THE ELEMENTARY PARTICLES AS QUANTUM KNOTS
IN ELECTROWEAK THEORY

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Abstract. We explore a knot model of the elementary particles that is compatible with electroweak physics. The knots are quantized and their kinematic states are labelled by $D_{jmm'}$, irreducible representations of $SU_q(2)$, where $j = N/2$, $m = w/2$, $m' = (r + 1)/2$ and $(N, w, r)$ designate respectively the number of crossings, the writhe, and the rotation of the knot. The knot quantum numbers $(N, w, r)$ are related to the standard isotopic spin quantum numbers $(t, t_3, t_0)$ by $(t = N/6, t_3 = -w/6, t_0 = -(r + 1)/6)$, where $t_0$ is the hypercharge. In this model the elementary fermions are low lying states of the quantum trefoil ($N = 3$) and the gauge bosons are ditrefoils ($N = 6$). The fermionic knots interact by the emission and absorption of bosonic knots. In this framework we have explored a slightly modified standard electroweak Lagrangian with a slightly modified gauge group which agrees closely but not entirely with standard electroweak theory.

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1 The Knot Model.

The possibility that the elementary particles are knots has been suggested by many authors, going back as far as Kelvin. There are many different field theoretic ways of constructing classical knots. In particular, a model related to the Skyrme soliton has been described by Fadeev, and by Fadeev and Niemi. Here we study a knot model that is independent of its particular field theoretic realization. In this model the field quanta of standard electroweak theory are to be understood not as point particles but as quantum knots defined by $SU_q(2)$. Since any knot model must at least be compatible with electroweak physics and must also be defined by its symmetry group namely $SU_q(2)$, these are the minimal requirements of a knot model. Then the field quanta are quantum knots; and the most elementary particles (the elementary fermions) are the simplest quantum knots (the quantum trefoils). There are indeed four trefoils and there are four classes of elementary fermions with three fermions in each class. It is then possible to posit four fermionic quantum trefoils, where each trefoil has three states of excitation. Then the electron, muon, and tauon appear as three states of excitation of the leptonic trefoil. In this model the fermionic knots interact by the emission and absorption of bosonic knots.

2 The Characterization of Oriented Knots.

Three-dimensional knots are described in terms of their projections onto a two-dimensional plane where they appear as two-dimensional curves with 4-valent vertices. At each vertex (crossing) there is an overline and an underline. We shall be interested here in oriented knots. The crossing sign of the vertex is +1 or -1 depending on whether the orientation of the overline is carried into the orientation of the underline by a counter-clockwise or clockwise rotation respectively. The sum of all the crossing signs is termed the writhe, $w$, a topological invariant. There is a second topological invariant, the rotation, $r$, the number of rotations of the tangent in going once around the knot.

Let $K$ and $K'$ be oriented knot diagrams with the same writhe and rotation
\[ w(K) = w(K') \]
\[ r(K) = r(K') \]

Then \( K \) is topologically equivalent (regularly isotopic) to \( K' \).

We may label an oriented knot by the number of crossings \((N)\), its writhe \((w)\), and rotation \((r)\). The writhe and rotation are integers of opposite parity.

The symmetry algebra of the unoriented knot is \( SL_q(2) \). We shall describe the oriented knot \((N, w, r)\) by the subgroup \( SU_q(2) \). To make its connection with \( SU_q(2) \) explicit, we may label a knot by the elements of the irreducible representation of \( SU_q(2) \) as follows:

\[ D_{jmm'}^i = D_{\frac{N}{2}w^2r+1}^{\frac{N}{2}} \]

### 3 The Quantum Mechanical Knot.

A **physical** knot as a classical dynamical system will have two topological integrals of the motion: the writhe \((w)\) and the rotation \((r)\). We shall consider systems where \( N \), the number of crossings, is a dynamical integral of the motion as well. We shall assume that the quantum mechanical knot has the same integrals of the motion as the classical knot. Then we label the states of the quantum mechanical knot by these same integrals of the motion.

Since the knot symmetry may be represented by \( SU_q(2) \), we may take the kinematical quantum states to be elements of the irreducible representations of \( SU_q(2) \). These are designated by \( D_{jmm'}^i \) where \( 2j + 1 \) is the dimensionality.

To label the states \( D_{jmm'}^i \) by the integrals of the motion \((N, w, r)\) we set

\[ j = \frac{N}{2w} \]
\[ m = \frac{w}{2} \]
\[ m' = \frac{r + 1}{2} \]

These linear relations between \((jmm')\) and \((N, w, r)\) are the simplest that permit half-integer representations and also respect the difference in parity between \( w \) and \( r \). Then

\[ D_{jmm'}^i = D_{\frac{N}{2}w^2r+1}^{\frac{N}{2}} \]
These are by definition the kinematical states of the quantum mechanical knot \((N, w, r)\).

The procedure that we have just followed resembles that followed for a quantum mechanical top. There the integrals of motion are the components of the angular momentum, the symmetry of the spherical spinning top is described by \(SU(2)\), and the quantum mechanical states are the irreducible representations of \(SU(2)\), again labelled \(D^j_{mn}\), where the indices in that case refer to the angular momentum. For example, the quantum states of the “spinning electron” are labelled by the fundamental representation of \(SU(2)\).

The quantum mechanical description of the knot is not complete at this point however since there is as yet no Hamiltonian and there are no operators for \((N, w, r)\), but these will be supplied in due course.

4 The Knot Algebra.

One way of seeing that \(SL_q(2)\) is the appropriate algebra of the knot is to observe on the one hand that the Kauffman algorithm\(^2\) (for generating the Kauffman or the Jones polynomial that characterizes a knot) may be expressed in terms of the matrix

\[
\epsilon_q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad \epsilon_q^2 = -1 \quad (4.1)
\]

and on the other hand that \(\epsilon_q\) is also the invariant matrix of \(SL_q(2)\) since

\[
T^t \epsilon_q T = T \epsilon_q T^t = \epsilon_q \quad (4.2)
\]

where \(T\) belongs to a two-dimensional representation of \(SL_q(2)\).

We shall now describe this algebra. Let

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (4.3)
\]

Then the matrix elements of \(T\) satisfy the following algebra

\[
ab = qba \quad bd = qdb \quad ad - qbc = 1 \quad bc = cb \\
ac = qca \quad cd = qdc \quad da - q_1 cb = 1 \quad q_1 = q^{-1} \quad (A)
\]
In the discussion of electroweak we need only the unitary subalgebra obtained by setting
\[ d = \bar{a} \]
\[ c = -q_1 \bar{b} \]

Then (A) reduces to the following
\[ ab = qba \quad \bar{a} \bar{b} + b \bar{b} = 1 \quad b \bar{b} = \bar{b} b \]
\[ \bar{a} \bar{b} = q \bar{b} a \quad \bar{a} a + q_1^2 \bar{b} b = 1 \]
\[ (A)' \]

For the physical applications we need the higher representations of $SU_q(2)$. The $2j + 1$-dimensional unitary irreducible representations of the $SU_q(2)$ algebra $(A)'$ are
\[ D^j_{mm'} = \sum_{s,t} A^j_{mm'}(s, t) \delta(s + t, n'_+ \bar{a}^s b^{n_+ - s} b'^{t} \bar{a}^{n_+ - t}) \]
\[ (4.4) \]

where
\[ A^j_{mm'}(s, t) = \left[ \frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n_+ \rangle_1! \langle n_- \rangle_1!} \right] \frac{\langle n_+ \rangle_1}{s} \frac{\langle n_- \rangle_1}{t} q^{t(n_+ - s + 1)} (-1)^t \]
and
\[ n_\pm = j \pm m \quad \langle n \rangle_1 = \frac{\langle n \rangle_1!}{\langle s \rangle_1! \langle n - s \rangle_1!} \]
\[ n'_\pm = j \pm m' \quad \langle n \rangle_1 = \left( \frac{q^n - 1}{q^n - 1} \right) \]

Every term of (4.4) contains a product of non-commuting factors that may be reduced (after dropping numerical factors) to the form
\[ a^{n_a} \bar{a}^{n_\bar{a}} b^{n_b} \bar{b}^{n_\bar{b}} \]

The $\delta$-function in (4.4) requires that
\[ n_a - n_\bar{a} = m + m' \]
\[ n_b - n_\bar{b} = m - m' \]
\[ (4.5) \quad (4.6) \]

where $n_a, n_b, n_\bar{a}, n_\bar{b}$ are the exponents of $a, b, \bar{a}, \bar{b}$, respectively.

