Searching for an Attractive Force in Holographic Nuclear Physics

Vadim Kaplunovsky
Physics Theory Group, University of Texas
1 University Station, C1608, Austin, TX 78712, USA

Jacob Sonnenschein
School of Physics and Astronomy,
The Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel Aviv University, Ramat Aviv, 69978, Israel.

Abstract

We are looking for a holographic explanation of nuclear forces, especially the attractive forces. Recently, the repulsive hard core of a nucleon-nucleon potential was obtained in the Sakai–Sugimoto model, and we show that a generalized version of that model — with an asymmetric configuration of the flavor D8 branes — also has an attractive potential. While the repulsive potential stems from the Chern–Simons interactions of the $U(2)$ flavor gauge fields in 5D, the attractive potential is due to a coupling of the gauge fields to a scalar field describing fluctuations of the flavor branes’ geometry. At intermediate distances $r$ between baryons — smaller than $R_{KK} = O(1)/M_{\text{meson}}$ but larger than the radius $\rho \sim R_{KK}/\sqrt{\lambda_{\text{t Hooft}}}$ of the instanton at the core of a baryon — both the attractive and the repulsive potentials behave as $1/r^2$, but the attractive potential is weaker: Depending on the geometry of the flavor D8 branes, the ratio $C_{a/r} = -V^{\text{attr}}(r)/V^{\text{rep}}(r)$ ranges from 0 to $1/9$. The 5D scalar fields also affect the isovector tensor and spin-spin forces, and the overall effect is similar to the isoscalar central forces, $V(r) \rightarrow (1 - C_{a/r}) \times V(r)$.

At longer ranges $r \gtrsim R_{KK}$, we find that the attractive potential decays faster than the repulsive potential, so the net potential is always repulsive. This unrealistic behavior may be peculiar to the Sakai–Sugimoto-like models, or it could be a general problem of the $N_c \rightarrow \infty$ limit inherent in holography.

*vadim@physics.utexas.edu
†cobi@post.tau.ac.il
1 Introduction

In recent years, holography or gauge-gravity duality gave us a new approach to hadronic physics (see [1] for a review). It has been spectacularly successful at explaining many features of the quark-gluon plasma such as its low viscosity [2], and there are some interesting results concerning the high-density nuclear matter [3, 4]. Motivated by this success, the authors wanted to apply holography to one of the oldest problems of nuclear physics: *The interactions between nucleons are very strong, so why isn’t the nuclear matter relativistic?* Instead, the bulk binding energy of the nuclear matter is only 1.7% of $Mc^2$, about 16 MeV per nucleon.

The usual explanation of this puzzle involves a near-cancellation between the attractive and the repulsive nuclear forces: The attractive potential is only a little bit stronger than the repulsive potential, and the difference is rather small. For example, in the Walecka’s mean-field model [5], the attractive potential due to $\sigma$-meson field is 400 MeV while the repulsive potential due to $\omega$-meson field is 350 MeV; there is also the Fermi motion energy of about 35 MeV/nucleon, so the net binding energy is only 16 MeV/nucleon. There have been many similar (but more elaborate) models since Walecka, but they all beg the same question:
Why is the attractive $-\langle \overline{\Psi} \Psi \rangle^2$ interaction between nucleons only a little bit stronger than the repulsive $+\langle \overline{\Psi} \gamma^\mu \Psi \rangle^2$ interaction? Is this a coincidence depending on quarks having precisely 3 colors and the right masses for the $u$, $d$, and $s$ flavors? Or is this a more robust feature of QCD that would persist for different $N_c$ and any quark masses (as long as two flavors are light enough)?

The most direct way of applying holography to these issues would be to build a holographic model of the bulk nuclear matter. Unfortunately, this approach is troubled by the large $N_c$ limit which is inherent in the Holographic QCD. Indeed, even taking the leading $1/N_c$ corrections into account is very hard in HQCD because it requires doing string loop calculations on the “gravity” side of the gauge-gravity duality. But for large $N_c$, the low-temperature low-pressure phase of the bulk nuclear matter becomes a crystalline solid instead of the Fermi liquid for the real-life $N_c = 3$, and its other properties — such as density or the binding energy — could also be quite different.

Paradoxically, direct holographic modeling works for exotic phases of nuclear matter — such as the quark-gluon plasma and maybe the high-density solid phase, if it exists but not for the good old nuclei themselves. However, we may still use holography to obtain phase-independent features of nucleons and nuclear forces, but relating those features to the experimental properties of real nuclei has to be done by some other methods. So in this article, instead of trying to model whole nuclei, we focus on a rather humble problem of obtaining an attractive two-body nuclear force from the Holographic QCD.

The first holographic model of a baryon appeared in [8] and [9] in the AdS$_5 \times S^5$ context: A D5 brane wrapping the $S^5$ had $N_c$ strings attached to it, while the opposite ends of those strings connected to the $N_c$ external quarks at the boundary of the AdS$_5$ space. Similar “external” baryons were constructed in confining backgrounds in [10]. To make a baryon out of dynamical rather than external quarks one needs to add $N_f$ flavor branes to the holographic model; usually one takes $N_f \ll N_c$ so the flavor branes act as probes of the background created by the color branes. A prototypical model of this kind was constructed by Sakai and Sugimoto [11]: Starting with the Witten’s model of $N_c$ D4–branes on a

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1Similar to the liquid helium solidifying under pressure, the $N_c = 3$ nuclear matter may also have a crystalline high-pressure phase. Although at very high pressures and densities, the nucleons are believed to merge into a quark liquid, and it is not clear if the nucleons form a lattice before merging, of if there is a direct transition from the nuclear liquid to the quark liquid. If the solid nuclear phase exists at all and behaves like a semi-classical crystal, then its structure should not depend much on the $N_c$ and it could be modeled using large–$N_c$ methods such as the skyrmion lattices of [6], or the holographic instanton lattices of [7]. Alas, judging the phenomenological success of such models is rather difficult because the high-pressure nuclear matter is hard to study experimentally; the best data comes from modeling neutron stars, and we still do not know for sure if their interiors are solid or liquid.
circle (with antiperiodic boundary conditions for the fermions to break the $\mathcal{N} = 4$ SUSY to $\mathcal{N} = 0$), they have added $N_f$ D8 and $N_f$ \( \overline{D8} \) branes. On the gravity side of the duality, the 10D geometry is warped $\mathbb{R}^{1,3} \times S^4 \times \text{a cigar}$, while the D8 and \( \overline{D8} \) branes connect to each other and span $\mathbb{R}^{1,3} \times S^4 \times \text{a U-shaped line on the cigar}$ (see figure (1) on page 20). A holographic baryon comprises a D4 brane wrapping a compact $S^4$ and $N_c$ open strings connecting this D4 to the flavor D8 branes. To minimize the baryon’s energy, the D4 brane acting as a baryonic vertex becomes embedded in the D8 branes [13], and for $N_f > 1$ it dissolves into an instanton of the $U(N_f)$ gauge theory on the flavor branes.

Sakai, Sugimoto et al wrote several papers [14, 15] about properties of such holographic baryons, and eventually [16] worked out a repulsive force between two such baryons. But they could not get an attractive force because of the accidental $\mathbb{Z}_2$ symmetry of the antipodal configuration of the flavor branes. To see the connection, note that in the large $N_c$ limit, the nuclear forces are dominated by the single-meson-exchange diagrams; the repulsive central forces come from exchanges of the vector mesons while the attractive central forces come from the scalar mesons. In holography, the 4D vector mesons are modes of the gauge fields living on the flavor branes, while the 4D scalar fields are modes of the scalar fields parametrizing transverse motion of those branes. In the original version of the Sakai–Sugimoto model, the D8 and \( \overline{D8} \) branes cross the color D4 branes at antipodal points of the $S^1$ circle, hence the name “antipodal model”. On the gravity side of the gauge-gravity duality, the $S^1$ becomes the circular dimension of the cigar, and the combined D8 + \( \overline{D8} \) branes stretch along the cigar’s diameter. The two sides of this diameter are symmetric, and this leads to the $\Phi \rightarrow -\Phi$ symmetry of the 9D scalar fields parametrizing the transverse motion of the flavor branes; in 4D terms, this $\mathbb{Z}_2$ symmetry flips the signs of all the scalar meson fields, $\phi_i(x) \rightarrow -\phi_i(x)$. But this symmetry does not affect the gauge fields on the flavor branes and hence the 4D vector mesons or the holographic baryons. Consequently, in the antipodal model, the baryons have Yukawa couplings to the vector mesons but not to the scalar mesons, and that’s why there are no attractive nuclear forces but only the repulsive forces.

In this article, we investigate nuclear forces in the non-antipodal version [17] of the Sakai–Sugimoto model. Without the accidental $\mathbb{Z}_2$ symmetry, the baryons should have Yukawa couplings to both vector and scalar mesons, and indeed we find both repulsive and attractive forces. Unfortunately, the attractive forces are too weak and the net force is repulsive at all distances, so our model of HQCD is not too realistic. Specifically, at intermediate distances $r$ between two nucleons — shorter than the ranges of the 4D Yukawa forces but longer than
the size of a baryon’s core — both the repulsive and the attractive potentials behave like 5D Coulomb potential and scale like $1/r^2$. But the attractive potential has a smaller coefficient,

$$C_{a/r} \equiv \frac{-V_{\text{attractive}}}{V_{\text{repulsive}}} = \frac{1}{9} \times (1 - \zeta^{-3}), \quad (1.1)$$

where $\zeta \geq 1$ parametrizes the geometry of the flavor D8 branes: The near-antipodal models have $\zeta \approx 1$ and $C_{a/r} \ll 1$ while the far-from-antipodal models have $\zeta \gg 1$ and $C_{a/r} \approx \frac{1}{9}$. In any case, $C_{a/r} < \frac{1}{9} < 1$ and the attractive nuclear potential is weaker than the repulsive.

At longer distances, nuclear forces are dominated by the 4D Yukawa potentials of the lightest mesons with the right quantum numbers, $J^{PC} = 1^{--}$ for the repulsive force and $J^{PC} = 0^{++}$ for the attractive force, thus

$$V_{\text{repulsive}} \propto \exp\left(-\frac{r \times m_{\text{vector}}^{\text{lightest}}}{r}\right), \quad V_{\text{attractive}} \propto -\exp\left(-\frac{r \times m_{\text{scalar}}^{\text{lightest}}}{r}\right). \quad (1.2)$$

In real life, the lightest isoscalar scalar meson $\sigma(600)$ is lighter than the lightest isoscalar vector meson $\omega(787)$, so at long distances the attraction wins over the repulsion. But in the Sakai–Sugimoto models — antipodal or non-antipodal — the lightest scalar meson has more than twice the mass of the lightest vector. Consequently, the attractive force has a shorter range than the repulsive force, and the net nuclear force is repulsive at all distances.

We don’t know why the meson spectra — and hence the nuclear forces — in the Sakai–Sugimoto model are so unrealistic. It could be something peculiar to the model’s setup, hopefully to be remedied by some future holographic models. But it could also be a general problem of the large $N_c$ limit; indeed, the QCD origin of the $\sigma(600)$ meson is poorly understood, and it’s not clear if for $N_c \to \infty$ it continues to exist or disappears from the spectrum. The best way to resolve this issue would be to find the $\sigma$ resonance and its mass in a lattice QCD calculation for several values of $N_c$, then extrapolate to $N_c \to \infty$. Alternatively, once we have several different holographic models, we can compare their predictions for the meson spectra in general and for the lightest scalar meson in particular. Either way, this issue will have to wait for future research.

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2 In all versions of the Sakai–Sugimoto model, the 5D instanton at the core of a baryon has a very small size $\rho \sim R_{KK}/\sqrt{\lambda}$ where $R_{KK}$ is the Kaluza–Klein scale of extra dimensions and $\lambda = N_c g^2_{\text{YM}} \gg 1$ is the ’t Hooft coupling. On the other hand, the 4D mesons have masses $M_{\text{meson}} \sim 1/R_{KK}$ so the Yukawa forces have ranges $R_{\text{Yukawa}} \sim R_{KK} \gg \rho$. This is quite different from the real-life mesons and baryons where $\rho_{\text{baryon}} \sim 4R_{\text{Yukawa}}$.

3 Actually, since in real life $N_c = 3 \ll \infty$, the longest-range component of the attractive force is not the single-sigma-meson exchange but rather the double-pion exchange. The Yukawa range of this force is $1/2m_\pi$, which is significantly longer than $1/m_\sigma$. 

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The rest of this paper is organized as follows. In the next section (§2) we explain the problems with the large \( N_c \) limit of nuclear physics. First, we explain why large \( N_c \) makes the nuclear matter solid rather than liquid. Next, we discuss the \( N_c \to \infty \) limit of the nuclear forces and what happens to the \( \sigma(600) \) meson. Finally, we bring up the issue of separating nucleons from other baryonic species such as \( \Delta \).

Section 3 is a review of the Sakai–Sugimoto model and its antipodal and non-antipodal versions. In particular, we derive the effective 5D Lagrangian for the \( U(2) \) flavor gauge fields (for simplicity we work with two flavors), then realize a holographic baryon as a lowest-energy YM instanton and calculate its mass and radius. Section 4 explains general properties of the holographic nuclear forces in the near, intermediate, and far zones; the three zones are illustrated in the diagram (4) on page 23. We also summarize the calculation by Hashimoto et al [16] of the repulsive force in the intermediate zone.

Section 5 is the core of this paper, that’s where we calculate the attractive and repulsive nuclear forces in the intermediate and far zones. In §5.1 we derive the effective 5D theory of scalar and vector fields living on the flavor branes. We show that the abelian vector and scalar fields give rise to 5D Coulomb forces between \( SU(2) \) instantons. In the near and intermediate zones, both the repulsive potential due to abelian vector and the attractive potential due to the abelian scalar have the same \( 1/r^2 \) dependence, but the attractive potential has a smaller coefficient as in eq. (1.1). In §5.2 we leverage this result to obtain both isoscalar and isovector forces between two spinning nucleons at intermediate distances from each other. The isovector spin-spin and tensor forces stem from the small overlap between the \( SU(2) \) instantons implementing the two nucleons and their interactions with the abelian vector and scalar fields. Our analysis follows Hashimoto et al [16], but taking the scalar fields into account reduces the isovector forces by the same overall factor \( 1 - C_a/r \) as the net isoscalar repulsive – attractive force. Thus,

\[
V_{\text{net}}(r, I_1, I_2, J_1, J_2) = (1 - C_a/r) \times V_{\text{net}}(r, I_1, I_2, J_1, J_2)[\text{without scalar fields}]. \tag{1.3}
\]

In §5.3 we consider the attractive forces in the far zone. Since in the Sakai–Sugimoto model the lightest scalar meson is heavier than the lightest vector meson, the attractive force decays with distance \( r \) faster than the repulsive force, so the net isoscalar central force is always repulsive. We also consider the long-range isovector tensor force due to pions. Although the pions are zero modes of the 5D vector fields and have nothing to do with the 5D scalar fields, the pion-nucleon coupling depends on the baryon’s radius \( \rho \) which is affected by the scalar-mediated forces in the near zone. Consequently, the isovector force due to pion exchange is

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reduced by the overall factor \((1 - C_{a/r})\). Likewise, all other isovector forces in the far and intermediate zones are reduced by the same overall factor.

Our calculation in §5 are based on Yang–Mills approximation for the effective Lagrangian for the flavor gauge fields. In section 6 we investigate the validity of this approximation by working with a complete non-abelian DBI+CS Lagrangian for the fields on the flavor branes. We show that although the \(SU(2)\) gauge fields become strong \((2\pi\alpha'F_{MN} \sim g_{MN})\) near the center of a baryon, the self-duality of those fields (in four non-compact space dimensions of the D8 branes) leads to cancellation of all the higher-order \(\text{tr}(F^4), \text{etc.}\), terms in the expansion of the DBI Lagrangian. Consequently, all our calculation in §5 are valid to the leading order in \(1/\lambda\). The leading post–YM effect in 5D is a small \((O(1/\lambda))\) correction to the self-duality condition for the \(SU(2)\) gauge fields due to abelian vector and scalar fields. To see how this correction affects a stand-alone semiclassical baryon, in §6 we minimize the DBI + CS action of an \(SO(4)\)–symmetric instanton-like field configuration with a general radial profile and show that the minimum is very close to good old YM instanton of the same radius as we had in §5. Calculating the non-abelian DBI action involves computing a symmetrized trace; this is done in the Appendix.

Finally, section 7 summarizes our results and makes suggestions for future research.

### 2 Limitations of the \(N_c \to \infty\) Limit and Holography

The large \(N_c\) limit is inherent in all holographic QCD methods, and this poses a problem for the aspects of nuclear physics that are different between the small \(N_c\) and the large \(N_c\) regimes. In particular, the bulk nuclear matter at zero temperature and pressure (but finite density) forms a quantum liquid for small \(N_c\) — such as real-life \(N_c = 3\) — but becomes a crystalline solid for large \(N_c\).

To see how this works, consider a condensed matter analogy — some atoms which attract to each other at long or medium distances but have repulsive hard cores. Semi-classically, at zero temperature and pressure such atoms always form some kind of a crystal; it takes strong quantum effects to put the atoms into some other phase such as liquid or super-solid. Of particular importance is the kinetic energy of the zero-point quantum motion of atoms confined to narrow potential wells,

\[
K \sim \frac{\pi^2 h^2}{2M_{\text{atom}}(\text{well diameter})^2},
\]  

(2.1)
or rather its ratio $K/U$ to the potential binding energy $U$ per atom. According to Newton Bernardes [18], this ratio is related to the de Bour parameter $\Lambda_B$ of the inter-atomic potential as

$$\frac{K}{U} \approx 11\Lambda_B^2,$$  

where $r_c$ is the radius of the atomic hard core and $\epsilon$ is the maximal depth of the potential. For small de Bour parameters, the quantum corrections to the semi-classical approximation are weak and the crystal remains stable at zero pressure. For larger $\Lambda_B$, the quantum corrections due to kinetic energy become important, and when $\Lambda_B$ exceeds a critical value somewhere between 0.2 and 0.3 [19], the crystal melts into a quantum liquid. For example, helium atoms have $\Lambda_B = 0.306$ and hence $K/U \approx 1$ while neon atoms have $\Lambda_B = 0.063$ and hence $K/U \approx 0.05$; consequently, at zero temperature and zero pressure helium is a quantum liquid while neon is a crystalline solid.

To see how the $K/U$ ratio of the nuclear matter depends on the number of colors, we note that in the large $N_c$ limit, the leading nuclear forces are proportional to $N_c$. Specifically, according to Kaplan and Manohar [20],

$$V(\vec{r}, I_1, I_2, J_2; N_c) = N_c \times A_C(r) + N_c \times A_S(r)(I_1 I_2)(J_1 J_2)$$

$$+ N_c \times A_T(r)(I_1 I_2) \left[ 3(nJ_1)(nJ_2) - (J_1 J_2) \right] + O(1/N_c).$$ (2.3)

for the same $N_c$-independent radial profiles $A_C, A_S, A_T$ of the central, spin-spin, and tensor potentials. Classically, such potentials would like to arrange a many-nucleon system in some kind of a crystal with $N_c$-independent nearest-neighbor distance $\sim 1$ fm, while the binding energy of a nucleon in such a crystal would be proportional to the $N_c$. Indeed, all models of nuclear matter based on semi-classical models of nucleons form such crystals, for example skyrmion crystals of ref. [6]. In the quantum theory, nucleons in such a lattice have zero-point kinetic energies (2.1) where the well diameter is independent on $N_c$ while the nucleon’s mass $M \propto N_c$, hence $K \propto 1/N_c$ and

$$\frac{K}{U} \propto \frac{N_c^{-1}}{N_c^{+1}} = \frac{1}{N_c^2}.$$ (2.4)

4Melting releases the individual atoms form narrow potential wells, which significantly lowers their kinetic energies (2.1). It also, moves the atoms away from the minima of the attractive potential, which lowers the potential binding energy $U$. The overall effect on the net $K-U$ energy per atom depends on the $K/U$ ratio: For low ratios the potential energy is more important and the crystal is stable, but for high ratios lowering the kinetic energy becomes advantageous and the crystal melts.
We may estimate the coefficient of this proportionality using the de Bour parameter $\Lambda_B$. The maximal depth of the central potential between two nucleons is about 100 MeV for $N_c = 3$, so we take it to be $\epsilon \sim N_c \times 30$ MeV for large $N_c$. Likewise, we take the nucleon mass to be $M_N \sim N_c \times 300$ MeV and hard-core radius $r_c \sim 0.7$ fm regardless of $N_c$. Consequently,

$$\Lambda_B = \frac{\hbar}{r_c \sqrt{2M\epsilon}} \sim \frac{2}{N_c} \Rightarrow \frac{K}{U} \sim \frac{45}{N_c^2}$$ \hspace{1cm} (2.5)$$

and hence liquid nuclear matter for $N_c \lesssim 8$ and solid nuclear matter for $N_c \gtrsim 8$.

