Classification of higher mobility closed-loop linkages

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Abstract
We provide a complete classification of paradoxical closed-loop $n$-linkages, where $n \geq 6$, of mobility $n - 4$ or higher, containing revolute, prismatic or helical joints. We also explicitly write down strong necessary conditions for $nR$-linkages of mobility $n - 5$. Our main new tool is a geometric relation between a linkage $L$ and another linkage $L'$ resulting from adding equations to the configuration space of $L$. We then lift known classification results for $L'$ to $L$ using this relation.

Keywords Dual quaternion · Absolute cone · Bond theory · Paradoxical linkage

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1 Introduction

A closed-loop linkage is a mechanical device consisting of rigid bodies coupled together by joints and forming a single loop. Each joint allows for a different type of motion, namely, revolute (rotational around a fixed axis), prismatic (translational along a fixed direction) and helical (rotational and translational, where the rotation angle and the translation length are dependent). We call these $R$, $P$, and $H$ joints. A closed linkage with exactly $n$-joints is denoted by $n$-linkage. If all joints are of the same type, revolute, we call it an $nR$-linkage.

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Closed linkages are a crucial and basic object in the modern theory and applications of kinematics [1–4]. They have been studied using algebraic and geometric methods in classical works such as Cayley [5], Chebyshev [6], Sylvester [7], and Kempe [8].

The mobility (see Definition 3.2 for a precise description) of a closed-loop linkage is generically $n - 6$ by the Chebychev–Grübler–Kutzbach criterion [4, 9], where $n$ is the number of 1-degree-of-freedom (DOF) joints in the closed-loop. A closed-loop linkage whose mobility is higher than $n - 6$ is called a paradoxical or overconstrained linkage. Mobile linkages with 4 joints of type $R$, $P$ or $H$ have been classified in [10]. In [11], the authors settle the classification problem for 5-linkages with at least one helical joint, which complements the classification of mobile linkages with five joints of type $R$, $P$ or $H$. Since the first mobile linkage with only revolute joints (6R-linkage) of Sarrus [12], numerous paradoxical 6R-linkages were discovered (see reviews in [13–17]) but the classification of the paradoxical 6R-linkages is still open. Furthermore, the classification of paradoxical linkages with $n \geq 6$ joints of type $R$, $P$ or $H$ is open too.

However, there has not been a systematic study yet for linkages with more than six joints or mobility higher than two. Our results constitute a significant step further in the solution to this problem. For a closed linkage, if the motions of all joints are in a Schönflies subgroup of $SE(3)$ which has dimension 4, then the linkage has mobility $n - 4$, see [18–20]. However, the converse is not true. For instance, the Goldberg 5R linkages of mobility one [21] and 6R linkages of mobility two [22] are counter-examples. We completely classify higher mobility closed-loop $n$-linkages with mobility $n - 4$.

**Theorem 1.1** (Main Theorem I, see Theorems 5.5, 5.7 and 5.9) A 6-linkage of mobility 2 is one of the following cases:

1. All joints are revolute with zero offsets and equal Bennett ratios;
2. There are exactly two P-joints and two pairs of neighbouring parallel R or H-joints.
3. All axes of revolute or helical joints are parallel.

**Theorem 1.2** (Main Theorem II, see Proposition 5.1 and Theorems 5.10, 5.12 and 5.13) The mobility of an $n$-linkage is at most $n - 3$.

1. An $n$-linkage, where $n > 6$, has mobility $n - 4$ if and only if all its revolute and helical axes are parallel.
2. An $n$-linkage, where $n \geq 5$, has mobility $n - 3$ only if all the axes of revolute or helical joints are concurrent.

Moreover, we give necessary conditions that an $nR$-linkage of mobility $n - 5$ satisfies:

**Theorem 1.3** (Main Theorem III, see Theorem 6.3) Every $nR$-linkage of mobility $n - 5$ with $n > 6$ has parallel neighbouring axes or triples of consecutive axes that satisfy a Bennett condition.

Note that there are known 7R-linkages of mobility two and 8R-linkages of mobility three constructed using factorization of motion polynomials [23–25] and other synthesis methods [26, 27]. A necessary condition when $n = 6$ is also given in [28].

The main new tool we use is the relation between a linkage $L$ with high enough mobility and a new linkage $L'$ obtained by adding equations to the configuration space of $L$. Each of
these equations corresponds in practice to the freezing of a joint in $L$ at a generic position, see Lemma 3.4. Consequently, we reduce the mobility by one, which allows us to obtain classification results for $L$ by using previously known classification results for $L'$. Furthermore, the relation between $L$ and $L'$ is achieved by a new formulation of the Bennett and conditions for concurrency on revolute axes in terms of points in the absolute quadric cone, see Lemma 3.3, which is interesting in itself. Finally, we use new considerations on bond theory for linkages with higher mobility, see Sect. 4, and an extension of the $abc$-lemma to dual quaternions, see Lemma 2.3.

2 Preliminaries

The real quaternions, $\mathbb{H}$, are the unique 4-dimensional associative division algebra over $\mathbb{R}$. An element in $\mathbb{H}$ can be uniquely written as $p = p_0 + p_1i + p_2j + p_3k$, where $p_i \in \mathbb{R}$ and $i^2 = j^2 = k^2 = ijk = -1$ are the usual Hamiltonian relations. Its conjugate is $\bar{p} := p_0 - p_1i - p_2j - p_3k$. The algebra of quaternions is normed, and the norm is a multiplicative function given by

$$N : \mathbb{H} \rightarrow \mathbb{R}$$

$$p \mapsto p\bar{p} = p_0^2 + p_1^2 + p_2^2 + p_3^2.$$ 

The quaternions are a good model for the 3D rotation group $SO(3)$, but it turns out that it is not enough to model the whole group of direct isometries of $\mathbb{R}^3$, $SE(3)$. For that, we need dual quaternions. Let $\mathbb{D}H$ be the 8-dimensional associative algebra over $\mathbb{R}$ given by

$$\mathbb{D}H := \mathbb{H}[\varepsilon]/(\varepsilon^2).$$

That is, an element in $\mathbb{D}H$ is of the form $h = p + \varepsilon d$ where $p$ and $d$ are quaternions and $\varepsilon^2 = 0$. The quaternions $p$ and $d$ are called the primal and dual parts of $h$, respectively. The dual quaternions are normed and the norm is given by

$$N : \mathbb{D}H \rightarrow \mathbb{D}$$

$$p + \varepsilon d \mapsto (p + \varepsilon d)(\bar{p} + \varepsilon \bar{d}) = p\bar{p} + \varepsilon(p\bar{d} + d\bar{p})$$

$$= p_0^2 + p_1^2 + p_2^2 + p_3^2 + 2\varepsilon(p_0d_0 + p_1d_1 + p_2d_2 + p_3d_3),$$

where $\mathbb{D} = \{a + be \mid a, b \in \mathbb{R} \text{ and } \varepsilon^2 = 0\}$. The subset of unit dual quaternions, $\mathbb{D}H^u$, given by elements with norm 1, has a group structure.

The dual quaternions act on $\mathbb{R}^3$ and, in fact, restricting the action to the unit dual quaternions yields a surjection to $SE(3)$. Such a map is not injective, however, since any two unit dual quaternions differ by a sign map to the same isometry. Hence, it is natural to identify $h$ with any nonzero real multiple of $h$, therefore, get a one-to-one correspondence

$$S^* := \{h \in \mathbb{D}^7 \mid N(h) \in \mathbb{R}^*\} \leftrightarrow SE(3).$$

Classically, it is usual to write $S^*$ as $S \setminus E$ where

$$S = \{(p_0 : p_1 : p_2 : p_3 : d_0 : d_1 : d_2 : d_3) \in \mathbb{P}^7 \mid p_0d_0 + p_1d_1 + p_2d_2 + p_3d_3 = 0\}$$

is the Study Quadric. Inside $S$ is the exceptional linear 3-space.
The subset $S \setminus E$ has a group structure inherited from the one in the unit dual quaternions. It is a classic result that $S \setminus E$ is isomorphic to $SE(3)$. See [29, Section 2.4]. In fact, the isomorphism can be given by

$$\psi : S \setminus E \to SE(3)$$

such that

$$x \mapsto 1 + \epsilon p + e d.$$ 

Here, a point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is written in the form $1 + \epsilon x = 1 + x_1 \epsilon i + x_2 \epsilon j + x_3 \epsilon k$.

**Example 2.1** Table 1 shows the correspondence between points in $S \setminus E$ and isometries of $\mathbb{R}^3$.

Very often, we work in the complexification of the real algebra $\\mathbb{D}\\mathbb{H}$. We can extend the dual quaternions to the algebra of complex dual quaternions, $\\mathbb{D}\\mathbb{H}_C$, defined as $\\mathbb{D}\\mathbb{H} \otimes_\mathbb{R} \mathbb{C}$.

Inside $\\mathbb{D}\\mathbb{H}_C$ there is the subalgebra of complex quaternions, $\\mathbb{H}_C = \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$. The following lemma says that it has a linear structure.

**Lemma 2.1** There is an algebra isomorphism,

$$\\mathbb{H}_C \cong M_2(\mathbb{C}).$$

**Proof** This is a special case of Wedderburn’s theorem, see [30], and follows directly from the fact that $\\mathbb{H}_C$ is a simple algebra over $\mathbb{C}$. $\square$

**Lemma 2.2** (*abc*-lemma) Suppose that $a$, $b$ and $c$ are norm zero complex quaternions. Then

$$abc = 0 \implies ab = 0 \quad \text{or} \quad bc = 0.$$ 

**Proof** Let $X = \{ x \in \mathbb{H}_C \mid abx = 0 \}$ and $Y = \{ x \in \mathbb{H}_C \mid bx = 0 \}$. Clearly, $Y \subseteq X$. Both $X$ and $Y$ are right ideals of $\mathbb{H}_C$ and, therefore, linear subspaces of $M_2(\mathbb{C})$. Suppose that $ab \neq 0$. Then $X \neq M_2(\mathbb{C})$ and $X$ and $Y$ are 2–dimensional. Hence, $X = Y$. Since $c \in X$ by hypothesis, it follows that $c \in Y$. $\square$

The *abc*-lemma is one of the main tools in our results concerning linkages of mobility 2 or higher. It was originally proven in [17] to help with classifying the bond

| Points in $S \setminus E$ | Isometries in $\mathbb{R}^3$ |
|---------------------------|-----------------------------|
| $(1 : 0 : 0 : 0 : t_1/2 : t_2/2 : t_3/3)$ | Translation by $(t_1, t_2, t_3)$ |
| $(t : u_1 : u_2 : u_3 : 0 : 0 : 0 : 0)$ | Rotation by $a$ around the axis $u = (u_1, u_2, u_3)$ and $u_1^2 + u_2^2 + u_3^2 = 1$. |

where $t = \cot(\frac{1}{2})$. 

