Abstract. A gauged $SU_q(2)$ theory is characterized by two dual algebras, the first lying close to the Lie algebra of $SU(2)$ while the second introduces new degrees of freedom that may be associated with non-locality or solitonic structure. The first and second algebras, here called the external and internal algebras respectively, define two sets of fields, also called external and internal. The gauged external fields agree with the Weinberg-Salam model at the level of the doublet representation but differ at the level of the adjoint representation. For example the $g$-factor of the charged $W$-boson differs in the two models. The gauged internal fields remain speculative but are analogous to color fields.
1. Introduction.

In an earlier note we discussed a modification of the Weinberg-Salam model suggested by gauging $SU_q(2)_L \times U(1)$.\textsuperscript{1} It is reasonable to do this since $SU(2)$, unlike the Poincaré group, is a phenomenological group, and $SU_q(2)$ may also be phenomenologically useful.

In taking this step one finds that the Lie algebra gets replaced by two dual algebras: the first lying close to and approaching the original Lie algebra in a correspondence limit ($q = 1$) while the second algebra is new and introduces new degrees of freedom.

We propose to study the replacement of the point-particle classical field theory by a soliton field theory described in the two complementary ways that correspond to the two dual algebras. In the first (macroscopic) picture the particles are regarded as point-like but subject to the first algebra. In the complementary (microscopic) picture the solitons are regarded as composed of preons subject to the second (dual) algebra. The first algebra is little different from the $SU(2)$ Lie algebra and will be called the external algebra. The second algebra is exotic and will be called the internal algebra since it governs the dynamics of the constituent particles.

2. Irreducible Representations of $SU_q(2)$.

We shall first summarize the necessary information about $SU_q(2)$.

The two-dimensional representation of $SL_q(2)$ may be defined by

\[ TeT^t = T^t\epsilon T = \epsilon \] (2.1)

where $t$ means transpose and

\[ \epsilon = \begin{pmatrix} 0 & q_1^{1/2} \\ -q_1^{1/2} & 0 \end{pmatrix} \] (2.2)

Set

\[ T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \] (2.3)

Then

\[ \alpha\beta = q\beta\alpha \quad \alpha\gamma = q\gamma\alpha \quad \alpha\delta - q\beta\gamma = 1 \]

\[ \delta\beta = q_1\beta\delta \quad \delta\gamma = q_1\gamma\delta \quad \delta\alpha - q_1\beta\gamma = 1 \]

\[ \beta\gamma = \gamma\beta \] (2.4)

If $q = 1$, Eqs. (2.4) are satisfied by complex numbers and $T$ is defined over a continuum, but if $q \neq 1$, then $T$ is defined only over this algebra–a non-commuting space.
A two-dimensional representation of $SU_q(2)$ may be obtained by going to a matrix representation of (2.4) and setting \(^2\)

$$\gamma = -q_1 \bar{\beta} \quad \delta = \bar{\alpha}$$

where the bar means Hermitian conjugate. Then

$$\alpha \beta = q \bar{\beta} \alpha \quad \alpha \bar{\alpha} + \beta \bar{\beta} = 1 \quad \beta \bar{\beta} = \bar{\alpha} \alpha$$

and $T$ is unitary:

$$\bar{T} = T^{-1}$$

If $q = 1$, (A) may be satisfied by complex numbers and $T$ is a $SU(2)$ unitary-simplectic matrix. If $q \neq 1$, there are no finite representations of (A) unless $q$ is a root of unity. We shall assume that $q$ is real and $q < 1$.

