilon \psi$ is in $\tilde{V}(S, T)$.

(ii) For almost all choices of signs $\varepsilon$ in $\{-1, 1\}^{S \times T}$ the pointwise product $\varepsilon \psi$ is in $\tilde{V}(S, T)$. 

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(iii) There is a constant $C$ and a partition $S \times T = \Gamma_1 \cup \Gamma_2$ such that

$$\sup_{s \in S} \sum_{t \in T} |\psi(s, t)|^2 \leq C^2 \quad \text{and} \quad \sup_{t \in T} \sum_{s \in S} |\psi(s, t)|^2 \leq C^2.$$ 

(iii)' There is a decomposition $\psi = \psi_1 + \psi_2$ with

$$\sup_{s \in S} \sum_{t \in T} |\psi_1(s, t)|^2 < \infty \quad \text{and} \quad \sup_{t \in T} \sum_{s \in S} |\psi_2(s, t)|^2 \leq C.$$

(iv) There is a constant $C$ such that for all finite subsets $E \subset S, F \subset T$ with $|E| = |F| = N$, we have

$$\sum_{(s, t) \in E \times F} |\psi(s, t)|^2 \leq C^2 N.$$ 

The deepest implication in Theorem 0.2 is (ii) $\Rightarrow$ (iii). The equivalence of (iii) and (iii)' is obvious and (iii)' $\Rightarrow$ (i) is rather easy (by duality, it follows from Khintchine’s inequality). The equivalence (iii) $\Leftrightarrow$ (iv) is a remarkable fact of independent interest. The decompositions of the form (iii) are related to some early work of Littlewood and the matrices admitting the decomposition (iii) are often called Littlewood tensors, following Varopoulos’s terminology. We note in passing that (ii) $\Rightarrow$ (iii) (and in fact a slightly stronger result) can be obtained as an application of Slepian’s comparison principle for Gaussian processes in the style of S. Chevet (see [C] théorème 3.2). However, we do not see how to exploit this approach in our more general context.

We will prove below a result which contains Theorem 0.2 as a particular case and implies Theorem 0.1 in the group case. Roughly our result gives a necessary condition (analogous to the above (iii)) for a random series $\sum_{n=1}^{\infty} \varepsilon_n \psi_n$ with random signs $\varepsilon_n = \pm 1$ and arbitrary coefficients $\psi_n$ in $\ell_\infty \hat{\otimes} \ell_\infty$ to define a.s. an element of $\ell_\infty \hat{\otimes} \ell_\infty$. The implication (ii) $\Rightarrow$ (iii) in Theorem 0.2 corresponds to the particular case when $\psi_n$ is of the form $\psi_n = \alpha_n e_{i_n} \otimes e_{j_n}$ where $\alpha_n \in \mathbb{C}$ and $n \rightarrow (i_n, j_n)$ is a bijection of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$.

Our necessary condition can be stated as follows: there is a sequence of scalars $\alpha_m$ with $\sum_{m \geq 0} |\alpha_m| < \infty$ and scalar coefficients

$$a_m^n(i), b_m^n(j), c_m^n(i), d_m^n(j).$$
such that
\[ \psi_n(i, j) = \sum_{m \geq 0} \alpha_m [a_m^a(i)b_m^b(j) + c_m^c(i)d_m^d(j)] \]
and such that for all \( m \)
\[ \sup_i (\sum_n |a_m^a(i)|^2)^{1/2} \leq 1, \quad \sup_j |b_m^b(j)| \leq 1 \]
\[ \sup_i |c_m^c(i)| \leq 1, \quad \sup_j (\sum_n |d_m^d(j)|^2)^{1/2} \leq 1. \]
In other words, the condition expresses that the sequence \( (\psi_n) \) can be written (up to a multiplicative norming constant) as an element of the closed convex hull of special sequences of the form
\[ \psi_n(i, j) = a_n^a(i)b(j) + c(i)d_n^d(j) \]
with
\[ \sum_n |a_n^a(i)|^2 \leq 1, |b(j)| \leq 1, |c(i)| \leq 1, \sum_n |d_n^d(j)|^2 \leq 1 \]
for all \( i \) and \( j \).

This will be stated below (cf. Theorems 2.1 and 2.2) in the more precise (and concise) language of tensor products.

To emphasize the content of Theorem 0.1, we now state an application in terms of Schur multipliers. For any sets \( S, T \) a function \( \psi : S \times T \to \mathbb{C} \) is called a Schur multiplier of \( B(\ell_2(S), \ell_2(T)) \) if for any \( u \in B(\ell_2(S), \ell_2(T)) \) with associated matrix \( (u(s, t)) \) the matrix \( (\psi(s, t)u(s, t)) \) is the matrix of an element of \( B(\ell_2(S), \ell_2(T)) \). It is known that the set of all Schur multipliers \( \psi : S \times T \to \mathbb{C} \) coincides with the space \( \tilde{V}(S, T) \). This essentially goes back to Grothendieck [G]. We give more background on Schur multipliers in section 1. The next statement is an application of Theorem 0.1 (and the easier implication (iii) \( \Rightarrow \) (i) in Theorem 0.2).

**Corollary 0.3.** Assume that \( \varphi \) satisfies (i) in Theorem 0.1. Then for all choices of signs \( \xi \in \{-1, 1\}^{G \times G} \) (indexed by \( G \times G \) this time) the product
\[ (s, t) \to \xi(s, t)\varphi(st) \]
is in \( \tilde{V}(G, G) \) hence it defines a Schur multiplier of \( B(\ell_2(G), \ell_2(G)) \).

Actually, the group structure plays a rather limited role in the preceding statement and in Theorem 0.1. To emphasize this point we state (see also Remark 2.4 below)
Corollary 0.4. Let $G$ be any set. Suppose given a map

$$p : G \times G \to G$$

such that for all fixed $(s_0, t_0)$ in $G \times G$ the maps $s \to p(s, t_0)$ and $t \to p(s_0, t)$ are bijective. (Actually it suffices to assume that there is a fixed finite upper bound on the cardinality of the sets $\{s|p(s, t_0) = x\}$ and $\{t|p(s_0, t) = x\}$ when $x, s_0, t_0$ run over $G$). Let $\varphi : G \times G \to \mathbb{C}$ be a function on $G \times G$. Assume that for all (actually “almost all” is enough) choices of signs $(\varepsilon_x)_{x \in G}$ the function

$$(s, t) \to \varepsilon_{p(s, t)} \varphi(p(s, t))$$

is a Schur multiplier of $B(\ell_2(G), \ell_2(G))$. Then, for all choices of signs $\varepsilon_{s, t}$ (indexed by $G \times G$ this time) the function

$$(s, t) \to \varepsilon_{s, t} \varphi(p(s, t))$$

is a Schur multiplier of $B(\ell_2(G), \ell_2(G))$.

The results stated above are proved in section 2. In section 3, we apply them to study a class of “lacunary subsets” of a discrete group which is analogous of the class of finite unions of Hadamard-lacunary subsets of $\mathbb{N}$. We give a combinatorial characterization of these sets which we call $L$-sets, but we leave as a conjecture a stronger result (see conjecture 3.5 below).
1. Preliminary Background.

We refer to [Pa] for more information on completely bounded maps.

Let $S$ be any set. As usual we denote by $\ell_\infty(S)$ the space of all complex valued bounded functions on $S$, equipped with the sup-norm.

For any Banach space $E$, we will also use the space $\ell_\infty(S, E)$ of all $E$-valued bounded functions $x : S \to E$ equipped with the norm $\|x\| = \sup_{s \in S} \|x(s)\|_E$.

When $S = \mathbb{N}$, we write simply $\ell_\infty$. In particular, we will use below the space $\ell_\infty(\ell_2)$ which also can be regarded as the space of all matrices $x(j, k)$ such that

$$\sup_j (\sum_k |x(j, k)|^2)^{1/2} < \infty.$$ 

Let $X, Y$ be Banach spaces and let $X \otimes Y$ be their linear tensor product. We recall the definition of the projective norm and of several other important tensor norms (cf. [G]).

For any $u$ in $X \otimes Y$, let

$$\|u\|_\wedge = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| \left| u = \sum_{i=1}^n x_i \otimes y_i, \ x_i \in X, \ y_i \in Y \right. \right\}$$

We will also need

$$\gamma_2(u) = \inf \left\{ \sup_{\xi \in B_{X^*}} \left( \sum_{i=1}^n |\xi(x_i)|^2 \right)^{1/2} \sup_{\eta \in B_{Y^*}} \left( \sum_{i=1}^n |\eta(y_i)|^2 \right)^{1/2} \right\}$$

where the infimum runs again over all possible representations of the form $u = \sum_{i=1}^n x_i \otimes y_i$.

Equivalently $\gamma_2(u)$ is the “norm of factorization through a Hilbert space” of the associated operator $u : X \to Y^*$.

We will also need a generalization of the $\gamma_2$-norm considered in [K] for Banach lattices. Recall that a Banach lattice $X$ is called 2-convex if we have

$$\forall \ x, y \in X \quad \left\| (\|x\|^2 + \|y\|^2)^{1/2} \right\| \leq (\|x\|^2 + \|y\|^2)^{1/2},$$

see e.g. [LT] for more information.

Let $X, Y$ be two 2-convex Banach lattices. For $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, we define

$$\gamma(u) = \inf \left\{ \left\| \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|_X \left\| \left( \sum_{i=1}^n |y_i|^2 \right)^{1/2} \right\|_Y \right\}$$

where the infimum runs over all representations of $u$. 

It is easy to see that the 2-convexity of $X$ and $Y$ implies that this is a norm on $X \otimes Y$.

Note that $\ell_\infty$ (or more generally $L_p$ for $2 \leq p \leq \infty$) is an example of a 2-convex Banach lattice. In the case of the product $\ell_\infty \otimes \ell_\infty$ it is easy to check that (1.2) and (1.3) are identical, so that

(1.4) \quad \gamma = \gamma_2 \text{ on } \ell_\infty \otimes \ell_\infty

Indeed, if $x_1, \ldots, x_n \in \ell_\infty(S)$ over an index set $S$ we have clearly

$$\sup_{\xi \in B_{\ell_\infty}} (\sum |(\xi(x_i)|^2)^{1/2} = \sup \sum |\alpha_i x_i|_{\ell_\infty} = \sup_{s \in S} (\sum |x_i(s)|^2)^{1/2} = \left\| \left( \sum |x_i|^2 \right)^{1/2} \right\|_{\ell_\infty}$$

Let $S$ and $T$ be two index sets. Consider $u$ in $\ell_\infty(S) \otimes \ell_\infty(T)$ with associated matrix $u(s,t) = \langle \delta_s \otimes \delta_t, u \rangle$ (we denote by $(\delta_s)$ and $(\delta_t)$ the Dirac masses at $s$ and $t$ respectively, viewed as linear functionals on $\ell_\infty(S)$ and $\ell_\infty(T)$). Then we have $\gamma_2(u) \leq 1$ iff there are maps $x : S \to \ell_2$ and $y : T \to \ell_2$ such that $\sup_{s \in S} \| x(s) \| \leq 1$, $\sup_{t \in T} \| y(t) \| \leq 1$ and

$$\forall \ s, t \in T \quad u(s,t) = \langle x(s), y(t) \rangle.$$

This is very easy to check.

The following result is well known

**Proposition 1.1.** Let $S, T$ be arbitrary sets. Let $\varphi : S \times T \to \mathbb{C}$ be a function. We consider the Schur multiplier

$$M_\varphi : B(\ell_2(S), \ell_2(T)) \to B(\ell_2(S), \ell_2(T))$$

defined in matrix notation by $M_\varphi((a(s,t))) = (\varphi(s,t)a(s,t))$. The following are equivalent

(i) $\| M_\varphi \| \leq 1$.

(ii) There are vectors $x(s), y(t)$ in a Hilbert space such that $\sup_s \| x(s) \| \leq 1, \sup_t \| y(t) \| \leq 1$ and $\varphi(s,t) = \langle x(s), y(t) \rangle$.

(iii) For all finite subsets $E \subset S$ and $F \subset T$ we have

$$\left\| \sum_{(s,t) \in E \times F} \varphi(s,t) e_s \otimes e_t \right\|_{\ell_\infty(S) \hat{\otimes}_2 \ell_\infty(T)} \leq 1.$$

Moreover if $S$ and $T$ are finite sets then (i) (ii) and (iii) are equivalent to
(iv) \( \left\| \sum_{(s,t) \in S \times T} \varphi(s,t)e_s \otimes e_t \right\|_{\ell_\infty(S) \otimes_{\gamma_2} \ell_\infty(T)} \leq 1 \) where \((e_s)\) and \((e_t)\) denote the canonical bases of \(\ell_\infty(S)\) and \(\ell_\infty(T)\) respectively.

