A Coupled Compressive Sensing Scheme for Uncoordinated Multiple Access

Vamsi K. Amalladinne, Student Member, IEEE, Avinash Vem, Dileep Kumar Soma, Krishna R. Narayanan, Member, IEEE, and Jean-Francois Chamberland, Member, IEEE.

Abstract—This article introduces a novel communication scheme for the uncoordinated multiple-access communication problem. The proposed divide-and-conquer approach leverages recent advances in compressive sensing and forward error correction to produce an uncoordinated access scheme, along with a computationally efficient decoding algorithm. Within this framework, every active device first partitions its data into several sub-blocks and, subsequently, adds redundancy using a systematic linear block code. Compressive sensing techniques are then employed to recover sub-blocks up to a permutation of their order, and the original messages are obtained by connecting pieces together using a low-complexity, tree-based algorithm. Explicit closed form expressions are derived to characterize the error probability and computational complexity of this access paradigm. An optimization framework, which exploits the trade-off between error probability and computational complexity, is developed to assign parity check bits to each sub-block. Specifically, two different check bit allocation strategies are discussed and their performances are analyzed in the limit as the number of active users and their corresponding payloads tend to infinity. The number of channel uses needed and the computational complexity associated with these allocation strategies are explicitly characterized for various scaling regimes. In terms of error performance, it is shown that the proposed scheme fails with vanishing probability in the asymptotic setting where the number of active users grows unbounded. Numerical results show that this novel scheme outperforms other existing practical coding strategies. Measured performance lies approximately 4.3 dB away from the Polyanisiky achievable bound, which is derived in the absence of complexity constraints.

Index Terms—Communication, forward error correction, uncoordinated multiple-access, compressive sensing.

I. INTRODUCTION

Uncoordinated and unsourced multiple access communication (MAC) is a novel formulation for non-orthogonal multiple access. This framework, which was introduced by Polyanskiy in [1], is particularly relevant in the context of the Internet of Things (IoT). It is closely related to coded random access [2] and the many access channel [3]. In this new paradigm, a wireless network is composed of \( K_{\text{tot}} \) users, out of which a smaller group of \( K_a \) users are active at any given time. These \( K_a \) active users each wish to transmit a \( B \)-bit message to the access point in an uncoordinated fashion. The access point is tasked with recovering the set of transmitted messages, without regard for the identities of the corresponding sources. The total number of users \( K_{\text{tot}} \), can be very large, whereas parameters \( K_a \) and \( B \) are envisioned to be orders of magnitude smaller than \( K_{\text{tot}} \).

For typical IoT applications, the message length \( B \) is envisioned to remain small and, in this regime, asymptotic information-theoretic results may not offer much insight. Rather, finite-length bounds are more meaningful. Along these lines, Polyanskiy [1] derived finite block-length (FBL), achievability bounds based on random Gaussian codebooks and maximum-likelihood (ML) decoding. While this work provides a benchmark to evaluate coding schemes, random coding with ML decoding is computationally infeasible in most practical situations and, hence, there is an important need for computationally-efficient coding and decoding schemes aimed at the unsourced MAC.

In [4], Ordeentlich and Polyanskiy show that many existing multiple access strategies, including treating inference as noise (TIN) and ALOHA perform poorly in this context, especially when \( K_a \) exceeds 100. They also propose the first low-complexity coding scheme tailored to this setting. In their scheme, the transmission period is divided into sub-blocks, or slots, and the system operates in a synchronous fashion. Specifically, all the users are aware of slot boundaries. Within this framework, every active user transmits a codeword during a randomly chosen slot. A data block is formed with a concatenated code that is designed for a \( T \)-user, real-addition Gaussian multiple access channel (\( T \)-GMAC); values for \( T \) range from 2 to 5. Although this scheme performs significantly better than ALOHA and TIN, there remains an important gap of approximately 20 dB between this realized performance and the aforementioned achievability limit associated with the unsourced MAC [4]. In subsequent work [5], Vem et al. introduce a low-complexity coding scheme that relies on a similar slotted structure. This latter framework consists of an improved, close-to-optimal coding strategy for the \( T \)-GMAC; coupled to the application of successive interference cancellation across slots to reduce the performance degradation caused by overcrowded slots. The combination of these two features constitutes a significant improvement over [4], with a performance curve that lies only approximately 6 dB away from the above mentioned achievability limit. Both schemes discussed above adopt a channel coding viewpoint, wherein the \( K_a \)-user GMAC is reduced to multiple smaller \( T \)-GMAC channel problems.
In contrast, in this article, we develop an alternate compressive sensing (CS) approach tailored to the uncoordinated and unsourced MAC problem. Enabling uncoordinated multiple access to a massive number of users has a strong connection to the problem of support recovery in noisy compressive sensing [1]. Conceptually, decoding an instance of the uncoordinated MAC entails finding the support of an unknown vector of length $2^B$. A naive CS solution requires operations on sensing matrices with $2^B$ columns, which is computationally intractable for values of $B$ on the order of 100. Consequently, any pragmatic solution to this problem needs to have a computational cost that is sub-linear in the dimension of the problem.

To this end, we propose a novel CS algorithm, referred to as the coupled compressive sensing algorithm (CCS) for the uncoordinated MAC problem. This algorithm achieves sub-linear time complexity using a divide-and-conquer approach. Information blocks of the users are divided into smaller sub-blocks such that each sub-block is amenable to a CS recovery. Before transmission, redundancy is added to individual sub-blocks using a systematic linear block code. The collection of sub-blocks transmitted within a slot are recovered using a CS algorithm. Once this is achieved, individual segments of the original messages need to be pieced together. This is accomplished via a low-complexity tree-based algorithm. The overall structure of this communication architecture yields better performance compared to other existing algorithms with comparable computational complexity. Following are the main contributions of this paper.

A. Contributions

1) A novel low complexity compressive sensing algorithm is proposed to solve the uncoordinated multiple access problem.

2) Explicit closed form expressions are provided to characterize the error probability and the average computational complexity of this scheme.

3) It is demonstrated that the parameters of this algorithm allow us to trade-off complexity with error performance. An optimization framework is developed to exploit this trade-off.

4) In the finite block length regime, the algorithm is shown to perform close to the FBL achievability bounds.

5) In the asymptotic regime where $K_n$ and $B$ approach infinity, bounds are provided for the average computational complexity and number of channel uses needed for this scheme to be asymptotically reliable. It is demonstrated that the algorithm has an average complexity which is sub-linear in the dimension of the problem.

In the section below, we review the connection between compressive sensing and multiple access and also the recent developments in the field of compressive sensing.

B. Compressive Sensing and Multiple Access

The connection between compressive sensing and multiple access has been explored in the literature [6, 7, 8]. The work in [6] provides necessary and sufficient conditions for exact support recovery by interpreting sparse recovery problem as a Gaussian MAC coding problem. In [7], the LASSO algorithm is used for user identification in a random multiple access scenario. However, the complexity of this algorithm does not scale well in many regimes of interest. Another closely related area where compressive sensing techniques are applied for random access is the problem of discovering the access points within the range of a wireless device in a network, also known as the neighbor discovery problem [8]. In this problem, each node wishes to identify the network interface addresses of nodes within a single hop. In [8], the authors propose two compressive sensing schemes based on group testing and second-order Reed Muller codes followed by chirp decoding. The second scheme is shown to have sub-linear computational cost. In [9], a low complexity neighbor discovery scheme, referred to as sparse-orthogonal frequency division multiplexing (sparse-OFDM) (which is based on the recent developments in sparse Fourier transform [10]) is proposed for the asynchronous neighbor discovery problem. However, it is not straightforward to extrapolate these schemes to the uncoordinated MAC problem and hence, their performance gap from the FBL bounds remains uncertain. In one of our recent works, we make a performance comparison between the proposed CCS scheme and the sparse-OFDM scheme in [9] and demonstrate that the CCS scheme significantly outperforms sparse-OFDM scheme for asynchronous neighbor discovery (These results are not included in this paper).

