Local realizations of $q$-Oscillators in Quantum Mechanics

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Abstract

Representations of the quantum $q$-oscillator algebra are studied with particular attention to local Hamiltonian representations of the Schrödinger type. In contrast to the standard harmonic oscillators such systems exhibit a continuous spectrum. The general scheme of realization of the $q$-oscillator algebra on the space of wave functions for a one-dimensional Schrödinger Hamiltonian shows the existence of non-Fock irreducible representations associated to the continuous part of the spectrum and directly related to the deformation. An algorithm for the mapping of energy levels is described.

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1. Introduction

The $q$-deformed symmetries have been introduced [1], [2] to characterize the integrability of some lattice spin models (Yang-Baxter equations) and conformal field theories [3], [4]. Recently there have been many attempts to find examples of a $q$-deformed symmetry algebra in various quantum systems. The simplest algebra which is important in quantum physics, the oscillator Heisenberg-Weyl algebra has been deformed in several ways [3], [8] and its representations have been formally classified [6], [7], [9]. The $q$-oscillators are often used as a tool to construct more complicated quantum algebras by means of the Schwinger ansatz [6], [9]. On the other hand, because of the significance of the conventional oscillator in many areas of modern quantum physics, any new realization of the $q$-oscillator algebra on the wave function space of a particular dynamical model is instructive as to the role of this algebra for the physical systems.

It is our goal to describe different implementations of the $q$-oscillator as a quantum system obeying the Schrödinger equation. As well as for the harmonic oscillator ($q = 1$), the connection between the values of different energy levels and the corresponding wave functions [11] (or reflection and transmission coefficients [10]) is a consequence of the $q$-oscillator algebra. One realization was given in [11] by means of the Darboux transformation [12] supplemented with a dilatation of coordinates. Since the Hamiltonians intertwined by a Darboux transformation form a super-Hamiltonian obeying the quantum mechanical superalgebra (in general, a polynomial superalgebra, see [10], [13] and references therein) the additional dilatation leads to a $q$-deformation of the supersymmetry itself. In order to reproduce a $q$-oscillator the potentials in the Darboux connected Hamiltonians must coincide up to a constant. This self-similarity property holds for the conventional harmonic oscillator explaining its equidistant energy spectrum. The dilated self-similarity condition [11], [14] naturally selects the potentials which yield the energy spectra and wave functions of a $q$-deformed oscillator.

Here, we analyze the possible realizations of the $q$-oscillator on the space of wave functions for a one-dimensional Schrödinger Hamiltonian. In our approach the local Hamiltonian, in general, is not bilinear in creation and annihilation operators but rather belongs to the universal enveloping $q$-oscillator algebra, i.e. to the algebra of polynomials (analytic functions) of the generators. Thereby the $q$-oscillator relations are considered as a sort of $q$-deformed (nonlinear) dynamical algebra. In Sec. 2, we recall the algebra in the form [6], convenient to build its representations in quantum mechanics. The classification of representations is done in terms of the central element and the second $q$-commutator of creation and annihilation operators is derived. In Sec. 3, the decomposition of the Hamiltonian realization introduced in [11] into irreducible $q$-oscillator representations is developed and its polynomial generalization provided by the $q$-deformation of a polynomial superalgebra is obtained. Two types of $q$-oscillator representations appear and, while the Fock representation refers to the bound states, the non-Fock representations cover the continuous spectrum.

In Sec. 4, we formulate systematically a general scheme of constructing a local Hamiltonian of Schrödinger type with deformed spectrum generating algebras and the related mapping of energy levels is examined. The different forms of $q$-oscillator algebra are described and the constraints on their realization in terms of a Schrödinger Hamiltonian are obtained. In Sec. 5, we outline possible applications and discuss extensions of our approach
onto other $q$-deformed algebras.

