Continuous Virasoro algebra in open string field theory

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It is known that the Takahashi-Tanimoto identity based solution in open string field theory derives a kinetic operator which is a sum of twisted Virasoro generators. Applying the infinite circumstance description of conformal field theory, we derive continuous Virasoro algebra associated with the kinetic operator. Fock space expansion of the OSFT Hilbert space is given in terms of continuous Fourier modes. We show emptiness of the BRST cohomology in this Fock space.

1 Introduction

Understanding the nature of the tachyon vacuum has been an important issue in open string field theory (OSFT). It is well known that the wedge-based analytic solution [1, 2, 3] successfully explains the absence of open strings around the tachyon vacuum [4]. However, the physics at the tachyon vacuum has not yet been fully understood. Usually, an OSFT shifted by a classical solution defines a boundary conformal field theory (BCFT) which is different from the reference BCFT associated with the original OSFT. However, such BCFT for the tachyon vacuum is expected to be irregular since there are no more boundaries due to the absence of open strings. This leads to a question: what kind of physics described by the OSFT at the tachyon vacuum? More precisely, is there any (two dimensional) field theory associated with the tachyon vacuum? The analysis performed in [4] do not provide any information about this question since the cohomology simply vanishes. Number of attempts had been made to answer this question. They are explanation in terms of shrunken boundaries [5, 6].

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deformation in ghost sector \[12\] and D-gD pair \[13\]. In spite of these efforts, it is fair to say that we do not yet have definite answer to the question raised above.

In this paper, we add one more attempt to the above list by applying the technique which has been developed in rather different context. That is so-called sine square deformation (SSD) which was originally introduced to reduce the boundary effect of the one dimensional open spin lattice \[14\]. The authors of \[14\] examined a specific boundary condition by deforming the open lattice Hamiltonian. They found that the ground state energy becomes almost identical to that of periodic lattice. The coincidence between open and periodic systems observed in SSD somewhat resembles to tachyon condensation, where OSFT is expected to be deformed into closed string theory.

Later, the deformed Hamiltonian on the open spin lattice was interpreted as a Hamiltonian of bulk conformal field theory \[15\]:

\[
H = \left( L_0 - \frac{1}{2} L_1 - \frac{1}{2} L_{-1} \right) + \left( \bar{L}_0 - \frac{1}{2} \bar{L}_1 - \frac{1}{2} \bar{L}_{-1} \right),
\]

where \( L_n \) is the Virasoro generator. The coincidence of ground state energy between open and periodic boundary conditions is explained by the fact that \( L_1 \) and \( L_{-1} \) vanish on the \( SL(2, C) \) invariant vacuum. Further, the spectrum of this Hamiltonian was explored in \[16\], \[17\], \[18\]. In \[18\], the authors developed the formalism of dipolar quantization in which bulk conformal fields are expanded by continuous mode numbers instead of discrete one. The Fourier mode of energy momentum tensor corresponds to Virasoro generator with continuous label; the generators obey the commutation relation

\[
[\mathcal{L}_\kappa, \mathcal{L}_\lambda] = (\kappa - \lambda)\mathcal{L}_{\kappa+\lambda},
\]

where \( \kappa \) and \( \lambda \) are real numbers rather than integers. And also, they called their formalism “infinite circumstance limit” of a CFT since the continuous label of Fourier modes indicates a system with infinite size. In fact, the authors of \[18\] presented a formula that embeds the infinite parameter to the complex plane and drew equal time contours derived from the formula.

It is not difficult to find resemblance between SSD \[1\] and OSFT. It is known that the identity based scalar solution of Takahashi and Tanimoto \[19\] yields following kinetic operator upon gauge fixing \[8\]:

\[
\mathcal{L}_0' = \frac{1}{2} L'_0 - \frac{1}{4} L'_2 - \frac{1}{4} L'_{-2} + \frac{3}{2}
\]

where \( L'_n = L_n + nq_n + \delta_{n,0} \) is the twisted Virasoro generator \[5\]. In fact, this kinetic operator exactly coincides with the \( Z_2 \) symmetric deformation studied in SSD literature \[18\], \[20\]. The infinite circumstance formalism can be applied to this kinetic operator since \( L'_0, L'_2, L'_{-2} \) form \( SL(2, R) \) algebra as \( L_0, L_1, L_{-1} \) of SSD does.

Aim of this paper is to investigate the Hilbert space defined by the cohomology of modified BRST charge generated by the Takahashi-Tanimoto solution, by employing
the formalism developed in [18]. This aim is accomplished by identifying eigenmodes of gauge fixed kinetic operator $L_0'$, which turn out have continuous mode numbers. Outline of this paper is as follows. In section 2, we review the results of [8]. The kinetic operator (3) is obtained by gauge fixing the identity based solution of [19]. Section 3 and 4 are devoted to quantum analysis along the line with [18]. Section 3 provides basic tools for our investigation. Worldsheet geometry generated by $L_0'$ (3) is described in detail. Then, continuous Fourier modes for conformal fields are introduced. Continuous Virasoro generators are derived from the modified BRST charge. Finally, it is shown that the continuous generators obey Virasoro algebra without anomaly. Section 4 is devoted to identification of the cohomology of the modified BRST charge. Eigenmode expansion of the modified BRST charge is given. We find that the eigenmode expansion looks like “continuum version” of the undeformed BRST charge except for anomalous constant. Using this expansion, we prove the absence of the BRST cohomology. We conclude in section 5 with further discussions.

Note added

While completing the manuscript, we found a paper by Kishimoto, Kitade and Takahashi [21], which deals with same classical solution using SSD formalism.

