PROJECTIVITY VIA THE DUAL K"AHLER CONE - HUYBRECHTS' CRITERION

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ABSTRACT. In this note we give an elementary proof for a remarkable criterion due to Daniel Huybrechts for a K"ahler surface to be projective.

INTRODUCTION

One of the main idea in higher dimensional algebraic geometry is to study varieties through numerical properties of their cones, of which origin probably goes back to the Kleiman criterion, the duality between the ample cone and the so-called Kleiman-Mori cone, the cone of effective curves (cf. [KMM]).

Quite recently, Daniel Huybrechts took this idea into his study of hyperk"ahler manifolds and stated as a byproduct the following remarkable criterion to distinguish projective surfaces from the cone theoretical view point:

Huybrechts’ Criterion ([Hu, Remark 3.12 (iii)]). A compact K"ahler surface is projective if and only if the dual cone of the K"ahler cone contains an inner integral point. (For the precise definitions, see (1.4) in section 1.)

However his original proof relies on powerful but highly advanced techniques in complex analysis (Demailly’s singular Morse theory) and he himself asked in the same paper whether it is possible or not to prove this in a more elementary way.

The aim of this short note is to answer his question by giving a proof based on more or less familiar results on surfaces found now in standard books, [Beu], [BPV] and [GH]. Our proof is based on the notion of algebraic dimension while it is almost free from the classification of surfaces.

Of course it is very interesting to ask whether Huybrechts’ criterion also holds in higher dimensions. We will address to this question in a second part of this paper [OP].

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1. Preliminaries

(1.0). Thoughout this note, the term surface means a compact, connected complex manifold of dimension two. Let $S$ be a surface. The transcendental degree of the meromorphic function field of $S$ over $\mathbb{C}$ is called the algebraic dimension and is denoted by $a(S)$. It is well known that $a(S) \in \{0, 1, 2\}$ and $S$ is projective if and only if $a(S) = 2$.

(1.1). A Hermitian metric $g$ on $S$ is called Kähler if the associated positive real $(1, 1)$ form $\omega_g$ is $d$–closed. We call $\omega_g$ a Kähler form if $g$ is a Kähler metric. A surface is called Kähler if it admits at least one Kähler metric. Note that every projective surface is Kähler but the converse is not true in general.

Let $S$ be a Kähler surface.

(1.2). By definition, any Kähler metric $g$ on $S$ determines a de Rham cohomology class $[\omega_g]$. This class lies in the real $(1, 1)$ part $H^{1,1}(S, \mathbb{R})$ of the Hodge decomposition of $H^2(S, \mathbb{C})$. We often abbreviate $H^{1,1}(S, \mathbb{R})$ by $H^{1,1}$. We call an element $\eta \in H^{1,1}$ a Kähler class if it is represented by a Kähler form, that is, in the case where there exists a Kähler metric $g$ such that $\eta = [\omega_g]$.

(1.3). The real vector space $H^{1,1}$ carries a natural symmetric bilinear form $(\cdot, \cdot)$ induced by the cup product on the integral cohomology group $H^2(S, \mathbb{Z})$. It is well known that $(\cdot, \cdot)$ on $H^{1,1}$ is non-degenerate and is of signature $(1, h^{1,1}(S) - 1)$. We also regard the finite dimensional real vector space $H^{1,1}$ as a linear topological space using some norm $|\cdot|$. Therefore we can speak of the closure $\overline{A}$ of $A \subset H^{1,1}$. For $a \in H^{1,1}$ and for a positive real number $\epsilon > 0$, we set

$$B_{\epsilon}(a) := \{x \in H^{1,1} | |x - a| \leq \epsilon\}.$$ 

Furthermore, let $U_{\epsilon}(a)$ be the interior of $B_{\epsilon}(a)$ and $\partial B_{\epsilon}(a)$ its boundary.

(1.4).

1. The Kähler cone $\mathcal{K}(S)$ of $S$ is the subset of $H^{1,1}$ consisting of the Kähler classes of $S$. By definition, $\mathcal{K}(S)$ is a convex cone of $H^{1,1}$. It is also well known that $\mathcal{K}(S)$ is an open subset of $H^{1,1}$.

2. The dual cone $\mathcal{K}^*(S)$ of the Kähler cone $\mathcal{K}(S)$ is the set of elements $x \in H^{1,1}$ such that $(x, \eta) > 0$ for any $\eta \in \mathcal{K}(S)$.

