APPORXIMATION OF THE TRAJECTORY ATTRACTOR OF
THE 3D SMECTIC-A LIQUID CRYSTAL FLOW EQUATIONS

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(Communicated by Shouchuan Hu)

Abstract. In this paper, we first establish the existence of trajectory attractors for the 3D smectic-A liquid crystal flow system and 3D smectic-A liquid crystal flow–α model, and then prove that the latter trajectory attractor converges to the former one as the parameter α → 0+

1. Introduction. In this paper, we consider a hydrodynamic smectic-A liquid crystal flow model which was proposed, starting from a dynamic continuum theory, in [20, 10]. The system of smectic-A liquid crystal flow is a model which describes the usual director inconsistency with the unit layer normal. The system of nematic liquid crystal flow is a model which describes the molecules director d which tends to align along a preferred direction with no positional order of centers of mass. The smectic-A liquid crystal equation is composed of the Navier-Stokes equations for the fluid velocity u and a fourth-order equation for the layer variable ϕ. The general smectic-A liquid crystal flow equations take the following form:

\[
\begin{align*}
\rho_t + \rho \cdot \nabla u &= 0, \\
\rho u_t + \rho u \cdot \nabla u + \nabla P &= \nabla \cdot (\sigma^e + \sigma^d), \\
\text{div } u &= 0, \\
\partial_t \varphi + (u \cdot \nabla) \varphi &= \lambda \left( \nabla \cdot (\xi \nabla \varphi) - K \Delta^2 \varphi \right),
\end{align*}
\]

where

\[
\sigma^e = -\xi d \otimes d + K \nabla (\nabla \cdot d) \otimes d - K (\nabla \cdot d) \nabla \nabla \varphi,
\]

2010 Mathematics Subject Classification. Primary: 76N10, 76N15; Secondary: 35M13, 35Q35, 53C35.

Key words and phrases. Liquid crystal flow, trajectory attractors, weak solutions.

This paper was supported in part by the NNSF of China with contract numbers 11671075, 11801068, 11971110 and the Graduate Innovation Fund Project of Donghua University with contract number CUSF-DH-D-2020077.

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\[
\sigma^d = \mu_1 (d^T D(u)d) d \otimes d + \mu_4 D(u) + \mu_5 (D(u) d \otimes d + d \otimes D(u)d).
\]

In the above system, \( \rho \) is the density of the liquid crystal material, \( u \) is the flow velocity, \( P \) is the fluid pressure and \( \varphi \) denotes the layer variable; the unit vector \( d \) stands for the molecules orientation of liquid crystal. In the smectic-A phase, the molecular direction and layer variable are such that \( d = \nabla \varphi \). Furthermore, \( \sigma^e \) and \( \sigma^d \) are the elastic stress tensor and the dissipative stress tensor respectively, \( \mu_1, \mu_4, \mu_5 \geq 0 \) are dissipative constant coefficients and \( D(u) \) denotes the symmetric matrix of the velocity gradient \( D(u) = \frac{1}{2}(\nabla u + \nabla^T u) \).

In the literature, one usually considers the case \( \xi d = f(d) \), where \( f(d) \) is the Ginzburg-Landau penalization function \( f(d) = \frac{1}{4\epsilon}(|d|^2 - 1) d \) \((0 < \epsilon \leq 1)\) and the penalization function \( F(d) = \frac{1}{4\epsilon}(|d|^2 - 1)^2 \) is such that \( f(d) = \frac{\partial F}{\partial d} \). In this paper, for simplicity, we take \( \mu_1 = \mu_5 = 0, \rho = \lambda = K = \epsilon = 1, \nu = \frac{\mu_4}{2} \), and then arrive at the following equations:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla P &= \mu \nabla \varphi, \\
\text{div } u &= 0, \\
\partial_t \varphi + (u \cdot \nabla) \varphi &= -\mu, \\
\mu &= \Delta^2 \varphi - \nabla \cdot f(\nabla \varphi),
\end{align*}
\tag{1.2}
\]

where \( F(\nabla \varphi) = \frac{1}{4}(|\nabla \varphi|^2 - 1)^2 \), \( f(\nabla \varphi) = (|\nabla \varphi|^2 - 1) \nabla \varphi \).

We now consider (1.2) subject to the initial and boundary conditions

\[
\begin{align*}
(u(x,0), \varphi(x,0)) &= (u_0(x), \varphi_0(x)), & \text{div } u_0 = 0, & x \in \Omega, \\
\varphi(x,t) &= 0, & \frac{\partial \varphi}{\partial n}(x,t) = \frac{\partial \Delta \varphi}{\partial n}(x,t) = 0, & x \in \partial \Omega.
\end{align*}
\tag{1.3}
\]

The first well-known result for the smectic-A liquid crystal flow was established in [16]. Liu [16] considered an incompressible system like (1.1) and the initial and boundary conditions like (1.3) in a bounded domain \( \Omega \), established the energy dissipation relation of the system and proved the existence of global weak solutions by applying a Galerkin procedure in both 2D and 3D. Moreover, he established the existence of classical solutions and the regularity of the weak solutions for \( \mu_4 \) large enough. Some regularity and stability results were also discussed in [16]. Later on, an approximated system like (1.2) was studied in [7], in which the authors considered the initial-boundary problem for arbitrary initial data, obtaining the existence of weak solutions which are bounded up to infinity in time. The existence of time-periodic weak solutions is also proved. The authors in [7] obtained the existence and uniqueness of regular solutions (up to infinity in time), but assuming a dominant viscosity coefficient in the linear part of the diffusion tensor. One year later, an hydrodynamic system like (1.1), with constant density function for the long-time behavior of the solutions, was analyzed in [19]. The authors proved that problem (1.1) possesses a global attractor in some phase space and possesses an exponential attractor which entails that the global attractor has finite fractal dimension in 2D. Corresponding results in 3D were also obtained; the authors showed that system (1.1) possesses a unique global strong solution when the initial data are small. Recently, it was proved in [15] that the system (1.2) has a compact absorbing set which is contained in \( V \times H^4 \), and that the associated dynamical system has a global attractor in two dimensional bounded domains. However, there is no result on attractors in 3D.
Although there are many useful results on the smectic-A liquid crystal flow, there are few results in 3D without any assumption. In this paper, we mainly study the long-time behavior in 3D. The notion of a trajectory attractor for evolutionary partial differential equations was developed in [3, 4, 5, 6]. Trajectory attractors can describe the long-time behavior of solutions to evolution equations for which the uniqueness of solutions is not proved yet or fails. We know that the existence of global weak solutions for the smectic-A liquid crystal flow in 3D has been proved in [16]. However, the uniqueness of solutions is still unknown. We study the relation between trajectory attractors of the 3D smectic-A liquid crystal flow system and the 3D smectic-A liquid crystal-α model, as well as the convergence of trajectory attractors as α → 0⁺. Similar results were established in [1, 2, 8, 12, 22] for the Navier-Stokes-α model, the 3D MHD-α model, the 3D Binary fluid mixtures and the 3D Navier-Stokes-Voight model, in which the authors also established the existence of trajectory attractors and studied the relation between trajectory attractors of the fluid system and their α model. However, there are some differences in this paper. Indeed, the trajectory space that we choose in this paper is more complicated than those chosen in [1, 2, 8, 22]. We overcome such difficulties by more accurate calculations, and our results on trajectory spaces extend those in [1, 2, 8, 22].

The article is structured as follows. Section 1 is the introduction. In Section 2, we prove the existence of weak solutions for the 3D smectic-A liquid crystal flow equations by energy estimates. In Section 3 and 4, we prove existence of the trajectory attractor for the 3D smectic-A liquid crystal flow equations and the trajectory attractor of the 3D smectic-A liquid crystal flow-α model, respectively. In Section 5, we show that the trajectory attractor of the 3D smectic-A liquid crystal flow-α model converges to the trajectory attractor of the 3D smectic-A liquid crystal flow system when α → 0⁺ in the topology Θ⁺ loc.

2. Existence of global weak solutions and their trajectory attractors.

2.1. Preliminaries. We first introduce some functional spaces and operators. Set

\[ \mathcal{V} = \{ \xi(x) = (\xi^1(x), \xi^2(x), \xi^3(x)) \in (C_0^\infty(\overline{\Omega}))^3, \ \nabla \cdot \xi = 0, \ \xi|_{\partial \Omega} = 0 \}; \]

\[ \mathcal{H} = \text{closure of } \mathcal{V} \text{ in } (L^2(\Omega))^3 \text{ with norm } \| \xi \|_{L^2}, \ \mathcal{H}' \text{ is the dual space of } \mathcal{H}; \]

\[ \mathcal{V} = \text{closure of } \mathcal{V} \text{ in } (H^1(\Omega))^3 \text{ with norm } \| \nabla \xi \|_{L^2}, \ \mathcal{V}' \text{ is the dual space of } \mathcal{V}. \]

Here we denote \( \| \cdot \| = \| \cdot \|_{L^2} \) for short.

The operator \( P : [L^2(\Omega)]^3 \to \mathcal{H} \) denotes the orthogonal projector, and \( A \) is the operator with domain \( D(A) = (H^2(\Omega))^3 \cap \mathcal{V} \),

\[ Au = -P \Delta u. \]

It is easy to see that the operator \( A \) is a self-adjoint and positive operator. The bilinear operator \( B(u, u) \) on \( \mathcal{V} \times \mathcal{V} \to \mathcal{V}' \) is defined as \( \langle B(u, v), w \rangle = b(u, v, w) \), and the trilinear form \( b : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \),

\[ b(u, v, w) = \int_\Omega (u \cdot \nabla)v \cdot w \, dx = \sum_{i,j=1}^3 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \]

satisfies

\[ b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0, \quad u, v, w \in \mathcal{V}, \quad (2.1) \]

and

\[ |\langle B(u, v), w \rangle| \leq c\|u\|_3 \|\nabla v\| \|\nabla w\|, \quad \|B(u, v)\|_{\mathcal{V}'} \leq c\|u\|_3 \|v\|. \quad (2.2) \]
We recall the Poincaré inequality,
\[ \|u\| \leq c\|\nabla u\|, \forall u \in \mathcal{V}. \quad (2.3) \]

Lemma 2.1 ([1]). For any \( f_n(t) \in L^p(0, M; \mathcal{V}') \), assume the operator \( A \) is self-adjoint and positive and \( f \) weakly in the space \( L^p(0, M; \mathcal{V}') \), \( p > 1 \) as \( n \to \infty \).

Let also \( \alpha_n \to 0^+ \) as \( n \to \infty \). Then there holds that
\[ (1 + \alpha^2_n A)^{-\frac{1}{2}} f_n(t) \to f(t) \text{ weakly in } L^p(0, M; \mathcal{V}') \text{ as } n \to \infty. \]

2.2. Existence of weak solutions for the 3D smectic-A liquid crystal flow equations.

Definition 2.2. We say that a pair of functions \((u, \varphi)\) is a weak solution of problem (1.2)-(1.3) on \([0, M]\) if
(i) \( u \) and \( \varphi \) satisfy
\[ u \in L^2(0, M; \mathcal{V}) \cap L^\infty(0, M; \mathcal{H}), \quad u_t \in L^\frac{4}{2}(0, M; \mathcal{V}'), \]
\[ \varphi \in L^2(0, M; H^4) \cap L^\infty(0, M; H^2), \quad \varphi_t \in L^\frac{4}{2}(0, M; \mathcal{H}), \]
(ii) \((u, \varphi)\) satisfies, in the sense of distributions,
\[ \langle u_t, v \rangle + \langle B(u, u), v \rangle + \nu \langle \nabla u, \nabla v \rangle = \langle \mu \nabla \varphi, v \rangle, \quad \text{for each } v \in \mathcal{V}, \]
\[ \varphi_t + (u \cdot \nabla) + (\Delta^2 \varphi - \nabla \cdot f(\nabla \varphi)) = 0, \quad \text{a.e. in } \Omega, \]
for almost every \( t \in (0, M) \), and
(iii) \( u(0) = u_0, \varphi(0) = \varphi_0 \).

