SOME RESULTS ON COMPLEX $m$–SUBHARMONIC CLASSES

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Abstract. In this paper we study the class $E_m(\Omega)$ of $m$–subharmonic functions introduced by Lu in [8]. We prove that the convergence in $m$–capacity implies the convergence of the associated Hessian measure for functions that belong to $E_m(\Omega)$. Then we extend those results to the class $E_{m,\chi}(\Omega)$ that depends on a given increasing real function $\chi$. A complete characterization of those classes using the Hessian measure is given as well as a subextension theorem relative to $E_{m,\chi}(\Omega)$.

1. Introduction

In complex analysis, the Monge-Ampere operator represents the objective of several studies since Bedford and Taylor [1, 2] demonstrated that the operator $(dd^c)^n$ is well defined on the set of locally bounded plurisubharmonic (psh) functions defined on an hyperconvex domain $\Omega$ of $\mathbb{C}^n$. This domain was extended by Cegrell [12, 13] by introducing and investigating the classes $E_0(\Omega)$, $F(\Omega)$ and $E(\Omega)$ that contain unbounded psh functions. He proved that $E(\Omega)$ is the largest domain of definition of the complex Monge-Ampere operator if we want the operator to be continuous for decreasing sequences. These works were taken up by Lu [8, 9] to define the complex Hessian operator $H_m$ on the set of $m$–subharmonic functions which coincides with the set of psh functions in the case $m = n$. By giving an analogy to Cegrell’s classes, Lu studied some analogous classes denoted by $E^0_m(\Omega)$, $F_m(\Omega)$ and $E_m(\Omega)$. One of the most well-known problems in this direction is the link between the convergence in capacity $Cap_m$ and the convergence of the complex Hessian operator. The paper is organized as follows: In section 2 we recall some preliminaries on the pluripotential theory for $m$–subharmonic function as well as the different energy classes which will be studied throughout the paper.

In section 3 we will be interested on giving a connection between the convergence in capacity $Cap_m$ of a sequence of $m$-subharmonic functions $f_j$ toward $f$, $\liminf_j f_j H_m(f_j)$ and $H_m(f)$ when the function $f \in E_m(\Omega)$. More precisely we prove the following theorem

Theorem A.

If $(f_j)_j$ is a sequence of $m$–subharmonic function that belong to $E_m(\Omega)$ and satisfies $f_j \to f \in E_m(\Omega)$ in $Cap_m$-capacity. Then

$$1\{f \geq -\infty\} H_m(f) \leq \liminf_{j \to +\infty} H_m(f_j).$$

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As a consequence of Theorem A we obtain several results of convergence and especially we prove that if we modify the sufficient condition in the previous theorem, one may obtain the weak convergence of $H_m(f_j)$ to $H_m(f)$.

In Section 4, we will study the classes $E_{m,\chi}(\Omega)$ introduced by Hung [16] for a given increasing function $\chi$. Those classes generalized the weighted pluricomplex energy classes investigated by Benelkourchi, Guedj and Zeriahi [4] and studied by [3, 5, 17]. We prove first the class $E_{m,\chi}(\Omega)$ is fully included in the Cegrell class $E_m(\Omega)$ and hence the Hessian operator $H_m(f)$ is well defined for every $f \in E_{m,\chi}(\Omega)$. Then we will be interested on giving several results of the class $E_{m,\chi}(\Omega)$ depending on some condition on the function $\chi$. Those results generalizes well known works in [3] and [4] it suffices to take $m = n$ to recover them. The most important result that we prove in this context is the given of a complete characterization for functions that belong to $E_{m,\chi}(\Omega)$ using the class $N_m(\Omega)$. In other words we show that

$$E_{m,\chi}(\Omega) = \{ f \in N_m(\Omega) / \chi(f) \in L^1(H_m(f)) \}.$$  

In the end we extend Theorem A to the class $E_{m,\chi}(\Omega)$ by proofing the following result

**Theorem B.**

Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a continuous increasing function such that $\chi(-\infty) > -\infty$ and $f, f_j \in E_m(\Omega)$ for all $j \in \mathbb{N}$. Suppose that there is a function $g \in E_m(\Omega)$ satisfying $f_j \geq g$ then:

1. If $f_j$ converges to $f$ in $\text{Cap}_{m-1}$-capacity then $\liminf_{j \to +\infty} -\chi(f_j)H_m(f_j) \geq -\chi(f)H_m(f)$.

2. If $f_j$ converges to $f$ in $\text{Cap}_m$-capacity then $-\chi(f_j)H_m(f_j)$ converges weakly to $-\chi(f)H_m(f)$.

### 2. Preliminaries

#### 2.1. m-subharmonic functions.

This section is devoted to recall some basic properties of $m$–subharmonic functions introduced by Blocki [11]. Those functions are admissible for the complex Hessian equation. Throughout this paper we denote by $d := \partial + \bar{\partial}$, $dx := i(\bar{\partial} - \partial)$ and by $\Lambda_p(\Omega)$ the set of $(p, p)$–forms in $\Omega$. The standard Kähler form defined on $\mathbb{C}^n$ will be denoted as $\beta := dd^c|z|^2$.

**Definition 2.1.** [11]

Let $\zeta \in \Lambda_1(\Omega)$ and $m \in \mathbb{N}\cap[1, n]$. The form $\zeta$ is called $m$–positive if it satisfies

$$\zeta^j \wedge \beta^{n-j} \geq 0, \quad \forall j = 1, \cdots, m$$

at every point of $\Omega$.

**Definition 2.2.** [11]

Let $\zeta \in \Lambda_p(\Omega)$ and $m \in \mathbb{N}\cap[p, n]$. The $\zeta$ is said to be $m$–positive on $\Omega$ if and only if the measure

$$\zeta \wedge \beta^{m-\psi_1} \wedge \cdots \wedge \psi_{m-p}$$

is positive at every point of $\Omega$ where $\psi_1, \cdots, \psi_{m-p} \in \Lambda_1(\Omega)$.
We will denote by $\Lambda_p^m(\Omega)$ the set of all $(p,p)-$forms on $\Omega$ that are $m-$positive. In 2005, Blocki [11] introduced the notion of $m-$subharmonic functions and developed an analogous pluripotential theory. This notion is given in the following definition:

**Definition 2.3.** Let $f : \Omega \to \mathbb{R} \cup \{-\infty\}$. The function $f$ is called $m-$subharmonic if it satisfies the following:

1. The function $f$ is subharmonic.
2. For all $\zeta_1, \cdots, \zeta_{m-1} \in \Lambda_1^m(\Omega)$ one has
   $$dd^c f \wedge \beta^{n-m} \wedge \zeta_1 \wedge \cdots \wedge \zeta_{m-1} \geq 0$$

We denote by $SH_m(\Omega)$ the cone of $m-$subharmonic functions defined on $\Omega$. 

**Remark 2.4.** In the case $m = n$ we have the following

1. The definition of $m-$positivity coincides with the classic definition of positivity given by Lelong for forms.
2. The set $SH_n(\Omega)$ coincides with the set of psh functions on $\Omega$.

One can refer to [11], [19], [6] and [8] for more details about the properties of $m-$subharmonicity.