Exact support recovery in noisy compressive sensing has been studied extensively in the literature [11, 12, 13]. In [11], the authors establish a connection between problem dimension, sparsity index, and the number of observations needed for exact support recovery using the LASSO algorithm. [12] and [13] provide necessary and sufficient conditions for exact support recovery in the presence of noise. A key result common among these studies is that $O(k \log (p/k))$ measurements are sufficient to recover a $k$ out of $p$ sparse vector in the presence of noise. Conventional compressive sensing solvers like LASSO [14] and iterative hard thresholding [15] are known to achieve this scaling when $k$ scales sub-linear with $p$. However, most of these algorithms admit computational complexity that scales as $\text{poly}(p)$, which precludes the application of these schemes to the uncoordinated MAC problem.

Several works in the field of data stream computations [16, 17, 18] aim to recover a $k$ out of $p$ sparse signal from a low dimensional sketch. The algorithms therein achieve measurement cost of $O(k \log (p/k))$ and computational cost of $O(k \text{polylog}(p))$. However, these algorithms admit a failure probability which is bounded away from zero and hence are not asymptotically reliable.

C. Organization and Notation

The remainder of this article is organized as follows. In Section II, we describe the system model and introduce the various parameters used in this context. Section III provides detailed descriptions of the encoding and decoding operations employed within our proposed scheme. The performance of this system is analyzed in terms of error probability and
average complexity in Section IV. Also, an optimization framework is established to exploit the trade-off between these two quantities. Bounds for the average computational cost and the number of channel uses needed for our scheme to work reliably in the asymptotic regime \((K_a, B \to \infty)\) are derived in Section V. Simulation results are given in Section VI to demonstrate the performance of this scheme in the finite-block length regime and finally, we conclude the paper in Section VII.

At this stage, it is worth review the notation we adopt throughout. We employ \(\mathbb{R}_{+}, \mathbb{Z}_{+}\), and \(N\) to represent the non-negative real numbers, non-negative integers, and the natural numbers, respectively. For any \(a, b \in \mathbb{Z}_{+}\) with \(a \leq b\), we use \([a : b]\) to denote \(\{c \in \mathbb{Z}_{+} : a \leq c \leq b\}\). For any \(a, b \in \mathbb{R}_{+}\) with \(a \leq b\), we use \([a, b]\) to denote \(\{c \in \mathbb{R}_{+} : a \leq c \leq b\}\). We write \(X \sim B(n, p)\) if a random variable \(X\) possesses a binomial distribution with parameters \(n\) and \(p\). We employ \(|A|\) for the cardinality of set \(A\), and we use \([x]\) to designate the closest integer to \(x\). We say \(f(n) = O(g(n))\) if there are constants \(c, n_0\) such that \(f(n) \leq cg(n)\) for all \(n \geq n_0\).

II. SYSTEM MODEL

Let \(S_{\text{tot}}\) be the collection of devices within a network, and let \(S_a\) denote the subset of active devices within a communication round, \(S_a \subset S_{\text{tot}}\). We can label the size of these sets as \(|S_{\text{tot}}| = K_{\text{tot}}\) and \(|S_a| = K_a\). Every active device wishes to communicate \(B\) bits of information to a base station and, collectively, these data transfers must take place through an uncoordinated uplink transmission scheme. That is, transmissions are not scheduled centrally and, consequently, active devices must act independently of one another. The number of channel uses dedicated to this process is \(N\), and we employ \(W = \{\hat{w}_k : k \in S_a\}\) to represent the collection of \(B\)-bit message vectors associated with the active devices. In our proposed scheme, we assume that devices pick their message vectors independently and uniformly at random from the set of binary sequences \(\{0, 1\}^B\).

The base station facilitates a slotted structure for multiple access on the uplink through coarse synchronization. As such, the signal available at the receiver assumes the form

\[
y = \sum_{k \in S_a} \hat{x}_k + z,
\]

where \(\hat{x}_k\) is the \(N\)-dimensional vector sent by device \(k\), and \(z\) represents additive white Gaussian noise. The signal sent by a device is power constrained, i.e., \(||\hat{x}_k||_2^2 \leq Np\) for \(k \in S_a\), a scenario akin to [1]. The energy-per-bit is then equal to \(\frac{E}{N_0} \triangleq \frac{NP}{2}\). The receiver produces an estimate \(\hat{W}(\hat{y})\) for the list of transmitted binary vectors \(W\) with \(||\hat{W}(\hat{y})|| \leq K_a\). The per-user error probability of the system is defined by

\[
P_e = \frac{1}{K_a} \sum_{k \in S_a} Pr \left( \hat{w}_k \notin \hat{W}(\hat{y}) \right).
\]

In words, there is a penalty when a sent message is missed by the base station. Moreover, the total number of vectors in \(\hat{W}(\hat{y})\) is subject to a hard constraint, which prevents the base station from admitting an excessive number of guesses. The following key features distinguish this paradigm from the traditional multiple access paradigm:

1) Users transmissions are uncoordinated in nature and, hence, all the active users share the same codebook.
2) Decoding is done up to permutation of messages.
3) Error probability is defined on a per user basis, as opposed to a success corresponding to receiving all the messages correctly.

We note that (1) can be expressed in matrix equation form as

\[
y = S\hat{b} + z,
\]

where \(S \in \mathbb{R}^{N \times 2^B}\) denotes the common codebook (where the \(i^{th}\) column \(S(:, i) = \hat{x}_i\)) used by the active users and \(\hat{b} \in \{0, 1\}^{2^B}\) contains the list of all transmitted codewords with \(\|\hat{b}\|_0 = K_a\). The above formulation allows us to view this problem as a compressive sensing problem with \(S\) taking the role of a sensing matrix and \(\hat{b}\) being an unknown \(K_a\)-sparse vector. As pointed out in the previous section, the matrix \(S\) has \(2^B\) columns \((B \approx 100)\) and hence a naive CS algorithm cannot be employed to recover the sparse vector \(\hat{b}\). Our goal is to devise an encoding and decoding scheme that achieves \(P_e \leq \varepsilon\), where \(\varepsilon\) is the target error probability, and does so with manageable computational complexity. Table I summarizes the important parameters encountered in this article, along with proper notation.

III. PROPOSED SCHEME

A notional diagram of the proposed system is shown in Fig. 1 and a top level overview of the proposed CCS scheme is illustrated in Fig. 2.

A. Encoder

The transmission strategy features two parts: a systematic linear block code based on random parity checks, which we refer to as the tree encoder, and a CS encoder.

\[
\begin{array}{cccc}
\text{Free Encoder} & & \text{CS Encoder} \\
\text{Tree Encoder} & & \text{Tree Encoder} \\
\end{array}
\]

Fig. 1: This is a schematic of the proposed scheme. Original messages are split into sub-blocks, and redundancy is added to individual components. Transmitted sub-signals are then determined via a CS matrix, and sent over a MAC channel. A CS decoder recovers lists of sub-blocks, and a tree decoder reconstructs the original messages.

Tree Encoder: Every \(B\)-bit binary message vector \(\hat{w}\) is partitioned into \(n\) sub-blocks, with the \(i^{th}\) sub-block consisting of \(m_i\) message bits with \(\sum_{i=0}^{n-1} m_i = B\). The tree encoder appends \(l_i\) parity check bits to sub-block \(i\), bringing its total length to \(m_i + l_i = J = M/n\) symbols. This ensures that all
the coded sub-blocks have the same length. The first block is always chosen to have $m_0 = J$ message bits, with $l_0 = 0$. The parity check bits in each sub-block are constructed as follows. Let $(p_0(i), p_1(i), \ldots, p_{l_i-1}(i))$ denote the parity bits in sub-block $i$. These bits are selected to satisfy random parity check constraints for all the message bits preceding their respective sub-block. To this end, we concatenate the message bits of all the sub-blocks $k \in [0 : l]$ and index them with the set $[0 : \sum_{k=0}^l m_k - 1]$. We then choose $l_i$ subsets $A_{j}^{(i)} \subseteq [0 : \sum_{k=0}^l m_k - 1] \forall j \in [0 : l_i - 1]$ uniformly at random. Parity check $p_{j}^{(i)}$ is chosen as the modulo-2 sum of all the message bits indexed by the set $A_{j}^{(i)}$. In effect, $p_{j}^{(i)}$ acts as a parity check constraint for some randomly chosen message bits preceding it. In Section IV-A we describe an optimization framework for the choice of parity length vector $l = (l_0 = 0, l_1, \ldots, l_{n-1})$.

