LQG MEAN FIELD GAMES WITH A MARKOV CHAIN AS ITS COMMON NOISE∗

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Abstract. This paper studies Mean Field Games with a common noise given by a continuous time Markov chain under a quadratic cost structure. The theory reveals the Markovian structure of the random equilibrium measure flow, which can be used to characterize the equilibrium via a finite dimensional system. The counterpart is the N-player game characterized via an ODE system, whose dimension has a polynomial growth in the number of players. The convergence of the N-player game towards Mean Field Games is proved by explicitly embedding the N-player game into one specific probability space.

Key words. Mean Field Games, Regime-switching diffusion, N-player game, Convergence, Common noise, Riccati system

AMS subject classifications. 91A16, 93E20

1. Introduction. Mean Field Game (MFG) theory has attracted resurgent attention from numerous researchers in probability after its pioneering works of [15, Lasry and Lions] and [14, Huang, Caines, and Malhame]. An important recent development in this direction is Mean Field Games with a common noise and we refer to comprehensive descriptions to the book [5, Carmona and Delarue] and the references therein.

Along another line, Linear-Quadratic (LQ) control problems have been widely recognized in the stochastic control theory due to their broad applications. The optimal path is Gaussian under the LQ structure and the problem is also called LQG to emphasize this Gaussian property, see for instance [19]. More importantly, LQ structure leads to solvability in a closed form, namely the Ricatti system, and this usually sheds light on many fundamental properties of the control theory. For this reason, LQ structure has also been studied in MFGs with or without common noises for its importance. The related literature include major and minor LQ Mean Field Games system ([12, 16, 11, 7]); social optimal in LQG Mean Field Games ([13, 6]); the LQG Mean Field Games with different model settings ([9, 3, 10]); and LQG Graphon Mean Field Games ([8]).

This paper studies a class of Mean Field Game problems and their N-player game counterparts in the context of LQ structure with a common noise. Different from the aforementioned works, the common noise in this paper is a continuous time Markov chain (CTMC) instead of Brownian motion. By using CTMC, the applications aim to model less frequently changing common noises, such as government policies implemented by two different regimes. Recently, LQ Mean Field Games but with a Brownian motion as the common noise has been studied in ([1, 18]) with restrictions of the dependence of measure on its mean alone. Another similar setting to our paper is the recent work [17], which studies a Mean Field Control problem essentially different from our game problem setting. Yet, there is no discussion of convergence for the N-player game to MFG among all aforementioned papers.

One advantage of using CTMC is that the underlying problem is more explainable relative to the Brownian motion as the common noise. However, the current problem is challenging enough since it shares the same difficulty with MFGs with the Brownian motion as the common noise, that is, the equilibrium measure exhibits a path-dependent feature on the common noise. Therefore, the characterization of MFGs with the common noise often leads to an infinite dimensional system, which includes the Master equation, stochastic HJB-FPK system, forward-backward stochastic differential equations (FBSDE). We refer to more comprehensive analyses to Volume II of [5].

One of our two major contributions in this paper is the characterization of MFG equilibrium by a finite dimensional system. It is noted that the first and second moment of equilibrium measure are both path-dependent on the common noise, and therefore there exists no finite dimensional system directly available to characterize the equilibrium. Interestingly, our research reveals the Markovian structure of the random equilibrium measure flow, which becomes the key step leading us to a finite dimensional system. In other words, instead of searching directly for the infinite dimensional system for the equilibrium, one can characterize the equilibrium by an Itô diffusion process with its coefficients determined by a
finite system of equations. As a result, although the generic player’s underlying path is not originally
given regime-switching dependent, the path at MFG equilibrium turns out to be a regime-switching
Ornstein–Uhlenbeck process with its coefficients determined by a 2-dimensional ODE system.

Some results relevant to our first contribution can be found in the recent papers [1, 18] in the context
of MFGs with the Brownian motion as the common noise. However, both papers have their mean field
term only via the mean process, but not the second moment as in our paper. By bringing in the second
moment, the underlying control problem is not a typical LQG setting due to the dependence of moment
processes on the state being a quadratic function, which gives extra difficulties. Moreover, [18] provides
the solution via FBSDE, which is an infinite dimensional system. In this context, [1] is more close to our
result in that it provides a finite dimensional system for the solution. However, their approach is different
from ours since they derive the finite dimensional ODE system from the Master equation.

Our second contribution, yet the more important one compared to the first one, is the proof for the
convergence of the $N$-player game to MFGs. One of the common convergence results in MFG theory is
that the equilibrium strategy of MFGs provides an $\epsilon$-equilibrium to the $N$-player game for sufficiently
large $N$, see Theorem 3.8 of [4]. Our convergence is different from the existing $\epsilon$-equilibrium result.
We show that the generic player’s state of the $N$-player game is convergent in distribution to that of
MFGs, which provides fundamentally different perspectives. None of the aforementioned papers include
discussions on the convergence in this sense. Roughly speaking, our convergence result is natural and
intuitive: The generic player’s behavior in $N$-player game is similar to the one of MFG for a large $N$
and this is exactly what the MFG is designed for as an asymptotic setting of $N$-player game.

To prove this weak convergence, we embed the generic player’s path of the $N$-player game to the
sample space of MFG setting, which enables us to compare the difference of paths between the $N$-player
game and MFGs in almost sure sense. Having said that, the main difficulty is that this embedding is
indeed not always possible since the sample space for the $N$-player game generated by $N$-dimensional
Brownian motion is much richer than the one for MFGs with only two-dimensional Brownian motion. In
addition, the generic player of the $N$-player game follows a path generated by $N$-dimensional Brownian
motion through coefficients determined from a huge ODE system consisting of $O(N^3)$ equations. Never-
theless, an intriguing phenomenon we want to highlight is that, there exists a hidden algebraic pattern
to the coefficients invariant to the number of players. This pattern eventually enables us to reduce the
representation of the generic path from a functional of $N$-dimensional Brownian motion to a functional of
two-dimensional Brownian motion, no matter how large the number $N$ is. Indeed, the pattern leading to
the success of the above embedding procedure is precisely accounted for the dimension-invariant feature
of the mean field terms at the equilibrium, which provides a new insight distinguished from the existing
results in the literature.

The rest of this paper is outlined as follows: Section 2 presents a precise formulation of the problem
and two main results. Section 3 is devoted to the derivation of our first result: the equilibrium of MFGs.
In Section 4, we show in detail the convergence of the $N$-player game to MFGs, which yields our second
main result. Section 5 demonstrates the convergence by some numerical examples. Section 6 is an
appendix, collecting some related facts to support our main theme.

2. Problem setup and Main results. First, we collect common notations used in this paper in
Subsection 2.1. Then, we set up problems on MFGs and the $N$-player game separately in Subsections 2.2
and 2.3. The main results are presented in Subsection 2.4 and some interpretations on our main results
are added in Subsection 2.5.

2.1. Notations. Let $T > 0$ be a fixed terminal time and $(\Omega, \mathcal{F}_T, \mathbb{P} = \{\mathcal{F}_t : 0 \leq t \leq T\}, \mathbb{P})$
be a completed filtered probability space satisfying the usual conditions, on which $W$ and $B$ are two
independent standard Brownian motions, and $Y$ is a continuous time Markov chain (CTMC) independent
of $(W, B)$ taking values in $\{0, 1\}$ with a generator

\begin{equation}
Q = \begin{bmatrix}
-\gamma_0 & \gamma_0 \\
\gamma_1 & -\gamma_1
\end{bmatrix},
\end{equation}

for some $\gamma_0 \geq 0, \gamma_1 \geq 0$. The Brownian motion $B$ does not play any role in MFG problem formulation
until the convergence proof of the $N$-player game to MFGs.
By $L^p := L_p^p(\Omega, \mathbb{P})$, we denote the space of random variables $X$ on $(\Omega, \mathcal{F}_T, \mathbb{P})$ with finite $p$-th moment with norm $\|X\|_p = (\mathbb{E} \left[ |X|^p \right])^{1/p}$. We also denote by $L^p_T := L_p^p([0, T] \times \Omega)$ the space of all $\mathbb{P}$-progressively measurable random processes $\alpha = (\alpha_t)_{0 \leq t \leq T}$ satisfying

$$\mathbb{E} \left[ \int_0^T |\alpha_t|^p dt \right] < \infty.$$ 

For any polish (complete separable metric) space $(S, \mathcal{B}(S), d)$, we use $\delta_x$ to denote the Dirac measure on the point $x \in S$. Then, the collection of all probabilities $m$ on $(S, \mathcal{B}(S), d)$ having finite $k$-th moment is denoted by $\mathcal{P}_k(S)$, i.e.

$$[m]_k := \int x^k m(dx) < \infty, \quad \forall m \in \mathcal{P}_k(S).$$

The equilibrium of MFGs with the common noise yields the conditional distribution. For real valued random variables $X$ and $Z$ in $(\Omega, \mathcal{F}_T, \mathbb{P})$, we denote the distribution of $X$ conditional on $\sigma(Z)$ by $\mathcal{L}(X|Z)$, or equivalently

$$\mathcal{L}(X|Z)(A) = \mathbb{E}[I_A(X)|Z], \quad \forall A \in \mathcal{F}_T.$$ 

Note that $\mathcal{L}(X|Z)(A) : \Omega \mapsto \mathbb{R}$ is a $\sigma(Z)$-measurable random variable, therefore, $\mathcal{L}(X|Z)$ is $\sigma(Z)$-measurable random probability distribution with $k$-th moment $[\mathcal{L}(X|Z)]_k = \mathbb{E}[X^k|Z]$, if it exists. We refer to more details on the conditional distribution in Volume II of [5]. Next proposition provides embedding approach to prove a convergence in distribution, which will be used later in the convergence of the $N$-player game to MFGs.

**Proposition 2.1.** Suppose $(\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{P}^{(N)})$ is a complete probability space. Let $X^{(N)}$ and $X$ be random variables of $\Omega^{(N)} \rightarrow S$ and $\Omega \rightarrow S$, respectively. Then, $X^{(N)}$ is convergent in distribution to $X$, denoted by $X^{(N)} \Rightarrow X$, if there exists $Z^N : \Omega \mapsto S$ satisfying $\mathcal{L}(Z^N) = \mathcal{L}(X^{(N)})$, such that $Z^N \rightarrow X$ holds almost surely, i.e.

$$\lim_{N \rightarrow \infty} d(Z^N, X) = 0, \quad \text{almost surely in } \mathbb{P},$$

where $d$ represents the metric in $S$.

In this paper, we formulate the $N$-player game in the completed filtered probability space

$$(\Omega^{(N)}, \mathcal{F}_T^{(N)}, \mathbb{P}^{(N)} := \{ \mathcal{F}_t^{(N)} : 0 \leq t \leq T \}, \mathbb{P}^{(N)}),$$

and $Y^{(N)}$ is the continuous time Markov chain in $\Omega^{(N)}$ with the same generator given by (2.1) and $W^{(N)} = (W_i^{(N)} : i = 1, \ldots, N)$ is a $N$-dimensional standard Brownian motion. We assume $Y^{(N)}$ and $W^{(N)}$ are independent of each other.

For better clarity, we use the superscript $(N)$ for a random variable to emphasize the probability space $\Omega^{(N)}$ it belongs to. For example, Proposition 2.1 denotes a random variable in $\Omega^{(N)}$ by $X^{(N)}$, while its distribution copy in $\Omega$ by $Z^N$, but not by $Z^{(N)}$.

**2.2. The equilibrium of MFGs.** In this section, we define the equilibrium of MFGs associated to a generic player’s stochastic control problem in the probability setting $\Omega$, see Section 2.1.

