STRUCTURE OF THE MORDELL-WEIL GROUP OVER THE $\mathbb{Z}_p$-EXTENSIONS

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Abstract. We study the $\Lambda$-module structure of the Mordell-Weil, Selmer, and Tate-Shafarevich groups of an abelian variety over $\mathbb{Z}_p$-extensions.

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1. Introduction

1.1. Overview and the questions. The Iwasawa theory of abelian varieties (elliptic curves) was initiated by Mazur in his seminal paper [Maz72], where he proved that the $p^\infty$-Selmer groups of an abelian variety $A$ defined over a number field $K$ are “well-controlled” over $\mathbb{Z}_p$-extensions if $A$ satisfies certain reduction condition at places dividing $p$. More precisely, let $K_\infty$ be a $\mathbb{Z}_p$-extension of $K$ and $K_n$ be the $n$-th layer, and $\text{Sel}_{K_n}(A)_p$ be the classical $p^\infty$-Selmer group over $K_n$ attached to $A$. For a natural restriction map

$$S_n^A : \text{Sel}_{K_n}(A)_p \to \text{Sel}_{K_\infty}(A)_p[\omega_n],$$

we have the following celebrated theorem.

Theorem 1.1.1 (Control Theorem). If either

- (Mazur, [Maz72]) $A$ has a good ordinary reduction at all places of $K$ dividing $p$

or

- (Greenberg, [Gre99, Proposition 3.7]) $K = \mathbb{Q}$ and $A$ is an elliptic curve having multiplicative reduction at $p$

then $\text{Coker}(S_n^A)$ is finite and bounded independent of $n$.

Here the word “control” means that the Selmer group $\text{Sel}_{K_n}(A)_p$ at each layer $K_n$ can be described by the one object. Hence we can expect that the each Selmer group over $K_n$ should behave in a certain regular way governed by the “limit” Selmer group $\text{Sel}_{K_\infty}(A)_p$. For instance, we have the following consequence of the above theorem.

Proposition 1.1.2. Assume either one of the condition of Theorem 1.1.1 and also assume that both $A(K_n)$ and $\text{III}_{1K_n}(A)_p$ are finite for all $n$. Then there exists $\mu, \lambda, \nu$ such that

$$|\text{Sel}_{K_n}(A)_p| = |\text{III}_{1K_n}(A)_p| = p^{e_n} \quad (n >> 0)$$

where

$$e_n = p^n \mu + n\lambda + \nu.$$
For the proof, see [Gre01, Corollary 4.11]. The value \( p^n \mu + n \lambda + \nu \) in the Proposition 1.1.2 naturally appears from the structure theory of \( \Lambda \)-modules. For a finitely generated \( \Lambda \)-module \( M \), there is a \( \Lambda \)-linear map

\[
M \to \Lambda^r \oplus \left( \bigoplus_{i=1}^n \frac{\Lambda}{g_i^{e_i}} \right) \oplus \left( \bigoplus_{j=1}^m \frac{\Lambda}{p^{f_j}} \right)
\]

with finite kernel and cokernel where \( r, n, m \geq 0, e_1, \ldots, e_n, f_1, \ldots, f_m \) are positive integers, and \( g_1, \ldots, g_n \) are distinguished irreducible polynomial of \( \Lambda \). The quantities \( r, e_1, \ldots, e_n, f_1, \ldots, f_m, g_1, \ldots, g_n \) are uniquely determined, and we call

\[
E(M) := \Lambda^r \oplus \left( \bigoplus_{i=1}^n \frac{\Lambda}{g_i^{e_i}} \right) \oplus \left( \bigoplus_{j=1}^m \frac{\Lambda}{p^{f_j}} \right)
\]

as an elementary module of \( M \) following [NSWS00, Page 292]. The \( \lambda \)-invariant \( \lambda(M) \) is defined as \( \sum_{i=1}^n e_i \cdot \deg g_i \) and the \( \mu \)-invariant \( \mu(M) \) is defined as \( f_1 + \ldots + f_m \).

In Proposition 1.1.2, the constants \( \lambda \) and \( \mu \) are indeed the \( \lambda \) and \( \mu \)-invariants of the module \( \text{Sel}_{K_1}(A)^{\vee} \), respectively and hence we can say that the module \( E \left( \text{Sel}_{K_1}(A)^{\vee} \right) \) gives the information about the arithmetic of the \( A \) at each finite level at once. Hence it is quite natural to ask questions about the \textit{shape} of the \( \Lambda \)-module \( E \left( \text{Sel}_{K_1}(A)^{\vee} \right) \). We can also consider the same question for the Mordell-Weil group and the Tate-Shafarevich group, which fit into a natural exact sequence

\[
0 \to A(K) \otimes \mathbb{Q}_p / \mathbb{Z}_p \to \text{Sel}_{K_1}(A)_p \to \text{III}_{K_1}(A)_p \to 0.
\]

The main goal of this paper is studying

\[
E \left( (A(K) \otimes \mathbb{Z}_p \mathbb{Q}_p / \mathbb{Z}_p)^{\vee} \right), \ E \left( \text{Sel}_{K_1}(A)^{\vee} \right), \ E \left( \text{III}_{K_1}(A)^{\vee} \right).
\]

More precisely, we can ask the following questions:

**Question 1.1.3** (Describing the elementary modules).
- Can we describe the structure of \( E \left( (A(K) \otimes \mathbb{Z}_p \mathbb{Q}_p / \mathbb{Z}_p)^{\vee} \right), E \left( \text{Sel}_{K_1}(A)^{\vee} \right), E \left( \text{III}_{K_1}(A)^{\vee} \right) \)? Or can we find some relations among these three modules?

**Question 1.1.4** ("Smallness" of the \( \text{III}^1 \)). Under the finiteness assumption of the groups \( \text{III}_{K_1}^1(A)_p \),
- Is the group \( \text{III}_{K_1}^1(A)_p \) \( \Lambda \)-cotorsion?
- Can we find an estimate (even conjecturally) of \( |\text{III}_{K_1}^1(A)_p| \) in terms of \( n \)?

**Question 1.1.5** (Algebraic Functional Equation). If \( A^t \) is the dual abelian variety of \( A \), then can we find any relation between

- \( E \left( \text{Sel}_{K_1}(A)^{\vee} \right) \) and \( E \left( \text{Sel}_{K_1}(A^t)^{\vee} \right) \)?
- \( E \left( \text{III}_{K_1}^1(A)_p^{\vee} \right) \) and \( E \left( \text{III}_{K_1}^1(A^t)_p^{\vee} \right) \)?
- \( E \left( (A(K) \otimes \mathbb{Z}_p \mathbb{Q}_p / \mathbb{Z}_p)^{\vee} \right) \) and \( E \left( (A^t(K) \otimes \mathbb{Z}_p \mathbb{Q}_p / \mathbb{Z}_p)^{\vee} \right) \)?

We will try to answer these questions in this paper. For the \( \Lambda \)-corank of the module \( \text{Sel}_{K_1}(A)_p \), we have the following famous theorem.

**Theorem 1.1.6** (Kato-Rohrlich). If \( A \) is an elliptic curve defined over \( \mathbb{Q} \) with good ordinary reduction or multiplicative reduction at \( p \), \( K \) is an abelian extension of \( \mathbb{Q} \) and \( K_\infty \) is a cyclotomic \( \mathbb{Z}_p \)-extension of \( K \), then the \( \text{Sel}_{K_1}(A)_p \) is a cotorsion \( \Lambda \)-module.

**Remark 1.1.7.** The cotorsionness of \( \text{Sel}_{K_1}(A)_p \) is known only (at this moment) under the assumptions of Theorem 1.1.6.

**Remark 1.1.8.** The \textit{main novelty} of this paper are two folds:
• First of all, instead of the characteristic ideals of the modules above (which are usually studied because of its connection with the Iwasawa Main Conjecture), we study their Λ-module structure. Here the word “structure” means that we study the elementary modules of
\[(A(K_\infty) \otimes \mathbb{Z}_p \mathbb{Q}_p/\mathbb{Z}_p)\,^\vee, \, \text{Sel}_{K_\infty}(A)_p, \, \text{III}^1_{K_\infty}(A)_p\,^\vee.\]

• Another point is that we do not assume the Λ-cotorsionness of the Selmer group. We solely assume the control of the Selmer group and study the three Λ-modules \((A(K_\infty) \otimes \mathbb{Z}_p \mathbb{Q}_p/\mathbb{Z}_p)\,^\vee, \, \text{Sel}_{K_\infty}(A)_p, \, \text{III}^1_{K_\infty}(A)_p\,^\vee\). Moreover, our technique using the functors \(\mathfrak{T}\) and \(\mathfrak{S}\) (which will be introduced in Appendix) gives another proof of the known algebraic functional equation results. See 1.3 for the comparison with former works.

We hope that our results enable us to determine the structure of the various arithmetic groups (e.g. Mordell-Weil, Selmer and \(\text{III}^1\)) as modules over the group ring at each finite layer \(K_n\).

1.2. Main theorems and consequences.

Theorem A (Theorem 2.1.2). We have a Λ-linear injection
\[(A(K_\infty) \otimes \mathbb{Z}_p \mathbb{Q}_p/\mathbb{Z}_p)\,^\vee \hookrightarrow \Lambda^r \bigoplus_{n=1}^{t} \frac{\Lambda}{\omega_{b_{n+1}, b_n}}\]
with finite cokernel for some integers \(r, b_1, \cdots, b_n\) where \(\omega_{n+1} := \frac{(1+T)^{n+1}-1}{(1+T)^{n+1}-1}\).

Remark 1.2.9. (1) Hence the direct factors of \(E ((A(K_\infty) \otimes \mathbb{Z}_p \mathbb{Q}_p/\mathbb{Z}_p))\,^\vee\) are only of the form \(\frac{\Lambda}{\omega_{k+1, k}}\) for some \(k\). For instance, neither \(\frac{\Lambda}{\omega_{2, p}}\) nor \(\frac{\Lambda}{\omega_{1, p}}\) can not be a direct factor of \(E ((A(K_\infty) \otimes \mathbb{Z}_p \mathbb{Q}_p/\mathbb{Z}_p))\,^\vee\).

(2) The same proof also works if \(K\) is a finite extension of \(\mathbb{Q}_p\). (See Page 7 for the proof.)