**CS Encoder:** Let $A = [\bar{a}_1, \ldots, \bar{a}_{2^n}] \in \{\pm \sqrt{\mathcal{P}}\}^{N \times 2^n}$, where $N = N/n$, denote a compressed sensing matrix that is designed to recover any $K_a$-sparse binary vector from $N$ noisy observations with a low probability of error. The $J$ bits in a sub-block are encoded using a function $f : \{0, 1\}^J \rightarrow \{\bar{a}_j, j \in [1 : 2^J]\}$, which maps each sub-block to a column in $A$. That is, a column of $A$ is a potentially transmitted sub-block.

**B. Decoder**

The decoding scheme consists of two components: a CS decoder that operates over each sub-block, and a tree decoder operating across sub-blocks. These components are detailed below.

**CS Decoder:** The aggregate signal received at the base station during the $i^{th}$ sub-block can be expressed as $\bar{y}_i = A\bar{x}_i + \bar{z}_i$, where $\bar{y}_i \in \{0, 1\}^{2^n}$ is a $K_a$-sparse binary vector that indicates the list of $i^{th}$ sub-blocks transmitted by the active users. The task of the CS decoder is to provide an estimate of the sparse vector $\hat{b}_i$ from the received signal $\bar{y}_i$ during the corresponding time slot. This is accomplished by first applying a conventional CS decoding algorithm (e.g. non-negative least squares (NNLS), LASSO etc.) to get an estimate $\hat{b}_i^{(cs)}$ of vector $\tilde{b}_i$. Yet, this does not ensure that the entries of vector $\hat{b}_i^{(cs)}$ are binary. The desired binary estimate $\hat{b}_i$ is obtained by setting the $K$ largest entries of the vector $\hat{b}_i^{(cs)}$ to one and the remaining $2^J - K$ entries to zero. The number $K$ is chosen as $K = K_a + K_\delta$, where $K_\delta$ is a small positive integer. Although the list output by the CS decoder is larger than $K_a$, the quantity $K_\delta$ is carefully chosen such that the erroneously decoded sub-blocks are very unlikely to satisfy the parity check constraints associated with encoding process.

**Tree Decoder:** The tree decoder seeks to recover the original messages transmitted by all the users by piecing together
valid sequences of elements drawn from the various CS lists. Towards this end, the access point constructs a decoding tree for every candidate message as follows. We fix a sub-block from the list of all possible initial sub-blocks supplied by the CS decoder, and we view it as the root node for a tree. Once this sub-block from stage zero is determined, there are $K$ possible choices for the following sub-block, and these are the nodes which appear in stage one of the tree. Similarly, there are $K$ possible choices for each sub-block for each possibility within the preceding sub-block and, hence, $K^2$ nodes in stage two. This process continues until the $(n - 1)^{th}$ stage is reached; at this point, the tree has $K^{n-1}$ leaves. Every path connecting the root node to a leaf becomes a possible message. If there exist a single valid path at the end that meets the parity checks, then the decoder outputs the corresponding message; otherwise, it reports a failure.

The number of possible paths increases exponentially with the stages of the tree and, hence, a naive search through all the leaf nodes is infeasible. In practice, invalid paths are pruned iteratively through the parity check constraints. Specifically, at stage $i \geq 1$, the decoder retains only nodes that satisfy the $i$ parity constraints on all the message bits preceding that stage. This iterative procedure continues until the $(n - 1)^{th}$ stage is reached. The complexity of this decoding scheme depends on the number of nodes surviving each stage, since parity checks only need to be enforced only on the children of surviving nodes in subsequent stages of the tree decoding process. Fig. 3 gives a step-by-step description of various stages involved in the tree decoding algorithm.

**Iterative Extension:** The framework of CCS algorithm allows for an iterative extension using successive cancellation. At the end of tree decoding, the access point produces an estimate $\tilde{b} \in \{0, 1\}^{n\times K}$ of the binary sparse vector $b$ (which contains the list of all transmitted codewords). A transmitted codeword will not be present in $\tilde{b}$ if either one of the sub-blocks corresponding to this message is erroneously decoded by the CS decoder or if the tree decoder fails to decode the message correctly. Hence, with a high probability, we have $\|\tilde{b}\|_0 \leq \|\tilde{b}\|_0 = K_\lambda$. For notational convenience, we denote $\|\tilde{b}\|_0 = K_\lambda$. The contribution of this successful output $\tilde{b}$ can be subtracted from the received signal $\hat{y}$ (see (3)) to yield,

$$\hat{y} = \hat{y} - S\tilde{b} = S(\hat{b} - \tilde{b}) + z.$$  

Similar to (3), the above equation resembles the standard form of a noisy binary compressive problem, where $S$ is the sensing matrix and $\hat{b} - \tilde{b}$ is the unknown binary vector with sparsity index given by $\|\hat{b} - \tilde{b}\|_0 = K_\lambda - K_\lambda$. An improved estimate of the vector $b$ can now be obtained by solving the above problem using the divide-and-conquer approach, where the CS decoder operates at sub-block level (to recover the valid sub-blocks the CCS algorithm failed to decode in the previous iteration) and the tree decoder at block-level. The list size output by the CS decoder at sub-block level for this iteration is set to $K' = K_\lambda - K_\lambda + K_\delta$, where $K_\delta$ is a small positive integer. This successive interference cancellation method can...

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**Fig. 3:** Illustration of the tree decoding algorithm
be repeated iteratively, leading to significant potential gains in performance, particularly for the first few iterations.

IV. PERFORMANCE ANALYSIS

In this section, we analyze the performance of the CS multiple access scheme. We rely mostly on established results to characterize the CS aspect of the scheme. On the other hand, the analysis of the tree encoding and decoding is new and remarkably tractable, as seen below.

Suppose that the list output by the CS decoder contains sub-block $i$ transmitted by user $k$ with probability $1 - p_{cs}$; alternatively, with probability $p_{cs}$, this sub-block is erroneously replaced by a vector chosen uniformly at random from all other candidates in the set $\{0, 1\}^l$. Let $E_k$ denote the event that the transmitted binary message from user $k$ is not present in the list output by the tree decoder. Similarly, let $C_k$ be the event that all the sub-blocks corresponding to this user are present on the lists output by the CS decoder. With this notation, probability $P(E_k)$ can be computed as

$$P(E_k) = P(E_k|C_k)P(C_k) + P(E_k|\overline{C_k})P(\overline{C_k}). \tag{4}$$

If the CS decoder fails to decode at least one of the sub-blocks belonging to a particular user, then the output of the tree decoder would not contain the original message originating from that user. Thus, we have $P(E_k|\overline{C_k}) = 1$. Given that the CS sub-blocks are decoded independently, $P(C_k)$ can be computed as $P(C_k) = (1 - p_{cs})^m$. We denote the event that the tree decoder declares a failure because of more than one path surviving the tree decoding process by $E_kC_k$, and we write the probability corresponding to this event as $p_{tree}$. When there are no iterations involved in the decoding process, the quantity $P_e$ is the same as $P(E_k)$; they can be computed using (4) along with the aforementioned observations,

$$P_e = p_{tree}(1 - p_{cs})^n + (1 - (1 - p_{cs})^n). \tag{5}$$

In view of this expression, we turn our attention to better understanding $p_{tree}$. Let $L_i$ denote the random variable corresponding to the number of erroneous paths that survive stage $i \in [1 : n - 1]$ of the tree decoding process. The following results hold.

