Generalized Hukuhara conformable fractional derivative and its application to fuzzy fractional partial differential equations

Manizheh Ghaffari1 · Tofigh Allahviranloo2 · Saeid Abbasbandy3 · Mahdi Azhini1

Accepted: 30 November 2021 / Published online: 28 January 2022 © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract
The main focus of this paper is to develop an efficient analytical method to obtain the traveling wave fuzzy solution for the fuzzy generalized Hukuhara conformable fractional equations by considering the type of generalized Hukuhara conformable fractional differentiability of the solution. To achieve this, the fuzzy conformable fractional derivative based on the generalized Hukuhara differentiability is defined, and several properties are brought on the topic, such as switching points and the fuzzy chain rule. After that, a new analytical method is applied to find the exact solutions for two famous mathematical equations: the fuzzy fractional wave equation and the fuzzy fractional diffusion equation. The present work is the first report in which the fuzzy traveling wave method is used to design an analytical method to solve these fuzzy problems. The final examples are asserted that our new method is applicable and efficient.

Keywords
Generalized Hukuhara conformable fractional derivative · Fuzzy traveling wave solution · Generalized partial Hukuhara differentiability · The fuzzy fractional wave equation · The fuzzy fractional diffusion equation

1 Introduction
During the last decade, the interest of mathematicians in fuzzy differential equations has been rapidly increasing. The main reason for this development is that using these problems will lead to a much more effective and elegant way of treating real-life issues. A particular subgroup of fuzzy differential equations is described with operators of fractional nature. Fractional calculus is a set of methods and hypotheses that extend the concept of a derivative operator from integer-order \( n \) to arbitrary order \( \alpha \). Modeling like biological population models, the predator-prey models and infectious diseases models, etc., are generalized to fractional order. Fractional calculus is not only a productive and emerging field, but it also represents a new philosophy, how to construct and apply a certain type of nonlocal operator to real-world problems (Ghobaei-Arani et al. 2019; Ghobaei-Arani and Souri 2019; Ghobaei-Arani et al. 2021; Khorsand et al. 2018; Mazandarani and Xiu 2021; Shahidinejad et al. 2020; Shahidinejad and Ghobaei-Arani 2020).

The interest in fractional fuzzy differential equations arose in 2012 with a paper by Agarwal et al. (2012). The existence and uniqueness of a fuzzy solution for fractional differential equations are proved in Arshad and Lupulescu (2011). The concepts of fuzzy fractional integral and Caputo partial differentiability based on generalized Hukuhara differentiability for the fuzzy multivariable functions have been introduced by Viet Long et al. (2017). Hoa et al. (2019) introduced the fuzzy Caputo–Katugampola fractional differential equations in fuzzy space, and under generalized Lipschitz condition, the existence and uniqueness of the solution are proved. The analytical solutions to some linear fractional partial fuzzy differential equations under certain conditions were investigated in Shaha et al. (2020). The perturbation-iteration algorithm was developed for numerical solutions of some types of fuzzy fractional partial differential equations with generalized Hukuhara derivative Senol et al. (2019). Zureigat et al. (2020) analyzed the compact Crank-Nicholson scheme to solve the fuzzy time diffusion equation with fractional-order \( 0 < \alpha \leq 1 \). Some new methods and useful materials concerning fuzzy fractional differential and fuzzy fractional
partial differential equations are introduced in Allahviranloo (2021).

Recently, a new well-behaved simple fractional derivative called “the conformable fractional derivative” is defined by Khalil et al. (2014); Arqub and Al-Smadi (2020). This new definition seems to be a natural extension of the usual derivative. Researchers started to combine this new definition with fuzzy calculus (Harir et al. 2020, 2021; Martynyuk et al. 2020). They used the concept of H-differentiability or strongly generalized differentiability. But it is well-known that the usual Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions (Diamond 2002; Bede and Gal 2005). To overcome this shortcoming, we will introduce the fuzzy conformable fractional derivative under generalized Hukuhara differentiability and prove some important properties for this kind of differentiability.

Consider the following generic form of second-order fuzzy fractional partial differential equation defined based on generalized Hukuhara conformable fractional derivative

\[ \mathcal{D}_x^{\alpha} v = F \left( v, v_{xgH}, v_{xxgH}, D_{ttgH}^{2\alpha} v \right), \]

with \( 0 < \alpha \leq 1 \). The main contribution of this paper is to find the wave traveling solutions for problem (1). For this purpose, the concept of generalized Hukuhara conformable fractional differentiability is introduced thoroughly in the fuzzy functions. Next, the fuzzy fractional wave equation and fuzzy diffusion equation are introduced based on the generalized Hukuhara conformable fractional differentiability. Finally, we discuss the fuzzy traveling wave solutions for these equations by considering the type of \( \alpha_{gH} \)-differentiability.

We now give a brief outline of the main sections of the paper and state the aims and objectives of each section. Section 3 deals with aspects of background knowledge in fuzzy mathematics and fuzzy derivatives with emphasis on the generalized Hukuhara differentiability. In Sect. 4, generalized Hukuhara conformable fractional differentiability is studied. Some properties for this concept of differentiability are proved. A fuzzy fractional wave equation and a fuzzy fractional diffusion equation under generalized Hukuhara conformable fractional differentiability are introduced in Sects. 5 and 6, respectively. Moreover the fuzzy traveling wave solutions of these equations are investigated in different scenarios. Finally in Sect. 7, the conclusions are given.

Puri and Ralescu (1986) suggested the fuzzy differential equations modeling with uncertainty under the concept of H-differentiability. Subsequently, Kaleva in Kaleva (1987) proposed fuzzy differential equations using the Hukuhara derivative, and some other authors developed it. But for some fuzzy differential equations in this framework, the diameter of the solution is unbounded as the time \( t \) increases (Diamond 2002).

To overcome this shortcoming, Bede and Gal introduced the weakly generalized differentiability and the strongly generalized differentiability for the fuzzy functions (Bede and Gal 2005). Moreover, they presented a more general definition of derivatives for the fuzzy functions and their applications for solving fuzzy differential equations (Bede and Gal 2005). Stefanini and Bede, by the concept of generalization of the Hukuhara difference of compact convex set, introduced generalized Hukuhara differentiability Stefanini and Bede (2009) for interval-valued functions. They showed that this concept of differentiability has relationships with weakly generalized differentiability and strongly generalized differentiability. The disadvantage of the strongly generalized differentiability of a function compared to H-differentiability is that the fuzzy differential equation has no unique solution (Bede and Gal 2005). Also, in Chalco-Cano et al. (2011) the authors studied relationships between the strongly generalized differentiability and the generalized Hukuhara differentiability, showing the equivalence between these two concepts when the set of switching points of the interval-valued function is finite (Table 1).