2 Modified BRST charge

The identity based solution of Takahashi and Tanimoto [19] is obtained by integrating primary fields multiplied by specific functions along “left” half of an open string:

$$\Psi_{TT} = \left[ \int_{\gamma_L} \frac{dz}{2\pi i} (F(z) - 1) j_B(z) - \int_{\gamma_L} \frac{dz}{2\pi i} \left( \frac{\partial F(z)}{F(z)} \right)^2 c(z) \right] |I\rangle ,$$

where $z$ represents the worldsheet coordinate of open string BCFT, which is taken to be entire complex plane in virtue of the doubling trick, and $j_B(z)$ and $c(z)$ are BRST current and conformal ghost respectively. $|I\rangle$ is the identity string field. The path $\gamma_L$ is taken to be right half of the unit circle. The function $F(z)$ is explicitly chosen to be

$$F(z) = 1 - \frac{1}{4} \left( z + \frac{1}{z} \right)^2 = \frac{1}{2} - \frac{1}{4} z^2 - \frac{1}{4} z^{-2}.$$

An advantage of this solution is the use of left half integrated operators and identity string field, which identifies noncommutative star product between string fields with conventional operator algebra of BCFT. It can be shown that this solution satisfies the equation of motion of OSFT [19]. The OSFT action expanded around this solution is characterized by the modified BRST charge

$$Q' = \int \frac{dz}{2\pi i} F(z) j_B(z) - \int \frac{dz}{2\pi i} \left( \frac{\partial F(z)}{F(z)} \right)^2 c(z).$$
where \( \gamma \) is the unit circle enclosing the origin \( z = 0 \). Note that this circle represents a equal time contour in radial quantization therefore can be shrunk to arbitrary small small radii. The contour integral can be evaluated by expanding \( j_B(z) \) and \( c(z) \) into Laurant series and picking up pole residues:

\[
Q' = \frac{1}{2} Q_B - \frac{1}{4} (Q_2 + Q_{-2}) + 2c_0 + c_2 + c_{-2},
\]

where \( j_B(z) = \sum_n Q_n z^{-m-1} \) and \( c(z) = \sum_n c_n z^{-n+1} \). The gauge fixed kinetic operator in Siegel gauge is derived from the commutator between \( Q' \) and antighost zero mode \( b_0 \):

\[
\mathcal{L}'_0 = \{ Q', b_0 \}
\]

\[
= \frac{1}{2} L'_0 - \frac{1}{4} (L'_2 + L'_{-2}) + \frac{3}{2},
\]

where \( L'_n \) is the twisted Virasoro generator defined by

\[
L'_n = L_n + nq_n + \delta_{n,0},
\]

and \( q_n \) is the \( n \)th mode of the ghost number current defined by

\[
j_g(z) = b(z)c(z) = \sum_n q_n z^{-n+1}.
\]

The total central charge of the matter and twisted ghost CFT is 24 rather than being zero. This value of central charge can be derived from \( (9) \) directly. Alternatively, it can be derived from the OPE between twisted energy momentum tensor in \( \rho = \log z \) coordinate:

\[
T'(\rho) = T(\rho) - \partial_\rho j_g(\rho),
\]

or in \( z \) coordinate

\[
T'(z) = T(z) - \frac{1}{z} \partial(z j_g(z)).
\]

The twisted energy momentum tensor defined above is consistent with twisted ghost pair \( c'(z) \) and \( b'(z) \) rather than the conventional one. The twisted and untwisted ghost pairs are related by

\[
c'(z) = z^{-1} c(z) = \sum_n c_n z^{-n},
\]

\[
b'(z) = z b(z) = \sum_n b_n z^{-n-1}.
\]

This correspondence between twisted and untwisted ghost CFT will be used frequently.
The spectrum of the deformed theory corresponds to the cohomology of modified charge $Q'$. The authors of [8] have derived the cohomology with the help of the $SL(2, R)$ symmetry. In order to derive the cohomology, following identity is crucial:

$$Q' = -\frac{1}{4} U' Q_B^{(2)} U'^{-1}$$

(15)

where $U' = e^{1/2L_{-2}}$ is a finite conformal transformation, and $Q_B^{(2)}$ is the shifted charge obtained by applying the replacement

$$c_n \rightarrow c_{n+2}, \quad b_n \rightarrow b_{n-2}$$

(16)

to the original BRST charge. The cohomology of $Q'$ is obtained by mapping the cohomology of $Q_B^{(2)}$, which is nothing but a shifted version of the original cohomology of $Q_B$. In this way, the cohomology of $Q'$ is identified as

$$|\Psi\rangle_{TZ} = U' (|P\rangle \otimes b_{-2} |0\rangle + |P'\rangle \otimes |0\rangle) ,$$

(17)

where $|P\rangle$ and $|P'\rangle$ are DDF states in matter CFT and $|0\rangle$ is the conventional $SL(2, R)$ vacuum of the ghost CFT defined by $c_n |0\rangle = 0 \ (n \geq 2)$ and $b_n |0\rangle = 0 \ (n \geq -1)$. Surprisingly, the existence of nontrivial cohomology does not contradict with the Sen’s conjecture that identifies the classical solution (4) as the tachyon vacuum. This is simply because the cohomology (17) does not contributes to any pertubative amplitudes due to mismatch of ghost number [19].

