Linear commuting maps and skew-symmetric biderivations of the deformative Schrödinger-Virasoro Lie algebras

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Abstract: In this paper, we investigate the skew-symmetric biderivations of the deformative Schrödinger-Virasoro Lie algebras which contain the twisted and original deformative Schrödinger-Virasoro Lie algebras. As an application, we give the explicit form of each linear commuting map on the deformative Schrödinger-Virasoro Lie algebras. In particular, we obtain that there exist non-inner biderivations and non-standard linear commuting maps for the certain deformative Schrödinger-Virasoro Lie algebras.

Key words: biderivations, commuting maps, Schrödinger-Virasoro Lie algebra, deformative Schrödinger-Virasoro Lie algebras

Mathematics Subject Classification (2010): 17B05, 17B40, 17B65, 17B68.

1 Introduction

Throughout this paper, we denote by \( \mathbb{Z}, \mathbb{C} \) the sets of integers, complex numbers respectively. We assume that all vector spaces are based on \( \mathbb{C} \), unless otherwise stated.

It is well known that the infinite-dimensional Schrödinger Lie algebras and the Virasoro algebra play important roles in many areas of mathematics and physics. In order to investigate the free Schrödinger equations, the twisted and original Schrödinger-Virasoro Lie algebras were introduced by [10] in the context of non-equilibrium statistical physics. In [19], the author introduced the deformations of the Schrödinger-Virasoro Lie algebras. In this paper, we consider the following Lie algebras, which are referred to as deformative Schrödinger-Virasoro Lie algebras \( \mathcal{L}(\lambda, \mu, s) \). Fix the complex numbers \( \lambda, \mu \) and \( s = 0 \) or \( \frac{1}{2} \), the Lie algebra \( \mathcal{L}(\lambda, \mu, s) \) has \( \mathbb{C} \)-basis \( \{L_n, M_n, Y_{n+s} \mid n \in \mathbb{Z}\} \) with the following nontrivial Lie brackets:

\[
[L_n, L_m] = (m-n)L_{n+m},
\]

\[
[L_n, M_m] = (m-n+2\mu)M_{n+m},
\]

\[
[L_n, Y_{m+s}] = (m+s - \frac{\lambda + 1}{2} n + \mu)Y_{n+m+s},
\]

\[
[Y_{n+s}, Y_{m+s}] = (m-n)M_{n+m+2s},
\]

where \( n, m \in \mathbb{Z} \). Obviously, \( \mathcal{L}(\lambda, \mu, s) \) is a generalization of the Schrödinger-Virasoro Lie algebras. Note that \( \mathcal{L}(\lambda, \mu, s) \) contains the Schrödinger-Virasoro Lie algebras and their deformations. For instance,

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Supported by the National Natural Science Foundation of China (No. 11431010, 11371278).

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• \( \mathcal{L}(0,0,s) \) is the well-known Schrödinger-Virasoro Lie algebra introduced in [10]. Its structures and representations have been widely studied in [8,15,21].

• \( \mathcal{L}(\lambda,\mu,0) \) is called the twisted deformative Schrödinger-Virasoro Lie algebras whose structure theories studied in [17,22].

• \( \mathcal{L}(\lambda,\mu,\frac{1}{2}) \) is called the original deformative Schrödinger-Virasoro Lie algebras. In [13,16], the authors obtained some results about structures.

• \( \mathcal{L}(\lambda,0,s) \) is also a class of the deformative Schrödinger-Virasoro Lie algebras. In [5], the author investigated the Lie bialgebra structures.

As is well known, derivations and generalized derivations are very important subjects in the research of both algebras and their generalizations. In recent years, biderivations have aroused many scholars’ great interests in [1–4,6,7,9,20,21,23,24]. In [2], Bréchar showed that all biderivations on commutative prime rings are inner biderivations and they also determined the biderivations of semiprime rings. The notation of biderivation of Lie algebras was introduced in [20]. In addition, in [3,21] the authors obtained that the skew-symmetric biderivations of the Schrödinger-Virasoro algebra and a simple generalized Witt algebra are inner biderivations. Furthermore, in [9] the authors determined all the skew-symmetric biderivations of \( W(a,b) \) and found that there exist non-inner biderivations. In 1957, the first important result on linear (or additive) commuting maps was introduced by Posners in [18]. The author of [2] described that commuting maps on an associative algebra have significant application to other important problems (e.g., Lie derivations, biderivations, etc). Moreover, linear commuting maps of some Lie algebras were investigated in [3,4,9,21]. Recently, the theory of biderivations and linear commuting maps have become one of research focuses in the Lie theory. Motivated by this reason, we attempt to investigate the linear commuting maps and skew-symmetric biderivations of some important Lie algebras. We know that the linear commuting maps and skew-symmetric biderivations of \( \mathcal{L}(0,0,0) \) were studied in [21]. Thus, the results of this paper are more generic.

This paper is organized as follows. In Section 2, we review some definitions and conclusions about biderivations of Lie algebras. In Section 3, we compute the skew-symmetric biderivations of \( \mathcal{L}(\lambda,\mu,s) \). In particular, we obtain that there exist non-inner biderivations for the certain \( \mathcal{L}(\lambda,\mu,s) \). In Section 4, we give the explicit form of each linear commuting map on \( \mathcal{L}(\lambda,\mu,s) \).

