Berezin Quantization of Gauged WZW and Coset Models*

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Abstract

Gauged WZW and coset models are known to be useful to prove holomorphic factorization of the partition function of WZW and coset models. In this note we show that these gauged models can be also important to quantize the theory in the context of the Berezin formalism. For gauged coset models Berezin quantization procedure also admits a further holomorphic factorization in the complex structure of the moduli space.

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* This work is dedicated to Professor Michel Ryan on the occasion of his 60th birthday.
1. INTRODUCTION

The application of diverse quantization methods to physical systems gives, in many cases, complementary information about these systems. Wess-Zumino-Witten (WZW) models are very interesting kind of two dimensional conformal field theories (CFT) representing exactly solvable models which have been studied intensively some years ago. These models have been studied in the context of different quantization procedures as canonical quantization and Feynman path integral (for a review see, for instance, [2, 3] and references therein).

The coupling of WZW and coset models to gauge fields constitutes the gauged WZW and coset models [4]. In the context of nonsupersymmetric theories these models have been discussed in various global contexts in [2, 3, 5, 6, 7]. In particular, in Ref. [5], Witten have used them to give an alternative proof of the holomorphic factorization of the partition function of WZW and coset models. In this proof the methods of geometric quantization and its relation to the canonical quantization of Chern-Simons theory [8] have been very useful. In the present note we use some of Witten results of Ref. [5], to quantize gauged WZW and coset models through the Berezin formalism [9] (for recent developments, see for instance, [10, 11, 12, 13, 14, 15, 16, 17, 18] to carry over this procedure. In particular, we will use some of the results obtained in [18]. Throughout this paper we follow the notation and conventions given by Witten in Ref. [5].

To begin with we first recall the structure of the WZW model described by the Lagrangian

\[ L(g) = -\frac{k}{8\pi} \int_{\Sigma} d^2\sigma \sqrt{\rho} \rho^{ij} \text{Tr} (g^{-1} \partial_i g \cdot g^{-1} \partial_j g) - \frac{ik}{12\pi} \int_{M} d^3\sigma \varepsilon^{ijk} \text{Tr} (g^{-1} \partial_i g \cdot g^{-1} \partial_j g \cdot g^{-1} \partial_k g), \]

where \( k \in \mathbb{Z} \) is the level, \( \rho_{ij} \) is the worldsheet metric, \( g : \Sigma \to G \) is a map of a compact and orientable Riemann surface \( \Sigma \) (without boundary) into a simple, compact, connected and simply connected Lie group \( G \) and \( M \) is a three-dimensional manifold whose boundary \( \partial M \) is \( \Sigma \). The partition function of the WZW is defined as \( Z_{WZW}(\Sigma) = \int \mathcal{D} g e^{-L(g)} \).

The WZW Lagrangian \( L(g) \) is invariant under the global action of \( G_L \times G_R \) on \( G \) given by \( g \to agb^{-1} \) with \( a \in G_L \) and \( b \in G_R \). Here \( G_L \) and \( G_R \) are copies of \( G \). However if one gauge out a subgroup \( F \) of \( G_L \times G_R \), for instance \( F = G_R \). This WZW action gives rise to
the coupling between the fields $g$ and $F$ gauge fields on $\Sigma$ through the following action

$$L(g, A) = L(g) + \frac{k}{2\pi} \int_\Sigma d^2z \text{Tr} A_z g^{-1} \partial_z g - \frac{k}{4\pi} \int_\Sigma d^2z \text{Tr} A_z A_z, \quad (2)$$

which is not, in general, a gauge invariant extension. The map $g$ is now generalized to a section $g \in \Gamma(\Sigma, X)$ of the $F$-bundle over $\Sigma$: $F \to X \xrightarrow{\pi} \Sigma$. The introduced gauge connection is a $f$-valued connection one-form on $X$, transforming in the adjoint representation of the gauge group $F$, where $f$ is the Lie algebra of $F$. Only for special “anomaly free” gauge groups $F$ there exists such a gauge invariant extension. But we will consider, in the present paper, ‘anomalous’ gauge groups $F$’s such that under the infinitesimal gauge transformation

$$\delta g = -gu, \quad \delta A_i = -D_iu = -\partial_iu - [A_i, u], \quad (3)$$

$L(g, A)$ differs from zero in the form

$$\delta L(g, A) = \frac{ik}{4\pi} \int_\Sigma d^2z \text{Tr} A_z.$$

Following Witten, one can define the functional

$$\psi(A) = \int \mathcal{D}ge^{-L(g,A)}$$

$$= \int \mathcal{D}g \exp \left( -L(g) - \frac{k}{2\pi} \int_\Sigma d^2z \text{Tr} A_z g^{-1} \partial_z g + \frac{k}{4\pi} \int_\Sigma d^2z \text{Tr} A_z A_z \right). \quad (5)$$

This functional obeys the following equations

$$\left( \frac{\delta}{\delta A_z} - \frac{k}{4\pi} A_z \right) \psi(A) = 0, \quad (6)$$

and

$$\left( D_z \frac{\delta}{\delta A_z} + \frac{k}{4\pi} D_z A_z - \frac{k}{2\pi} F_{zz} \right) \psi(A) = 0, \quad (7)$$

where $F_{zz} = \partial_z A_z - \partial_z A_z$. If one define the operators: $\frac{D}{DA_z} = \frac{\delta}{\delta A_z} - \frac{k}{4\pi} A_z$ and $\frac{D}{DA_z} = \frac{\delta}{\delta A_z} + \frac{k}{4\pi} A_z$, we can rewrite Eqs. (6) and (7) as

$$\frac{D}{DA_z} \psi(A) = 0, \quad (8)$$

and

$$\left( D_z \frac{D}{DA_z} - \frac{k}{2\pi} F_{zz} \right) \psi(A) = 0. \quad (9)$$

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\[
\frac{D}{Dz} \text{ and } \frac{D}{DA}\]
can be regarded as a gauge connection on a unitary line bundle \(L^\otimes k\) over the space of all connections \(\mathcal{A}\) over \(\Sigma\). The curvature of this connection can be computed by using the quantization condition: \(\left[ \frac{D}{Dz}, \frac{D}{DA}(w) \right] = \frac{k}{2\pi}\delta(z, w)\) and it yields \(-i\omega\), where \(\omega = k\omega_0\) is the symplectic form on \(\mathcal{A}\) with \(\omega_0 = \frac{1}{2\pi}\int_\Sigma \text{Tr} \delta A \wedge \delta A\). Then \(\omega\) gives to \(\mathcal{A}\) the structure of a symplectic manifold \((\mathcal{A}, \omega)\) and this suggest the geometric prequantization of \(\mathcal{A}\). As \(\mathcal{A}\) is topologically trivial, the prequantum line bundle \(L^\otimes k\) can be identified with the trivial holomorphic line bundle: \(\mathcal{P} = \mathcal{A} \times \mathbb{C}\), whose \(L^2\)-completion of holomorphic sections constitutes the Hilbert space represented by \(H^0_{L^2}(\mathcal{A}, L^\otimes k)\) with Hermitian inner product
\[
\langle \chi | \psi \rangle_{L^2} = \frac{1}{\text{vol}(F)} \int_\mathcal{A} D\mathcal{A} \langle \chi(A) | \psi(A) \rangle.
\]

The measure \(DA\) on \(\mathcal{A}\) can be determined by the symplectic structure \(\omega\) and it can be written as
\[
\langle \chi | \psi \rangle_{L^2} = \frac{1}{\text{vol}(F)} \int_\mathcal{A} \bar{\chi}(A) \psi(A) \exp \left( -\Phi \right) \omega^n \frac{\omega}{n!},
\]
where we have divided by the volume of the gauge group \(F = G\) and where \(\Phi\) is the Kähler potential of the metric on \(\mathcal{A}\). Many of these results about the differential geometry of this moduli space were firstly described in Ref. [19].

