A stability result for translating space-like graphs in Lorentz manifolds

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Abstract

In this paper, we investigate space-like graphs defined over a domain \( \Omega \subset M^n \) in the Lorentz manifold \( M^n \times \mathbb{R} \) with the metric \( -ds^2 + \sigma \), where \( M^n \) is a complete Riemannian \( n \)-manifold with the metric \( \sigma \), \( \Omega \) has piecewise smooth boundary, and \( \mathbb{R} \) denotes the Euclidean 1-space. We can prove an interesting stability result for translating space-like graphs in \( M^n \times \mathbb{R} \) under a conformal transformation.

1 Introduction

Recent years, the study of submanifolds of constant curvature in product manifolds attracts many geometers’ attention. For instance, Hopf in 1955 discovered that the complexification of the traceless part of the second fundamental form of an immersed surface \( \mathbb{H}^2 \), with CMC \( H \), in \( \mathbb{R}^3 \) is a holomorphic quadratic differential \( Q \) on \( \mathbb{H}^2 \), and then he used this observation to get his well-known conclusion that any immersed CMC sphere \( S^2 \to \mathbb{R}^3 \) is a standard distance sphere with radius \( 1/H \). By introducing a generalized quadratic differential \( \tilde{Q} \) for immersed surfaces \( \mathbb{H}^2 \) in product spaces \( S^2 \times \mathbb{R} \) and \( \mathbb{H}^2 \times \mathbb{R} \), with \( S^2, \mathbb{H}^2 \) the 2-dimensional sphere and hyperbolic surface respectively, Abresch and Rosenberg [1] can extend Hopf’s result to CMC spheres in these target spaces. Meeks and Rosenberg [12] successfully classified stable properly embedded orientable minimal surfaces in the product space \( N \times \mathbb{R} \), where \( N \) is a closed orientable Riemannian surface. In fact, they proved that such a surface must be a product of a stable embedded geodesic on \( N \) with \( \mathbb{R} \), a minimal graph over a region of \( N \) bounded by stable geodesics, \( N \times \{t\} \) for some \( t \in \mathbb{R} \), or is in a moduli space of periodic multigraphs parameterized by \( P \times \mathbb{R}^+ \), where \( P \) is the set of primitive (non-multiple) homology classes in \( H_1(N) \). Mazet, Rodríguez and Rosenberg [11] analyzed properties of periodic minimal or CMC surfaces in the product manifold \( \mathbb{H}^2 \times \mathbb{R} \), and they also construct examples of periodic minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \). In [13], Rosenberg, Schulze and Spruck showed

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that a properly immersed minimal hypersurface in $N \times \mathbb{R}^+$ equals some slice $N \times \{c\}$ when $N$ is a complete, recurrent $n$-dimensional Riemannian manifold with bounded curvature. Very recently, Gao, Mao and Song [9] proved the existence and uniqueness of solutions to the CMC equation with nonzero Neumann boundary data in product manifold $N^n \times \mathbb{R}$, where $N^n$ is an $n$-dimensional ($n \geq 2$) complete Riemannian manifold with nonnegative Ricci curvature. Equivalently, this conclusion gives the existence of CMC graphic hypersurfaces defined over a compact strictly convex domain $\Omega \subset N^n$ and having nonvanishing contact angle. Of course, for more information, readers can check references therein of these papers. Hence, it is interesting and important to consider submanifolds of constant curvature in the product manifold of type $N^n \times \mathbb{R}$.

Inspired by Shahriyari’s progress on complete translating graphs in $\mathbb{R}^3$ (see [16] for details) and the Jenkins-Serrin theory on minimal graphs and CMC graphs, Zhou [18] considered complete translating, minimal and CMC graphs in $3$-dimensional product manifold and the Jenkins-Serrin theory on minimal graphs and CMC graphs, Zhou [18] considered complete, recurrent $n$-submanifolds of constant curvature in the product manifold of type $M^n \times \mathbb{R}$ where $M^n$ is a complete Riemannian surface, and successfully showed the boundary behavior of $\Omega$. This conclusion extends some of Shahriyari’s conclusions in [16] from the Euclidean $3$-space $\mathbb{R}^3$ to the setting of $3$-dimensional product space $N^2 \times \mathbb{R}$.