**Proof.** Let us first assume that \(S\) and \(T\) are finite sets. The equivalence of (ii) (iii) and (iv) is then obvious. Assume (i). This means exactly that for any \(a : \ell_2(S) \to \ell_2(T)\) with \(\|a\| \leq 1\) and for any \(\alpha\) and \(\beta\) in the unit ball respectively of \(\ell_2(S)\) and \(\ell_2(T)\) we have

\[
|\sum_{s,t} \varphi(s,t)a(s,t)\alpha(s)\beta(t)| \leq 1.
\]

In other words \(\|M_\varphi\| \leq 1\) means that \(\varphi\) lies in the polar of the set \(C_1\) of all matrices of the form \((\alpha(s)a(s,t)\beta(t))\) with \(a, \alpha, \beta\) as above. But it turns out that this set \(C_1\) is itself the polar of the set \(C_2\) of all matrices \((\varphi(s,t))\) such that \(\|\sum \psi(s,t)e_s \otimes e_t\|_{\ell_\infty(S) \otimes_{\gamma_2} \ell_\infty(T)} \leq 1\).

(Indeed, this follows from the known factorization property which describes the norm \(\gamma_2^*\) which is dual to the norm \(\gamma_2\), cf. e.g. [Kw] or [P1] chapter 2.b). In conclusion \(\varphi\) belongs to \(C^{00}_2 = C_2\) iff (i) holds, and this proves the equivalence of (i) and (iv) in the case \(S\) and \(T\) are finite sets.

In the general case of arbitrary sets \(S\) and \(T\), we note that (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (i) is obvious by passing to finite subsets. It remains to prove (i) \(\Rightarrow\) (ii), but this is immediate by a compactness argument. Indeed, if (i) holds there is obviously a net \((\varphi^i)\) tending to \(\varphi\) pointwise and formed with finitely supported functions on \(S \times T\) such that \(\|M_{\varphi^i}\| \leq 1\). Then by the first part of the proof, each \(\varphi^i\) satisfies (ii) and it is easy to conclude by an ultraproduct argument that \(\varphi\) also does. \(\blacksquare\)

**Remark.** As observed by Uffe Haagerup (see [H3]) Proposition 1.1 implies that the completely bounded norm of \(M_\varphi\) coincides with its norm. Indeed, it is easy to deduce from (ii) that \(\|M_\varphi\|_{cb} \leq 1\).

In the harmonic analysis literature, the c.b. multipliers of \(C_\lambda(G)\) are sometimes called Herz-Schur multipliers. They were considered by Herz (in a dual framework, as multipliers on \(A(G)\)) before the notion of complete boundedness surfaced. The next result from [BF] (see also [H3]) clarifies the relation between the various kinds of multipliers.

**Proposition 1.2.** Let \(G\) be a discrete group. Consider a function \(\varphi : G \to \mathbb{C}\). We define then complex functions \(\varphi_1, \varphi_2\) and \(\varphi_3\) on \(G \times G\) by setting

\[
\forall (s,t) \in G \times G \quad \varphi_1(s,t) = \varphi(st^{-1}), \varphi_2(s,t) = \varphi(s^{-1}t), \varphi_3(s,t) = \varphi(st).
\]
We consider the corresponding Schur multipliers $M_{\varphi_1}, M_{\varphi_2}$ and $M_{\varphi_3}$ on $B(\ell_2(G), \ell_2(G))$. Then

(i) ([BF]) The Schur multiplier $M_{\varphi_1}$ is bounded iff the linear operator $T_\varphi : C_\lambda^*(G) \rightarrow C_\lambda^*(G)$ which maps $\lambda(x)$ to $\varphi(x)\lambda(x)$ is completely bounded. Moreover, $\|M_{\varphi_1}\| = \|T_\varphi\|_{cb}$.

(ii) Moreover, $M_{\varphi_1}$ is bounded iff $M_{\varphi_2}$ (resp. $M_{\varphi_3}$) is bounded and we have

$$\|M_{\varphi_1}\| = \|M_{\varphi_2}\| = \|M_{\varphi_2}\|_{cb} = \|M_{\varphi_3}\| = \|M_{\varphi_3}\|_{cb}.$$  

Proof. The last assertion is immediate (note that if $\psi(s, t)$ is a bounded Schur multiplier on $S \times T$ then for any bijections $f : S \rightarrow S$ and $g : T \rightarrow T$ $\psi(f(s), g(t))$ also is a bounded Schur multiplier with the same norm). Note that $M_{\varphi_1}$ leaves $C_\lambda^*(G)$ invariant and its restriction to $C_\lambda^*(G)$ coincides with $T_\varphi$. Hence by the preceding remark we have

$$\|T_\varphi\|_{cb} \leq \|M_{\varphi_1}\|_{cb} = \|M_{\varphi_1}\|.$$  

Conversely, if $\|T_\varphi\|_{cb} \leq 1$ then the factorization theorem of c.b. maps due to Wittstock (Haagerup [H3] and Paulsen proved it independently, see [Pa]) says that there is a Hilbert space $H$, a representation $\pi : B(\ell_2(G)) \rightarrow B(H)$ and operators $V_1$ and $V_2$ from $\ell_2(G)$ into $H$ with $\|V_1\| \leq 1, \|V_2\| \leq 1$ such that $\forall a \in C_\lambda^*(G)$ $T_\varphi(a) = V_2^* \pi(a)V_1$.

In particular we have $\varphi(x)\lambda(x) = T_\varphi(\lambda(x)) = V_2^* \pi(\lambda(x))V_1$, which implies

$$\forall s, t \in G \quad \varphi(st^{-1}) = <\delta_s, T_\varphi(\lambda(st^{-1}))\delta_t>$$

$$= <\pi(\lambda(s))^*V_2\delta_s, \pi(\lambda(t^{-1}))V_1\delta_t>$$

This shows that $\varphi_1$ satisfies (ii) in Proposition 1.1, hence $\|M_{\varphi_1}\| \leq 1$. 

Grothendieck [G] proved that $\gamma_2$ and $\|\|_{\wedge}$ are equivalent norms on $\ell_\infty \otimes \ell_\infty$ (or on $\ell_\infty(S) \otimes \ell_\infty(T)$, more precisely there is a constant $K_G$ such that

$$\forall u \in \ell_\infty \otimes \ell_\infty \quad \|u\|_{\wedge} \leq K_G \gamma_2(u).$$

The exact numerical value of the best constant $K_G$ in (1.5) is still an open problem (see [P1] for more recent results).

Grothendieck’s striking theorem admits many equivalent reformulations. In the context of Banach lattices, Krivine [K] emphasized the following one. Let $X, Y$ be 2-convex Banach lattices, then $\gamma$ and $\|\|_{\wedge}$ are equivalent norms on $X \otimes Y$ and we have

$$\forall u \in X \otimes Y \quad \|u\|_{\wedge} \leq K_G \gamma(u).$$
Note that in (1.5) and (1.6) the converse inequality is trivial (since \( \| \| \wedge \) is the “greatest cross-norm”), we have \( \gamma_2(u) \leq \| u \| \wedge \) and \( \gamma(u) \leq \| u \| \wedge \) for all \( u \) in \( X \otimes Y \).

The reader should note that the equality \( \gamma = \gamma_2 \) on \( \ell_\infty \otimes \ell_\infty \) is a special property of \( \ell_\infty \) spaces. If \( X = Y = \ell_2 \) for instance then on \( X \otimes Y \) \( \gamma_2 \) is the injective norm (i.e. the usual operator norm) while \( \gamma \) is identical to the projective norm (i.e. the trace class norm).

We refer the reader to [P2] for the discussion of a more general class of cross-norms which behave like \( \gamma \) and \( \gamma_2 \).

While the proof of Proposition 1.1 uses nothing more than the Hahn-Banach theorem, the next result is a reformulation of Grothendieck’s theorem one more time, it was observed in some form already in [G] (Prop. 7, p. 68), and was later rediscovered and extended by various authors, notably J.Gilbert in harmonic analysis (see [GL],[Be]) and U.Haagerup in operator algebras (see [H3] the unpublished preliminary version of [H2]).

**Theorem 1.3.** In the case when \( S,T \) are finite sets in the same situation as Proposition 1.1 we have

\[
\frac{1}{K_G} \left\| \sum \varphi(s,t)e_s \otimes e_t \right\|_{\ell_\infty(S) \hat{\otimes} \ell_\infty(T)} \leq \left\| M_\varphi \right\| \leq \left\| \sum \varphi(s,t)e_s \otimes e_t \right\|_{\ell_\infty(S) \hat{\otimes} \ell_\infty(T)}.
\]

Moreover, when \( S,T \) are arbitrary sets, the space of all bounded Schur multipliers of \( B(\ell_2(S), \ell_2(T)) \) coincides with the space \( \hat{V}(S,T) \).

**Proof.** This follows immediately from Grothendieck’s inequality (1.5) and Proposition 1.1. \( \blacksquare \)

Perhaps a more intuitive formulation is as follows. Let us call “simple multipliers” the Schur multipliers of the form

\[ \varphi(s,t) = \varepsilon_s \eta_t \]

with \( \varepsilon_s, \eta_t \in \mathbb{C} \) such that \( |\varepsilon_s| \leq 1, |\eta_t| \leq 1 \). These are obviously such that \( \| M_\varphi \| \leq 1 \), but precisely Theorem 1.3 says that any multiplier \( \varphi \) with \( \| M_\varphi \| \leq \frac{1}{K_G} \) lies in the convex hull of the set of simple multipliers if \( S,T \) are finite sets and if \( S,T \) are infinite sets, then \( \varphi \) lies in the pointwise closure of the convex hull of the set of simple multipliers.

**Remark 1.4 :** Consider again a “simple Schur multiplier” of the form \( \varphi(s,t) = \varepsilon_s \eta_t \) as above with \( |\varepsilon_s| \leq 1, |\eta_t| \leq 1 \). Then we have

\[ \forall A \in B(\ell_2(S), \ell_2(T)) \quad M_\varphi(A) = v_1 Av_2 \]
where \( v_1 : \ell_2(S) \to \ell_2(S) \) and \( v_2 : \ell_2(T) \to \ell_2(T) \) are the diagonal operators of multiplication by \((\varepsilon_s)\) and \((\eta_t)\) respectively. Therefore, it is obvious that \( \|M_\varphi\|_{cb} \leq 1 \). (see [Pa] for more information.)

We need to consider “sums” of Banach spaces which are usually not direct sums. Although we will mainly work with natural concrete Banach spaces \( X \) and \( Y \) for which saying that an element belongs to \( X + Y \) will have a clear meaning, we recall the following formal definition of \( X + Y \).

Assume that \( X, Y \) are both continuously injected in a larger topological vector space \( \mathcal{X} \). Then \( X + Y \) is defined as the subspace of \( \mathcal{X} \) of all elements of the form \( \sigma = x + y \) with \( x \in X, y \in Y \), equipped with the norm

\[
\|\sigma\|_{X+Y} = \inf \{ \|x\|_X + \|y\|_Y | \sigma = x + y \}.
\]

Equipped with this norm, \( X + Y \) is a Banach space (and its dual can be identified with \( X^* \cap Y^* \) under some mild compatibility assumption on \( X, Y \)).

Alternately, one may consider the direct sum \( X \oplus_1 Y \) equipped with the norm \( \|(x, y)\| = \|x\| + \|y\| \) together with the closed subspace \( N \subset X \oplus_1 Y \) of all elements \((x, y)\) which satisfy the identity \( x + y = 0 \) when injected into \( \mathcal{X} \).

Then the quotient space \( \sum = (X \oplus_1 Y)/N \) can be identified with \( X + Y \).

It will be convenient at some point to use the following elementary fact.

**Lemma 1.5.** Let \( dm(t) = \frac{dt}{2\pi} \) be the normalized Haar measure on \( T \). Then for any integer \( N \) and any continuous function \( f : T^N \to \mathbb{R} \) we have

\[
\int f(t_1, \ldots, t_N)dm(t_1)\ldots dm(t_N) \geq \inf \int f(e^{in_1t}, e^{in_2t}, \ldots, e^{in_Nt})dm(t)
\]

where the infimum on the right side runs over all sets of integers \( n_1, n_2, \ldots, n_N \) with \( 2n_1 < n_2, 2n_2 < n_3, \ldots, 2n_{N-1} < n_N \).