The curvature of the connection compatible with the Hermitian structure is given by \(\bar{\partial}\partial (-\Phi) = -i\omega\). Of course the existence of a prequantization bundle implies that \(\left[ \frac{\omega}{2\pi} \right] \in H^2(\mathcal{A}_J, \mathbb{Z})\). If one picks a complex structure on \(\Sigma\) it induces on \(\mathcal{A}\) a fixed complex structure \(J\) giving rise to a complex Kähler manifold \(\mathcal{A}_J\). Complex structure \(J\) also induces a Kähler polarization on \(L^\otimes k\) which completes the geometric quantization of the Kähler manifold \(\mathcal{A}_J\).

The prequantum line bundle can be pushed-down as follows. The symplectic action of the gauge group \(\hat{F}_C\) defined by: \(\hat{F}_C := \{ f : \Sigma \to F_C\}\) (here \(F_C\) is the complexification of the gauge group \(F\)) on \(\mathcal{A}_J\) can be lifted in such a way that it preserves the connection and the Hermitian inner product [10] or [11]. One may define the pushdown line bundle \(\hat{\mathcal{L}}^\otimes k\) by stating that its sections \(H^0_{L^2}(\mathcal{M}_J, \hat{\mathcal{L}}^\otimes k) = H^0_{L^2}(F_J^{-1}(0), \mathcal{L}^\otimes k)^{\hat{f}_C}\), constitutes a \(\hat{F}_C\)-invariant subspace of the space of sections \(\Gamma(\mathcal{A}_J, \mathcal{L})\). Here \(\mathcal{M}_J = F_J^{-1}(0)/\hat{F}_C\) is the Marsden-Weinstein quotient with \(F_J^{-1}(0) \subset \mathcal{A}_J\) is an \(\hat{F}_C\)-invariant subspace, where \(F_J : \mathcal{A}_J \to \mathfrak{f}_C\) is the moment map, with \(\mathfrak{f}_C\) being the dual of the Lie algebra \(\mathfrak{f}_C\) of \(\hat{F}_C\). Equations (8) and (9) implies that the wave function \(\psi(A)\) is a holomorphic and gauge invariant section and therefore it belongs to \(H^0_{L^2}(\mathcal{M}_J, \hat{\mathcal{L}}^\otimes k)\). The connection also can be pushed-down and it satisfies \(\hat{\nabla} \psi = \nabla_V \psi\).
Meanwhile the curvature of the connection $\tilde{\nabla}$ is $-i\tilde{\omega}$. Thus the pushed-down prequantization is given by $(\tilde{\mathcal{L}}, \tilde{\nabla}, \langle \cdot | \cdot \rangle_{\tilde{\mathcal{L}}})$, where $\langle \cdot | \cdot \rangle_{\tilde{\mathcal{L}}}$ is the $\hat{F}_C$-invariant inner product $\langle \cdot | \cdot \rangle_{\tilde{\mathcal{L}}}$.

If one varies the complex structure over the space $\mathcal{Z}$ of all conformal classes of worldsheet metrics $\rho$. Then one can construct the following vector bundle: $H^0_{L^2}(\mathcal{A}_J, \mathcal{L}^{\otimes k}) - \nabla \mapsto \mathcal{Z}$. This bundle can be also pushed-down to the Kähler quotient $\mathcal{M}_J$ such that one get:

$$H^0_{L^2}(\mathcal{M}_J, \tilde{\mathcal{L}}^{\otimes k}) - \tilde{\nabla} \mapsto \tilde{\mathcal{Z}}.$$  \hfill (12)

This bundle admits a projectively flat connection which helps to show that the geometric quantization procedure is independent on the complex structure \[8\]. A covariantly constant section of the bundle $(12)$ in an arbitrary basis $\{e_\ell\}$ where $\ell = 1, \ldots, \dim(\tilde{\mathcal{V}}')$ (with $\tilde{\mathcal{V}}' = \tilde{\mathcal{V}}'_\rho = H^0_{L^2}(\mathcal{M}_J, \tilde{\mathcal{L}}^{\otimes k})$) can be written as:

$$\psi(A; \rho) = \sum_{\ell=1}^{\dim(\tilde{\mathcal{V}}')} e_\ell(A; \rho) \cdot \tilde{T}_\ell(\rho),$$  \hfill (13)

for some expansion coefficients $\tilde{T}_\ell(\rho)$ which are standard anti-holomorphic functions on $\tilde{\mathcal{Z}}$.

The partition function of WZW models $Z_{WZW}(\Sigma; \rho)$ can be expressed in terms of these coefficients as follows \[5\]

$$Z_{WZW}(\Sigma; \rho) = \sum_{\ell=1}^{\dim(\tilde{\mathcal{V}}')} |f_\ell(\rho)|^2.$$  \hfill (14)

2. Berezin Quantization of Gauged WZW Models

In this section we describe the Berezin’s quantization of the Kähler quotient $(\mathcal{M}_J, \tilde{\omega})$ where $\mathcal{M}_J = F_J^{-1}(0)/\hat{F}_C$. That means we will find an associative and noncommutative family of algebras $(\hat{\mathcal{S}}_B, *_B)$ with $\hat{\mathcal{S}}_B \subset C^\infty(\mathcal{M}_J)$ being the space of covariant symbols (which are indexed with a real and positive parameter $\hbar$ in order to recover the classical limit $\hbar \to 0$) and $*_B$ is the Berezin star product. In order to do that we first Berezin quantize $(\mathcal{A}_J, \omega)$ and then project out all relevant quantities to be $\hat{F}_C$-invariant one finally get $(\hat{\mathcal{S}}_B, *_B)$ \[18, 20\], where $\hat{\mathcal{S}}_B \subset C^\infty(\mathcal{A}_J)\hat{F}_C \equiv C^\infty(F_J^{-1}(0)/\hat{F}_C)$.

Consider a given prequantization $(\tilde{\mathcal{L}}, \tilde{\nabla}, \langle \cdot | \cdot \rangle_{\tilde{\mathcal{L}}})$ of the Kähler manifold $\mathcal{M}_J$, which can be regarded as the pushed-down prequantization with $\tilde{\mathcal{L}} = L\hat{F}_C$ being the $\hat{F}_C$-invariant unitary line bundle over $\mathcal{A}_J$. The inner product $\langle \cdot | \cdot \rangle_{\tilde{\mathcal{L}}}$ is the $\hat{F}_C$-invariant inner product compatible
with the connection $\tilde{\nabla}$ constructed from $\langle \cdot | \cdot \rangle_\mathcal{L}$ and it is given by

$$
\langle \tilde{\chi} | \tilde{\psi} \rangle_\mathcal{L} = \int_{\mathcal{M}_J} \langle \tilde{\chi} | \tilde{\psi} \rangle \frac{\tilde{\omega}}{n!} = \langle \chi | \psi \rangle_\mathcal{L} = \langle \chi | \psi \rangle_\mathcal{L},
$$

where $\langle \chi | \psi \rangle_\mathcal{L}$ given by Eqs. (10) or (14). This inner product is defined for all $\tilde{\chi}, \tilde{\psi} \in H^0_{\tilde{L}_2}(\mathcal{M}_J, \tilde{\mathcal{L}}^{\otimes \chi}) = H^0_{\tilde{L}_2}(F^{-1}_J(0), \mathcal{L}^{\otimes \chi})_{\tilde{F}_c}$ where $\langle \tilde{\chi} | \tilde{\psi} \rangle = \langle \chi | \psi \rangle$ with $\langle \chi | \psi \rangle = \exp(-\Phi)\tilde{\nabla} \psi$ and $\Phi$ is the Kähler potential. Also $\tilde{\omega}$ is preserved by the action of $\tilde{F}_c$, i.e. $\omega$ is $\tilde{F}_c$-invariant. The norm of an element $\tilde{\psi}$ of $H^0_{\tilde{L}_2}(F^{-1}_J(0), \mathcal{L}^{\otimes \chi})_{\tilde{F}_c}$ is defined by $\langle |||\tilde{\psi}|||^2 \rangle_\mathcal{L} = \langle \tilde{\psi} | \tilde{\psi} \rangle_\mathcal{L} = |||\psi|||^2_\mathcal{L}.$

