Embeddings of right-angled Artin groups into higher dimensional Thompson groups

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Abstract

In this paper, we construct embeddings of right-angled Artin groups into higher dimensional Thompson groups. In particular, we embed every right-angled Artin groups into $n$-dimensional Thompson groups, where $n$ is the number of complementary edges in the defining graph. It follows that $\mathbb{Z}^n \ast \mathbb{Z}$ embeds into $nV$ for every $n \geq 1$.

1 Introduction

The Thompson group $V$ is an infinite simple finitely presented group, which is described as a subgroup of the homeomorphism group of the Cantor set $C$. Brin [1] defined higher dimensional Thompson groups as generalizations of the Thompson group $V = 1V$. By definition, $n$-dimensional Thompson group $n_1V$ embeds into $n_2V$ when $n_1 \leq n_2$. Brin [1] showed that $V$ and $2V$ are not isomorphic. Bleak and Lanoue [3] showed $n_1V$ and $n_2V$ are isomorphic if and only if $n_1 = n_2$.

In [4], Bleak and Salazar-Díaz proved that $\mathbb{Z}^2 \ast \mathbb{Z}$ does not embed in $V$. Recently, Corwin and Haymaker [6] determined which right-angled Artin groups embed into $V$. Using the nonembedding result of [4], they showed that $\mathbb{Z}^2 \ast \mathbb{Z}$ is the only obstruction for a right-angled Artin group to be embedded into $V$. On the other hand, Belk, Bleak and Matucci [2] proved that a right-angled Artin group embeds in $nV$ with sufficiently large $n$. They took $n$ to be the sum of the number of vertices and the number of complementary edges in the defining graph. They conjectured that a right-angled Artin group embeds into $(n - 1)V$ if and only if the right-angled Artin group does not contain $\mathbb{Z}^n \ast \mathbb{Z}$. Corwin [5] constructed embeddings of $\mathbb{Z}^n \ast \mathbb{Z}$ into $nV$ for every $n \geq 2$. It follows that every $nV$ with $n \geq 2$ does not embed into $V$.

In this paper, we give another construction of embeddings of right-angled Artin groups into higher-dimensional Thompson groups. In particular, we may embed a right-angled Artin group into $nV$, where $n$ is the number of
complementary edges in the defining graph. We may construct embeddings of $\mathbb{Z}^n * \mathbb{Z}$ into $nV$ in this way.

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2 Right-angled Artin groups

Let $\Gamma$ be a finite graph with a vertex set $V(\Gamma) = \{v_i\}_{1 \leq i \leq m}$ and an edge set $E(\Gamma)$. The corresponding right-angled Artin group, denoted by $A_\Gamma$, is a group defined by the presentation

$$A_\Gamma = \langle g_1, \ldots, g_m \mid g_ig_j = g_jg_i \text{ for all } \{v_i, v_j\} \in E(\Gamma) \rangle.$$

In the following, we let

$$\bar{E}(\Gamma) = \{\{v_i, v_j\} \mid v_i \neq v_j \in V(\Gamma) \text{ are not connected by edges}\}.$$

We call the elements of $\bar{E}(\Gamma)$ complementary edges.

We use the following theorem, known as the ping-pong lemma for the right-angled Artin groups.

**Theorem 2.1 ([7]).** Let $A_\Gamma$ be a right-angled Artin group with generators $\{g_i\}_{1 \leq i \leq m}$ acting on a set $X$. Suppose that there exist subsets $S_i$ ($1 \leq i \leq m$) of $X$, with divisions $S_i = S_i^+ \coprod S_i^-$, satisfying the following conditions:

1. $g_i(S_i^+) \subset S_i^+$ and $g_i^{-1}(S_i^-) \subset S_i^-$ for all $i$.
2. If $g_i$ and $g_j$ commute, then $g_i(S_j) = S_j$.
3. If $g_i$ and $g_j$ do not commute, then $g_i(S_j) \subset S_j^+$ and $g_i^{-1}(S_j) \subset S_j^-$. 
4. There exists $x \in X - \bigcup_{i=1}^m S_i$ such that $g_i(x) \in S_i^+$ and $g_i^{-1}(x) \in S_i^-$ for all $i$.

Then this action is faithful.

3 Embedding right-angled Artin groups into $nV$

We use the following notations in [1]. We let $I$ be a half-open interval $[0, 1)$. An $n$-dimensional rectangle is an affine copy of $I^n$ in $I^n$, constructed by
repeating “dyadic divisions”. An \( n \)-dimensional pattern is a finite set of \( n \)-dimensional rectangles, with pairwise disjoint, non-empty interiors and whose union is \( I^n \). A numbered pattern is a pattern with a one-to-one correspondence to \( \{0, 1, \ldots, r-1\} \) where \( r \) is the number of rectangles in the pattern.

