Do static sources outside a Schwarzschild black hole radiate?

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Abstract

We show that static sources coupled to a massless scalar field in Schwarzschild spacetime give rise to emission and absorption of zero-energy particles due to the presence of Hawking radiation. This is in complete analogy with the description of the bremsstrahlung by a uniformly accelerated charge from the coaccelerated observers’ point of view. The response rate of the source is found to coincide with that in Minkowski spacetime as a function of its proper acceleration. This result may be viewed as restoration of the equivalence
principle by the Hawking effect.
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The relation between radiation from accelerated charges and the equivalence principle has for some time been the source of much confusion and discussion. A particularly interesting question is how to reconcile the following two facts (in Minkowski spacetime): On the one hand, an accelerated charge is known to radiate when it is seen from the viewpoint of inertial observers. On the other hand, according to the equivalence principle, the same charge is seen by comoving observers as a static charge in a uniform “gravitational field”, and, hence, is not expected to radiate. In the classical context, this question has been answered first by Rohrlich [1] and further clarified by Boulware [2], who has shown that the presence of a horizon for the collection of comoving observers, who perceive the charge as static, serves to explain the apparent paradox. This resolution is based on the fact that the radiation zone (as described by the Minkowski observers) lies beyond the comoving observers’ horizon and is thus unobservable by them. In the quantum mechanical context, a solution to the apparent paradox (which is now cast in terms of photon emission rates) has been given by the authors [3], by recalling that, as seen by the comoving observers, the static charge (which has in fact constant proper acceleration) is immersed in the Fulling-Davies-Unruh (FDU) thermal bath in Rindler spacetime [4]. That is, the interaction of the static charge with this thermal bath results in the absorption and stimulated emission of photons with zero Rindler energy and this completely accounts for the bremsstrahlung due to a uniformly accelerated charge in quantum electrodynamics. (Here, the Rindler energy means the energy corresponding to the boost Killing vector field with respect to which Rindler spacetime is static.)

The purpose of this Letter is to note that, in complete analogy to the result obtained in the case of the static charge in Rindler spacetime as described before, the analysis of a static charge in a static black-hole spacetime, which interacts with Hawking radiation [5], yields a finite response rate. In fact we will see that the total response rate is exactly the same as that of a uniformly accelerated source in Minkowski spacetime as a function of the proper acceleration.

We first review the general formalism for computing the response rate of a classical source in a static spacetime and the result of Ref. [3] in the context of massless scalar field [9]. Then
we present our result for Schwarzschild spacetime.

Let us consider a globally-hyperbolic static spacetime described by the metric $ds^2 = f(x)dt^2 - h_{ij}(x)dx^i dx^j$. We will study a real scalar field $\Phi$ that interacts with a classical source $j(x)$ ($x = (t, x)$) and is described by the action $S = \int d^4x \sqrt{|h|} \left( \frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi + j \Phi \right)$, where $h(x) = \det h_{ij}(x)$. Let

$$u_{\omega \lambda}(x) = \sqrt{\frac{\omega}{\pi}} U_{\omega \lambda}(x) \exp(-i\omega t)$$

(1)

with $\omega > 0$ and their complex conjugates, $u_{\omega \lambda}(x)^*$, be solutions to $\Box u = 0$, where $s = (s_1, \cdots, s_n)$ is a set of continuous labels and $\lambda$ is a discrete label for the complete set of modes. We have assumed $\omega$ to be continuous because this is the case in the spacetimes we study, and adopted it as one of the labels. The factor of $\sqrt{\omega/\pi}$ has been inserted for later convenience. Let these solutions be Klein-Gordon orthonormalized:

$$i \int d\Sigma n^\mu (u_{\omega \lambda}^{*} \nabla_\mu u_{\omega' \lambda'} - \nabla_\mu u_{\omega \lambda}^{*} \cdot u_{\omega' \lambda'}) = \delta(\omega - \omega') \delta(s - s') \delta_{\lambda \lambda'},$$

(2)

$$i \int d\Sigma n^\mu (u_{\omega \lambda} \nabla_\mu u_{\omega' \lambda'} - \nabla_\mu u_{\omega \lambda} \cdot u_{\omega' \lambda'}) = 0,$$