**Example 2.5.**
1. If $\zeta := i(4dz_1 \wedge d\overline{z}_1 + 4dz_2 \wedge d\overline{z}_2 - dz_3 \wedge d\overline{z}_3)$ then $\zeta \in \Lambda_1^2(\mathbb{C}^3) \setminus \Lambda_3^1(\mathbb{C}^3)$.
2. If $f(z) := -|z_1|^2 + 2|z_2|^2 + 2|z_3|$ then $f \in SH_2(\mathbb{C}^3) \setminus SH_3(\mathbb{C}^3)$. It is easy to see that $f \in SH_2$. However, the restriction of $f$ on the line $(z_1,0,0)$ is not subharmonic so $f$ is not a plurisubharmonic.

Following Bedford and Taylor [2], one can define, by induction a closed non-negative current when the function $f$ is $m-$sh functions and locally bounded as follows:

$$dd^c f_1 \wedge \cdots \wedge dd^c f_k \wedge \beta^{n-m} := dd^c(f_1 dd^c f_2 \wedge \cdots \wedge dd^c f_k \wedge \beta^{n-m}),$$

where $f_1,\ldots, f_k \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$. In particular, for a given $m-$sh function $f \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$, we define the nonnegative Hessian measure of $f$ as follows

$$H_m(f) = (dd^c f)^m \wedge \beta^{n-m}.$$ 

### 2.2. Cegrell classes of $m-$sh functions and $m-$capacity.

**Definition 2.6.**
1. A bounded domain $\Omega$ in $\mathbb{C}^n$ is said to be $m-$hyperconvex if the following property holds for some continuous $m-$sh function $\rho : \Omega \to \mathbb{R}^-$:

   $$\{ \rho < c \} \in \Omega,$$

for every $c < 0$.

2. A set $M \subset \Omega$ is called $m-$polar if there exist $u \in SH_m(\Omega)$ such that

   $$M \subset \{ u = -\infty \}.$$

Throughout the rest of the paper, we denote by $\Omega$ a $m-$hyperconvex domain of $\mathbb{C}^n$. In [8] and [9], Lu introduced the following classes of $m-$sh functions to generalize Cegrell’s classes. We recall below the definitions of those classes.
Definition 2.7. We denote by:
\[ \mathcal{E}^0_m(\Omega) = \{ f \in \mathcal{SH}_m(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \xi} f(z) = 0 \ \forall \xi \in \partial \Omega, \int_\Omega H_m(f) < +\infty \}, \]
\[ \mathcal{F}_m(\Omega) = \{ f \in \mathcal{SH}_m(\Omega) : \exists (f_j) \subset \mathcal{E}^0_m, f_j \searrow f \text{ in } \Omega \sup_j \int_\Omega H_m(f_j) < +\infty \}. \]

and
\[ \mathcal{E}_m(\Omega) = \{ f \in \mathcal{SH}_m(\Omega) : \forall U \Subset \Omega, \exists f_U \in \mathcal{F}_m(\Omega); f_U = f \text{ on } U \}. \]

Definition 2.8. A function \( f \in \mathcal{SH}_m(\Omega) \) is said to be \( m \)-maximal if for every \( g \in \mathcal{SH}_m(\Omega) \) such that \( g \leq f \) outside a compact subset of \( \Omega \) then \( g \leq f \) in \( \Omega \).

The previous notion represents an essential tool in the study of the Hessian operator since Blocki \[11\] showed that every \( m \)-maximal function \( f \in \mathcal{E}_m(\Omega) \) satisfies \( H_m(f) = 0 \). Take \((\Omega_j)_j\), a sequence of strictly \( m \)-pseudoconvex subsets of \( \Omega \) such that \( \Omega_j \Subset \Omega_{j+1}, \bigcup_{j=1}^\infty \Omega_j = \Omega \) and for every \( j \) there exists a smooth strictly \( m \)-subharmonic function \( \varphi \) in a neighborhood \( V \) of \( \Omega_j \) such that \( \Omega_j := \{ z \in V/\varphi(z) < 0 \} \).

Definition 2.9. Let \( f \in \mathcal{SH}_m(\Omega) \) and \((\Omega_j)_j\) be the sequence defined above. Take \( f^j \) the function defined by:
\[ f^j = \sup \{ \psi \in \mathcal{SH}_m(\Omega) : \psi|_{\Omega_j} \leq f \} \in \mathcal{SH}_m(\Omega), \]
and define \( \tilde{f} := (\lim_{j \to +\infty} f^j)^* \), called the smallest maximal \( m \)-subharmonic function majorant of \( f \).

It is clear that \( f \leq f^j \leq f^{j+1} \), so \( \lim_{j \to +\infty} f^j \) exists on \( \Omega \) except at an \( m \)-polar set, we deduce that \( \tilde{f} \in \mathcal{SH}_m(\Omega) \). Moreover, if \( f \in \mathcal{E}_m(\Omega) \) then by [9] and [11] \( \tilde{f} \in \mathcal{E}_m(\Omega) \) and it is \( m \)-maximal on \( \Omega \). We denote \( \mathcal{MSH}_m(\Omega) \) the family of \( m \)-maximal functions in \( \mathcal{SH}_m(\Omega) \).

We cite below some useful properties of \( \mathcal{MSH}_m(\Omega) \).

Proposition 2.10. \[11\] Let \( f, g \in \mathcal{E}_m(\Omega) \) and \( \alpha \in \mathbb{R}, \alpha \geq 0 \), then we have

1. \( \tilde{f} + g \geq \tilde{f} + \tilde{g} \).
2. \( \alpha \tilde{f} = \tilde{\alpha f} \).
3. If \( f \leq g \) then \( \tilde{f} \leq \tilde{g} \).
4. \( \mathcal{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega) = \{ f \in \mathcal{E}_m : \tilde{f} = f \} \).

In [20], author introduced a new Cegrell class \( \mathcal{N}_m(\Omega) := \{ f \in \mathcal{E}_m : \tilde{f} = 0 \} \).

It is easy to check that \( \mathcal{N}_m(\Omega) \) is a convex cone satisfying
\[ \mathcal{E}^0_m(\Omega) \subset \mathcal{F}_m(\Omega) \subset \mathcal{N}_m(\Omega) \subset \mathcal{E}_m(\Omega). \]

Definition 2.11. Let \( \mathcal{L}_m \in \{ \mathcal{F}_m, \mathcal{N}_m, \mathcal{E}_m \} \). We define
\[ \mathcal{L}^0_m(\Omega) := \{ f \in \mathcal{L}_m : H_m(f)(P) = 0, \forall P \text{ } m \text{-polar set} \}. \]

Definition 2.12. (1) Let \( E \) be a Borel subset of \( \Omega \). The \( \text{Cap}_s \)-capacity of a \( E \) with respect to \( \Omega \) is given as follows:
\[ \text{Cap}_s(E) = \text{Cap}_s(E, \Omega) = \sup \left\{ \int_E H_s(f) : f \in \mathcal{SH}_m(\Omega), -1 \leq f \leq 0 \right\} \]

where \( 1 \leq s \leq m \).
Proposition 3.1. \( \text{(See where)} \) For every non-negative measures \( \nu \), \( \nu = \mu + \nu \), satisfying \( \langle \mu + \nu \rangle(\Omega) < \infty \) and \( \int_{\Omega} f \, d\mu \geq \int_{\Omega} f \, d\nu \) for all \( f \in \mathcal{E}^a_m(\Omega) \), one has \( \mu(K) \geq \nu(K) \) for all complete \( m \)-polar subsets \( K \) in \( \Omega \).

