Generalized Macaulay representations and the flag \( f \)-vectors of generalized colored complexes

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Abstract

A colored complex of type \( \mathbf{a} = (a_1, \ldots, a_n) \) is a simplicial complex \( \Delta \) on a vertex set \( V \), together with an ordered partition \((V_1, \ldots, V_n)\) of \( V \), such that every face \( F \) of \( \Delta \) satisfies \(|F \cap V_i| \leq a_i\). For each \( \mathbf{b} = (b_1, \ldots, b_n) \leq \mathbf{a} \), let \( f_{\mathbf{b}} \) be the number of faces \( F \) of \( \Delta \) such that \(|F \cap V_i| = b_i\). The array of integers \( \{f_{\mathbf{b}}\}_{\mathbf{b} \leq \mathbf{a}} \) is called the fine \( f \)-vector of \( \Delta \), and it is a refinement of the \( f \)-vector of \( \Delta \). In this paper, we generalize the notion of Macaulay representations and give a numerical characterization of the fine \( f \)-vectors of colored complexes of arbitrary type, in terms of these generalized Macaulay representations.

As part of the proof, we introduce the property of \( a \)-Macaulay decomposability for simplicial complexes, which implies vertex-decomposability, and we show that every pure color-shifted balanced complex \( \Delta \) of type \( \mathbf{a} \) is \( a \)-Macaulay decomposable. Combined with previously known results, we also obtain a numerical characterization of the flag \( f \)-vectors of completely balanced Cohen-Macaulay complexes.

1 Introduction and Overview

In topological combinatorics, a fundamental combinatorial invariant of a \((d-1)\)-dimensional triangulated space \( \Delta \) is its \( f \)-vector \((f_0, f_1, \ldots, f_{d-1})\), where \( f_i \) denotes the number of \( i \)-dimensional faces. Around the 1960s, Schützenberger [27], Kruskal [18] and Katona [17] independently gave a numerical characterization of the \( f \)-vectors of simplicial complexes in terms of Macaulay representations (also known as \( k \)-binomial expansions). This became (inaccurately) known as the Kruskal-Katona theorem. Subsequently in the 1970s and 1980s, following the influential work of McMullen ([21] [22]) and Stanley ([28] [29] [31], etc.), the study of the \( f \)-vectors of several important classes of complexes was reformulated in terms of \( h \)-vectors.

The \( h \)-vector \((h_0, h_1, \ldots, h_d)\) of \( \Delta \) is another closely related combinatorial invariant of \( \Delta \) with algebraic significance, and its study allows the use of techniques in commutative algebra (see, e.g., [31]) to understand enumerative properties of \( \Delta \). Setting \( f_{-1} = 1 \), the \( h \)-vector and \( f \)-vector of \( \Delta \) are related by

\[
\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x - 1)^{d-i},
\]

and it is easy to see that the \( f \)-vector and \( h \)-vector of \( \Delta \) determine each other completely.

Much work has been done finding numerical characterizations analogous to the Kruskal-Katona theorem. A common strategy is to first prove that several classes of complexes share the same \( f \)-vectors or \( h \)-vectors, and then find an explicit numerical characterization for one of these classes. For example, Stanley [29] showed that the collections of \( h \)-vectors of Cohen-Macaulay complexes, constructible complexes and shellable complexes are each identical to the collection of \( f \)-vectors of multicomplexes, and he observed that Macaulay’s theorem [20] yields an explicit numerical characterization of the \( f \)-vectors of multicomplexes. Wegner [35] showed that the \( f \)-vectors of polyhedral complexes have the same numerical characterization as that of the \( f \)-vectors of simplicial complexes already established by the Kruskal-Katona theorem, while Nevo [25] subsequently proved a simultaneous generalization of Wegner’s theorem and Macaulay’s theorem. In fact, the crucial step in several proofs of the Kruskal-Katona theorem is to show that the \( f \)-vector of a simplicial
complex is the $f$-vector of a compressed simplicial complex, and then give a numerical characterization for this smaller class of complexes. For a good exposition on the Kruskal-Katona theorem, see [14, Section 8].

Colored complexes and completely balanced Cohen-Macaulay complexes are two important classes of complexes for which complete numerical characterizations of their fine $f$-vectors were previously unknown. Our main goal is to give numerical characterizations for these two classes. In this paper, we define a colored complex of type $a = (a_1, \ldots, a_n)$ to be a simplicial complex $\Delta$ on a non-empty vertex set $V$, together with an ordered partition $(V_1, \ldots, V_n)$ of $V$, such that every face $F$ of $\Delta$ satisfies $|F \cap V_i| \leq a_i$. Note that the usual notion of colored complexes many authors use is equivalent to our definition of colored complexes of type $(1, \ldots, 1)$. A $(d - 1)$-dimensional balanced complex of type $a$ is a colored complex $\Delta$ of type $a$ satisfying $\sum a_i = d$, while a $(d - 1)$-dimensional completely balanced complex is a balanced complex of type $1_d := (1, \ldots, 1)$. For each $b = (b_1, \ldots, b_n) \leq a$, let $f_b$ be the number of faces $F$ of $\Delta$ such that $|F \cap V_i| = b_i$.

The array of integers $\{f_b\}_{b \leq a}$ is called the fine $f$-vector of $\Delta$, and it is a refinement of the $f$-vector of $\Delta$. Although some authors call this array the flag $f$-vector, we instead reserve the notion of ‘flag $f$-vector’ to mean something else that is closely related. For brevity, we write ‘CM’, ‘$a$-balanced’ and ‘$a$-colored’ to mean ‘Cohen-Macaulay’, ‘balanced of type $a$’ and ‘colored of type $a$’ respectively. All other relevant terminology that we use in the rest of this section will be defined in later sections.

In an attempt to glean information about the fine $f$-vectors of balanced CM complexes, Björner, Frankl and Stanley [1, 30] proved the following theorem:

**Theorem 1.1.** Let $a \in \mathbb{P}^n$, and let $f = \{f_b\}_{b \leq a}$ be an array of integers. The following are equivalent:

(i) $f$ is the fine $h$-vector of an $a$-balanced CM complex.

(ii) $f$ is the fine $h$-vector of an $a$-balanced pure shellable complex.

(iii) $f$ is the fine $f$-vector of an $a$-colored multicomplex.

(iv) $f$ is the fine $f$-vector of a color-compressed $a$-colored multicomplex.

Subsequently, Babson and Novik [1] proved that the fine $h$-vector of an $a$-balanced CM complex is the fine $h$-vector of a color-shifted $a$-balanced CM complex, and that a color-shifted balanced complex is CM if and only if it is pure. Also, Biermann and Van Tuyl [2] recently proved that the $h$-vector of a completely balanced CM complex is the $h$-vector of a pure completely balanced vertex-decomposable complex. In fact, we prove the stronger statement that the fine $h$-vector of an $a$-balanced CM complex is the fine $h$-vector of a pure $a$-balanced vertex-decomposable complex (Corollary 4.13). Since a color-compressed colored multicomplex is necessarily color-shifted, it follows that the fine $h$-vector of an $a$-balanced CM complex is also the fine $f$-vector of a color-shifted $a$-colored multicomplex. Thus, finding a numerical characterization for any of these classes of complexes would yield important enumerative information for all eight of them.1 See Corollary 4.13 for a full list of equivalences.

The Frankl-Füredi-Kalai theorem [11], also known as the colored Kruskal-Katona theorem, extends the Kruskal-Katona theorem, and it gives a numerical characterization of the $f$-vectors of $(1, \ldots, 1)$-colored complexes, or equivalently, the $f$-vectors of $(1, \ldots, 1)$-colored multicomplexes. This theorem is one of the only two known instances of numerical characterizations involving a generalized notion of Macaulay representations (the other being a numerical version of the Clements-Lindström theorem [7; see 11]), and when combined with Theorem 1.1 it gives a numerical characterization of the $h$-vectors of completely balanced CM complexes. However, since the Frankl-Füredi-Kalai theorem [11] was published in 1988, very little progress has been made towards a numerical characterization of the more refined fine $h$-vectors.

As pointed out in [1], part of the difficulty lies in the non-uniqueness of color-compressed multicomplexes with a given fine $f$-vector. Frohmader [12] showed that it is not possible to make further progress towards a numerical characterization of the fine $f$-vectors of color-shifted $(1, \ldots, 1)$-colored complexes through stronger restrictions on the color-selected subcomplexes. As for specific small cases, Walker [34] gave a numerical characterization of the fine $f$-vectors of $(1, 1)$-colored complexes, while Frohmader [13] gave a numerical

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1The eight classes: balanced CM complexes, color-shifted balanced CM complexes, pure color-shifted balanced complexes, pure balanced vertex-decomposable complexes, pure balanced shellable complexes, colored multicomplexes, color-compressed colored multicomplexes, color-shifted colored multicomplexes.
characterization of the fine $f$-vectors of $\binom{1,1,1}{n}$-colored complexes. Both numerical characterizations by Walker and Frohmader are given in terms of (non-linear) inequalities that the $f$-vectors of the fine $f$-vector must satisfy. In particular, Frohmader’s complicated proof involves the brute-force check of many cases.

To address these difficulties, we define a suitably generalized notion of Macaulay representations. Such generalized Macaulay representations are non-unique, and we have our first main result:

**Theorem 1.2.** Let $a \in \mathbb{P}^n$, and let $f = \{f_b\}_{b \leq a}$ be an array of integers. The following are equivalent:

(i) $f$ is the fine $f$-vector of an $a$-colored complex.

(ii) $f$ is the fine $f$-vector of a color-compressed $a$-colored complex.

(iii) $f$ is the fine $f$-vector of a color-shifted $a$-colored complex.

(iv) $f_0 = 1$, $f_{a'} = 0$ for all $a' \not\geq 0$, $f_{\delta_{i,n}} > 0$ for all $i \in [n]$, and there exists an array $\{\alpha_b\}_{0_n < b \leq a}$ such that each $\alpha_b$ is a generalized $b$-Macaulay representation of $f_b$, and $\alpha_b \preceq \alpha_{b'}$, for all $b, b'$ satisfying $0_n < b' < b \leq a$.

In statement (iv) $\delta_{i,n}$ denotes the $i$-th unit coordinate vector in $\mathbb{N}^n$, and $\preceq$ is a partial order defined on generalized Macaulay representations (see Section 5.4). For $n = 1$, this statement specializes to the usual numerical characterization of the $f$-vectors of simplicial complexes given by the Kruskal-Katona theorem (see Corollary 5.29). As for the case $a = 1_n$, the numerical version of the Frankl-Füredi-Kalai theorem gives a numerical characterization of the $f$-vectors of $1_n$-colored complexes, whereas Theorem 1.2 gives a numerical characterization of the more refined fine $f$-vectors of $1_n$-colored complexes; see Corollary 5.30.

Note that Claim 4.2(ii) in [11] is not true, hence the original proof of the Frankl-Füredi-Kalai theorem in [11] is incorrect as stated. Nevertheless, London [19] gave a different (and essentially correct) proof, Engel [10, Chapter 8] gave a proof using properties of Macaulay posets, while Mermin and Murai [23] gave a proof in [11] is incorrect as stated. Nevertheless, London [19] gave a different (and essentially correct) proof, Engel [10, Chapter 8] gave a proof using properties of Macaulay posets, while Mermin and Murai [23] gave a proof in [11], which makes the $\partial_{\langle k \rangle}(\cdot)$ operator introduced in [11] not well-defined. The version of the Frankl-Füredi-Kalai theorem stated in [3] includes a fix suggested by J. Eckhoff and agrees with our numerical characterization.

Next, we state our second main result, which combines results in [1, 4, 30].

**Theorem 1.3.** Let $d \in \mathbb{P}$, let $f = \{f_S\}_{S \subseteq [d]}$ be an array of integers, and define $\phi : 2^d \to \{0,1\}^d$ by $S \mapsto (s_1, \ldots, s_d)$, where $s_i = 1$ if and only if $i \in S$. Then the following are equivalent:

(i) $f$ is the flag $f$-vector of a $(d - 1)$-dimensional completely balanced CM complex.

(ii) $f$ is the flag $f$-vector of a $(d - 1)$-dimensional color-shifted completely balanced CM complex.

(iii) $f$ is the flag $f$-vector of a $(d - 1)$-dimensional pure color-shifted completely balanced CM complex.

(iv) $f$ is the flag $f$-vector of a $(d - 1)$-dimensional pure completely balanced vertex-decomposable complex.

(v) $f$ is the flag $f$-vector of a $(d - 1)$-dimensional pure completely balanced shellable complex.

(vi) $f_d > 0$, and there exists a generalized $1_d$-Macaulay representation $\alpha$ of $f_d$ such that $f_S = \partial_{\phi(S) - 1_d}(\alpha)$ for every $S \subseteq [d]$.

In particular, our contributions are the addition of statements (iv) and (vi) to this list. For statement (iv) see Corollary 4.13 while for statement (vi) see Theorem 5.23. A non-trivial example of Theorem 1.3 is given in Example 5.24. In view of Theorem 1.1 and Theorem 1.2, there is also a different but equivalent numerical characterization of the flag $h$-vectors of completely balanced CM complexes; see Corollary 5.31.

The rest of the paper is organized as follows. Section 2 covers the preliminaries. Section 3 deals with color compression and color shifting, as well as a review of the theory of colored algebraic shifting introduced by Babson and Novik [1]. In particular, we show that color compression preserves the fine $f$-vectors of colored complexes (Corollary 3.4).
As part of the proofs of Theorems 1.2 and 1.3, we introduce the property of $\alpha$-Macaulay decomposability for simplicial complexes in Section 1. An $\alpha$-Macaulay decomposable simplicial complex is vertex-decomposable (Proposition 1.2), and we show every pure color-shifted $\alpha$-balanced complex is $\alpha$-Macaulay decomposable and hence vertex-decomposable (Theorem 1.9). This allows a geometric interpretation of ‘decomposing’ $\alpha$-balanced complexes into pieces we can better understand, and in Section 2 we define the purely numerical notion of generalized $\alpha$-Macaulay representations using $\alpha$-Macaulay decomposability and relate them to the fine $f$-vectors of $\alpha$-colored complexes and completely balanced CM complexes.

2 Preliminaries

Let $\mathbb{N}$ and $\mathbb{P}$ denote the non-negative integers and positive integers respectively, and for convenience, let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}, \overline{\mathbb{P}} = \mathbb{P} \cup \{\infty\}$. For $n \in \mathbb{P}$ and $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ in $\mathbb{Z}^n$, let $|a|$ be the set \{1, \ldots, $n$\}, let $|a|_i = a_i + \cdots + a_n$, and write $a \preceq b$ if $a_i \leq b_i$ for all $i \in [n]$. Set $[0] = \emptyset$. By convention, $a < b$ if $a \leq b$ and $a \neq b$. Also, write $a < b$ if $a < b$ and $|b| = |a| + 1$. For brevity, we use $0_n$ and $1_n$ to mean the $n$-tuples $(0, \ldots, 0)$ and $(1, \ldots, 1)$ respectively. Denote the Kronecker delta function by $\delta(i, j)$, i.e. $\delta(i, j) = 1$ if $i = j$, $\delta(i, j) = 0$ if $i \neq j$, and define $\delta_{i,n} = (\delta(i, 1), \delta(i, 2), \ldots, \delta(i, n)) \in \mathbb{N}^n$. If $f$ is a function whose image is contained in $\mathbb{Z}^n$ (for some $n \in \mathbb{P}$), then $f$ has $n$ component functions, and for each $t \in [n]$, denote the $t$-th component function of $f$ by $f[t]$, i.e. $f[t](x)$ is the $t$-th entry of the $n$-tuple $f(x)$.

Let $n \in \mathbb{P}$ and let $\pi = (X_1, \ldots, X_n)$ be an ordered sequence of (not necessarily non-empty) subsets of a set $X$. For any $W \subseteq X$, define $\pi \cap W := (X_1 \cap W, \ldots, X_n \cap W)$. If the subsets $X_1, \ldots, X_n$ are not all empty, and if $1 \leq i_1 < \cdots < i_m \leq n$ are all the indices such that $X_{i_t} \neq \emptyset$ for each $t \in [m]$, then define $\pi := (X_{i_1}, \ldots, X_{i_m})$. Analogously, given $x = (x_1, \ldots, x_n) \in \mathbb{N}^n \setminus \{0_n\}$, if $1 \leq i_1 < \cdots < i_m \leq n$ are all the indices such that $x_{i_t} > 0$ for each $t \in [m]$, then define $x := (x_{i_1}, \ldots, x_{i_m}) \in \mathbb{P}^m$.

An order ideal of a poset $(P, \preceq)$ is a subset $I \subseteq P$ such that if $x \in P$ and $y \preceq x$, then $y \in I$. If $I$ is a well-ordering on $P$, then the order ideal $I$ is called an initial segment of $P$ (with respect to $\preceq$). An ordered partition of a non-empty set $S$ is a finite sequence $(S_1, \ldots, S_n)$ of non-empty, pairwise disjoint subsets of $S$ satisfying $S = S_1 \cup \cdots \cup S_n$. For each $k \in \mathbb{N}$, let $(S)_k$ denote the set of all $k$-subsets of $S$, which by default is empty if $|S| < k$. If $S$ is linearly ordered, then the colex (colexicographic) order $\leq_{clex}$ on $(S)_k$ is defined by

$$A <_{clex} B \iff \max\{i : i \in A - B\} < \max\{j : j \in B - A\}.$$  

Equivalently, if $A \Delta B$ is the symmetric difference of $A$ and $B$, then $A <_{clex} B$ if and only if $\max(A \Delta B) \in B$. This order is sometimes known as the reverse lexicographic order or the squashed order.

Let $X = \{x_1, x_2, \ldots\}$ be a set of variables linearly ordered by $x_1 < x_2 \leq \ldots$ and let $\mathcal{M}^d_X$ be the collection of all monomials in variables $X$ of degree $d \in \mathbb{N}$. The lex (lexicographic) order $\leq_{lex}$ on $\mathcal{M}^d_X$ induced by this linear order on $X$ is defined by

$$x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots <_{lex} x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots \iff \alpha_i < \beta_i \text{ for the largest } i \in \mathbb{P} \text{ such that } \alpha_i \neq \beta_i.$$  

The rev-lex (reverse lexicographic) order $\leq_{rlex}$ on $\mathcal{M}^d_X$ induced by this linear order on $X$ is defined by

$$x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots <_{rlex} x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots \iff \alpha_i > \beta_i \text{ for the smallest } i \in \mathbb{P} \text{ such that } \alpha_i \neq \beta_i.$$  

By identifying each squarefree monomial $x_{i_1}\cdots x_{i_d} \in \mathcal{M}^d_X$ with the $(\underline{d})$-tuple, the induced lex order on the subcollection of all squarefree monomials in $\mathcal{M}^d_X$ is equivalent to the colex order defined on $(\underline{d})$.  

\footnote{This order $\leq_{rlex}$ is sometimes called the reverse lexicographic order presumably because of the following reason. In commutative algebra, the usual convention (e.g., in [9 Section 15.2]) is to fix the linear order $x_1 > x_2 > \ldots$ on $X$, and the corresponding reverse lexicographic order $\leq_{rlex}$ on $\mathcal{M}^d_X$ induced by this linear order is defined by

$$x_1^{a_1}x_2^{a_2}x_3^{a_3}\cdots <_{rlex} x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots \iff \alpha_i > \beta_i \text{ for the largest } i \in \mathbb{P} \text{ such that } \alpha_i \neq \beta_i,$$

which is the reverse of $\leq_{lex}$. However, we choose not to follow this convention and instead use the linear order $x_1 < x_2 < \ldots$, so that our definition of shifted complexes with vertex set $X$ coincides with its usual definition, e.g. in [10].}
2.1 Colored Complexes and Balanced Complexes

A simplicial complex $\Delta$ on a vertex set $V$ is a collection of subsets of $V$ such that \{v\} $\in \Delta$ for all $v \in V$ and $\Delta$ is closed under set inclusion (i.e. $F \in \Delta, F' \subseteq F \Rightarrow F' \in \Delta$). In this paper, we always assume $V$ is finite and non-empty. Elements of $\Delta$ are called faces. The dimension of each $F \in \Delta$ is $\dim F = |F| - 1$, and the dimension of $\Delta$, denoted by $\dim \Delta$, is the maximum dimension of its faces. In particular, 0-dimensional faces of $\Delta$ are called vertices, while maximal faces of $\Delta$ are called facets. If all facets of $\Delta$ have the same dimension, then we say $\Delta$ is pure. Given another simplicial complex $\Delta'$ with vertex set $V'$, we say $\Delta$ and $\Delta'$ are isomorphic if there exists a bijection $\nu: V \rightarrow V'$ such that $F \in \Delta$ if and only if $\nu(F) \in \Delta'$.

Let $\Delta$ be a $(d-1)$-dimensional simplicial complex on a vertex set $V$. The $f$-vector of $\Delta$ is $(f_0, \ldots, f_d)$, where each $f_i$ is the number of $i$-dimensional faces of $\Delta$. By default, set $f_{-1} = 1$, which corresponds to the empty face $\emptyset \in \Delta$. The $h$-vector of $\Delta$ is $(h_0, \ldots, h_d)$, defined by

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}, \quad 0 \leq i \leq d. \quad (2)$$

Note that the $f$-vector and $h$-vector of $\Delta$ determine each other completely, since (2) is invertible, i.e.,

$$f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{d-i} h_j, \quad 0 \leq i \leq d. \quad (3)$$

Given an arbitrary collection $\mathcal{F} = \{F_1, \ldots, F_n\}$ of non-empty subsets of $V$, there is a (unique) smallest simplicial complex, denoted by $\langle \mathcal{F} \rangle$, which contains all $F_i$. This simplicial complex is said to be generated by $\mathcal{F}$, and a simplicial complex generated by one face, i.e. $|F| = 1$, is called a simplex.

A subcomplex of $\Delta$ is a subcollection of $\Delta$ that is also a simplicial complex. The $k$-skeleton of $\Delta$ is the subcomplex $\{F \in \Delta : \dim F \leq k\}$. The deletion of a face $\sigma$ from $\Delta$ is the subcomplex

$$\text{dl}_\Delta(\sigma) := \{\tau \in \Delta : \sigma \not\subseteq \tau\}.$$ 

The link of a face $\sigma$ in $\Delta$ is the subcomplex

$$\text{lk}_\Delta(\sigma) := \{\tau \in \Delta : \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}.$$ 

The join of two simplicial complexes $\Delta_1, \Delta_2$ with disjoint vertex sets is the simplicial complex

$$\Delta_1 \ast \Delta_2 := \{\sigma_1 \cup \sigma_2 : \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}.$$ 

Let $\mathbb{F}$ be a field, and consider the polynomial ring $R = \mathbb{F}[x_1, \ldots, x_n]$. Given a simplicial complex $\Delta$ with vertex set $V = \{v_1, \ldots, v_n\}$, the Stanley-Reisner ring (also known as the face ring) of $\Delta$ (over $\mathbb{F}$) is the ring $\mathbb{F}[\Delta] := R/I_{\Delta}$, where $I_{\Delta}$ is the Stanley-Reisner ideal of $\Delta$ defined by

$$I_{\Delta} := \langle \{x_{i_1} \ldots x_{i_m} : \{v_{i_1}, \ldots, v_{i_m}\} \not\in \Delta\} \rangle.$$ 

We say $\Delta$ is Cohen-Macaulay (over $\mathbb{F}$) if $\mathbb{F}[\Delta]$ is a Cohen-Macaulay ring. For brevity, write ‘CM’ to mean ‘Cohen-Macaulay’. For a detailed treatment of CM complexes and their significance, see [31].

Recall that an ordered partition of a non-empty set $V$ is a finite sequence $(V_1, \ldots, V_n)$ of non-empty, pairwise disjoint subsets of $V$ satisfying $V = V_1 \cup \cdots \cup V_n$.

**Definition.** Let $a = (a_1, \ldots, a_n) \in \mathbb{P}^n$. A colored complex of type $a$ is an ordered pair $(\Delta, \pi)$ satisfying:

i) $\Delta$ is a simplicial complex on a vertex set $V$;

ii) $\pi = (V_1, \ldots, V_n)$ is an ordered partition of $V$;

iii) $|F \cap V_i| \leq a_i$ for every $F \in \Delta$ and every $i \in [n]$. 


Note that the usual notion of colored complexes is equivalent to our definition of colored complexes of type \(1_n\), and we say a simplicial complex \(\Delta\) on a vertex set \(V\) is *colorable of type* \(a\) if there exists an ordered partition \(\pi\) of \(V\) such that \((\Delta, \pi)\) is a colored complex of type \(a\). A *balanced complex of type* \(a \in \mathbb{P}^n\) is a colored complex \(\Delta\) of type \(a\) such that \(|a| = \dim \Delta + 1\), and a *completely balanced complex* is a balanced complex of type \(1_n\). In contrast to the original definition as introduced by Stanley [30], we do not assume balanced complexes to be pure. Note also that some authors use the notion “balanced” to mean what we call completely balanced. For brevity, write ‘\(a\)-colored’ and ‘\(a\)-balanced’ to mean ‘colored of type \(a\)’ and ‘balanced of type \(a\)’ respectively.

Given a poset \(P\), the set of chains of \(P\) is a simplicial complex denoted by \(\Delta(P)\), and it is called the *order complex* of \(P\). Its vertex set can be partitioned into \(\rho = (P_1, \ldots, P_n)\), whereby \(x \in P_i\) if and only if \(i - 1\) is the length of the longest chain \(x_0 < x_1 < \cdots < x_i = x\) of \(P\) with top element \(x\), hence \((\Delta(P), \rho)\) is a completely balanced complex. We can also produce examples of balanced complexes of arbitrary type \(a \in \mathbb{P}^n\) from a completely balanced complex \((\Delta, \pi)\) satisfying \(\dim \Delta = |a| - 1\). More specifically, let \(d = |a|\) and let \(\pi = (V_1, \ldots, V_n)\) be the ordered partition of vertex set \(V\). Then any surjective map \(\phi: [d] \to [n]\) induces the ordered partition \(\pi' = (V'_1, \ldots, V'_n)\) of \(V\) such that \(v \in V_i\) implies \(v \in V'_{\phi(i)}\), and we check that \((\Delta, \pi')\) is an \(a\)-balanced complex. However, as pointed out by Swartz [32] Section 3, not all balanced complexes arise from completely balanced complexes in this manner.

Let \((\Delta, \pi)\) be a \((d-1)\)-dimensional colored complex of type \(a = (a_1, \ldots, a_n) \in \mathbb{P}^n\), where \(\pi = (V_1, \ldots, V_n)\) is the ordered partition of vertex set \(V\). For each \(b = (b_1, \ldots, b_n) \in \mathbb{N}^n\) satisfying \(b \leq a\), let \(f_b\) be the number of faces \(F\) of \(\Delta\) such that \(|F \cap V_i| = b_i\). The array of integers \(\{f_b\}_{b \leq a}\) is called the *fine \(f\)-vector* of \((\Delta, \pi)\), and it is a refinement of the \(f\)-vector \((f_0, \ldots, f_{d-1})\) of \(\Delta\) in the sense that

\[
f_i = \sum_{|b|=i+1} f_b, \quad 0 \leq i \leq d-1.
\]

The *fine \(h\)-vector* of \((\Delta, \pi)\) is the array of integers \(\{h_b\}_{b \leq a}\) defined in terms of the fine \(f\)-vector by

\[
h_b := \sum_{c \leq b} f_c \prod_{i=1}^{n} (-1)^{b_i-c_i} \left(\frac{a_i - c_i}{b_i - c_i}\right), \quad 0 \leq b \leq a. \tag{4}
\]

**Theorem 2.1** [30]. Let \((\Delta, \pi)\) be a \((d-1)\)-dimensional balanced complex of type \(a \in \mathbb{P}^n\), and let \(\{h_b\}_{b \leq a}\) be its fine \(h\)-vector. If \((h_0, \ldots, h_d)\) is the \(h\)-vector of \(\Delta\), then

\[
h_i = \sum_{|b|=i} h_b. \tag{5}
\]

If \((\Delta, \pi)\) is completely balanced, i.e. of type \(1_d\), we can identify each \(b = (b_1, \ldots, b_d) \in \mathbb{N}^d\) satisfying \(b \leq 1_d\) with the subset \(\{i \in [d] : b_i = 1\}\) of \([d]\). Under this identification, the corresponding arrays \(\{f_S\}_{S \subseteq [d]}\) and \(\{h_S\}_{S \subseteq [d]}\) are called the *flag \(f\)-vector* and *flag \(h\)-vector* of \((\Delta, \pi)\) respectively. In this case, (4) becomes

\[
h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_S, \quad S \subseteq [d], \tag{6}
\]

and by Theorem 2.1 the flag \(h\)-vector is a refinement of the \(h\)-vector.

