Tropical Krichever construction for the non-periodic box and ball system

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Abstract
A solution for an initial value problem of the box and ball system is constructed from a solution of the periodic box and ball system. The construction is done through a specific limiting process based on the theory of tropical geometry. This method gives a tropical analogue of the Krichever construction, which is an algebro-geometric method to construct exact solutions to integrable systems, for the non-periodic system.

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1. Introduction

The box and ball system (BBS) is a cellular automaton expressed by an infinite array of ‘.’ (empty box with capacity 1) and ‘1’ (ball) \cite{21, 22}. If we put \(U^t_n := \{ \text{the number of ball in the } n \text{th box at time } t \} \in \{0, 1\}\), the evolution equation of the BBS is given by

\[
U^{t+1}_n = \min \left[ 1 - U^t_n \sum_{k=-\infty}^{n-1} U^t_k - \sum_{k=-\infty}^{n+1} U^{t+1}_k \right],
\]

where we suppose that the number of ‘1’ is finite. It is proved that any state of the BBS consists of only solitons. The BBS provides a simple illustration for the soliton interaction:

\[
\begin{align*}
\text{t=0} & : \ldots 111 \ldots 11 \ldots 1 \ldots \\
\text{t=1} & : \ldots \ldots \ldots \ldots 11 \ldots 11 \ldots 1 \ldots \\
\text{t=2} & : \ldots \ldots \ldots \ldots 11 \ldots 11 \ldots 11 \ldots \\
\text{t=3} & : \ldots \ldots \ldots \ldots 11 \ldots 1 \ldots 111 \ldots
\end{align*}
\]
Since the BBS has the infinite number of conserved quantities, we may regard it as an integrable cellular automaton. Some integrable extensions of the BBS are proposed, that is, the BBS with arbitrary capacity of each box and/or species of the ball [22, 26] and/or with a carrier [18, 23].

In [25], Tokihiro et al. clarified that the BBS is directly obtained from the discrete Korteweg–de Vries (dKdV) equation through the technique of ultradiscretization. The exact solution to the BBS,

\[
U^t_n = T^t_n + T^{t+1}_{n+1} - T^{t+1}_{n+1} - T^t_{n+1}, \quad T^t_n = \min_\lambda \{a_\lambda n + b_\lambda t + c_\lambda\},
\]

was also given by ultradiscretizing the N-soliton solution of the dKdV equation. On the other hand, the BBS can be constructed by applying the crystallization technique to a quantum integrable system. Many results from this context are also reported (see, for example, [1]).

We know some approaches to solve the initial value problem of the BBS. The method in [20] is based on the application of representation theory. A combinatorial method employing the theory of soliton equations is developed in [14]. Recently, an analogue of the dressing method is also reported [28]. Moreover, many generalizations of BSS are constructed and solved by a representation-theoretical method [12]. These generalizations are based on various types of Lie algebras. As far as the authors know, no (tropical) geometric counterpart is known at the moment.

Another extension of the BBS imposing the periodic boundary condition [27]

\[ U^t_{n+L} = U^t_n \]

is actively studied. It is called the periodic BBS (pBBS). We refer to the period L as system size. Its exact solution is given in terms of the tropical theta function (see the appendix) as

\[ U^t_n = \Theta^t_n + \Theta^{t+1}_{n+1} - \Theta^{t+1}_{n+1} - \Theta^t_n. \]

A solution of the initial value problem of the pBBS is also studied from various contexts. As is the case in the BBS, the method using representation theory [10, 11] and the combinatorial method [13] are known. The relationship between these two methods is discussed in [8]. An interesting method extending the solution for the open boundary system to that for the periodic boundary system is reported in [4]. In addition, an analytical method employing the theory of tropical geometry shows remarkable development [2, 5–7]. Since each method is based on its own theory, we may consider that the BBS and pBBS give a junction of various mathematical theories. In particular, construction of solutions of the BBS from that of the pBBS through a limiting procedure is proposed in some earlier literatures.

Our aim is to give a tropical-geometric proof of theorem 3.10, which has been proved by means of combinatorics, representation theory, etc.

**Theorem** (Theorem 3.10). The general solution \( \tilde{U}^t_n \) of the pBBS (section 2) is expressed as

\[ \tilde{U}^t_n = \lim_{L \to +\infty} U^t_n = T^t_n + T^{t+1}_{n+1} - T^{t+1}_{n+1} - T^t_{n+1}, \]

where \( T^t_n \) is a piecewise-linear function in \( n, t \) (equation (10), section 3.2).

This paper is organized as follows. We start with the results for the pBBS based on the theory of tropical curves (section 2). In section 3, we study the system-size dependence of its solution. Then evaluating the limit as the system size tends to infinity, we show that the solution of the pBBS reduces to that of the BBS. A non-trivial limit of important geometrical objects is discussed in this process. This process exactly gives a tropical analogue of the Krichever construction, which is an algebro-geometric method to construct exact solutions to integrable systems [9]. Finally, we give concluding remarks in section 4.
2. Review of the periodic box and ball system

In this section, we review the theory of the pBBS (see, for example [2, 5, 7] for details).

2.1. Spectral curve

Let \( K = \bigcup_{d \in \mathbb{Z}_{>0}} \mathbb{C}((q^{1/d})) \) be the Puiseux series field of indeterminate \( q \). Let
\[
\text{val} : K \to \mathbb{Q} \cup \{ +\infty \}
\]
be the valuation with \( \text{val}(0) = +\infty \), \( \text{val}(q) = 1 \). The sub-ring \( R = \{ x \in K | \text{val}(x) \geq 0 \} \) is called the valuation ring. Then, \( \text{val}(x) = \max\{r \in \mathbb{Q} | q^{-r}x \in R\} \) holds.

We define \( 2 \times 2 \) matrices \( T(y) \) and \( H(y) \) by
\[
T(y) = \begin{pmatrix} q & 0 \\ y & 1 \end{pmatrix}, \quad H(y) = \begin{pmatrix} 1 & 1 \\ y & q \end{pmatrix} \in \text{Mat}(2, \mathbb{Z}[y]).
\]

For a given initial state of the pBBS, we define \( \mathcal{X}(y) \in \text{Mat}(2; \mathbb{Z}[y]) \) as follows:
\[
\ldots 11 \ldots 1 \ldots \Rightarrow \mathcal{X}(y) := HTTHHHTHHH.
\]

In other words, we replace ‘.’ and ‘1’ with \( H(y) \) and \( T(y) \), respectively, and define \( \mathcal{X}(y) \) as their product.

We regard the characteristic polynomial of \( \mathcal{X}(y) \), say \( \Phi(x, y) := \det(\mathcal{X}(y) - xE) \) as a polynomial in \( x \) and \( y \) over \( K \). The algebraic curve defined by \( \Phi(x, y) = 0 \) is called the spectral curve.