Lemma 1. The expected values for $L_i$ can be computed as

$$E[L_i] = \sum_{m=i}^{i} \binom{K - 1}{m} \prod_{j=m}^{i} p_j \tag{6}$$

where $p_i = 2^{-i}$, $q_i = 1 - p_i$, and $i \in [1 : n - 1]$.

Proof: Let $\vec{v}$ denote a vector chosen uniformly at random from the set $\{0, 1\}^m$ for some $m \in \mathbb{N}$. If $\vec{v}$ includes $l$ linear parity-check constraints ($l < m$), then admissible $\vec{v}$ must lie in a subspace of dimension $2^{m-l}$. This dimensionality reduction arises from the fact $l$ bits within $\vec{v}$ are completely determined by the remaining $m - l$ bits through the parity constraints $\vec{v}$ satisfies. Hence, the probability that a randomly chosen vector $\vec{v}$ satisfies these $l$ constraints is given by $p = 2^{m-l}/2^m = 2^{-l}$. For convenience, we introduce the notation $p_i = 2^{-i}$ and $q_i = 1 - p_i$ where $i \in [1 : n - 1]$. At stage 1, the probability that any binary random vector of length $2f$ satisfies $l_1$ randomly chosen parity checks is given by $p_1 = 2^{-l_1}$. Thus, the probability that $k$ out of $K - 1$ erroneous paths survive after stage 1 is given by

$$P(L_1 = k) = \binom{K - 1}{k} q_i^{K-1-k} p_1^k. \tag{6}$$

In this context, random variable $L_1$ possesses binomial distribution $B(K - 1, p_1)$.

Moving forward, we can extend this analysis to subsequent stages. Suppose $i \geq 2$ and assume $L_{i-1}$ is given, then $L_i$ becomes the sum of $L_{i-1} + 1$ independent binomial random variables; $L_{i-1}$ of them have parameters $(K, p_i)$, whereas the last one features parameters $(K - 1, p_i)$. It follows that the distribution of $L_i|L_{i-1}$ is given by $B((L_{i-1} + 1)K - 1, p_i)$. The expected value $E[L_i]$ then admits the recursive form

$$E[L_i] = E[E[L_i|L_{i-1}]] = E[(L_{i-1} + 1)K - 1]p_i \quad \tag{7}$$

where (7) is due to the fact that $E[L_i|L_{i-1}]$ is the mean of binomial random variable $L_i|L_{i-1}$. The above equation can be solved recursively using initial condition $E[L_1] = (K - 1) p_1$; this yields the closed-form expression in (7).

Having gained a handle on the expected growth in the number of erroneous paths, we are in a position to analyze the performance of the tree decoder.

Lemma 2. The probability of error for the tree decoder $p_{tree}$ is given by $p_{tree} = 1 - G_{L_{n-1}}(0)$ where $G_{L_{n-1}}(z) = \prod_{i=0}^{n-2} f_{i-1}^{K-1}(z)$ and

$$f_k(z) = \begin{cases} q_k + p_k f_{k+1}(z) & k \in [1 : n - 1] \\ z & k = n. \end{cases} \tag{8}$$

As in Lemma 1, we have $p_i = 2^{-i}$ and $q_i = 1 - p_i$.

Proof: The quantity $p_{tree}$, which denotes the probability that more than one path survives the last stage of tree decoding process, can be represented as $P(L_{n-1} \geq 1)$. To compute this value, we first derive the probability generating function (PGF) $G_{L_{n-1}}(z)$ of random variable $L_{n-1}$. The function $G_{L_{n-1}}(z)$ is defined as

$$G_{L_{n-1}}(z) = E[z^{L_{n-1}}] = \sum_{k=0}^{K-1} P(L_{n-1} = k) z^k. \tag{9}$$

Leveraging the fact that $L_{n-1}|L_{n-2} \sim B((L_{i-2} + 1)K - 1, p_{n-1})$, the above expression can be computed as

$$G_{L_{n-1}}(z) = E[z^{L_{n-1}}] = E[E[z^{L_{n-1}|L_{n-2}}]] = E[(q_{n-1} + p_{n-1})^{L_{n-2}+1}K-1] = (q_{n-1} + p_{n-1}z)^{K-1} G_{L_{n-2}}((q_{n-1} + p_{n-1}z)^K), \tag{10}$$
where the third equality applies because \( \mathbb{E}[z^{L_{n-1}} | L_{n-2}] \) is the PGF of binomial random variable \( L_{n-1} | L_{n-2} \). The above equation can be solved recursively with initial condition \( G_{L_1}(z) = (q_1 + p_1 z)K^{-1} \) to yield a closed-form solution for the PGF found in (8). Using (9), the quantity \( p_{\text{tree}} \) can then be computed as

\[
p_{\text{tree}} = P(L_{n-1} \geq 1) = 1 - P(L_{n-1} = 0) = 1 - G_{L_{n-1}}(0),
\]

where \( G_{L_{n-1}}(0) \) is obtained by evaluating (8) at \( z = 0 \).

We define the computational complexity \( C_{\text{tree}} \) of the tree decoder as the total number of parity check constraints that need to be verified. The expected value of this quantity is capture in Lemma 3 below.

**Lemma 3.** A closed-form expression for computing the expected computational complexity \( \mathbb{E}[C_{\text{tree}}] \) is given by

\[
\mathbb{E}[C_{\text{tree}}] = K \left[ \sum_{i=1}^{n-1} l_i + \sum_{i=1}^{n-2} l_{i+1} \sum_{m=1}^{i} K^{i-m} (K-1) \prod_{j=m}^{i} p_j \right]
\]

where \( p_i = 2^{-l_i} \) and \( q_i = 1 - p_i \).

**Proof:** For each non-leaf node that survives stage \( i \), a total of \( l_{i+1} \) parity checks need to be verified for each of its \( K \) children. Hence, computational complexity and its expected value can be expressed as

\[
C_{\text{tree}} = K l_1 + \sum_{i=1}^{n-2} (L_i + 1) K l_{i+1}
\]

\[
\mathbb{E}[C_{\text{tree}}] = K l_1 + \sum_{i=1}^{n-2} \mathbb{E}[L_i] + 1 \] \( K l_{i+1} \). (13)

Substituting (6) into (13), the closed-form expression of (11) is obtained.

### A. Choice of the Parity Length Vector

The rate of the tree code is intrinsically determined by the number of parity constraints added to the information bits. Once the rate is fixed, one must decide where to position these parity bits. Allocating more parity bits towards the initial stages of the tree will limit expected complexity. However, this will come at the expense of a higher probability of decoding failure. On the other hand, pushing the parity bits towards later stages will have the dual effect of reducing the error rate and increasing computational complexity. Hence, the location of parity bits should be selected judiciously as a means to tradeoff performance and complexity. To this end, we formulate the constrained optimization problem of minimizing the expected complexity subject to the probability of decoding failure being less than a carefully chosen threshold \( \varepsilon_{\text{tree}} \).

\[
\text{minimize} \quad \mathbb{E}[C_{\text{tree}}] \]

\[
\text{subject to} \quad p_{\text{tree}} \leq \varepsilon_{\text{tree}},
\]

\[
\sum_{i=1}^{n-1} \log_2 \left( \frac{1}{p_i} \right) = M - B,
\]

\[
\log_2 \left( \frac{1}{p_i} \right) \in [0 : J] \forall i \in [1 : n - 1],
\]

where \( \mathbb{E}[C_{\text{tree}}] \) is given in (11) and \( p_{\text{tree}} \) comes from (8). Since the parity lengths are non-negative integers, the above problem can be difficult to solve. As such, we relax the problem to \((l_1, l_2, \ldots, l_{n-1}) \in \mathbb{R}^{n-1}_+ \). Also, we replace the constraint \( p_{\text{tree}} \leq \varepsilon_{\text{tree}} \) with \( \mathbb{E}[L_{n-1}] \leq \varepsilon_{\text{tree}} \) for the purpose of mathematical tractability. Even with these relaxations, this problem remains challenging because of the non-convex nature of the objective function. Hence, instead of minimizing the average number of parity check constraints that need to be verified, we minimize the average number of nodes in the tree on which parity check constraints need to be verified. Let \( C_{\text{tree}} \) denote the total number of nodes on which parity check constraints need to be verified. Then, similar to (11), the quantity \( \mathbb{E} [C_{\text{tree}}'] \) can be computed as

\[
\mathbb{E} [C_{\text{tree}}'] = K + \sum_{i=1}^{n-2} \mathbb{E}[L_i] + 1 \] \( K l_{i+1} \).