Given a random measure flow $m : [0, T] \times \Omega \mapsto \mathcal{P}_2(\mathbb{R})$, consider a generic player who wants to minimize her expected accumulated cost on $[0, T]$:

$$J(y, x, \alpha) = \mathbb{E} \left[ \int_0^T \frac{1}{2} \alpha_s^2 + F(Y_s, X_s, m_s)ds + G(Y_T, X_T, m_T) \bigg| Y_0 = y, X_0 = x \right]$$

with some given cost functions $F, G : \{0, 1\} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ and underlying random processes $(Y, X) : [0, T] \times \Omega \mapsto \{0, 1\} \times \mathbb{R}$. Among three processes $(Y, X, m)$, the generic player can control the process $X$ via $\alpha$ in the form of

$$X_t = X_0 + \int_0^t \alpha_s ds + W_t, \quad \forall t \in [0, T].$$
We assume that the initial state $X_0$ is independent of $Y$. The process $Y$ of (2.1) represents the common noise and $m = (m_t)_{0 \leq t \leq T}$ is a given random density flow normalized up to total mass one.

The objective of the control problem for the generic player is to find its optimal control $\hat{\alpha} \in A := L^4_T$ to minimize the total cost, i.e.

$$V(y, x) = J(y, x, \hat{\alpha}) \leq J(y, x, \alpha), \quad \forall \alpha \in A.$$  

(2.4)

Associated to the optimal control $\hat{\alpha}$, we denote the optimal path by $\hat{X} = (\hat{X}_t)_{0 \leq t \leq T}$. To introduce MFG Nash equilibrium, it is often convenient to highlight the dependence of the optimal path and optimal control of the generic player and its associated value on the underlying density flow $m$, which are denoted by

$$\hat{X}_t[m], \hat{\alpha}_t[m], \text{ and } V[m],$$

respectively. Now, we present the definition of the equilibrium below, see also Volume II-P127 of [5] for a general setup with a common noise.

**Definition 2.2.** Given an initial distribution $\mathcal{L}(X_0) = m_0 \in \mathcal{P}_2(\mathbb{R})$, a random measure flow $\hat{m} = \hat{m}(m_0)$ is said to be a MFG equilibrium measure if it satisfies fixed point condition

$$\hat{m}_t = \mathcal{L}(\hat{X}_t | \hat{m}), \quad \forall 0 < t \leq T, \text{ almost surely in } \mathbb{P}$$  

(2.5)

The path $\hat{X}$ and the control $\hat{\alpha}$ associated to $\hat{m}$ is called as the MFG equilibrium path and equilibrium control, respectively. The value function of the control problem associated to the equilibrium measure $\hat{m}$ is called as MFG value function, denoted by

$$U(m_0, y, x) = V[\hat{m}](y, x).$$  

(2.6)

The flowchart of MFGs diagram is given in Figure 1. It is noted from the optimality condition (2.4) and the fixed point condition (2.5) that

$$J[\hat{m}](y, x, \hat{\alpha}) \leq J[\hat{m}](y, x, \alpha), \quad \forall \alpha$$

holds for the equilibrium measure $\hat{m}$ and its associated equilibrium control $\hat{\alpha}$, while it is not

$$J[m](y, x, \hat{\alpha}) \leq J[m](y, x, \alpha), \quad \forall \alpha, m.$$  

Otherwise this problem turns into a McKean-Vlasov control problem discussed in [17].
2.3. Equilibrium of the $N$-player game. The discrete counterpart of MFGs is $N$-player game, which is formulated below in the probability space $\Omega^{(N)}$, see Section 2.1 for more details on the probability setup.

Recall that, $W^{(N)}_{it}$ and $W^{(N)}_{jt}$ are independent Brownian motions for $j \neq i$ and the common noise $Y^{(N)}$ is the continuous time Markov chain in $\Omega^{(N)}$ with the generator given by (2.1). Let the player $i$ follow the dynamic, for $i = 1, 2, \ldots, N$,

\begin{equation}
-dX^{(N)}_{it} = \alpha^{(N)}_{it} dt + dW^{(N)}_{it}, \quad X^{(N)}_{i0} = x_i^N.
\end{equation}

In the above, the initial state is denoted by $x_i^N$ instead of $x_i^{(N)}$, since $x_i^N$ is independent of the choice of the sample $\omega^{(N)} \in \Omega^{(N)}$ as a constant.

The cost function for player $i$ associated to the control $\alpha^{(N)} = (\alpha_i^{(N)} : i = 1, \ldots, N) \in A^N$ is

\begin{equation}
J_i^N(y, x^N, \alpha^{(N)}) = \mathbb{E} \left[ \int_0^T \left( \frac{1}{2} |\alpha_i^{(N)}|^2 + F(Y_i^{(N)}, X_i^{(N)}, \rho(X_i^{(N)})) \right) dt + G(Y_T^{(N)}, X_T^{(N)}, \rho(X_T^{(N)})) \right],
\end{equation}

where $x^N = (x_1^N, x_2^N, \ldots, x_N^N) \in \mathbb{R}^N$ is the initial state for $N$ players and

$$\rho(x^N) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i^N}$$

is the empirical measure of a vector $x^N$ with Dirac measure $\delta$. We use the notation for the control $\alpha^{(N)} = (\alpha_1^{(N)}, \alpha_2^{(N)}, \ldots, \alpha_N^{(N)})$.

Definition 2.3. 1. The value function of player $i$ for $i = 1, 2, \ldots, N$ of the Nash game is defined by $V^N = (V_i^N : i = 1, 2, \ldots, N)$ satisfying the equilibrium condition

\begin{equation}
(2.9) \quad V_i^N(y, x^N) = J_i^N(y, x^N, \hat{\alpha}_i^{(N)}, \hat{\alpha}_{-i}^{(N)}) \leq J_i^N(y, x^N, \alpha_i^{(N)}, \hat{\alpha}_{-i}^{(N)}), \quad \forall \alpha_i^{(N)} \in A.
\end{equation}

2. The equilibrium path of the $N$-player game is the random path $\hat{X}_i^{(N)} = (\hat{X}_{it}^{(N)}, \ldots, \hat{X}_{Ni}^{(N)})$ driven by (2.7) associated to the control $\hat{\alpha}_i^{(N)}$ satisfying the equilibrium condition of (2.9).

3. The generic player’s path at equilibrium is $X_i^{(N)}$, where $u := u^{(N)}$ is a uniform random variable on the set $\{1, 2, \ldots, N\}$ in $\Omega^{(N)}$ independent of $(W^{(N)}, Y^{(N)})$.

2.4. The main result with quadratic cost structures. We consider the following two functions $F, G : \{0, 1\} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R}$ in the cost functional (2.2):

\begin{equation}
(2.10) \quad F(y, x, m) = h(y) \int_\mathbb{R} (x - z)^2 m(dz),
\end{equation}

and

\begin{equation}
(2.11) \quad G(y, x, m) = g(y) \int_\mathbb{R} (x - z)^2 m(dz),
\end{equation}

for some $h, g : \{0, 1\} \rightarrow \mathbb{R}^+$. In this case, the $F$ and $G$ terms in (2.8) of the $N$-player game can be written by

$$F(Y_t^{(N)}, X_i^{(N)}, \rho(X_i^{(N)})) = \frac{h(Y_t^{(N)})}{N} \sum_{j=1}^N (X_j^{(N)} - X_{jt}^{(N)})^2,$$

and

$$G(Y_T^{(N)}, X_i^{(N)}, \rho(X_i^{(N)})) = \frac{g(Y_T^{(N)})}{N} \sum_{j=1}^N (X_j^{(N)} - X_{jt}^{(N)})^2,$$
respectively.

First, we note that $F$ and $G$ possess the quadratic structures in $x$. Secondly, the coefficients $h(y)$ and $g(y)$ provide the sensitivity to the mean field effects, which depend on the current CTMC state. For another remark, let us consider the scenario where sensitivities are invariant, say

$$h(0) = h(1) = h, \quad g(0) = g(1) = 0.$$ 

Then the cost function and hence the entire problem is free from the common noise. Interestingly, as shown in the Appendix 6.1, there is no global solution for MFGs when $h < 0$, while there is global solution when $h > 0$. Therefore, we require positive values for all sensitivities for simplicity. It is of course an interesting problem to investigate the explosion when some sensitivities take negative.

Wrapping up the above discussions, we impose the following assumptions:

(A1) The cost functions are given by (2.10)-(2.11) with $h, g > 0$; the initial $X_0$ of MFGs satisfies $\mathbb{E}[X_0^2] < \infty$.

(A2) In addition to (A1), as $N \to \infty$, the initial $\rho(x^N)$ of the $N$-player game is weakly convergent to the initial $\mathcal{L}(X_0)$ of MFGs.

Our objective of this paper is to understand the Nash equilibrium of MFGs and its connection to the $N$-player game equilibrium:

(P1) With Assumption (A1), characterize the MFG equilibrium path $\hat{X}$ and the value function $U$, as well as associated equilibrium measure $\hat{\mu}$ along the Definition 2.2.

(P2) With Assumption (A2), prove the convergence of $\hat{X}^{(N)}_{ut}$ from the $N$-player game in Definition 2.3 to $\hat{X}$ from MFGs in Definition 2.2.

For our first main result, we present the Riccati system for $(a_y, b_y, c_y, k_y : y = 0, 1)$:

$$
\begin{bmatrix}
  a_y' - 2a_y^2 - \gamma_y a_y + \gamma_y a_{1-y} + h_y = 0, \\
  b_y' - 4a_y b_y - \gamma_y b_y + \gamma_y b_{1-y} + h_y = 0, \\
  c_y' + a_y + b_y - \gamma_y c_y + \gamma_y c_{1-y} = 0, \\
  k_y' - 2a_y^2 + 4a_y b_y - \gamma_y k_y + \gamma_y k_{1-y} = 0, \\
  a_y(T) = b_y(T) = c_y(T) = k_y(T) = 0,
\end{bmatrix}
$$

(2.12)

where $h_y = h(y)$, $g_y = g(y)$ for $y = 0, 1$.

**Theorem 2.4 (Riccati).** Under (A1), there exists a unique solution $(a_y, b_y, c_y, k_y : y = 0, 1)$ for the Riccati system (2.12). With these solutions, the MFG equilibrium path $\hat{X} = \hat{X}[\hat{\mu}]$ is given by

$$d\hat{X}_t = 2a_{Y_t}(t)(\mathbb{E}[X_0] - \hat{X}_t)dt + dW_t, \quad \hat{X}_0 = X_0,$$

(2.13)

with equilibrium control

$$\hat{\alpha}_t = 2a_{Y_t}(t)(\mathbb{E}[X_0] - \hat{X}_t).$$

(2.14)

Moreover, the value function $U$ is

$$U(m_0, y, x) = a_y(0)x^2 - 2a_y(0)x[m_0]_1 + k_y(0)[m_0]_2^2 + b_y(0)[m_0]_1 + c_y(0), \quad y = 0, 1.$$ 

**Theorem 2.5 (Convergence).** Under Assumption (A2), $(\hat{X}^{(N)}_{ut}, Y^{(N)}_t)$ of the $N$-player game converges in distribution to the MFG equilibrium $(\hat{X}_t, Y_t)$ for any $t \in (0, T]$.

2.5. Remarks on the main results. The Nash equilibrium of the $N$-player game can be considered as $N$-coupled stochastic control problem. With the presence of the quadratic cost, the problem can be solved in the framework of LQG via Riccati system (4.1) in the subsequent part. However, the number of unknowns in this Riccati system is in the order of $O(N^3)$, which means the complexity of the solution has a polynomial growth in the number of players $N$.

