CMB anisotropies induced by tensor modes in Massive Gravity

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ABSTRACT: We study Gravitational Waves (GWs) in the context of Massive Gravity, an extension to General Relativity (GR) where the fluctuations of the metric have a nonzero mass, and specifically investigate the effect of the tensor modes on the Cosmic Microwave Background (CMB) anisotropies. We first study the time evolution of the tensor modes in Massive Gravity and show that there is a graviton mass limit $m_l = 10^{-66} g \sim 10^{-29} cm^{-1}$, so that for masses $m \leq m_l$ the tensor perturbations in Massive Gravity are indistinguishable from the corresponding ones in GR. Also, we show that short wavelength massive modes behave almost indistinguishably from their massless counterparts. Later on, we show that massive gravitons with masses within the range $m = 10^{-27} cm^{-1} - m = 10^{-26} cm^{-1}$ would leave a clear signature on the lower multipoles ($\ell < 30$) in the CMB anisotropy power spectrum. Hence, our results indicate that CMB anisotropies measurements might be decisive to show whether the tensor modes are massive or not.

KEYWORDS: cosmology of theories beyond the SM, gravity waves / theory, CMBR theory.
1. Introduction and Summary

Over the last years we have been witnessing a great evolution in the observation of the universe. An accurate statistical analysis of the anisotropies of the CMB using the five-year WMAP data [1] has shown that the most favored cosmological model to fit the data is the flat ΛCDM model, which includes not only the very well known baryonic matter, but the mysterious dark matter (DM) and dark energy (DE) as well. The introduction of dark matter does not require a modification of GR (despite some alternative models do); in the pure general-relativistic case, all we have to do is to add further terms to the energy-momentum tensor on the right-hand side of the GR field equations. The DM candidates must pass some important tests [2] to ensure that the resulting model is physically consistent. However, despite the good theoretical candidates available in [2], there is no observational evidence so far to support any of them as the actual components of DM.

However, the inclusion of DE is not that simple. Ordinary matter, either baryonic or dark, cannot accelerate the universe as shown by observations [3], [4]. This discovery posed one of the deepest questions in modern cosmology: is the universe accelerating due to a repulsive gravity (caused by the quantum energy of the vacuum, for example, as in the ΛCDM model), or does General Relativity (GR) break down on cosmological scales [5]?
Many attempts have been made addressing these two possible cases, either by introducing new features into GR, or by modifying it (see [6] for a review of the models for DE).

The two key points introduced above are just to illustrate the need of studying alternative theories of gravity in parallel with the improvement of models in GR itself. As alternative theories we mean modifications of GR, and the simplest possibility for modifying GR is the introduction of a mass for the graviton. It was performed for the first time by the pioneering work of M. Fierz and W. Pauli [7], where they considered a linearized field theory of spin-two massive particles. The Lorentz invariance of the Fierz-Pauli (FP) lagrangian yields a spin-two massive state with five polarization modes (states with helicities $\pm 2, \pm 1$ and 0), differing from GR where one finds only a spin-two state with the two tensor polarization modes (helicities $\pm 2$). Such extra degrees of freedom yield an additional contribution of one vector and one real scalar massless particles with helicities $\pm 1$ and 0, respectively. The scalar particle couples to the trace of the stress energy-momentum tensor, causing a discontinuity in the propagator when one switches from the massive to the massless regime. This is the so-called van Dam-Veltman-Zakharov (vDVZ) discontinuity [8], whose net effect for a theory of a massive spin-two graviton is catastrophic: it would not even pass the solar-system tests for a theory of gravity (the prediction of the angle concerning the bending of the light by the Sun, for example).

However, in a full theory of gravity, we must consider nonlinear effects; the FP theory is valid only in the linear approximation. Nonlinear effects eliminate the vDVZ discontinuity in the classical level [10], [11], so that classically we may reconcile the massive theory with the GR predictions.

However, at the quantum level, the nonlinear interactions appear at the loop diagrams, so that the theory becomes strongly coupled above the energy scale $\Lambda = (m^4 M_{Pl})^{1/5}$, where $m$ is the graviton mass and $M_{Pl}$ is the Planck mass [12], [13]. For masses $m \sim H_0$, where $H_0$ is the present-day value of the Hubble parameter, the energy scale $\Lambda$ is too small, well below the expected value, $\Lambda = (m M_{Pl})^{1/2}$. In brane-world models ([14], [15], [16], [17]) a similar problem occurs: either they have ghosts [18], [19], [20] and [21], or are strong coupled at low energies [15], [19], [20] and [22].

A great step forward was taken in the works [23] and [24]. In reference [23] the authors proposed a consistent modification of gravity in the infrared as an analog of the Higgs mechanism in GR. In this model, Lorentz invariance is spontaneously broken and the graviton, as a result, acquires a mass. In reference [24] the author introduces a Lorentz-violating massive gravity model in which the vDVZ discontinuity, ghosts and the low strong coupling scale are absent. In reference [25] the author studies the most general Lorentz-violating gravitational theory with massive gravitons, showing that there is a number of different regions in the mass parameter space of this theory in which it can be described by a consistent low-energy effective theory without instabilities and the vDVZ discontinuity.

Therefore, the theory of Massive Gravity, as developed in [24] and [25], gives rise to physical propagating modes, and is free of the pathologies mentioned above. It is a potential candidate to provide the proper answers to the open questions in cosmology as mentioned earlier. There is a number of works studying cosmology in the context of Massive Gravity [20], [27], [28]; in the present work, we aim at extending this discussion by
analyzing the anisotropies of the CMB induced by tensor perturbations in Massive Gravity. We have analyzed tensor and vector perturbations in theories of gravitation with massive gravitons in a previous work [23], and in this paper we focus specifically on the signatures of the massive tensor modes. Henceforth we shall use the terms graviton and tensor modes interchangeably.

We know from a number of sources [30], [31], [32], [33], that primordial GW might leave a signature in the anisotropies and polarization spectrum of CMB, generated by the influence of such GWs on the photon redshifts. So, if the gravitons do have a mass, we expect that they will leave a different signature on the CMB anisotropy spectrum. Therefore, the main goals of this present work are twofold: develop solutions for massive GWs and analyze their signatures on the CMB polarization spectrum.

To this end, the present paper is organized as follows: in section 2 we review the basics of Massive Gravity and its cosmological tensor perturbations. In section 3 we start reviewing the solutions for primordial GW in GR, and right after we present our results in Massive Gravity; furthermore, we compare the results of both theories for different graviton masses, and discuss the differences between them. In section 4 we review the basics of radiative transfer in the presence of weak gravitational fields, deriving the relevant Boltzmann equations. In section 5 we derive the expressions for CMB anisotropies and polarization, with the corresponding correlation functions. In section 6 we discuss the solutions to the Volterra integral equation which provides the functions that enables us to evaluate the coefficients to the harmonic expansion the modes introduced in section 5. In section 7 we apply all these theoretical tools to Massive Gravity, obtaining the individual power spectra for different wavenumbers and graviton masses, and we compare these results with the predictions of GR. At the end of this paper we discuss the obtained results and make the corresponding conclusions.

