Supplementary material for “Multiplicative Bell Inequalities”

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This supplementary text for the Letter “Multiplicative Bell Inequalities” includes additional details and proofs regarding some mathematical statements within the manuscript.

I. Bell and Tsirelson bounds for the case of $n = 2$

Proof. From the inequality of geometric and arithmetic means:

$$B_2 = (c_{11} - c_{21})(c_{12} + c_{22}) \leq \left(\frac{c_{11} - c_{21} + c_{12} + c_{22}}{2}\right)^2$$

where on the LHS we have $B_2$, and on the RHS the numerator is equivalent to the well-known Bell-CHSH parameter. By plugging the known Bell and Tsirelson bounds for the Bell-CHSH parameter [1, 2], it can be shown that the respective bounds for $B_2$ are 1 and 2. In order to show that these bounds can be saturated, let us demonstrate a local hidden-variable strategy: $a_1 = b_1 = b_2 = 1$, $E[a_2] = 0$. And the quantum strategy should be same as the one which saturates Tsirelson’s bound for Bell-CHSH, where $c_{ij} = \frac{1}{\sqrt{2}}(1 - 2\delta_{i,2}\delta_{j,1})$.

In general, our $n$-device multiplicative Bell parameter can be associated with an additive one in a similar way. Again, from the inequality of geometric and arithmetic means:

$$\sqrt[n]{B_n} = \sqrt[n]{\prod_{j=1}^{n} u_j \cdot c_j} \leq \sqrt[n]{\frac{\sum_{j=1}^{n} u_j \cdot c_j}{n}}$$

Thus, let us define the associated $n$-device additive Bell parameter as:

$$B'_n \triangleq \sum_{j=1}^{n} u_j \cdot c_j$$

and by plugging (3) into (2), we have:

$$|B_n| \leq \left(\frac{B'_n}{n}\right)^n$$

II. Construction of the Multiplicative Bell Parameter for $n$ measurement devices

Let $X$ be the following vector of quantum operators:

$$X = \begin{bmatrix} B_j \\ A_1 \\ \vdots \\ A_n \end{bmatrix}$$

where $i, j \in \{1, 2, \ldots, n\}$ are Alice and Bob’s inputs, and $A_i, B_j$ are operators with spectrum $\{\pm 1\}$, which represent the outcomes of Alice and Bob’s measurements, respectively.

The second moment matrix of $X$ is defined by the following relation:

$$\Sigma_{ij} = \langle X_i X_j \rangle.$$ 

It immediately follows that $\langle B_j B_j \rangle = \langle A_i A_i \rangle = 1$. Since $A_i, A_j$ are non-commuting linear operators (for $i \neq j$), the expected value $\langle A_i A_j \rangle$ cannot be measured in any experiment, and is generally a complex number. We shall denote it by $r_{ij}$:

$$r_{ij} \triangleq \langle A_i A_j \rangle = \langle A_j A_i \rangle^*.$$ 


The expected values of $A_i B_j$ are the aforementioned two-point correlators between Alice and Bob’s measurement results, and had been denoted by $c_{ij}$:

$$c_{ij} \triangleq \langle A_i \otimes B_j \rangle \quad (8)$$

note that $c_{ij} \in [-1, 1]$. Therefore, the second moment matrix $\Sigma$ is equal to:

$$\Sigma = \begin{bmatrix}
1 & c_{11} & c_{12} & \cdots & c_{1n} \\
c_{11} & 1 & r_{12} & \cdots & r_{1n} \\
c_{12} & r_{21} & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
c_{nj} & r_{n1} & \cdots & r_{n,n-1} & 1
\end{bmatrix} \succeq 0 \quad (9)$$

and it is positive semi-definite \[?\].

Using Schur’s complement, we obtain:

$$R_A \triangleq \begin{bmatrix}
1 & \cdots & r_{ij} \\
\vdots & \ddots & \vdots \\
r_{ji} & \cdots & 1
\end{bmatrix} \succeq \begin{bmatrix}
c_{1j} \\
\vdots \\
c_{nj}
\end{bmatrix} \begin{bmatrix}
c_{1j} & \cdots & c_{nj}
\end{bmatrix} \quad (10)$$

We denote:

$$c_j \triangleq \begin{bmatrix}
c_{1j} \\
\vdots \\
c_{nj}
\end{bmatrix}, C_j \triangleq c_j c_j^T \quad (11)$$

Now, let us construct the following $n \times n$ matrix:

$$\Gamma \triangleq \begin{bmatrix}
1 & r & \cdots & r \\
r & 1 & r & \vdots \\
\vdots & r & \ddots & r \\
r & \cdots & r & 1
\end{bmatrix} \quad (12)$$

where $r \in \mathbb{R}$. Note that $\Gamma$ is real and symmetric. Its orthogonal eigenvectors are:

$$u_j(l) = \begin{cases}
1 & \text{if } l \leq j \\
-j & \text{if } l = j + 1 \\
0 & \text{if } l > j + 1
\end{cases} \quad \text{for } 1 \leq j < n, \quad u_n = \begin{bmatrix} 1 \\
\vdots \\
1 \end{bmatrix}$$

Now, we shall use the vectors $u_k$ in order to construct our multiplicative Bell parameter:

$$B_n \triangleq \prod_{j=1}^n c_j \cdot u_j \quad (13)$$

### III. Proof of main result

In order to prove this result, it is required to prove two parts:

(a) $|B_n| \leq n!$

(b) $|B_n| = n!$ can be achieved in quantum mechanics.