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diagrams for 6R-linkages with mobility 1 with at most four connections in the sense of bond theory. We shall make extensive usage of this result. There is, in fact, a generalisation of the abc-lemma for complex dual quaternions that we will also use:

**Lemma 2.3** (Extended abc-lemma) Suppose that $a$, $b$ and $c$ are norm zero complex dual quaternions, none of which is a quaternion multiple of $\epsilon$. Then $abc = 0$ if and only if at least one of the following is true:

1. $ab = 0$;
2. $bc = 0$;
3. $ab$ and $bc$ are both multiples of $\epsilon$.

**Proof** It is clear that $ab = 0$ or $bc = 0$ implies $abc = 0$. Suppose that both $ab$ and $bc$ are multiples of $\epsilon$. Then, for every $x \in \mathbb{DH}_C$, we have that $abxbc = 0$. We have thus to prove that there exists an element $x$ for which $bxb = b$. Any norm zero complex dual quaternion is of the form $u(i - i)v$ (by [31, Lemma 1]) where $u$ and $v$ are invertible dual quaternions and the $i$ is the complex imaginary unit. It follows that one only has to show that there is an element $x$ such that $(i - i)xu(i - i) = (i - i)$. We take $x = \frac{1}{2N(u)N(v)}(uv)$.

To prove the converse, we look at the left annihilator $\text{Ann}_f$ of $bc$ in $\mathbb{DH}_C$. If $bc$ is not a multiple of $\epsilon$, we have $a \in \text{Ann}_f(bc) = \mathbb{DH}_C bc \subseteq \mathbb{DH}_C b = \text{Ann}_f(b)$, which proves our claim.

### 3 Linkages

Algebraically, we use the following definition of a linkage, which is taken from [11]:

**Definition 3.1** Let $n \in \mathbb{N}$. An $n$-linkage is a finite sequence of joints ($j_1$, …, $j_n$) where each joint is represented by an element in $S \setminus E$ in the following way:

If $j_k$ is an $R$-joint it is represented by the map $m_k : \mathbb{P}^1 \to S \setminus E$ such that $t_k \mapsto t_k - h_k$. The element $h_k$ is a fixed unit dual quaternion such that $h_k = p_k + e q_k$ and $h_k^2 = -1$ and is of the form $h_k = (0 : p_k : p_k : q_k : q_k : q_k : k_3)$. Hence, $t_k - h_k = (t_k - p_k : -p_k : -p_k : 0 : -q_k : -q_k : -q_k)$. The parameter $t_k$ corresponds to the angle of rotation $2\arccot(t_k)$.

If $j_k$ is a $P$-joint it is represented by the map $m_k : \mathbb{P}^1 \setminus \{0\} \to S \setminus E$ such that $t_k \mapsto t_k - e p_k$. Hence, $t_k - e p_k = (t_k : 0 : 0 : 0 : -p_k : -p_k : -p_k)$. The parameter $t_k$ corresponds to the extent of translation $-\frac{N(p_k)}{h_k}$.

If $j_k$ is an $H$-joint it is represented by the map $m_k : \mathbb{R} \to S \setminus E$ where $a_k \mapsto (1 - e g_k a_k p_k)(\cot(\frac{g_k}{2}) - h_k)$. The number $g_k \in \mathbb{R}$ is a fixed constant called the pitch and $h_k^2 = -1$.

If $L$ is a linkage, we impose that for each parameter $t_k$, the loop-closure equation.
is satisfied.

The set of all \((t_1, \ldots, t_n)\) satisfying the loop equation (1) for a linkage \(L\) is called the configuration set of \(L\), denoted by \(K_L\). Hence, for instance, a linkage consisting of 6 revolute joints has a configuration space in \((\mathbb{P}^1)^6\). Finally, the joints \(j_k\) and \(j_{k+1}\) are connected via a mechanical part of the linkage which is called a link and is denoted by link \(k+1\). For a closed-loop linkage, the joint \(j_n\) and joint \(j_1\) are connected via the link 1.

**Definition 3.2** The Zariski closure of all the complex configurations of an \(n\)-linkage is an algebraic variety which is denoted by \(\overline{K}_L\), and its dimension is called the mobility.

With this dimension, we mean the complex dimension, and we are more interested in the special complex solutions in \(\overline{K}_L\), which will be introduced in the next section. We say that a linkage \(L\) is mobile if \(\dim \overline{K}_L > 0\) and, in particular, we assume that no joint is frozen, that is, there is no \(t_i\) for which the projection \(\pi : \overline{K}_L \to \pi(\overline{K}_L)\) to \(t_i\) is zero-dimensional. Namely, we always consider the component of \(\overline{K}_L\) with the highest dimension where no joint is frozen. Otherwise, we can treat the linkage as a reduced linkage with fewer joints.

**Remark 3.3** Although we consider the complex dimension for the configuration space of \(L\), its real dimension is the same, provided that there is a real non-singular point in the highest complex dimensional components of the configuration space. If not, the real dimension is less than the complex dimension for the configuration space.

We explain how to associate several linkages to a given linkage \(L\).

**Definition 3.4** [11] Let \(L\) be an \(n\)-linkage.

- We define the spherical projection linkage as the linkage \(L_s\) obtained from \(L\) by the map \(\pi_s : \mathbb{D}\mathbb{H} \to \mathbb{H}\) given by taking dual quaternions modulo \(\epsilon\);
- If \(L\) has at least one helical joint, we associate the linkage \(L_r\) where every \(H\)-joint is replaced by an \(R\)-joint with the same axis. Similarly, we define \(L_p\) where every \(H\)-joint is replaced by a \(P\)-joint with the direction parallel to the axis of the \(H\)-joint.

**Remark 3.5** By Theorem 9 in [11] and the paragraph right after that theorem, the mobility of \(L_r\) (or \(L_p\)) is greater than or equal to the mobility of \(L\).

**Denavit–Hartenberg Parameters and the Absolute Quadric Cone:**

We present geometric conditions of consecutive rotation joints. First, we introduce the set of Denavit–Hartenberg parameters of consecutive rotation joints. For indices \(i = 1, \ldots, n\), let \(l_i\) be the rotation axis of the \(i\)-th joint. The angle \(\alpha_i\) is defined as the angle of the direction vectors of \(l_i\) and \(l_{i+1}\) (with some choice of orientation) and is called the twist angle between \(l_i\) and \(l_{i+1}\). We also set \(c_i := \cos(\alpha_i)\). The number \(d_i\) is defined as the orthogonal distance of the lines \(l_i\) and \(l_{i+1}\) and is called the twist distance between \(l_i\) and \(l_{i+1}\). Note that \(d_i\) may be negative; this depends on some choice of the orientation of rotation axes. We set \(b_i := \frac{d_i}{\sin(\alpha_i)}\) as Bennett ratios (see [32]). Finally, we define the offset \(o_i\) as the signed distance of the two foot points on \(l_i\).

We recall two definitions [33, Definition 3 and Definition 4] as:
Definition 3.6 [33] For a sequence \( h_i, h_{i+1}, \ldots, h_j \) of consecutive joints, we define the coupling space \( L_{i,i+1,\ldots,j} \) as the linear subspace of \( \mathbb{R}^8 \) generated by all products \( h_{k_1} h_{k_2} \cdots h_{k_i}, i \leq k_1 < \cdots < k_j \leq j \). (Here, we view dual quaternions as real vectors of dimension eight.) The empty product is included, its value is 1.

Definition 3.7 [33] The dimension of the coupling space \( L_{i,i+1,\ldots,j} \) will be called the coupling dimension. We denote it by \( l_{i,i+1,\ldots,j} = \dim L_{i,i+1,\ldots,j} \).

Two rotation quaternions with the same axis are called compatible. A collection of rotation quaternions is called concurrent if their axes intersect at a common point or are parallel. Unless stated otherwise, we always assume our axes to be non-compatible.

The following is a crucial geometric arrangement found by [34].

Definition 3.8 Suppose \( h_1, h_2 \) and \( h_3 \) are non-compatible rotation quaternions. We say that the triple \((h_1, h_2, h_3)\) satisfies Bennett conditions if the normal feet of \( h_1 \) and \( h_3 \) on \( h_2 \) coincide and \( b_1 = b_2 \) or \( b_1 = -b_2 \). In this case, we call \((h_1, h_2, h_3)\) a Bennett triple.

Recall the following result:

Theorem 3.1 [33] If \( h_1, h_2, \ldots, h_n \) are rotation quaternions such that \( h_i \) and \( h_{i+1} \) are not compatible for \( i = 1, \ldots, n-1 \), the following statements hold true:

- All coupling dimensions \( l_{1,\ldots,i} \) with \( 1 \leq i \leq n \) are even.
- The equation \( l_{1,2} = 4 \) always holds. Moreover, \( L_{1,2} \subset S \) if and only if the axes of \( h_1 \) and \( h_2 \) are concurrent.
- If \( \dim L_{1,2,3} = 4 \), then the axes of \( h_1, h_2 \) and \( h_3 \) are concurrent.
- If \( \dim L_{1,2,3} = 6 \), then the axes of \( h_1, h_2 \) and \( h_3 \) satisfy the Bennett conditions.

Remark 3.9 Three axes intersecting at a point can be thought of as a degenerate Bennett triple, where the normal distances are zero. Three axes being all parallel can be thought of as a degenerate triple, where the twist angles are zero.