Remark 2.13. For a given subset \( E \subset \Omega \), the outer \( s \)-capacity \( \text{Cap}^*_s \) of \( E \) is defined as

\[ \text{Cap}^*_s(E, \Omega) := \inf \{ \text{Cap}_s(F, \Omega) ; \ E \subset F \text{ and } F \text{ is an open subset of } \Omega \} . \]

3. Convergence in \( \text{Cap}_m \)-Capacity

Proposition 3.2. \( \text{(See [6] and [7])} \)

1. For every \( f, g \in \mathcal{E}_m(\Omega) \), such that \( g \leq f \) one has

\[ 1_{\{f = -\infty\}} H_m(f) \leq 1_{\{g = -\infty\}} H_m(g) \]

2. If \( f \in \mathcal{E}_m(\Omega) \), and \( g \in \mathcal{E}^a_m(\Omega) \) then

\[ 1_{\{f + g = -\infty\}} H_m(f + g) \leq 1_{\{f = -\infty\}} H_m(f) \]

Proposition 3.3. \( \text{(1) If } f \in \mathcal{SH}^{-}_m(\Omega), g \in \mathcal{P}_m(\Omega) \text{ and } f \geq g \text{ then } f \in \mathcal{P}_m(\Omega) \).

(2) If \( f, g \in \mathcal{P}_m(\Omega) \) then \( f + g \in \mathcal{P}_m(\Omega) \).
Proof. (1) Since \( f \in \mathcal{E}_m(\Omega) \) so is \( g \). Now assume that there exists \( P_1, \ldots, P_n \) polar in \( \mathbb{C} \) such that \( 1_{(g=-\infty)}H_m(g)(\Omega \setminus P_1 \times \ldots \times P_n) = 0 \). Then by proposition 3.1, we deduce that
\[
1_{(f=-\infty)}H_m(f)(\Omega \setminus P_1 \times \ldots \times P_n) = 0.
\]
It follows that \( f \in \mathcal{P}_m(\Omega) \). The proof of the first assertion is completed.

(2) Using [9], the set \( \mathcal{E}_m(\Omega) \) is a convex cone. Hence if \( f, g \in \mathcal{E}_m(\Omega) \) so is \( f + g \).

Take \( P_1, \ldots, P_n \) polar in \( \mathbb{C} \) such that \( 1_{(g=-\infty)}H_m(g)(\Omega \setminus P_1 \times \ldots \times P_n) = 0 \). We have
\[
H_m(f + g) = \sum_{k=0}^{m} \binom{m}{k} (dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m}.
\]
If we fix \( k \in \{0, \ldots, m\} \) then by lemma 1 in [17] we obtain the following writing
\[
(dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m} = \mu + 1_{(f=g=-\infty)}(dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m}
\]
where \( \mu \) is a measure that has no mass on \( m \)-polar sets. We deduce that
\[
1_{(f+g=-\infty)}H_m(f + g) = \sum_{k=0}^{m} \binom{m}{k} 1_{(f=g=-\infty)}(dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m}.
\]
It follows by Lemma 5.6 in [6] that
\[
\int_{\Omega \setminus (P_1 \times \ldots \times P_n)} 1_{(f+g=-\infty)}H_m(f + g)
\]
\[
= \sum_{k=0}^{m} \binom{m}{k} \int_{\Omega \setminus (P_1 \times \ldots \times P_n)} 1_{(f=g=-\infty)}(dd^c f)^k \wedge (dd^c g)^{m-k} \wedge \beta^{n-m}
\]
\[
\leq 2^m \left( \int_{\Omega \setminus (P_1 \times \ldots \times P_n) \cap \{f=g=-\infty\}} H_m(f) \right)^{\frac{1}{m}} \cdot \left( \int_{\Omega \setminus (P_1 \times \ldots \times P_n) \cap \{f=g=-\infty\}} H_m(g) \right)^{\frac{1}{m}}
\]
\[
= 0.
\]
We conclude that \( f + g \in \mathcal{P}_m(\Omega) \). \( \square \)

The following theorem represents the first main result in this paper.

**Theorem 3.4.** If \( f_j \) is a sequence of \( m \)-subharmonic function that belong to \( \mathcal{E}_m(\Omega) \) and satisfies \( f_j \to f \in \mathcal{E}_m(\Omega) \) in \( \Cap_m \)-capacity. Then
\[
1_{(f_j=-\infty)}H_m(f_j) \leq \liminf_{j \to +\infty} H_m(f_j).
\]

**Proof.** Take \( 0 \leq \varphi \in C_{0}^{0}(\Omega) \) and \( \Omega_1 \Subset \Omega \) such that \( \text{supp} f \Subset \Omega_1 \), it suffices to show that
\[
\liminf_{j \to +\infty} \int_{\Omega} \varphi H_m(f_j) \geq \int_{\Omega} 1_{(f=-\infty)} \varphi H_m(f).
\]
For each \( a > 0 \) one has that
\[
\int_{\Omega} \varphi H_m(f_j) - \int_{\Omega} 1_{(f=-\infty)} \varphi H_m(f) = A_1 + A_2 + A_3,
\]
where
\[
A_1 = \int_{\Omega} \varphi (H_m(f_j) - H_m(\max(f_j, -a))) + \int_{\Omega} 1_{(f=-\infty)} \varphi H_m(f)
\]
\[
A_2 = \int_{\Omega} \varphi (H_m(\max(f_j, -a)) - H_m(f_j))
\]
\[
A_3 = \int_{\Omega} \varphi (H_m(\max(f, -a)) - H_m(f)).
\]
Using Theorem 3.6 in [6] we obtain that
\[ \int \{ f_j \leq -a \} \varphi(H_m(f_j) - H_m(\max(f_j, -a))) + \int \Omega 1_{\{ f = -\infty \}} \varphi H_m(f) \]
\[ \geq - \int \{ f_j \leq -a \} \varphi H_m(\max(f_j, -a)) + \int \Omega 1_{\{ f = -\infty \}} \varphi H_m(f) \]
\[ \geq - \int_{\{ f_j \leq -a \} \cap \{(f_j - f) \leq 1\}} \varphi H_m(\max(f_j, -a)) - \int_{\{ (f_j - f) > 1 \}} \varphi H_m(\max(f_j, -a)) + \int \Omega 1_{\{ f = -\infty \}} \varphi H_m(f) \]
\[ \geq - \int_{\{ f < -a + 2 \} \cap \Omega} \varphi H_m(\max(f_j, -a)) - a^n \text{Cap}_m(\{ |f_j - f| > 1 \} \cap \Omega) + \int \Omega 1_{\{ f = -\infty \}} \varphi H_m(f). \]

If we let \( j \to +\infty \) then by Theorem 3.8 in [6] we obtain
\[ \liminf_{j \to +\infty} A_1 \geq \int \Omega h_{\{ f < -a + 2 \} \cap \Omega} \varphi H_m(\max(f_j, -a)) + \int \Omega 1_{\{ f = -\infty \}} \varphi H_m(f). \]

