Abstract. $CWH, CWN$ stand for collectionwise Hausdorff and collectionwise normal respectively. We analyze the statement “there is a $\lambda - CWH$ not $CWH$ first countable (Hausdorff topological) space”. We prove the existence of such a space under various conditions, show its equivalence to: there is a $\lambda$-CWN not CWN first countable space and give an equivalent set theoretic statement; the nicest version we can obtain is in §4. The author had a flawed proof of the existence of such spaces in ZFC, for some $\lambda > \aleph_1$, in June of 1992; still we decided that there is some interest in the correct part and some additions.

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We shall deal mainly with first countable topological spaces. All spaces will be Hausdorff.

0.1 Definition. 1) A space $X$ is metrizable if the topology on $X$ is induced by a metric.

2) A space $X$ is $(< \lambda)$-metrizable if for each $Y \subseteq X$, $|Y| < \lambda$, the induced topology on $Y$ is metrizable. Let $\mu$-metrizable mean $(< \mu^+)$-metrizable.

3) A space $X$ is CWH (collectionwise Hausdorff) if for every subspace $Y$ on which the induced topology is discrete (i.e. every subset is open) there is a sequence $\langle u_y : y \in Y \rangle$ of pairwise disjoint open subsets of $X$, such that for every $y \in Y$ we have $y \in u_y$.

4) A space $X$ is $(< \lambda)$-CWH if for every $Y \subseteq X$ of cardinality $< \lambda$, $Y$ (with the induced topology) is CWH.

$\mu$-CWH means $(< \mu^+) – CWH$.

5) A space is CWN (collectionwise normal) when: if $\langle Y_i : i < \alpha \rangle$ is a sequence of pairwise disjoint subsets of $X$, and each $Y_i$ is clopen in $X \upharpoonright (\bigcup_{j<\alpha} Y_j)$, then we can find pairwise disjoint open $\langle U_i : i < \alpha \rangle$ in $X$ such that $Y_i \subseteq U_i$.

6) A space is $(< \lambda)^-*$ CWN if every subspace with $< \lambda$ points is CWN (we use the $*$ because there may be a bound $\alpha < \lambda$ such that all relevant subspaces are of size $< \alpha$).

$n^-*$ CWN means $(< \mu^+)^-*$ CWN.

0.2 Question. (ZFC) 1) Are there $\aleph_1$-metrizable not metrizable (first countable Hausdorff topological) spaces?

2) Are there $\aleph_1 – CWH$ not CWH first countable spaces?

We shall also consider analogous questions with $\aleph_1$ replaced by any $\lambda > \aleph_0$.

Note: $\lambda$-metrizable $\implies \lambda – CWH$. Also, metrizable $\implies$ CWN $\implies$ CWH.

0.3 Observation. 1) Assume $X$ is a space with character $\chi \leq \lambda$ (i.e. every point has a neighborhood basis of cardinality $\leq \chi$).

Then:

(a) $X$ is $\lambda – CWH$ if and only for every subspace $Y$ of cardinality $\leq \lambda$ on which the induced topology is discrete there is a sequence $\langle u_y : y \in Y \rangle$ of pairwise disjoint open subsets of $X$, $y \in u_y$.

(b) In (a), for any fixed $\mu \leq r$, we can restrict ourselves (on both sides) to discrete subsets of cardinality $\mu$.

2) If $X$ is CWN then $X$ is CWH.

Proof. 1) The implication $\iff$ is immediate. For the implication $\Rightarrow$ assume that $Y \subseteq X$, $|Y| \leq \lambda$ and $X \upharpoonright Y$ is the discrete topology. Let $\langle U_i^y : i < i^y \leq \chi \rangle$ be a neighborhood basis in $X$ for $y \in Y$; choose for $y^1, y^2 \in Y, i_1 < i^{y^1}, i_2 < i^{y^2}$ a point $z[y^1, y^2, i_1, i_2]$ which is in $U_{i_1}^{y_1} \cap U_{i_2}^{y_2}$, if this intersection is non-empty. By the assumption $X \upharpoonright Y_1$ is CWH, where $Y_1 = Y \cup \{ z[y^1, y^2, i_1, i_2] : y^1 \in Y, y^2 \in Y, i_1 < i^{y^1}, i_1 < i^{y^2} \}$.
§1 Analysis of “$\aleph_1 - CHW$ but not $CHW$”

1.1 Lemma. 1) Assume

$(\ast)_\lambda \ cf(\lambda) = \aleph_0 < \lambda, \eta_\alpha \in \omega_\lambda$ for $\alpha < \lambda^+$, and for each $\beta < \lambda^+$, we can find pairwise disjoint end segments for $\langle \eta_\alpha : \alpha < \beta \rangle$

(e.g. $\exists h_\beta : \beta \rightarrow \omega$ such that

$\alpha_1 < \alpha_2 < \beta \land k > h_\beta(\alpha_1) \land k > h_\beta(\alpha_2) \Rightarrow \eta_{\alpha_1} \upharpoonright k \neq \eta_{\alpha_2} \upharpoonright k$).

Then 1) the space $\omega_\lambda \cup \{ \eta_\alpha : \alpha < \lambda^+ \}$ with the topology given below is

$(\alpha)$ first countable and Hausdorff

$(\beta)$ $\lambda - CHW$, even $\lambda$-metrizable

$(\gamma)$ not $\lambda^+-CHW$.

The topology is the obvious one each $\eta \in \omega_\lambda$ is isolated, and for each $\alpha < \lambda^+$, the neighborhood basis of $\eta_\alpha$ is $\{\{\eta_\alpha \upharpoonright \ell : k < \ell \leq \omega \} : k < \omega\}$.

2) Moreover, the space is not metrizable but is $\lambda$-metrizable.

Proof. Straightforward. $\square_{1.1}$

1.2 Conclusion. 1) If the answer to 0.2(1) or 0.2(2) is “no”, then $(\ast)_\lambda$ of 1.1 is not true for any $\lambda$.

2) If $(\ast)_\lambda$ of 1.1 fails for all $\lambda$, then

$(\ast) \ cf(\lambda) = \aleph_0 < \lambda \Rightarrow pp(\lambda) = \lambda^+$

(by [Sh355,1.5A]).

3) If 2)'s conclusion holds, then for every $\lambda$ singular we have $pp(\lambda) = \lambda^+$. (By [Sh371, 1.10] or [Sh371, 1.10A(6)] or [Sh355, 2.4(1)]), hence for $\theta < \mu$

$
\text{cov}(\mu, \theta^+), \theta^+, 2) \leq \mu^+$ (by [Sh430, 1.1]).

4) If 3)'s conclusion holds then:

$(\ast)$ if $\lambda$ is singular strong limit then

(a) $2^\lambda = \lambda^+$

hence

(b) $\diamond^+_S$, where $S_\lambda = \{ \delta < \lambda^+ : cf(\delta) \neq cf(\lambda) \}$, and $\diamond^+_S$ means that there is a $\langle P_\delta : \delta \in S \rangle$, $P_\alpha \subseteq [\alpha]^\lambda$, $|P_\alpha| = \lambda$ such that

$(\forall X \subseteq \lambda^+) \exists \text{club } C \left[ \bigwedge_{\delta \in S \cap C} (X \cap \delta) \in P_\delta \right]$

(by [Sh108] and see there on earlier work of Gregory).

So clearly $\diamond^+_S \ & S_1 \subseteq S \Rightarrow \diamond^+_S$.

(5) Not only $pp(\lambda) > \lambda^+$ and $\lambda > \aleph_0 = cf(\lambda)$ implies $(\ast)_\lambda$ (from 1.11); but assume we have $\langle \lambda_n : n < cf(\lambda) \rangle$, $\sum_n \lambda_n = \lambda$, $\lambda_n = cf(\lambda_n)$, tcf($\Pi \lambda_n / J^b_\omega$) = $\lambda^+$ exemplified by $\bar{f} = f_{\lambda \downarrow \alpha} < \lambda^+$ such that
⊕ if \( \aleph_0 < \text{cf}(\delta) = \kappa < \lambda \), then there is a closed unbounded \( A \subseteq \delta \) and 
\( n_\alpha < \text{cf}(\lambda) \) for \( \alpha \in A \) such that 
\( n_\alpha, n_\beta < n < \text{cf}(\lambda) \Rightarrow f_\alpha(n) < f_\beta(n) \).

Then using \( \oplus \) rather than \((*)_\lambda\), in 1.1 we get a \( \kappa^+\)-CWH, \( \kappa^+\)-metrizable first countable space (see [Sh:355,§6]). \( \square_{1.2} \)

1.3 Construction. Assume \( \lambda = \beth_\omega \) (or just \( \lambda \) is a strong limit, \( \text{cf}(\lambda) \neq \aleph_0 \)), 
\( 2^\lambda = \lambda^+ \) and \( S \) is a stationary subset of \( \lambda^+ \),
\( S \subseteq \{ \delta < \lambda^+ : \text{cf}(\delta) = \aleph_0 \) and \( \omega^2 \) divides \( \delta \} \) (the existence of an \( S \) like that such that \( \diamond_S \) suffices).

We shall build a space with the set of points \( \{ x_\alpha, y_\alpha : \alpha < \lambda^+ \} \). Each \( x_\alpha \) will be isolated in \( X \) and each \( y_\beta \) will have a countable neighborhood basis in \( X \). We shall have \( \{ u_{\alpha,n} : n < \omega \} \) as a neighborhood base of \( y_\alpha \) with \( u_{\alpha,n} \) decreasing in \( n \) and 
\( u_{\alpha,n} = \{ y_\alpha \} \cup \{ x_\beta : f_\alpha(\beta) > n \} \) where \( f_\alpha(\beta) \in \omega \).

Note that each \( Y_\alpha \) is isolated in the space restricted to \( \{ Y_\alpha : \alpha < \lambda^+ \} \).

The only thing left is to define \( f \).

We set \( f_\alpha(\beta) = 0 \) except in some specified cases. For the space to be Hausdorff it is enough to have:

for \( \alpha < \beta \) there is an \( m = m(\alpha, \beta) < \omega \) such that
\( \neg(\exists \gamma)[f_\alpha(\gamma) \geq m \& f_\beta(\gamma) \geq m] \). We shall make a stronger condition:

\((*) \quad \alpha < \beta \Rightarrow (\exists \lambda < 1)(f_\alpha(\gamma) \geq 1 \& f_\beta(\gamma) \geq 1] \).

Remember that \( \diamond_S \) holds as \( 2^\lambda = \lambda^+ \) and \( \text{cf}(\lambda) > \aleph_0 \). So there is a \( \langle g_\alpha : \alpha \in S \rangle \) \( g_\alpha : \alpha \to \omega \) such that

\((\forall g \in \lambda^+)(\exists \alpha \in S)(g_\alpha = g | \alpha) \).

Now, if the space is CWH then there is a \( g : \lambda^+ \to \omega \) such that \( \langle u_{\alpha,g(\alpha)} : \alpha < \lambda^+ \rangle \) are pairwise disjoint.

We define by induction on \( \alpha \) a limit \( \alpha < \lambda^+\), \( f_i(j) \) for \( i, j < \alpha \). Call the sequence \( \langle f_i : i < \alpha \rangle \) in \( \alpha^\omega \to ^\alpha \), so if \( \alpha < \beta \), then \( f_\alpha \) is an initial segment of \( f_\beta \). Usually we just give value zero to \( f_i(j) \).

If \( \alpha \in S \), and \( g_\alpha \) looks as a candidate for \( g \), i.e. \( \langle u_{i,g_\alpha(i)} : i < \alpha \rangle \) are pairwise disjoint, where \( u_{i,k}^\alpha = \{ \beta < \alpha : f_i(\beta) > k \} \), and if for some \( m = m_\alpha \), 
\( \text{otp}(\{ \beta < \alpha : g_\alpha(\beta) = m \}) = \alpha \), then choose

(a) \( \beta_\alpha^n < \beta_\alpha^{n+1} \cdots < \alpha = \bigcup_n \beta_\alpha^n \)
(b) \( g_\alpha(\beta_\alpha^n) = m \)

and define \( f_\alpha^{\alpha+\omega} \) (extending \( f_\alpha \)) by

\( f_\alpha^{\alpha+\omega}(\alpha + n) = n \)

and

\( f_\beta^{\alpha+\omega}(\alpha + n) = m + 1 \)

(other values of \( f_\alpha^{\alpha+\omega} \) are zero). If \( g_\alpha \) fails the conditions above, choose \( m_\alpha = 0 \), 
\( \beta_\alpha^n \) satisfying conditions (a) above and extend \( f_\alpha \) as just described.

So we cannot extend \( g_\alpha \) to \( \alpha^+ \) if \( g(\alpha) = k \). We get
So the space is not CWH (hence not metrizable). For simplicity, we can request that $\beta_n^\alpha \notin \bigcup_{\gamma \in S} [\gamma, \gamma + \omega)$. Suppose the space is not $\aleph_1 - CWH$. So for some $U \in [\lambda^+]^{\aleph_1}$, 

$$X \upharpoonright \{x_\alpha, y_\alpha : \alpha \in U\}$$

is not CWH.

So without loss of generality if 

$$\alpha \in S \cap U$$

then

$$\alpha + n \in U \text{ and } \beta_n^\alpha \in U.$$ 

So

$$\otimes \text{ for every } g : U \to \omega \text{ (candidate to give the separation), we get: for some } \alpha \in S \cap U, (\exists n) g(\beta_n^\alpha) \leq m_\alpha.$$ 

This is a contradiction. \qed

1.4 Comments.

(1) The space constructed in 1.3 does not have neighborhood bases consisting of countable sets, so is not excluded by the earlier consistency results from [JShS320].

(2) But $\Vdash_{\text{Levy}(\aleph_1, \lambda^+)} "X \text{ is not } \aleph_1 - \text{CWH}"$ may fail unless we put more restrictions on the $\beta_n^\alpha$. See (3).