The numerical coefficient in eq. (2.5) and hence our estimate $N_c^{\text{crit}} \sim 8$ for the dividing line between liquid and solid bulk nuclear matter (at low pressures and temperatures) should be taken with a large grain of salt. Also, the transition between liquid nuclear matter for $N_c = 3$ and crystalline nuclear matter for large $N_c$ may go through some exotic phases at intermediate values of $N_c$, perhaps something like a quantum supersolid, perhaps something more exotic without known condensed-matter analogues. But regardless of the details of this transition, in the large $N_c$ limit the potential energy of interacting near-static nucleons becomes much larger than the nucleons’ kinetic energies, and the bulk nuclear matter at $T = 0, P = 0$ conditions becomes a conventional semi-classical crystal. The structure of such crystals can be modeled holographically — and indeed there is active research in this direction (for instance [7]) — but we have no experimental data to compare to the models because real-life nuclei with $N_c = 3$ are liquid rather than solid.

Meanwhile, instead of trying to build holographic models of complete nuclei we focus on holographic models of the nuclear forces. But even at the level of the two-body forces, the large $N_c$ limit maybe different from the real-life case of just 3 colors. Of particular concern is the isoscalar attractive force due to exchanges of the $\sigma(600)$ scalar mesons between the nucleons. In real life, this is a major component of the net attractive force — especially at the medium-long distances between the nucleons — but in the large $N_c$ limit this component may weaken or disappear because the $\sigma(600)$ meson itself may become heavier or even disappear from the scalar meson spectrum.

The $\sigma(600)$ (also known as $f_0(600)$) is the lightest isoscalar true-scalar meson. In real life, it appears as a very broad resonance of two pions — so broad that its central mass is somewhat controversial and different experimentalists locate it anywhere between 400 meV and 700 MeV, and sometimes even higher, cf. references in the Particle Data Group’s listing [21]. But the real controversy about the $\sigma(600)$ resonance is its physical origin. Unlike the heavier $I^G = 0^+, J^{CP} = 0^{++}$ mesons $f_0(980), f_0(1370)$, etc., the $\sigma(600)$ meson does not exist
in the non-relativistic quark model so for many years R. L. Jaffe and others [22, 23, 24, 25] were claiming that the \( \sigma(600) \) is not a true \( q\bar{q} \) meson but a \( qq\bar{q}\bar{q} \) tetraquark. Specifically, it’s a molecule-like bound state of two pions which exists because the \( \rho \)-meson exchanges in the \( t \)-channel induce an attractive \( s \)-channel force between the pions. If this claim is true, then the \( \sigma \) resonance goes away in the large \( N_c \) limit because the forces between pions become weak as \( 1/N_c \).

But many other authors (see [26] for a sample) identify the \( \sigma(600) \) with the \( \sigma \) field of the linear sigma model of the chiral symmetry breaking. Or rather, the massive \( \sigma(x) \) field parametrizing fluctuations of magnitude of the symmetry-breaking VEV \( \langle \Psi\bar{\Psi} \rangle \) gives rise to primordial sigma quanta, while the real sigma mesons \( \sigma(600) \) are quantum mixtures of those primordial quanta with the \( |\pi\pi\rangle \) states (and to lesser extent with the other \( I^G = 0^+, \ J^{PC} = 0^{++} \) mesons). From this point of view, the non-relativistic quark model is irrelevant because the quarks do not become non-relativistic until after the chiral symmetry has already been broken. Indeed, the NRQM does not see that the pions are (pseudo) Goldstone bosons, so the fact that it does not see the sigma meson at all is simply another limitation of the NRQM as far as the chiral symmetry breaking is concerned. If this point of view is right, then the sigma meson exits for all \( N_c \). For large \( N_c \) limit, this meson is mostly a quantum of the \( \sigma(x) \) field — its mixing with \( |\pi\pi\rangle \) and other states becomes weak — and it’s a narrow resonance rather than a broad hump we have for \( N_c = 3 \), but it remains a dominant resonance in the \( I^G = 0^+, \ J^{PC} = 0^{++} \) \( \pi\pi \) channel, and its mass should not be too different from the real-life 600 MeV.

The other mesons — scalar or vector, isoscalar or isovector — are unlikely to be disturbed by the large \( N_c \) limit, so their contributions to the nuclear forces would be similar to real-life QCD. IF the \( \sigma(600) \) meson remains in the spectrum in the large \( N_c \) limit and if its mass remains similar to the real-life 600 MeV, then the entire nuclear potential (2.3) for \( N_c \to \infty \) would be similar to what it is in real life, except for the overall factor \( N_c \). In particular, the net central potential \( V_c(r) \) would be repulsive at short distances (the hard core) but

\footnote{In the non-relativistic quark model, all \( 0^{++} \) mesons have \( S = 1 \) and \( L = 1 \). Consequently, the lightest \( 0^{++} \) meson should be heavier than the lightest \( 1^{-} \) mesons \( \rho(770) \) or \( \omega(787) \) that have \( S = 1 \) but \( L = 0 \). Depending on the assumptions one makes about the forces between the quark and the antiquark, this argument identifies the lightest true \( q\bar{q} \) meson with \( I^G = 0^+ \) and \( J^{PC} = 0^{++} \) as either \( f_0(980) \) or \( f_0(1370) \). In any case, the \( \sigma(600) \) resonance is way too light to be a p-wave \( q\bar{q} \) state, so it has to be something else.}
attractive at medium and long distances:

\[ V_c(r) \]

\[ N_c \to \infty \text{ limit with a light } \sigma \text{ meson} \quad (2.6) \]

On the other hand, if the \( \sigma(600) \) meson disappears from the spectrum for large \( N_c \), or if it becomes heavier than the lightest vector meson, then the dominant attractive force would become shorter-ranged than the repulsive force, and the net force at medium and long distances would be repulsive rather than attractive:

\[ V_c(r) \]

\[ N_c \to \infty \text{ limit without the } \sigma \text{ meson} \quad (2.7) \]

In this scenario, at large \( N_c \) the nuclear force is repulsive at all distances, and there are no bound nuclei at all, liquid or crystalline.

So what really happens to the sigma-meson and to the nuclear forces at large \( N_c \)? The best way to settle this controversy would be to find the \( \sigma \) resonance and its mass in a lattice QCD calculation for several values of \( N_c \). Such a calculation would require a realistic pion mass (unlike most present-day lattice calculations extrapolating from \( m_\pi \geq 350 \text{ MeV} \)) and rather large lattices to distinguish the sigma resonance from the two-pion continuum, so it
may be too hard for the present-day computers. But thanks to the Moore’s Law, finding the \( \sigma \) resonance on a lattice should become possible in a not-too-distant future.

Alternatively, we may try to resolve the issue using holography. Although a holographic model of real QCD — or rather, of QCD with large \( N_c \) — is yet to be constructed, several known models seem to be qualitatively similar, so we can compare their predictions for the meson spectra in general, and for the lightest true scalar meson in particular. However, the models that seem qualitatively similar to QCD may not be similar enough, and their predictions could be widely off target. Indeed, the predictions of different models have turned out to be quite different from each other. For example, in the Sakai–Sugimoto model which we use in this article, the lightest true scalar meson is more than twice as heavy as the lightest vector meson. Consequently — as we shall see in painful detail in section 5 — the net nuclear force is everywhere repulsive and looks like (2.7) rather than like (2.6). On the other hand, in the highly-non-antipodal version of the Dymarsky–Kuperstein–Sonnenschein model \[27\], the lightest \( J^{CP} = 0^{++} \) meson is much lighter than any other mesons (except the pions) \[29\]. However, this lightest scalar is a pseudo-Goldstone boson of the approximate conformal symmetry of the flavor sector, so it is not clear how much attractive force it can mediate. As of this writing, it is not clear if the net nuclear potential in this model looks like the real-life potential (2.6) or like the everywhere-repulsive potential (2.7) we calculate in this paper for the Sakai–Sugimoto model.

But suppose tomorrow somebody discovers a holographic model of the real QCD and — miracle of miracles — it has a realistic spectrum of mesons, including the \( \sigma (600) \) resonance, and even the realistic Yukawa couplings of those mesons to the baryons. Even for such a model, the two-body nuclear forces would not be quite as in the real world because the semi-classical holography limits \( N_c \to \infty, \lambda \to \infty \) suppress the multiple meson exchanges between baryons. Although in this case, the culprit is not the large number of colors but the large ’t Hooft coupling \( \lambda = N_c g_{YM}^2 \).

Indeed, from the hadronic point of view, nuclear forces arise from the nucleons exchanging one, two, or more mesons, and in real life the double-meson exchanges are just as important as the single-meson exchanges. In particular, since the lightest mesonic state with \( I^G = 0^+, J^{CP} = 0^{++} \) quantum numbers is a pair of un-bound pions, the longest-range isoscalar attractive force between nucleons comes from exchanges of two pions rather than of any single mesons. In holography, the single-meson exchanges happen at the tree level of the string theory while the multiple meson exchanges involve string loops \((k - 1 \text{ loops for } k \text{ mesons})\), and the loop amplitudes are suppressed by the powers of \( 1/\lambda \) relative to the tree
amplitudes.

Naively, one would expect the loop amplitudes to carry additional factors of \(1/N_c\) (which is dual to the string coupling) rather than \(1/\lambda\), or maybe both \(1/\lambda\) and \(1/N_c\) factors, but the naive power-of-\(N_c\) counting does not work for loop amplitudes involving baryons made of \(N_c\) quarks.\(^6\) Indeed, in honest QCD with a large number of colors, the multi-meson-exchange contributions to the non-relativistic effective potential for the baryons are **not** suppressed by powers of \(1/N_c\) [27]. However, the extra powers of \(N_c\) due to \(N_c\) quarks in a baryon are not accompanied by the extra powers of \(\lambda\), so in holography, the contributions of the multiple meson exchanges are suppressed, albeit by powers of \(1/\lambda\) rather than \(1/N_c\).

To see how this works in a general holographic model of QCD with \(N_c \gg N_f\), note that such a model starts with a string-theoretic construction where the colors and the flavors live on separate branes. For large \(N_c\) and large \(\lambda\), the color branes become black branes producing curvature and fluxes through the bulk, which provide a non-trivial background for degrees of freedom living in the bulk itself as well as on the flavor branes. The bulk degrees of freedom are dual to the pure-color sector of QCD (glueballs, *etc.*), while the vector and scalar fields living on the flavor branes are dual to the \(q\bar{q}\) mesons. The flavor fields have rather weak couplings to each other: in 5D terms,

\[
g_{5D,\text{flavor}} \sim \frac{\sqrt{R_{KK}}}{\sqrt{\lambda N_c}}
\]

so the 4D mesons — which are modes of the 5D vector and scalar fields with wave functions \(\psi \sim R_{KK}^{-1/2}\) — have couplings to each other of the order

\[
g_{MMM} \sim \frac{1}{\sqrt{\lambda N_c}}.
\]  

A holographic baryon is made from some brane spanning only the compact dimensions that is connected to the flavor branes by \(N_c\) strings, although this construction is often equivalent to an instanton of the 5D flavor gauge fields. Consequently, the baryon-meson coupling is

\(^6\)The authors thank Alexei Cherman and Thomas Cohen for bringing this fact to our attention after we made a mistake in the first version of this paper.
enhanced by an extra factor of $N_c$,

$$g_{MBB} \sim N_c \times \frac{1}{\sqrt{\lambda} N_c} = \frac{\sqrt{N_c}}{\sqrt{\lambda}}. \quad (2.10)$$

At the tree level of the baryon-meson theory, scattering of two baryons proceeds through a single-meson exchange, which produces a $O(N_c/\lambda)$ amplitude,

$$A^{\text{tree}} \sim g_{MBB}^2 \sim \frac{N_c}{\lambda}. \quad (2.11)$$

At the one-loop level, there are two types of diagrams, the triangle diagrams such as

$$A^\Delta \sim g_{MBB}^3 \times g_{MMM} \sim \frac{N_c}{\lambda^2} \quad (2.12)$$

and the box and crossed-box diagrams

$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
with amplitudes

\[ A^\Box \sim g^4_{MBB} \sim \frac{N_c^2}{\lambda^2}. \tag{2.14} \]

that carry an extra power of \( N_c \). However, Banerjee et al showed \cite{27} that for non-relativistic baryons, the box and the crossed-box diagrams almost cancel each other from the effective potential between the baryons, with the un-canceled part having a lower power of the \( N_c \). Banerjee et al did not pay any attention to the powers of \( \lambda \), but clearly the un-canceled sub-leading terms in the box and crossed-box diagrams cannot carry higher powers of the 't Hooft coupling than the leading terms \( (2.14) \), thus

\[ A_{\text{uncanceled}}^\Box \sim \frac{N_c}{\lambda^2} \sim A^\Delta \sim \frac{1}{\lambda} A^{\text{tree}}. \tag{2.15} \]

In other words, the contribution of the double-meson exchange and other one-loop processes to the 2–body nuclear potential carries the same power of \( N_c \) but is suppressed by a factor \( 1/\lambda \) compared to the tree-level single-meson exchange.

To be precise, the large \( \lambda \) limit suppresses exchanges of the \textit{un-bound} meson pairs but not of the meson-meson resonances — which become narrow (because of weak \( g_{MMM} \)) and act as single mesons exchanged between the two baryons. In particular, in the isoscalar \( 0^{++} \) channel that gives rise to the dominant attractive force between nucleons, the \( \lambda \to \infty \) limit suppresses the contribution of the unbound two-pion continuum, but it replaces it with a discrete set of \( f_0 \) resonances. In a good holographic model of QCD (which alas has not been found yet), the \textit{overall strength} of the \( 0^{++} \) channel should be similar to the real QCD, so it would produce a similar isoscalar attractive force \textit{at short distances}. However, the range of this attractive force would be significantly shorter: Instead of decaying with distance like \( \exp(-2m_\pi r) \) as in real life, the holographic attractive force decays as \( \exp(-m_0 r) \) where \( m_0 \) is the mass of the lightest isoscalar \( 0^{++} \) meson, presumably \( \sigma(600 \text{ MeV}) \).

In principle, the isoscalar \( 1^{--} \) channel that gives rise to the dominant repulsive force suffers from similar corrections in the \( \lambda \to \infty \) limit. But in practice, the strongest and the longest-range contribution to this channel comes from exchanges of a single \( \omega(787) \) meson, so suppressing the multi-meson exchanges in this channel would not make a qualitative difference. Thus altogether, the net effect of large 't Hooft coupling on the central nuclear potential — besides the overall \( 1/\lambda \) factor — is the shortening of the attractive tail at long
distances:

\[ \lambda \times V_c(r) \]

blue: real QCD, \( \lambda \sim 1 \)
red: best possibility for holographic QCD, \( \lambda \gg 1 \) \hfill (2.16)

However, this optimistic picture presumes a holographic model of QCD that correctly reproduces (a) the overall strength of the isoscalar \( 0^{++} \) and \( 1^{--} \) channels, and (b) the mass spectra of vector and scalar mesons, especially the masses of the lightest \( 0^{++} \) and \( 1^{--} \) mesons \( \sigma(600) \) and \( \omega(787) \). But thus far, no known model satisfies these requirements, not even approximately, so the nuclear forces they produce could be much more different from the real life than \hfill (2.16). In particular, the nuclear force we calculate in this paper for the Sakai–Sugimoto model turns out to be everywhere repulsive:

\[ V_c(r)/N_c \]

blue: real QCD
red: Sakai–Sugimoto model \hfill (2.17)

Now let’s go back to the large \( N_c \) limit — in holography or in honest QCD — and consider yet another general problem with baryons made from many quarks: How to separate the nucleons with \( I = J = \frac{1}{2} \) from the other kinds of baryons such as \( \Delta \) with \( I = J = \frac{3}{2} \)? In real life, there is a large mass gap between the nucleons and the \( \Delta \) baryons — almost 300 MeV — but for large \( N_c \) this gap shrinks as \( 1/N_c \). At the same time, the two-baryon
potential grows like $N_c$, so for large $N_c$ it becomes stronger than the gap. Consequently, two interacting nucleons may “forget” their individual spins and isospins and mix up with other baryonic species such as $\Delta$. In fact, for large $N_c$ there is a whole lot of baryonic species with $I = J$ ranging from $\frac{1}{2}$ (for odd $N_c$) or 0 (for even $N_c$) all the way up to $N_c/2$, and a strongly-interacting nucleon might mix up with all of them. While such mixing would not affect the isoscalar spin-blind central force between two baryons, it might significantly enhance the isovector spin-spin and tensor forces.

Therefore, comparing the two-baryon forces in the large $N_c$ limit to the real-life two-nucleon forces is rather tricky. One has to carefully keep track of the spin and isospin degrees of freedom of the two baryons, expand the interaction Hamiltonian into central, spin-spin, and tensor forces as in eq. (2.3), and then compare the radial profiles $A_C(r)$, $A_S(r)$, and $A_T(r)$. Moreover, the spin and isospin degrees of freedom require quantum mechanical treatment because semi-classically, we do not get definite spins or isospins even for stand-alone single baryons. Instead, we get skyrmions, or instantons, or some other kind of solitons with a definite orientation of the $SU(2)_{\text{isospin}}$ relative to the $SU(2)_{\text{spin}}$; in quantum terms, they become superpositions of baryons with all possible $I = J = \frac{1}{2}, \frac{3}{2}, \ldots, \infty$. Consequently, a force between two such semiclassical baryons is not a force between two nucleons but rather a superposition of forces between different baryonic species.

This problem affected the first holographic calculation of the nuclear forces by K. Y. Kim and I. Zahed [30]. Their baryons were semiclassical instantons in the Sakai–Sugimoto model, so instead of definite $|I, I_z, J, J_z\rangle$ they had a definite direction $\mathbf{n}$ in $S^3 = SU(2)_{\text{isospin}} \times SU(2)_{\text{spin}} / SU(2)_{\text{common}}$. Consequently, Kim and Zahed [30] found that the force between two baryons depends on the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ — it was attractive for some angles and repulsive for other — but they could not interpret this angular dependence in terms of the isovector spin-spin and tensor forces. By comparison, Hashimoto, Sakai, and Sugimoto [16] made a similar calculation using properly quantized collective coordinates for each instanton. Consequently, they obtained the force between two nucleons rather than some mixed-up baryons, and they could see how this force depends on each nucleon’s $I_z$ and $J_z$. In particular, they saw that at medium-short distances, the net force between two nucleons is always repulsive. Evidently, the attraction Kim and Zahed saw for some relative orientations of semiclassical baryons happens only for high spins and isospins, but not for nucleons with $I = J = \frac{1}{2}$.

