Separatrix Reconnections in Chaotic Regimes

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In this paper we extend the concept of separatrix reconnection into chaotic regimes. We show that even under chaotic conditions one can still understand abrupt jumps of diffusive-like processes in the relevant phase-space in terms of relatively smooth realignments of stable and unstable manifolds of unstable fixed points.

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Two dimensional nonmonotonic conservative maps are recognized to be of relevance in modeling a number of nonlinear systems, as for instance laser acceleration of charged particles [1,2], and the nonlinear flow of magnetic field lines in fusion machines like tokamaks and others [3,4]. As opposed to the more traditional monotonic versions, nonmonotonic maps are characterized by frequency curves that are not monotonic functions of the action variable. In laser accelerators, nonmonotonicity arises as a result of the relativistic mass variation of the accelerating particles [5]; in tokamaks, it arises as a result of the geometrical peculiarities of the relevant background magnetic fields. In any case, nonmonotonicity has strong influence on the types of bifurcations that can occur in the associated nonlinear dynamics. Indeed, monotonic maps typically allow only period doubling bifurcations of fixed points but if one adds nonmonotonicity, tangent bifurcations involving pairs of elliptic and hyperbolic fixed points also become possible in the appropriate phase-space.

Period doubling cascades of periodic orbits generally precede a transition to chaotic regimes of these orbits, but tangent bifurcations bear no direct relationship to nonintegrability. Indeed, it has been argued that the most noticeable effects of a tangent bifurcation are to be seen while the system lies in predominantly integrable regimes. Although we will show that this sort of argument is questionable, consider the process depicted in the integrable case of Fig.(1a). Two chains of fixed points undergo tangent bifurcation, or actually an inverse tangent bifurcation, starting from the leftmost panel down to the rightmost. Before the bifurcation itself where elliptic fixed points collapse against hyperbolic points, the separatrices defining the upper chain undergo a reconnecting process with those defining the lower chain - this is seen in Fig.(1b). It is precisely due to the smoothness caused by integrability that the reconnection can be seen so clearly. This is why reconnection is thought to be of relevance only in integrable cases. In contrast, the process is generally regarded as of little significance in chaotic regimes because in those situations all separatrices - which shall be correctly called stable and unstable manifolds then - would be already interleaved with little global response as relevant control parameters are varied. Speaking in more precise terms, effects associated with reconnections are thought to be unobservable when the elliptic fixed points of the reconnecting chains undergo full cascades of period doublings, before any sort of mutual contact of the relevant manifolds takes place.

While it is true that reconnections are not easily visualized in chaotic regimes, it is our purpose here to show that their effect can still be quite appreciable. What happens is that even in chaotic regimes the unstable or stable manifolds of originally hyperbolic points may still make a transition from a situation where their mutual crossings are absent - or, actually, relatively infrequent - to a situation where the mutual crossings become very frequent. We shall illustrate the process with a set of figures from which one will be able to see that this change in topology is a reminiscence of the corresponding behavior of regular regimes. The macroscopic result of this type of transition is that as soon as the crossings become frequent, stochastic diffusion undergoes an abrupt jump from slower to faster rates.

From this point on, the discussion relies on more technical grounds. Therefore we introduce here the model map we shall be working with. The map is called the nonmonotonic twist map and reads:

\[ p_{n+1} = p_n - k \sin \phi_n, \quad \phi_{n+1} = \phi_n + f(p_{n+1}), \quad (1) \]

where \((p, \phi)\) is a pair of discrete canonical variables - \(p\) representing an action and \(\phi\) a \(2\pi\)-periodic angular coordinate - and where the map itself is totally simpletic given that the left-hand-side of the second Eq.(1) depends on \(p_{n+1}\) instead of \(p_n\). Function \(f(p)\) is of foremost importance here. It is in fact a measure of the frequency with which the discrete orbits move on the \((p, \phi)\) phase-space.