In Ref. [1] it was shown that one can identify the complex conjugate $\tilde{\psi}(A)$ of $\psi(A)$ with $\chi(A)$, if $\chi(A)$ is defined by $\chi(A) = \int \mathcal{D}h e^{-L'(h,A)}$, where $L'(h,A)$ is given by

$$
L'(h,A) = L(h) - \frac{k}{2\pi} \int_S d^2z \text{Tr} A_z \partial_h h^{-1} - \frac{k}{4\pi} \int_S d^2z \text{Tr} A_z A_z.
$$

The computation of the norm $\langle |||\tilde{\psi}|||^2 \rangle_\mathcal{L} = |||\psi|||^2_\mathcal{L}$ by integrating out with respect to $A$ (with an appropriate regularization procedure) and the use of the formula of Polyakov and Wiegman ensures the holomorphic factorization of the partition function of the WZW model, i.e. $Z_{WZW}(\Sigma) = |||\psi(A)|||^2_\mathcal{L}$.

For future reference, we proceed to give a global set up for the Berezin quantization [10, 11, 12]. We take $\tilde{\mathcal{Q}} \in \tilde{\mathcal{L}}^{\otimes \chi}, \pi[\tilde{\mathcal{Q}}] = \tilde{A} \in \mathcal{M}_J$, where $\tilde{\mathcal{L}}^{\otimes \chi}$ is a complex unitary line bundle $\tilde{\mathcal{L}}^{\otimes \chi}$ without the zero section. Now consider $\tilde{\psi}(\tilde{A}) = \tilde{\psi}[\pi(\tilde{\mathcal{Q}})] = \tilde{L}_Q[\tilde{\psi}]\tilde{\mathcal{Q}}$ with $\tilde{L}_Q[\tilde{\psi}]$ being a linear functional of $\tilde{\psi}$. The group $\tilde{F}_c$ acts on the space of sections $H^0_{\tilde{L}_2}(F^{-1}_J(0), \mathcal{L}^{\otimes \chi})_{\tilde{F}_c}$ in the equivariant form $(\tilde{\Gamma} \tilde{\psi})(\tilde{A}) \equiv \tilde{\Gamma} \tilde{\psi}(\tilde{A}^{-1} \tilde{A})$, where $\tilde{\Gamma} \in \tilde{F}_c, \tilde{A} \in \mathcal{M}_J$ and $\tilde{\psi} \in H^0_{\tilde{L}_2}(F^{-1}_J(0), \mathcal{L}^{\otimes \chi})_{\tilde{F}_c}$.

On the other hand Riesz theorem implies the existence of a section $\tilde{e}_\mathcal{Q} \in H^0_{\tilde{L}_2}(F^{-1}_J(0), \mathcal{L}^{\otimes \chi})_{\tilde{F}_c}$ such that $\tilde{L}_Q[\tilde{\psi}] = \langle \tilde{e}_\mathcal{Q} | \tilde{\psi} \rangle_{\tilde{L}_0}$. $\tilde{e}_\mathcal{Q}$ is a $\tilde{F}_c$-invariant section called the equivariant generalized coherent state.

Now let $\tilde{O}_{\tilde{F}_c} : H^0_{\tilde{L}_2}(F^{-1}_J(0), \mathcal{L}^{\otimes \chi})_{\tilde{F}_c} \to H^0_{\tilde{L}_2}(F^{-1}_J(0), \mathcal{L}^{\otimes \chi})_{\tilde{F}_c}$ be a bounded operator. The covariant symbol of this operator is defined as

$$
O_{\tilde{F}_c} A = \langle \tilde{e}_\mathcal{Q} | \tilde{O}_{\tilde{F}_c} | \tilde{e}_\mathcal{Q} \rangle_{\tilde{L}_0} = \frac{\langle e_\mathcal{Q} | \tilde{O}_{\tilde{F}_c} | e_\mathcal{Q} \rangle_{\tilde{L}_0}}{|||e_\mathcal{Q}|||^2_{\tilde{L}_0}},
$$

where $|||e_\mathcal{Q}|||^2_{\tilde{L}_0}$ is given by

$$
|||e_\mathcal{Q}|||^2_{\tilde{L}_0} = \langle e_\mathcal{Q} | e_\mathcal{Q} \rangle_{\tilde{L}_0} = \frac{1}{\text{vol}(\tilde{F}_c)} \int_{\mathcal{M}_J} \mathcal{D} \tilde{A} \tilde{L}_Q(\tilde{A}) e_\mathcal{Q}(\tilde{A}),
$$

(17)
where $e_Q \in H^0_{L^2}(F^{-1}_J(0), \mathcal{L}_0^{\otimes k})$ and $\tilde{\mathcal{O}} : H^0_{L^2}(F^{-1}_J(0), \mathcal{L}_0^{\otimes k}) \to H^0_{L^2}(F^{-1}_J(0), \mathcal{L}_0^{\otimes k})$ is a bounded operator.

The space of covariant symbols $\hat{S}_B = S_B^{\hat{F}_C}$ is defined as the pushing-down of $S_B$. Each covariant symbol can be analytically continued to the open dense subset of $\mathcal{M}_J \times \mathcal{M}_J$ in such a way $\langle e_Q|\bar{e}_Q \rangle_{\tilde{\mathcal{C}}} \neq 0$ with $\pi(\tilde{\mathcal{C}}) = \tilde{\mathcal{A}}$ (with local complex coordinates $\{\tilde{A}_z, \tilde{A}_{\bar{z}}\}$) and $\pi(\tilde{\mathcal{C}}') = \tilde{\mathcal{A}}'$ (with local complex coordinates $\{\tilde{A}'_w, \tilde{A}'_{\bar{w}}\}$), which is holomorphic in the first entry and anti-holomorphic in the second one. This analytic continuation is reflected in the covariant symbol in the form

$$\mathcal{O}^{\hat{F}_C}_B(\tilde{A}_{\bar{z}}, \tilde{A}'_w) = \frac{\langle e_Q|\bar{e}_Q \rangle^{\hat{F}_C}_{\mathcal{L}_0}}{\langle e_Q|\bar{e}_Q \rangle^{\hat{F}_C}_{\mathcal{L}_0}}.$$  

The operator $\tilde{\mathcal{O}}^{\hat{F}_C}$ can be obtained from its symbol in the form

$$\tilde{\mathcal{O}}^{\hat{F}_C}(\tilde{A}_{\bar{z}}) = \langle e_Q|\bar{e}_Q \rangle^{\hat{F}_C}_{\mathcal{L}_0}.$$  