3. An element $x$ of $\mathcal{K}^*(S)$ is called integral if $x \in \mathcal{K}^*(S) \cap \imath^*H^2(S, \mathbb{Z})$, where $\imath : \mathbb{Z} \to \mathbb{R}$ is a natural inclusion of sheaves. An integral element is nothing but an element of $\mathcal{K}^*(S) \cap NS(S)$ (cf.(1.7)).
An element $x$ of $\mathcal{K}^*(S)$ is called an inner point if there exists a positive real number $\epsilon > 0$ such that $U_\epsilon(x) \subset \mathcal{K}(S)^\ast$.

**Lemma (1.5).** Let $(H, |\cdot|)$ be a finite dimensional, real normed vector space equipped with a real valued, non-degenerate bilinear form $(\cdot, \cdot)$. Let $K \subset H$ be a non-empty convex subset such that $0 \notin K$. Set $K^* \subset H$ to be the dual of $K$ with respect to $(\cdot, \cdot)$. Let $x \in H$. Then $x$ is an inner point of $K^*$ if and only if there exists a positive real number $r > 0$ such that $(x, \eta) \geq r|\eta|$ for all $\eta \in K^*$.

**Proof.** This will follow from the compactness of the space $B_\epsilon(x) \times (\overline{K} \cap \partial B_1(0))$. □

The following direct consequence is crucial for our proof:

**Corollary (1.6).** Let $x \in H^{1,1}$. Then $x$ is an inner point of $\mathcal{K}^*(S)$ if and only if there exists a positive real number $r > 0$ such that $(x, \eta) \geq r|\eta|$ for all $\eta \in \mathcal{K}(S)$. □

(1.7). The group $H^{1,1} \cap \iota^*H^2(S, \mathbb{Z})$ is called the Néron-Severi group of $S$ and is denoted by $NS(S)$. The rank of $NS(S)$ is called the Picard number of $S$ and is written by $\rho(S)$. By the Lefschetz $(1, 1)$ Theorem, each element of $NS(S)$ is represented by the first Chern class of some line bundle. However, contrary to the projective case, the natural map from the group of Cartier divisors to the Picard group is not surjective in general. So, we CAN NOT say that each element of $NS(S)$ is represented by a divisor in the Kähler category.

2. Kähler cones of K3 surfaces and complex tori

**Theorem (2.1) [Bea, Page 123, Theorem 2].** Let $S$ be a K3 surface, that is, a (Kähler) surface such that $K_S = 0$ in $\text{Pic}(S)$ and that $\pi_1(S) = \{1\}$. Let $C^+(S)$ be the connected component of the space $\{x \in H^{1,1}|(x, x) > 0\}$ which contains the Kähler classes. Then the Kähler cone $\mathcal{K}(S)$ coincides with the subspace $\mathcal{K}^\ast(S)$ of $C^+(S)$ defined by $(x, [C]) > 0$ for all non-singular rational curves $C$ in $S$, that is,

$$\mathcal{K}(S) = \mathcal{K}^\ast(S) := \{x \in C^+(S)|(x, [C]) > 0 \text{ for all } C \simeq \mathbb{P}^1 \text{ in } S\}. \square$$

**Remark.** It is clear that $\mathcal{K}(S) \subset \mathcal{K}^\ast(S)$. However, the other inclusion $\mathcal{K}^\ast(S) \subset \mathcal{K}(S)$ is highly non-trivial. For details, we refer to [Bea].

**Theorem (2.2).** Let $S$ be a complex torus of dimension 2. Let $C^+(S)$ be the connected component of the space $\{x \in H^{1,1}|(x, x) > 0\}$ which contains the Kähler classes. Then $\mathcal{K}(S) = C^+(S)$.

**Remark.** This result should be known. However, the authors could not find any references. The present proof was communicated to us by D. Huybrechts; our original proof is more complicated and works by reduction to the algebraic case.

**Proof.** First notice that any $(1, 1)$– class can be represented by a form with constant coefficients. Suppose $\mathcal{K}(S) \neq C^+(S)$. Since $\mathcal{K}(S) \subset C^+(S)$, we find a constant $(1, 1)$–form
\[ \phi \in C^+(S) \cap \partial K(S). \] Then \( \phi \) is semipositive but not positive. Therefore \( \phi^2 = 0 \), contradiction. \( \square \)

**Lemma (2.3).** Let \( S \) be a minimal Kähler surface. Assume that \( a(S) = 0 \). Then \( S \) is either a K3 surface or a complex torus of dimension 2.