To reduce the complexity, one can solve MFGs instead of solving the huge Riccati system. In our case, the MFG equilibrium control (2.14) of Theorem 2.4 suggests that a player in the $N$-player game
shall steer towards the population center
\[ \tilde{x}^N = \frac{1}{N} \sum_{i=1}^{N} x_i^N \approx E[X_0] \]
at a velocity proportional to her distance to the center \( \tilde{x}^N - X_{it}^{(N)} \) with the proportionality \( 2a_y(t) \) simply determined from an ODE system of two equations:
\[ a_y' - 2a_y^2 - \gamma_y a_y + \gamma_1 a_{1-y} + b_y = 0, \quad a_y(T) = g_y, \quad y = 0, 1. \]

The above fact is exactly the essence of the MFG as its presence as the asymptotic version of \( N \)-player game and it can be demonstrated by numerical computations, see Section 5. Theorem 2.5 provides the infinitely many conditions:
\[ m(\text{procedure is called embedding and it is not a trivial matter. To see this, since } \Omega \text{ is much richer than } \Omega \text{ having only 2-dimensional Brownian motion } W,B, \text{ it is in general impossible to replicate the distribution of any random variable from } \Omega^{(N)} \text{ to } \Omega. \]
The reason having such embedding is exactly due to the dimension-invariant feature of the mean field \( N \)-player game for its approximation. To prove the convergence in distribution, we construct \( Z_t^N \) in \( \Omega \) such that \( \mathcal{L}(Z_t^N, Y) = \mathcal{L}(X_{ut}^{(N)}, Y^{(N)}) \) and then prove the almost sure convergence \( Z_t^N \rightarrow X_t \). This procedure is called embedding and it is not a trivial matter. To see this, since \( \Omega^{(N)} \) accommodating \( N \)-dimensional Brownian motion \( W^{(N)} \) is much richer than \( \Omega \) having only 2-dimensional Brownian motion \( (W,B) \), it is in general impossible to replicate the distribution of any random variable from \( \Omega^{(N)} \) to \( \Omega \). The reason having such embedding is exactly due to the dimension-invariant feature of the mean field terms at equilibrium, see more details in the proof of Lemma 4.3. The crucial observation towards this decomposition is the pattern of \( N \times N \) matrix \( A_{iy} \) described in Table 1 and equation (4.4).

3. Riccati system for MFGs. This section is devoted to the proof of the first main result Theorem 2.4 on the MFG solution. First, we outline the scheme based on the Markovian structure of the equilibrium by reformulating the MFG problem in Subsection 3.1. Next, we solve the underlying control problem in Subsection 3.2 and provide the corresponding Riccati system. Finally, Subsection 3.3 proves Theorem 2.4 by checking the fixed point condition of MFG problem.

3.1. Overview. By Definition 2.3, to solve for the equilibrium measure, one shall search the infinite dimensional space of the random measure flows \( m : (0, T) \times \Omega \rightarrow \mathcal{P}_2(\mathbb{R}) \), until a measure flow satisfies the fixed point condition \( m_t = \mathcal{L}(\hat{X}_t|Y), \forall t \in (0, T] \), see Figure 1, which is equivalent to check the following infinitely many conditions:
\[ [m_t]_k = E[\hat{X}_t^k|Y], \quad \forall k = 1, 2, \ldots, \]
if they exist.

The first observation is that the cost functions \( F \) and \( G \) in (2.10)-(2.11) are dependent on the measure \( m \) only via the first two moments:
\[ F(y, x, m) = h(y)(x^2 - 2x[m]_1 + [m]_2), \]
\[ G(y, x, m) = g(y)(x^2 - 2x[m]_1 + [m]_2). \]
Therefore, the underlying stochastic control problem for MFGs can be entirely determined by the input given by \( \mathbb{R}^2 \)-valued random process \( \mu_t = [m]_1 \) and \( \nu_t = [m]_2 \), which implies that the fixed point condition can be effectively reduced to check two conditions only:
\[ \mu_t = E[\hat{X}_t|Y], \quad \nu_t = E[\hat{X}_t^2|Y]. \]
This observation effectively reduces our search from the space of random measure-valued processes \( m : (0, T) \times \Omega \rightarrow \mathcal{P}_2(\mathbb{R}) \) to the space of \( \mathbb{R}^2 \)-valued random processes \( (\mu, \nu) : (0, T) \times \Omega \rightarrow \mathbb{R}^2 \).

Note that, if underlying MFGs have no common noise \( Y \), then \( (\mu, \nu) \) is a deterministic mapping \( [0, T] \mapsto \mathbb{R}^2 \) and the above observation is enough to reduce the original infinite dimensional MFGs into a finite dimensional system. However, the following example shows that this is not the case for MFGs with a common noise and it becomes the main drawback to characterize MFGs via a finite dimensional system.
To illustrate, we consider the following uncontrolled mean field dynamics: Let the mean field term
\[ \mu_t := \mathbb{E}[\hat{X}_t | Y], \]
where the underlying dynamic is given by
\[ d\hat{X}_t = -\mu_t Y_t dt + dW_t. \]

- \( \mu_t \) is path dependent on \( Y \), i.e.
  \[ \mu_t = \mu_0 \exp \left\{ -\int_0^t Y_s ds \right\}. \]

This implies that no finite dimensional system is possible to characterize the process \( \mu_t \), since the \((t, Y) \mapsto \mu_t \) is a function on an infinite dimensional domain.

- \( \mu_t \) is Markovian, i.e.
  \[ d\mu_t = -Y_t \mu_t dt. \]

It might be possible to characterize \( \mu_t \) via a function \((t, Y_t, \mu_t) \mapsto \frac{d\mu_t}{dt} \) on a finite dimensional domain.

To solidify the above idea, we need to postulate the Markovian structure for the first and second moment of the MFG equilibrium. More precisely, our search for the equilibrium will be confined to the space \( \mathcal{M} \) of measure flows whose first and second moment exhibits Markovian structure.

**Definition 3.1.** The space \( \mathcal{M} \) is the collection of all \( \mathcal{F}_t^Y \)-adapted measure flows \( m : [0, T] \times \Omega \rightarrow \mathcal{P}_2(\mathbb{R}) \), whose first moment \([m_t]_1 := \mu_t\) and second moment \([m_t]_2 := \nu_t\) satisfy

\[
\begin{align*}
\mu_t &= \mu_0 + \int_0^t (w_0(Y_s, s)\mu_s + w_1(Y_s, s)) \, ds, \quad \forall t \in [0, T] \\
\nu_t &= \nu_0 + \int_0^t (w_2(Y_s, s)\mu_s + w_3(Y_s, s)\nu_s + w_4(Y_s, s)\mu_s^2 + w_5(Y_s, s)) \, ds, \quad \forall t \in [0, T]
\end{align*}
\]

for some smooth deterministic functions \((w_i : i = 0, \ldots, 5)\).

![Fig. 2: Equivalent MFGs diagram with \( \mu_0 = [m_0]_1 \) and \( \nu_0 = [m_0]_2 \).](image)

The flowchart for our equilibrium is depicted in Figure 2. Subsection 3.2 covers the derivation of the Riccati system for the LQG system with a given population measure flow \( m \in \mathcal{M} \), which provides the key building block to MFGs. In Subsection 3.3, we check the fixed point condition and provide a finite dimensional characterization of MFGs, which gives the first main result Theorem 2.4.

**3.2. The generic player’s control with a given population measure.** The advantage of the generic player’s control problem associated with \( m \in \mathcal{M} \) is that its optimal path can be characterized via the following classical stochastic control problem:
Next, we turn to the solution of the control problem (P3). Denote function $v$ by $(3.3)$:

$$v_{\alpha}(y,x,t,\bar{\mu},\bar{\nu})$$

Furthermore, the optimal control has to admit the feedback form of $y$ whenever they have a variable $(3.2)$:

$$V_{\alpha}(y,x,t,\bar{\mu},\bar{\nu})$$

underlying the processes $(Y,X,\mu,\nu)$ defined through (2.1)-(2.3)-(3.1) with the finite dimensional cost functions: $F,G: \mathbb{R}^4 \rightarrow \mathbb{R}$ given by

$$F(y, x, \bar{\mu}, \bar{\nu}) = h(y)(x^2 - 2x\bar{\mu} + \bar{\nu}),$$

$$G(y, x, \bar{\mu}, \bar{\nu}) = g(y)(x^2 - 2x\bar{\mu} + \bar{\nu}),$$

where $\bar{\mu}, \bar{\nu}$ are scalars, while $\mu, \nu$ are used as processes.

**Lemma 3.2.** Given $m \in \mathcal{M}$ associated with $w = (w_i : i = 0, \ldots, 5)$, the player's value (2.4) under assumption (A1) is

$$U[m_0](y,x) = V(y, x, 0, [m_0]_1, [m_0]_2),$$

and the optimal control has a feedback form

$$\bar{\alpha}_t = \bar{\alpha}(Y_t, X_t, t, \mu_t, \nu_t)$$

underlying the processes $(Y, X, \mu, \nu)$ defined through (2.1)-(2.3)-(3.1), whenever there exists a feedback optimal control $\bar{\alpha}$ for the problem (P3).

**Proof.** This is due to the quadratic cost structure in (2.10)-(2.11)

Next, we turn to the solution of the control problem (P3).

3.2.1. HJB equation. For the simplicity of notations, for each $i \in \{0, 1, 2, 3, 4, 5\}$ and $y \in \{0, 1\}$, denote function $v(y, x, t, \bar{\mu}, \bar{\nu})$ as $v_y$, and denote $w_i(y, t)$ as $w_{iy}$. We apply similar notations for other functions whenever they have a variable $y \in \{0, 1\}$. Formally, under enough regularity conditions, the value function $V$ defined in (P3) is the solution $v$ of the following coupled HJBs

$$\begin{aligned}
\partial_t v_0 &+ \frac{1}{2} \partial_{xx} v_0 - \frac{1}{2} (\partial_x v_0)^2 + \partial_t v_0 (w_{00} \bar{\mu} + w_{10}) + \partial_t v_0 (w_{20} \bar{\mu} + w_{30} \bar{\nu}) + w_{40} \bar{\mu}^2 + w_{50} \bar{\nu} - \gamma_0 v_0 + \gamma_0 v_1 + F_0 = 0, \\
\partial_t v_1 &+ \frac{1}{2} \partial_{xx} v_1 - \frac{1}{2} (\partial_x v_1)^2 + \partial_t v_1 (w_{01} \bar{\mu} + w_{11}) + \partial_t v_1 (w_{21} \bar{\mu} + w_{31} \bar{\nu}) + w_{41} \bar{\mu}^2 + w_{51} \bar{\nu} - \gamma_1 v_1 + \gamma_1 v_0 + F_1 = 0,
\end{aligned}
$$

(3.2)

Furthermore, the optimal control has to admit the feedback form of

$$\begin{aligned}
\bar{\alpha}_t &\leftarrow -\partial_x v(Y_t, X_t, t, \mu_t, \nu_t).
\end{aligned}
$$

Next, we identify what conditions are needed for equating control problem and HJB equation. Denote

$$S = \{ v \in L^\infty : \| \partial_{xx} v \|_\infty + \| \partial_t v \|_\infty < \infty, \| \partial_\mu v \|_\infty < \infty, \| \partial_\nu v \|_\infty < \infty \}.$$

**Lemma 3.3.** Consider the control problem (P3) with some given smooth $w$.

1. (Verification theorem) Suppose there exists a solution $v \in S$ of (3.2). Then, $v_y(x, t, \bar{\mu}, \bar{\nu}) = V(y, x, t, \bar{\mu}, \bar{\nu})$ holds, and an optimal control is provided by (3.3).