(14)

After these modifications, the optimization framework for allocating parity lengths becomes

\[
\text{minimize} \quad \mathbb{E}[C_{\text{tree}}']
\]

\[
\text{subject to} \quad \mathbb{E}[L_{n-1}] \leq \varepsilon_{\text{tree}},
\]

\[
\sum_{i=1}^{n-1} \log_2 \left( \frac{1}{p_i} \right) = M - B,
\]

\[
p_i \in \left[ \frac{1}{M^2}, 1 \right] \forall i \in [1 : n - 1].
\]

The above problem is a geometric program (19), and it can be solved using standard convex solvers. In implementing the proposed scheme, we adopt an allocation scheme based on the solution to the above optimization problem. Specifically, if \((\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_{n-1})\) is the aforementioned solution, then we allocate parity check bits as \( l_i = \lfloor \log_2 (1/\hat{p}_i) \rfloor \) for \( i \in [1 : n - 1] \).

### V. Asymptotic Analysis

As is often the case with problems of inference and communication systems, we seek better insight by studying the CCS algorithm in the context of large settings. In particular, we provide bounds for the average computational complexity and the number of channel uses needed for CCS to be asymptotically reliable. Throughout this section, we assume that both the number of active users \( K_a \) and the number of bits transmitted by each user \( (B) \) jointly tend to infinity. We examine two different regimes for parameters \( K_a \) and \( B \), which will be referred henceforth as logarithmic regime and linear regime, respectively:

(i) \( B = a \log_2 K_a \) for some fixed constant \( a > 1 \),
(ii) \( B = a K_a \) for some fixed constant \( a > 0 \).

The scaling in logarithmic regime corresponds to scenarios akin to (8) where every active user sends its identity as data. If the number of active users \( K_a \) scales sub-linearly with the
total number of users $K_{\text{tot}}$, i.e., $K_a = K_{\text{tot}}^{1/\alpha}$ for $\alpha > 1$, then each active user needs to send $B = \log_2 K_{\text{tot}} = \alpha \log_2 K_a$ bits to identify itself. In the linear regime, we treat the special case $B = K_a$; this assumption greatly simplifies the exposition for the relation. Still, the results we provide are valid for any linear factor $\alpha > 0$. At this stage, it is pertinent to emphasize that implementing a naive CS solution is computationally infeasible for these regimes because the problem is $K_a$ out of $K_a^\alpha$ ($\alpha > 1$) sparse in the logarithmic regime, and $K_a$ out of $2^{K_a}$ sparse in linear regime. Also, we note that these regimes are selected, partly, to enable the comparison of our algorithm with prior art.

From Section [IV-A] we already know that the positioning of parity-check bits can act as a mechanism to tradeoff performance and computational complexity. Again, allocating parity-check bits towards the later stages of CCS improves performance at the expense of complexity, whereas placing bits in early stages has the reverse effect. In view of this principle, we consider two distinct parity-check bit allocation strategies for asymptotic regime, one for each of the extreme behaviors outlined above:

i. All parity check-bits are appended towards the end of the tree.

ii. Parity check bits are equally distributed among all the non-root nodes of the tree.

In this context, we provide bounds for the number of parity check bits, the number of channel uses, and the average computational complexity needed to achieve asymptotic reliable communication ($P_e \to 0$ as $K_a \to \infty$). This is accomplished for trailing and equally distributed parities in both logarithmic and linear regimes, leading to four distinct cases.

For both the parity-check profiles mentioned above, we show that the number of channel uses (i.e., sample complexity in the compressive sensing literature) scales in an order-optimal fashion with respect to $K_a$ and $B$. In particular, the average computational complexity scales sub-linearly with the dimension of the original compressive sensing problem, $2^B$, for both cases. We also demonstrate that, although the number of channel uses is order optimal in both scenarios, the preconstant in the Big-$O$ terms is larger for equally distributed parity-check bits than it is for the trailing parity allocation. Of course, this gain in terms of channel uses in trailing parity allocation comes at the cost of higher average computational cost. These findings are formalized in Theorem 5 and Theorem 6.

**Remark 4** (Number of channel uses and computational cost of conventional CS solvers). If conventional CS solvers (e.g., LASSO, OMP, etc.) are employed to solve a global uncoordinated MAC problem of dimension $2^B$, then the computational complexity scales polynomially (at least linearly) with $2^B$; specifically, complexity will be on the order $O(2^{2B})$ for some $\gamma \geq 1$. Moreover, the number of channel uses needed for this scheme to work equals the optimal number of samples needed to solve a $K_a$-sparse problem of dimension $2^B$, which is given by $N_{\text{opt}} = O(K_a \log_2 2^B)$. Thus, the optimal number of samples scales as $O(K_a \log_2 K_a)$ and $O(K_a^2)$ for logarithmic and linear regimes, respectively.

Moving forward, the various parameters employed in our asymptotic analysis are chosen as follows. The list size output by the CS decoders at the sub-block level is set to $K = K_a$. The length of each coded sub-block is selected to be $J = c_1 \log_2 K_a$ for some fixed constant $c_1 > 1$ ($c_1 < \alpha$ in logarithmic regime). We note that $c_1$ can be chosen arbitrarily close to one from the right. With this scaling, the sparsity index admits a sub-linear scaling with the dimension of the compressive sensing problems at the sub-block level; that is, $J = \frac{K}{a} = K_a^{-(c_1-1)} \to 0$ as $K_a \to \infty$. We take the outer code to have a fixed rate, $nJ = O(B)$. Hence, the number of sub-blocks $n$ is of the form $n = c_2$ and $n = c_2 \frac{K_a}{\log_2 K_a}$ for logarithmic and linear regimes, respectively. Constant $c_2$ depends on $c_1$ and the total number of parity-check bits.

From (5), we get an upper bound on the overall error probability,

$$P_e \leq n \rho_{\text{cs}} + \rho_{\text{tree}}.$$

Thence, asymptotic reliability is guaranteed whenever $n \rho_{\text{cs}}$ and $\rho_{\text{tree}}$ decay to zero as $K_a \to \infty$. In the upcoming proofs, we show that the former condition can be ensured by allocating the required number of channel uses needed to solve a $K_a^2$ out of $2^{2J}$ sparse problem for the CS sub-problems. Once this is achieved, it then suffices to identify conditions under which $\rho_{\text{tree}} \to 0$ asymptotically. This proof strategy is applied to the two scenarios of interest below.

**A. All the parity check-bits are appended towards the end of the tree**

Let $C_{\text{cs}}$, $E[C_{\text{tree}}]$, $E[C]$ and $N$ denote the computational complexity of solving the $n$ CS sub-problems, the average tree decoding complexity, the average overall decoding complexity, and the total number of channel uses, respectively. Then, the following theorem holds.

**Theorem 5.** The CCS algorithm, without successive cancellation, can decode a user with vanishing error probability $P_e \to 0$ asymptotically in $K_a$ whenever a total of $P$ parity-check bits are appended at the end of the tree if $P = (n + \delta - 1) \log_2 K_a$ for some fixed constant $\delta > 0$, with $N, C_{\text{cs}}, E[C_{\text{tree}}]$ and $E[C]$ as given in Table [II].