(3)

where $d\Sigma$ is the volume element of a Cauchy surface and where $n^\mu$ is the future-pointing unit normal to it. The in-field $\Phi^{\text{in}}$ satisfying the free field equation $\Box \Phi^{\text{in}} = 0$ can now be expanded as

$$\Phi^{\text{in}}(x) = \sum_{\lambda} \int d\omega d^p s \left[ u_{\omega \lambda}(x) a_{\omega \lambda}^{\text{in}} + H.c. \right].$$

Let the initial state be the in-vacuum state $|0\rangle_{\text{in}}$ defined by $a_{\omega \lambda}^{\text{in}}|0\rangle_{\text{in}} = 0$ for all $\omega$, $s$ and $\lambda$.

We will be interested in static sources. However, as we will see later, we need to introduce oscillation as a regulator in order to avoid the appearance of intermediate indefinite results. Therefore we consider a source of the form $j_{\omega}(x) = J(x) \cos \omega_0 t$. The rate of spontaneous emission with fixed $s$ and $\lambda$ can now be found to lowest order in perturbation theory:

$$R_{sp}(\omega_0; s, \lambda) d^p s = \frac{\omega_0}{2} |\tilde{J}(\omega_0, s, \lambda)|^2 d^p s,$$
where \( \tilde{J}(\omega_0, s, \lambda) = \int d^3x \sqrt{h(x)}f(x) J(x)U_{\omega_0 s \lambda}(x) \). We note that Eq. (1) gives the emission rate per unit *coordinate* time. Later we will convert it into the rate per unit *proper* time for point sources.

If the source is immersed in a thermal bath of inverse temperature \( \beta = 1/k_B T \), the rates of absorption and *induced* emission are both \( R_{sp}(\omega_0; s, \lambda) / (\exp \beta \omega_0 - 1) \). Summing the absorption rate and the spontaneous and induced emission rates, we find the total *response* rate:

\[
R(\omega_0; s, \lambda) = \frac{\omega_0}{2} \coth \frac{\beta \omega_0}{2} |\tilde{J}(\omega_0, s, \lambda)|^2.
\]

In the case of interest here, i.e. for \( \omega_0 \to 0 \), we have

\[
R(0; s, \lambda) = \beta^{-1} |\tilde{J}(0, s, \lambda)|^2.
\] (4)

Let us now review how the bremsstrahlung rate due to a uniformly accelerated source (in Minkowski spacetime) is reproduced from the Rindler-spacetime point of view by taking the FDU thermal bath into account. First we present the conventional result for the emission rate, which is to be compared with the Rindler-spacetime result. We define the Rindler coordinates \( \tau \) and \( \xi \) in terms of the usual Minkowski coordinates by

\[
t = a^{-1}e^{a\xi} \sinh a\tau,
\]
\[
z = a^{-1}e^{a\xi} \cosh a\tau,
\]
and consider the classical source \( j_0 = q\delta(\xi)\delta(x)\delta(y) \). This source has constant proper acceleration \( a \). Using the standard method (see, e.g., Ref. [10]), we obtain the rate of spontaneous emission of particles with fixed transverse momentum \( (k_x, k_y) \):

\[
R_{sp}^M(k_x, k_y)dk_xdk_y = \int_{-\infty}^{+\infty} dw \Delta_{k_\perp} \left( \frac{2}{a} \sinh \frac{aw}{2} \right) \frac{dk_xdk_y}{(2\pi)^2} = \frac{1}{4\pi^3 a} [K_0(k_\perp / a)]^2 dk_xdk_y,
\] (5)

where \( k_\perp = \sqrt{k_x^2 + k_y^2} \). (We refer the reader to Ref. [11] for formulas involving special functions used in this Letter.) The function \( \Delta_m(\sqrt{\sigma}) = -\frac{1}{\sqrt{3}} N_0(m\sqrt{\sigma}) \), where \( \sigma = t^2 - z^2 \), is the symmetrized two-point function of massive scalar field in two dimensions with \( \sigma > 0 \).