2. $q$-oscillator algebra and its representations

The deformation of a bosonic oscillator can be defined in terms of the $q$-commutator, $aa^+ - q^2 a^+ a$ where $a, a^+ = (a)^+$ are annihilation and creation operators and $q$ is a real number which can be taken positive without loss of generality. It can be closed in different forms among which we select the following,

$$aa^+ - q^2 a^+ a = 1.$$  (1)

It is supplemented with the number operator $N$ such that,

$$[N, a^+] = a^+; \quad [N, a] = -a$$  (2)

where the usual commutators are implied. For the harmonic oscillator when $q = 1$, the ground state is a zero mode of the annihilation operator, $a\psi_0 = 0$ and the number operator $N$ can be normalized to be $N = a^+ a$ so that the zero occupation number is assigned for the ground state.

If $q \neq 1$ the number operator is no more bilinear in $a, a^+$. However there is a central element given by,

$$\hat{\zeta} = q^{-2N} \left( [N]_q - a^+ a \right),$$  (3)

where the $q$-symbol $[N]_q$ is defined as

$$[N]_q = \frac{1 - q^{2N}}{1 - q^2}.$$  

The operator $\hat{\zeta}$ commutes with all generators of the $q$-oscillator algebra as can be shown by the identities,

$$a^+ F(N) = F(N - 1)a^+; \quad aF(N) = F(N + 1)a.$$  (4)

Therefore its eigenvalues $\zeta$ enumerate the representations and, for a given representation with a chosen $\zeta$, one can find the connection between the bilinear operators $a^+ a, aa^+$ and the number operator $N$,

$$a^+ a = [N]_q - \zeta q^{2N},$$

$$aa^+ = [N + 1]_q - \zeta q^{2N+2}$$  (5)

where the $q$-commutator (1) has been used. These operators commute with $N$ and their spectra are generated by the spectrum of $N$. From Eqs.(3) the commutator can be evaluated,

$$aa^+ - a^+ a = q^{2N} \left( 1 - \frac{\zeta}{\zeta_c} \right),$$  (6)

\footnote{In fact, it is not unique since any periodic function $\phi(N) = \phi(N \pm 1)$ also belongs to the central element subspace which thereby consists of any algebraic combinations of $\hat{\zeta}$ and $\phi(N)$. But for further purposes it is sufficient to select out only one central element in the form (3).}
where $\zeta_c$, the critical value of $\zeta$ that sets the commutator to zero, is

$$\zeta_c = -\frac{1}{1-q^2}.$$ 

Any representation can be described in the basis of eigenfunctions of the number operator with eigenvalues $\nu_n$, $N\psi_n = \nu_n\psi_n$. As in the case of the conventional harmonic oscillator, due to relations (2), all eigenstates can be built from one selected state, $\psi_0$ by means of the ladder operators, $\psi_{n+1} \simeq a^+\psi_n$, $\psi_{n-1} \simeq a\psi_n$ and hence $\nu_{n\pm 1} = \nu_n \pm 1$. For a chosen $\psi_0$ the nonequivalent representations are parametrized by the values of $\nu_0$ running within the unit interval, $0 \leq \nu_0 < 1$ since the shift on an integer number maps one state to another one of the same representation. We can redefine the number operator, $N = \tilde{N} + \nu_0$ so that the eigenvalues of $\tilde{N}$ become integer numbers. This redefinition is compatible with the basic commutation relations (1), (2) and due to Eq.(3) corresponds to the following change of the central element,

$$\hat{\zeta}' = \hat{\zeta}q^{2\nu_0} - \zeta_c \left(q^{2\nu_0} - 1\right).$$

For $\zeta \neq \zeta_c$ any representation characterized by two parameters $\nu_0, \zeta$ is equivalent to the representation labelled by $\nu'_0 = 0, \zeta'$. In this case it is sufficient to shift eigenvalues of the number operator to integer numbers and to study eigenvalues of the central element. For $\zeta = \zeta_c$ the representations are labelled by values of $\nu_0$.

Concerning the classification of $q$-oscillator representations in terms of $\zeta$, there are three types of nonequivalent representations: the Fock representation $\zeta > \zeta_c$, non-Fock representations for $\zeta < \zeta_c$ and the special representation for $\zeta = \zeta_c$.

