RIEMANNIAN TRUST-REGION BASED ADAPTIVE KALMAN FILTER WITH UNKNOWN NOISE COVARIANCE MATRICES

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ABSTRACT

The problem of adaptive Kalman filtering for a discrete observable linear time-varying system with unknown noise covariance matrices is addressed in this paper. The measurement difference autocovariance method is used to formulate a linear least squares cost function containing the measurements and the process and measurement noise covariance matrices. Subsequently, a Riemannian trust-region optimization approach is designed to minimize the least squares cost function and ensure symmetry and positive definiteness for the estimates of the noise covariance matrices. The noise covariance matrix estimates, under sufficient excitation of the system, are shown to converge to their unknown true values. Saliently, the exponential stability and convergence guarantees for the proposed adaptive Kalman filter to the optimal Kalman filter with known noise covariance matrices is shown to be achieved under the relatively mild assumptions of uniform observability and uniform controllability. Numerical simulations on a linear time-varying system demonstrate the effectiveness of the proposed adaptive filtering algorithm.

Keywords Adaptive Kalman filtering · Manifold optimization · Riemannian Trust-Region method · Covariance Estimation

1 Introduction

The Kalman filter for linear systems plays a foundational role for modern statistical estimation theory [1,2]. One of the important assumptions that allows for a reliable state estimation algorithm is that the covariance matrices (CM) for the process and measurement noise signals entering the system are perfectly known. This assumption is rarely true in practice given the difficulty associated with obtaining perfect models and characterization for noise parameters. Traditionally, the CMs are either predetermined through extensive experimentation or are artificially inflated to adopt a conservative strategy. The case when inaccurate noise covariances are used is known to cause filter divergence [3,4,5,6,7,8]. These challenges motivate adaptive algorithms for state estimation while simultaneously estimating the noise CMs.

Over the years, the noise CM estimation methods can be broadly classified into Bayesian methods [9,10,11], maximum likelihood methods [12,13,14], correlation methods [15,16,17,18,19], covariance matching methods [20,21,22], subspace methods [23], and predictor error methods [24] to name a few. A recent survey describes and compares noise CM estimation methods [25]. Of all these various existing solutions, the correlation methods have received significant attention because they require the least a priori information, have less computational requirements, are known to produce unbiased estimates, and in some formulations are independent of the state estimates. For that reason, correlation methods have also been termed as feedback-free in Ref. [25].

Among the various implementations of correlation methods, the autocovariance least squares (ALS) approach evaluates the autocovariance of the innovation sequence which is linearly dependent on the noise CM [18,19]. The Measurement Autocovariance Method (MACM) method similarly builds a linear equation in the noise CMs by evaluating the
correlation of a modified measurement model formed by stacking measurements in time\cite{26}. This method was also shown to estimate the cross-covariance of the noises\cite{27}. Recently, the noise CM estimates found using a measurement difference autocovariance (MDA) approach were shown to converge to their true values in the mean squares sense for linear time-varying (LTV) systems with mild requirements of observability of the system\cite{28}. The noise CM estimation technique developed in this paper introduces certain judiciously determined modifications to the MDA approach to ensure a stable adaptive Kalman filter (AKF) formulation.

In spite of notable advances in the estimation of noise CM, the adaptive filtering problem that uses the noise CM estimates to estimate the states has not been fully addressed. Our previous work derived a convergent AKF for detectable linear time-invariant (LTI) systems\cite{29}. The convergence of the noise covariance estimates and the state error covariance were established based on the full rank condition of the coefficients of the noise CMs in the linear equation obtained from the MACM method. Although the MDA method\cite{28} produced convergent noise CM estimates for LTV systems, the stability of the AKF formulated using these noise CM estimates was not analyzed. It needs to be noted that LTV systems represent a larger class of systems that are ubiquitous in engineering applications. Additionally, nonlinear systems with and extended Kalman filter formulation are also represented as linear time-varying systems. As a result, an exponentially stable AKF formulation for LTV systems with unknown noise CM has potential for positively impacting many applications and is the topic of this paper.

The stability of the AKF is a non-trivial problem. In most correlation methods, a least squares formulation estimates the noise CMs by vectorizing the linear equation formed using the MACM, ALS, and MDA approaches. A recursive version of the least squares problem allows for estimating the noise CMs on line. In this setting, although the estimates of the noise CM are guaranteed to be symmetric at all times, there are no assurances on their positive definiteness. In fact, during the transients, the noise CM can sometimes violate positive definiteness which can potentially lead to inconsistencies in the AKF. In order to circumvent this issue, Ref.\cite{29} adopts a convenient remedy, that is to revert back the CM estimate to the most recent (prior) value when it was symmetric and positive definite (SPD). In spite of the poor transient performance, the overall convergence result still holds with this heuristic in place as the estimates are proved to be SPD in the limit with probability one. However, this motivates the question whether or not the adaptive estimator can be modified to guarantee the SPD property for the CM estimates at all times.

In this paper, we provide a positive answer to the foregoing question by adopting a differential geometric approach that ensures SPD noise CM estimates. Geometric optimization methods have gained popularity in the past decade because firstly, the conformity of the optimized values to the geometry of the set over which they are optimized is guaranteed and secondly, convergence to the optimal value is shown to be faster as compared to their Euclidean counterparts\cite{30,31}. Specifically, the Riemannian trust-region optimization methods provide superior convergence assurances compared to most other geometric optimization techniques that have been used in various aspects of identification theory\cite{32,33,34}.

The set of SPD matrices form a Riemannian manifold when endowed with an appropriate metric. Given a cost function, the Riemannian versions of the gradient and Hessian enable geometric optimization techniques to be applied that ensure SPD matrix estimates.

In this paper, a Riemannian trust-region (RTR) optimization based adaptive Kalman filter is introduced that minimizes the recursive least squares cost function and estimates the unknown noise CMs while simultaneously estimating the states. Exponential stability of the adaptive Kalman filter has been established under the uniform observability and uniform controllability assumptions of the system regardless of whether or not the noise CM estimates converge. To the best of our knowledge, such a stable formulation of an adaptive Kalman filter has never been proved. Under the persistence of excitation condition, the RTR based noise CM estimates are shown to converge in probability to its true value. Under this condition, the state error covariance sequence of the adaptive Kalman filter is also shown to converge to the classical non-adaptive Kalman filter employing known values of the noise CMs.

