Incomplete information and correlated electrons

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Although Gödel’s incompleteness theorem made mathematician recognize that no axiomatic system could completely prove its correctness and that there is an eternal hole between our knowledge and the world, and in spite of the work of Poincaré of about 100 years ago and the further development of the theory of chaos, the dream of man to conquer nature and to know everything about nature refuse to die away. Physicists continue this ambition in working so far on the approaches based on the hypothesis to completely or approximately know the systems of interest. In this paper, however, I review the recent development of a different approach, a statistical theory based upon the notion of incomplete information. Incomplete information means that, with complex systems whose interactions cannot be completely written in its hamiltonian or whose equation of motion does not have exact solution, the information needed to specify the systems is not completely accessible to us. This consideration leads to generalized statistical mechanics characterized by an incompleteness parameter $\omega$ which equals unity when information is complete. The mathematical and physical bases of the information incompleteness are discussed.

The application of the concomitant incomplete fermion statistics to correlated electron systems is reviewed. By comparison with some numerical results for correlated electron systems, it is concluded that, among several other generalizations of Fermi-Dirac distribution, only the incomplete one is
suitable for describing this kind of systems. The extensive incomplete fermion distribution \( n = 1/\{\exp[\omega(e - e_f)/k_B T] + 1\} \) gives very good description of weakly correlated electrons with about \( 0.003 < \omega < 1 \), the normalization index in \( \sum_i p_i^\omega = 1 \) where \( p_i \) is probability distribution. On the other hand, the nonextensive fermion distribution, \( n = 1/\{1 + (\omega - 1)(e - e_f)/k_B T]^\omega/(\omega-1) + 1\} \), does not show weak correlation behaviors of electrons and is only suitable to describe strong correlated heavy fermion systems showing strong increase of Fermi momentum with increasing correlations for \( 0 < \omega < 1 \).

I. INTRODUCTION

As the study of complexity advanced, scientists have realized that chaotic and fractal behaviors were ubiquitous in nature and the simple phenomena described by deterministic or quasi-deterministic [1] physical sciences considering only simple interactions or predictable linear behaviors were only a few special or accidental cases. It was also realized that patching up was fundamentally useless within the conventional physics theories that break down once applied to complex systems having long range interactions or showing nonlinear behavior related to chaotic or fractal phase space structure. Generalization of these theories would be necessary. Driven by the increasing knowledge about chaos and fractals, the attempt of generalization has been rapidly focused on the problems relative to information and statistics theory [2-9]. The development of the nonextensive statistical mechanics (NSM) [7,8,11,12], among others [13], is a good example of this tendency in physics.

Though considered by some to have a weak point due to the lack of clear physical significations of its generalization parameter \( q \), the probability distributions of NSM has been proved to be surprisingly useful for describing complex systems having long term interactions or correlations for which Boltzmann-Gibbs statistics (BGS) is no more valid. NSM generalizes BGS with a distribution function called \( q \)-exponential given by \( \exp_q(x) = [1+(1-q)x]^{1/(1-q)} \). The latter is the inverse function of a generalized logarithm \( \ln_q(x) = \frac{x^{1-q} - 1}{1-q} \) which can be
used as a generalization of Hartley logarithmic information measure to obtain the q-entropy

\[ S_q = -k \frac{1}{1-q} \sum_i p_i^q \quad (q \in R) \] proposed by Tsallis \cite{7}. When \( q = 1 \), These above two
generalized functions become the usual ones and the q-entropy becomes Shannon one.

In the present paper, I will review our recent efforts to find consistent foundation for NSM
distribution functions and to give satisfactory answers to some fundamental questions. These
efforts are based on a notion which is both new and old: incomplete information \cite{8,9}. New
because scientists always claimed, in constructing physics theories, that their theories contain
all necessary information for specifying the systems under consideration. This is the case of
all the conventional physical theories: from Newtonian to quantum physics, in passing by
Einstein, Boltzmann and Shannon (certainly, a theory containing only partial information
about the system of interest is a little bit discouraging). Old because since the discovery
of, e.g., irrational numbers, mathematicians know that, within arithmetical system, they
lose some information about the world and that one could not know everything with infinite
precision. In 1931, Gödel shown \cite{4–6} that mathematics system (or any axiomatic system)
is incomplete in the sense that within any such axiomatic system there is never sufficient
information to prove all possible statements of the theory \cite{6}. If a non negligible amount of
information is not accessible to us, BGS theory has to be modified. Incomplete information
theory is a kind of modification (generalization) of BGS suggested by this consideration as
well as by some difficulties encountered within NSM in the last decade \cite{8,15}.

