NON-REDUCEDNESS OF THE HILBERT SCHEMES OF FEW POINTS

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ABSTRACT. We use generalised Bialynicki-Birula decomposition, apolarity and obstruction theories to prove non-reducedness of the Hilbert scheme of 13 points on $\mathbb{A}^6$. Our argument doesn’t involve computer calculations and gives an example of a fractal-like structure on this Hilbert scheme.

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1. INTRODUCTION

Hilbert schemes of points have many applications in different areas of mathematics, for example Haiman proved the $n!$ conjecture using their geometry (see [Hai01]). Fogarty [Fog68] considered these schemes when the ambient variety was a smooth surface and showed that then the Hilbert scheme of points is smooth as well. If we start from a three or more dimensional scheme this is not the case and many questions arise, such as when the Hilbert scheme of points is reduced, Cohen-Maculay, etc. Recently, Jelisiejew closed a long-standing open question in this area and showed that the Hilbert scheme of points on an affine space is not reduced [Jel20]; this problem was open for more than fifty years. Still the question about the smallest counter-example to reducedness remains. Methods of Jelisiejew were partially implicit and the non-reducedness witness was represented by a scheme of degree 5082 on $\mathbb{A}^{14}$. In practise, people analyse the Hilbert schemes of $d$ points, where $d$ is much smaller, usually $d \leq 20$; see for example [BCR19] or [DJ+17]. The question remains whether non-reducedness is present in this range.

The aim of this paper is to provide simpler example of a non-reduced point on the Hilbert scheme with both the dimension of an affine space and the number of points much smaller than in [Jel20]. More precisely we show that the Hilbert scheme of $d$ points on $\mathbb{A}^n$ is non-reduced as soon as $d \geq 13, n \geq 6$, which immediately follows from non-reducedness of $\text{Hilb}_{\mathbb{A}^n/k}^{13}$. In particular we give:

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Example 1.1 (Corollary 3.17). Let $I$ be the ideal generated by polynomials $f$ in variables $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_6}$ such that $f$ applied to $F = x_1 x_2 x_4 - x_1 x_2^2 + x_2 x_3^2 + x_3 x_5 x_6 + x_4 x_5^2$ is zero or $f$ is of degree greater or equal three. Then $[I] \in \Hilb^{13}_{\mathbb{A}^6/k}$ is a non-reduced point.

In order to prove non-reducedness of points similar to the above we use the $G_m$-action on $\mathcal{H} := \Hilb^d_{\mathbb{A}^n/k}$ coming from the action of $G_m$ on $\mathbb{A}^n$ by multiplication. We define the barycenter scheme $\mathcal{B} \subset \mathcal{H}$ which consists of points represented by tuples of points in $\mathbb{A}^n$ such that their weighted average is 0. Next we note that $\mathcal{H} \simeq \mathbb{A}^n \times \mathcal{B}$. Moreover, the $G_m$-action on $\mathcal{H}$ restricts to $\mathcal{B}$. The point from Example 1.1 is an example of a hedgehog point:

Definition 1.2. A $G_m$-invariant point $[I] \in \mathcal{B}$ is a hedgehog point if it satisfies the following conditions:

- $(T_{[I]}\mathcal{B})_{>0} = 0$,
- $T_{[I]}\mathcal{B} \neq (T_{[I]}\mathcal{B})_0$,
- there is no closed point $[I'] \neq [I]$ such that $\lim_{t \to 0} t \cdot [I'] = [I]$.

The last dot can be stated in other terms. Let $V$ be the subscheme of $\mathcal{B}$ consisting of points for which their $G_m$-limit at zero is $[I]$. We call it the negative spike at $[I]$. The last condition above is equivalent to 0-dimensionality of $V$ or to the fact that $V$ is topologically a point. This justifies the name "hedgehog point", as $\mathcal{B}$ looks like $\mathcal{B}^G_{\mathbb{A}^n}$ with infinitesimal spikes, near $[I]$. A more precise definition of the negative spike and being a hedgehog point is given in Definition 2.29. We prove that:

Theorem 1.3 (Hedgehog point theorem 3.1). A hedgehog point $[I] \in \mathcal{B}$ is non-reduced. Hence $(0,[I]) \in \mathbb{A}^n \times \mathcal{B} \simeq \mathcal{H}$ is non-reduced as well.

The geometric idea behind the proof is as follows:

1. By a generalisation of Białynicki-Birula decomposition for the $G_m$-action on the Barycenter scheme we get that there is an open neighbourhood of $[I]$ such that the barycenter looks like an affine $\mathbb{A}^1$-cone over $G_m$-fixed points. Here the fact that $(T_{[I]}\mathcal{B})_{>0} = 0$ is crucial.

2. The fiber over $[I]$ of this cone is $V$ and it is 0-dimensional as $[I]$ is a hedgehog point. Assuming (for a contradiction) reducedness of $[I]$ in the barycenter scheme, we get that the aforementioned affine cone is trivial i.e. the barycenter scheme is isomorphic to the fix-points subscheme near $[I]$. This follows from semi-continuity of dimension of fibers for projective morphisms.

3. The tangent space to the fixed-point subscheme of $\mathcal{B}$ is $T_{[I]}\mathcal{B}$ so it is different than $T_{[I]}\mathcal{B}$, because $[I]$ is a hedgehog point. Hence, we get a contradiction with $[I]$ being reduced.

The key tool is the generalisation of Białynicki-Birula decomposition from [JS19]. It reduces the whole problem to only a few facts about tangents and the fiber $V$.

To construct hedgehog points we start with a cubic in six variables $F \in k[x_1, \ldots, x_6]$ and proceed as in Example 1.1. We produce an ideal $I$ which is defined as $F^+ + S_{\geq 3}$ where $F^+ \subset S = k[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_6}]$ is the set of differential polynomials, such that when applied to $F$ they give 0. The cubic $F$ is general enough if it lies in the open (and non-empty) set of cubics defined precisely in Definition 3.2. We prove that:

Theorem 1.4 (Corollary 3.16). For general enough cubic $F$ the given point $[I] \in \Hilb^{13}_{\mathbb{A}^6/k}$ lies in the barycenter subscheme and is a hedgehog point. In particular it is non-reduced.

Non-emptiness of the set of general enough cubics is witnessed by $F$ from Example 1.1. Due to the paper [Sha01] algebras with their Hilbert function $(1, 6, 6)$ form an irreducible component of the barycenter scheme. The point $[I]$ described above lies only in that component, what follows from the fact that $\mathcal{B}$ and $\mathcal{B}^G_{\mathbb{A}^n}$ are homeomorphic near $[I]$ (by point (2) in the above sketch of the proof of Theorem 3.1, but without assuming reducedness) and from the fact that $\mathcal{B}^G_{\mathbb{A}^n}$ consists of algebras of type $(1, 6, 6)$ near $[I]$. Moreover, a general point of the component containing $[I]$ is smooth, so non-reducedness here is of the "embedded component" type.
After proving theorems (1.3) and (1.4) we determine the negative spike $V$ over $[I]$ coming from a general enough cubic $F$. We prove:

**Theorem 1.5 (Theorem 3.27).** If $I = F^⊥ + S_{≥3}$ for a general enough cubic $F$, then the negative spike at $[I]$ is isomorphic to $\text{Spec}(S/F^⊥)$.

As a subscheme of the Hilbert scheme $V$ is induced by a deformation. We describe a related deformation and call it the fractal family. We chose the name fractal because the fiber over the unique closed point and the base space of this deformation are almost the same, so it is like a deformation over itself. This exhibits an interesting phenomenon: the Hilbert scheme which parametrises finite schemes, naturally contains some of them (or very close to such), as closed subschemes. Overall, the isomorphism between $\text{Spec}(S/F^⊥)$ and $V$ comes from the composition of the morphism to $\mathcal{H}$ defined by the fractal family and the projection $\mathcal{H} \cong \mathbb{A}^n \times B \to B$ from the Hilbert scheme to its barycenter subscheme.

We briefly discuss the contents of the paper. In Section 2 we introduce Hilbert schemes of points, describe a few groups acting on Hilbert schemes of points on affine spaces and define Białynicki-Birula decomposition. Next we introduce obstruction theories, describe the barycenter points, describe a few groups acting on Hilbert schemes of points on affine spaces and define

2. Preliminaries

We assume familiarity with schemes and their functorial description (e.g. as in [FGI+05, Part 1, Chapter 2]). We work in an algebraically closed field $k$ of characteristic 0, although the characteristic and algebraic closedness play a significant role only at a few places.

We will be working a lot with group-scheme actions. A good reference to this topic is again [FGI+05, Part 1, Chapter 2].

Now we list some of the notations appearing later in the paper. The reason for two notations - $S$ and $P$ - denoting the polynomial algebra will become clear after Definition 2.30.

- $k$ denotes the algebraically closed field of characteristic 0.
- $X$ denotes scheme locally of finite type over $k$.
- For $x \in X(k)$ the tangent space to $X$ at $x$ will be denoted $T_xX$. Moreover, $T'_xX$ will denote the cotangent space to $X$ at $x$.
- $T$ denotes a test scheme for a functor.
- $S$ denotes a polynomial algebra $k[α_1, \ldots, α_n]$ where $n$ is usually clear from the context.
- $P$ denotes a polynomial algebra $k[x_1, \ldots, x_n]$ isomorphic to the one above, but with different names for variables.
- If $A$ is a $k$-algebra, then $S_A$ denotes $S \otimes_k A = A[α_1, \ldots, α_n]$.
- If $n$ is fixed, we will write $x$ or $α$ for respectively $x_1, \ldots, x_n$ or $α_1, \ldots, α_n$.
- $k[ε]$ denotes the ring $\frac{k[ε]}{(ε^2)}$, so whenever something is denoted by $ε$ we have $ε^2 = 0$.
- $\mathbb{A}^n / \mathbb{P}^n$, $G_m$ denote respectively the affine/projective space of dimension $n$ and the algebraic torus $\text{Spec}(k[t^{±1}])$.
- If $G_m$ acts on a vector space $V$, then the induced grading on $V$ is given by $V_t := \{v \in V : t \in G_m(k) \text{ acts by multiplying by } t^{-n}\}$.
- If $B$ is a non-negatively graded $k$-algebra then its Hilbert function is a function $\mathbb{N} \to \mathbb{N}$ defined by $n \mapsto \dim_k(B_n)$.
- For a linear space $W$ let $\text{Sym}_n(W)$ be a subspace of $W \otimes W$ consisting of symmetric tensors. The canonical symmetric bilinear $k$-vector space homomorphism from $W \times W$ to
Sym$_2(W)$ is defined by the formula:

\[(w, v) \mapsto \frac{1}{2} (w \otimes v + v \otimes w).\]

This yields a natural duality between Sym$_2(W)$ and Sym$_2(W^\vee)$ induced by the natural duality between $W \otimes W$ and $W^\vee \otimes W^\vee$. More precisely we define a map:

\[
\cdot : \text{Sym}_2(W^\vee) \times \text{Sym}_2(W) \to k
\]

by the formula $(\eta \mu) \cdot (w v) = \left( \frac{1}{2} (\eta \otimes \mu + \mu \otimes \eta) \right) \cdot (w \otimes v + v \otimes w) := \frac{1}{4} (\eta(w)\mu(v) + \eta(v)\mu(w) + \mu(w)\eta(v) + \mu(v)\eta(w))$. This gives an identification of Sym$_2(W^\vee)$ with Sym$_2(W^\vee)$.

- $\delta_{ij}$ denotes the Kronecker delta.

2.1. Hilbert schemes. We recall the definition of the Hilbert functor of points on a scheme.

**Definition 2.1.** Let $X$ be a scheme over $k$. Then the functor $\text{Hilb}^d_{\text{X/k}} : (\text{Sch}/k)^{\text{op}} \to \text{Set}$ given by:

\[
\text{Hilb}^d_{\text{X/k}}(T) = \{ Z \subset X_T \text{ closed subscheme such that } Z \to T \text{ is finite locally free of degree } d \}
\]

is called a Hilbert functor of $d$ points on $X$. For the definition of finite locally free of degree $d$ see [sta17, Tag 02K9]. Alternatively one could write finite of degree $d$ and flat.

**Example 2.2.** If $X = \mathbb{A}^n$ then for a $k$-algebra $A$ we have:

\[
\text{Hilb}^d_{\mathbb{A}^n/k}(\text{Spec}(A)) = \{ I \subset S_A \text{ ideal such that } S_A/I \text{ is a locally free } A\text{-module of rank } d \}
\]

In this section we fix $n$ and $d$ and write $\mathcal{H}$ for both this functor and the scheme representing it. If we want to emphasize the number of points $d$ we write $\mathcal{H}^d$. Also, we will denote by $[I]$ an $A$-point corresponding to an ideal $I \subset S_A$ such that $S_A/I$ is locally free of rank $d$.

In the whole paper we will be only concerned with the Hilbert schemes of points on affine spaces. To study local properties of these functors/schemes the following description of the tangent space will be crucial:

**Theorem 2.3.** Fix a $k$-point $[I] \in \mathcal{H}$. Then the tangent space $T_{[I]} \mathcal{H}$ is isomorphic to the $k$-vector space $\text{Hom}_S(I, S/I)$.

The full proof of this theorem can be found in [FGI+05, Corollary 6.4.10]. We will use this identification a few times, so we recall how to obtain a homomorphism in $\text{Hom}_S(I, S/I)$ from an element of $T_{[I]} \mathcal{H}$. First, using the fact that $T_{[I]} \mathcal{H} = \text{Mor}(\text{Spec}(k[e]), \mathcal{H} \mid \{\text{pt.}\} \mapsto [I])$ we see that an element of $T_{[I]} \mathcal{H}$ is given by a diagram:

\[
\begin{array}{c}
0 \rightarrow I' \rightarrow S[e] \rightarrow \frac{S[e]}{I'} \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow I \rightarrow S \rightarrow \frac{S}{I} \rightarrow 0
\end{array}
\]
where we use a convention that $S[\epsilon] := S[k[\epsilon]]$. The corresponding element of $\text{Hom}_S(I, S/I)$ is given by $\delta$ on the diagram below:

\[ \begin{array}{cccccc}
0 & \to & I' & \to & S[\epsilon] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I & \to & S & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & & & 0 & & \\
\end{array} \]

Here $s$ is just a natural embedding of $S$ into $S[\epsilon]$. The homomorphism $\beta \circ s \circ \alpha$ composed with $\gamma$ is 0 (by diagram chasing) thus we get the existence of $\delta$.

**Remark 2.4.** On the other hand, if we are given $\delta \in \text{Hom}_S(I, S/I)$, then the corresponding ideal $I' \subset S[\epsilon]$ is given by:

\[(3) \quad I' = (f - \epsilon \cdot \delta(f) : f \in I).\]

A precise explanation can be found in [FGI+05].

When dealing with locally free/flat modules over local rings as $k[\epsilon]$ the following algebraic lemma is useful:

**Lemma 2.5.** Suppose that $(B, m)$ is a local $k$-algebra with residue field $k$ and $C$ is a free $B$-module of rank $d$. Assume that $(c_1, \ldots, c_d)$ is a tuple of elements of $C$ such that the tuple $(c_1/m, \ldots, c_d/m)$ is a basis of $C/mC$. Then $(c_1, \ldots, c_d)$ is a basis of $C$.

**Proof.** Omitted. \qed

2.2. **Groups acting on $\mathcal{H}$**. Generally, if an algebraic commutative group $G$ acts on a scheme $X$, then there is an induced action of $G$ on $\text{Hilb}_X$. Indeed, take $T$ - a test scheme. Then the $G$-action $G \times X \to X$ yields (by the base change to $T$) a morphism $G_T \times X_T \to X_T$. If we take $t \in G(T) = \text{Mor}(T, G) = \text{Mor}_T(T, G_T)$, then we get the diagram:

\[ \begin{array}{ccc}
G_T \times X_T & \longrightarrow & X_T \\
\downarrow \scriptstyle{(t, id)} & & \downarrow \scriptstyle{t} \\
X_T & \longrightarrow & X_T \\
\end{array} \]

where the vertical map $X_T \to G_T \times X_T$ is the identity on the second coordinate and factors through the $T$-point $t$ on the first one. Now if $Z \subset X_T$ represents a $T$-point of $\text{Hilb}_{X/k}^{d}$, then the action of $t$ on $Z$ is defined by the image of map $(t \cdot) : X_T \to X_T$. This map is an isomorphism (because $G$ is an algebraic group), so $t \cdot Z := (t \cdot)(Z) \subset X_T \to T$ is again locally finite of degree $d$. Below we provide two examples of lifting actions from $\mathbb{A}^n$ to $\mathcal{H}$.

**Example 2.6.** Consider the $G_m$-action on $\mathbb{A}^n$ given by the standard grading $\text{deg}(a_i) = 1$. Let us describe the induced morphism $G_m \times \mathcal{H} \to \mathcal{H}$ constructed above on its $T$-points for $T = \text{Spec}(A)$ where $A$ is a $k$-algebra. We are interested in the map:

$G_m(\text{Spec}(A)) \times \mathcal{H}(\text{Spec}(A)) \to \mathcal{H}(\text{Spec}(A))$

Take $a \in G_m(\text{Spec}(A)) = A^*$ (invertible elements). As above, $a$ yields an isomorphism:

\[ \begin{array}{ccc}
\mathbb{A}_A^n & \longrightarrow & \mathbb{A}_A^n \\
\downarrow & & \downarrow \\
\mathcal{H} & \longrightarrow & \mathcal{H} \\
\end{array} \]
which on the level of algebras comes from the $A$-algebra homomorphism \((a)^\#: S_A = A[k] \to A[\bar k]\) sending $a_i$ to $a\alpha$. Now, a point \([Z] \in \mathcal{H}(\text{Spec}(A))\) is represented by an ideal $I \subset S_A$ and the new point is given by the ideal $a \cdot I := \ker(q \circ (a)^\#)$ where $q: S_A \to \frac{S_A}{\bar I}$ is the quotient map. The justification of this formula follows from the diagrams below (the left hand side is the definition of the action and the right hand side is the corresponding diagram on the level of algebras):

\[
\begin{array}{ccc}
\mathbb{A}^n_A & \xrightarrow{a} & \mathbb{A}^n_A \\
\uparrow & & \uparrow \\
Z & \xrightarrow{a \cdot Z} & \mathbb{A}^n_A \\
\downarrow & & \downarrow \\
S_A & \xrightarrow{(a)^\#} & S_A \\
\downarrow & & \downarrow \\
\frac{S_A}{\bar I} & \xleftarrow{\bar I} & \frac{S_A}{\bar I} \\
\end{array}
\]

Using the above description of $G_m$-action on $\mathcal{H}$ one can prove the following result:

**Proposition 2.7.** Suppose that $A$ is a $\mathbb{Z}$-graded $k$-algebra and consider a morphism $\phi: \text{Spec}(A) \to \mathcal{H}$ given by a subscheme $Z \subset \mathbb{A}^n_A$ cut by the ideal $I \subset S_A$. Consider a $\mathbb{Z}$-grading on $S_A = S \otimes_k A$ given by $(S_A)_l = \bigoplus_{i+j=l} S_i \otimes_k A_j$. If $I$ is homogeneous with respect to this grading, then $\phi: \text{Spec}(A) \to \mathcal{H}$ is $G_m$-equivariant.

**Proof.** Left to the reader. \qed

**Example 2.8.** Now we analyse the induced $\mathbb{A}^n = G_m^n$-action on $\mathcal{H}$ coming from the action on $\mathbb{A}^n$ by translations. Again, for a $k$-algebra $A$ we get a map:

$\mathbb{A}^n(\text{Spec}(A)) \times \mathcal{H}(\text{Spec}(A)) \to \mathcal{H}(\text{Spec}(A))$

but now it will be defined by $((a_1, \ldots, a_n), I) \mapsto \ker(q \circ (a+)^\#)$ where $(a+)^\#: S_A \to S_A$ is a $A$-homomorphism sending $a_i$ to $a_i + a_i$.

From Example 2.6 we can see that a $k$-point $[I] \in \mathcal{H}$ is $G_m$-invariant if and only if $I$ is homogeneous with respect to the standard grading on $S$. Thus, if $I$ is homogeneous, then $G_m$ acts on its tangent space $T_{[I]} \mathcal{H}$. We describe this action in the theorem below:

**Theorem 2.9.** Let $[I]$ be a $G_m$-invariant $k$-point of $\mathcal{H}$ (so $I \subset S$ is homogeneous). Then the $G_m$-action on the tangent space $T_{[I]} \mathcal{H} = \text{Hom}_S(I, S/I)$ induces a grading $\text{Hom}_S(I, S/I) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_S(I, S/I)_k,$

where:

(4) \[ \text{Hom}_S(I, S/I)_k = \{ \delta \in \text{Hom}_S(I, S/I) : \delta(I_n) \subset (S/I)_{n+k} \text{ for all integers } n \}. \]

**Proof.** Omitted. \qed

There is no analog of this theorem for the $\mathbb{A}^n$ additive action on $\mathcal{H}$ (as in Example 2.8), because there is no $\mathbb{A}^n$-invariant point in $\mathcal{H}$.

