Algebraic properties of a disordered asymmetric Glauber model

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Abstract. We consider an asymmetric variant of disordered Glauber dynamics of Ising spins on a one-dimensional lattice, where each spin flips according to the relative state of the spin to its left. Moreover, each bond allows for two rates: flips which equalize nearest neighbour spins, and flips which ‘unequalize’ them. In addition, the leftmost spin flips depending on the spin at that site. We explicitly calculate all eigenvalues of the transition matrix for all system sizes and conjecture a formula for the normalization factor of the model. We then analyse two limits of this model, which are analogous to ferromagnetic and antiferromagnetic behaviour in the Ising model, for which we are able to prove an analogous formula for the normalization factor.

Keywords: algebraic structures of integrable models, rigorous results in statistical mechanics, disordered systems (theory), stationary states

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1. Introduction

The Ising model with Glauber dynamics [1] has been an extremely important source for understanding time-dependent behaviour in the Ising model. Moreover, the simplicity of the model has inspired the creation of a number of related models which have been amenable to rigorous combinatorial and probabilistic techniques. Most of the studies, with the exception of [2], have been on the probabilistic side, with systems being studied on the continuum, on infinite lattices in arbitrary dimension, or considering asymptotics of large finite lattices. The literature on Glauber dynamics in the Ising model is exceedingly large. When bonds between neighbouring sites have random strengths, there seem to be a few studies for large or infinite systems, e.g. [3]–[6]. We know of very few combinatorial results, e.g. [7,8], and they correspond to situations where the bond strengths are not arbitrary. In what follows, we show that a model with Glauber-like dynamics and arbitrary bond strengths exhibits rich algebraic and combinatorial structure.

We consider a simplified one-dimensional version of the Ising model with spin-flip dynamics, on sites labelled 1 to \( L \), and where there are two kinds of spins, which we label as occupation numbers \( \{0, 1\} \). The transition rules are given by an asymmetric version of the usual rule. Each site \( i \) looks only to the site to its left, \( i - 1 \), and the \( i \)th site switches its state with a rate that depends only on whether both sites are in the same state or not. If the state of both sites is different, the transition occurs with rate \( \alpha_i \), and if they are the same, it occurs with rate \( \beta_i \). In other words, \( \alpha_i \)'s tries to ferromagnetize the system and \( \beta_i \)'s tries to antiferromagnetize it. In addition there are boundary interactions at the first site, which flips with rate \( \alpha_1 \) if it contains a particle and \( \beta_1 \) if it does not. There is a mention of a directed one-dimensional Glauber model with open boundary condition in [9], albeit with very special rates.

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We note some important differences between the usual Glauber dynamics for the Ising model (also known as the kinetic Ising model) and the asymmetric as well as disordered version studied here. First, we list the conceptual differences. For the most part, we are concerned with formulae valid for all system sizes, and not only in the thermodynamic limit. In spite of the fact that the dynamics of this model does not admit detailed balance with respect to the steady state, unlike the kinetic Ising model, it exhibits remarkable algebraic structure. This structure makes the system more tractable, in a certain sense. This allows us to compute the spectral gap, which contains information about the long-time relaxation to equilibrium for an arbitrary system size, for instance. Note that this is in spite of the fact that there are many more arbitrary (disordered) parameters here than in the kinetic Ising model.

We now move on to more specific differences. The kinetic Ising model has been extensively studied on a ring \([1, 2, 10]\), unlike the open boundary condition here. Similarly, the asymmetric model studied here does not admit a simple formula for the spectrum in the case of periodic boundary conditions. Therefore, comparison of the two models is somewhat arbitrary. This model is very far from equilibrium because of a lack of detailed balance. This makes it much harder to compute correlation functions. We have explicit formulae only for the density, and a conjecture for the normalization factor. Finally, in our case, we are able to state all eigenvalues of the transition matrix even with disordered rates, unlike the original case, where one knows that the eigenvalues lie between 0 and \(-L\) for an \(L\)-sized system \([2]\) for very special rates. One does not know if the techniques applied here work for the kinetic Ising model with constant bulk rates (much less disordered rates looked at here) with open boundary conditions. Studies from looking at small systems suggest otherwise.

One of the new features here is that the nonequilibrium ‘partition function’\(^{1}\) of this model grows superexponentially in some special cases explored in section 7, like \(2^L\), unlike what is normally seen in equilibrium systems. The physical relevance of this observation is not entirely clear. The intuition that the normalization factor is the ratio between the most observed and least observed configurations suggests that some configurations are exceedingly rare, suppressed by a factor \(c^{-L^2}\).

