Deconstructing Stellar Consensus
(Extended Version)
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Abstract
Some of the recent blockchain proposals, such as Stellar and Ripple, allow for open membership while using quorum-like structures typical for classical Byzantine consensus with closed membership. This is achieved by constructing quorums in a decentralised way: each participant independently chooses whom to trust, and quorums arise from these individual decisions. Unfortunately, the consensus protocols underlying such blockchains are poorly understood, and their correctness has not been rigorously investigated. In this paper we rigorously prove correct the Stellar Consensus Protocol (SCP), with our proof giving insights into the protocol structure and its use of lower-level abstractions. To this end, we first propose an abstract version of SCP that uses as a black box Stellar’s federated voting primitive (analogous to reliable Byzantine broadcast), previously investigated by García-Pérez and Gotsman [7]. The abstract consensus protocol highlights a modular structure in Stellar and can be proved correct by reusing the previous results on federated voting. However, it is unsuited for realistic implementations, since its processes maintain infinite state. We thus establish a refinement between the abstract protocol and the concrete SCP that uses only finite state, thereby carrying over the result about the correctness of former to the latter. Our results help establish the theoretical foundations of decentralised blockchains like Stellar and gain confidence in their correctness.

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1 Introduction
Permissioned blockchains are becoming increasingly popular due to the low-energy consumption and hard guarantees they provide on when a transaction can be considered successfully committed. Such blockchains are often based on classical Byzantine fault-tolerant (BFT) consensus protocols, like PBFT [4]. In these protocols consensus is reached once a quorum of participants agrees on the same decision. Quorums can be defined as sets containing enough nodes in the system (e.g., $2f + 1$ out of $3f + 1$, assuming at most $f$ failures) or by a more general structure of a Byzantine quorum system (BQS) [12]. Unfortunately, defining quorums in this way requires fixing the number of participants in the system, which prevents decentralisation.

Some of the recent blockchain proposals, such as Stellar [13] and Ripple [15], allow for open membership while using quorum-like structures typical for classical Byzantine consensus with closed membership. This is achieved by constructing quorums in a decentralised way: each protocol participant independently chooses whom to trust, and quorums arise from these individual decisions. In particular, in Stellar trust assumptions are specified using a federated Byzantine quorum system (FBQS), where each participant selects a set of quorum slices—sets of nodes each of which would convince the participant to accept the validity of a given statement (§2). Quorums are defined as sets of nodes $U$ such that each node in $U$ has some quorum slice fully within $U$, so that the nodes in a quorum can potentially reach an agreement. Consensus is then implemented by a fairly intricate protocol whose key component is federated voting—a protocol similar to Bracha’s protocol for reliable Byzantine
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Unfortunately, even though Stellar has been deployed as a functioning blockchain, the structure of the consensus protocol underlying it is poorly understood and its correctness has not been rigorously investigated. In this paper we aim to close this gap, rigorously defining and proving correct the Stellar Consensus Protocol (SCP). Apart from giving more confidence in the correctness of the protocol, our proof is structured in such a way as to give insights into its structure and its use of lower-level abstractions.

In more detail, the guarantees provided by SCP are nontrivial. When different participants in an FBQS choose different slices, only a subset of the participants may take part in a subsystem in which every two quorums intersect in a correct node—a property required for achieving consensus. The system may partition into such subsystems, and SCP will guarantee agreement within each of them. In blockchain terms, the blockchain may fork, but in this case each fork will be internally consistent, a property that is enough for business applications of the Stellar blockchain. The subsystems where agreement is guaranteed are characterised by Mazières et al. through the notion of intact sets. Our proof of correctness establishes safety and liveness properties of SCP relative to such intact sets (§3).

As a stepping stone in the proof, we first propose an abstract version of SCP that uses as a black box Stellar’s federated voting primitive (analogous to reliable Byzantine broadcast) previously investigated by García-Pérez and Gotsman (§5). This abstract formulation allows specifying the protocol concisely and highlights the modular structure present in it. This allows proving the protocol by reusing the previous results on federated voting (re-reviewed in §4). However, the abstract protocol is unsuited for realistic implementations, since its processes maintain infinite state. To address this, we formulate a realistic version of the protocol—a concrete SCP—that uses only finite state. We then prove a refinement between the abstract and concrete SCP, thereby carrying over the result about the correctness of former to the latter (§6).

A subtlety in SCP is that its participants receive information about quorum slices of other participants directly from them. Hence, Byzantine participants may lie to others about their choices of quorum slices, which may cause different participants to disagree on what constitutes a quorum. Our results also cover this realistic case (§7).

Overall, our results help establish the theoretical foundations of decentralised blockchains like Stellar and gain confidence in their correctness. Proofs of the lemmas and theorems in the paper are given in the appendices.

2 Background: System Model and Federated Byzantine Quorum Systems

System model. We consider a system consisting of a finite universe of nodes $V$ and assume a Byzantine failure model where faulty nodes can deviate arbitrarily from their specification. All other nodes are called correct. Nodes that are correct, or that only deviate from their specification by stopping execution, are called honest. Nodes that deviate from their specification in ways other than stopping are called malicious. We assume that any two nodes can communicate over an authenticated perfect link. We assume a partial synchronous network, which guarantees that messages arrive within bounded time after some unknown, finite global stabilisation time ($GST$). Each node has a local timer and a timeout service that can be initialised with an arbitrary delay $\Delta$. We assume that after GST the clock skew of correct nodes is bounded, i.e., after GST two correct nodes can only disagree in the duration of a given delay $\Delta$ by a bounded margin.

Federated Byzantine quorum systems. Given a finite universe $V$ of nodes, a federated Byzantine quorum system (FBQS) is a function $S : V \rightarrow 2^V \setminus \{\emptyset\}$ that specifies a
non-empty set of quorum slices for each node, ranged over by \( q \). We require that a node belongs to all of its own quorum slices: \( \forall v \in V, \forall q \in \mathcal{S}(v), v \in q \). Quorum slices reflect the trust choices of each node. A non-empty set of nodes \( U \subseteq V \) is a quorum in an FBQS \( S \) iff \( U \) contains a slice for each member, i.e., \( \forall v \in U, \exists q \in \mathcal{S}(v), q \subseteq U \).

For simplicity, for now we assume that faulty nodes do not equivocate about their quorum slices, so that all the nodes share the same FBQS. In \( \mathbb{N}^2 \) we consider the more realistic subjective FBQS \( \mathbb{S} \), where malicious nodes may lie about their slices and different nodes have different views on the FBQS. There we also lift the results on the subsequent sections of the paper to subjective FBQSes.

**Example 1.** Consider a universe \( V \) with \( 3f + 1 \) nodes, and consider the FBQS \( S \) where for every node \( v \in V \), the set of slices \( \mathcal{S}(v) \) consists of every set of \( 2f + 1 \) nodes that contains \( v \) itself. \( S \) encodes the classical cardinality-based quorum system of \( 3f + 1 \) nodes with failure threshold \( f \), since every set of \( 2f + 1 \) or more nodes is a quorum.

**Example 2.** Let the universe \( V \) contain four nodes \( v_1 \) to \( v_4 \), and consider the FBQS \( S \) in the diagram below.

For each node, all the outgoing arrows with the same style determine one slice. Node \( v_2 \) has two slices, determined by the solid and dashed arrow styles respectively. The rest of the nodes have one slice. \( S \) has the following set of quorums \( \mathcal{Q} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\} \).

A consensus protocol that runs on top of an FBQS may not guarantee global agreement, because when nodes choose slices independently, only a subset of the nodes may take part in a subsystem in which every two quorums intersect in at least one correct node—a basic requirement of a Byzantine quorum system \( \mathbb{I} \) to ensure agreement. To formalise which parts of the system may reach agreement internally, we borrow the notions of intertwined nodes and of intact set from \( \mathbb{I} \). Two nodes \( v_1 \) and \( v_2 \) are intertwined iff they are correct and every quorum containing \( v_1 \) intersects every quorum containing \( v_2 \) in at least one correct node. Consider an FBQS \( S \) and a set of nodes \( I \). The projection \( \mathcal{S}|_I \) of \( S \) to \( I \) is the FBQS over universe \( I \) given by \( \mathcal{S}|_I(v) = \{q \cap I | q \in \mathcal{S}(v)\} \). For a given set of faulty nodes, a set \( I \) is an intact set iff \( I \) is a quorum in \( S \) and every member of \( I \) is intertwined with each other in the projected FBQS \( S|_I \). The intact sets characterise those sets of nodes that can reach consensus, which we later show using the following auxiliary result.

**Lemma 3.** Let \( S \) be an FBQS and assume some set of faulty nodes. Let \( I \) be an intact set in \( S \) and consider any two quorums \( U_1 \) and \( U_2 \) in \( S \) such that \( U_1 \cap I \neq \emptyset \) and \( U_2 \cap I \neq \emptyset \). Then the intersection \( U_1 \cap U_2 \) contains some node in \( I \).

The maximal intact sets are disjoint with each other:

**Lemma 4.** Let \( S \) be an FBQS and assume some set of faulty nodes. Let \( I_1 \) and \( I_2 \) be two intact sets in \( S \). If \( I_1 \cap I_2 \neq \emptyset \) then \( I_1 \cup I_2 \) is an intact set in \( S \).
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In SCP the system may split into different partitions—i.e., the maximal intact sets—that may be inconsistent with each other, but which constitute independent systems each of which can reach consensus.

Consider the $S$ from Example 1 which encodes the cardinality-based quorum system of $3f + 1$ nodes, and let $f = 1$, so that the universe $V$ contains four nodes $v_1$ to $v_4$. If we assume that node $v_3$ is faulty, then the set $I = \{v_1, v_2, v_4\}$ is the only maximal intact set: $I$ is a quorum in $S$, and $S|I$ contains the quorums $\{\{v_1, v_2\}, \{v_2, v_4\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\}$, which enjoy quorum intersection. This ensures that every two nodes in $I$ are intertwined in the projected system $S|I$.

Now consider the $S$ from Example 2. If we assume that node $v_3$ is faulty, then the sets $I = \{v_1, v_2\}$ and $I' = \{v_4\}$ are the maximal intact sets: $I$ and $I'$ are quorums in $S$, and the projected systems $S|I$ and $S|I'$ enjoy quorum intersection—$S_I$ contains quorums $\{v_1, v_2\}$ and $\{v_2\}$, and $S_{I'}$ contains quorum $\{v_4\}$—which ensures that every two nodes in either $I$ or $I'$ are intertwined in the projected systems $S|I$ and $S_{I'}$ respectively. It is easy to check that adding any set of correct nodes to either $I$ or $I'$ results in sets that are not quorums in $S$, or in projected systems that contain some pairs of nodes that are not intertwined.

3 Specifications

Assume a set $\text{Val}$ of consensus values. In the consensus protocols that we study in Sections 3 and 4 each correct node proposes some $x \in \text{Val}$ through an invocation $\text{propose}(x)$, and each node may decide some $x' \in \text{Val}$ through an indication $\text{decide}(x')$. We consider a variant of the weak Byzantine consensus specification in [2] that we call non-blocking Byzantine consensus for intact sets, which is defined as follows. Given a maximal intact set $I$,

(Integrity) no correct node decides twice,
(Agreement for intact sets) no two nodes in $I$ decide differently,
(Weak validity for intact sets) if all nodes are honest and every node proposes $x$, then no node in $I$ decides a consensus value different from $x$; furthermore, if all nodes are honest and some node in $I$ decides $x$, then $x$ was proposed by some node, and
(Non-blocking for intact sets) if a node $v$ in $I$ has not yet decided in some run of the protocol, then for every continuation of that run in which all the malicious nodes stop, node $v$ eventually decides some consensus value.

The usual Weak validity property of consensus [2] ensures that if all nodes are correct and they propose the same consensus value, then no node can decide a consensus value different from the proposed one; and that if all nodes are correct, then a node can only decide a consensus value proposed by some node. Our Weak validity for intact sets above adapts this requirement to the nodes in a maximal intact set, and weakens its condition by assuming that all nodes are honest instead of correct. Notice that if every two quorums intersect our property entails the usual Termination guarantee that every correct node eventually decides some consensus value [2]. Instead, we consider the weaker liveness guarantee of Non-blocking for intact sets, which we have obtained by adapting the Non-blocking property in [16]. Non-blocking requires that some continuation of a given run exists in which every correct node
terminates. Our Non-blocking for intact sets adapts this requirement to the nodes in a maximal intact set, and requires that they terminate in every continuation of the run in which malicious nodes are stopped. It is easy to check that if every correct node is in an intact set, then Non-blocking for intact sets entails Non-blocking in \[16\]. For instance, this condition holds in the cardinality-based quorum systems \((3f + 1)\). Besides, if every correct node is honest, then Non-blocking for intact sets entails the usual Termination property that guarantees that every correct node eventually decides some consensus value.

The non-blocking Byzantine consensus for intact sets above entails the weak Byzantine consensus specification \([2]\) in the cardinality-based quorum systems \((3f + 1)\), which guarantees the Integrity property above, as well as the usual Agreement property that ensures that no two correct nodes decide differently, and the usual Weak validity and Termination properties that we have recalled in the paragraphs above.

One of the core components of the consensus protocol in \([5]\) is federating voting (FV) \([13, 14]\). Assume a set of voting values \(A\) that could be disjoint with the set \(\text{Val}\) of consensus values (we typically let \(A\) be the set of Booleans \(\text{Bool} \equiv \{\text{true}, \text{false}\}\)). FV allows each correct node to vote for some \(a \in A\) through an invocation \(\text{vote}(a)\), and each node may deliver some \(a' \in A\) through an indication \(\text{deliver}(a')\). The interface of FV is akin to that of consensus, where each node activates itself through the primitive \(\text{vote}(a)\). However, FV has weaker liveness guarantees than consensus, which are reminiscent to those of Byzantine reliable broadcast from \([2]\) and weakly reliable Byzantine broadcast from \([7]\). Here, we consider a variant of the latter specification that we call reliable Byzantine voting for intact sets, which is defined as follows. Given a maximal intact set \(I\),

- \((\text{No duplication})\) every correct node delivers at most one voting value,
- \((\text{Totality for intact sets})\) if a node in \(I\) delivers a voting value, then every node in \(I\) eventually delivers a voting value,
- \((\text{Consistency for intertwined nodes})\) if two intertwined nodes \(v\) and \(v'\) deliver \(a\) and \(a'\) respectively, then \(a = a'\), and
- \((\text{Validity for intact sets})\) if all nodes in \(I\) vote for \(a\), then all nodes in \(I\) eventually deliver \(a\).

The ability of each node to activate itself independently in the specification above simulates a malicious sender that may send different voting values to each node in the specification of weakly reliable Byzantine broadcast from \([2]\).

## 4 Federated Voting

In this section we recall federated voting (FV) from \([13, 14]\), which also corresponds to the Stellar broadcast considered in \([2]\). We prove that FV implements the specification of reliable Byzantine voting for intact sets, thereby generalising the results of \([2]\) to the case of multiple intact sets within the system. The consensus protocol that we study in the next section uses multiple instances of FV independent from each other. This is done by letting each node run a distinct process for each instance of FV, which is identified by a tag \(t\) from some designated set \(\text{Tag}\) of tags.

Algorithm 1 below depicts FV over an FBQS \(S\) with set of quorums \(Q\). A node \(v\) runs a process \(\text{federated-voting}(v, t)\) for each tag \(t\). The messages exchanged by such a process are also tagged with \(t\), in order to distinguish them from the messages exchanged for instances of FV associated with tags different from \(t\).

FV adapts Bracha’s protocol for reliable Byzantine broadcast \([1]\), which works over the cardinality-based quorum systems of \(3f + 1\) nodes, to the federated setting of the FBQSs. In FV nodes process each other’s messages in several stages, where for each tag \(t\) progress is denoted by several Boolean flags (line 2 of Algorithm 1). When a node \(v\) votes \(a\) for tag
Algorithm 1: Federated voting (FV) over an FBQS $S$ with set of quorums $Q$.

1. process federated-voting($v \in V, t \in \text{Tag}$)
2. voted, ready, delivered ← false ∈ Bool;
3. vote($a$)
4. if not voted then
5. voted ← true;
6. send VOTE($t, a$) to every $v' \in V$;
7. when received VOTE($t, a$) from every $u \in U$ for some $U \in Q$ such that $v \in U$ and not ready
8. ready ← true;
9. send READY($t, a$) to every $v' \in V$;
10. when received READY($t, a$) from every $u \in B$ for some $v$-blocking $B$ and not ready
11. ready ← true;
12. send READY($t, a$) to every $v' \in V$;
13. when received READY($t, a$) from every $u \in U$ for some $U \in Q$ such that
14. $v \in U$ and not delivered
15. delivered ← true;
16. trigger deliver($a$);

$t$ for the first time, the node sends VOTE($t, a$) to every node (including itself, for uniformity; lines 3–6). When a node $v$ receives a VOTE($t, a$) message from a quorum to which $v$ itself belongs, it sends a READY($t, a$) message to every node, signalling its willingness to deliver the value $a$ for tag $t$ (lines 7–9). Note that, for each tag $t$, two nodes in the same intact set $I$ cannot send READY messages with two different voting values through the rule in lines 7–9. Indeed, this would require two quorums of VOTE messages, each with a node in $I$, with different voting values for the same tag. But by Lemma 3 these quorums would intersect in a node in $I$, which is by definition correct and cannot send contradictory VOTE messages for the same tag. When a node $v$ receives the message READY($t, a$) from a quorum to which $v$ itself belongs, it delivers $a$ for tag $t$ (lines 13–15).

The exchange of READY messages in the protocol is necessary to establish liveness guarantees. It ensures that, if a node in an intact set $I$ delivers a voting value for some tag, other nodes in $I$ have enough information to also deliver a voting value for the same tag. This relies on the rule in lines 10–12 which uses the notion of $v$-blocking set [12]. Given a node $v$, a set $B$ is $v$-blocking iff $B$ overlaps each of $v$’s slices, i.e., $\forall q \in S(v). q \cap B \neq \emptyset$. (To illustrate this notion, in Example 1 every set of $f + 1$ nodes is $v$-blocking for every $v$, and in Example 2 the set \{v_1, v_3\} is $v_2$-blocking and the set \{v_2\} is $v_1$-blocking.) Lines 10–12 allow a node to send a READY($t, a$) message even if it previously voted for a different voting value for tag $t$: this is done if $v$ receives READY($t, a$) from each member of a $v$-blocking set. If $v$ is in an intact set $I$, the following lemma guarantees that in this case $v$ has received at least one READY($t, a$) message from some node in $I$.

Lemma 5. Let $S$ be an FBQS and assume a set of faulty nodes. Let $I$ be an intact set in $S$ and $v \in I$. Then, no $v$-blocking set $B$ exists such that $B \cap I = \emptyset$.
Figure 1 Execution of the instance of FV for tag t.

By Lemma 5 the first node in I to ever send a READY(t, a) message for a tag t has to do it through the rule in lines 7 and 13 of Algorithm 1. Therefore, the value a has been cross-checked by a quorum.

If the condition v ∈ U in lines 7 and 13 of Algorithm 1 was dropped, this could violate Agreement for intact sets as follows. Take the S from Example 2 and consider a run of FV for some tag t where v3 is malicious. Node v3 could respectively send READY(t, a) and READY(t, a') with a ≠ a' to correct nodes v1 and v2. Since {v1} ∈ Q, these nodes will respectively deliver a and a' by lines 13 and 15 of Algorithm 1 without condition v ∈ U.

Our first contribution is to generalise the results of [7] to establish the correctness of FV within each of the maximal intact sets of an FBQS, as captured by Theorem 6 below.

Theorem 6. Let S be an FBQS and t be a tag. The instance for t of FV over S satisfies the specification of reliable Byzantine voting for intact sets.

By Lemma 7 guarantees that it is finite.

Lemma 7. Let S be an FBQS and t be a tag. Consider an execution of the instance for t of FV over S. Let I be an intact set in S and assume that GST has expired. If a node v ∈ I delivers a voting value then every node in I will deliver a voting value within bounded time.

We write δI for the time that a node in I takes to deliver some voting value after GST and provided that some other node in I already delivered some voting value. The delay δI—which is determined by S and I—is unknown, but Lemma 7 guarantees that it is finite.

Example 8. Consider the S from Example 1 which encodes the cardinality-based quorum system 3f + 1, and let f = 1 such that the universe V contains four nodes v1 to v4. Every set of three or more nodes is a quorum, and every set of two or more nodes is v-blocking for every v ∈ V. Let us fix a tag t and consider an execution of the instance of FV for tag t where we let the voting values be the Booleans. Assume that nodes v1, v2 and v4 are correct, which constitute the maximal intact set. In the execution, nodes v1 and v2 vote false, and node v4 votes true. Malicious node v3 sends the message VOTE(t, false) to every node (highlighted in red) thus helping the correct nodes to deliver false.