We introduce a useful interpretation of the above results. Any plane in \( \mathbb{R}^3 \) is given by an equation \( a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 = 0 \) which is uniquely determined up to scalar multiplication. The coefficients of this equation are in one to one correspondence with points in \( \mathbb{P}^3^* \), the dual projective space, except for the case when \( a_0 = a_1 = a_2 = 0 \) and \( a_3 = 1 \) which corresponds to the plane at infinity.

We define the dual absolute quadric cone \( A : (a_0^2 + a_1^2 + a_2^2 = 0) \subset \mathbb{P}^3^* \). The set \( A \) comes with an involution \( \gamma : A \to A \) given by \( p = (a_0 : \ldots : a_3) \mapsto p^- = (\overline{a_0} : \ldots : \overline{a_3}) \), where \( \overline{a_i} \) denotes the complex conjugate of \( a_i \). Let \( e = (0 : 0 : 0 : 1) \). Then \( e \in A \) is the only real point of \( A \), its only singular point and is the only point of \( A \) fixed by \( \gamma \).

Let \( R = \{ h \in \mathbb{D}^3 \mid h^2 = -1 \} \) be set of rotation quaternions. Then, \( R \) also comes with an involution \( \tau : R \to R \) given by \( h \mapsto -h \), which coincides with quaternion conjugation. For every line \( l \in \mathbb{R}^3 \), there is a corresponding dual line \( l^* \in \mathbb{P}^3^* \) which intersects \( A \) in two non-real points. Notice that the revolute joints \( h \) and \( -h \) have the same fixed line of rotation. Then there is a 1 to 1 correspondence between the sets
{Points in $A \setminus \text{modulo } \gamma}$ $\leftrightarrow$ {Lines in $\mathbb{R}^3$} $\leftrightarrow$ {Dual quaternions in $R$ modulo $\tau$.}

We explain that this correspondence lifts to

{Points in $A \setminus e$} $\leftrightarrow$ {Oriented lines in $\mathbb{R}^3$} $\leftrightarrow$ {Dual quaternions in $R$}.

The primal part of $h$ induces an orientation on the line of rotation. Hence a rotation quaternion corresponds to an oriented line in $\mathbb{R}^3$. We now explain the first correspondence. Let $p = (a : b : c : d)$ and its complex conjugate $p^-$ be points in $A \setminus e$ corresponding to a line $l$ in $\mathbb{R}^3$. To associate an orientation of $l$ for the point $p$, we take the point $(a, b, c) \in \mathbb{C}^3$. We write

\[
\begin{pmatrix}
 a \\
 b \\
 c \\
 d
\end{pmatrix} = \begin{pmatrix}
 a_1 + a_2 i \\
 b_1 + b_2 i \\
 c_1 + c_2 i \\
 d_1 + d_2 i
\end{pmatrix} = \begin{pmatrix}
 a_1 \\
 b_1 \\
 c_1 \\
 d_1
\end{pmatrix} + \begin{pmatrix}
 a_2 \\
 b_2 \\
 c_2 \\
 d_2
\end{pmatrix} i.
\]

Let $u = a_1 i + b_1 j + c_1 k$ and $v = a_2 i + b_2 + j + c_2 k$. Since $p \in A \setminus e$, it follows that $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$ and $u^2 = v^2$. Therefore, $uv = (b_1 c_2 - c_1 b_2)i + (c_1 a_2 - a_1 c_2)j + (a_1 b_2 - b_1 a_2)k$. The unit quaternion of $uv$ defines an orientation for the line $l$. Let $\lambda = \lambda_1 + \lambda_2 i \in \mathbb{C}^2$. Then $\lambda(u + vi) = (\lambda_1 u - \lambda_2 v) + i(\lambda_2 u + \lambda_1 v)$ and

\[
(\lambda_1 u - \lambda_2 v)(\lambda_2 u + \lambda_1 v) = \lambda_1 \lambda_2 u^2 - \lambda_1 \lambda_2 v^2 + \lambda_1^2 uv - \lambda_2^2 uv = \lambda_1^2 uv - \lambda_2^2 uv = (\lambda_1^2 + \lambda_2^2)uv.
\]

Hence, the orientation is independent of the choice of representative in $\mathbb{P}^3$. Similarly, for $p^-$, we have the orientation defined by the unit vector of $-uv$. Moreover, we can associate an element of $R$ to the point $p$. Indeed, we associate to $p$ the element $h := uv - e(d_1 v - d_2 u)$. Notice that with a different representative of $p$, the dual quaternion $h$ only differs by a positive scalar. Hence, there is a positive scalar $\mu = \sqrt{u^2 v^2}$ for which $\frac{h}{\mu} \in R$.

**Lemma 3.2** The orbit of $h_1$ around $h_2$ is contained in the plane spanned by $p_1, p_2, p_2^-$ in $A \setminus e$.

**Proof** To prove this claim, we may assume that $p_2 = (1 : i : 0 : 0)$. Then the action of rotation by the angle $\alpha$ to $p_1 = (x : y : z : w)$ gives $(\cos(\alpha)x - \sin(\alpha)y : \sin(\alpha)x + \cos(\alpha)y : z : w)$ -- this is just the action of the transpose of the projective rotation matrix around the third coordinate axis. Hence the orbits are contained in planes with an equation $\mu z - \lambda w = 0$ for some suitable $\lambda, \mu$, and these are planes through $p_2, p_2^-$. \qed

**Lemma 3.3**

1. The axes of joints $h_1$ and $h_2$ are concurrent if and only if the four points $p_1, p_1^-, p_2$ and $p_2^-$ are on a conic in $A$.
2. The joints $h_1$, $h_2$ and $h_3$ form a Bennett triple with $b_1 = b_2$ if and only if $p_1, p_2, p_2^-, p_3$ are on a conic in $A \setminus e$. Similarly, $h_1$, $h_2$ and $h_3$ form a Bennett triple with $b_1 = -b_2$ if and only if $p_1^-, p_2, p_2^-, p_3$.
3. If the joints $h_1$, $h_2$, $h_3$ form a Bennett triple and the joints $h_1$, $h_2$, $h_4$ form a Bennett triple, then joints $h_4$, $h_2$, $h_3$ form a Bennett triple.

**Proof**

1. The axes of $h_1$ and $h_2$ span a plane in $\mathbb{P}^3$. This plane is a point in the dual space contained in the dual lines $l_1^*$ and $l_2^*$. Hence, $l_1^*$ and $l_2^*$ also span a plane. The intersection of this plane with $A \setminus e$ is a conic that contains in particular $p_0, p_1, p_2$ and $p_2^-$. When the two axes are parallel, the conic is degenerate and it consists of two lines passing through the vertex $e$.

2. We translate the lemma in coordinates assuming $b_1 = b_2$. The case $b_1 = -b_2$ is analogous. The statements remain unchanged if we apply an Euclidean isometry an all 3 lines $h_1$, $h_2$ and $h_3$ simultaneously. Therefore, we may assume w.l.o.g. that $h_2$ is the first axis and that the common normal between $h_1$ and $h_2$ is the third axis. The point $p_2$ corresponding to $h_2$ is $(0 : 1 : i : 0)$, and the point $p_2^*$ corresponding to $h_2$ with reversed orientation is $(0 : 1 : -i : 0)$. Assume $h_1$ corresponds to the point $p_1 = (x_1 : y_1 : z_1 : w_1)$ and that $h_3$ corresponds to the point $p_3 = (x_3 : y_3 : z_3 : w_3)$. Note that $x_1^2 + y_1^2 - z_1^2 = x_3^2 + y_3^2 + z_3^2 = 0$. Because the common normal between $h_1$ and $h_2$ is the third axis, we can choose projective coordinates such that $x_1, y_1$ are purely imaginary (i.e. real multiplies of $i$) and $z_1$ is real. The orbit of rotation of $h_3$ around $h_2$ is contained in the plane spanned by $p_2$, $p_2^*$ and $p_3$ by Lemma 3.2. Hence the first statement in the lemma is not changed if we apply a rotation around $h_2$ to $h_3$. Therefore, we may assume w.l.o.g. that the common normal between $h_2$ and $h_3$ is parallel to the third axis and that $x_3, y_3$ are purely imaginary and $z_3$ is real. We now compute the angles between $l_2$, $l_1$ and $l_2$, $l_3$. W.l.o.g., we may assume $x_1^2 + y_1^2 = 1$, $z_1 = i$. If $\alpha_1$ is the angle between $l_2$ and $l_1$, then we have $x_1 = \sin(\alpha_1)$, $y_1 = \cos(\alpha_1)$. Similarly, if $\alpha_2$ is the angle between $l_2$ and $l_3$, then $x_3 = \sin(\alpha_2)$, $y_3 = \cos(\alpha_2)$. Computation of distances and the offset: the normal distance $d_1$ between $h_2$ and $h_1$ is the imaginary part of $w_1$ divided by $i$. Similarly, the normal distance $d_2$ between $h_2$ and $h_3$ is the imaginary part of $w_3$ divided by $i$. The real part of the $w_1$ is zero, and the real part of $w_3$ is the offset $o$. Hence we have $w_1 = d_1 i$ and $w_3 = d_2 i + o$. Collecting everything about $p_1$, $p_2$ and $p_3$, we get:

$$p_1 = (\sin(\alpha_1) : \cos(\alpha_1) : i : d_1 i)$$
$$p_2 = (0 : 1 : i : 0)$$
$$p_3 = (\sin(\alpha_2) : \cos(\alpha_2) : i : d_2 i + o).$$

Now it is clear that the first statement of the lemma is equivalent to

$$x_1 w_3 - x_3 w_1 = \sin(\alpha_1) o + (\sin(\alpha_1)d_2 - \sin(\alpha_2)d_1)i = 0$$

and this is equivalent to the second statement.