This model can be interpreted as a voter model \([11]\) within a hierarchical demographic, that is to say individual \(i-1\) influences the opinion of individual \(i\), but not vice versa. The rate \(\alpha_i\) governs the tendency of \(i-1\) to be a usual campaigner, whereas \(\beta_i\) governs the tendency of \(i-1\) to be a double agent. This model can also be thought of as a variant of the East model studied in \([12, 8]\) on the one-dimensional lattice. In the East model, a spin was allowed to flip only if the site to its left was occupied, and the rate of the flip depended on whether the site was occupied or not.

Crisanti and Sompolinsky \([6]\) considered a mean-field model of Glauber dynamics on Ising spins with asymmetric bonds with random strengths to understand properties of neural networks. This work is not related to it, but the idea is similar. In their work, asymmetry refers to partial asymmetry, instead of the total asymmetry assumed here. In their case, one does not have the rich structure detailed here.

In an earlier work, the steady state of the border process for this model with \(\beta_i = 0\) and \(\alpha_i = 1\) for \(i > 0\), the asymmetric annihilation process, was obtained \([13]\) by using

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\(^{1}\) This is an abuse of notation. It is more properly called the normalization factor.
a transfer matrix ansatz. The spectrum of the transition matrix for $\alpha_i$ unequal was calculated in [14]. This model, even with $\beta_i = 0$, is a clear generalization because one can distinguish here between particles and holes, whereas the former could not. In addition, there is no analogue of the rate for $\beta_i$ in the earlier works because that would have amounted to creation of particles in the bulk.

We will write down the precise statement of the Markov chain in section 2. The main result of the paper is an explicit formula for the eigenvalues of the Markov chain for any size $L$, which we prove in section 3. This leads, in an obvious way, to a formula for the spectral gap of the system.

We will show that the transition matrix of the model exhibits a simple recurrence relation allowing one to express the matrix for a system of size $L$ in terms of that of size $L - 1$ in section 4. We prove a formula for the density in this disordered model in section 5. We conjecture a formula for the normalization factor for the distribution in section 6. Lastly, we consider two special cases, which we naturally label the ferromagnetic and the antiferromagnetic limits, in section 7.

2. The model

We consider a nonequilibrium system on a finite lattice with $L$ sites labelled from 1 to $L$. Each site is occupied by one of two spins 1 and 0. The evolution rule in the bulk, for site $i$ depending on the spin at site $i - 1$, is given by

$$
10 \rightarrow 11 \quad \text{and} \quad 01 \rightarrow 00 \quad \text{with rate } \alpha_i, \\
11 \rightarrow 10 \quad \text{and} \quad 00 \rightarrow 01 \quad \text{with rate } \beta_i. 
$$

The evolution of the first site is given by

$$
1 \rightarrow 0 \quad \text{with rate } \alpha_1, \\
0 \rightarrow 1 \quad \text{with rate } \beta_1. 
$$

The rule at the left boundary (2) is constructed by supposing that there is a virtual site labelled 0 to the left of the first site which is always empty. The left boundary conditions are deduced from the bulk rules (1) by looking at the second component of the bond.

Remark 1. There are two symmetries of the model; the first is manifested by interchanging all $\alpha_i$s and $\beta_i$s as well as flipping spins at all odd sites, and the second by leaving $\alpha_1$ and $\beta_1$ as they are, interchanging all other $\alpha_i$s and $\beta_i$s and flipping spins at all even sites.

Remark 2. This model has the property that the correlation functions $\langle \eta_i \cdots \eta_n \rangle$ do not depend on the state of $\eta_{n+1}$, simply because site $i_n + 1$ cannot influence any site less than or equal to $i_n$. Further, as is well-known for Glauber dynamics, correlation functions of $n$ sites like the one above depend only on those of sites less than or equal to $n$. 

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3. Spectrum of the transition matrices

Recall that the transition matrix of a continuous-time Markov chain is a matrix labelled by the configuration space whose \((i, j)\)th entry (for \(j \neq i\)) records the rate of the transition from state \(j\) to state \(i\). The diagonal entry is then the value such that the column sums of the matrix are zero. The probabilities of the steady state of the Markov chain are given by the right eigenvector of the transition matrix.