The Fock representation is characterized by the existence of a ground state of the number operator, zero mode of the annihilation operator, $a\psi_0 = 0$. Since from Eqs.(1) and (3) and $\nu'_0 = 0$, we have $\zeta' = 0$, the Fock representation is unique and can be built for any $0 < q < \infty$.

The non-Fock representations may appear only if $0 < q \leq 1$ and thereby $\zeta_c < 0$. The spectrum of $\tilde{N}$ is unbounded from below. Consistency with Eqs.(3) requires $\zeta < \zeta_c$. Due to the relation (3) there is a one-parameter family of irreducible non-Fock representations. Their parametrization can be realized either with the help of the central element by fixing $\nu_0 = 0$ or by means of the parameter $0 \leq \nu_0 < 1$ for a fixed value of $\zeta$. In the following we make use of the second part of the alternative and for definiteness we set $\zeta = 2\zeta_c$.

In the special representation (again for $0 < q \leq 1$) the creation and annihilation operators commute (see Eq.(3)) and it follows from Eq.(1) that the bilinear operators (3) become $c$-numbers,

$$aa^+ = a^+a = \frac{1}{1-q^2} \equiv \rho^2.$$ 

This representation is generated by an unitary operator $U$ so that $a = \rho U$, $a^+ = \rho U^+$. The powers of creation and annihilation operators form a discrete subgroup of the $U(1)$ group. In this representation the number operator cannot be expressed as a function of $a, a^+$.

Thus any Hamiltonian realization of the $q$-oscillator algebra can be decomposed into the above irreducible representations. In the following section we restrict ourselves to the $q$-oscillator model with a local (in $x$) Hamiltonian of the Schrödinger type.

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6There are also possibilities to construct non-local Hamiltonians (see, for instance [3], [13]) which we do not discuss in the present letter.
3. Local Hamiltonian model of $q$-oscillator

Let us construct the creation and annihilation operators by means of Darboux operators (of first order in derivatives), supplemented with the dilatation operator,

$$a^+ = (-\partial_x + W(x))T_q; \quad a = T_q^{-1}(\partial_x + W(x)) \tag{9}$$

where $W(x)$ is a real function and the unitary dilatation operator is defined by,

$$T_q = \exp\left(\frac{1}{2}\eta\{x, \partial_x\}\right), \quad T_q\psi(x) = \sqrt{q}\psi(qx); \quad \eta \equiv \ln q.$$

Then the hermitian bilinear operators take the Schrödinger Hamiltonian form,

$$a^+a = H_1 + k_1; \quad aa^+ = q^2H_2 + k_2; \quad H_{1,2} = -\partial_x^2 + V_{1,2} \tag{10}$$

where $k_i$ are constants. When $k_1 = k_2$ the Hamiltonians are linked by the intertwining relations,

$$H_1 a^+ = q^2 a^+ H_2; \quad q^2 H_2 a = a H_1. \tag{11}$$

The relation between potentials and the ”superpotential” $W(x)$ are standard for Supersymmetrical Quantum Mechanics (SSQM) apart from the dilatation contribution [10],

$$V_1(x) = W^2(x) - W'(x) - k; \quad V_2(x) = \frac{1}{q^2}W^2\left(\frac{x}{q}\right) + \frac{1}{q}W'\left(\frac{x}{q}\right) - \frac{k}{q^2}.$$ 

If we impose the conditions

$$V_1 = V_2, \quad k = \frac{1}{1 - q^2}$$

the operators $a, a^+$ satisfy the $q$-commutator relation (14) which leads to the $q$-self-similarity equation for $W(x)$:

$$W'(x) + qW'(qx) + W^2(x) - q^2W^2(qx) = 1 \tag{12}$$

where the prime stands for the derivative with respect to $x$. This equation can be considered on the entire axis, $x \in (-\infty, +\infty)$, or on the semiaxis, $x \in [0, \infty)$.

