FROM GROUPS TO SYMMETRIC SPACES

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We fix an algebraically closed field $k$ of characteristic exponent $p$. (We assume, except in §18, that either $p = 1$ or $p \gg 0$.) We also fix a symmetric space that is a triple $(G, \theta, K)$ where $G$ is a connected reductive algebraic group over $k$, $\theta : G \to G$ is an involution and $K$ is the identity component of the fixed point set of $\theta$ ($K$ is a connected reductive algebraic group). We shall often write $(G, K)$ instead of $(G, \theta, K)$. Let $g = \text{Lie}(G)$, $\mathfrak{k} = \text{Lie}(K)$, $\mathfrak{p} = g/\mathfrak{p}$. Note that $K$ acts naturally on $\mathfrak{p}$ by the adjoint action.

If $H$ is a connected reductive algebraic group over $k$ then $H$ gives rise to a symmetric space $(H \times H, H)$ where $H$ is imbedded in $H \times H$ as the diagonal; here $\theta(a, b) = (b, a)$. (Such a symmetric space is said to be diagonal.)

In this paper we examine various properties/constructions which are known for groups (or for diagonal symmetric spaces) and we do some experiments to see to what extent they generalize to non-diagonal symmetric spaces.

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1. Notation. Let $\mathcal{B}$ be the variety of Borel subgroups of $G$.

An element $x \in p$ is said to be nilpotent if the closure of the $K$-orbit of $x$ in $p$ contains 0. Let $\mathcal{N}$ be the set of nilpotent elements in $p$ (a closed $K$-stable subvariety of $p$).

Let $l$ be a prime number invertible in $k$ and let $\overline{\mathbb{Q}}_l$ be an algebraic closure of $\mathbb{Q}_l$.

We denote by $\overline{\mathbb{F}}_q$ a finite field with $q$ elements; $\overline{\mathbb{F}}_q$ denotes an algebraic closure of $\mathbb{F}_q$; $\psi : \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_l^*$ denotes a fixed nontrivial character.

For a finite set $X$, we denote by $|X|$ the cardinal of $X$.

In the remainder of this subsection we assume that $k = \overline{\mathbb{F}}_q$. For an algebraic variety $X$ over $k$ let $\mathcal{D}(X)$ be the bounded derived category of constructible $\overline{\mathbb{Q}}_l$-sheaves on $X$. If $f : X \rightarrow k$ is a morphism let $L_f$ be the inverse image under $f$ of the Artin-Schreier local system on $k$ defined by $f$.

Let $V$ be an $n$-dimensional vector space over $k$ and let $V^*$ be its dual. Let $\langle , \rangle : V^* \times V \rightarrow k$ be the canonical pairing. Then the local system $\mathcal{L}_{\langle , \rangle}$ on $V^* \times V$ is well defined. Consider the diagram $V^* \xrightarrow{a} V^* \times V \xrightarrow{b} V$ where $a$ and $b$ is the first and second projection. If $K \in \mathcal{D}(V)$ we set $\mathfrak{z}(K) = a_!(b^!(\mathcal{L}_{\langle , \rangle})[n]) \in \mathcal{D}(V^*)$ (Deligne-Fourier transform). If $V$ is endowed with a nondegenerate symmetric bilinear form then we can use this to identify $V$ and $V^*$ and to regard $\mathfrak{z}(K)$ as an object of $\mathcal{D}(V)$.

2. Almost diagonal symmetric spaces. Let $Z$ be the centre of $G$. Let $(G, \theta, K)$ be the symmetric space such that $G = G/Z$ and $\theta : G \rightarrow G$ is the involution induced by $\theta$. We say that $(G, \theta, K)$ is almost diagonal (resp. quasi-split) if for any involution $\theta' : G \rightarrow G$ such that $\theta, \theta'$ induce the same involution of the Weyl group of $G$ we have dim($K$) $\geq$ dim($K'$) (resp. dim($K$) $\leq$ dim($K'$)); here $K'$ is the identity component of the fixed point set of $\theta'$. We say that $(G, \theta, K)$ is almost diagonal (resp. quasi-split) if $(G, \theta, K)$ is almost diagonal (resp. quasi-split).

For example, if $(G, K)$ is diagonal then it is almost diagonal and quasi-split. In addition,

(a) $(GL_{2n}, Sp_{2n})$,
(b) $(SO_{2n}, SO_{2n-1}), (n \geq 2)$,
(c) $(E_6, F_4)$

are almost diagonal. Also, $(G, G)$ is almost diagonal.

We say that $(G, K)$ is of equal rank if $G, K$ contain a common maximal torus.
3. **A generalization of Schur’s lemma.** In this subsection we assume that \( k = \mathbb{F}_q \) and that we are given an \( \mathbb{F}_q \)-rational structure on \( G \) compatible with \( \theta \). Then the finite groups \( G(\mathbb{F}_q), K(\mathbb{F}_q) \) are well defined. For any irreducible representation \( \rho \) of \( G(\mathbb{F}_q) \) over \( \mathbb{C} \) let \( \rho_0 \) be the space of \( K(\mathbb{F}_q) \)-invariant vectors in \( \rho \). If \( (G, K) = (H \times H, H) \) is diagonal and the \( \mathbb{F}_q \)-structure on \( G \) comes from an \( \mathbb{F}_q \)-structure on \( H \) then \( \dim \rho_0 \in \{0, 1\} \). Indeed we have \( \rho = \rho' \otimes \rho'' \) where \( \rho', \rho'' \) are irreducible representations of \( H(\mathbb{F}_q) \) and the claim follows from Schur’s lemma for \( H(\mathbb{F}_q) \). If \( (G, K) \) is as in 2(a) then again \( \dim \rho_0 \in \{0, 1\} \) (see [BKS]); if \( (G, K) \) is as in 2(b), 2(c) then \( \dim \rho_0 \in \{0, 1\} \) (R. Lawther). For general \( (G, K) \), it is not true that \( \dim \rho_0 \leq 1 \); but one can show that \( \dim \rho_0 \leq c \) where \( c \) is a constant depending only on \( (G, K) \) not on \( q \) or \( \rho \) (see [L3]). Hence the algebra of double cosets \( C[K(\mathbb{F}_q) \backslash G(\mathbb{F}_q)/K(\mathbb{F}_q)] \) is a direct sum of matrix algebras of sizes \( \leq c \).

4. **Elliptic curves arising from a symmetric space.** Let \( \mathcal{O}, \mathcal{O}' \) be two \( K \)-orbits on \( \mathcal{B} \) and let \( gK \in G/K \). Following [Gi] we consider the subvariety \( \mathcal{O} \cap g\mathcal{O}' \) of \( \mathcal{B} \). This clearly depends only on the coset \( gK \). (In the case where \( (G, K) \) is diagonal, the variety \( \mathcal{O} \cap g\mathcal{O}' \) is a special case of a variety which appears in [L1, 2.5].) We will give an example (which I have found around 1990) where \( \mathcal{O} \cap g\mathcal{O}' \) is an elliptic curve with finitely many points removed. Let \( V \) be a 3-dimensional \( k \)-vector space with a fixed nondegenerate symmetric bilinear form \( (, ) \). Let \( SO(V) \) be the corresponding special orthogonal group. We take \( (G, K) = (GL(V), SO(V)) \). For any subspace \( V' \) of \( V \) we set \( V'^\perp = \{ x \in V; (x, V') = 0 \} \). Let \( Q \) be the set of lines \( L \) in \( V \) such that \( L \subseteq L^\perp \). Let \( Q' \) be the set of planes \( P \) in \( V \) such that \( P^\perp \subseteq P \). We identify \( \mathcal{B} \) with the set of pairs \( L, P \) where \( L \) is a line in \( V \), \( P \) is a plane in \( V \) and \( L \subseteq P \). There are four \( K \)-orbits on \( \mathcal{B} \):

\[
\mathcal{O}_0 = \{ L, P; L \subseteq P, L \in Q, P \in Q' \}, \quad \mathcal{O}_1 = \{ L, P; L \subseteq P, L \in Q, P \not\subseteq Q' \}, \\
\mathcal{O}_2 = \{ L, P; L \subseteq P, L \not\subseteq Q, P \in Q' \}, \quad \mathcal{O}_3 = \{ L, P; L \subseteq P, L \not\subseteq Q, P \not\subseteq Q' \}.
\]