*Proof (a)* First, let us denote: $\hat{u}_j \triangleq \frac{u_j}{\|u_j\|}$. $B_n \leq n!$ follows from:

1. $\prod_{j=1}^n \|u_j\| = n!$

2. $\forall j \in \{1, 2, \ldots n\}, \quad u_j^T c_j c_j^T u_j \leq u_j^T R_A u_j$
3. $\prod_{j=1}^{n} \hat{u}_j^T \mathcal{R}_A \hat{u}_j \leq \left( \sum_{i=1}^{n} \frac{\hat{u}_i^T \mathcal{R}_A \hat{u}_i}{n} \right)^n$

4. $\sum_{j=1}^{n} \hat{u}_j^T \mathcal{R}_A \hat{u}_j = n$

Let us calculate the product of all the eigenvectors’ squared norms:

$$ ||\mathbf{u}_n||^2 \prod_{k=1}^{n-1} ||\mathbf{u}_k||^2 = n \prod_{k=1}^{n-1} (k^2) = (n!)^2 \quad (14) $$

which proves $[1]$ is a direct result of $[111]$, and $[3]$ follows immediately from the inequality of arithmetic and geometric means. So all we have left is to prove $[4]$

$$ \sum_{j=1}^{n} \hat{u}_j^T \mathcal{R}_A \hat{u}_j = \sum_{j=1}^{n} \sum_{l=1}^{n} \hat{u}_j (l) \mathcal{R}_A (l, k) \hat{u}_j (k) = \sum_{j=1}^{n-1} \sum_{l=1}^{n} \hat{u}_j (l) r_{lk} \hat{u}_j (k) + \sum_{l=1}^{n} \hat{u}_n (l) r_{lk} \hat{u}_n (k) = $$

$$ \sum_{j=1}^{n-1} \frac{1}{||\mathbf{u}_j||^2} \left[ \sum_{l=1}^{j} \left( \sum_{k=1}^{l} u_{jk} (l) r_{lk} u_{jk} (k) + u_{jl} (l) r_{l,j+1} u_{jl} (j+1) \right) + \sum_{k=1}^{j} u_{j} (j+1) r_{j+1,k} u_{j} (k) + [u_{j} (j+1)]^2 \right] + \frac{1}{n} \sum_{j=1}^{n} \sum_{k=1}^{n} r_{lk} = $$

$$ \sum_{j=1}^{n-1} \frac{1}{j (j+1)} \left[ \sum_{l=1}^{j} \left( \sum_{k=1}^{l} r_{lk} - j r_{l,j+1} \right) - \sum_{k=1}^{j} j r_{j+1,k} + j^2 \right] + \frac{1}{n} \left[ \sum_{l=1}^{n} 1 + \sum_{1 \leq l < k \leq n} (r_{lk} + r_{lk}) \right] = $$

$$ \sum_{j=1}^{n-1} \frac{1}{j (j+1)} \left[ j + 2 \sum_{1 \leq l < k \leq j} Re \{r_{lk}\} - j \sum_{l=1}^{j} r_{l,j+1} - j \sum_{k=1}^{j} r_{j+1,k} + j^2 \right] + \frac{1}{n} \left[ n + 2 \sum_{1 \leq l < k \leq n} Re \{r_{lk}\} \right] = $$

$$ n - 1 + \sum_{j=1}^{n-1} \frac{1}{j (j+1)} \left[ 2 \sum_{1 \leq l < k \leq j} Re \{r_{lk}\} - 2 j \sum_{l=1}^{j} Re \{r_{l,j+1}\} \right] + \frac{2}{n} \sum_{1 \leq l < k \leq n} Re \{r_{lk}\} = $$

$$ n + \sum_{j=1}^{n-1} \frac{2}{j (j+1)} \left[ \sum_{1 \leq l < k \leq j} Re \{r_{lk}\} - j \sum_{l=1}^{j} Re \{r_{l,j+1}\} \right] + \frac{2}{n} \sum_{1 \leq l < k \leq n} Re \{r_{lk}\} = $$

$$ n + \sum_{1 \leq l < k \leq n-1} Re \{r_{lk}\} \left[ \sum_{j=k}^{n-1} \frac{2}{j (j+1)} - 2 k + \frac{2}{n} \right] + \sum_{l=1}^{n} Re \{r_{ln}\} \left( - \frac{2}{n} + \frac{2}{n} \right) = n $$

where the transitions follow by computing the coefficient of $Re \{r_{lk}\}$ for each pair $l, k$ s.t. $l < k$, where we separated the cases of $k < n$ and $k = n$. The final transition uses the sum: $\sum_{j=1}^{n} \frac{1}{j (j+1)} = \frac{n}{n+1}$.

