A NOTE ON THE CONE OF MOBILE CURVES

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Abstract. S. Boucksom, J.-P. Demailly, M. Păun and Th. Peternell proved that the cone of mobile curves $\overline{ME}(X)$ of a projective complex manifold $X$ is dual to the cone generated by classes of effective divisors and conjectured an extension of this duality in the Kähler set-up. We show that their conjecture implies that $\overline{ME}(X)$ coincides with the cone of integer classes represented by closed positive smooth $(n-1, n-1)$-forms. Without assuming the validity of the conjecture we prove that this equality of cones still holds at the level of degree functions.

Let $X$ be a smooth complex projective variety of dimension $n$. A curve $C$ on $X$ is called mobile if it is the member of an algebraic family of (generically) irreducible curves covering $X$. Let $\overline{ME}(X)$ denote the closed convex cone generated by classes of mobile curves inside $N_1(X) := (H^{n-1,n-1}_R(X) \cap H^{2n-2}(X,\mathbb{Z})/\text{Tors}) \otimes \mathbb{Z} \mathbb{R}$. We shall call the elements of $\overline{ME}(X)$ mobile classes.

In [BDPP] it is shown that the following cones in $N_1(X)$ coincide:

1. the cone $\overline{ME}(X)$ of mobile curves,
2. the cone $\mathcal{M}_{NS} := \mathcal{M} \cap N_1$, where $\mathcal{M} \subset H^{n-1,n-1}_R(X)$ is the closure of the convex cone generated by cohomology classes of currents of the type $\nu_* (\tilde{\omega}_1 \wedge \ldots \wedge \tilde{\omega}_{n-1})$ for Kähler forms $\tilde{\omega}_1, \ldots, \tilde{\omega}_{n-1}$ on a modification $\nu : \tilde{X} \to X$ de $X$,
3. the dual cone $(\mathcal{E}_{NS})^\vee$ of the cone $\mathcal{E}_{NS}$ of pseudo-effective divisors on $X$.

It was known from [Dem92] that $\mathcal{E}_{NS} = \mathcal{E} \cap NS_R$, where $\mathcal{E}$ is the cone of classes of positive closed currents of type $(1, 1)$ and $NS_R(X) := (H^{1,1}_R(X) \cap H^2(X,\mathbb{Z})/\text{Tors}) \otimes \mathbb{Z} \mathbb{R}$. In [BDPP] Conj. 2.3 it is further conjectured that the cones $\mathcal{M}$ and $\mathcal{E}$ are dual.

In this note we compare $\overline{ME}(X)$ to the closed convex cone $P^{n-1,n-1}$ in $H^{n-1,n-1}_R(X)$ generated by closed positive smooth $(n-1, n-1)$-forms. From the above statements it is clear that $\overline{P^{n-1,n-1}} \cap N_1(X) \subset \overline{ME}(X)$. The converse inclusion will follow from our arguments if we admit the conjecture

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of Boucksom, Demailly, Păun, Peternell. If not we still get an equality at the level of degree functions as follows.

Any mobile class $\alpha$ gives rise to a degree function

$$\deg_\alpha : \text{Pic}(X) \to \mathbb{R}, \quad \deg_\alpha L := c_1(L)\alpha$$

and further to a notion of stability of torsion-free sheaves on $X$ which generalizes the classical case $\alpha = H^{n-1}$ for a class $H$ of an ample divisor, cf. [CaPe].

On the other side we consider semi-Kähler metrics on $X$, i.e. such that their associated Kähler forms $\omega$ satisfy $d\omega^{n-1} = 0$. Such a metric gives likewise a degree function:

$$\deg_\omega : \text{Pic}(X) \to \mathbb{R}, \quad \deg_\omega L := c_1(L)[\omega^{n-1}].$$

Then we can state:

Theorem. For any mobile class $\alpha$ in the interior of the mobile cone $\overline{ME(X)}$ there exists a semi-Kähler metric on $X$ with associated Kähler form $\omega$ such that $\deg_\alpha = \deg_\omega$ on $\text{Pic}(X)$.

By [LT] Cor. 5.3.9 this has the following consequence on the moduli space of stable vector bundles:

Corollary. If $\alpha$ is a class in the interior of $\overline{ME(X)}$, then the smooth part of the moduli space of stable vector bundles with respect to $\alpha$ admits a natural Kähler structure.

The following notation will be used:

We denote by $\mathcal{D}^{p,q}$ the space of currents of bidegree $(p, q)$ (and bidimension $(n-p, n-q)$) on $X$. Let

$$V_d = V_d(X) := \{ T \in \mathcal{D}^{1,1}_\mathbb{R} |dT = 0\}/\text{dd}^c\mathcal{D}^{0,0}_\mathbb{R},$$

$$V_{dd^c} = V_{dd^c}(X) := \{ T \in \mathcal{D}^{1,1}_\mathbb{R} | \text{dd}^cT = 0\}/\{ \partial S + \partial \bar{S} | S \in \mathcal{D}^{1,0}_\mathbb{R}\},$$

$V_d^+, V_{dd^c}^+$ the cones generated by positive currents in $V_d$ and $V_{dd^c}$ respectively and

$V_{d,NS}^+, V_{dd^c,NS}^+$ their intersections with $N_{S\mathbb{R}}(X) := (H^{1,1}_{\mathbb{R}}(X) \cap H^2(X, \mathbb{Z})/\text{Tors}) \otimes \mathbb{Z}\mathbb{R}.$

We first prove a

Lemma. The natural map $j : V_d \to V_{dd^c}$ induces a bijection between positive cones

$$V_{d,NS}^+ \to V_{dd^c,NS}^+.$$. 
Proof. It is known that in the case of compact Kähler manifolds the map $j$, which associates to a class of a bidegree $(1,1)$ closed current $\{T\}_d \in V_d$ its class $\{T\}_{dd^c} \in V_{dd^c}$, is well defined and bijective.