**Proof.** If \( f \) is a trigonometric polynomial, this is obvious by choosing \((n_k)\) lacunary enough. By density, this must remain true for all real valued \( f \) in \( C(T^N) \). Let us consider the infinite dimensional torus \( T^\mathbb{N} \). We denote by \( z_j \) the \( j \)-th coordinate on \( T^\mathbb{N} \) and by \( \mu \) the normalized Haar measure on \( T^\mathbb{N} \). The following is a reformulation of the main result of [LPP].
Theorem 1.6. Let \( a_1, \ldots, a_n \) be elements of a von Neumann algebra \( M \), let \( \xi_1, \ldots, \xi_n \) be elements of the predual \( M_* \). Then

\[
| \sum_{j=1}^{n} \langle \xi_j, a_j \rangle | \leq \int \left[ \| \sum z_j a_j \|_{M_*} \, d\mu(z) [ \| (\sum a_j^* a_j)^{1/2} \|_M + \| (\sum a_j a_j^*)^{1/2} \|_M ] \right].
\]

Proof. Two approaches are given in [LPP]. The first one proves this result using the factorization of analytic functions in \( H^1 \) with values in \( M_* \). Actually, in [LPP] (1.7) is stated with \( \int \| \sum z_j \xi_j \|_{M_*} \, d\mu(z) \) replaced by \( \int \| \sum e^{in_j t} \xi_j \|_{M_*} \, dm(t) \) for any lacunary sequence \( n_j \) such that \( n_j > 2n_{j-1} \). Using the preceding lemma, it is then easy to obtain (1.7) as stated above. (Moreover, it is possible to use the factorization argument of [LPP] directly in \( T^N \), see the following remark.) A second approach is given in the appendix of [LPP]. There it is shown that (1.7), with some additional numerical factor, can be deduced from (and is essentially equivalent to) the non-commutative Grothendieck inequality due to the author (see [P1], Theorem 9.4 and Corollary 9.5).

Remark. The reader may find the use of a lacunary sequence \((n_k)\) in the preceding proof a bit artificial. Actually, we can use directly the independent sequence \((z_k)\) on \( T^N \) equipped with \( \mu \). Indeed, the classical factorization theory of \( H^1 \) functions as products of two \( H^2 \) functions extends to this setting, provided one considers \( T^N \) as a compact group with ordered dual in the sense e.g. of [R] chapter 8. Here the dual of \( T^N \) is ordered lexicographically. The factorization of matrix valued functions (as used in [P2] Appendix B) also extends to this setting, so that the main results of [P2] also remain valid in this setting. This approach is described in [P3]. We chose the more traditional “one dimensional” torus presentation to provide more precise and explicit references for the reader.

We will use the following well known consequence of the Hahn-Banach Theorem (cf. [Kw], see also e.g. Lemma 1.3 in [P2]).

Lemma 1.7. Let \( S, T \) be finite sets. Let \( u : \ell_\infty(S, \ell_2) \hat{\otimes}_\gamma \ell_\infty(T) \to \mathbb{C} \) be a linear form of norm \( \leq 1 \) on \( \ell_\infty(S, \ell_2) \hat{\otimes}_\gamma \ell_\infty(T) \). Then there are probabilities \( P, Q \) on \( S \) and \( T \) such that \( \forall \varphi \in \ell_\infty(S, \ell_2) \forall \eta \in \ell_\infty(T) \).

\[
| \langle u, \varphi \otimes \eta \rangle | \leq \left( \int \| \varphi(s) \|^2_{\ell_2} \, dP(s) \right)^{1/2} \left( \int |\eta(t)|^2 \, dQ(t) \right)^{1/2}.
\]

Proof. (Sketch). By assumption and by definition (1.3) we have for all finite sequences
\( \varphi_k \in \ell_\infty(S, \ell_2), \eta_k \in \ell_\infty(T) \)

\[
\sum_k | < u, \varphi_k \otimes \eta_k > | \leq \sup_{s \in S} (\sum \| \varphi_k(s) \|^2)^{1/2} \sup_{t \in T} (\sum | \eta_k(t) |^2)^{1/2} \\
\leq \frac{1}{2} \sup_{s \times T} \{ \sum \| \varphi_k(s) \|^2 + | \eta_k(t) |^2 \}
\]

Let \( C \) be the convex cone in \( C(S \times T) \) formed by all the functions of the form

\[
(s, t) \to \frac{1}{2} \sum \| \varphi_k(s) \|^2 + | \eta_k(t) |^2 - | < u, \varphi_k \otimes \eta_k > |.
\]

Then \( C \) is disjoint from the open cone \( C_\ominus = \{ \varphi \mid \max \varphi < 0 \} \), hence (Hahn-Banach) there is a hyperplane in \( C(S \times T) \) which separates \( C \) and \( C_\ominus \). By an obvious adjustment, this yields a probability \( \lambda \) on \( S \times T \) such that \( \int f(s, t) d\lambda \geq 0 \) for any \( f \) in \( C \). Hence letting \( P \) (resp. \( Q \)) be the projection of \( \lambda \) on the first (resp. second) coordinate we obtain

\[
| < u, \varphi \otimes \eta > | \leq \frac{1}{2} (\int \| \varphi(s) \|^2 dP(s) + \int | \eta(t) |^2 dQ(t)).
\]

Finally applying this to \( (\varphi^{\theta^{-1}}) \otimes (\theta \eta) \) and minimizing the right hand side over all \( \theta > 0 \), we obtain the announced result (1.8). \( \blacksquare \)

**Remark 1.8** Let \( S, T \) be finite sets and let \( P, Q \) be probabilities on \( S \) and \( T \) respectively. Let \( J_P : \ell_\infty(S) \to L_2(P) \) and \( J_Q : \ell_\infty(T) \to L_2(Q) \) be the canonical inclusions. Then we have \( \forall \psi \in \ell_\infty(S) \otimes \ell_\infty(T) \)

\[
(1.9) \quad \|(J_P \otimes J_Q)(\psi)\|_{L_2(P) \otimes L_2(Q)} \leq \|\psi\|_{\ell_\infty(S) \otimes \ell_\infty(T)}. 
\]

Indeed, this is elementary. For any \( x_1, ..., x_n \) in \( \ell_\infty(S) \), \( y_1, ..., y_n \) in \( \ell_\infty(T) \), we have

\[
\sum \| x_i \|^2_{L_2(P)}^{1/2} \leq \| (\sum | x_i |^2 )^{1/2} \|_{\ell_\infty(S)} \quad \text{and} \quad \sum \| y_i \|^2_{L_2(P)}^{1/2} \leq \| (\sum | y_i |^2 )^{1/2} \|_{\ell_\infty(T)}. 
\]

This clearly implies (1.9).

**Remark 1.9** Let us denote simply by \( H_1(T; M_\ast) \) the subspace of \( L_1(T, dm; M_\ast) \) formed of all the functions \( f \) such that the \( (M_\ast\text{-valued}) \) Fourier transform is supported on the non-negative integers. Similarly, we can denote by \( H_1(T^\mathbb{N}; M_\ast) \) the subspace of \( L_1(T^\mathbb{N}, m^\mathbb{N}; M_\ast) \) formed by the functions with Fourier transform supported by the non-negative elements of \( \mathbb{Z}^\mathbb{N} \) ordered lexicographically. We again denote by \( z_j \) the \( j \)-th coordinate on \( T^\mathbb{N} \) and we let \( \hat{f}(z_j) = \int f \bar{z}_j \). In [LPP] the following refinement of (1.7) is proved.

Assume that there is a function \( f \) in the unit ball of \( H_1(T; M_\ast) \) (resp. \( H_1(T^\mathbb{N}; M_\ast) \)) such that \( \hat{f}(3^j) = a_j \) (resp. \( \hat{f}(z_j) = a_j \)) for all \( j \), then we have

\[
(1.10) \quad | \sum_{1}^{n} \xi_j, a_j > | \leq \left\| \left( \sum a_j^* a_j \right)^{1/2} \right\|_M + \left\| \left( \sum a_j^* a_j \right)^{1/2} \right\|_M.
\]
§ 2. Main results.

Our main result is a general statement which does not use the group structure at all, it can be viewed as a generalization of Varopoulos’s result stated above as Theorem 0.2.

Theorem 2.1. Let \( S \) and \( T \) be arbitrary sets, and let

\[
\psi_n \in \ell_\infty(S) \hat{\otimes} \ell_\infty(T)
\]

be a sequence such that the series

\[
\sum_{n=1}^{\infty} \varepsilon_n \psi_n
\]

converges in \( \ell_\infty(S) \hat{\otimes} \ell_\infty(T) \) for almost all choice of signs \( \varepsilon_n = \pm 1 \). Then, if we denote by \( (e_n) \) the canonical basis of \( \ell_2 \), the series

\[
\sum_{n=1}^{\infty} e_n \otimes \psi_n
\]

is convergent in the space \( \ell_\infty(S, \ell_2) \hat{\otimes} \ell_\infty(T) + \ell_\infty(S) \hat{\otimes} \ell_\infty(T, \ell_2) \).

Note. The spaces \( \ell_\infty(S, \ell_2) \hat{\otimes} \ell_\infty(T) \) and \( \ell_\infty(S) \hat{\otimes} \ell_\infty(T, \ell_2) \) are naturally continuously injected into \( \ell_\infty(S \times T, \ell_2) \), which is used to define the above sum.

Notation. Let \( \Omega = T^\mathbb{N} \). Let \( \mu \) be the normalized Haar measure on \( \Omega \), i.e. \( \mu = (\frac{dt}{2\pi})^\mathbb{N} \). We denote by \( z = (z_k)_{k \in \mathbb{N}} \) a generic point of \( \Omega \) (and we consider the \( k \)-th coordinate \( z_k \) as a function of \( z \)).

We will denote \( \ell_\infty \) instead of \( \ell_\infty(\mathbb{N}) \) and \( \ell_\infty(\ell_2) \) instead of \( \ell_\infty(\mathbb{N}, \ell_2) \).

With this notation, we can state a more precise version of Theorem 2.1.

Theorem 2.2. In the same situation as Theorem 2.1, let \( \psi_1, ..., \psi_n \) be a finite sequence in \( \ell_\infty(S) \hat{\otimes} \ell_\infty(T) \).

(i) Assume

\[
(2.1) \quad \int \gamma_2(\sum_{k=1}^{n} z_k \psi_k) d\mu(z) < 1.
\]

Then there is a decomposition in \( \ell_\infty(S) \hat{\otimes} \ell_\infty(T) \) of the form

\[
\psi_k = A_k + B_k
\]
such that (with $\gamma$ as defined in (1.3))

$$\left\| \sum_{1}^{n} e_k \otimes A_k \right\|_{\ell_{\infty}(S, \ell_{2}) \otimes \ell_{\infty}(T)} < 1$$

and

$$\left\| \sum_{1}^{n} e_k \otimes B_k \right\|_{\ell_{\infty}(S) \otimes \ell_{\infty}(T, \ell_{2})} < 1$$

(ii) Assume

$$\int \left\| \sum_{1}^{n} z_k \psi_k \right\|_{\ell_{\infty}(S) \otimes \ell_{\infty}(T)} d\mu(z) < 1$$

then there is a decomposition $\psi_k = A_k + B_k$ such that

$$\left\| \sum_{1}^{n} e_k \otimes A_k \right\|_{\ell_{\infty}(S, \ell_{2}) \otimes \ell_{\infty}(T)} < K_G$$

and

$$\left\| \sum_{1}^{n} e_k \otimes B_k \right\|_{\ell_{\infty}(S) \otimes \ell_{\infty}(T, \ell_{2})} < K_G,$$

where $K_G$ is the Grothendieck constant.

**Proof.** The proof is based on the main result of [LPP] reformulated above as Theorem 1.6. By a standard Banach space technique, Theorem 2.2 can be reduced to the case when $S$ and $T$ are finite sets. (Use the fact that $\ell_{\infty}$ is a $\mathcal{L}_{\infty}$ space, more precisely it can be viewed as the closure of the union of an increasing family of finite dimensional sublattices each isometric to $\ell_{\infty}(S)$ for some finite set $S$).

We will denote by $\alpha_1$ (resp. $\alpha_2$) the norm on $\ell_1(S, \ell_{2}^{n}) \otimes \ell_1(T)$ (resp. $\ell_1(S) \otimes \ell_1(T, \ell_{2}^{n})$) which is dual to the norm in $\ell_{\infty}(S, \ell_{2}^{n}) \otimes \ell_{\infty}(T)$ (resp. $\ell_{\infty}(S) \otimes \ell_{\infty}(T, \ell_{2}^{n})$).

Let $(e_k)$ be the canonical basis of $\ell_{2}^{n}$. Let $A_k \in \ell_1(S) \otimes \ell_1(T)$ and let $\Phi = \sum e_k \otimes A_k$. We will make the obvious identifications permitting to view $\Phi$ as an element either of $\ell_1(S, \ell_{2}^{n}) \otimes \ell_1(T)$ or of $\ell_1(S) \otimes \ell_1(T, \ell_{2}^{n})$. Then, by duality Theorem 2.2 (i) is equivalent to the following inequality.