In this way, they use the LU-parametric representation of fuzzy numbers and fuzzy-valued functions to obtain valid approximations of fuzzy generalized Hukuhara derivative and solve fuzzy differential equations (Bede and Stefanini 2011). Allahviranloo et al. (2015) introduced the fuzzy generalized Hukuhara partial differentiability and solved the fuzzy heat equation under generalized Hukuhara differentiability. Moreover, in Moghaddam and Allahviranloo (2018) the authors obtained the fuzzy solutions of the fuzzy Poisson equation under generalized Hukuhara differentiability. Recently, Chalco-Cano et al. (2020) provided a new characterization of the switching points for generalized Hukuhara differentiability and shown that the set of all switching points is at most countable.

### 2 Related works

The concept of the fuzzy derivative was first introduced by Chang and Zadeh (1972). The starting point of the topic in the set-valued differential equation and also fuzzy differential equation is Hukuhara’s paper Hukuhara (1967).

### 3 Preliminaries

In the following, we focus on the basic definitions and the necessary notation which will be used throughout the paper. Let \( E \) is the set of fuzzy numbers and \( T \subset E \) shows the set of all triangular fuzzy numbers.
Let \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \) are two triangular fuzzy numbers, so the generalized Hukuhara difference, \( a \ominus_{gH} b \), is defined as follows (Bede 2013)

\[
\begin{align*}
\ominus_{gH} b &= c \\
\iff \begin{cases} 
(i). & c = (a_1 - b_1, a_2 - b_2, a_3 - b_3), \\
(ii). & c = (a_3 - b_3, a_2 - b_2, a_1 - b_1).
\end{cases}
\end{align*}
\]

Actually

\[
\begin{align*}
\ominus_{gH} b &= \left( \min\{a_1 - b_1, a_3 - b_3\}, a_2 - b_2, \\
&\quad \max\{a_1 - b_1, a_3 - b_3\} \right).
\end{align*}
\]

In this article, we assume that \( a \ominus_{gH} b \in T \).

Let \( f : [a, b] \to T \) and its first \( k \) generalized Hukuhara derivatives are continuous fuzzy triangular functions without any switching points on domain \( I := [a, b] \) (Bede 2013).

**Definition 3.1** (See Anastassiou 2010). Let \( f : I \to E \) be a fuzzy function and \( t_0 \in I \). If

\[
\forall \epsilon > 0 \; \exists \delta > 0 \; \forall t \left( 0 < |t - t_0| < \delta \Rightarrow D(f(t), L) < \epsilon \right),
\]

Here, \( D \) is the Hausdorff distance. Then, we say that \( L \in E \) is limit of \( f \) in \( t_0 \), which is denoted by \( \lim_{t \to t_0} f(t) = L \).

Also the fuzzy function \( f \) is said to be fuzzy continuous if

\[
\lim_{t \to t_0} f(t) = f(t_0),
\]

**Theorem 3.2** (See Armand et al. 2018) Let \( f, g : I \to E \) be two fuzzy functions. If \( \lim_{t \to c} f(t) = L_1 \) and \( \lim_{t \to c} g(t) = L_2 \), then

\[
\lim_{t \to c} (f(t) + g(t)) = L_1 + L_2.
\]

\( \square \) Springer
\[ L_2, L_1, L_2 \in \mathbb{E} \text{ then} \]
\[ \lim_{t \to \infty} [f(t) \oslash g(t)] = L_1 \oslash g H L_2. \]

**Definition 3.3** (See Bede 2013) The fuzzy function \( f(t) \) is generalized Hukuhara differentiable ([\( g H \)]-differentiable) at \( t_0 \in \mathbb{T} \) if
\[ f'_H(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) \oslash g H f(t_0)}{h}, \]
belongs to \( \mathbb{E} \). In addition we can say that \( f(t) \) is
- \([i] - \) \([g H \)]-differentiable function if and only if for all \( t \in \mathbb{T} \)
\[ f'_H(t) = (f'_1(t), f'_2(t), f'_3(t)), \]
defines a triangular fuzzy number.
- \([ii] - \) \([g H \)]-differentiable function if and only if for all \( t \in \mathbb{T} \)
\[ f'_{ii,H}(t) = (f'_1(t), f'_2(t), f'_3(t)), \]
is a triangular fuzzy number.

**Proposition 3.4** (See Shahsavari et al. 2020) Let \( \lambda_1 \) and \( \lambda_2 \) are two real constants such that \( \lambda_1, \lambda_2 \geq 0 \) (or \( \lambda_1, \lambda_2 < 0 \)). If \( f(t) \) is a triangular fuzzy function, then
\[ \lambda_1 f(t) \oslash g H \lambda_2 f(t) = (\lambda_1 - \lambda_2) f(t). \]

**Definition 3.5** (See Bede 2013) Let \( f : \mathbb{T} \to \mathbb{T} \) is a fuzzy function and \( f(t) = (f_1(t), f_2(t), f_3(t)) \) and \( t_0 \in \mathbb{T} \) then
\[ \int_a^b f(t) \, dt = \left( \int_a^b f_1(t) \, dt, \int_a^b f_2(t) \, dt, \int_a^b f_3(t) \, dt \right). \]

**Theorem 3.6** (See Bede 2013) If \( f \) is a \( g H \)-differentiable fuzzy function with no switching point in the interval \( \mathbb{T} \), then we have
\[ \int_a^b f'_H(t) \, dt = f(b) \oslash g H f(a). \]

**Lemma 3.7** (See Viet Long et al. 2017) If \( f : \mathbb{T} \to \mathbb{T} \) be a triangular fuzzy function with no switching point in interval \( \mathbb{T} \), then we have
1. If \( f(t) \) is \([i] - g H\)-differentiable , then
\[ \int_a^b f'_{i,H}(t) \, dt = f(b) \oslash f(a). \]

2. If \( f(t) \) is \([ii] - g H\)-differentiable , then
\[ \int_a^b f'_{ii,H}(t) \, dt = (-1) f(a) \oslash (-1) f(b). \]

**Lemma 3.8** (See Moghaddam and Allahviranloo 2018) \[ \int_a^b f(t) \, dt = \oslash \int_a^b f(t) \, dt \] where \( \oslash \) denotes Hukuhara difference and \( f(t) \) be a fuzzy function.