Stability plays an important role in the study of minimal or CMC hypersurfaces in Euclidean space or, more generally, product manifolds. For instance, if stability assumption was made, nice curvature estimates or classification results for minimal or CMC surfaces can be obtained – see, e.g., [7, 8, 12, 15, 17, 18].

The famous Bernstein theorem (holds only for $n \leq 7$) in the Euclidean space says that the entire nonparametric minimal hypersurfaces in $\mathbb{R}^{n+1}$, $n \leq 7$, are hyperplanes (see [14]). Calabi [3] (for $n \leq 4$), Cheng-Yau [4] (for all $n$) proved that a complete maximal spacelike hypersurface in the flat Lorentz-Minkowski $(n+1)$-space $L^{n+1} \equiv \mathbb{R}^{n+1}_1$ is totally geodesic. Therefore, specially, the only entire nonparametric maximal space-like hypersurfaces in $\mathbb{R}^n_1$ are space-like hyperplanes. This interesting example shows that it is meaningful to ask whether classical results in Riemannian geometry (or specially the Euclidean space) can be transplanted to pseudo-Riemannian geometry (or specially the pseudo-Euclidean space) or not. This example also shows that, in some aspect, there exists essential difference between the Euclidean space and the pseudo-Euclidean space.

Motivated by the previous experience, we try to get stability conclusions in Lorentz manifolds of type $M^n \times \mathbb{R}$. Fortunately, so far, we get one – see Theorem 1.1 for details. In order to state our conclusion clearly, we need to introduce some notions first.

Throughout this paper, denote by $M^n \times \mathbb{R}$, with the metric $-ds^2 + \sigma$, an $(n+1)$-dimensional $(n \geq 2)$ Lorentz manifold where $M^n$ is a complete Riemannian $n$-manifold with the metric $\sigma$. For a domain $\Omega \subset M^n$ with piecewise smooth boundary, a translating space-like graph in the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$ is the space-like graph of $u(x)$, where $u(x) : \Omega \rightarrow \mathbb{R}$ is a solution of the following mean curvature type equation

$$\text{div} \left( \frac{Du}{\sqrt{1 - |Du|^2}} \right) = \frac{c}{\sqrt{1 - |Du|^2}}, \quad (1.1)$$

where $D$ is a covariant derivative operator on $M^n$, $\text{div}(\cdot)$ denotes the divergence operator, and $c$ is a constant. Translating space-like graphs by mean curvature flow (MCF for short) in the Lorentz manifold $M^n \times \mathbb{R}$ are translating surfaces that can be viewed as a space-like graph of a function over a domain. In fact, let $\{x, u(x)\}$ be a space-like graphic surface defined over $\Omega \subset M^n$ in the Lorentz manifold $M^n \times \mathbb{R}$, and then, since the mean curvature of the space-like surface is (see [5],
the graph of $u$ is a vertically translating space-like with constant speed $c$ if and only if $u$ is a solution to the equation (1.1). Recently, Mao and his collaborators [5] showed that along the nonparametric MCF with prescribed contact angle boundary condition in the Lorentz 3-manifold $M^2 \times \mathbb{R}$, if $M^2$ has nonnegative Gaussian curvature, then the evolution of space-like graphs over compact strictly convex domains in $M^2$ exists for all the time and solutions of the flow converge to ones moving only by translation. Translating solutions play an important role in the study of type-II singularities of the MCF. For instance, Angenent and Velázquez [2, 3] gave some examples of convergence which by translation. Translating solutions play an important role in the study of type-II singularities of the MCF there are modeled by translating surfaces.

Denote by $M^n \times \mathbb{R}$ the $(n+1)$-dimensional pseudo-Riemannian manifold
\[
\{(x,s) | x \in M^n, s \in \mathbb{R}\}
\]
equipped with the weighted metric $e^{cs}(-ds^2 + \sigma_{ij}dx^idx^j)$. Clearly, $\widehat{M^n} \times \mathbb{R}$ can be achieved by the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$ with a conformal transformation to its Lorentzian metric. Here, we have used Einstein summation convention, that is, summation should be done to repeated subscripts and superscripts. In the sequel, without specification, Einstein summation convention will be always used. We can prove a stability result for translating space-like graphs as follows:

**Theorem 1.1.** Assume that $u(x)$ is a solution to (1.1). Then $\Sigma = \{x, u(x) | x \in \Omega\}$ is a stable, maximal space-like graph in $M^n \times \mathbb{R}$.