For all $\psi_k$ in $\ell_{\infty}(S) \otimes \ell_{\infty}(T)$

$$\left| \sum A_k \otimes \psi_k \right| \leq \int \gamma_2(\sum z_k \psi_k) d\mu(z) [\alpha_1(\Phi) + \alpha_2(\Phi)].$$

To check this, by homogeneity we may assume (2.1) and also

$$\alpha_1(\Phi) + \alpha_2(\Phi) = 1.$$
Then, by Lemma 1.7, there are probabilities $P_1, P_2$ on $S$ and $Q_1, Q_2$ on $T$ such that (with obvious identifications).

$$\forall \varphi \in \ell_\infty(S, \ell_2^n) \quad \forall \eta \in \ell_\infty(T)$$

$$| < \varphi \otimes \eta, \Phi > | \leq \alpha_1(\Phi) \left( \int \| \varphi(s) \|_{\ell_2^n}^2 dP_1(s) \int |\eta(t)|^2 dQ_1(t) \right)^{1/2}$$

and

$$\forall \beta \in \ell_\infty(S) \quad \forall \omega \in \ell_\infty(T, \ell_2^n)$$

$$| < \beta \otimes \omega, \Phi > | \leq \alpha_2(\Phi) \left( \int |\beta(s)|^2 dP_2(s) \int \| \omega(t) \|_{\ell_2^n}^2 dQ_2(t) \right)^{1/2}.$$

Now let $P = \alpha_1(\Phi)P_1 + \alpha_2(\Phi)P_2, Q = \alpha_1(\Phi)Q_1 + \alpha_2(\Phi)Q_2$. By (2.3) these are probabilities.

Then

(2.4) $$| < \varphi \otimes \eta, \Phi > | \leq \left( \int \| \varphi(s) \|_{\ell_2^n}^2 dP(s) \right)^{1/2} \left( \int |\eta(t)|^2 dQ(t) \right)^{1/2}$$

and

(2.5) $$| < \beta \otimes \omega, \Phi > | \leq \left( \int |\beta(s)|^2 dP(s) \right)^{1/2} \left( \int \| \omega(t) \|_{\ell_2^n}^2 dQ(t) \right)^{1/2}.$$

This means that $A_k$ defines a bounded linear operator $a_k : L_2(P) \rightarrow L_2(Q)^*$ such that $< a_k(\beta), \eta > = < \beta \otimes \eta, A_k >$. Moreover (2.5) and (2.4) imply respectively $\| \sum a_k^* a_k \| \leq 1$ and $\| \sum a_k a_k^* \| \leq 1$. Let $J_P : \ell_\infty(S) \rightarrow L_2(P)$ and $J_Q : \ell_\infty(T) \rightarrow L_2(Q)$ be the canonical inclusions. Let $\xi_k = (J_P \otimes J_Q)(\psi_k) \in L_2(P) \otimes L_2(Q)$. By (1.9) we have

$$\int \left\| \sum z_k \xi_k \right\|_{L_2(P) \otimes L_2(Q)} d\mu(z) < 1.$$ 

Note that $< \xi_k, a_k > = < \psi_k, A_k >$. Hence applying (1.7) we obtain the desired inequality (2.2). This concludes the proof of the first part. The second part is an immediate consequence of the first one by (1.6). ■

Remark: It is also possible to deduce Theorem 2.2 directly from the factorization Theorem of [P2] (see Corollary 1.7 or Theorem 2.3 in [P2]), which applies in particular to functions in $H^1$ with values in $\ell_\infty(S) \otimes_{\gamma_2} \ell_\infty(T)$. Using this, the argument of [LPP] then gives the
decomposition of Theorem 2.2 in a somewhat more explicit fashion as a formula in terms of the factorization of the “analytic” function $z \to \sum z_k \psi_k$.

Remark 2.3. Let $A_k$ be as above such that

$$ (2.9) \quad \left\| \sum_{1}^{n} e_k \otimes A_k \right\|_{\ell_{\infty}(S,\ell_2) \hat{\otimes}_2 \ell_{\infty}(T)} < 1. $$

Then for any $n$-tuple $t_1, ..., t_n$ in $T$ we have

$$ (2.10) \quad \sup_{s \in S} \left( \sum_{k=1}^{n} |A_k(s,t_k)|^2 \right)^{1/2} < 1. $$

A similar remark holds for $\sum_{1}^{n} e_k \otimes B_k$.

Indeed, by the definition (1.3), (2.9) means that there is a Hilbert space $H$ and elements $\alpha$ in $\ell_{\infty}(S,\ell_2(H))$ and $\beta$ in $\ell_{\infty}(T,H)$ each with norm $< 1$ such that

$$ (2.11) \quad \forall \ k = 1, ..., n \quad A_k(s,t) = \langle \alpha_k(s), \beta(t) \rangle $$

(where $\alpha(s) \in \ell_2(H)$ and $\alpha_k(s)$ denotes the $k$–th coordinate of $\alpha(s)$). Then (2.10) is an immediate consequence of (2.11).

We now derive Theorem 0.1 from Theorems 2.1 and 2.2.

**Proof of Theorem 0.1.**

(i) $\Rightarrow$ (ii) is trivial. Assume (ii). By Remark 1.2 for almost all choices of signs $\varepsilon$ in $\{-1,1\}^G$ the function $(s,t) \to \varepsilon(st)\varphi(st)$ defines a c.b. Schur multiplier $M_{\varepsilon\varphi}$ of $B(\ell_2(G),\ell_2(G))$.

We can assume $\|M_{\varepsilon\varphi}\|_{cb} \leq F(\varepsilon)$ for some measurable function $F(\varepsilon)$ finite almost everywhere on $\{-1,1\}^G$. A fortiori for each finite subsets $S \subset G$ and $T \subset G$, the function $(s,t) \to \varepsilon(st)\varphi(st)$ restricted to $S \times T$ is a c.b. Schur multiplier of $B(\ell_2(S),\ell_2(T))$ with norm $\leq F(\varepsilon)$. By Proposition 1.1, this means that we have for all finite subsets $S \subset G, T \subset G$

$$ \left\| \sum_{(s,t) \in S \times T} \varepsilon(st)\varphi(st)e_s \otimes e_t \right\|_{\ell_{\infty}(S) \hat{\otimes}_2 \ell_{\infty}(T)} \leq F(\varepsilon), $$

where we have denoted by $(e_s)$ and $(e_t)$ the canonical bases of $\ell_{\infty}(S)$ and $\ell_{\infty}(T)$.

Equivalently we have

$$ (2.12) \quad \left\| \sum_{x \in S \times T} \varepsilon(x)\varphi(x) \sum_{(s,t) \in S \times T, \ s_t = x} e_s \otimes e_t \right\|_{\ell_{\infty}(S) \hat{\otimes}_2 \ell_{\infty}(T)} \leq F(\varepsilon). $$
By a classical integrability result of Kahane (cf. [Ka]) we can assume that $F$ is integrable over $\{-1, 1\}^G$, so that there is a number $C > 0$ such that the average over $\epsilon$ of the left side of (2.12) is less than $C$. By a simple elementary reasoning (decompose into real and imaginary parts, use the triangle inequality and the unconditionality of the average over $\epsilon$), it follows from (2.12) that if $\mu_G$ denotes the normalized Haar measure on $T^G$ and if $z = (z_x)_{x \in G}$ denotes a generic point of $T^G$, we have

$$\mu_G \left( \left\| \sum_{x \in ST} z_x \varphi(x) \sum_{(s,t) \in S \times T} e_s \otimes e_t \right\|_{\ell\infty(S) \otimes \ell\infty(T)} \right) < 2C.$$ 

Let $\psi_x = \varphi(x) \sum_{(s,t) \in S \times T} e_s \otimes e_t$ for all $x$ in $ST$ and let $\psi_x = 0$ otherwise.

Then by Theorem 2.2 and Remark 2.3 we have a decomposition

$$\psi_x = A_x + B_x$$

in $\ell\infty(S) \otimes \ell\infty(T)$ such that

$$\sup_{s \in S} \left( \sum_{t \in T} |A_{st}(s,t)|^2 \right)^{1/2} < 2C$$

and

$$\sup_{t \in T} \left( \sum_{s \in S} |B_{st}(s,t)|^2 \right)^{1/2} < 2C$$

This yields functions $\varphi_1$ and $\varphi_2$ on $S \times T$ such that $\varphi_1(s,t) = A_{st}(s,t)$, $\varphi_2(s,t) = B_{st}(s,t)$ and

$$\forall (s,t) \in S \times T \quad \varphi(st) = \psi_{st}(s,t) = \varphi_1(s,t) + \varphi_2(s,t).$$

Hence

$$\sup_{s \in S} \left( \sum_{t \in T} |\varphi_1(s,t)|^2 \right)^{1/2} < 2C \sup_{t \in T} \left( \sum_{s \in S} |\varphi_2(s,t)|^2 \right)^{1/2} < 2C.$$ 

Let us denote by $\varphi_1^{S,T}$ and $\varphi_2^{S,T}$ the functions obtained on $G \times G$ by extending $\varphi_1$ and $\varphi_2$ by zero outside $S \times T$.

Now if we let $S \times T$ tend to $G \times G$ along the set of all products of finite sets directed by inclusion and if we let $\Phi_1, \Phi_2$ be pointwise cluster points of the corresponding sets ($\varphi_1^{S,T}$) and ($\varphi_2^{S,T}$), we obtain finally two functions $\Phi_1$ and $\Phi_2$ on $G \times G$ such that $\varphi(st) = \Phi_1(s,t) + \Phi_2(s,t)$ for all $(s,t)$ in $G \times G$ and satisfying

$$\sup_{s \in G} \left( \sum_{t \in G} |\Phi_1(s,t)|^2 \right)^{1/2} \leq 2C \sup_{t \in G} \left( \sum_{s \in G} |\Phi_2(s,t)|^2 \right)^{1/2} \leq 2C$$
Let then \( \Gamma_1 \cup \Gamma_2 = G \times G \) be a partition defined by
\[
\Gamma_1 = \{(s, t) \in G \times G \mid |\Phi_1(s, t)| \geq |\Phi_2(s, t)|\}
\]
\[
\Gamma_2 = \{(s, t) \in G \times G \mid |\Phi_1(s, t)| < |\Phi_2(s, t)|\}
\]
It is then clear that \( \varphi \) satisfies the property (iii) in Theorem 0.1. This shows (ii) \( \Rightarrow \) (iii). The equivalence (iii) \( \Leftrightarrow \) (iv) is part of Theorem 0.2 (due to Varopoulos). (Note that the implication (iii) \( \Rightarrow \) (i) also follows from the implication (iii) \( \Rightarrow \) (i) in Varopoulos’s Theorem 0.2.)

We now show (iii) \( \Rightarrow \) (v). Assume (iii). Let \( a(x) \) be as in (v) and let \( g \) and \( h \) be in the unit ball of \( \ell_2(G, H) \). Assume
\[
\max \left\{ \left\| \left( \sum a(x)^* a(x) \right)^{1/2} \right\|, \left\| \left( \sum a(x)a(x)^* \right)^{1/2} \right\| \right\} \leq 1.
\]
It clearly suffices to show that
\[
| \sum_{s, t \in G \times G} \varphi(st^{-1}) < h(s), a(st^{-1})g(t) > | \leq 2C.
\]
Let \( \Sigma_1 \) (resp. \( \Sigma_2 \)) be the left side of (2.14) with the summation restricted to \( (s, t^{-1}) \in \Gamma_1 \) (resp. \( (s, t^{-1}) \in \Gamma_2 \)). Observe that \( \sum_{(s, t^{-1}) \in \Gamma_1} |\varphi(st^{-1})|^2 \|h(s)\|^2 \leq C^2 \), hence by Cauchy-Schwarz and (2.14)
\[
|\Sigma_1| \leq C \left( \sum_{(s, t^{-1}) \in \Gamma_1} \|a(st^{-1})g(t)\|^2 \right)^{1/2} \leq C \left( \sum_{(s, t) \in G \times G} <a(st^{-1})^* a(st^{-1})g(t), g(t) > \right)^{1/2}
\]
\[
\leq C \left( \sum_t \|g(t)\|^2 \sum_s a(st^{-1})^* a(st^{-1}) \right)^{1/2} \leq C.
\]
A similar argument yields \( |\Sigma_2| \leq C \) hence (2.15) follows and the proof of (iii) \( \Rightarrow \) (v) is complete.

Finally we show (v) \( \Rightarrow \) (i). We start by recalling that for any finitely supported function \( a : G \to B(H) \) we have the elementary inequality
\[
\max \left\{ \left\| \left( \sum a(x)^* a(x) \right)^{1/2} \right\|, \left\| \left( \sum a(x)a(x)^* \right)^{1/2} \right\| \right\} \leq \left\| \sum \lambda(x) \otimes a(x) \right\|_{B(\ell_2(G, H))}.
\]
Now assume (v). We have then by (2.16) if \( \sup_{x} |\varepsilon(x)| \leq 1 \)
\[
\left\| \sum_{x \in G} \varepsilon(x)\varphi(x)\lambda(x) \otimes a(x) \right\| \leq C \left\| \sum_{x \in G} \lambda(x) \otimes a(x) \right\|,
\]
hence the multiplier of $C^*_\lambda(G)$ defined by $\varepsilon\varphi$ is completely bounded with norm $\leq C$. This proves $(v) \Rightarrow (i)$. ■

Proof of Corollary 0.4. Let $p$ and $\varphi$ be as in Corollary 0.4.