Figure 2 depicts a possible execution of FV described above, from which a trace can be constructed as follows: all the events in each row may happen concurrently, and any two events in different rows happen in real time, where time increases downwards; in those cells that are tagged with a message, the node sends the message to every node, and in a given cell a node has received all the messages from every node in the rows above it. (These conventions are only for presentational purposes, and should not be mistaken with the perfectly synchronised round-based model of [8], which we do not use.) The quorum {v1, v2, v3} sends VOTE(t, false) to every node, which makes nodes v1 and v2 send READY(t, false) to every node through lines 7 and 9 of Algorithm 1. However, there exists not a quorum U such that v4 ∈ U and every member of U sends a message VOTE(t, a) with the same Boolean a, and thus node
v_4$ sends $\text{READY}(t, false)$ through lines 10–12 of Algorithm 1 only after receiving corresponding ready messages from the $v_2$-blocking set \{v_1, v_2\}. Observe how node $v_4$ changes its original vote $true$ and sends $false$ in the $\text{READY}$ message. After every correct node receives $\text{READY}(t, false)$ from the quorum \{v_1, v_2, v_4\}, they all deliver $false$.

5 Abstract Stellar Consensus Protocol

In this section we introduce the abstract SCP (ASCP), which concisely specifies the mechanism of SCP \cite{13, 14} and highlights the modular structure present in it\footnote{More precisely, in this paper we focus on Stellar’s core balloting protocol, which aims to achieve consensus. We abstract from Stellar’s nomination protocol—which tries to converge (best-effort) on a value to propose—by assuming arbitrary proposals to consensus.}. Like Paxos \cite{8}, ASCP uses ballots—pairs $⟨n, x⟩$, where $n \in N^+$ a natural positive round number and $x \in \text{Val}$ a consensus value. We assume that $\text{Val}$ is totally ordered, and we consider a special null ballot $⟨0, ⊥⟩$ and let $\text{Ballot} = (N^+ \times \text{Val}) \cup \{⟨0, ⊥⟩\}$ be the set of ballots. (We write $b.n$ and $b.x$ respectively for the round and consensus value of ballot $b$.) The set $\text{Ballot}$ is totally ordered, where we let $b < b′$ iff either $b.n < b′.n$, or $b.n = b′.n$ and $b.x < b′.x$.

To better convey SCP’s mechanism, we let the abstract protocol use FV as a black box where nodes may hold a binary vote on each of the ballots: we let the set of voting values $\mathbf{V}$ be the set of Booleans and the set of tags $\mathbf{Tag}$ be the set of ballots, and let the protocol consider a separate instance of FV for each ballot. A node voting for a Boolean $a$ for a ballot $b$ that carries the consensus value $b.x$ encodes the aim to either $\text{abort}$ the ballot (when $a = false$) or to $\text{commit}$ it (when $a = true$) thus deciding the consensus value $b.x$. From now on we will unambiguously use ‘Booleans’, ‘ballots’ and ‘values’ instead of ‘voting values’, ‘tags’ and ‘consensus values’, respectively.

We have dubbed ASCP ‘abstract’ because, although it specifies the protocol concisely, it is unsuited for realistic implementations. On the one hand, each node $v$ maintains infinite state, because it stores a process federated-voting($v$, $b$) for each of the infinitely many ballots $b$ in the array $\text{ballots}$ (line 2 of Algorithm 2). On the other hand, each node $v$ may need to send or receive an infinite number of messages in order to progress (lines 6, 8, 15 and 21 of Algorithm 2) which are explained in the detailed description of ASCP below. This is done by assuming a batched network semantics (BNS) in which the network exchanges batches, which are (possibly infinite) sequences of messages, instead of exchanging individual messages: the sequence of messages to be sent by a node when processing an event is batched per recipient, and each batch is sent at once after the atomic processing of the event; once a batch is received, the recipient node atomically processes all the messages in the batch in sequential order. By convention, we let the statement forall in lines 7 and 21 of Algorithm 2 consider the ballots $b′$ in ascending ballot order. In §6 we introduce a ‘concrete’ version of SCP that is amenable to implementation, since nodes in it maintain finite state and exchange a finite number of messages; however, this version does not use FV as a black box.

ASCP uses the following below-and-incompatible-than relation on ballots. We say ballots $b$ and $b′$ are compatible (written $b \sim b′$) iff $b.x = b′.x$, and incompatible (written $b \not\sim b′$) otherwise, where we let $\bot \not= x$ for any $x \in \text{Val}$. We say ballot $b$ is below and incompatible than ballot $b′$ (written $b \not<_B b′$) iff $b < b′$ and $b \not\sim b′$. In a nutshell, ASCP works as follows: each node uses FV to prepare a ballot $b$ which carries the candidate value $b.x$, this is, it aborts every ballot $b′ \not<_B b$, which prevents any attempt to decide a value different from $b.x$ at a round smaller than $b.n$: once $b$ is prepared, the node uses FV again to commit ballot $b$, thus deciding the candidate value $b.x$. 

\[\text{ballot} \langle \text{ballot} \rangle \text{than} \]
Algorithm 2: Abstract SCP (ASCP) over an FBQS $S$ with set of quorums $Q$.

1. process abstract-consensus($v \in V$)
2. \hspace{1em} ballots ← [new process federated-voting($v, b$)]$_{b \in \text{Ballot}}$;
3. \hspace{1em} candidate, prepared ← $\langle 0, \bot \rangle \in \text{Ballot}$;
4. \hspace{1em} round ← $0 \in \mathbb{N}^+ \cup \{0\}$;
5. \hspace{1em} proposse($x$)
6. \hspace{2em} candidate ← $\langle 1, x \rangle$;
7. \hspace{2em} for all $b' \preceq$ candidate do ballots[$b'$].vote(false);
8. \hspace{1em} when triggered ballots[$b'$].deliver(false) for every $b' \preceq b$ and prepared $< b$
9. \hspace{2em} prepared ← $b$;
10. \hspace{2em} if candidate $\leq$ prepared then
11. \hspace{3em} candidate ← prepared;
12. \hspace{3em} ballots[candidate].vote(true);
13. \hspace{1em} when triggered ballots[$b$].deliver(true)
14. \hspace{2em} trigger decide($b,x$);
15. \hspace{1em} when exists $U \in Q$ such that $v \in U$ and for each $u \in U$ exist $M_u \in \{VOTE, READY\}$ and $b_u \in \text{Ballot}$ such that round $< b_u.n$ and either received $M_u(b_u, true)$ from $u$ or received $M_u(b', false)$ from $u$ for every $b' \in [z_u, b_u)$ with $z_u < b_u$
16. \hspace{2em} round ← min\{$b_u.n \mid u \in U$\};
17. \hspace{2em} start-timer($F$(round));
18. \hspace{1em} when triggered timeout
19. \hspace{2em} if prepared $= \langle 0, \bot \rangle$ then candidate ← $\langle \text{round} + 1, \text{candidate}.x \rangle$;
20. \hspace{2em} else candidate ← $\langle \text{round} + 1, \text{prepared}.x \rangle$;
21. \hspace{2em} for all $b' \preceq$ candidate do ballots[$b'$].vote(false);

ASCP is depicted in Algorithm 2 above. We assume that each node $v$ creates a process federated-voting($v, b$) for each ballot $b$, which is stored in the infinite array ballots[$b$] (line 2). The node keeps fields candidate and prepared, which respectively contain the ballot that $v$ is trying to commit and the highest ballot prepared so far. Both candidate and prepared are initialised to the null ballot (line 3). The node also keeps a field round that contains the current round, initialised to 0 (line 4). Once $v$ proposes a value $x$, the node assigns the ballot $\langle 1, x \rangle$ to candidate and tries to prepare it by invoking FV’s primitive vote(false) for each ballot below and incompatible than candidate (lines 5-7). This may involve sending an infinite number of messages, which by BNS requires sending finitely many batches. Once $v$ prepares some ballot $b$ by receiving FV’s indication deliver(false) for every ballot below and incompatible than $b$, and if $b$ exceeds prepared, the node updates prepared to $b$ (lines 8-9). The condition in line 8 may concern an infinite number of ballots, but it may hold after receiving a finite number of batches by BNS. If prepared reaches or exceeds candidate, then the node updates candidate to prepared, and tries to commit it by voting true for that ballot (lines 10-12). Once $v$ commits some ballot $b$ by receiving FV’s indication deliver(true) for ballot $b$, the node decides the value $b.x$ (lines 13-14) and stops execution.

If the candidate ballot of a node $v$ can no longer be aborted nor committed, then $v$ resorts
to a timeout mechanism that we describe next. The primitive start-timer(Δ) starts the node’s local timer, such that a timeout event will be triggered once the specified delay Δ has expired. (Invoking start-timer(Δ’), while the timer is already running has the effect of restarting the timer with the new delay Δ’.) In order to start the timer, a node v needs to receive, from each member of a quorum that contains v itself, messages that endorse either committing or preparing ballots with rounds bigger than round (line 15 of Algorithm 2). Since the domain of values can be infinite, the condition in line 15 requires that for each node u in some quorum U that contains v itself, there exists a ballot b_u with round b_u.n > round, and either u receives from v a message endorsing to commit b_u, or otherwise v receives from u messages endorsing to abort every ballot in some non-empty, right-open interval [z_u, b_u), whose upper bound is b_u. This condition may require receiving an infinite number of ballots, but it may hold after receiving a finite number of batches by BNS. Once the condition in line 15 holds, the node updates round to the smallest n such that every member of the quorum endorses to either commit or prepare some ballot with round bigger or equal than n, and (re-)starts the timer with delay F(round), where F is an unbound function that doubles its value with each increment of n (lines 16-17). If the candidate ballot can no longer be aborted or committed, then timeout will be eventually triggered (line 18) and the node considers a new candidate ballot with the current round increased by one, and with the value candidate.x if the node never prepared any ballot yet (line 19) or the value prepared.x otherwise (line 20). Then v tries to prepare the new candidate ballot by voting false for each ballot below and incompatible than it (line 21). This may involve sending an infinite number of messages, which by BNS requires sending finitely many batches.

The condition for starting the timer in line 15 does not strictly use FV as a black box. However, this use is warranted because line 15 only ‘reads’ the state of the network. ASCP makes every other change to the network through FV’s primitives.

ASCP guarantees the safety properties of non-blocking Byzantine consensus in §3. Since a node stops execution after deciding some value, Integrity for intact sets holds trivially. The requirement in lines 8-12 of Algorithm 2 that a node prepares the candidate ballot before voting for committing it, enforces that if a voting for committing some ballot within the nodes of an intact set I succeeds, then some node in I previously prepared that ballot:

**Lemma 9.** Let S be an FBQS and consider an execution of ASCP over S. Let I be an intact set in S. If a node v_1 \ belong to I commits a ballot b, then some node v_2 \ belong to I prepared b.

Aborting every ballot below and incompatible than the candidate one prevents that one node in an intact set I prepares a ballot b_1, and concurrently another node in I sends READY(b_2, true) with b_2 below and incompatible than b_1:

**Lemma 10.** Let S be an FBQS and consider an execution of ASCP over S. Let I be an intact set in S. Let v_1 and v_2 be nodes in I and b_1 and b_2 be ballots such that b_2 \ belong to I prepares the bigger of the two ballots by Lemma 9, which results in a contradiction by Lemma 10.

**Lemma 11.** Let S be an FBQS and consider an execution of ASCP over S. Let b_1 be the largest ballot prepared by some node v_1 at some moment in the execution. If all nodes are
honest, then some node $v_2$ proposed $b_1.x$.

Now we examine the liveness properties of non-blocking Byzantine consensus in §3, which ASCP also meets. Recall from §3 the bounded interval $\delta_I$ that a node in an intact set $I$ takes to deliver some Boolean for a given ballot, provided that some other node in $I$ has already delivered a Boolean for the same ballot. Let $v$ be a node in $I$ that prepares some ballot $b$ such that no other node in $I$ has ever prepared a ballot with round bigger or equal than $b.n$. We call the interval of duration $\delta_I$ after $v$ prepares $b$ the window for intact set $I$ of round $b.n$. Lemma 12 below guarantees that after some moment in the execution, no two consecutive windows ever overlap.

Lemma 12. Let $S$ be an FBQS and consider an execution of ASCP over $S$. Let $I$ be an intact set in $S$ and assume that all faulty nodes eventually stop. There exists a round $n$ such that either every node in $I$ decides some value before reaching round $n$, or otherwise the windows for $I$ of all the rounds $m \geq n$ never overlap with each other, and in each window of round $m$ the nodes in $I$ that have not decided yet only prepare ballots with round $m$.

Lemma 12 helps to establish Non-blocking for intact sets as follows. After the moment where no two consecutive windows overlap, either every node in $I$ has the same candidate ballot at the beginning of the window of some round, or otherwise the highest ballots prepared by each node in $I$ during that window coincide with each other. In either case all the nodes in $I$ will eventually have the same candidate ballot, and they will decide a value in bounded time.

Correctness of ASCP is captured by Theorem 13 below:

Theorem 13. Let $S$ be an FBQS. The ASCP protocol over $S$ satisfies the specification of non-blocking Byzantine consensus for intact sets.

6 Concrete Stellar Consensus Protocol

In this section we introduce concrete SCP (CSCP) which is amenable to implementation because each node $v$ maintains finite state and only needs to send and receive a finite number of messages in order to progress. CSCP relies on bunched voting (BV) in Algorithm 3, which generalises FV and embodies all of FV’s instances for each of the ballots. CSCP considers a single instance of BV, and thus each node $v$ keeps a single process bunched-voting($v$) (line 2) of Algorithm 4. In BV, nodes exchange messages that contain two kinds of statements: a prepare statement $prep\ b$ encodes the aim to abort the possibly infinite range of ballots that are lower and incompatible than $b$; and a commit statement $cmt\ b$ encodes the aim to commit ballot $b$.

Algorithm 3 depicts BV over an FBQS $S$ with set of quorums $Q$. A node $v$ stores the highest ballot for which $v$ has respectively voted, readied, or delivered a prepare statement in fields max-voted-prep, max-readied-prep, and max-delivered-prep (line 2). It also stores the set of ballots for which $v$ has respectively voted, readied, or delivered a commit statement in fields ballots-voted-cmt, ballots-readied-cmt, and ballots-delivered-cmt (line 3). All these fields are finite and thus $v$ maintains only finite state. When a node $v$ invokes prepare($b$), if $b$ exceeds the highest ballot for which $v$ has voted a prepare, then the node updates max-voted-prep to $b$ and sends VOTE($prep\ b$) to every other node (lines 4-7). The protocol then proceeds with the usual stages of FV, with the caveat that at each stage of the protocol only the maximum ballot is considered for which the node can send a message—or deliver an indication—with a prepare statement. In particular, when there exists a ballot $b$ that exceeds max-readied-prep and such that $v$ received a message VOTE($prep\ b_u$) from each member $u$
Algorithm 3: Bunched voting (BV) over an FBQS $S$ with set of quorums $Q$.

1. process bunched-voting($v \in V$)
2. \quad max-voted-prep, max-readied-prep, max-delivered-prep $\leftarrow (0, \bot) \in \text{Ballot}$;
3. \quad ballots-voted-cmt, ballots-readied-cmt, ballots-delivered-cmt $\leftarrow \emptyset \in 2^{\text{Ballot}}$;
4. \quad prepare($b$)
5. \hspace{1em} if max-voted-prep < $b$ then
6. \hspace{2em} max-voted-prep $\leftarrow b$;
7. \hspace{2em} send VOTE$(\text{PREP max-voted-prep})$ to every $v' \in V$;
8. \hspace{1em} when exists maximum $b$ such that max-voted-prep < $b$ and exists $U \in Q$
9. \hspace{2em} such that $v \in U$ and for every $u \in U$ received VOTE$(\text{PREP } b_u)$ where $b' \lessdot b_u$
10. \hspace{2em} for every $b' \lessdot b$
11. \hspace{2em} max-readied-prep $\leftarrow b$;
12. \hspace{2em} send READY$(\text{PREP max-readied-prep})$ to every $v' \in V$;
13. \hspace{1em} when exists maximum $b$ such that max-readied-prep < $b$ and exists $V$-blocking $B$
14. \hspace{2em} such that for every $u \in B$ received READY$(\text{PREP } b_u)$ where $b' \lessdot b_u$
15. \hspace{2em} for every $b' \lessdot b$
16. \hspace{2em} max-delivered-prep $\leftarrow b$;
17. \hspace{2em} prepared(max-delivered-prep);
18. \quad commit($b$)
19. \hspace{1em} if $b \notin$ ballots-voted-cmt and max-voted-prep = $b$ then
20. \hspace{2em} ballots-voted-cmt $\leftarrow$ ballots-voted-cmt $\cup \{b\}$;
21. \hspace{2em} send VOTE$(\text{CMT } b)$ to every $v' \in V$;
22. \hspace{1em} when received VOTE$(\text{CMT } b)$ from every $u \in U$ for some $U \in Q$ such that $v \in U$ and $b \notin$ ballots-readied-cmt
23. \hspace{2em} ballots-readied-cmt $\leftarrow$ ballots-readied-cmt $\cup \{b\}$;
24. \hspace{2em} send READY$(\text{CMT } b)$ to every $v' \in V$;
25. \hspace{1em} when received READY$(\text{CMT } b)$ from every $u \in B$ for some $v$-blocking $B$ and $b \notin$ ballots-readied-cmt
26. \hspace{2em} ballots-readied-cmt $\leftarrow$ ballots-readied-cmt $\cup \{b\}$;
27. \hspace{2em} send READY$(\text{CMT } b)$ to every $v' \in V$;
28. \hspace{1em} when received READY$(\text{CMT } b)$ from every $u \in U$ for some $U \in Q$ such that $v \in U$ and $b \notin$ ballots-delivered-cmt
29. \hspace{2em} ballots-delivered-cmt $\leftarrow$ ballots-delivered-cmt $\cup \{b\}$;
30. \hspace{1em} committed($b$);
of some quorum to which \( v \) belongs, then the node proceeds as follows: it checks that each \( b' \) lower and incompatible than \( b_i \) is also lower and incompatible than \( b \) (line 8). If \( b \) is the maximum ballot passing the previous check for every member \( u \) of the quorum, then the node updates the field \( \text{max-readied-prep} \) to \( b \) and sends the message \( \text{READY}(\text{prep} \ b) \) to every other node (lines 9\( \leftarrow \) 11). The node \( v \) checks similar conditions for the case when it receives messages \( \text{READY}(\text{prep} \ b_u) \) from each member \( u \) of a \( v \)-blocking set, and proceeds similarly by updating \( \text{max-readied-prep} \) to \( b \) and sending \( \text{READY}(\text{prep} \ b) \) to every other node (lines 11\( \leftarrow \) 13). The node will update \( \text{max-delivered-prep} \) and trigger the indication \( \text{prepared}(b) \) when the same conditions are met after receiving messages \( \text{READY}(\text{prep} \ b_u) \) from each member \( u \) of a quorum to which \( v \) belongs (lines 14\( \leftarrow \) 16). When a node \( v \) invokes \( \text{commit}(b) \) then the protocol proceeds with the usual stages of FV with two minor differences (lines 17\( \leftarrow \) 20).

First, a node \( v \) only votes commit for the highest ballot for which \( v \) has voted a prepare statement (condition \( \text{max-voted-prep} = b \) in line 18). Second, the protocol uses the sets of ballots \( \text{ballots-voted-cmt}, \text{ballots-readied-cmt} \) and \( \text{ballots-delivered-cmt} \) in order to keep track of the stage of the protocol for each ballot.

The structure of CSCP in Algorithm 4 directly relates to ASCP in Algorithm 2. A node proposes a value \( x \) in line 5. A node tries to prepare a ballot \( b \) by invoking \( \text{prepare}(b) \) in line 7 and receives the indication \( \text{prepared}(b) \) in line 8. A node tries to commit a ballot \( b \) by invoking \( \text{commit}(b) \) in line 12 and receives the indication \( \text{committed}(b) \) in line 13. A node decides a value \( x \) in line 14. Timeouts are set in lines 15\( \leftarrow \) 17 and triggered in line 18.

Next we establish a correspondence between CSCP in Algorithm 4 and ASCP in Algorithm 2. First, a node \( v \) only votes commit for the highest ballot for which \( v \) has voted a prepare statement (condition \( \text{max-voted-prep} = b \) in line 18). Second, the protocol uses the sets of ballots \( \text{ballots-voted-cmt}, \text{ballots-readied-cmt} \) and \( \text{ballots-delivered-cmt} \) in order to keep track of the stage of the protocol for each ballot.