The consideration of the completeness condition $1 = \int_{\mathcal{A}_J} |e_Q| \langle e_Q| \exp \left( - \Phi(A_z, A_{\bar{z}}) \right) \frac{a^n}{n!} (A_z, A_{\bar{z}})$ in the computation of $\langle e_Q|\tilde{\mathcal{O}}(\psi) \rangle_{\mathcal{L}_0} Q$ yields

$$\tilde{\mathcal{O}}(\psi)(A_{\bar{z}}) = \int_{\mathcal{A}_J} \mathcal{D} A' \mathcal{O}_B(\mathcal{A}_{\bar{z}}, A'_w) \mathcal{B}_Q(\mathcal{A}_{\bar{z}}, A'_w) \psi(A'_w) Q,$$

or in terms of the symplectic structure we have

$$\tilde{\mathcal{O}}(\psi)(A_{\bar{z}}) = \int_{\mathcal{A}_J} \mathcal{O}_B(\mathcal{A}_{\bar{z}}, A'_w) \mathcal{B}_Q(\mathcal{A}_{\bar{z}}, A'_w) \psi(A'_w) \exp \left( - \Phi(A'_w, A'_w) \right) \frac{\omega^n}{n!} (A'_w, A'_w) Q,$$

where $\psi(A'_w) = \langle e_Q|\psi \rangle_{\mathcal{L}_0}$ and $\mathcal{B}_Q(A_{\bar{z}}, A'_w) \equiv \langle e_Q|e_Q \rangle_{\mathcal{L}_0}$. $\mathcal{B}_Q(A_{\bar{z}}, A'_w)$ is the generalized Bergman kernel. Finally, taking the $\hat{F}_C$-invariant part of the above expression we get Eq. (20).

Similar considerations apply to other formulas. But an essential difference with respect to the quantization of $(A_J, \omega)$ is that, in the present case, the Kähler quotient is topologically nontrivial and therefore the line bundle $\tilde{\mathcal{L}}^{\otimes k}$ is also nontrivial. It is only locally trivial i.e. $\tilde{\mathcal{L}}^{\otimes k} = \mathcal{W}^{(j)} \times \mathbb{C}$ for each dense open subset $\mathcal{W}^{(j)} \subset \mathcal{M}_J$ with $j = 1, 2, \ldots, N$. Analog global formulas found on $\mathcal{L}$, can be applied only on each local trivialization of $\tilde{\mathcal{L}}^{\otimes k}$. Of course, transition functions on $\mathcal{W}^{(i)} \cap \mathcal{W}^{(j)}$ with $i \neq j$ are very important and sections and other relevant quantities like the Bergman kernel, Kähler potential, covariant symbols, etc., transform nicely under the change of the open set. Thus in a particular trivialization...
\[ \tilde{L}^{\otimes k}, \text{ the function } \mathcal{O}^{(j)}_{B(0)}(\tilde{A}_w, \tilde{A}_w) \in C^\infty(\mathcal{W}_j^{(j)}) \text{ is called the covariant symbol of the operator } \tilde{O}_{0}^{(j)} \text{ acting on } H^0_L(\mathcal{W}_j^{(j)}, \tilde{L}^{\otimes k})]. \] 

Now if \( \mathcal{O}^{(j)}_{B(0)}(\tilde{A}_w, \tilde{A}_w) \) and \( \mathcal{O}^{(j)}_{B(0)}(\tilde{A}_w, \tilde{A}_w) \) are two covariant symbols associated to \( \tilde{O}_{0}^{(j)} \) and \( \tilde{O}_{0}^{(j)} \), respectively, then the covariant symbol of \( \tilde{O}_{0}^{(j)} \tilde{O}_{0}^{(j)} \) is given by the Berezin-Wick star product \( \mathcal{O}^{(j)}_{B(0)} \ast_B \mathcal{O}^{(j)}_{B(0)} \)

\[
(O^{(j)}_{B(0)} \ast_B O^{(j)}_{B(0)})(\tilde{A}_w, \tilde{A}_w)
= \int_{\mathcal{W}_j^{(j)}} \mathcal{O}^{(j)}_{B(0)}(\tilde{A}_w, \tilde{A}_w) \mathcal{O}^{(j)}_{B(0)}(\tilde{A}_w, \tilde{A}_w) \frac{B^{(j)}(\tilde{A}_w, \tilde{A}_w) B^{(j)}(\tilde{A}_w, \tilde{A}_w)}{B^{(j)}(\tilde{A}_w, \tilde{A}_w)} \exp \left\{ -\Phi^{(j)}(\tilde{A}_w, \tilde{A}_w) \right\} \frac{\tilde{\omega}}{n!}(\tilde{A}_w, \tilde{A}_w),
\]

where \( \mathcal{K}^{(j)}(\tilde{A}_w, \tilde{A}_w, \tilde{A}_w, \tilde{A}_w) := \Phi^{(j)}(\tilde{A}_w, \tilde{A}_w) + \Phi^{(j)}(\tilde{A}_w, \tilde{A}_w) - \Phi^{(j)}(\tilde{A}_w, \tilde{A}_w) - \Phi^{(j)}(\tilde{A}_w, \tilde{A}_w) \) is called the Calabi diastatic function on \( \mathcal{W}_j^{(j)} \). This construction is valid for all local pre-quantizations: \( (\tilde{L}^{\otimes k}, \tilde{\nabla}^{(j)}, \langle \cdot | \cdot \rangle_{\tilde{L}^{(j)}}) \) with \( j = 1, \ldots, N \). Finally, this structure given by the pair \( (\tilde{S}_B, \ast_B) \) constitutes the Berezin quantization of \( (M_j, \tilde{\omega}) \) which is determined by the \( \tilde{\mathcal{F}}_\mathbf{C} \)-gauged WZW model.

3. BEREZIN QUANTIZATION OF COSET MODELS

3.1. The \( G/H \) Model

Coset models \( G/H \) are CFT’s which are equivalent to the gauged WZW models by gauging an anomaly-free subgroup \( F \) of \( G_L \times G_R \). For instance consider any subgroup \( H \subset G_{\text{adj}} \) with \( G_{\text{adj}} \) being the diagonal subgroup of \( G_L \times G_R \). In this case \( H \) is always anomaly-free.

We now consider the case of a subgroup \( F \subset G_L \times G_R \), which is not anomaly-free. In addition we take \( H \subset G_L \), such that we have \( F = G_R \times H_L \). Then an \( F \) connection consist of a pair \( (A, B) \) of two connections: one \( H \)-valued connection \( B \) and a \( G \)-valued connection \( A \). If \( G \) has a non-trivial center \( Z(G) \) (diagonally embedded in \( G_L \times G_R \)), then the symmetry group is \( G_L \times G_R / Z(G) \). Therefore the subgroup that acts faithfully is not \( F = G_R \times H_L \) but \( F' = G_R \times H_L / Z \), where \( Z = H \cap Z(G) \).