2.2. Tropical theta function solutions

In this subsection, we introduce tropical theta function solutions to the pBBS [7]. See appendix for fundamental results of the tropical geometry.

2.2.1. Spectral curve and theta function solutions

Let \( \Phi = \Phi(x, y) \) be the characteristic polynomial of \( \mathcal{X} \), and \( \Gamma \) be the tropical curve defined by \( \Phi \). Denote by \( \iota : \Gamma \to \Gamma^0 \) the surjection defined in appendix A.1. Note that \( \Gamma^0 \) is a piecewise-linear subset of \( \mathbb{R}^2 \). The tropical curve \( \Gamma \) is given by means of the following propositions.

**Proposition 2.1** ([2]). For a given initial state of the pBBS with system size \( L \), let \( S_1 \leq S_2 \leq \cdots \leq S_g \) be the lengths of solitons. Then, \( \Gamma^0 \) is illustrated as figure 1, where \( A_i := \sum_{i=1}^g \min[S_i, S_t] \).

Define the segments \( \gamma^+, \gamma^- : [0, S_g] \to \Gamma^0 \) by
\[
\gamma^+(t) = \left( \min_{i} [S_i, t], t \right), \quad \gamma^-(t) = \left( \min \left[ (L-g)t, L-\sum_{i} \min[S_i, t] \right], t \right)
\]
and \( \theta^0_{i} : (0, L-2A_i) \to \Gamma^0 \) for \( i = 1, 2, \ldots, g \) by \( \theta^0_{i}(t) = (t+A_i, S_i) \). Moreover, we define the half-lines \( \sigma_1, \ldots, \sigma_4 : (0, +\infty) \to \Gamma^0 \) as
\[
\sigma_1(t) = (-Lt, -2t), \quad \sigma_2(t) = (t + L - g, 1),
\]
\[
\sigma_3(t) = (A_x, t + S_g), \quad \sigma_4(t) = (L - A_x, t + S_g).
\]

Then, it follows that \( \Gamma^0 = \gamma^+ \cup \gamma^- \cup \bigcup_{i=1}^4 \theta^0_{i} \cup \bigcup_{i=1}^4 \sigma_i \). Note that \( \theta^0_{i} \) coincides with \( \theta^0_{i+4} \) if there exists \( i \) such that \( S_i = S_{i+1} \).
Proposition 2.2. Let $\Gamma$ be the tropical curve and $\iota : \Gamma \to \Gamma^0$ be the above surjection. Then, $\Gamma$ is the unique connected finite graph such that (i) $\Gamma = (\gamma^+ \cup \gamma^-) \cup \left( \bigcup_{i=1}^{g} \theta_i \right) \cup \left( \bigcup_{i=1}^{4} \sigma_i \right)$, (ii) $\iota(\gamma^\pm) = \gamma^\pm$, $\iota(\theta_i) = \theta_i$, $\iota(\sigma_i) = \sigma_i$.

Let $\beta_i \in H_1(\Gamma, \mathbb{Z})$ ($i = 1, \ldots, g$) be $g$ cycles over $\Gamma$ (see figure 1) expressed as

$$\beta_i(t) = \begin{cases} \gamma^+(t) & 0 \leq t \leq S_i \\ \theta_i(t - S_i) & S_i < t < L - 2A_i + S_i \\ \gamma^-(-t + L - 2A_i + 2S_i) & L - 2A_i + S_i \leq t \leq L - 2A_i + 2S_i \end{cases}$$

Note that $\beta_1, \ldots, \beta_g$ give a basis of $H_1(\Gamma, \mathbb{Z})$. For the basis $\{\beta_i\}$, the period matrix $B$ and the Jacobi variety $\text{Jac} \Gamma := \mathbb{R}^g/B\mathbb{Z}^g$ are defined (see definition appendix A.5).

Corollary 2.3. We have $B = ((L - 2A_i)\delta_{i,j} + 2\min[S_i, S_j])_{i,j}$.

For a sufficiently large real number $U > 0$, we put

$$O : (X, Y) = (0, 0), \quad Q : (X, Y) = (U, 1), \quad R : (X, Y) = (A_i, U).$$

Furthermore, we define real vectors $\mu, \omega \in \text{Jac} \Gamma$ by

$$\mu := -A(Q), \quad \omega := -A(R),$$

where $A : \text{Pic} \Gamma \to \text{Jac} \Gamma$ is the tropical Abel–Jacobi mapping with initial point $O$. We note that $\mu$ and $\omega$ are independent of $U$. From proposition 2.1, we have

$$\mu \equiv (1, 1, \ldots, 1), \quad \omega \equiv -^t(S_1, S_2, \ldots, S_g), \quad (\text{mod } B\mathbb{Z}^g).$$

Remark 2.4. Using the expression $B = (b_1, \ldots, b_g)$, we have $L\mu = ^t(L, L, \ldots, L) = b_1 + b_2 + \cdots + b_g \equiv 0 \pmod{B\mathbb{Z}^g}$, which reflects the periodic boundary condition $U_{n+L} = U_n$. See theorem 2.5.
The general solution to the pBBS is represented by the tropical theta function associated with the tropical curve. Let $\Theta(z; B)$ be the tropical theta function defined in \textit{appendix}.

\textbf{Theorem 2.5} ([16]). The general solution to the pBBS is expressed as
\[ U_n^l = \Theta_n^l + \Theta_{n+1}^{l+1} - \Theta_{n+1}^l, \quad \Theta_n^l = \Theta(\mu n + \omega t + c_0; B), \]
where $c_0 \in \text{Jac} \Gamma$ is determined from an initial state through the eigenvector mapping (described below).

2.2.2. Eigenvector mapping. Here, we review the eigenvector mapping [17], which maps an initial data of a discrete integrable system to a positive divisor on some algebraic curve. We here consider only a special case necessary for our purpose.

Let $\mathcal{X} = \mathcal{X}(y)$ be the $2 \times 2$ matrix defined in section 2.1 and $C$ be the algebraic curve defined from the algebraic equation $f(x, y) = \det(\mathcal{X}(y) - xE) = 0$. Denote the genus of $C$ by $\tilde{g}$. It is known that the genus of the tropical curve $\Gamma$ associated with $f$ is equal or less than $\tilde{g}$.

\textbf{Lemma 2.6} ([17]). There exists an unique point $\infty \in C$ expressed as $(x, y) = (\infty, \infty)$.

Due to the definition of $C$, the equation
\[ \mathcal{X}(y)^l(1, h(x, y)) = x^l(1, h(x, y)) \] (4)
determines the rational function $h$ on $C$ uniquely.