2. Suppose that the value function $V$ belongs to $S$, and then $\bar{V}_y(x, t, \bar{\mu}, \bar{\nu}) := V(y, x, t, \bar{\mu}, \bar{\nu})$ solves HJB equation (3.2). Moreover, $\bar{\alpha}$ of (3.3) is an optimal control.
Proof. 1. We first prove the verification theorem. Since $v \in S$, for any admissible $\alpha \in L^2_\mu$, the process $X^\alpha$ is well defined and one can use Dynkin’s formula given by Lemma 6.5 to write
\[
\mathbb{E}[v(Y_T, X_T, T, \mu_T, \nu_T)] = v(y, x, t, \bar{\mu}, \bar{\nu}) + \mathbb{E}\left[\int_t^T G^\alpha(s)v(Y_s, X_s, s, \mu_s, \nu_s)ds\right],
\]
where
\[
G^\alpha f(y, x, s, \bar{\mu}, \bar{\nu}) = \left(\partial_t + a\partial_x + \frac{1}{2}\partial_{xx} + Q + (w_{0y}\bar{\mu} + w_{1y})\partial_{\bar{\mu}} + (w_{2y}\bar{\mu} + w_{3y}\bar{\nu} + w_{4y}\bar{\mu}^2 + w_{5y})\partial_{\bar{\nu}}\right)f(y, x, s, \bar{\mu}, \bar{\nu}).
\]
Note that HJB actually implies that
\[
\inf_{\alpha} \left\{G^\alpha v + \frac{1}{2}a^2\right\} = -\bar{F},
\]
which again implies
\[
-G^\alpha v \leq \frac{1}{2}a^2 + \bar{F}.
\]
Hence, we obtain that for all $\alpha \in L^2_\mu$,
\[
v(y, x, t, \bar{\mu}, \bar{\nu}) = \mathbb{E}\left[\int_t^T G^\alpha(s)v(Y_s, X_s, s, \mu_s, \nu_s)ds\right] + \mathbb{E}\left[\int_t^T \frac{1}{2}a^2(s) + \bar{F}(Y_s, X_s, \mu_s, \nu_s)\right] ds + \mathbb{E}\left[\bar{G}(Y_T, X_T, \mu_T, \nu_T)\right]
\]
\[
= J(y, x, t, \alpha, \bar{\mu}, \bar{\nu}).
\]
In the above, if $\alpha$ is replaced by $\bar{\alpha}$ given by the feedback form (3.3), then since $\partial_x v$ is Lipschitz continuous in $x$, there exists corresponding optimal path $\bar{X} \in L^2_\mu$. Thus, $\bar{\alpha}$ is also in $L^2_\mu$. One can repeat all above steps by replacing $X$ and $\alpha$ by $\bar{X}$ and $\bar{\alpha}$, and $\leq$ sign by $\leq$ sign to conclude that $v$ is indeed the optimal value.

2. The opposite direction of the verification theorem follows by taking $\theta \to t$ for the dynamic programming principle, for all stopping time $\theta \in [t, T]$,
\[
\bar{V}(y, x, t, \bar{\mu}, \bar{\nu}) = \mathbb{E}\left[\int_t^\theta \frac{1}{2}a^2 + \bar{F}(Y_s, X_s, \mu_s, \nu_s)ds + \bar{V}(Y_{\theta}, X_{\theta}, \theta, \bar{\mu}, \bar{\nu})\right] X_t = x, Y_t = y, \mu_t = \bar{\mu}, \nu_t = \bar{\nu},
\]
which is valid under our regularity assumptions on all the partial derivatives.

3.2.2. LQG solution. Note that, the costs $\bar{F}$ and $\bar{G}$ of (P3) are quadratic functions in $(x, \bar{\mu}, \bar{\nu})$, while the drift function of the process $\nu$ of (3.1) is not linear in $(x, \bar{\mu}, \bar{\nu})$. Therefore, the control problem (P3) does not fall into the standard LQG control framework. Nevertheless, similar to the LQG solution, we guess the value function as a quadratic function in the form of
\[
v_y(x, t, \bar{\mu}, \bar{\nu}) = a_y(t)x^2 + d_y(t)x + e_y(t)\bar{\mu} + f_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{\nu} + c_y(t), \quad y = 0, 1.
\]
With the above setup, for $t \in [0, T]$, the optimal control is
\[
\hat{\alpha}_t = -\partial_x v(Y_t, \bar{X}_t, t, \mu_t, \nu_t) = -2a_y(t)\bar{X}_t - d_y(t) - f_y(t)\mu_t,
\]
and the optimal path $\hat{X}$ is

$$d\hat{X}_t = \left(-2a_y(t)\hat{X}_t - d_y(t) - f_y(t)\mu_t\right)dt + dW_t.$$  

Denote the following ODE systems for $y,z$ with terminal conditions

$$d\hat{X}_t = \left(-2a_y(t)\hat{X}_t - d_y(t) - f_y(t)\mu_t\right)dt + dW_t.$$  

(3.6)

with terminal conditions

$$a_y(T) = g_y, \quad b_y(T) = g_y, \quad c_y(T) = 0, \quad d_y(T) = 0, \quad e_y(T) = 0, \quad f_y(T) = -2g_y, \quad k_y(T) = 0.$$  

(3.8)

**Lemma 3.4.** Suppose there exists a unique solution $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y = 0,1)$ to the ODE system (3.7)-(3.8) on $[0, T]$. Then the value function of $(P2)$ is

$$V(y, x, t, \mu, \bar{v}) = v_y(x, t, \bar{\mu}, \bar{v})$$  

(3.9)

for $y = 0,1$ and the optimal control and optimal path are given by (3.5) and (3.6), respectively.

**Proof.** With the form of value function $v_y$ given in (3.4) and the first and second moment of the conditional population density given in (3.1), we have

$$\partial_t v_y = a_y(t)x^2 + d_y(t)x + e_y(t)\bar{\mu} + f_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{v} + c_y(t),$$  

$$\partial_{x} v_y = 2ax_y(t) + d_y(t) + f_y(t)\bar{\mu},$$  

$$\partial_{x^2} v_y = 2a_y(t),$$  

$$\partial_{\bar{\mu}} v_y = e_y(t) + f_y(t)x + 2k_y(t)\bar{\mu},$$  

$$\partial_{\bar{v}} v_y = b_y(t),$$

for $y = 0,1$. Plugging them back to the coupled HJBs in (3.2), we get a system of ODEs in (3.7) by equating $x, \bar{\mu}, \bar{v}$-like terms in each equation.

Therefore, any solution $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y = 0,1)$ of ODE system (3.7) leads to the solution of HJB (3.2) in the form of the quadratic function given by (3.9). Since the $(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y = 0,1)$ are differentiable functions on the closed set $[0, T]$, they are also bounded, and the regularity condition $||\partial_{xx} v||_\infty + ||\partial_t v||_\infty + ||\partial_{x} v||_\infty + ||\partial_{\bar{v}} v||_\infty < \infty$ is valid. Finally, we invoke the verification theorem given by Lemma 3.3 to conclude the desired result.

**3.3. Fix point condition and the proof of Theorem 2.4.** Going back to the ODE system (3.7), there are 14 (7 pairs) equations, while we have total 26 deterministic functions of $[0, T] \times \mathbb{R}$ to be determined to characterize MFGs. Those are

$$(a_y, b_y, c_y, d_y, e_y, f_y, k_y : y = 0,1)$$  

and $(w_{iy} : i = 0, \ldots, 5, \; y = 0,1)$.

In this below, we identify the missing 12 equations by checking the fixed point condition:

$$\mu_s = \mathbb{E}\left[\hat{X}_s\right]Y, \quad \nu_s = \mathbb{E}\left[\hat{X}_s^2\right]Y, \quad \forall s \in [0, T],$$  

(3.10)

where $\mu$ and $\nu$ are two auxiliary processes $(\mu, \nu)[w]$ defined in (3.1), see Figure 2. This leads to a complete characterization of the equilibrium for MFGs (P1).
Note that based on the dynamic of the optimal $\hat{X}$ defined in (3.6), the fixed point condition (3.10) implies that the first moment $\hat{\mu}_s := \mathbb{E} \left[ \hat{X}_s \mid Y \right]$ and the second moment $\hat{\nu}_s := \mathbb{E} \left[ \hat{X}^2_s \mid Y \right]$ of the optimal path conditioned on $Y$ satisfy

\[
\begin{align*}
\hat{\mu}_s &= \hat{\mu} + \int_t^s (-2aY(r) + fY(r)) \mu_r - dY(r) dr, \\
\hat{\nu}_s &= \hat{\nu} + \int_t^s (1 - 4aY(r)\nu_r - 2dY(r)\hat{\mu}_r - 2fY(r)\hat{\mu}_r^2) dr,
\end{align*}
\]

for $s \geq t$. Note that under the optimal control in (3.5), comparing the terms in (3.1) and (3.11), we obtain another 12 equations:

\[
\begin{align*}
w_{0y} &= -2a_y - f_y, \quad w_{1y} = -d_y, \quad w_{2y} = -2d_y, \quad w_{3y} = -4a_y, \quad w_{4y} = -2f_y, \quad w_{5y} = 1.
\end{align*}
\]

Using further algebraic structures, one can reduce the ODE system of 26 equations composed by (3.7) and (3.12) into a system of 8 equations of the form (2.12) for the MFG characterization in Theorem 2.4.

**Proof of Theorem 2.4.** Since $a_y (y = 0, 1)$ has the same expressions as (2.12), its existence, uniqueness and boundedness are shown in Lemma 6.8. Given $a_y (y = 0, 1)$ and smooth bounded $w$'s, $(b_y, d_y, e_y, f_y : y = 0, 1)$ is a coupled linear system, and their existence, uniqueness and boundedness is shown by Theorem 12.1 in [2]. Similarly, given $(b_y, d_y, f_y : y = 0, 1), (k_y, c_y : y = 0, 1)$ is a linear system, and their existence and uniqueness is also guaranteed by Theorem 12.1 in [2].

The ODE system (3.7) can be rewritten by

\[
\begin{align*}
a'_y - 2a_y^2 - \gamma_y a_y + \gamma_y a_z + h_y &= 0, \\
d'_y - 2a_y d_y - f_y d_y - \gamma_y d_y + \gamma_y d_z &= 0, \\
e'_y - d_y f_y - 2k_y d_y - e_y(2a_y + f_y) - 2b_y d_y - \gamma_y e_y + \gamma_y e_z &= 0, \\
f'_y - 2a_y f_y - f_y(2a_y + f_y) - \gamma_y f_y + \gamma_y f_z - 2h_y &= 0, \\
k'_y - \frac{1}{2} f_y^2 - 2k_y(2a_y + f_y) - 2b_y f_y - \gamma_y k_y + \gamma_y k_z &= 0, \\
c'_y + a_y - \frac{1}{2} d_y^2 - e_y d_y + b_y - \gamma_y c_y + \gamma_y c_z &= 0,
\end{align*}
\]

with the terminal conditions

\[
a_y(T) = g_y, \quad b_y(T) = g_y, \quad c_y(T) = 0, \quad d_y(T) = 0, \quad e_y(T) = 0, \quad f_y(T) = 0, \quad k_y(T) = 0.
\]

Since $b_y (y = 0, 1)$ has the same expressions as (2.12), its existence, uniqueness and boundedness are shown in Lemma 6.8. Meanwhile, with the given $(a_y, b_y : y = 0, 1)$, we denote $l_y = 2a_y + f_y$, and then

\[
l'_y - \frac{l_y^2}{2} - \gamma_y l_y + \gamma_y l_z = 0, \quad l_y(T) = 0.
\]

By Lemma 6.6 and Lemma 6.7 in Appendix, there exists a unique solution for $l_y (y = 0, 1)$, which is $l_y = 0, y = 0, 1$. This gives $f_y = -2a_y$ and $d'_y - \gamma_y d_y + \gamma_y d_z = 0$, which implies $d_y = 0, y = 0, 1$. Then, the equation for $e_y$ can be simplified as $e'_y - \gamma_y e_y + \gamma_y e_z = 0$, which indicates that $e_y = 0, y = 0, 1$. For $k_y, c_y$, with the given of $(a_y, b_y, f_y : y = 0, 1)$, we have

\[
k'_y - \gamma_y k_y + \gamma_y k_z - 2a_y^2 + 4a_y b_y = 0, \quad k_y(T) = 0,
\]

\[
e'_y + a_y - \frac{1}{2} d_y^2 - e_y d_y + b_y - \gamma_y c_y + \gamma_y c_z = 0.
\]

The existence and uniqueness of the solution for $k_y, c_y (y = 0, 1)$ are yielded by Theorem 12.1 in [2].