The paper is organized as follows. Section 2 presents the adaptive Kalman filter formulation using the MDA method. The RTR-based noise CM estimation method is described in Section 3. Convergence of the RTR-based noise CM estimates to their true values and the stability of the adaptive Kalman filter formulated using the RTR-based noise CM estimates is discussed in Section 4. Section 5 presents numerical simulations that demonstrate the effectiveness of the RTR-based adaptive Kalman filter (RTRAKF). Finally, the paper is concluded in Section 6 and future work is discussed.

Notation: The set of symmetric, and symmetric and positive definite matrices are denoted by $S_n$, and $\mathbb{P}_n$ respectively. The identity matrix of size $n$ is denoted by $I_n$. The operator $\text{vech}(\cdot): S_n \rightarrow \mathbb{R}^{n(n+1)/2}$ returns a vector with the unique elements of a symmetric matrix and the operator $\text{vec}(\cdot): \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n^2}$ returns a vector of all the elements of a matrix. The Kronecker product, denoted by $\otimes$, operates on a matrix product as $\text{vec}(AXB) = (B^T \otimes A)\text{vec}(X)$. A modified Kronecker product taking one sided symmetry into account for $X \in S_n$ is given by $\text{vec}(AXB) = (B^T \otimes_u A)\text{vech}(X)$. Another modified Kronecker product taking two-sided symmetry into account for $X \in S_n$ is given by $\text{vech}(AXA^T) = (A \otimes_k A)\text{vech}(X)$. The operator $\text{Exp}$ denotes the matrix exponential and $\text{Tr}(\cdot)$ denotes the Trace operation on a square matrix. The Block Trace operation, denoted by $B\text{Tr}(\cdot)$ is defined as $\text{Tr}(A(I_n \otimes B)) = \text{Tr}(B\text{Tr}(A)B)$ wherein
Assumption 3. Let the pair \( (w_k) \) wherein the process noise \( A \in \mathbb{R}^{n_2 \times n_2} \) and \( B \in \mathbb{R}^{n \times n} \). The symbol \( k \to \infty \) represents the convergence as \( k \to \infty \), and \( \frac{k}{\to} \) denotes the convergence in probability as \( k \to \infty \). The notation \( \Pr\{\cdot\} \) denotes the probability of occurrence of an event.

2 Adaptive Filter Formulation

The basic structure of the adaptive filter formulated in this section follows from correlation based techniques [29, 28, 27].

2.1 Problem Formulation

A discrete linear time-varying (LTV) system is considered here with the system equations given by

\[
\begin{align*}
x_{k+1} &= F_k x_k + G_k u_k + w_k \\
y_k &= H_k x_k + v_k
\end{align*}
\]

wherein the process noise \( w_k \sim \mathcal{N}(0_{n \times 1}, Q) \) and the measurement noise \( v_k \sim \mathcal{N}(0_{p \times 1}, R) \) are uncorrelated white Gaussian noises with constant noise covariance matrices. Let \( \phi_{i,k} \) be the associated state transition matrix such that \( \phi_{k+1,k} = F_k \) and \( \phi_{k,t} = \phi_{k,q} \phi_{q,t} \) for any \( q \in \mathbb{Z} \). No restrictions are made for the matrices \( F_k, G_k \) and \( H_k \) except for uniform observability of pair \((F_k, H_k)\) and uniform controllability of the \((F_k, Q_k^{1/2})\) pair. To that end, the definition for uniform observability and uniform controllability is given by [36, Chapter 7.5]

**Definition 1.** The pair \((F_k, H_k)\) is uniformly observable if there exists an integer \( s \geq 0 \) and constants \( 0 < \alpha_1 < \alpha_2 \) such that

\[ \alpha_1 I \preceq M_{k+s,k} \preceq \alpha_2 I \]  

(2)

wherein,

\[ M_{k+s,k} = \sum_{i=k}^{k+s} \phi_{i,k}^T H_i^T H_i \phi_{i,k} \]  

(3)

and let \( M_{k+1,k} \) for \( l < s \) be the partial observability Gramian.

**Definition 2.** The pair \((F_k, E_k)\) is uniformly controllable if there exists an integer \( s \geq 0 \) and constants \( 0 < \beta_1 < \beta_2 \) such that

\[ \beta_1 I \preceq Y_{k+s,k} \preceq \beta_2 I \]  

(4)

wherein,

\[ Y_{k+s,k} = \sum_{i=k}^{k+s} \phi_{k+s+1,i+1} E_i^T E_i \phi_{k+s+1,i+1} \]  

(5)

Assumption 3. Let the pair \((F_k, H_k)\) be uniformly observable and the pair \((F_k, Q_k^{1/2})\) be uniformly controllable.

The above assumption ensures the Kalman filter is exponentially stable [36, Theorem 5.3]. Subsequently, the following assumption is made on the \( Q \) and \( R \) noise covariance matrices.

**Assumption 4.** The noise covariance matrices \( Q \) and \( R \) are both assumed to be constant and unknown.

The aim of this paper is to estimate the unknown \( Q \) and \( R \) matrices while simultaneously estimating the states.

2.2 Measurement Difference Autocovariance approach

Since the pair \((F_k, H_k)\) is uniformly controllable with constants \( s \geq 0 \) and \( 0 < \alpha_1 < \alpha_2 \), consider \( m \geq s \) measurements that are aggregated in time to form a linear time series. Such a development is described in the following result.

**Proposition 5.** For the LTV system given by Eq. (1) with the Assumptions and a \( m \geq s \) from Definition, the measurements \( y_k \) of the system follow a linear time series given by

\[ \sum_{i=0}^{m} A_i^k y_{k-i} - \sum_{i=1}^{m} B_i^k G_{k-i} u_{k-i} = \sum_{i=1}^{m} B_i^k w_{k-i} + \sum_{i=0}^{m} A_i^k v_{k-i} \]  

(6)

wherein, the coefficients \( A_i^k \) and \( B_i^k \) are completely determined from the system matrices \( F_k \) and \( H_k \).
wherein,

time series given by

two equations given by

The proof follows from much of the past work on the measurement difference methods \[28, 29\]. We begin

Proof. The proof follows from much of the past work on the measurement difference methods \[28, 29\]. We begin by accumulating measurements by stacking them one on top of the other to form a modified measurement model given in Eq. \(7\). Defining \(W_{k-1,k-m+1} = \begin{bmatrix} w_{k-1}^T & \cdots & w_{k-m+1}^T \end{bmatrix}^T\) and pre multiplying by \(O_{k,k-m+1}\), the invertible observability Gramian \(M_{k,k-m+1}\) defined in Eq. \(6\) is recovered as shown below.