### 2.2 Colored Multicomplexes

A *multicomplex* \(M\) on a set of variables \(X\) is a collection of monomials in these variables that is closed under divisibility (i.e. \(m \in M\), \(m' | m \Rightarrow m' \in M\)). In this paper, we always assume \(M\) is non-empty, and we do not require every \(x \in X\) to be in \(M\). A *subcomplex* of \(M\) is a subcollection of \(M\) that is also a multicomplex. Given a subset \(Y \subseteq X\) and any monomial \(m = \prod \{x^{(c)} : x \in X\}\) in \(M\), let \(m_Y := \prod \{x^{(c)} : x \in Y\}\), let the degree of \(m\) be \(\deg(m) := \sum_{x \in X} c(x)\), and define the subcomplex \(M_Y := \{m_Y : m \in M\}\). By default, we
always have $1 \in M_Y$. For each $d \in \mathbb{N}$, denote the collection of all monomials in $M$ of degree $d$ by $M^d$. The $f$-vector of $M$ is $(f_0, f_1, \ldots)$, where $f_i = |M^i|$ for each $i \in \mathbb{N}$. Note that $f_0 = 1$, corresponding to $1 \in M$.

Let $X = \{x_1, x_2, \ldots\}$ be a set of variables linearly ordered by $x_1 < x_2 < \ldots$. Denote the collection of all monomials in variables $X$ by $\mathcal{M}_X$, and for every $d \in \mathbb{N}$, denote the subcollection of all monomials in $\mathcal{M}_X$ of degree $d$ by $\mathcal{M}_X^d$. Given any map $\phi : X \to \mathbb{P}$, let $e = (e_1, e_2, \ldots) := (\phi(x_1), \phi(x_2), \ldots)$, and let $\mathcal{M}_X(e)$ be the collection of all monomials $x_1^{e_1} x_2^{e_2} \cdots$ such that $0 \leq c_i \leq e_i$ for each $i$. Also, for every $d \in \mathbb{N}$, let $\mathcal{M}_X^d(e)$ be the subcollection of all monomials in $\mathcal{M}_X(e)$ of degree $d$. In particular, $\mathcal{M}_X(1_n)$ is the collection of all squarefree monomials on variables $x_1, \ldots, x_n$. Note that every simplicial complex $\Delta$ on a vertex set $V \subseteq X$ corresponds bijectively to a finite multicomplex $M \subseteq \mathcal{M}_X(1_n)$ via $x_i \cdots x_i \in \Delta \iff x_i \cdots x_i \in M$, hence multicomplexes can be considered as generalizations of simplicial complexes. Note however that the $f$-vector of $\Delta$ and the $f$-vector of $M$ differ by a shift in the indexing.

**Definition.** Let $a = (a_1, \ldots, a_n) \in \mathbb{P}^n$. A colored multicomplex of type $a$ is an ordered pair $(M, \pi)$ satisfying:

(i) $M$ is a multicomplex on set $X$;
(ii) $\pi = (X_1, \ldots, X_n)$ is an ordered partition of $X$;
(iii) $\deg(m_{X_i}) \leq a_i$ for every $m \in M$ and every $i \in [n]$.

The multidegree of a monomial $m \in M$ is the vector $\deg(m) := (\deg(m_{X_1}), \ldots, \deg(m_{X_n})) \in \mathbb{N}^n$. For each $b \in \mathbb{N}^n$, let $f_b$ be the number of monomials $m$ in $M$ such that $\deg(m) = b$. The array of integers $\{f_b\}_{b \leq a}$ is called the fine $f$-vector of $(M, \pi)$, and it is a refinement of the $f$-vector $(f_0, f_1, \ldots)$ of $M$ in the sense that

$$f_i = \sum_{|b|=i} f_b, \quad i \in \mathbb{N}.$$  

For type $a = 1_n$, define the flag $f$-vector of $(M, \pi)$ analogously as in the case of simplicial complexes.

Fix a set of variables $X$, and let $\pi = (X_1, \ldots, X_n)$ be an ordered partition of $X$ such that each $X_i = \{x_{i,1}, x_{i,2}, \ldots\}$ is linearly ordered by $x_{i,1} < x_{i,2} < \ldots$. For each $i \in [n]$, let $\phi_i : X_i \to \mathbb{P}$ be any map, and let $e_i = (\phi_i(x_{i,1}), \phi_i(x_{i,2}), \ldots)$. Let $\mathcal{M}_\pi(e_1, \ldots, e_n)$ denote the collection of all monomials $m$ in variables $X$ such that $m_{X_i} \in \mathcal{M}_X(e_i)$ for every $i \in [n]$. When $e_i = (\infty, \infty, \ldots)$ for every $i \in [n]$, we simply write $\mathcal{M}_\pi(e_1, \ldots, e_n)$ as $\mathcal{M}_\pi$. If a colored multicomplex $(M, \pi)$ satisfies $M \subseteq \mathcal{M}_\pi(e_1, \ldots, e_n)$, then by abuse of notation, we say $(M, \pi)$ is in $\mathcal{M}_\pi(e_1, \ldots, e_n)$. Note that it is implicitly assumed every colored multicomplex in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ has $\pi$ as its corresponding ordered partition, and the length of each sequence $e_i$ is the cardinality of $X_i$.

### 2.3 Macaulay Representations

Given $N, i \in \mathbb{P}$, it is easy to show there exists a unique expansion

$$N = \binom{N_i}{i} + \binom{N_{i-1}}{i-1} + \cdots + \binom{N_j}{j},$$  

such that $N_i > N_{i-1} > \cdots > N_j$ with $j \geq 1$. (See, e.g., [14] Section 8, for a proof.) Such an expansion is called the $i$-th Macaulay representation of $N$ (or the $i$-binomial expansion of $N$).

Since this expansion is unique for each $N \in \mathbb{P}$, we can define functions $\partial^i, \partial^{(i)}, \partial_i, \partial_{(i)}$ on $\mathbb{N}$ by

$$\partial^i(N) = \binom{N_i}{i+1} + \binom{N_{i-1}}{i} + \cdots + \binom{N_j}{j+1}, \quad \partial^{(i)}(N) = \binom{N_i + 1}{i+1} + \binom{N_{i-1} + 1}{i} + \cdots + \binom{N_j + 1}{j+1},$$

$$\partial_i(N) = \binom{N_i}{i-1} + \binom{N_{i-1}}{i-2} + \cdots + \binom{N_j}{j-1}, \quad \partial_{(i)}(N) = \binom{N_i - 1}{i-1} + \binom{N_{i-1} - 1}{i-2} + \cdots + \binom{N_j - 1}{j-1},$$

for all $N \in \mathbb{P}$, and $\partial^0(0) = \partial^{(i)}(0) = \partial_i(0) = \partial_{(i)}(0) = 0$. By convention, define $\binom{n}{k}$ to be 0 if $k < 0$ or $k > n$, and set $\binom{0}{0} = 1$. These functions allow us to numerically characterize the $f$-vectors of simplicial complexes and multicomplexes as follows:
2.4 Graph Theory Terminology

Because graph theory terminology varies widely in the literature, we will describe the particular terminology that we use, which will be pertinent when defining generalized Macaulay representations in Section 5.

All graphs in this paper are assumed to be finite and simple, i.e. a graph is an ordered pair \( G = (V, E) \), where \( V \) is a finite set called the vertex set, whose elements are called vertices, and \( E \) is a subset of \( \binom{V}{2} \), whose elements are called edges. A vertex \( v \in V \) is incident to an edge \( e \in E \) if \( e = \{u, v\} \) for some vertex \( u \neq v \). The degree of \( v \in V \), denoted by \( \deg(v) \), is the number of edges incident to \( v \). We say \( u, v \in V \) are adjacent if \( \{u, v\} \in E \). A labeling of a subset \( U \subseteq V \) is a map \( \phi : U \to \mathbb{L} \), where \( \mathbb{L} \) is an arbitrary set whose elements are called \( \phi \)-labels, or simply labels if the context is clear. A vertex labeling of \( G \) is a labeling of \( V \). A subgraph of \( G \) is a graph \( G' = (V', E') \) such that \( V' \subseteq V \) and \( E' \subseteq E \). Given a subset of vertices \( S \subseteq V \), the subgraph \( G' = (S, E \cap \binom{S}{2}) \) is called the subgraph of \( G \) induced by \( S \). Given an edge \( e = \{u, v\} \) of \( G \), the graph \( G \setminus e := (V, E \setminus \{e\}) \) is said to be obtained from \( G \) by deleting the edge \( e \), and the graph constructed from \( G \) by identifying the vertices \( u \) and \( v \) and then deleting edge \( e \), which we denote by \( G/e \), is said to be obtained from \( G \) by contracting \( e \). If \( \phi \) is a vertex labeling of \( G \) such that \( \phi(u) = \phi(v) \), then \( \phi \) induces a vertex labeling of \( G/e \) in the obvious way, and we denote this vertex labeling of \( G/e \) by \( \phi/e \). For convenience, we sometimes identify a graph as a simplicial complex of dimension \( \leq 1 \).

Given \( u, v \in V \), a walk from \( u \) to \( v \) is a sequence \( v_0, v_1, \ldots, v_n \) of (not necessarily distinct) vertices such that \( \{v_0 = u, v_n = v\} \), and any two consecutive vertices are adjacent. The length of this walk is \( n \). If the \( n \) edges \( \{v_0, v_1\}, \ldots, \{v_{n-1}, v_n\} \) are distinct, we call this walk a path. If we also have \( n \geq 3 \) and \( u = v \), then we call this path a cycle. We say \( G \) is connected if there is a path from \( x \) to \( y \) for every \( x, y \in V \), and we say \( G \) is acyclic if it has no cycles. A connected acyclic graph is called a tree, and for any vertices \( x, y \) of a tree, there exists a unique path from \( x \) to \( y \).

Given a tree \( G = (V, E) \), a vertex \( v \in V \) is called a leaf if \( \deg(v) = 1 \). If \( \deg(u) = 3 \) for every non-leaf \( u \in V \), then \( G \) is called trivalent. We say \( G \) is rooted if it has a distinguished vertex \( r_0 \), which we call the root. Given this root \( r_0 \) and any vertices \( x, y \in V \) (possibly \( x = y \)), if \( x \) is contained in the path from \( r_0 \) to \( y \), then we say \( x \) is an ancestor of \( y \), and \( y \) is a descendant of \( x \). If in addition \( \{x, y\} \in E \), then we say \( x \) is a parent of \( y \), and \( y \) is a child of \( x \). Note that in a rooted tree, the root has no parents, and every other vertex has a unique parent. A planted tree is a rooted tree whose root is a leaf, i.e. the root has a unique child. A rooted tree is called binary if every vertex has \( \leq 2 \) children. In this paper, we assume all rooted trees are planar, i.e. we specify an ordering of the children (if any) of every vertex in rooted trees, or equivalently, we embed every rooted tree in the plane.

Let \( G = (V, E) \) be a trivalent planted binary tree with root \( r_0 \). Let \( Y \) be the set of all trivalent vertices in \( V \), and let \( L \) be the set of all leaves in \( V \) distinct from \( r_0 \), whose elements are called terminal vertices. For convenience, define the \((r_0, 1, 3)\)-triple of \( G \) as the ordered triple \((r_0, L, Y)\), and for any vertex \( x \) of \( G \), let \( D_G(x) \) denote the set of all descendants of \( x \) in \( G \). By definition, every \( y \in Y \) has exactly 2 children, which we shall call the left child and right child. The distinction between the left and right children of every \( y \in Y \) is well-defined by the assumption of planarity, and we always attach subscripts ‘left’ and ‘right’ to \( y \) (i.e. \( y_{\text{left}} \) and \( y_{\text{right}} \)) to denote the left and right children respectively of \( y \). The depth-first order on \( V \) is the linear order \( \leq \) on \( V \) such that \( r_0 \) is the unique minimal element, and \( y < y_{\text{left}} < y_{\text{right}} \) for every

Theorem 2.2 (Kruskal-Katona theorem \cite{27} \cite{18} \cite{17}). A vector \((f_0, \ldots, f_{d-1}) \in \mathbb{Z}^d \) is the \( f \)-vector of some \((d - 1)\)-dimensional simplicial complex if and only if one of the following equivalent conditions holds:

(i) \( 0 < f_i \leq \partial^i(f_{i-1}) \) for every \( i \in [d - 1] \).
(ii) \( 0 < \partial_{i+1}(f_i) \leq f_{i-1} \) for every \( i \in [d - 1] \).

Theorem 2.3 (Macaulay theorem \cite{20} \cite{28}). A vector \((f_0, f_1, \ldots) \) of integers is the \( f \)-vector of some multi-complex if and only if one of the following equivalent conditions holds:

(i) \( f_0 = 1 \) and \( 0 \leq f_{i+1} \leq \partial^i(f_i) \) for every \( i \in \mathbb{P} \).
(ii) \( f_0 = 1 \) and \( 0 \leq \partial_{i+1}(f_{i+1}) \leq f_i \) for every \( i \in \mathbb{P} \).
3 Color Compression and Color Shifting

Both color compression and color shifting are natural generalizations of the notions of compression and shifting (usually defined for simplicial complexes) to colored complexes and colored multicomplexes, and the ideas involved are not new. Color compression was first introduced by Björner, Frankl and Stanley [4] in 1987 in the context of colored multicomplexes, albeit under the same original name of ‘compression’, while color shifting was, to the best of our knowledge, first considered by Babson and Novik [1] in 2006 in the context of a-colored complexes. In fact, Babson and Novik developed a remarkable theory of colored algebraic shifting, which is a colored analog of symmetric algebraic shifting proposed by Kalai [15], and they studied properties of the subclass of colored shifted complexes obtained by colored algebraic shifting. One important property of colored algebraic shifting is that it preserves the Cohen-Macaulayness of an a-balanced CM complex, provided a suitable linear order of its vertices is chosen; see Theorem 3.12.

Throughout this section, let \( X \) be a set of variables, and let \( \pi = (X_1, \ldots, X_n) \) be an ordered partition of \( X \) such that each \( X_i = \{x_{i,1}, x_{i,2}, \ldots \} \) is linearly ordered by \( x_{i,1} < x_{i,2} < \ldots \). For every \( i \in [n] \), let \( \phi_i : X_i \to \mathbb{P} \) be an arbitrary map, and define \( e_i = (e_{i,1}, e_{i,2}, \ldots) := (\phi_i(x_{i,1}), \phi_i(x_{i,2}), \ldots) \). The purpose of this section is to extend the definitions and relevant results of color compression and color shifting to multicomplexes in \( \mathcal{M}_{\pi}(e_1, \ldots, e_n) \), analogous to how the Clements-Lindström theorem [7] extends both the Kruskal-Katona theorem and the Macaulay theorem.

3.1 Color Compression

In this subsection, we show that color compression preserves the fine \( f \)-vectors of colored multicomplexes in \( \mathcal{M}_{\pi}(e_1, \ldots, e_n) \). Our proof uses the Clements-Lindström theorem and follows from a slight modification of the proof of [4, Theorem 1], which considered colored multicomplexes in \( \mathcal{M}_a \).

First of all, observe that for every \( i \in [n] \), \( d \in \mathbb{N} \), we can order the monomials in \( \mathcal{M}_{X_i}^d(e_i) \) by treating \( \mathcal{M}_{X_i}^d(e_i) \) as a subposet of \( \mathcal{M}_{X_i}^d \) with the induced lex order. This induced lex order is clearly a well-ordering on \( \mathcal{M}_{X_i}^d(e_i) \), and we write \( \mathcal{T}_{X_i}^d(e_i)(N) \) to denote the lex initial segment of \( \mathcal{M}_{X_i}^d(e_i) \) of size \( N \).

Let \( (M, \pi) \) be a colored multicomplex of type \( a \in \mathbb{P}^n \) in \( \mathcal{M}_{\pi}(e_1, \ldots, e_n) \), and fix some \( t \in [n] \). Note that every monomial \( m \in M \) can be factorized as \( m = m_{X_t} m_{X-X_t} \), hence we can write \( M \) as the disjoint union

\[
M = \bigcup_{\tilde{m} \in M_{X-X_t}} \left( \bigcup_{d \in \mathbb{N}} \left\{ m \in M : m_{X-X_t} = \tilde{m}, \deg(m/\tilde{m}) = d \right\} \right).
\] (8)

For each \( d \in \mathbb{N} \), \( \tilde{m} \in M_{X-X_t} \), let \( f_d(\tilde{m}) \) be the number of \( m \in M \) such that \( m_{X-X_t} = \tilde{m} \) and \( \deg(m/\tilde{m}) = d \). Define the operation \( \mathcal{C}_t \) on \( M \) by

\[
\mathcal{C}_t(M) := \bigcup_{\tilde{m} \in M_{X-X_t}} \left( \bigcup_{d \in \mathbb{N}} \left\{ m \tilde{m}^* : m^* \in \mathcal{T}^d(e_t)(f_d(\tilde{m})) \right\} \right).
\] (9)

When \( M_{X_t} = M \), this specializes to the usual notion of compression for multicomplexes, and (9) becomes

\[
\mathcal{C}_t(M) = \bigcup_{d \in \mathbb{N}} \mathcal{T}_{X_t}^d(|M^d|).
\] (10)
The Clements-Lindström theorem [7] is equivalent to the following statement.

**Theorem 3.1.** If $e_{t,1} \geq e_{t,2} \geq \ldots$, and $M_{X_i} = M$, then $C_t(M)$ is a multicomplex.

Using this version of Clements-Lindström theorem, we prove the following important lemma.

**Lemma 3.2.** If $e_{t,1} \geq e_{t,2} \geq \ldots$, then $C_t(M)$ is a multicomplex.

**Proof.** Let $p$ be an arbitrary monomial in $M_\pi(e_1, \ldots, e_n)$, and suppose there is some $x \in X$ such that $p' := px$ is in $C_t(M)$. To show $C_t(M)$ is a multicomplex, we have to show that $p \in C_t(M)$.

Suppose $x \in X - X_i$. Let $q := p_{X - X_i}, q' := p'_{X - X_i}, d := \deg(p/q)$, and note that $q' = qx$, $\deg(p'/q') = d$. Every $m \in M$ satisfying $m_{X - X_i} = q'$ must also satisfy $m/x \in M$ and $(m/x)_{X - X_i} = q$, hence $f_d(q') \leq f_d(q)$. This means $T_{X_i}^{e_{X_i}}(f_d(q')) \subseteq T_{X_i}^{d_{X_i}}(f_d(q))$, therefore $p \in C_t(M)$.

Suppose instead $x \in X_i$. Again let $q := p_{X - X_i}$, and observe that $p'_{X - X_i} = q$. For any subcollection $M' \subseteq M_\pi$, define $M'/q := \{m/q : m \in M', M_{X - X_i} = q\}$. We check that $M/q$ is a subcomplex of $M$ satisfying $(M/q)_{X_i} = M/q$. Since $C_t(M/q) = C_t(M)/q$, it follows from $p' \in C_t(M)$ and $p'_{X - X_i} = q$ that $p'/q \in C_t(M/q)$. Finally, since $e_{t,1} \geq e_{t,2} \geq \ldots$, the Clements-Lindström theorem says $C_t(M/q)$ is a multicomplex, thus $p/q \in C_t(M/q)$, which implies $p \in C_t(M)$. □

It is clear from Lemma 3.2 and the definition of $C_t$ that $(C_t(M), \pi)$ is an $a$-colored multicomplex in $M_\pi(e_1, \ldots, e_n)$ having the same fine $f$-vector as $(M, \pi)$. If $C_t(M) = M$ for every $t \in [n]$, then we say $(M, \pi)$ is **color-compressed**. Equivalently, we have the following definition.

**Definition.** Let $a \in \mathbb{P}^n$, and let $(M, \pi)$ be an $a$-colored multicomplex in $M_\pi(e_1, \ldots, e_n)$. We say $(M, \pi)$ is **color-compressed** if for every $m \in M$ and every $t \in [n]$, the set

$$\left\{ m'_{X_i} : m' \in M, \deg(m'_{X_i}) = \deg(m_{X_i}), m'_{X - X_i} = m_{X - X_i} \right\}$$

is a lex initial segment of $M_\pi^{\deg(m_{X_i})}(e_t)$.

**Theorem 3.3.** Let $a \in \mathbb{P}^n$, and let $f = \{f_b\}_{b \leq a}$ be an array of integers. If $e_{t,1} \geq e_{t,2} \geq \ldots$ for every $i \in [n]$, then the following are equivalent:

(i) $f$ is the fine $f$-vector of an $a$-colored multicomplex in $M_\pi(e_1, \ldots, e_n)$.

(ii) $f$ is the fine $f$-vector of a color-compressed $a$-colored multicomplex in $M_\pi(e_1, \ldots, e_n)$.

**Proof.** Let $(M, \pi)$ be an $a$-colored multicomplex in $M_\pi(e_1, \ldots, e_n)$ that is not color-compressed. Starting with $M_0 = M$, iteratively construct a sequence of multicomplexes $M_0, M_1, M_2, \ldots$ such that $M_i = C_t(M_{i-1})$ for each $i \geq 1$, where $t_i \in [n]$ is chosen so that $M_i \neq M_{i-1}$. Given any $t \in [n]$ and any $a$-colored multicomplex $(M', \pi)$ in $M_\pi(e_1, \ldots, e_n)$, observe that

$$\sum_{p \in M'} \sum_{i \in [n]} \{|q \in M_{X_i}(e_i) : q \leq_{\ell \pi} p_{X_i} | \geq \sum_{p \in C_t(M')} \sum_{i \in [n]} \{|q \in M_{X_i}(e_i) : q \leq_{\ell \pi} p_{X_i} |, \tag{11} \right.$$  

with equality holding if and only if $C_t(M') = M'$. Thus, the sequence $M_0, M_1, M_2, \ldots$ must terminate, say with last term $M_k$, and $(M_k, \pi)$ is a color-compressed $a$-colored multicomplex in $M_\pi(e_1, \ldots, e_n)$ having the same fine $f$-vector as $(M, \pi)$, hence proving [i] $\Rightarrow$ [ii]. The converse [ii] $\Rightarrow$ [i] is trivial. □

**Definition.** Let $(\Delta, \rho)$ be an $a$-colored complex for some $a \in \mathbb{P}^n$, where $\rho = (V_1, \ldots, V_n)$ is an ordered partition of the vertex set $V$ of $\Delta$, and assume $V_i$ is linearly ordered for each $i \in [n]$. Then we say $(\Delta, \rho)$ is **color-compressed** if for every $F \in \Delta$ and every $t \in [n]$, the set

$$\left\{ F' \cap V_i : F' \in \Delta, |F' \cap V_i| = |F \cap V_i|, F' \cap (V \setminus V_i) = F \cap (V \setminus V_i) \right\}$$

is a colex initial segment of $(|F \cap V_i|)$. 10
Corollary 3.4. Let $a \in \mathbb{P}^n$, and let $f = \{f_b\}_{b \leq a}$ be an array of integers. The following are equivalent:

(i) $f$ is the fine $f$-vector of an $a$-colored complex.

(ii) $f$ is the fine $f$-vector of a color-compressed $a$-colored complex.

Proof. This immediately follows from Theorem 3.3 by setting $e_i = (1, 1, \ldots)$ for each $i \in [n]$. ☐

In the proof of Theorem 3.3, we started with an $a$-colored multicomplex $(M, \pi)$ in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ that is not color-compressed, and we constructed a finite sequence of multicomplexes in $\mathcal{M}_\pi(e_1, \ldots, e_n)$:

$$M, \mathcal{C}_{t_1}(M), \mathcal{C}_{t_2}(\mathcal{C}_{t_1}(M)), \ldots, \mathcal{C}_{t_k}(\mathcal{C}_{t_{k-1}}(\cdots \mathcal{C}_{t_1}(M))).$$

All the terms in this sequence are distinct, and the last term corresponds to a color-compressed colored multicomplex. Clearly if $n = 1$ (i.e., in the “uncolored” case), the last term is uniquely determined by $M$, known as the compression of $M$. However, if $n > 1$, then in general, different choices of $t_1, \ldots, t_k$ determine different color-compressed colored multicomplexes, as the following example shows.

Example 3.5. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ be sets of variables, each linearly ordered by $x_1 < x_2 < x_3$ and $y_1 < y_2 < y_3$ respectively, and set $\pi = (X, Y)$. Consider the colored multicomplex $(M, \pi)$ of type $(1, 1)$ in $\mathcal{M}_\pi(1, 1, 1, 1, 1, 1, 1, 1)$, given by $M = \{x_1 y_1, x_1 y_2, x_1 y_3, x_2 y_1, x_2 y_2, x_3 y_1, x_3 y_2, x_3 y_3\}$. Note that $(\mathcal{C}_1(M), \pi)$ and $(\mathcal{C}_2(M), \pi)$ are both color-compressed, yet $\mathcal{C}_1(M) \neq \mathcal{C}_2(M)$.

Definition. Let $(M, \pi)$ and $(M', \pi)$ be colored multicomplexes in $\mathcal{M}_\pi(e_1, \ldots, e_n)$, and suppose $(M', \pi)$ is color-compressed. If $M' = \mathcal{C}_{t_1}(\mathcal{C}_{t_2}(\cdots \mathcal{C}_{t_k}(M)))$ for some finite sequence $t_1, \ldots, t_k$ of integers in $[n]$, then we say $(M', \pi)$ is a color compression of $(M, \pi)$.

Remark 3.6. It is easy to see that the integers $t_1, \ldots, t_k$ in (12) are distinct, hence every colored multicomplex in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ has at most $n!$ color compressions. We leave the reader to verify that if each $X_i$ is infinite, then in fact almost every colored multicomplex in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ has $n!$ color compressions.

For $n \neq 1$, the non-uniqueness of color compressions of colored multicomplexes (Remark 3.6), together with the non-uniqueness of color-compressed multicomplexes with a given fine $f$-vector [3], suggests that the usual definition of the $i$-th Macaulay representation of a positive integer $N$ (which is uniquely determined given $N$ and $i$) is inadequate for the task of numerically characterizing the fine $f$-vectors of color-compressed $a$-colored multicomplexes in $\mathcal{M}_\pi(e_1, \ldots, e_n)$. Later in Section 5, we will introduce generalized Macaulay representations to address these non-uniqueness issues.

3.2 Color Shifting

The notions of shifting and color shifting were previously defined only for simplicial complexes, and we now extend these notions to multicomplexes. Let $Y = \{y_1, y_2, \ldots\}$ be a set of variables linearly ordered by $y_1 < y_2 < \ldots$, let $\phi : Y \to \mathbb{P}$ be an arbitrary map, and define $e = (e_1, e_2, \ldots) := (\phi(y_1), \phi(y_2), \ldots)$. A multicomplex $M' \in \mathcal{M}_\pi(e)$ is called shifted (in $\mathcal{M}_\pi(e)$) if every $m' \in M'$ satisfies the following property:

$$y_j \text{ divides } m' \text{ and } y_i^{e_i} \text{ does not divide } m' \text{ for some integers } 1 \leq i < j \implies m' y_i/y_j \in M'.$$

When $M'$ is finite and $e = (1, 1, \ldots)$, then $M'$ can be identified with a simplicial complex, and this notion of ‘shifted’ coincides with the usual notion of a shifted complex; see, e.g., [16]. A colored multicomplex $(M, \pi)$ of type $a \in \mathbb{P}^n$ in $\mathcal{M}_\pi(e_1, \ldots, e_n)$ is called color-shifted if $M_{\chi_i}$ is shifted in $\mathcal{M}_{\chi_i}(e_i)$ for every $i \in [n]$. A color-shifted colored complex of arbitrary type can be analogously defined as follows.

Definition. Let $(\Delta, \rho)$ be an $a$-colored complex for some $a \in \mathbb{P}^n$, and let $\rho = (V_1, \ldots, V_n)$. For each $i \in [n]$, assume $V_i = \{v_{i,1}, \ldots, v_{i,\lambda_i}\}$ has $\lambda_i$ elements linearly ordered by $v_{i,1} < \cdots < v_{i,\lambda_i}$. Then we say $(\Delta, \rho)$ is color-shifted if every $F \in \Delta$ and $i \in [n]$ satisfy the following property:

$$v_{i,j} \in F \text{ and } v_{i,j'} \notin F \text{ for some integers } 1 \leq j' < j \leq \lambda_i \implies (F \setminus \{v_{i,j}\}) \cup \{v_{i,j'}\} \in \Delta.$$
Remark 3.7. Let \((M, \pi)\) be a colored multicomplex of type \(a \in \mathbb{P}^n\) in \(M_\pi(e_1, \ldots, e_n)\). It clearly follows from definition that if \((M, \pi)\) is color-compressed, then \((M, \pi)\) is color-shifted. In general, the converse is not true. However, if \(a = 1_n\), then \((M, \pi)\) being color-shifted is equivalent to \((M, \pi)\) being color-compressed, and we leave this easy exercise to the reader.

The rest of this section deals with the notion of colored algebraic shifting introduced by Babson and Novik [1]. Recall we have defined a linear order on \(X_i\) for each \(i \in [n]\), so that \(X\) as a poset is a disjoint union of \(n\) chains. Let \(<\) be any linear extension of this partial order on \(X\), and let \(<_{\text{rev-lex}}\) be the rev-lex order on \(M_\pi\) induced by \(<\). The main idea of colored algebraic shifting is that given \(<\), \(a \in \mathbb{P}^n\), and any \(a\)-colored complex \((\Gamma, \pi \cap V)\) (where \(V \subseteq X\) denotes the vertex set of \(\Gamma\)), we can construct another simplicial complex \(\Delta_\prec(\Gamma)\) with vertex set \(V' \subseteq X\), such that \((\Delta_\prec(\Gamma), \pi \cap V')\) is a color-shifted \(a\)-colored complex having the same fine \(f\)-vector as \((\Gamma, \pi \cap V)\). This map \((\Gamma, \pi \cap V) \mapsto (\Delta_\prec(\Gamma), \pi \cap V')\) is called colored algebraic shifting, and the \(a\)-colored complex \((\Delta_\prec(\Gamma), \pi \cap V')\) is called the colored algebraic shifting of \((\Gamma, \pi \cap V)\) (with respect to \(<\)). If the context is clear, we simply say \(\Delta_\prec(\Gamma)\) is the colored algebraic shifting of \(\Gamma\).

To define \(\Delta_\prec(\Gamma)\), we first have to fix some notation. Let \(|X_i| = \lambda_i \in \mathbb{P}\) for each \(i \in [n]\), and write \(\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{P}^n\). Let \(F\) be a field of characteristic zero, and let \(F[X]\) denote the polynomial ring over \(F\) with the set of variables \(X\), such that \(F[X] = \mathbb{P}^n\)-graded by multidegree \(\text{Deg}\). A monomial order on \(F[X]\) is a linear order \(<\) on the monomials of \(F[X]\) such that if \(m_1, m_2, m'\) are monomials of \(F[X]\) satisfying \(m' \neq 1\), then \(m_1 > m_2\) implies \(m'm_1 > m'm_2 > m_2\). For example, the degree-rev-lex order, induced by a given linear order \(<\) on \(X\), is a monomial order \(<_{\text{deg-rev-lex}}\) on \(F[X]\) that extends the rev-lex order \(<_{\text{rev-lex}}\) induced by \(<\), and it is defined by \(m_1 <_{\text{deg-rev-lex}} m_2\) if \(\deg(m_1) < \deg(m_2)\), or if \(\deg(m_1) = \deg(m_2)\) and \(m_1 <_{\text{rev-lex}} m_2\). Given \(f \in F[X]\) and a monomial order \(<\) on \(F[X]\), define the initial term of \(f\), written \(\text{in}_<(f)\), to be the greatest term of \(f\) with respect to \(<\). If \(I \subseteq F[X]\) is a homogeneous ideal, then the initial ideal of \(I\) is the monomial ideal \(\text{in}_<(I)\) generated by \(\{\text{in}_<(f) : f \in I\}\).

Let \(G := GL_{\lambda_1}(F) \times \cdots \times GL_{\lambda_n}(F)\). Every \(g = (g_1, \ldots, g_n) \in G\) defines an \(\mathbb{P}^n\)-graded automorphism of \(F[X]\) via \(g(x_{i,j}) = \sum_{t=1}^{\lambda_i} g_t^{(i,j)} x_{i,t,j}\) for all \(i \in [n]\), \(j \in [\lambda_i]\), where \(g_t^{(i,j)}\) denotes the \((i, j)\)-th entry of the matrix \(g_t \in GL_{\lambda_i}(F)\).

Theorem 3.8 ([1] Theorem 5.3). Let \(I \subseteq F[X]\) be a homogeneous ideal, and let \(<\) be any monomial order on \(F[X]\). Then there exists a Zariski open set \(U \subseteq G\) and an ideal \(J\) such that \(\text{in}_<(uI) = J\) for all \(u \in U\).

The monomial ideal \(J\) defined in Theorem 3.8 is called the \(G\)-generic initial ideal of \(I\) (with respect to the monomial order \(<\) ), and we denote it by \(\text{gin}_G^{\prec}(I)\). In the case \(n = 1\), \(G\)-generic initial ideals are called generic initial ideals; see [9] Section 15.9) for a good introduction to generic initial ideals.

Definition. Given \(m \in M_\pi\), define \(\text{Max}(m)\) as the \(n\)-tuple \((c_1, \ldots, c_n) \in [\lambda_1] \times \cdots \times [\lambda_n]\), where \(c_i = \max\{t \in [\lambda_i] : x_{i,t} \text{ divides } m_{X_i}\}\) for each \(i \in [n]\). Define the set of monomials

\[A_\pi := \left\{ m \in M_\pi : \text{Deg}(m) + \text{Max}(m) \leq \lambda + 1_n \right\}.\]

The color-squarefree map \(\bar{\Phi} : A_\pi \rightarrow M_\pi(1_{\lambda_1}, \ldots, 1_{\lambda_n})\) is the map given by

\[
\prod_{i \in [n]} \left( x_{i,t_{i,1}} x_{i,t_{i,2}} \cdots x_{i,t_{i,\mu_i}} \right) \mapsto \prod_{i \in [n]} \left( x_{i,t_{i,1}} x_{i,t_{i,2}+1} x_{i,t_{i,3}+2} \cdots x_{i,t_{i,\mu_i}+\mu_i-1} \right),
\]

where \(t_{i,1} \leq \cdots \leq t_{i,\mu_i}\) for all \(i \in [n]\). It is easy to see \(\bar{\Phi}\) is a bijective map that preserves the \(\mathbb{P}^n\)-grading.

Definition. Let \(a \in \mathbb{P}^n\), let \(<\) be a linear extension of the partial order on \(X\), and let \(<_{\text{rev-lex}}\) be the degree-rev-lex order on \(M_\pi\) induced by \(<\). If \(\Gamma\) is a simplicial complex on a vertex set \(V \subseteq X\) such that \((\Gamma, \pi \cap V)\) is an \(a\)-colored complex, and \(I_\Gamma \subseteq F[X]\) is the Stanley-Reisner ideal of \(\Gamma\), then define \(\Delta_\prec(\Gamma) := \bar{\Phi}(m) : m \in \left(M_\pi - \text{gin}_{<_{\text{rev-lex}}}^{\prec}(I_\Gamma)\right) \cap A_\pi\).
Lemma 4.1. Any \( V \) and if \( \Delta \) is pure, then for a fixed \( a \)-vertex. In general, if \( \Delta \) is a simplex, then an \( F \)-by all faces.

Definition. Let this definition specializes to the original definition given by Provan and Billera [26].

Remark 3.10. Note that the proof of Theorem 3.9 as given in [1, Theorem 6.5] requires the assumption that \( F \) is a field of characteristic zero. Also, the significance of the rev-lex order satisfies two key properties: If \( x_0 \) is the smallest element in \( X \) with respect to \( \prec \), then \( \inf_{\prec_{r\ell}}(I + (x_0)) = \inf_{\prec_{r\ell}}(I) + (x_0) \) and \( \inf_{\prec_{r\ell}}(I : x_0) = (\inf_{\prec_{r\ell}}(I) : x_0) \) for every homogeneous ideal \( I \subseteq \mathbb{F}[X] \), where the ideal \( (I : x_0) \) is defined by \( \{m \in \mathbb{F}[X] : mx_0 \in I \} \).

Theorem 3.11 ([24]). If \( \Gamma \) is a simplicial complex on a vertex set \( V \subseteq X \) such that \( (\Gamma, \pi \cap V) \) is a color-shifted \( a \)-colored complex, then \( \Delta_\prec(\Gamma) = \Gamma \) for any linear extension \( \prec \) of the partial order on \( X \).

Theorem 3.12 ([1, Theorem 6.5]). If \( \Gamma \) is a simplicial complex on a vertex set \( V \subseteq X \) such that \( (\Gamma, \pi \cap V) \) is an \( a \)-balanced CM complex, then there exists a particular linear extension \( \prec \) of the partial order on \( X \) such that \( \Delta_\prec(\Gamma) \) (computed over a field of characteristic zero) is Cohen-Macaulay.

4 Macaulay Decomposability

In this section, we introduce the notion of \( a \)-Macaulay decomposability for simplicial complexes and show that \( a \)-Macaulay decomposable simplicial complexes are vertex-decomposable. As the main result of this section, we prove that pure color-shifted \( a \)-balanced complexes are \( a \)-Macaulay decomposable and hence vertex-decomposable.

Note that vertex-decomposability was originally defined for pure complexes by Provan and Billera [26] and subsequently generalized to non-pure complexes by Björner and Wachs [6, Section 11] as follows.

Definition. A simplicial complex \( \Delta \) on a vertex set \( V \) is called vertex-decomposable if

1. \( \Delta \) is a simplex or \( \Delta = \{\emptyset\} \); or
2. there exists a vertex \( x \in V \), called a shedding vertex of \( \Delta \), such that
   - \( \text{dl}_\Delta(x) \) and \( \text{lk}_\Delta(x) \) are vertex-decomposable, and
   - no facet of \( \text{lk}_\Delta(x) \) is a facet of \( \text{dl}_\Delta(x) \).

In this paper, we always use this generalized definition for vertex-decomposability. If \( \Delta \) is pure, then this definition specializes to the original definition given by Provan and Billera [26].

Definition. Let \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n \), and let \( \Delta \) be a simplicial complex on a vertex set \( V \). A subcomplex \( \Delta' \) of \( \Delta \) is called an \( \mathbf{a} \)-rib of \( \Delta \) if there exists an ordered partition \( (V_1, \ldots, V_n) \) of \( V \) such that \( \Delta' \) is generated by all faces \( F \in \Delta \) of the form \( F = F_1 \cup \cdots \cup F_n \), where \( F_i \in \binom{V_i}{a_i} \) for each \( i \in [n] \), i.e.

\[
\Delta' = \left\{ \{F_1 \cup \cdots \cup F_n \in \Delta : F_i \in \binom{V_i}{a_i} \text{ for each } i \in [n] \} \right\}.
\]

For example, if \( p, q \in \mathbb{P} \), then the complete bipartite graph \( K_{p,q} \) is a \((1,1)\)-rib of a simplex with \( p + q \) vertices. In general, if \( \Delta \) is a simplex, then an \( \mathbf{a} \)-rib of \( \Delta \) is the subcomplex \( \langle \binom{V_1}{a_1}, \ldots, \binom{V_n}{a_n} \rangle \) for some ordered partition \( (V_1, \ldots, V_n) \). By definition, an \( \mathbf{a} \)-rib of any simplicial complex \( \Delta \) is always pure of dimension \( |\mathbf{a}| - 1 \), and if \( \Delta \) is pure, then for a fixed \( \mathbf{a} \in \mathbb{P}^n \), the union of all \( \mathbf{a} \)-ribs of \( \Delta \) (i.e. over all possible ordered partitions of \( V \)) forms the \((|\mathbf{a}| - 1)\)-skeleton of \( \Delta \). In the case \( n = 1 \), the \( \mathbf{a} \)-rib of every \( \Delta \) is unique.

Lemma 4.1. Any \( \mathbf{a} \)-rib of a simplex is vertex-decomposable.

More precisely, \( \Delta_\prec(\Gamma) \subseteq \mathcal{M}_a(1_{\lambda_1}, \ldots, 1_{\lambda_n}) \) is a multicomplex consisting of only squarefree monomials, hence \( \Delta_\prec(\Gamma) \) can be identified with a simplicial complex whose vertex set is contained in \( X \), and we simply say \( \Delta_\prec(\Gamma) \) is a simplicial complex without any ambiguity.
Proof. Let \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n \), let \( \Delta \) be a simplex on a vertex set \( V \), and let \( \Gamma \) be the \( \mathbf{a} \)-rib of \( \Delta \) corresponding to the ordered partition \( (V_1, \ldots, V_n) \) of \( V \). Given simplicial complexes \( \Sigma_1, \Sigma_2 \) with disjoint vertex sets, \cite{26} Proposition 2.4] yields \( \Sigma_1 \ast \Sigma_2 \) is vertex-decomposable if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are both vertex-decomposable. Since \( \Gamma = (\langle V_1 \rangle) \ast \cdots \ast (\langle V_n \rangle) \) by definition, it then suffices to show that the \( k \)-skeleton (any \( k \in \mathbb{N} \)) of a simplex is vertex-decomposable, which is straightforward and left to the reader. \( \square \)

For \( x, y \in \mathbb{Z}^n \), recall that \( x < y \) if \( x \leq y \) and \( x \neq y \). Also, recall that \( x < y \) if \( x < y \) and \( |y| = |x| + 1 \).

Definition. Let \( \mathbf{a} \in \mathbb{N}^n \). A simplicial complex \( \Delta \) on a vertex set \( V \) is called \( \mathbf{a} \)-Macaulay decomposable if

(i) \( \Delta \) is an \( \mathbf{a} \)-rib of a simplex; or
(ii) there exists a vertex \( x \in V \) and some \( \mathbf{a}' \in \mathbb{N}^n \), \( \mathbf{a}' < \mathbf{a} \), such that
   (a) \( \text{dl}_\Delta(x) \) is \( \mathbf{a} \)-Macaulay decomposable,
   (b) \( \text{lk}_\Delta(x) \) is \( \mathbf{a}' \)-Macaulay decomposable, and
   (c) no facet of \( \text{lk}_\Delta(x) \) is a facet of \( \text{dl}_\Delta(x) \).

Such a vertex \( x \) in \( (ii) \) is called a Macaulay shedding vertex of \( \Delta \). As the following proposition shows, a Macaulay shedding vertex is a shedding vertex of a vertex-decomposable simplicial complex.

Proposition 4.2. Let \( \mathbf{a} \in \mathbb{N}^n \), and let \( \Delta \) be a simplicial complex. If \( \Delta \) is \( \mathbf{a} \)-Macaulay decomposable, then \( \Delta \) is vertex-decomposable, and a Macaulay shedding vertex of \( \Delta \) is a shedding vertex of \( \Delta \).

Proof. We shall prove by double induction on \( |\mathbf{a}| \) and \( |V| \), where \( V \) denotes the vertex set of \( \Delta \). The cases \( |\mathbf{a}| = 0 \) and \( |V| \leq 2 \) are trivial, and in view of Lemma 4.1, we assume \( \Delta \) is not an \( \mathbf{a} \)-rib of any simplex.

Let \( x \) be a Macaulay shedding vertex of \( \Delta \). Note that \( \text{dl}_\Delta(x) \) is an \( \mathbf{a} \)-Macaulay decomposable simplicial complex on a vertex set contained in \( V \setminus \{x\} \), while \( \text{lk}_\Delta(x) \) is \( \mathbf{a}' \)-Macaulay decomposable for some \( \mathbf{a}' < \mathbf{a} \), hence \( \text{dl}_\Delta(x) \) and \( \text{lk}_\Delta(x) \) are both vertex-decomposable by induction hypothesis. Furthermore, no facet of \( \text{lk}_\Delta(x) \) is a facet of \( \text{dl}_\Delta(x) \) by the definition of \( \mathbf{a} \)-Macaulay decomposability, hence \( \Delta \) is vertex-decomposable.

Finally, it easily follows from definition that \( x \) is also a shedding vertex. \( \square \)

Corollary 4.3. Let \( \Delta \) be a \((d-1)\)-dimensional simplicial complex. Then \( \Delta \) is vertex-decomposable if and only if \( \Delta \) is \((d)\)-Macaulay decomposable.

Proof. Let \( \Delta \) be vertex-decomposable with vertex set \( V \). In view of Proposition 4.2, we only need to show that \( \Delta \) is \((d)\)-Macaulay decomposable, which easily follows by induction on \(|V|\). \( \square \)

Definition. A (not necessarily pure) simplicial complex is called shellable if its facets can be arranged in a linear order \( F_1, \ldots, F_t \) such that the subcomplex \( \langle \{F_1, \ldots, F_k\} \rangle \cap \langle \{F_{k+1}\} \rangle \) is pure of dimension \( \dim F_{k+1} - 1 \) for all \( k \in [t−1] \). Such an ordering of the facets is called a shelling.

Similar to vertex-decomposability, the notion of shellability was originally defined for pure complexes. The definition we use here follows the generalized notion of shellability introduced by Björner and Wachs \cite{3}, whereby shellable simplicial complexes are not necessarily pure. In this generalized non-pure sense, vertex-decomposable simplicial complexes are shellable. The pure case was proven by Provan and Billera \cite{26}, and the non-pure case was proven by Björner and Wachs \cite{3} Theorem 11.3. Together with Proposition 4.2, we get the following implications.

Corollary 4.4. Let \( \mathbf{a} \in \mathbb{N}^n \). Then \( \mathbf{a} \)-Macaulay decomposable \( \implies \) vertex-decomposable \( \implies \) shellable.

Note that the second implication is strict, even for pure complexes (see \cite{26} Remark 3.4.5)). Corollary 4.3 gives a partial converse to the first implication in the case \( n = 1 \). However for \( n > 1 \), the first implication is strict; see Proposition 4.7.

Proposition 4.5. Let \( n, m \in \mathbb{P} \), let \( \mathbf{a} \in \mathbb{N}^n \), \( \mathbf{b} \in \mathbb{N}^m \), and let \( \Delta, \Delta' \) be simplicial complexes with disjoint vertex sets \( V, V' \) respectively. If \( \Delta \) is \( \mathbf{a} \)-Macaulay decomposable and \( \Delta' \) is \( \mathbf{b} \)-Macaulay decomposable, then \( \Delta \ast \Delta' \) is \((\mathbf{a}, \mathbf{b})\)-Macaulay decomposable.
Proof. We shall prove by double induction on $|a| + |b|$ and $|V| + |V'|$, where the cases $|a| + |b| \leq 2$ or $|V| + |V'| \leq 2$ are trivial. Suppose $\Delta$ is not an $a$-rib of a simplex, and let $x$ be a Macaulay shedding vertex of $\Delta$. Since $dl_{\Delta}(x)$ is an $a$-Macaulay decomposable simplicial complex on a vertex set contained in $V \setminus \{x\}$, while $lk_{\Delta}(x)$ is $a'$-Macaulay decomposable for some $a' < a$, it follows from induction hypothesis that $dl_{\Delta}(x) * \Delta'$ is $(a,b)$-Macaulay decomposable and $lk_{\Delta}(x) * \Delta'$ is $(a',b)$-Macaulay decomposable. It is easy to show that $dl_{\Delta}(x) * \Delta' = dl_{\Delta + \Delta'}(x) \setminus \Delta_x$ and $lk_{\Delta}(x) * \Delta' = lk_{\Delta + \Delta'}(x) \setminus \Delta_x$. Also, note that if a facet $F$ of $lk_{\Delta + \Delta'}(x)$ is a facet of both $dl_{\Delta}(x)$ and $lk_{\Delta}(x)$, which is a contradiction. Consequently, $\Delta * \Delta'$ is $(a,b)$-Macaulay decomposable by definition in this case. Similarly, if instead $\Delta'$ is not a $b$-rib of a simplex, then $\Delta * \Delta'$ is $(a,b)$-Macaulay decomposable.

Finally, if $\Delta$ is an $a$-rib of a simplex $\Sigma$ with vertex set $X$, and $\Delta'$ is a $b$-rib of a simplex $\Sigma'$ with vertex set $X'$, then $\Delta * \Delta' = \{\{X_i\}\} \cdots \{\{X_n\}\} * \{\{X'_1\}\} \cdots \{\{X'_m\}\}$, where $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_m)$, and $(X_1, \ldots, X_n), (X'_1, \ldots, X'_m)$ are the corresponding ordered partitions of $X, X'$ respectively. Thus, $\Delta * \Delta'$ is an $(a,b)$-rib of the simplex with vertex set $X \cup X'$ and hence $(a,b)$-Macaulay decomposable.

Definition. Let $m,n \in \mathbb{P}$ satisfy $m \leq n$, and let $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $b = (b_1, \ldots, b_m) \in \mathbb{N}^m$. If there are integers $0 = i_0 < i_1 < \cdots < i_{m-1} < i_m = n$ such that $b_i = a_{i+t-1} + \cdots + a_i$ for all $t \in [m]$, then we say $a$ is an ordered refinement of $b$. In particular, $a$ is an ordered refinement of $b$ implies $|a| = |b|$. If there exists a permutation $\phi$ on $[n]$ such that $(a_{\phi(1)}, \ldots, a_{\phi(n)})$ is an ordered refinement of $b$, then we say $a$ is a permuted refinement of $b$, and we denote this by $a \leq_{pr} b$.

Definition. Let $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let $V = \{(i,j) \in \mathbb{P}^2 : j \in [n], i \in [a_j + 1]\}$, and let $\Sigma$ be the simplex with vertex set $V$. For each $t \in [n]$, let $V_t$ denote the subset $\{(i,t) \in \mathbb{P}^2 : i \in [a_t + 1]\} \subseteq V$. Then the unique $a$-rib of $\Sigma$ corresponding to the ordered partition $(V_1, \ldots, V_n)$ of $V$ is called the $a$-rib cage.

For example, the $(2)$-rib cage is isomorphic to the empty triangle $\langle \{1,2\}, \{2,3\}, \{3,1\}\rangle$, while the $(1,1)$-rib cage is isomorphic to the complete bipartite graph $K_{2,2}$.

Lemma 4.6. Let $b \in \mathbb{P}^m$ and let $\Delta$ be the $b$-rib cage. If $a \in \mathbb{P}^n$ is an ordered refinement of $b$ such that $\Delta$ is $a$-Macaulay decomposable, then $a = b$.

Proof. Given an arbitrary simplicial complex $\Sigma$ with vertex set $U$, define $u \in U$ to be a cone-point of $\Sigma$ if $u \in F$ for all facets $F$ of $\Sigma$. Assume $a = (a_1, \ldots, a_n) \in \mathbb{P}^n$ is an ordered refinement of $b$, and suppose $\Delta$ is $a$-Macaulay decomposable corresponding to the ordered partition $(V_1', \ldots, V_n')$ of the vertex set $V$ of $\Delta$. Note that $|V| = |b| + m = |a| + m$, so if $n > m$, then there exists some $j \in [n]$ such that $|V_j'| = a_j$, which implies $V_j'$ is a non-empty set of cone-points of $\Delta$. However, by the definition of the $b$-rib cage, $\Delta$ has no cone-points. Therefore, we conclude that $n = m$, which then forces $a = b$.

Proposition 4.7. Let $m,n \in \mathbb{P}$, let $a \in \mathbb{N}^n$, $b \in \mathbb{N}^m$, and let $\Delta$ be a simplicial complex. If $a \leq_{pr} b$, then $\Delta$ is $a$-Macaulay decomposable implies $\Delta$ is $b$-Macaulay decomposable. Furthermore, if $a \in \mathbb{P}^n$, $b \in \mathbb{P}^m$ and $n \neq m$, then this implication is strict.

Proof. First of all, note that if we can show any $a$-rib of a simplex is $b$-Macaulay decomposable for all $b \geq_{pr} a$, then by using the definition of Macaulay decomposability, we can prove the first assertion by double induction on $|a|$ and the number of vertices of $\Delta$.

Let $a = (a_1, \ldots, a_n)$, let $\Delta'$ be a simplex with vertex set $V$, and let $\Gamma$ be an $a$-rib of $\Delta'$ corresponding to the ordered partition $(V_1', \ldots, V_n')$ of $V$. Choose some $b = (b_1, \ldots, b_m) \geq_{pr} a$. Without loss of generality, assume $a$ is an ordered refinement of $b$, and let $0 = i_0 < i_1 < \cdots < i_{m-1} < i_m = n$ such that $b_t = a_{i_{t-1} + 1} + \cdots + a_i$ for all $t \in [m]$. For each $t \in [m]$, define $\Gamma_t := \langle V_{i_{t-1}+1} (a_{i_{t-1}+1}) \rangle \# \cdots \# \langle V_i (a_i) \rangle$, and note that $\Gamma_t$ is an $(a_{i_{t-1}+1}, \ldots, a_i)$-rib of the simplex with vertex set $V_{i_{t-1}+1} \cup \cdots \cup V_i$, so $\Gamma_t$ is vertex-decomposable by Lemma 4.4 and hence $(b_t)$-Macaulay decomposable by Corollary 4.3. Now since $\Gamma = \Gamma_1 * \cdots * \Gamma_m$, Proposition 4.5 yields $\Gamma$ is $b$-Macaulay decomposable, thus proving the first assertion.
Finally, the second assertion follows immediately from Lemma 4.6.

**Proposition 4.8.** Let \( a \in \mathbb{N}^n \). If \( \Delta \) is an \( a \)-Macaulay decomposable simplicial complex, then \( \dim \Delta = |a| - 1 \).

**Proof.** The assertion follows by double induction on \(|a|\) and \(|V|\), where \( V \) denotes the vertex set of \( \Delta \).

An \( a \)-Macaulay decomposable simplicial complex is not necessarily pure. For example, the non-pure simplicial complex \( \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 5\}, \{2, 5\}\} \) is \( (2, 1) \)-Macaulay decomposable with 5 as a Macaulay shedding vertex. For pure simplicial complexes, we have the following useful equivalent definition.

**Definition.** Let \( a \in \mathbb{N}^n \). A pure simplicial complex \( \Delta \) on a vertex set \( V \) is an \textit{\( a \)-Macaulay decomposable} if

1. \( \Delta \) is an \( a \)-rib of a simplex; or
2. there exists a vertex \( x \in V \) and some \( a' \in \mathbb{N}^n \), \( a' \prec a \), such that
   - \( \text{dl}_\Delta(x) \) is \( a \)-Macaulay decomposable, and
   - \( \text{lk}_\Delta(x) \) is \( a' \)-Macaulay decomposable.

Furthermore, if \( \Delta \) is pure and \( a \)-Macaulay decomposable, then \( \text{dl}_\Delta(x) \) and \( \text{lk}_\Delta(x) \) must both be pure. Conversely, given a simplicial complex \( \Delta' \), if there exists some vertex \( x' \) of \( \Delta' \) such that (i): \( \text{dl}_{\Delta'}(x') \) is pure and \( a \)-Macaulay decomposable, and (ii): \( \text{lk}_{\Delta'}(x') \) is pure and \( a' \)-Macaulay decomposable for some \( a' \in \mathbb{N}^n \), \( a' \prec a \), then \( \Delta' \) is pure and \( a \)-Macaulay decomposable.

**Theorem 4.9.** Let \( a \in \mathbb{P}^n \) and let \((\Delta, \pi)\) be a pure \( a \)-balanced complex. If \((\Delta, \pi)\) is color-shifted, then \( \Delta \) is \( a \)-Macaulay decomposable.

**Proof.** Write \( a = (a_1, \ldots, a_n) \), \(|a| = k \), and let \( \pi = (V_1, \ldots, V_n) \) be the ordered partition of the vertex set \( V \) of \( \Delta \). For each \( i \in [n] \), assume \( V_i = \{v_{i,1}, \ldots, v_{i,\lambda_i}\} \) has \( \lambda_i \) elements linearly ordered by \( v_{i,1} < \cdots < v_{i,\lambda_i} \).

Without loss of generality, assume \( \Delta \) is not an \( a \)-rib of a simplex, which implies \( \lambda_t > a_t \) for some \( t \in [n] \). Define \( a' := a - \delta_n \), and let \( W, W' \) be the vertex sets of \( \text{dl}_\Delta(v_t, \lambda_t) \) and \( \text{lk}_\Delta(v_t, \lambda_t) \) respectively. Since \((\Delta, \pi)\) is color-shifted, it follows that \( (\text{dl}_\Delta(v_t, \lambda_t), \pi \cap W) \) is a color-shifted \( a \)-colored complex, while \((\text{lk}_\Delta(v_t, \lambda_t), \pi \cap W')\) is a color-shifted \( a' \)-colored complex. We now prove this theorem by double induction on \( k \) and \(|V|\).

Suppose there is a facet \( F \) of \( \text{dl}_\Delta(v_t, \lambda_t) \) such that \( \dim F < k - 1 \). Since \( \Delta \) is pure of dimension \( k - 1 \), we have \( F \cup \{v_{t,\lambda_t}\} \subseteq \Delta \), so \((\Delta, \pi)\) is color-shifted implies \( F \cup \{v_{t,j}\} \in \Delta \) for every \( j \in [\lambda_t - 1] \) satisfying \( v_{t,j} \notin F \).

Such a \( j \) exists since \( \lambda_t > a_t \), and in particular, \( F \cup \{v_{t,j}\} \in \text{dl}_\Delta(v_t, \lambda_t) \), which contradicts the assumption that \( F \) is a facet of \( \text{dl}_\Delta(v_t, \lambda_t) \). This means \((\text{dl}_\Delta(v_t, \lambda_t), \pi \cap W)\) is a pure color-shifted \( a \)-balanced complex, hence \( \text{dl}_\Delta(v_t, \lambda_t) \) is \( a \)-Macaulay decomposable by induction hypothesis. A similar argument shows \( \text{lk}_\Delta(v_t, \lambda_t) \) is pure of dimension \( k - 2 \) and hence \( a' \)-Macaulay decomposable by induction hypothesis and Proposition 4.7, therefore \( \Delta \) is \( a \)-Macaulay decomposable.

**Remark 4.10.** Both the hypotheses in Theorem 4.9, i.e. that \((\Delta, \pi)\) is pure and \( a \)-balanced (instead of just \( a \)-colored), are necessary. Murai [24] gave an example of a non-pure \( 1_5 \)-balanced color-shifted complex that is not shellable, so since all \( 1_5 \)-Macaulay decomposable simplicial complexes are shellable (Corollary 4.4), this same example fails to be \( 1_5 \)-Macaulay decomposable. Also, the pure simplicial complex \( \{(1, 2), (3, 4)\} \), together with the ordered partition \( \{(1), (2), (3), (4)\} \) of its vertex set \( [4] \), is clearly color-shifted and \( 1_4 \)-colored, although it is not \( 1_4 \)-balanced. However, it fails to be \( 1_4 \)-Macaulay decomposable. In general, Proposition 4.8 says every \( a \)-colored complex that is not \( a \)-balanced can never be \( a \)-Macaulay decomposable.

Theorem 4.9 generalizes Murai’s result [24, Proposition 4.2], which states that a pure \( a \)-balanced color-shifted complex is shellable. From Theorem 4.9 and its proof, we get the following useful corollaries.

**Corollary 4.11.** Let \( a = (a_1, \ldots, a_n) \in \mathbb{P}^n \), let \((\Delta, \pi)\) be a pure color-shifted \( a \)-balanced complex, and let \( \pi = (V_1, \ldots, V_n) \) be the corresponding ordered partition of its vertex set. If \( \Delta \) is not an \( a \)-rib of a simplex, then the maximal element of \( V_i \) is a Macaulay shedding vertex of \( \Delta \) for every \( i \in [n] \) satisfying \( \lvert V_i \rvert > a_i \).

**Proof.** See the proof of Theorem 4.9.
Corollary 4.12. Let \( a \in \mathbb{P}^n \), and let \((\Delta, \pi)\) be a color-shifted \(a\)-balanced complex. Then the following are equivalent:

(i) \( \Delta \) is pure.
(ii) \( \Delta \) is pure \(a\)-Macaulay decomposable.
(iii) \( \Delta \) is pure vertex-decomposable.
(iv) \( \Delta \) is pure shellable.
(v) \( \Delta \) is Cohen-Macaulay (over any field).

Proof. Theorem 4.10 yields \((i) \Rightarrow (ii)\), the implications \((ii) \Rightarrow (iii) \Rightarrow (iv)\) follow from Corollary 4.4, while the implications \((iv) \Rightarrow (v) \Rightarrow (i)\) are well-known (see, e.g., [31]). \(\square\)

Corollary 4.13. Let \( a \in \mathbb{P}^n \), and let \( f = \{ f_b \}_{b \leq a} \) be an array of integers. Then the following are equivalent:

(i) \( f \) is the fine \( h \)-vector of an \( a \)-balanced CM complex.
(ii) \( f \) is the fine \( h \)-vector of an \( a \)-balanced pure shellable complex.
(iii) \( f \) is the fine \( h \)-vector of an \( a \)-balanced pure vertex-decomposable complex.
(iv) \( f \) is the fine \( h \)-vector of an \( a \)-balanced pure \( a \)-Macaulay decomposable complex.
(v) \( f \) is the fine \( h \)-vector of an \( a \)-balanced pure color-shifted complex.
(vi) \( f \) is the fine \( h \)-vector of an \( a \)-balanced color-shifted CM complex.
(vii) \( f \) is the fine \( h \)-vector of an \( a \)-balanced color-shifted pure shellable complex.
(viii) \( f \) is the fine \( h \)-vector of an \( a \)-balanced color-shifted pure vertex-decomposable complex.
(ix) \( f \) is the fine \( h \)-vector of an \( a \)-balanced color-shifted pure \( a \)-Macaulay decomposable complex.
(x) \( f \) is the fine \( f \)-vector of an \( a \)-colored multicomplex.
(xi) \( f \) is the fine \( f \)-vector of an \( a \)-colored color-shifted multicomplex.
(xii) \( f \) is the fine \( f \)-vector of an \( a \)-colored color-compressed multicomplex.

Proof. Let \((\Gamma, \pi)\) be an \( a \)-balanced CM complex, and assume each \( V_i \) in the ordered partition \( \pi = (V_1, \ldots, V_n) \) is linearly ordered, so that \( V := \bigcup_{i \in [n]} V_i \) as a poset is a disjoint union of \( n \) chains. By Theorem 3.19 and Theorem 3.12, there exists a particular linear extension \( < \) of this partial order on \( V \) such that the colored algebraic shifting \((\Delta_{\prec}(\Gamma), \pi')\) is a color-shifted \(a\)-balanced CM complex with the same fine \( h \)-vector as \((\Gamma, \pi)\), hence \((i) \iff (vi)\). Corollary 4.12 then gives the equivalences \((v) \iff (vi) \iff (vii) \iff (viii) \iff (ix)\) and the implication \((ix) \Rightarrow (iv)\) is trivial, while the implications \((iv) \Rightarrow (iii) \Rightarrow (ii)\) follow from Corollary 4.4.

Finally, Theorem 1.1 yields the equivalences \((i) \iff (ii) \iff (x) \iff (xi)\). The implication \((xi) \Rightarrow (xi)\) follows from definition, while \((xi) \Rightarrow (x)\) is trivial. \(\square\)

Remark 4.14. Corollary 4.12 generalizes a result by Babson and Novik [1, Theorem 6.2], which states that given \((\Delta, \pi)\) a color-shifted \(a\)-balanced complex, \(\Delta\) is Cohen-Macaulay (over any field) if and only if \(\Delta\) is pure. Also, Corollary 4.13 generalizes a result by Biermann and Van Tuyl [2], which deals with \(h\)-vectors of the more refined \(h\)-vectors, and is equivalent to the statement that the \(h\)-vector of a completely balanced CM complex is the \(h\)-vector of a completely balanced pure vertex-decomposable complex of the same dimension. The proof by Biermann and Van Tuyl uses a generalization of the “whiskering” construction of graphs introduced by Villarreal [33], and Cook and Nagel [8].

5 Generalized Macaulay Representations

In this section, we introduce the notion of generalized Macaulay representations and use them to obtain numerical characterizations of the fine \(f\)-vectors of \(a\)-colored complexes and the flag \(f\)-vectors of completely balanced CM complexes. Our proof of these two characterizations can be divided into four main steps, so we divide the rest of this section into four subsections.

In Section 5.1, we start with a pure color-shifted \(a\)-balanced complex \((\Delta, \pi)\) and construct a labeled trivalent planted binary tree \((T, \phi)\), which we call the shedding tree of \((\Delta, \pi)\). Next, in Section 5.2, we define a purely numerical notion of an \(a\)-Macaulay tree of \(N\) for any \(n, N \in \mathbb{P}, a \in \mathbb{N}^n \backslash \{0_n\}\). If \(a \in \mathbb{P}^n\), then
every pure color-shifted a-balanced complex with N facets corresponds to an a-Macaulay tree of N, while conversely, every a-Macaulay tree of N corresponds to a pure a-balanced complex, although not necessarily color-shifted.

In Section 5.2, we define what it means for an a-Macaulay tree to be compressed-like and compatible, and we characterize pure color-compressed a-balanced complexes in terms of certain compressed-like compatible a-Macaulay trees. Finally, in Section 5.3, we define the notion of a generalized a-Macaulay representation of N for any n ∈ P, N ∈ N, a ∈ N\{0\}, and we complete our two desired numerical characterizations, thus finishing the proofs of Theorems 1 and 1.3.

Throughout these subsections unless otherwise stated, let a = (a1, ..., an), let V be the vertex set of Σ, and let π = (V1, ..., Vn) be the corresponding ordered partition of V. For each i ∈ [n], write λi = |Vi|, and let Vi = {vi,1, vi,2, ..., vi,λi} be linearly ordered by vi,1 < vi,2 < ... < vi,λi. Set λ = (λ1, ..., λn). For every b ∈ (b1, ..., bn) ∈ Zn and every subcomplex ∆' of ∆ with vertex set V', let Fb(∆') be the set of all faces F ∈ ∆' such that |F ∩ V| = bi for all i ∈ [n], and define fb(∆') = |Fb(∆')|. By default, set fb(∆) = 0 if b /∈ Nn. Observe that {fb(∆')}b≤a is the fine f-vector of (∆', π ∩ V'). For x = (x1, ..., xn), y = (y1, ..., yn) in Zn, define (x) := (x1, y1)(x2, y2)...(xn, yn), and recall that x < y if x ≤ y and x ≠ y. Note that x < y if and only if x = y − δi,n for some i ∈ [n].

5.1 Construction of Shedding Trees

Let a ∈ Pn, and let (∆, π) be a pure color-shifted a-balanced complex. If ∆ is an a-rib of a simplex, then the fine f-vector of (∆, π) is easy to compute: Since ∆ = (Vn)1*a1*···*(Vn)an, we get fb(∆) = (a) for every b ∈ Nn satisfying b ≤ a. If ∆ is not an a-rib of a simplex, then by Corollary 4.11 there exists some i ∈ [n] such that λi > ai, and vii,λi is a Macaulay shedding vertex of ∆. Note that ∆ = dl(∆) ∪ {F ∪ {v} : F ∈ lk(∆)} for every vertex v of ∆, hence

\[ Fb(∆) = Fb(dl(v,λi)) \cup \left\{ F ∪ \{v,λi\} : F ∈ Fb−δi,n(lk(\{v,λi\})) \right\} \]  \hspace{1cm} (14)

for all b ∈ Nn satisfying b ≤ a, which yields

\[ fb(∆) = fb(dl(v,λi)) + fb−δi,n(lk(\{v,λi\})) \]  \hspace{1cm} (15)

for all b ∈ Nn satisfying b ≤ a. Furthermore, if we denote the vertex sets of dl(\{v,λi\}) and lk(\{v,λi\}) by W and W' respectively, then Corollary 4.11 says (dl(\{v,λi\}), π ∩ W) is a pure color-shifted a-balanced complex, while (lk(\{v,λi\}), π ∩ W') is a pure color-shifted (a − δi,n)-balanced complex, so these two balanced subcomplexes are a-Macaulay decomposable and (a − δi,n)-Macaulay decomposable respectively by Theorem 4.3 and Proposition 4.7.

The main idea of this subsection is to start with the pure color-shifted a-balanced complex (Δ, π), and for every pure color-shifted a'-balanced subcomplex (Δ', π') of (Δ, π) that is not an a'-rib of a simplex (where π' = (V1', ..., Vm'), a' = (a1', ..., am') ∈ Pm and m ≤ n), choose some t ∈ [m] such that |Vt'| > at', replace (Δ', π') with the two pure color-shifted balanced subcomplexes (dl(Δ',Vt',max), π' ∩ W) and (lk(Δ',Vt',max), π' ∩ W') (where Vt',max is the maximal element of Vt', W and W' are the vertex sets of dl(Δ',Vt',max), lk(Δ',Vt',max) respectively), and repeat the process until every remaining pure color-shifted balanced subcomplex, say of type a", is an a"-rib of a simplex. By repeatedly using (14) on these subcomplexes of Δ, we can then compute fb(Δ) for each b ∈ Nn satisfying b ≤ a. We now make this idea rigorous.

Definition. Let S = (S1, ..., Sn) be an n-tuple of sets, let x = (x1, ..., xn) ∈ Nn, and let t ∈ [n]. A simplicial complex Σ is called t-factorizable with respect to (S, x) if Σ = (S1) * Σ' for some (non-empty) simplicial complex Σ'. If x ∈ Pn, then define the n-tuple κ(S, x) ∈ Pn by κ[S](S, x) := |Si| + δ(|Si|, 0) * xi for each i ∈ [n].

Let T0 be the planted tree with 2 vertices, where r0 is the root, and r1 is the unique child of r0. Define the vertex labeling φ0 of T0 by φ0(r0) = φ0(r1) = (a, Δ, π, λ), i.e. the φ0-labels are 4-tuples, where the
first entry is in \( \mathbb{N}^n \), the second entry is a simplicial complex, the third entry is an \( n \)-tuple of subsets of \( V \), and the last entry is in \( \mathbb{P}^n \). Starting with the pair \((T_0, \phi_0)\), we shall construct a sequence ‘seq’ of pairs \((T_0, \phi_0), (T_1, \phi_1), (T_2, \phi_2), \ldots \) via the following algorithm:

\[
\text{seq} \leftarrow (T_0, \phi_0).
\]

\[
t \leftarrow n.
\]

while \( t > 0 \) do

\[
(T', \phi') \leftarrow \text{last pair in seq}.
\]

\[
\mathcal{L} \leftarrow \text{set of terminal vertices of } T'.
\]

if \( \exists v \in \mathcal{L} \) with \( \phi'(v) = (\Delta', \pi', \lambda') \) such that \( \Delta' \) is not \( t \)-factorizable with respect to \( (\pi', \lambda') \) then

Choose any such \( v \) with \( \phi'(v) = (\Delta', \pi', \lambda') \).

Write \( \alpha' = (a'_1, \ldots, a'_n) \), \( \pi' = (V'_1, \ldots, V'_n) \), and let \( v'_{t, \text{max}} \) be the maximal element of \( V'_i \).

Let \( W, W' \) denote the vertex sets of \( d\Delta'(v'_{t, \text{max}}), \text{lk}\Delta'(v'_{t, \text{max}}) \) respectively.

Construct a binary tree \( T \) from \( T' \) by adding two children \( v_{\text{left}} \) and \( v_{\text{right}} \) to \( v \).

Treat \( T' \) as a subgraph of \( T \). Let \( V' \) denote the vertex set of \( T' \).

Define the vertex labeling \( \phi \) of \( T \) by

\[
\phi(u) = \begin{cases} 
\phi'(u), & \text{if } u \in V' \setminus \{v\}; \\
t, & \text{if } u = v; \\
(\alpha' \cap W, \text{lk}\Delta'(v'_{t, \text{max}}), \pi' \cap W, \kappa(\pi' \cap W, \lambda' - \delta_{t,n}')), & \text{if } u = v_{\text{left}}; \\
(\alpha' - \delta_{t,n}, \text{lk}\Delta'(v'_{t, \text{max}}), \pi' \cap W', \kappa(\pi' \cap W', \lambda' - \delta_{t,n}')), & \text{if } u = v_{\text{right}}.
\end{cases}
\]

\[
\text{seq} \leftarrow (\text{seq}, (T, \phi)) \ (\text{i.e. seq with the term } (T, \phi) \text{ appended}).
\]

else

\[
t \leftarrow t - 1.
\]

end if

end while

Clearly \( |V'_i| = \alpha'_i \) or \( \alpha'_i = 0 \) implies \( \Delta' \) is \( t \)-factorizable with respect to \( (\pi', \lambda') \), so if \( \Delta' \) is not \( t \)-factorizable with respect to \( (\pi', \lambda') \), then \( |V'_i| > \alpha'_i > 0 \), and Corollary [11] implies \( v'_{t, \text{max}} \) is a Macaulay shedding vertex of \( \Delta' \). Since \( \Delta \) is an \( \text{a-Macaulay decomposable} \), this algorithm must eventually terminate, independent of the choices of \( v \) made, i.e. the sequence ‘seq’ has finitely many terms. Let \((T_0, \phi_0), (T_1, \phi_1), \ldots, (T_k, \phi_k)\) be a sequence constructed from \((\Delta, \pi)\). It is easy to see that the last pair \((T_k, \phi_k)\) (possibly \((T_0, \phi_0)\)) is uniquely determined by \((\Delta, \pi)\), independent of the choices of \( v \) made. Note however that the other pairs in the sequence do depend on the choices of \( v \) at each iteration in the algorithm. We call any such sequence a \textit{shedding sequence} of \((\Delta, \pi)\), and we say the last pair \((T_k, \phi_k)\) is the \textit{shedding tree} of \((\Delta, \pi)\).

By construction, each \( T_i \) is a trivalent planted binary tree with \( T_{i-1} \) as a subgraph, and each \( \phi_i \) is a vertex labeling of \( T_i \), thus each pair \((T_i, \phi_i)\) is a labeled trivalent planted binary tree. The trivalent vertices of \( T_i \) are labeled with integers in \([n]\), such that \( \phi_i(y) \geq \phi_i(y') \) for every pair \( y, y' \) of trivalent vertices in \( T_i \) satisfying \( y' \in D_{T_i}(y) \), i.e. \( y' \) is a descendant of \( y \). Also, every terminal vertex of \( T_i \) is labeled with a 4-tuple \((\alpha', \Delta', \pi', \lambda')\), where \( \alpha' = (a'_1, \ldots, a'_n) \in \mathbb{N}^n, \Delta' \) is a \( (n-\text{empty}) \) simplicial complex, \( \pi' = (V'_1, \ldots, V'_n) \) is an \( n \)-tuple of subsets of \( V \), and \( \lambda' \in \mathbb{P}^n \). We can check that \((\Delta', \pi')\) is a pure \( \alpha' \)-balanced color-shifted complex, and \( \Delta' \) is an \( \text{a-Macaulay decomposable} \). Furthermore, if \( (T_i, \phi_i) = (T_k, \phi_k) \) is the shedding tree of \((\Delta, \pi)\), then \( \Delta' \) is \( t \)-factorizable with respect to \((\pi', \lambda')\) for every \( t \in [n] \), hence \( \Delta' \) is the \( \alpha' \)-rib of a simplex corresponding to the ordered partition \( \pi' \) in this case.

Example 5.1. Let \( \tau = (P, Q) \) such that \( P = \{p_1, p_2, p_3\} \) and \( Q = \{q_1, q_2, q_3, q_4\} \) satisfy \( p_1 < p_2 < p_3 \) and \( q_1 < q_2 < q_3 < q_4 \). Consider the pure color-shifted \((1, 1)\)-balanced complex \((\Sigma, \tau)\), where

\[
\Sigma = \left\{ \{p_1, q_1\}, \{p_2, q_1\}, \{p_3, q_1\}, \{p_1, q_2\}, \{p_2, q_2\}, \{p_3, q_2\}, \{p_1, q_3\}, \{p_1, q_4\} \right\}.
\]

Then its shedding tree is given in Figure [1]

Definition. Let \((T_0, \phi_0), (T_1, \phi_1), \ldots, (T_k, \phi_k)\) be a shedding sequence of \((\Delta, \pi)\), and let \((T_i, \phi_i)\) be any term in this sequence. The restriction of \( \phi_i \) to the terminal vertices of \( T_i \) has four component functions, and we denote them as \( \phi_i^1, \phi_i^2, \phi_i^3, \phi_i^4 \) respectively.
Let \((T, \phi) = (T_k, \phi_k)\) be the shedding tree of \((\Delta, \pi)\), and let \((r_0, L, Y)\) be the (root, 1, 3)-triple of \(T\). Note that \(\phi^{[L,1]}(u) \in \mathbb{N}^n\) and \(\phi^{[L,4]}(u) \in \mathbb{P}^n\) for all \(u \in L\). By the construction of \((T, \phi)\), it is easy to show that \(0_n < \phi^{[L,1]}(u) \leq \phi^{[L,4]}(u)\) for all \(u \in L\). Also, by the repeated use of (15), we have the following.

**Proposition 5.2.** For each \(b \in \mathbb{N}^n\) satisfying \(b \leq a\),

\[
f_b(\Delta) = \sum_{u \in L} \left( \phi^{[L,4]}(u) + b - a \right).
\]

For example, the balanced complex \((\Sigma, \tau)\) in Example 5.1 has \(f_{(1,1)}(\Sigma) = 8\), \(f_{(1,0)}(\Sigma) = 3\), \(f_{(0,1)}(\Sigma) = 4\), and we check that indeed \(8 = \left(\frac{3}{2}\right)_{(1,1)} + \left(\frac{1}{2}\right)_{(1,0)} + \left(\frac{1}{3}\right)_{(1,0)}\), \(3 = \left(\frac{3}{2}\right)_{(1,0)} + \left(\frac{1}{2}\right)_{(1,1)} + \left(\frac{1}{1}\right)_{(1,-1)}\), and \(4 = \left(\frac{3}{2}\right)_{(0,1)} + \left(\frac{1}{2}\right)_{(0,0)} + \left(\frac{1}{3}\right)_{(0,0)}\).

**Remark 5.3.** The entire Section 5.1 is still true if we replace every instance of “color-shifted” with “color-compressed”. In particular, if \((\Delta, \pi)\) is color-compressed and \((T_i, \phi_i)\) is a term in any shedding sequence of \((\Delta, \pi)\), then \(\phi^{[L,2]}_i(u), \phi^{[L,3]}_i(u)\) is also color-compressed for every terminal vertex \(u\) of \(T_i\).

### 5.2 Macaulay Trees

Let \(a \in \mathbb{N}^n \setminus \{0_n\}\). Given any pure color-shifted \(\mathfrak{P}\)-balanced complex \((\Delta, \pi)\), it is clear from Proposition 5.2 that the shedding tree \((T, \phi)\) of \((\Delta, \pi)\) encodes superfluous information for computing the fine \(f\)-vector of \(\Delta\). Motivated by the desire to retain only the necessary numerical information needed to compute the fine \(f\)-vector of \(\Delta\), we shall define the notion of an \(a\)-Macaulay tree of \(N\) for any \(N \in \mathbb{P}\), which does not depend on the existence of any \(\mathfrak{P}\)-balanced complexes. First, we introduce a series of related definitions.

**Definition.** Let \(a \in \mathbb{N}^n\). An \(a\)-**splitting tree** is an ordered triple \((T, \mu, \nu)\) such that

(i) \(T = (V, E)\) is a trivalent planted binary tree with (root, 1, 3)-triple \((r_0, L, Y)\);

(ii) \(\mu : Y \to [n]\) is a labeling of the trivalent vertices of \(T\); and

(iii) \(\nu : V \to \mathbb{Z}^n\) is a vertex labeling of \(T\) recursively defined as follows.

(a) Define \(\nu(r_0) = \nu(r_1) = a\), where \(r_1\) is the unique child of the root \(r_0\).

(b) For every \(y \in Y\), if \(\nu(y)\) was already defined, while \(\nu(y_{\text{left}}), \nu(y_{\text{right}})\) are both not yet defined, then define \(\nu(y_{\text{left}}) = \nu(y), \nu(y_{\text{right}}) = \nu(y) - \delta_{\mu(y),n}\).

Note that \(\nu : V \to \mathbb{Z}^n\) is completely determined by \(T, \mu, a\), and we say \((T, \mu, \nu)\) is the \(a\)-**splitting tree induced** by \((T, \mu)\).

**Definition.** Let \(T\) be a trivalent planted binary tree with (root, 1, 3)-triple \((r_0, L, Y)\), and let \(\varphi\) be any vertex labeling of \(T\) such that \(\varphi(Y) \subseteq [n]\) and \(\varphi(L) \subseteq \mathbb{P}^n\). The **left-weight labeling** of \((T, \varphi)\) is the map \(\omega : Y \cup L \to \mathbb{P}^n\) defined by

\[
\omega(v) := \left(\varphi(v_m) + \sum_{i=1}^{m-1} \delta_{\varphi(v_i),n}\right),
\]

where \(v_1, \ldots, v_m\) is the left relative sequence of \(v\) in \(T\), and we say \(\omega(v)\) is the **left-weight** of \(v\) in \((T, \varphi)\). In particular, if \(v \in L\), then \(\omega(v) = \varphi(v)\). We say \(\omega\) is **proper** if \(\omega(y_{\text{left}}) \geq \omega(y_{\text{right}})\) for every \(y \in Y\).
Definition. Let \( n, N \in \mathbb{P}, a \in \mathbb{N}^n \setminus \{0_n\} \). An a-Macaulay tree of \( N \) is any ordered pair \((T, \varphi)\) such that

(i) \( T \) is a trivalent planted binary tree with (root, 1, 3)-triple \((r_0, L, Y)\); and

(ii) \( \varphi \) is a vertex labeling of \( T \) satisfying all of the following conditions.