Theorem 2.1. Let \( u_0 \in \mathcal{H}, \varphi_0 \in H^2 \). Then the problem (1.2)-(1.3) admits a weak solution \( u(t) \in L^\infty(0, M; \mathcal{H}) \cap L^2(0, M; \mathcal{V}), \varphi(t) \in L^\infty(0, M; H^2) \cap L^2(0, M; H^4) \) such that \( u(0) = u_0, \varphi(0) = \varphi_0 \), for any function \( \psi(t) \in C_0^\infty(0, M) \), there holds
\[ \begin{align*}
- \int_0^M \left( \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|\Delta \varphi(t)\|^2 + \int_\Omega F(\nabla \varphi) dx \right) \psi'(t) dt \\
+ \int_0^M \left( \nu \|\nabla u\|^2 + \|\Delta^2 \varphi - \nabla \cdot f(\nabla \varphi)\|^2 \right) \psi(t) dt \leq 0. \quad (2.4)
\end{align*} \]

Proof. The proof of Theorem 2.1 for a weak solution \((u(t), \varphi(t))\) is carried out by using a Galerkin scheme. We look for an approximated solution \((u_m(t), \varphi_m(t))\) such that
\[ u_m(t) = \sum_{i=1}^m a_{j,m}(t)w_j(x), \quad \varphi(t) = \sum_{i=1}^m b_{j,m}(t)q_j(x), \]
where \( \{w_j\} \subset \mathcal{V} \) is a sequence which is dense and orthogonal in \( \mathcal{V} \), and \( \{q_j\} \) is a sequence of eigenfunctions of the minus Laplace operator \( A \). Furthermore \( a_{j,m}, b_{j,m} \) are functions to be determined in \( C^1(0, M) \) in such a way that \((u_m(t), \varphi_m(t))\) satisfies the following problem:
\[ \begin{align*}
\frac{d u_m}{dt} + \nu P_{0,m} A u_m + P_{0,m} B(u_m, u_m) - P_{0,m} B(\mu_m, \varphi_m) &= 0, \\
\frac{d \varphi_m}{dt} + P_{1,m} B(u_m, \varphi_m) + P_{1,m} \mu_m &= 0, \\
\frac{d u_m}{dt} &= u_{0,m}, \quad \varphi_m(0) = \varphi_{0,m}. \quad (2.5)
\end{align*} \]
Here \( P_{0,m} \) and \( P_{1,m} \) are orthogonal projectors from \( \mathcal{H} \) and \( L^2(\Omega) \) onto the linear space spanned by \( u_1, u_2, \ldots, u_m \) and \( \varphi_1, \varphi_2, \ldots, \varphi_m \), respectively. Moreover, we have
\[ u_{0,m} = P_{0,m} u_0, \quad \varphi_m = P_{1,m} \varphi_0. \]
Next we take the scalar product in $\mathbb{H}$ of equation (2.5) with $u_m$, then the scalar product of equations (2.5) with $\mu_m$. Observe in particular that
\[
\langle B(u_m, u_m), u_m \rangle = 0, \quad \langle P_0 B(u_m, \varphi_m), u_m \rangle = \langle P_1 B(u_m, \varphi_m), \mu_m \rangle.
\]
Since $\mu_m = \Delta^2 \varphi_m - \nabla \cdot f(\nabla \varphi_m)$, and using the equivalent of (2.12) below, it follows that
\[
\frac{d}{dt} \langle \frac{d\varphi_m}{dt}, \mu_m \rangle = \frac{d}{dt} \langle \frac{d\varphi_m}{dt}, \Delta^2 \varphi_m \rangle - \langle \frac{d\varphi_m}{dt}, \nabla \cdot f(\nabla \varphi_m) \rangle = \frac{1}{2} \frac{d}{dt} \|\Delta \varphi_m\|^2 + \nu \int_\Omega F(\nabla \varphi_m) dx.
\]

(2.6)

Add then the resulting relations (performing similar calculations as in (2.10)-(2.12) below). Then, we obtain the following energy equality:
\[
\frac{d}{dt} \left( \frac{1}{2} \|u_m(t)\|^2 + \frac{1}{2} \|\Delta \varphi_m(t)\|^2 + \int_\Omega F(\nabla \varphi_m) dx \right) + \nu \|\nabla u_m\|^2 + \|\mu_m\|^2 = 0.
\]

(2.7)

Integrating both sides of (2.7) between 0 and $M$, we deduce that $\{u_m\}$ is bounded in $L^\infty(0, M; \mathbb{H}) \cap L^2(0, M; \mathbb{V})$ and $\{\varphi_m\}$ is bounded in $L^\infty(0, M; H^2) \cap L^2(0, M; H^4)$. Therefore, we can assume that there exists a function $u(s) \in L^\infty(0, M; \mathbb{H}) \cap L^2(0, M; \mathbb{V})$ such that $u_m \to u$, weakly in $L^2(0, M; \mathbb{V})$, weakly-star in $L^\infty(0, M; \mathbb{H})$, $m \to \infty$, and there exists a function $\varphi(s) \in L^\infty(0, M; H^2) \cap L^2(0, M; H^4)$ such that $\varphi_m \to \varphi$, weakly in $L^2(0, M; H^2)$, weakly-star in $L^\infty(0, M; H^4)$, and $m \to \infty$. It follows that $\partial_t u_m(s)$ and $\partial_t \varphi_m(s)$ are bounded in $L^2(0, M; \mathbb{V}')$ and $L^\infty(0, M; \mathbb{H})$, respectively. Hence we can assume that $\partial_t u_m \to \partial_t u$, weakly in $L^2(0, M; \mathbb{V}')$ and $\partial_t \varphi_m \to \partial_t \varphi$, weakly in $L^2(0, M; \mathbb{H})$. To sum up, using well-known compactness arguments, we can find a pair $(u, \varphi)$ such that
\[
u \|\nabla u_m\|^2 + \|\mu_m\|^2 = 0.
\]

(2.8)

Next, multiplying (2.7) by $\psi(t)$, integrating over $[0, M]$, and repeating the computations performed in (2.24)-(2.29) below, we prove that inequality (2.4) holds for any function $\psi(t) \in C_0^\infty(0, M)$. \qed

2.3. Energy estimates.

Lemma 2.3. If $(u(t), \varphi(t))$ is a weak solution of problem (1.2)-(1.3), then it satisfies
\[
\frac{d}{dt} \left( \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|\Delta \varphi(t)\|^2 + \int_\Omega F(\nabla \varphi) dx \right) + \nu \|\nabla u\|^2 + \|\mu\|^2 = 0.
\]

(2.9)

Proof. Multiplying (1.2) by $u$, and integrating over $\Omega$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u(t)|^2 dx + \nu \int_\Omega |\nabla u(t)|^2 dx = \int_\Omega \mu \nabla \cdot u dx.
\]

(2.10)

Multiplying (1.2) by $\mu$, and integrating over $\Omega$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta \varphi|^2 dx + \int_\Omega |\mu|^2 dx + \int_\Omega (u \cdot \nabla) \varphi dx = -\int_\Omega \partial \varphi \nabla \cdot f(\nabla \varphi) dx.
\]

(2.11)
By definition of $F(\nabla \varphi)$ and $f(\nabla \varphi)$, this yields
\[
\frac{d}{dt} F(\nabla \varphi) = (|\nabla \varphi|^2 - 1) \nabla \varphi \cdot \nabla \varphi_t = f(\nabla \varphi) \cdot \nabla \varphi_t, 
\] (2.11)
and integrating (2.11) over $\Omega$ by parts, we get
\[
\frac{d}{dt} \int_{\Omega} F(\nabla \varphi) dx = -\int_{\Omega} \partial_t \varphi \nabla \cdot f(\nabla \varphi) dx. 
\] (2.12)
Combining (2.9), (2.10), and (2.12) yields (2.8). The proof is thus complete.

Lemma 2.4. If $(u(t), \varphi(t))$ is a weak solution of problem (1.2)-(1.3), then $(u(t), \varphi(t))$ satisfies (2.8), and the following inequalities hold:
\[
\|u(t)\|^2 + \|\Delta \varphi(t)\|^2 \leq \left(\|u(0)\|^2 + \|\Delta \varphi(0)\|^2\right) + 2 \int_{\Omega} F(\nabla \varphi(0)) dx \ e^{-kt} + \frac{2C_0}{k}, 
\] (2.13)
\[
\nu \int_t^{t+1} \|\nabla u(s)\|^2 ds + \int_t^{t+1} \|\mu(s)\|^2 ds \leq \left(\frac{1}{2} \left(\|u(0)\|^2 + \|\Delta \varphi(0)\|^2\right) + \int_{\Omega} F(\nabla \varphi(0)) dx \right) \ e^{-kt} + \frac{C_0}{k}, 
\] (2.14)
where $k, C_0$ only depend on $\nu, \Omega$.

Proof. Multiplying (1.2)$_4$ by $\varphi$, and integrating over $\Omega$, we get
\[
\int_{\Omega} \mu \cdot \varphi dx = \int_{\Omega} \|\Delta \varphi\|^2 dx + \int_{\Omega} f(\nabla \varphi) \cdot \nabla \varphi dx 
= \int_{\Omega} \|\Delta \varphi\|^2 dx + \int_{\Omega} |\nabla \varphi|^4 dx - \int_{\Omega} |\nabla \varphi|^2 dx. 
\] (2.15)
On the other hand, by the H"older inequality, we obtain
\[
\int_{\Omega} \mu \cdot \varphi dx \leq \|\mu\| \|\varphi\| \leq \|\mu\|^2 + \frac{1}{4} \|\varphi\|^2. 
\] (2.16)
By definition of $F(\nabla \varphi)$, we have
\[
k \int_{\Omega} F(\nabla \varphi) dx = k \int_{\Omega} |\nabla \varphi|^4 dx - \frac{k}{2} \int_{\Omega} |\nabla \varphi|^2 dx + \frac{|\Omega|}{4}. 
\] (2.17)
Let
\[
E(t) = \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|\Delta \varphi(t)\|^2 + \int_{\Omega} F(\nabla \varphi) dx. 
\]
Thus it follows from to (2.8), (2.15) and (2.17) that
\[
\frac{d}{dt} E(t) + k E(t) + \|\mu\|^2 + \frac{k}{2} \int_{\Omega} |\nabla \varphi|^2 dx 
= \frac{k}{2} \int_{\Omega} |u|^2 dx + \frac{k}{2} \int_{\Omega} \|\Delta \varphi\|^2 dx + \frac{k}{4} \int_{\Omega} |\nabla \varphi|^4 dx - \nu \|\nabla u\|^2 
+ \int_{\Omega} \mu \cdot \varphi dx - \int_{\Omega} \|\Delta \varphi\|^2 dx - \int_{\Omega} |\nabla \varphi|^4 dx + \int_{\Omega} |\nabla \varphi|^2 dx + \frac{|\Omega|}{4}. 
\] (2.18)
By the Sobolev inequality, Young inequality and (2.16), we have
\[
\int_{\Omega} |u|^2 dx \leq C_\Omega \int_{\Omega} |\nabla u|^2 dx, 
\] (2.19)
\[
\int_{\Omega} |\nabla \varphi|^2 dx \leq \frac{1}{4} \int_{\Omega} |\nabla \varphi|^4 dx + |\Omega|, 
\] (2.20)
\[
\frac{1}{4} \| \varphi \|^2 \leq \frac{C_\Omega}{4} \int_\Omega |\nabla \varphi|^2 dx \leq \frac{1}{4} \int_\Omega |\nabla \varphi|^4 dx + \frac{C_\Omega^3}{16}. \tag{2.21}
\]