However, one can still ask what happens locally or how does the orbit map (of a point) look like. The following theorem answers the question on the level of tangent spaces:

**Theorem 2.10.** Let $[I]$ be a $k$-point of $\mathcal{H}$ and let $\mathbb{A}^n \to \mathcal{H}$ be its orbit map under the additive action of $\mathbb{A}^n$. Then the induced tangent map $\eta: T_{[0]} \mathbb{A}^n \to T_{[I]} \mathcal{H}$ sends the $i$'th basis vector $\partial_i$ to $\eta(\partial_i) \in T_{[I]} \mathcal{H} = \text{Hom}_S(I, S/I)$ given by the formula:

(5) \[ \eta(\partial_i)(f) = \frac{\partial f}{\partial a_i}. \]

Thus, by abuse of notation we will write $\partial_i$ for $\eta(\partial_i)$.

**Proof.** We leave it as an exercise for the reader. \qed

Suppose additionally that $[I]$ is a $G_m$-invariant point of $\mathcal{H}$. From the description of the grading on $T_{[I]} \mathcal{H}$ in Theorem 2.9 we see that The image of $\eta$ above lies in $\text{Hom}_S(I, S/I)_{-1} \subset \text{Hom}_S(I, S/I)_{<0}$. Thus it is natural to consider the definition:

**Definition 2.11.** A $G_m$-invariant $k$-point $[I] \in \mathcal{H}$ has **trivial negative tangents** (abbreviated TNT), if the map $\eta: \text{lin}(\partial_1, \ldots, \partial_n) \to \text{Hom}_S(I, S/I)_{<0}$ from Theorem 2.10 is surjective.
This just means that the only negative tangents at \([l]\) come from \(A^n\) additive action and are given by derivatives.

This definition is taken from [Jel19]. Note that because the image of \(\eta\) is contained in \(\text{Hom}_S(I, S/I)_{-1}\) and \(\delta_1, \ldots, \delta_n \in \text{Hom}_S(I, S/I)_{-1}\) are linearly independent it follows that \([l]\) as in the definition has TNT if and only if \(\dim_k \text{Hom}_S(I, S/I)_{<0} = n\).

2.3. Białynicki-Birula decomposition. The torus action can be useful to study geometry of Hilbert schemes. To exploit it we will need the theory of Białynicki-Birula decompositions which originates from the papers [BB73] and [BB76]. Later in the paper, we will work with the Białynicki-Birula decomposition induced by the \(G_m\)-action from Example 2.6.

In this subsection we present necessary tools from the paper [JS19] which generalises Białynicki-Birula’s result. In the paper the authors work with linearly reductive groups, so the class that contains \(G_m\), but we will present the results from there only for the \(G_m\)-case.

Definition 2.12. Let \(X\) be a \(G_m\)-scheme. Consider a functor \(X^+ : (\text{Sch}/k)^{op} \to \text{Set}\):

\[
X^+(T) = \left\{ f : A^1 \times T \to X \mid f \text{ is } G_m\text{-equivariant} \right\},
\]

where \(G_m\) acts on \(A^1 \times T\) by the natural action of \(G_m\) on \(A^1\) (i.e. by multiplication). It is called the Białynicki-Birula decomposition of \(X\). Moreover, Białynicki-Birula decomposition is functorial, which means that if \(X, Y\) are \(G_m\)-schemes and \(g : X \to Y\) is \(G_m\)-equivariant, then there is \(g^+ : X^+ \to Y^+\) such that \(g^+ : X^+(T) \to Y^+(T)\) is given by \(g^+(T)(f : A^1 \times T \to X) = (g \circ f : A^1 \times T \to Y)\) for any test scheme \(T\).

From the Białynicki-Birula decomposition of a scheme \(X\) we get three natural transformations:

- \(X^+(T) \to X(T)\) given by restricting \(f\) to \(1 \times T\). This is called forgetting about the limit and will be denoted by \(i_X\).
- \(X^+(T) \to X^{G_m}(T)\) given by restricting \(f\) to \(0 \times T\). This is called restricting to the limit and will be denoted by \(\pi_X\).
- \(X^{G_m}(T) \to X^+(T)\) given by \((f : T \to X^{G_m}) \mapsto ((j \circ f \circ pr_1) : T \times A^1 \to X)\), where \(j : X^{G_m} \to X\) is the closed embedding. This morphism is a section of \(\pi_X\) and will be denoted by \(s_X\).

Under this notation the following holds:

Theorem 2.13 (Existence of Białynicki-Birula decompositions [JS19, Theorem 1.1]). Let \(X\) be a \(G_m\)-scheme locally of finite type over \(k\). Then the functor \(X^+\) defined by (6) is represented by a scheme locally of finite type over \(k\), denoted also by \(X^+\). Moreover, the scheme \(X^+\) has a natural \(A^1\)-action. The map \(\pi_X : X^+ \to X^{G_m}\) is affine of finite type and equivariant.

Moreover, if \(X\) is separated, then \(X^+\) can be seen as a subset of \(X\), because we have the following fact:

Fact 2.14. [JS19, lemma 5.8] For separated \(X\) the morphism \(i_X : X^+ \to X\) is a monomorphism of functors.

Hence, for any scheme \(T\) we will treat \(X^+(T)\) as a subset of \(X(T)\), as long as \(X\) is separated. In particular, for a \(k\)-point \(x \in X^{G_m}\), identifying \(x\) with \(s(x) \in X^+\) and with \(x \in X\) we can treat the tangent space to \(X^+\) at \(x\) as a subset of \(T_x X\):

\[
T_x X^+ = X^+ \left( \text{Spec}(k[x]) \right)_{\{pt\} \to X} \subset X \left( \text{Spec}(k[x]) \right)_{\{pt\} \to X} = T_x X.
\]

This fact also justifies why \(X^+\) is called a decomposition of \(X\). Geometrically speaking, \(k\)-points of \(X^+\) are these points \(p\) of \(X\) for which the \(G_m\)-orbit of \(p\) can be extended to a map from \(A^1\) to \(X\). The image of 0 of this extended map is \(G_m\)-invariant and it is equal \(\pi_X(p)\). One can think of \(\pi_X(p)\) as the limit \( \lim_{t \to 0} t \cdot p \) where dot represents the action of \(G_m\) on \(X\). For a proper scheme such limits always exist. A good example to bear in mind is the Białynicki-Birula decomposition of \(G_m\)-action on \(\mathbb{P}^1\) described below:
Example 2.15. Consider the case when \( X = \mathbb{P}^1 \) and \( G_m \) fixes \( \infty \) and acts on \( \mathbb{P}^1 \setminus \{ \infty \} = \mathbb{A}^1 \) by multiplication. Then \( X^+ = \mathbb{A}^1 \cup \{ \infty \} \) and \( X^{G_m} = \{ 0, \infty \} \). The morphism \( \pi_X \) takes \( \mathbb{A}^1 \) to 0 and \( \infty \) to \( \infty \). On the other hand, \( i_X \) embeds \( \mathbb{A}^1 \) onto \( \mathbb{P}^1 \setminus \{ \infty \} \) and sends \( \infty \) to \( \infty \) in \( \mathbb{P}^1 \).

In fact for any smooth \( X \) the map \( \pi_X \) is locally an affine bundle and this was proved by Białynicki-Birula in [BB73], [BB76].

As can be seen in the above example, near \( 0 \) the morphism \( i_X \) is even an isomorphism. Thus one can study the local geometry of \( \mathbb{P}^1 \) near \( 0 \) by looking at \( (\mathbb{P}^1)^+ \). This is not a coincidence as seen in the following proposition which will be crucial for our purposes:

Proposition 2.16. [JS19, Proposition 1.6] Let \( X \) be separated and locally of finite type. Let \( x \in X^{G_m} \) be a fixed \( k \)-point such that the cotangent space \( T^v_x X \) has no negative degrees (with respect to the grading induced by the \( G_m \)-action on \( X \)). Then the map \( i_X : X^+ \to X \) is an open embedding near \( x' = s_X(x) \in X^+ \). More precisely, there exists an affine open \( \mathbb{A}^1 \)-stable neighbourhood \( U \) of \( x' \) such that \( (i_X)^{|U} : U \to X \) is an open embedding. In particular, \( x \in X \) has an affine open \( G_m \)-stable neighbourhood. Conversely, if a fixed point \( x \) has an affine open \( \mathbb{A}^1 \)-stable neighbourhood in \( X \), then \( T^v_x X \) has no negative degrees.

Remark 2.17. The condition of \( T^v_x X \) having no negative degrees is equivalent to \( T_x X \) having no positive degrees, because the \( G_m \)-grading on the dual space is reversed.

It is worth to mention here, that the idea of using Białynicki-Birula decomposition in order to study Hilbert schemes is not new. However, before the tools only applied to smooth varieties, so these applications were concerning mostly Hilbert schemes of points on smooth surfaces, which by [Fog68] are smooth. An example of such an application is presented in the paper [ES87] by Ellingsrud and Strømme where they calculate the homology groups of the complex manifolds corresponding to Hilbert schemes of points on the projective plane over complex numbers.

2.4. Obstruction theories. We will need obstruction theories so we briefly introduce them here. We follow presentation from [JS21]. For a more detailed explanation see [Har10].

We will start with some intuition. Let \( x \in X \) be a \( k \)-point in a scheme \( X \) locally of finite type over \( k \). Then \( x \) is an image of a morphism \( \text{Spec}(k) \to X \) and one could ask how does this morphism extends to spectras of Artinian rings other then \( k \).

It is natural then to consider the functor \( D_{X,x} : \text{Art}_k \to \text{Set} \) from the category \( \text{Art}_k \) of local Artinian \( k \)-algebras with the residue field \( k \) given by:

\[
D_{X,x}(R) = \{ \eta : \text{Spec}(R) \to X \mid \eta(m_R) = x \}.
\]

It is called the deformation functor of \( X \) at \( x \) and it is pro-representable by the completion of the local ring \( \mathcal{O}_{X,x} \), i.e. \( D_{X,x}(R) \simeq \text{Hom}(\mathcal{O}_{X,x}, R) \).

Let \( \text{Spec}(k) \to X \) in a purely algebraic way: if \( (A, m) \) is a Noetherian complete local \( k \)-algebra (e.g. \( \mathcal{O}_{X,x} \)), when can we lift a morphism \( \eta_0 : A \to B_0 \) to a morphism \( \eta : A \to B \) for a local surjection of Artinian rings \( B \to B_0 \) as on the diagram below?

\[
\begin{array}{ccc}
B & \longrightarrow & B_0 \\
\uparrow & \searrow & \\
A & \nearrow & \\
\end{array}
\]

For technical reasons we assume that \( m_B \cdot \ker(B \to B_0) = 0 \) - this is called a small extension. Every local surjection of Artinian rings is a composition of small extensions so by restricting to such morphisms we don’t lose any information.

Let \( D = \text{Hom}(A, -) \). By the Cohen structure theorem [Coh46] \( A \) can be presented as a quotient of \( \hat{S} := k[a_1, \ldots, a_d] \) for \( d = \operatorname{dim}(A) \) of the form \( A = \hat{S}/I \) where \( I \subset \hat{m}_\hat{S}^2 \). We can always lift
the morphism $k[a_1, \ldots, a_d] \rightarrow B_0$ to $B$ (not necessarily uniquely), so we obtain a diagram:

$$
\begin{array}{cccc}
0 & \rightarrow & I & \rightarrow & \hat{S} & \rightarrow & A & \rightarrow & 0 \\
\downarrow{\mu|_I} & & \downarrow{\mu} & & \downarrow{\eta_0} & & \\
0 & \rightarrow & K & \rightarrow & B & \rightarrow & B_0 & \rightarrow & 0 \\
\end{array}
$$

The restriction $\mu|_I$ doesn’t depend on the choice of $\mu$ and $m_B \cdot K = 0$, so we get a homomorphism $I/mI \rightarrow K$. In other words we get an element of $(1/mI)^\vee \otimes_k K$ which vanishes if and only if the restriction $A \rightarrow B$ exists. If we denote $Ob = (1/mI)^\vee$ and $D(C) = \text{Hom}(A, C)$ then we get a functorial (under morphisms of small extensions) sequence:

$$D(B) \rightarrow D(B_0) \rightarrow Ob \otimes_k K \tag{8}$$

It is exact which in this case means that $\eta_0 \in D(B_0)$ goes to 0 if and only if there exists $\eta \in D(B)$ which restricts to $\eta_0$. In general obstruction theories provide such exact sequences for every small extension.

**Definition 2.18.** Let $D$ be a functor $\text{Art}_k \rightarrow \text{Set}$ such that $D(k) = \{ \text{pt.} \}$ and let $T_D := D(k[\varepsilon])$. An obstruction theory for $D$ is a finitely dimensional vector space $Ob_D$ over $k$ such that for all small extensions $0 \rightarrow K \rightarrow B \rightarrow B_0 \rightarrow 0$ there is given a map $ob : D(B_0) \rightarrow Ob_D \otimes_k K$ with the following properties:

1. for all $\eta \in D(B_0)$, $ob(\eta_0) = 0$ if and only if $\eta_0$ extends to $\eta \in D(B)$,
2. if $\eta_0 \in D(B_0)$, then the set of extensions of $\eta_0$ to $\eta \in D(B)$ is affine over $T_D \otimes_k K$ (we will not use this condition so we do not explain it).

We also require that $ob$ is functorial with respect to morphisms of small extensions which means that if we have a commutative diagram where horizontal lines are small extensions:

$$
\begin{array}{cccc}
0 & \rightarrow & K & \rightarrow & B & \rightarrow & B_0 & \rightarrow & 0 \\
\downarrow{\rho|_K} & & \downarrow{\rho} & & \downarrow{\rho_0} & & \\
0 & \rightarrow & K' & \rightarrow & B' & \rightarrow & B_0' & \rightarrow & 0 \\
\end{array}
$$

then we get the following commutative diagram:

$$
\begin{array}{ccc}
D(B) & \rightarrow & D(B_0) & \rightarrow & Ob \otimes_k K \\
\downarrow{D(\rho|_K)} & & \downarrow{D(\rho_0)} & & \downarrow{D(\rho|_K)} \\
D(B') & \rightarrow & D(B_0') & \rightarrow & Ob \otimes_k K' \\
\end{array}
$$

**Example 2.19.** Now we will discuss primary obstructions. Let $D \simeq \text{Hom}(A, -)$, be pro-representable by $(A, m)$ where $A = \hat{S}/I$ for some $I \subset m_S^2$. Then $A/m^2 \simeq \hat{S}/m_S^2$ and we get a homomorphism $\eta : A \rightarrow A/m^2 \simeq \hat{S}/m_S^2$. We can try to prolong it to a morphism $\mu : A \rightarrow \hat{S}/m_S^3$. Then assuming that $D$ has an obstruction theory $Ob$ we get a sequence as in (8) with $B_0 = \hat{S}/m_S^2$, $B = \hat{S}/m_S^3$ and $K = m_S^2/m_S^3$. Hence, an extension $\mu$ exists if and only if $ob(\eta) \in Ob \otimes_k K$ is 0. However, in this case an element of $Ob \otimes_k K$ can be interpreted as a homomorphism $ob_0 : K^\vee \rightarrow Ob$. Consider the identifications:

$$K^\vee = (m_S^2/m_S^3)^\vee \simeq \text{Sym}_2(m_S/m_S^3)^\vee \simeq \text{Sym}_2(\frac{m_S}{m_S^3}) = \text{Sym}_2(TmA) \tag{9}$$

where the middle isomorphism comes from duality defined by equation (2). By identifying $K^\vee$ with $\text{Sym}_2(TmA)$ we can see $ob_0$ as a map $\text{Sym}_2(TmA) \rightarrow Ob$ or a bilinear symmetric map $TmA \times TmA \rightarrow Ob$.

Now suppose that $(A', m')$ is another Noetherian complete local $k$-algebra and we have a surjection $\xi : A \rightarrow A'$. Assume also that $A' = \hat{T}/J$ for $\hat{T} := k[\gamma_1, \ldots, \gamma_m]$, $m_{\hat{T}} := (\gamma)$, and $J \subset m_{\hat{T}}$. We will describe how to use primary obstructions to get some information about $J$. 

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First note that we can present \( A \) as a quotient of \( k[[\gamma, \beta]] \) by an ideal contained in \((\gamma, \beta)^2\) so that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\zeta} & A' \\
\uparrow & & \uparrow \\
k[[\gamma, \beta]] & \xrightarrow{p} & k[[\gamma]]
\end{array}
\]

where \( p \) is defined by fixing \( \gamma \) and taking \( \beta \) to 0. After dividing by squares of maximal ideals we get:

\[
\begin{array}{ccc}
A/\mathfrak{m}^2 & \longrightarrow & A'/(\mathfrak{m}')^2 \\
\uparrow \cong & & \uparrow \cong \\
k[[\gamma, \beta]]/(\gamma, \beta)^2 & \xrightarrow{p} & k[[\gamma]]/(\gamma)^2
\end{array}
\]

Next we invert isomorphisms to get:

\[
\begin{array}{ccc}
A/\mathfrak{m}^2 & \longrightarrow & A'/(\mathfrak{m}')^2 \\
\uparrow \eta' & & \uparrow \mu' \\
k[[\gamma, \beta]]/(\gamma, \beta)^2 & \xrightarrow{p} & k[[\gamma]]/(\gamma)^2
\end{array}
\]

where \( \mu' \) is just the composition. We define \( \eta : A \to k[[\gamma, \beta]]/(\gamma, \beta)^2 \) and \( \mu : A \to k[[\gamma]]/(\gamma)^2 \) as compositions of the quotient map \( A \to A/\mathfrak{m}^2 \) with \( \eta' \) and \( \mu' \) respectively. Consider now the diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & K := (\gamma, \beta)^2/(\gamma, \beta)^2 & \longrightarrow & k[[\gamma, \beta]]/(\gamma, \beta)^2 & \longrightarrow & k[[\gamma, \beta]]/(\gamma, \beta)^2 \\
\downarrow r & & \downarrow & & \downarrow & & \downarrow p \\
0 & \longrightarrow & K' = (\gamma)^2/(\gamma)^2 & \longrightarrow & k[[\gamma]]/(\gamma)^2 & \longrightarrow & k[[\gamma]]/(\gamma)^2 
\end{array}
\]

We can apply the functor \( D \) and get:

\[
\begin{array}{ccccccc}
D(k[[\gamma, \beta]]/(\gamma, \beta)^2) & \longrightarrow & D(k[[\gamma, \beta]]/(\gamma, \beta)^2) & \longrightarrow & Ob \otimes_k K \\
\downarrow D(p) & & \downarrow & & \downarrow id \otimes r \\
D(k[[\gamma]]/(\gamma)^2) & \longrightarrow & D(k[[\gamma]]/(\gamma)^2) & \longrightarrow & Ob \otimes_k K'
\end{array}
\]

By the fact that \( D(p)(\eta) = p \circ \eta = \mu \) we get that \((id \otimes r)(ob(\eta)) = ob'(\mu)\). This yields the following commutative diagram:

\[
\begin{array}{ccc}
K^\vee & \xrightarrow{ob(\eta)} & Ob \\
\uparrow r^\vee & & \uparrow ob'(\mu) \\
(K')^\vee & & 
\end{array}
\]

By identifying \( K^\vee \) with \( \text{Sym}_2(T_mA) \) and \((K')^\vee \) with \( \text{Sym}_2(T_{m'}A') \) as in equation (9) we get the diagram:

\[
\begin{array}{ccc}
\text{Sym}_2(T_mA) & \xrightarrow{ob(\eta)} & Ob \\
\uparrow r^\vee & & \uparrow ob'(\mu) \\
\text{Sym}_2(T_{m'}A') & & 
\end{array}
\]
The map \( \text{ob}(\eta) \) is the primary obstruction map \( \text{ob}_0 \) by its definition. Hence, \( \text{ob}'(\mu) \) is the restriction of primary obstructions to \( \text{Sym}_2(T_mA') \). This allows us to give a lower bound for the second degree part of \( J \) if we can calculate the restriction of primary obstruction map \( \text{ob}_0 \) to \( \text{Sym}_2(T_mA') \):

**Proposition 2.20.** Consider the map \( \text{ob}'(\mu) : \text{Ob} \to K \). Then:

\[
(10) \quad \text{im}(\text{ob}'(\mu)^\vee) \subset \frac{(\gamma_1, \ldots, \gamma_m)^3 + J}{(\gamma_1, \ldots, \gamma_m)^2}.
\]

**Proof.** Consider the commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\zeta} & A' \\
\downarrow{\nu} & & \downarrow{\mu} \\
k[\gamma_1, \ldots, \gamma_m] & \xrightarrow{\text{id}} & k[\gamma_1, \ldots, \gamma_m] + J \\
\end{array}
\]

where maps from \( A' = k[\gamma_1, \ldots, \gamma_m] \) to \( k[\gamma_1, \ldots, \gamma_m] + J \) and \( k[\gamma_1, \ldots, \gamma_m] \) are quotients, and \( \nu, \mu \) are appropriate compositions with \( \zeta \). Now look at the map of extensions:

\[
\begin{array}{ccc}
0 & \to & K' = \frac{(\gamma_1, \ldots, \gamma_m)^2}{(\gamma_1, \ldots, \gamma_m)^3} + J \\
\downarrow{u} & & \downarrow{=} \\
0 & \to & K'' = \frac{(\gamma_1, \ldots, \gamma_m)^2}{(\gamma_1, \ldots, \gamma_m)^3} + J \\
\end{array}
\]

By applying \( D \), we get the following:

\[
\begin{array}{ccc}
D(B) & \xrightarrow{=} & D(B_0) \\
\downarrow & & \downarrow{\text{id} \otimes u} \\
D(B') & \xrightarrow{=} & D(B'_0) \\
\end{array}
\]

The fact that \( (\text{id} \otimes u)(\text{ob}'(\mu)) = 0 \) translates into:

\[
\begin{array}{ccc}
(K')^\vee \xrightarrow{\text{ob}'(\mu)} & \text{Ob} \\
\downarrow{u^\vee} & & \downarrow{=} \\
(K')^\vee & \to & 0 \\
\end{array}
\]

So \( \text{ob}'(\mu) \circ u^\vee = 0 \), which means that \( \text{im}(\text{ob}'(\mu)^\vee) \subset \ker u = \frac{(\gamma_1, \ldots, \gamma_m)^3 + J}{(\gamma_1, \ldots, \gamma_m)^2} \). □

The following lemma will be useful for calculating primary obstructions. In its statement we use the fact that in the category of rings over \( k \) product of \( k[x] \) and \( k[y] \) is \( k[x,y] \).