The irreducible representations of $SU_q(2)$ are as follows:\(^3\)

$$D^j_{mm'}(\alpha, \bar{\alpha}, \beta, \bar{\beta}) = \Delta^j_{mm'} \sum_{s,t} \left\langle \frac{n_+}{s} \right\rangle_1 \left\langle \frac{n_-}{t} \right\rangle_1 q_1^{t(n_+ - s + 1)} (-1)^t \delta(s + t, n'_+) \alpha^s \beta^{n_+ - s} \bar{\beta}^t \bar{\alpha}^n \right\rangle_t$$

(2.7)

where

$$n_\pm = j \pm m \quad \left\langle \frac{n}{s} \right\rangle_1 = \frac{\langle n \rangle_1!}{\langle s \rangle_1! \langle n - s \rangle_1!} \quad \langle n \rangle_1 = \frac{q_1^{2n} - 1}{q_1^2 - 1}$$

(2.8)

$$\Delta^j_{mm'} = \left[ \frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!} \right]^{1/2} q_1 = q^{-1}$$

Here the arguments of (2.7) satisfy the (A) algebra. In the limit $q = 1$ $D^j_{mm'}$ become the Wigner functions, $D^j_{mm'}(\alpha \beta \gamma)$, the irreducible representation of $SU(2)$. The orthogonality properties of the $D^j_{mm'}$ may be expressed as follows:\(^2\)

$$h(D^j_{mn} D^j_{m'n'}) = \delta^{jj'} \delta_{mm'} \delta_{nn'} \frac{q_1^{2n}}{[2j + 1]q}$$

(2.9)

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q_1}$$

(2.10)

Here $h$ is a linear operator introduced by Woronowicz having the property that $h[D^j_{mn} D^{j'}_{m'n'}]$ for $SU_q(2)$ corresponds to the integral over the group manifold of $SU(2)$. The coefficients describing the decomposition of the product of two irreducible representations into the Clebsch-Gordan series may be computed with the aid of (2.9).\(^4\)
3. The Dual Algebras.

The dual algebras may be exhibited in the following way. The two-dimensional representation may be Borel factored:

\[ \mathcal{D}^{1/2}(\alpha, \bar{\alpha}, \beta, \bar{\beta}) = e^{B\alpha} e^{\lambda \beta} e^{C\bar{\alpha}} \]  

The algebra \((A)\) of \((\alpha, \beta, \bar{\alpha}, \bar{\beta})\) is then inherited by \((B, C, \theta)\) as

\[ (B, C) = 0 \quad (\theta, B) = B \quad (\theta, C) = C \]  

\[ \lambda = \ln q \]

The 2\(j+1\) dimensional irreducible representation of \(SU_q(2)\) shown in (2.7) may be rewritten in terms of \((B, C, \theta)\). Then by expanding to terms linear in \((B, C, \theta)\) one has

\[ \mathcal{D}^j_{mm'}(B, C, \theta) = \mathcal{D}^j_{mm'}(0, 0, 0) + B(J_B^j)_{mm'} + C(J_C^j)_{mm'} + 2\lambda \theta (J_\theta^j)_{mm'} + \ldots \]

where the non-vanishing matrix coefficients \((J_B^j)_{mm'}, (J_C^j)_{mm'}, \) and \((J_\theta^j)_{mm'}\) are

\[ \langle m - 1|J_B^j|m \rangle = (\langle j + m \rangle_1 \langle j - m + 1 \rangle_1)^{1/2} \]  

\[ \langle m + 1|J_C^j|m \rangle = (\langle j - m \rangle_1 \langle j + m + 1 \rangle_1)^{1/2} \]  

\[ \langle m|J_\theta^j|m \rangle = m \]

and where \(\langle n \rangle_1\), as given by (2.8), is a basic integer corresponding to \(n\). The \((B, C, \theta)\) and \((J_B, J_C, J_\theta)\) are generators of two dual algebras satisfying the following commutation rules:

\[ (J_B, J_\theta) = -J_B \quad (J_C, J_\theta) = J_C \quad (J_B, J_C) = q_1^{2j-1}[2J_\theta] \]  

\[ (B, C) = 0 \quad (\theta, B) = B \quad (\theta, C) = C \]

Here \([x]\) is given by (2.10). The commutation relations (3.6) follow from (3.5).

