The Superspace WZNW Action
for 4D, N=1 Supersymmetric QCD

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ABSTRACT

We discuss features of the 4D, N = 1 WZNW term expressed as a function of chiral superfields and defined in a manner appropriate to calculate phenomenological matrix elements.

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(I.) Introduction

A possible interpretation of the recent experimental observation of a “light” Higgs particle\[1\] at 114.5 GeV is that the probability of finding supersymmetry in Nature grows ever larger. It is, therefore, appropriate that techniques be developed and results worked out with an eye toward developing (in principle at least) experimentally verifiable tests. One place in which this might be done is within the context of effective actions for 4D, \( N = 1 \) SUSY QCD. This is an area that has attracted our attention during the course of the past few years \[2\]. We have explored the possibility of a non-conventional description of the 4D, \( N = 1 \) supersymmetric WZNW action and its possible role as an effective action for low-energy 4D, \( N = 1 \) SUSY QCD; we have also investigated the more conventional description and within this approach written the first practical description \[3, 4\] of the BGJ anomaly action and the corresponding expression for the gauged 4D, \( N = 1 \) supersymmetric WZNW action \[3\]. In this paper we discuss some features of this latter result.

For many years a treatment of the 4D, \( N = 1 \) supersymmetric WZNW action given in the literature by Nemeschansky and Rohm (N-R) \[5\] has been the standard basis for considering properties of the low-energy 4D, \( N = 1 \) SUSY QCD effective action related to the anomaly (see for example \[6\]). This description possesses an infinite number of unspecified constants that appear in an undetermined function (denoted by \( \beta_{ij} \)) in the N-R work.\[6\] These constants enter in the calculations of matrix elements based upon the N-R WZNW action. Any physically testable process described by the N-R WZNW model is largely unspecified due to this. In the N-R work \[5\] a discussion was given of complex geometry, Stein manifolds and the relevant Dolbeault cohomology groups related to a tensor such as \( \beta_{ijk} \). Unfortunately, since this tensor was not determined, one could not derive specific physical consequences or predictions from the N-R action.

On the other hand, our recent work \[3\] based on the minimal homotopy of 4D, \( N = 1 \) supersymmetric Yang-Mills theory \[3, 4\] is quite predictive. We have presented in these references the entire non-linear and therefore non-perturbative structure of the 4D, \( N = 1 \) anomaly and WZNW term. We will use these new results to explore the formulation of the WZNW terms in the low-energy effective 4D, \( N = 1 \) supersymmetric QCD action.

\[6\] Even prior to its publication, the N-R action was in the unpublished notes of one of the present authors (SJG) where it remained due to the lack of an explicit expression for the \( \beta \)-coefficient.
(II.) Explicit 4D, $N = 1$ Superspace WZNW Action

In previous work [3, 4] it was shown that the consistent anomaly in 4D, $N = 1$ supersymmetric Yang-Mills theories can be cast in the form

$$A_{BGJ}(\Lambda; e^V) = \frac{1}{4\pi^2} \text{Im} \left[ \int d^4x \ d^2\theta \ d^2\tilde{\theta} \ \mathcal{P}(\Lambda; e^V) \right],$$

where

$$\mathcal{P}(\Lambda; e^V) = \text{Tr} \left[ \Lambda \left( \Gamma^\alpha W_\alpha - \int_0^1 dy \ y \ ( [ W^\alpha, \pi_\alpha ] e^V \mathcal{G} 
+ \{ \tilde{W}^{\dot{\alpha}}, I - e^V \mathcal{G} \} \tilde{\pi}_{\dot{\alpha}} ) \right) \right]$$

is a function of the chiral gauge parameter $\Lambda$ and of the following geometric objects:

$$W_\alpha \equiv i \mathcal{D}^2 ( \mathcal{G} D_\alpha \mathcal{G}^{-1} ) , \quad \tilde{W}\dot{\alpha} \equiv i \mathcal{G} \{ D^2 ( \mathcal{G}^{-1} \mathcal{D}_{\dot{\alpha}} \mathcal{G} ) \} \mathcal{G}^{-1} ,$$

$$\pi_\alpha \equiv \mathcal{G}^2 e^V \Gamma_\alpha \quad , \quad \tilde{\pi}_{\dot{\alpha}} \equiv e^V \mathcal{G} \tilde{\Gamma}_{\dot{\alpha}} \mathcal{G} ,$$

$$\Gamma_\alpha \equiv i e^{-V} D_\alpha e^V \quad , \quad \tilde{\Gamma}_{\dot{\alpha}} \equiv e^{-V} \Gamma_{\dot{\alpha}} e^V ,$$

$$\mathcal{G} \equiv \left[ I + y ( e^V - I ) \right]^{-1} .$$

A set of parametric gauge orbits of the Yang-Mills superfields which appear in this action are defined by the equations

$$\Gamma'_\alpha \equiv U^{-1} \Gamma_\alpha U + i U^{-1} D_\alpha U \quad , \quad W'_\alpha \equiv U^{-1} W_\alpha U ,$$

$$e^{V'} \equiv U^\dagger e^V U \quad , \quad U \equiv e^{-i\omega\Lambda/f} , \quad \mathcal{D}_{\dot{\alpha}} \Lambda = 0 .$$