We can now compare the rate (5) with the rate obtained in the Rindler point of view, where the variable \( \tau \) is adopted as time. We first note that from this perspective the source
is immersed in the FDU thermal bath. This source absorbs particles from the heat bath, which also gives rise to induced emission. Since the particle concept depends on the timelike Killing vector that one uses to define it, emission of a Minkowski particle (i.e. one defined with respect to $\partial/\partial t$) can correspond either to absorption or to emission of a Rindler particle (i.e. one defined with respect to $\partial/\partial \tau$) \cite{12}. However, the rate of response, i.e. emission plus absorption, must be independent of the description that one uses. Therefore, the rate of spontaneous emission given by (5) should equal the total response rate of the source $j_0$ computed in Rindler spacetime with the FDU thermal bath.

There is a technical complication with the verification of the above statement due to the fact that the spontaneous emission rate vanishes because the source is now static whereas the density of states in the thermal bath diverges in the zero-frequency limit. As a result, we encounter an expression of the form $0 \times \infty$ in the process of computing the response rate using the particle concept in Rindler spacetime. For this reason we regularize the calculation by considering

$$j = \sqrt{2} q \cos \omega_0 \tau \delta(\xi) \delta(x) \delta(y)$$

and taking the limit $\omega_0 \to 0$ in the end. The factor of $\sqrt{2}$ is necessary to make the time average of the squared charge equal $q^2$. The source (6) is then equivalent to the source $j_0$ in the limit $\omega_0 \to 0$ because the rate is proportional to the squared charge at the lowest order.

Now we verify explicitly that the $\omega_0 \to 0$ limit of the total response rate of the source (6), which is obtained from (4) with $\beta^{-1} = a/2\pi$, coincides with the rate (5). The positive-frequency modes with respect to $i\partial/\partial \tau$ are given by

$$u_{\omega k_x k_y}(\tau, \xi, x, y) = \sqrt{\omega \pi} \psi_{\omega k_\perp}(\xi) \times \frac{e^{ik_x x + ik_y y - i\omega \tau}}{2\pi},$$

where

$$\left[ -\frac{d^2}{d\xi^2} + k_\perp^2 e^{2a\xi} \right] \psi_{\omega k_\perp}(\xi) = \omega^2 \psi_{\omega k_\perp}(\xi),$$

and where $k_\perp = \sqrt{k_x^2 + k_y^2}$. Requiring that $\psi_{\omega k_\perp}(\xi)$ decrease for $\xi \to +\infty$, we find that $\psi_{\omega k_\perp}(\xi) \propto K_{i\omega/a}(k_\perp/a e^{a\xi})$. By the usual method of turning the normalization integral into
a surface term (see, e.g., Ref. [3]), we find that the function \( u_{\omega k_x k_y} \) is normalized according to (2) if for large and negative \( \xi \)

\[
\psi_{\omega k_\perp} (\xi) \approx -\frac{1}{\omega} \sin [\omega \xi + \alpha(\omega)].
\]  

(9)

This determines \( \psi_{\omega k_\perp} (\xi) \):

\[
\psi_{\omega k_\perp} (\xi) = \sqrt{\sinh (\frac{\pi \omega}{a})} K_{i\omega/a} ((k_\perp/a)e^{a\xi}).
\]

Consequently, we find

\[
\psi_{0k_\perp} (\xi) = a^{-1} K_0 ((k_\perp/a)e^{a\xi}).
\]  

(10)

We note here that \( \psi_{0k_\perp} (\xi) \approx -\xi + \text{const.} \) for large and negative \( \xi \). This can be understood as the \( \omega \to 0 \) limit of (9). In fact, one can directly determine the normalization factor of \( \psi_{0k_\perp} \) by requiring this behavior without referring to the solutions with nonzero \( \omega \). We will use this method for the Schwarzschild black-hole case.

Using (7) with (10) in (4), one finds that the total response rate in the thermal bath of temperature \( \beta^{-1} = a/2\pi \) in Rindler spacetime is indeed equal to \( R_{sp}^M (k_x, k_y) \) given by (3). We compute the integrated response rate given by the integral over the transverse momentum for later use:

\[
R_{sp, \text{tot}}^M = \int dk_x dk_y R_{sp}^M (k_x, k_y) = \frac{q^2}{4\pi^2} a.
\]  

(11)

Now we turn our attention to the Schwarzschild case. In particular, we determine the response rate of a point source analogous to (3) in the limit \( \omega_0 \to 0 \). We use the standard Schwarzschild metric, \( ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \), where \( f(r) = 1 - 2M/r \).