In the fundamental and adjoint representations of the external algebra the commutation relations (3.6) simplify as follows:\(^1\)

(a) \(J = 1/2\) \quad (fundamental)

\[ (J_B, J_\theta) = -J_B \quad (J_C, J_\theta) = J_C \quad (J_B, J_C) = 2J_\theta \]

(b) \(J = 1\) \quad (adjoint)

\[ (J_B, J_\theta) = -J_B \quad (J_C, J_\theta) = J_C \quad (J_B, J_C) = \langle 2 \rangle_1 J_\theta \]
The right-hand side of (3.6) is not linear in the \( J_\theta \) generators unless \( J = 1/2 \) or \( J = 1 \), and only in these cases do we speak of a “Lie algebra” or structure constants.

For these two cases let us introduce Hermitian generators as follows:

\[
\begin{align*}
J_B &= J_1 + i J_2 \\
J_C &= J_1 - i J_2 \\
J_\theta &= J_3
\end{align*}
\] (3.11)

Then

\[
\begin{align*}
J = 1/2 : & \quad (J_m, J_n) = i \epsilon_{mnp} J_p \\
J = 1 : & \quad (J_1, J_2) = i \frac{(2)}{2} J_3 \\
& \quad (J_2, J_3) = i J_1 \\
& \quad (J_3, J_1) = i J_2
\end{align*}
\] (3.12)

(3.13)

For the fundamental and the adjoint representations we may write

\[
\begin{align*}
(J_a, J_b) &= f_{ab}^m J_m \\
g_{ab} &= \text{Tr} \ J_a J_b
\end{align*}
\] (3.14)

(3.15)

where \( f_{ab}^m \) and \( g_{ab} \) correspond to the usual structure constants and group metric and where

\[
\begin{align*}
f_{abc} &= f_{ab}^m g_{mc} \\
\end{align*}
\] (3.16)

If \( J = 1/2 \)

\[
\begin{align*}
f_{abc} &= i \epsilon_{abc} \quad (3.17)
\end{align*}
\]

\[ J = 1 \]

\[
\begin{align*}
f_{abc} &= i \langle 2 \rangle_1 \epsilon_{abc} \quad (3.18)
\end{align*}
\]

In both cases \( f_{abc} \) has the important property of being completely antisymmetric.1

The metric \( g_{ab} \sim \delta_{ab} \) in the fundamental representation but in the adjoint representation

\[
\begin{align*}
g_{ab} &= g_a \delta_{ab}
\end{align*}
\] (3.19)

where

\[
\begin{align*}
g_1 &= g_2 = \langle 2 \rangle_1 \quad \text{and} \quad g_3 = 2
\end{align*}
\] (3.20)
4. The Internal Algebra and the Microscopic Picture.

Let us expand a generic field operator in the irreducible representations (2.7) as follows:

\[ \psi(x, \{\alpha\}) = \sum_{jmn} \varphi_{mn}^j(x) D_{mn}^j(\{\alpha\}) \] (4.1)

where \{\alpha\} is an abbreviation for \{\alpha\bar{\beta}\bar{\beta}\} and where \( D_{mn}^j \), and therefore \( \psi(x) \), lies in the algebra. Here \( \varphi_{mn}^j(x) \) is expanded in Fock annihilation and creation operators:

\[ \varphi_{mn}^j(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \left[ e^{-ipx} a_{mn}^j(\vec{p}) + e^{ipx} \bar{a}_{mn}^j(\vec{p}) \right] \] (4.2)

where the Lorentz tensor indices have been suppressed.

Under a gauge transformation \( T \)

\[ T\psi = \sum \varphi_{mn}^j(x) T D_{mn}^j(\{\alpha\}) \] (4.3)

where \( T D_{mn}^j \) still lies in the internal algebra. Then

\[ T D_{mn}^j = \sum \langle jmn|T|j'n'm'\rangle D_{m'n'}^{j'} \]

\[ T\psi = \sum \varphi_{mn}^j(x) \langle jmn|T|j'n'm'\rangle D_{m'n'}^{j'} \] (4.4)

where

\[ \frac{q_{n'}^{2n'}}{[2j' + 1]} \langle j'n'm'|T|jmn\rangle = h(D_{m'n'}^{j'} T D_{mn}^j) \] (4.5)

and \( h \) is the linear operator appearing in (2.9). If, for example, \( T = D_{m'n'n'}^{j''} \) then \( \langle j'n'm'|T|jmn\rangle = h(D_{m'n'n'}^{j''} D_{m'n'}^{j'} D_{mn}^j) \) is a \( q \)-Clebsch-Gordan coefficient.\(^4\)