We have introduced a constant $f$ with the dimensions of mass so as to allow $\Lambda$ to have canonical engineering dimensions. It thus follows that the 4D, $N = 1$ supersymmetric gauged WZNW term takes the form

$$\mathcal{S}^{\text{gauged}}_{\text{WZNW}}(\Lambda; e^{V'}) = C_0 \left( \frac{1}{4\pi f} \right) \text{Im} \left[ \int d^4x \ d^2\theta \ d^2\tilde{\theta} \int_0^1 dw \ \mathcal{P}(\Lambda; U^\dagger e^V U) \right]$$

$$= C_0 \left( \frac{1}{4\pi f} \right) \text{Re} \left[ \int d^4x \ d^2\theta \ d^2\tilde{\theta} \int_0^1 dw \ \mathcal{P}(U^{-1} \partial_w U; U^\dagger e^V U) \right].$$

\footnote{The second line of (2.5) corrects a misprint in the corresponding expression in [3].}

\footnote{Since ultimately it is our desire for this action to be as close as possible to the form of a useful phenomenological action, we have introduced a normalization constant $C_0/f$ to multiply the result of our previous work. The actual value of the dimensionless constant $C_0$ is determined by looking at the purely bosonic limit of this action.}
In the remainder of this work we only consider the ungauged version of this action obtained by setting the Yang-Mills superfield to zero (i.e. evaluate the anomaly on a pure gauge orbit)

\[
S_{WZNW}(U) = C_0 \left( \frac{1}{4 \pi^2} \right) \text{Re} \left[ \int d^4x \, d^2\theta \, d^2\bar{\theta} \int_0^1 dw \, \int_0^1 dy \, y \, Q(U) \right],
\]  
(2.6)

where the integrand takes the form

\[
Q(U) = \text{Tr} \left[ U^{-1} (\partial_w U) \left( [Y^\alpha, u_\alpha] U^\dagger U \hat{G} + \{ \tilde{V}^{\hat{\alpha}}, \tilde{H} \} \tilde{u}_{\hat{\alpha}} \right) \right],
\]  
(2.7)

\[
\begin{align*}
Y_\alpha &\equiv iyD^2 \left( \hat{G} U^\dagger U \Pi^L_\alpha \right), \quad \tilde{V}_{\hat{\alpha}} \equiv -iy\hat{G}D^2 \left( \overline{\Pi}^L_\alpha U^\dagger U \hat{G} \right) \hat{G}^{-1}, \\
u_\alpha &\equiv iy\hat{G}^2 U^\dagger U \Pi^L_\alpha, \quad \tilde{u}_{\hat{\alpha}} \equiv -iy\overline{\Pi}^L_{\hat{\alpha}} \hat{G} U^\dagger U, \\
\hat{G} &\equiv \left[ \mathbf{1} + y \left( U^\dagger U - \mathbf{1} \right) \right]^{-1}, \quad \tilde{H} \equiv 1 - U^\dagger U \hat{G}.
\end{align*}
\]  
(2.8)

We have defined

\[
\tilde{V}_{\hat{\alpha}} = \hat{G} \left( -Y_\alpha \right)^\dagger \hat{G}^{-1}, \quad \tilde{u}_{\hat{\alpha}} = \hat{G} \left( -u_\alpha \right)^\dagger \hat{G}^{-1},
\]  
(2.9)

where the \( \dagger \)-operation involves superspace conjugation.

The superfield \( \hat{G} \) is the vanishing gauge superfield limit of the minimal homotopy \( G \). In the above equations we have also introduced the left- and right-invariant Maurer-Cartan forms as well as their hermitian conjugates via the definitions

\[
\begin{align*}
U^{-1}dU &\equiv \Pi^L, \quad [dU^\dagger][U^\dagger]^{-1} \equiv \overline{\Pi}^L, \\
(dU)U^{-1} &\equiv \Pi^R, \quad [U^\dagger]^{-1}[dU^\dagger] \equiv \overline{\Pi}^R,
\end{align*}
\]  
(2.10)

for any derivative operator \( d \). So for example, \( \Pi^L_{\alpha} = U^{-1}D_{\alpha}U \), etc. The results in (2.6 - 2.8) define an explicit 4D, \( N = 1 \) supersymmetric WZNW action with two free parameters. The quantities \( Y_\alpha \) and \( \tilde{V}_{\hat{\alpha}} \) can be further expanded by using

\[
\begin{align*}
D_{\alpha}\hat{G} &= -y\hat{G} \left[ U^\dagger U \Pi^L_{\alpha} \right] \hat{G}, \quad \overline{D}_{\hat{\alpha}}\hat{G} = -y\hat{G} \left[ \overline{\Pi}^L_{\hat{\alpha}} U^\dagger U \right] \hat{G}, \\
\partial_{\alpha}\hat{G} &= -y\hat{G} \left[ \overline{\Pi}^L_{\alpha} U^\dagger U + U^\dagger U \Pi^L_{\alpha} \right] \hat{G}.
\end{align*}
\]  
(2.11)