**Lemma 2.21.** Let \( \delta : A \to \frac{k[x]}{(e)} \), \( \delta' : A \to \frac{k[y]}{(e')} \) and denote by also \( \delta, \delta' \) the corresponding elements in \( T_mA \). Suppose that the induced homomorphism \( (\delta \times \delta') : A \to \frac{k[x', x]}{(e', e')} \) can be extended to a homomorphism to \( \frac{k[x']}{(e')} \):

\[
\begin{array}{ccc}
A & \xrightarrow{\delta \times \delta'} & \frac{k[x', x]}{(e', e')} \\
\downarrow{\gamma} & & \downarrow{=} \\
\frac{k[x']}{(e')} & & \frac{k[x']}{(e')} \\
\end{array}
\]

Then the primary obstruction map \( \text{ob}_0 \) vanishes on \( \delta \cdot \delta' \in \text{Sym}_2(T_mA) \).
Proof. Consider a morphism of small extensions:

\[
0 \rightarrow \frac{(e \circ e')^2}{(e' \circ e')} \rightarrow \frac{k[e']}{(e' \circ e')} \rightarrow \frac{k[e']}{(e' \circ e')} \rightarrow 0
\]

\[
0 \rightarrow \frac{(e \circ e')^2}{(e' \circ e')} \rightarrow \frac{k[e']}{(e' \circ e')} \rightarrow \frac{k[e']}{(e' \circ e')} \rightarrow 0
\]

By applying \( D = \text{Hom}(A, -) \) on it we get:

\[
\begin{array}{cccc}
D\left(\frac{k(e')}{(e' \circ e')}\right) & \rightarrow & D\left(\frac{k(e')}{(e' \circ e')}\right) & \rightarrow & \text{ob} \otimes \frac{(e' \circ e')^2}{(e' \circ e')} & \rightarrow & \delta \times \delta' & \rightarrow & \text{ob}(\delta \times \delta') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
D\left(\frac{k(e')}{(e' \circ e')}\right) & \rightarrow & D\left(\frac{k(e')}{(e' \circ e')}\right) & \rightarrow & \text{ob} \otimes \frac{(e' \circ e')^2}{(e' \circ e')} & \rightarrow & \delta \times \delta' & \rightarrow & 0
\end{array}
\]

The fact that \((\text{id} \otimes s)(\text{ob}(\delta \times \delta')) = 0\) translates to:

\[
\begin{array}{ccc}
\frac{(e \circ e')^2}{(e' \circ e')} & \text{ob}(\delta \times \delta') & \rightarrow \text{Ob} \\
\gamma \rightarrow & \delta \times \delta' & \rightarrow 0
\end{array}
\]

By the discussion before Proposition 2.20 for \( A' = \frac{k(e')}{(e' \circ e')} \) we get that \( \text{ob}(\delta \times \delta') \) is the restriction of the primary obstruction map \( \text{ob}_0 : \text{Sym}_2(T_m A) \rightarrow \text{Ob} \) to the image of \( \text{Sym}_2(T(e', d') \frac{k(e')}{(e' \circ e')}^2) \) by the tangent map \( d(\delta \times \delta') \). However, by definition of \( \delta \times \delta' \) we have:

(11) \[ d(\delta \times \delta')(e^\vee) = \delta \]

and

(12) \[ d(\delta \times \delta')(e^\vee) = \delta' \]

The map \( s^\vee \) on the above diagram is the embedding of \( k(e \cdot e')^\vee \) into \( \text{Sym}_2(k(e \cdot e') \oplus k(e' \cdot e')^\vee) \). Thus, using the identifications (11) and (12) we get that \( \text{ob}_0(\delta \cdot \delta') = 0 \). This finishes the proof. \( \square \)

Example 2.22. Fix a k-point \( x \in \mathcal{H} \). Then \( x \) is given by an ideal \( I \subset S \) such that \( S/I \) of degree \( d \). Consider the functor \( D_{\mathcal{H}, X} \) pro-representable by \( \tilde{O}_{\mathcal{H}, X} \). Its tangent space is \( \text{Hom}_S(I, S/I) \) by Theorem 2.3. It is also known that \( D_{\mathcal{H}, X} \) has an obstruction theory with the obstruction space \( \text{Ob} = \text{Ext}_I^1(I, S/I) \). For a proof see [FGH05, Corollary 6.4.10]. Thus in this case the primary obstruction map is a function \( \text{ob}_0 : \text{Hom}_S(I, S/I) \rightarrow \text{Ext}_I^1(I, S/I) \).

We will need the method of calculating primary obstructions for the functor in Example 2.22.

It comes from the paper [JS21]. Consider an ideal \( I \subset S \) and its free resolution as an S-module \( (F_i)_{i \geq 1} \). Then the S-module \( S/I \) has a free resolution which starts with the quotient map \( q : F_0 = S \rightarrow S/I \) and later is continued by the resolution \( (F_i)_{i \geq 1} \) using the composition \( F_1 \rightarrow I \subset S = F_0 \). For a given \( \delta \in \text{Hom}_S(I, S/I) \), we can lift it to S-module homomorphisms as on the diagram below (not necessarily uniquely):

\[
\begin{array}{ccccccc}
0 & \rightarrow & I & \rightarrow & F_0 & \rightarrow & F_1 & \rightarrow & F_2 & \rightarrow & F_3 & \rightarrow & \cdots \\
\delta & \circ & q & \rightarrow & s_1(\delta) & \rightarrow & s_2(\delta) & \rightarrow & s_3(\delta) & \rightarrow & \cdots
\end{array}
\]

Under this notation the following theorem holds:
Theorem 2.23. [JS21, Theorem 4.18] Consider a point represented by an ideal $I \subset S$ on the Hilbert scheme $\mathcal{H}$ and the associated obstruction theory with obstruction group $\text{Ob} = \text{Ext}^1_{S}(I, S/I)$. Then for $\delta_1, \delta_2 \in \text{Hom}_{S}(I, S/I)$ the primary obstruction map sends the class of $\delta_1 \cdot \delta_2 \in \text{Sym}_2(\text{Hom}_{S}(I, S/I))$ to the class of:

$$q \circ (s_1(\delta_1) \circ s_2(\delta_2) + s_1(\delta_2) \circ s_2(\delta_1))$$

in $\text{Ext}^1_{S}(I, S/I)$.

Note that $q \circ s_1(\delta_1) \circ s_2(\delta_2) \circ d_2 = \delta_1 \circ d_0 \circ d_1 \circ d_3(\delta_2) = \delta_1 \circ d_0 \circ d_3(\delta_2) = 0$, so (13) indeed yields an element of $\text{Ext}^1_{S}(I, S/I)$.

2.5. Barycenter. It will be convenient for us not to work with the Hilbert scheme $\mathcal{H}$ itself, but with its closed subset consisting of tuples which are centered around 0. In order to construct it we need to know that the support map which takes a point of the Hilbert scheme and associates with its closed subset consisting of tuples which are centered around 0. In order to construct it we need to know that the support map which takes a point of the Hilbert scheme and associates with its support (in the symmetric product of the ambient variety) is a morphism of schemes.

This is summarised in the following theorem from [FGI+05, Theorem 7.1.14]:

Theorem 2.24. There is a morphism of schemes (called the Hilbert-Chow morphism):

$$\theta : \mathcal{H} \to (\mathbb{A}^n)^d / S_d$$

which on the level of sets is given by: $[Z] \mapsto \sum_p \deg(\mathcal{O}_{Z,p})[p]$, where $Z$ is a subscheme of $\mathbb{A}^n$ as in Definition 2.1.

Here the quotient $(\mathbb{A}^n)^d / S_d$ is by the definition the spectrum of the algebra of global sections of $(\mathbb{A}^n)^d$ invariant under the action of the symmetric group $S_d$ acting by permuting points. It is a categorical quotient because $(\mathbb{A}^n)^d$ is an affine scheme.

Now we will define the barycenter scheme. The morphism $(\mathbb{A}^n)^d \to \mathbb{A}^n$ given by taking the average of $d$ vectors is $S_d$-equivariant, so it descends to a morphism $Av : (\mathbb{A}^n)^d / S_d \to \mathbb{A}^n$. By taking composition with the Hilbert-Chow morphism we get the barycenter morphism

$$\text{Bar} : \mathcal{H} \to \mathbb{A}^n.$$

Definition 2.25. The barycenter scheme is $\mathcal{B} := \text{Bar}^{-1}(\{0\})$. If we want to emphasize $d$ we write $\mathcal{B}^d$.

To study the barycenter scheme we give a more explicit definition of the Hilbert-Chow morphism, more precisely a formula for the composition $\text{Spec}(A) \to \mathcal{H} \to (\mathbb{A}^n)^d / S_d$ for a ring $A$ such that $\text{Spec}(A) \to \mathcal{H}$ is given by a free family $R$ over $A$. For $r \in R$ let $\text{Nm}_{R/A}(r)$ and $\text{Tr}_{R/A}(r)$ be respectively the determinant and the trace of the endomorphism of $R$ (as an $A$-module) given by multiplying by $r$. Then the following theorem from the notes [Ber08, Theorem 2.16, Corollary 2.18] holds:

Theorem 2.26. Suppose a morphism $\text{Spec}(A) \to \mathcal{H}$ is given by a free $A$-algebra $R$. Then the composition:

$$\begin{align*}
\text{Spec}(A) & \longrightarrow \mathcal{H} \longrightarrow (\mathbb{A}^n)^d / S_d \\
(k[\alpha] \otimes_k \cdots \otimes_k k[\alpha])^{S_d} & \longrightarrow (R \otimes_A \cdots \otimes_A R)^{S_d} \xrightarrow{\text{LNm}} A
\end{align*}$$

where the first map is the homomorphism $k[\alpha] \to k[\alpha] \otimes_k A \to R$ tensored $d$ times, and the second map $\text{LNm}$ is the unique $A$-algebra homomorphism that satisfies the equation:

$$\text{LNm}(r \otimes \cdots \otimes r) = \text{Nm}_{R/A}(r).$$

Moreover, if we consider the composition of the above morphism $\text{Spec}(A) \to (\mathbb{A}^n)^d / S_d$ with the average morphism $Av : (\mathbb{A}^n)^d / S_d \to \mathbb{A}^n$ we get a morphism $\text{Spec}(A) \to \mathbb{A}^n = \text{Spec}(k[\alpha])$ which on the level of algebras is given by the formula:

$$a_j \mapsto \frac{1}{d} \text{Tr}_{R/A}(a_j)$$

where on the right hand side we identify $a_j$ with its image in $R$. 

Every point of the Hilbert scheme can be uniquely shifted so that it is centered at 0, so there is an isomorphism $\mathcal{H} \cong \mathbb{A}^n \times B$ as one can prove using for example Theorem 2.26.

**Proposition 2.27.** The barycenter morphism $\text{Bar} : \mathcal{H} \to \mathbb{A}^n$ is equivariant under the action of the $n$-dimensional additive group $G^\mathbb{A}_n$. Hence $\mathcal{H} \cong \mathbb{A}^n \times B$.

**Proof.** We omit the proof of the $\mathbb{A}^n$-equivariance of $\text{Bar}$. The isomorphism $\mathcal{H} \cong \mathbb{A}^n \times B$ is given by the left vertical map on the diagram:

$$
\begin{array}{ccc}
\mathbb{A}^n \times B & \xrightarrow{id \times 0} & \mathbb{A}^n \times \{0\} \\
\downarrow{id \times j} & & \downarrow{id \times 0} \\
\mathbb{A}^n \times \mathcal{H} & \xrightarrow{id \times \text{Bar}} & \mathbb{A}^n \times \mathbb{A}^n \\
\downarrow{\mu} & & \downarrow{\mu^+} \\
\mathcal{H} & \xrightarrow{\text{Bar}} & \mathbb{A}^n
\end{array}
$$

where $j : B \to \mathcal{H}$ is the closed embedding. The bottom square is commutative because $\text{Bar}$ is $\mathbb{A}^n$-equivariant. The upper square is a pullback diagram, because of the definition of $B$. The bottom square is also a pullback, which follows from the existence of group inverse in $G^\mathbb{A}_n \cong \mathbb{A}^n$. Thus the left hand side map $\mathbb{A}^n \times B \to \mathcal{H}$ is a pullback of an isomorphism, so it is also an isomorphism and we get the conclusion. The details are left to the reader. $\square$

The isomorphism $\mathcal{H} \cong \mathbb{A}^n \times B$ respects torus action:

**Proposition 2.28.** The barycenter morphism $\text{Bar} : \mathcal{H} \to \mathbb{A}^n$ is $G_m$-equivariant. Moreover, the isomorphism $\mathcal{H} \cong \mathbb{A}^n \times B$ from Proposition 2.27 is $G_m$-equivariant, where the torus action on $B$ is the one induced from $G_m$-action on $\mathcal{H}$.

**Proof.** One should use Theorem 2.26. Details are left as an exercise for the reader. $\square$

The $G_m$-action on $B$ inherited from $\mathcal{H}$ allows us to perform the Białynicki-Birula decomposition on $B$ and get the diagram:

$$
\begin{array}{ccc}
B^+ & \xrightarrow{i} & B \\
\downarrow{s} & & \downarrow{\pi} \\
B^{G_m}
\end{array}
$$

For a $G_m$-invariant point $[l] \in B$ we identify it with $[l] \in B^{G_m}$ and $s([l]) \in B^+$ in the spirit of Fact 2.14. Now we can finally give a precise definition of a hedgehog point:

**Definition 2.29.** The negative spike at a $G_m$-invariant point $[l]$ is the fiber $V = \pi^{-1}([l]) \subset B^+$. The point $[l]$ is said to be a *hedgehog point* if the following conditions are satisfied:

1. $(T[l]B)_0 = 0$,
2. $T[l]B \neq (T[l]B)_0$,
3. $V$ is 0-dimensional.

The equivalence of this formulation and and the one in Definition 1.2 follows from the fact that $V$ has an $\mathbb{A}^1$ action, so it is 0-dimensional if and only if it is a point (topologically).

### 2.6. Macaulay duality

In order to find hedgehog points on the barycenter scheme we will use the construction of *apolar algebras*. More thorough introduction of this topic can be found in [Jel17].

**Definition 2.30.** For $P = k[x_1, \ldots, x_n]$ let $\alpha_1 := \frac{\partial}{\partial x_1}, \ldots, \alpha_n := \frac{\partial}{\partial x_n}$. Then the polynomial algebra $S = k[\alpha_1, \ldots, \alpha_n]$ generated by commuting derivatives acts on $P$ by taking derivatives. This action is called evaluation and is denoted by:

$$
\text{ev}_f : S \times P \to P.
$$

For $f \in P$ we also define $\text{ev}_f : S \to P$ by the formula $\text{ev}_f(h) := h \downarrow f$. If $I \subset P$ is a $P$-submodule we define its dual as $I^\perp := \text{Ann}(I) = \{h(\alpha_1, \ldots, \alpha_n) \in k[\alpha_1, \ldots, \alpha_n] : \text{ev}_f(h) = 0 \text{ for all } f \in I\}$. 
The algebra $\text{Apolar}(I) := k[a_1, \ldots, a_n]/I^\perp$ is called the apolar algebra of $I$. When $I$ is generated by a single polynomial $F$ we simply write $\text{Apolar}(F)$ for $\text{Apolar}(I)$ and $F^\perp$ for $I^\perp$.

**Example 2.31.** If $I$ in the above definition is generated by $F$, then the ideal $I^\perp = F^\perp$ is given by the kernel of the map $ev_F : k[a_1, \ldots, a_n] \to k[x_1, \ldots, x_n]$. The quotient $\text{Apolar}(F) = k[a_1, \ldots, a_n]/F^\perp$ is isomorphic (as a linear space over $k$) to the image

$$ev_F(k[a_1, \ldots, a_n]) = \operatorname{lin}_k(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial^2 F}{\partial x_i \partial x_j}, \ldots, 1)$$

For example if $F = x^2 + y^2$, then $\text{Apolar}(F) \cong \operatorname{lin}_k(x^2 + y^2, x, y, 1)$ - it is four-dimensional, and so it gives a point in $\mathcal{H}^4$. Moreover, if $F$ is homogeneous, then so is $F^\perp$ hence $\text{Apolar}(F)$ is graded.

Later, in Subsection 3.3 we will need the relative Macaulay duality, so we introduce it here. We restrict ourselves to the case $n = 6$.

**Notation 2.32.** We define rings of new variables $k[\beta_1, \ldots, \beta_6]$ and $k[y_1, \ldots, y_6]$ so that $\beta_i := \frac{\partial}{\partial y_i}$. Additionally, we define a new free variable $t$ which will correspond to the torus action. In this case we get the relative evaluation map:

$$ev : k[t][a_1, \ldots, a_6, \beta_1, \ldots, \beta_6] \times k[t][x_1, \ldots, x_6, y_1, \ldots, y_6] \to k[t][x_1, \ldots, x_6, y_1, \ldots, y_6]$$

The action of $k[t, \lambda, \vec{\beta}]$ on $k[t, \vec{x}, \vec{y}]$ will be also denoted by $\lambda$ so that $ev((a, b)) = a \cdot \lambda b$. Moreover, if $b \in k[t, \vec{x}, \vec{y}]$ then $ev_b : k[t, \vec{x}, \vec{y}] \to k[t, \vec{x}, \vec{y}]$ is the map $ev_b(a) := a \cdot \lambda b$.

Fix a cubic $F$. We write $F_x := F(x), F_y := F(y)$ and $F_x^\perp \subset k[a_1, \ldots, a_6], F_y^\perp \subset k[\beta_1, \ldots, \beta_6]$ for the dual ideals. We will also use polynomials $F_x \cdot F_y$ or $F(t \cdot \vec{x} + \vec{y})$ of mixed variables and in such cases we will usually omit in which ring we are calculating those, unless it is unclear.

The following polynomial will later yield the fractal family mentioned in the introduction.