Let \( P = \{P_i\}_{0 \leq i \leq r-1} \) and \( Q = \{Q_i\}_{0 \leq i \leq r-1} \) be numbered patterns. We define \( v(P, Q) \) to be a map from \( I^n \) to itself which takes each \( P_i \) onto \( Q_i \) affinely so as to preserve the orientation. The \( n \)-dimensional Thompson group \( nV \) is the set of partially affine, partially orientation preserving right-continuous bijections from \( I^n \) to itself.

Using these notations, we give a construction of embeddings of right-angled Artin groups into higher dimensional Thompson groups.

**Theorem 3.1.** Let \( \Gamma \) be a graph with the vertex set \( V(\Gamma) = \{v_i\}_{1 \leq i \leq m} \). Suppose that there are nonempty subsets \( \{D_i\}_{1 \leq i \leq m} \) of \( \{1, \ldots, n\} \), such that \( D_i \cap D_j = \emptyset \) if and only if \( v_i \) and \( v_j \) are connected by an edge. Then the right-angled Artin group \( A_\Gamma \) embeds into \( nV \).

For a nonempty subset \( D \) of \( \{1, \ldots, n\} \), a \( D \)-slice of \( I^n \) is an \( n \)-dimensional rectangle \( S = \prod_{d=1}^{n} I_d \), where \( d \in D \) if and only if \( I_d \) is properly contained in \([0, 1)\).

**Lemma 3.2.** Let \( D \) be a nonempty subset of \( \{1, \ldots, n\} \). For every \( D \)-slice \( S \) of \( I^n \) and every division \( S = S^+ \bigcup S^- \) where \( S^+ \) and \( S^- \) are again \( D \)-slices, there is \( h \in nV \) satisfying

\[
(1) \quad h \text{ changes } d \text{-th coordinate of } I^n \text{ if and only if } d \in D.
\]

\[
(2) \quad h(I^n - S^-) = S^+ \text{ and } h^{-1}(I^n - S^+) = S^-.
\]

**Proof.** There is an \( n \)-dimensional pattern which contains \( S \) as a rectangle and consists of \( D \)-slices. We fix one of such pattern \( P \), and consider \( I^n - S \) as a disjoint union of \((|P| - 1)\)-many \( D \)-slices.

We divide \( S^+ \) into mutually disjoint \(|P|\)-many \( D \)-slices. We choose one of those \( D \)-slices in \( S^+ \) and name it \( S^{++} \). We consider \( S^+ - S^{++} \) as a disjoint union of \((|P| - 1)\)-many \( D \)-slices. Similarly, we choose a \( D \)-slices \( S^{--} \) in \( S^- \), and consider \( S^- - S^{--} \) as a disjoint union of \((|P| - 1)\)-many \( D \)-slices.

We define \( h \in nV \) as follows:

0. \( h \) maps \( I^n - S \) to \( S^+ - S^{++} \).

1. \( h \) maps \( S^+ \) to \( S^{++} \).

2. \( h \) maps \( S^- - S^{--} \) to \( I^n - S \).
3. $h$ maps $S^-$ to $S^-$.

This $h$ satisfies conditions (1) and (2).

We show the construction of the map $h$ in the following figure, in the case where $n = 2$, $D = \{1, 2\}$, $S = [0, 1/2) \times [0, 1/2)$ and $S^+ = [0, 1/4) \times [0, 1/2)$.

\[ \begin{array}{ccc}
0_1 & 0_2 \\
1 & 2_1 & 2_2 \\
3 & 2_3 & 0_3 \\
\end{array} \] \[ \begin{array}{ccc}
2_1 & 2_2 \\
0_1 & 0_2 \\
1 & 0_3 & 3 \\
& 2_3 \\
\end{array} \]

Remark 3.1. We take $h \in nV$ as in Lemma 3.2 with respect to a $D$-slice $S$ and some division $S = S^+ \bigsqcup S^-$. Let $S'$ be a $D'$-slice with $D \cap D' = \emptyset$. We may observe that $h(S') = S'$, because $S'$ is determined only by $d'$-th coordinates for $d' \in D'$, which are unchanged by $h$.

Lemma 3.3. For nonempty subsets $\{D_i\}_{1 \leq i \leq m}$ of $\{1, \ldots, n\}$, there is a set of $n$-dimensional rectangles $\{S_i\}_{1 \leq i \leq m}$ satisfying

1. For every $i$, $S_i$ is a $D_i$-slice of $I^n$.
2. $S_i \cap S_j = \emptyset$ if and only if $D_i \cap D_j \neq \emptyset$.
3. $\bigcup_{i=1}^{m} S_i \subsetneq I^n$.

Proof. We fix a dyadic division $I = \bigsqcup_k J_k$, where $k \geq m + 1$. We define $S_i = \bigsqcup_{d \in D_i} I_d$ by setting $I_d = J_i$ when $d \in D_i$, and $I_d = I$ otherwise.