## 2 Preliminaries and main results

Firstly, we shall recall some definitions and conclusions about biderivations of Lie algebras in [3,21]. In this section, we assume that \( L \) is a Lie algebra over \( \mathbb{C} \).

**Definition 2.1.** A bilinear map \( \phi: L \times L \to L \) is called skew-symmetric if \( \phi(x,y) = -\phi(y,x) \) for all \( x,y \in L \).
Definition 2.2. A bilinear map \( \phi : L \times L \rightarrow L \) is called a \textit{biderivation} if it satisfies the following two axioms:

\[
\phi([x,y],z) = [x,\phi(y,z)] + [\phi(x,z),y] \\
\phi(x,[y,z]) = [\phi(x,y),z] + [y,\phi(x,z)]
\]

for any \( x, y, z \in L \).

Example 2.3. Let \( \lambda \in \mathbb{C} \), then the map \( \phi_\lambda : L \times L \rightarrow L \), sending \( (x,y) \) to \( \lambda [x,y] \), is a biderivation of \( L \). All biderivations of this kind are called \textit{inner biderivations} of \( L \).

Remark 2.4. It is straight to check that every inner biderivation \( \phi_\lambda \) is a skew-symmetric biderivation.

By [3,21], the following two results are straightforward to verify.

Lemma 2.5. Let \( \phi \) be a skew-symmetric biderivation on \( L \), then

\[
[\phi(x,y),[u,v]] = [[x,y],\phi(u,v)]
\]

for any \( x,y,u,v \in L \). In particular, \( [\phi(x,y),[x,y]] = 0 \).

Lemma 2.6. Let \( \phi \) be a skew-symmetric biderivation on \( L \). If \( [x,y] = 0 \), then \( \phi(x,y) \in C_L([L,L]) \), where \( C_L([L,L]) \) is the centralizer of \( [L,L] \).

3 Skew-symmetric biderivations of \( \mathcal{L} \)

In this section, we would like to compute the skew-symmetric biderivations of \( \mathcal{L}(\lambda,\mu,s) \), which is denoted by \( \mathcal{L} \) for simplicity.

Lemma 3.1. (1) The center \( Z(\mathcal{L}) \) of \( \mathcal{L} \) is given by

\[
Z(\mathcal{L}) = \begin{cases} 
\mathbb{C}M_{-2\mu} & \text{if } \lambda = 0, \mu \in \frac{1}{2}\mathbb{Z} ; \\
0 & \text{otherwise}.
\end{cases}
\]

(2) For \( x \in \mathcal{L} \), denote \( \bar{x} = x + [\mathcal{L},\mathcal{L}] \). Then

\[
\mathcal{L}/[\mathcal{L},\mathcal{L}] = \begin{cases} 
\mathbb{C}M_{-\mu} & \text{if } \lambda = -3, \mu \in s+\mathbb{Z} ; \\
0 & \text{otherwise}.
\end{cases}
\]

(3) The centralizer of \([\mathcal{L},\mathcal{L}]\) coincides with \( Z(\mathcal{L}) \), i.e., \( C_{\mathcal{L}}([\mathcal{L},\mathcal{L}]) = Z(\mathcal{L}) \).

Proof. It can be easily obtained by the Lie brackets of \( \mathcal{L} \). \( \square \)
For convenience, we first introduce two kinds of skew-symmetric biderivations.

(1) For $\mathcal{L}$ with $\lambda = 1, \mu \in s + \frac{1}{2}\mathbb{Z}$, define the following skew-symmetric bilinear map:

$$
\phi_0 : \mathcal{L} \times \mathcal{L} \to \mathcal{L}, \quad (L_n, L_m) \mapsto (m - n)M_{n + m - 2\mu}, \quad (3.1)
$$

the others map to 0.

(2) For $\mathcal{L}$ with $\lambda = 1, \mu \in s + \mathbb{Z}$, define the following skew-symmetric bilinear map:

$$
\phi_1 : \mathcal{L} \times \mathcal{L} \to \mathcal{L}
\begin{align*}
(L_n, L_m) &\mapsto (m - n)Y_{n + m - \mu}, \\
(L_n, Y_{m+s}) &\mapsto (m + s - n + \mu)M_{n + m + s - \mu}, \\
(Y_{m+s}, L_n) &\mapsto (n - m - s - \mu)M_{n + m + s - \mu},
\end{align*}
$$

the others map to 0.

Obviously, we see that $\phi_0$ and $\phi_1$ are skew-symmetric non-inner biderivations of $\mathcal{L}$.

**Theorem 3.2.** Let $\phi$ be a skew-symmetric biderivation of $\mathcal{L}$, then we have

$$
\phi(x, y) = \begin{cases} 
\alpha[x, y] + \beta\phi_0(x, y) & \text{if } \lambda = 1, \mu \in s + \frac{1}{2} + \mathbb{Z}; \\
\alpha[x, y] + \beta\phi_0(x, y) + \gamma\phi_1(x, y) & \text{if } \lambda = 1, \mu \in s + \mathbb{Z}; \\
\alpha[x, y] & \text{otherwise},
\end{cases}
$$

for all $x, y \in \mathcal{L}$, where $\alpha, \beta, \gamma \in \mathbb{C}$, $\phi_0$ and $\phi_1$ are given by (3.1) and (3.2).