The paper is organized as follows. In Section 2, some useful formulas for space-like hypersurfaces in a Lorentz manifold will be recalled. The proof of Theorem 1.1 will be given in Section 3. Meanwhile, as a byproduct, a convergence result related to maximal, CMC or translating space-like graphs in Lorentz manifolds will also be shown. In Section 4, examples related to the existence of translating space-like graphs in the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$ will be introduced.

## 2 Geometry of space-like hypersurfaces in a Lorentz manifold

Given an $(n+1)$-dimensional Lorentz manifold $(\overline{M}^{n+1}, \overline{g})$, with the metric $\overline{g}$, and its space-like hypersurface $M^n$. For any $p \in M^n$, one can choose a local Lorentzian orthonormal frame field $\{e_0, e_1, e_2, \ldots, e_n\}$ around $p$ such that, restricted to $M^n$, $e_1, e_2, \ldots, e_n$ form orthonormal frames tangent to $M^n$. Taking the dual coframe field $\{w_0, w_1, w_2, \ldots, w_n\}$ such that the Lorentzian metric $\overline{g}$ can be written as $\overline{g} = -w_0^2 + \sum_{i=1}^n w_i^2$. Making the convention on the range of indices
\[
0 \leq \alpha, \beta, \gamma, \ldots \leq n; \quad 1 \leq i, j, k \ldots \leq n,
\]
and doing differentials to forms $w_\alpha$, one can easily get the following structure equations
\[
\begin{align*}
\text{(Gauss equation)} & \quad R_{ijkl} = \overline{R}_{ijkl} - (h_{ij}h_{kl} - h_{ik}h_{jl}), \\
\text{(Codazzi equation)} & \quad h_{ij,k} - h_{ik,j} = \overline{R}_{ikjl}, \\
\text{(Ricci identity)} & \quad h_{ij,kl} - h_{ij,lk} = \sum_{m=1}^n h_{mj}R_{mkil} + \sum_{m=1}^n h_{im}R_{mjkl}.
\end{align*}
\]
and the Laplacian of the second fundamental form $h_{ij}$ of $M^n$ as follows

$$
\Delta h_{ij} = \sum_{k=1}^{n} \left( h_{kk,ij} + \overline{R}_{0ik,j} + \overline{R}_{0ij,k} \right) + \sum_{k=1}^{n} \left( h_{kk} \overline{R}_{0ij,0} + h_{ij} \overline{R}_{00,k} \right) + \\
\sum_{m,k=1}^{n} \left( h_{mj} \overline{R}_{mk} + 2h_{mk} \overline{R}_{mj} + h_{mi} \overline{R}_{mk} \right) + \\
- \sum_{m,k=1}^{n} \left( h_{mi} h_{mj} h_{kk} + h_{km} h_{mj} h_{ik} - h_{km} h_{mk} h_{ij} - h_{mi} h_{mk} h_{ij} \right),
$$

(2.4)

where $R$ and $\overline{R}$ are the curvature tensors of $M^n$ and $\overline{M}^{n+1}$ respectively, $A := h_{ij} w_i w_j$ is the second fundamental form with $h_{ij}$ the coefficient components of the tensor $A$, $\Delta$ is the Laplacian on the hypersurface $M^n$, and, as usual, the comma “,” in subscript of a given tensor means doing covariant derivatives – this convention will also be used in the sequel. For detailed derivation of the above formulae, we refer readers to, e.g., [10, Section 2].

Clearly, in our setting here, all formulas mentioned in this section can be used directly with $\overline{M}^{n+1} = M^n \times \mathbb{R}$.