**Definition 2.33.** Suppose $g \in k[\lambda]$ is such that $g \cdot F = 1$ and $(Q_i)_{1 \leq i \leq 6}$ are such that $Q_i \cdot F = x_i$. Let $\Gamma(i) \in k[t, \lambda, \vec{\beta}]$ be defined by:

$$\Gamma(i) := g(\vec{\lambda}) + t \cdot \sum \beta_i \cdot Q_i(\vec{\lambda}) + t^2 \cdot \sum a_i \cdot Q_i(\vec{\beta}) + t^3 \cdot g(\vec{\beta}).$$

**Lemma 2.34.** The following identity holds:

$$\Gamma(t) \cdot (F_x \cdot F_y) = F(t \cdot \vec{x} + \vec{y}).$$

**Proof.** The idea is to use Taylor’s formula. First note that:

$$F(\vec{x} + \vec{y}) = F(\vec{x}) + \sum_i \frac{\partial F}{\partial x_i}(\vec{x}) \cdot y_i + \sum_{ij} \frac{\partial^2 F}{\partial x_i \partial x_j}(\vec{x}) \cdot y_i y_j + \ldots$$

and also:

$$F(\vec{x} + \vec{y}) = F(\vec{y}) + \sum_i \frac{\partial F}{\partial y_i}(\vec{y}) \cdot x_i + \sum_{ij} \frac{\partial^2 F}{\partial y_i \partial y_j}(\vec{y}) \cdot x_i y_j + \ldots$$

On the other hand, $F$ is a cubic, so $F(\vec{x} + \vec{y})$ can be think of as a cubic with variable $\vec{x}$ and coefficients in $k[\vec{y}]$. From the second Taylor expansion we see that the part of $x$-degree zero is $F(\vec{y})$ and the part of $x$-degree one is $\sum \frac{\partial F}{\partial y_i}(\vec{y}) \cdot x_i$. Moreover, using the first Taylor expansion we get that the part of $F(\vec{x} + \vec{y})$ of $x$-degree at least two is $F(\vec{x}) + \sum \frac{\partial^2 F}{\partial x_i \partial x_j}(\vec{x}) \cdot y_i y_j$. Thus in fact:

$$F(\vec{x} + \vec{y}) = F(\vec{x}) + \sum_i \frac{\partial F}{\partial x_i}(\vec{x}) \cdot y_i + \sum_i \frac{\partial F}{\partial y_i}(\vec{y}) \cdot x_i + F(\vec{y})$$

so also:

$$F(t \cdot \vec{x} + \vec{y}) = t^3 \cdot F(\vec{x}) + t^2 \cdot \sum \frac{\partial F}{\partial x_i}(\vec{x}) \cdot y_i + t \cdot \sum \frac{\partial F}{\partial y_i}(\vec{y}) \cdot x_i + F(\vec{y})$$

By the assumptions about $g$’s and $Q_i$’s action on $F$ from Definition 2.33 we get the conclusion. □

The following observation will be useful:
**Lemma 2.35.** Let \((F_x \cdot F_y)^\perp \subset k[t, \bar{\alpha}, \bar{\beta}]\) and \(F_x^\perp \subset k[\bar{\alpha}], F_y^\perp \subset k[\bar{\beta}]\). Then:

\[(F_x \cdot F_y)^\perp = (F_x^\perp, F_y^\perp).\]

**Proof.** Consider the composition of the evaluation maps on the diagram:

\[
\begin{array}{ccc}
k[t, \bar{\alpha}, \bar{\beta}] & \xrightarrow{ev_F(g)} & k[t, \bar{\alpha}, \bar{\beta}] \\
& \xrightarrow{ev_F(g)} & k[t, \bar{\alpha}, \bar{\beta}] \\
& \xrightarrow{ev_F(g)} & k[t, \bar{\alpha}, \bar{\beta}] \\
\end{array}
\]

Note that \(\ker ev_F(g) = k[t, \bar{\beta}] \cdot F_x^\perp\), \(\ker ev_F(g) = k[t, \bar{\alpha}] \cdot F_y^\perp\) and \((ev_F(x))(k[t, \bar{\alpha}] \cdot F_y^\perp) = k[t, \bar{\alpha}] \cdot F_y^\perp\).

Take \(f \in (ev_F(x))^{-1}(k[t, \bar{\alpha}] \cdot F_y^\perp)\) i.e. in the kernel of the composition. This means that \(ev_F(x)(f) \in k[t, \bar{\alpha}] \cdot F_y^\perp\). Take \(h \in k[t, \bar{\alpha}] \cdot F_y^\perp\) such that \(ev_F(x)(h) = ev_F(x)(f)\). Then \(f - h \in \ker ev_F(x) = k[t, \bar{\beta}] \cdot F_x^\perp\). We present \(f\) as:

\[(27) \quad f = (f - h) + h \in k[t, \bar{\beta}] \cdot F_x^\perp + k[t, \bar{\alpha}] \cdot F_y^\perp.
\]

Thus, we have an inclusion \((ev_F(x))^{-1}(k[t, \bar{\alpha}] \cdot F_y^\perp) \subset k[t, \bar{\beta}] \cdot F_x^\perp + k[t, \bar{\alpha}] \cdot F_y^\perp\), since \(f\) was an arbitrary element of the left hand side. The other inclusion is immediate.

On the other hand the composition of \(ev_F(x)\) and \(ev_F(y)\) is \(ev_F(x,y)\) so by the definition its kernel is \((F_x \cdot F_y)^\perp\). This finishes the proof. \(\Box\)

Also, note that:

**Lemma 2.36.** The \(k[t, \bar{\beta}]\)-algebras \(\frac{k[t, \bar{\alpha}, \bar{\beta}]}{F(t \cdot \bar{x} + g)}\) and \(\frac{k[t, \bar{\beta}]}{F_y^\perp}\) are isomorphic.

**Proof.** Consider the natural \(k\)-algebra homomorphism \(\mu : k[t, \bar{\beta}] \rightarrow \frac{k[t, \bar{\alpha}, \bar{\beta}]}{F(t \cdot \bar{x} + g)}\). First we show that this map is onto. Note that:

\[(28) \quad (\alpha \cdot t \cdot \beta) \cdot F(t \cdot \bar{x} + g) = (\frac{\partial}{\partial x_i} - t \cdot \frac{\partial}{\partial y_i})(F(t \cdot \bar{x} + g)) = (\frac{\partial F}{\partial x_i})(t \cdot \bar{x} + g) \cdot t - t \cdot (\frac{\partial F}{\partial x_i})(t \cdot \bar{x} + g) = 0
\]

so \(\alpha \equiv t \cdot \beta (mod \ (t \cdot \bar{x} + g)^\perp)\), and we get surjectivity. We will prove now that the kernel of \(\mu\) is \(F_y^\perp\). Take \(f = f(\bar{\beta}, t) \in \ker \mu\). This means that:

\[(29) \quad f(\bar{\beta}, t) \cdot F(t \cdot \bar{x} + g) = 0
\]

Let \(f(\bar{\beta}, t) = \sum_i f_i(\bar{\beta}) \cdot t^i\). Then:

\[(30) \quad f(\bar{\beta}, t) \cdot F(t \cdot \bar{x} + g) = \sum_i (f_i(\bar{\beta}) \cdot F(t \cdot \bar{x} + g)) \cdot t^i = \sum_i (f_i(\bar{\beta}) \cdot F_y)(t \cdot \bar{x} + g) \cdot t^i.
\]

by the chain rule. Suppose (for a contradiction) that there exist \(i\) such that \(f_i(\bar{\beta}) \cdot F_y \neq 0\). Take the smallest such \(i\). Then the polynomial in variables \(x, \bar{y}\) near \(t^i\) in \(f(\bar{\beta}, t) \cdot F(t \cdot \bar{x} + g)\) has to be 0, so \((f_i(\bar{\beta}) \cdot F_y)(t \cdot \bar{x} + \bar{y})\) has all coefficients near \(x' \bar{y}'\) zero. But this is only possible if \(f_i(\bar{\beta}) \cdot F_y = 0\) which gives a contradiction. Thus all functions \(f_i\) belong to \(F_y^\perp\) and so \(f \in F_y^\perp\) which finishes the proof. \(\Box\)

This lemma yields an immediate corollary:

**Corollary 2.37.** The algebra \(\frac{k[t, \bar{\alpha}, \bar{\beta}]}{F(t \cdot \bar{x} + g)^\perp}\) is a free \(k[t]\)-module, because:

\[(31) \quad \frac{k[t, \bar{\alpha}, \bar{\beta}]}{F(t \cdot \bar{x} + g)^\perp} \simeq \frac{k[t, \bar{\beta}]}{F_y^\perp} \simeq k[t] \otimes_k \frac{k[\bar{\beta}]}{F_y^\perp}.
\]
3. THE MAIN EXAMPLE

In this section we fix symbols \( i, s, \pi \) for the maps:

\[
\begin{array}{ccc}
B^+ & \xrightarrow{i} & B \\
\downarrow \pi & & \downarrow \\
BG_w & & \\
\end{array}
\]

on the Białynicki-Birula decomposition for the \( G_w \)-action on \( B \) induced by the \( G_m \)-action on \( H \).

This makes sense as \( B \subset H \) is \( G_m \)-invariant by Proposition 2.28.

We start by proving the Hedgehog point theorem which we recall here:

**Theorem 3.1 (Hedgehog point theorem).** A hedgehog point \([I] \in B\) is non-reduced. Hence \((0,[I]) \in \mathbb{A}^n \times B \simeq H\) is non-reduced as well.

*Proof.* Assume on the contrary that \( B \) is reduced at \([I]\). By Proposition 2.16 the map \( i \) is an isomorphism on some open neighbourhood of \([I]\), because \( (T_{[I]}B)_{>0} = 0 \) as \([I] \) is a hedgehog point. Hence, \( B^+ \) is also reduced at \([I]\). From affineness of \( \pi \), see Theorem 2.13, we get that \( B^+ \) corresponds to some sheaf \( M \) of \( O_{G_m} \)-algebras and \( \pi \) is the natural morphism from the relative Spec of \( M \) to \( B^w \). Moreover, the sheaf \( M \) has an \( N \)-grading (as an \( O_{G_m} \)-module) coming from \( \mathbb{A}^1 \)-action. Thus, with a convention that \( \dim \emptyset = -1 \), for a point \( p \in B^w \) we get:

\[
\dim \pi^{-1}(p) = \dim \text{Spec}_{O_{G_m}}(M)|_p = \dim \text{Proj}_{O_{G_m}}(M)|_p + 1
\]

(32)

Thanks to the third condition of being a hedgehog point we know that the fiber of \( \pi \) over \([I]\) is 0-dimensional, so \( \dim \text{Proj}_{O_{G_m}}(M)|_{[I]} = -1 \). By upper semi-continuity of fibers of projective morphisms (e.g. [Har77, Theorem 12.8]) there is an open neighbourhood \( U \) of \([I]\) inside \( B^w \) such that for every \( p \in U \) we have \( \dim \text{Proj}_{O_{G_m}}(M)|_p = -1 \), so \( \dim \pi^{-1}(p) = 0 \). These fibers all have a unique fixed point of \( \mathbb{A}^1 \)-action so they are all topologically points. Thus we get a morphism:

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\pi^{-1}|_{\pi^{-1}(U)}} & U \\
\downarrow \pi^{-1}|_{\pi^{-1}(U)} & & \\
U & = & U \\
\end{array}
\]

which is affine and on the level of sets is a bijection. Note that the section \( s|_U \) is the inverse to \( \pi^{-1}|_{\pi^{-1}(U)} \) on the level of topological spaces, so \( \pi^{-1}|_{\pi^{-1}(U)} \) is in fact a homeomorphism. Consider the diagram:

\[
\begin{array}{ccc}
U & \xrightarrow{s|_U} & \pi^{-1}(U) \\
\downarrow \pi^{-1}|_{\pi^{-1}(U)} & & \downarrow \\
\pi^{-1}(U) & = & \pi^{-1}(U) \\
\end{array}
\]

The morphism \( \pi \) is affine, so separated. Thus, by Cancellation theorem [Vak17, 10.1.19] we get that \( s|_U \) is a closed embedding. But it also a homeomorphism, so it is in fact an isomorphism of schemes. Hence, \( T_{[I]}B^w = T_{[I]}U' \simeq \pi^{-1}|_{\pi^{-1}(U')} = T_{[I]}B^+ = T_{[I]}B \). This gives us a contradiction, because \( T_{[I]}B^w = (T_{[I]}B)_0 \) and \([I]\) is a hedgehog point so \( T_{[I]}B \neq (T_{[I]}B)_0 \).

Now we concentrate on the search for hedgehog points. We fix \( n = 6 \) and \( d = 13 \). Therefore \( H = \text{Hilb}_{\mathbb{A}^6/k}^{13} \) and \( B \) is the corresponding barycenter scheme so \( H \simeq \mathbb{A}^6 \times B \). Moreover, \( H^{14} \) denotes the Hilbert scheme of 14 points on \( \mathbb{A}^6 \). We start with the definition of being general enough:

**Definition 3.2.** Let \( F \) be a cubic in six variables \( x_1, \ldots, x_6 \). We call \( F \) general enough if the following conditions are satisfied:

1. \( \langle F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_6}, x_1, \ldots, x_6, 1 \rangle \) is a basis of \( \text{im}(ev_F) \) so that \( F^\perp \) gives a point \([F^\perp] \in H^{14} \).
(2) The minimal graded free resolution of $F^⊥$ starts with:

\[
0 \leftarrow F^⊥ \leftarrow S^{⊙15}(-2) \leftarrow S^{⊙35}(-3) \leftarrow \ldots
\]

(3) $[F^⊥] \in H^{14}$ has TNT (which is equivalent to $\dim(T_{[F^⊥]}H^{14})_{<0} = 6$, see Definition 2.11).

Let us elaborate about this definition for a while. First, condition (1) assures that $[F^⊥]$ gives us a point in $H^{14}$. Secondly, the condition about the minimal graded free resolution encodes two properties: it implies that (a) $F^⊥$ is generated by its second degree $(F^⊥)_2$ (which is 15-dimensional, because $\dim_k(F^⊥)_2 = \dim_k S_2 - \dim_k(S/F^⊥)_2 = 21 - 6 = 15$), and (b) the kernel of the surjection $F^⊥ \leftarrow S^{⊙15}(-2)$ is generated by the third degree, so just by the relations between generators of $F^⊥$ multiplied by linear forms. More thorough analysis of this resolution will appear in Subsection 3.1. The TNT condition assures the tame behaviour of tangent space to $[F^⊥]$.

Notice that all of these conditions are open (on the moduli space of cubics). Indeed, condition (1) is open, because it can be translated into a statement about the rank of some minors of $ev_F : S \rightarrow P$. The openness of condition (2) follows from semi-continuity of Betti numbers (for the definition of Betti numbers see [Eis05, Chapter 1B]). At last, the openness of condition (3) is implied by the semi-continuity of dimension of a fixed degree to a point in $(H^{14})^G_m$. It is because if we restrict the tangent sheaf to $(H^{14})^G_m$ it decomposes as a sum of sheaves of integer degrees and the result follows from half-continuity of dimension of a coherent sheaf on a scheme. Moreover, the set of general enough cubics is not only open, but also non-empty, as the below example shows:

**Example 3.3.** Let $F = x_1x_2x_4 - x_1x_5^2 + x_2x_3^2 + x_3x_5x_6 + x_4x_5^2$ as in Example 1.1. One can check by hand, or using a computer, that this $F$ satisfies conditions (1) and (2) in Definition 3.2. The fact that it satisfies the condition (3) can be deduced from [Jel18, lemma 3.6].

Now we describe a procedure of making a point $[I] \in H$ from a general enough cubic $F$. By the definition of being general enough the $k$-linear space $\text{link}_k(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_6}, x_1, \ldots, x_6, 1)$ is 14-dimensional and is generated by a basis $(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_6}, x_1, \ldots, x_6, 1)$. Then it follows that the algebra $S/F^⊥$ has a natural grading with the Hilbert function $(1, 6, 6, 1)$. Hence the third degree of $S/F^⊥$ is generated by a single element $g$ and we can assume that $ev_F(g) = 1$. We fix a representative of the class of $g$ in $S$ and denote it also by $g$. Let $I = (F^⊥, g) = F^⊥ + S_{\geq 3} \subset S$. Then $S/I \simeq \text{Apol}(F)/(g)$ is 13-dimensional, so it represents a point $[I] \in H$. Note that the ideal $I$ is homogeneous and the only maximal ideal which contains it is the ideal of $0 \in A^6$. Thus, $[I]$ is in fact an element of $B^G_m \subset B \subset H$ by the description of the Hilbert-Chow morphism in Theorem 2.24 and Definition 2.25.

In this section we fix a general enough cubic $F$ and $g$ such that $ev_F(g) = 1$. Also, $[I]$ denotes a point of $H$ constructed from $F$ by taking $I = (F^⊥, g)$. For our applications, the analysis of the $π$-fiber over $[I]$ will be important, so we fix a notation $V := π^{-1}([I])$. Recall that by Theorem 2.13, the fiber $V$ has a natural $A^1$-action and it is affine as the inverse image of an affine subscheme of $B^G_m$.

The structure of this section is as follows. First we study the tangent space $T_{[I]}H$ and check that it satisfies assumptions of Proposition 2.16. Then we take a closer look at the resolution of the ideal $I$ and using primary obstructions we prove that $V$ is 0-dimensional. At the end we study a deformation of $S/I$ coming from the fractal family to determine the structure of $V$ completely.

### 3.1. The analysis of the point $[I]$.

We start with the following proposition:

**Proposition 3.4.** The induced action of $G_m$ on $T_{[I]}H$ has no positive degrees.

**Proof.** Suppose $k ≥ 1$ and take $δ ∈ \text{Hom}_S(I, S/I)_k$. Then $δ(I_2) ⊂ (S/I)_{2+k} ⊂ (S/I)_{≥3} = 0$ because $S/I$ has its Hilbert function $(1, 6, 6)$. Similarly, $δ(I_3) = δ(I_4) = \cdots = 0$. Moreover $I$ is generated by its second and third degree (more precisely by $(F^⊥)_2$ and $g$, because $F$ is general enough) so $δ(I) = 0$. Thus we have proved that $\text{Hom}_S(I, S/I)_k = 0$ for $k ≥ 1$.

**Corollary 3.5.** The induced action of $G_m$ on $T_{[I]}B$ has no positive degrees.
Proof. This follows from the Proposition 3.4, because $T[I]B$ is a linear subspace of $T[I]\mathcal{H}$. □

**Corollary 3.6.** The map $i : B^+ \to B$ is an open embedding near $[I]$ i.e. there exist an open $G_m$-invariant neighbourhood $U \subset B^+$ of $s([I])$ such that $i|_U : U \to B$ is an open immersion. In particular, we identify $T[I]B^+ = T[I]B$.

**Proof.** This follows from Corollary 3.5 and Proposition 2.16. □

To study a neighbourhood of $[I] \in \mathcal{H}$ and make use of obstruction theory on $\mathcal{H}$ we need to analyse resolutions of $F^\perp$ and $I$. First, we take the minimal graded free resolution of the $S$-module $F^\perp$ which is of the form below, because $F$ is general enough (Definition 3.2):

$$
0 \leftarrow F^\perp \leftarrow I \leftarrow S^{\oplus 15}(-2) \leftarrow S^{\oplus 35}(-3) \leftarrow \ldots
$$

Here $d_0$ sends the natural basis $(E_i)_{1 \leq i \leq 15}$ of $S^{\oplus 15}(-2)$ to fifteen generators of $F^\perp$. The ideal $F^\perp$ is generated by $(F^\perp)_2$ (by $F$ being general enough) which is 15-dimensional as $\dim_k (F^\perp)_2 = \dim_k S_2 - \dim_k (S/F^\perp)_2 = 21 - 6 = 15$. The number 35 appears, because if we look at the third degree of $0 \leftarrow F^\perp \leftarrow S^{\oplus 15}(-2)$ then we get $0 \leftarrow (F^\perp)_3 \leftarrow S_1 \oplus_k S^{\oplus 15} \otimes_k \mathfrak{g}$, so the kernel of the map $d_0$ is at least $6 \cdot 15 - \dim_k (F^\perp)_3 = 90 - (\dim_k S_3 - \dim_k (S/F^\perp)_3) = 90 - (8/3 - 1) = 35$-dimensional.

Next we readjust this resolution so that it works for $I$:

$$
0 \leftarrow I \leftarrow I \leftarrow S^{\oplus 15}(-2) \oplus S(-3) \leftarrow S^{\oplus 35}(-3) \oplus S^{\oplus 6}(-4) \leftarrow d_2 \ldots
$$

Here for $G$ a generator of $S(-3)$, we define $d_0(G) = g$, and for $(H_k)_{1 \leq k \leq 6}$ - generators of $S^{\oplus 6}(-4)$, we put $d_1 (H_k) = \sum_{i=1}^{15} b_{ki} \cdot E_i + a_k \cdot G$ for quadrics $b_{ki}$'s such that $a_k \cdot g + \sum_{i=1}^{15} b_{ki} \cdot d_0(E_i) = 0$. The existence of such quadrics follows from the fact that $(F^\perp)_4 = S_4$. The ideal $I$ is generated by $F^\perp$ and $g$, so $d_0$ is surjective. Moreover, for $f \in \ker d_0$ there exists $h$ in $S^{\oplus 6}(-4)$ such that $f - d_1(h) \in \ker d_0 \cap S^{\oplus 15}(-2)$. But ker $d_0 \cap S^{\oplus 15}(-2) = d_1(S^{\oplus 35}(-3))$, because we have started from a resolution of $F^\perp$.