The closure of \( \mathcal{O}_1 \) is \( \bar{\mathcal{O}}_1 = \{ L, P; L \subseteq P, L \in Q \} \). The closure of \( \mathcal{O}_2 \) is \( \bar{\mathcal{O}}_2 = \{ L, P; L \subseteq P, P \in Q' \} \). We can find \( g \in GL(V) \) such that the quadrics \( Q, gQ \) in the projective plane are in general position that is \( Q \cap gQ \) consists of four distinct points. Let \( E = \bar{\mathcal{O}}_1 \cap \bar{g\mathcal{O}}_2 = \{ L, P; L \subseteq P, L \in Q, P \in gQ' \} \). The non-singular surfaces \( \{ L, P; L \subseteq P, L \in Q \}, \{ L, P; L \subseteq P, P \in gQ' \} \) in \( \mathcal{B} \) intersect transversally hence \( E \) is a non-singular curve in \( \mathcal{B} \). For any \( L \in Q \) let \( F_L \) be the fibre of the first projection \( E \to Q \). Now \( F_L \) may be identified with the set of tangents to the quadric \( gQ \) passing through \( L \). It consists of two elements if \( L \) is not one of the four points in \( Q \cap gQ \) and is a single element otherwise. Thus \( E \) is a double covering of the projective line \( Q \) branched at four points. Hence \( E \) is an elliptic curve. Let \( E_{ij} = \bar{\mathcal{O}}_1 \cap \bar{g\mathcal{O}}_2 \) \((i, j \in [0, 3])\). We have

\[
E = E_{00} \sqcup E_{02} \sqcup E_{10} \sqcup E_{12}.
\]

We have

\[
E_{00} = \{ L, P; L \subseteq P, L \in Q \cap gQ, P \in Q' \cap gQ' \} = \emptyset.
\]

(If \( L \) is one of the four elements of \( Q \cap gQ \) and \( P \) is one of the four elements of
$Q' \cap gQ'$ then $L \not\subset P.$) We have

$$E_{02} = \{L, P; L \subset P, L \not\subset Q, L \not\subset gQ, P \in Q' \cap gQ'\}$$

$$= \{L, P; L \subset P, L \not\subset Q, P \in Q' \cap gQ'\}.$$ 

Now for any of the four elements $P \in Q' \cap gQ'$ there is a unique $L \in Q$ such that $L \subset P$. Hence $E_{02}$ consists of four points. Similarly, $E_{10}$ consists of four points. It follows that $E_{12} = E - (E_{10} \cup E_{02})$ is an elliptic curve with eight points removed.

Note that if $(G, K)$ is diagonal then the variety $\mathcal{O} \cap g\mathcal{O}'$ (with $gK \in G/K$) can never be an elliptic curve with finitely many points removed.

5. Dimension of a nilpotent orbit. If $(G, K)$ is diagonal, any $K$-orbit in $\mathcal{N}$ has even dimension. The same holds if $(G, K)$ is almost diagonal, but not in the general case.

Consider for example a $k$-vector space $V$ of dimension $N \geq 4$ with a fixed non-degenerate symmetric bilinear form $(,)$ and let $U$ be a codimension one subspace of $V$ such that $(,)$ is nondegenerate when restricted to $U$. Let $SO(V)$, $SO(U)$ be the corresponding special orthogonal groups. We have an obvious imbedding $SO(U) \subset SO(V)$ which makes $(SO(V), SO(U))$ into a symmetric space. In this case $p$ may be identified with $U$ with the obvious action of $SO(U)$. We have $\mathcal{N} = \{x \in U; (x, x) = 0\}$. This is the union of two $SO(U)$-orbits $\{0\}$ and $\mathcal{N} - \{0\}$. Note that $\dim(\mathcal{N} - \{0\}) = N - 2$ is even precisely when $N$ is even that is, precisely when $(SO(V), SO(U))$ is almost diagonal.

6. Some intersection cohomology sheaves. Let $C$ be a $K$-orbit in $\mathcal{N}$. Let $IC(C, Q_l) \in \mathcal{D}(p)$ be the intersection cohomology complex of the closure $\tilde{C}$ of $C$ with coefficients in $Q_l$, extended by 0 on $p - \tilde{C}$. Let $\mathcal{H}^i IC(C, Q_l)$ be the $i$-th cohomology sheaf of $IC(C, Q_l)$ and let $\mathcal{H}^i IC(C, Q_l)$ be its stalk at $x \in p$.

Assume for example that $(G, K) = (SO(V), SO(U))$, with $V, U, N$ as in §5. Let $C = \mathcal{N} - \{0\}$. Then $\tilde{C} = \mathcal{N}$. Moreover:

(a) If $N$ is even $\geq 4$ then $IC(C, Q_l) = Q_l$.

(b) If $N$ is odd $\geq 3$ then $\mathcal{H}^0 IC(C, Q_l) = Q_l$, $\dim \mathcal{H}^{N-3} IC(C, Q_l)$ is 1 if $x = 0$ and is 0 if $x \in p - \{0\}$; $\mathcal{H}^i IC(C, Q_l) = 0$ if $i \neq 0, N - 3$.

Now let $(G, K) = (GL(V' \times V''), GL(V') \times GL(V''))$ where $V', V''$ are $k$-vector spaces of dimension 2. In this case we may identify $p$ with $\text{Hom}(V', V'') \times \text{Hom}(V'', V')$ with the obvious action of $K$. Let $C_1 \subset p$ be the set of pairs $A, B$ where $A \in \text{Hom}(V', V''), B \in \text{Hom}(V'', V')$ are such that $\ker A = B(V'')$ is 1-dimensional and $\ker B = A(V')$ is one dimensional. Note that $C_1$ is a $K$-orbit in $\mathcal{N}$ of dimension 4. Moreover $\tilde{C}_1$ is the set of pairs $A, B$ where $A \in \text{Hom}(V', V''), B \in \text{Hom}(V'', V')$ are such that $A$ and $B$ are singular and $AB = 0, BA = 0$. We have

(c) $\mathcal{H}^0 IC(\tilde{C}_1, Q_l) = Q_l$, $\dim \mathcal{H}^{2} IC(\tilde{C}_1, Q_l)$ is 2 if $x = 0$ and is 0 if $x \in p - \{0\}$; $\mathcal{H}^i IC(\tilde{C}_1, Q_l) = 0$ if $i \neq 0, 2$.

7. Generalization of a theorem of Steinberg. In this subsection we assume that $k = \mathbf{F}_q$. Assume also that we are given an $\mathbf{F}_q$-rational structure on $G$. 


compatible with \( \theta \). Then \( p \) and \( N \) have a natural \( \mathbf{F}_q \)-structure. Steinberg [S] has shown that if \( (G, K) \) is diagonal, then the number of elements in \( N(\mathbf{F}_q) \) is an even power of \( q \). The same holds if \( (G, K) \) is almost diagonal, but not in the general case.

For example, if \( (G, K) \) is as in 2(a) then \( |N(\mathbf{F}_q)| = q^{2n^2 - 2n} \). (Assume for simplicity that \( G \) is \( \mathbf{F}_q \)-split. The \( K(\mathbf{F}_q) \)-orbits in \( N(\mathbf{F}_q) \) are indexed by the partitions of \( n \) in such a way that the stabilizer in \( K(\mathbf{F}_q) \) of an element in \( N(\mathbf{F}_q) \) has cardinal equal to the cardinal of the stabilizer in \( GL_n(\mathbf{F}_q) \) of a unipotent element of the corresponding type (with \( q \) replaced by \( q^2 \)), see [BKS]. Hence the desired equality follows from the corresponding equality in the diagonal case.)

If \( V, U, (\cdot) \) are as in §5 with \( V, U, (\cdot) \) defined over \( \mathbf{F}_q \) and \( (G, K) = (SO(V), SO(U)) \) then \( |N(\mathbf{F}_q)| \) is equal to \( q^{N-2} \) if \( N \) is even and to \( q^{N-2} \pm (q^{(N-1)/2} - q^{(N-3)/2}) \) if \( N \) is odd. Note that \( (SO(V), SO(U)) \) is almost diagonal precisely when \( N \) is even.

8. \( \mathfrak{g} \)-thin, \( \mathfrak{g} \)-thick nilpotent orbits. In this subsection we assume that \( k = \overline{\mathbf{F}}_q \). Let \( C \) be a \( K \)-orbit in \( N \). Then the Deligne-Fourier transform \( \mathfrak{g}(IC(\overline{C}, \mathbb{Q}_l)) \in \mathcal{D}(\mathfrak{p}) \) is up to shift a simple perverse sheaf on \( p \). (We can choose a \( K \)-invariant non-degenerate symmetric bilinear form on \( p \).