To summarize, $[3]$ and $[4]$ show that $\prod_{j=1}^{n} \hat{u}_j^T \mathcal{R}_A \hat{u}_j \leq 1$, which when combined with $[2]$ implies:

$$ 1 \geq \prod_{j=1}^{n} |c_j \cdot \hat{u}_j|^2 = \prod_{j=1}^{n} \frac{1}{||\mathbf{u}_j||^2} \prod_{j=1}^{n} |c_j \cdot \mathbf{u}_j|^2 \quad (15) $$

Finally, substituting $[4]$ into $[15]$ gives us $[a]$. $\square$

Proof $[b]$ First, let us assume that Alice and Bob each have a qubit, and their outputs $A_i, B_j$ are results of measurements on their respective qubits. Following this assumption, we begin by computing the quantum expected value of $A_i A_k$, where $\hat{a}_i$ is a normalized vector which signifies Alice’s $i$th measurement direction, i.e., $A_i = \hat{a}_i \cdot \hat{\sigma}$. We denote Alice’s density matrix as $\mathcal{R}_A$.

$$ \langle A_i A_k \rangle = tr \{ \mathcal{R}_A A_i A_k \} = tr \{ \mathcal{R}_A (\hat{a}_i \cdot \hat{\sigma}) (\hat{a}_k \cdot \hat{\sigma}) \} = tr \{ [\hat{a}_i \cdot \hat{a}_k] \mathcal{R}_A + i \mathcal{R}_A (\hat{a}_i \times \hat{a}_k) \cdot \hat{\sigma} \} = \hat{a}_i \cdot \hat{a}_k + i \cdot tr \{ \mathcal{R}_A (\hat{a}_i \times \hat{a}_k) \cdot \hat{\sigma} \} $$

Note that $tr \{ \mathcal{R}_A (\hat{a}_i \times \hat{a}_k) \cdot \hat{\sigma} \}$ is real, meaning that it is the imaginary part of $\langle A_i A_k \rangle$. Thus, for any state, the matrix $\mathcal{R}_A$ is as follows:

$$ (\mathcal{R}_A)_{ik} = \hat{a}_i \cdot \hat{a}_k + i \cdot tr \{ \mathcal{R}_A (\hat{a}_i \times \hat{a}_k) \cdot \hat{\sigma} \} \quad (16) $$
Let us denote:

\[ \mathcal{V}_A \triangleq \begin{bmatrix} | & \hat{a}_1 & \cdots & | & \hat{a}_n \end{bmatrix} \]  

(17)

and also \( T_A \) shall be a \( n \times n \) matrix, s.t.

\[ (T_A)_{ik} \triangleq Im \{ (\mathcal{R}_A)_{ik} \} = tr [ \rho_A (\hat{a}_i \times \hat{a}_k) \cdot \sigma ] . \]

Note that \( T_A \) is anti-symmetric (this follows immediately from anti-commutativity of the vector product). From the last two definitions, we have:

\[ \mathcal{R}_A = \mathcal{V}_A^T \mathcal{V}_A + iT_A \]  

(18)

Our proof for [a] also shows that the quantum limit is reached if and only if the following two inequalities are saturated:

1. \( \prod_{j=1}^n \hat{u}_j^T \mathcal{R}_A \hat{u}_j \leq 1 \)

2. \( \forall j \in \{1, 2, \ldots, n\}, (c_j \cdot \mathcal{V}_A) \leq u_j^T \mathcal{R}_A u_j \)

Here we will show that Alice can always choose her measurement directions \( \{\hat{a}_i\} \) s.t. [1] is saturated, and Bob can always choose his measurement directions \( \{\hat{b}_j\} \) s.t. [2] is saturated.

We start by proving the following identity for every vector \( u \in \mathbb{R}^n \):

\[ u^T \mathcal{R}_A u = u^T \mathcal{V}_A^2 u + i u^T T_A u = ||\mathcal{V}_A u||^2 \]  

(19)

the last transition follows from \( u^T T_A u = 0 \), which is true for all real vectors \( u \) since \( T_A \) is anti-symmetric.

Let us show that:

\[ \exists \mathcal{V}_A : ||\mathcal{V}_A u_j||^2 = ||u_j||^2 , \forall j \in [n] \]  

(20)

For \( j \neq n \):

\[ ||\mathcal{V}_A u_j||^2 = (\mathcal{V}_A u_j) \cdot (\mathcal{V}_A u_j) = \left( \sum_{i=1}^n u_j (i) \hat{a}_i \right) \cdot \left( \sum_{k=1}^n u_j (k) \hat{a}_k \right) = \sum_{i=1}^n \sum_{k=1}^n u_j (i) u_j (k) \hat{a}_i \cdot \hat{a}_k = \]

\[ ||u_j||^2 + 2 \sum_{i=1}^j \left( \sum_{k=i+1}^j \hat{a}_i \cdot \hat{a}_k \right) - j \hat{a}_i \cdot \hat{a}_{j+1} \]

and for \( j = n \):

\[ ||\mathcal{V}_A u_n||^2 = ||u_n||^2 + 2 \sum_{i=1}^{n-1} \sum_{k=i+1}^n \hat{a}_i \cdot \hat{a}_k \]  

(21)

which implies that in order to obtain the equalities \( ||\mathcal{V}_A u_j||^2 = ||u_j||^2 \) for all \( j \in [n] \), it is sufficient to choose \( \{a_i\} \) s.t.: 

\[ \begin{cases} \forall j \in [n-1], \sum_{i=1}^j \hat{a}_i \cdot \left( \sum_{k=i+1}^j \hat{a}_k - j \hat{a}_{j+1} \right) = 0 \\ \sum_{i=1}^{n-1} \sum_{k=i+1}^n \hat{a}_i \cdot \hat{a}_k = 0 \end{cases} \]  

(22)

this can be achieved as follows:

(i) Choose \( \hat{a}_1 \) arbitrarily.

