On the constrained classical capacity of infinite-dimensional covariant channels

A. S. Holevo
Steklov Mathematical Institute, Moscow

Abstract

The additivity of the minimal output entropy and that of the $\chi$-capacity are known to be equivalent for finite-dimensional irreducibly covariant channels. In this paper we formulate a list of conditions allowing to establish similar equivalence for infinite-dimensional irreducibly channels with constrained input. This is then applied to Bosonic Gaussian channels with quadratic input constraint to extend the classical capacity results of the recent paper [2] to the case where the complex structures associated with the channel and with the constraint operator need not commute. In particular, this implies a multimode generalization of the ”threshold condition”, obtained for single mode in [9]) and the proof of the fact that under this condition the classical ”Gaussian capacity” resulting from optimization over Gaussian inputs is equal to the full classical capacity.

1 Introduction: finite dimensions

For the background of this section we refer to [5], [4]. Let $\Phi$ be a quantum channel in $d$-dimensional Hilbert space $\mathcal{H}$. A quantum analog of the Shannon capacity is the $\chi$-capacity

$$C_\chi(\Phi) = \max_{\pi} \left\{ S\left( \Phi \left[ \sum_x \pi(x) \rho(x) \right] \right) - \sum_x \pi(x) S(\Phi[\rho(x)]) \right\},$$

where $S(\rho) = -\text{Tr} \rho \log \rho$ is the von Neumann entropy, and the maximum is over state ensembles i.e. finite probability distributions $\pi$ ascribing probabilities $\pi(x)$ to density operators $\rho(x)$.

The classical capacity of the quantum channel $\Phi$, defined as the maximal transmission rate per use of the channel, with coding and decoding chosen for increasing number $n$ of independent uses of the channel

$$\Phi^{\otimes n} = \Phi \otimes \cdots \otimes \Phi$$
such that the error probability goes to zero as $n \to \infty$, is equal to

$$C(\Phi) = \lim_{n \to \infty} (1/n)C_{\chi}(\Phi^\otimes n).$$

In the case where the $\chi$-capacity is additive,

$$C_{\chi}(\Phi^\otimes n) = nC_{\chi}(\Phi), \quad (2)$$

one has $C(\Phi) = C_{\chi}(\Phi)$. An obvious upper estimate for $C_{\chi}(\Phi)$ is

$$C_{\chi}(\Phi) \leq \max_{\rho} S(\Phi[\rho]) - \tilde{S}(\Phi), \quad (3)$$

where the minimal output entropy of the quantum channel $\Phi$ is defined as

$$\tilde{S}(\Phi) = \min_{\rho} S(\Phi(\rho)).$$

For some channels $\tilde{3}$ may become equality, allowing to reduce the additivity of $C_{\chi}$ to the additivity of the minimal output entropy

$$\tilde{S}(\Phi^\otimes n) = n\tilde{S}(\Phi). \quad (4)$$

This is the case for irreducibly covariant channels. Channel $\Phi$ is covariant if there is a continuous (projective) unitary representation $g \to V_g$ of a symmetry group $G$ in $\mathcal{H}$ such that

$$\Phi[V_g\rho V_g^*] = U_g\Phi[\rho]U_g^*, \quad (5)$$

where $U_g$ are unitary operators, and irreducibly covariant if the representation $g \to V_g$ is irreducible. Assuming compactness of $G$, one may show, see e.g. $\tilde{3}$, that for irreducibly covariant channel

$$\max_{\rho} S(\Phi[\rho]) = S(\Phi[I/d]), \quad (6)$$

and

$$C_{\chi}(\Phi) = \max_{\rho} S(\Phi[\rho]) - \tilde{S}(\Phi) = S(\Phi[I/d]) - \tilde{S}(\Phi), \quad (7)$$

making (3) the equality. The optimal (generalized) ensemble for $C_{\chi}(\Phi)$ is $\{\pi^0(dg), V_g\rho_0 V_g^*\}$ where $\pi^0$ is the invariant probability measure on $G$ and $\rho_0$ is a minimizer for $S(\Phi(\rho))$.