2. A quick overview of Massive Gravity

As we have pointed out in the Introduction, the key ingredient to construct a physically-consistent theory of gravitation with massive gravitons lies on the spontaneous violation of the Lorentz symmetry. As in the Higgs analog in the Standard Model of electroweak interactions, we introduce, following [25] and [26], a set of four scalar Goldstone fields \( \phi^0(x), \phi^i(x) \), such that the action for Massive Gravity is written as

\[
S = \int d^4x \sqrt{-g} \left[ -M_P^2 R + \Lambda^4 F(X, V^i, W^{ij}) + \mathcal{L}_{\text{matter}} \right],
\]

where the first term on the right-hand side represents the usual Einstein-Hilbert action, and \( F \) is an arbitrary function of the metric components, their derivatives, and the Goldstone fields. The lagrangian for ordinary matter, \( \mathcal{L}_{\text{matter}} \), is assumed to be minimally coupled to the metric. The simplest way to combine the derivatives of the Goldstone fields to enter the argument of \( F \) is given by the set of scalar quantities

\[
X = \Lambda^{-4} g^{\alpha\beta} \partial_\alpha \phi^0 \partial_\beta \phi^0,
\]
\[ V^i = \Lambda^{-4} g^{\alpha\beta} \partial_\alpha \phi^0 \partial_\beta \phi^i, \]
\[ W^{ij} = \Lambda^{-4} g^{\alpha\beta} \partial_\alpha \phi^i \partial_\beta \phi^j - \frac{V^i V^j}{X}, \]
where \( \Lambda \) is the parameter which characterizes the cutoff scale of the theory. The second term on the right-hand side of (2.1) is invariant under the spatial reparametrization symmetry \( x^i(t) \rightarrow x^i(t) + \xi^i(t) \) and rotations.

We now introduce the “vacuum” solutions for the model (2.1),
\[ g_{\alpha\beta} = a^2 \eta_{\alpha\beta}, \quad \phi^0 = \Lambda^2 t, \quad \phi^i = \Lambda^2 x^i, \]
which corresponds to the flat FRW space; in the “unitary gauge” described by (2.5) the action will depend solely on the metric components. Now, in order to study linear cosmological perturbations around a flat Friedmann-Robertson-Walker (FRW) space, we spontaneously break the Lorentz symmetry of the model by fixing the Goldstone fields to the vacuum (2.5), so that the only remaining perturbations are given by
\[ g_{\alpha\beta} = a^2 \eta_{\alpha\beta} + \delta g_{\alpha\beta}, \]
where \( \eta_{\alpha\beta} = diag\{+,-,-,-\} \), \( a(\eta) \) is the scale factor, and \( \delta g_{\alpha\beta} \) is a metric perturbation whose components are given by (2.5),
\[ \delta g_{00} = 2a^2 \varphi, \quad \delta g_{0i} = a^2 (S_i - \partial_i B), \]
\[ \delta g_{ij} = a^2 [ -h_{ij} - \partial_i Q_j - \partial_j Q_i + 2(\psi \delta_{ij} - \partial_i \partial_j E) ], \]
where \( \varphi, \psi, B, E \) are scalar fields, \( Q_i \) and \( S_i \) are vector fields, and \( h_{ij} \) is a tensor field. The constraints satisfied by the vector and tensor fields are (2.9),
\[ h_{ij}, j = 0, \quad h^i_i = 0, \quad Q^i, i = S^i, i = 0. \]

Now, in the unitary gauge (2.5) we expand \( \sqrt{-g + \delta g} \), \( X(g + \delta g), V^i(g + \delta g), W^{ij}(g + \delta g) \) and \( F(g + \delta g) \) in powers of the metric perturbation \( \delta g \), and substitute these results into the massive term in (2.1), so that the lagrangian for the second-order perturbations reads
\[ \mathcal{L}_m = \frac{M^2_{Pl}}{2} \left[ m_0^2 \delta g_{00}^2 + 2m_1^2 \delta g_{0i}^2 + m_2^2 \delta g_{ij}^2 + m_3^2 \delta g_{00} \delta g_{00} + 2m_4^2 \delta g_{00} \delta g_{0i} \right], \]
where \( m_0, m_1, m_2, m_3 \) and \( m_4 \) are parameters related to the function \( F \) and its derivatives,
\[ m_0^2 = \frac{\Lambda^4}{M^2_{Pl}} \left[ X F_X + 2X^2 F_{XX} \right], \quad m_1^2 = \frac{2\Lambda^4}{M^2_{Pl}} \left[ -X F_X + W F_W + \frac{1}{2} X W F_{VV} \right], \]
\[ m_2^2 = \frac{2\Lambda^4}{M^2_{Pl}} \left[ W F_W - 2W^2 F_{WW} \right], \quad m_3^2 = \frac{\Lambda^4}{M^2_{Pl}} \left[ W F_W + 2W^2 F_{WW} \right], \]
\[ m_4^2 = -\frac{\Lambda^4}{M^2_{Pl}} \left[ X F_X + 2X W F_{XW} \right], \]
where $W = -1/3\delta_{ij}W^{ij}$ and

$$F_X = \frac{\partial F}{\partial X}, \quad F_{XX} = \frac{\partial^2 F}{\partial X^2}, \quad F_{V V} \delta_{ij} = \frac{\partial^2 F}{\partial V^i \partial V^j}.$$  \tag{2.14}

$$F_W \delta_{ij} = \frac{\partial F}{\partial W^i}, \quad F_{XX} \delta_{ij} = \frac{\partial^2 F}{\partial X \partial W^i}.$$  \tag{2.15}

$$\frac{\partial^2 F}{\partial W^i \partial W^j} = F_{WW} \delta_{ij} + F_{WW} \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}.$$  \tag{2.16}

(see Appendix A in references [26] and [28] for details). The spatial indices in (2.10) are summed over and, as argued in the reference [24], the mass parameters $m_i$ are proportional to some scale denoted by $m$.

The Einstein equations for the model (2.1), with the Goldstone fields in the unitary gauge (2.5), and metric (2.6) read (for computational details, see appendix A of the references [26] and [28]),

$$3\mathcal{H}^2 = \frac{a^2}{M_{Pl}^2} (\rho_m + \rho_\phi + \rho_\Lambda),$$  \tag{2.17}

$$2\mathcal{H'} + \mathcal{H}^2 = -\frac{a^2}{M_{Pl}^2} (p_m + p_\phi + p_\Lambda),$$  \tag{2.18}

$$\partial_0 (a^3 F_X X^{1/2}) = 0,$$  \tag{2.19}

where $\mathcal{H} = a'/a$, $\rho_m$ and $p_m$ stand for the density and pressure for the ordinary matter respectively, and

$$\rho_\phi = \Lambda^4 X F_X, \quad p_\phi = \Lambda^4 W F_W,$$  \tag{2.20}

$$\rho_\Lambda = -\frac{\Lambda^4}{2}, \quad p_\Lambda = \frac{\Lambda^4}{2}.$$  \tag{2.21}

The prime represents a derivative with respect to the conformal time $\eta$.

Once we have established the dynamical equations for the background, let us now turn our attention to the metric perturbations (2.8). The steps toward obtaining the dynamical equations for the massive metric perturbations are quite similar to those referred in [34], and they can be found in details in the Appendix A 3 in [28]; here we simply quote the results. Since in this paper we are interested solely in the tensor perturbations represented by the element $h_{ij}$ in (2.8), we write its dynamical equation as [28]

$$h_{ij}'' - \nabla^2 h_{ij} + 2\mathcal{H} h_{ij} + a^2 m_2^2 h_{ij} = 0.$$  \tag{2.22}

Since we deal only with the mass $m_2$ throughout this paper, we henceforth drop the subscript 2 and write it simply as $m$.

