THE SYMPLECTIC NATURE OF THE SPACE OF DORMANT INDIGENOUS BUNDLES ON ALGEBRAIC CURVES

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Abstract. We study the symplectic nature of the moduli stack classifying dormant curves over a field $K$ of positive characteristic, i.e., proper hyperbolic curves over $K$ equipped with a dormant indigenous bundle. The central objects of the present paper are the following two Deligne-Mumford stacks. One is the cotangent bundle $\mathcal{T}_g,K^{\text{red}}$ of the moduli stack $\mathcal{M}_g,K^{\text{red}}$ classifying ordinary dormant curves over $K$ of genus $g$. The other is the moduli stack $\mathcal{S}_g,K^{\text{red}}$ classifying ordinary dormant curves over $K$ equipped with an indigenous bundle. These Deligne-Mumford stacks admit canonical symplectic structures respectively. The main result of the present paper asserts that a canonical isomorphism $\mathcal{T}_g,K^{\text{red}} \rightarrow \mathcal{S}_g,K^{\text{red}}$ preserves the symplectic structure. This result may be thought of as a positive characteristic analogue of the works of S. Kawai (in the paper entitled “The symplectic nature of the space of projective connections on Riemann surfaces”), P. Arés-Gastesi, I. Biswas, and B. Loustau. Finally, as its application, we construct a Frobenius-constant quantization on the moduli stack $\mathcal{S}_g,K^{\text{red}}$.

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Introduction

0.1. The purpose of the present paper is to study symplectic geometry of indigenous bundles in positive characteristic. Here, recall that the notion of an indigenous bundle was originally introduced and studied by R. C. Gunning in the context of Riemann surfaces (cf. [18]). Roughly speaking, an indigenous bundle on a connected compact hyperbolic Riemann surface $X$ is a...
projective line bundle (or equivalently, a PGL$_2$-torsor) on $X$ together with a connection and a global section satisfying a certain transversality condition. It may be thought of as an algebraic object that encodes the (analytic, i.e., non-algebraic) uniformization data for the $X$. Various equivalent mathematical objects, including certain kinds of differential operators between line bundles (related to Schwarzian derivatives), have been investigated by many mathematicians. For example, indigenous bundles on $X$ correspond bijectively to certain projective structures on the underlying topological space $\Sigma$ of $X$. A projective structure on $\Sigma$ is, by definition, the equivalence class of an atlas covered by coordinate charts on $\Sigma$ such that the transition functions are expressed as Möbius transformations. (In particular, each projective structure determines a unique Riemann surface structure on $\Sigma$.) If we are given a projective structure, then the collection of its transition functions specifies a PGL$_2$-torsor equipped with a connection, which forms an indigenous bundle. In this way, we obtain a bijective correspondence between the set of projective structures on $\Sigma$ defining the Riemann surface $X$ and the set of (isomorphism classes of) indigenous bundles on $X$.

0.2. Now, we shall recall the works of S. Kawai, P. Arés-Gastesi, I. Biswas, and B. Loustau (cf. [23]; [1]; [2]; [25]) related to natural symplectic structures on certain moduli spaces of projective structures, or equivalently, indigenous bundles.

Let $\Sigma$ be, as above, a connected orientable closed surface of genus $g > 1$. Write $\mathfrak{T}^\Sigma$ for the Teichmüller space associated with $\Sigma$, i.e., the quotient space
\[
\mathfrak{T}^\Sigma := \text{Conf}(\Sigma)/\text{Diff}^0(\Sigma),
\]
where Conf$(\Sigma)$ denotes the space of all conformal structures on $\Sigma$ compatible with the orientation of $\Sigma$, and Diff$^0(\Sigma)$ denotes the group of all diffeomorphisms of $\Sigma$ homotopic to the identity map of $\Sigma$. In particular, it is a covering space of the moduli stack $\mathcal{M}^\text{an}_{g,c}$ classifying connected compact Riemann surfaces of genus $g$. Also, write
\[
\mathfrak{S}^\Sigma := \text{Proj}(\Sigma)/\text{Diff}^0(\Sigma),
\]
where Proj$(\Sigma)$ denotes the space of all projective structures on $\Sigma$. It is known that the cotangent bundle $T^\vee_{\mathfrak{T}^\Sigma}$ of $\mathfrak{T}^\Sigma$ (resp., the quotient space $\mathfrak{S}^\Sigma$) admits a structure of complex manifold of dimension $6g - 6$, equipped with a holomorphic symplectic structure $\omega^\text{can}_{\mathfrak{T}^\Sigma}$ (resp., $\omega^\text{PGL}_{\mathfrak{T}^\Sigma}$ (cf. [15])). Consider an analytic section
\[
\sigma^\text{unif} : \mathfrak{T}^\Sigma \to \mathfrak{S}^\Sigma
\]
of the natural projection $\mathfrak{S}^\Sigma \to \mathfrak{T}^\Sigma$ arising from the uniformization in the sense of either Bers, Schottky, or Earle (cf. [4]; [8]; [13]). (Note that the Bers uniformization are determined after choosing a specific point of $\mathfrak{T}^\Sigma$.) Because of a natural affine structure on $\mathfrak{S}^\Sigma$, the section $\sigma^\text{unif}$ may be extended to an isomorphism
\[
\Psi^\text{unif} : T^\vee_{\mathfrak{T}^\Sigma} \sim \to \mathfrak{S}^\Sigma
\]
whose restriction to the zero section $\mathfrak{T}^\Sigma \to T^\vee_{\mathfrak{T}^\Sigma}$ coincides with $\sigma^\text{unif}$. It follows from [23], Theorem, [1], Theorem 1.1, [1], Remark 3.2, and [25], Theorem 6.10, that $\Psi^\text{unif}$ preserves the symplectic structure up to a constant factor, i.e.,
\[
\Psi^\text{unif}_* (\omega^\text{PGL}_{\mathfrak{T}^\Sigma}) = \sqrt{-1} \cdot \omega^\text{can}_{\mathfrak{T}^\Sigma}.
\]
(Notice that we use the conventions concerning $\omega_{PGL}^T$ chosen in [25]. With these conventions, Kawai’s result may be described as the equality (2), as B. Loustau mentioned in a footnote of loc. cit. In Kawai’s original paper, he asserted the equality $\Psi^* \unif(\omega_{PGL}^T) = \pi \cdot \omega_{\can}^T$.)

0.3. Our aim in the present paper is to address the question whether a similar result holds for hyperbolic (algebraic) curves of positive characteristic. Just as in the case of the theory over $\mathbb{C}$, one may define the notion of an indigenous bundle in characteristic $p > 0$ and their moduli space. Various properties of such objects were discussed in the context of the $p$-adic Teichmüller theory developed by S. Mochizuki (cf. [27]; [28]). One of the key ingredients in the development of this theory is the study of the $p$-curvature of indigenous bundles. Recall that the $p$-curvature of a connection measures the obstruction to the compatibility of $p$-power structures that appear in certain associated spaces of infinitesimal (i.e., “Lie”) symmetries. We say that an indigenous bundle is dormant (cf. Definition 3.1.1 (i)) if its $p$-curvature vanishes identically. This condition implies that the underlying projective line bundle with connection is locally trivial in the Zariski topology.

In many aspects (including the aspect just explained), dormant indigenous bundles may be thought of as reasonable (algebraic) products used to develop an analogous theory of indigenous bundles on Riemann surfaces. As explained in §0.2 each connected compact hyperbolic Riemann surface $X$ (with marking) of genus $g > 1$ admits a canonical indigenous bundle $P^\otimes_X$ determined by the section $\sigma_{\unif}$. Thus, the Teichmüller space $\tilde{\Sigma}^\Sigma$ may be identified with the moduli space classifying such $X$’s equipped with a specific nice indigenous bundle (i.e., $P^\otimes_X$). With that in mind, we consider the moduli stack classifying proper hyperbolic curves of characteristic $p$ equipped with a dormant indigenous bundle as a characteristic $p$ analogue of such a covering space of $\mathcal{M}^\an_{g,\mathbb{C}}$. On the basis of this perspective, we give an affirmative answer to the above question.

0.4. In what follows, we shall describe the main result of the present paper. Let $K$ be a field of characteristic $p > 2$ and $g$ an integer $> 1$. Denote by $\mathfrak{S}^{\zar\ldots}_{g,K}$ (resp., $\mathfrak{G}^{\zar\ldots}_{g,K}$) (cf. (29) and (40)) the moduli stack classifying ordinary dormant curves (cf. Definition 3.1.1 (ii)) of genus $g$ over $K$ (resp., ordinary dormant curves of genus $g$ over $K$ equipped with an indigenous bundle). Also, denote by $\mathcal{T}^{\zar\ldots}_{g,K}$ (cf. (40)) the cotangent bundle of $\mathfrak{S}^{\zar\ldots}_{g,K}$. It is known (cf. Propositions 2.8.1 and 3.2.1) that $\mathfrak{G}^{\zar\ldots}_{g,K}$ (resp., $\mathcal{T}^{\zar\ldots}_{g,K}$) may be represented by a geometrically connected smooth Deligne-Mumford stack over $K$ of dimension $6g - 6$. As we will discuss in §4.4, there exists a canonical symplectic structure on $\mathfrak{G}^{\zar\ldots}_{g,K}$ (resp., $\mathcal{T}^{\zar\ldots}_{g,K}$), which we denote by $\omega_{\otimes}^{PGL}$ (resp., $\omega_{\otimes}^{\can}$).

Then, the main result of the present paper is described as follows.

**Theorem A** (cf. Theorem 4.4.1). The canonical isomorphism

$$\Psi_{g,K} : \mathfrak{S}^{\zar\ldots}_{g,K} \xrightarrow{\sim} \mathfrak{G}^{\zar\ldots}_{g,K}$$
(cf. (41)) preserves the symplectic structure, i.e.,
\[ \Psi^*(\omega_{PGL}^{\bullet}) = \omega_{can}^{\bullet}. \]

In particular, the above theorem implies that the image of the canonical section \( \mathfrak{M}_{g,K}^{\zeta} \to \mathfrak{G}_{g,K}^{\zeta} \) is Lagrangian (cf. Remark 4.4.2) with respect to the symplectic structure \( \omega_{PGL}^{\bullet} \). Finally, as an application of Theorem A, we construct (cf. Corollary 6.0.4) a so-called Frobenius-constant quantization on the moduli stack \( \mathfrak{G}_{g,K}^{\zeta} \); such an additional structure will be of interest in the context of symplectic geometry.

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1. **Preliminaries**

In this section, we shall review some definitions and facts concerning our discussion.

1.1. **Ground ring.** Throughout the present paper, we fix an integer \( g > 1 \) and a commutative ring \( R \) over \( \mathbb{Z} \). The assumption that 2 is invertible in the ground ring \( R \) will be necessary to construct the symplectic structure \( \omega_{PGL}^{\bullet} \) introduced later (cf. (32)) and to apply previous results on indigenous bundles in positive characteristic (cf. [28]) to our discussion. Also, we shall write \( \mathbf{Set} \) for the category of (small) sets.

1.2. **Sheaves and complexes.** Let \( S \) be a Deligne-Mumford stack over \( R \) (cf. [24] for the definition and basic properties of Deligne-Mumford stacks). Unless stated otherwise, the structure sheaf \( \mathcal{O}_S \) of \( S \) and all \( \mathcal{O}_S \)-modules are considered as sheaves in the étale topology. If \( \nabla : \mathcal{K}^0 \to \mathcal{K}^1 \) is a morphism of sheaves (in the étale topology) of abelian groups on \( S \), then we often think of it as a complex concentrated at degree 0 and 1. Denote this complex by
\[ \mathcal{K}^\bullet[\nabla]. \]

Also, any abelian sheaf \( \mathcal{F} \) may be thought of as a complex concentrated at degree 0. For \( n \in \mathbb{Z} \), we define the complex
\[ \mathcal{F}[n] \]
to be \( \mathcal{F} \) shifted down by \( n \), so that \( \mathcal{F}[n]^{-n} := \mathcal{F} \) and \( \mathcal{F}[n]^i := 0 \) for each \( i \neq -n \).

If, moreover, \( X \) is a Deligne-Mumford stack over \( S \), then we shall write \( \Omega_{X/S} \) for the sheaf of 1-forms of \( X \) over \( S \), \( \bigwedge^i \Omega_{X/S} \) for its \( i \)-th exterior power, and \( \mathcal{T}_{X/S} \) for the dual \( \mathcal{O}_X \)-module \( \Omega^\vee_{X/S} \) of \( \Omega_{X/S} \) (i.e., the sheaf of derivations on \( \mathcal{O}_X \) over \( S \)).
1.3. **Complex analytic spaces.** Suppose that $X$ is a complex analytic space (resp., a complex analytic space over a complex analytic space $S$). Then, we shall write $\mathcal{O}_X$ for the structure sheaf of $X$ consisting of holomorphic functions, $\Omega_X$ (resp., $\Omega_{X/S}$) for the sheaf of holomorphic 1-forms of $X$ (resp., the sheaf of holomorphic 1-forms of $X$ over $S$), and $\mathcal{T}_X$ (resp., $\mathcal{T}_{X/S}$) for the dual $\mathcal{O}_X$-module of $\Omega_X$ (resp., $\Omega_{X/S}$).

Next, let us suppose that $X$ is a scheme (resp., a Deligne-Mumford stack) of finite type over $\mathbb{C}$. Then, we shall write $X^{an}$ for the complex analytic space (resp., the complex analytic stack, i.e., the stack in groupoids over the category of complex analytic spaces equipped with the analytic topology) associated with $X$.

1.4. **Symplectic structures.** Let $X$ be a smooth Deligne-Mumford stack over $R$ of relative dimension $n > 0$ (resp., a smooth Deligne-Mumford complex analytic stack (cf. 3.3) of dimension $n > 0$). A **symplectic structure** on $X$ is, by definition, a nondegenerate closed 2-form $\omega \in \Gamma(X, \bigwedge^2 \Omega_{X/R})$ (resp., $\omega \in \Gamma(X, \bigwedge^2 \Omega_X)$). Here, a 2-form $\omega$ is called **nondegenerate** if the morphism $\Omega_{X/R} \to \mathcal{T}_{X/R}$ (resp., $\Omega_X \to \mathcal{T}_X$) induced naturally by $\omega$ is an isomorphism.

Given an $\mathcal{O}_X$-module $\mathcal{F}$, we shall write

$$\mathbb{A}(\mathcal{F})$$

for the total space of $\mathcal{F}$. In the non-resp’d case, $\mathbb{A}(\mathcal{F})$ forms the relative affine scheme $\text{Spec}(\mathcal{S}(\mathcal{F}^\vee))$ over $X$, where $\mathcal{S}(\mathcal{F}^\vee)$ denotes the symmetric algebra on the dual $\mathcal{F}^\vee$ of $\mathcal{F}$ over $\mathcal{O}_X$. Denote by $T_X^\vee$ the total space of $\Omega_{X/R}$ (resp., $\Omega_X$), i.e.,

$$T_X^\vee := \mathbb{A}(\Omega_{X/R}) \ (\text{resp., } T_X^\vee := \mathbb{A}(\Omega_X)),$$

which is a smooth Deligne-Mumford stack over $R$ of relative dimension $2n$ (resp., a smooth Deligne-Mumford complex analytic stack of dimension $2n$); we shall refer to $T_X^\vee$ as the **cotangent bundle** of $X$. Denote by

$$\pi_T^\vee: T_X^\vee \to X, \quad 0_X: X \to T_X^\vee$$

the natural projection and the zero section respectively. It is well-known that there exists a unique 1-form

$$\lambda_X \in \Gamma(T_X^\vee, \Omega_{T_X^\vee/R}) \ (\text{resp., } \lambda_X \in \Gamma(T_X^\vee, \Omega_{T_X^\vee}))$$

on $T_X^\vee$ determined by the following condition: if $\lambda_\sigma$ is the 1-form on an open subscheme $U$ of $X$ corresponding to a local section $\sigma: U \to T_X^\vee$ of $T_X^\vee$, then the equality $\sigma^*(\lambda_X) = \lambda_\sigma$ holds. We shall refer to $\lambda_X$ as the **Liouville form** on $T_X^\vee$. By construction, the Liouville form $\lambda_X$ lies in $\Gamma(T_X^\vee, \pi_T^\vee(\Omega_{X/R})) \subseteq \Gamma(T_X^\vee, \Omega_{T_X^\vee/R})$ (resp., $\Gamma(T_X^\vee, \pi_T^\vee(\Omega_X)) \subseteq \Gamma(T_X^\vee, \Omega_{T_X^\vee})$). Its exterior derivative

$$(3) \quad \omega_X^{\text{can}} := d\lambda_X \in \Gamma(T_X^\vee, \bigwedge^2 \Omega_{T_X^\vee/R}) \quad (\text{resp., } \omega_X^{\text{can}} := d\lambda_X \in \Gamma(T_X^\vee, \bigwedge^2 \Omega_{T_X^\vee}))$$

defines a symplectic structure on $T_X^\vee$. If $q_1, \ldots, q_n$ are local coordinates in $X$, then the dual coordinates $p_1, \ldots, p_n$ in $T_X^\vee$ are the coefficients of the decomposition of the 1-form $\lambda_X$ into linear combination of the differentials $dq_i$, i.e., $\lambda_X = \sum_{i=1}^n p_i dq_i$. Hence, $\omega_X^{\text{can}}$ may be expressed locally as $\omega_X^{\text{can}} = \sum_{i=1}^n dp_i \wedge dq_i$. 

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1.5. Curves and their moduli space. Let \( S \) be a Deligne-Mumford stack over \( R \). By a curve over \( S \), we mean a geometrically connected smooth relative scheme \( f : X \to S \) over \( S \) of relative dimension 1. Here, a morphism \( f : X \to S \) between Deligne-Mumford stacks is called a relative scheme if it is a schematic morphism. Also, we shall say that a proper curve \( f : X \to S \) over \( S \) is of genus \( g \) if the direct image \( f_*(\Omega_X/S) \) of \( \Omega_X/S \) is a locally free \( \mathcal{O}_S \)-module of constant rank \( g \). Write

\[
\mathcal{M}_{g,R}
\]

for the moduli stack classifying proper curves of genus \( g \) over \( R \); it may be represented by a geometrically connected smooth Deligne-Mumford stack over \( R \) of relative dimension \( 3g - 3 \). Also, write \( f_{g,R} : C_{g,R} \to \mathcal{M}_{g,R} \) for the tautological curve over \( \mathcal{M}_{g,R} \) and

\[
\mathcal{Gsch}/\mathcal{M}_{g,R}
\]

for the category of relative schemes over \( \mathcal{M}_{g,R} \).

It follows from Serre duality that for any proper curve \( f : X \to S \), the \( \mathcal{O}_S \)-module \( R^1f_*(\Omega_X/S) \) is isomorphic to \( \mathcal{O}_S \). Throughout the present paper, we fix a specific choice of an isomorphism

\[
\int_{C_{g,z,\frac{1}{2}}}: R^1f_{g,z,\frac{1}{2}}(\Omega_{C_{g,z,\frac{1}{2}}}/\mathcal{M}_{g,z,\frac{1}{2}}) \sim \mathcal{O}_{g,z,\frac{1}{2}}
\]

of \( \mathcal{O}_{g,z,\frac{1}{2}} \)-modules (i.e., the trace map). Note that the isomorphism \( \int_{C_{g,z,\frac{1}{2}}} \) will be used to determine both the symplectic structure \( \omega_{g,R}^{\text{PGL}} \) (cf. (32)) and the morphism denoted by \( \Psi_{g,K} \) (cf. (41)); we have to choose them in such a coherent way in order to obtain the equality asserted in Theorem A.

