Complex Networks of Functions

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Abstract

Functions correspond to one of the key concepts in mathematics and science, allowing the representation and modeling of several types of signals and systems. The present work develops an approach for characterizing the coverage and interrelationship between discrete signals that can be fitted by a set of reference functions, allowing the definition of transition networks between the considered discrete signals. While the adjacency between discrete signals is defined in terms of respective Euclidean distances, the property of being adjustable by the reference functions provides an additional constraint leading to a surprisingly diversity of transition networks topologies. First, we motivate the possibility to define transitions between parametric continuous functions, a concept that is subsequently extended to discrete functions and signals. Given that the set of all possible discrete signals in a bound region corresponds to a finite number of cases, it becomes feasible to verify the adherence of each of these signals with respect to a reference set of functions. Then, by taking into account also the Euclidean proximity between those discrete signals found to be adjustable, it becomes possible to obtain a respective transition network that can be not only used to study the properties and interrelationships of the involved discrete signals as underlain by the reference functions, but which also provide an interesting complex network theoretical model on itself, presenting a surprising diversity of topological features, including modular organization coexisting with more uniform portions, tails and handles, as well as hubs. Examples of the proposed concepts and methodologies are provided respectively with respect to three case examples involving power, sinusoidal and polynomial functions.

1 Introduction

Functions are to mathematics as sentences are to linguistics, constituting basic resources for developing more complete mathematical systems and models. The importance of functions is reflected in their widespread applications not only to the physical sciences, but to virtually every scientific field.

Traditionally, the mathematical study of functions and their properties has been approached in continuous vector spaces, involving infinite instances of a given type of function. While this constitutes an effective and important approach, most of the signals in practical applications have discrete nature, being represented as discrete signals or vectors. This follows as a consequence of the sampling of physical signals by using acquisition systems that inherently implies the signals to be quantized along their domain and magnitude.

Though discrete functions are systematically studied in areas as digital signal processing (e.g. [1, 2]), emphasis is often placed on aspects of quantization errors and representations in the frequency domain, employing the Fourier series or transform (e.g. [3]). However, relatively lesser attention is typically focused on the relationship between the discrete signals, or on how they can be approximated by specific functions. Though the latter subject constitutes one of the main motivation of the areas of numerical methods (e.g. [4]) and numerical analysis (e.g. [5]), this subject is typically approached from the perspective of function approximation, not often addressing the interrelationship between functions.

The present work develops an approach aimed at characterizing not only which discrete signals in a discrete
region $\Omega \subset \mathbb{R}^2$ can be adjusted by a given set of reference functions $g_i(x)$, $i = 1, 2, \ldots, N$, but also how such adjustable discrete signals interrelate one another in the sense of being similar, or adjacent. The consideration of a pre-specified set of function types happens frequently in science, especially when fitting data or studying dynamic systems. In particular, the solution of linear systems of differential equations is often approached in terms of linear combinations of a set of eigenfunctions (e.g. [6]), which could also be taken as the reference functions considered in this work.

The concepts and methods developed in the present article are interesting not only theoretically while studying how distinct types of functions are related, but also from several application perspectives, such as characterizing specific discrete spaces, discrete signal approximation, morphing of functions (i.e. transforming a function into another through incremental changes), controlling systems underlain by specific types of function, among many other possibilities. In a sense, functions can be approached as a way to constrain, in specific respective manners, the adjacency between continuous signals in a given region or space. For instance, the function sine restricts all possible continuous signals in a given space.

In addition to the relevance of the described developments respectively to the aforementioned mathematical aspects, they also provide several contributions to the area of network science (e.g. [7]). Indeed, as it will be seen along this work, the transition networks derived from discrete signal spaces with respect to sets of reference functions are characterized by a noticeably rich topological structure that can involve modularity, hubs, symmetries, handles and tails [8], as well as coexistence from several application perspectives, such as characterization of complex networks, but also constitute a model or benchmark that can be used as reference in studies aimed at investigating the classification and robustness of networks, as well as investigations addressing the particularly challenging relationship between network topology and possible implemented dynamics.

In order to obtain the means for quantifying how discrete signals in a region can be approximated by reference functions, and how these signals interrelate one another, we develop several respective concepts and methods. More specifically, after defining the problem in a more formal manner, we proceed by suggesting how to define a system of adjacencies between continuous functions, in terms of the identification of respective transition points. These concepts are then transferred to discrete signals, allowing the proposal of indices for quantifying the coverage of the discrete signals by adopted reference functions. Subsequently, we adapt the concept of adjacency between functions to discrete signals, allowing the derivation of a methodology for obtaining transition networks expressing how the discrete signals in a region can be transformed one another while being approximated as instances of the reference functions. Several studies involving the obtained transition networks are then described, including the identification of shortest paths between two adjustable discrete signals, random walks related to the unfolding of dynamics on the network, as well as the possibility of identifying discrete signals that are more central regarding the interrelationships represented in the transition networks.

The developed concepts and methods are then illustrated with respect to three main case examples involving (i) four power functions; (ii) a single complete polynomial of forth order; and (iii) two sets of hybrid reference functions involving combinations of power functions and sinusoidals. Several remarkable results are identified and discussed.

## 2 Defining the Problem

Consider the region $\Omega \subset \mathbb{R}^2$ in Figure 1, which corresponds to the Cartesian product of the intervals $x_{\text{min}} \leq x \leq x_{\text{max}}$ and $y_{\text{min}} \leq y \leq y_{\text{max}}$.

![Figure 1: A region $\Omega \subset \mathbb{R}^2$ delimited as $x_{\text{min}} \leq x \leq x_{\text{max}}$ and $y_{\text{min}} \leq y \leq y_{\text{max}}$, and an example of a function $y = f(x)$ completely comprised in this region.](image)

Let $y = f(x)$ correspond to a generic signal, which can be associated to a function, completely bound in $\Omega$, in the sense of having all its points comprised within $\Omega$. No requirement, such as continuity or smoothness, are whatsoever imposed on these functions.