**Definition 3.9** (See Allahviranloo et al. 2015) Let \( (x_0, t_0) \in \mathbb{D} \subseteq \mathbb{R}^2 \), then the first generalized Hukuhara partial derivative ([\( g H - p \)]-derivative for short) of a fuzzy value function \( \nu(x, t) : \mathbb{D} \to \mathbb{E} \) at \( (x_0, t_0) \) with respect to variables \( x, t \) is the fuzzy functions \( \partial_{x,H} \nu(x_0, t_0) \) and \( \partial_{t,H} \nu(x_0, t_0) \) given by
\[ \partial_{x,H} \nu(x_0, t_0) = \lim_{k \to 0} \frac{\nu(x_0, t_0 + k)}{k}, \]
\[ \partial_{t,H} \nu(x_0, t_0) = \lim_{k \to 0} \frac{\nu(x_0 + k, t_0)}{k}, \]
provided that \( \partial_{x,H} \nu(x_0, t_0) \) and \( \partial_{t,H} \nu(x_0, t_0) \in \mathbb{E} \).

**Definition 3.10** (See Allahviranloo et al. 2015) A triangular fuzzy function \( \nu(x, t) = (\nu_1(x, t), \nu_2(x, t), \nu_3(x, t)) \), without any switching points on \( \mathbb{D} \) is called
- \([i] - p\)-differentiable with respect to \( t \) at \((x_0, t_0)\) if and only if
\[ \nu_{i,H} (x_0, t_0) = \left( \frac{\partial \nu_1 (x, t)}{\partial t}, \frac{\partial \nu_2 (x, t)}{\partial t}, \frac{\partial \nu_3 (x, t)}{\partial t} \right) \bigg|_{x=x_0, t=t_0}, \]
defines a triangular fuzzy number, and
- it’s \([ii] - p\)-differentiable if and only if
\[ \nu_{ii,H} (x_0, t_0) = \left( \frac{\partial^2 \nu_1 (x, t)}{\partial t^2}, \frac{\partial^2 \nu_2 (x, t)}{\partial t^2}, \frac{\partial^2 \nu_3 (x, t)}{\partial t^2} \right) \bigg|_{x=x_0, t=t_0}, \]
defines a triangular fuzzy number.

Moreover, if \( \nu_1(x, t) \) is \([g H - p]\)-differentiable at \((x, t)\) with respect to \( x \) without any switching point on \( \mathbb{D} \) and
- if the type of \([g H - p]\)-differentiability of both \( \nu(x, t) \) and \( \nu_1(x, t) \) is the same, then \( \nu_1(x, t) \) is \([i] - p\)-differentiable w.r.t \( x \) and
\[ \nu_{x,H} (x_0, t_0) = \left( \frac{\partial^2 \nu_1 (x, t)}{\partial x^2}, \frac{\partial^2 \nu_2 (x, t)}{\partial x^2}, \frac{\partial^2 \nu_3 (x, t)}{\partial x^2} \right) \bigg|_{x=x_0, t=t_0}, \]
– if the type of \([gH - p]\)-differentiability \(v(x, t)\) and \(v_{x}(x, t)\) is different, therefore \(v_{x}(x, t)\) is \((ii) - p\)-differentiable w.r.t \(x\) and
\[
\nu_{x, gH}(x_{0} , t_{0}) = \left( \frac{\partial^{2} \nu_{2}(x, t)}{\partial x^{2}} , \frac{\partial^{2} \nu_{2}(x, t)}{\partial x^{2}} , \frac{\partial^{2} \nu_{1}(x, t)}{\partial x^{2}} \right)_{|_{x = x_{0}, t = t_{0}}}
\]

4 Generalized Hukuhara conformable fractional derivative

In this section, we are going to introduce conformable fractional derivative based on the generalized Hukuhara derivative. Moreover, we will prove several properties for this kind of differentiability.

Definition 4.1 Let \(f : [0, \infty) \rightarrow \mathbb{E}\) be a triangular fuzzy function. The generalized Hukuhara conformable fractional derivative of \(f\) of order \(\alpha\) is defined by
\[
T_{gH}^{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) \ominus_{gH} f(t)}{\epsilon},
\]
provided that \(T_{gH}^{\alpha}(f)(t) \in \mathbb{E}\). If the generalized Hukuhara conformable fractional derivative of \(f\) of order \(\alpha\) exists, then we simply say \(f\) is \(\alpha_{gH}\)-differentiable.

Theorem 4.2 If a fuzzy function \(f : [0, \infty) \rightarrow \mathbb{E}\) is \(\alpha_{gH}\)-differentiable at \(t_{0} > 0, \alpha \in (0, 1]\), then \(f\) is continuous at \(t_{0}\).

Proof We have \(f(t + \epsilon t^{1-\alpha}) \ominus_{gH} f(t) = \frac{f(t + \epsilon t^{1-\alpha}) \ominus_{gH} f(t)}{\epsilon} \). By using Theorem 3.2, we conclude that
\[
\lim_{\epsilon \to 0} f(t + \epsilon t^{1-\alpha}) \ominus_{gH} f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) \ominus_{gH} f(t)}{\epsilon} \lim_{\epsilon \to 0} \epsilon,
\]
then
\[
\lim_{\epsilon \to 0} f(t + \epsilon t^{1-\alpha}) \ominus_{gH} f(t) = T_{gH}^{\alpha}(f)(t) \ominus_{gH} f(t).
\]
Now, let \(h = \epsilon t_{0}^{1-\alpha}\), therefore
\[
\lim_{h \to 0} f(t + h) \ominus_{gH} f(t) = 0.
\]
Therefore, according to Definition 3.1, it can be concluded that the function \(f\) is fuzzy continuous.

Definition 4.3 Let \(f : [0, \infty) \rightarrow \mathbb{E}\) be a triangular fuzzy function. The generalized Hukuhara conformable fractional derivative of \(f\) of order \(\beta \in (1, 2]\) is defined by
\[
T_{gH}^{\beta}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{2-\beta}) \ominus_{gH} f(t)}{\epsilon},
\]
provided that \(T_{gH}^{\beta}(f)(t) \in \mathbb{E}\). If the generalized Hukuhara conformable fractional derivative of \(f\) of order \(\beta\) exists, then we simply say \(f\) is \(\beta_{gH}\)-differentiable.