It follows by Theorem 3.8 in [6] that for all \( s > 0 \) one has
\[ \liminf_{a \to +\infty} \liminf_{j \to +\infty} A_1 \geq \liminf_{a \to +\infty} \liminf_{j \to +\infty} h_{\{ f < -s \} \cap \Omega} \varphi H_m(\max(f_j, -a)) + \int \Omega 1_{\{ f = -\infty \}} \varphi H_m(f) \]
\[ \geq \liminf_{a \to +\infty} \liminf_{j \to +\infty} h_{\{ f < -s \} \cap \Omega} \varphi H_m(\max(f_j, -a)) + \int \Omega 1_{\{ f = -\infty \}} \varphi H_m(f) \]
\[ = \int \Omega h_{\{ f < -s \} \cap \Omega} \varphi H_m(f) + \int \Omega 1_{\{ f = -\infty \}} \varphi H_m(f). \]

Since \( \lim_{s \to +\infty} \text{Cap}_m(\{ f < -s \} \cap \Omega) = 0 \) then there exists a subset \( A \) of \( \Omega \) with \( \text{Cap}_m(A) = 0 \) such that the function \( h_{\{ f < -s \} \cap \Omega} \) increases to 0 as \( s \to +\infty \) on \( \Omega \setminus A \). Now by a decomposition theorem in [9] we get that if \( s \to +\infty \)
\[ \liminf_{a \to +\infty} \liminf_{j \to +\infty} A_1 \geq \int \Omega -1_E \varphi H_m(f) + \int \Omega 1_{\{ f = -\infty \}} \varphi H_m(f) \geq 0. \]

It follows by Theorem 3.8 in [6] that
\[ \liminf_{j \to +\infty} \left( \int \Omega \varphi H_m(f_j) - \int \Omega 1_{\{ f > -\infty \}} \varphi H_m(f) \right) \]
\[ \geq \liminf_{a \to +\infty} \liminf_{j \to +\infty} A_1 + \liminf_{a \to +\infty} A_3 \geq 0. \]

\[ \square \]

**Corollary 3.5.** Let \((f_j)_j \subset \mathcal{E}_m(\Omega)\) such that \( f_j \to f \in \mathcal{E}_m(\Omega) \) in \( \text{Cap}_m \)-capacity. If \((f_j, f) \in \mathcal{Q}_m(\Omega)\) for all \( j \geq 1 \). Then
\[ H_m(f) \leq \liminf_{j \to +\infty} H_m(f_j). \]
Now let \( \Omega \), by Theorem 3.4 we obtain that

\[
1_{\{f=\infty\}} H_m(f) \leq 1_{\{f_j=\infty\}} H_m(f_j) \leq H_m(f_j).
\]

The result follows using Theorem 3.4. \( \square \)

**Corollary 3.6.** Let \((f_j)_j \subset F_m(\Omega)\) such that \(f_j \to f \in F_m(\Omega)\) in \(C_{m}\)-capacity. If \((f_j, f) \in Q_m(\Omega)\) for all \(j \geq 1\), and

\[
\lim_{j \to +\infty} \int_{\Omega} H_m(f_j) = \int_{\Omega} H_m(f).
\]

Then \(H_m(f_j) \to H_m(f)\) weakly as \(j \to +\infty\).

Proof. Without loss of generality one can assume that \(H_m(f_j) \to \mu\) weakly as \(j \to +\infty\). Using Corollary 3.3 we obtain that \(\mu(\Omega) \leq \liminf_{j \to +\infty} \int_{\Omega} H_m(f_j) = \int_{\Omega} H_m(f)\). Hence without loss of generality one can assume that there exists a positive measure \(\mu\) such that \(H_m(f_j) \to \mu\) weakly as \(j \to +\infty\). The proof will be completed if we show that \(\mu = H_m(f)\) on \(\Omega_1\). For this take \(u \in E_m^0(\Omega_1)\), then by Stokes’ theorem we obtain that

\[
\int_{\Omega_1} -u d\mu = \lim_{j \to +\infty} \int_{\Omega_1} -u H_m(f_j) \geq \lim_{j \to +\infty} \int_{\Omega_1} -u H_m(f) = \lim_{j \to +\infty} \int_{\Omega_1} -u H_m(f_j).
\]

Moreover by Proposition 3.2 and [15] we get

\[
H_m(f)(K) \leq \mu(K). \quad (*)
\]

for all compact subsets \(K\) of \(E_1, \ldots, E_n\). We deduce that \(\mu \geq 1_{\{f=\infty\}} H_m(f)\). So by Theorem 3.4 we obtain

\[
H_m(f) \leq \mu \text{ on } \Omega_1.
\]

Now let \(\Omega_2\) be a domain satisfying \(D \subset \Omega_2 \subset \Omega_1\). By Stokes theorem we obtain that

\[
H_m(f) \leq \mu \text{ on } \Omega_1.
\]
\[ \mu(\Omega_2) \leq \liminf_{j \to +\infty} \int_{\Omega_2} H_m(f_j) = \liminf_{j \to +\infty} \int_{\Omega_2} H_m(\tilde{f}_j) \leq \int_{\Omega_2} H_m(\tilde{f}) \leq \int_{\Omega_1} H_m(\tilde{f}) = \int_{\Omega_1} H_m(f). \]

It follows that
\[ \mu(\Omega_1) \leq H_m(f)(\Omega_1). \tag{**} \]

Using (\*) and (**) we deduce that \( \mu = H_m(f) \) on \( \Omega_1 \). \( \Box \)

The following lemma will be useful in the proof of several results in this paper.

Lemma 3.8. Fix \( f \in F_m(\Omega) \). Then for all \( s > 0 \) and \( t > 0 \), one has
\[ t^m \text{Cap}_m(f < -s - t) \leq \int_{\{f < -s\}} H_m(f) \leq s^m \text{Cap}_m(f < -s). \tag{3.1} \]

Proof. Let \( t, s > 0 \) and \( K \) be a compact subset satisfying \( K \subset \{f < -s - t\} \). We have
\[ \text{Cap}_m(K) = \int_{\Omega} H_m(h^*_K) = \int_{\{f < -s - t\}} H_m(h^*_K) \]
\[ = \int_{\{f < -s + th_K\}} H_m(h_K) = \frac{1}{t^m} \int_{\{f < g\}} H_m(g), \]

Using Theorem 3.6 in [6] we obtain that
\[ \frac{1}{t^m} \int_{\{f < g\}} H_m(g) = \frac{1}{t^m} \int_{\{f < \max(f, g)\}} H_m(\max(f, g)) \leq \]
\[ \frac{1}{t^m} \int_{\{f < \max(f, g)\}} H_m(f) = \frac{1}{t^m} \int_{\{f < -s + th_K\}} H_m(f) \leq \frac{1}{t^m} \int_{\{f < -s\}} H_m(f). \]

The left hand inequality of (3.1) follows by taking the supremum over all compact sets \( K \subset \Omega \).

For the right hand inequality, we have
\[ \int_{\{f \leq -s\}} H_m(f) = \int_{\Omega} H_m(f) - \int_{\{f > -s\}} H_m(f) \]
\[ = \int_{\Omega} H_m(\max(f, -s)) - \int_{\{f > -s\}} H_m(\max(f, -s)) \]
\[ = \int_{\{f \leq -s\}} H_m(\max(f, -s)) \leq s^m \text{Cap}_m\{f \leq -s\}. \]

The result follows. \( \Box \)

Remark 3.9. Using the previous lemma we deduce the following results

(1) \( f \in F_m(\Omega) \) if and only if \( \limsup_{s \to 0} s^m \text{Cap}_m(\{f < -s\}) < +\infty \).