Note that in this case, since $2a_y + f_y = 0$ and $d_y = 0$ for $y = 0, 1$, from (3.11) we have $\hat{\mu}_s = \hat{\mu}$ for all $s \in [t, T]$. Then

\[
\hat{\nu}_s = \hat{\nu} + \int_t^s \left( 1 - 4aY_r(r)\hat{\mu}_r^2 - 4aY_r(r)\hat{\mu}_r \right) dr.
\]
Plugging \( d_y = 0 \) for \( y = 0, 1 \) and \( \mu_s = \bar{\mu} \) back to (3.5), we obtain the optimal control by
\[
\hat{\alpha}_s = 2a_{Y_i}(s)(\bar{\mu} - \hat{X}_s).
\]

Since we have \( d_y = 0 \) for \( y = 0, 1 \) and \( \mu_s = \bar{\mu} \) for \( s \in [t, T] \), the value function can be simplified from (3.4) to
\[
v_y(x, t, \bar{\mu}, \bar{\nu}) = a_y(t)x^2 - 2a_y(t)x\bar{\mu} + k_y(t)\bar{\mu}^2 + b_y(t)\bar{\nu} + c_y(t).
\]

By the equivalence Lemma 3.2, it yields the value function \( U \) of Theorem 2.4. Moreover, since \( f_y = -2a_y \) and \( k_y \neq 0 \), the ODE system (3.7) together with (3.12) can be reduced into (2.12). From the Lemma 6.8, the existence and uniqueness of \((a_y, b_y, c_y, k_y : y = 0, 1)\) in (2.12) is guaranteed.

4. The \( N \)-Player Game and its Convergence to MFGs. In this section, we show the convergence of the \( N \)-player game to MFGs. To simplify the presentation, we omit the superscript \( (N) \) for the processes in the probability space \( \Omega(N) \), whenever there is no confusion. First, we solve the \( N \)-player game in Subsection 4.1, which provides a Riccati system consisting of \( O(N^3) \) equations. Subsection 4.2 reduces the corresponding Riccati system into an ODE system whose dimension is independent to \( N \). This becomes the key building block of the convergence proved in Subsection 4.3.

4.1. Characterization of the \( N \)-player game by Riccati system. The \( N \)-player game is indeed an \( N \)-coupled stochastic LQG problem by its very own definition. Therefore, the solution can be derived via Riccati system given below: For \( i = 1, 2, \ldots, N, \ y = 0, 1, \)

\[
\begin{aligned}
A'_{iy} - 2A'_{iy}e_ie_i^T A_{iy} - 4 \sum_{j \neq i} A_{iy}^T e_j e_j^T A_{iy} - \gamma_y A_{iy} + \gamma_y A_{i(1-y)} + & h_y \sum_{j \neq i} (e_i - e_j) (e_i - e_j)^T = 0, \\
B'_{iy} - 2A'_{iy}e_ie_i^T B_{iy} - 2 \sum_{j \neq i} (A_{iy}^T e_j e_j^T B_{ijy} + A_{ijy}^T e_j e_j^T B_{iy}) - \gamma_y B_{iy} + & \gamma_y B_{i(1-y)} = 0, \\
C'_{iy} - 1/2 B'_{iy} e_i e_i^T B_{iy} - \sum_{j \neq i} B_{ijy}^T e_j e_j^T B_{iy} + & tr(A_{iy}) - \gamma_y C_{iy} + \gamma_y C_{i(1-y)} = 0, \\
A_{iy}(T) = g_y N A_i, \\ B_{iy}(T) = C_{iy}(T) = 0,
\end{aligned}
\]

(4.1)

where the solutions consist of \( N \times N \) symmetric matrices \( A_{iy}'s, N \)-dimensional vectors \( B_{iy}'s, \) and \( C_{iy} \in \mathbb{R} \). In the above, \( A_i' \)'s are \( N \times N \) matrices with diagonal 1 except \((A_i)_{ii} = N - 1, (A_i)_{ij} = (A_i)_{ji} = -1\) for any \( j \neq i \) and the rest entries as 0, and \( e_i \)'s are the \( N \)-dimensional natural basis.

**Lemma 4.1.** Suppose \((A_{iy}, B_{iy}, C_{iy} : i = 1, 2, \ldots, N; \ y = 0, 1)\) is the solution of (4.1). Then, the value functions of \( N \)-player game defined by (2.9) is
\[
V_i(y, x_N) = (x_N)^T A_{iy}(0)x_N^N + (x_N)^T B_{iy}(0) + C_{iy}(0), \quad i = 1, \ldots, N.
\]

Moreover, the path and the control under the equilibrium are
\[
(4.2) \quad d\hat{X}_i = \left(-2(A_{iy})_{ii}^T \hat{X}_i - (B_{iy})_{ii}\right) dt + dW_i, \quad i = 1, \ldots, N,
\]

and
\[
\hat{\alpha}_i = -2(A_{iy})_{ii}^T \hat{X}_i - (B_{iy})_{ii},
\]

where \((A)_i\) denotes the \( i \)-th column of matrix \( A \), \((B)_i\) denotes the \( i \)-th entry of vector \( B \) and \( \hat{X}_i = \left[\hat{X}_{1i} \hat{X}_{2i} \cdots \hat{X}_{Ni}\right]^T \).
Proof. It is standard that, under the enough regularities, the players’ value function \( V(y, x^N) = (V_1, \ldots, V_N)(y, x^N) \) can be lifted to the solution \( v_{iy}(x^N, t) \) of the following system of HJB equation, for \( i = 1, \ldots, N \),

\[
\begin{aligned}
\partial_t v_{i0} - \frac{1}{2} (\partial_i v_{i0})^2 - \sum_{j \neq i} \partial_j v_{j0} \partial_j v_{i0} + \frac{1}{2} \Delta v_{i0} - \gamma_0 v_{i0} + \gamma_0 v_{i1} + \frac{h_0}{N} \sum_{j \neq i} ((e_i - e_j)^\top x^N)^2 = 0, \\
\partial_t v_{i1} - \frac{1}{2} (\partial_i v_{i1})^2 - \sum_{j \neq i} \partial_j v_{j1} \partial_j v_{i1} + \frac{1}{2} \Delta v_{i1} - \gamma_1 v_{i1} + \gamma_1 v_{i0} + \frac{h_1}{N} \sum_{j \neq i} ((e_i - e_j)^\top x^N)^2 = 0, \\
v_{iy}(x^N, T) = \frac{g_y}{N} \sum_{j \neq i} ((e_i - e_j)^\top x^N)^2.
\end{aligned}
\]

Then, the value functions \( V \) of \( N \)-player game defined by (2.9) is \( V_i(y, x^N) = v_{iy}(x^N, 0) \) for all \( i = 1, \ldots, N \). Moreover, the path and the control under the equilibrium are

\[
d\hat{X}_t = -\partial_i v_{iY_i}(\hat{X}_t, t)dt + dW_t, \quad i = 1, \ldots, N,
\]

and

\[
\hat{\alpha}_t = -\partial_i v_{iY_i}(\hat{X}_t, t).
\]

The proof is the application of Dynkin’s formula and the details are omitted here. Due to its LQG structure, the value function leads to a quadratic function of the form

\[
v_{iy}(x^N, t) = (x^N)^\top A_{iy}(t)x^N + (x^N)^\top B_{iy}(t) + C_{iy}(t).
\]

For each \( i = 1, 2, \ldots, N \), after plugging \( V_{iy} \) into (4.3), and matching the coefficient of variables, we get the desired results.

4.2. Reduced Riccati form for the equilibrium. The \( N \)-player game can be characterized by Riccati system (4.1). Our objective is the convergence of \( (\hat{X}^{(N)}, Y^{(N)}) \) generated by (4.2) relying on the solution of Riccati system (4.1) to the \( (\hat{X}_t, Y_t) \) of (2.13). Note that \( \hat{X}_t \) relies only on two functions \( (a_y : y = 0, 1) \) while \( \hat{\rho}(\hat{X}^{(N)}) \) depends on \( O(N^3) \) functions from \( (A_{iy} : i = 1, 2, \ldots, N, \ y = 0, 1) \), which can be solved from a huge Riccati system. Therefore, it is almost a hopeless task to see the connection between these two processes without gaining further insight on the structure of Riccati system (4.1).

To proceed, let us first observe some hidden patterns from a numerical result for the solution of Riccati (4.1). Table 1 shows the \( A_{20} \) at \( t = 1 \) for \( N = 5 \) with same parameters as in figure 3 and figure 4 in section 5.1.

|       |       |       |       |
|-------|-------|-------|-------|
| 0.1319| -0.1924| 0.0202| 0.0202|
| -0.1924| 0.7696| -0.1924| -0.1924|
| 0.0202| -0.1924| 0.1319| 0.0202|
| 0.0202| -0.1924| 0.0202| 0.1319|
| 0.0202| -0.1924| 0.0202| 0.0202|

Table 1: \( A_{20}(1) \) for \( N = 5 \)

Inspired by the numerical example of \( A_{iy} \) in Table 1, we may want to believe a pattern

\[
(A_{iy})_{pq} = \begin{cases} 
    a_{1y}(t), & \text{if } p = q = i, \\
    a_{2y}(t), & \text{if } p = q \neq i, \\
    a_{3y}(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\
    a_{4y}(t), & \text{otherwise.}
\end{cases}
\]

The next result justifies the above pattern: the \( N^2 \) entries of the matrix \( A_{iy} \) can be embedded to a 4-dimensional vector space no matter how big \( N \) is.
LEMMA 4.2. There exists a unique solution \((a_{1y}^N, a_{2y}^N)\) from the ODE system (4.5).

\begin{align}
\begin{cases}
a_{1y} - \frac{2(N-1)}{N-1} a_{2y} - \gamma_y a_{1y} + \gamma_y a_{1(1-y)} + \frac{N-1}{N} h_y = 0, \\
a_{2y} - \frac{2(N-2)}{N-2} a_{1y} - \gamma_y a_{2y} + \gamma_y a_{2(1-y)} + \frac{h_y}{N} = 0, \\
a_{1y}(T) = \frac{N-1}{N} g_y, \ a_{2y}(T) = \frac{g_y}{N}.
\end{cases}
\end{align}

for \(y = 0, 1\). Moreover, the path and the control of player \(i\) under the equilibrium are

\begin{align}
d\hat{X}^{(N)}_{it} &= -2a_{1y}^N (A_{1y}^N)_{ij} (X^{(N)}_{jt} - \frac{1}{N-1} \sum_{j\neq i}^N X^{(N)}_{jt}) dt + dW^{(N)}_{it}, \quad i, 1, \ldots, N,
\end{align}

and

\begin{align}
\hat{a}_{it}^{(N)} &= -2a_{1y}^N (A_{1y}^N)_{ij} (X^{(N)}_{jt} - \frac{1}{N-1} \sum_{j\neq i}^N X^{(N)}_{jt}).
\end{align}

Proof. It is obvious to see that \(B_{iy} = 0\) for all time \(t \in [0, T]\). Note that in this case, for \(i = 1, 2, \ldots, N\), the optimal control is given by

\[ \hat{a}_i = -2 \sum_{j=1}^N (A_{1y}^N)_{ij} \hat{X}^{(N)}_{jt} = -2 (A_{1y}^N)_{i}^\top \hat{X}^{(N)}_{i} \]