Proof. This is of course well known, see e.g. [BPV]. We give a proof to convince the reader that no deep result from classification theory is involved. Since \( a(S) = 0 \), we have \( \kappa(S) = 0 \) or \( -\infty \), where \( \kappa(S) \) is the Kodaira dimension of \( S \). Moreover, if \( h^0(K_S) = 0 \), then by the Serre duality \( h^2(O_S) = 0 \) and \( S \) is then projective by the Kodaira criterion, a contradiction. Therefore \( K_S = 0 \) in \( \text{Pic}(S) \) by the minimality of \( S \). Since \( S \) is Kähler, this gives the result. \( \square \)

In order to prove Huybrechts’ criterion, we also need to know the structure of the Néron-Severi groups of K3 surfaces and complex tori of algebraic dimension zero.

**Proposition (2.4).** Let \( S \) be a K3 surface. Assume that \( a(S) = 0 \). Then,

1. \( \text{Pic}(S) \) and \( \text{NS}(S) \) are torsion free and are isomorphic under \( c_1 \). Moreover \( \text{NS}(S) \otimes \mathbb{R} \) is negative definite with respect to \( (*,*) \).
2. \( S \) contains at most 19 distinct smooth rational curves and contains no other curves.

Proof of (1). The first part of the assertion is well known. Using \( a(S) = 0 \) and the Riemann-Roch Theorem, we readily see that \( L^2 < 0 \) for all \( L \in \text{Pic}(S) - \{0\} \). Since \( (*,*) \) is defined over \( \mathbb{Z} \), this implies the result. \( \square \)

Proof of (2). Let \( C \) be an irreducible curve on \( S \). Then \( C \cong \mathbb{P}^1 \), because \( 0 > C^2 = (K_S + C.C) = 2p_a(C) - 2 \) by (1) and the adjunction formula.

**Claim 1.** Let \( C_1, ..., C_m \) be \( m \) distinct irreducible curves on \( S \). Then \([C_1], ..., [C_m]\) are linearly independent in \( \text{NS}(S) \otimes \mathbb{R} \).

Proof. Since the classes \([C_i]\) defined over \( \mathbb{Z} \), it is enough to show that if \( \sum_{i \in I} p_i[C_i] = \sum_{j \in J} q_j[C_j] \), where \( I \cap J = \emptyset \), \( p_i \in \mathbb{Z}_{\geq 0} \) and \( q_j \in \mathbb{Z}_{\geq 0} \) then \( p_i = q_j = 0 \).

By (1), we have

\[
0 \geq (\sum_{i \in I} p_i[C_i], \sum_{i \in I} p_i[C_i]) = (\sum_{i \in I} p_i[C_i], \sum_{j \in J} q_j[C_j]) \geq 0.
\]

Therefore, \( (\sum_{i \in I} p_i[C_i], \sum_{i \in I} p_i[C_i]) = 0 \). Then again by (1), we have

\[
[\sum_{i \in I} p_i C_i] = \sum_{i \in I} p_i[C_i] = 0
\]

in \( \text{NS}(S) \) and \( \sum_{i \in I} p_i C_i = 0 \) in \( \text{Pic}(S) \). This is possible only in the case where \( p_i = 0 \) for all \( i \in I \). Similarly, \( q_j = 0 \) for all \( j \in J \). \( \square \)
Claim 2. $S$ contains at most 19 distinct $\mathbb{P}^1$’s.

Proof. Recall that $(H^{1,1}, (\ast, \ast))$ is of dimension 20 and of signature $(1, 19)$. Assume that $S$ contains more than or equal to 20 distinct $\mathbb{P}^1$’s among them. Then, since $\mathbb{R}([C_1], \ldots, [C_{20}]) \subset NS(S) \otimes \mathbb{R} \subset H^{1,1}$ and $\dim_{\mathbb{R}} \mathbb{R}([C_1], \ldots, [C_{20}]) = 20 = \dim_{\mathbb{R}} H^{1,1}$ by Claim 1, we get $NS(S) \otimes \mathbb{R} = H^{1,1}$. However, $NS(S) \otimes \mathbb{R}$ is of signature $(0, 20)$ by (1) while $H^{1,1}$ is of signature $(1, 19)$, a contradiction. □

Now we are done. □

Remark. For each integer $m$ such that $0 \leq m \leq 19$, there actually exists a K3 surface of $a(S) = 0$ which contains exactly $m$ distinct $\mathbb{P}^1$’s and no other curves.

