Quantum entanglement swapping with spontaneous parametric down conversion

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Abstract

Two remote parties that have never interacted each other can be entangled through entanglement swapping operation done by a third party. Currently existing entanglement swapping experiments are done probabilistically by post-selection, i.e., once a successful swapping is verified, the resultant entanglement is destructed. We propose a simple non-post-selection scheme to demonstrate the high quality quantum entanglement swapping with the spontaneous parametric down conversion (SPDC) process. Our scheme only requires the normal photon detectors which only distinguish the vacuum and non-vacuum Fock states.

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Introduction. Entanglement plays an important role in quantum mechanics. It is at the central role in the non-locality of quantum mechanics [1] including Einstein-Podolsky-Rosen paradox, Bell’s theorem and so on. Entanglement is perhaps the most important resource in quantum computation and information [2]. To set up entanglement between particle A and B, one may straightforwardly consider the method of collecting them from the same source or of having them interact each other and then obtaining the entangled state of A and B after an appropriate non-trivial time evolution. However, one can also obtain the entanglement through jointly measuring two particles (or light beams) in Bell basis as shown in Fig. 1. Note that this measurement does not have to be done directly on particle A and B: When A is entangled with particle A’, B is entangled with B’, a Bell measurement on A’ and B’ will project A and and B into an entangled state. That is to say, it is possible to entangle the two remote particles A and B without any interaction by the method of entanglement swapping [3–5]. The first entanglement swapping experiment was done by the Innsbruck group some years ago [6]. However, similar to the case of the quantum teleportation experiments [7,9], the result there is a post-selectin result: once a successful entanglement swapping is verified, the swapped entangled state is destructed already (picture A in Fig. 2). Recently, the entanglement swapping has also been tested in photon number space [8]. Again it is a post-selection test unless a sophisticated photon detector to distinguish the one photon and two photons is used (picture B of Fig. 2). Such a sophisticated photon detector is generally believed to be rather rare by our current technology therefore it’s not likely to really implement such a sophisticated photon detector in the experiment.

In this work, we report a very simple and robust non-post-selection experimental scheme for the entanglement swapping based on the weakly entangled states initially. Before going into details of our scheme, we examine the post-selection nature of some existing experiments. Post-selection nature of currently existing experiments. The existing experimental set-up in polarization space is schematically shown in Fig.2 A. An emitted pair will be in the maximally entangled state in the polarization space if we only collect the beam lights in
crossing points of two emission cones [10]. In the experiment, the pump light passes through the crystal twice. The un-normalized total emitted state used there is

\[ |X\rangle_{1324} + |Y\rangle_{13} + |Y\rangle_{24} \tag{1} \]

where

\[ |X\rangle_{1324} = (|H\rangle_1|V\rangle_3 - |V\rangle_1|H\rangle_3)(|H\rangle_2|V\rangle_4 - |V\rangle_2|H\rangle_4); \tag{2} \]

\[ |Y\rangle_{ij} = |2H\rangle_i|2V\rangle_j + |2V\rangle_i|2H\rangle_j - |HV\rangle_i|HV\rangle_j. \tag{3} \]

Only for the state \( |X\rangle_{1234} \), the maximally entangled state will be swapped to beam 1 and 4, provided that both D2 and D3 are clicked. However, the initially emitted state contains the constitutes of \( |Y\rangle \). For this term, after D2 and D3 are both clicked, either beam 1 or beam 4 will contain nothing therefore they are not entangled. To overcome this, the 4 fold detection is carried out in the experiment [6]. However, this will destroy all swapped entanglement between 1 and 4. The existing experimental set-up in vacuum-one-photon space is shown in Fig.2 B. After passing through the beam splitter, the state is

\[ |00\rangle_{2'3'}|11\rangle_{14} + \frac{1}{2}(|10\rangle_{2'3'}|\Psi^+\rangle_{14} + |01\rangle_{2'3'}|\Psi^-\rangle_{14} \] + \frac{1}{\sqrt{2}}|20\rangle_{2'3'}|00\rangle_{14} - \frac{1}{\sqrt{2}}|02\rangle_{2'3'}|00\rangle_{14}. \tag{4} \]