If \( u : T \to \mathcal{M}_{g,R} \) is an object of \( \mathcal{Gsch}/\mathcal{M}_{g,R} \), then we shall write

\[
f_T : C_T \to T
\]

for the curve over \( T \) classified by \( u \), i.e., \( C_T := C_{g,R} \times_{f_{g,R},\mathcal{M}_{g,R}} T \). We obtain an isomorphism

\[
\int_{C_T} : R^1f_*(\Omega_{C_T/T}) \sim \mathcal{O}_T
\]

defined as the pull-back of \( \int_{C_{g,z,\frac{1}{2}}} \) via the composite \( T \Rightarrow \mathcal{M}_{g,R} \to \mathcal{M}_{g,z,\frac{1}{2}} \). For a vector bundle \( \mathcal{E} \) on \( C_T \) (i.e., a locally free coherent \( \mathcal{O}_{C_T} \)-module), denote by

\[
\int_{C_T,\mathcal{E}} : R^1f_*(\Omega_{C_T/T} \otimes \mathcal{E}^\vee) \sim f_T^*(\mathcal{E})^\vee
\]

the isomorphism induced from the pairing

\[
\begin{array}{cccc}
R^1f_*(\Omega_{C_T/T} \otimes \mathcal{E}^\vee) & \otimes & f_T^*(\mathcal{E}) & \longrightarrow \ U \\
\downarrow & & \downarrow & \\
R^1f_*(\Omega_{C_T/T} \otimes (\mathcal{E}^\vee \otimes \mathcal{E})) & \longrightarrow & R^1f_*(\Omega_{C_T/T}) & \\
\int_{C_T} & & & \longrightarrow \mathcal{O}_T,
\end{array}
\]

where the second arrow denotes the morphism arising from the natural pairing \( \mathcal{E}^\vee \otimes \mathcal{E} \to \mathcal{O}_{C_T} \).
1.6. Connections. Let $S$ be a relative scheme over $\mathcal{M}_{g,R}$, i.e., an object of $\mathfrak{Sch}_{[\mathfrak{M}_{g,R}]}$, which classifies a proper curve $f_S : C_S \to S$ over $S$. Let $G$ be a connected smooth algebraic group over $R$ with Lie algebra $\mathfrak{g}$ and $\pi : \mathcal{P} \to C_S$ a (right) $G$-torsor over $C_S$. Write

$$\text{ad}(\mathcal{P})$$

for the adjoint vector bundle associated with $\mathcal{P}$. That is to say, $\text{ad}(\mathcal{P})$ is the vector bundle obtained from $\mathcal{P}$ via change of structure group by the adjoint representation $\text{Ad} : G \to \text{GL}(\mathfrak{g})$. The direct image $\pi_*(\mathcal{T}_{\mathcal{P}/S})$ has a $G$-action arising from the $G$-action on $\mathcal{P}$, and hence, we obtain its subsheaf

$$\tilde{\mathcal{T}}_{\mathcal{P}/S} := (\pi_*(\mathcal{T}_{\mathcal{P}/S}))^G$$

consisting of $G$-invariant sections. The differential of $\pi$ gives a short exact sequence of $\mathcal{O}_{C_S}$-modules

$$0 \to \text{ad}(\mathcal{P}) \to \tilde{\mathcal{T}}_{\mathcal{P}/S} \xrightarrow{\alpha_\mathcal{P}} \mathcal{T}_{C_S/S} \to 0. \tag{6}$$

An $S$-connection on $\mathcal{P}$ is, by definition, a split injection $\nabla_{\mathcal{P}} : \mathcal{T}_{C_S/S} \to \tilde{\mathcal{T}}_{\mathcal{P}/S}$ of the short exact sequence $\textbf{(6)}$, i.e., $\alpha_\mathcal{P} \circ \nabla_{\mathcal{P}} = \text{id}$. Since $C_S$ is of relative dimension 1 over $S$, any such $S$-connection is necessarily integrable, which means that it is compatible with the respective Lie bracket structures on $\mathcal{T}_{C_S/S}$ and $\tilde{\mathcal{T}}_{\mathcal{P}/S} = (\pi_*(\mathcal{T}_{\mathcal{P}/S}))^G$. By a flat $G$-torsor over $C_S$, we mean a pair $(\mathcal{P}, \nabla_{\mathcal{P}})$ consisting of a $G$-torsor $\mathcal{P}$ over $C_S$ and an $S$-connection $\nabla_{\mathcal{P}}$ on $\mathcal{P}$.

If $G = \text{GL}_m$ for some $m \geq 1$, then the notion of an $S$-connection on a $\text{GL}_m$-torsor $\mathcal{P}$ recalled here may be identified with the usual definition of an $S$-connection (cf. [22], §1.0) on the associated vector bundle $\mathcal{P} \times_{\text{GL}_m} (\mathcal{O}_{C_S}^{\oplus m})$ (cf. Remark [17.1] for a detailed discussion); in this situation, we shall not distinguish between these notions of connections.

Given an $S$-connection $\nabla_{\mathcal{P}}$ on $\mathcal{P}$, we denote by

$$\nabla_{\mathcal{P}}^{\text{ad}} : \text{ad}(\mathcal{P}) \to \Omega_{C_S/S} \otimes \text{ad}(\mathcal{P}) \tag{7}$$

the $S$-connection on $\text{ad}(\mathcal{P})$ induced by $\nabla_{\mathcal{P}}$ via the change of structure group by $\text{Ad} : G \to \text{GL}(\mathfrak{g})$. More explicitly, $\nabla_{\mathcal{P}}^{\text{ad}}$ is the connection uniquely determined by the condition that $\langle \partial_1, \nabla_{\mathcal{P}}^{\text{ad}}(\partial_2) \rangle = [\nabla_{\mathcal{P}}(\partial_1), \partial_2]$ for any local sections $\partial_1 \in \mathcal{T}_{C_S/S}$ and $\partial_2 \in \text{ad}(\mathcal{P})$, where $\langle - , - \rangle$ denotes the pairing $\mathcal{T}_{C_S/S} \times (\Omega_{C_S/S} \otimes \text{ad}(\mathcal{P})) \to \text{ad}(\mathcal{P})$ arising from the natural pairing $\mathcal{T}_{C_S/S} \times \Omega_{C_S/S} \to \mathcal{O}_{C_S}$.

1.7. Ring of differential operators. Recall (cf. [7], §1.2) that the sheaf of crystalline differential operators on $C_S$ over $S$ is the Zariski sheaf

$$\mathcal{D}_{C_S/S}$$

on $C_S$ generated, as a sheaf of noncommutative rings, by $\mathcal{O}_{C_S}$ and $\mathcal{T}_{C_S/S}$ subject to the relations

$$f_1 \ast f_2 = f_1 \cdot f_2, \quad f_1 \ast \xi_1 = f_1 \cdot \xi_1, \quad \xi_1 \ast \xi_2 - \xi_2 \ast \xi_1 = [\xi_1, \xi_2], \quad \xi_1 \ast f_1 - f_1 \ast \xi_1 = \xi_1(f_1)$$

for any local sections $f_1, f_2 \in \mathcal{O}_{C_S}$ and $\xi_1, \xi_2 \in \mathcal{T}_{C_S/S}$, where $\ast$ denotes the multiplication in $\mathcal{D}_{C_S/S}$. In a usual sense, the order $(\geq 0)$ of a given crystalline differential operator, i.e., a local section of $\mathcal{D}_{C_S/S}$, is well-defined. Hence, $\mathcal{D}_{C_S/S}$ admits, for each $j \geq 0$, the subsheaf

$$\mathcal{D}_{C_S/S}^{\leq j} \subseteq \mathcal{D}_{C_S/S}$$

consisting of local sections of $\mathcal{D}_{C_S/S}$ of order $\leq j$. The sheaf $\mathcal{D}_{C_S/S}$ (resp., $\mathcal{D}_{C_S/S}^{\leq j}$ for each $j = 0, 1, 2, \cdots$) admits two different structures of $\mathcal{O}_{C_S}$-module — one as given by left multiplication,
where we denote this $\mathcal{O}_{C_S}$-module by $^lD_{C_S/S}$ (resp., $^rD_{C_S/S}^{\leq j}$), and the other given by right multiplication, where we denote this $\mathcal{O}_{C_S}$-module by $^rD_{C_S/S}$ (resp., $^rD_{C_S/S}^{\leq j}$) —. In particular, we have $^lD_{C_S/S}^{\leq 0} = ^rD_{C_S/S}^{\leq 0} = \mathcal{O}_{C_S}$ (as $\mathcal{O}_{C_S}$-modules). The set $\{D_{C_S/S}^{\leq j}\}_{j \geq 0}$ forms an increasing filtration on $D_{C_S/S}$ satisfying that

\[ \bigcup_{j \geq 0} D_{C_S/S}^{\leq j} = D_{C_S/S}, \quad \text{and} \quad D_{C_S/S}^{\leq j}/D_{C_S/S}^{\leq (j-1)} \cong T_{C_S/S}\]  

for every $j \geq 1$.

Let $F$ be a vector bundle on $C_S$. In what follows, we shall regard the tensor product $D_{C_S/S}^{\leq j} \otimes F := \mathcal{O}_{C_S} \otimes F$ (resp., $F \otimes D_{C_S/S}^{\leq j} := F \otimes \mathcal{O}_{C_S} \otimes F$) as being equipped with a structure of $\mathcal{O}_{C_S}$-module arising from the structure of $\mathcal{O}_{C_S}$-module $^lD_{C_S/S}$ (resp., $^rD_{C_S/S}^{\leq j}$) on $D_{C_S/S}^{\leq j}$.

Next, let $\nabla_F$ be an $S$-connection on $F$. It induces a structure of left $D_{C_S/S}$-module

\[ \hat{\nabla}_F : D_{C_S/S} \otimes F \to F \]

on $F$ determined uniquely by the condition that $\hat{\nabla}_F(\partial \otimes v) = (\partial, \nabla_F(v))$ for any local sections $v \in F$ and $\partial \in T_{C_S/S}$, where $(-, -)$ denotes the pairing $T_{C_S/S} \times (\Omega_{C_S} \otimes F) \to F$ induced by the natural paring $T_{C_S/S} \times \Omega_{C_S} \to \mathcal{O}_{C_S}$. The assignment $\nabla_F \mapsto \hat{\nabla}_F$ determines a bijective correspondence between the set of $S$-connections on $F$ and the set of structures of left $D_{C_S/S}$-module on $F$.

**Remark 1.7.1.** As mentioned in §1.6, there exists a bijective correspondence between $S$-connections on a $\text{GL}_m$-torsor $P$ and $S$-connections (in the classical sense) on the corresponding vector bundle $F := P \times^{\text{GL}_m} (\mathcal{O}_{C_S}^m)$. In this remark, we describe this correspondence in somewhat detail.

Let us write

\[ D_{\text{diff}}^{\leq 1}_{\hat{\nabla}, F} := \mathcal{H}om_{\mathcal{O}_{C_S}}(F, F \otimes D_{C_S/S}^{\leq 1}), \]

i.e., the sheaf of first order differential operators from $F$ to $F$ itself. The natural surjection $D_{C_S/S}^{\leq 1} \to T_{C_S/S}$ (cf. (8)) induces a surjection

\[ q : D_{\text{diff}}^{\leq 1}_{\hat{\nabla}, F} \to \mathcal{E}nd_{\mathcal{O}_{C_S}}(F) \otimes T_{C_S/S}. \]

Note that the morphism $T_{C_S/S} \hookrightarrow \mathcal{E}nd_{\mathcal{O}_{C_S}}(F) \otimes T_{C_S/S}$ given by assigning $\partial \mapsto \text{id}_F \otimes \partial$ for any local section $\partial \in T_{C_S/S}$ is injective; we shall regard $T_{C_S/S}$ as an $\mathcal{O}_{C_S}$-submodule of $\mathcal{E}nd_{\mathcal{O}_{C_S}}(F) \otimes T_{C_S/S}$ by this injection. Let us write

\[ D_{\text{diff}}^{\leq 1\bullet}_{\hat{\nabla}, F} := q^{-1}(T_{C_S/S}) \quad (\subseteq D_{\text{diff}}^{\leq 1}_{\hat{\nabla}, F}), \]

which admits a surjection

\[ \alpha_{\hat{\nabla}} := q|_{D_{\text{diff}}^{\leq 1\bullet}_{\hat{\nabla}, F}} : D_{\text{diff}}^{\leq 1\bullet}_{\hat{\nabla}, F} \to T_{C_S/S}. \]

Recall (cf. [10], § 2, (2.2)) that there exists a canonical isomorphism

\[ \xi : T_{P/S} \cong D_{\text{diff}}^{\leq 1\bullet}_{\hat{\nabla}, F}. \]
of \(O_{CS}\)-modules making the following diagram commute:

\[
\begin{array}{ccc}
\tilde{T}_{P/S} & \xrightarrow{\xi} & \text{Diff}^{\leq 1, \cdot}_{\partial, F} \\
\downarrow{\alpha_P} & & \downarrow{\alpha_F} \\
T_{CS/S} & & 
\end{array}
\]

Now, suppose that we are given an \(S\)-connection \(\nabla^1_P : T_{CS/S} \to \tilde{T}_{P/S}\) on the GL\(_m\)-torsor \(P\). Then, one may find an \(S\)-connection \(\nabla^1_F : F \to \Omega_{CS/S} \otimes F\) on \(F\) satisfying the equality 
\((\xi \circ \nabla^1_P)(\partial)(v) - v \otimes \partial = \langle \partial, \nabla^1_F(v) \rangle\) for any local sections \(\partial \in T_{CS/S}\) and \(v \in F\), where \(\langle - , - \rangle\) denotes the pairing \(T_{CS/S} \times (\Omega_{CS/S} \otimes F) \to F\) arising from the natural pairing \(T_{CS/S} \times \Omega_{CS/S} \to O_{CS}\).

Conversely, let \(\nabla^2_F : F \to \Omega_{CS/S} \otimes F\) be an \(S\)-connection on the vector bundle \(F\) and \(\partial\) a local section of \(T_{CS/S}\). Then, the assignment \(v \mapsto \langle \partial, \nabla^2_F(v) \rangle + v \otimes \partial\) (where \(v \in F\)) determines a local section \(\delta_{\partial}\) of \(\text{Diff}^{\leq 1, \cdot}_{\partial, F}\). The morphism \(\nabla^2_P : T_{CS/S} \to \tilde{T}_{P/S}\) given by assigning \(\partial \mapsto \xi^{-1}(\delta_{\partial})\) forms an \(S\)-connection on \(P\).

One verifies that the assignments \(\nabla^1_P \mapsto \nabla^1_F\) and \(\nabla^2_F \mapsto \nabla^2_P\) determine a bijective correspondence between the set of \(S\)-connections on \(P\) and the set of \(S\)-connections on \(F\), as desired.

## 2. Indigenous Bundles

In this section, we recall the notion of an indigenous bundle on a curve and some properties concerning indigenous bundles. We refer the reader to [27], [28], [31], [32] for detailed discussions involved.

### 2.1. Let us fix a relative scheme \(S\) over \(\mathcal{M}_{g,R}\), which classifies a proper curve \(f_S : C_S \to S\) of genus \(g\). In what follows, let \(G\) be the projective linear group over \(R\) of rank 2, i.e., \(G := \text{PGL}_2\). Since 2 is invertible in \(R\), its Lie algebra \(\mathfrak{g}\) is naturally isomorphic to \(\text{sl}_2\). Write \(B\) for the Borel subgroup of \(G\) defined to be the image of upper triangular matrices via the natural quotient \(\text{GL}_2 \twoheadrightarrow G\). We recall from [14], §4, or [27], Chap.I, §2, Definition 2.2, the following definition of an indigenous bundle:

**Definition 2.1.1.**

(i) Let \(\mathcal{P}^\circ := (\mathcal{P}_B, \nabla_{\mathcal{P}_B})\) be a pair consisting of a (right) \(B\)-torsor \(\mathcal{P}_B\) over \(C_S\) and an \(S\)-connection \(\nabla_{\mathcal{P}_B}\) on the \(G\)-torsor \(\mathcal{P}_G := \mathcal{P}_B \times^B G\) induced by \(\mathcal{P}_B\). We shall say that \(\mathcal{P}^\circ\) is an **indigenous bundle** on \(C_S/S\) if the composite

\[(9) \quad \nabla_{\mathcal{P}_G} : T_{CS/S} \xrightarrow{\nabla_{\mathcal{P}_B}} \tilde{T}_{P_B/S} \to \tilde{T}_{P_G/S}/\tilde{\iota}(\tilde{T}_{P_B/S})\]

is an isomorphism, where \(\tilde{\iota}\) denotes the natural injection \(\tilde{T}_{P_B/S} \hookrightarrow \tilde{T}_{P_G/S}\).

(ii) Let \(\mathcal{P}^\circ := (\mathcal{P}_B, \nabla_{\mathcal{P}_B})\) and \(\mathcal{Q}^\circ := (\mathcal{Q}_B, \nabla_{\mathcal{Q}_B})\) be indigenous bundles on \(C_S/S\). An **isomorphism of indigenous bundles** from \(\mathcal{P}^\circ\) to \(\mathcal{Q}^\circ\) is an isomorphism \(\mathcal{P}_B \simeq \mathcal{Q}_B\) of \(B\)-torsors such that the induced isomorphism \(\mathcal{P}_G \simeq \mathcal{Q}_G\) of \(G\)-torsors is compatible with the respective \(S\)-connections.
Then, we already know that the following proposition holds.

**Proposition 2.1.2** (cf. [27], Chap. I, §2, Theorem 2.8). *Any indigenous bundle on $C_S/S$ does not have nontrivial automorphisms.*

2.2. In what follows, we shall construct a canonical filtration on the adjoint vector bundle associated with the underlying $G$-torsor of an indigenous bundle. Let $\mathcal{P}_G := (\mathcal{P}_B, \nabla_{\mathcal{P}_G})$ be an indigenous bundle on $C_S/S$. Consider the morphism of short exact sequences

$$0 \rightarrow \text{ad}(\mathcal{P}_B) \rightarrow \tilde{T}_{\mathcal{P}_B/S} \xrightarrow{\alpha_{\mathcal{P}_B}} T_{C_S/S} \rightarrow 0 \quad \text{arising from the change of structure group } B \hookrightarrow G.$$ 

This diagram yields an isomorphism

$$\text{ad}(\mathcal{P}_G)/\iota(\text{ad}(\mathcal{P}_B)) \sim \tilde{T}_{\mathcal{P}_G/S}/\tilde{\iota}(\tilde{T}_{\mathcal{P}_B/S}).$$

Let us define a 3-step decreasing filtration \{\text{ad}(\mathcal{P}_G)^i\}_{i=0}^3 on the rank 3 vector bundle ad($\mathcal{P}_G$) as follows:

- $\text{ad}(\mathcal{P}_G)^0 := \text{ad}(\mathcal{P}_G),$
- $\text{ad}(\mathcal{P}_G)^1 := \iota(\text{ad}(\mathcal{P}_B)),$
- $\text{ad}(\mathcal{P}_G)^2 := \text{Ker}\left(\text{ad}(\mathcal{P}_G)^1 \xrightarrow{\nabla_{\mathcal{P}_G}^{\text{ad}(\mathcal{P}_G)^1}} \Omega_{C_S/S} \otimes \text{ad}(\mathcal{P}_G) \rightarrow \Omega_{C_S/S} \otimes (\text{ad}(\mathcal{P}_G)/\text{ad}(\mathcal{P}_G)^1)\right),$
- $\text{ad}(\mathcal{P}_G)^3 := 0$

(cf. (7) for the definition of $\nabla_{\mathcal{P}_G}^{\text{ad}(\mathcal{P}_G)}$). It follows from the definition of an indigenous bundle that

$$\nabla_{\mathcal{P}_G}^{\text{ad}(\mathcal{P}_G)^j+1} : \text{ad}(\mathcal{P}_G)^{j+1}/\text{ad}(\mathcal{P}_G)^j \rightarrow \Omega_{C_S/S} \otimes (\text{ad}(\mathcal{P}_G)^j/\text{ad}(\mathcal{P}_G)^{j+1})$$

for any $j \in \{0, 1\}$ and the $\mathcal{O}_{C_S}$-linear morphism

$$\nabla_{\mathcal{P}_G}^{\text{ad}(\mathcal{P}_G)^j} : \text{ad}(\mathcal{P}_G)^j/\text{ad}(\mathcal{P}_G)^{j+1} \rightarrow \Omega_{C_S/S} \otimes (\text{ad}(\mathcal{P}_G)^j/\text{ad}(\mathcal{P}_G)^{j+1})$$

induced by $\nabla_{\mathcal{P}_G}^{\text{ad}(\mathcal{P}_G)}$ is an isomorphism. Denote by

$$\nabla_{\mathcal{P}_G} : (\text{ad}(\mathcal{P}_G)/\text{ad}(\mathcal{P}_G)^1) \rightarrow \Omega_{C_S/S} \sim T_{C_S/S}$$

the composite of (10) and $\nabla_{\mathcal{P}_G}^{-1} : \tilde{T}_{\mathcal{P}_G/S}/\tilde{\iota}(\tilde{T}_{\mathcal{P}_B/S}) \sim T_{C_S/S}$ (cf. (9)). Also, denote by $\nabla_{\mathcal{P}_G}^d$ the composite isomorphism

$$\nabla_{\mathcal{P}_G}^d : \Omega_{C_S/S} \sim \Omega_{C_S/S} \xrightarrow{id_{\Omega_{C_S/S}}} \Omega_{C_S/S} \otimes (\text{ad}(\mathcal{P}_G)/\text{ad}(\mathcal{P}_G)^1) \xrightarrow{id_{\Omega_{C_S/S}} \otimes (\nabla_{\mathcal{P}_G}^{\text{ad}(\mathcal{P}_G)})^{-1}} \Omega_{C_S/S} \otimes (\text{ad}(\mathcal{P}_G)^1/\text{ad}(\mathcal{P}_G)^2) \xrightarrow{(\nabla_{\mathcal{P}_G}^{\text{ad}(\mathcal{P}_G)})^{-1}} \text{ad}(\mathcal{P}_G)^2.$$
where the first arrow denotes the automorphism of $Ω_{Cs/S}$ given by multiplication by 2, which is invertible in $R$ by assumption. Note that the first arrow in this composite should be added in order to conclude Lemma 5.5.1 described later.