To end this section let us discuss an important aspect concerning the mass parameters of Massive Gravity. As we have pointed out in the Introduction, there are regions in the mass parameter space in which this theory is free of ghosts and instabilities; this means that the mass parameters $m_0, m_1, m_2, m_3$ and $m_4$ cannot be chosen arbitrarily, but they have to satisfy some constraints [24], [25]. Since in this paper we deal only with the mass parameter $m_2$, there is a number of choices on these parameters in which the model is
physically healthy; therefore, any of these choices would produce a physically acceptable theory. We simply assume that the mass parameters in our work are within the region in which the pathologies are absent.

Specific restrictions on the function $F$ are discussed in [26]. In this reference, the authors demonstrate the existence of a wide class of functions $F$ for which expanding cosmological solutions are compatible with constant graviton masses and allow for the effective field theory description. Therefore, we may simply restrict $F$ in such a way the mass $m_2$ is constant along the story of the universe, which we assume to hold throughout this paper.

3. Primordial Gravitational Waves in GR and Massive Gravity

3.1 Primordial Gravitational Waves in GR

In GR, a primordial GW is described by the cosmological tensor perturbation whose dynamical evolution is governed by the equation [34]

$$h''_{ij} - \nabla^2 h_{ij} + 2\mathcal{H}h'_{ij} = 0.$$ (3.1)

Before going into the Fourier space to solve this equation, it is convenient to introduce a new parametrization into this model [35]. Let us write down the present-day scale factor $a(\eta_0)$ as a quantity with dimension of length; then, setting $R_H = c/H_0$ as the Hubble radius, we then define $a(\eta_0) = 2R_H$. Now, since the GW wavenumber $k$ is very small for primordial GWs (in the frequency range which could produce a signature on CMB), with wavelength comparable to the present-day Hubble radius $R_H$, we introduce a dimensionless time-independent vector $n$ which has the same direction of $k$, and whose modulus is exactly the proportionality factor between the modulus of $k$ and $R_H$:

$$n = 2R_H k.$$ (3.2)

Since the early cosmological perturbations are of quantum-mechanical origin, we construct the tensor $h_{ij}$ as a quantum-mechanical operator, whose Fourier expansion is given by

$$h_{ij}(\eta, \mathbf{x}) = \frac{\sqrt{16\pi\ell_{Pl}^2}}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{d^3n}{\sqrt{2n}} \sum_{r=1,2} [\varepsilon_{ij}^r(n)h_n^r(\eta)e^{in\cdot\mathbf{x}}\hat{a}_n^r + \varepsilon_{ij}^{r*}(n)h_n^{r*}(\eta)e^{-in\cdot\mathbf{x}}\hat{a}_n^{r\dagger}],$$ (3.3)

where $r$ stands for the polarization mode of the GW, $\varepsilon_{ij}^r$ is the GW polarization tensor, and $\ell_{Pl}$ is the Planck length. The annihilation and creation operators $\hat{a}_n^r$ and $\hat{a}_n^{r\dagger}$ satisfy the well known commutation relations

$$[\hat{a}_n^r, \hat{a}_{n'}^{s\dagger}] = \delta_{rs}\delta^{(3)}(\mathbf{n} - \mathbf{n'}),$$ (3.4)

and, for the vacuum state $|0\rangle$,

$$\hat{a}_n^r |0\rangle = 0.$$ (3.5)
Now, substituting the expression (3.3) into (3.1), and redefining the conformal time derivative as \( d/d\eta = (a/c) d/dt \), we get the equation governing the dynamics of GR tensor modes (dropping the polarization index for a while)

\[
h''_n + 2\mathcal{H}h'_n + n^2h_n = 0.
\]

Then, defining the quantity \( \mu_n(\eta) = a(\eta)h_n(\eta) \) [33], we obtain an equation for a parametrically disturbed oscillator

\[
\mu''_n + \left[n^2 - \frac{a''}{a}\right] \mu_n = 0.
\]

To solve the simplified equation (3.7) we have to specify the scale factor. Since we are mostly interested in the time of recombination, we use the scale factor for a flat universe filled with radiation and matter, whose expression is [33]

\[
a(\eta) = 2R_H \left( \frac{1 + z_{eq}}{2 + z_{eq}} \right) \eta \left( \eta + \frac{2\sqrt{2} + z_{eq}}{1 + z_{eq}} \right),
\]

where \( z_{eq} \) is the redshift associated with the epoch of radiation-matter equality, whose value is \( z_{eq} \sim 3 \times 10^3 \), and the corresponding conformal instant \( \eta_{eq} \) is given by

\[
\eta_{eq} = (\sqrt{2} - 1) \frac{\sqrt{2} + z_{eq}}{1 + z_{eq}} \sim 7.6 \times 10^{-3}.
\]

It is easy to see that (3.8) reduces to a scale factor for a radiation-dominated and matter-dominated universe

\[
a(\eta) = \begin{cases} 
4R_H \eta, & \eta \leq \eta_{eq}; \\
2R_H (\eta + \eta_{eq})^2, & \eta \geq \eta_{eq}
\end{cases}
\]

respectively, so that it comprises the whole period we are interested in.

Substituting the scale factor (3.8) into (3.7), we can obtain exact analytical solutions for the functions \( \mu_n(\eta) \) [33]. Following [33], we normalize the GW amplitudes \( h_n(\eta) \) in terms of its value at \( \eta_r = 10^{-6} \) (in terms of redshift, \( z_r \sim 3 \times 10^7 \)); the resulting numerical solutions are displayed in the figure 1.

### 3.2 Primordial Gravitational Waves in Massive Gravity

Once we have reviewed the properties and evolution of GW amplitudes in GR, let us now analyze the same issues in Massive Gravity. First of all, we treat the tensor perturbations in the massive case quantum-mechanically, so that the Fourier expansion for the massive tensor field \( h_{ij}(\eta, \mathbf{r}) \) has an analog expression as in (3.3):

\[
h^{(m)}_{ij}(\eta, \mathbf{x}) = \frac{\sqrt{16\pi\ell_P^4}}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{d^3 n}{\sqrt{2E_n}} \sum_{r=1,2} [\epsilon^{(m)r}_{ij}(\mathbf{n})h^{(m)r}(\eta)e^{in\cdot x} a^{(m)r}_n + \epsilon^{(m)r\star}_{ij}(\mathbf{n})h^{(m)r\star}_n(\eta)e^{-in\cdot x} a^{(m)r\star}_n],
\]

(3.11)
Figure 1: The time evolution of the normalized GW amplitudes $h_n(\eta)/h_n(\eta_r)$. Compare with Figure 1 of [33].

where the superscript $(m)$ stands for massive, and $E_n$ denotes the energy of the mode $n$. Now, plugging (3.11) into (2.22) we get

$$h^{(m)''}_n + 2H h^{(m)'}_n + \left(n^2 + m^2 a^2\right)h^{(m)}_n = 0.$$  \hspace{1cm} (3.12)

In (3.12) we have dropped the GW polarization indices $r$ because we are going to treat only the tensor modes in this paper. Using the same strategy as in the GR case, we introduce a function $\mu^{(m)}_n(\eta) = a(\eta)h^{(m)}_n(\eta)$, so that equation (3.12) becomes

$$\mu^{(m)''}_n + \left[n^2 + m^2 a^2 - \frac{a''}{a}\right] \mu^{(m)}_n = 0.$$  \hspace{1cm} (3.13)

As we have discussed in the section above, the scale factor (3.8) represents very well the periods of the universe considered in this paper, so that it makes sense to employ it in Massive Gravity as well, since we may expect that the contribution of massive gravitons to the expansion of the universe is negligible in its early epochs; then, as a first approximation, we may neglect the contribution of the components $\rho_\phi$, equation (2.20), and $\rho_\Lambda$, equation (2.21), in (2.17).