On the other hand, the analysis of Hashimoto et al was limited to the first-order perturbation theory for nucleons that are far enough from each other to avoid the strong mixing
of spins and isospins. This approach will not work for the hard-core region at very short distances where the interactions are much stronger than the gaps between states of the individual baryons. In the hard core, the semiclassical analysis of Kim and Zahed might work better than the perturbative expansion of Hashimoto et al, although comparing the semi-classical large–$N_c$ results to the real-life nuclear forces might be problematic.

To summarize, the large $N_c$ limit of nuclear physics suffers from three major problems. The third problem of baryon mixing is only technical, and it can be solved — at least for the medium and long distances between the nucleons — by following Hashimoto et al rather than Kim and Zahed. But there are no ways around the first problem of different phase structures of nuclear matter with $N_c = 3$ and with $N_c \to \infty$. Even at high pressures and densities, there is a difference: For $N_c = 3$, squeezing nucleons together makes them merge into a quark liquid, while for $N_c \to \infty$ the nucleons always retain their individual identities and a would-be quark liquid suffers from the “chiral density wave” instability. It is possible that at some intermediate pressures and densities the $N_c = 3$ nucleons form a crystal — just like helium solidifies at high pressures — before merging into a quark liquid. If this intermediate-pressure phase of real nuclear matter is ever observed in a lab, or can be reliably shown to exist in some exotic but observable places like inferiors of neutron stars, it would be very interesting to compare its properties to the holographic models. Until then, we can only speculate.

Finally, the second problem — concerning the fate of the $\sigma(600)$ resonance in the large $N_c$ limit and its effect on the attractive nuclear force — is solvable in principle, but it has not been solved yet. In holography, this problem is aggravated by using QCD-like models in lieu of the presently unknown holographic dual of the real QCD. The meson spectra of such models are not quite realistic; for example, in the Sakai–Sugimoto model (both antipodal and non-antipodal versions) there is no $\sigma$ resonance and the lightest scalar meson has more than twice the mass of the lightest vector meson. Consequently, we shall see in section 5 that in this model, the attractive force is both weaker and shorter-ranged than the repulsive force, so the net nuclear force is always repulsive. This could be a peculiar failing of the Sakai–Sugimoto model, or it could be the general problem of holography or even of the large $N_c$ limit. Hopefully, future research will resolve this issue.
3 Baryons in the Non-antipodal Sakai-Sugimoto Model

The Sakai–Sugimoto [11] model is based on placing a set of $N_f$ D8 and anti-D8 flavor branes into the gravitational background of $N_c$ coincident near-extremal D4 branes [12]. We take the $N_c \gg N_f$ limit, so the flavor branes are treated as probes. The color D4 branes span the Minkowski spacetime an a compact circle of radius $R$ in the direction $x_4$. In the holographic limit, these branes become a background comprised of the following metric, RR 4–flux, and dilaton:

$$
\begin{align*}
    ds^2 &= \left( \frac{u}{R_{D4}} \right)^{3/2} \left[ -dt^2 + \delta_{ij} dx^i dx^j + f(u) dx_4^2 \right] + \left( \frac{R_{D4}}{u} \right)^{3/2} \left[ \frac{du^2}{f(u)} + u^2 d\Omega_4^2 \right] \\
    F_4 &= \frac{2\pi N_c}{V_4} \times \epsilon_4, \quad e^\phi = g_s \left( \frac{u}{R_{D4}} \right)^{3/4} 
\end{align*}
$$

(3.1)

where

$$
\begin{align*}
    R_{D4}^3 &= \pi g_s N_c l_s^3, \quad f(u) = 1 - \left( \frac{u \Lambda}{u} \right)^3, \\
    V_4 &= \text{the volume of the unit sphere } \Omega_4 \text{ and } \epsilon_4 \text{ is its 4–volume form, } l_s = \sqrt{\alpha'} \text{ is the string length, and } g_s \text{ is the string coupling.}
\end{align*}
$$

(3.2)

The manifold spanned by the coordinates $u$ and $x_4$ has the topology of a cigar with tip at $u_{\text{min}} = u_\Lambda$; the flavor branes span a continuous line on that cigar, see figure [11]. In order to avoid a conical singularity at the tip of the cigar, the radius $R$ of the $x_4$ circle has to satisfy

$$
2\pi R = \frac{4\pi}{3} \left( \frac{R_{D4}^3}{u_\Lambda} \right)^{1/2}.
$$

(3.3)

Consequently, the Kaluza–Klein scale of the model is

$$
M_\Lambda = \frac{1}{R} = \frac{3}{2} \frac{u_\Lambda^{1/2}}{R_{D4}^{3/2}}.
$$

(3.4)

while the confining string tension [31] is

$$
T_{\text{str}} = \left. \frac{1}{2\pi \ell_s^2} \sqrt{g_{xx} g_{tt}} \right|_{u = u_\Lambda} = \frac{1}{2\pi \ell_s^2} \left( \frac{u_\Lambda}{R_{D4}} \right)^{3/2}.
$$

(3.5)

Besides the line on the $(u, x_4)$ cigar, the D8 branes span the Minkowski spacetime and the compact $S^4$ sphere. All $N_f$ branes are coincident, and the action of this brane stack has
a Dirac–Born–Infeld (DBI) term and a Chern–Simons (CS) term,

\[ S_{\text{D8}} = S_{\text{DBI}} + S_{\text{CS}}. \]  

(3.6)

The DBI term is

\[ S_{\text{DBI}} = T_8 \int d^9 x \, e^{-\phi} \text{str} \left( \sqrt{-\det(g_{mn} + 2\pi\alpha' F_{mn})} \right) \]  

(3.7)

where \( T_8 = (2\pi)^{-8} l_s^{-9} \) is the D8-brane’s tension, \( g_{mn} \) is the nine-dimensional induced metric on the branes,

\[ dS_{\text{D8}}^2 = \left( \frac{u}{R} \right)^{\frac{3}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + \left[ \left( \frac{u}{R} \right)^{\frac{3}{2}} f(u) + \left( \frac{R}{u} \right)^{\frac{3}{2}} \frac{1}{f(u)} \left( \frac{du}{dx_4} \right)^2 \right] dx_4^2 + \left( \frac{R}{u} \right)^{\frac{3}{2}} u^2 d\Omega_4^2, \]  

(3.8)

\[ F_{MN} \] is the \( U(N_f) \) gauge field strength on the worldvolume of the D8-branes, and \( \text{str} \) is the symmetrized trace over the flavor indices. The 9D Chern–Simons term involves the 5D gauge fields and the RR flux on the \( S^4 \); after integration over the \( S^4 \) it becomes the usual 5D CS term,

\[ S_{\text{CS}} = \frac{N_c}{24\pi^2} \int \text{tr} \left( A F^2 - \frac{i}{2} A^3 F - \frac{1}{10} A^5 \right). \]  

(3.9)

In the geometry of the cigar, the \( D8 \) and the \( \bar{D}8 \) branes cannot continue indefinitely in the negative \( u \) directions, and they have no place to end. Instead, the branes connect to the antibranes and form continuous U-shaped lines, whose geometry satisfies classical equation of motion for the DBI action (3.7). The solutions form a family parametrized by the \( u_0 \geq u_\Lambda \), the lowest point on the brane stack. Figure (1) below illustrates two such solutions: The physical meaning of the \( u_0 \) parameter is the “string endpoint mass” of a quark defined such that the mass of a stringy meson involving a non-spinning string of length \( L \) is \( M_{\text{meson}} = T_{\text{str}} L + 2m_{q}^{\text{SEP}} \). In terms of the brane geometry, this endpoint mass is \( m_{q}^{\text{SEP}} = \frac{1}{2\pi\alpha'} \int_{u_\Lambda}^{u_0} \sqrt{-g_{uu} g_{uu}} \, du = \frac{1}{2\pi\alpha'} \int_{u_\Lambda}^{u_0} f^{-1/2}(u) \, du. \)  

(3.10)

The original Sakai–Sugimoto model assumed the antipodal configuration of the flavor branes, as shown on the right side of the figure (1); for this configuration \( u_0 = u_\Lambda \) and \( m_{q}^{\text{SEP}} = 0 \). But comparing holographic predictions to for the meson and baryon masses to the experimental data shows that non-zero end-point mass gives a better fit. Likewise, the width of a meson due to its decay into two mesons \( [33] \), the best fit to Regge trajectories, and matching baryonic
properties \[13\], all favor non-zero end-point masses and hence non-antipodal models depicted on the left side of the figure \[11\].

In section 6 we shall consider the full DBI + CS action for the flavor branes, but most of our analysis will be based on the weak-gauge-field approximation

\[
S_{\text{DBI}}[g,F] = S_{\text{DBI}}^0[g] + S_{\text{YM}} + \mathcal{O}(F^4) \tag{3.11}
\]

where

\[
S_{\text{DBI}}^0 = \frac{N_f T_8 V_4}{g_s} \int d^4x \sqrt{\eta^{\mu\nu}} \left( u^8 f(u) + \frac{P_{D4}^3}{f(u)} \left( \frac{du}{dx_4} \right)^2 \right) \tag{3.12}
\]

is the action for the metric with switched-off gauge fields. To simplify this action — as well as the YM action for the gauge fields — it’s convenient to change the cigar coordinate along the branes from \(u\) or \(x_4\) to \(z\) defined according to \[13\]

\[
u = u_A (\zeta^3 + \zeta z^2)^{1/3}, \quad \zeta = \frac{u_0}{u_A}. \tag{3.13}
\]

The reason for this choice is that both \(x_4(z)\) and \(u(z)\) are single-valued smooth functions (unlike the \(x_4(u)\) function which is double-valued and singular at \(u = u_0\)). Unlike in \[11\], here
$z$ is dimensionless; its values run from $-\infty$ to $+\infty$. The $\zeta$ parameter is 1 for the antipodal model while the non-antipodal models have $\zeta > 1$.

The Yang-Mills part of the action (3.11) is obtained by expanding (3.7) to the lowest non-trivial order in gauge fields. In terms of the $z$ coordinate,

$$S_{YM} = -\kappa \int d^4x \, dz \, \text{tr}\left( \frac{1}{2} h(z; \zeta) F_{\mu\nu}^2 + M_A^2 k(z; \zeta) F_{\mu z}^2 \right)$$  \hspace{1cm} (3.14)$$

where

$$\kappa = \lambda N_c \frac{1}{(216 \pi^3)} , \quad \lambda = N_c g^2_{YM}, \quad h(z; 1) = (1 + z^2)^{-1/3}, \quad k(z; 1) = 1 + z^2, \quad (3.15)$$

and the full expressions for $h(z; \zeta)$ and $k(z; \zeta)$ with $\zeta > 1$ are given in equation (4.2) of [13].

For the case of two flavors, the $U(2)$ gauge fields can be decomposed as

$$A^m = A^m_{SU(2)} + \frac{1}{2} \hat{A}^m_{U(1)} = \frac{1}{2} A^m a \times \tau^a + \frac{1}{2} \hat{A}^m \times 1_{2 \times 2}$$  \hspace{1cm} (3.16)$$

where $A^m$ are the $SU(2)$ gauge fields, $\tau^a$ are the Pauli matrices, and $\hat{A}^m$ is the abelian gauge field in $U(1) \subset U(2)$. For a baryon, the $SU(2)$ gauge fields form an instanton in the 4 space dimensions $(1, 2, 3, z)$, while Chern–Simons coupling induces the abelian electric field $\hat{A}_0$. After rescaling the coordinates and truncating equations of motion to the leading terms in $1/\lambda$, the gauge fields take form [16] [13]

$$A^c^l_M(x^i, \tilde{z}) = iv(x) \times g \, \partial_M g^{-1}$$

$$v(x) = \frac{\xi^2}{\xi^2 + \rho^2}, \quad \xi = \sqrt{(x^i - X^i)^2 + (\tilde{z} - \tilde{Z})^2},$$

$$g(x^i, \tilde{z}) = \frac{(\tilde{z} - \tilde{Z}) \, 1 - i(x^i - X^i) \, \tau_i}{\xi} (i = 1, 2, 3),$$

$$\hat{A}^c^l_0 = \frac{27 \pi}{\lambda \zeta} \frac{\xi^2 + 2 \rho^2}{(\xi^2 + \rho^2)^2},$$

$$A^c^l_M = \hat{A}^c^l_M = 0,$$  \hspace{1cm} (3.17)$$

where $M, N = 1, 2, 3, z$ and $\tilde{z} = \frac{3}{8^4 \pi^5} \times \frac{z}{M_A}$. Note that there is a critical value for $\zeta$, namely, $\zeta = (5/8)^{1/3}$, but this critical value is unphysical — all the Sakai–Sugimoto models have $\zeta \geq 1$ since the branes cannot go below the tip of the cigar. Substituting the fields (3.17) into the YM + CS action and minimizing with respect to the baryon radius $\rho$ gives the
baryon’s classical mass

$$M_{\text{cl}} = 8\pi^2 \kappa \left( \zeta + \frac{18\pi}{\lambda \zeta} \sqrt{\frac{8\zeta^3 - 5}{10}} \right)$$

for

$$\rho_{\text{cl}}^2 = \frac{81\pi}{\lambda} \sqrt{\frac{2}{40\zeta^3 - 25}}$$

(3.18)

Note that similar to the antipodal case, the baryon’s radius scales as $\lambda^{-1/2}$, only the numerical coefficient is different for $\zeta \neq 1$.

By repeating the analysis of [14] for the non-antipodal models, Seki and Sonnenschein [13] obtained the mass spectrum for baryons with different $I = J$ and radial quantum numbers, the mean-square charge radii, the isoscalar and isovector magnetic moments, etc., etc., as functions of $\zeta$ and $M_{\Lambda}$. The found the best fit to experimental data obtain for an un-physical $\zeta \approx 0.942$, which indicates that the generalized Sakai–Sugimoto model is not a very accurate description of real-life baryons. Nevertheless, in this article we shall stick with the non-antipodal Sakai–Sugimoto models simply because its the only model of holographic QCD we know in detail.

4 Summary of the Repulsive Force

In real life, the nucleon has a fairly large radius compared to the ranges of mesonic Yukawa forces (except pion’s), $R_{\text{nucleon}} \sim 4/M_{\rho_{\text{meson}}}$. But in the holographic nuclear physics with $\lambda \gg 1$, we have the opposite situation: While the meson masses are $O(M_{\Lambda})$ and the 4D Yukawa forces have $O(1/M_{\Lambda})$ ranges, the baryon has a much smaller radius $R_{\text{baryon}} \sim \lambda^{-1/2}/M_{\Lambda}$, cf. eq. (3.18). Thanks to this hierarchy, the nuclear forces between two baryons
at distance \( r \) from each other fall into 3 distinct zones:

\[ R_{\text{baryon}} \sim 1/M_\Lambda \sqrt{\lambda} \]
\[ M_{\text{meson}}^{-1} \sim 1/M_\Lambda \]

(4.1)

In the near zone \( r \lesssim R_{\text{baryon}} \ll (1/M_\Lambda) \), the two baryons overlap and cannot be approximated as two separate instantons of the \( SU(2) \) gauge field; instead, we need the ADHM solution of instanton number = 2 in all its complicated glory. On the other hand, in the near zone, the nuclear force is five-dimensional: the curvature of the fifth dimension \( z \) does not matter at short distances, so we may treat the \( U(2) \) gauge fields as living in a flat 5D spacetime. To leading order in \( 1/\lambda \), the \( SU(2) \) fields are given by the ADHM solution, while the abelian \( \hat{A}_0(\vec{x},z) \) is the 5D Coulomb field coupled to the instanton density \( (1/(32 \pi^2)) \epsilon^{0KLMN} \text{tr}(F_{KL}F_{MN})_{\text{ADHM}} \). Unfortunately, for two overlapping baryons this density has a rather complicated profile, which makes calculating the near-zone nuclear force rather difficult.

The far zone \( r \gtrsim (1/M_\Lambda) \gg R_{\text{baryon}} \) poses the opposite problem: The curvature of the 5D space and the \( z \)-dependence of the gauge coupling becomes very important at large distances. At the same time, the two baryons become well-separated instantons which may be treated as point sources of the 5D abelian field \( \hat{A}^0 \). In 4D terms, the baryons act as point sources for all the massive vector mesons \( A^\mu_n(x) \) comprising the massless 5D vector field \( A^\mu(x,z) \), hence
the nuclear force in the far zone is the sum of 4D Yukawa forces,

\[
V(r) = \frac{N_c^2}{4\kappa} \sum_n |\psi_n(z = 0)|^2 \times \frac{e^{-m_n r}}{4\pi r} \tag{4.2}
\]

where \(m_n = O(M_\Lambda)\) are the vector meson’s masses and \(\psi_n(z)\) are their wave functions in the curved fifth dimension. At the inner edge of the far zone, all the 4D vector mesons contribute to the potential (4.2) but for larger distances, the lightest vector meson becomes dominant.

In the intermediate zone \(R_{\text{baryon}} \ll r \ll (1/M_\Lambda)\), we have the best of both situations: The baryons do not overlap much and the fifth dimension is approximately flat. At first blush, the nuclear force in this zone is simply the 5D Coulomb force between two point sources,

\[
V(r) = \frac{N_c^2}{4\kappa} \times \frac{1}{4\pi^2 r^2} = \frac{27\pi N_c}{2\lambda M_\Lambda} \times \frac{1}{r^2} \quad \text{(for } \zeta = 1) \tag{4.3}
\]

This \(1/r^2\) behavior of the repulsive potential suggests that the intermediate zone of the holographic nuclear force corresponds to the repulsive hard core of the real-life nucleons.

The real-life hard-core repulsion has both isoscalar and isovector components of comparable strengths, but the potential (4.3) is purely isoscalar. The reason for this discrepancy is that the point-source approximation of holographic baryons is too crude for the intermediate zone where two instantons of size \(\rho = R_{\text{baryon}}\) have \(O((R_{\text{baryon}}/r)^2)\) effects on each other. Moreover, since the size of a stand-alone baryon is a compromise between two subleading effects — the \(U(1)\) Coulomb repulsion and the \(z\)-dependence of the gauge couplings \(\kappa h(z)\) and \(\kappa k(z)\) — the baryons are linearly sensitive to anything affecting the leading \(SU(2)\) gauge fields. Hence, the overlap between two baryons gives rise to an additional \(1/r^2\) nuclear force of strength comparable to (4.3), and since the overlap depends on the baryon’s relative isospins, this extra force has an isovector component.

To properly account for the baryon-baryon overlap, Hashimoto, Sakai, and Sugimoto [16] (and also Kim and Zahed [30]) set the \(SU(2)\) gauge fields to the self-dual ADHM solution of instanton number = 2. The instanton density \(I(\vec{x}, z) = (1/32\pi^2)e^{0KLMN}\text{tr}(F_{KL}F_{MN})_{\text{ADHM}}\) of this solution deviates by \(O((R_{\text{baryon}}/r)^2)\) from the sum of two separate instantons, and that has two effects: (A) the \(U(1)\) Coulomb energy is significantly different from (4.3), and (B) the width of the instanton density in \(z\) direction is different, which changes the \(SU(2)\) field’s energy since the gauge coupling depends on \(z\). Somehow, the two effects cancel out from the isoscalar components of the hard-core potential, but they do give rise to isovector forces of comparable magnitude. Specifically, for baryons of spin = isospin = \(\frac{1}{2}\) — i.e., for
two nucleons — Hashimoto et al obtained

\[ V(r, n) = \frac{27\pi N_c}{2\lambda M_A} \times \frac{1}{r^2} \times \left( 1_{\text{naive}} + \frac{64}{5}(I_1 \cdot I_2)(n \cdot J_1)(n \cdot J_2) \right) \quad (n = \frac{\vec{r}}{r}) \quad (4.4) \]

for the intermediate-zone distances \( r \). Note that the isoscalar component of this hard-core potential is precisely as in the naive eq. (4.3), it’s the isovector component that has really needed all the hard work.

5 Attractive Forces in the Non-Antipodal Model

5.1 5D Scalars and their Interactions.