We first define several notions required to formalise our refinement result. A history is a sequence of the events \( v.\text{propose}(x) \) and \( v.\text{decide}(x) \), where \( v \) is a correct node and \( x \) a value. The specification of consensus assumes that \( v \) triggers an event \( v.\text{propose}(x) \), thus a history will have \( v.\text{propose}(x) \) for every correct node \( v \). A concrete trace \( \tau \) is a sequence of events that subsumes histories, and contains events \( v.\text{prepare}(b) \), \( v.\text{commit}(b) \), \( v.\text{prepared}(b) \), \( v.\text{committed}(b) \), \( v.\text{start-timer}(n) \), \( v.\text{timeout} \), \( v.\text{send}(m, v') \), and \( v.\text{receive}(m, v') \), where \( v \) is a correct node and \( v' \) any node, \( b \) is a ballot, \( m \) is a message in \{\text{VOTE}(s), \text{READY}(s)\} with \( s \) a statement in \{\text{prep} \ b, \text{cmt} \ b\}, and \( n \) is a round. An abstract trace \( \tau \) is a sequence of events that subsumes histories, and contains events \( v.\text{start-timer}(n) \), \( v.\text{timeout} \), and batched events \( v.\text{vote-batch}([b_i], a) \), \( v.\text{deliver-batch}([b_i], a) \), \( v.\text{send-batch}([m_i], v') \), and \( v.\text{receive-batch}([m_i], v') \), where \( v \) is a correct node and \( v' \) any node, \( n \) is a round, \( [b_i] \) is a sequence of ballots, \( a \) is a Boolean, and \([m_i]\) is a sequence of messages in \{\text{VOTE}(b, a), \text{READY}(b, a)\}.

The sequences of ballots and messages above, which represent a possibly infinite number of ‘batched’ events, ensure that the length of any abstract trace is bounded by \( \omega \). We may omit the adjective ‘concrete/abstract’ from ‘trace’ when it is clear from the context. Given a trace \( \tau \), a history \( H(\tau) \) can be uniquely obtained from \( \tau \) by removing every event in \( \tau \) different from \( v.\text{propose}(x) \) or \( v.\text{decide}(x) \).

An execution of CSCP (respectively, ASCP) entails a concrete trace (respectively, abstract trace) \( \tau \) iff for every invocation and indication as well as for every send or receive primitive in an execution of the protocol in Algorithm 4 (respectively, for every invocation, indication and primitive in an execution of the protocol in Algorithm 2) where the vote,
Algorithm 4: Concrete SCP (CSCP) over an FBQS $S$ with set of quorums $Q$.

1. \textbf{process} \text{concrete-consensus}(v \in V)
2. \hspace{1em} brs \leftarrow \text{new process} \text{bunched-voting}(v);
3. \hspace{1em} candidate, prepared \leftarrow (0, \bot) \in \text{Ballot};
4. \hspace{1em} round \leftarrow 0 \in \mathbb{N}^+ \cup \{0\};
5. \hspace{1em} \text{propose}(x)
6. \hspace{1em} \hspace{1em} candidate \leftarrow (1, x);
7. \hspace{1em} \hspace{1em} \text{brs.prepare}(candidate);
8. \hspace{1em} \text{when triggered} \text{brs.prepared}(b) \text{and prepared} < b
9. \hspace{1em} \hspace{1em} prepared \leftarrow b;
10. \hspace{1em} \hspace{1em} \text{if candidate} \leq \text{prepared} \text{then}
11. \hspace{1em} \hspace{1em} candidate \leftarrow \text{prepared};
12. \hspace{1em} \hspace{1em} \text{brs.commit}(candidate);
13. \hspace{1em} \text{when triggered} \text{brs.committed}(b)
14. \hspace{1em} \hspace{1em} \text{trigger} \text{decide}(b,x);
15. \hspace{1em} \text{when exists} U \in Q \text{such that} v \in U \text{and for each} u \in U \text{exist}
16. \hspace{1em} \hspace{1em} M_u \in \{\text{VOTE, READY}\} \text{and} b_u \in \text{Ballot} \text{such that} \text{round} < b_u.n \text{and received}
17. \hspace{1em} \hspace{1em} M_u(s_u b_u) \text{from} u \text{with} s_u \in \{\text{cmt, prep}\}
18. \hspace{1em} \hspace{1em} \text{round} \leftarrow \min\{b_u.n \mid u \in U\};
19. \hspace{1em} \hspace{1em} \text{start-timer}(F(\text{round}));
20. \hspace{1em} \text{when triggered} \text{timeout}
21. \hspace{1em} \hspace{1em} \text{if} \text{prepared} = (0, \bot) \text{then} \text{candidate} \leftarrow (\text{round} + 1, \text{candidate}.x);
22. \hspace{1em} \hspace{1em} \text{else candidate} \leftarrow (\text{round} + 1, \text{prepared}.x);
23. \hspace{1em} \hspace{1em} \text{brs.prepare}(candidate);

deliver, send and receive events are batched together), $\tau$ contains corresponding events in the same order.

We are interested in traces that are relative to some intact set $I$. Given a trace $\tau$, the $I$-projected trace $\tau|_I$ is obtained by removing the events $v.ev \in \tau$ such that $v \notin I$.

\textbf{Theorem 14.} Let $S$ be an FBQS and $I$ be an intact set. For every execution of CSCP over $S$ with trace $\tau$, there exists an execution of ASCP over $S$ with trace $\rho$ and $H(\tau|_I) = H(\rho|_I)$.

\textbf{Proof sketch.} We define a simulation function $\sigma$ from concrete to abstract traces. Theorem 14 can be established by showing that, for every finite prefix $\tau$ of a trace entailed by CSCP, the simulation $\sigma(\tau)$ is a prefix of a trace entailed by ASCP.

Every execution of ASCP enjoys the properties of Integrity, Agreement for intact sets, Weak validity for intact sets and Non-blocking for intact sets, and so does every execution of CSCP by refinement.

\textbf{Corollary 15.} Let $S$ be an FBQS. The CSCP protocol over $S$ satisfies the specification of non-blocking Byzantine consensus for intact sets.
Lying about Quorum Slices

So far we have assumed the unrealistic setting where faulty nodes do not equivocate their quorum slices, so all nodes share the same FBQS $S$. We now lift this assumption. To this end, we use a generalisation of FBQS called subjective FBQS \[\text{subjective FBQS}\], which allows faulty nodes to lie about their quorum slices. Assuming that $V_{\text{ok}}$ is the set of correct nodes, the subjective FBQS $\{S_v\}_{v \in V_{\text{ok}}}$ is an indexed family of FBQSes where the different FBQSes agree on the quorum slices of correct nodes, i.e., $\forall v_1, v_2, v \in V_{\text{ok}}. S_{v_1}(v) = S_{v_2}(v)$. For each correct node $v$, the FBQS $S_v$ is the view of node $v$, which reflects the choices of trust communicated to $v$. We can run either ASCP or CSCP over a subjective FBQS $\{S_v\}_{v \in V_{\text{ok}}}$ by letting each correct node $v$ act according to its view $S_v$.

We generalise the definition of intact set to subjective FBQSes, and we lift our results so far to the subjective FBQSes. Let $\{S_v\}_{v \in V_{\text{ok}}}$ be a subjective FBQS. A set $I$ is an intact set if for each $v \in V_{\text{ok}}$ the set $I$ is a quorum in $S_v$ that only contains correct nodes, and every member of $I$ is intertwined with each other in the projected FBQS $S_v|I$.

\begin{lemma}
Let $\{S_v\}_{v \in V_{\text{ok}}}$ be a subjective FBQS. For any node $v \in V_{\text{ok}}$, a set $I$ is an intact set in $S_v$ iff $I$ is an intact set in $\{S_v\}_{v \in V_{\text{ok}}}$.
\end{lemma}

Since Lemma \[\text{Lemma 16}\] above guarantees that every view has the same intact sets, which also coincide with the intact sets of the subjective FBQS, from now on we may say ‘an intact set $I$’ and omit to which system (a particular view, or the subjective FBQS) $I$ belongs.

Using the fact that nodes agree on the slices of correct nodes, we can prove Lemma \[\text{Lemma 17}\] below, which is the analogue to Lemma \[\text{Lemma 3}\] and states sufficient safety conditions for the nodes in an intact set $I$ to reach agreement when each node acts according to its own view.

\begin{lemma}
Let $\{S_v\}_{v \in V_{\text{ok}}}$ be a subjective FBQS and for each correct node $v$ let $Q_v$ be the set of quorums in the view $S_v$. Let $I$ be an intact set and consider two quorums $U_1$ and $U_2$ in $\bigcup_{v \in V_{\text{ok}}} Q_v$. If $U_1 \cap I \neq \emptyset$ and $U_2 \cap I \neq \emptyset$, then $U_1 \cap U_2 \cap I \neq \emptyset$.
\end{lemma}

Using arguments similar to those in the previous sections, we can establish the correctness of ASCP and CSCP over subjective FBQSes.

\begin{theorem}
Let $\{S_v\}_{v \in V_{\text{ok}}}$ be a subjective FBQS. The ASCP protocol over $\{S_v\}_{v \in V_{\text{ok}}}$ satisfies the specification of non-blocking Byzantine consensus for intact sets.
\end{theorem}

\begin{theorem}
Let $\{S_v\}_{v \in V_{\text{ok}}}$ be a subjective FBQS and $I$ be an intact set. For every execution of CSCP over $\{S_v\}_{v \in V_{\text{ok}}}$ with trace $\tau$, there exists an execution of ASCP over $\{S_v\}_{v \in V_{\text{ok}}}$ with trace $\rho$ and $H(\tau|I) = H(\rho|I)$.
\end{theorem}

\begin{corollary}
Let $\{S_v\}_{v \in V_{\text{ok}}}$ be a subjective FBQS. The CSCP protocol over $\{S_v\}_{v \in V_{\text{ok}}}$ satisfies the specification of non-blocking Byzantine consensus for intact sets.
\end{corollary}

Related Work

García-Pérez and Gotsman \[\text{[7]}\] have previously investigated Stellar’s federated voting and its relationship to Bracha’s broadcast over classical Byzantine quorum systems. They did not address the full Stellar consensus protocol. Our proof of SCP establishes the correctness of federated voting by adjusting the results in \[\text{[3]}\] to multiple intact sets within the system.

Losa et al. \[\text{[10]}\] have also investigated consensus over FBQSs. They propose a generalisation of Stellar’s quorums that does not prescribe constructing them from slices, yet allows different participants to disagree on what constitutes a quorum. They then propose a
protocol solving consensus over intact sets in this setting that provides better liveness guarantees than SCP, but is impractical. Losa et al.’s work is orthogonal to ours: they consider a more general setting than Stellar’s and a theoretical protocol, whereas we investigate the practical protocol used by Stellar.

The advent of blockchain has given rise to a number of novel proposals of BFT protocols; see [3] for a survey. Out of these, the most similar one to Stellar is Ripple [15]. In particular, Ripple have recently proposed a protocol called Cobalt that allows for a federated setting similar to Stellar’s [11]. We hope that our work will pave the way to investigating the correctness of this and similar protocols.

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Lemma 23 (Analogous to Lemma 36 in \cite{8} for intact sets). Let $S$ be an FBQS and $t$ be a tag, and consider an execution of the instance for $t$ of $FV$ over $S$. Let $I$ be an intact set in $S$. The first node $v \in I$ that sends a $\text{READY}(t,a)$ message first needs to receive a $\text{VOTE}(t,a)$ message from every member of a quorum $U$ to which $v$ belongs.

Proof. Let $v$ be any node in $I$. By Lemma 5 no $v$-blocking set $B$ exists such that $B \cap I = \emptyset$. Therefore, the first node $v \in I$ that sends a $\text{READY}(t,a)$ message does it through lines 4-9 of Algorithm 4 which means that $v$ received $\text{VOTE}(t,a)$ messages from every member of a quorum $U$ to which $v$ belongs.

Lemma 24. Let $S$ be an FBQS and assume some set of faulty nodes. Let $I$ be an intact set in $S$, and consider $B$ of nodes. If $B$ is not $\nu$-blocking for any $v \in I \setminus B$, then either $B \supseteq I$ or $I \setminus B$ is a quorum in $S|I$.

Proof. Straightforward, since every node in $I$ is intertwined with each other by definition of intact set.

Proof of Lemma 3. Assume $V_{ok}$ is the set of correct nodes, and let $I_1$ and $I_2$ be sets such that $I_1 \cap I_2 \neq \emptyset$. Assume towards a contradiction that $I_1 \cup I_2$ is not an intact set, and therefore there exist nodes $v_1 \in I_1$ and $v_2 \in I_2$ that are not intertwined. By definition of intertwined, this assumption entails that there exist quorums $U_1$ and $U_2$ such that $v_1 \in U_1$ and $v_2 \in U_2$, and $(U_1 \cap U_2) \cap V_{ok} = \emptyset$. Since both $U_1$ and $I_2$ are quorums in $S$ and both have non-empty intersection with the intact set $I_1$, we have $(U_1 \cap I_2) \cap I_1 \neq \emptyset$ by Lemma 3 and we can conclude that $U_1$ has non-empty intersection with the intact set $I_2$. Thus, we know that $(U_1 \cap U_2) \cap I_2 \neq \emptyset$ again by Lemma 3. But this results in a contradiction because $(U_1 \cap U_2) \cap I_2 = \emptyset$ by assumptions, since $I_2$ contains only correct nodes.

Proof of Lemma 5. Since $I$ is a quorum in $S$ and by the definition of quorum, for every node $v \in I$ there exists one slice of $v$ that lies within $I$, and the required holds.

Proof of Lemma 22. Let $S$ be an FBQS and assume some set of faulty nodes. Let $I$ be an intact set in $S$. Every two quorums in $S|I$ have non-empty intersection.

Proof. Straightforward, since every node in $S|I$ is intertwined with each other by definition of intact set.

Proof of Lemma 21. $U_1 \cap I$ and $U_2 \cap I$ are quorums in $S|I$ by Lemma 21. Since every two quorums in $S|I$ have non-empty intersection by Lemma 22, we have $(U_1 \cap I) \cap (U_2 \cap I) = (U_1 \cap U_2) \cap I \neq \emptyset$. Therefore the intersection $U_1 \cap U_2$ contains some node in $I$. \hfill $\blacksquare$

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**Proof.** Assume \( B \) is not \( v \)-blocking for any \( v \in I \setminus B \). If \( B \supseteq I \) then we are done. Otherwise, for every node \( v \) in \( I \setminus B \), there exists a slice \( q \in S(v) \) such that \( q \cap B = \emptyset \). We know that \( q \cap I \neq \emptyset \) since \( v \in q \) by definition of FBQS. We also know that \( q \cap I \in S|_{I}(v) \) by definition of \( S|_{I} \), and since \( q \cap B = \emptyset \), the intersection \( q \cap I \) is a subset of \( I \setminus B \). Since for each node \( v \in I \) there exists a slice \( q \in S(v) \) such that \( q \cap I \) is a subset of \( I \setminus B \), the set \( I \setminus B \) is a quorum in \( S|_{I} \), as required.  

- **Lemma 25** (Analogous to Lemma 16 in [8] for intact sets). Let \( S \) be an FBQS and \( t \) be a tag, and consider an execution of the instance for \( t \) of FV over \( S \). Let \( I \) be an intact set in \( S \). If two nodes in \( I \) send respectively messages \( \text{READY}(t, a) \) and \( \text{READY}(t, a') \), then \( a = a' \).

**Proof.** Assume that two nodes in \( I \) send respectively messages \( \text{READY}(t, a) \) and \( \text{READY}(t, a') \). By Lemma 23 some node \( v \in I \) has received \( \text{VOTE}(t, a) \) from a quorum \( U \) to which \( v \) belongs, and some node \( v' \in I \) has received \( \text{VOTE}(t, a') \) from a quorum \( U' \) to which \( v' \) belongs. By Lemma 3 the intersection \( U \cap U' \) contains some node in \( I \), so that this node has sent \( \text{VOTE}(t, a) \) and \( \text{VOTE}(t, a') \). But due to the use of the guard variable \( \text{voted} \) in lines 2 and 3 of Algorithm 1 a node can only vote for one value per tag, and thus it cannot vote different values for the same tag. Hence, \( a = a' \).

- **Lemma 26** (Analogous to Lemma 17 in [8] for intact sets). Let \( S \) be an FBQS and assume some set of faulty nodes. Let \( I \) be an intact set in \( S \). Assume that \( I = V^+ \cup V^- \) and for some quorum \( U \) we have \( U \cap I \subseteq V^- \). Then either \( V^- = \emptyset \) or there exists some node \( v \in V^- \) such that \( V^+ \) is \( v \)-blocking.

**Proof.** Assume that \( V^+ \) is not \( v \)-blocking for any \( v \in V^- \). By Lemma 24 either \( V^- = \emptyset \) or \( V^- \) is a quorum in \( S|_{I} \). In the former case we are done, while in the latter we get a contradiction as follows. By Lemma 21 the intersection \( U \cap I \) is a quorum in \( S|_{I} \). Since every two quorums in \( S|_{I} \) have non-empty intersection by Lemma 22 we have \( (U \cap I) \cap V^- \neq \emptyset \). But this is impossible, since \( U \cap I \subseteq V^+ \) and \( V^+ \cap V^- = \emptyset \).

**Proof of Theorem 5.** We prove that the instance for tag \( t \) of FV over \( S \) enjoys each of the properties that define the specification of reliable Byzantine voting for intact sets.

**No duplication:** Straightforward by the use of the guard variable \( \text{delivered} \) in line 25 of Algorithm 1.

**Totality for intact sets:** Assume some node in \( I \) delivers a value \( a \) for tag \( t \). By the condition in line 7 of Algorithm 1 the node has received \( \text{READY}(t, a) \) messages from every member in a quorum \( U \). Since \( U \cap I \) contains only correct nodes, these nodes send \( \text{READY}(t, a) \) messages to every node. By the condition in line 10 of Algorithm 1 any correct node \( v \) sends \( \text{READY}(t, a) \) messages if it receives \( \text{READY}(t, a) \) from every member in a \( v \)-blocking set. Hence, the \( \text{READY}(t, a) \) messages from the nodes in \( U \cap I \) may convince additional correct nodes to send \( \text{READY}(t, a) \) messages to every node. Let these additional correct nodes send \( \text{READY}(t, a) \) messages until a point is reached at which no further nodes in \( I \) can send \( \text{READY}(t, a) \) messages. At this point, let \( V^+ \) be the set of nodes in \( I \) that sent \( \text{READY}(t, a) \) messages (where \( U \cap I \subseteq V^+ \)), and let \( V^- = I \setminus V^+ \). By Lemma 25 the nodes in \( V^- \) did not send any \( \text{READY}(t, \_ \_ ) \) messages at all. The set \( V^+ \) cannot be \( v \)-blocking for any node \( v \) in \( V^- \), or else more nodes in \( I \) could come to send \( \text{READY}(t, a) \) messages. Then by Lemma 26 we have \( V^- \neq \emptyset \), meaning that every node in \( I \) has sent \( \text{READY}(t, a) \) messages. Since \( I \) is a quorum, all the nodes in \( I \) will eventually deliver a Boolean for tag \( t \) due to the condition in line 7 of Algorithm 1.
**Consistency for intertwined nodes**: Assume that two intertwined nodes \( v \) and \( v' \) deliver values \( a \) and \( a' \) for tag \( t \) respectively. By the condition in line 13 the nodes received a quorum of \( \text{READY}(t, a) \), respectively, \( \text{READY}(t, a') \) messages. Since the two nodes are intertwined, there is a correct node \( u \) in the intersection of the two quorums, which sent both \( \text{READY}(t, a) \) and \( \text{READY}(t, a') \). By the use of the guard variable \( \text{readied} \) in line 7 of Algorithm 1 node \( u \) can only send one and the same ready message for tag \( t \) to every other node, and thus \( a = a' \) as required.

**Validity for intact sets**: Assume every node in an intact set \( I \) votes for value \( a \). Since \( I \) is a quorum, every node in \( I \) will eventually send \( \text{READY}(t, a) \) by the condition in line 14 of Algorithm 1. By Lemma 25 these messages cannot carry a value different from \( a \). Then by the condition in line 13 of Algorithm 1 every node in \( I \) will eventually deliver the value \( a \) for tag \( t \). Due to **Consistency for intact sets**, no node delivers a value different from \( a \). 