In the first case it has been shown [11], [14] that for $0 < q \leq 1$ a nonsingular solution of this equation exists; its asymptotics,

$$W(x)||_{|x| >> 1} = \pm \frac{1}{\sqrt{1 - q^2}} + O\left(\frac{1}{x^2}\right)$$

leads to a decreasing potential, $V(x) \sim 1/x^2, \ |x| >> 1$. The spectrum of the Hamiltonians of this type must thus have a continuous part. If the solution $W(x)$ is chosen to take a positive constant value for $x \to +\infty$ and a negative one for $x \to -\infty$ it follows from Eqs.(13) that the normalizable ground state $\psi_0 \sim \exp(-\int dx W(x))$ exists and represents the zero-mode of the annihilation operator $a$: in this case the $q$-oscillator model contains a bound state spectrum.

Let us decompose the $q$-oscillator representation given by the model (9) - (12) into irreducible representations. According to the results of Sec. 2 the bound state spectrum having
the true ground state forms the Fock representation, the continuous spectrum consists of
the set of non-Fock representations. From the \(q\)-oscillator relations (5) we find the number
operator as a function of the Hamiltonian for both cases,

\[
N = \frac{\ln[(1 - q^2)^2 H^2]}{4 \ln q},
\]

where the nonlinear operator relation can be interpreted in terms of the spectral decompo-
sition for the Hamiltonian \(H\). For the Fock representation this connection was found in [11]
and here we extend it on the entire set of non-Fock representations for positive energies.
Accordingly the central element can be defined as follows,

\[
\hat{\zeta} = \zeta_c (1 + \text{sign} H), \quad H \psi = E \psi.
\]

and in the Fock representation \(\zeta = 0\), \(E < 0\) and \(E_{n+1} = q^2 E_n\) while in the non-Fock
representation one has \(\zeta = 2\zeta_c\), \(E > 0\). Both discrete and continuous sequences of energy
levels have \(E = 0\) as the accumulation point.

The special representation could be realized on zero-energy states (at the treshold between
discrete and continuous spectra) where \(\zeta = \zeta_c\). But for this particular model it can be proven
that the physical states\(^6\) for zero energy do not exist because the two zero energy solutions
have increasing asymptotic behavior \(\sim \sqrt{x}\). Therefore the special representation does not
appear in the decomposition of the space of physical wave functions.

We have thus shown that the \(q\)-oscillator system with the local Hamiltonian (10) is
composed of two types of irreducible representations, in particular, the continuous part of
the spectrum is covered by non-Fock representations parametrized by the second invariant
within the interval \(0 \leq \nu_0 < 1\) corresponding to the energy interval \(|\zeta_c| \geq E > q^2 |\zeta_c|\),

\[
\mathcal{H} = \mathcal{H}_F \bigoplus \int_{0 \leq \nu_0 < 1} d\mu(\nu_0) \mathcal{H}^{\nu_0}_{NF}.
\]

where \(\mathcal{H}\) denotes the appropriate Hilbert space of wave functions (of both bound states and
scattering states). The \(q\)-oscillator generators act on scattering wave functions as pseudo-
differential operators in accordance with (9). It can be easily shown [10] that their action
preserves the scattering boundary conditions. A more rigorous analysis will be done else-
where.

We now present a novel generalization of the above \(q\)-oscillator model based on the \(q\-
deformation of the polynomial supersymmetric algebra developed in [10]. This algebra is
generated by a Darboux differential operator of higher order in derivatives dressed by a
dilatation operator similarly to (\(\partial\)). In order to reproduce the \(q\)-oscillator algebra we are
forced to select the following algebras,

\[
a^+a = H_1^n + \frac{1}{1 - q^{2n}}, \quad aa^+ = q^{2n} H_2^n + \frac{1}{1 - q^{2n}}.
\]

If \(H_1 = H_2\) then

\[
aa^+ - q^{2n} a^+ a = 1.
\]