Assume moreover that additivity of the minimal output entropy (4) holds, then

$$n \left[ \max_{\rho} S(\Phi[\rho]) - \tilde{S}(\Phi) \right] = nC_{\chi}(\Phi) \leq C_{\chi}(\Phi^\otimes n) \leq \max_{\rho(n)} S(\Phi^\otimes n[\rho^{(n)}]) - \tilde{S}(\Phi^\otimes n) = n \left[ \max_{\rho} S(\Phi[\rho]) - \tilde{S}(\Phi) \right], \quad (8)$$

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where the first equality follows from (7), the first inequality – from the definition of \( C_\chi \), the second inequality – from (3) applied to \( \Phi \otimes n \), the second equality – from the equality

\[
\max_{\rho^{(n)}} S(\Phi \otimes n \left[ \rho^{(n)} \right]) = n \max_{\rho} S(\Phi [\rho])
\]

valid for all channels, and from the condition (4). Thus \( C_\chi(\Phi \otimes n) = n C_\chi(\Phi) = n \left[ S(\Phi[I/d]) - \tilde{S}(\Phi) \right] \) (8) and

\[
C(\Phi) = C_\chi(\Phi) = S(\Phi[I/d]) - \tilde{S}(\Phi).
\]

The equality (8) is a simple corollary of (sub)additivity of the von Neumann entropy with respect to tensor products (cf. lemma 2 below). For irreducibly covariant channels it is just a consequence of (4).

2 Infinite-dimensional case

Let \( \mathcal{H} \) be a separable complex Hilbert space, \( \mathcal{L}(\mathcal{H}) \) the algebra of all bounded operators in \( \mathcal{H} \), \( \mathcal{S}(\mathcal{H}) \) be the space of trace-class operators, and \( \mathcal{D}(\mathcal{H}) \) be the convex set of density operators in \( \mathcal{H} \). Quantum channel is a linear completely positive trace-preserving map \( \Phi \) in \( \mathcal{S}(\mathcal{H}) \).

Generalized ensemble is a pair \( \{ \pi(dx), \rho(x) \} \) where \( \pi \) is a probability measure on a standard Borel space \( \mathcal{X} \) and \( x \to \rho(x) \) is a measurable map from \( \mathcal{X} \) to \( \mathcal{D}(\mathcal{H}) \). The average state of the generalized ensemble \( \pi \) is defined as the barycenter of the probability measure

\[
\bar{\rho}_\pi = \int_\mathcal{X} \rho(x) \pi(dx).
\]

The conventional ensembles correspond to finitely supported measures.

In the infinite-dimensional case one usually has to consider the input constraints to avoid infinite values of the capacities. Let \( H \) be a positive selfadjoint operator in \( \mathcal{H} \), which usually represents energy of the input. We consider the input states with constrained energy: \( \text{Tr} \rho H \leq E \), where \( E \) is a fixed positive constant. Since the operator \( H \) can be unbounded, care should be taken in defining the trace; we put \( \text{Tr} \rho H = \int_0^\infty \lambda \, dm_\rho(\lambda) \), where \( m_\rho(\lambda) = \text{Tr} \rho E(\lambda) \), and \( E(\lambda) \) is the spectral function of the selfadjoint operator \( H \). Then the constrained \( \chi \)-capacity is given by the following generalization of the expression (1):

\[
C_\chi(\Phi, H, E) = \sup_{\pi : \text{Tr} \bar{\rho}_\pi H \leq E} \chi(\pi),
\]

where

\[
\chi(\pi) = S(\Phi[\bar{\rho}_\pi]) - \int_\mathcal{X} S(\Phi[\rho(x)]) \pi(dx)
\]

To ensure that this expression is defined correctly, certain additional conditions upon the channel \( \Phi \) and the constraint operator \( H \) should be imposed (see [4], [5], Sec. 11.5), which are always fulfilled in the Gaussian case we consider below.
Denote $H^{(n)} = H \otimes I \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes H$, then the **constrained classical capacity** is given by the expression

$$C(\Phi, H, E) = \lim_{n \to \infty} \frac{1}{n} C_\chi(\Phi^{\otimes n}, H^{(n)}, nE).$$  \hspace{1cm} (11)

Consider the following constrained set of states

$$\mathcal{E}_E = \{ \rho : \text{Tr} \rho H \leq E \},$$

We have an obvious estimate

$$C_\chi(\Phi, H, E) \leq \sup_{\rho \in \mathcal{E}_E} S(\Phi(\rho)) - \inf_{\rho} S(\Phi(\rho)).$$  \hspace{1cm} (12)