**Proof of Lemma 7** We show how to strengthen the proof of **Totality for intact sets** above to prove the required. In order to reach the point at which no further nodes in \( I \) can send \( \text{READY}(b, a) \) messages, each node has to send and receive a finite number of messages. This number is unknown, but its upper bound is determined by the size of the system and the topology of the slices, and thus it is constant and bounded. Since GST has expired, the point at which no further nodes in \( I \) can send \( \text{READY}(b, a) \) messages is reached in bounded time. Therefore, every node in \( I \) sends \( \text{READY}(b, a) \) to every other node in \( I \) in bounded time. These messages arrive in bounded time too, after which every node in \( I \) delivers a value, and the required holds.

### C Proofs in §5

In the remainder of the appendix, we say ballot \( b_1 \) is **below and compatible than** ballot \( b_2 \) (written \( b_1 \preceq b_2 \)) iff \( b_1 \leq b_2 \) and \( b_1 \sim b_2 \).

**Proof of Lemma 9** Assume that a node \( v_1 \in I \) commits ballot \( b \). By line 7 of Algorithm 1 node \( v_1 \) received \( \text{READY}(b, true) \) from every member of a quorum to which \( v_1 \) belongs. By Lemma 28 the first node to do so received \( \text{VOTE}(b, true) \) messages from every member of a quorum \( U \) to which \( v_1 \) belongs. Since \( v_1 \) is intertwined with every other node in \( I \), there exists a correct node \( v_2 \) in the intersection \( U \cap I \) that sent \( \text{VOTE}(b, true) \). The node \( v_2 \) can send \( \text{VOTE}(b, true) \) only through line 6 of Algorithm 1, which means that \( v_2 \) triggers \( \text{brs}[b], \text{vote}(true) \) in line 12 of Algorithm 2. By line 8 of the same figure, this is only possible after \( v_2 \) has aborted every \( b' \not\preceq b \), and the lemma holds.

**Proof of Lemma 10** Assume towards a contradiction that \( v_1 \) prepares \( b_1 \), and that \( v_2 \) sends \( \text{READY}(b_2, true) \). By definition of prepare, node \( v_1 \) aborted every ballot \( b \preceq b_1 \). By line 7 of Algorithm 1 node \( v_1 \) received \( \text{READY}(b, false) \) from every member of a quorum \( U_b \) for each ballot \( b \preceq b_1 \). By assumptions, \( b_2 \preceq b_1 \), and therefore \( v_2 \) received \( \text{READY}(b_2, false) \) from every member of the quorum \( U_{b_2} \). By Lemma 28 the first node \( u_1 \in I \) that sent \( \text{READY}(b_2, false) \) received \( \text{VOTE}(b_2, false) \) from a quorum \( U_1 \) to which \( u_1 \) belongs. Since \( v_2 \) sent \( \text{READY}(b_2, true) \) and by Lemma 28 the first node \( u_2 \in I \) that sent \( \text{READY}(b_2, true) \) received \( \text{VOTE}(b_2, true) \) from a quorum \( U_2 \) to which \( u_2 \) belongs. Since \( u_1 \) and \( u_2 \) are intertwined, the intersection \( U_1 \cap U_2 \) contains some correct node \( v \), which sent both \( \text{VOTE}(b_2, false) \) and \( \text{VOTE}(b_2, true) \) messages. By the use of the Boolean \( \text{voted} \) in line 3 of Algorithm 1 this results in a contradiction and we are done.
Lemma 27. Let $S$ be an FBQS and consider an execution of ASCP over $S$. Let $I$ be an intact set in $S$. If a node $v_1 \in I$ commits a ballot $b_1$, then the largest ballot $b_2$ prepared by any node $v_2 \in I$ before $v_1$ commits $b_1$ is such that $b_1 \sim b_2$.

Proof. Assume node $v_1$ commits ballot $b_1$. By the guard in line 13 of Algorithm 1 node $v_1$ received the message $\text{READY}(b_1, \text{true})$ from every member of a quorum to which $v_1$ belongs, which entails that node $v_1$ received $\text{READY}(b_1, \text{true})$ from itself. By Lemma 23 the first node $u \in I$ that send $\text{READY}(b_1, \text{true})$ needs to receive a $\text{VOTE}(b_1, \text{true})$ message from every member of some quorum to which $u$ belongs. Thus, $u$ itself triggered $\text{brs}[b_1].\text{vote}(\text{true})$, which by lines 7 and 21 of Algorithm 2 means that $u$ prepared ballot $b_1$. Hence, the largest ballot $b_2$ such that there exists a node $v_2 \in I$ that triggers $\text{brs}[b_2].\text{vote}(\text{true})$ before $v_1$ commits $b_1$, is bigger or equal than $b_1$. If $b_2 = b_1$, then $b_2.x = b_1.x$ and by lines 8–12 of Algorithm 2 node $v_2$ prepares $b_2$ before it triggers $\text{brs}[b_2].\text{vote}(\text{true})$ and the lemma holds.

If $b_2 > b_1$, then we assume towards a contradiction that $b_2.x \neq b_1.x$. By lines 8–12 of Algorithm 2 node $v_2$ prepared $b_2$. But this results in a contradiction by Lemma 10 because $v_1$ and $v_2$ are intertwined and $v_1$ sent $\text{READY}(b_1, \text{true})$, but $b_1 \lesssim b_2$. Therefore $b_2.x = b_1.x$, and by lines 8–12 of Algorithm 2 node $v_2$ prepares $b_2$ before it triggers $\text{brs}[b_2].\text{vote}(\text{true})$. \hfill \blacktriangleleft

We define the ready-tree for Boolean $a$ and ballot $b$ at node $v$, which characterises the messages that need to be exchanged by the FV protocol in order for node $v$ to send a $\text{READY}(b, a)$ message, under the assumption that all nodes are honest. The ready-tree for Boolean $a$ and ballot $b$ at node $v$ is the tree computed recursively as follows:

- If $v$ sent the message after receiving $\text{VOTE}(b, a)$ from every member of a quorum $U$, then let $U$ be the root of the tree, which has no children.
- If $v$ sent the message after receiving $\text{READY}(b, a)$ from every member of a $v$-blocking set $B$, then let $B$ be the root of the tree, and let its children be the ready-trees for Boolean $a$ and ballot $b$ at each of the members of $B$.

For short, we may say ‘ready-tree at node $v$’ when the Boolean $a$ and the ballot $b$ are clear from the context.

A ready-tree is always of finite height, or otherwise some node would have faked ready messages by lines 7–10 and 13–12 of Algorithm 11 which contradicts the assumption that all nodes are honest.

Lemma 28. Let $S$ be an FBQS and $b$ be a ballot, and consider and execution of the instance for ballot $b$ of FV over $S$. Assume all nodes are honest. If a node $v$ sends $\text{READY}(b, a)$ then there exists a quorum $U$ such that every member of $U$ sent $\text{VOTE}(b, a)$.

Proof. Assume that a node $v$ sends $\text{READY}(b, a)$, and consider the ready-tree at node $v$. The lemma holds since each leaf of the ready-tree is a quorum whose members sent $\text{VOTE}(b, a)$.

Proof of Lemma 11. Assume all nodes are honest, which entails that each node in the system sends the same set of batches to every node. We proceed by induction on the number of batches sent by the nodes in the execution so far.

Since $v_1$ prepares $b_1$, and by lines 13–15 of Algorithm 11 for each ballot $b_i \lesssim b_1$ there exists a quorum $U_i$ such that $v_1$ receives $\text{READY}(b_i, \text{false})$ from every member of $U_i$. Since the number of quorums to which $v_1$ belongs is finite and by BNS, there exists a quorum $U_R$ among the $U_i$ such that every member of $U_R$ sent a batch containing a message $\text{READY}(b_j, \text{false})$ for each $b_j$ in some right-open interval $[b, b_1)$ with $b < b_1$ where the ballot $b$ is determined by the ballots that $v_1$ had aborted before preparing $b_1$. If $v_1$ had aborted every ballot $b_k \lesssim b_1$ before preparing $b_1$, or if $v_1$ had received from each member of $U_R$ a message $\text{READY}(b_k, \text{false})$ for each ballot $b_k \lesssim b_1$ such that $v_1$ did not abort $b_k$ before preparing $b_1$, then $v_1$ would...
have prepared a ballot bigger than $b_1$ by BNS and by lines 13 and 15 of Algorithm 1 which contradicts the assumptions. Therefore, there exists a ballot $b_0 \leq b_1$ and a node $u \in U_R$ such that $v_1$ did not abort $b_0$ before preparing $b_1$, and such that $v_1$ received from $u$ a batch that contains a message $\text{READY}(b_j, \text{false})$ for each $b_j \in [b_0, b_1)$, but which does not contain the message $\text{READY}(b_0, \text{false})$.

By Lemma 28 for each $b_j \in [b_0, b_1)$ there exists a quorum $U_j$ such that $u$ received $\text{VOTE}(b_j, \text{false})$ from each member of $U_j$. Without loss of generality, we fix each $U_j$ to be the union of the quorums at the leaves of the ready-tree for Boolean $\text{false}$ and ballot $b_j$ at node $u$, which is itself a quorum since quorums are closed by union. What follows mimics the argument used in the previous paragraph in order to show that some node exists that sent enough vote messages to prepare $b_1$, and not so many as to prepare a ballot bigger than $b_1$. Since the number of quorums to which $u$ belongs is finite and by BNS, there exists a quorum $U_V$ among the $U_j$ such that every member of $U_V$ sent a batch containing a message $\text{VOTE}(b_k, \text{false})$ for each $b_k$ in some right-open interval $[b', b_1)$ with $b' < b_1$, where the ballot $b'$ is determined by the messages sent by $u$ before sending the batch that contains the ready messages described in the paragraph above. If $u$ had sent $\text{READY}(b_k, \text{false})$ for every ballot $b_k \leq b_1$ before sending the batch described in the paragraph above, or if $u$ had received from each member of $U_V$ a message $\text{VOTE}(b_k, \text{false})$ for each ballot $b_k \leq b_1$ such that $u$ did not send $\text{READY}(b_k, \text{false})$ before sending the batch described in the paragraph above, then the ballot $b_0$ would not exist by the definition of ready-tree, by BNS, and by lines 13 and 15 of Algorithm 1 which would contradict the facts established in the paragraph below. Therefore, there exists a ballot $b'_0 \leq b_1$ and a node $v_2 \in U_V$ such that $u$ did not send $\text{READY}(b'_0, \text{false})$ before sending the batch described in the paragraph above, and such that $u$ received from $v_2$ a batch that contains a message $\text{VOTE}(b_k, \text{false})$ for each $b_k \in [b', b_1)$, but which does not contain the message $\text{VOTE}(b'_0, \text{false})$.

By BNS and by line 4 of Algorithm 1 and lines 7 and 21 of Algorithm 2 in order for $v_2$ to send a batch containing the vote messages described in the previous paragraph, the node necessarily tried to prepare some ballot $b_2 \geq b_1$ such that $b_2, x \notin \{b, x \mid b \in [b', b_1]\}$—which results in $v_2$ triggering $\text{brs}[b_i].\text{vote}(b_i, \text{false})$ for each ballot $b_i \not< b_2$, thus triggering $\text{brs}[b_i].\text{vote}(b_i, \text{false})$ for each ballot $b_i \in [b', b_1)$—and either

(i) $b_i \not< b_2$, or otherwise

(ii) $b_1 \not< b_2$ and $v_2$ triggered $\text{brs}[b_i].\text{vote}(b_i, \text{true})$ before triggering the $\text{brs}[b_i].\text{vote}(b_i, \text{false})$ for each $b_i \not< b_2$,

which prevents that $v_2$ triggers $\text{brs}[b_i].\text{vote}(b_i, \text{false})$.

We show that (ii) above always holds. Assume (ii) towards a contradiction. Let us focus on the ballots $b_3 = (1, b_1, x) \leq b_1$ and $b_4 = (1, b_2, x) \leq b_2$. By BNS node $v_2$ triggered each $\text{brs}[b_j].\text{vote}(b_j, \text{true})$ before triggering any $\text{brs}[b_i].\text{vote}(b_i, \text{false})$ with $i \geq j$, and thus it triggered $\text{brs}[b_3].\text{vote}(b_3, \text{true})$ before triggering the $\text{brs}[b_i].\text{vote}(b_i, \text{false})$ with $b_i \geq b_3$ and $b_i \not< b_4$. Since $v_2$ triggers $\text{brs}[b_3].\text{vote}(b_3, \text{true})$, and by lines 8 and 12 of Algorithm 2 the largest ballot candidate prepared by $v_2$ before triggering $\text{brs}[b_3].\text{vote}(b_3, \text{false})$ is such that candidate $\leq b_3$. Since every node proposes some value when the execution of the protocol starts by the specification of consensus, the value candidate $x$ is bigger or equal than the value proposed by $v_2$ by lines 8 and 11 of Algorithm 2. Since $v_2$ triggered $\text{brs}[b_i].\text{vote}(b_i, \text{false})$ for each $b_i \not< b_4$, and by BNS and by line 4 of Algorithm 2 node $v_2$ proposed a value bigger or equal than $b_4, x$. But since candidate $\leq b_3 < b_4$, this contradicts the fact we established earlier that candidate $x$ is bigger or equal than the value proposed by $v_2$. Therefore, we know that $b_2 \geq b_1$.

Now we distinguish the following cases. If $b_2, n = 1$ then, by line 7 of Algorithm 2 node
v_2 proposed b_2.x = b_1.x and the lemma holds. If b_2.n > 1 then, by BNS and by lines 18 \( \text{and} \) 21 of Algorithm 2, node v_2 triggered a timeout event when its current round was b_2.n – 1. If v_2 never prepared any ballot when this timeout expired, then by line 14 of Algorithm 2 the node proposed b_2.x and the lemma holds. Otherwise, by line 20 of Algorithm 2 the value b_2.x is equal to h.x where h is the largest ballot prepared by v_2 when that timeout expired, and the required holds by applying the induction hypothesis.

Recall from §4 the window for intact set I of round n, which is the interval of duration \( \delta_I \) in which every node in I that has not decided any value yet prepares a ballot of round n. Now we introduce some additionally terminology for the proofs in the remainder of the appendix. Assume all faulty nodes have stopped, and consider the window for intact set I of round n, where we let v_n be the first node in I that prepares a ballot b_n with round n. By line 8 of Algorithm 2 and lines 13 \( \text{and} \) 14 of Algorithm 4 for each ballot b_j \( \subseteq b_n \) there exists a quorum U_j such that node v_n received \( \text{READY}(b_j, \text{false}) \) from each member of U_j. We call the prepare-footprint for intact set I of round n, written \( P^I_n \), to the set computed as follows:

- Let v_n \( \in I \) be the first node in I that prepares a ballot b_n with round n.
- Let h be the highest ballot prepared by v_n before preparing b_n.
- Compute the set of ballots that v_n needs to abort in order to prepare b_n and that v_n did not abort previously. This is, take the ballots b_j \( \subseteq b_n \) such that either b_j > h, or b_j \( \leq \) h and v_n never aborted b_j yet.
- For each b_j in the set computed in the previous step, and for each member u \( \in U_j \), compute the ready-tree of Boolean \( \text{false} \) and ballot b_j at node u.
- Let \( P^I_n \) be the union of the leaves of all the ready-trees computed in the previous step.

Since the leaves of any ready-tree are quorums, and since quorums are closed under union, the prepare-footprint \( P^I_n \) is a quorum. By the definition of ready-tree, node v_n prepares b_n because of the vote messages sent by each member of \( P^I_n \).

For each window for intact set I and round n, we consider the abort-interval for intact set I of round n, which is the interval in which the nodes in \( P^I_n \) send the vote messages needed for v_n to prepare b_n. We write \( \delta^I_n \) for the duration of the abort-interval for intact set I of round n.

Since the prepare-footprint \( P^I_n \) of any round n in any execution of ASCP is finite, and since messages arrive in bounded time after GST expires, after the last node in the prepare footprint \( P^I_n \) sends its messages, the first node v_n \( \in I \) that prepares a ballot b_n with round n does so in bounded time. We write \( \delta^I_n \) for the finite delay that v_n takes in preparing b_n after every node in \( P^I_n \) has sent their messages, once GST has expired. Since the number of nodes in the universe is finite, all the \( \delta^I_n \) with any round n are bounded by some finite delay \( \delta^I \).

From now on we may omit the ‘for intact set I’ qualifier from the window, the abort-interval and the prepare-footprint of round n when the intact set is clear from the context.

**Proof of Lemma 12.** If every node in I decides a value before reaching some round n then we are done. Otherwise, assume that GST has expired and let \( \delta \) be the network delay after GST. Without loss of generality, assume that all faulty nodes have already stopped. Let n_0 be a round such that the abort-interval of round n_0 happens entirely after GST. (We accompany the proof with Figure 2 to illustrate the intervals that we describe in the remainder.) After the abort-interval of n_0, the first node v_{n_0} \( \in I \) to ever prepare a ballot b_{n_0} with round n_0 does it less than \( \delta^I \) time after the abort-interval of n_0 terminates. By Totality for intact nodes, each node in I that has not decided any value yet prepares some ballot with round n_0 during the window of round n_0, and thus by lines 15 \( \text{and} \) 17 of Algorithm 2.
each node in $I$ that has not decided any value yet sets its timeout to $F(n_0)$ in the period of time between the beginning of the abort-interval of $n_0$ and the end of the window of $n_0$. (Remember the primitive start-timer has the effect of restarting the timeout if a node’s local timer was already running.) This period of time has a duration bounded by $\delta_1^{n_0} + \delta_2^r + \delta_1$.

Now consider any round $n \geq n_0$ and assume that $F(n) > \delta_1^{n_0} + \delta_2^r + \delta_1$. As shown in Figure 2, the abort-interval of round $n + 1$ happens entirely after the window of round $n$ ends. Since the window of round $n + 1$ can only happen once the nodes in the abort-interval of round $n + 1$ have sent their messages, the window of round $n + 1$ does not overlap the window of round $n$. And since the delay function $F$ doubles its value with each increment of the round, the same happens with any subsequent window thereafter. By the definition of window of round $n$, no node in $I$ ever prepares a ballot with round equal or bigger than $n$ before the window starts. Since faulty nodes are stopped, and by lines 18–21 of Algorithm 2 no messages supporting to prepare a ballot of round $n + 1$ are ever sent before the abort-interval of round $n + 1$ starts. Therefore, each window of round $n + 1$ with $n \geq n_0$ happens after the immediately preceding window of round $n$, and no two of such windows overlap with each other, and the lemma holds.

**Corollary 29.** Let $m \geq n$ and let $b_{\text{max}}$ be the maximum ballot prepared by any node in $I$ before the abort-interval of round $m + 1$ starts. Every node in $I$ prepares $b_{\text{max}}$ before the abort-interval of round $m + 1$ starts.

**Proof.** Since the set-timeout interval is of length at most $\delta$, ballot $b_{\text{max}}$ is prepared by some
node in I within $\delta + \delta_1^P$ time after the set-timeout interval of round $m$ starts. By Lemma 7, every node in I prepares $b_{\text{max}}$ within $\delta + \delta_1^P + \delta_1$ time after the set-timeout interval of round $m$ starts. The required holds since $F(m) > \delta + \delta_1^P + \delta_1$.

**Proof of Theorem 13.** We prove that ASCP over $S$ enjoys each of the properties that define the specification of non-blocking Byzantine consensus for intact sets.

**Integrity:** Straightforward by definition since each node $v$ stops execution once $v$ decides some value.

**Agreement for intact sets:** Assume towards a contradiction that two nodes $v_1$ and $v_2$ in I decide respectively through ballots $b_1$ and $b_2$ such that $b_1.x \neq b_2.x$. Without loss of generality, we assume $b_1 < b_2$. By line 13 of Algorithm 2 nodes $v_1$ and $v_2$ respectively committed ballots $b_1$ and $b_2$. Since $v_1$ has committed ballot $b_1$ and by lines 13, 15 of Algorithm 4 node $v_1$ received $\text{READY}(b_1, true)$ from a quorum to which $v_1$ belongs. And since $v_2$ has committed ballot $b_2$ and by Lemma 9 we know that some correct node $v' \in I$ has prepared $b_2$. But this results in a contradiction by Lemma 10 and we are done.

**Weak validity for intact sets:** Assume all nodes are honest. To prove the first part of the property, let every node propose value $x$. Now assume towards a contradiction that a node $v_1\in I$ decides value $y \neq x$. By Lemma 27, the largest ballot $b_2$ prepared by any node $v_2\in I$ before $v_1$ decides $y$ is such that $b_2.x = y$. By Lemma 11 there exists a node that proposed value $y$. But this contradicts our assumption that every node proposed value $x$.

To prove the second part of the property, let a node $v_1\in I$ decide value $x$. By Lemma 27 the largest ballot $b$ prepared by any node $v_2\in I$ before $v_1$ decides $x$ is such that $b.x = x$. By Lemma 11 there exists a node that proposed value $x$, and we are done.