Then inserting (2.16), (2.19)–(2.21) into (2.18) yields
\[
\frac{d}{dt} E(t) + k E(t) \leq - \left( \nu - \frac{k C_\Omega}{2} \right) \int_\Omega |\nabla u|^2 dx - \frac{2 - k}{2} \| \varphi \|^2 dx
\]
\[
- \frac{2 - k}{4} \int_\Omega |\nabla \varphi|^4 dx + |\Omega| + \frac{k |\Omega|}{4} + \frac{C_\Omega^3}{16}, \tag{2.22}
\]
where \( C_\Omega \) only depends on \( \Omega \). Let \( k = \min \left\{ 2, \frac{2 \nu}{C_\Omega} \right\} \), \( C_0 = |\Omega| + \frac{k |\Omega|}{4} + \frac{C_\Omega^3}{16} \). Then (2.22) leads to
\[
\frac{d}{dt} E(t) + k E(t) \leq C_0. \tag{2.23}
\]

Applying the Gronwall inequality to (2.23) implies
\[
\frac{1}{2} \| u(t) \|^2 + \frac{1}{2} \| \varphi(t) \|^2 + \int_0^t F(\nabla \varphi(t)) dx \leq \left( \frac{1}{2} \| u(0) \|^2 + \frac{1}{2} \| \varphi(0) \|^2 + \int_0^t F(\nabla \varphi(0)) dx \right) e^{-kt} + \frac{C_0}{k}, \tag{2.24}
\]
which gives (2.13). Integrating (2.8) on \([t, t + 1]\), we have
\[
\frac{1}{2} \| u(t + 1) \|^2 + \frac{1}{2} \| \varphi(t + 1) \|^2 + \int_t^{t+1} F(\nabla \varphi(t + 1)) dx
\]
\[
+ \nu \int_t^{t+1} \| \nabla u(s) \|^2 ds + \int_t^{t+1} \| \mu(s) \|^2 ds \leq \frac{1}{2} \| u(t) \|^2 + \frac{1}{2} \| \varphi(t) \|^2 + \int_{0}^{t} F(\nabla \varphi(t)) dx, \tag{2.25}
\]
which, together with (2.24), gives (2.14).

\[\square\]

**Lemma 2.5.** If \((u(t), \varphi(t))\) is a weak solution of problem (1.2)–(1.3), then \((u(t), \varphi(t))\) satisfies (2.8), and the following inequality holds:
\[
\int_{t}^{t+1} \| \Delta \varphi \|^2 ds \leq C_1 \left( \frac{1}{2} \| u(0) \|^2 + \frac{1}{2} \| \varphi(0) \|^2 + \int_0^t F(\nabla \varphi(0)) dx \right) e^{-kt} + R_1^2, \tag{2.26}
\]
where \( C_1, R_1 \) only depend on \( \nu, \Omega, \) and \( E(0) \).

**Proof.** According to the Sobolev embedding inequality and (2.13), we have
\[
\| \nabla \varphi \|_{L^6}^2 \leq c \| \Delta \varphi \|^2 \leq 2E(0) + \frac{2C_0}{k}. \tag{2.27}
\]

By definition of \( f(\nabla \varphi) \), we get
\[
\nabla \cdot f(\nabla \varphi) = (|\nabla \varphi|^2 - 1) \Delta \varphi + 2 |\nabla \varphi|^2 \cdot \nabla \varphi, \tag{2.28}
\]
so that
\[
\int_\Omega |\nabla \cdot f(\nabla \varphi)|^2 dx \leq \int_\Omega |\nabla \varphi|^2 |\Delta \varphi|^2 dx + 2 \int_\Omega |\nabla \varphi|^2 |\nabla \Delta \varphi|^2 dx + \int_\Omega |\Delta \varphi|^4 dx
\]
\[
\leq c \left( \int_\Omega |\nabla \varphi|^6 dx \right)^{\frac{2}{3}} \left( \int_\Omega |\Delta \varphi|^2 dx \right)^{\frac{3}{2}} + \frac{3}{5} \int_\Omega |\Delta \varphi|^2 dx + \frac{2\Omega}{5}
\]
\[
\leq c \left( 2E(0) + \frac{2C_0}{k} \right)^{\frac{2}{3}} + \frac{3}{5} \left( 2E(0) + \frac{2C_0}{k} \right) + \frac{2\Omega}{5}, \tag{2.29}
\]
which yields
\[
\| \nabla \cdot f(\nabla \varphi) \|_{L^\infty(t, t+1; L^2_\infty(\Omega))} \leq c \left( 2E(0) + \frac{2C_0}{k} \right)^\frac{\gamma}{2} + \frac{3}{5} \left( 2E(0) + \frac{2C_0}{k} \right) + \frac{2\Omega}{5}. \quad (2.30)
\]
According to (1.2)_4, (2.14) and (2.30), we have
\[
\int_t^{t+1} \| \Delta^2 \varphi \|_{L^2_\infty}^2 \, ds \leq 2 \int_t^{t+1} \| \mu(s) \|_{L^2_\infty}^2 \, ds + 2 \int_t^{t+1} \| \nabla \cdot f(\nabla \varphi) \|_{L^2_\infty}^2 \, ds \\
\leq 2E(0) + \frac{2C_0}{k} + 2c \left( 2E(0) + \frac{2C_0}{k} \right)^\frac{\gamma}{2} + \frac{6}{5} \left( 2E(0) + \frac{2C_0}{k} \right) + \frac{4\Omega}{5}. \quad (2.31)
\]
According to the interpolation inequality \( W^{4,5}(\Omega) \rightarrow W^{1,\infty}(\Omega) \rightarrow W^{1,6}(\Omega) \), we get
\[
\| \nabla \varphi \|_{L^\infty} \leq c \| \Delta^2 \varphi \|_{L^2_\infty}^\frac{1}{2} \| \nabla \varphi \|_{L^6_\infty}^\frac{1}{2}. \quad (2.32)
\]
Using (2.27), (2.31) and (2.32), we obtain
\[
\int_t^{t+1} \| \nabla \varphi \|_{L^\infty}^2 \, ds \leq c \int_t^{t+1} \| \Delta^2 \varphi \|_{L^2_\infty}^2 \| \nabla \varphi \|_{L^6_\infty}^2 \, ds \\
\leq 2c \left( 2E(0) + \frac{2C_0}{k} \right) \left[ E(0) + \frac{C_0}{k} + c \left( 2E(0) + \frac{2C_0}{k} \right)^\frac{\gamma}{2} + \frac{6}{5} \left( 2E(0) + \frac{2C_0}{k} \right) + \frac{4\Omega}{5} \right]. \quad (2.33)
\]
On the other hand,
\[
\int_\Omega \| \nabla \cdot f(\nabla \varphi) \|^2 \, dx \leq \int_\Omega \| \nabla \varphi \|_{L^\infty}^2 \| \Delta \varphi \|^2 \, dx + 2 \int_\Omega \| \nabla \varphi \|_{L^2_\infty}^2 \| \nabla^2 \varphi \|^2 \, dx + \int_\Omega |\Delta \varphi|^2 \, dx \\
\leq c \| \nabla \varphi \|_{L^\infty}^4 \int_\Omega |\Delta \varphi|^2 \, dx + \int_\Omega |\Delta \varphi|^2 \, dx. \quad (2.34)
\]
Therefore
\[
\int_t^{t+1} \| \nabla \cdot f(\nabla \varphi) \|^2 \, ds \leq c \int_t^{t+1} \left( \| \nabla \varphi \|_{L^\infty}^4 \int_\Omega |\Delta \varphi|^2 \, dx \right) \, ds + \int_t^{t+1} \int_\Omega |\Delta \varphi|^2 \, dx \, ds. \quad (2.35)
\]
Since \( \Delta^2 \varphi = \mu + \nabla \cdot f(\nabla \varphi) \), and using (2.35), we obtain
\[
\int_t^{t+1} \| \Delta^2 \varphi \|^2 \, ds \leq 2 \int_t^{t+1} \| \mu \|^2 \, ds + 2 \int_t^{t+1} \| \nabla \cdot f(\nabla \varphi) \|^2 \, ds \\
\leq 2 \int_t^{t+1} \| \mu \|^2 \, ds + 2 \left( c \int_t^{t+1} \| \nabla \varphi \|_{L^\infty}^4 \, ds + 1 \right) \times \sup_\Omega |\Delta \varphi|^2 \, dx. \quad (2.36)
\]
Inserting (2.13), (2.14), (2.33) into (2.36) gives (2.26). The proof is complete. \( \square \)

3. Existence of trajectory attractors.

3.1. Trajectory space. We consider the spaces \( \mathcal{X}_+^b \) defined by
\[
\mathcal{X}_+^b = \{(u(t), \varphi(t)) \mid (u, \varphi)(\cdot) \in L^2_0(\mathbb{R}_+; \mathbb{V} \times H^1) \cap L^\infty(\mathbb{R}_+; \mathbb{H} \times H^2), \forall t \}
\]
\[
(\partial_t u, \partial_t \varphi)(\cdot) \in L^2_\infty(\mathbb{R}_+; \mathbb{V}' \times \mathbb{H}),
\]
with its norm
\[
\|(u, \varphi)\|_{\mathcal{X}_+^b} = \|(u, \varphi)\|_{L^2_0(\mathbb{R}_+; \mathbb{V} \times H^1)} + \|(u, \varphi)\|_{L^\infty(\mathbb{R}_+; \mathbb{H} \times H^2)} + \|(\partial_t u, \partial_t \varphi)\|_{L^2_\infty(\mathbb{R}_+; \mathbb{V}' \times \mathbb{H})},
\]
where
\[
\|\{u, \varphi\}\|_{L^2_t(\mathbb{R}_+; \mathbb{V} \times H^4)}^2 = \sup_{t \geq 0} \int_t^{t+1} \|\nabla u(s)\|^2 ds + \sup_{t \geq 0} \int_t^{t+1} \|\Delta^2 \varphi(s)\|^2 ds,
\]
\[
\|\{u, \varphi\}\|_{L^\infty_t(\mathbb{R}_+; \mathbb{H} \times H^2)} = \text{ess sup}_{t \geq 0} \|u(t)\| + \text{ess sup}_{t \geq 0} \|\Delta \varphi(t)\|,
\]
\[
\|\partial_t u, \partial_t \varphi\|_{L^\infty_t(\mathbb{R}_+; \mathbb{V} \times \mathbb{H})}^\frac{4}{3} = \sup_{t \geq 0} \int_t^{t+1} \|\partial_t u(s)\|^\frac{4}{3} ds + \sup_{t \geq 0} \int_t^{t+1} \|\partial_t \varphi(s)\|^\frac{4}{3} ds.
\]
The difference between this paper and [1, 2, 8] is that, there the authors considered the space $\mathcal{F}_+^{loc}$ as
\[
\{(u(t), \varphi(t)) \mid (u, \varphi) \in L^2_t(\mathbb{R}_+; \mathbb{V} \times H^4) \cap L^\infty_t(\mathbb{R}_+; \mathbb{H} \times H^2),
\]
\[
(\partial_t u, \partial_t \varphi)(\cdot) \in L^\frac{4}{3}_t(\mathbb{R}_+; D(A)' \times \mathbb{V}').
\]
We know that $\mathcal{F}_+^{loc}$ with its norm $\|\cdot\|_{\mathcal{F}_+^{loc}}$ is a Banach space. Similarly, we define the space $\mathcal{F}_+^{loc}$ as
\[
\mathcal{F}_+^{loc} = \{(u, \varphi) : (u, \varphi)(\cdot) \in L^2_t(\mathbb{R}_+; \mathbb{V} \times H^4) \cap L^\infty_t(\mathbb{R}_+; \mathbb{H} \times H^2),
\]
\[
(\partial_t u, \partial_t \varphi)(\cdot) \in L^\frac{4}{3}_t(\mathbb{R}_+; D(A)' \times \mathbb{V}'),
\]
and we define a topology $\Theta_+^{loc}$ on $\mathcal{F}_+^{loc}$: for a sequence $\{(u_n, \varphi_n)(\cdot)\} \subset \mathcal{F}_+^{loc}$, $(u_n, \varphi_n) \rightarrow (u, \varphi)$ in the topology $\Theta_+^{loc}$ if
\[
(u_n, \varphi_n) \rightarrow (u, \varphi), \text{ weakly in } L^2(0, M; \mathbb{V} \times H^4),
\]
\[
\text{and weakly-star in } L^\infty(0, M; \mathbb{H} \times H^2), n \rightarrow \infty,
\]
\[
(\partial_t u_n, \partial_t \varphi_n) \rightarrow (\partial_t u, \partial_t \varphi), \text{ weakly in } L^\frac{4}{3}(0, M; \mathbb{V}' \times \mathbb{H}), n \rightarrow \infty.
\]
Let $\{T(h) \mid h \geq 0\}$ denote the time translation operator acting on the trajectory space,
\[
T(h)(u, \varphi)(t) = (u(t + h), \varphi(t + h)).
\]

3.2. Existence of the trajectory attractors of the 3D smectic-A liquid crystal flow equations. The 3D smectic-A liquid crystal flow equations can be rewritten as
\[
\begin{cases}
\partial_t u + \nu Au + B(u, u) = B(\mu, \varphi), \quad x \in \Omega, \ t > 0, \\
\partial_t \varphi + B(u, \varphi) = -\mu,
\end{cases}
\]
where $B(u, v)$ is defined in subsection 2.1.