It can be seen from the above theorem that, in the linear regime, even thought the average complexity scales sub-linearly in $2^{K_a}$, it is still huge as the gain in complexity is only in the exponent $\frac{1}{c_1}$ of $2^{\frac{1}{c_1}}$. Decoding complexity for this scenario is high because the tree is not pruned during the initial stages. For a constant fraction of the depth of the tree, the number of parity-check bits allocated to each node is zero. Consequently, the tree grows exponentially until the first parity-check bits appear. To avoid an exponential growth, the tree has to be pruned even during the initial stages by adding extra parity bits.

**Proof:** For mathematical convenience, we assume that the total number of parity-check bits $P$ is an integer multiple of the length of coded sub-blocks $J$. Since all the parity-check bits are appended towards the end of the tree, the final $\frac{P}{J}$ sub-blocks contain only parity-check bits, and the initial $n - \frac{P}{J}$
sub-blocks are composed of information bits. In other words, the number of parity-check bits in sub-block $i$ is given by

$$l_i = \begin{cases} 0 & \text{if } i \in [1 : n - 1 - P/J] \\ J & \text{if } i \in [n - P/J : n - 1]. \end{cases} \quad (17)$$

With this parity-check profile, we find conditions on $P$ for which $\mathbb{E}[L_{n-1}]$ decays to zero in the limit as $K_a \to \infty$. This immediately results in a vanishing tree decoding error probability because $P_{\text{tree}} = \mathbb{P}(L_{n-1} \geq 1) \leq \mathbb{E}[L_{n-1}]$ by the Markov inequality.

The quantity $\prod_{j=m}^{n-1} p_j$ appearing in (6) can be computed using (17) as

$$\prod_{j=m}^{n-1} p_j = \begin{cases} \frac{K_a^P - K_a^{P-m-1}}{2J} & \text{if } m \in [1 : n - P/J] \\ \frac{2J(n-m)}{2J} & \text{if } m \in [n + 1 - P/J : n - 1]. \end{cases}$$

Substituting the above expression into (6), we can upper bound the expected value $\mathbb{E}[L_{n-1}]$ by

$$\mathbb{E}[L_{n-1}] \leq \sum_{m=1}^{n-P} \frac{K_a^{P-m-1}(K_a - 1)}{2J} + \sum_{m=n-P+1}^{n-1} \frac{K_a^{P-m-1}(K_a - 1)}{2J(n-m)}$$

$$= \frac{K_a^P}{2J} \left[ K_a^{P-m-1} - 1 \right] + \frac{K_a - 1}{2J} \left[ 1 + \frac{K_a^{P-m-1}}{K_a^{P-m-1}} \right]$$

$$\leq \frac{K_a^{P-m-1}}{2P} + \frac{K_a - 1}{2J} + \frac{1}{K_a^{P-m-1} - 1}.$$

The quantity $\frac{K_a^{P-m-1}}{2P}$ goes to zero as $K_a \to \infty$ since $c_1 > 1$. The quantity $\frac{K_a^{P-m-1}}{2P}$ goes to zero if $P = (n - 1) \log_2 K_a + f(K_a)$, where $f(x)$ is any function that increases with $x$. Specifically, if we choose $f(x) = \delta \log_2 x$ for some fixed constant $\delta > 0$, we get $p_{\text{tree}} \leq \frac{K_a^P}{c_1 + 1 - 1} + \frac{1}{K_a^{P-m}}$, which implies $p_{\text{tree}} \to 0$ as $K_a \to \infty$.

When the total number of parity-check bits is $P = (n - 1 + \delta) \log_2 K_a$, we have $M = nJ = B + (n - 1 + \delta) \log_2 K_a$. Substituting $J = c_1 \log_2 K_a$ into the above equation, we obtain

$$n = \frac{B + (n - 1 + \delta) \log_2 K_a}{c_1 \log_2 K_a}.$$  

Hence, in logarithmic regime, we have

$$n = c_2 \frac{K_a^{c_1 + 1}}{c_1 - 1}. $$

In logarithmic regime, this leads to $n = c_2 \log_2 K_a$. These results imply that, in logarithmic regime, the number of sub-blocks needed is $O(1)$, whereas in linear regime, it scales nearly linearly with $K_a$.

Let $N_s$ denote the number of channel uses allocated to each CS sub-problem. Standard results in the compressive sensing literature offer sufficient conditions on $N_s$ for exact support recovery using conventional CS algorithms such as LASSO and OMP [11], [20]. For instance, if the measurement matrix is from the uniform Gaussian ensemble, then $N_s = O(K_a \log_2(2^J - K_a)) = O(K_a J)$ channel uses are sufficient to ensure that the error probability of support recovery is $O(\beta_1 \exp(-\beta_2 \min(2^J, K_a)))$ when the standard LASSO algorithm is used with a proper choice of the regularization parameter; $\beta_1$ and $\beta_2$ are fixed constants in the expression above (11). Altogether, having $N_s = \lambda K_s J$ for some fixed constant $\lambda$ ensures that $N_p = 0$ for both logarithmic and linear regimes.

Since there are $n$ CS sub-problems, $N_s$ and the total number of channel uses $N_s$ are related as $N = nN_s = n\lambda K_a J$. This simplifies to $N = \lambda \frac{\alpha^p}{\beta^p} K_a \log_2 K_a$ for logarithmic regime and $N \approx \lambda \frac{c_1}{c_1 - 1} K_a^2$ for linear regime, both of which are consistent with the statement of the theorem. The computational complexity of this scheme arises from its two components: the complexity of solving the CS sub-problems, and the expected complexity associated with the tree decoder process.

**CS Sub-Problems:** The computational complexity of conventional CS solvers (e.g., LASSO, OMP, nnLS) scales polynomially in the length of the unknown vector. Since we have $n$ sub-problems, the computational complexity of the CS sub-problems $C_{cs}$ is given by:

$$C_{cs} = O \left( n \left( \frac{2^J}{p} \right)^p \right) \text{ for some } p > 1$$

$$= O \left( n K_s^{P+1} \right).$$

The above expression simplifies to $C_{cs} = O \left( K_a^{P+1} \right)$ for logarithmic regime and $C_{cs} = O \left( \frac{K_a^{P+1} + 1}{\log_2 K_a} \right)$ for linear regime. Hence, the computational cost of solving the CS sub-problems is bounded by a polynomial in $K_a$ for both regimes. If CS solvers with linear complexity in the length of the unknown vector are employed for the sub-problems (i.e., $p = 1$), then $C_{cs}$ scales as $K_a$ and $K_a^2$ in logarithmic and linear regimes, respectively.

**Tree Decoder:** Substituting $J = c_1 \log_2 K_a$ and $P = (n - 1 + \delta) \log_2 K_a$ into (17), we get

$$l_i = \begin{cases} 0 & \text{if } i \in [1 : \alpha_1 n + \alpha_2] \\ \frac{1}{c_1} & \text{if } i \in [\alpha_1 n + \alpha_2 + 1 : n - 1], \end{cases} \quad (18)$$