**Non-blocking for intact sets:** Assume all faulty nodes eventually stop. If some node in I decides some value, then by Lemma 7 and by Agreement for intact sets every other node in I will decide the same value within bounded time. Without loss of generality, assume that no node in I has decided any value and that GST has expires and every malicious node has stopped. By Lemma 9 no node can ever send a $\text{READY}(b, true)$ message for a ballot $b$ below and incompatible than any ballot that is already prepared, and thus no node can block itself by signalling that its willingness to commit a ballot that can no longer be committed. By Lemma 12 there exists a round $n$ such that any two windows of rounds bigger or equal than $n$ never overlap. Without loss of generality, assume that no node in I has decided any value before the window of round $n$. By Corollary 23 for every window of round $m$ bigger or equal than $n$, every node in I that has not decided any value yet prepares the same ballot $b_{\text{max}}$ before the abort-interval of round $m + 1$ starts. If every node in I updates its candidate ballot to $b_{\text{max}}$ before the abort-interval of round $m + 1$ starts, then every node in I will try to commit $b_{\text{max}}$ by lines 8, 12 of Algorithm 2 and they all will decide value $b_{\text{max}}.x$ in bounded time by lines 3, 4, 7, 9 and 13, 15 of Algorithm 4 and lines 13, 14, 13 of Algorithm 2. Otherwise, every node in I will update its candidate ballot to $\langle m + 1, b_{\text{max}}.x \rangle$ in the abort-interval of round $m + 1$, and all will try to commit $\langle m + 1, b_{\text{max}}.x \rangle$ and decide value $b_{\text{max}}.x$ in bounded time for reasons similar to the ones above.

Now we introduce a notation for sequence comprehension that we will use intensively in the remainder of the appendices. We write $[e_1, \ldots, e_m]$ for a sequence of elements, where each element $e_i$ is a ballot, an event or a message. We write $[]$ for the empty sequence and $\tau_1 \cdot \tau_2$ for the concatenation of sequences. The notation $[e(b), P(b)]$ stands for the sequence
\[ [e_l(b_1), \ldots, e_l(b_m)] \] where each element \( e_l(b_i) \) depends on a ballot \( b_i \) that meets predicate \( P \), and where the elements are ordered in ascending ballot order, this is, \( e_l(b_i) \) occurs before \( e_l(b_j) \) in \( [e_l(b), \ P(b)] \) iff \( i < j \).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Node $v_1$ & Node $v_2$ & Node $v_3$ & Node $v_4$
\hline
\begin{itemize}
\item propose(3)
\item vote-batch([b, b \leq (1, 3), false])
\item \texttt{VOTE}(b, false), b \leq (1, 3)
\end{itemize}
& \begin{itemize}
\item propose(3)
\item vote-batch([b, b \leq (1, 3), false])
\item \texttt{VOTE}(b, false), b \leq (1, 3)
\end{itemize}
& \begin{itemize}
\item \texttt{VOTE}(b, false), b \leq (1, 2)
\end{itemize}
& \begin{itemize}
\item propose(1)
\item vote-batch([b, b \leq (1, 1), false])
\item \texttt{VOTE}(0, false)
\end{itemize}
\hline
\begin{itemize}
\item \texttt{start\-timer}(F(1))
\item \texttt{READY}(b, false), b \leq (1, 2)
\end{itemize}
& \begin{itemize}
\item \texttt{start\-timer}(F(1))
\item \texttt{READY}(b, false), b \leq (1, 2)
\end{itemize}
& \begin{itemize}
\item \texttt{start\-timer}(F(1))
\item \texttt{READY}(0, false)
\end{itemize}
& \begin{itemize}
\item \texttt{start\-timer}(F(1))
\item \texttt{READY}(0, false)
\end{itemize}
\hline
\begin{itemize}
\item \texttt{deliver\-batch}([b, b \leq (1, 1), false])
\end{itemize}
& \begin{itemize}
\item \texttt{deliver\-batch}([b, b \leq (1, 1), false])
\end{itemize}
& \begin{itemize}
\item \texttt{deliver\-batch}([b, b \leq (1, 1), false])
\end{itemize}
& \begin{itemize}
\item \texttt{deliver\-batch}([b, b \leq (1, 1), false])
\end{itemize}
\hline
\begin{itemize}
\item \texttt{timeout}
\item \texttt{vote\-batch}([b \leq (2, 2), false])
\item \texttt{VOTE}(b, false), (1, 3) \leq b \leq (2, 2)
\end{itemize}
& \begin{itemize}
\item \texttt{timeout}
\item \texttt{vote\-batch}([b \leq (2, 2), false])
\item \texttt{VOTE}(b, false), (1, 3) \leq b \leq (2, 2)
\end{itemize}
& \begin{itemize}
\item \texttt{timeout}
\item \texttt{vote\-batch}([b \leq (2, 2), false])
\item \texttt{VOTE}(b, false), (1, 3) \leq b \leq (2, 2)
\end{itemize}
& \begin{itemize}
\item \texttt{timeout}
\item \texttt{vote\-batch}([b \leq (2, 2), false])
\item \texttt{VOTE}(b, false), (1, 3) \leq b \leq (2, 2)
\end{itemize}
\hline
\begin{itemize}
\item \texttt{start\-timer}(F(2))
\item \texttt{READY}(b, false), (1, 3) \leq b \leq (2, 2)
\end{itemize}
& \begin{itemize}
\item \texttt{start\-timer}(F(2))
\item \texttt{READY}(b, false), (1, 3) \leq b \leq (2, 2)
\end{itemize}
& \begin{itemize}
\item \texttt{start\-timer}(F(2))
\item \texttt{READY}(b, false), (1, 3) \leq b \leq (2, 2)
\end{itemize}
& \begin{itemize}
\item \texttt{start\-timer}(F(2))
\item \texttt{READY}(b, false), (1, 3) \leq b \leq (2, 2)
\end{itemize}
\hline
\begin{itemize}
\item \texttt{deliver\-batch}([b, (1, 3) \leq b \leq (2, 2), false])
\end{itemize}
& \begin{itemize}
\item \texttt{deliver\-batch}([b, (1, 3) \leq b \leq (2, 2), false])
\end{itemize}
& \begin{itemize}
\item \texttt{deliver\-batch}([b, (1, 3) \leq b \leq (2, 2), false])
\end{itemize}
& \begin{itemize}
\item \texttt{deliver\-batch}([b, (1, 3) \leq b \leq (2, 2), false])
\end{itemize}
\hline
\begin{itemize}
\item \texttt{READY}(2, true)
\item \texttt{VOTE}(2, true)
\end{itemize}
& \begin{itemize}
\item \texttt{READY}(2, true)
\item \texttt{VOTE}(2, true)
\end{itemize}
& \begin{itemize}
\item \texttt{READY}(2, true)
\item \texttt{VOTE}(2, true)
\end{itemize}
& \begin{itemize}
\item \texttt{READY}(2, true)
\item \texttt{VOTE}(2, true)
\end{itemize}
\hline
\begin{itemize}
\item \texttt{deliber\-batch}([2, true])
\end{itemize}
& \begin{itemize}
\item \texttt{deliber\-batch}([2, true])
\end{itemize}
& \begin{itemize}
\item \texttt{deliber\-batch}([2, true])
\end{itemize}
& \begin{itemize}
\item \texttt{deliber\-batch}([2, true])
\end{itemize}
\hline
\begin{itemize}
\item \texttt{decide}(2)
\end{itemize}
& \begin{itemize}
\item \texttt{decide}(2)
\end{itemize}
& \begin{itemize}
\item \texttt{decide}(2)
\end{itemize}
& \begin{itemize}
\item \texttt{decide}(2)
\end{itemize}
\hline
\end{tabular}
\caption{Execution of ASCP.}
\end{table}
Example 30. Consider the FBQS from Example 8 where the universe contains four nodes $v_1$ to $v_4$ and every set of three or more nodes is a quorum, and every set of two or more nodes is $v$-blocking for any $v \in V$. Consider an execution of ASCP where the node $v_3$ is faulty. The FBQS has the intact set $I = \{v_1, v_2, v_4\}$.

We assume that the set of values coincides with $\mathbb{N}^+$, which we write in boldface. In the execution, nodes $v_1$ and $v_2$ propose value 3, and node $v_4$ propose value 1. Faulty node $v_3$ sends a batch containing the messages $\text{VOTE}(0, \bot, \text{false})$ and $\text{VOTE}(1, 1, \text{false})$ to every correct node, thus helping them to prepare ballot $(1, 2)$. Since $(1, 2)$ exceeds $v_4$'s candidate ballot $(1, 1)$, node $v_4$ will try to commit both $(1, 1)$ and $(1, 2)$. However, neither of $v_1$ or $v_2$ will try to commit any ballot since $(1, 2)$ is smaller than their candidate ballot $(1, 3)$, and therefore no quorum exists that tries to commit a ballot. Consequently, the timeout at round 1 of every correct node will expire, and since all of them managed to prepare $(1, 2)$, they all will try to prepare the increased ballot $(2, 2)$, and will ultimately commit that ballot and decide value 2. Notice that value 2 was not proposed by any correct node, but nevertheless all of them agree on the same decision. To the nodes in $I$, node $v_3$ being faulty is indistinguishable from the situation where node $v_3$ is correct but slow, and it proposes 2. Therefore the nodes in $I$ cannot detect whether the decided value was proposed by some node in $I$ or not.

Figure 3 depicts the trace of the execution of ASCP described above. In each cell, we separate by a dashed line the events (above the line) that are triggered atomically, if any, from the batches of messages (below the line) that are sent by the node, if any. By BNS, the sending of every batch happens atomically with the events above the dashed line. At each cell, a node has received every batch in the rows above it. (For convenience, above the dashed line, we depict ‘batched’ events vote-batch and deliver-batch, which are defined in §6. Under the dashed line, we save the ‘batched’ send and receive primitives, and we depict one batch of messages per line.)

In the first row of Figure 3, the correct nodes $v_1$, $v_2$ and $v_4$ try to prepare the ballots that they propose (lines 3–7 of Algorithm 2), and lines 3–5 of Algorithm 1), which results in each of the $v_1$, $v_2$ and $v_4$ sending a $\text{VOTE}(b, \text{false})$ message for each $b \subseteq \{1, x\}$, where $x$ is respectively 3, 3 and 1. (Faulty node $v_3$ sends a $\text{VOTE}(b, \text{false})$ message for each $b \subseteq \{1, 2\}$.) Notice the use of the sequence comprehension notation to denote sequences of events triggered in a cell, as well as sequences of messages in a batch. To wit, node $v_1$ triggers propose(3) followed by the batched event vote-batch[$b, b \subseteq \{1, 3\}, \text{false}$], which stands for

[batches{0, 1}, vote(0, 1, false), ballots{1, 1}, vote(1, 1, false), 

balls[1, 2]; vote(1, 2, false)],

and it sends a batch with the sequence of messages $\text{VOTE}(b, \text{false}), b \subseteq \{1, 3\}$, which stands for

[\text{VOTE}(0, 1, false), VOTE(1, 1, false), VOTE(1, 2, false)].

In the second row of Figure 3, nodes $v_1$, $v_2$, and $v_4$ start the timer with delay $F(1)$, since there exist ballot $(1, 1)$ and open interval $[(0, \bot), (1, 1))$ such that the quorum $\{v_1, v_2, v_4\}$ receives from itself a message $\text{VOTE}(0, \bot, \text{false})$, and $[(0, \bot), (1, 1))$ is the singleton containing the null ballot $(0, \bot)$ (lines 15–17 of Algorithm 2). This means that all correct nodes receive from themselves vote messages that support preparing ballots with rounds bigger or equal than 1. In addition to this, nodes $v_1$ and $v_2$ send the batch $\text{READY}(b, \text{false}), b \subseteq (1, 2)$, since they receive a message $\text{VOTE}(b, \text{false})$ for each $b \subseteq (1, 2)$ from the quorum $\{v_1, v_2, v_3\}$, to which they belong (lines 7–9 of Algorithm 1). And similarly, node $v_4$ sends
a READY((0, ⊥), false), since it receives the message VOTE((0, ⊥), false) from all nodes, which constitute a quorum to which v4 belongs. Notice that node v4 cannot send READY((1, 1), false) because no quorum to which v4 belongs exists that sends VOTE((1, 1), false).

In the third row of of Figure 3, nodes v1, v2 and v4 deliver false for ballot ⟨1, 1⟩, since they receive the message READY((0, ⊥), false) from the quorum {v1, v2, v4} to which they all belong (lines 10–12 of Algorithm 1), which results in each of those nodes preparing ballot ⟨1, 1⟩ and triggering lines 8–12 of Algorithm 2. Since the prepared ballot ⟨1, 1⟩ reaches v4’s candidate ballot, then v4 triggers the batched event vote-batch([(1, 1)], true) and prepares a batch with the message VOTE((1, 1), true) that it will send later (lines 8–12 of Algorithm 2 and lines 9–10 of Algorithm 1). In addition to this, node v4 also prepares a batch with the message READY((1, 1), false) that it will also send later, since it receives READY((1, 1), false) from the v4-blocking set {v1, v2} (lines 10–12 of Algorithm 1). Recall that the rule in lines 10–12 of Algorithm 1 allows a node to send a ready message with some Boolean even if the node previously voted a different Boolean for the same ballot. Finally, node v4 sends the two batches before atomically.

In the fourth row of Figure 3 nodes v1, v2 and v4 deliver false for ballot ⟨1, 2⟩, since they receive a message READY(b, false) for each b ≤ ⟨1, 2⟩ from the quorum {v1, v2, v4} to which they all belong (lines 13–15 of Algorithm 1), which results in each of those nodes preparing ballot ⟨1, 2⟩ and triggering lines 8–12 of Algorithm 2. Since the prepared ballot ⟨1, 2⟩ exceeds v4’s candidate ballot, then v4 updates its candidate ballot to ⟨1, 2⟩ and triggers vote-batch([(1, 2)], true), which results in v4 sending VOTE((1, 2), true) (lines 8–12 of Algorithm 2). Recall that lines 8–12 of Algorithm 1 also prepares a batch with the message READY((1, 1), false) that it will also send later, since it receives READY((1, 1), false) from the v4-blocking set {v1, v2} (lines 10–12 of Algorithm 1). Finally, node v4 sends the two batches before atomically.

At this point no node can decide any value, because there exists not any ballot such that a quorum of nodes votes true for it, and the timeouts of all correct nodes will expire after F(1) time.

In the sixth row of Figure 3 nodes v1, v2 and v4 trigger timeout, and since they all prepared ballot ⟨1, 2⟩, they update their candidate ballot to ⟨2, 2⟩ and trigger the batched event vote-batch([(b, b ≤ ⟨2, 2⟩)], false) (lines 18–20 of Algorithm 2). Nodes v1, v2 and v4 send the batch [VOTE((2, 2), false), ⟨1, 3⟩ ≤ b ≤ ⟨2, 2⟩]], which contains infinitely many messages that are sent at once by BNS.

In the seventh row of Figure 3 nodes v1, v2 and v4 start the timer with delay F(2), since there exist ballot ⟨2, 2⟩ and open interval [(1, 2), ⟨2, 2⟩] such that the quorum {v1, v2, v4} receives from itself the infinitely many messages VOTE(b, false) with b ∈ [(1, 2), ⟨2, 2⟩)] (lines 15–17 of Algorithm 2), which are received at once by BNS. This means that all correct nodes receive from themselves vote messages that support preparing ballots with rounds bigger or equal than 2. Then, nodes v1, v2 and v4 send the batch [READY(b, false), ⟨1, 3⟩ ≤ b ≤ ⟨2, 2⟩]], since they receive a message VOTE(b, false) for each b such that ⟨1, 3⟩ ≤ b ≤ ⟨2, 2⟩] from the quorum {v1, v2, v3} to which they belong (lines 10–12 of Algorithm 1). The batch contains infinitely many messages, which are sent at once by BNS.

In the eighth row of Figure 3 nodes v1, v2 and v4 trigger vote-batch(b, ⟨1, 3⟩ ≤ b ≤ ⟨2, 2⟩], false), which stands for a vote false for each b below and incompatible than ⟨2, 2⟩ for which the node didn’t vote any Boolean yet, since they receive a message READY(b, false) for each of such b’s from the quorum {v1, v2, v4} to which they all belong (lines 15–16 of Algorithm 1). Since the prepared ballot ⟨2, 2⟩ reaches the candidate ballot of all correct nodes, they trigger the event vote-batch([(2, 2)], true) and send a VOTE((2, 2), true) (lines 8–12 of Algorithm 2 and lines 9–10 of Algorithm 1).

In the ninth row of Figure 3 nodes v1, v2 and v4 send the batch [READY⟨2, 2⟩, true]), since they all received VOTE((2, 2), true) from the quorum {v1, v2, v4} to which all belong.
Definition 31. We define \( \phi(\tau) = \max\{b \mid \forall b' \leq b. v.b'.deliver(false) \in \tau\} \) and let \( op \in \{send, receive\} \) and \( \mathcal{M} \in \{VOTE, READY\} \).

\[
\begin{align*}
\sigma([]) &= [] \\
\sigma(\tau \cdot [v\text{-}prepare(b)]) &= \sigma(\tau) \cdot v\text{-}vote\text{-}batch([b'], b' \leq b, false) \\
\sigma(\tau \cdot [v\text{-}commit(b)]) &= \sigma(\tau) \cdot v\text{-}vote\text{-}batch([b', \phi(\sigma(\tau)) \leq b, true]) \\
\sigma(\tau \cdot [v\text{-}prepared(b)]) &= \sigma(\tau) \cdot v\text{-}deliver\text{-}batch([b', b' \leq b) \\
& \quad \land \forall v.\text{deliver\text{-}batch}(bs) \in \sigma(\tau). (b', false) \notin \{b\}, false) \\
\sigma(\tau \cdot [v\text{-}committed(b)]) &= \sigma(\tau) \cdot \text{deliver\text{-}batch}([b, true]) \\
\sigma(\tau \cdot [v\text{-}op(VOTE(\text{PREP } b), u)]) &= \sigma(\tau) \cdot v\text{-}op\text{-}batch([\sigma(M', false), b' \leq b) \\
& \quad \land \forall v.\text{op\text{-}batch}(ms, u) \in \sigma(\tau). \sigma(M', a) \notin \{ms\}, u) \\
\sigma(\tau \cdot [v\text{-}op(READY(\text{PREP } b), u)]) &= \sigma(\tau) \cdot v\text{-}op\text{-}batch([VOTE(b', true), \phi(\sigma(\tau)) < b' \leq b], u) \\
\sigma(\tau \cdot [v\text{-}op(READY(cmt b), u)]) &= \sigma(\tau) \cdot v\text{-}op\text{-}batch([READY(b, true), u]) \\
\sigma(\tau \cdot [e]) &= \sigma(\tau) \cdot [e] \quad \text{otherwise}
\end{align*}
\]

Lemma 32. Let \( \mathcal{S} \) be an FBQS with some intact set \( I \), \( v \) be a node with \( v \in I \), and \( \tau \) be a trace entailed by an execution of CSBP. If \( \sigma(\tau) \) is a trace entailed by an execution of ASCP, then \( v\text{-}round, v\text{-}prepared, \) and \( v\text{-}candidate \) coincide in both executions.

Proof. We prove the statement by induction on \( \tau \). For the base case, it suffices to observe, that \( v\text{-}candidate, v\text{-}prepared, \) and \( v\text{-}round \) coincide when initialised in line 8 and 9 of Algorithm 1 and line 3 and 4 of Algorithm 2. For the step case \( \tau = \tau' \cdot e \) we consider only the interesting cases, where \( v\text{-}candidate, v\text{-}prepared, \) or \( \tau' \) are modified in line 6, line 11, line 16, line 19, or line 20 of Algorithm 2. For the other events in the concrete trace \( \tau \), the fields are not modified and the statement holds.

Case \( e = v\text{-}propose(x) \): By definition \( \sigma(\tau) \) contains \( v\text{-}propose(x) \), and by line 6 of Algorithm 1 and line 6 of Algorithm 2 \( v\text{-}candidate \) coincides.

Case \( e = v\text{-}prepared(b) \): By definition \( \sigma(\tau) \) contains \( v\text{-}deliver\text{-}batch([b', b' \leq b], false) \). By induction hypothesis \( v\text{-}prepared \) coincide, and therefore \( v\text{-}prepared \) < \( b \). Then by line 9 of Algorithm 1 and line 9 of Algorithm 2 \( v\text{-}prepared \) coincides. Again, by induction hypothesis \( v\text{-}candidate \) coincide, and therefore \( v\text{-}candidate \leq v\text{-}prepared \) coincide.