In addition, consider the difference between two generic functions $y = f(x)$ and $y = h(x)$, both comprised in $\Omega$, as corresponding to the following root mean square distance (or error):

$$\delta(f, h) = \sqrt{\frac{1}{x_{\text{max}} - x_{\text{min}}} \int_{x_{\text{min}}}^{x_{\text{max}}} [f(x) - h(x)]^2 dx} \quad (1)$$
A possible manner to quantify the similarity between \( f(x) \) and \( h(x) \) is as

\[
\sigma(f, h) = e^{-\alpha \delta(f, h)}
\]  

(2)

for some chosen value of \( \alpha \).

Let \( y = g_i(x), i = 1, 2, \ldots, N \) be a finite set of specific functions \( \text{types} \) taken as a reference for our analysis. For instance, we could have \( g_1(x) = a_1x + a_0, g_2(x) = a_1x^2 + a_0, \) and \( g_3(x) = a_1x^3 + a_0 \) and \( g_4(x) = a_1x^4 + a_0 \), with \( a_0, a_1, a_2, a_3, a_4 \subseteq \mathbb{R} \).

An interesting question regards the identification, among all the possible signals \( y = \tilde{f}(x) \) in \( \Omega \), of which of these signals can be expressed as \( g_i(x) \) for \( i = 1, 2, \ldots, N \), by yielding zero difference or unit similarity between \( \tilde{f}(x) \) and \( g_i(x) \). For each of the reference functions \( g_i(x) \), we obtain a respective set \( S_i \) containing all functions \( \tilde{f}(x) \) that can be exactly expressed in terms of \( g_i(x) \).

It is also interesting to allow for some tolerance by taking these two functions to be related provided:

\[
\delta(\tilde{f}, g_i) \leq \tau_d
\]  

(3)

or, considering their similarity, as:

\[
\sigma(\tilde{f}, g_i) \geq \tau_s
\]  

(4)

with:

\[
\tau_s = e^{-\alpha \tau_d}
\]  

(5)

The identification of the sets \( S_i \) can provide interesting insights regarding the relative density of each type of the reference functions in the specified region \( \Omega \), paving the way to the identification of reference functions with more general fitting capability as well as the interrelationship between these functions, in the sense of their proximity.

It should be observed that the obtained \( S_i \) will also depend on the specific size (or even shape) of \( \Omega \), as a consequence of the requirement of all functions to be completely bound in that region. The alternative approach of allowing the clipping of functions can also be considered, but this is not developed in the present work.

In this work, we focus on \( \text{discrete signals} \), which are typically handled in scientific applications and technology. These signals are sampled along their domain and quantized in their magnitude (see Section 4). In order to identify the adjacency between discrete signals, given an \( \Omega \) and a set of reference functions \( g_i(x) \), first we identify (by using linear least squares) the discrete functions that can be approximated, within a tolerance, by the reference functions, therefore defining the sets \( S_i \), and then link these functions by considering their pairwise Euclidean distance. The thus obtained network \( \Gamma \) or network can be verified to be undirected and to contain a total of nodes equal to the sum of the cardinality of the obtained sets \( S_i, i = 1, 2, \ldots, N \).

In addition, each of the nodes becomes intrinsically associated to the respective reference functions that were found to provide a good respective approximation. In case a discrete function \( \tilde{f} \) is found to be adjusted by two or more of the reference functions, only that corresponding to the best fitting may be associated to \( \tilde{f} \), therefore avoiding replicated labeling. The reference function thus associated to each node of \( \Gamma \) is henceforth called the node \( \text{type} \).

The transition network \( \Gamma \) provides a systematic representation of the relationships between the discrete functions in \( \Omega \) that can be reasonably approximated by the reference functions. Several concepts and methods from the area of network science (e.g. \[7, 9\]) can then be applied in order to characterize the topological properties of the obtained network. For instance, the average degree of a node can provide an interesting indication about how that function can be transformed (or ‘morphed’), by a minimal perturbation, into other functions in \( \Omega \).

The definition of a system of adjacencies between the functions of \( \Omega \) as proposed above also paves the way for performing respective random walks (e.g. \[10\]). Starting at a given node, adjacent nodes are subsequently visited according to a given criterion (e.g. uniform probability), therefore defining sequences of incremental transformations of the original function. These trajectories of functions can provide insights about how a function can be progressively transformed into another (morphing), to define minimal distances between any of the adjustable functions in \( \Omega \) or, when associated to energy landscapes, to investigate the properties of respectively associated dynamical systems (e.g. \[11\]), including possible oscillations (cycles) and chaotic behavior.

3 Continuous Function Adjacency

A mathematical function often involves parameters, corresponding to values determining its respective instantiation. For instance, the function:

\[
g(x) = a_1x + a_0
\]  

(6)

corresponds to a straight line function whose inclination and translation is specified by the parameters \( a_0 \) and \( a_1 \), respectively.

Given two generic functions in the region \( \Omega \), a particularly interesting question is whether one of them can be made identical to the other, which will be henceforth be expressed as these functions being mutually adjacent, in the sense of providing an interface between these two functions, which can be therefore transitioned. More specifically, let the two following functions \( g_i(x) \) and \( g_j(x) \), with
respective parameters $a_0^1, a_1^i, \ldots, a_i^j, \text{ and } a_0^j, a_1^j, \ldots, a_j^j$:

$$g_i(x; a_0^i, a_1^i, \ldots, a_i^j)$$

$$g_j(x; a_0^j, a_1^j, \ldots, a_j^j)$$

It should be kept in mind that, throughout this work, the superscript value $j$ in the terms $a_0^j$ corresponds to an index associated to the respective reference function, not corresponding to the $j$-power of $a$.

The functions $g_i()$ and $g_j()$ can be said to be adjacent provided it is possible to find respective configurations of parameters $\tilde{a}_0^i, \tilde{a}_1^i, \ldots, \tilde{a}_i^j$ and $\tilde{a}_0^j, \tilde{a}_1^j, \ldots, \tilde{a}_j^j$ so that:

$$g_i(x; \tilde{a}_0^i, \tilde{a}_1^i, \ldots, \tilde{a}_i^j) = g_j(x; \tilde{a}_0^j, \tilde{a}_1^j, \ldots, \tilde{a}_j^j)$$

for every value of $x$ in $\Omega$.