**Proof.** We shall complete the proof by verifying the following ten claims.

**Claim 1.** There exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$
\phi(L_n, L_m) \equiv \begin{cases} 
\alpha[L_n, L_m] + c_{-\mu}Y_{-\mu} \mod Z(\mathcal{L}) & \text{if } \lambda = -1, \mu \in s + \mathbb{Z}; \\
\alpha[L_n, L_m] + \beta(m - n)M_{n + m - 2\mu} \mod Z(\mathcal{L}) & \text{if } \lambda = 1, \mu \in s + \frac{1}{2} + \mathbb{Z}; \\
\alpha[L_n, L_m] + \beta(m - n)M_{n + m - 2\mu} + \gamma(m - n)Y_{n + m - \mu} \mod Z(\mathcal{L}) & \text{if } \lambda = 1, \mu \in s + \mathbb{Z}; \\
\alpha[L_n, L_m] \mod Z(\mathcal{L}) & \text{otherwise},
\end{cases}
$$

for all $n, m \in \mathbb{Z}$, where $c_{-\mu} \in \mathbb{C}$.

If $n = m$, then the fact $[L_n, L_m] = 0$ gives that $\phi(L_n, L_m) \in Z(\mathcal{L})$ by Lemmas 2.6, 3.1(3). For $n \neq m$, suppose that $\phi(L_n, L_m) = \sum_{i \in \mathbb{Z}}(a_iL_i + b_iM_i + c_{i+s}Y_{i+s})$ for some $a_i, b_i, c_{i+s} \in \mathbb{C}$. According to Lemma 2.5 one has

$$
0 = \frac{1}{m-n}[[L_n, L_m], \phi(L_n, L_m)] = [L_{n+m}, \sum_{i \in \mathbb{Z}}(a_iL_i + b_iM_i + c_{i+s}Y_{i+s})] = \sum_{i \in \mathbb{Z}}(a_i(i - n - m)L_{n+m+i} + b_i(i - \lambda(n + m) + 2\mu)M_{n+m+i} + c_{i+s}(i + s - \frac{\lambda+1}{2}(n + m) + \mu)Y_{n+m+i+s}),
$$
which follows that
\[ a_i = 0 \quad \text{if } i \neq n + m, \]
\[ b_i = 0 \quad \text{if } i \neq \lambda(n + m) - 2\mu, \]
\[ c_{i+s} = 0 \quad \text{if } i + s \neq \frac{k+1}{k}(n + m) - \mu. \]  
(3.3)

Furthermore, by Lemma 2.5 and the equation (3.3), for any \( k \neq 0 \), then we get

\[
0 = \left[ \phi(L_n, L_m), [L_k, L_0] \right] - \left[ [L_n, L_m], \phi(L_k, L_0) \right]
= \left[ a_{n+m}L_{n+m} + b_{\lambda(n+m)-2\mu}M_{\lambda(n+m)-2\mu} + c_{\frac{k+1}{k}(n+m)-\mu}M_{\frac{k+1}{k}(n+m)-\mu}, -kL_k \right]
- [(m-n)L_{n+m}, a_kL_k + b_{k\lambda-2\mu}M_{k\lambda-2\mu} + c_{\frac{k+1}{k}+k-\mu}M_{\frac{k+1}{k}+k-\mu}, kL_k].
\]

Thus,
\[
k(n+m-k)a_{n+m}L_{n+m+k} = (m-n)(k-n-m)a_kL_{n+m+k}, \quad \text{(3.4)}
\]
\[
k\lambda(n+m-k)b_{\lambda(n+m)-2\mu}M_{\lambda(n+m)-2\mu+k} = \lambda(m-n)(k-n-m)b_{k\lambda-2\mu}M_{n+m+k\lambda-2\mu}, \quad \text{(3.5)}
\]
\[
\frac{k+1}{2}k(n+m-k)c_{\frac{k+1}{k}(n+m)-\mu}M_{\frac{k+1}{k}(n+m)-\mu+k}
= \frac{k+1}{2}(m-n)(k-n-m)c_{\frac{k+1}{k}+k-\mu}M_{n+m+k\lambda-2\mu \mu}. \quad \text{(3.6)}
\]

Due to the arbitrariness of \( k \), we can conclude \( a_{n+m} = \alpha(m-n) \) from (3.4) for some \( \alpha \in \mathbb{C} \).

In the following we shall consider the coefficients of \( M \) and \( Y \) in (3.5) and (3.6). It can be divided into four cases by choosing different \( \lambda \).

**Case 1.** \( \lambda = 0 \). One can deduce that \( c_{\frac{k+1}{k}(n+m)-\mu} = 0 \) by the arbitrariness of \( k \). If \( \mu \in \frac{1}{2}\mathbb{Z} \), then we get \( \phi(L_n, L_m) = \alpha[L_n, L_m] + b_{-2\mu}M_{-2\mu} \); otherwise, \( \phi(L_n, L_m) = \alpha[L_n, L_m] \).

**Case 2.** \( \lambda = -1 \). It is easy to get that \( b_{\lambda(n+m)-2\mu} = 0 \). If \( \mu \notin s + \mathbb{Z} \), then \( c_{-\mu} = 0 \). Furthermore, we obtain that \( \phi(L_n, L_m) = \alpha[L_n, L_m] + c_{-\mu}Y_{-\mu} \) if \( \mu \in s + \mathbb{Z} \).