**Definition 3.1.** The trajectory space $\mathcal{K}_+$ is the union of Leray-Hopf weak solutions $(u(t), \varphi(t))$ of problem (1.2)-(1.3) with $(u(t), \varphi(t)) \in L^\infty(0, +\infty; \mathbb{H} \times H^2) \cap L^2(0, +\infty; \mathbb{V} \times H^4)$ such that (2.4), (2.8) are valid.

**Lemma 3.2.** If $(u(t), \varphi(t))$ is a weak solution of problem (3.1), then $(u(t), \varphi(t))$ satisfies (2.8), and the following inequality holds:
\[
\int_t^{t+1} \|\partial_t u(s)\|^\frac{4}{3} ds + \int_t^{t+1} \|\partial_t \varphi(s)\|^\frac{4}{3} ds 
\]
\[
\leq C_2 \left( \frac{1}{2} \|u(0)\|^2 + \frac{1}{2} \|\Delta \varphi(0)\|^2 + \int_\Omega F(\nabla \varphi(0)) \phi dx \right) e^{-kt} + R_2^2,
\]
where $C_2, R_2$ only depend on $\nu, \Omega, \text{and } E(0)$. 

Proof. According to system (3.1), we easily have
\[
\int_t^{t+1} \| \partial_t u(s) \|^\frac{4}{3} \, ds \leq 2^{\frac{1}{2}} \nu^{\frac{1}{2}} \int_t^{t+1} \| Au(s) \|^\frac{4}{3} \, ds + 2^{\frac{1}{2}} \int_t^{t+1} \| B(u, u) \|^\frac{4}{3} \, ds \\
+ 2^{\frac{1}{2}} \int_t^{t+1} \| B(\mu, \varphi) \|^\frac{4}{3} \, ds,
\]
(3.3)
and
\[
\int_t^{t+1} \| \partial_t \varphi(s) \|^\frac{4}{3} \, ds \leq \int_t^{t+1} \| \partial_t \varphi(s) \|^2 \, ds \leq 2^{\frac{1}{2}} \int_t^{t+1} \| B(u, \varphi) \|^2 \, ds + 2^{\frac{1}{2}} \int_t^{t+1} \| \mu \|^2 \, ds.
\]
(3.4)
First, using the inequality (2.14), we get
\[
\nu \int_t^{t+1} \| Au(s) \|^2 \, ds = \nu \int_t^{t+1} \| \nabla u(s) \|^2 \, ds \\
\leq \left( \frac{1}{2} \| u(0) \|^2 + \frac{1}{2} \| \Delta \varphi(0) \|^2 + \int_{\Omega} F(\nabla \varphi(0)) \, dx \right) e^{-kt} + C_0 \frac{\nu}{k}.
\]
(3.5)
which, along with (2.2), (2.13)-(2.14), yields
\[
\int_t^{t+1} \| B(u, u) \|^{\frac{3}{2}} \, ds \\
\leq c \int_t^{t+1} \| u \|^{\frac{2}{3}} \| \nabla u \|^2 \, ds \leq c \sup_{s \in [t, t+1]} \| u(s) \|^{\frac{2}{3}} \int_t^{t+1} \| \nabla u(s) \|^2 \, ds \\
\leq \frac{c}{\nu} \left( \| u(0) \|^2 + \| \Delta \varphi(0) \|^2 + 2 \int_{\Omega} F(\nabla \varphi(0)) \, dx + \frac{2C_0}{k} \right)^{\frac{1}{2}} \\
\times \left[ \left( \frac{1}{2} \| u(0) \|^2 + \frac{1}{2} \| \Delta \varphi(0) \|^2 + \int_{\Omega} F(\nabla \varphi(0)) \, dx \right) e^{-kt} + C_0 \frac{\nu}{k} \right].
\]
(3.6)
Next,
\[
\| B(\mu, \varphi) \|^{\frac{3}{2}} \leq c (\| \nabla \varphi \|_{L^\infty} \| \mu \|)^{\frac{3}{2}} = c \| \nabla \varphi \|_{L^2}^{\frac{3}{2}} \| \mu \|^{\frac{3}{2}}. 
\]
(3.7)
Applying the interpolation inequality $W^{4,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \hookrightarrow W^{2,2}(\Omega)$, we obtain
\[
\| \nabla \varphi \|_{L^\infty} \leq c \| \Delta \varphi \|_{L^2}^{\frac{1}{2}} \| \Delta \varphi \|_{L^2}^{\frac{1}{2}}. 
\]
(3.8)
Hence using (3.7)-(3.8), we get
\[
\int_t^{t+1} \| B(\mu, \varphi) \|^{\frac{3}{2}} \, ds \leq c \int_t^{t+1} \| \Delta^2 \varphi \|^{\frac{3}{2}} \| \Delta \varphi \| \| \mu \|^{\frac{3}{2}} \, ds \\
\leq c \left( \int_t^{t+1} \| \mu \|^2 \, ds \right)^{\frac{2}{3}} \cdot \left( \int_t^{t+1} \| \Delta^2 \varphi \| \| \Delta \varphi \|^{3} \, ds \right)^{\frac{1}{3}} \\
\leq c \sup_{s \in [t, t+1]} \| \Delta \varphi(s) \|^{\frac{3}{2}} \cdot \left( \int_t^{t+1} \| \mu \|^2 \, ds \right)^{\frac{2}{3}} \cdot \left( \int_t^{t+1} \| \Delta^2 \varphi \|^2 \, ds \right)^{\frac{1}{2}},
\]
(3.9)
which, along with (2.13)-(2.14), becomes
\[
\int_t^{t+1} \| B(\mu, \varphi) \|^{\frac{3}{2}} \, ds \leq c \left[ \left( \frac{1}{2} \| u(0) \|^2 + \frac{1}{2} \| \Delta \varphi(0) \|^2 + \int_{\Omega} F(\nabla \varphi(0)) \, dx \right) e^{-kt} + C_0 \frac{\nu}{k} \right] \\
\times \left( \| u(0) \|^2 + \| \Delta \varphi(0) \|^2 + 2 \int_{\Omega} F(\nabla \varphi(0)) \, dx + \frac{2C_0}{k} \right)^{\frac{1}{2}}.
\]
(3.10)
Finally,
\[ \int_t^{t+1} \|B(\mu, \varphi)\|^2 \, ds \leq c \left( \frac{1}{2} \|u(0)\|^2 + \frac{1}{2} \|\varphi(0)\|^2 + \int_\Omega F(\nabla \varphi(0)) \, dx \right) e^{-\kappa t} + \frac{C_0}{k} \]
\[ \times \left( \|u(0)\|^2 + \|\varphi(0)\|^2 + 2 \int_\Omega F(\nabla \varphi(0)) \, dx + \frac{2C_0}{k} \right)^{\frac{1}{2}}. \] (3.11)

Finally,
\[ \int_t^{t+1} \|B(u, \varphi)\|^2 \, ds \leq c \int_t^{t+1} \|\nabla u\|^2 \|\varphi\|^2 \, ds \]
\[ \leq c \sup_{s \in [t, t+1]} \|\varphi(s)\|^2 \cdot \int_t^{t+1} \|\nabla u(s)\|^2 \, ds \]
\[ \leq c \left( \frac{1}{2} \|u(0)\|^2 + \frac{1}{2} \|\varphi(0)\|^2 + \int_\Omega F(\nabla \varphi(0)) \, dx \right) e^{-\kappa t} + \frac{C_0}{k} \]
\[ \times \left( \|u(0)\|^2 + \|\varphi(0)\|^2 + 2 \int_\Omega F(\nabla \varphi(0)) \, dx + \frac{2C_0}{k} \right)^{\frac{1}{2}}. \] (3.12)

Inserting (3.5), (3.6), (3.11) into (3.3), and inserting (3.12), (2.14) into (3.4), we obtain (3.2).

**Proposition 3.1.** For any function \((u, \varphi)(\cdot) \in K^+\), then

(i) \((u, \varphi)(\cdot) \in F^b_+\);

(ii) there holds that \(T(h)(u, \varphi)(\cdot) \in K^+\), i.e.,

\[ \|T(h)(u, \varphi)(\cdot)\|_{F^b_+} \leq C \left( \frac{1}{2} \|u(0)\|^2 + \frac{1}{2} \|\varphi(0)\|^2 + \int_\Omega F(\nabla \varphi(0)) \, dx \right) e^{-\kappa t} + R, \] (3.13)

where \(C, R\) only depend on \(\nu, \Omega, \) and \(E(0)\).

**Proof.** (3.13) is clearly established by Lemmas 2.4-2.5, 3.2. Let \((u_1(s), \varphi_1(s)) \to (u_2(s), \varphi_2(s))\) in \(F^b_+\). Then \((u_1(s+h), \varphi_1(s+h)) \to (u_2(s+h), \varphi_2(s+h))\) in \(F^b_+\) and the translation group \(\{T(h)\}\) is continuous in \(F^b_+\). The proof similar to [6] XII.1.2.

**Proposition 3.2.** The trajectory space \(K^+\) is closed in the topology \(\Theta^l_{oc}\).

**Proof.** We consider an arbitrary sequence \(\{(u_n, \varphi_n)\} \subset K^+\), such that \((u_n, \varphi_n) \to (u, \varphi)\) in the topology \(\Theta^l_{oc}\) as \(n \to \infty\). We need to prove that \((u, \varphi) \in K^+\). Since \((u_n, \varphi_n) \to (u, \varphi)\) in the topology \(\Theta^l_{oc}\), by definition of \(\Theta^l_{oc}\),

\[ u_n \to u, \text{ weakly in } L^2(0, M; \mathbb{V}), \text{ and weakly-star in } L^\infty(0, M; \mathbb{H}), n \to \infty; \] (3.14)
\[ \partial_t u_n \to \partial_t u, \text{ weakly in } L^\frac{4}{3}(0, M; \mathbb{V}'), n \to \infty; \] (3.15)
\[ \varphi_n \to \varphi, \text{ weakly in } L^2(0, M; H^4), \text{ and weakly-star in } L^\infty(0, M; H^2), n \to \infty; \] (3.16)
\[ \partial_t \varphi_n \to \partial_t \varphi, \text{ weakly in } L^\frac{4}{3}(0, M; \mathbb{H}), n \to \infty. \] (3.17)