Definition 4.4 Let \(\alpha \in (0, 1)\) and \(f\) is \(\alpha_{gH}\)-differentiable at a point \(t > 0\). We can say that \(f(t)\) is
\[
\alpha_{t, gH}\text{-differentiable function if and only if for all } t > 0 \quad T_{gH}^{\alpha}(f)(t) = \left( T^{\alpha}(f_{1})(t), T^{\alpha}(f_{2})(t), T^{\alpha}(f_{3})(t) \right),
\]
defines a triangular fuzzy number.

\[
\alpha_{ii, gH}\text{-differentiable function if and only if for all } t > 0 \quad T_{gH}^{\alpha}(f)(t) = \left( T^{\alpha}(f_{3})(t), T^{\alpha}(f_{2})(t), T^{\alpha}(f_{1})(t) \right),
\]
be a triangular fuzzy number.

Here, \(T^{\alpha}(f_{i})(t), i = 1, 2, 3\) is the conformable fractional derivative for the real-valued function \(f_{i}(t)\) (Khalil et al. 2014).

Definition 4.5 We say that a point \(\xi_{0} \in (0, \infty)\) is a switching point for the differentiability of \(f\), if in any neighborhood \(V\) of \(\xi_{0}\) there exist points \(\xi_{1} < \xi_{0} < \xi_{2}\) such that

Type I. at \(\xi_{1}\) (5) holds, while (6) does not hold and at \(\xi_{2}\) (6) holds and (5) does not hold, or

Type II. at \(\xi_{1}\) (6) holds, while (5) does not hold and at \(\xi_{2}\) (5) holds and (6) does not hold.

Theorem 4.6 Let \(\alpha \in (0, 1)\) and \(f\) be \(\alpha_{gH}\)-differentiable at a point \(t > 0\). Then
\[
T_{gH}^{\alpha}(f)(t) = t^{1-\alpha} f_{gH}^{'}(t).
\]

Proof In Definition 4.1, let \(h = \epsilon t^{1-\alpha}\) and then \(\epsilon = t^{\alpha-1}h\). Hence
\[
T_{gH}^{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) \ominus_{gH} f(t)}{\epsilon}
= \lim_{h \to 0} \frac{f(t + h) \ominus_{gH} f(t)}{h^{\alpha-1}}
= t^{1-\alpha} \lim_{h \to 0} \frac{f(t + h) \ominus_{gH} f(t)}{h}
= t^{1-\alpha} f_{gH}^{'}(t).
\]
Remark 4.7 Using Theorem 4.6 and Definition 4.5, it can be similarly easily shown that for $\beta \in (1, 2)$

$$T_{gH}^\beta(f)(t) = t^{2-\beta} f''_g(t),$$

where $f$ is $gH$-differentiable of second order.

**Example 4.8** Consider the fuzzy function $f : [0, \pi] \to \mathbb{E}$ defined by

$$f(t) = (1.3 \sin(t), 5.2 \sin(t), 9.6 \sin(t)).$$

We have the following $\alpha_{gH}$-derivatives of $f(t)$

$$T_{gH}^\frac{3}{2}(f)(t) = (1.3t^\frac{3}{2} \cos(t), 5.2t^\frac{3}{2} \cos(t), 9.6t^\frac{3}{2} \cos(t)) \ t \in [0, \frac{\pi}{2}],$$

$$T_{gH}^1(f)(t) = (9.6t^\frac{1}{2} \cos(t), 5.2t^\frac{1}{2} \cos(t), 1.3t^\frac{1}{2} \cos(t)) \ t \in [\frac{\pi}{2}, \pi].$$

Therefore, the fuzzy function $f(t)$ is $\alpha_{gH}$-differentiable function on $t \in [0, \frac{\pi}{2}]$. This function is switched to $\alpha_{gH}$-differentiable at $t = \frac{\pi}{2}$. Hence, the point $t = \frac{\pi}{2}$ is a switching point of Type I for the differentiability of $f$ (Fig. 1).

**Theorem 4.9** Let $g : \mathbb{R} \to \mathbb{E}$ be a fuzzy function such that $f$ is $gH$-differentiable at the point $g(t)$ without any switching points, and $\alpha \in (0, 1)$.

- Assume $f(t)$ is $[(i) - gH]$-differentiable at $g(t)$, then function $(f \circ g)(t)$ is $\alpha_{gH}$-differentiable if

$$T_{i,gH}^\alpha(f \circ g)(t) = \begin{cases} t^{1-\alpha} g'(t) \odot f'_i(g(t)), & \text{If } g'(t) > 0, \\ \ominus(-1)t^{1-\alpha} g'(t) \odot f'_i(g(t)), & \text{If } g'(t) < 0. \end{cases}$$

- Let $f(t)$ is $[(ii) - gH]$-differentiable at $g(t)$, then the function $(f \circ g)(t)$ is $\alpha_{gH}$-differentiable if

$$T_{ii,gH}^\alpha(f \circ g)(t) = \begin{cases} t^{1-\alpha} g'(t) \odot f''_i(g(t)), & \text{If } g'(t) > 0, \\ \ominus(-1)t^{1-\alpha} g'(x) \odot f''_i(g(x)), & \text{If } g'(t) < 0. \end{cases}$$

**Proof** First let $f(t)$ is $[(i) - gH]$-differentiable at $g(t)$. We have the following cases

i. If $g'(t) > 0$. Hence by attention to Theorem 4.6 we have

$$t^{1-\alpha} g'(t) \odot \left( f'_i(g(t)), f'_2(g(t)), f'_3(g(t)) \right)$$

$$= \left( t^{1-\alpha} g'(t) f'_i(g(t)), t^{1-\alpha} g'(t) f'_2(g(t)), t^{1-\alpha} g'(t) f'_3(g(t)) \right)$$

$$= T_{i,gH}^\alpha(f \circ g)(t).$$

ii. If $g'(t) < 0$, then

$$\ominus(-1)t^{1-\alpha} g'(t) \odot \left( f'_i(g(t)), f'_2(g(t)), f'_3(g(t)) \right)$$

$$= \ominus(-1) \left( t^{1-\alpha} g'(t) f'_i(g(t)), t^{1-\alpha} g'(t) f'_2(g(t)), t^{1-\alpha} g'(t) f'_3(g(t)) \right)$$

$$= \ominus \left( -t^{1-\alpha} g'(t) f'_i(g(t)), -t^{1-\alpha} g'(t) f'_2(g(t)), -t^{1-\alpha} g'(t) f'_3(g(t)) \right)$$

$$= \left( t^{1-\alpha} g'(t) f'_i(g(t)), t^{1-\alpha} g'(t) f'_2(g(t)), t^{1-\alpha} g'(t) f'_3(g(t)) \right)$$

$$= T_{i,gH}^\alpha(f \circ g)(t).$$

We can use the same procedure when $f(t)$ is $[(ii) - gH]$-differentiable at $g(t)$.