\textbf{Theorem 2.7} ([17]). Let $\text{Div} C := \bigoplus_{p \in C} \mathbb{Z} \cdot p$ be the divisor group of $C$, which is a free Abelian group generated by the points in $C$. Then, there exists a general positive divisor $D \in \text{Div} C$ of degree $\tilde{g}$ such that the zero divisor $(h)_0$ of $h$ is expressed as
\[ (h)_0 = D + \infty. \] (5)

In general, the mapping $\mathcal{X}(y) \mapsto D$ is called the eigenvector mapping. Denote $D = p_1 + p_2 + \cdots + p_{\tilde{g}}$ ($p_k \in C$) and the $(x, y)$-coordinate of the point $p_k$ by $(x_k, y_k)$. Let $x_k := \text{val}(x_k)$, $y_k := \text{val}(y_k)$. We regard the point $P_k : (X, Y) = (x_k, y_k)$ in the real plane $\mathbb{R}^2$ as the tropicalization of the point $p_k$. It immediately follows that $P_k \in \Gamma$ (see, for example, [7]). We denote $p \leadsto P$ if $P$ is the tropicalization of $p$.

The element $c_0 \in \text{Jac} \Gamma$ in \textit{Theorem 2.5} is calculated from the configuration of $P_k$.

\textbf{Proposition 2.8} ([16]). Let $\kappa \in \text{Jac} \Gamma$ be the Riemann constant of $\Gamma$ (section A.1.5). Then, we have
\[ c_0 = \kappa - A(P_1 + \cdots + P_{\tilde{g}}). \] (6)

2.2.3. Riemann constant. For general tropical curves, it is hard to calculate the Riemann constant $\kappa$ explicitly. However, for our tropical curves, we have the following simple expression.

\textbf{Proposition 2.9}.

\[ \kappa \equiv \frac{L}{2} \cdot \gamma(1, 1, \ldots, 1). \]

To prove the proposition, we need some ideas on the tropical geometry introduced in the \textit{appendix}. Let $\tilde{A} : \Gamma \rightarrow \mathbb{R}^\tilde{g}$ be the multi-valued function defined in section A.1.5. Define the multi-valued function $f : \Gamma \rightarrow \mathbb{R}$ by $f(P) := \Theta(\tilde{A}(P); B)$.

Denote the point $\theta(L/2 - A_0)$ in $\Gamma$ by $Q_i$. The set $U = \Gamma \setminus \{Q_1, \ldots, Q_i\}$ is a simply connected open subset of $\Gamma$. Let $A_0 : U \rightarrow \mathbb{R}^\tilde{g}$ be the restriction of $A$ to $U$, where we take the branch as $A_0(O) = 0$. Then $A_0$ is single-valued. Define a single-valued function $f_0 : U \rightarrow \mathbb{R}$ by $f_0(P) := \Theta(A_0(P); B)$. 

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Lemma 2.10. The point \( Q_i \) \((i = 1, 2, \ldots, g)\) is a zero of \( f \) of degree 1.

**Proof.** We may take the branch of \( f \) such that \( 0 < t < L/2 - A_i \Rightarrow f(\theta_i(t)) = f_0(\theta_i(t)) \) without loss of generality. The value of tropical Abel–Jacobi mapping is calculated as \((\theta_i(t)) = (t\delta_{ij} + \min[S_i, S_j])_{j=1}^d. \) From the equation
\[
(t\delta_{ij} + \min[S_i, S_j])_{j=1}^d = \frac{1}{2}B_i + (t - L/2 + A_i)e_i,
\]
we have \( f(\theta_i(t)) = \Theta \left( \frac{1}{2}B_i + (t - L/2 + A_i)e_i; B \right) \). By lemma A.11, we obtain the result. \( \square \)

**Proof of the proposition.** By lemma A.13, the set of zeros of \( f \) is \( \{Q_1, \ldots, Q_g\} \). It gives the result. \( \square \)

2.3. Example

We consider the pBBS of system size \( L = 10 \) and the initial state given in (1). We have \( S = 2, S_1 = 1, S_2 = 2, \) and therefore obtain \( A_1 = 2, A_2 = 3 \) and
\[
B = \begin{pmatrix} 8 & 2 \\ 2 & 8 \end{pmatrix}.
\]
By calculating \( h(x, y) \) in (4) associated with \( \mathcal{X}(y) \) in (1), one can find \( P_1 : (2, 1) \) and \( P_2 : (5, 1) \). Their images by the mapping \( \tilde{A} \) are \((1, 1)\) and \((4, 1)\), respectively. Hence, we have \( c_0 = \{0, 3\} \). Now, we obtain the tropical theta function
\[
\Theta^f_n = \Theta \left( n \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} : \begin{pmatrix} 8 \\ 2 \\ 8 \end{pmatrix} \right), \quad (7)
\]
which expresses the solution for this initial state through theorem 2.5.

3. Solutions to the non-periodic system

In this section, we propose the method to construct solutions to the BBS from the general solutions to the pBBS. Let \( \mathcal{X} = \mathcal{X}(y) \) be the \( 2 \times 2 \) matrix defined in the previous section. We denote the tropical curve associated with the pBBS by \( \Gamma \), and its period matrix by \( B \). Moreover, denote \( \mathcal{X}[\mathcal{M}] := \mathcal{X}^H \mathcal{M} (\mathcal{M} \gg 1) \). We are interested in the behavior of solutions to the periodic system when \( \mathcal{M} \to +\infty \), or equivalently \( L \to +\infty \). More concretely, we are interested in the behavior of the function \( \Theta^f_n = \Theta(\mu n + \omega t + c_0; B) \) when \( L \to +\infty \).

Let \( S = -2 \text{diag}(A_1, \ldots, A_g) + (2 \min[S_i, S_j])_{i, j} \). Then, we have \( B = LE + S \) from corollary 2.3. Define \( z = \mu n + \omega t + c_0. \) From (3), the vectors \( \mu \) and \( \omega \) do not depend on \( L \). Due to (6), the problem is reduced to considering the asymptotic behavior of the eigenvector mapping \( \mathcal{X}(y) \mapsto D = p_1 + \cdots + p_g \) (theorem 2.7) when \( L \to +\infty \).

Recall that the \( x \) - and \( y \)-coordinates of the points \( p_1, \ldots, p_g \) are characterized by the relations (4) and (5). Moreover, their tropicalizations also can be characterized in terms of \( \mathcal{X}(y) \). Let \( \bar{P}_i \) be the tropicalization of \( p_i (p_i \mapsto \bar{P}_i) \).

**Lemma 3.1.** Denote the \((i, j)\)-component of \( \mathcal{X}(y) \) by \( a_{i,j}(y) \in K[y] \). The point \((X, Y) \in \Gamma \) is contained in \([P_1, \ldots, P_g]\) if and only if: (i) the polynomial \( a_{2,1}(y) \) has a root \( y^* \in K \) such that \( \text{val}(y^*) = Y \), (ii) \( X = \text{val}(a_{1,1}(y^*)) \).
Proof. By (4) and (5), we have $(X, Y) \in \Gamma$ is contained in $\{P_1, \ldots, P_3\}$ iff there exist two elements $x^*, y^* \in K$ such that $X = \text{val}(x^*), Y = \text{val}(y^*)$ and $X(y^*) \cdot i(1, 0) = x^* \cdot i(1, 0)$.