We say that \( C \) is \( \mathfrak{g} \)-thin if the support of \( \mathfrak{g}(IC(\overline{C}, \mathbb{Q}_l)) \) is contained in \( N \) (hence is the closure of a single \( K \)-orbit in \( N \), by the finiteness of the number of \( K \)-orbits in \( N \)). We say that \( C \) is \( \mathfrak{g} \)-thick if the support of \( \mathfrak{g}(IC(\overline{C}, \mathbb{Q}_l)) \) is equal to \( p \) (hence \( \mathfrak{g}(IC(\overline{C}, \mathbb{Q}_l)) \) restricted to some open dense subset of \( p \) is up to shift an irreducible local system).

Note that the \( K \)-orbit 0 is always \( \mathfrak{g} \)-thick. The statement that

"any \( K \)-orbit in \( N \) is \( \mathfrak{g} \)-thick"

is true if \( (G, K) \) is diagonal. It is also true if \( (G, K) \) is as in 2(a) (see §14) and if \( (G, K) \) is as in 2(b) (in this last case this follows from the computations in §12). It is however false for general \( (G, K) \).

For example if \( (G, K) = (SO(V), SO(U)) \) with \( V, U, N \) as in §5, \( N \) odd, and if \( C = N - \{0\} \) then \( C \) is not \( \mathfrak{g} \)-thick but \( \mathfrak{g} \)-thin (this follows from the computations in §12).

If \( (G, K) = (GL(V' \times V''), GL(V') \times GL(V'')) \) and \( C_1 \) are as in §6 then \( C_1 \) is \( \mathfrak{g} \)-thick (this follows from the computations in §13) but no \( K \)-orbit in \( N \) other than 0 and \( C_1 \) is \( \mathfrak{g} \)-thick.

From these and other examples it appears that, \( \mathfrak{g} \)-thin \( K \)-orbits in \( N \) can exist only if \( (G, K) \) is of equal rank.

9. \( \mathfrak{g} \)-thin nilpotent orbits and affine canonical bases. In this subsection we assume that \( (G, K) = (GL(V' \times V''), GL(V') \times GL(V'')) \) where \( V', V'' \) are \( k \)-vector spaces of dimension \( N', N'' \). In this case we may identify \( p \) with \( E := \text{Hom}(V', V'') \times \text{Hom}(V'', V') \) with the obvious action of \( K \).

Now \( N \) consists of all \( (A, B) \in E \) such that \( AB : V'' \to V'' \) and \( BA : V' \to V' \) are nilpotent. We describe the classification of \( K \)-orbits in \( N \). This is the same
as the (known) classification of isomorphism classes of nilpotent representations of fixed degree of a cyclic quiver with 2 vertices. Let \( P = \{1, -1\} \times \mathbb{Z}_{>0} \). For \((r, m) \in P\) we define \( g_{r,m} \in \mathbb{N} \times \mathbb{N}\) by \( g_{r,m} = (m/2, m/2) \) if \( m \) is even, \( g_{r,m} = ((m+r)/2, (m-r)/2) \) if \( m \) is odd. Let \( \mathcal{P} = \mathcal{P}_{N', N''} \) be the set of maps \( \sigma : \mathcal{P} \to \mathbb{N}\) such that \( \sigma \) has finite support and \( \sum_{(r, m) \in P} \sigma (r, m) g_{r, m} = (N', N'') \). For \( \sigma \in \hat{\mathcal{P}} \) let \( \mathcal{N}_\sigma \) be the set of all \((A, B) \in E\) with the following property: there exists direct sum decompositions \( V' = \oplus_{i=1}^s V'_i, V'' = \oplus_{i=1}^t V''_i \) such that
\[
AV'_i \subset V''_i, BV''_i \subset V'_i, \qquad \dim (V'_i \oplus V''_i) = m_i;
\]
the linear map \( T_i : V'_i \oplus V''_i \to V'_i \oplus V''_i, (\xi', \xi'') \mapsto (B\xi'', A\xi') \) is nilpotent with a single Jordan block; set \( r_i = 1 \) if \( T_i^{m_i-1} V'_i \neq 0 \), \( r_i = -1 \) if \( T_i^{m_i-1} V''_i \neq 0 \);
for any \((r, m) \in P\), the number of \( i \in [1, s]\) such that \( (r_i, m_i) = (r, m) \) is equal to \( \sigma (r, m) \).

Then \((\mathcal{N}_\sigma)_{\sigma \in \mathcal{P}}\) are precisely the \( K\)-orbits on \( \mathcal{N}\).

Let \( \bar{\mathcal{P}}^0 \) be the set of all \( \sigma \in \hat{\mathcal{P}} \) such that for any \( m \geq 1 \) we have either \( \sigma (1, m) = 0 \) or \( \sigma (-1, m) = 0 \).

In the remainder of this subsection we assume that \( k = \mathbb{F}_q \). We show:

(a) If \( \sigma \in \bar{\mathcal{P}}^0 \) then the \( K\)-orbit \( \mathcal{N}_\sigma \) is \( \mathfrak{g}\)-thin.

The proof is based on the theory of (affine) canonical bases. From [L4, 5.9] we see that \( \mathcal{K} = IC(\mathcal{N}_\sigma, Q_l) \) is (up to shift) an element of the canonical basis attached to the cyclic quiver with 2 vertices. Using [L5, 10.2.3] we see that \( \mathfrak{g}\mathcal{K} \) is (up to shift) also an element of the canonical basis attached to the same cyclic quiver with the opposite orientation. Using again [L4, 5.9] we see that \( \mathfrak{g}\mathcal{K} = IC(\mathcal{N}_\sigma, Q_l) \) (up to shift) for a well defined \( \sigma' \in \hat{\mathcal{P}} \) (we use an isomorphism of the cyclic quiver with the one with opposed orientation). In particular the support of \( \mathfrak{g}\mathcal{K} \) is contained in \( \mathcal{N} \). This proves (a).

We will show elsewhere that, conversely, if \( \sigma \in \hat{\mathcal{P}} \) and \( \mathcal{N}_\sigma \) is \( \mathfrak{g}\)-thin, then \( \sigma \in \bar{\mathcal{P}}^0 \).

10. Character sheaves on \( p \). In this subsection we assume that \( k = \mathbb{F}_q \).

A simple perverse sheaf on \( p \) is said to be orbital if it is \( K\)-equivariant and its support is the closure of a single \( K\)-orbit in \( \mathcal{N} \). A simple perverse sheaf on \( p \) is said to be anti-orbital (or a character sheaf) if it is of the form \( \mathfrak{g}(A) \) for some orbital simple perverse sheaf \( A \) on \( p \). (These two definitions are identical to the definitions of an orbital or anti-orbital complex on \( g \) given in [L2].) For example, if \( C \) is an \( \mathfrak{g}\)-thin nilpotent \( K\)-orbit in \( p \) then \( IC(C, Q_l) \) is up to shift both an orbital and an antiorbital simple perverse sheaf.

11. Computation of a Fourier transform. We set \( k = \mathbb{F}_q \). Let \( U \) be a \( k\)-vector space of dimension \( M \geq 3 \) with a fixed nondegenerate symmetric bilinear form \( (, ) : U \times U \to k \). When \( M \) is even we set \( \delta = 1 \) if \( (, ) \) is split over \( k \) and \( \delta = -1 \) otherwise. When \( M \) is odd we set \( \delta = 0 \). Define \( f : U \to \bar{Q}_l \) by \( f(x) = 0 \) if \( (x, x) \neq 0 \), \( f(x) = 1 \) if \( (x, x) = 0, x \neq 0 \), \( f(x) = 1 + \delta q^{(M-2)/2} \) if \( x = 0 \).

Define \( \hat{f} : U \to \bar{Q}_l \) by \( \hat{f}(x) = q^{-M/2} \sum_{y \in U} \psi ((x, y)) f(y) \) for \( x \in U \) (a Fourier transform). When \( M \) is odd we define a function \( \zeta : \{x \in U; (x, x) \neq 0\} \to \{1, -1\} \)
by \( \zeta(x) = 1 \) if \((,)\) is split on the subspace \( \{ x' \in U; (x, x') = 0 \} \) and \( \zeta(x) = -1 \) otherwise. We show:

(a) For \( M \) even we have \( \hat{f} = f \).

(b) For \( M \) odd we have \( \hat{f}(x) = \zeta(x) \) if \((x, x) \neq 0\), \( \hat{f}(x) = 0 \) if \((x, x) = 0, x \neq 0\), \( \hat{f}(x) = q^{(M-2)/2} \) if \( x = 0 \).