We observe that the characteristic polynomial of the transition matrices factorizes into linear factors and has an explicit formula. This is reminiscent of the formula for the asymmetric annihilation process conjectured in [13] and proved in [14]. Let \(B_L\) be the set of binary vectors of size \(L\), that is vectors of length \(L\) whose elements are either 0 or 1. Let us denote \(M_L\) as the transition matrix, whose coordinates are labelled by vectors \(b \in B_L\).

**Theorem 1.** The characteristic polynomial of \(M_L\) is given by

\[
|M_L - \lambda I_L| = \prod_{b \in B_L} \left( \lambda + \sum_{i=1}^{L} b_i (\alpha_i + \beta_i) \right) .
\]  
(3)

For example, when \(L = 2\), we have

\[
|M_2 - \lambda I_2| = \lambda (\lambda + \alpha_1 + \beta_1)(\lambda + \alpha_2 + \beta_2)(\lambda + \alpha_1 + \alpha_2 + \beta_1 + \beta_2) .
\]  
(4)

The proof of theorem 1 will turn out to be a simpler version of the proof of the eigenvalues in the asymmetric annihilation process [14]. We will first define a slight rearrangement of a Hadamard matrix. Let \(B_L\) be ordered lexicographically. For \(b, c \in B_L\), define the square matrix \(H_L\) of size \(2^L\) by

\[
H_L = \frac{1}{2^{L/2}}((-1)^{\hat{b}}c)_{b,c \in B_L},
\]  
(5)

where \(\hat{b}\) is the reverse of \(b\). Note that \(H_L\) is symmetric because

\[(H_L)_{b,c} = (-1)^{\hat{b}}c = (-1)^{\hat{b}}\hat{c} = (H_L)_{\hat{c},b}.\]  
(6)

This is different from the usual definition in which elements of \(H_L\) are written as \((-1)^{b\cdot c}\). For example, the matrix for \(L = 3\) is given by

\[
H_3 = \frac{1}{2\sqrt{2}} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1
\end{pmatrix} .
\]  
(7)
Lemma 2. $H_L^2 = I_L$.

Proof. We first look at the diagonal terms in $H_L^2$:
\[
(H_L^2)_{b,b} = \sum_{c \in \mathbb{B}_L} (H_L)_{b,c}(H_L)_{c,b} = \frac{1}{2^L} \sum_{c \in \mathbb{B}_L} (-1)^{2b-c} = 1. \tag{8}
\]

Now suppose $b \neq d$. Then
\[
(H_L^2)_{b,d} = \sum_{c \in \mathbb{B}_L} (H_L)_{b,c}(H_L)_{c,d} = \frac{1}{2^L} \sum_{c \in \mathbb{B}_L} (-1)^{(b+d)\cdot \hat{c}}, \tag{9}
\]
where the addition of binary vectors is the usual xor-addition: $x + (1-x) = 1$ by definition and $x + x = 0$ if $x$ is a bit.

Now, for any fixed $b, d$, we will construct a sign-reversing involution on $\mathbb{B}_L$. Since $b \neq d$, $b + d$ contains at least one entry equal to 1. Consider the leftmost such entry, at position $i$, say. To any element $c$, associate another element $\rightarrow c' = (c_1, \ldots, c_{L-i}, 1 - c_{L-i+1}, c_{L-i+2}, \ldots, c_L)$.

Clearly, this is an involution and, moreover,
\[
(-1)^{(b+d)\cdot \hat{c}} + (-1)^{(b+d)\cdot \hat{c}'} = 0.
\]

We have therefore partitioned $\mathbb{B}_L$, for any fixed $b, d, b \neq d$, such that the number of terms which contribute $+1$ to the sum is exactly the same as the number of terms which contribute $-1$. Hence $(H_L^2)_{b,d} = 0$. \hfill \Box

We now write down the transition matrix $M_L$ in this language. We use the convention that $(M_L)_{b,c}$ represents the transition rate from $c \rightarrow b$, with the end result that $M_L|v_L\rangle = 0$ for the steady state column vector $|v_L\rangle$:
\[
(M_L)_{b,c} = \begin{cases} 
\alpha_i, & b + c = 0 \cdots 0 \underbrace{1}_i 0 \cdots 0 \text{ and } c_{i-1} + c_i = 1, \\
\beta_i, & b + c = 0 \cdots 0 \underbrace{1}_i 0 \cdots 0 \text{ and } c_{i-1} = c_i, \\
- \sum_{\substack{i=1 \atop c_{i-1}+c_i=1}}^L \alpha_i - \sum_{\substack{i=1 \atop c_{i-1}=c_i}}^L \beta_i, & b = c, \\
0, & \text{otherwise},
\end{cases} \tag{10}
\]

Lemma 3. $H_L M_L H_L$ is lower triangular.