In 4D, the attractive forces between two baryons emerge from exchanges of virtual mesons with even spins and positive parity, especially the true scalars \( 0^+ \). In the holographic theory, the baryons are instantons of the 5D non-abelian gauge fields, while the 4D scalar mesons are modes of the 5D scalar field \( \Phi(x) \), so to get an attractive nuclear force we need a 5D scalar-vector coupling of the form

\[ S_{5D} \supset \int d^5x \, \Phi \times \text{tr}(F_{MN}^2). \quad (5.1) \]

In the Sakai–Sugimoto model, the 5D scalar \( \Phi(x) \) describes deviations of the D8 brane stack from its equilibrium position in the \((u, x^4)\) plane.\(^7\) For the non-antipodal version of the model, such deviations have a \( \delta u \) component, and since the local 5D gauge coupling depends on \( u \), we get \( \delta u(\Phi) \times \text{tr}(F_{MN}^2) \) interactions as in eq. (5.1) and hence the attractive force. Unfortunately, for the antipodal model worked out by Sakai and Sugimoto themselves, the equilibrium brane stack lies along the radius \( x^4 = \text{const} \) (or rather two opposite radii), so the first-order deviations are in the \( x^4 \) direction only and have no \( \delta u \) component. Consequently, there is no linear \( \Phi \times \text{tr}(F_{MN}^2) \) coupling in 5D — in fact, there is exact \( Z_2 \) symmetry \( \Phi \rightarrow -\Phi \) which forbids it — and that’s why there is no attractive nuclear force in the antipodal model.

In this section, we shall derive the effective 5D Lagrangian — including the crucial coupling (5.1) — for the non-antipodal model, and then use it to derive the attractive nuclear force for the intermediate distance. Our first step is a precise definition of the scalar

\(^7\)In addition to the isosinglet scalar \( \Phi(x) \) which describes the motion of the whole D-brane stack, there are also isotriplet scalar fields \( \Phi^a(x) \) which describe the relative motion of the two D8 branes. For the moment, let us focus on the isosinglet \( \Phi \), we shall return to the isotriplets later in this section.
field $\Phi(x)$ in terms of the D8–brane stack deviation from its equilibrium position. Using some kind of a non-singular coordinate $w$ along the brane stack, we define:

In equilibrium:  
$$u = \bar{u}(w), \quad x^4 = \bar{x}^4(w)$$  
(5.2)

development:  
$$u(w, x^\mu) = \bar{u}(w) + \pi\alpha' \Phi(w, x^\mu),$$  
(5.3)

$$x^4(w, x^\mu) = \bar{x}^4(w) - \beta(w) \times \pi\alpha' \Phi(w, x^\mu),$$  
(5.4)

where $$\beta(w) = \frac{g_{w4}(\bar{u}) \times (d\bar{u}/dw)}{g_{44}(\bar{u}) \times (d\bar{x}^4/dw)}.$$  
(5.5)

The last formula here assures that to first order in $\Phi$, the deviation of the stack is locally perpendicular to the stack itself.

In eqs. (5.2–5.5) $w$ is a generic non-singular coordinate along the un-perturbed brane stack, for example $\bar{z} = [(\bar{u}^3 - u_0^3)/u_0a^2\Lambda^3]^{1/2}$ (but not the original $z$ which would be affected by the deviation field $\Phi$). However, to simplify the 5D notations we would like to have the same metric for all five dimension, $g_{ww} = g_{11}$, at least for $\Phi = 0$. This calls for

$$dw^2 = f(\bar{u})(d\bar{x}^4(w))^2 + \frac{R_{D4}^3}{\bar{u}^3 f(\bar{u})}(d\bar{u}(w))^2,$$  
(5.6)

which together with the brane equilibrium equation

$$\left(\frac{d\bar{x}^4}{d\bar{u}}\right)^2 = \frac{R_{D4}^3}{\bar{u}^3 f^2(\bar{u})} \times \frac{u_0^8 f(u_0)}{\bar{u}^8 f(\bar{u}) - u_0^8 f(u_0)}$$  
(5.7)

gives us

$$\frac{d\bar{u}}{dw} = \sqrt{\frac{\bar{u}^8 f(\bar{u}) - u_0^8 f(u_0)}{\bar{u}^5 R_{D4}^3}}, \quad \frac{d\bar{x}^4}{dw} = \frac{u_0^4 \sqrt{f(u_0)}}{\bar{u}^4 f(\bar{u})},$$  
(5.8)

implicitly defining the $w$ coordinate. We could not solve these equations analytically, but fortunately we would not need the explicit formulae in this paper. All we will need to know is that

for small $w$,  
$$\bar{u}(w) = \zeta u_\Lambda \left(1 + \frac{8\zeta^3 - 5}{9\zeta^2} \times (M_\Lambda w)^2 + O((M_\Lambda w)^4)\right).$$  
(5.9)
For the above definitions, the 5D metric for \( x^M = (x^\mu, w) \) becomes
\[
d s_{5d}^2 = \left( \frac{\bar{u} + \pi \alpha' \Phi}{R} \right)^{3/2} \times \left[ \eta_{MN} \, d x^M \, d x^N + \frac{\bar{u}^5 R_{D4}^3 (\pi \alpha')^2}{u_0^8 f(u_0)} \times (\partial_M \Phi \, d x^M)^2 
- \frac{8 \pi \alpha' \Phi}{\bar{u}} \times d w^2 + O(\Phi^2) \right]
\]
while the \( S^4 \) radius and the dilaton depend on \( \Phi \) according to
\[
\text{radius}[S^4] = R_{D4}^{3/4} (\bar{u} + \pi \alpha' \Phi)^{1/4}, \quad e^\phi = g_s \left( \frac{\bar{u} + \pi \alpha' \Phi}{R_{D4}} \right)^{3/4}.
\]
Consequently, expanding the DBI action
\[
S_{\text{DBI}} = T_8 \int d^5 x \, e^{-\phi} \text{Vol}(S^4) \text{str} \left( \det (g_{MN} + 2 \pi \alpha' F_{MN}) \right)^{1/2}
\]
to the second power in \( F_{NM} \) and \( \partial_M \Phi \) and to the first power in \( \Phi \) itself, we get
\[
S_{\text{DBI}} = \int d^4 x \, d w \left( \text{const} + \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}} \right), \tag{5.13}
\]
\[
\mathcal{L}_{\text{kin}} = \frac{R_{D4}^3}{48 \pi^4 g_s \ell_s^9} \left\{ \bar{u}(w) \times \frac{1}{2} \text{tr}(F_{KL}^2) + \frac{\bar{u}(w)^9}{u_0^8 f(u_0)} \times \frac{1}{2} (\partial_M \Phi)^2 \right\} \tag{5.14}
\]
\[
= \frac{N_c \lambda M_N \zeta}{216 \pi^3} \left\{ \left( \frac{\bar{u}(w)}{u_0} \right) \times \frac{1}{2} \text{tr}(F_{KL}^2) + \frac{(\bar{u}(w)/u_0)^9}{1 - \zeta^{-3}} \times \frac{1}{2} (\partial_M \Phi)^2 \right\}
\]
\[
\langle \langle \text{using the } \eta^{MN} \text{ to contract the 5D indices}. \rangle \rangle
\]
\[
\mathcal{L}_{\text{int}} = \frac{N_c}{48 \pi^2} \left( -3 \Phi \times \frac{1}{2} \text{tr}(F_{\mu \nu}^2) + 5 \Phi \times \text{tr}(F_{\mu \nu}^2) \right) + O(\Phi^2). \tag{5.15}
\]

Thus far, we have focused only on the isosinglet scalar field \( \Phi \) describing the common motion of the two flavor D8 branes but ignored the isotriplet scalars \( \Phi^a \) describing the relative motion of the two D8 branes. Fortunately, we may easily add the \( \Phi^a \) to the 5D theory by applying the \( U(2) \) symmetry to the effective Lagrangian (5.14–5.15). Thus, without re-
expanding the DBI action for two separated branes, we immediately obtain

\[ L_{\text{kin}} = \frac{N_c \lambda M \Lambda \zeta}{216 \pi^3} \left\{ \left( \frac{\bar{u}(w)}{u_0} \right) \times \left( \frac{1}{4} (F_{MN}^a)^2 + \frac{1}{4} \hat{F}_{MN}^2 \right) + \frac{\left( \frac{\bar{u}(w)}{u_0} \right)^9}{1 - \zeta^{-3}} \times \left( \frac{1}{2} (D_M \Phi^a)^2 + \frac{1}{2} (\partial_M \Phi)^2 \right) \right\}, \]  

\[ L_{\text{int}} = \frac{N_c}{48 \pi^2} \text{tr} \left( (\Phi = \Phi + \Phi^a \tau^a) \times ( - \frac{3}{4} (F^2_{\mu\nu}) + 5 (F^2_{\mu w})) \right) + O(\Phi^2) \]

\[ = \frac{N_c}{48 \pi^2} \left\{ \Phi \times \left( - \frac{3}{4} (F^a_{\mu\nu})^2 + \frac{5}{2} (F^a_{\mu w})^2 \right) + \Phi \times \left( - \frac{3}{4} (\hat{F}_{\mu\nu})^2 + \frac{5}{2} (\hat{F}_{\mu w})^2 \right) + 2 \Phi^a \times \left( - \frac{3}{4} F^a_{\mu\nu} \hat{F}_{\nu\mu} + \frac{5}{2} F^a_{\mu w} \hat{F}_{\mu w} \right) \right\} + O(\Phi^2). \]  

(5.17)

However, for the two-baryon system we are interested in, the isotriplet scalars fields \( \Phi^a \) are much weaker than the isosinglet field \( \Phi \) because they have much weaker sources. Indeed, for two static baryons, the \( SU(2) \) gauge fields are purely magnetic in 5D sense, \( i.e., F_{0i}^a = F_{0w}^a = 0 \), while the \( U(1) \) gauge fields are purely electric, \( \hat{F}_{ij} = \hat{F}_{iw} = 0 \). Consequently, on the last line of eq. (5.17), both \( F^a_{\mu\nu} \hat{F}_{\nu\mu} = 0 \) and \( F^a_{\mu w} \hat{F}_{\mu w} = 0 \), which leaves the isotriplet scalars \( \Phi^a \) with no source at all.

For the baryons with non-zero spins, the \( SU(2) \) electric fields do not exactly vanish, but they are much weaker than their magnetic counterparts. Specifically,

\[ \frac{F_{\text{el}}^{SU(2)}}{F_{\text{mag}}^{SU(2)}} \sim \left( \frac{E_{\text{spin}}}{E_{\text{static}}} \right)^{1/2} \sim \frac{J}{M_{\text{baryon}} R_{\text{baryon}}} \sim \frac{1}{\sqrt{\lambda}} \times \frac{J}{N_c} \ll 1. \]  

(5.18)

At the same time, the abelian electric fields are also much weaker than the non-abelian magnetic fields. Indeed, were it not for the Chern–Simons interactions

\[ \mathcal{L}_{\text{CS}} = \frac{N_c}{96 \pi^2} \text{tr} (\epsilon^{JKLMN} A_J F_{KL} F_{MN} + \cdots) \]

\[ = \frac{N_c}{64 \pi^2} \hat{A}_J \times \epsilon^{JKLMN} (\text{tr}(F_{KL} F_{MN}) + \frac{1}{3} \hat{F}_{KL} \hat{F}_{MN}) \]

\[ = \frac{1}{2} N_c \hat{A}_J \times (\text{instanton current})^J + \cdots, \]  

(5.19)

the baryons would not generate any abelian fields at all. As it is, for baryons of radius \( \rho \),
the non-abelian magnetic fields are

\[ F_{\text{mag}}^{SU(2)} = \begin{cases} O(1/\rho^2) & \text{in the near zone,} \\ O(\rho/r^3) & \text{in the intermediate zone,} \end{cases} \tag{5.20} \]

while the abelian electric fields are

\[ \hat{F}_{\text{el}}^{U(1)} = \begin{cases} O(N_c/\kappa \rho^3) & \text{in the near zone,} \\ O(N_c/\kappa r^3) & \text{in the intermediate zone,} \end{cases} \tag{5.21} \]

where \( \kappa = O(N_c \lambda M) \) is the 5D kinetic-energy coefficient, thus in both zones we have

\[ \frac{\hat{F}_{\text{el}}^{U(1)}}{F_{\text{mag}}^{SU(2)}} \sim \frac{N_c}{\kappa \rho} \sim \frac{1}{\lambda M \rho} \sim \frac{1}{\sqrt{\lambda}} \ll 1. \tag{5.22} \]

As to the abelian magnetic fields, they are generated by the Chern–Simons terms involving \( F_{\text{mag}}^{SU(2)} \times F_{\text{el}}^{SU(2)} \), so they are even weaker than the electric abelian fields. Altogether, we have a hierarchy of gauge fields

\[ F_{\text{mag}}^{SU(2)} \ll F_{\text{el}}^{U(1)} \ll F_{\text{el}}^{SU(2)} \ll F_{\text{mag}}^{U(1)}. \tag{5.23} \]

Consequently, the scalar-vector interaction Lagrangian (5.17) provides a much stronger source for the isosinglet scalar field \( \Phi \) than to the isotriplet fields \( \Phi^a \), which leads to the scalar field hierarchy

\[ \frac{\Phi^a}{\Phi} \sim \frac{1}{\lambda} \times \frac{J}{N_c} \ll 1. \tag{5.24} \]

Hence, the nuclear forces due to the triplet \( \Phi^a \) are much smaller then the forces due to the singlet \( \Phi \), so we shall disregard the \( \Phi^a \) through the rest of this article.

Focusing on the singlet scalar \( \Phi \), we see that the dominant source for it comes from the \( SU(2) \) magnetic fields, so to the leading order in \( 1/\lambda \) we may approximate

\[ \mathcal{L}_{\text{int}} \approx \frac{N_c}{48\pi^2} \Phi \times \left( -\frac{3}{4}(F_{ij})^2 + \frac{5}{2}(F_{iw})^2 \right) + \text{unimportant}. \tag{5.25} \]

Moreover, in the near and intermediate zones where the 5D gauge coupling is approximately constant — \( \kappa(w) = \kappa_0 \times (\bar{u}(w)/u_0) \approx \kappa_0 \) for \( |w| \ll M_\Lambda^{-1} \), cf. eq. (5.9), — the \( SU(2) \) fields
are self-dual in the 4 space dimensions,

$$F_{ij}^a(\vec{x}, w) = \epsilon_{ijk} F_{kw}^a(\vec{x}, w)$$  \hspace{1cm} (5.26)

Their specific form is given by the ADHM self-dual solution with instanton number = 2, but fortunately we don’t need the gory details of this solution here. All by itself, the self-duality assures us that

$$\frac{1}{4} (F_{\mu\nu}^a)^2 = \frac{1}{2} (F_{\mu\nu})^2 = \frac{1}{16} \epsilon^{0KLMN} (F_{KL}^a F_{MN}^a) \equiv 4\pi^2 \times \text{instanton density } I(\vec{x}, w)$$  \hspace{1cm} (5.27)

and hence, the scalar field $\Phi$ couples to the same instanton number density as the abelian vector potential $\hat{A}_0$,

$$\mathcal{L}_{\text{int}} = \frac{N_c}{48\pi^2} \Phi \times \left( -3 \times 4\pi^2 I + 5 \times 4\pi^2 I \right) = \frac{N_c}{6} \Phi \times I, \hspace{1cm} (5.28)$$

$$\text{cf. } \mathcal{L}_{\text{SC}} = \frac{N_c}{2} \hat{A}_0 \times I.$$  \hspace{1cm} (5.29)

Therefore, in the near and intermediate zones where both the $\hat{A}^\mu$ and the $\Phi$ have approximately constant kinetic terms

$$\mathcal{L}_{\text{kin}}[\hat{A}_\mu, \Phi] \approx \frac{N_c \lambda M \zeta}{216\pi^3} \left\{ \frac{1}{4} \hat{F}_{MN}^2 + \frac{1}{1 - \zeta^{-3}} \times \frac{1}{2} (\partial_M \Phi)^2 \right\}, \hspace{1cm} (5.30)$$

both $\hat{A}_0$ and $\Phi$ are 5D Coulomb fields of the same charge density $I(\vec{x}, w)$,

$$\hat{A}_0(\vec{x}, w) = \frac{72\pi}{N_c \lambda M \zeta} \int d^3 \vec{x}' d w' \frac{I(\vec{x}', w')}{(\vec{x}' - \vec{x})^2 + (w' - w)^2}, \hspace{1cm} (5.31)$$

$$\Phi(\vec{x}, w) = -\frac{1 - \zeta^{-3}}{3} \times \hat{A}_0(\vec{x}, w). \hspace{1cm} (5.32)$$

This fixed $\Phi/\hat{A}_0$ ratio has profound consequences for the attractive nuclear force: For any geometry of the instanton density $I(\vec{x}, w)$ in the near and intermediate zones, there is a fixed ratio between the attractive force due to $\Phi$ and the repulsive force due to $\hat{A}_0$, namely

$$\frac{V^{\text{attractive}}(r)}{V^{\text{repulsive}}(r)} = -C_{a/r} = -\frac{1 - \zeta^{-3}}{3^2} = \text{const.}$$  \hspace{1cm} (5.33)

Note that this ratio vanishes for $\zeta = 1$ — there is no attractive scalar force in the antipodal model. For the non-antipodal models, the attractive/repulsive force ratio $C_{a/r}$ increases
with $\zeta$, but it never gets larger than $1/9$. Thus, in the near and intermediate zones of the Sakai–Sugimoto model, the attractive nuclear force is always weaker than the repulsive force.

### 5.2 Attractive Forces in the Intermediate Zone

In light of eq. (5.33), calculating the attractive force between two nucleons at an intermediate distance from each other

$$\rho = R_{\text{baryon}} \sim \frac{\zeta^{-3/4} \lambda^{-1/2}}{M_A} \ll r \ll w_{\text{max}} \sim \frac{\zeta^{-1/2}}{M_A}, \quad (5.34)$$

seems like a simple exercise. Approximating both baryons as point sources of 5D Coulomb fields $\hat{A}_0$ and $\Phi$, we get

$$V^{\text{repulsive}}(r) = + \frac{N_c^2}{4\kappa(\zeta)} \times \frac{1}{4\pi^2 r^2} = + \frac{27\pi N_c}{2\zeta \lambda M_A} \times 1/r^2, \quad (5.35)$$

$$V^{\text{attractive}}(r) = - \frac{1 - \zeta^{-3}}{9} \times V^{\text{repulsive}}(r) = - \frac{3\pi N_c (1 - \zeta^{-3})}{2\zeta \lambda M_A} \times 1/r^2, \quad (5.36)$$

and the net potential is a repulsive $+1/r^2$ hard core. Note that both the repulsive and the attractive potentials (5.35), (5.36) are blind to spins and isospins of the two baryons. However, we saw in section 4 that such blindness is an artefact of treating the two baryons as point sources. A better approximation makes the repulsive potential sensitive to the baryon’s spins and isospins, and we shall see momentarily that the attractive potential has a similar sensitivity.

