(2+1)-Gravity Solutions with Spinning Particles

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Abstract

We derive, in 2+1 dimensions, classical solutions for metric and motion of two or more spinning particles, in the conformal Coulomb gauge introduced previously. The solutions are exact in the N-body static case, and are perturbative in the particles’ velocities in the dynamic two-body case. A natural boundary for the existence of our gauge choice is provided by some “CTC horizons” encircling the particles, within which closed timelike curves occur.
1 Introduction

Interest in the classical solutions of 2+1-Gravity [1]-[10] has recently revived [11]-[13] because of the discovery of exact moving particle solutions [12]-[13] in a regular gauge of conformal type [11]. Simplifying features of such a gauge are the instantaneous propagation (which makes the ADM decomposition of space-time explicit and particularly simple) and the conformal factor of Liouville type (which can be exactly found at least in the two-body case).

We have already provided in Ref. [12], hereafter referred to as [BCV], the main results for the case of $N$ moving spinless particles. The purpose of the present paper is to extend the BCV gauge choice [11] to spinning particles and to provide solutions for the metric and the motion in some particular cases.

Localized spin $s$, in 2 + 1 dimensions [6], is characterized by the fact that a Minkowskian frame set up in the neighbourhood of the particle has a multivalued time, which is shifted by the amount $\delta T = -s$, when turning around it in a closed loop. A consequence of this jump (which is backwards in time for a proper loop orientation) is that there are closed timelike curves (CTC’s) [14] around the particles at a distance smaller than some critical radius $R_0 \sim O(s)$.

This feature suggests that the single-valued time of our gauge, which is synchronized in a global way, cannot be pushed too close to the particles themselves. Indeed we shall find that there are “CTC horizons” around the particles of radii $R_i \sim s_i$ which cannot be covered by our gauge choice [15]. Nevertheless, we will be able to describe the motion of the particles themselves on the basis of our “external solutions” to the metric and to the DJH matching conditions.

Technically, the existence of the time shifts mentioned before modifies the number of “apparent singularities” which appear in the Riemann-Hilbert problem [16] for the analytic function providing the mapping to Minkowskian coordinates. Such singularities are not branch points of the mapping function, but nevertheless appear as poles in its Schwarzian derivative.

While for $N$ spinless particles there are $2N - 1$ singularities ($N$ for the particles, 1 at infinity and $N-2$ apparent singularities), in the spinning case there are $3N-1$, corresponding to one more apparent singularity per particle. This means that explicit exact solutions are more difficult to find.

In the spinless case we found exact solutions for the two-body problem with any speed (3 singularities) and for $N$ bodies with small speed. In the spinning case we find here an
exact solution only for the static ($N$-body) case, and we discuss the two-body problem, which corresponds to five singularities, for the case of small speed only.

The outline of the paper is as follows. In Sec. 2 we recall the general features of our method in the conformal Coulomb gauge in both first-order and ADM [17] formalisms. In particular, we show how the metric can be found once the mapping function $f(z, t)$ and the meromorphic function $N(z, t)$ are given. In Sec. 3 we give an exact solution for $f$ and $N$, in the case of spinning particles at rest, characterized by the fact that $N$ has double poles at the particle sites, with residues proportional to the spins. We show that such double poles, related to an energy-momentum density of $\delta'$-type, are at the origin of the time shifts, of the apparent singularities, and of the CTC horizons close to the particles. In Sec. 4 we discuss the two-body problem, corresponding to 5 singularities, at both first-order and second-order in the velocities. The second-order solution corresponds to the non-relativistic limit and is of particular interest, even in the spinless case.

Our results and conclusions are summarized in Sec. 5, and some technical details are contained in Appendices $A$ and $B$.

2 General features and gauge choice

2.1 From Minkowskian to single-valued coordinates

In [BCV] we have proposed a non-perturbative solution for the metric and the motion of $N$ interacting spinless particles in (2+1)-gravity, based on the introduction of a new gauge choice which yields an instantaneous propagation of the gravitational force.

Our gauge choice is better understood in the first-order formalism which naturally incorporates the flatness property of (2+1) space-time outside the sources. This feature allows to choose a global Minkowskian reference system $X^a \equiv (T, Z, \overline{Z})$, which however is in general multivalued, due to the localized curvature at the particle sources. In order to have well-defined coordinates, cuts should be introduced along tails departing from each particle, and a Lorentz transformation should relate the values of $dX^a$'s along the cuts, so that the line element $ds^2 = \eta_{ab}dX^a dX^b$ is left single-valued.

The crucial point of our method is to build a representation of the $X^a$'s starting from a regular coordinate system $x^\mu = (t, z, \overline{z})$, as follows:

$$dX^a = E^a_\mu dx^\mu = E^a_0 dt + E^a_z dz + E^a_{\overline{z}} d\overline{z} \quad (2.1)$$

Here the dreibein $E^a_\mu$ is multivalued and satisfies the integrability condition:
\[ \partial_{\mu} E^a_{\nu} = 0 \]  

(2.2)

which implies a locally vanishing spin connection, outside the particle tails [4].

Let us choose to work in a Coulomb gauge:

\[ \partial \cdot E^a = \partial_z E^a_z + \partial \pi E^a_z = 0 \]  

(2.3)

which, together with the equations of motion (2.2), implies

\[ \partial_z E^a_z = \partial \pi E^a_z = 0, \]  

(2.4)

so that \( E^a_z(E^a_{\pi}) \) is analytic ( antianalytic ).

Multiplying (2.4) by \( E^a_z \) we also get \( \partial_z g_{zz} = 0 \); we choose to impose the conformal condition \( g_{zz} = g_{\pi \pi} = 0 \) in order to avoid arbitrary analytic functions as components of the metric. Hence we can parametrize \( E^a_z, E^a_{\pi} \) in terms of null-vectors:

\[ E^a_z = NW^a, \quad E^a_{\pi} = \overline{N\tilde{W}}^a \]  

(2.5)

where \( W^2 = \tilde{W}^2 = 0 \), and we can assume \( N(z,t) \) to be a single-valued meromorphic function ( with poles at \( z = \xi_i \), as we shall see ).

We have to build \( W^a, \tilde{W}^a \) in order to represent the DJH [3] matching conditions of the \( X^a \) coordinates, around the particle sites \( z = \xi_i(t) \):

\[ (dX^a)_I \rightarrow (dX^a)_II = (L_i)^b_i(dX^b)_I \quad i = 1,2,...,N \]  

(2.6)

where \( L_i = exp(iJ_a P^a_i) \) ( \( (iJ_a)_{bc} = \epsilon_{abc} \) ) denote the holonomies of the spin connection, which is treated here in a global way, in order to avoid distributions.

The simplest realization of such \( O(2,1) \) monodromies is given by a spin \( \frac{1}{2} \) projective representation:

\[ f(z,t) \rightarrow a_i f(z,t) + b_i \]  

(2.7)

where the mapping function \( f(z,t) \) is an analytic function with branch-cuts at \( z = \xi_i(t) \) and \( V_i = P_i/E_i \) ( \( \gamma_i = (1 - |V_i|^2)^{-\frac{1}{2}} \) ) are the constant Minkowskian velocities.
Since the $W$ vectors must transform according to the adjoint representation of $O(2, 1)$, the natural choice, constructed out of the mapping function $f$ defined before, is to set:

$$W^a = \frac{1}{f} (f, 1, f^2), \quad \tilde{W}^a = \frac{1}{f} (\tilde{f}, \tilde{f}^2, 1)$$

(2.8)

which gives for the spatial component $g_{z\bar{z}}$ of the metric tensor the expression

$$-2g_{z\bar{z}} = e^{2\phi} = \left| \frac{N}{f} \right|^2 (1 - |f|^2)^2$$

(2.9)

in which we recognize the general solution of a Liouville-type equation [18].

We can now integrate (2.1) out of particle 1:

$$X^a = X_1^a(t) + \int_{\xi_1}^{z} dz \, N W^a(z, t) + \int_{\bar{\xi}_1}^{\bar{z}} d\bar{z} \tilde{N} \tilde{W}^a(\bar{z}, t)$$

(2.10)

in terms of the parametrization $X_1^a(t)$ of one Minkowskian trajectory, which is left arbitrary.

The $X^a = X^a(x)$ mapping is at this point uniquely determined once the solution to the monodromy problem (2.7) is found. Since the coefficients $(a_i, b_i)$ are constants of motion, the monodromy problem can be recast into a Riemann-Hilbert problem [16] for an appropriate II order differential equation with Fuchsian singularities, whose solutions are quoted in [BCV] for the spinless case.