\[
O_{k,k-m+1}^T Y_{k,k-m+1} = M_{k,k-m+1} x_{k-m+1} + \sum_{i=1}^{m} M_{k,m}^{-1} O_{k,k-m+1}^T U_{k-1,k-m+1} + O_{k,k-m+1}^T V_{k,k-m+1}
\]

Inverting \(M_{k,k-m+1}\) and using a one step predictor for the state \(x_{k-m+1}\), a linear time series can be formed from the two equations given by

\[
M_{k,k-m+1}^{-1} O_{k,k-m}^T Y_{k,k-m} = x_{k-m+1} + M_{k,k-m+1}^{-1} O_{k,k-m+1}^T U_{k-1,k-m+1} + M_{k,k-m+1}^{-1} O_{k,k-m+1}^T V_{k,k-m+1}
\]

\[
M_{k-1,k-m}^{-1} O_{k-1,k-m}^T Y_{k-1,k-m} = x_{k-m} + M_{k-1,k-m}^{-1} O_{k-1,k-m}^T U_{k-2,k-m} + M_{k-1,k-m}^{-1} O_{k-1,k-m}^T V_{k-1,k-m}
\]

Substituting \(x_{k-m+1} = F_{k-m} x_{k-m} + G_{k-m} u_{k-m} + w_{k-m}\) and eliminating the state by subtraction, we get a linear time series given by

\[
A_k Y_{k,k-m} = B_k U_{k-1,k-m} + A_k V_{k,k-m}
\]

wherein,

\[
A_k = [M_{k,k-m+1}^{-1} O_{k,k-m+1}^T, 0_{n \times p}] - [0_{n \times p}, F_{k-m} M_{k-1,k-m}^{-1} O_{k-1,k-m}^T]
\]

and

\[
B_k = [M_{k,k-m+1}^{-1} O_{k,k-m+1}^T M_{k-1,k-m+1}, I_{n \times n}] - [0_{n \times n}, F_{k-m} M_{k-1,k-m}^{-1} O_{k-1,k-m}^T M_{k-2,k-m}^{-1}]
\]

Separating out individual coefficients of the coefficients of \(y_k\)

\[
A_0 = M_{k,k-m+1}^{-1} \phi_{k,k-m+1}^T H_k^T
\]

\[
A_i = M_{k,k-m+1}^{-1} \phi_{k-i,k-m+1}^T H_k^T - F_{k-m} M_{k-1,k-m}^{-1} F_{k-m}^T \phi_{k-i,k-m+1}^T H_k^T
\]

\[
A_m = -F_{k-m} M_{k-1,k-m}^{-1} F_{k-m}^T H_{k-m}
\]

wherein, \(i = 1, \ldots, m - 1\) above and the coefficients of \(w_k\) is given by

\[
B_1 = M_{k,k-m+1}^{-1} \phi_{k,k-m+1} M_{k,k}
\]

\[
B_i = M_{k,k-m+1}^{-1} \phi_{k-i+1,k-m+1} M_{k,k} - F_{k-m} M_{k-1,k-m}^{-1} F_{k-m}^T \phi_{k-i+1,k-m+1}^T M_{k-1,k-m+1}
\]

\[
B_m = I_{n \times n} - F_{k-m} M_{k-1,k-m}^{-1} F_{k-m}^T M_{k-1,k-m+1}
\]

wherein, \(i = 2, \ldots, m - 1\) above. The statement of the proposition follows.
Defining $Z_k$ as the left hand side of Eq. (6), the autocovariance function of $Z_k$ is given by

$$C_{k,k-p} = E[Z_kZ_{k-p}^T] = \sum_{i=p+1}^{m} B_i^k Q B_{i-p}^k + \sum_{i=p}^{m} A_i^k R A_{i-p}^k$$

(11)

wherein $p = 0, \ldots, m$. Notice that the autocovariance $C_{k,k-p} = 0_{n\times n}$ for $p > m$ vanishes. As long as the number of past measurements $y_k$ stored at every time instant is greater than $m + 1$, the autocovariance function can be estimated.

### 2.3 Covariance Matrix Estimation

The autocovariance is estimated using a single measurement as $\hat{C}_{k,k-p} = \hat{Z}_k\hat{Z}_{k-p}^T$. The elements of the autocovariance function can be rearranged using the $\text{vech}(\cdot)$ operation as follows.

$$\begin{bmatrix}
\text{vech}(\hat{C}_{k,k}) \\
\text{vec}(\hat{C}_{k,k-1}) \\
\vdots \\
\text{vec}(\hat{C}_{k,k-p})
\end{bmatrix} =
\begin{bmatrix}
\sum_{i=1}^{m} B_i^k \otimes_h B_i^k & \sum_{i=0}^{m} A_i^k \otimes_h A_i^k \\
\sum_{i=2}^{m} B_{i-1}^k \otimes_u B_i^k & \sum_{i=1}^{m} A_{i-1}^k \otimes_u A_i^k \\
\vdots & \vdots \\
\sum_{i=P+1}^{m} B_{i-p}^k \otimes_u B_i^k & \sum_{i=P}^{m} A_{i-p}^k \otimes_u A_i^k
\end{bmatrix}
\begin{bmatrix}
\text{vech}(\hat{Q}) \\
\text{vech}(\hat{R})
\end{bmatrix}$$

(12)

A recursive least squares (RLS) estimation technique starting from an initial guess $(\hat{\theta}_0, \Psi_0)$ is given by

$$\begin{align}
\hat{\theta}_{k+1} &= \hat{\theta}_k + L_k (b_k - D_k \hat{\theta}_k) \\
\Psi_{k+1} &= (I - L_k D_k) \Psi_k (I - L_k D_k)^T + L_k R W L_k^T \\
L_k &= \Psi_k D_k^T (R W + D_k \Psi_k D_k^T)^{-1}
\end{align}$$

(13)

The convergence of the estimate has been established in Ref. [28, Theorem 8].

### 2.4 Adaptive Kalman Filter

Using the estimates $\hat{Q}_k$ and $\hat{R}_k$ of the noise covariance matrices, the following equations constitute the adaptive Kalman filter equations.

$$\begin{align}
\hat{x}_{k|k-1} &= F_{k-1} \hat{x}_{k-1} + G_{k-1} u_{k-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + \hat{K}_k (y_k - H_k \hat{x}_{k|k-1}) \\
\hat{P}_{k|k-1} &= F_{k-1} \hat{P}_{k-1|k-1} F_{k-1}^T + \hat{Q}_k \\
\hat{K}_k &= \hat{P}_{k|k-1} H_k^T (H_k \hat{P}_{k|k-1} H_k^T + \hat{R}_k)^{-1} \\
\hat{P}_{k|k} &= (I - \hat{K}_k H_k) \hat{P}_{k|k-1} (I - \hat{K}_k H_k)^T + \hat{K}_k \hat{R}_k \hat{K}_k^T
\end{align}$$

(14)

wherein, $\hat{P}_{k|k}$, $\hat{P}_{k|k-1}$, and $\hat{K}_k$ are the estimates of the quantities in the nominal Kalman filter [35, Chapter 7].