We have

\[
\hat{f}(0) = q^{-M/2} |\{ y \in U; (y, y) = 0 \}| + \delta q^{-1} = q^{-M/2}(q^{M-1} + \delta(q^{M/2} - q^{(M-2)/2})) + \delta q^{-1} = q^{(M-2)/2} + \delta.
\]

Now assume that \( x \in U - \{0\}, (x, x) = 0 \). We can find \( x' \in U \) such that \((x', x') = 0, (x, x') = 1\). Let \( U' = \{ z \in U; (z, x) = 0, (z, x') = 0 \} \). Any \( y \in U \) such that \((y, y) = 0\) can be written uniquely in the form \( y = ax + bx' + z \) where \( a, b \in k, z \in U' \) and \( 2ab + (z, z) = 0 \). We have

\[
\hat{f}(x) = q^{-M/2} \sum_{a, b \in k, z \in U'; 2ab + (z, z) = 0} \psi(b) + \delta q^{-1}.
\]

If \( b \neq 0 \) and \( z \in U' \) then \( a \) in the last sum is determined by \( a = -(z, z)/(2b) \).

Hence

\[
\hat{f}(x) = q^{-M/2} \sum_{b \in k^*} |U'| \psi(b) + q^{-M/2} \sum_{a \in k, z \in U'; (z, z) = 0} 1 + \delta q^{-1} = -q^{(M-4)/2} + q^{(M-2)/2}(q^{M-3} + \delta(q^{(M-2)/2} - q^{(M-4)/2})) + \delta q^{-1} = \delta.
\]

Since \( f \mapsto \hat{f} \) is an isometry, we have

\[
\sum_{x \in U; (x, x) \neq 0} \hat{f}(x)\overline{f(x)} = \sum_{x \in U} f(x)\overline{f(x)} - \sum_{x \in U; (x, x) = 0} \hat{f}(x)\overline{f(x)}
\]

\[
= \left| \{ x \in U; (x, x) = 0, x \neq 0 \} \right| + (1 + \delta q^{(M-2)/2})^2
\]

\[
- \left| \{ x \in U; (x, x) = 0, x \neq 0 \} \right| \delta^2 - q^{(M-2)/2} + \delta)^2.
\]

This is 0 if \( M \) is even hence in this case \( \hat{f}(x) = 0 \) for any \( x \in U \) such that \((x, x) \neq 0\), proving (a).

In the remainder of this subsection we assume that \( M \) is odd. Since \( f \) is invariant under the \( O(U) \times k^* \)-action on \( V \) \((k^* \) acts by homothety) the same holds for \( \hat{f} \). Hence there exists \( \lambda, \mu \in \mathbb{Q}_l \) such that \( \hat{f}(x) = \lambda \) if \( x \in \zeta^{-1}(1), \hat{f}(x) = \mu \) if \( x \in \zeta^{-1}(-1) \). We fix a 2-dimensional subspace \( R \) of \( U \) such that \((, )\) is nondegenerate, split on \( R \). Let \( U' = \{ x \in U; (x, R) = 0 \} \). For any \( c \in k^* \) we can find \( x, x' \in R \) such that \((x, x) = c, (x', x') = 0, (x, x') = 1\). Any \( y \in U \)
such that \((y, y) = 0\) can be written uniquely in the form \(y = ax + bx' + z\) where \(a, b \in k, z \in U'\) and \(a^2 c + 2ab + (z, z) = 0\). We have

\[
\hat{f}(x) = \sum_{a, b \in k, z \in U'; a^2 c + 2ab + (z, z) = 0} \psi(ac + b).
\]

Thus \(\hat{f}(x)\) does not depend on \(x\) but only on \(c\) (and \(U'\)). We denote it by \(\phi(c)\). If \(a \neq 0\) and \(z \in U'\) then \(b\) in the last sum is determined uniquely by \(b = -(a^2 c + (z, z))/(2a)\). Hence

\[
\phi(c) = \sum_{a \in k^*, z \in U'} \psi(ac - (a^2 c + (z, z))/(2a)) + \sum_{b \in k; z \in U'; (z, z) = 0} \psi(b)
= \sum_{a \in k^*, z \in U'} \psi((ac - (z, z)a^{-1})/2).
\]

We compute

\[
\sum_{c \in k^*} \phi(c) = \sum_{z \in U'} \sum_{a \in k^*} (\psi(-(z, z)a^{-1})/2) \sum_{c \in k^*} \psi(ac/2))
= -\sum_{z \in U'} \sum_{a \in k^*} \psi(-(z, z)a^{-1})/2) = -\sum_{z \in U' \setminus \{(z, z) \neq 0\}} (q - 1) + \sum_{z \in U' \setminus \{(z, z) = 0\}} (q - 1) = (q^M - q^{M-1}) - q^{M-1}(q - 1) = 0.
\]

Note also that \(\phi(c)\) depends only on the image of \(c\) in \(k^*/k^*\). The last sequence of equalities shows that \(\phi(c) = -\phi(c')\) if \(c, c' \in k^*, c'/c \notin k^*/k^*\). It follows that \(\lambda = -\mu\).

By a property of Fourier transform we have \(\sum_{x \in U} \hat{f}(x) = q^{M/2}f(0)\) hence \(\sum_{x \in U; (x, x) \neq 0} \hat{f}(x) = q^{M/2} - q^{(M-2)/2}\) that is

\[
\lambda|\zeta^{-1}(1)| + \mu|\zeta^{-1}(-1)| = |\zeta^{-1}(1)| - |\zeta^{-1}(-1)|.
\]

Since \(\lambda = -\mu\) and \(|\zeta^{-1}(1)| - |\zeta^{-1}(-1)| \neq 0\) it follows that \(\lambda = 1, \mu = -1\). This proves (b).

12. Another computation of a Fourier transform. We set \(k = F_q\). Let \(V', V''\) be two \(k\)-vector spaces of dimension 2. Let \(E = \text{Hom}(V', V'') \times \text{Hom}(V'', V')\) Let \(E^0\) be the set of all \((A; B) \in E\) such that \(A\) and \(B\) are singular and \(AB = 0, BA = 0\). (We write \((A; B)\) instead of \((A, B)\) to avoid confusion with an inner product.) Define a nondegenerate symmetric bilinear form \(, : E \times E \to k\) by \((A; B), (\hat{A}; \hat{B}) = \text{tr}(AB, V'') + \text{tr}(B\hat{A}, V')\). Define \(f : E \to Q_4\) by \(f(A; B) = 1\).
if \((A; B) \in E^\circ - \{0; 0\}\), \(f(0; 0) = 1 + 2q\), \(f(A; B) = 0\) if \((A; B) \in E - E^\circ\). Define \(\hat{f} : E \to \mathbb{Q}_l\) by

\[
\hat{f}(\tilde{A}; \tilde{B}) = q^{-4} \sum_{(A; B) \in E} \psi(((A; B), (\tilde{A}; \tilde{B}))) f(A; B).
\]

(Fourier transform).

Let \(E_{rs}\) be the set of all \((A; B) \in E\) such that \(A\) and \(B\) are nonsingular and \(AB : V'' \to V'', \ BA : V' \to V'\) are regular semisimple. Define \(\zeta : E_{rs} \to \mathbb{Q}_l\) by \(\zeta(A : B) = 1\) if the two eigenvalues of \(AB\) (or \(BA\)) are in \(F_q^*\); \(\zeta(A : B) = -1\) if the two eigenvalues of \(AB\) (or \(BA\)) are not in \(F_q^*\). We show that for any \((A; B) \in E_{rs}\) we have

(a) \[
\hat{f}(A; B) = q^{-2}\zeta(A; B).
\]

For \((A : B) \in E\) let \(f_1(A : B)\) be the number of \((L'; L'')\) where \(L'\) is a line in \(V'\), \(L''\) is a line in \(V''\) and \(BV'' \subset L' \subset \ker A\), \(AV'' \subset \ker B\). This defines a function \(f_1 : E \to \mathbb{Q}_l\). Note that \(f_1(A; B) = f(A; B)\) if \((A; B) \neq (0; 0)\) and \(f_1(0; 0) = (q+1)^2 = f(0; 0) + q^2\). Let \(\hat{f}_1 : E \to \mathbb{Q}_l\) be the Fourier transform of \(f_1\).

