ON COMPACT HYPERBOLIC MANIFOLDS OF EULER CHARACTERISTIC TWO

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Dedicated to the memory of Colin Maclachlan

ABSTRACT. We prove that for \( n > 4 \) there is no compact arithmetic hyperbolic \( n \)-manifold whose Euler characteristic has absolute value equal to 2. In particular, this shows the nonexistence of arithmetically defined hyperbolic rational homology \( n \)-sphere with \( n \) even different than 4.

1. MAIN RESULT AND DISCUSSION

1.1. Smallest hyperbolic manifolds. Let \( \mathbb{H}^n \) be the hyperbolic \( n \)-space. By a hyperbolic \( n \)-manifold we mean an orientable manifold \( M = \Gamma \backslash \mathbb{H}^n \), where \( \Gamma \) is a torsion-free discrete subgroup \( \Gamma \subset \text{Isom}^+ (\mathbb{H}^n) \). The set of volumes of hyperbolic \( n \)-manifolds being well ordered, it is natural to try to determine for each dimension \( n \) the hyperbolic manifolds of smallest volume. For \( n = 3 \) this problem has recently been solved in [15], the smallest volume being achieved by a unique compact manifold, the Weeks manifold. When \( n \) is even the volume is proportional to the Euler characteristic, and this allows to formulate the problem in terms of finding the hyperbolic manifolds \( M \) with smallest \(|\chi(M)|\). In particular this observation solves the problem in the case of surfaces. For \( n > 3 \), noncompact hyperbolic \( n \)-manifolds \( M \) with \(|\chi(M)| = 1 \) have been found for \( n = 4, 6 \) [14].

In the present paper we consider the case of compact manifolds of even dimension. In particular, such manifolds have even Euler characteristic (see [17, Theorem 1.2]). We restrict ourselves to the case of arithmetic manifolds, where Prasad’s formula [20] can be used to study volumes. We complete the proof of the following result.

Theorem 1. Let \( n > 5 \). There is no compact arithmetic manifold \( M = \Gamma \backslash \mathbb{H}^n \) with \(|\chi(M)| = 2 \).

The result for \( n > 10 \) already follows from the work of Belolipetsky [4, 5], also based on Prasad’s volume formula. More precisely, Belolipetsky determined the smallest Euler characteristic \(|\chi(\Gamma)|\) for arithmetic orbifold quotients \( \Gamma \backslash \mathbb{H}^n \) (\( n \) even). This smallest value grows fast with the dimension \( n \), and for compact quotients we have \(|\chi(\Gamma)| > 2 \) for \( n > 10 \). That the
result of nonexistence holds for \( n \) high enough is already a consequence of Borel-Prasad’s general finiteness result [9], which was the first application of Prasad’s formula. The proof of Theorem 1 for \( n = 6,8,10 \) requires a more precise analysis of the Euler characteristic of arithmetic subgroups \( \Gamma \subset \text{PO}(n,1) \), and in particular of the special values of Dedekind zeta functions that appear as factors of \( \chi(\Gamma) \).

For \( n = 4 \), the corresponding problem is not solved, but there is the following result [5].

**Theorem 2** (Belolipetsky). If \( M = \Gamma \backslash \mathbb{H}^4 \) is a compact arithmetic manifold with \( \chi(M) \leq 16 \), then \( \Gamma \) arises as a (torsion-free) subgroup of the following hyperbolic Coxeter group:

\[
W_1 = \begin{array}{c}
\circ & \circ & \circ & \circ & \circ \\
\end{array} \]

An arithmetic (orientable) hyperbolic 4-manifold of Euler characteristic 16 has been first constructed by Conder and Maclachlan in [12], using the presentation of \( W_1 \) to obtain a torsion-free subgroup with the help of a computer. Further examples with \( \chi(M) = 16 \) have been obtained by Long in [18] by considering a homomorphism from \( W_1 \) onto the finite simple group \( \text{PSp}_4(4) \).

1.2. **Hyperbolic homology spheres.** Our original motivation for Theorem 1 was the problem of existence of hyperbolic homology spheres. A **homology n-sphere** (resp. **rational homology n-sphere**) is a \( n \)-manifold \( M \) that possesses the same integral (resp. rational) homology as the \( n \)-sphere \( S^n \). This forces \( M \) to be compact and orientable.

Rational homology \( n \)-spheres \( M \) have \( \chi(M) = 2 \) if \( n \) is even. On the other hand, for \( M = \Gamma \backslash \mathbb{H}^n \) with \( n = 4k + 2 \) we have \( \chi(M) < 0 \) (cf. [25, Proposition 23]), and this exclude the possibility of hyperbolic rational homology spheres for those dimensions. For \( n \) even, Wang’s finiteness theorem [28] implies that there is only a finite number of hyperbolic rational homology \( n \)-spheres. Theorem 1 shows the nonexistence of arithmetic rational homology spheres for \( n > 5 \) even.