(ii) For each \( i = 2, 3, \ldots, n \), choose \( \hat{a}_i \) which is orthogonal to the sum: \( \sum_{j=1}^{i-1} \hat{a}_j \)

Let us prove that this construction satisfies (22) by induction with respect to \( j \).

**Basis** - \( j = 1 \): \( \hat{a}_1 \cdot (-\hat{a}_2) = 0 \)

**Step** - we assume the claim is satisfied for \( j \) and prove it for \( j + 1 \):
the following quantum state $|\psi\rangle$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

(23)

which is actually the known $|\beta_0\rangle$ Bell state.

Let us compute the correlations $c_{ij}$:

$$c_{ij} = \langle \psi | A_i \otimes B_j | \psi \rangle = \frac{1}{2} [0|A_i|0] [0|B_j|0] + |1|A_i|0] [1|B_j|0] + |0|A_i|1] [0|B_j|1] + |1|A_i|1] [1|B_j|1] = \frac{1}{2} tr (A_i^T B_j)$$

(24)

Where $B_j = \hat{b}_j \cdot \hat{\sigma}$. Note that the expression $\frac{1}{2} tr (A_i^T B_j)$ is similar to the Frobenius inner product; however, for complex matrices the Hermitian conjugate of the first matrix should be taken (rather than its transpose):

$$\langle A, B \rangle_F \triangleq tr (A^T B)$$

where the overline denotes (element-wise) complex conjugation. By replacing order of transposition and conjugation, we have:

$$\langle A, B \rangle_F = tr (A^T B) = tr (A^T B)$$

which implies that $c_{ij} = \frac{1}{2} \langle A_i, B_j \rangle_F$. We take a closer look at $A_i$:

$$A_i = \overline{a_i \cdot \hat{\sigma}} = \overline{a_i \cdot \hat{\sigma}}$$

since $a_i$ is a real vector. The Pauli matrices $\sigma_z, \sigma_x$ are also real, and $\sigma_y$ is purely imaginary, so $\overline{\sigma_y} = -\sigma_y$, and it follows that $A_i = R_y \hat{a}_i \cdot \hat{\sigma}$, where:

$$R_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(25)

$R_y$ is a real, orthogonal symmetric which flips the sign of a vector’s $y$ component, i.e., $R_y$ is a reflection relative to the $X - Z$ plane. Using $R_y$, we shall obtain our final expression for $c_{ij}$:

$$c_{ij} = \frac{1}{2} tr \left[ (R_y \hat{a}_i \cdot \hat{\sigma}) (\hat{b}_j \cdot \hat{\sigma}) \right] = (R_y \hat{a}_i) \cdot \hat{b}_j = \hat{a}_i \cdot \hat{b}_j$$

(26)

i.e., for $|\psi\rangle = |\beta_0\rangle$, the correlation $c_{ij}$ is simply the dot product between Alice and Bob’s respective measurement directions, with one of them reflected relative to the $X - Z$ plane. We use this in order to find an expression for the
Moreover, plugging (27) and (18) into (10) shows that:
\[ c_j \triangleq \begin{bmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{bmatrix} = \begin{bmatrix} \hat{a}_1 \cdot R_y \hat{b}_j \\ \vdots \\ \hat{a}_n \cdot R_y \hat{b}_j \end{bmatrix} = \begin{bmatrix} -\hat{a}_1 \\ \vdots \\ -\hat{a}_n \end{bmatrix} R_y \begin{bmatrix} \hat{b}_j \\ \vdots \\ \hat{b}_j \end{bmatrix} = V_A^T R_y \hat{b}_j \]

By recalling the way we constructed our Bell parameter, it follows that:
\[ B_n = \prod_{j=1}^{n} u_j \cdot c_j = \prod_{j=1}^{n} u_j \cdot V_A^T R_y \hat{b}_j = \prod_{j=1}^{n} \hat{b}_j \cdot (R_y V_A u_j) \]

which implies that for a given \( V_A \), the following choice for Bob’s measurement directions would maximize \( B_n \):
\[ \hat{b}_j = \frac{R_y V_A u_j}{\|R_y V_A u_j\|} \]

Moreover, plugging (27) and (18) into (10) shows that:
\[ V_A^T V_A + i T_A \geq c_j c_j^T = V_A^T R_y \begin{bmatrix} \hat{b}_j \hat{b}_j^T \end{bmatrix} R_y V_A \]

for all \( j \in \{1, ..., n\} \).