This leads to

\[
Y_\alpha = -y(1 - y) \left[ \hat{G} \overline{\Pi}_{\alpha} U^\dagger U \hat{G} \Pi_{\alpha} - \frac{i}{2} \hat{G} \left( \overline{D}_{\hat{\alpha}} \Pi_{\alpha} - \overline{\Pi}_{\hat{\alpha}} \Pi_{\alpha} \right) U^\dagger U \hat{G} \Pi_{\alpha} \right.

\left. - i(1 - y) \hat{G} \overline{\Pi}_{\alpha} \hat{G} \overline{\Pi}_{\hat{\alpha}} U^\dagger U \hat{G} \Pi_{\alpha} \right],
\]  
(2.12)

\[
\tilde{V}_{\hat{\alpha}} = y(1 - y) \left[ \hat{G} \overline{\Pi}_{\alpha} U^\dagger U \hat{G} \Pi_{\alpha} - \frac{i}{2} \hat{G} \overline{\Pi}_{\alpha} U^\dagger U \hat{G} \left( D_{\alpha} \Pi_{\alpha} - \Pi_{\alpha} \Pi_{\alpha} \right) \right.

\left. - i(1 - y) \hat{G} \overline{\Pi}_{\alpha} U^\dagger U \hat{G} \Pi_{\alpha} \Pi_{\alpha} \right].
\]  
(2.13)
All of the superfield Maurer-Cartan forms in (2.12, 2.13) are left-invariant forms but we have dropped the superscripts $L$.

Expressed in terms of Maurer-Cartan forms and their derivatives, the function $Q(U)$ is given by

$$Q(U) = iy(1 - y) \, \text{Tr} \left[ U^\dagger U \hat{G} \Pi_w \, \chi \right] ,$$

$$\chi \equiv \left[ \hat{G} \Pi_a \, \Pi^\dagger U^\dagger \hat{G} \Pi_a , \, U^\dagger U \hat{G} \Pi_a \right] + \left\{ \hat{G} \Pi_a \, \Pi^\dagger U^\dagger \hat{G} \Pi_a \right\} \hat{G} \Pi^\dagger_a$$

$$+ \frac{i}{2} \left\{ \hat{G} \Pi^\dagger_a \, \Pi^\dagger \hat{G} \Pi_a U^\dagger \hat{G} \Pi^\dagger U^\dagger \hat{G} \Pi^\dagger \hat{G} \Pi_a \right\}$$

$$+ i \left( 1 - y \right) \left\{ \hat{G} \Pi^\dagger \Pi^\dagger \hat{G} \Pi^\dagger U^\dagger \hat{G} \Pi^\dagger \, U^\dagger U \hat{G} \Pi_a \right\}$$

$$+ i \left( 1 - y \right) \left\{ \hat{G} \Pi^\dagger \Pi^\dagger \hat{G} \Pi^\dagger \hat{G} \Pi^\dagger , \, \hat{H} \right\} \hat{G} \Pi^\dagger \hat{G} \Pi^\dagger \hat{G} \Pi_a . \quad (2.14)$$

Eqs. (2.6, 2.14) represent our explicit superspace form of the supersymmetric WZNW action.

In order to make a comparison with the action in [5] we can introduce the holomorphic “Lie-group frames” $L_j^k(\Lambda)$ and $R_j^k(\Lambda)$ as well as their anti-holomorphic hermitian conjugates via the superfield Maurer-Cartan forms in the equations

$$\Pi_L^L \equiv -\frac{i}{f} w(d\Lambda^j) \, L_j^k \, t_k , \quad \Pi_L^R \equiv -\frac{i}{f} w(d\Lambda^j) \, \hat{T}_j^k \, t_k ,$$

$$\Pi_R^L \equiv -\frac{i}{f} w(d\Lambda^j) \, R_j^k \, t_k , \quad \Pi_R^R \equiv -\frac{i}{f} w(d\Lambda^j) \, \hat{R}_j^k \, t_k . \quad (2.15)$$