**Case 3.** \( \lambda = 1 \). If \( \mu \in s + \mathbb{Z} \), then we have \( \phi(L_n, L_m) = \alpha[L_n, L_m] + b_{n+m-2\mu}M_{n+m-2\mu} + c_{n+m-\mu}Y_{n+m-\mu} \). Moreover, if \( \mu \in s + \frac{1}{2} + \mathbb{Z} \), one has \( \phi(L_n, L_m) = \alpha[L_n, L_m] + b_{n+m-2\mu}M_{n+m-2\mu} \). Comparing the coefficients in (3.5) and (3.6), we conclude \( b_{n+m-2\mu} = \frac{b_{-2\mu}}{k}(n-m) \) and \( c_{n+m-\mu} = \frac{c_{-\mu}}{k}(n-m) \) by the arbitrariness of \( k \). Hence, we could assume \( b_{n+m-2\mu} = \beta(m-n) \) and \( c_{n+m-\mu} = \gamma(m-n) \) for some \( \beta, \gamma \in \mathbb{C} \). We have obtained the desired result.

**Case 4.** \( \lambda \neq 0, \pm 1 \). We also obtain that \( \phi(L_n, L_m) = \alpha[L_n, L_m] \).

**Claim 2.**

\[
\phi(L_n, M_m) \equiv \left\{ \begin{array}{ll}
\alpha[L_n, M_m] + f_{-\mu}Y_{-\mu} \pmod{Z(L)} & \text{if } \lambda = -1, \quad \mu \in s + \mathbb{Z}; \\
\alpha[L_n, M_m] \pmod{Z(L)} & \text{otherwise},
\end{array} \right.
\]

for all \( n, m \in \mathbb{Z} \), where \( f_{-\mu} \in \mathbb{C} \).
For any fixed $n, m \in \mathbb{Z}$, suppose $\phi(L_n, M_m) = \sum_{i \in \mathbb{Z}} (d_i L_i + e_i M_i + f_i + Y_{i+s})$ for some $d_i, e_i, f_{i+s} \in \mathbb{C}$. By Lemma 2.5 and Claim 1, for any $k \neq 0$, we have

$$0 = [(\phi(L_n, M_m), [L_0, L_k]) - ([L_n, M_m], \phi(L_0, L_k)]]$$

$$= \left( \sum_{i \in \mathbb{Z}} (d_i L_i + e_i M_i + f_{i+s} Y_{i+s}), kL_k \right) - [(m - \lambda n + 2\mu)M_{n+m}, k\alpha L_k]$$

$$= \sum_{i \in \mathbb{Z}} \left( k(k-i)d_{i+k} - k(i-k\lambda + 2\mu)e_i M_{i+k} - k(i+s - \frac{\lambda+1}{2}k + \mu)f_{i+s} Y_{i+s+k} \right)$$

$$+ k\alpha (m - \lambda n + 2\mu)(n + m - k\lambda + 2\mu)M_{n+m+k},$$

which implies that

$$d_i = 0 \text{ if } i \neq k, \quad (3.7)$$

$$(i - k\lambda + 2\mu)e_i = 0 \text{ if } i \neq n + m, \quad (3.8)$$

$$(n + m - k\lambda + 2\mu)(e_i - \alpha(m - \lambda n + 2\mu)) = 0 \text{ if } i = n + m, \quad (3.9)$$

$$f_{i+s} = 0 \text{ if } i + s \neq \frac{\lambda+1}{2}k - \mu. \quad (3.10)$$

Firstly, one can easily get that $d_i = 0$ by the arbitrariness of $k$ in (3.7). If $\lambda \neq 0$, one shows that $e_i = 0$ if $i \neq n + m$ and $e_{n+m} = \alpha(m - \lambda n + 2\mu)$ by (3.8) and (3.9). If $\lambda \neq -1$, we obtain that $f_{i+s} = 0$ from (3.10). Thus,

**Case 1.** $\lambda = -1$. We have $\phi(L_n, M_m) = \alpha[L_n, M_m] + f_{-\mu} Y_{-\mu}$. If $\mu \notin s + \mathbb{Z}$, then $f_{-\mu} = 0$.

**Case 2.** $\lambda = 0$. According to (3.8) and (3.9), we have $\phi(L_n, M_m) = e_{-2\mu} M_{-2\mu}$ if $n + m = -2\mu$. Furthermore, $\phi(L_n, M_m) = \alpha[L_n, M_m] + e_{-2\mu} M_{-2\mu}$ if $n + m \neq -2\mu$. Thanks to Lemma 3.1(1), this claim holds.

**Case 3.** $\lambda \neq 0, -1$. One shows that $\phi(L_n, M_m) = \alpha(m - \lambda n + 2\mu)M_{n+m} = \alpha[L_n, M_m]$. This completes the claim.