### 3 Stability

Similar to the calculation in [5, Sect. 1], for the space-like graph $\Sigma = \{(x, u(x)) | x \in \Omega\}$, defined over $\Omega \subset M^n$, in the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$ with the metric $\overline{g} := \sigma_{ij} dw^i \otimes dw^j - ds \otimes ds$, tangent vectors are given by

$$X_i = \partial_i + D_i u \partial_s, \quad i = 1, 2, \ldots, n,$$

and the corresponding upward unit normal vector is given by

$$\overline{v} = \frac{1}{\sqrt{1 - |Du|^2}} \left( \partial_s + D^j u \partial_j \right),$$

where $D^j u = \sigma^{ij} D_i u$. Denote by $\nabla$ the gradient operator on $M^n \times \mathbb{R}$, and then the second fundamental form $h_{ij} dw^i \otimes dw^j$ of $\Sigma$ is given by

$$h_{ij} = -\langle \nabla X_i, X_j, \overline{v} \rangle = \frac{1}{\sqrt{1 - |Du|^2}} D_i D_j u.$$

Moreover, the scalar mean curvature of $\Sigma$ is

$$H = \sum_{i=1}^{n} h_{ii} = \frac{1}{\sqrt{1 - |Du|^2}} \left( \sum_{i,k=1}^{n} g^{ik} D_k D_i u \right) = \frac{\sum_{i,k=1}^{n} \left( \sigma^{ik} + \frac{D^i u D_k u}{1 - |Du|^2} \right) D_k D_i u}{\sqrt{1 - |Du|^2}} = \text{div} \left( \frac{Du}{\sqrt{1 - |Du|^2}} \right),
$$

(3.1)

where $g^{ik}$ is the inverse of the induced Riemannian metric $g$ on the space-like graph $\Sigma$. Denote by $\Theta$ the angle function of $\Sigma$, and then using (1.1), the above equality can be written equivalently as

$$H = -c \Theta = -c \langle \overline{v}, \partial_s \rangle.
$$

(3.2)
Proof of Theorem 1.1. The area functional of $\tilde{M}^n \times \mathbb{R}$ is given by

$$F(\Sigma) = \int_\Sigma e^{cs}d\mu,$$

where $d\mu$ is the volume element of $\Sigma$ induced by the metric $g$ of the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$. Let $\Sigma_r$ be a family of surfaces satisfying

$$\frac{\partial \Sigma_r}{\partial r} \bigg|_{r=0} = \phi \vec{v} \quad \text{with} \quad \Sigma_0 = \Sigma,$$

(3.3)

where $\phi(x)$ is a smooth function defined on $\Sigma$ with compact support. Treating $\Sigma_r$ as a curvature flow of $\Sigma$ in the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$, and by direct calculation, it follows that:

Lemma 3.1. Along the curvature flow (3.3), we have

$$\frac{\partial \vec{v}}{\partial r} \bigg|_{r=0} = \nabla \phi,$$

$$\frac{\partial H}{\partial r} \bigg|_{r=0} = \Delta \phi - (|A|^2 + \text{Ric}(\vec{v}, \vec{v})) \phi,$$

(3.4)

where, following the convention used in Section 2, $\nabla$ and $\Delta$ denote the covariant derivative and the Laplacian of $\Sigma$ respectively, and $\text{Ric}(\cdot, \cdot)$ stands for the Ricci tensor of the ambient space $M^n \times \mathbb{R}$.

Proof. First, we have

$$\frac{\partial \vec{v}}{\partial r} \bigg|_{r=0} = \left\langle \frac{\partial \vec{v}}{\partial r}(\Sigma_r),_i \right\rangle g^{ik}(\Sigma_r)_k \bigg|_{r=0}$$

$$= - \left\langle \vec{v}, (\phi \vec{v}),_i \right\rangle g^{ik}(\Sigma_r)_k \bigg|_{r=0}$$

$$= \phi, g^{ik}(\Sigma_r)_k \bigg|_{r=0} = \nabla \phi,$$

where, following the convention used in Section 2, $(\cdot)_k$ means doing covariant derivative with respect to the tangent vector $X_k$ on the translating space-like graph $\Sigma$.