These relations (4.5) and (4.6) hold for every term of (4.4) and are independent of $j$. 
5 Gauge Group $U_a(1) \times U_b(1)$ of the $SU_q(2)$ Algebra.

The $SU_q(2)$ algebra $(A)'$ is invariant under the following gauge transformations

$$a' = e^{i\varphi_a}a$$
$$b' = e^{i\varphi_b}b$$

(5.1)

Then every term $\sim a^n \bar{a}^n b^m \bar{b}^m$ of $D^{ij}_{mm'}$ is multiplied by

$$e^{i\varphi_a(n_a-n_b)}e^{i\varphi_b(n_b-n_b)} = e^{i\varphi_a(m+m')}e^{i\varphi_b(m-m')}$$

(5.2)

by (4.5) and (4.6). Then (5.1) induces the following gauge transformations on $D^{ij}_{mm'}$

$$D^{ij}_{mm'} = e^{i\varphi_a(m+m')}e^{i\varphi_b(m-m')}D^{ij}_{mm'}$$

(5.3)

$$\bar{D}^{ij}_{mm'} = e^{-i\varphi_a(m+m')}e^{-i\varphi_b(m-m')}\bar{D}^{ij}_{mm'}$$

(5.4)

By analogy with the electric charge we may define two “knot charges” $Q_a$ and $Q_b$ determined by the writhe and rotation as follows:

$$Q_a \equiv -k(m + m') = -k\left(\frac{w + r + 1}{2}\right)$$

(5.5)

$$Q_b \equiv -k(m - m') = -k\left(\frac{w - r - 1}{2}\right)$$

(5.6)

then (5.3) and (5.4) become

$$D^{ij}_{mm'} = U_a U_b D^{ij}_{mm'}$$

(5.7)

$$\bar{D}^{ij}_{mm'} = U^*_a U^*_b \bar{D}^{ij}_{mm'}$$

(5.8)

where

$$U_a = e^{-ik^{-1}Q_a\varphi_a}$$

(5.9)

$$U_b = e^{-ik^{-1}Q_b\varphi_b}$$

(5.10)

Then $U_a$ and $U_b$ are two independent gauge transformations on the irreducible representations $D^{ij}_{mm'}$ of $SU_q(2)$ and therefore also on the quantum states $D^{N/2}_{w,r+1}$ of the knot $(N, w, r)$. By (5.8) $D^{ij}_{mm'}$ and $\bar{D}^{ij}_{mm'}$ have opposite charges.

We shall now examine the possibility that the elementary particles are quantum knots described by the kinematic states $D^{N/2}_{w+1, r+1}$. 

6
6 The Knot Conjecture.

We shall explore the possibility that the elementary particles are quantum knots (knotted flux tubes). In such a model the simplest knots (trefoils) should correspond to the most elementary particles (fermions). Indeed there are 4 trefoils and there are 4 families of fermions. Each trefoil is labelled by \((N, w, r)\) where \(N = 3\) and each family of fermions is labelled by \((t, t_3, Q)\) where \(t = 1/2\). Is there a meaningful correspondence \((N, w, r) \leftrightarrow (t, t_3, Q)\)? i.e.

(a) between single trefoil solitons and single fermion families?

(b) between states of a trefoil soliton and members of a fermion family?

We may try to establish this correspondence by labelling both the trefoil solitons and the fermion families by the irreducible representations of \(SU_q(2)\), namely \(D^{j}_{mm'}\) as follows:

\[(Nwr) \leftrightarrow jmm' \leftrightarrow (t, t_3, Q)\]

\[
\begin{array}{c c c c}
\uparrow & \uparrow & \uparrow \\
\text{trefoils} & \text{SU}_q(2) & \text{fermion family}
\end{array}
\]

where \(D^j_{mm'} = D^{N/2}_{\frac{w^2 + 1}{2}}\) and \(N = 3, j = \frac{3}{2}, \) and \(t = \frac{1}{2}\).

The Fermionic Solitons as Trefoils.\(^2\)

In order to put each trefoil in correspondence with one class of fermions we shall compare the knot charges, \(Q_a = -k(m + m')\), of the 4 trefoils with the electric charges of the 4 classes as shown in Table 1:

\[
\begin{array}{c c c c c c}
\text{Trefoils} (w, r) & \frac{D^{N/2}_{\frac{w^2 + 1}{2}}}{\frac{3}{2}} & Q_a & \text{Fermion Class} & Q_f \\
(-3, 2) & D^{3/2}_{\frac{3}{2}} & 0 & (\nu_e \nu_\mu \nu_\tau) & 0 \\
(3, 2) & D^{3/2}_{\frac{3}{2}} & -3k & (e^-, \mu^-, \tau^-) & -e \\
(3, -2) & D^{3/2}_{\frac{3}{2}} & -k & (d, s, b) & -\frac{1}{3} e \\
(-3, -2) & D^{3/2}_{-\frac{3}{2}} & 2k & (u, c, t) & \frac{2}{3} e
\end{array}
\]
where \( N = 3 \) and
\[
Q_a \equiv -k(m + m') \quad (6.1)
\]
\[
Q_f = \text{electric charge of the fermion class} \quad (6.2)
\]

There is a unique mapping and single value of \( k \) that permits one to match the trefoils with the fermion classes by satisfying
\[
Q_a(w, r) = Q_f \text{ (fermion class)} \quad (6.3)
\]

namely
\[
k = \frac{e}{3} \quad (6.4)
\]

Then
\[
Q_a = -\frac{e}{3} (m + m') \quad (6.5)
\]
or
\[
Q_a = -\frac{e}{6} (w + r + 1) \quad (6.6)
\]

Then \( Q_a \) may be considered the electric charge of the quantum trefoil \( D_{\frac{3}{2} \pi \pi} \).

It is interesting that this same correspondence between trefoils and fermion classes was found in our earlier phenomenological work \(^3,^4 \) before the natural appearance of \( Q_a \) as \(-\frac{e}{3} (m + m') \) or \( -\frac{e}{6} (w + r + 1) \) was noticed. Other mappings of the trefoils onto the fermion families are possible, but there is only one mapping with a single value of \( k \).

One therefore identifies \( Q_a \) with the electric charge. One could also attempt the match with \( Q_b \), but in that case the neutrinos would be assigned to the \((3,2)\) knot, in contradiction to earlier work \(^3,^4 \) that associates neutrinos with \((-3,2)\). We defer the interpretation of \( Q_b \).

Since \( Q_a \sim m + m' = n_a - n_{\bar{a}}, \) note that the vanishing of \( Q_a \) implies
\[
n_a = n_{\bar{a}}
\]
and therefore that \( a \) and \( \bar{a} \) may be eliminated from every term of \( D_{mn'} \) with the aid of
\[
a^n\bar{a}^n = \prod_{s=0}^{n-1} (1 - q^{2s}b\bar{b}) \quad n \geq 1 \quad (6.7)
\]
as follows from \((A)'\). Therefore electrically neutral states (neutrinos and neutral bosons) lie entirely in the \((b, \bar{b})\) subalgebra.
Note also that \( \bar{D}^j_{mm'} \) has opposite charges from \( D^j_{mm'} \) and may therefore be identified as the state of the antiparticle.

Given the match in Table 1 we may now compare all the quantum numbers \((t, t_3, Q)\) labelling the different classes of fermions in the standard representation with the quantum numbers \((N, w, r)\) labelling the corresponding quantum knots.