3. This follows directly from the definition of Bennett triples.

---

**Lemma 3.4** Let $L$ be an $nR$-linkage with mobility $m \geq 2$ and joints $\{h_1, \ldots, h_n\}$. Suppose that freezing joint $h_n$ in a general position gives a linkage $L'$ such that $\{h_{n-1}, h_1, h_2\}$ form a Bennett triple. Then $\{h_{n-1}, h_n, h_1, h_2\}$ are concurrent.
Proof The rotation around \( h_n \) is a one-parameter subgroup of Euclidean congruence transformations, which acts on the set of lines and therefore also on the absolute cone. Every orbit is contained in a plane passing through \( p_n \) and \( p_n^{-} \) by Lemma 3.2. It follows that the orbit of \( p_{n-1} \) is contained in the plane spanned by \( p_{n-1}, p_n, p_n^{-} \). For almost every point \( x \) in the orbit, we also have that the line corresponding to \( x \) together with \( h_1, h_2 \) satisfies Bennett’s condition. By Lemma 3.3, this implies that \( x \) is contained in the plane spanned by \( p_1, p_2, p_2^{-} \). But the orbit of \( p_{n-1} \) does span a unique plane, and hence this unique plane must contain the points \( p_{n-1}, p_n, p_n^{-}, p_1, p_2, p_2^{-} \). Coplanarity of \( p_n, p_n^{-}, p_1, p_2, p_2^{-} \) is equivalent to the lines \( h_n \) and \( h_{n-1} \) being coplanar. If three lines \((l_1, l_2, l_3)\) form a Bennett triple, and \( l_1, l_2 \) are coplanar, then it follows all three lines are concurrent. In our situation, this implies that all four lines \( \{h_{n-1}, h_n, h_1, h_2\} \) must be concurrent. \(\square\)

4 Bond theory

In this section, we define bonds associated to a linkage. We use it only for the results in Sect. 6.

Recall that a linkage is a sequence of joints \((j_1, \ldots, j_n)\), see Definition 3.1.

**Definition 4.1** Let \( L \) be a closed-loop linkage and \( S \) a subsequence of \( L \).

- We define the **length** of \( S \), denoted by \(|S|\), to be the number of elements of \( S \);
- We say that \( S \) forms a **chain** if it is a finite sequence of consecutive joints of \( L \).

Let \( L \) be the linkage \((j_1, \ldots, j_n)\) with \( n \) joints. To a chain \( S = (j_r, \ldots, j_s) \), where indices are taken mod \( n \), we associate the map \( \Phi_S : K_L \rightarrow \mathbb{C}^n \) given by \( m_r(t) \cdots m_s(t) \). We define the complementary chain to \( S \), \( \bar{S} = (j_{s+1}, \ldots, j_{r-1}) \) with indices taken modulo \( n \). The loop equation Equation 1 can be written as:

\[
\Phi_S(t) \Phi_{\bar{S}}(t) \in \mathbb{C}^n \quad \forall \ t \in K_L.
\]

Moreover,

\[
\Phi_{\bar{S}}(t) = m_r(t)m_{r+1}(t) \cdots m_{s-1}(t)m_s(t).
\]

The images of the maps \( \Phi_S \) and \( \Phi_{\bar{S}} \) correspond to the relative motion between the link \( r \) and link \( s + 1 \). We want to point out that it is also very interesting to look at what is the motion behaviour at the compactification of \( SE(3) \), see [33], i.e. the complex solutions of \( \overline{K}_L \): We extend \( \Phi_S \) to \( \overline{K}_L \) and define the **set of bonds** of \( L, B \), as

\[
B = \{ t \in \overline{K}_L \mid m_1(t) \cdots m_n(t) = 0 \}.
\]

The following lemma, which is adapted from [33], is a consequence of the Affine Dimension Theorem [35, Proposition I.7.1,] and is the basic consequence of higher mobility from the perspective of bonds:

**Lemma 4.1** The set \( B \) is an algebraic hypersurface of \( \overline{K}_L \). Moreover, its complex dimension is one less than the mobility of the linkage.
Each bond component of an \( m \)-dimensional irreducible component of \( \mathcal{K}_L \) has dimension \( m - 1 \). In this paper, we always consider such irreducible components where no joint is frozen. There might be lower dimensional components in \( \mathcal{K}_L \).

**Lemma 4.2** Suppose \( C \subset \mathbb{P}^1 \times \mathbb{P}^1 \) is a curve given by
\[
(i - h_i)(t_1 - h_{s+1})(t_2 - h_{s+2})(i - h_{s+3}) = 0
\]
where \( h_i \) are non-parallel R-joints. Then, there is a Bennett triple between \((h_s, h_{s+1}, h_{s+2})\) or \((h_{s+1}, h_{s+2}, h_{s+3})\).

**Proof** Since \( C \) is a curve, there is a surjective map \( C \to \mathbb{P}^1 \) to one of the components, say, to the first one. In particular, the pre-image of \( i \in \mathbb{P}^1 \) is \((i, t)\) for some fixed \( t \in \mathbb{P}^1 \). That is, we have, over \( i \),
\[
(i - h_i)(i - h_{s+1})(t - h_{s+2})(i - h_{s+3}) = 0.
\]
Let \( a = i - h_i \), \( b = i - h_{s+1} \), \( c = (t - h_{s+2})(i - h_{s+3}) \). Since \( h_s \) and \( h_{s+1} \) are not parallel, \( ab \neq 0 \) and \( ab \) is not a quaternion multiple of \( c \). By the extended abc-lemma Lemma 2.3, it follows that \( (i - h_{s+1})(t - h_{s+2})(i - h_{s+3}) = 0 \). Hence, we have \( \dim L_{s+1,s+2,s+3} = 4 \) or \( \dim L_{s+1,s+2,s+3} = 6 \). Since the axes are not parallel, by Theorem 3.1, this is equivalent to a Bennett condition between \((h_{s+1}, h_{s+2}, h_{s+3})\).

The following definition is adapted from [33, 36]:

**Definition 4.2** Let \( \beta \) be a bond of \( L \). Recall that \( \beta \) is attached to a joint \( j_k \) if \( N(m_k(\beta)) = 0 \), where \( m_k = t_k - ep_k \) if \( j_k \) is a P-joint and \( m_k = t_k - h_k \) if \( j_k \) is an R-joint. We call \( A_\beta \) the subsequence of joints to which \( \beta \) is attached. Hence, \( \beta \in B \) if and only if there is a \( k \) such that \( j_k \in A_\beta \).

It was proven in [36] that the mobility of a specific joint in a linkage is equivalent to the existence of a bond attached to that joint. Hence, for a mobile linkage, there is always some \( \beta \) for which \( A_\beta \) is non-empty. In fact, it is known that \( |A_\beta| \geq 2 \) in general. See [33]. Furthermore, we have the following result:

**Lemma 4.3** Let \( L \) be an \( nR \)-linkage and suppose that \( A_\beta \) is a chain. Then there is at least one pair of parallel joints.

**Proof** Without loss of generality we can assume \( A_\beta = (j_1, \ldots, j_r) \). Then,
\[
\Phi_{A_\beta}(\beta) = (i - h_1) \cdots (i - h_r) = 0,
\]
where \( h_i = p_i + eq_i \) is a rotation dual quaternion, that is, \( h_i^2 = -1 \). Multiplying by \( e \) we get \((i - p_1) \cdots (i - p_r) = 0 \). By repeated usage of the \( abc \)-lemma there is an index \( s \) for which \((i - p_s)(i - p_{s+1}) = 0 \) which implies that \( p_s \) and \( p_{s+1} \) are parallel. \( \square \)

We can see these parallel properties from the following example (see Fig. 1) which is a concrete known mobile \( P4R \)-linkage taken from [32, 37].
Example 4.3 This is an example of a $P4R$-linkage with two pairs of parallel neighbouring axes. It is constructed using the geometry constraints from [32, 37]. Five configurations of the linkage are shown in Fig. 1. Set

$$h_0 = \epsilon j + \epsilon k, \quad h_1 = k + \epsilon j, \quad h_2 = k + \epsilon i, \quad h_3 = j + \epsilon i + 2\epsilon k, \quad h_4 = j + \epsilon k.$$ 

Both sides of

$$(t_0 - h_0)(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) \equiv 1$$

are written as 8-dimensional vectors, which yields a system of 7 equations. The 5th coordinate is a redundant condition because it is already satisfied when the other six equations are fulfilled due to the Study condition [29]. In order to exclude “unwanted” solutions, that is, those such that

$$t_0(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1) = 0$$

in general we add an extra equation

$$t_0(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)u - 1 = 0,$$

Using Gröbner basis on Maple, an elimination ideal with respect to $u$ (the Rabinowitsch trick, [38]) is

$$I = \langle t_1 - t_3, t_1 + t_2, t_1 + t_4, t_0t_1 + t_1^2 + t_0 + 1 \rangle$$

which defines the closure of the configuration set $K_L$. The set of bonds are then defined by adding the equation $t_0(t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1) = 0$. They are

$$\{t_0 = 0, t_1 = i, t_2 = -i, t_3 = i, t_4 = -i\}$$

and its conjugate, where $i$ is the complex imaginary unit.

Fig. 1 Five configurations of a $P4R$-linkage
5 n-linkages with mobility at least $n - 4$

In this section, we explore the geometric constraints that arise from requiring higher mobility in a linkage, i.e. mobility at least $n - 4$ for an $n$-linkage. When we freeze one joint of a linkage $L$ of mobility $m$, we always do it at a generic position. In this case, the mobility of the linkage reduces by 1 since we are adding one equation not in the ideal defining $K_L$. Equivalently, we have a new linkage $L'$ whose mobility is $m - 1$.

The structure of this section is as follows: We begin by looking at the case of 6R-linkages in Sect. 5.1. This is Theorem 5.5. We then analyse 6-linkages which include $P$ and $H$-joints in Sect. 5.2. These are Theorem 5.7 and Theorem 5.9, respectively. We extend this analysis to $n$-linkages where $n > 6$ in Sects. 5.3 and 5.4. For $nR$-linkages see Theorem 5.10, for $n$-linkages with $P$-joints Theorem 5.12 and for $n$-linkages with $H$-joints Theorem 5.13.