For \((\tilde{A}; \tilde{B}) \in E\) we have

\[
\hat{f}_1(\tilde{A}; \tilde{B}) = q^{-4} \sum_{L', L''} \sum_{\substack{(A; B); \ AV' \subset L'', AL' = 0, \ BV'' \subset L', BL'' = 0}} \psi(\text{tr}(A\tilde{B}, V'') + \text{tr}(B\tilde{A}, V'))
\]

where \(L'\) runs through the lines in \(V'\) and \(L''\) runs through the lines in \(V''\). For fixed \(L', L''\), the set of \((A; B)\) in the last sum is a \(k\)-vector space of dimension 2 and \((A, B) \mapsto \text{tr}(A\tilde{B}, V'') + \text{tr}(B\tilde{A}, V')\) is a linear form on this vector space which is zero if \(\tilde{A}L' \subset L'', \tilde{B}L'' \subset L'\) and is zero otherwise. Thus

\[
\hat{f}_1(\tilde{A}; \tilde{B}) = q^{-2}|\{L'; L''; \tilde{A}L' \subset L'', \tilde{B}L'' \subset L'\}|.
\]

Assume now that \((\tilde{A}; \tilde{B}) \in E_{rs}\). The condition that \(\tilde{A}L' \subset L''\), \(\tilde{B}L'' \subset L'\) is equivalent to \(\tilde{B}\tilde{A}L' = L', \tilde{A}L' = L''\). Hence

\[
\hat{f}_1(\tilde{A}; \tilde{B}) = q^{-2}|\{L'; \tilde{B}\tilde{A}L' = L'\}| = q^{-2}(\zeta(\tilde{A}; \tilde{B}) + 1).
\]

We have

\[
\hat{f}(\tilde{A}; \tilde{B}) = \hat{f}_1(\tilde{A}; \tilde{B}) - q^{-2} = q^{-2}\zeta(\tilde{A}; \tilde{B})
\]

and (a) is proved.
13. Computation of a Deligne-Fourier transform. The results in this subsection were found around 1990.

Let $V$ be a $k$-vector space of dimension $2n$ with a fixed nondegenerate symplectic form $\langle , \rangle : V \times V \to k$. Let
\[ E = \{ T \in \text{End}(V); \langle T(x), y \rangle = \langle x, T(y) \rangle \quad \forall x, y \in V \}. \]

The non-degenerate symmetric bilinear form $(,) : \text{End}(V) \times \text{End}(V) \to k$ given by $T, T' \mapsto \text{tr}(TT')$ remains nondegenerate when restricted to $E$. The symplectic group $Sp(V)$ acts naturally on $E$, preserving $(,)$. Let $E'$ be the set of all $T \in E$ such that $T : V \to V$ is semisimple and any eigenspace of $T$ is 2-dimensional. Note that $E'$ is open dense in $E$. Let $E_0$ be the set of all $T \in E$ such that $T : V \to V$ is nilpotent. Note that $E_0$ is $Sp(V)$-stable and the set of $Sp(V)$-orbits on $E_0$ is in natural bijection with the set of partitions of $n$. (Any element of $E_0$ has Jordan blocks of sizes $n_1, n_1, n_2, n_2, \ldots, n_t, n_t$ where $n_1 \geq n_2 \geq \cdots \geq n_t$ is a partition of $n$.)

Let $F$ be the set of all flags $V_\ast = (V_0 \subset V_1 \subset V_2 \ldots \subset V_{2n})$ in $V$ such that $\dim V_i = i$, $V_{2n-i} = \{ x \in V; \langle x, V_i \rangle = 0 \}$ for all $i \in [0, 2n]$. For $T \in E, V_\ast \in F$ we write $T \vdash V_\ast$ instead of $TV_i \subset V_i$ for all $i$.

Let $V_\ast \in F$. Let $E_{V_\ast} = \{ T \in E; T \vdash V_\ast \}$. Let $E_{V_\ast}^0 = \{ T \in E_0; T \vdash V_\ast \}$. We have:

(a) $E_{V_\ast}$ is exactly the orthogonal of $E_{V_\ast}^0$ with respect to $\langle , \rangle : E \times E \to k$.

If $T \in E_{V_\ast}, T' \in E_{V_\ast}^0$ then $TV_i \subset V_i, T'V_i \subset V_{i-1}$ for all $i \geq 1$. Hence $T'TV_i \subset V_{i-1}$ for all $i$ so that $\text{tr}(TT') = 0$ and $(T, T') = 0$. Thus $E_{V_\ast}$ is contained in the orthogonal of $E_{V_\ast}^0$ with respect to $\langle , \rangle : E \times E \to k$. From the definitions we see that
\[ \dim E_{V_\ast} + \dim E_{V_\ast}^0 = n^2 + (n^2 - n) = 2n^2 - n = \dim E \]
and (a) follows.

Let $\tilde{E} = \{(T, V_\ast) \in E \times F; T \vdash V_\ast \}$. Define $\pi : \tilde{E} \to E$ by $(T, V_\ast) \mapsto T$. Let $\mathcal{K} = \pi_! \mathbb{Q}_l \in \mathcal{D}(E)$. Let $\tilde{E}_0 = \{(T, V_\ast) \in E_0 \times F; T \vdash V_\ast \}$. Define $\pi_0 : \tilde{E}_0 \to E$ by $(T, V_\ast) \mapsto T$. Let $\mathcal{K}_0 = \pi_0! \mathbb{Q}_l \in \mathcal{D}(E)$.

In the remainder of this subsection we assume that $k = \mathbb{F}_q$. We show:

(b) $\mathfrak{F}(\mathcal{K}_0) \cong \mathcal{K}[n]$.

Consider the diagram $E \leftarrow^a E \times E \rightarrow^b E$ where $a$ (resp. $b$) is the first (resp. second) projection. By definition we have
\[ \mathfrak{F}(\mathcal{K}_0) = a_! (b^*(\pi_0! \mathbb{Q}_l) \otimes \mathcal{L}_f)[2n^2 - n]. \]

We have $b^*(\pi_0! \mathbb{Q}_l) = \rho_!(\mathbb{Q}_l)$ where $\rho : E \times \tilde{E}_0 \to E \times E$ is given by $(T; (T', V_\ast)) \mapsto (T; T')$ and
\[ \mathfrak{F}(\mathcal{K}_0) = a_!(\rho_!(\mathcal{L}_f))[2n^2 - n] = \tilde{\rho}_! \mathcal{L}_f[2n^2 - n] \]
where \( \hat{\rho} = a\rho : E \times \mathcal{E}_0 \to E \) and \( f : E \times \mathcal{E}_0 \to k \) is given by \( (T; (T', V_\omega)) \mapsto (T, T') \).

We have a partition \( E \times \mathcal{E}_0 = Z' \sqcup Z'' \) where \( Z' = \{(T; (T', V_\omega)) \in E \times \mathcal{E}_0; T \not\subset V_\omega\} \), \( Z'' = \{(T; (T', V_\omega)) \in E \times \mathcal{E}_0; T \not\subset V_\omega\} \). Let \( \hat{\rho}' : Z' \to E; \hat{\rho}'' : Z'' \to E \) be the restrictions of \( \hat{\rho}; \) let \( f' : Z' \to k, f'' : Z'' \to E \times E \) be the restrictions of \( f \). We have a distinguished triangle

\[
(\hat{\rho}''_f, L_{\hat{f}'}, \hat{\rho}_f L_{f'})
\]

in \( \mathcal{D}(E) \). We show that \( \hat{\rho}''_f L_{f''} = 0 \). For each \( T \in E \) the fibre \( Z''_T \) of \( \hat{\rho}'' \) at \( T \) may be identified with \( \{T'; V_\omega \in \mathcal{E}_0; T \not\subset V_\omega\} \) and it is enough to show that \( H^i_c(Z''_T; L_{f''}) = 0 \) for all \( T \in E \) and all \( i \). We can map \( Z''_T \) to \( \{V_\omega \in F; T \not\subset V_\omega\} \) by \( (T', V_\omega) \mapsto V_\omega \). For each \( T \in E \) and \( V_\omega \in F \) such that \( T \not\subset V_\omega \), let \( Z''_{T,V_\omega} \) be the fibre of the last map at \( V_\omega \). It is enough to show that \( H^i_c(Z''_{T,V_\omega}; L_{f''}) = 0 \) for all \( T \in E \), \( V_\omega \in F \) such that \( T \not\subset V_\omega \) and all \( i \). This follows from [L5, 8.1.13] since \( Z''_{T,V_\omega} \) is a \( k \)-vector space and the restriction of \( f'' \) to this vector space (that is \( T' \mapsto (T, T') \)) is a nonzero linear form (see (a)).

Thus we have \( \hat{\rho}''_f L_{f''} = 0 \). From the distinguished triangle above it follows that \( \hat{\rho}''_f L_f = \hat{\rho}_f L_{f'} \). From (a) we see that \( f' \) is identically zero. Hence \( L_{f'} = Q_{\omega} \). Moreover \( f' \) factors as \( Z' \twoheadrightarrow E \twoheadrightarrow E \) where \( s(T; (T', V_\omega)) = (T, V_\omega) \). Thus \( \hat{\rho}_f L_f = \pi_1 s_{\omega} Q_{\omega} \). Note that \( s \) is a vector bundle; its fibre over \( (T, V_\omega) \) may be identified with \( \{T' \in E_0; T' \not\subset V_\omega\} \), a vector space of dimension \( n^2 - n \). Hence \( s_{\omega} Q_{\omega} = \langle Q_{\omega}[−2n^2 + 2n] \rangle \) (we ignore the Tate twist). This proves (b).

