Pointwise estimates for the ground state of singular
Dirichlet fractional Laplacian

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Abstract
We establish sharp pointwise estimates for the ground states of some singular fractional Laplace operators on relatively compact Euclidean subsets. The considered operators are of the type $(-\Delta)^{\alpha/2}|_{\Omega} - c|x|^{-\alpha}$, where $(-\Delta)^{\alpha/2}|_{\Omega}$ is the fractional Laplacian on an open subset $\Omega$ of $\mathbb{R}^d$ with zero exterior condition and $0 < c \leq c^* := \frac{2\Gamma(d/2 + \alpha)}{\Gamma(d/2 - \alpha)}$. The intrinsic ultracontractivity property for such an operators is discussed as well and a sharp large time asymptotic for their heat kernels is derived.

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1. Introduction
The fractional Laplacian is the prototype for operators of jump type, where the Brownian motion (Laplacian) is replaced by a Levy process. Furthermore, such an operator has many applications in physics such as relativistic Schrödinger operators, or as operators describing superdiffusive behavior of particles (Levy flights).

In this paper, we shall be concerned with some properties of ‘localized’ fractional Laplacian with some negative perturbations.

Let $L_0 := (-\Delta)^{\alpha/2}|_{\Omega}$, $0 < \alpha < \min(2,d)$ be the fractional Laplacian on an open bounded subset $\Omega \subset \mathbb{R}^d$ with zero exterior condition in $L^2(\Omega, dx)$. It is well known that $L_0$ has purely discrete spectrum $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_k \to \infty$ and that the associated semigroup $T_t := e^{-tL_0}$, $t > 0$ is irreducible. Hence, $L_0$ has a unique strictly positive normalized ground state $\varphi_0$. Moreover, according to Chen–Kim–Song [CKS10, equation (4.1)], if $\Omega$ is a $C^{1,1}$ domain (in the sense defined in [CKS10]), then $\varphi_0$ enjoys the property of being comparable to the function $\delta^{\alpha/2}(x)$, where $\delta(x)$ is the Euclidian distance function between $x$ and $\Omega$. In other words

$$\varphi_0 \sim \delta^{\alpha/2} \text{ on } \Omega.$$  \hfill (1.1)

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Furthermore, Kulczycki proved in [Kul98] that the semigroup $T_t$, $t > 0$ is intrinsically ultracontractive (IUC) regardless of the regularity of $\Omega$, i.e., there is a constant $c_t$, such that
\[ p_t(x, y) \leq c_t \phi_0(x) \phi_0(y) \quad \forall (x, y) \in \Omega \times \Omega, \quad \forall t > 0. \tag{1.2} \]

The latter property induces among others the large time asymptotics for the heat kernel $p_t$ of $e^{-tL_0}$, $t > 0$:
\[ p_t(x, y) \sim e^{-\lambda t} \phi_0(x) \phi_0(y), \quad \text{on } \Omega \times \Omega. \tag{1.3} \]

Such types of estimates are very important in the sense that they give precise information on the local behavior of the ground state and the heat kernel (for large $t$) as well as on their respective rates of decay at the boundary.

Set $G$ the Green's kernel of $L_0^{-1}$ and let $V \geq 0$ a.e. be a potential in $L^1_{\text{loc}}$, such that
\[ \int_{\Omega} G(x, y) V(y) \, dy < 1, \tag{1.4} \]
(a Kato potential for example) and define
\[ L_V := L_0 - V, \tag{1.5} \]
in the quadratic forms sense. Owing to a resolvent formula, it is possible to show that $L_V$ still has the same spectral properties as $L_0$. In particular, it has a unique, strictly positive (continuous if $V$ is in the Kato class) and bounded ground state $\phi^V_0$. On the other hand, if $\Omega$ is a $C_1$ domain, then by [Han06, proposition 9.3, theorem 2.4], we conclude that the Green kernel $G^V$ of $L_V^{-1}$ is comparable to $G$. Hence, we still have
\[ \phi^V_0 \sim \phi_0 \sim \delta^{\alpha/2} \text{ if } \Omega \text{ is } C^{1,1}. \tag{1.6} \]

The comparability between $\phi^V_0$ and $\phi_0$ still holds true if one replaces $V$ by a positive measure which is in the Kato class or is potentially small, i.e.,
\[ K^V := \int_{\Omega} G(., y) \, d\mu(y) < 1, \tag{1.7} \]
provided the Green kernels $G$ and $G^V$ are comparable. For conditions ensuring comparability of Green functions, we refer the reader to [Han06].

Potentials of the type
\[ \frac{c}{|x|^r}, \quad 0 \leq r < \alpha \tag{1.8} \]
enter in the latter category.

However, for the limiting power $r = \alpha$, they do not fit into any of the mentioned type of potentials, if $0 \in \Omega$. Indeed, for this case, one has
\[ \text{ess sup}_{\Omega} \int_{\Omega} G(., y)|y|^{-\alpha} \, dy = \infty. \tag{1.9} \]

One of our aims in this paper is to prove that in the latter case, the ground state has singularities, describe them as well as their decay at the boundary for an open bounded subset $\Omega$ containing the origin.

Precisely, we shall prove that for
\[ V(x) = \frac{c}{|x|^\alpha}, \quad 0 < c \leq c^* := \frac{2^\alpha \Gamma^2 \left( \frac{d+\alpha}{2} \right)}{\Gamma^2 \left( \frac{d+2}{2} \right)}, \quad L_V := L - V, \tag{1.10} \]
the operator $L_V$ still has discrete spectrum, a unique normalized ground state $\phi^V_0 > 0$ a.e. and there is $0 < s \leq \frac{d-\alpha}{4}$, such that
\[ \phi^V_0 \sim \delta^{\alpha/2} |.|^{-s} \text{ if } \Omega \text{ is } C^{1,1}. \tag{1.11} \]
We shall however prove that the intrinsic ultracontractivity property is still preserved for domains which are less than $C^{1,1}$ domains. Namely, the operator $e^{-tL_V}$, $t > 0$ has a kernel $p^V_t$ with the following property: There is a constant $c_0$, such that
\[ p^V_t(x,y) \leq c_0 \psi_0^V(x) \psi_0^V(y) \quad \text{for a.e.} \quad (x,y) \in \Omega \times \Omega, \quad \forall t > 0. \quad (1.12) \]
Equivalently, this means that the $V_0$-transformed semigroup $(\psi_0^V)^{-1} e^{-tL_V} \psi_0^V$ is bounded from $L^1(\Omega, (\psi_0^V)^2) dx$ into $L^\infty(\Omega, dx)$.

Let us stress that the IUC property has many interesting consequences, such as deriving a large time asymptotics of the heat kernel, a universal bound for the eigenfunctions in term of the ground state solely and a lower bound for the Green kernel. For more details about the subject, we refer the reader to the standard book [Dav89, chapter 4.2].

Although we shall focus on the very special case $V = V^+ - V^-$ such that $V^+$ is of the type described in 1.

1. For positive potentials $V$, such that $V$ is bounded away from the origin and $\kappa' V^\kappa < V \leq \kappa V^\kappa$, a.e. near $0$, $\kappa \leq 1$. (1.13)

2. For signed potentials $V = V^+ - V^- \in L^1_0$, such that $V^+$ is of the type described in 1.

Our method relies basically on an improved Sobolev inequality together with a transformation argument which leads to a generalized ground state representation.

The paper is organized as follows. In section 2, we give the backgrounds together with some preparing results. For the comparability of the ground states, we shall consider two situations separately: the subcritical (section 3), with theorem 3.2 as the main result and the critical case (section 4), with theorem 4.2 as the main result. The last section is devoted to intrinsic ultracontractivity.

2. Preparing results

We first give some preliminary results that are necessary for the later development of the paper. Some of them are known. However, for the convenience of the reader, we shall give new proofs for them.