The set of parameters $\tilde{a}_0^i, \tilde{a}_1^i, \ldots, \tilde{a}_i^j$ and $\tilde{a}_0^j, \tilde{a}_1^j, \ldots, \tilde{a}_j^j$ are henceforth understood to represent a transition point in the parameter space $[\tilde{a}_0^i, \tilde{a}_1^i, \ldots, \tilde{a}_i^j, \tilde{a}_0^j, \tilde{a}_1^j, \ldots, \tilde{a}_j^j]$, namely:

$$P_{g_i \leftrightarrow g_j} : [\tilde{a}_0^i, \tilde{a}_1^i, \ldots, \tilde{a}_i^j, \tilde{a}_0^j, \tilde{a}_1^j, \ldots, \tilde{a}_j^j]$$

Observe that each transition point defines a respective instantiation of both involved functions, therefore also corresponding to a specific instantiated function in $\Omega$.

As an example, let’s consider the following four parametric power functions:

$$g_1(x) = a_1^1 x + a_0^1$$

$$g_2(x) = a_2^2 x^2 + a_0^2$$

$$g_3(x) = a_3^3 x^3 + a_0^3$$

$$g_4(x) = a_4^4 x^4 + a_0^4$$

with $a_0^1, a_0^2, a_0^3, a_0^4, a_1^1, a_2^2, a_3^3, a_4^4 \in \mathbb{R}$.

All pairwise combinations of these functions $g_i()$ and $g_j()$ have respective transition points corresponding to $\tilde{a}_1^i = \tilde{a}_1^j = 0$ for any values of $a_0^i$ and $a_0^j$ for which the functions remain completely comprised within $\Omega$. Though the four reference functions above have an infinite number of pairwise transition points, each of them defines a respective transition network $\Gamma$ as presented in Figure 2.

It is interesting to observe that, in this particular example, each of the transition points corresponds to the constant functions $g(x) = a_0$ for $a_0 = a_3^3 = a_4^4$, which therefore acts as a quadruple transition point for each $a_0 \in [y_{\text{min}}, y_{\text{max}}]$ with $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = 0$:

$$P : [\tilde{a}_0 = \tilde{a}_2 = \tilde{a}_3 = a_0, \tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 0]$$

Observe that other sets of reference functions can present many other types of transition points, which can be of types other than the null function. Actually, any shared term between two parametric functions potentially corresponds to a transition point.

In addition to transitions between types of reference functions as developed above, it is also possible to have transitions between incrementally different instances of a same type of function. This can be achieved by adopting a tolerance $\tau$ regarding the similarity of two instances of the same type of function $g(x)$, i.e.:

$$\int_{x_{\text{min}}}^{x_{\text{max}}} \left[ g(x; a_0, a_1, \ldots, a_{N-1}) - g(x; a_0 + \delta_0, a_1 + \delta_1, \ldots, a_N + \delta_{N-1}) \right] dx \leq \tau$$

In this manner, it is possible to obtain long sequences of transitions between instances of a same function as the respective parameters are incrementally varied ($\delta$), typically giving rise to handles and tails [8] in respectively obtained network representations.

Given that the approach reported in this work is respective to discrete signals and functions given a tolerance $\tau$, both types of function transitions identified in this section are expected to be taken into account and incorporated into the respectively derived transition networks.

4 The Discrete Case

Though interesting in itself, the above described problem involves infinite and non-countable sets $S_i$. Though this could be approached by using specific mathematical resources, in the present work we focus on regions $\Omega$ that are discrete in both $x$ and $y$, taken with respective resolutions:

$$\Delta x = \frac{x_{\text{max}} - x_{\text{min}}}{N_x - 1}$$

$$\Delta y = \frac{y_{\text{max}} - y_{\text{min}}}{N_y - 1}$$ (11)
where \( N_x \) and \( N_y \) correspond to the number of discrete values taken for representing \( x \) and \( y \), respectively.

The so-obtained discretized region \( \Omega \) is depicted in Figure 3.

![Figure 3: A discretized region \( \Omega \subset \mathbb{R}^2 \), with \( N_x \) values along the \( x \)-axis and \( N_y \) values along the \( y \)-axis.](image)

More specifically, we now have that:

\[
\begin{align*}
X_j &= (j - 1) \Delta x - x_{\min} \\
Y_k &= (k - 1) \Delta y - y_{\min}
\end{align*}
\]

(12)

for \( j = 1, 2, \ldots, N_x \) and \( k = 1, 2, \ldots, N_y \).

Now, the possible functions in \( \Omega \) can be expressed as the finite set of vectors or discrete signals:

\[
\vec{f} = [f_1 f_2 \ldots f_{N_x}]^T
\]

with \( f_j \) taking values in the set \( \{Y_k\} \) respectively to the abscissae \( X_j \).

It is assumed henceforth, typically with little loss of generality, that \( x_{\min} = -1, x_{\max} = 1, y_{\min} = -1, y_{\max} = 1 \).

The total number of possible vectors in the discretized region \( \Omega \) can now be calculated as being given as corresponding to the number of permutations:

\[
N_T = N_x^{N_y}
\]

(14)

Henceforth, we identify each of the \( N_T \) possible discrete signals (or functions) in \( \Omega \) in terms of a respective label \( n = 1, 2, \ldots, N_T \). In case \( N_y \) is relatively small, it is possible to implement this association by deriving the discrete signal from its respective label \( n \) by first representing this value in radix \( N_y \), yielding the number \([p_{N_y-1} \ldots p_1 p_0]_{N_y}\), and then making:

\[
Y_{i+1} = p_i \Delta y + y_{\min}
\]

(15)

for \( i = 0, 1, \ldots, N_x - 1 \).

The difference between two discrete functions \( \vec{f} \) and \( \vec{h} \) can now be expressed in terms of the following root mean square error:

\[
\delta(\vec{f}, \vec{h}) = \sqrt{\frac{1}{x_{\max} - x_{\min}} \sum_{j=1}^{N_x} [f[x_j] - h[x_j]]^2}
\]

(16)

while the similarity between those functions can still be gauged by using Equation 2.