Similarly to the case of gauged WZW model we can consider the case of an ‘anomalous’ extension of the \( G/H \) model called gauged coset \( G/H \) model. In this case one can define
also a holomorphic and gauge invariant wave function

\[ \chi(A, B) = \int \mathcal{D}g e^{-L(g, A, B)}, \]

where

\[ L(g, A, B) = L(g) + \frac{k}{2\pi} \int \Sigma d^2z \text{Tr} A_z^{-1} \partial_z g - \frac{k}{2\pi} \int \Sigma d^2z \text{Tr} B_z \partial_z g \cdot g^{-1} + \frac{k}{2\pi} \int \Sigma d^2z \text{Tr} B_z g A_z g^{-1} - \frac{k}{4\pi} \int \Sigma d^2z \text{Tr}(A_z A_z + B_z B_z). \]

These wave function satisfies two copies of the system given by Eqs. (8) and (9) for both connections \( A \) and \( B \) with opposite complex structures. These equations implies the existence of a connection on a trivial holomorphic line bundle \( L \) where both connections \( A \) and \( B \) are regarded as the quotient space of the \( R \) coset models and they are finite dimensional if also a holomorphic and gauge invariant wave function

where \( z \) is the dual space to \( \tilde{\rho}(R) \) and the fact that \( \psi(A_z, B_z) \) can be regarded as a holomorphic section of this prequantum line bundle. Here \( \tilde{\mathcal{B}}_j \) is \( \mathcal{B}_j \) with the opposite complex structure. The product space \( \mathcal{C}_j = \mathcal{A}_j \times \tilde{\mathcal{B}}_j \) has the structure of a symplectic manifold with symplectic structure given by: \( \omega = k\omega_0 \), where \( \omega_0 = \frac{1}{2\pi} \int \Sigma \text{Tr}(\delta A \wedge \delta A) - \frac{1}{2\pi} \int \Sigma \text{Tr}(\delta B \wedge \delta B) \).

The corresponding prequantization over this product \( \mathcal{C}_j = \mathcal{A}_j \times \tilde{\mathcal{B}}_j \) is given by \( (\mathcal{L}^{\otimes k}, \nabla, \langle \cdot | \cdot \rangle_{\mathcal{L}}) \), with \( \mathcal{L}^{\otimes(k)} = \mathcal{L}^{\otimes(k)}(\mathcal{A}_j) \otimes \mathcal{L}^{\otimes(-k)}(\mathcal{B}_j) \), where \( \mathcal{L}^{\otimes(k)}(\mathcal{A}_j) \) is the line bundle over \( \mathcal{A}_j \) and \( \mathcal{L}^{\otimes(k)}(\mathcal{B}_j) \) is the line bundle over \( \mathcal{B}_j \). Let \( \mathcal{M}_j = \mathcal{A}_j / \tilde{G} \) and \( \mathcal{N}_j = \mathcal{B}_j / \tilde{H} \) be the moduli spaces and \( \mathcal{R}_j = \mathcal{C}_j / \tilde{\mathcal{F}}_C^1 \) be the quotient space, where \( \tilde{\mathcal{F}}_C^1 = \tilde{G}_C \times \tilde{H}_C / Z \). When the group \( Z \) is trivial, \( \mathcal{R}_j \) is the product manifold \( \mathcal{R}_j = \mathcal{M}_j \times \tilde{\mathcal{N}}_j \) which is a Kähler manifold and can be regarded as the quotient space of the \( \mathcal{A}_j \times \tilde{\mathcal{B}}_j \) by the diagonal action of the group \( \tilde{G}_C \times \tilde{H}_C \).

For the general case of non-trivial \( Z \) the pushed-down prequantization with \( \tilde{L} = \tilde{\mathcal{F}}_C^1 \), being the \( \tilde{\mathcal{F}}_C^1 \)-invariant complex unitary line bundle over \( \mathcal{R}_j \). For trivial \( Z \) we will consider the space of holomorphic sections on \( \tilde{L}_1(\mathcal{A}_j) \otimes \tilde{L}_2(\mathcal{B}_j) = \tilde{L}_1^{\otimes(k)}(\mathcal{A}_j) \otimes \tilde{L}_2^{\otimes(-k)}(\mathcal{B}_j) \), which are \( \tilde{\mathcal{F}}_C^1 \)-invariant. This is given by

\[ \tilde{W} = H^0_{L^2}(\mathcal{R}_j, \tilde{L}^{\otimes k}) = H^0_{L^2}(\mathcal{A}_j \times \tilde{\mathcal{B}}_j, \mathcal{L}^{\otimes(k)} \tilde{G}_C \times \tilde{H}_C) \]

\[ = H^0_{L^2}(\mathcal{A}_j \times \tilde{\mathcal{B}}_j, \mathcal{L}^{\otimes(k)}(\mathcal{A}_j) \otimes \mathcal{L}^{\otimes(-k)}(\mathcal{B}_j)) \tilde{G}_C \times \tilde{H}_C = \tilde{V}_G \otimes \tilde{V}_{H^*}, \]

where \( \tilde{V}_G = H^0_{L^2}(\mathcal{A}_j, \mathcal{L}^{\otimes(k)} \tilde{G}_C), \tilde{V}_H = H^0_{L^2}(\mathcal{B}_j, \mathcal{L}^{\otimes(-k)} \tilde{H}_C) \) and \( \tilde{V}_{H^*} = H^0_{L^2}(\mathcal{B}_j, \mathcal{L}^{\otimes(-k)} \tilde{H}_C) \). \( \tilde{V}_{H^*} \) is the dual space to \( \tilde{V}_H \). These spaces are precisely the spaces of conformal blocks of the coset models and they are finite dimensional if \( \mathcal{R} \) is compact. For non-trivial \( Z \) the space of
sections $\tilde{W}^{G \times H}$ should be modified to take the $Z'$-invariant part $\tilde{W}^{Z'} = (\tilde{V}_G \otimes \tilde{V}_{H^*})^{Z'}$, where $Z'$ arises in the exact sequence: $0 \to i(\hat{F}_C) \to \hat{F}_C' \to Z' \to 0$. Then $Z'$ can be defined as the quotient: $Z' = \hat{F}_C'/i(\hat{F}_C)$, where $i : \hat{F}_C \to \hat{F}_C'$ is the natural projection map \[5\].

The inner product $\langle \cdot | \cdot \rangle_{\tilde{\mathcal{L}}}$ on $\tilde{W}^{Z'}$ is the $Z'$-invariant inner product compatible with the connection $\tilde{\nabla}$ constructed from $\langle \cdot | \cdot \rangle_{\mathcal{L}}$ and it is given by

\[
\langle \tilde{\chi} | \tilde{\psi} \rangle_{\tilde{\mathcal{L}}(1) \otimes \tilde{\mathcal{L}}(2)} = \langle \chi | \psi \rangle_{\mathcal{L}(1) \otimes \mathcal{L}(2)} = \langle \chi | \psi \rangle_{\mathcal{L}(1) \otimes \mathcal{L}(2)}
\]

\[
= \frac{1}{\text{vol}(\hat{G}_C) \times \text{vol}(\hat{H}_C)} \int_{A_j \times B_j} \mathcal{D}A \mathcal{D}B \chi^A(A, B) \psi^B(A, B), \tag{27}
\]

where $\mathcal{D}A$ and $\mathcal{D}B$ can be written in terms of the symplectic form $\omega$ it yields:

\[
\langle \chi | \psi \rangle_{\mathcal{L}(1) \otimes \mathcal{L}(2)} = \frac{1}{\text{vol}(\hat{G}_C) \times \text{vol}(\hat{H}_C)} \int_{A_j \times B_j} \langle \tilde{\chi} | \tilde{\psi} \rangle \tilde{\omega}^n / n!.
\]  

By using the previous definitions, the inner product of the tensor product space can be carried over to the form

\[
\langle \tilde{\chi}(A) \otimes \tilde{\chi}(B) | \tilde{\psi}(A) \otimes \tilde{\psi}(B) \rangle_{\tilde{\mathcal{L}}(1) \otimes \tilde{\mathcal{L}}(2)} = \langle \chi(A) | \chi(B) \rangle_{\mathcal{L}(1) \otimes \mathcal{L}(2)} \cdot \langle \tilde{\chi}(A) | \tilde{\psi}(A) \rangle_{\tilde{\mathcal{L}}(1)} \cdot \langle \tilde{\chi}(B) | \tilde{\psi}(B) \rangle_{\tilde{\mathcal{L}}(2)}
\]

\[
= \langle \tilde{\chi}(A) | \tilde{\psi}(A) \rangle_{\tilde{\mathcal{L}}(1)} \cdot \langle \tilde{\chi}(B) | \tilde{\psi}(B) \rangle_{\tilde{\mathcal{L}}(2)}, \tag{29}
\]

for all $\tilde{\chi}(A, B), \tilde{\psi}(A, B) \in (\tilde{V}_G \otimes \tilde{V}_{H^*})^{Z'}$. Then the inner product factorizes into two pieces corresponding to the group factors $\hat{G}_C$ and $\hat{H}_C$. In fact, this is the unique $Z'$-invariant inner product in $\tilde{W}^{Z'}$ \[21\].