2.3. Denote by

$$\tilde{\nabla}^{ad}_{P_G} : \tilde{T}_{P_G/S} \to Ω_{Cs/S} \otimes \text{ad}(P_G)$$

a unique $f_S^{-1}(O_S)$-linear morphism determined by the condition that

$$\langle \partial_1, \tilde{\nabla}^{ad}_{P_G}(\partial_2) \rangle = [\nabla_{P_G}(\partial_1), \partial_2] - \nabla^e([\partial_1, \alpha_{P_G}(\partial_2)])$$

for any local sections $\partial_1 \in T_{Cs/S}$ and $\partial_2 \in \tilde{T}_{P_G/S}$, where $\langle -, - \rangle$ denotes the pairing $T_{Cs/S} \times (Ω_{Cs/S} \otimes \text{ad}(P_G)) \to \text{ad}(P_G)$ arising from the natural pairing $T_{Cs/S} \times Ω_{Cs/S} \to O_{Cs}$. This morphism fits into the following short exact sequence of complexes:

$$0 \longrightarrow \text{ad}(P_G) \longrightarrow \tilde{T}_{P_G/S} \underset{\alpha_{P_G}}{\longrightarrow} T_{Cs/S} \longrightarrow 0$$

where the upper horizontal sequence is (10). Since $\tilde{\nabla}^{ad}_{P_G} \circ \nabla_{P_G} = 0$, the connection $\nabla_{P_G} : T_{Cs/S} \to \tilde{T}_{P_G/S}$ induces a morphism $T_{Cs/S}[0] \to \mathcal{K}^1[\tilde{\nabla}^{ad}_{P_G}]$ defining a split injection of (11). This implies that the short exact sequence

$$0 \longrightarrow \mathbb{R}^1 f_*(\mathcal{K}^1[\tilde{\nabla}^{ad}_{P_G}]) \longrightarrow \mathbb{R}^1 f_*(\mathcal{K}^1[\tilde{\nabla}^{ad}_{P_G}]) \longrightarrow \mathbb{R}^1 f_*(T_{Cs/S}) \longrightarrow 0$$

obtained from (11) by applying the functor $\mathbb{R}^1 f_*(\cdot)$ is exact.

Next, let us consider the restriction

$$\tilde{\nabla}^{ad}_{P_B} := \tilde{\nabla}^{ad}_{P_G}\mid_{\tilde{T}_{P_B}} : \tilde{T}_{P_B} \to Ω_{Cs/S} \otimes \text{ad}(P_G)$$

of $\tilde{\nabla}^{ad}_{P_G}$; it fits into the following morphism of short exact sequences:

$$0 \longrightarrow \tilde{T}_{P_B/S} \underset{\text{incl.}}{\longrightarrow} \tilde{T}_{P_G/S} \underset{\alpha'_{P_G}}{\longrightarrow} T_{Cs/S} \longrightarrow 0$$

$$0 \longrightarrow Ω_{Cs/S} \otimes \text{ad}(P_G) \underset{\text{id}}{\longrightarrow} Ω_{Cs/S} \otimes \text{ad}(P_G) \longrightarrow 0 \longrightarrow 0,$$

where $\alpha'_{P_G}$ denotes the composite of the natural quotient $\tilde{T}_{P_G/S} \to \tilde{T}_{P_G/S}/\tilde{\imath}(\tilde{T}_{P_B/S})$ and $\nabla_{P_G} : \tilde{T}_{P_G/S}/\tilde{\imath}(\tilde{T}_{P_B/S}) \to T_{Cs/S}$ (cf. (9)). Since $f_*(T_{Cs/S}) = 0$, the diagram (11) induces an injection

$$\mathbb{R}^1 f_*(\mathcal{K}^1[\tilde{\nabla}^{ad}_{P_B}]) \hookrightarrow \mathbb{R}^1 f_*(\mathcal{K}^1[\tilde{\nabla}^{ad}_{P_G}]).$$
2.4. Denote by
\[ \eta : \tilde{T}_{PB/S} \to \text{ad}(P_G) \]
the \( O_{CS} \)-linear morphism given by \( \partial \mapsto \partial - (\nabla_{P_G} \circ \alpha_{P_G})(\partial) \) for any local section \( \partial \in \tilde{T}_{PB/S} \). One verifies that \( \eta \) is an isomorphism and satisfies \( \tilde{\alpha}^\text{ad}_{PB} \circ \eta = \nabla^\text{ad}_{P_G} \circ \eta \). Hence, the pair of morphisms \( (\eta, \text{id}_{O_{CS/S} \otimes \text{ad}(P_G)}) \) specifies an isomorphism \( \eta^* : K^*[\tilde{\alpha}^\text{ad}_{PB}] \sim K^*[\nabla^\text{ad}_{P_G}] \). This isomorphism fits into the following isomorphism of sequences of complexes:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^2_{CS/S}[-1] & \longrightarrow & K^*[\tilde{\alpha}^\text{ad}_{PB}] & \longrightarrow & T_{CS/S}[0] & \longrightarrow & 0 \\
0 & \longrightarrow & \Omega^2_{CS/S}[-1] & \longrightarrow & K^*[\nabla^\text{ad}_{P_G}] & \longrightarrow & T_{CS/S}[0] & \longrightarrow & 0,
\end{array}
\]

where
- \( \alpha'_{PB} \) denotes the morphism given by \( \alpha'_{PB} \) (cf. (13));
- both the upper and lower second arrows arise from the composite
  \[
  \Omega^2_{CS/S} \xrightarrow{\text{id} \otimes \nabla_{P_G}} \Omega_{CS/S} \otimes \text{ad}(P_G)^2 \hookrightarrow \Omega_{CS/S} \otimes \text{ad}(P_G)^1;
  \]
- the lower third arrow arises from the composite
  \[
  \text{ad}(P_G) \to \text{ad}(P_G)/\text{ad}(P_G)^1 \xrightarrow{\nabla_{P_G}} T_{CS/S}.
  \]

By applying the functor \( \mathbb{R}^1 f_*(-) \) to this diagram, we obtain the following isomorphism of sequences of \( O_S \)-modules:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & f_*\left(\Omega^2_{CS/S}\right) & \longrightarrow & \mathbb{R}^1 f_*\left(K^*[\tilde{\alpha}^\text{ad}_{PB}]\right) & \longrightarrow & \mathbb{R}^1 f_*\left(T_{CS/S}\right) & \longrightarrow & 0 \\
0 & \longrightarrow & f_*\left(\Omega^2_{CS/S}\right) & \longrightarrow & \mathbb{R}^1 f_*\left(K^*[\nabla^\text{ad}_{P_G}]\right) & \longrightarrow & \mathbb{R}^1 f_*\left(T_{CS/S}\right) & \longrightarrow & 0.
\end{array}
\]

(16)

Since the lower horizontal sequence is exact (cf. [27], Chap. I, § 2, Theorem 2.8 (1)), the upper sequence turns out to be exact. Finally, we remark that the lower sequence in (16) is obtained by taking cohomology of the differentials in the \( E_1 \)-term of the spectral sequence

\[ E_1^{p,q} = \mathbb{R}^q f_* (K^p[\nabla^\text{ad}_{P_G}]) \to \mathbb{R}^{p+q} f_* (K^*[\nabla^\text{ad}_{P_G}]). \]

2.5. Next, let us introduce the notion of an indigenous bundle of canonical type. Write \( \pi^\dagger_{GL_2} : P^\dagger_{GL_2} \to C_S \) for the (right) \( GL_2 \)-torsor over \( C_S \) associated with the rank 2 vector bundle \( rD^\leq 1 \). More precisely, \( P^\dagger_{GL_2} \) is the \( C_S \)-scheme representing the functor

\[ \text{Isom}_{O_{CS}}(O^\leq 2_{CS}, rD^\leq 1_{CS}) : \text{Et}_{CS} \to \text{Set}, \]

classifying locally defined isomorphisms \( O^\leq 2_{CS} \cong rD^\leq 1_{CS} \), where \( \text{Et}_{CS} \) denotes the small étale site on \( C_S \). Also, write

\[ \pi^\dagger_G : P^\dagger_G \to C_S \]
for the \( G \)-torsor induced by \( \mathcal{P}^1_{\text{GL}_2} \) via the change of structure group by the quotient \( \text{GL}_2 \to G \). Let us consider \( \mathcal{O}_{C_S} \) as an \( \mathcal{O}_{C_S} \)-submodule of \( \mathcal{O}_{C_S}^\otimes 2 \) via the injection \( \mathcal{O}_{C_S} \hookrightarrow \mathcal{O}_{C_S}^\otimes 2 \) into the first factor. The subfunctor of \( \mathcal{I} \text{som}_{\mathcal{O}_{C_S}}(\mathcal{O}_{C_S}^\otimes 2, \tau D_{C_S}^{\leq 1}) \) consisting of locally defined isomorphisms \( w : \mathcal{O}_{C_S}^\otimes 2 \nrightarrow \tau D_{C_S}^{\leq 1} \) with \( w(\mathcal{O}_{C_S}) \subseteq \tau D_{C_S}^{\leq 0} / S \) may be represented by a \( B \)-reduction

\[
\pi^1_B : \mathcal{P}^1_B \to C_S
\]
of \( \mathcal{P}^1_G \). The adjoint vector bundle \( \text{ad}(\mathcal{P}^1_G) \) is canonically isomorphic to the sheaf \( \mathcal{E}nd_{\mathcal{O}_{C_S}}(\tau D_{C_S}^{\leq 1}) \) of \( \mathcal{O}_{C_S} \)-linear endomorphisms of \( \tau D_{C_S}^{\leq 1} / S \) with vanishing trace.

Let us take a scheme \( U \) equipped with an étale morphism \( U \to C_S \) and a section \( x \in \Gamma(U, \mathcal{O}_{C_S}) \) such that \( dx \) generates \( \Omega_{U/S} = (\Omega_{C_S/S})|_U \). We shall refer to such a pair \((U, x)\) as a **local chart** of \( C_S \) relative to \( S \). The decomposition \( \tau D_{C_S/S}^{\leq 1}|_U \nrightarrow \mathcal{O}_U \oplus \mathcal{O}_U \cdot \partial_x \) by means of \( (U, x) \), where \( \partial_x \in \Gamma(U, T_{C_S/S}) \) denotes the dual base of \( dx \), gives an isomorphism

\[
\tau_{(U, x)} : \mathcal{P}^1_G|_U = (U \times_{C_S} \mathcal{P}^1_G) \nrightarrow U \times_R G
\]
of \( G \)-torsors which induces an isomorphism

\[(17) \quad \tilde{\tau}^\text{ad}_{(U, x)} : \tilde{T} \mathcal{P}^1_{G/S}|_U \nrightarrow T_{U/S} \oplus (\mathcal{O}_U \otimes_R \mathfrak{g}).\]

**Definition 2.5.1.** We shall say that an indigenous bundle \( \mathcal{P}^\otimes \) on \( C_S/S \) is **of canonical type** if it is of the form \( \mathcal{P}^\otimes = (\mathcal{P}^\otimes_B, \nabla_{\mathcal{P}^\otimes_B}) \), where \( \nabla_{\mathcal{P}^\otimes_B} \) is an \( S \)-connection on \( \mathcal{P}^\otimes_G \), satisfying the following condition: for any local chart \((U, x)\) of \( C_S \) relative to \( S \), the restriction \( \nabla_{\mathcal{P}^\otimes_B}|_U \) of \( \nabla_{\mathcal{P}^\otimes} \) to \( U \) may be expressed, via \((17)\), as the map \( U \to T_{U/S} \oplus (\mathcal{O}_U \otimes_R \mathfrak{g}) \) determined by

\[
1 \cdot \partial_x \mapsto \left( 1 \cdot \partial_x, \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \right)
\]
for some \( a \in \Gamma(U, \mathcal{O}_{C_S}) \).

2.6. In §§2.6-2.7 we describe indigenous bundles in terms of differential operators between line bundles. Recall that a **theta characteristic** (in other words, a **spin structure**) of \( C_S/S \) is, by definition, a line bundle \( \mathcal{L} \) on \( C_S \) together with an isomorphism \( \mathcal{L}^\otimes 2 \cong \Omega_{C_S/S} \). The curve \( C_S/S \) necessarily admits, at least étale locally on \( S \), a theta characteristic (cf. [31], Remark 2.2.1 (i)).

Let \( \mathcal{L} \) be a theta characteristic of \( C_S/S \). We shall write

\[
\text{Diff}_\mathcal{L}^{\leq 2} := \mathcal{H}om_{\mathcal{O}_{C_S}}(\mathcal{L}^\vee, \mathcal{L}^\vee \otimes \Omega_{C_S/S}^\otimes 2 \otimes D_{C_S/S}^{\leq 2}), \quad \mathcal{D}_{\mathcal{L}}^{\leq 2} := \mathcal{H}om_{\mathcal{O}_{C_S}}(\mathcal{T}_{\mathcal{C}_S/S}^\otimes 2 \otimes \mathcal{L}, \mathcal{D}_{\mathcal{C}_S/S}^2 \otimes \mathcal{L}),
\]

where \( \mathcal{L}^\vee \otimes \Omega_{C_S/S}^\otimes 2 \otimes D_{C_S/S}^2 \) and \( \mathcal{D}_{\mathcal{C}_S/S}^2 \otimes \mathcal{L} \) may be considered as being equipped with the structures of \( \mathcal{O}_{C_S} \)-module defined in §1.7. A second order differential operator from \( \mathcal{L}^\vee \to \mathcal{L}^\vee \otimes \Omega_{C_S/S}^\otimes 2 \) is nothing but a global section of \( \text{Diff}_\mathcal{L}^{\leq 2} \). By passing to the composite injection

\[
\Omega_{C_S/S}^\otimes 2 \nrightarrow \mathcal{H}om_{\mathcal{O}_{C_S}}(\mathcal{L}^\vee, \mathcal{L}^\vee \otimes \Omega_{C_S/S}^\otimes 2 \otimes \mathcal{O}_{C_S}) \left(= \mathcal{H}om_{\mathcal{O}_{C_S}}(\mathcal{L}^\vee, \mathcal{L}^\vee \otimes \Omega_{C_S/S}^\otimes 2 \otimes \mathcal{D}_{C_S/S}^2) \right) \hookrightarrow \text{Diff}_\mathcal{L}^{\leq 2},
\]
we identify \( \Gamma(C_S, \Omega_{C_S/S}^\otimes 2) \) with a submodule of \( \Gamma(C_S, \text{Diff}_\mathcal{L}^{\leq 2}) \).
Next, let us define
\[ \mathcal{D} \text{iff}_L^{≤2} \]
to be the subsheaf of \( \mathcal{D} \text{iff}_L^{≤2} \) consisting of differential operators such that the principal symbol is 1 and the subprincipal symbol is 0. That is to say, a second order differential operator \( D \) from \( \mathcal{L}^\vee \) to \( \mathcal{L}^\vee \otimes \Omega_{C_S/S}^{≤2} \) lies in \( \mathcal{D} \text{iff}_L^{≤2} \) if and only if whenever we choose a local chart \( (U, x) \) of \( C_S \) and a trivialization \( \mathcal{L}|_U \cong \mathcal{O}_U \cdot (dx)^{≤2} \), the operator \( D \) may be given, on \( U \), by assigning
\[
(dx)^{≤(-\frac{1}{2})} \mapsto (dx)^{≤(-\frac{1}{2})} \otimes (dx)^{≤2} \otimes \partial_x^2 + (dx)^{≤(-\frac{1}{2})} \otimes a \otimes 1
\]
for some \( a \in \Gamma(U, \Omega_{C_S/S}^{≤2}) \). For each \( D \in \Gamma(C_S, \mathcal{D} \text{iff}_L^{≤2}) \) and \( A \in \Gamma(C_S, \mathcal{D} \text{iff}_L^{≤2}) \), the sum \( D + A \) lies in \( \Gamma(C_S, \mathcal{D} \text{iff}_L^{≤2}) \) if and only if \( A \in \Gamma(C_S, \Omega_{C_S/S}^{≥2}) \). Thus, \( \Gamma(C_S, \mathcal{D} \text{iff}_L^{≤2}) \) admits a canonical structure of \( \Gamma(C_S, \Omega_{C_S/S}^{≤2}) \)-torsor.