In order to verify if a given function \( \vec{f} \) can be approximated by a reference function \( g_i(x) \), we apply the linear least squares methodology (e.g. [12]). This approach provides the set of fit parameters (e.g. the coefficients of a polynomial) so as to minimize the error of the fitting as expressed by the sum of the square of the differences between \( \vec{f} \) and \( g_i(x) \) (taken at the abscissae \( X_f \)). For instance, if \( g_i(x) \) is a third degree polynomial and \( N_x = 5 \), we first obtain the matrix:

\[
A = \begin{bmatrix}
1 & X_1 & X_1^2 & X_1^3 \\
1 & X_2 & X_2^2 & X_2^3 \\
1 & X_3 & X_3^2 & X_3^3 \\
1 & X_4 & X_4^2 & X_4^3 \\
1 & X_5 & X_5^2 & X_5^3
\end{bmatrix}
\]

and then express the respective coefficients in terms of the vector:

\[
\vec{p} = [a_0 a_1 a_2 a_3]^T
\]

So that the fitting can be represented in terms of the following overdetermined system:

\[
\vec{f} = A \vec{p}
\]

(17)

The respective solution can be obtained in terms of the pseudo-inverse of \( A \) as:

\[
\vec{p} = (A^T A)^{-1} A^T \vec{f}
\]

(18)

![Figure 4: Example of the linear least squares methodology for fitting a discrete signal \( Y_i = f(X_i) \), with \( N_x = 7 \) and \( N_y = 5 \), by a reference function of the type \( a_1 x^3 + a_0 \). The respectively obtained root mean square error was \( \tau_d = 0.136 \), implying a similarity of \( \tau_s = e^{-10 \tau_d} = 0.256 \) (for \( \alpha = 10 \)).](image)
5 Discrete Signals Coverage

The discretization of Ω implies that not all signals in $\tilde{f}$ can be expressed with full accuracy in terms of reference functions $\tilde{g}_i$, so that it becomes important to adopt some difference tolerance $\tau_d$, or respectively associated similarity tolerance $\tau_s$. Henceforth, every discrete signal $\tilde{f}$ that can be approximated by a reference function $g_i$ within a given tolerance $\tau$ will be said to be *adjustable* by that reference function.

It is important to keep in mind that, when a tolerance is allowed, more than one of the reference functions can be verified to provide a good enough (i.e. with error smaller than the specified tolerance) approximation, in which case a same function $\tilde{f}$ will be identified as being adjustable by more than one reference function, which is reasonable given that this actually happens in discrete domains. However, in case the mapping is required to be made unique, it is possible to keep only one of the fittings for each possible $\tilde{f}$, such as that corresponding to the smallest approximation error. In this work, however, multiple adjustments will be considered.

The sets $S_i(\tau_d)$, which are defined by $\tau_d$, will now contain a *finite* number of discrete functions. Thus, given a discrete signal $\tilde{f}$ and a set of reference functions $g_i$, $i = 1, 2, \ldots, N$, the total number of adjustable signals $N_a$ can be expressed as:

$$N_a = \sum_{k=1}^{N} S_k(\tau_d) \quad (19)$$

We can now take the relative frequency of each reference function $g_i$ with respect to the whole of adjustable functions as:

$$r(g_i, \tau_d) = \frac{\# \, \{S_i(\tau_d)\}}{N_a} \quad (20)$$

where $\# \, \{S_i(\tau_d)\}$ corresponds to the cardinality of the set $S_i(\tau_d)$.

This measurement, which is henceforth referred to as *relative coverage*, can be used to compare the fitting potential of each of the considered reference functions.

It is also possible to consider the following densities relative to the total number of functions in $\Omega$ as:

$$q(g_i, \tau_d) = \frac{\# \, \{S_i(\tau_d)\}}{N_T} \quad (21)$$

In case only one fitting is associated to each possible discrete signal $\tilde{f}$ in $\Omega$, we will have that $0 \leq q(g_i, \tau_d) \leq 1$ and that $\sum_{k=1}^{N} q(g_i, \tau_d) = 1$. This can be achieved by considering the sets $S_i = S_i - \bigcup_{k=1}^{N} S_k \neq i$ instead of $S_i$ in Equation 21. Otherwise, this measurement may take values larger than 1 and we will also have that $\sum_{k=1}^{N} q(g_i, \tau_d) \geq 1$, indicating that the possible discrete functions in $\Omega$ is being covered in excess.

The relative density $q(g_i, \tau_d)$, henceforth called the *coverage index* of $g_i$ provides a means to quantify of how well the reference function $g_i$ covers the discrete signals in the given region $\Omega$ and resolution $\tau_d$. Larger values of $q(g_i, \tau_d)$ will typically be observed when $\tau_d$ is increased (or $\tau_s$ is decreased).

Also, observe that the above relative densities also depend on the choice of the discretization resolutions $\Delta x$ and $\Delta y$, with $\# \, \{S_i(\tau_d)\}$ increasing substantially with $N_x$ and $N_y$.

6 Discrete Functions Adjacency

While the relative densities $r(g_i, \tau_d)$ can provide interesting insights about the generality of each considered reference function $g_i$, these measurements can provide no information about the proximity or interrelationship between the discrete functions $\tilde{f}$ as fitted by a set of reference functions $g_i$, $i = 1, 2, \ldots, N$. However, it is possible to quantify the proximity between all the possible discrete functions in $\Omega$ in terms of some distance between the respective vectors and then define links between the pairs of functions that have respective distances smaller than a given threshold $L$.

Consider the Euclidean distance between two discretized functions $f[i]$ and $\tilde{f}[j]$ in the region $\Omega$ as:

$$\omega \left( f[i], \tilde{f}[j] \right) = \sqrt{\sum_{k=1}^{N_T} \left[ f_k[i] - f_k[j] \right]^2} \quad (22)$$

The whole set of Euclidean distances between every possible pair of functions in a given $\Omega$ can then be represented in terms of the following distance matrix:

$$W_{i,j} = \omega \left( f[i], \tilde{f}[j] \right) \quad (23)$$

The symmetric matrix $W$ can be immediately understood as providing the strength of the links between the nodes of a graph, each of these nodes being associated to one of the possible $N_T$ discrete functions in a given $\Omega$. However, such a graph would express the *distances* between functions, not their *proximity*. Though these distances could be transformed into similarity measurements by adopting an expression analogous to Equation 2, therefore yielding a weighted respective graph, in this work we adopt the alternative approach of understanding two discrete functions as being *adjacent* provided the respective Euclidian distance as defined above is smaller or equal to a given threshold $L$.