**Proposition 5.1** The maximum possible mobility of an $n$-linkage where $n \geq 4$ is $n - 3$.

**Proof** By the classification of 4-linkages, see [10], this is well known. We proceed by strong induction on $n$. Let $L$ be a linkage with mobility $m$ and $n$ joints, with $m \geq n - 2$. If we freeze one joint, then we get a linkage $L'$ of mobility $m - 1$ and $n - l$ active joints, where $l \geq 1$. This is a contradiction by the induction hypothesis. $\square$

**Proposition 5.2** A closed $n$-linkage with $n \geq 5$ and $P$ or $R$ joints has mobility $n - 3$ if and only if the motions of all joints are in one of the two three dimensional groups, $SE(2)$ and $SO(3)$.

**Proof** We prove the result by induction on $n$. If $n = 5$, the result is well known, see [11]. We proceed by strong induction on $n$. Let $L$ be a linkage with mobility $m$ and $n$ joints, with $m = n - 3$. If we freeze one joint, then we get a linkage $L'$ of mobility $m - 1$ and $n - l$ active joints, where $l \geq 1$. By the induction hypothesis and Proposition 5.1, it follows that $l = 1$. Moreover, all the motions of joints of $L'$ are in $SE(2)$ and $SO(3)$. The motion of the joint we froze must also be in the same group due to the closure condition. $\square$

**Remark 5.1** For $n = 4$, the statement is not true by several examples: The Bennett 4R-linkage is one of them.

**Example 5.2** Consider an $nR$-linkage, $L$, where all revolute axes are parallel. All the joint motions are in $SE(2)$; therefore, the mobility of $L$ is $n - 3$. Let $L'$ be a new linkage formed by replacing the $R$-joints by $H$-joints with the same pitch. We claim that there is an isomorphism of configurations of $L$ and $L'$, therefore the mobility is the same. If we have a configuration of $L'$, then we can ignore all the translations perpendicular to the plane where all the motions are taking place and this gives a configuration of $L$. In the other direction, we add translations perpendicular to this plane, proportional to the rotation angles. A priori, it is not clear that these translations add up to zero. But they do, since the sum of all rotation angles in $L$ is constant.

A closed linkage has mobility at least $n - 4$ if the motions of all joints are in a Schönflies subgroup of $SE(3)$ which has dimension 4, see [18–20]. However, the converse is not true. For instance, the Goldberg 5R is one counter example.
Recall that if \( L \) has two neighbouring \( R \)-joints with equal axes or two neighbouring \( P \)-joints with equal directions, then we say that \( L \) is \textit{degenerate}. We also assume that \( L \) is not degenerate. No joint parameters are constant during the linkage’s motion (the highest dimensional components of the configuration space) (otherwise, one could easily make \( n \) smaller). We say that a \( P \)-joint is \textit{perpendicular} to an \( R \)-joint if the direction of the \( P \)-joint is perpendicular to the orientation vectors of the \( R \)-joint.

Throughout this section, we use the following fact when we freeze one joint of an \( n \)-linkage with mobility \( n-4 \).

\textbf{Lemma 5.3} \textit{Let} \( L \) \textit{be an} \( n \)-\textit{linkage, where} \( n > 5 \), \textit{of mobility} \( n-4 \). \textit{If we freeze one joint of} \( L \), \textit{then there is at most one more joint being frozen simultaneously in the resulting linkage} \( L' \).

\textbf{Proof} \textit{Recall that we freeze a joint of} \( L \) \textit{at a generic position. If there are other two or more joints being frozen simultaneously, then the resulting linkage} \( L' \) \textit{has} \( j \leq n-3 \) \textit{joints with mobility} \( n-5 \) \textit{which is impossible}. \( \square \)

\subsection*{5.1 6R-linkages}

In this section, we classify paradoxical 6R-linkages with mobility at least 2. It is known that all axes of a 6R-linkage with mobility at least 3 are concurrent: This is a spherical linkage or a planar linkage. If a 5R-linkage has mobility 2, then, by Proposition 5.2, it is either spherical or planar. If a 4R-linkage has mobility 1, then, by [10], it can be spherical, planar or Bennett. For a Bennett 4R-linkage, its Denavit–Hartenberg parameters fulfil:

\begin{align*}
    b_1 &= b_2 = b_3 = b_4, \\
    c_1 &= c_3, \quad c_2 = c_4, \\
    o_1 &= o_2 = o_3 = o_4 = 0.
\end{align*}

If a 5R-linkage has mobility 1, then, by [33, 39], it is a Goldberg 5R-linkage. For a Goldberg 5R-linkage, its Denavit–Hartenberg parameters fulfil:

\begin{align*}
    b_1 &= b_2 = b_3 = b_4, \\
    c_1 &= c_4, \\
    o_2 &= o_3 = o_4 = 0,
\end{align*}

and some further complicated equational conditions which are not used in this paper, see [16]. We give the classification of paradoxical 6R-linkage of mobility 2 in the following lemma.

\textbf{Lemma 5.4} \textit{Let} \( L = (j_1, j_2, j_3, j_4, j_5, j_6) \) \textit{be a} 6R-\textit{linkage with mobility} 2. \textit{Suppose we freeze joint} \( j_1 \) \textit{at a generic position, and} \( j_2 \) \textit{gets frozen as a consequence. Then the relative position of} \( j_3 \) \textit{and} \( j_6 \) \textit{does not change and the resulting linkage is a} Bennett 4R-\textit{linkage}.

\textbf{Proof} \textit{Let} \( L' \) \textit{be the linkage resulting from freezing joint} \( j_1 \). \textit{We move} \( L \) \textit{and freeze} \( j_1 \) \textit{in another position. Again,} \( j_2 \) \textit{gets frozen and we call the resulting} 4R-\textit{linkage} \( L'' \). \textit{Then,} \( L' \)
and $L''$ have the same Denavit–Hartenberg parameters since they are both mobile 4R-linkages with partially coinciding parameters. If $L'$ is not a Bennett linkage, then $j_3, j_4, j_5, j_6$ are concurrent.

**Case I** $j_3, j_4, j_5, j_6$ intersect at a common point. Since we can replace these four joints by a spherical joint (S), the resulting linkage, denoted by $\tilde{L}$, is a 2RS-linkage with mobility at least 1 because, otherwise, joints $j_1$ and $j_2$ would not move. Call the centre of the spherical joint $o$. We can compute the orbit of $o$ in the frame of the link with axes $j_1$ and $j_2$. If $o$ is not on the joint $j_1$, then the orbit is a circle around $j_1$ and similarly for $j_2$. These two circles must coincide; otherwise, the 2RS-linkage would not be mobile and we conclude that $L$ is a degenerate linkage. On the other hand, if $o$ is on $j_1$, then all axes are concurrent or one joint does not move.

**Case II** $j_3, j_4, j_5, j_6$ are parallel By spherical projection, $j_1$ and $j_2$ are also parallel. Suppose $j_1$ and $j_3$ are not parallel. In that case, if we freeze joint $j_4$ of $L$, the resulting linkage has two pairs/triples of parallel axes which is not possible due to the classification of 4R and 5R-linkages. Therefore, $j_1$ and $j_3$ are parallel. Moreover, the mobility of $L$ needs to be three which contradicts our assumption.

\[\square\]

**Theorem 5.5** Let $L = (j_1, j_2, j_3, j_4, j_5, j_6)$ be a 6R-linkage with mobility 2. Then its Denavit–Hartenberg parameters are: all Bennett ratios are the same, i.e. $b_s = b_{s+1}$, all offsets are zero, i.e. $c_s = 0$, and the cosines $c_s$ of twist angles $\alpha_s$ fulfil one of the following two cases (by a cyclic shift of indices):

1. $c_1 = c_4$, $c_2 = c_6$, $c_3 = c_5$. This is a composition of two Bennett Linkages and one common link and two joints.
2. $c_1 = c_4$, $c_2 = c_5$, $c_3 = c_6$.

**Proof** Fix a 2-dimensional irreducible component of $K_L$. Suppose we freeze one joint. Since $L$ has mobility 2, we still have a mobile linkage $L'$. The number of joints in $L'$ is either 4 or 5 by Theorem 5.3.

**Case I** $L'$ has 4 mobile joints We can assume that the other frozen joint is $j_1$. As the resulting linkage $L'$ is a 4R-linkage, $L'$ must be a Bennett linkage or a linkage with four concurrent axes. It is not possible that another non-neighbour joint gets frozen, because, by Lemma 3.4, all six axes must be concurrent which contradicts the fact that the mobility is 2. Therefore, it is only possible that one of the neighbours of $j_1$ is frozen and we can assume it is joint $j_2$. Then, by Lemma 5.4, the relative position of axes of joints $j_3$ and $j_6$ does not change when we move $L$ and $L'$ is a Bennett linkage. Therefore we can introduce an extra link connecting $j_3$ and $j_6$ and the linkage $L$ is the composition of two Bennett linkages stacked on top of each other, having one link and two joints in common. See Fig. 2 for an example.

**Case II** $L'$ has 5 mobile joints The linkage $L'$ becomes a Goldberg 5R-linkage. The equational conditions of case (2) can be obtained by cyclically freezing joint by joint using the Denavit–Hartenberg parameter constraints in Eq. 3.

\[\square\]

We show the following numeric example of a 6R-linkage of mobility 2.