**Claim 3.**

$$\phi(L_n, Y_{m+s}) = \left\{ \begin{array}{ll}
    l_{-\mu} Y_{-\mu} (\text{mod } Z(\mathcal{L})) & \text{when } n + m + s = -\mu \\
    \alpha[L_n, Y_{m+s}] + l_{-\mu} Y_{-\mu} (\text{mod } Z(\mathcal{L})) & \text{when } n + m + s \neq -\mu \\
    \alpha[L_n, Y_{m+s}] + (m + s - n + \mu)M_{n+m+s-\mu} (\text{mod } Z(\mathcal{L})) & \text{if } \lambda = 1, \mu \in s + \mathbb{Z}; \\
    \alpha[L_n, Y_{m+s}] (\text{mod } Z(\mathcal{L})) & \text{otherwise,}
\end{array} \right.$$  

for $n, m \in \mathbb{Z}$, where $l_{-\mu} \in \mathbb{C}$.

For any fixed $n, m \in \mathbb{Z}$, suppose $\phi(L_n, Y_{m+s}) = \sum_{i \in \mathbb{Z}} (g_i L_i + h_i M_i + l_{i+s} Y_{i+s})$ for some $g_i, h_i, l_{i+s} \in \mathbb{C}$. 

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Case 1. \( \lambda = -1, \mu \in s + \mathbb{Z} \). According to Lemma 2.5 and Claim [1] for any \( k \neq 0 \), then we get
\[
0 = \phi (L_n, Y_{m+s}) - \left[ \phi (L_0, L_{k}) \right] = \left[ \sum_{i \in \mathbb{Z}} (g_i L_i + h_i M_l + i_{s+i} Y_{i+s}), k L_k \right] - \left[ (m + s + \mu) Y_{n+m+s}, k \alpha L_k + c_{-\mu} Y_{-\mu} \right]
\]
\[
= \sum_{i \in \mathbb{Z}} \left( k(k-i)g_i L_{i+k} - k(i+k+2\mu)h_i M_{i+k} - k(i + s + \mu)l_{i+s} Y_{i+s+k} \right) + c_{-\mu} (m + s + \mu)(n + m + s + \mu) M_{n+m+s-\mu} + k \alpha (m + s + \mu)(n + m + s + \mu) Y_{n+m+s+k}.
\]
Firstly, by the arbitrariness of \( k, n \) and \( m \), one can easily get from the above formula that
\[
c_{-\mu} = g_i = h_i = 0, \quad (3.11)
\]
\[
(i + s + \mu) l_{i+s} = 0 \quad \text{if} \quad i \neq n + m, \quad (3.12)
\]
\[
(n + m + s + \mu)(l_{i+s} - \alpha(m + s + \mu)) = 0 \quad \text{if} \quad i = n + m. \quad (3.13)
\]
By (3.12) and (3.13), we conclude that \( \phi (L_n, Y_{m+s}) = l_{-\mu} Y_{-\mu} \) if \( n + m + s = \mu \), and \( \phi (L_n, Y_{m+s}) = \alpha L_n, Y_{m+s} + l_{-\mu} Y_{-\mu} \) if \( n + m + s \neq -\mu \).

Case 2. \( \lambda = 1, \mu \in s + \mathbb{Z} \). Using the similar method of Case 1, we conclude the following equalities:
\[
g_i = 0 \quad \text{if} \quad i \neq k, \quad (3.14)
\]
\[
(i - k + 2\mu) h_i = 0 \quad \text{if} \quad i \neq n + m - s - \mu, \quad (3.15)
\]
\[
(n + m + s - k + \mu)(h_i - \gamma(m - n + s + \mu)) = 0 \quad \text{if} \quad i = n + m - s - \mu, \quad (3.16)
\]
\[
(i + s - k + \mu) l_{i+s} = 0 \quad \text{if} \quad i \neq n + m, \quad (3.17)
\]
\[
(n + m + s - k + \mu)(l_{i+s} - \alpha(m - n + s + \mu)) = 0 \quad \text{if} \quad i = n + m. \quad (3.18)
\]
Obviously, it can be easily obtained that \( g_i = 0 \) by (3.14). According to (3.15)-(3.18), we have \( \phi (L_n, Y_{m+s}) = \alpha L_n, Y_{m+s} + \gamma(m - n + s + \mu) M_{n+m+s-\mu} \).

Case 3. Otherwise. By Lemma 2.5 and Claim [1] for any \( k \neq 0 \), we have
\[
0 = \phi (L_n, Y_{m+s}) - \left[ \phi (L_0, L_{k}) \right] = \left[ \sum_{i \in \mathbb{Z}} (g_i L_i + h_i M_l + i_{s+i} Y_{i+s}), k L_k \right] - \left[ (m - \lambda + \frac{1}{2} n + s + \mu) Y_{n+m+s}, k \alpha L_k \right]
\]
\[
= \sum_{i \in \mathbb{Z}} \left( k(k-i)g_i L_{i+k} - k(i-k+2\mu)h_i M_{i+k} - k(i + s - \lambda + \frac{1}{2} k + \mu)l_{i+s} Y_{i+s+k} \right) + k \alpha (m - \lambda + \frac{1}{2} n + s + \mu)(n + m + s - \lambda + \frac{1}{2} k + \mu) Y_{n+m+s+k},
\]
which implies that
\[
g_i = 0 \quad \text{if} \quad i \neq k, \quad (3.19)
\]
\[
(i - k\lambda + 2\mu) h_i = 0 \quad \text{if} \quad i \neq k\lambda - 2\mu, \quad (3.20)
\]
\[
(i + s - \lambda + \frac{1}{2} k + \mu) l_{i+s} = 0 \quad \text{if} \quad i \neq n + m, \quad (3.21)
\]
\[
(n + m + s - \lambda + \frac{1}{2} k + \mu)(l_{i+s} - \alpha(m - \lambda + \frac{1}{2} n + s + \mu)) = 0 \quad \text{if} \quad i = n + m. \quad (3.22)
\]
We get that $g_i = 0$ by the arbitrariness of $k$ in (3.19). If $\lambda = 0$, we obtain that $\phi(L_n, Y_{m+s}) = \alpha[L_n, Y_{m+s}] + h_{-2\mu}M_{-2\mu}$. If $\lambda \neq 0$, by the arbitrariness of $k$ in (3.20), it follows $h_i = 0$. Since $\lambda \neq -1$ or $\mu \notin s + \mathbb{Z}$, we get $l_{i+s} = 0$ if $i \neq n + m$ by (3.21) and (3.22). Thus, $\phi(L_n, Y_{m+s}) = \alpha[L_n, Y_{m+s}]$.

