Nutation dynamics and commensurate time crystal in a many-body seesaw

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We present a new mechanism for realizing of time crystal phase with arbitrary period, termed as commensurate time crystal (CTC), in a quantum seesaw constructed by an interacting Bose-Hubbard model. By periodically modulating the Hamiltonian, we find that the wave function will recover to its initial condition when the modulating frequency is commensurate with the initial energy level spacing between the ground and first excited levels. The period is determined by the driving frequency and commensurate ratio. In this case, the wave function will almost be restricted to the lowest two instantaneous energy levels. By projecting the wave function to these two relevant states, the dynamics is exactly the same as that for a spin precessing dynamics and nutating dynamics about an rotating axis. The commensurate motion between these two dynamics give rise to CTC. We map out the corresponding phase diagram and show that in the low frequency regime the state is thermalized and in the strong modulating limit, the dynamics is determined by the effective Floquet Hamiltonian. This CTC is realizable between these two limits. The measurement of this dynamics from mean position and mean momentum in phase space is also discussed.

The ultracold atoms provide an important platform for exploring several fundamental concepts in statistical physics. The ergodic theory, initiated from 1871 by Ludwig Boltzmann, asserts that the long-time average along trajectories equals the space average[1, 2]. Recently, this theory can be examined using the ultracold atoms, showing that the integrable and non-integral models can exhibit totally different behaviors[3–5]. The phase transitions in these systems can be revealed from the entanglement properties. For instance, the bipartite entanglement may become non-analytical at the phase boundary[6, 7]. The von Neumann entropy can even be used to measure the topological properties of the ground states[8–10], which satisfies the area’s law[11, 12].

The recent interest about these models are focused on many-body localization[13–15], out-of-time ordered correlation[16, 17] and even time crystal[18–21]. Due to lacking of ergodicity, the localized and delocalized phases have totally different level spacing distributions, which can be well described by random matrix theory[22–24]. These two phases may also be distinguished from their long-time dynamics in entanglement entropy and out-of-time ordered correlators. This platform can also be used to realize the time crystal phases, in which under periodic modulation the dynamics of the wave functions spontaneously break the translation symmetry of time[25–27]. The intimate relation between them is still a hot topic in recent years[20, 28–30].

In this work, we provide a new mechanism for the realization of time crystal phase. We investigate the many-body dynamics in a modulating Bose-Hubbard model and realize a novel commensurate time crystal (CTC) state. (I) When the modulating frequency is commensurate with the level spacing between the ground state and first excited state of the Hamiltonian at time \( t = 0 \), periodic recovery of the wave function can be realized, with period determined by the tunable commensurate ratio. This is totally different from the previous models in which the period of time crystal is two times the driving period[30–32]. (II) By projecting the wave function to these two lowest states, the model is reduced to a spin model precession about an oscillating magnetic field, which yields a new mechanism to the realization of the CTC via nutation dynamics[33–35]. This picture even has a single particle and classical analogous. (III) This phase can be realized only when the commensurate ratio is large enough. In the low modulating limit and beyond the adiabatic limit, the many-body state will quickly approaches the thermalized state. In the high frequency limit, this nutation dynamics is suppressed, and the dynamics is determined by the Floquet Hamiltonian. We have also discussed the experimental detection of these phases, and discuss their stability with disorder and non-
integrability interactions.

Physical model and dynamics. We consider the following modulating Bose-Hubbard model in a finite chain

\[
H = -J \sum_{i=1}^{L-1} (\hat{b}_i^\dagger \hat{b}_{i+1} + \text{h.c.}) + \frac{U}{2} \sum_{i=1}^{L} n_i (n_i - 1) + \alpha \sum_{i=1}^{L} \sin(\omega t)(i - \frac{L + 1}{2}) n_i, \tag{1}
\]