(3) If we build $X$ as above, let $P = \text{Levy}(\aleph_1, \lambda^+)$ and we build a $P$-name $\dot{g}$ such that

$$\Vdash_P "\dot{g} : \lambda \to \omega \text{ witnesses that } X \text{ is CWH},"$$

then $X$ is $\aleph_1$-CWH.

[Why? given a $Y \in [\lambda^+]^{\aleph_1}$, we can find $(p_i : i < \omega_1)$ increasing in $P$ such that 

$$\bigwedge_{\alpha \in Y} \bigvee_i p_i \Vdash \dot{g}(\alpha) = \text{something}.$$

1.5 Definition. We say that the space $X$ is $\lambda - WCH$ if for any discrete set of $\lambda$ points, some subset of cardinality $\lambda$ can be separated by disjoint open sets. 

1.5A Remark. By a theorem of Foreman and Laver for first countable spaces we have the consistency of: $\aleph_1 - WCH \Rightarrow \aleph_2 - WCH$.

On the other hand, e.g. namely in [FoLa], starting with a huge embedding $j : V \to M$ with critical point $\kappa$ and $j(\kappa) = \lambda$, the following is obtained:
There is a forcing notion $P \ast R$ such that $P$ is $\kappa$-c.c., $|P| = \kappa, V[G_P] \models \kappa = \omega_1$, $R \in V[G_P]$ is $\lambda$-c.c., of cardinality $\lambda$ and $(<\kappa)$-closed and $V[G_{P \ast R}] \models \lambda = \omega_2$. In addition, there is a regular embedding $h : (P \ast R) \rightarrow j, P$ with $h(p) - p$ for all $p \in P$ and the master condition property holds for $h, j$. $P \ast R$. Finally, if $G$ is $(P \ast R)$-generic, then in $V[G], j_P/h''(G)$ is $\kappa$-centered.

The consistency of $\aleph_1 - WCWH \rightarrow \aleph_2 - WCWH$ for first countable spaces clearly follows from the above result of [FoLa]. For the convenience of the reader we include the following easy Claim 1.5B which shows this implication.

1.5B Claim. Suppose $X$ is a first countable topological space and $|X| = \kappa^+$, while $Y_0 \subseteq X$ is a discrete subspace of $X$, with $|Y_0| = \kappa^+$. If $P$ is a $\kappa^+$-c.c., even $\kappa$-centered forcing notion such that $\Vdash P \{\bar{x}_\gamma : \gamma < \lambda\}$ is separated, then in $V$, there is a $Y \subseteq Y_0, |Y| = |Y_0|$ and $Y$ is separated in $X$.

Proof. Without loss of generality, the set of points of $Y_0$ in $V_P$ is $\kappa^+$, and we denote $\lambda = \kappa^+$. We may fix a set $\{\bar{x}_\gamma : \gamma < \lambda\}$ of $P$-names such that $\Vdash P \{\bar{x}_\gamma : \gamma < \lambda\}$ is separated.

We can also assume that there are no repetitions among the $x_\gamma$, and that $x_\gamma \geq \gamma$. Suppose that in $V$, the neighborhood bases for points in $Y_0$ are given by

$$\langle \langle u^n_y : n < \omega : y \in Y_0 \rangle :$$

So, without loss of generality $\{u^n_y : n < \omega : y \in Y_0\}$ are pairwise disjoint, in $V^P$.

Now, let $P = \bigcup_{i < \kappa} P_i$ where each $P_i$ is directed.

For each $\alpha < \lambda$, there is a forcing value to $x_\alpha$, say $\beta_\alpha$. So, there is an $i(*) < \kappa$ such that $A = \{\alpha : \beta_\alpha \in P_i(*)\}$ is unbounded in $\lambda$.

Therefore, $\{\beta_\alpha : \alpha \in A\}$ is separated by

$$\{u^n_{\beta_\alpha} : \alpha \in A\}.$$ (So, having that any two members of $P_i$ are compatible, or that out of any $\lambda$ elements of $P$ there are $\lambda$ pairwise compatible, i.e. $P$ is $\lambda$-Knaster, suffices). □

1.6 Claim. There is a first countable Hausdorff space $X$ which is $(2^{\aleph_1})^+ - WCWH$ but is not $WCWH$.

Proof. Let $\lambda = \sum_{n<\omega} \lambda_n, \lambda_n^{\aleph_0} < \lambda_{n+1}. Let \langle \eta_\alpha : \alpha < \lambda^+\rangle, \eta_\alpha \in \omega \lambda, \alpha < \beta and \eta_\alpha < J_{\text{post}} \eta_\beta.$

Topology: as in 1.1.
Proof of not $\lambda^+ - WCWH$: if $\mathcal{U} \in [\lambda^+]^{\lambda^+}$, $\langle \eta_\alpha : \alpha \in \mathcal{U} \rangle$ cannot be separated as $|\{ \eta_\alpha \upharpoonright \ell : \ell < \omega, \alpha \in \mathcal{U} \}| \leq \lambda$.

If $\mathcal{U} \in [\lambda^+]^{(2^{\aleph_0})^+}$, without loss of generality $\text{otp}(\mathcal{U}) = (2^{\aleph_0})^+$; set $\mathcal{U} = \{ \alpha_\zeta : \zeta < (2^{\aleph_0})^+ \}$. Now for some $Y \in [(2^{\aleph_0})^+][(2^{\aleph_0})^+]$ and $n$, $\langle \eta_{\alpha_\zeta} \upharpoonright [n, \omega) : \zeta \in Y \rangle$ is strictly increasing (not just modulo $J^b_\omega$ but in every coordinate (see [Sh355,§6], [Sh400,§5], [Sh430,§6]).

1.7 Remark. We can prove other Claims like 1.6 (see the references above).
2.1 Definition. For an ordinal $\gamma$ let us define

\((\ast)^1_\gamma\) there is a $S \subseteq \{\delta < \gamma : cf(\delta) = \aleph_0\}$ and, for $\delta \in S$, a sequence $\langle \beta^\delta_n : n < \omega \rangle$ strictly increasing with limit $\delta$, and a $m_\delta < \omega$, such that $(\forall g \in \gamma)(\exists \delta \in S)(\exists \infty n) [g(\beta^\delta_n) \leq m_\delta]$. 

2.2 Claim. (1) If the answer to 0.2 is no (or much less), then for some $\gamma < \omega_2$, $(\ast)^1_\gamma$ holds.

(2) If $\text{MA} + \neg \text{CH}$, then $\gamma < 2^{\aleph_0} \Rightarrow \neg (\ast)^1_\gamma$.

(3) Without loss of generality, in $(\ast)^1_\gamma$, each $\beta^\delta_n$ is a successor ordinal.

Proof. 1) By the proof of 1.3 and 1.2.

(2) Check. Use the natural forcing $\{p : p \text{ is a finite function from } \gamma \text{ to } \omega\}$ with $p \leq g$ iff $p \subseteq g$ & $(\forall \delta)(\delta \in S \cap \text{Dom}(p) \rightarrow (\forall n) [\beta_n \in \text{Dom}(g) \setminus \text{Dom}(p) \rightarrow g(\beta_n) > n_\delta])$. (3) Check.

2.2A Conclusion. If $\text{MA} + \neg \text{CH}$ then the answer to 0.2 is yes. In fact, there is an $\aleph_1$-metrizable (hence $\aleph_1$-CWH) not CWH (hence not metrizable) first countable space.

Proof. By 2.2(1) and 2.2(2).

2.3 Claim. If $(\ast)^1_\gamma$ for some $\gamma < \omega_2$, then $(\ast)^1_{\omega_1}$.

Proof. Choose $\gamma^* < \omega_2$ minimal such that $(\ast)^1_{\gamma^*}$. Clearly $\gamma^* \geq \omega_1$.

If $\gamma^* = \omega_1$ we are done. So assume $\gamma^* > \omega_1$, and we shall get a contradiction. We fix an $S \subseteq \gamma$ and $mJ$, $\langle \beta^J_n : n < \omega \rangle$ for $J \in S_1$, which exemplify $(\ast)^1_{\gamma^*}$. Note that for every $\gamma < \gamma^*$ there is a $g_\gamma \in \gamma^\omega$ such that:

$$\otimes \text{ if } \delta \in S \cap \gamma \text{ then } \{n : g_\gamma(\beta^\delta_n) \leq m_\delta\} \text{ is finite.}$$

Case 1. $\gamma^* = \gamma + 1$, $\gamma \notin S$.

Extend $g_\gamma$ by $\{\langle \gamma, 0 \rangle\}$.

Case 2. $\gamma^* = \gamma + 1$, $\gamma \in S$:

define $g \in \gamma^* \omega$:

- if $\beta \in \gamma, \beta \notin \{\beta^\gamma_n : n < \omega\}$ then $g(\beta) = g_\gamma(\beta)$
- if $\beta = \gamma$ then $g(\beta) = 0$
- if $\beta = \beta^\gamma_n$ then $g(\beta) = \text{Max}\{g_\gamma(\beta), n + 8, m_\gamma + 8\}$.

So $g$ gives a contradiction.
Case 3. \( \text{cf}(\gamma^*) = \aleph_0 \).

Let \( \gamma^* = \bigcup_{n<\omega} \gamma_n \), \( \gamma_0 = 0 \), \( \gamma_n < \gamma_{n+1} \), and each \( \gamma_{n+1} \) is a successor of a successor ordinal.

Let \( g = \bigcup \{ g_{\gamma_{n+1}} \restriction [\gamma_n, \gamma_{n+1}) : n < \omega \} \) - it gives a contradiction.

Case 4. \( \text{cf}(\gamma^*) = \omega_1 \).

Let \( \langle \gamma_i : i < \omega_1 \rangle \) be increasing continuous with limit \( \gamma^* \), \( \gamma_0 = 0 \), \( \gamma_{i+1} \) a successor of a successor ordinal.

Let \( S' = \{ \gamma_i : \gamma_i \in S \) (so \( i \) is a limit ordinal)\}.

Subcase A. \( \gamma^*, \langle < \beta_n^i : n < \omega > : \gamma \in S' \rangle, \langle m_\gamma : \gamma \in S' \rangle \) does not exemplify \((*)_{\gamma^*}^\gamma\).

So some \( g^* \in \gamma^* \omega \) shows this. Define \( g \) by:

\[
\text{if } \beta \in [\gamma_i, \gamma_{i+1}) \text{ then } g(\beta) = \max \{ g_{\gamma_{i+1}}(\beta), g^*(\beta) \}
\]

So \( g \) gives a contradiction.

Subcase B. \( \langle < \beta_n^i : n < \omega > : \gamma \in S' \rangle, \langle m_\gamma : \gamma \in S' \rangle \) exemplifies \((*)_{\gamma^*}^\gamma\).

Let \( S^* = \{ i < \omega_1 : i \) a limit, \( \gamma_i \in S \) \} (necessarily stationary).

Let \( \gamma^* = \bigcup_{i < \omega_1} a_i, a_i \) countable increasing continuous, such that \( a_0 = \emptyset \),

\[
a_i \cap \{ j : j < \omega_1 \} = \{ j : j < i \}, a_i \subseteq \gamma_i \text{ and } \gamma_j \in a_i \land j \in S^* \Rightarrow \bigwedge_n \beta_n^j \in a_i.
\]

For \( i \in S^* \) let \( u_i = \{ n < \omega : \beta_n^i \in a_i \} \).

Note

\( \odot \) if \( i \in S^* \) and \( j < i \), then \( \{ n \in u_i : \beta_n^j \in a_j \} \) is finite, as it is included in \( \{ n < \omega : \beta_n^i < \gamma_j \} \). [Why? Remember \( a_j \subseteq \gamma_j \).]

Let \( S^{**} = \{ i \in S^* : u_i \) is infinite and \( i \) is a limit ordinal\} \). So we already know

\( \oplus \) for every \( g \in \gamma^* \omega \), for some \( i \in S^* \), for infinitely many \( n < \omega \), \( g(\beta_n^i) \leq m_{\gamma_i} \).

We claim

\( \oplus^+ \) for every \( g \in \gamma^* \omega \) for some \( i \in S^{**} \), for infinitely many \( n \in u_i \) we have \( g(\beta_n^i) \leq m_{\gamma_i} \).

Otherwise, for some \( g^* \in \gamma^* \omega \) this fails and we define \( g \):

\[
\text{let } \beta \in a_{i+1} \setminus a_i \text{ (there is one and only one such } i) ,
\text{ then } g(\beta) = \max \{ g^*(\beta), m_{\gamma_i} + 8, m_{\gamma_{i+1}} + 8 \}
\]

As \( g \) gives a contradiction to \( \oplus \), clearly \( \oplus^+ \) holds.

Now let \( h \) be a one to one function from \( \omega_1 \) onto \( \gamma^* \) such that for \( i \) limit, \( h \) maps \( \{ j : j < i \} \) into \( a_i \).

Let for \( i \in S^{**} \), \( \{ j_n^i : n < \omega \} \) enumerate \( \{ j < i : h(j) \in \{ \beta_n^i : n \in u_i \} \} \), and \( m_i^* = m_{\gamma_i} \) for \( i \in S^{**} \).

Now \( \langle < j_n^i : n < \omega > : i \in S^{**} \rangle, \langle m_i^* : i \in S^{**} \rangle \) exemplifies that \( \gamma^* \) could have been chosen to be \( = \omega_1 \), as required. \( \square_{2.3} \)

We define the combinatorial property we actually use.
2.4 Definition. 1) $\text{INCWH}(\lambda) = \text{INCWH}^1(\lambda)$ means:

$\lambda$ is regular $> \aleph_0$ and for some stationary $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \aleph_0\}$ we have $\langle m_\delta < \beta^\delta_n : n < \omega : \delta \in S \rangle$ such that:

$m_\delta < \omega$, $\beta^\delta_n < \beta^\delta_{n+1} < \delta = \bigcup_{n<\omega} \beta^\delta_n$, $\beta^\delta_n$ is a successor and:

(a) for every $g \in \lambda$, for some $\delta \in S$, for infinitely many $n$, $g(\beta^\delta_n) \leq m_\delta$

(b) for every $U \subseteq \lambda$, $|U| < \lambda$, for some $g \in U$, for every $\delta \in S \cap U$, for every $n < \omega$ large enough, $g(\beta^\delta_n) > m_\delta$.