The field quanta associated with \( \varphi_{mn}^j(x) \) or \( D_{mn}^j(\{\alpha\}) \) may be interpreted as composite particles while the constituent or preon fields may be associated with the generators of the internal algebra. The total field operator may be interpreted as an expansion in solitons, as we shall now show.\(^5\)

Let us illustrate with the global Hamiltonian of a scalar field. We assume

\[ H^q = \frac{1}{2} h \int : \sum_{k=0}^{3} \partial_k \bar{\psi} \partial_k \psi + m_o^2 \bar{\psi} \psi : \ d\vec{x} \] (4.6)

Here \( h \), defined in (2.9), is an average over the algebra.

At the global level \( H^q \) is invariant under gauge transformations since \( T \) is unitary and therefore

\[ (\bar{\psi}\psi)' = (\bar{\psi}\hat{T}\psi) = \bar{\psi}\psi \]

\[ (\partial_k \bar{\psi} \partial_k \psi)' = \partial_k \bar{\psi} \partial_k \psi \] (4.7)
By (4.1) and (4.2)

\[ H^q = \int d\vec{p} \ p_o \sum_{j_{mn}} h(\mathcal{D}_{mn} \mathcal{D}'_{m'n'}) \frac{1}{2} : [\bar{a}_{jmn}(p)a_{j'n'}(p) + a_{j'm'n'}(p)a_{jm}(p)] \]  

(4.8)

Evaluate \( H^q \) on the state \( |N(p); jmn\rangle \). Then by (2.9)

\[ H^q |N(\bar{p}); jmn\rangle = \sum_{jmn} \frac{Np_o q^{2n}}{[2j + 1]_q} |N(\bar{p}); jmn\rangle \]  

(4.9)

Therefore the rest mass of a single field quantum associated with the field \( \varphi^j_{mn} \) is

\[ \frac{m_o q^{2n}}{[2j + 1]_q} \]  

(4.10)

If \( q = 1 \) the rest mass of a particle with quantum numbers \( (jmn) \) is

\[ \frac{m_o}{2j + 1} \]

and does not depend on \( n \). If \( q \neq 1 \) the mass depends on \( n \) and if \( q \approx 1 \) there is an approximate harmonic oscillator fine structure. On the other hand a point particle has no mass spectrum and the existence of such a spectrum here implies an extended object. Since the spectrum is approximately that of a \( q \)-harmonic oscillator, one may assume that the extension is approximately described by a \( q \)-harmonic oscillator wave function. It is in this sense that we describe the field quanta of \( \psi \) as solitons.

5. The External Algebra and \( q \)-Electroweak.

In the Weinberg-Salam model the Lagrangian density is\(^6\)

\[ \mathcal{L} = -\frac{1}{4}(G^{\mu\nu} \cdot G_{\mu\nu} + H^{\mu\nu} H_{\mu\nu}) + i(\bar{L} i \gamma L + \bar{R} i \gamma R) \]

\[ + (\bar{D} \phi) \cdot (D \phi) - V(\bar{\varphi} \varphi) - \frac{m}{\rho_o}(\bar{L} \varphi R + \bar{R} \varphi L) \]  

(5.1)

where the covariant derivative is

\[ D = \partial + ig \bar{\mathcal{W}}^i + ig' \omega a t_a \]  

(5.2)

Here \( \bar{\mathcal{W}}^\mu \) and \( \mathcal{W}_\rho^\mu \) are the connection fields of \( SU(2)_L \) and \( U(1) \), the chiral isotopic spin and hypercharge groups with independent coupling constants \( g \) and \( g' \), while \( G \) and \( H \) are
the corresponding field strengths. The Lagrangian (5.1) also contains the contribution of one lepton doublet and the mass generating Hibbs doublet $\varphi$.