The inclusion $j(V^+_d) \subset V^+_{dd^c}$ being obvious, we consider a $dd^c$-closed positive current $T$ with $\{T\}_{dd^c} \in V_{dd^c,NS}$. Let $\eta := j^{-1}(\{T\}_{dd^c}) \in V_d$. We shall show that $\eta \in V^+_d$.

By the cited result of [BDPP] it suffices to check that for any modification $\nu : \tilde{X} \to X$ and Kähler forms $\tilde{\omega}_1, ..., \tilde{\omega}_{n-1}$ on $\tilde{X}$ one has

$$\eta \nu_*(\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1}) \geq 0.$$  

But

$$\eta \nu_*(\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1}) = \nu^*(\eta)[\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1}] = \nu^*(\{T\}_{dd^c})[\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{n-1}]$$

and Theorem 3 of [AlBa95] asserts that $\nu^*(\{T\}_{dd^c}) \in V^+_{dd^c}(\tilde{X})$, whence the desired inequality. \qed

We can now give the proof of the Theorem.

Proof. Let $\alpha$ be an element in the interior of the cone of mobile curves. We denote by $D^{1,1}_+ \subset D^{1,1}_+$ the cone of positive currents inside the space $D^{1,1}_+$ of bidegree $(1,1)$ currents on $X$. We fix a Kähler form $\sigma$ on $X$ and set $D^{1,1}_{+,\sigma} := \{T \in D^{1,1}_+ \mid \int_X T \wedge \sigma^{n-1} = 1\}$. This is a compact set for the weak topology on $D^{1,1}_+$, [D] III.1.23. Let $\beta_1, ..., \beta_k \in H^{n-1,n-1}_{\mathbb{R}}(X)$ be such that $V_{dd^c,NS} = \{t \in V_{dd^c} \mid t\beta_1 = 0, ..., t\beta_k = 0\}$ and set $W := \{T \in D^{1,1}_+ \mid dd^cT = 0, \{T\}_{dd^c}\alpha = 0, \{T\}_{dd^c}\beta_1 = 0, ..., \{T\}_{dd^c}\beta_k = 0\}$. Remark that $W$ and $D^{1,1}_{+,\sigma}$ are disjoint. Indeed, if $T \in D^{1,1}_{+,\sigma}$ were $dd^c$-closed and $\{T\} \in V^+_{dd^c,NS}$, then by the Lemma there would exist a $d$-closed positive current $S \in D^{1,1}_+$ such that $\{T\}_{dd^c} = j(\{S\}_d)$ and in particular

$$\{T\}_{dd^c}\alpha = \{S\}_d\alpha > 0.$$  

The Hahn-Banach theorem then implies the existence of a functional on $D^{1,1}_+$, which vanishes on $W$ and is positive on $D^{1,1}_{+,\sigma}$. This functional is thus given by a real $(n-1,n-1)$-form $u$ on $X$. The form $u$ is strictly positive on $X$ since the functional is positive on $D^{1,1}_{+,\sigma}$. The vanishing on $W$ implies that $u$ is also $d$-closed. As functionals on $NS_{\mathbb{R}}(X)$, $[u]$ and $\alpha$ have the same kernel, hence they coincide up to some multiplicative constant.
It is enough now to take a positive \((n-1)\)-st root \(\omega\) of \(u\). For the convenience of the reader we show how this is done. Remark first that
\[
(1), \quad (i \sum_{1 \leq i,j \leq n} a_{ij} dz_i \wedge dz_j)^{n-1} = (n-1)! ^{(n-1)^2} \sum_{1 \leq i,j \leq n} (-1)^{i+j} c_{ij} dz_i \wedge dz_j,
\]
where we denoted by \(c_{ij}\) the cofactor of \(a_{ij}\) in the matrix \(A = (a_{ij})_{1 \leq i,j \leq n}\) and
\[
dz_i := dz_1 \wedge ... \wedge dz_{i-1} \wedge dz_{i+1} \wedge ... \wedge dz_n, \quad \hat{dz}_j := dz_1 \wedge ... \wedge dz_{j-1} \wedge dz_{j+1} \wedge ... \wedge dz_n.
\]
The relation \(t^CA = \det(A)I_n\) for the matrix of cofactors \(C = (c_{ij})_{1 \leq i,j \leq n}\) implies
\[
A = \frac{1}{n-1} \sqrt{\det(C)} C^{-1}
\]
when \(A\) is positive definite. When the matrix \(C\) is positive definite, one obtains an unique positive definite solution \(A\) of equation (1).

\[\square\]

References

[AlBa95] Alessandrini, L., Bassanelli, G.: Modifications of compact balanced manifolds, C. R. Acad. Sci. Paris Sr. I Math. 320(1995), 1517–1522

[BDPP] Boucksom S., Demailly J.-P., Păun M., Peternell Th.: The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, arXiv:math/0405285

[CaPe] Campana F., Peternell Th. (with an appendix by Matei Toma): Geometric stability of the cotangent bundle and the universal cover of a projective manifold, arXiv:math/0405093

[Dem92] Demailly J.-P: Regularization of closed positive currents and intersection theory. J. Algebraic Geom. 1 (1992), 361–409

[D] Demailly J.P.: Complex analytic and algebraic geometry, http://www-fourier.ujf-grenoble.fr/~demailly/books.html

[LT] Lübke, M., Teleman, A. The Kobayashi-Hitchin correspondence, World Scientific Publishing Co., Inc., River Edge, NJ, 1995

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