**Example 5.3** Here is an example of a 6R-linkage without any parallel neighbouring axes with mobility 2. It is constructed using factorization of motion polynomials as in [22].
fact, it is a combination of two Bennett 4R-linkages which have two consecutive rotational joints in common. Set

\[
\begin{align*}
  h_1 &= i, \\
  h_2 &= \left(\frac{1}{3} + \frac{4}{9} \varepsilon \right)i + \left(\frac{2}{3} + \frac{2}{9} \varepsilon \right)j - \left(\frac{2}{3} - \frac{4}{9} \varepsilon \right)k, \\
  h_3 &= \left(\frac{41}{105} + \frac{4288}{11025} \varepsilon \right)i + \left(\frac{88}{105} - \frac{16}{11025} \varepsilon \right)j - \left(\frac{8}{21} - \frac{872}{2205} \varepsilon \right)k, \\
  h_4 &= \left(\frac{33}{35} + \frac{68}{1225} \varepsilon \right)i - \left(\frac{6}{35} - \frac{274}{1225} \varepsilon \right)j - \left(\frac{2}{7} - \frac{12}{245} \varepsilon \right)k, \\
  h_5 &= \left(\frac{1093}{1365} \varepsilon + \frac{313072}{1863225} \right)i - \left(\frac{52}{105} \varepsilon + \frac{16}{11025} \right)j + \left(\frac{92}{273} \varepsilon - \frac{149572}{372645} \right)k, \\
  h_6 &= \left(\frac{29}{39} + \frac{340}{1521} \varepsilon \right)i - \left(\frac{2}{3} - \frac{2}{9} \varepsilon \right)j + \left(\frac{2}{39} - \frac{536}{1521} \varepsilon \right)k.
\end{align*}
\]

We have that the four joints defined by \( h_1, h_2, h_3, h_4 \) are a Bennett 4R-linkage. At the same configuration the \( h_1, h_5, h_6, h_4 \) are another Bennett 4R-linkage [22]. Using Gröbner basis from Maple, an elimination ideal for the Zariski closure \( K_L \) (See [28]) is found to be

![Fig. 2 Five configurations of the 6R-linkage in Example 5.3](image-url)
\[ I = \langle 2t_2 + 2t_3 + 1, t_6 + t_5 + 1, t_1t_5 + t_3t_5 + 1, t_4t_5 - t_3 + 3t_4 - t_5 - 1, t_1t_3 + 2t_1t_5 + 2t_3t_4 + 2t_4t_5 - 2t_3 + 3t_4 - t_5 - 1, 2t_1t_3 + 2t_1^2t_4 - t_1t_5 + 3t_1t_4 - 2t_3^2 + 3t_3t_4 + t_1 - t_3 + 2t_4 - 1 \rangle. \]

By adding the equation \((t_1^2 + 1)(t_2^2 + 1)(t_3^2 + 1)(t_4^2 + 1)(t_5^2 + 1) + 1 = 0\) and using primary decomposition, the 1-dimensional bonds are defined by the ideals:

\[
B_1 = \langle t_1t_3 - t_1t_6 - t_1 - t_3t_6 - t_3 - 1, t_2 + t_3 + 1/2, t_4 + t_6, t_5 + t_6 + 1, t_6^2 + 1 \rangle,
\]

\[
B_2 = \langle t_1t_6^2 + 3t_1t_6 + 13/4t_1 + 4t_6^2 + 2t_4t_6 + 2t_4 - 1/2t_6^2 - 5/4t_6 - 1/4, t_2 + t_4, t_4t_6 - t_1t_6 - 3/2t_1 - t_4t_6 - t_4 + 1/2t_6 - 1/2, t_3 - t_4 + 1/2, t_4^2 + 1, t_5 + t_6 + 1 \rangle,
\]

\[
B_3 = \langle t_1 + 1/3t_6 - 1/3, t_2 + t_6, t_3 - t_6 + 1/2, t_5 + t_6 + 1, t_6^2 + 1 \rangle,
\]

\[
B_4 = \langle t_1t_6^2 + 3t_1t_6 + 13/4t_1 + 4t_6^2 + 2t_4t_6 + 2t_4 - 1/2t_6^2 - 5/4t_6 - 1/4, t_2 + t_4, t_4t_6 - t_1t_6 - 3/2t_1 - t_4t_6 - t_4 + 1/2t_6 - 1/2, t_3 - t_4 + 1/2, t_4^2 + 1, t_5 + t_6 + 1 \rangle,
\]

\[
B_5 = \langle t_1 + t_6 - 1, t_2 + t_3 + 1/2, t_3t_4 + t_3t_6 - t_4t_6 + 1/2t_4 + 1/2t_6 + 1, t_3t_6^2 + t_3 - t_4t_6^2 + t_4t_6 - 1/2t_4^2 - 1/4t_6 - 1/2, t_4t_6 - t_1t_6 + 5/4t_4^2 - t_1t_6^2 + 1/2t_4t_6 + t_4 + 5/4t_6^2 + t_6 + 1, t_5 + t_6 + 1 \rangle,
\]

\[
B_6 = \langle t_2 + t_6 + 3/2, t_3 - t_6 - 1, t_4 - 1/3t_6 - 2/3, t_5 + t_6 + 1, t_6^2 + 2t_6 + 2 \rangle.
\]

Five configurations of the mobility 2 linkage are shown in Fig. 2.

**Example 5.4** There is a cheap way to obtain another linkage when the original linkage has a Bennett triple. This is called the isomeric mechanism by [40]. Namely, one can replace a joint axis (which is the middle one of a Bennett triple) by another new axis where the new axis and the original three axis make a Bennett loop. There are six consecutive Bennett triples in Example 5.3. Using the isomerization for some consecutive Bennett triples, we will obtain 6R-linkages with frozen joints which are not mainly interesting here. One can obtain other three isomeric mechanisms by replacing one (or two) of the joints \(h_1\) and \(h_4\) by new joint axes \(h'_1\) and \(h'_4\)

\[
h'_1 = \left( \frac{101}{117} + \frac{112}{1521}i \right) + \frac{4}{9}j - \left( \frac{28}{117} - \frac{404}{1521}i \right)k,
\]

\[
h'_4 = \left( \frac{3301}{4095} + \frac{240628}{1863225}i \right) + \left( \frac{86}{315} + \frac{274}{1225}i \right)j - \left( \frac{430}{819} - \frac{117232}{372645} \right)k.
\]

With this replacement, we have no Bennett 4R-linkage for any consecutive four joints.

### 5.2 6-linkages

In this section, we classify 6-linkages of mobility 2 or higher containing revolute joints and at least one prismatic or helical joint. Mobile linkages with 4 joints of type \(R, P\) or \(H\) have been classified in [10]. Mobile linkages with 5 joints of type \(R, P\) or \(H\) have been classified in [11].

The following result settles the case for 5-linkages with \(P\)-joints and \(R\)-joints:
Theorem 5.6 [11, Theorem 6] Let \( L \) be a mobile 5-linkage with at least one \( P \)-joint and all other joints of type \( R \). Then the following two cases are possible.

1. Up to cyclic shift, \( j_0 \) is the only \( P \)-joint, \( h_1 \parallel h_2 \) and \( h_3 \parallel h_4 \), and \( t_1 = \pm t_2 \) and \( t_3 = \pm t_4 \) is fulfilled on the configuration curve.
2. All axes of \( R \)-joints are parallel.

Remark 5.5 Let \( L \) be a 5-linkage with at least one \( P \)-joint and all other joints are parallel of type \( R \). If the mobility of \( L \) is 2 then all \( P \)-joints are perpendicular to the \( R \)-joints. If the mobility of \( L \) is 1 then there are at least two \( P \)-joints which are not perpendicular to the \( R \)-joints.

Complementing Theorem 5.5, we solve the classification of linkages with 6 joints of mobility 2 with joints type \( R \) and \( P \):

Theorem 5.7 Let \( L \) be a 6-linkage of mobility 2 with at least one \( P \)-joint and all other joints of type \( R \). Then one of two following cases occur:

1. All axes of \( R \)-joints are parallel and \( L \) has at least two \( P \)-joints.
2. Up to cyclic shift, \( j_0 \) and \( j_3 \) are the only two \( P \)-joints, \( h_1 \parallel h_2 \), \( h_5 \parallel h_4 \), \( p_0 \parallel p_3 \), and \( t_1 = \pm t_2 \) and \( t_5 = \pm t_4 \) is fulfilled on the configuration curve.

Proof We make a case distinction based on the number of \( P \)-joints.

Case I \( L \) has one \( P \)-joint We claim that this case is not possible. We prove the claim by finding contradictions. Since \( L \) has mobility 2, we can find an \( R \)-joint such that the \( P \)-joint is still active when we freeze this \( R \)-joint. If we freeze such an \( R \)-joint, we still have a mobile linkage \( L' \). We now have two possibilities. Either no other joint is frozen, or another \( R \)-joint gets frozen. If no other joint is frozen, then the resulting linkage is a \( P4R \)-linkage of mobility 1. Moreover, by the classification in Theorem 5.6, we have two pairs of parallel joints, or all axes are parallel by Theorem 5.6 and Remark 5.5.

- All axes of \( R \)-joints are parallel. Then all axes of \( R \)-joints in \( L \) are parallel using the spherical projection. Moreover, \( L \) has mobility two if and only if the \( P \)-joint is frozen because, otherwise, the mobility is three by Proposition 5.2 which contradicts the fact that all joints of \( L \) are movable.
- The \( L \) has exactly two pairs of parallel joints. Therefore, we can freeze the \( P \)-joint of \( L \), resulting in a \( 5R \)-linkage with two pairs of parallel joints in different directions which is not possible by the classification of \( 5R \)-linkages [33, 39].

On the other hand, if another \( R \)-joint is frozen, then the resulting mobile 4-linkage \( L' \) will become a 4-linkage with at least one \( P \)-joint. Therefore, all axes of \( R \)-joints are parallel in \( L' \) by [10]. In addition, if we freeze the \( P \)-joint, all axes of \( R \)-joints are parallel in \( L \). Hence, the \( P \)-joint must be perpendicular to the \( R \)-joints. Then the mobility of \( L \) needs to be three, which contradicts our assumption.

Case II \( L \) has two \( P \)-joints We freeze one \( P \)-joint (say, joint \( j_6 \)) and call the resulting linkage \( L' \). If the other \( P \)-joint is frozen too, then the resulting \( 4R \)-linkage is mobile when all axes are parallel. A Bennett or spherical \( 4R \)-linkage is not possible because freezing a \( P \)-joint can change the normal feet or twist distances. If the other \( P \)-joint is active, then we
claim that the resulting mobility of one linkage $L'$ has five active joints. In other words, we have two pairs of neighbouring parallel joints in $L'$ by Theorem 5.6 and Remark 5.5. Therefore, in this case, we have two possibilities, either all axes of $R$-joints are parallel, or there are two pairs of parallel $R$-joints as in Theorem 5.6 (see Fig. 1). In addition, we know that the $P$-joints are not perpendicular to the $R$-joints in $L'$, and the same holds for $L$. We prove the claim by contradiction. Assume that if we freeze the joint $j_6$, another $R$ joint is frozen simultaneously. Then the resulting $L'$ is a mobile 4-linkage with a $P$-joint by Lemma 5.3. The $P$-joint is perpendicular to the $R$-joints in $L'$ the by the classification of 4-linkages. The axes of $R$-joints of $L$ must be parallel by spherical projection. In addition, we have the contradictions that the mobility of $L$ is 3 when the joint $j_6$ is active, perpendicular to the $R$-joints, or the joint $j_6$ is not an active joint in $L$. Furthermore, we claim that the two $P$-joints have the same direction when there are two pairs of parallel $R$-joints in $L$ (see Fig. 3). Otherwise, if we freeze an $R$-joint in $L$ and denote the resulting linkage by $L''$, then a neighbouring $R$-joint must be frozen because of two pairs of parallel $R$-joints. Hence, the two remaining $R$-joints must generate a circular translation, and the resulting linkage is a 4-linkage with two $P$-joints. Then the $P$-joints are perpendicular $R$-joints in $L''$, which is a contradiction.