**5 Fuzzy traveling wave solution of the fuzzy fractional wave equation**

We want to find traveling wave fuzzy solution of the fuzzy one-dimensional homogeneous fractional wave equation. Consider this problem as follows

$$\begin{cases} \mathcal{D}_{i,gH} 2^\alpha \nu \ominus gH k^2 \odot \nu_{x,t} = 0, \ (x, t) \in \mathbb{R} \times [0, \infty), \\ \nu(x, 0) = f(x), \ \mathcal{D}_{t}^\alpha \nu(x, 0) = g(x) \end{cases} \tag{7}$$

where $\alpha \in (\frac{1}{2}, 1)$ and $\mathcal{D}_{i,gH} 2^\alpha$ is the generalized Hukuhara conformable fractional partial derivatives with respect to $t$ and $\nu_{x,t}$ is the generalized Hukuhara partial derivative with respect to $x$. Here, $f(x), g(x)$ are given continuous fuzzy functions. We will find the triangular analytical fuzzy solutions of Eq. (7) by using traveling wave method provided that the types of $\alpha_{gH}$-differentiability of $\nu(x, t)$ with respect to $t$ and $[gH - p]$-differentiability with respect to $x$ are the same. By considering the type of $\alpha_{gH}$-differentiability of $\nu(x, t)$ with respect to $t$, we have different cases as follows:
Generalized Hukuhara conformable fractional derivative and its application to fuzzy fractional…

1-1-gH. Let \( \nu(x, t) \) and \( \frac{\partial x}{\partial H} \nu \) are \( \alpha \times gH \)-differentiable with respect to \( t \), and \( \nu(x, t) \) and \( \nu_{x,gH} \) are \( [i - p] \)-differentiable with respect to \( x \). Then \( \nu(\xi) \) is a \((i - gH)\)-differentiable fuzzy. Here, we outline the main steps of traveling wave method function.

Step 1 Consider fuzzy one-dimensional homogeneous fractional wave Eq. (7)

\[
\nu(x, t) = \mathcal{U}(\xi), \quad \text{where} \quad \xi = x - \frac{t^\alpha}{\alpha},
\]

which can be analyzed through a change of variables \( \nu(x, t) = \mathcal{U}(\xi) \). Here, \( \mathcal{U} \) is a continuous function and \( gH \)-differentiable in \( \xi \) and \( \gamma \) is a positive real constant.

Step 2 We have

\[
\frac{\partial \xi}{\partial t} = -\gamma t^{\alpha - 1} < 0, \quad \frac{\partial \xi}{\partial x} = 1 > 0,
\]

therefore, by using Theorem 4.9, the fuzzy multivariate chain rule (Moghaddam and Allahviranlo 2018), we have

\[
\frac{\partial \xi}{\partial t} = \Theta \gamma \circ \frac{d_1 \mathcal{U}}{d \xi}, \quad \Rightarrow \frac{\partial \xi}{\partial t} = \gamma \mathcal{U} \circ \frac{d^2 \mathcal{U}}{d \xi^2}.
\]

Hence Eq. (7) is reduced to the following fuzzy ordinary differential equations of \( \xi \)

\[
\gamma^2 \frac{d^2 \mathcal{U}}{d \xi^2} \circ gH \kappa^2 \frac{d^2 \mathcal{U}}{d \xi^2} = 0.
\]

Step 3 To find fuzzy solutions for ordinary differential Eqs. (8) and (17), we need some auxiliary boundary conditions, which in this article, we consider the following auxiliary boundary conditions

\[
\lim_{\xi \to \pm \infty} \mathcal{U}(\xi) = 0, \quad \lim_{\xi \to \pm \infty} \frac{d \mathcal{U}}{d \xi} = 0,
\]

\[
\lim_{\xi \to \pm \infty} \frac{d^2 \mathcal{U}}{d \xi^2} = 0.
\]

By using Proposition 3.4, Eq. (17) can also be written as follows

\[
(\gamma^2 - \kappa^2) \frac{d^2 \mathcal{U}}{d \xi^2} = 0.
\]

One possibility is for \( \frac{d^2 \mathcal{U}}{d \xi^2} = 0 \). In which case we have

\[
\mathcal{U}(\xi) = \mathcal{C}_1 \oplus \mathcal{C}_2 \xi \Rightarrow \nu(x, t)
\]

\[
= \mathcal{C}_1 \oplus \mathcal{C}_2 (x - \gamma \frac{t^\alpha}{\alpha})
\]

where \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are fuzzy integral constants. But boundary conditions (9) cannot be satisfied unless \( \mathcal{C}_2 = 0 \). Thus the only traveling solution is a fuzzy constant.

Another possibility is for \( \gamma^2 = \kappa^2 \). In this case

\[
\nu(x, t) = \mathcal{U}(x - \kappa \frac{t^\alpha}{\alpha}), \quad \nu(x, t) = \mathcal{U}(x + \kappa \frac{t^\alpha}{\alpha})
\]

(11)
are traveling wave solutions of the fuzzy fractional wave equation and $U$ can be any two $g_H$-differentiable function. In general, it follows that any solution to the fuzzy fractional wave equation can be obtained as a superposition of two traveling waves,

$$
\nu(x, t) = \mathcal{F}(x + \kappa \frac{t^a}{\alpha}) \oplus \mathcal{G}(x - \kappa \frac{t^a}{\alpha})
$$

(12)

Since Eq. (12) is a fuzzy solution for Eq. (7), then it must apply to the initial conditions of Eq. (7)

$$
\nu(x, 0) = f(x), \quad D_{t^H}^\alpha \nu(x, 0) = g(x).
$$

(13)

Hence, the initial condition $\nu(x, 0) = f(x)$ concludes

$$
\mathcal{F}(x) \oplus \mathcal{G}(x) = f(x).
$$

(14)