In (5.2), the expression for the covariant derivative, the matrices $\vec{t}$ and $t_o$ are the generators of the $SU(2)$ and $U(1)$ groups. If we now pass to $SU_q(2)$ without changing $U(1)$, Eqs. (5.2) will be unchanged in the doublet representation since Eqs. (4.1) and (4.2) hold for both $SU(2)$ and $SU_q(2)$. Therefore at the level of the doublet representation there is no divergence between the standard $SU(2)$ theory and the corresponding $SU_q(2)$ theory and one again obtains the standard relations

$$e = g \sin \theta_W = g' \cos \theta_W$$

$$M_W = M_Z \cos \theta_W$$

where $g$ and $g'$ are the coupling constants of the chiral isotopic spin group and the hypercharge group respectively while $M_W$ and $M_Z$, the masses of the charged and neutral bosons, are also related by $\theta_W$ the Weinberg angle. The argument leading to these results is not changed since the form of $D$ in (5.2) is not changed on interpreting the $\vec{t}$ matrices as belonging to the fundamental representation of $SU_q(2)$ instead of $SU(2)$.

6. Gauge Invariance of the External Sector.

All matrices and fields in the external sector are numerically valued. Consider a general field transformation:

$$\psi' = T\psi$$

By definition the covariant derivative, $D_\mu \psi$, then transforms as follows:

$$(D_\mu \psi)' = T(D_\mu \psi)$$

Hence

$$D'_\mu = T D_\mu T^{-1}$$

In terms of $D_\mu$ the vector connection $W_\mu$ and the field strength $G_{\mu\lambda}$ are defined by

$$W_\mu = D_\mu - \partial_\mu$$

$$G_{\mu\lambda} = (D_\mu, D_\lambda)$$

Then

$$W'_\mu = TW_\mu T^{-1} + T\partial_\mu T^{-1}$$

$$G'_{\mu\lambda} = TG_{\mu\lambda} T^{-1}$$
The field invariant may be chosen to be

\[ I = \text{Tr} \, G^{\mu \lambda} G_{\mu \lambda} \]  
(6.7)

Assume now

\[ W_\mu = ig \, W^a_\mu t_a \]  
(6.8)

where the numerically valued \( t_a \) belong to the adjoint representation and satisfy equations of the form (3.14). Then

\[ G_{\mu \lambda} = G^a_{\mu \lambda} t_a \]  
(6.9)

By (6.7) and (6.9)

\[ I = \text{Tr} \, G^a_{\mu \lambda} G^{b \mu \lambda} t_a t_b \]

\[ = g_{ab} G^a_{\mu \lambda} G^{b \mu \lambda} \]  
(6.10)

In the Weinberg-Salam model

\[ g_{ab} = 2 \delta_{ab} \]  
(6.11)

In that case one may write

\[ I = 2 G^a_{\mu \lambda} G^{a \mu \lambda} = 2G_{\mu \lambda} \cdot G^{\mu \lambda} \]  
(6.12)

as in (5.1). Here however we must retain \( g_{ab} \) since by (3.20) it is not isotropic. Other terms like \( \bar{L}DL \) are also invariant since \( T \) is unitary and therefore

\[ L'D'L' = (LT^{-1})(TDT^{-1})(TL) = LDL \]  
(6.13)

Hence the full (5.1) is gauge-invariant with the external \( SU_q(2)_L \) substituted for \( SU(2)_L \). As already remarked this \( q \)-theory leads to the same consequences as the Weinberg-Salam theory at the doublet level. However there will be differences at the adjoint level. In particular the couplings of \( W^+_\mu \) and \( W^-_\mu \) appearing in \( D_\mu \) depend on \( q \) but those of \( A_\mu \) and \( Z_\mu \) do not. Since the full theory remains gauge invariant, however, there will still be Ward identities.