Now, using the above arguments and consequently the scale factor (3.8), we can solve numerically equation (3.13) for different wavenumbers $n$ and masses $m$. We choose the graviton masses $m$ using the following argument: in GR, only GW with frequencies $\nu$ within the range $10^{-15}$ to $10^{-18}$ Hz may leave a signature on CMB polarization [37]; these frequencies correspond to wavenumbers $k$ within the range $10^{-25}$ cm$^{-1}$ ($n \sim 5 \times 10^3$) to $10^{-28}$ cm$^{-1}$ ($n \sim 10$). For Massive Gravity, we use the same values for $k$, but now we vary the frequencies in order to obtain constant nonzero graviton masses through the dispersion relation

$$\omega^2 = k^2 + m^2,$$  \hspace{1cm} (3.14)
which comes straight from (2.22), where now $\omega = 2\pi \nu$. As a result, we find that if the values of the mass $m$ lie within the range $10^{-66} - 10^{-62} g$, the corresponding frequencies have values very close to the expected in GR. In particular, we’ve found that if the graviton mass is $m = 10^{-66} g \sim 10^{-29} cm^{-1}$, the behavior of the GWs in Massive Gravity is exactly the same of GWs in GR. Therefore, if the graviton mass is equal or less than the graviton mass limit $m_l = 10^{-66} g$, the effects of Massive Gravity are indistinguishable from GR.

It is important to mention that there has been a lot of efforts to constrain the masses of the tensor modes over the past few decades. For instance, Goldhaber and Nieto [38] have found a limit $m < 2.0 \times 10^{-62} g$ analyzing the motion of galaxies in clusters. Later on, Talmadge et al. [39] studied the variations of Kepler’s third law when compared with the orbits of Earth and Mars, and found a limit $m < 7.68 \times 10^{-55} g$. Recently, Finn and Sutton [40] calculated the decay of the orbital period of the binary pulsars PSR B1913+16 (Hulse and Taylor pulsar) and PSR B1534+12 due to emission of massive gravitons, and found $m < 1.4 \times 10^{-52} g$. Cooray and Seto [41] investigated the variation of the speed of gravity when compared to the speed of light due to a massive tensor mode, and determined an upper limit of $\sim 10^{-56} g$ for its mass, by using the measurements of a sample of close white dwarf binaries detectable with the Laser Interferometer Space Antenna (LISA), together with a optical light curve data related to binary eclipses from meter-class telescopes for the same sample. A recent and comprehensive review of the methods to determine the bounds for the masses of gravitons and photons can be found in [42], which we refer to for further details.

Since we are interested in investigating signatures of massive gravitons, we shall consider only graviton masses higher than the limit $m = 10^{-66} g$; the numerical solutions to the massive tensor perturbation equations (3.13) are depicted in the Figure 2 below. For sake of comparison we depict the general-relativistic GW amplitudes in each graph as well. We have used the same normalization as [33], and the plots start at $\eta = \eta_r = 10^{-6}$. The mass $m = 2.843 \times 10^{-28} cm^{-1}$ correspond to $m = 10^{-65} g$, and so forth.

Let us now analyze in detail the behavior of massive gravitons in the light of equation (3.13). In the very early universe, before the time of equality radiation-matter, the value of $a(\eta)$ is very low, and then the $m^2 a^2$ on the left-hand side of (3.13) can be dropped; therefore, we recover the characteristic tensor mode equation of GR, (3.7), and the behavior of massless and massive gravitons are the same. On superhorizon scales, $n \ll a''/a$, the resulting equation for the tensor modes is

$$\mu_n^{(m)''} - \frac{a''}{a} \mu_n^{(m)} = 0,$$

whose solution is given by $\mu_n^{(m)} = f(n)a$, which means that the tensor amplitudes are “frozen”, no matter the gravitons are massless or not. This particular behavior can be clearly seen from figures 2 - 7, where the amplitudes are constant for all the modes considered prior to decoupling.

However, as the universe evolves, the tensor modes “fall” into the horizon, so that their amplitudes are no longer constant; on subhorizon scales, $n \gg a''/a$, we can neglect
the effect of the term $a''/a$, so that we are left with

$$\mu_n^{(m)''} + \left[n^2 + m^2 a^2\right] \mu_n^{(m)} = 0. \quad (3.16)$$

On subhorizon scales the massive term becomes dominant over low values of $n$, so that it “enforces” the tensor modes to fall into the horizon earlier than in the massless case. It is clear from equation (3.16) that the heavier the gravitons, the earlier their modes fall into the horizon.

This effect can be clearly seen in the figures 2, 3, and 4 where the $n$ values are sufficiently low to account for this effect. However, for larger values of $n$, this effect weakens, since $n^2 \gg m^2 a^2$ in the time of decoupling, and the massive term will be predominant only for low redshifts, as can be seen in figures 5, 6, and 7.

In particular, for low $n$ (corresponding to tensor modes with long wavelengths), its
constant contribution to (3.16) can be completely neglected, so that we are left with
\[
\mu_n^{(m)''} + \left[n^2 + m^2 a^2 - \frac{a''}{a}\right] \mu_n^{(m)} = 0,
\]
and then the oscillatory behavior is strikingly different from the massless case, as shown in figures 6, 7. For higher \(n\) (that is, tensor modes with short wavelengths), tough, this effect is not so strong, but induces a slight phase difference in the oscillatory behavior of the tensor modes. Such phase difference is stronger for higher masses; as an example of it, we have zoomed in the “tail” of figure 6 to show this fact. This is presented in figure 8.

Hence, from this analysis we may conclude that the tensor modes of Massive Gravity behave similarly to the massless modes of GR, but the heavier the tensor modes are, the more distinct are their physical evolution if compared to the massless modes. Nevertheless, if massive gravitons do exist, they likely have left a signature on some physical observable;
then, by comparing the predicted signatures of the massless and the massive modes with the observed ones, one should be able to determine whether they possess or not a nonzero mass. In our view, the best laboratory to test this assumption is the anisotropies measurement of CMB, for reasons that will become clear in section 7. Before doing so, though, we take some time to review the basic aspects of the theory of CMB anisotropies and polarization.