Let $0 < \alpha < \min(2, d)$. Consider the quadratic form $\mathcal{E}^\alpha$ defined in $L^2 := L^2(\mathbb{R}^d, dx)$ by
\[
\mathcal{E}^\alpha(f,g) = \frac{1}{2} \mathcal{A}(d, \alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\alpha}} \, dx \, dy,
\]
\[
D(\mathcal{E}^\alpha) = W^{\alpha/2,2}(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}^d) : \mathcal{E}^\alpha[f] := \mathcal{E}^\alpha(f,f) < \infty \},
\]
where
\[
\mathcal{A}(d, \alpha) = \alpha \Gamma \left( \frac{d+\alpha}{2} \right) \frac{2^d \pi^{d/2}}{\Gamma(1 - \frac{\alpha}{2})}. \quad (2.2)
\]
It is well known that $E^\alpha$ is a transient Dirichlet form and is related (via Kato representation theorem) to the self-adjoint operator, commonly named the $\alpha$-fractional Laplacian on $\mathbb{R}^d$ which we shall denote by $(-\Delta)^{\alpha/2}$.

Alternatively, the expression of the operator $(-\Delta)^{\alpha/2}$ is given by (see [BBC03, equation (3.11)])

$$(-\Delta)^{\alpha/2} f(x) = A(d, \alpha) \lim_{\epsilon \to 0^+} \int_{\{y \in \mathbb{R}^d : |y-x| > \epsilon\}} \frac{f(x) - f(y)}{|x-y|^{d+\alpha}} dy,$$

provided the limit exists and is finite.

From now on, we shall ignore in the notations the dependence on $\alpha$ and shall set $\cdots$ as shorthand for $\int_{\mathbb{R}^d} \cdots$. The notation $\omega$ means quasi everywhere with respect to the capacity induced by $E$.

For every open subset $\Omega \subset \mathbb{R}^d$, we denote by $L_0 := (-\Delta)^{\alpha/2}|_\Omega$ the localization of $(-\Delta)^{\alpha/2}$ on $\Omega$, i.e., the operator whose Dirichlet form in $L^2(\Omega, dx)$ is given by

$$D(E_\Omega) = W^{\alpha/2}_0(\Omega) := \{ f \in W^{\alpha/2,2}(\mathbb{R}^d) : f = 0 \text{ q.e. on } \Omega^c \}$$

$$E_\Omega(f, g) = \frac{1}{2} A(d, \alpha) \int_\Omega \int_\Omega \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{d+\alpha}} \, dx \, dy$$

$$= \frac{1}{2} A(d, \alpha) \left( \int_\Omega \int_\Omega \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{d+\alpha}} \, dx \, dy + \int_\Omega f(x)g(x)\kappa^{(\alpha)}_\Omega(x) \, dx \right),$$

where

$$\kappa^{(\alpha)}_\Omega(x) := A(d, \alpha) \int_\Omega \frac{1}{|x-y|^{d+\alpha}} \, dy.$$

The Dirichlet form $E_\Omega$ coincides with the closure of $E$ restricted to $C^\infty_c(\Omega)$, and is therefore regular and furthermore transient.

Let us emphasize at this stage that the operator $L_0$ is not the $\alpha/2$-power of the Dirichlet–Laplacian on $\Omega$, unless $\Omega = \mathbb{R}^d$. Indeed, if $\Omega$ is bounded, connected and regular, the ground state of $L_0$ behaves like $\delta^{\alpha/2}$, whereas the one corresponding to the $\alpha/2$-power of the Dirichlet–Laplacian (which is nothing else but the ground state of the Dirichlet–Laplacian) behaves like $\delta$. Let us also mention that the spectral properties of the $\alpha/2$-power of the Dirichlet–Laplacian are well understood, thanks to the spectral calculus, while less is known about $L_0$ even in one dimension.

Another aspect of differences between the mentioned operators can be read from the irreducibility property. While $L_0$ is irreducible even when $\Omega$ is disconnected (see [BBC03, p 93]), the $\alpha/2$ power of the Dirichlet–Laplacian is not irreducible if $\Omega$ is disconnected.

If moreover $\Omega$ is bounded, thanks to the well-known Sobolev embedding,

$$\left( \int_{\Omega} |f|^{\frac{2d}{d-\alpha}} \, dx \right)^{\frac{d-\alpha}{2d}} \leq C(\Omega, d, \alpha) E_\Omega[f], \forall f \in W^{\alpha/2,2}_0(\Omega),$$

the operator $L_0$ has compact resolvent (that we shall denote by $K := L_0^{-1}$) which together with the irreducibility property imply that there is a unique continuous bounded, $L^2(\Omega, dx)$ normalized function $\varphi_0 > 0$ and $\lambda_0 > 0$, such that

$$L_0 \varphi_0 = \lambda_0 \varphi_0 |_{\Omega}.$$

We shall prove that this property of $L_0$ is still preserved by perturbations of the form minus $c|x|^{-\alpha}$, where $0 < c \leq c^*$. However, singularities will appear for the ground state of the perturbed operator provided $\Omega$ contains the origin.

Owing to the sharpness of the constant (see for example [Yaf99])

$$\frac{1}{c^*} := \frac{\Gamma^2\left(\frac{d-\alpha}{d}\right)}{2^\alpha \Gamma^2\left(\frac{d+\alpha}{2}\right)},$$

4.
in the Hardy’s inequality

$$\int_{\mathbb{R}^d} |x|^{-\alpha} f^2(x) \, dx \leq C_{d,\alpha} E[f], \quad \forall f \in W^{\alpha/2,2}(\mathbb{R}^d),$$

(2.8)

we derive that for every $0 \leq c \leq c^*$, the quadratic form $E_c$ defined by

$$D(E_c) = W^{\alpha/2,2}(\Omega), \quad E_c[f] = E[f] - c \int_{\Omega} \frac{f^2(x)}{|x|^{2\alpha}} \, dx, \quad \forall f \in W^{\alpha/2,2}(\Omega)$$

(2.9)
is positive.

Roughly speaking, the function $w$ will play the role of a ‘desingularizing’ weight when transferring the problem from $L^2(\Omega, dx)$ into $L^2(\Omega, w^2dx)$.

The existence of such a function $w$ can be proved by using Fourier transform, for example (see [FLS08]). However, for later use (especially in lemmata 3.1 and 4.1) and for the sake of completeness, we shall give an alternative proof which is based on potential theoretical tools.

In order to rewrite equation (2.10) in a more suitable manner, we set $K_{d,\alpha} := (-\Delta)^{-\alpha/2}$, the Riesz kernel operator given by

$$K_{d,\alpha} f := A(d, \alpha) \int |\cdot - y|^{2-d} f(y) \, dy,$$

(2.12)

where

$$A(d, \alpha) := \left(\frac{2}{\pi}\right)^{d/2} \frac{\Gamma\left(d-\alpha\right)}{2^\alpha \Gamma\left(\frac{d}{2}\right)} .$$

(2.13)

Then, equation (2.10) is equivalent to

$$K_{d,\alpha} (\cdot |^{-\alpha} w)(x) = c^{-1} w(x) \text{ a.e.}$$

(2.14)

Clearly, the potential candidates for satisfying (2.10) are radially symmetric functions. Thus, we shall look for $w(x)$ of the type $w(x) := |x|^{-\beta}$, where $\beta > 0$.