\[ \text{From now on, the parameter } q \text{ will be replaced by } \omega \text{ and Tsallis entropy by } S_\omega = -k \frac{1}{1-\omega} \sum_i p_i^\omega. \]

The above generalized functions will be called \( \omega \)-exponential and \( \omega \)-logarithm. I make this replace-
ment for the simple reason that, though it often gives similar forms of functions as \( q, \omega \) defined in
the framework of the theory I review here does not have the same physical content as the parameter
\( q \) in Tsallis version of NSM. So I prefer to use \( \omega \) to avoid confusions. By definition, \( \omega \) has clear
physical meaning as the reader will find in this paper.
II. COMPLETE INFORMATION ASSUMPTION

In this section, I will briefly review the well known information theory founded by Shannon et al \cite{16}. It should be remember that *information* about a real system is not our knowledge about it. It is our ignorance. The ignorance of something to which we may have access. A mail address, as a state of physical system, may be an information if we do not know it. More we know about a system, less there is information in its description. So in a deterministic theory (e.g., classical mechanics), information is null. In statistical theory, there is information because we ignore something so that we are not sure of the exact state at any given moment of the system under consideration. So information can be related to the uncertainty due to the ignorance or to the *probability* of finding the system at different states. It should be noted that, as mentioned above, up to now, we always suppose that the information we address in any statistical theory is complete or completely accessible. That is if we obtain it, we can answer all questions which can be asked about the system. This certainty is reflected by the following postulate:

\[
\sum_{i=1}^{v} p_i = 1, \quad (1)
\]

where \( v \) must be the number of all the possible states of the system under consideration. As a result, the arithmetic average of \( \xi \) is given by

\[
\bar{x} = \sum_{i=1}^{v} p_i x_i.
\]

By some analysis of the information properties, it is supposed \cite{2,16} that the information is given by the well known Hartley formula \( \ln(N) \) \cite{17} needed to specify \( N \) elements, or by \( \ln(1/p_i) \), the information needed to specify that an element will be found at the state \( i \). If we perfectly know all the \( v \) possible states, then the complete information measure \( I \) is given by averaging all \( \ln(1/p_i) \):

\[
I = \sum_{i=1}^{v} p_i \ln(1/p_i). \quad (2)
\]

It should be emphasize that the above definition of information or entropy needs the harsh condition that the interactions in the system of interest are of short range or limited between
the walls of the containers of subsystems which are consequently independent of each other. To see this, it suffices to consider the assumption of information additivity, i.e., for a system $C$ containing two subsystems $A$ and $B$, it is supposed $I(C) = I(A) + I(B)$. This additivity is valid if and only if the information $I(C)$ needed in order to specify simultaneously $A$ and $B$ is given by $\ln[N(A)N(B)]$ where $N(A)$ and $N(B)$ are respectively the number of elements in $A$ and $B$. This is as if we had a system $C$ containing $N(A)N(B)$ elements. This result needs that the states of the elements of $A$ do not depend on the states of $B$. In other words, these is no interactions between the elements of $A$ and those of $B$. There may be interactions between the elements on the walls of the containers of $A$ and $B$, but most of the elements inside $A$ and $B$ must be independent. This is a case of short range interaction where we have not only additive information or entropy, but also additive energy and other extensive thermodynamic variables.

I would like to recall in passing here that the total information $\ln[N(A)N(B)]$ implies

$$p_{ij}(C) = p_i(A)p_j(B)$$

where $p_{ij}(C)$ is the probability that the composite system $C$ is at the product state $ij$ when $A$ is at the state $i$ with probability $p_i(A)$ and $B$ at $j$ with $p_j(B)$. Eq.(3) symbolizes the independence of the noninteracting subsystems having additive physical quantities. But for interacting subsystems, it symbolizes totally different physical reality. This product law has been widely employed and discussed in the last decade in connection with equilibrium and many body problems [18–24] and caused much confusion within NSM because it paradoxically independence of subsystems and additive energy for nonextensive interacting systems. Very recently, we shown that Eq.(3) was nothing but the consequence of the existence of thermodynamic equilibrium in interacting systems described by $\omega$-entropy and did not need independence of the subsystems. This conclusion allows to exactly define equilibrium parameters such as temperature, pressure and chemical potential for nonextensive systems and to obtain the exact one body quantum distributions [21–24].

According to above discussions, we can say that, if there are long range interactions
between $A$ and $B$, the information about $C$ will be different from $\ln[N(A)N(B)]$ because the elements are correlated and can no more occupy their states independently. According to the nature of the correlation, there may be *more or less* information than in the noninteracting case. In general, we should write $I(C) = I(A) + I(B) + f[I(A), I(B)]$, a case treated by NSM. Now Eq. (3) becomes questionable, yet it is a crucial relationship for any statistical mechanics, for it’s applications to many-body systems and it’s thermodynamics connection. The reader will find detailed discussions on this issue below.