Since \((u_n, \varphi_n) \subset K^+, (u_n, \varphi_n)\) satisfies

\[ \begin{cases}
\partial_t u_n + \nu Au_n + B(u_n, u_n) = B(\mu_n, \varphi_n), \quad x \in \Omega, \ t > 0, \\
\partial_t \varphi_n + B(u_n, \varphi_n) = -\mu_n.
\end{cases} \] (3.18)
Noting that $u_n$ is bounded in $L^2(0, M; \mathcal{V}) \cap L^\infty(0, M; \mathbb{H})$, and $u_n \to u$ strongly in $L^2(0, M; \mathbb{H})$, then $\{u_n\}$ contains a subsequence which we still denote by $\{u_n\}$ such that
\[
B(u_n, u_n) \to B(u, u) \text{ weakly in } L^\frac{4}{3}(0, M; \mathcal{V}').
\] (3.19)
We can compute $B(\mu_n, \varphi_n), B(u_n, \varphi_n), \mu_n$ as in (3.10), (3.12), (2.14), which implies that $B(\mu_n, \varphi_n), B(u_n, \varphi_n), \mu_n$ are bounded in $L^\frac{4}{3}(0, M; \mathbb{H})$, and there exists a subsequence such that
\[
B(\mu_n, \varphi_n) \to B(\mu, \varphi) \text{ weakly in } L^\frac{4}{3}(0, M; \mathbb{H}),
\] (3.20)
\[
B(u_n, \varphi_n) \to B(u, \varphi) \text{ weakly in } L^\frac{4}{3}(0, M; \mathbb{H}),
\] (3.21)
\[
\mu_n \to \mu \text{ weakly in } L^\frac{4}{3}(0, M; \mathbb{H}).
\] (3.22)
Taking the limit of (3.18) and using (3.14)-(3.17), (3.20)-(3.22), then $(u, \varphi)$ is a weak solution of problem (3.1). Next we prove that $(u, \varphi)$ satisfies the energy inequality (2.4). Since $(u_n, \varphi_n) \subset \mathcal{K}^+$, then $(u_n, \varphi_n)$ satisfies
\[
- \int_0^M \left( \frac{1}{2} \|u_n(t)\|^2 + \frac{1}{2} \|\nabla \varphi_n(t)\|^2 + \int_\Omega F(\nabla \varphi_n)dx \right) \psi'(t)dt
+ \int_0^M (\nu \|\nabla u_n\|^2 + \|\mu\|^2) \psi(t)dt \leq 0.
\] (3.23)
Noticing (3.14)-(3.17), we know that $(u_n, \varphi_n) \to (u, \varphi)$ strongly in $L^2(0, M; \mathbb{H} \times H^2)$, i.e.,
\[
\int_0^M \|u_n(t) - u(t)\|^2 dt \to 0,
\int_0^M \|\nabla \varphi_n(t) - \nabla \varphi(t)\|^2 dt \to 0,
\]
\[n \to \infty. \quad (3.24)\]
Since $\psi(t) \in C^\infty_0(0, M)$, and (3.24), we obtain
\[
\lim_{n \to \infty} \int_0^M \left( \frac{1}{2} \|u_n(t)\|^2 + \frac{1}{2} \|\nabla \varphi_n(t)\|^2 \right) \psi'(t)dt
= \int_0^M \left( \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|\nabla \varphi(t)\|^2 \right) \psi'(t)dt.
\] (3.25)
Next, we have
\[
\left| \int_0^M \int_\Omega F(\nabla \varphi_n)dx \psi'(t)dt - \int_0^M \int_\Omega F(\nabla \varphi)dx \psi'(t)dt \right|
= \frac{1}{4} \left[ \int_0^M \int_\Omega \left( \|\nabla \varphi_n\|^2 + \|\nabla \varphi\|^2 - 2 \right) \left( \|\nabla \varphi_n\|^2 - \|\nabla \varphi\|^2 \right) dx \psi'(t)dt \right]
\leq \frac{1}{4} \left[ 2 \int_\Omega \left( \|\nabla \varphi_n\|^4 + \|\nabla \varphi\|^4 + 4 \right) dx \right]^\frac{1}{2} \left[ \int_\Omega \|\nabla \varphi_n - \nabla \varphi\|^4 dx \right]^\frac{1}{2} \psi'(t)dt
\leq \frac{1}{2} \sup_{t \in [0, M]} \left( \|\psi'(t)\| \left( \|\nabla \varphi_n(t)\|^2 + \|\nabla \varphi(t)\|^2 \right) \right) \int_0^M \|\nabla \varphi_n(t) - \nabla \varphi(t)\|^2 dt
\to 0, \quad n \to \infty.
\] (3.26)
Notice that $u_n \sqrt{\psi} \to u \sqrt{\psi}$ weakly in $L^2(0, M; \mathcal{V})$, $n \to \infty$. Therefore
\[
\nu \int_0^M \|\nabla u(t)\|^2 \psi dt \leq \nu \lim \inf_{n \to \infty} \int_0^M \|\nabla u_n(t)\|^2 \psi dt.
\] (3.27)
We compute $\mu_n$ again as in (2.14) which implies that $\mu_n$ is bounded in $L^2(0, M; \mathbb{H})$, and there exists a subsequence such that $\mu_n \to \mu$ weakly in $L^2(0, M; \mathbb{H})$. Then
\[ \mu_n \sqrt{\psi} \rightharpoonup \mu \sqrt{\psi} \text{ weakly in } L^2(0, M; \mathbb{H}), \text{ i.e., for any } w \in \mathbb{H}, \int_0^M (\mu \sqrt{\psi}, w) dt = \lim_{n \to \infty} \int_0^M (\mu_n \sqrt{\psi}, w) dt. \]

Letting \( w = \mu \sqrt{\psi} \), we have
\[
\int_0^M ||\mu \sqrt{\psi}||^2 dt \leq \lim_{n \to \infty} \left( \int_0^M ||\mu_n \sqrt{\psi}||^2 dt \right)^\frac{1}{2} \left( \int_0^M ||\mu \sqrt{\psi}||^2 dt \right)^\frac{1}{2},
\]
and, since \( \psi(t) \) is independent of \( x \), we obtain
\[
\int_0^M ||\mu(t)||^2 dt \leq \lim_{n \to \infty} \int_0^M ||\mu_n(t)||^2 dt. \tag{3.28}
\]

Taking the limit on both sides of equation (3.23), and using (3.25)-(3.28), we have
\[
- \int_0^M \left( \frac{1}{2} ||u(t)||^2 + \frac{1}{2} ||\Delta \varphi(t)||^2 + \int_\Omega F(\nabla \varphi) dx \right) \psi'(t) dt
+ \int_0^M (\nu ||\nabla u||^2 + ||\mu||^2) \psi(t) dt \leq 0. \tag{3.29}
\]

Thus we have proved that \((u, \varphi) \in \mathcal{K}^+\).

In a last step, we give a detailed proof of the convergence (3.22). Since \( \varphi_n \rightharpoonup \varphi \), weakly in \( L^2(0, M; H^1) \), a.e., \( \forall w \), the following relation holds:
\[
\lim_{n \to \infty} \int_0^M \langle \Delta^2 \varphi_n, w \rangle dt = \int_0^M \langle \Delta^2 \varphi, w \rangle dt. \tag{3.30}
\]

Our purpose is to prove the following limit:
\[
\lim_{n \to \infty} \int_0^M \langle \mu_n - \mu, w \rangle dt = 0. \tag{3.31}
\]

Using (3.30) and the Hölder inequality for \( q = \frac{3}{4} \), we have
\[
\lim_{n \to \infty} \int_0^M \langle \mu_n - \mu, w \rangle dt
= \lim_{n \to \infty} \int_0^M \langle \Delta^2 \varphi_n - \nabla \cdot f(\nabla \varphi_n) - \Delta^2 \varphi + \nabla \cdot f(\nabla \varphi) \rangle dt
= 0 - \lim_{n \to \infty} \int_0^M \langle \nabla \cdot f(\nabla \varphi) - \nabla \cdot f(\nabla \varphi), w \rangle dt
\leq \lim_{n \to \infty} \int_0^M \left( ||\nabla \varphi||^2 + ||\Delta \varphi_n - \Delta \varphi||_{L^2(\Omega)} \right) ||w|| dt
\leq \lim_{n \to \infty} \int_0^M \left( ||\nabla \varphi||^{\frac{3}{2}}_{L^\infty(\Omega)} + 1 \right) ||\Delta \varphi_n - \Delta \varphi||_{L^2(\Omega)} ||w|| dt
+ \lim_{n \to \infty} \int_0^M \left( ||\nabla \varphi_n - \nabla \varphi||^{\frac{3}{2}}_{L^2(\Omega)} \right) ||\Delta \varphi_n - \Delta \varphi||_{L^\infty(\Omega)} ||w|| dt
\leq \lim_{n \to \infty} \left( \int_0^M \left( ||\nabla \varphi||^{\frac{3}{2}}_{L^\infty(\Omega)} + 1 \right) ||\Delta \varphi_n - \Delta \varphi||_{L^2(\Omega)} \right)^{\frac{1}{2}} \left( \int_0^M ||w||^4 dt \right)^{\frac{1}{4}}
+ \lim_{n \to \infty} \left( \int_0^M ||\nabla \varphi_n - \nabla \varphi||^{\frac{3}{2}}_{L^2(\Omega)} ||\Delta \varphi_n - \Delta \varphi||_{L^\infty(\Omega)} ||w||_{L^\infty(\Omega)} \right)^{\frac{1}{2}} \left( \int_0^M ||w||^4 dt \right)^{\frac{1}{4}}
\times \left( \int_0^M ||w||^4 dt \right)^{\frac{1}{4}}
subject to initial and boundary conditions,

\[
T \left\{ \begin{align*}
\text{(i)} & \quad \Delta \varphi_n - \Delta \varphi_n = \Delta \varphi - \Delta \varphi, \\
\text{(ii)} & \quad \Delta \varphi_n - \Delta \varphi, \\
\text{(iii)} & \quad \Delta \varphi_n - \Delta \varphi
\end{align*} \right. 
\]

This completes the proof.

**Definition 3.3.** A set \( P \subseteq K^+ \) is said to be an attracting set of the semigroup \( \{T(h)\} \) in the topology \( \Theta^{loc} \) if for any bounded \( B \subseteq K^+ \) in \( F^b \), the set \( P \) is such that \( T(h)B \subseteq P \) as \( h \to +\infty \), i.e., for every bounded set \( B \subseteq K^+ \) in the topology \( \Theta^{loc} \), there is an \( h_1 = h_1(B; \mathcal{O}) \geq 0 \) such that \( T(h)B \subseteq \mathcal{O}(P) \) for all \( h \geq h_1 \).

**Definition 3.4.** A set \( A \subseteq K^+ \) is said to be a trajectory attractor of the semigroup \( \{T(h)\} \) in the topology \( \Theta^{loc} \) if

(i) \( A \) is compact in the topology \( \Theta^{loc} \),

(ii) \( A \) is strictly invariant, i.e., \( T(h)A = A \) for all \( h \geq 0 \),

(iii) \( A \) is a minimal uniformly attracting set for \( F^b_+ \).

**Theorem 3.3.** The translation semigroup \( \{T(h)\} \) acting on \( K^+ \) has a trajectory attractor \( A \). The set \( A \) is bounded in \( F^b_+ \) and compact in \( \Theta^{loc} \). Moreover

\[
A = \Pi_+ K,
\]

and the set \( K \) is bounded in \( F^b \) and compact in \( \Theta^{loc} \).

**Proof.** It is clear that \( \{T(h)\}K^+ \subseteq K^+ \), \( h \geq 0 \). Thanks to (3.13), the set \( P = \{(u, \varphi) \in F^b ||(u, \varphi)(\cdot)||_{F^b} \leq 2R\} \) is an absorbing set for \( K^+ \). The ball \( P \) is compact in \( \Theta^{loc} \) and bounded in \( F^b_+ \). Thus, the conditions of Theorem XII.2.1 and Theorem XII.2.2 in [6] are satisfied and hence Theorem 3.3 is proved readily.