Construction. By [OZ], there exists a projective K3 surface $T$ which contains 19 $\mathbb{P}^1$’s, say, $C_1, \ldots, C_{19}$ whose intersection matrix $(C_i, C_j)$ is of type $A_{19}$. Let $f : \mathcal{X} \to \mathcal{B}$ be the Kuranishi family of $T$ and identify the base space $\mathcal{B}$ with an open set $\mathcal{U}$ of the period domain $\mathcal{P}$ of the K3 surfaces under some marking $\tau : R^2 f_* \mathbb{Z} \simeq \Lambda_{K3} \times K$:

$$\mathcal{B} \simeq \mathcal{U} \subset \mathcal{P} := \{[\omega] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) | (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0\}.$$ 

Let $c_i$ be the element of the K3 lattice $\Lambda_{K3}$ which corresponds to the class $[C_i]$ under the marking $\tau$. Define the subset $c_i^+ \subset \mathcal{U}$ by $c_i^+ := \{[\omega] \in \mathcal{U} | (\omega, c_i) = 0\}$. Let $0 \leq m \leq 19$ and choose a very general point $P$ of the space $c_1^+ \cap \ldots \cap c_m^+$. (This space is of dimension $20 - m > 0$ by (2.4)). Then the fiber $\mathcal{X}_P$ is a K3 surface which contains exactly $m$ distinct $\mathbb{P}^1$’s whose intersection matrix is of type $A_m$ and has no other curves. This also implies $a(\mathcal{X}_P) = 0$. □

Proposition (2.5). Let $S$ be a complex torus of dimension 2. Assume that $a(S) = 0$. Then, $NS(S) \otimes \mathbb{R}$ is negative semi-definite with respect to $(\ast, \ast)$.

Proof. Obvious. □

3. PROOF OF HUYBRECHTS’ CRITERION

Proof of the “only if” part. Any ample class gives a desired point. □

Proof of the “if” part. Let $S$ be a Kähler surface which has an inner integral point of $K^*(S)$. It is sufficient to show that $a(S) \neq 0, 1$.

Lemma (3.1). $a(S) \neq 1$.

Proof. Assume to the contrary that $a(S) = 1$ and take the algebraic reduction $f : S \to C$, which, in the surface case, is a surjective morphism to a non-singular curve with connected fibers. Let $F$ be a general fiber of $f$ and set $f(F) = P$. Then $[F] = [f^*(P)] \in NS(S)$, where $P$ is regarded as a divisor on $C$. Since $P$ is ample on $C$, the class $[P]$ is represented by a positive definite real $d$–closed $(1, 1)$ form $\theta$. Set $\Theta := f^* \theta$. Then $[F] = [\Theta]$ and $\Theta$
is positive semi-definite at each point \( Q \in S \). Therefore for a Kähler form \( \omega \) and for any \( \epsilon > 0 \), we have \( [\Theta + \epsilon \omega] \in K(S) \). Thus, \( [F] = \lim_{\epsilon \to 0} [\Theta + \epsilon \omega] \in \overline{K(S)} \). Moreover, since \((|F|,|\omega|) = \int_F \omega > 0\), we see that \([F] \neq 0\). Let \( M \) be an inner integral point of \( K^*(S) \). Then, by (1.6), we have \((M,F)^2 > 0\), whence \((M + n[F])^2 = M^2 + 2n(M,F) > 0\) for a large integer \( n \). Since \( M + n[F] \in NS(S) \), this implies \( a(S) = 2 \), a contradiction. \( \Box \)

The next Lemma reduces our problem to the case of minimal surfaces.

**Lemma (3.2).** Let \( \tau : S \to T \) be the blow down of a \((-1)\)-curve \( E \). Then,

1. \( S \) is projective if and only if \( T \) is projective.
2. \( S \) is Kähler if and only if \( T \) is Kähler.
3. Assume that there exists an inner integral point \( x \) of \( K^*(S) \). Then there also exists an inner integral point of \( K^*(T) \).

**Proof.** The assertions (1) and (2) are well known. (However, it might be worth reminding here that the “only if” part of both (1) and (2) is false in general if dimension is three or higher and the center is not a point. One of instructive counterexamples is found in [Og].)