(2) If \( f \in F_m(\Omega) \) then
\[ \int_{\Omega} H_m(f) = \lim_{s \to 0} s^m \text{Cap}_m(\{f < -s\}) \]
and
\[ \int_{\{f = -\infty\}} H_m(f) = \lim_{s \to +\infty} s^m \text{Cap}_m(\{f < -s\}). \]
(3) The function \( f \in \mathcal{F}_m^*(\Omega) \) if and only if 
\[
\lim_{s \to +\infty} s^\alpha \text{Cap}_m(\{f < -s\}) = 0.
\]
Indeed it is known that if \( f \) is an \( m \)-sh function on \( \Omega \) then \( H_m(f)(P) = 0 \) for every \( m \)-polar set \( P \subset \Omega \) if and only if \( H_m(f)\{f = -\infty\} = 0 \) which follows directly from the previous assertion of this remark.

4. THE CLASS \( \mathcal{E}_{m,\chi}(\Omega) \)

Throughout this section \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) will be an increasing function. In [10] Hung introduced the class \( \mathcal{E}_{m,\chi}(\Omega) \) to generalize the fundamental weighted energy classes introduced firstly by Benelkourchi, Guedj, and Zeriahi [4]. Such class is defined as follows:

**Definition 4.1.** We say that \( f \in \mathcal{E}_{m,\chi}(\Omega) \) if and only if there exits \( (f_j)_j \subset E_0^0(\Omega) \) such that \( f_j \searrow f \) in \( \Omega \) and
\[
\sup_{j \in \mathbb{N}} \int_{\Omega} (-\chi(f_j)) H_m(f_j) < +\infty.
\]

**Remark 4.2.** It is clear that the class \( \mathcal{E}_{m,\chi}(\Omega) \) generalizes all analogous Cegrell classes defined by Lu in [8] and [9]. Indeed

1. \( \mathcal{E}_{m,\chi}(\Omega) = \mathcal{F}_m(\Omega) \) when \( \chi(0) \neq 0 \) and \( \chi \) is bounded.
2. \( \mathcal{E}_{m,\chi}(\Omega) = \mathcal{E}_m(\Omega) \) in the case when \( \chi(t) = -(-t)^p \);
3. \( \mathcal{E}_{m,\chi}(\Omega) = \mathcal{F}_m^0(\Omega) \) in the case when \( \chi(t) = -1 - (-t)^p \).

Note that if we take \( m = n \) in all the previous cases we recover the classic Cegrell classes defined in [12] and [13].

Note that in the case \( \chi(0) \neq 0 \) one has that \( \mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{F}_m(\Omega) \) so the Hessian operator is well defined in and is with finite total mass on \( \Omega \). So in the rest of this paper we will always consider the case \( \chi(0) = 0 \).

In the following Theorem we will prove that the Hessian operator is well defined on \( \mathcal{E}_{m,\chi}(\Omega) \). Note that this result was proved in [10] but with an extra condition \((\chi(2t) \leq a \chi(t))\). Here we omit that condition and the proof of such result is completely different.

**Theorem 4.3.** Assume that \( \chi \neq 0 \). Then
\[
\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega).
\]
So for every \( f \in \mathcal{E}_{m,\chi}(\Omega) \), \( H_m(f) \) is well defined and \( -\chi(f) \in L^1(H_m(f)) \).

**Proof.** Since \( \chi \neq 0 \) so there exists \( t_0 > 0 \) such that \( \chi(-t_0) < 0 \). Take \( \chi_1 \) an increasing function satisfying \( \chi'_1 = \chi''_1 = 0 \) on \([-t_0,0] \), \( \chi_1 \) is convex on \([-\infty,-t_0] \) and \( \chi_1 \geq \chi \). Let \( g \in \mathcal{SH}_m^-(\Omega) \), then
\[
\frac{d^\infty \chi_1(g) \wedge \beta^{n-m}}{\beta^{n-m}} = \chi''_1(g) dg \wedge d^\infty g \wedge \beta^{n-m} + \chi'_1(g) d^e \chi_1(g) \wedge \beta^{n-m} \geq 0.
\]
So the function \( \chi_1(g) \in \mathcal{SH}_m^-(\Omega) \). Now consider \( f \in \mathcal{E}_{m,\chi}(\Omega) \), then by definition there exists a sequence \( (f_j)_j \subset E_0^0(\Omega) \) that decreases to \( f \) and satisfying
\[
\sup_{j \in \mathbb{N}} \int_{\Omega} -\chi(f_j) H_m(f_j) < \infty.
\]
By definition of the class \( \mathcal{E}_m(\Omega) \), it remains to prove that \( f \) coincides locally with a function in \( \mathcal{F}_m(\Omega) \). For this take \( G \Subset \Omega \) be a domain and consider the function
\[
f_j^G := \sup\{g \in \mathcal{SH}_m^-(\Omega); g \leq f_j \text{ on } G\}.
\]
We have \( f^G_j \in \mathcal{E}^0_{m}(\Omega) \) and \( f^G_j \searrow f \) on \( G \). Take \( \varphi \in \mathcal{E}^0_{m}(\Omega) \) such that \( \chi_1(f_1) \leq \varphi \). We obtain using integration by parts that

\[
\sup_{j \in \mathbb{N}} \int_{\Omega} -\varphi H_m(f^G_j) \leq \sup_{j \in \mathbb{N}} \int_{\Omega} -\varphi H_m(f_j) \\
\leq \sup_{j \in \mathbb{N}} \int_{\Omega} -\chi_1(f_1) H_m(f_j) \\
\leq \sup_{j \in \mathbb{N}} \int_{\Omega} -\chi_1(f_j) H_m(f_j) \\
\leq \sup_{j \in \mathbb{N}} \int_{\Omega} -\chi(f_j) H_m(f_j) < \infty.
\]

We deduce that

\[
\sup_{j \in \mathbb{N}} \int_{\Omega} H_m(f^G_j) \leq (\sup_{G} \varphi)^{-1} \sup_{j \in \mathbb{N}} \int_{\Omega} -\varphi H_m(f^G_j) < \infty.
\]

It follows that the limit \( \lim_{j \to +\infty} f^G_j \in F_m(\Omega) \) and therefore \( f \in \mathcal{E}_m(\Omega) \).

For the second assertion, we have that every \( f \in \mathcal{E}_{m,\chi}(\Omega) \) is upper semicontinuous, so the sequence of measures \( \mu_j := -\chi(f_j) H_m(f_j) \) is bounded. Take \( \mu \) a cluster point of \( \mu_j \) then \( -\chi(f) H_m(f) \leq \mu \). Hence \( \int_{\Omega} -\chi(f) H_m(f) < \infty \) and the desired result follows.