The symbol $t_k$ denotes the hermitian group generators. The left-invariant holomorphic frames can be calculated as power series from

$$L_j^k(\Lambda) = (C_2)^{-1} \text{Tr} \{ \alpha^k [g_2(\frac{1}{2}L\Lambda) \, t_j] \} \quad , \quad L_\Lambda t_j \equiv \frac{i}{f} \left[ \Lambda^k t_k , \, t_j \right] , \quad (2.16)$$

where $g_2(x) = e^{-x} \sinh(x)/x$ and the constant $C_2$ is chosen so that $L_j^k(0) = \delta_j^k$. The right-invariant holomorphic frames arise as $R_j^k(\Lambda) = L_j^k(-\Lambda)$. (If we integrate with respect to a real variable the determinant of the Lie-group frames over one copy of the group elements, this calculates the volume of the group, for any compact group.) By use of (2.15) an alternative form of the 4D, $N = 1$ supersymmetric ungauged WZNW action is

$$S_{WZNW} = C_0 \left( \frac{1}{4\pi^2} \right) \text{Re} \left\{ \int d^8 z \left[ T_{ijk}^\dagger (D^a \Lambda^i) \, (\partial_2 L^j) \, (\overleftarrow{D}_a \overleftarrow{X}^k) + T_{ijk}^\dagger (D^2 \Lambda^i) \, (\overleftarrow{D}_a \overleftarrow{X}^j) \, (\overleftarrow{D}_a \overleftarrow{X}^k) \right] \right\} , \quad (2.17)$$
where the transcendental coefficients $T_{ij}(\Lambda, \overline{\Lambda})$, $T_{ij}(\Lambda, \overline{\Lambda})$ and $T_{ijk}(\Lambda, \overline{\Lambda})$ can be obtained explicitly from (2.14) \[7\].

(III.) Concluding Remarks

The next major step to take along the direction of this research is to solve the problem of gauging both left and right symmetries in the context of the action in (2.14) or (2.17). This would allow the systematic investigation of supersymmetrical electro-weak couplings to the supersymmetric hadronic model that we have described. We also note that the 4D, $N = 1$ SUSY WZNW model is invariant under the set of variations

$$\delta \Lambda^j = -i f [ \bar{\alpha}^k (L^{-1})_k^j - \alpha^k (R^{-1})_k^j ] ,$$

with constant real parameters $\bar{\alpha}^k$ and $\alpha^k$ and these transformations correspond to the $SU_L(3) \otimes SU_R(3)$ flavor symmetries of ordinary QCD. This has implications for the metric term of the actual 4D, $N = 1$ SUSY low-energy QCD model. Unlike the case of 4D, $N = 2$ SUSY YM theory \[8\], there is no direct paradigm for the non-perturbative structure of the unbroken $N = 1$ theory. It is therefore of some interest to see if the corresponding $N = 2$ vector multiplet or hypermultiplet effective action $\sigma$-model terms are capable of shedding some light on the $N = 1$ problem. In particular, it will be important to see if the $N = 1$ truncations of the $N = 2$ models also possess these symmetries.

A simple examination of the non-supersymmetric WZNW model shows that it is on the pure gauge orbit of the appropriate consistent anomaly\[9\]. By its method of construction the action in (2.6, 2.14) is also on the pure gauge orbit of an anomaly which, as derived in the work of \[3, 4\], is guaranteed to solve the consistency condition. Our result is an exact non-perturbative superfield result for the 4D, $N = 1$ Yang-Mills effective action. Therefore all the terms in (2.17) must be included in order to describe the actual 4D, $N = 1$ superspace WZNW action related to the non-Abelian consistent anomaly. Our work also likely implies that there is no choice of the function $\beta_{ij}$ in \[5\] such that the resulting N-R WZNW action is on the supersymmetric gauge orbit of a superspace anomaly that satisfies the Wess-Zumino consistency condition. This is because, although one can choose $\beta_{ij} = T_{ij}$, there are no local redefinitions that begin solely with the N-R WZNW action and produce the $T_{ij}$ and $T_{ijk}$ terms in (2.17).

Our work is based on a particular choice $G$ of the homotopy function used in determining the consistent anomaly. Although this choice is not unique, it has allowed

\[9\]By way of comparison, the pure gauge orbit of the covariant anomaly is zero!
us to obtain an explicit expression for the anomaly and the corresponding WZNW action. As shown in ref. [4] any other choice leads to cohomologically equivalent results. Aside from this, we have introduced two free parameters \( f \) and \( C_0 \). One free parameter, \( f \), appears exactly as in the nonsupersymmetric case where it corresponds to the pion-decay constant. The other constant \( C_0 \), which is proportional to the number of QCD colors, is determined by the normalization in the non-supersymmetric case. Although our final expressions are not simple, they can be used for actual superspace calculations of supersymmetric processes which may ultimately make experimentally verifiable predictions.

“I resent your insinuendoes.”
– Richard J. Daley

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