Second, we have

$$\frac{\partial g_{lm}}{\partial r} \bigg|_{r=0} = \frac{\partial}{\partial r} \left\langle (\Sigma_r)_l, (\Sigma_r)_m \right\rangle \bigg|_{r=0}$$

$$= 2 \left\langle (\phi \vec{v}),_l, (\Sigma_r)_m \right\rangle \bigg|_{r=0}$$

$$= -2 \phi \left\langle \vec{v}, (\Sigma_r),_l m \right\rangle \bigg|_{r=0} = 2 \phi h_{ij},$$
and
\[
\frac{\partial h_{ij}}{\partial r} \bigg|_{r=0} = -\frac{\partial}{\partial r} \left( \frac{\phi_i(\Sigma_r), m g^{ml}}{\phi_j(\Sigma_r), ij} \right) \bigg|_{r=0}
\]
\[
= -\left( \phi_i(\Sigma_r), m g^{ml}, \phi_j(\Sigma_r), ij \right) \bigg|_{r=0} - \left( \phi_j(\Sigma_r), ij \right) \bigg|_{r=0}
\]
\[
= -\left( \phi_i(\Sigma_r), m g^{ml}, \Gamma^k_{ij}(\Sigma_r) + h_{ij} \nu \right) \bigg|_{r=0} - \left( \phi_j(\Sigma_r), ij \right) \bigg|_{r=0}
\]
\[
= -\Gamma^k_{ij} \phi_{k} + \phi_{ij} + \phi h_{ij} g^{lm} h_{im}
\]
\[
= \nabla_i \nabla_j \phi + \phi h_{ij} g^{lm} h_{im},
\]
where, as usual, \(\Gamma^k_{ij}\) denote Christoffel symbols determined by the metric \(g\). By (2.1), (2.2), (2.4) and Simon’s identity of \(\phi\), we have
\[
g^{ij} \frac{\partial h_{ij}}{\partial r} \bigg|_{r=0} = \Delta \phi + |A|^2 \phi - \overline{\text{Ric}}(\nu, \nu) \phi,
\]
and then
\[
\frac{\partial H}{\partial r} \bigg|_{r=0} = \frac{\partial}{\partial r} \left( g^{ij} h_{ij} \right) \bigg|_{r=0}
\]
\[
= -g^{il} \frac{\partial g^{lm}}{\partial r} |_{r=0} g^{mj} h_{ij} + g^{il} \frac{\partial h_{ij}}{\partial r} |_{r=0}
\]
\[
= -2\phi |A|^2 + \Delta \phi + |A|^2 \phi - \overline{\text{Ric}}(\nu, \nu) \phi
\]
\[
= \Delta \phi - (|A|^2 + \overline{\text{Ric}}(\nu, \nu)) \phi.
\]
This completes the proof of Lemma 3.1.

By (3.2) and (3.4), it is not hard to obtain
\[
\frac{\partial F(\Sigma_r)}{\partial r} \bigg|_{r=0} = \int_{\Sigma} \phi \left( H + c \langle \nu, \partial_s \rangle \right) e^{cs} d\mu = 0,
\]
\[
\frac{\partial^2 F(\Sigma_r)}{\partial^2 r} \bigg|_{r=0} = \int_{\Sigma} \phi \left[ \Delta \phi - (|A|^2 + \overline{\text{Ric}}(\nu, \nu)) \right] + c \langle \nabla \phi, \partial_s \rangle e^{cs} d\mu.
\]
(3.5)

Define an elliptic operator \(L\) as follows
\[
L \phi = \Delta \phi - (|A|^2 + \overline{\text{Ric}}(\nu, \nu)) \phi + c \langle \nabla \phi, \partial_s \rangle.
\]
(3.6)

Therefore, putting (3.6) into the second equality of (3.5) yields
\[
\frac{\partial^2 F(\Sigma_r)}{\partial^2 r} \bigg|_{r=0} = \int_{\Sigma} \phi L \phi e^{cs} d\mu.
\]
(3.7)
Now, we only need to show that the RHS of (3.7) is non-positive. Since Σ is a space-like graph, its angle function satisfies $\Theta = \langle \vec{v}, \partial_r \rangle < 0$. Thus we can write $\phi = \eta \Theta$, where $\eta$ is another function over $\Sigma$ with compact support. Then it follows that

$$\phi L \phi = \eta \Theta (\eta L \Theta + \Theta \Delta \eta + 2 \langle \nabla \eta, \nabla \Theta \rangle + c \Theta \langle \nabla \eta, \partial_s \rangle).$$  \hspace{1cm} (3.8)

The reason why we adapt this form is based on a general formula of $\Delta \Theta$ as follows.