**Lemma 2.1.** Let $w(x) = |x|^{-\beta}$, $x \neq 0$. Then, for every $0 < \beta < d - \alpha$, we have

$$K_{d,\alpha} (\cdot |^{-\alpha} w)(x) = C_{d,\alpha,\beta} w(x) \text{ a.e.,}$$

(2.15)

where

$$C_{d,\alpha,\beta} = 2^{2\beta} \frac{\Gamma\left(\frac{\alpha+\beta}{2}\right) \Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{d-\alpha}{2}\right) \Gamma\left(\frac{d+\beta}{2}\right)} .$$

(2.16)

**Proof.** A straightforward computation yields

$$K_{d,\alpha} (\cdot |^{-\alpha} w)(x) = Cw(x) \text{ a.e. } \iff w(x) = C^{-1} A(d, \alpha) \int |x - y|^{2-d} |y|^{-(\alpha+\beta)} \, dy$$

$$= C^{-1} A(d, \alpha) \int |x - y|^{2-d} |y|^{d-(\alpha+\beta)-\alpha} \, dy$$
Observing that $F$ of Lemma 2.2.

Proof. We are going to prove that the derivative of $F$ vanishes in $(0, d - \alpha)$ only at the point $\beta^*$. Since $F \geq 0$, $F(0) = F(d - \alpha) = 0$, we derive that $F$ achieves its maximum at $\beta^*$, which together with Lemma 2.1 would complete the proof.

Let $\beta \in (0, d - \alpha)$. A direct computation yields

\[
F'(\beta) = 0 \iff \Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{d - \beta}{2}\right)\left[\Gamma'\left(\frac{\alpha + \beta}{2}\right)\Gamma\left(\frac{d - (\alpha + \beta)}{2}\right)
+ \Gamma\left(\frac{\alpha + \beta}{2}\right)\Gamma'\left(\frac{d - (\alpha + \beta)}{2}\right)\right] - \Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{d - \alpha}{2}\right) \\
\times \left[\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{d - \beta}{2}\right) + \Gamma\left(\frac{\beta}{2}\right)\Gamma'\left(\frac{d - \beta}{2}\right)\right] = 0.
\]

(2.20)
Dividing by
\[ \Gamma \left( \frac{\beta}{2} \right) \Gamma \left( \frac{\alpha + \beta}{2} \right) \Gamma \left( \frac{d - \beta}{2} \right) \Gamma \left( \frac{d - (\alpha + \beta)}{2} \right) \]
leads to
\[ F'(\beta) = 0 \iff \left[ \Gamma' \left( \frac{\alpha + \beta}{2} \right) + \Gamma' \left( \frac{d - (\alpha + \beta)}{2} \right) \right] - \left[ \Gamma' \left( \frac{d - \beta}{2} \right) + \Gamma' \left( \frac{\beta}{2} \right) \right] = 0. \tag{2.21} \]

Set
\[ \Phi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{and} \quad \Psi(z) := \Phi \left( \frac{\alpha + \beta}{2} \right) - \Phi \left( \frac{\beta}{2} \right). \]

Then, equation (2.21) is equivalent to
\[ \left[ \Phi \left( \frac{1}{c} \right) - \Phi \left( \frac{\beta}{2} \right) \right] - \left[ \Phi \left( \frac{d - \beta}{2} \right) - \Phi \left( \frac{d - (\alpha + \beta)}{2} \right) \right] = \Psi(\beta) - \Psi(d - (\alpha + \beta)) = 0. \]

Now, recalling the known formula for the digamma-function \( \Gamma'/\Gamma \)
\[ \Phi(\beta) = -\gamma + \int_0^1 \frac{1 - t^{\beta - 1}}{1 - t} \, dt = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-t^\beta}}{1 - e^{-t}} \, dt, \tag{2.22} \]
we obtain
\[ \Psi(\beta) = \int_0^\infty \frac{e^{-x^{\beta}} - e^{-x^{\alpha + \beta}}}{1 - e^{-x}} \, dx \]
\[ = \int_0^\infty \frac{e^{-x^{\beta}} - e^{-x^{\alpha}}}{1 - e^{-x}} \, dx, \tag{2.23} \]
and \( \Psi'(\beta) = -\frac{2}{\beta} \Psi(\beta) < 0 \forall \beta \in (0, d - \alpha). \) Thus, for \( \beta \in (0, d - \alpha), \) \( F'(\beta) = 0 \) if and only if \( \beta = d - (\alpha + \beta), \) yielding \( \beta = \beta^*, \) which finishes the proof. \( \square \)

**Remark 2.1.** By the way, we note that lemma 2.1, lemma 2.2 together with [Fit00, theorem 1.9] or [BB12, theorem 3.1] yield a proof for Hardy’s inequality (2.8) with constant larger than \( \frac{1}{c}. \)

From now on, for every \( c \in (0, c^*), \) we set
\[ \beta_c := F^{-1}(c) \quad \text{and} \quad V_c = \frac{c}{|x|^\alpha}. \tag{2.24} \]

3. The subcritical case \( 0 < c < c^* \)

We fix \( c \in (0, c^*), \) neglect the dependence on \( c, \) set \( V := V_c, \beta = \beta_c \) and suppose that \( \Omega \) is bounded and contains zero.

It follows from the above considerations together with Hardy’s inequality (2.8) that
\[ \int f^2(x)V(x) \, dx \leq \frac{c}{c^*} \mathcal{E}_c[f]. \quad \forall f \in W_0^{\alpha/2,2}(\Omega). \tag{3.1} \]

Having in mind that \( 0 < \frac{1}{c} < 1, \) we conclude that the quadratic form which we denote by \( \mathcal{E}^V \) and which is defined by
\[ D(\mathcal{E}^V) = W_0^{\alpha/2,2}(\Omega), \quad \mathcal{E}^V[f] = \mathcal{E}_c[f] - \int f^2(x)V \, dx, \quad \forall f \in W_0^{\alpha/2,2}(\Omega), \tag{3.2} \]
Having Hardy’s inequality in mind and observing that

Thus, using Fubini’s together with dominated convergence theorem, we achieve

\[ \| \phi_0^V \|_{L^2} = 1, \quad \phi_0^V > 0 \text{ q.e. and } L_V \phi_0^V = \lambda_0^V \phi_0^V. \]  

(3.3)

In the goal of obtaining the precise behavior of the ground state, we proceed to transform the form \( \mathcal{E}^V \) into a Dirichlet form on \( L^2(\Omega, w^2dx) \), where \( w(x) = |x|^{-\beta} \).

Let \( Q \) be the \( w \)-transform of \( \mathcal{E}^V \), i.e., the quadratic form defined in \( L^2(\Omega, w^2dx) \) by

\[ D(Q) := \{ f : w f \in W^{2/2}(\Omega) \} \subset L^2(\Omega, w^2dx), \quad Q[f] = \mathcal{E}^V[w,f], \; \forall f \in D(Q). \]  

(3.4)

**Lemma 3.1.** The form \( Q \) is a regular Dirichlet form and

\[ Q[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} w(x)w(y) \, dx \, dy, \; \forall f \in D(Q). \]  

(3.5)

**Proof.** Obviously, \( Q \) is closed and densely defined as it is unitary equivalent to the closed densely defined form \( \mathcal{E}^V \). Let us prove (3.5).

Writing

\[ w(x)w(y) \left( \frac{g(x)}{w(x)} - \frac{g(y)}{w(y)} \right)^2 = (g(x) - g(y))^2 + g^2(x) \frac{w(y) - w(x)}{w(x)} + g^2(y) \frac{w(x) - w(y)}{w(y)}, \]  

(3.6)

and setting \( g = w f \), we obtain

\[ Q[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} w(x)w(y) \, dx \, dy \]
\[ + A(d, \alpha) \int \int \frac{w(x) - w(y)}{|x-y|^{d+\alpha}} f^2(x)w(x) \, dx \, dy \]
\[ - \int f^2(x)w^2(x)V(x) \, dx \; \forall f \in D(Q). \]  

(3.7)

Having Hardy’s inequality in mind and observing that

\[ Q[f] \geq A(d, \alpha) \int \int \frac{w(x) - w(y)}{|x-y|^{d+\alpha}} f^2(x)w(x) \, dx \, dy \]
\[ - \int f^2(x)w^2(x)V(x) \, dx \; \forall f \in D(Q), \]  

(3.8)

we derive in particular that the integral

\[ \int \int \frac{w(x) - w(y)}{|x-y|^{d+\alpha}} f^2(x)w(x) \, dx \, dy \]  

is finite.