Then there is a constant $C$ such that for all finite subsets $S, T$ of $G$ we have

\[
\int \left\| \sum_{x \in p(S,T)} \varphi(x) z_x ( \sum_{p(s,t)=x} e_s \otimes e_t ) \right\|_{\ell_\infty(G) \otimes \ell_\infty(G)} < C.
\]

Reasoning as above in the proof of $(ii) \Rightarrow (iii)$ in Theorem 0.1, we find a decomposition of the form

\[
\varphi(p(s,t)) = A_{p(s,t)}(s,t) + B_{p(s,t)}(s,t)
\]

and using Remark 2.3 and the bounds

\[
(2.17) \sup_{s,x} |\{t\mid p(s,t) = x\}| < \infty \quad \text{and} \quad \sup_{t,x} |\{s\mid p(s,t) = x\}| < \infty
\]

we can obtain that for some constant $C'$

\[
\sup_{s \in S} \sum_{t \in T} |A_{p(s,t)}(s,t)|^2 \leq C'
\]

\[
\sup_{t \in T} \sum_{s \in S} |A_{p(s,t)}(s,t)|^2 \leq C'.
\]

We then conclude the proof as in the proof of Theorem 0.1 by a pointwise compactness argument, showing that $(s,t) \to \varphi(p(s,t))$ satisfies $(iii)'$ in Theorem 0.2. hence (recall Proposition 1.1 or Theorem 1.3) for all bounded complex functions $(s,t) \to \varepsilon_{s,t}$ the function $(s,t) \to \varepsilon_{s,t} \varphi(p(s,t))$ is a Schur multiplier of $B(\ell_2(G), \ell_2(G))$.

Remark 2.4. Let $S, T, X$ be arbitrary sets and let $p : S \times T \to X$ be a map satisfying (2.17). Consider a function $\varphi : X \to \mathbb{C}$ and let $\psi : S \times T \to \mathbb{C}$ be defined by

\[
\psi(s,t) = \varphi(p(s,t)).
\]

Let $K(T)$ be the linear span of the canonical basis of $\ell_2(T)$. For any $x$ in $X$, let $\Lambda(x) : K(T) \to \mathbb{C}^S$ be the operator defined by the matrix $\Lambda_x(s,t)$ defined by $\Lambda_x(s,t) = 1$ if $p(s,t) = x$ and $\Lambda_x(s,t) = 0$ otherwise. Then we can generalize Varopoulos's theorem as follows. The following are equivalent
(i) For all bounded functions $\varepsilon : S \times T \to \mathbb{C}$ the pointwise product $\varepsilon \psi$ is a bounded Schur multiplier of $B(\ell_2(S), \ell_2(T))$.

(ii) For almost all choices of signs $\varepsilon \in \{-1, 1\}^{S \times T}$, the product $\varepsilon \psi$ is a bounded Schur multiplier of $B(\ell_2(S), \ell_2(T))$.

(iii) There is a partition of $S \times T$ say $S \times T = \Gamma_1 \cup \Gamma_2$ such that

$$\sup_{s \in S} \sum_{t \in T} |\psi(s, t)|^2 1_{\{(s, t) \in \Gamma_1\}} < \infty$$

$$\sup_{t \in T} \sum_{s \in S} |\psi(s, t)|^2 1_{\{(s, t) \in \Gamma_2\}} < \infty.$$ 

(iv) There is a constant $C$ such that for any Hilbert space $H$ and for any finitely supported function $a : X \to B(H)$ we have

$$\left\| \sum_{x \in X} \varphi(x) \Lambda(x) \otimes a(x) \right\|_{B(\ell_2(S, H), \ell_2(T, H))} \leq C \max \left\{ \left\| \left( \sum a(x)^* a(x) \right)^{1/2} \right\|, \left\| \left( \sum a(x) a(x)^* \right)^{1/2} \right\| \right\}.$$ 

This statement is proved exactly as above. This applies in particular when $p$ is the product map on a semigroup. When $S = T = X = \mathbb{N}$ and $p(s, t) = s + t$ we recover results already obtained in [B2].
§ 3. Lacunary sets.

The study of “thin sets” such as Sidon sets in discrete non Abelian groups has been developed by several authors, namely Leinert [L1,L2] Bożejko ([B1, 2, 3]), Figà-Talamanca and Picardello [FTP] and others. (See [LR] for the theory of Sidon sets in Abelian groups).

In this section, we apply the preceding results to a class of “lacunary” sets which we call “L-sets”. There is a striking analogy between the “L-sets” defined below and a class of subsets of \( \mathbb{N} \) which we will call Paley sets. A subset \( \Lambda \subset \mathbb{N} \) will be called a Paley set if there is a constant \( C \) such that for all \( f = \sum_{n=0}^{\infty} a_n e^{i n t} \) in \( H^1 \) we have

\[
(3.1) \quad \left( \sum_{n \in \Lambda} |\hat{f}(n)|^2 \right)^{1/2} \leq C \|f\|_1.
\]

It is well known (cf. e.g. [R]) that Paley sets are simply the finite unions of Hadamard-lacunary sequences, i.e. of sequences \( \{n_k\} \) such that \( \lim \inf_{k \to \infty} (n_{k+1}/(n_k)) > 1 \).

Equivalently, \( \Lambda \) is a Paley set iff there is a constant \( C \) such that

\[
\forall n > 0 \quad |\Lambda \cap [n, 2n[| \leq C.
\]

In [LPP], it is proved that if \( \Lambda \subset \mathbb{N} \) is a Paley set there is a constant \( C \) such that for any \( f \) in \( H^1(\hat{H} \otimes H) \) there is a decomposition in \( \hat{H} \otimes H \) of the form \( \hat{f}(n) = a(n) + b(n) \quad \forall n \in \Lambda \) such that

\[
(3.2) \quad \text{tr} \left( \sum_{n \in \Lambda} a(n)^* a(n) \right)^{1/2} + \text{tr} \left( \sum_{n \in \Lambda} b(n)^* b(n) \right)^{1/2} \leq C \|f\|_{H^1(\hat{H} \otimes H)}
\]

Moreover, when \( \hat{f} \) is supported by \( \Lambda \) this inequality becomes an equivalence.

The papers [LPP] and [HP] suggest that there is a strong analogy between Paley sequences and free subsets of a discrete group \( G \). To explain this we introduce more notation. Let \( H \) be a Hilbert space. We denote by \( A(G, \hat{H} \otimes H) \) the set of all functions \( f : G \to \hat{H} \otimes H \) such that for some \( g, h \) in \( \ell_2(G, H) \) we have

\[
\forall x \in G \quad f(x) = \sum_{st=x} g(s) \otimes h(t).
\]

Let

\[
\|f\|_{A(G, \hat{H} \otimes H)} = \inf \{ \|g\|_{\ell_2(G, H)} \|h\|_{\ell_2(G, H)} \}
\]
where the infimum runs over all possible representations. Then (see [HP]) if \( G \) is the free group on \( n \) generators \( g_1, ..., g_n \), we have the following analogue of (3.1). For any \( f \) in \( A(G, H \hat{\otimes} H) \) then there is a decomposition \( f(g_k) = a_k + b_k \) in \( H \hat{\otimes} H \) such that

\[
tr(\sum a_k^* a_k)^{1/2} + tr(\sum b_k b_k^*)^{1/2} \leq 2 \|f\|_{A(G, H \hat{\otimes} H)}.
\]

Moreover, when \( f \) is supported by \( \Lambda \) this inequality becomes an equivalence.

This motivated the following

**Definition 3.1.** A subset \( \Lambda \) of a discrete group \( G \) will be called an \( L \)-set if there is a constant \( C \) such that for any \( H \) and for any \( f \) in \( A(G, H \hat{\otimes} H) \) we have

\[
\inf_{f(x)=a(x)+b(x)} \left\{ tr \left( \sum_{x \in \Lambda} a(x)^* a(x) \right)^{1/2} + tr \left( \sum b(x)^* b(x) \right)^{1/2} \right\} \leq C \|f\|_{A(G, H \hat{\otimes} H)},
\]

where the infimum runs over all possible decompositions \( f(x) = a(x) + b(x) \) in \( H \hat{\otimes} H \).

**Proposition 3.2.** The following properties of a subset \( \Lambda \subset G \) are equivalent.

(i) \( \Lambda \) is an \( L \)-set.

(ii) There is a constant \( C \) such that for any Hilbert space \( H \) and for any finitely supported function \( a : \Lambda \to B(H) \) we have

\[
\left\| \sum_{x \in \Lambda} \lambda(x) \otimes a(x) \right\|_{B(\ell_2(G, H))} \leq C \max \left\{ \left\| (\sum a(x)^* a(x))^{1/2} \right\|, \left\| (\sum a(x)a(x)^*)^{1/2} \right\| \right\}.
\]

(iii) For any bounded sequence \( \varepsilon \) in \( \ell_\infty(G) \) supported by \( \Lambda \), the associated multiplier defined on \( C^*_\Lambda(G) \) by

\[
\sum f(x)\lambda(x) \to \sum f(x)\varepsilon(x)\lambda(x)
\]

is completely bounded on \( C^*_\Lambda(G) \).

**Proof.** (i) and (ii) are clearly equivalent. They are but a dual reformulation of each other. (ii) and (iii) are equivalent by Theorem 0.1 applied to the indicator function of \( \Lambda \).

**Remark.** By Theorem 0.1, the preceding properties are also equivalent to the property (iii)', obtained by requiring that the property (iii) holds only for almost all choice of signs \((\varepsilon(x))_{x \in \Lambda} \) in \( \{-1, 1\}^\Lambda \).
Remark. The preceding result shows that Λ is an $L$-set iff Λ is a strong 2-Leinert set in the sense of Božejko [B1]. Leinert [L1,L2] first constructed infinite sets of this kind in free noncommutative groups. Leinert’s results were clarified in [AO]. Moreover, in [H1] several related important inequalities were obtained for the operator norm of the convolution on the free group by a function supported by the words of a given fixed length in the generators. In particular, it was known to Haagerup (see [HP]) that any free subset of a discrete group is an $L$-set. For instance the generators (or the words of length one) on the free group with countably many generators form an $L$-set. On the other hand it is rather easy to see that the set of words of a fixed length $k > 1$ is not an $L$-set (for instance it clearly does not satisfy (ii) in Theorem 3.3).

The main result of this section is the following

**Theorem 3.3.** Let $G$ be an arbitrary discrete group. Let $Λ ⊂ G$ be a subset. Let $R_Λ ⊂ G × G$ be defined by

$$R_Λ = \{(s, t) ∈ G × G | st ∈ Λ\}.$$ 

The following properties of $Λ$ are equivalent

(i) $Λ$ is an $L$-set.

(ii) There is a constant $C$ such that for any finite subsets $E, F ⊂ G$ with $|E| = |F| = N$ we have

$$|R_Λ ∩ (E × F)| ≤ CN.$$ 

(iii) There is a constant $C$ and there is a partition $R_Λ = Γ_1 ∪ Γ_2$ such that

$$\sup_{s ∈ G} \sum_{t ∈ G} 1_{(s, t) ∈ Γ_1} ≤ C \quad \text{and} \quad \sup_{t ∈ G} \sum_{s ∈ G} 1_{(s, t) ∈ Γ_2} ≤ C.$$ 

**Proof:** This is an immediate consequence of Theorem 0.1 applied to the indicator function of $Λ$.

Remark. As already mentioned, the equivalence between (ii) and (iii) is due to Varopoulos [V1]. Our result shows that $Λ ⊂ G$ is an $L$-set iff $R_Λ$ determines a $V$-set for $ℓ_∞(G) ⊗ ℓ_∞(G)$ in the sense of Varopoulos (see [LP] and [V2]). Equivalently, let $G' = T^G$ and let $(γ_s)_{s ∈ G}$ be the coordinates on $G$. Then $Λ$ is an $L$-set in $G$ iff the set $\{γ_s × γ_t | st ∈ Λ\}$ is a Sidon set in the dual of $G' × G'$ i.e. in $\mathbb{Z}^{(G)} × \mathbb{Z}^{(G)}$. 

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Remark. Taking Remark 2.4 into account, we can extend the notion of $L$-set to the case when $\Lambda$ is a subset of a semi-group $G$ embeddable into a group (for instance $G = \mathbb{N}$). In that case we will say that $\Lambda$ is an $L$-set if it satisfies the equivalent properties of Theorem 3.3 (or the analogue of (ii) in Proposition 3.2). This provides a common framework for Paley sets and $L$-sets. Note however that all the $L$-subsets of $\mathbb{Z}$ (or of any amenable group) are finite, while the $L$-subsets of $\mathbb{N}$ are exactly the Paley sets. Thus this notion depends of the choice of the semi-group containing $\Lambda$. If we remain in the category of groups this difficulty does not arise, if $H$ is a subgroup of a group $G$ and if $\Lambda \subset H$, then $\Lambda$ is an $L$-set in $H$ iff it is an $L$-set in $G$ (this is easy to check e.g. by Proposition 3.2).

Note that $L$-sets are clearly stable under finite unions. Moreover the translate of an $L$-set is again an $L$-set. The only known examples of $L$-sets seem to be finite unions of translates of free sets. The sets which are translates of a free set (more precisely translates of a free set augmented by the unit element) are characterized in [A0] as those which have the Leinert property. In analogy with Paley sequences we formulate the following.