In standard monotonic maps it reads \(f(p) = p\), but as we wish to incorporate nonmonotonic features we add a quadratic term such that \(f\) becomes \(f(p) = p - \alpha p^2\) as in Ref. [6]. In this case one has effectively a nonmonotonic
frequency curve with maximum located at $p = 1/(2\alpha)$, where $df(p)/dp = 0$. The map (1) has several families of fixed points. Let us focus here on the first order family (period one orbits) which is characterized by $p_{n+1} = p_n$ and $\phi_{n+1} = \phi_n + 2m\pi$, with $m$ as an arbitrary positive or negative integer; we shall refer to the fixed points as $(p^\ast_m, \phi^\ast_m)$. Eq. (1) informs that the fixed points are located at $\phi^\ast_m = 0, \pi$ and $p^\ast_m = [1 \pm \sqrt{1 - 8m^2\alpha/(2\alpha)}]/\alpha$. In this case of period one orbits, let us have a brief look at the distribution of the various fixed points over the phase-space. For $m = 0$ one has four points located at $p^\ast_0 = 0, 1/\alpha$. If $m > 0$ the fixed points lie in the finite interval $0 < p < 1/\alpha$, and if $m < 0$ the points lie in any of the intervals $-\infty < p < 0$ or $1/\alpha < p < +\infty$. 

The existence of points located in the finite interval must satisfy the condition $m < 1/(8\pi\alpha)$, so it may well happen that no fixed point can be actually found there if $\alpha$ is large enough. What happens is that as $\alpha$ grows from some small value, all the fixed points originally located in the sub-interval $0 < p < 1/(2\alpha)$, collapse, at $p = 1/(2\alpha)$, against the corresponding points originally located in the sub-interval $1/(2\alpha) < p < 1/\alpha$ via a sequence of inverse tangent bifurcations. Meanwhile, the points placed externally to the interval $(0, 1/\alpha)$, as well as the points of the pair $m = 0$, simply approach each other but never touch. We point out that although this latter two types of fixed points never undergo inverse tangent bifurcation, their manifolds can naturally undergo the reconnection processes. In previous works the $m = 1$ case has been investigated. It has been shown that when chaos is absent, in the sense that elliptic points of the various chains have not yet period doubled to chaos, initial conditions at negative values of $p$ do not move up to positive values unless the separatrices of the two chains corresponding to the $m = 1$ resonances touch each other. In addition it has been argued that for those situations where the elliptic fixed points have already undergone full cascades of period doublings, no detectable difference in the global aspects of the dynamics should be observed as arising from a possible contact of separatrices. We now proceed to show that this is not quite exact; even under chaotic conditions, some noticeable effects resulting from manifold reconnections can be in fact observed. Specifically we shall show that after what could be best called a reconnection-like process, diffusion makes a somewhat abrupt jump from lower to higher values.

Let us focus the discussion on the $m = 0$ case because this resonance is the largest one in the system. The linear stability of the fixed points can be examined from the characteristic equation $\lambda = [(2 - \kappa) \pm \sqrt{(\kappa - 2)^2 - 4}]/2$, with $\kappa = \cos(\phi^\ast_0)(1 - 2\alpha p^\ast_0)k$. $\lambda$ is the eigenvalue of the linearized map; if complex (purely real) the corresponding fixed point is unstable (stable). One then sees that for the chain located at $p^\ast_0 = 0$, the fixed point at $\phi^\ast_0 = \pi$ is always unstable, while that at $\phi^\ast_0 = 0$ is unstable only when $k > 4$, being stable otherwise. As for the chain at $p^\ast_0 = 1/\alpha$, the point at $\phi^\ast_0 = 0$ is always unstable, while that at $\phi^\ast_0 = \pi$ is unstable when $k > 4$; in both cases, de-stabilization occurs via period doubling of the elliptic points. One can also make an estimate of the condition to be observed for separatrix touching. To do so, let us first imagine that we are working in a situation where $k$ is small and the dynamics is therefore mostly regular. What happens then is that the advance within any particular resonant island tends to be slow. One can thus approximate $p_{n+1} - p_n$ and $\phi_{n+1} - \phi_n$ by their respective infinitesimal increments $dp$ and $d\phi$ and finally write an expression valid in the vicinity of a $m = 0$ chain: $dp/d\phi = -k\sin(\phi)/f(p)$. Then, with obvious notation, one obtains an expression for the separatrix of the lower chain: 

$$\int^{p_{sep}} f(p)dp = k(1 + \cos(\phi_{sep})).$$

(2)