2) We can replace $m_\delta$ by $\langle m^\delta_n : n < \omega \rangle$, requesting $q(\beta^\delta_n) \leq m^\delta_n$ in (a) and $q(\beta^\delta_n) > m^\delta_n$ in (b). In this way we obtain a weaker property, which we call $\text{INCWH}^2(\lambda)$.

For other versions of the principle, as well as the connections between the various versions, see §3.

2.4A Discussion. 1) If $\text{INCWH}(\lambda)$, then there is a space (as in 1.3) which is Hausdorff first countable with $\lambda$ points, not metrizable, not even CWH, but every subspace of smaller cardinality is metrizable.

2) So if we prove $(\exists \lambda > \aleph_1)\text{INCWH}(\lambda)$ we have solved the original problem 0.2.

3) $(b)_\kappa$ means that we require $|U| < \kappa$. Note that $(b)_{\aleph_1}$ holds trivially and that $n \leq \kappa \& (b)_\kappa \Rightarrow (b)_n$.

More formally

2.5 Claim. If $\text{INCWH}(\lambda)$ then $\text{SINCWH}(\lambda)$ (even exemplified by a $(< \lambda)$-metrizable space) where:

2.6 Definition. $\text{SINCWH}(\lambda)$ means that there is a first countable $T_2$-space $X$ with $\lambda$ points which is $(< \lambda)$-CWH (i.e. for every discrete subset of cardinality $< \lambda$ we can choose pairwise disjoint open neighborhoods) but not $\lambda$-CWH.

Proof of 2.5. The points of $X$ are $y_\alpha$ $(\alpha < \lambda)$ and $x_{\alpha,\beta}$ $(\beta < \alpha < \lambda)$ which have neighborhood bases $\langle u_{\alpha,n} : n < \omega \rangle$:

If $\alpha \in S$: $u_{\alpha,n} = \{y_\alpha\} \cup \{x_{\alpha,\beta} : \text{for some } k > n, \beta = \beta^\alpha_k\}$

If $\alpha \notin S$: $u_{\alpha,n} = \{y_\alpha\} \cup \{x_{\delta,\alpha} : \alpha < \delta \in S, \text{ for some } k, \alpha = \beta^\delta_k, \text{ and } n \leq m^\delta\}$.

Here, $S$ is a fixed stationary $\subseteq \{\delta < \lambda : \text{cf}(\delta) = \aleph_0\}$ which exemplifies $\text{INCWH}(\lambda)$, together with $\langle m_\delta, (\beta^\delta_n : n < \omega : \delta \in S) \rangle$.

Checking of “$X$ not CWH”

Let $Y = \{Y_\alpha : \alpha < \lambda\}$. Note that $X \upharpoonright Y$ is a discrete subspace of $X$. Let $\{u_{\alpha,n} : n < \omega\}$ be the neighborhood basis of $y_\alpha$, there is $\langle y_{\alpha} \subseteq \{x_{\alpha,\beta} : \beta < \lambda\} \rangle$ a sequence of pairwise disjoint sets, for
some $g \in \lambda^\omega$. As $u_{\alpha,g(\alpha)} \cap u_{\beta,g(\beta)} = \emptyset$ for $\alpha \neq \beta(< \lambda)$ clearly for $\alpha \in S, \beta = \beta^2_n$ we get $n > g(\alpha) \Rightarrow g(\beta) > m_\alpha$ (since otherwise $x_{\alpha,\beta} \in u_{\alpha,g(\alpha)} \cap u_{\beta,g(\beta)}$).

So $g$ contradicts (a) of $INCW(H)(\lambda)$.

Checking of “$X$ is $(< \lambda) - CWH$”

Let $Z \subseteq X$, $|Z| < \lambda$ and $X \upharpoonright Z$ is discrete. Let

$Z_0 = \{x_{\alpha,\beta} : \beta < \alpha < \lambda\} \cap Z, Z_1 = \{y_\alpha : \alpha \in \lambda \setminus S\} \cap Z, Z_2 = \{y_\alpha : \alpha \in S\} \cap Z,$

so $\langle Z_1, Z_2, Z_3 \rangle$ is a partition of $Z$. Let $U = \{\alpha \in S : y_\alpha \in Z_2\}$, so $|U| < \lambda, U \subseteq \lambda$

hence by the assumption, there is a $g_0 \in \lambda^\omega$ as in $(b)_\lambda$.

We define $u_z$, a neighborhood of $z$ for $z \in Z$:

\[
\begin{align*}
&\text{if } z \in Z_0, u_z = \{x_{\alpha,\beta}\} \\
&\text{if } z = y_\alpha \in Z_1, u_z = u_{\alpha,n(\alpha)} \text{ where } \\
&\quad n(\alpha) = \text{Min}\{n : n \geq g(\alpha) + 8 \text{ and } u_{\alpha,n} \cap Z_0 = \emptyset\} \\
&\text{if } z = y_\delta \in Z_2, u_z = u_{\delta,n(\delta)} \text{ where } \\
&\quad n(\delta) = \text{Min}\{n : n \geq m_\delta + 8 \text{ and } u_{\delta,n} \cap Z_0 = \emptyset\}.
\end{align*}
\]

Now check. $\square_{2.5}$

2.7 Claim. Assume $\lambda, \langle m_\delta, \langle \beta^\delta_n : n < \omega \rangle : \delta \in S \rangle$, are as in 2.4 but we require $\lambda$ just to be an ordinal, and weaken $(b)_\lambda$ to

$(b)_\kappa$ for every $U \subseteq \lambda$; $|U| < \kappa$, for some $g \in U^\omega$

for every $\delta \in S \cap U$, for every $n$ large enough $g(\beta^\delta_n) > m_\delta$.

Then for some regular $\mu$, $\kappa \leq \mu \leq \lambda$ we have $INCW(H)(\mu)$.

Proof. If we allow $\mu$ in the definition of $INCW(H)(\mu)$ to be an ordinal: straightforward (and suffices for our main interest). Namely, we choose a $U$ such that

\[
\begin{align*}
&(\alpha) \quad U \subseteq \lambda, \\
&(\beta) \quad \text{there is no } g \in \lambda^\omega \text{ such that for every } \delta \in S \cap U \text{ for every } n \text{ large enough } \\
&\quad g(\beta^\delta_n) > m_\delta, \\
&(\gamma) \quad \text{under } (\alpha) + (\beta) \text{ the order type of } U \text{ is minimal.}
\end{align*}
\]

Clearly $\text{otp}(U) \leq \lambda$ and $\text{otp}(U) \geq \kappa$. By the same proof of 2.3, $\text{otp}(U)$ is a regular cardinal, we call it $\mu$ and with the $a_i$’s as in the proof of 2.3, we get $INCW(H)(\mu)$. $\square_{2.7}$

2.8 Conclusion. If $\lambda = cf(\lambda) > \aleph_0$, $\diamondsuit_{\delta < \lambda : cf(\delta) = \aleph_0}$ then for some regular uncountable $\lambda' \leq \lambda$, $INCW(H)(\lambda')$. This follows by the proof of 1.3 and $(b)_{\aleph_1}$ and 2.7.

2.9 Observation. If $S_1 \subseteq S_2 \subseteq \{\delta < \lambda : cf(\delta) = \aleph_0\}$, $\langle m_\delta, \langle \beta^\delta_n : n < \omega \rangle : \delta \in S_1 \rangle$ witness $INCW(H)(\lambda)$, then we can find a $\langle m'_\delta, \langle \gamma^\delta_n : n < \omega \rangle : \delta \in S_2 \rangle$ witnessing $INCW(H)(\lambda)$.

2.10 Remark. 1) We can replace in our discussion $\aleph_0$ by $\theta$. Toward this we define a family of spaces.
2.11 Definition. \( X \in T^\ell_\theta \) if \( X \) is a Hausdorff space with each point \( x \) having a neighborhood basis \( \{u_{x,\alpha} : \alpha < \alpha^*\} \) such that:

1. (a) \( \ell = 0 \) and \( \alpha^* \leq \theta \)
2. (b) \( \ell = 1, \alpha^* \leq \theta \) and \( \langle u_{x,\alpha} : \alpha < \alpha^* \rangle \) is decreasing.
3. (c) \( \ell = 2, \alpha^* = \theta \), and \( \langle u_{x,\alpha} : \alpha < \alpha^* \rangle \) is decreasing.

2.12 Definition. We define also the principles
\( INCWH(\lambda, \theta) = INCWH^1(\lambda, \theta) \) and \( INCWH^2(\lambda, \theta) \) as in 2.4.

2.13 Claim.

\( (\alpha) \) if \( \lambda > cf(\lambda) = \theta, pp(\lambda) > \lambda^+ \) (or the parallel of 1.2(5)), then

\( \otimes \) there is an \( X \in T^2_\theta, |X| = \lambda^+, X \) is \( \lambda - CWH \), \( X \) has a discrete subspace of size \( \lambda^+ \), but for some \( X' \subseteq X, |X'| = \lambda, cl(X') = X \) (so \( |cl(X')| > \lambda \) (this is a strong form of \( X \) is not \( \lambda^+ - CWH \)).

\( (\beta) \) if \( \lambda > cf(\lambda) = \theta, \lambda \) is a strong limit and \( 2^\lambda = \lambda^+ \), then: \( INCWH(\lambda', \theta) \) for some \( \lambda' = cf(\lambda') \in [\theta^+, \lambda^+] \).

Proof. Similar to the above. \( \square_{2.13} \)
§3 Variants of Freeness

3.1 Definition. 1) \(\text{INCwh}(\lambda) = \text{INCwh}^1(\lambda)\) is defined as in 2.4 except that \(\langle \beta_n^\delta : n < \omega \rangle\) is not required to be increasing with limit \(\delta\), just \([n \neq m \Rightarrow \beta_n^\delta \neq \beta_m^\delta]\). 2) \(\text{INCwh}^2(\lambda)\) is defined as in (1) but we use \(\langle m_n^\delta : n \in \omega \rangle\) rather than a single \(m_\delta^\delta\).

3.2 Claim. 0) \(\text{INCWH}^\ell(\lambda) \Rightarrow \text{INCwh}^\ell(\lambda), \text{INCWH}^1(\lambda) \Rightarrow \text{INCWH}^2(\lambda),\)
(1) \(\text{INCwh}^1(\lambda) \Rightarrow \text{INCwh}^2(\lambda)\).
1) \(\text{INCwh}^2(b)\) (where \(b = \operatorname{Min}\{|F| : f \subseteq \omega \omega \) and for no \(g \in \omega \omega \) for every \(f \in F, f <^* g\}\).
2) Assume \(\lambda \leq 2^{\aleph_0}\) and for \(\alpha < r\), \(f_\alpha\) is a partial function from \(\omega\) to \(\omega\), \(\text{Dom}(f_\alpha)\) is infinite and \(U \subseteq \lambda \& |u| < \lambda \Rightarrow (\exists f \in \omega \omega) \bigwedge_{\alpha \in U} f_\alpha \leq^* f\) but for no \(f \in \omega \omega\),

\[
\bigwedge_{\alpha < \lambda} f_\alpha <^* f, \\
\text{then } \text{INCwh}^2(\lambda).
\]
3) It does not matter in 3.1 if we demand “\(\beta_n^\delta\) is a successor ordinal”.

Proof. 0) Check.
1) By 2).
2), 3) Check. \(\square_{3.2}\)

Questions. 1) Are there such examples for \(\lambda\) singular?
2) Suppose in the definition we allow for each \(\alpha\) a filter on \(\text{Dom}(f_\alpha)\) generated by \(\aleph_0\) sets; do we get an equivalent principle?

3.3 Claim. Assume \(\text{INCwh}^2(\kappa), \lambda > \kappa, \lambda = cf(\lambda), S \subseteq \{\delta < \lambda : cf(\delta) = \aleph_0\}\) is stationary and \(\diamondsuit_S\) holds.
Then (1) there is a \(\langle (m_n^\delta, \beta_n^\delta : n < \omega) : \delta \in S \rangle\) as in 2.4(2), but only (a) and (b) hold.
2) For some regular \(\lambda' \in [\kappa, \lambda]\), we have \(\text{INCWH}^2(\lambda')\).
3) We can replace \(\text{INCwh}^2(\lambda'), \text{INCWH}^2(\lambda)\) by \(\text{INCwh}^1(\lambda), \text{INCWH}^1(\lambda')\) respectively.

Proof. Now (2) follows from (1) as in 2.7 and we leave (3) to the reader. The proof of 3.3(1) is like the proof of 1.3 with one twist. Let \(h : \lambda \rightarrow \kappa\) be such that for every \(\zeta < \kappa, h^{-1}(\{\zeta\})\) has cardinality \(\lambda\). Let \(\langle (m_n^\delta, * \beta_n^\delta : n < \omega) : \delta \in S^* \rangle\) witness \(\text{INCwh}^2(\kappa)\).

Let \(\langle g_\delta : \delta \in S \rangle\) witnesses \(\diamondsuit_S\) i.e. \(g_\delta \in \delta \omega\) and for every \(g \in \lambda \omega\) for stationarily many \(\delta \in S, g_\delta = g \upharpoonright \delta\).
For each \(\delta \in S\) we define a function \(g_\delta^* \in \kappa \omega:\)

\[
g_\delta^*(\zeta) = \operatorname{Min}\left\{m : \text{for arbitrarily large } \alpha < \delta \text{ we have } : m = g_\delta(\alpha) \text{ and } h(\alpha) = \zeta\right\}, \text{ if defined.}
\]
If for some $\zeta < \kappa$, $g_\delta^*(\zeta)$ is not defined (i.e. there is no such $m$) - we do nothing. If $g_\delta^* \in \kappa$ is defined we know that for some $\zeta(\delta) \in S^*$, $(\exists \infty n)(g_\delta^*(\beta_n^\zeta(\delta)) \leq m_n^\delta)$. (Such a $\zeta(\delta)$ exists by the choice of $\langle m_n^\delta, * \beta_n^\zeta : n < \omega : \zeta \in S^* \rangle$). We fix such a $\delta$.