2.7. Now, let \( \mathcal{D} \) be an element of \( \Gamma(C_S, \mathcal{D} \text{iff}_L^{≤2}) \). Denote by \( \mathrm{C} \mathcal{D} \) : \( \mathcal{T}_{C_S/S}^{≤2} \otimes \mathcal{L} \rightarrow \mathcal{D}_{C_S/S}^{≤2} \otimes \mathcal{L} \)
the morphism defined as the image of \( \mathcal{D} \) via the composite
\[
\mathcal{D} \text{iff}_L^{≤2} \cong \mathcal{L}^{σ3} \otimes \mathcal{D}_{C_S/S}^{≤2} \otimes \mathcal{L} \cong \mathrm{C} \mathcal{D} \mathcal{D} \text{iff}_L^{≤2}
\]
of two natural isomorphisms arising from the definitions of \( \mathcal{D} \text{iff}_L^{≤2} \) and \( \mathrm{C} \mathcal{D} \mathcal{D} \text{iff}_L^{≤2} \). Since \( \mathcal{D} \in \Gamma(C_S, \mathcal{D} \text{iff}_L^{≤2}) \), the composite
\[
\mathcal{D}_{C_S/S}^{≤1} \otimes \mathcal{L} \mapsto \mathcal{D}_{C_S/S} \otimes \mathcal{L} \mapsto (\mathcal{D}_{C_S/S} \otimes \mathcal{L})/(\operatorname{Im}(\mathrm{C} \mathcal{D} \mathcal{D}))
\]
is an isomorphism, where \( (\mathcal{D}_{C_S/S} \otimes \mathcal{L})/(\operatorname{Im}(\mathrm{C} \mathcal{D} \mathcal{D})) \) denotes the quotient of \( \mathcal{D}_{C_S/S} \otimes \mathcal{L} \) by the left \( \mathcal{D}_{C_S/S} \)-submodule generated by \( \operatorname{Im}(\mathrm{C} \mathcal{D} \mathcal{D}) \). The structure of left \( \mathcal{D}_{C_S/S} \)-module on \( \mathcal{D}_{C_S/S}^{≤1} \otimes \mathcal{L} \) transposed from \( (\mathcal{D}_{C_S/S} \otimes \mathcal{L})/(\operatorname{Im}(\mathrm{C} \mathcal{D} \mathcal{D})) \) via \( \mathcal{D}_{C_S/S}^{≤1} \otimes \mathcal{L} \) corresponds (cf. §L.7) to an \( S \)-connection
\[ \nabla_\mathcal{D} : \mathcal{D}_{C_S/S}^{≤1} \otimes \mathcal{L} \rightarrow \Omega_{C_S/S} \otimes (\mathcal{D}_{C_S/S}^{≤1} \otimes \mathcal{L}) \]
on \( \mathcal{D}_{C_S/S}^{≤1} \otimes \mathcal{L} \). Note that (since the subprincipal symbol of \( \mathcal{D} \) is 0) the \( S \)-connection \( \operatorname{det}(\nabla_\mathcal{D}) \) on the determinant \( \operatorname{det}(\mathcal{D}_{C_S/S}^{≤1} \otimes \mathcal{L}) \) induced by \( \nabla_\mathcal{D} \) coincides, via \( \operatorname{det}(\mathcal{D}_{C_S/S}^{≤1} \otimes \mathcal{L}) \cong \mathcal{T}_{C_S/S} \otimes \Omega_{C_S/S}^{≤1} \cong \mathcal{O}_{C_S} \), with the universal derivation \( d : \mathcal{O}_{C_S} \rightarrow \Omega_{C_S/S} \). The \( G \)-torsor associated with \( \mathcal{D}_{C_S/S}^{≤1} \otimes \mathcal{L} \) via projectivization is canonically isomorphic to \( \mathcal{P}_{G}^{\dagger} \). Hence, \( \nabla_\mathcal{D} \) yields an \( S \)-connection \( \nabla_{\mathcal{D},G} \) on \( \mathcal{P}_{G}^{\dagger} \). It follows from the various definitions involved that the pair
\[ \mathcal{P}^{\dagger} \mathcal{D} := (\mathcal{P}_{B}, \nabla_{\mathcal{D},G}) \]
forms an indigenous bundle on \( C_S/S \) of canonical type.

**Proposition 2.7.1.** (i) Suppose that there exists a theta characteristic \( \mathcal{L} \) of \( C_S/S \). Then, the assignment \( \mathcal{D} \mapsto \mathcal{P}^{\dagger} \) constructed above defines a bijective correspondence between the set \( \Gamma(C_S, \mathcal{D} \text{iff}_L^{≤2}) \) and the set of isomorphism classes of indigenous bundles on \( C_S/S \).

(ii) For any indigenous bundle \( \mathcal{P}^{\dagger} \) on \( C_S/S \), there exists a unique indigenous bundle \( \mathcal{P}^{\dagger} \) on \( C_S/S \) of canonical type which is isomorphic to \( \mathcal{P}^{\dagger} \).
Proof. Let us consider assertion (i). First, we shall prove the injectivity of the assignment $\mathbb{D} \mapsto P_{\mathbb{D}}$. Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be elements of $\Gamma(C_S, \mathcal{D}iff^\le_{\Sigma} \mathbb{L})$, and suppose that there exists an isomorphism $\xi : P_{\mathbb{D}_1} \cong P_{\mathbb{D}_2}$. Let $(U, x)$ be a local chart of $C_S$ relative to $S$, which gives an identification $\tau_{(U,x)} : P_{\mathbb{D}_1} \cong U \times_R G$. After possibly replacing $U$ with its étale covering, one may find an element $R := \begin{pmatrix} b & c \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(U)$ such that the restriction of $\xi$ to $U$ may be described, via $\tau_{(U,x)}$, as the assignment $v \mapsto R \cdot v$ (for any $v \in B(U)$), where $R$ denotes the image of $R$ via the quotient $\text{SL}_2 \to G$. Here, observe that the restrictions $\nabla_{\mathbb{D}_1|U}, \nabla_{\mathbb{D}_2|U}$ of the $S$-connections $\nabla_{\mathbb{D}_1}, \nabla_{\mathbb{D}_2}$ may be expressed, via $\tau_{(U,x)}^\text{ad}$ (cf. (17)), as the maps determined, respectively, by

$$1 \cdot \partial_x \mapsto \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, \quad 1 \cdot \partial_x \mapsto \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}$$

for some $a_1, a_2 \in \Gamma(U, \mathcal{O}_{C_S})$. Since $\xi$ is compatible with the respective $S$-connections $\nabla_{\mathbb{D}_1}$ and $\nabla_{\mathbb{D}_2}$, the following equalities hold in $\Gamma(U, \mathcal{O}_{C_S} \otimes_R gl_2)$:

$$\begin{pmatrix} 0 & a_2 \\ 1 & 0 \end{pmatrix} = R \cdot \begin{pmatrix} 0 & a_1 \\ 1 & 0 \end{pmatrix} \cdot R^{-1} + R \cdot d(R^{-1}) = \begin{pmatrix} -\frac{b^2 + c}{b} & a_1 b^2 - c^2 - b c' + b' c \\ 1 & b \end{pmatrix}.$$

It follows that $(b^2 = 1, c = 0, a_1 = a_2)$. In particular, by applying the above discussion to various local charts $(U, x)$, we obtain the equality $\mathbb{D}_1 = \mathbb{D}_2$. This completes the proof of the injectivity.

Next, let us prove the surjectivity of the assignment $\mathbb{D} \mapsto P_{\mathbb{D}}$. Let $P_s := (P_B, \nabla_{P_G})$ be an indigenous bundle on $C_S/S$. It follows from Proposition 2.1.2 that, by means of descent with respect to étale morphisms, we are always free to replace $S$ by any étale covering of $S$. Hence, we may assume that there exists a rank 2 vector bundle $\mathcal{V}$ on $C_S$ with trivial determinant such that the $G$-torsor associated with $\mathcal{V}$ via projectivization is isomorphic to $P_G$. The $B$-reduction $P_B$ of $P_G$ determines a line subbundle $\mathcal{N}$ of $\mathcal{V}$, which satisfies that $\mathcal{V}/\mathcal{N} \cong \mathcal{V}^\vee$. Here, write $P_{GL_2}$ for the $GL_2$-torsor corresponding to $\mathcal{V}$ and $P_{G_m}$ for the $G_m$-torsor induced from $P_{GL_2}$ via the change of structure group by the determinant map $\det : GL_2 \to G_m$. Since the map $(q, \det) : GL_2 \to G \times_R G_m$, where $q$ denotes the natural quotient $GL_2 \to G$, is étale, the natural morphism

$$\tilde{T}_{PGL_2}/S \cong \tilde{T}_{P_G}/S \times_{\alpha_{P_G} : \alpha_{P_G}, \alpha_{P_{G_m}}} \tilde{T}_{P_{G_m}}/S$$

is an isomorphism. By this isomorphism, the pair $(\nabla_{P_G}, d)$ of the $S$-connection $\nabla_{P_G}$ on $P_G$ and the trivial $S$-connection $d$ on $\mathcal{O}_{C_S}$ determines an $S$-connection $\nabla_\mathcal{V}$ on $\mathcal{V}$. For each $j = 0, 1, 2$, we shall write $\zeta^j$ for the composite

$$\zeta^j : D^\le_{C_S/S} \otimes \mathcal{N} \hookrightarrow D_{C_S/S} \otimes \mathcal{V} \xrightarrow{\hat{\nabla}_\mathcal{V}} \mathcal{V},$$

where the first arrow arises from the inclusions $D^\le_{C_S/S} \hookrightarrow D_{C_S/S}$ and $\mathcal{N} \hookrightarrow \mathcal{V}$. In particular, $\zeta^0$ coincides with the inclusion $\mathcal{N} \hookrightarrow \mathcal{V}$ under the natural identification $D^\le_{C_S/S} \otimes \mathcal{N} = \mathcal{N}$. It follows from the definition of an indigenous bundle that $\zeta^1$ is an isomorphism and $\zeta^2$ is surjective. Moreover, $\zeta^1$ induces, by taking determinants, an isomorphism

$$(20) \quad \mathcal{T}_{C_S/S} \otimes \mathcal{N} \cong \text{det}(D^\le_{C_S/S} \otimes \mathcal{N}) \cong (\det(\mathcal{V}) \cong \mathcal{O}_{C_S}).$$
That is to say, the line bundle $\mathcal{N}$ together with an isomorphism $\mathcal{N}^{\otimes 2} \xrightarrow{\sim} \Omega_{C_S/S}$ induced by (20) specifies a theta characteristic of $C_S/S$. The square of $\mathcal{L}':=\mathcal{L} \otimes \mathcal{N}^{\vee}$ is trivial, so there exists a unique $S$-connection $\nabla_{\mathcal{L}'}$ on $\mathcal{L}'$ whose square coincides with $d$. By tensoring $(\mathcal{V}, \nabla_{\mathcal{V}})$ with $(\mathcal{L}', \nabla_{\mathcal{L}'})$, we may assume that the vector bundle $\mathcal{V}$ was taken to satisfy $\mathcal{N} = \mathcal{L}$.

Let us consider the short exact sequence

\begin{equation}
0 \to \mathcal{D}^{\leq 1}_{C_S/S} \otimes \mathcal{L} \to \mathcal{D}^{\leq 2}_{C_S/S} \otimes \mathcal{L} \to \mathcal{T}^{\otimes 2}_{C_S/S} \otimes \mathcal{L} \to 0
\end{equation}

(cf. (8)). The composite $\left((\zeta^1)^{-1} \circ \zeta^2\right): \mathcal{D}^{\leq 2}_{C_S/S} \otimes \mathcal{L} \to \mathcal{D}^{\leq 1}_{C_S/S} \otimes \mathcal{L}$ specifies a split surjection of (21). This split surjection determines a split injection $\mathcal{T}^{\otimes 2}_{C_S/S} \otimes \mathcal{L} \to \mathcal{D}^{\leq 2}_{C_S/S} \otimes \mathcal{L}$ of (21), and hence, determines a differential operator $\mathcal{D}: \mathcal{L}'^\vee \to \mathcal{L}'^\vee \otimes \Omega^{\otimes 2}_{C_S/S} \otimes \mathcal{D}^{\leq 2}_{C_S/S}$ via (18). The isomorphism $\zeta^1$ is verified to be compatible with the respective $S$-connections $\nabla_\mathcal{D}$ and $\nabla_{\mathcal{V}}$, as well as with the respective filtrations $\mathcal{D}^{\leq 1}_{C_S/S} \otimes \mathcal{L} \supseteq \mathcal{D}^{\leq 0}_{C_S/S} \otimes \mathcal{L} \supseteq 0$ and $\mathcal{V} \supseteq \mathcal{L} \supseteq 0$. Consequently, $\zeta^1$ determines an isomorphism $\mathcal{P}^{\#\dagger} \xrightarrow{\sim} \mathcal{P}^\#$ of indigenous bundles. This implies that the assignment $\mathcal{D} \mapsto \mathcal{P}^{\#\dagger}$ is surjective, and we finish the proof of assertion (i).

Next, let us consider assertion (ii). Since indigenous bundles (of canonical type) may be constructed by means of descent with respect to étale morphisms, we are always free to replace $S$ with its étale covering (cf. Proposition 2.1.2). In particular, the problem is reduced to the case where $C_S/S$ admits a theta characteristic. Hence, assertion (ii) follows from assertion (i). \hfill \Box

2.8. Let us introduce notations concerning moduli functors classifying indigenous bundles. Denote by

$$\mathcal{S}_{g,R}: \mathcal{S}_{\text{ch}/\mathcal{M}_{g,R}} \to \text{Set}$$

the $\text{Set}$-valued functor on $\mathcal{S}_{\text{ch}/\mathcal{M}_{g,R}}$ which, to any object $T \to \mathcal{M}_{g,R}$ of $\mathcal{S}_{\text{ch}/\mathcal{M}_{g,R}}$, assigns the set of isomorphism classes of indigenous bundles on the curve $f_T: C_T \to T$. Also, denote by

$$\pi^{\mathcal{S}}_{g,R}: \mathcal{S}_{g,R} \to \mathcal{M}_{g,R}$$

the natural projection. Given an object $S \to \mathcal{M}_{g,R}$ of $\mathcal{S}_{\text{ch}/\mathcal{M}_{g,R}}$, we obtain

$$\mathcal{S}_S := \mathcal{S}_{g,R} \times_{\mathcal{M}_{g,R}} \mathcal{M}_{g,R} S,$$

which has the projection

$$\pi^{\mathcal{S}}_S: \mathcal{S}_S \to S.$$

In what follows, let us consider a natural affine structure on $\mathcal{S}_S$ by means of modular interpretation. Let $S$ be as above. Also, let $\mathcal{P}^\# := (\mathcal{P}_B, \nabla_{\mathcal{P}_B})$ be an indigenous bundle on $C_S/S$ and $A$ an element of $\Gamma(C_S, \Omega^{\otimes 2}_{C_S/S})$, or equivalently a global section of the projection $A: (f_S(\Omega^{\otimes 2}_{C_S/S})) \to S$. By Propositions 2.1.2 and 2.7.1 (ii), there exist a unique indigenous bundle $\mathcal{P}^{\#\dagger} := (\mathcal{P}^{\dagger}_B, \nabla^{\dagger}_{\mathcal{P}_B})$ on $C_S/S$ of canonical type and a unique isomorphism $\mathcal{P}^\# \xrightarrow{\sim} \mathcal{P}^{\#\dagger}$. By passing to the composite

$$\Omega^{\otimes 2}_{C_S/S} \xrightarrow{\text{id} \otimes \nabla^\sharp} \Omega_{C_S/S} \otimes \text{ad}(\mathcal{P}_G^\dagger)^2 \hookrightarrow \Omega_{C_S/S} \otimes \tilde{T}^{\dagger}_{\mathcal{P}_G/S} \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_S}(\mathcal{T}_{C_S/S}, \tilde{T}^{\dagger}_{\mathcal{P}_G/S}),$$
one may identify $A$ with an $\mathcal{O}_{C_S}$-linear morphism $\mathcal{T}_{C_S/S} \to \tilde{T}_{P_G^+/S}$. Hence, the sum $\nabla_{P_G}^+ + A : \mathcal{T}_{C_S/S} \to \tilde{T}_{P_G^+/S}$ makes sense and specifies an $S$-connection on $P_G^+$. Moreover, it follows from the definition of an indigenous bundle (of canonical type) that the pair

$$P_{+A}^\pm := (P_G^+, \nabla_{P_G}^+ + A)$$

forms an indigenous bundle on $C_S/S$ of canonical type. Notice that if there exist a theta characteristic $L$ of $C_S/S$ and an element $\mathbb{D}$ of $\Gamma(C_S, \text{Diff}_{\mathcal{C}}^{\otimes 2\star})$ with $P_{+A}^\pm = P_{+A}^\pm$ (cf. Proposition 2.7.1 (ii)), then $P_{+A}^\pm$ may be described as $P_{+A}^\pm = P_{D+A}^\pm$. The assignment $(P_G^+, A) \mapsto P_{+A}^\pm$ is well-defined and functorial with respect to $S$. Therefore, it determines an action

$$\mathcal{G}_S \times S \mathcal{A}(f_{S*}(\Omega_{C_S/S}^{\otimes 2})) \to \mathcal{G}_S$$

of the relative affine space $\mathcal{A}(f_{S*}(\Omega_{C_S/S}^{\otimes 2}))$ on $\mathcal{G}_S$. By Proposition 2.7.1 (i) and (ii), the following proposition holds.

**Proposition 2.8.1** (cf. [27], Chap. I, § 2, Corollary 2.9). The functor $\mathcal{G}_S$ may be represented by an $\mathcal{A}(f_{S*}(\Omega_{C_S/S}^{\otimes 2}))$-torsor over $S$ with respect to the $\mathcal{A}(f_{S*}(\Omega_{C_S/S}^{\otimes 2}))$-action just discussed. In particular, if, moreover, $S$ is a geometrically connected smooth Deligne-Mumford stack over $R$ of relative dimension $n$, then the functor $\mathcal{G}_S$ may be represented by a geometrically connected smooth Deligne-Mumford stack over $R$ of relative dimension $n + 3g - 3$.

2.9. In this subsection, we shall consider a cohomological expression of the deformation space of an indigenous bundle. Let $S$ and $P_G^+$ be as above.

Until just before Proposition 2.9.1 we impose the assumption that $S$ is affine. Then, the sequence (12) reads

$$0 \to \mathbb{H}^1(\mathcal{K}^* [\nabla_{P_G}^\text{ad}]) \to \mathbb{H}^1(\mathcal{K}^* [\tilde{\nabla}_{P_G}^\text{ad}]) \xrightarrow{\alpha_{P_G}} H^1(C_S, \mathcal{T}_{C_S/S}) \to 0,$$

where, given a complex $\mathcal{K}^*$, we shall denote by $\mathbb{H}^1(\mathcal{K}^*)$ its 1-st hypercohomology group. In what follows, we shall write $R_\epsilon := R[\epsilon]/(\epsilon^2)$, and denote the base-changes to $R_\epsilon$ of objects over $R$ by means of a subscripted $\epsilon$. Write $v : S \to \mathcal{G}_{g,R}$ for the classifying morphism of $(C_S, P_G^+)$ and $\overline{v} : S \to \mathcal{M}_{g,R}$ for the classifying morphism of $C_S$, i.e., $\overline{v} := \pi_{g,R} \circ v$.

The tangent space

$$T_{C_S} := \Gamma(S, \overline{v}^*(\mathcal{T}_{\mathcal{M}_{g,R}}/R))$$

of $\mathcal{M}_{g,R}/R$ at $\overline{v}$ may be identified with the deformation space of the curve $C_S$ over $S_\epsilon$. We shall denote by

$$T_{C_S, P_G, \nabla_{P_G}}$$

the deformation space of $(C_S, P_G, \nabla_{P_G})$ (as a data consisting of a curve and a flat $G$-torsor over it) over $S_\epsilon$; it has the subspace

$$T_{P_G, \nabla_{P_G}}$$

classifying deformations whose underlying curves are the trivial deformation. By arguments similar to the arguments in [17], § 4 (e.g., the proof of Proposition 4.4), there are canonical
isomorphisms

\[ t_{C_S} : T_{C_S} \sim H^1(C_S, T_{C_S/S}), \]
\[ t_{C_S,P_G,\nabla_{P_G}} : T_{C_S,P_G,\nabla_{P_G}} \sim H^1(\mathcal{K}^\bullet[\nabla_{P_G}^{\text{ad}}]), \]
\[ t_{P_G,\nabla_{P_G}} : T_{P_G,\nabla_{P_G}} \sim H^1(\mathcal{K}^\bullet[\nabla_{P_G}^{\text{ad}}]) \]

(cf. \$2.10$ for their precise constructions) making the following diagram commute:

\[
\begin{array}{ccc}
T_{P_G,\nabla_{P_G}} & \xrightarrow{\text{incl.}} & T_{C_S,P_G,\nabla_{P_G}} \\
\downarrow t_{P_G,\nabla_{P_G}} & & \downarrow t_{C_S,P_G,\nabla_{P_G}} & \downarrow t_{C_S} \\
H^1(\mathcal{K}^\bullet[\nabla_{P_G}^{\text{ad}}]) & \longrightarrow & H^1(\mathcal{K}^\bullet[\nabla_{P_G}^{\text{ad}}]) & \longrightarrow & H^1(C_R, T_{C_S/S}),
\end{array}
\]

(23)

where the right-hand arrow in the upper horizontal sequence is obtained by forgetting the data of deformations of \((P_G, \nabla_{P_G})\) and the lower horizontal arrows are the morphisms in \(22\). Indeed, \(t_{C_S}\) is nothing but the Kodaira-Spencer map of \(C_S/S\) (cf. [30], \$10, p.122).