**Case III** $L$ has $k = 3$ or more $P$-joints Then all rotation axes are parallel. Because the spherical projection of $L$ with $6 - k \leq 3$ revolute joints is degenerate: all axes are coinciding. Therefore, the mobility of $L$ is 2 when at least two $P$-joints are not perpendicular to the $R$-joints.

**Remark 5.6** Let $L$ be a 6-linkage of mobility 2 or 3 with at least one $P$-joint and all other joints are parallel $R$-joints. If the mobility of $L$ is 3 then all $P$-joints are perpendicular to the $R$-joints. If the mobility of $L$ is 2 then there are at least two $P$-joints which are not perpendicular to the $R$-joints. Compare with Remark 5.5.

Mobile linkages with 5 joints of type $R$, $P$ or $H$ have been classified in [11].

---

**Fig. 3** Five configurations of a $PRRPRR$-linkage
**Theorem 5.8** [11, Theorem 10] Let $L$ be a non-degenerate mobile 5-linkage with $R$-, $P$-, and $H$-joints, with at least one $H$-joint. Up to cyclic permutation, the following cases are possible.

1. All axes of $R$- and $H$-joints are parallel.
2. There is one $P$-joint $j_0$, all other joints are of type $H$ or $R$, $h_1 \parallel h_2$ and $h_3 \parallel h_4$.

We solve the case for linkages with 6 joints of mobility 2 with joints type $R$, $P$ or $H$:

**Theorem 5.9** Let $L$ be a 6-linkage of mobility 2 or higher with at least one $H$-joint. Up to cyclic permutation, the following cases are possible.

1. All axes of $R$- and $H$-joints are parallel.
2. There are two $P$-joints $j_0$ and $j_3$ and $p_0 \parallel p_3$. All other joints are of type $H$ or $R$, $h_1 \parallel h_2$ and $h_4 \parallel h_5$.

**Proof** Recall that if two axes of a linkage are parallel then they coincide in the spherical projection. Let $k$ be the number of blocks of neighbouring equal axes of the spherical projection $L_s$. By Remark 3.5, we know that mobility of $L_s$ is greater than or equal to the mobility of $L$. Then the mobility of $L_s$ is at least two. Furthermore, if the rotation joints of $L_s$ are concurrent, then they must be parallel because at least one $R$-joint comes from an $H$-joint. Therefore, by Theorem 5.5 and Theorem 5.7, there are at most two blocks of equal axes of the spherical projection $L_s$. The proof proceeds by case distinction on $k$.

**Case I** $k = 1$. Then all axes of $L_s$ are equal; hence all axes of $H$- and $R$-joints of $L$ are parallel; this is possibility (1) of the theorem, and the motion of each joint belongs to the same Schönflies motion subgroup of $SE(3)$.

**Case II** $k = 2$. Then each of the two blocks of $R$-joints in $L_s$ has at least two joints. Indeed, if a block has a single joint, by the spherical projection, it must be parallel to other $R$-joints which contradicts our assumption of two blocks. The linkage $L_s$ is movable and, therefore, has at least four joints that move. In particular, it cannot happen that all axes of $L_s$ that move are parallel. After removing the fixed joints, $L_s$ still has two blocks of parallel axes. By comparing with Theorem 5.7, it follows that $L_s$ is a $PRPRR$-linkage; if, say, $j_0$ and $j_3$ are prismatic joints, then $h_1 \parallel h_2$ and $h_4 \parallel h_5$ which is the possibility (2) of the theorem. 

5.3 $nR$-linkages

In this section, we classify paradoxical $nR$-linkages with mobility of $n - 4$, where $n > 6$.

**Theorem 5.10** Let $L$ be an $nR$-linkage, where $n > 6$, of mobility $n - 4$ or higher, then all axes are concurrent. Moreover, $L$ has mobility $n - 3$.

**Proof** We prove this result by induction on the number of active joints. For $n = 7$, we assume that $L$ is a $7R$-linkage with mobility 3. Then, freezing the joint $j_n$ of $L$, we get a linkage $L'$ with mobility 2.
No other joint is frozen By Theorem 5.5, the triples \((h_2, h_1, h_{n-1})\) and \((h_1, h_{n-1}, h_{n-2})\) satisfy Bennett conditions and, in fact, all the Bennett ratios are equal. As we freeze the joint \(j_n\) at an arbitrary position, we get that \(h_2, h_1, h_n, h_{n-1}\) are concurrent by Lemma 3.4. Therefore, all Bennett ratios are equal to zero, and \(L'\) has mobility 3. Then \(L\) has mobility 4, which is a contradiction.

Another joint is frozen The linkage \(L'\) becomes a 5R-linkage with mobility at least 2. Then all axes of \(L'\) are concurrent. We can freeze another joint of \(L\) different from \(j_n\) and, in a similar way, we conclude that all axes of the resulting linkage are concurrent. Therefore, all axes are concurrent, and the mobility of \(L\) is 4.

Fix \(n \geq 8\). Suppose that all axes of any \((n - k)R\)-linkage with \(k \geq 1\) whose mobility is at least \(n - 5\) are concurrent. Let \(L\) be an \(nR\)-linkage with mobility at least \(n - 4\). Then, freezing the joint \(j_n\) of \(L\), we get an \((n - k)R\)-linkage \(L'\) with mobility at least \(n - 5\). By strong induction hypothesis, \(L'\) has mobility \(n - 4\) and all axes of \(L'\) (and hence of \(L\)) are concurrent. □

5.4 n-linkages

In this section, we classify paradoxical \(n\)-linkages with mobility \(n - 4\), where \(n > 6\). If the mobility of an \(n\)-linkage with at least one \(P\)-joint and other \(R\)-joints is \(n - 3\), then all the revolute joints are parallel. Compare the following lemma with Theorem 5.7.

Lemma 5.11 Let \(L\) be an \(7\)-linkage of mobility 3 with at least one \(P\)-joint and other \(R\)-joints. Then all axes of \(R\)-joints are parallel and the number of \(P\)-joints is at least 2.

Proof We make a case distinction based on the number of \(P\)-joints.

Case I \(L\) has one \(P\)-joint Since \(L\) has mobility 3, we can find an \(R\)-joint such that the \(P\)-joint is still active when we freeze this \(R\)-joint. If we freeze such an \(R\)-joint, we still have a mobile linkage \(L'\) with mobility 2. If no other joint is frozen, then we have a contradiction by Theorem 5.7. On the other hand, by Lemma 5.3, if another \(R\)-joint is frozen, then the resulting mobile 5-linkage \(L'\) has mobility 2 with one active \(P\)-joint. Hence, all axes of \(R\)-joints need to be parallel in \(L'\). If we have just one group of parallel joints in \(L\), where the two frozen \(R\)-joints are parallel to the axes of \(R\)-joints, i.e. all axes of \(R\)-joints in \(L\) are parallel, or parallel to each other and neighbouring in \(L\), which contradicts the assumption. If we have two groups of parallel joints in \(L\), we freeze the \(P\)-joint of \(L\). In that case, the resulting linkage with at most six revolute joints has mobility of two and two groups of parallel joints by Lemma 5.3, which is impossible by the classification of 5R-linkages and 6R-linkages with mobility 2, c.f. Theorem 5.5. Therefore, the \(L\) with one \(P\)-joint is impossible by above arguments.

Case II \(L\) has two \(P\)-joints If we freeze a \(P\)-joint, we still have a mobile linkage \(L'\) with mobility 2. If the other \(P\)-joint is frozen, then the resulting \(L'\) is a 5R-linkage of mobility 2. Then all axes of \(R\)-joints must be parallel because the \(P\)-joint would change the intersecting point if all axes of \(R\)-joints were concurrent. Notice that the \(P\)-joints are not perpendicular to the \(R\)-joints. On the other hand, if the other \(P\)-joint is not frozen, then the resulting linkage must be a 5-linkage by Theorem 5.7 and Proposition 5.2 and all axes of \(R\)-joints are parallel in \(L'\). Therefore, the frozen \(R\)-joint must be parallel to the axes of \(R\)-joints in \(L'\), i.e. all axes of \(R\)-joints in \(L\) are parallel, and the direction of one \(P\)-joint is parallel to the plane that is perpendicular to the axes of \(R\)-joints which contradicts that the mobility of \(L\) is 3.
Case III L has three P-joints If we freeze a P-joint (named \(j_n\)), we still have a mobile linkage \(L'\) with mobility 2. The other two P-joints cannot be frozen simultaneously. Otherwise, the resulting linkage \(L'\) is an impossible 4R-linkage of mobility 2. On the one hand, if another P-joint is frozen, then the resulting mobile 5-linkage \(L'\) has mobility 2 with one P-joint by Lemma 5.3. Then all axes of R-joints are parallel in \(L'\), and all axes of R-joints must be parallel in \(L\). On the other hand, if the two P-joints are both active, then either all axes of R-joints are parallel in \(L'\), or we have two pairs of parallel joints and two P-joints with the same direction by Theorem 5.7. Assume that we have only one group of parallel joints, all axes of R-joints are parallel in \(L\). Assume that we have two pairs of parallel joints, we can freeze another P-joint of \(L\) different from \(j_n\) and, in a similar way, we conclude that the directions of all three P-joints are the same and two of them are neighbouring joints by Theorem 5.7. Then \(L\) must degenerate, which is a contradiction. Therefore, all axes of R-joints are parallel in \(L\), and there are at least two P-joints such that they are not perpendicular to the R-joints.