For each $n < \omega$ choose $\xi(\delta, n) < \kappa$ such that:

\begin{itemize}
  \item[\begin{itemize}
    \item[(*)] for arbitrarily large $\gamma < \delta$ we have
      \begin{itemize}
        \item[\begin{itemize}
          \item[(\text{\textup{*}})_1] \text{for arbitrarily large } \gamma < \delta \text{ we have}
          \begin{itemize}
            \item[\begin{itemize}
              \item[(\text{\textup{*}})_2] \text{Choose } \gamma^\delta \text{ such that:}
              \begin{itemize}
                \item[(a)] $\gamma_n^\delta < \delta, h(\gamma_n^\delta) = \beta_n^\zeta(\delta) \land h_0(\gamma) = \beta_n^\zeta(\delta) \land h_1(\gamma) = \xi(\delta, n)$
                \item[(b)] $\delta = \bigcup_{n<\omega} \gamma_n^\delta$ and $\gamma_n^\delta < \gamma_n^\delta + 1$.
              \end{itemize}
            \end{itemize}
          \end{itemize}
        \end{itemize}
      \end{itemize}
    \end{itemize}
  \end{itemize}
\end{itemize}

We claim $\langle m_n^\zeta(\delta), \gamma_n^\delta : n < \omega : \delta \in S \rangle$ witness the conclusion. Looking at Definition 2.4, the preliminary properties hold.

We have to prove clause (a) of 2.7.

**Proof of (a).** Let $g \in \lambda \omega$. For each $\zeta < \kappa$, $\{\alpha < \lambda : h(\alpha) = \zeta\}$ has cardinality $\lambda$, so

$$g^*(\zeta) = \text{Min}\{m : (\exists \lambda \alpha)[h(\alpha) = \zeta \land g^\zeta(\alpha) = m]\}$$

is well defined. Let

$$A =: \{ (\zeta, m) : (\exists \lambda \alpha < \lambda)[g(\alpha) = m \land h(\alpha) = \zeta] \text{ and } \zeta < \kappa, m < \omega \}.$$ 

Then

$$E =: \\{ \delta < \lambda : \text{for every } (\zeta, m) \in A, \text{ for } \lambda \text{ many}
\begin{align*}
\alpha < \lambda, g^\zeta(\alpha) &= m, h(\alpha) = \zeta, \text{ and for every } \\
(\zeta, m) &\in (\kappa \times \omega) \setminus A, \text{ we have } \\
\delta &> \sup \{ \alpha < \lambda : g^\zeta(\alpha) = g^*(\zeta) \land h(\alpha) = \zeta \}
\end{align*}
\}$$

is a club of $\lambda$.

For stationarily many $\delta \in S$, $g_\delta \subseteq g$ so there is such a $\delta \in E \cap S$.

Now check: $g_\delta^* = g^*(g_\delta^* \text{ was defined earlier})$. The rest is also easy to check.

**Proof of (b) i.e. $(< \kappa)$-freeness.** Let $u \subseteq \lambda$, $|u| < \kappa$, hence $v = \{h(\alpha) : \alpha \in u\}$ is a subset of $\kappa$ of cardinality $\kappa$, so by the choice of $\langle m_\delta^\zeta, * \beta_\delta^\zeta : n < \omega, \delta \in S \rangle$ there is a $f^* : v \rightarrow \omega$ as required.

Choose $f : u \rightarrow \omega$ by $f(\alpha) = f^*(h(\alpha))$, now $f$ is as required. \qed

### 3.4 Discussion

1) Probably $\text{INCWH}(\lambda)$ should mean just there is a first countable $(< \lambda)$-CWH not $\lambda$-CWH, as this is actually the two notions which speak on $m_\alpha, \beta_\delta^\zeta(n < \omega)$ or $m_\alpha^\delta, \beta_\delta^\zeta(n < \omega)$ and they should be named $\text{INCWH}^\ell(\lambda), \ell = 1, 2$, respectively.

So, $(\exists \lambda \geq \mu)\text{INCWH}^\ell(\lambda)$ is equivalent to $(\exists \lambda \geq \mu)\text{INCWh}^\ell(\lambda)$ (for $\ell = 1, 2$).
3.5 Definition. 1) $\text{INCWH}^3(\lambda)$ means: there are $S \subseteq \lambda$ and $f : \lambda \times \lambda \to \omega$ such that if we define the spaces as before, i.e.

the points of $X$ are $y_\alpha, x_{\alpha,\beta}, (\alpha < \beta < \lambda)$

each $x_{\alpha,\beta}$ is isolated

$$u_{\alpha,n} = \{y_\alpha\} \cup \{x_{\alpha,\beta} : f(\alpha,\beta) \leq n, \alpha < \beta, \alpha \notin S, \beta \in S\}$$

$$\cup \{x_{\beta,\alpha} : f(\beta,\alpha) \leq n, \beta < \alpha, \beta \notin S, \alpha \in S\}$$

for $n \in \omega$ is a neighborhood base at $Y_\alpha$ such that:

(a) $\alpha < \beta < \lambda$, $u_{\alpha,n} \cap u_{\beta,m} \neq \emptyset \Rightarrow \beta \in S \land \alpha \notin S$

(b) for every $\alpha < \beta \in S$ and for some $n$ we have: $\alpha < \gamma < \beta \Rightarrow u_{\beta,n} \cap u_{\gamma,0} = \emptyset$,

then

(c) the space $X$ is not CWH but is $(< \lambda)$-CWH.

2) $\text{INCWH}^4(\lambda)$ means: there is a symmetric two-place function $f$ from $\lambda$ to

$\{v : v \subseteq \omega \times \omega \text{ is finite, and } (n, m) \in v, n', m' \leq n, m' \leq m \Rightarrow (n', m') \in v\}$

which is not free (i.e. for any $g : \lambda \to \omega$ for some $\alpha < \beta$, $(g(\alpha), g(\beta)) \in f(\alpha, \beta)$), but is $\lambda$-free (i.e. for every $A \subseteq \lambda$, $|A| < \lambda$, there is a $g : A \to \omega$ with no such $\alpha < \beta$ which are from $A$.

The point is that

3.6 Claim. 1) $\text{INCWH}^1(\lambda) \Rightarrow \text{INCWH}^2(\lambda) \Rightarrow \text{INCWH}^3(\lambda) \Rightarrow \text{INCWH}^4(\lambda)$.

2) If $\lambda = \text{cf}(\lambda) > \aleph_0$ and $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \aleph_0\}$ is stationary not reflecting, then $\text{INCWH}^3(\lambda)$.

3.7 Lemma. In 2.5 we can weaken $\text{INCWH}^1_\lambda$ to $\text{INCWH}^3(\lambda)$.

Comment. The $\text{INCWH}^\ell(\lambda)$ are not so artificial: we can translate $\text{INCWH}(\lambda)$ to a similar statement.

3.8 Claim. $\text{SINCWH}(\lambda) \Rightarrow \text{INCWH}^4(\lambda)$.

Proof. Let the space $X$ exemplify $\text{SINCWH}(\lambda)$. Let $\{y_\alpha : \alpha < \lambda\} \subseteq X$ exemplifies “$X$ not $\lambda$-CWH” i.e. it is discrete not separated and $\alpha \neq \beta \Rightarrow y_\alpha \neq y_\beta$.

Let $u_{\alpha,n} \supseteq u_{\alpha,n+1}, \{u_{\alpha,n} : n < \omega\}$ be a neighborhood basis of $y_\alpha$. Now for each $\alpha, n, \beta, m$ choose if possible $x_{\alpha,n,\beta,m} \in u_{\alpha,n} \cap u_{\beta,m}$. Let $f(\alpha, \beta) = \{(n, m) : x_{\alpha,n,\beta,m} \text{ is defined}\}$. This $f$ exemplifies $\text{INCWH}^4(r)$. $\square_{3.8}$

3.8A Remark. The $\Leftarrow$ holds as well.
4.0 Definition. For $\lambda > cf(\lambda) = \theta$ let $(\ast)_\lambda$ means: there is a $\{\eta_\alpha : \alpha < \lambda^+\} \subseteq \theta \lambda$ which is $\lambda$-free (see (c) in 4.1(1) below).

4.1 Definition. 1) For $\theta$ a regular cardinal and $\sigma \geq 1$ (if $\sigma = 1$ we omit it) let:

$$SP_{\theta,\sigma} = \left\{ \lambda : \text{there is a family } H \text{ such that:} \right\}$$

(a) every $h \in H$ is a partial function from ordinals to $\theta$
(b) $h \in H \Rightarrow |Dom(h)| = \theta$
(c) every $H' \subseteq H$ of cardinality $< \lambda$ is $\sigma$-free which means that it can be represented as a union $\bigcup_{i<i(\ast)} H'_i$, $i(\ast) < 1 + \sigma$, and each $H'_i$ is free. For $H'_i$ to be free means that there is a $g$, a function from ordinals to $\theta$ such that

$$\forall h (\exists \xi < \theta)[h \in H'_i \rightarrow (\forall \alpha \in \text{Dom}(h)[h(\alpha) \leq g(\alpha) \lor h(\alpha) \leq \xi])]$$

(d) $H$ is not $\sigma$-free, $|H| = \lambda$

2)

$$SP_{d_{\theta,\sigma}} = \{ \lambda : \text{there is an } H \text{ satisfying (a)-(d) above and (e) each } h \in H \text{ is one to one}\}.$$
3) \[ SP_{\theta, \sigma} = \left\{ \lambda : \text{there is a family } H \text{ such that} : \right\] 

(a) if \((h, \bar{u}) \in H\) then \(h\) is a function from ordinals to \(\theta\)
(b) if \((h, \bar{u}) \in H\), then \(\bar{u} = \langle u_\varepsilon : \varepsilon < \theta \rangle\) is a decreasing sequence of subsets of \(\text{Dom}(h)\)
(c) every pair \((H^1, Z^1)\), with \(Z^1 \subseteq \text{ordinals}, |Z^1| < r\) and \(H^1 \subseteq H\) of cardinality \(< \lambda\) is \(\sigma\)-free, which means it can be represented as \(\bigcup_{i < i(\ast)} (H^1_i, Z^1_i), i(\ast) < 1 + \sigma\) and each \(H' \cap Z\) is free. This means that there are functions \(g, f\) with \(g : H' \to \theta\) and \(f\) from ordinals to \(\theta\)

such that for every \((h, \bar{u}) \in H', \) for every \(\alpha \in Z' \cap \text{Dom}(h)\) we have \(h(\alpha) \leq \max\{f(\alpha), g(h)\}\).

(d) \(H\) is not \(\sigma\)-free, \(|H| = \lambda\)

4.2 Observation. 0) In Definition 4.1, if each \(h \in H\) converges to \(\theta\), in clause (c) of 4.1(1) we can just demand \(\bigcup_{h \in H} \text{Dom}(h) \subseteq \lambda\) and in 4.1(3) without loss of generality \(\bigcup_{(h, u) \in H} \text{Dom}(h) \subseteq \lambda\). Also, without loss of generality \(\text{Dom}(g) = \lambda\).

1) In Definition 4.1(1) without loss of generality \(\bigcup_{h \in H} \text{Dom}(h) \subseteq \lambda\) and in 4.1(3) without loss of generality \(\bigcup_{(h, u) \in H} \text{Dom}(h) \subseteq \lambda\).

2) Note \(\theta^+ \cap SP_\theta = \emptyset\) [why? if \(H = \{h_\zeta : \zeta < \zeta^* \leq \theta\}, \bigcup_{\zeta} \text{Dom}(h_\zeta) = \{\alpha_i : i < \theta\}, \) let \(g(\alpha_i) = \sup\{h_\zeta(\alpha_i) : \zeta < i, \alpha_i \in \text{Dom}(h_\zeta)\}\). This also follows from 4.1(B1) and 4.2(2).

3) \(SP_\theta \cap [\theta^+, 2^\theta] \neq \emptyset\) [this follows from 4.2(4) below].
4) We let 
\[ b[\theta] = \text{Min}\{|F| : F \subseteq \theta, \text{ and for no } g \in \theta \text{ do we have } (\forall f \in F)(\exists \zeta < \theta)(f \upharpoonright \zeta, \theta) < g \upharpoonright [\zeta, \theta])\} \]
if \( \sigma \leq \theta^+ \) then clearly \( b[\theta] \in SP_{\theta, \sigma} \).

5) In Definition 4.1(3) without loss of generality for \((h, \overline{u}) \in H, \bigcap_{\zeta < \theta} u_\zeta = \emptyset \). Also without loss of generality, for \((k, \overline{u}) \in H, u_\zeta = \{ \alpha \in \text{Dom}(h) : h(\alpha) \geq \zeta \} \) (we say: \( \overline{u} \) is standard for \( h \)).

6) Suppose that \( H \) is as in 4.1(3). In c), if we set \( Z' = \theta \) and assume that \( \overline{u} \) is standard, we obtain:
For every \( H' \subseteq H \) with \( |H'| < \lambda \), there are sets \( H'_i \) for \( i < i(*) < 1 + \sigma \) such that \( H' = \bigcup_{i < i(*)} H'_i \) and for each \( i < i(*) \), there is a function \( g_i : H'_i \rightarrow \theta \) with the following property.
For every \( (k, \overline{u}) \in H'_i \)
\[ \exists \xi < \theta, \exists \zeta < \theta, \forall \alpha \in u_\zeta[k(\alpha) \leq \max\{\xi, g_i(\alpha)\}] \].

7) Note also that we can without loss of generality assume that \( Z' \subseteq \bigcup_{k \in H'} \text{Dom}(k) \), for 4.1.3c).