Next, denote by

\[ T_{C_S,P^\bullet} \]

the deformation space of \((C_S, P_B, \nabla_{P_G})\) (as a data consisting of a curve, a \(B\)-torsor over it, and an \(S\)-connection on the \(G\)-torsor associated to this \(B\)-torsor) over \(S_\epsilon\). Just as in the case of \(T_{C_S,P_G,\nabla_{P_G}}\), there exists a canonical bijection

\[ t_{C_S,P^\bullet} : T_{C_S,P^\bullet} \sim H^1(\mathcal{K}^\bullet[\nabla_{P_B}^{\text{ad}}]), \]

which makes the following square diagram commute:

\[
\begin{array}{ccc}
T_{C_S,P^\bullet} & \longrightarrow & T_{C_S,P_G,\nabla_{P_G}} \\
t_{C_S,P^\bullet} & \downarrow t_{C_S,P_G,\nabla_{P_G}} & \downarrow t_{C_S,P_G,\nabla_{P_G}} \\
H^1(\mathcal{K}^\bullet[\nabla_{P_B}^{\text{ad}}]) & \longrightarrow & H^1(\mathcal{K}^\bullet[\nabla_{P_G}^{\text{ad}}]),
\end{array}
\]

(24)

where the upper horizontal arrow arises from the change of structure group by \(B \hookrightarrow G\). Moreover, \(T_{C_S,P^\bullet}\) may be identified with the tangent space \(\Gamma (S, v^* (\mathcal{T}_{g,R})_*)\) of \(\mathcal{G}_{g,R}/R\) at \(v\), i.e., the deformation space of \((C_S, \mathcal{P}^\bullet)\) (as a pair of a curve and an indigenous bundle on it) over \(S_\epsilon\). Indeed, let \((C_S', \mathcal{P}_B', \nabla_{P_G}'\)) be the deformation classified by an element \(v\) of \(T_{C_S,P^\bullet}\), where we shall write \(P_G' := \mathcal{P}_B' \times_B G\) and \(v^* : \tilde{T}_{P_G}^{\text{ad}}/S_\epsilon \hookrightarrow \tilde{T}_{P_G}^{\text{ad}}/S_\epsilon\). Then, the composite

\[
\nabla_{P_G'} : T_{C_S'/S_\epsilon} \xrightarrow{v^*} \tilde{T}_{P_G}^{\text{ad}}/S_\epsilon \rightarrow \tilde{T}_{P_G}^{\text{ad}}/S_\epsilon / v^* (\tilde{T}_{P_G}^{\text{ad}}/S_\epsilon)
\]

is an isomorphism since it becomes the isomorphism \(\nabla_{P_G}\) when restricted to the closed subscheme \(C_S' \subseteq \mathcal{C}_S\). It follows that the pair \((\mathcal{P}_B', \nabla_{P_G}')\) forms an indigenous bundle on \(C_S'/S_\epsilon\), as desired.

On the other hand, the structure of \(\mathbb{A}(f_{S*}(\Omega^{\otimes 2}_{C_S/S}))\)-torsor on \(\mathcal{G}_S\) (cf. Proposition 2.8.1) yields a canonical bijection

\[ t_{\nabla_{P_G}} : T_{\nabla_{P_G}} \sim (\sim \Gamma(S, v^* (\mathcal{T}_{g,R}/g_{\alpha,R})) \sim \Gamma (C_S, \Omega^{\otimes 2}_{C_S/S}). \]
We have the following morphism of short exact sequences:

\[
0 \longrightarrow T_{\nabla P_G} \longrightarrow T_{C_S, P^\oplus} \longrightarrow T_{C_S} \longrightarrow 0
\]

where the upper horizontal sequence is obtained by differentiating the smooth morphism \(\tau^g_{T_G}: \mathcal{E}_g^R \rightarrow \mathcal{M}_g^R\).

Finally, we remark that the various isomorphisms obtained above are functorial, in the natural sense, with respect to \(S\). Hence, by combining (16) and the above isomorphism of short exact sequences for various affine schemes \(S\), we obtain the following proposition.

**Proposition 2.9.1.** Let us keep the above notation (but the affineness assumption on \(S\) are not imposed now). Then, there exists a canonical isomorphism of short exact sequences of \(O_S\)-modules:

\[
0 \longrightarrow v^*(T_{\tilde{E}_{g,R}/\mathcal{M}_{g,R}}) \longrightarrow v^*(T_{\tilde{E}_{g,R}/R}) \longrightarrow T_{\tilde{E}_{g,R}/R} \longrightarrow 0
\]

where the upper horizontal sequence is obtained by differentiating the smooth morphism \(\tau^g_{\tilde{E}_{g,R}}: \mathcal{E}_{g,R} \rightarrow \mathcal{M}_{g,R}\).

2.10. We shall describe explicitly the deformations of data discussed above in terms of Čech cohomology. To do this, it suffices, by taking account of the commutative diagrams (23) and (24), to consider (the inverse of) the bijective correspondence \(\text{cohomology}\). To do this, it suffices, by taking account of the commutative diagrams (23) and (24), to consider (the inverse of) the bijective correspondence \(\text{cohomology}\).

Let us take an affine open covering \(U := \{U_\alpha\}_{\alpha \in I}\) of \(C_S\), where \(I\) is an index set. We shall write \(I_2\) for the set of pairs \((\alpha, \beta) \in I \times I\) with \(U_{\alpha \beta} := U_\alpha \cap U_\beta \neq \emptyset\). One may calculate \(\mathbb{H}^1(K^\bullet[\nabla_{\tilde{P}_G}])\) as the total cohomology of the Čech double complex \(\text{Tot}^*(\tilde{C}^\bullet(U, K^\bullet[\nabla_{\tilde{P}_G}]))\) associated to \(K^\bullet[\nabla_{\tilde{P}_G}]\). Each element \(v\) of \(\mathbb{H}^1(K^\bullet[\nabla_{\tilde{P}_G}])\) may be given by a 1-cocycle of \(\text{Tot}^*(\tilde{C}^\bullet(U, K^\bullet[\nabla_{\tilde{P}_G}]))\), i.e., a collection of data

\[
v = \{(a_{\alpha \beta}, \alpha), \{b_\alpha\}_\alpha\}
\]

consisting of a Čech 1-cocycle \(\{a_{\alpha \beta}\}_{\alpha \beta} \in \tilde{C}^1(U, \nabla_{\tilde{P}_G}/S)\), where \(a_{\alpha \beta} \in \Gamma(U_{\alpha \beta}, \nabla_{\tilde{P}_G}/S)\), and a Čech 0-cochain \(\{b_\alpha\}_\alpha \in \tilde{C}^0(U, \Omega_{C_S/S} \otimes \text{ad}(P_G))\), where \(b_\alpha \in \Gamma(U_\alpha, \Omega_{C_S/S} \otimes \text{ad}(P_G)) = \text{Hom}_{P_G}(\mathcal{T}_{U_\alpha/S}, \text{ad}(P_G)|_{U_\alpha})\), which agree under \(\nabla_{\tilde{P}_G}\) and the Čech coboundary map. The elements in \(\mathbb{H}^1(K^\bullet[\nabla_{\tilde{P}_G}])\) (resp., \(\mathbb{H}^1(K^\bullet[\nabla_{\tilde{P}_G}])\)) may be represented by \(v\) as above such that \(\{a_{\alpha \beta}\}_{\alpha \beta} \in \tilde{C}^1(U, \text{ad}(P_G))\) (resp., \(\{a_{\alpha \beta}\}_{\alpha \beta} \in \tilde{C}^1(U, \nabla_{\tilde{P}_G}/S)\)). The \(S\)-schemes \(U_{\alpha, \epsilon}(:= U_\alpha \times_S S_\epsilon)\) for various \(\alpha \in I\) may be glued together by means of the isomorphisms

\[
\tau_{C_S, \alpha \beta}^\epsilon := \text{id}_{U_{\alpha \beta, \epsilon}} + \epsilon \cdot \alpha_{P_G}(a_{\alpha \beta}) : U_{\beta, \epsilon}|_{U_{\alpha \beta}} \cong U_{\alpha, \epsilon}|_{U_{\alpha \beta}}
\]
for \((\alpha,\beta)\in I_2\). The resulting \(S_c\)-scheme, which we denote by \(C^\ast_S\), specifies the deformation corresponding to \(\alpha^\ast_H(v)\) \((\in H^1(C_S,\mathcal{T}_{C_S/S})\) via \(t_{C_S}\). Moreover, the flat \(G\)-torsors \((\mathcal{P}_G,\nabla^\epsilon)\) may be glued together by means of the isomorphisms

\[
\tau_{P_G,\alpha}\beta := \text{id}_{\mathcal{P}_G,\epsilon}|_{u_{\alpha\beta}} + \epsilon \cdot a_{\alpha\beta} : (\mathcal{P}_G,\nabla_{\mathcal{P}_G,\epsilon}|_{u_{\alpha\beta}} + \epsilon \cdot b_{\alpha\beta})|_{u_{\alpha\beta}} \sim (\mathcal{P}_G,\nabla_{\mathcal{P}_G,\epsilon}|_{u_{\alpha\beta}} + \epsilon \cdot b_{\alpha\beta})|_{u_{\alpha\beta}}
\]

over \(\tau_{C_S,\alpha\beta}\) for \((\alpha,\beta)\in I_2\). The curve \(C^\ast_S\) together with the resulting flat \(G\)-torsor, which we denote by \((\mathcal{P}_G,\nabla_{\mathcal{P}_G})\), specifies the deformation of \((C_S,\mathcal{P}_G,\nabla_{\mathcal{P}_G})\) classified by \(t_{C_S,\mathcal{P}_G,\nabla_{\mathcal{P}_G}}^{-1}(v)\). That is to say, the assignment \(v \mapsto (C^\ast_S,\mathcal{P}_G,\nabla_{\mathcal{P}_G})\) gives the inverse of the bijection \(t_{C_S,\mathcal{P}_G,\nabla_{\mathcal{P}_G}}^{-1}\).

2.11. In this subsection, we discuss deformations of indigenous bundles on a Riemann surface. If, in Definition 2.1.1 (i), “\(C_S\)” is replaced by a Riemann surface and the words “\(B\)-torsor” and “connection” are understood in the analytic sense, then one obtains the notion of an indigenous bundle on a Riemann surface. Notice that the complex analytic stack \(\mathfrak{S}_{g,C}\) associated with \(\mathfrak{S}_{g,C}\) may be thought of as the moduli stack classifying connected compact Riemann surfaces of genus \(g\) equipped with an indigenous bundle.

Let \(C_C\) be a proper curve over \(\mathbb{C}\) of genus \(g\) and \(\mathcal{P}_B^\ast := (\mathcal{P}_B,\nabla_{\mathcal{P}_B})\) an indigenous bundle on \(C_C\). Then, \(\mathcal{P}_B^\ast\) gives rise to an indigenous bundle \(\mathcal{P}_B^{\ast,\text{an}} := (\mathcal{P}_B^{\text{an}},\nabla_{\mathcal{P}_B^{\text{an}}})\) on the Riemann surface \(C_C^{\text{an}}\) via the GAGA principle. Just as in the case of the algebraic setting discussed before, a \(\mathbb{C}\)-connection \(\nabla_{\mathcal{P}_B^{\text{an}}}\) on the (holomorphic) adjoint vector bundle \(\text{ad}(\mathcal{P}_B^{\text{an}})\) of the \(G\)-torsor \(\mathcal{P}_B^{\text{an}}\) are defined. Also, we obtain a \(\mathbb{C}\)-linear morphism \(\nabla_{\text{ad}}^{\mathcal{P}_B^{\text{an}}} : \mathcal{T}_{\mathcal{P}_B^{\text{an}}} \to \Omega_{C_C} \otimes \text{ad}(\mathcal{P}_B^{\text{an}})\) and an isomorphism of complexes \(\eta_{\text{an}}^\ast : \mathcal{K}^\ast[\nabla_{\text{ad}}^{\mathcal{P}_B^{\text{an}}}] \sim \mathcal{K}^\ast[\nabla_{\mathcal{P}_B^{\text{an}}}].\)

Next, denote by

\[
T_{C_C^{\text{an}},\mathcal{P}_B^{\ast,\text{an}}}
\]

the deformation space of \((C_C^{\text{an}},\mathcal{P}_B^{\ast,\text{an}},\nabla_{\mathcal{P}_B^{\ast,\text{an}}}^{\text{an}})\) (as a data consisting of a Riemann surface \(C_C^{\text{an}}\), a \(B\)-torsor over it, and a \(\mathbb{C}\)-connection on the \(G\)-torsor associated to this \(B\)-torsor) over \(pt := \text{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))^{\text{an}}\). Then, it may be identified with the tangent space of \(\mathfrak{S}_{g,C}^{\text{an}}\) at the point classified by \((C_C^{\text{an}},\mathcal{P}_B^{\ast,\text{an}})\). Also, there exists a canonical isomorphism

\[
t_{C_C^{\text{an}},\mathcal{P}_B^{\ast,\text{an}}} : T_{C_C^{\text{an}},\mathcal{P}_B^{\ast,\text{an}}} \sim \mathbb{H}^1(\mathcal{K}^\ast[\nabla_{\text{ad}}^{\mathcal{P}_B^{\text{an}}}])
\]

which makes the following diagram commute:

\[
\begin{array}{c}
\begin{array}{ccc}
T_{C_C^{\text{an}},\mathcal{P}_B^{\ast,\text{an}}} & \xrightarrow{t_{C_C^{\text{an}},\mathcal{P}_B^{\ast,\text{an}}}} & \mathbb{H}^1(\mathcal{K}^\ast[\nabla_{\text{ad}}^{\mathcal{P}_B^{\ast}}])
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\mathbb{H}^1(\mathcal{K}^\ast[\nabla_{\text{ad}}^{\mathcal{P}_B^{\ast}}]) & \xrightarrow{\mathbb{H}^1(\mathcal{K}^\ast[\nabla_{\text{ad}}^{\mathcal{P}_B^{\ast}}])} & \mathbb{H}^1(\mathcal{K}^\ast[\nabla_{\mathcal{P}_G^{\ast}}])
\end{array}
\end{array}
\]

where \(\mathbb{H}^1(\eta^\ast)\) denotes the isomorphism induced from \(\eta^\ast\) and the vertical arrows are obtained naturally via the GAGA principle. In fact, the explicit construction of this isomorphism can be given in the same way as discussed in the previous subsection. Since the rightmost vertical arrow is an isomorphism (cf. [12], II, Théorème 6.13), the remaining vertical arrows are in fact isomorphisms.
3. Dormant Indigenous Bundles

In this section, we recall the definition of a dormant indigenous bundle and discuss various moduli functors concerning dormant indigenous bundles.

3.1. Let $S, G,$ and $B$ be as in §2.1. Suppose further that $R = K$ for a field of characteristic $p > 2$. First, we recall the definition of $p$-curvature map. Let $\pi : \mathcal{P} \to C_S$ be a $G$-torsor over $C_S$ and $\nabla : \mathcal{T}_{C_S/S} \to \mathcal{T}_{\mathcal{P}/S}$ an $S$-connection on $\mathcal{P}$. If $\partial$ is a derivation corresponding to a local section of $\mathcal{P}$, then we shall denote by $\partial^{[p]}$ the $p$-th iterate of $\partial$, which specifies a derivation corresponding to a local section of $\mathcal{T}_{C_S/S}$.

Since the equality $\alpha\pi(\partial^{[p]}) = (\alpha\pi(\partial))^{[p]}$ holds for any local section $\partial$ of $\mathcal{T}_{C_S/S}$, we shall refer to this morphism $\psi_{\mathcal{P},\nabla} : \mathcal{T}^{\otimes p}_{C_S/S} \to \text{ad}(\mathcal{P})$ determined by assigning $\partial \mapsto (\nabla(\partial^{[p]}) - (\nabla(\partial))^{[p]})$ as the $p$-curvature map of $(\mathcal{P}, \nabla_{\mathcal{P}})$.

**Definition 3.1.1.**

(i) We shall say that an indigenous bundle $\mathcal{P}^\otimes := (\mathcal{P}_B, \nabla_{\mathcal{P}_G})$ on $C_S/S$ is **dormant** if the $p$-curvature map $\psi_{\mathcal{P}_G,\nabla_{\mathcal{P}_G}}$ of $(\mathcal{P}_G, \nabla_{\mathcal{P}_G})$ vanishes identically on $C_S$.

(ii) Let $T$ be a $K$-scheme. A **dormant curve** over $T$ of genus $g$ is a pair

$X^\otimes_{T} := (X/T, \mathcal{P}^\otimes)$

consisting of a proper curve $X$ over $T$ of genus $g$ and a dormant indigenous bundle $\mathcal{P}^\otimes$ on $X/T$.

(iii) Let $T$ be a $K$-scheme, and let $X^\otimes_{T} := (X/T, \mathcal{P}^\otimes := (\pi_{\mathcal{P}} : \mathcal{P}_B \to X, \nabla_{\mathcal{P}_G}))$ and $Y^\otimes := (Y/T, \mathcal{Q}^\otimes := (\pi_{\mathcal{Q}} : \mathcal{Q}_B \to Y, \nabla_{\mathcal{Q}_G}))$ be dormant curves over $T$ of genus $g$. An **isomorphism of dormant curves** from $X^\otimes_{T}$ to $Y^\otimes$ is a pair $(h, h_B)$ consisting of an isomorphism $h : X \sim Y$ of $T$-schemes and an isomorphism $h_B : \mathcal{P}_B \sim \mathcal{Q}_B$ that makes the square diagram

\[
\begin{array}{ccc}
\mathcal{P}_B & \xrightarrow{h_B} & \mathcal{Q}_B \\
\downarrow\pi_{\mathcal{P}} & & \downarrow\pi_{\mathcal{Q}} \\
X & \xrightarrow{h} & Y
\end{array}
\]

commute and is compatible with both the $B$-actions and the $S$-connections in the evident sense.

Next, we shall introduce the notion of ordinarity for dormant curves. If $\mathcal{P}^\otimes := (\mathcal{P}_B, \nabla_{\mathcal{P}_G})$ is an indigenous bundle on $C_S/S$, i.e., $C^\otimes_{S/S} := (C_S/S, \mathcal{P}^\otimes)$ form a dormant curve, then the
natural morphism $\text{Ker}(\nabla_{P_G})[0] \to \mathcal{K}^\bullet[\nabla_{P_G}]$ determines, via the functor $\mathbb{R}^1f_{S*}(-)$, a morphism

$$\gamma_P^\sharp : \mathbb{R}^1f_{S*}(\text{Ker}(\nabla_{P_G})) \to \mathbb{R}^1f_{S*}(\mathcal{K}^\bullet[\nabla_{P_G}])$$

of $\mathcal{O}_S$-modules. By composing it and $\gamma_P^\flat$ (cf. (16)), we obtain a morphism

$$\gamma_P^\diamond : \mathbb{R}^1f_{S*}(\text{Ker}(\nabla_{P_G})) \to \mathbb{R}^1f_{S*}(\mathcal{T}_{CS/S})$$

of $\mathcal{O}_S$-modules. Notice that $\gamma_P^\diamond$ coincides with the morphism obtained by applying the functor $\mathbb{R}^1f_{S*}(-)$ to the natural composite

$$\text{Ker}(\nabla_{P_G}) \hookrightarrow \text{ad}(P_G) \twoheadrightarrow \text{ad}(P_G)/\text{ad}(P_G)^1 \xrightarrow{v_{P_G}} \mathcal{T}_{CS/S}.$$ 

**Definition 3.1.2.** We shall say that $C_{S/S} = (C_S/S, \mathcal{P}^\circ)$ is **ordinary** if $\gamma_P^\diamond$ is an isomorphism.