Overall, obtaining the transition network for a set of reference functions $g_i(x)$ and a respective discrete region $\Omega$, with $N_x$ and $N_y$, involves the following 3 main processing stages:

6
• Assign a label $n$ to each of the $N_T = N_x N_y$ possible discrete signals in $\Omega$;

• For each value $n = 1, 2, \ldots, N_T$, obtain the respective function $\vec{f}_n = [Y_{N_x-1}, \ldots, Y_2, Y_1]$ by using Equation 15 and apply least square approximation respectively to each of the reference functions $g_i(x)$, $i = 1, 2, \ldots, N$. In case the similarity between $\vec{f}_n$ and $g_i$ as obtained by applying Equations 16 and then 2, is larger or equal to $\tau_s$, assign a respective node with label $n$, also incorporating the type $i$ of the respective approximating function $g_i(x)$;

• Interconnect all pairs of nodes obtained in the previous step which have Euclidean distance smaller than $L$, therefore yielding the transition network $\Gamma$.

It is also important to keep in mind that one so obtained transition network can be understood as constraining the overall adjacency network between all possible functions $\vec{f}$ in $\Omega$ so that only the nodes associated to cases that can be adjusted with good accuracy by a respective reference function $g_i$ are maintained. In brief, the transition network therefore provides a representation of the adjacency between the possible discrete functions that can be adjusted by the reference functions.

7 Optimized Transitions and Random Walks

The derivation of the transition network $\Gamma$ respective to a set of reference functions and a discrete region $\Omega$ paves the way to several interesting analysis and simulations, some of which are discussed in this section.

One first interesting possibility is, given two functions $\vec{f}_i$ and $\vec{f}_j$ in $\Gamma$, to identify the shortest paths between the respective nodes. We mean paths in the plural because it may happen that more than one shortest path exist between any two nodes of a network. Each of these obtained shortest paths indicate the smallest number of successive transitions from $\vec{f}_i$ to $\vec{f}_j$ that are necessary to take one of those functions into the other (or vice-versa) while using only instances of the considered reference functions. This result is potentially interesting for several applications, including implementing optimal controlling dynamics underlain by the reference functions, or optimal morphing between two or more signals underlain by the respectively considered reference functions.

Given a transition network $\Gamma$ and all the shortest paths between its pairs of nodes, it also becomes interesting to consider statistics of the length of those paths, such as their average and standard deviation, which can provide interesting information about the overall potential of the reference functions for implementing optimal transitions and morphings as mentioned above.

Another interesting approach considering a transition network consists in performing random walks (e.g. [10]) along its nodes. Several types of random walks can be adopted, including uniformly random and preferential choice of nodes according to several local topological properties of the network nodes, such as degree and clustering coefficient. These random walks can be understood as implementing respective types of dynamics in the network. For instance, a random walk with uniform transition probabilities is intrinsically associated to diffusion in the network. In this manner, random walks on transition networks provide means for simulating and characterizing properties related to dynamics involving transition between the discrete signals in $\Gamma$.

Yet another interesting perspective allowed by the derivation of the transition matrices $\Gamma$ concerns studies involving betweenness centrality (e.g. [9]) or accessibility (e.g. [13]) of edges and nodes in $\Gamma$, which can complement the two aforementioned analyses. For instance, it could be interesting to use the accessibility to identify the discrete signals in $\Omega$, as underlain by the reference functions $g_i(x)$, leading to the largest and smallest number of nodes, therefore providing information about the role of those nodes regarding influencing or being influenced by other nodes. The accessibility measurement can also be applied in order to identify the center and periphery of the obtained transition networks [14].

8 Case Example 1: Power Functions

This section presents a case example of the proposed methodology assuming the four power functions in Equation 9. First, we consider the region $\Omega$ as being sampled by $N_x = 5$ abscissa values and $N_y = 7$ coordinate samples, assuming $\tau_s = 0.2$, $\alpha = 10$, and $L = 0.6$. The resulting transition network is depicted in Figure 5, as visualized by the Fruchterman-Reingold methodology [15].

Several remarkable features can be identified in the obtained transition network. First, we find the nodes organized according to a well-defined bilateral symmetry, which can be verified to correspond to the sign of the coefficients $a^i_1$, $i = 1, 2, 3, 4$. In addition, the nodes corresponding to approximations by the power functions $g_1$ and $g_3$, both of which presenting odd parity, tend to be adjacent one another, with a similar tendency being observed for the nodes respective to the evenly symmetric power functions $g_2$ and $g_4$. Five main clusters of nodes can also be identified along the diagonal of the figure running from bottom-left to top-right, each of which with a
Figure 5: Visualization, by using the Fruchterman-Reingold method, of the transition network obtained for the reference power functions in Eq. 9, \( N_x = 5 \) and \( N_y = 5 \), assuming \( \tau_s = 0.2 \) and \( L = 0.6 \). The colors indicate, according to the legend, the respective type of power function approximating the discrete signals. The two semiplanes of the bilateral symmetry corresponds to the sign of the coefficients \( a_{1}^1 \). The five main clusters of nodes correspond to the constant functions \( a_{0}^1 = -1, -0.5, 0, 0.5, 1 \). Observe the hubs at the center of each of the 5 clusters of nodes. See text for more information.

respective central hub. These hubs correspond to the constant functions \( a_{0}^1 = -1, -0.5, 0, 0.5, 1 \) which, as discussed in Section 3, represent transition points of the adopted set of reference functions and \( \Omega \). As could be expected, these hubs and surrounding clusters of nodes, are characterized by the presence of all the four types of considered power functions. The other, smaller, clusters of nodes are associated to transition points allowed by the adoption of a non-null tolerance, and possibly reflect the intrinsic structure of the discrete space \( \Omega \).

It is also possible to derive a reduced version of the above transition network. Basically, all nodes associated to each of the 4 categories of nodes (i.e. the adopted 4 reference functions) are subsumed by a respective node, while the interconnections between all the original nodes are also collected into the links between the agglomerated nodes. Figure 6 illustrates the reduced version of the transition network in Figure 5.