8) We restrict our attention to the case \( \sigma \leq \theta^+ \). Actually, the main interest is in \( \sigma = 1 \). For \( \sigma \) large enough the definition of \( \sigma \)-free sets as it stands would imply that all relevant \( H \) are \( \sigma \)-free, if \( |H| = \lambda \).

**Notation.** For \( a \subseteq \theta \times \theta : a \) is pic if \((\zeta_1, \xi_1) \neq (\zeta_2, \xi_2) \in a \Rightarrow \neg(\zeta_1, \xi_1) \leq (\zeta_2, \xi_2)\) coordinatewise.

Pic\((\theta \times \theta) = \{a : a \subseteq \theta \times \theta \text{ and } a \text{ is pic (hence finite)}\}\)
CL\((a) = \{((\zeta, \xi) \in (\theta \times \theta) : (\exists x \in a)(x \leq (\zeta, \xi) \text{ coordinatewise})\}, \text{ for } a \subseteq \theta \times \theta.\)
4.1A Definition. 1) For \( \theta \) a regular cardinal and \( \sigma \geq 1 \) (if \( \sigma = 1 \) we omit it) let:

\[
SQ_{\theta,\sigma} = \left\{ \lambda : \text{there is a family } H \text{ such that :} \right. \\
(a) \text{ every } h \in H \text{ is a partial function from the ordinals} \\
(b) h \in H \Rightarrow |Dom(h)| = \theta, Rang(h) \subseteq Pic(\theta \times \theta) \\
(c) \text{ every } H' \subseteq H \text{ of cardinality } < \lambda \text{ is } \sigma\text{-free which means that} \\
\text{it can be represented as a union } \bigcup_{i<\iota(\sigma)} H'_i, i(\sigma) < 1 + \sigma, \\
\text{and each } H'_i \text{ is free. For } H'_i \text{ to be free means that there is a } g, \text{ a function from ordinals to } \theta \text{ such that} \\
(\forall h)(\exists \xi < \theta)[h \in H'_i \rightarrow (\forall \alpha \in Dom(h))[(g(\alpha), \Xi) \in Cl(k(\alpha))] \\
(d) H \text{ is not } \sigma\text{-free, } |H| = \lambda \left. \right\}.
\]

2)

\[
SQ_{d,\theta,\sigma} = \left\{ \lambda : \text{there is an } H \text{ satisfying (a)-(d) above and} \right. \\
(e) \text{ each } h \in H \text{ is simple, which means: there is an} \\
\text{enumeration } \text{Dom}(h) = \{\alpha_\zeta : \zeta < \theta\} \text{ with no repetitions,} \\
\text{such that } h(\alpha_g) = \{(\zeta_1, \zeta_2) : (\zeta_1, \zeta_2) \not\in (\beta_\zeta, \gamma_\zeta)\} \\
\text{for some } \langle \gamma_\zeta : \zeta < \theta \rangle \text{ which are strictly increasing and} \\
\bigcup_{\zeta < \zeta} \beta_\zeta < \gamma_\zeta \left. \right\}.
\]
3) 

\[\text{SQ}_{\theta, \sigma} \lambda = \left\{ \lambda : \text{there is a family } H \text{ such that :} \right\} \]

(a) if \((h, \bar{u}) \in H\) then \(h\) is a function from ordinals to \(\text{Pie}(\theta \times \theta)\)

(b) if \((h, \bar{u}) \in H\), then \(\bar{u} = \{u_\varepsilon : \varepsilon < \theta\}\) is a decreasing sequence of subsets of \(\text{Dom}(h)\)

(c) every pain \((H^1, Z^1)\), with \(Z^1 \leq\) ordinals, \(|Z^1| < r\) and \(H' \subseteq H\) of cardinality \(< \lambda\) is \(\sigma\)-free, which means it can be represented as \(\bigcup_{i<i(\ast)} (H'_i, Z'_i)\), \(i(\ast) < 1 + \sigma\) and each \((H'_i, Z'_i)\) is free. This means that there are functions \(g, f\) with \(g : H'_i \to \theta\) and \(f\) from ordinals to \(\theta\) such that for every \((h, \bar{u}) \in H'_i\), for every \(\bar{z} \in Z'_i \cap \text{Dom}(h)\) we have \((g(h), f(z)) \in c\ell(k(z))\)

(d) \(H\) is not \(\sigma\)-free, \(|H| = \lambda\)

(e) \((k, \bar{u}) \in H \Rightarrow \bigcap_{\varepsilon < \theta} u_\varepsilon = \emptyset\) \]

**Note:** 1) In 4.1A3)c), we can assume that \(Z' \subseteq \bigcup_{h \in H'} \text{Dom}(h)\).

2) As in 4.1, we consider only the case \(\sigma \leq \theta +\).

3) \(SP_{x\theta, \sigma}\) can be understood as a particular case of \(SQ_{x\theta, \sigma}\), where \(\text{Rang}(h)\) is restricted to \(\{(\zeta, \zeta) : \zeta < \theta\}\). Here, \(x \in \{w, d\}\) or \(x\) is omitted.

**4.1B Fact.** 1) \(\lambda \in SP_{\theta, \sigma}\) implies that \(\lambda \in SQ_{\theta, \sigma}\)

\(\lambda \in SPd_{\theta, \sigma}\) implies that \(\lambda \in SQd_{\theta, \sigma}\), and

\(\lambda \in SPw_{\theta, \sigma}\) implies that \(\lambda \in SQw_{\theta, \sigma}\).

2) \(\lambda \in SQt_{\theta, \sigma}\) implies that \(\lambda \in SP_{\theta, \sigma}\).

**Proof.** \(H\) exemplifies that \(\lambda \in SP_{\theta, \sigma}\), let \(H^\odot = \{h^\times : h \in H\}\), where for \(h \in H\), \(h^\odot\)

is a function with domain \(\text{Dom}(h)\) and

\(h^\odot(\alpha) = \{(h(\alpha), h(\alpha))\}\).

Similarly for \(SPd_{\theta, \sigma}\).

If \(H\) exemplifies that \(\lambda \in SQw_{\theta, \sigma}\), let \(H^\odot = \{(h^\odot, \bar{u}) : (h, \bar{u}) \in H\}\).

2) Let \(H = \{h_j : j < \lambda\}\) exemplifies that \(\lambda \in SQd_{\theta, \sigma}\), Let us enumerate \(\text{Dom}(h_j) \leq \{\alpha^j_\zeta : \zeta < \theta\}\) for \(j < \lambda\), as in clause (e) of 4.1A(2).

Then we know that

\(h_j(\alpha^j_\zeta) = \left\{ (\varepsilon_1, \varepsilon_2) : \varepsilon_1 < \theta \text{ and } \varepsilon_2 < \theta \text{ and } (\varepsilon_1, \varepsilon_2) \notin (\beta^j_\gamma, \gamma^j_\delta) \right\}\)
for some $(\gamma_\zeta : \zeta < \theta)$ which is strictly increasing and $\gamma_\zeta > \bigcup_{\xi < \zeta} \beta_\xi$.

Let $h_j^\oplus$ be the function with domain $\text{Dom}(h_j') = \{\alpha_\zeta : \zeta < \theta\}$ and defined by $h_j^\oplus(\alpha_\zeta) = \beta_\zeta$. Then $H^\oplus = \{h_j^\oplus : j < \lambda\}$ exemplifies that $\lambda \in \text{SP}_{\theta, \sigma}$.

**Notation.** For a function $h$ from a subset of ordinals to $P_i \epsilon(\theta \times \theta)$, we say that $h$ converges to $\theta$, if

$$(\forall \beta < \theta)(\exists \alpha)(\forall \gamma \in \text{Dom}(h) \setminus \alpha) \left[ (\varepsilon_1, \varepsilon_2) \in h(\gamma) \Rightarrow \varepsilon_1 > \beta \text{ and } \varepsilon_2 > \beta \right].$$

**4.2A Observation.** 0) In Definition 4.1A, if each $h \in H$ converges to $\theta$, in clause (c) of 4.1A(2) we can just demand $$(\forall h)[h \in H' \Rightarrow \theta \supseteq \{\alpha : \exists (\varepsilon_1, \varepsilon_2) \in h(\alpha)[\varepsilon_1 > g(\alpha) \lor \varepsilon_2 > g(\alpha)]\}].$$

1) In Definition 4.1(1) without loss of generality $\bigcup_{h \in H} \text{Dom}(h) \subseteq \lambda$ and in 4.1A(3) without loss of generality $\bigcup_{\langle h(\bar{u}) \in H \rangle} \text{Dom}(h) \subseteq \lambda$. Also, without loss of generality, $\text{Dom}(g) = \lambda$.

2) Note $\theta^+ \cap SQ_\theta = \emptyset$ [why? if $H = \{h_\zeta : \zeta < \zeta^* \leq \theta\}, \bigcup_{\zeta} \text{Dom}(h_\zeta) = \{\alpha_i : i < \theta\}$, let $g(\alpha_i) = \sup\{\max h_\zeta(\alpha_i) : \zeta < i, \alpha_i \in \text{Dom}(h_\zeta)\}]$.

3) $SQ_\theta \cap [\theta^+, 2^\theta] \neq \emptyset$ [this follows from 4.2.3) and 4.1B1)]. Actually, $b[\theta] \in SQ_\theta$.

4) In 4.1A3c), if we set $Z' = \theta$, we obtain the following property.

For every $H' \subseteq H$ of cardinality $< \lambda$, there are sets $H'_i$ for $i < i(*) < 1 + \sigma$, such that there are functions $(q_i : i < i(*)), q_i : H'_i \rightarrow \theta$ satisfying: if $(h, \bar{u}) \in H'_i$, then $(\exists \xi < \theta)(\exists \xi < \theta)(\forall \alpha \in \zeta_\xi)[(q_i(\alpha), \xi) \in c\ell(h(\alpha))].$

**4.3 Claim.** 1) If there is an $H$ as in (a), (b) of 4.1(1) which is $(< \mu) - \sigma$-free not $\lambda - \sigma$-free then there is a $\lambda' \in [\mu, \lambda] \cap \text{SP}_{\theta, \sigma}$. Similarly for 4.1(2), 4.1(3).

2) If $pp_{\Gamma(\theta)}(\lambda) > \lambda^\sigma$, $\lambda > cf(\lambda) = \theta$ (or just $(*)_\lambda$ of 4.0) and $\lambda \geq \sigma$ then $\text{SP}_{\theta, \sigma} \cap [\lambda^\sigma, \lambda^\theta] \neq \emptyset$.

**Proof.** 1) Straightforward.

2) Let for $\{\eta_\alpha : \alpha < \lambda^+\} \subseteq \lambda$ be $\theta$-free, without loss of generality $(\{\eta_\alpha(\zeta) : \alpha < \lambda^+\} : \zeta < \theta)$ are pairwise disjoint and let

$$H = \left\{h : \text{ for some } \alpha < \lambda^+ \text{ and } a \subseteq \lambda^+, \text{otp}(a) = \theta, \text{Dom}(h) = a, \text{h is strictly increasing and for } \beta \in a \Rightarrow h(\beta) = \sup\{\varepsilon : \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon)\} \right\}.$$
with \( \zeta \) such that
\[ \cup \{ \text{Rang}(\eta_\alpha) : \alpha \in A \cap \alpha^*_\zeta \} = \cup \{ \text{Rang}(\eta_\alpha) : \alpha \in A \setminus \alpha^*_\zeta \}. \]
Next choose \( \alpha \in A \cap \alpha^*_\zeta \) and \( \beta \in A \cap \alpha^*_\zeta \) such that \( \eta_\beta(\zeta) = \eta_\alpha(\zeta) \) and let \( a = \{ \beta, \zeta < \theta \}, h \in a, h(\beta) = \sup \{ \varepsilon : \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon) \} \geq \zeta \), so \( h \in H \). As \( \beta \in A \), \( g(\beta) = \varepsilon = \text{constant} \), so if \( \zeta < \theta \), \( \{ \beta : \beta \in \text{Dom}(h) : h(\beta) \geq g(\beta), \zeta \} \) include \( \{ \beta : \zeta, \varepsilon < \zeta < \theta \} \), which is a contradiction.

On the other hand, \( H \) is \( \lambda^+ \)-free. For suppose \( H' \subseteq H \), \( |H'| \leq \lambda \). For \( h \in H' \) choose \( \alpha_h, a_h \) witnessing \( h \in H \). Then \( h = \cup \{ \{ \alpha_h \} \cup a_h : h \in H' \} \) is a subset of \( \lambda^+ \) of cardinality \( \leq \lambda \), hence we can find \( (\varepsilon_\alpha : \alpha \in b) \) such that \( (\text{Rang}(\eta_\alpha \upharpoonright [\varepsilon_\alpha, \theta])) : \alpha \in b \) is a sequence of pairwise disjoint subsets of \( \lambda \). Let us define a \( g : \lambda^+ \rightarrow \theta \) such that \( \alpha \in b \Rightarrow g(\alpha) = \varepsilon_\alpha \). Now if \( h \in H' \), let \( a_h = \{ \beta : \zeta < \theta \} \) (increasing with \( \zeta \)), so
\[
\begin{align*}
P(\beta) &= \sup \{ \varepsilon : \eta_\alpha(\varepsilon) = \eta_\beta(\varepsilon) \} \\
&= \max \{ \varepsilon_\alpha, \varepsilon_\beta \} = \max \{ g(\alpha), g(\beta) \}
\end{align*}
\]
So choose \( \xi = g(\alpha) \) and we get the desired conclusion.
To finish we use part (1).
\( \square_{4.3} \)

4.3A Claim. 1) If there is an \( H \) as in (a), (b) of 4.1A(1) which is \( (\mu) - \sigma \)-free not \( \lambda - \sigma \)-free then there is a \( \lambda' \in [\mu, \lambda] \cap \text{SQ}_{\theta, \sigma} \). Similarly for 4.1A(2), 4.1A(3).
2) If \( pp_{\Gamma(\theta)}(\lambda) > \lambda^+ \), \( \lambda > cf(\lambda) = \theta \) (or just \( (\ast)_\lambda \) of 4.0) and \( \lambda \geq \sigma \) then \( \text{SQ}_{\theta, \sigma} \cap [\lambda^+, \lambda^0] \neq \emptyset \).