Let \( \mathcal{A}_0 \) be the set of isomorphism classes of simple perverse sheaves on \( E \) which appear (possibly with a shift) as a direct summand of \( \mathcal{K}_0 \). Let \( \mathcal{A} \) be the set of isomorphism classes of simple perverse sheaves on \( E \) which appear (possibly with a shift) as a direct summand of \( \mathcal{K} \). From (b) we see that

(c) \( \mathcal{A}_0 \) defines a bijection \( \mathcal{A}_0 \cong \mathcal{A} \).

Let \( \mathcal{A}_0' \) be the set of isomorphism classes of \( Sp(V) \)-equivariant simple perverse sheaves on \( E \) with support contained in \( E_0 \). Clearly

(d) \( \mathcal{A}_0 \subset \mathcal{A}_0' \).

Let \( \pi' : \pi^{-1}(E') \to E' \) be the restriction of \( \pi \). Let \( \mathcal{A}' \) be the set of isomorphism classes of simple perverse sheaves on \( E \) which appear (possibly with a shift) as a direct summand of \( \pi_{\omega} Q_{\omega} \). Clearly we have a natural injective map

(e) \( \mathcal{A}' \to \mathcal{A} \).

From (c),(d),(e) we see that

(f) \( |\mathcal{A}'| \leq |\mathcal{A}| = |\mathcal{A}_0| \leq |\mathcal{A}_0'| \).

Now \( \pi' \) is a composition \( \pi^{-1}(E') \xrightarrow{\sigma} E'' \xrightarrow{\tau} E' \) where \( E'' \) is the set of pairs \( (T, \omega) \) where \( T \in E' \) and \( \omega \) is an indexing \( E_{\omega(1)}, E_{\omega(2)}, \ldots, E_{\omega(n)} \) of the eigenspaces of \( T \) by \( [1, n] \); \( \sigma \) is defined by \( (T, \omega) \mapsto T \); \( \tau \) is given by \( (T, V_\omega) \mapsto (T, \omega) \) with \( \omega \) defined as follows: for \( i \in [1, n] \) we have \( V_i = L_1 + L_2 + \cdots + L_i \) where \( L_1, L_2, \ldots L_i \) are lines in \( E_{\omega(1)}, E_{\omega(2)}, \ldots, E_{\omega(n)} \) respectively. Note that \( \tau \) is a fibre bundle with fibres isomorphic to a product of \( n \) projective lines. Moreover \( \sigma \) is a finite principal covering with group \( S_n \), the symmetric group in \( n \) letters. It follows that \( \tau(Q_{\omega}) \)
is a direct sum of shifts of $Q_l$. Hence the objects of $A'$ are (up to shift) the same as the direct summands of $\sigma(Q_l)$. Now $E''$ is irreducible (since $E'$ is so). It follows that the direct summands of $\sigma(Q_l)$ (up to shift and isomorphism) are in natural bijection with the irreducible representations of $S_n$ (up to isomorphism). Thus, $|A'| = p_n$ the number of partitions of $n$. Now the objects of $A'_0$ are in bijection with the $Sp(V)$-orbits on $E_0$ (the stabilizer in $Sp(V)$ of an element in $E_0$ is connected). Hence $|A'_0| = p_n$. Using now (f) we see that $|A'| = |A| = |A_0| = |A'_0| = p_n$. In particular we see that any object in $A'_0$ is in $A_0$, any object in $A$ has support equal to $E$ and $\mathcal{F}$ defines a bijection between $A'_0$ and $A$. Thus,

(g) if $A$ is an $Sp(V)$-equivariant simple perverse sheaf on $E$ with support contained in $E_0$, then $\mathcal{F}(A)$ has support equal to $E$.

14. Nilpotent $K$-orbits and conjugacy classes in a Weyl group. In this subsection we assume that $p = 1$. We show how the construction of [KL, 9.1] extends to the case of symmetric spaces.

We identify $p$ with $\{x \in g; \theta(x) = -x\}$ in the obvious way. Let $T$ be the set of subspaces $t$ of $p$ such that for some torus $T$ in $G$ the Lie algebra of $T$ equals $t$ and such that $t$ has maximum possible dimension. It is known that $K$ acts transitively on $T$ by conjugation. For $t \in T$ let $W$ be the group of components of the normalizer of $t$ in $K$. Let $W$ be the set of conjugacy classes in the finite group $W'$; this is independent of the choice of $t$.

Let $\Phi = k((\epsilon))$, $\phi = k[[\epsilon]]$ where $\epsilon$ is an indeterminate. Let $\mathfrak{g}_\phi = \Phi \otimes \mathfrak{p}$, $\mathfrak{p}_\Phi = \Phi \otimes \mathfrak{p}$, $\mathfrak{p}_\phi = \phi \otimes \mathfrak{p}$. Also, the groups $G(\Phi), K(\Phi)$ are well defined and the set $T(\Phi)$ of $\Phi$-points of $T$ is well defined (it is a set of subspaces of $\mathfrak{p}_\phi$).

The group $K(\Phi)$ acts naturally by conjugation on $T(\Phi)$; as in [KL, §1, Lemma 2] we see that the set of $K(\Phi)$-orbits on $T(\Phi)$ is naturally in 1–1 correspondence with the set $W$. For $\gamma \in W$ let $O_\gamma$ be the $K(\Phi)$-orbit on $T(\Phi)$ corresponding to $\gamma$.

Let $\mathfrak{p}_{\Phi,rs}$ be the set of all $\xi \in \mathfrak{p}_\Phi$ for which there is a unique $t' \in T(\Phi)$ such that $\xi \in t'$; we then set $t'_\xi = t'$. For $\gamma \in W$ let $\mathfrak{p}_{\Phi,rs,\gamma}$ be the set of all $\xi \in \mathfrak{p}_{\Phi,rs}$ such that $t'_\xi \in O_{\gamma}$.

Let $x \in N$. As in [KL, 9.1] there is a unique element $\gamma \in W$ such that the following holds: there exists a ”Zariski open dense” subset $V$ of $x + \mathfrak{p}_\phi$ (a subset of $\mathfrak{p}_\phi$) such that $V \subset \mathfrak{p}_{\Phi,rs,\gamma}$. We set $\Psi(x) = \gamma$. Now $\Psi$ is a map $N \rightarrow W$. This map is constant on $K$-orbits hence it defines a map from the set of $K$-orbits on $N$ to $W$ denoted again by $\Psi$. (In the case where $(G, K) = (H \times H, H)$ is diagonal, $\Psi$ reduces to the map defined in [KL, 9.1].)

If $(G, K)$ is diagonal then $\Psi$ is expected to be injective (this is known to be true in almost all cases). If $(G, K)$ is as in 2(a) then $\Psi$ is bijective (see §15). If $(G, K)$ is as in 2(b) then $\Psi$ is bijective (see §16). For general $(G, K)$, $\Psi$ is neither injective nor surjective (see §17).

15. Example: $(GL_{2n}, Sp_{2n})$. In this subsection we assume that $(G, K) = (GL(V), Sp(V))$ with $k = C$ and we keep the notation of §13. Let $\Phi, \phi$ be as in
§14. Let $\Phi$ be an algebraic closure of $\Phi$. Define $v : \Phi \to \mathbb{Q} \cup \{\infty\}$ by $v(0) = \infty$ and $v(a_0 v^f + \text{higher powers of } v) = f$ if $a_0 \in \mathbb{C}^*$, $f \in \mathbb{Q}$. Let $E_\Phi = \Phi \otimes E, E_\phi = \phi \otimes E, V_\Phi = \bigotimes_{i<j} V_{\sigma_i}$. Let $E'_\Phi$ be the set of all $T \in E_\Phi$ such that $T$ defines a semisimple endomorphism of $V_\Phi$ whose eigenspaces are all 2-dimensional. For any conjugacy class $\sigma$ in the symmetric group in $n$ letters let $E'_{\Phi,\sigma}$ be the set of all $T \in E'_\Phi$ such that, if the eigenvalues of $T$ are denoted by $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \ldots, \lambda_{2n-1} = \lambda_{2n}$ (elements of $\mathbb{C}((\epsilon^{1/N}))$ for some $N \geq 1$) then the element $\gamma$ of $\text{Gal}(\mathbb{C}((\epsilon^{1/N}))/\Phi)$ which maps $\epsilon^{1/N}$ to $\exp(2\pi \sqrt{-1}/N)\epsilon^{1/N}$ permutes $\lambda_2, \lambda_4, \ldots, \lambda_{2n}$ according to a permutation in $\sigma$. Note that $E'_\Phi = \sqcup_{\sigma} E'_{\Phi,\sigma}$.