### III. COMPLEXITY AND MATHEMATICS

Certainly, complete information is possible whenever all possible states are well known so that we can count them to carry out the calculation of probability and information. In physics, this requires that we can find the *exact hamiltonian* and also the *exact solutions of the equation of motion* to know all the possible states and to obtain the exact values of physical quantities dependent on the hamiltonian. The reader will see that these two “exact” conditions of complete information are almost impossible to satisfy.

Let us begin by asking some questions about the mathematical basis of physical theory. What is the A basic field of mathematics is the classical arithmetic. From the epistemological point of view, arithmetic is a theory based on a model of world resulted from the direct intuition of human beings. This is a simple model for fragmented world containing only isolated, distinct and independent parts. So you have $1 = 1$, $1 + 1 = 2$ and a series of rules, theorems and generalizations. No matter how complicated are the immense mathematical constructions developed from arithmetic, their validity is always limited by these initial conditions imposed by the crude data of our senses and direct intuition. Indeed, our senses, luckily, have the capability of filtering the complex world into separated and discernible parts. If not, scientific knowledge would be impossible. But these harsh constraints imposed by this filtration, as claimed by Poincaré [25], should not be forgotten. We have to ask the following question : how far he can go with the concepts formed through the
filtration in the real messy world or complex systems including interacting, entangled and overlapped parts, especially when the interactions can no more be neglected.

So in some sense, it can be said that mathematics is an approximate theory containing finite amount of information about the world which is surely incomplete because some information is lost by our senses through the formation of the axioms. Any formation of axiomatic systems is necessarily made through a kind of filtration of the world. The results of the filtration are not wrong, but they are only partially true. Something about the connection of different parts of the world is rejected by the filtration. In my opinion, this is why axiomatic systems, as stated by the incompleteness theorem of Gödel, inevitably fail to prove some statements, especially those about their axioms. There is no enough information for that. The missing information is just what rejected by the formation of axioms.

A mathematician is rather interested by the coherence of his logical systems based on axioms. He may put aside the missing information and work within the logical systems without being connected to physical reality. But for a physicist, the connection of his theory to the outside world is the most important thing he mind. He possibly ask : My physical theory is in fact an application of a incomplete mathematical theory. If the information I am handling is not complete, how can I apply it to the world whose description probably needs more information?

In what follows, we will try to answer this question in recognizing that the incompleteness of all axiomatic systems discovered by Gödel has put an end to the ambition of establishing physical theories containing or capable of treating complete information about any system in the world. In this sense, any physics theory is incomplete by definition. This is the very reason for the introduction of “incomplete information” into statistical physics. This introduction needs in addition other considerations I am presenting below.
IV. COMPLEXITY AND INCOMPLETE INFORMATION

Now let us look at the information problem from the physical viewpoint. I will try to show that, due to the omnipresent complexity in the world, we cannot have access to all the necessary information for complete description of a system. Here “complexity” means that the systems show nonlinear behaviors which are extremely sensible to initial conditions and unpredictable. This is the famous chaos observed almost everywhere in the world [3–6].

A complex system is not necessarily a complicated system with a large number of freedoms. A one dimensional oscillator with well known nonlinear interaction (with potential $\propto x^4$, for example) or a three body system with gravitation ($\propto 1/r$) can behave chaotically. These two cases are just very good examples of the impossibility of the two “exact” conditions of complete information mentioned above. In the case of the three body problem, we know (at least we believe that we know) the exact interaction of the system (Newtonian gravitation). But Poincaré showed that the exact and predictable solution of the equation of motion was not possible [4,5]. There are in fact infinite number of periodic and aperiodic solutions. The movement is chaotic and unpredictable and the attractors of the chaotic structures formed by the trajectories in phase space are fractal. This means that we never know all possible states of the system and that complete information treatment becomes impossible. We even have to redefine probability distribution in order to calculate it in chaotic or fractal phase space.

Above conclusion is for hamiltonian systems whose interactions is à priori well known. When the hamiltonian cannot be exactly written, the situation is more complicated. Even the exact and predictable solutions of equation of motion are not complete due to the incomplete hamiltonian. This may happen if, for a isolated closed system, the interactions are too complex to be written, or, for a system with simple interactions, the effects of the external perturbations are not negligible. Sometimes negligible perturbations may have drastic consequences if the system is sensitive to initial conditions. In this sense, the omission of small interactions may make enormous information unaccessible to the theory. This
incompleteness due to neglected interactions simply adds to the incompleteness mentioned previously.