which is shown in Fig. 1. Here \( \hat{b}_i^\dagger \) (\( \hat{b}_i \)) are the creation (annihilation) operators at the \( i \)-th lattice site, \( n_i = \hat{b}_i^\dagger \hat{b}_i \) is the number operator, \( J \) is the tunneling strength and \( U > 0 \) is the on-site many-body interaction. In following, we set \( J = 1 \) as the basic energy scale. In experiments, \( J \approx 2\pi \times \mathcal{O}(0.1) \) kHz\cite{36,37,38,39}. The last term represents the quantum seesaw, which can be realized by an tilting magnetic field with modulating frequency \( \omega \) and degree of tilting \( \alpha \). This tilted potential has been realized in experiments\cite{40,41,42,43}. The hard wall boundary condition has been realized with a box potential in several groups, with \( L \) typically from 10 to 100\cite{44,45,46,47}. The data we will present are obtained by exact diagonalization (ED) and time-evolving block decimation (TEBD) methods\cite{48,49}.

In simulation, we choose \( \omega = \beta \Delta E_{12}, \) where \( \Delta E_{12} = E_2 - E_1 \) is the level spacing between the ground state and first excited state of \( H(t = 0) \) and \( \beta \) is a rational number, termed as commensurate ratio.

Let us first consider the dynamics in a small chain \( (L = 21, \beta = 15) \) and a long chain \( (L = 60, \beta = 33) \) in Fig. 2. The similar physics can be found by choosing other parameters. To measure whether the wave function will recover to its initial state, we measure the overlap between them via \( P(t) = \langle \psi(0) \mid \psi(t) \rangle \)\cite{50,51} (see Fig. 2a - b), which is related to the Ramsey interferometry in experiments\cite{52,53,54}. In both case, the wave function will almost recover to its initial state with \( P > 93\% \). To determine the period, we have also calculated the Fourier spectrum in time domain with two frequencies \( \omega_\pm = \omega \pm \omega' \), where \( \omega \) is the driving frequency of the seesaw. We find that \( T = 2\pi/\omega' = \beta T_0 \), with \( T_0 = 2\pi/\omega \) being the period of the driving field. This dynamics can persist for an extraordinary long time, which is a typical feature of time crystal by spontaneous translation symmetry breaking in time domain. In previous literature, \( T = 2T_0 \)\cite{30,31,32}; while in this work \( T/T_0 \) can take arbitrary integer numbers. Here we only consider the case that \( \beta \) to be integer values for simplicity, and a more general proof will be presented below, from which one can see our conclusion to be general.

To detect this dynamics, we investigate the mean position \( x_c \) and mean momentum \( p_c \) in the phase space in Fig. 2e - f, by defining \( x_c(t) = \sum_m \langle \psi(t) \mid b_m^\dagger b_m \psi(t) \rangle m/N \), where \( N \) is the total number of particle and \( p_c(t) = dx_c(t)/dt \). In the phase space, these two variables construct an almost closed trajectory after one period \( T \). For the two sets of parameters in Fig. 2, the real space displacement is roughly one or two lattice sites, and the change of mean momentum is slightly bigger. These two variables can be measured in both real and momentum spaces from the time-of-flight spectroscopy\cite{55,56,57,58}.

We next explore how the parameters influence the mean position displacement, by defining

\[
X_c = \frac{\max(x_c) - \min(x_c)}{2} \sim s^\eta, \tag{2}
\]

where \( s \) may refer to \( \beta, U, N, L, \alpha \) etc.. The exponents for these five cases are \( \eta \approx -1.0, -0.8, -1.0, 3.0 \) and 1.0, respectively. This means, in experiments, the larger center of mass displacement can be found with relative smaller modulating frequency, interaction, total number of particle, and relative larger number of lattice site \( L \), degree of tilting \( \alpha \). We also measure the area of the trajectory enclosed by \( p_c \) and \( x_c \) using \( I = \frac{1}{2\pi} \oint p_c dz_c \sim s^\eta \). This quality has a number of interesting features. In the adiabatic limit, it should be quantized (in unit of planck constant), which has played an fundamental role.
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major terms. Let \( \psi(t) \) to be solution of the following

equivalent Hamiltonian,

\[
H_{eq}(t) = \epsilon(t) + \sum_{i=x,y,z} h_i(t) \sigma_i = \epsilon(t) + B(t) \cdot \sigma,
\]

where the Pauli matrices act on the subspace constructed

by the lowest two states in Eq. 3. In this manner, we

map the many-body model to the spin dynamics in an

oscillating magnetic field \( B(t) \), in which the spin exhibits

both procession and nutation dynamics (see Fig. 4b).