Table 2.

| Standard Representation | Knot Representation |
|-------------------------|---------------------|
| \((e\mu\tau)_L\)       | \((\nu_e\nu_\mu\nu_\tau)_L\) |
| \((d_{sb})_L\)          | \((u_{ct})_L\)       |
| \(\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -e\) | \(\frac{1}{2} \quad \frac{1}{2} \quad 0\) |
| \(\frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{3} \quad \frac{1}{3} e\) | \(\frac{1}{2} \quad -\frac{1}{2} \quad 3 \quad -2 \quad D^{3/2}_{-\frac{1}{3},\frac{1}{3}} \quad -\frac{1}{3} e\) |
| \(-3 \quad 3 \quad \frac{1}{3}\) | \(-2 \quad D^{3/2}_{-\frac{1}{3},\frac{1}{3}} \quad \frac{1}{3} e\) |
| \(\frac{3}{2}\) | \(-\frac{1}{2}\) |

One then reads off the following relations from Table 2.

\[
t = \frac{N}{6} \tag{6.8}
\]

since \(N = 3\) for trefoils.

Also \(t_3\) is proportional to \(w\) (not to \(r\)) and

\[
t_3 = -\frac{w}{6} \tag{6.9}
\]

Since \(m = \frac{w}{2}\)

\[
t_3 = -\frac{m}{3} \tag{6.10}
\]

Finally in the knot representation the electric charge is

\[
Q = -\frac{e}{3} \left(m + m'\right) \tag{6.11}
\]

But in the standard theory (point particle representation)

\[
Q = (t_3 + t_0)e \tag{6.12}
\]

Since (6.11) and (6.12) must agree, we have

\[
t_3 + t_0 = -\frac{1}{3} \left(m + m'\right) \tag{6.13}
\]
By (6.10) and (6.13) the hypercharge is

\[ t_0 = -\frac{1}{3} m' \]  

(6.14)

Therefore alternative forms of the quantum state of the fermionic knots are

\[ D^{\frac{N}{2}}_{\frac{w}{6} + 1} \text{ or } D^{3t}_{-3t_3 - 3t_0} \]  

(6.15)

Therefore the invariance group of the algebra, namely, \( U_a(1) \times U_b(1) \) defines the charge and hypercharge. By (5.8) the adjoint representations carry opposite charge.

The relation that we have just found between isotopic spin and knot quantum numbers is

\[ t = \frac{N}{6} \]
\[ t_3 = -\frac{w}{6} \]
\[ t_0 = -\frac{r + 1}{6} \]  

(6.16)

Table 2 refers to the L-chiral field. The R-chiral field is unknotted since \( t = 0 \).

7 Quantum Operators for \((N, w, r)\) and \(Q\).

Denote the operators whose eigenvalues are \((N, w, r + 1)\) by \((\mathcal{N}, \mathcal{W}, \mathcal{R})\), i.e.

\[ \mathcal{N} \; D^{\frac{N}{2}}_{\frac{w}{6} + 1} = N \; D^{\frac{N}{2}}_{\frac{w}{6} + 1} (a, \bar{a}, b, \bar{b}) \]  

(7.1)
\[ \mathcal{W} \; D^{\frac{N}{2}}_{\frac{w}{6} + 1} = w \; D^{\frac{N}{2}}_{\frac{w}{6} + 1} (a, \bar{a}, b, \bar{b}) \]  

(7.2)
\[ \mathcal{R} \; D^{\frac{N}{2}}_{\frac{w}{6} + 1} = (r + 1) \; D^{\frac{N}{2}}_{\frac{w}{6} + 1} (a, \bar{a}, b, \bar{b}) \]  

(7.3)

where the \( D^{\frac{N}{2}}_{\frac{w}{6}} \) are functions of \((a, \bar{a}, b, \bar{b})\). These operators have the following forms in the basic algebra:

\[ \mathcal{N} = \omega_a + \omega_{\bar{a}} + \omega_b + \omega_{\bar{b}} \]  

(7.4)
\[ \mathcal{W} = \omega_a - \omega_{\bar{a}} + \omega_b - \omega_{\bar{b}} \]  

(7.5)
\[ \mathcal{R} = \omega_a - \omega_{\bar{a}} - \omega_b + \omega_{\bar{b}} \]  

(7.6)
where the $\omega_x$ are dilatation operators defined by their action on every term of $D^j_{mm'}$ according to
\[ \omega_x(\ldots x^n \ldots) = n_x(\ldots x^n \ldots) \quad x = (a, \bar{a}, b, \bar{b}) \] (7.7)
i.e. $\omega_x$ acts like $x \frac{\partial}{\partial x}$.

The four knot eigenstates of $N$ with eigenvalue $N = 3$ are the four trefoils $D^{3/2}_{3/2} = \frac{a}{2} + \frac{1}{2}$.

The charge operator is
\[
Q = -\frac{e}{3} \left( W + \mathcal{R} \right) = -\frac{e}{3} \left( m + m' \right) \] (7.8)
\[
= -\frac{e}{3} (\omega_a - \omega_a) \] (7.9)

Then
\[
Q D^{3t}_{-3t} - 3t_0 = -\frac{e}{3} (\omega_a - \omega_a) D^{3t}_{-3t} - 3t_0 \] (7.10)
\[
= -e(t_3 + t_0)D^{3t}_{-3t} - 3t_0 \] (7.11)

The 3 eigenfunctions of $Q$ with $N = 3$ and $t = \frac{1}{2}$ are $D^{3/2}_{-3t} - 3t_0$ (the four trefoils) and the 4 eigenvalues $-e(t_3 + t_0)$ are the charges of the 4 solitons.

One may compare (7.10) and (7.11) with similar equations for the angular momentum of a top, namely
\[
L_3 D^j_{mm'} = \frac{i \hbar}{2} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) D^j_{mm'} \] (7.12)
\[
= -m\hbar D^j_{mm'} \] (7.13)
\[
K_3 D^j_{mm'} = -m'\hbar D^j_{mm'} \] (7.14)

According to (7.10) and (7.12) the charge is quantized in units of $\frac{e}{3}$ while the angular momentum is quantized in units of $\hbar$.

The complete $L$-chiral wave function of the fermionic soliton is now
\[
F(\vec{p}, \vec{s}, \vec{t}) D^{3t}_{3t} (q|a\bar{a}b\bar{b}) \] (7.15)
where the first factor is the $L$-chiral standard Dirac wave function for a point particle with momentum $\vec{p}$, spin $\vec{s}$, and isotopic spin $\vec{t}$ and where the second factor is the internal knot state.
8 The State Space of the Solitons.

The 4 solitons are eigenstates of $N, W$, and $R$, but they are functions of $(a, \bar{a}, b, \bar{b})$. They may be numerically evaluated on the state space of the algebra. Let us next describe this state space.

Since $b$ and $\bar{b}$ commute, they have common eigenvalues. Let $|0\rangle$ be designated as a ground state and let

$$
\begin{align*}
b|0\rangle &= \beta|0\rangle \quad (8.1) \\
\bar{b}|0\rangle &= \beta^*|0\rangle \quad (8.2)
\end{align*}
$$

and

$$\bar{b}b|0\rangle = |\beta|^2|0\rangle \quad (8.3)$$

where $\bar{b}b$ is Hermitian with real eigenvalues and orthogonal eigenstates.

One finds by $(A)'$ that

$$\bar{b}b|n\rangle = E_n|n\rangle \quad (8.4)$$

where

$$|n\rangle \sim \bar{a}^n|0\rangle \quad (8.5)$$

and

$$E_n = q^{2n}|\beta|^2 \quad (8.6)$$

$\bar{b}b$ resembles the Hamiltonian of an oscillator but with eigenvalues arranged in geometrical progression and with $|\beta|^2$ corresponding to $\frac{1}{2} \hbar \omega$. If we take $H(\bar{b}b)$ to be the Hamiltonian of the knot, where the functional form of $H$ is left unspecified, it will have the same eigenstates as $\bar{b}b$.