Case \( e = v\text{-}start\text{-}timer(n) \): By line 15 of Algorithm 2 \( v' \) contains \( v\text{-}receive(M_u(su, bw), u) \) from \( u \) with \( su \in \{\text{CMT, PREP}\} \) for a \( U \in \mathcal{Q} \) such that \( v \in U \) and for each \( u \in U \) exists \( M_u \in \{VOTE, READY\} \) and \( bw \in \mathcal{B} \) such that \( round < bw, n \).

Sub-case \( M_u(\text{PREP } b_u) \). By definition \( \sigma(\tau') \) contains a batch with \( M_u(b_u, false) \) for every \( b' \leq b_u \) and every \( M_u(\text{PREP } b_u) \).

Sub-case \( M_u(\text{CMT } b_u) \). By definition \( \sigma(\tau') \) contains a batch with \( M_u(b_u, true) \) for every \( M_u(\text{CMT } b_u) \).
By induction hypothesis, round and therefore round < \text{\texttt{b}}_w.n coincides. By line [16] of Algorithm [4] and by line [16] of Algorithm [2] round coincides.

**Case e = timeout:** By definition \(\sigma(\tau)\) contains \text{\texttt{v.timeout}}, and by induction hypothesis candidate, prepared, and round coincide. Then by line [16] and [20] of Algorithm [4] and line [16] and [20] of Algorithm [2] candidate, prepared, and round coincide.

\[ \Rightarrow \]

**Lemma 33.** Let \(S\) be an FBQS with some intact set \(I\), \(v\) be a node with \(v \in I\), and \(\tau\) be a trace entailed by an execution of CSCP. Then for every ballot \(b \in v.balloots-delivered-cmt\) (respectively, \(b \in v.balloots-readied-cmt\)) holds \(b \leq v.max-delivered-prep\) (respectively, \(b \leq v.max-delivered-prep\)).

**Proof.** Assume towards a contradiction, that there is a ballot \(b \in v.balloots-delivered-cmt\) (respectively, \(b \in v.balloots-readied-cmt\)) such that \(b > v.max-delivered-prep\) (respectively, \(b > v.max-delivered-prep\)). This is only possible if \(v\) sent \text{\texttt{READY}}(\text{\texttt{prep}} b') and \text{\texttt{READY}}(\text{\texttt{cmt}} b) to itself where \(b' < b\) (lines [15] and [16] and lines [27] and [28] of Algorithm [9]). But then \(v\) sent contradicting messages, which contradicts that \(v \in I\).

\[ \Rightarrow \]

**Lemma 34.** Let \(S\) be an FBQS with some intact set \(I\), \(v\) be a node with \(v \in I\), and \(\tau\) be a trace entailed by an execution of CSCP. If \(\sigma(\tau)\) is a trace entailed by an execution of ASCP, for every \(b > v.max-delivered-prep\) holds \(v.brs[b].delivered\) is false.

**Proof.** Assume towards a contradiction, that \(v.brs[b].delivered\) is true. By lines [18] and [15] of Algorithm [11] this is only possible if \(\sigma(\tau)\) contains an event \(v.send\text{\texttt{-batch}}(ms, u)\) with \text{\texttt{READY}}(b, a) \(\in ms\) for \(a \in \{true, false\}\) from every \(u\) in a quorum \(U\). Assume \text{\texttt{READY}}(b, true) \(\in ms\). Then by definition \(\sigma(\tau)\) contains \(v.send\text{\texttt{(READY}(cmt b), u)\) and by lines [27] and [28] of Algorithm [9] \(b \in v.balloots-delivered-cmt\). But then \(b \leq v.max-delivered-prep\) by Lemma [33]. As \(b > v.max-delivered-prep\), \(\sigma(\tau)\) contains an event \(v.send\text{\texttt{-batch}}(ms, u)\) with \text{\texttt{READY}}(b, false) \(\in ms\) and by lines [18] and [15] of Algorithm [11] this is only possible if \(\sigma(\tau)\) contains an event \(v.send\text{\texttt{-batch}}(ms, u)\) where \text{\texttt{READY}}(b, false) \(\in ms\) from every \(u\) in a quorum \(U\) where \(v \in U\). Again, by definition of \(\sigma\) and BNS this entails that \(\tau\) contains \(v.receive\text{\texttt{(READY}(prep b_u), u)\) for \(b' \leq b_u\) for every \(b' \leq b\). But then, by lines [14] and [15] of Algorithm [11] \(v.max-delivered-prep\) is assigned to \(b\) and this contradicts \(b > v.max-delivered-prep\).

\[ \Rightarrow \]

**Lemma 35.** Let \(S\) be an FBQS with some intact set \(I\), \(v\) be a node with \(v \in I\), and \(\tau\) be a trace entailed by an execution of CSCP. If \(\sigma(\tau)\) is a trace entailed by an execution of ASCP, for every \(b > v.max-readied-prep\) holds \(v.brs[b].ready\) is false.

**Proof.** Assume towards a contradiction, that \(v.brs[b].ready\) is true. By lines [10] and [12] of Algorithm [11] this is only possible if \(\sigma(\tau)\) contains an event \(v.send\text{\texttt{-batch}}(ms, u)\) with \text{\texttt{READY}}(b, a) \(\in ms\) for \(a \in \{true, false\}\) for every \(u\) in either a quorum \(U\) or a \(v\)-blocking set \(B\). Assume \text{\texttt{READY}}(b, true) \(\in ms\). Then by definition \(\sigma(\tau)\) contains \(v.send\text{\texttt{(READY}(cmt b), u)\) for every \(u\) and by lines [8] and [9] or lines [11] and [12] of Algorithm [11] \(b \in v.balloots-readied-cmt\). But then \(b \leq v.max-readied-prep\) by Lemma [33]. As \(b > v.max-readied-prep\), \(\sigma(\tau)\) contains an event \(v.send\text{\texttt{-batch}}(ms, u)\) with \text{\texttt{READY}}(b, false) \(\in ms\) for every \(u\) in either a quorum \(U\) or a \(v\)-blocking set \(B\). Assume \text{\texttt{READY}}(b, true) \(\in ms\). Then again, by definition of \(\sigma\) and BNS this entails that \(\tau\) contains \(v.receive\text{\texttt{(READY}(prep b_u), u)\) for \(b' \leq b_u\) for every \(b' \leq b\) for every \(u\) in either a quorum \(U\) or a \(v\)-blocking set \(B\). But then, by lines [8] and [9] or lines [11] and [12] of Algorithm [11] of Algorithm [3] \(v.max-readied-prep\) is assigned to \(b\) and this contradicts \(b > v.max-readied-prep\).

\[ \Rightarrow \]
Lemma 36. Let $S$ be an FBQS with some intact set $I$, $v$ be a node with $v \in I$, and $\tau$ be a trace entailed by an execution of CSCP. If $\sigma(\tau)$ is a trace entailed by an execution of ASCP and $b \notin \text{ballots-delivered-cmt}$ then $b.\text{delivered}$ is false.

Proof. Assumes towards a contradiction that $b.\text{delivered}$ is true. By lines 13–15 of Algorithm 4 and BNS, this is only possible if $\sigma(\tau)$ contains an event $v.\text{receive\nobreakdash-batch}(ms, u)$ with $\text{READY}(b, a) \in ms$ for $a \in \{\text{true}, \text{false}\}$ from a quorum $U$ such that $v \in U$. If $a$ is true, then by definition of $\sigma$, $\tau$ contains $v.\text{receive} (\text{READY}(\text{CMT} b), u)$ from a quorum $U$ such that $v \in U$. By lines 27 and 28 in Algorithm 4, $b \in \text{ballots\nobreakdash-delivered\nobreakdash-cmt}$ and this contradicts $b \notin \text{ballots\nobreakdash-delivered\nobreakdash-cmt}$. If $a$ is false, then $v.\text{receive}(\text{READY}(\text{PREP} b_n), u)$ from a quorum $U$ such that $v \in U$ and $b' \preceq b_n$ for every $b' \preceq b$. Then by lines 13 and 15 of Algorithm 3, $\text{max\nobreakdash-delivered\nobreakdash-prep}$ is assigned to $b$ and $b.\text{delivered}$ is true contradicts Lemma 34.

Lemma 37. Let $S$ be an FBQS with some intact set $I$ and $\tau$ be a trace entailed by an execution of CSCP. For every finite prefix $\tau'$ of the projected trace $\tau |_I$, the simulated $\rho' = \sigma(\tau')$ is the prefix of a trace entailed by an execution of ASCP.

Proof. We proceed by induction on the length of $\tau'$. The case $\tau' = []$ is trivial since $\sigma([]) = []$ is the prefix of any trace. We let $\tau' = \tau'_1 \cdot [e]$ and consider the following cases:

Case $e = v.\text{propose}(b)$. For any execution of the CSCP with trace $\tau'_1$, the prefix $\tau'_1$ contains either the event $v.\text{propose}(b,x)$ by lines 5 and 7 of Algorithm 4 or the event $v.\text{timeout}$ by lines 13 and 21 of Algorithm 4. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains either $v.\text{propose}(b,x)$ or $v.\text{timeout}$. By the induction hypothesis, the simulated prefix $\rho'_1$ is entailed by an execution of ASCP. Let the subtrace that simulates event $e$ be $\rho'_e = v.\text{vote\nobreakdash-batch}([b', b' \preceq b], \text{false})$. We show that $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP.

Sub-case $v$ proposes $b.x$. By lines 5–7 of Algorithm 4, $v$ triggers $v'.\text{vote} (\text{false})$ for every $b' \preceq (1, b, x)$ is in the execution of ASCP.

Sub-case $v$ triggers timeout. By line 21 of Algorithm 4, ballot $b$ equals $\text{candidate}$ and by Lemma 32 $\text{candidate}$ coincides. By lines 13–21 of Algorithm 4, $v.\text{vote} (\text{false})$ for every $b' \preceq b$ is in the execution of ASCP.

As $v$ triggered $\text{vote} (\text{false})$ for every $b' \preceq b$ in both cases. When batched, this results in the event $v.\text{vote\nobreakdash-batch}([b', b' \preceq b], \text{false})$, and $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP.

Case $v.\text{commit}(b)$. By lines 3 and 12 of Algorithm 4, for any execution of CSCP with trace $\tau'$, the prefix $\tau'_1$ contains the event $v.\text{prepared}(b)$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains the event $v.\text{deliver\nobreakdash-batch}([b', b' \preceq b], \text{false})$. By the induction hypothesis, the simulated prefix $\rho'_1$ is entailed by an execution of ASCP. Let the subtrace that simulates event $e$ be $\rho'_e = v.\text{vote\nobreakdash-batch}([b'', \phi(\tau'_1) < b'' \leq b], \text{true})$. We show that $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP. Fix a ballot $b''$ where $\phi(\tau'_1) < b'' \leq b$. By definition $\phi(\tau'_1)$ equals $\text{prepared}$ and for every $b''$ holds $\text{prepared} < b''$. Since $\rho'_1$ contains the event $v.\text{deliver\nobreakdash-batch}([b, b' \preceq b], \text{false})$, $v$ triggered $b'.\text{deliver\nobreakdash-false}$ for each $b' \preceq b$, and $\text{candidate}$ and $\text{prepared}$ coincide by Lemma 32 the guard at line 3 of Algorithm 4 holds after any of such executions of ASCP. We can reason in the same fashion for every $b''$ in $\phi(\tau'_1) < b'' \leq b$. By processing $b''$ in increasing order of ballots, $\text{candidate}$ increases monotonically and triggers $v.\text{vote}(b'', \text{true})$ for every ballot $b''$. As $v$ triggered $\text{vote}(b'', \text{true})$ for every $\phi(\tau'_1) < b'' \leq b$. When batched, this results in the event $v.\text{vote\nobreakdash-batch}([b'', \phi(\tau'_1) < b'' \leq b], \text{true})$, and therefore $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP.
Case $e = \texttt{v.prepared}(b)$. By lines [14] and [16] of Algorithm 3, for any execution of CSCP with trace $\tau'$ there exists a maximum $b$ such $b > \text{max-delivered-prep}$ and a quorum $U$ that contains node $v$ and for each $u \in U$ node $v$ received $\text{READY} (\text{PREP} b_u)$ where $b' \preceq b_u$ for every $b' \preceq b$. Therefore the prefix $\tau'_1$ contains for every $u \in U$ the event $v.\text{receive}(\text{READY} (\text{PREP} b_u), u)$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains the event $v.\text{receive-batch} ([\text{READY} (b'_u, \text{false}), b'_u \preceq b_u], u)$ for each $v.\text{receive}(\text{READY} (\text{PREP} b_u), u)$ that occurs in $\tau'_1$. By the induction hypothesis, the simulated prefix $\rho'_1$ is entailed by an execution of ASCP. Let the subtrace that simulates event $e$ be $\rho'_c = v.\text{deliver-batch} ([b''], b'' \preceq b \land \forall v.\text{deliver-batch} (bs) \in \sigma(\tau). (b', \text{false}) \not\in bs], \text{false})$. We show that $\rho'_1 \cdot \rho'_c$ is the prefix of a trace entailed by an execution of ASCP. Fix a ballot $b''$ where $b'' \preceq b$ and there is no batch $v.\text{deliverBatch} (bs)$ with $b'' \in bs$ in $\rho'_1$. For each node $u \in U$, we know that $\rho'_1$ contains an event $v.\text{receive-batch} ([\text{READY} (b'_u, \text{false}), b'_u \preceq b_u], u)$. As for every $b' \preceq b$ we know $b' \preceq b_u$, we have $b'' \preceq b_u$. Thus and by BNS, we know that $v$ received $\text{READY}(b'', \text{false})$ from $u$. By Lemma [34] and by $b > \text{max-delivered-prep}$, we know that $b'' \cdot \text{delivered}$ is false. Therefore, by lines [13] and [15] of Algorithm 1 triggers $v.b'.\text{deliver}(b'', \text{false})$. We can reason in the same fashion for every ballot $b'$ and batch the delivers in the event $v.\text{deliver-batch} ([b''], b'' \preceq b \land \forall v.\text{deliver-batch} (bs) \in \sigma(\tau). (b', \text{false}) \not\in bs], \text{false})$, and therefore $\rho'_1 \cdot \rho'_c$ is the prefix of a trace entailed by an execution of ASCP.

Case $e = \texttt{v.committed}(b)$. By lines [27] and [29] of Algorithm 3, for any execution of CSCP with trace $\tau'$, there exists a quorum $U$ that contains node $v$ which is such that $v$ receives $\text{READY}(\text{cmt} b)$ from every $u \in U$ and $b \not\in \text{ballots-delivered-cmt}$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains an event $v.\text{receive-batch} ([\text{READY} (b, \text{true}), u])$ for each $v.\text{receive}(\text{READY}(\text{cmt} b), u)$ that occurs in $\tau'_1$. By the induction hypothesis, the simulated prefix $\rho'_1$ is entailed by an execution of ASCP. Let the subtrace that simulates the event $e$ be $\rho'_c = v.\text{deliver-batch} ([b], \text{true})$. We show that $\rho'_1 \cdot \rho'_c$ is the prefix of a trace entailed by an execution of ASCP. As $v$ received $\text{READY}(b, \text{true})$ from a quorum $U$ where $v \in U$. As $b \not\in \text{delivered}$ by Lemma [39] deliver is false, and by lines [7] and [9] of Algorithm 1 triggers $v.\text{deliver}(b, \text{true})$. When batched, this results in the event $v.\text{deliver-batch} ([b], \text{true})$, and therefore $\rho'_1 \cdot \rho'_c$ is the prefix of a trace entailed by an execution of ASCP.

Case $e = \texttt{v.send}(\texttt{VOTE} (\text{PREP} b), u)$. By lines [4] and [7] of Algorithm 3, for any execution of CSCP with trace $\tau'$, the prefix $\tau'_1$ contains the event $\texttt{v.prepare}(b)$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains the event $\texttt{v.send-batch} ([\texttt{VOTE}(b', \text{false}), b' \preceq b \land \forall a \in \text{Bool}. \forall v.\text{send-batch} (ms, u) \in \sigma(\tau). M(b', a) \not\in ms], u)$. We show that $\rho'_1 \cdot \rho'_c$ is the prefix of a trace entailed by an execution of ASCP. Fix a ballot $b' \preceq b$ such that $\rho'_1$ does not contain the an event $\text{v.send-batch} (ms, u)$ with $\text{VOTE}(b', \text{false}) \not\in ms$. Then by lines [2] and [5] of Algorithm 1 we know that the Boolean voted is false. Hence, the condition in line [3] of the same figure is satisfied, and since $v.\texttt{v.send-batch} ([b', b' \not\preceq b], \text{false}), v \texttt{v.send-batch} (ms, u)$ with $\text{VOTE}(b', \text{false}) \not\in ms$ results in a trace entailed by an execution of ASCP by line [6] of the same figure. We can reason in the same fashion for every ballot $b' \preceq b$ and conclude together with BNS that $\rho'_1 \cdot \rho'_c$ is the prefix of a trace entailed by an execution of ASCP.

Case $e = \texttt{v.receive} (\texttt{VOTE}(\text{PREP} b), u)$. By assumption the network does not create or drop messages, hence $v$ receives $\texttt{VOTE}(\text{PREP} b)$ only after $u$ previously sent the same message and the prefix $\tau'_1$ contains the event $u.\texttt{send}(\texttt{VOTE}(\text{PREP} b), v)$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains an event with $u.\texttt{send-batch} ([\texttt{VOTE}(b', a), b' \preceq b], v)$ for $a \in \text{Bool}$. By the induction hypothesis, the simulated prefix $\rho'_1$ is entailed by
an execution of ASCP. Let the subtrace that simulates event $e$ be
\[
\rho'_e = v.\text{receive-batch}(\text{VOTE}(b', \text{false}), b' \leq b) \\
\wedge \forall a \in \text{Bool}. \forall v.\text{receive-batch}(ms, u) \in \sigma(\tau). \text{VOTE}(b', a) \notin ms], u)
\]
We show that $\rho'_1 = \rho'_e$ is the prefix of a trace entailed by an execution ASCP. By the ascending-ballot-order convention, it is enough to show that each $b' \leq b$, $v$ receives a batch with \text{VOTE}(b', a)$ for $a \in \text{Bool}$ exactly once in $\rho'$. For a fixed $b'$, an event with $v.\text{receive-batch}(ms, u)$ with \text{VOTE}(b', \text{false}) \in ms$ is in $\rho'_e$ only if $v.\text{receive}(\text{VOTE}(b', a), u)$ is not in $\rho'_1$. On the other hand, $u$ sent a batch event with $u.b'.\text{send}(\text{VOTE}(b', a), v)$ for each $b' \leq b$. Hence, $\rho'_1 : \rho'_e$ is the prefix of a trace entailed by an execution of ASCP.

Case $e = v.\text{send}(\text{READY}(\text{PREP } b), u)$. For any execution of the CSCP with trace $\tau'_1$, the node $v$ sends \text{READY}(\text{PREP } b)$ either after hearing from a quorum in line 10 of Algorithm 3 or after hearing from a $v$-blocking set in line 13 of the same figure. We consider both cases:

Sub-case $v$ sends \text{READY}(\text{PREP } b) after hearing from a quorum. By lines 15-10 of Algorithm 3 there exists a maximum ballot $b$ such that $\text{max-readied-prep } < b$ and there exists a quorum $U$ such that $v \in U$ and for every node $u \in U$ the node $v$ received \text{VOTE}(\text{prep } b_u)$ where $b' \leq b_u$ for every $b' \leq b$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains an event with $v.\text{receive-batch}(\text{VOTE}(b', a), b_u \leq b_u) \wedge \forall a \in \text{Bool}. \forall v.\text{receive-batch}(ms, u) \in \sigma(\tau). \text{VOTE}(b', a) \notin ms], u)$ for each node $u \in U$ and each event $v.\text{receive}(\text{VOTE}(\text{PREP } b_u), u)$. By the induction hypothesis, the simulated prefix $\rho'_1$ is a trace entailed by an execution of ASCP. Let the subtrace that simulates event $e$ be $\rho'_e = v.\text{send-batch}(\text{READY}(b', \text{false}), b' \leq b) \wedge \forall a \in \text{Bool}. \forall v.\text{send-batch}(ms, u) \in \sigma(\tau). \text{READY}(b', a) \notin ms], u)$. We show that $\rho'_1 : \rho'_e$ is the prefix of a trace entailed by an execution of ASCP. If b.n = 1 then by b maximal and $b \leq b_u$, $v$ received a batch with $b'.\text{VOTE}(b', \text{false})$ for $b' \leq b$ from every $u \in U$ such that $v \in U$. Then, by lines 15-10 in Algorithm 3 by BNS a batch with \text{READY}(b_j, a)$ is in $\rho'_1$. If b.n > 1, and as $v$ is correct, by lines 15-10 and lines 10-10 of Algorithm 3 $v$ prepared the ballot $b'_e = (b.n - 1, b'o.x)$ in the previous round. By lines 10-10 of Algorithm 3 $v$ sends \text{READY}(\text{PREP } b'_e)$. Hence by definition of $\sigma$, a batch with \text{READY}(b_j, a)$ is in $\rho'_1$ for every $b_j \leq b'_e$. It remains to show that a batch with $v.\text{send-batch}(\text{READY}(b_j, a), b'_e \leq b_j < b', u)$ is in $\rho'_1 : \rho'_e$. By assumption, for each node $u$ and $b'_e \leq b_u$ the node $v$ received \text{VOTE}(b'_u, a)$. It suffices to show that the node $v$ receives \text{VOTE}(b_j, a)$ from every $u \in U$ for every ballot $b_j$. Then, by lines 10-10 and BNS in Algorithm 3 a batch with \text{READY}(b_j, a)$ is in $\rho'_1$. By Lemma 32 and $b' > b > \text{max-readied-prep }$ ready is false for $b'$.