**Conjecture 3.5.** Every $L$-set $\Lambda$ can be written as a finite union $x_1 F_1 \cup \ldots \cup x_n F_n$ where $x_1, \ldots, x_n \in G$ and $F_1, \ldots, F_n$ are free subsets of $G$. (Here, the subset reduced to the unit element is considered free, so that a singleton is a translate of a free set.)

It is possible to check that if $\Lambda$ satisfies (iii) in Theorem 3.3 with $C = 1$ then it satisfies the Leinert property in the sense of [AO], hence it is a translate of a free set augmented by the unit element, a fortiori it is the union of two translates of free sets. Therefore to verify the above conjecture it suffices to prove that any set $\Lambda$ satisfying (iii) in Theorem 3.3 with some constant $C$ can be written as a finite union of sets satisfying the same property with $C = 1$. 

§ 4. A more general framework.

Actually, Theorem 2.2 can be extended to a rather general situation already considered in [P2]. We describe this briefly since it is easy to adapt the preceding ideas to this setting.

Let $X$ be a Banach space. We will identify an element $u$ in $X \otimes \ell_2$ with an operator $u : X^* \to \ell_2$ (of finite rank and weak-* continuous). Hence, for any $\xi$ in $X^*$, $u(\xi) \in \ell_2$.

A norm $\delta$ on $X \otimes \ell_2$ will be called 2-convex if there is a constant $c > 0$ such that for any $u$ in $X \otimes \ell_2$ (4.1)
\[ c \|u\| = c \sup_{\xi \in X^*, \|\xi\| \leq 1} \|u(\xi)\| \leq \delta(u) \]
and such that for all $u, u_1, u_2$ in $X \otimes \ell_2$ satisfying
\[ \forall \xi \in X^* \quad \|u(\xi)\| \leq \left(\|u_1(\xi)\|^2 + \|u_2(\xi)\|^2\right)^{1/2}, \]
we have
\[ \delta(u) \leq (\delta(u_1)^2 + \delta(u_2)^2)^{1/2}. \]

Note that if $\|u(\xi)\| = \|u_1(\xi)\|$ for all $\xi$ in $X^*$, we must have $\delta(u) = \delta(u_1)$, moreover for all $T : \ell_2 \to \ell_2$ we have $\delta(Tu) \leq \|T\|\delta(u)$.

Now let $X, Y$ be two Banach spaces and let $\delta_1$ (resp. $\delta_2$) be a 2-convex norm on $X \otimes \ell_2$ (resp. $Y \otimes \ell_2$). We can introduce a norm $\Gamma$ on $X \otimes Y$ by setting $\forall u \in X \otimes Y, u = \sum_{i=1}^{n} x_i \otimes y_i$
\[ \Gamma(u) = \inf \left\{ \delta_1(\sum_{i=1}^{n} x_i \otimes e_i) \delta_2(\sum_{i=1}^{n} y_i \otimes e_i) \right\} \]
where the infimum runs over all possible decompositions of $u$ and where $e_i$ denotes the canonical basis of $\ell_2$. It is easy to see that this is a norm. We denote by $X \hat{\otimes}_\Gamma Y$ the completion of $X \otimes Y$ for this norm.

We also denote by $X \hat{\otimes}_{\delta_1} \ell_2$ and $Y \hat{\otimes}_{\delta_2} \ell_2$ the completions of $X \otimes \ell_2$ and $Y \otimes \ell_2$ for the norms $\delta_1$ and $\delta_2$.

Assume that $X$ and $Y$ are continuously injected in a Banach space $Z$. Then, by (4.1) $X \hat{\otimes}_{\delta_1} \ell_2$ and $Y \hat{\otimes}_{\delta_2} \ell_2$ are both continuously injected into $Z \hat{\otimes} \ell_2$ (the injective tensor product), so that we can give a meaning to the sum $X \hat{\otimes}_{\delta_1} \ell_2 + Y \hat{\otimes}_{\delta_2} \ell_2$.

For simplicity, we denote
\[ X[\ell_2] = X \hat{\otimes}_{\delta_1} \ell_2 \text{ and } Y[\ell_2] = Y \hat{\otimes}_{\delta_2} \ell_2. \]
We can now equip \( X[\ell_2] \otimes \ell_2 \) with a 2-convex norm \( \Delta_1 \) as follows. For any \( v = \sum_{i=1}^{n} v_i \otimes e_i \) with \( v_i \in X[\ell_2] \) there is clearly an operator \( w \in X \hat{\otimes}_{\delta_1} \ell_2 \) (not necessarily unique) such that
\[
\|w(\xi)\| = \left( \sum_{i=1}^{n} \|v_i(\xi)\|^2 \right)^{1/2} \quad \forall \, \xi \in X^*.
\]
We then define
\[
\Delta_1(v) = \delta_1(w).
\]
By (4.2) this does not depend on the particular choice of \( w \). Clearly, this defines (by the density of \( \cup \ell^n_2 \) in \( \ell_2 \)) a 2-convex norm \( \Delta_1 \) on \( X[\ell_2] \otimes \ell_2 \). Now using the pair \((\Delta_1, \delta_2)\) (instead of the pair \((\delta_1, \delta_2)\)) we can define the space
\[
X[\ell_2] \hat{\otimes}_Y Y
\]
exactly as above for \( X \hat{\otimes}_Y Y \).

Similarly, we can define \( \Delta_2 \) on \( Y[\ell_2] \otimes \ell_2 \) and using the pair \((\delta_1, \Delta_2)\) we construct the space
\[
X \hat{\otimes}_Y [\ell_2]
\]
exactly as above for \( X \hat{\otimes}_Y Y \).

Assume that \( X, Y \) are both continuously injected in a Banach space \( Z \). Then, by (4.1), it is easy to check that \( X \hat{\otimes}_{\delta_1} \ell_2 \) and \( Y \hat{\otimes}_{\delta_2} \ell_2 \) are both continuously injected into the injective tensor products \( X \bar{\otimes} \ell_2 \) and \( Y \bar{\otimes} \ell_2 \), and consequently also \( X[\ell_2] \hat{\otimes}_Y Y \) and \( X \hat{\otimes}_Y [\ell_2] \) are both continuously injected into \( X \bar{\otimes} Y \bar{\otimes} \ell_2 \), so that using this inclusion we may consider the sum
\[
X[\ell_2] \hat{\otimes}_Y Y + X \hat{\otimes}_Y [\ell_2],
\]
with its natural norm (see section 1).

Then Theorems 2.1 and 2.2 can be generalized as follows.

**Theorem 4.1.** With the preceding notation, consider elements \( \psi_n \) in \( X \hat{\otimes}_Y Y \) such that the series \( \sum_{n=1}^{\infty} \varepsilon_n \psi_n \) converges for almost all choice of signs \( \varepsilon_n = \pm 1 \). Then necessarily the series \( \sum_{n=1}^{\infty} e_n \otimes \psi_n \) converges in the space \( X[\ell_2] \hat{\otimes}_Y Y + X \hat{\otimes}_Y [\ell_2] \).

**Theorem 4.2.** Let \( \psi_1, \ldots, \psi_N \) be a finite sequence in \( X \hat{\otimes}_Y Y \) such that
\[
\int \Gamma(\sum_{k=1}^{N} z_k \psi_k) d\mu(z) < 1.
\]
Then there is a decomposition in $X \hat{\otimes}_\Gamma Y$ of the form

$$\psi_k = A_k + B_k$$

such that

$$\left\| \sum_{1}^{N} e_k \otimes A_k \right\|_{X[\ell_2] \hat{\otimes}_\Gamma Y} < 1$$

and

$$\left\| \sum_{1}^{N} e_k \otimes B_k \right\|_{X \hat{\otimes}_\Gamma Y[\ell_2]} < 1.$$ (Note. Here of course $\sum_{k} e_k \otimes A_k$ is identified with an element of $X[\ell_2] \hat{\otimes}_\Gamma Y$ in the obvious natural way and similarly for $\sum_{k} e_k \otimes B_k$.)

The proof is the same as for Theorems 2.1 and 2.2. We leave the details to the reader.

**Remark 4.3.** In particular, with the notation and terminology of [LPP] we find (without any UMD assumption) that if $X, Y$ are two 2-convex Banach lattices, then there is a natural inclusion $\text{Rad}(X \hat{\otimes} Y) \subset X(\ell_2) \hat{\otimes}_\Gamma Y + X \hat{\otimes} Y(\ell_2)$, so that if $X, Y$ are both of finite cotype we have

$$\text{Rad}(X \hat{\otimes} Y) \approx \text{Rad}(X) \hat{\otimes} Y + X \hat{\otimes} \text{Rad}(Y).$$

This is a slight refinement of some of the results of [LPP].

**Remark 4.4.** As in Remark 1.9, let us denote by $H_1(T; X)$ (resp. $H_1(T^N; X)$) the subspace of $L_1(T, m; X)$ (resp. $L_1(T^N, m^N; X)$) formed by the functions with Fourier transform supported by the non-negative elements. We define similarly (for short) the space $H_\infty(T; X)$ (resp. $H_\infty(T^N; X)$). Using Remark 1.9 we obtain the same conclusion as Theorem 4.2 whenever there is a function $f$ in the interior of the unit ball of $H_1(T; X \otimes_\Gamma Y)$ (resp. $H_1(T^N; X \otimes_\Gamma Y)$) such that $\hat{f}(3^k) = \psi_k$ (resp. $\hat{f}(z_k) = \psi_k$) for all $k = 1, \ldots, n$. Of course this remark applies in particular to Theorem 2.2. In the case of Theorem 2.2 this remark seems useful because it turns out the converse is true. More precisely using a classical inequality (cf. [R] p.222) it can be proved that, in the situation of Theorem 2.2, for every element in the unit ball of $\ell_\infty(S, \ell_2) \hat{\otimes}_\ell_\infty(T)$ with a finitely supported sequence of coefficients $(A_k)$ in $\ell_\infty(S) \hat{\otimes}_\ell_\infty(T)$ there is a function $f$ in $H_\infty(T; \ell_\infty(S) \hat{\otimes}_\ell_\infty(T)) \subset H_1(T; \ell_\infty(S) \hat{\otimes}_\ell_\infty(T))$ with norm less than an absolute constant $C$ such that

$$\hat{f}(3^k) = A_k.$$
We can treat similarly any element in the unit ball of $ℓ_∞(S)⊗ℓ_∞(T,ℓ_2)$, hence the same conclusion holds for any element (with a finitely supported sequence of coefficients) in the unit ball of the space

$$ℓ_∞(S,ℓ_2)⊗ℓ_∞(T) + ℓ_∞(S)⊗ℓ_∞(T,ℓ_2).$$

In other words, the point of the present remark is that it yields a characterization of the sequences $(ψ_k)$ for which the conclusion of Theorem 2.2 holds.
§ 5. More applications to completely bounded maps.

As emphasized in [BP] (see also [P4]) the \( cb \) norm on \( B(M_n, B(H)) \) can be viewed as an example of the \( \Gamma \) norms discussed in section 4. In particular we can obtain an analogue of Theorem 2.2 for \( cb \) maps.

**Theorem 5.1.** Consider Hilbert spaces \( H_1 \) and \( H \) and completely bounded maps

\[
u_1, \ldots, u_N : B(H_1) \to B(H).
\]

Assume that \( \int \left\| \sum_{k=1}^{N} z_k u_k \right\|_{cb} d\mu(z) < 1 \). Then there is for some Hilbert space \( K \) a representation

\[
\pi : B(H_1) \to B(K)
\]

and operators \( V_k : H \to K, W_k : H \to K, \ V : H \to K, \ W : H \to K \) such that

\[
\|V\| \leq 1, \|W\| \leq 1, \left\| \sum_{1}^{N} V_k^* V_k \right\| \leq 1, \left\| \sum_{1}^{N} W_k^* W_k \right\| \leq 1
\]

and such that for all \( k = 1, \ldots, N \)

\[
\forall x \in B(H_1) \; u_k(x) = V_k^* \pi(x) W + V^* \pi(x) W_k.
\]

**Remark 5.2.** First observe that it is enough to find representations \( \pi' : B(H_1) \to B(K') \), \( \pi'' : B(H_1) \to B(K'') \) such that \( u_k(.) = V_k^* \pi'(.) W + V^* \pi''(.) W_k \) since we can replace each of \( \pi' \) and \( \pi'' \) by \( \pi' \oplus \pi'' \).