where $\alpha_1 = 1 - \frac{1}{c_1}$ and $\alpha_2 = \frac{\delta c_2}{c_1} - 1$. The product $\prod_{j=m}^{n-1} p_j$
\[
E[C_{\text{tree}}] = K_a \left[ (n - 1 + \delta) \log_2 K_a + c_1 \log_2 K_a \sum_{m=1}^{\alpha_1 n + \alpha_2} K_a^{\alpha_1 n + \alpha_2 - m} (K_a - 1) \right.
\]
\[
+ \sum_{i=\alpha_1 n + \alpha_2 + 1}^{n-2} c_1 \log_2 K_a \sum_{m=1}^{\alpha_1 n + \alpha_2} K_a^{i-m} (K_a - 1) \frac{1}{K_a^{c_1 (i-\alpha_1 n - \alpha_2)}} \]
\[
+ \sum_{i=\alpha_1 n + \alpha_2 + 1}^{n-2} c_1 \log_2 K_a \sum_{m=\alpha_1 n + \alpha_2 + 1}^{i} K_a^{i-m} (K_a - 1) \frac{1}{K_a^{c_1 (i-m+1)}} \]
\[
= K_a \left[ (n - 1 + \delta) \log_2 K_a + c_1 \log_2 K_a \left[ K_a^{\alpha_1 n + \alpha_2 - 1} + 1 \right] \right.
\]
\[
+ \sum_{i=\alpha_1 n + \alpha_2 + 1}^{n-2} \frac{c_1 (K_a - 1) \log_2 K_a}{K_a (K_a^{c_1 - 1} - 1)} \sum_{i=\alpha_1 n + \alpha_2 + 1}^{n-2} \frac{K_a^{c_1 - 1)(i-\alpha_1 n - \alpha_2)}{K_a^{c_1 (i-\alpha_1 n - \alpha_2) - 1}} \]
\[
\leq K_a \left[ (n - 1 + \delta) \log_2 K_a + c_1 K_a^{\alpha_1 n + \alpha_2} \log_2 K_a \right.
\]
\[
+ c_1 K_a^{\alpha_1 n + \alpha_2} \log_2 K_a \sum_{i=\alpha_1 n + \alpha_2 + 1}^{n-2} \frac{1}{K_a^{c_1 (i-\alpha_1 n - \alpha_2) - 1}} + \frac{c_1 \log_2 K_a}{K_a^{c_1 - 1} - 1} (n(1 - \alpha_1) - 2 - \alpha_2) \]
\[
\leq (n - 1 + \delta) K_a \log_2 K_a + c_1 K_a^{\alpha_1 n + \alpha_2 + 1} \log_2 K_a \frac{K_a^{c_1 - 1} \log_2 K_a}{K_a^{c_1 - 1} - 1} + \frac{c_1 \log_2 K_a}{K_a^{c_1 - 1} - 1} (n(1 - \alpha_1) - 2 - \alpha_2).
\]

(19)

According to (11), \(J = c_1 \log_2 K_a\), and \(P = (n - 1 + \delta) \log_2 K_a\), the expected complexity for this scenario can be upper bounded as (19). In the logarithmic regime, since \(n = \frac{\alpha_1 + \delta - 1}{c_1}\), the above expression reduces to

\[
E[C_{\text{tree}}] = O \left( K_a \log_2 K_a \right) + O \left( K_a^{c_1 - \frac{1}{\log_2 K_a}} \log_2 K_a \right)
\]
\[
+ O \left( \frac{K_a^{c_1 - \frac{1}{\log_2 K_a}} \log_2 K_a}{K_a^{c_1 - 1} - 1} \right).
\]

It can be seen that the above expression is dominated by the second term in the sum. Consequently, the average complexity of tree decoding for logarithmic regime scales as \(O \left( K_a^{c_1 - \frac{1}{\log_2 K_a}} \log_2 K_a \right)\).

In the linear regime, since \(n = \frac{1}{c_1 - \log_2 K_a} + \frac{\delta - 1}{c_1 - 1}\), (19) can be simplified to

\[
E[C_{\text{tree}}] \leq O \left( K_a^2 \right) + O \left( K_a \log_2 K_a \right) + O \left( 2 \frac{1}{c_1} K_a \log_2 K_a \right)
\]
\[
+ O \left( \frac{2^{\frac{1}{c_1}} K_a \log_2 K_a}{K_a^{c_1 - 1} - 1} \right) + O \left( \frac{K_a^{c_1 - \frac{1}{\log_2 K_a}} \log_2 K_a}{K_a^{c_1 - 1} - 1} \right)
\]
\[
+ O \left( \frac{K_a \log_2 K_a}{K_a^{c_1 - 1} - 1} \right).
\]

where we leverage the identity \(x^{rac{1}{c_1}} = 2^x\) for all \(x > 0\). The above expression is dominated by the third term in the sum. Thus, the average complexity of tree decoding for the linear regime is of the order \(O \left( 2^{\frac{1}{c_1}} K_a \log_2 K_a \right)\).

\[\blacksquare\]

### B. Equal Number of Parity-Check Bits per Sub-Block

Again, we use \(C_{\text{cs}}, E[C_{\text{tree}}], E[C]\), and \(N\) to denote the computational complexity of solving the \(n\) CS sub-problems, the average tree decoding complexity, the average overall decoding complexity, and the total number of channel uses, respectively. Recall that having an equal number of parity-check bits per sub-block, except for the root block, limits complexity at the expense of performance. We present our findings below in the form of a theorem.

**Theorem 6.** The CCS algorithm, without successive cancellation, can decode a user with a vanishing error probability \(P_e \to 0\) asymptotically in \(K_a\) whenever a total of \(P\) parity-check bits are allocated to the end of every non-root sub-block of the tree if \(P = c(n - 1) \log_2 K_a\) for some fixed constant \(c \in (1, c_1)\) with \(N, C_{\text{cs}}, E[C_{\text{tree}}]\) and \(E[C]\) as given in Table 11.

**Proof:** For ease of exposition, we take the total number of parity-check bits \(P\) to be an integer multiple of \(n - 1\). With careful accounting, we note that it is possible to relax this assumption and retain the character of our findings. Yet this structure simplifies the proof considerably. For the current scenario, the number of parity-check bits per non-root sub-block is given by

\[
l_i = \frac{P}{n - 1} \quad i \in [1 : n - 1].
\]

(20)
The quantity \( \prod_{j=m}^{n-1} p_j \) appearing in (6) can then be computed using (20),
\[
\prod_{j=m}^{n-1} p_j = \left( \frac{1}{2^{n-m}} \right)^{n-m} = \frac{1}{2^{n-m}}.
\]
Substituting this expression into (6), we get an upper bound for the expected value \( E[L_{n-1}] \),
\[
E[L_{n-1}] = \frac{n-1}{2^{n-m}} - \left( \frac{K_a - 1}{2^{n-m}} - \frac{K_a - 1}{2} \right)
\]
\[
\leq \frac{K_a - 1}{2^{n-m}} - K_a.
\]
If we select \( P = c(n - 1) \log_2 K_a \) for some fixed constant \( c \in (1, c_1) \), then we get
\[
E[L_{n-1}] \leq \frac{K_a}{2^{c(n - 1) \log_2 K_a}} - K_a
\]
\[
= \frac{1}{K_a^{c - 1} - 1}.
\]
The quantity \( \frac{1}{K_a^{c - 1}} \) → 0 as \( K_a \to \infty \) because \( c > 1 \). Thus, \( p_{\text{tree}} \to 0 \) as \( K_a \to \infty \).

When the total number of parity check bits is \( P = c(n - 1) \log_2 K_a \), we have \( M = nJ = B + c(n - 1) \log_2 K_a \). Substituting \( J = c_1 \log_2 K_a \) into the equation above, we obtain
\[
n = \frac{B - c \log_2 K_a}{(c_1 - c) \log_2 K_a}.
\]
Hence, we have \( n = c_2 \frac{K_a}{\log_2 K_a} \) in logarithmic regime, and we get \( n = c_2 \frac{K_a}{\log_2 K_a} \) where \( c_2 \approx \frac{1}{c_1 - c} \) for large \( K_a \) in linear regime.

In a manner akin to the previous case, the number of channel uses for the CS sub-problem is chosen as \( N_s = \lambda K_a J \). This ensures that \( n_{PC} \to 0 \) for both the regimes. The total number of channel uses \( N \) is then given by \( N = nN_s = n \lambda K_a J \). This simplifies to \( N = \lambda \frac{(c_1 - c)}{(c_1 - c)} K_a \log_2 K_a \) for logarithmic regime and \( N \approx \lambda n c_2 K_a^2 \) for linear regime, both of which are consistent with the statement of the theorem.