(3.9)

Thus, using Fubini’s together with dominated convergence theorem, we achieve

\[ Q[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^2}{|x-y|^{d+\alpha}} w(x)w(y) \, dx \, dy \]
\[ + A(d, \alpha) \int f^2(x)w(x) \left( \lim_{\epsilon \to 0} \int_{|x-y| < \epsilon} \frac{w(x) - w(y)}{|x-y|^{d+\alpha}} \, dy \right) \, dx \]
\[ - \int f^2(x)w^2(x)V(x) \, dx \; \forall f \in D(Q). \]  

(3.10)
Now, owing to the fact that \( w \) is a solution of the equation

\[
(−Δ)^2 w − V w = 0,
\]

having equation (2.3) at hand and substituting in (3.10), we obtain formula (3.5) from which we read that \( Q \) is Markovian and hence a Dirichlet form.

**Regularity:** relying on expression (3.5) of \( Q \), we learn from [FOT94, example 1.2.1.] that \( C_{C0}^∞(\Omega) \subset D(Q) \) if and only if

\[
J := \int_\Omega \int_\Omega \frac{|x−y|^2}{|x−y|^{d+\alpha}} w(x) w(y) \, dx \, dy < ∞.
\]

(3.12)

For \( d = 1 \), the latter assumption is obviously verified.

Assume that \( d \geq 2 \). Set \( r = 2 − \alpha \). Then, \( 0 < r < d \).

We rewrite \( J \) as

\[
J = \int_\Omega w(x) \left( \int_\Omega \frac{|y|^{-r}|y|^{-\beta}}{|x−y|^{d−r}} |y|^{-\beta}|y|^{r} \, dy \right) dx
\]

\[
\leq |\Omega|^{1−2−\beta} \, \frac{1}{A(d, \alpha)} \int_\Omega w(x) K_{d, r}(|·|−r, ·−\beta) \, dy
\]

\[
\leq \kappa \int_\Omega w(x)|x|^{-\frac{\beta}{2}} < ∞,
\]

(3.13)

where \( \kappa \) is a finite constant and \( J \) is finite. Here, from the first to the second inequality, we used lemma 2.1.

Hence, from the Beurling–Deny–LeJan formula (see [FOT94, theorem 3.2.1, p 108]) together with identity (3.5), we learn that \( Q \) is regular, which completes the proof. \( \square \)

**Remark 3.1.** At this stage, we quote that a generalized ground state representation for fractional Laplacian on the whole space was proved in [FLS08] using a different method from ours.

We designate by \( L^w \) the operator associated with \( Q \) in the weighted Lebesgue space \( L^2(Ω, \omega^2 \, dx) \) and \( T^w_t, t > 0 \) its semigroup. Then,

\[
L^w = w^{-1}L^w, \quad \text{and} \quad T^w_t = w^{-1}e^{-tL^w}w, t > 0.
\]

(3.14)

**Theorem 3.1.** Set \( r = \frac{d}{d−\alpha} \).

\[
A := \frac{1}{C(\Omega, d, \alpha)^{1/r}} \left( 1 – \frac{c}{c^2} \right)^{1/r} \left( \frac{1}{\lambda} \sup_{x\inΩ} |x|^2 \right)^{1−1/r}
\]

\( \text{and } q = 2 − 1/r. \)

Then,

\[
\|f^2\|_{L^q(\omega^2 \, dx)} \leq A Q[f], \forall f \in D(Q).
\]

(3.15)

It follows that for every \( t > 0 \), the operator \( T^w_t \) is ultracontractive, i.e.,

\[
\|T^w_t\|_{L^1(Ω, \omega^2 \, dm), L^\infty} \leq A \left( \frac{t}{2} \right)^{-r/2−2} e^{\lambda_0^2}, \text{ a.e., } \forall t > 0.
\]

(3.16)

**Proof.** By Hölder’s inequality, we obtain for every \( f \in D(Q) \)

\[
\int \omega^2 f^2(2−1/r) \leq \left( \int \omega^2 f^2 \right)^{1/r} \left( \int f^2 \right)^{1−1/r}.
\]

(3.17)
Having Sobolev inequality (2.5) at hand, we obtain
\[
\left( \int w^{2r} f^{2r} \right)^{1/r} \leq \frac{1}{C(\Omega, d, \alpha)} \left( 1 - \frac{c}{c^1} \right) Q[f]. \tag{3.19}
\]

On the other hand we have,
\[
\int g^2 |x|^{2\beta} \, dx \leq \sup_{x \in \Omega} |x|^{2\beta} \int g^2 \, dx \leq \frac{1}{\lambda_0} \sup_{x \in \Omega} |x|^{2\beta} \mathcal{E}^V[g], \quad \forall \, g \in W_0^{\alpha/2, 2}(\Omega). \tag{3.20}
\]

Setting \( g = uf \), we obtain
\[
\int f^2 \leq \frac{1}{\lambda_0} \sup_{x \in \Omega} |x|^{2\beta} Q[f], \quad \forall \, f \in \mathcal{C}. \tag{3.21}
\]

Combining (3.18), (3.19) and (3.21), we obtain (3.16).

Finally, since \( Q \) is a Dirichlet form, it is known that a Sobolev embedding for the domain of a Dirichlet form yields the ultracontractivity of the related semigroup (see [SC02, theorems 4.1.2, 4.1.3]), which ends the proof. \( \square \)

Theorem 3.1 leads to an upper bound for \( \phi_0^V \), giving thereby partial information about the singularities of \( \phi_0^V \).

**Corollary 3.1.** The following upper bound holds true
\[
\phi_0^V \leq \left( A \left( \frac{t}{2} \right) \right)^{-r/(r-2)} e^{t\lambda_1} w, \quad \text{a.e., } \forall \, t > 0. \tag{3.22}
\]

**Proof.** The proof is quite standard so we omit it (see the proof of [Dav89, theorem 4.2.3]). \( \square \)

As long as the distance function is involved (decay at the boundary) in any accurate description of \( \phi_0^V \), one needs among others a representation of \( \phi_0^V \) involving the Green kernel \( G \) or \( G^V \). This is established in the following.

**Lemma 3.2.** The following identity holds true
\[
\phi_0^V = K (V \phi_0^V) + \lambda_0^V K \phi_0^V \text{ a.e.} \tag{3.23}
\]

**Proof.** Set
\[
u = \phi_0^V - K (V \phi_0^V) - \lambda_0^V K \phi_0^V. \tag{3.24}
\]

Owing to the fact that \( \phi_0^V \) lies in \( W_0^{\alpha/2, 2}(\Omega) \) and hence lies in \( L^2(V \, dx) \), we obtain that the measure \( \phi_0^V \) has finite energy integral with respect to the Dirichlet form \( \mathcal{E}_\Omega \), i.e.,
\[
\int |f \phi_0^V | \, dx \leq \gamma (E_{\Omega}[f])^{1/2}, \quad \forall \, f \in C_c^\infty(\Omega), \tag{3.25}
\]

and therefore \( K(V \phi_0^V) \) lies in \( W_0^{\alpha/2, 2}(\Omega) \). Thus, \( u \in W_0^{\alpha/2, 2}(\Omega) \) and satisfies identity
\[
\mathcal{E}_\Omega(u, g) = \mathcal{E}_\Omega (\phi_0^V, g) - \int \phi_0^V g \, dx - \lambda_0^V \int \phi_0^V g \, dx
= \mathcal{E}_V (\phi_0^V, g) - \lambda_0^V \int \phi_0^V g \, dx = 0, \quad \forall \, g \in W_0^{\alpha/2, 2}(\Omega). \tag{3.26}
\]

Since \( \mathcal{E} \) is positive definite, we conclude that \( u = 0 \) a.e., which yields the result. \( \square \)

We state now the main theorem of this section.
Theorem 3.2. Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain containing zero. Let \( V, \beta \) and \( \beta^* \) be as defined in the beginning of this section and in (2.24). Let \( \psi^0_V \) be the ground state of the operator \( L_V \) associated with the form \( E_V \) (as defined in (3.3)) and let \( w(x) = |x|^{-\beta} \) solve \((\Delta)^2 w - Vw = 0 \) (see lemma 2.2).