3 Continuous Virasoro algebra

3.1 Geometrical analysis

In order to reformulate the system described by the kinetic operator (8), we will identify the nature of time evolution generated by it. The twist involved in (8) is irrelevant for this purpose thus we only need to consider the untwisted generator

$$\mathcal{L}_0 = \frac{1}{2} L_0 - \frac{1}{4} (L_2 + L_{-2}) .$$

(18)

Following [18], we introduce a classical representation of $\mathcal{L}_0$:

$$l_0 = -g(z) \frac{\partial}{\partial z} ,$$

(19)

where the function $g(z)$ is chosen to be

$$g(z) = zF(z) = \frac{1}{2} z^2 - \frac{1}{4} z^3 - \frac{1}{4} z^{-1} .$$

(20)

Next, we will find an eigenfunction of $l_0$ which satisfies

$$g(z) \partial_z f_\kappa(z) = \kappa f_\kappa(z) .$$

(21)
A solution of the above equation is easily found to be

\[ f_\kappa(z) = e^{\kappa \int \frac{dz'}{\bar{\Sigma}(z')}} = e^{\frac{2\kappa}{z^2-1}}, \]  

(22)

here \( z \) is the radial coordinate of conventional CFT. Note that the eigenfunction is regular for any real \( \kappa \). Therefore \( l_0 \) shows continuous spectrum. Further, the eigenfunctions \( f_\kappa(z) \) can be used to define continuously indexed generators

\[ l_\kappa = -g(z)f_\kappa(z)\frac{\partial}{\partial z}. \]  

(23)

It is easily confirmed that they form continuous Witt algebra

\[ [l_\kappa, l_\lambda] = (\kappa - \lambda)l_{\kappa + \lambda}. \]  

(24)

Let us now describe time evolution generated by \( l_0 \). Note that \( l_0 \) is the generator of time translation which acts on a conformal field. We also note that the worldsheet time \( t \) should be paired with another parameter \( s \) along a string to define complex coordinate \( \rho = t + is \). We require

\[ \frac{\partial}{\partial \rho} = g(z)\frac{\partial}{\partial z}. \]  

(25)

This defines a relation between complex coordinates \( z \) and \( \rho \). The \( z \) dependence of \( \rho \) can be obtained by rewriting above equation to

\[ \frac{d\rho}{dz} = \frac{1}{g(z)}. \]  

(26)

and integrating this with respect to \( z \). Thus we obtain

\[ \rho = \frac{2}{z^2 - 1}. \]  

(27)

Let us investigate the equal time contours of (27). Decomposing right hand side of (27) into real and imaginary parts with \( z = x + iy \) and comparing them to left hand side, we obtain

\[ t = \frac{2(x^2 - y^2 - 1)}{(x^2 + y^2)^2 - 2(x^2 - y^2) + 1} \]  

(28)

\[ s = -\frac{4xy}{(x^2 + y^2)^2 - 2(x^2 - y^2) + 1}. \]  

(29)

We identify the worldsheet of a string as a whole \( \rho \) plane, i.e., \( -\infty < t < \infty \) and \( -\infty < s < \infty \). From (27), we see that only half of the \( z \) plane is covered by the trajectories of a string. How it is covered depends on a choice of branch cut on the \( z \) plane. We would like to choose the upper half \( z \) as an image of whole \( \rho \) plane as
this choice is compatible with the result of [8]. The contour is plotted in Fig. 1. z = 1 and z = −1 corresponds to s = −∞ and s = ∞ respectively. They are remnants of open string boundaries. They are kept fixed and do not evolve in time. The negative and positive axes are identified. The global structure of contours are nontrivial. At t ≤ −2, contour splits into two closed curves within the unit circle |z| = 1. At t > 0, two curves are placed outside of the unit circle. Other values of t looks a bit different. The region 0 < t < −2 has only one contour. The contour at t = 0 is a hyperbola $x^2 - y^2 = 1$. We stress that the global structure of the contours is consistent with the physical expectation, namely, the theory defines closed string vacuum where open string endpoint (D-branes) vanish.

### 3.2 Mode expansion

Next we introduce the mode expansion of conformal fields according to the prescription of [18]. Consider a primary field $\phi(z)$ with weight $h$. The Fourier mode of this field is now continuously labeled

$$\phi_\kappa = \oint \frac{dz}{2\pi i} g(z)^{h-1} f_\kappa(z) \phi(z),$$

where the integral is performed along a constant $t$ contour according to (28). We also have the inverse relation

$$\phi(z) = g(z)^{-h} \int d\kappa \phi_\kappa f_{-\kappa}(z).$$

These relations correspond to Fourier transformation and its inverse rather than discrete Fourier series. In our case, relevant fundamental fields are $\partial X^\mu(z)$, $c(z)$ and $b(z)$.
Their Fourier modes can be pulled out by applying (30) to each of them. Thus we have

\[ A_\kappa = i \sqrt{\frac{2}{\alpha'}} \int \frac{dz}{2\pi i} f_\kappa(z) \partial X^\mu(z), \]  

(32)

\[ C_\kappa = \int \frac{dz}{2\pi i} g(z)^{-2} f_\kappa(z)c(z), \]  

(33)

\[ B_\kappa = \int \frac{dz}{2\pi i} g(z)f_\kappa(z)b(z). \]  

(34)

Further, we introduce the twisted versions of (33) and (34). Using the fact that 
\[ c'(z) = zc(z) \] and 
\[ b'(z) = z^{-1}b(z) \] have weight 0 and 1 respectively, we can write Fourier modes for them as

\[ B'_\kappa = \int \frac{dz}{2\pi i} f_\kappa(z)zb(z), \]  

(35)

\[ C'_\kappa = \int \frac{dz}{2\pi i} g(z)^{-1} f_\kappa(z)z^{-1}c(z). \]  

(36)

Inverse formulas for the Fourier modes can be obtained by applying (31) to each fields.