Proof. 1) Straightforward.
2) This follows from 4.3.2) and 4.1B1.
\( \square_{4.3A} \)

4.4 Claim. 1) The following implications hold for any \( \lambda \):
\[
(a) \Rightarrow (b) \Leftrightarrow (b)^+ \Leftrightarrow (c) \Rightarrow (c)^+ \Rightarrow (d),
\]
where
\[
\begin{align*}
(a) & : \lambda \in \text{SQ}_{\lambda_{\emptyset}} \\
(b) & : \text{There is a} \ (<\lambda) \text{-CWH not} \ \lambda \text{-CWH first countable space} \\
(b)^+ & : \text{There is a space like in} \ (b), \text{which is in addition} \ (<\lambda) \text{-metrizable} \\
(c) & : \text{There is a} \ (<\lambda)^{-*} \text{CWN first countable space with} \ \lambda \text{ points.} \\
(c)^+ & : \text{There is a space like in} \ (c), \text{which is in addition} \ (<\lambda) \text{-metrizable} \\
(d) & : \lambda \in \text{SQ}_{\lambda_{\emptyset}}.
\end{align*}
\]
2) \( \lambda \in \text{SQ}_{\theta, \sigma} \Rightarrow \lambda \in \text{SQ}_{\lambda_{\emptyset}} \Rightarrow [\lambda, \lambda^0] \cap \text{SQ}_{\theta, \sigma} \neq \emptyset \) for \( \sigma \leq \theta^+ \).
3) \( \lambda \in \text{SP}_{\theta, \sigma} \Rightarrow \lambda \in \text{SP}_{\lambda_{\emptyset}} \Rightarrow [\lambda, \lambda^0] \cap \text{SP}_{\theta, \sigma} \neq \emptyset \) for \( \sigma \leq \theta^+ \).
4) Similarly for \( \theta_{\emptyset} \).

Proof. 1) 4(a) implies (b), (b)^+, (c), (c)^+. First implication - assume \( H \) exemplifies that \( \lambda \in \text{SQ}_{\theta} \), we can use the space
\[
X = \{ y_i : i < \lambda \} \cup \{ z_h : h \in H \} \cup \{ x_{h,i} : h \in H, i \in \text{Dom}(h) \},
\]
and for \( \zeta < \theta \) let \( u_\zeta(z_h) = \{ z_h \} \cup \{ x_{h,i} : i \in \text{Dom}(h), (\zeta, \zeta) \notin cf(h(i)) \} \), \( u_\zeta = u_\zeta \upharpoonright \mathcal{P} \cup \{ h \in H : h \in \text{Dom}(h), (\zeta, \zeta) \notin cf(h(i)) \} \), and \( x \) is isolated.
Suppose $H' \subseteq H$, $|H'| < \lambda$ and let
\[ X[H'] = \{y_i : i < \lambda\} \cup \{z_h : h \in H'\} \cup \{x_{h,i} : h \in H, i \in \text{Dom}(h)\}. \]
Let $g : \lambda \to \theta$ be such that for every $h \in H'$, for some $\zeta[h] < \theta$ we have
\[ i \in \text{Dom}(h) \Rightarrow g(i), \zeta[h] \in \text{cl}(h(i)). \]
Let us choose for $t \in X[H']$ a neighborhood $v_t$:
\[
\begin{align*}
\text{if} & \quad t = x_{h,i} \quad \text{then} \quad v_t = \{x_{h,i}\} \\
\text{if} & \quad t = y_i \quad \text{then} \quad v_t = u_{g(i)}[y_i] \\
\text{if} & \quad t = z_h \quad \text{then} \quad v_t = u_{\zeta[h]}[z_h].
\end{align*}
\]
Now
\[
\langle v_{y_i} : i < \lambda \rangle \setminus \langle v_{z_h} : h \in H' \rangle \setminus \langle v_{x_{h,i}} : i < \lambda, h \in H \text{ and } x_{h,i} \notin \bigcup_{j < \lambda} v_{y_j} \cup \bigcup_{h \in H'} v_{h} \rangle
\]
is a partition of $X[H']$ to pairwise disjoint open sets. In each basic open set there is at most one point which is not isolated, and if so it has a neighborhood base consisting of a decreasing sequence of (open) sets of length $\theta$.
This suffices to show that $X$ is ($< \lambda$)-metrizable when $\theta = \aleph_0$ and as required generally (for 4)).

As for showing that $X$ is not $CWH$ (hence not metrizable and not normal), note that $\{y_i : i < \lambda\} \cup \{z_h : h \in H\}$ is a discrete subspace.
If it is separated, we have a sequence of pairwise disjoint neighborhoods:
\[
\langle u_{g(i)}[y_i] : i < \lambda \rangle \setminus \langle u_{\zeta(h)}[z_h] : h \in H \rangle.
\]
But $H$ is not free (in the sense of Definition 4.1.4.1)) and we get a contradiction.

$\quad (b)^+ \Rightarrow (b)$.
Trivial.

$(b) \Rightarrow (b)^+$. Let $X$ exemplify the second clause so without loss of generality $|X| = \lambda$. Let $Y$ be a discrete subspace of cardinality $\lambda$ which cannot be separated. Let $X^+$ be the topology $X$ on the set of points of $X$ generated by basic open sets of $X$ and $\{\{x\} : x \in X \setminus Y\}$.
Now $X^+$ is not $\lambda - CWH$ ($Y$ still exemplifies it). But $X^+$ is ($< \lambda$)-metrizable as:

If $Z \subseteq X$, $|Z| < \lambda$, then we can find a sequence $\langle u_z : z \in Z \cap Y \rangle$ of pairwise disjoint open sets, and in $X \upharpoonright u_z$, every point is isolated except $z$, which has a neighborhood basis of cardinality $\aleph_0$, and every $x \in Z \setminus \bigcup_{z \in Z \cap Y} u_z$ is isolated.
This is enough.

$(b)^+ \Rightarrow (c)^+$
Trivial (as ($< \lambda$)-metrizable $\Rightarrow (\lambda)^{- \ast} CWN$).

$(c)^+ \Rightarrow (c)$
Trivial.
(c) ⇒ (b)⁺

If $X, \langle Y_i : i < \alpha \rangle$ exemplifies clause (c) in (1) with $\langle u_\zeta(y) : \zeta < \theta \rangle$ a decreasing neighborhood basis of $y$; we can get another example $X'$ to the third clause, as follows.

We are, without loss of generality, assuming that $|X| = \lambda$. Then

$$X' = \bigcup_{i<\alpha} Y_i \cup \left\{ x_{y,z,\zeta,\xi} : \text{for some } i \neq j < \alpha, y \in Y_i, z \in Y_j, u_\zeta[y] \cap u_\zeta[z] \neq \emptyset \right\}$$

with the neighborhood bases for $y, z \in \bigcup_{i<\alpha} Y_i$ given by

$$u_\zeta'[t] = \{ y \} \cup \left\{ x_{y,z,\zeta,\xi} : x_{y,z,\zeta,\xi} \in X', t = y \wedge \varepsilon \leq \zeta \text{ or } t = z \wedge \varepsilon \leq \xi \right\}$$

and $x_{y,z,\zeta,\xi}$ isolated.

Clearly $Y =: \bigcup_{i<\alpha} Y_i$ is discrete. Assume that $\langle u_\varepsilon'[y] : y \in Y \rangle$ is a sequence of pairwise disjoint open sets. Then let

$$U_i = \bigcup \{ u_\varepsilon(y) : y \in Y_i \}.$$ 

So in $X, U_i$ is an open set (as a union of open sets),

$$Y_i \subseteq U_i \text{ as } y \in u_\varepsilon(y)[y]$$

$$i \neq y \Rightarrow U_i \cap U_j = \emptyset \Rightarrow \exists y \in U_i, \exists z \in U_j (U_\varepsilon(y) \cap U_\varepsilon(z) \neq \emptyset)$$

$$\Rightarrow X_{y,\varepsilon(y),\varepsilon(\zeta)} \text{ is well defined}$$

$$\Rightarrow \text{ in } X' \text{ we have that } U_\varepsilon'[y] \cap U_\varepsilon'[z] \neq \emptyset.$$ 

This is a contradiction.

So we conclude that $y$ cannot be separated in $X'$, so $X'$ is not $X - CW H$.

Next, assume that $Z \subseteq X', |Z| < \lambda$, so in $Z, \langle Y_i \cap Z : i < \alpha, y_i \cap z \neq \emptyset \rangle$ can be separated, say by $\langle U_i : i < \alpha, Y_i \cap Z \neq \emptyset \rangle$. So for $y \in Y \cap Z$, there is an $\varepsilon(y)$, such that $U_\varepsilon'[y] \in U_i$ (the isolated points in $X' \cap Z \setminus Y$ can be taken care of easily so we ignore them).

Now, if $Y_1 \neq z \in Y \cap Z$ then:

$$\text{if } (\exists i)(y, z \in X) \text{ then } x_i' \cap x_i' = \emptyset$$
(for any choice of $\varepsilon(y), \varepsilon(z)$ if)
\[ y \in Y_i, z \in Y_j, i \neq j, \text{ if } u'_{\varepsilon(y)} \cap u'_{\varepsilon(z)} \neq \emptyset \]
then \( X_{y,z,\varepsilon(y),\varepsilon(z)} \) exists, so
\[ \emptyset \neq u_{\varepsilon(y)} \cap U_{\varepsilon(z)} \subseteq U_i \cap U_j \]
which is a contradiction.
That \( X' \) is \((< \lambda)\)-metrizable now follows as in \((b) \Rightarrow (b)^+\).

\((c) \Rightarrow (d)\).
Assume that \( X \) is a normal first countable \((< \lambda)\) - *CW-N not \( \lambda - * \) CW-N-space, without loss of generality with the set of points \( \lambda \), so there is a sequence \( \langle Y_i : i < \alpha \rangle \) of pairwise disjoint subsets of \( X, Y_i \neq \emptyset, Y_i \) is clopen in \( X \upharpoonright (\bigcup_{j < \alpha} Y_j) \) and
\[ \langle Y_i : i < \alpha \rangle \]
cannot be separated. For \( y \in Y =: \bigcup_{i < \lambda} Y_i \) let \( \bar{u}Y = \langle u_{\zeta}[y] : \zeta < \theta \rangle \) be a neighborhood basis of the topology for \( y \), and without loss of generality \( \varepsilon < \zeta < \theta \Rightarrow u_{\zeta}[y] \subseteq u_{\varepsilon}[y] \). Let
\[ H = \left\{(h, \bar{u}) : \text{for some } i < \alpha \text{ and for some } y \in Y_i, \right. \]
\[ (k, \bar{u}) = (y, \bar{u}_y), \text{ which means :} \]
\[ \text{Dom}(h) = \bigcup_{j \neq i} Y_j, c\ell(h(z)) = \{((\zeta, \xi) \in \theta \times \theta : u_{\zeta}[y] \cap u_{\xi}[z] = \emptyset \} \]
\[ \text{and } \bar{u} \text{ is } \langle u_{\xi}(y) \cap \text{Dom}(h) : \zeta < \theta \rangle \left\} \right. \]
Note that \( h(z) \) is uniquely determined by \( c\ell(h(z)), c\ell_Q(k(z)) \). As we check that \( H \) exemplifies \( SQ_{w_{\aleph_0}}, \) i.e. the clauses in 4.1A(3). Clauses \((a), (b)\) are immediate. As for clause \((c)\), let \( H' \subseteq H, |H'| < \lambda, \) and \( Z' \subseteq \bigcup \{\text{Dom}(h) : (h, \bar{u}) \in H\}, |Z'| < \lambda, \) let \( Y' =: \{y : y \in \bigcup_{i < \alpha} Y_i, \text{ and } y \in Z' \text{ or } (h, \bar{u}_y) \in H'\}, \) so \(|Y'| < \lambda; \) we can find
\[ X' \subseteq X, |X'| \leq |Y'| + \theta < \lambda \text{ such that } Y' \subseteq X', \text{ and for every } y, z \in Y', \zeta < \theta, \xi < \theta, \]
we have \( u_{\zeta}[y] \cap u_{\xi}[z] \neq \emptyset \Rightarrow u_{\zeta}[y] \cap u_{\xi}[z] \cap X' \neq \emptyset \). As \(|X'| < \lambda \) we know that \( X' \) (i.e. \( X \cap X' \)) is CW-N, and \( \langle Y_i \cap X' : i < \alpha \rangle \) is a discrete sequence of closed sets in \( X' \) hence there is a function \( g : Y' \to \theta \) such that
\[ (*) \text{ if } i < j < \alpha, y \in Y' \cap Y_i, z \in Y' \cap Y_j, \text{ then } \]
\[ u_{g(y)}[y] \cap u_{g(z)}[z] = \emptyset \text{ (intersecting with } X' \text{ is immaterial).} \]
Hence by the choice of \( g \)
\[ (** \text{ if } i \neq j(i < \alpha, j < \alpha), y \in Y' \cap Y_i, z \in Y' \cap Y_j \]
then \( \langle g(y), g(z) \rangle \in c\ell(h(z)) \).
This is enough.

We are left with proving that $H$ is not free, so suppose $f, g : Y \to \theta$ satisfies

\[ \forall y \in Y, \quad \forall z \in \text{Dom}(h_y), (g(y), f(z)) \in c\ell(h_y(z)), \]

so without loss of generality $f = g$.

For $i < \alpha$ let

\[ U_i = \bigcup \{ U_{g(y)}[y] : y \in Y_i \}. \]

So $U_i$, being the union of open sets is open.