**Remark 7.** From the above expressions, the ratio of number of samples required for equally distributed parity allocation and trailing parity allocation strategies with the context of logarithmic and linear regimes are given by \( (c_1 - c)/(c_1 - c + 1) \) and \( c_1/c_2 \), respectively. These ratios can be shown to be strictly greater than one using the fact that \( 1 < c < c_1 < c_2 \). Hence, although the sample complexity has the same order for both the allocation strategies in both the regimes, the pre-scaling constants are larger for equally distributed parity allocation strategy than trailing parity allocation strategy. It is worth noting that the gain in the number of channel uses comes at the cost of computations in trailing parity allocation scheme.

Again, computational complexity for the CCS algorithm arises from two components: the complexity of solving the CS sub-problems, and the average complexity associated with the tree decoder.

**CS Sub-Problems:** This analysis is very similar to that in Section V-A. We assume that the CS sub-problems have polynomial complexity in the dimension of the problem. The complexity of the CS sub-problems \( C_{cs} \) can then be expressed as
\[
C_{cs} = O\left( n \left( 2^J \right)^p \right) \text{ for some } p > 1
\]
\[
= O\left( nK_a^{pc} \right).
\]
Thus, \( C_{cs} \) scales as \( O\left( K_a^{pc} \right) \) and \( O\left( K_a^{pc + 1} \right) \) in logarithmic and linear regimes, respectively, when a CS algorithm of linear complexity is used to solve the CS sub-problems.

**Tree Decoder:** Substituting \( l_i = c(n - 1) \log_2 K_a \) from (6), \( n = c_2 \log_2 K_a \), and \( c_2 = \frac{1}{c_1 - c} \) into (13), an upper bound for the average computational complexity of tree decoding for this scenario can be computed as (21). It can be seen from (21) that the average complexity scales as \( O\left( nK_a \log_2 K_a \right) \). Hence, average complexity of tree decoding scales as \( O\left( K_a \log_2 K_a \right) \) and \( O\left( K_a^2 \right) \) for logarithmic and linear regimes, respectively.

**VI. SIMULATION RESULTS**

In this section, we study the performance of the proposed framework and we provide comparisons with existing schemes in literature. We consider a system with \( K_a \in [25 : 300] \) active users, each having \( B = 75 \) bits of information to transmit. We divide these bits into \( n = 11 \) sub-blocks and the quantity \( J \), which denotes the length of each sub-block, is chosen depending on \( K_a \). Specific values appear in Table IV. Similar to (5), we use sensing matrices that are constructed based on BCH codes for the compressed sensing problem. In particular, we select a subset \( C^0 \) of codewords of size \( |C^0| = 2^J \) from the (2047,23) BCH codebook \( C \) with the following properties: (i) \( \bar{c} \in C^0 \implies \bar{c} \in C \subseteq C \in C^0 \), where \( \bar{c} \in \bar{c} \) denotes the one’s complement of \( \bar{c} \); (ii) \( c_1, c_2 \in C^0 \implies c_1 + c_2 \in C^0 \); (iii) \( 0 \in C^0 \) where \( 0 \) denotes the all zero codeword. We then choose the sensing matrix as \( A = [a_0, a_1, \ldots, a_{2^J - 1}] \), with dimension \( 2047 \times 2^J \) with \( a_i = \sqrt{P}(2c_i - 1) \) and \( c_i \in C^0 \) for \( i \in [0 : 2^J - 1] \). The total number of channel uses is therefore given by \( N = 11 \times 2047 = 22,517 \). The target error probability of the system is fixed at \( e = 0.05 \). We set list size \( K \) for the NNLS CS problem to \( K = K_a + 10 \). For each \( K_a \in [25 : 300] \), we solve the optimization problem (15) using the CVX solver (21), and the resulting solution dictates the choice of the parity length vector. The selected value for \( \varepsilon_{\text{tree}} \) as a function of \( K_a \) is given in Table IV.
$$\mathbb{E}[C_{\text{tree}}] = K_a c (n - 1) \log_2 K_a + \sum_{i=1}^{n-2} c \log_2 K_a \sum_{m=1}^{i} K_a^{i-m} (K_a - 1) \prod_{j=m}^{i} \frac{1}{K_a^j}$$

$$\leq K_a c (n - 1) \log_2 K_a + K_a c \log_2 K_a \sum_{i=1}^{n-2} \sum_{m=1}^{i} \frac{1}{K_a^{(c-1)(i-m+1)}}$$

$$= K_a c (n - 1) \log_2 K_a + \frac{K_a c \log_2 K_a}{K_a^{c-1} - 1} \sum_{i=1}^{n-2} \left[ 1 - \frac{1}{K_a^{(c-1)i}} \right]$$

$$\leq K_a c n \log_2 K_a + \frac{K_a c \log_2 K_a n}{K_a^{c-1} - 1}. \quad (21)$$

| $K_a$ | 25 | 50 | 75 | 100 | 125 | 150 | 175 | 200 | 225 | 250 | 275 | 300 |
|-------|----|----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $J$   | 14 | 14 | 14 | 14  | 14  | 15  | 15  | 15  | 15  | 15  | 15  | 15  |
| $\varepsilon_{\text{tree}}$ | 0.0025 | 0.0045 | 0.006 | 0.01 | 0.0125 | 0.0055 | 0.0065 | 0.007 | 0.008 | 0.01 | 0.0125 | 0.0175 |

**TABLE IV: Various parameters used in simulations.**

![Fig. 4: Minimum $E_b/N_0$ required to achieve $P_e \leq 0.05$ vs. number of active users for various schemes. Results for 2 and 3 iterations are represented by ‘x’ and ‘o’. Observe that the SNR gains diminish with each iteration.](image)

Parameters $B$ and $N$ are chosen such that the rate $\frac{B}{N} = \frac{25}{225}$ is approximately equal to the rate resulting from the choice of parameters $B = 100$ and $N = 30,000$ in [4], [5]. This enables a fair comparison between the schemes contained therein and our proposed approach. We emphasize that the choice of $B$ and $N$ for our simulations is motivated by the existence of good compressive sensing matrices based on BCH codes. When these parameters are proportionally scaled up, performance of the system can only improve, as the finite block length effects are more pronounced for lower values of $B$ and $N$. Table [V] captures the trade-off between the error performance and average computational cost as a function of parity length vector for $K_a = 200$. These results are consistent with the theory developed in this paper. In Fig. 4 the $E_b/N_0$ required to achieve a target error probability of 0.05 is plotted as a function of $K_a$ for various schemes. The bottom most curve corresponds to Polynomials’ achievability bound [1] on the performance of a finite block length (FBL) code for this model. The curves labelled $T = 2, T = 4$, and 4-fold ALOHA, which correspond to the performance curves in [3] and [4], assume the existence of a code for the $T$-user MAC channel which achieves the bound in [1]. It can be seen from Fig. 4 that our proposed approach with just one extended round of iteration outperforms existing schemes for $K_a \in [75 : 300]$. 

**TABLE V: Results from the optimization framework developed in the paper.** Error probability is least when parity check bits are pushed to end and average computational complexity is least when equal parity-check bits are allocated per sub-block.

**VII. Conclusion**

In this article, we proposed a novel compressive sensing architecture for the uncoordinated massive random access problem. The proposed scheme efficiently combines techniques from compressive sensing and forward error correction to yield an algorithm with sub-linear computational cost. While the algorithm was developed for the needs of uncoordinated access, the technical approach may be relevant to many compressive sensing applications where the unknown sparse...
vector is very large such as IoT neighbor discovery or heavy hitters detection in wired networks [9], [16]. Specifically, it would be interesting to see how this algorithmic framework fares in comparison to state-of-the-art techniques, especially in scenarios that require sub-linear decoding complexity. Still, the treatment of such problems would be a significant departure from the content of this article and, as such, the aforementioned questions are left as potential avenues for future research.

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