### 3.3 Virasoro generator

Now we are ready to introduce continuous Virasoro generators, which of our main interest. First, we note that \( b_0 \) is a zero mode of \( b'(z) \) obtained through

\[ b_0 = \int \frac{dz}{2\pi i} zb(z) = \int \frac{dz}{2\pi i} b'(z) = B'_0. \]  

(37)

Therefore we can write

\[ L'_0 = \{Q', b_0\} = \{Q', B'_0\}. \]  

(38)

Analogously, we will obtain the formula for nonzero mode

\[ L'_\kappa = \{Q', B'_\kappa\}. \]  

(39)

We will show \( L'_\kappa \) form continuous Virasoro algebra as expected. First, we derive the explicit form of the twisted generator. This is done in similar fashion as that of conventional CFT [18], by integrating operator product expansion around a single pole. As a warming up, let us begin with \( L'_0 \). First, we rewrite (5) in terms of \( g(z) \):

\[ Q' = \int \frac{dz}{2\pi i} z^{-1}g(z)j_B(z) - \int \frac{dz}{2\pi i} z \left( \partial(z^{-1}g(z)) \right)^2 g(z)c(z), \]  

(40)

where the integral is preformed along constant \( t \) contour. Then, we obtain

\[ L'_0 = \{Q', B'_0\} \]

\[ = \int d\omega \int_{t=0} dz dw (z^{-1}g(z))wT(j_B(z)b(w)) \]

\[ - \int d\omega \int_{t=0} dz dw \frac{z \left( \partial(z^{-1}g(z)) \right)^2}{g(z)} wT(c(z)b(w)), \]  

(41)

8
where $T$ denotes time ordering. Each time ordered product in the last line is replaced with operator product expansion that takes form of a Laurent expansion around $z = w$:

$$T (j_B(z) b(w)) = \frac{3}{(z-w)^3} + \frac{j_g(w)}{(z-w)^2} + \frac{T(w)}{(z-w)},$$

(42)

$$T (c(z) b(w)) = \frac{1}{z-w}.$$  

(43)

Then, by picking up pole residues, we obtain the result which can summarized into a compact notation

$$L'_0 = \left[ gT + \frac{3}{2} \frac{g_k}{g} - \frac{h^2}{g} \right],$$

(44)

where the square bracket simply denotes a contour integral of a product of functions or fields,

$$[ab] = \oint_{t=0} \frac{dz}{2\pi i} a(z)b(z),$$

(45)

and functions $h(z)$ and $k(z)$ are defined by

$$h(z) = z \frac{d}{dz} \left( \frac{g(z)}{z} \right), \quad k(z) = z \frac{d^2}{dz^2} \left( \frac{g(z)}{z} \right).$$

(46)

The generator with nonzero $\kappa$ can be derived in similar manner. In this case, it is soon realized that the result can be obtained just by inserting $f_\kappa$ in (44). Thus we have

$$L'_{\kappa} = \{ Q', B'_{\kappa} \}$$

(47)

$$= [f_\kappa gT] + [f_\kappa hj_g] + \left[ \frac{f_\kappa}{g} \left( \frac{3}{2} \frac{g_k}{g} - \frac{h^2}{g} \right) \right]$$

(48)

Note that last term in the final expression (48) is a constant. Explicit evaluation of this constant requires convenient choice of $t = 0$ as was done in [18]. With this choice, we can convert the contour integral to the one along a straight line as

$$[f_\kappa a] = \int \frac{dz}{2\pi} \frac{f_\kappa(z)}{g(z)} g(z) a(z)$$

(49)

$$= \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{is} a \left( \sqrt{1 + \frac{2}{is}} \right) a \left( \sqrt{1 + \frac{2}{is}} \right).$$

(50)

where we have used $\rho = is = 2/(1-z^2)$. In this way, a contour integral is evaluated by inverse Fourier transform of $g(\sqrt{1 + 2/(is)}) a(\sqrt{1 + 2/(is)})$. This quantity can be unambiguously evaluated if $a(z)$ involve even powers of $z$ only. Fortunately, this is the case for last term of (48):

$$g(z) k(z) = \frac{(z^4 + 3)(z^2 - 1)^2}{8z^4} = \frac{-2(s^2 - is - 1)}{s^2(s - 2i)^2},$$

(51)
\[ h(z)^2 = \frac{(z^4 - 1)^2}{4z^4} = \frac{-4(s - i)^2}{s^2(s - 2i)^2}. \] (52)

Fourier transform of these functions can be evaluated analytically, although the results turn out to be distributions rather than ordinary functions:

\[ \left[ \frac{f_\kappa}{g} k \right] = \frac{3}{2} e^{-2\kappa} \theta(\kappa) + \frac{1}{4} \kappa e(\kappa), \] (53)

\[ \left[ \frac{f_\kappa}{h^2} \right] = e^{-2\kappa}(\kappa - 1)\theta(\kappa) + \frac{1}{2}(\kappa + 1)e(\kappa), \] (54)

where \( \theta(\kappa) \) and \( e(\kappa) \) are Heaviside step function and sign function respectively. Plugging these back to (48), we arrive at final expression

\[ L'_\kappa = [f_\kappa gT] + [f_\kappa h j_g] + a_\kappa, \] (55)

where

\[ a_\kappa = \frac{1}{4}(5\kappa + 4)e^{-2\kappa}\theta(\kappa) - \frac{1}{8}(\kappa + 4)e(\kappa). \] (56)