One consequence of this Theorem in \(p\)-adic local case is the following:

Corollary 1.2.10 (Theorem 2.2.7). Let \(L\) be a finite extension of \(\mathbb{Q}_p\). If \(A/L\) has potentially supersingular reduction and \(L_\infty/L\) is a ramified \(\mathbb{Z}_p\)-extension, then \(A(L_\infty)[p^\infty]\) is finite.

To analyze the Λ-module structure of the Tate-Shafarevich group, we introduce a functor \(\mathfrak{S}\). For a finitely generated Λ-module \(X\), we define \(\mathfrak{S}(X) := \varinjlim_n \left( \frac{X}{\omega_n^r} \right)\). See the appendix-Proposition A.2.12 for the explicit description of this functor. In particular, \(\mathfrak{S}(X)\) is a finitely generated torsion Λ-module.

We have the following theorem which provides an answer for Question 1.1.3 and Question 1.1.4.

Theorem B (Theorem 3.0.4). If Coker(\(S^A_n\)) and \(\text{III}^1_{K_\infty}(A)_p\) are finite for all \(n\), then we have an isomorphism
\[\text{III}^1_{K_\infty}(A)_p^\vee \cong \mathfrak{S}(\text{Sel}_{K_\infty}(A)_p^\vee)\]
of Λ-modules. In particular, \(\text{III}^1_{K_\infty}(A)_p^\vee\) is a cotorsion Λ-module.

Remark 1.2.11. (1) This theorem can be regarded as a Λ-adic analogue of the Tate-Shafarevich conjecture. More precisely, if the control theorem (of the Selmer groups) holds, the \(Z_p\)-cotorsionness of \(\text{III}^1_{K_\infty}(A)_p\) (i.e. finiteness) at finite level can be lifted to the Λ-cotorsionness of the \(\text{III}^1_{K_\infty}(A)_p\).

(2) This theorem also describes the Λ-module structure of \(\text{III}^1_{K_\infty}(A)_p^\vee\) by the structural data of the \(\text{Sel}_{K_\infty}(A)_p^\vee\); If we know the elementary module of \(\text{Sel}_{K_\infty}(A)_p^\vee\), then we can explicitly write the elementary module of the \(\text{III}^1_{K_\infty}(A)_p^\vee\) and \((A(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)\,^\vee\). (For the precise statement, see Corollary 3.0.6)

In this sense, this result distinguishes the Mordell-Weil group and the Tate-Shafarevich group from the Selmer group.
For an estimate of the Tate-Shafarevich group at each finite layer $K_n$, we have the following theorem. This generalizes [Gre99, Theorem 1.10].

**Theorem C** (Theorem 4.0.1). Suppose that $\text{Coker}(S_n^A)$ is finite and bounded independent of $n$, and also suppose that $\text{III}_{K_n}(A)_p$ is finite for all $n$. Then there exists an integer $\nu$ independent of $n$ such that

$$|\text{III}_{K_n}(A)_p| = p^{\nu n} \quad (n >> 0)$$

where

$$e_n = p^n \mu \left( \text{III}_{K_n}(A)^\vee_p \right) + n \lambda \left( \text{III}_{K_n}(A)^\vee_p \right) + \nu.$$ 

Our last main result deals with Question 1.1.5. The combination of the (cyclotomic) Iwasawa Main Conjecture with the analytic functional equation between the (conjectural) $p$-adic $L$-functions of $A$ and $A^t$ suggests us to expect the equality

$$\text{char}_\Lambda \left( \text{Sel}_{K_n}(A)^\vee_p \right) = \text{char}_\Lambda \left( \text{Sel}_{K_n}(A^t)^\vee_p \right)$$

of characteristic ideals. Note that if the groups $\text{Sel}_{K_n}(A)^\vee_p$ and $\text{Sel}_{K_n}(A^t)^\vee_p$ have positive $\Lambda$-rank, the above equality of ideals is vacuous.

The below theorem refines this equality of ideals to the statement about the isomorphism classes, which is a generalization of [Gre99, Theorem 1.14].

**Theorem D** (Theorem 5.3.3). If $\text{Coker}(S_n^A)$ and $\text{Coker}(S_n^{A^t})$ are finite for all $n$, then we have an isomorphism

$$E \left( \text{Sel}_{K_n}(A)^\vee_p \right) \simeq E \left( \text{Sel}_{K_n}(A^t)^\vee_p \right)^\iota$$

of $\Lambda$-modules. Here $\iota$ is an involution of $\Lambda$ satisfying $\iota(T) = \frac{1}{1+T} - 1$.

**Remark 1.2.12.** (1) As we mentioned earlier, our result generalizes [Gre99, Theorem 1.14] in two aspects. Firstly, this result is not just an equality of ideals, but rather a statement about the isomorphism classes. Secondly, we removed the $\Lambda$-cotorsion assumption of the Selmer groups $\text{Sel}_{K_n}(A)^\vee_p$ and $\text{Sel}_{K_n}(A^t)^\vee_p$. (See Remark 1.1.7)

(2) By combining Theorem C and Theorem D, we can compare the $\Lambda$-module structure of $(A(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$ with that of $(A^t(K_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$. The same statement holds for $\text{III}_{K_n}(A)^\vee_p$ and $\text{III}_{K_n}(A^t)^\vee_p$ also. See Proposition 5.3.4 for the precise statement.

1.3. Comparison with the former work.

- Imai [Ima75] proved that $A(L(\mu_{p^n}))_{\text{tor}}$ is finite where $A$ is an abelian variety over a $p$-adic local field $L$ with good reduction. Kato [Kat04, Page 233] proved that $A(\mathbb{Q}_p(\mu_{p^n})) / [p^n]$ is finite when $A$ is an elliptic curve defined over $\mathbb{Q}$ with no additional assumptions about the reduction type of $A$. Corollary 1.2.10 (Theorem 2.2.7) proves the finiteness of the $p^{\infty}$-torsion group over any ramified $\mathbb{Z}_p$-extensions (of a $p$-adic local field) when an abelian variety has good supersingular reduction. Our approach depends heavily on Theorem A and totally different with those of [Ima75] and [Kat04].

- Theorem C is proved in [Gre99, Theorem 1.10] under the additional assumption that $\text{Sel}_{K_n}(A)^\vee_p$ is a cotorsion $\Lambda$-module. His formulation used different $\lambda$ and $\mu$, but one can show that our result implies the Greenberg’s formula. See Remark 4.0.2 for this issue.

- In the same paper [Gre99], Greenberg proved the equality of characteristic ideals between $\text{Sel}_{K_n}(A)^\vee_p$ and $\text{Sel}_{K_n}(A^t)^\vee_p$. Under the assumption that $\text{Sel}_{K_n}(A)^\vee_p$ is a cotorsion $\Lambda$-module. See [Gre99, Theorem 1.14].

- The technique we will use for the proof of Theorem D gives another proof of the previous algebraic functional equation results, for instance [Rub90, Theorem 8.2], [Gre89, Proposition 1, 2, Theorem 2], [JP14, Theorem 3.8], [JM15, Theorem 2.10]. Especially for [JP14, Theorem 3.8] and [JM15, Theorem 2.10], not only we can remove the cotorsionness assumption about the Selmer groups, but also we can upgrade the statements as the comparison between elementary modules of two Selmer groups.
1.4. Organization of the paper.
- We record the definitions (Definition A.2.1 and the Definition A.2.8), basic properties and explicit descriptions (See Proposition A.1.6 and Proposition A.2.12) of the functors $\mathfrak{F}$ and $\mathfrak{G}$ in the appendix.
- In section 2, we prove Theorem A (Theorem 2.1.2) and its consequence Corollary 1.2.10 (Theorem 2.2.7) by using a functor $\mathfrak{G}$.
- In section 3, we define the Selmer group and the Tate-Shafarevich group. We prove Theorem B (Theorem 3.0.4) and its consequence about the separating the Mordell-Weil and the Tate-Shafarevich groups (Corollary 3.0.6).
- In section 4, we find an estimate on $|\text{III}_{K_n}^1(A)\rangle_p$ and explain why our result is a generalization of the Greenberg’s one [Gre99, Theorem 1.10].
- In section 5, we briefly recall the Greenberg-Wiles formula [DDT95, Theorem 2.19] and the properties of the pairing of the Flach [Fla90]. Then we finally prove Theorem D (Theorem 5.3.3).

1.5. Notations. We fix the notations below throughout the paper.
- We fix one rational odd prime $p$, and we let $\Lambda := \mathbb{Z}_p[T]$, the one variable power series ring over $\mathbb{Z}_p$. We also define $\omega_n = \omega_n(T) := (1 + T)^n - 1$ and $\omega_{n+1,n} := \frac{\omega_n(T)}{\omega_n(T)}$. Note that $\omega_{n+1,n}$ is a distinguished irreducible polynomial in $\mathbb{Z}_p[T]$.
- For a locally compact Hausdorff continuous $\Lambda$-module $M$, we define $M^\vee := \hom_{cts}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ which is also a locally compact Hausdorff. $M^\vee$ becomes a continuous $\Lambda$-module via the action defined by $(f \cdot \phi)(m) := \phi(f \cdot m)$ where $f \in \Lambda$, $m \in M$, $\phi \in M^\vee$. We also define $M^!$ to be the same underlying set $M$ whose $\Lambda$-action is twisted by an involution $\iota : T \to \frac{-1}{T} - 1$ of $\mathbb{Z}_p[T]$.
- For a cofinitely generated $\mathbb{Z}_p$-module $X$, we define $X_{\div} := \frac{X}{X_{\div}}$ where $X_{\div}$ is the maximal $p$-divisible subgroup of $X$. Note that $X_{\div} \simeq \lim_{\longleftarrow} X/p^nX$ and $(X_{\div})^\vee \simeq X^\vee[p^\infty]$.
- For a finitely generated $\Lambda$-module $M$, there are prime elements $g_1, \cdots, g_n$ of $\Lambda$, non-negative integer $r$, positive integers $e_1, \cdots, e_n$ and a pseudo-isomorphism $M \to \Lambda^r \bigoplus \left( \bigoplus_{i=1}^n \frac{\Lambda}{g_i^{e_i}} \right)$. We call $E(M) := \Lambda^r \bigoplus \left( \bigoplus_{i=1}^n \frac{\Lambda}{g_i^{e_i}} \right)$ as an elementary module of $M$ following [NSWS00, Page 292].