4. The 3D smectic-A liquid crystal flow-\( \alpha \) model and its trajectory attractors.

4.1. The 3D smectic-A liquid crystal flow-\( \alpha \) model. In this section, we introduce the approximated smectic-A liquid crystal flow equations to problem (1.2)-(1.3). The model can be written as follows:

\[
\begin{align*}
\partial_t v + (u \cdot \nabla) v + \sum_{j=1}^{3} u_j \nabla v_j - \nu \Delta v + \nabla P' &= \mu \nabla \varphi, \\
v &= (1 - \alpha^2 \Delta) u, \quad \text{div } v = 0, \quad \text{div } u = 0, \\
\partial_t \varphi + (u \cdot \nabla) \varphi &= -\mu, \\
\mu &= \Delta^2 \varphi - \nabla \cdot f(\nabla \varphi),
\end{align*}
\]

subject to initial and boundary conditions,
\[
\begin{aligned}
(u(x,0), \varphi(x,0)) &= (u_0(x), \varphi_0(x)), \quad \text{div } u_0 = 0, \ x \in \Omega, \\
(u(x,t), \partial_t \varphi, \partial_n \varphi(x,t)) &= \partial_t \varphi \big|_{\partial \Omega} = \frac{\partial \Delta \varphi}{\partial n}(x,t) = 0, \ x \in \partial \Omega.
\end{aligned}
\] (4.2)

We denote
\[
\tilde{B}(u,v) = -P((u \times (\nabla \times v)), \ u,v \in \mathcal{V},
\] (4.3)

the bilinear operator. In fact, since
\[
-(u \times (\nabla \times v)) = \sum_{j=1}^{3}(u_j \partial_j v - u_j \nabla v_j) = (u \cdot \nabla)v - \sum_{j=1}^{3}u_j \nabla v_j,
\] (4.4)

we have
\[
\{\tilde{B}(u,v), w\} = \langle B(u,v), w \rangle - \langle B(w,v), u \rangle.
\] (4.5)

For any \(u,v,w \in \mathcal{V}\), letting \(w = u\), we obtain
\[
\{\tilde{B}(u,v), u\} = 0.
\] (4.6)

Using (4.5) and \(v = u - \alpha^2 \Delta u\), we get
\[
\langle \tilde{B}(u,v), w \rangle = \langle B(u,v), w \rangle - \langle B(w,v), u \rangle = -(B(u,w),v) + \langle B(w,u), v \rangle = -\langle B(u,w),u \rangle + \langle B(u,w), \alpha^2 \Delta u \rangle - \langle B(w,u), \alpha^2 \Delta u \rangle,
\]

and
\[
\begin{aligned}
|\langle \tilde{B}(u,v), w \rangle| &\leq c\|\nabla w\|_2 \left(\|u\|_2^2 + \alpha^2 \|\Delta u\|_\infty + \alpha^2 \|\nabla u\|_2 \right) \\
&\leq c\|\nabla w\|_2 \left(\|u\|_2^2 \|\nabla u\|_2 + \alpha^2 \|\Delta u\|_2 \right).
\end{aligned}
\] (4.7)

The equality (2.1) follows from the identity
\[
\tilde{B}(u,u) = B(u,u).
\] (4.8)

The 3D smectic-A liquid crystal flow-\(\alpha\) model can be rewritten as
\[
\begin{aligned}
(1 + \alpha^2 A)\partial_t u + \nu(1 + \alpha^2 A)Au + \tilde{B}(u,v) &= B(\mu, \varphi), \ x \in \Omega, \ t > 0, \\
\partial_t \varphi + B(u, \varphi) &= -\mu.
\end{aligned}
\] (4.9)

**Lemma 4.1.** If \((u(t), \varphi(t))\) is a weak solution of problem (4.9), then \((u(t), \varphi(t))\) satisfies
\[
\frac{d}{dt} \left(\frac{1}{2}\|u(t)\|^2 + \alpha^2 \|\nabla u(t)\|^2 + \|\Delta \varphi(t)\|^2\right) + \int_{\Omega} F(\nabla \varphi)dx \\
+ \nu(\|\nabla u\|^2 + \alpha^2 \|\Delta u\|^2) + \|\mu\|^2 = 0.
\] (4.10)

**Proof.** The proof is the same as in Lemma 2.3, so we omit it. \(\square\)

**Lemma 4.2.** If \((u(t), \varphi(t))\) is a weak solution of problem (4.9), then \((u(t), \varphi(t))\) satisfies (4.10), and the following inequalities hold:
\[
\begin{aligned}
\|u(t)\|^2 + \alpha^2 \|\nabla u(t)\|^2 + \|\Delta \varphi(t)\|^2 &\leq \left(\|u(0)\|^2 + \alpha^2 \|\nabla u(0)\|^2 + \|\Delta \varphi(0)\|^2\right) + 2\int_{\Omega} F(\nabla \varphi(0))dx e^{-2C_0 k} + \frac{2C_0}{k}, \quad (4.11) \\
\int_{t}^{t+1} \nu(\|\nabla u(s)\|^2 + \alpha^2 \|\Delta u(s)\|^2)ds &+ \int_{t}^{t+1} \|\mu(s)\|^2ds + \int_{t}^{t+1} \|\Delta^2 \varphi(s)\|^2ds
\end{aligned}
\]
where }k, C_0\text{ only depend on }\nu, \Omega, \text{ and } \bar{C}_1, \bar{R}_1\text{ depend on }\nu, \Omega, \bar{E}(0), \text{ but are independent of }\alpha.

Proof. Let

\[ E_\alpha(t) = \frac{1}{2} (\|u(t)\|^2 + \alpha^2 \|\nabla u(t)\|^2 + \|\nabla \varphi(t)\|^2) + \int_\Omega F(\nabla \varphi) dx, \]  

(4.13)

\[ \bar{E}(0) = \frac{1}{2} (\|u(0)\|^2 + \|\nabla u(0)\|^2 + \|\nabla \varphi(0)\|^2) + \int_\Omega F(\nabla \varphi)(0) dx. \]  

(4.14)

Since }\alpha \in (0, 1], \text{ then } E_\alpha(0) \leq \bar{E}(0). \text{ According to (2.18), we also have}

\[
\frac{d}{dt} E_\alpha(t) + k E_\alpha(t) + \|\mu\|^2 + \frac{k}{2} \int_\Omega |\nabla \varphi|^2 dx \\
= \frac{k}{2} \left( \int_\Omega |u|^2 dx + \alpha^2 \int_\Omega |\nabla u|^2 dx \right) + \frac{k}{2} \int_\Omega |\nabla \varphi|^2 dx + \frac{k}{4} \int_\Omega |\nabla \varphi|^4 dx \\
- \nu (\|\nabla u\|^2 + \alpha^2 \|\nabla \varphi(t)\|^2) - \int_\Omega \mu \cdot \varphi dx - \int_\Omega |\nabla \varphi|^2 dx \\
- \int_\Omega |\nabla \varphi|^4 dx + \int_\Omega |\nabla \varphi|^2 dx + \frac{k |\nabla \varphi|^4}{4}.
\]  

(4.15)

Inserting (2.16), (2.19)-(2.21) into (4.15), we obtain

\[
\frac{d}{dt} E_\alpha(t) + k E_\alpha(t) \leq - \left( \nu - \frac{k C_0}{2} \right) \left( \int_\Omega |\nabla u|^2 dx + \alpha^2 \int_\Omega |\nabla u|^2 dx \right) + |\Omega| + \frac{k |\nabla \varphi|^4}{4} \\
- \left( 1 - \frac{k}{2} \right) |\nabla \varphi|^2 dx - \left( \frac{1}{2} - \frac{k}{4} \right) \left( \int_\Omega |\nabla \varphi|^4 dx + \frac{C_0^3}{16} \right),
\]  

(4.16)

where }C_\Omega \text{ only depends on }\Omega. \text{ Let}

\[ k = \min \{2, \frac{2\nu}{C_\Omega}\}, \quad C_0 = |\Omega| + \frac{k |\nabla \varphi|^4}{4} + \frac{C_0^3}{16}. \]

Applying the Gronwall inequality to (4.16) gives

\[ E_\alpha(t) \leq E_\alpha(0)e^{-kt} + \frac{C_0}{k}. \]  

(4.17)

Obviously, we get (4.11). Integrating (4.10) on }[t, t+1] \text{ and adding (2.26), we have}

\[ E_\alpha(t+1) + \int_t^{t+1} \nu (\|\nabla u(s)\|^2 + \alpha^2 \|\nabla \varphi(s)\|^2) ds + \int_t^{t+1} \|\mu(s)\|^2 ds + \int_t^{t+1} \|\nabla \varphi(s)\|^2 ds \leq E_\alpha(t) + \int_t^{t+1} \|\Delta^2 \varphi(s)\|^2 ds, \]  

(4.18)

which, together with (4.17) and (2.26), gives (4.12). \qed

4.2. The energy estimates for the 3D smectic-A liquid crystal flow-\alpha model. \text{ We define the function } w = (1 + \alpha^2 A)^{-\frac{1}{2}} u \text{ and easily have}

\[ v = (1 + \alpha^2 A) u = (1 + \alpha^2 A)^{\frac{1}{2}} w, u = (1 + \alpha^2 A)^{-\frac{1}{2}} w, \]

that is,

\[ u = (1 + \alpha^2 A)^{-\frac{1}{2}} w, \quad v = (1 + \alpha^2 A)^{\frac{1}{2}} w. \]
We note that the function \((w, \varphi)\) satisfies the following equations:

\[
\begin{align*}
\partial_t w + \nu Aw + (1 + \alpha^2 A)^{-\frac{1}{2}} \tilde{B}((1 + \alpha^2 A)^{-\frac{1}{2}} w, (1 + \alpha^2 A)^{\frac{1}{2}} w) \\
= (1 + \alpha^2 A)^{-\frac{1}{2}} B(\mu, \varphi),
\end{align*}
\]

\[\tag{4.19}\]

Lemma 4.3. If \((w(t), \varphi(t))\) is a weak solution of problem (4.19), then the following inequalities hold:

\[
\|w(t)\|^2 + \|\Delta \varphi(t)\|^2 \leq \left(\|w(0)\|^2 + \|\Delta \varphi(0)\|^2\right) + 2\int \Omega F(\nabla \varphi(0)) \, dx \, e^{-kt} + \frac{2C_0}{k}, \tag{4.20}
\]

\[
\int_t^{t+1} (\nu \|\nabla w(s)\|^2 + \|\mu(s)\|^2) \, ds \leq \left(\frac{1}{2} (\|w(0)\|^2 + \|\Delta \varphi(0)\|^2) + \int \Omega F(\nabla \varphi(0)) \, dx \right) e^{-kt} + \frac{C_0}{k}, \tag{4.21}
\]

\[
\int_t^{t+1} \|\Delta^2 \varphi\|^2 \, ds \leq \tilde{C}_2 (\frac{1}{2} (\|w(0)\|^2 + \|\Delta \varphi(0)\|^2) + \int \Omega F(\nabla \varphi(0)) \, dx) e^{-kt} + \tilde{R}_2, \tag{4.22}
\]

\[
\int_t^{t+1} \|\partial_t w(s)\|^\frac{4}{3},ds \leq \tilde{C}_3 (\frac{1}{2} (\|w(0)\|^2 + \|\Delta \varphi(0)\|^2) + \int \Omega F(\nabla \varphi(0)) \, dx) e^{-kt} + \tilde{R}_3, \tag{4.23}
\]

\[
\int_t^{t+1} \|\partial_t \varphi(s)\|^\frac{4}{3} \, ds \leq \tilde{C}_4 (\frac{1}{2} (\|w(0)\|^2 + \|\Delta \varphi(0)\|^2) + \int \Omega F(\nabla \varphi(0)) \, dx) e^{-kt} + \tilde{R}_4, \tag{4.24}
\]

where \(k, C_0\) only depend on \(\nu, \Omega\), and \(\tilde{C}_i, \tilde{R}_i\) \((i = 2, 3, 4)\) depend on \(\nu, \Omega, E(0)\), but are independent of \(\alpha\).