Case IV L has \(k = 4\) or more P-joints Since the spherical projection \(L_s\) has \(7 − k \leq 3\) revolute joints, it is degenerate: all axes are coinciding. Therefore, all rotation axes are parallel and at least two P-joints are not perpendicular to the R-joints.  

\[\square\]

Theorem 5.12 Let \(L\) be an \(n\)-linkage, where \(n > 6\), of mobility \(n − 4\) with at least one P-joint and other R-joints. Then all axes of R-joints are parallel and the number of P-joints is at least 2.

Proof We prove this result by induction on the number of active joints of \(L\). The base of the induction is Lemma 5.11.

Fix \(n \geq 8\). Suppose that all axes of the R-joints in any \((n − k)\)-linkage of mobility \(n − 5\) with at least one P-joint are parallel, where \(1 \leq k \leq 2\) by Lemma 5.3. The number of P-joints is at least two when \(k = 1\), and at least two P-joints are not perpendicular to the R-joints. All P-joints are perpendicular to the R-joints when \(k = 2\). Let us freeze an R-joint (say \(j_n\)) of \(L\) such that one P-joint is always active. We get an \((n − l)\)-linkage \(L'\) with mobility at least \(n − 5\), where \(1 \leq l \leq 2\) by Lemma 5.3.

Case I \(l = 1\) By strong induction hypothesis, all axes of R-joints of \(L'\) (hence of \(L\)) are parallel and the number of P-joints is at least 2 in \(L'\) (hence in \(L\)).

Case II \(l = 2\) Let \(j_m\) be the other frozen joint. By strong induction hypothesis, all axes of R-joints of \(L'\) are parallel and the directions of P-joints are perpendicular to the direction of the axes of R-joints.

- If the joint \(j_m\) is a P-joint, then the axis of the frozen R-joint \(j_n\) must be parallel to all axes of R-joints of \(L'\). Hence, all axes of R-joints of \(L\) are parallel. As P-joints are perpendicular to the R-joints in \(L'\) (hence of R-joints in \(L\)), the P-joint \(j_m\) must be perpendicular to the R-joints in \(L\). Then the mobility of \(L\) is \(n − 3\), which is a contradiction.

- If the joint \(j_m\) is an R-joint, then the axes of the frozen R-joints \(j_m,j_n\) must be parallel. We can freeze a P-joint of \(L\) such that \(j_m\) and \(j_n\) are not frozen simultaneously, and the resulting linkage we call \(L''\). If \(L''\) has a P-joint, in a similar way, we conclude that all axes of R-joints in \(L''\) are parallel. Then all axes of R-joints in \(L\) are parallel. Then the mobility of \(L\) is \(n − 3\), which is a contradiction. Hence \(L''\) has no P-joint, by Theorem 5.10, we conclude that all axes of R-joints in \(L''\) are parallel. Then all axes of R-joints in \(L\) are parallel. Then the mobility of \(L\) is \(n − 3\), which is a contradiction.
Theorem 5.13 Let $L$ be an $n$-linkage, $n \geq 7$, of mobility $n - 4$ and with at least one $H$-joint. Then all axes of $R$- and $H$-joints are parallel.

Proof By Remark 3.5, we know that the mobility of $L$, $n$, is greater than or equal to the mobility of $L$, $n$. Then the mobility of $L$, $n$, is at least $n - 4$. Furthermore, if the rotation joints of $L$, $n$, are concurrent, then they must be parallel because at least one $R$-joint comes from an $H$-joint. Therefore, by Theorem 5.10 and Theorem 5.12, all axes of $L$, $n$, are equal; hence, all axes of $H$- and $R$-joints of $L$ are parallel; this is possible, and the motion of each joint belongs to the same Schönflies motion subgroup of $SE(3)$. □

6 nR-linkages with mobility $n - 5$

In this section, we present necessary conditions that the geometry of a paradoxical nR-linkage with mobility $n - 5$ must satisfy, where $n > 6$.

Lemma 6.1 Let $L$ be a $7R$-linkage of mobility 2 with no parallel axes and no Bennett conditions are satisfied. Let $\beta$ be a bond. Then, up to symmetry, we have $A_\beta = (j_1, j_3)$.

Proof We exclude all the possibilities where $|A_\beta| \geq 3$ up to symmetry. Notice that whenever $A_\beta$ is a chain there is at least one pair of parallel axes, see Lemma 4.3.

1. $|A_\beta| \geq 6$: This is clearly not possible since in this case $A_\beta$ is a chain.

2. $|A_\beta| = 5$: There are two cases up to symmetry: $A_\beta = (j_1, j_2, j_3, j_4, j_6)$ and $A_\beta = (j_1, j_2, j_3, j_5, j_6)$. Suppose that $A_\beta = (j_1, j_2, j_3, j_4, j_6)$. By assumption, there are no parallel axes. By Lemma 2.3, it follows that $(i - h_3)(t_5 - h_5)(i - h_6) = 0$, and $(h_4, h_5, h_6)$ is a Bennett triple. The argument is the same when $A_\beta = (j_1, j_2, j_3, j_5, j_6)$ and we omit it.

3. $|A_\beta| = 4$: There three cases up to symmetry: $A_\beta = (j_1, j_2, j_3, j_5)$, $A_\beta = (j_1, j_4, j_5)$ and $A_\beta = (j_1, j_3, j_5)$. In the first two cases, Lemma 2.3 implies that the linkage has a Bennett triple. Suppose $A_\beta = (j_1, j_3, j_5)$. If $h_1$ and $h_3$ are not parallel, then

\[(i - h_1)(t_2 - h_3)(i - h_3)(t_4 - h_4)(i - h_5) = 0.
\]

Since $L$ has mobility 2, this defines a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ by Corollary 4.1. By Lemma 4.2, either $(h_1, h_2, h_3)$ or $(h_3, h_4, h_5)$ is a Bennett triple.

4. If $|A_\beta| = 3$, then the subsequence $A_\beta$ does not contain consecutive elements of $L$. Indeed, suppose $A_\beta = (j_1, j_2, j_3)$. Since $h_1$ and $h_2$ are not parallel, we have the conditions

\[C_1 : (i - h_2)(t_3 - h_3)(t_4 - h_4)(i - h_5) = 0,
\]

\[C_2 : (i + h_1)(t_7 + h_7)(t_8 + h_8)(i + h_8) = 0.
\]

At least one of the $C_i$ is a curve since $L$ has mobility 2. Then there is a Bennett triple by Lemma 4.2. Thus $A_\beta = (j_1, j_3, j_5)$: By Lemma 2.3, this is only possible if $(i - h_1)(t_2 - h_3)(i - h_5) \in E$. If that was the case then, $i - h_5 \in E$ which is impossible.

5. $|A_\beta| = 2$: The previous arguments apply to exclude bonds such that $A_\beta = (j_1, j_2)$. If $A_\beta = (j_1, j_3)$ then $(h_1, h_2, h_3)$ would be a Bennett triple.
**Theorem 6.2** Let $L$ be a mobile $7R$-linkage with mobility at most 2 and with no concurrent axes or Bennett triples. Then, $L$ has mobility 1.

**Proof** By Lemma 6.1, joints $j_1$ and $j_2$ are not simultaneously in $A_\beta$. Hence, the projection

$$\pi : \overline{K_L} \to \mathbb{P}^1 \times \mathbb{P}^1$$

$$(t_1, \ldots, t_7) \mapsto (t_1, t_2)$$

is not surjective and the image is contained in a curve. It follows that $t_1$ and $t_2$ are algebraically dependent. In the same way, $t_2$ and $t_3$ are algebraically dependent as well and so on and, therefore, all $t_i$ are so. It follows that the mobility of $L$ is 1.

We now explain how to generalise this result to any number of joints.

**Theorem 6.3** Let $L$ be an $nR$-linkage with $n \geq 7$ and no concurrent axes or Bennett triples. Then $L$ has mobility at most $n - 6$ (i.e. it does not have paradoxical mobility).

**Proof** We prove this result by induction on the number of active joints where the base case is Lemma 6.2. Fix $n \geq 8$. We freeze a joint in $L$ and we have an $(n-1)R$-linkage, say $L'$. We proceed by looking at the number of joints that get frozen as a result (in each case we are going to prove that $L'$ has mobility at most $n - 7$).

**Case I:** One joint gets frozen

We claim that $L'$ has no concurrent axes or Bennett triples. Otherwise, by Lemma 3.4, we get the contradiction that $L$ has concurrent axes or Bennett triples. By the induction hypothesis, $L'$ has mobility at most $n - 7$.

**Case IIa:** Two joints get frozen and $n = 8$ The linkage $L'$ is a $6R$-linkage. By Theorem 5.5, the mobility of $L'$ is at most 1. Otherwise, there would be at least one Bennett triple in $L$.

**Case IIb:** Two joints get frozen and $n \geq 9$ The linkage $L'$ cannot have all $5$ axes concurrent. By Proposition 5.2, it follows that $L'$ has mobility at most 1.

**Case IIIa:** Three joints get frozen and $n = 8$ The linkage $L'$ cannot have all $6$ axes concurrent. By Proposition 5.2, it follows that $L'$ has mobility at most 2.

**Case IIIb:** Three joints get frozen and $n = 9$ (this case is very similar to Case IIIa). The linkage $L'$ cannot have all 6 axes concurrent. By Proposition 5.2, it follows that $L'$ has mobility at most 2.

**Case IIIc:** Three joints get frozen and $n \geq 10$ The linkage $L'$ cannot have all $n - 3$ axes concurrent. By Theorem 5.10, it follows that $L'$ has mobility at most $n - 7$.

**Case IV:** Four or more joints get frozen Then the linkage $L'$ is an $(n-s)R$-linkage. By Proposition 5.1, it follows that the mobility of $L'$ is at most $n - s - 3 \leq n - 7$.

In all cases, we have shown that the mobility of $L'$ is at most $n - 7$. This implies that $L$ has mobility at most $n - 6$.

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