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2. Structure of limit Mordell-Weil group

In this section, we study the structure of $\Lambda$-adic Mordell-Weil group and prove our first main theorem (Theorem 2.1.2). We first start with a lemma. Hereafter, $^\vee$ means the Pontryagin dual.

Lemma 2.0.1. Let $F$ be a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_l$ for some prime $l$ and let $A$ be an abelian variety defined over $F$. Take any $\mathbb{Z}_p$-extension $F_\infty$ and consider the module $X := (A(F_\infty)[p^\infty])^\vee$.

1. $X$ is a finitely generated torsion $\Lambda$-module with $\mu = 0$, and $\text{char}_A X$ is coprime to $\omega_n$ for all $n$.
2. The modules $\frac{A(F_\infty)[p^\infty]}{\omega_n A(F_\infty)[p^\infty]}$ and $\frac{A(F_\infty)[p^\infty]}{\omega_n A(F_\infty)[p^\infty]}$ are finite and bounded independent of $n$.
3. For the natural maps $MW_n^A : A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[\omega_n]$ and $S_n^A : \text{Sel}_{F_\infty}(A)_p \to \text{Sel}_{F_\infty}(A)_p[\omega_n]$, the groups $\text{Ker}(MW_n^A)$ and $\text{Ker}(S_n^A)$ are finite and bounded independent of $n$.

We first remark that (2) is a direct consequence of the (1) by the structure theorem of the finitely generated $\Lambda$-modules. Hence we prove (1) and (3) only.
Proof. For (1), it suffices to show that \( \frac{X}{pX} \) and \( \frac{X}{\omega_n X} \) are finite. Since we have isomorphisms
\[
\frac{X}{pX} \simeq (A(F_\infty)[p])^\vee, \quad \frac{X}{\omega_n X} \simeq (A(F_n)[p^\infty])^\vee
\]
and the groups \( A(F_\infty)[p], A(F_n)[p^\infty] \) are finite, we get (1).

For (3), by the definition of the Selmer group, we have injections
\[
\text{Ker}(MW^n_F) \hookrightarrow \text{Ker}(S^n_F) \hookrightarrow A(F_\infty)[p^\infty]/\omega_n A(F_\infty)[p^\infty].
\]
Hence the (3) follows from (2). □

2.1. Proof of Theorem A. We first define a functor \( \mathcal{G} \). (More detailed explanation is given in the appendix.) For a finitely generated \( \Lambda \)-module \( X \), we defined
\[
\mathcal{G}(X) := \lim_{\leftarrow n} \left( \frac{X}{\omega_n X}[p^\infty] \right).
\]
We have the following description of the functor \( \mathcal{G} \). (For the proof, see the second subsection of the appendix.)

- \( \mathcal{G}(\Lambda) = 0 \). This shows that the \( \mathcal{G}(X) \) is a torsion \( \Lambda \)-module.
- \( \mathcal{G}(\frac{\Lambda}{\omega_n^{m+1,m}}) = \begin{cases} \frac{\Lambda}{\omega_n^{m+1,m}} & e \geq 2, \\ 0 & e = 1 \end{cases} \)
- \( \mathcal{G} \) is a covariant functor and preserves the pseudo-isomorphism.

Theorem 2.1.2. Let \( F \) be a finite extension of \( \mathbb{Q} \) or \( \mathbb{Q}_p \) and let \( F_\infty \) be any \( \mathbb{Z}_p \)-extension of \( F \). We identify \( \mathbb{Z}_p[\text{Gal}(F_\infty/F)] \) with \( \Lambda \) by fixing a generator \( \gamma \) of \( \text{Gal}(F_\infty/F) \) and identifying with \( 1 + T \). Then we have:

1. \( \mathcal{G} \left( (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \right) = 0. \)
2. There is a \( \Lambda \)-linear injection
\[
(A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \hookrightarrow \Lambda^r \oplus \left( \bigoplus_{n=1}^{t} \frac{\Lambda}{\omega_n^{b_n+1,b_n}} \right)
\]
with finite cokernel for some integers \( r, b_1, \cdots, b_n \).

We call \( A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \) the limit Mordell-Weil group.

Remark 2.1.3. (1) Hence the direct factors of \( E \left( (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \right)_{\Lambda\text{-tor}} \) are only of the form \( \frac{\Lambda}{\omega_n^{k+1,k}} \) for some \( k \).

(2) If \( F \) is a number field, then for any integer \( e \) and any irreducible distinguished polynomial \( h \in \Lambda \) coprime to \( \omega_n \) for all \( n \), the natural injection
\[
\mathbb{M}_F^\infty(A)^{\vee}[h^e] \hookrightarrow \text{Sel}_{F_\infty}(A)^{\vee}[h^e]
\]
has a finite cokernel.

(3) As it will be clear in the proof, Theorem 2.1.2 holds for any \( p \)-adic local field \( F \) also. This produces one consequence about the finiteness of the \( p^\infty \)-torsion of an abelian variety \( A/F \) over \( \mathbb{Z}_p \)-extension. (See Theorem 2.2.7)

Note that \( (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \) has no non-trivial finite \( \Lambda \)-submodule since it is \( \mathbb{Z}_p \)-torsion-free. Due to Proposition A.2.12, it is enough to show the assertion (1) only. As a preparation for the proof, we record the following lemma.
Lemma 2.1.4. (1) Let $R$ be an integral domain and $Q(R)$ be the quotient field of $R$. Consider an exact sequence of $R$-modules $0 \to A \to B \to C \to 0$ where $A$ is an $R$-torsion module. Then we have a short exact sequence $0 \to A_{R-\text{tor}} \to B_{R-\text{tor}} \to C_{R-\text{tor}} \to 0$.

(2) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}_p$-modules. If $A$ has finite cardinality, then we have a short exact sequence $0 \to A = A[p^\infty] \to B[p^\infty] \to C[p^\infty] \to 0$.

(3) If $0 \to X \to Y \to Z \to W \to 0$ is an exact sequence of finitely generated $\mathbb{Z}_p$-modules with finite $W$, then the sequence $X_{/\text{div}} \to Y_{/\text{div}} \to Z_{/\text{div}} \to W_{/\text{div}} \to 0$ is exact.

Proof. We prove (1) and (3) only, since (2) is a direct consequence of (1). Note that for an $R$-module $M$, we have $\text{Tor}_1^R(M, Q(R)/R) \cong M_{R-\text{tor}}$. Since $A$ is an $R$-torsion module, we have $A \otimes_R Q(R) = 0$. Now (1) follows from the long exact sequence associated with functor $\text{Tor}_1^R(-, Q(R)/R)$.

For (3), break up the sequence to $0 \to X \to Y \to T \to 0$ and $0 \to T \to Z \to W \to 0$. Applying $p^\infty$-torsion functor to the Pontryagin dual of the first short exact sequence gives $X_{/\text{div}} \to Y_{/\text{div}} \to T_{/\text{div}} \to 0$. By (2), we get $0 \to T_{/\text{div}} \to Z_{/\text{div}} \to W_{/\text{div}} \to 0$. Combining these two sequences proves the assertion. □

Proof of Theorem 2.1.2. Let $C_n$ be the cokernel of the natural map

$$MW_n^A : A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \to A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p [\omega_n].$$

Since $\lim_n A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p = A(\mathbb{F}_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p$ by definition, we get $\lim_n C_n = 0$ and $\lim(C_n)_{/\text{div}} = 0$.

From an exact sequence

$$A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \to A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p [\omega_n] \to C_n \to 0,$$

we have

$$0 = (A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p)_{/\text{div}} \to (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p [\omega_n])_{/\text{div}} \to (C_n)_{/\text{div}} \to 0$$

by Lemma 2.1.4-(3). Taking direct limit to this sequence shows

$$\lim_n (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p [\omega_n])_{/\text{div}} = 0$$

since $\lim(C_n)_{/\text{div}} = 0$. If we take the Pontryagin dual, we get

$$\lim_n \left( A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \right)^{\vee} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p [p^\infty] = 0.$$

Hence we get

$$\mathcal{E} \left( (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p)^{\vee} \right) := \lim_n \left( A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \right)^{\vee} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p [p^\infty] = 0.$$

Next we state the control result of the limit Mordell-Weil group under the $\Lambda$-cotorsion assumption. For the second statement about the $\lambda$-invariant, we need to use Theorem 2.1.2.

Theorem 2.1.5. Let $F$ be a number field, $F_\infty$ be a $\mathbb{Z}_p$-extension of $F$ and $F_n$ be the $n$-th layer. The following two assertions are equivalent:

- The sequence $\{\text{rank}_\mathbb{Z} A(F_n)\}_{n \geq 0}$ is bounded.
- $A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p$ is a cotorsion $\Lambda$-module.

If this equivalent condition holds, the natural map

$$A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \to A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p [\omega_n]$$

is surjective for almost all $n$ and $\text{rank}_\mathbb{Z} A(F_n)$ stabilizes to the $\lambda \left( (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p)^{\vee} \right)$. 
Proof. By Lemma 2.0.1, the natural map $A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[\omega_n]$ has finite kernel for all $n$. If $A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ is a cotorsion $\Lambda$-module, then by the $\Lambda$-module theory, $\mathbb{Z}_p$-corank of modules $A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[\omega_n]$ are bounded, and hence $\text{rank}_\mathbb{Z} A(F_n)$ is bounded.

Conversely, assume that $\text{rank}_\mathbb{Z} A(F_n)$ is bounded and take $n$ so that

\[ \text{rank}_\mathbb{Z} A(F_{n+k}) = \text{rank}_\mathbb{Z} A(F_n) \]

for all $k \geq 0$. Now consider the following diagram:

\[
\begin{array}{ccc}
A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p & \xrightarrow{r} & A(F_{n+1}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \\
\downarrow{s} & & \downarrow{t} \\
A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[\omega_n] & \xrightarrow{\iota} & A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[\omega_{n+1}] \\
\end{array}
\]

Since the map $s$ has finite kernel, $\text{Ker}(r)$ is also finite. Considering the $\mathbb{Z}_p$-corank shows that $r$ is surjective. By taking direct limit, we get a surjection

\[ A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \twoheadrightarrow A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p. \]

Since this map factors through the natural inclusions

\[ A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[\omega_n] \hookrightarrow A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p, \]

the above inclusion is an isomorphism indeed, and hence $A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ is a cotorsion $\Lambda$-module. This also proves the assertion about the stabilized value of the sequence $\{\text{rank}_\mathbb{Z} A(F_n)\}_{n \geq 0}$. \qed

Remark 2.1.6. (1) In above theorem, the condition that $A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ is a cotorsion $\Lambda$-module is not enough to guarantee that

\[ \text{rank}_\mathbb{Z} A(F_n) = \lambda \left( (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \right) \quad (n >> 0). \]

Indeed, we need to use Theorem 2.1.2 that the direct factors of $E \left( (F(K_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \right)_{\Lambda-\text{tor}}$ are only of the form $\Lambda_{\omega_{n+1},x}$ for some $k$.