(a) \( \varphi(r_0) = a, \varphi(Y) \subseteq [n] \) and \( \varphi(L) \subseteq \mathbb{P}^n \).
(b) If \( y, y' \in Y \) satisfies \( y' \in D_T(y) \), i.e. \( y' \) is a descendant of \( y \), then \( \varphi(y) \geq \varphi(y') \).
(c) If \( y \in Y \) satisfies \( \varphi(y) < n \), then \( \varphi[i](u) = \varphi[i](u') \) for all \( u, u' \in L \cap D_T(y) \) and all \( t \in [n] \setminus [\varphi(y)] \).
(d) The left-weight labeling of \((T, \varphi)\) is proper.
(e) The a-splitting tree \((T, \varphi[y, \nu])\) induced by \((T, \varphi[Y])\) satisfies \( \varphi(u) \geq \nu(u) > 0_n \) for all \( u \in L \).
(f) If \( y \in Y \) satisfies \( \nu[\varphi(y)](y) = 1 \), then \( \omega[\varphi(y)](x) = \omega[\varphi(y)](y) - 1 \) for all \( x \in D_T(y_{\text{right}}) \).
(g) \( N \) can be written as the sum

\[
N = \sum_{u \in L} \frac{\varphi(u)}{\nu(u)}.
\]

An a-Macaulay tree is an a-Macaulay tree of \( N \) for some \( N \in \mathbb{P} \), and a Macaulay tree is an a-Macaulay tree for some \( a \in \mathbb{N}^n \setminus \{0_n\} \). Suppose \((T, \varphi)\) is a Macaulay tree, and let \((r_0, L, Y)\) be the (root, 1, 3)-triple of \( T \). The weight of \((T, \varphi)\) is the left-weight of the unique child of \( r_0 \). Given any \( t \in \{0, 1, \ldots, n\} \), we say \( x \in Y \cup L \) is a \( t \)-leading vertex of \((T, \varphi)\) if it satisfies the following conditions:

(i) If \( x \in Y \), then \( \varphi(x) \leq t \).
(ii) If \( \bar{x} \) is the parent of \( x \) and \( \bar{x} \neq r_0 \), then \( \varphi(\bar{x}) > t \).

Define \( L_{(T, \varphi)}(t) \) as the collection of all \( t \)-leading vertices of \((T, \varphi)\). Note that a \( t \)-leading vertex could possibly be a \( t' \)-leading vertex for distinct \( t, t' \in \{0, 1, \ldots, n\} \), while not every vertex in \( Y \cup L \) is necessarily a \( t \)-leading vertex for some \( t \in \{0, 1, \ldots, n\} \). It is easy to show that for every \( t \in \{0, 1, \ldots, n\} \) and every path \( v_0, v_1, \ldots, v_t \) from \( v_0 = r_0 \) to some terminal vertex \( v_t \in L \), there exists a unique \( i_t \in [t] \) such that \( v_{i_t} \) is a \( t \)-leading vertex. In particular, the unique child of \( r_0 \) is always an \( n \)-leading vertex, and it is the only \( n \)-leading vertex in \( Y \cup L \), while every terminal vertex is a 0-leading vertex, i.e. \( L_{(T, \varphi)}(0) = L \).

Definition. Let \((T, \varphi)\) be a Macaulay tree, and let \((r_0, L, Y)\) be the (root, 1, 3)-triple of \( T \). A trivalent vertex \( y \in Y \) is called a cloning vertex of \((T, \varphi)\) if it satisfies the following two conditions:

(i) The two subgraphs of \( T \) induced by \( D_T(y_{\text{left}}) \) and \( D_T(y_{\text{right}}) \) are isomorphic as rooted trees, and they have the same corresponding \( \varphi \)-labels, i.e. if \( \beta : D_T(y_{\text{left}}) \rightarrow D_T(y_{\text{right}}) \) is the corresponding isomorphism, then \( \varphi(\beta(x)) = \varphi(x) \) for all \( x \in D_T(y_{\text{left}}) \).
(ii) Both \( y_{\text{left}} \) and \( y_{\text{right}} \) are \( (\varphi(y) - 1) \)-leading vertices of \((T, \varphi)\).

If \((T, \varphi)\) has no cloning vertices, then we say \((T, \varphi)\) is condensed. In particular, \((T, \varphi)\) is condensed implies \( \varphi(y_{\text{left}}) > \varphi(y_{\text{right}}) \) for every \( y \in Y \) such that \( y_{\text{left}} \) and \( y_{\text{right}} \) are both leaves.

Definition. Let \((T, \varphi)\) be an a-Macaulay tree of \( N \) for some \( a \in \mathbb{N}^n \setminus \{0_n\}, N \in \mathbb{P} \). The condensation of \((T, \varphi)\) is a condensed a-Macaulay tree \((\tilde{T}, \tilde{\varphi})\) of \( N \) that is constructed from \((T, \varphi)\) via the following algorithm:

\[
(\tilde{T}, \tilde{\varphi}) \leftarrow (T, \varphi).
\]

while \( \exists \) a cloning vertex of \((\tilde{T}, \tilde{\varphi})\) do

Choose any cloning vertex \( y \) of \((\tilde{T}, \tilde{\varphi})\).
\( \tilde{\varphi}(y) \leftarrow \tilde{\varphi}(y_{\text{left}}) \).
\( \tilde{V} \) \rightarrow vertex set of \( \tilde{T} \).
\( \tilde{\varphi} \leftarrow \tilde{\varphi}|_{\tilde{V}\setminus D_T(y_{\text{right}})} \).
\( \tilde{T} \leftarrow \) subgraph of \( \tilde{T} \) induced by \( \tilde{V}\setminus D_T(y_{\text{right}}) \).
\( e \leftarrow \{y, y_{\text{left}}\} \).
(\( \tilde{T}, \tilde{\varphi} \) \leftarrow (\tilde{T}/e, \tilde{\varphi}/e) \).
end while
In defining the condensation \((\tilde{T}, \varphi)\) of \((T, \varphi)\), the number of vertices of \(\tilde{T}\) becomes strictly smaller in each iteration of the while loop, hence the while loop is not an infinite loop. It is also easy to see that the pair \((\tilde{T}, \varphi)\) obtained from the above algorithm is uniquely determined by \((T, \varphi)\), independent of the choice of the cloning vertex in each iteration of the while loop, hence \((\tilde{T}, \varphi)\) is well-defined, and in particular, the condensation of a condensed Macaulay tree is itself. The fact that \((\tilde{T}, \varphi)\) is a condensed a-Macaulay tree of \(N\) is straightforward and left to the reader as an exercise.

For the rest of this subsection, let \(a \in \mathbb{P}^n\), let \((\Delta, \pi)\) be a pure color-shifted a-balanced complex, and let \((T_0, \phi_0), (T_1, \phi_1), \ldots, (T_k, \phi_k)\) be a shedding sequence of \((\Delta, \pi)\), where \((r_0, L_1, Y_1)\) denotes the (root, 1, 3)-triple of each \(T_i\). Clearly \(r_0\) is the common root of \(T_0, T_1, \ldots, T_k\), and observe that

\[
Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_k; \quad Y_0 \cup L_0 \subseteq Y_1 \cup L_1 \subseteq \cdots \subseteq Y_k \cup L_k.
\]

(17)

For every \(y \in Y_k\), let \(\alpha(y)\) be the largest integer \(k\) such that \(y \in L_{\alpha(y)}\), while for every \(u \in L_k\), set \(\alpha(u) = k\). Also, for each \(i \in \{0, 1, \ldots, k\}\), define the vertex labeling \(\varphi_i\) of \(T_i\) as follows.

\[
\varphi_i(u) = \begin{cases} 
  a, & \text{if } u = r_0; \\
  \phi_i(u), & \text{if } u \in Y_i; \\
  \phi_i^{[L, A]}(u), & \text{if } u \in L_i.
\end{cases}
\]

By construction, every \(u \in Y_i\) is labeled with an integer in \([n]\), while every \(u \in L_i\) is labeled with an \(n\)-tuple in \(\mathbb{P}^n\), hence the left-weight labeling of \((T_i, \varphi_i)\) is well-defined, and we denote this left-weight labeling by \(\omega_i\).

**Lemma 5.4.** If \(v \in Y_k\), then \(\omega_{\alpha(v)}(v) = \omega_{\alpha(v)+1}(v)\).

**Proof.** Write \(\phi_{\alpha(v)}(v) = (a', \Delta', \pi', \lambda')\), where \(a' = (a_1', \ldots, a_n')\) and \(\pi' = (V_1', \ldots, V_n')\) is an ordered partition of the vertex set \(V'\) of \(\Delta'\). Let \(t = \phi_{\alpha(v)+1}(v)\), let \(v_{t,\text{max}}\) be the maximal element of \(V'_t\), and denote the vertex set of \(d(\Delta') v_{t,\text{max}}\) by \(W\). Also, write \(\pi' \cap W = (V_1', \ldots, V_m')\).

Since \(\Delta'\) is color-shifted, every vertex \(v'' < v_{t,\text{max}}\) in \(V'_t\) must be in \(W\), hence \(|V'_t| = |V''_t| = 1\). Note that \(|V'_t| > a'_t > 0\) yields \(|V''_t| > 0\), while every vertex in \(V''_t\) is in \(W\), thus \(\kappa(\pi' \cap W, \lambda' - \delta_{t,n}) = \lambda' - \delta_{t,n}\). Now, \(\varphi_{\alpha(v)+1}(v_{\text{left}}) = \phi_{\alpha(v)+1}^{[L, A]}(v_{\text{left}}) = \kappa(\pi' \cap W, \lambda' - \delta_{t,n})\) by construction, and \(v, v_{\text{left}}\) is the left relative sequence of \(v\) in \(T_{\alpha(v)+1}\), therefore \(\omega_{\alpha(v)+1}(v) = \varphi_{\alpha(v)+1}(v_{\text{left}}) + \delta_{t,n} = \lambda' = \phi_{\alpha(v)}(v) = \varphi_{\alpha(v)}(v) = \omega_{\alpha(v)}(v)\).

**Proposition 5.5.** For every \(i \in \{0, 1, \ldots, k\}\), we have the following:

(i) \(\omega_i(v) = \omega_j(v)\) for all \(v \in Y_i \cup L_i\) and all integers \(j\) satisfying \(i \leq j \leq k\).

(ii) \(\omega_i\) is proper.

(iii) If the children of some \(y \in Y_i\) are both in \(L_k\), then \(\varphi_i(y_{\text{left}}) > \varphi_i(y_{\text{right}})\).

(iv) If \(\varphi_i(x) < n\) for some \(x \in Y_i\), then \(\omega_i^{[t]}(x') = \omega_i^{[t]}(x)\) for all \(x' \in D_{T_i}(x)\) and all \(t \in [n]\) \([\varphi_i(x)]\).

**Proof.** First of all, observe that statement (i) follows from Lemma 5.4 by the algorithmic construction of a shedding sequence. Next, we prove statements (ii), (iii), and (iv) concurrently by induction on \(i\), with the base case \(i = 0\) being vacuously true since \(Y_0 = \emptyset\).

Choose an arbitrary \(y \in Y_i\) and let \(q := \alpha(y) < i\). If \(q < i - 1\), then both \(y_{\text{left}}\) and \(y_{\text{right}}\) are vertices in \(T_{q+1}\), and \(\omega_{q+1}\) is proper by induction hypothesis, hence \(\omega_{q+1}(y_{\text{left}}) \geq \omega_{q+1}(y_{\text{right}})\). Using statement (i), we thus get \(\omega_i(y_{\text{left}}) \geq \omega_i(y_{\text{right}})\). Furthermore, if \(y_{\text{left}}\) and \(y_{\text{right}}\) are both in \(L_k\), then the induction hypothesis also yields \(\varphi_{q+1}(y_{\text{left}}) > \varphi_{q+1}(y_{\text{right}})\), which implies \(\varphi_i(y_{\text{left}}) > \varphi_i(y_{\text{right}})\).

Suppose \(q = i - 1\). Let \(\phi_q(y) = (a', \Delta', \pi', \lambda')\), and write \(a' = (a_1', \ldots, a_n')\), \(\pi' = (V_1', \ldots, V_m')\). Let \(t = \phi_i(y)\), let \(v_{t,\text{max}}\) be the maximal element of \(V'_t\), and denote the vertex sets of \(d(\Delta') v_{t,\text{max}}\) by \(W, W'\) respectively. By the maximality of \(q = \alpha(y)\), we have

\[
\omega_i(y_{\text{left}}) = \varphi_i(y_{\text{left}}) = \kappa(\pi' \cap W, \lambda' - \delta_{t,n});
\]

(18)

\[
\omega_i(y_{\text{right}}) = \varphi_i(y_{\text{right}}) = \kappa(\pi' \cap W', \lambda' - \delta_{t,n});
\]

(19)
Clearly $\text{lk}_{\Delta'}(v_{t,\text{max}}') \subseteq \text{dl}_{\Delta'}(v_{t,\text{max}}')$ implies $W' \subseteq W$. Note that $\text{dl}_{\Delta'}(v_{t,\text{max}}')$ is $\overline{a}$-balanced, while $\text{lk}_{\Delta'}(v_{t,\text{max}}')$ is $(\overline{a} - \delta_{t,n})$-balanced, so $|V_i'| > a_i' > 0$ implies the $0$ entries of $\pi' \cap W$ and $\pi' \cap W'$ are identical, thus $\kappa(\pi' \cap W, \lambda' - \delta_{t,n}) \geq \kappa(\pi' \cap W', \lambda' - \delta_{t,n})$, i.e., $\omega_i(y_{\text{left}}) \geq \omega_i(y_{\text{right}})$.

Suppose further that $y_{\text{left}}$ and $y_{\text{right}}$ are both in $L_k$, and recall we already have

$$\varphi_i(y_{\text{left}}) = \omega_i(y_{\text{left}}) = \omega_i(y_{\text{right}}) = \varphi_i(y_{\text{right}}).$$

If $\varphi_i(y_{\text{left}}) = \varphi_i(y_{\text{right}})$, then $\pi' \cap W = \pi' \cap W'$. Since $y_{\text{left}}, y_{\text{right}} \in L_k$ implies $\text{dl}_{\Delta'}(v_{t,\text{max}}')$ is an $\overline{a}$-rib of a simplex and $\text{lk}_{\Delta'}(v_{t,\text{max}}')$ is an $(\overline{a} \pm \delta_{t,n})$-rib of a simplex, it follows that $\Delta'$ is an $\overline{a}$-rib of a simplex, which contradicts the fact that $y$ is not a terminal vertex in $T_k$. Hence, we must have $\varphi_i(y_{\text{left}}) < \varphi_i(y_{\text{right}})$ in this case, therefore completing the induction step for statements (ii) and (iii).

Finally, we prove statement (iv). Fix some $i > 0$, assume $\varphi_i(x) < n$ for some $x \in Y_i$, and write $m := \varphi_i(x)$. Let $\phi_{\alpha(x)}(x) = (\overline{a}, \overline{\Delta}, \overline{\pi}, \overline{\lambda})$, write $\overline{\alpha} = (\overline{a}_1, \ldots, \overline{a}_n)$, $\overline{\pi} = (\overline{V}_1, \ldots, \overline{V}_n)$, $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_n)$, and let $\overline{V} := \bigcup_{i \in [m]} \overline{V}_i$ be the vertex set of $\overline{\Delta}$. From the construction of the shedding sequence, $\overline{\Delta}$ is $t$-factorizable with respect to $(\overline{\pi}, \overline{\alpha})$ for all $t \in [n]\setminus[m]$, hence

$$\overline{\Delta} = \left( \begin{array}{c} \overline{V}_n \\ \overline{a}_n \end{array} \right) \ast \left( \begin{array}{c} \overline{V}_{n-1} \\ \overline{a}_{n-1} \end{array} \right) \ast \cdots \ast \left( \begin{array}{c} \overline{V}_{m+1} \\ \overline{a}_{m+1} \end{array} \right) \ast \Sigma$$

for some (non-empty) simplicial complex $\Sigma$ with vertex set $\overline{V}_1 \cup \cdots \cup \overline{V}_m$, which yields

$$\text{dl}_{\overline{\Delta}}(v) = \left( \begin{array}{c} \overline{V}_n \\ \overline{a}_n \end{array} \right) \ast \left( \begin{array}{c} \overline{V}_{n-1} \\ \overline{a}_{n-1} \end{array} \right) \ast \cdots \ast \left( \begin{array}{c} \overline{V}_{m+1} \\ \overline{a}_{m+1} \end{array} \right) \ast \text{dl}_\Sigma(v);$$

$$\text{lk}_{\overline{\Delta}}(v) = \left( \begin{array}{c} \overline{V}_n \\ \overline{a}_n \end{array} \right) \ast \left( \begin{array}{c} \overline{V}_{n-1} \\ \overline{a}_{n-1} \end{array} \right) \ast \cdots \ast \left( \begin{array}{c} \overline{V}_{m+1} \\ \overline{a}_{m+1} \end{array} \right) \ast \text{lk}_\Sigma(v);$$

for all $v \in \overline{V}_1 \cup \cdots \cup \overline{V}_m$. For every $y \in D_{\Delta}(x) \cap Y$, since $\varphi_i(y) \leq \varphi_i(x) = m$ implies the Macaulay shedding vertex used to construct the children of $y$ in $T_{\alpha(x)+1}$ is chosen from $\overline{V}_1 \cup \cdots \cup \overline{V}_m$, it then follows inductively that for every $x' \in D_{\Delta}(x)$, the simplicial complex $\phi_{\alpha(x')}^{[L,\Delta]}(x')$ satisfies

$$\phi_{\alpha(x')}^{[L,\Delta]}(x') = \left( \begin{array}{c} \overline{V}_n \\ \overline{a}_n \end{array} \right) \ast \left( \begin{array}{c} \overline{V}_{n-1} \\ \overline{a}_{n-1} \end{array} \right) \ast \cdots \ast \left( \begin{array}{c} \overline{V}_{m+1} \\ \overline{a}_{m+1} \end{array} \right) \ast \Sigma'$$

for some (non-empty) simplicial complex $\Sigma'$ (dependent on the choice of $x'$) with its vertex set contained in $\overline{V}_1 \cup \cdots \cup \overline{V}_m$. Consequently, statement (i) yields $\omega_i^{[t]}(x') = \omega_i^{[t]}(x') = \phi_{\alpha(x')}^{[L,\Delta]}(x') = \overline{\lambda}_t = \omega_i^{[t]}(x)$ for all $x' \in D_{\Delta}(x)$ and all $t \in [n]\setminus[m]$. □

**Theorem 5.6.** The pair $(T_i, \varphi_i)$ is an $a$-Macaulay tree for every $i \in \{0,1,\ldots,k\}$. Furthermore, if $\Delta$ has $N$ facets, then $(T_k, \varphi_k)$ is a condensed $a$-Macaulay tree of $N$.

**Proof.** Choose an arbitrary $i \in \{0,1,\ldots,k\}$, and let $(T_i, \varphi_i|Y_i, \nu_i)$ be the $a$-splitting tree induced by $(T_i, \varphi_i|Y_i)$. Clearly $\varphi_i(r_0) = a$, $\varphi_i(Y_i) \subseteq [n]$ and $\varphi_i(L_i) \subseteq \mathbb{P}^n$ by the definition of $\varphi_i$, while the construction of a shedding sequence yields $\varphi_i(y) \geq \varphi_i(y')$ for every $y, y' \in Y_i$ satisfying $y' \in D_{\Delta}(y)$. It is also easy to show that $\varphi_i(u) = \phi_{\alpha}^{[L,\Delta]}(u) = \nu_i(u) > 0$ for all $u \in L_i$. Proposition 5.5 says $\omega_i$ is proper, and if $\varphi_i(x) < n$ for some $x \in Y_i$, then $\omega_i^{[t]}(x') = \omega_i^{[t]}(x)$ for all $x' \in D_{\Delta}(x)$ and all $t \in [n]\setminus[\varphi_i(x)]$. This implies $\varphi_i^{[t]}(u) = \nu_i^{[t]}(u')$ for all $u, u' \in L_i \cap D_{\Delta}(x)$ and all $t \in [n]\setminus[\varphi_i(x)]$. Furthermore, if $y \in Y_i$ satisfies $\nu_i^{[\varphi_i(y)]}(y) = 1$, then $\nu_i^{[\varphi_i(y)]}(y_{\text{right}}) = 0$, hence by the definition of $\phi_{\alpha}^{[L,\Delta]}$ and $\kappa$, we get

$$\omega_i^{[\varphi_i(y)]}(y_{\text{right}}) = \phi_{\alpha(y)+1}^{[\varphi_i(y)]}(y_{\text{right}}) = \varphi_i(y)$-th entry of $\phi_{\alpha(y)+1}(y_{\text{right}}) = \varphi_i(y) - 1 = \omega_i^{[\varphi_i(y)]}(y) - 1,$$
and define the simplicial complex \( \Psi \) 

Next, suppose \( (T_k, \varphi_k) \) is not condensed, let \( z \) be a cloning vertex of \( T_k, \varphi_k \), and let \( \varphi_k(z) = q \). Also, let \( \phi_k(z) = (a', \Delta', \pi', X') \), and write \( a' = (a'_1, \ldots, a'_n) \), \( \pi' = (V'_1, \ldots, V'_q) \), \( X' = (X'_1, \ldots, X'_q) \). If \( z_{\text{left}} \) and \( z_{\text{right}} \) are both in \( L \), then statement (iii) of Proposition 5.5 says \( \varphi_k(z_{\text{left}}) > \varphi_k(z_{\text{right}}) \), which contradicts the assumption that \( z \) is a cloning vertex. Consequently, \( z_{\text{left}}, z_{\text{right}} \in Y \) and \( \varphi_k(z_{\text{left}}) = \varphi_k(z_{\text{right}}) < q \). Now by assumption, the two subgraphs of \( T_k \) induced by \( D_{T_k}(z_{\text{left}}) \) and \( D_{T_k}(z_{\text{right}}) \) are isomorphic as rooted trees, and they have the same corresponding \( \varphi_k \)-labels, hence it follows from statement (iv) of Proposition 5.5 that \(|\omega_k(x)| = |\omega_k(x)| = \lambda_q - 1\) for all \( x \in (D_{T_k}(z)) \setminus \{z\} \). Now, let \( v'_{q, \lambda_q} \) be the maximal element in \( V'_q \). Then

\[
dl_{\Delta'}(v'_{q, \lambda_q}) = \left( \binom{V'_{\lambda_q}}{a_{q+1}} \right) \ast \left( \binom{V'_{q}}{a_{q}} \right) \ast \Sigma; \\
\lk_{\Delta'}(v'_{q, \lambda_q}) = \left( \binom{V'_{\lambda_q}}{a_{q+1}} \right) \ast \left( \binom{V'_{q}}{a_{q}} \right) \ast \left( \binom{V'_{q}}{\lambda_q} \right) \ast \Sigma;
\]

for some common (non-empty) simplicial complex \( \Sigma \) with vertex set \( V'_1 \cup \cdots \cup V'_{q-1} \). This implies

\[
\Delta' = \left( \binom{V'_{\lambda_q}}{a_{q+1}} \right) \ast \left( \binom{V'_{q}}{a_{q}} \right) \ast \Sigma,
\]

thus \( \Delta' \) is \( q \)-factorizable with respect to \((\pi', a')\). However, the construction of the shedding sequence would then force \( \varphi(z) \neq q \), which is a contradiction, therefore \( (T_k, \varphi_k) \) must be condensed. Finally, if \( \Delta \) has \( N \) facets, then Proposition 5.2 yields

\[
N = f_a(\Delta) = \sum_{u \in L_k} \phi_k^{[L,4]}(u) = \sum_{u \in L_k} \varphi_k(u).
\]

Remark 5.7. In view of Theorem 5.6, we say \((T, \varphi)\) is the Macaulay tree induced by \((T, \varphi_i)\).

For any \( m \in \mathbb{P}, t \in [n] \), define \([m]_t := \{(1, t), (2, t), \ldots, (m, t)\} \subseteq \mathbb{P} \times [n] \). If \( x = (x_1, \ldots, x_n) \in \mathbb{P}^n \) and \( y = (y_1, \ldots, y_n) \in \mathbb{N}^n \) satisfy \( x \geq y \), define the simplicial complex

\[
\langle x | y \rangle := \langle \binom{x_1}{y_1} | \binom{x_2}{y_2} | \cdots | \binom{x_n}{y_n} \rangle.
\]

Definition. Let \( a \in \mathbb{N}^n \setminus \{0_n\} \), let \((T, \varphi)\) be an \( a \)-Macaulay tree, and denote the (root, 1, 3)-triple of \( T \) by \((r_0, L, Y)\). Also, let \((T, \varphi|_Y, \nu)\) be the \( a \)-splitting tree induced by \((T, \varphi|_Y)\), and let \( \omega \) be the left-weight labeling of \((T, \varphi)\). Recall that \( \omega^{[j]} \) denotes the \( j \)-th component function of \( \omega \) for any \( j \in [n] \). For each \( u \in L \), define the set

\[
\psi_{(T, \varphi)}(u) := \{(\omega^{[\varphi(v)]}(v), \varphi(v)) \in \mathbb{P} \times [n] : v \in Y, \text{ both } v \text{ and its right child are ancestors of } u\},
\]

and define the simplicial complex \( \Psi_{(T, \varphi)}(u) := \langle \psi_{(T, \varphi)}(u) \rangle \ast \langle \psi_{(T, \varphi)}(u) \rangle \), whose vertex set is a subset of \( \mathbb{P} \times [n] \). In particular, \( \langle \psi_{(T, \varphi)}(u) \rangle \) and \( \langle \psi_{(T, \varphi)}(u) \rangle \) have disjoint vertex sets, so \( \Psi_{(T, \varphi)}(u) \) is well-defined. Let \( \Delta_{(T, \varphi)} \) denote the union of all simplicial complexes \( \Psi_{(T, \varphi)}(u) \) over all possible terminal vertices \( u \in L \). Let \( (s_1, \ldots, s_n) \in \mathbb{P}^n \) be the weight of \((T, \varphi)\), and define \( \pi_{(T, \varphi)} \) as the ordered partition \( \pi_{(T, \varphi)} := \langle [s_1], \ldots, [s_n] \rangle \).

Proposition 5.8. Let \( n, N \in \mathbb{P}, a \in \mathbb{P}^n \), and let \((T, \varphi)\) be an \( a \)-Macaulay tree of \( N \). Then \( \Delta_{(T, \varphi)}, \pi_{(T, \varphi)} \) is a pure \( a \)-balanced complex with \( N \) facets.

Proof. Follow the notation as above. Choose an arbitrary \( u \in L \), and let \( V(u) \) be the vertex set of \( \Psi_{(T, \varphi)}(u) \). For each \( t \in [n] \), define \( b_t(u) := \{ (i, j) \in \psi_{(T, \varphi)}(u) : j = t \} \) and \( B_t(u) := \{ (i, j) \in \psi_{(T, \varphi)}(u) : j = t \} \). Observe
that \( (b_1(u), \ldots, b_n(u)) = a - \nu(u) \) by the construction of an \( a \)-splitting tree, while \( \langle \varphi(u) \rangle \) is the \( \nu(u) \)-rib of a simplex corresponding to the ordered partition \((\varphi(1)(u)_1, \varphi(2)(u)_2, \ldots, \varphi(n)(u)_n) \). Since \( B_i(u) \subseteq [s_i]_i \) for each \( i \in [n] \) implies \( \pi(T, \varphi) \cap V(u) = (B_1(u), \ldots, B_n(u)) \), it follows that \( (\Psi(T, \varphi)(u), \pi(T, \varphi) \cap V(u)) \) is an \( a \)-balanced complex. Also, the \( \nu(u) \)-rib \( \langle \varphi(u) \rangle \) and the simplex \( \langle \{\psi(T, \varphi)(u)\} \rangle \) are both pure, so \( \Psi(T, \varphi)(u) \) is pure. Furthermore, \( \psi(T, \varphi)(u) = \psi(T, \varphi)(u') \) if and only if \( u = u' \), thus \[
\Delta(T, \varphi) = \bigcup_{u \in L} \Psi(T, \varphi)(u),
\]
which has \( \sum_{u \in L} \langle \varphi(u) \rangle = N \) facets, by the definition of \( (T, \varphi) \). Finally, let \( r_1 \) denote the unique child of \( r_0 \), let \( v_1, \ldots, v_m \) be the left relative sequence of \( r_1 \), and let \( u_i \in L \) be the right-most relative of \( v_i \) for each \( i \in [m] \). Since \( \omega(r_1) = (s_1, \ldots, s_n) \), the construction of \( \omega \) yields \[
\bigcup_{i \in [n]} [s_i]_i \subseteq \bigcup_{i \in [m]} V(u_i) \subseteq \bigcup_{i \in [n]} [s_i]_i,
\]
therefore \( (\Delta(T, \varphi), \pi(T, \varphi)) \) is a pure \( a \)-balanced complex with \( N \) facets.