Indeed, if beam 2' or beam 3' contains exactly one photon, beam 1 and 4 will be maximally entangled. However, since the photon detectors do not distinguish one photon and two photon cases, the final result on beam 1 and beam 4 will be distorted by the constitute \( \frac{1}{\sqrt{2}}|20\rangle_{2'3'}|00\rangle_{14} - \frac{1}{\sqrt{2}}|02\rangle_{2'3'}|00\rangle_{14} \). That is to say, whenever one detector is clicked, beam 1 and 4 is actually in a mixture of a maximally entangled state and vacuum, instead of a pure maximally entangled state. To overcome this, beam 1 and beam 4 are also detected in the experiment [8], again, this post-selection operation will destroy all swapped entanglement between 1 and 4 whenever.

**Proposal for non-post-selection entanglement swapping.** Consider the initial state

\[ |\Psi^-\rangle_{1234} = |\Psi^-\rangle_{12} \otimes |\Psi^-\rangle_{34}. \tag{5} \]
Obviously, none of particle 1,2 is entangled with any particle of 3,4 at this stage. However, if we jointly measure particle 2 and 3 in Bell basis, particle 1 and 4 will be projected to one of the 4 Bell state depending on the measurement result of particle 2 and 3. Explicitly, Eq.(5) can be recast into the following form:

\[
|\Psi\rangle_{1234} = \frac{1}{2} \left( |\Psi^+\rangle_{23} |\Psi^+\rangle_{14} - |\Psi^-\rangle_{23} |\Psi^-\rangle_{14} - |\Phi^+\rangle_{23} |\Phi^+\rangle_{14} + |\Phi^-\rangle_{23} |\Phi^-\rangle_{14} \right) 
\]

(6)

and \(|\Phi^\pm\rangle_{ij} = \frac{1}{\sqrt{2}} (|0\rangle_i |0\rangle_j \pm |1\rangle_i |1\rangle_j)\), |\Psi^\pm\rangle_{ij} = \frac{1}{\sqrt{2}} (|0\rangle_i |1\rangle_j \pm |1\rangle_i |0\rangle_j)\). This shows that whenever particle 2,3 is collapsed to a certain Bell state, particle 1,4 is projected to the same Bell state therefore the maximal entanglement between particle 1 and 4(Alice and Bob) is set up. In Eq.(5) we have used the initial state of product of two antisymmetric states, actually, a product of arbitrary two maximally entangled state will cause the similar result: after a joint measurement to particle 2,3 in Bell basis, particle 1,4 will be projected to a maximally entangled state. It has been shown in Ref. [5] that, even though we start from a product of non-maximally entangled states, we can still probabilistically obtain the maximal entanglement between 1 and 4 after the joint measurement to particle 2,3. For example, we consider the following initial state

\[
|\Psi'\rangle_{1234} = |\theta\rangle_{12} |\theta\rangle_{34} 
\]

(7)

and \(|\theta\rangle_{ij} = \cos \theta |0\rangle_i |0\rangle_j + \sin |1\rangle_i |1\rangle_j\). This state cab be recast to

\[
|\Psi'\rangle_{1234} = \frac{1}{2} \sin \theta \cos \theta \left( |\Psi^+\rangle_{23} |\Psi^+\rangle_{14} - |\Psi^-\rangle_{23} |\Psi^-\rangle_{14} \right) + \\
\frac{1}{2} \left[ |\Phi^+\rangle_{23} (\cos^2 \theta |0\rangle_1 |0\rangle_4 + \sin^2 \theta |1\rangle_1 |1\rangle_4) + |\Phi^-\rangle_{23} (\cos^2 \theta |0\rangle_1 |0\rangle_4 - \sin^2 \theta |1\rangle_1 |1\rangle_4) \right]. 
\]