In any case, complete information description of complex systems is only a science fiction. Although we cannot say that all these systems have chaotic or fractal nature, a common feature of them is that *a part of their phase space is unknown so that complete and exhaustive exploit of the phase space is impossible*. The calculable information is inevitably limited by this incompleteness of knowledge. That is evident. The treatments of these systems based on the assumption of complete information and probability distribution are not well founded. They are legitimate only when unaccessible part of the information is negligible with respect to the accessible information and to the desired precision of observation or theoretical description.

In what follows, we will try to introduce the notion of incompleteness of information into physics through statistical method. It was with this method that man began to overcome the obstacle of his limited knowledge in supposing, on the basis of Newtonian or quantum mechanics, that the missing knowledge (information) is mathematically accessible or, equivalently, that the calculated probability must sum to one. Now if we say that we cannot have access to every information we need or to every point of the phase space, a serious impact on the normalization of probability, the very first stone in the construction of statistics, will be inevitable.

### V. CHAOS AND INCOMPLETE PROBABILITY DISTRIBUTIONS

#### A. Incomplete normalization

What can we do for probability and information calculation if we do not know how many states the system of interest has? When we deal with a chaotic system having fractal attractor in phase space [3], it is as if we toss a coin which often comes down, neither tails nor heads, but standing on the side without, in addition, being observed. All calculations
based on Eq. (1) with \( v = 2 \) would lead to aberrant results because we have now \( \sum_{i=1}^{v} p_i = Q \neq 1 \). In this case, \( p_i \) is referred to as incomplete distribution [2] and \( Q \) is a constant depending on and characterizing the incompleteness of the system and provides a possible key to introduce incompleteness of information into physics theory. It should be supposed \( Q = 1 \) if information is complete.

The philosophy of incomplete information theory we developed is to keep the methods of classical complete probability theory for incomplete information or probability distribution by introducing empirical parameters in order to characterize the incompleteness. This is just the same methodology as in the theory of chaos or fractals introducing fractal dimension to characterize the structures of space time. In this sense, we can refer to the parameter \( \omega \) introduced below as incompleteness parameter.

First of all, we need a “normalization” for incomplete distribution \( p_i \) in order to take advantage of the conventional probability theory. This is an occasion to introduce a parametrization function \( F_\omega \) and to write

\[
\sum_i F_\omega(p_i) = 1
\]  

which can be called generalized or incomplete normalization. \( F_\omega \) should depend on the nature of the system and become identity function whenever information is supposed complete (\( Q = 1 \)). The arithmetic average should now be given by \( \bar{x} = \sum_i F_\omega(p_i)x_i \). \( F_\omega \) can be determined if the information measure and the distribution law are given. For example, with Hartley information measure and exponential distribution, \( F_\omega \) can be showed to be identity function [3]. In general, by entropy maximization through the functional

\[
\delta \left[ \sum_i F_\omega(p_i)I(p_i) + \beta \sum_i F_\omega(p_i)x_i \right] = 0
\]  

we get:

\[
\frac{\partial \ln F_\omega(p_i)}{\partial p_i} = \frac{\partial I/\partial p_i}{I + \beta f_\omega^{-1}(p_i)}
\]  

or \( F_\omega(p_i) = C \exp\left[ \int \frac{\partial I/\partial p_i}{I + \beta f_\omega^{-1}(p_i)} dp_i \right] \) where \( \beta \) is the multiplier of Lagrange connected to expectation, \( I(p_i) \) is the information measure, \( p_i = f_\omega(x_i) \) the distribution function depending on
the parameter \( \omega \), \( C \) the normalization constant of \( F_\omega \).

**B. Incomplete normalization of NSM**

In my previous papers \[8,9\], in order to find coherent foundation for \( \omega \)-exponential distribution on the basis of \( \omega \)-logarithm information measure, \( F_\omega(p_i) = p_i^{\omega} \) was postulated. So that

\[
\sum_i p_i^{\omega} = 1.
\] (7)

In what follows, I will try to show that the conjecture of power law incomplete normalization in the previous section is inevitable in a chaotic or fractal space time.