\(^7\)By physical states we understand the wave functions which remain bounded at infinity.
The equations for $N$ and $\zeta$ are the same as for $n = 1$ provided that one makes the substitution $q \rightarrow q^n; \quad H \rightarrow H^n$. For $n \geq 2$ the polynomial in the right hand side of (16) has always complex roots which corresponds to the primitive Darboux transformations of second-order in derivatives (for details see [10]). There are two nonequivalent sets of $q$-oscillator models corresponding to $n$ even or odd. The Hamiltonians belonging to an odd algebra in general have the representation content of the $n = 1$ model. In the even case it is clear that the Fock representation is not involved into the decomposition of the related $q$-oscillator wave function space since $\|a^+a\| > 1/(1-q^n)$.

Let us consider, in particular, the case $n = 2$ where the creation and annihilation operators read,

$$a^+ = (a)^+ = \left(\partial_x^2 - 2f(x)\partial_x + b(x)\right)T_q.$$  

From the intertwining relations (11) one finds,

$$b(x) = f(x)^2 - f'(x) - \frac{f''(x)}{2f(x)} + \left(\frac{f'(x)}{2f(x)}\right)^2 + \frac{1}{4f(x)^2(1-q^4)}.$$  

The $q$-self-similarity equation now reads,

$$f(x)^2 + 2f'(x) + \frac{f''(x)}{2f(x)} - \left(\frac{f'(x)}{2f(x)}\right)^2 - \frac{1}{4f(x)^2(1-q^4)} =$$

$$q^2 f(qx)^2 - 2qf'(qx) + \frac{f''(qx)}{2f(qx)} - \left(\frac{f'(qx)}{2f(qx)}\right)^2 - \frac{q^2}{4f(qx)^2(1-q^4)}.$$  

This equation is essentially more complicated than (12) and the existence of its regular solution has not been yet established though for the semi-axis problem the linearization method seems to be applicable and convergent.

Evidently from a regular nodeless solution $f(x)$ of this equation one would obtain the $q$-oscillator model with non-negative Hamiltonian $H = -\partial_x^2 + V(x)$ where

$$V(x) = f(x)^2 - 2f'(x) + \frac{f''(x)}{2f(x)} - \left(\frac{f'(x)}{2f(x)}\right)^2 - \frac{1}{4f(x)^2(1-q^4)}.$$  

Negative energy levels would only be allowed for Hamiltonians unbounded from below because of the properties of non-Fock representations. Such Hamiltonians would then be associated with singular potentials.

Now we derive the systematic description of deformed dynamical algebras which are realized on wave functions of a Shr¨odinger operator and, in general, induce a nonlinear mapping of energy levels.

4. Generalized realizations of $q$-oscillator models

To describe the $q$-oscillator algebra in a generalized form we introduce new creation and annihilation operators $A^+, A$ by means of a transformation preserving the commutation relations with the number operator (2),

$$a = F(N)A; \quad a^+ = A^+ F^*(N).$$
The function $F(x)$ can be chosen to be real since its phase factor does not play any role in the relations to be derived below. From (4) and (1) the $q$-commutator with a new deformation parameter is reproduced if
\[ F^2(N - 1) = C_q F^2(N) \tag{21} \]
where $C_q$ is a positive $c$-number; this choice preserves the bosonic character of the algebra. The solution of the previous equation reads,
\[ F^2(N) = C_q^{-N} \phi^{-1}(N); \quad \phi(N - 1) = \phi(N). \]
The function $\phi$ is periodic, positive and in fact takes a definite value for a particular representation, thereby being an invariant. From Sec. 2 it follows that $\phi = \phi(\nu_0)$. The basic $q$-commutator takes the form,
\[ AA^+ - q^2 C_q A^+ A = C_q^N \phi(N). \tag{22} \]
The central element is modified as follows,
\[ \hat{\zeta} = q^{-2N} \left( [N]_q - C_q^{-N} \phi^{-1}(N) A^+ A \right). \tag{23} \]
The bilinear operators are modified as well,
\[ A^+ A = C_q^{N-1} \phi(N) a^+ a; \quad AA^+ = C_q^N \phi(N) aa^+, \tag{24} \]
as compared with Eqs.(5). The additional $q$-commutator is built with a new deformation parameter,
\[ AA^+ - C_q A^+ A = \left( 1 - \frac{\zeta}{\zeta_c} \right) C_q^N q^{2N} \phi(N) \tag{25} \]
The classification of representations is similar to that in Sec. 2 since we have not introduced any new algebra but have chosen different elements of the same universal enveloping algebra as basic generators. The special choice $C_q = 1/q$ leads to the algebra in [5] whereas $C_q = 1/q^2$ leads to the algebra introduced in [17].