The obtained reduced graph corroborates the predominance of transitions between odd (\( g_1 \) and \( g_3 \)) and even (\( g_2 \) and \( g_4 \)) power functions. In addition, we also have that the largest number of transitions is observed between instances of \( g_1 \), and that the smallest number of transitions takes place between instances of \( g_3 \) and \( g_4 \). The largest number of transitions between odd and even functions take place between \( g_1 \) and \( g_2 \).

One interesting question regards to what an extent the transition networks may vary with respect to distinct values of the tolerance \( \tau_d \) or \( \tau_s \). Figure 7 depicts 9 additional transition networks obtained for the same configuration adopted in the previous example, with respect to several different values of \( \tau_s \) respectively indicated above each network.

As illustrated in Figure 7, the size and connectivity of the transition network decreases steadily with \( \tau_s \), and several markedly distinct types of networks, most of which presenting bilateral symmetry, are respectively observed. Given that the more generalized ability of the power functions to adjust the discrete signals when larger tolerance values are allowed (i.e. small values of \( \tau_s \)), the initial networks tend to present a more widespread and uniform interconnectivity. Observe also that the networks split into two or more connected components for values of \( \tau_s \) larger than approximately 0.2.

The relative coverage and coverage index (see Section 5) of the four considered power functions for \( \tau_s = 0.01, 0.02, \ldots, 0.5 \) are shown in Figures 8(a) and (b), respectively.

Similar values of relative coverage can be observed for the four power functions, with oscillations along \( \tau_s \) that tend to increase from left to right in Figure 8(a), up to a point, near \( \tau_s = 0.35 \), where the relative coverages become nearly constant and markedly distinct between the 4 considered types of reference functions.

As expected, the coverage index decreased steadily with \( \tau_s \) for all the four considered reference functions, also presenting values similar. Observe that only a small per-
Figure 7: Additional examples of transition networks obtained for the same configuration used in the previous example, but with respect to several other values of $\tau_s$, as respectively indicated in each network. The colors follow the same convention as in Fig. 5. Network visualizations obtained by using the Fruchterman-Reingold method.

percentage of the possible discrete signals are adjustable at $\tau_s = 0.2$.

Let’s now consider the shortest path between two functions in the above transition network. Figure 9 illustrates the shortest sequence of transitions between the functions respectively identified by the numbers 53 and 105.

Figure 10 shows the first 23 steps of a possible self avoiding random walk along the above transition network, starting at signal $n = 53$.

Self avoiding operation was adopted in not to repeat nodes. Observe the relatively smooth transition, involving minimal modifications of the discrete signals, along each
Figure 8: The relative coverage (a) and coverage index (b) for the case example 1 with $N_x = 5$ and considering the reference functions in Equation 9, as obtained for similarity tolerance values $\tau_s = 0.01, 0.02, \ldots, 0.5$.

Figure 9: The minimal sequence of transitions in the transition network in Figure 5, assuming $N_x = 5$ and $N_y = 5$, leading from signal $n = 53$ to signal $n = 105$. The numbers within parenthesis indicate the type of respectively fitted power function ($1 = g_1$, $2 = g_2$, $3 = g_3$, and $4 = g_4$).

of the implemented transitions.

Figure 11 depicts the transition network obtained for the same situation above, but now with $N_x = 7$ instead of $N_x = 5$.

The resulting transition network again presents several interesting features. As before, we have the bilateral symmetry corresponding to the sign of the coefficients associated to the $x$-term. In addition, clusters and respective central hubs have again be obtained, corresponding to the constant (null) transition functions as observed before. However, unlike the network obtained for $N_x = 5$, now we most of the nodes separated also along the up-down orientation, corresponding to interactions between blue-yellow (up) and red-green (down). These two portions of the transition network can therefore be understood as being directly associated to the odd/even parity of the involved reference functions. Of particular interest is the fact that the discrete signals associated to the blue nodes, associated to the reference function $g_1(x) = a_1^1x + a_0^1$, define a relatively regular pattern of interconnection that is markedly distinct to the more sequential pattern of interconnections observed for the 3 other reference functions. Observe that this transition network also incorporates several handles, corresponding to relatively long sequences of links [8]. Such sequences are associated to incrementally distinct instances of the same type of reference function, as discussed in Section 3.

9 Case Example 2: Polynomials

While the previous case example assumed power functions containing only two terms, we now address the more general situation where only one complete polynomial of order $P$ is adopted as reference function, i.e.:

$$g_1 = a_P x^P + \ldots + a_2 x^2 + a_1 x + a_0$$  \hspace{1cm} (24)

Figure 12 illustrates the transition network obtained for the above polynomial reference function assuming $P = 4$, $N_x = 7$, $N_y = 5$, $\tau_s = 0.4$, $\alpha = 10$, and $L = 0.6$.

Interestingly, a completely different topology is now observed for the polynomial transition network as compared to the previous the networks respective to power functions. The main distinguishing features are two: (i) a much larger number of nodes are now observed; and (ii) their interconnectivity is much more uniform, without present of well-defined heterogeneities such as clusters, hubs, tails or handles. All these properties can be understood as being consequence of the substantially higher flexibility that a complete polynomial function has for adjusting signals as compared to those of the more specific power functions considered previously in this work. As a consequence of this enhanced adjusting property, many more discrete signals could be fitted with reasonably accuracy, hence the larger network size obtained. The observed uniformity of connections also follows from the flex-
Figure 10: One of the many possible random walks with 23 steps in the transition network shown in Figure 5, considering self-avoiding uniform transition probabilities. Observe the incremental change implemented in the involved discrete signals at each successive step. The numbers within parenthesis indicate the type of respectively fitted power function (1 = $g_1$, 2 = $g_2$, 3 = $g_3$, and 4 = $g_4$).

Figure 11: Visualization, by using the Fruchterman-Reingold method, of the transition network obtained for the reference power functions in Eq. 9, $N_x = 7$ and $N_y = 5$, assuming $\tau_s = 0.2$ and $L = 0.6$. The colors indicate, according to the legend, the respective type of power function approximating the discrete signals. See text for more information.