The positive-frequency solutions to the massless scalar field equation in this spacetime can be written as

\[
u_{\omega lm} = \sqrt{\frac{\omega}{\pi}} \frac{\psi_{\omega l}(r)}{r} \times Y_{lm}(\theta, \varphi) e^{-i\omega t}.
\]  

(12)

Here \( \psi_{\omega l}(r) \) is the solution to the differential equation.
\begin{equation}
\left\{-f(r) \frac{d}{dr} \left[ f(r) \frac{d}{dr} \right] + V_{\text{eff}}(r) \right\} \psi_{\omega l}(r) = \omega^2 \psi_{\omega l}(r), \tag{13}
\end{equation}

where \( V_{\text{eff}}(r) = (1 - 2M/r) [2M/r^3 + l(l + 1)/r^2] \). For given \( \omega, l \) and \( m \) there are two independent and orthogonal solutions of (13). One is purely incoming from the past horizon \( H^- \) and the other is purely incoming from past null infinity \( J^- \).

In the Unruh vacuum [5], which corresponds to the physical black hole formed by gravitational collapse, a thermal flux of temperature \( \beta^{-1} = 1/(8\pi M) \) comes out from \( H^- \). In the Hartle-Hawking vacuum [13] there is an additional thermal flux coming from \( J^- \). We concentrate on the Unruh vacuum in this Letter.

The regularized classical source we consider is
\begin{equation}
 j(x) = \sqrt{2} q f(r_0)^{1/2} \frac{1}{r_0^2 \sin \theta_0} \cos \omega_0 t \delta(r - r_0) \delta(\theta - \theta_0) \delta(\varphi - \varphi_0). \tag{14}
\end{equation}

This source and the source (6) have the same strength in the sense that they give the same value when integrated over the hypersurface of constant time.

Using Eq. (14) and introducing the correction factor \( f(r_0)^{-1/2} \) to convert the rate per coordinate time into that per proper time, we find that the response rate per proper time of the source (14) with fixed angular momentum in the limit \( \omega_0 \to 0 \) is given by
\begin{equation}
 R_{lm} = \frac{q^2}{4\pi MR_0^2} f(r_0)^{1/2} |\psi_{\omega l}(r_0)|^2 |V_{lm}(\theta_0, \varphi_0)|^2, \tag{15}
\end{equation}

provided that the function \( u_{\omega lm} \) in (12) is normalized according to (2). Now our task is to find the function \( \psi_{\omega l} \) incoming from \( H^- \) and corresponding to this normalization. (Strictly speaking, we need to prove that \( \psi_{\omega l}(r) \to \psi_{\omega l}(r) \) as \( \omega \to 0 \).)

It is useful to introduce the dimensionless Wheeler tortoise coordinate \( x = y + \ln(y - 1) \), where \( y = r/2M \). Eq. (13) can then be rewritten as
\begin{equation}
 \left[- \frac{d^2}{dx^2} + (2M)^2 V_{\text{eff}}(x) \right] \psi_{\omega l} = (2M\omega)^2 \psi_{\omega l}. \tag{16}
\end{equation}

In the limit \( \omega \to 0 \), the incoming wave from the white-hole horizon is totally reflected towards the black-hole horizon. This implies that the Klein-Gordon normalization (2) is
achieved for \( u_{\omega lm} \) with \( M\omega \ll 1 \) if \( \psi_{\omega l} \approx -\omega^{-1}\sin[2M\omega x + \alpha(\omega)] \) for large and negative \( x \).

Thus, in the limit \( \omega \to 0 \), we must normalize the solution \( \psi_{\omega} \) so that

\[
\psi_{\omega l} \approx -2Mx + \text{const.} \quad (x < 0, \ |x| \gg 1).
\]  

(17)

Now Eq. (13), or equivalently Eq. (16), can be solved explicitly for \( \omega = 0 \). The general solution is

\[
\psi_{0l}(y) = C_1 y P_l(2y - 1) + C_2 y Q_l(2y - 1),
\]

where \( P_l(z) \) and \( Q_l(z) \) are Legendre functions of the first and second kinds with the branch cut \((-\infty, 1]\) for \( Q_l(z) \). Note that \( P_l(z) \sim z^l \) and \( Q_l(z) \sim z^{-l-1} \) for large \( z \) and that the solution we seek must decrease for large \( y \) since the wave is totally reflected back to the horizon. From these facts and the condition (17) we find

\[
\psi_{0l} = 4MyQ_l(2y - 1).
\]