This implies the lemma. □

Let $\Gamma[M]$ be the tropical curve defined from $\mathcal{X}[M](y), p_1[M] + \ldots + p_3[M]$ be the image of $\mathcal{X}[M](y)$ by the eigenvector mapping and $P_1[M], \ldots, P_3[M] \in \Gamma[M]$ be their tropicalization. We have the following theorem, the proof of which is given in the next subsection.

**Theorem 3.2.** There exists some large number $m_0 > 0$ such that for all $M > m_0$, the value $A_0(P_1[M] + \cdots + P_3[M]) \in \mathbb{R}^8$ is constant.

### 3.1. Proof of theorem 3.2

First, we introduce some notations for the proof of the theorem. Let $R := \{x \in K \mid \text{val}(x) \geq 0\}$ be the valuation ring of $K$. Define two subsets $K^+, R^+$ of $K$ by $K^+ := \{c_0q^{n/d} + c_1q^{(n+1)/d} + \cdots \mid c_0 \geq 0\}$ and $R^+ := R \cap K^+$. These four sets naturally have an algebraic structure as below:

- $K$: field,
- $R$: ring,
- $K^+$: semi-field,
- $R^+$: semi-ring.

The semi-ring $R^+$ will not be used in the following.

Let $\mathcal{O}$ be a $K$-algebra. Naturally, the algebra $\mathcal{O}$ is also regarded as an $R$-algebra. For an $R$-subalgebra $\mathcal{O}_R \subset \mathcal{O}$, we define the mapping $v(\cdot; \mathcal{O}_R) : \mathcal{O} \to \mathbb{R} \cup \{+\infty\}$ by $v(x; \mathcal{O}_R) := \sup\{r \mid q^{-r}x \in \mathcal{O}_R\}$. For example, if $\mathcal{O} = K$ and $\mathcal{O}_R = R$, then $v(x; R) = \text{val}(x)$.

We fix $\mathcal{O} = K[y]$ in the rest of the paper. For a rational number $p$, we denote by $\mathcal{O}_K(p)$ the $R$-subalgebra of $K[y]$ generated by $q^{-py}$. Let $v_p(\cdot) := v(\cdot; \mathcal{O}_K(p))$. It immediately follows that $v_p(q^n) = m + np$ for any rational number $m$ and any integer $n$.

For any two elements $f, g \in K[y]$, we have (i) $v_p(fg) = v_p(f) + v_p(g)$, (ii) $v_p(f + g) \leq \min\{v_p(f), v_p(g)\}$. Let $K^+[y]$ be the $K^+$-subsemifield of $K[y]$ generated by $y$. By restricting $v_p$ on $K^+[y]$, the above inequality (ii) is improved as follows:

$$f, g \in K^+[y] \Rightarrow v_p(f + g) = \min\{v_p(f), v_p(g)\}. \quad (8)$$

Because $\mathcal{O}_K(p)$ is an $R$-algebra, the substitution $q \mapsto 0$ is well defined over $\mathcal{O}_K(p)$. For example, for the element $y \in \mathcal{O}_K(p) (p \geq 0)$, the substitution procedure is expressed as

$$y = q^p (q^{-py}) \mapsto \begin{cases} y, & p = 0, \\ 0, & p > 0. \end{cases}$$

For $f \in \mathcal{O}_K(p)$, we denote by $f_{(p)}$ the element of $\mathbb{C}[q^{-py}]$ which is obtained by substitution $q \mapsto 0$. For example, $y_{(0)} = y, (q^{-1}y + 2y^2 + 3q^{-2}y^2)_{(1)} = q^{-1}y + 3q^{-2}y^2$, etc.

**Remark 3.3.** Let $l = \{x \in R \mid \text{val}(x) > 0\}$ be the maximal ideal of $R$. Then, the substitution $q \mapsto 0$ is equivalent to the algebraic mapping $\mathcal{O}_K(p) \to \mathcal{O}_K(p) \otimes_R (R/l); f \mapsto f \otimes 1$. In fact, under the natural isomorphism $\mathcal{O}_K(p) \otimes_R (R/l) \simeq \mathbb{C}[q^{-py}]$, we have $f \otimes 1 \simeq f_{(p)}$. It would be more natural to express the substitution procedure by the algebraic mapping. However, we do not use this expression to avoid redundant notations hereafter.

**Remark 3.4.** By the definition of $v_p$, we have $q^{-v_p(f)} : f \in \mathcal{O}_K(p)$ for any non-zero $f \in K[y]$. We can check $(q^{-v_p(f)} \cdot f)_{(p)} \neq 0$, which is not always true for arbitrary $K$-algebra $\mathcal{O}$ and $R$-subalgebra $\mathcal{O}_R$. In fact, this is not true for $\mathcal{O} = K((y)), \mathcal{O}_R = R((y))$.

**Lemma 3.5.** For non-zero $f \in K[y]$, there exists a unique non-zero polynomial $\in_p(f) \in \mathbb{C}[q^{-py}]$ such that $v_p(f - q^{-v_p(f)} \cdot \in_p(f)) > v_p(f)$. 

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Proof. It is sufficient to put \( \text{in}_p(f) := (q^{-v_p(f)} f)(\rho) \). In fact, we have \( (q^{-v_p(f)} f \circ \text{in}_p(f))(\rho) = 0 \), which implies \( v_p(q^{-v_p(f)} f \circ \text{in}_p(f)) > v_p(q^{-v_p(f)} f) \). \( \-box \)

Remark 3.6. In the situation as in lemma 3.5, we denote \( f = q^{v(R)} \cdot \text{in}_p(f) + o_p(q^{v(R)}). \)

Lemma 3.7. For \( f \in K[y] \), the following (i) and (ii) are equivalent.

(i) There exists some \( y^* \in K \) such that \( f(y^*) = 0 \) and \( \text{val}(y^*) = p. \)

(ii) The polynomial \( \text{in}_p(f) \) has a non-zero root.