For the sake of simplicity, let us consider a phase space in which the trajectory of a chaotic system forms a simple self-similar fractal structure, say, Sierpinski carpet (Figure 1). This means that the state point of the system can be found only on the black rectangular segments whose number is \( W_k = 8^k \) at \( k^{th} \) iteration. Hence the total surface at this stage is given by \( S_k = W_k s_k \) where \( s_k = l_0/3^k \) is the surface of the segments at \( k^{th} \) iteration and \( l_0 \) the length of side of the square space at \( 0^{th} \) iteration. If the segments do not have same surface, we should write \( S_k = \sum_{i=1}^{W_k} s_k(i) \). We suppose that the density of state is identical everywhere on the segments and that the distribution is microcanonical, so that the probability for the system to be in the \( i^{th} \) segment may be defined as usual by \( p_i = s_k(i)/S_k \). This probability is obviously normalized. The problem is that, as discussed in \[3\], \( S_k \) is an indefinite quantity as \( k \to \infty \) and, strictly speaking, can not be used to define exact probability definition. In addition, \( S_k \) is not differentiable and contains inaccessible points. Thus the probability defined above makes no sense.

Alternatively, the probability may be reasonably defined on an integrable and differentiable support, say, the Euclidean space containing the fractal structure. To see how to do this, we write \( S_k = l_0^2(1/3^k)^{d-d_f} \) for identical segments or, for segments of variable size,

\[
\sum_{i=1}^{W_k} \left[ \frac{s_k(i)}{S_0} \right]^{d_f/d} = 1
\] (8)
where $S_0 = l_0^d$ (here $d = 2$ for Sierpinski carpet) a characteristic volume of the fractal structure embedded in a $d$-dimension Euclidean space, $d_f = \frac{\ln n}{\ln m}$ is the fractal dimension, $n = 8$ the number of segments replacing a segment of the precedent iteration and $m = 3$ the scale factor of the iterations. The microcanonical probability distribution at the $k^{th}$ iteration can be defined as $p_i = \frac{s_k(i)}{S_k}$ so that $\sum_{i=1}^{W_k} p_i^{d_f/d} = 1$ which is just Eq.(7) with $\omega = d_f/d$. The conventional normalization $\sum_{i=1}^{W_k} p_i = 1$ can be recovered when $d_f = d$.

It should be noticed that, in Eq.(8), the sum over all the $W_k$ segments at the $k^{th}$ iteration does not mean the sum over all possible states of the system under consideration. This is because that the segment surface $s_k(i)$ does not represent the real number of state points on the segment which, as expected for any self-similar structure, evolves with $k$ just as $S_k$. So at any given order $k$, the complete summation over all possible segments is not a complete summation over all possible states. But in any case, whatever is $k$, Eq.(8) and $\sum_{i=1}^{W_k} p_i^{\omega} = 1$ always holds for $\omega = d_f/d$.

In this simple case with self-similar fractal structure, the incompleteness of the normalization Eq.(7) is measured by the parameter $\omega = d_f/d$. If $d_f > d$, there are more state points than $W_k$, the number of accessible states at given $k$. If $d_f < d$, the number of accessible states is less than $W_k$. When $d_f = d$, the summation is complete at any order $k$, corresponding to complete information calculation.

VI. INCOMPLETENESS PARAMETER $\omega$

Here I will discuss in a detailed way the incompleteness parameter $\omega$ and its physical meanings. Incomplete statistics gives to the empirical parameter $\omega$ a clear physical signification: *measure of the incompleteness of information or of chaos*. Let us illustrate this by the simple case of self-similar fractal phase space with segments of equal size.
A. $\omega$ and Phase Space Expansion

As discussed in the case of chaotic phase space, $\omega = \ln n/d \ln m$ gives a measure of the incompleteness of the state counting in the $d$-dimension phase space. $\omega = 1$ means $d_f = d$ or $n = m^d$. In other word, at the $k^{th}$ iteration, a segment of volume $s_k$ is completely covered (replaced) by $n$ segments of volume $s_{k+1} = s_k/m^d$. So the summation over all segments is equivalent to the sum over all possible states, making it possible to calculate complete information.

When $\omega > 1$ (or $\omega < 1$), $n > m^d$ (or $n < m^d$) and $s_k$ is replaced by $n$ segments whose total volume is more (or less) than $s_k$. So there is expansion (or negative expansion) of state volume when we refine the phase space scale. An estimation of this expansion at each scale refinement can be given by the ratio $r = n s_{k+1}/s_k = n/m^d - 1 = \left(\frac{1}{m^d}\right)^{1-\omega} - 1 = (\omega - 1) \frac{(m^d)^{\omega-1}-1}{\omega-1}$.