![Figure 2: An example of a (1, 1)-Macaulay tree \((T, \varphi)\) such that \((\Delta(T, \varphi), \pi(T, \varphi))\) is not color-shifted.](image)

**Remark 5.9.** The pure \( a \)-balanced complex \((\Delta(T, \varphi), \pi(T, \varphi))\) is not necessarily color-shifted. For example, the (1, 1)-Macaulay tree \((T, \varphi)\) in Figure 2 yields \[
\Delta(T, \varphi) = \left\{ \{(1, 1), (1, 2)\}, \{(2, 1), (1, 2)\}, \{(3, 1), (1, 2)\}, \{(1, 1), (2, 2)\}, \{(1, 1), (3, 2)\}, \{(2, 1), (3, 2)\} \right\},
\]
yet \( \{(2, 1), (2, 2)\} \notin \Delta(T, \varphi) \), which implies \((\Delta(T, \varphi), \pi(T, \varphi))\) is not color-shifted. In this example, \((T, \varphi)\) is condensed, so \((T, \varphi)\) being condensed does not necessarily imply \((\Delta(T, \varphi), \pi(T, \varphi))\) is color-shifted.

![Figure 3: Distinct (2, 2)-Macaulay trees \((T, \varphi)\) and \((T', \varphi')\) such that \((\Delta(T, \varphi), \pi(T, \varphi)) = (\Delta(T', \varphi'), \pi(T', \varphi'))\).](image)

**Remark 5.10.** If \((\Delta, \pi)\) is a pure color-shifted \( a \)-balanced complex such that \((\Delta, \pi) = (\Delta(T, \varphi), \pi(T, \varphi))\) for some \( a \)-Macaulay tree \((T, \varphi)\), then knowing \((\Delta, \pi)\) does not uniquely determine \((T, \varphi)\). For example, in Figure 3 \((T, \varphi)\) and \((T', \varphi')\) are two distinct (2, 2)-Macaulay trees such that \((\Delta(T, \varphi), \pi(T, \varphi)) = (\Delta(T', \varphi'), \pi(T', \varphi'))\). However, the condensations of \((T, \varphi)\) and \((T', \varphi')\) are identical, and by the definition of a shedding sequence, it is straightforward to show (e.g. by induction on the number of vertices of \( \Delta \)) that the condensation of \((T, \varphi)\) is the \( a \)-Macaulay tree induced by the shedding tree of \((\Delta(T, \varphi), \pi(T, \varphi))\).
5.3 Compressed-like Compatible Macaulay Trees

Let \( a \in \mathbb{P}^n \). Given any pure color-shifted \( a \)-balanced complex \((\Delta, \pi)\), we can construct the \( a \)-Macaulay tree induced by the shedding tree of \((\Delta, \pi)\), and we showed this Macaulay tree must be condensed. Conversely, given any \( a \)-Macaulay tree \((T, \varphi)\) that is not necessarily condensed, we can construct the pure \( a \)-balanced complex \((\Delta(T, \varphi), \pi(T, \varphi))\). As shown in Remark 5.9, \((\Delta(T, \varphi), \pi(T, \varphi))\) is not necessarily color-shifted, independent of whether \((T, \varphi)\) is condensed. In this subsection, we introduce the notions of ‘compressed-like’ and ‘compatible’ for \( a \)-Macaulay trees, and we will establish the bijection

\[
\begin{aligned}
\{ \text{pure color-compressed} \} & \quad \leftrightarrow \quad \{ \text{compatible a-Macaulay trees of } N \} \\
\{ \text{a-balanced complexes} \} & \quad \leftrightarrow \quad \{ \text{condensed compressed-like} \}
\end{aligned}
\]

with the corresponding maps

\[
(\Delta, \pi) \mapsto \text{ a-Macaulay tree induced by the shedding tree of } (\Delta, \pi),
\]

\[
(\Delta(T, \varphi), \pi(T, \varphi)) \mapsto (T, \varphi).
\]

**Definition.** Let \( a \in \mathbb{N}^n \setminus \{0_n\} \) and let \((T, \varphi)\) be an \( a \)-Macaulay tree. Denote the left-weight labeling of \((T, \varphi)\) by \( \omega \), and let \((r_0, L, Y)\) be the \((root, 1, 3)\)-triple of \( T \). We say \((T, \varphi)\) is compressed-like if the following conditions hold for every \( t \in [n] \) and every \( y \in Y \) satisfying \( \varphi(y) = t \).

(i) If \( y \) is a descendant of \( x_{\text{left}} \) for some \( x \in Y \) satisfying \( \varphi(x) = t \), and \( x_0, x_1, \ldots, x_t \) is the path (of length \( \ell \geq 2 \)) from \( x_0 = x \) to \( x_t = y_{\text{right}} \), then \( \omega^{[\ell]}(x_i) = \omega^{[\ell]}(x) - i \) for all \( i \in [\ell] \).

(ii) Let \( y_1, \ldots, y_m \) be the right relative sequence of \( y_{\text{left}} \) in \( T \). If \( k \in [m] \) is the smallest integer such that \( y_k \in L \) or \( \varphi(y_k) \neq t \), then \( \omega^{[t']}(y_k) \geq \omega^{[t']}(y_{\text{right}}) \) for all \( t' \in [n] \setminus \{t\} \).

**Lemma 5.11.** Let \( k, N \in \mathbb{P} \), and let \((\Delta, V)\) be a pure compressed \((k-1)\)-dimensional simplicial complex. Then the Macaulay tree induced by the shedding tree of \((\Delta, V)\) is compressed-like.

**Proof.** Without loss of generality, assume \( V = [n] \) and \( n > k \). Let \((T, \varphi)\) be the \( k \)-Macaulay tree induced by the shedding tree of \((\Delta, V)\), let \((r_0, L, Y)\) be the \((root, 1, 3)\)-triple of \( T \), and denote the unique child of \( r_0 \) by \( r_1 \). By definition, \( dl_\Delta(n) \) and \( lk_\Delta(n) \) are both pure compressed simplicial complexes, and in particular, every \( k \)-set in \({\binom{n}{k}}\) is in \( \Delta \), i.e. \( dl_\Delta(n) = \{\binom{n-1}{k}\} \). This means the left child of \( r_1 \) is a terminal vertex, and by induction on \( n \), the left child of every \( y \in V \) is a terminal vertex, so \((T, \varphi)\) is trivially compressed-like.

**Proposition 5.12.** Let \( k, N \in \mathbb{P} \). Then there exists a unique condensed compressed-like \( k \)-Macaulay tree \((T, \varphi)\) of \( N \), and the left child of every trivalent vertex of \( T \) is a terminal vertex.

**Proof.** First of all, the existence of a condensed compressed-like \( k \)-Macaulay tree \((T, \varphi)\) of \( N \) follows from Lemma 5.11 and Theorem 5.6. Let \((r_0, L, Y)\) be the \((root, 1, 3)\)-triple of \( T \), and note that \( \varphi(y) = 1 \) for all \( y \in Y \). Let \((T, \varphi|_Y, \nu)\) be the \( k \)-splitting tree induced by \((T, \varphi|_Y)\), let \( \omega \) be the left-weight labeling of \((T, \varphi)\), and let \( u_1, \ldots, u_m \) be all the terminal vertices (i.e. \( |L| = m \)), arranged in depth-first order. It is easy to show that \( |Y| = m - 1 \), and we let \( y_1, \ldots, y_{m-1} \) be all the trivalent vertices arranged in depth-first order.

Next, we show that the left child of every trivalent vertex in \( T \) is a terminal vertex. Suppose not, and say \( z := y_{\text{left}} \notin L \) for some \( y \in Y \). Let \( z = z_1, z_2, \ldots \) be the right relative sequence of \( z \) in \( T \) for some \( \ell \geq 2 \), and let \( z' \) be the left child of \( z_{\ell-1} \). Note that \( z_\ell \) is the right child of \( z_{\ell-1} \), so \((T, \varphi)\) is condensed-like implies \( \omega(z_\ell) = \omega(z_{\ell-1}) - 1 \), yet the construction of \( \omega \) yields \( \omega(z') = \omega(z_{\ell-1}) - 1 \), which contradicts the assumption that \((T, \varphi)\) is condensed. Therefore, the left child of every \( y \in Y \) is in \( L \) as claimed.
Consequently, \( y_i \) is the parent of \( u_i \) for every \( i \in [m-1] \), while \( y_{m-1} \) is the parent of \( u_m \). Furthermore, by letting \( y_m := u_m \), the right relative sequence of \( y_1 \) is precisely \( y_1, \ldots, y_m \). Thus, \( T \) is uniquely determined by \( m \), and \( \nu(u_i) = k + 1 - i \) for every \( i \in [m] \). Since \( \omega \) is proper implies \( \omega(u_i) \geq \omega(y_{i+1}) \) for all \( i \in [m-1] \),

\[
\varphi(u_i) = \omega(u_i) \geq \omega(y_{i+1}) > \omega(y_{i+1}) - 1 = \omega(u_{i+1}) = \varphi(u_{i+1})
\]

for all \( i \in [m-2] \). Also, \( (T, \varphi) \) is condensed, so \( \varphi(u_{m-1}) = \omega(u_{m-1}) > \omega(u_m) = \varphi(u_m) \). This means

\[
N = \sum_{i=1}^{m} \left( \varphi(u_i) \right) = \sum_{i=1}^{m} \left( \frac{\varphi(u_i)}{k + 1 - i} \right),
\]

(34)

where \( \varphi(u_1) > \cdots > \varphi(u_m) \geq \nu(u_m) \geq 1 \), therefore the uniqueness of \( (T, \varphi) \) follows from the uniqueness of the \( k \)-th Macaulay representation of \( N \).

**Theorem 5.13.** If \( (\Delta, \pi) \) is a pure color-compressed \( \mathbf{a} \)-balanced complex for some \( \mathbf{a} \in \mathbb{P}^n \), then the Macaulay tree induced by the shedding tree of \( (\Delta, \pi) \) is compressed-like.

**Proof.** For convenience, identify each vertex \( v_{i,j} \) of \( \Delta \) with the pair \( (j, i) \), so that each \( V_i \) in \( \pi = (V_1, \ldots, V_n) \) is identified with \([\lambda_i]_i\). Choose any shedding sequence \((T_0, \phi_0), (T_1, \phi_1), \ldots, (T_k, \phi_k)\) of \((\Delta, \pi)\), let \((v_0, L_i, Y_i)\) be the \((\text{root}, 1, 3)\)-triple of each \( T_i \), let \((T, \phi)\) be the Macaulay tree induced by \((T, \phi) := (T_k, \phi_k)\), and denote the left-weight labeling of \((T, \phi)\) by \( \omega \). For every \( y \in Y_k \), let \( \alpha(y) \) be the largest integer < \( k \) such that \( y \in L_\alpha(y) \), while for every \( u \in L_k \), set \( \alpha(u) = k \).

Clearly \((T, \varphi)\) is compressed-like when \( k = 0 \), so assume \( k \geq 1 \). Fix \( t \in [n] \), choose some \( x \in Y_k \) such that \( \varphi(x) = t \), and let \( x_1, \ldots, x_t \) be the right relative sequence of \( x_{\text{left}} \). If \( x_{\text{left}} \notin L_k \), then choose any \( \ell' \in \llbracket \ell - 1 \rrbracket \) such that \( \varphi(x_{\ell'}) = \omega^{[t]}(x) = \omega^{[t]}(x) - i \) for all \( i \in \llbracket \ell' + 1 \rrbracket \).

The claim is trivially true if \( x_{\text{left}} \notin L_k \), so assume \( x_{\text{left}} \in L_k \). By the construction of a shedding sequence, we have \( \varphi(x_j) = t \) for every \( j \in \llbracket \ell' \rrbracket \). Let \( \phi_{\alpha(x_j)}(x) = (\mathbf{a}'', \Delta'', \pi'', \mathcal{X}'') \), where \( \mathbf{a}'' = (a_1'', \ldots, a''_n) \) and \( \mathbf{X}'' = (X_1'', \ldots, X''_n) \). Define \( \Delta_i := \phi_{\alpha(x_i)}(x) \) for each \( i \in \llbracket \ell \rrbracket \), and note that \( \Delta_i = d_{\Delta'} ((\lambda_i', t)) \). Since \((\Delta', \pi')\) is color-compressed, every \( a''_i \)-set in \((\lambda''_{\ell''} - 1 \llbracket \ell'' \rrbracket)_i \) is contained in \((\text{some facet of}) \Delta_i \), thus by induction on \( \ell' \), every \( (a''_i + 1 - i) \)-set in \((\lambda''_{\ell''} - 1 \llbracket \ell'' \rrbracket)_i \) is contained in \((\text{some facet of}) \Delta_i \). This yields

\[
\omega^{[t]}(x_j) = \lambda''_{\ell''} - i = \omega^{[t]}(x) - i
\]

for all \( i \in \llbracket \ell' \rrbracket \), so our claim is true in both cases. The first condition in the definition of ‘compressed-like’ then holds by repeatedly using this claim and the definition of \( \omega \).

The second condition in the definition of ‘compressed-like’ is vacuously true if \( n = 1 \), as assumed \( n > 1 \). Fix some \( t' \in [n] \setminus \{t\} \), choose any \( y \in Y \) such that \( \varphi(y) = t' \), let \( y_1, \ldots, y_m \) be the right relative sequence of \( y_{\text{left}} \), and let \( q \in [m] \) be any integer such that \( y_q \in L \) or \( \varphi(y_q) = \omega^{[t']}((y_q)_{\text{left}}) \). If this second condition holds, we need to show that \( \omega^{[t']}((y_q)_{\text{left}}) \geq \omega^{[t']}((y_q)_{\text{right}}) \). If \( q = 1 \), then \( \omega \) is proper implies \( \omega^{[t']}((y_q)_{\text{left}}) = \omega^{[t']}((y_q)_{\text{left}}) \geq \omega^{[t']}((y_q)_{\text{right}}) \), and we are done, so assume \( q \geq 2 \). For the rest of this proof, let \( \phi_{\alpha(y_{\text{right}})} = (\mathbf{a}''', \Delta''', \pi''', \mathcal{X}''') \) and let \( \phi_{\alpha(y)} = (\tilde{\mathbf{a}}, \Delta, \pi, \mathcal{X}) \), where \( \mathbf{a}''' = (a''_1, \ldots, a''_n), \tilde{\mathbf{a}} = (\tilde{a}_1, \ldots, \tilde{a}_n), \pi''' = (V_1', \ldots, V''_n), \pi'' = (\tilde{V}_1, \ldots, \tilde{V}_n), \mathcal{X}''' = (X_1'', \ldots, X''_n), \text{ and } \mathcal{X} = (X_1, \ldots, X_n) \).

If \( V''_t = \emptyset \), then \( a''_t = 0 \). Since \( \varphi(y) \neq t' \) implies \( \tilde{a}_t = 0 \), this means \( \tilde{V}_t = \emptyset \), hence for all descendants \( u \) of \( y \) in \( T \), the \( t' \)-th entry in the \( n \)-tuple \( \phi_{\alpha(u)}^{[\ell]}(u) \) must be the empty set. It then follows from the construction of a shedding sequence that \( \omega^{[t']}(u) = t' \)-th entry of \( \phi_{\alpha(u)}^{[\ell]}(u) = \tilde{\lambda}_t \) for all descendants \( u \) of \( y \). In particular, \( \omega^{[t']}(y) = \omega^{[t']}(y_{\text{right}}) \), and we are done.

If instead \( \tilde{V}_t' \neq \emptyset \), then \( (\lambda_t', t') \) is the maximal element of \( V''_t \), and we choose a facet \( F \) of \( \Delta'' \) that contains this vertex \( (\lambda_t', t') \). Note that \( \Delta'' = \text{lk}_\Delta(\omega^{[t]}(y), t)) \) is \((\tilde{a} - \delta_{t,n})\)-balanced, which implies \( F \cup \{\omega^{[t]}(y), t\} \) is a
facets of $\Delta$, thus the definition of $q$ and the construction of a shedding sequence together yield $|F \cap V_t| \geq q - 2$. Let $F'$ be the set of the $q - 2$ largest elements in $F \cap V_t$ and let $u_i = (w_i(t)(y_i), t)$ for each $i \in [q - 1]$. Since $\Delta$ is color-compressed, $(F \setminus F') \cup \{u_1, \ldots, u_{q-1}\}$ must be a facet of $\Delta$, thus $F \setminus F'$ is a facet of $\phi_{\alpha(y_q)}(y_q)$. In particular, $(\lambda'', \nu')$ is contained in $F \setminus F'$, so $\phi_{\alpha(y_q)}(y_q)$ is color-compressed implies the $\nu'$-entry of $\phi_{\alpha(y_q)}(y_q)$ is a set containing $[\lambda'']_{\nu'}$. Consequently, $\omega^{\nu'} = \nu'$-th entry of $\phi_{\alpha(y_q)}(y_q) \geq \lambda''_{\nu'} = \omega^{\nu'}(y_{\text{right}})$. □

For the rest of this subsection, let $(T, \varphi)$ be an $a$-Macaulay tree for some $a = (a_1, \ldots, a_n) \in \mathbb{N}^n \setminus \{0_n\}$. Denote the (root, 1, 3)-triple of $T$ by $(r_0, L, Y)$, denote the left-weight labeling of $(T, \varphi)$ by $\omega$, and let $(T, \varphi|_Y, \nu)$ be the $a$-splitting tree induced by $(T, \varphi|_Y)$. For any $x \in Y \cup L$, recall that $\mathcal{D}_T(x)$ denotes the set of all descendants of $x$ in $T$.

**Definition.** Suppose $t \in [n]$ and $x \in Y \cup L$. Let $x_1, \ldots, x_t$ be the right relative sequence of $x$ in $T$, and let $k$ be the smallest integer in $[t]$ such that $x_k \in L$ or $\varphi(x_k) \neq t$. We then define the following sets:

$$
\psi(T, \varphi)(x) := \{ (\omega^{\nu}(v), \varphi(v)) \in \mathbb{P} \times [n] : v \in Y, \text{ both } v \text{ and its right child are ancestors of } x \};
$$

$$
\psi^t(T, \varphi)(x) := \{ s \in \mathbb{P} : (s, t) \in \psi(T, \varphi)(x) \};
$$

$$
\hat{\psi}(T, \varphi)(x) := \{ \omega^{\nu}(x_i) : i \in [k] \};
$$

$$
\hat{\xi}(T, \varphi)(x) := \psi^t(T, \varphi)(x) \cup \hat{\psi}(T, \varphi)(x) \cup \xi(T, \varphi)(x).
$$

Observe that the sets $\psi(T, \varphi), \psi^t(T, \varphi), \hat{\psi}(T, \varphi), \hat{\xi}(T, \varphi)$ are pairwise disjoint, hence $\xi(T, \varphi)(x) \in (P_a)$, and we call this $a$-set $\xi(T, \varphi)(x)$ the $t$-signature of $x$ in $(T, \varphi)$. In particular, if $x$ is a $t$-leading vertex, then $\psi^t(T, \varphi)(x) = \emptyset$.

**Definition.** Let $t \in \{0, 1, \ldots, n - 1\}$, and let $u_1, \ldots, u_m$ (for some $m \in \mathbb{P}$) be all the $t$-leading vertices in $L_{T, \varphi}(t)$ arranged in depth-first order. For every $u \in L_{T, \varphi}(t)$ such that the $(t + 1)$-th entry of $a - \nu(u)$ is positive, let $j$ be the unique integer in $[m]$ satisfying $u = u_j$, and define $\xi^{t+1}(u) := u_{j-1}$. In particular, since $\nu(u_1) = a$, we necessarily have $j > 1$, hence $\xi^{t+1}(u)$ is well-defined.

Let $v$ be the unique common ancestor of $u_{j-1}$ and $u_j$ in $Y \cup L$ such that neither child of $v$ is a common ancestor of $u_{j-1}$ and $u_j$. The path from $r_0$ to the right-most relative of $v_{\text{left}}$ and the path from $r_0$ to the left-most relative of $v_{\text{right}}$ share the common vertex $v$, and these two paths must each contain some $t$-leading vertex. The definition of $t$-leading vertices implies $\varphi(v) > t$, so by the definition of the depth-first order, $u_{j-1}$ is contained in the right relative sequence of $v_{\text{left}}$ in $T$, and $u_j$ is contained in the left relative sequence of $v_{\text{right}}$ in $T$. Furthermore, since the $(t + 1)$-th entry of $a - \nu(u_j)$ is positive, the definition of a Macaulay tree forces $\varphi(v) = t + 1$. Consequently, if $(T, \varphi)$ is compressed-like, then the second condition in the definition of ‘compressed-like’ yields $\xi^{t+1}(u_j) \leq_{ct} \xi^{t+1}(u_{j-1})$, or equivalently, $\xi^{t+1}(u) \leq_{ct} \xi^{t+1}(u_{j-1})$.

**Definition.** Let $i, j \in \mathbb{N}$ satisfy $j + 1 < i \leq n$. For every $u \in L_{T, \varphi}(j)$ such that the $i$-th entry of $a - \nu(u)$ is positive, assume $\xi^{i+1}(u)$ has already been defined, and suppose $\xi^{i+1}(u) \leq_{ct} \xi^{i+1}(u_{j+1})$. Then for any $x \in L_{T, \varphi}(j)$, let $\bar{x}$ be the unique ancestor of $x$ that is a $(j + 1)$-leading vertex, set $\bar{y} := \xi^{j+1}_{i+1}(\bar{x})$, define

$$
U := \{ z \in L_{T, \varphi}(j) \cap \mathcal{D}_T(\bar{y}) : \xi^{i+1}_{i+1}(z) \geq_{ct} \xi^{i+1}_{i+1}(\bar{y}) \},
$$

and define $\xi^{j+1}_{i+1}(\bar{x})$ to be the smallest vertex in $U$ with respect to the depth-first order.

Note that if $y$ is the unique $j$-leading vertex in the right relative sequence of $\bar{y}$, then the definition of a $(j + 1)$-signature yields

$$
\xi^{i+1}_{i+1}(\bar{x}) \leq_{ct} \xi^{i+1}_{i+1}(\bar{y}) \leq_{ct} \xi^{i+1}_{i+1}(y) = \xi^{i+1}_{i+1}(y),
$$

thus $y \in U$, i.e. $U$ is non-empty, which implies $\xi^{j+1}_{i+1}(\bar{x})$ is well-defined.
Remark 5.14. Let \( i, j \in \mathbb{N} \) satisfy \( j \leq i \leq n \), and let \( x \in \mathcal{L}_{(T, \varphi)}(j) \) such that the \( i \)-th entry of \( a - \nu(x) \) is positive. If \( \zeta^s_j(x) \) is well-defined, then \( \zeta^s_j(x) \) can be determined from \( x \) via the following algorithm.

\[
\ell \leftarrow \text{length of path from } r_0 \text{ to } x.
\]
\[
(x_0, x_1, \ldots, x_\ell) \leftarrow \text{path from } r_0 \text{ to } x.
\]
\[
g \leftarrow \text{largest integer in } [\ell - 1] \text{ such that } \varphi(x_g) = i \text{ and } x_{g+1} \text{ is the right child of } x_g.
\]
\[
z \leftarrow \text{left child of } x_g.
\]
\[
\ell' \leftarrow \text{number of terms in the right relative sequence of } z.
\]
\[
(y_1, \ldots, y_{\ell'}) \leftarrow \text{right relative sequence of } z.
\]
\[
k \leftarrow \text{smallest integer in } [\ell'] \text{ such that } y_k \in L \text{ or } \varphi(y_k) \neq i.
\]
\[
\text{if } y_k \in L \text{ then }
\]
\[
\zeta^s_j(x) \leftarrow y_k.
\]
\[
\text{else }
\]
\[
s \leftarrow \varphi(y_k).
\]
\[
y \leftarrow y_k.
\]
\[
\text{while } s \geq t \text{ do }
\]
\[
U \leftarrow \left\{ u \in \mathcal{L}_{(T, \varphi)}(s-1) \cap \mathcal{D}_T(y) : \xi^{(s)}_{(T, \varphi)}(u) \geq \xi^{(s)}_{(T, \varphi)}(x) \right\}.
\]
\[
y \leftarrow \text{smallest vertex in } U \text{ with respect to the depth-first order.}
\]
\[
\text{if } y \in L \text{ then }
\]
\[
s \leftarrow t.
\]
\[
\text{else }
\]
\[
s \leftarrow \varphi(y).
\]
\[
\text{end if}
\]
\[
\text{end while}
\]
\[
\zeta^s_j(x) \leftarrow y.
\]
\[
\text{end if}
\]

Notice that \( \zeta^s_j(x) \) is well-defined only if the set \( U \) in each iteration of the while loop is non-empty.

Proposition 5.15. Let \( (T, \varphi) \) be compressed-like, let \( s, t \in \mathbb{N} \) satisfy \( t < s \leq n \), and let \( x \in \mathcal{L}_{(T, \varphi)}(t) \) such that the \( s \)-th entry of \( a - \nu(x) \) is positive. If \( \zeta^s_t(x) \) is well-defined, then we have the following:

(i) \( \psi^{(s)}_{(T, \varphi)}(x) = \{q_1, \ldots, q_m\} \) for some \( m \in [a_s] \) such that \( a_s - m + 1 < q_1 < \cdots < q_m \).

(ii) \( \xi^{(s)}_{(T, \varphi)}(u) = \{q_1 - i : i \in [a_s - m + 1]\} \cup \{q_2, \ldots, q_m\} \) for every \( u \in \mathcal{D}_T(\zeta^s_t(x)) \).

(iii) If \( i \) is an integer satisfying \( s < i \leq n \), then \( \xi^{(i)}_{(T, \varphi)}(u) = \xi^{(i)}_{(T, \varphi)}(x) \) for every \( u \in \mathcal{D}_T(\zeta^s_t(x)) \).

Proof. First of all, since the \( s \)-th entry of \( a - \nu(x) \) is positive, \( \psi^{(s)}_{(T, \varphi)}(x) \) must be non-empty, so we can write \( \psi^{(s)}_{(T, \varphi)}(x) \) as the set \( \{q_1, \ldots, q_m\} \) for some \( m \in [a_s] \), such that \( q_1 < \cdots < q_m \). For each \( i \in [m] \), let \( y_i \in Y \) be the unique ancestor of \( x \) such that \( (\omega^{[\varphi(y_i)]}(y_i), \varphi(y_i)) = (q_i, s) \). Also, let \( z \) be the left child of \( y_1 \). The subset consisting of the \( m \) largest elements in \( \xi^{(s)}_{(T, \varphi)}(x) \) \((\omega_{a_s}^{(s)}(u))\) is precisely \( \psi^{(s)}_{(T, \varphi)}(x) \)), hence \( q_1 \geq a_s - m + 1 \).

If \( q_1 = a_s - m + 1 \), then the definition of the left-weight labeling \( \omega \) yields \( \omega^{[n]}(z) = a_s - m \), while the construction of the \( a \)-splitting tree \( (T, \varphi|_Y, \nu) \) yields \( \nu^{[n]}(z) = a_s - m + 1 \). This means the left-most relative of \( z \), which we denote by \( z_{\text{left-most}} \), satisfies \( \varphi(z_{\text{left-most}}) \geq \nu(z_{\text{left-most}}) \), hence contradicting the definition of a Macaulay tree. Consequently, \( q_1 > a_s - m + 1 \), i.e. statement (i) is true.