For instance, in the two-body case, there are 3 singularities, which can be mapped to $\zeta_1 = 0, \zeta_2 = 1, \zeta_\infty = \infty$, where $\zeta = \frac{z - \xi_1}{\xi_2 - \xi_1}$, and the mapping function is the ratio of two hypergeometric functions

$$f(\zeta) = \frac{\gamma_{12} V_{12}}{\gamma_{12} - 1} \zeta^{\mu_1} \tilde{F} \left( \frac{1}{2} (1 + \mu_\infty + \mu_1 - \mu_2), \frac{1}{2} (1 - \mu_\infty + \mu_1 - \mu_2), 1 + \mu_1; \zeta \right)$$

$$\tilde{F}(a, b; c; z) \equiv \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} F(a, b; c; z), \quad \gamma_{12} \equiv \frac{P_1 P_2}{m_1 m_2}$$

(2.11)

whose difference of exponents are $\mu_i = m_i/2\pi$ (i = 1, 2) and $\mu_\infty = M/2\pi - 1$, where $M$ is the total mass

$$\cos\left(\frac{M}{2}\right) = \cos\left(\frac{m_1}{2}\right) \cos\left(\frac{m_2}{2}\right) - \frac{P_1 \cdot P_2}{m_1 m_2} \sin\left(\frac{m_1}{2}\right) \sin\left(\frac{m_2}{2}\right)$$

(2.12)
In general, we can set
\[ f = \frac{y_1}{y_2}, \quad y_i'' + q(\zeta_i) y_i = 0, \quad (2.13) \]
where the potential
\[ 2q(\zeta) = \{ f, \zeta \} = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2, \quad (2.14) \]
is a meromorphic function with double and simple poles at the singularities \( \zeta = \zeta_i \).

It turns out that, for more than two particles, “apparent singularities” must be added into the differential equation in order to preserve the constancy of the monodromy matrix with moveable singularities, as firstly noticed by Fuchs [19]. Such singularities are zeros of \( f' \), rather than branch points of \( f \), and their position is related to the ones of the particles in a generally complicated way, determined by the monodromies. The total number of singularities for \( N \) particles is \( 2N - 1 \), and the mapping function was found for \( N \geq 3 \) in [BCV] in the limiting case of small velocities.

In order to determine the metric completely, we derive (2.10) with respect to time, and we obtain
\[
E_0^a = \partial_t X^a = \partial_t X_1^a + \partial_t \left( \int_{\xi_1}^z dz \ N W^a + \int_{\bar{\xi}_1}^\zeta d\bar{z} \ N \bar{W}^a \right) =
\]
\[
= e^t(t) + \int_{\xi_1}^z dz \ \partial_t (NW^a) + \int_{\bar{\xi}_1}^{\zeta} d\bar{z} \ \partial_t (N\bar{W}^a)
\]
\[ (2.15) \]

In terms of the vectors \( E_0^a = \partial_t X^a \), \( E_z^a = NW^a \), \( E_\bar{z}^a = N\bar{W}^a \), the components of the metric are given by:

\[
-2g_{z\bar{z}} = e^{2\phi} = |N|^2(-2W \cdot \bar{W}),
\]
\[
g_{zz} = \frac{1}{2} \beta e^{2\phi} = NW_a E_0^a, \quad g_{z\bar{z}} = \frac{1}{2} \beta e^{2\phi} = N\bar{W}_a E_0^a
\]
\[
g_{00} = \alpha^2 - |\beta|^2 e^{2\phi} = E_0^a E_0^a, \quad \alpha = V_a E_0^a
\]
\[ (2.16) \]

so that the line element takes the form
\[
\begin{align*}
\alpha^2 dt^2 - e^{2\phi} |dz - \beta dt|^2.
\end{align*}
\]
\[ (2.17) \]
Here we have defined the unit vector

$$V^a = \frac{1}{1 - |f|^2} (1 + |f|^2, 2\overline{f}, 2f) = \epsilon^a_{bc} W^b \overline{W}^c (W \cdot \overline{W})^{-1}$$

(2.18)

which represents the normal with respect to the surface $X^a = X^a(t, z, \overline{z})$, embedded at fixed time in the Minkowskian space-time $ds^2 = \eta_{ab} dX^a dX^b$. The tangent plane is instead generated by the vectors

$$\partial_z X^a = N W^a, \quad \partial_{\overline{z}} X^a = \overline{N} \overline{W}^a.$$  

(2.19)

We notice that it is not a priori warranted to have such a well defined foliation of space-time in terms of surfaces at fixed time. This probably requires the notion of a universal global time, which is not valid for universes with closed time-like curves, as it happens in the case of spinning sources. Hence we can anticipate that we will have problems in defining our gauge globally for spinning sources.

### 2.2 The Einstein equations in the ADM formalism

Quite similarly to what we have discussed now, the starting point of the ADM formalism is to assume that space-time can be globally decomposed as $\Sigma(t) \otimes R$, where $\Sigma(t)$ is a set of space-like surfaces. The $(2 + 1)$-dimensional metric is then split into “space” and “time” components:

$$g_{00} = \alpha^2 - e^{2\phi} \beta \overline{\gamma}, \quad g_{0z} = \frac{1}{2} \beta e^{2\phi}, \quad g_{0\overline{z}} = \frac{1}{2} \beta e^{2\phi} \quad g_{zz} = -\frac{1}{2} e^{2\phi}$$

(2.20)

where $\alpha$ and $\beta$ provide the same parametrization as in Eq. (2.17). The lapse function $\alpha$ and the shift functions $g_{0i}$ have the meaning of Lagrange multipliers in the Hamiltonian formalism, since their conjugate momenta are identically zero.

The ADM space-time splitting can be worked out from the Einstein-Hilbert action by rewriting the scalar curvature $R^{(3)}$ into its spatial part $R^{(2)}$, intrinsic to the surfaces $\Sigma(t)$, and an extrinsic part, coming from the embedding, as follows

$$S = -\frac{1}{2} \int \sqrt{|g|} R^{(3)} = -\frac{1}{2} \int \sqrt{|g|} \left[ R^{(2)} + (Tr K)^2 - Tr (K^2) \right] d^3 x,$$

(2.21)

where the equivalence holds up to a boundary term. Here we have introduced the extrinsic curvature tensor $K_{ij}$, or second fundamental form of the surface $\Sigma(t)$, given by:

$$K_{ij} = \frac{1}{2} \sqrt{\left| g_{ij} \right| / |g|} \left( \nabla_i^{(2)} g_{0j} + \nabla_j^{(2)} g_{0i} - \partial_0 g_{ij} \right)$$

(2.22)
where we denote by $\nabla^{(2)}_i$ the covariant derivatives with respect to the spatial part of the metric. The momenta $\Pi^{ij}$, conjugate to $g_{ij}$ are proportional to $K^{ij} - g^{ij}K$, which therefore complete the canonical coordinate system of the Hamiltonian formalism.

We can generate the ADM decomposition starting from the first-order formalism foliation $X^a = X^a(t, z, \bar z)$. The Coulomb gauge condition, imposed to fix such a mapping, can be directly related to the gauge condition of vanishing “York time” [20].

$$g_{ij} \Pi^{ij} = K(z, \bar z, t) = K_{z\bar z} = \frac{1}{2\alpha} (\partial_z g_{0\bar z} + \partial_{\bar z} g_{0z} - \partial_0 g_{z\bar z}) = 0 \tag{2.23}$$

In fact, by rewriting this combination in terms of the dreibein we get, by using eq. (2.2):

$$E_0^a \cdot (\partial_z E_\alpha^0 + \partial_{\bar z} E_\beta^0) + E_{\alpha}^0 \cdot (\partial_z E_\alpha^0 + \partial_\beta E_{\bar z}^0) + E_{\beta}^0 \cdot (\partial_{\bar z} E_\alpha^0 + \partial_0 E_{\bar z}^0) = 0 \tag{2.24}$$

We thus see that our gauge choice is defined by the conditions

$$g_{zz} = g_{\bar z\bar z} = K = 0 \tag{2.25}$$

and thus corresponds to a conformal gauge, with York time $g_{ij} \Pi^{ij} = 0$.

By combining the above conditions, we obtain a new action without time derivatives, demonstrating that the propagation of the fields $\alpha, \beta, \phi$ can be made instantaneous, as it appears from the equations of motion of Ref. [11]:

$$\nabla^2 \phi + \frac{e^{2\phi}}{\alpha^2} \partial_z \bar \beta \partial_{\bar z} \beta = \nabla^2 \phi + N \nabla e^{-2\phi} = -|g| e^{-2\phi} T^{00},$$

$$\partial_{\bar z} \left( \frac{e^{2\phi}}{\alpha} \partial_z \beta \right) = \partial_{\bar z} N = -\frac{1}{2} \alpha^{-1} |g| (T^{0z} - \beta T^{0\bar z}),$$

$$\nabla^2 \alpha - 2 \frac{e^{2\phi}}{\alpha} \partial_z \bar \beta \partial_{\bar z} \beta = \alpha^{-1} |g| (T^{z\bar z} - \beta T^{0\bar z} - \bar \beta T^{0z} + \beta \bar \beta T^{00}). \tag{2.26}$$

We understand from Eq. (2.26) that the sources of the meromorphic $N$ function are given by a combination of $\delta$-functions. For two particles, this leads to the solution of [BCV],

$$N = \frac{C(z_1)^{-1-\frac{M}{2\bar \phi}}}{\zeta(1-\zeta)} \tag{2.27}$$

which shows simple poles at $z = \xi_1$ and $z = \xi_2$, and no pole at $\zeta = \infty$. The $N$ function so determined provides also a feedback in the first of Eqs. (2.26), in which it modifies the sources of the “Liouville field” $e^{2\tilde \phi} = e^{2\phi}/|N|^2$, which is determined by the mapping function $f$ in Eq. (2.9).
The $t$-dependence of the trajectories is now provided by the covariant conservation of the energy-momentum tensor, which in turn implies the geodesic equations

$$ \frac{d^2 \xi^\mu_i}{ds_i^2} + (\Gamma^\mu_{\alpha\beta})_i \frac{d\xi^\alpha_i}{ds_i} \frac{d\xi^\beta_i}{ds_i} = 0 \quad i = 1, \ldots, N $$

(2.28)

Remarkably, we can completely solve these geodesic equations in the first-order formalism, by measuring the distance between two particles in the $X^a$ coordinates:

$$ X_2^a(t) - X_1^a(t) = B_2^a - B_1^a + V_2^a T_2 - V_1^a T_1 = $$

$$ = \int_{\xi_1}^{\xi_2} d\xi N W^a(\xi, t) + \int_{\xi_1}^{\xi_2} d\xi \tilde{N} \tilde{W}^a(\xi, t). $$

(2.29)

The explicit solution is obtained by inverting these relations to obtain the trajectories $\xi_i(t)$ as functions of the constants of motion $B_i^a, V_i^a$.