4. The Radiative Transfer Equation in the presence of Weak Gravitational Fields - an overview

The first account of the effect of primordial GWs in polarizing the CMB photons was introduced in a seminal paper by Polnarev [30]. To begin with, let us consider a given beam of radiation characterized by its Stokes parameters \{I, Q, U, V\}, where I is the total intensity of the wave, the parameters Q and U measure the linear polarization of the wave, and V measures its circular polarization. They are integrated over all radiation frequencies, so that there is a set of Stokes parameters for each monochromatic component wave of the radiation beam \{I(\nu, \theta, \varphi), Q(\nu, \theta, \varphi), U(\nu, \theta, \varphi), V(\nu, \theta, \varphi)\}. Associated with the Stokes parameters are the components of the photon distribution function \(n\), which can be cast in a symbolic vector of the form [30], [43],

\[
\hat{n} = \frac{1}{2} \frac{c^2}{h\nu^3} \begin{pmatrix}
I + Q \\
I - Q \\
2U \\
-2U
\end{pmatrix}.
\]

Now, the transfer equation for the photon distribution functions subject to a weak GW-field is given by

\[
\frac{\partial \hat{n}}{\partial \eta} + \dot{\varepsilon}^i \frac{\partial \hat{n}}{\partial x^i} + \frac{\partial \hat{n}}{\partial \nu} \frac{d\nu}{d\eta} = C[\hat{n}],
\]

where \(\dot{\varepsilon}^i\) is the unit vector along the photon geodesic, and \(C[\hat{n}]\) is the scattering term given by

\[
C[\hat{n}] = -\sigma_T N e a(\eta) \left\{ \hat{n}(\eta, r, \nu, \mu, \varphi) \frac{1}{4\pi} \int_{-1}^{1} d\mu' d\varphi' P(\mu, \varphi, \mu', \varphi') \hat{n}(\eta, r, \nu, \mu', \varphi') \right\},
\]

\[
P(\mu, \varphi, \mu', \varphi') = Q \left\{ P^0(\mu, \mu') + \sqrt{1 - \mu^2} \sqrt{1 - \mu'^2} P^1(\mu, \varphi, \mu', \varphi') + P^2(\mu, \varphi, \mu', \varphi') \right\},
\]

where

\[
Q = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix},
\]

\[
P^0 = \frac{3}{4} \begin{pmatrix}
2(1 - \mu^2)(1 - \mu'^2) + \mu^2 \mu'^2 & \mu^2 & 0 & 0 \\
\mu^2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu \mu'
\end{pmatrix},
\]
\[ P^1 = \frac{3}{4} \begin{pmatrix} 
4\mu \cos \psi & 0 & -2\mu \sin \psi & 0 \\
0 & 0 & 0 & 0 \\
2\mu' \sin \psi & 0 & \cos \psi & 0 \\
0 & 0 & 0 & \cos \psi 
\end{pmatrix}, \] (4.7)

\[ P^2 = \frac{3}{4} \begin{pmatrix} 
\mu^2 \mu'^2 \cos 2\psi & -\mu^2 \cos 2\psi & -\mu^2 \mu' \sin 2\psi & 0 \\
-\mu'^2 \cos 2\psi & \cos 2\psi & \mu' \sin 2\psi & 0 \\
\mu \mu'^2 \sin 2\psi & -\mu \sin 2\psi & \mu \mu' \cos 2\psi & 0 \\
0 & 0 & 0 & \cos \psi 
\end{pmatrix}, \] (4.8)

\[ \sigma_T \] is the Thomson scattering cross-section, \( N_e(\eta) \) is the number of free electrons in the unit comoving volume, \( \mu = \cos \theta \), and we have defined \( \psi := \varphi - \varphi' \) \[30\].

In order to get the expression of the Boltzmann equation for our problem, let us decompose the vector \( \hat{n} \) into its zeroth-order contribution, \( \hat{n}^{(0)} \), i.e., in the absence of GW, and its first-order correction \( \hat{n}^{(1)} \),

\[ \hat{n} = \hat{n}^{(0)} + \hat{n}^{(1)}, \] (4.9)

where \( n^{(0)} \) is the blackbody radiation function

\[ n^{(0)}(\nu) = \frac{1}{e^{h\nu/kT_0} - 1}, \] (4.10)

and \( T_0 \sim 2.725 \) K is the present-day value of the CMB temperature. Since equation (4.2) is linear, we can expand \( \hat{n}^{(1)} \) in the same way as we did in (3.3),

\[ \hat{n}^{(1)}_{\eta, \nu, \hat{e}} = \frac{\sqrt{16\pi} \ell}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{d^3 n}{2n} \sum_{r=+,-} \left\{ \hat{n}^{(1)}_{n,r}(\eta, \nu, \hat{e}) e^{i n \cdot \hat{a}_n^r} + \hat{n}^{(1)*}_{n,r}(\eta, \nu, \hat{e}) e^{-i n \cdot \hat{a}_n^r} \right\}; \] (4.11)

now, introducing the basis for Thomson scattering \[30\],

\[ \hat{a}_r(\mu, \varphi) = \frac{1}{2} \left( 1 - \mu^2 \right) e^{\pm 2i\varphi} \hat{u}, \quad \hat{b}_r(\mu, \varphi) = \frac{1}{2} \left( \frac{1 + \mu^2}{1 + \mu^2} - \frac{1 + \mu^2}{1 + \mu^2} \right) e^{\pm 4i\mu}, \] (4.12)

where \( r = 1 \) corresponds to a left-hand polarization, and \( r = 2 \) to the right-hand one, we may further expand the functions \( \hat{n}^{(1)}_{n,r}(\eta, \nu, \hat{e}) \) as

\[ \hat{n}^{(1)}_{n,r}(\eta, \nu, \mu, \varphi) = \frac{1}{2} f(\nu) \left[ \alpha_{n,r}(\eta, \nu, \mu, \varphi) \hat{a}_r(\mu, \varphi) + \beta_{n,r}(\eta, \nu, \mu, \varphi) \hat{b}_r(\mu, \varphi) \right], \] (4.13)

where \( \alpha_{n,r}(\eta, \mu) \) and \( \beta_{n,r}(\eta, \mu) \) are functions to be determined by the solutions to the Boltzmann equation, and

\[ f(\nu) = \nu \frac{dn^{(0)}}{d\nu}. \] (4.14)

Now, plugging relation (4.13) into (4.2), using the geodesic equation for the photon,

\[ \frac{d\nu}{d\lambda} = -\nu \left[ \mathcal{H} + \frac{1}{2} \frac{\partial h_{ij}}{\partial \eta} p^i p^j \right] \frac{d\eta}{d\lambda}, \] (4.15)
where $\lambda$ is an affine parameter, and $\mathcal{H}$ is the Hubble parameter in conformal time, and expanding the photon momenta in spherical coordinates around the GW direction $\hat{n}$, we obtain the Boltzmann equations for the radiative transfer in the presence of weak gravitational fields (dropping the indices $n, r$ for the sake of simplicity) [30], [33],

$$
\frac{\partial}{\partial \eta} \beta(\eta, \mu) + [q(\eta) + i n \mu] \beta(\eta, \mu) = \frac{3}{16} q(\eta) I(\eta),
$$

and

$$
\frac{\partial}{\partial \eta} \xi(\eta, \mu) + [q(\eta) + i n \mu] \xi(\eta, \mu) = \frac{d}{d \eta} h(\eta),
$$

where we have defined

$$
\xi(\eta, \mu) = \alpha(\eta, \mu) + \beta(\eta, \mu),
$$

and introduced scattering rate $q(\eta)$ defined by

$$
q(\eta) = \sigma T N_e(\eta) a(\eta).
$$

The solutions to the Boltzmann equations (4.16) and (4.17), given by the functions $\alpha(\eta, \mu)$ and $\beta(\eta, \mu)$, are the essential elements for the computation of the anisotropies and polarization of the CMB, as we sketch in the next section.