The norm of an element $\tilde{\psi}(A, B)$ of $(\tilde{V}_G \otimes \tilde{V}_{H^*})^{Z'}$ has shown to factorize holomorphically being equal to the partition function of the $G/H$ model \[3\]

\[
Z_{G/H}(\Sigma) = \langle \tilde{\psi} | \tilde{\psi} \rangle_{\tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}}} \equiv \| \tilde{\psi}(A) \otimes \tilde{\psi}(B) \|^2_{\tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}}}.
\]

\[
= \frac{1}{\text{vol}(\hat{G}_C) \times \text{vol}(\hat{H}_C)} \int \mathcal{D}A \mathcal{D}B \mathcal{D}g \mathcal{D}h e^{-L(g, A, B) - L'(h, A, B)} \tag{30}
\]

where

\[
L'(h, A, B) = L(h) + \frac{k}{2\pi} \int_\Sigma d^2z \text{Tr} B_z h^{-1} \partial_z h - \frac{k}{2\pi} \int_\Sigma d^2z \text{Tr} A_z \partial_z h \cdot h^{-1}
\]
\[ + \frac{k}{2\pi} \int d^2z \text{Tr} A_z h \bar{B}_z h^{-1} - \frac{k}{4\pi} \int d^2z \text{Tr} (A_z A_z + B_z B_z). \]  

(31)

From this it is easy to show that the partition function of the \( G/H \) model reduces to the product of factors

\[ Z_{G/H}(\Sigma) = [||\tilde{\psi}(\tilde{A})||^2]_{\mathcal{E}'} \cdot [||\tilde{\psi}(\tilde{B})||^2]_{\mathcal{E}''}, \]

(32)
corresponding to the group factors of \( \tilde{G}_\Sigma \times \tilde{H}_\Sigma \). Finally one has to take the \( Z' \)-invariant part.

Removing the zero section we take \( \tilde{Q} \in \tilde{L}^{\otimes (k)}_{0(1)} \otimes \tilde{L}^{\otimes (-k)}_{0(2)}, \pi[\tilde{Q}] = (\tilde{A}, \tilde{B}) \in \mathcal{M}_J \times \mathcal{N}_J \)

with local complex coordinates \( \{\tilde{A}_z, \tilde{A}_{\bar{z}}; \tilde{B}_z, \tilde{B}_{\bar{z}}\} \) and \( \pi[\tilde{Q}'] = (\tilde{A}', \tilde{B}') \in \mathcal{M}_J \times \mathcal{N}_J \) with local complex coordinates \( \{\tilde{A}'_w, \tilde{A}'_{\bar{w}}; \tilde{B}'_w, \tilde{B}'_{\bar{w}}\} \). Now consider the holomorphic section \( \tilde{\psi}(\tilde{A}_z, \tilde{B}_z) = \tilde{\psi}[\pi(\tilde{Q})] = \tilde{L}_Q[\tilde{\psi}] \tilde{Q} \) with \( \tilde{L}_Q[\tilde{\psi}] \) being a linear functional of \( \tilde{\psi} \).

In the present situation there exists also a section \( \tilde{c}_Q = \tilde{c}^{(1)}_Q \otimes \tilde{c}^{(2)}_Q \in (\tilde{V}_G \otimes \tilde{V}_{H^*})^{Z'} \), According to Eq. (32) it has a norm

\[ [||\tilde{c}_Q(A) \otimes \tilde{c}_Q(B)||^2]_{\mathcal{L} \otimes \mathcal{L}'} = [||\tilde{c}_Q(A)||^2]_{\mathcal{L}} \cdot [||\tilde{c}_Q(B)||^2]_{\mathcal{L}'} . \]

(33)

The bounded operator \( \tilde{\mathcal{O}}^{Z'} = \tilde{\mathcal{O}}^{G_c} \otimes \tilde{\mathcal{O}}^{H_c} : (\tilde{V}_G \otimes \tilde{V}_{H^*})^{Z'} \rightarrow (\tilde{V}_G \otimes \tilde{V}_{H^*})^{Z'} \) can be recovered from its symbol in the form \( \tilde{\mathcal{O}}^{Z'} \tilde{\psi}(\tilde{A}_z, \tilde{B}_z) = \langle c_Q | \tilde{\mathcal{O}}(\tilde{\psi}) \rangle_{\mathcal{L} \otimes \mathcal{L}'} p \) this yields

\[ \tilde{\mathcal{O}}\psi(A_z, B_z) = \tilde{\mathcal{O}}^{G_c} \otimes \tilde{\mathcal{O}}^{H_c} \left( \psi(A_z) \otimes \psi(B_z) \right) = \tilde{\mathcal{O}}^{G_c} \psi(A_z) \otimes \tilde{\mathcal{O}}^{H_c} \psi(B_z). \]

(34)

The covariant symbol is defined as

\[ \tilde{\mathcal{O}}^{Z'}_{\mathcal{H}}(\tilde{A}_z, \tilde{B}_z) = \frac{\langle c_Q | \tilde{\mathcal{O}}^{Z'} | c_Q \rangle_{\mathcal{L}(1) \otimes \mathcal{L}(2)}}{[||c_Q||^2]_{\mathcal{L}(1) \otimes \mathcal{L}(2)}}. \]

(35)

Using the properties (29) and (33) one can show that

\[ \tilde{\mathcal{O}}^{Z'}_{\mathcal{H}}(\tilde{A}_z, \tilde{B}_z) = \frac{\langle c_Q | \tilde{\mathcal{O}}(c_Q) \rangle_{\mathcal{L}(1) \otimes \mathcal{L}(2)}}{[||c_Q||^2]_{\mathcal{L}(1) \otimes \mathcal{L}(2)}} = \frac{\langle c_Q | \tilde{\mathcal{O}}(c_Q) \rangle_{\mathcal{L}(1)}}{[||c_Q||^2]_{\mathcal{L}(1)}} \cdot \frac{\langle c_Q | \tilde{\mathcal{O}}(c_Q) \rangle_{\mathcal{L}(2)}}{[||c_Q||^2]_{\mathcal{L}(2)}}. \]

(36)

In other worlds it factorizes holomorphically

\[ \tilde{\mathcal{O}}^{Z'}_{\mathcal{H}}(\tilde{A}_z, \tilde{B}_z) = \tilde{\mathcal{O}}^{G_c}_{\mathcal{H}}(\tilde{A}_z) \cdot \tilde{\mathcal{O}}^{H_c}_{\mathcal{H}}(\tilde{B}_z). \]