1. Such $S_i$ is a $D_i$-slice.
2. If $D_i \cap D_j \neq \emptyset$, then $S_i \cap S_j = \emptyset$ since $I_d \cap I_d' = J_i \cap J_j = \emptyset$ for all $d \in D_i \cap D_j$. The converse follows from the observation that a $D$-slice and a $D'$-slice always intersect when $D \cap D' = \emptyset$.
3. Since we took $J_k$ small enough, $\bigcup_{i=1}^{m} S_i$ is properly contained in $I^n$.

Therefore, $\{S_i\}_{1 \leq i \leq m}$ satisfies conditions required in Lemma 3.3.

Proof of Theorem 3.1. Let $\Gamma$ be a finite graph with vertices $\{v_i\}_{1 \leq i \leq m}$. Let $\{D_i\}_{1 \leq i \leq m}$ be nonempty subsets of $\{1, \ldots, n\}$ such that $D_i \cap D_j = \emptyset$ if and only if $v_i$ and $v_j$ are connected by an edge.

According to Lemma 3.3, we take $\{S_i\}_{1 \leq i \leq m} \subset I^n$ with respect to $\{D_i\}_{1 \leq i \leq m}$. For every $i$, we fix $D_i$-slices $S_i^+$ and $S_i^-$ satisfying $S_i = S_i^+ \bigsqcup S_i^-$. We define $h_i$ to be $h$ of Lemma 3.2 which is defined with respect to $S_i^+$, $S_i^+$, and $S_i^-$. 

We may define a homomorphism \( \phi : A_\Gamma \to nV \) which maps each generator \( g_i \), corresponding to the vertex \( v_i \), to \( h_i \). This homomorphism is well-defined, since \( h_i \) and \( h_j \) commute when \( v_i \) and \( v_j \) are connected by an edge, according to the first condition of Lemma 3.2.

We consider an action of \( A_\Gamma \) on \( I^n \), which is defined by \( g \cdot x = \phi(g)(x) \).

In the following, we show that this action is faithful, and thus \( \phi \) is injective.

1. By the definition of \( h_i \), \( h_i(S_i^+) \subseteq S_i^+ \) and \( h_i^{-1}(S_i^-) \subseteq S_i^- \) for all \( i \).

2. According to Remark 3.1, \( h_i(S_j) = S_j \) when \( g_i \) and \( g_j \) commute.

3. When \( g_i \) and \( g_j \) do not commute, \( v_i \) and \( v_j \) are not connected by an edge, and \( S_i \) and \( S_j \) are disjoint. Therefore \( h_i(S_j) \subseteq h_i(I^n - S_i) \subseteq S_i^+ \) and \( h_i^{-1}(S_j) \subseteq h_i^{-1}(I^n - S_i) \subseteq S_i^- \).

4. Since \( \bigcup_{i=1}^{m} S_i \nsubseteq I^n \), there is \( x_0 \in I^n - S_i \) for all \( i \). Such \( x_0 \) satisfies \( h_i(x_0) \in S_i^+ \) and \( h_i^{-1}(x_0) \in S_i^- \), for all \( i \).

By Theorem 2.1, \( \phi \) is injective and an embedding of \( A_\Gamma \) into \( nV \).

Corollary 3.4. A right-angled Artin group \( A_\Gamma \) embeds into \( n \)-dimensional Thompson groups, where \( n \) is the number of complementary edges in \( \Gamma \).

Proof. We may assume that every vertex of \( \Gamma \) contributes to a complementary edge. In fact, if we let

\[
V_0(\Gamma) = \{ v \in V(\Gamma) \mid \text{For all } v \neq v' \in V(\Gamma), \{v, v'\} \in E(\Gamma) \},
\]

then \( A_\Gamma = \mathbb{Z}^{V_0(\Gamma)} \times A_{\Gamma'} \) for some subgraph \( \Gamma' \) satisfying the assumption and \( E(\Gamma) = \bar{E}(\Gamma') \). In general, if two groups \( G \) and \( H \) embed in \( nV \), then \( G \times H \) again embeds in \( nV \). Therefore, it is enough to consider whether \( A_{\Gamma'} \) embeds into \( nV \) or not.

Given \( A_\Gamma \) satisfying our assumption, we let \( V(\Gamma) = \{v_i\}_{1 \leq i \leq m} \) be the vertex set and \( \bar{E}(\Gamma) = \{ \bar{e}_k\}_{1 \leq k \leq n} \) be the set of complementary edges. For every \( i \in \{1, \ldots, m\} \), we let

\[
D_i = \{ k \in \{1, \ldots, n\} \mid \bar{e}_k \text{ contains } v_i \text{ as an endpoint.} \}.
\]

We associate \( D_i \) with \( v_i \).

\( \Gamma \) satisfies the condition required in Theorem 3.1 with respect to the subsets \( \{D_i\}_{1 \leq i \leq m} \) of \( \{1, \ldots, n\} \).

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