Plugging the pattern (4.4) into the differential equation of \(A_{iy}\), we have

\begin{align*}
a_{1y} - 2a_{1y}^2 - 4(N-1)a_{3y}^2 - \gamma_y a_{1y} + \gamma_y a_{1(1-y)} + \frac{N-1}{N} h_y &= 0, \\
a_{2y} - 2a_{2y}^2 - 4a_{1y} a_{2y} - 4(N-2)a_{3y} a_{4y} - \gamma_y a_{2y} + \gamma_y a_{2(1-y)} + \frac{h_y}{N} &= 0, \\
a_{3y} - 2a_{1y} a_{3y} - 4a_{1y} a_{3y} - 4(N-2)a_{3y}^2 - \gamma_y a_{3y} + \gamma_y a_{3(1-y)} - \frac{h_y}{N} &= 0, \\
a_{4y} - 2a_{1y} a_{4y} - 4a_{2y} a_{4y} - 4(N-2)a_{3y} a_{4y} - \gamma_y a_{4y} + \gamma_y a_{4(1-y)} &= 0,
\end{align*}

which gives \(a_{1y} + (N-2)a_{3y} = a_{2y} + (N-2)a_{4y}\) since two expressions for \(a_{3y}\) should be equal. This implies that \(a_{1y} + (N-2)a_{3y} = (a_{2y} + (N-2)a_{4y})^t\) or

\begin{align*}
2a_{1y}^2 + 2(N-2)a_{1y} a_{3y} + 4(N-1)a_{3y}^2 + 4(N-2)a_{2y} a_{4y} + 4(N-2)a_{3y} a_{4y} + 4(N-2)a_{3y} a_{4y} \\
+ \gamma_y (a_{1y} + (N-2)a_{3y}) - \gamma_y (a_{1(1-y)} + (N-2)a_{3(1-y)}) - \frac{h_y}{N} \\
= 2(N-1)a_{3y}^2 + 4a_{1y} a_{2y} + 4(N-2)a_{2y} a_{3y} + 4(N-2)a_{3y} a_{4y} + 4(N-2)a_{1y} a_{4y} \\
+ 4(N-2)(a_{3y} a_{4y} + \gamma_y (a_{2y} + (N-2)a_{4y}) - \gamma_y (a_{2(1-y)} + (N-2)a_{4(1-y)}) - \frac{h_y}{N}.
\end{align*}

After combining terms and substituting \(a_{2y} + (N-2)a_{4y}\) with \(a_{1y} + (N-2)a_{3y}\), we get \(a_{1y} + (N-2)a_{3y} - (N-1)a_{3y} = 0\), which yields \(a_{3y} = a_{1y}\) or \(a_{3y} = -\frac{1}{N-1} a_{1y}\). Note that \(a_{3y} \neq a_{1y}\) due to their different differential equations. Hence, we can conclude that \(a_{3y} = -\frac{1}{N-1} a_{1y}\). In conclusion, for \(i = 1, 2, \ldots, N\), \(A_{iy}\) \((y = 0, 1)\) has the following expressions:

\[ (A_{iy})_{pq} = \begin{cases} 
a_{1y}(t), & \text{if } p = q = i, \\
a_{2y}(t), & \text{if } p = q \neq i, \\
-\frac{1}{N-1} a_{1y}(t), & \text{if } p \neq q, p = i \text{ or } q = i, \\
\frac{1}{(N-1)(N-2)} a_{1y}(t) - \frac{1}{N-2} a_{2y}(t), & \text{otherwise}.
\end{cases} \]
The existence and uniqueness of (4.1) is equivalent to the existence and uniqueness of (4.5). For $a_{1y}$, the existence and uniqueness can be deduced from Lemma 6.6 and 6.7. Given $a_{1y}$'s, $a_{2y}$'s are linear equations, thus their existence and uniqueness are guaranteed by Theorem 12.1 in [2]. Together with previous discussions, we conclude the results.

4.3. Convergence. Let us reiterate our goal (P2) for the convergence based on the current progress. Our objective is the convergence in distribution of $(\hat{X}_{nt}^{(N)}, Y_{t}^{(N)})$ generated by (4.6) in the sample space $\Omega^{(N)}$ towards the $(X_{t}, Y_{t})$ of (2.13) in $\Omega$.

The key towards this proof is to provide an explicit embedding of $(\hat{X}_{nt}^{(N)}, Y_{t}^{(N)})$ to the same probability space $(\Omega, \mathcal{F}_{T}, \mathbb{P})$. Note that, no matter how large $N$ is, our objective is to copy the distribution of $X_{t}^{(N)}$ from $\Omega^{(N)}$ having $N$-dimensional Brownian motion $W^{(N)}$ to a smaller space $\Omega$ having only two Brownian motions $W$ and $B$. In general, the space of random processes generated by $N$-dimensional Brownian motion is much richer than the one generated by 2-dimensional Brownian motion whenever $N > 2$. However, it is possible for our case to copy the distribution $(\hat{X}_{nt}^{(N)}, Y_{t}^{(N)})$ due to the nature of the mean field effect. Next we present the coupling result.

**Lemma 4.3.** Let $Z^{N}$ be the solution of

\[
Z^{N}_{t} = x^{N}_{u} - \int_{0}^{t} 2a_{1y}^{N}(s) \left( Z^{N}_{s} - (\bar{x}^{N} + \frac{\sqrt{N-1}}{N}B_{s} + \frac{1}{N}W_{s}) \right) ds + W_{t},
\]

where $u$ is the random variable uniformly distributed on set $\{1, 2, \ldots, N\}$, $W$ and $B$ are Brownian motions on the $(\Omega, \mathcal{F}_{T}, \mathbb{P})$ defined in Section 2, and

\[
\hat{a}^{N}_{1y} = \frac{N}{N-1} a^{N}_{1y},
\]

where $a^{N}_{1y}$ is from the ODE system (4.5). Then, $(Z^{N}_{t}, Y_{t})$ in $(\Omega, \mathcal{F}_{T}, \mathbb{P})$ has the same distribution as $(\hat{X}_{nt}^{(N)}, Y_{t}^{(N)})$ in $(\Omega^{(N)}, \mathcal{F}_{T}^{(N)}, \mathbb{P}^{(N)})$.

**Proof.** Continued from the Lemma 4.2, player $i$'s path in the $N$-player game follows

\[
\hat{X}_{it}^{(N)} = x^{N}_{i} - \int_{0}^{t} 2a_{1y}^{N}(s) \left( \hat{X}_{is}^{(N)} - \frac{1}{N-1} \sum_{j \neq i}^{N} \hat{X}_{js}^{(N)} \right) ds + W_{it}^{(N)}.
\]

With the notation

\[
\hat{X}_{is}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \hat{X}_{is}^{(N)},
\]

one can rewrite the path by

\[
\hat{X}_{it}^{(N)} = \bar{x}^{N} + \frac{1}{N} \sum_{i=1}^{N} W_{it}^{(N)} = \bar{x}^{N} + \frac{\sqrt{N-1}}{N} \left( \sqrt{N-1} \bar{W}_{it}^{(N)} \right) + \frac{1}{N} W_{it}^{(N)},
\]

where $\bar{W}_{it}^{(N)} := \frac{1}{N-1} \sum_{j \neq i}^{N} W_{jt}^{(N)}$.

Finally, to see the distribution of $Z^{N}_{t}$ in the space $\Omega$ identical distribution to $\hat{X}_{nt}^{(N)}$ in $\Omega^{(N)}$, we follow the following steps:

- Embed $Y^{(N)}$ from $\Omega^{(N)}$ to $Y$ from $\Omega$;
- Replace the index $i$ of (4.8) by uniform random variable $u$;
Proof of Theorem 2.5. We define
\[ \mathcal{E}_t(b) = \exp \left\{ \int_0^t b_s ds \right\}. \]
and
\[ G_t(x, b, W) = \mathcal{E}_t(-b)x + \mathcal{E}_t(-b) \int_0^t \mathcal{E}_s(b) dW_s. \]
Then, we can rewrite the process \( Z_N \) of (4.7) by
\[ Z_N^t = \bar{x}^N + \frac{\sqrt{N-1}}{N} B_t + \frac{1}{N} W_t + G_t(x_u^N, 2\hat{a}_1^N(Y, \cdot), W) \]
and write \( \hat{X} \) by
\[ \hat{X}_t = \mathbb{E}[X_0] + G_t(X_0, 2a(Y, \cdot), W) \]
1. We recall that the term \( \bar{x}^N = \frac{1}{N} \sum_{i=1}^N x_i^N \) is a deterministic real number. Since \( \rho(x^N) \) is weakly convergent to the law of \( X_0 \) by (A2), one can have
\[ \langle \phi, \rho(x^N) \rangle \to \mathbb{E}[\phi(X_0)], \]
for all test functions \( \phi \). If we use \( \phi(x) = x \), then it yields \( \bar{x}^N \to \mathbb{E}[X_0] \). Hence, we have
\[ \bar{x}^N + \frac{\sqrt{N-1}}{N} B_s + \frac{1}{N} W_t \to \mathbb{E}[X_0] \text{ almost surely.} \]
2. Note that from (4.5), the convergence \( a_{1y}^N \to a_y \) holds in \( L^\infty[0, T] \), where \( a_y \) is the solution from (2.12). Therefore, we have the almost sure convergence
\[ \lim_{N \to \infty} ||\hat{a}_1^N(Y, \cdot) - a(Y, \cdot)||_\infty = 0, \text{ almost surely}. \]
By (A2), \( x_u^N \) converges to \( X_0 \) in distribution. By Skorohod representation theorem, we can have a replication \( x_u^N \) and \( X_0 \) in the same probability space with almost sure convergence. For the simplicity of notation, we assume that
\[ x_u^N \to X_0 \text{ almost surely}. \]
Finally, since \( G_t \) is continuous on \( \mathbb{R} \times L^\infty \times L^\infty \), we have
\[ G_t(x_u^N, 2\hat{a}_1^N(Y, \cdot), W) \to G_t(X_0, 2a(Y, \cdot), W), \text{ almost surely}. \]
Therefore, we conclude that \( Z_u^N \to \hat{X}_t \) almost surely. Combine with Lemma 4.3 and Proposition 2.1, we conclude the desired weak convergence.

5. Numerical results.

5.1. Simulations of Riccati system, the value function and optimal control of the generic player. We have derived a 8 dimensional Riccati ODE system (2.12) to determine the parameter functions
\[ (a_y, b_y, c_y, k_y : y = 0, 1) \]
needed for the characterization of the equilibrium and the value function. Meanwhile, we also show the solvability of the Riccati ODE system in Section 3.

As mentioned earlier, different from the MFG characterization with the common noise, the derived Riccati system is essentially finite dimensional. In this subsection, we present a numerical experiment and show some numerical results for solving Riccati system to demonstrate its computational advantages.

For the illustration purpose, assume the finite time horizon is given with $T = 5$ and that the coefficients of the dynamic equation are listed below

\begin{align*}
\gamma_0 &= 0.5, \gamma_1 = 0.6, \\
h_0 &= 2, h_1 = 5, g_0 = 3, g_1 = 1, \\
\mu_0 &= 0, \nu_0 = 2.
\end{align*}

Firstly, using Euler’s forward difference method with the step size $\delta = 10^{-2}$, we can obtain trajectories of $(a_y, b_y, c_y : y = 0, 1)$, which is the solution of ODE system (2.12). Next, using the trajectories of the parameter functions and Markov chain $Y_t$, we can achieve the simulations for $\hat{\alpha}_t$ and $\hat{X}_t$. The Matlab code can be found at https://github.com/JiaminJIAN/Regime_switching_MFG.