Now we proceed to nonlinear realizations of the spectrum generating deformed algebras in Quantum Mechanics. Such realizations are of interest for the algebraic description of physical systems which spectra are only approximately related to the harmonic or $q$-harmonic oscillators. We start from the generalized intertwining relations,
\[ H_1 A^+ = A^+ g(H_2); \quad g(H_2) A = AH_1 \tag{26} \]
where $H_{1,2}$ are Hamiltonians of two quantum systems with related energy spectra and wave functions, $H_i \psi_i = E_i \psi_i; \quad E_1 = g(E_2), \quad \psi_1 = A^+ \psi_2$. In the case of polynomial SSQM we have $g(x) = q^2 x$. In virtue of (26), $[A^+ A, H_1] = [AA^+, g(H_2)] = 0$. If we assume in addition that the function $g$ is invertible, we have $[AA^+, H_2] = 0$; hence the bilinear operators commute with the Hamiltonians and represent, in general, symmetry operators [18]. In one-dimensional QM, they are functions of Hamiltonians,
\[ A^+ A = \sigma_1(H_1); \quad AA^+ = \sigma_2(H_2) \tag{27} \]
where $\sigma_i$ are arbitrary invertible functions. It follows then that for analytic functions $f(z)$

$$f(\sigma_1(H_1))A^+ = \sum_{m=0}^{\infty} c_m \sigma_1^m(H_1)A^+ = A^+ \sum_{m=0}^{\infty} c_m \sigma_2^m(H_2) = A^+ f(\sigma_2(H_2))$$  \hspace{1cm} (28)

Let us now define the inverse functions,

$$\sigma_i(\pi_i(z)) = \pi_i(\sigma_i(z)) = z,$$ \hspace{1cm} (29)

and choose $f(z) = \pi_i(z)$. Then the mapping function $g$ and its inverse function $g^-$ are determined by these functions,

$$g(z) = \pi_1(\sigma_2(z)), \quad g^-(z) = \pi_2(\sigma_1(z)).$$ \hspace{1cm} (30)

With the help of the inverse function one derives the second set of intertwining relations,

$$A^+ H_2 = g^-(H_1)A^+.$$ \hspace{1cm} (31)

To find links between energy levels of the same dynamical system we impose the self-similarity condition, $H_1 = H_2$. In this way we discover a deformed dynamical algebra of a Hamiltonian.

The ladder procedure connects different levels and eigenstates belonging to the following sequence,

$$A^+: \cdots \longrightarrow g^-(g^-(E)) \longrightarrow g^-(E) \longrightarrow E \longrightarrow g(E) \longrightarrow g(g(E)) \longrightarrow \cdots$$

$$A: \quad \cdots \leftarrow g^-(g^-(E)) \leftarrow g^-(E) \leftarrow E \leftarrow g(E) \leftarrow g(g(E)) \leftarrow \cdots$$ \hspace{1cm} (32)

For physical systems of oscillator type with Hamiltonians bound from below this operators may generate the entire spectrum of a model whereas on the continuous part of spectrum they connect only subsets of levels.