Let us show the assertion (3). Recall that \( H^2(S,K) = \tau^*H^2(T,K) \oplus K[E] \simeq H^2(T,K) \oplus K[E] \) for \( K = \mathbb{Z}, \mathbb{R} \). Moreover, this equality and isomorphism are compatible with the cup product and the Hodge decompositions. Let us regard \( H^{1,1}(S) \) as a normed space by the product norm of \( H^{1,1}(T) \) and \( \mathbb{R}[E] \). Set \( e := [E] \). Then the inner integral point \( x \in K^*(S) \) of the form \( x = \tau^*y + ae \) where \( y \in NS(T) \) and \( a \in \mathbb{Z} \). We show that \( y \) is an inner point of \( K^*(T) \). Let \( \sigma \in K(T) \). Then \( \tau^*\sigma \neq 0 \) and \( \tau^*\sigma - ee \in K(S) \) for all sufficiently small positive real numbers \( \epsilon \). Therefore \( \tau^*\sigma \in \overline{K(S)} \). Since \( x \) is an inner point of \( K^*(S) \), there exists \( r > 0 \) such that \((x,\eta) \geq r|\eta| \) for all \( \eta \in K(S) \) by (1.6). In particular, \((x,\tau^*\sigma) \geq r|\tau^*\sigma| \). On the other hand, using \( x = y + ae \) and applying the projection formula, we calculate \((x,\tau^*\sigma) = (y,\sigma) \). This together with the compatibility of the norms implies \((y,\sigma) \geq r|\sigma| \) for all \( \sigma \in K(T) \), hence for all \( \sigma \in \overline{K(T)} \). \( \Box \)

By virtue of (2.3), (3.1) and (3.2), in order to conclude the “if” part, it is now sufficient to show the following:

**Lemma (3.3).**

1. Let \( S \) be a K3 surface. Assume that \( K^*(S) \) contains an inner integral point \( x \). Then \( a(S) \neq 0 \).
2. Let \( S \) be a complex torus of dimension 2. Assume that \( K^*(S) \) contains an inner integral point \( x \). Then \( a(S) \neq 0 \).

**Proof of (1).** Assume to the contrary that \( a(S) = 0 \). Let \( C_1, \ldots, C_m \) \((0 \leq m \leq 19)\) denote the distinct smooth rational curves on \( S \) ((2.4)(2)). We argue dividing into two cases:

   Case 1. \( x \in \mathbb{R}([C_1], \ldots, [C_m]) \);
   Case 2. \( x \not\in \mathbb{R}([C_1], \ldots, [C_m]) \).
Case 1. By (2.4), the subspace of $H^{1,1}$

$$[C_1] \perp \cap ... \cap [C_m] \perp$$

is of signature $(1, 19 - m)$ (where $\perp$ is taken with respect to $(\ast, \ast)$). Therefore

$$[C_1] \perp \cap ... \cap [C_m] \perp \cap C^+(S) \neq \emptyset.$$  

Let $\eta$ be an element of this set. Then by (2.1), $\eta \in \overline{K(S)}$ and $\eta \neq 0$. On the other hand, by our assumption, we have $(x.\eta) = 0$. This contradicts (1.6).

Case 2. In this case $m \leq 18$. Indeed, otherwise we would have $\mathbb{R}\langle x, [C_1], ..., [C_{19}] \rangle = NS(S) \otimes \mathbb{R} = H^{1,1}$ and would get the same contradiction as in Claim 2 of (2.4). Therefore the subspace

$$x^+ \cap [C_1] \perp \cap ... \cap [C_m] \perp$$

is of signature $(1, 19 - m - 1)$ and then

$$x^+ \cap [C_1] \perp \cap ... \cap [C_m] \perp \cap C^+(S) \neq \emptyset.$$  

Let $\eta$ be an element of $x^+ \cap [C_1] \perp \cap ... \cap [C_m] \perp \cap C^+(S)$. Then by (2.1), $\eta \in \overline{K(S)}$ and $\eta \neq 0$. On the other hand, by the choice of $\eta$, we have $(x.\eta) = 0$, a contradiction.  

Proof of (2). Note that $(H^{1,1}, (\ast, \ast))$ is non-degenerate and of signature $(1, 3)$. Assume to the contrary that $a(S) = 0$. Then $x^2 \leq 0$ and $x \neq 0$ by (2.5). We argue dividing into two cases:

Case 1. $x^2 = 0$;

Case 2. $x^2 < 0$.

Case 1. Since $x^2 = 0$ and $(x.\eta) > 0$ for all $\eta \in \overline{K(S)}$, we have $x \in \overline{C^+(S)}$, whence $x \in \overline{K(S)}$ by (2.2). However, then $x^2 = (x.x) > 0$ by (1.6), a contradiction.

Case 2. Since $x^2 < 0$, the subspace $x^+ \subset H^{1,1}$ is of index $(1, 2)$. Combining this with (2.2), we have $x^+ \cap \overline{K(S)} = x^+ \cap C^+(S) \neq \emptyset$. Therefore there exists an element $\eta \in \overline{K(S)}$ such that $(x.\eta) = 0$. However, this contradicts $x \in \overline{K^*(S)}$.  

□
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