**Proposition 1** Consider the following assumptions:

1. $\sup_{\rho \in \mathcal{E}_E} S(\Phi(\rho))$ is attained on a state $\rho_0^E$;
2. $\inf_{\rho} S(\Phi(\rho))$ is attained on a state $\rho_0$;
3. $\Phi$ is a covariant channel in the sense (5), and there exists a Borel probability measure $\pi^0_\mathcal{E}$ on $G$ such that

$$\rho_0^E = \int_G V_g \rho_0 V_g^* \pi^0_\mathcal{E}(dg).$$

4. the minimal output entropy of the channel $\Phi$ is additive in the sense (4),

Then under the conditions 1-3

$$C_\chi(\Phi, H, E) = \sup_{\rho \in \mathcal{E}_E} S(\Phi(\rho)) - \inf_{\rho} S(\Phi(\rho)) = S(\Phi[\rho_0^E]) - S(\Phi[\rho_0]),$$  \hspace{1cm} (13)

the optimal ensemble for $C_\chi$ consisting of the states $V_g \rho_0 V_g^*$ with the probability distribution $\pi^0_\mathcal{E}(dg)$.

If, in addition the condition 4 holds, then

$$C_\chi(\Phi^{\otimes n}, H^{(n)}, nE) = n C_\chi(\Phi, H, E)$$

and

$$C(\Phi, H, E) = C_\chi(\Phi, H, E) = S(\Phi[\rho_0^E]) - S(\Phi[\rho_0]).$$

**Proof.** To prove the first statement it is sufficient to substitute the ensemble $\{\pi^0_\mathcal{E}(dg), V_g \rho_0 V_g^*\}$ into the expression (11). For covariant channels the integral term is equal to $S(\Phi[\rho_0])$, thus we obtain that the right-hand side of (12) is also a lower estimate for $C_\chi(\Phi, H, E)$.

To prove the second statement we use lemma 11.20 of [5]

**Lemma 2**

$$\sup_{\rho^{(n)} : \text{Tr} \rho^{(n)} H^{(n)} \leq nE} S(\Phi^{\otimes n}[\rho^{(n)}]) = n \sup_{\rho : \text{Tr} \rho H \leq E} S(\Phi[\rho]).$$
Proof. We give the proof here for completeness. We first show that
\[
\sup_{\rho^{(n)}: \text{Tr} \rho^{(n)} H^{(n)} \leq nE} S(\Phi^{\otimes n}[\rho^{(n)}]) \leq n \sup_{\rho: \text{Tr} \rho H \leq E} S(\Phi[\rho]).
\] (14)

Indeed, denoting by \(\rho_j\) the partial state of \(\rho^{(n)}\) in the \(j\)-th tensor factor of \(\mathcal{H}_A^{\otimes n}\) and letting \(\bar{\rho} = \frac{1}{n} \sum_{j=1}^{n} \rho_j\), we have
\[
S(\Phi^{\otimes n}[\rho^{(n)}]) \leq \sum_{j=1}^{n} S(\Phi[\rho_j]) \leq nS(\Phi[\bar{\rho}]),
\]
where in the first inequality we used subadditivity of the quantum entropy, while in the second – its concavity. Moreover, \(\text{Tr} \bar{\rho} H = \frac{1}{n} \text{Tr} \rho^{(n)} H^{(n)} \leq E\), hence (14) follows. In the opposite direction, take \(\rho^{(n)} = \rho_0^{E\otimes n}\) and use additivity of entropy for product states.

Then we have similarly to the finite-dimensional case
\[
n \left[ S \left( \Phi \left[ \rho_E \right] \right) - S(\Phi \left[ \rho_0 \right]) \right] = n C_\chi(\Phi, H, E) \leq C_\chi(\Phi^{\otimes n}, H^{(n)}, nE)
\]
\[
\leq \max_{\rho^{(n)}: \text{Tr} \rho^{(n)} H^{(n)} \leq nE} \left( S(\Phi^{\otimes n}[\rho^{(n)}]) - \min_{\rho} S(\Phi^{\otimes n}[\rho]) \right)
\]
\[
= n \left[ \max_{\rho: \text{Tr} \rho H \leq E} S \left( \Phi \left[ \rho \right] \right) - \min_{\rho} S \left( \Phi \left[ \rho \right] \right) \right]
\]
\[
= n \left[ S \left( \Phi \left[ \rho_E \right] \right) - S(\Phi \left[ \rho_0 \right]) \right],
\]
where the first equality follows from (13), the first inequality from the definition of \(C_\chi\), the second inequality from (12) applied to \(\Phi^{\otimes n}\), the second equality from lemma [2] and the condition 4. Summarizing,
\[
C_\chi(\Phi^{\otimes n}, H^{(n)}, nE) = n C_\chi(\Phi, H, E) = n \left[ S \left( \Phi \left[ \rho_E \right] \right) - S(\Phi \left[ \rho_0 \right]) \right],
\]
hence the second statement follows.