**Remark 5.3.** Assume that we have a net \( (u_k^\alpha)_{k \leq N} \) of \( N \)-tuples of maps from \( B(H_1) \) into \( B(H) \) such that for all \( x \) in \( B(H_1) \) \( u_k^\alpha(x) \to u_k(x) \) when \( \alpha \to \infty \) and satisfying the conclusions of Theorem 5.1, \( i.e. \) such that there is a Hilbert space \( K_\alpha \), a representation \( \pi_\alpha : B(H_1) \to B(K_\alpha) \) and operators \( V_k^\alpha, V^\alpha, W_k^\alpha, W^\alpha \) such that \( V^\alpha, W^\alpha, \sum_k V_k^\alpha V_k^\alpha \) and \( \sum_k W_k^\alpha W_k^\alpha \) are all of norm \( \leq 1 \) and we have for all \( x \) in \( B(H_1) \)

\[
u_k^\alpha(x) = V_k^\alpha \pi_\alpha(x) W_k + V^\alpha \pi_\alpha(x) W_k^\alpha.
\]

Then \( \{u_k\} \) satisfies the conclusions of Theorem 5.1. Indeed, we can take for \( K \) an ultraproduct of the Hilbert spaces \( K_\alpha \), and similarly for \( \pi \) and for the operators \( V, W, V_k, W_k \).

**Remark 5.4.** Assume \( H_1 \) and \( H \) both finite dimensional. Then any operator \( u : B(H_1) \to B(H) \) can be identified with a linear operator \( \tilde{u} : H_1 \otimes H \to (H_1 \otimes H)^* \) defined by

\[
\forall x_1, y_1 \in H_1 \quad \forall x, y \in H
\]
\[ < \tilde{u}(x_1 \otimes x), y_1 \otimes y > = < \overline{y}, u(x_1 \otimes y_1)x > . \]

(Note: We denote by \( y \to \overline{y} \) an anti isometry of \( H \) onto itself; note that on the left side we have a bilinear pairing while the scalar product appearing on the right side is antilinear in the first variable).

Consider a factorization of \( \tilde{u} \) of the form

\[ \tilde{u} = \sum_{i=1}^{i=n} x_i \otimes y_i \]

with \( x_i, y_i \in (H_1 \otimes H)^* \)

We define

\[ \delta_1(\sum_{i=1}^{n} x_i \otimes e_i) = \sup\{ |\sum_{i=1}^{n} < x_i, h_i \otimes h > | h_i \in H_1, \sum ||h_i||^2 \leq 1, h \in H, ||h|| \leq 1 \}. \]

We may identify an element \( x_i \) in \( (H_1 \otimes H)^* \) with a linear operator \( V_i : H \to H_1 \) by setting

\[ \forall k \in H, \forall h \in H \quad < x_i, k \otimes h > = < k, V_ih > \]

Then (5.1) becomes

\[ \delta_1(\sum_{i=1}^{n} x_i \otimes e_i) = \sup\{ (\sum ||V_i h||^2)^{1/2} | h \in H, ||h|| \leq 1 \} \]

\[ = \left\| \left( \sum V_i^* V_i \right)^{1/2} \right\| \]

Let \( X = (H_1 \otimes H)^* \) and let \( \mathcal{V} \) be the linear span of the basis vectors of \( \ell_2 \). Clearly the formula (5.3) defines a 2-convex norm on \( X \otimes \ell_2 \) (by density, say, of \( X \otimes \mathcal{V} \) in \( X \otimes \ell_2 \)).

We set \( Y = X \) and we define \( \delta_2 = \delta_1 \) on \( Y \otimes \ell_2 \). Then we can define the norm \( \Gamma \) associated to \( \delta_1 \) and \( \delta_2 \) as in section 4, and also the norms \( \Delta_1 \) and \( \Delta_2 \) and the spaces \( X[\ell_2] \otimes \Gamma Y \) and \( X[\ell_2] \otimes \Gamma Y \).

By well known results on the factorization of c.b. maps (cf. [Pa]) we have then

\[ \|u\|_{cb} = \Gamma(\tilde{u}). \]

Moreover, if \( u_1, ..., u_N \) are given c.b. maps from \( B(H_1) \) into \( B(H) \), and if \( \tilde{u}_1, ..., \tilde{u}_N \) denote the corresponding elements of \( X \otimes Y \) (with \( X = Y = (H \otimes H_1)^* \)). We claim that

\[ \left\| \sum_{k=1}^{N} \tilde{u}_k \otimes e_k \right\|_{X[\ell_2] \otimes \Gamma Y} < 1 \]
iff there are operators \( \{ W_i^k | k \leq N, i \leq n \} \) and \( \{ V_i | i \leq n \} \) such that \( \forall x \in B(H_1) \)

\[
 u_k(x) = \sum_{i=1}^n V_i^* x W_i^k \quad \text{and} \quad \left\| \sum_i V_i^* V_i \right\| < 1 \quad \left\| \sum_{i,k} W_i^k W_i^k \right\| < 1.
\]

Indeed (5.4) holds iff we can write

\[
 \sum_{k=1}^N \tilde{u}_k \otimes e_k = \sum_{i=1}^n \xi_i \otimes \eta_i
\]

with \( \xi_i \in X \otimes \ell_2^N \) and \( \eta_i \in Y \) such that

\[
 \Delta_1 (\sum \xi_i \otimes e_i) < 1
\]

and

\[
 \delta_2 (\sum \eta_i \otimes e_i) < 1.
\]

Let \( \xi_i = \sum_{k \leq N} \xi_i^k \otimes e_k \) with \( \xi_i^k \in X \) and let \( V_i^k \) and \( W_i \) be the operators associated to \( \xi_i^k \) and \( \eta_i \) by the correspondence (5.2). We then obtain the above claim.

This remark shows that in the case when both \( H_1 \) and \( H \) are finite dimensional, Theorem 5.1 can be viewed as a particular case of Theorem 4.1.

**Proof of Theorem 5.1.** We assume \( H \) finite dimensional until the last step of the proof. The case when \( H_1 \) is also finite dimensional has been checked in the preceding remark. Assume \( H_1 \) infinite dimensional assume that each \( u_1, u_N \) is weak*-continuous, i.e. continuous from \( \sigma(B(H_1), B(H_1)^*) \) into \( B(H) \). We may use the fact that there is an increasing set \( B(H_\alpha) \subset B(H_1) \) with \( \dim H_\alpha < \infty \) such that \( \bigcup B(H_\alpha) \) is weak*-dense in \( B(H_1) \). Applying the first part of the proof to the restrictions \( u_k|_{B(H_\alpha)} \) for each \( \alpha \) and passing to the limit in a standard way (as in remark 5.3) we obtain Theorem 5.1 in that case also.

Next when \( H_1 \) is arbitrary and \( H \) finite dimensional we can involve the local reflexivity principle (cf. e.g. [D]) to claim that there is a net \( (u_\alpha^k)_{k \leq N} \) of maps which are weak*-continuous from \( B(H_1) \) into \( B(H) \), which tend pointwise to \( (u_k)_{k \leq N} \) when \( \alpha \to \infty \) and which satisfy

\[
 \int \left\| \sum_{\alpha} z_k u_\alpha^k \right\|_{cb} d\mu(z) < 1.
\]

By remark 5.3 we obtain Theorem 5.1 in that case also.

Finally we remove the assumption that \( H \) is finite dimensional.
Let $H_\alpha$ be an increasing net of finite dimensional subspaces of $H$ with $\bigcup H_\alpha = H$.

For each $k$ and $\alpha$ let

$$u_\alpha^k(x) = P_{H_\alpha} u_k(x) |_{H_\alpha} \in B(H_\alpha).$$

By the first part of the proof the conclusion of Theorem 5.1 holds for $(u_\alpha^k)_{k \leq N}$ for each $\alpha$.

Using an ultraproduct argument as in Remark 5.3 we conclude one more that $u_1,...,u_N$ satisfy the conclusions of Theorem 5.1.

We now give some consequences of Theorem 5.1. We denote again by $\ell^n_2$ the $n$ dimensional Hilbert space equipped with its canonical basis $(e_i)_{i \leq n}$. We will identify $M_n$ with $B(\ell^n_2)$ and $e_{ij}$ with $e_i \otimes e_j$.

We will need the following notation.

Let $H_1, H_2$ be two Hilbert spaces and let $S_1 \subset B(H_1), S_2 \subset B(H_2)$ be closed subspaces (“operator spaces”). We will denote by $S_1 \otimes_{\text{min}} S_2$ the completion of $S_1 \otimes S_2$ equipped with the norm induced by $B(H_1 \otimes_2 H_2)$, where $H_1 \otimes_2 H_2$ is the Hilbertian tensor product. The space $S_1 \otimes_{\text{min}} S_2$ is called the minimal (or the spatial) tensor product of $S_1$ and $S_2$. In particular we have obviously $M_n(B(H)) = M_n \otimes_{\text{min}} B(H)$.

We will denote by $BR_n$ (resp. $BC_n$) the subspace of $M_n \otimes_{\text{min}} M_n$ formed by all elements of the form

$$y_1 = \sum_i e_{ii} \otimes \sum_j x_{ij}e_{ij}$$

(resp. $$y_2 = \sum_j e_{jj} \otimes \sum_i x_{ij}e_{ij}.$$)

Note that $\|y_1\| = \sup \left(\sum_i |x_{ij}|^2\right)^{1/2}$ and $\|y_2\| = \sup \left(\sum_j |x_{ij}|^2\right)^{1/2}$ so that $BR_n$ (resp. $BC_n$) is naturally isometric with the space of matrices with “bounded rows” (resp. “bounded columns”), which explains our notation.

Let $J_1 : M_n \to BR_n$ (resp. $J_2 : M_n \to BC_n$) be the map defined by

$$J_1(x) = \sum_i e_{ii} \otimes \sum_j x_{ij}e_{ij}$$

(resp. $J_2(x) = \sum_j e_{jj} \otimes \sum_i x_{ij}e_{ij}$).

Obviously we have $\|J_1\| \leq 1$ and $\|J_2\| \leq 1$. Moreover a simple verification shows that

$$(5.5) \quad \|J_1\|_{cb} \leq 1 \quad \text{and} \quad \|J_2\|_{cb} \leq 1.$$
Let us denote here \( G = T_n^2 \) and let \( \mu \) be the normalized Haar measure on \( G \). By a simple computation one can check that for any Hilbert space \( H \) and for any \( x_{ij} \) in \( B(H) \) we have

\[
\left\| \sum_i e_{ii} \otimes \sum_j e_{ij} \otimes x_{ij} \right\|_{BR_n \otimes \min B(H)} = \sup_i \left\| \sum_j e_{ij} \otimes x_{ij} \right\|_{M_n(B(H))} = \sup_i \left\| \left( \sum_j x_{ij} x_{ij}^* \right)^{1/2} \right\|_{B(H)}
\]

and similarly

\[
\left\| \sum_j e_{jj} \otimes \sum_i e_{ij} \otimes x_{ij} \right\|_{BC_n \otimes \min B(H)} = \sup_j \left\| \left( \sum_i x_{ij} x_{ij}^* \right)^{1/2} \right\|_{B(H)}
\]

Observe that the preceding expressions do not change if we replace \((x_{ij})\) by \((z_{ij} x_{ij})\) with \( |z_{ij}| = 1 \), i.e. with \( z = (z_{ij}) \in G \). Hence if we denote by \( T_z : M_n \to M_n \) the Schur multiplier defined by

\[ T_z((a_{ij})) = (a_{ij} z_{ij}), \]

we find using (5.5) and this observation that for all \( z = (z_{ij}) \) in \( G \) we have

\[
\| J_1 T_z \|_{cb} \leq 1 \quad \| J_2 T_z \| \leq 1.
\]

We can now state

**Corollary 5.5.** Let \( H \) be a Hilbert space. Consider an operator \( u : M_n \to B(H) \). Let \( u_z = u T_z : M_n \to B(H) \).