(8)

From this we can see that, even in the case that \(\theta\) is very small, we can still set up the maximal entanglement between particle 1 and 4 with a small probability through swapping operation. As we shall show it soon, this small value of \(\theta\) can be an important advantage in a real experiment with imperfect entanglement source and limited power of practically existing devices. By making use of the small value of \(\theta\), one may test entanglement swapping without post-selection.
Our strategy is to build up the entanglement between Alice and Bob, given two copies of weakly entangled state, e.g.

\[ |\chi\rangle = \frac{1}{\sqrt{1 + \epsilon^2}}(|00\rangle + \epsilon|11\rangle) \]  

which weakly entangles Alice and Clare, Bob and Clare respectively. After a Bell type measurement in Clare’s subspace and certain specific result (the coincidence event) is observed, we believe we have create a state \( \rho_{AB} \) between Alice and Bob satisfying one of the following two equations

\[ \langle \Psi^\pm | \rho_{AB} | \Psi^\pm \rangle \sim 1 \]  

and \( |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle) \). Our first experimental scheme is schematically shown in fig.3 A. The spontaneous parametric down conversion process may happen after the pump light passes through the nonlinear crystal. The total state for all the four beams can be written in the following form (in a good approximation)

\[ |\chi_0\rangle = \frac{1}{1 + |\tau|^2}(|0\rangle_1|0\rangle_4 + |1\rangle_1|1\rangle_4)(|0\rangle_2|0\rangle_3 + |1\rangle_2|1\rangle_3) \]

where the subscripts indicate the specific modes (subspaces) and \( |\tau| << 1 \). Here beam 4 and beam 3 are interpreted as in the subspaces of Alice and Bob respectively, while beam 1 and beam 2 both belong to the subspace of Clare. In the experiment we should arrange the optical paths of beam 2 and beam 3 appropriately so that they reach the beam splitter simultaneously. Either of the following two events indicates a successful creation of a maximal entanglement in beam 4 and 3.

**Event 1.** Detector D_1 is fired and D_2 is silent. Such an event indicates that an entangled state \( |\Psi^+\rangle \) is prepared on beam 3 and beam 4.

**Event 2.** Detector D_2 is fired and D_1 is silent. Such an event indicates that an entangled state \( |\Psi^-\rangle \) is prepared on beam 3 and beam 4.

We denote \( U_B \) as the time evolution operator of our beam splitter. We assume the following properties for the (balanced) beam splitters in the Schrödinger picture [11,12].
\[ U_B(a_1^\dagger, a_2^\dagger)U_B = \frac{1}{\sqrt{2}}(a_1^\dagger, a_2^\dagger) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \]  

(12)

Here \( a_1^\dagger \) and \( a_2^\dagger \) are creation operators of mode 1 and mode 2 respectively. There are many other forms of balanced beam splitters, but all of them will essentially cause the same result on entanglement swapping by our scheme therefore in this paper we only consider the one defined above. Note that here we are using the Schrodinger picture and we simply distinguish different mode around the beam splitter by the propagation direction only [12]. For example, beam 1 and beam 1’ are in the same mode but different state due to the nontrivial time evolution in the two mode space caused by the beam splitter. Using all this, we know that the total state after the beam splitter is

\[ |\chi_1\rangle = U_B |\chi_0\rangle = \frac{1}{1 + |\tau|^2} \left[ |0000\rangle + \frac{\tau}{\sqrt{2}}(|10\rangle(|10\rangle + |01\rangle) - \frac{\tau}{\sqrt{2}}|01\rangle(|10\rangle - |01\rangle) + \tau^2 U_B |1111\rangle \right]. \]  

(13)