**Lemma 3.2.** For any $C^2$ space-like hypersurface $S$ in the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$, it holds that

$$\Delta \Theta - (\|A\|^2 + \text{Ric}(\vec{v}, \vec{v})) \Theta - \langle \nabla H, \partial_s \rangle = 0,$$ \hspace{1cm} (3.9)

where $A$ is the second fundamental form of $S$.

**Proof.** Fix a point $p \in S$. Suitably choose an orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ on $S$ such that $\nabla e_i e_j(p) = 0$ and $\langle e_i, e_j \rangle = \delta_{ij}$. Then $\nabla e_i e_j(p) = h_{ij} \vec{v}$, where, following the convention used in Section 2, $\nabla$ denotes the covariant derivative of the ambient space $M^n \times \mathbb{R}$ and $\vec{v}$ is the unit normal vector of $S$. It is easy to know that for any smooth vector field $X$, $\nabla_X \partial_s = 0$. By direct calculation, one has

$$\Delta \Theta(p) = \nabla_{e_i} \nabla_{e_i} \langle \partial_s, \vec{v} \rangle - \nabla_{e_i e_i} \Theta(p) = e_i \langle \partial_s, h_{ik} e_k \rangle(p) = h_{ik,i} \langle \partial_s, e_k \rangle + \|A\|^2 \Theta. \hspace{1cm} (3.10)$$

Using the Codazzi equation (2.2) directly yields

$$h_{ik,i} = h_{ii,k} + \text{Ric}(\vec{v}, \vec{v}).$$

Hence, it gives

$$h_{ik,i} \langle \partial_s, e_k \rangle = \langle \nabla H, \partial_s \rangle + \text{Ric}(\vec{v}, \langle \partial_s, e_k \rangle e_k).$$ \hspace{1cm} (3.11)

Since $\nabla_X \partial_s = 0$ for any vector $X$, we know

$$\langle \partial_s, e_k \rangle e_k = \partial_s + \Theta \vec{v}$$

and $\text{Ric}(\vec{v}, \partial_s) = 0$. Putting these two facts into (3.11) implies

$$h_{ik,i} \langle \partial_s, e_k \rangle = \langle \nabla H, \partial_s \rangle + \text{Ric}(\vec{v}, \vec{v}) \Theta.$$

The assertion of this lemma follows by combing the above equality with (3.10) directly. \hfill $\square$

Let us go back to the proof of Theorem 1.1. Since $\Sigma$ is a translating space-like graph in the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$, one has $H = -c \Theta$, and then (3.9) can be rewritten as

$$L \Theta = 0.$$  

Therefore (3.8) becomes

$$\phi L \phi = \eta \Theta (\Theta \Delta \eta + 2 \langle \nabla \eta, \nabla \Theta \rangle + c \Theta \langle \nabla \eta, \partial_s \rangle).$$
On the other hand, the divergence of $\eta \Theta^2 \nabla_\eta e^{cs}$ is
\[
\text{div} \left( \eta \Theta^2 \nabla_\eta e^{cs} \right) = \eta \Theta e^{cs} (\Theta \Delta \eta + 2(\nabla_\eta, \nabla \Theta) + e \Theta (\nabla \eta, \partial_s)) + \Theta^2 |\nabla \eta|^2 e^{cs}
= \phi e^{cs} L \phi + \Theta^2 |\nabla \eta|^2 e^{cs}.
\]
(3.12)

Combining (3.12) with (3.7) and applying the divergence theorem result in
\[
\frac{\partial^2 F(\Sigma_r)}{\partial^2 r} \bigg|_{r=0} = - \int_\Sigma \Theta^2 |\nabla \eta|^2 e^{cs} d\mu \leq 0.
\]

Then we conclude that the translating space-like graph $\Sigma$ is stable and maximal in $\widetilde{M^n \times \mathbb{R}}$. 

Applying Lemma 3.2, we can obtain the following interesting rigidity result.