### 3.4 Virasoro algebra

We would like to derive the commutator between continuous Virasoro generator \( L'_\kappa \). We divide (48) into untwisted part and the remaining:

\[ L'_\kappa = L_\kappa + \delta L_\kappa + a_\kappa \] (57)

where

\[ L_\kappa = \left[ f_\kappa gT \right] \] (58)

\[ \delta L_\kappa = \left[ f_\kappa h j_g \right] \] (59)

\[ a_\kappa = \left[ \frac{3}{2} \frac{f_\kappa}{g} k - \frac{f_\kappa}{h} h^2 \right] \] (60)

Then, the commutator is expanded as

\[ [L'_\kappa, L'_\lambda] = [L_\kappa, L_\lambda] + [L_\kappa, \delta L_\lambda] + [\delta L_\kappa, L_\lambda] + [\delta L_\kappa, \delta L_\lambda]. \] (61)

According to [18], the untwisted generators satisfy Virasoro algebra:

\[ [L_\kappa, L_\lambda] = (\kappa - \lambda) L_{\kappa+\lambda}. \] (62)

There is no central term since we consider matter plus ghost CFT with vanishing total central charge. Next we would like to evaluate first order term in \( \delta \):

\[ [L_\kappa, \delta L_\lambda] + [\delta L_\kappa, L_\lambda] = [L_\kappa, \delta L_\lambda] - [L_\kappa, \delta L_\lambda] \] (63)
Finally, the right hand side of the above equation can be obtained by antisymmetrizing the first term:

\[
[L_\kappa, \delta L_\lambda] = \oint \oint \frac{dz}{2\pi i} \frac{dw}{2\pi i} g(z)f_\kappa(z)h(w)f_\lambda(w)T(z)j_\delta(w) = \oint \oint \frac{dz}{2\pi i} \frac{dw}{2\pi i} g(z)f_\kappa(z)h(w)f_\lambda(w) \left( \frac{-3}{(z-w)^3} + \frac{j_\delta(w)}{(z-w)^2} + \frac{\partial j_\delta(w)}{z-w} \right)
\]

\[
= -\frac{3}{2} \left[ (g f_\kappa)'(h f_\lambda) + (g f_\kappa)' h f_\lambda j_\delta \right] + \left[ g h f_\kappa f_\lambda \partial j_\delta \right]
\]

Then, we have

\[
[L_\kappa, \delta L_\lambda] + [\delta L_\kappa, L_\lambda] = (\kappa - \lambda) \left[ h f_{\kappa+\lambda} j_\delta \right] + \left[ f_{\kappa+\lambda} \left( -\frac{3}{2}(\kappa - \lambda)g' h - \frac{3}{2}(\kappa^2 - \lambda^2)h \right) \right]
\]

(64)

Finally, \( \delta^2 \) term is evaluated as

\[
[\delta L_\kappa, \delta L_\lambda] = \oint \oint \frac{dz}{2\pi i} \frac{dw}{2\pi i} h(z)f_\kappa(z)h(w)f_\lambda(w)j_\delta(z)j_\delta(w) = \oint \oint \frac{dz}{2\pi i} \frac{dw}{2\pi i} h(z)f_\kappa(z)h(w)f_\lambda(w) \frac{1}{(z-w)^2}
\]

\[
= \left[ (h f_\kappa)' h f_\lambda \right] + \left[ h' f_\kappa h f_\lambda + h f_\kappa' h f_\lambda \right] = \left[ f_{\kappa+\lambda} \left( g h h' + \kappa h^2 \right) \right]
\]

(66)

Our result can be summarized into

\[
[L_\kappa', L_\lambda'] = (\kappa - \lambda) \left( L_{\kappa+\lambda} + \delta L_{\kappa+\lambda} \right) + u(\kappa, \lambda)
\]

(67)

where

\[
u(\kappa, \lambda) = \left[ f_{\kappa+\lambda} \left( -\frac{3}{2}(\kappa - \lambda)g' h - \frac{3}{2}(\kappa^2 - \lambda^2)h + g h h' + \kappa h^2 \right) \right].
\]

(68)
The constant $u(\kappa, \lambda)$ can be explicitly evaluated in terms of Fourier transformation. The result turns out to be

$$ u(\kappa, \lambda) = (\kappa - \lambda) \left\{ \frac{1}{4} (5\kappa + 5\lambda + 4)e^{-2(\kappa+\lambda)} \theta(\kappa + \lambda) - \frac{1}{8} (\kappa + \lambda + 4)e(\kappa + \lambda) \right\} $$

$$ = (\kappa - \lambda) a_{\kappa+\lambda}, $$

where $a_{\kappa+\lambda}$ is already defined in (56). Putting back this result to (67), we obtain

$$ [\mathcal{L}^\prime_\kappa, \mathcal{L}^\prime_\lambda] = (\kappa - \lambda) (\mathcal{L}_{\kappa+\lambda} + \delta \mathcal{L}_{\kappa+\lambda}) + (\kappa - \lambda) a_{\kappa+\lambda} $$

$$ = (\kappa - \lambda) \mathcal{L}_{\kappa+\lambda}^\prime. $$

(70)

Thus we have shown that the generator $\mathcal{L}^\prime_\kappa$ satisfy Virasoro algebra without anomaly although it is defined in terms of twisted generators.