By considering Theorem 4.9 we have

$$
D_{t^H}^\alpha \nu(x, t) = \kappa \odot \mathcal{F}_{t^H}(x + \kappa \frac{t^a}{\alpha})
$$

$$
\odot \kappa \odot \mathcal{G}_{t^H}(x - \kappa \frac{t^a}{\alpha}),
$$

By the initial condition $D_{t^H}^\alpha \nu(x, 0) = g(x)$, we can write

$$
\mathcal{F}_{t^H}(x) \odot \mathcal{G}_{t^H}(x) = \frac{1}{\kappa} g(x)
$$

After integration by using Lemma 3.7

$$
\left(\mathcal{F}(x) \odot \mathcal{F}(0)\right) \odot \left(\mathcal{G}(x) \odot \mathcal{G}(0)\right)
$$

$$
= \frac{1}{\kappa} \int_0^x g(s)ds \Rightarrow \mathcal{F}(x) \odot \mathcal{G}(x)
$$

$$
= \left(\mathcal{F}(0) \odot \mathcal{G}(0)\right) \oplus \frac{1}{\kappa} \int_0^x g(s)ds
$$

(15)

The following system of equations is obtained by Eqs. (19) and (15)

$$
\left\{
\begin{array}{l}
\mathcal{F}(x) \oplus \mathcal{G}(x) = f(x), \\
\mathcal{F}(x) \odot \mathcal{G}(x) = \left(\mathcal{F}(0) \odot \mathcal{G}(0)\right) \oplus \frac{1}{\kappa} \int_0^x g(s)ds,
\end{array}
\right.
$$

such that this system of equations has the following fuzzy solutions

$$
\mathcal{F}(x) = \frac{1}{2} f(x) \oplus \frac{1}{2} \left(\mathcal{F}(0) \odot \mathcal{G}(0)\right) \oplus \frac{1}{2\kappa} \int_0^x g(s)ds,
$$

$$
\mathcal{G}(x) = \frac{1}{2} f(x) \odot \frac{1}{2} \left(\mathcal{F}(0) \odot \mathcal{G}(0)\right) \odot \frac{1}{2\kappa} \int_0^x g(s)ds,
$$

On the other hand, according to Lemma 3.8, $\mathcal{G}(x)$ can be rewritten as follows

$$
\mathcal{G}(x) = \frac{1}{2} f(x) \odot \frac{1}{\kappa} \int_0^x g(s)ds
$$

By substituting these equations for $\mathcal{F}$ and $\mathcal{G}$ into general solution (12), the fuzzy traveling wave solution is obtained as follows

$$
\nu(x, t) = \frac{1}{2} \left(f(x + \kappa \frac{t^a}{\alpha}) \oplus f(x - \kappa \frac{t^a}{\alpha})\right)
$$

$$
\odot \frac{1}{\kappa} \int_0^{x+\kappa \frac{t^a}{\alpha}} g(s)ds
$$

(16)

Here, $\nu(x, t)$ and $D_{t^H}^\alpha \nu$ are $\alpha_{i, g^H}$-differentiable with respect to $t$, and $\nu(x, t)$ and $\nu_{x^H}$ are $[(i) - g^H]$-differentiable with respect to $x$

2-2-gH. Let $\nu(x, t)$ and $D_{t^H}^\alpha \nu$ are $\alpha_{i, g^H}$-differentiable with respect to $t$ and $\nu(x, t)$ and $\nu_{x^H}$ are $[(ii) - g^H]$-differentiable with respect to $x$ then $\mathcal{U}(\xi)$ is a $[(ii) - g^H]$-differentiable fuzzy function. In this case, the main steps of the fuzzy traveling wave method are as follows

**Step 1** Let we can analyze fuzzy one-dimensional homogeneous fractional wave Eq. (7) through the following change variables

$$
\nu(x, t) = \mathcal{U}(\xi), \quad \text{where} \quad \xi = x - \gamma \frac{t^a}{\alpha},
$$

where $\mathcal{U}$ is a continuous function and $g^H$-differentiable in $\xi$ and $\gamma$ is a positive real constant.

**Step 2** We have

$$
\frac{\partial \xi}{\partial t} = -\gamma t^{a-1} < 0, \quad \frac{\partial \xi}{\partial x} = 1 > 0,
$$

therefore, by using Theorem 4.9, the fuzzy multivariate chain rule (Moghaddam and Allahviranloo 2018), we have

$$
D_{\gamma \xi^{H}}^\alpha \nu = t^{1-a} \odot \frac{d_{\gamma \xi^{H}} \mathcal{U}}{d\xi} \odot \frac{\partial \xi}{\partial t} = \odot \gamma \odot \frac{d_{\gamma \xi^{H}} \mathcal{U}}{d\xi},
$$

$$
\Rightarrow D_{\gamma \xi^{H}}^2 \nu = \gamma^2 \odot \frac{d^2_{\gamma \xi^{H}} \mathcal{U}}{d\xi^2},
$$

$$
\nu_{\gamma \xi^{H}} = \frac{d_{\gamma \xi^{H}} \mathcal{U}}{d\xi} \odot \frac{\partial \xi}{\partial x} = \frac{d_{\gamma \xi^{H}} \mathcal{U}}{d\xi},
$$

$$
\Rightarrow \nu_{\gamma \xi^{H}} = \frac{d^2_{\gamma \xi^{H}} \mathcal{U}}{d\xi^2}.
$$
Hence Eq. (7) is reduced to the following fuzzy ordinary differential equations of $\xi$

$$\nu^2 \frac{d^2 i_{i;sH}}{d\xi^2} \oplus_{gH} \kappa^2 \frac{d^2 i_{i;gH}}{d\xi^2} = 0. \tag{17}$$

**Step 3** As we explained in case $1-1-gH$, any solution of the fuzzy fractional wave equation can be obtained as follows

$$v(x,t) = F\left(x + \kappa \frac{t^\alpha}{\alpha}\right) \oplus F\left(x - \kappa \frac{t^\alpha}{\alpha}\right). \tag{18}$$

Equation (18) is a fuzzy solution for Eq. (7); then, the initial condition $v(x,0) = f(x)$ yields

$$F(x) \oplus F(x) = f(x). \tag{19}$$

On the other hand, using Theorem 4.9 and the initial value $D_{i;H}^\alpha v(x,0) = g(x)$, we have

$$v_{ii;gH}(x,t) = \kappa \odot F'_{ii;gH}(x + \kappa \frac{t^\alpha}{\alpha}) \oplus \kappa \odot F'_{ii;gH}(x - \kappa \frac{t^\alpha}{\alpha}),$$

$$\Rightarrow \kappa F'_{ii;gH}(x) \oplus \kappa F'_{ii;gH}(x) = g(x).$$

Integrate each side of the above equation by using Lemma 3.7, therefore

$$\left(-1\right) F(0) \oplus \left(-1\right) F(x) \oplus \left(-1\right) F(0) \oplus \left(-1\right) F(x) = \frac{1}{\kappa} \int_0^x g(s)ds,$$

and

$$F(x) \oplus F(x) = \left(F(0) \oplus F(0) \oplus \left(-1\right) \int_0^x g(s)ds.\right.$$