This kind of dynamics is analogous to the three-body
dynamics in astrophysics[65, 66], in which precession and

nutation dynamics can be seen everywhere.

Let us illustrate the results in Fig. 2 using the above

projection. We find that the dynamics can be reproduced

by the following effective magnetic field \( B(t) \) with

\[
B(t) = (g \sin \omega t + \lambda, 0, \epsilon + A \sin \omega t).
\]

For example, we find that when \( \lambda = A = 0, g = 0.84,
\epsilon = \Delta E_{12}/2, \omega = 13\Delta E_{12} \), and we can get the same
behaviors for Fig. 2a. Let us assume the initial state to be
\( | \downarrow \rangle = \phi_1(0) \), and to the leading order via perturbation

FIG. 3. (Color online) (a)-(e) The maximal displacement

\( X_c \), (see Eq. 2) as a function of \( \beta, U, N, L \) and \( \alpha \). (f) The

area enclosed by the trajectory in phase space as a function

of \( \beta \). Parameters for open symbols \( \bigcirc \) are \( L = 21, N = 10, \)

\( U = \sqrt{2}, \alpha = 1/(3\sqrt{5}) \); \( \Box \) are \( L = 30, N = 15, U = 0.8, \)

\( \alpha = 0.1 \) and \( \Delta \) are \( L = 39, N = 22, U = \sqrt{7}/2, \alpha = 0.05 \).

Mapping to a spin vector about a rotating magnetic

field. This model provides a new mechanism for realiza-

tion of time crystal with controllable commensurate pe-

iod, which will be termed as commensurate time crys-

tal (CTC) phase. To pin down the underlying mechanism

for this dynamics, we now project the wave function

\( \psi(t) \) from the time-dependent Schrödinger equation
to the instantaneous eigenstates \( \phi_n \), where \( H(t)\phi_n(t) = \epsilon_n(t)\phi_n(t) \) with \( \epsilon_n \) to be arranged in increasing order.
The idea is quite similar to the derivation of geometry

phase in topological physics[61–64], but now we need to

consider a few low-lying eigenstates. In this way,

\[
\psi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots
\]

We will focus in the regime when \( \omega \) is smaller than the

whole band width (see the eigenvalues of \( H(t) \) in Fig. 4a,

with band width \( W_b = \max(\epsilon_n) - \min(\epsilon_n) = 11.33844 \)).

We find that \( |c_1|^2 + |c_2|^2 \) is always greater than 0.9, in-

indicating that almost all the wave functions, during time-
evolution, is restricted to the lowest two instantaneous

eigenstates. In this way, we can keep only these two

FIG. 4. (Color online) (a) Instantaneous eigenvalues for

\( L = 7, N = 3, U = 0.8, \alpha = 0.5, \omega = 11\Delta E_{12} \) and \( \Delta E_{12} = 0.518804 \).

(b) Nutation and procession dynamics for a spin

about an modulating magnetic field. The corresponding spin

vector is defined as \( \mathbf{n}(t) = (\psi(t)|\sigma|\psi(0)) \), with Pauli matrices

act on the lowest two levels \( \phi_{1,2} \). (c) The dynamics of the

spin vector \( \mathbf{n} \) in Bloch sphere, with \( A = \lambda = 0, g = \sqrt{2}, \epsilon = 0.53, \omega = 10\Delta E_{12} \). (d) (f) Show the nutation dynamics of spin,

with the arrows mark the evolution in one full period.
theory we find

$$\psi(t) = \frac{1}{\sqrt{|c_1(t)|^2 + |c_2(t)|^2}}(c_1(t)|\downarrow\rangle + c_2(t)|\uparrow\rangle),$$