3.2. Denote by

$$\mathcal{M}^{\text{zax}}_{g,K} \quad (\text{resp., } \mathcal{M}^{\circ\text{zax}}_{g,K})$$

the stack classifying dormant curves (resp., ordinary dormant curves) over $K$ of genus $g$. We obtain a natural sequence of stacks

$$\mathcal{M}^{\circ\text{zax}}_{g,K} \to \mathcal{M}^{\text{zax}}_{g,K} \to \mathcal{S}_{g,K}.$$ 

We here recall the following result from the $p$-adic Teichmüller theory studied by S. Mochizuki.

**Proposition 3.2.1.**

(i) The stack $\mathcal{M}^{\text{zax}}_{g,K}$ may be represented by a nonempty, geometrically connected, and smooth Deligne-Mumford stack over $K$ of dimension $3g-3$, and forms a closed substack of $\mathcal{S}_{g,K}$. Moreover, the projection $\mathcal{M}^{\text{zax}}_{g,K} \to \mathcal{M}_{g,K}$ is finite and faithfully flat.

(ii) The stack $\mathcal{M}^{\circ\text{zax}}_{g,K}$ may be represented by a dense open substack of $\mathcal{M}^{\text{zax}}_{g,K}$ and coincides with the étale locus of $\mathcal{M}^{\text{zax}}_{g,K}$ over $\mathcal{M}_{g,K}$. In particular, $\mathcal{M}^{\circ\text{zax}}_{g,K}$ is a nonempty, geometrically connected, and smooth Deligne-Mumford stack over $K$ of dimension $3g-3$.

**Proof.** See [28], Chap. II, §2.3, Lemma 2.7 and Theorem 2.8 (and its proof). \qed

Finally, by means of cohomological expressions, we describe the differential of the closed immersion $\mathcal{M}^{\text{zax}}_{g,K} \hookrightarrow \mathcal{S}_{g,K}$.

**Proposition 3.2.2.** Let $\mathcal{P}^\circ$ be an indigenous bundle on $C_S/S$ and let $v : S \to \mathcal{S}_{g,K}$ be (as in (29)) the $S$-rational point of $\mathcal{S}_{g,K}$ classifying $(C_S/S, \mathcal{P}^\circ)$. Suppose further that $v$ factors through the closed immersion $\mathcal{M}^{\text{zax}}_{g,K} \to \mathcal{S}_{g,K}$, i.e., that $\mathcal{P}^\circ$ is dormant. Denote by $\bar{v} : S \to \mathcal{M}^{\text{zax}}_{g,K}$ the resulting $S$-rational point of $\mathcal{M}^{\text{zax}}_{g,K}$. Then, there exists a canonical isomorphism

$$\bar{t}_{C_S,\mathcal{P}^\circ} : \bar{v}^*(\mathcal{T}_{\mathcal{M}^{\text{zax}}_{g,K}/K}) \sim \mathbb{R}^1f_{S*}(\text{Ker}(\nabla_{P_G})).$$
which makes the following square diagram commute:

\[
\begin{array}{ccc}
\mathbb{R}^1f_*(\ker(\nabla_{P_G}^{ad})) & \xrightarrow{\kappa_{P}} & \mathbb{R}^1f_*(K^*[\nabla_{P_G}^{ad}]), \\
i_{C,S,G} & \downarrow & \downarrow \iota \\
M_{g,K}^{2g-2} & \xrightarrow{\iota} & S_{g,K}^{2g-2}
\end{array}
\]

where the upper horizontal arrow denotes the \(O_S\)-linear morphism obtained by differentiating the closed immersion \(M_{g,K}^{2g-2} \hookrightarrow S_{g,K}^{2g-2}\) and the right-hand vertical arrow is the middle vertical arrow in (25).

**Proof.** See the proof of [28], Chap. II, §2.3, Theorem 2.8. \(\square\)

### 4. Comparison of Symplectic Structures

In this section, we construct a natural symplectic structure (cf. (32)) on the moduli stack \(S_{g,R}\) appearing in the statement of Theorem A (= Theorem 4.4.1).

4.1. Let \(R\) be as in §1.1 and \(G, \mathcal{B}\) as in §2.1. Also, let \(\pi : S \to \mathcal{M}_{g,R}\) be an object of \(\mathcal{S}ch_{/\mathcal{M}_{g,R}}\) and \(\mathcal{P}^\otimes := (\mathcal{P}, \nabla_{P_G})\) an indigenous bundle on \(C_S/S\). Denote by \(v : S \to S_{g,R}\) the \(S\)-rational point of \(S_{g,R}\) classifying the pair \((C_S, \mathcal{P}^\otimes)\). Recall that the Killing form on \(g = \mathfrak{sl}_2\) is a nondegenerate symmetric bilinear map \(\kappa : g \times g \to k\) defined by \(\kappa(a, b) = \frac{1}{4} \cdot \text{tr}(\text{ad}(a) \cdot \text{ad}(b)) \) (= tr(ab)) for any \(a, b \in g\), where we recall the assumption that 2 is invertible in \(R\). The adjoint bundle \(\text{ad}(\mathcal{P}_G)\) has an \(O_S\)-bilinear map \(\kappa_{\mathcal{P}} : \text{ad}(\mathcal{P}_G) \otimes \text{ad}(\mathcal{P}_G) \to O_{C_S}\), obtained by the change of structure group via \(\kappa\); it induces an isomorphism

\[
\kappa_{\mathcal{P}}^\otimes : \text{ad}(\mathcal{P}_G) \xrightarrow{\sim} \text{ad}(\mathcal{P}_G)^\vee.
\]

Let us write \(\nabla_{P_G}^{ad, \otimes 2}\) for the \(S\)-connection on the tensor product \(\text{ad}(\mathcal{P}_G) \otimes \text{ad}(\mathcal{P}_G)\) induced naturally by \(\nabla_{P_G}^{ad}\). The morphism \(\kappa_{\mathcal{P}}\) is compatible with the respective \(S\)-connections \(\nabla_{P_G}^{ad, \otimes 2}\) and \(d\). By composing the morphism \(\kappa_{\mathcal{P}}\) and the cup product in the de Rham cohomology, we obtain a skew-symmetric \(O_S\)-bilinear map on \(\mathbb{R}^1f_*(K^*[\nabla_{P_G}^{ad}])\):

\[
\mathbb{R}^1f_*(K^*[\nabla_{P_G}^{ad}]) \otimes \mathbb{R}^1f_*(K^*[\nabla_{P_G}^{ad}]) \to \mathbb{R}^2f_*(K^*[\nabla_{P_G}^{ad, \otimes 2}]) \\
\to \mathbb{R}^2f_*(K^*[d]) \\
\sim \mathbb{R}^1f_*(\Omega_{C_S/S}) \\
\xrightarrow{f_{C_S}} O_S
\]

(31) \(\text{(cf. [21], Corollary 5.6, for the definition of the third arrow). The \(O_S\)-bilinear maps (31) of the case where \((S, \mathcal{P}^\otimes) = (S_{g,R}, \mathcal{P}_{g,R}^\otimes)\), where \(\mathcal{P}_{g,R}\) denotes the tautological indigenous bundle on the curve \(C_{\mathcal{E}_{g,R}}\) over \(S_{g,R}\), determines a 2-form}

\[
\omega_{PGL}^{\mathcal{P}_{g,R}}
\]
4.2. In this subsection, we shall prove that $\omega^\mathrm{PGL}_{g,R}$ forms a symplectic structure on $\mathfrak{S}_{g,R}$ (cf. Proposition 4.2.2). To this end, we first prepare for describing Lemma 4.2.1 below.

Let us consider the case where $R = \mathbb{C}$. Let $C^\mathrm{an}$ be a connected compact Riemann surface of genus $g$, $\Sigma$ its underlying orientable closed surface, and $\pi := \pi_1(\Sigma, z)$ the fundamental group of $\Sigma$ with respect to a fixed base-point $z \in \Sigma$. The space $\text{Hom}(\pi, G)$ of representations $\pi \to G$ has a canonical $G$-action obtained by composing representations with inner automorphisms of $G$; we shall denote by $\text{Hom}(\pi, G)/G$ the orbit space. The subspace

$$\mathfrak{R} \subseteq \text{Hom}(\pi, G)/G$$

consisting of all irreducible representations forms a complex manifold of dimension $6g - 6$. The Riemann-Hilbert correspondence gives a canonical identification between $\mathfrak{R}$ with the moduli space of irreducible flat $G$-torsors (in the analytic sense) over $C^\mathrm{an}$ (cf. [9], §2). By taking the monodromy representations of indigenous bundles, we obtain (cf. [20], Theorem) a local biholomorphic map

$$\mu : \mathfrak{S}_{g,C}^\mathrm{an} \to \mathfrak{R}.$$  

Now, let us take a representation $\pi \to G$, or equivalently, an irreducible flat $G$-torsor $(\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}})$ over $C^\mathrm{an}$, classified by a point of $\mathfrak{R}$. Denote by

$$T_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}}$$

the tangent space of $\mathfrak{R}$ at this point, i.e., the deformation space of $(\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}})$ over $C^\mathrm{an}$ ($:= C^\mathrm{an} \times \text{pt}_\epsilon$). Just as in the algebraic case (i.e., $t_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}}$) discussed in §2.9 there exists a canonical isomorphism

$$t_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}} : T_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}} \to \mathbb{H}^1(\mathcal{K}^\bullet [\nabla_{\mathcal{P}_G^\mathrm{an}}]).$$

Then, we have the following lemma.

**Lemma 4.2.1.** The following square diagram is commutative:

$$\begin{array}{ccc}
T_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}} & \stackrel{t_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}}}{\longrightarrow} & \mathbb{H}^1(\mathcal{K}^\bullet [\nabla_{\mathcal{P}_G^\mathrm{an}}]) \\
\downarrow d\mu_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}} & & \downarrow i \\
T_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}} & \stackrel{\sim}{\longrightarrow} & \mathbb{H}^1(\mathcal{K}^\bullet [\nabla_{\mathcal{P}_G^\mathrm{an}}])
\end{array}$$

(cf. [27] for the definition of $t_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}}$), where the left-hand vertical arrow $d\mu_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}}$ denotes the isomorphism arising from the local biholomorphic map $\mu$.

**Proof.** Let us take an element $v$ of $\mathbb{H}^1(\mathcal{K}^\bullet [\nabla_{\mathcal{P}_G^\mathrm{an}}])$, which may be represented by a 1-cocycle $(\{a_{\alpha, \beta}\}_{\alpha, \beta}, \{b_{\alpha}\}_\alpha)$ of the total complex $\text{Tot}^\bullet(\mathcal{K}^\bullet(\mathcal{U}, \mathcal{K}^\bullet[\nabla_{\mathcal{P}_G^\mathrm{an}}]))$ (as in [26]) with respect to a suitable analytic open covering $\mathcal{U} := \{U_\alpha\}_{\alpha \in \mathcal{E}}$ of $C^\mathrm{an}$. Denote by $(C^\mathrm{an}, v, \mathcal{P}_B^\mathrm{an}, \nabla_{\mathcal{P}_B^\mathrm{an}})$ the deformation classified by the point of $T_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}}$ corresponding to $v$ via $t_{\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}}}$. In particular, the underlying Riemann surface $C^\mathrm{an}$ of $(\mathcal{P}_G^\mathrm{an}, \nabla_{\mathcal{P}_G^\mathrm{an}})$, where $\mathcal{P}_G^\mathrm{an} := \mathcal{P}_B^\mathrm{an} \times B G$, corresponds to the element of $H^1(C^\mathrm{an}, T_{\mathcal{P}_G^\mathrm{an}})$ represented by the 1-cocycle $\{\alpha_{\alpha, \beta}(a_{\alpha, \beta})\}_{\alpha, \beta}$. We shall denote...
by \((\hat{\mathcal{P}}^\text{an}_G, \nabla_{\hat{\mathcal{P}}^\text{an}_G})\) the deformation of \((\mathcal{P}^\text{an}_G, \nabla_{\mathcal{P}^\text{an}_G})\) classified the image of \(v\) via \(d\mu_{C^\text{an}_G, P^\text{an}_G}\). Then, after possibly replacing \(\mathfrak{U}\) with its refinement, the element of \(H^1(\mathcal{K}^\bullet(\nabla_{\mathcal{P}^\text{an}_G}^\text{ad}))\) corresponding to this deformation via \(t_{\mathcal{P}^\text{an}_G, \nabla_{\mathcal{P}^\text{an}_G}}\) may be represented by a 1-cocycle \(\bar{v} := \{(\bar{a}_{\alpha\beta})_{\alpha, \beta}, \{\bar{b}_\alpha\}_\alpha\}\) of \(\text{Tot}^\bullet(\hat{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{K}^\bullet(\nabla_{\hat{\mathcal{P}}^\text{an}_G}^\text{ad})))\). By the definition of \(d\mu_{C^\text{an}_G, P^\text{an}_G}\), the pair \((\hat{\mathcal{P}}^\text{an}_G, \nabla_{\hat{\mathcal{P}}^\text{an}_G})\) may be characterized as a unique deformation of \((\mathcal{P}^\text{an}_G, \nabla_{\mathcal{P}^\text{an}_G})\) over \(C^\text{an}\) having the same monodromy as \((\mathcal{P}^\text{an}_G, \nabla_{\mathcal{P}^\text{an}_G})\). Hence, one verifies from the construction of the Riemann-Hilbert correspond (cf. \[11\], Proposition 5.2.1) that the equality \(a_{\alpha\beta} - \nabla_{\mathcal{P}^\text{an}_G}(\alpha P^\text{an}_G(a_{\alpha\beta})) = \bar{a}_{\alpha\beta}\) holds for any \((\alpha, \beta) \in I_2\). This implies the equality \(H^1(\eta^\text{an})(v) = \bar{v}\) by the definition of \(\eta^\text{an}\), and hence, completes the proof of the lemma.

By means of the above lemma, one may prove the following proposition.

**Proposition 4.2.2.** \(\omega^\text{PGL}_{g,R}\) is a symplectic structure on \(S_{g,R}\).

**Proof.** First, let us prove that \(\omega^\text{PGL}_{g,R}\) is nondegenerate. Let \(S\) be a scheme over \(R\) and \(v : S \to S_{g,R}\) an \(S\)-rational point, which classifies a pair \((C_S, \mathcal{P}^\text{an}_G)\) as before. The isomorphisms \(\int_\text{C_{\text{ad}}(\mathcal{P}_G)}\) and \(\int_\text{C_{\text{ad}}(\mathcal{P}_G)}\) (cf. [11]) induces, under the identification \(\text{ad}(\mathcal{P}_G) = \text{ad}(\mathcal{P}_G)^\vee\) arising from \(\kappa_{\text{ad}}\), isomorphisms

\[
R^q f_{S*}(K^p(\nabla_{\mathcal{P}_G}^\text{ad})) \sim R^{1-q} f_{S*}(K^{1-p}(\nabla_{\mathcal{P}_G}^\text{ad}))^\vee
\]

for the pairs \((p, q)\) with \(0 \leq p, q \leq 1\). The collection of these isomorphisms determines an isomorphism of spectral sequences from \(\text{E}^\text{p,q}_1 = R^q f_{S*}(K^p(\nabla_{\mathcal{P}_G}^\text{ad})) \Rightarrow R^{p+q} f_{S*}(K^{1-p}(\nabla_{\mathcal{P}_G}^\text{ad}))^\vee\) to \(\text{E}^\text{p,q}_1 = R^{1-q} f_{S*}(K^{1-p}(\nabla_{\mathcal{P}_G}^\text{ad}))^\vee \Rightarrow R^{2-p-q} f_{S*}(K^{1-p}(\nabla_{\mathcal{P}_G}^\text{ad}))^\vee\). Hence, it induces an isomorphism \(R^1 f_{S*}(K^p(\nabla_{\mathcal{P}_G}^\text{ad})) \sim R^1 f_{S*}(K^{1-p}(\nabla_{\mathcal{P}_G}^\text{ad}))^\vee\). One verifies immediately that the morphism \(R^1 f_{S*}(K^p(\nabla_{\mathcal{P}_G}^\text{ad}))^\vee \to \mathcal{O}_S\) associated with this isomorphism coincides with \([31]\). This implies that \(\omega^\text{PGL}_{g,R}\) is nondegenerate.

Next, we consider the closeness of \(\omega^\text{PGL}_{g,R}\). Since the closeness of a differential form is preserved under base-change via \(\text{Spec}(R) \to \text{Spec}(\mathbb{Z}[\frac{1}{2}])\), it suffices to verify the case where \(R = \mathbb{Z}[\frac{1}{2}]\). But, \(\bigwedge^3 \mathcal{O}_{S_{g,R}}[\mathbb{Z}[\frac{1}{2}]]\) is flat over \(\mathbb{Z}[\frac{1}{2}]\), so the assertion of the case where \(R = \mathbb{Z}[\frac{1}{2}]\) follows from that of the case where \(R = \mathbb{C}\). According to the discussion in [13], §1.4, §1.7, and Theorem (or [9], §3.2), \(\mathfrak{X}\) admits a holomorphic symplectic structure

\[
\omega_{\mathfrak{X}} \in \Gamma(\mathfrak{X}, \bigwedge^2 \mathcal{O}_{\mathfrak{X}}).
\]

It follows from [9], Theorem 3.2, that, if \(\phi\) is a point of \(\mathfrak{X}\) and \((\mathcal{P}^\text{an}_G, \nabla_{\mathcal{P}^\text{an}_G})\) denotes the corresponding flat \(G\)-torsor, then the pairing

\[
H^1(\mathcal{K}^\bullet(\nabla_{\mathcal{P}_G}^\text{ad})) \times H^1(\mathcal{K}^\bullet(\nabla_{\mathcal{P}_G}^\text{ad})) \to \mathbb{C}
\]
determined by \(\omega_{\mathfrak{X}}\) under the isomorphism \(t_{\mathcal{P}^\text{an}_G, \nabla_{\mathcal{P}^\text{an}_G}}\) may be obtained (up to a constant factor) in the same manner as \([31]\). In particular, the pull-back \(\mu^*(\omega_{\mathfrak{X}})\) defines a symplectic structure on \(S_{g,C}\). By Lemma 4.2.1 together with the commutativity of the diagram (28), the 2-form on \(S_{g,C}\) corresponding to \(\mu^*(\omega_{\mathfrak{X}})\) via the GAGA principle coincides with \(\omega^\text{PGL}_{g,C}\). In particular, \(\omega^\text{PGL}_{g,C}\) turns out to be closed. This completes the proof of the proposition. \(\square\)
4.3. Let us keep the notation in §4.1. Suppose further that $\varphi : S \to \mathcal{M}_{g,R}$ is étale, hence $\mathfrak{G}_S$ is a smooth Deligne-Mumford stack over $R$ of relative dimension $6g - 6$. Consider the composite isomorphism

$$\Omega_{S/R} \quad (= \mathcal{T}_{S/R}^\vee) \xrightarrow{\sim} \varphi^* (\mathcal{T}_{\mathcal{M}_{g,R}/R}^\vee)$$

$$\xrightarrow{\sim} \varphi^* (\mathcal{R}_1 f_{g,R*} (\mathcal{T}_{C_{g,R}/\mathcal{M}_{g,R}})^\vee)$$

$$\xrightarrow{\sim} \varphi^* (f_{g,R*} (\Omega_{C_{g,R}/\mathcal{M}_{g,R}}^{\otimes 2}))$$

$$\xrightarrow{\sim} f_{S*} (\Omega_{C_S/S}^{\otimes 2}),$$

where

- the first isomorphism follows from the étaleness of $\varphi$;
- the second and third isomorphisms arise from the Kodaira-Spencer map of $C_{g,R}/\mathcal{M}_{g,R}$ (i.e., the rightmost vertical arrow in (25)) and $\int_{C_{g,R}/\mathcal{M}_{g,R}} (\Omega_{C_{g,R}/\mathcal{M}_{g,R}}^{\otimes 2})$ (cf. (1)) respectively.