$r$ describes how much unaccessible states increase at each step of the iteration or of the refinement of phase space. The physical content of $\omega$ is clear if we note that $\omega > 1$ and $\omega < 1$ correspond to an expansion ($r > 0$) and a negative expansion ($r < 0$), respectively, of the state volume at each step of the iteration. When $\omega = 0$, we have $d_f = 0$ and $n = 1$, leading to $r = \frac{1}{m^d} - 1$. The iterate condition $n \geq 1$ means $\omega \geq 0$, as proposed in references [8]. $\omega < 0$ is impossible since it means $d_f < 0$ or $n < 1$ which obviously makes no sense. We can also write : $\omega - 1 = \ln(r + 1)/\ln(m^d) = \ln(n s_{k+1}/s_k)/\ln(m^d)$, which implies that it is the difference $\omega - 1$ which is a direct measure of the state space expansion through the scale refinement.

B. $\omega$ and Information Growth

The expansion of the state volume of a system in its phase space during the scale refinement should be interpreted as follows : the extra state points $\Delta = n s_{k+1} - s_k$ acquired at $(k+1)^{th}$ order iterate with respect to $k^{th}$ order are just the number of unaccessible states at $k^{th}$ order. $\Delta > 0$ (or $\Delta < 0$) means that we have counted less (or more) states at $k^{th}$ order
than we should have done. \( \Delta \) contains the \textit{accessible information gain} (AIG) through the \((k + 1)^{th}\) iterate.

To illustrate the relation between this “hidden information” and the parameter \( \omega \), let us first consider the Hartley logarithm information in the simple case where the distribution is microcanonical and scale-invariant \([26]\). At the iterate of order \( k \), the average information contained on \( s_k \) is given by

\[
I_k = \int_{s_k} p^\omega \ln(1/p) ds.
\]

At \( k + 1 \) order, \( I_{k+1} = \int_{ns_k+s_k} p^\omega \ln(1/p) ds \).

Hence AIG is just \( \Delta I = I_{k+1} - I_k = \int_{ns_k+s_k} p^\omega \ln(1/p) ds = \sigma I \Delta \), where \( \sigma = p^\omega \ln(1/p) \) is the information density or the average information carried by each state. The relative AIG is given by \( \Delta I/I_k = r = (1 - \omega)(1/m^d)^{1-\omega-1} \) which is independent of scale but dependent on scale changes. For given scaling factor \( m \), the magnitude of \( \Delta I \) or \( r \) increases with increasing difference \(|1 - \omega|\). The sign of \( r \) (or AIG) was discussed earlier. For given \( \omega \), \(|\Delta I|\) increases with decreasing scaling. For \( \omega = 1 \) or \( m = 1 \), there is no information gain, corresponding to the case of complete information.

According to the relationship \( \omega = d_f/d \) and the above discussions, it can be concluded that the incompleteness parameter \( \omega \) may be considered as a measure of chaos. Certainly this is a conclusion on the basis of simple models and the relation between \( \omega \) and the degree of chaos or fractal may be more complicated with more complex chaos and fractals, but it is consequent to say that more a system is chaotic, more its information is incomplete and more \( \omega \) is different from unity.

VII. NONADDITIVE INCOMPLETE DISTRIBUTIONS

To get the nonextensive distribution in \( \omega \)-exponential as mentioned above, we can maximize the entropy \( S_\omega = -k \sum_{i=1}^{\omega} \frac{p_i - \sum_{i=1}^{\omega} p_i}{1-\omega} \) according to the Jaynes principle \([27]\) with the constraints \( U = \sum_i p_i E_i \) and \( N = \sum_i p_i N_i \) for grand-canonical ensemble, where \( U \) is the internal energy, \( N \) the average particle number, \( E_i \) the energy and \( N_i \) the particle number at the state \( i \) of the system. We obtain:

\[
p_i = \left[ 1 - (1-\omega)\beta(E_i - \mu N_i) \right]^\frac{1}{1-\omega}.
\]
where \( Z^\omega = \sum_i [1 - (1 - \omega)\beta(e_i - \mu N_i)]^\frac{1}{\omega} \). \([x]_+ = x \) if \( x > 0 \) and \([x]_- = 0 \) otherwise. \( \beta \) is the inverse temperature and \( \mu \) the chemical potential. This distribution function has been proved particularly useful for systems showing non gaussian distribution functions (for detailed information, see [11] and references there-in). Considering Eq.(3), the product probability law at thermodynamic equilibrium, the one-particle distribution from Eq.(9) can be rewritten as

\[
p_k = \frac{1}{z}[1 - (1 - \omega)\beta(e_k - \mu)]^\frac{1}{\omega}[1 + (1 - \omega)\beta(\epsilon_k - \nu)]^\frac{1}{\omega} = \frac{1}{Z} e^{-\beta'(e_k - \nu)}
\]