Summarising, we got a resolution of $I$. Using it, we can produce new negative tangents to $[I] \in \mathcal{H}$. Take $Q \in (S/I)_2$. Let $\mu_Q : S^{\oplus 15}(-2) \oplus S(-3) \to S/I$ be an $S$-module homomorphism defined by $\mu_Q (E_i) = 0$ and $\mu_Q (G) = Q$. Consider the following diagram:

![Diagram](image)

Then $\mu_Q \circ d_1 = 0$, because $\mu_Q(d_1 (S^{\oplus 35}(-3))) \subset \mu_Q(S^{\oplus 15}(-2)) = 0$ and $\mu_Q(d_1 (H_k)) = \mu_Q(\sum_{i=1}^{15} b_{ki} \cdot E_i + a_k \cdot G) = a_k \cdot Q = 0$ as $(S/I)_3 = 0$. Hence, by the exactness of the sequence, we get that $\mu_Q$ factors and gives us an $S$-module homomorphism which we will call $r_Q$.

**Definition 3.7.** Let $X := \operatorname{lim}_k (R_Q : Q \in (S/I)_2) < \operatorname{Hom}_S(I, S/I)$ for $r_Q$ defined as above, so:

$$
\begin{align*}
\forall Q : I &\to S/I \\
(F^\perp)_2 &\mapsto 0 \\
G &\mapsto Q
\end{align*}
$$

Fix $Q_1, \ldots, Q_6 \in S$ such that $ev_f(Q_1) = x_i$. Such elements exist because $F$ is general enough. Their classes in $S/I$, which we will denote also by $Q_1, \ldots, Q_6$, form a basis of $(S/I)_2$. It follows that $r_{Q_1} \ldots \cdot r_{Q_6}$ is a basis of $X$. We will also write $r$ for $r_{Q_6}$.

**Example 3.8.** We will determine the element $[I'] \in \operatorname{Mor} (\operatorname{Spec}(k[x]), \mathcal{H} | \{pt.\} \to [I]) = T[I] \mathcal{H}$ which corresponds to $r_Q \in W$. The point $[I']$ is given by an ideal $I' \subset S[\epsilon]$. By Remark 2.4 we have:

$$
I' = (f - r_Q(f) \cdot \epsilon : f \in I).
$$
But since \( I \) is generated by \( F^\perp \) and \( g \) we get:

\[
(f - r_Q(f) \cdot \varepsilon : f \in I) = F^\perp \cdot S[\varepsilon] + (g - \varepsilon Q) \cdot S[\varepsilon].
\]

So \( I' = F^\perp \cdot S[\varepsilon] + (g - \varepsilon Q) \cdot S[\varepsilon] \).

The \( k \)-algebra \( k[\varepsilon] \) is local, so \( S[\varepsilon] \) as a flat, finitely generated \( k[\varepsilon] \)-module is free. Consider elements \((1, a_1, \ldots, a_6, Q_1(\bar{a}), \ldots, Q_6(\bar{a}))\). Their classes mod \( \varepsilon \) form a basis of \( S[\varepsilon] \), so by Lemma 2.5 we get that \((1, a_1, \ldots, a_6, Q_1(\bar{a}), \ldots, Q_6(\bar{a}))\) is a basis of the \( k[\varepsilon] \)-module \( \frac{S[\varepsilon]}{I} \).

**Remark 3.9.** The evaluation map allows us to define an isomorphism:

\[\lambda : X \to k[x_1, \ldots, x_6]_1\]

by sending \( r_Q \) to \( ev_F(Q) \in k[x_1, \ldots, x_6]_1 \). This identification is natural and under it tangents \( r_1, \ldots, r_5 \) correspond to \( x_1, \ldots, x_6 \).

The map \( \lambda \) from Remark 3.9 allows us to naturally shift subspace \( X \):

**Definition 3.10.** Let \( \nabla : k[x_1, \ldots, x_6]_1 \to k[\partial_1, \ldots, \partial_6]_1 \) be an isomorphism sending \( x_i \) to \( \partial_i \). We define:

\[Y := \text{lin}_k(r_Q - \frac{1}{13} \nabla(\lambda(r_Q)) : Q \in (S/I)_2)\]

In other words \( Y = \text{lin}_k(\eta_i - \frac{1}{13} \partial_i : i = 1, \ldots, 6) \). We will write \( \eta_i \) for \( \eta_i - \frac{1}{13} \partial_i \) so that \( Y = \text{lin}_k(\eta_i : i = 1, \ldots, 6) \).

Now we are ready to prove the following geometric characterization of the subspace \( Y \):

**Proposition 3.11.** The subspace \( Y \) is the tangent space to the fiber of \( \pi : B^+ \to B^{G_m} \) over \( I \in B^{G_m} \).

In other words \( Y = T[I]\) under the chain of inclusions:

\[T[I] V \subset T[I]B^+ = T[I]B = T[I]\mathcal{H} = \text{Hom}_S(I, S/I).\]

**Proof.** From Proposition 2.27 we know that \( \mathcal{H} = B \times A^6 \), so \( T[I]\mathcal{H} = T[I]B \oplus T[I]A^6 \) and the additive action of \( v \in A^6(k) \) on \([I] = ([I], 0) \in B \times A^6\) is given by \( v + ([I], 0) = ([I], v) \). Thus the subspace \( T[I]A^6 \subset T[I]\mathcal{H} \) is the image of the tangent map of the morphism of \([I]'s A^6\)-orbit in \( \mathcal{H} \).

By Theorem 2.10 we know explicit formulas for elements of this subspace, so we get:

\[
T[I]\mathcal{H} = T[I]B \oplus \text{lin}_k(\partial_1, \ldots, \partial_6).
\]

By Corollary 3.6 we have \( T[I]B^+ = T[I]B \). The morphisms \( \pi, s \) giving the diagram:

\[
\begin{array}{ccc}
B^+ & \xrightarrow{s} & X \\
\downarrow{\pi} & & \downarrow \chi \\
B^{G_m} & & \\
\end{array}
\]

are \( G_m \)-equivariant and induce a \( G_m \)-invariant decomposition \( T[I]B^+ = \ker d\pi \oplus \text{im} ds \) where a \( d \) means the induced morphism on tangent spaces at \([I]\). By the functorial description of \( B^{G_m} \) we have \( T[I]B^{G_m} \simeq \text{im} ds = (T[I]B)_0 \). Thus, by \( G_m \)-invariance \( \ker d\pi = (T[I]B)_{\neq 0} = (T[I]B)_{<0} \) where the last equality holds, because \( T[I]B \) doesn’t have any positive degrees by Corollary 3.5. By the definition of \( V \) we see that \( T[I]V = \ker d\pi < T[I]B \). Hence, overall we get that:

\[
T[I]V = (T[I]B)_{<0} < T[I]\mathcal{H}.
\]

In order to determine \( (T[I]B)_{<0} \) we calculate \( (T[I]\mathcal{H})_{<0} \). Take an arbitrary \( \delta \in \text{Hom}_S(I, S/I) \) homogeneous of negative degree. It induces the maps from the resolution of \( I \):

\[
0 \leftarrow I \leftarrow S^{\oplus 15}(\bar{-2}) \oplus S(\bar{-3}) \leftarrow S^{\oplus 35}(\bar{-3}) \oplus S^{\oplus 6}(\bar{-4}) \leftarrow \ldots
\]
where \( \delta \) is just the composition. By restricting \( \delta \) to \( F^\perp \) and \( \delta \) to \( S^{\oplus 15}(-2) \) we get the diagram:

\[
\begin{array}{ccccc}
0 & \xleftarrow{\delta} & F^\perp & \xleftarrow{d_0} & S^{\oplus 15}(-2) & \xleftarrow{d_1} & S^{\oplus 35}(-3) & \rightarrow & \ldots \\
& & & & & & & & \\
& & \delta \downarrow & & \downarrow & & & & \\
& & S/I & & S/I & & & & \\
& & & & & \mu & & & & \\
& & & & & & & & \\
& & S/F^\perp & & & & & & \\
\end{array}
\]

where \( \mu \) is the natural map. The homomorphism \( \delta \) is of negative degree so \( \delta \) sends generators \( E_i \)'s of \( S^{\oplus 15}(-2) \) to \( (S/I)_0 \). However, the quotient \( \mu \) is an isomorphism in degrees \( \leq 2 \). So, in fact \( \delta \) factor through \( S/F^\perp \) and the composition with \( d_1 \) is still 0. Thus, there exists the unique \( \delta' \) such that the above diagram commutes. But this \( \delta' \) belongs to \( \text{Hom}_S(F^\perp, S/F^\perp) \cong T_{F^\perp} \mathcal{H}^{14} \) so it can only be a linear combination of derivatives \( \delta_i \) (because \( F \) is general enough so in particular \([F^\perp] \in \mathcal{H}^{14} \) has TNT).

Now we will consider two cases:

1. \( \delta \) is of degree \( \leq -2 \): Then \( \delta' \) also of degree \( \leq -2 \) so \( \delta' = 0 \), because partial derivatives are of degree \(-1\) and \( \delta' \) is their linear combination. Thus, \( \delta|_{(F^\perp)_2} = 0 \) or equivalently \( \delta|_{S^{\oplus 15}(-2)} = 0 \). Let \( \delta(G) = \delta(g) =: s \in (S/I)_2 \), we know that \( \delta \circ d_1 = 0 \), so:

\[
0 = \delta \circ d_1(H_k) = \delta(\sum_{i=1}^{15} b_{ij} \cdot E_i + \alpha_k \cdot G) = \delta(\alpha_k \cdot G) = \alpha_k \cdot s \in (S/I)_2
\]

Note that \( (S/I)_2 \cong (S/F^\perp)_2 \), so an element from \((S/I)_2\) is 0 if and only if its evaluation on \( F \) is 0. Hence, we get:

\[
ev_F(\alpha_k \cdot s) = \frac{\partial}{\partial x_k} ev_F(s) = 0 \text{ for all } k = 1, \ldots, 6
\]

The polynomial \( ev_F(s) \) is of degree \( \geq 2 \), because \( s \in (S/I)_1 \). But this means that \( ev_F(s) = 0 \) because it has all partial derivatives equal to 0. Thus \( s = 0 \) in \( S/I \) and so in this case \( \delta = 0 \).

2. \( \delta \) is of degree \(-1\): A homomorphism \( \delta' \in \text{Hom}_S(F^\perp, S/F^\perp) \) is a linear combination of partial derivatives in this case, because \([F^\perp] \in \mathcal{H}^{14} \) has TNT. If \( \partial := \sum_{i=1}^{6} c_i \partial_i \) is defined using the same coefficients, but for derivatives \( \partial_i \in \text{Hom}_S(I, S/I) \), then \( \delta - \partial = 0 \) under the restriction to \((F^\perp)_2\). Thus \( \delta - \partial \) has the properties that \([F^\perp]_2 \rightarrow 0 \) and \( g \rightarrow s \) for some element \( s \) of \((S/I)_2 \), because both \( \delta \) and \( \partial \) are of degree \(-1 \). Hence \( \delta - \partial \in X \) and we see that \( \text{Hom}_S(I, S/I)_{-1} = X \oplus \text{lin}_k(\partial_1, \ldots, \partial_6) \), as we have presented \( \delta \) as a sum of elements from \( \text{lin}_k(\partial_1, \ldots, \partial_6) \) and \( X \) in a unique way, namely \( \delta = \partial + (\delta - \partial) \).

Summarising, we have proved that:

\[
(T[H])_{<0} = (T[H])_{-1} = X \oplus \text{lin}_k(\partial_1, \ldots, \partial_6).
\]

By Definition 3.7 and Definition 3.10 we get that \( X \oplus \text{lin}_k(\partial_1, \ldots, \partial_6) = Y \oplus \text{lin}_k(\partial_1, \ldots, \partial_6) \) so we have:

\[
(T[H])_{<0} = (T[H])_{-1} = Y \oplus \text{lin}_k(\partial_1, \ldots, \partial_6).
\]

We will now show that \( Y \) lies in \( T[F^\perp]B \).

By Example 3.8 we have an explicit description of morphisms \( f_O \) as elements of \( \mathcal{H}(\text{Spec}(k[\epsilon])) \). We apply Theorem 2.26. Fix \( f_i \in X \) and consider the composition starting with the morphism coming from \( f_i \):

\[
\text{Spec}(k[\epsilon]) \longrightarrow \mathcal{H} \longrightarrow (\mathbb{A}^6)^{13} \rightarrow S_{13} \longrightarrow \mathbb{A}^6
\]
The composition, on the level of rings, gives a homomorphism \( k[\bar{a}] \to k[\bar{e}] \) which is the composition of:

\[
\begin{align*}
    k[\bar{a}] & \longrightarrow (k[\bar{a}] \otimes_k \cdots \otimes_k k[\bar{a}])^S_l \\
    & \longrightarrow (S^l_{I}[\bar{e}] \otimes_{k[\bar{e}]} \cdots \otimes_{k[\bar{e}]} S^l_{I}[\bar{e}])_{LNM} \to k[\bar{e}]
\end{align*}
\]

where \( I' = F^{-1} \cdot S[\bar{e}] + (g - \varepsilon Q_1) \) is the ideal from Example 3.8. By Theorem 2.26 this whole composition takes \( a_j \) to \( \frac{1}{13} \text{Tr}_{S_{I[\bar{e}]}[\bar{e}]/k[\bar{e}]}(a_j) \). By Example 3.8 we can take \( (a_1, a_2, \ldots, a_6, Q_1, \ldots, Q_6) \) as a basis of \( S^l_{I[\bar{e}]} \) over \( k[\bar{e}] \). Let \( A = (a_{lm}) \) be the matrix of multiplying by \( a_j \) with respect to this basis.

We analyse its diagonal coefficients:

- \( a_j \cdot 1 = a_j \) so \( a_{11} = 0 \).
- \( a_j \cdot a_k \) is a \( k \)-linear combination of polynomials \( Q_k \) modulo \( F^\perp \), so for \( k = 1, \ldots, 6 \) we have \( a_{kk} = 0 \).
- For \( k \neq j \) the polynomial \( a_j \cdot Q_k \) lies is \( F^\perp \) because \( ev_F(a_j \cdot Q_k) = ev_{x_j}(a_j) = 0 \). Hence \( a_j \cdot Q_k = 0 \) in \( S^l_{I[\bar{e}]} \) and here the diagonal coefficients are also zero.
- For \( k = j \) the polynomial \( a_j \cdot Q_j \) is equal to \( g \) modulo \( F^\perp \), because \( ev_F(a_j \cdot Q_j) = ev_{x_j}(a_j) = 1 = ev_F(\bar{g}) \). Thus, the coefficient near \( Q_j \) is \( \delta_{jj} \cdot \varepsilon \) in this case.

Overall, \( \frac{1}{13} \text{Tr}_{S_{I[\bar{e}]}[\bar{e}]/k[\bar{e}]}(a_j) = \frac{1}{13} \delta_{jj} \cdot \varepsilon \), so the induced tangent vector in \( T_{[I]} \mathbb{A}^6 \) is \( \frac{1}{13} \partial_j \). Thus the image of \( \eta_j = \varepsilon_j - \frac{1}{13} \partial_j \) under that tangent map of Bar is 0. This implies that \( Y < \ker dBar = T_{[I]} \mathcal{B} \).

Now we bring everything together. First, the equation (37) implies that:

(43) \( (T_{[I]} \mathcal{H})_{<0} = (T_{[I]} \mathcal{B})_{<0} \oplus \text{lin}(\partial_1, \ldots, \partial_6) \).

Furthermore, the equation (42) gives as:

(44) \( (T_{[I]} \mathcal{H})_{<0} = Y \oplus \text{lin}(\partial_1, \ldots, \partial_6) \).

But \( Y < T_{[I]} \mathcal{B} \) so also \( Y < (T_{[I]} \mathcal{B})_{<0} \) as elements of \( Y \) are of degree \(-1\). Hence we get:

(45) \( Y = (T_{[I]} \mathcal{B})_{<0} = T_{[I]} V \)

where the last equality is the equation (38). \( \square \)

By Example 2.22 the functor \( D_{\mathcal{H}, [I]} \) pro-representable by \( \hat{\mathcal{O}}_{\mathcal{H}, [I]} \) has an obstruction theory with obstruction space \( Ob = \text{Ext}^1_{\mathcal{O}}(I, S/1) \). Since we want to determine primary obstructions to \( V \), the following proposition will make the calculations easier:

**Proposition 3.12.** Let \( \text{ob}_0 : \text{Sym}_2(T_{[I]} \mathcal{H}) \to \text{Ext}^1_{\mathcal{O}}(I, S/1) \) be the primary obstruction map. Take \( \delta \in \text{lin}(\partial_1, \ldots, \partial_6) \subset T_{[I]} \mathcal{H} \) and \( \delta \in T_{[I]} \mathcal{H} \). Then \( \text{ob}_0(\delta \cdot \delta) = 0 \).

**Proof:** The idea is to use \( \mathbb{A}^n \)-action to extend \( \delta \) and then use Lemma 2.21.

Suppose that \( \delta \) correspond to the morphism \( \delta : \text{Spec}(\frac{k[\bar{e}]}{(\bar{e})^2}) \to \mathcal{H} \). By composing:

\[
\mathbb{A}^n \times \text{Spec}(\frac{k[\bar{e}]}{(\bar{e})^2}) \xrightarrow{id \times \delta} \mathbb{A}^n \times \mathcal{H} \xrightarrow{\mu} \mathcal{H}
\]

we get a \( \mathbb{A}^n \)-equivariant morphism which extends \( \delta \). The tangent vector \( \partial \) is in \( \text{lin}(\partial_1, \ldots, \partial_6) \) so it is induced by the \( \mathbb{A}^n \)-action on \( \mathcal{H} \). Take a morphism \( \bar{\delta} : \text{Spec}(\frac{k[\bar{e}]}{(\bar{e})^2}) \to \mathbb{A}^n \) such that its composition with the orbit map of \([I]\) is \( \partial : \text{Spec}(\frac{k[\bar{e}]}{(\bar{e})^2}) \to \mathcal{H} \). Consider the composition \( \eta \):

\[
\text{Spec}(\frac{k[\bar{e}]}{(\bar{e})^2}) \times \text{Spec}(\frac{k[\bar{e}]}{(\bar{e})^2}) \xrightarrow{\delta \times id} \mathbb{A}^n \times \text{Spec}(\frac{k[\bar{e}]}{(\bar{e})^2}) \xrightarrow{id \times \delta} \mathbb{A}^n \times \mathcal{H} \xrightarrow{\mu} \mathcal{H}
\]

\( \eta \)

\[
\text{Spec}(\frac{k[\bar{e}]}{(\bar{e})^2}) \xrightarrow{\delta \times id} \mathbb{A}^n \times \text{Spec}(\frac{k[\bar{e}]}{(\bar{e})^2}) \xrightarrow{id \times \delta} \mathbb{A}^n \times \mathcal{H} \xrightarrow{\mu} \mathcal{H}
\]
By its definition \( \eta \) restricted to Spec\( ( \frac{k[\epsilon]}{(\epsilon')} ) \) is \( \partial \) and \( \eta \) restricted to Spec\( ( \frac{k[\epsilon]}{(\epsilon')} ) \) is \( \delta \). Thus the following diagram commutes:

\[
\begin{array}{c}
\text{Spec} \left( \frac{k[\epsilon']}{(\epsilon')} \right) \quad \eta \quad \text{Spec} \left( \frac{k[\epsilon]}{(\epsilon')} \right) \\
\downarrow \quad \delta \quad \downarrow \\
\text{Spec} \left( \frac{k[\epsilon]}{(\epsilon')} \right) \sqcup \text{Spec} \left( k \right) \text{Spec} \left( \frac{k[\epsilon]}{(\epsilon')} \right)
\end{array}
\]

On the level of algebras this diagram is dual to the diagram in the statement of Lemma 2.21. Thus the existence of \( \eta \) implies that \( \partial b_0 (\partial \cdot \delta) = 0 \) by Lemma 2.21.