Indeed, let’s follow Hashimoto et al [16] and take the $SU(2)$ gauge fields to be exactly self-dual, i.e., the two-instanton ADHM solution. The instanton density of this solution

$$I(\vec{x}, w) = \frac{\epsilon^{0KLMN}}{32\pi^2} \text{tr} \left( F_{LK} F_{MN} \right)_{\text{ADHM}} = I(1)(\vec{x}, w) + I(2)(\vec{x}, w) + \Delta I^{\text{overlap}}(\vec{x}, w) \quad (5.37)$$

differs from two separate instantons by $\Delta I^{\text{overlap}} = O((\rho/r)^2)$, which has two effects: First, the repulsive Coulomb self-interaction of the instanton density (5.37)

$$V[U(1)] = + \frac{27\pi N_c}{2\zeta \lambda M_A} \times \left\{ \frac{1}{r^2} + \int d^3 \vec{x} dw \frac{\Delta I^{\text{overlap}}(\vec{x}, w)}{\left( \vec{x} - \vec{X}_1 \right)^2 + w^2} + \left( \vec{X}_1 \rightarrow \vec{X}_2 \right) + O \left( (\Delta I)^2 \right) \right\} \quad (5.38)$$

is more complicated than (5.35), and second, for a $w$-dependent gauge coupling $\kappa(w) =$
\( \kappa(0) \times (\bar{u}(w)/u_0) - \text{cf. eqs. } (5.16) \text{ and } (5.9) \) — re-distribution of the \( SU(2) \) gauge fields in the \( w \) direction changes their energy by

\[
\Delta E[SU(2)] = \frac{N_c \lambda \Lambda \zeta}{216 \pi^3} \left\{ \int d^3\vec{x} \, dw \left( \bar{u}(w)/u_0 \right) \times \frac{1}{2} \text{tr} \left( F_{MN}^2 \right) - 2 \times 8\pi^2 \right\}
\]

\[
= \frac{N_c \lambda \Lambda \zeta}{81 \pi} \times \frac{8\zeta^3 - 5}{3\zeta} \times \int d^3\vec{x} \, dw \, w^2 \times \Delta \text{overlap}(\vec{x}, w). \tag{5.39}
\]

Both effects were evaluated in careful detail by Hashimoto et al.\([16]\), so let us simply adapt their results to the present situation. For two nucleons — i.e., baryons of spin = isospin = \( \frac{1}{2} \) — and \( N_c \gg 1 \) (which suppresses the quantum fluctuations of the baryons’ sizes or locations), they found

\[
\Delta E[SU(2)] = \frac{A}{r^2} \times \left( 1 + \frac{16}{9} (I_1 \cdot I_2) \times \left[ 2(n \cdot J_1)(n \cdot J_2) - (J_1 \cdot J_2) \right] \right) \tag{5.40}
\]

where

\[
A = \frac{\lambda N_c M^3}{162 \pi} \times \frac{8\zeta^3 - 5}{3\zeta} \times \rho^4. \tag{5.41}
\]

\[
V[U(1)] = \frac{B}{r^2} \times \left( 1_{\text{naive}} - \frac{2}{5} + \frac{32}{45} (I_1 \cdot I_2) \times (J_1 \cdot J_2) \right) \tag{5.42}
\]

where

\[
B = \frac{27\pi N_c}{2\zeta \lambda M}. \tag{5.43}
\]

Or rather, they obtained these formulae for the antipodal modal, but the extra \( \zeta \)-dependent factors for the non-antipodal models are obvious from eqs. (5.39) and (5.38).

Now consider the attractive scalar force between two baryons. Since the scalar field \( \Phi \) couples to the same non-trivial instanton density (5.37) as the abelian gauge field \( \hat{A}_0 \), the attractive force has the same complicated spin and isospin dependence as the \( U(1) \) force (5.42): Instead of the naive eq. (5.36), we get

\[
V[\Phi] = -C_{a/r} \times V[U(1)] = -\frac{C_{a/r} B}{r^2} \times \left( 1_{\text{naive}} - \frac{2}{5} + \frac{32}{45} (I_1 \cdot I_2) \times (J_1 \cdot J_2) \right) \tag{5.44}
\]

where

\[
C_{a/r} = \frac{1}{9} (1 - \zeta^{-3}) \text{ as in eq. } (5.33).
\]

But besides the direct contribution (5.44) of the scalar force to the two-baryon potential, it also affects the baryon radius \( \rho \) — which in turn affects the \( SU(2) \) force according to eqs. (5.40), (5.41). To see how this works, consider a stand-alone \( SU(2) \) instanton with

\[
I(\vec{x}, w) = \frac{6\rho^4}{\pi(\vec{x}^2 + w^2 + \rho^2)^4}. \tag{5.45}
\]

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The classical energy of this instanton is

\[ E(\rho) = E[SU(2)] + \Delta E[U(1)] + \Delta E[\Phi] \] (5.46)

where

\[ E[SU(2)] = \frac{N_c \lambda M \zeta}{27\pi} \int d^3\vec{x} dw I(\vec{x}, w) \times \frac{\bar{u}(w)}{u_0} = \frac{N_c \lambda M \zeta}{27\pi} + \frac{\lambda N_c M^3}{162\pi} \times \frac{8\zeta^3 - 5}{3\zeta} \times \rho^2, \] (5.47)

the \( \hat{A}_0 \)-mediated Coulomb self-interaction is

\[ \Delta E[U(1)] = +B \times \frac{2}{5\rho^2} \] (5.48)

where \( B \) is as in eq. (5.43), and the \( \Phi \)-mediated Coulomb self-interaction is

\[ \Delta E[\Phi] = -C_{a/r} \times \Delta E[U(1)] = -C_{a/r} B \times \frac{2}{5\rho^2}. \] (5.49)

Note that the scalar / abelian vector force ratio (5.33) works at short distances as well as intermediate.

Assembling all the contributions to the classical instanton energy, we get

\[ E(\rho) = \frac{N_c \lambda M \zeta}{27\pi} + \frac{\lambda N_c M^3}{162\pi} \times \frac{8\zeta^3 - 5}{3\zeta} \times \rho^2 + \frac{27\pi N_c}{5 \lambda M} \times \frac{1 - C_{a/r}}{\zeta} \times \frac{1}{\rho^2}, \] (5.50)

and minimizing this formula with respect to \( \rho \) gives us the classical mass and radius of the baryon:

\[ R_{\text{baryon}} = \rho_{\text{at min}} = \left( \frac{81\pi \sqrt{2/15}}{\lambda M^2} \right)^{1/2} \times \left( \frac{3}{8\zeta^3 - 5} \right)^{1/4} \times (1 - C_{a/r})^{1/4}, \] (5.51)

\[ M_{\text{baryon}} = \frac{N_c \lambda M \zeta}{27\pi} + \frac{M_A}{\sqrt{30}} \left( \frac{8\zeta^3 - 5}{3\zeta^2} \right)^{1/2} \times (1 - C_{a/r})^{1/2}, \] (5.52)

Note that these formulae differ from eqs. (3.18) by factors \((1 - C_{a/r})^{\text{some power}}\): These factors are due to \( \Phi \)-mediated attractive force which we didn’t take into account back in section 3. In particular, the scalar force reduces the baryon radius by a factor \((1 - C_{a/r})^{1/4}\). Substituting this radius into eq. (5.41) gives us

\[ A = \frac{2B}{5} \times (1 - C_{a/r}), \] (5.53)
which means that the indirect effect of the scalar force reduces the $SU(2)$-mediated nuclear force by the factor $(1 - C_{a/r})$. At the same time, the direct effect (5.44) of the scalar force reduces the net Coulomb force by exactly the same factor,

$$V[U(1)] + V[\Phi] = V[U(1)] \times (1 - C_{a/r}).$$

(5.54)

Altogether, we end up with this $(1 - C_{a/r})$ factor multiplying the whole nuclear force in all its complicated glory,

$$V_{\text{net}}(r) = \Delta E[SU(2)] + V[U(1)] + V[\Phi]$$

$$= \frac{2B}{5r^2} \times (1 - C_{a/r}) \times \left(1 + \frac{16}{9}(I_1 \cdot I_2) \times \left[2(n \cdot J_1)(n \cdot J_2) - (J_1 \cdot J_2)\right]\right)$$

$$+ \frac{B}{r^2} \times (1 - C_{a/r}) \times \left(1_{\text{naive}} - \frac{2}{5} + \frac{32}{45}(I_1 \cdot I_2) \times (J_1 \cdot J_2)\right)$$

$$= \frac{B}{r^2} \times (1 - C_{a/r}) \times \left(1_{\text{naive}} + \frac{64}{45}(I_1 \cdot I_2) \times (n \cdot J_1)(n \cdot J_2)\right).$$

(5.55)

To summarize, the net holographic nuclear force in the intermediate zone is precisely as in Hashimoto, Sakai, and Sugimoto [16]. The only effect of the attractive force due to the scalar field $\Phi$ is to reduce the whole force by a constant overall factor $(1 - C_{a/r})$. Depending on the $\zeta$ parameter of the model, this factor varies between $\frac{8}{9}$ (reduction by 11%) and 1 (no reduction at all), but it is always positive so the net force in the intermediate zone is always repulsive.

### 5.3 Attractive Forces in the Far Zone

In the far zone, the $w$-dependence of the gauge coupling

$$\kappa(w) = \frac{N_c \lambda M_A \zeta}{216 \pi^3} \times \frac{\bar{u}(w)}{u_0}$$

(5.56)

becomes important and the lowest-energy two-instanton solution of the $SU(2)$ equation $D_M(\kappa F^{MN}) = 0$ is no longer self-dual. Consequently, working out the overlap between two distant instantons becomes rather difficult and we are reduced to a cruder point-source approximation. Earlier, we saw that in the intermediate zone, this approximation has yielded a correct isoscalar central force but missed the isovector spin-spin and tensor forces. By analogy, we expect that in the far zone, the point-source approximation would give us a
correct isoscalar force, but the isovector forces could be wrong.

Let’s focus on the isoscalar forces. In the point-source approximation, they follow the Green’s functions of the \( \hat{A}_0(\vec{x}, w) \) and \( \Phi(\vec{x}, w) \) fields in the far zone,

\[
V_{\text{repulsive}}(r) = + \frac{54 \pi^3 N_c}{\zeta \lambda M_A} \times \left\langle \vec{x}, 0 \right| \left[ \left( \frac{\bar{u}(w)}{u_0} \right) \vec{\nabla}^2 + \frac{\partial}{\partial w} \left( \frac{\bar{u}(w)}{u_0} \right) \frac{\partial}{\partial w} \right]^{-1} \left| \vec{0}, 0 \right>,
\]

\[
V_{\text{attractive}}(r) = -C_{a/r} \times \frac{54 \pi^3 N_c}{\zeta \lambda M_A} \times \left\langle \vec{x}, 0 \right| \left[ \left( \frac{\bar{u}(w)}{u_0} \right)^9 \vec{\nabla}^2 + \frac{\partial}{\partial w} \left( \frac{\bar{u}(w)}{u_0} \right)^9 \frac{\partial}{\partial w} + \mu^2(w) \right]^{-1} \left| \vec{0}, 0 \right>,
\]

where

\[
\mu^2(w) = -\frac{2 \zeta M_A^2}{9} \times \left[ 35 \left( \frac{\bar{u}(w)}{u_0} \right)^{10} - 32 \zeta^{-3} \left( \frac{\bar{u}(w)}{u_0} \right)^7 + 13(1 - \zeta^{-3}) \left( \frac{\bar{u}(w)}{u_0} \right)^2 \right]
\]

(5.59)

is the effective mass term for the \( \Phi \) in the expansion of the DBI Lagrangian in terms of 5D coordinate \((x^\mu, w)\). Expanding the 5D fields \( \hat{A}_0(\vec{x}, w) \) and \( \Phi(\vec{x}, w) \) in terms of the 4D vector and scalar mesons, we may express their Green’s functions — and hence the potentials (5.57–5.58) — as sums of Yukawa potentials

\[
V_{\text{repulsive}}(r) = + \frac{27 \pi^2 N_c}{2 \zeta \lambda M_A} \times \sum_{n=1}^{\infty} \left| \Psi_n^V(w = 0) \right|^2 \times \frac{\exp(-m_n^V r)}{r},
\]

\[
V_{\text{attractive}}(r) = -C_{a/r} \times \frac{27 \pi^2 N_c}{2 \zeta \lambda M_A} \times \sum_{n=1}^{\infty} \left| \Psi_n^S(w = 0) \right|^2 \times \frac{\exp(-m_n^S r)}{r}.
\]

(5.61)

(5.62)

Here \( \Psi_n^V(w) \) and \( \Psi_n^S(w) \) are the wave functions in the fifth dimension of the respective vector or scalar mesons. Note that only the odd–\( n \) modes — which give rise to the true vector \( 1^- \) and true scalar \( 0^+ \) mesons — contribute to the sums (5.61, 5.62). The even–\( n \) modes —

\[\text{The negative sign of } \mu^2(w) \text{ is an artefact of the } w \text{-dependent normalization of the scalar field; a canonically normalized } \Phi \text{ has positive mass}^2\]

\[
\mu_{\text{can}}^2(w) = + \frac{\zeta M_A^2}{9} \times \left[ 20 \left( \frac{\bar{u}(w)}{u_0} \right) + \zeta^{-3} \left( \frac{\bar{u}(w)}{u_0} \right)^{-2} - 44(1 - \zeta^{-3}) \left( \frac{\bar{u}(w)}{u_0} \right)^{-7} \right].
\]

(5.60)
responsible for the axial-vector $1^+$ and pseudo-scalar $0^-$ mesons — don’t contribute because their wave functions vanish at $w = 0$.

We do not have analytical formulae for the meson’s masses or wave functions, but numerical calculations [11, 32] show that for Sakai–Sugimoto models with any $\zeta \geq 1$, the vector mesons are always lighter than the scalar mesons,

$$\forall n, \quad m_n^V < m_n^S. \quad (5.63)$$

Numerical calculation of the mesons’ wave functions at $w = 0$ is still in progress, but there does not seem to be much difference between vector and scalar mesons, so it is reasonable to assume that

for odd $n, \quad \frac{|\Psi_n^V(0)|^2}{|\Psi_n^S(0)|^2} > C_{a/r} = \frac{1 - \zeta^{-3}}{9}. \quad (5.64)$

If this assumption is correct, then every term in the repulsive potential $(5.61)$ is stronger than the corresponding term in the attractive potential $(5.62)$, and the net isoscalar force is repulsive throughout the far zone.

At the outer end of the far zone, we don’t need the assumption $(5.64)$ to show that the repulsion is stronger than attraction, all we need to know is that the lightest vector meson is lighter than the lightest scalar meson, $m_1^V < m_1^S$. Indeed, for $r \gg (1/m_{\text{meson}}) \sim (1/M_\Lambda \sqrt{\zeta})$, each sum $(5.61, 5.62)$ is dominated by the slowest-decaying Yukawa term belonging to the lightest vector or scalar meson,

$$V^{\text{repulsive}}(r) \sim +O(N_c/\lambda) \times \frac{\exp(-m_1^V r)}{r}, \quad V^{\text{attractive}}(r) \sim -O(N_c/\lambda) \times \frac{\exp(-m_1^S r)}{r}. \quad (5.65)$$

Regardless of the pre-exponential factors here, the longer-ranged force always wins over the shorter-ranged force at long distances, and since $m_1^V < m_1^S$ in all versions of the Sakai–Sugimoto model — antipodal and non-antipodal with any $\zeta > 1$ — the net isoscalar force is repulsive at long distances $r \gg 1/(M_\Lambda \sqrt{\zeta})$.

Note that this behavior of the Sakai–Sugimoto model is very different from the real-life nuclear physics. Indeed, in reality the lightest scalar meson $\sigma$ is lighter then the lightest vector meson, $M_\sigma \sim 600$ MeV while $M_\omega \approx 787$ MeV, and consequently the real-life isoscalar nuclear force is attractive rather than repulsive at short distances. We do not know the reason for this discrepancy; perhaps it’s a peculiar bad feature of the Sakai–Sugimoto scheme and could be fixed by an alternative holographic model. But perhaps it’s an inherent problem of the large–$N_c$ nuclear physics. As explained in section 2, the QCD origin of the $\sigma(600)$
resonance is rather controversial — maybe it’s a true meson originating in the $\sigma$ field of the linear-sigma-model-like chiral symmetry breaking, or maybe it’s just a two-pion resonance which would not exist without strong $\pi\pi$ interactions. In the first scenario, the large $N_c$ limit would make $\sigma(600)$ into a narrow resonance, but it would still be there and lighter than the lightest vector meson $\omega$, so the attractive nuclear force would have a longer range than the repulsive force. But in the second scenario, there would be no $\sigma(600)$ resonance in the large $N_c$ limit, the lightest remaining scalar meson would be heavier than $\omega$, and the dominant isoscalar nuclear force at long distances would be repulsive rather than attractive.

The best way to settle this issue would be to find the $\sigma$ resonance and its mass in a lattice QCD calculation for several values of $N_c$, which would hopefully allow us to extrapolate to $N_c \to \infty$. Alternatively, once we have several different holographic models, we can compare their predictions for the meson spectra in general and for the lightest scalar to lightest vector mass ratio in particular. Either way, this issue will have to wait for future research.

Another issue that would have to wait for future research involves nuclear forces arising from nucleons exchanging pairs of un-bound mesons — especially pairs of pions — rather than single mesons or resonances. In the real-life nuclear physics, the double pion exchange generates the longest-ranging attractive isoscalar force, which makes a significant contribution to the bulk binding energy of the nuclei. In the large $N_c$ limit, the double-pion exchange decreases as $1/N_c$ relatively to the single-meson exchanges, but it would remain significant at longer distances where the $\exp(-m r)$ factors for single mesons like $\sigma$ or $\omega$ are even smaller than $1/N_c$. Unfortunately, in holographic duals of QCD, the double-meson exchanges happen at the one-string-loop level, so they cannot be be calculated in terms of an effective semi-classical 5D gravity — or even 10D gravity plus other local fields — but require a fully quantum string theory. Since string perturbation theory in curved backgrounds is rather hard, we leave the double-meson exchanges for future research.

Instead, let us now address the dominant nuclear force at the longest distances, namely the isovector force due to single pion exchange between two nucleons,

$$V^\pi(r) = -\frac{g_A^2}{\pi f_\pi^2} (I_1 I_2) \left[ (J_1 J_2) \times \frac{m_\pi^2}{3r} + T_{12}(n) \times \left( \frac{m_\pi^2}{3r} + \frac{m_\pi}{r^2} + \frac{1}{r^3} \right) \right] e^{-m_\pi r}$$

$$\mathcal{L}^\pi \to -\frac{g_A^2}{\pi f_\pi^2} (I_1 I_2) T_{12}(n) \times \frac{1}{r^3} \quad (5.66)$$

9 Probably the $f_0(980)$, or maybe even the $f_0(1450)$ — which is the best fit for the lightest scalar in the Sakai–Sugimoto model — in case the $f_0(980)$ is also a two-pion resonance that would go away when $N_c \to \infty$. Either way, the lightest surviving scalar would be heavier than $\omega(787)$. 