Here $\bar{a}$ and $a$ are raising and lowering operators respectively

$$
\begin{align*}
\bar{a}|n\rangle &= \lambda_n|n+1\rangle \quad (8.7) \\
a|n\rangle &= \mu_n|n-1\rangle \quad (8.8)
\end{align*}
$$

One finds

$$
\begin{align*}
|\lambda_n|^2 &= 1 - q^{2n}|\beta|^2 \quad (8.9) \\
|\mu_n|^2 &= 1 - q^{2(n-1)}|\beta|^2 \quad (8.10)
\end{align*}
$$
If there is a highest state \((M)\) then
\[
1 - q^{2M}|\beta|^2 = 0 \quad (8.11)
\]
If there is a lowest state \((M')\) then
\[
1 - q^{2(M'-1)}|\beta|^2 = 0 \quad (8.12)
\]
If there are both a highest and lowest state, then
\[
q^{2M} = q^{2(M'-1)} \quad (8.13)
\]
and if \(q\) is real as we assume
\[
M = M' - 1 \quad (8.14)
\]
but this is not possible since \(M' \leq M\). Therefore if \(q\) is real, there is no finite representation of the elements of this algebra. This may be a lowest or a highest state but not both.

For the physical application however, we may require finite representations. If this is so, it must be possible to cut off the \(q\)-oscillator spectrum by imposing physical boundary conditions at one or both of the upper and lower bounds. Indeed, insofar as the present model is an electroweak model that excludes gluon and gravitational forces, one must expect that the neglected physics will impose boundary conditions on this model.

### 9 The Fermion States.

We shall propose that the separate fermion states are the ground and low lying excited states of the fermionic soliton and their state functions are
\[
D_{-3t_3 -3t_0}|n\rangle \quad n = 0, 1, 2 \quad (9.1)
\]
where \(|n\rangle\) is the \(n^{th}\) level of the \(q\) oscillator. If there are only 3 fermions in each family, we shall assume that they occupy the lowest levels so that \(n\) takes on the values 0,1,2. We interpret \(|0\rangle\) as the state of lowest energy.

Then the complete \(L\)-chiral wave function of a fermion becomes, by \((7.15)\) and \((9.1)\)
\[
F(\vec{p}, \vec{s}, \vec{t})D_{-3t_3 -3t_0}|n\rangle \quad (9.2)
\]
For the internal state of the antiparticle we propose the adjoint representation, namely

\[ \bar{D}^{3/2} - 3t_3 |n\rangle \]  

(9.3)

By choosing the adjoint, we guarantee that the charge of the antisoliton is opposite to that of the soliton as shown in (3.9).

The states |n\rangle are not only the eigenstates of the q-oscillator but they are also eigenstates of any Hamiltonian of the form

\[ H = f(\bar{b}b) \]

This \( H \) commutes with the integrals of the motion:

\[ [H, \mathcal{N}] = [H, \mathcal{W}] = [H, \mathcal{R}] = 0 \]

as required.

The relative masses of the fermions are determined by an effective Hamiltonian \( H(\bar{b}b) \). Let the energy of the state \( \psi_n = D^{3t_3} - 3t_0 |n\rangle \) be \( E_n \). Then

\[ H(\bar{b}b)\psi_n = E_n\psi_n \]  

(9.4)

where the \( E_n \) determines the relative masses of the Fermions. At the field-theoretic level the masses of the Fermions are determined by their various interactions with other fields, but within the limitations of the knot model we may assume that there is an effective \( H \) of the following unknown form:

\[ H = f(\bar{b}b) \]  

(9.5)

Then

\[ H\psi_n = f(\bar{b}b)D^{ij}_{mm'} |n\rangle \]

\[ = D^{ij}_{mm'} f(q^{6Q/e} \bar{b}b) |n\rangle \]  

(9.6)

where \( Q \) is the electric charge of \( D^{ij}_{mm'} \) and differs among the four solitons. Then

\[ H\psi_n = D^{ij}_{mm'} f(q^{6Q/e} q^{2n} |\beta|^2) |n\rangle \]

\[ = f(q^{6Q/e+2n} |\beta|^2)\psi_n \]  

(9.7)
and

\[ E_n = f(q^{6Q/e} q^{2n} |\beta|^2) \]  

(9.8)

By choosing \( H \) to agree with the effective mass term in the Higgs Lagrangian of the standard theory, one arrives at \(^3,^4\)

\[ f(\bar{b}b) = \bar{D}^i_{mm'}D^i_{mn} \]  

(9.9)

10 The Fermion-Boson Interactions.

In this model the fermion solitons interact by the emission and absorption of bosonic solitons. We denote the generic fermion-boson interaction by

\[ \mathcal{F}_3 \mathcal{B}_2 \mathcal{F}_1 \]  

(10.1)

where

\[ \mathcal{F}_1 = F_1(\vec{p}, \vec{s}, \vec{t}) D^{3/2}_{m_1 m'_1} (a\bar{a}b\bar{b}) |n_1\rangle \]  

(10.2)

\[ \mathcal{F}_3 = \langle n_3 | \bar{D}^{3/2}_{m_3 m'_3} (a\bar{a}b\bar{b}) \bar{F}_3(\vec{p}, \vec{s}, \vec{t}) \]  

(10.3)

\[ \mathcal{B}_2 = B_2(\vec{p}, \vec{s}, \vec{t}) D^j_{m_2 m'_2} (a\bar{a}b\bar{b}) \]  

(10.4)

Here \( F(\vec{p}, \vec{s}, \vec{t}) \) and \( B(\vec{p}, \vec{s}, \vec{t}) \) are the standard fermionic and bosonic normal modes. Then (10.1) becomes

\[ (\mathcal{F}_3 \mathcal{B}_2 \mathcal{F}_1) \langle n_3 | \bar{D}^{3/2}_{m_3 m'_3} D^j_{m_2 m'_2} D^{3/2}_{m_1 m'_1} |n_1\rangle \]  

(10.5)

The correction to the standard matrix elements appears in the second factor, namely

\[ \langle n_3 | \bar{D}^{3/2}_{m_3 m'_3} D^j_{m_2 m'_2} D^{3/2}_{m_1 m'_1} |n_1\rangle \]  

(10.6)

If there are \( M \) generations of fermions, then \( n_1 \) and \( n_3 \) take on values \( 0 \ldots M - 1 \).

We must require that the basic internal interaction be invariant under gauge transformations, \( U_a(1) \times U_b(1) \), of the underlying algebra, i.e.

\[ (\bar{D}^{3/2}_{m_3 m'_3})' (D^j_{m_2 m'_2})' (D^{3/2}_{m_1 m'_1})' = \bar{D}^{3/2}_{m_3 m'_3} D^j_{m_2 m'_2} D^{3/2}_{m_1 m'_1} \]  

(10.7)
Here both charges are conserved:

\[ \exp[ik\phi_a](-Q_a(3) + Q_a(2) + Q_a(1)) = 1 \] (10.8)

\[ \exp[ik\phi_b](-Q_b(3) + Q_b(2) + Q_b(1)) = 1 \] (10.9)

Therefore both charges are conserved:

\[ Q(1) + Q(2) = Q(3) \] (10.10)

Then

\[ m_3 = m_1 + m_2 \] (10.11)

\[ m'_3 = m'_1 + m'_2 \] (10.12)

and the possible values of \((j, m, m')\) for the intermediate boson are restricted by the known values of \((j, m, m')\) for the initial and final fermions.