Let $x \in E_0$. Recall that $x : V \to V$ has Jordan blocks of sizes

$$n_1, n_1, n_2, n_2, \ldots, n_t, n_t$$

where $n_1 \geq n_2 \geq \cdots \geq n_t$ is a partition of $n$. The following statement is easily verified (it can be reduced to the case where $t = 1$).

(a) There exists $J \in E$ such that $x + \epsilon J : V_\Phi \to V_\Phi$ has eigenvalues $\exp(2\pi \sqrt{-1}s/n_i) a_i \epsilon^{1/n_i} \in \Phi$ ($i = 1, 2, \ldots, 2t, s = 1, 2, \ldots, n_i$) where $a_2, a_4, \ldots, a_{2t}$ are distinct in $\mathbb{C}$ and $a_1 = a_2, a_3 = a_4, \ldots, a_{2t-1} = a_{2t}$. In particular we have $x + \epsilon J \in E'_\Phi$.

Now let $X = X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots \in E_\phi$ where $X_0, X_1, \ldots$ are in $E$. Let $\bar{x} = x + \epsilon X \in E_\Phi$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ be the eigenvalues of $\bar{x}$ in $\Phi$. We can assume that $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \ldots, \lambda_{2n-1} = \lambda_{2n}$ and $v(\lambda_2) \leq v(\lambda_4) \leq \cdots \leq v(\lambda_{2n})$. Let $\mu_1, \mu_2, \ldots, \mu_{2n}$ be the eigenvalues of $x + \epsilon J$. We can assume that $\mu_1 = \mu_2, \mu_3 = \mu_4, \ldots, \mu_{2n-1} = \mu_{2n}$ and $v(\mu_2) \leq v(\mu_4) \leq \cdots \leq v(\mu_{2n})$. Applying [KL, 9.4] we have $v(\lambda_1 \lambda_2 \ldots \lambda_s) \geq v(\mu_1 \mu_2 \ldots \mu_s)$ for $s = 1, 2, \ldots, 2n$. Using this for $s = 2, 4, \ldots, 2n-2$ and adding the resulting inequalities we deduce

$$v(\lambda_2^{2n-2} \lambda_4^{2n-4} \ldots \lambda_{2n-2}^2) \geq v(\mu_2^{2n-2} \mu_4^{2n-4} \ldots \mu_{2n-2}^2)$$

hence

$$v(\lambda_2^{n-1} \lambda_4^{n-2} \ldots \lambda_{2n-2}) \geq v(\mu_2^{n-1} \mu_4^{n-2} \ldots \mu_{2n-2}).$$

For any $z \in E_\Phi$ we denote by $\Theta(z) \in \Phi$ the trace of the $(4n^2 - 4n)$-th exterior power of $\text{ad}(z) : \text{End}(V_\Phi) \to \text{End}(V_\Phi)$. We have $\Theta(\bar{x}) = \prod_{1 \leq i < j \leq n} (\lambda_{2i} - \lambda_{2j})^8$.

Note that

$$v(\Theta(\bar{x})) = 8 \sum_{i<j} v(\lambda_{2i} - \lambda_{2j}) \geq 8 \sum_{i<j} v(\lambda_{2i})$$

$$= 8v(\lambda_2^{n-1} \lambda_4^{n-2} \ldots \lambda_{2n-2}) \geq v(\mu_2^{n-1} \mu_4^{n-2} \ldots \mu_{2n-2})$$

$$= 8 \sum_{i<j} v(\mu_{2i}) = 8 \sum_{i<j} v(\mu_{2i} - \mu_{2j}) = v(\Theta(x + \epsilon J)).$$

Moreover if $v\Theta(\bar{x}) = v\Theta(x + \epsilon J)$ then $v(\lambda_{2i} - \lambda_{2j}) = v(\lambda_{2i})$ for all $i < j$ in $[1,n]$ and $v(\lambda_1 \lambda_3 \ldots \lambda_s) = v(\mu_1 \mu_3 \ldots \mu_s)$ for $s = 2, 4, \ldots, 2n-2$ hence $v(\lambda_2) = v(\mu_2)$ that is, $v(\lambda_i) = v(\mu_i)$ for $i = 2, 4, \ldots, 2n-2$. 

Let \( S \) be the set of all \( \tilde{x} \) in \( x + \epsilon E_\phi \) such that \( v(\Theta(\tilde{x})) = v(\Theta(x + \epsilon J)) \). Then \( S \neq \emptyset \) (it contains \( x + \epsilon J \)) and is "Zariski open" in \( x + \epsilon E_\phi \). Let \( \tilde{x} \in S \) and let \( \lambda_1, \ldots, \lambda_{2n} \) be as above. We have

(a) \( v(\lambda_1) = v(\lambda_2) = \cdots = v(\lambda_{2n_1}) = 1/n_1, \)
(b) \( v(\lambda_{2n_1+1}) = v(\lambda_{2n_1+2}) = \cdots = v(\lambda_{2n_1+2n_2}) = 1/n_2, \)

except that when \( n_1 = 1, v(\lambda_{2n-1}) = v(\lambda_{2n}) \) is an integer (or \( \infty \)) not necessarily equal to \( v(\mu_{2n}) \). As in [KL, p.156] we see that

\[
v(\lambda_i) = 1/m \implies \lambda_i \in \mathbb{C}[\epsilon^{1/m}].
\]

(In [KL, p.156] lines -7, -8 one should replace "\( t \) not divisible by \( m \)" by "\( t \) not a divisor of \( m \)."

16. Example: \((SO_{2n}, SO_{2n-1})\). In this subsection we assume that \( k = \mathbb{C} \) and that \((G, K) = (SO(V), SO(U))\) with \( V, U, (, ) \) as in §5 and with \( \dim(U) \) odd. Let \( \Phi, \phi, \Phi, \psi \) be as in §15. Let \( U_\Phi = \Phi \otimes U, U_\phi = \phi \otimes U, U_\Phi = \Phi \otimes U \). Let \( U_\Phi' \) be the set of all \( x \in U_\Phi \) such that \( (x, x) \neq 0 \). Let \( U_\Phi', U_\Phi' \) be the set of all \( x \in U_\Phi' \) such that \( v((x, x)) \in 2 \mathbb{Z} \) (resp. \( v((x, x)) \in 2 \mathbb{Z} + 1 \). We have \( U_\Phi' = U_\Phi' \cup U_\Phi' \).

Let \( U_0 = \{x \in U_0, (x, x) = 0\} \). Let \( x \in U_0 \). Let \( X = u_0 + \epsilon u_1 + \cdots \in U_\Phi \) where \( u_0, u_1, \ldots \) are in \( U \). We have

\[
(x + \epsilon X, x + \epsilon X) = 2(x, u_0)\epsilon + (2(x, u_1) + (u_0, u_0))\epsilon^2 + a\epsilon^3
\]

with \( a \in \phi \). If \( x = 0 \) let \( S \) be the set of all \( x + \epsilon X \) (with \( X \) varying as above) such that \( (u_0, u_0) \neq 0 \).

If \( x \in U_0 - \{0\} \) let \( S \) be the set of all \( x + \epsilon X \) (with \( X \) varying as above) such that \( (x, x) \neq 0 \). In any case \( S \) is a nonempty "Zariski open" subset of \( x + \epsilon U_\Phi \).

If \( x = 0 \) we have \( S \subset U_\Phi' \). If \( x \in U_0 - \{0\} \) we have \( S \subset U_\Phi', U_\Phi' \).

We see that the map \( \Psi \) in §14 is bijective in this case.

17. Example: \((GL_4, GL_2 \times GL_2)\). In this subsection we assume that \((G, K) = (GL(V', V''), GL(V') \times GL(V''))\) where \( V', V'' \) are \( k \)-vector spaces of dimension 2 and \( k = \mathbb{C} \). We may identify \( \mathfrak{p} \) with \( \text{Hom}(V', V'') \times \text{Hom}(V'', V') \) with the obvious action of \( K \). Let \( \Phi, \phi, \Phi, \psi \) be as in §15. Let \( V'_\Phi = \Phi \otimes V', V''_\Phi = \Phi \otimes V' \). In our case \( p_{\Phi, rs} \) consists of all \((A, B) \in \text{Hom}(V'_\Phi, V''_\Phi) \times \text{Hom}(V''_\Phi, V'_\Phi)\) such that \( AB : V''_\Phi \rightarrow V''_\Phi \) is a regular semisimple automorphism (or equivalently, \( BA : V'_\Phi \rightarrow V'_\Phi \) is a regular semisimple automorphism). The group \( \mathcal{W} \) in §14 is in our case a dihedral group of order 8; hence \( \mathcal{W} \) has five elements, say \( \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \). We describe the corresponding partition \( p_{\Phi, rs} = \bigsqcup_{j \in \{1, 3\}} p_{\Phi, rs, \gamma j} \).