Now we work on the resolution of the ideal \( I \) in order to use Theorem 2.23 to calculate primary obstruction map restricted to Sym\( _2 (T_{[\bar{l}]}) \). Fix \( Q \in (S/I)_2 \). For \( r_Q : I \rightarrow S/I \) we will construct a lift of \( r_Q \) to a homogeneous chain complex map as in the diagram above Theorem 2.23. The liftings are not unique but we choose them so that they are amenable to calculations. We have:

\[
\begin{array}{c}
0 \\ r_Q \\ S/I \\ 0 \\
\downarrow \\
S \quad \downarrow q \\
S^\oplus 15 (-2) \oplus S(-3) \quad \downarrow d_0 \\
\text{Spec} \left( \frac{k[\epsilon']}{(\epsilon')} \right) \\
\downarrow \\
\text{Spec} \left( \frac{k[\epsilon]}{(\epsilon')} \right) \\
\downarrow \\
\text{Spec} \left( \frac{k[\epsilon]}{(\epsilon')} \right)
\end{array}
\]

The maps here are defined in the following way:

- We define \( s_1 (r_Q) : S^\oplus 15 (-2) \oplus S(-3) \rightarrow S \) which takes \( S^\oplus 15 (-2) \) to 0 and the generator \( G \) of \( S(-3) \) to \( S \) (by abuse of notation we denote by \( Q \) both an element of \( S/I \) and some quadric element in the inverse image \( q^{-1} (Q) \subseteq S \)). Then it is obvious that \( r_Q \circ d_0 = q \circ s_1 (r_Q) \).
- The second map to define is \( s_2 (r_Q) \). We declare it to be 0 on \( S^\oplus 35 (-3) \). Now it is enough to define it on the basis \( (H_k)_{1 \leq k \leq 6} \) of \( S^\oplus 6 (-4) \). Recall that \( (E_i)_{1 \leq i \leq 15} \) is the natural basis of \( S^\oplus 15 (-2) \). By the definition \( s_1 (r_Q) (E_i (s)) = 0 \), so \( s_1 (r_Q) \circ d_1 (H_k) = s_1 (r_Q) (\sum b_{ki} \cdot E_i + a_k \cdot G) = s_1 (r_Q) (\sum b_{ki}) \cdot E_i = a_k \cdot G \in S \) for \( b_{ki} \)'s as in the description of \( d_1 \). Note that \( a_k := ev_f (a_k, Q) \in k \) because \( a_k, Q \in S \). Hence \( ev_f (a_k \cdot Q - a_k \cdot g) = 0, \) so \( a_k \cdot Q - a_k \cdot g \in (F^+) \) and there exist linear forms \( a_k \), such that \( a_k \cdot Q - a_k \cdot g \) is the image of \( \sum a_k E_i \) by \( d_0 \) (because \( F^+ \) is generated by \( F^+ 2 \) by the fact that \( F \) is general enough). We fix them and denote by \( a_{k} = a_k (Q), \) since they depend on \( Q \). Now we define \( s_2 (r_Q) (H_k) := \sum a_k E_i + a_k G \). The constant \( a_k \) also depends on \( Q \), so we denote it \( a_k (Q) = ev_f (a_k \cdot Q) \).

This resolution gives us a canonical identification \( \text{Ext}^2_k (I, S/I) = \frac{\ker d_1}{\text{im} d_0} \), where

\[
d_1^i : \text{Hom}_S (S^\oplus 15 (-2) \oplus S(-3), S/I) \rightarrow \text{Hom}_S (S^\oplus 35 (-3) \oplus S^\oplus 6 (-4), S/I)
\]

is composed with \( d_1 \) from the right and

\[
d_2^* : \text{Hom}_S (S^\oplus 35 (-3) \oplus S^\oplus 6 (-4), S/I) \rightarrow \text{Hom}_S (F_3, S/I)
\]

is composed with \( d_2 \) from the right.

It will turn out that for our application only a part of the obstruction space will be needed. More precisely, the \( S \)-module \( \text{Hom}_S (S^\oplus 35 (-3) \oplus S^\oplus 6 (-4), S/I) \) splits as

\[
\text{Hom}_S (S^\oplus 35 (-3), S/I) \oplus \text{Hom}_S (S^\oplus 6 (-4), S/I)
\]

Let \( i_2 : S^\oplus 6 (-4) \rightarrow S^\oplus 35 (-3) \oplus S^\oplus 6 (-4) \) be the natural embedding. It induces

\[
i_2^* : \text{Hom}_S (S^\oplus 35 (-3) \oplus S^\oplus 6 (-4), S/I) \rightarrow \text{Hom}_S (S^\oplus 6 (-4), S/I).
\]
Consider the projection:

\[
\Ext^1_{\mathfrak{I}}(I, S/\mathfrak{I}) = \frac{\ker d_2^I}{\text{im } d_1^I} \to i_2^* (\ker d_2^I) \to i_2^* (\text{im } d_1^I).
\]

We have the following proposition, which will be crucial for proving 0-dimensionality of the scheme \(\mathcal{V}^1\):

**Proposition 3.13.** The primary obstruction map \(\text{ob}_0 : \text{Sym}_2(\mathfrak{T}_1 \mathcal{H}) \to \Ext^1_{\mathfrak{I}}(I, S/\mathfrak{I})\) restricted to \(\text{Sym}_2(\mathfrak{T}_1 \mathcal{V}) \subset \text{Sym}_2(\mathfrak{T}_1 \mathcal{H})\) and composed with the projection (46) yields a \(k\)-linear map:

\[
\Omega : \text{Sym}_2(\mathfrak{T}_1 \mathcal{V}) \to \frac{i_2^* (\ker d_2^I)}{i_2^* (\text{im } d_1^I)},
\]

whose kernel is \(\text{lin}_k(\frac{\partial F}{\partial x_i}(\eta_1, \ldots, \eta_6), \ldots, \frac{\partial F}{\partial x_b}(\eta_1, \ldots, \eta_6)) < \text{Sym}_2(\mathfrak{T}_1 \mathcal{V})\).

**Proof.** Fix \(D \in \text{Sym}_2(\mathfrak{T}_1 \mathcal{V})\) and write it in the form \(D = \sum_{1 \leq l, m \leq 6} d_{lm} \eta_l \eta_m\), \(d_{lm} = d_{ml}\) (we can assume symmetry of \(d_{lm}\) since the characteristic of the base field is not two). Recall that \(\eta_l = \frac{b_l}{n} \partial_i\). Using this equation we expand \(D\) and get:

\[
D = \sum_{1 \leq l, m \leq 6} d_{lm} \eta_l \eta_m = \sum_{1 \leq l, m \leq 6} d_{lm} \eta_l \eta_m + \sum_i \partial_i \cdot \delta_i
\]

for some \(\delta_i \in \mathfrak{T}_1 \mathcal{H}\). By \(k\)-linearity of \(\Omega\) and Proposition 3.12 we get that:

\[
\Omega(D) = \Omega(\sum_{1 \leq l, m \leq 6} d_{lm} \eta_l \eta_m).
\]

We will use the formula (13) from Theorem 2.23 to determine the right hand side of this equation. First we take any \(Q, Q' \in (S/\mathfrak{I})_2\) and calculate:

\[
q \circ s_1(x_Q) \circ s_2(x_{Q'}) (H_k) = q \circ s_1(x_Q) (\sum_i a_i (Q') E_i + a_k (Q') G) = q (a_k (Q') \cdot Q) = a_k (Q') \cdot Q,
\]

so:

\[
q \circ (s_1(x_Q) \circ s_2(x_{Q'})) (H_k) = a_k (Q') \cdot Q + a_k (Q) \cdot Q'.
\]

If \(Q = Q_j\), then by the definition \(a_k (Q) = a_k (Q_j) = ev_{x_j} (a_k (Q)) \equiv ev_{x_j} (a_k) = \delta_{kj}\). The third equality follows from the fact that \(ev_{x_j} (Q_j) = x_j\) and the Definition 2.30. Hence for \(Q = Q_j, Q' = Q_j\) we have:

\[
q \circ (s_1(x_Q) \circ s_2(x_{Q'})) (H_k) = \delta_{ij} \cdot Q_j + \delta_{kj} \cdot Q_i.
\]

By equation (49), \(k\)-linearity of \(\Omega\) and Theorem 2.23 we get that \(\Omega(D) = \sum_{i, m} d_{im} \Omega(\eta_l \eta_m)\) is a class (modulo \(i_2^* (\text{im } d_1^I)\)) of a homomorphism:

\[
r : H_k \mapsto \sum_{l, m} d_{lm} (a_k (Q_l) \cdot Q_m + a_k (Q_m) \cdot Q_l),
\]

but

\[
\sum_{l, m} d_{lm} (a_k (Q_l) \cdot Q_m + a_k (Q_m) \cdot Q_l) = \sum_{l, m} d_{lm} (\delta_{lk} \cdot Q_m + \delta_{km} \cdot Q_l =
\]

\[
\sum_{m} d_{km} Q_m + \sum_{l} d_{lk} Q_l = 2 \sum_{l} d_{lk} Q_l,
\]

so \(D \in \ker \Omega\) if and only if a homomorphism \(r : H_k \mapsto 2 \sum_{l} d_{lk} Q_l\) is in \(i_2^* (\text{im } d_1^I)\).

The condition \(r \in i_2^* (\text{im } d_1^I)\) is equivalent to the existence of a map \(p\) making the below diagram commute (with \(r\)|\(_{\text{sym}(3)_{(-3)}} = 0\)):
In other words:

(56) \( D \in \ker \Omega \iff r \in i^*_2(\im d^*_1) \iff \) there exists \( p \) making the above diagram commute.

Such a map \( p \) is determined by images of \( (E_i)_{1 \leq i \leq 15} \) and \( G \). Hence it exists if and only if there exist \( (s_i)_{1 \leq i \leq 15} \) and \( s \) in \( S/I \) such that \( p(E_i) = s_i \) and \( p(G) = s \) makes the above diagram commute. This imposes the following equations on \( (s_i)_{1 \leq i \leq 15} \) and \( s \):

(57) \[
\sum b_{kl} \cdot s_i + \alpha_k \cdot s = 2 \sum d_{lk} Q_i \quad \text{for} \quad k = 1, \ldots, 6 \quad \text{and} \quad p|_{d^*_1(\mathbb{S}^{\oplus 35}(-3))} = 0.
\]

Note that \( 2 \sum d_{lk} Q_i \) is of degree 2 in \( S/I \), so we can look only at \( s_i \)'s of degree 0 (because by the definition \( b_{kl}'s \) are of degree two) and \( s \) of degree 1. In this case \( p|_{\mathbb{S}^{\oplus 15}(-2)} \) factors through \( S/F^\perp \). Now, using the fact that \( 0 \leftarrow F^\perp \leftarrow \mathbb{S}^{\oplus 15}(-2) \leftarrow \mathbb{S}^{\oplus 35}(-3) \) is exact (because \( F \) is general enough), we see that the condition \( p|_{d^*_1(\mathbb{S}^{\oplus 35}(-3))} = 0 \) implies that \( p|_{\mathbb{S}^{\oplus 15}(-2)} \) factors through \( F^\perp \).

We get the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & F^\perp \\
& | & | \\
& \downarrow & \downarrow \rho \\
& S/I & \rightarrow S/F^\perp \\
& | & | \\
\downarrow & & \downarrow \\
S/\mathbb{S}^{\oplus 35}(-3) & \leftarrow & S/\mathbb{S}^{\oplus 15}(-3)
\end{array}
\]

where \( \rho \) is the unique homogeneous factorisation. Again, by the fact that \( F \) is general enough we know that such \( \rho \) must be 0. Indeed, that is because \( \text{Hom}_S(F^\perp, S/F^\perp)_2 = 0 \) as \( [F^\perp] \in \mathcal{H}^{14} \) has TNT (see Definition 2.11). Hence, the constants \( s_i \) must be all 0.

This means that the existence of \( p \) satisfying our assumptions is equivalent to the existence of a linear polynomial \( s \) such that for \( k = 1, \ldots, 6 \):

(58) \[
\alpha_k \cdot s = 2 \sum d_{lk} Q_i
\]

Note that the equality (58) takes place in \( (S/I)_2 = (S/F^\perp)_2 \), so we can apply the isomorphism \( \text{ev}_F : S/F^\perp \rightarrow \text{ev}_F(S) \) and get:

(59) \[
\text{ev}_F(\alpha_k \cdot s) = 2 \sum d_{lk} x_l \quad \text{for} \quad k = 1, \ldots, 6
\]

Moreover, if we find \( s \) such that the above equation holds, then also equation (58) is satisfied by it, as the evaluation function is an isomorphism on the second degree of \( S/F^\perp \). The equation (59) can be rewritten as:

(60) \[
\frac{\partial}{\partial x_k}(\text{ev}_F(s)) = \frac{\partial}{\partial x_k}(\sum_{l,m} d_{lm} x_l x_m) \quad \text{for} \quad k = 1, \ldots, 6
\]

and rearranging the terms, we get:

(61) \[
\frac{\partial}{\partial x_k}(\sum_{l,m} d_{lm} x_l x_m - \text{ev}_F(s)) = 0 \quad \text{for} \quad k = 1, \ldots, 6
\]

The quadric \( \sum_{l,m} d_{lm} x_l x_m - \text{ev}_F(s) \) has all partial derivatives 0 if and only if it is 0 itself. Thus \( p \) exists iff \( \sum_{l,m} d_{lm} x_l x_m = \text{ev}_F(s) \) for some \( s \in S_1 \) iff \( \sum_{l,m} d_{lm} x_l x_m \in \text{ev}_F(S_1) = \text{im}(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_6}). \)

The existence of \( p \) is equivalent to the fact that \( D \in \ker \Omega \), so we get that:

(62) \[
\ker \Omega = \text{im}(\frac{\partial}{\partial x_1}(\eta_1, \ldots, \eta_6), \ldots, \frac{\partial}{\partial x_6}(\eta_1, \ldots, \eta_6)),
\]

which concludes the proof. \( \square \)
3.2. **Zero-dimensionality of V.** We will use Proposition 3.13 to prove that \( V \) is 0-dimensional. Let us start with some analysis of \( V \). Recall that it is defined as \( \pi^{-1}( [I] ) \) for \( \pi : B^+ \to B^{G_m} \) and that it is affine with \( \mathbb{A}^1 \)-action (by multiplication) where the unique fixed point is \( [I] \). Since it is affine let \( V = \text{Spec } B \). The action of \( \mathbb{A}^1 \) on \( \text{Spec } B \) translates to \( \mathbb{N} \)-grading on \( B \) so \( B = \bigoplus_{n=0} B_n \) with \( B_0 = k \) as we have only one fixed point. Moreover, we know that \( T_{[I]} V \) is 6-dimensional (Proposition 3.11 and Definition 3.7), so for \( m := \bigoplus_{n \geq 3} B_n \) which is the fix-point ideal of \( B \), we get that \( \dim m/m^2 = 6 \)-dimensional. Fix six homogeneous representants of \( m/m^2 \) and call them \( y_1 \), \ldots, \( y_6 \). Assume they are dual to generators \( \eta_1, \ldots, \eta_6 \) of the tangent space at \( [I] \). Note that (by what is usually called a graded Nakayama lemma):

\[
\begin{align*}
\eta_1 \, & \doteq (\eta_1^�, \ldots, \eta_6^�), \\
\eta_2 \, & \doteq (\eta_1^�, \ldots, \eta_5^�), \\
\eta_3 \, & \doteq (\eta_1^6, \ldots, \eta_5^6), \\
\eta_4 \, & \doteq (\eta_1^6, \ldots, \eta_4^6), \\
\eta_5 \, & \doteq (\eta_1^6, \ldots, \eta_3^6), \\
\eta_6 \, & \doteq (\eta_1^6, \ldots, \eta_2^6),
\end{align*}
\]

so \( m = (y_1^6, \ldots, y_6^6) \). Also, all \( y_i^6 \)'s are homogeneous of degree 1, because \( \eta_i \)'s are homogeneous of degree 1. Thus we can define a graded homomorphism \( k[y_1, \ldots, y_6] \to B \) sending \( y_i \) to \( \eta_i \). It is onto by an induction on the degree argument. If we denote by \( J \) the kernel of this homomorphism, then we get an identification of graded rings \( B \simeq k[y_1, \ldots, y_6]/J \) with \( J \) -homogeneous. Note that also \( J \subset (y_1, \ldots, y_6)^2 \), because the tangent space at \( [I] \) is \( 6 \)-dimensional. So to prove that \( V \) is 0-dimensional it is enough to prove that \( k[y_1, \ldots, y_6]/J \) is 0-dimensional. We will do so by obtaining a lower bound on the ideal \( J \).

First consider the composition of inclusions and morphisms below:

\[
V \subset B^+ \subset B \subset H
\]

By looking near \( [I] \) we get a homomorphism \( \mathcal{O}_{H,[I]} \to \mathcal{O}_{V,[I]} \). It is surjective as \( V \subset B^+ \) is a closed inclusion, \( B^+ \to B \) is an isomorphism near \( [I] \) and \( B \subset H \) is a closed inclusion. We complete it and get a map \( \zeta : \hat{\mathcal{O}}_{H,[I]} \to \hat{\mathcal{O}}_{V,[I]} \) which is also surjective. Using the map \( \zeta \) we will prove that:

**Proposition 3.14.** The vector space \( (F^\perp)_{2}(\gamma) := \{ g(\gamma_1, \ldots, \gamma_6) \in k[\gamma_1, \ldots, \gamma_6] : g(\alpha_1, \ldots, \alpha_6) \in (F^\perp)_{2} \} \) is contained in \( I_2 \).

But before giving the proof of this proposition we give an immediate corollary:

**Corollary 3.15.** \( V \) is 0-dimensional.

**Proof.** The cubic \( F \) is general enough, so \( F^\perp \) is generated by its second degree \( (F^\perp)_2 \). By Proposition 3.14 \( (F^\perp)_2(\gamma) \subset J \), so also \( F^\perp(\gamma) \subset J \). Hence we get a surjection:

\[
\frac{k[\gamma]}{F^\perp(\gamma)} \longrightarrow \frac{k[\gamma]}{J}
\]

The left hand side ring is 0-dimensional (with respect to the Krull dimension), so the other one also is. Since \( V \simeq \text{Spec} (\frac{k[\gamma]}{J}) \) we get the conclusion. \( \square \)

**Proof of the Proposition 3.14.** Let \( Ob = \text{Ext}^2_1 (I, S/I) \) and \( ob_0 : \text{Sym}_2(T_{[I]} H) \to Ob \) be the primary obstruction map for \( \hat{\mathcal{O}}_{H,[I]} \). Let \( ob_v : \text{Sym}_2(T_{[I]} V) \to Ob \) be the restriction of \( ob_0 \) to \( \text{Sym}_2(T_{[I]} V) \). By Proposition 2.20 with \( \zeta : \hat{\mathcal{O}}_{H,[I]} \to \hat{\mathcal{O}}_{V,[I]} \) we get:

\[
\text{im } ob_v \subset \frac{(\gamma)^3 + I}{(\gamma)^3} = \frac{(\gamma)^3 + J_2}{(\gamma)^3}
\]

Hence, it is enough to prove that \( (F^\perp)_2(\gamma) \subset \text{im } ob_v \). Note that if we compose \( ob_v \) from the right with \( Ob \to \frac{\text{ker } d_2}{\text{im } d_1} \) we get the map \( \Omega \) from Proposition 3.13. Thus we end up with the sequence:

\[
\begin{array}{c}
\text{ker } \Omega \longrightarrow \text{Sym}_2(T_{[I]} V) \longrightarrow Ob \longrightarrow \frac{\text{ker } d_2}{\text{im } d_1}
\end{array}
\]
so after dualizing we get:

\[(\ker \Omega)^\vee \leftarrow j^! \text{Sym}_2(T_{[l]} V)^\vee \leftarrow \text{Ob}^\vee \leftarrow \left(\frac{i_2^!(\ker \Omega_2)}{i_2^!(\text{im} d_1^!)}\right)^\vee\]

Hence \(\Omega^\vee \subset \text{im} \text{ob}^\vee \subset I_2\). Note that \(j\) is the kernel of \(\Omega\), so \(\Omega^\vee = ker j^!\).