(2) If $F$ is a number field, consider a $\Lambda$-linear injection

\[ (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \hookrightarrow \Lambda^t \oplus \bigoplus_{n=1}^{t} \Lambda_{\omega_{n+1},b_n} \]

in Theorem 2.1.2. Assume that $\text{Coker}(S_n^\Lambda)$ and $\Omega_{F_n}^\Lambda(A)_{p}$ are finite for all $n$. Then by the snake lemma, the natural map

\[ A(F_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[\omega_n] \]

has finite kernel and cokernel for all $n$. By comparing the $\mathbb{Z}_p$-coranks of the two modules, we get

\[ r = \lim_{n \to \infty} \frac{\text{rank}_\mathbb{Z} A(F_n)}{p^n}, \quad a_n = \frac{\text{rank}_\mathbb{Z} A(F_{n+1}) - \text{rank}_\mathbb{Z} A(F_n)}{p^{n-1}(p-1)} - r \quad (n \geq 1) \]

where $a_n$ is defined as the number of $1 \leq i \leq t$ satisfying $b_i = n - 1$. Hence for this case, we can describe $E \left( (A(F_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \right)$ by using the sequence $\{\text{rank}_\mathbb{Z} A(F_n)\}_{n \geq 0}$.

2.2. Consequence of Theorem A. We prove one interesting consequence of Theorem 2.1.2 about the finiteness of the $p^\infty$-torsion group of an abelian variety over a $p$-adic local field. Let $L$ be a finite extension of $\mathbb{Q}_p$, $L_\infty$ be a $\mathbb{Z}_p$-extension of $L$, $L_n$ be a $n$-th layer, and $A$ be an abelian variety defined over $L$.

We consider $A(L_\infty)[p^\infty]$, which is a cofinitely generated cotorsion $\Lambda$-module by Lemma 2.0.1. If $A$ has (potentially) good reduction over $L$, Imai [Ima75] proved that the torsion subgroup of the $A(L(\mu_{p^\infty}))$ is finite. By using Theorem 2.1.2, we prove that if $A$ has potentially supersingular reduction, $A(L_\infty)[p^\infty]$ is finite for the general ramified $\mathbb{Z}_p$-extension $L_\infty/L$.

Theorem 2.2.7. If $A/L$ has potentially supersingular reduction and $L_\infty/L$ is a ramified $\mathbb{Z}_p$-extension, then $A(L_\infty)[p^\infty]$ is finite.
Proof. We may assume that A has supersingular reduction over L. Let F be a formal group of A. Since A/L has supersingular reduction, we have an isomorphism

\[ H^1(L_\infty, F) \cong H^1(L_\infty, A)[p^\infty]. \]

Since \( L_\infty / L \) is a ramified \( \mathbb{Z}_p \)-extension, by [CG96, Proposition 2.10, Theorem 2.13] we have

\[ H^1(L_\infty, F) = 0. \]

Hence from the Kummer sequence, we have an isomorphism

\[ H^1(L_\infty, A[p^\infty])^\vee \cong (A(L_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p)^\vee. \]

Since \( \Lambda \)-torsion part of \( H^1(L_\infty, A[p^\infty])^\vee \) is pseudo-isomorphic to \((A'(L_\infty)[p^\infty])^\vee \) by [Gre89, Proposition 3.1] (up to twisting by \( \iota \)), combining Theorem 2.1.2 and Lemma 2.0.1 shows the desired assertion. \( \square \)

**Remark 2.2.8.** (1) In the proof of the above Theorem, we have also proved that \((A(L_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p)^\vee \) is a torsion-free \( \Lambda \)-module.

(2) Kato [Kat04, Page 233] proved that \( A(\mathbb{Q}_p(\mu_{p^\infty})) / p^\infty \) is finite when \( A \) is an elliptic curve defined over \( \mathbb{Q} \). By using Theorem 2.2.7, we can generalize this result to the finiteness of \( A(L_\infty)[p^\infty] \) where \( A \) is an elliptic curve with potentially good reduction and \( L_\infty / L \) is a any ramified \( \mathbb{Z}_p \)-extension.

The proof of the potentially supersingular case follows from Theorem 2.2.7. For the potentially ordinary case, one can use the filtration on the Tate module \( T_pA \).

3. Selmer and \( \mathbb{III}^1 \)

We study the Selmer group and the Tate-Shafarevich group in this section. We will describe the \( \Lambda \)-adic Tate-Shafarevich group under mild assumptions. (See Theorem 3.0.4)

**Definition 3.0.1.** Let \( F \) be a number field and \( A \) be an abelian variety over \( F \). Let \( S \) be a finite set of places of \( F \) containing the places over \( p \), infinite places and places of bad reductions of \( A \). We define the Selmer group and the Tate-Shafarevich group as follows:

1. \( \text{Sel}_F(A)_p := \text{Ker} \left( H^1(F^S / F, A[p^\infty]) \to \prod_{v \in S} H^1(F_v, A) \right) \).

2. \( \mathbb{III}_p^1(A)_p := \text{Ker} \left( H^1(F^S / F, A)[p^\infty] \to \prod_{v \in S} H^1(F_v, A)[p^\infty] \right) \).

**Remark 3.0.2.** By [Mil06, Corollary I.6.6], this definition is independent of the choice of \( S \) as long as \( S \) contains infinite places, primes over \( p \) and primes of bad reduction of \( A \). Moreover, all modules in the above definition are cofinitely generated \( \mathbb{Z}_p \)-modules.

**Notation 3.0.3.** Hereafter,

1. we let \( A \) be an abelian variety over a number field \( K \) and let \( S \) be a finite set of places of \( K \) containing the places over \( p \), infinite places and places of bad reductions of \( A \). We fix a \( \mathbb{Z}_p \)-extension \( K_\infty \) of \( K \).

2. we identify \( \mathbb{Z}_p[\text{Gal}(K_\infty / K)] \) with \( \Lambda \) by fixing a generator \( \gamma \) of \( \text{Gal}(K_\infty / K) \) and identifying with \( 1 + T \).

3. we let \( S^A_n : \text{Sel}_{K_\infty}(A)_p \to \text{Sel}_{K_\infty}(A)_p[\omega_n] \)

be the natural restriction map.

Next we state the result that under the control of Selmer groups and Tate-Shafarevich conjecture, \( \mathbb{III}_{K_\infty}^1(A)_p \) is a cotorsion \( \Lambda \)-module. This can be regarded as a \( \Lambda \)-adic analogue of Tate-Shafarevich conjecture. (Recall that \( \Theta(X) = \lim_{\gamma} \left( \frac{X}{\omega_n X[p^\infty]} \right) \).
Theorem 3.0.4. If \( \text{Coker}(S_n^A) \) and \( \Sha_{K_n}(A)_p \) are finite for all \( n \), then we have an isomorphism

\[
\Sha_{K_n}(A)_p \cong \mathfrak{S}(\Sel_{K_n}(A)_p)
\]

of \( \Lambda \)-modules. In particular, \( \Sha_{K_n}(A)_p \) is a cotorsion \( \Lambda \)-module.

Hence under the Tate-Shafarevich conjecture, if the control theorem for the Selmer groups holds, then we can describe the \( \Lambda \)-module structure of \( \Sha_{K_n}(A)_p \) by using \( \Sel_{K_n}(A)_p \).

Remark 3.0.5. (1) Since we have an explicit description of the functor \( \mathfrak{S} \) in the appendix (Proposition A.2.12), if we have data of \( E(\Sel_{K_n}(A)_p^\vee) \), we can disassemble the factors of \( (A(K_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee \) and the factors of \( \Sha_{K_n}(A)_p^\vee \) from \( E(\Sel_{K_n}(A)_p^\vee) \) completely. See Corollary 3.0.6 below.

(2) If we make the assumption only about the \( \text{Coker}(S_n^A) \), then we have the similar statement for the \( \Lambda \)-adic Bloch-Kato’s Tate-Shafarevich group (which is defined to be finite at finite layers \( K_n \)). This remark can be applied to Corollary 3.0.6, Theorem 4.0.1, Proposition 5.3.4, and Theorem 5.3.5.

Proof of Theorem 3.0.4. We start from the natural map \( S_n^A : \Sel_{K_n}(A)_p \to \Sel_{K_n}(A)_p[\omega_n] \). Note that \( \lim_n \text{Ker}(S_n^A) = \lim_n \text{Coker}(S_n^A) = 0 \) by definition. Hence we get

\[
\lim_n \text{Ker}(S_n^A)^\vee[p^\infty] = \lim_n \text{Coker}(S_n^A)^\vee[p^\infty] = 0.
\]

On the other hand, since \( \text{Coker}(S_n^A) \) is finite, by Lemma 2.1.4-(3) we get an exact sequence

\[
0 \to \text{Coker}(S_n^A)^\vee[p^\infty] \to \frac{\Sel_{K_n}(A)_p^\vee}{\omega_n \Sel_{K_n}(A)_p^\vee}[p^\infty] \to \Sel_{K_n}(A)_p^\vee[p^\infty] \to \text{Ker}(S_n^A)^\vee[p^\infty]
\]

where \( \Sel_{K_n}(A)_p^\vee[p^\infty] \) is isomorphic to \( \Sha_{K_n}(A)_p^\vee \) since \( \Sha_{K_n}(A)_p \) is finite. Now taking projective limit to the above sequence gives

\[
\mathfrak{S}(\Sel_{K_n}(A)_p^\vee) := \lim_n \frac{\Sel_{K_n}(A)_p^\vee}{\omega_n \Sel_{K_n}(A)_p^\vee}[p^\infty] \cong \Sha_{K_n}(A)_p^\vee.
\]

Under the same conditions with the above theorem, we can separate the Mordell-Weil group and the Tate-Shafarevich group from the Selmer group. More precisely, we describe the elementary modules of the Mordell-Weil group and the Tate-Shafarevich group by using the elementary module of the Selmer group.