\( p_{\Phi, rs, \gamma j} \) consists of all \((A, B) \in p_{\Phi, rs}\) such that the eigenvalues \( \lambda, \mu \) of \( AB \) in \( \Phi \) satisfy the following condition:

\[
\begin{align*}
j = 1: & \lambda, \mu \in \Phi, v(\lambda) \in 2 \mathbb{Z}, v(\mu) \in 2 \mathbb{Z}.
\end{align*}
\]

\[
\begin{align*}
j = 2: & \lambda, \mu \in \Phi, v(\lambda) \in 2 \mathbb{Z} + 1, v(\mu) \in 2 \mathbb{Z} + 1.
\end{align*}
\]

\[
\begin{align*}
j = 3: & \lambda, \mu \in \Phi, v(\lambda) \in 2 \mathbb{Z}, v(\mu) \in 2 \mathbb{Z} + 1.
\end{align*}
\]
Let $e_1, e_2$ be a basis of $V'$; let $e_3, e_4$ be a basis of $V''$. The following elements form a set of representatives for the $K$-orbits on $N$.

\begin{align*}
N_1 & : e_1 \mapsto e_4, e_2 \mapsto e_3, e_3 \mapsto 0, e_4 \mapsto e_2; \\
N_2 & : e_1 \mapsto 0, e_2 \mapsto e_4, e_3 \mapsto e_2, e_4 \mapsto e_1; \\
N_3 & : e_1 \mapsto e_4, e_2 \mapsto 0, e_3 \mapsto 0, e_4 \mapsto e_2; \\
N_4 & : e_1 \mapsto 0, e_2 \mapsto e_4, e_3 \mapsto 0, e_4 \mapsto e_1; \\
N_5 & : e_1 \mapsto e_4, e_2 \mapsto e_3, e_3 \mapsto 0, e_4 \mapsto 0; \\
N_6 & : e_1 \mapsto 0, e_2 \mapsto 0, e_3 \mapsto e_2, e_4 \mapsto e_1; \\
N_7 & : e_1 \mapsto e_4, e_2 \mapsto 0, e_3 \mapsto e_2, e_4 \mapsto 0; \\
N_8 & : e_1 \mapsto e_4, e_2 \mapsto 0, e_3 \mapsto 0, e_4 \mapsto 0; \\
N_9 & : e_1 \mapsto 0, e_2 \mapsto 0, e_3 \mapsto e_1, e_4 \mapsto 0; \\
N_{10} & = 0.
\end{align*}

For $i \in [1, 10]$ we consider $(A_i, B_i) = N_i + \epsilon \xi$ where $\xi \in \mathfrak{p}_\phi$ is given by

\begin{align*}
 e_1 & \mapsto a e_3 + b e_4, e_2 \mapsto c e_3 + d e_4, e_3 \mapsto x e_1 + y e_2, e_4 \mapsto z e_1 + u e_2
\end{align*}

and $a, b, c, d, x, y, z, u \in \Phi$. Define $a_0, b_0, c_0, d_0, x_0, y_0, z_0, u_0 \in \mathbb{C}$ to be the constant terms of $a, b, c, d, x, y, z, u$. The characteristic polynomial $P_i$ of $A_i B_i$ is given by

\begin{align*}
P_1 & = X^2 - O(\epsilon) X + \epsilon x_0 + O(\epsilon^2); \\
P_2 & = X^2 - O(\epsilon) X + \epsilon a_0 + O(\epsilon^2); \\
P_3 & = X^2 - (\epsilon(z_0 + d_0) + O(\epsilon^2)) X + \epsilon^2 c_0 x_0 + O(\epsilon^3); \\
P_4 & = X^2 - (\epsilon(b_0 + u_0) + O(\epsilon^2)) X + \epsilon^2 a_0 y_0 + O(\epsilon^3); \\
P_5 & = X^2 - (\epsilon z_0 + y_0) + O(\epsilon^2)) X + \epsilon^2(-x_0 u_0 + y_0 z_0) + O(\epsilon^3); \\
P_6 & = X^2 - (\epsilon(b_0 + c_0) + O(\epsilon^2)) X + \epsilon^2(-a_0 d_0 + b_0 c_0) + O(\epsilon^3); \\
P_7 & = X^2 - (\epsilon(z_0 + c_0) + O(\epsilon^2)) X + \epsilon^2 c_0 z_0 + O(\epsilon^3); \\
P_8 & = X^2 - (\epsilon x_0 + O(\epsilon^2)) X + \epsilon^3 c_0 (x_0 u_0 - y_0 z_0) + O(\epsilon^4); \\
P_9 & = X^2 - (\epsilon a_0 + O(\epsilon^2)) X + \epsilon^3 u_0 (a_0 d_0 - b_0 c_0) + O(\epsilon^4); \\
P_{10} & = X^2 - (\epsilon^2(a_0 x_0 + b_0 z_0 + c_0 y_0 + d_0 u_0) + O(\epsilon^3)) X + \epsilon^4(a_0 d_0 - b_0 c_0)(x_0 u_0 - y_0 z_0) + O(\epsilon^5),
\end{align*}

where $O(\epsilon^m)$ denotes an element of $\epsilon^m \Phi$. We see that:

\begin{itemize}
  \item if $x_0 \neq 0$ then $N_1 + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_1}$;
  \item if $a_0 \neq 0$ then $N_2 + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_1}$;
  \item if $c_0 x_0 ((z_0 + d_0)^2 - 4 c_0 x_0) \neq 0$, then $N_3 + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_2}$;
  \item if $a_0 y_0 ((b_0 + u_0)^2 - 4 a_0 y_0) \neq 0$, then $N_4 + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_2}$;
  \item if $(x_0 u_0 - y_0 z_0) ((z_0 - y_0)^2 + 4 x_0 u_0) \neq 0$, then $N_5 + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_2}$;
  \item if $(a_0 d_0 - b_0 c_0) ((b_0 - c_0)^2 + 4 a_0 d_0) \neq 0$, then $N_6 + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_2}$;
  \item if $c_0 z_0 (z_0 - c_0) \neq 0$, then $N_7 + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_2}$;
  \item if $c_0 z_0 (x_0 u_0 - y_0 z_0) \neq 0$, then $N_8 + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_3}$;
  \item if $a_0 u_0 (a_0 d_0 - b_0 c_0) \neq 0$, then $N_9 + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_3}$;
  \item if $(a_0 d_0 - b_0 c_0)(x_0 u_0 - y_0 z_0) \neq 0$ and $(a_0 x_0 + b_0 z_0 + c_0 y_0 + d_0 u_0)^2 - 4(a_0 d_0 - b_0 c_0)(x_0 u_0 - y_0 z_0) \neq 0$, then $N_{10} + \epsilon \xi \in \mathfrak{p}_{\Phi, rs, \gamma_3}$.
\end{itemize}

Thus we have
\[ \Psi(N_1) = \Psi(N_2) = \gamma_4, \]
\[ \Psi(N_3) = \Psi(N_4) = \Psi(N_5) = \Psi(N_6) = \Psi(N_7) = \gamma_2, \]
\[ \Psi(N_8) = \Psi(N_9) = \gamma_3, \]
\[ \Psi(N_{10}) = \gamma_1. \]

**18. Final comments.** We want to define the notion of symmetric space without any assumption on \( p \). Let \( G \) be a connected reductive group over \( k \) and let \( K \) be a closed connected reductive subgroup of \( G \). For simplicity we assume that \( G, K \) contain a common maximal torus \( T \). (A similar definition can be given without this assumption.) We have inclusions \( R_K \subset R_G \subset X(T) \) where \( X(T) \) is the character group of \( T \) and \( R_K \) (resp. \( R_G \)) is the set of roots of \( K \) (resp. \( G \)). We say that \((G,K)\) is a symmetric space (of equal rank) if the inclusions \( R_K \subset R_G \subset X(T) \) are the same as the corresponding inclusions for a symmetric space of equal rank in characteristic 0. Thus \( K \) does not necessarily come from an involution of \( G \). For example if \( p = 2 \) we can take \( G \) of type \( E_8 \) and \( K \) a subgroup of type \( D_8 \) (such a subgroup exists but it is not the fixed point set of an involution of \( G \)). It would be interesting to see how many of the basic properties of symmetric spaces in characteristic 0 extend to this more general case (including the case \( p = 2 \)).

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