In the above equation we have omitted all subscripts for the mode indicators. In all the state vectors in the format of \( |wxyz\rangle \) or in the format of \( |wx\rangle|yz\rangle \), we always assume that the symbol in the first, second, third and fourth position from the left to the right are for the quantum state in beam 1’, 2’, 3, 4 respectively. Equation(13) shows that once event 1 or event 2 happens, we have obtained the state \( |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_3|0\rangle_4 - |0\rangle_3|1\rangle_4) \) or \( |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|1\rangle_3|0\rangle_4 + |0\rangle_3|1\rangle_4) \) with a probability of \( p = 1 - |\tau|^2 \). Suppose \( |\tau|^2 = 10^{-3} \), this probability is about 99.9%. Note that the photon detectors used here need not be capable of distinguishing one photon and two photons. Moreover, we even don’t have to worry about the efficiency of the photon detectors due to the very small value of \( |\tau|^2 \).

In an experimental test, we need to verify that beam 3 and beam 4 are indeed entangled after we observed the event 1 or event 2 successfully. Doing so is quite simple. First we check the probability distribution. The state \( |\Psi^\pm\rangle \) will give the equal classical probability of one photon on beam 4 and beam 3. To check this, we only need to place extra photon detectors on those two beams. We then check the phase information which distinguishes \( |\Psi^+\rangle \) and \( |\Psi^-\rangle \). To do so we just let beam 3 and beam 4 pass through another balanced beam splitter defined by eq.(12). For state\( |\Psi^+\rangle \), we always find a photon left to the beam.
splitter, while for the state $|\Psi^-\rangle$, we always find a photon right to the beam splitter. That is to say, in order to verify the phase information, we just observe the coincidence that $D_1$ is always fired together with $D_3$, and that $D_2$ is always fired together with $D_4$ in fig.4B. Due to the limited efficiency of photon detectors, normally we cannot always observe the theoretically expected coincidence. That is to say, whenever $D_1$ is clicked, $D_3$ clicks only in a probability of $\eta$, where $\eta$ is the photon detector efficiency. But the fact that whenever $D_1$ clicks, $D_4$ never (or rarely) clicks will be a strong evidence of phase coherence between beam 3 and beam 4.

In the above scheme, the pump light there has to pass through the nonlinear crystal twice. This may increase technical difficulty in synchronization. To avoid this, we also propose the following alternative scheme shown in Fig. 4 where the pump light only pass through the nonlinear crystal once. The unbalanced beam splitter is almost transparent. Its time evolution operator satisfies

$$U(a_1^\dagger, a_2^\dagger)U^\dagger = \frac{1}{\sqrt{1+\epsilon^2}} (a_1^\dagger, a_2^\dagger) \begin{pmatrix} 1 & \epsilon \\ \epsilon & -1 \end{pmatrix}. \quad (14)$$

This shows that, in the case that one pair $|1\rangle_u|1\rangle_l$ is emitted from the nonlinear crystal, after passing through unbalanced beam splitters, the initial state is evolved to

$$|\chi'\rangle_{1234} = \frac{1}{1+\epsilon^2} (|1\rangle_1|0\rangle_2 + \epsilon |0\rangle_1|1\rangle_2) \otimes (\epsilon |1\rangle_3|0\rangle_4 + |0\rangle_3|1\rangle_4). \quad (15)$$

This can be recast to

$$|\chi'\rangle_{1234} = \frac{1}{1+\epsilon^2} \left[ |1\rangle|0\rangle|1\rangle + \epsilon (|0\rangle|1\rangle|0\rangle) + |1\rangle|0\rangle|1\rangle) + \epsilon^2|0\rangle|1\rangle|0\rangle \right]. \quad (16)$$

where we have omitted the subscripts for all beams. We just keep in the mind that for each term the subscripts are from 1 to 4 from left to right. One can easily show that whenever $D_2$ or $D_3$ is clicked, beam 1, 4 must be projected onto a Bell state. The first term in the right hand side of Eq.(16) will never cause any clicking because beam 2 and beam 3 contain nothing there. The last term can cause the clicking of $D_2$ or $D_3$, but the probability is very small because the value of $\epsilon^2$ is much smaller than $\epsilon$. Therefore one need only consider
the middle term, i.e., the term with a factor of $\epsilon$. The second term can be written in the equivalent form of

$$\frac{\epsilon}{2} \left( |\Psi^+\rangle_{23} |\Psi^+\rangle_{14} - |\Psi^-\rangle_{23} |\Psi^-\rangle_{14} \right).$$