3 The case of Bosonic Gaussian channels

In the papers [2], [8], [1] a solution of the quantum Gaussian optimizers conjecture was given for gauge-covariant or contravariant Bosonic Gaussian channels. In particular, the constrained classical capacity was computed under the assumption that the constraint operator is gauge-invariant with respect to the same complex structure as the channel. Basing on observations of previous section and using the fact that a general Bosonic Gaussian channel is irreducibly covariant under the group of displacements (the Weyl group), we can relax this restriction.

In this section we systematically use the notations and some results from the book [5] where further references are given. Let \(\mathcal{H}\) be the space of an irreducible
representation \( \{ z \to V(z); z \in Z \} \) of Canonical Commutation Relations, where \( Z \) is a finite-dimensional symplectic space \((\mathbb{R}^2s, \Delta)\), where

\[
\Delta = \begin{bmatrix}
0 & 1 & & & \\
-1 & 0 & & & \\
& & \ddots & & \\
& & & 0 & 1 \\
& & & -1 & 0
\end{bmatrix} = \text{diag} \begin{bmatrix} 0 & 1 \\
-1 & 0 \end{bmatrix}.
\] (15)

A centered Gaussian state \( \rho \) on \( \mathcal{L}(\mathcal{H}) \) is determined by its covariance matrix \( \alpha \) which is a real symmetric \( s \times s \)-matrix satisfying

\[
\alpha \geq \pm \frac{i}{2} \Delta.
\]

The entropy of \( \rho \) is equal to

\[
S(\rho) = \frac{1}{2} \text{Sp} \ g \left( \text{abs}(\Delta^{-1} \alpha) - \frac{I}{2} \right),
\] (16)

where \( \text{Sp} \) is used to denote trace of a matrix as distinct from the trace \( \text{Tr} \) of operators in the underlying Hilbert space \([7]\). The operator \( A = \Delta^{-1} \alpha \) has the eigenvalues \( \pm i \alpha_j \). Hence its matrix is diagonalizable (in the complex domain). For any diagonalizable matrix \( M = U \text{diag}(m_{ij}) U^{-1} \), we denote \( \text{abs}(M) = U \text{diag}(|m_{ij}|) U^{-1} \).

Let \( \Phi \) be a centered Bosonic Gaussian channel, while \( H = \sum_{j,k=1}^{s} \epsilon_{jk} R_j R_k \), where \( \epsilon = [\epsilon_{jk}] \) is a symmetric positive definite matrix, is a quadratic energy operator. Notice that it always has an associated complex structure \( J_H \) satisfying \([\epsilon \Delta, J_H] = 0\). This is the orthogonal operator from the polar decomposition of the operator \(-\epsilon \Delta\), see sec. 12.2.3 \([5]\) for detail.

Any centered Gaussian channel \( \Phi \) is characterized by the matrix parameters \((K, \mu)\), satisfying

\[
\mu \geq \pm \frac{i}{2} (\Delta - K^t \Delta K).
\]

Its action on the centered Gaussian state with a covariance matrix \( \alpha \) is described by the equation

\[
\alpha \to K^t \alpha K + \mu.
\]

The condition 1 follows from the argument of sec. 12.5 \([5]\), moreover \( \rho^0_E \) is a centered Gaussian state with a covariance matrix

\[
\alpha^0_E = \arg \max_{\alpha: \text{Sp} \alpha \leq E} \text{Sp} \ g \left( \text{abs}(\Delta^{-1} [K^t \alpha K + \mu]) - \frac{I}{2} \right).
\]

A Bosonic Gaussian channel is irreducibly covariant with respect to the representation \( \{ z \to V(z) \} \) in the sense

\[
\Phi[V(z)\rho V(z)^*] = V(K^s z)\Phi[\rho]V(K^s z)^*, \quad z \in Z,
\]
where \( K^* = \Delta^{-1} K^t \Delta \), see e.g. sec. 12.4.2 in [5].