where \( \beta' = \frac{\beta}{1 - (q - 1)\beta\mu}, \mu' = \mu[1 - (q - 1)\beta\mu] \) which imply \( \beta' \mu' = \beta \mu, \nu = \frac{\ln[1 + (1 - q)\beta\mu]}{(1 - q)\beta\mu} \) and \( \epsilon_k = \frac{\ln[1 + (q - 1)\beta\mu]}{(q - 1)\beta\mu} \). The exponential distribution Eq.(10) makes it possible to straightforwardly obtain the \textit{exact quantum distribution} \textsuperscript{23} (EQD) given by

\[
\bar{n}_k = \frac{1}{e^{\epsilon_k / \beta'} \pm 1} = \frac{1}{[1 + (\omega - 1)\beta(e_k - \mu)]^\frac{1}{\omega} \pm 1} \quad (11)
\]

where \( \bar{n}_k \) is the occupation number of the one-particle state \( k \). ")" is for fermions and ")" for bosons. These distribution can be compared to the \textit{approximate quantum distributions} (AQD) of NSM \textsuperscript{20} \( \bar{n}_k = e^{\epsilon_k / \beta'} \pm 1 = \frac{1}{[1 + (q - 1)\beta(e_k - \mu)]^\frac{1}{\omega} \pm 1} \) given within a factorization approximation using additive energy. At first glance, EQD and AQD are not very different from each other if we put \( \omega = q \). But Figure 2 shows that they are two very different distributions. AQD remains approximately the same as the conventional Fermi-Dirac distribution for whatever \( q \) value. So its Fermi energy \( e_f \) is almost constant with changing \( q \). On the contrary, EQD changes drastically with \( \omega \). The Fermi energy \( e_f \) shows a strong increase with decreasing \( \omega \) up to two times \( e_f_0 \) of the conventional Fermi-Dirac distribution when \( \omega \to 0 \). This \( e_f \) increase has been indeed noticed through numerical calculations for strongly correlated heavy electrons on the basis of tight-binding Kondo lattice model \textsuperscript{28,29} as shown in Figure 3. Increasing correlation corresponds to decreasing \( \omega \) from unity (zero correlation).
This implies that EQD based on incomplete information has its merit in the description of heavy electron systems. Further investigation is needed to know the connection between the correlation and the nonextensive parameter \(1 - \omega\).

\section*{VIII. ADDITIVE INCOMPLETE DISTRIBUTIONS}

Although the nonextensive EQD accounts for an important aspect of correlated electrons, i.e., the correlation induced Fermi energy increase, another important aspect of the weak correlation is missing in the description of nonextensive EQD. This is the flattening of \(n\) drop at \(e_f\) \cite{28-31}. That is, the correlation, even at low temperature, drives electrons above \(e_f\) so that the \(n\) discontinuity becomes less and less sharp as the correlation increases. Curiously, this flattening of \(n\) discontinuity at \(e_f\) is completely absent in EQD of NSM. From Figure 2, we see that the sharp \(n\) drop at \(e_f\) is independent of \(\omega\) or correlations.

In what follows, I will present an additive incomplete statistical mechanics. It is assumed that the additive Hartley information measure still holds. So with respect to the conventional Shannon information theory and BGS, only the normalization is changed according to Eq.(7) \cite{9,13}. The additive incomplete entropy is given by

\[ S = k \sum_{i=1}^{w} p_i^{\omega} \ln\left(1/p_i\right). \]

When \(\omega \to 1\), \(S\) is Shannon entropy, which identifies \(k\) to Boltzmann constant.

For \textit{grand canonical ensemble}, the usual entropy maximization procedure leads to

\[ p_i = e^{-\omega \beta (E_i - \mu N_i)}/Z \] where partition function is given by

\[ Z = \left\{ \sum_{i=1}^{w} e^{-\omega \beta (E_i - \mu N_i)} \right\}^{1/\omega}. \]

For quantum particle systems, we have

\[ \bar{n}_k = \frac{1}{e^{\omega \beta (E_k - \mu)} \pm 1}. \quad (12) \]

The fermion distribution given by Eq.(12) is plotted in Figure 3 for different \(\omega\) values in comparison with some numerical simulation results. We note that IFD reproduces well the numerical results for about \(J < 1\). When coupling is stronger, a long tail in the KLM distributions begins to develop at high energy. At the same time, a new Fermi surface at \(k = k_{f_0} + \pi/2 = 0.75\pi\) starts to appear and a sharp \(n\) drop takes place at the new
Fermi momentum. At $J = 4$, KLM distribution (x-marks) is very different from IFD (e.g. $\omega = 0.0011$). The solid line fitting better the $J = 4$ KLM distribution is given by the incomplete statistics version of fractional exclusion distribution $(1/n-\alpha)^\alpha(1/n-\alpha+1)^{1-\alpha} = e^{\omega \beta (e-e_f) \; [32,33]}$ with $1/\alpha = 0.85$ due to the KLM occupation number smaller than 0.5 at low momentum $k$.