Proposition 4.4. Then the following statements are equivalent:

\[(1) \quad \chi(-\infty) = -\infty \]
\[(2) \quad \mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_a^0(\Omega).\]

Proof. We will prove that (1) \( \Rightarrow \) (2). For this assume that \( \chi(-\infty) = -\infty \) and take \( f \in \mathcal{E}_{m,\chi}(\Omega) \). By definition of the class \( \mathcal{E}_{m,\chi}(\Omega) \), there exists a sequence \( \{f_j\} \subset \mathcal{E}^0_{m} \) such that \( f_j \searrow f \) and

\[
\sup_{j} \int_{\Omega} -\chi(f_j) H_m(f_j) < +\infty.
\]

Since \( \chi \) is increasing then for all \( t > 0 \)

\[
\int_{\{f_j < -t\}} H_m(f_j) \leq \int_{\{f_j < -t\}} \frac{\chi(f_j)}{\chi(-t)} H_m(f_j) \\
\leq (\chi(-t))^{-1} \sup_{j} \int_{\Omega} \chi(f_j) H_m(f_j).
\]

Since the sequence \( \{f_j < -t\} \) is increasing to \( \{f < -t\} \) then by letting \( j \to \infty \) we get

\[
\int_{\{f < -t\}} H_m(f) \leq (\chi(-t))^{-1} \sup_{j} \int_{\Omega} \chi(f_j) H_m(f_j).
\]

Now if we let \( t \to +\infty \) we deduce that

\[
\int_{\{f = -\infty\}} H_m(f) = 0.
\]

Hence, \( f \in \mathcal{E}_a^0(\Omega) \).
(2) ⇒ (1) Assume that $\chi(-\infty) > -\infty$, then $\mathcal{F}_m(\Omega) \subset \mathcal{E}_{m,\chi}(\Omega)$. But it is known that $\mathcal{F}_m(\Omega)$ is not a subset of $\mathcal{E}_{m}^a(\Omega)$. We deduce that $\mathcal{E}_{m,\chi}(\Omega) \not\subset \mathcal{E}_{m}^a(\Omega)$. □

The rest of this section will be devoted to give a connection between the class $\mathcal{E}_{m,\chi}(\Omega)$ and the $\text{Cap}_{m}^{-}$-capacity of sublevels $\text{Cap}_{m}(\{f < -t\})$. As a consequence we deduce a complete characterization of the class $\mathcal{E}_{m}^p(\Omega)$ introduced by Lu [8] in term of the $\text{Cap}_{m}^{-}$-capacity of sublevel. For this we introduce the class $\hat{\mathcal{E}}_{m,\chi}(\Omega)$ as follows:

**Definition 4.5.**

$$\hat{\mathcal{E}}_{m,\chi}(\Omega) := \left\{ \varphi \in \mathcal{S}H_{m}^{-}(\Omega) / \int_0^{+\infty} t^m \chi'(−t)\text{Cap}_{m}(\{\varphi < −t\})dt < +\infty \right\}.$$  

The previous class coincides with the class $\hat{\mathcal{E}}_{\chi}(\Omega)$ given by Benelkourchi, Guedj, and Zeriahi [4], it suffices to take $m = n$ to recover it. In the following proposition we cite some properties of $\hat{\mathcal{E}}_{m,\chi}(\Omega)$ and we give a relationship between $\mathcal{E}_{m,\chi}(\Omega)$ and $\hat{\mathcal{E}}_{m,\chi}(\Omega)$:

**Proposition 4.6.**

1. The classe $\hat{\mathcal{E}}_{m,\chi}(\Omega)$ is convex.
2. For every $f \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$ and $g \in \mathcal{S}H_{m}^{-}(\Omega)$, one has that $\max(f, g) \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$.
3. $\mathcal{E}_{m,\chi}(\Omega) \subset \hat{\mathcal{E}}_{m,\chi}(\Omega)$.
4. If we denote by $\hat{\chi}(t)$ the function defined by $\hat{\chi}(t) := \chi(t/2)$, then $\mathcal{E}_{m,\chi}(\Omega) \subset \hat{\mathcal{E}}_{m,\chi}(\Omega)$.

**Proof.**

1) Let $f, g \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$ and $0 \leq \alpha \leq 1$. Since we have 
$$\{\alpha f + (1 - \alpha)g < -t\} \subset \{f < -t\} \cup \{g < -t\}$$
then $f + \alpha g \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$. The result follows.

2) The proof of this assertion is obvious.

3) Take $f \in \hat{\mathcal{E}}_{m,\chi}(\Omega)$. It remains to construct a sequence $f_j \in \mathcal{E}_{m}^0(\Omega)$ satisfying
$$\int_\Omega -\chi(f_j) H_m(f_j) < \infty.$$ 
For this, we may assume without loss of generality that $f \leq 0$. If we set $f_j := \max(f, -j)$ then $f_j \in \mathcal{E}_{m}^0(\Omega)$. Using Lemma [3,8] we get that
$$\int_\Omega -\chi(f_j) H_m(f_j) = \int_0^{+\infty} \chi'(-t)H_m(f_j)(f_j < -t)dt \leq \int_0^{+\infty} \chi'(-t)t^m\text{Cap}_{m}(f < -t)dt < +\infty.$$ 
It follows that $f \in \mathcal{E}_{m,\chi}(\Omega)$.

4) The proof of this assertion follows directly using the same argument as in 3) and the second inequality in Lemma [3,8] for $t = s$. □

**Proposition 4.7.** Assume that for all $t < 0$ one has $\chi(t) < 0$, then for all $f \in \mathcal{E}_{m,\chi}(\Omega)$ one has
$$\lim_{z \to w} f(z) = 0, \quad \forall w \in \partial \Omega.$$
Proof. Since by hypothesis we have for all $t < 0$: $\chi(t) < 0$ so we can assume, without loss of generality, that the length of the set $\{t > 0; t < t_0$ and $\chi'(t) \neq 0\}$ is positive for all $t_0 > 0$. We suppose by contradiction that there is $w_0 \in \partial \Omega$ such that $\limsup f(z) = \varepsilon < 0$. Then there is a ball $B_0$ centered at $w_0$ satisfying $B_0 \cap \Omega \subset \{f < \frac{\varepsilon}{2}\}$. If we consider $(K_j)_j$ to a sequence of regular compact subsets so that for all $j$ one has $K_j \subset K_{j+1}$ and $B_0 \cap \Omega = \bigcup K_j$. Then the extremal function $h_{K_j,\Omega}$ belongs to $\mathcal{E}^0_m(\Omega)$ and decreases to $h_{E,\Omega}$. It is easy to check that $h_{E,\Omega} \notin \mathcal{F}_m(\Omega)$. By the definition of the class $\mathcal{F}_m(\Omega)$ we obtain

$$\sup_j \text{Cap}_m(K_j) = \sup_j \int_{\Omega} H_m(f_{K_j,\Omega}) = +\infty.$$ 

So

$$\text{Cap}_m(B_0 \cap \Omega) = +\infty.$$ 

We deduce that

$$\text{Cap}_m(\{f < -s\}) = +\infty, \forall s \leq -\varepsilon/2,$$

hence

$$\int_0^{+\infty} t^m \chi'(t) \text{Cap}_m(\{f < -t\})dt = +\infty.$$ 

We get a contradiction with the fact that $\mathcal{E}_{m,\chi}(\Omega) \subset \hat{\mathcal{E}}_{m,\chi}(\Omega)$.

\[\square\]

**Proposition 4.8.** Assume that $\chi \neq 0$. If there exists a sequence $(f_k) \subset \mathcal{E}^0_m(\Omega)$ such that

$$\sup_{k \in \mathbb{N}} \int_{\Omega} -\chi(f_k) H_m(f_k) < \infty,$$

then the function $f := \lim_{k \to +\infty} f_k \neq -\infty$ and therefore $f \in \mathcal{E}_{m,\chi}(\Omega)$.