Proof. First we note the fact that, for fixed \( M > 0 \) such that \( a_i \) is expressed as a Laurent series of \( q^{1/M} \) for any \( i \). By the transformation \( y \mapsto y^p \), one can assume \( p = 0 \) without loss of generality. From remark 3.6, we have \( f = \text{in}_0(f) + o(1). \) (i) \( \Rightarrow \) (ii): because \( \text{val}(y^*) = 0 \), the substitution \( y^* \mapsto y^*|_{y=0} \) is well defined and \( y^*|_{y=0} \neq 0 \). Therefore, we have \( \text{in}_0(f)(y^*|_{y=0}) = 0. \) (ii) \( \Rightarrow \) (i): if \( f = \text{in}_0(f) \), then there is nothing to prove. We assume \( f \neq \text{in}_0(f) \). Let \( \alpha \in \mathbb{C}^* \) be a root of \( \text{in}_0(f) \) such that \( \text{in}_0(f) = (y-\alpha)^f f_1 \), where \( f_1 \in \mathbb{C}[y] \) and \( f_1(\alpha) \neq 0 \). On the other hand, by assumption, there exist some \( r \in \mathbb{Q}_0 \) and \( m \in \mathbb{Z}_{\geq 0} \) such that \( f = \text{in}_0(f) + q^r(y-\alpha)^m(f_0 + o(1)) \), where \( f_0 \in \mathbb{C}[y] \) and \( f_0(\alpha) \neq 0 \) (this implies \( \text{val}(f(\alpha)) = r \)). If \( m > 0 \), \( f \) has a root \( \alpha \) with \( \text{val}(\alpha) = 0 \). When \( m = 0 \), by putting \( y = \alpha = q^r y_1 \), we obtain \( f = q^r (y_1 Y_1 + C_0 + o(1)) \), where \( Y_1 = \alpha y_1 \) be a non-zero solution of \( y_1 Y_1 + C_0 = 0 \). In a similar way, we obtain some positive rational number \( r_1 \) such that \( \text{val}(f(\alpha + q^{r_1} \alpha_1)) = r + r_1 \). Recursively, we get sequences \( \alpha_2, \alpha_3, \ldots; r_2, r_3, \ldots; n_1, n_2, n_3, \ldots \) with \( \text{val}(f(\alpha + q^{r_i/n_i} \alpha_1 + q^{r_2/n_2} \alpha_2 + \cdots + q^{r_{n_1}/n_{n_1}} \alpha_{n_1+1})) = r + r_1 + \cdots + r_{n_1+1} \).

Because there exists some large \( M > 0 \) with \( M r_i \in \mathbb{Z} \) for all \( i \), we have \( \lim_{N \to \infty} \text{val}(f(\alpha + q^{r_i/n_i} \alpha_1 + q^{r_2/n_2} \alpha_2 + \cdots + q^{r_{n_1}/n_{n_1}} \alpha_{n_1+1})) = +\infty \), which implies \( y^* = \alpha + q^{r_i/n_i} \alpha_1 + q^{r_2/n_2} \alpha_2 + \cdots \) is a root of \( f \).

Next, we prove the following lemma.

Lemma 3.8.

(i) There exists some \( y_0 \) such that \( \text{in}_y(a_{i,j}[M]) \) (i = 1, 2, M = 0, 1, 2, \ldots) is a monomial for all \( Y > y_0 \).

(ii) If \( M > Y > 0 \), then \( \text{in}_y(a_{i,j}[M]) = \text{in}_y(a_{i,j}[Y + 1]). \)

(iii) A point \( P_i[M] \in \Gamma \) contained in \( \{(X, Y)|Y < 0\} \) satisfies \( A_0(P_i[M]) = 0. \)

Proof. Proof of (i). Note that for any \( f = \gamma^m(a_0 + a_1 y + \cdots + a_0 y^m) \in K[y], (a_i \in K, a_0 \neq 0), \) the inequality \( p > \max_i (\frac{1}{2} \text{val}(a_0) - \text{val}(a_i)) \) implies \( \text{in}_p(f) = \text{in}_p(a_0 y^m). \) Because of the expression \( H, T = [\text{an upper triangle matrix over} K^+] + [\text{an lower triangle matrix over} K^+] \), we have the following relations for sufficiently large \( \xi \).

\[
Y > \xi \quad \Rightarrow \quad \text{in}_y(a_{1,1}[M]) = \text{in}_y(a_1), \quad \text{in}_y(a_{1,2}[M]) = \text{in}_y(\beta), \quad \text{in}_y(a_{2,1}[M]) = \text{in}_y(\gamma y), \quad \text{in}_y(a_{2,2}[M]) = \text{in}_y(\delta + \epsilon y),
\]

where \( \alpha, \beta, \gamma, \delta, \epsilon \in K^+ \).

Define inductively \( a[M], b[M], \ldots \in K^+ (M = 0, 1, 2, \ldots) \) by the formulas \( a[M+1] = a[M] + a[M], \quad b[M+1] = a[M] + b[M], \quad \gamma[M+1] = a[M] + \gamma[M] + b[M], \quad \delta[M+1] = a[M] + \gamma[M] + b[M]. \) Because of the expression \( H, T = [\text{an upper triangle matrix over} K^+] + [\text{an lower triangle matrix over} K^+] \), we have the following relations.

\[
Y > y_0 \quad \Rightarrow \quad \text{in}_y(a_{1,1}[M]) = \text{in}_y(a[M]), \quad \text{in}_y(a_{1,2}[M]) = \text{in}_y(b[M]), \quad \text{in}_y(a_{2,1}[M]) = \text{in}_y(\gamma[M]), \quad \text{in}_y(a_{2,2}[M]) = \text{in}_y(\delta[M] + \epsilon[M]).
\]
Proof of (ii). Let $h_{i,j}[M]$ be the $(i, j)$-component of $H^M$. Then, it follows that

$$
M > Y \Rightarrow \begin{cases} 
 h_{1,1}[M] = 1 + o_Y(1), \\ h_{1,2}[M] = 1 + o_Y(q^Y), \\ h_{2,1}[M] = y + o_Y(q^Y), \\ h_{2,2}[M] = y + o_Y(q^Y). 
\end{cases}
$$

This implies $v_Y(a_{i,1}[M + 1]) = v_Y(a_{i,2}[M + 1]) = v_Y(a_{i,1}[M] + v(a_{i,2}[M]), (i = 1, 2)$. Again using (9), we obtain

$$
in_Y(a_{i,1}[M + 2]) = in_Y(a_{i,1}[M] + v(a_{i,2}[M + 1]) = in_Y(a_{i,1}[M + 1]), \ i = 1, 2.
$$

Therefore, $M > Y \Rightarrow in_Y(a_{i,1}[M]) = in_Y(a_{i,1}[Y + 1])$. □

Proof of (iii). It is straightforward from proposition 2.1.

Now, we proceed for the proof of the theorem.

**Proof.** It follows that any point $P_i[M]$ is contained in the domain $\{Y \leq Y_0\}$ from (i) and lemma 3.1. By (iii), we need not consider any points $P_i[M]$ contained in the domain $\{Y < 0\}$. From (ii) and lemma 3.1, the $X$- and $Y$-coordinates of points $P_i[M]$ contained in the domain $\{0 \leq Y \leq Y_0\}$ are constant for any $M > Y_0$. Then, we obtain the theorem when we put $m_0 := Y_0$. □

**Remark 3.9.** The proofs of above lemmas are considered as the calculation for tropical pole divisors, which is the key procedure for the Krichever construction.