Corollary 3.0.6. Suppose that \( \text{Coker}(S_n^A) \) and \( \Sha_{K_n}(A)_p \) are finite for all \( n \) and let

\[
E(\Sel_{K_n}(A)_p^\vee) \cong \Lambda^r \oplus \left( \bigoplus_{i=1}^d \frac{\Lambda}{g_i^e} \right) \oplus \left( \bigoplus_{m=1}^f \frac{\Lambda}{\omega_{\alpha_{m+1},a_m}} \right) \oplus \left( \bigoplus_{n=1}^t \frac{\Lambda}{\omega_{b_{n+1},b_n}} \right)
\]

where \( r \geq 0, g_1, \ldots, g_d \) are prime elements of \( \Lambda \) which are coprime to \( \omega_n \) for all \( n \), \( d \geq 0, l_1, \ldots, l_d \geq 1, f \geq 0, e_1, \ldots, e_f \geq 2 \) and \( t \geq 0 \). Then we have isomorphisms

\[
E(\Sha_{K_n}(A)_p^\vee) \cong \Lambda^r \oplus \left( \bigoplus_{m=1}^f \frac{\Lambda}{\omega_{\alpha_{m+1},a_m}} \right)
\]

and

\[
E(\Sel_{K_n}(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee) \cong \Lambda^r \oplus \left( \bigoplus_{m=1}^f \frac{\Lambda}{\omega_{\alpha_{m+1},a_m}} \right) \oplus \left( \bigoplus_{n=1}^t \frac{\Lambda}{\omega_{b_{n+1},b_n}} \right).
\]
Proof. The isomorphism for $\mathfrak{III}^1_{K_∞}(A)_{p}^{∨}$ is a direct consequence of Theorem 3.0.4. For the second isomorphism, since $\mathfrak{III}^1_{K_∞}(A)_{p}^{∨}$ is a torsion $Λ$-module by Theorem 3.0.4, the exact sequence

$$0 \to \mathfrak{III}^1_{K_∞}(A)_{p}^{∨} \to \text{Sel}_{K_∞}(A)_{p}^{∨} \to (A(K_∞) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{∨} \to 0$$

induces another exact sequence

$$0 \to (\mathfrak{III}^1_{K_∞}(A)_{p}^{∨})_{Λ-tor} \to (\text{Sel}_{K_∞}(A)_{p}^{∨})_{Λ-tor} \to (A(K_∞) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{∨}_{Λ-tor} \to 0$$

by Lemma 2.1.4-(1). Now Theorem 3.0.4 shows the desired isomorphism. \qed

4. Estimates on the size of $\mathfrak{III}^1$

Now we find an estimate of $|\mathfrak{III}^1_{K_∞}(A)_{p}|$, which is an analogue of the Iwasawa’s class number formula [Gre99, Theorem 1.1].

**Theorem 4.0.1.** Suppose that $\text{Coker}(S_n^A)$ is finite and bounded independent of $n$ and also suppose that $\mathfrak{III}^1_{K_∞}(A)_{p}$ is finite for all $n$. Then there exists an integer $\nu$ independent of $n$ such that

$$|\mathfrak{III}^1_{K_∞}(A)_{p}| = p^{e_n} \quad (n >> 0)$$

where

$$e_n = p^n \mu (\mathfrak{III}^1_{K_∞}(A)_{p}^{∨}) + n\lambda (\mathfrak{III}^1_{K_∞}(A)_{p}^{∨}) + \nu.$$ 

**Remark 4.0.2.** (1) This theorem can be an evidence for the control of the Tate-Shafarevich group over the tower of fields $\{K_n\}_{n≥0}$. More precisely, if the characteristic ideal of $\mathfrak{III}^1_{K_∞}(A)_{p}$ is coprime to $\omega_n$ for all $n$, (Note that by Theorem 3.0.4, $\mathfrak{III}^1_{K_∞}(A)_{p}$ is a $Λ$-torsion under the assumption of Theorem 4.0.1) then by the $Λ$-module theory, we get

$$|\mathfrak{III}^1_{K_∞}(A)_{p}[\omega_n]| = p^{e_n}$$

for the same $e_n$ appearing in the above theorem. Hence we could expect the natural map

$$\mathfrak{III}^1_{K_∞}(A)_{p} \to \mathfrak{III}^1_{K_∞}(A)_{p}[\omega_n]$$

to have bounded kernel and cokernel.

(2) This theorem generalizes [Gre99, Theorem 1.10] by removing the cotorsionness assumption of $\text{Sel}_{K_∞}(A)_{p}$. If Selmer group is a $Λ$-cotorsion, then our formula recovers that of [Gre99, Theorem 1.10]. We give a brief explanation here.

In [Gre99, Theorem 1.10], the Greenberg’s formula was

$$|\mathfrak{III}^1_{K_∞}(A)_{p}| = p^{f_n} \quad (n >> 0)$$

for $f_n = p^n\mu_A + n \cdot (\lambda_A - \lambda_A^{MW}) + \nu$ where

- $\mu_A$ is a $μ$-invariant of the Selmer group $\text{Sel}_{K_∞}(A)_{p}^{∨}$,
- $\lambda_A$ is a $λ$-invariant of the Selmer group $\text{Sel}_{K_∞}(A)_{p}^{∨}$,
- $\lambda_A^{MW}$ is the stabilized value of $\{\text{rank}_2 A(K_n)\}_{n≥0}$.

Since $(A(K_∞) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{∨}$ is a $Λ$-torsion for this case, $\lambda_A^{MW}$ is same as the $λ$-invariant of $(A(K_∞) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{∨}$ by Theorem 2.1.5. Hence by the multiplicative property of characteristic ideals, we get

$$\lambda_A - \lambda_A^{MW} = \lambda \left(\mathfrak{III}^1_{K_∞}(A)_{p}^{∨}\right).$$

For the $μ$-invariant, by Theorem 2.1.2, $(A(K_∞) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{∨}$ has $μ$-invariant zero. Hence we get

$$\mu_A = \mu \left(\mathfrak{III}^1_{K_∞}(A)_{p}^{∨}\right)$$

which justifies the claim.
Proof of Theorem 4.0.1. By the straight forward calculation, we can check that for a finitely generated $\Lambda$-module $Y$, we have
\[
\log_p \left| \frac{Y}{\omega_n Y} \right| = p^n \mu (\mathcal{G}(Y)) + n \lambda (\mathcal{G}(Y)) + \nu
\]
for all $n >> 0$.

If we let $Y = \text{Sel}_{K_n}(A)^\vee_p$, this gives an estimate for the group $\text{Sel}_{K_n}(A)^\vee_p [p^n \mathbb{Z}]$. Since $\text{Ker}(S^1_n)$ and $\text{Coker}(S^1_n)$ are bounded independent of $n$, we get the desired assertion since we have isomorphisms
\[
\text{Sel}_{K_n}(A)^\vee_p [p^n \mathbb{Z}] \simeq \mathfrak{M}^1_{K_n}(A)^\vee_p \quad \text{and} \quad \mathfrak{M}^1_{K_n}(A)^\vee_p \simeq \mathfrak{G} (\text{Sel}_{K_n}(A))^\vee_p.
\]
The second isomorphism holds due to Theorem 3.0.4. \hfill \Box

5. Algebraic functional equation

In this section, we want to compare two modules $E (\text{Sel}_{K_n}(A)^\vee_p)$ and $E (\text{Sel}_{K_n}(A)^\vee_p)^t$ under the control of the Selmer groups of $A$ and $A^t$. The strategy of the proof is the following:
- By using the Greenberg-Wiles formula [DDT95, Theorem 2.19], we compare the $\mathbb{Z}_p$-corank of Selmer groups of $A$ and $A^t$ at each finite layer $K_n$. (See section 5.1 below)
- Using two functors $\mathfrak{G}$ and $\mathfrak{H}$, we can lift the duality between Selmer groups of $A$ and $A^t$ (induced by the Flach’s pairing: See section 5.2) to the $\Lambda$-adic setting.

5.1. The Greenberg-Wiles formula. We recall the Greenberg-Wiles formula in [DDT95, Theorem 2.19], which compares the cardinalities of two finite Selmer groups. We define $\text{Sel}_{K_n,p^m}(A)$ as the kernel of the natural restriction map
\[
H^1(K^S/K_n, A[p^m]) \to \prod_v H^1(K_n, A)
\]
where $v$ runs through the primes of $K_n$ over the primes in $S$.

Corollary 5.1.1. (1) (Greenberg-Wiles formula) For a fixed $n$, $\frac{|\text{Sel}_{K_n,p^m}(A)|}{|\text{Sel}_{K_n,p^m}(A^t)|}$ becomes stationary as $m \to \infty$.

(2) We have $\text{corank}_{\mathbb{Z}_p} \text{Sel}_{K_n}(A)_p = \text{corank}_{\mathbb{Z}_p} \text{Sel}_{K_n}(A^t)_p$ for all $n$.

Proof. For the proof of (1), see [DDT95, Theorem 2.19]. For (2), consider the following two natural exact sequences:

\[
0 \to \frac{A(K_n)[p^\infty]}{p^mA(K_n)[p^\infty]} \to \text{Sel}_{K_n,p^m}(A) \to \text{Sel}_{K_n,p^m}(A)[p^m] \to 0
\]

\[
0 \to \frac{A^t(K_n)[p^\infty]}{p^mA^t(K_n)[p^\infty]} \to \text{Sel}_{K_n,p^m}(A^t) \to \text{Sel}_{K_n,p^m}(A^t)[p^m] \to 0
\]

Since $A(K_n)[p^\infty]$ and $A^t(K_n)[p^\infty]$ are finite groups, the size of the groups $\frac{A(K_n)[p^\infty]}{p^mA(K_n)[p^\infty]}$, $\frac{A^t(K_n)[p^\infty]}{p^mA^t(K_n)[p^\infty]}$ are of bounded order as $m$ varies. If we consider the ratio between two groups $\text{Sel}_{K_n,p^m}(A)$ and $\text{Sel}_{K_n,p^m}(A^t)$, we get the (2) from (1). \hfill \Box
5.2. Flach’s pairing on Selmer groups. We briefly recall properties of the pairing of Flach.