Claim 4.

$$\phi(Y_{n+s}, Y_{m+s}) \equiv \begin{cases} \alpha[Y_{n+s}, Y_{m+s}] + r_{-\mu}Y_{-\mu}(\mod \mathcal{L}) & \text{if } \lambda = -1, \mu \in s + \mathbb{Z}; \\ \alpha[Y_{n+s}, Y_{m+s}](\mod \mathcal{L}) & \text{otherwise}, \end{cases}$$

for all $n,m \in \mathbb{Z}$, where $r_{-\mu} \in \mathbb{C}$.

For any fixed $n,m \in \mathbb{Z}$, assume $\phi(Y_{n+s}, Y_{m+s}) = \sum_{i \in \mathbb{Z}} (p_i L_i + q_i M_i + r_{i+s} Y_{i+s})$ for some $p_i, q_i, r_{i+s} \in \mathbb{C}$. By Lemma 2.5 and Claim 1, for any $k \neq 0$, we have

$$0 = [\phi(Y_{n+s}, Y_{m+s}), [L_0, L_k]] - [[Y_{n+s}, Y_{m+s}], \phi(L_0, L_k)]$$

$$= \sum_{i \in \mathbb{Z}} (p_i L_i + q_i M_i + r_{i+s} Y_{i+s}) - (m-n)M_{n+m+2s}, k\alpha L_k$$

$$= \sum_{i \in \mathbb{Z}} (k(k-i)p_i L_{i+k} - k(i-k\lambda + 2\mu)q_i M_{i+k} - k(i+s-\frac{\lambda+1}{2}k + \mu)r_{i+s} Y_{i+s+k})$$

$$+ k\alpha(m-n)(n+m+s-k\lambda + 2\mu)M_{n+m+2s+k},$$

which implies that

$$p_i = 0 \text{ if } i \neq k,$$

$$q_i = 0 \text{ if } i \neq n + m + 2s,$$

$$(m-n)(n+2s-k\lambda + 2\mu)(q_i - \alpha(m-n)) = 0 \text{ if } i = n + m + 2s,$$

$$r_{i+s} = 0 \text{ if } i+s \neq \frac{\lambda+1}{2}k - \mu.$$
which infers that
\[ f_{-\mu} = l_{-\mu} = 0. \] (3.28)

If \( m \neq 0 \) and \( n + m + s = -\mu \), by Claims 2, 3 and (3.27), it is enough to see that
\[ \alpha m(s + \mu)M_{s-\mu} = \alpha(s + \mu)(m-n)M_{s-\mu} - (s + \mu)l_{-\mu}M_{s-\mu}, \]
which implies that \( l_{-\mu} = -\alpha n \) if \( \mu \neq -s \). Thus, one shows that
\[
\phi(L_n, Y_{m+s}) = \begin{cases} 
  l_s Y_s & \text{if } n + m = 0; \\
  \alpha[L_n, Y_{m+s}] & \text{if } n + m \neq 0,
\end{cases} \tag{3.29}
\]
for any \( n, m \in \mathbb{Z} \).

On the other hand, for any \( n, m \in \mathbb{Z} \), we have
\[
\phi([L_0, Y_{n+s}], Y_{m+s}) = [L_0, \phi(Y_{n+s}, Y_{m+s})] + [\phi(L_0, Y_{m+s}), Y_{n+s}]. \tag{3.30}
\]
If \( m \neq 0 \) and \( n \neq -s - \mu \), according to Claim 4 (3.29) and (3.30), we can easily get that
\[ r_{-\mu} = 0. \tag{3.31} \]

If \( m = 0 \) and \( n \neq 0 \), by Claim 4 (3.29)-(3.31), one shows that
\[
-\alpha n(n + s + \mu)M_{n+2s} = -\alpha n(n + 2s + 2\mu)M_{n+s} + n\lambda M_{n+s},
\]
which implies that \( l_s = \alpha(s + \mu) \). Hence, for \( n, m \in \mathbb{Z} \), we have
\[
\phi(L_n, Y_{m+s}) = \alpha[L_n, Y_{m+s}]. \tag{3.32}
\]

**Claim 5.** There exist \( \alpha, \beta, \gamma \in \mathbb{C} \) such that
\[
\phi(L_n, L_m) \equiv \begin{cases} 
  \alpha[L_n, L_m] + \beta \phi_0(L_n, L_m) (\text{mod} Z(L)) & \text{if } \lambda = 1, \mu \in s + \frac{1}{2} + \mathbb{Z}; \\
  \alpha[L_n, L_m] + \beta \phi_0(L_n, L_m) + \gamma \phi_1(L_n, L_m) (\text{mod} Z(L)) & \text{if } \lambda = 1, \mu \in s + \mathbb{Z}; \\
  \alpha[L_n, L_m] (\text{mod} Z(L)) & \text{otherwise},
\end{cases}
\]
for all \( n, m \in \mathbb{Z} \), where \( \phi_0 \) and \( \phi_1 \) are given by (3.1) and (3.2).