Consequently, we find that

$$\left\{ \begin{array}{l}
F(x) \oplus F(x) = f(x), \\
F(x) \oplus F(x) = \left(F(0) \oplus F(0) \oplus \left(-1\right) \int_0^x g(s)ds.\right.
\end{array} \right.$$

By solving this system and using Lemma 3.8, the following results are obtained

$$F(x) = \frac{1}{2} f(x) \oplus \frac{1}{2} \left(F(0) \oplus \left(-1\right) \int_0^x g(s)ds.\right. \tag{20}$$

where $v(x,t)$ and $D_{i;H}^\alpha v$ are $a_{ii;gH}$-differentiable with respect to $t$, and $v(x,t)$ and $v_{ss;H}$ are $\left(iii\right) - gH$-differentiable with respect to $x$.

**Example 5.1** Consider the following fuzzy fractional wave equation

$$\left\{ \begin{array}{l}
D_{i;H}^\frac{3}{5} v \oplus_{gH} v_{ss;H} = 0, \\
v(x,0) = \left(3.9x, 6.7x, 9.5x\right), \quad D_{i;H}^\frac{3}{5} v(x,0) = \left(3.9, 6.7, 9.5\right).
\end{array} \right.$$

To find a $1-1-gH$-differentiable solution for this problem, we use Eq. (16)

$$v(x,t) = \frac{1}{2} \left(f\left(x + \kappa \frac{t^\alpha}{\alpha}\right) \oplus f\left(x - \kappa \frac{t^\alpha}{\alpha}\right) \oplus \frac{1}{2} \int_0^x g(s)ds.\right.$$

$$= \left(\frac{3.9}{2} \left((x + \frac{8}{7}t^\frac{7}{7}) + (x - \frac{8}{7}t^\frac{7}{7})\right), \right.$$

$$\frac{6.7}{2} \left((x + \frac{8}{7}t^\frac{7}{7}) + (x - \frac{8}{7}t^\frac{7}{7})\right),$$

$$\frac{9.5}{2} \left((x + \frac{8}{7}t^\frac{7}{7}) + (x - \frac{8}{7}t^\frac{7}{7})\right).$$

$$\oplus \left(\frac{1}{2} \int_0^x +\frac{7}{7}t^\frac{7}{7} \right) 3.9ds, \frac{1}{2} \int_0^x +\frac{7}{7}t^\frac{7}{7} 6.7ds,$$

$$\frac{1}{2} \int_0^x +\frac{7}{7}t^\frac{7}{7} 9.5ds$$

$$= \left(\frac{31.2}{7}t^\frac{7}{7} + 3.9x, \frac{53.6}{7}t^\frac{7}{7} + 6.7x, \frac{76}{7}t^\frac{7}{7} + 9.5x\right).$$

**Example 5.2** Consider the following fuzzy fractional wave equation

$$\left\{ \begin{array}{l}
D_{i;H}^\frac{3}{5} v \oplus_{gH} v_{ss;H} = 0, \\
v(x,0) = (2.1x^2, 5x^2, 7.9x^2), \quad D_{i;H}^\frac{3}{5} v(x,0) = 0.
\end{array} \right.$$
We want to find a $1 - 1 - gH$-differentiable solution for this problem. By Eq. (16) we have

$$v(x, t) = \frac{1}{2} \left( f \left( x + \frac{4}{3}t \right) \oplus f \left( x - \frac{4}{3}t \right) \right)$$

$$= \left( 1.05, 2.5, 3.95 \right) \left( \left( x + \frac{4}{3}t \right)^2 + \left( x - \frac{4}{3}t \right)^2 \right).$$

We plot this solution in Fig. 2.

**Example 5.3** Consider the following fuzzy fractional wave equation

$$\begin{align*}
\mathcal{D}_{gH}^{\alpha} v &= \mathcal{K} \odot v_{xx} \quad (21) \\
v(x, 0) &= f(x),
\end{align*}$$

with the initial condition

$$v(x, 0) = f(x),$$

where $f(x) \in \mathbb{E}$.

**Step 1** To find a traveling wave solution for Eq. (21), consider

$$v(x, t) = \mathcal{W}(\xi), \quad \text{where} \quad \xi = x - \mathcal{K} \frac{t^\alpha}{\alpha},$$

where $\mathcal{W}$ is a continuous function and $gH$-differentiable in $\xi$.

**Step 2** We have

$$\frac{\partial \xi}{\partial t} = \mathcal{K} t^{\alpha-1} < 0, \quad \frac{\partial \xi}{\partial x} = 1.$$

Let $v(x, t)$ is $\alpha_{i.gH}$-differentiable with respect to $t$, and $v(x, t)$ and $v_{xi.gH}$ are $[(i) - p]$-differentiable with respect to $x$. Then $\mathcal{W}(\xi)$ is a $[(i) - gH]$-differentiable fuzzy function and

$$\mathcal{P}_{gH}^{\alpha} v = t^{1-\alpha} \odot \frac{d_{i.gH} \mathcal{W}}{d\xi} \odot \frac{\partial \xi}{\partial t} = \mathcal{K} \odot \frac{d_{i.gH} \mathcal{W}}{d\xi}.$$

$$v_{xi.gH} = \frac{d_{i.gH} \mathcal{W}}{d\xi} \odot \frac{\partial \xi}{\partial x} = \frac{d_{i.gH} \mathcal{W}}{d\xi},$$

$$\Rightarrow v_{xi.gH} = \frac{d_{i.gH} \mathcal{W}}{d\xi^2}.\]
Hence Eq. (21) is reduced to the following fuzzy ordinary differential equation of $\xi$

$$\frac{d^2 g_H U}{d\xi^2} \oplus \frac{d g_H U}{d\xi} = 0.$$  \hspace{1cm} (23)

**Step 3** To find fuzzy solutions for ordinary differential Eq. (23), we need some auxiliary boundary conditions, which in this article, we consider the following auxiliary boundary conditions

$$\lim_{\xi \to \pm \infty} U(\xi) = 0, \quad \lim_{\xi \to \pm \infty} \frac{d U}{d\xi} = 0,$$

$$\lim_{\xi \to \pm \infty} \frac{d^2 U}{d\xi^2} = 0.$$  \hspace{1cm} (24)

We integrate both sides of Eq. (23). According to the auxiliary boundary conditions expressed in Eq. (24), the integration constants are zero and

$$\frac{d g_H U}{d\xi} \oplus U = 0.$$  \hspace{1cm} (25)

This equation has the following fuzzy solution (Armand and Gouyandeh 2017)

$$U(\xi) = \mathcal{C} e^{-\xi},$$

which satisfies the condition $U(\xi) = 0$ when $\xi \to \infty$. Therefore

$$u(x, t) = \mathcal{C} e^{-(x-x^\alpha \frac{t}{\tau})}.$$  \hspace{1cm} (26)

We plot this solution in Fig. 4.