**Proof.** We know that

\[
\langle (1 + \alpha^2 A)^{-\frac{1}{2}} \tilde{B}((1 + \alpha^2 A)^{-\frac{1}{2}} w, (1 + \alpha^2 A)^{\frac{1}{2}} w), w \rangle
\]

\[
= \langle \tilde{B}((1 + \alpha^2 A)^{-\frac{1}{2}} w, (1 + \alpha^2 A)^{\frac{1}{2}} w), (1 + \alpha^2 A)^{-\frac{1}{2}} w \rangle = 0. \tag{4.25}
\]

Here we have used the identity (4.6), and the fact that

\[
\langle (1 + \alpha^2 A)^{-\frac{1}{2}} B(\mu, \varphi), w \rangle = \langle B((1 + \alpha^2 A)^{-\frac{1}{2}} w, \varphi), \mu \rangle. \tag{4.26}
\]

Multiplying (4.19)\_1 by \(w\), multiplying (4.19)\_2 by \(\mu\), then integrating over \(\Omega\) and using (4.25)-(4.26), we obtain

\[
\frac{d}{dt} \left(\frac{1}{2} \|w(t)\|^2 + \frac{1}{2} \|\Delta \varphi(t)\|^2 \right) + \nu \|\nabla w\|^2 + \|\mu\|^2 = 0. \tag{4.27}
\]

We can then get the estimates (4.20)-(4.22) as in the proof of Lemmas 2.5-2.6. Next, we have

\[
\int_t^{t+1} \|\partial_t w(s)\|_{V'}^\frac{4}{3},ds \leq 2\frac{\nu}{3} \int_t^{t+1} \|Aw(s)\|_{V'}^\frac{4}{3},ds + 2\frac{\nu}{3} \int_t^{t+1} \|(1 + \alpha^2 A)^{-\frac{1}{2}} B(\mu, \varphi)\|_{V'}^\frac{4}{3},ds
\]

\[
+ 2\frac{\nu}{3} \int_t^{t+1} \|(1 + \alpha^2 A)^{-\frac{1}{2}} \tilde{B}((1 + \alpha^2 A)^{-\frac{1}{2}} w, (1 + \alpha^2 A)^{\frac{1}{2}} w)\|_{V'}^\frac{4}{3},ds
\]

\[
\leq 2\frac{\nu}{3} (\int_t^{t+1} \|\nabla w(s)\|_{V'}^\frac{4}{3},ds + \int_t^{t+1} \|\tilde{B}(u, v)\|_{V'}^\frac{4}{3},ds + \int_t^{t+1} \|B(\mu, \varphi)\|_{V'}^\frac{4}{3},ds) \tag{4.28}
\]
Proposition 4.1. For any function $l$ solutions (weak solution of problem (1.1)), we consider the trajectory space $K_0^+$ as the union of weak solutions $(w(t), \varphi(t))$ of problem (4.19), where $w(t) = (1 + \alpha^2 A)^{\frac{1}{2}} u(t)$; here $u(t)$ is a weak solution of problem (1.2)-(1.3).

Proposition 4.1. For any function $(w, \varphi)(\cdot) \in K_0^+$, then
(i) for all $\psi(t) \in C_0^\infty(\mathbb{R}_+)$, there holds
\[
- \int_0^M \left( \frac{1}{2} \|w(t)\|^2 + \|\Delta \varphi(t)\|^2 \right) + \int_\Omega F(\nabla \varphi) \, dx \psi'(t) \, dt \\
+ \int_0^M \left( \|\nabla w\|^2 + \|\Delta^2 \varphi - \nabla \cdot f(\nabla \varphi)\|^2 \right) \psi(t) \, dt \leq 0,
\] (4.32)
(ii) there holds that $T(h)(w, \varphi)(\cdot) \in K_0^+$, i.e.,
\[
\|T(h)(w, \varphi)(\cdot)\|_{\mathcal{F}_+} \leq \tilde{C} \left( \frac{1}{2} \|w(0)\|^2 + \|\Delta \varphi(0)\|^2 \right) + \int_\Omega F(\nabla \varphi(0)) \, dx e^{-kt} + \tilde{R},
\] (4.33)
where $\tilde{C}, \tilde{R}$ only depend on $\nu, \Omega, \tilde{E}(0)$, but are independent of $\alpha$.

Proof. Multiply (4.27) by the test function $\psi(t) \in C_0^\infty(\mathbb{R}_+)$, and integrate in time over $(0, +\infty)$. Then integrating by part in the first integral term, we obtain (4.32). The proof of (4.33) is clearly established by Lemma 4.3.

Proposition 4.2. The trajectory space $K_0^+$ is closed in the topology $\Theta_0^\loc$.

Proof. The proof is the same as in Proposition 3.2.
Theorem 4.3. The translation semigroup \( \{T(h)\} \) acting on \( K_\alpha^+ \) has a trajectory attractor \( \mathfrak{A}_\alpha \). The set \( \mathfrak{A} \) is bounded in \( F^b_+ \) and compact in \( \Theta^{\text{loc}}_+ \). Moreover, 
\[
\mathfrak{A}_\alpha = \Pi + K_\alpha,
\]
where the set \( K_\alpha \) is bounded in \( F^b \) and compact in \( \Theta^{\text{loc}} \).

Proof. It is clear that \( \{T(h)\}K_\alpha^+ \subseteq K_\alpha^+ \), \( h \geq 0 \). Thanks to (4.33), the set \( \tilde{P} = \left\{ (w, \varphi) \in F^b \bigl\| (w, \varphi) \bigr\|_{F^b_+} \leq 2R \right\} \) is an absorbing set for \( K_\alpha^+ \). Then Theorem 4.3 is proved. \( \square \)

5. Approximation of the trajectory attractors of the 3D smectic-A liquid crystal flow equations.

Lemma 5.1. Let two sequences \( \{u_n(t)\} \subset F^b_+ \) and \( \{\alpha_n\} \subset [0,1] \) be given such that \( \alpha_n \to 0^+ \) as \( n \to \infty \). We denote \( w_n = (1 + \alpha_n^2 A)^{\frac{1}{2}} u_n \) for \( n \in \mathbb{N}^+ \). We assume that the sequence \( \{w_n(t)\} \) is bounded in \( F^b_+ \) and \( w_n(t) \to w(t) \) in \( \Theta^{\text{loc}}_+ \) as \( n \to \infty \). Then the sequence \( \{w_n(t)\} \) is bounded in \( F^b_+ \) and \( w_n(t) \to w(t) \) in \( \Theta^{\text{loc}}_+ \) as \( n \to \infty \).

Proof. The proof was given in [1]. \( \square \)

Theorem 5.1. Let a sequence \( \{w_n(t), \varphi_n(t)\} \subset K_\alpha^+ \), \( \alpha_n \to 0^+ \) \( (n \to \infty) \), be such that \( (w_n(t), \varphi_n(t)) \to (w(t), \varphi(t)) \) in the topology \( \Theta^{\text{loc}}_+ \) as \( n \to \infty \). Then \( (w(t), \varphi(t)) \) is a weak solution of the 3D smectic-A liquid crystal flow equations such that \( (w(t), \varphi(t)) \) satisfies the inequality (2.4), i.e., \( (w(t), \varphi(t)) \in K^+ \), where \( K^+ \) is the trajectory space of problem (1.2)-(1.3).

Proof. By Lemma 5.1, we clearly have \( w = u \), so that 
\[
w_n(t) \to u(t) \text{ in } \Theta^{\text{loc}}_+ \text{ as } n \to \infty.
\]
Since \( (w_n(t), \varphi_n(t)) \in K_\alpha^+ \), we have 
\[
\|(w_n(t), \varphi_n(t))\|_{F^b_+} \leq C.
\]
We set \( u_n = (1 + \alpha_n^2 A)^{\frac{1}{2}} w_n \), and inequality (5.2) implies that 
\[
\text{ess sup}_{t \geq 0} \left( \|u_n(t)\|^2 + \alpha_n^2 \|\nabla u_n(t)\|^2 \right) \leq C,
\]
\[
\text{sup}_{t \geq 0} \int_t^{t+1} (\nu \|\nabla u_n(s)\|^2 + \alpha_n^2 \|\Delta u_n(s)\|^2)ds \leq C.
\]
Since \( (w_n(t), \varphi_n(t)) \in K_\alpha^+ \), i.e., \( (w_n(t), \varphi_n(t)) \) is a weak solution of 
\[
\left\{ \begin{array}{l}
\partial_t w_n + \nu A w_n + (1 + \alpha_n^2 A)^{-\frac{1}{2}} \tilde{B}(u_n, v_n) = (1 + \alpha_n^2 A)^{-\frac{1}{2}} \tilde{B}(\mu_n, \varphi_n), \\
\partial_t \varphi_n + B(u_n, \varphi_n) = -\mu_n,
\end{array} \right.
\]
and noticing (5.1), \( (w_n(t), \varphi_n(t)) \to (u(t), \varphi(t)) \) in the topology \( \Theta^{\text{loc}}_+ \) as \( n \to \infty \). Taking the limit of (5.5) in \( L^2(0, M; \mathbb{V}') \) and using Lemma 2.1 , we get 
\[
\left\{ \begin{array}{l}
\partial_t u + \nu A u + \lim_{n \to \infty} (1 + \alpha_n^2 A)^{-\frac{1}{2}} \tilde{B}(u_n, v_n) = B(\mu, \varphi), \\
\partial_t \varphi + B(u, \varphi) = -\mu;
\end{array} \right.
\]
here we have used (3.20)-(3.22). Next we establish that 
\[
(1 + \alpha_n^2 A)^{-\frac{1}{2}} \tilde{B}(u_n, v_n) \to B(u, u) \text{ weakly in } L^2(0, M; \mathbb{V}').
\]
We know that
\[
\tilde{B}(u_n, v_n) = \tilde{B}(u_n, u_n + \alpha_n^2Au_n) = B(u_n, u_n) + \alpha_n^2\tilde{B}(u_n, Au_n);
\]
here we have used (4.8). Using (4.7), we see that
\[
\|\alpha_n^2\tilde{B}(u_n, Au_n)\|_{\mathcal{V}} \leq c\|\nabla u_n\|^\frac{2}{p} \|Au_n\|^\frac{\beta}{2}. \tag{5.9}
\]
Fixing an arbitrary \(\beta, \beta > 1\), applying the H"{o}lder inequality with \(\frac{1}{p} + \frac{1}{q} = 1\), we obtain the following inequality
\[
\int_0^M \|\alpha_n^2\tilde{B}(u_n, Au_n)\|_\mathcal{V}^\beta dt \leq c^\beta \alpha_n^{2\beta} \int_0^M \|\nabla u_n\|^{\frac{2}{p}} \|Au_n\|^{\frac{\beta}{2}} dt
\]
\[
\leq c^\beta \alpha_n^{2\beta} \left(\sup_{t \in [0,M]} \|\nabla u_n\|^\gamma\right) \left(\int_0^M \|\nabla u_n\|^{\frac{2}{p\gamma}} \|Au_n\|^{\frac{\beta}{2}} dt\right)^\frac{1}{\gamma}
\]
\[
\leq c^\beta \alpha_n^{2\beta} \left(\sup_{t \in [0,M]} \|\nabla u_n\|^\gamma\right) \left(\int_0^M \|Au_n\|^{\frac{2}{p\gamma}} dt\right)^\frac{1}{\gamma} \left(\int_0^M \|\nabla u_n\|^{q(\frac{2}{p\gamma}-\gamma)} dt\right)^\frac{1}{\gamma}, \tag{5.10}
\]
where \(\gamma\) is an arbitrary constant such that \(0 < \gamma < \frac{\beta}{2}\) and \(\frac{1}{p} + \frac{1}{q} = 1\). We now set \(\frac{3}{2}\beta p = 2\), that is \(p = \frac{4}{3\beta}\), so that \(q = \frac{4}{4 - 3\beta}\). We set \(q(\frac{2}{p} - \gamma) = 2\), i.e.,
\[
q = \frac{4}{4 - 3\beta} \left(\frac{\beta}{2} - \gamma\right) = 2 \Leftrightarrow \gamma = 2(\beta - 1).
\]
We know that \(\gamma\) satisfies the inequality \(0 < \gamma < \frac{\beta}{2}\),
\[
\gamma = 2(\beta - 1) < \frac{\beta}{2} \Leftrightarrow \beta < \frac{4}{3}.
\]
Replacing \(p, q, \gamma\) by \(\beta\) in (5.10), then we obtain the following inequality
\[
\int_0^M \|\alpha_n^2\tilde{B}(u_n, Au_n)\|_\mathcal{V}^\beta dt
\]
\[
\leq c^\beta \alpha_n^{2\beta} \left(\sup_{t \in [0,M]} \alpha_n^2 \|\nabla u_n\|^\gamma\right)^{\frac{1}{\gamma}} \left(\int_0^M \alpha_n^2 \|Au_n\|^2 dt\right)^\frac{\beta}{2} \left(\int_0^M \|\nabla u_n\|^2 dt\right)^\frac{\beta}{2}, \tag{5.11}
\]
Now inserting (5.3)-(5.4) into (5.11) yields
\[
\int_0^M \|\alpha_n^2\tilde{B}(u_n, Au_n)\|_\mathcal{V}^\beta dt \leq C^\beta \alpha_n^{2\beta - \frac{2\beta}{3}}, \quad 1 < \beta < \frac{4}{3}. \tag{5.12}
\]
Therefore, the term
\[
\alpha_n^2\tilde{B}(u_n, Au_n) \to 0 \text{ strongly in } L^\beta(0, M; \mathcal{V}'), \quad 1 < \beta < \frac{4}{3} \tag{5.13}
\]
In Section 3, we have proved that
\[
B(u_n, u_n) \to B(u, u) \text{ as } n \to \infty \text{ weakly in } L^\beta(0, M; \mathcal{V}'). \tag{5.14}
\]
Noting (5.8), combining with (5.13)-(5.14) and using Lemma 2.1, we find that
\[
(1 + \alpha_n^2A)^{-\frac{1}{2}}\tilde{B}(u_n, v_n) \to B(u, u) \text{ weakly in } L^\beta(0, M; \mathcal{V}'). \tag{5.15}
\]
We have proved that \(w = u\) in \(\Theta^\text{loc}_+\). Then replacing \(u\) by \(w\) in (5.6) and (5.15), we see that \((w, \varphi)\) satisfies
\[
\begin{align*}
\partial_t w + \nu Aw + B(u, u) &= B(\mu, \varphi), \\
\partial_t \varphi + B(u, \varphi) &= -\mu,
\end{align*} \tag{5.16}
\]
that is, \((w(t), \varphi(t))\) satisfies the 3D smectic-A liquid crystal flow equations. According to (4.32), we know that \((w_n(t), \varphi_n(t)) \in K_{\alpha_n}^+\), so that \((w_n(t), \varphi_n(t))\) satisfies