For any finite extension $\mathfrak{f}$ of $K$ contained in $K^S$, Flach ([Fla90]) constructed a $\text{Gal}(\mathfrak{f}/K)$-equivariant bilinear pairing

$$F_\mathfrak{f} : \text{Sel}_\mathfrak{f}(A)_p \times \text{Sel}_\mathfrak{f}(A^t)_p \to \mathbb{Q}_p/\mathbb{Z}_p$$

whose left kernel (resp. right kernel) is the maximal $p$-divisible subgroup of $\text{Sel}_\mathfrak{f}(A)_p$ (resp. $\text{Sel}_\mathfrak{f}(A^t)_p$). Here $\text{Gal}(\mathfrak{f}/K)$-equivariance means the property

$$F_\mathfrak{f}(g \cdot x, g \cdot y) = F_\mathfrak{f}(x, y)$$

for all $g \in \text{Gal}(\mathfrak{f}/K)$ and $x \in \text{Sel}_\mathfrak{f}(A)_p, y \in \text{Sel}_\mathfrak{f}(A^t)_p$.

If we have two finite extensions $\mathfrak{f}_1 \geq \mathfrak{f}_2$ of $K$ in $K^S$, we have the following functorial diagram:

$$\begin{array}{ccc}
\text{Sel}_{\mathfrak{f}_1}(A)_p & \times & \text{Sel}_{\mathfrak{f}_2}(A^t)_p \\
\downarrow \text{Cor} & & \downarrow \text{Res} \\
\text{Sel}_{\mathfrak{f}_1}(A)_p & \times & \text{Sel}_{\mathfrak{f}_2}(A^t)_p \end{array}$$

$$\overset{F_{\mathfrak{f}_2}}{\longrightarrow} \mathbb{Q}_p/\mathbb{Z}_p \quad \overset{F_{\mathfrak{f}_1}}{\longrightarrow} \mathbb{Q}_p/\mathbb{Z}_p$$

By this functoriality, we get a perfect pairing

$$\lim_n (\text{Sel}_{K_n}(A)_p/\text{div} \times \lim_n (\text{Sel}_{K_n}(A^t)_p/\text{div}) \to \mathbb{Q}_p/\mathbb{Z}_p$$

which is $\Lambda$-equivariant. Hence we have an isomorphism

$$\lim_n (\text{Sel}_{K_n}(A)_p/\text{div}) \cong \lim_n (\text{Sel}_{K_n}(A^t)_p)^\vee [p^\infty])$$

of $\Lambda$-modules.

5.3. Proof of Theorem D. We first mention two technical lemmas without proof. This can be proved by using the explicit description of the functors $\mathfrak{F}$ and $\mathfrak{G}$. (See Proposition A.1.6 and Proposition A.2.12.)

Lemma 5.3.2. 1) Let $M$ and $N$ be finitely generated torsion $\Lambda$-modules. If there are $\Lambda$-linear maps $\phi : M \to N$ and $\psi : N \to M$ with finite kernels, then $M$ and $N$ are pseudo-isomorphic.

2) Let $X$ and $Y$ be finitely generated $\Lambda$-modules. Suppose that $\text{rank}_{\mathbb{Z}_p} \mathfrak{X}_n^X = \text{rank}_{\mathbb{Z}_p} \mathfrak{Y}_n^Y$ holds for all $n$, and that there are two $\Lambda$-linear maps $\mathfrak{G}(X) \to \mathfrak{F}(Y)$, $\mathfrak{G}(Y) \to \mathfrak{F}(X)$ with finite kernels. Then $E(X)$ and $E(Y)$ are isomorphic as $\Lambda$-modules.

Now we state the functional equation result between $\Lambda$-adic Selmer groups of $A$ and $A^t$. As we remarked in the introduction (Remark 1.2.12), the theorem below is a generalization of [Gre99, Theorem 1.14] in two directions.

Theorem 5.3.3. If $\text{Coker}(S_n^A)$ and $\text{Coker}(S_n^{A^t})$ are finite for all $n$, then we have an isomorphism

$$E \left( \text{Sel}_{K_\infty}(A)_p^\vee \right) \cong E \left( \text{Sel}_{K_\infty}(A^t)_p^\vee \right)$$

of $\Lambda$-modules. Here $\iota$ is an involution of $\Lambda$ satisfying $\iota(T) = \frac{1}{1+T} - 1$.

Proof of Theorem 5.3.3. By Lemma 5.3.2, it suffices to show the following two assertions:

- $\text{Sel}_{K_\infty}(A)_p[\omega_n]$ and $\text{Sel}_{K_\infty}(A^t)_p[\omega_n]$ have same $\mathbb{Z}_p$-corank for all $n$.
- There is a $\Lambda$-linear map $\mathfrak{G}(\text{Sel}_{K_\infty}(A^t)_p^\vee) \to \mathfrak{F}(\text{Sel}_{K_\infty}(A)_p^\vee)$ with the finite kernel.
The first assertion follows from our assumption about the finiteness of $\text{Coker}(S^n_A), \text{Coker}(S^n_{A'})$ and Corollary 5.1.1. Now we prove the second statement by using Flach’s pairing.

By the same method as the proof of Theorem 3.0.4, we have an exact sequence

$$0 \to \text{Coker}(S^n_{A'}) [p^\infty] \to \frac{\text{Sel}_{K_\infty}(A')}{\omega_n \text{Sel}_{K_\infty}(A')_p}[p^\infty] \to \text{Sel}_{K_n}(A')[p^\infty] \to \text{Ker}(S^n_{A'})[p^\infty],$$

and isomorphisms

$$\lim_n \text{Ker}(S^n_{A'})[p^\infty] = \lim_n \text{Coker}(S^n_{A'})[p^\infty] = 0.$$

(Note that for the exact sequence, we used the finiteness of $\text{Coker}(S^n_{A'})$.) Now taking projective limit to the above sequence gives an isomorphism

$$(1) \quad \mathfrak{G}(\text{Sel}_{K_\infty}(A')_p) := \lim_n \frac{\text{Sel}_{K_\infty}(A')_p}{\omega_n \text{Sel}_{K_\infty}(A')_p}[p^\infty] \simeq \lim_n \text{Sel}_{K_n}(A')[p^\infty].$$

Now consider a natural exact sequence

$$0 \to \text{Ker}(S^n_A) \to \text{Sel}_{K_n}(A)_p \to \text{Sel}_{K_\infty}(A)_p[\omega_n] \to \text{Ker}(S^n_A) \to 0.$$

By Lemma 2.1.4-(3) and the finiteness of the $\text{Coker}(S^n_A)$, we get another exact sequence

$$\text{Ker}(S^n_A) \to (\text{Sel}_{K_n}(A)_p)_{/\text{div}} \to (\text{Sel}_{K_\infty}(A)_p[\omega_n])_{/\text{div}} \to \text{Ker}(S^n_A) \to 0.$$

(Note that $\text{Coker}(S^n_A)$ is finite.) By taking projective limit, we get an exact sequence

$$(2) \quad \lim_n \text{Ker}(S^n_A) \to \lim_n (\text{Sel}_{K_n}(A)_p)_{/\text{div}} \to \lim_n (\text{Sel}_{K_\infty}(A)_p[\omega_n])_{/\text{div}}.$$

We now analyze the three terms in this sequence (2).

- By the proof of Lemma 2.0.1, the first term $\lim_n \text{Ker}(S^n_A)$ injects into $\lim_n A(K_\infty)[p^\infty]$ which is a finite group. (This follows from the structure theorem of the $\Lambda$-modules and Lemma 2.0.1-(1).)

- The middle term in (2) is isomorphic to $\lim_n (\text{Sel}_{K_n}(A')_p)[p^\infty]$ by the remark mentioned before this subsection (The functorial property of the Flach’s pairing), which is also isomorphic to $\mathfrak{G}(\text{Sel}_{K_\infty}(A')_p)$ by (1).

- Lastly, the third term in (2) is isomorphic to $\mathfrak{F}(\text{Sel}_{K_\infty}(A')_p)$ by definition.

Hence the sequence (2) becomes a $\Lambda$-linear map $\mathfrak{G}(\text{Sel}_{K_\infty}(A')_p) \to \mathfrak{F}(\text{Sel}_{K_\infty}(A')_p)$ with the finite kernel. \(\square\)

**Proposition 5.3.4.** Suppose that $\text{Coker}(S^n_A), \text{Coker}(S^n_{A'})$ are finite for all $n$. If $\text{III}_{K_n}(A)_p$ is finite for all $n$, then we have isomorphisms

$$E(\text{III}_{K_\infty}(A)_p) \simeq E(\text{III}_{K_\infty}(A')_p)^t$$

and

$$E((A(K_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee) \simeq E((A'(K_\infty) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^\vee)^t$$

of $\Lambda$-modules.

We remark here that the finiteness of $\text{III}_{K_n}(A)_p$ implies the finiteness of $\text{III}_{K_n}(A')_p$ by [Mil06, Lemma I.7.1].

**Proof.** This follows from Corollary 3.0.6 and Theorem 5.3.3. \(\square\)

As a consequence of Theorem 5.3.3, we can compare the size of $\text{III}_{K_n}(A)_p$ and $\text{III}_{K_n}(A')_p$ over the tower of fields $\{K_n\}_{n \geq 0}$. 


Theorem 5.3.5. Suppose that Coker(S^A_n), Coker(S^{A'}_n) are finite and bounded independent of n. If \( \text{III}_{K_n}(A)_p \) are finite for all n, then the ratios \( \frac{\text{III}_{K_n}(A)_p}{\text{III}_{K_n}(A')_p} \) and \( \frac{\text{III}_{K_n}(A)_p}{\text{III}_{K_n}(A')_p} \) are bounded independent of n.