Sub-case $v$ sends \text{READY}(\text{PREP } b) after hearing from a $v$-blocking set. By lines 11-12 of Algorithm 3 there exists a maximum ballot $b$ such that $\text{max-readied-prep } < b$ and there exists a $v$-blocking set $B$ such that for every $u \in B$ the node $v$ received \text{READY}(\text{PREP } b_u)$ where $b' \leq b_u$ for every $b' \leq b$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains an event $v.\text{receive-batch}(\text{READY}(b'_u, \text{false}), b'_u \leq b_u)]$, u) for each node $u \in B$ and each event $v.\text{receive}(\text{READY}(\text{PREP } b_u), u)$. By the induction hypothesis, the simulated prefix $\rho'_1$ is entailed by an execution of ASCP. Let the subtrace that simulates event $e$ be $\rho'_e = v.\text{send-batch}(\text{READY}(b', \text{false}), b' \leq b) \wedge \forall v.\text{send-batch}(ms, u) \in \sigma(\tau). \text{READY}(b', a) \notin ms], u)$. Fix a ballot $b'' \in B$ such that $b'' \leq b$ and for a batch with $v.b''.\text{send}(\text{READY}(b'', \text{false}), u) \notin \sigma(\tau'_1)$. By Lemma 33 and $b'' > b > \text{max-readied-prep }$ ready is false for $b''$. We have to show that $v$ received \text{READY}(b'', \text{false})$ from every $u$ in the $v$-blocking set $B$. Then by lines 10-12 in Algorithm 3 $v$ send \text{READY}(b'', \text{false})$ to $u$. As for every $b' \leq b$ we know $b' \leq b_u$, we have $b'' \leq b_u$. Thus, we know that a batch with \text{READY}(b'', \text{false})$ is in $\rho'_1$. 
Both cases show that for the subtrace $\rho'$ that simulates event $e$, the trace $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP.

**Case e = v.receive(READY(PREP b), u).** Analogue to case $v.receive(VOTE(PREP b), u)$.

**Case e = v.send(VOTE(CMT b), u).** By lines 17 and 20 of Algorithm 3, for any execution of ASCP with trace $\tau$ the prefix $\tau'$ contains the event $v.commit(b)$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains the event $v.vote$-batch($b', \phi(\rho'_1) < b', b [true]$). By the induction hypothesis, the simulated prefix $\rho'_1$ is the prefix of a trace entailed by an execution of ASCP. Let the subtrace that simulates event $e$ be $\rho'_e = v.vote$-batch($v.vote(true), \phi(\rho'_1) < b', b, u$). We show that $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP. Fix a ballot $b'$ such that $\rho'_1$ does not contain a $v.vote$-batch($ms, u$) with $v.vote(b', true) \in ms$. By line 18 of Algorithm 3 we know that $b' = max_voted$-prep, and by lines 17 of the same figure, $v$ did not send $v.vote$-prep $b''$ for any $b'' > max_voted$-prep. By definition of $\sigma$, a batch event with $v.b'.send($v.vote($b', false), u$) \not\in \sigma(\tau)$ for $b' > b$. As $b' \not\in ballots$-voted-cmt, again by definition of $\sigma$, a batch event with $v.b'.send($v.vote($b', true), u$) \not\in \sigma(\tau)$. Therefore we know that the Boolean $v.vote$ is false. Hence, the condition in line 18 of the same figure is satisfied. Since $v.vote$-batch($b', \phi(\rho'_1) < b', b, u$), $v$ sends $v.vote(true)$, appending an event $v.send$-batch($ms, u$) with $v.vote(b', true) \in ms$ results in the prefix of a trace entailed by an execution of ASCP by line 19. We can reason in the same fashion for every $b'$ in $\phi(\sigma(\tau)) < b' \leq b$, and therefore and by BNS $\rho'_1 \cdot \rho'_e$ is a trace entailed by an execution of ASCP.

**Case e = v.receive(VOTE(CMT b), u).** By assumption the network does not create or drop messages, hence $v$ receives $v.vote$-batch($v.vote(true)$, u) only after $u$ previously sent the same message and the prefix $\tau'_1$ contains the event $u.send(v.vote(CMT b), v)$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains an event with $u.send$-batch($v.vote(b, false)$, v). By induction hypothesis $\rho'_1$ is the prefix of a trace entailed an execution of ASCP. Let the subtrace that simulates event $e$ be $\rho'_e = v.vote$-batch($v.vote(true), b' = \{b' \mid \phi(\rho'_1) < b', b\}, u)$. We show that $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP. As $u$ sends $v.vote$-batch($v.vote(true)$, v) to $v$, we know that $v$ receives $v.vote$-batch($v.vote(true)$) exactly once for every $b' \in \{b' \mid \phi(\rho'_1) < b' \leq b\}$ and the batch is exactly once in $\rho'$. Hence, $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP.

**Case e = v.send(READY(CMT b), u).** For any execution of ASCP with trace $\tau'_1$, the node $v$ sends $READY$($CMT b$) either after hearing from a quorum in line 21 of Algorithm 3 or after hearing from a $v$-blocking set in line 22 of the same figure. We consider both cases:

**Sub-case v sends READY(CMT b) after hearing from a quorum.** By lines 21 and 23 of Algorithm 3 there exists a quorum $Q$ such that $v \in Q$ and for every node $u \in Q$ the node $v$ received $v.vote$-batch($v.vote(CMT b), b$) and $b \not\in ready$ and $b \geq max_ready$-prep. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains an event $v.vote$-batch($v.vote(b, a), \phi(\rho'_1) < b', b, u$) and for every $u \in Q$ such that $v \in Q$ for every event $v.receive$($v.vote(CMT b), u$). By the induction hypothesis, the simulated prefix $\rho'_1$ is a trace entailed by an execution of ASCP. Let the subtrace that simulates event $e$ be $v$-send$($v.vote(true), true), u$. We show that $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP. If $v$ received $v.vote(b, true)$ from a quorum $Q$ such that $v \in Q$ and $readied$ in Algorithm 1 is false, then by lines 18 in Algorithm 1 a batch with $v.vote(b, true)$ is in $\tau'_1$. Assume a $v.receive$-batch($ms, u$) with $v.vote(b, false) \in ms$ is in $\rho'_1$. By definition of $\sigma$ this is only possible, if $v$ received $v.receive$-prep $b_u$ for some $b_u > b$. As $v$ processed $v.receive$($v.vote(CMT b), u$) and as $v$ is correct, $v$ cannot have processed $v.receive$($v.vote(true), u$). Hence $v$ received $v.vote(b, true)$ from $u$, and as $v$ has not received $v.vote(b, false)$, readied in Algorithm 1 is false.
Sub-case $v$ sends $\text{READY}(\text{CMT } b)$ after hearing from a $v$-blocking set. By lines 24–26 of Algorithm 4 there exists a maximum ballot $b$ and a $v$-blocking set $B$ such that $v$ received $\text{READY}(\text{CMT } b)$ from every node $u \in B$ and $b \notin \text{readied}$ and $b \geq \text{max-readied-prep}$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains the event $v.b.\text{receive}(\text{READY}(b, a), u)$ for $a \in \{\text{true}, \text{false}\}$ for every $u \in B$. By the induction hypothesis, $\rho'_1$ is the prefix of a trace entailed by an execution of ASCP. Let the subtrace that simulates $e$ be $v.\text{send-batch}([\text{READY}(b, \text{true})], u)$. We show that $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP. We have to show that $v$ received $\text{READY}(b, \text{true})$ from a $v$-blocking set $B$ and $\text{readied}$ in Algorithm 1 is false. Then by lines 14–17 of Algorithm 1, $v.\text{send-batch}(\text{msgs}, u)$ with $\text{READY}(b, \text{false})$ is in $\rho'_1$. Assume $v$ received $\text{READY}(b, \text{false})$ from $u$. By definition of $\sigma$ this is only possible, if $v$ received $\text{READY}(\text{PREP } b_u)$ for some $b_u > b$. As $v$ processed $v.\text{receive}(\text{READY}(\text{CMT } b), u)$ and as $v$ is correct, $v$ cannot have processed $v.\text{receive}(\text{READY}(\text{PREP } b_u), u)$. Hence $v$ received $\text{READY}(b, \text{true})$ from $u$, and as $v$ has not received $\text{VOTE}(b, \text{false})$, $\text{readied}$ in Algorithm 1 is false.

Both cases show that for the subtrace $\rho'_e$ that simulates event $e$, the trace $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP.

Case $e = v.\text{receive}(\text{READY}(\text{CMT } b), u)$. Analogue to case $v.\text{receive}(\text{VOTE}(\text{CMT } b), u)$.

Case $e = v.\text{propose}(x)$. Straightforward by definition of $\sigma$, since $\tau$ contains $v.\text{propose}(x)$ iff the simulated $\rho = \sigma(\tau)$ contains $v.\text{propose}(x)$.

Case $e = v.\text{decide}(x)$. By lines 13–14 in Algorithm 4, for any execution of CSCP with trace $\tau'$ the node $v$ decides value $x$ only after $v$ triggers committed$(b)$ for a ballot $b$ with $b.x = x$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains the event $v.\text{deliver-batch}([b], \text{true})$. By induction hypothesis $\rho'_1$, the simulated prefix $\rho'_1$ is entailed by an execution of ASCP. Let the subtrace that simulates event $e$ be $\rho'_e = [v.\text{decide}(x)]$. We show that $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP. As $v.\text{deliver-batch}([b], \text{true})$, $v$ triggered deliver$(\text{true})$ for ballot $b$, by lines 13 and 14 of Algorithm 2, $v.\text{decide}(x)$ is in the execution of ASCP and $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP.

Case $e = v.\text{start}-\text{timer}(n)$: By lines 15–17 of Algorithm 4 for any execution of CSCP with trace $\tau'_1$, there exists a quorum $U$ which is such that $v$ receives $M(s_b_u)$ where $M \in \{\text{VOTE}, \text{READY}\}$ and $s \in \{\text{CMT, PREP}\}$ from every $u \in U$ and round $< b_u.n$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains the event $v.\text{receive-batch}(M'(b', \text{false}), b' \leq b_u, u)$ for every $v.\text{receive}(M(\text{PREP } b_u), u)$ that occurs in $\tau'_1$, or $v.\text{receive-batch}(M(b_u, \text{true}), u)$ for every $v.\text{receive}(M(\text{CMT } b_u), u)$ that occurs in $\tau'_1$. By induction hypothesis $\rho'_1$ is the prefix of a trace entailed by an execution of ASCP. Let the subtrace that simulates the event $e$ be $\rho'_e = [v.\text{start}-\text{timer}(n)]$. We show that $\rho'_1 \cdot \rho'_e$ is the prefix of trace entailed by an execution of ASCP.

By Lemma 32 coincides round and by assumption $n < b_u.n$ round holds. We have distinguished two cases:

Sub-case $v$ received $M(\text{PREP } b_u)$ from $u$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains an event with $v.\text{receive-batch}(M_u(b'_u, \text{false}), b'_u \leq b_u, u)$ and every $M_u(\text{PREP } b_u)$.

Sub-case $M_u(\text{CMT } b_u)$. The definition of $\sigma$ entails that the simulated prefix $\rho'_1 = \sigma(\tau'_1)$ contains a $v.\text{receive-batch}(M_u(b_u, \text{true}), u)$ for every $M_u(\text{CMT } b_u)$.

Combining the cases leads to the conditions in line 15 in Algorithm 2 satisfied. Thus, by line 17 of the same figure, $v.\text{start}-\text{timer}(n)$ is in the execution of ASCP and $\rho'_1 \cdot \rho'_e$ is the prefix of a trace entailed by an execution of ASCP.
Case \( e = v.\text{timeout} \): Straightforward by definition of \( \sigma \), since \( \tau \) contains \( v.\text{timeout} \) iff the simulated \( \rho = \sigma(\tau) \) contains \( v.\text{timeout} \).

Proof of Theorem 14. Let \( \tau \) be the trace entailed by an execution of CSCP over \( S \). We prove that there exists a trace \( \rho \) entailed by an execution of ASCP such that \( H(\tau|_I) = H(\rho|_I) \) by reductio ad absurdum. Assume towards a contradiction that for all traces \( \rho \), if \( H(\tau|_I) = H(\rho|_I) \) then \( \rho \) is not a trace entailed by an execution of ASCP. Fix the trace \( \rho \) to be \( \sigma(\tau|_I) \), which entails that \( H(\tau|_I) = H(\rho|_I) \) by definition of \( \sigma \) and \( H \). Since the number of events in a trace entailed by ASCP is bounded by \( \omega \), we denote the \( i \)th event of \( \rho \) as \( e_i \), with \( i \) a natural number. Since \( \rho \) is not a trace entailed by an execution of ASCP by assumptions, there exists \( i \geq 0 \) such that the prefix \([e_0, \ldots, e_i]\) of \( \rho \) is a trace entailed by ASCP, but the prefix \([e_0, \ldots, e_{i+1}]\) of \( \rho \) is not a trace entailed by ASCP. Since \( \sigma \) maps one event of a concrete trace into one event of an abstract trace, there exists a finite prefix \( \tau' \) of \( \tau|_I \) such that \( \sigma(\tau') = [e_0, \ldots, e_{i+1}] \). But this leads to a contradiction because \( \sigma(\tau') \) is a trace of an execution of ASCP by Lemma 37. Therefore, there exists a trace \( \rho \) entailed by an execution of ASCP such that \( H(\tau|_I) = H(\rho|_I) \), and we are done.

Proof of Corollary 15. Let \( \tau \) be the trace entailed by an execution of CSCP over \( S \). Assume towards a contradiction that the execution does not satisfy some of the properties of Integrity, Agreement for intact sets, Weak validity for intact sets, or Non-blocking for intact sets. By Theorem 14 there exists a trace \( \rho \) entailed by an execution of ASCP over \( S \) such that \( H(\tau|_I) = H(\rho|_I) \). By definition of history, \( H(\tau|_I) \) and \( H(\rho|_I) \) coincide in their respective propose and decide events. Since \( \rho|_I \) is entailed by an execution of ASCP, this execution fails to satisfy some of the properties of Integrity, Agreement for intact sets, Weak validity for intact sets, or Non-blocking for intact sets, which contradicts Theorem 13.
Example 38.

| Node \(v_1\) | Node \(v_2\) | Node \(v_3\) | Node \(v_4\) |
|-------------|-------------|-------------|-------------|
| propose(3)  | propose(3)  | propose(1)  | propose(1) |
| brs.prepare((1, 3)) | brs.prepare((1, 3)) | brs.prepare((1, 1)) | brs.prepare((1, 1)) |
| VOTE(PREP (1, 3)) | VOTE(PREP (1, 3)) | VOTE(PREP (1, 1)) | VOTE(PREP (1, 1)) |
| start-timer(F(1)) | start-timer(F(1)) | start-timer(F(1)) | start-timer(F(1)) |
| READY(PREP (1, 2)) | READY(PREP (1, 2)) | READY(PREP (1, 1)) | READY(PREP (1, 1)) |
| brs.prepared((1, 1)) | brs.prepared((1, 1)) | brs.prepared((1, 1)) | brs.prepared((1, 1)) |
| brs.prepared((1, 2)) | brs.prepared((1, 2)) | brs.prepared((1, 2)) | brs.prepared((1, 2)) |
| brs.committed((2, 2)) | brs.committed((2, 2)) | brs.committed((2, 2)) | brs.committed((2, 2)) |
| decide(2) | decide(2) | decide(2) | decide(2) |

Figure 4 Execution of CSCP.

intermixed in such a way that the assumptions on atomic and batched semantics of \(§3\) are met. It is routine to check that \(H(\tau|_{1,2,4}) = H(\rho|_{1,2,4})\) and that \(\rho|_{1,2,4} = \sigma(\tau|_{1,2,4})\).

Proofs in \(§7\)

Since in a subjective FBQS \(\{S_v\}_{v \in \mathcal{V}_{sk}}\) each node \(v\) runs the consensus protocols according to its own view \(S_v\), the set of quorums \(Q\) in the pseudo-code of Algorithm 1 and Algorithm 2–4 coincides with the set of quorums in the view \(S_v\). The notion of \(v\)-blocking set in these figures also coincides with the usual notion in the view \(S_v\). To fix terminology, we say that \(U\) is a quorum known by \(v\) in a subjective FBQS \(\{S_v\}_{v \in \mathcal{V}_{sk}}\) if \(U\) is a quorum in \(v\)’s view \(S_v\), i.e., \(\forall v' \in U, \exists q \in S_v(v'), q \subseteq U\). We say that a set \(B\) is \(v\)-blocking in a subjective FBQS \(\{S_v\}_{v \in \mathcal{V}_{sk}}\) if \(B\) overlaps each of \(v\)’s slices in \(v\)’s view \(S_v\), i.e., \(\forall q \in S_v(v), q \cap B \neq \emptyset\).

Proof of Lemma \([16]\) Since the nodes in \(I\) are correct, they never equivocate their quorum slices, and thus \(I\) is a quorum in each view \(S_v\) with \(v \in I\) if \(I\) is a quorum in \(\{S_v\}_{v \in \mathcal{V}_{sk}}\). For the same reason, all the views \(S_v|_I\) with \(v \in I\) coincide with each other. Thus, every node in \(I\) is intertwined with each other in the all the views \(S_v\) with \(v \in I\) if every node in \(I\) is intertwined with each other in \(\{S_v\}_{v \in \mathcal{V}_{sk}}\), and the lemma holds.

Proof of Lemma \([17]\) Since \(U_1\) and \(U_2\) are in \(\bigcup_{v \in \mathcal{V}_{sk}} S_v\), there exist correct nodes \(v_1\) and \(v_2\) such that \(U_1\) is a quorum known by \(v_1\) and \(U_2\) is a quorum known by \(v_2\). By Lemma \([21]\) the sets \(U_1 \cap I\) and \(U_2 \cap I\) are quorums in \(S_{v_1}|_I\) and \(S_{v_2}|_I\) respectively. Since the nodes
in $I$ never equivocate their slices, all the views $S_v|_I$ with $v \in I$ coincide with each other. Since every node is intertwined with each other in such a view, every two quorums have non-empty intersection by Lemma 22. Therefore, $(U_1 \cap I) \cap (U_2 \cap I) = (U_1 \cap U_2) \cap I \neq \emptyset$, and the intersection $U_1 \cap U_2$ contains some node in $I$.

Lemma 39. Assume that $V_{vak}$ is the set of correct nodes and let $\{S_v\}_{v \in V_{vak}}$ be a subjective FBQS with some intact set $I$. Let $v \in I$. Then, no $v$-blocking set $B$ exists such that $B \cap I = \emptyset$.

Proof. Straightforward by the definition of $v$-blocking in $\{S_v\}_{v \in V_{vak}}$ and by Lemma 4.

Lemma 40 (Analogous to Lemma 46 in [8] for intact sets). Let $\{S_v\}_{v \in V_{vak}}$ be a subjective FBQS and $t$ be a tag, and consider an execution of the instance for $t$ of $FV$ over $\{S_v\}_{v \in V_{vak}}$. Let $I$ be an intact set in $\{S_v\}_{v \in V_{vak}}$. The first node $v \in I$ that sends a $\text{READY}(t,a)$ message first needs to receive a $\text{VOTE}(t,a)$ message from every member of a quorum $U$ known by $v$ and to which $v$ belongs.