**Proof.** Using the hypothesis we observe that the length of the set $\{t > 0; t < t_0$ and $\chi'(t) \neq 0\}$ is positive. By lemma 3.8 we get

$$\sup_{k \in \mathbb{N}} \int_{\Omega} \chi(f_k) H_m(f_k) < \infty.$$ 

Then

$$\int_0^{+\infty} t^m \chi'(t) \text{Cap}_m(\{f < -t\})dt = \lim_{k \to +\infty} \int_0^{+\infty} t^m \chi'(t) \text{Cap}_m(\{f < -t\})dt$$

$$\leq \lim_{k \to +\infty} 2^m \int_0^{+\infty} \chi'(t) \int_{\{f < -t\}} H_m(f_k)dt$$

$$\leq 2^m \sup_{k \in \mathbb{N}} \int_{\Omega} \chi(f_k) H_m(f_k) < \infty.$$ 

Note that in the previous inequality we have used the convergence monotone theorem. We conclude that $f \neq -\infty$ and therefore $f \in \mathcal{E}_{m,\chi}(\Omega)$.

\[\square\]

**Theorem 4.9.** Assume that for all $t < 0$ one has $\chi(t) < 0$. Then

$$\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{N}_m(\Omega).$$

**Proof.** By proposition 4.8 it suffices to prove that every maximal function $f \in \mathcal{E}_{m,\chi}(\Omega)$ is identically equal to 0. Take a sequence $f_j \in \mathcal{E}^0_m(\Omega)$ as in the definition of the class $\mathcal{E}_{m,\chi}(\Omega)$. So we obtain using Lemma 3.8 that

$$\int_0^{+\infty} \chi'(\frac{-s}{2}) f^m \text{Cap}_m(\{f < -s\})ds = \lim_{j \to +\infty} \int_0^{+\infty} \chi'(\frac{-s}{2}) s^m \text{Cap}_m(\{f_j < -s\})ds$$
Proof. (1) Take a test function \( \rho \) for all \( j \in \mathbb{N} \). Let \( \rho \) exist. Assume that Theorem 4.11. guarantee that for all \( j \in \mathbb{N} \) it suffices to prove the reverse inclusion 4.9. It suffices to prove the reverse inclusion \( \{ f \in \mathcal{N}_m(\Omega) \} \subset \mathcal{E}_{m,\chi}(\Omega) \). Take \( f \in \mathcal{N}_m(\Omega) \) satisfying \( \int_\Omega -\chi(f)H_m(f) < \infty \). It suffices to construct sequence \( f_j \in \mathcal{E}_{m,\chi}(\Omega) \) that decreases to \( f \) and satisfies

\[
\sup_j \int_\Omega -\chi(f_j)H_m(f_j) < \infty.
\]

Let \( \rho \) be an exhaustion function for \( \Omega \) (\( \Omega = \{ \rho < 0 \} \)). The theorem 5.9 in [6] guarantee that for all \( j \in \mathbb{N} \), there is a function \( f_j \in \mathcal{E}_{m,\chi}(\Omega) \) satisfying \( H_m(f_j) = 1_{\{f_j > \rho\}}H_m(f) \). We have \( H_m(f_j) \leq H_m(f_{j+1}) \leq H_m(f) \), so we get that \( f_j \geq f_{j+1} \) using the comparison principle and \( (f_j) \) converges to a function \( f \). It is easy to check that \( f \geq f \). Now following the proof of Theorem 4.10 we deduce the existence of a negative \( m \)-sh function \( g \) satisfying \( \int_\Omega -gH_m(f) < \infty \). If follows by Theorem 2.10 [7] that \( g = f \). Thus the monotone convergence theorem gives

\[
\int_\Omega -\chi(f_j)H_m(f_j) = \int_\Omega -\chi(f_j)1_{\{f_j > \rho\}}H_m(f) \to \int_\Omega -\chi(f)H_m(f) < \infty.
\]

Now we will extend the theorem A to the class \( \mathcal{E}_{m,\chi}(\Omega) \).

**Theorem 4.11.** Assume that \( \chi \) is continuous, \( \chi(-\infty) > -\infty \) and \( f, f_j \in \mathcal{E}_{m,\chi}(\Omega) \) for all \( j \in \mathbb{N} \). If there exists \( g \in \mathcal{E}_{m,\chi}(\Omega) \) satisfying \( f_j \geq g \) on \( \Omega \) then:

(1) If \( f_j \) converges to \( f \) in \( \text{Cap}_{m-1} \) capacity then \( \liminf_{j \to +\infty} -\chi(f_j)H_m(f_j) \geq -\chi(f)H_m(f) \).

(2) If \( f_j \) converges to \( f \) in \( \text{Cap}_m \) capacity then \( -\chi(f_j)H_m(f_j) \) converges weakly to \( -\chi(f)H_m(f) \).

**Proof.** (1) Take a test function \( \varphi \in C_{0}^{\infty}(\Omega) \) such that \( 0 \leq \varphi \leq 1 \). Using [9] there exist \( \psi_k \in \mathcal{E}_{m,\chi}(\Omega) \cap C(\Omega) \) with \( \psi_k \geq f \) and \( \psi_k \nabla f \) in \( \Omega \). For a fixed integer \( k \geq 1 \)
there exists, by [14], \( j_0 \in \mathbb{N} \) such that \( f_j \geq \psi_k \) on \( \text{supp} \varphi \) for all \( j \geq j_0 \). So by Theorem 3.10 in [6], we obtain that for all \( k \geq 1 \) one has

\[
\liminf_{j \to +\infty} \int_{\Omega} -\varphi(f_j)H_m(f_j) \geq \liminf_{j \to +\infty} \int_{\Omega} -\varphi(\psi_k)H_m(f_j) = \int_{\Omega} -\varphi(\psi_k)H_m(f).
\]

Now if we let \( k \) tends to \( +\infty \) then by the Lebesgue monotone convergence theorem, we get

\[
\liminf_{j \to +\infty} \int_{\Omega} -\varphi(f_j)H_m(f_j) \geq \int_{\Omega} -\varphi(f)H_m(f).
\]

The result follows.

(2) Without loss of generality one can assume that \( \chi(-\infty) = -1 \). Let \( \varphi \in C^\infty_0(\Omega) \) such that \( 0 \leq \varphi \leq 1 \). We claim that

\[
\limsup_{j \to +\infty} \int_{\Omega} -\varphi(f_j)H_m(f_j) \leq \int_{\Omega} -\varphi(f)H_m(f). \quad (*)
\]