1. There exist finite constants \( C_1^V, C_2^V > 0 \), such that
\[
C_1^V w \leq \psi^0_V \leq C_2^V w \text{ a.e. near } 0.
\]

2. Assume that \( \Omega \) satisfies the uniform interior ball condition (as defined in [CKS10]), then there is a finite constant \( C^\Omega > 0 \), such that
\[
\psi^0_V \geq C^\Omega \delta^{\alpha/2} \text{ a.e.,}
\]
where \( \delta(x) = \text{dist}(x, \Omega^c) \).

3. If furthermore \( \Omega \) is \( C^{1,1} \), then there are finite constants \( C_1^V, C_2^V \), such that
\[
C_1^V w^{\alpha/2} \leq \psi^0_V \leq C_2^V w^{\alpha/2} \text{ a.e.}
\]

Proof.

1. As \( \psi^0_V > 0 \) a.e., there is \( \epsilon > 0 \) and \( \eta > 0 \), such that
\[
\psi^0_V > \epsilon \text{ a.e. in } B_{2\eta} \subset \Omega.
\]
Choose \( C_\eta > 0 \) so that \( \epsilon > C_\eta \rho^{-\beta} \) for \( r \geq \eta \).
Let \( 0 < j < 1 \) be a mollifier, such that
\[
j \in C_0^\infty((0, \infty)). \quad \text{Supp } j \subset [0, 2] \text{ and } j = 1 \text{ on } [0, 1].
\]

Now, define
\[
\theta(x) := \psi^0_V(x) - C_\eta j \left( \frac{|x|}{\eta} \right) w(x) \text{ a.e. and } v_\eta(x) := j \left( \frac{|x|}{\eta} \right) w(x).
\]

Then, \( \theta^- = 0 \), a.e. on \( \mathcal{H}_\eta \) and by [FLS08, lemma 3.3], \( \theta \in W_0^{\alpha/2,2}(\Omega) \). Thus, \( \theta^- \in W_0^{\alpha/2,2}(B_\eta) \).

Activating Sobolev inequality (2.5) and utilizing the fact that \( \psi^0_V \) is the ground state of \( L_V \), we obtain
\[
\| \theta^- \|^2_{L^2} \leq C \left( \frac{1}{2} A(d, \alpha) \int \frac{(\theta^-(x) - \theta^-(y))^2}{|x - y|^{d+\alpha}} \dx \dy - \int V(x)(\theta^-)^2(x) \dx \right)
\]
\[
\leq -C \left( \frac{1}{2} A(d, \alpha) \int \frac{(\theta(x) - \theta(y))(\theta^-(x) - \theta^-(y))}{|x - y|^{d+\alpha}} \dx \dy - \int V(x)\theta(x)\theta^-(x) \dx \right)
\]
\[
= -C \left( \frac{1}{2} A(d, \alpha) \int \frac{(\psi^0_V(x) - \psi^0_V(y))(\theta^-(x) - \theta^-(y))}{|x - y|^{d+\alpha}} \dx \dy - \int V(x)\psi^0_V(x)\theta^-(x) \dx \right)
\]
\[
+ C_\eta C \left( \frac{1}{2} A(d, \alpha) \int \frac{(v_\eta(x) - v_\eta(y))(\theta^-(x) - \theta^-(y))}{|x - y|^{d+\alpha}} \dx \dy - \int V(x)v_\eta(x)\theta^-(x) \dx \right)
\]
\[
= -C^\Omega \int \psi^0_V(x)\theta^-(x) \dx + CC_\eta E_V(v_\eta, \theta^-).
\]
In the ‘passage’ from the first to the second inequality, we used the fact that for any Dirichlet form \( \mathcal{D} \), one has \( \mathcal{D}(f^+, f^-) \leq 0 \) (see [MR92, theorem 4.4-i]).

Let \( 0 \leq \psi \in C_c^\infty(\mathbb{R}^d) \) having support in \( B_{\eta} \). Set \( v_\eta := v - w \). Then, using identity (2.3) together with lemma 2.1

\[
\phi_v^V (v_\eta, \phi) = \frac{1}{2} A(d, \alpha) \int \int \left( \frac{(v_\eta(x) - v_\eta(y)) (\psi(x) - \psi(y))}{|x - y|^{d+\alpha}} \right) dx \; dy 
- \int V(x) v_\eta(x) \psi(x) \; dx 
= \int v_\eta(x) (-\Delta)^{\alpha/2} \psi(x) \; dx - \int V(x) v_\eta(x) \psi(x) \; dx 
= \int v_\eta(x) (-\Delta)^{\alpha/2} \psi(x) \; dx - \int V(x) w(x) \psi(x) \; dx 
= \int v_\eta(x) (-\Delta)^{\alpha/2} \psi(x) \; dx - \int (-\Delta)^{\alpha/2} w(x) \psi(x) \; dx 
= \int (v_\eta(x) - w(x))(-\Delta)^{\alpha/2} \psi(x) \; dx = \int v_\eta(x) (-\Delta)^{\alpha/2} \psi(x) 
= A(d, \alpha) \int \int \frac{v_\eta(x) \phi(y)}{|x - y|^{d+\alpha}} \; dx \; dy \leq 0, \tag{3.34}
\]

where the last identity is obtained by using Fourier transform (see [FLS08, lemma 3.1]). Thus, approximating \( \theta^- \) by a sequence of positive functions \( \{\psi_k\} \subset C_c^\infty(\mathbb{R}^d) \) having supports in \( B_{\eta} \), w.r.t. the \( E_1 \) norm, we conclude that \( \phi_v^V (v_\eta, \theta^-) \leq 0 \). We thereby obtain

\[
\|(-\Delta)^{\alpha/2} \phi_0^V \|_{L^2} \leq -C_0 \int \psi_0^V (x) \theta^- (x) \; dx \leq 0. \tag{3.35}
\]

Finally, we are led to \( \theta^- = 0 \) in \( B_{\eta} \) yielding \( \theta = \theta^+ \) in \( B_{\eta} \), and hence

\[
\phi_0^V \leq C_0 w \text{ a.e. on } B_{\eta}, \tag{3.36}
\]

which together with corollary 3.1 leads to \( \phi_0^V \sim w \) near the origin.

2. Relying on [CKS10, theorem 3.1], one can prove by mean of a direct computation that for every bounded domain fulfilling the uniform interior ball condition, the Green function satisfies the lower bound

\[
G(x, y) \geq C|x - y|^{-d+\alpha} \left( 1 + \frac{\delta^{\alpha/2}(x) \delta^{\alpha/2}(y)}{|x - y|^\alpha} \right), \; x \neq y. \tag{3.37}
\]

Hence,

\[
G(x, y) \geq C \delta^{\alpha/2}(x) \delta^{\alpha/2}(y). \tag{3.38}
\]

Now the desired lower bound for \( \phi_0^V \) is obtained by using formula (3.2).

3. It suffices to prove that there is a finite constant \( C \), such that \( \phi_0^V \leq C \delta^{\alpha/2} \) away from the origin.