By this composite isomorphism, the cotangent bundle $T_{S/S}^\vee = \Lambda (\Omega_{S/R})$ may be thought of as the trivial $\Lambda (f_{S*} (\Omega_{C_S/S}^{\otimes 2}))$-torsor over $S$. Hence, for each global section $\sigma : S \to \mathfrak{G}_S$ of the projection $\pi_S^\mathfrak{G} : \mathfrak{G}_S \to S$, there exists (cf. Proposition 2.8.1) a unique isomorphism

$$\Psi_S : T_{S/S}^\vee \xrightarrow{\sim} \mathfrak{G}_S$$

over $S$ which is compatible with the respective $\Lambda (f_{S*} (\Omega_{C_S/S}^{\otimes 2}))$-actions and whose restriction $\Psi_S |_{0_S}$ to the zero section $0_S : S \to T_{S/S}^\vee$ coincides with $\sigma$. In particular, $\Psi_S$ induces an isomorphism of short exact sequences

$$0 \longrightarrow 0_{S}^*(T_{S/S}^\vee) \longrightarrow 0_{S}^*(T_{S/S}^\vee/R) \longrightarrow T_{S/R} \longrightarrow 0$$

$$0 \longrightarrow \sigma^* (\mathcal{T}_{\mathfrak{G}_S/S}) \longrightarrow \sigma^* (\mathcal{T}_{\mathfrak{G}_S/S/R}) \longrightarrow T_{S/R} \longrightarrow 0.$$

Next, write $\varphi^\mathfrak{G} : \mathfrak{G}_S \to \mathfrak{G}_{g,R}$ for the base-change of $\varphi$ via the projection $\pi^\mathfrak{G}_{g,R} : \mathfrak{G}_{g,R} \to \mathcal{M}_{g,R}$. Since $\mathfrak{G}_S$ is étale, which implies that $\mathfrak{G}^\mathfrak{G}_{g,R} (\Omega_{\mathfrak{G}_{g,R}/R}) \cong \Omega_{\mathfrak{G}_S}/R$, the pull-back

$$\omega^\mathfrak{G}_{\mathfrak{PGL}} := \varphi^\mathfrak{G}^* (\omega_{g,R})$$

determines a symplectic structure on $\mathfrak{G}_S$ (cf. Proposition 4.2.2).

4.4. Let $K$ be a field of characteristic $p > 2$. We shall write

$$\mathcal{O}^\mathfrak{G}_{T_{g,K}^{\otimes 2as}} := \mathcal{T}_{g,K}^{\otimes 2as} \quad (= \mathcal{T}_{g,K}^{\otimes 2as} \times_{\mathfrak{M}_{g,K}^{\otimes 2as}} \mathfrak{M}_{g,K}^{\otimes 2as}),$$

$$\mathcal{O}^\mathfrak{G}_{\mathfrak{G}_{g,K}^{2as}} := \mathfrak{G}_{g,K} \times_{\mathfrak{M}_{g,K}^{\otimes 2as}} \mathfrak{M}_{g,K}^{\otimes 2as}.$$

The stack $\mathcal{O}^\mathfrak{G}_{\mathfrak{G}_{g,K}^{2as}}$ admits a section $\sigma_{g,K} : \mathfrak{M}_{g,K}^{2as} \to \mathcal{O}^\mathfrak{G}_{\mathfrak{G}_{g,K}^{2as}}$ arising from the natural immersion $\mathfrak{M}_{g,K}^{2as} \to \mathfrak{G}_{g,K}$. By Proposition 3.2.1, one may apply the discussion in the previous subsection to the case where the data “$(R, S, \sigma : S \to \mathfrak{G}_S)$” is taken to be $(K, \mathcal{O}^\mathfrak{G}_{g,K}^{2as}, \sigma_{g,K})$. Thus, we obtain an isomorphism

$$\Psi_{g,K} : \mathcal{O}^\mathfrak{G}_{T_{g,K}^{\otimes 2as}} \xrightarrow{\sim} \mathcal{O}^\mathfrak{G}_{\mathfrak{G}_{g,K}^{2as}}.$$
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over \( \mathfrak{M}^{\text{ zar}}_{g,K} \) (cf. (38)) and a symplectic structure

\[
\omega_{\odot}^{\text{PGL}}
\]
on \( \mathfrak{S}^{\text{ zar}}_{g,K} \) (cf. (39)). On the other hand, recall that \( \otimes T^{\vee}_{g,K} \) admits a canonical symplectic structure \( \omega_{\odot}^{\text{can}} := \omega_{\odot}^{\text{can}}_{g,K} \) (cf. (3)). The main assertion of the present paper, i.e., Theorem A, is described as follows.

**Theorem 4.4.1.** The isomorphism \( \Psi_{g,K} \) preserves the symplectic structure, i.e., the following equality holds:

\[
\Psi_{g,K}^* (\omega_{\odot}^{\text{PGL}}) = \omega_{\odot}^{\text{can}}.
\]

**Remark 4.4.2.** Let \( X \) be a smooth Deligne-Mumford stack over \( K \) equipped with a symplectic structure \( \omega \). We shall say that a smooth substack \( Y \) of \( X \) is Lagrangian (with respect to \( \omega \)) if \( \omega|_Y = 0 \) and \( \dim(Y) = \frac{1}{2} \cdot \dim(X) \). For example, if \( X = T^{\vee}_Y \), where \( Y \) denotes a smooth Deligne-Mumford stack over \( K \) considered as a closed substack of \( X \) by the zero section \( 0_Y : Y \to T^{\vee}_Y = (X) \), then \( Y \) is Lagrangian with respect to the symplectic structure \( \omega_{\odot}^{\text{can}} \). This example together with the above theorem shows that the sub stack \( \otimes \mathfrak{M}^{\text{ zar}}_{g,K} \) of \( \mathfrak{S}^{\text{ zar}}_{g,K} \) is Lagrangian with respect to the symplectic structure \( \omega_{\odot}^{\text{PGL}} \).

5. Proof of Theorem A

This section is devoted to the proof of Theorem 4.4.1.

5.1. To begin with, we shall discuss the translations of the symplectic structures \( \omega_{\odot}^{\text{PGL}} \), \( \omega_{\odot}^{\text{can}} \) with respect to the affine structures on the respective underlying spaces. Let \( R \) be as in §1.4 and \( \pi : S \to \mathfrak{M}^{\text{ zar}}_{g,R} \) an étale relative scheme over \( \mathfrak{M}^{\text{ zar}}_{g,R} \). In the following, \( (-) \) denotes either \( T^{\vee} \) or \( \mathfrak{S} \). We shall write

\[
(-)^{T^{\vee}}_S := (-)_S \times_S T^{\vee}_S,
\]

which has the projections

\[
\pi^{(-)}_{S,1} : (-)^{T^{\vee}}_S \to (-)_S \quad \text{and} \quad \pi^{(-)}_{S,2} : (-)^{T^{\vee}}_S \to T^{\vee}_S
\]
ono onto the first and second factors respectively. Define \( \tau^{(-)}_S \) to be the composite

\[
\tau^{(-)}_S : (-)^{T^{\vee}}_S \to (-)^{T^{\vee}}_S \times S \mathfrak{A}(\Omega_{S/R}) \cong (-)^{T^{\vee}}_S \times S \mathfrak{A}(f^*_S(\Omega_{C_S/S}^{\otimes 2})) \to (-)^{T^{\vee}}_S,
\]

where

- the first arrow denotes the product of the identity morphism of \( (-)_S \) and the section \( T^{\vee}_S \to T^{\vee}_S \times S \mathfrak{A}(\Omega_{S/R}) \) corresponding to the Liouville form \( \lambda_S \in \Gamma(T^{\vee}_S, \pi^*_S(\Omega_{S/R})) \) (cf. §1.4) on \( T^{\vee}_S \);
the second arrow denotes the product of the identity morphism of \((-)^T\) and the isomorphism \(A(\Omega_{S/R}) \simeq A(f_{S*}(\Omega^2_{S/S}))\) (cf. (37));

- the third arrow arises from the structure of \(A(f_{S*}(\Omega^2_{S/S}))\)-torsor (cf. (37) and Proposition 2.8.1) on the first factors in \((-)^T\).

For each \(A \in \Gamma(S, \Omega_{S/R})\) (\(= \Gamma(C_S, \Omega^2_{S/S})\)), we denote by

\[
\tau^{(-)}_{S,A} : (-)_S \to (-)_S
\]

the automorphism of \((-)_S\) determined, via its own affine structure, by the translation by \(A\).

Also, denote by

\[
\sigma^{T'}_{S,A} : S \to T^\vee_S
\]

the section of \(\pi^{T'}_S\) corresponding to \(A\). In particular, \(0_S = \sigma^{T'}_{S,0}\). By the various definitions involved, the equality

\[
(42) \quad \tau^{(-)}_{S,A} = \pi^{(-)}_{S,1} \circ \tau^{(-)}_S \circ (\text{id}_{(-)_S} \times \sigma^{T'}_{S,A})
\]

holds. By replacing \(S\) with a covering space \(S'\) of \(\mathcal{M}_{g,c}^\text{an}\) (e.g., \(\mathcal{T}^\Sigma\)), we obtain, in the same manner as above, a complex analytic stack \((-)^T_{S'}\) and various morphisms \(\pi^{(-)}_{S',1}, \pi^{(-)}_{S',2},\) and \(\tau^{(-)}_{S'}\).

(They will be used in the proof of Proposition 5.1.2.)

Then, the following Propositions 5.1.1 and 5.1.2 hold.

**Proposition 5.1.1.** The following equality in \(\Gamma(T^\vee_{S'}T^\vee, \wedge^2 \Omega_{T^\vee_{S'}T^\vee/R})\) holds:

\[
(\pi^{T'}_{S,1} \circ \tau^{T'}_S)^*(\omega^\text{can}_S) = \pi^{T'}_{S,1}(\omega^\text{can}_S) + \pi^{T'}_{S,2}(\omega^\text{can}_S).
\]

In particular, for each \(A \in \Gamma(S, \Omega_{S/R})\), the following equality holds:

\[
\tau^{T'}_{S,A}(\omega^\text{can}_S) = \omega^\text{can}_S + \pi^{T'}_{S,A}(dA).
\]

**Proof.** Let us consider the former assertion. Let \(q_1, \cdots, q_{3g-3}\) be local coordinates in \(S\), and let \(p_1, \cdots, p_{3g-3}\) and \(p_1', \cdots, p_{3g-3}'\) be its dual coordinates in the first and second factors, respectively, of the product \(T^\vee_S \times S T^\vee_S\). Then, locally on \(T^\vee_{S'}T^\vee\), we have

\[
(\pi^{T'}_{S,1} \circ \tau^{T'}_S)^*(\omega^\text{can}_S) = (\pi^{T'}_{S,1} \circ \tau^{T'}_S)^*(\sum_{i=1}^{3g-3} dp_i \wedge dq_i)
\]

\[
= \sum_{i=1}^{3g-3} d(p_i + p_i') \wedge dq_i
\]

\[
= \left(\sum_{i=1}^{3g-3} dp_i \wedge dq_i\right) + \left(\sum_{i=1}^{3g-3} dp_i' \wedge dq_i\right)
\]

\[
= \pi^{T'}_{S,1}(\omega^\text{can}_S) + \pi^{T'}_{S,2}(\omega^\text{can}_S).
\]

This completes the proof of the former assertion.
Proposition 5.1.2. The following equality in $\Gamma(\mathcal{S}^\vee_\mathcal{S}, \bigwedge^2 \Omega^\vee_{\mathcal{S}^\vee_\mathcal{S}}/R)$ holds:

$$\pi^*_{S,A}(\omega^\mathrm{PGL}_S) = \pi^*_{S,1}(\omega^\mathrm{PGL}_S) + \pi^*_{S,2}(\omega^\mathrm{can}_S).$$

In particular, for each $A \in \Gamma(S, \Omega_{S/R})$, the following equality holds:

$$\tau^*_{S,A}(\omega^\mathrm{PGL}_S) = \omega^\mathrm{PGL}_S + \pi^*_{A}(dA).$$

Proof. We shall prove the former assertion. The equality (44) may be obtained as the pull-back, via the composite $S \to \mathfrak{m}_{g,R} \to \mathfrak{m}_{g,\mathbb{Z}[\frac{1}{2}])}$, of the equality (14) in the case where $(R, S)$ is taken to be $(\mathbb{Z}[\frac{1}{2}], \mathfrak{m}_{g,\mathbb{Z}[\frac{1}{2}]}).$ Hence, the proof is reduced to the case where $(R, S) = (\mathbb{Z}[\frac{1}{2}], \mathfrak{m}_{g,\mathbb{Z}[\frac{1}{2}]}).$

Notice here that the morphism

$$\Gamma(\mathcal{S}^\vee_{g,\mathbb{Z}[\frac{1}{2}]} : \bigwedge^2 \Omega^\vee_{\mathcal{S}^\vee_{g,\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}[\frac{1}{2}]}), \Gamma(\mathcal{S}^\vee_{g,\mathbb{C}} : \bigwedge^2 \Omega^\vee_{\mathcal{S}^\vee_{g,\mathbb{C}}/\mathbb{C}}),$$

where $\mathcal{S}^\vee_{g,\mathbb{Z}[\frac{1}{2}]} := \mathcal{S}^\vee_{g,\mathbb{Z}[\frac{1}{2}]}$, $\mathcal{S}^\vee_{g,\mathbb{C}} := \mathcal{S}^\vee_{g,\mathbb{C}}$, arising from base-change via $\text{Spec}(\mathbb{C}) \to \text{Spec}(\mathbb{Z}[\frac{1}{2}])$ is injective because $\bigwedge^2 \Omega^\vee_{\mathcal{S}^\vee_{g,\mathbb{Z}[\frac{1}{2}]}/\mathbb{Z}[\frac{1}{2}]}$ is flat over $\mathbb{Z}[\frac{1}{2}]$. Thus, it suffices to prove the equality (44) of the case where $(R, S) = (\mathbb{C}, \mathfrak{m}_{g,\mathbb{C}}).$ Moreover, by applying the analytification functor and replacing $\mathfrak{m}_{g,\mathbb{C}}$ by the Teichmüller space $\mathcal{T}^\Sigma$, i.e., its universal covering, one can reduce the problem to proving the equality

$$\pi^*_{S,1}(\omega^\mathrm{PGL}_S) + \pi^*_{S,2}(\omega^\mathrm{can}_S)$$

of holomorphic 2-forms on $\mathcal{T}^\Sigma$.

Since $\mathcal{T}^\Sigma$ is simply connected, there exists a complex differential 1-form $\delta$ on $\mathcal{T}^\Sigma$ satisfying the equality $(1 + \sqrt{-1}) \cdot \omega_{WP} = -d\delta$, where $\omega_{WP}$ denotes the Weil-Petersson symplectic form on $\mathcal{T}^\Sigma.$ Denote by $\sigma^\mathrm{unif}$ and $\sigma^\mathrm{B}$ the sections $\mathcal{T}^\Sigma \to \mathcal{S}^\Sigma\mathrm{E}$ determined by the Fuchsian and Bers uniformizations respectively (cf. [13; 11, §3.1, (3.1)]). Because of the $\mathfrak{A}((\Omega_{\mathcal{T}^\Sigma})$-torsor structure on $\mathcal{S}^\Sigma\mathrm{E}$, it makes sense to speak of the sum

$$\sigma^\mathrm{B}_{\mathrm{unif}, \delta} := \sigma^\mathrm{B}_{\mathrm{unif}} + \delta,$$

which specifies a section $\mathcal{T}^\Sigma \to \mathcal{S}^\Sigma\mathrm{E}.$ The differences $\sigma^\mathrm{unif}_{\mathrm{F}} - \sigma^\mathrm{B}_{\mathrm{unif}}$, and $\sigma^\mathrm{unif}_{\mathrm{F}} - \sigma^\mathrm{B}_{\mathrm{unif}, \delta}$ may be thought of as elements of $\Gamma(\mathcal{T}^\Sigma, \Omega_{\mathcal{T}^\Sigma}).$ Then,

$$d(\sigma^\mathrm{unif}_{\mathrm{F}} - \sigma^\mathrm{B}_{\mathrm{unif}, \delta}) = d(\sigma^\mathrm{unif}_{\mathrm{F}} - \sigma^\mathrm{B}_{\mathrm{unif}}) - d\delta = -\sqrt{-1} \cdot \omega_{WP} + (1 + \sqrt{-1}) \cdot \omega_{WP} = \omega_{WP},$$
where the second equality follows from the definition of \( \delta \) and [26], Theorem 1.5. Thus, it follows from [25], Theorem 6.8, that the diffeomorphism \( \Psi_{\sigma^B_{\text{unif}, \delta}} : T^\vee_{\Sigma} \to \mathcal{G}_{\Sigma} \) induced by \( \sigma^B_{\text{unif}, \delta} \) satisfies the equality \( \Psi^*_{\sigma^B_{\text{unif}, \delta}} (\omega^\text{PGL}_{\Sigma}) = \omega^\text{can}_{\Sigma} \). Hence, the following sequence of equalities holds:

\[
\begin{align*}
(\Psi^*_{\sigma^B_{\text{unif}, \delta}} \times \text{id}_{T^\vee_{\Sigma}})^* ((\pi^\mathcal{E}_{\Sigma, 1} \circ \tau^\mathcal{E}_{\Sigma})^* (\omega^\text{PGL}_{\Sigma})) &= (\pi^\mathcal{E}_{\Sigma, 1} \circ \tau^\mathcal{E}_{\Sigma} \circ (\Psi^*_{\sigma^B_{\text{unif}, \delta}} \times \text{id}_{T^\vee_{\Sigma}}))^* (\omega^\text{PGL}_{\Sigma}) \\
&= (\pi^\mathcal{E}_{\Sigma, 1} \circ (\Psi^*_{\sigma^B_{\text{unif}, \delta}} \times \text{id}_{T^\vee_{\Sigma}}) \circ \tau^\mathcal{E}_{\Sigma})^* (\omega^\text{PGL}_{\Sigma}) \\
&= (\Psi^*_{\sigma^B_{\text{unif}, \delta}} \circ \pi^\mathcal{E}_{\Sigma, 1} \circ \tau^\mathcal{E}_{\Sigma})^* (\omega^\text{PGL}_{\Sigma}) \\
&= (\pi^\mathcal{E}_{\Sigma, 1} \circ \tau^\mathcal{E}_{\Sigma})^* (\omega^\text{can}_{\Sigma}).
\end{align*}
\]

On the other hand, we have the following equalities:

\[
\begin{align*}
(\Psi^*_{\sigma^B_{\text{unif}, \delta}} \times \text{id}_{T^\vee_{\Sigma}})^* (\pi^\mathcal{E}_{\Sigma, 1}(\omega^\text{PGL}_{\Sigma}) + \pi^\mathcal{E}_{\Sigma, 2}(\omega^\text{can}_{\Sigma})) &= (\pi^\mathcal{E}_{\Sigma, 1} \circ (\Psi^*_{\sigma^B_{\text{unif}, \delta}} \times \text{id}_{T^\vee_{\Sigma}}))^* (\omega^\text{PGL}_{\Sigma}) + (\pi^\mathcal{E}_{\Sigma, 2} \circ (\Psi^*_{\sigma^B_{\text{unif}, \delta}} \times \text{id}_{T^\vee_{\Sigma}}))^* (\omega^\text{can}_{\Sigma}) \\
&= (\Psi^*_{\sigma^B_{\text{unif}, \delta}} \circ \pi^\mathcal{E}_{\Sigma, 1})^* (\omega^\text{PGL}_{\Sigma}) + \pi^\mathcal{E}_{\Sigma, 2}(\omega^\text{can}_{\Sigma}) \\
&= \pi^\mathcal{E}_{\Sigma, 1}(\omega^\text{can}_{\Sigma}) + \pi^\mathcal{E}_{\Sigma, 2}(\omega^\text{can}_{\Sigma}).
\end{align*}
\]

By (46) and (47), the equality (45) holds if and only if the equality

\[
\pi^\mathcal{E}_{\Sigma, 1}(\omega^\text{can}_{\Sigma}) = \pi^\mathcal{E}_{\Sigma, 2}(\omega^\text{can}_{\Sigma})
\]

holds. But, the equality (48) follows from an argument similar to the argument in the proof of the former assertion of Proposition 5.1.1. This completes the proof of the former assertion.