If the rules for connecting \(m\) and \(m'\) to \(t_3\) and \(t_0\) are extended without change from fermions to the intermediate boson, then the conservation of \(Q_a\) and \(Q_b\) by the basic interaction implies the conservation of \(t_3\) and \(t_0\) by the same interaction. Therefore we adopt for bosonic knots the same rules as for fermionic knots:

\[
\begin{align*}
    m &= -3t_3 \\
    m' &= -3t_0 \quad \text{or} \quad r + 1 = -6t_0 \\
    j &= 3t \\
    N &= 6t
\end{align*}
\] (10.13)

Applied to the vector bosons these rules imply Table 3:

**Table 3.**

| \(t\) | \(t_3\) | \(t_0\) | \(D_{-3t_3-3t_0}^{3t}\) |
|-------|---------|---------|---------------------|
| \(W^+\) | 1 | 1 | 0 | \(D_{-3t_3}^{-3t_0}^{3t}\) |
| \(W^-\) | 1 | -1 | 0 | \(D_{3t_0}^{3t}\) |
| \(W^3\) | 1 | 0 | 0 | \(D_{0\ 0}^{3t}\) |
| \(W^0\) | 0 | 0 | 0 | \(D_{0\ 0}^{0}\) |

The first 3 columns of the Table express the fact that \(\vec{W}\) is an is triplet and \(W^0\) is an isosinglet in the standard theory.
The fourth column $D_{-3t_{3} -3t_{0}}^{3t}$ labels the internal states of the four vector bosons. If $t = 1, j = 3$; and if $j = \frac{N}{2}$, as we have assumed, then $N = 6$ and $\tilde{W}$ is a ditrefoil consistent with the pair production of fermions by $\tilde{W}$.

Since $W^{0}$ is coupled only to $U(1)$ in the standard theory, there is no self-coupling, i.e., it itself carries neither electric nor hypercharge. The assignment of $j$ to $W^{0}$ is also restricted in the internal matrix element by the $q$-Clebsch-Gordan rules. If $j = 0$ and we maintain the relation $j = N/2$, then $N = 0$ and $W^{0}$ is an unknotted clockwise loop.

The possibility of extending the conservation laws and the same rule for associating $Q_{a}$ and $Q_{b}$ with $(m, m')$ to all solitons depends on the fact that $Q_{a}$ and $Q_{b}$ are independent of $j$.

The conservation of $Q_{a}$ and $Q_{b}$, i.e. the invariance of the action under $U_{a}(1) \otimes U_{b}(1)$ is more fundamental in the present model than the conservation of $t_{3}$ and $t_{0}$. Electric charge and isospin in this model are simply characterizations of the geometry of the knotted soliton.

The complete matrix elements are of the following form:

$$(\bar{F}_{3}B_{2}F_{1})\langle n_{3}|\bar{D}_{m_{3}n_{3}}^{3/2} D_{m_{2}n_{2}}^{j_{2}} D_{m_{1}n_{1}}^{3/2}|n_{1}\rangle$$

Corrections to the standard matrix elements appear in the second factor and have been computed in the phenomenological work. These corrections are small, however, and it is necessary to make more refined calculations and also to go to higher order to judge their significance. In addition some of the assumptions made in the earlier work may be dropped and some need to be modified in light of the present paper.

We have therefore explored a field theoretic formulation of these ideas. The following sections repeat much of Ref. 5 and are included for completeness.

11 Field Theory.

In passing from the standard to the knot field theory we shall modify both the symmetry group and the Lagrangian. The symmetry of standard electroweak local is $SU(2) \otimes$ local $U(1)$. We shall assume a slightly expanded symmetry to characterize the knot model,
namely:

\[ [\text{local } SU(2) \otimes \text{ local } U(1)] \otimes [\text{global } (U_a(1) \otimes U_b(1))] \]

The vector connection in the standard theory is

\[ W_+ t_+ + W_- t_- + W_3 t_3 + W_0 t_0 \]  \hspace{1cm} (11.1)

where \((t_+, t_-, t_3, t_0)\) are the generators of the standard electroweak theory in the charge representation and

\[
t_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]  \hspace{1cm} (11.2)

We now replace (11.1) by

\[ W_+ \tau_+ + W_- \tau_- + W_3 \tau_3 + W_0 \tau_0 \]  \hspace{1cm} (11.3)

where

\[ \tau_k = c_k(q, \beta) t_k D_k \quad k = (+, -, 3, 0) \]  \hspace{1cm} (11.4)

and the \(D_k\) are the charge states of the four vector mesons

\[
D_+ \equiv D^3_{30}/N_+ = \bar{b}a^3 \\
D_- \equiv D^2_{03}/N_- = a^3b^3 \\
D_3 \equiv D^3_{00} = f_3(\bar{b}b) \\
D_0 \equiv D^0_{00} = 1
\]

The \(c_k\) are numerical functions of the parameters \((q, \beta)\) that are fixed by relations between the masses of the vector bosons.

Here are the two-dimensional representation of the new generators:

\[
(\vec{\tau}, \tau_0) = \begin{pmatrix} 0 & D_+ \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ D_- & 0 \end{pmatrix}, \begin{pmatrix} D_3 & 0 \\ 0 & -D_3 \end{pmatrix}, \begin{pmatrix} D_0 & 0 \\ 0 & D_0 \end{pmatrix}
\]  \hspace{1cm} (11.5)

For the fermions and Higgs-like fields one replaces the numerically valued 2-rowed spinors of isotopic spin \(SU(2)\) by the following operator valued spinors

\[
\begin{pmatrix} D^{3/2}_\nu \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ D^{3/2}_\ell \end{pmatrix}, \begin{pmatrix} D^{3/2}_a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ D^{3/2}_d \end{pmatrix}
\]  \hspace{1cm} (11.6)
where $D^{3/2}_r$ is an abbreviation for the irreducible representation associated with the $r^{th}$ soliton and where $r = (\nu, \ell, u, d)$.

We also introduce $\psi_{Ari}$ defined by

$$\psi_{1ri} = \psi_1 D_r(a\bar{a}b\bar{b}|i) \quad A = 1 \quad r = \nu, \ell \quad i = 1, 2, 3$$

$$\psi_{2ri} = \psi_2 D_r(a\bar{a}b\bar{b}|i) \quad A = 2 \quad r = u, d \quad i = 1, 2, 3$$

where the lepton and neutrino solitons are combined into one isotopic spinor, $\psi_{1ri}$, and the up and down quarks into a second isotopic spinor, $\psi_{2ri}$ and where $i$ runs over the three states of the soliton. Then

$$\psi_1 = \begin{pmatrix} \psi_\nu D^{3/2}_\nu |i\rangle \\ \psi_\ell D^{3/2}_\ell |i\rangle \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} \psi_u D^{3/2}_u |i\rangle \\ \psi_d D^{3/2}_d |i\rangle \end{pmatrix}$$

### 12 $\tau$-Commutators, Gauge Fields, Field Strengths and Interactions.

(a) $\tau$-Commutators.

$$\tau_k = c_k t_k D_k \quad k = (+, -, 3)$$

$$[\tau_k, \tau_\ell] = c_k c_\ell [t_k D_k, t_\ell D_\ell] \quad [t_k, D_\ell] = 0$$

$$= c_k c_\ell ([t_k, t_\ell] D_k D_\ell + t_\ell t_k [D_k, D_\ell])$$

where

$$[t_k, t_\ell] = c_{k\ell} s t_s \quad t_k t_\ell = \gamma_{k\ell} s t_s + \gamma_{k\ell} \gamma = \frac{1}{2} \delta(k, \pm) \delta(\ell, \mp)$$

$$[D_k, D_\ell] = \hat{c}_{k\ell} s D_s \quad D_k D_\ell = \hat{\gamma}_{k\ell} s D_s$$

$$[\tau_k, \tau_\ell] = \frac{c_k c_\ell}{c_s} \hat{C}_{k\ell} s \tau_s + c_k c_\ell \gamma_{k\ell} \hat{c}_{k\ell} s D_s$$

$$\hat{C}_{k\ell} s = c_{k\ell} s \hat{\gamma}_{k\ell} s + \gamma_{k\ell} s \hat{c}_{k\ell} s$$
The coefficients \( c_{k\ell}^* \) and \( \gamma_{k\ell}^* \) are numerically valued while \( \hat{c}_{k\ell} \) and \( \hat{\gamma}_{k\ell} \) are functions of \( \bar{b}b \). The structure coefficients \((\hat{C}, \hat{c})\) of the \( \tau \)-algebra are therefore functions of \( \bar{b}b \).