Proof. Let $v$ be any node in $I$. By Lemma 40 no $v$-blocking set $B$ exists such that $B \cap I = \emptyset$. Therefore, the first node $v \in I$ that sends a $\text{READY}(t,a)$ message does it through lines 7–9 of Algorithm 1, which means that $v$ received $\text{VOTE}(t,a)$ messages from every member of a quorum $U$ known by $v$ and to which $v$ belongs.

Lemma 41. Assume that $V_{vak}$ is the set of correct nodes and let $\{S_v\}_{v \in V_{vak}}$ be a subjective FBQS with some intact set $I$. Consider a set $B$ of nodes. If $B$ is not $v$-blocking in $\{S_v\}_{v \in V_{vak}}$ for any $v \in I \setminus B$, then either $B \supseteq I$ or $I \cap B$ is a quorum in $S_v|_I$ for every $v \in B$.

Proof. Assume $B$ is not $v$-blocking in $\{S_v\}_{v \in V_{vak}}$ for any $v \in I \setminus B$. If $B \supseteq I$ then we are done. Otherwise, for every node $v$ in $I \setminus B$, there exists a slice $q \in S_v(v)$ such that $q \cap B = \emptyset$. We know that $q \cap I = \emptyset$ since $v \in q$ by definition of subjective FBQS. We also know that $q \cap I = S_v|_I(v)$ by definition of $S_v|_I$, and since $q \cap B = \emptyset$, the intersection $q \cap I$ is a subset of $I \setminus B$. Since for each node $v \in I$ there exists a slice $q \in S_v(v)$ such that $q \cap I$ is a subset of $I \setminus B$, the set $I \setminus B$ is a quorum in $S_v|_I$. Since the nodes in $I$ are correct and never equivocate their slices, we know that $S_{v_1}|_I = S_{v_2}|_I$ for every two nodes $v_1$ and $v_2$ in $I$, and the required holds.

Lemma 42 (Analogous to Lemma 23 of [8] for intact sets). Let $\{S_v\}_{v \in V_{vak}}$ be a subjective FBQS and consider an execution of ASCP over $\{S_v\}_{v \in V_{vak}}$. Let $I$ be an intact set in $\{S_v\}_{v \in V_{vak}}$ and $b$ be a ballot. If two nodes in $I$ send respectively messages $\text{READY}(b,a)$ and $\text{READY}(b,a')$, then $a = a'$.

Proof. Assume that two nodes in $I$ send respectively messages $\text{READY}(b,a)$ and $\text{READY}(b,a')$. By Lemma 40 some node $v \in I$ has received $\text{VOTE}(b,a)$ from a quorum $U$ known by $v$ to which $v$ belongs, and some node $v' \in I$ has received $\text{VOTE}(b,a')$ from a quorum $U'$ known by $v'$ to which $v'$ belongs. By Lemma 11 the intersection $U \cap U'$ contains some node in $I$, so that this node has sent $\text{VOTE}(t,a)$ and $\text{VOTE}(t,a')$. But due to the use of the guard variable voted in lines 2 and 10 of Algorithm 1, a node can only vote for one value per tag, and thus it cannot vote different values for the same tag. Hence, $a = a'$.

Lemma 43 (Analogous to Lemma 24 in [8] for intact sets). Assume that $V_{vak}$ is the set of correct nodes and let $\{S_v\}_{v \in V_{vak}}$ be a subjective FBQS with some intact set $I$. Assume that $I = V^+ \cup V^-$ and for some quorum $U$ known by a node $v$ we have $U \cap I \subseteq V^+$. Then either $V^- = \emptyset$ or there exists some node $v' \in V^-$ such that $V^+$ is $v'$-blocking in $\{S_v\}_{v \in V_{vak}}$. 
Proof. Since every one in $I$ is correct, all the views $S_v|_I$ with $v \in I$ coincide with each other. Since $V^+$ and $V^-$ only contain nodes in $I$, they both lie within the projection $S_v|_I$. Thus, for every $v \in V^+$, $V^-$ is $v$-blocking in $\{S_v\}_{v \in V_{t, a}}$ iff $V^-$ is $v$-blocking in any $S'_{v'}|_I$ with $v' \in I$. Assume that $V^+$ is not $v$-blocking in $\{S_v\}_{v \in V_{t, a}}$ for any $v \in V^-$. By Lemma 24 either $V^- = \emptyset$ or $V^-$ is a quorum in $S|_I$. In the former case we are done, while in the latter we get a contradiction as follows. By Lemma 21 the intersection $U \cap I$ is a quorum in $S|_I$. Since every two quorums in $S|_I$ have non-empty intersection by Lemma 22 we have $(U \cap I) \cap V^- \neq \emptyset$. But this is impossible, since $U \cap I \subseteq V^+$ and $V^+ \cap V^- = \emptyset$.

Lemma 44. Let $\{S_v\}_{v \in V_{t, a}}$ be a subjective FBQS and $t$ be a tag. The instance for $t$ of FV over $\{S_v\}_{v \in V_{t, a}}$ satisfies the specification of reliable Byzantine voting for intact sets.

Proof. We prove that the instance for tag $t$ of FV over $\{S_v\}_{v \in V_{t, a}}$ enjoys each of the properties that define the specification of reliable Byzantine voting for intact sets.

No duplication: Straightforward by the use of the guard variable delivered in line 28 of Algorithm 4.

Totality for intact sets: Assume some node $v$ in $I$ delivers a value $a$ for tag $t$. By the condition in line 7 of Algorithm 4, the node $v$ has received READY$(t, a)$ messages from every member in a quorum $U$ known by $v$. Since $U \cap I$ contains only correct nodes, these nodes send READY$(t, a)$ messages to every node. By the condition in line 10 of Algorithm 4 any correct node $v'$ sends READY$(t, a)$ messages if it receives READY$(t, a)$ from every member in a $v'$-blocking set. Hence, the READY$(t, a)$ messages from the nodes in $U \cap I$ may convince additional correct nodes to send READY$(t, a)$ messages to every node. Let these additional correct nodes send READY$(t, a)$ messages until a point is reached at which no further nodes in $I$ can send READY$(t, a)$ messages. At this point, let $V^+$ be the set of nodes in $I$ that sent READY$(t, a)$ messages (where $U \cap I \subseteq V^+$), and let $V^- = I \setminus V^+$. By Lemma 12 the nodes in $V^-$ did not send any READY$(t, \_)$ messages at all. The set $V^+$ cannot be $v'$-blocking for any node $v'$ in $V^-$, or else more nodes in $I$ could come to send READY$(t, a)$ messages. Then by Lemma 13 we have $V^- = \emptyset$, meaning that every node in $I$ has sent READY$(t, a)$ messages. Since $I$ is a quorum known by every node in $I$, all the nodes in $I$ will eventually deliver a Boolean for tag $t$ due to the condition in line 7 of Algorithm 4.

Consistency for intertwined nodes: Assume that two nodes $v$ and $v'$ in an intact set $I$ deliver values $a$ and $a'$ for tag $t$ respectively. By the condition in line 13 of Algorithm 4, the nodes received READY$(t, a)$ messages from a quorum known by $v$, respectively, READY$(t, a')$ messages from a quorum known by $v'$. Since the two nodes are intertwined, there is a correct node $u$ in the intersection of the two quorums, which sent both READY$(t, a)$ and READY$(t, a')$. By the use of the guard variable readied in line 7 of Algorithm 4, node $u$ can only send one and the same ready message to every other node, and thus $a = a'$ as required.

Validity for intact sets: Assume every node in an intact set $I$ votes for value $a$. Since $I$ is a quorum known by every member of $I$, every node in $I$ will eventually send READY$(t, a)$ by the condition in line 7 of Algorithm 4. By Lemma 12 these messages cannot carry a value different from $a$. Then by the condition in line 13 of Algorithm 4 every node in $I$ will eventually deliver the value $a$ for tag $t$. Due to Consistency for intact sets, no node delivers a value different from $a$.

Lemma 45. Let $\{S_v\}_{v \in V_{t, a}}$ be a subjective FBQS and $t$ be a tag, and consider an execution of the instance for $t$ of FV over $\{S_v\}_{v \in V_{t, a}}$. Let $I$ be an intact set in $\{S_v\}_{v \in V_{t, a}}$ and assume that GST has expired. If a node $v \in I$ delivers a voting value then every node in $I$ will deliver a voting value within some bounded time.
Proof. Analogous to the proof of Lemma 37.

Let \( \{ S_v \}_{v \in V_a} \) be a subjective FBQS and \( t \) be a tag, and consider an execution of the instance for tag \( t \) of FV over \( \{ S_v \}_{v \in V_a} \). Let \( I \) be an intact set in \( \{ S_v \}_{v \in V_a} \). As we did in 37 we write \( \delta_I \) for the finite time that a node in \( I \) takes to deliver some voting value after GST and provided that some other node in \( I \) already delivered some voting value. Lemma 38 guarantees that \( \delta_I \) is finite.

\begin{lemma}
Let \( \{ S_v \}_{v \in V_a} \) be a subjective FBQS and consider an execution of ASCP over \( \{ S_v \}_{v \in V_a} \). Let \( I \) be an intact set in \( \{ S_v \}_{v \in V_a} \). If a node \( v_1 \in I \) commits a ballot \( b \), then some node \( v_2 \in I \) prepared \( b \).
\end{lemma}

Proof. Assume that a node \( v_1 \in I \) commits ballot \( b \). By line 7 of Algorithm 11 node \( v_1 \) received \( \text{READY}(b, \text{true}) \) from every member of a quorum known by \( v_1 \) and to which \( v_1 \) belongs. By Lemma 39 the first node \( u \in I \) to do so received \( \text{VOTE}(b, \text{true}) \) messages from every member of a quorum \( U \) known by \( u \) and to which \( u \) belongs. Since \( v_1 \) is intertwined with every other node in \( I \), there exists a correct node \( v_2 \) in the intersection \( U \cap I \) that sent \( \text{VOTE}(b, \text{true}) \). The node \( v_2 \) can send \( \text{VOTE}(b, \text{true}) \) only through line 6 of Algorithm 11, which means that \( v_2 \) triggers \( \text{brs}[b].\text{vote}(\text{true}) \) in line 12 of Algorithm 11. By line 11 of the same figure, this is only possible after \( v_2 \) has aborted every \( b' \not< b \), and the lemma holds.

\begin{lemma}
Let \( \{ S_v \}_{v \in V_a} \) be a subjective FBQS and consider an execution of ASCP over \( \{ S_v \}_{v \in V_a} \). Let \( I \) be an intact set in \( S \). Let \( v_1 \) and \( v_2 \) be nodes in \( I \) and \( b_1 \) and \( b_2 \) be ballots such that \( b_2 \not< b_1 \). The following two things cannot both happen: node \( v_1 \) prepares \( b_1 \) and node \( v_2 \) sends \( \text{READY}(b_2, \text{true}) \).
\end{lemma}

Proof. Assume towards a contradiction that \( v_1 \) prepares \( b_1 \), and that \( v_2 \) sends \( \text{READY}(b_2, \text{true}) \).

By definition of prepare, node \( v_1 \) aborted every ballot \( b \not< b_1 \). By line 4 of Algorithm 11 node \( v_2 \) received \( \text{READY}(b, \text{false}) \) from every member of a quorum \( U_b \) known by \( v_1 \) for each ballot \( b \not< b_1 \). By assumptions, \( b_2 \not< b_1 \), and therefore \( v_1 \) received \( \text{READY}(b_2, \text{false}) \) from every member of the quorum \( U_{b_2} \). By Lemma 39 the first node \( u_1 \in I \) that sent \( \text{READY}(b_2, \text{false}) \) received \( \text{VOTE}(b_2, \text{false}) \) from a quorum \( U_{b_1} \) known by \( u_1 \) and to which \( u_1 \) belongs. Since \( v_2 \) sent \( \text{READY}(b_2, \text{true}) \) and by Lemma 39 the first node \( u_2 \in I \) that sent \( \text{READY}(b_2, \text{true}) \) received \( \text{VOTE}(b_2, \text{true}) \) from a quorum \( U_{b_2} \) known by \( u_2 \) and to which \( u_2 \) belongs. Since \( u_1 \) and \( u_2 \) are intertwined, the intersection \( U_1 \cap U_2 \) contains some correct node \( v \), which sent both \( \text{VOTE}(b_2, \text{false}) \) and \( \text{VOTE}(b_2, \text{true}) \) messages. By the use of the Boolean \( \text{voted} \) in line 13 of Algorithm 11 this results in a contradiction and we are done.

\begin{lemma}
Let \( \{ S_v \}_{v \in V_a} \) be a subjective FBQS and consider an execution of ASCP over \( \{ S_v \}_{v \in V_a} \). Let \( I \) be an intact set in \( S \). If a node \( v_1 \in I \) commits a ballot \( b_1 \), then the largest ballot \( b_2 \) prepared by any node \( v_2 \in I \) before \( v_1 \) commits \( b_1 \) is such that \( b_1 \sim b_2 \).
\end{lemma}

Proof. Assume node \( v_1 \) commits ballot \( b_1 \). By the guard in line 13 of Algorithm 11 node \( v_1 \) received the message \( \text{READY}(b_1, \text{true}) \) from every member of a quorum known by \( v_1 \) and to which \( v_1 \) belongs, which entails that node \( v_1 \) received \( \text{READY}(b_1, \text{true}) \) from itself. By Lemma 10 the first node \( u \in I \) that send \( \text{READY}(b_1, \text{true}) \) needs to receive an \( \text{VOTE}(b_1, \text{true}) \) message from every member of some quorum known by \( u \) and to which \( u \) belongs. Thus, \( u \) itself triggered \( \text{brs}[b_1].\text{vote}(\text{true}) \), which by lines 7 and 24 of Algorithm 2 means that \( u \) prepared ballot \( b_1 \). Hence, the largest ballot \( b_2 \) such that there exists a node \( v_2 \in I \) that triggers \( \text{brs}[b_2].\text{vote}(\text{true}) \) before \( v_1 \) commits \( b_1 \), is bigger or equal than \( b_1 \). If \( b_2 = b_1 \),...
Lemma 49. Since straightforward by Lemma 28, since all nodes are honest and thus all the views coincide with each other.

Lemma 50. Since straightforward by Lemma 50, since all nodes are honest and thus all the views coincide with each other.

The definition of ready-tree for Boolean a and ballot b at node v from Appendix C can be lifted to the subjective FBQSs straightaway, since the definition assumes that all nodes are honest, and thus they do not equivocate their quorum slices and all the views coincide with each other.

Lemma 49. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS b be a ballot, and consider an execution of the instance for ballot b of FV over \( \{S_v\}_{v \in V_{ok}} \). Assume all nodes are honest. If a node v sends \( \text{READY}(b, a) \) then there exists a quorum \( U \) known by every node such that every member of \( U \) sent \( \text{VOTE}(b, a) \).

Proof. Straightforward by Lemma 28 since all nodes are honest and thus all the views coincide with each other.

Lemma 50. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS and consider an execution of ASCP over \( \{S_v\}_{v \in V_{ok}} \). Let \( b_1 \) be the largest ballot prepared by some node \( v_1 \) at some moment in the execution. If all nodes are honest, then some node \( v_2 \) proposed \( b_1 \).

Proof. Straightforward by Lemma 50 since all nodes are honest and thus all the views coincide with each other.

Lemma 51. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS and consider an execution of ASCP over \( \{S_v\}_{v \in V_{ok}} \). Let I be a maximal intact set in \( \{S_v\}_{v \in V_{ok}} \) and assume that GST has expired. Let v be a node in I that prepares some ballot b such that no other node in I has ever prepared a ballot with round bigger or equal than b.n. In the interval of duration \( \delta_1 \) after \( u \) prepares \( b \), every node in I that has not decided any value yet, either decides a value or prepares a ballot with round b.n.

Proof. Since \( v \) has prepared \( b \), then \( v \) has delivered \( \text{false} \) for every ballot \( b_1 \leq b \). Let \( u \in I \) be a node different from \( v \) that has not decided any value yet. By assumptions, \( u \) has neither prepared any ballot with round bigger or equal than b.n. Since GST has expired and by Lemma 51, node \( u \) will deliver \( \text{false} \) for every ballot \( b_1 \leq b \) within \( \delta_1 \), and the lemma holds.

Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS and consider an execution of ASCP over \( \{S_v\}_{v \in V_{ok}} \). Let I be an intact set in \( \{S_v\}_{v \in V_{ok}} \). As in 35, we say the window for intact set I of round n for the interval in which every node in an intact set I that has not decided any value yet prepares a ballot of round n. As in Appendix C, we let \( v_0 \) be the first node in I that ever prepares a ballot \( b_n \) with round n. The definition of prepare-footprint of ballot \( b_n \) from Appendix C can be lifted to the subjective FBQSs straightaway, since the definition assumes that all faulty nodes have stopped, and thus the remaining correct nodes agree on the slices of every other correct node, and all the quorums that are not stopped belong to all the views. We also distinguish the abort interval for intact set I of round n and the duration \( \delta_A \), whose definitions can be lifted to subjective FBQSs straightaway. As in Appendix C, we may omit the ‘for intact set I’ qualifier when the intact set is clear from the context.
Lemma 52. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS and consider an execution of ASCP over \( \{S_v\}_{v \in V_{ok}} \). Let I be a maximal intact set in \( \{S_v\}_{v \in V_{ok}} \) and assume that all faulty nodes eventually stop. There exists a round \( n \) such that either every node in I decides some value before reaching round \( n \), or otherwise the windows of all the rounds \( m \geq n \) happen consecutively and never overlap with each other, and in each of the consecutive windows of round \( m \) the nodes in I that have not decided any value yet only prepare ballots with round \( m \).

Proof. Analogous to the proof of Lemma 12.

Corollary 53. Let \( m \geq n \) and let \( b_{\text{max}} \) be the maximum ballot prepared by any node in I before the abort interval of round \( m + 1 \) starts. Every node in I prepares \( b_{\text{max}} \) before the abort interval of round \( m + 1 \) starts.

Proof. Analogous to the proof of Corollary 20.

Proof of Theorem 18. Analogous to the proof of Theorem 13 by using Lemmas 45–52 and Corollary 53.

Lemma 54. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS with some intact set \( I \), \( v \) be a node with \( v \in I \), and \( \tau \) be a trace entailed by an execution of CSCP. If \( \sigma(\tau) \) is a trace entailed by an execution of ASCP, then \( v.\text{round} \), \( v.\text{prepared} \), and \( v.\text{candidate} \) coincide in both executions.

Proof. Analogous to the proof of Lemma 39.

Lemma 55. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS with some intact set \( I \), \( v \) be a node with \( v \in I \), and \( \tau \) be a trace entailed by an execution of CSCP. Then for every ballot \( b \in v.\text{ballots-delivered-cmt} \) (respectively, \( b \in v.\text{ballots-readied-cmt} \)) holds \( b \leq v.\text{max-delivered-prep} \) (respectively, \( b \leq v.\text{max-delivered-prep} \)).

Proof. Analogous to the proof of Lemma 39.

Lemma 56. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS with some intact set \( I \), \( v \) be a node with \( v \in I \), and \( \tau \) be a trace entailed by an execution of CSCP. If \( \sigma(\tau) \) is a trace entailed by an execution of ASCP, for every \( b > v.\text{max-delivered-prep} \) holds \( v.\text{brs}[b].\text{delivered} \) is false.

Proof. Analogous to the proof of Lemma 39 by using Lemma 55.

Lemma 57. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS with some intact set \( I \), \( v \) be a node with \( v \in I \), and \( \tau \) be a trace entailed by an execution of CSCP. If \( \sigma(\tau) \) is a trace entailed by an execution of ASCP, for every \( b > v.\text{max-readied-prep} \) holds \( v.\text{brs}[b].\text{ready} \) is false.

Proof. Analogous to the proof of Lemma 39 by using Lemma 55.

Lemma 58. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS with some intact set \( I \), \( v \) be a node with \( v \in I \), and \( \tau \) be a trace entailed by an execution of CSCP. If \( \sigma(\tau) \) is a trace entailed by an execution of ASCP and \( b \notin \text{ballots-delivered-cmt} \) then \( b.\text{delivered} \) is false.

Proof. Analogous to the proof of Lemma 39 by using Lemma 55.

Lemma 59. Let \( \{S_v\}_{v \in V_{ok}} \) be a subjective FBQS with some intact set \( I \) and \( \tau \) be a trace entailed by an execution of CSCP. For every finite prefix \( \tau' \) of the projected trace \( \tau|_I \), the simulated \( \rho' = \sigma(\tau') \) is the prefix of a trace entailed by an execution of ASCP in Algorithm 2.

Proof. Analogous to the proof of Lemma 37 by using Lemmas 54–58.
Proof of Theorem 19. Analogous to the proof of Theorem 14 by using Lemma 59.

Proof of Corollary 20. Analogous to the proof of Corollary 15 by using Theorem 19.