$$c_1(t) = 1 + \frac{A}{i\omega}(1 - \cos(\omega t)),$$

$$c_2(t) = \frac{ig[\omega + \epsilon_1 i\Delta E_{12}^2(\epsilon_1 \Delta E_{12} \sin(\omega t) - \omega \cos(\omega t))] \epsilon_2}{(\Delta E_{12})^2 - \omega^2},$$

where $\Delta E_{12} = 2\sqrt{c^2 + \lambda^2}$. Using this wave function, we are able to compute the overlap $P(t) = \langle \psi(t) | \downarrow \rangle$, which is shown in Fig. 2a - b with blue dotted line. The agreement between the exact many-body solution is excellent. We also describe this dynamics on the Bloch sphere, showing in Fig. 4c and the three directions in Fig. 4d - f, respectively. From Eq. 6, one sees that the wave function will exactly recover to its initial state after a period $T$ determined by $\omega T = 2\pi p$ and $\Delta E_{12} T = 2\pi q$, where $p$ and $q$ are relatively prime numbers with $p > q$. In this case $\beta = p/q$ and the whole period is determined by $T = pT_0$. This result is also true with higher-order perturbation approximation. Thus by controlling this ratio, our results can be used to realize CTC states with different period. This period is determined solely by $\beta$, and is independent of other parameters, such as non-integrability terms and time-independent disorders, thus it has the same robustness of period as that discussed in Refs. [67, 68].

This projection yields a new picture to understand the many-body dynamics. While the precession in classical mechanics has found wide applications in quantum mechanics, that is, Rabi oscillation[69-71], the mutation is rarely discussed in quantum mechanics[72-74]. We show that the commensurate dynamics between them can be used for CTC phase. Intriguingly, this result shows that this CTC may be realized even with classical systems and non-interacting two-level systems. The latter setup can be immediately implemented using the state-of-art architectures for quantum computations, such as quantum dots[75, 76], NV color center[77, 78], trapped ions[79, 80] and superconducting qubits[81, 82].

Phase diagram. We finally address the phase diagram for the searching of CTC phase. From the above analysis the modulating frequency matters. This modulating frequency has a number of important consequences to the dynamics. In the extremely low modulating limit, the system follows adiabatically the dynamics of the Hamiltonian. In the high frequency limit, that is, $\omega \gg \max(\epsilon_n) - \min(\epsilon_n)$, the effective dynamics can be well described by the effective Floquet Hamiltonian defined by $U(T_0) = \exp(-iH_F T_0)$[83, 84]. To the third-order approximation[85, 86], $H_F = -J_F \sum_n (b_n^\dagger b_{n+1} + h.c.) + \frac{\lambda}{2} \sum_n n_{\uparrow}(n_{\downarrow} - 1)$, where $J_F = (1 - \frac{\lambda}{\Delta E_{12}})J$. The latter case was widely used in literatures for the searching of novel phases, including the spin-orbit coupling[87-89] in lattice models as well as some exotic topological phases[90-92]. We are mainly interested in the physics between these two limits. In Fig. 5, we show that in the relative low modulating limit, the wave function is thermalized. With the increasing of modulating frequency, it enters the mixed phase, which exhibits non-normal oscillation. From its Fourier spectrum, one may see multiply modulating frequencies due to coupling of a lot of low-lying eigenstates. In this way, the period oscillation is hard to be developed in the experimental accessible time window. With the further increasing of modulating frequency, the dynamics will finally be restricted to the lowest two states, in which almost perfect oscillation can be found. The crossover between these three cases is somewhat smooth, probably due to the finite size effect. We also find the on-set of the CTC depends strongly on the degree of tilting $\alpha$. The larger the degree of tilting is, the larger the commensurate ratio is required to be.

To conclude, we present a new mechanism to realize the time crystal with controllable period using a quantum seesaw based on an interacting Bose-Hubbard model. This phase is realized by commensurate between precession dynamics and mutation dynamics in the many-body system. In this model, the periodic driving field is used to restrict the dynamics of wave function to the lowest two energy levels, which may also be realized in other modulating systems. In this sense, this behavior should be quite general in a lot of periodic driving system before fully entering the effective Floquet Hamiltonian regime. From this picture, this kind of dynamics is robust against disorders and non-integrability interactions.

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