Multiplying (31) by \( u_j^T \) and \( u_j \) on the left and right respectively, we have:
\[ u_j^T V_A^T \left( I - R_y \begin{bmatrix} \hat{b}_j \hat{b}_j^T \end{bmatrix} R_y \right) V_A u_j \geq 0 \]

which is simply a rearrangement of (2) where (27) and (18) (for \( u = u_j \)) have been plugged in. This shows that the inequality (2) is saturated if and only if \( V_A u_j \) belongs to the kernel of \( \left( I - R_y \begin{bmatrix} \hat{b}_j \hat{b}_j^T \end{bmatrix} R_y \right) \). Since \( R_y \begin{bmatrix} \hat{b}_j \hat{b}_j^T \end{bmatrix} R_y \) is a projection into \( \text{span} \{ R_y \hat{b}_j \} \), this occurs if and only if \( V_A u_j \) is parallel to \( R_y \hat{b}_j \), which indeed is satisfied by (29). \( \square \)

IV. \( P_n \) is an harmonic function

Proof. First, let us denote: \( g_i (\vec{\mu}) \triangleq \sum_{j=1}^{i} \mu_j - i \cdot \mu_{i+1} \). Note that:
\[ P_n (\vec{\mu}) = \exp \left[ \ln (P_n (\vec{\mu})) \right] \]

Now, we find the second partial derivative of \( P_n \) relative to \( \mu_k \). (33) implies:
\[ \frac{\partial^2}{\partial \mu_k^2} P_n = \left( \frac{\partial^2}{\partial \mu_k^2} \ln P_n + \left[ \frac{\partial}{\partial \mu_k} \ln P_n \right]^2 \right) \cdot P_n \]

We shall show that \( \sum_{k=1}^{n} \frac{\partial^2}{\partial \mu_k^2} P_n (\vec{\mu}) = 0 \), by demonstrating that
\[ \sum_{k=1}^{n} \left\{ \frac{\partial^2}{\partial \mu_k^2} \ln P_n + \left[ \frac{\partial}{\partial \mu_k} \ln P_n \right]^2 \right\} = 0 \]

In order to do so, we start by finding the first partial derivatives of \( \ln P_n (\vec{\mu}) \).
\[ \frac{\partial}{\partial \mu_k} \ln P_n = \frac{1}{\sum_{i=1}^{n} \mu_i} - \frac{k-1}{g_k} + \sum_{i=1}^{n} \frac{1}{g_i} \]
and the second partial derivatives:

\[ \frac{\partial^2}{\partial \mu_k^2} \ln P_n = -\frac{1}{(\sum_{i=1}^{n} \mu_i)^2} \left( \frac{k-1}{g_{k-1}} \right)^2 - \sum_{i=1}^{n-1} \frac{1}{g_i^2} \]  

thus,

\[ \sum_{k=1}^{n} \left\{ \frac{\partial^2}{\partial \mu_k^2} \ln P_n + \left[ \frac{\partial}{\partial \mu_k} \ln P_n \right]^2 \right\} = 2 \sum_{k=1}^{n} \left\{ \frac{1}{\sum_{i=1}^{n} \mu_i} \left[ \sum_{i=k}^{n-1} \frac{1}{g_i} - \frac{k-1}{g_{k-1}} \right] + \sum_{i=k}^{n-1} \frac{1}{g_i} \left[ \sum_{j=i+1}^{n} \frac{1}{g_j} - \frac{k-1}{g_{k-1}} \right] \right\} = 0 \]

which ends our proof. \(\square\)

[1] J. Clauser, M. Horne, A. Shimony, and R. Holt, Phys. Rev. Lett 23, 880 (1969).
[2] B. S. Cirel’son, Lett. Math. Phys. 4, 93 (1980).
Bell inequalities are important tools in contrasting classical and quantum behaviors. To date, most Bell inequalities are linear combinations of statistical correlations between remote parties. Nevertheless, finding the classical and quantum mechanical (Tsirelson) bounds for a given Bell inequality in a general scenario is a difficult task which rarely leads to closed-form solutions. Here we introduce a new class of Bell inequalities based on products of correlators that alleviate these issues. Each such Bell inequality is associated with a unique coordination game. In the simplest case, Alice and Bob, each having two random variables, attempt to maximize the area of a rectangle and the rectangle’s area is represented by a certain parameter. This parameter, which is a function of the correlations between their random variables, is shown to be a Bell parameter, i.e. the achievable bound using only classical correlations is strictly smaller than the achievable bound using non-local quantum correlations. We continue by generalizing to the case in which Alice and Bob, each having now \( n \) random variables, wish to maximize a certain volume in \( n \)-dimensional space. We term this parameter a multiplicative Bell parameter and prove its Tsirelson bound. Finally, we investigate the case of local hidden variables and show that for any deterministic strategy of one of the players the Bell parameter is a harmonic function whose maximum approaches the Tsirelson bound as the number of measurement devices increases. Some theoretical and experimental implications of these results are discussed.

Bell inequalities are mathematical instruments, which enable to find out whether correlations between distant experimenters are stronger than those allowed by local hidden variable theories. In other words, a violation of some Bell inequality implies that the observed system exhibits a quantum behavior. Since their first appearance in Bell’s paper, Bell inequalities have revolutionized our understanding of quantum nonlocality in particular, and quantum theory in general. Many variations and generalizations of Bell’s original inequality have appeared.

Related research avenues in the foundations of quantum mechanics have been the search for bounds on the strength of quantum correlations, as well as finding some deeper physical reasons for these bounds. Tsirelson bounds set the maximal possible values for Bell parameters in quantum mechanics, i.e. they tell us to which extent a Bell’s inequality can be violated by a quantum mechanical system.

In a novel approach has been employed for constructing the Bell-CHSH parameter and deriving a richer Tsirelson bound for it, which depends on local uncertainty relations. Their prescription is as follows: one can begin by writing down a certain covariance matrix (encoding generalized uncertainty relations), continue by assuming that it is positive-semidefinite, and then use the sum of quadratic forms in order to infer the inequality.