![Fig. 3: Simulations for $a_y, V, \alpha$ and $\nu$.](image1)

![Fig. 4: Simulations for $b_y$ and $c_y$.](image2)
As shown in figure 3, people tend to centralize since the conditional second moment of the population density $\nu_t$ is always decreasing.

5.2. Convergence of the $N$-player game. In section 4, we showed that the generic player’s path for $N$-player game is convergent to the generic player’s path for MFGs. In this subsection, we demonstrate the convergence of the conditional first moment, conditional second moment and the value functions of the $N$-player game to the corresponding terms of the generic player in Mean Field Game setup by using some numerical examples.

The following figures show the value functions, $\mu^{(N)}(t)$ and $\nu^{(N)}(t)$ under $N \in \{10, 20, 50, 100\}$ with the same parameters’ settings as in figure 3 and figure 4 in section 5.1. We can clearly see the convergence to the solution of the generic player.

![Graphs showing convergence](image)

(a) $\mu_t$: conditional mean of the population density
(b) $\nu_t$: conditional 2nd moment of the population density

Fig. 5: Simulations for $\mu_t$ and $\nu_t$.

![Graph showing value function](image)

Fig. 6: Simulation of player 1’s optimal value function $V$.

6. Appendix.

6.1. Some explicit solutions on LQG-MFGs. In this part, we only provide explicit solutions to some LQG-MFGs without the common noise. The methodology could be the utilization of the standard Stochastic Maximum Principle or Dynamic Programming approach, and all proofs will be omitted.
Suppose the position of a generic player $X_t$ follows
\[
dX_t = \alpha_t dt + \sigma dW_t, \quad X_0 \sim \mathcal{N}(0, 1).
\]
The goal of the generic player is to minimize the running cost
\[
\inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[ \int_0^T \left( \frac{1}{2} \alpha_t^2 + h \int_{\mathbb{R}} (X_t - y)^2 m(t, dy) \right) dt \right],
\]
subject to
\[
m_t = \text{Law}(X_t), \quad \forall t \in [0, T],
\]
where $h \in \mathbb{R}$ is a constant.

Denote
\[
V(x, t) = \inf_{\alpha} \mathbb{E} \left[ \int_t^T \left( \frac{1}{2} \alpha_s^2 + h \int_{\mathbb{R}} (X_s - y)^2 m(s, dy) \right) ds \right] \bigg| X_t = x.
\]
Note that the model can be characterized by Hamilton-Jacobian-Bellman equation coupled by Fokker-Planck-Kolmogorov equation:
\[
\begin{aligned}
&\partial_t V + \frac{1}{2} \sigma^2 \partial_{xx} V - \frac{1}{2} (\partial_x V)^2 + F(x, m) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \\
&\partial_t m - \frac{1}{2} \sigma^2 \partial_{x,m} m - \partial_x (m \partial_x V) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \\
&m_0 \sim \mathcal{N}(0, 1), V(x, T) = 0, \quad x \in \mathbb{R},
\end{aligned}
\]
where $F(x, m) = h \int_{\mathbb{R}} (x - y)^2 m(t, dy)$.

The monotonicity condition on the source term $F$ in the variable $m$ plays crucial role for the uniqueness of the MFG system. A monotone function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be increasing if it satisfies $(f(x_1) - f(x_2))(x_1 - x_2) \geq 0$, and decreasing if $-f$ is increasing. This definition can be generalized to an infinite dimensional function $F(x, m)$.

**Definition 6.1.** The real function $F$ on $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$ is said to be monotone, if, for all $m \in \mathcal{P}_2(\mathbb{R})$, the mapping $\mathbb{R} \ni x \mapsto F(x, m)$ is at most of quadratic growth, and for all $m_1, m_2$ it satisfies
\[
\int_{\mathbb{R}} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) \geq 0.
\]

$F$ is said to be anti-monotone, if $(-F)$ is monotone.

According to [4], if $F$ is monotone, then MFGs have at most one solution. Interestingly, the monotonicity of $F$ is dependent on the sign of $h$.

**Lemma 6.2.** $F(x, m) = h \int_{\mathbb{R}} (x - y)^2 m(t, dy)$ is monotone if $h < 0$, and anti-monotone if $h > 0$.

A natural question is that, how the MFG system behaves differently to the monotonicity of $F$?

**6.1.1. Case I: $h > 0$.**

**Lemma 6.3.** For $h > 0$, there exists a solution (may not be unique) to the MFG system in the form of $V(x, t) = f_1(t)x^2 + f_3(t)$ and $m(t) \sim \mathcal{N}(0, \gamma(t))$, where
\[
\begin{aligned}
f_1(t) &= \sqrt{\frac{h}{2}} \int_t^T e^{-2 \sqrt{2h}(T-t)} \left( 1 + \int_0^t \sigma^2 e^{2 \sqrt{2h}s} ds \right), \\
f_3(t) &= \int_t^T \sigma^2 f_1(s) + h \gamma(s) ds.
\end{aligned}
\]
6.1.2. Case II: $h < 0$.

**Lemma 6.4.** For $h < 0$, there exists a unique solution in $[t_0,T]$ to the MFG system in the form of $V(x,t) = g_1(t)x^2 + g_3(t)$ and $m(t) \sim \mathcal{N}(0,\lambda(t))$, where
\[
    g_1(t) = -\sqrt{-\frac{h}{2}} \tan \left( \sqrt{-2h}(T-t) \right), \quad \lambda(t) = e^{-\int_0^t 4g_1(s)ds} \left( 1 + \int_0^t \sigma^2 e^{\int_0^s 4g_1(u)du}ds \right),
\]
\[
    g_3(t) = \int_t^T (\sigma^2 g_1(s) + h\lambda(s))ds, \quad t_0 = \max \left( 0, T - \frac{1}{\sqrt{-2h}} \frac{\pi}{2} \right).
\]

6.1.3. Remark. When $h > 0$, the cost is anti-monotone, and there exists at least one global solution. When $h < 0$, the cost is monotone, and there exists at most one solution. Unfortunately, this solution lives in a short period of time. Lemma 6.4 coincides with the notes in Section 3.8 of [5] saying that due to the opposite time evolution of the system of HJB-FPK, the existence of the solution may exist for only a short period of time.

6.2. Dynkin’s formula for a regime-switching diffusion with a quadratic function. Since the running cost (2.10) has a quadratic growth in the state variable, the value function $V[\hat{\mu}](y,x,t)$ is expected to possess similar growth. Throughout this subsection, we will use $K$ in various places as a generic constant which varies from line to line. The notions of this subsection is independent to other parts of the paper.

**Lemma 6.5.** Let $X$ be the solution of
\[
    dX_t = \alpha_t dt + \sigma_t dW_t,
\]
where $X$, $\alpha$ and $\sigma$ are bounded and take value in $\mathbb{R}^3$. $Y$ is CTMC with a generator
\[
    Y \sim Q = (q_{ij})_{i,j=1,2,\ldots,n}.
\]
If $X_0 \in L^4$, $\alpha \in L^2$ and $f : \mathbb{R}^3 \mapsto \mathbb{R}$ satisfies $\|\Delta f\|_{\infty} + \|\partial f\|_{\infty} < \infty$, then the following identity holds for all $t \in [0,T]$:
\[
    \mathbb{E} \left[ f(Y_t,X_t,t) \right] = \mathbb{E} \left[ f(Y_0,X_0,0) \right] + \mathbb{E} \left[ \int_0^t (\partial_t + \mathcal{L} + \mathcal{Q})f(Y_s,X_s,s)ds \right],
\]
where
\[
    \mathcal{L}f(y,x,s) = \left( \frac{1}{2} \text{Tr}(\sigma_s\sigma_s^\top \Delta) + \alpha_s \cdot \nabla_x \right) f(y,x,s)
\]
and
\[
    \mathcal{Q}f(y,x,s) = \sum_{i=1}^n q_{yi}f(i,x,s).
\]

**Proof.** It’s enough to show that the local martingale defined by Itô’s formula
\[
    (6.1) \quad M_t^f = f(Y_t,X_t,t) - f(Y_0,X_0,0) - \int_0^t (\partial_t + \mathcal{L} + \mathcal{Q})f(Y_s,X_s,s)ds
\]
is uniformly integrable, hence is a true martingale.

First, $X_t$ is $L^4$ bounded uniformly in $t$ from the following inequality due to our assumptions on $X_0$ and $\alpha$:
\[
    \sup_{t \in [0,T]} \mathbb{E} \left[ \|X_t\|^4 \right] \leq K \mathbb{E} \left[ \|X_0\|^4 + \int_0^T \|\alpha_s\|^4 ds + \int_0^T \|\sigma_s W_s\|^4 ds \right] \leq K,
\]
where $K$ is a generic constant which varies from line to line.
On the other hand, since \( \Delta f \) is uniformly bounded, \( f \) is at most quadratic growth, i.e.
\[
|f(x)| \leq K(x^2 + 1), \forall x \text{ for some large } K.
\]
Hence, we conclude that \( f(Y_t, X_t, t) \) is uniformly bounded in \( L^2 \) in \( t \) from the fact
\[
\sup_{t \in [0, T]} \mathbb{E} \left[ f^2(Y_t, X_t, t) \right] \leq K \sup_{t \in [0, T]} \mathbb{E} \left[ \|X_t\|^4 \right] + K \leq K.
\]
The uniform \( L^2 \)-boundedness of \( \int_0^t \partial_t f(Y_s, X_s, s)ds \) follows from our assumption on \( \partial_t f \). Similarly, since \( Qf \) has a quadratic growth uniformly in \( y \) and \( t \),
\[
\left\{ \int_0^t Qf(Y_s, X_s, s)ds : 0 \leq t \leq T \right\}
\]
is \( L^2 \) bounded. At last, we have
\[
\mathbb{E} \left[ \left( \int_0^t \mathcal{L}f(Y_s, X_s, s)ds \right)^2 \right] \\
\leq K \mathbb{E} \left[ \int_0^t (\alpha_s \cdot \nabla f + \frac{1}{2} \text{Tr}(\sigma_s \sigma_s^\top \Delta f))^2(Y_s, X_s, s)ds \right] \\
\leq K \mathbb{E} \left[ \int_0^t \|\alpha_s\|^2 \|\nabla f\|^2(Y_s, X_s, s)ds \right] + K \mathbb{E} \left[ \int_0^t \frac{1}{4} \text{Tr}(\sigma_s \sigma_s^\top \Delta f))^2(Y_s, X_s, s)ds \right] \\
\leq K \mathbb{E} \left[ \int_0^t \|\alpha_s\|^4 ds \right] + K \mathbb{E} \left[ \int_0^t \|\nabla f\|^4(Y_s, X_s, s)ds \right] + K \mathbb{E} \left[ \int_0^t \frac{1}{4} \|\Delta f\|^2(Y_s, X_s, s)ds \right].
\]
Since \( \nabla f \) is linear growth in \( x \), the second term \( \sup_{t \in [0, T]} \mathbb{E} \left[ \int_0^t \|\nabla f\|^4(Y_s, X_s, s)ds \right] \) is finite. Together with assumptions on \( \Delta f \) and \( \alpha \), we have uniform \( L^2 \)-boundedness of \( \int_0^t \mathcal{L}f(Y_s, X_s, s)ds \).

As a result, each term of the right hand side of (6.1) is uniformly \( L^2 \)-bounded in \( t \), and thus \( M^f_t \) belongs to \( L^2_2 \) and this implies the uniformly integrability.