It is obvious by Claim 4 and (3.11).

**Claim 6.** \( \phi(L_n, M_m) \equiv \alpha[L_n, M_m] (\text{mod} Z(L)) \) for all \( n, m \in \mathbb{Z} \).

It is easily obtained by Claim 2 and (3.28).

**Claim 7.**
\[
\phi(L_n, Y_{m+s}) \equiv \begin{cases} 
  \alpha[L_n, Y_{m+s}] + \gamma \phi_1(L_n, Y_{m+s}) (\text{mod} Z(L)) & \text{if } \lambda = 1, \mu \in s + \mathbb{Z}; \\
  \alpha[L_n, Y_{m+s}] (\text{mod} Z(L)) & \text{otherwise},
\end{cases}
\]
for all \( n, m \in \mathbb{Z} \), where \( \phi_1 \) is given by (3.2).
It can be easily obtained by Claim 8 and (3.32).

Claim 8. \( \phi(Y_{n+s}, Y_{m+s}) \equiv \alpha[Y_{n+s}, Y_{m+s}] \mod Z(\mathcal{L}) \) for \( n, m \in \mathbb{Z} \).

It is obvious by Claim 4 and (3.31).

Claim 9. \( \phi(M_n, M_m) \equiv 0 \mod Z(\mathcal{L}) \) for \( n, m \in \mathbb{Z} \).

For any fixed \( n, m \in \mathbb{Z} \), since \( [M_n, M_m] = 0 \), it is enough to see that \( \phi(M_n, M_m) \in C_\mathcal{L}([\mathcal{L}, \mathcal{L}]) \) by Lemma 2.6. Thus, this claim holds by Lemma 3.1(3).

Claim 10. \( \phi(M_n, Y_{m+s}) \equiv 0 \mod Z(\mathcal{L}) \) for \( n, m \in \mathbb{Z} \).

It can be obtained by using the similar method of claim 9.

Now, by Claims 5-10, for any \( x, y \in \mathcal{L} \), we have

\[
\phi(x, y) = \begin{cases} 
\alpha[x, y] + \beta \phi_0(x, y) \mod Z(\mathcal{L}) & \text{if } \lambda = 1, \mu \in s + \frac{1}{2} + \mathbb{Z}; \\
\alpha[x, y] + \beta \phi_0(x, y) + \gamma \phi_1(x, y) \mod Z(\mathcal{L}) & \text{if } \lambda = 1, \mu \in s + \mathbb{Z}; \\
\alpha[x, y] \mod Z(\mathcal{L}) & \text{otherwise.}
\end{cases}
\]

If \( \lambda \neq 0 \) or \( \mu \notin \frac{1}{2}\mathbb{Z} \), we get \( Z(\mathcal{L}) = 0 \) by Lemma 3.1(1). Hence, Theorem 3.2 holds. Then, consider the case \( \lambda = 0, \mu \in \frac{1}{2}\mathbb{Z} \). By Lemma 3.1(1) and (3.33), we may assume that \( \phi(x, y) = \alpha[x, y] + \theta(x, y)M_{-2\mu} \), where \( \theta \) is a bilinear function from \( \mathcal{L} \times \mathcal{L} \) to \( \mathbb{C} \). We need to show that \( \theta \) is the zero function. In fact, by

\[
\phi([x, y], z) = [x, \phi(y, z)] + [\phi(x, z), y],
\]

we have that \( \theta([x, y], z) = 0 \) for all \( x, y, z \in \mathcal{L} \). Note that \( \mathcal{L} = [\mathcal{L}, \mathcal{L}] \) in this case by Lemma 3.1(2). It follows that \( \theta \) is exactly the zero function, as desired.

### 4 Linear commuting maps on \( \mathcal{L} \)

Let \( \mathcal{A} \) be an associative algebra, a map \( \psi : \mathcal{A} \rightarrow \mathcal{A} \) is called a commuting map if \( \psi(x)x = x\psi(x) \) for all \( x \in \mathcal{A} \). If we denote \([x, y] = xy - yx\) for \( x, y \in \mathcal{A} \), then a commuting map \( \psi \) on \( \mathcal{A} \) can also be defined as \([\psi(x), x] = 0 \) for all \( x \in \mathcal{A} \).