(b) **Gauge Fields and Field Strengths.**

We introduce the four vector boson fields by defining

\[
\mathcal{W}_{rs} \equiv ig \, \vec{W} \tau_{rs} + ig_0 \, W^0(\tau_0)_{rs} \tag{12.7}
\]

where

\[
(\tau_\pm)_{rs} = c_\pm(t_\pm)_{rs}D_\pm
\]
\[
(\tau_3)_{rs} = c_3(t_3)_{rs}D_3
\]
\[
(\tau_0)_{rs} = c_0(t_0)_{rs}D_0
\]

and \( \vec{W}, W^0 \) replace the components of the standard boson field while \( W \) lies in the internal algebra. Here \((D_+, D_-, D_3, D_0) \equiv (\bar{b}^3\bar{a}^3, a^3\bar{b}^3, f_3(\bar{b}b), 1)\).

The covariant derivative is now

\[
\nabla_{rs} = \delta_{rs} \partial + \mathcal{W}_{rs} \tag{12.9}
\]

The corresponding field strengths are

\[
\mathcal{W}_{\mu\lambda} = (\nabla_\mu, \nabla_\lambda) \tag{12.10}
\]

(c) **Boson-Fermion Interactions.**

We shall introduce the direct boson-fermion interactions as follows:

\[
(\bar{\psi}_A)_{ri} \, \nabla_{rs} U_A(\psi_A)_{si'} \quad A = 1, 2 \quad (r, s) = (\nu, \ell) \text{ or } (u, d) \text{ and } i, i' = 1, 2, 3 \tag{12.11}
\]

where \( A = 1 \) labels the \((\nu, \ell)\) doublet and \( A = 2 \) labels the quark doublet \((u, d)\).

The \( U_A \) are unitary matrices. The form of \( U_1 \) is restricted by the “universal Fermi interaction”, while \( U_2 \) replaces the Kobayashi-Maskawa matrix. Here \( U_2 \) “rotates” the initial state \((\psi_A)_{si'}\).

Here is a tentative choice: \( U_1 = 1 \quad U_2 = e^{i\varphi(a+a')} \)
13 The Field Invariant.

We choose as the invariant of the non-Abelian vector field

\[ I = \langle 0 | \text{Tr} \ W_{\mu \lambda} W^{\mu \lambda} | 0 \rangle \quad (13.1) \]

which differs from the standard \( I \) in the use of an expectation value over \( |0\rangle \) and the meaning of \( W_{\mu \lambda} \).

The non-Abelian contribution to the field strength is

\[ W_{\mu \lambda} = [\nabla_\mu, \nabla_\lambda] \quad (13.2a) \]

\[ = [\partial_\mu + W_\mu, \partial_\lambda + W_\lambda] \quad (13.2b) \]

where

\[ W_\mu = W_\mu^s \tau_s \quad s = (+, -, 3) \quad (13.2c) \]

By (13.2) and the \( \tau \)-algebra one finds

\[ W_{\mu \lambda} = W_{\mu \lambda}^s \tau_s + \hat{W}_{\mu \lambda}^s \mathcal{D}_s \quad s = (+, -, 3) \quad (13.3) \]

where the new field strengths are

\[ W_{\mu \lambda}^s = ig(\partial_\mu W_\lambda^s - \partial_\lambda W_\mu^s) - \frac{g^2 c_m c_\ell}{c_s} \hat{C}_{m \ell}^s W_\mu^m W_\lambda^\ell \quad (13.4) \]

\[ \hat{W}_{\mu \lambda}^s = -\frac{1}{2} g^2 c_m c_\ell \hat{c}_{m \ell}^s W_\mu^m W_\lambda^\ell \delta(m, \pm) \delta(\ell, \mp) \delta(s, 3) \quad (13.5) \]

\[ \hat{C}_{m \ell}^s = c_k \ell \hat{\gamma}^s_{k \ell} + \gamma_{k \ell} \hat{c}_s_{k \ell} \quad (13.6) \]

and

\[ I = \langle 0 | \text{Tr}[W_{\mu \lambda}^s \tau_s + \hat{W}_{\mu \lambda}^s \mathcal{D}_s][W_{\mu \lambda}^{\mu \lambda k} \tau_k + \hat{W}_{\mu \lambda}^{\mu \lambda k} \mathcal{D}_k]|0 \rangle \quad (13.7) \]

by (13.1) and (13.3).

Here the \( W_{\mu \lambda}^s \) and \( \hat{W}_{\mu \lambda}^s \) are functions of \( \bar{b} b \). In terms of the new field strengths \( W_{\mu \lambda}^s \) and \( \hat{W}_{\mu \lambda}^s \) the field invariant becomes

\[ I = \text{Tr}(\langle 0 | W_{\mu \lambda}^s W_{\tau_s \tau_r}^{\mu \lambda} + \hat{W}_{\mu \lambda}^s \hat{W}_{\tau_s \tau_r}^{\mu \lambda} \mathcal{D}_s \mathcal{D}_r + (W_{\mu \lambda}^s \tau_s)(\hat{W}_{\tau_s}^{\mu \lambda} \mathcal{D}_r) + (\hat{W}_{\mu \lambda}^s \mathcal{D}_s)(W_{\tau_s}^{\mu \lambda} \mathcal{D}_r) ) |0 \rangle \]
Since $\text{Tr} \, \tau_s = 0$, $I$ may be reduced to the following expression:

$$I = \text{Tr} \langle 0 | W_{\mu \lambda}^s W^{\mu \lambda r} \tau_s \tau_r + \hat{W}_{\mu \lambda}^s \hat{W}^{\mu \lambda r} D_s D_r |0\rangle$$

(13.8)

Note that $W_{\mu \lambda}^s \sim \delta(s, 3)$ by (13.5). After the insertion of a complete set of intermediate states, the field invariant becomes

$$I = \sum_n \langle 0 | W_{\mu \lambda}^s W^{\mu \lambda r} |n\rangle \langle n | \text{Tr} \, \tau_s \tau_r |0\rangle + \sum_n \langle 0 | \hat{W}_{\mu \lambda}^s \hat{W}^{\mu \lambda r} |n\rangle \langle n | D_s D_r |0\rangle$$

(13.9)

This expression simplifies since $W_{\mu \lambda}^s$ and $\hat{W}_{\mu \lambda}^s$ are functions of $\bar{b}b$, and therefore have no off-diagonal elements. Then

$$I = \langle 0 | W_{\mu \lambda}^s W^{\mu \lambda r} |0\rangle \langle 0 | \text{Tr} \, \tau_s \tau_r |0\rangle + \langle 0 | \hat{W}_{\mu \lambda}^s \hat{W}^{\mu \lambda r} |0\rangle \langle 0 | D_s D_r |0\rangle$$

(13.10)

where

$$\langle 0 | \text{Tr} \, \tau_s \tau_r |0\rangle = c_s c_r \langle 0 | \text{Tr} \, t_s t_r D_s D_r |0\rangle = c_s c_r \langle 0 | D_s D_r |0\rangle$$

(13.11)

$I$ is now reduced to

$$I = \sum_{s,r} \langle 0 | A_{sr} W_{\mu \lambda}^s W^{\mu \lambda r} + 2 \hat{W}_{\mu \lambda}^s \hat{W}^{\mu \lambda r} |0\rangle \langle 0 | D_s D_r |0\rangle$$

(13.12)

where

$$A_{sr} = c_s c_r \text{Tr} \, t_s t_r$$

$$= c_s c_r \left[ \delta(s, \pm) \delta(r, \mp) + 2 \delta(s, 3) \delta(r, 3) \right]$$

(13.13)

We have now reduced (13.1) to (13.12) with the following properties. In this expression $\langle 0 | D_s D_r |0\rangle = 0$ unless $s$ and $r$ represent either opposite or zero charge, so that $D_s D_r$ is neutral.

$W_{\mu \lambda}^s$ has the same form as in the standard theory, but the structure coefficients $C_{m \ell}^s$ differ from those of the Lie algebra of $SU(2)$ in that they are not numerically valued but depend on $\bar{b}b$.