Assuming that \( \Phi \) is gauge-covariant or contravariant with respect to a complex structure \( J \) in \( Z \), the conditions 2 and 4 follow from the results of the paper [2] concerning the minimal output entropy. Moreover, \( \rho_0 \) can be taken as the vacuum state related to the complex structure \( J \). It is shown in sec. 12.3.2 of [5] that the vacuum state related to the complex structure \( J \) is pure centered Gaussian state with the covariance matrix \( \frac{1}{2} \Delta J \).

Then the condition 3 is fulfilled provided
\[
\alpha_0 \geq \frac{1}{2} \Delta J. \tag{17}
\]
In this case
\[
\rho^0_E = \int_Z V(z) \rho_0 V(z)^* \pi^0_E(dz),
\]
where \( \pi^0_E(dz) \) is centered Gaussian distribution on \( Z \) with the covariance matrix \( \alpha_0 - \frac{1}{2} \Delta J \). One can check this by comparing the quantum characteristic functions of both sides. The optimizing ensemble consists thus of the \( J \)-coherent states \( V(z) \rho_0 V(z)^* \) with the probability distribution \( \pi^0_E(dz) \).

The constrained classical capacity of the channel \( \Phi \) is equal to
\[
C(\Phi; H, E) = C^*_\chi(\Phi; H, E) = \frac{1}{2} \max_{\alpha; \text{Sp} \alpha \leq E} \text{Sp} g \left( \text{abs} (\Delta^{-1} [K^t \alpha K + \mu]) - \frac{I}{2} \right) - \frac{1}{2} \text{Sp} g \left( \text{abs} (\Delta^{-1} \left[ \frac{1}{2} K^t \Delta J K + \mu \right]) - \frac{I}{2} \right). \tag{18}
\]

Gauge-covariance of the channel \( \Phi \) with respect to a complex structure \( J \) is equivalent to the conditions
\[
[K, J] = 0, \quad [\Delta^{-1} \mu, J] = 0. \tag{19}
\]
Given a symmetric \( \mu \geq 0 \), one can always find a complex structure \( J \), satisfying \( [\Delta^{-1} \mu, J] = 0 \); it is just the orthogonal operator from the polar decomposition of the operator \( \Delta^{-1} \mu \) in the Euclidean space \( (Z, \mu) \). Then the first equation becomes a restriction for admissible \( K \). For gauge-contravariant channels it is replaced by \( \{K, J\} = 0 \).

In the paper [9] the Gaussian capacities obtained by optimization over Gaussian inputs where computed for a generic non-degenerated single-mode channel when the input signal energy is above certain threshold. Our observations imply in particular that these Gaussian capacities are in fact equal to the full classical capacities, and the inequality (17) appears as the multimode generalization of the threshold condition in [9].

Let us confirm this by calculation of the example of squeezed noise channel. The channel is described by the parameters
\[
K = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}; \quad \mu_1 \mu_2 \geq \frac{1}{4} |k^2 - 1|.
\]
This describes attenuation ($0 < k < 1$), amplification ($1 < k$) and additive noise ($k = 1$) channels, with the background squeezed noise. Take the Hamiltonian $H = q^2 + p^2$ with the corresponding complex structure $J_H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The complex structure of the channel satisfying (19) is given by

$$J = \begin{bmatrix} \frac{\sqrt{\mu_1/\mu_2}}{\sqrt{\mu_1/\mu_2}} & -\mu_2/\mu_1 \\ \mu_2/\mu_1 & 0 \end{bmatrix},$$

which does not commute with $J_H$ unless $\mu_1 = \mu_2$. The covariance matrix of the squeezed vacuum is

$$\frac{1}{2} \Delta J = \frac{1}{2} \begin{bmatrix} \frac{\sqrt{\mu_1/\mu_2}}{\sqrt{\mu_1/\mu_2}} & 0 \\ 0 & \frac{\sqrt{\mu_2/\mu_1}}{\sqrt{\mu_2/\mu_1}} \end{bmatrix}.$$

The eigenvalues of the matrix

$$\Delta^{-1} \left[ \frac{1}{2} K^t \Delta J K + \mu \right] = \begin{bmatrix} 0 & \mu_2 + k^2/2 \\ -\left( \mu_1 + k^2/2 \right) & 0 \end{bmatrix}$$

are equal to $\pm i \left( \sqrt{\frac{\mu_1}{\mu_2}} + k^2/2 \right)$, hence the second term in (18) is $g \left( \sqrt{\frac{\mu_1}{\mu_2}} + \left( k^2 - 1 \right)/2 \right)$.