For odd dimensions, \( \chi(M) = 0 \) and \textit{a priori} the volume is not a limitation for the existence of hyperbolic (rational) homology spheres. In fact, an infinite tower of covers by hyperbolic integral homology 3-spheres has been constructed by Baker, Boileau and Wang in [3]. In [10] Calegari and Dunfield constructed an infinite tower of hyperbolic rational homology 3-spheres that are arithmetic and obtained by congruence subgroups. Note that a recent conjecture of Bergeron and Venkatesh predicts a lot of torsion in the homology groups of such a “congruence tower” of arithmetic \( n \)-manifolds with \( n \) odd [7].

1.3. **Locally symmetric homology spheres.** Instead of considering hyperbolic homology spheres, one can more generally look for homology spheres
that are locally isometric to a given symmetric space of nonpositive nonflat sectional curvature. Such a symmetric space \( X \) is called of noncompact type, and it is classical that \( X \) can be written as \( G/K \), where \( G \) is a connected real semisimple Lie group with trivial center with \( K \subset G \) a maximal compact subgroup. Moreover, \( G \) identifies as a finite index subgroup in the group of isometries of \( X \) (of index two if \( G \) is simple).

Let us explain why the case \( X = H^n \) is the main source of locally symmetric rational homology spheres (among \( X \) of noncompact type). Let \( M \) be a compact orientable manifold locally isometric to \( X \). Then \( M \) can be written as \( \Gamma \backslash X \), where \( \Gamma \cong \pi_1(M) \) is a discrete subgroup of isometries of \( X \). We will suppose that \( \Gamma \subset G \), for \( G \) as above. Let \( X_u \) be the compact dual of \( X \). We have the following general result (see [8, Sections 3.2 and 10.2]).

**Proposition 3.** For each \( j \) there is an injective homomorphism \( H^j(X_u, \mathbb{C}) \to H^j(\Gamma \backslash X, \mathbb{C}) \).

In particular, if \( \Gamma \backslash X \) is a rational homology sphere, then so is \( X_u \). Note that the compact dual of \( X = H^n \) is the genuine sphere \( S^n \). By looking at the classification of compact symmetric spaces, Johnson showed the following in [16, Theorem 7].

**Corollary 4.** If \( M = \Gamma \backslash X \) is a rational homology \( n \)-sphere with \( \Gamma \subset G \), then \( X \) is either the hyperbolic \( n \)-space \( H^n \) (with \( n \neq 4k + 2 \)), or \( X = \text{PSL}_3(\mathbb{R})/\text{PSO}(3) \) (which has dimension 5).

Proposition 3 shows that the correct problem to look at – rather than homology spheres – is the existence of locally symmetric spaces \( \Gamma \backslash X \) with the same (rational) homology as the compact dual \( X_u \). When \( X \) is the complex hyperbolic plane \( H^2_C \), the compact dual is the projective plane \( P^2_C \), and the quotients \( \Gamma \backslash X \) are compact complex surfaces called fake projective planes. Their classification was recently obtained by the work of Prasad–Yeung [21], together with Cartwright–Steger [11] who performed the necessary computer search. Later, Prasad and Yeung also considered the problem of the existence of more general arithmetic fake Hermitian spaces [22, 23].

The present paper uses the same methodology as in Prasad and Yeung’s work, the main ingredient being the volume formula.

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2. **Proof of Theorem 1**

Let \( G = \text{PO}(n,1)^{\circ} \cong \text{Isom}^+(H^n) \), and consider the universal covering \( \phi : \text{Spin}(n,1) \to G \). For our purpose it will be easier to work with lattices in \( \text{Spin}(n,1) \). A lattice \( \Gamma \subset G \) is arithmetic exactly when \( \Gamma = \phi^{-1}(\Gamma) \) is an arithmetic subgroup of \( \text{Spin}(n,1) \). Since the covering \( \phi \) is twofold, we have \( \chi(\Gamma) = \frac{1}{2} \chi(\Gamma) \), where \( \chi \) is the Euler characteristic in the sense of C.T.C. Wall. In particular, if \( M = \Gamma \backslash H^n \) is a manifold with \( |\chi(M)| = 2 \), then \( |\chi(\Gamma)| = 1 \).
Thus, Theorem 1 is an obvious consequence of the following proposition. The proof relies on the description of arithmetic subgroups with the help of Bruhat-Tits theory, as done for instance in [9] and [20]. An introduction can be found in [13]. We also refer to [27] for the needed facts from Bruhat-Tits theory.