6.3. Proof of the existence and uniqueness of the ODE system. Consider the following ODE system
\[
\begin{align*}
\begin{cases}
    a_0^t - Ca_0^2 - \gamma_0(a_0 - a_1) + h_0 = 0, \\
    a_1^t - Ca_1^2 + \gamma_1(a_0 - a_1) + h_1 = 0, \\
    a_0(T) = g_0, a_1(T) = g_1,
\end{cases}
\end{align*}
\] (6.2)
where \( C, h_i, g_i \) \((i = 0, 1)\) are in \( \mathbb{R}^+ \). We need to show the existence and uniqueness of the solution to (6.2). Define \( T_0^{(N)} \) and \( T_1^{(N)} \) as
\[
T_0^{(N)}[a](t) = \left( g_0 + \int_t^T \left( h_0 - Ca_0^2(s) - \gamma_0(a_0(s) - a_1(s)) \right) ds \right) \wedge N \vee 0,
T_1^{(N)}[a](t) = \left( g_1 + \int_t^T \left( h_1 - Ca_1^2(s) - \gamma_1(a_1(s) - a_0(s)) \right) ds \right) \wedge N \vee 0,
\]
where \( a = \begin{bmatrix} a_0 & a_1 \end{bmatrix}^\top \). Let \( D = \{ f \in C([0,T]) : 0 \leq \sup_{t \in [0,T]} f(t) \leq N \} \). Note that \( T_y^{(N)}(y = 0, 1) \) maps \( D^2 \) to \( D^2 \).

**Lemma 6.6.** For fixed \( N \), there exists a unique solution in \( C([0,T]) \) to
\[
a = \begin{bmatrix} T_0^{(N)}[a] \\ T_1^{(N)}[a] \end{bmatrix}.
\] (6.3)
Next, we want to show that for large enough $k > 1$, the closest time to $\tau$, $t \in [0, T]$ is such that $a_0(t) = 0$, which gives the desirable point $\hat{t}$. Then for all $t \in [\hat{t}, T)$, we prove the positiveness of $a_0(t)$. Since $a_0(t) = 0$, we have $a_0(t) = a_0(T) = g_0 > 0$, which gives the desirable point $\tau$. Then for all $t \in (\tau, T]$, we prove the positiveness of $a_0(t)$. For simplicity of notations, $a_i$ is used instead of $a_i^{(N)}$ for $i = 0, 1$ if there is no confusion.

First, for $i = 0, 1$, we prove the positiveness of $a_i$ by contradiction. Suppose $a_i (i = 0, 1)$ are not positive functions on $[0, T]$. Since $a_0$ is continuous and $a_0(T) = g_0 > 0$, there exists some $\tau \in [0, T]$ as the closest time to $T$ such that $a_0(\tau) = 0$. Note that finding such a $\tau$ is possible. Let $t_n \in [0, T]$ be a non-decreasing sequence such that $a_0(t_n) = 0$, there exists some $\tau$ such that $t_n \to \tau < T$ as $n \to \infty$ since $a_0$ is continuous and $a_0(T) = g_0 > 0$. By the continuity of $a_0$, we have $a_0(\tau) = 0$, which gives the desirable point $\tau$. Then for all $t \in (\tau, T]$, we prove the positiveness of $a_0(t)$.

In this case, plugging $t = \tau$ to (6.2), we have $a_0'(\tau) = -h_0 - \gamma_0 a_1(\tau) > 0$, which yields $a_1(\tau) < 0$. Since $a_1$ is continuous on $[0, T]$ and $a_1(T) = g_1 > 0$, from the intermediate value theorem, there exists some $\tilde{\tau} \in (\tau, T)$ such that $a_1(\tilde{\tau}) = 0$ and $a_1'(\tilde{\tau}) > 0$. However, this indicates that $a_1'(\tilde{\tau}) = -h_0 - \gamma_0 a_0(\tilde{\tau}) > 0$ by plugging $t = \tau$ back to (6.2), and it implies $a_0(\tilde{\tau}) < 0$, which contradicts with the fact that $a_0(t) > 0$ for all $t \in (\tau, T]$. Thus the positiveness of $a_0$ and $a_1$ is obtained.

Next, we prove the upper bound for the integral in (6.4). Note that for all $t \in [0, T]$, 

$$(a_0 + a_1)'(T - t) = (h_0 + h_1) - C(a_0^2 + a_1^2)(T - t) - (\gamma_0 - \gamma_1)a_0(T - t) + (\gamma_0 - \gamma_1)a_1(T - t) \leq (h_0 + h_1) + (\gamma_0 + \gamma_1)(a_0 + a_1)(T - t),$$

Similarly, we have 

$$\|e^{kt}(a_0^{(n+1)}(t) - a_0^{(n)}(t))\|_\infty \leq \frac{2CN + 2\gamma_1}{k} \|a^{(n)} - a^{(n-1)}\|_k,$$

Choosing $k > 2CN + 2\max\{\gamma_0, \gamma_1\}$, then

$$\|a^{(n+1)} - a^{(n)}\|_k \leq \frac{2CN + 2\max\{\gamma_0, \gamma_1\}}{k} \|a^{(n)} - a^{(n-1)}\|_k,$$

which gives us a contraction mapping from $D^2$ to $D^2$. Hence, by the Banach fixed point theorem, there exists a unique solution to (6.3).

Next, we want to show that for large enough $N$, the solution to (6.3) is also the solution to (6.2).

**Lemma 6.7.** For $N \geq e^{(\gamma_0 + \gamma_1)T} ((h_0 + h_1)T + (g_0 + g_1))$, the solution $a^{(N)}$ to (6.3) satisfies the inequalities

$$(6.4) \quad 0 \leq g_i + \int_0^T \left(h_i - 2(a_i^{(N)}(s))^2 - \gamma_i (a_i^{(N)}(s) - a_j^{(N)}(s))\right) ds \leq N$$

for all $t \in [0, T]$, where $i, j \in \{0, 1\}$ and $i \neq j$.

**Proof.** For simplicity of notations, $a_i$ is used instead of $a_i^{(N)}$ for $i = 0, 1$ if there is no confusion.

First, for $i = 0, 1$, we prove the positiveness of $a_i$ by contradiction. Suppose $a_i (i = 0, 1)$ are not positive functions on $[0, T]$. Since $a_0$ is continuous and $a_0(T) = g_0 > 0$, there exists some $\tau \in [0, T]$ as the closest time to $T$ such that $a_0(\tau) = 0$. Note that finding such a $\tau$ is possible. Let $t_n \in [0, T]$ be a non-decreasing sequence such that $a_0(t_n) = 0$, there exists some $\tau$ such that $t_n \to \tau < T$ as $n \to \infty$ since $a_0$ is continuous and $a_0(T) = g_0 > 0$. By the continuity of $a_0$, we have $a_0(\tau) = 0$, which gives the desirable point $\tau$. Then for all $t \in (\tau, T]$, we prove the positiveness of $a_0(t)$.

In this case, plugging $t = \tau$ to (6.2), we have $a_0'(\tau) = -h_0 - \gamma_0 a_1(\tau) > 0$, which yields $a_1(\tau) < 0$. Since $a_1$ is continuous on $[0, T]$ and $a_1(T) = g_1 > 0$, from the intermediate value theorem, there exists some $\tilde{\tau} \in (\tau, T)$ such that $a_1(\tilde{\tau}) = 0$ and $a_1'(\tilde{\tau}) > 0$. However, this indicates that $a_1'(\tilde{\tau}) = -h_0 - \gamma_0 a_0(\tilde{\tau}) > 0$ by plugging $t = \tau$ back to (6.2), and it implies $a_0(\tilde{\tau}) < 0$, which contradicts with the fact that $a_0(t) > 0$ for all $t \in (\tau, T]$. Thus the positiveness of $a_0$ and $a_1$ is obtained.
with \((a_0 + a_1)(T) = g_0 + g_1\). By Grönwall’s inequality,
\[
(a_0 + a_1)(T - t) \leq e^{(\gamma_0 + \gamma_1)T}((g_0 + g_1) + (h_0 + h_1)T), \quad \forall t \in [0, T].
\]

Hence \(a_i(t) \leq e^{(\gamma_0 + \gamma_1)T}((g_0 + g_1) + (h_0 + h_1)T)\) for all \(t \in [0, T], i = 0, 1\). Hence, when \(N \geq e^{(\gamma_0 + \gamma_1)T}((g_0 + g_1) + (h_0 + h_1)T)\), (6.4) holds.

**Lemma 6.8.** With the given of \(h_y, g_y \in \mathbb{R}^+, y = 0, 1\), there exists a unique solution to the Riccati system (2.12).

**Proof.** The existence, uniqueness and boundedness of the solution to \(a_y (y = 0, 1)\) are shown in Lemma 6.6 and Lemma 6.7. Given \((a_y : y = 0, 1)\), the coefficient functions \(b_y (y = 0, 1)\) form a linear ordinary differential equation system. Applying the Theorem 12.1 in [2], we can obtain the existence and uniqueness of \(c_y, k_y (y = 0, 1)\).

**6.4. Multidimensional Problem.** In this subsection we consider the multidimensional problem, which is a straightforward extension of the previous one-dimensional setup. The same type of Riccati system to characterize the equilibrium and the value function is obtained, and we have a similar result as the Theorem 2.4.

Suppose that \(X_t, W_t\) and \(\alpha_t\) take values in \(\mathbb{R}^d\), and all components of \(W_t\) are independent. Suppose that the dynamic of the generic player is given by
\[
dX_t = \alpha_t dt + dW_t.
\]

Consider the cost function
\[
J[m](y, x, t, \mu, \nu) = \mathbb{E} \left[ \int_t^T \left( \frac{1}{2} \|\sigma_s\|^2 + h(Y_s) \right) ds + \left( g(Y_T) \int_{\mathbb{R}^d} \|X_t - z\|^2 m(dz) \right) \right] = \mathbb{E} \left[ \int_t^T \frac{1}{2} \sigma_s \sigma_s + h(Y_s) \right] \left( X_s X_s \right) \left( X_s - 2 \mu_s X_s + \nu_s \cdot 1 \right) ds + \left( g(Y_T) \right) \left( X_T - 2 \mu_T X_T + \nu_T \cdot 1 \right) \left( X_t = x, Y_t = y, \mu_t = \hat{\mu}, \nu_t = \hat{\nu} \right),
\]

where \(m\) is the joint density function in \(\mathbb{R}^d\), and \(\mu, \nu\) take value in \(\mathbb{R}^d\). For \(y = 0, 1\), define
\[
\begin{align}
\alpha_y' - 2a_y^2 - \gamma_y a_y + \gamma_y a_{1-y} + h_y &= 0, \\
b_y' - 4a_y b_{1-y} - \gamma_y b_y + \gamma_y b_z + h_y &= 0, \\
c_y' + da_y + db_y - \gamma_y c_y + \gamma_y c_{1-y} &= 0, \\
k_y' - 2a_y^2 + 4a_y b_y - \gamma_y k_y + \gamma_y k_{1-y} &= 0, \\
a_y(T) = g_y, b_y(T) = g_y, c_y(T) = 0.
\end{align}
\]

**Theorem 6.9 (Verification theorem for MFGs).** There exists a unique solution \((a_y, b_y, c_y, k_y : y = 0, 1)\) for the Riccati system (6.5). With these solutions, for \(t \in [0, T]\), the MFG equilibrium path follows \(\hat{X} = \hat{X}[\hat{m}]\) is given by
\[
d\hat{X}_t = 2 a_y(t)(\mathbb{E}[X_0] - \hat{X}_t) dt + dW_t, \quad \hat{X}_0 = X_0,
\]

with equilibrium control \(\hat{\alpha}_t = 2 a_y(t)(\mathbb{E}[X_0] - \hat{X}_t)\). Moreover, the value function \(U\) is
\[
U(m_0, y, x) = a_y(0)x - 2a_y(0)x[1 + k_y(0)] + b_y(0)[m_0] + c_y(0)
\]
for $y = 0, 1$.

The proof is similar to the one-dimensional problem, and we don’t show the details here.

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