\[
- \int_0^M \left( \frac{1}{2} \left( \|w_n(t)\|^2 + \|\Delta \varphi_n(t)\|^2 \right) + \int_\Omega F(\nabla \varphi_n) dx \right) \psi'(t) dt \\
+ \int_0^M \left( \nu \|\nabla w_n\|^2 + \|\Delta^2 \varphi_n - \nabla \cdot f(\nabla \varphi_n)\|^2 \right) \psi(t) dt \leq 0.
\]  

By assumption, \((w_n(t), \varphi_n(t)) \rightarrow (w(t), \varphi(t))\) in the topology \(\Theta_{+}^{loc}\) as \(n \rightarrow \infty\), i.e.,

\[
w_n \rightarrow w, \text{ weakly in } L^2(0, M; V), \text{ and weakly-star in } L^\infty(0, M; H^1), n \rightarrow \infty,
\]

\[
\partial_t w_n \rightarrow \partial_t w, \text{ weakly in } L^2(0, M; V'), n \rightarrow \infty;
\]

\[
\varphi_n \rightarrow \varphi, \text{ weakly in } L^2(0, M; H^1), \text{ and weakly-star in } L^\infty(0, M; H^1), n \rightarrow \infty,
\]

\[
\partial_t \varphi_n \rightarrow \partial_t \varphi, \text{ weakly in } L^2(0, M; H^1), n \rightarrow \infty,
\]

and we can obtain

\[
- \int_0^M \left( \frac{1}{2} \left( \|w(t)\|^2 + \|\Delta \varphi(t)\|^2 \right) + \int_\Omega F(\nabla \varphi) dx \right) \psi'(t) dt \\
+ \int_0^M \left( \nu \|\nabla w\|^2 + \|\Delta^2 \varphi - \nabla \cdot f(\nabla \varphi)\|^2 \right) \psi(t) dt \leq 0.
\]  

The proof of (5.22) is the same as in Proposition 3.2. Thus \((w(t), \varphi(t))\) satisfies the inequality (2.4), i.e., \((w(t), \varphi(t)) \in K^+\), and the proof is now complete. 

We introduce the set

\[B_\alpha = \{(w(t), \varphi(t)), \ t \geq 0\}, \quad 0 < \alpha \leq 1;\]

here \(w(t) = (1 + \alpha^2 A)^{\frac{1}{2}} u(t)\), and \((u(t), \varphi(t))\) is a solution of problem (1.2)-(1.3).

**Theorem 5.2.** Let \(B_\alpha = \{(w(t), \varphi(t)), \ t \geq 0\}, \ 0 < \alpha \leq 1, \) be the bounded sets of solutions of the 3D smectic-A liquid crystal flow-\(\alpha\) system (4.19) that satisfy the inequality

\[\|(w(t), \varphi(t))\|_{\mathcal{F}_+^*} \leq R, \quad \forall \alpha \in (0, 1].\]  

Then the sets of shifted solutions \(\{T(h) B_\alpha\}\) convergence holds:

\[T(h)B_\alpha \rightarrow \mathfrak{A} \text{ in the topology } \Theta_{+}^{loc} \text{ as } h \rightarrow +\infty, \ \alpha \rightarrow 0^+,\]

where \(\mathfrak{A}\) is the trajectory attractor of the 3D smectic-A liquid crystal flow equation (1.2)-(1.3).

**Proof.** There exists a sequence \(\alpha_n \rightarrow 0^+, \) as \(n \rightarrow +\infty.\) Let \((w^{\alpha_n}(t), \varphi^{\alpha_n}(t)) \in B_\alpha,\)

where \((w^{\alpha_n}(t), \varphi^{\alpha_n}(t))\) is a solution of the 3D smectic-A liquid crystal flow-\(\alpha\) system (4.19), i.e.,

\[(w^{\alpha_n}(t), \varphi^{\alpha_n}(t)) \in K_{\alpha_n}^+;\]

\[T(h)(w^{\alpha_n}(t), \varphi^{\alpha_n}(t)) = (w^{\alpha_n}(t+h), \varphi^{\alpha_n}(t+h)).\]

Since \(T(h)K_{\alpha_n}^+ \subseteq K_{\alpha_n}^+\), then

\[(w^{\alpha_n}(t+h), \varphi^{\alpha_n}(t+h)) \in K_{\alpha_n}^+;\]

i.e.,

\[
\begin{align*}
\partial_t w^{\alpha_n}(t+h) + \nu Aw^{\alpha_n}(t+h) + (1 + \alpha^2 A)^{\frac{1}{2}} \tilde{B}(w^{\alpha_n}(t+h), v^{\alpha_n}(t+h)) \\
= (1 + \alpha^2 A)^{\frac{1}{2}} \tilde{B}(\mu^{\alpha_n}(t+h), \varphi^{\alpha_n}(t+h)),
\end{align*}
\]

\[
\partial_t \varphi^{\alpha_n}(t+h) + B(u^{\alpha_n}(t+h), \varphi^{\alpha_n}(t+h)) = -\mu^{\alpha_n}(t+h).
\]

\[\Box\]
which implies
\[ \|(w^{\alpha_n}(t+h), \varphi^{\alpha_n}(t+h))\|_{\mathcal{P}^+} \leq R. \]  
(5.26)

By Theorem 5.1, there exists \((w(t+h), \varphi(t+h))\) in \(\mathcal{F}^{loc}_+\), such that \((w^{\alpha_n}(t+h), \varphi^{\alpha_n}(t+h)) \to (w(t+h), \varphi(t+h))\) in the topology \(\Theta^{loc}_+\) as \(n \to +\infty\), and 
\((w(t+h), \varphi(t+h))\) is a weak solution of the 3D smetic-A liquid crystal flow equations, i.e.,
\[ \lim_{n \to +\infty} (w^{\alpha_n}(t+h), \varphi^{\alpha_n}(t+h)) = (w(t+h), \varphi(t+h)) \text{ in the topology } \Theta^{loc}_+. \]

Owing to (5.26), \((w(t+h), \varphi(t+h))\) solves
\[ \|(w(t+h), \varphi(t+h))\|_{\mathcal{P}^+} \leq R. \]  
(5.27)

There exists a sequence \(h_n \to +\infty\) as \(n \to +\infty\) such that
\[ \lim_{n \to +\infty} (w(t+h_n), \varphi(t+h_n)) = (w(t+h), \varphi(t+h)) \text{ in the topology } \Theta^{loc}_+. \]  
(5.28)

Next, we prove that \(\lim_{n \to +\infty} (w(t+h_n), \varphi(t+h_n)) \in \mathfrak{A}\). By definition of \(\cdot\|_{\mathcal{P}^+}\), (5.27)-(5.28) means that
\[ \|(w(t+h_n), \varphi(t+h_n))\|_{\mathcal{P}^+} \]
\[ \sup_{t \geq h_n} \|w(t)\| + \sup_{t \geq h_n} \|\Delta \varphi(t)\| + \left( \sup_{t \geq h_n} \int_t^{t+1} \|\nabla w(s)\|^2 ds \right)^\frac{1}{2} \]
\[ + \left( \sup_{t \geq h_n} \int_t^{t+1} \|\Delta \varphi(s)\|^2 ds \right)^\frac{1}{2} + \left( \sup_{t \geq h_n} \int_t^{t+1} \|\partial_s w(s)\|^\frac{4}{3} ds \right)^\frac{3}{4} \]
\[ + \left( \sup_{t \geq h_n} \int_t^{t+1} \|\partial_s \varphi(s)\|^\frac{4}{3} ds \right)^\frac{3}{4} \leq R, \]  
(5.29)

which implies
\[ \lim_{n \to +\infty} \|(w(t+h_n), \varphi(t+h_n))\|_{\mathcal{P}^+} \]
\[ \sup_{t \in \mathbb{R}} \|w(t)\| + \sup_{t \in \mathbb{R}} \|\Delta \varphi(t)\| + \left( \sup_{t \in \mathbb{R}} \int_t^{t+1} \|\nabla w(s)\|^2 ds \right)^\frac{1}{2} \]
\[ + \left( \sup_{t \in \mathbb{R}} \int_t^{t+1} \|\Delta \varphi(s)\|^2 ds \right)^\frac{1}{2} + \left( \sup_{t \in \mathbb{R}} \int_t^{t+1} \|\partial_s w(s)\|^\frac{4}{3} ds \right)^\frac{3}{4} \]
\[ + \left( \sup_{t \in \mathbb{R}} \int_t^{t+1} \|\partial_s \varphi(s)\|^\frac{4}{3} ds \right)^\frac{3}{4} \leq R, \]  
(5.30)
i.e., for all \(t \in \mathbb{R}\),
\[ \lim_{n \to +\infty} (w(t+h_n), \varphi(t+h_n)) = (w(t), \varphi(t)) \in \mathcal{F}^b, \]  
(5.31)
and thus \((w(t), \varphi(t))\) is a weak solution of the 3D smetic-A liquid crystal flow equations, i.e., \((w(t), \varphi(t)) \in \mathfrak{A}\).

Owing to the above argument, we set \(\mathcal{B}_\alpha = \mathfrak{A}_\alpha\), and we have proved the following theorem.

**Theorem 5.3.** According to the definition of \(\mathfrak{A}_\alpha\), the following convergence holds:
\( \mathfrak{A}_\alpha \to \mathfrak{A} \) in the topology \(\Theta^{loc}_+\) as \(\alpha \to 0^+\).  
(5.32)
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Received November 2019; revised February 2020.

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