Let \( z_1, \ldots, z_\ell \) be the right relative sequence of \( z \), and let \( k \) be the smallest integer in \([\ell]\) such that \( z_k \in L \) or \( \varphi(z_k) \neq s \). Since \( (T, \varphi) \) is compressed-like, the first condition in the definition of ‘compressed-like’ yields

\[
\psi^{(s)}_{(T, \varphi)}(\zeta^s_t(x)) = \{q_2, \ldots, q_m\} \cup \left\{ \omega^{[n]}(z_i) : i \in [k-1] \right\} = \{q_2, \ldots, q_m\} \cup \left\{ q_1 - i : i \in [k-1] \right\}.
\]

Next, let \( z' \) be the left-most relative of \( z_k \). The vertex \( z' \) and all vertices in \( \mathcal{D}_T(\zeta^s_t(x)) \cap L \) are contained \( \mathcal{D}_T(z_k) \cap L \), so it follows from the definition of a Macaulay tree that \( \varphi^{[n]}(u) = \varphi^{[n]}(z') = \omega^{[n]}(z_k) = q_1 - k \) for every \( u \in \mathcal{D}_T(\zeta^s_t(x)) \) \cap L. Since \( t < s \) and \( \zeta^s_t(x) \) is by construction a \( t \)-leading vertex of \( (T, \varphi) \), we get \( \xi^{(s)}_{(T, \varphi)}(u) = \{q_1 - i : i \in [a_s - m + 1]\} \cup \{q_2, \ldots, q_m\} \) for every \( u \in \mathcal{D}_T(\zeta^s_t(x)) \), thus proving statement (ii).
Finally, if \( i \) is an integer satisfying \( s < i \leq n \), then since both \( \zeta_t^s(x) \) and \( x \) are in \( D_T(y_1) \), it follows from the definition of a Macaulay tree and \( \phi(y_1) = s < i \) that \( \psi_{(T,\phi)}^{(i)}(u) = \psi_{(T,\phi)}^{(i)}(x) \) and \( \omega^{[i]}(u) = \omega^{[i]}(x) \) for all \( u \in D_T(\zeta_t^s(x)) \), therefore statement (iii) follows.

**Definition.** Define \((T,\varphi)\) to be always \( n\)-compatible. Suppose \( n > 1 \), \( t \in [n-1] \), and \((T,\varphi)\) is compressed-like. Then we say \((T,\varphi)\) is \( t\)-compatible if

1. \((T,\varphi)\) is \( s\)-compatible for all \( s \in [n]\ \mid \lceil t \rceil \); and
2. \( \zeta_t^s(x) \leq \zeta_t^s(x) \) for every \( x \in L(T,\varphi)(t) \) and every \( i \in [n] \ \mid \lceil t \rceil \) such that the \( i\)-th entry of \( a - \nu(x) \) is positive.

In particular, the condition that \((T,\varphi)\) is \( s\)-compatible for all \( s \in [n] \ \mid \lceil t \rceil \) ensures \( \zeta_t^s(x) \) is well-defined. If \((T,\varphi)\) is \( 1\)-compatible, which is identical to \((T,\varphi)\) being \( j\)-compatible for all \( j \in [n] \), then we say \((T,\varphi)\) is compatible.

Equivalently, if \((T,\varphi)\) is compressed-like, then define \((T,\varphi)\) to be compatible if \( \zeta_t^s(x) \leq \zeta_t^s(x) \) for every \( i, t \in [n] \) satisfying \( s < i \leq n \) and every \( x \in L(T,\varphi)(t) \) such that the \( i\)-th entry of \( a - \nu(x) \) is positive.

**Lemma 5.16.** Let \( s, t \in \mathbb{N} \) satisfy \( t < s \leq n \), and let \( x \in L(T,\varphi)(t) \) such that the \( s\)-th entry of \( a - \nu(x) \) is positive. Suppose \((T,\varphi)\) is compressed-like and \((t+1)\)-compatible. Then \( \zeta_t^s(x) \) is well-defined, and for every integer \( k \) that satisfies \( t < k < s \), we have the following:

(i) \( \psi_{(T,\varphi)}^{(k)}(u) \subseteq \zeta_t^s(x) \) for all \( u \in D_T(\zeta_t^s(x)) \).

(ii) \( \zeta_t^s(x) \geq \zeta_t^s(x) \) for all \( u \in D_T(\zeta_t^s(x)) \).

**Proof.** First of all, the definition of \((t+1)\)-compatible implies \( \zeta_t^s(x) \) is well-defined. Let \( x_0, x_1, \ldots, x_{\ell} \) be the path from \( r_0 \) to \( x \) in \( T \), and let \( q \) be the largest integer in \([\ell-1] \) such that \( \varphi(x_q) = s \) and \( x_{q+1} \) is the right child of \( x_q \). For each integer \( i \) satisfying \( t < i < n \), let \( q_i \) be the unique integer in \([\ell] \) such that \( x_{q_i} \) is an \( i\)-leading vertex. Also, denote the left child of \( x_q \) by \( z \), let \( z_1, \ldots, z_m \) be the path in \( T \) from \( z \) to \( \zeta_t^s(x) \), and for each integer \( i \) satisfying \( t < i < s \), let \( j_i \) be the unique integer in \([m] \) such that \( z_{j_i} \) is an \( i\)-leading vertex.

For example, \( q_{\ell} = \ell \) and \( j_{\ell} = m \), i.e. \( x_{q_{\ell}} = x \) and \( z_{j_{\ell}} = \zeta_t^s(x) \).

Suppose \( k \in \mathbb{P} \) satisfies \( t < k < s \). Then \((T,\varphi)\) is \((k+1)\)-compatible implies \( \zeta_t^s(x_{q_k}) \) is well-defined, and Remark 5.14 yields \( \zeta_t^s(x_{q_k}) = j_{q_k} \). Furthermore, since \((T,\varphi)\) is \( k\)-compatible, we have

\[
\zeta_t^s(x_{q_k}) = \zeta_t^s(x_{q_k}) \geq \zeta_t^s(x_{q_k}) \geq \zeta_t^s(x_{q_k}).
\] (38)

Now, define vertex \( z' \) algorithmically as follows:

\[
z' \leftarrow z_{j_{q_k}}.
\]

**while** \( z' \in L(T,\varphi)(k-1) \) **do**

**if** \( \omega^{[k]}(z') \in \zeta_t^s(x) \) **then**

\[
z' \leftarrow z'_{\text{right}}.
\]

**else**

\[
z' \leftarrow z'_{\text{left}}.
\]

**end if**

**end while**

Using Remark 5.14 it then follows from (38) and the first condition in the definition of ‘compressed-like’ that \( z' = z_{j_{k-1}} \), thus \( \psi_{(T,\varphi)}^{(k)}(z_{j_{k-1}}) = \psi_{(T,\varphi)}^{(k)}(z') \subseteq \zeta_t^s(x) \). Since \( z_{j_{k-1}} \) is a \((k-1)\)-leading vertex, we get \( \psi_{(T,\varphi)}^{(k)}(u) \subseteq \zeta_t^s(x) \) for all \( u \in D_T(z_{j_{k-1}}) \). Statement (i) then follows from the fact that \( \zeta_t^s(x) \in D_T(z_{j_{k-1}}) \).

Next, we prove statement (iii) if \( \psi_{(T,\varphi)}^{(k)}(\zeta_t^s(x)) = a_k \), then it follows from statement (i) that \( \zeta_t^s(x) = \psi_{(T,\varphi)}^{(k)}(u) = \zeta_t^s(x) \) for all \( u \in D_T(\zeta_t^s(x)) \) and we are done. Assume \( \psi_{(T,\varphi)}^{(k)}(\zeta_t^s(x)) < a_k \), and define

\[
p := \max \left( \zeta_t^s(x) \right) \psi_{(T,\varphi)}^{(k)}(\zeta_t^s(x)) \right).
\]
The construction of $\zeta^i(x)$ yields $\xi^{(k)}_{\mathcal{T}(T,\varphi)}(z_{j_{k-1}}) \geq_{ct} \xi^{(k)}_{\mathcal{T}(T,\varphi)}(x)$, so in particular, $\omega^{[k]}(z_{j_{k-1}}) \geq p$. Since $z_{j_{k-1}}$ is a $(k-1)$-leading vertex, it then follows from the definition of a Macaulay tree that the left-most relative of $z_{j_{k-1}}$, which we denote by $z''$, satisfies $\omega^{[k]}(z'') = \omega^{[k]}(z_{j_{k-1}}) \geq p$. Consequently, $\omega^{[k]}(u) \geq p$ for all $u \in \mathcal{D}_T(z_{j_{k-1}})$, therefore the definition of a $k$-signature, together with statement (i) yields $\xi^{(k)}_{\mathcal{T}(T,\varphi)}(u) \geq_{ct} \xi^{(k)}_{\mathcal{T}(T,\varphi)}(x)$ for all $u \in \mathcal{D}_T(\zeta^i(x)) \subseteq \mathcal{D}_T(z_{j_{k-1}})$. \hfill \Box

**Theorem 5.17.** Let $a \in \mathbb{P}^n$. If $(\Delta, \pi)$ is a pure color-compressed $a$-balanced complex, then the $a$-Macaulay tree induced by the shedding tree of $(\Delta, \pi)$ is compatible.

**Proof.** For convenience, identify each vertex $v_{i,j}$ of $\Delta$ with the pair $(j,i)$, so that each $V_i$ in $\pi = (V_1, \ldots, V_n)$ is identified with $[\lambda_i]$. Let $(T_0, \varphi_0), (T_1, \varphi_1), \ldots, (T_k, \varphi_k)$ be a shedding sequence of $(\Delta, \pi)$. For each $i \in [k]$, let $(r_0, L_i, Y_i)$ denote the (root, 1,3)-triple of $T_i$, let $(T_i, \varphi_i)$ be the Macaulay tree induced by $(T_i, \varphi_i)$, and let $\omega_i$ be the left-weight labeling of $(T_i, \varphi_i)$. Let $(T_k, \varphi_k | Y_k, \nu)$ be the $a$-splitting tree induced by $(T_k, \varphi_k | Y_k)$. Also, for every $y \in Y$, let $\omega(y)$ be the largest integer $< k$ such that $y \in L_\omega(y)$, while for every $u \in L_k$, set $\omega(u) = k$. Note that for $(T_k, \varphi_k)$ to be compatible, there is an implicit assumption of $(T_k, \varphi_k)$ being compressed-like, which we can assume by Theorem 5.13.

Suppose $(T_k, \varphi_k)$ is not compatible, and let $t \in [n-1]$ be maximal such that $(T_k, \varphi_k)$ is not $t$-compatible. Choose some $x \in \mathcal{L}_{(T_k, \varphi_k)}(t)$, $s \in [n] \setminus [t]$ such that the $s$-th entry of the $r$-tuple $a - \nu(x)$ is positive, and $\xi^{(t)}_{(T_k, \varphi_k)}(x) \not\leq_{ct} \xi^{(t)}_{(T_k, \varphi_k)}(\zeta^i(x))$. In particular, $\zeta^i(x)$ is still well-defined since $(T_k, \varphi_k)$ is $(t+1)$-compatible by the maximality of $t$. Next, define

$$F' := \bigcup_{j \in [n] \setminus [t]} \{(i,j) : i \in \xi^{(j)}_{(T_k, \varphi_k)}(x)\}.$$ 

By statement (iv) of Proposition 5.5, $\omega^{[j]}(u) = \omega^{[j]}(x)$ for all $j \in [n] \setminus [t]$ and all $u \in \mathcal{D}_{T_k}(x)$, hence $F \cup F' \in \Delta$ for all $u \in \mathcal{D}_{T_k}(x)$ and all $F \in \phi^{[2]}_{\alpha(u)}(x)$ satisfying $F \subseteq \bigcup_{i \in [t]} V_i$. Note that $F_t := \{(i,t) : i \in \xi^{(t)}_{(T_k, \varphi_k)}(x)\} \subseteq V_t$ is a face of $\phi^{[2]}_{\alpha(x)}(x)$, so $F' \cup F_t \in \Delta$.

Recall the definition of the operation $C_j$ (for any $j \in [n]$) on colored multicouples that was introduced in Section 5.1.1 and define $C_{j}$ on colored complexes analogously. Statement (ii) of Proposition 5.15 yields $\xi^{(s)}_{(T_k, \varphi_k)}(\zeta^i(x)) \leq_{ct} \xi^{(s)}_{(T_k, \varphi_k)}(\zeta^i(x))$, so since $(\Delta, \pi)$ is color-compressed implies $C_{\alpha}(\Delta) = \Delta$, it then follows from $F' \cup F_t \in \Delta$ that $F \cup F_t \in \Delta$, where

$$\tilde{F} := (F' \setminus \{(i,s) : i \in \xi^{(s)}_{(T_k, \varphi_k)}(x)\}) \cup \{(i,s) : i \in \xi^{(s)}_{(T_k, \varphi_k)}(\zeta^i(x))\}.$$ 

Now, statement (ii) of Lemma 5.16 says $\psi^{(i)}_{(T_k, \varphi_k)}(\zeta^i(x)) \subseteq \xi^{(i)}_{(T_k, \varphi_k)}(x)$ for all integers $i$ satisfying $t < i < s$ (if any), while statement (iii) of Proposition 5.15 says $\psi^{(i)}_{(T_k, \varphi_k)}(\zeta^i(x)) = \psi^{(i)}_{(T_k, \varphi_k)}(x)$ for all integers $i$ satisfying $s < i \leq n$ (if any). Consequently, $\psi_{(T_k, \varphi_k)}(\zeta^i(x)) \subseteq \tilde{F}$, hence it follows from $F \cup F_t \in \Delta$ that

$$\left(\tilde{F} \setminus \psi^{(i)}_{(T_k, \varphi_k)}(\zeta^i(x))\right) \cup F_t \in \phi^{[2]}_{\alpha(\zeta^i(x))}(\zeta^i(x)).$$

Looking at vertices in $V_t$, this means $\xi^{(t)}_{(T_k, \varphi_k)}(\zeta^i(x)) \geq_{ct} \xi^{(t)}_{(T_k, \varphi_k)}(x)$, which contradicts our original assumption that $\xi^{(t)}_{(T_k, \varphi_k)}(\zeta^i(x)) \not\leq_{ct} \xi^{(t)}_{(T_k, \varphi_k)}(x)$. \hfill \Box

**Lemma 5.18.** Let $(s_1, \ldots, s_n) \in \mathbb{P}^n$ be the weight of $(T, \varphi)$. For each $t \in [n]$, let $G_t \in \left\{\begin{array}{ll} [s_1] & \\
 a_i & \end{array}\right.$ and define $F_t := \{(i,t) : i \in G_t\}$. If there exists some $u \in L$ such that $\psi^{(i)}_{(T, \varphi)}(u) \subseteq G_t$ for all $t \in [n]$, then $F_1 \cup \cdots \cup F_n$ is contained in $\Psi_{(T, \varphi)}(u)$ if and only if $\xi^{(t)}_{(T, \varphi)}(u) \geq_{ct} G_t$ for all $t \in [n]$.
Proof. Suppose there exists some \( u \in L \) as described. Since \( \Psi(T, \varphi)(u) = \langle \nu(u) \rangle \ast \langle \psi(T, \varphi)(u) \rangle \) by definition, we have \( F_i \cup \cdots \cup F_n \in \Psi(T, \varphi)(u) \) if and only if

\[
F_i \setminus \{ (i, t) : i \in \psi(T, \varphi)(u) \} \in \left\{ \left[ \nu^t(u) \right]_i \right\}
\]

for all \( t \in [n] \). Clearly, (40) is true if \( \psi(T, \varphi)(u) = a_t \), and note that \( |\psi(T, \varphi)(u)| = a_t - \nu^t(u) \) by construction. Thus, to prove this lemma, it suffices to show that \( \max(G_i \setminus \psi(T, \varphi)(u)) \leq \varphi^t(u) = \omega^t(u) \) for all \( t \in [n] \) satisfying \( |\psi(T, \varphi)(u)| < a_t \), which is equivalent to the condition \( G_t \leq \xi^t(T, \varphi)(u) \).

\( \square \)

Theorem 5.19. Let \( (T, \varphi) \) be a compressed-like, compatible \( \alpha \)-Macaulay tree. Then \( (\Delta_{T, \varphi}, \pi(T, \varphi)) \) is a pure color-compressed \( \alpha \)-balanced complex.

Proof. Denote the \((1, 1, 3)\)-triple of \( T \) by \((\tau_0, Y, L)\), let \((T, \varphi|Y, \nu)\) be the \( \alpha \)-splitting tree induced by \((T, \varphi|Y)\), and let \( \omega \) be the left-weight labeling of \((T, \varphi)\). We know from Proposition 5.38 that \((\Delta_{T, \varphi}, \pi(T, \varphi))\) is a pure \( \alpha \)-balanced complex, so we are left to show that \((\Delta_{T, \varphi}, \pi(T, \varphi)) \) is color-compressed.

Let \( F \) be an arbitrary facet of \( \Delta_{T, \varphi} \). By construction, there exists a unique terminal vertex in \( L \), which we denote by \( u_F \), such that \( F \in \Psi(T, \varphi)(u_F) \). Recall that \( [s]_i := \{ (1, i), (2, i), \ldots, (s, i) \} \) for any \( s \in \mathbb{P}, i \in [n] \), and note that \( \pi(T, \varphi) = [s]_1, \ldots, [s]_n \), where \( (s_1, \ldots, s_n) \in \mathbb{P}^n \) is the weight of \((T, \varphi)\). For each \( i \in [n] \), define \( F_i := F \cap [s]_i \in (s]_i \), and define \( \hat{F}_i := \{ s \in \mathbb{P} : (s, i) \in F_i \} \subseteq [s]_i \). If \( \hat{F}_i \neq [a_i] \), then let \( \hat{F}_i' < \hat{F}_i \) denote the immediate predecessor of \( \hat{F}_i \) in \((\mathbb{P}, a_i)\) with respect to the color order, while if \( \hat{F}_i = [a_i] \), then let \( \hat{F}_i' := [a_i] \). In either case, define \( F_i' := \{ (s, i) : s \in \hat{F}_i' \} \) and \( F'(i) := (F \setminus F_i) \cup F_i' \). Let

\[
S(T, \varphi) := \left\{ F \in \Delta_{T, \varphi} : F \text{ is a facet of } \Delta_{T, \varphi}, \text{ and there exists some } i \in [n] \text{ such that } F(i) \notin \Delta_{T, \varphi} \right\}.
\]

By the definition of color compression, \((\Delta_{T, \varphi}, \pi(T, \varphi)) \) is color-compressed if and only if \( S(T, \varphi) = \emptyset \).

Suppose \( S(T, \varphi) \neq \emptyset \). Choose any \( F \in S(T, \varphi) \) such that \( u_F \) is minimal with respect to the depth-first order, and let \( t \in [n] \) satisfy \( F'_t \notin \Delta_{T, \varphi} \). If \( \psi(T, \varphi)(u_F) = \emptyset \), then \( \hat{F}_t \subseteq [\varphi^t(u_F)] \) by definition, which implies \( F'_t \subseteq [\varphi^t(u_F)] \), thus \( F'_t \in \Psi(T, \varphi)(u_F) \), which is a contradiction. Consequently, we can assume \( \psi(T, \varphi)(u_F) \neq \emptyset \), so write \( \psi(T, \varphi)(u_F) = \{ q_1, \ldots, q_m \} \) (where \( 1 \leq m \leq a_t \)) such that \( q_1 < \cdots < q_m \). For each \( i \in [m] \), let \( y_i \in Y \) be the unique ancestor of \( u_F \) such that \( \omega^t(y_i)(y_i) = (q_i, t) \). If \( m < a_t \) and \( F'_t \setminus \psi(T, \varphi)(u_F) \neq [a_t - m] \), then \( \psi(T, \varphi)(u_F) \subseteq \hat{F}_t' \) by the definition of the color order, which means \( F'_t \in \Psi(T, \varphi)(u_F) \), and again we have a contradiction, hence we can further assume that \( \hat{F}_t = \{ q_1, \ldots, q_m \} \cup [a_t - m] \).

Note that \( \psi(T, \varphi)(u_F) \neq \emptyset \) implies the \( t \)-th entry of \( \alpha - \nu(u_F) \) is positive, so \( \zeta^0_0(u_F) \) is well-defined, and statement (i) of Proposition 5.15 tells us \( q_1 > a_t - m + 1 \). By the definition of the color order, we then get

\[
\hat{F}_t' = \left\{ q_1 - i : i \in [a_t - m + 1] \right\} \cup \{ q_2, \ldots, q_m \}.
\]

Consequently, statement (ii) of Proposition 5.15 yields \( \zeta^t(T, \varphi)(\zeta^0_0(u_F)) = \hat{F}_t' \). Furthermore, statement (iii) of Proposition 5.15 says \( \psi(T, \varphi)(\zeta^t_0(u_F)) = \xi^t(T, \varphi)(u_F) \) for all integers \( i \) satisfying \( t < i \leq n \) (if any), while Lemma 5.16 says \( \psi(T, \varphi)(\zeta^t_0(u_F)) \subseteq \xi^t(T, \varphi)(u_F) \) and \( \psi(T, \varphi)(\zeta^t_0(u_F)) \geq \xi^t(T, \varphi)(u_F) \) for all \( i \in [t - 1] \). Now, define

\[
G := F'_t \cup \bigcup_{i \in [n] \setminus \{ t \}} \{ (s, i) : s \in \xi^t_i(T, \varphi)(\zeta^t_0(u_F)) \}.
\]
and observe that \((F'_{(t)})_i \leq_{ct} \hat{G}_t\) for all \(i \in [n]\). However, Lemma \([5.18]\) yields \(G \in (\Psi_{(T,\varphi)}(\zeta'_0(u_F)), \text{ and } \zeta'_0(u_F)\) is strictly smaller than \(u_F\) in the depth-first order, so the minimality of \(u_F\) implies \(F''_{(t)} \in \Delta(T,\varphi)\) (although not necessarily in \(\Psi_{(T,\varphi)}(\zeta'_0(u_F))\)), which is a contradiction. \(\square\)

**Theorem 5.20.** Let \((T, \varphi)\) be an \(a\)-Macaulay tree of \(N\) for some \(a \in \mathbb{P}^n\), \(N \in \mathbb{P}\). Then \((T, \varphi)\) is the \(a\)-Macaulay tree induced by the shedding tree of some pure color-compressed \(a\)-balanced complex with \(N\) facets if and only if \((T, \varphi)\) is condensed, compressed-like, and compatible.

**Proof.** If there exists a pure color-compressed \(a\)-balanced complex \((\Delta, \pi)\) with \(N\) facets, such that \((T, \varphi)\) is the \(a\)-Macaulay tree induced by the shedding tree of \((\Delta, \pi)\), then Theorem \([5.5]\) says \((T, \varphi)\) is condensed, Theorem \([5.13]\) yields \((T, \varphi)\) is compressed-like, and Theorem \([5.17]\) tells us \((T, \varphi)\) is compatible. Conversely, if instead \((T, \varphi)\) is condensed, compressed-like, and compatible, then Proposition \(5.8\) gives the pure \(a\)-balanced complex \((\Delta(T,\varphi), \pi(T,\varphi))\) with \(N\) facets, Theorem \([5.19]\) tells us \((\Delta(T,\varphi), \pi(T,\varphi))\) is color-compressed, while by Remark \([5.10]\) the \(a\)-Macaulay tree induced by the shedding tree of \((\Delta(T,\varphi), \pi(T,\varphi))\) is precisely \((T, \varphi)\). \(\square\)

### 5.4 Generalized \(a\)-Macaulay Representations

**Definition.** Let \(N \in \mathbb{N}\), \(a \in \mathbb{N}^n \setminus \{0_n\}\). If \(N > 0\), then a **generalized \(a\)-Macaulay representation** of \(N\) is a condensed compressed-like compatible \(a\)-Macaulay tree of \(N\). Explicitly, it is an ordered pair \((T, \varphi)\) such that

1. \(T\) is a trivalent planted binary tree with (root, 1, 3)-triple \((r_0, L, Y)\).
2. \(\varphi\) is a vertex labeling of \(T\) satisfying all of the following conditions.
   - (a) \(\varphi(r_0) = a\), \(\varphi(Y) \subseteq [n]\) and \(\varphi(L) \subseteq \mathbb{P}^n\).
   - (b) If \(y, y' \in Y\) satisfies \(y' \in D_T(y)\), i.e. \(y'\) is a descendant of \(y\), then \(\varphi(y) \geq \varphi(y')\).
   - (c) If \(y \in Y\) satisfies \(\varphi(y) < n\), then \(\varphi^{[\ell]}(u) = \varphi^{[\ell]}(u')\) for all \(u, u' \in L \cap D_T(y)\) and all \(\ell \in [n]\) \([-\varphi(y)\].
   - (d) The left-weight labeling \(\omega\) of \((T, \varphi)\) satisfies \(\omega(y_{left}) \geq \omega(y_{right})\) for every \(y \in Y\).
   - (e) The \(a\)-splitting tree \((T, \varphi | Y, \nu)\) induced by \((T, \varphi | Y)\) satisfies \(\varphi(u) \geq \nu(u) > 0_n\) for all \(u \in L\).
   - (f) If \(y \in Y\) satisfies \(\nu^{[\varphi(y)]}(y) = 1\), then \(\omega^{[\varphi(y)]}(x) = \omega^{[\varphi(y)]}(y) - 1\) for all \(x \in D_T(y_{right})\).
   - (g) \(N\) can be written as the sum
     \[
     N = \sum_{u \in L} \left( \varphi(u) \nu(u) \right).
     \]
3. There does not exist any \(y \in Y\) satisfying both of the following two conditions:
   - (a) The two subgraphs of \(T\) induced by \(D_T(y_{left})\) and \(D_T(y_{right})\) are isomorphic as rooted trees, and they have the same corresponding \(\varphi\)-labels, i.e. if \(\beta : D_T(y_{left}) \to D_T(y_{right})\) is the corresponding isomorphism, then \(\varphi(\beta(x)) = \varphi(x)\) for all \(x \in D_T(y_{left})\).
   - (b) Both \(y_{left}\) and \(y_{right}\) are \((\varphi - 1)\)-leading vertices of \((T, \varphi)\).
4. For every \(t \in [n]\) and every \(y \in Y\) such that \(\varphi(y) = t\), the following conditions hold:
   - (a) If \(y\) is a descendant of \(x_{left}\) for some \(x \in Y\) satisfying \(\varphi(x) = t\), and \(x_{0}, x_{1}, \ldots, x_{\ell}\) is the path (of length \(\ell \geq 2\)) from \(x_{0} = x\) to \(x_{\ell} = y_{right}\), then \(\omega^{[\ell]}(x_{i}) = \omega^{[\ell]}(x) - i\) for all \(i \in [\ell]\).
   - (b) Let \(y_1, \ldots, y_m\) be the right relative sequence of \(y_{left}\) in \(T\). If \(k \in [m]\) is the smallest integer such that \(y_k \in L\) or \(\varphi(y_k) \neq t\), then \(\omega^{[\ell]}(y_k) \geq \omega^{[\ell]}(y_{right})\) for all \(t' \in [n]\) \([-\varphi(y_k)\].
   - (v) \(\xi^{(t)}_{(T,\varphi)}(x) \leq_{ct} \xi^{(t)}_{(T,\varphi)}(\zeta'_t(x))\) for every \(i, t \in [n]\) satisfying \(t < i\) and every \(t\)-leading vertex \(x\) of \((T, \varphi)\) such that the \(i\)-th entry of \(\alpha - \nu(x)\) is positive.

As for the case \(N = 0\), the **generalized \(a\)-Macaulay representation** of \(0\) is unique, and it is defined to be the pair \(\varphi_0 := (T_0, \varphi_0)\), where \(T_0\) is the null graph (i.e. the unique graph with zero vertices), and \(\varphi_0 : \emptyset \to \emptyset\) is the (unique) function whose domain and codomain are empty sets. A **generalized \(a\)-Macaulay representation** is a generalized \(a\)-Macaulay representation of \(N\) for some \(N \in \mathbb{N}\), and a **generalized Macaulay representation** is a generalized \(a\)-Macaulay representation for some \(a \in \mathbb{N}^n \setminus \{0_n\}\). The \(n\)-tuple \(a\) is called the **type** of
this generalized Macaulay representation. A generalized Macaulay representation $\alpha$ is called non-trivial if $\alpha \neq \alpha_0$, and called trivial if $\alpha = \alpha_0$.