**7 Conclusion**

In this paper, we have defined the generalized Hukuhara conformable fractional derivative and the type of differentiability of this derivative is studied in detail, and we have proved some novel properties for it. The fuzzy traveling wave solution of the fractional wave equation and fractional diffusion equation was obtained by considering the type of $\mathcal{A}_{gH}$-differentiability. To demonstrate the efficiency of the method, the fuzzy traveling wave solutions of some examples were obtained. All results show that this method is a compelling and efficient method for obtaining an analytical solution for the fuzzy linear fractional partial differential equation.

**Acknowledgements** The authors wish to thank the anonymous referees for their comments and criticism. All of their comments were taken into account in the revised version of the paper, resulting in a substantial improvement with respect to the original submission.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.
References

Agarwal RP, Arshad S, ÖRegan D, Lupulescu V (2012) Fuzzy fractional integral equations under compactness type condition. Fract Calc Appl Anal 15:572–590

Allahviranloo T (2021) Fuzzy fractional differential operators and equations. In: Kaprzyk J (eds) Studies in fuzziness and soft computing, vol 397. Springer, Cham. https://doi.org/10.1007/978-3-030-51272-9

Allahviranloo T, Gouyandeh Z, Armand A, Hasanoglu A (2015) On fuzzy solutions for heat equation based on generalized hukuhara differentiability. Fuzzy Sets Syst 265:1–23

Anastassiou GA (2010) Fuzzy mathematics: approximation theory, studies in fuzziness and soft computing. Springer, Berlin

Armand A, Gouyandeh Z (2017) On fuzzy solution for exact second order fuzzy differential equation. Int J Ind Math 9:279–288

Armand A, Allahviranloo T, Gouyandeh Z (2018) Some fundamental results on fuzzy calculus. Iran J Fuzzy Syst 15(3):27–46

Arqub OA, Al-Smadi M (2020) Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions. Soft Comput 24:12501–12522

Arshad S, Lupulescu V (2011) On the fractional differential equations with uncertainty. Nonlinear Anal 75:3685–3693

Bede B (2013) Mathematics of fuzzy sets and fuzzy logic. Springer, London

Bede B, Gal SG (2005) Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. Fuzzy Sets Syst 151:581–599

Bede B, Stefanini L (2011) Solution of fuzzy differential equations with generalized differentiability using ŁU-parametric representation. EUSFLAT 1:785–790

Chalco-Cano Y, Roman-Flores H, Jimenez-Gamero MD (2011) Generalized derivative and π-derivative for set-valued functions. Inf Sci 181:2177–2188

Chalco-Cano Y, Costab TM, Román-Floresc H, Rufián-Lizana A (2020) New properties of the switching points for the generalized Hukuhara differentiability and some results on calculus. Fuzzy Sets Syst. https://doi.org/10.1016/j.fss.2020.06.016

Chang S, Zadeh L (1972) On fuzzy mapping and control. IEEE Trans Syst Cybern 2:30–34

Diamond P (2002) Brief note on the variation of constants formula for fuzzy differential equations. Fuzzy Sets Syst 129:65–71

Ghobaei-Arani M, Souri A (2019) LP-WSC: a linear program- ming approach for autonomic resource provisioning of multitier applications in cloud computing environments. Softw Pract Exp 48:2147–2173. https://doi.org/10.1002/spe.2627

Martynyuk AA, Stamov GT, Stamova IM (2020) Fractional-like Hukuhara derivatives in the theory of set-valued differential equations. Chaos Solitons Fractals 131:109487

Mazandarani M, Xiu L (2021) A review on fuzzy differential equations. IEEE Access 9:62195–62211. https://doi.org/10.1109/ACCESS.2021.3074245

Mohgaddam RG, Allahviranloo T (2018) On the fuzzy Poisson equation. Fuzzy Sets Syst 347:105–128

Puri ML, Ralescu DA (1986) Differentials of fuzzy functions. J Math Anal Appl 114:409–422

Senol M, Atpinar S, Zararsiz Z, Salahshour S, Ahmadian A (2019) Approximate solution of time-fractional fuzzy partial differential equations. Comput Appl Math. https://doi.org/10.1007/s40314-019-0796-6

Shah K, Seadawy AR, Arfana M (2020) Evaluation of one dimensional fuzzy fractional partial differential equations. Alex Eng J In Press. https://doi.org/10.1016/j.aej.2020.05.003

Shahidinejad A, Ghobaei-Arani M (2020) Joint computation offloading and resource provisioning for edge-cloud computing environment: a machine learning-based approach. Softw Pract Exp 50:2212–2230. https://doi.org/10.1002/spe.2888

Shahidinejad A, Ghobaei-Arani M, Esmaeili L (2020) An elastic controller using Colored Petri Nets in cloud computing environment. Clust Comput 23:1045–1071. https://doi.org/10.1007/s10586-019-02972-8

Shahsavari Z, Allahviranloo T, Abbaspandy S, Rostamy-Malkhalifeh M (2020) The traveling wave solution of the fuzzy linear partial differential equation. Appl Appl Math Int J 15(1):Article 23

Stefanini L, Bede B (2009) Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. Nonlinear Anal 71:1311–1328

Viet Long H, Thi Kim Son N, Thi Thanh Tam H (2017) The solvability of fuzzy fractional partial differential equations under Caputo gH-differentiability. Fuzzy Sets Syst 309:35–63

Zureigat H, Ismail AI, Sathasivam S (2020) A compact Crank-Nicholson scheme for the numerical solution of fuzzy time fractional diffusion equations. Neural Comput Appl 32:6405–6412. https://doi.org/10.1007/s00521-019-04148-2

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.