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where $T_{12}(n) = 3(nJ_1)(nJ_2) - (J_1J_2)$ is the direction dependence of the tensor force. The overall coefficient of this force was calculated by Hashimoto et al. [15, 16] for the antipodal model. Adapting their method to non-antipodal models gives us

$$\frac{g_A^2}{\pi f_\pi^2} = \frac{8N_c\lambda\zeta^{3/2}M_A^2}{3^6\pi F(\zeta)} \times \rho^4$$

(5.67)

where $\rho$ is the classical radius of the baryon and $F^{-1/2}$ is the normalization factor of the pion’s wave function

$$\Psi_\pi(w) = \frac{1}{\sqrt{\kappa_0(\zeta)F(\zeta)}} \times \frac{u_0}{\bar{u}(w)},$$

(5.68)

$$F(\zeta) = \frac{M\sqrt{\zeta}}{3} \int_{-w_{\text{max}}}^{+w_{\text{max}}} dw \frac{u_0}{\bar{u}(w)} = \int_1^\infty dy \left( \frac{y^3}{y^8 - \zeta^3y^5 - (1 - \zeta^{-3})} \right)^{1/2},$$

(5.69)

$$\frac{\pi}{3} < F(\zeta) < \frac{\sqrt{\pi}\Gamma(3/16)}{4\Gamma(11/16)} \approx 1.65.$$  

(5.70)

The pions themselves are modes of the $SU(2)$ gauge fields $A_{\alpha M}(x, w)$ and don’t know the 5D scalar field $\Phi(x, w)$ from the hole in the ground. However, their interactions with baryons depend on the classical baryon radius $\rho$, and because of the $\Phi$–mediated attractive forces in the near zone, this radius is smaller than it would have been otherwise,

$$\rho^4 = \left( \frac{81\pi\sqrt{2/15}}{\lambda M_A^2} \right)^2 \times \left( \frac{3}{8\zeta^3 - 5} \right) \times (1 - C_{a/r}).$$

(5.51)

Consequently, the pion-mediated nuclear force in the far zone becomes sensitive to the $\Phi$-mediated attractive force in the near zone,

$$V^\pi(r) = -\frac{(I_1I_2)T_{12}(n)}{r^3} \times \frac{48N_c}{5\lambda M_A^2} \times \frac{\pi\zeta^{3/2}}{(8\zeta^3 - 5)F(\zeta)} \times (1 - C_{a/r}).$$

(5.71)

What about the other isovector mesons’ contributions to the nuclear force? In the Sakai–Sugimoto model we have (pseudo) scalar isovector mesons coming from the modes of the 5D scalar fields $\Phi^\alpha(x, w)$, but their couplings to baryons are too small to create an appreciable force. In addition, we have vector and axial vector isovector mesons $\rho$ (770 MeV), $a_1$ (1260 MeV), etc., coming from the non-zero modes of the $SU(2)$ gauge fields $A_{\alpha M}(x, w)$. Their contributions to the isovector spin-spin and tensor forces were calculated

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by Hashimoto et al [10] as

\[ V_{\text{isovector}}(r) = -\frac{128\pi^3}{27}\kappa_0^2\rho^4(\mathbf{I}_1\mathbf{I}_2)\sum_{n=1}^{\infty} Q_n e^{-m_n r} \left[ \frac{m_n^2}{3r} + T_{12}(n) \left( \frac{1}{r^3} + \frac{m_n}{r^2} + \frac{m_n^2}{3r} \right) \right] \]  

(5.72)

where

\[ Q_n = \begin{cases} 
+ |\Psi_n(w = 0)|^2 & \text{for odd } n, \\
-\frac{1}{m_n^2} |\Psi_n'(w = 0)|^2 & \text{for even } n.
\end{cases} \]  

(5.73)

Note the overall factor \( \rho^4 \) for all the isovector forces. Since the baryon radius is affected by the attractive forces in the near zone, they have indirect effect on all the isovector forces in the far zone,

\[ V_{\text{isovector}}(r) \propto (1 - C_{a/r}). \]  

(5.74)

This rule applies even to the effects of the baryon-baryon overlap that Hashimoto et al could not calculate in the far zone. We cannot calculate their \( r \) dependence either, but their \( \rho \) dependence should be the same as in the intermediate zone,

\[ V_{\text{overlap}}(r) \propto \rho^4 \propto (1 - C_{a/r}). \]  

(5.75)

6 Full DBI Action in the Near Zone

In the previous section we studied nuclear forces in the intermediate and far zones, but since those forces depend on the baryon radius \( \rho \), we had to stick our noses into the near zone to see how \( \rho \) is affected by the scalar-mediated forces. In the intermediate and far zones, all the gauge fields are weak enough so we could expand the DBI Lagrangian in power of \( \mathcal{F}_{MN} \) and truncate the expansion after the leading Yang–Mills term as we did in eqs. (5.13–5.15),

\[ L_{\text{DBI}} \propto \text{str} \sqrt{\det(g + 2\pi\alpha'\mathcal{F})} \approx \sqrt{-\det(g)} \times \left( 2 + \frac{(2\pi\alpha')^2}{4} \text{tr} \left( \mathcal{F}_{MN}\mathcal{F}^{MN} \right) \right). \]  

(6.1)

for \( r \gg \rho \approx \frac{1}{\sqrt{\lambda M_A}} \), \( 2\pi\alpha'\mathcal{F}_{MN} \ll G_{MN} \).
However, in the near zone \( r \sim \rho \) the \( SU(2) \) gauge fields become too strong for the YM approximation,

\[
\text{for } r \sim \rho \quad 2\pi\alpha' F_{MN} \sim g_{MN} \quad \Rightarrow \quad L_{\text{DBI}} \not\approx \sqrt{-\det(g)} \left( 2 + \frac{(2\pi\alpha')^2}{4} \text{tr} (F_{MN}F^{MN}) \right),
\]

which casts doubt on accuracy of eqs. (5.51–5.52) for the baryon’s mass and radius and hence of all the \( \rho \)-dependent formulae for isovector nuclear forces.

In this section we shall see that despite the higher-order DBI interactions beyond the Yang–Mills approximation, to leading order in \( 1/\lambda \) the baryon radius remains exactly as in eq. (5.51) and we do not need to make any leading-order corrections to the nuclear forces we have computed in the previous section 4. And at this happy note, the readers impatient with technical details may skip the rest of this section and go straight to the summary section 7.

Our main point is that while the Yang-Mills approximation \( (6.1) \) to the full DBI action \( ^{10} \) does not work for generic strong gauge fields, it may work ‘by accident’ for some special \( F_{MN}(x) \) configurations. In particular, for self-dual gauge fields living on a flat D4 brane stack, the YM approximation not only works but happens to be exact, regardless of the number of instantons or their sizes \( ^{35} \),

\[
\text{for } F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \quad \text{str} \sqrt{\det(1_{4\times4} + 2\pi\alpha' F)} = 2 + \frac{(2\pi\alpha')^2}{4} \text{tr} (F_{MN}F^{MN}) \quad (6.3)
\]

In our case, the 5D metric is not flat, and besides the (approximately) self-dual \( SU(2) \) magnetic fields we also have the abelian electric field. Nevertheless, eq. (6.3) continues to hold exactly for a slightly modified self-duality condition for the \( SU(2) \) magnetic fields.

Indeed, consider a more general case of some non-flat but static metric

\[
ds^2 = -|g_{00}(x)|dt^2 + g_{mn}(x)dx^m dx^n \quad (m, n = 1, 2, 3, 4), \quad (6.4)
\]

arbitrary but purely-electric abelian fields \( F_{\mu0}(x) \) and purely-magnetic \( SU(2) \) fields \( F^a_{mn}(x) \).

In general, Tseytlin’s non-abelian version \( ^{34} \) of the DBI Lagrangian works like this: First, one calculates the determinant of the \( g_{MN} + 2\pi\alpha' F_{MN} \) matrix (in space-time indices) while completely ignoring their gauge indices or the fact that they don’t commute with each other.

\(^{10}\)Actually, the DBI + CS action is also incomplete — for multiple D-branes, there are additional terms involving covariant derivatives \( D_L F_{MN} \) of the non-abelian gauge fields. Fortunately, for instanton-like gauge fields of a baryon, such covariant derivatives are relatively small — even when the gauge fields themselves are large as in eq. (6.2) — so following Tseytlin \( ^{34} \) we shall limit our analysis to the non-abelian DBI action.
Second, one expands the square root of this determinant into a formal power series in the gauge fields. Finally, for each term in this determinant one takes a symmetrized trace over the gauge indices, and then tries to re-sum the series. For the static case at hand we have

$$g_{MN} + 2\pi \alpha' F_{MN} = \begin{pmatrix} -|g_{00}| & -\pi \alpha' \hat{F}_{m0} \\ +\pi \alpha' \hat{F}_{0m} & g_{mn} + 2\pi \alpha' F_{mn} \end{pmatrix}$$  \hspace{1cm} (6.5)$$

and hence

$$- \det_{5D}(g_{MN} + 2\pi \alpha' F_{MN}) = |g_{00}| \times \det_{4D}(\hat{g}_{mn} + 2\pi \alpha' F_{mn})$$  \hspace{1cm} (6.6)$$

where

$$\hat{g}_{mn}(x) = g_{mn}(x) - (\pi \alpha')^2 |g^{00}| \hat{F}_{m0} \hat{F}_{n0}. \hspace{1cm} (6.7)$$

Note that while the determinants on both sides of eq. (6.6) are formal — they ignore the non-commutativity of the magnetic fields $F_{mn} = \frac{1}{2} \tau^a F^a_{mn}$ — but the modified metric (6.7) does not have any non-commutativity problems because the electric fields $\hat{F}_{m0}$ are purely abelian. When those electric fields are too strong, we can get a different problem of $\hat{g}_{mn}$ matrix loosing positive-definiteness, but fortunately the $U(1)$ fields of a baryon never get that strong: Even in the near zone,

$$(\pi \alpha')^2 |g^{00}| g^{mn} \hat{F}_{m0} \hat{F}_{n0} \lesssim O(1/\lambda) \ll 1,$$  \hspace{1cm} (6.8)$$

so the modified metric $\hat{g}_{mn}$ remains safely positive-definite.

On the right hand side of eq. (6.6), the 4D determinant evaluates to

$$\det(\hat{g}_{mn} + 2\pi \alpha' F_{mn}) = \det(\hat{g}_{mn}) \times \left[ 1 + 2(\pi \alpha')^2 \hat{g}^{mp} \hat{g}^{nq} F_{mn} F_{pq} + (\pi \alpha')^4 \left( \hat{g}^{mp} \hat{g}^{nq} F_{mn} \tilde{F}_{pq} \right)^2 \right]$$  \hspace{1cm} (6.9)$$

where

$$\tilde{F}_{pq} = \frac{1}{2} \sqrt{\det(\hat{g})} \epsilon_{pqrs} \hat{g}^{rm} \hat{g}^{sn} F_{mn} \hspace{1cm} (6.10)$$

is the Hodge dual of the $F_{mn}$ with respect to the modified metric $\hat{g}_{mn}$. When the $SU(2)$ magnetic field is self-dual (or anti-self-dual) with respect to that metric, $\tilde{F}_{mn} = \pm F_{mn}$, the 4D determinant (6.9) becomes a full square, hence

$$\sqrt{- \det_{5D}(g_{MN} + 2\pi \alpha' F_{MN})} = \sqrt{|g_{00}|} \sqrt{\det(\hat{g})} \times \left[ 1 + (\pi \alpha')^2 \hat{g}^{mp} \hat{g}^{nq} F_{mn} F_{pq} \right]$$

$$= \sqrt{|g_{00}|} \left[ \sqrt{\det(\hat{g})} \pm \frac{1}{2} (\pi \alpha')^2 \epsilon_{mpnq} F_{mn} F_{pq} \right], \hspace{1cm} (6.11)$$
and the symmetrized trace becomes the ordinary matrix trace (over the $U(2)$ gauge indices). Consequently, the complete DBI action for all the gauge and metric fields splits into the ordinary Yang–Mills action for the (self-dual or anti-self-dual) $SU(2)$ magnetic fields, plus the abelian DBI action for the metric and the $U(1)$ electric fields only,

$$L_{DBI} = -T_8 e^{-\phi} \text{Vol}(S^4) \sqrt{|g_{00}|} \times \frac{(\pi\alpha')^2}{4} (\pm \varepsilon^{mpnq}) F^a_{mn} F^a_{pq}$$

$$-2 T_8 e^{-\phi} \text{Vol}(S^4) \times \sqrt{-\det_{5D} (g_{MN} + \pi\alpha' \hat{F}_{MN})}$$

(6.12)

(where on the second line we have used $|g_{00}| \times \det_{4D}(\hat{g}_{mn}) = -\det_{5D}(g_{MN} + \pi\alpha' \hat{F}_{MN})$).

For the baryon, the $SU(2)$ magnetic fields become strong in the near zone, but the $U(1)$ electric fields they induce (via the Chern–Simons interactions) are relatively weak ($O(1/\sqrt{\lambda})$ or weaker) in all zones, and the perturbations of the 5D metric due to the scalar field $\Phi(x)$ are also weak in all zones. Consequently, the second line of the Lagrangian (6.12) gives rise to the usual kinetic terms for the scalar and the abelian gauge fields, while all the higher-order terms are suppressed by negative powers of the 't Hooft coupling $\lambda$ (and outside the near zone also by positive powers of $(\rho/r) \ll 1$). At the same time, the first line of the Lagrangian (6.12) gives rise to the Yang–Mills Lagrangian for the $SU(2)$ gauge fields, and also to their interactions with the scalar field $\Phi$ and with the background metrics’s curvature via the $u$–dependence of the 5D gauge coupling

$$\kappa(x,w) = 2(\pi\alpha')^2 T_8 e^{-\phi} \text{Vol}(S^4) \sqrt{|g_{00}|} = \frac{N_c}{48\pi^2} \left( \frac{\bar{u}(w)}{\pi\alpha'} + \Phi(x,w) \right).$$

(6.13)

However, we do not get any terms with higher powers of the strong $SU(2)$ gauge fields, and everything works precisely as in §5 — except that the $F^a_{mn}(\vec{x},w)$ fields should be self dual with respect to the modified metric $\hat{g}_{mn}$ rather than the true metric $g_{mn}$.

To see the effect of this modification of the self-duality condition, consider an $SO(4)$ spherically symmetric instanton-like field configuration

$$A_0(\vec{x},w) = 0, \quad A_m(\vec{x},w) = h(r) \times -i U^\dagger \partial_m U, \quad U(\vec{x},w) = \frac{w + i\vec{r} \cdot \vec{x}}{r}, \quad r = \sqrt{\vec{x}^2 + w^2}$$

(6.14)

where $m = 1, 2, 3, w$ and $h(r)$ is some smooth function of the 4D radius. The corresponding magnetic field strength has form

$$F_{mn}(\vec{x},w) = -i\partial[m h \times U^\dagger \partial_n U + ih(1 - h) \times [U^\dagger \partial_m U, U^\dagger \partial_n U]].$$

(6.15)
In particular, along the $w$ axis we have

$$F^a_{wi} = -F^a_{iw} = \frac{2}{r} \frac{dh}{dr} \times \delta^a_i, \quad F^a_{ij} = \frac{4h(1-h)}{r^2} \times \epsilon^{aij},$$  \hspace{1cm} (6.16)$$

while in other directions we have similar fields rotated by appropriate $SO(4) \times SU(2)$ symmetries. The instanton density of such fields is

$$I = \frac{\epsilon^{klmn} F^a_{kl} F^a_{mn}}{64\pi^2} = \frac{3}{\pi^2} \frac{h(1-h)}{r^3} \frac{dh}{dr},$$  \hspace{1cm} (6.17)$$

so the net instanton number is

$$\#\text{Instantons} = \int_0^\infty I(r) \times 2\pi^2 r^3 \, dr = \int_{h(0)=0}^{h(\infty)=1} 6h(1-h) \, dh = 1$$  \hspace{1cm} (6.18)$$

for any radial profile $h(r)$ that satisfies the boundary conditions.

The magnetic fields (6.15) become self-dual $F^a_{mn} = \frac{1}{2} \epsilon_{mnpq} F^a_{pq}$ with respect to the flat 4D metric $\delta_{mn}$ when

$$\frac{2}{r} \frac{dh}{dr} = \frac{4h(1-h)}{r^2};$$  \hspace{1cm} (6.19)$$

solving this differential equation gives us the usual instanton profile

$$h_{\text{instanton}}(r) = \frac{r^2}{r^2 + \rho^2} \text{ for some constant } \rho.$$  \hspace{1cm} (6.20)$$

The self-duality condition with respect to the modified 4D metric $\hat{g}_{mn}$ calls for a slightly different profile

$$h_{\text{mod}}(r) = \frac{r^2}{r^2 + \rho^2 \times (1 + \delta(r))}$$  \hspace{1cm} (6.21)$$

where the correction $\delta(r)$ depends on the abelian and the scalar fields. We shall see momentarily that this correction is rather small, $\delta(r) = O(1/\lambda)$, so the baryon profile (6.21) is approximately the usual instanton profile (6.20).

To obtain the modified metric $\hat{g}_{mn}$ and the corresponding self-duality condition we need the ordinary 5D metric $g_{MN}$. Eq. (5.10) gives us this metric to first order in the scalar field $\Phi$, and that’s a good enough approximation even in the near zone where

$$\Phi(\vec{x}, w) = O(M_\Lambda) \ll \frac{\tilde{u}(w)}{\alpha'} = O(\lambda M_\Lambda).$$  \hspace{1cm} (6.22)$$

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Consequently, expanding the modified metric

\[ d\hat{s}^2 \equiv \hat{g}_{mn} dx^m dx^n = g_{mn} dx^m dx^n - |g^{00}| (\pi \alpha' \partial_m \hat{A}_0 dx^m)^2 \]  

(6.23)

in powers of $1/\lambda$ (in the near zone where $\delta \bar{u}(w) \equiv \bar{u}(w) - u_0 \lesssim (u_0/\lambda)$), we get

\[ d\hat{s}^2 = \left(\frac{u_0}{R_{D4}}\right)^{3/2} \left[ d\vec{x}^2 + dw^2 + O(1/\lambda) \right] \]  

(6.24a)

\[ = \left(\frac{u_0}{R_{D4}}\right)^{3/2} \left[ \left(1 + \frac{3\delta \bar{u} + 3\pi \alpha' \Phi}{2u_0}\right) d\vec{x}^2 + \left(1 + \frac{3\delta \bar{u} - 13\pi \alpha' \Phi}{2u_0}\right) dw^2 \right] \]  

(6.24b)

\[ + (\pi \alpha')^2 \left(\frac{R_{D4}}{u_0}\right)^3 \left(\frac{1}{1 - \zeta^{-3}} (\partial_m \Phi dx^m)^2 - (\partial_m \hat{A}_0 dx^m)^2 \right) \]  

(6.24c)

\[ + O(1/\lambda^2) \]  

(6.24d)

Note that to the zeroth order in $1/\lambda$, the modified metric (6.24a) of the near zone is simply flat, so the modified self-duality condition is just the good old flat-space self-duality condition $F_{mn}^a = \frac{1}{2} \epsilon_{mpq} F_{pq}^a$. And that’s why the DBI correction $\delta(r)$ to the baryon profile (6.21) is $O(1/\lambda)$, hence

\[ R_{\text{baryon}}[\text{full DBI}] = R_{\text{baryon}}[\text{as in eq. 5.51}] \times (1 + O(1/\lambda)), \]  

(6.25)

and all our results of §5 for the intermediate-zone and far-zone nuclear forces are indeed correct to leading order in $1/\lambda$.

For completeness sake, let’s calculate the modified baryon profile (6.21) to first order in $1/\lambda$ using the modified metric (6.24a–d). Since the overall conformal factor of $\hat{g}_{nm}$ does not affect the self-duality condition (6.10), let’s factor it out as

\[ C = \left(\frac{\bar{u}(w)}{R_{D4}}\right)^{3/2} \left(1 - \frac{2\pi \alpha' \Phi}{\bar{u}} + \cdots \right) \]  

(6.26)

and focus on the remaining deviations of the $\hat{g}_{mn}$ from flatness,

\[ d\hat{s}^2 = C \times \left[ (d\vec{x}^2 + dw^2) + D(r) \times dr^2 + \frac{2\pi \alpha' \Phi}{\bar{u}} \times (d\vec{x}^2 - 3dw^2) + O(1/\lambda^2) \right] \]  

(6.27)

where

\[ D(r) = (\pi \alpha')^2 \left(\frac{R_{D4}}{u_0}\right)^3 \times \left(\frac{(d\Phi/dr)^2}{1 - \zeta^{-3}} - (d\hat{A}_0/dr)^2 \right). \]  

(6.28)
Note that the first two terms inside the square brackets in (6.27) are spherically symmetric in all 4 space dimensions, but this $SO(4)$ symmetry between the $\vec{x}$ and $w$ coordinates is broken by the third term. Consequently, a single static baryon is $SO(4)$ symmetric only to the leading order of the $1/\lambda$ expansion but the sub-leading terms spoil this symmetry. To see this asymmetry we should use a more complicated ansatz than (6.14) for the non-abelian gauge fields, so let’s leave this issue for future research. For now let us focus on the baryon’s radial profile, and in first-order perturbation theory this profile depends only on the spherically symmetric part

$$d\hat{s}_{\text{symm}}^2 = C \times \left( \left( 1 + \mathcal{D}(r) \right) \times dr^2 + r^2 \times d\Omega_3 \right) \quad (6.29)$$

of the modified metric $\hat{g}_{mn}$.