We will prove that \(ker j^! = (F^\perp)_2(\gamma) \subset \text{Sym}_2(T_{[l]} V)^\vee\) under the identification \(\text{Sym}_2(T_{[l]} V)^\vee \simeq \text{Sym}_2(T_{[l]} V)^\vee\) coming from the natural duality \(\cdot : \text{Sym}_2(W)^\vee \times \text{Sym}_2(W) \rightarrow k\) for \(W = T_{[l]} V\) (see equation \((2)\)). However, for the proof it will be easier to start with a differentuality. First let us write \(\text{Sym}_2(T_{[l]} V) = k[y_1, \ldots, y_6]\). Consider the evaluation map from Definition 2.30 with variables \(x_i\) changed to \(\eta_i\):

\[(66) \quad \iota : k[x_1, \ldots, x_6] \times k[y_1, \ldots, y_6] \rightarrow k[y_1, \ldots, y_6]\]

In other words \(a_i\) acts as \(\frac{\partial}{\partial x_i}\). By restricting this map to degree two we get a perfect pairing:

\[(67) \quad \iota : k[a_1, \ldots, a_6] \times k[y_1, \ldots, y_6]_2 \rightarrow k\]

This gives us an identification \(k[a_1, \ldots, a_6]_2 \simeq (k[y_1, \ldots, y_6]_2)^\vee \simeq \text{Sym}_2(T_{[l]} V)^\vee\). We will now calculate \(ker j^! \subset \text{Sym}_2(T_{[l]} V)^\vee \simeq k[a_1, \ldots, a_6]_2\). Take \(D \in k[a_1, \ldots, a_6]_2\). We have:

\[(68) \quad D \in ker j^! \text{ if and only if } D|_{\ker \Omega} = 0,\]

From Proposition 3.13 we know that \(\ker \Omega = \text{im} (\frac{\partial F}{\partial x_k}(y_1, \ldots, y_6), \ldots, \frac{\partial F}{\partial x_k}(y_1, \ldots, y_6))\), so:

\[(69) \quad D|_{\ker \Omega} = 0 \iff D \cdot \left(\frac{\partial F}{\partial x_k}(y_1, \ldots, y_6)\right) = 0 \text{ for all } k = 1, \ldots, 6.\]

We transform this condition using the evaluation action:

\[D \cdot \left(\frac{\partial F}{\partial x_k}(y_1, \ldots, y_6)\right) = (D \cdot a_k) \cdot (F(y_1, \ldots, y_6)) = (a_k \cdot D) \cdot (F(y_1, \ldots, y_6)) = \frac{\partial}{\partial y_k}(D \cdot F).\]

Since \(D \cdot F\) is a linear form, the vanishing of \(\frac{\partial}{\partial y_k}(D \cdot F)\) for all \(k = 1, \ldots, 6\) is equivalent to \(D \cdot F = 0\). Thus we get:

\[(70) \quad D \in ker j^! \iff D|_{\ker \Omega} = 0 \iff D \cdot F = 0.\]

Hence \(ker j^! = (F^\perp)_2 \subset k[a_1, \ldots, a_6]_2\). The only thing left is to check what happens under the isomorphisms:

\[(71) \quad k[a_1, \ldots, a_6]_2 \simeq \text{Sym}_2(T_{[l]} V)^\vee \simeq \text{Sym}_2(T_{[l]} V) = \text{Sym}_2(k \gamma_1 \oplus \cdots \oplus k \gamma_6).\]

Take \(a, a_j \in k[a_1, \ldots, a_6]_2\) and \(\eta^a, \eta^j \in \text{Sym}_2(T_{[l]} V)\) where \((\eta^a_j)\) is the dual basis to \((\eta)_j\) \(\subset T_{[l]} V\). Consider the action of these elements on \(\text{Sym}_2(T_{[l]} V)\):

- \(a, a_j \cdot \eta, \eta_j\) is 1 if \(i \neq j\) and 2 if \(i = j\),
- \(\eta^a_i \eta^j_j \cdot \eta, \eta_j\) = \(\frac{1}{2}(\eta^a_i(\eta), \eta^j_j(\eta_j) + \eta^j_j(\eta_j), \eta^a_i(\eta_j))\) so it is \(\frac{1}{2}\) if \(i \neq j\) and 1 if \(i = j\) (in both cases we use Formula \((2)\)).

It is easy to check that the action on \(\eta_0 \eta_i\) for \(\{a, b\} \neq \{i, j\}\) is zero in both cases. Thus under the above isomorphism \(a, a_j \simeq 2\eta^a_i \eta^j_j\). Hence, after two isomorphisms everything is multiplied by 2 (when considered in bases \((a, a_j)\) and \((\eta^a_i \eta^j_j)\)). Now, by the definition of \(\gamma^a_i\)’s (they were coming exactly from the dual basis to \((\eta)_j\)) and homogeneity of \((F^\perp)_2\), we get the conclusion.

**Corollary 3.16.** The ideal \(I = (F^\perp, g) = F^\perp + S_{\geq 3}\) yields a hedgehog point \([l] \in B\). In particular \([l] \in B\) is a non-reduced point of \(B\) and \(H\).

**Proof.** The point \([l] \in B\) lies in \(B\) as it is supported at \(0 \in \mathbb{A}^6\). It is \(G_m\)-invariant, because \(I\) is homogeneous. Now we check three properties from Definition 2.29:

1. \((T_{[l]} B)_{\geq 0} = 0\) by Corollary 3.5,
(2) \( T_0 \mathcal{B} \neq (T_0 \mathcal{B})_0 \) by Proposition 3.11 and Definition 3.7,
(3) the negative spike \( V \) at \([I]\) is 0-dimensional by Corollary 3.15.

The non-reducedness follows from Theorem 3.1. □

Hence, by Example 3.3 we get:

**Corollary 3.17.** The ideal \( I = (F^+, g) = F^+ + S_{\geq 3} \) for \( F = x_1x_2x_4 - x_1x_3^2 + x_2x_3^2 + x_3x_5^2 + x_4x_6^2 \) yields a non-reduced point \([I] \in \mathcal{H}\).

### 3.3. Fractal family and its flatness.

Since we proved that \( V \) is 0-dimensional in Corollary 3.15, we know that the composition of morphisms:

\[
V \subset B^+ \to B \subset \mathcal{H}
\]

is a closed embedding. Indeed, that is because \( V \) is concentrated on just one point, the first and the last morphisms are closed embeddings, and \( i : B^+ \to B \) is an isomorphism on some open neighbourhood of \([I] \in B^+\). The subscheme \( V = \text{Spec}(B) \hookrightarrow \mathcal{H} \) yields a deformation of \([I]\) over \( B \). In Subsection 3.2 we presented \( B \) as a quotient \( \frac{k[\gamma_{i,j}, \ldots, \gamma_{k,l}]}{J} \), where \( J \) was a homogeneous ideal. Moreover, we proved that \( F^+(\gamma) \subset J \) in Proposition 3.14 and this was the heart of our argument for 0-dimensionality of \( V \) in Corollary 3.15. Thus, a natural suspicion is that in fact \( J = F^+(\gamma) \), or more generally \( V \simeq \text{Spec}(S/F^+) \). If this was true, then there would exist a morphism \( \text{Spec}(S/F^+) \to \mathcal{H} \) onto \( V \). However, such a morphism would induce a deformation of \( S/I \) over \( S/F^+ \) - almost over itself! Moreover, since it would factor through \( B^+ \) we should have been able to prolong it to an element of \( \mathcal{H}(A^1 \times \text{Spec}(S/F^+)) \). Suspecting the existence of such a structure, we get the other way around: we find a deformation over \( A^1 \times \text{Spec}(S/F^+) \) which yields a morphism to \( \mathcal{H} \) such that after composing it with the projection \( \mathcal{H} = A^1 \times B \to B \) the induced morphism to \( B^+ \) is an isomorphism onto \( V \).

**Definition 3.18.** Consider a \( \frac{k[\delta, \beta]}{\mathfrak{f}_d} \) - algebra \( M \) of the form:

\[
\frac{k[\delta, \beta]}{\mathfrak{f}_d} \to M := \frac{k[\delta, \beta]}{(F_+^d F_0^d x(t))}
\]

where we take \( \Gamma(t) \) from Definition 2.33 with \( g \) and \( Q_i \)'s as in Subsection 3.1. We call it the fractal family. The name refers to the self-similarity of the base space and the special fiber. The rest of this subsection aims to prove flatness of the fractal family.

Before stating the next algebraic preparation step, we recall some notion from commutative algebra.

**Definition 3.19.** If \( C \) is a finite dimensional \( k[t] \)-algebra, then the \( k[t] \)-module \( \text{Hom}_{k[t]}(C, k[t]) \) has a \( C \)-module structure given by \((c \cdot \Phi)(d) := \Phi(c \cdot d)\) for \( \Phi \in \text{Hom}_{k[t]}(C, k[t]) \) and \( c, d \in C \). Moreover, if \( C = \frac{k[\delta, \beta]}{H} \) for some \( H \in k[t, \delta] \), then there is a natural \( C \)-modules homomorphism:

\[
\Phi : C \to \text{Hom}_{k[t]}(C, k[t])
\]

sending \( c \in C \) to \( \Phi_c := \Phi(c) \) given by the formula:

\[
\Phi_c(d) = (ev_{c \cdot H}(d))(0).
\]

The evaluation at \( \delta = 0 \) at the end makes sense as \((ev_{c \cdot H}(d)) \in k[t, \delta] \) and the result lies indeed in \( k[t] \).

**Proposition 3.20.** Let \( H \in k[t, \delta] \) and consider the algebra \( C := \frac{k[t, \delta]}{H^+} \). Suppose that \( C \) is a free \( k[t] \)-module and for all \( t_0 \in k \) the dimension of the \( k \)-linear space \( \frac{k[t]}{H(t_0)^+} \) is constant. Then \( \Phi : C \to \text{Hom}_{k[t]}(C, k[t]) \) is a \( C \)-module isomorphism.

Before the proof let us cite a lemma which is a translation of [Jel18, Proposition 2.12] to our context:
Lemma 3.21. Assume that $k$ is algebraically closed. If $C = \frac{k[t, x]}{H}$ for $H \in k[t, x]$ is such that for all $t_0 \in k$ the dimension of the $k$-linear space $\frac{k[t, x]}{H(t_0)}$ is constant, then for all $t_0 \in k$ there is an isomorphism between $\frac{k[t, x]}{H(t_0)}$ and $\frac{k[t, x]}{H(t_0)}$ induced by the map:

$$k[t, x] \rightarrow \frac{k[t, x]}{H(t_0)}$$

which maps $f$ to $ev_{H(t_0)}(f)(t_0)$.

Proof of Proposition 3.20. First we prove that $\ker \Phi = 0$:

$$\Phi(c) = 0 \iff \text{im} ev_{c \perp H} \subset (x) \iff c \perp H = 0 \iff c = 0 \in \frac{k[t, x]}{H^+}.$$  

The middle equivalence holds, because if $c \perp H \neq 0$, then there exists some partial derivative such that after applying it to $c \perp H$ we get a non-zero element in $k[t]$. Now our goal is to show that $\Phi$ is onto. Suppose for a contradiction that it is not. Let $D := \text{Hom}_{k[t]}(C, k[t])$. Then we get the following exact sequence of $C$-modules:

$$0 \rightarrow C \rightarrow D \rightarrow E \rightarrow 0$$

where $E \neq 0$. This is also a sequence of $k[t]$-modules and since $E \neq 0$ is finitely generated, there exist an ideal $m = (t - t_0) \subset k[t]$ (for some $t_0 \in k$) such that the quotient $E/(t - t_0)E$ is non-zero (take $m$ such that $E_m \neq 0$ and use Nakayama’s lemma). The ideal $m$ is of this form as $k$ is algebraically closed. We tensor everything by $\frac{k[t]}{(t - t_0)}$ and we get:

$$C/(t - t_0)C \rightarrow D/(t - t_0)D \rightarrow E/(t - t_0)E \rightarrow 0$$

To get the final contradiction, we will prove that $C/(t - t_0)C \rightarrow D/(t - t_0)D$ is an isomorphism. First, by Lemma 3.21 we get that $C/(t - t_0)C \simeq \frac{k[t]}{H(t_0)}$. On the other hand, $D/(t - t_0)D \simeq \text{Hom}_k(C/(t - t_0)C, k)$. Indeed, if we apply the functor $\text{Hom}_{k[t]}(C, -)$ on a short exact sequence:

$$0 \rightarrow (t - t_0) \rightarrow k[t] \rightarrow \frac{k[t]}{(t - t_0)} \rightarrow 0$$

we get:

$$0 \rightarrow \text{Hom}_{k[t]}(C, (t - t_0)) \rightarrow \text{Hom}_{k[t]}(C, k[t]) \rightarrow \text{Hom}_{k[t]}(C, \frac{k[t]}{(t - t_0)}) \rightarrow 0$$

where exactness follows from the fact that $C$ is a free $k[t]$-module. Now, the last sequence identifies with:

$$0 \rightarrow (t - t_0) \text{Hom}_{k[t]}(C, k[t]) \rightarrow \text{Hom}_{k[t]}(C, k[t]) \rightarrow \text{Hom}_k(C/(t - t_0)C, \frac{k[t]}{(t - t_0)}) \rightarrow 0$$

so we get the desired isomorphism $D/(t - t_0)D \simeq \text{Hom}_k(C/(t - t_0)C, k)$. Now, note that by its definition, the homomorphism

$$\frac{k[t]}{H(t_0)} \simeq C/(t - t_0)C \rightarrow D/(t - t_0)D \simeq \text{Hom}_k(C/(t - t_0)C, k) \simeq \text{Hom}_k(\frac{k[t]}{H(t_0)}^+, k)$$

is given by the same function $\Phi$ but coming from the Maculay’s duality without the variable $t$. The same argument as in the beginning of this proof shows that the obtained map $\frac{k[t]}{H(t_0)}^+ \rightarrow \text{Hom}_k(\frac{k[t]}{H(t_0)}^+, k)$ is injective. Now, its surjectivity follows from the equality of dimensions over $k$. But this contradicts the fact that we have an exact sequence:

$$C/(t - t_0)C \rightarrow D/(t - t_0)D \rightarrow E/(t - t_0)E \rightarrow 0$$

with $E/(t - t_0)E \neq 0$. This finishes the proof of surjectivity of $\Phi$. □
We come back to the analysis of the fractal family. Since \( \Gamma(t) \neq (F_x \cdot F_g) \) by Lemma 2.34, we have \( (F_x \cdot F_g)^\perp \subset F(t \cdot \bar{x} + \bar{y})^\perp \subset k[t, \bar{x}, \bar{y}]. \) We get a short exact sequence:

\[
0 \to L \to \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \to \frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp} \to 0
\]

where \( L \) is the kernel of the natural quotient. This sequence splits, as the right hand side module is a free \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-module (isomorphic to \( \frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp} \) by Lemma 2.36). After applying the functor \( \text{Hom}_{k[t]}(-, k[t]) \) we obtain a sequence of \( k[t] \)-modules:

\[
0 \leftarrow \text{Hom}_{k[t]}(L, k[t]) \leftarrow \text{Hom}_{k[t]}\left(\frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp}, k[t]\right) \leftarrow \text{Hom}_{k[t]}\left(\frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp}, k[t]\right) \leftarrow 0
\]

However, by Definition 3.19

\[
\text{Hom}_{k[t]}\left(\frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp}, k[t]\right)
\]

has a \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-module structure and

\[
\text{Hom}_{k[t]}\left(\frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp}, k[t]\right)
\]

has a \( \frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp} \)-module structure. In particular both of these \( k[t] \)-modules have a structure of \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-module. Thus the above sequence can be considered as a sequence of \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-modules. If we fix a splitting on the first sequence:

\[
\frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp} \simeq L \oplus \frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp}
\]

we get the induced splitting of \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-modules:

\[
\text{Hom}_{k[t]}\left(\frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp}, k[t]\right) \simeq \text{Hom}_{k[t]}(L, k[t]) \oplus \text{Hom}_{k[t]}\left(\frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp}, k[t]\right)
\]

Consider the \( k \)-algebra \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \). It is a free \( k[t] \)-module, because

\[
\frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} = \frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp} \simeq \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \otimes_k k[t]
\]

where the first equality follows from Lemma 2.35. Moreover, the polynomial \( (F_x \cdot F_g) \) does not depend on \( t \), so the assumptions of Proposition 3.20 are satisfied. Thus, the left hand side of \( \text{equation (77)} \) is isomorphic to \( \frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp} \) as a \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-module, so it is a free \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-module. This means that \( \text{Hom}_{k[t]}(L, k[t]) \) is a projective \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-module (as it is a direct factor in a free \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-module). We will now prove that the fractal family is flat by showing it is isomorphic to \( \text{Hom}_{k[t]}(L, k[t]) \) as a \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-module:

**Proposition 3.22.** \( \text{Hom}_{k[t]}(L, k[t]) \) is isomorphic to the fractal family (as a \( \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp} \)-module).

**Proof.** Let \( C := \frac{k[t, \bar{x}, \bar{y}]}{(F_x \cdot F_g)^\perp}, C_0 := \frac{k[t, \bar{x}, \bar{y}]}{(F(t \cdot \bar{x} + \bar{y})^\perp} \) and \( q : C \to C_0 \) be the quotient map. From Proposition 3.20 we know that \( \text{Hom}_{k[t]}(C, k[t]) \simeq C, \text{Hom}_{k[t]}(C_0, k[t]) \simeq C_0 \) by homomorphisms as in Definition 3.19. Thus in order to prove the proposition, it is enough to interpret the image of \( \text{Hom}_{k[t]}(C_0, k[t]) \) inside \( C \simeq \text{Hom}_{k[t]}(C, k[t]) \) as the ideal \( \Gamma(t) \).
Consider \( c_0 \in C_0 \) and let \( \tilde{c}_0 \in C \) be a lifting of \( c_0 \). We will calculate the image of \( \tilde{c}_0 \) coming from the morphism \( C_0 \simeq \text{Hom}_{k[t]}(C_0,k[t]) \to \text{Hom}_{k[t]}(C,k[t]) \simeq C \). First, look at the diagram:

\[
\begin{array}{c}
0 \\ \\
\downarrow \\
\downarrow \\
\downarrow \\
0 \\
\end{array}
\]

\[
\begin{array}{c}
L \\
\downarrow \\
C \\
\downarrow q \\
k[t] \\
\end{array} \\
\begin{array}{c}
\phi_0 \\
\downarrow \\
C_0 \\
\downarrow \\
0 \\
\end{array}
\]

here \( \phi_0 \) comes from Definition 3.19, so it is equal to \( z \circ ev_{c_0} \circ F(t,x+\tilde{y}) \) where \( z : k[t,x,\tilde{y}] \to k[t] \) is the map sending \( x \) and \( \tilde{y} \) to 0. We claim that:

\[
(79) \quad \phi_0 \circ q = \phi_{c_0} \circ \Gamma(t)
\]

Indeed, that it because:

\[
(80) \quad \phi_0 \circ q = z \circ ev_{c_0} \circ F(t,x+\tilde{y}) \circ q = z \circ ev_{c_0} \circ \Gamma(t) \circ (F_t+F_{\tilde{y}}) = \phi_{c_0} \circ \Gamma(t)
\]

So the induced homomorphism \( C_0 \simeq \text{Hom}_{k[t]}(C_0,k[t]) \to \text{Hom}_{k[t]}(C,k[t]) \simeq C \) sends \( c_0 \mapsto \tilde{c}_0 \cdot \Gamma(t) \) (and this doesn’t depend on the choice of \( \tilde{c}_0 \)). Hence the cokernel of this homomorphism is isomorphic to \( C/\Gamma(t) \) so to the fractal family by Lemma 2.35. □

This yields the following:

**Corollary 3.23.** The fractal family is a flat \( \text{Hom}_{k[t]}(L,k[t]) \)-module.

**Proof.** From Proposition 3.22 we know that fractal family is isomorphic to \( \text{Hom}_{k[t]}(L,k[t]) \) which is a projective \( \frac{k[t]}{F_t} \)-module by equation (78) and the fact that \( \frac{k[t]}{F_t} \) satisfies the assumptions of Proposition 3.20. In particular, the fractal family is flat. □

3.4. Complete description of the fiber \( V \). In this subsection we prove that the morphism induced by the fractal family to \( \mathcal{H} \) composed with the projection \( \mathcal{H} \to B \) is in fact an isomorphism onto \( V \). Let \( \mathcal{Z} \) be the universal family over \( \mathcal{H} \). The fractal family \( \mathcal{M} \) gives a morphism \( \psi \):

\[
\begin{array}{c}
\text{Spec}(M) \\
\downarrow \\
\mathcal{A}^1 \times \text{Spec}(\frac{k[\tilde{y}]}{F_{\tilde{y}}}) \\
\downarrow \\
\mathcal{Z} \\
\end{array} \xrightarrow{\psi} \mathcal{H}
\]

By Proposition 2.7 a morphism \( \mathcal{A}^1 \times \text{Spec}(\frac{k[\tilde{y}]}{F_{\tilde{y}}}) \to \mathcal{H} \) is \( \mathbb{G}_m \)-equivariant if and only if the associated ideal in \( S[\frac{k[\tilde{y}]}{F_{\tilde{y}}}] \) is homogeneous with respect to variables \( \tilde{a} \) and \( t \) all of degree one.