10 Case Example 3: Hybrid Functions

As with power functions and polynomials, also sinusoidal functions are extensively applied in mathematics, physics, and science in general, constituting the basic components of the flexible Fourier series. The third case example considered in the present work adopts a set of reference functions containing two power functions and two sinusoidals, more specifically:

$$
g_1(x) = a_1^1 x + a_0^1$$
$$g_2(x) = a_1^2 x^2 + a_0^2$$
$$g_3(x) = a_1^3 \sin(3x) + a_0^3$$
$$g_4(x) = a_1^4 \sin(5x) + a_0^4$$

Figure 13 illustrates the transition network obtained for the above hybrid reference functions assuming $P = 4$, $N_x = 5$, $N_y = 7$, $\tau_s = 0.2$, $\alpha = 10$, and $L = 0.6$. 
Figure 12: Visualization, by using the Fruchterman-Reingold method, of the transition network obtained for a complete polynomial of order $P = 4$ as single reference function, and $N_x = 7$, $N_y = 5$, $\tau_s = 0.4$, $\alpha = 10$, and $L = 0.6$. The colors are assigned so that increasing values are represented from cyan to magenta color tones.

Figure 13: Visualization, by using the Fruchterman-Reingold method, of the transition network obtained for the reference hybrid (two power and two sinusoidal) functions in Eq. 25, for $N_x = 7$ and $N_y = 5$, $\tau_s = 0.2$, and $L = 0.6$. The colors indicate, according to the legend, the respective type of power function approximating the adjustable discrete signals. In addition to the 5 clusters observed in the previous examples involving power functions, now also tails and handles are obtained. See text for additional discussion.

A particularly interesting structure is observed for this example. First, the five main clusters, corresponding to respective constant transition points, are again observed in analogous manner with the other examples involving power functions. Bilateral symmetry is again observed, being related to the sign of the coefficients $a_i^1$, $i = 1, 2, 3, 4$. Now, surrounding those clusters of nodes, which incorporate all four types of reference functions, we also discern a relatively regular subnetwork involving the first order power $g_1$ (blue) and the lower frequency sinusoidal $g_3$ (yellow), both of which have odd parity. These nodes also tend to form handles at the border of the obtained transition network.

The nodes associated to the second order power function $g_2$ (red) results mostly distributed along the six projecting tails at the periphery of the network, which correspond to incremental instantiations of the same type of function. Contrariwise, the nodes corresponding to discrete signals adjustable by the high frequency sinusoidal $g_4$ (green) are found concentrated in the three most central clusters of nodes of the network, despite the fact that both $g_2$ and $g_4$ share even parity.

In order to study the effect of extending a set of reference functions on the topology of the respectively defined transition network, we incorporate two additional power functions, respective to third and forth orders, into the set of reference functions adopted in the previous example (Eq. 25), yielding the following extended set of reference functions:

$$
\begin{align*}
  g_1(x) &= a_1^1 x + a_0^1 \\
  g_2(x) &= a_2^1 x^2 + a_0^2 \\
  g_3(x) &= a_3^1 x + a_0^3 \\
  g_4(x) &= a_4^1 x^2 + a_0^4 \\
  g_5(x) &= a_5^1 \sin(3x) + a_0^5 \\
  g_6(x) &= a_6^1 \sin(5x) + a_0^6
\end{align*}
$$

Figure 14 illustrates the transition network obtained for the above hybrid reference functions assuming $P = 4$, $N_x = 5$, $N_y = 7$, $\tau_s = 0.2$, $\alpha = 10$, and $L = 0.6$.

It is particularly interesting to contrast the obtained transition network in Figure 14 with the networks in Figure 11, obtained for four power functions, and that in Figure 13, which considers two power functions and two sinusoidal functions. Therefore, it could be expected that network in Figure 14, respective to the union of the reference functions in the two aforementioned sets, inherits some of their respective topological features.

Indeed, the network in Figure 14 incorporates some features from both the related structures. First, we again observe the bilateral symmetry also common to those previous networks. In addition, the obtained network can be understood, to a good extent, to the structure in Figure 11 to which peripheral subnetworks corresponding to the two sinusoids ($g_5$ in pink and $g_6$ in cyan) have been incorpo-
Figure 14: Visualization, by using the Fruchterman-Reingold method, of the transition network obtained for the second case of reference hybrid (four power and two sinusoidal) functions as in Eq. 26, for \( N_x = 7 \) and \( N_y = 5 \), \( \tau_s = 0.2 \) and \( L = 0.6 \). The colors indicate, according to the legend, the respective type of power function approximating the adjustable discrete signals. In addition to the 5 clusters observed in the previous examples involving power functions, now also tails and handles are obtained. See text for additional information.

rated, being characterized by several respective handles. Also of particular interest is the fact of the tails in Figure 14 being assimilated into the inner structure of the network.

11 Concluding Remarks

Functions can be understood as essential mathematical concepts, being widely used both from the theoretical and applied points of view in science and technology. As a consequence of their great importance, whole areas of mathematics and other major areas have been dedicated to their study and applications, including calculus, mathematical physics, linear algebra, functional analysis, numerical methods, numerical analysis, dynamic systems, and signal processing, to name but a few examples.

The present work situates at the interface between several of these areas, also encompassing other areas, including network science, computer graphics, and shape analysis. More specifically, we aimed at developing the issue of how well all possible signals in a given region \( \Omega \) correspond to instances of a given set of reference functions. Given that infinite sets of adjustable functions would be obtained when working with continuous functions, we focused instead on addressing the aforementioned problem in discrete regions, leading to finite sets of adjustable functions to be obtained. In particular, if the region \( \Omega \) is sampled by \( N_x \times N_y \) values, the total number of possible discrete signals in that region is necessarily equal to \( N_x^{N_y} \).

The adoption of discrete signals also paves the way to verify if each of them can be adjusted, given a pre-specified tolerance, as instances of the reference parametric functions by using the least linear squares methodology.

Having identified the sets of adjustable discrete signals respectively to each of the adopted reference functions, it becomes possible not only to study their relative density, but to approach the particularly interesting issue of transitions between adjacent functions, yielding respective transition networks. The adjacency between two functions, as understood in this work, was first characterized with respect to continuous parametric functions as corresponding to respective instances leading to the identity between the two functions, being subsequently adapted to discrete signals and functions by taking into account the Euclidian distances smaller than a specified threshold \( L \).

A number of interesting possible investigations can then be performed with basis on these obtained networks, including studies of optimal sequence of transitions, random walks potentially associated to dynamical systems, as well the identification of particularly central signals in terms of betweenness centrality and accessibility.