**Definition 4.1.** Let \( L \) be a Lie algebra, a map \( \psi : L \rightarrow L \) is called commuting if

\[
[\psi(x), x] = 0 \quad \forall x \in L.
\]

**Definition 4.2.** Define the following map

\[
\psi(x) = \alpha x + f(x), \quad \forall x \in \mathcal{L}
\]
on \( \mathcal{L} \), where \( \alpha \in \mathbb{C}, f : \mathcal{L} \rightarrow Z(\mathcal{L}) \). Obviously, \( \psi(x) \) is a linear commuting map. We call such a map a **standard linear commuting map** on \( \mathcal{L} \). Other linear commuting maps are called **non-standard**.
Now we apply Theorem 3.2 to describe linear commuting maps on the deformative Schrödinger-Virasoro Lie algebras $L$. Obviously, the identity map is a standard linear commuting map. For $L$ with $\lambda = 0, \mu \in \frac{1}{2}Z$ and some linear function $f$ from $L$ to $C$, by Lemma 3.1(1), the map $x \mapsto f(x)M_{-2\mu}$ is also a standard linear commuting map.

For convenience, we first introduce two kinds of linear commuting maps.

(1) For $L$ with $\lambda = 1, \mu \in s + \frac{1}{2}Z$, define the following linear map:

$$
\psi_0 : L \to L, \quad L_n \mapsto M_{n-2\mu},
$$

(4.1)

the others map to 0.

(2) For $L$ with $\lambda = 1, \mu \in s + Z$, define the following linear map:

$$
\psi_1 : L \to L, \quad L_n \mapsto Y_{n-\mu}, \quad Y_{n+s} \mapsto M_{n+s-\mu},
$$

(4.2)

others map to 0.

It can be easily checked that $\psi_0$ and $\psi_1$ are non-standard linear commuting maps.

**Theorem 4.3.** Each linear commuting map $\psi$ on $L$ is one of the following forms:

$$
\psi(x) = \begin{cases} 
\alpha(x) + \beta \psi_0(x) & \text{if } \lambda = 1, \mu \in s + \frac{1}{2}Z; \\
\alpha(x) + \beta \psi_0(x) + \gamma \psi_1(x) & \text{if } \lambda = 1, \mu \in s + Z; \\
\alpha(x) + f(x)M_{-2\mu} & \text{if } \lambda = 0, \mu \in \frac{1}{2}Z; \\
\alpha(x) & \text{otherwise},
\end{cases}
$$

for any $x \in L$, where $\alpha, \beta, \gamma \in C$, $\psi_0$ and $\psi_1$ are given by (4.1) and (4.2), and $f$ is a linear function form $L$ to $C$.

**Proof.** We assume that $\psi$ is a linear commuting map on $L$. Define

$$
\phi : L \times L \to L, \quad (x, y) \mapsto [\psi(x), y] \quad \forall x, y \in L.
$$

(4.3)

Thus, one has

$$
\phi(x, [y, z]) = [\phi(x, y), z] + [y, \phi(x, z)] \quad \forall x, y, z \in L.
$$

Obviously, $\phi$ is a derivation with respect to the second component. Since $[\psi(x), y] = [x, \psi(y)]$, we conclude that $\phi$ is also a derivation with respect to the first component. Thus, $\phi$ is a biderivation of $L$. From (4.3), we know that $\phi$ is skew-symmetric. According to Theorem 3.2 there exist $\alpha, \beta, \gamma \in C$ such that

$$
\phi(x, y) = \begin{cases} 
\alpha[x, y] + \beta \phi_0(x, y) & \text{if } \lambda = 1, \mu \in s + \frac{1}{2}Z; \\
\alpha[x, y] + \beta \phi_0(x, y) + \gamma \phi_1(x, y) & \text{if } \lambda = 1, \mu \in s + Z; \\
\alpha[x, y] & \text{otherwise},
\end{cases}
$$
for all $x, y \in \mathcal{L}$, where $\phi_0$ and $\phi_1$ are given by (3.1) and (3.2). Furthermore, by (4.3), then we obtain that

$$
[\psi(x) - \alpha x, y] = \begin{cases} 
\beta \phi_0(x, y) & \text{if } \lambda = 1, \mu \in s + \frac{1}{2} + \mathbb{Z}; \\
\beta \phi_0(x, y) + \gamma \phi_1(x, y) & \text{if } \lambda = 1, \mu \in s + \mathbb{Z}; \\
0 & \text{otherwise.}
\end{cases}
$$

**Case 1.** $\lambda = 1, \mu \in s + \frac{1}{2} + \mathbb{Z}$. By the definition of $\phi_0$ given by (3.1), we conclude that $\psi(x) - \alpha x = \beta \psi_0(x)$. The conclusion holds.

**Case 2.** $\lambda = 1, \mu \in s + \mathbb{Z}$. By the definition of $\phi_0$ and $\phi_1$ given by (3.1) and (3.2), one follows that $\psi(x) - \alpha x = \beta \psi_0(x) + \gamma \psi_1(x)$.

**Case 3.** $\lambda \neq 1$ or $\mu \notin s + \frac{1}{2} \mathbb{Z}$. Obviously, we have $\psi(x) - \alpha x \in \mathbb{Z}(\mathcal{L})$. By Lemma (3.1), we only need to consider the case $\lambda = 0, \mu \in \frac{1}{2} \mathbb{Z}$. Define $f$ by letting $\psi(x) - \alpha x = f(x)M_{-2\mu}$, then $f$ is a linear function from $\mathcal{L}$ to $\mathbb{C}$, and $\psi(x) = \alpha x + f(x)M_{-2\mu}$. This completes the proof.

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