### 2.3 Spinning particle metric

The metric of a spinning particle at rest [6] is related to the large distance behaviour of the one of two moving particles which carry orbital angular momentum $J$. In the latter case, from the two-body solution [12], we find that the Minkowskian time $T$ at large distance from the sources changes by

$$ \Delta T = -8\pi G J = -J \quad (8\pi G = 1) $$

(2.30)

when the angular variable $\theta$ changes by $2\pi$, so that we must identify times which differ by $8\pi G J$ to preserve single-valuedness of the $X^a$ mapping.

Analogously, it was realized long ago [3] that $T$ has a shift proportional to the spin $s$, when turning around a spinning source, while the spatial component $Z$ rotates by the deficit angle $m$. By the transformation

$$ T = t - \frac{s}{2\pi} \theta, \quad Z = \frac{w^{1-\mu}}{1-\mu}, \quad \mu = \frac{m}{2\pi}, $$

(2.31)

the Minkowskian metric becomes the one of Ref. [3]:

$$ ds^2 = \left( dt - \frac{s}{2\pi} d\theta \right)^2 - |w|^{-2\mu} dw d\bar{w} $$

(2.32)

which, however, is not conformal ($g_{zz} \neq 0$).
Hence, in order to switch to a conformal type gauge, we must modify the $Z$ mapping of Eq. (2.31) by allowing another term, dependent on $\overline{z}$, which preserves the polydromy properties of the first term. If we set

$$Z = \frac{1}{1 - \mu} \left( z^{1-\mu} + A^2 \overline{z}^{\mu-1} \right) + B$$

(2.33)

(where $B$ is an additional arbitrary constant), we can get a cancellation of the $g_{zz}$ term in the metric [13]:

$$ds^2 = \left( dt - \frac{s}{4\pi} \left( \frac{dz}{iz} - \frac{d\overline{z}}{i\overline{z}} \right) \right)^2 - |z^{-\mu}dz - A^2\overline{z}^{\mu-2}d\overline{z}|^2$$

(2.34)

by choosing $A = -s/4\pi$.

The one-particle solution now looks like:

$$ds^2 = dt^2 + \frac{s}{2\pi} (dt^2 - \frac{i}{z} dz + h.c.) - e^{2\phi} dz d\overline{z}$$

(2.35)

where the conformal factor $e^{2\phi}$ is given by:

$$e^{2\phi} = |z|^{-2\mu} (1 - A^2 |z|^{2(\mu-1)})^2 = \frac{N^2}{f'} (1 - |f|^2)^2$$

(2.36)

In our general solution of Eqs. (2.8)-(2.10) this expression implies the following choice for the analytic functions $N(z), f(z)$:

$$N(z) = -\frac{iA(\mu - 1)}{z^2}, \quad \mu \equiv \frac{m}{2\pi}$$

$$f(z) = -iAz^{\mu-1}$$

(2.37)

Let us first note that $f$ has a singular behaviour for $z \to 0$, compared to the one of Eq. (2.11), and takes therefore large values for $z$ small. This implies that in $z$ coordinates, there is a “horizon” surrounding the spinning particle which corresponds to $|f| = 1$, and thus to a vanishing determinant $\sqrt{|g|} \simeq \alpha e^{2\phi} = 0$ in Eq. (2.36). It is therefore given by the circle

$$|z|^2 = r_0^2 = A^{1-\mu}$$

(2.38)

or, in $Z$ coordinates, by the circle

$$Z_0 = \frac{2A}{1 - \mu} e^{i\theta(1-\mu)} \Rightarrow |Z_0| = R_0 = \frac{s}{2\pi(1 - \mu)}.$$  

(2.39)
The meaning of this “horizon” is that values of $Z$ with $|Z| < R_0$ cannot be obtained from the parametrization (2.33) for any values of $z$. This in turn is related to the fact that, due to the symmetry $z \rightarrow A^{\frac{1}{1-\mu}} / z$, the inverse of the mapping (2.33) is not single-valued, and in fact we shall choose the determination of $z$ such that $z \simeq Z^{1-\mu}$ for $Z \rightarrow \infty$.

The fact that our gauge is unable to describe the internal region $|Z| < R_0$ is related to the existence, in that region, of CTC’s [14], which do not allow a global time choice. Indeed, closed time-like curves can be built when the negative time-jump $\Delta T = -s$ cannot be compensated by the time occurring to a light signal to circle the particle at distance $R$, which is given by $T_{\text{travel}} = 2\pi (1 - \mu) R$, thus implying

$$R < \frac{s}{2\pi (1 - \mu)} = R_0,$$

i.e., the same critical radius as in Eq. (2.39). For this reason we shall call the sort of horizon just found a “CTC horizon”.

Secondly, we note that the ratio $N/f'$ is similar to the spinless case, but the behaviour of $N(z)$ and $f'(z)$ separately is more singular, i.e.

$$N \simeq \frac{\sigma}{z^2}, \quad f' \simeq z^{\mu-2}.$$  

(2.41)

This is to be expected because the source for $N(z)$ in the equation of motion (2.26) is a more singular distribution, i.e.

$$\partial_\tau N(z) \propto \frac{s}{2\pi} \delta'(r)$$

(2.42)

in order to allow for a localized angular momentum. The same $z^{-2}$ behaviour for $N(z)$ can be found in the large distance limit of the two-body problem.

This more singular behaviour of the metric is an obstacle to define the $X^a$ coordinates in the vicinity of the particle at rest. Nevertheless the particle site can be unambiguously obtained by looking at the center of rotation of the DJH matching conditions [2] arising from (2.33), when turning around $z = 0$ in the region outside the CTC horizon, i.e.,

$$Z - B \rightarrow e^{-2i\pi \mu} (Z - B).$$

(2.43)

In the following, we shall use this procedure in order to identify the value of the $Z$ coordinate at the particle site (Cfr. Appendix A ).
3 Spinning particles at rest

The function $N(z, t)$ plays an important role in the following discussion because its polar structure determines the time shift around each particle, and from this we can get information about the apparent singularities which appear in the spinning case.

Let us recall the form (2.27) of $N(z)$ for the two-body problem in the spinless case:

$$
N(z, t) = -\frac{R(\xi(t))}{(z - \xi_1)(z - \xi_2)} = \frac{R(\xi)}{\xi^2} \frac{1}{\zeta(1 - \zeta)}
$$

(3.1)

where $\xi \equiv \xi_{21} = \xi_2 - \xi_1$ is the interparticle coordinate, and $R(\xi)$ was determined [12] to be

$$
R(\xi) = C\xi(t)^{1 - \frac{M}{2\pi}}
$$

(3.2)

The imaginary part of this coefficient is related to the asymptotic time shift by

$$
\Delta T \simeq -R \int \frac{dz}{z^2} \frac{f}{f'} + (h.c.) = -\frac{4\pi}{1 - \frac{M}{2\pi}} ImR = -J
$$

(3.3)

where $M$ is the total mass of Eq. (2.12).

Therefore, by Eq. (2.30) $R$ is determined in terms of the total angular momentum of the system, which for the spinless case is purely orbital ($J = L$), and given by [12]

$$
2\gamma_1 |V_2 - V_1| B_{21} \frac{\sin \pi \mu_1 \sin \pi \mu_2}{\sin \frac{M}{2}} = L = \frac{4\pi}{1 - \frac{M}{2\pi}} Im(R)
$$

(3.4)

In the spinning case, we can assume that at large distances $N(z, t)$ has the same $z^{-2}$ behaviour as in the spinless case. However, around each particle $N$ should have double poles, as found in Eqs. (2.37) and (2.41). We take therefore the ansatz

$$
N(z, t) = \frac{R(\xi)}{\xi^2} \left( \frac{1 - \sigma_1 - \sigma_2}{\zeta(1 - \zeta)} - \frac{\sigma_1}{\zeta^2} - \frac{\sigma_2}{(\zeta - 1)^2} \right),
$$

(3.5)

where $R\sigma_i$ are the double pole residues.