(37)

into the product of the two holomorphic symbols \( \tilde{\mathcal{O}}^{G_c}_{\mathcal{H}}(\tilde{A}_z) \) and \( \tilde{\mathcal{O}}^{H_c}_{\mathcal{H}}(\tilde{B}_z) \). These symbols correspond to the linear operators: \( \tilde{\mathcal{O}}^{G_c} : \tilde{V}_G \rightarrow \tilde{V}_G \) and \( \tilde{\mathcal{O}}^{H_c} : \tilde{V}_{H^*} \rightarrow \tilde{V}_{H^*} \).
This implies that the space of covariant symbols $\tilde{S}_B$ is actually the tensor product $(\tilde{S}_B^{Gc} \otimes \tilde{S}_B^{Hc})^{Z'}$. This tensor product can be analytically continued to the open dense subset of $\mathcal{M}_j \times \mathcal{M}_j \times \mathcal{N}_j \times \mathcal{N}_j$ in such a way that it can be written

$$O^{Z'}_B(\tilde{A}_z, \tilde{B}_z; \tilde{A}'_w, \tilde{B}'_w) = \frac{\langle e_Q|\tilde{O}|e_{Q'}\rangle^{Z'}_{L(1) \otimes L(2)}}{\langle e_Q|e_{Q'}\rangle^{L(1)}_{L(1) \otimes L(2)}}. \tag{38}$$

Following a similar procedure as in getting (36), we obtain that the extended symbol also factorizes holomorphically

$$O^{Z'}_B(\tilde{A}_z, \tilde{B}_z; \tilde{A}'_w, \tilde{B}'_w) = \frac{\langle e_Q|\tilde{O}|e_{Q'}\rangle^{Gc}_{L(1)}}{\langle e_Q|e_{Q'}\rangle^{Gc}_{L(1)}} \cdot \frac{\langle e_Q|\tilde{O}|e_{Q'}\rangle^{Hc}_{L(2)}}{\langle e_Q|e_{Q'}\rangle^{Hc}_{L(2)}}, \tag{39}$$

where $O^{Gc}_B(\tilde{A}_z, \tilde{A}'_w)$ and $O^{Hc}_B(\tilde{B}_z, \tilde{B}'_w)$ are the corresponding extended symbols. Here we have used the fact that the Bergman kernel $B_Q(\tilde{A}_z, \tilde{B}_z; \tilde{A}'_w, \tilde{B}'_w) = \langle e_{Q'}|e_{Q'}\rangle_{L(1) \otimes L(2)}$ can be also factorized as $\langle e_{Q'}^1|e_{Q'}^1\rangle_{L(1)} \cdot \langle e_{Q'}^2|e_{Q'}^2\rangle_{L(2)}$ and consequently

$$B_Q(\tilde{A}_z, \tilde{B}_z; \tilde{A}'_w, \tilde{B}'_w) = B_Q(\tilde{A}_z, \tilde{A}'_w) \cdot B_Q(\tilde{B}_z, \tilde{B}'_w). \tag{40}$$

Now if $O_{B(0)}(\tilde{A}_z, \tilde{B}_z; \tilde{A}'_w, \tilde{B}'_w)$ and $O'_{B(0)}(\tilde{A}'_w, \tilde{B}'_w; \tilde{A}_z, \tilde{B}_z)$ are two covariant symbols of $\tilde{O}_0$ and $\tilde{O}'_0$, respectively, then the covariant symbol of $\tilde{O}_0 \tilde{O}'_0$ is given by the Berezin-Wick star product

$$(O_{B(0)}^{(j)} \tilde{O}'_{B(0)}^{(j)})(\tilde{A}_z, \tilde{B}_z; \tilde{A}'_w, \tilde{B}'_w)$$

$$= \int_{W^{(j)}_j \times W^{(j)}_j} O_{B(0)}^{(j)}(\tilde{A}_z, \tilde{B}_z; \tilde{A}'_w, \tilde{B}'_w) O_{B(0)}^{(j)}(\tilde{A}'_w, \tilde{B}'_w; \tilde{A}_z, \tilde{B}_z)$$

$$\times \frac{B^{(j)}(\tilde{A}_z, \tilde{B}_z; \tilde{A}_w, \tilde{B}'_w) B^{(j)}(\tilde{A}_w, \tilde{B}'_w; \tilde{A}_z, \tilde{B}_z)}{B^{(j)}(\tilde{A}_z, \tilde{B}_z; \tilde{A}_w, \tilde{B}_z)} \exp \left\{ -\Phi^{(j)}(\tilde{A}_w, \tilde{B}_w; \tilde{A}'_w, \tilde{B}'_w) \right\} \frac{\tilde{w}}{n!} (\tilde{A}_w, \tilde{B}_w; \tilde{A}'_w, \tilde{B}'_w)$$

$$= \int_{W^{(j)}_j \times W^{(j)}_j} O_{B(0)}^{(j)}(\tilde{A}_z, \tilde{B}_z; \tilde{A}_w, \tilde{B}'_w) O_{B(0)}^{(j)}(\tilde{A}_w, \tilde{B}'_w; \tilde{A}_z, \tilde{B}_z)$$

$$\times \exp \left\{ \tilde{K}^{(j)}(\tilde{A}_z, \tilde{B}_z; \tilde{A}_w, \tilde{B}_w; \tilde{A}'_w, \tilde{B}'_w) \right\} \frac{\tilde{w}}{n!} (\tilde{A}_w, \tilde{B}_w; \tilde{A}'_w, \tilde{B}'_w) \tag{41}$$

where $\tilde{K}^{(j)}(\tilde{A}_z, \tilde{B}_z; \tilde{A}_w, \tilde{B}_w; \tilde{A}'_w, \tilde{B}'_w) := \Phi^{(j)}(\tilde{A}_z, \tilde{B}_z; \tilde{A}_w, \tilde{B}_w) + \Phi^{(j)}(\tilde{A}_w, \tilde{B}_w; \tilde{A}_z, \tilde{B}_z) - \Phi^{(j)}(\tilde{A}_z, \tilde{B}_z; \tilde{A}_w, \tilde{B}_w) - \Phi^{(j)}(\tilde{A}_w, \tilde{B}_w; \tilde{A}_z, \tilde{B}_w)$ is called the Calabi diastatic function on $W^{(j)}_j \times W^{(j)}_j$. This construction is valid for all local prequantization $(\tilde{L}^{(j)}, \tilde{\nabla}^{(j)}, \langle \cdot , \cdot \rangle^{(j)})$. Finally, this
structure leads to the pair \((\tilde{S}_B, \tilde{∗}_B)\) which constitutes the Berezin quantization of \((\mathcal{R}_J, \tilde{ω})\).

It an easy matter to see, using all previous results, that Berezin product also factorizes holomorphically

\[
(\mathcal{O}_{B(0)}^{(j)} \mathcal{O}_{B(0)}^{(j)})(\tilde{A}_z, \tilde{B}_z; \tilde{A}_\bar{z}, \tilde{B}_\bar{z}) = \left( \mathcal{O}_{B(0)}^{(j)} *_B \mathcal{O}_{B(0)}^{(j)} \right)(\tilde{A}_z, \tilde{A}_\bar{z}) \cdot \left( \mathcal{O}_{B(0)}^{(j)} *_B \mathcal{O}_{B(0)}^{(j)} \right)(\tilde{B}_z, \tilde{B}_\bar{z}).
\]

(42)

3.2. The \(G/G\) Model

In the present subsection we specialize the discussion in the previous subsection to the case \(\hat{H}_C = \hat{G}_C\). In this case very interesting features about the gauged coset model \(G/G\) arises. First of all this case corresponds to a topological quantum field theory discovered by Witten in [5] and applied in [6].