If $i < j, y \in Y_i, z \in Y_j$ then

\[ u_{g(y)}[y] \cap u_{g(z)}[z] \neq \emptyset \Rightarrow (g(y), g(z)) \in c\ell(h_y(z)) \]

\[ \Rightarrow (g(y), f(z)) = (g(y), g(z)) \in c\ell(h_y(z)). \]

Contradiction, by the choice of $f$ and $g$.

So $u_{g(y)}[y] \cap u_{g(z)}[z] = \emptyset$, as $y \in Y_i, z \in Y_j$ were arbitrary, $U_i \cap U_j = \emptyset$.

We conclude that $\langle Y_i : i < \lambda \rangle$ can be separated, which is a contradiction.

2) We prove each implication

(A) $\lambda \in SQd_{\theta, \sigma} \Rightarrow \lambda \in SQ_{\theta, \sigma} \Rightarrow \lambda \in SQw_{\theta, \sigma}$. Obvious.

(B) $\lambda \in SQw_{\theta, \sigma} \Rightarrow SQd_{\theta, \sigma} \cap [\lambda, \theta^\omega] \neq \emptyset$ when $\sigma \leq \theta^+$.

Assume that $H$ exemplifies $\lambda \in SQw_{\theta, \sigma}$. By the definition $(h, u) \in H \Rightarrow u_\xi \subseteq \text{Dom}(h)$ & $\bigcap_{\xi < \theta} u_\xi = \emptyset$. Let for each $(h, u) \in H$,

\[ H^*_{(h, u)} = \left\{ f : f \text{ is a function from ordinals to } Pie(\theta \times \theta) \text{ and } \text{Dom}(f) = \nu \right\} \]

for some set $v, v \subseteq \text{Dom}(h), |v| = \theta$, but $\zeta < \theta \Rightarrow |v \setminus u_\zeta| < \theta$, and

\[ (\forall \alpha \in \nu)[c\ell(f(\alpha)) \supseteq c\ell(h(\alpha))], \text{ and } f \text{ is simple} \]

and $H^* = \bigcup \{ H^*_{(h, u)} : (h, u) \in H \}$.

It is easy to check that $H^*$ satisfies clauses (a) and (b) from 4.1A(1) and (e) of 4.1A(2) and $|H^*| = \lambda^\theta$.

As for clause (c) of 4.1A(1), let $H' \subseteq H, |H'| < \lambda$, let $H' = \{ f_j : j < j(*) \}, j(*) < \lambda$, and $(h_j, v_j)$ as in the definition of $H^*_{(h, u)}$ for some $(h_j, u_j) \in H$. Define $H'' = \{(h_j, u_j) : j < j(*)\}, Y = \bigcup_{j < j(*)} v_j$. Now $H''$ is a subset of $H$ of cardinality $< \lambda$, $Y \subseteq \text{Ord}$ and $|Y| < \lambda$ so as $H$ exemplifies $\lambda \in SQw_{\theta, \sigma}$, we can find a $\langle g_i : i < i(*) \rangle$, $i(*) < \sigma, g_i \in \lambda^\theta$ and for every $(h_j, u_j) \in H''$ for some $i = i(j) < i(*)$ we have

\[ (\exists \xi < \theta)(\exists \xi < \theta)(\forall \alpha \in u_{j, \zeta} \cap Y)[(g_i(\alpha), \xi) \in c\ell(h_j(\alpha))]. \]

Now $\langle g_i : i < i(*) \rangle$ are O.K. for $H'$, too, as $c\ell(f_j(\alpha)) \supseteq c\ell(h_j(\alpha))$ and $|v_j \setminus u_{i, \zeta}| < \theta$.
We are left with clause (d) of 4.1A(1), so assume \( i(*) < \sigma \) and \( g_i \in {}^\lambda \theta \) for \( i < i(*) \) exemplifies \( H^* \) is \( \sigma \)-free. By the choice of \( H \) for some \( (h, \bar{u}) \in H \) we have \( \bigwedge_i \neg (\exists \zeta < \theta)(\exists \xi < \theta)(\forall \alpha \in u_\zeta)[(g_i(\alpha), \xi) \in c\ell(h(\alpha))]. \)

Let \( \langle a_i : i < i(*) \rangle \) be a partition of \( \theta \) to unbounded subsets, and we choose by induction on \( \zeta < \theta \), an ordinal \( \alpha_\zeta \in u_\zeta \) and \( \gamma_\xi < \theta \) such that if \( \alpha_\zeta \in a_i \) then

\[
\Upsilon_\zeta \in \theta \setminus \left[ \bigcup_{i < \sigma} \left( g_i(\Upsilon_\xi) \cup \Upsilon_\epsilon \right) + 1 \right]
\]

\[(g_i(\alpha_\zeta), \Upsilon_\zeta) \notin c\ell(h(\alpha_\zeta))\]

and let \( f(\alpha_\zeta) \) be such that

\[
\text{c}\ell(f(\alpha_\zeta)) = \{(\gamma_1, \gamma_2) : \gamma_1 < \theta, \gamma_2 < \theta, \text{ and } (\gamma_1, \gamma_2) \notin (g_i(\alpha_\zeta), \Upsilon_\zeta)\}.
\]

Let \( v = \{\alpha_\eta : \zeta < \theta\} \), so \( f \in H^*(h, \bar{u}) \subseteq H^* \) exemplifies that \( \langle g_i : i < i(*) \rangle \) exemplify that \( H^* \) is \( \sigma \)-free. We can finish by 4.3A(1).

(3) As in 2), \( \lambda \in SP\theta,\sigma \Rightarrow \lambda \in SP\theta,\sigma \Rightarrow \lambda \in SPw_{\theta,}\sigma \) is obvious.

We need to prove that \( \lambda \in SPw_{\theta,}\sigma \Rightarrow SP\theta,\sigma \cap [\lambda, \lambda^\theta] \neq \emptyset \) when \( \sigma \leq \theta^+ \).

The proof if similar to that of (2). We start with \( H \) exemplifying that \( \lambda \in SPw_{\theta,}\sigma \).

We assume that for each \( (h, \bar{u}) \in H, \bar{u} \) is standard. So for \( (h, \bar{u}) \in H, \) we define

\[
H^*_\langle h, \bar{u} \rangle = \left\{ f : f \text{ is a function from ordinals to } \theta \text{ and } f \text{ is } 1 - 1, \right. \\
\left. \text{ and for some set } v \subseteq \text{Dom}(h), \text{ we have that } |v| = \theta, \text{ but } \zeta < \theta \Rightarrow |v \setminus u_\zeta| < \theta, \text{ while } (\forall \alpha \in v)f(\alpha) \leq h(\alpha) \right\}.
\]

Let \( H^* = \cup\{H^*_\langle h, \bar{u} \rangle : (h, \bar{u}) \in H \} \).

Checking that this \( H^* \) is as required is similar to (2). For example, to see 4.1.1)d), suppose that \( i(*) < \sigma \) and \( \langle g_i : i < i(*) \rangle \) exemplify that \( H^* \) is free. By the choice of \( H^* \), there is an \( (h, \bar{u}) \in H \) such that

\[
\bigwedge_i \neg (\exists \zeta < \theta)(\exists \xi < \theta)(\forall \alpha \in u_\zeta)[h(\alpha) \leq \max\{g_i(\alpha), \xi\}] .
\]

Let \( \langle a_i : i < i(*) \rangle \) be as in (2), and we choose by induction on \( \zeta < \theta \), an ordinal \( \alpha_\zeta \in u_\zeta \) and \( \Upsilon_\zeta \in \theta \) such that

\[
\alpha_\zeta \in a_i \Rightarrow \Upsilon_\zeta \in \theta \setminus \left[ \bigcup_{i < \sigma} \left( g_i(\alpha_\xi) \cup \Upsilon_\xi \right) + 1 \right]
\]

and

\[
h(\alpha_\zeta) > \max\{g_i(\alpha_\xi), \Upsilon_\xi\}.
\]
Then we let \( f(\alpha \zeta) \) be such that
\[
f(\alpha \zeta) \leq \max\{g_i(\alpha \zeta, \gamma \zeta) \}
\]
but
\[
f(\alpha \zeta) \notin \{f(\alpha \xi) : \xi < \zeta \}.
\]

4) Included in the proof of (1).

4.5 Claim. Assume \( \lambda \in SP_{\theta, \sigma}, \mu \) is a strong limit with \( cf(\mu) > \theta \), and \( 2^\mu = \mu^+ > \lambda \).
Then there is a \( \kappa \in [\lambda, \mu^+] \), a regular cardinal such that \( \kappa \in SP_{\theta, \sigma}^+ \) where

4.6 Definition. 1) \( \kappa \in SP_{\theta, \sigma}^+ \) means that \( \kappa \) is regular > \( \theta \) and we can find an \( S \subseteq \{\delta < \kappa : cf(\delta) = \theta \} \) stationary, \( \bar{\eta} = \langle \eta_\delta : \delta \in S \rangle, \bar{h} = \langle h_\delta : \delta \in S \rangle \), such that

(a) \( \eta_\delta \) is a strictly increasing sequence of ordinals
of length \( \theta \) with limit \( \delta \)
(b) \( h_\delta : Rang(\eta_\delta) \to \theta \) is strictly increasing
(c) \( H = \{h_\delta : \delta \in S \} \) is \( (< \kappa)-\sigma \)-free not \( \sigma \)-free (in 4.1’s sense).

2) \( \kappa \in SP_{\theta, \sigma}^* \) if in the above we add:
(d) \( h_\delta(\eta_\delta(\varepsilon)) \) depend on \( \eta_\delta(\varepsilon) \) only
(e) \( \bar{\eta} \) is tree like, i.e. \( \eta_{\delta_1}(\varepsilon_1) = \eta_{\delta_2}(\varepsilon_2) \Rightarrow \varepsilon_1 = \varepsilon_2 \) & \( \eta_{\delta_1} \upharpoonright \varepsilon_1 = \eta_{\delta_2} \upharpoonright \varepsilon_2 \).

Remark. The assumption “\( \mu^+ = 2^\mu \)” (in 4.5) is very reasonable because of 4.2(2) (and 4.2(3) from the topological point of view).

4.6A Observation. 1) \( SP_{\theta, \sigma}^* \subseteq SP_{\theta, \sigma}^+ \subseteq SP_{\theta, \sigma} \).
2) If \( \langle h_\delta, \eta_\delta : \delta \in S \rangle, \kappa \) satisfies the preliminary requirements and clauses (a), (b) of 4.5 and \( H \) is \( (< \kappa_1)-\)free, \( \kappa_1 > \theta \) then for some \( \mu \in [\kappa_1, \kappa], \mu \in SP_{\theta}^+ \).
3) Similarly for \( SP_{\theta, \sigma}^* \).

Proof. Like 2.3 or 2.4.

4.6B Conclusion. For \( \lambda > \theta = cf(\theta), \chi = \beth_\chi > \lambda \), the following are equivalent:

(a) for some \( \mu \in [\lambda, \chi], \mu \in SP_{\theta} \)
(b) for some \( \mu \in [\lambda, \chi], \mu \in SP_{\theta}^+ \).
(c) for some \( \mu \in [\lambda, \chi], \mu \in SP_{\theta}^* \).

Proof. By 4.5, (b) \( \Rightarrow \) (a), as for (a) \( \Rightarrow \) (b), let \( \mu = \beth_{\lambda^+(\theta^+)} \), if \( pp(\mu) > \mu^+ \) use 4.2(2) and if \( pp(\mu) = \mu^+ \) use 4.4.

Proof of 4.5. Use \( \diamond_{\{\delta < \mu^+: cf(\delta) = \theta \}} \) and imitate 3.3.
4.7 Claim. Assume $\theta = \theta^{< \theta}$ or $\exists F \subseteq \theta \theta$ which is cofinal in $\theta \theta$ and $|\{f \mid \zeta : f \in F, \zeta < \theta\}| \leq \theta$. Let $\langle h_\delta, \eta_\delta : \delta \in S \rangle$ exemplify $\lambda \in SP^*_\theta$ (even omitting “$\eta_\delta$ converge to $\delta, \eta_\delta$ strictly increasing”). Then any $\theta^+$-complete forcing preserves the non-freeness of $\{h_\delta : \delta \in S\}$.