As a consequence of the double poles, logarithmic contributions to $T$ appear around each particle, which give rise to a time shift proportional to each spin $s_i$:

$$
\Delta T_i = \oint_{C_i} \frac{dz}{f'} \frac{Nf}{f'} + (h.c.) = -4\pi \frac{Im(R\sigma_i)}{1 - \mu_i} = -s_i, \quad \mu_i \equiv \frac{m_i}{2\pi}
$$

(3.6)
where we have used the power behaviour \( f|_{i} \simeq z^{\mu_i - 1} \). Eqs. (3.6) and (3.3) determine the values of the \( \sigma_i \)'s in terms of spins and angular momentum:

\[
\sigma_i = \frac{(1 - \mu_i) s_i}{(1 - \frac{M}{2\pi}) J} \tag{3.7}
\]

Furthermore, eq. (3.5) can be rewritten so as to show the presence of two zeros of \( N \), at \( \zeta = \eta_1, \zeta = \eta_2 \), i.e.,

\[
N(z, t) = -\frac{R(\xi) (\zeta - \eta_1)(\zeta - \eta_2)}{\xi^2 \zeta^2(\zeta - 1)^2} \tag{3.8}
\]

with

\[
\eta_{1,2} = \frac{1}{2} \left( 1 + \sigma_1 - \sigma_2 \pm \sqrt{1 - 2(\sigma_1 + \sigma_2) + (\sigma_1 - \sigma_2)^2} \right) \tag{3.9}
\]

Such zeros turn out to be the apparent singularities of our system. In fact, in order to avoid zeros of the metric determinant we have to cancel them by having \( f' \) to vanish at \( \zeta = \eta_i \) too. Therefore, even if \( f \) is analytic around \( \zeta = \eta \), its Schwarzian derivative has extra double poles with differences of indices \( \bar{\mu} = 2 \), corresponding to the general parametrization

\[
\{ f, z \} = \frac{1}{2} \frac{\mu_1(2 - \mu_1)}{\zeta^2} + \frac{1}{2} \frac{\mu_2(2 - \mu_2)}{(\zeta - 1)^2} - \frac{3}{2} \frac{1}{(\zeta - \eta_1)^2} - \frac{3}{2} \frac{1}{(\zeta - \eta_2)^2} + \frac{\beta_1}{\zeta} + \frac{\beta_2}{\zeta - 1} + \frac{\gamma_1}{\zeta - \eta_1} + \frac{\gamma_2}{\zeta - \eta_2} \tag{3.10}
\]

with the “accessory parameters” \( \beta' \)'s and \( \gamma' \)'s so far undetermined.

### 3.1 The two-body case

There is no general known solution to the Fuchsian problem of Eq. (3.10), since it contains five singularities. However in a few limiting cases a particular solution can be obtained. For example in the static two-body case under consideration, the form of \( f' \) is determined by its zeros, and by the known behaviour around \( \zeta = 0 \) and \( \zeta = 1 \), as follows:

\[
f' = -\frac{K}{\xi} (\zeta - \eta_1)(\zeta - \eta_2) \zeta^{\mu_1 - 2}(1 - \zeta)^{\mu_2 - 2}. \tag{3.11}
\]

Furthermore, it should be integrable to a function \( f \) with static monodromy matrix, and behaviour \( f \simeq \zeta^{\mu_1 + \mu_2 - 1} \) at \( \zeta = \infty \), of the form:
\[ f = \frac{K}{\mu_1 + \mu_2 - 1} \zeta^{\mu_1 - 1}(1 - \zeta)^{\mu_2 - 1}(\zeta - \tau). \] (3.12)

The consistency of Eqs. (3.11) and (3.12) gives a constraint on the possible values of \( \eta_1, \eta_2 \):

\[ \eta_1 \eta_2 = \sigma_1 = \frac{(1 - \mu_1)\tau}{1 - \mu_1 - \mu_2}, \quad (1 - \eta_1)(1 - \eta_2) = \sigma_2 = \frac{(1 - \mu_2)(1 - \tau)}{1 - \mu_1 - \mu_2} \] (3.13)

which, by Eq. (3.7), is satisfied if the total angular momentum is simply the sum of the two spins \( s_i \), i.e. \( J = s_1 + s_2 = S \), or

\[ \frac{\sigma_1}{1 - \mu_1} + \frac{\sigma_2}{1 - \mu_2} = \frac{1}{1 - \mu_1 - \mu_2}, \] (3.14)

as expected in the static case. This condition also determines the value of

\[ R = i \frac{S}{4\pi} (1 - \mu_1 - \mu_2) \] (3.15)

In order to determine the constant \( K \) in Eq. (3.12) we must use the analog of Eq. (2.29) which defines the Minkowskian interparticle distance

\[
B_{21} = Z_2 - Z_1 = \int_1^2 \, dz \, \frac{N}{f'} + \int_1^2 \, dz \, \frac{N f'^2}{f^2} = \left( \frac{R}{K} \right) \int_0^1 \, d\zeta \, \zeta^{-\mu_1}(1 - \zeta)^{-\mu_2} + \frac{R K}{(\mu_1 + \mu_2 - 1)^2} \int_0^1 \, d\zeta \, \zeta^{\mu_1 - 2}(1 - \zeta)^{\mu_2 - 2}(\zeta - \tau)^2
\] (3.16)

We can see that the second integral is not well defined in the physical range

\[ 0 < \mu_i < 1, \quad 0 < \mu_1 + \mu_2 < 1 \] (3.17)

for which, as shown in I, there are no CTC’s at large distances. This fact reflects the existence of CTC horizons close to the particles, (cfr. Sec. (2.3)), in which the mapping to Minkowskian coordinates is not well defined.

The rigorous way of overcoming this problem is to solve for the DJH matching conditions of type (2.43) outside the CTC horizons, thus defining \( B_{21} \) as the relevant translational parameter, as explained in Appendix A.

Here we just notice that the integral in question can be defined by analytic continuation from the region \( 2 > \mu_i > 1 \) to the region (3.17), so as to yield, by Eqs. (3.15) purely imaginary values for \( K \) and \( R \), with
\[
\frac{1}{|K|} B(1 - \mu_1, 1 - \mu_2) - \frac{|K|B(\mu_1, \mu_2)}{(\mu_1 + \mu_2 - 1)^2} \left( 1 - \tau \frac{1 - \mu_1 - \mu_2}{1 - \mu_1} - (1 - \tau) \frac{1 - \mu_1 - \mu_2}{1 - \mu_2} + \right.
\]
\[
\left. + \tau(1 - \tau) \frac{(1 - \mu_1 - \mu_2)(2 - \mu_1 - \mu_2)}{(1 - \mu_1)(1 - \mu_2)} \right) = \frac{4\pi \, B_{21}}{(1 - \mu_1 - \mu_2)S} \tag{3.18}
\]

where \( S = J = s_1 + s_2 \) denotes the total spin. In this equation the smaller branch of \( K \) should be chosen for the solution to satisfy \(|f| < 1\) closer to the particles.

In particular, if \( S/B_{12} \ll 1 \), the acceptable branch of the normalization \( K \) becomes small and of the same order. In general, however \( K \) is not a small parameter, and thus \( f \) is not small, unlike the spinless case in which \( f \) is of the order of \( L/B_{12} \) and is thus infinitesimal in the static limit.

Whatever the value of \( K \), for \( z \) sufficiently close to the particles, the critical value \(|f| = 1\) is reached, because of the singular behaviour of Eq. (3.12) for \( \mu_i < 1 \). Therefore we have two horizons encircling each particle, which may degenerate in one encircling both for sufficiently large values of \( S/B_{12} \).

Having found an explicit solution for \( f \), its Schwarzian derivative is easily computed from Eq. (2.14). By using the notation

\[
\tilde{\mu}_i = \mu_i - 1 \quad (i = 1, 2) \quad , \quad \tilde{\mu}_3 = \tilde{\mu}_4 = 2 \\
\eta_1 = \zeta_3 \quad , \quad \eta_2 = \zeta_4 \quad , \quad \gamma_1 = \beta_3 \quad , \quad \gamma_2 = \beta_4
\]

we find that the residues \( \beta_i \) at the single poles of Eq. (3.10) (called the accessory parameters) take the quasi-static form of BCV, i.e.

\[
\beta_i = -(1 - \tilde{\mu}_i) \sum_{j \neq i} \frac{(1 - \tilde{\mu}_j)}{\zeta_i - \zeta_j}, \quad (i, j = 1, ..., 4). \tag{3.20}
\]

### 3.2 The static metric

From the two-body static solution for \( f \) and \( N \) we can build the static vierbein of Eqs. (2.5) and (2.15) which has the components:

\[
E^0_0 = (1, 0, 0) \\
E^a_z = NW^a = \frac{R}{\xi(\mu_1 + \mu_2 - 1)} \left( \frac{\zeta - \tau}{\zeta(1 - \zeta)} \frac{\mu_1 + \mu_2 - 1}{K_0} \zeta^{1 - \mu_1} (1 - \zeta)^{-\mu_2} \right.
\]
\[
\left. \frac{K_0}{(\mu_1 + \mu_2 - 1)} \zeta^{\mu_2 - 2}(1 - \zeta)^{\mu_2 - 2}(\zeta - \tau)^2 \right) \tag{3.21}
\]
The corresponding metric has the form

\[
\begin{align*}
g_{00} &= 1 \\
g_{0z} &= \frac{Nf}{f'} = \frac{R}{\xi(\mu_1 + \mu_2 - 1)} \frac{\zeta - \tau}{\zeta(1 - \zeta)} \\
-2g_{z\zeta} &= e^{2\phi} = \frac{R^2}{\xi^2|\xi_0|^2} (\zeta \bar{\zeta})^{-\mu_1} ((1 - \zeta)(1 - \bar{\zeta}))^{-\mu_2} \\
&\quad \cdot \left[ 1 - \frac{K_0^2(\zeta \bar{\zeta})^{\mu_1-1}}{(\mu_1 + \mu_2 - 1)^2} ((1 - \zeta)(1 - \bar{\zeta}))^{\mu_2-1} (\zeta - \tau)(\bar{\zeta} - \tau) \right]^2 \tag{3.22}
\end{align*}
\]

and is degenerate whenever \( e^{2\phi} = 0 \), revealing explicitly the presence of a singularity line on which the determinant of the metric is vanishing.