The ‘anomalous’ Lagrangian is basically given by Eq. (25) with the addition that now \(B\) is like \(A\) a \(G\)-valued connection. In this case Lagrangian (25) has the additional symmetry:

\[
z \leftrightarrow \bar{z}, \quad A \leftrightarrow B, \quad g \leftrightarrow g^{-1}.
\]

(43)

Thus, complex conjugate \(\overline{\chi}(A, B)\) of \(\chi(A, B)\) can be computed leaving the complex structure fixed but changing \(A \leftrightarrow B\) and \(g \leftrightarrow g^{-1}\). Remember that, \(\overline{\chi}(\tilde{A}, \tilde{B}) = \tilde{\chi}(\tilde{A}) \otimes \tilde{\chi}(\tilde{B}) \in (\tilde{V}_G \otimes \tilde{V}_{G^*})^Z\). In our present case \(\hat{H}_C = \hat{G}_C\), we have \(\overline{\chi}(A, B) = \overline{\chi}(A) \otimes \overline{\chi}(B) \in (\tilde{V}_G \otimes \tilde{V}_{G^*})^Z = \left( \text{Hom}(\tilde{V}_G, \tilde{V}_G) \right)^Z\). This fact and the symmetry \(A \leftrightarrow B\) implies that \(\overline{\chi}(A, B)\) is Hermitian, i.e. \(\overline{\chi}(A, B) = \overline{\chi}(B, A)\). Also it satisfies the property: \(\overline{\chi}^2 = \chi\), which corresponds to an orthogonal projector of the bundle \(\mathcal{V}\) onto a finite rank sub-bundle \(\mathcal{V}'\). This implies that for any finite, holomorphic and orthonormal basis \(\{e_i(A; \rho)\}\) of \(\mathcal{V}'\), any section \(\chi\) can be written diagonally as

\[
\chi(A, B; \rho) = \sum_{i,j=1}^{\dim \mathcal{V}'} \delta_{ij} e_i(A; \rho) \otimes \overline{e}_j(B; \rho).
\]

(44)

By the usual rules of tensor products of Hilbert spaces [21] we have the topological invariant

\[
Z_{G/G}(\Sigma) = |\chi|^2 = \dim(\mathcal{V}'),
\]

(45)

where \(\mathcal{V}'\) is the space of conformal blocks of the WZW model.
The generalized coherent states can be also expressed in this basis depending on the complex structure

$$\tilde{e}_Q(\tilde{A}_z, \tilde{B}_z; \rho) = \sum_{i=1}^{\dim \mathcal{V}'} \mathbf{e}_i(\tilde{A}_z; \rho) \otimes \overline{\mathbf{e}}_i(\tilde{B}_z; \rho).$$

(46)

It is easy to see that the norm of $e_Q(A, B; \rho)$ is given by

$$||\tilde{e}_Q(A; \rho) \otimes \tilde{e}_Q(B; \rho)||^2_{\text{Hom}(\tilde{\mathcal{L}}, \tilde{\mathcal{L}})} = ||\tilde{e}_Q(A; \rho)||^2_{\tilde{\mathcal{L}}^*} \cdot ||\tilde{e}_Q(B; \rho)||^2_{\tilde{\mathcal{L}}} = \dim \mathcal{V}'.$$  

(47)

The covariant symbol is defined as before but taking $\hat{H}_C = \hat{G}_C$ in Eq. (36). After some computations we finally get

$$\mathcal{O}^{(i)}_{B}(\tilde{A}_z, \tilde{B}_z; \rho) = \frac{1}{(\dim \mathcal{V}')^2} \sum_{k, \ell=1}^{\dim \mathcal{V}'} \mathcal{O}^{G_c}_{B, kl}(\tilde{A}_z, \rho) \overline{\mathcal{O}}^{G_c}_{B, kl}(\tilde{B}_z, \rho),$$

(48)

which coincides with the holomorphic factorization (47). The space of covariant symbols $\tilde{S}_B$ is actually the tensor product $\left(\tilde{S}^{G_c}_B \otimes (\tilde{S}^{G_c}_B)^*\right)^{Z'}$. This is isomorphic to the space $\left(\text{Hom}(\tilde{S}^{G_c}_B, \tilde{S}^{G_c}_B)\right)^{Z'}$, the space of linear matrices $\dim \mathcal{V} \times \dim \mathcal{V}'$. This space can be analytically continued in order to define the extended symbols. They also satisfy the holomorphic factorization condition

$$\mathcal{O}^{(i)}_{B}(\tilde{A}_z, \tilde{B}_z; \rho) = \mathcal{O}^{G_c}_{B}(\tilde{A}_z, \tilde{B}_z) \cdot \overline{\mathcal{O}}^{G_c}_{B}(\tilde{B}_z, \tilde{A}_z),$$

(49)

where $\mathcal{O}^{G_c}_{B}(\tilde{A}_z, \tilde{B}_z)$ and $\overline{\mathcal{O}}^{G_c}_{B}(\tilde{B}_z, \tilde{A}_z)$ are the corresponding extended symbols.

The Bergmann kernel for the $G/G$ model can be written in the basis (46) as follows

$$B_Q(\tilde{A}_z, \tilde{B}_z; \tilde{A}_w, \tilde{B}_w) = \left(\sum_{k=1}^{\dim \mathcal{V}'} \mathbf{e}_k(\tilde{A}_z; \rho) \mathbf{e}_k(\tilde{A}_w; \rho)\right) \cdot \left(\sum_{\ell=1}^{\dim \mathcal{V}'} \overline{\mathbf{e}}_\ell(\tilde{B}_z; \rho) \overline{\mathbf{e}}_\ell(\tilde{B}_w; \rho)\right).$$

(50)

According to this result and the factorization of the extended symbols (49) we finally get the holomorphic factorization of the Berezin product

$$(\mathcal{O}^{(i)}_{B(0)} \ast_B \mathcal{O}^{(j)}_{B(0)})(\tilde{A}_z, \tilde{B}_z; \tilde{A}_z, \tilde{B}_z) = \left(\mathcal{O}^{(i)}_{B(0)} \ast_B \mathcal{O}^{(j)}_{B(0)}\right)(\tilde{A}_z, \tilde{A}_z) \cdot \left(\overline{\mathcal{O}}^{(j)}_{B(0)} \ast_B \overline{\mathcal{O}}^{(i)}_{B(0)}\right)(\tilde{B}_z, \tilde{B}_z).$$

(51)

4. FINAL REMARKS

In this paper we have applied the Berezin quantization global procedure to the gauged WZW and coset models. Our description has been formal. For the gauged $G/H$ model
we have found that holomorphic factorization of the partition function of the corresponding model can be carried over to the Berezin quantization procedure. Covariant symbols, extended covariant symbols and the Berezin-Wick star product factorizes into two pieces corresponding to the group factors $\hat{G}_C \times \hat{H}_C$. For the topological $G/G$ model, its corresponding Berezin quantization leads to the quantization of the space of linear matrices $\dim \mathcal{V}' \times \dim \mathcal{V}'$ on the space of conformal blocks of the associated CFT of the $G/G$ model.

It would be interesting to extend the cases considered in this paper to $\mathcal{N} = 1$ and $\mathcal{N} = 1$ supersymmetric cosets models and to topological Kazama-Suzuki models and its coupling to topological gravity \cite{2, 22}. It would be equally interesting to see if the Berezin quantization procedure is also applicable to string theory in the version of \cite{23}. Finally, the application of some recents results \cite{24} to the quantization of gauged WZW and coset models deserves further study.


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