### 3.2 Limiting procedure for the tropical theta function solution

By the results of the previous sections, we have the following expression for sufficient large $L$.

$$
B = LE + S, \quad z = \frac{L}{2} \cdot (1, 1, \ldots, 1) + \mu n + \omega + c,
$$

where $c$ is independent of $L$. Substituting this to the tropical theta function, we obtain

$$
\Theta(z; B) = \min_{r \in \mathbb{Z}^d} \left[ \frac{L}{2} (r, r) + \langle r, e \rangle + \frac{1}{2} \langle r, Si \rangle + \langle r, \mu n + \omega + c \rangle \right],
$$

where $e = e_1 + \ldots + e_g$. When $L \to +\infty$, it follows that

$$
\Theta(z; B) = \min_{r \in [0,1]^d} \left[ \frac{L}{2} (r, r) + \langle r, e \rangle + \frac{1}{2} \langle r, Si \rangle - \langle r, \mu n + \omega + c \rangle \right]
$$

Let

$$
T_n^i = \min_{r \in [0,1]^d} \left[ \frac{1}{2} (r, Si) - \langle r, \mu n + \omega + c \rangle \right]. \quad (10)
$$

Then, we obtain the main theorem in this paper.

**Theorem 3.10.** Let $U_n^i$ be the general solution of the pBBS. The $\tilde{U}_n^i$ below satisfies the BBS:

$$
\tilde{U}_n^i := \lim_{L \to +\infty} U_n^i = T_n^i + T_{n+1}^{i+1} - T_{n-1}^i - T_n^{i-1}.
$$

**Remark 3.11.** This limiting procedure is a tropical analogue of the method in [16, part III b, section 5].
3.3. Example

From the solution of the \( pBBS \) (7), we obtain

\[
T_i^t = \min\{0, -n + t - 3, -n + 2t - 1, -2n + 3t - 8\},
\]

which gives the two-soliton solution of the BBS.

4. Concluding remarks

In this paper, starting with a given initial state of the \( pBBS \), we have constructed the solution of the box and ball system (BBS). Its initial state is given by inserting infinite number of empty boxes into that of the \( pBBS \). Theorem 3.2, which evaluates the asymptotic behavior of the eigenvector mapping, plays an essential role when the system size tends to infinity. The obtained solution is identical to the well-known soliton solution of the BBS. Moreover, our method gives a tropical analogue of the Krichever construction and establishes an application of tropical geometry to integrable cellular automata.

Our method can be readily extended to a system represented by Lax pair, for example, the two-dimensional BBS and the discrete Painlevé equations [19, 24]. It is a future problem to study such systems in the context of the tropical geometry.

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Appendix. Review of the tropical geometry

In this section, we introduce some ideas of the tropical geometry. For details, see [3].

A.1. Definitions

A.1.1. Tropical curve. Let \( K \) be the Puiseux series field of indeterminate \( q \) over \( \mathbb{C} \), and \( \text{val} : K \rightarrow \mathbb{Q} \cup \{+\infty\} \) be the valuation with \( \text{val}(0) = +\infty \), \( \text{val}(q) = 1 \). For a polynomial

\[
\Phi(x, y) = \sum_{w=w_1,w_2}\in\mathbb{Z}^2\ a_wx^{w_1}y^{w_2} \quad (a_w = 0 \text{ except for finitely many } w)
\]

in \( x \) and \( y \) over \( K \), we define \( \text{Val}_\Phi(X, Y) := \min_{w}\{\text{val}(a_w) + w_1X + w_2Y\}, \) \( X, Y \in \mathbb{R} \). Let \( \Gamma^0 \) be the subset of \( \mathbb{R}^2 \) defined by

\[
\Gamma^0 = \{P \in \mathbb{R}^2 | \text{the continuous map } \text{Val}_\Phi : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ is indifferentiable at } P \}.
\]

For a point \( P \in \Gamma^0 \), define the finite set \( \Lambda(P) \subset \mathbb{Z}^2 \) by

\[
\Lambda(P) := \{w \in \mathbb{Z}^2 | \text{Val}_\Phi(X, Y) = \text{val}(a_w) + w_1X + w_2Y\}, \quad P = (X, Y).
\]

Let \( \Phi_P \) be the polynomial over \( K \) defined by \( \Phi_P := \sum_{w \in \Lambda(P)} a_wx^{w_1}y^{w_2} \).

We regard the polynomial \( \Phi_P \) as an element of the extended ring \( K[x, x^{-1}, y, y^{-1}] \). The multiplicity of the point \( P \) is a positive number \( \partial(P) \) which is defined by the following formula:

\[
\partial(P) := \# [\text{irreducible components of } \Phi_P \in K[x^{\pm 1}, y^{\pm 1}]].
\]
**Definition A.1.** The tropical curve associated with $\Phi$ is a connected finite graph which may have edges of infinite length such that (i) there exists a surjection $\iota : \Gamma \to \Gamma^0$, (ii) there exists a finite subset $\Delta \subset \Gamma^0$ such that $\iota$ is a finite covering over $\Gamma^0 \setminus \Delta$, (iii) $\iota^{-1}(P) = 0$ for all $P \in \Gamma^0$. Such $\Gamma$ exists and is unique.

Note that the slope of edges of $\Gamma^0$ is a rational number. By using the lattice length of $\mathbb{R}^2$, we equip $\Gamma^0$ with the structure of metric graph. Pulling back by the almost locally isomorphic map $\iota$, we equip $\Gamma$ with metric as well. For two points $P, Q \in \Gamma$, denote by $\text{dist}(P, Q)$ the distance between $P$ and $Q$.

### A.1.2. Piecewise-linear functions over $\Gamma$.

A continuous function $f : \Gamma \to \mathbb{R}$ is piecewise linear if the limit

$$D_\gamma(f) := \lim_{t \to 0^+} \frac{f(\gamma(t)) - f(\gamma(0))}{\text{dist}(\gamma(t), \gamma(0))}$$

is an integer for any path $\gamma : [0, 1] \to \Gamma$. Let $\mathcal{L}$ be the set of piecewise-linear functions over $\Gamma$. For a point $P \in \Gamma$, we define the subset $\mathcal{T}_P \subset \mathcal{L}$ by $\mathcal{T}_P := \{D_\gamma : \mathcal{L} \to \mathbb{Z} | \gamma(0) = P\}$. Note that the set $\mathcal{T}_P$ is finite for any $P$.

The degree of $f$ at $P$ is defined by $\text{ord}_P(f) := -\sum_{D_\gamma \in \mathcal{T}_P} D_\gamma f$. By definition, we have $\text{ord}_P(f + g) = \text{ord}_P(f) + \text{ord}_P(g)$, $\text{ord}_P(kf) = k \cdot \text{ord}_P(f)$ ($k \in \mathbb{Z}$).