(i) Assume \( \int \| u_z \|_{cb} \, d\mu(z) < 1 \). Then there are operators \( a_1 : BR_n \to B(H) \) and \( a_2 : BC_n \to B(H) \) such that

\[
\| a_1 \|_{cb} \leq 1 \quad \| a_2 \|_{cb} \leq 1 \quad \text{and} \quad u = a_1 J_1 + a_2 J_2.
\]

(ii) Conversely if (5.8) holds we have

\[
\int \| u_z \|_{cb} \, d\mu(z) \leq \sup_{z \in G} \| u_z \|_{cb} \leq 2.
\]

**Proof.** Note that (5.9) is an obvious consequence of (5.7) so it suffices to prove the first part. Let \( u_{ij} : M_n \to B(H) \) be defined by \( u_{ij}(x) = x_{ij} u(e_{ij}) \) so that \( u_z = \sum_{ij} z_{ij} u_{ij} \). By
Theorem 5.1 we can find a Hilbert space $K$ operators $V^{ij}, W^{ij}, V, W$ from $H$ into $K$ and a representation $\pi : M_n \to B(K)$ such that for all $x$ in $M_n$

$$u_{ij}(x) = V^{ij*}\pi(x)W + V^{*}\pi(x)W^{ij}$$

and

$$\|V\| \leq 1, \|W\| \leq 1, \left\| \sum_{ij} V^{ij*}V^{ij} \right\| \leq 1, \left\| \sum_{ij} W^{ij*}W^{ij} \right\| \leq 1.$$ 

Let $T_{ij} = u(e_{ij})$. We deduce from (5.10)

$$T_{ij} = V^{ij*}\pi(e_{ij})W + V^{*}\pi(e_{ij})W^{ij}.$$ 

Let $a_1 : BR_n \to B(H)$ and $a_2 : BC_n \to B(H)$ be defined by

$$a_1(e_{ii} \otimes e_{ij}) = V^{*}\pi(e_{ij})W^{ij} \quad \text{and}$$

$$a_2(e_{jj} \otimes e_{ij}) = V^{ij*}\pi(e_{ij})W.$$ 

Clearly by (5.11) we have $u = a_1J_1 + a_2J_2$. We claim that $\|a_1\|_{cb} \leq 1$ and $\|a_2\|_{cb} \leq 1$. To check this we will use the well known inequality

$$\left\| \sum a_k^*b_k \right\| \leq \left\| \left( \sum a_k^*a_k \right)^{1/2} \right\| \left\| \left( \sum b_k^*b_k \right)^{1/2} \right\|$$

valid for $a_k, b_k \in B(H)$. For any $\xi_{ij}$ in $B(K_1)$ ($K_1$ an arbitrary Hilbert space) we have

$$\left\| \sum a_1(e_{ii} \otimes e_{ij}) \otimes \xi_{ij} \right\| = \left\| \sum V^{*}\pi(e_{ij})W^{ij} \otimes \xi_{ij} \right\|$$

$$\leq \left\| \left( \sum W^{ij*}W^{ij} \right)^{1/2} \right\| \left\| \left( \sum V^{*}\pi(e_{ij}) \otimes \xi_{ij} \right) \left( V^{*}\pi(e_{ij}) \otimes \xi_{ij}^* \right)^{1/2} \right\|$$

$$\leq \|V\| \left\| \left( \sum_{ij} \pi(e_{ij}) \pi(e_{ij})^* \otimes \xi_{ij} \xi_{ij}^* \right)^{1/2} \right\|$$

$$\leq \left\| \left( \sum \pi(e_{jj}) \otimes \sum_{i} \xi_{ij}^* \xi_{ij}^* \right)^{1/2} \right\|$$

$$\leq \sup_{j} \left\| \left( \sum_{i} \xi_{ij} \xi_{ij}^* \right)^{1/2} \right\|$$

$$= \left\| \sum_{ij} e_{ii} \otimes e_{ij} \otimes \xi_{ij} \right\|_{BR_n \otimes \text{min} B(K_1)}.$$
Taking $B(K_1) = M_n$ with $n \geq 1$ arbitrary, we obtain (recall (5.6))

$$\|a_1\|_{cb} \leq 1.$$ 

Similarly we have $\|a_2\|_{cb} \leq 1$. This concludes the proof.

We now turn to a generalized version of Schur multipliers.

We consider an operator $T : M_n(B(H_1)) \to M_n(B(H))$ (where $H, H_1$ are Hilbert spaces) of the special form

(5.13) \[ T((x_{ij})) = (T_{ij}(x_{ij})) \]

where $T_{ij} : B(H_1) \to B(H)$ are operators.

**Remark 5.6.** Assume $\|T\|_{cb} \leq 1$. Then there is a Hilbert space $K$, a representation $\pi : B(H_1) \to B(K)$ and operators $x_i, y_j : H \to K$ such that for all $i, j$

(5.14) \[ \forall x \in B(H_1) \quad T_{ij}(x) = x_i^*\pi(x)y_j \text{ and } \|x_i\| \leq 1, \|y_j\| \leq 1. \]

Conversely, it is clear that (5.14) implies $\|T\|_{cb} \leq 1$.

This statement generalizes Proposition 1.1 to the present setting.

Such a statement is a simple consequence of the factorization theorem of c.b. maps (cf. [Pa]) and of the particular form (5.13) of the map $T$.

**Corollary 5.7.** Let $T : M_n(B(H_1)) \to M_n(B(H))$ be an operator of the form (5.13) (generalized Schur multiplier). As above, let $G = T^n$ and let $\mu$ be the normalized Haar measure on $G$. Let $T_z : M_n(B(H_1)) \to M_n(B(H))$ be the operator defined by

$$\forall z = (z_{ij}) \in G \quad T_z((x_{ij})) = (z_{ij}T_{ij}(x_{ij})).$$

Assume $\int \|T_z\|_{cb} \, d\mu(z) < 1$. Then there is a decomposition $T = \alpha_1 + \alpha_2$ where $\alpha_1$ and $\alpha_2$ are each of the form (5.13) and moreover there are operators

$$\tilde{\alpha}_1 : BR_n \otimes_{\min} B(H_1) \to M_n(B(H))$$

and

$$\tilde{\alpha}_2 : BC_n \otimes_{\min} B(H_1) \to M_n(B(H))$$

satisfying

$$\|\tilde{\alpha}_1\|_{cb} \leq 1, \|\tilde{\alpha}_2\|_{cb} \leq 1,$$

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and such that

\begin{equation}
\alpha_1 = \tilde{\alpha}_1(J_1 \otimes I_{B(H_1)}) \quad \alpha_2 = \tilde{\alpha}_2(J_2 \otimes I_{B(H_1)}). \tag{5.15}
\end{equation}

Conversely, if such a decomposition holds then necessarily

\[ \int \| T_z \|_{cb} \, d\mu(z) \leq \sup_{z \in G} \| T_z \|_{cb} \leq 2. \]

**Proof.** Let us define again \( \tilde{T}_{ij} : M_n(B(H_1)) \to M_n(B(H)) \) by the identity

\[ T_z = \sum_{ij} z_{ij} \tilde{T}_{ij}. \]

Then, by Theorem 5.1 there are a Hilbert space \( \tilde{K} \) and a representation \( \tilde{\pi} : M_n(B(H_1)) \to B(\tilde{K}) \) together with operators \( V, W, V^{ij}, W^{ij} \) from \( \ell^2_n(H) \) into \( \tilde{K} \) such that

\begin{equation}
\forall \xi \in M_n(B(H_1)) \quad \tilde{T}_{ij}(\xi) = V^{ij*} \tilde{\pi}(\xi)W + V^* \tilde{\pi}(\xi)W^{ij*} \tag{5.16}
\end{equation}

and such that

\begin{equation}
\| V \| \leq 1, \| W \| \leq 1, \left\| \sum_{ij} V^{ij*} V^{ij} \right\| \leq 1, \left\| \sum_{ij} W^{ij*} W^{ij} \right\| \leq 1. \tag{5.17}
\end{equation}

By standard arguments, we can assume w.l.o.g. that \( \tilde{K} = \ell^2_n(K) \) for some Hilbert space \( K \) and that \( \tilde{\pi} : M_n(B(H_1)) \to B(\tilde{K}) = M_n(B(K)) \) is of the form \( \tilde{\pi} = I_{M_n} \otimes \pi \) for some representation \( \pi : B(H_1) \to B(K) \). Once we identify \( \tilde{K} \) with \( \ell^2_n(K) \) we may identify each of \( V, W, V^{ij}, W^{ij} \) with an \( n \times n \) matrix of operators from \( H \) into \( K \).

Thus we identify \( V \) with \( (V(k, \ell))_{k,\ell \leq n} \) \( V^{ij} \) with \( (V^{ij}(k, \ell))_{k,\ell \leq n} \), and so on.

Let now \( x \) be arbitrary in \( B(H_1) \). We have by (5.16)

\[ T_{ij}(x) = (\tilde{T}_{ij}(e_{ij} \otimes x))_{ij} \]

\[ = [V^{ij*}(e_{ij} \otimes \pi(x))W + V^*(e_{ij} \otimes \pi(x))W^{ij}]_{ij} \]

hence

\begin{equation}
T_{ij}(x) = V^{ij}(i, i)^* \pi(x)W(j, j) + V(i, i)^* \pi(x)W^{ij}(j, j). \tag{5.18}
\end{equation}

By (5.17) we have

\[ \| W(j, j) \| \leq 1, \| V(i, i) \| \leq 1. \]
Moreover, we claim that (5.17) implies

\[
(5.19) \quad \sup_i \left\| \sum_j V^{ij}(i,i)^* V^{ij}(i,i) \right\| \leq 1 \quad \text{and} \quad \sup_j \left\| \sum_i W^{ij}(j,j)^* W^{ij}(j,j) \right\| \leq 1.
\]

Indeed, (5.17) implies that for each \( \ell = 1, 2, \ldots, n \)

\[
\left\| \sum_{ij} V^{ij}(\ell,\ell)^* V^{ij}(\ell,\ell) \right\| \leq 1
\]

hence a fortiori for each \( i \) (taking \( \ell = i \))

\[
\left\| \sum_j V^{ij}(i,i)^* V^{ij}(i,i) \right\| \leq 1.
\]

The other estimate is proved similarly, hence the above claim.

Finally, let

\[
\alpha_1((x_{ij})) = (\alpha^1_{ij}(x_{ij})) \quad \text{and} \quad \alpha_2((x_{ij})) = (\alpha^2_{ij}(x_{ij}))
\]

where we define for all \( x \) in \( B(H_1) \)

\[
\alpha^1_{ij}(x) = V(i,i)^* \pi(x) W^{ij}(j,j) \quad \text{and} \quad \alpha^2_{ij}(x) = V^{ij}(i,i)^* \pi(x) W(j,j).
\]

Clearly we can write (5.15) for some uniquely defined operators \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) as in Corollary 5.7. We have

\[
\forall x \in B(H_1) \quad \tilde{\alpha}_1(e_{jj} \otimes e_{ij} \otimes x) = e_{ij} \otimes \alpha^1_{ij}(x)
\]

and

\[
\tilde{\alpha}_2(e_{ii} \otimes e_{ij} \otimes x) = e_{ij} \otimes \alpha^2_{ij}(x).
\]

Finally, it remains to check that \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) acting on the spaces indicated in Corollary 5.7 are of c.b. norm at most 1.

Let \( y_1 \in BR_n \otimes_{\min} B(H_1) \) be of norm \( \leq 1 \). Let \( y_1 = \sum_i e_{ii} \otimes \sum_j e_{ij} \otimes x_{ij} \) with \( x_{ij} \in B(H_1) \).

We have

\[
\tilde{\alpha}_1(y_1) = \sum_{ij} e_{ij} \otimes V(i,i)^* \pi(x_{ij}) W^{ij}(j,j).
\]

Observe that for all \( y_{ij} \) in \( B(H) \)

\[
\left\| \sum e_{ij} \otimes y_{ij} \right\|_{M_n(B(H))} = \left\| \sum e_{ij} \otimes e_{ij} \otimes y_{ij} \right\|_{M_n(M_n(B(H)))}.
\]
Now let \( y_{ij} = V(i,i)^* \pi(x_{ij}) W^{ij}(j,j) \). We have
\[
\sum_{ij} e_{ij} \otimes e_{ij} \otimes y_{ij} = \left( \sum_{ij} e_{ii} \otimes e_{ij} \otimes V(i,i)^* \pi(x_{ij}) \right) \left( \sum_{ij} e_{ij} \otimes e_{jj} \otimes W^{ij}(j,j) \right).
\]
Hence we have
\[
\|\tilde{\alpha}_1(y_1)\| \leq \left\| \sum_{ij} e_{ii} \otimes e_{ij} \otimes V(i,i)^* \pi(x_{ij}) \right\| \cdot \left\| \sum_{ij} e_{ij} \otimes e_{jj} \otimes W^{ij}(j,j) \right\|
\]
hence by (5.6)', (5.6)" and (5.19).
\[
\|\tilde{\alpha}_1(y_1)\| \leq \sup_i \left( \|V(i,i)\| \left\| \sum_j x_{ij} x_{ij}^* \right\|^{1/2} \right) \leq \|y_1\|_{BR_n \otimes \min B(H)}
\]
This shows that \( \|\tilde{\alpha}_1\|_{cb} \leq 1 \). Similarly we have \( \|\tilde{\alpha}_2\|_{cb} \leq 1 \). This concludes the proof.

**Final Remark:** Recently, C. Le Merdy has shown that Theorem 5.1 and corollary 5.7 remain valid if \( B(H_1) \) is replaced by an arbitrary \( C^* \)-algebra \( A \subset B(H_1) \). Indeed, he has proved (cf. [LeM]) that any bounded analytic function with values in the space \( CB(A, B(H)) \) (i.e. the space of all c.b. maps from \( A \) into \( B(H) \)) can be extended to a bounded analytic function (with the same \( H^\infty \) norm) with values in the space \( CB(B(H_1), B(H)) \). In other words, there is a way to extend c. b. maps from \( A \) into \( B(H) \) to c. b. maps defined on the whole of \( B(H_1) \) which preserves analyticity. This extends a result due to Haagerup and the author corresponding to the particular case when \( H \) is of dimension 1. Using Le Merdy’s result and Remark 4.4 it is rather easy to adapt the proof of Theorem 5.1 (or corollary 5.7) with \( B(H_1) \) replaced by any \( C^* \)-subalgebra \( A \subset B(H_1) \).
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