Proof. By Proposition 5.3.4, we have

\[
\mu\left(\text{III}_{K_n}(A)_p\right) = \mu\left(\text{III}_{K_n}(A')_p\right)
\]

\[
\lambda\left(\text{III}_{K_n}(A)_p\right) = \lambda\left(\text{III}_{K_n}(A')_p\right).
\]

Now applying the estimate of Theorem 4.0.1 to both groups \( \text{III}_{K_n}(A)_p \) and \( \text{III}_{K_n}(A')_p \) gives the desired assertion. \( \square \)

### Appendix A. Functors \( \mathcal{F} \) and \( \mathcal{G} \)

A.1. Functor \( \mathcal{F} \). We first recall our convention on Pontryagin dual. For a locally compact Hausdorff continuous \( \Lambda \)-module \( M \), we define \( M^\vee := \text{Hom}_{\text{cts}}(M, \mathbb{Q}_p/\mathbb{Z}_p) \) which is also a locally compact Hausdorff. \( M^\vee \) becomes a continuous \( \Lambda \)-module via action defined by \( (f \cdot \phi)(m) := \phi(f \cdot m) \) where \( f \in \Lambda, m \in M, \phi \in M^\vee \).

If \( M, N \) are two locally compact Hausdorff continuous \( \Lambda \)-modules with a perfect pairing \( P: M \times N \to \mathbb{Q}_p/\mathbb{Z}_p \) satisfying \( P(f \cdot x, y) = P(x, \iota(f) \cdot y) \), then \( P \) induces \( \Lambda \)-module isomorphisms \( M \simeq N^\vee \) and \( N \simeq M^\vee \).

**Definition A.1.1.** For any finitely generated \( \Lambda \)-module \( X \), define \( \mathcal{F}(X) = \left( \lim_{\longrightarrow} \frac{X}{\omega_nX} \right)^\vee \). Here direct limit is taken with respect to norm maps \( \frac{X}{\omega_nX} \to \frac{X}{\omega_{n+1}X} \). \( \mathcal{F} \) is contravariant and preserves finite direct sums.

Our goal is examining \( \Lambda \)-module structure of \( \mathcal{F}(X) \) for a finitely generated \( \Lambda \)-module \( X \) (Proposition A.1.6). We will use the following proposition crucially.

**Lemma A.1.2.** (1) Let \( M \) be a finitely generated \( \Lambda \)-module, and let \( \{\pi_n\} \) be a sequence of non-zero elements of \( \Lambda \) such that \( \pi_0 \in m, \pi_{n+1} \in \pi_n \), \( \frac{M}{\pi_nM} \) is finite for all \( n \) where \( m \) is the maximal ideal of \( \Lambda \). Then we have an isomorphism

\[
\left( \lim_{\longrightarrow} \frac{M}{\pi_nM} \right)^\vee \simeq \text{Ext}_\Lambda^1(M, \Lambda)^\vee
\]

as \( \Lambda \)-modules.

(2) For a finitely generated \( \Lambda \)-module \( X \), we have an isomorphism \( \lim_{\longrightarrow} T_p(X^\vee[\omega_n]) \simeq \text{Hom}_\Lambda(X, \Lambda)^\vee \).

**Proof.** We only need to prove (2) since (1) is [NSWS00, Proposition 5.5.6]. By (1), we have an isomorphism \( T_p(X^\vee[\omega_n]) \simeq \text{Ext}_\Lambda^1(\frac{X}{\omega_nX}, \Lambda)^\vee \) and this last group is isomorphic to

\[
\text{Hom}_\Lambda(\frac{X}{\omega_nX}, \Lambda)^\vee \simeq \text{Hom}_\Lambda(X, \frac{\Lambda}{\omega_n\Lambda})^\vee \simeq \text{Hom}_\Lambda(X, \Lambda)^\vee.
\]

Hence \( \lim_{\longrightarrow} T_p(X^\vee[\omega_n]) \simeq \lim_{\longrightarrow} \text{Hom}_\Lambda(X, \frac{\Lambda}{\omega_n\Lambda})^\vee \simeq \text{Hom}_\Lambda(X, \Lambda)^\vee \).

**Lemma A.1.3.** (1) If \( K \) is finite, then \( \mathcal{F}(K) = 0 \)

(2) \( \mathcal{F}(\Lambda) = 0 \)

(3) If \( g \) is a prime element of \( \Lambda \) coprime to \( \omega_n \) for all \( n \geq 0 \), then \( \mathcal{F}(\frac{\Lambda}{g^e}) = \frac{\Lambda}{\omega_n \lambda_{e+1,m}} \) for all \( e \geq 1 \)

(4)

\[
\mathcal{F}(\frac{\Lambda}{\omega_n^{m+1,m}}) = \begin{cases} 
\frac{\Lambda}{\omega_n^{m+1,m}} & e \geq 2, \\
0 & e = 1.
\end{cases}
\]
Proof. For (2), \( \frac{\Lambda}{\omega_n K} \) is \( \mathbb{Z}_p \)-torsion-free so \( \mathfrak{F}(\Lambda) = 0 \). For (3), note that \( \frac{\Lambda}{(g', \omega_n)} \) is finite if \( g \) is a prime element of \( \Lambda \) coprime to \( \omega_n \) for all \( n \geq 0 \). So by Lemma A.1.2-(1), we get
\[
\mathfrak{F}\left( \frac{\Lambda}{g'} \right) \simeq \text{Ext}^1_\Lambda\left( \frac{\Lambda}{g'}, \Lambda \right) \simeq \frac{\Lambda}{\iota(g')}. 
\]
For (4), we only prove for \( e \geq 2 \). Let \( h = \omega_{m+1, m} \) and consider the following exact sequence for \( n \geq m + 1 \):
\[
0 \rightarrow \frac{\Lambda}{(h^{n-1}, \omega_n)} \rightarrow \frac{\Lambda}{(h^n, \omega_n)} \rightarrow \frac{\Lambda}{h} \rightarrow 0.
\]
Since \( \frac{\Lambda}{h} \) is \( \mathbb{Z}_p \) torsion-free, we get \( \frac{\Lambda}{(h^{n-1}, \omega_n)} \simeq \frac{\Lambda}{(h^n, \omega_n)} \). So we get
\[
\mathfrak{F}\left( \frac{\Lambda}{h^n} \right) = \left( \lim_{n} \frac{\Lambda}{(h^n, \omega_n)} \right)^{\vee} = \left( \lim_{n} \frac{\Lambda}{(h^{n-1}, \omega_n)} \right)^{\vee} = \left( \lim_{n} \frac{\Lambda}{(h^{n-1}, \omega_n)} \right)^{\vee} \quad (\text{Since } h \text{ is coprime to } \frac{\omega_n}{h} \text{ for all } n \geq m + 1 )
\]
\[
\simeq \text{Ext}^1_\Lambda\left( \frac{\Lambda}{h^{n-1}}, \Lambda \right)^{\vee} \quad (\text{By Lemma A.1.2-(1)})
\]
which shows the assertion. \( \square \)

Lemma A.1.4. Let \( 0 \rightarrow K \rightarrow X \xrightarrow{\phi} Z \rightarrow 0 \) be a short exact sequence of finitely generated \( \Lambda \) modules where \( K \) is finite. Then the natural map \( \mathfrak{F}(\phi) : \mathfrak{F}(Z) \rightarrow \mathfrak{F}(X) \) is an isomorphism.

Proof. Since the tensor functor is right exact, we have \( \frac{\omega_n K}{\omega_n Z} \rightarrow \frac{X}{\omega_n X} \rightarrow \frac{Z}{\omega_n Z} \rightarrow 0 \).

Let \( A_n, B_n \) be the kernel and image of \( \frac{K}{\omega_n K} \rightarrow \frac{X}{\omega_n X} \), respectively. Note that both are finite modules since \( K \) is finite. We have the following two short exact sequences
\[
(3) \quad 0 \rightarrow A_n \rightarrow \frac{K}{\omega_n K} \rightarrow B_n \rightarrow 0
\]
\[
(4) \quad 0 \rightarrow B_n \rightarrow \frac{X}{\omega_n X} \rightarrow \frac{Z}{\omega_n Z} \rightarrow 0
\]

By Lemma 2.1.4-(2), two sequences remain exact after taking \( p^\infty \)-torsion parts. Since \( \mathfrak{F}(K) = 0 \) by Lemma A.1.3, we get \( \left( \lim_{n} B_n[p^\infty] \right)^{\vee} = 0 \) from (3). Applying this to (4) gives an isomorphism
\[
\mathfrak{F}(\phi) : \mathfrak{F}(Z) \xrightarrow{\mathfrak{F}(\phi)} \mathfrak{F}(X).
\]
\( \square \)

Lemma A.1.5. Let \( 0 \rightarrow Z \xrightarrow{\psi} S \rightarrow C \rightarrow 0 \) be a short exact sequence of finitely generated \( \Lambda \) modules where \( C \) is finite. Then the natural map \( \mathfrak{F}(\psi) : \mathfrak{F}(S) \rightarrow \mathfrak{F}(Z) \) is an injection with finite cokernel.

Proof. By the snake lemma, we have \( C[\omega_n] \rightarrow \frac{Z}{\omega_n Z} \rightarrow \frac{S}{\omega_n S} \rightarrow C[\omega_n] \rightarrow 0 \).

If we let \( X_n = \text{Ker}(C[\omega_n] \rightarrow \frac{Z}{\omega_n Z}), E_n = \text{Im}(C[\omega_n] \rightarrow \frac{Z}{\omega_n Z}), D_n = \text{Ker}(\frac{S}{\omega_n S} \rightarrow C[\omega_n]) \), then we get the following three short exact sequences:
\[
(5) \quad 0 \rightarrow X_n \rightarrow C[\omega_n] \rightarrow E_n \rightarrow 0
\]
\[
(6) \quad 0 \rightarrow E_n \rightarrow \frac{Z}{\omega_n Z} \rightarrow D_n \rightarrow 0
\]
\[
(7) \quad 0 \rightarrow D_n \rightarrow \frac{S}{\omega_n S} \rightarrow C[\omega_n] \rightarrow 0
\]
Since $C$ is finite, $X_n$ and $E_n$ are finite for all $n$. So the sequences (5), (6) remain exact after taking $p^\infty$-torsion parts due to Lemma 2.1.4-(2). On the other hand, we have $\lim\frac{C}{\omega_n C} = 0$ and $\lim\frac{C[\omega_n]}{C} = C$. So by taking direct limit and Pontryagin dual for the sequences (5), (6), (7), we get

$$0 \to \mathfrak{S}(S) \to \mathfrak{S}(Z) \to C^\vee.$$  

□

Now we can prove our main goal.