Let us remark that the zeros of \( f' \), the apparent singularities, are geometrically saddle points for the modulus \( |f|^2 \), which instead diverges on the particle sites. It is easy to realize that in the range \( S/B_{12} \ll 1 \), where \( K_0 \simeq S/B_{12} \), the curve \( |f| = 1 \) defines two distinct horizons, one for each particle.

In the complementary range \( S \geq B_{12} \), with \( K_0 \) satisfying (3.18) in its generality, the curve \( |f| = 1 \) defines a line surrounding both particles.

As a consequence, we can distinguish the two particles and set up the scattering problem only in the case where \( S/B_{12} \) is at most of order \( O(1) \). This restriction is physically motivated by the presence of closed timelike-curves, which make impossible to reduce the impact parameter without entering in causality problems.

### 3.3 The N-body static case

In the general static case, with \( N \) bodies, we can also provide a solution for the mapping function by algebraic methods, following the pattern described above.

Firstly, the meromorphic \( N \) function, having simple and double poles at \( \zeta = \zeta_i \) can be parametrized as

\[
N = R \left( - \sum_{i=1}^{N} \frac{\sigma_i}{(\zeta - \zeta_i)^2} + \sum_{i=1}^{N} \frac{\nu_i}{\zeta - \zeta_i} \right) = R \frac{\prod_{i=1}^{2N-2}(\zeta - \eta_i)^2}{\prod_{i=1}^{N}(\zeta - \zeta_i)^2} \tag{3.23}
\]

with the following conditions

\[
\sum_{i} \nu_i = 0 \quad , \quad (N \sim -Rz^{-2} \text{ for } z \to \infty)
\]
\[ \sum_i \sigma_i - \sum_i \zeta_i \nu_i = 1, \quad (\text{normalization of } R). \quad (3.24) \]

Therefore, there are \(2N - 2\) zeros (or apparent singularities) given in terms of the \(\sigma\)'s and of \(N - 2\) \(\nu\)-type parameters. Furthermore, the \(\sigma\)'s are given by the time shifts in terms of \(s_i/J\) as in Eq. (3.7).

Secondly, \(f'\) shows the same \(2N - 2\) zeros at \(\zeta = \eta_j\) in the form

\[
\frac{df}{dz} = \frac{1}{\xi} f'(\xi) = \frac{K}{\xi} \prod_{i=1}^{N} (\zeta - \zeta_i)^{\mu_i - 2} \prod_{j=1}^{2N-2} (\zeta - \eta_j) \quad (3.25)
\]

while the mapping function, having static monodromy and behaviour \(\zeta \sum_{i} \mu_i^{-1}\) at \(\zeta = \infty\), has only \((N - 1)\) zeros with the form

\[
f = \frac{K}{\sum_{i} \mu_i - 1} \prod_{i=1}^{N} (\zeta - \zeta_i)^{\mu_i - 1} \prod_{k=1}^{N-1} (\zeta - \tau_k). \quad (3.26)
\]

At this point, the integrability condition, that (3.25) is just the \(z\)-derivative of (3.26) provides \(2N - 2\) conditions for a total of \(N - 2 + N - 1 = 2N - 3\) parameters. Therefore, all the \(\eta\) parameters are determined as function of the \(\sigma\)'s, and there is one extra condition among the \(\sigma\)'s, namely that

\[
\sum_{i=1}^{n} \frac{\sigma_i}{1 - \mu_i} = \frac{1}{1 - \sum_{i} \mu_i} \quad (3.27)
\]

which is verified, by Eq. (3.7) because \(\sum_i s_i = S = J\) in the static case.

Finally, the normalization \(K\) and the \(N - 2\) “shape parameters” \(\zeta_j = \frac{\zeta_j}{\xi_{21}}\) are determined from the \(N - 1\) “equations of motion”

\[
B_j - B_1 = \int_1^j \frac{N}{f'} dz + \int_1^j \frac{Nf'^2}{f}, \quad (3.28)
\]

similarly to the two-body case.

We conclude that the static \(N\)-body case with spin provides a solvable example of non-vanishing mapping function with static monodromies and total mass \(M = \sum_{i=1}^{N} m_i\), having a Schwarzian with a total of \(3N - 1\) singularities.
4 Spinning particles in slow motion

For two moving spinning particles, the fuchsian Riemann-Hilbert problem for the mapping function is in principle well defined by Eqs. (2.13), (2.14) and (3.10). Indeed, the location of the apparent singularities is fixed in general by Eqs. (3.5), (3.8) and (3.9) and it is possible to see that all accessory parameters in the Schwarzian of Eq. (3.10) are also fixed in terms of the invariant mass $M$ of Eq. (2.12) and of the spins.

The fact that the potential of the Fuchsian problem is determined follows from some general conditions that the accessory parameters should satisfy, which were described in I, and are the following.

Firstly, the point at $\zeta = \infty$ is regular, with difference of exponents given by $\mu_\infty = \frac{M}{2\pi} - 1$. This yields two conditions:

$$\sum_{i=1}^{2} (\beta_i + \gamma_i) = 0,$$
$$\sum_{i=1}^{2} \mu_i (2 - \mu_i) - 6 + 2 \sum_{i=1}^{2} (\beta_i \zeta_i + \gamma_i \eta_i) = 1 - \mu_\infty^2. \quad (4.1)$$

Secondly, there is no logarithmic behaviour [16] of the solutions $y_i$ at the apparent singularities $\eta_j$. This yields two more conditions:

$$-\gamma_1^2 = \sum_{j=1}^{2} \frac{\mu_j (2 - \mu_j)}{(\eta_1 - \zeta_j)^2} - \frac{3}{(\eta_1 - \eta_2)^2} + \sum_j \frac{2\beta_j}{\eta_1 - \zeta_j} + \frac{2\gamma_2}{\eta_1 - \eta_2},$$
$$-\gamma_2^2 = \sum_{j=1}^{2} \frac{\mu_j (2 - \mu_j)}{(\eta_2 - \zeta_j)^2} - \frac{3}{(\eta_1 - \eta_2)^2} + \sum_j \frac{2\beta_j}{\eta_2 - \zeta_j} + \frac{2\gamma_1}{\eta_2 - \eta_1}. \quad (4.2)$$

The four algebraic (non linear) Eqs. (4.1) - (4.2) determine $\beta_1, \beta_2$ and $\gamma_1, \gamma_2$ in terms of $M$ and of the $\sigma$’s, similarly to what happens in the (much simpler) spinless case. However, since no general solution to this Fuchsian problem with 5 singularities is known, one should resort to approximation methods in order to provide the mapping function explicitly.

The idea is to expand $f$ in the (small) Minkowskian velocities $V_i << 1$, around the static solution, which is exactly known. One should, however, distinguish two cases, according to whether $S/B_{12} \ll 1$ is of the same order as the $V$’s, or instead $S/B_{12} = O(1)$, where $B_{12} = B_1 - B_2$ is defined as the relative impact parameter of the Minkowskian trajectories.
\[ Z_1 = B_1 + V_1 T_1 \quad Z_2 = B_2 + V_2 T_2. \] (4.3)

In the first case, that we call “peripheral”, the expansion we are considering is effectively in both the \( V_i \)'s and \( f \) itself, at least in a region sufficiently far away from the horizons, which do not overlap, as noticed before. This is a situation of peripheral scattering with respect to the scale provided by the horizon, and will be treated in the following to first (quasi-static) and second (non-relativistic) order.

The second case, that we call “central”, \( (S/B_{12} = O(1)) \) has non-linear features even to first order in the velocities and the horizons may overlap. We shall only treat the quasi-static case.

4.1 Peripheral quasi-static case \(( S \ll B_{12} )\)

Since in this case both \( f \) and \( V_i \) can be considered as small, at first order we have just to look for a mapping function which solves the linearized monodromies of Eq. (2.7) around the particles \((i = 1, 2)\)

\[
\tilde{f}_i = \frac{a_i}{a^*_i} f(\zeta) + \frac{b_i}{a^*_i} \quad (a_i = e^{i \pi \mu_i}, \quad b_i = -i V_i \sin\pi \mu_i) \quad (4.4)
\]

and, furthermore, has the two apparent singularities in Eq. (3.9) and the behaviour in Eq. (2.41) at \( \zeta = 0 \) and \( \zeta = 1 \). Since, by Eq. (4.4), \( f' \) has static monodromies, we can set

\[
f' = K \zeta^{\mu_1-2}(1 - \zeta)^{\mu_2-2}(\zeta - \eta_1)(\zeta - \eta_2), \quad (4.5)
\]

as in Eq. (3.11).

However, in the moving case, \( f' \) is not integrable and \( f \) has the quasi-static monodromy (4.4), which contains a translational part. We then set

\[
f(\zeta) = f(i) + \int_i^\zeta dt \ f'(t) \quad (4.6)
\]

where the ith-integral is understood as analytic continuation from \( \mu_i > 1 \), as explained in Appendix A.