**Theorem 3.3.** Let $\{\Sigma_n\}_{n=1}^\infty$ be a sequence of smooth connected space-like graphs in the Lorentz $(n + 1)$-manifold $M^n \times \mathbb{R}$ with diameter $\rho$ converging uniformly to a connected space-like hypersurface $\Sigma$ in the $C^2$ sense. If all $\Sigma_n$ are translating space-like graphs in the interior of $\Sigma$, the angle function $\Theta$ satisfies that $\Theta < 0$ or $\Theta \equiv 0$. The conclusion is also true in the case of maximal or CMC space-like graphs.

**Proof.** Without loss of generality, we assume $\Theta < 0$. By continuity, we know that in the interior of all $\Sigma_n$, $|A|^2 < \beta_1$ holds for some positive constant $\beta_1$ depending only on $M^n$.

Now, first, we assume that $\Sigma_n$ are maximal or CMC space-like graphs. Then $\nabla H \equiv 0$. By Lemma 3.2, we have
\[
\Delta \Theta - (|A|^2 + \overline{\text{Ric}}(\bar{v}, \bar{v})) \Theta = 0
\]
(3.13)
on all $\Sigma_n$. Since
\[
\overline{\text{Ric}}(\bar{v}, \bar{v}) = \frac{u_k^2 (k_i^i_k k_i^i l + \Gamma_i^i_{k k} k_i^i l - \Gamma_i^i_{k l} k_i^i l)}{1 - |D u|^2}, \quad i, k, l = 1, 2, \ldots, n,
\]
there exists a positive constant $\beta_2$ only depending on $M^n$ such that $\overline{\text{Ric}}(\bar{v}, \bar{v}) \leq \beta_2$ in the interior of all $\Sigma_n$. By (3.13) we have $\Delta \Theta \geq (\beta_1 + \beta_2) \Theta$ on all $\Sigma_n$. Because $\Sigma$ is the $C^2$ uniform limit of $\Sigma_n$ as $n \to \infty$, it follows that $\Theta \leq 0$ and $\Delta \Theta \geq (\beta_1 + \beta_2) \Theta$. By the strong maximum principle of second-order elliptic equations, we can obtain that $\Theta \equiv 0$ or $\Theta < 0$ on $\Sigma$.

Second, assume that $\Sigma_n$ are translating space-like graphs. Then $H \equiv -c \Theta$ by (5.2). Similar argument gives
\[
\Delta \Theta \geq (\beta_1 + \beta_2) \Theta - c \langle \nabla \Theta, \partial_s \rangle
\]
on all $\Sigma_n$. Based on the strong maximum principle and the fact that $\Theta \leq 0$ on $\Sigma$, we also have $\Theta \equiv 0$ or $\Theta < 0$ on $\Sigma$.

$\square$
4 Examples of translating space-like graphs

In this section, we construct some examples of translating space-like graphs to MCF when the hypersurface $M^n$ has a domain with certain warped product structure.

Suppose that $M^n$ is an $n$-dimensional ($n \geq 2$) complete Riemannian manifold with a metric $\sigma$ containing a domain $M^n_0$ equipped with the following coordinate system:

$$\{ \theta = (\theta_2, \theta_3, \ldots, \theta_n) \in S^{n-1}, r \in [0, r_0) \} \quad \text{with} \quad \sigma = dr^2 + h^2(r)d\theta^2,$$

where $d\theta^2$ is the round metric on the unit $(n-1)$-sphere $S^{n-1}$, $h(r)$ is a positive function satisfying $h(0) = 0$, $h'(0) = 1$ with $h'(r) \neq 0$ for all $r \in (0, r_0)$.

Now, with the help of examples constructed below, we can somehow show the existence of translating space-like graphs in the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$ with the structure (4.1) and the metric $\bar{g}$.

**Theorem 4.1.** Let $M^n$ be a complete Riemannian $n$-manifold mentioned above. Let $u(r) : [0, r_0) \to \mathbb{R}$ be a $C^2$ solution of the following ordinary differential equation (ODE for short)

$$\frac{u_{rr}}{1 - u_r^2} + (n - 1) \frac{h'(r)}{h(r)} u_r = c,$$

with $u_r(0) = 0$ for $r \in [0, r_0)$ and $|u_r| < 1$. Then $\Sigma = (x, u(r))$ for $r \in [0, r_0)$ is a translating space-like graph in the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$, where $x = (r, \theta) \in M^n_0$ given by (4.1). If $r_0 = \infty$, then $\Sigma$ is complete.