(17)

We know that state $|\Psi^+\rangle_{23}$ will cause detector D2 being clicked and state $|\Psi^-\rangle_{23}$ will cause the detector D3 being clicked. Therefore, in the setup given by Fig.4, whenever D2 is clicked, beam 1,4 have been projected to $|\Psi^+\rangle_{14}$; whenever D3 is clicked, beam 1,4 have been projected to $|\Psi^-\rangle_{14}$. We can choose to use the polarizing beam splitters (PBS) instead of the unbalanced beam splitters there. We need only to rotate beam u and beam l appropriately therefore the polarization in both beams are a bit deviate from the horizontal one, i.e. initially we produce a state of

$$|H\rangle_u \otimes |H\rangle_l = \frac{1}{1+\epsilon^2} (|H\rangle_u + \epsilon|V\rangle_u) \otimes (|H\rangle_l + \epsilon|V\rangle_l).$$

(18)

Now the two unbalanced beam splitters are replaced by two polarizing beam splitters. The state for beam 1,2,3 and 4 will be identical to that given by Eq.(15) therefore all the rest results are the same.

Concluding remark. In summary, we have given a simple proposal to do the quantum entanglement swapping experiment without postselection. Since the post-selection experiments with similar or more complicated technical setup have been carried out already [6,8], we believe our scheme can be carried out easily with the current technology. We don’t know how to scale the method to the case of many entangled pairs.

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FIG. 1. Schematic diagram of entanglement swapping. Initially, beam 1 and beam 2 are in EPR state, beam 3 and beam 4 are in another EPR state. After a joint measurement on beam 2 and 3 in Bell basis, beam 1 and beam 4 are projected into a Bell state.

FIG. 2. Currently existing entanglement swapping experiments are post-selection. A. The set-up in polarization space. Because it’s possible that beam 1 and 3 contains 2 pairs (or nothing) and beam 2 and 4 contain nothing (or 2 pairs). To make sure beam 1 and 4 are entangled, one must also detect beam 1 and 4. B. The set-up in vacuum-one photon space. The photon detector here does not distinguish 1 photon or 2 photons. When a detector is clicked, it’s also possible that the beam contains 2 photons therefore the actual state for beam 1 and 4 is vacuum. To remove such types of events, one must also detect beam 1 and 4, therefore the state of beam 1 and 4 is destructed. NC: nonlinear crystal used in SPDC process. BS: beam splitter. M: mirror. D: photon detector.
FIG. 3. A. A schematic diagram for the experimental set-up of non-post-selection quantum entanglement swapping. Whenever we find the coincidence that D$_1$ is fired (silent) and D$_2$ is silent (fired), we have remotely prepared the entangled state of $|\Psi^+\rangle (|\Psi^-\rangle)$ on beam 3 and 4. B. Phase information verification of the entanglement swapping. The fact that detectors D$_1$ (D$_2$) and D$_3$ (D$_4$) will be always both fired (silent) or both silent (fired) verifies the maximal entanglement of the state prepared by the entanglement swapping.

FIG. 4. A. An alternative scheme for non-post-selection entanglement swapping. UBS: unbalanced beam splitter; BS: balanced beam splitter. After the clicking of either D$_2$ or D$_3$, beam 1 and beam 4 are in the maximally single photon entangled state in the two level space of vacuum-one-photon state. B. The unbalanced beam splitters in A can be replaced by polarizing beam splitters.