The deviations from the standard theory can also be seen by examining the self-interactions of the vector fields, namely

\[ -\frac{1}{4g^2} g_{mn} G^m_{\mu \nu} G^n_{\mu \nu} \]  
(6.14)

where

\[ G^a_{\mu \nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + f_{bc}^q W^b_\mu W^c_\nu \]  
(6.15)
and $g$ is the weak coupling constant appearing in (6.8). The trilinear couplings are then

$$\sim g_{mn} f_{bc} W^b \nu^c (\partial^\mu W^{\nu n} - \partial^\nu W^{\mu n})$$

$$= f_{nb} W^b \nu^c (\partial^\mu W^{\nu n} - \partial^\nu W^{\mu n}) \quad (6.16)$$

where

$$f_{nb} = i(2) \epsilon_{nb} \quad \text{by (3.18)}$$

$$= i(1 + q_1^2) \epsilon_{nb} \quad (6.17)$$

Hence the asymmetry expressed by $f_{bc}^m$ may be removed in these terms.

The quartic couplings are on the other hand

$$\sim g_{mn} f_{bc} f_{kl}^n W^b \nu^c W^k \mu W^\ell \nu \quad (6.18)$$

Here

$$g_{mn} f_{bc} f_{kl}^n = f_{nb} f_{lk}^n \quad (6.19)$$

At this point the asymmetry arising from (3.19) and expressed by $f_{kl}^n$ can no longer be hidden. It distinguishes one preferred direction in isotopic spin space, and in principle should be experimentally detectable.

There is in fact already a theoretically detectable divergence from the Weinberg-Salam theory buried in the trilinear terms. By (6.16) and (6.17) the trilinear terms are

$$(1 + q_1^2) g \epsilon_{nb} W^b \nu^c (\partial^\mu W^{\nu n} - \partial^\nu W^{\mu n}) \quad (6.20)$$

The electromagnetic part of this interaction contains the term

$$-ie(1 + q_1^2) W^\mu W^\nu F_{\mu \nu} \quad (6.21)$$

which is obtained by use of the Weinberg-Salam relations:

$$W^3_\mu = A_\mu \sin \theta \quad (6.22a)$$

and

$$e = g \sin \theta \quad (6.22b)$$

These relations also hold here since their derivation is at the level of the doublet representation.

A general term of this kind, namely

$$-ie \kappa W^\mu W^\nu F_{\mu \nu} \quad (6.23)$$
gives rise to a magnetic moment
\[(1 + \kappa) \frac{e}{2m_W} \vec{s}\] (6.24)
where \(\vec{s}\) is the spin vector.

The \(g\)-factor of the \(W\) boson in the \(q\)-model is then \(1 + q_1^2\) rather than \(g = 2\), the value in the Weinberg-Salam model.\(^6\)

It is remarkable that the \(gGG\) in (6.14) remains gauge invariant although both the cubic and quartic terms are changed. Since gauge invariance is still preserved with the new structure constants, the good formal properties of the standard theory are also preserved. Except for the appearance of structure constants depending on \(q\) the field Lagrangian is standard.

In the Weinberg-Salam theory the component fields \((W_1, W_2, W_3)\) appear in two ways: first, in interaction with chiral fermions the three fields are associated with matrix elements of the two-dimensional fundamental matrices; and second, in the description of free fields the \(W_i\) are associated with the three-dimensional generators \(t_i\) rather than with the matrix elements of the \(t_i\). We have followed the same pattern here, but there remains a difference between the fundamental and adjoint representation in the two formalisms. In the Standard Model these two representations are related by the Clebsch-Gordan coefficients of \(SU(2)\). Here on the other hand \(D^{1/2}(\alpha, \bar{\alpha}, \beta, \bar{\beta})\) and \(D^1(\alpha, \bar{\alpha}, \beta, \bar{\beta})\) in the internal algebra are related by the Clebsch-Gordan coefficients computed from the algebra \((A)\).\(^4\) Next one may go from the fundamental to the adjoint representation of the external algebra indirectly by going through the internal algebra and then making use of (3.4). Alternatively one may make use of the co-product defined by:\(^7\)