Since the ith-integral has purely static monodromy around the i-th particle we obtain from Eq. (4.6)

\[
(\tilde{f}(\zeta) - f(i)) = e^{2 i \pi \mu_i} (f(\zeta) - f(i)) \quad (4.7)
\]
and thus Eq. (4.4) is satisfied if \( f(i) = -\frac{v_i}{2} \), thus yielding the condition

\[
\frac{v_2 - v_1}{2} = K \int_0^1 dt \ t_{\mu_1}^{-2}(1 - t)^{\mu_2 - 2}(t - \eta_1)(t - \eta_2) = KB(\mu_1, \mu_2) \left( 1 - \sigma_1 \frac{1 - \mu_1}{1 - \mu_1 - \mu_2} - \sigma_2 \frac{1 - \mu_2}{1 - \mu_1 - \mu_2} \right)
\]

(4.8)

By then using Eq. (3.7), we determine the normalization

\[
K = \frac{V_{21}}{2} B^{-1}(\mu_1, \mu_2) \left[ 1 - \frac{s_1 + s_2}{J} \right]^{-1} = B^{-1}(\mu_1, \mu_2) J \frac{V_{21}}{L} \]

(4.9)

where \( L \) is the orbital angular momentum of the system, given in Eq. (3.4).

Eqs. (4.5), (4.6) and (4.9) yield the ( peripheral ) quasi-static solution with spin. In particular, Eq. (4.9) shows the existence of two regimes, according to whether the spin is small or large with respect to the orbital angular momentum \( L \).

If the spin \( S \) is small ( \( S \ll L \) ), so are the \( \sigma \)'s, the apparent singularities of Eq. (3.9) become degenerate with the particles

\[
\eta_1 \simeq \sigma_1, \quad \eta_2 \simeq 1 - \sigma_2
\]

(4.10)

and the normalization \( K \sim V_{21} \) is vanishingly small in the static limit, thus recovering the spinless quasi-static limit of I.

If instead \( B_{21} \gg S \gg L \) the parameters \( \sigma \) and \( \eta \) are of order unity, and , by Eq. (3.4), the normalization becomes

\[
K \simeq B^{-1}(\mu_1, \mu_2) \frac{S}{2L} V_{21} \left( \frac{1 - \mu_1 - \mu_2} {B_{21}} \frac{B(1 - \mu_1, 1 - \mu_2)}{B_{21}} \right)
\]

(4.11)

in agreement with the static case relation (3.18) for \( S \ll B_{21} \). In the latter case the mapping function becomes nontrivial in the static limit, as discussed in the previous section.

In the general case, for any \( S/L \) of order unity, the mapping function \( f \) is of first order in the small parameters and its Schwarzian derivative does not change with respect to the static case, except for the actual values of the \( \sigma \)'s and \( \eta \)'s, so that the accessory parameters are provided by Eq. (3.20). The first non-trivial change of \{\( f, \zeta \)\} is at second order in the small parameters, as we shall see ( Secs. 4.2 and 4.3 ).

4.2 Central quasi-static case ( \( S/B_{12} \simeq O(1) \) )
In this case we have to expand the monodromies in Eq. (2.7) in the \( b_i \) parameters only, thus keeping possible non linear terms in \( f \). By expanding around the quasi-static solution \( f^{(0)} \) of Eq. (4.5) we can write

\[
f = f^{(0)} + \delta f + \ldots
\]

(4.12)

where, around the generic particle,

\[
\tilde{f} = \frac{a_i}{a_i^*} f + \frac{b_i}{a_i^*} - \frac{a_i b_i^*}{a_i^* a_i^*(f_s^{(0)})^2}, \quad i = 1, 2
\]

(4.13)

and we have introduced in the last term the static limit \( f_s^{(0)} \) of Eq. (3.12). These monodromy conditions, unlike the ones in Eq. (4.4), are nonlinear. However it is not difficult to check that they linearize for the function

\[
h = \frac{1}{f_s^{(0)}} f',
\]

(4.14)

which satisfies the first-order monodromy conditions

\[
\tilde{h} = e^{-2i\pi\mu_i} h - V_i (1 - e^{-2i\pi\mu_i}).
\]

(4.15)

Furthermore, from the boundary conditions for \( f \), we derive the following ones for \( h \)

\[
h \simeq \begin{cases} 
-V_i + O((\zeta - \zeta_i)^{1-\mu_i}) , & (\zeta \simeq \zeta_i, \ i = 1, 2), \\
\zeta^{1-\mu_1-\mu_2} , & (\zeta \to \infty) \\
(\zeta - \tau)^{-1} , & (\zeta \to \tau)
\end{cases}
\]

(4.16)

The solution for \( h \) can be found from the ansatz

\[
h = -V_1 + (V_1 - V_2) I_0 (\zeta) + A \frac{\zeta^{1-\mu_1}(1 - \zeta)^{1-\mu_2}}{K_0(\zeta - \tau)}
\]

\[
I_0 = \frac{1}{B(1 - \mu_1, 1 - \mu_2)} \int_0^\zeta dt \ t^{-\mu_1} (1 - t)^{-\mu_2}
\]

(4.17)

by noticing that the first two terms automatically satisfy the boundary conditions at \( \zeta = 0, 1, \infty \). The last term is proportional to \((f_s^{(0)})^{-1}\), contains the pole at \( \zeta = \tau \), and the constant \( A \) is determined so as to satisfy the translational part of the monodromy (4.13). Similarly to Eq. (4.8) we get the equation for \( A \).
\[ \frac{V_2 - V_1}{2} = A \int_0^1 f_s^{(0)}(\zeta) + \int_0^1 d\zeta f_s^{(0)}(\zeta)[ -V_1 f_s^{(0)}(\zeta) + (V_1 - V_2) f_s^{(0)}(\zeta) I_0(\zeta)] \]  
(4.18)

where the integrals are understood as analytic continuations from \( 2 > \mu_i > 1 \), and the non-linear terms represent higher order contributions in the parameter \( S/B_{12} \).

By using the normalization condition (4.8) we then obtain by an integration by parts and for velocities along the \( x \)-axis,

\[ A = 1 - \int_0^1 d\zeta (f_s^{(0)})^2(\zeta) I'_0 \]  
(4.19)

This means that the coefficient of the first-order quasi-static solution \( f^{(0)} \) in Eq. (4.12) is renormalized by higher orders in \( S/B_{12} \). Furthermore from the expression (4.14) of \( f'/f_0' = h f_s^{(0)} \) we obtain the correction to the Schwarzian derivative

\[ \{ f, \zeta \} - \{ f, \zeta \}^{(0)} \simeq (h f_s^{(0)})'' s - \frac{f''(0)}{f'_0(0)} (h f_s^{(0)})' = \]

\[ \frac{K_0(V_1 - V_2)}{(\mu_1 + \mu_2 - 1)B(1 - \mu_1, 1 - \mu_2)} \frac{\zeta - \tau}{\zeta(1 - \zeta)} \left( \frac{2}{\zeta - \tau} - \frac{1}{\zeta - \eta_1} - \frac{1}{\zeta - \eta_2} \right) \]  
(4.20)

which turns out to be of order \( O(V) \cdot O(S/B_{12}) \), i.e. formally of 1st order in both parameters.

The quasi-static solution for general spin values just obtained is particularly interesting because it allows to understand how the trajectory equations (4.3) can make sense, despite the multivaluedness of the Minkowskian time.

In fact, we can solve for the mapping from regular to Minkowskian coordinates by expanding around some arbitrary point \( \xi_0 \neq \xi_i \) to get, instead of Eq. (2.10)

\[ X^a = X_0^a(t) + \int_{\xi_0}^\xi dz NW^a + \int_{\xi_0}^{\bar{\xi}} d\bar{\zeta} \bar{N} \bar{W}^a. \]  
(4.21)

We can then explicitly check that, to first order in the velocities, the combinations \( Z - V_1 T (Z - V_2 T) \) are well defined at particle 1 ( particle 2 ).

For this we need the static time

\[ T = t + \int_{\xi_0}^\xi dz \frac{N f_0^{(0)}}{f'_0} + (c.c) = t - \frac{R}{1 - \mu_1 - \mu_2} \left( (1 - \tau) \log 2(1 - \zeta) + \tau \log 2\zeta \right) + (h.c.) \]  
(4.22)

which shows the logarithmic singularities at \( z = \xi \), and we also need the \( Z \) coordinate up to first order in \( V \)
\[ Z = Z_0(t) + \int_{\xi_0}^{\xi} dz \frac{N}{f'(0)} \left( 1 - \frac{\delta f'}{f'(0)} \right) + \int_{\tilde{\xi}_0}^{\xi} d\tilde{\xi} \frac{N}{f'(0)} \left( \frac{\tilde{f}^2}{f'(0)} + 2\tilde{f}^{(0)} \delta \tilde{f} - \delta \tilde{f} \tilde{f}' \right) \] (4.23)