5. Harmonic analysis on a 2-sphere

To begin with, let us first construct the polarization tensor associated with the Stokes parameters $Q(\theta, \varphi)$ and $U(\theta, \varphi)$, where the coordinates $(\theta, \varphi)$ describe the position of a given region of the sky. The Stokes parameters $Q$ and $U$ can be cast into the symmetric trace-free (STF) polarization tensor [32], [44]

$$
P_{ab}(\theta, \varphi) = \frac{1}{2} \begin{pmatrix} Q & -U \sin \theta \\ -U \sin \theta & -Q \sin^2 \theta \end{pmatrix}.
$$

On the two-sphere tensor analysis can be easily implemented; the “divergence” and “curl” of a symmetric rank-2 tensors are respectively given by $T^{ab, :ab}$ and $T^{ab, :ac} \varepsilon^{c}_b$, where “:” denotes covariant differentiation, $g_{ab}$ and $\varepsilon_{ab}$ are respectively the two-dimensional metric and antisymmetric tensors on the 2-sphere, given by

$$
g_{ab}(\theta, \varphi) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \quad \varepsilon_{ab}(\theta, \varphi) = \sin \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

With these elements on hand, we introduce invariants which can be built up from the polarization tensor $P_{ab}$ and its derivatives. From the symmetric tensor on the 2-sphere we construct two of the invariants, and from the second derivatives we construct the other two in the form of a “divergence” and a “curl” [33],

$$
I = g^{ab} P_{ab}, \quad V = i \varepsilon^{ab} P_{ab}, \quad E = -2 P^{,ab, :ab}, \quad B = -2 P^{,bc, \varepsilon}_c.
$$
With these invariants we get a very convenient way to completely characterize the radiation beam, since they do not depend on the reference frame chosen. We now proceed to expand the invariants \((I, E, B, V)\) in spherical harmonics in order to perform an analysis on the each multipole of the radiation field \[33\]

\[
I(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}^T Y_{\ell m}(\theta, \varphi),
\]

\[
E(\theta, \varphi) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \frac{(\ell + 2)!}{(\ell - 2)!} \right]^{\frac{1}{2}} a_{\ell m}^E Y_{\ell m}(\theta, \varphi),
\]

\[
B(\theta, \varphi) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \left[ \frac{(\ell + 2)!}{(\ell - 2)!} \right]^{\frac{1}{2}} a_{\ell m}^B Y_{\ell m}(\theta, \varphi),
\]

\[
V(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}^V Y_{\ell m}(\theta, \varphi).
\]

It is important to stress that these expansions are consistent with the similar definitions in the literature \[31\], \[32\].

We are now in position to write down the \(I, E\) and \(B\) functions \((5.4), (5.5)\) and \((5.6)\) in terms of the functions \(\alpha(\mu, \varphi)\) and \(\beta(\mu, \varphi)\) introduced in \((4.13)\). From \((4.1)\) we obtain for a monochromatic radiation beam

\[
I(\eta, \nu, \theta, \varphi) = \frac{\hbar \nu^3}{c^2} \left[ n_1(\eta, \nu, \theta, \varphi) + n_2(\eta, \nu, \theta, \varphi) \right],
\]

\[
Q(\eta, \nu, \theta, \varphi) = \frac{\hbar \nu^3}{c^2} \left[ (n_1(\eta, \nu, \theta, \varphi) - n_2(\eta, \nu, \theta, \varphi)) \right],
\]

\[
U(\eta, \nu, \theta, \varphi) = -4 \frac{\hbar \nu^3}{c^2} n_3(\eta, \nu, \theta, \varphi),
\]

so that from \((4.12), (4.13)\) and \((5.8)\) we get (restoring the \(n\)-dependence of the Fourier expansion),

\[
I_{n,r}(\eta, \nu, \theta, \varphi) = \frac{\hbar \nu^3}{c^2} [n_{1}(\nu) + f(\nu)\alpha_{n,r}(\eta, \mu)(1 - \mu^2)e^{\pm 2i\varphi}],
\]

\[
Q_{n,r}(\eta, \nu, \theta, \varphi) = \frac{\hbar \nu^3}{c^2} f(\nu)\beta_{n,r}(\eta, \mu)(1 + \mu^2)e^{\pm 2i\varphi},
\]

\[
U_{n,r}(\eta, \nu, \theta, \varphi) = \mp 2 \frac{\hbar \nu^3}{c^2} f(\nu)\beta_{n,r}(\eta, \mu)e^{\pm 2i\varphi}.
\]

From equations \((5.9), (5.10), (5.11)\) we may readily evaluate the expressions for \(I, E\) and \(B\), using \((5.1), (5.2), (5.3)\) and \((5.9), (5.10), (5.11)\); then, integrating over photon frequencies, we obtain

\[
I_{n,r}(\mu, \varphi) = \gamma \left[ (1 - \mu^2) \alpha_{n,r}(\eta, \mu)e^{\pm 2i\varphi} \right],
\]

\[
E_{n,r}(\mu, \varphi) = -\gamma \left[ (1 - \mu^2) \left( 1 + \mu^2 \frac{d^2}{d\mu^2} + 8\mu \frac{d}{d\mu} + 12 \right) \beta_{n,r}(\eta, \mu)e^{\pm 2i\varphi} \right],
\]

\[
B_{n,r}(\mu, \varphi) = \mp \gamma \left[ 2(1 - \mu^2) \left( \mu \frac{d^2}{d\mu^2} + 4i \frac{d}{d\mu} \right) \beta_{n,r}(\eta, \mu)e^{\pm 2i\varphi} \right],
\]

(5.12)
where we have defined
\[ \gamma = \int d\nu \frac{h\nu^3}{c^2} f(\nu). \] (5.13)

Once we have the key expressions for evaluating the power spectrum correlation function all we must do now is solving the Boltzmann equations (4.16) and (4.17), which we handle in the next section.

6. The solutions to the Boltzmann equations

In the paper [33] the authors discuss an analytical method for solving the Volterra equation represented by (4.16) in terms of a series expansion, and compare their results with the exact numerical solutions. Here we follow only their numerical approach, which we sketch below. To do so, we introduce first the functions
\[ \Phi(\eta) = \frac{3}{16} g(\eta) \mathcal{I}(\eta), \] (6.1)
\[ H(\eta) = e^{-\tau(\eta)} \frac{dh(\eta)}{d\eta}, \] (6.2)
where the function \( \tau(\eta) \) represents the optical depth of the universe, and is defined within a time interval \( \eta' \) and \( \eta \):
\[ \tau(\eta, \eta') = \int_{\eta'}^{\eta} d\eta'' q(\eta''); \] (6.3)
g(\eta) is the visibility function, written as
\[ g(\eta) = q(\eta) e^{-\tau(\eta)} = \frac{d}{d\eta} e^{-\tau(\eta)}. \] (6.4)

Taking \( \eta' = \eta_0 \), we further write the optical depth from a given conformal instant \( \eta \) to the present as \( \tau(\eta_0, \eta) = \tau(\eta) \), that is
\[ \tau(\eta) = \int_{\eta}^{\eta_0} d\eta' q(\eta'). \] (6.5)

Now, using these definitions, the formal solutions to the equations (4.16) and (4.17) are given by the integral relations [33]
\[ \beta(\eta, \mu) = e^{\tau(\eta)-in\mu\eta} \int_{0}^{\eta} d\eta' \Phi(\eta') e^{in\mu\eta'}, \] (6.6)
\[ \xi(\eta, \mu) = e^{\tau(\eta)-in\mu\eta} \int_{0}^{\eta} d\eta' H(\eta') e^{in\mu\eta'}. \] (6.7)