**Proposition A.1.6.** Let $X$ be a finitely generated $\Lambda$-module. If we let

$$E(X) \simeq \Lambda^r \oplus \left(\bigoplus_{i=1}^{d} \frac{\Lambda}{g_i} \bigoplus_{\substack{m=1 \ldots \omega_m \geq 1 \ldots \omega_m \geq 1 \ldots f \geq 0 \ldots e_1, \ldots, e_f \geq 2 \ldots t \geq 0}} \frac{\Lambda}{\omega_{a_m+1, a_m}} \bigoplus_{n=1}^{t} \frac{\Lambda}{l(\omega_{b_n+1, b_n})} \right)$$

where $r \geq 0, g_1, \cdots, g_d$ are prime elements of $\Lambda$ which are coprime to $\omega_n$ for all $n$, $d \geq 0$, $l_1, \cdots, l_d \geq 1$, $f \geq 0$, $e_1, \cdots, e_f \geq 2$ and $t \geq 0$, then we have an injection

$$\left(\bigoplus_{i=1}^{d} \frac{\Lambda}{l(g_i)} \bigoplus_{\substack{m=1 \ldots \omega_m \geq 1 \ldots \omega_m \geq 1 \ldots f \geq 0 \ldots e_1, \ldots, e_f \geq 2 \ldots t \geq 0}} \frac{\Lambda}{l(\omega_{a_m+1, a_m})} \right) \hookrightarrow \mathfrak{S}(X)$$

with finite cokernel. In particular, $F(X)$ is a finitely generated $\Lambda$-torsion module.

**Proof.** By the structure theorem of the finitely generated $\Lambda$-modules, we have an exact sequence

$$0 \to K \to X \to E(X) \to C \to 0$$

where $K, C$ are finite modules. Let $Z$ be the image of $X \to E(X)$. Now Lemma A.1.3, Lemma A.1.4, Lemma A.1.5 applied to two short exact sequences $0 \to K \to X \to Z \to 0$ and $0 \to Z \to E(X) \to C \to 0$ give the desired assertion. □

**Corollary A.1.7.** Let $X$ be a finitely generated $\Lambda$-module and let $X_{\Lambda-tor}$ be the maximal $\Lambda$-torsion submodule of $X$. If characteristic ideal of $X_{\Lambda-tor}$ is coprime to $\omega_n$ for all $n$, then there is a pseudo-isomorphism $\phi : X \to \Lambda^{\text{rank} X} \oplus \mathfrak{S}(X)^\vee$. If we assume additionally that $X$ does not have any non-trivial finite submodules, then $\phi$ is an injection.

A.2. Functor $\mathfrak{S}$. Now we consider another functor $\mathfrak{S}$.

**Definition A.2.8.** For a finitely generated $\Lambda$-module $X$, define $\mathfrak{S}(X) = \lim\frac{X}{\omega_n X[p^\infty]}$. Note that $\mathfrak{S}$ is covariant and preserves finite direct sums.

**Lemma A.2.9.** (1) If $\frac{X}{\omega_n X}$ is finite for all $n$, then $\mathfrak{S}(X) \simeq X$. In particular, $\mathfrak{S}\left(\frac{\Lambda}{g}\right) \simeq \frac{\Lambda}{g}$ if $g$ is coprime to $\omega_n$ for all $n$.

(2) $\mathfrak{S}(\Lambda) = 0$.

(3)

$$\mathfrak{S}\left(\frac{\Lambda}{\omega_{m+1,m}}\right) = \begin{cases} \frac{\Lambda}{\omega_{m+1,m}} & e \geq 2, \\ 0 & e = 1. \end{cases}$$

**Proof.** For (1), since $\frac{X}{\omega_n X}$ is finite for all $n$, we get $\mathfrak{S}(X) = \lim\frac{X}{\omega_n X}$ which is isomorphic to $X$. Since $\frac{\Lambda}{\omega_n \Lambda}$ is $\mathbb{Z}_p$-torsion-free, $\mathfrak{S}(\Lambda) = 0$. Proof of (3) is almost same as that of Lemma A.1.3-(4). □

**Lemma A.2.10.** Let $0 \to K \to X \to Z \to 0$ be a short exact sequence of finitely generated $\Lambda$-modules where $K$ is finite. Then we have a short exact sequence $0 \to K \simeq \mathfrak{S}(K) \to \mathfrak{S}(X) \to \mathfrak{S}(Z) \to 0$. 

Proof. By the snake lemma, we get \( Z[\omega_n] \to \frac{K}{\omega_n K} \to \frac{X}{\omega_n X} \to \frac{Z}{\omega_n Z} \to 0 \). Define \( E_n, A_n, B_n \) as
\[
E_n = \text{Ker}(Z[\omega_n] \to \frac{K}{\omega_n K}), \quad A_n = \text{Im}(Z[\omega_n] \to \frac{K}{\omega_n K}), \quad B_n = \text{Ker}(\frac{X}{\omega_n X} \to \frac{Z}{\omega_n Z}).
\]
Then we get the following three short exact sequences:
\[
(8) \quad 0 \to E_n \to Z[\omega_n] \to A_n \to 0
\]
\[
(9) \quad 0 \to A_n \to \frac{K}{\omega_n K} \to B_n \to 0
\]
\[
(10) \quad 0 \to B_n \to \frac{X}{\omega_n X} \to \frac{Z}{\omega_n Z} \to 0
\]
Since all the terms in sequence (8), (9), (10) are compact, the sequences (8), (9), (10) remain exact after taking projective limit.

We can easily show that \( \lim_{n \to \infty} A_n = 0 \) and \( \lim_{n \to \infty} \frac{K}{\omega_n K} \simeq \lim_{n \to \infty} B_n \). Since \( B_n \) is finite, \( \therefore K \) is finite, the sequence (10) remains exact after taking \( p^\infty \)-torsion parts. By taking projective limit, we get
\[
0 \to \lim_{n \to \infty} \frac{K}{\omega_n K} \simeq \lim_{n \to \infty} B_n \to \lim_{n \to \infty} \frac{X}{\omega_n X}[p^\infty] \to \lim_{n \to \infty} \frac{Z}{\omega_n Z}[p^\infty] \to 0
\]
which gives the assertion. \( \square \)

**Lemma A.2.11.** Let \( 0 \to Z \to S \to C \to 0 \) be a short exact sequence of finitely generated \( \Lambda \) modules where \( C \) is finite. Then we have an exact sequence \( 0 \to \mathfrak{S}(Z) \to \mathfrak{S}(S) \to \mathfrak{S}(C) \simeq C \).

**Proof.** By the snake lemma, we get \( C[\omega_n] \to \frac{M}{\omega_n M} \to \frac{S}{\omega_n S} \to \frac{C}{\omega_n C} \to 0 \). Let
\[
A_n = \text{Ker}(C[\omega_n] \to \frac{M}{\omega_n M}), \quad B_n = \text{Im}(C[\omega_n] \to \frac{M}{\omega_n M}), \quad E_n = \text{Ker}(\frac{S}{\omega_n S} \to \frac{C}{\omega_n C}).
\]
Then we get the following three short exact sequences:
\[
(11) \quad 0 \to A_n \to C[\omega_n] \to B_n \to 0
\]
\[
(12) \quad 0 \to B_n \to \frac{M}{\omega_n M} \to E_n \to 0
\]
\[
(13) \quad 0 \to E_n \to \frac{S}{\omega_n S} \to \frac{C}{\omega_n C} \to 0
\]
We can easily show that \( \lim_{n \to \infty} B_n = 0 \). Taking \( p^\infty \)-torsion parts from (12), (13) gives the following two short exact sequences:
\[
(14) \quad 0 \to B_n \to \frac{M}{\omega_n M}[p^\infty] \to E_n[p^\infty] \to 0
\]
\[
(15) \quad 0 \to E_n[p^\infty] \to \frac{S}{\omega_n S}[p^\infty] \to \frac{C}{\omega_n C}[p^\infty]
\]
Here, (14) is exact due to Lemma 2.1.4.

Since all the terms in (14), (15) are finite, taking projective limit preserves those two short exact sequences. Combining with \( \lim_{n \to \infty} B_n = 0 \) gives the assertion. \( \square \)

Now we can prove our main goal for \( \mathfrak{S} \), which is an analogue of Proposition A.1.6.

**Proposition A.2.12.** Let \( X \) be a finitely generated \( \Lambda \)-module. If we let
\[
E(X) \simeq \Lambda^\ast \oplus \left( \bigoplus_{i=1}^d \frac{\Lambda}{gl^i} \right) \oplus \left( \bigcap_{m=1}^f \frac{\Lambda}{\omega_{a_m+1} \omega_{a_{m+1}+2}} \right) \oplus \left( \bigcap_{n=1}^t \frac{\Lambda}{\omega_{b_n+1} b_n} \right)
\]
where \( r \geq 0, g_1, \cdots, g_d \) are prime elements of \( \Lambda \) which are coprime to \( \omega_n \) for all \( n \), \( d \geq 0, l_1, \cdots, l_d \geq 1, f \geq 0, e_1, \cdots, e_f \geq 2 \) and \( t \geq 0 \), then we have a pseudo-isomorphism

\[
\mathfrak{G}(X) \xrightarrow{\mathfrak{F}(\phi)} \left( \bigoplus_{i=1}^{d} \frac{\Lambda}{g_i} \right) \oplus \left( \bigoplus_{m=1}^{f} \frac{\Lambda}{\omega_{m}^{e_m-1}a_m} \right).
\]

In particular, \( \mathfrak{G}(X) \) is a finitely generated \( \Lambda \)-torsion module.

\textbf{Proof.} By the structure theorem of the finitely generated \( \Lambda \)-modules, we have an exact sequence

\[
0 \to K \to X \to E(X) \to C \to 0
\]

where \( K, C \) are finite modules. Let \( Z \) be the image of \( X \to E(X) \). Now Lemma A.2.9, Lemma A.2.10, Lemma A.2.11 applied to two short exact sequences \( 0 \to K \to X \to Z \to 0 \) and \( 0 \to Z \to E(X) \to C \to 0 \) give the desired assertion. \( \square \)

Combining Proposition A.1.6 and Proposition A.2.12, we get the following corollary.

\textbf{Corollary A.2.13.} For any finitely generated \( \Lambda \)-module \( X \), two \( \Lambda \)-torsion modules \( \mathfrak{F}(X)^\prime \) and \( \mathfrak{G}(X) \) are pseudo-isomorphic.

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