4 Spectral analysis

4.1 Commutation relations

We present the algebra satisfied by Fourier modes with continuous algebra. As an example, let us evaluate the commutator between $\mathcal{B}_\kappa$ and $\mathcal{C}_\lambda$. The commutator can be evaluated in similar way as the derivation of continuous Virasoro algebra, where OPE and contour integral is used:

$$ \{\mathcal{B}_\kappa, \mathcal{C}_\lambda\} = \oint \frac{dz}{2\pi i} \oint_{t=0} \frac{dw}{2\pi i} f_\kappa(z) f_\lambda(w) g(z) T(b(z)c(w)) $$

$$ = \oint_{t=0} \frac{dz}{2\pi i} f_{\kappa+\lambda}(z) g(z) $$

$$ = \oint_{t=0} \frac{ds}{2\pi} e^{i\kappa s} $$

$$ = \delta(\kappa + \lambda). $$

(71)

where we transformed variable $z$ into $s$ by choosing $t = 0$ contour. The commutation relation for the twisted pairs $\mathcal{B}'_\kappa$ and $\mathcal{C}'_\lambda$ yields exactly same result since the extra weight factors ($z^{-1}$ for $b'(z)$ and $z$ for $c(z)$) do not change the commutator. Thus we have

$$ \{\mathcal{B}'_\kappa, \mathcal{C}'_\lambda\} = \delta(\kappa + \lambda). $$

(72)

The commutator for $\mathcal{A}_\kappa^\mu$ is evaluated similarly as

$$ [\mathcal{A}_\kappa^\mu, \mathcal{A}_\lambda^\nu] = \kappa \eta^{\mu\nu} \delta(\kappa + \lambda). $$

(73)

Note that the commutation relations derived here can be understood as “continuum version” of the discrete one

$$ \{b'_m, c'_n\} = \delta_{m+n}, \quad [\alpha^\mu_m, \alpha^\mu_n] = m \eta^{\mu\nu} \delta_{m+n}, $$

(74)
where $m$ and $n$ are integers.

We can also derive commutators between $L'_\kappa$ and other modes. This can be done in much the same way as we did for Virasoro algebra of commutation relations between Fourier modes. In this case, relevant OPEs are those between $T(z)$ or $j_g(z)$ with other fundamental fields. Explicitly, they are

$$T(z) \partial X^\mu(w) \sim \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{(z-w)} + \cdots,$$

(75)

$$T(z) \partial c(w) \sim -\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)} + \cdots,$$

(76)

$$T(z) \partial b(w) \sim \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{(z-w)} + \cdots,$$

(77)

$$j_g(z) c(w) \sim \frac{c(w)}{z-w} + \cdots,$$

(78)

$$j_g(z) b(w) \sim \frac{-b(w)}{z-w} + \cdots.$$

(79)

We have the following result

$$[L'_\kappa, A^\mu_{\lambda}] = -\lambda A^\mu_{\kappa+\lambda},$$

(80)

$$[L'_\kappa, B'_\lambda] = (\kappa - \lambda)B'_{\kappa+\lambda},$$

(81)

$$[L'_\kappa, C'_\lambda] = (-2\kappa - \lambda)C'_{\kappa+\lambda}.$$  

(82)

The correspondence between discrete and continuous algebras is worth to mention. The commutators for $B'_\lambda$ and $C'_\lambda$ turns out to be “continuum version” of the untwisted commutators rather than twisted ones:

$$[L_m, b_n] = (m-n)b_{m+n}, \quad [L_m, c_n] = (-2m-n)c_{m+n}. $$

(83)

This is surprising but consistent with the fact that $L'_\kappa$ obeys Virasoro algebra without anomaly.

### 4.2 Mode expansion of Virasoro generators

We would like to derive the Fourier mode expansion of the Virasoro generator

$$L'_\kappa = L'^m_m + L'^g_g + a_\kappa,$$

(84)

where

$$L'^m_m = \oint \frac{dz}{2\pi i} f_\kappa g T^m$$

$$= -\frac{1}{\alpha'} [f_\kappa g : \partial X^\mu \partial X_\mu :].$$

(85)
\[ L'^g_\kappa = -\int \frac{dz}{2\pi i} f_\kappa g : \partial bc + 2b \partial c : = -\int \frac{dz}{2\pi i} f_\kappa h : cb : \]
\[ = -\left[ f_\kappa g : \partial bc + 2b \partial c : \right] - \left[ f_\kappa h : bc : \right]. \quad (86) \]

Here the normal ordering is defined through the time ordering prescription we have already worked out. Fourier mode expansion of the Virasoro generator is obtained by replacing each field in the Virasoro generator with the inverse Fourier expansion according to (31):
\[ \partial X^\mu(z) = -i \frac{\alpha'}{2} g(z)^{-1} \int d\kappa \mathcal{A}_\kappa^\mu f_{-\kappa}(z), \quad (87) \]
\[ c'(z) = \int d\kappa C_\kappa' f_{-\kappa}(z), \quad (88) \]
\[ b'(z) = g(z)^{-1} \int d\kappa B_\kappa' f_{-\kappa}(z). \quad (89) \]

Evaluation of matter part proceed straightforwardly. Two \( g^{-1}(z) \) from \( \partial X^\mu \) and another \( g(z) \) in the generator multiplies to total weight \( g^{-1}(z) \). And also, two \( f_\kappa(z) \) from \( \partial X^\mu \)s and another one in the generator give rise to a delta function
\[ \int \frac{dz}{2\pi i} \frac{f_{\kappa - \kappa_1 - \kappa_2}(z)}{g(z)} = \delta(\kappa - \kappa_1 - \kappa_2). \quad (90) \]

Then this delta function is integrated with the oscillator \( \mathcal{A}_\kappa^\mu \) and yields
\[ L^m_\kappa = \frac{1}{2} \int d\kappa' : \mathcal{A}_\kappa^{\mu,\mu'} \mathcal{A}_{\kappa',\kappa'} : , \quad (91) \]

which is merely a continuum version of the discrete expression.