Indeed, by the quasicontinuity of \( f \) and \( g \) with respect to the capacity \( \text{Cap}_m \), we obtain that for every \( k \in \mathbb{N} \) there exist an open subset \( O_k \) of \( \Omega \) and a function \( \tilde{f}_k \in C(\Omega) \) such that \( \text{Cap}_m(O_k) \leq \frac{1}{2^k} \) and \( \tilde{f}_k = f \) on \( \Omega \setminus O_k \) and \( g \geq -\alpha_k \) on \( \text{supp}\varphi \setminus O_k \) for some \( \alpha_k > 0 \). Let \( \varepsilon > 0 \), then by Theorem 3.6 in [15] one has

\[
\int_{\Omega} -\varphi(f_j)H_m(f_j) = \int_{\Omega \setminus O_k} -\varphi(f_j)H_m(f_j) + \int_{O_k} -\varphi(f_j)H_m(f_j) \\
\leq \int_{\Omega \setminus O_k} -\varphi(f_j)H_m(f_j) + \int_{O_k} -\varphi H_m(f_j) \\
\leq \int_{\{f_j \leq f - \varepsilon\} \setminus O_k} -\varphi(f_j)H_m(f_j) \\
+ \int_{\{f_j > f - \varepsilon\} \setminus O_k} -\varphi(f_j)H_m(f_j) + \int_{O_k} -\varphi H_m(f_j) \\
\leq \int_{\{f_j \leq f - \varepsilon\} \setminus O_k} -\varphi H_m(f_j) \\
+ \int_{\Omega \setminus O_k} -\varphi(f_{\text{max}}(f_j, -\alpha_k))H_m(f_j) \\
+ \int_{O_k} -\varphi(\tilde{f}_k - \varepsilon)H_m(f_j) + \int_{\Omega} -\varphi h_{O_k, \Omega} H_m(f_j) \\
\leq \alpha_k^m \text{Cap}_m(\{f_j < f - \varepsilon\} \cap \text{supp}\varphi) \\
+ \int_{\Omega \setminus O_k} -\varphi(\tilde{f}_k - \varepsilon)H_m(f_j) + \int_{\Omega} -\varphi h_{O_k, \Omega} H_m(f_j).\]

If we let \( j \) goes to \( +\infty \), we get using theorem 3.8 [6] that

\[
\limsup_{j \to +\infty} \int_{\Omega} -\varphi(f_j)H_m(f_j) \leq \int_{\Omega \setminus O_k} -\varphi(\tilde{f}_k - \varepsilon)H_m(f) + \int_{\Omega} -\varphi h_{O_k, \Omega} H_m(f).
\]

If we let \( \varepsilon \to 0 \), we obtain

\[
\limsup_{j \to +\infty} \int_{\Omega} -\varphi(f_j)H_m(f_j) \leq \int_{\Omega \setminus \{f = -\infty\}} -\varphi(f)H_m(f) + \int_{\Omega} -\varphi h_{\bigcup_{l=k}^\infty O_l, \Omega} H_m(f) \quad (**)
\]
Now as \( \bigcup_{l=k}^{\infty} O_l \setminus O \) when \( k \to +\infty \) then

\[
\text{Cap}_m(O) \leq \lim_{k \to \infty} \text{Cap}_m \left( \bigcup_{l=k}^{\infty} O_l \right) \leq \lim_{k \to \infty} \sum_{l=k}^{\infty} \text{Cap}_m(O_l) \leq \lim_{k \to \infty} \frac{1}{2^{k-1}}
\]

so there exists an \( m \)-polar set \( M \) such that \( h_{\bigcup_{l=k}^{\infty} O_l, \Omega} \neq 0 \) when \( k \to +\infty \) on \( \Omega \setminus M \). So if we take \( k \to +\infty \) in (**) we obtain

\[
\limsup_{j \to +\infty} \int_{\Omega} -\varphi(f_j)H_m(f_j) \leq \int_{\Omega \setminus \{f = -\infty\}} -\varphi(f)H_m(f) \quad + \quad \int_{M} \varphi H_m(f)
\]

\[
\leq \int_{\Omega \setminus \{f = -\infty\}} -\varphi(f)H_m(f) \quad + \quad \int_{\{f = -\infty\}} -\varphi(f)H_m(f)
\]

\[
= -\int_{\Omega} -\varphi(f)H_m(f).
\]

This proves the claim (*). Moreover since \( f_j \) converges in \( \text{Cap}_m \)-capacity so it converges in \( \text{Cap}_{m-1} \)-capacity. Using the assertion \((a)\) we obtain

\[
\liminf_{j \to +\infty} \int_{\Omega} -\varphi(f_j)H_m(f_j) \geq \int_{\Omega} -\varphi(f)H_m(f).
\]

If we combine the last inequality with (**) we get

\[
\lim_{j \to +\infty} \int_{\Omega} -\varphi(f_j)H_m(f_j) = \int_{\Omega} -\varphi(f)H_m(f),
\]

for every \( \varphi \in C_0^\infty(\Omega) \) with \( 0 \leq \varphi \leq 1 \). Hence we get the desired result. \( \square \)

Now we will be intrusted to the problem of subextention in the class \( \mathcal{E}_{m,\chi}(\Omega) \). For \( \Omega \in \Omega \in \mathbb{C}^n \) and \( f \in \mathcal{E}_{m,\chi}(\Omega) \), we say that \( \tilde{f} \in \mathcal{E}_{m,\chi}(\tilde{\Omega}) \) is a subextension of \( f \) if \( \tilde{f} \leq f \) on \( \Omega \). In the following theorem we prove that every function \( f \in \mathcal{E}_{m,\chi}(\Omega) \) has a subextension.

**Theorem 4.12.** Let \( \Omega \) be a \( m \)-hyperconvex domain such that \( \Omega \in \tilde{\Omega} \in \mathbb{C}^n \). If \( \chi(t) < 0 \) for all \( t < 0 \) and \( f \in \mathcal{E}_{m,\chi}(\Omega) \) then is \( \tilde{f} \in \mathcal{E}_{m,\chi}(\tilde{\Omega}) \) satisfying

\[
\int_{\Omega} -\chi(\tilde{f})H_m(\tilde{f}) \leq \int_{\Omega} -\chi(f)H_m(f)
\]

and \( \tilde{f} \leq f \) on \( \Omega \).

**Proof.** Let \( f \in \mathcal{E}_{m,\chi}(\Omega) \) and \( f_k \in \mathcal{E}_{m}^0(\Omega) \) be the sequence as in the definition of the class \( \mathcal{E}_{m,\chi}(\Omega) \). We obtain using lemma 3.2 in [18] that for every \( k \in \mathbb{N} \), there exists a subextension \( \tilde{f}_k \) of \( f_k \). It follows that

\[
\int_{\tilde{\Omega}} -\chi(\tilde{f}_k)H_m(\tilde{f}_k) = \int_{\{f_k = \tilde{f}_k \} \cap \Omega} -\chi(\tilde{f}_k)H_m(\tilde{f}_k)
\]

\[
\leq \int_{\{f_k = \tilde{f}_k \} \cap \Omega} -\chi(f_k)H_m(f_k)
\]

\[
\leq \int_{\Omega} -\chi(f_k)H_m(f_k).
\]

So we obtain

\[
\sup_k \int_{\tilde{\Omega}} -\chi(\tilde{f}_k)H_m(\tilde{f}_k) \leq \int_{\Omega} -\chi(f)H_m(f) < \infty. \quad (*)
\]
Using the proposition 4.8 we get that the function \( \tilde{f} = \lim_{k \to \infty} \tilde{f}_k \neq -\infty \) and \( \tilde{f} \in E_{m,\chi}(\tilde{\Omega}) \). Then by (\( \ast \))

\[
\int_{\tilde{\Omega}} -\chi(\tilde{f}) H_m(\tilde{f}) \leq \int_{\Omega} -\chi(f) H_m(f) < \infty.
\]

It follows by the Comparison Principle that for all \( k \in \mathbb{N} \) one has \( \tilde{f}_k \leq f_k \) on \( \Omega \). If we let \( k \) goes to \( \infty \), we deduce that \( \tilde{f} \leq f \) on \( \Omega \). \( \square \)

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