IX. CONCLUSION

Summing up, I have discussed the philosophical basis of incomplete information from both the viewpoints of mathematical and physics. The information we deal with in scientific theories can not be complete in the sense that a part of the information necessary for complete description of the system under consideration is not accessible to our theory or knowledge. This part of information is rejected from scientific knowledge by the formation of concepts, axioms and models. The amount of rejected information is particularly important for complex systems having chaotic behaviors and fractal phase space. A parameterized normalization $\sum_i p_i^{\omega} = 1$ is proposed for this kind of systems, where $\omega$ is the incompleteness parameter characterizing the inaccessibility of phase space points or of the information of the system. It also offers a measure of the degree of chaos.

The wide drop in the fermion occupation number and the sharp cutoff of occupation number at $e_f$ showing strong increase with increasing interaction can be interpreted by the nonextensive incomplete fermion distribution with decreasing $\omega$ value. On the other hand, it fails to describe weak correlation effect on electrons which is well accounted for by additive incomplete fermion distribution. But the additive distribution does not show the sharp cutoff at $e_f$ when correlation is strong. This result suggests to combine these two partially valid models to describe correlated electrons in a global way. Further results of this current work will be presented in other papers of ours.
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Figure caption:

FIG. 1. A simple model of fractal phase space in Sierpinski carpet (or sponge). At $k^{th}$ iteration, the side of the squares (black or white) is $l_k = l_0/2^k$ and their number is $W_k = 8^k$, $l_k$ being the length of the side at at $0^{th}$ iteration. The total surface at $k^{th}$ iteration is $S_k = W_k s_k$ or $W_k s_k / S_k = 1$. The classical probability definition by relative frequency of visits of each point by the system must be modified because the total number of visits (proportional to black surface $S_k$ of the carpet) is no more a finite quantity. (Construction of Sierpinski carpet. First iteration c(1) : removing the central square formed by the straight lines cutting each side into three segments of equal size. Repeat this operation on the 8 remaining squares of equal size and so on.)

FIG. 2. Nonextensive fermion distributions of AQD and EQD of incomplete statistical mechanics. AQD distribution is only slightly different from that at $q = 1$ (conventional Fermi-Dirac distribution) even with $q$ very different from unity. But EQD changes drastically with decreasing $\omega$. As $\omega \to 0$, the occupation number tends to $1/2$ for all states below $e_f$ which increases up to 2 times $e_{f0}$, the conventional fermi energy at $T = 0$.

FIG. 3. Comparison of additive incomplete fermion distribution (IFD, lines) with the numerical results (symbols) of Eder et al on the basis of Kondo lattice $t - J$ model (KLM) for different coupling constant $J$ [Phys. Rev. B, 55(1997)6109]. In my calculations, the density of electrons is chosen to give $k_{f0} = 0.25\pi$ in the first Brillouin zone. We note that IFD reproduces well the numerical results for about $J < 1$. When coupling is stronger, a long tail in the KLM distributions begins to develop at high energy. At the same time, a new Fermi surface at $k = k_{f0} + \pi/2 = 0.75\pi$ starts to appear and a sharp $n$ drop takes place at the new Fermi momentum. At $J = 4$, KLM distribution (x-marks) is very different from IFD (e.g. $\omega = 0.0011$). The solid line fitting better the $J = 4$ KLM distribution is given by the incomplete statistics version of fractional exclusion distribution $(1/n - \alpha)^\alpha (1/n - \alpha + 1)^{1-\alpha} = e^{\omega \beta (e-e_f)}$ [Yong-Shi Wu, Phys. Rev. Lett., 73(1994)922] with $1/\alpha = 0.85$ due to the KLM occupation number smaller than 0.5 at low momentum $k$. 

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Figure 1: A simple model of fractal phase space in Sierpinski carpet. At $k^{th}$ iteration, the side of the squares (black or white) is $l_k = l_0/2^k$ and their number is $W_k = 8^k l_k$, $l_k$ being the length of the side at $0^{th}$ iteration. The total surface at $k^{th}$ iteration is $S_k = W_k s_k$ or $W_k s_k/S_k = 1$. The classical probability definition by relative frequency of visits of each point by the system must be modified because the total number of visits (proportional to black surface $S_k$) is no more a finite quantity.

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