The latter assertion follows from the former assertion and the argument in the proof of the latter assertion of Proposition 5.1.1 where \( T^\vee \) is replaced by \( \mathcal{G} \). This completes the proof of the proposition.

\[\Box\]

5.2. Now, let us consider the case where \( R = K \) for a field \( K \) of characteristic \( p > 2 \). Suppose further that \( \varpi : S \to \mathcal{M}_{g,K} \) factors through the projection \( \mathfrak{z} \mathcal{M}_{g,K} \to \mathcal{M}_{g,K} \) and that the resulting morphism \( \tilde{\varpi} : S \to \mathfrak{z} \mathcal{M}_{g,K} \) is étale. Denote by \( \sigma_S : S \to \mathcal{G}_S \) (resp., \( \Psi_S : T^\vee_S \to \mathcal{G}_S \)) the restriction of \( \sigma_{g,K} \) (resp., \( \Psi_{g,K} \)) to \( S \). Since the natural map

\[
\Gamma(\mathfrak{c} \mathcal{M}_{g,K}^{2z \ldots}, \bigwedge^2 \Omega_{\mathfrak{c} \mathcal{M}_{g,K}^{2z \ldots} / K}) \to \Gamma(T^\vee_S; \bigwedge^2 \Omega_{T^\vee_S / K})
\]

induced from \( \tilde{\varpi} \) is injective, the proof of Theorem 4.4.1 is reduced to proving the equality

\[
\Psi^*_S(\omega^\text{PGL}_S) = \omega^\text{can}_S.
\]

Moreover, for the same reason, we are always free to replace \( S \) by any étale covering of \( S \).
5.3. It follows from Propositions 5.1.1 and 5.1.2 that for each \( A \in \Gamma(S, \Omega_{S/K}) \), we have the following sequence of equalities:

\[
\begin{align*}
\sigma^*_S(\omega^\text{PGL}_S) - 0^*_S(\omega^\text{can}_S) & = \sigma^*_S(\tau^*_{S,A}(\omega^\text{PGL}_S) - dA) - 0^*_S(\tau^*_{S,A}(\omega^\text{can}_S) - dA) \\
& = (0^*_S(\Psi^*_S(\tau^*_{S,A}(\omega^\text{PGL}_S))) - dA) - (0^*_S(\tau^*_{S,A}(\omega^\text{can}_S)) - dA) \\
& = 0^*_S(\tau^*_{S,A}(\Psi^*_S(\omega^\text{PGL}_S))) - \tau^*_{S,A}(\omega^\text{can}_S)) \\
& = \sigma^*_{T^v_{S,A}}(\Psi^*_S(\omega^\text{PGL}_S) - \omega^\text{can}_S).
\end{align*}
\]

After possibly replacing \( S \) by its étale covering, we may assume that \( S \) is affine and the vector bundle \( \Omega_{S/K} \) is free. Under this assumption, \( \Psi^*_S(\omega^\text{PGL}_S) - \omega^\text{can}_S = 0 \) if and only if \( \sigma^*_{T^v_{S,A}}(\Psi^*_S(\omega^\text{PGL}_S) - \omega^\text{can}_S) = 0 \) for all \( A \in \Gamma(S, \Omega_{S/K}) \). Thus, in order to complete the proof of Theorem 4.4.1, it suffices (by (49)) to prove the equality

\[
\sigma^*_S(\omega^\text{PGL}_S) = 0^*_S(\omega^\text{can}_S).
\]

5.4. On the one hand, let us consider the right-hand side of the required equality (50), i.e., \( 0^*_S(\omega^\text{can}_S) \). The differential of the zero section \( 0_S : S \to T^v_{S} \) specifies a split injection \( T_{S/K} \hookrightarrow 0^*_S(T^v_{T^v_{S}/K}) \) of the natural short exact sequence

\[
0 \to 0^*_S(T^v_{T^v_{S}/S}) \to 0^*_S(T^v_{T^v_{S}/K}) \to T_{S/K} \to 0.
\]

This split injection gives a decomposition

\[
0^*_S(T^v_{T^v_{S}/K}) \cong T_{S/K} \oplus 0^*_S(T^v_{T^v_{S}/S}) \cong T_{S/K} \oplus \Omega_{S/K} \cong \mathbb{R}^1f_*\Omega^2_{(C_S/S)} \oplus f_*\Omega^2_{(C_S/S)},
\]

where the last isomorphism follows from (37). The \( O_S \)-bilinear map on \( 0^*_S(T^v_{T^v_{S}/K}) \) determined by \( \omega^\text{can}_S \) is given via this decomposition, by the pairing \( \langle - , - \rangle : \mathbb{R}^1f_*\Omega^2_{(C_S/S)} \times f_*\Omega^2_{(C_S/S)} \to O_S \) arising from \( f_*\Omega^2_{(C_S/S)} \) (cf. [31]). More precisely, this bilinear map may be expressed, via (51), as the map given by assigning

\[
((a, b), (a', b')) \mapsto \langle a, b' \rangle - \langle a', b \rangle
\]

do local sections \( a, a' \in \mathbb{R}^1f_*\Omega^2_{(C_S/S)} \) and \( b, b' \in f_*\Omega^2_{(C_S/S)} \).

5.5. On the other hand, let us consider the left-hand side of (50), i.e., \( \sigma^*_S(\omega^\text{PGL}_S) \). The differential of the section \( \sigma_S (= \Psi_S \circ 0_S) : S \to \mathfrak{g}_S \) specifies a split injection \( T_{S/K} \hookrightarrow \sigma^*_S(\mathfrak{g}_S/K) \) of the short exact sequence

\[
0 \to \sigma^*_S(\mathfrak{g}_S/K) \to \sigma^*_S(\mathfrak{g}_S/K) \to T_{S/K} \to 0.
\]

If \( C_S^{2aa...} = (C_S/S, P^\circ = (P_B, \nabla_{P_G})) \) denotes the ordinary dormant curve classified by \( \hat{\gamma} \) (hence \( \gamma^*_{P_G} \) is an isomorphism), then it follows from Proposition 2.9.1 that this split injection corresponds to a split injection

\[
\mathbb{R}^1f_*\Omega^2_{(C_S/S)} \hookrightarrow \mathbb{R}^1f_*\Omega^2_{(C_S/S)} \to 0.
\]

of the short exact sequence

\[
0 \to f_*\Omega^2_{(C_S/S)} \xrightarrow{\gamma^*_{P_G}} \mathbb{R}^1f_*\Omega^2_{(C_S/S)} \xrightarrow{\gamma^*_{P_G}} f_*\Omega^2_{(C_S/S)} \to 0.
\]
Moreover, if we identify \( \mathbb{R}^1 f_* (\mathcal{T}_{CS/s} / s) \) with \( \mathbb{R}^1 f_* (\text{Ker}(\nabla^\text{ad}_{\mathcal{P}^G})) \) via the isomorphism \( \gamma^\times_{\mathcal{P}^G} \), then it follows from Proposition 3.2.2 that the injection (53) coincides with \( \gamma^\times_{\mathcal{P}^G} : \mathbb{R}^1 f_* (\text{Ker}(\nabla^\text{ad}_{\mathcal{P}^G})) \to \mathbb{R}^1 f_* (\mathcal{K}^\text{ad}_{\mathcal{P}^G})) \). Consider the resulting decomposition

\[
\mathbb{R}^1 f_* (\mathcal{K}^\text{ad}_{\mathcal{P}^G})) \cong \mathbb{R}^1 f_* (\mathcal{T}_{CS/s}) \oplus f_* (\Omega^2_{CS/s}).
\]

Because of the discussion in the previous subsection, the proof of Theorem 4.4.1 is reduced to the following lemma.

**Lemma 5.5.1.** The \( \mathcal{O}_S \)-bilinear map on \( \mathbb{R}^1 f_* (\mathcal{K}^\text{ad}_{\mathcal{P}^G})) \) corresponding to \( \omega_{\mathcal{P}^G} \) is given, via the decomposition (54), by the pairing \( \mathbb{R}^1 f_* (\mathcal{T}_{CS/s}) \times f_* (\Omega^2_{CS/s}) \to \mathcal{O}_S \) arising from \( \int_{CS} \omega^2_{CS} \) (in the sense of (52)).

**Proof.** The subsheaf \( \text{ad} (\mathcal{P}^G) \subseteq \text{ad} (\mathcal{P}^G) \) is isotropic with respect to \( \kappa_{\mathcal{P}^G} : \text{ad} (\mathcal{P}^G) \otimes \mathcal{O}_S \to \mathcal{O}_S \) (cf. (32)). Hence, \( \kappa_{\mathcal{P}^G} \) induces an \( \mathcal{O}_S \)-bilinear map \( \kappa_{\mathcal{P}^G} : (\text{ad} (\mathcal{P}^G) / \text{ad} (\mathcal{P}^G)) \times \text{ad} (\mathcal{P}^G)^2 \to \mathcal{O}_S \). One verifies from a straightforward calculation that the diagram

\[
\begin{align*}
\xymatrix{
\mathcal{T}_{CS/s} \times \Omega_{CS/s} \ar[r]^-{\nabla^\times_{\mathcal{P}^G}} \ar[dr]_{\langle -, - \rangle} & (\text{ad} (\mathcal{P}^G) / \text{ad} (\mathcal{P}^G)) \times \text{ad} (\mathcal{P}^G)^2 \\
& \mathcal{O}_S \ar[l]_-{\pi_{\mathcal{P}^G}}}
\end{align*}
\]

is commutative, where \( \langle -, - \rangle \) denotes the natural paring \( \mathcal{T}_{CS/s} \times \Omega_{CS/s} \to \mathcal{O}_S \). Therefore, the assertion follows from this observation together with, e.g., the explicit description of \( \mathbb{R}^1 f_* (\mathcal{K}^\text{ad}_{\mathcal{P}^G})) \) in terms of the Čech double complex (cf. \( \S 2.10 \)). \( \square \)

Consequently, we have proved Theorem 4.4.1 (i.e., Theorem A), as desired.

## 6. Appendix: Application of Theorem A

As an application of Theorem 4.4.1, we construct certain additional structures on \( \mathfrak{g}^{2x \times} \). Let \( K \) be a field of characteristic \( p > 2 \), \( X \) a smooth Deligne-Mumford stack over \( K \), and \( \omega \) a symplectic structure on \( X \). The 2-form \( \omega \) corresponds to a nondegenerate pairing \( \mathcal{T}_{X/K} \otimes \mathcal{O}_X \to \mathcal{O}_X \) on \( \mathcal{T}_{X/K} \) and gives an identification \( \mathcal{T}_{X/K} \cong \mathcal{T}^\vee_{X/K} = \Omega_{X/K} \). Under this identification, \( \omega \) may be thought of as a nondegenerate pairing \( \omega^{-1} : \Omega_{X/K} \otimes \mathcal{O}_X \to \mathcal{O}_X \). Thus, we obtain a skew-symmetric \( K \)-bilinear map

\[\{ -, - \}_\omega : \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X\]

defined by \( \{ f, g \}_\omega = \omega^{-1}(df, dg) \). One verifies from the closedness of \( \omega \) that \( \{ -, - \}_\omega \) defines a Poisson bracket.

**Definition 6.0.1.** A **restricted structure** on the pair \( (X, \omega) \) is a map \( -[^p] : \mathcal{O}_X \to \mathcal{O}_X \) such that the triple \( (\mathcal{O}_X, \{ -, - \}_\omega, -[^p]) \) forms a sheaf of restricted Poisson algebras over \( K \) (cf. [6], Definition 1.8, for the definition of a restricted Poisson algebra).
In addition, we shall recall the definition of a Frobenius-constant quantization (cf. [6], Definition 3.3; [6], Definition 1.1 and Definition 1.4). We shall denote by \( X^{(1)} \) the Frobenius twist of \( X \) over \( K \), i.e., the base-change \( X \times_{K,F_K} K \) of the \( K \)-stack \( X \) via the absolute Frobenius morphism \( F_K \) of \( K \). Also, denote by \( F : X \to X^{(1)} \) the relative Frobenius morphism of \( X \) over \( K \).

**Definition 6.0.2.**

(i) Let us consider a pair \((\mathcal{O}_X^h, \tau)\) consisting of

- a Zariski sheaf \( \mathcal{O}_X^h \) of flat \( k[[h]] \)-algebras on \( X \) complete with respect to the \( h \)-adic filtration;
- an isomorphism \( \tau : \mathcal{O}_X^h/h \cdot \mathcal{O}_X^h \cong \mathcal{O}_X \) of sheaves of algebras.

Then, we shall say that the pair \((\mathcal{O}_X^h, \tau)\) is a quantization on the pair \((X, \omega)\) if the commutator in \( \mathcal{O}_X^h \) is equal, via \( \tau \), to \( h \cdot \{ - , - \} \mod h^2 \cdot \mathcal{O}_X^h \).

(ii) A Frobenius-constant quantization on \((X, \omega)\) is a collection of data

\[
\mathcal{O}_X^h = (\mathcal{O}_X^h, \tau, s)
\]

consisting of a quantization \((\mathcal{O}_X^h, \tau)\) of \((X, \omega)\) and a morphism \( s : \mathcal{O}_X^{(1)} \to \mathcal{Z}^h \subseteq \mathcal{O}_X^h \), where \( \mathcal{Z}^h \) denotes the center of \( \mathcal{O}_X^h \), of sheaves of algebras such that the composite

\[
\mathcal{O}_X^{(1)} \xrightarrow{s} \mathcal{Z}^h \xrightarrow{incl} \mathcal{O}_X^h \xrightarrow{\text{quot},} \mathcal{O}_X^h/h \cdot \mathcal{O}_X^h \xrightarrow{\tau} \mathcal{O}_X
\]

coincides with the morphism \( F^* : \mathcal{O}_X^{(1)} \hookrightarrow \mathcal{O}_X \) induced by \( F \).

See [6], the discussion at the end of §1.2 or Theorem 1.23, for relationships between restricted structures on \((X, \omega)\) and Frobenius-constant quantizations on \((X, \omega)\).

**Example 6.0.3.** Let \( S \) be a smooth Deligne-Mumford stack over \( K \). In a natural manner, one may construct a restricted structure and a Frobenius-constant quantization on the cotangent bundle \( T_S^\vee \) equipped with the symplectic structure \( \omega_S^\text{can} \), as follows.

(i) If \( f \) is a local function on \( T_S^\vee \) lifted from an open substack of \( S \), then we set \( f^{[p]} := f^p \).

Also, if \( \partial \) is a fiber-wise linear function on \( T_S^\vee \) over an open substack of \( S \), then we define \( \partial^{[p]} \) to be the \( p \)-th iterate of \( \partial \). Then, these assignments \((-) \mapsto (-)^{[p]}\) define an endomorphism

\[
(-)^{[p]} : \mathcal{O}_{T_S^\vee} \to \mathcal{O}_{T_S^\vee}
\]

of \( T_S^\vee \). By the discussion in (ii) below and [6], the discussion at the end of §1.2, the map \((-)^{[p]}\) forms a restricted structure on \((T_S^\vee, \omega_S^\text{can})\).

(ii) Assume that \( S \) is affine. The sheaf of asymptotic differential operators \( D^h(S) \) on \( S \) (cf. [6], Example 3.1) is the \( h \)-completion of the \( K[[h]] \)-algebra generated by \( \Gamma(S, \mathcal{O}_S) \) and \( \Gamma(S, \mathcal{T}_{S/K}) \) subject to the relations

\[
f_1 \ast f_2 = f_1 \cdot f_2, \quad f_1 \ast \xi_1 = f_1 \cdot \xi_1, \quad \xi_1 \ast \xi_2 - \xi_2 \ast \xi_2 = h \cdot [\xi_1, \xi_2], \quad \xi_1 \ast f_1 - f_1 \ast \xi_1 = h \cdot \xi_1(f_1).
\]

for any local sections \( f_1, f_2 \in \Gamma(S, \mathcal{O}_S) \) and \( \xi_1, \xi_2 \in \Gamma(S, \mathcal{T}_{S/K}) \), where \( \ast \) denotes the multiplication in \( D^h(S) \). We have a natural isomorphism

\[
\tau(S) : D^h(S)/h \cdot D^h(S) \cong \Gamma(T_S^\vee, \mathcal{O}_{T_S^\vee}).
\]
If $Z^h(S)$ denotes the center of $D^h(S)$, then there exists (cf. [5], the proof of Proposition 3.5) a morphism

$$s(S) : \Gamma(T_S^{\vee(1)}, O_{T_S^{\vee(1)}}) \to Z^h(S)$$

determined, under the identification $\Gamma(T_S^{\vee(1)}, O_{T_S^{\vee(1)}}) = \Gamma(T_S^{\vee}, O_{T_S^{\vee}})$ via $\text{id}_{T_S^{\vee}} \times F_{K}^{abfs} : T_S^{\vee(1)} \sim T_S^{\vee}$, by

$$s(S)(f) = f^p, \quad s(S)(\partial) = \partial^p - h^{p-1} \cdot \partial^{[p]}$$

for any $f \in \Gamma(S, O_S)$, and $\partial \in \Gamma(S, T_S/K)$. By applying a natural noncommutative localization procedure called Ore localization (cf. [16], § 9, p.143, Definition), we obtain, from the triple $(D^h(S), \tau(S), s(S))$, a Frobenius-constant quantization

(55)

$$(D_S^h, \tau, s)$$

on $(T_S^\vee, \omega_S^{can})$ (cf. [5], Proposition 3.5).

In general, let $S$ be an arbitrary smooth Deligne-Mumford stack. Then, the Frobenius-constant quantizations constructed above for various affine schemes having an open immersion into $S$ may be glued together to obtain a Frobenius-constant quantization on $(T_S^\vee, \omega_S^{can})$.

By applying the discussion in Example 6.0.3 to the case $S = \mathcal{M}^{Zas}_{g,K}$, we obtain a restricted structure as well as a Frobenius-constant quantization on $(\mathcal{M}^{Zas}_{g,K}, \omega_{\PGL})$. Such additional structures may be evidently transported into $(\mathcal{G}^{Zas}_{g,K}, \omega_{\PGL})$ via the isomorphism $\Psi_{g,K}$. The isomorphism $\Psi_{g,K}$ preserves the symplectic structure as asserted in Theorem 4.4.1, so we have the following corollary.

**Corollary 6.0.4.** There exist a canonical restricted structure and a Frobenius-constant quantization on $(\mathcal{G}^{Zas}_{g,K}, \omega_{\PGL})$.

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