**Proposition 5.** Let $n > 4$. There is no cocompact arithmetic lattice $\Gamma \subset \text{Spin}(n,1)$ such that $\chi(\Gamma)$ is a reciprocal integer, i.e., such that $\chi(\Gamma) = 1/q$ for some $q \in \mathbb{Z}$.

**Proof.** We can assume that $n$ is even. Let $\Gamma \subset \text{Spin}(n,1)$ be a cocompact lattice. Clearly, it suffices to prove the proposition for $\Gamma$ maximal. In this case, $\Gamma$ can be written as the normalizer $\Gamma = N_{\text{Spin}(n,1)}(\Lambda)$ of some principal arithmetic subgroup $\Lambda$ (see [9, Proposition 1.4]). By definition, there exists a number field $k \subset \mathbb{R}$ and a $k$-group $G$ with $G(\mathbb{R}) \cong \text{Spin}(n,1)$ such that $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$, for some coherent collection $(P_v)_{v \in V_f}$ of parahoric subgroups $P_v \subset G(k_v)$ (indexed by the set $V_f$ of finite places of $k$). It follows from the classification of algebraic groups (cf. [26]) that $G$ is of type $B_r$ with $r = n/2$ ($> 2$), the field $k$ is totally real, and (using Godement’s criterion) $k \neq \mathbb{Q}$. Let us denote by $d$ the degree $[k : \mathbb{Q}]$.

Let $T \subset V_f$ be the set of places where $P_v$ is not hyperspecial. By Prasad’s volume formula (see [20] and [9, Section 4.2]), we have:

$$|\chi(\Lambda)| = 2|D_k| r^2 + r/2 C(r)^d \prod_{j=1}^{r} \frac{\zeta_k(2j)}{\prod_{v \in T} \lambda_v},$$

with $D_k$ (resp. $\zeta_k$) the discriminant (resp. Dedekind zeta function) of $k$; the constant $C(r)$ is given by

$$C(r) = \prod_{j=1}^{r} \frac{(2j - 1)!}{(2\pi)^{2j}};$$

and each $\lambda_v$ is given by the formula

$$\lambda_v = \frac{1}{(q_v)^{(\dim M_v - \dim M_v^0)/2}} \frac{|\mathcal{M}(f_v)|}{|\mathcal{M}_v(f_v)|},$$

where $f_v$ is the residue field of $k_v$, of size $q_v$, and the reductive $f_v$-groups $M_v$ and $\mathcal{M}_v$ associated with $P_v$ are those described in [20]. By definition $M_v$ is semisimple of type $B_r$.

A necessary condition for $\Gamma = N_{G(\mathbb{R})}(\Lambda)$ to be maximal is that each $P_v$ defining $\Lambda$ has maximal type in the sense of [24]. We list in Table 1 the factors $\lambda_v$ corresponding to parahoric subgroups $P_v$ of maximal types (to improve the readability we set $q_v = q$ in the formulas). This list of maximal type and the formulas for $\lambda_v$ are essentially the same as in [4, Table 1]: the only difference is a factor 2 in the denominator of some $\lambda_v$, which can be explained from the fact that Belolipetskii did not work with $G$ simply connected.
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\[ \begin{array}{lll}
  \mathbf{G}/k_v & \text{isogeny type of } \mathbf{M}_v & \lambda_v \\
  \text{split:} & B_{r-1} \times (\text{split } \text{GL}_1) & \frac{q^{2r-1}}{q-1} \\
  & D_i \times B_{r-i} \ (i = 2, \ldots, r-1) & \frac{(q+1)^i \prod_{k=1}^{i+1} (q^{2k}-1)}{\prod_{k=1}^{i-1} (q^{2k}-1)} \\
  & 1D_r & q^r + 1 \\
  \text{non-split:} & B_{r-1} \times (\text{nonsplit } \text{GL}_1) & \frac{q^{2r-1}}{q+1} \\
  & 2D_{i+1} \times B_{r-i-1} \ (i = 1, \ldots, r-2) & \frac{(q^{i+1}-1)^k \prod_{k=1}^{i+2} (q^{2k}-1)}{\prod_{k=1}^{i-1} (q^{2k}-1)} \\
  & 2D_r & q^r - 1 \\
\end{array} \]

Table 1. \( \lambda_v \) for \( P_v \) of maximal type

From [9, Section 5] (cf. also [13, Chapter 12]) we can deduce that the index \([\Gamma : \Lambda]\) of \( \Lambda \) in its normalizer has the following property:

\[ [\Gamma : \Lambda] \text{ divides } h_k 2^d 4^{#T}. \] (5)