### 3 Riemannian Trust-Region method

Although the recursive least squares successfully estimates the elements of the noise covariance matrices, it does not guarantee SPD estimates of the covariance matrix. The convergence of the estimates to the true covariance matrices is guaranteed provided the matrix $D_k$ is persistently excited. However, the transients are important when the covariance estimate is concurrently used to estimate the state vector. In this case, the filter may run into a problem of loss of observability or worse, provide negative information updates to the filter by virtue of a non positive definite noise covariance matrix estimate. As a result, having a SPD noise covariance matrix estimate is crucial to obtain a stable adaptive Kalman filter. To this end, a geometric optimization approach that respects the geometry of SPD matrices is introduced here. A brief summary of the geometry of SPD matrices is provided below (for a comprehensive review, see, e.g., [37] for SPD matrices and [30] for Riemannian optimization methods).
3.1 Geometry of Covariance Matrices

The space $\mathbb{P}_n$ forms a manifold with its tangent space at a point $X \in \mathbb{P}_n$ denoted by $T_X \mathbb{P}_n$ and identified with $\mathbb{S}_n$, the set of symmetric matrices. The affine invariant metric at $X \in \mathbb{P}_n$ defined by

$$\langle V_1, V_2 \rangle_X = \text{Tr}\{X^{-1}V_1X^{-1}V_2\} \quad V_1, V_2 \in T_X \mathbb{P}_n$$

(15)

turns the manifold into a Riemannian manifold. The shortest path on the manifold between two points $X, Y \in \mathbb{P}_n$ is called the geodesic curve and is parameterized as

$$\gamma(s) = X^{1/2}\left(X^{-1/2}YX^{-1/2}\right)^s X^{1/2} \quad s \in [0, 1]$$

(16)

wherein, $\gamma(0) = X$ and $\gamma(1) = Y$ denote the end points of the geodesic. A geodesic curve emanating from a point $X \in \mathbb{P}_n$ in the direction $V \in T_X \mathbb{P}_n$ is parameterized by

$$\gamma_{X,V}(s) = X^{1/2}\text{Exp}\left(sX^{-1/2}VX^{-1/2}\right)X^{1/2}$$

(17)

and resides within $\mathbb{P}_n$ for any $s \in \mathbb{R}$. Given a smooth function $f : \mathbb{P}_n \rightarrow \mathbb{R}$, $\tilde{f}$ as the extension of $f$ to $\mathbb{R}^{n \times n}$, a smooth geodesic curve $\gamma : \mathbb{R} \rightarrow \mathbb{P}_n$ such that $\gamma(0) = X \in \mathbb{P}_n$ and $\gamma(0) = V \in T_X \mathbb{P}_n$, the Euclidean gradient $\nabla \tilde{f}$ defined using the directional derivative $D\tilde{f}(X)[V]$ of $\tilde{f}$ at $X$ in the direction $V$ is given as

$$\text{Tr}(V \nabla \tilde{f}(X)) = D\tilde{f}(X)[V]$$

(18)

The Riemannian gradient of $f$ at $X$, denoted by $\text{grad}\ f(X) \in T_X \mathbb{P}_n$ is similarly defined as

$$\langle V, \text{grad}\ f(X) \rangle_X = \frac{d}{dt} f(\gamma(t)) |_{t=0}$$

(19)

Note that the Riemannian gradient is obtained from the Euclidean gradient by

$$\text{grad}\ f(X) = X\text{sym}(\nabla \tilde{f}(X))X$$

(20)

From [38, Section 4.1.4], the Riemannian Hessian of $f$ defined as a map $\text{Hess} f(X) : T_X \mathbb{P}_n \rightarrow T_X \mathbb{P}_n$ is given by

$$\text{Hess} f(X)[V] = D(\text{grad} f(X))[V] - \text{sym}(\text{grad} f(X)X^{-1}V)$$

(21)

Using the above expressions for $\text{grad} f$, the Hessian can be expressed in terms of the extension $\tilde{f}$ as

$$\text{Hess} f(X)[V] = X\text{sym}(D(\nabla \tilde{f})(X)[V])X + \text{sym}(V\text{sym}(\nabla \tilde{f})X)$$

(22)

3.2 Cost function, Gradient and Hessian

The cost function for the recursive least squares minimization from Eq. (13) is minimized with a Riemannian optimization framework. The recursive least squares cost function is given by

$$J_k(\theta) = \frac{1}{2}(D_k\theta - b_k)^TR_k^{-1}(D_k\theta - b_k) + \frac{1}{2}(\theta - \hat{\theta}_{k-1})^T\Psi_{k-1}^{-1}(\theta - \hat{\theta}_{k-1})$$

(23)

Before evaluating the Riemannian gradient and the Riemannian Hessian, the cost function must be reformatted to explicitly depend on $Q$ and $R$ matrices. Such reformatting is possible via simple algebraic manipulation. The unique elements of a SPD matrix are given by

$$\text{vech}(X) = \begin{bmatrix} I_0^0 X e_i \\ \vdots \\ I_n^{n-1} X e_n \\ \hat{b}X_n \in \mathbb{R}^{(n+1)/2 \times n^2} \end{bmatrix} = \begin{bmatrix} I_0^0 & 0 & \cdots & 0 \\ 0 & I_1^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n^{n-1} \end{bmatrix} (I_n \otimes X)\text{vec}(I_n)$$

(24)

wherein, $X \in \mathbb{S}_n$, $e_i \in \mathbb{R}^n$ is the $i^{th}$ canonical basis vector and $I_i \in \mathbb{R}^{(n-i) \times n}$ is formed by deleting the first $i$ rows of $I_n$, the identity matrix. The following statement provides the expressions for the gradient of the least squares cost function.
Lemma 6. Given the cost function in Eq. (25), its Riemannian gradients at $Q$ and $R$ are given by
\[
\nabla_Q \J_k(Q, R) = (Q \nabla_Q \J_k Q, R \nabla_R \J_k R)
\] (25)
wherein, $\J_k$ is the Euclidean extension of the cost $J_k$ and the expressions for $\nabla_Q \J_k$ and $\nabla_R \J_k$ are given by
\[
\nabla_Q \J_k = B \text{Tr}_n \left\{ \text{sym} \left( \begin{bmatrix} \text{vec}(I_n) \left( D_k^Q R_k W^{-1}(D_k \theta - b_k) + [I_m, 0_{m \times m_r}, \Psi^{-1}_{k-1}(\theta - \hat{\theta}_{k-1})]^T \right) \end{bmatrix} \right) \right\}
\] (26)
\[
\nabla_R \J_k = B \text{Tr}_p \left\{ \text{sym} \left( \begin{bmatrix} \text{vec}(I_p) \left( D_k^R R_k W^{-1}(D_k \theta - b_k) + [0_{m_r \times m}, I_m, \Psi^{-1}_{k-1}(\theta - \hat{\theta}_{k-1})]^T \right) \end{bmatrix} \right) \right\}
\] (27)
wherein, $D_k = [D_k^Q, D_k^R], \theta = [\text{vech}(Q)^T, \text{vech}(R)^T]^T, m_q = n(n+1)/2$ and $m_r = p(p+1)/2$.

Proof. Consider a geodesic $\gamma_{Q,V}(t)$ as defined in Eq. (17). From the definition of the gradient in Eq. (13), the expression for $\nabla_Q \J_k$ is given by
\[
D\J_k(Q, R)[V_Q, V_R] = \frac{\partial \J_k(\gamma_{Q,V}(s), \gamma_{R,V}(s))}{\partial s}\bigg|_{s=0} = \text{Tr}\{\nabla_Q \J_k V_Q\} + \text{Tr}\{\nabla_R \J_k V_R\}
\]
Separating the expressions into the parts containing $Q$ and $R$, we get
\[
\text{Tr}\{\nabla_Q \J_k V_Q\} = (D_k \theta - b_k)^T R_k W^{-1} \left( D_k^Q \text{Tr}_n I_n \otimes V_Q \text{vec}(I_n) \right) + (\theta - \hat{\theta}_{k-1})^T \Psi^{-1}_{k-1}[I_m, 0_{m \times m_r}]^T \text{Tr}_n I_n \otimes V_Q \text{vec}(I_n)
\]
\[
= \text{Tr}\{\text{vec}(I_n) \left( D_k^Q R_k W^{-1}(D_k \theta - b_k) + [I_m, 0_{m \times m_r}, \Psi^{-1}_{k-1}(\theta - \hat{\theta}_{k-1})]^T \right) \text{Tr}_n I_n \otimes V_Q\}
\]
Further simplification results in an expression given by
\[
\text{Tr}\{\nabla_Q \J_k V_Q\} = B \text{Tr}_n \left\{ \text{vec}(I_n) \left( D_k^Q R_k W^{-1}(D_k \theta - b_k) + [I_m, 0_{m \times m_r}, \Psi^{-1}_{k-1}(\theta - \hat{\theta}_{k-1})]^T \right) \text{Tr}_n I_n \otimes V_Q\right\}
\]
wherein, the expression $\text{Tr}\{A(I_n \otimes B)\} = \text{Tr}\{B \text{Tr}_n \{A\} B\}$ is used. Comparing the expressions on both sides of the equations gives the result of the lemma. The expression for $\nabla_R \J_k$ results from a derivation similar to the one above and is omitted.

The expression for the Riemannian Hessian of $J_k$ can be derived from Eq. (22) and is given through the following statement.

Lemma 7. The expression for the Riemannian Hessian of $J_k$ from Eq. (22) is given by
\[
\text{Hess} \J_k(Q, R)[V_Q, V_R] = \left(\text{Hess}_Q \J_k(Q, R)[V_Q, V_R], \text{Hess}_R \J_k(Q, R)[V_Q, V_R]\right)
\] (28)
wherein
\[
\text{Hess}_Q \J_k(Q, R)[V_Q, V_R] = Q \text{sym}(D(\nabla_Q \J_k)(Q, R)[V_Q, V_R])Q + \text{sym}(V_Q \text{sym}(\nabla_Q \J_k)Q)
\]
\[
\text{Hess}_R \J_k(Q, R)[V_Q, V_R] = R \text{sym}(D(\nabla_R \J_k)(Q, R)[V_Q, V_R])R + \text{sym}(V_R \text{sym}(\nabla_R \J_k)R)
\]
The expressions for the directional derivatives of the Euclidean gradients are given by
\[
D(\nabla_Q \J_k)(Q, R)[V_Q, V_R] = B \text{Tr}_n \left\{ \text{sym} \left( \begin{bmatrix} \text{vec}(I_n) \left( D_k^Q R_k W^{-1} D_k \theta - b_k \right) + [I_m, 0_{m \times m_r}, \Psi^{-1}_{k-1}(\theta - \hat{\theta}_{k-1})]^T \right) \text{Tr}_n I_n \otimes V_Q\right\}
\] (29)
and
\[
D(\nabla_R \J_k)(Q, R)[V_Q, V_R] = B \text{Tr}_p \left\{ \text{sym} \left( \begin{bmatrix} \text{vec}(I_p) \left( D_k^R R_k W^{-1} D_k \theta - b_k \right) + [0_{m_r \times m}, I_m, \Psi^{-1}_{k-1}(\theta - \hat{\theta}_{k-1})]^T \right) \text{Tr}_p I_p\right\}
\] (30)
wherein, $\theta_v = [\text{vech}(Q)^T, \text{vech}(R)^T]^T$.

Proof. The directional derivative of $\nabla_Q \J_k(Q, R)[V_Q, V_R]$ is obtained as
\[
D(\nabla_Q \J_k)(Q, R)[V_Q, V_R] = \frac{\partial \nabla_Q \J_k(\gamma_{Q,V}(s), \gamma_{R,V}(s))}{\partial s}\bigg|_{s=0}
\] (31)
Since the gradient is affine in $Q$ and $R$, the directional derivative is independent of the points $Q$ and $R$ where it is evaluated. The expression is obtained trivially by substituting $V_Q$ and $V_R$ in place of $Q$ and $R$ and removing the constant terms.
3.3 Riemannian Trust-Region Method

The Riemannian trust-region (RTR) method is used to solve the quadratic least squares cost function in Eq. (23). At each step the RTR method performs an inner iteration that minimizes a quadratic approximation of a cost function at \( Q, R \) given by

\[
m_{(Q,R)}(V_Q, V_R) = J_k(Q, R) + \langle \text{grad} J_k(Q, R), (V_Q, V_R) \rangle_{(Q,R)} + \frac{1}{2} \langle \text{Hess} J_k(Q, R) | (V_Q, V_R), (V_Q, V_R) \rangle_{(Q,R)}
\]

The optimal \( V_Q^* \in T_Q \mathbb{P}_n \) and \( V_R^* \in T_R \mathbb{P}_p \) are obtained subject to a norm constraint on step size given by

\[
\| (V_Q, V_R) \|_{(Q,R)} = \sqrt{\langle (V_Q, V_R), (V_Q, V_R) \rangle_{(Q,R)}} \leq \Delta
\]

wherein, \( \Delta \) is the trust-region radius. A truncated conjugate gradient (tCG) method [30, Algorithm 11] solves the inner iteration at each step. Then a verification step evaluates the decrease in the true and approximate cost function given by the ratio

\[
\rho = \frac{J_k(Q, R) - J_k(\gamma Q, V_Q^*(1), \gamma R, V_R^*(1))}{\tilde{m}_{(Q,R)}(0_{n \times n}, 0_{p \times p}) - \tilde{m}_{(Q,R)}(V_Q^*, V_R^*)}
\]

and decide whether the optimal \( (V_Q^*, V_R^*) \) are accepted and whether the radius \( \Delta \) should be decreased. Algorithm [I] describes the RTR algorithm. The constants used are taken from [30, Algorithm 10].

**Algorithm 1** Riemannian Trust-Region Method

| Input: \( Q_{k-1}, R_{k-1}, \Psi_{k-1}, D_k, b_k, R_W, \Delta > 0, \Delta_1 \in (0, \bar{\Delta}), \rho \in [0, \frac{1}{2}) \) |
| Initialization: \( (Q_k)_1 = Q_{k-1}, (R_k)_1 = R_{k-1} \) |
| Output: \( Q_k, R_k \) |
| for \( k = 1 \rightarrow n \) do |
| Minimize \( \tilde{m}_{(Q,R)}(V_Q, V_R) \) subject to norm constraint with \( \Delta_i \) \( \triangleright \) Eq. (32) |
| if \( \rho_i < \frac{1}{4} \) then |
| \( \Delta_{i+1} = \frac{1}{2} \Delta_i \) |
| else if \( \rho_i > \frac{3}{4} \) and \( \| (V_Q^*)_i, (V_R^*)_i \| = \Delta_i \) then |
| \( \Delta_{i+1} = \min(2 \Delta_i, \Delta) \) |
| else |
| \( \Delta_{i+1} = \Delta_i \) |
| end if |
| if \( \rho_i > \rho_{\min} \) then |
| \( ((Q_k)_{i+1}, (R_k)_{i+1}) = (\gamma Q, V_Q^*(1), \gamma R, V_R^*(1)) \) |
| else |
| \( ((Q_k)_{i+1}, (R_k)_{i+1}) = ((Q_k)_i, (R_k)_i) \) |
| end if |
| end for |

**Remark 8.** The RTR algorithm ensures that the estimates are symmetric and positive definite. However, in practice, the SPD noise covariance estimates may be arbitrarily close to semidefiniteness. This may create numerical errors in the filter updates. To avoid this situation, the minimum eigenvalue of \( Q_k \) and \( R_k \) is lower bounded by a small positive constant \( \epsilon > 0 \). The modified optimization variable is given by

\[
Q_k = \epsilon I_n + \hat{Q}_k
\]

\[
R_k = \epsilon I_p + \hat{R}_k
\]

Such a modification ensures that the eigenvalues of the noise covariance estimates obtained by the RTR method are lower bounded by \( \epsilon \) instead of zero. Such a modification merely results in a shift of the origin and does not affect the RLS solution.

The algorithm for the RTR-based AKF is summarized below.

4 Stability Analysis

In this section, the main contributions of this paper, i.e., stability of the RTR-based covariance estimation scheme and the adaptive Kalman filter using the RTR-based covariance estimates is presented.
Algorithm 2 Riemannian Trust-Region based Adaptive Kalman Filter (RTRAKF)

Input: \( \hat{x}_0, Q_0, \hat{R}_0, \hat{P}_0, \Psi_0, m, P, y_i, i = 1, 2, \ldots \)

Output: \( \hat{Q}_k, \hat{R}_k, \hat{P}_k, \hat{x}_k \)

for \( k = 1 \rightarrow n \) do
  if \( i > m + P \) then
    \( \Rightarrow \) Eq. (12)
  end if
  Calculate \( D_k \) and \( b_k \)
  Calculate the Riemannian Gradient
  Calculate the Riemannian Hessian
  Use Algorithm 1 to obtain \( \hat{Q}_k \) and \( \hat{R}_k \)
  Update \( \hat{x}_k \) and \( \hat{P}_k \)
  \( \Rightarrow \) Eq. (14)
end for

4.1 Convergence of the noise covariance estimates

The RTR method, by design, ensures that \( \hat{Q}_k \) and \( \hat{R}_k \) are SPD. Starting from SPD initial guesses the following results establish the convergence of the RTR-based noise covariance estimators by comparing them to the RLS solution.

**Proposition 9.** Given that \( D_k \) is persistently excited, \( \Pr\{\hat{Q}_i^{RLS} \in \mathbb{P}_n, \hat{R}_i^{RLS} \in \mathbb{P}_p, \forall i > k\} \xrightarrow{k \to \infty} 1. \)

**Proof.** The convergence of the batch least squares estimate \( \hat{\theta}_k \) in the mean squared sense to the true value \( \theta^* \) was established given that the combined coefficient matrix \( D = [D_1^T, D_2^T, \ldots]^T \) is full column rank [28, Theorem 8]. Since, \( D_k \) is persistently excited, the full rank condition is automatically satisfied. Since, convergence in the mean squared sense implies convergence in probability, we know that \( \Pr\{\|\hat{\theta}_k - \theta^*\| > 0\} \to 0. \) Consequently, for any constant \( \delta > 0, \Pr\{\|\hat{\theta}_k - \theta^*\| < \delta\} \xrightarrow{k \to \infty} 1. \) We know that the true \( Q \) and \( R \) which are formed from the \( \theta^* \) elements are SPD. Hence, there exists a \( \delta \) such that \( \forall \hat{\theta}_k : \|\hat{\theta}_k - \theta^*\| < \delta, \hat{\theta}_k \) is such that the matrices \( \hat{Q}_k \) and \( \hat{R}_k \) formed by its elements are SPD. Picking such a \( \delta \) ensures that \( \Pr\{\hat{Q}_k^{RLS} \in \mathbb{P}_n, \hat{R}_k^{RLS} \in \mathbb{P}_p\} \xrightarrow{k \to \infty} 1 \) which in turn ensures the following statement.

\[ \Pr\{\exists i > k, \hat{Q}_i^{RLS} \notin \mathbb{P}_n, \hat{R}_i^{RLS} \notin \mathbb{P}_p\} \xrightarrow{k \to \infty} 0. \]

The negation of the above statement proves the statement of the proposition. \( \square \)

**Proposition 10.** Given \( \hat{Q}_k \in \mathbb{P}_n, \hat{R}_k \in \mathbb{P}_p, \Psi_k \in \mathbb{P}_{m+n+m}, \) and the one step RLS and RTR solutions denoted by \( (\hat{Q}_k^{RLS}, \hat{R}_k^{RLS}) \) and \( (\hat{Q}_k^{RTR}, \hat{R}_k^{RTR}) \) respectively, if \( \hat{Q}_k^{RLS} \in \mathbb{P}_n \) and \( \hat{R}_k^{RLS} \in \mathbb{P}_p \) then \( (\hat{Q}_{k+1}^{RTR}, \hat{R}_{k+1}^{RTR}) = (\hat{Q}_k^{RTR}, \hat{R}_k^{RTR}). \)

**Proof.** The cost function given in Eq. (23) is quadratic with a positive definite Euclidean Hessian and is hence convex in the argument \( \theta. \) As a result, the recursive least squares minimizer produces unique solutions up to the error due to the stopping criterion. Similarly, the choice of constants in Algorithm 1 and the usage of exact Hessian ensures that \( \lim_{k \to \infty} \text{grad} J_k = 0 \) for the RTR algorithm [30, Theorem 7.4.4]. Since, \( Q_k \in \mathbb{P}_n \) and \( R_k \in \mathbb{P}_p, \text{grad} J_k = 0 \implies (\nabla_{\hat{Q} J_k}, \nabla_{\hat{R} J_k}) = (0, 0). \) Hence, this solution exactly matches the solution from the RLS step up to the error induced due to the stopping criterion. The statement of the proposition follows. \( \square \)

**Theorem 11.** Given that \( D_k \) is persistently excited, the sequences \( \hat{Q}_k^{RTR} \) and \( \hat{R}_k^{RTR} \) found using the Algorithm 2 converge to their true values, \( Q \) and \( R \) respectively, in probability.

**Proof.** From Proposition 9 we know that \( \Pr\{\hat{Q}_i^{RLS} \in \mathbb{P}_n, \hat{R}_i^{RLS} \in \mathbb{P}_p, \forall i > k\} \xrightarrow{k \to \infty} 1. \) Hence, from Proposition 10 we get that \( \Pr\{(\hat{Q}_k^{RLS}, \hat{R}_k^{RLS}) \neq (\hat{Q}_k^{RTR}, \hat{R}_k^{RTR})\} \xrightarrow{k \to \infty} 0. \) Since the RLS solution converges to the true value in the mean squares sense and the RTR solution matches the least squares solution with probability 1 as \( k \to \infty, \) the RTR solution converges in probability to \( (Q, R). \) \( \square \)
4.2 Convergence of the state error covariance matrix

The stability properties of the adaptive Kalman filter using the RTR-based noise covariance estimates are established through the following statement.

Proposition 12. Given the noise covariance estimates \( \hat{Q}_k \) and \( \hat{R}_k \) from the RTR Algorithm described in Algorithm 7 the adaptive Kalman filter from Eq. (14) is exponentially stable.

Proof. The proof follows from [36, Theorem 5.3] and Definitions 1 and 2. The observability Gramian for the pair \((F_k, \hat{R}_k^{-1/2} H_k)\) corresponding to the adaptive Kalman filter is given by

\[
\hat{M}_{k+m, k} = \sum_{i=k}^{k+m} \phi_{i,k}^T H_i^T \hat{R}_i^{-1} H_i \phi_{i,k}
\]

Since, \( \hat{R}_k \succ 0 \) and the observability Gramian \( \hat{M}_{k+m, k} \) for the known case is SPD, the observability Gramian \( \hat{M}_{k+m, k} \) for the adaptive Kalman filter using the RTR noise covariance matrix estimates is also SPD. Hence, the pair \((F_k, \hat{R}_k^{-1/2} H_k)\) is uniformly observable. Given that \( \hat{Y}_{k+s,k} \succ 0 \), the controllability Gramian for the pair \((F_k, \hat{Q}_k^{1/2})\) corresponding to the adaptive Kalman filter for the same \( s > 0 \) is given by

\[
\hat{Y}_{k+s,k} = \sum_{i=k}^{k+s} \phi_{i,k} \hat{Q}_i \phi_{i,k}^T
\]

Since \( \hat{Q}_k \succ 0 \), the \( \hat{Y}_{k+s,k} \) is non-singular and the pair \((F_k, \hat{Q}_k^{1/2})\) is uniformly controllable. Hence from [36 Theorem 5.3], the adaptive Kalman filter is exponentially stable.

The following statement establishes the convergence of the state error covariance matrix of the adaptive Kalman filter.

Theorem 13. The state error covariance matrix sequence \( \hat{P}_k \) of the adaptive Kalman filter converges to the state error covariance matrix \( ^oP_k \) of the optimal Kalman filter with probability 1.

Proof. Consider three covariance sequences \( \hat{P}_k \), \( P_k \) and \( ^oP_k \) given by [29]

\[
\hat{P}_{k+1} = \hat{F}_k \hat{P}_k \hat{F}_k^T + \hat{K}_k \hat{R}_k \hat{K}_k^T + \hat{Q}_k \\
P_{k+1} = \hat{F}_k P_k \hat{F}_k^T + \hat{K}_k P_k \hat{K}_k^T + Q \\
^oP_{k+1} = \hat{F}_k \hat{P}_k \hat{F}_k^T + \hat{K}_k P_k \hat{K}_k^T + Q
\]

wherein, \( \hat{F}_k = F_k - \hat{K}_k H_k \), \( \hat{F}_k = F_k - \hat{K}_k H_k \),

\[
\hat{K}_k = F_k \hat{P}_k H_k^T (H_k \hat{P}_k H_k^T + \hat{R}_k)^{-1} \\
\hat{K}_k = F_k P_k H_k^T (H_k P_k H_k^T + R)^{-1}
\]

Each of the three sequences denotes the one-step predictor state covariance matrix. The sequence \( \hat{P}_k \) denotes the apparent state error covariance matrix of the adaptive Kalman filter and uses the noise covariance matrix estimates for its propagation. The sequence \( P_k \) denotes the actual state error covariance matrix of the adaptive Kalman filter and uses the Kalman gain from the apparent covariance sequence along with the true noise covariance matrices. The sequence \( ^oP_k \) denotes the optimal state error covariance which represents the case when \( Q \) and \( R \) are fully known. We will first prove the equivalence of \( \hat{P}_k \) and \( P_k \) in the limit. Assuming the same error covariance at the initial time, the sequence formed by differencing \( \hat{P}_k \) and \( P_k \) is given by

\[
\hat{P}_{k+1} - P_{k+1} = (F_k - \hat{K}_k H_k)(\hat{P}_k - P_k)(F_k - \hat{K}_k H_k)^T + \hat{K}_k(\hat{R}_k - R) \hat{K}_k^T + (\hat{Q}_k - Q)
\]

Since the RTR method ensures that \( \hat{Q}_k \) and \( \hat{R}_k \) are SPD, we conclude that \( F_k - \hat{K}_k H_k \) is exponentially stable from Proposition 12. Additionally, from Theorem 11 both \( \hat{Q}_k \) and \( \hat{R}_k \) converge in probability to \( Q \) and \( R \) respectively as \( k \to \infty \). Hence, the exponentially stable matrix sequence converges to zero in probability, i.e., \( \hat{P}_k \xrightarrow{\text{p}} P_k \). From the expression for \( \hat{K}_k \), since \( \hat{P}_k \xrightarrow{\text{p}} P_k \) and \( \hat{R}_k \xrightarrow{\text{p}} R \), we get

\[
\hat{K}_k \xrightarrow{\text{p}} F_k P_k H_k^T (H_k P_k H_k^T + R)^{-1}
\]
as \( k \to \infty \). Hence, the matrix sequence for \( P_k \) and \( \overset{\text{c}}{P}_k \) is identical in the limit as \( k \to \infty \) with probability 1. Invoking Proposition 12 the state transition matrix \( F_k - K_k H_k \) is exponentially stable. The matrix sequence \( \overset{\text{c}}{P}_k \) has a unique limit [35 Theorem 7.5]. Hence, \( \overset{\text{c}}{P}_k \xrightarrow{k \to \infty} \overset{\text{c}}{P}_k \) as \( k \to \infty \). Finally, we conclude that \( \overset{\text{c}}{P}_k \xrightarrow{k \to \infty} \overset{\text{c}}{P}_k \) as \( k \to \infty \).

5 Numerical Simulations

A LTV system is simulated in this section to demonstrate the RTR based adaptive Kalman filter. The dynamics of the system given in Eq. (1) with the system matrices given below [28].

\[
F_k = \begin{bmatrix} 0 & 1 \\ -a_k b_k & -(a_k + b_k) \end{bmatrix} \quad G_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad H_k = \begin{bmatrix} 1 & d_k \end{bmatrix}
\]

wherein, \( \{a_k, b_k\} = c_k \pm i(0.4 + 0.2 \sin(2\pi k/\tau)) \), \( c_k = -0.7 + 0.2 \cos(2\pi k/\tau) \), \( d_k = 2\sin(10\pi k/\tau) \), and \( i \) is the imaginary unit. The measurements are assumed to be available every \( 1/\tau \) second with \( \tau = 10^4 \). The true noise covariance matrices are given by \( Q = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \) and \( R = 2 \). For the purposes of the simulation, the control inputs \( u_k \) are assumed to be drawn from a unit normal distribution. The number of measurements stacked at every time step are \( m = 3 \). Fig. 1 shows the Frobenius norm of the error in estimating the \( Q \) matrix with the RTR and the RLS method. The estimates from both methods are shown to converge to zero. A similar trend is seen in the estimation error Frobenius norms in estimating \( R \) in Fig. 2 and the error between the state covariances of the adaptive Kalman filter and the optimal Kalman filter with known noise covariance matrices shown in Fig. 3. The difference between the two methods is seen when comparing the transient \( \overset{\text{Q}}{\hat{Q}}_k \) eigenvalues shown in Fig. 4. The RLS method sometimes leads to a negative eigenvalue while RTR method lower bounds the eigenvalue by a prescribed minimum value of 0.1 (Remark 8). Since the \( R \) matrix is a scalar, A similar trend is seen in Fig. 5 that shows the time history of its estimate \( \overset{\text{R}}{\hat{R}}_k \). The result of negative eigenvalues of the noise covariance matrix estimates culminates as an inconsistent non positive definite error state covariance \( \overset{\text{P}}{\hat{P}}_{k+1|k} \) shown in Fig. 6.

![Figure 1: The Frobenius norm of the Q estimation error.](image1)

![Figure 2: The Frobenius norm of the R estimation error.](image2)
Figure 3: The Frobenius norm of the estimation error in the estimated state error covariance matrix $\hat{P}_{k|k-1}$ and the optimal $P_{k|k-1}$.

Figure 4: The transient eigenvalues of $\hat{Q}_k$, the true eigenvalues of $Q$, and $\lambda_{min} = 0.1$ from Remark 8.

6 Conclusion

A Riemannian trust-region (RTR) based adaptive Kalman filter to estimate the states as well as the process and measurement noise covariance matrices (CM) for a discrete observable linear time-varying system is presented in this paper. Rigorous convergence guarantees are provided for the noise CM estimate as well as the state error CM of the adaptive Kalman filter formulated using the noise CM estimates. To ensure convergence, the only assumptions required in this formulation are uniform observability, uniform controllability, and sufficient excitation of the system matrices. In fact, if the system matrices are not sufficiently excited, the adaptive Kalman filter is still exponentially stable by virtue of positive definite noise CM estimates. The results provided in this paper can be extended to handle detectable systems by using a state transformation to reveal and ignore the unobservable subspace of the system. The convergence of the RTR optimization method to the optimal value is theoretically fast and confirmed through numerical simulations because of the quadratic nature of the cost function. However, the RTR method is an iterative procedure and may be computationally expensive for problems of larger dimensionality. Formulating non-iterative schemes to ensure positive definite estimates presents a potential direction of future research.
Figure 5: The transient values of $\hat{R}_k$, the true $R$, and $\lambda_{min} = 0.1$ from Remark 8.

Figure 6: The transient eigenvalues of predicted state error covariance for the RTR-based adaptive Kalman filter.
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