For a spherically symmetric metric (6.29) the self-duality condition (6.10) becomes very simple: The $SU(2)$ magnetic fields $F_{mn}^a$ with one radial and one tangential index should be $\sqrt{1 + \mathcal{D}}$ times stronger than the fields with two tangential indices, for example along the $w$ axis we should have

$$F_{wi}^a = \sqrt{1 + \mathcal{D}} \times \frac{1}{2} \epsilon_{ijk} F_{jk}^a. \quad (6.30)$$

In terms of the radial profile $h(r)$ of the spherically symmetric $SU(2)$ fields (6.14), the modified self-duality condition amounts to

$$\frac{2}{r} \frac{dh}{dr} = \sqrt{1 + \mathcal{D}(r)} \times \frac{4h(1-h)}{r^2}. \quad (6.31)$$

Solutions of this differential equation have general form

$$\frac{h(r)}{1-h(r)} = \int \frac{2dr}{r} \sqrt{1 + \mathcal{D}(r)} + \text{const}, \quad (6.32)$$

and for $\mathcal{D}(r) = O(1/\lambda) \ll 1$ as in eq. (6.28) we may approximate these solutions as

$$h(r) = \frac{r^2}{r^2 + \rho^2(1 + \delta(r))} \quad (6.21)$$

where

$$\delta(r) = \int_r^\infty \frac{\mathcal{D}(r') \, dr'}{r'}. \quad (6.33)$$

In the first-order perturbation theory we may calculate $\mathcal{D}(r)$ using the scalar and electric...
fields of the un-perturbed baryon. In the near zone
\[
\hat{A}_0(r) = \frac{27\pi}{\lambda M_\Lambda \zeta} \times \frac{r^2 + 2\rho^2}{(r^2 + \rho^2)^2},
\]
(6.34)
\[
\Phi(r) = -\frac{1 - \zeta^{-3}}{3} \times \hat{A}_0(r),
\]
(6.35)

hence
\[
\mathcal{D}(r) = -(1 - C_{a/r}) \times \left( \frac{R_{D4}}{u_0} \right)^2 \left( \frac{54\pi^2}{\lambda M_\Lambda} \right)^2 \times \frac{r^2(r^2 + 3\rho^2)^2}{(r^2 + \rho^2)^6}
\]
(6.36)
\[
\delta(r) = -\frac{\pi}{\lambda} \times \frac{(40\zeta^3 - 25)^{3/2}}{2^{3/2}\zeta^5 \sqrt{1 - C_{a/r}}} \times \frac{32 + 25(r/\rho)^2 + 5(r/\rho)^4}{120(1 + (r/\rho)^2)^5}.
\]
(6.37)

As promised, this correction to the baryon profile is small in the near zone — \(\delta(r \sim \rho) = O(1/\lambda)\) — and becomes even smaller at larger radii.

** ***

Thus far, we have merely assumed the self-duality of the \(SU(2)\) magnetic fields with respect to the modified 4D metric \(\hat{g}_{mn}\). To justify this assumption, we are now going to show that the lowest-energy field configuration with \(N_{\text{instanton}} = 1\) is indeed self-dual, or at least approximately self-dual to leading order in \(1/\lambda\).

For simplicity, we look for the minimum of energy among the \(SO(4)\) symmetric field configurations only, i.e. we restrict the \(SU(2)\) gauge fields to the ansatz (6.14) but allow for generic radial profiles \(h(r)\) (subject to boundary conditions \(h(0) = 0, h(\infty) = 1\)). Likewise, we take the modified 4D metric \(\hat{g}_{mn}\) to be spherically symmetric as in eq. (6.29). For such configurations, the 4D determinant (6.9) becomes
\[
\det(\hat{g}_{mn} + 2\pi\alpha' F_{mn}) = \det(\hat{g}_{mn}) \times \left[ 1 + 2(\pi\alpha')^2 \hat{g}^{mp} \hat{g}^{nq} F_{mn} F_{pq} + (\pi\alpha')^4 \left( \hat{g}^{mp} \hat{g}^{nq} F_{mn} F_{pq} \right)^2 \right]
\]
\[
= C^4(1 + \mathcal{D}) \times (1 + \alpha^2 \bar{\tau}^2)(1 + \beta^2 \bar{\tau}^2)
\]
(6.38)

where
\[
\alpha = \frac{2\pi\alpha'}{C} \times \frac{2h(1 - h)}{r^2}, \quad \beta = \frac{2\pi\alpha'}{C} \times \frac{1}{r} \frac{dh}{dr} \times (1 + \mathcal{D})^{-1/2},
\]
(6.39)

and \(\bar{\tau}^2 = \tau_1^2 + \tau_2^2 + \tau_3^2\). Please note that we are not allowed to use the Pauli matrix algebra.
and set $\vec{\tau}^2 = 3$ while we calculate the determinant \((6.38)\); instead, we should to treat the $\tau_i$ as if they were independent and un-constrained commuting numbers.

Given the determinant \((6.38)\), we need to expand its square root in powers of $\alpha \tau_i$ and $\beta \tau_i$, then for each term in this expansion we should restore the $SU(2)$ gauge indices of the $\tau_{1,2,3}$ factors and calculate the symmetrized trace, and then we need to re-sum the series. This is easy to do in the self-dual case of $\alpha = \beta$ when the determinant \((6.38)\) is a full square, but for general $\alpha \neq \beta$ this calculation takes a few pages. We present it in the Appendix to this article; here is the end result:

$$\text{str} \sqrt{(1 + \alpha^2 \vec{\tau}^2)(1 + \beta^2 \vec{\tau}^2)} = 2 + 4\alpha^2 + 4\beta^2 + 6\alpha^2\beta^2.$$  \hfill (6.40)

For a given product $\alpha \times \beta$ this expression is minimized for self-dual $\alpha = \beta$ — this is the DBI version of the BPS lower bound on the Young–Mills action for gauge fields of a given instanton number. For convenience, we may rewrite eq. \((6.40)\) as

$$\text{str} \sqrt{(1 + \alpha^2 \vec{\tau}^2)(1 + \beta^2 \vec{\tau}^2)} = 2 + 6\alpha\beta + P(\alpha, \beta) \times (\alpha - \beta)^2$$ \hfill (6.41)

where $P(\alpha, \beta)$ is some complicated expression. It’s gory details will not be important in the following, expect for the special case of approximately self-dual $\alpha \approx \beta$ when

$$P(\alpha \approx \beta) \approx \frac{(3 + \alpha\beta)}{(1 + \alpha^2)(1 + \beta^2)}. \hfill (6.42)$$

More generally, $P(\alpha, \beta)$ is always positive and never greater then 3. For weak fields $\alpha, \beta \gg 1$, $P \approx 3$ — indeed, the Yang–Mills Lagrangian $\text{tr}(1 + \frac{1}{2}(\alpha + \beta)^2 \vec{\tau}^2) = 2 + 3(\alpha + \beta)^2$ has form \((6.41)\) for $P = 3$ — so the difference $3 - P$ measures the importance of the higher-order DBI corrections beyond the Yang–Mills approximation. For example, the approximately self-dual baryon with $h \approx r^2/(r^2 + \rho^2)$ for $\rho$ as in eq. \((5.51)\) has

$$\alpha(r) \approx \beta(r) \approx \frac{2\pi\alpha'}{C} \times \frac{2\rho^2}{(r^2 + \rho^2)^2} = \frac{\sqrt{k(\zeta)} \times \rho^4}{(r^2 + \rho^2)^2} \quad \text{for} \quad k(\zeta) = \frac{40\zeta^3 - 25}{64\zeta^3 + 8} \hfill (6.43)$$

and hence

$$P(r) \equiv P(\alpha(r), \beta(r)) = 3 - \frac{5k}{(1 + (r/\rho)^2)^4 + k} + \frac{2k^2}{[(1 + (r/\rho)^2)^4 + k]^2}. \hfill (6.44)$$

At the very center of the baryon, $P(r = 0)$ dips to 2.20 for the antipodal model — and
even lower to 1.37 for the non-antipodal models with $\zeta \gg 1$ — which indicates fairly strong higher-order DBI corrections. However, just one unit of $\rho$ outside the center, $P(r = \rho)$ climbs back to 2.8 for the $\zeta \gg 1$ models and even higher for the antipodal model, so the higher-order DBI corrections are important only in the inner core $r \lesssim \rho$ of the baryon’s near zone. Outside this inner core — from the outer side of the near zone all the way to the far zone — the higher-order DBI terms are small and one may safely use the Yang–Mills approximation to the $SU(2)$ fields’ Lagrangian.

Plugging eq. (6.41) into the non-abelian DBI Lagrangian, we obtain

$$-L_{DBI} = T_8 e^{-\phi} \text{Vol}(S^4) \sqrt{|g_{00}|} \times \text{str} \sqrt{\det(g_{mn} + 2\pi\alpha' F_{mn})}$$

(6.45a)

$$= \frac{T_8 \text{Vol}(S^4)}{g_s} \times C^2 \sqrt{1 + D} \times \left(2 + 6\alpha\beta + P(\alpha, \beta)(\alpha - \beta)^2\right)$$

(6.45b)

$$= \frac{T_8 \text{Vol}(S^4)}{g_s} \times \left[2C^2 \sqrt{1 + D} + 12(2\pi\alpha')^2 \times \frac{h(1 - h)}{r^3} \frac{dh}{dr} + (2\pi\alpha')^2 \frac{P(\alpha, \beta)}{\sqrt{1 + D}} \times \left(\frac{1}{r} \frac{dh}{dr} - \frac{2h(1 - h)}{r^2} \times \sqrt{1 + D}\right)^2\right]$$

(6.45c)

Inside the brackets on the bottom line (6.45c) of this formula, the first term gives rise to the DBI Lagrangian for the scalar and abelian vector fields but does not affect the $SU(2)$ gauge fields. The other two terms generate the Hamiltonian for the static $SU(2)$ magnetic fields in a fixed background of the other fields $\Phi(r)$ and $\hat{A}_0(r)$. The second term provides the minimal BPS energy of the self-dual fields, while the third term is the energy cost of deviations from self-duality: It vanishes precisely when the modified self-duality equation (6.31) is satisfied. Adding the Chern–Simons term

$$\mathcal{H}_{CS} = \frac{N_c}{2} \hat{A}_0(r) \times I(r) = \frac{3N_c\hat{A}_0(r)}{2\pi^2} \times \frac{h(1 - h)}{r^3} \frac{dh}{dr}$$

(6.46)

to the Hamiltonian for magnetic $SU(2)$ fields in a fixed background of $\hat{A}_0$ and $\Phi$ fields, we obtain

$$\mathcal{H} = \text{h–independent terms}$$

$$+ \frac{N_c}{2\pi^2} \left(\frac{\bar{u}}{\pi\alpha'} + \Phi + 3\hat{A}_0\right) \times \frac{h(1 - h)}{r^3} \frac{dh}{dr}$$

$$+ \frac{N_c}{24\pi^2} \left(\frac{\bar{u}}{\pi\alpha'} + \Phi\right) \frac{P(\alpha, \beta)}{\sqrt{1 + D}} \times \left(\frac{1}{r} \frac{dh}{dr} - \frac{2h(1 - h)}{r^2} \times \sqrt{1 + D}\right)^2.$$
Using spherically averaged brane geometry

\[\bar{u}(x) = u_0 + \delta \bar{u}(x),\]

\[\delta \bar{u}(w \to r) \approx \frac{16\zeta^3 - 10}{81\zeta} \lambda \alpha' M_A^3 \times \left( w^2 \to \frac{r^2}{4} \right) \]

\[\approx M_A \alpha' \sqrt{\frac{8\zeta^3 - 5}{10\zeta^2}} \times \left( 1 - C_{a/r} \right) \times (r/\rho)^2 \]

in the near zone, we integrate this Hamiltonian density to

\[H[h(r)] = h-\text{independent} + \frac{N_c u_0}{\pi \alpha'} \int_0^\infty dr h(1-h) \frac{dh}{dr} \]

\[+ N_c \int_0^\infty dr h(1-h) \frac{dh}{dr} \times \left( \frac{\delta \bar{u}(r)}{\pi \alpha'} + 3 \hat{A}_0(r) + \Phi(r) \right) \]

\[+ N_c \int_0^\infty dr \frac{P(\alpha(r), \beta(r))}{\sqrt{1 + \mathcal{D}}} \times \left( \frac{dh}{dr} - 2h(1-h) \times \frac{\sqrt{1 + \mathcal{D}}}{r} \right)^2 \times \frac{u_0(1 + O(1/\lambda))}{12\pi \alpha'} \].

(6.49)

Note that the integral on the top line (6.49a) has the same value \(\frac{1}{6}\) for any baryon profile \(h(r)\) satisfying the boundary conditions; this integral provides the leading contribution \((\lambda N_c M_A \zeta)/27\pi\) to the baryon’s mass. The second line (6.49b) corrects the BPS energy of self-dual \(SU(2)\) fields in a non-uniform background of \(\hat{A}_0(r)\) and \(\Phi(r)\) fields, and also brane curvature \(\delta u(r)\). This extra energy has a non-trivial dependence on the baryon’s radius and profile; its overall magnitude is \(O(N_c M_A) = O(\lambda^{-1} M_{\text{baryon}})\). Finally, the third line (6.49c) is the energy cost of deviation from self-duality. The \(u\)-dependent factor here is \(u_0\) (rather than much smaller \(\delta \bar{u} + \pi \alpha'(3 \hat{A}_0 + \Phi)\) on the second line), so the energy cost of a major non-self-duality would be \(O(M_{\text{baryon}})\). And minimizing this extra energy is precisely why the baryon profile is approximately self-dual.

Indeed, minimizing the Hamiltonian (6.49) as a functional of the instanton profile \(h(r)\) gives us a rather messy differential equation

\[\left( \frac{d}{dr} + \frac{2(1 - 2h)}{r} \sqrt{1 + \mathcal{D}} \right) \left[ P(\alpha(r), \beta(r)) \times \left( \frac{r}{\sqrt{1 + \mathcal{D}}} \frac{dh}{dr} - 2h(1-h) \right) \right] = -6h(1-h) \times \frac{d}{dr} \frac{\delta \bar{u} + \pi \alpha'(3 \hat{A}_0 + \Phi)}{u_0}. \]

(6.50)
In the near zone, the ratio \((\delta u + \cdots)/u_0\) on the right hand side is \(O(1/\lambda)\) small, so on the left hand side we should have a similarly small deviation from self-duality with respect to the modified metric \(\hat{g}_{mn}\). The net energy cost of this deviation is \(O(\lambda^{-2}M_{\text{baryon}})\), which is too small to be concerned with at the present level of analysis. Likewise, the effects of self-duality violation on the baryon’s average radius — and hence on the nuclear forces in the intermediate and far zones — are minor corrections of relative order \(O(1/\lambda)\) to the leading-order effects we have calculated in the previous section 5.

To conclude this section, we notice that while the deviation of the baryon profile from the modified self-duality condition is \(O(1/\lambda)\) small, it is comparable to the modification of flat-space self-duality condition due to \(D(r) \neq 0\). So for completeness sake, we would like to calculate both effects on the baryon profile to the same order \(O(1/\lambda)\). Let’s parametrize deviations of \(h(r)\) from a flat-space instanton using \(\delta(r)\) as in

\[
h(r) = \frac{r^2}{r^2 + \rho^2 \times (1 + \delta(r))}.
\]

Substituting this formula into the differential equation (6.50) and expanding to first order in \(\delta(r)\) and \(D(r)\), we get

\[
\left(\frac{d}{dr} + \frac{2(\rho^2 - r^2)}{r(\rho^2 + r^2)} \right) \left[ \frac{\rho^2 r^2 P(r)}{(\rho^2 + r^2)^2} \times \left( \frac{d\delta}{dr} + D \right) \right] = \frac{6\rho^2 r^2}{(\rho^2 + r^2)^2} \times \frac{d}{dr} \frac{\delta u + \pi \alpha'(3\hat{A}_0 + \Phi)}{u_0},
\]

where on the left hand side \(P(r)\) is as in eqs. (6.43) (6.44) and on the right hand side

\[
\frac{\delta u + \pi \alpha'(3\hat{A}_0 + \Phi)}{u_0} = T \left[ \frac{r^2}{5\rho^2} + \frac{\rho^2(2\rho^2 + r^2)}{(\rho^2 + r^2)^2} \right]
\]

where \(T = \frac{9\pi}{\lambda \zeta^2} \sqrt{\frac{40\zeta^3 - 25}{8}} \sqrt{1 - \frac{C_{n/r}}{}} = O(1/\lambda),\)

Solving the differential equation (6.52) is a straightforward exercise in calculus. Integrating the outer differential operators gives us

\[
r \frac{d\delta}{dr} + D = -T P(r) \times \frac{2r^2(3r^4 + 12r^2\rho^2 + 14\rho^4)}{5\rho^2(r^2 + \rho^2)^2}.
\]
Note that the right hand side here is of the same magnitude as the $D$ term on the right hand side which modifies the self-duality condition; indeed, in present notations

$$D = -2kT \times \frac{\rho^6 r^2 (3\rho^2 + r^2)^2}{(r^2 + \rho^2)^6}. \quad (6.56)$$

Altogether,

$$\delta(r) = T \int \frac{2r \, dr}{(r^2 + \rho^2)^6} \left( k\rho^6 (3\rho^2 + r^2)^2 - \frac{[3(r^2 + \rho^2)^2 + 2\rho^4] \times [(\rho^2 + r^2)^4 + k\rho^8]^2}{5\rho^2 [3(\rho^2 + r^2)^4 + k\rho^8]} \right) \quad (6.57)$$

where the analytic form of the integral is rather unwieldy. Instead of writing it as a formula, let’s plot $\delta(r)$ for 3 representative values of $\zeta$:

The differences between the three colored lines here are due to the higher-order (beyond YM) terms in the DBI action whose effect depends on the $\zeta$ parameter. The low-order interactions — Yang–Mills, Chern–Simons, $\Phi F^2$, and the D8-brane curvature $\delta \bar{u}(r)$ — have the same radial dependence (in units of $\rho$) for all the Sakai–Sugimoto models, so when the higher-order interactions pucker out at $r \gtrsim \rho$, the deviations from self-duality become $\zeta$–independent and all the colored lines converge to the same black-dotted line

$$\delta_0(r) \approx \frac{T}{5} \left(\frac{-r^2}{\rho^2} - 2 \log \frac{r^2 + \rho^2}{\rho^2} + \frac{(5/3)\rho^2}{r^2 + \rho^2}\right). \quad (6.59)$$
This line shows that the low-order interactions themselves make the $SU(2)$ fields deviate from self duality. In particular, at large radii we have a growing deviation $\delta \propto -r^2$ due to brane curvature $\delta \bar{u} \propto +r^2$ (and hence the 5D gauge coupling growing with $r$). Consequently, self-duality of the $SU(2)$ fields becomes less accurate in the intermediate zone $r \gg \rho$, and eventually breaks down in the far zone $r \gtrsim (\rho/\sqrt{T}) \sim \rho \sqrt{\lambda} \sim M_{\Lambda}^{-1}$.

7 Summary and Open Questions

Viable holographic nuclear physics obviously requires both attractive and repulsive nuclear forces. In holographic context, the hard core repulsive potential was found in [16], and in this article we saw that the non-antipodal version of the Sakai–Sugimoto model gives rise to the attractive potential as well as repulsive. Let us summarize our main results:

- We argue that nuclear physics in the $N_c \to \infty$ limit could be quite different from the real-life case of $N_c = 3$, which limits the applicability of holography to zero-temperature nuclear physics. In particular, the ratio of kinetic to potential energy of the bulk nuclear matter scales with $N_c$ as $1/N_c^2$, and consequently nuclear matter is a Fermi liquid for small $N_c$ but becomes a crystalline solid for large $N_c$. We estimate the transition between liquid and solid phases happens for $N_c \sim 8$, but this estimate is rather crude and should be taken with a large grain of salt.