In the present work we utilize a similar approach, except that we utilize a product of quadratic forms rather than a sum. The Bell parameter obtained in this procedure has a Tsirelson bound which can be readily found. We are also able to conceive a game, which describes a specific computational task that is equivalent to maximizing a certain parameter. In more detail, Alice and Bob are engaged in a two-player coordination game, where their decisions control the movement of a walker over a two-dimensional grid. The objective is to maximize the average area of the rectangle covered by the walker, which is proportional to the value of our Bell parameter. If in this game Alice and Bob share an EPR state, they may cover an average area double in size compared to the outcome of a classical strategy.

This procedure can be naturally generalized to construct Bell parameters which describe the case where Alice and Bob each have multiple measurement devices. These parameters correspond to appropriately generalized games, where the objective is to maximize the average volume of a hyperrectangle. Remarkably, as the dimensionality increases, classical and quantum strategies become asymptotically similar, with the ratio of corresponding parameters being at least \( \sqrt{\pi/2e} \).

The volume maximization game.— Two remote parties, Alice and Bob, are engaged in a game in which they attempt to maximize an area of a rectangle (see the left part of Fig. 1). The parties are assumed not to communicate by any means. A single round of the game proceeds as follows. Alice and Bob each have a distinct pair of
backward along the same direction if
a
forward in Bob’s chosen direction if
to onto Bob’s vector, and subsequently moves one step
step size as the length of projection of, say, Alice’s vec-
receiving the inputs from both players the walker sets its
his side does the same: he gets one of his vectors from the
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s
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those generated in the
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for Alice, and
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j
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vector
vector onto Bob’s vector, and subsequently moves one step
Bob’s chosen direction if
a = b, and one step
backward along the same direction if
a \neq b.
For Alice, and
1
j
A
vector
vector onto Bob’s vector, and subsequently moves one step
Bob’s chosen direction if
a = b, and one step
backward along the same direction if
a \neq b.

where \( s_{ij} (a,b) \) is the walker’s step when Alice uses her
th measuring instrument (vector) and Bob uses his
jth measuring instrument, for \( i,j \in \{ 1,2 \} \). After a number
of rounds the walker’s position expressed in Bob’s coor-
dinate system (represented by his pair of vectors) de-
fining the following rectangle: one of its vertices is at the ori-
gin, the edges that meet at the origin coincide with the
axes, and another vertex is at the walker’s position. The
goal of both parties is to maximize the average area (that
is, over many rounds) of this rectangle by choosing the
values of \( a, b \).

Suppose Alice and Bob play a total of
rounds. On
each round,
, \( i,j \) are chosen randomly, uniformly and inde-
pendently of other rounds. Let \( S_T \) be a random variable
equal to the normalized area after \( T \) rounds,


where \( s (\tau) \) is the step vector \( s_{ij} (a,b) \), where \( a, b, i, j \) are
those generated in the \( \tau \)th round. Then,


is the average normalized area, where \( c_{ij} \triangleq E[ab | i,j] \) is
the two-point correlator of Alice and Bob’s binary out-
comes when Alice uses her \( i \)th measuring instrument
(vector) and Bob uses his \( j \)th measuring instrument. Thus, we take our multiplicitive two-device Bell param-
eter to be:


While the setting of the present game is similar in spirit
to the Bell-CHSH scenario, its objective leads to a whole
new class of Bell inequalities whose Bell parameters are
average volumes of hyperrectangles. The above story de-
scribes the simplest of such games with two parties and
a two-dimensional walker.

If in this game Alice and Bob share an EPR state they
may cover an average area as twice as large than any
classical strategy would allow. In other words, the Bell
limit for \( B_2 \) is 1, and the Tsirelson limit is 2, which can be
shown using the well-known inequality of geometric and
arithmetic means together with the bounds on the (ad-
ditive) Bell-CHSH parameter \([17, 19]\). The formal proof
can be found in part I of the supplementary material.

Multiplicative Bell parameter for \( n \) possible measure-
ment choices.—The above game naturally extends to
any number of dimensions. The class of Bell inequalities
associated with such games is characterized by products
of sums of two-point correlators, which are measures of
the extent of coordination between Alice and Bob’s deci-
sions. Define


as the \( n \)-device multiplicative Bell parameter. Geometri-
cally, \(|B_n|\) is proportional to the volume of a particular
\( n \)-dimensional hyperrectangle. The construction of these
parameters is described as follows:


where \( \{ u_j | j = 1, \ldots, n \} \) is an orthogonal set of vectors,
and \( c_j \) is the vector of correlators between Alice’s mea-
surement outcomes and Bob’s \( j \)th outcome. A more de-
tailed description of this construction can be found in
part II of the supplementary material.

As an aside, we note that our multiplicitive Bell param-
eters can be associated with additive Bell parameters
(denoted by \( B_n' \)) via the following relation:


which is a result of the inequality of geometric and arith-
matic means. For \( n = 2 \), this inequality is tight, and \( B_2' \)
is the well-known Bell-CHSH parameter. More details
can be found in part I of the supplementary material.