**Definition A.2.** A piecewise-linear function $f : \Gamma \to \mathbb{R}$ is rational if (i) $f$ is bounded (ii) $\text{ord}_P(f) \neq 0$ for finitely many $P$. We say that $P$ is a zero of $f$ if $\text{ord}_P(f) > 0$, and that $P$ is a pole of $f$ if $\text{ord}_P(f) < 0$.

### A.1.3. Divisor group and Picard group.

Let $\text{Div} \Gamma := \bigoplus_{P \in \Gamma} \mathbb{Z} \cdot P$ be the free Abelian group generated by points in $\Gamma$. For a rational function $f$, define the divisor $(f)$ by $(f) := \sum_{P \in \Gamma} \text{ord}_P(f) \cdot P \in \text{Div} \Gamma$.

**Definition A.3.** The quotient group of $\text{Div} \Gamma$ divided by the equivalence relation $D_1 \sim D_2 \iff D_1 - D_2 \equiv (f)$. $(f)$ is a rational function is called the Picard group of $\Gamma$. Denote the Picard group of $\Gamma$ by Pic $\Gamma$.

A degree of a divisor $D = \sum_{P} n_P \cdot P$ is the integer $\sum n_P$. By the following theorem, we can define the degree of an element of Pic $\Gamma$ as well.

**Theorem A.4.** Let $D \in \text{Div} \Gamma$. Then, $D \sim 0 \Rightarrow$ the degree of $D$ is 0.

### A.1.4. Period matrix, Jacobi variety.

Let $\gamma : [a, b] \to \Gamma$ be a piecewise-linear map. Define

$$||\gamma|| := \lim_{N \to \infty} \sum_{n=0}^{N-1} \text{dist} \left( \gamma \left( \frac{nh+(N-n)b}{N} \right) , \gamma \left( \frac{(n+1)h+(N-n-1)b}{N} \right) \right).$$

For piecewise-linear paths $\gamma_1 : [a_1, b_1] \to \Gamma$, $\gamma_2 : [a_2, b_2] \to \Gamma$, we define a real number $(\gamma_1, \gamma_2)$ as follows: (i) let $U_1 := [a_1, b_1], U_2 := [a_2, b_2]$. If $\gamma_1, \gamma_2$ are injective and $\gamma_1(U_1) \cap \gamma_2(U_2)$ is connected, put $(\gamma_1, \gamma_2) := (\pm 1) ||\gamma_1(U_1) \cap \gamma_2(U_2)||$, where the signature is $+$ (resp. $-$) if the direction of $\gamma_1^{-1} \circ \gamma_2 : \mathbb{R} \to \mathbb{R}$ is positive (resp. negative). (ii) In general case, divide the paths as $\gamma_1, k : [t_{k-1}, t_k] \to \Gamma, \gamma_2, k : [s_{k-1}, s_k] \to \Gamma$ ($a_1 = t_0 < t_1 < \cdots < t_N = b_1$, $a_2 = s_0 < s_1 < \cdots < s_M = b_2$) such that $\gamma_1, k, \gamma_2, k$ are injective and $\gamma_1, k \cap \gamma_2, k$ are connected, and define $(\gamma_1, \gamma_2) := \sum_{k \in \mathbb{Z}} ||(\gamma_1, k, \gamma_2, k)||$.

We call the first Betti number $g$ of $\Gamma$ the genus of $\Gamma$. In the following, we fix a $\mathbb{Z}$-basis $\beta_1, \ldots, \beta_g$ of $H_1(\Gamma, \mathbb{Z})$. 


Definition A.5. The period matrix of $\Gamma$ is the $g \times g$ real symmetric matrix $B$ defined by $B := (\beta_i, \beta_j)_j$. We call the real variety $\text{Jac } \Gamma := \mathbb{R}^g/B\mathbb{Z}^g$ the Jacobi variety of $\Gamma$.

Lemma A.6. The matrix $B$ is non-degenerate and positive definite.

Choose and fix a point $O \in \Gamma$. Let $\gamma_P$ be a path on $\Gamma$ which starts from $O$ and ends at $P$.

Definition A.7. The Abel–Jacobi mapping starting at $O$ is the mapping $A : \Gamma \to \text{Jac } \Gamma; \quad P \mapsto ((\beta_1, \gamma_P), (\beta_2, \gamma_P), \ldots, (\beta_g, \gamma_P)) \mod B\mathbb{Z}^g$. The value of $A$ does not depend on the choice of $\gamma_P$. We can extend the Abel–Jacobi mapping over $\text{Div } \Gamma$ linearly:

\[ A : \text{Div } \Gamma \to \text{Jac } \Gamma; \quad A(\sum n_P P) = \sum n_P A(P). \]

Theorem A.8 ([15]). The mapping $A$ induces the homomorphism of Abelian groups $\text{Pic } \Gamma \to \text{Jac } \Gamma$.

A.1.5. Tropical theta function and Riemann constant. We introduce a tropical analogue of the theta functions over Riemann surfaces.

Definition A.9. The following real function $\Theta$ over $\mathbb{R}^g$ is called the tropical theta function associated with $\Gamma$:

\[ \Theta(z; B) := \min_{m \in \mathbb{Z}} \left[ \frac{1}{2}(m, Bm) + \langle m, z \rangle \right], \quad z \in \mathbb{R}^g, \]

where $\langle v, w \rangle := \sum_{i=1}^g v_i w_i$ ($v, w \in \mathbb{R}^g$), and $B$ is the period matrix of $\Gamma$.

Because $B$ is positive definite, the tropical theta function is well defined over $\mathbb{R}^g$. In fact, $\Theta$ is a piecewise-linear convex function over $\mathbb{R}^g$.

Lemma A.10.

(i) Let $r \in \mathbb{Z}^g$. Then,

\[ \Theta(z + Br; B) = \left( -\frac{1}{2}r, Br \right) - \langle z, r \rangle + \Theta(z; B), \]

(A.1)

(ii)

\[ \Theta(-z; B) = \Theta(z; B). \]

Proof. It is straightforward by definition.

Now we note the behavior of $\Theta$ around $\frac{1}{2}Be_i \in \mathbb{R}^g$, where $e_i = i(0, \ldots, 1, \ldots, 0)$.

Lemma A.11. Let $\gamma$ be the map $\gamma : \mathbb{R} \to \mathbb{R}^g; \quad t \mapsto \frac{1}{2}Be_i + te_i$. Then, the function $f(t) = \Theta(\gamma(t); B)$ satisfies $\text{ord}_0(f) = 1$.