Since $I$ is evaluated on the state $|0\rangle$, however, all expressions $F(\bar{b}b)$ become $F(|\beta|^2)$. Therefore the structure coefficients $C_{m \ell}^s(\bar{b}b)$ become $C_{m \ell}^s(|\beta|^2)$.
The final reduced form of \( \langle 0 | \text{Tr} \ W_{\mu\lambda} W^{\mu\lambda} | 0 \rangle \) will have one part \( \sim W_{\mu\lambda} s W^{\mu\lambda} r \), essentially the same as the standard theory but with structure coefficients depending on \( |\beta|^2 \).

There is also a second part \( \sim \hat{W}_{\mu\lambda} s \hat{W}^{\mu\lambda} r \) which is also dependent on \( q \) and \( \beta \). The sum of these two parts is multiplied by \( \langle 0 | D_s D_r | 0 \rangle \), again a function of \( q \) and \( \beta \), the two parameters of the theory. These expressions also depend on the numerical coefficients \( (c_+, c_-, c_3, c_0) \) introduced in the definition of the \( \tau \). The dependence of these coefficients on \( q \) and \( \beta \) will be fixed in the Higgs sector.

## 14 Gauge Invariance.

The new gauge group is generated by the following unitary transformations:

\[
S = S \otimes s
\]

(14.1)

where \( S \) is the standard symmetry:

\[
S \in \text{local}[SU(2) \otimes U(1)]
\]

(14.2)

and \( s \) is the gauge symmetry of the knot:

\[
s \in \text{global} \ U_a(1) \otimes U_b(1)
\]

(14.3)

or

\[
S = e^{i\tilde{\theta}(x) e^{i\theta_0(x)}
\]

(14.4)

and

\[
s = e^{iQ_a \theta_a} e^{iQ_b \theta_b}
\]

(14.5)

where \( \theta_a \) and \( \theta_b \) are independent of \( x \). Then

\[
S \psi_1 = S \begin{pmatrix} D_{\nu} \\ D_{\ell} \end{pmatrix} = S \otimes s \begin{pmatrix} D_{\nu} \\ D_{\ell} \end{pmatrix}
\]

(14.6)

\[
= S \begin{pmatrix} D'_{\nu} \\ D'_{\ell} \end{pmatrix}
\]

(14.7)

\[
= e^{i\tilde{\theta}(x) e^{i\theta_0(x)}} \begin{pmatrix} D'_{\nu} \\ D'_{\ell} \end{pmatrix}
\]

(14.8)
where

\[ D'_k = e^{iQ_a(k)\theta_a} e^{iQ_b(k)\theta_b} D_k \quad k = (\nu, \ell) \]  

(14.9)

The interaction terms will transform as

\[ (\bar{\psi}_A)' \mathcal{\nabla}'(U_A\psi_A)' = (\bar{\psi}_A\bar{S}) \mathcal{\nabla}'(SU_A\psi_A) \]  

(14.10)

Since \( S \) is unitary

\[ = \bar{\psi}_A S^{-1} \mathcal{\nabla}'(SU_A\psi_A) \]  

(14.11)

Then the interaction terms are invariant if

\[ \mathcal{\nabla}' = S \mathcal{\nabla} S^{-1} \]  

(14.12)

and since \( \mathcal{\nabla} = \mathcal{\beta} + \mathcal{W} \).

\[ \mathcal{W}' = S \mathcal{W} S^{-1} + S \mathcal{\beta} S^{-1} \]

\[ = (Ss) \mathcal{W}(s^{-1}S^{-1}) + S \mathcal{\beta} S^{-1} \]  

(14.13)

since \( s \) is global. The field strengths transform as follows:

\[ W'_{\mu\lambda} = [\nabla'_\mu, \nabla'_\lambda] \]  

(14.14)

\[ = S[\nabla_\mu, \nabla_\lambda] S^{-1} \]  

(14.15)

Then

\[ W'_{\mu\lambda} = S W_{\mu\lambda} S^{-1} \]  

(14.16)

and the non-Abelian field invariant will transform as

\[ \text{Tr} (W'_{\mu\lambda} W'^{\mu\lambda})' = \text{Tr} S W_{\mu\lambda} W^{\mu\lambda} S^{-1} \]  

(14.17)

where the trace is on the \( t \) matrices. Then

\[ \text{Tr} S(W_{\mu\lambda} W^{\mu\lambda}) S^{-1} = \text{Tr} s S(W_{\mu\lambda} W^{\mu\lambda}) S^{-1} s^{-1} \]  

(14.18)

\[ = s \cdot \text{Tr} S(W_{\mu\lambda} W^{\mu\lambda}) S^{-1} \cdot s^{-1} \]  

(14.19)

\[ = s \cdot \text{Tr} W_{\mu\lambda} W^{\mu\lambda} \cdot s^{-1} \]  

(14.20)
Let us next expand the trace as follows:

\[
\text{Tr} \, \mathcal{W}_\mu\mathcal{W}^{\mu^\lambda} = \text{Tr}[W_{\mu\lambda}^s W^{\mu^\lambda \tau_s \tau_r} + \hat{W}_{\mu\lambda}^s \hat{W}^{\mu^\lambda \tau_s \tau_r} D_s D_r]
\]

\[
= c_m c_p W_{\mu\lambda}^m W^{\mu^\lambda \tau_p} (\text{Tr} \, t_m t_p D_m D_p)
\]

\[
+ 2 \hat{W}_{\mu\lambda}^m \hat{W}^{\mu^\lambda \tau_p} D_m D_p
\]  

(14.21)

Then, since \( W_{\mu\lambda}^m \) and \( \hat{W}_{\mu\lambda}^m \) are functions of \( \bar{b}b \) only and since

\[
s \bar{b}b s^{-1} = \bar{b}b
\]  

(14.22)

one has

\[
s \text{Tr} \, \mathcal{W}_\mu\mathcal{W}^{\mu^\lambda}/s^{-1} = c_m c_p W_{\mu\lambda}^m W^{\mu^\lambda \tau_p} (\text{Tr} \, t_m t_p) (s D_m D_p s^{-1})
\]

\[
+ 2 \hat{W}_{\mu\lambda}^m \hat{W}^{\mu^\lambda \tau_p} (s D_m D_p s^{-1})
\]  

(14.23)

Here \( \text{Tr} \, t_m t_p \) vanishes unless \((m, p) = (\pm, \mp)\) or \((m, p) = (3, 3)\). Hence the first term on the right side of (14.23) vanishes unless \((m, p) = (\pm, \mp)\) or \((m, p) = (3, 3)\) or

\[
s D_m D_p s^{-1} = s D_{\mp} D_{\mp} s^{-1} \quad \text{or} \quad s D_3 D_3 s^{-1}
\]  

(14.24)

But \( D_3 D_3 \) as well as \( D_{\mp} D_{\mp} \) are neutral (zero \( Q_a \) and \( Q_b \) charges) and are therefore invariant under the \( s \)-transformation.

The second term on the right is invariant for the same reason since \( \hat{W}_{\mu\lambda}^m \hat{W}^{\mu^\lambda \tau_p} \) vanishes unless \((m, p) = (3, 3)\). Therefore

\[
S(\text{Tr} \, \mathcal{W}_\mu\mathcal{W}^{\mu^\lambda}) S^{-1} = \text{Tr} \, \mathcal{W}_\mu\mathcal{W}^{\mu^\lambda}
\]  

(14.25)

Hence the non-Abelian field invariant (13.1) is gauge-invariant. By (14.10) and (14.12) the fermion-boson interaction terms (12.11) are also gauge-invariant. Finally we will maintain the gauge-invariance of the Higgs sector in the usual way.

## 15 The Higgs Sector.

(a) The Vector Masses.