\[
\Delta(J_\pm^{1/2}) \overset{\text{def}}{=} 1 \otimes J_\pm^{1/2} + J_\pm^{1/2} \otimes 1
\]
\[
\Delta(J_z^{1/2}) \overset{\text{def}}{=} \hat{q}^{\frac{1}{2}J_z} \otimes J_z^{1/2} + J_z^{1/2} \otimes \hat{q}^{\frac{1}{2}J_z}
\] (6.25)
and decomposing the 4-dimensional representation so obtained into the adjoint and trivial representations. The two procedures are equivalent.

Here note the following relation between \(\hat{q}\) and \(q\):

\[
[2]_q = \langle 2 \rangle_1
\] (6.26)

This shift from \(\hat{q}\) to \(q\) results from our use of \(\langle \ \rangle_1\) in (3.5) instead of \([ \ ]_q\).

In spite of these differences between the external \(q\)-algebra and the Lie algebra, the external physical fields, as well as their Lagrangian and transformation laws, are all numerically valued and can be treated by the standard procedures that we have followed here.
7. Gauge Invariance of the Internal Sector.

To seriously pursue the \( q \)-theory one must discuss the dual algebra generated by \((B, C, \theta)\) or alternatively by \((\alpha, \bar{\alpha}, \beta, \bar{\beta})\). Since this algebra is not a Lie algebra, any gauge theory based on the dual algebra must be quite different from the gauge theory based on \((J_B, J_C, J_\theta)\). In particular there is no analogue of \( g_{ab} \).

Nevertheless one may still define a vector connection \( V_\mu \) in terms of the covariant derivative \( \nabla_\mu \) as before:

\[
\nabla_\mu = \partial_\mu + V_\mu \quad (7.1)
\]

and the corresponding field strengths:

\[
V_{\mu\nu} = (\nabla_\mu, \nabla_\nu) \quad (7.2)
\]

The earlier stated transformation laws (6.2) and (6.6) still hold

\[
\nabla'_\mu = T \nabla_\mu T^{-1} \quad (7.3)
\]

\[
V'_{\mu\nu} = TV_{\mu\nu} T^{-1} \quad (7.4)
\]

where \( T \) lies in the internal algebra \( A \). Then

\[
(V_{\mu\nu}V^{\mu\nu})' = T(V_{\mu\nu}V^{\mu\nu})T^{-1} \quad (7.5)
\]

In the standard theory the field invariant may be expressed as either \( \text{Tr} \ G^{\mu\nu}G_{\mu\nu} \) or \( g_{ab}G_{\mu\nu}^a G^{b\mu\nu} \). That is not possible here because both \( V^{\mu\nu} \) and \( T \) lie in the \( A \)-algebra. The trace is therefore not invariant since in general

\[
((V^{\mu\nu})_{ab}, T_{cd}) \neq 0
\]

Therefore we choose as field invariant

\[
I = h(\Phi^+ V_{\mu\lambda} V^{\mu\lambda} \Phi) \quad (7.6)
\]

where

\[
\Phi' = T\Phi \quad (7.7)
\]

\[
\Phi'^+ = \Phi'^+ T^{-1} \quad (7.8)
\]

and \( T \) is a unitary transformation lying in the \( A \)-algebra.
The new factor $\Phi$ may be taken to be a Higgs field. Then

$$I = \sum h[\varphi_{mn}^j (D_{mn}^j)^+ \cdot (V_{\mu\lambda})^j_{m'n'} (V_{\mu\lambda})^{j'}_{m'n'} \cdot (V_{\mu\lambda})^{j''}_{m'n''} (D_{m'n''}^{j''})^+ \cdot \varphi_{m'n''}^{j'''} (D_{m'n''}^{j'''}))]$$

$$= \sum (\varphi^+)_{mn}^j (V_{\mu\lambda})^j_{m'n'} (V_{\mu\lambda})^{j''}_{m'n''} (D_{m'n''}^{j''})^+ h[(D_{mn}^j)^+ D_{m'n''}^{j'''} D_{m'n''}^{j'''} D_{m'n''}]$$