The evaluation of ghost part is rather involved. We first replace the ghost pairs to the twisted ones in terms of the relation
\[ c(z) = z c'(z), \quad b(z) = z^{-1} b'(z). \quad (92) \]

This replacement reads
\[ -\left[ f_\kappa g : \partial bc + 2b \partial c : \right] - \left[ f_\kappa h : bc : \right] = -\left[ f_\kappa g : \partial b' c' + 2b' \partial c' : \right] - \left[ f_\kappa h : b' c' : \right] \]
\[ = \left[ f_\kappa z^{-1} : b' c' : \right]. \quad (93) \]

Then, by replacing \( b' \) and \( c' \) with the Fourier expansion (88) and (89), and evaluate each term carefully leads to rather simple result
\[ L'^g_\kappa = \int d\kappa'(2\kappa - \kappa') : B_\kappa' C_\kappa' : . \quad (94) \]
Note that this is again a continuous version of the un-twisted ghost Virasoro generator

$$L^g_m = \sum_k (2m - k)b_k c_{m-k}. \quad (95)$$

This result is again convinced from the fact that the continuous generator satisfy un-twisted algebra. In summary, the total Virasoro generator is continuum version of the un-twisted one up to the constant $a_\kappa$:

$$\mathcal{L}_\kappa' = \frac{1}{2} \int d\kappa' : A^\mu_{\kappa-k', \kappa} A_{\mu, \kappa'} : + \int d\kappa' (2\kappa - \kappa') : B'_{\kappa} C'_{\kappa-k'} : + a_\kappa. \quad (96)$$

### 4.3 Mode expansion of modified BRST charge

Having obtained mode expansion of fundamental fields, we next derive the mode expansion of the modified BRST charge, which will be used to investigate physical state. This can be done straightforwardly by inserting the expanded fields into the original expression of the modified BRST charge. First we rewrite the original expression of (6) in terms of twisted ghosts. Explicit expression of the BRST current in terms and matter and ghost CFT is

$$j_B(z) = c T^m - c b \partial c + \frac{3}{2} \partial^2 c$$

$$= z c' T^m - z c' b' \partial c' + \frac{3}{2} \partial^2 (z c'), \quad (97)$$

where we have used the relation $c(z) = z c'(z)$ and $b(z) = z^{-1} b'(z)$. Using this expression, we can write the modified BRST charge as

$$Q' = \oint \frac{dz}{2\pi i} z g(z) j_B(z) = \oint \frac{dz}{2\pi i} z^2 \partial (z^{-1} g(z)) g(z) c(z)$$

$$= [g c' T^m] - [g : c' b' \partial c' :] + \left[ + \frac{3}{2} \partial^2 (z c') - \frac{z^2 \partial (z^{-1} g(z))}{g(z)} z c' \right]$$

$$= [g c' T^m] - [g : c' b' \partial c' :] + \left[ + \frac{3}{2} - \frac{h^2}{g} \right] c', \quad (99)$$

where $k(z)$ and $h(z)$ are those defined in (46). The Fourier mode expansion is obtained by inserting expanded fields into this expression,

$$T^m(z) = g(z)^{-2} \int d\kappa \mathcal{L}^m_{\kappa} f_{-\kappa}(z), \quad (100)$$

$$c'(z) = \int d\kappa C'_{\kappa} f_{-\kappa}(z), \quad (101)$$

$$b'(z) = g(z)^{-1} \int d\kappa B'_{\kappa} f_{-\kappa}(z). \quad (102)$$
After some algebra, we arrive at following expression:

\[
Q' = \int d\kappa C'_{\kappa} L_{-\kappa}^m + \frac{1}{2} \int d\lambda d(\lambda - \kappa) : C'_{\kappa} B'_{-\kappa - \Lambda} C'_{\lambda} : + \int d\kappa C'_{\kappa} a_{\kappa}.
\]  

(103)

Further, this result can be compared to the expansion with Virasoro generator (96). As similar to the discrete case, the BRST charge can be expressed in terms of ghost Virasoro as

\[
Q' = \int d\kappa \left( C'_{\kappa} L_{-\kappa}^m + \frac{1}{2} : C'_{\kappa} L_{-\kappa}^{g} : \right) + \int d\kappa C'_{\kappa} a_{\kappa}.
\]  

(104)

This expression can be confirmed explicitly by inserting (96) into (103). Again, this can be compared to the expression of continuum version

\[
Q_B = \sum_n \left( c_n L_{-n}^m + \frac{1}{2} : c_n L_{-n}^{g} : \right).
\]  

(105)

In closing this section, we would like to summarize our results. Fourier mode expansion of \( L'_{\kappa} \) and \( Q' \) are obtained from those of \( L_n \) and \( Q_B \) by following procedure:

1. Replace \( \alpha^\mu_n, c_n, b_n \) to their continuum counterpart \( A^\mu_{\kappa}, C'_{\kappa}, B'_{\kappa} \).
2. Replace sum to the \( \kappa \) integral.
3. Include a constant \( a_{\kappa} \) to the Virasoro generator.

And also, we found that commutation relations are also continuum version of the discrete algebra of undeformed theory.