The neutral coupling in the knot model may be chosen as

\[
g J W^3 \tau_3 + g_0 J W^0 \tau_0
\]  

(15.1)
Introducing the physical fields \((A, Z)\) in the standard way we have

\[
W_0 = A \cos \theta - Z \cos \theta \quad (15.2)
\]
\[
W_3 = A \sin \theta + Z \cos \theta \quad (15.3)
\]

Then the neutral couplings (15.1) becomes

\[\mathcal{A}A + ZZ\]  \quad (15.4)

where

\[
\mathcal{A} = i(g \tau_3 \sin \theta + g_0 \tau_0 \cos \theta) \quad (15.5)
\]
\[
Z = i(g \tau_3 \cos \theta - g_0 \tau_0 \sin \theta) \quad (15.6)
\]

Since there is no interaction between photons and neutrinos, one has by (15.4)

\[
\left( \bar{D}^{3/2}_\nu 0 \right) \mathcal{A} \begin{pmatrix} \bar{D}^{3/2}_\nu \\ 0 \end{pmatrix} = \bar{D}^{3/2}_\nu \mathcal{A} \frac{1}{\bar{A}_3} \bar{D}^{3/2}_\nu = 0 \quad (15.7)
\]

According to (15.5) the preceding equation is satisfied by

\[
\mathcal{A} = i(g \tau_3 \sin \theta + g_0 \tau_0 \cos \theta) = 0 \quad (15.8)
\]

and by (15.6) and (15.8)

\[
Z = ig \frac{1}{\cos \theta} \tau_3 \quad (15.9)
\]

Then the covariant derivative of any neutral state is

\[
\nabla = \partial + ig \left[ W_+ \tau_+ + W_- \tau_- + \frac{Z \tau_3}{\cos \theta} \right] \quad (15.10)
\]

Denote the neutral Higgs scalar by

\[
\varphi = \rho(x) D_\nu|0\rangle \quad (15.11)
\]

where \(D_\nu\) is the neutral trefoil; namely, \((-3,2)\), carrying the representation \(D_{-3/2}^{3/2}\).
We now replace the kinetic energy term of the neutral Higgs of the standard model by the corresponding term of the knot theory as follows:

\[
\frac{1}{2} \text{Tr}(\nabla_\mu \varphi \nabla^\mu \varphi) = \frac{1}{2} \text{Tr}(0 | D_\nu [\partial_\mu \bar{\rho} \partial^\mu \rho + g^2 \bar{\rho}^2 (W_+^\mu W_+ + W_-^\mu W_- + Z^\mu Z_\mu \cos^2 \theta \tau^3)] D_\nu | 0) \]

\[
= I \partial_\mu \bar{\rho} \partial^\mu \rho + g^2 \bar{\rho}^2 \left( I_{++} W_+^\mu W_+ + I_{--} W_-^\mu W_- + \frac{I_{33}}{\cos^2 \theta} Z^\mu Z_\mu \right)
\]

where

\[
I = \frac{1}{2} \text{Tr}(0 | D_\nu D_\nu | 0)
\]

\[
I_{++} = \frac{1}{2} \text{Tr}(0 | D_\nu \bar{\tau}_+ \tau_+ D_\nu | 0)
\]

\[
I_{--} = \frac{1}{2} \text{Tr}(0 | D_\nu \bar{\tau}_- \tau_- D_\nu | 0)
\]

\[
I_{33} = \frac{1}{2} \text{Tr}(0 | D_\nu \bar{\tau}_3 \tau_3 D_\nu | 0)
\]

To agree with the mass relations according to the standard theory, the expressions (5.14)-(5.18) must be reduced to the following

\[
\partial_\mu \bar{\rho} \partial^\mu \rho + g^2 \bar{\rho}^2 \left( W_+^\mu W_+ + W_-^\mu W_- + \frac{1}{\cos^2 \theta} Z^\mu Z_\mu \right)
\]

where

\[
\bar{\rho} = I^{1/2} \rho
\]

To achieve this reduction we impose the following relations

\[
\frac{I_{kk}}{I} = 1 \quad k = (+, -, 3)
\]

or

\[
\frac{\text{Tr}(0 | \bar{D}_\nu (\bar{\tau}_k \tau_k) D_\nu | 0)}{\text{Tr}(0 | D_\nu D_\nu | 0)} = 1
\]

since \( \tau_k = c_k t_k \tau_k \), one has

\[
|c_k|^2 = \frac{\langle 0 | \bar{D}_\nu (\bar{D}_k D_k) D_\nu | 0 \rangle}{\langle 0 | D_\nu D_\nu | 0 \rangle} \quad k = (+, -, 3)
\]
The coefficients \((c_{\pm}, c_3)\) were introduced as arbitrary functions. They are now fixed by (15.23) as definite functions of \(q\) and \(\beta\) as follows:

\[
|c_-|^2 = \frac{1}{2}\beta^{6}\prod_{1}^{3}(1 - q^{2}\beta^{2})
\]

\[
|c_+|^2 = \frac{1}{2}\beta^{6}\prod_{0}^{2}(1 - q^{2}\beta^{2})
\]

\[
|c_3|^2 = \langle 0|\overline{D}^3_{00}D^3_{00}|0 \rangle
\]

\(c_0\) determined by (15.8) is

\[
c_0 = \frac{\langle 0|\overline{D}^3_{00}|0 \rangle}{\langle 0|\overline{D}^3_{00}|0 \rangle}
\]

where the Weinberg relation

\[
\tan \theta = \frac{g_0}{g}
\]

has been assumed. Here \(\overline{D}^3_{00}\) is a polynomial in \(\overline{b}b\), and \(\langle 0|\overline{D}^3_{00}|0 \rangle\) is a polynomial in \(|\beta|^{2}\).

(b) The Fermion Masses.

The mass term of the Weinberg-Salam Lagrangian is

\[
M \sim \overline{L}\varphi R + \overline{R}\varphi L
\]

One way of implementing (15.28) is to assume, in addition to the usual \(SU(2)\) assignments that \(L\) is a \(SU_q(2)\) trefoil and \(R\) is a \(SU_q(2)\) singlet while \(\varphi\) is also a \(SU_q(2)\) trefoil identical to \(L\).

One would then find for the mass of the \(n\)th fermion as an excited state of the \((w, r)\) soliton the following:

\[
M_n(w, r) = \rho(w, r)\langle n|\overline{D}^{3/2}_{\frac{3}{2} \frac{3}{2}}D^{3/2}_{\frac{3}{2} \frac{3}{2}}|n \rangle
\]

The interpretation of (15.28) that leads to (15.29) is based on the existence of 4 Higgs doublets, one for each trefoil, and implies that the Higgs potential has 4 minima. In the present model one may suppose that the Higgs potential is a scalar function defined over the \(SU_q(2)\) algebra and that it may be realized as the expectation value \(\langle 0|\overline{V}(\varphi\varphi)|0 \rangle\) where the minima of the expectation value lie at the 4 points where \(\varphi\) is a monomial labelling a trefoil. These four trefoils may be realized both as four fermionic solitons and also as four Higgs scalar solitons. Instead of the unitarity gauge, where the standard Higgs doublet
becomes \( \begin{pmatrix} 0 \\ \rho \end{pmatrix} \), one postulates a privileged gauge where the four independent doublets \( \Phi_i (i = \nu, \ell, u, d) \) are

\[
\begin{pmatrix} \varphi_\nu D_\nu \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_\ell D_\ell \end{pmatrix}, \begin{pmatrix} \varphi_u D_u \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \varphi_d D_d \end{pmatrix}
\]

(15.30)

representing the four Higgs partners of the four fermionic knots. Then adapting (15.28) one has

\[
\mathcal{M}_i = \bar{L}_i \Phi_i \mathcal{R}_i + \bar{R}_i \Phi_i L_i \quad i = \nu, \ell, u, d
\]

(15.31)

or

\[
\mathcal{M}_i = \varphi_i (\bar{L}_i R_i + \bar{R}_i L_i)(\bar{D}_i D_i)
\]

(15.32)

leading to (15.29).

This speculative expression, a special case of \( H = f(\bar{b}b) \), is discussed in Refs. 3 and 4.

**Remarks.**

The present model has been constructed to agree closely with the standard model where both are well defined. However, neither the standard model nor the knot model as here presented describes the origin of the fermionic spectrum or the origin of the Kobayashi-Maskawa matrix. In addition the number of Higgs particles is also left indefinite in both approaches. On the other hand, as we have shown, the additional degrees of freedom of the knot model provide a possible formal basis for filling these gaps.

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