- Holographically, the attractive forces between nucleons arise from the coupling of the gauge fields living on the flavor branes to the scalar fields parametrizing fluctuations of the those branes’ geometry; in 5D, this coupling has form

$$S_{5D} \supset \int d^5 x \bar{\Phi} \times \text{tr}(F_{MN}^2). \quad (7.1)$$

The antipodal Sakai–Sugimoto model has an accidental $\Phi \to -\Phi$ symmetry which forbids this coupling, so in that model there are no attractive forces. But the non-antipodal models don’t have this symmetry, and consequently they do have the scalar-vector coupling (7.1) and hence the attractive nuclear forces.

- At intermediate distances $r$ between the two nucleons — larger than the nucleon radius $\rho \sim R_{KK}/\sqrt{\lambda}$ but smaller than the Kaluza–Klein scale $R_{KK}$, see diagram [11] on page 23 — both the attractive and the repulsive potentials have the 5D Coulomb
form, \( V(r) \propto 1/r^2 \). But the attractive potential has a smaller coefficient, so the net central potential is repulsive,

\[
C_{a/r} \equiv \frac{-V_{\text{attractive}}}{V_{\text{repulsive}}} = \frac{1 - \zeta^{-3}}{9} < \frac{1}{9} < 1 \implies V_{\text{net}}(r) > 0. \quad (7.2)
\]

- There are similar attractive forces between different parts of the same baryon, and they reduce the baryon’s radius by a factor \((1 - C_{a/r})^{1/4}\). Consequently, the isovector spin-spin and tensor forces between two baryons are reduced by an overall factor \((1 - C_{a/r})\).

- At longer distances \( r \gtrsim R_{KK} \sim 1/M_{\text{meson}} \), the nuclear forces are dominated by 4D Yukawa forces due to the lightest meson with appropriate quantum numbers: The vector isosinglet for the repulsive central forces and the scalar isosinglet for the attractive central forces. In the Sakai–Sugimoto model, the lightest scalar meson is heavier than the lightest vector meson, and consequently the attractive force has a shorter range than the repulsive force. It is not clear whether this un-realistic behavior is peculiar to the Sakai–Sugimoto models or a general problem of the large \( N_c \) limit. Indeed, in real life the lightest scalar meson is \( \sigma(600) \) but it’s QCD origin is not clear. If it happens to be a bound state of two pions rather than a true \( q\bar{q} \) meson, then in the large \( N_c \) limit this bound state will fall apart and the lightest surviving scalar meson would be heavier than the lightest vector meson \( \omega(787) \).

- The vector fields living on the flavor branes are governed by the DBI + CS action, but usually the DBI part of the action is truncated to the lowest-order Yang–Mills terms,

\[
\mathcal{L}_{\text{DBI}} \propto \text{str} \sqrt{\det(g + 2\pi \alpha' \mathcal{F})} \approx \sqrt{-\det(g)} \times \left( N_f + \frac{(2\pi \alpha')^2}{4} \text{tr} (\mathcal{F}_{MN} \mathcal{F}^{MN}) \right). \quad (7.3)
\]

In the Sakai–Sugimoto model with \( \lambda \gg 1 \), this approximation is valid at intermediate and long distances from baryons, but inside the instanton core of a baryon the non-abelian gauge fields become too strong to neglect the higher-order terms such as \( \text{tr}(\mathcal{F}^4) \).

We argue that the self-duality of the non-abelian magnetic fields saves the day and leads to approximate cancellation of the higher-order terms. This was known for instantons in flat space, and we show that this is also true for the Sakai–Sugimoto baryons; all we need to do is to slightly modify the self-duality condition for the non-abelian fields to account for their coupling to the abelian electric and scalar fields. The effect of this modification on the baryon’s radial profile is quite small: \( O(1/\lambda) \) near the baryon’s center and even smaller for \( r \gtrsim \rho \).
We have also computed numerically the deviation of the baryon’s profile from self-duality due to curvature of the flavor branes. The net deviation from a simple YM instanton is plotted on figure 6.58 on page 51.

Our work gives rise to several open questions, the biggest of which is “What happens to the $\sigma(600)$ meson in the large $N_{c}$ limit?”. The best answer for this question would be a lattice calculation of $m_\sigma$ for several values of $N_{c}$, although it’s not clear if such a calculation is possible with-present-day lattice sizes. (But thanks to Moore’s law, it should be possible in a few years.) Alternatively, we can try several different models of holographic QCD and compare their predictions for the lightest scalar to lightest vector mass ratio. As of this writing, all known models have ratios > 1, with one exception [29] — but in that model, the lightest scalar meson is is a pseudo-Goldstone boson and probably does not couple to the other particles like the real $\sigma(600)$ meson. If future holographic models show the same pattern — the lightest true scalar meson is either heavier than the lightest vector or else is a pseudo-Goldstone boson whose couplings are suppressed — then most likely, in the large $N_{c}$ limit of QCD there is no sigma meson and the attractive nuclear force has a shorter range than the repulsive force. But if we see a wide variation of the lightest-scalar-to-lightest-vector mass ratios between different models, then we wouldn’t know what really happens in large-$N_{c}$ QCD, but on the other hand, a holographic model with $m_\sigma < m_\omega$ might also have a semi-realistic nuclear potential — repulsive at shorter distances but attractive at longer distances.

We expect different holographic models to have different attractive / repulsive force ratios at intermediate and short distances, and it would be interesting to see if any model has $C_{a/r} > 1$. In such a model, net attraction between different parts of the same baryon would make it collapse to a singular point. Or rather, a classical baryon would collapse to a point, but quantum corrections would keep its size finite, perhaps $O(R_{KK}/\lambda)$, but much smaller than in the Sakai–Sugimoto model. Consequently, the net force between two such baryons would be repulsive at very short distances $r \lesssim \rho \sim R_{KK}/\lambda$ but attractive at intermediate distances $\rho \ll r \ll R_{KK}$. At longer distances $r \gtrsim R_{KK}$, the net force could be either attractive or repulsive, depending on the meson spectrum.

Another open question concerns the dependence of the effective 5D action on the ’t Hooft coupling $\lambda$ in the effective 5D action. In the Sakai–Sugimoto model, the flavor gauge coupling $^2 \propto 1/\lambda$, but this power of $\lambda$ could be different in other holographic models. It would be interesting to find models where the flavor physics does not depend on $\lambda$ at all and to explore the baryons and the nuclear forces in such models. In particular, we would like
to see if such models have largish baryon radii, $\rho \sim 1/M_{\text{meson}}$ like in real life, rather than $\rho \ll 1/M_{\text{meson}}$ as in the Sakai–Sugimoto model. For such large-radius baryons there would be no intermediate zone of distances; instead, the near zone (where two baryons overlap) would connect directly to the far zone dominated by 4D Yukawa forces. Consequently, for $r \sim \rho$ both the baryon overlap and the curvature of the fifth dimension would be important, and the nuclear forces in this regime would be quite different from anything in the Sakai–Sugimoto model.

There are also more general issues concerning meson spectra in holographic models. Apart from the specific mass ratios that are probably model dependent, there are general differences from the real-life mesons found in the Particle Data Book. For example, the 5D scalar fields give rise to both scalar and pseudoscalar mesons in 4D (depending on the mode number in the fifth dimension), and their charge conjugation signs follow from parity, C-positive scalars and C-negative pseudoscalars. But in real life, all pseudoscalar mesons are C-positive rather than C-negative.

Also, the high-spin mesons in holographic models have different physical origin from the low-spin mesons and consequently much larger masses. While the $J = 0$ and $J = 1$ 4D mesons are modes of 5D scalar or vector fields, the $J \geq 2$ mesons are semi-classical rotating open strings which start and end on flavor branes but at different points in space [33]. But in real life, both low-spin and high-spin mesons belong to the same Regge trajectories

$$M^2 = \alpha' \times J + \text{const}$$

and there are no essential differences between them.

We don’t know what makes the holographic meson spectra so different from the real life, but it’s almost certainly not the large $N_c$ limit. It would be interesting to see if this problem is common to all holographic models — perhaps because it’s inherent in the $\lambda \to \infty$ limit — or if there are some model with more realistic meson spectra. If we can find such a model, maybe it would also have a light scalar meson and hence attractive net nuclear force at longer distances.

Yet another open question concerns nuclear forces stemming from double-meson exchanges, especially the double-pion exchange which produces a long-range attractive force. In holography, the double meson exchanges happen at the one-string-loop level while single meson exchanges happen at the tree level. This makes the double exchanges smaller by a factor $1/\lambda$, and also much harder to calculate. But if such calculation is feasible for some
holographic model, it would be very interesting to compare its result to the real-life nuclear force due to double-pion exchanges.

Finally, there is a long-standing open problem concerning sensitivity of nuclear forces — and hence of the nuclear binding energy — to the pion’s mass. Hopefully, holography can shed some new light on this old problem. Although holographic models usually have coincident flavor branes and hence zero current quark masses and massless pions, there are ways [36, 37, 38, 39] to explicitly break the chiral symmetry and give the pions a small mass. It would be interesting to see if a small but non-zero $m_\pi^2$ would affect the isoscalar central force between two nucleons, and whether such effect would happen at the tree level of string theory or only at the loop levels.

Acknowledgments

The authors would like to thank Ofer Aharony, Jacques Distler, Shigenori Seki, and Shimon Yankielowicz for many fruitful conversations. We also thank Alexei Cherman and Tom Cohen for explaining to us how to count the powers of $N_c$ in nuclear forces involving multiple meson exchanges.

The research presented in this article was supported by: The US–Israel Binational Science Foundation (both authors), the US National Science Foundation (V. K., grant #PHY–0455649), the Israel Science Foundation (J. S., grant #1468/06), the German–Israeli Project Cooperation (J. S., grant #DIP H52), and German–Israeli Foundation (J. S.).

A Symmetrized Trace of the Non-Abelian DBI Action

According to Tseytlin [34], the non-abelian version of the Dirac–Born–Infeld Lagrangian $\sqrt{\det(g_{MN} + 2\pi\alpha' F_{MN})}$ works like this: First we focus on the spacetime indices of the $g_{MN} + 2\pi\alpha' F_{MN}$ and formally calculate the determinant of this $d \times d$ matrix while completely ignoring the gauge indices of the $F_{MN}$ fields or the fact that they don’t commute with each other. In other words, at this stage of the calculation, we treat each component $F_{MN}$ as if it was a just real number rather than a generator of some non-abelian group. Second, we expand the square root of the determinant into a power series in the $F_{MN}$ fields; again, we ignore the fields’ non-commutativity and treat them as real numbers. Third, for each term in the expansion, we restore the gauge indices of the fields (in the fundamental representation.
of a $U(N)$ group, symmetrize the product of non-commuting fields, and take the trace,

$$\text{str}\left(F_{M_1 N_1} F_{M_2 N_2} \cdots F_{M_k N_k}\right) = \frac{1}{k!} \sum \text{tr}(\text{all permutations of } F_{M_1 N_1} F_{M_2 N_2} \cdots F_{M_k N_k}).$$

(A.1)

Finally, we try to re-sum the power series in the $F$ fields; if we are lucky, it might have a nice analytic form.

In section 6, we had spherically-symmetric (in 4D) $SU(2)$ fields (6.15), and we had calculated the DBI determinant for those fields as

$$\det\left(\hat{g}_{mn} + 2\pi \alpha' F_{mn}\right) = \det(\hat{g}_{mn}) \times (1 + \alpha^2 \bar{\tau}^2) \left(1 + \beta^2 \bar{\tau}^2\right).$$

(6.38)

In this Appendix, we calculate the symmetrized trace of the square root of this determinant and show that

$$\text{str} \sqrt{(1 + \alpha^2 \bar{\tau}^2) (1 + \beta^2 \bar{\tau}^2)} = \frac{2 + 4\alpha^2 + 4\beta^2 + 6\alpha^2\beta^2}{\sqrt{(1 + \alpha^2)(1 + \beta^2)}}.$$

(6.40)

Clearly, expanding the square root on the LHS into powers of $\alpha$ and $\beta$ produces all powers of $\tau_i \tau_i$, so our first step is to symmetrize the product $(\bar{\tau}^2)^n$ with respect to all distinct permutations of the $2n$ Pauli matrices.

**Lemma:**

$$\begin{align*}
\left[(\bar{\tau}^2)^n\right]_{\text{symm}} & \overset{\text{def}}{=} \frac{1}{(2n - 1)!!} \sum_{\text{distinct permutations}} \tau_{i_1} \tau_{i_1} \tau_{i_2} \tau_{i_2} \cdots \tau_{i_n} \tau_{i_n} \\
& = (2n + 1) \times \mathbf{1}_{2 \times 2}
\end{align*}$$

(A.2)

where

$$(2n - 1)!! = \frac{(2n)!}{2^n n!}$$

(A.3)

is the number of distinct permutations of $2n$ matrices that come in $n$ identical pairs. For the purpose of symmetrization, $\tau_i$ and $\tau_j$ carrying different isovector indices $i$ and $j$ count as distinct; the summation over $i, j, \ldots = 1, 2, 3$ is done after the symmetrization.

The lemma (A.2) is trivially true for $n = 0$ and $n = 1$; indeed, for $n = 1$ there is nothing to symmetrize and $\left[\tau_i \tau_i\right]_{\text{symm}} = \tau_i \tau_i = \delta_{ii} \times \mathbf{1} = 3 \times \mathbf{1}$. For the first non-trivial case $n = 2$, we have
the lemma works according to

\[
\left[(\vec{\tau}^2)^2 = \tau_i \tau_i \tau_j \tau_j\right]_{\text{symm}} = \frac{1}{3} \left(\tau_i \tau_i \tau_j \tau_j + \tau_i \tau_j \tau_i \tau_j + \tau_j \tau_i \tau_i \tau_j\right)
\]

\[
= \frac{1}{3} \left((\tau_i \tau_i)^2 + \{\tau_i, \tau_j\} \times \tau_i \tau_j\right)
\]

\[
= \frac{1}{3} \left((3)^2 + 2\delta_{ij} \times \tau_i \tau_j = 9 + 2 \times 3 = 15\right)
\]

\[
= 5 \ (i.e., \ 5 \times 1).
\]  

(A.4)

For higher \(n > 2\) the simplest proof of the lemma is recursive. Let’s group the \((2^n - 1)!!\) distinct permutations of the \(2n\) matrices \(\tau_i \tau_i \tau_j \tau_j \cdots\) into two sets according to the two leftmost matrices being similar or distinct: \((2^n - 3)!!\) of the permutations start with \(\tau_i \tau_i\) (for the same \(i\)) while the remaining \((2n - 2) \times (2n - 3)!!\) permutations start with \(\tau_i \tau_j\) — or equivalently \(\tau_j \tau_i\) — with distinct \(i\) and \(j\). Consequently,

\[
(2n - 1)!! \times \left[(\vec{\tau}^2)^n\right]_{\text{symm}} = \sum_{\text{all distinct permutations of } \tau_i \tau_i \tau_j \tau_j \tau_k \tau_k \cdots}^{(2n-3)!!} \tau_i \tau_i \times \sum_{(2n-2)(2n-3)!!} \text{permutations of } \tau_j \tau_j \tau_k \tau_k \cdots
\]

\[
+ \frac{1}{2}\{\tau_i, \tau_j\} \times \sum_{(2n-2)(2n-3)!!} \text{permutations of } \tau_i \tau_j \tau_k \tau_k \tau_\ell \tau_\ell \cdots
\]

\[
= 3 \times (2n - 3)!! \left[(\vec{\tau}^2)^{n-1}\right]_{\text{symm}}
\]

\[
+ \delta_{ij} \times (2n - 2)(2n - 3)!! \left[\tau_i \tau_j (\vec{\tau}^2)^{n-2}\right]_{\text{symm}}
\]

\[
= 3(2n - 3)!! \times \left[(\vec{\tau}^2)^{n-1}\right]_{\text{symm}}
\]

\[
+ (2n - 2)(2n - 3)!! \times \left[(\vec{\tau}^2)^{n-1}\right]_{\text{symm}}
\]

\[
= (3 + 2n - 2) \times (2n - 3)!! \left[(\vec{\tau}^2)^{n-1}\right]_{\text{symm}}
\]

(A.5)

and hence

\[
\left[(\vec{\tau}^2)^n\right]_{\text{symm}} = \left[(\vec{\tau}^2)^{n-1}\right]_{\text{symm}} \times \left(\frac{(3 + 2n - 2) \times (2n - 3)!!}{(2n - 1)!!}\right) = \frac{2n + 1}{2n - 1}.
\]

(A.6)
Applying this formula recursively, we get

\[
\left[(\vec{\tau}^2)^n\right]_{\text{symm}} = \frac{2n+1}{2n-1} \times \left[(\vec{\tau}^2)^{n-1}\right]_{\text{symm}} = \frac{2n+1}{2n-1} \times \frac{2n-1}{2n-3} \times \cdots \times \frac{5}{3} \times \left[(\vec{\tau}^2)^1\right]_{\text{symm}}
\]

\[
= \frac{2n+1}{3} \times 3 \times 1 = (2n+1) \times 1.
\] (A.7)

Quod erat demonstrandum.

In terms of symmetrized traces, the Lemma tells us that

\[
\text{str} \left[(\vec{\tau}^2)^n\right] \equiv \text{tr} \left[\left[(\vec{\tau}^2)^n\right]_{\text{symm}}\right] = 2(2n+1),
\] (A.8)

which leads us to the following

**Theorem:** the symmetrized trace of any analytic function

\[
f(\vec{\tau}^2) = \sum_{n=0}^{\infty} C_n (\vec{\tau}^2)^n
\] (A.9)

of $\vec{\tau}^2$ can be evaluated as

\[
\text{str} \left[f(\vec{\tau}^2)\right] = \sum_n C_n \times \left(\text{str} \left[(\vec{\tau}^2)^n\right] = 2(2n+1)\right) = \left(4x \frac{\partial}{\partial x} + 2\right) f(x) \bigg|_{x=1}.
\] (A.10)

In particular, the square root of the DBI determinant (6.38) has symmetrized trace

\[
\text{str} \sqrt{(1 + \alpha^2 \vec{\tau}^2)(1 + \beta^2 \vec{\tau}^2)} = \left(4\alpha^2 \frac{\partial}{\partial \alpha^2} + 4\beta^2 \frac{\partial}{\partial \beta^2} + 2\right) \sqrt{(1 + \alpha^2)(1 + \beta^2)}
\]

\[
= \frac{2 + 4\alpha^2 + 4\beta^2 + 6\alpha^2\beta^2}{\sqrt{(1 + \alpha^2)(1 + \beta^2)}},
\] (A.11)

as promised in eq. (6.40).

To conclude this appendix, we note that the symmetrized trace (A.11) is bounded from above by the low-order tension+Yang–Mills limit $2 + 3(\alpha^2 + \beta^2)$ and from below by its value $2 + 6\alpha\beta$ for the self-dual fields, thus

\[
\forall \alpha, \beta \geq 0, \quad 2 + 6\alpha\beta \leq \frac{2 + 4\alpha^2 + 4\beta^2 + 6\alpha^2\beta^2}{\sqrt{(1 + \alpha^2)(1 + \beta^2)}} \leq 2 + 3(\alpha^2 + \beta^2).
\] (A.12)
In terms of eq. (6.41), these bounds amount to limits on $P(\alpha, \beta)$,

$$\text{str} \sqrt{(1 + \alpha^2 \tau^2)(1 + \beta^2 \tau^2)} = 2 + 6 \alpha \beta + P(\alpha, \beta) \times (\alpha - \beta)^2, \quad 0 \leq P(\alpha, \beta) \leq 3. \quad (A.13)$$
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