Quantum correlations allow exceeding the classical Bell
bound only up to a certain limit known as the Tsirelson
bound \([17]\). Deriving this quantum bound is generally a

orthogonal vectors in \( \mathbb{R}^2 \),

\[
\begin{align*}
    v_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
    v_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
    u_1 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \\
    u_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\end{align*}
\]

and a (potentially random) binary variable, \( a \in \{-1,1\} \)
for Alice, and \( b \in \{-1,1\} \) for Bob. A referee gives Al-
ise one of the vectors she owns, and consequentially she
chooses the sign for \( a \) and passes both (vector and \( a \)) to a
walker that travels in the two-dimensional plane. Bob, on
his side does the same: he gets one of his vectors from the
referee, chooses \( b \) and passes both to the walker. Upon
receiving the inputs from both players the walker sets its
step size as the length of projection of, say, Alice’s vec-

The above game necessarily extends to
any number of dimensions. The class of Bell inequalities
associated with such games is characterized by products
of sums of two-point correlators, which are measures of
the extent of coordination between Alice and Bob’s deci-
sions. Define

\[
B_n \triangleq (c_{1n} + \cdots + c_{nn}) \prod_{j=1}^{n-1} (c_{jj} + \cdots + c_{jj} - jc_{j+1,j}),
\]

as the \( n \)-device multiplicitive Bell parameter. Geometri-
cally, \(|B_n|\) is proportional to the volume of a particular
\( n \)-dimensional hyperrectangle. The construction of these
parameters is described as follows:

\[
B_n = \prod_{j=1}^{n} u_j \cdot c_j,
\]

where \( \{ u_j | j = 1, \ldots, n \} \) is an orthogonal set of vectors,
and \( c_j \) is the vector of correlators between Alice’s mea-
surement outcomes and Bob’s \( j \)th outcome. A more de-
tailed description of this construction can be found in
part II of the supplementary material.

As an aside, we note that our multiplicitive Bell param-
eters can be associated with additive Bell parameters
(denoted by \( B_n' \)) via the following relation:

\[
|B_n| \leq \left( \frac{B_n'}{n} \right)^n,
\]

which is a result of the inequality of geometric and arith-
matic means. For \( n = 2 \), this inequality is tight, and \( B_2' \)
is the well-known Bell-CHSH parameter. More details
can be found in part I of the supplementary material.

Quantum correlations allow exceeding the classical Bell
bound only up to a certain limit known as the Tsirelson
bound \([17]\). Deriving this quantum bound is generally a

difficult problem \footnote{1}. However, in our case we have the following elegant closed-form expression for the Tsirelson bound, which can be computed efficiently:

\textbf{Main result.}— The Tsirelson bound on the multiplicative $n$-device Bell parameter is $n$ factorial,

$$|B_n| \leq n!.$$  (9)

The bound follows from the positive semi-definiteness of a certain second moment matrix, which, by Schur’s complement, yields the following matrix inequality:

$$R_A \succeq c^T c.$$  

This inequality means that the difference between $R_A$ (Alice’s local uncertainty matrix \footnote{1}) and the outer product of the correlations vector with itself is a semidefinite positive matrix. A detailed proof, including the players’ strategies which saturate this bound, appears in part \textit{III} of the supplementary material.

\textbf{Bell limit.}— The Bell limit is the maximal value of a Bell parameter in local hidden variables theories. Generally, this classical bound cannot be computed efficiently for ordinary (additive) Bell inequalities \footnote{1}. Let us examine \footnote{2} while assuming there exists a joint probability distribution for $a_i, b_j$:

$$B_n = E \left[ b_n \sum_{k=1}^{n} a_k \prod_{j=1}^{n-1} E \left[ b_j \left( \sum_{k=1}^{j} a_k - j a_{j+1} \right) \right] \right],$$  (10)

where $a_i, b_j$ are random variables corresponding to the values of $a, b$ when Alice and Bob’s inputs are $i, j$ respectively. We suspect that finding a tight bound on \footnote{2} is also difficult. Therefore, we examine the special case where Bob adopts a deterministic strategy. In this case, $B_n$ is an $n$-variable function of $\mu_i$,

$$\mathcal{P}_n (\mu_1, \ldots, \mu_n) \triangleq \left( \sum_{i=1}^{n} \mu_i \right) \cdot \prod_{j=1}^{n-1} \left( \sum_{i=1}^{j} \mu_i - j \cdot \mu_{j+1} \right),$$  (11)

where the local statistics at Alice’s site is represented by the one-point correlators, $\mu_i \triangleq E[a | i], |\mu_i| \leq 1$.

As it turns out, $\mathcal{P}_n$ is an $n$-variable harmonic function. This means that the Laplacian of $\mathcal{P}_n$ vanishes, which implies that its maximum is achieved on the faces of the $n$-dimensional hypercube. The proof appears in part \textit{IV} of the supplementary material. Unfortunately, we have not succeeded in establishing an efficient way to find the maximum of $\mathcal{P}_n$.

Consider the strategy where Alice and Bob’s choices are not only independent but also deterministic. In this case the values achieved by $\mathcal{P}_n$ constitute a special corner of the $n$-dimensional hypercube:

$$\mu_i = \begin{cases} (-1)^i & i \leq i_c \\ 1 & i > i_c \end{cases}$$  (12)

i.e., Bob always chooses 1, and Alice alternates between $\pm 1$ until the index reaches some integer cutoff. For indexes larger than this cutoff, Alice always chooses 1. We shall choose $i_c$ as the greatest even number which is no
larger than \( n - \sqrt{n} \).