Imposing relations (22), (25) on the functions $\sigma_{1,2}$ given by (27), we realize the $q$-oscillator algebra provided that the following constraints for a representation $[\zeta, \nu_0]$ are fulfilled,

for $C_q \neq 1$,

$$\ln(\sigma_2(z) - q^2 \sigma_1(z)) = \left(1 + 2 \frac{\ln q}{\ln C_q}\right) \ln \left[\sigma_2(z) - q^2 C_q \sigma_1(z)\right] - \frac{2 \ln q}{\ln C_q} \ln \phi(\nu_0) + \ln \left(1 - \frac{\zeta}{\zeta_c}\right),$$ \hspace{1cm} (33)

and for $C_q = 1$, $\phi = 1$,

$$\sigma_2(z) - q^2 \sigma_1(z) = 1.$$ \hspace{1cm} (34)

In the latter case the energy mapping reads,

$$E \longrightarrow \pi_1 \left(q^2 \sigma_1(E) + 1\right).$$ \hspace{1cm} (35)

In particular, for $q = 1$ this mapping represents the generalization of the harmonic oscillator spectrum.

The content of irreducible representations of the $q$-oscillator algebra in a particular model depends on the position of fixed points of the mapping (32). If a fixed-point energy value is higher than the ground state energy the Fock representation exists and it is realized in between them. If there is no fixed points for finite energies then the Fock representation spans the whole Hilbert space of wave functions. On the contrary if the fixed point coincides with the ground state energy the non-Fock representations only appear in the decomposition into irreducible representations.
5. Conclusions and perspectives

1. The general strategy suggested by our approach is to find the spectrum generating algebra from the properties of the bound state spectrum and scattering coefficients, i.e. to determine the functions $\sigma_{1,2}$ satisfying the constraints (33). In practice, it can be done only approximately and the required perturbation theory will be studied elsewhere. In order to find the related potential the inverse scattering method is required to be extended onto potentials with $1/x^2$ asymptotics at infinity [14].

2. There exist other generalizations [19] of the $q$-oscillator algebra of the following form

$$aa^+ - \Phi_1^2(N)a^+a = \Phi_2^2(N).$$

(36)

where $\Phi_i$ are supposed to be real functions sufficiently regular. In fact, one can redefine the basic elements of the universal enveloping algebra,

$$a = M(N)A; \quad a^+ = A^+M(N); \quad M^* = M;$$

(37)

so to replace the function $\Phi_1(N)$ by a constant $c$. We restrict ourselves to transformations which do not change the commutation relations with the number operator. The required function $M(N)$ obeys the following equation:

$$\Phi_1(N)M(N + 1) = cM(N).$$

(38)

In a particular representation $\{\zeta, \nu_0\}$ its solution is

$$M(n + \nu_0) = M(\nu_0)c^{-n} \prod_{l=0}^{n-1} \Phi_1(l + \nu_0),$$

(39)

where $M(\nu_0)$ is an arbitrary function for $0 \leq \nu_0 < 1$. Thereby one comes to the $q$-deformed algebra [22] but with the arbitrary function $\Phi_2(N)$ on the righthand side in qualitative agreement with [19]. We stress however the additional freedom to modify $\Phi_2(N)$ [37], which does not change the enveloping algebra and was not considered in [19]. Thus non-equivalent $q$-deformed algebras are only those which cannot be related by these “gauge” transformations.

3. We believe that the analysis of fixed points of functional mappings [32] can lead to a better understanding of the physical signatures (see [19] and references therein) of a $q$-deformed algebra. The number of fixed points might represent a sort of topological invariant under “smooth” perturbations of a potential. Still the problem of the implementation of the special representation on wave functions of a local Hamiltonian (a fixed point of energy mapping) remains open.

4. We do not see any obstacles to extend our approach to the half-line (radial) problem, at least, for the $S$-wave. In this case one is allowed also to use solutions with singularities at negative $x$.

5. Finally we would like to mention the possibility to vary the deformation parameter according to an external dynamics (for example, a time-dependent $q = q(t)$ as an order parameter describing the transition from ”confined”, $q \geq 1$, to ”deconfined”, $q < 1$, phase).
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After completing our paper we received the paper [20] where a similar decomposition has been described without however introducing the notions of the central element and $N$-operator.

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