Owing to the \( C^{1,1} \) regularity of \( \Omega \), one has by [CKS10, corollary 1.2] the following sharp estimate for the Green kernel of \( L_0^C \):

\[
G(x, y) \sim |x - y|^{-d+\alpha} \left( 1 + \frac{\delta^{\alpha/2}(x) \delta^{\alpha/2}(y)}{|x - y|^\alpha} \right), \; x \neq y. \tag{3.39}
\]

We use once again formula (3.2). Fix \( \rho > 0 \). Let \( x \in \Omega \) be such that \( |x| > \rho \). We decompose the integral

\[
K \phi_v^V = \int_{|\gamma| < \rho} G(x, y) \phi_v^V (y) \; dy + \int_{|\gamma| \geq \rho} G(x, y) \phi_v^V (y) \; dy 
=: I_1 + I_2 \tag{3.40}
\]
If $|x - y| \geq \varepsilon$, then by (3.39), there is a constant $0 < C_{\text{sim}} < \infty$ depending solely on $\Omega$, $d$ and $\alpha$, such that
\begin{equation}
I_2 \leq C_{\text{sim}} e^{-d+2\alpha} \delta(x)^{\alpha/2} \int_{\Omega} \delta^{\alpha/2}(y) \phi_0^V(y) \, dy.
\end{equation}

Observe that
\begin{equation}
A^V_2 := \int_{\Omega} \delta^{\alpha/2}(y) \phi_0^V(y) \, dy < \infty.
\end{equation}

On the other hand, owing to the fact that $\delta(x) \leq \delta(y) + |x - y|$, we obtain
\begin{equation}
\left( 1 \wedge \frac{\delta(x)\delta(y)}{|x - y|^2} \right) \leq 2 \left( 1 \wedge \frac{\delta(x)}{|x - y|} \right) \left( 1 \wedge \frac{\delta(y)}{|x - y|} \right), \quad x \neq y.
\end{equation}

Having the upper estimate on $\phi_0^V$ in mind (corollary 3.1), we therefore obtain
\begin{equation}
I_1 \leq C_{\text{sim}} \epsilon \delta^{\alpha/2}(x) \int_{|x - y| \leq \epsilon} \frac{|y|^{-2\beta}}{|x - y|^{d-\frac{2\beta}{\alpha}}} \, dy.
\end{equation}

Now, if $|x - y| < \varepsilon$, then
\begin{equation}
|y| \geq |x| - |x - y| \geq \varepsilon - \varepsilon > 0
\end{equation}
yielding
\begin{equation}
I_1 \leq C_{\text{sim}} \epsilon \delta^{\alpha/2}(x) \sup_{x \in \Omega} \int_{|x - y| \leq \epsilon} \frac{dy}{|x - y|^{d-\frac{2\beta}{\alpha}}}.
\end{equation}

with
\begin{equation}
\sup_{x \in \Omega} \int_{|x - y| \leq \epsilon} \frac{dy}{|x - y|^{d-\frac{2\beta}{\alpha}}} < \infty.
\end{equation}

Finally, the term $K^V \phi_0^V$ can be estimated from above in the same manner to obtain
\begin{align}
K^V \phi_0^V & \leq C_{\text{sim}} e^{-d+2\alpha} \delta(x)^{\alpha/2} \int_{\Omega} \delta^{\alpha/2}(y) \phi_0^V(y) V(y) \, dy \\
& + C_{\text{sim}} \epsilon \delta^{\alpha/2}(x) \sup_{x \in \Omega} \int_{|x - y| \leq \epsilon} \frac{dy}{|x - y|^{d-\frac{2\beta}{\alpha}}},
\end{align}

away from the origin and
\begin{equation}
A^V_1 := \int_{\Omega} \delta^{\alpha/2}(y) \phi_0^V(y) V(y) \, dy < \infty.
\end{equation}

Matching all together yields the claim, which completes the proof. □

**Remark 3.2.** From the proof of the latter theorem, we learn that the $C^{1,1}$ assumption is involved only to determine the optimal decay rate of the ground state at the boundary. It is not clear for us whether this assumption is necessary or not.

**4. The critical case $c = c^*$**

We still assume in this section that $\Omega$ is an open bounded subset of $\mathbb{R}^d$ containing zero. However, we shall replace the potential function $V_c$ by $V_{c^*}$, where $V_{c^*} = \frac{c^*}{|x|^{2\beta}}$ and $c^*$ is defined by (1.10), which we call the critical case.

The critical case differs in some respects from the subcritical one. The most apparent difference is that the critical quadratic form is no longer closed on the starting fractional Sobolev space $W^{0,2,2}_0(\Omega)$. Consequently, the proof of lemma 3.2 is no longer valid to express the ground state for the simple reason that it may not belong to $W^{0,2,2}_0(\Omega)$. We shall in fact
show by the end of this section that \( q_{\mu}^* \notin W_{0}^{2/2}(\Omega) \) in the critical case. Also, the reasoning in the proof of assertion 1 of the latter theorem breaks down.

We shall however prove that the critical form is closable and has compact resolvent. An approximation process will then lead to extend the identity of lemma 3.2 helping therefore to obtain the sharp estimate of the ground state.

From now on, we set
\[
w_\ast(x) = |x|^{-\beta} = |x|^{-\frac{4\alpha}{d-\alpha}}, \quad x \neq 0,
\]
and recall that \( V_{\ast} = \frac{c_{\ast}}{|x|^{\beta}} \). Thus, by lemma 2.1, \( w_{\ast} \) is a solution of equation
\[
(-\Delta)^{2} w - V_{\ast} w = 0.
\]

Let \( \tilde{\mathcal{E}}_{\ast} \) be the quadratic form defined by
\[
D(\tilde{\mathcal{E}}_{\ast}) = W_{0}^{1/2,2}(\Omega), \quad \tilde{\mathcal{E}}_{\ast}[f] = \mathcal{E}_{\ast}[f] - c_{\ast} \int \frac{f^{2}(x)}{|x|^{\beta}} \, dx, \quad \forall f \in W_{0}^{1/2,2}(\Omega).
\]
By analogy to the subcritical case, we define the \( w_{\ast} \)-transform of \( \tilde{\mathcal{E}}_{\ast} \), which we denote by \( \tilde{Q}_{\ast} \) and is defined by
\[
D(\tilde{Q}_{\ast}) := \{ f : w_{\ast}f \in W_{0}^{1/2,2}(\Omega) \} \subset L^{2}(\Omega, w_{\ast}^{2} \, dx), \quad \tilde{Q}_{\ast}[f] = \tilde{\mathcal{E}}_{\ast}[w_{\ast}f], \quad \forall f \in D(\tilde{Q}_{\ast}).
\]

Following the computations made in the proof of lemma 3.1, we realize that \( \tilde{Q}_{\ast} \) has the following representation:
\[
\tilde{Q}_{\ast}[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^{2}}{|x - y|^{d+\alpha}} w_{\ast}(x) w_{\ast}(y) \, dx \, dy, \quad \forall f \in D(\tilde{Q}_{\ast}).
\]

**Lemma 4.1.** The form \( \tilde{Q}_{\ast} \) is closable in \( L^{2}(\Omega, w_{\ast}^{2} \, dx) \). Furthermore, its closure is a Dirichlet form in \( L^{2}(\Omega, w_{\ast}^{2} \, dm) \).

It follows in particular that \( \tilde{\mathcal{E}}_{\ast} \) is closable.

**Proof.** We first mention that since \( \tilde{\mathcal{E}}_{\ast} \) is densely defined then \( \tilde{Q}_{\ast} \) is densely defined as well.