The upper separatrix of the $(p_o = 0, \phi_o = \pi)$ fixed point touches the $(p_o = 1/\alpha, \phi_o = 0)$ fixed point of the corresponding upper chain (see Fig.1) when $k = k_\alpha = 1/(12\alpha^2)$; such value for $k$ is also known as the reconnection threshold in regular regimes. We shall extend this definition for the threshold into chaotic regimes. If we do that we can draw Fig.2. The figure is similar to corresponding figures shown in Refs. 1, 2. It displays simultaneously the threshold and period doubling curves in the parameter space. The period doubling curve, in the $m = 0$ case analyzed here, is simply a horizontal line at $k = 4$. It is thus seen that for values of $\alpha$ below $\alpha \sim 0.144$ period doublings occur before reconnection. Previous works have focused interest on the region $\alpha > 0.144$ because in that region one could clearly speak in terms of reconnection - recall that in this region reconnection takes place before period doublings. In the present paper we shall concentrate efforts to see what happens deep into chaotic regimes when $\alpha < 0.144$.

To start with the investigation, let us consider the proximity of the threshold curve at $k = 5$. For this value of $k$, elliptic points have been totally destroyed by full cascades of period doublings. Therefore, as we increase $\alpha$ the theoretical reconnection threshold can be attained while the system remains in a deep chaotic regime. Let us try to examine how the relevant manifolds of the $m = 0$ resonances behave on the phase-space. The analysis is made with help of the panels of Fig.3 were we focus attention on the upper unstable and lower stable manifolds (respective orientation indicated by arrows) of the originally hyperbolic points (points indicated by black dots) of the lower and upper chain respectively; in order to draw the manifolds we launch 1000 initial conditions along the linearized manifolds, iterating the dynamics forward or backwards according to the case. First of all we note that in integrable cases separatrices describe homoclinic loops. Now, even in our nonintegrable case, when $\alpha$ is small it is seen that the tendency of the unstable manifold of the lower chain is to follow the homoclinic loop of the integrable approximation. Of course, due to the nonintegrable features the unstable manifold eventually starts to execute increasingly large

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oscillations after it first intersects the stable manifold of the same point. As the orbit is about to complete the homoclinic loop the oscillations grow and as the oscillations grow it may happen that the unstable manifold of the lower chain crosses the stable manifold of the upper chain as well. But what must be observed here is that this latter intersection occurs only after the stable manifold of the lower chain crosses the unstable manifold of the same lower chain many times. In this situation one can safely look at the process as closely resembling the situation of the lower chain crossing the unstable manifold of this latter chain as well. But what must be observed here is that this latter intersection occurs only after the stable manifold of the lower chain crosses the unstable manifold of the same lower chain many times. In this situation one can safely look at the process as closely resembling the integrable case, although, as mentioned before, one lies in a deeply chaotic regime; this feature is somewhat puzzling and shall be considered in detail later.

Then, when one increases the value of $\alpha$, the overall topology of manifold crossings appears to undergo a substantial change. This can be observed in Fig. 5b where it is seen that this change in topology is in fact very similar to what happens in the purely integrable model; to our knowledge this had not been realized before. Here the unstable manifold of the hyperbolic point of the lower chain makes direct connection with the stable manifold of the upper chain. This leads to the opening of a new diffusive channel connecting the regions located below the lower chain and above the upper chain. In other words, although one lies in a deep chaotic regime since the original elliptic points have fully period doubled to chaos, noticeable changes can be expected as a result of clear alterations on the topology of the manifolds. This behavior of the manifolds is rather conservative in the sense that while the elliptic points have bifurcated, the manifolds still try to preserve some aspects of integrability. As it appears this feature takes place because near the midpoint between the two chains, the orbital frequency attains a maximum $df(p)/dp = 0$ in view of nonmonotonicity. As a result, the local dynamics is relatively linear and nonintegrable effects are relatively smaller than in other regions of the phase-space.