Proof. By lemma A.10 (i), we have $f(t) = \Theta(\frac{1}{2}Be_i + te_i; B) = \Theta(-\frac{1}{2}Be_i + te_i; B) = -t + \Theta(-\frac{1}{2}Be_i + te_i; B)$. Therefore, from lemma A.10 (ii), it follows that $\Theta(\frac{1}{2}Be_i + te_i; B) - \Theta(\frac{1}{2}Be_i - te_i; B) = -t$. Then, $f(t) = f(-t) = -t$. Because $f$ is piecewise linear, we obtain $\text{ord}_0(f) = 1$ by taking the limit $t \to 0^+$.

We construct a new function over $\Gamma$ by using the Abel–Jacobi mapping and the tropical theta function. Define the multi-valued function $\tilde{A} : \Gamma \to \mathbb{R}^g$ by $A : P \mapsto ((\beta_1, \gamma_P), \ldots, (\beta_g, \gamma_P))$, where $\gamma_P$ is a path over $\Gamma$ from $O$ to $P$. By definition, the $\tilde{A}$ is a lift of $A$. Consider the multi-valued function $\tilde{f} : P \mapsto \Theta(\tilde{A}(P); B)$ over $\Gamma$. 

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Lemma A.12. The degree $\text{ord}_P(f)$ does not depend on the choice of the branch of $f : \Gamma \to \mathbb{R}$. 

Proof. For a fixed point $P \in \Gamma$, let $z, z' \in \mathbb{R}^d$ be two values of the multi-valued function $\widetilde{A}$ at $P$. Then, there exists some $r \in \mathbb{Z}^d$ such that $z' = z + Br$. From (A.1), it is sufficient to prove

$$\text{ord}_P \left( \frac{1}{2} (r, Br) - (z, r) \right) = 0, \quad z = \widetilde{A}(P)$$

for any $r$. Let $e_i \in \mathbb{R}^d$ be the $i$th fundamental vector. Due to the equation $(z, e_i) = (\beta_i, \gamma_P)$ ($\gamma_P$ is a path from $O$ to $P$), the problem boils down to prove

$$\text{ord}_P((\beta_i, \gamma_P)) = 0.$$ 

Let $F$ be the multi-valued function $Q \mapsto (\beta_i, \gamma_Q)$. Take a small neighborhood $V$ of $P$. By retaking smaller $V$ if needed, we can assume $V = \bigcup_{\gamma} \gamma_0$ and $\gamma_P = \{D_{\gamma_P}\}$. Because $\beta_i$ is a closed path, there exist $2n$ indexes $\alpha_1', \ldots, \alpha_n', \alpha_1'', \ldots, \alpha_n''$ such that (i) $\beta_i$ comes to $P$ along to $\gamma_{\alpha_i'}$ (ii) $\beta_i$ goes from $P$ along to $\gamma_{\alpha_i''}$. By definition of $F$, we have

$$D_{\alpha_i'} F = -1, \quad D_{\alpha_i''} F = 1. $$

Therefore, $\text{ord}_P(F) = \sum_{i=1}^n (-1) + \sum_{i=1}^n (+1) = 0$. □

By this lemma, we can define the degree of multi-valued function $f$. We call a point $P$ which satisfies $\text{ord}_P(f) > 0$ a zero of $f$.

Lemma A.13 ([15]). The number of zeros of $f$ is $g = \text{genus } \Gamma$ (counting multiplicities).

Definition A.14. Let $Q_1, Q_2, \ldots, Q_k$ be the zeros of $f$ with multiplicity. The Riemann constant $\kappa$ is the element of $\text{Jac } \Gamma$ defined by $\kappa := A(Q_1 + \cdots + Q_k)$.

References

[1] Hatayama G, Hikami K, Inoue R, Kuniba A, Takagi T and Tokihiro T 2001 J. Math. Phys. 42 274–308
[2] Inoue R and Iwao S 2011 Tropical curve theory and integrable piecewise linear map arXiv:11115771
[3] Ilyenborg S, Mikhalkin G and Shustin E 2009 Tropical Algebraic Geometry (Oberwolfach Seminars vol 35) (Berlin: Birkhäuser)
[4] Iwao S, Mada J, Izuzumi M and Tokihiro T 2009 J. Phys. A: Math. Theor. 42 315209
[5] Iwao S 2010 J. Phys. A: Math. Theor. 43 155208
[6] Iwao S 2010 PhD Thesis Tokyo University
[7] Iwao S 2011 Two dimensional periodic box-ball system and its fundamental cycle arXiv:11024392
[8] Kirillov A N and Sakamoto R 2009 Lett. Math. Phys. 89 51–65
[9] Kirichev I M 1977 Funct. Anal. Appl. 11 12–26
[10] Kuma A, Takagi T and Takenouchi A 2006 Nucl. Phys. B 747 354–97
[11] Kuma A and Sakamoto R 2006 J. Stat. Mech. P09005
[12] Kuniba A, Sakamoto R and Yamada Y 2007 Nucl. Phys. B 786 207–66
[13] Mada J, Ishimi M and Tokihiro T 2006 J. Phys. A: Math. Gen. 39 L617
[14] Mada J, Izuzumi M and Tokihiro T 2008 J. Phys. A: Math. Theor. 41 175207
[15] Mikhalkin G and Zarkov I 2006 Tropical curve, their Jacobians and $\Theta$-functions arXiv:0612267
[16] Mumford D 1993 Tata Lectures on Theta II (Progress in Mathematics vol 43) (Boston, MA: Birkhäuser)
[17] Moerbeke P and Mumford D 1979 Acta Math. 143 93–154
[18] Murata M, Isojima S, Nobe A and Satsuma J 2006 J. Phys. A: Math. Gen. 39 L27–34
[19] Murata M 2009 J. Phys. A: Math. Theor. 42 115201
[20] Takagi T 2005 Nucl. Phys. B 707 577–601
[21] Takahashi D and Satsuma J 1990 J. Phys. Soc. Japan 59 3514–9
[22] Takahashi D 1992 On a fully discrete soliton system Nonlinear Evolution Equations and Dynamical Systems: Proc. NEEDS’91 (Baia Verde) (River Edge, NJ: World Science Publisher) pp 245–9

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[23] Takahashi D and Matsukidaira J 1997 J. Phys. A: Math. Gen. 30 L733–9
[24] Tsuda T 2008 Lett. Math. Phys. 85 65–78
[25] Tokihiro T, Takahashi D, Matsukidaira J and Satsuma J 1996 Phys. Rev. Lett. 76 3247–50
[26] Tokihiro T, Takahashi D and Matsukidaira J 2000 J. Phys. A: Math. Gen. 33 607–19
[27] Yura F and Tokihiro T 2002 J. Phys. A: Math. Gen. 35 3787–801
[28] Willox R, Nakata Y, Satsuma J, Ramani A and Grammaticos B 2010 J. Phys. A: Math. Theor. 43 482003