Since, by Eq. (4.14) and (4.17) \( \delta f'/f'(0) = -V_1 f^{(0)} + O((\zeta - \zeta_1)^0) \) and \( \delta \tilde{f} = -V_1/2 + O(f^2_0 V) \), the \( Z \)-coordinate also has logarithmic singularities, which cancel in the combination

\[
\lim_{\xi \to \xi_1} (Z - V_1 T) = B_1 = Z_0(t) - V_1 t + \int_{\xi_0}^{\xi_1} dz \frac{N}{f'(0)} (1 + O(V)) + \int_{\tilde{\xi}_0}^{\xi_1} d\tilde{\xi} \frac{N\tilde{f}^2}{f'(0)} (1 + O(V))
\] (4.24)

which is thus well defined. From the similar relation from particle 2 and using the expression of \( R \) in Eq. (3.2) we obtain the quasi-static equations of motion

\[
i(B_1 - B_2) -(V_1 - V_2) t = \left( \int_{\xi_2}^{\xi_1} dz \frac{N}{f'(0)} + \int_{\tilde{\xi}_2}^{\tilde{\xi}_1} d\tilde{\xi} \frac{N\tilde{f}^2}{f'(0)} \right) (1 + O(V)) = \\
= \alpha \xi_1^{1-\frac{\alpha}{2\pi}} + \beta \xi_1^{1-\frac{\beta}{2\pi}} = \\
= \frac{C}{K_0} \xi_1^{1-\frac{\alpha}{2\pi}} B(1 - \mu_1, 1 - \mu_2) + \frac{C K_0 \xi_1^{1-\frac{\alpha}{2\pi}} B(\mu_1, \mu_2)}{(\mu_1 + \mu_2 - 1)^2} \left( 1 - \tau \frac{1 - \mu_1 - \mu_2}{1 - \mu_1} \right) - \\
- (1 - \tau) \frac{1 - \mu_1 - \mu_2}{1 - \mu_2} + \tau (1 - \tau) \frac{1 - \mu_1 - \mu_2 (2 - \mu_1 - \mu_2)}{(1 - \mu_1)(1 - \mu_2)}
\] (4.25)

where we have assumed the velocities along the \( x \) axis, and the impact parameters along the \( y \) axis. The equation (4.25) can be inverted to give

\[ C \xi_1^{1-\frac{\alpha}{2\pi}} = \frac{V_{21} t}{\alpha + \beta} + i \frac{S}{4\pi} \frac{1 - \mu_1 - \mu_2}{1 - \mu_1 - \mu_2}
\] (4.26)

From Eq. (4.26) we can learn that the spins renormalize the constants describing the trajectory but not the exponent determining instead the scattering angle, which is therefore unaffected, and given by \( \theta = \frac{\Delta R}{2} (1 - \frac{\Delta R}{2\pi})^{-1} \) as in the spinless case. The constant term in the r.h.s. of Eq. (4.26) is expected to be proportional to \( J \), but only the spin part is here determined, because we have neglected the \( O(V) \) terms in Eq. (4.25).

### 4.3 The peripheral non relativistic case \( (S \ll B_{12}) \)

We now expand the projective monodromy transformations of Eq. (2.7) up to next nontrivial order in both \( f \) and \( V_1 \). By referring to a generic particle and by defining
with similar notation for $a$’s and $b$’s, we obtain

$$\tilde{f}(1) = \frac{a}{a^*} f^{(0)}(1) + \frac{b}{a^*} f^{(1)},$$

$$\tilde{f}(3) = \frac{a}{a^*} f^{(0)}(3) - \frac{ab}{a} f^{(1)}(1) \left( f^{(1)} \right)^2 + \left( \frac{a}{a^*} - \frac{|b|^2}{a^*} \right) f^{(2)}(1) + \frac{b}{a^*}.$$  

The first equation yields the quasi-static solution described in Sec. 4.1. The second equation (which is down by two orders) is non-linear, but this time it linearizes for the function $h_1 = \frac{1}{f^{(1)}} \left( f^{(3)}(1)' \right)'$.

which satisfies the 1-st order monodromy conditions

$$\tilde{h}_1 \mid_i = e^{-2i\pi\mu_i} h_1 - V_i (1 - e^{-2i\pi\mu_i}).$$

Furthermore, from the boundary conditions for $f$ we derive the following ones for $h_1$

$$h_1 \sim \begin{cases} 
-V_i + O((\zeta - \zeta_i)^{2-\mu_i}), & \zeta \simeq \zeta_i, \ i = 1, 2 \\
\zeta^{1-\mu_1-\mu_2}, & \zeta \rightarrow \infty \\
(\zeta - \eta_i)^{-1}, & \zeta \rightarrow \eta_i 
\end{cases}$$

where the values $h_1 \mid_i = -V_i$ are needed to realize the translational part of the monodromy (4.28).

The solution for $h_1$ can be found from the ansatz

$$h_1 = -V_1 + (V_1 - V_2) \left[ I_0(\zeta) + \frac{1}{B(1-\mu_1, 1-\mu_2)} \frac{\zeta^{1-\mu_1} (1 - \zeta)^{1-\mu_2} (A_1(\zeta - 1) + A_2\zeta)}{(\zeta - \eta_1)(\zeta - \eta_2)} \right]$$

by noticing that the first two terms automatically satisfy the translational part of the monodromy (4.28), so that the third one should have the purely static monodromies $e^{-2i\pi\mu_i}$, besides the poles at $\zeta = \eta_i$. The constants $A_1$ and $A_2$ can then be chosen so as to satisfy $h + V_i \sim (\zeta - \zeta_i)^{2-\mu_i}$. Since, by (3.9), $\eta_1\eta_2 = \sigma_1$, $(1 - \eta_1)(1 - \eta_2) = \sigma_2$, we find
\[ A_1 = \frac{\sigma_1}{1 - \mu_1} = \frac{s_1}{(1 - \mu_1 - \mu_2)J}, \quad A_2 = \frac{\sigma_2}{1 - \mu_2} = \frac{s_2}{(1 - \mu_1 - \mu_2)J} \] (4.33)

thus making the spinless limit particularly transparent.

From the form (4.29) of \( h_1 \), from its definition and from the form of \( f^{(1)} \) in Eqs. (4.5) and (4.6), we then find the result

\[
\frac{f^{(3)}}{f^{(1)}} = -V_1 f^{(1)} + \text{const.} + \frac{(1 - \mu_1 - \mu_2) \delta M}{2\pi} \int_0^\zeta dt \left( -\frac{A_1}{t} + \frac{A_2}{1 - t} + t^{\mu_1 - 2}(1 - t)^{\mu_2 - 2} (t - \eta_1) (t - \eta_2) \int_0^t d\tau \tau^{-\mu_1} (1 - \tau)^{-\mu_2} \right),
\] (4.34)

where we have defined the parameter

\[
\delta M = |V_{21}|^2 \frac{\sin \pi \mu_1 \sin \pi \mu_2}{\sin \pi (\mu_1 + \mu_2)} = [M - (m_1 + m_2)]^{(2)},
\] (4.35)

representing the nonrelativistic contribution to the total invariant mass. In fact, the perturbative large \( \zeta \) behaviour of \( f' \) provided by Eq. (4.31) is

\[
\frac{f'_{(1)} + f'_{(3)}}{f'_{(1)}} \sim 1 + \frac{\delta M}{2\pi} \log \zeta \sim \zeta^{\frac{\delta M}{2\pi}}
\] (4.36)

as expected from the behaviour \( f' \sim \zeta^{\frac{\delta M}{2\pi}} \) of the full solution.

We further notice that Eq. (4.31) provides a non-trivial change of the Schwarzian derivative, similarly to what was noticed in the previous sections. Since

\[
\{f, \zeta\} = L'' + \frac{1}{2} L'^2, \quad L \equiv \log f^{(1)} + \frac{f''}{f'} + ... \] (4.37)

we have, after some algebra, the non-relativistic correction to the Schwarzian (Cfr. App. B)

\[
\{f, \zeta\} - \{f, \zeta\}^{(0)} = f^{(1)} h' + ... = \frac{\delta M}{2\pi} \frac{1}{1 - \frac{2}{J}} \left[ -\frac{1}{\zeta(1 - \zeta)} \left( 1 - \mu_1 - \mu_2 - (3 - \mu_1 - \mu_2) \frac{s_1 + s_2}{J} \right) + \left( \frac{s_1}{J} \frac{1}{\zeta - \eta_1} - \frac{s_2}{J} \frac{1}{1 - \zeta} \right) \left( \frac{1}{\zeta - \eta_1} + \frac{1}{\zeta - \eta_2} \right) \right],
\] (4.38)

where \( \{f, \zeta\}^{(0)} \) denotes the static expression in Eqs. (3.10) and (3.20).
The same expression (4.38) could have been obtained by expanding Eqs. (4.1)-(4.2) in the parameter $\delta \mathcal{M}$, around the static solution of Eq. (3.20) (Appendix B).

Let us note that the solution here considered contains more information than simply the terms of order $O(V^2)$, because the expansion of the $1 - \frac{S}{T}$ denominator can give rise in the small velocity limit to a mixed perturbation both in $O(V^2)$ and in $O(V) \cdot O(S/B_{12})$. In fact from Eq. (4.38) we can rederive, to this order, the Schwarzian given in Eq. (4.20).

5 Discussion

We have shown here that the BCV gauge can be extended to the case of spinning particles in 2+1-Gravity, in the region external to some "CTC horizons" that occur around the particles themselves.

Let us note that the existence of the BCV gauge is in general related to the lack of CTC’s. In fact, in our conformal Coulomb gauge the proper time element is of the form

$$ds^2 = \alpha^2 dt^2 - e^{2\phi} |dz|^2 - \beta dt|^2$$  \hspace{1cm} (5.1)

and, if some instantaneous motion is possible ($dt = 0$) the proper time

$$ds^2 = -e^{2\phi} |dz|^2 \quad (dt = 0)$$  \hspace{1cm} (5.2)

is necessarily spacelike, unless

$$e^{2\phi} \simeq |g| \simeq (1 - |f|^2)^2 = 0,$$  \hspace{1cm} (5.3)

a case in which it becomes light-like.