**Remark 4.2.** Clearly, (4.2) is a second-order ODE whose component of the second-order derivative term does not degenerate under the assumption $|u_r| < 1$. The existence of its solution is obvious.

**Proof.** If $r_0 = \infty$, then $M^n_0$ is simply connected and should be a whole $M^n$. Thus $\Sigma$ is complete.

In the rest part, we show that $\Sigma$ is a translating space-like graph. By (4.2), we know that here it is sufficient to derive the identity

$$H = -c\Theta,$$

where $H$ is the mean curvature of $\Sigma$ and $\bar{\Theta}$ is its upward normal vector.

Fix a point $(x, u(x))$ on $\Sigma$, where $x \in M^n_0$ and the polar coordinate of $x$ in $M^n_0$ is not $(0, 0, \ldots, 0)$. Clearly, the polar coordinate system on $M^n_0$ given by (4.1) determines a frame field $\{\partial_r, \partial_{\theta_2}, \ldots, \partial_{\theta_n}\}$ naturally. For the space-like graph $\Sigma$ determined by $u(x) = u(r)$ in the Lorentz $(n+1)$-manifold $M^n \times \mathbb{R}$, denote by $u_r$ and $u_{\theta_i}$, $i = 2, 3, \ldots, n$, the partial derivatives of $u$. Since here $u(r)$ is a radial function, $u_{\theta_i} \equiv 0$, $i = 2, 3, \ldots, n$. Therefore, on $\Sigma$, a natural frame $\{e_1 = \partial_r + u_r \partial_\theta, e_i = \partial_{\theta_i}\}$, $i = 2, \ldots, n$ can be obtained, where, as before, $\partial_\theta$ denotes the vector field tangent to $\mathbb{R}$. Then the Riemannian metric on $\Sigma$ and the upward unit normal vector of $\Sigma$ are given by

$$g_{11} = \langle e_1, e_1 \rangle = 1 - u_r^2, \quad g_{ki} = g_{lk} = \langle e_i, e_k \rangle = 0, \quad k \neq l,$$

$$g_{ii} = \langle e_i, e_i \rangle = h^2(r), \quad i = 2, \ldots, n,$$
\[ \vec{v} = \frac{\partial_s + u_r \partial_r}{\sqrt{1 - u_r^2}}. \]

By direct calculation, its second fundamental forms are
\[ h_{11} = -\left\langle \nabla_{e_1} e_1, \vec{v} \right\rangle = \frac{u_{rr}}{\sqrt{1 - u_r^2}}, \]
and
\[ h_{ii} = -\left\langle \nabla_{e_i} e_i, \vec{v} \right\rangle = -\left\langle -h(r)h'(r) \partial_r, \vec{v} \right\rangle = \frac{h'(r)h(r) u_r}{\sqrt{1 - u_r^2}}, \quad i = 2, \ldots, n. \]

where we use the fact
\[ \left\langle \nabla_{e_i} e_i, \partial_r \right\rangle = -h'(r)h(r), \quad i = 2, \ldots, n. \]

Then, by (4.2), the mean curvature of \( \Sigma \) with respect to \( \vec{v} \) is
\[ H = g^{11} h_{11} + g^{22} h_{22} + \ldots + g^{nn} h_{nn} = \frac{1}{\sqrt{1 - u_r^2}} \left( \frac{u_{rr}}{1 - u_r^2} + (n - 1) \frac{h'(r)}{h(r)} u_r \right) = c \frac{1}{\sqrt{1 - u_r^2}}. \]

On the other hand, we have
\[ \Theta = \left\langle \vec{v}, \partial_s \right\rangle = \left\langle \frac{\partial_s + u_r \partial_r}{\sqrt{1 - u_r^2}}, \partial_s \right\rangle = -\frac{1}{\sqrt{1 - u_r^2}}. \]

Hence in our case here we have \( H = -c \Theta \), which implies \( \Sigma \) is a translating space-like graph. The proof is finished.

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