To compute the first term, we can restrict to diagonal covariance matrices

$$\alpha = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}, \quad \alpha_1 + \alpha_2 \leq E, \quad \alpha_1\alpha_2 \geq \frac{1}{4}.$$

The matrix

$$\Delta^{-1} \left[ K' \alpha K + \mu \right] = \begin{bmatrix} 0 & \mu_2 + k^2/2 \\ -\left( \mu_1 + k^2/2 \right) & 0 \end{bmatrix}$$

has the eigenvalues $\pm i \sqrt{\left( \mu_1 + k^2/2 \right) \left( \mu_2 + k^2/2 \right)}$, so that the maximized expression is $g \left( \sqrt{\left( \mu_1 + k^2/2 \right) \left( \mu_2 + k^2/2 \right) - 1/2} \right)$. Since $g(x)$ is increasing, we have to maximize $\left( \mu_1 + k^2/2 \right) \left( \mu_2 + k^2/2 \right)$ under the constraints $\alpha_1 + \alpha_2 \leq E, \quad \alpha_1\alpha_2 \geq \frac{1}{4}$. The first constraint gives the values

$$\alpha_1^0 = E/2 + (\mu_2 - \mu_1)/2k^2, \quad \alpha_2^0 = E/2 - (\mu_2 - \mu_1)/2k^2$$

corresponding to the maximal value of the first term

$$g \left( \frac{1}{2} \left( k^2 E + (\mu_1 + \mu_2) - 1 \right) \right).$$

The second constraint will be automatically fulfilled provided we impose the condition (17) which amounts to $\alpha_1^0 \geq \frac{k}{2} \sqrt{\mu_1/\mu_2}$, $\alpha_2^0 \geq \frac{k}{2} \sqrt{\mu_2/\mu_1}$, or, introducing the squeezing parameter $\eta = \sqrt{\mu_2/\mu_1}$

$$E \geq \frac{1}{2} \left[ \eta + \eta^{-1} + |\eta - \eta^{-1}| \left( 1 + \frac{2}{k^2 \sqrt{\mu_1/\mu_2}} \right) \right].$$
Under this condition

\[ C(\Phi; H, E) = C_\chi(\Phi; H, E) \]

\[ = g \left( \frac{1}{2} (k^2 E + (\mu_1 + \mu_2) - 1) \right) - g \left( \sqrt{\mu_1 \mu_2} + (k^2 - 1)/2 \right). \]

These values up to notations coincide with those computed in [9], Cor. 2.

References

[1] V. Giovannetti, R. García-Patron, N. J. Cerf, A. S. Holevo, Ultimate classical communication rates of quantum optical channels, Nature Photonics, 2014, 216, 6 pp.

[2] V. Giovannetti, A. S. Holevo, R. García-Patron, A solution of the Gaussian optimizer conjecture, arXiv:1312.2251; accepted for publication in Commun. Math. Phys.

[3] A. S. Holevo, Additivity conjecture and covariant channels. Int. J. Quant. Inform., 3, (2005), 41-48.

[4] A. S. Holevo and V. Giovannetti, Quantum channels and their entropic characteristics, Rep. Prog. Phys. 75 (2012), 046001.

[5] A. S. Holevo, Quantum systems, channels, information. A mathematical introduction, De Gruyter, Berlin–Boston, 2012.

[6] A. S. Holevo and M. E. Shirokov, Continuous ensembles and the \(\chi\)-capacity of infinite-dimensional channels, Probab. Theory and Appl. 50 (2005), 86-98.

[7] A. S. Holevo, M. Sohma, O. Hirota, Error exponents for quantum channels with constrained inputs, Rep. Math. Phys., 46 (2000), 343-358.

[8] A. Mari, V. Giovannetti, and A. S. Holevo, Quantum state majorization at the output of bosonic Gaussian channels, Nature Communications, 5 (2014), 3826, 5 pp.

[9] J. Schäfer, E. Karpov, R. García-Patrón, O. V. Pilyavets, and N. J. Cerf, Equivalence Relations for the Classical Capacity of Single-Mode Gaussian Quantum Channels, Phys. Rev. Lett. 111, (2013) 030503.