Therefore, the (closed) curves defined by $|f| = 1$ are, at the same time, the boundary for the existence of CTC’s, and also for the existence of the gauge itself, because the metric determinant vanishes. This is not surprising, because our gauge allows the definition of a single-valued global time, which is expected to be impossible in the case of CTC’s.

We have provided explicit solutions for the mapping function in various cases. Firstly for $N$ spinning particles at rest, a case in which the Schwarzian shows $3N - 1$ singularities, and in particular for $N = 2$, a case in which we have also given a closed form for the metric (Eq. 3.22).

Secondly we have also described metric and motion for two spinning particles, in the quasi-static and the non-relativistic cases. In particular, we have shown that it is possible
to determine the motion of the particle sites $\xi_i(t)$ (as singularities of the Schwarzian) by imposing the $O(2,1)$ monodromies on the exterior solution (Sec. 4.2 and Appendix A).

Actually, since the Minkowskian coordinates are sums of analytic and antianalytic functions of the regular ones in the BCV gauge, it turns out that each one of them can be continued in the interior region. This suggests that perhaps the gauge can be extended to the interior region by relaxing the conformal condition.

We feel however that the clear delimitation of CTC horizons, with sizes related to the spins, is actually a quite physical feature of our gauge and points in the direction that a pointlike spin in (2+1)-Gravity is not really a self consistent concept.

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A Analytic continuation of monodromy conditions

We sketch here the proof that the monodromy conditions can be solved by the analytic continuation in the mass range of the integrals defining $f(z)$ and the $X$-mapping.

Let us first discuss the question of defining the polydromy of $f(z)$. Since in the physical mass range $f(z)$ has a divergent behaviour around the particles, its behaviour for $z \to \zeta_i$ cannot be used directly to test the monodromy conditions

$$f(z) \to \frac{a}{a^*} f(z) + \frac{b}{a^*}$$

(A.1)

For example, the quasi-static two-particle solution for $f(z)$ is given by the expression

$$f_{(\xi_0)} = \int_{\xi_0}^z f'(t) dt = K \int_{\xi_0}^z (t - \eta_1)(t - \eta_2)(t - \xi_1)^{\mu_1-2}(\xi_2 - t)^{\mu_2-2} dt$$

(A.2)

which diverges for $t \to \xi_i$ in the physical range $0 < \mu_i < 1$. In order to take the limit $\xi_0 \to \xi_i$ and to verify the monodromy conditions, this integral can be defined by the expression

$$f_{(\xi_0)}(z) \equiv -\frac{1}{(1-\mu_1)(1-\mu_2)} \frac{\partial^2}{\partial \xi_1 \partial \xi_2} K \int_{\xi_0}^z (t - \eta_1)(t - \eta_2)(t - \xi_1)^{\mu_1-1}(\xi_2 - t)^{\mu_2-1} dt$$

(A.3)

which is now convergent at the endpoints $\xi_i$.  

We can then decompose it as follows:

\[ f(\xi_0)(z) = f(\xi_0)(\xi_1) + f(\xi_1)(z) \]  

(A.4)

Imposing the monodromy conditions (A1) we obtain

\[ f(\xi_0)(\xi_i) = -\frac{V_i}{2} \]  

(A.5)

which implies

\[ \frac{\overline{V}_{12}}{2} = -\frac{K}{(1-\mu_1)(1-\mu_2)} \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \int_{\xi_1}^{\xi_2} dt (t-\eta_1)(t-\eta_2)(t-\xi_1)^{\mu_1-1}(t-\xi_2)^{\mu_2-2} \biggr|_{\eta_1,\eta_2 \text{ const.}} \]  

(A.6)

Evaluating this expression in the particular case \( \xi_1 = 0, \xi_2 = 1 \), and substituting the relations (3.13) we obtain:

\[ \frac{\overline{V}_{12}}{2} = -K \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)(\mu_2)} \left[ 1 - \sigma_1 \frac{(1-\mu_1-\mu_2)}{1-\mu_1} - \sigma_2 \frac{(1-\mu_1-\mu_2)}{1-\mu_2} \right] \]

\[ = -K B(\mu_1, \mu_2) \left( 1 - \frac{s_1 + s_2}{J} \right) \]  

(A.7)

which coincides with the result in Eq. (4.8), obtained by analytic continuation in the parameters \( \mu_i \) from \( 2 > \mu_i > 1 \).

Analogously, the \( X \)-mapping for two particles is defined by an integral which is divergent at the particle sites. However we can measure the distance \( X_1 - X_2 \) in the \( X \)-coordinates by looking at the center of rotation of the monodromy around the particle sites.

Consider for example the two-particle static case, where the \( Z \)-mapping can be defined from a generic point \( \zeta_0 \):

\[ Z - Z_{\zeta_0} = \frac{R}{K} \int_{\zeta_0}^{\zeta} dt \, t^{-\mu_1}(1-t)^{-\mu_2} + \overline{R K} \int_{\zeta_0}^{\zeta} dt \, t^{\mu_1-2} (1-t)^{\mu_2-2} (t - \tau)^2 \]  

(A.8)

Integrating by part around particle 1, we can distinguish a divergent term which has simple polydromy properties from a remaining series of convergent integrals:

\[ Z - Z_{\zeta_0} = \frac{R}{K} \int_{\zeta_0}^{\zeta} dt \, t^{-\mu_1}(1-t)^{-\mu_2} + \overline{R K} \frac{\zeta^{\mu_1-1}}{\mu_1-1} (\zeta-1)^{\mu_2-2} (\zeta - \eta)^2 \]
$$- \frac{R K}{\mu_1 - 1} \zeta_0^{\mu_1 - 1} (\zeta_0 - 1)^{\mu_2 - 2} (\zeta_0 - \eta)^2 - \frac{R K}{\mu_1 - 1} \int_{\zeta_0}^{\tilde{\zeta}} dt \ t^{\mu_1 - 1} \frac{\partial}{\partial t} ((1 - t)^{\mu_2 - 2}(t - \eta))$$

Repeating the same reasoning for particle 2 we obtain the distance $Z_{12}$

$$Z_1 - Z_2 = -\frac{R}{K} \int_0^1 dt t^{\mu_1 - 1} (1 - t)^{-\mu_2} - \frac{R K}{\mu_2 - 1} \zeta_0^{\mu_1 - 1} (1 - \zeta_0)^{\mu_2 - 2}$$

$$- \frac{\zeta_0^{\mu_1 - 2} (1 - \zeta_0)^{\mu_2 - 1}}{\mu_2 - 1} + \frac{R K}{\mu_2 - 1} \int_{\zeta_0}^1 dz (1 - z)^{\mu_2 - 1} \frac{\partial}{\partial z} (z^{\mu_1 - 2}(z - \eta)^2) -$$

$$- \frac{R K}{\mu_1 - 1} \int_{\zeta_0}^0 dz z^{\mu_1 - 1} \frac{\partial}{\partial z} ((1 - z)^{\mu_2 - 2}(z - \eta)^2)$$

This difference is a finite measure of the distance between the particles in Minkowskian coordinates. It seems to depend on the choice of the arbitrary point $\xi_0$ but in fact it is independent, because of the relation:

$$\frac{\partial}{\partial \xi_0} (Z_1 - Z_2) = 0$$

In virtue of this property, the complete evaluation coincides with the analytic continuation of the integral in Eq. (4.25) from a range where the mass parameters give a finite result.

### B Perturbative expansion of the Schwarzian Derivative

Given the Schwarzian derivative as

$$\left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 = 2q(z) = \frac{1}{2} \left( \frac{1 - \mu_i^2}{(z - z_i)^2} + \frac{2 \beta_i}{z - z_i} \right)$$

we can define the lowest order $2q_0(z)$, given by the choice of the parameter $\mu_\infty = \sum_i \mu_i - 1$, which defines a function $f(0)$. In general, by taking as development parameter

$$\delta = \frac{M}{2\pi} - \sum_i \mu_i$$

the correction term in the Schwarzian at first order in $\delta$ is given by a sum of poles:
which defines a first order correction to \( f(z) \) given by:

\[
\begin{align*}
2q_1(z) &= \frac{\beta_i^{(1)} - \beta_i^{(0)}}{z - z_i} \\
\end{align*}
\] (B.3)

This equation can be easily solved to give the general first-order solution

\[
\begin{align*}
g(z) &= c_1 + c_2 f_0(z) + \int_{\xi_1}^{z} dt f_0'(t) \int_{\xi_1}^{t} \frac{2q_1(w)}{f_0'(w)} dw \\
\end{align*}
\] (B.5)

The constants \( c_1, c_2 \) and the normalization of \( f_0 \) have to be chosen by imposing the monodromy conditions for \( f \), which gives, for example, \( c_2 = -V_1 \). This representation of the solution to the Schwarzian contains explicit logarithmic terms around the apparent singularities which cancel only in the case that the residues \( \beta_i^{(1)} \)'s satisfy the non-logarithmic conditions (4.2).

In that case, a simple integration by parts allows to eliminate the poles in the apparent singularities \( \eta_i \) inside the integral (B.5) reproducing the results (4.17), (4.38) given in the text.

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