Now, since the function \( H(\eta) \) is known, we can obtain a single integral equation for the function \( \Phi(\eta) \) by plugging (6.6) and (6.7) into (4.19), so that
\[ \mathcal{I}(\eta) = e^{\tau(\eta)} \int_{-1}^{1} d\mu d\eta' \left\{ (1 + \mu^2) \Phi(\eta') - \frac{1}{2} (1 - \mu^2)^2 H(\eta') \right\} e^{in\mu(\eta'-\eta)}; \] (6.8)
such expression can be further simplified by introducing the kernels $K_{\pm}(\eta - \eta')$, defined as

$$K_{\pm}(\eta - \eta') = \int_{-1}^{1} d\mu(1 \pm \mu^2)^2 e^{i\mu(\eta - \eta')}$$

so that expression (6.8) yields

$$I(\eta) = e^{-\tau(\eta)} \int_{0}^{\eta} d\eta' \left\{ K_{+}(\eta - \eta')\Phi(\eta') - \frac{1}{2}K_{-}(\eta - \eta')H(\eta') \right\}.$$  

(6.10)

The final equation for $\Phi(\eta)$ is obtained by multiplying both sides of this equality by $(3/16)q(\eta)e^{-\tau(\eta)}$ and using the expression (6.1), so that

$$\Phi(\eta) = \frac{3}{16}q(\eta) \int_{0}^{\eta} d\eta' I(\eta')K_{+}(\eta - \eta') + G(\eta),$$

(6.11)

where $G(\eta)$ is related to the function (6.2),

$$G(\eta) = -\frac{3}{32}q(\eta) \int_{0}^{\eta} d\eta' H(\eta')K_{-}(\eta - \eta').$$

(6.12)

The solution to Volterra integral equation (6.11) provides the values of the functions $\alpha$ and $\beta$ for every conformal instant $\eta$; in particular, to the present-day $\eta_0$, the expressions $\alpha(\eta_0, \mu) = \alpha(\mu)$ and $\beta(\eta_0, \mu) = \beta(\mu)$ are respectively given by

$$\alpha_{n,r}(\mu) = \int_{0}^{\eta_0} d\eta \left( H_{n,r}(\eta) - \Phi_{n,r}(\eta) \right) e^{-i\mu\zeta}, \quad \beta_{n,r}(\mu) = \int_{0}^{\eta_0} d\eta \Phi_{n,r}(\eta) e^{-i\mu\zeta},$$

(6.13)

where we have introduced the variable $\zeta = n(\eta_0 - \eta)$. From (6.13) we compute the coefficients $a_{\ell m r}^{X}$: to do that we substitute expressions (5.4-5.6) and (6.13) into (5.12), and integrate over angular variables, so that

$$a_{\ell m r}^{T} = (-i)^{\ell-2} \left( \delta_{2,m}\delta_{1,r} + \delta_{-2,m}\delta_{2,r} \right) a_{\ell m r}^{T},$$

$$a_{\ell m r}^{E} = (-i)^{\ell-2} \left( \delta_{2,m}\delta_{1,r} + \delta_{-2,m}\delta_{2,r} \right) a_{\ell m r}^{E},$$

$$a_{\ell m r}^{B} = (-i)^{\ell-2} \left( \delta_{2,m}\delta_{1,s} - \delta_{-2,m}\delta_{2,s} \right) a_{\ell m r}^{B},$$

(6.14)

where

$$a_{\ell m r}^{T} = \gamma \sqrt{4\pi(2\ell + 1)} \int_{0}^{\eta_0} d\eta \left( H_{n,r}(\eta) - \Phi_{n,r}(\eta) \right) T_{\ell}(\zeta),$$

(6.15)

$$a_{\ell m r}^{E} = \gamma \sqrt{4\pi(2\ell + 1)} \int_{0}^{\eta_0} d\eta \Phi_{n,r}(\eta) E_{\ell}(\zeta),$$

(6.16)

$$a_{\ell m r}^{B} = \gamma \sqrt{4\pi(2\ell + 1)} \int_{0}^{\eta_0} d\eta \Phi_{n,r}(\eta) B_{\ell}(\zeta),$$

(6.17)

and $T_{\ell}(\zeta)$, $E_{\ell}(\zeta)$, $B_{\ell}(\zeta)$ are the multipole projection functions which appear after the integration over the angular variables, whose form are given by

$$T_{\ell}(\zeta) = \sqrt{\frac{(\ell + 2)!}{(\ell - 2)!}} \frac{j_{\ell}(\zeta)}{\zeta^2},$$

$$E_{\ell}(\zeta) = \left[ 2 - \frac{l(l-1)}{\zeta^2} \right] j_{\ell}(\zeta) - 2 \zeta j_{\ell-1}(\zeta),$$

$$B_{\ell}(\zeta) = 2 \left[ -\frac{(\ell - 1)}{\zeta} j_{\ell}(\zeta) + j_{\ell-1}(\zeta) \right].$$

(6.18)
Once we have obtained the expressions for (6.14) we can evaluate the most important tool of CMB physics: the correlation function $C_{\ell}^{XX'}$, where $X, X' = E, B$. Since each $a_{\ell m}^X$ in (5.5), (5.6), depends upon the wavenumber $n$ and polarization state $r$, we write $a_{\ell m}^X$ as $a_{\ell m, nr}^X$, so that the correlation function is given by [33]:

$$
C_{\ell}^{XX'} = \frac{C^2}{4\pi^2(2\ell + 1)} \int n dn \sum_{r=1,2} \sum_{m=-\ell}^{\ell} [a_{\ell m, nr}^X a_{\ell m, nr}^{X*} + a_{\ell m, nr}^{X*} a_{\ell m, nr}^X].
$$

To conclude this section, let us say some words about the ionization history of the universe. We focus here on the time of decoupling, were the CMB radiation was released from the primordial nuclei. At decoupling, which took place around redshift $z \sim 1088$, the universe underwent a transition from a completely ionized state to a state in which neutral hydrogen and helium atoms were formed. In this process the radiation decoupled from the matter, originating the CMB radiation and a neutral pre-galactic baryonic medium (PGM). Then, at some redshift between $14 < z < 6$ the PGM was ionized again by the UV radiation from the first luminous objects, leaving the intergalactic medium (IGM) ionized [45]. Such process is called reionization, and would leave observable imprints on the CMB polarization spectrum due to the interactions of the CMB photons with the free electrons now available due to the reionized medium [43], [47]. However, the reionization epoch is still not fully understood, and many models have been proposed to shed a light on the physics of this process (see [48] and references therein), which can be homogeneous models with a sudden reionization (e. g. as discussed in [49], [50]), or extended models with double reionization [51], among others (see [48] for a more comprehensive list of papers).

In the present paper we shall consider solely the epoch of recombination, whose physical process is very well understood. Despite reionization is fundamental to understand the low-multipole behavior of CMB polarization, it can be neglected in a first-approximation to study temperature anisotropies generated by the tensor modes. Next, as for the decoupling epoch, the formulae for the density of free electrons $N_e(\eta)$ and the fraction of ionized electrons are given by [22]

$$
N_e(\eta) = \left( 1 - \frac{Y_p}{2} \right) \frac{X_e(\eta)\Omega_b\rho_c}{m_p} \left( \frac{a(\eta)}{a(\eta_0)} \right)^3,
$$

and [53]

$$
X_e(\eta) = \left( 1 - \frac{Y_p}{2} \right)^{-1} \left( \frac{c_2}{1000} \right) \left( \frac{m_p}{2\sigma_T R_H\rho_c} \right) \Omega_b^{c_1-1} \left( \frac{z}{1000} \right)^{c_2-1} \left( \frac{a^'}{a} \right) (1 + z)^{-1}.
$$

In these formulae $Y_p \approx 0.23$ is the primordial helium mass fraction, $\Omega_b$ is the baryon content, and $m_p$ is the mass of a proton. The constants are given by $c_1 = 0.43$, $c_2 = 16 + 1.8 \ln \Omega_B$; we take $\Omega_b = 0.046$ [44].