Substituting this in (15), we have

\[
R_{lm} = \frac{q^2}{\pi M f(r_0)^{1/2}} |Q_l(z_0)|^2 |Y_{lm}(\theta_0, \varphi_0)|^2,
\]

where \( z_0 = r_0/M - 1 \). It is possible to sum over \( l \) and \( m \) using the formulas

\[
\sum_{l=-l}^{l} |Y_{lm}(\theta, \varphi)|^2 = (2l + 1)/4\pi \quad \text{and} \quad \sum_{l=0}^{\infty} (2l + 1) |Q_l(z)|^2 = 1/(z^2 - 1).
\]

The result is

\[
R_{\text{tot}} = \sum_{l,m} R_{lm} = \frac{q^2}{4\pi^2} a(r_0),
\]

(18)

where \( a(r_0) = M f(r_0)^{-1/2}/r_0^2 \) is the proper acceleration of the static source. Note that this is identical with (11) as a function of proper acceleration.

We have not rigorously proved the validity of our approach where we directly work with the \( \omega = 0 \) modes satisfying the normalization condition (17) instead of explicitly taking the \( \omega \to 0 \) limit of the modes with \( \omega \neq 0 \). However, the exact agreement of (11) and (18) itself and the fact that we have reproduced with this method precisely the results of Ref. \[9\] serve as consistency checks of our approach. We will present elsewhere \[14\] a detailed analysis about how the functions \( \psi_{\omega l} \) approach \( \psi_{0l} \) in the \( \omega \to 0 \) limit. Here we will present another consistency check using a model \[15\] where the effective potential \( V_{\text{eff}} \) is replaced by a simpler but similar potential \( V_{\text{eff}}^{(s)}(x) = l(l + 1)\theta(x - 1)/(2Mx)^2 \) \( (l \neq 0) \). With this replacement, the function \( \psi_{\omega l}^{(s)} \) incoming from \( H^- \) and corresponding to \( \psi_{\omega l} \) can be found explicitly for any value of \( \omega \), and we have
where $\omega = 2M\omega$. Continuity of the value and the first derivative at $x = 1$ gives $\beta_{wl}^{(\pm)} = (1/2)e^{\mp i\tilde{\omega}}[(1 \mp i/\tilde{\omega})h_{l}^{(1)}(\tilde{\omega}) \mp i(h_{l}^{(1)})'(\tilde{\omega})]$. The normalization condition leads to $a_{wl}\beta_{wl}^{(\pm)} = i/2\omega$ up to a phase factor. Then the mode functions with $M\omega \ll 1$ can be approximated by

$$
\psi_{wl}^{(s)}(x) = 2M(1 - x + l^{-1}) + O(\omega^2) \quad (x < 1),
$$

$$
= 2Ml^{-1}x^{-l} + O(\omega^2) \quad (x > 1).
$$

Thus, we see explicitly that the $\omega \to 0$ limit of $\psi_{wl}^{(s)}(x)$ is indeed $\psi_{0l}^{(s)}(x)$ satisfying the normalization condition (1.7).

Note that the response rate of a static source will vanish in the absence of the Hawking effect, i.e. in the Boulware vacuum [10], whereas the source in Minkowski spacetime with the corresponding acceleration radiates. In this sense the equivalence principle is violated in the Boulware vacuum. Since the Hawking effect is closely related to the absence of singularity of the quantum state on the future horizon [17,18], it is reasonable to expect that this effect plays a crucial role in “restoring the equivalence principle” near the horizon. It is therefore not surprising that the rate of response (1.8) agrees with the corresponding result (1.11) in Minkowski spacetime in the limit $r_0 \to 2M$. However, the fact that they coincide for all $r_0$ was rather unexpected. It would be interesting to see if this fact, i.e. the complete “restoration of the equivalence principle” for a static scalar source by the Hawking effect, is a special case of a more general phenomenon.

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