Proof. Instead of the domain of the functions $h_\delta$ being a subset of $\lambda$, we can assume that it is $T = \{\eta_\delta \cup \zeta : \delta \in S, \zeta < \theta$ a successor} (identify $\eta_\delta(\zeta)$ with $\eta_\delta \upharpoonright (\zeta + 1)$, so $\text{Dom}(h_\delta) = \{\eta_\delta \cup \zeta : \zeta < \theta$ is a successor ordinal $\}$). Suppose $Q$ is a $\theta^+$-complete forcing notion, $p \in Q$ and $p \Vdash "g : T \to \theta$ exemplifies $\{h_\delta : \delta \in S\}$ is free”. We now define by induction on $\ell g(\eta) < \theta$ a sequence $\langle p_{\eta,t}, \varepsilon_{\eta,t}, \nu : t \in T_\eta \rangle$ for $\eta \in T$ such that:

\begin{itemize}
  \item[(a)] $T_\eta \subseteq \ell g(\eta) \geq \theta$, is closed under initial segments
  \item[(b)] $t \triangleleft s \in T_\eta \Rightarrow p_{\eta,t} \leq_Q p_{\eta,s}$
  \item[(c)] If $t \in T_\eta$, either $\bigwedge_{\zeta < \theta} t^\zeta < \zeta > \in T_\eta$ or $\bigwedge_{\zeta < \theta} t^\zeta < \zeta > \notin T_\eta$
  \item[(d)] If $t \in \ell g(\eta) \geq \theta, \ell g(t)$ is a limit ordinal and $(\forall \zeta < \ell g(t))(t \upharpoonright \zeta \in T_\eta)$, then $t \in T_\eta$.
  \item[(e)] If $\nu \triangleleft \eta$ then $T_\nu \subseteq T_\eta$ and $t \in T_\nu \Rightarrow (p_{\eta,t}, \varepsilon_{\eta,t}) = (p_{\nu,t}, \varepsilon_{\nu,t})$
  \item[(f)] If $\ell g(\eta)$ is a limit ordinal then $T_\eta = \{t : t \in \bigcup_{\nu \triangleleft \eta} T_\nu \text{ or } \ell g(t) \text{ is a limit ordinal and } (\forall s)[s \triangleleft t \Rightarrow s \in \bigcup_{\nu \triangleleft \eta} T_\nu]\}$
  \item[(\eta)] Assume $\eta = \nu^\dagger \triangleleft \alpha > s$ is a $\triangleleft$-maximal element of $T_\nu$, then:
    \begin{itemize}
      \item[(a)] if $\{\zeta < \theta : p_{\eta,s} \not\leq_Q "g(\eta) \neq \zeta\"\}$ is bounded in $\theta$
        then $s$ is a $\triangleleft$-maximal element of $T_\eta$.
      \item[(b)] if $A = \{\zeta < \theta : p_{\eta,s} \not\leq_Q "g(\eta) \neq \zeta\"\}$ is unbounded in $\theta$, then for every
        $\zeta < \theta, s^\dagger \triangleleft \zeta > is a maximal member of $T_\eta$, and $p_{s^\dagger \triangleleft \zeta}> forces a value$
        $\varepsilon_{s^\dagger \triangleleft \zeta} > \ell g(\eta)$ to $g(\eta)$.
    \end{itemize}
\end{itemize}

We can carry this definition.

\begin{itemize}
  \item[(\*)] if $\delta \in S$ then for some $\zeta = \zeta_\delta < \theta$ and $t = t_\delta \in T_{\eta_\delta \upharpoonright \zeta}$ we have: $t$ is a $\triangleleft$-maximal member of $T_{\eta_\delta \upharpoonright \zeta}$ for every $\zeta \in [\zeta, \theta)$.
\end{itemize}

[why? otherwise we can construct a $t \in \theta \theta$ such that $(\forall s)[s \triangleleft t \Rightarrow s \in \bigcup_{\xi \triangleleft \theta} T_{\eta_\delta \upharpoonright \xi}]$, $t(\varepsilon) > \varepsilon$ and for unboundedly many $\xi < \theta$, for some $s^\dagger \triangleleft \zeta > \triangleleft t$ we have $s^\dagger \triangleleft \zeta > \in T_{\eta_\delta \upharpoonright (\xi + 1)} \setminus T_{\eta_\delta \upharpoonright \xi}, s^\dagger \triangleleft \zeta > > h_\delta(\eta_\delta \upharpoonright (\xi + 1)), \xi$.

Now $\{p_{\nu,s} : \nu \triangleleft \eta_\delta, s \triangleleft t, s \in T_\nu\}$ has an upper bound in $Q, p^*$. Then $p^*$ forces for $g(\eta \upharpoonright (\xi + 1))$ a value $> h_\delta(\eta \upharpoonright (\xi + 1)), \xi$; this is a contradiction to $p \Vdash "g$ exemplifies the freeness of $\{h_\delta : \delta \in S\}".$

\[\square_{4.7}\]
4.8 Theorem. Assume $\lambda < \mu$, $(\forall \kappa < \mu)[\chi^{\aleph_0} < \mu]$ (possibly $\mu = \infty$). Then the following are equivalent:

(A) There is a space $X$ such that:
   (a) $X$ is $(< \lambda)$-metrizable
   (b) $X$ is not metrizable
   (c) $X$ has $< \mu$ points.

(B) There is a first countable Hausdorff space $X$ such that:
   (a) $X$ is $(< \lambda)$-CWH
   (b) $X$ is not $\lambda$-CWH
   (c) $X$ has $< \mu$ points.

(B)\(^+\) There is a space $X$ like in (B), and in addition
   (a)\(^+\) $X$ is $(< \lambda)$-metrizable.

(C) There is a first countable Hausdorff space $X$ such that:
   (a) $X$ is $(< \lambda)$- CWN
   (b) $X$ is not $\lambda$- CWN
   (c) $X$ has $< \mu$ points.

(C)\(^+\) There is an $X$ like in (C), and in addition,
   (a)\(^+\) $X$ is $(< \lambda)$-metrizable.

(D) there is a family $H$ of functions with domains countable sets of ordinals and range $\subseteq \omega$ such that:
   (a) $H$ is $(< \lambda)$-free
   (b) $H$ is not free
   (c) $|H| < \mu$.

(D)\(^+\) as in (D) and
   (d) $\cup \{\text{Dom}(h) : h \in H\} = \lambda' \in [\lambda, \mu)$
   (e) each $h$ is one to one.

(D)\(^{'}) [\mu, \lambda] \cap SP_{\aleph_0} \neq \emptyset$
(D)\(^{''}) [\mu, \lambda] \cap SP_{w\aleph_0} \neq \emptyset$
(D)\(^{'''}) [\mu, \lambda] \cap SP_{d\aleph_0} \neq \emptyset$

(E) there is a $\bar{u} = (\langle u_{\alpha, n} : n < \omega > : \alpha \in v\rangle, u_{\alpha, n+1} < u_{\alpha, n} \subseteq v$, such that:
   (a) $\bar{u}$ is not free
   (b) for $v' \in [v]^{< \lambda}$, $\bar{u} \upharpoonright v'$ is free
   (c) $|v| < \mu$.

(E)\(^{'}) [\mu, \lambda] \cap SQ_{\aleph_0} \neq \emptyset$
(E)\(^{''}) [\mu, \lambda] \cap SQ_{w\aleph_0} \neq \emptyset$
(E)\(^{'''}) [\mu, \lambda] \cap SQ_{d\aleph_0} \neq \emptyset$. 
4.8A Theorem. In 4.8 if \( (\forall \kappa < \mu)(\beth_\kappa^+(\kappa) < \mu) \) (really \( (\forall \kappa < \mu)(\beth_{\omega_1}(\kappa) < \mu) \)) is O.K. Equivalently \( \mu = \beth_\delta = \theta^+(\delta) \) then we can add

\[ (F) \] for some regular \( \kappa \in [\lambda, \mu) \) we have \( \text{INCWH}(\kappa) \)
\[ (F)' \] \( \lambda \in SP_{\mathcal{N}_0}^+ \)
\[ (F)'' \] \( \lambda \in SP_{\mathcal{N}_0}^- \)

Proof of 4.8A. By 4.4(1) (the \((b) \iff (b)^+ \iff (c) \iff (c)^+\) part) we know the equivalence of (A), (B), (B)' \iff (C) \iff (C)^+.
By x.x \((D) \iff (D)\)'.
By 4.4(3) we have \((D)' \implies (D)^{''} \implies (D)^{'''}\).
By x.x \((E) \iff (E)\)'.
By 4.4(3A) \((E)^{'} \implies (E)^{''} \implies (E)^{'''}\).
By 4.1A(2) \((E)^{''} \implies (D)\)'.
By 4.1A(1) \((D)^{'} \implies (E)^{'} \implies (D)^{''} \implies (E)^{''} \implies (D)^{'''} \implies (E)^{''}\).

Together we get the equivalence of \((D), (E), (D)^{'} \implies (E)^{''} \implies (D)^{''} \implies (E)^{''}\).
By 4.4(1) \((E)^{'} \implies (A) \implies (E)^{''}\), so by the last sentence and the first paragraph we have finished the proof of 4.8. For 4.8A use 4.6B.

4.4 Fact. Let \( \lambda = \text{cf}(\lambda) > \theta = \text{cf}(\theta) \).

(A) There is \( H = \bigcup_{i<\lambda} H_i \) such that:
\[ (\alpha) \] \( H_i \) is increasing continuous
\[ (\beta) \] \( H_\theta \) is a family of functions \( h \), \( \text{Dom}(h) \) is a set of \( \theta \) ordinals, \( h \) is one to one
\[ (\gamma) \] each \( H_i \) is free, but \( H \) is not free.

(B) = \((B)^{r,\theta}\). Let \( X = X_{\lambda,\theta} = : \lambda \theta \)

\[ F = F_{\lambda,\theta} =: \left\{ f : f \text{ a partial function from } X \text{ to } \theta, |\text{Dom } f| = \theta \text{ and } \begin{align*} (\forall^* i < \lambda) (\forall^* \eta \in \text{Dom}(f))[f(\eta) \leq \eta(i)] \end{align*} \text{ and } f \text{ is one to one} \right\}. \]

Then there is no \( G : X \to \omega \) such that
\[ f \in F \implies f(\forall^* \eta \in \text{Dom}(f))[f(\eta) \leq G(\eta)]. \]
Then
\[ (A) \iff (B). \]

Proof. \((A) \implies (B)\).
Let $H, H_i (i < \lambda)$ exemplifies (A), let $A = \cup\{\text{Dom}(h) : h \in H\}$, and let $g_i : A = \theta$ exemplify “$H_i$ is free”.

We define an equivalence relation $E$ on $A : \alpha \exists_\beta \iff \bigwedge_{i < \lambda} g_i(\alpha) = g_i(\beta)$. If for some $h \in H$ and $\alpha, (\alpha/E) \cap \text{Dom}(h)$ has cardinality $\theta$, choose $i < \lambda$ such that $h \in H_i$, and $g_i$ cannot satisfy the requirement. let $h^\otimes$ be a function with domain $\text{Dom}(h)$, $h^\otimes(\alpha) = \sup\{h(\beta) : \beta \in \alpha/E\}$. Now $H' := \{h^\otimes : h \in H\}, H''\{h^\otimes : h \in H'_i\}$ exemplifies (A) too. So without loss of generality $E$ is the equality on $A$.

Next for each $\alpha \in A$ let $\eta_\alpha \in \lambda \theta (= X)$ be defined by $\eta_\alpha(i) = g_i(\alpha)$, so $\alpha \neq \beta \Rightarrow \eta_\alpha \neq \eta_\beta$. For $h \in H$ let $\text{Dom}(h) = \{\alpha_{h, \zeta} : \zeta < \theta\}$ such that $\langle h(\alpha_{h, \zeta}) : \zeta < \theta\rangle$ is strictly increasing. For $h \in H$ let the function $f_h$ be defined by:

$$\text{Dom}(f_h) = \{\eta_{\alpha_{h, \zeta}} : \zeta < \theta\}, f_h(\eta_{\alpha_{h, \zeta}}) = h(\alpha_{h, \zeta}).$$

Now

$$(*) \quad h \in H \Rightarrow f_h \in F.$$

[Why? Let $i(*) = \min\{i : h \in H_i\}$ (well defined as $H = \bigcup_{i < \lambda} H_i$), so $i \in [i(*), \lambda)$ implies $h \leq^* (g_i \restriction \text{Dom}(h))$. So for some $\zeta(*) < \theta$, for every $\zeta \in [\zeta(*), \theta]$ we have $h(\alpha_{h, \zeta}) \leq g_i(\alpha_{h, \zeta})$, but $f_h(\eta_{\alpha_{h, \zeta}}) = h(\alpha_{h, \zeta})$ and $g_i(\alpha_{h, \zeta}) = \eta_\alpha(i)$ so: for every $i < \lambda$ large enough for all but $\theta$ members $\eta = \eta_{\alpha_{h, \zeta}}$ of $\text{Dom} \ f_h$, $f_h(\eta) = h(\alpha_{h, \zeta}) \leq g_i(\alpha_{h, \zeta}) = \eta_{\alpha_{h, \zeta}}(i) = \eta(i)$ as required.]

So assume $G$ is a function from $X$ to $\omega$ such that

$$(**) \quad f \in F \Rightarrow (\forall^* \eta \in \text{Dom}(f))[f(\eta) \leq G(\eta)]$$

and we should get a contradiction. Let us define $g \in A\theta$ by $g(\alpha) = G(\eta_\alpha)$. So for $h \in H$, we have $f_h \in F$ hence by $(*) + (**) \Rightarrow \exists \zeta(*) < \theta, \zeta \in [\zeta(*), \theta) \Rightarrow f_h(\eta_{\alpha_{h, \zeta}}) \leq G(\eta_{\alpha_{h, \zeta}})$. But $f_h(\eta_{\alpha_{h, \zeta}}) = h(\alpha_{h, \zeta})$, and $g(\alpha_{h, \zeta}) = G(\eta_{\alpha_{h, \zeta}})$ so $\zeta \in [\zeta(*), \theta) \Rightarrow h(\alpha_{h, \zeta}) \leq g(\alpha_{h, \zeta})$. So $g$ shows that $H$ is free, contradiction. We have proved (B).

$(B) \Rightarrow (A)$

The demand $A = \bigcup_{h \in H} \text{Dom}(h) \subseteq \text{Ord}$ is immaterial, so let $A = X, H = F_{\lambda, \theta}$.

Lastly for $i < \lambda$ let $g_i : A \to \theta$ be $g_i(\eta) = \eta(i)$, and

$$H_i = \{f \in F : \text{for every } j \in [i, \lambda) \text{ we have } (\forall^* \eta \in \text{Dom}(f))[f(\eta) \leq \eta(i)]\}.$$

4.10 Conclusion. $\text{INCWH}(\lambda)$ implies $(B)_{\lambda, \theta}$ of Fact 4.9 implies $(\exists \mu)[\lambda \leq \mu \leq 2^\lambda \land \text{INCWH}(\mu)].$

4.11 Remark. It is well known that

$(*)$ if there is a real valued measure $m$ on $P(\lambda), \theta = \aleph_0$

$$G(f) = \min\{n : m(f^{-1}(\{n\}) > 0\}$$

then $G$ contradicts $(B)_{\lambda, \aleph_0}$.

Also, it is consistent that $\text{SP}_{\aleph_0} \subseteq (2^{\aleph_0})^+$. This follows from the consistency of the PMEA (Product Measure Extension Axiom) and Fact 4.9.

The consistency of PMEA is due to Kunen. See [Fl] for an exposition.
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