7. CMB Anisotropies in Massive Gravity

Let us now compute the Boltzmann equations for radiative transfer in the presence of weak gravitational fields in Massive Gravity. The anisotropy and polarization power spectra, as
discussed in sections 4, 5 and 6, is a combination of two physical processes appearing in the Boltzmann equation (4.2): Thomson scattering, represented by the collisional integral (4.3) on its right-hand side, and the gravitational redshift of the photon, represented by the geodesic equation (4.15) on its left-hand side. The collisional integral for Massive Gravity is the same as in GR, since the process of scattering involves only photons and electrons, and the lagrangian of matter is minimally coupled to the metric. The basic change is due to the different gravitational field strength represented by $h_{ij}$ in the geodesic equation, which now depends on the details of the underlying gravitational model. However, the general form of Boltzmann equations (4.16) and (4.17) are the same for both GR and Massive Gravity (see [29] for details), which we proceed to evaluate numerically in what follows.

We start our analysis of the possible signatures of massive gravitons on CMB anisotropies in the light of our discussion in section 3.2 and the results reviewed in section 6. For the sake of simplicity we consider first the case of GR, and then extend our discussion to Massive Gravity. From figure 1 we see that long wavelength tensor modes (that is, low $n$, in the figure corresponding to $n = 10$) remain frozen during decoupling, so that they won’t contribute significantly to CMB anisotropies and polarization at this epoch; this effect will be mainly due to the short wavelength tensor modes (high $n$, in the figure corresponding to $n = 10^2$ and $n = 10^3$; notice that tensor modes with $n = 10^4$ are “dead” at decoupling), which fell into the horizon at an earlier time. For late times the situation is inverse: long wavelength modes are now dead, and short wavelength modes are falling into the horizon. Therefore, this quick and simple analysis leads us to conclude that short wavelength tensor modes will be predominant over early times, and long wavelength modes at late times. In terms of multipoles of the radiation this feature can be understood in the following way: short wavelength modes and early times correspond to larger values of the variable $\zeta = n(\eta_0 - \eta)$, so that the spherical Bessel functions appearing in (6.18) will have nonzero values only for higher multipoles; conversely, long wavelengths and late times will give rise to smaller $\zeta$, and hence $j_\ell(\zeta)$ will be nonzero only for low multipoles.

Therefore, roughly speaking, in GR long wavelength tensor modes will influence mainly the CMB low multipoles, whereas short wavelength tensor modes will substantially contribute to the polarization on higher multipoles. This is also true in the case of Massive Gravity, but now with one substantial difference: long wavelength modes fall into horizon earlier than their massless counterparts, as seen in figures 2 and 3, altering then the form of the signature at low multipoles. Also, the oscillations noticed in the late-time evolution of the massive modes, valid not only for those possessing long wavelengths, but for the ones possessing short wavelengths as well (as seen in figure 8, for instance) would contribute to yield a distinct signature on low multipoles.

However, since we are not considering reionization in this paper, the distinct signatures discussed above will not show up on the polarization spectrum. This happens because in this case the visibility function is zero at this epoch, and then the source functions $\Phi$ (6.1) will be zero, since they depend linearly on $g(\eta)$. However, this fact does not affect CMB anisotropies, since the mode coefficients $a_{\ell m}^{T}$, given by (6.14), depends on $H(\eta)$, defined by (6.2), which is not zero even in the absence of reionization. Since the source function $H(\eta)$ depends on the tensor mode amplitudes, and massive and massless modes are different at
late times, we conclude that the massive modes would leave a distinct signature on CMB low multipoles even without reionization.

Let us now perform a numerical analysis in order to check the points discussed above. As we have discussed in Section 3.2 if the mass of the tensor mode is less or equal than $m_l$, Massive Gravity produces the same results as GR; we choose then masses within the range $m = 10^{-27} cm^{-1}$ - $m = 10^{-26} cm^{-1}$, whose associated anisotropies power spectrum is depicted in figure 9. Figure 10 show the low-multipole region of the correlation function in detail.

![CMB Anisotropy for tensor modes](image)

**Figure 9:** The correlation functions $C^{TT}_\ell$ for GR and Massive Gravity. Notice that the massive gravitons leave a signature on the spectrum for low multipoles.

This figure show distinct signatures for massless and massive gravitons, as we have argued above. Therefore, for the range of masses selected, massive tensor modes leave a clear signature on low multipoles $\ell < 30$. Since the heavier modes fall into the horizon earlier, they have the stronger signature, as shown. If we had chosen a different mass, say $m = 10^{-21} cm^{-1}$, the signature would be stronger, and possibly would appear for multipoles $\ell > 30$. This can be explained by simply analyzing the trend shown in figures 2 and 3, the heavier the mass, the earlier the modes fall into the horizon, which correspond to higher multipoles. However, even in this case, as the trend shown in figure 10 indicates, the signature will be particularly strong on low multipoles. Therefore, if the tensor modes of the metric fluctuations are massive, they could be detected directly by the CMB anisotropy power spectrum if their mass are greater than the limit $m_l \sim 10^{-29} cm^{-1}$, and their signatures would be noticeable specially on low multipoles.

Therefore, the results above indicate clearly that the future measurements on the TT correlation might be decisive for probing the existence of massive tensor modes, for the signature left by them could be strong enough to be distinguished from those of the massless modes.
CMB Anisotropy for tensor modes

Figure 10: The low-multipole “tail” in the TT correlation function. Notice the quite distinct signatures for $\ell < 30$ for the mass range selected.

8. Conclusions

In this work we have first studied the time evolution of massive tensor modes and shown that there is a graviton mass limit, $m_l \sim 10^{-29} cm^{-1}$, such that gravitons with masses $m \leq m_l$ behave indistinguishably from massless gravitons. The same happens to gravitons with short wavelengths (wavenumbers $n \geq 100$ in our example): their behavior is almost the same as of the massless gravitons for all the masses taken into account here.

We have also shown that long wavelength massive tensor modes fall into the horizon earlier their massless counterparts, whereas short wavelength modes behaves quite similar as in GR. The net effect of this behavior, as we have shown in the TT correlation function plotted in figures 9 and 10, is a distinguished signature on low multipoles; the heavier the mass of the mode, the stronger is its signature compared to that of massless gravitons. For the range of masses considered here, $m = 10^{-27} cm^{-1} - m = 10^{-26} cm^{-1}$, the signatures show up at $\ell < 30$; however, we have argued that such signatures might appear at $\ell > 30$ in the case of masses greater than $m = 10^{-25} cm^{-1}$.

Therefore, our results indicate that the future precise measurements of the CMB anisotropies induced by tensor modes might be decisive for probing the existence of massive gravitons, for the signature left by them could be strong enough to be distinguished from those of the massless modes.

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