(7.9)

In particular if $I$ is evaluated on the vacuum state of $\Phi$ one finds

$$I = [(\varphi^+)_{oo}^j (\varphi^+)_{oo}^j \sum (V_{\mu\lambda})^j_{m'n'} (V_{\mu\lambda})^{j''}_{m'n''} \left[ \delta^{j'}_{j''} \delta_{m'm'} \delta_{n'n''} \frac{1}{[2j' + 1]_q} \right]$$

(7.10)

or

$$I = (\varphi^+)_{oo}^j \sum (V_{\mu\lambda})^j_{m'n'} (V_{\mu\lambda})^{j}_{m'n'} \frac{g^{2n}}{[2j + 1]_q}$$

(7.11)

The contribution of the trivial representation to this sum is

$$I_o = \left| \varphi^+_{oo} \right|^2 (V_{\mu\lambda})^+_{oo} (V_{\mu\lambda})^+_{oo}$$

(7.12)

which resembles

$$\frac{1}{g^2} V_{\mu\lambda} V_{\mu\lambda}$$

(7.13)

for the Abelian case if we set

$$(\varphi^+)_{oo}^2 = \frac{1}{g^2}$$

(7.14)

The full $I$ contains contributions that are averaged over all representations.

One may choose the following gauge invariant Lagrangian\(^8\)

$$h \int \left\{ -\frac{1}{4} \Phi^+ V_{\mu\lambda} V_{\mu\lambda} \Phi + i \bar{\psi} \gamma^{\mu} \nabla_{\mu} \psi + \frac{1}{2} [\nabla_{\mu} \Phi^+ \nabla^{\mu} \Phi] + U(\Phi^+ \Phi) \right\}$$

(15.7)

This form differs from (4.2) in Ref. \(^8\) in two respects: (a) it is an average over the algebra and (b) the invariance transformations are unitary.

8. Macroscopic and Microscopic Pictures.

There are two possibilities suggested by the foregoing. Following the first or standard path one may expand the external gauge fields in the numerical matrices ($J_1, J_2, J_3$) as well as in the usual normal modes. Following the second path one may expand the internal fields in the $D_{mn}^j(\alpha, \bar{\alpha}, \beta, \bar{\beta})$ as well as in the usual normal modes. In the first case one has the standard classical point particle theory obeying the algebra (3.12). In the second case one has a classical soliton field theory lying in the algebra of the arguments of $D_{mn}^j(\alpha, \bar{\alpha}, \beta, \bar{\beta})$ as illustrated in (4.6) and the following discussion. The point particle picture is macroscopic.
while the soliton picture is microscopic. Analogous to the treatment of familiar composite particles by the separation into relative and center of mass coordinates, or into an internal and an external problem, we try to give one (internal) description based on the $D_{mn}^j$, which might be called the color description and a second (external) description based on the algebra of $(J_1, J_2, J_3)$ which might be called the flavor description.

To pass from the field operator in the microscopic description to the corresponding operator in the macroscopic description one averages the operator field of the soliton over the algebra as follows:

$$h[\psi(x)] = h[\sum \varphi_{mn}^j(x)D_{mn}^j] = \sum \varphi_{mn}^j(x)h(D_{mn}^j) = \varphi_{oo}^o(x)$$ (8.2)

We interpret $\varphi_{oo}^o(x)$ as the field operator in the point particle description.

In the standard quantum field theory the field quanta acquire mass and extension via clouds of virtual particles or renormalization of the bare mass. Solitons also arise in classical theory, both topologically and non-topologically, in many forms including the Prasad-Sommerfield model and in the context of Kaluza-Klein extensions, as strings and branes. The proposal described here offers another classical point of departure for a modification of quantum field theory. Finally we emphasize that the model here proposed is suggested by and derived from the quantum group $SU_q(2)$ but does not strictly adhere to the structure of the quantum group, as ordinarily understood.

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