**Remark 5.21.** Conditions \((\text{ii})\), \((\text{iii})\) and \((\text{iv})\) can be easily checked by hand (at least when $T$ is not too large), while condition \((\text{v})\) is somewhat tedious to check. For large values of $n$, we suggest checking these conditions algorithmically. Note that all these conditions involve only comparing the labels and left-weights of certain pairs of vertices in $T$, hence generalized Macaulay representations are purely numerical notions and do not depend on the existence of any colored complexes.

In general, a generalized a-Macaulay representation of $N$ is not uniquely determined by $a$ and $N$. However, in the case $n = 1$, since every compressed-like $k$-Macaulay tree is trivially compatible (for any $k \in \mathbb{P}$), Proposition 5.11 says the generalized $k$-Macaulay representation of $N$ is unique given $k, N \in \mathbb{P}$.

If $a \in \mathbb{P}^n$, then Theorem 5.20 gives the bijection

\[
\begin{align*}
\{ \text{isomorphism classes of pure color-compressed a-balanced complexes with $N$ facets} \} & \leftrightarrow \{ \text{generalized a-Macaulay representations of $N$} \},
\end{align*}
\]

where the corresponding maps are given by

\[
(\Delta, \pi) \mapsto \text{a-Macaulay tree induced by the shedding tree of} (\Delta, \pi),
\]

\[
(\Delta_{(T, \varphi)}, \pi_{(T, \varphi)}) \mapsto (T, \varphi).
\]

**Definition.** Let $\alpha = (T, \varphi)$ be an a-Macaulay tree of $N$ for some $a \in \mathbb{P}^n, N \in \mathbb{P}$, let the (root, 1, 3)-triple of $T$ be $(r_0, L, Y)$, and let $(T, \varphi|_Y, \nu)$ be the a-splitting tree induced by $(T, \varphi|_Y)$. Then for any $x \in \mathbb{Z}^n$, define

\[
\partial_{[x]}(\alpha) := \sum_{u \in L} \left( \varphi(u) \nu(u) + x \right).
\]

**Theorem 5.22.** Let $a \in \mathbb{P}^n$, and let $f = \{f_b\}_{b \leq a}$ be an array of integers. Then the following are equivalent:

\begin{enumerate}[(i)]
  \item $f$ is the fine $f$-vector of a pure color-compressed a-balanced complex.
  \item $f_a > 0$, and there exists a generalized a-Macaulay representation $\alpha$ of $f_a$ such that $f_b = \partial_{[b-a]}(\alpha)$ for every $b \leq a$.
\end{enumerate}

**Proof.** This immediately follows from Theorem 5.20 and Proposition 5.2. \qed

**Theorem 5.23.** Let $d \in \mathbb{P}$, let $f = \{f_S\}_{S \subseteq [d]}$ be an array of integers, and define $\phi : 2^{[d]} \to \{0, 1\}^d$ by $S \mapsto (s_1, \ldots, s_d)$, where $s_i = 1$ if and only if $i \in S$. Then the following are equivalent:

\begin{enumerate}[(i)]
  \item $f$ is the flag $f$-vector of a $(d - 1)$-dimensionally completely balanced CM complex.
  \item $f_{[d]} > 0$, and there exists a generalized $1_d$-Macaulay representation $\alpha$ of $f_{[d]}$ such that $f_S = \partial_{[\phi(S)-1_d]}(\alpha)$ for every $S \subseteq [d]$.
\end{enumerate}

**Proof.** As mentioned in Remark 3.7, the notion of ‘color-compressed’ is equivalent to the notion of ‘color-shifted’ in the specific case of type $1_d$, hence by Corollary 3.13, $f$ is the flag $f$-vector of a $(d - 1)$-dimensional completely balanced CM complex if and only if it is the flag $f$-vector of a pure color-compressed $1_d$-balanced complex, and the assertion then follows from Theorem 5.22. \qed

**Example 5.24.** By the Frankl-Füredi-Kalai theorem [3], the vector $(7, 11, 5)$ is the $f$-vector of a completely balanced CM complex, since the corresponding $h$-vector $(1, 4, 0, 0)$ is the $f$-vector (including the $f_{-1}$ term) of a $1_3$-colored complex. One possible refinement of $(7, 11, 5)$ is the array $F = \{F_S\}_{S \subseteq [3]}$, given by

\[
F_{\{1, 2, 3\}} = 5, \quad F_{\{1, 2\}} = 5, \quad F_{\{1, 3\}} = 3, \quad F_{\{2, 3\}} = 3, \quad F_{\{1\}} = 3, \quad F_{\{2\}} = 2, \quad F_{\{3\}} = 2, \quad F_{\emptyset} = 1,
\]
and we would like to know if $F$ is the flag $f$-vector of a completely balanced CM complex. Figure 4 gives a list of all 24 generalized $1_3$-Macaulay representations of 5. Using Theorem 5.23, we then conclude that up to permutations of the 3 colors such that $f_{\{1,2\}} \geq f_{\{1,3\}} \geq f_{\{2,3\}}$, the possible flag $f$-vectors $\{f_S\}_{S \subseteq [3]}$ of a 2-dimensional completely balanced CM complex satisfying $f_{[3]} = 5$ (and $f_0 = 1$) are given by

$$\begin{bmatrix} f_{\{1,2\}} & f_{\{1,3\}} & f_{\{2,3\}} \\ f_{\{1\}} & f_{\{2\}} & f_{\{3\}} \end{bmatrix} \in \left\{ \begin{bmatrix} 5 & 5 & 1 \\ 5 & 4 & 2 \\ 5 & 3 & 3 \\ 5 & 3 & 2 \\ 4 & 4 & 3 \\ 4 & 3 & 3 \end{bmatrix} \right\}.$$

Therefore $F$ is not the flag $f$-vector of a completely balanced CM complex.

**Definition.** Let $a, a' \in \mathbb{N}^n \setminus \{0_n\}$ satisfy $a' \leq a$, and let $\alpha = (T, \varphi)$ be an $a$-Macaulay tree. Denote the $(r_0, 1, 3)$-triple of $T$ by $(r_0, L, Y)$, and let $(T, \varphi|_Y, \nu)$ be the $a$-splitting tree induced by $(T, \varphi|_Y)$. Then, the $a'$-twin of $(T, \varphi)$ is the pair $(T', \varphi')$ constructed from $(T, \varphi)$ via the following algorithm:

1. $S \leftarrow \{x \in (Y \cup L \cup \{r_0\}) : \nu(x) > a - a'\}$.
2. $S_0 \leftarrow \{y \in Y : \nu(y) > a - a', \nu(y)_{\text{right}} \not\equiv a - a'\}$.
3. $T' \leftarrow$ subgraph of $T$ induced by $S$.
4. $\varphi' \leftarrow \varphi|_S$.
5. $\varphi'(r_0) \leftarrow a'$.

---

**Figure 4:** All 24 generalized $1_3$-Macaulay representations of 5.
while $S_0 = \emptyset$ do
  $z \leftarrow$ largest vertex in $S_0$ with respect to the depth-first order.
  $z_{\text{child}} \leftarrow$ unique child of $z$ in $T'$.
  $S_0 \leftarrow S_0 \setminus \{z\}$.
  $L' \leftarrow$ set of leaves of $T'$.
  if $z_{\text{child}} \in L'$ then
    $\varphi'(z) \leftarrow \varphi'(z_{\text{child}}) + \delta_{\varphi'(z), n}$.
    $\varphi'(z_{\text{child}}) = \varphi'(z)$.
  else
    $t' \leftarrow \varphi'(z)$.
    $\varphi'(z) \leftarrow \varphi'(z_{\text{child}})$.
    for $x \in D_{T'}(z_{\text{child}}) \cap L'$ do
      $(\varphi')^{t'}(x) \leftarrow (\varphi')^{t'}(x) + 1$.
    end for
  end if
  $e \leftarrow \{z, z_{\text{child}}\}$.
  $(T', \varphi') \leftarrow (T'/e, \varphi'/e)$.
end while

We leave the reader with the straightforward task of verifying that the pair $(T', \varphi')$ obtained from the above algorithm is an $\alpha'$-Macaulay tree of $\partial_{[\alpha' - \alpha]}(\alpha)$.

For the rest of this subsection, let $a, a' \in \mathbb{N}^n \setminus \{0_n\}$ satisfy $a' \leq a$, let $\alpha = (T, \varphi)$ be an $a$-Macaulay tree, let $\alpha' = (T', \varphi')$ be an $a'$-Macaulay tree, and let $\alpha'' = (T'', \varphi'')$ be the condensation of the $a'$-twin of $\alpha$. Denote the (root, 1, 3)-triples of $T'$ and $T''$ by $(r'_0, L', Y')$ and $(r''_0, L'', Y'')$ respectively. Next, construct the graph $\widetilde{T}$ as follows: Take the disjoint union of $T'$ and $T''$, identify the two roots $r'_0, r''_0$ as a single vertex $\tilde{r}$, then attach a leaf $\tilde{r}$ to $\tilde{r}$. Note that $\widetilde{T}$ is a trivalent planted binary tree with root $\tilde{r}$, and treat $T'$ and $T''$ as subgraphs of $\widetilde{T}$. Now, let $\tilde{a} := (a', 2) \in \mathbb{N}^{n+1} \setminus \{0_{n+1}\}$, and define the vertex labeling $\tilde{\varphi}$ of $\widetilde{T}$ as follows:

$$
\tilde{\varphi}(x) = \begin{cases} 
  \tilde{a}, & \text{if } x = \tilde{r}; \\
  n + 1, & \text{if } x = \tilde{r}; \\
  \varphi'(x), & \text{if } x \in Y'; \\
  (\varphi'(x), 2) \in \mathbb{P}^{n+1}, & \text{if } x \in L'; \\
  \varphi''(x), & \text{if } x \in Y''; \\
  (\varphi''(x), 1) \in \mathbb{P}^{n+1}, & \text{if } x \in L''.
\end{cases}
$$

It is easy to check that $(\widetilde{T}, \tilde{\varphi})$ is an $\tilde{a}$-Macaulay tree, and we denote this Macaulay tree by $\alpha \wedge \alpha'$. Furthermore, $\alpha \wedge \alpha'$ is compressed-like if both $\alpha$ and $\alpha'$ are compressed-like, and $\alpha \wedge \alpha'$ is condensed if $\alpha'$ is condensed.

(a) (1,1)-Macaulay tree $\alpha'$ (b) (2,2)-Macaulay tree $\alpha$ (c) (1,1)-twin $\alpha''$ of $\alpha$ (d) (1,1,2)-Macaulay tree $\alpha \wedge \alpha'$

Figure 5: Macaulay trees $\alpha$, $\alpha'$, $\alpha''$, and $\alpha \wedge \alpha'$. 

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Example 5.25. Figure 5 shows an example of a (1,1)-Macaulay tree \( \alpha' \), a (2,2)-Macaulay tree \( \alpha \), the (1,1)-twin \( \alpha'' \) of \( \alpha \) (which is its condensation), and the corresponding (1,1,2)-Macaulay tree \( \alpha \land \alpha' \).

Definition. Let \( \alpha, \alpha' \) be non-trivial generalized Macaulay representations. If \( \alpha \land \alpha' \) is compatible, or equivalently, if \( \alpha \land \alpha' \) is a generalized Macaulay representation, then we write \( \alpha \succeq \alpha' \). Also, write \( \alpha_0 \succeq \alpha'' \) for all generalized Macaulay representations \( \alpha'' \).

Proposition 5.26. \( \succeq \) is a partial order on generalized Macaulay representations.

Proof. Given \( n \)-tuples \( \rho = ([s_1]_1, \ldots, [s_n]_n) \) and \( \rho' = ([s'_1]_1, \ldots, [s'_n]_n) \) for some \( s_1, \ldots, s_n, s'_1, \ldots, s'_n \in \mathbb{P} \), write \( \rho \preceq \rho' \) to mean \( (s_1, \ldots, s_n) \preceq (s'_1, \ldots, s'_n) \). Let \( a, b, c \in \mathbb{N}^n \setminus \{0_n\} \) satisfy \( c \preceq b \preceq a \), and let \( \alpha, \beta, \gamma \) be arbitrary generalized Macaulay representations of types \( a, b, c \) respectively. Without loss of generality, assume \( \alpha, \beta, \gamma \) are all non-trivial. Clearly \( \alpha \succeq \alpha \), since \( \alpha \) is compatible implies \( \alpha \land \alpha \) is compatible.

Next, suppose \( \alpha \preceq \beta \) and \( \beta \preceq \alpha \). This means \( \alpha \land \beta \) and \( \beta \land \alpha \) are well-defined, which implies \( a = b \), hence the condensation of the \( a \)-twin of \( \beta \) is \( \beta \), and the condensation of the \( b \)-twin of \( \alpha \) is \( \alpha \). Since \( \alpha \land \beta \) is compressed-like and compatible, Theorem 5.19 yields \((\Delta_{\alpha \land \beta}, \pi_{\alpha \land \beta})\) is color-compressed, hence by the construction of \( \alpha \land \beta \), we get \( \Delta_{\alpha} \subseteq \Delta_{\beta} \) and \( \pi_{\alpha} \subseteq \pi_{\beta} \). A symmetric argument also gives \( \Delta_{\beta} \subseteq \Delta_{\alpha} \) and \( \pi_{\beta} \subseteq \pi_{\alpha} \), thus \((\Delta_{\alpha}, \pi_{\alpha}) = (\Delta_{\beta}, \pi_{\beta})\). Now \( \alpha \) and \( \beta \) are condensed, so Theorem 5.20 yields \( \alpha = \beta \).

Finally, suppose instead that \( \alpha \preceq \beta \) and \( \beta \preceq \gamma \). Let \( \alpha' \) be the \( b \)-twin of \( \alpha \), let \( \alpha \) be the \( c \)-twin of \( \alpha \), and let \( \beta \) be the \( c \)-twin of \( \beta \). Since \( \alpha \land \beta \) and \( \beta \land \gamma \) are compressed-like and compatible, Theorem 5.19 says \((\Delta_{\alpha \land \beta}, \pi_{\alpha \land \beta})\) and \((\Delta_{\beta \land \gamma}, \pi_{\beta \land \gamma})\) are both color-compressed. By the construction of \( \alpha \land \beta \), we get \( \Delta_{\alpha'} \subseteq \Delta_{\beta} \), \( \Delta_{\alpha'} \subseteq \Delta_{\beta} \), \( \pi_{\alpha'} \subseteq \pi_{\beta} \), and \( \pi_{\alpha'} \subseteq \pi_{\beta} \). Similarly, the construction of \( \beta \land \gamma \) yields \( \Delta_{\beta'} \subseteq \Delta_{\gamma} \) and \( \pi_{\beta'} \subseteq \pi_{\gamma} \), thus \( \Delta_{\beta'} \subseteq \Delta_{\gamma} \) and \( \pi_{\beta'} \subseteq \pi_{\gamma} \). Consequently, the construction of \( \alpha \land \gamma \) yields \((\Delta_{\alpha \land \gamma}, \pi_{\alpha \land \gamma})\) is a pure color-compressed \((c,2)\)-balanced complex, so since \( \alpha \land \gamma \) is condensed, Remark 5.10 tells us the Macaulay tree induced by the shedding tree of \((\Delta_{\alpha \land \gamma}, \pi_{\alpha \land \gamma})\) is precisely \( \alpha \land \gamma \), and Theorem 5.20 then yields \( \alpha \land \gamma \) is compatible, i.e. \( \alpha \preceq \gamma \).

Proposition 5.27. Let \( k, N, N' \in \mathbb{P} \), let \( \alpha \) be the generalized \((k+1)\)-Macaulay representation of \( N \), and let \( \alpha' \) be the generalized \( k \)-Macaulay representation of \( N' \). Then the following are equivalent:

(i) \( \alpha \preceq \alpha' \).
(ii) \( \partial_{k}(\alpha) \preceq \partial_{k}(N) \).
(iii) \( \partial_{k+1}(\alpha) \preceq N' \).

Proof. First of all, \( \partial_{k}(\alpha) = \partial_{k+1}(N) \) by definition, hence \( \text{[ii]} \Leftrightarrow \text{[iii]} \). Let \( \alpha'' \) be the condensation of the \( k \)-twin of \( \alpha \), and note that \( \alpha'' \) is by construction a generalized \( k \)-Macaulay representation of \( \partial_{k-1}(\alpha) \). By definition, \( \alpha \preceq \alpha' \) if and only if \( \alpha \land \alpha' \) is compatible, which is equivalent to the condition that \( \Delta_{\alpha''} \subseteq \Delta_{\alpha'} \). Now, \( \alpha', \alpha'' \) are both generalized \( k \)-Macaulay representations, so by Theorem 5.20 \( \Delta_{\alpha'}, \Delta_{\alpha''} \) are both pure compressed \((k-1)\)-dimensional simplicial complexes and hence each completely determined by its number of facets. This implies \( \Delta_{\alpha''} \subseteq \Delta_{\alpha'} \) if and only if \( f_{k-1}(\Delta_{\alpha''}) \leq f_{k-1}(\Delta_{\alpha'}) \), which by Proposition 5.8 is equivalent to \( \partial_{k-1}(\alpha) \preceq N' \).

Finally, we finish the proof of Theorem 1.2, which follows from Corollary 3.3, Remark 3.7, and the following theorem.

Theorem 5.28. Let \( a \in \mathbb{P}^n \), and let \( f = \{f_b\}_{b \leq a} \) be an array of integers. Then the following are equivalent:

(i) \( f \) is the fine \( f \)-vector of a color-compressed \( a \)-colored complex.
(ii) \( f_{\alpha} = 1 \), \( f_{\alpha'} = 0 \) for all \( \alpha', \alpha \neq \alpha' \), \( f_{\delta_{a,b}} > 0 \) for all \( i \in [n] \), and there exists an array \( \{\alpha_b\}_{b<\alpha_b\leq a} \) such that each \( \alpha_b \) is a generalized \( b \)-Macaulay representation of \( f_b \), and \( \alpha_b \preceq \alpha_{b'} \), for all \( b, b' \) satisfying \( 0_n < b' < b \leq a \).
Proof. Let \((\Delta, \pi)\) be an arbitrary colored-compressed \(a\)-colored complex, and assume without loss of generality that \(\pi = ([\lambda_1], \ldots, [\lambda_n])\), where \(\lambda_1, \ldots, \lambda_n \in \mathbb{P}\). Clearly \(f_{\emptyset, \emptyset}(\Delta) = 1\) and \(f_{\emptyset, \emptyset}(\Delta) = 0\) for all \(a' \geq 0\). By definition, \(f_{\delta, \emptyset}(\Delta) = \lambda_i > 0\) for each \(i \in [n]\). For any \(b \in \mathbb{N}^n\) satisfying \(0_b < b \leq a\), define \(\Delta_b := \langle f_{b}(\Delta) \rangle\), let \(U_b\) denote the vertex set of \(\Delta_b\), and let \(\pi_b := \pi \cap U_b\). By default, set \(U_b = \emptyset\) if \(F_b(\Delta) = \emptyset\). Notice that if \(F_b(\Delta) \neq \emptyset\), then \((\Delta_b, \pi_b)\) is a pure color-compressed \(b\)-balanced complex, thus by Theorem 5.20, the \(\mathcal{B}\)-Macaulay tree induced by the shedding tree of \((\Delta_b, \pi_b)\) is a generalized \(\mathcal{B}\)-Macaulay representation, which we denote by \(\overline{\pi_b} = (T_b, \overline{\pi_b})\). We can then extend each such \(\overline{\pi_b}\) to a generalized \(b\)-Macaulay representation \(\alpha_b\) as follows.

Given \(b = (b_1, \ldots, b_n)\), let \(m\) be the number of non-zero entries in \(b\), and let \(I := \{i_1, \ldots, i_m\} \subseteq [n]\) be the set of indices with \(i_1 < \cdots < i_m\), such that \(b_{i_1}, \ldots, b_{i_m}\) are all the non-zero entries of \(b\). Let \((r_0, L, Y)\) be the \((1, 3)\)-triple of \(T_b\), and define the vertex labeling \(\varphi_b\) of \(T_b\) so that:

- \(\varphi_b(\alpha_0) = b\);
- \(\varphi_b(y) = i_{\overline{\pi_b}(y)} \in I\) for all \(y \in Y\);
- \((\varphi_b)^{(t)}(u) = (\overline{\pi_b})^{(t)}(u)\) for all \(t \in [m], u \in L\);
- \((\varphi_b)^{(j)}(u) = \lambda_j\) for all \(j \in [n] \setminus I, u \in L\).

We can check that \(\alpha_b\) is a \(b\)-Macaulay tree of \(f_{b}(\Delta)\), and the fact that \(\overline{\pi_b}\) is a generalized Macaulay representation easily implies \(\alpha_b\) is also a generalized Macaulay representation. As for the case \(F_b(\Delta) = \emptyset\), define \(\alpha_b\) to be the trivial generalized Macaulay representation \(\alpha_b\).

Let \(b, b' \in \mathbb{N}^n\) satisfy \(0_b < b' < b \leq a\). We want to show that \(\alpha_b \preceq \alpha_{b'}\). Clearly, if \(F_{b'}(\Delta) = \emptyset\) implies \(F_{b'}(\Delta) = \emptyset\), we can thus assume \(F_{b'}(\Delta)\) and \(F_{b'}(\Delta)\) are non-empty, i.e. \(\alpha_b\) and \(\alpha_{b'}\) are both non-trivial. Let \(\alpha_{b'}\) denote the \(b'\)-twin of \(\alpha_b\). Theorem 5.20 tells us \(\Delta_{b'} = \Delta_{a'_{b'}}\) and \(\langle F_{b'}(\Delta) \rangle = \Delta_{\alpha_{b'}, b'}\), while it follows from definition that \(\langle F_{b'}(\Delta) \rangle \subseteq \Delta_{b'}\). Thus, by the construction of \(\alpha_b \land \alpha_{b'}\), we get that \((\Delta_{\alpha_b \land \alpha_{b'}}, \pi_{\alpha_b \land \alpha_{b'}})\) is a pure color-compressed \((b', 2)\)-balanced complex, therefore Theorem 5.20 tells us \(\alpha_b \land \alpha_{b'}\) is compatible, i.e. \(\alpha_b \preceq \alpha_{b'}\).

Conversely, suppose instead statement (ii) of the theorem is true. By assumption, \(\alpha_b \preceq \alpha_{b'}\) for every \(b, b' \in \mathbb{N}^n\) satisfying \(0_b < b' < b \leq a\), so since \(\preceq\) is a partial order (Proposition 5.26), it follows that \(\alpha_b \preceq \alpha_c\) for every \(b, c \in \mathbb{N}^n\) satisfying \(0_b < c \leq b \leq a\). Define \(\pi_{\Gamma} := ([f_{\delta, \emptyset}]_1, \ldots, [f_{\delta, \emptyset}]_n)\), and let \(\Gamma\) be the union of all simplicial complexes \(\Delta_{\alpha_b}\) over all \(n\)-tuples \(b\) satisfying \(0_b < b \leq a, f_b > 0\). For each such \(b\), the \(b\)-balanced complex \((\Delta_{\alpha_b}, \pi_{\overline{\alpha_b}})\) is color-compressed, hence \((\Gamma, \pi_{\Gamma})\) is a color-compressed \(a\)-colored complex.

Now, choose arbitrary \(b, c \in \mathbb{N}^n\) such that \(0_b < c \leq b \leq a\). Note that \(\alpha_b \preceq \alpha_b\), while \(\alpha_b \preceq \alpha_b\) if and only if \(f_b = 0\), so \(f_b = 0\) if and only if \(f_b(\Gamma) = 0\). Consequently, if \(f_b > 0\), then \(f_c > 0\) and \(F_{c}(\Delta_{\alpha_b}) \subseteq \Delta_{\alpha_c}\), which imply \(F_c(\Gamma) = F_{c}(\Delta_{\alpha_c})\). Therefore by Proposition 5.27 \(\alpha_b\) is a generalized \(b\)-Macaulay representation of \(f_{b}(\Gamma)\) for every \(b \in \mathbb{N}^n\) satisfying \(0_b < b \leq a\), i.e. \(\alpha_b\) is the fine \(f\)-vector of \((\Gamma, \pi_{\Gamma})\).

In the special case \(n = 1\), i.e. \(a = (a_1)\), an \(a\)-colored complex is precisely \(a\) \((k - 1)\)-dimensional simplicial complex for some \(k \subseteq [a_1]\), thus Theorem 1.2 and Proposition 5.27 together give the following corollary, which is equivalent to the Kruskal-Katona theorem (Theorem 2.2).

**Corollary 5.29.** Let \(d \in \mathbb{P}\), and let \(f = (f_0, \ldots, f_{d-1}) \in \mathbb{Z}^d\). Then the following are equivalent:

(i) \(f\) is the \(f\)-vector of a \((d - 1)\)-dimensional simplicial complex.
(ii) \(f\) is the \(f\)-vector of a compressed \((d - 1)\)-dimensional simplicial complex.
(iii) \(f\) is the \(f\)-vector of a shifted \((d - 1)\)-dimensional simplicial complex.
(iv) \(f \in \mathbb{P}^d\) and \(\alpha_d \geq \cdots \geq \alpha_1\), where each \(\alpha_k\) is the generalized \(k\)-Macaulay representation of \(f_{k-1}\).
(v) \(0 < \partial_{k+1}(f_k) \leq f_{k-1}\) for every \(k \in [d - 1]\).

As for the case \(a = 1_n\), Theorem 1.2 yields the following corollary. In particular, recall that all simplicial complexes in this paper are assumed to have non-empty vertex sets.

**Corollary 5.30.** Let \(n \in \mathbb{P}\), let \(f = \{f_S\}_{S \subseteq [n]}\) be an array of integers, and define \(\phi : 2^{[n]} \rightarrow \{0, 1\}^n\) by \(S \mapsto (s_1, \ldots, s_n)\), where \(s_i = 1\) if and only if \(i \in S\). Then the following are equivalent:
(i) $f$ is the flag $f$-vector of a $1_n$-colored complex.
(ii) $f$ is the flag $f$-vector of a color-compressed $1_n$-colored complex.
(iii) $f$ is the flag $f$-vector of a color-shifted $1_n$-colored complex.

(iv) $f_\emptyset = 1, f_{\{i\}} > 0$ for all $i \in [n]$, and there exists an array $\{\alpha_S\}_{S \subseteq [n], S \neq \emptyset}$ such that each $\alpha_S$ is a generalized $\phi(S)$-Macaulay representation of $f_S$, and $\alpha_S \preceq \alpha_{S'}$ for all non-empty $S, S' \subseteq [n]$ satisfying $|S| = |S'| + 1$.

By Theorem [1.1] the flag $h$-vector of a $(d-1)$-dimensional completely balanced CM complex is the flag $f$-vector of a $1_d$-colored complex, hence by Corollary [5.30] we get the following different but equivalent numerical characterization of completely balanced CM complexes.

**Corollary 5.31.** Let $d \in \mathbb{P}$, let $h = \{h_S\}_{S \subseteq [d]}$ be an array of integers, and define $\phi : 2^{[d]} \to \{0, 1\}^d$ by $S \mapsto (s_1, \ldots, s_d)$, where $s_i = 1$ if and only if $i \in S$. Then the following are equivalent:

(i) $h$ is the flag $h$-vector of a $(d-1)$-dimensional completely balanced CM complex.
(ii) $h_\emptyset = 1, h_{\{i\}} > 0$ for all $i \in [d]$, and there exists an array $\{\alpha_S\}_{S \subseteq [d], S \neq \emptyset}$ such that each $\alpha_S$ is a generalized $\phi(S)$-Macaulay representation of $h_S$, and $\alpha_S \preceq \alpha_{S'}$ for all non-empty $S, S' \subseteq [d]$ satisfying $|S| = |S'| + 1$.

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