Now we proceed to show that \( \tilde{Q}_{\ast} \) possesses a closed extension. To that end, we introduce the form \( \tilde{Q} \) defined by
\[
D(\tilde{Q}) := \left\{ f : f \in L^{2}(\Omega, w_{\ast}^{2} \, dx), \quad \int \int \frac{(f(x) - f(y))^{2}}{|x - y|^{d+\alpha}} w_{\ast}(x) w_{\ast}(y) \, dx \, dy < \infty \right\}
\]
\[
\tilde{Q}[f] = \frac{A(d, \alpha)}{2} \int \int \frac{(f(x) - f(y))^{2}}{|x - y|^{d+\alpha}} w_{\ast}(x) w_{\ast}(y) \, dx \, dy, \quad \forall f \in D(\tilde{Q}).
\]

Arguing as in the proof of lemma 3.1, we obtain that \( C^{\infty}(\Omega) \subset D(\tilde{Q}) \).

Hence from the Beurling–Deny–LeJan formula (see [FÔT94, theorem 3.2.1, p 108]), the form \( \tilde{Q} \) is the restriction to \( C^{\infty}(\Omega) \) of a Dirichlet form and therefore closable and Markovian. Since \( D(\tilde{Q}_{\ast}) \subset D(\tilde{Q}) \), we conclude that \( \tilde{Q}_{\ast} \) is closable and Markovian as well, yielding that its closure is a Dirichlet form. Now the closability of \( \tilde{\mathcal{E}}_{\ast} \) is an immediate consequence of the closability of \( \tilde{Q}_{\ast} \), which finishes the proof.

From now on, we set \( \mathcal{E}_{\ast} \), respectively \( \tilde{\mathcal{E}}_{\ast} \), the closure of \( \mathcal{Q}_{\ast} \) and \( \tilde{Q}_{\ast} \). The self-adjoint operator related to \( \mathcal{E}_{\ast} \), respectively \( \tilde{\mathcal{E}}_{\ast} \), the closure of \( \mathcal{Q}_{\ast} \) and \( \tilde{Q}_{\ast} \). Finally, \( T_{\ast} := e^{-iH_{\ast}}, \quad t > 0 \) and \( S_{\ast} := e^{-iH_{\ast}}, \quad t > 0 \). Obviously, \( H_{\ast} = \tilde{w}_{\ast}^{-1}L_{\ast}w_{\ast} \).

The development of this section depends heavily on the following improved Sobolev inequality due to Frank–Lieb–Seiringer [FLS08, theorem 2.3]. For every \( 2 < p < \frac{2d}{d-\alpha} \), there is a constant \( S_{d,\alpha}(\Omega) \), such that
\[
\left( \int |f|^{p} \, dx \right)^{2/p} \leq S_{d,\alpha}(\Omega) \mathcal{E}_{\ast}[f], \quad \forall f \in W_{0}^{2/2}(\Omega).
\]
Of course the latter inequality extends to the elements of $D(\mathcal{E}_*)$ with $\mathcal{E}_*$ replaced by $\mathcal{E}_s$. The idea of using improved Sobolev type inequality to obtain estimates for the ground state was already used in [BBB, DD03].

**Theorem 4.1.** For every $t > 0$, the operator $S_t$ is ultracontractive. It follows that

(i) the operators $S_t$, $t > 0$ and hence $T^*_t$, $t > 0$ are Hilbert–Schmidt operators and the operator $L_s$ has a compact resolvent;

(ii) set $\lambda^{(k)}_0$ the smallest eigenvalue of $L_s$, then $\lambda^{(k)}_0$ is nondegenerate, i.e., there is $\psi^{(k)}_0$ (the ground state), such that $\psi^{(k)}_0 > 0$ a.e. and $\ker(L_s - \lambda^{(k)}_0) = \mathbb{R}\psi^{(k)}_0$;

(iii) if $\Omega$ satisfies the uniform interior ball condition, then

\[
\psi^{(k)}_0(x) \geq \left(C_G \lambda^{(k)}_0 \int \delta(y)^{\alpha/2} \psi^{(k)}_0(y) \, dy \right) \delta(x)^{\alpha/2}, \text{ a.e. and (4.8)}
\]

(iv) let $w_*$ be as defined in (4.1), then $\psi^{(k)}_0 \leq C e^{L^2} w_*$ a.e..

**Proof.** The proof that $S_t$, $t > 0$ is ultracontractive runs as the one corresponding to the subcritical case with the help of lemma 4.1 and inequality (4.7) as the main ingredient.

(i) Every ultracontractive operator has an almost everywhere bounded kernel and since $w_* \in L^2(\Omega)$, one obtains that $S_t$, $t > 0$ is a Hilbert–Schmidt operator as well as $T^*_t$ and hence $L_s$ has compact resolvent.

(ii) Since $T^*_t$, $t > 0$ has a nonnegative kernel, it is irreducible and the claim follows from the well-known fact that the generator of every irreducible semigroup has a nondegenerate ground state energy with a.e. nonnegative ground state.

(iii) The fact that $T^*_t$ is a Hilbert–Schmidt operator yields that $L_s$ possesses a Green kernel, $G_s$ and that $G_s \geq G$. Writing

\[
\psi^{(k)}_0 = \lambda^{(k)}_0 \int G_s(\cdot, y) \psi^{(k)}_0(y) \, dy \geq \lambda^{(k)}_0 \int G(\cdot, y) \psi^{(k)}_0(y) \, dy \tag{4.9}
\]

and using the lower bound (3.38) yields the result.

(iv) Follows from the fact that $S_t$, $t > 0$ is ultracontractive.

Let $0 < c_k \uparrow c_*$. Then, $L_k := L - \frac{c_k}{\partial^2}$ increases in the strong resolvent sense to $L_s$. Since $L_s$ has compact resolvent, the latter convergence is even uniform (see [BAB11, lemma 2.5]). Thus, setting $\lambda^{(k)}$'s the ground state energy of the $L_k$’s and $\psi^{(k)}$ its associated ground state, we obtain

\[
\lambda^{(k)}_0 \to \lambda^{*}_0 \quad \text{and} \quad \psi^{(k)}_0 \to \psi^{*}_0 \text{ in } L^2(\Omega, \, dx). \tag{4.10}
\]

For an accurate description of the behavior of the ground state, we shall extend formula (3.2) to $\psi^{*}_0$.

**Lemma 4.2.** We have

\[
\psi^{*}_0 = K^V \psi^{*}_0 + \lambda^{*}_0 K \psi^{*}_0. \tag{4.11}
\]

**Proof.** Use formula (3.2) for $\psi^{(k)}_0$, $\lambda^{(k)}_0$’s then pass to the limit and use the fact that $K$ is bounded from $L^1$ into itself.

Now having assertion 4 of theorem 4.1 in hand together with lemma 4.2, imitating the final part of the proof of theorem 3.2 yields that there is a finite constant $C$ such that $\psi^{*}_0 \leq C \delta^{\alpha/2}$ away from the origin, provided that $\Omega$ is $C^{1,1}$. We thus achieve the following description of $\psi^{*}_0$ for $C^{1,1}$ domains

\[
\psi^{*}_0 \leq C \delta^{\alpha/2} w_* \text{ a.e. and } \psi^{*}_0 \sim \delta^{\alpha/2} \text{ away from } 0. \tag{4.12}
\]

Finally we summarize.
Theorem 4.2. Let \( \Omega \) be an open bounded domain containing the origin. Let \( \varphi_0^* \) and \( w_* \) be as in theorem 4.1. Then,

1. \( \varphi_0^* \sim w_* \) near the origin and
2. if furthermore \( \Omega \) is \( C^{1,1} \), then
\[
\varphi_0^* \sim \delta^{\alpha/2} w_*.
\]

(4.13)

Proof. Assertion 2 was already proved. To prove the first assertion, one uses theorem 3.2-3. together with the approximation result (4.10).