Moreover, a case by case analysis of the possible factor \( \lambda_v \) shows that \( \lambda_v > 4 \), so that \( 4^{-#T} \prod_{v \in T} \lambda_v \geq 1 \) (with equality exactly when \( T \) is empty). We thus have the following lower bound for the Euler characteristic of any maximal arithmetic subgroup \( \Gamma \subset \text{Spin}(n,1) \):

\[ |\chi(\Gamma)| \geq \frac{2}{h_k} \left( \frac{C(r)}{2} \right)^d |D_k|^{r^2 + r/2} \zeta_k(2) \cdots \zeta_k(2r) \] (6)

We make use of the following upper bound for the class number (see for instance [6, Section 7.2]):

\[ h_k \leq 16 \left( \frac{\pi}{12} \right)^d |D_k|, \] (7)

which together with the basic inequality \( \zeta_k(2j) > 1 \) transforms (6) into

\[ |\chi(\Gamma)| > \frac{1}{8} \left( \frac{6 \cdot C(r)}{\pi} \right)^d |D_k|^{r^2 + r/2 - 1}. \] (8)

Moreover, according to [19, Table 4], we have that for a degree \( d \geq 5 \) the discriminant of \( k \) is larger than \( (6.5)^d \). With this estimates we can check that for \( r \geq 3 \) and \( d \geq 5 \) we have \( |\chi(\Gamma)| > 1 \). For the lower degrees, if we suppose that \( |\chi(\Gamma)| \leq 1 \), we obtain upper bounds for \( |D_k| \) from Equation (8). This upper bounds exclude the existence of such a \( \Gamma \) for \( r \geq 6 \) (which is already clear from the work of Belolipetsky [4]). For \( r = 3 \) (where the
The special values of $\zeta_k$ can be computed with the software Pari/GP (cf. Remark 6). We list in Table 2 the values we need. We check that for every field $k$ under consideration a prime factor $> 2$ appear in the numerator of the product $\prod_{j=1}^{m} |\zeta_k(1 - 2j)|$. A direct computation for $r = 3, 4, 5$ shows that the formula in Table 1 is actually given by a polynomial in $q$ (this seems to hold for any $r$). In particular, we always have $\lambda_v \in \mathbb{Z}$, and we conclude from (9) that $|\chi(\Gamma)|$ cannot be a reciprocal integer.

Remark 6. The function `zetak` in Pari/GP allows to obtain approximate values for $\zeta_k(1 - 2j)$. On the other hand the size of the denominator of the product $\prod_{j=1}^{m} |\zeta_k(1 - 2j)|$ can be bounded by the method described in [25, Section 3.7]. By recursion on $m$, this allows to ascertain that the values $\zeta_k(1 - 2j)$ correspond exactly to the fractions given in Table 2.

Remark 7. The fact that for $|D_k| = 5$ the value $\zeta_k(-1)\zeta_k(-3)$ has trivial numerator explains why the proof fails for $n = 4$ (i.e., $r = 2$). And indeed

| degree | $|D_k|$ | $\zeta_k(-1)$ | $\zeta_k(-3)$ | $\zeta_k(-5)$ | $\zeta_k(-7)$ | $\zeta_k(-9)$ |
|--------|--------|----------------|----------------|----------------|----------------|----------------|
| $d = 2$ | 5      | $1/30$         | $1/60$         | $67/630$       | $361/120$      | $412751/1650$ |
|        | 8      | $1/12$         | $11/120$       | $361/252$      | $24611/240$    |                 |
|        | 12     | $1/6$          | $23/60$        | $1681/126$     |                 |                 |
|        | 13     | $1/6$          | $29/60$        | $33463/1638$   |                 |                 |
|        | 17     | $1/3$          | $41/30$        | $5791/63$      |                 |                 |
| $d = 3$ | 49     | $-1/21$        | $79/210$       | $-7393/63$     |                 |                 |
|        | 81     | $-1/9$         | $199/90$       | $-50353/27$    |                 |                 |

Table 2. Special values of $\zeta_k$
there is a principal arithmetic subgroup $\Gamma \subset \text{Spin}(4,1)$ with $|\chi(\Gamma)| = 1/14400$ and whose image in $\text{Isom}^+(H^4)$ is contained as an index 2 subgroup of the Coxeter group $W_1$. On the other hand, for $|D_k| > 5$ the appearance of a non-trivial numerator in $\zeta_k(-3)$ shows – at least for the fields considered in Table 2 – the impossibility of a $\Gamma$ defined over $k$ with $\chi(\Gamma)$ a reciprocal integer. This is the first step in Belolipetsky’s proof of Theorem 2.

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