We denote the value of \( P_n \) achieved by this strategy as \( FD_n \). Since it is a special case of the previous subsection, it is clear that:

\[
FD_n \leq \max P_n (\bar{\mu}) \leq \text{Bell limit} \leq n!
\]  

(13)

Let us write down an analytic expression for \( FD_n \). Plugging (12) into (11), shows that:

\[
FD_n = (2 \cdot 2 \cdot 4 \cdot 4 \cdot i_c \cdot i_c) \cdot \frac{i_c \cdots i_c}{n-i_c} \cdot (n-i_c)
\]

(14)

i.e., \( FD_n \) is a product of all the even numbers smaller or equal to the cutoff squared, multiplied by the cutoff value \((n-i_c-1)\) times, multiplied by the difference between \(n\) and the cutoff value. It can be shown that (14) is equivalent to:

\[
FD_n = 2^{i_c} \cdot \left[ \left( \frac{i_c}{2} \right) ! \right]^2 \cdot i_c^{n-i_c-1} \cdot (n-i_c).
\]

(15)

As the number of measuring devices grows indefinitely, \( n \to \infty \), the ratio between the Bell limit and Tsirelson limit is at least \( \sqrt{\frac{2}{2e}} \). This result follows from (13) upon noting that

\[
\lim_{n \to \infty} \frac{FD_n}{n!} = 2^{n - \sqrt{n} \cdot \left( \frac{n-\sqrt{n}}{2} \right)!} \cdot (n-\sqrt{n})^{n-1} \cdot \sqrt{n}
\]

\[
\lim_{n \to \infty} \frac{2^{n - \sqrt{n} \cdot \left( \frac{n-\sqrt{n}}{2} \right)!}}{n!} \cdot (n-\sqrt{n})^{n-1} \cdot \sqrt{n} = \sqrt{\frac{\pi}{2e}}
\]

(16)

This ratio is plotted in Fig. 2 for values of \( n \) up to 255.

**Theoretical and experimental significance.**— Linear Bell inequalities are suitable for analyzing local hidden variables, since the latter are described by convex sets created by intersection of linear constraints. However, the quantum set is not a polytope and therefore it can be insightful to study its structure with nonlinear bounds like the one suggested here.

Moreover, the multiplicative bipartite Bell parameter alleviates the detection loophole in actual Bell experiments. To see how, recall that in a Bell experiment with photon pairs the detector efficiency, \( \eta \), represents the fraction of incoming photons registered on the average by the detector. A detector with \( \eta = 1 \) is perfect but most of the actual detectors has \( \eta \) strictly less than 1. The detection efficiency influences the classical bound on the Bell-CHSH parameter, \( |B'_{2}| \leq 4/\eta - 2 \ [20] \). So in actual experiments the Bell bound is always greater than 2, and \( B'_{2} \) becomes virtually ineffective for \( \eta = 4/(2\sqrt{2} + 2) \approx 0.83 \), since it can no longer discern quantum behaviors. The robustness of \( B'_{2} \) to the detection loophole may be quantified as, \( \Delta' = 2\sqrt{2} - (4/\eta - 2) \), the interval allotted for quantum violations of the Bell-CHSH inequality. As it turns out, the multiplicative parameter \( B_{2} \) exhibits a greater robustness for all values of \( \eta \) from 0.83 to 1. In particular, according to (11) the parameter \( |B_{2}| \leq (2/\eta - 1)^2 \) and therefore its robustness is \( \Delta = 2 - (2/\eta - 1)^2 \), which as shown in Figure 3 is always greater than \( \Delta' \) for all values of \( \eta \in (0.83, 1) \).

**Simulation.**— We ran simulations of the volume maximization game for \( n = 2, 3 \) (see Fig. 4). For the classical strategy in \( n = 2 \), we used: \( a_1 = b_1 = b_2 = 1, E(a_2) = 0 \) (optimality of this strategy is proven in the supplementary material - see part I). For the classical strategy \( n = 3 \), we used the fully-deterministic strategy described in the former paragraph. For the quantum
strategies in both $n = 2, 3$, we used the “winning” strategies described in the proof of our main result.

![Diagram showing random paths in two and three dimensions.](image)

FIG. 4. The walker’s random paths in two and three dimensions (corresponding to the number of measuring instruments). The orange paths correspond to classical strategies, while the cyan paths correspond to the “winning” quantum strategies. Clearly, the statistics of the quantum paths differ from the classical ones. This premise alludes to a test of quantumness where the covariance or some other statistic is empirically evaluated over the paths generated in different trials of the volume maximization game.

**Conclusions.** — We have explored a new coordination game which favors quantum players. This has allowed us to find a new class of multiplicative Bell inequalities, as well as their corresponding Tsirelson bounds. It was shown that in quantum mechanics any $n$-device multiplicative Bell parameter is bounded by the volume of an $n$-dimensional ellipsoid representing the local uncertainty associated with the system of one of the players. Contrary to our intuition, as $n$ goes to infinity, the performances of classical and quantum players become comparable.

Some future generalizations of our game and associated parameter may include: multiple players; allowing non-binary measurement outcomes, both discrete and continuous; using a different set of vectors $u_j$ for the construction, possibly a non-orthogonal one. Another research direction might include a deeper investigation of the associated additive Bell parameters $B'_n$, and their relation to the multiplicative ones. Finally, stronger-than-quantum correlations within post-quantum models may be analyzed using the current approach.

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