Remark 4.1. Theorem 4.2 indicates that \( \varphi_0^* \not\in \mathcal{W}_{\alpha/2,2}^0(\Omega) \) and whence \( \mathcal{D}(\mathcal{E}_\alpha) \) contains strictly \( \mathcal{W}_{\alpha/2,2}^0(\Omega) \). Indeed, if \( \varphi_0^* \in \mathcal{W}_{\alpha/2,2}^0(\Omega) \), then
\[
\int |x|^{-\alpha} (\varphi_0^*)^2 \, dx < \infty.
\]
But for small \( r > 0 \), we have
\[
\int |x|^{-\alpha} (\varphi_0^*)^2 \, dx \geq C \int_{B_r} |x|^{-\alpha} |x|^{-d+\alpha} \, dx = \infty.
\]
(4.14)

5. Preservation of intrinsic ultracontractivity

In this section, we still assume that \( \Omega \) is bounded and shall discuss stability of the intrinsic ultracontractivity (IUC) property for the semigroups associated with the operators \( L_{c^*} \), \( 0 < c \leq c^* \) (as defined at the beginning of sections 3 and 4). Namely, the IUC property for the semigroups
\[
T_t^c := e^{-tL_{c^*}}, \quad t > 0, \quad 0 < c \leq c^* \quad \text{with} \quad T_{t^*}^c = T_t^c.
\]
(5.1)
The IUC means in our setting that if we set \( \varphi_0 \) the ground state of the operator \( L_{c^*} \) and \( p_t^c \), \( t > 0 \) the heat kernel of \( T_t^c \), then
\[
p_t^c(x,y) \leq C_{t}(\varphi_0(x)\varphi_0(y), \quad \text{a.e.,} \quad \forall \, t > 0.
\]
(5.2)

Kulczycki proved (see [Kul98]) that \( T_t := e^{-tL_{c^*}} \), \( t > 0 \) is IUC for every bounded domain, via log-Sob inequality. We shall prove that this property still holds true for \( T_t^c \) for a large class of bounded domains including, for instance, \( C^{1,1} \) domains.

One of the crucial ingredients for the proof is the following Hardy-type inequality which we assume for the rest of the paper: There is a finite constant \( C_H > 0 \), such that
\[
\int\frac{f^2(x)}{\varphi_0^2(x)} \, dx \leq C_H \mathcal{E}[f], \quad \forall \, f \in \mathcal{W}_{\alpha/2,2}^0(\Omega).
\]
(5.3)

Remark 5.1. The latter inequality holds true for bounded domains satisfying the uniform interior ball condition and \( d \geq 2, \; \alpha \neq 1 \). Indeed, for this class of domains, we already observed that
\[
\varphi_0 \geq C g^{\alpha/2},
\]
(5.4)
whereas [CS03, corollary 2.4] asserts that if \( \Omega \) is a Lipschitz domain, then for every \( \alpha \neq 1 \) and \( d \geq 2 \), we have
\[
\int \frac{f^2(x)}{\delta^\alpha(x)} \, dx \leq C_H \mathcal{E}[f], \quad \forall \, f \in \mathcal{W}_{\alpha/2,2}^0(\Omega).
\]
(5.5)
Combining the two inequalities yields (5.3).
Hereafter, we shall designate by $Q_0^\phi$ the $\phi_0^*$-transform of $E_\phi$ and $C_0^\phi$ its form core defined by

$$C_0^\phi := \{ f : w_\phi f \in W^{2,2}_0(\Omega) \}.$$  \hfill (5.6)

Following the spirit of the proof of lemma 3.1 together with lemma 4.1, we derive the expression of $Q_0^\phi$ on $C_0^\phi$:

$$Q_0^\phi[f] = \int \int \frac{(f(x) - f(y))^2}{|x - y|^{d+a}} \phi_0^*(x)\phi_0^*(y) \, dx \, dy + \lambda_0 \int (f\phi_0^*)^2(x) \, dx, \quad \forall f \in C_0^\phi.$$  \hfill (5.7)

We also recall the known fact that since $T_t$ is IUC, then there is a finite constant $C_G$, such that

$$G(x, y) \geq C_G\phi_0(x)\phi_0(y).$$  \hfill (5.8)

**Lemma 5.1.** There is a finite constant $C_G$, such that

$$\phi_0^* \geq C_G\phi_0 \text{ a.e.}$$  \hfill (5.9)

**Proof.** Lemma 4.2 together with observation (5.8) yield

$$\phi_0^* \geq \lambda_0^*K\phi_0 \geq \lambda_0^*C_G\phi_0 \int \phi_0(y)\phi_0^*(y) \, dy,$$  \hfill (5.10)

which was to be proved. \hfill $\Box$

**Theorem 5.1.** The semigroups $T_t^\phi$, $t > 0$, $0 < c \leq c_\star$ are IUC.

**Proof.** We shall give the proof only for $T_t^\phi$. For the other cases, the proof is similar.

We shall show that $T_t^\phi \equiv (\phi_0^*)^{-1}T_t^\phi \phi_0^*$, $t > 0$, the semigroup associated with $Q_0^\phi$, is ultracontractive. Various constants in this proof will be denoted simply by $C$.

On one hand, thanks to the improved Sobolev inequality together with formula (5.7), we obtain

$$\left( \int (f\phi_0^*)^p \, dx \right)^{2/p} \leq C \left( \int \int \frac{(f(x) - f(y))^2}{|x - y|^{d+a}} \phi_0^*(x)\phi_0^*(y) \, dx \, dy + \lambda_0 \int (f\phi_0^*)^2(x) \, dx \right), \quad \forall f \in C_0^\phi.$$  \hfill (5.11)

Meanwhile, lemma 5.1 together with the lower bound on $\phi_0$ in (5.4) leads to

$$\phi_0^* \geq C\phi_0,$$  \hfill (5.12)

which in conjunction with inequality (5.3) leads to

$$\int f^2(x) \, dx \leq C_H E[f\phi_0] = C \left( \int \int \frac{(f(x) - f(y))^2}{|x - y|^{d+a}} \phi_0(x)\phi_0(y) \, dx \, dy + \lambda_0 \int (f\phi_0)^2(x) \, dx \right) \leq CQ_0^\phi[f], \quad \forall f \in C_0^\phi.$$  \hfill (5.13)

Now, setting $r = 2(2 - p^{-1})$, where $p$ is the exponent of inequality (4.7), we achieve by Hölder inequality

$$\left( \int |f|^{r}(\phi_0^*)^2 \, dx \right)^{2/r} \leq CQ_0^\phi[f], \quad \forall f \in C_0^\phi,$$  \hfill (5.14)

yielding a Sobolev embedding for $D(Q_0^\phi)$, which in turns yields the ultracontractivity of $T_t^\phi$, which was to be proved. \hfill $\Box$
The latter theorem leads directly to the following.

**Corollary 5.1.** Assume that $\Omega$ is bounded and contains zero. Set $p^*_t$ the heat kernel of $T^*_t$ and $G_*$ the Green kernel of $L^{-1}_*$. Then,

(i) the following large time asymptotic for $p^*_t$ holds true:
\[
p^*_t(x, y) \sim e^{-\lambda^*_0 t} \delta^\alpha/2(x) \delta^\alpha/2(y) w_*(x) w_*(y), \quad \text{a.e. for large } t; \tag{5.15}
\]

(ii) Green kernel lower bound
\[
G_*(x, y) \geq C \delta^\alpha/2(x) \delta^\alpha/2(y) w_*(x) w_*(y), \quad \text{a.e. and} \tag{5.16}
\]

(iii) let $f \in L^2$, $f \geq 0$ a.e. and $u$ the weak solution of $L_* u = f$, then
\[
u \geq C \delta^\alpha/2 w_*, \quad \text{a.e.} \tag{5.17}
\]

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