We now proceed to show that reconnections in chaotic regimes may have direct influence on macroscopic processes, such as diffusion. We actually measure a fraction of particles that are transmitted across the region where $df(p)/dp \sim 0$ in a numerical experiment that goes as follows.

A set of 1000 initial conditions is launched in the region below the lower reconnecting chain and iterated many times. All particles arriving at the region above the upper reconnecting chain are reinjected into the initial lower region such as to create a steady state in the long run. The simulation is optimized by reflecting particles that move into the region below the injection momentum $p = -2.0$. The mental picture one can form of the system is that of a multicomponent gas placed in a vessel divided by a semi-permeable membrane whose role is played by the reconnecting chains. The gas is placed below the lower chain, and some of the particles are able to crossover up to the region above the upper chain. As time evolves, one reaches a steady state and we measure transmission by computing the number of particles in the upper region divided by the total number of particles.

Figure (4) shows the fraction of particles transmitted after many iterations, versus $\alpha$, for $k = 5$ and 6. On this range of $k$, as suggested by Fig. 2, there is no KAM curve in the phase space. In Fig. (5a) we see an increase in particle transmission starting at the reconnection. The starting point of the increase can be evaluated analytically using Eq. (5). For $k = 5$ one obtains $\alpha = 0.125$ in good agreement with the simulation. It is also observed that for $k = 5$ the transmission reaches a plateau around $\alpha = 0.15$; presumably at this value the reconnection is fully completed.

In Fig. (5b) we finally display surface of sections of 20 trajectories initially placed at $p = -2.0$ with uniform distribution along $\phi$. Fig. (6a) is made for a value of $\alpha$ prior to the reconnection, $\alpha = 0.10$, and Fig. (6b) with $\alpha$ past the reconnection threshold, $\alpha = 0.16$. The alterations in the diffusive pattern suggested by all the previous analysis can be seen in those figures as well - while in Fig. (6a) the particles remain mostly in the lower region, in Fig. (6b) particles can be easily transmitted across the barrier at $p \sim 0$.

To summarize, in this paper we have investigated the effect that reconnections involving unstable manifolds of deeply chaotic regimes can have on some macroscopically observable features like particle diffusion. We have used a nonmonotonic map to create reconnecting chains. Examining a particular family of fixed point for which $m = 0$ in the notation of the text, we have seen that the topology of manifolds of unstable fixed points may have a similar behavior as in an integrable approximation. As we vary convenient parameters, manifolds of different chains of islands can be clearly seen to undergo reconnection process even in deep chaotic regimes. This is somewhat unexpected, since if one lies in chaotic regimes where manifolds are already strongly interlaced, any reconnection would be expected of little influence in macroscopic features.

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FIGURE LEGENDS

FIG. 1. Inverse tangent bifurcation and the preceding reconnection in a purely integrable case. The figure is constructed with an integrable one-degree-of-freedom Hamiltonian of the type $H = p - \alpha p^2 + k \cos \phi$ with $(p, \phi)$ as continuous canonical variables; $k = 5$ and $\alpha = 0.10$ in (a), 0.128 in (b), and 0.16 in (c).

FIG. 2. Parameter space $(\alpha, k)$ for the period one $m = 0$ resonance, and the relevant threshold curves.

FIG. 3. The reconnection of stable and unstable manifolds in a deeply chaotic regime for which $k = 5$.

FIG. 4. Transmission fraction as a function of $\alpha$ in the deeply chaotic regimes $k = 5$ and $k = 6$.

FIG. 5. Poincaré plot of the $k = 5$ case for $\alpha = 0.10$ in (a) and for $\alpha = 0.16$ in (b).
Fig. 1: Corso and Rizzato
Fig. 4

particle transmission

\[ K=6.0 \quad \quad \quad \quad K=5.0 \]

\( \alpha \)
Fig. (2): Corso and Rizzato

\[ k = \frac{1}{12 \alpha^2} \] - threshold curve

\[ k = 4 \] - period doubling line

Fig. (3): Corso and Rizzato

(\( \alpha = 0.10 \))

Fig. (5): Corso and Rizzato

(\( \alpha = 0.13 \))