The fractal family is given by the ideal \( (F_t^{1+},\Gamma(t)) \subset \frac{k[\tilde{y}]}{F_{\tilde{y}}} \) so it is homogeneous with respect to this grading, see Definition 2.33. Let \( \psi : \text{Spec}(\frac{k[\tilde{y}]}{F_{\tilde{y}}}) \to \mathcal{H}^+ \) be the morphism induced from \( \psi : \mathcal{A}^1 \times \text{Spec}(\frac{k[\tilde{y}]}{F_{\tilde{y}}}) \to \mathcal{H}, \) see Definition 2.12. We write \( \psi|_{\{1\}} : \text{Spec}(\frac{k[\tilde{y}]}{F_{\tilde{y}}}) \to \mathcal{H} \) for the restriction of \( \psi \) to \( \{1\} \times \text{Spec}(\frac{k[\tilde{y}]}{F_{\tilde{y}}}). \)

By Proposition 2.27 we have \( \mathcal{H} \simeq \mathcal{A}^n \times \mathcal{B} \) and by Proposition 2.28 the projection \( p : \mathcal{H} \to \mathcal{B} \) is \( \mathbb{G}_m \)-equivariant. Hence, the composition \( \nu = p \circ \psi : \mathcal{A}^1 \times \text{Spec}(\frac{k[\tilde{y}]}{F_{\tilde{y}}}) \to \mathcal{B} \):
is $\mathbb{G}_m$-equivariant. The morphism $u$ induces a morphism to $\bar{u} : \text{Spec} (\frac{k[\hat{\beta}]}{F^+_\varphi}) \to B^+$ by Definition 2.12:

$$
\begin{align*}
\text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi}) & \xrightarrow{\bar{u}} B^+ \xrightarrow{\pi} B^{G_m} \\
\{1\} \times id & \quad \downarrow \quad \downarrow \\
\mathbb{A}^1 \times \text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi}) & \xrightarrow{u} B
\end{align*}
$$

In this subsection we prove that $\bar{u}$ is an isomorphism onto $V \subset B^+$. We start with the following fact:

**Lemma 3.24.** The composition $\pi \circ \bar{u}$ is a constant morphism onto the point $[I]$.

**Proof.** Consider the composition $w := \pi_H \circ \bar{v}$ for the restricting to the limit morphism $\pi_H$ for Białynicki-Birula decomposition for the $G_m$-action on $\mathcal{H}$. Then the following diagram commutes:

$$
\begin{array}{ccc}
\text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi}) & \xrightarrow{\theta} & \mathcal{H}^+ \xrightarrow{p^+} B^+ \\
& \searrow w & \downarrow \pi_H \\
& & \mathcal{H}^{G_m} \xrightarrow{\pi} B^{G_m}
\end{array}
$$

where $p^+$ and $p^{G_m}$ are morphism induced by $p$ on Białynicki-Birula decomposition functors and fix-point functors respectively. It is enough to prove that $w : \text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi}) \to \mathcal{H}^{G_m}$ is a constant map onto $[I]$, because $p^{G_m}([I]) = [I] \in B^{G_m}$.

By the functorial description of the map $\pi$ we see that $w = \pi_H \circ \bar{v} = v|_{\{0\} \times \text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi})}$, As $v$ is given by the fractal family $M$ its restriction to zero is given by $M/(t)$ so by the algebra:

$$M_0 := \frac{k[\hat{\alpha}, \hat{\beta}]}{(F^+_\varphi, F^+_\varphi, \Gamma(0))}$$

but $\Gamma(0) = g(\bar{\alpha})$ (see Definition 2.33). Hence $M_0$ is just the product family:

$$M_0 \simeq \frac{k[\hat{\alpha}]}{(F^+_\varphi, g(\bar{\alpha}))} \otimes_k \frac{k[\hat{\beta}]}{(F^+_\varphi)}$$

As $I = (F^+_\varphi, g(\bar{\alpha}))$ this family corresponds to the constant morphism onto $[I]$. This finishes the proof.

By Lemma 3.24 the morphism $\bar{u} : \text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi}) \to B^+$ factors through $V \subset B^+$, because by its definition $V = \pi^{-1}([I])$. We will write $\bar{u} : \text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi}) \to V$ for the factorisation. We would like to prove that $\bar{u}$ is an isomorphism. In order to do so, first we check that it is $\mathbb{G}_m$-equivariant, where the $\mathbb{G}_m$-action on $\text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi})$ comes from the natural grading of the ring $\frac{k[\hat{\beta}]}{F^+_\varphi}$ where $\text{deg}(\beta_i) = 1$.

We start with the following:

**Lemma 3.25.** The morphism $\bar{u} : \text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi}) \to V$ is $\mathbb{G}_m$-equivariant with respect to the natural $\mathbb{G}_m$-action on $\text{Spec}(\frac{k[\hat{\beta}]}{F^+_\varphi})$ and the usual $\mathbb{G}_m$-action on $\mathcal{H}$. 


We want to show that $\text{Spec}(\kappa|_{\{1\}}) \to \mathcal{H}$ is $\mathbb{G}_m$-equivariant. The fractal family restricted to $t = 1$ is given by the ideal $(\mathcal{F}_t^+, \Gamma(1)) \subset \mathcal{S}_{\mathbb{G}_m}$. The part $\mathcal{F}_t^+$ is homogeneous with respect to the standard grading on $S$ and $\Gamma(1) = \kappa(\alpha) + \sum_i \beta_i \cdot Q_i(\alpha) + \sum_i \alpha_i \cdot Q_i(\beta) + g(\beta)$ is homogeneous with respect to the grading with $\deg(\alpha_i) = \deg(\beta_i) = 1$. Thus, by Proposition 2.7 we get the $\mathbb{G}_m$-equivariance of $\nu|_{\{1\}} : \text{Spec}(\kappa|_{\{1\}}) \to \mathcal{H}$.

Now we prove that this suffices to $\mathbb{G}_m$-equivariance of $\nu$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
\text{Spec}(\kappa|_{\{1\}}) & \xrightarrow{\nu} & \mathcal{H}^+ \\
\downarrow & & \downarrow \\
\text{Spec}(\kappa|_{\{1\}}) & \xrightarrow{\nu|_{\{1\}}} & \mathcal{H} \\
\end{array}
$$

We know that $\nu|_{\{1\}}, i_{\mathcal{H}}, p^+$ are $\mathbb{G}_m$-equivariant. Thus, the $\mathbb{G}_m$-equivariance of $\nu$ will finish the proof. Denote by $\mu : \mathbb{G}_m \times \text{Spec}(\kappa|_{\{1\}}) \to \text{Spec}(\kappa|_{\{1\}})$ the $\mathbb{G}_m$-action on $\text{Spec}(\kappa|_{\{1\}})$ and $\nu, v^+$ the $\mathbb{G}_m$-action on $\mathcal{H}, \mathcal{H}^+$ respectively. We look at the following diagram:

$$
\begin{array}{ccc}
\mathbb{G}_m \times \text{Spec}(\kappa|_{\{1\}}) & \xrightarrow{id \times \nu|_{\{1\}}} & \mathbb{G}_m \times \mathcal{H}^+ \\
\downarrow & & \downarrow \\
\text{Spec}(\kappa|_{\{1\}}) & \xrightarrow{\nu} & \mathcal{H} \\
\end{array}
$$

We want to show that $v^+ \circ (id \times \sigma) = \sigma \circ \mu$. However, by Fact 2.14 $i_{\mathcal{H}}$ is a monomorphism of functors, so this is equivalent to the fact that

$$i_{\mathcal{H}} \circ v^+ \circ (id \times \sigma) = i_{\mathcal{H}} \circ \sigma \circ \mu$$

By the commutativity of the diagram this translates to $v \circ (id \times v|_{\{1\}}) = v|_{\{1\}} \circ \mu$. But this is the $\mathbb{G}_m$-equivariance of $\nu|_{\{1\}}$ so we are done. \hfill \Box

Now we investigate the behaviour of tangents under $\nu|_{\{1\}}$:

**Proposition 3.26.** Let $(\beta_i^\vee)_{1 \leq i \leq 6} \subset T(\beta) \text{Spec}(\kappa|_{\{1\}})$ be the dual basis to $(\beta_1, \ldots, \beta_6) \in (\beta)/(\beta)^2 = T(\beta) \text{Spec}(\kappa|_{\{1\}})$ be the dual basis to $(\beta_1, \ldots, \beta_6) \in (\beta)/(\beta)^2 = T(\beta) \text{Spec}(\kappa|_{\{1\}})$. The tangent map $dv|_{\{1\}} : T(\beta) \text{Spec}(\kappa|_{\{1\}}) \to T[\mathcal{H}] \mathcal{H}$ sends $\beta_i^\vee$ to $-\beta_i$ from Definition 3.7.

**Proof.** An element $\beta_i^\vee \in T(\beta) \text{Spec}(\kappa|_{\{1\}})$ comes from the morphism $\text{Spec}(\kappa|_{\{1\}}) \to \text{Spec}(\kappa|_{\{1\}})$ defined by taking $\beta_j$ to 0 for $j \neq i$ and $\beta_i$ to $\epsilon$. By composing with $\nu|_{\{1\}}$ we get a morphism $\text{Spec}(\kappa|_{\{1\}}) \to \mathcal{H}$ and it is given by the pullback of the family $\text{Spec}(M')$ over $\text{Spec}(\kappa|_{\{1\}})$ to $\text{Spec}(\kappa|_{\{1\}})$. Recall that $M' = \frac{\kappa|_{\{1\}}(\beta_i)\beta_i}{(\mathcal{F}_t^+, \Gamma(1))}$ and $\Gamma(1) = g(\alpha) + \sum_j \beta_j \cdot Q_j(\alpha) + \sum_i \alpha_i \cdot Q_i(\beta) + g(\beta)$. Thus, the
pullback of the fractal family through \( \text{Spec}(k[\varepsilon]) \to \text{Spec}(k[\beta]/F_\ell) \) gives the family:

\[
    k[\varepsilon] \xrightarrow{\delta} k[\varepsilon]_{(F_\ell, Q_0(\Delta))}
\]

By Example 3.8 this family corresponds to the tangent \( -\xi_i = -\xi Q_0 \).

As a corollary, we prove the following:

**Theorem 3.27.** The morphism \( \bar{\alpha} : \text{Spec}(k[\beta]/F_\ell) \to V \) is an isomorphism. Moreover, the ideal \( J \) from the isomorphism \( V \simeq \text{Spec}(k[\gamma]) \) described in the beginning of Subsection 3.2 is equal to:

\[
    (F^\bot)(\gamma) := \{ g(\gamma_1, \ldots, \gamma_6) \in k[\gamma_1, \ldots, \gamma_6] : g(\alpha_1, \ldots, \alpha_6) \in F^\bot \}.
\]

**Proof.** Consider the diagram:

```
\[
\begin{array}{ccc}
\text{Spec}(k[\beta]/F_\ell) & \xrightarrow{\bar{\alpha}} & \mathcal{H}^+ \\
\downarrow \delta & & \downarrow \beta \\
\mathbb{A}^1 \times \text{Spec}(k[\beta]/F_\ell) & \xrightarrow{\nu} & \mathcal{H} \\
\end{array}
\]
```

where \( r \) is the embedding of \( B \) in \( \mathcal{H} \) and morphisms \( p^+, r^+ \) come from \( p \) and \( r \) respectively, by Definition 2.12. We are interested in the morphism

\[
    q := (r \circ i \circ \bar{\alpha}) : \text{Spec}(k[\beta]/F_\ell) \to \mathcal{H}
\]

We calculate:

\[
    q = r \circ i \circ \bar{\alpha} = r \circ i \circ p^+ \circ \delta = r \circ p \circ i \mathcal{H} \circ \delta = r \circ p \circ \nu|_{\{1\}}
\]

Thus, the tangent map \( dq : T(\beta) \text{Spec}(k[\beta]/F_\ell) \to T|_\mathcal{H} \mathcal{H} \) is the composition of \( d\nu|_{\{1\}} : T(\beta) \text{Spec}(k[\beta]/F_\ell) \to T|_\mathcal{H} \mathcal{H} \) and the projection \( T|_\mathcal{H} \mathcal{H} \to T|_\mathcal{H} B < T|_\mathcal{H} \mathcal{H} \). By Proposition 3.26 if \( \beta^\nu_i \) is as in the statement of this proposition, then \( dq|_{\{1\}}(\beta^\nu_i) = -\xi_i \). Note that

\[
    -\xi_i = -(\xi_i - \frac{1}{13} \partial_i) - \frac{1}{13} \partial_i = -\eta_i - \frac{1}{13} \partial_i
\]

by Definition 3.10. By Proposition 3.11 \( \eta_i \in T|_\mathcal{H} V < T|_\mathcal{H} B \), so the projection of \( d\nu|_{\{1\}}(\beta^\nu_i) \) onto \( T|_\mathcal{H} B \) is \( -\eta_i \). because \( T|_\mathcal{H} \mathcal{H} = T|_\mathcal{H} B \oplus \text{lin}(\partial_1, \ldots, \partial_6) \). Hence

\[
    dq(\beta^\nu_i) = -\eta_i \text{ for } i = 1, \ldots, 6.
\]

After identifying \( V \) with the image of the closed embedding given by the composition (72) we get a factorisation:

\[
    q : \text{Spec}(k[\beta]/F_\ell) \to V \subset \mathcal{H}
\]

Recall that at the beginning of Subsection 3.2 we presented \( V \) as \( \text{Spec}(k[\delta]/F_\ell) \) and \( (\gamma_i)_{1 \leq i \leq 6} \) was a basis of \( T^\nu_\mathcal{H} V \) dual to \( (\eta_i)_{1 \leq i \leq 6} \). Thus, by equation (84), we get that:

\[
    (dq)^\nu(\gamma_i) = -\beta_i \text{ for } i = 1, \ldots, 6.
\]
Moreover, by Lemma 3.25 \( \bar{u} \) is \( G_m \)-equivariant, so \( q \) is \( G_m \)-equivariant as well. In this way we see that \( q : \text{Spec}(\frac{k[\bar{\beta}]}{\bar{F}_V}) \to \text{Spec}(\frac{k[\bar{\gamma}]}{F_{\bar{V}}}) \simeq V \) yields a graded homomorphism of algebras:

\[
\begin{array}{c}
\frac{k[\bar{\beta}]}{\bar{F}_V} \\
\downarrow \quad \downarrow \quad \downarrow \\
\frac{k[\bar{\gamma}]}{F_{\bar{V}}}
\end{array}
\]

and by equation (85) it is defined by \( q^\#(\gamma_i) = -\beta_i \) for \( i = 1, \ldots, 6 \). Thus if \( f(\bar{\gamma}) \in J \) then \( f(\bar{\beta}) \in F_{\bar{V}} \) and because \( J \) and \( F_{\bar{V}} \) are homogeneous it follows that \( J \subset F^\perp(\bar{\gamma}) \). From Proposition 3.14 we get that \( F^\perp(\bar{\gamma}) \subset J \) so these two ideals are in fact equal. Hence, \( q = r \circ i \circ \bar{u} \) is an isomorphism. The morphism \( i \) is an isomorphism on some open neighbourhood of \([I]\) by Corollary 3.6 and \( r \) is a closed embedding. Thus, \( \bar{u} \) is an isomorphism onto \( V \subset \mathcal{B}^+ \).

The point \([I]\) \( \in \mathcal{H} \) corresponds to the subscheme \( \text{Spec}(S/(F^\perp, g)) \subset \mathcal{A}^b \) and now we know that the fiber \( V \) at \([I]\) is isomorphic to \( \text{Spec}(\frac{k[\bar{\beta}]}{\bar{F}_V}) \) so it contains a copy of \( \text{Spec}(S/(F^\perp, g)) \) as a closed subscheme cut out only by \( g \). Hence, the negative spike at \([I]\) almost recovers the ideal \( I \). A similar phenomenon can be observed in the geometry of fractals. For example, the Mandelbrot set, which parametrizes functions \( f(z) = z^2 + c \) such that the sequence \( f(0), f(f(0)), \ldots \) is bounded, contains Julia sets of some functions \( f_\gamma(z) = z^2 + c_\gamma \). In other words, in the case of fractals the geometry of the Mandelbrot set contains subsets describing behaviour of a single function \( f \). It would be interesting to see why such phenomena arise and are these two worlds: (1) of moduli spaces of points and (2) of dynamical behaviour of complex valued functions, somehow related.

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REFERENCES

[AM69] Atiyah, M. F.; Macdonald, I. G. Introduction to commutative algebra. Student economy edition. For the 1969 original see [MR0242802]. Addison-Wesley Series in Mathematics. Westview Press, Boulder, CO, 2016. ix+128 pp. ISBN: 978-0-8133-5018-9; 0-201-40751-5

[BBC73] Białynicki-Birula, A. Some theorems on actions of algebraic groups. Ann. of Math. (2) 98 (1973), 480–497.

[BB76] Białynicki-Birula, A. Some properties of the decompositions of algebraic varieties determined by actions of a torus. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), no. 9, 667–674.

[BCR19] Cristina Bertone and Francesca Cioffi and Margherita Roggero, Smoothable Gorenstein Points Via Marked Schemes and Double-generic Initial Ideals, Experimental Mathematics, 1-18, 2019, Taylor & Francis.

[Ber08] Jose Bertin, The punctual Hilbert scheme: an introduction. https://www-fourier.ujf-grenoble.fr/sites/ifmaquette.ujf-grenoble.fr/files/bertin_rev.pdf

[Ber06] Borel, Armand Lie groups and linear algebraic groups. I. Complex and real groups. Lie groups and automorphic forms, 1–49, AMS/IP Stud. Adv. Math., 37, Amer. Math. Soc., Providence, RI, 2006.

[Coh87] Eisenbud, David The geometry of syzygies. A second course in commutative algebra and algebraic geometry. Graduate Texts in Mathematics, 229. Springer-Verlag, New York, 2005. xvi+243 pp. ISBN: 0-387-22215-4

[ES87] Ellingsrud, Geir; Strømme, Stein Arild On the homology of the Hilbert scheme of points in the plane. Invent. Math. 87 (1987), no. 2, 343–352.

[Fan05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. Fundamental algebraic geometry, volume 123 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005. Grothendieck’s FGA explained.

[FGI05] Fogarty, J. (1968). Algebraic Families on an Algebraic Surface. American Journal of Mathematics, 90(2), 511-521. doi:10.2307/2373541

[Hai01] Haiman, Mark Hilbert schemes, polygraphs and the Macdonald positivity conjecture. J. Amer. Math. Soc. 14 (2001), no. 4, 941–1006.

[Harr10] Hartshorne, Robin Deformation theory. Graduate Texts in Mathematics, 257. Springer, New York, 2010. viii+234 pp. ISBN: 978-1-4419-1595-5

[Harr66] Hartshorne, Robin Connectedness of the Hilbert scheme. Inst. Hautes Études Sci. Publ. Math. No. 29 (1966), 5–48.
[Har77] Hartshorne, Robin. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977. xvi+496 pp. ISBN: 0-387-90244-9

[Jel14] Jelisiejew, Joachim (2014). Deformations of zero-dimensional schemes and applications, https://arxiv.org/pdf/1307.8108.pdf.

[Jel17] Jelisiejew, Joachim (2017). Classifying local Artinian Gorenstein algebras. Collect. Math. 68 (2017) no. 1, 101-127.

[Jel18] Jelisiejew, Joachim. VSFs of cubic fourfolds and the Gorenstein locus of the Hilbert scheme of 14 points on $\mathbb{A}^6$. Linear Algebra Appl. 557 (2018), 265–286.

[Jel19] Jelisiejew, Joachim. Elementary components of Hilbert schemes of points. J. Lond. Math. Soc. (2) 100 (2019), no. 1, 249–272.

[Jel20] Jelisiejew, Joachim. Pathologies on the Hilbert scheme of points. Invent. Math. 220 (2020), no. 2, 581–610.

[JS19] Jelisiejew, Joachim; Sienkiewicz, Łukasz. Bialynicki-Birula decomposition for reductive groups. J. Math. Pures Appl. (9) 131 (2019), 290–325.

[JŠ21] Joachim Jelisiejew and Klemen Šivic (2021). Components and singularities of Quot schemes and varieties of commuting matrices, https://arxiv.org/pdf/2106.13137.pdf.

[MFK94] Mumford, D.; Fogarty, J.; Kirwan, F. Geometric invariant theory. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], 34. Springer-Verlag, Berlin, 1994. xiv+292 pp. ISBN: 3-540-56962-4

[Ree95] Reeves, Alyson A. The radius of the Hilbert scheme. J. Algebraic Geom. 4 (1995), no. 4, 639–657.

[Sha01] Shafarevich, I. R. Degeneration of semisimple algebras. Special issue dedicated to Alexei Ivanovich Kostrikin. Comm. Algebra 29 (2001), no. 9, 3943–3960.

[sta17] Stacks Project. http://math.columbia.edu/algebraic_geometry/stacks-git, 2017.

[Vak17] Ravi Vakil, THE RISING SEA Foundations of Algebraic Geometry. url http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf