A Simple Proof of Hardy-Lieb-Thirring Inequalities

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Abstract: We give a short and unified proof of Hardy-Lieb-Thirring inequalities for moments of eigenvalues of fractional Schrödinger operators. The proof covers the optimal parameter range. It is based on a recent inequality by Solovej, Sørensen, and Spitzer. Moreover, we prove that any non-magnetic Lieb-Thirring inequality implies a magnetic Lieb-Thirring inequality (with possibly a larger constant).

1. Introduction and main result

This paper is concerned with estimates on moments of negative eigenvalues of Schrödinger operators $(-\Delta)^s - C_{s,d}|x|^{-2s} - V$ in $L^2(\mathbb{R}^d)$ in terms of integrals of the potential $V$. Here

$$C_{s,d} := 2^{2s} \frac{\Gamma((d + 2s)/4)^2}{\Gamma((d - 2s)/4)^2}$$

is the sharp constant in the Hardy inequality

$$\int_{\mathbb{R}^d} |p|^{2s} |\hat{u}(p)|^2 dp \geq C_{s,d} \int_{\mathbb{R}^d} |x|^{-2s} |u(x)|^2 dx, \quad u \in C_0^\infty(\mathbb{R}^d),$$

which is valid for $0 < s < d/2$ [He] and we write $\hat{u}(p) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} u(p)e^{-ip \cdot x} dx$ for the Fourier transform of $u$. In [FrLiSe1] we have shown that for any $\gamma > 0, 0 < s \leq 1$ and $0 < s < d/2$ one has

$$\text{tr} \left( (-\Delta)^s - C_{s,d}|x|^{-2s} - V \right)^\gamma \leq L_{\gamma,d,s}^{\text{HLT}} \int_{\mathbb{R}^d} V(x)^{\gamma d/2s} dx$$

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with a constant $L^{\text{HLT}}_{\gamma,d,s}$ independent of $V$. Here and in the following, $t_{\pm} := \max\{ \pm t, 0 \}$ denote the positive and negative parts of a real number or a self-adjoint operator $t$. The case $s = 1$ in (1.3) has been shown earlier in [EkFr]. We refer to (1.3) as Hardy-Lieb-Thirring inequality since it is (up to the value of the constant) an improvement of the Lieb-Thirring inequality [LiTh]

$$\text{tr} \left( (-\Delta)^s - V \right)_-^{\gamma} \leq L_{\gamma,d,s} \int_{\mathbb{R}^d} V(x)^{\gamma + d/2s} \, dx.$$

(1.4)

It should be pointed out that if $0 < s < d/2$, then (1.4) is valid even for $\gamma = 0$ (as first shown by Cwikel, Lieb, and Rozenblum) while (1.3) is not. We refer to the surveys [LaWe,Hu] for background and references concerning (1.4).

The original motivation for (1.4) came from the problem of stability of non-relativistic matter (see [LiSe] for a textbook presentation). Likewise, our motivation for (1.3) was stability of relativistic matter in magnetic fields up to and including the critical value of the nuclear charge $\alpha Z = 2/\pi = C_{1/2,3}$; see [FrLiSe1] and also [FrLiSe2].

The purpose of this paper is fourfold.

1. We will give a new, much simpler proof of (1.3). While the method in [FrLiSe1] relied on rather involved relations between Sobolev inequalities and decay estimates on heat kernels, the present proof uses nothing more than (1.4) (with $\gamma = 0$ and with $s$ replaced by some $t < s$) and the generalization of a powerful (though elementary to prove) new inequality by Solovej, Sørensen and Spitzer [SoSoSp].

2. We will extend (1.3) to its optimal parameter range $0 < s < d/2$. For $d \geq 3$ and $1 < s < d/2$ this is a new result, even for integer values of $s$ when the operator is local. This result can not be attained with the method of [FrLiSe1], since positivity properties of the heat kernel break down for $s > 1$.

3. Though our new proof of (1.3) does not work in the presence of a magnetic field, we shall prove a new operator-theoretic result, which says that any non-magnetic Lieb-Thirring inequality implies a magnetic Lieb-Thirring inequality (with possibly a different constant). This recovers, in particular, that (1.3) holds if $(-\Delta)^s$ is replaced by $|D - A|^{2s}$ and $0 < s \leq 1$. (The reason for the restriction $s \leq 1$ at this point is that we need a diamagnetic inequality.) Another application of this result concerns the recent inequality in [KoVuWe] corresponding to the endpoint $\gamma = 0$ of (1.4) with $s = 1, d = 2$.

4. We show that an analog of inequality (1.3) for $s = 1/2, d = 3$ holds in a model for pseudo-relativistic electrons that includes spin. The difficulty here is that the potential energy is non-local. This new estimate simplifies some of the arguments in [FrSiWa] and will be, we believe, a crucial ingredient in the proof of stability of matter in this model.

Here is the precise statement of our result.

**Theorem 1.1.** Let $d \geq 1$, $0 < s < d/2$ and $\gamma > 0$. Then there is a constant $L^{\text{HLT}}_{\gamma,d,s}$ such that

$$\text{tr} \left( (-\Delta)^s - C_{s,d}|x|^{-2s} - V \right)_-^{\gamma} \leq L^{\text{HLT}}_{\gamma,d,s} \int_{\mathbb{R}^d} V(x)^{\gamma + d/2s} \, dx.$$

(1.5)
If \( d \geq 2, 0 < s \leq 1 \) and \((-\Delta)^s\) is replaced by \(|D - A|^{2s}\) for some \( A \in L_{2, \text{loc}}(\mathbb{R}^d, \mathbb{R}^d)\), then (1.5) remains valid if \( L_{\gamma, d, s}^{\text{H}} \) is replaced by \( L_{\gamma, d, s}^{\text{H}}(e/p)^p \Gamma(p+1) \) with \( p = \gamma + d/2s \).

The crucial ingredient in our proof of (1.5) is the following lower bound for the quadratic form

\[
h_s[u] := \int_{\mathbb{R}^d} |p|^{2s}|\hat{u}(p)|^2 dp - C_{s,d} \int_{\mathbb{R}^d} |x|^{-2s}|u(x)|^2 dx
\]

of the operator \((-\Delta)^s - C_{s,d}|x|^{-2s}\).

**Theorem 1.2.** Let \( 0 < t < s < d/2 \). Then there exists a constant \( \kappa_{d,s,t} > 0 \) such that for all \( u \in C_0^\infty(\mathbb{R}^d) \) one has

\[
h_s[u]^\theta \|u\|^{2(1-\theta)} \geq \kappa_{d,s,t} \|(-\Delta)^{t/2}u\|^2, \quad \theta := t/s.
\]

In the special case \( d = 3 \) and \( s = 1/2 \) this is a recent result by Solovej, Sørensen and Spitzer [SoSoSp, Thm. 11]. The results reported here are motivated by their work. Below we shall show that their proof extends to arbitrary \( 0 < s < d/2 \).

Our original proof of (1.5) in [FrLiSe1] for \( 0 < s \leq 1 \) relied on the Gagliardo-Nirenberg-type inequality

\[
h_s[u]^\theta \|u\|^{2(1-\theta)} \geq \sigma_{d,s,q} \|u\|_q^2, \quad \theta := \frac{d}{s} \left( \frac{1}{2} - \frac{1}{q} \right),
\]

for \( 2 < q < 2s/(d-2s) \). This is weaker than (1.6) in view of the Sobolev inequality [LiLo, Thms. 4.3 and 8.3]

\[
\|(-\Delta)^{t/2}u\|^2 \geq S_{d,t} \|u\|_q^2, \quad q = \frac{2d}{d-2t}.
\]

What makes (1.6) much easier to prove than (1.7) is that it is a linear inequality, that is, all norms are taken in \( L_2(\mathbb{R}^d) \). Indeed, (1.6) is easily seen to be equivalent to the operator inequality

\[
(-\Delta)^s - C_{s,d}|x|^{-2s} \geq K_{d,s,t} l^{-2(s-t)} (-\Delta)^t - l^{-2s}, \quad l > 0,
\]

where \( K_{d,s,t} = (s^{-t} l^t (s-t)^{s-t})^{1/s} \kappa_{d,s,t} \), and this is the way we shall prove it in the next section.

**2. Proof of Theorem 1.2**

Throughout this section we assume that \( 0 < s < d/2 \). Recall that for \( 0 < \alpha < d \) the Fourier transform of \(|x|^{-d+\alpha}\) is given by

\[
b_{d-\alpha}(\cdot | \cdot |^{-d+\alpha})^\wedge (p) = b_\alpha |p|^{-\alpha}, \quad b_\alpha := 2^{\alpha/2} \Gamma(\alpha/2);
\]

see, e.g., [LiLo, Thm. 5.9], where another convention for the Fourier transform is used, however. This implies that for \( 2s < \alpha < d \) one has

\[
\int_{\mathbb{R}^d} \frac{1}{|p - q|^{d-2s}|q|^{\alpha}} dq = \Psi_{s,d}(\alpha) \frac{1}{|p|^{\alpha-2s}},
\]

(2.2)
where
\[ \Psi_{s,d}(\alpha) := (2\pi)^{d/2} \frac{b_{2s} b_{\alpha-2s} b_{d-\alpha}}{b_{d-2s} b_{\alpha+2s} b_{\alpha}} = \frac{\pi^{d/2} \Gamma(s)}{\Gamma((d-2s)/2)} \frac{\Gamma((\alpha-2s)/2)}{\Gamma((d-\alpha)/2) \Gamma((d-\alpha+2s)/2) \Gamma((d-\alpha)/2)}. \]

We shall need the following facts about \( \Psi_{s,d}(\alpha) \) as a function of \( \alpha \in (2s, d) \).

**Lemma 2.1.** \( \Psi_{s,d} \) is an even function with respect to \( \alpha = (d+2s)/2 \) and one has
\[
\Psi_{s,d}((d+2s)/2) = (2\pi)^{d/2} \frac{b_{2s}}{b_{d-2s}} \frac{C_{s,d}^{-1}}{C_{s,d}}
\]
with \( C_{s,d} \) from (1.1). Moreover, \( \Psi_{s,d} \) is strictly decreasing on \( (2s, (d+2s)/2) \) and strictly increasing on \( ((d+2s)/2, d) \).

This is Lemma 3.2 from [FrLiSe1] in disguise.

**Proof of Lemma 2.1.** \( \Psi_{s,d}(\alpha) \) is obviously invariant under replacing \( \alpha \) by \( d+2s-\alpha \), and its value at \( \alpha = (d+2s)/2 \) follows immediately from definition (1.1). To prove the monotonicity we write
\[
\Psi_{s,d}(\alpha) = \frac{\pi^{d/2} \Gamma(s)}{\Gamma((d-2s)/2)} \frac{f(t)}{f(s+t)}, \quad t = (\alpha - 2s)/2,
\]
where \( T := (d-2s)/2 \) and \( f(t) := \Gamma(t)/\Gamma(T+s-t) \). We need to show that \( \log(f(t)/f(s+t)) \) is strictly decreasing in \( t \in (0, T/2) \). Noting that
\[
\frac{f'(t)}{f(t)} = \psi(t) + \psi(T+s-t)
\]
with \( \psi := \Gamma'/\Gamma \) the Digamma function, we have
\[
\frac{d}{dt} \log \frac{f(t)}{f(t+s)} = \psi(t) + \psi(T+s-t) - \psi(t+s) - \psi(T-t) = -\int_t^{t+s} h(\tau) \, d\tau
\]
with \( h(\tau) := \psi'(\tau) - \psi'(T+s-\tau) \) for \( 0 < \tau < T+s \). Since \( \psi' \) is strictly decreasing [AbSt, (6.4.1)], \( h \) is an odd function with respect to \( \tau = (T+s)/2 \) which is strictly positive for \( \tau < (T+s)/2 \). Since the midpoint of the interval \((t, t+s)\) lies to the left of \((T+s)/2\), the integral of \( h \) over this interval is strictly positive, which proves the claim.

\[ \square \]

Now we prove (1.8), following the strategy of Solovej, Sørensen and Spitzer [SoSøSp] in the special case \( d = 3 \), \( s = 1/2 \); see also [LiYa, Thm. 11] for a related argument.

**Proof of Theorem 1.2.** For technical reasons we prove the theorem only for \( 2s/3 \leq t < s \). In view of the inequality \((-\Delta)^t \leq (3t/2)^{2(s-3t/2)}(-\Delta)^{2s/3} + (1-3t/2s)^{l-2t} \) for \( t < 2s/3 \) and \( l > 0 \) this implies the result for all \( 0 < t < s \).

By a well-known argument (going back at least to Abel and, in the present context, to [KoPeSe]) based on the Cauchy-Schwarz inequality, one has for any positive measurable function \( h \) on \( \mathbb{R}^d \)
\[
(2\pi)^{d/2} \frac{b_{2s}}{b_{d-2s}} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2s}} \, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\tilde{u}(p) \tilde{u}(q)}{|p-q|^{d-2s}} \, dp \, dq \leq \int_{\mathbb{R}^d} t_h(p) |\tilde{u}(p)|^2 \, dp,
\]
where

\[ t_h(p) := h(p)^{-1} \int_{\mathbb{R}^d} \frac{h(q)}{|p - q|^{d-2s}} \, dq. \]

Below we shall choose \( h \) (depending on \( l > 0 \)) in such a way that for some positive constants \( A \) and \( B \) (depending on \( d, s \) and \( t \), but not on \( l \)) one has

\[ t_h(p) \leq \Psi_{s,d}((d + 2s)/2)|p|^{2s} - A l^{-2(s-t)} |p|^{2t} + B l^{-2s}. \tag{2.4} \]

(By scaling it would be enough to prove this for \( l = 1 \), but we prefer to keep \( l \) free.) Because of (2.3) this estimate proves (1.8).

We show that (2.4) holds with \( h(p) = (|p|^{d+2s}/2 + t^{\beta-(d+2s)/2}|p|^\beta)^{-1} \), where \( \beta \) is a parameter depending on \( t \) that will be fixed later. (Indeed, we shall choose \( \beta = 2t + (d - 2s)/2 \).) Since the derivatives of the function \( r \mapsto r^{-1} \) have alternating signs one has \((a + b)^{-1} \leq a^{-1} - a^{-2}b + a^{-3}b^2 \) and therefore

\[ \int_{\mathbb{R}^d} \frac{h(q)}{|p - q|^{d-2s}} \, dq \]

\[ \leq \int_{\mathbb{R}^d} \frac{1}{|p - q|^{d-2s}} \left( \frac{1}{|q|^{d+2s}/2} - \frac{t^{\beta-(d+2s)/2}}{|q|^{d+2s-\beta}} + \frac{t^{2\beta-d-2s}}{|q|^{3(d+2s)/2 - 2\beta s}} \right) \, dq. \]

If we assume that \((d + 6s)/4 < \beta < (3d + 2s)/4\) then the right side is finite and, using notation (2.2) with \( \Psi \) instead of \( \Psi_{s,d} \), equal to

\[ \psi \left( \frac{d + 2s}{2} \right) |p|^{2s} - \psi(d + 2s - \beta) \frac{t^{\beta-(d+2s)/2}}{|p|^{d-\beta}} \]

\[ + \psi \left( \frac{3(d + 2s)}{2} - 2\beta \right) \frac{t^{2\beta-d-2s}}{|p|^{3d/2 - 2\beta s + s}}. \]

Thus

\[ t_h(p) \leq \psi \left( \frac{d + 2s}{2} \right) |p|^{2s} - \left( \psi(d + 2s - \beta) - \psi \left( \frac{d + 2s}{2} \right) \right) \frac{t^{\beta-(d+2s)/2}}{|p|^{d-\beta}} \]

\[ + \left( \psi \left( \frac{3(d + 2s)}{2} - 2\beta \right) - \psi(d + 2s - \beta) \right) \frac{t^{2\beta-d-2s}}{|p|^{2\beta-2d}} \]

\[ + \psi \left( \frac{3(d + 2s)}{2} - 2\beta \right) \frac{t^{3\beta-3d/2 - 3s}}{|p|^{3\beta-3d/2 - s}}. \]

If we assume that \( \beta \leq (d + 2s)/2 \), then the exponents of \(|p|\) on the right side satisfy \(2s \geq \beta - (d - 2s)/2 \geq 2\beta - d \geq 3\beta - 3d/2 - s\), and if \( \beta \geq (3d + 2s)/6 \) then the last exponent is non-negative. Now we choose \( \beta = 2t + (d - 2s)/2 \), so that the exponent of the second term is \( 2t \) and the condition \( \beta \geq (3d + 2s)/6 \) is satisfied, since we are assuming that \( t \geq 2s/3 \). Moreover, according to Lemma 2.1, the coefficient of the second term is negative. Finally, we use that there are constants \( C_1 \) and \( C_2 \) such that for any \( \varepsilon > 0 \) one has

\[ |p|^{2\beta-d} \leq \varepsilon |p|^{\beta-(d-2s)/2} + C_1\varepsilon^{-\frac{2(2\beta+d)}{2d-2s-2\beta}}, \]

\[ |p|^{3\beta-3d/2-s} \leq \varepsilon |p|^{\beta-(d-2s)/2} + C_2\varepsilon^{-\frac{6\beta-3d-2s}{2d+2s-2\beta}}. \]

This concludes the proof of (2.4). \( \square \)
3. Proof of Theorem 1.1

We fix $0 < s < d/2$ and $\gamma > 0$ and write

$$\text{tr} \left( (-\Delta)^s - C_{s,d} |x|^{-2s} - V \right)^\gamma = \gamma \int_0^\infty N(-\tau, (-\Delta)^s - C_{s,d} |x|^{-2s} - V) \tau^{\gamma-1} d\tau,$$

where $N(-\tau, H)$ denotes the number of eigenvalues less than $-\tau$, counting multiplicities, of a self-adjoint operator $H$. We shall use (1.8) with $t^{-2s} = \sigma \tau$ and some $0 < t < s$ and $0 < \sigma < 1$ to be specified below. Abbreviating $K_t = K_{d,s,t}$ we find that

$$N(-\tau, (-\Delta)^s - C_{s,d} |x|^{-2s} - V) \leq N(0, K_t(\sigma \tau)^{(s-t)/s}(-\Delta)^t - V + (1 - \sigma)\tau)$$

$$= N \left( 0, (-\Delta)^t - K_t^{-1}(\sigma \tau)^{-(s-t)/s} (V - (1 - \sigma)\tau) \right).$$

Now we use (1.4) with $\gamma = 0$ and $s$ replaced by $t$ (see [Da] for $t \leq 1$ and [Cw] for $t < d/2$). Abbreviating $L_t = L_{0,d,t}$ we have

$$N(-\tau, (-\Delta)^s - C_{s,d} |x|^{-2s} - V) \leq L_t K_t^{-d/2t}(\sigma \tau)^{-d(s-t)/2st} \int_{\mathbb{R}^d} (V - (1 - \sigma)\tau)^{d/2t} dx$$

and

$$\text{tr} \left( (-\Delta)^s - C_{s,d} |x|^{-2s} - V \right)^\gamma$$

$$\leq \gamma L_t K_t^{-d/2t}(\sigma)^{-d(s-t)/2st} \int_{\mathbb{R}^d} dx \int_0^\infty d\tau \tau^{\gamma-1-d(s-t)/2st} (V - (1 - \sigma)\tau)^{d/2t} dx$$

$$= \gamma L_t K_t^{-d/2t}(\sigma)^{-d(s-t)/2st}(1 - \sigma)^{-\gamma + \frac{d(s-t)}{2st}} \frac{\Gamma(\gamma - \frac{d(s-t)}{2st})\Gamma(\frac{d}{2} + 1)}{\Gamma(\gamma + \frac{d}{2st} + 1)} \int_{\mathbb{R}^d} V^{\gamma+d/2s} dx.$$

Here we assumed that $t > ds/(2\gamma s + d)$ so that the $\tau$ integral is finite. Finally, we optimize over $0 < \sigma < 1$ by choosing $\sigma = d(s-t)/2st$ and over $ds/(2\gamma s + d) < t < s$ to complete the proof of (1.5).

The statement about the inclusion of $A$ follows from Example 4.1 and Theorem 4.2 in the following section.

4. Magnetic Lieb-Thirring Inequalities

In this section we discuss Lieb-Thirring inequalities for magnetic Schrödinger operators, that is, (1.4) (and its generalizations) with $(-\Delta)^s$ replaced by $|D - A|^{2s}$ for some vector field $A \in L_{2,\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$.

It is a remarkable fact that all presently known proofs of Lieb-Thirring inequalities, which allow for the inclusion of a magnetic field, yield the same constants in the magnetic case as in the non-magnetic case. It is unknown whether this is also true for the unknown sharp constants. Note that the diamagnetic inequality implies that the lowest eigenvalue does not decrease when a magnetic field is added, but there is no such result for, e.g., the number or the sum of eigenvalues; see [AvHeSi,Li]. Rozenblum [Ro] discovered, however, that any power-like bound on the number of eigenvalues in the non-magnetic case implies a similar bound in the magnetic case, with possibly a worse constant. Here we show the same phenomenon for moments of eigenvalues.
We work in the following abstract setting. Let \((X, \mu)\) be a sigma-finite measure space and let \(H\) and \(M\) be non-negative operators in \(L_2(X, \mu)\) such that for any \(u \in L_2(X, \mu)\) and any \(t > 0,\)

\[
| \exp(-tM)u(x)| \leq (\exp(-tH)|u|)(x) \quad \mu - \text{a.e. } x \in X. \tag{4.1}
\]

Note that this implies that \(\exp(-tH)\) is positivity preserving. We think of \(H\) as a non-magnetic operator, \(M\) a magnetic operator and (4.1) as a diamagnetic inequality. It might be useful to keep the following example in mind.

**Example 4.1.** Let \(X = \mathbb{R}^d\) with Lebesgue measure, \(H = (-\Delta)^s\), and \(M = |D - A|^{2s}\) for some \(0 < s \leq 1\) and \(A \in L_{2,\text{loc}}(\mathbb{R}^d)\). The diamagnetic inequality (4.1) in the case \(s = 1\) was shown in [Si1], and in the case \(0 < s < 1\) it follows from the \(s = 1\) result since the function \(\lambda \mapsto \exp(-\lambda^s)\) is completely monotone and hence by Bernstein’s theorem [Do] the Laplace transform of a positive measure. More generally, (4.1) holds for \(H = (-\Delta)^s + W\) and \(M = |D - A|^{2s} + W\) with \(s\) and \(A\) as before and a, say, bounded function \(W\). This can be seen using Trotter’s product formula. By an approximation argument the inequality holds also for \(W(x) = -C_{s,d}|x|^{-2s}\); see [FrLiSe1].

The main result in this section is

**Theorem 4.2.** Let \(H\) and \(M\) be as above and assume that there exist some constants \(L > 0, \gamma \geq 0, p > 0\) and a non-negative function \(w\) on \(X\) such that for all \(V \in L_p(V, w \, d\mu)\) one has

\[
\text{tr}(H - V)\gamma \leq L \int_X V_w^p \, w \, d\mu. \tag{4.2}
\]

Then one also has

\[
\text{tr}(M - V)\gamma \leq L \left( \frac{e}{p} \right)^p \Gamma(p + 1) \int_X V_w^p \, d\mu. \tag{4.3}
\]

We do not know whether the factor \((e/p)^p \Gamma(p + 1)\) in (4.3) can be omitted. Results from [FrLoWe] about the eigenvalues of the Landau Hamiltonian in a domain (but without potential) seem to indicate that a factor \(>1\) is necessary. Our proof of Theorem 4.2 uses some ideas from [Ro] where the case \(\gamma = 0\) was treated; see also [FrLiSe2] for a result about operators with discrete spectrum.

**Remark 4.3.** With the same proof one can deduce estimates on \(\text{tr } f(M)\) from estimates on \(\text{tr } f(H)\) for more general functions \(f\). For example, let \(d = 2\) and \(f(t) := |\ln |t||^{-1}\) if \(-e^{-1} < t < 0, f(t) := 1\) if \(t \leq -e^{-1}\), and \(f(t) := 0\) if \(t \geq 0\). Then there exists a constant \(L\) and for any \(q > 1\) a constant \(L_q\) such that for all \(l > 0\) and \(A \in L_{2,\text{loc}}(\mathbb{R}^2, \mathbb{R}^2),\)

\[
\text{tr } f \left( \int |(D - A)^2 - V| \right) \leq L \int_{|x| < l} |V(x)| |x| l \, dx + L_q \int_0^\infty \left( \int_S V(r \omega)^q \, d\omega \right)^{1/q} r \, dr.
\]

Indeed, this follows by Lemma 4.4 via integration from the \(A \equiv 0\) result of [KoVuWe].
The key ingredient in the proof of Theorem 4.2 is a bound on the negative eigenvalues of $M - V$ by those of $H - \alpha V$, averaged over all coupling constants $\alpha$. As before, we denote by $N(-\tau, A)$ the number of eigenvalues less than $-\tau$, counting multiplicities, of a self-adjoint operator $A$.

**Lemma 4.4.** Let $H$ and $M$ be non-negative self-adjoint operators satisfying (4.1) and let $V \geq 0$. Then for any $\tau \geq 0$ and $t > 0$ one has

$$N(-\tau, M - V) \leq t e^t \int_0^\infty N(-\tau, H - \alpha V) e^{-\alpha t} \, d\alpha.$$  \hfill (4.4)

**Proof.** Since (4.1) remains valid with $H + \tau$ and $M + \tau$ in place of $H$ and $M$ we only sketch the main idea. Indeed, for any $\sigma > 0$,

$$(m + \sigma)^{-1} = V^{1/2}(M + \sigma V)^{-1}V^{1/2} = \int_0^\infty V^{1/2} \exp(-s(M + \sigma V))V^{1/2} \, ds,$$

and by (4.1) and Trotter’s product formula $|\exp(-s(M + \sigma V))V^{1/2}u| \leq \exp(-s(H + \sigma V))V^{1/2}|u|$ a.e. Hence $|m + \sigma|^{-1} \leq (h + \sigma)^{-1}$ a.e. Iterating this inequality and recalling that $(1 + tm/n)^{-n} \to \exp(-tm)$ strongly as $n \to \infty$, we obtain (4.1) for $h$ and $m$.

By [Si2, Thm. 2.13] this analog of (4.1) implies that

$$\text{tr} \exp(-tm) = \|\exp(-tm/2)\|_2^2 \leq \|\exp(-th/2)\|_2^2 = \text{tr} \exp(-th)$$

with $\| \cdot \|_2$ the Hilbert-Schmidt norm, and hence by the Birman-Schwinger principle,

$$N(M - V) = N(1, m) \leq e^t \text{tr} \exp(-tm) \leq e^t \text{tr} \exp(-th).$$

Using the Birman-Schwinger principle once more, we find

$$\text{tr} \exp(-th) = t \int_0^\infty N(\alpha, h) e^{-t\alpha} \, d\alpha = t \int_0^\infty N(H - \alpha V) e^{-t\alpha} \, d\alpha,$$

proving (4.4). \hfill $\square$

**Proof of Theorem 4.2.** By the variational principle we may assume that $V \geq 0$. By Lemma 4.4 one has for any $t > 0$,

$$\text{tr}(M - V)_\gamma = \gamma \int_0^\infty N(-\tau, M - V) \tau^{\gamma-1} \, d\tau$$

$$\leq \gamma t e^t \int_0^\infty \int_0^\infty N(-\tau, H - \alpha V) \tau^{\gamma-1} \, d\tau \exp(-\alpha t) \, d\alpha$$

$$= t e^t \int_0^\infty \text{tr}(H - \alpha V)^\gamma \exp(-\alpha t) \, d\alpha,$$

and by assumption (4.2) the right-hand side can be bounded from above by

$$Lt e^t \left( \int_0^\infty \alpha^p \exp(-\alpha t) \, d\alpha \right) \int_X V^p w \, d\mu = Lt^{-p} e^t \Gamma(p + 1) \int_X V^p w \, d\mu.$$

Now the assertion follows by choosing $t = p$. \hfill $\square$
5. A Pseudo-Relativistic Model Including Spin

Throughout this section we assume that \( d = 3 \). The helicity operator \( h \) on \( L_2(\mathbb{R}^3, \mathbb{C}^2) \) is defined as the Fourier multiplier corresponding to the matrix-valued function \( p \mapsto \sigma \cdot p/|p| \), where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) denotes the triple of Pauli matrices. The properties of these matrices imply that \( h \) is a unitary and self-adjoint involution. The analog of the Hardy (or Kato) inequality (1.2) is

\[
\int_{\mathbb{R}^3} |\xi| |\hat{u}(\xi)|^2 d\xi \geq \tilde{C} \int_{\mathbb{R}^3} \frac{|u(x)|^2 + |(hu)(x)|^2}{2 |x|} dx, \quad u \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2),
\]

with the sharp constant

\[
\tilde{C} = \frac{2}{2/\pi + \pi/2};
\]

see [EvPeSi]. Note that this constant is strictly larger than

\[
C := \mathcal{C}_{1/2,3} = 2/\pi,
\]

which is the constant one would get if \( hu \) were replaced by \( u \) on the right side of (5.1).

For a function \( V \) on \( \mathbb{R}^3 \) taking values in the Hermitian \( 4 \times 4 \) matrices we introduce the non-local potential

\[
\Phi(V) := \frac{1}{2} \left( \frac{1}{L_2(\mathbb{R}^3, \mathbb{C}^2)} h \right)^* V \left( \frac{1}{L_2(\mathbb{R}^3, \mathbb{C}^2)} h \right),
\]

where \( \left( \frac{1}{L_2(\mathbb{R}^3, \mathbb{C}^2)} h \right) \) is considered as an operator from \( L_2(\mathbb{R}^3, \mathbb{C}^2) \) to \( L_2(\mathbb{R}^3, \mathbb{C}^4) \). The operator \( \sqrt{-\Delta} - \Phi(V) \) in \( L_2(\mathbb{R}^3, \mathbb{C}^2) \) has been suggested by Brown and Ravenhall as the Hamiltonian of a massless, relativistic spin-1/2 particle in a potential \(-V\). It results from projecting onto the positive spectral subspace of the Dirac operator. One of the advantages of this operator over the simpler \( \sqrt{-\Delta} - V \) is that it is well-defined for nuclear charges \( aZ \leq \tilde{C} \), which includes all known elements. We refer to [LiSe] for more background about this model. Despite the efforts in [LiSiSo,BaEv,HoSi] the problem of stability of matter for the corresponding many-particle system is not yet completely understood and the following result, we believe, might be useful in this respect.

**Theorem 5.1.** Let \( d = 3 \) and \( \gamma > 0 \). Then there is a constant \( \tilde{L}_\gamma^{\text{HLT}} \) such that

\[
\text{tr} \left( \sqrt{-\Delta} - \tilde{C} \Phi(|x|^{-1}) - \Phi(V) \right)^\gamma \leq \tilde{L}_\gamma^{\text{HLT}} \int_{\mathbb{R}^3} \text{tr} C_1 V(x)^{\gamma+3} dx.
\]

For the proof of this theorem we need some facts about the partial wave decomposition of the operator \( \sqrt{-\Delta} - \tilde{C} \Phi(|x|^{-1}) \) from [EvPeSi]. This operator commutes with the total angular momentum operator \( \mathbf{J} = \mathbf{L} + \frac{1}{2} \mathbf{S} \), where \( \mathbf{L} = -i \nabla \times x \), as well as with the operator \( \mathbf{L}^2 \). The subspace corresponding to total angular momentum \( j = 1/2 \) is of the form \( \mathcal{J}_{1/2,0} \oplus \mathcal{J}_{1/2,1} \), where the subspaces \( \mathcal{J}_{1/2,j} \) correspond to the eigenvalues \( l(l+1) \) of \( \mathbf{L}^2 \).

The next result, essentially contained in [FrSiWa], says that on the space \( \mathcal{J}_{1/2,0} \oplus \mathcal{J}_{1/2,1} \) the operator \( \sqrt{-\Delta} - \tilde{C} \Phi(|x|^{-1}) \) is controlled by the operator \( \sqrt{-\Delta} - C|x|^{-1} \) with the smaller coupling constant \( C \). (Strictly speaking, the latter operator should be tensored with \( 1_{\mathbb{C}^2} \), but we suppress this if there is no danger of confusion.)
Lemma 5.2. If $0 \neq \psi \in \mathcal{S}_{1/2,0} \cap C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$, then

$$\frac{2}{1 + (2/\pi)^2} \geq \frac{(\psi, (\sqrt{-\Delta - \tilde{\Phi}(|x|^{-1})) \psi)}{(\psi, (\sqrt{-\Delta - C|x|^{-1}) \psi)} \geq \frac{1}{1 + (2/\pi)^2}.$$  

If $0 \neq \psi \in \mathcal{S}_{1/2,1} \cap C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$, this bound is true provided $(\psi, (\sqrt{-\Delta - C|x|^{-1}) \psi)$ is replaced by $(h\psi, (\sqrt{-\Delta - C|x|^{-1}) h\psi).$

Proof of Lemma 5.2. We prove the assertion only for $l = 1$ since the lower bound for $l = 0$ is contained in [FrSiWa, Lemma 2.7] and the upper bound is proved as below. By orthogonality we may assume that the Fourier transform of $\psi$ is of the form $\hat{\psi}(\xi) = |\xi|^{-2}g(|\xi|)\Omega_{1/2,1,m}(\xi/|\xi|), \text{ where } m \in \{-1/2, 1/2\} \text{ and } \Omega_{1/2,1,m}$ are explicit functions in $L_2(S^2, \mathbb{C}^2).$ By the properties of these functions one has $h\hat{\psi}(\xi) = -|\xi|^{-2}g(|\xi|)\Omega_{1/2,0,m}(\xi/|\xi|).$ The ground state representation from [FrSiWa, Lemma 2.6] reads

$$\left(\psi, (\sqrt{-\Delta - \tilde{\Phi}(|x|^{-1})) \psi) = \frac{\tilde{c}}{2\pi} \int_0^\infty \int_0^\infty |g(p) - g(q)|^2 \tilde{k}(\frac{1}{2} \frac{p + q}{p}) \frac{dp}{p} \frac{dq}{q},$$

$$\left(h\psi, (\sqrt{-\Delta - C|x|^{-1}) h\psi) = \frac{c}{2\pi} \int_0^\infty \int_0^\infty |g(p) - g(q)|^2 k(\frac{1}{2} \frac{p + q}{p}) \frac{dp}{p} \frac{dq}{q},$$

where $\tilde{k}(t) = \frac{1}{2} (Q_0(t) + Q_1(t)), k(t) = Q_0(t)$, and $Q_i$ are the Legendre functions of the second kind [AbSt, 8.4]. The assertion now follows from the fact that $Q_0 \geq Q_1 \geq 0.$

Proof of Theorem 5.1. We first claim that for any $0 < t < 1/2$ there is a $\tilde{K}_i > 0$ such that

$$\sqrt{-\Delta - \tilde{\Phi}(|x|^{-1})} \geq \tilde{K}_i l^{-1+2t}(-\Delta)^l - l^{-1}, \quad l > 0. \tag{5.3}$$

Indeed, it follows from Lemma 5.2 and (1.8) that on $\mathcal{S}_{1/2,0} \oplus \mathcal{S}_{1/2,1}$ one has for any $0 < t < 1/2$,

$$\sqrt{-\Delta - \tilde{\Phi}(|x|^{-1})} \geq \left(1 + (2/\pi)^2\right)^{-1} \left(K_i l^{-1+2t}(-\Delta)^l - l^{-1}\right), \quad l > 0.$$

On the other hand, the arguments of [EvPeSi] show that there exists a constant $\tilde{C}' > \tilde{C}$ such that $\sqrt{-\Delta} \geq \tilde{C}' \Phi(|x|^{-1})$ on $(\mathcal{S}_{1/2,0} \oplus \mathcal{S}_{1/2,1})^\perp.$ Hence on that space

$$\sqrt{-\Delta - \tilde{\Phi}(|x|^{-1})} \geq \frac{\tilde{C}' - \tilde{C}}{\tilde{C}'} \sqrt{-\Delta} \geq \frac{\tilde{C}' - \tilde{C}}{\tilde{C}'} \left(\frac{1}{2t} l^{-1+2t}(-\Delta)^l - 1 - \frac{2t}{2t} l^{-1}\right), \quad l > 0.$$  

This proves (5.3).

Given (5.3), the proof of (5.2) is similar to that of (1.5). We may assume that $V(x) = v(x)I_{C^4}$ for a non-negative, scalar function $v$ (otherwise, replace $V(x)$ by $v(x)I_{C^4}$, where $v(x)$ is the operator norm of the $4 \times 4$ matrix $V(x)_+$. For a given $l > 0$ and
0 < t < 1/2 we introduce the operator \( H := \tilde{K}_t t^{1+2t} ( -\Delta )^t - v - t^{-1} \) in \( L_2(\mathbb{R}^3, \mathbb{C}) \). Then according to (5.3) one has

\[
N(-\tau, \sqrt{-\Delta} - \tilde{C} |x|^{-1} - \Phi (V)) \leq N(-\tau, \frac{1}{2} (H \otimes 1_{\mathbb{C}^2} + h (H \otimes 1_{\mathbb{C}^2}) h)) \leq 4N(-\tau, H).
\]

In the last inequality we used that \( N(-\tau, \frac{1}{2} (A + B)) \leq N(-\tau, A) + N(-\tau, B) \) for any self-adjoint, lower semi-bounded operators \( A \) and \( B \), which follows from the variational principle. Now one can proceed in the same way as in the proof of (1.5). \( \square \)

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