The potential of the reported concepts and methods were then illustrated with respect to three case examples respective to: (i) four power functions; (ii) a single complete polynomial of forth order; and (iii) two sets of hybrid reference functions involving combinations of power functions and sinusoidals.

As expected, the coverage index decreased steadily as \( \tau_s \) increased, while the four power functions presented similar potential for adjusting the discrete functions in the assumed region \( \Omega \).

In addition, the obtained transition networks gave rise to a surprising diversity of topologies, including combinations of modularity and regularity, as well as hubs, handles and tails. Several of the networks also were characterized by symmetries which have been found to be related to the sign of the reference function coefficients, as well as their parity. The power functions and sinusoids were found to lead to quite distinct patterns of interconnectivity in the resulting transition networks, with the latter leading to peripheral handles.

The intricate and diverse patterns of topological structure obtained for the transition networks are also influenced by the discrete aspects of the lattice underlying \( \Omega \). For instance, most of the case examples involving power and sinusoids for \( N_x = 5 \) were found to incorporate five clusters of nodes associated to the null discrete transi-
tion. Other topological heterogeneities of the obtained networks are also related to specific anisotropies of the lattice, as well as to the nature of the respective reference functions.

One particularly distinguishing aspect of the proposed approaches concerns the complete, exhaustive representation of every possible discrete signal in the region $\Omega$. As such, these approaches provide the basis for systematic studies in virtually every theoretical or applied areas involving discrete signals or functions. In particular, it would be interesting to revisit dynamic systems from the perspective of the described concepts and methods, associating each admissible signal to a respective node in the transition networks, and studying or modeling specific dynamics by considering these networks.

The generality of the concepts and methods developed along this work paves the way to many related further developments. For instance, it would be interest to extend the approach from 1D signals to higher dimensional scalar and vector fields, as well as to other types of regions possibly including non regular borders or even disconnected parts. It would also be interesting to study other types of functions such as exponential, logarithm, Fourier series, as well as several types of statistical distributions. In addition, the several types of obtainable transition networks can be applied as benchmark in approaches aimed at characterizing classifying complex networks, as well as for studies aimed at investigating the robustness of networks to attacks, and also from the particularly important perspective of relating topology and dynamics in network science. Another interesting possibility consists in applying the developed methodologies to the analysis of real data, such as time series, shapes and images.

Though the present work focused on undirected networks, it is possible to adapt the proposed concepts and methods for handling directed transition networks, therefore extending even further the possibly modeled shapes and dynamics. This can be done, for instance, by defining the concept of adjacency in an asymmetric manner, such as when one of the reference functions approaches, through incremental parameter variations, approaches another parameterless reference function, in which case the direction would extend from the former to the latter respectively associated nodes. Another possibility would be to establish the directions in terms of an external field, which could be possibly associated to a dynamical system.

Last but not least, the networks generated by the proposed methodology yield remarkable patterns when visualized into a geometric space, presenting shapes with diverse types of coexisting regularity, heterogeneity and symmetries. It has been verified that an even wider and richer repertoire of shapes can be obtained by the suggested method by varying the involved parameters. For instance, symmetries of types other than bilateral can be obtained by using reference functions containing 3 or more terms instead of the 2 terms as adopted in most of the examples in this work. One particularly interesting aspect of generating shapes in the described manner is that very few parameters are involved while determining structures with high levels of spatial and morphologic diversity and complexity. Actually, the only involved parameters specifying each of the possibly obtained shapes are $N_x$, $N_y$, the reference functions, $\alpha$ and $L$. This potential for producing such flexible shapes paves the way to several studies not only in shape and pattern generation and recognition, but also for development biology, in the sense that the obtained structures could represent a model of morphogenesis through gene expression control by the reference functions, while the spatial organization of the cells would be defined in a manner similar to the Fruchterman-Reingold method, i.e. nodes that are connected attract one another, while disconnected nodes tend to repel one another. These interactions could be associated to morphic fields (e.g. biochemical concentrations, electric fields, etc.) taking place during development.

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**References**

[1] A. V. Oppenheim and R. Schafer. *Discrete-Time Signal Processing*. Pearson, 2009.

[2] C. Phillips, J. Parr, and E. Riskin. *Signals, Systems and Transforms*. Pearson, 2013.

[3] E. O. Brigham. *Fast Fourier Transform and its Applications*. Pearson, 1988.

[4] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery. *Numerical Recipes in C*. Cambridge University Press, Cambridge, USA, second edition, 1992.

[5] R. L. Burden, G. D. Faires, and A. M. Burden. *Numerical Analysis*. Cengage Learning, 2015.

[6] R. K. Nagle, E. B. Saff, and A. D. Snider. *Fundamentals of Differential Equations*. Pearson, 2017.

[7] A.L. Barabási and Pósfai M. *Network Science*. Cambridge University Press, 2016.
[8] P. R. Villas Boas, F. A. Rodrigues, G. Travieso, and L. da F. Costa. Chain motifs: The tails and handles of complex networks. *Phys. Rev. E*, 77:026106, 2008.

[9] L. da F. Costa, F.A. Rodrigues, G. Travieso, and P.R. Villas Boas. Characterization of complex networks: A survey of measurements. *Advances in Physics*, 56(1):167–242, 2007.

[10] J. Zinn-Justing. *From random walks to random matrices*. Oxford University Press, 2019.

[11] K. Ogata. *System Dynamics*. Pearson, 2003.

[12] L. da F. Costa. Linear least squares: Versatile curve and surface fitting. Researchgate, 2019. https://www.researchgate.net/publication/337103890_Linear_Least_Squares_Versatile_Curve_and_Surface_Fitting_CDT-17. Online; accessed 20-Jan-2021.

[13] B. A. N. Travencolo and L. da F. Costa. Accessibility in complex networks. *Phys. Letts. A*, 373(1):89–95, 2008.

[14] B. A. N. Travencolo, M. P. Viana, and L. da F. Costa. Border detection in complex networks. *New J. Phys.*, 11:063019, 2009.

[15] T. M. J. Fruchterman and E. M. Reingold. Graph drawing by force directed placement. *J. Software: Pract. and Exp.*, 21(1):1129–1164, 1991.