### 4.4 Spectrum

Now that we have identified the eigenmodes of the gauge \( L'_{0} \), the Fock space can be built by applying these oscillators to the vacuum. We follow same strategy as that of [18] by assuming existence of the vacuum \( |\Omega\rangle \) on which negative modes of continuous Fourier modes vanish. Note that the new vacuum \( |\Omega\rangle \) is completely different from the SL(2,R) vacuum \( |0\rangle \) of the original OSFT. To begin with, we assume that \( |\Omega\rangle \) satisfies

\[
\mathcal{A}^\mu_{\kappa} |\Omega\rangle = 0 \quad (\kappa > 0), \quad C'_{\kappa} |\Omega\rangle = 0 \quad (\kappa > 0), \quad B'_{\kappa} |\Omega\rangle = 0 \quad (\kappa > 0).
\]  

(106)

These conditions are chosen so that each of \{\partial X^\mu(z), c'(z), b'(z)\} vanishes at \( t \to -\infty \). Then, the Hilbert space is composed by Fock states of the form

\[
\mathcal{A}^\mu_{-\kappa_1} \cdots C'_{\kappa_j} \cdots B'_{-\kappa_k} \cdots |\Omega\rangle.
\]  

(107)

\footnote{Of course, the relation between \( |\Omega\rangle \) and \( |0\rangle \) is important issue, and to be addressed, since the continuous Hilbert space arises due to the shift of background from the original OSFT. We leave it as a future task as we find it not so straightforward.}
It is clear that such Fock states are eigenstates of $L'_0$. This choice is natural, since $L'_0$ is the gauge fixed kinetic operator, and expanding the Hilbert space in terms of eigenstates of it is the standard strategy in quantum field theory.

The physical space is given by the cohomology of the BRST charge $Q'$. Let us try to identify the cohomology. An important ingredient is the Fourier mode expansion of (103), which we quote here again:

$$Q' = \int d\kappa C'_{\kappa} L^m_{-\kappa} + \frac{1}{2} \int d\kappa d\lambda (\lambda - \kappa) : C'_{\kappa} B'_{-\kappa - \lambda} C'_{\lambda} : + \int d\kappa C'_{\kappa} a_{\kappa}. \quad (108)$$

The physical state should a closed state of $Q'$. This requires

$$Q' |P\rangle = 0, \quad B'_0 |P\rangle = 0 \rightarrow \{Q', B'_0\} |P\rangle = L'_0 |P\rangle = 0. \quad (109)$$

Therefore, $|P\rangle$ is an eigenstate of $L'_0$. We further require $|P\rangle$ to be killed by $C'_\kappa$ and $B'_\kappa$ for positive $\kappa$. Then, we obtain the “Virasoro constraint”

$$(L^m_{\kappa} - a_{-\kappa}) |P\rangle = 0 \quad (\kappa \geq 0), \quad (110)$$

This can be compared to the discrete counterpart of

$$L^m_n |P\rangle = 0 \quad (n > 0). \quad (111)$$

where $|P\rangle$ is obtained by weight 1 primary field. However, the operator expression (91) immediately implies that the condition (110) cannot be satisfied since $L^m_{\kappa} |P\rangle = 0$, which contradicts with nonzero value of $a_{\kappa}$. Therefore, there is no closed state in the Fock space, and then, no physical state. This result contradicts with the nontrivial physical state of [8] but is consistent with Sen’s conjecture of no open strings.

### 5 Summary and Discussion

We analyzed the identity based solution of Takahashi and Tanimoto by adopting the infinite circumstance formalism. We obtained the continuous Virasoro algebra of matte plus ghost system. The oscillator expression of conformal fields is introduced, and it turns out it can be obtained by extending an integer number to continuous variable.

We would like to explain the strong resemblance between discrete and continuous theories by introducing another coordinate system. It is

$$Z = f_0(z) = e^{z^2} = e^\rho. \quad (112)$$

Applying standard transformation law of primary field, we obtain

$$\phi_{\kappa} = \oint_C dZZ^{\kappa + h - 1} \tilde{\phi}(Z), \quad (113)$$
where $\tilde{\phi}(Z)$ is a transformed fields. Note that this expression looks similar to that of conventional CFT

$$\phi_n = \oint dz z^{x+h-1} \phi(z),$$

except for the continuous mode number. However, it is also should be noted that the correspondence between $Z$ and $\rho$ is not one-to-one. The contour $C$ winds infinitely many times around $Z = 0$ because of the relation

$$\rho = \log Z$$

Therefore, the worldsheet in $Z$ coordinate is a Riemann surface composed by infinitely many sheets. This interpretation explains the resemblance between discrete and continuous algebras, since the commutation relation is evaluated by an product between two operators nearby, therefore only reflects local property within same sheet.

The noncompact Riemann surface of $\log Z$ could be a source of further speculation about the nature of the tachyon vacuum. Intuitively, the noncompact worldsheet can be interpreted as a collection of open string worldsheets. Therefore, the tachyon vacuum encodes infinitely many open strings in certain manner. More concretely, let us consider a subset of continuous Virasoro generators labeled by a positive integer $q$ and arbitrary integer $n$:

$$L_n^q.$$  

By keeping $q$ fixed, we introduce a rescaled generators

$$t_n^q = \frac{L_n^q}{q}. $$

It is obvious that these generators form subalgebra since $[t_n^q, t_n^q] = (m - n)t_{m+n}^q$. Therefore, infinitely many discrete algebras are embedded in the continuous algebra.

Another interpretation is possible by considering non-scaled generators. The spectrum of non-scaled generators has an interval $\lambda$ since

$$[L_0, L_{-\lambda n}] = \lambda n L_{-\lambda n}. $$

The spectrum becomes denser for small $\lambda$. This is reminiscent of the spectrum found in [22], reported as the landscape of boundary string field theory.

Our analysis revealed unexpected richness of the Hilbert space of OSFT. In particular, the noncompact worldsheet introduced in this section will be important to understand the nature of tachyon vacuum. We expect further progress in this direction. Also, it will also be interesting to extend our analysis to the wedge based analytic solutions.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.
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