Proof. We will show the result by explicitly demonstrating the structure of non-zero terms. By definition,
\[
(H_L M_L H_L)_{c,d} = \frac{1}{2^L} \sum_{e,f \in \mathbb{B}_L} (-1)^{\hat{c} \cdot \hat{e} + \hat{f} \cdot \hat{d}} (M_L)_{e,f}. \tag{11}
\]

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We will divide the proof into three parts. We first consider diagonal elements, then certain off-diagonal elements which are non-zero, and finally show that all other off-diagonal elements are zero by introducing another sign-reversing involution just as in lemma 2.

\begin{itemize}
  \item \( c = d \).
  
  Then \( c \cdot \hat{e} + f \cdot \hat{d} = c \cdot (\hat{e} + \hat{f}) \). Suppose \( e \) differs from \( f \) only at position \( i \), then \((M_L)_{e,f}\) contributes either \( \alpha_i \) or \( \beta_i \) and the sign is always the same: \(+1\) if \( \hat{c}_i = 0 \) and \(-1\) otherwise. On the other hand, if \( e = f \), this sum is identically 0 and therefore we get a contribution of \(-\alpha_i\) if \( f_{i-1} + f_i = 1 \) and \(-\beta_i\) if \( f_{i-1} = f_i \) from (10).

  To summarize, we get \( 2^{L-1} \) terms contributing \pm \( \alpha_i \), \pm \( \beta_i \), all of the same sign, from summands \( e \neq f \) and \( 2^{L-1} \) terms contributing \(-\alpha_i\), \(-\beta_i\) from summands \( e = f \). These will clearly cancel if \( \hat{c}_i = 0 \) and will give \( 2^{L-1}(-2\alpha_i - 2\beta_i) \) if \( \hat{c}_i = 1 \). Thus,

  \[
  (H_L M_L H_L)_{c,c} = -\sum_{i=1}^{L} \hat{c}_i (\alpha_i + \beta_i). \tag{12}
  \]

  \begin{itemize}
    \item \( c \neq d \).
  \end{itemize}

  For now, we look at conditions on \( c, d \) such that \((M_L)_{e,f} = \alpha_i\) and \((-1)^{e+\hat{f}d}\) always has the same sign. This happens when \( e \) differs from \( f \) only at position \( i \) and \( f_{i-1} + f_i = 1 \) \( \Rightarrow e_{i-1} = e_i \). Now,

  \[
  c \cdot \hat{e} + f \cdot \hat{d} = e \cdot \hat{c} + f \cdot \hat{d},
  
  = e \cdot (\hat{c} + \hat{d}) + (e + f) \cdot \hat{d},
  
  = f \cdot (\hat{c} + \hat{d}) + (e + f) \cdot \hat{c}. \tag{13}
  \]

  Since \( e + f \) is fixed, only the former term contributes to the change of sign. As we vary \( e \), this term does not change sign if and only if \( (\hat{c} + \hat{d})_{i-1} = (\hat{c} + \hat{d})_i \) and \( c_j = d_j \) for all other values of \( j \). Since we are assuming \( c \neq d \), the only way this can happen is if \( \hat{c} + \hat{d} = 0 \cdots 0110 \cdots 0 \) with 1s in the \( i - 1 \)th and \( i \)th positions. If this is the case, the sign of all these terms is determined just by \( \hat{d}_i \) or, equivalently, \( \hat{c}_i \). This gives us a total contribution of \( 2^{L-1} \alpha_i (-1)^{d_i} \).

  We also have contributions of \( \alpha_i \) from diagonal terms \( e = f \) and \( e_{i-1} + e_i = 1 \) from (10). In that case, \( c \cdot \hat{e} + f \cdot \hat{d} = e \cdot (\hat{c} + \hat{d}) \). Since we just argued that \( (\hat{c} + \hat{d})_{i-1} = (\hat{c} + \hat{d})_i \), diagonal terms always contribute a term of \((-1)^i (-\alpha_i)\), giving us a grand total of \( 2^{L-1} \alpha_i \).

  Therefore, diagonal contributions will cancel off-diagonal ones if \( \hat{d}_i = 1 \), \( \hat{c}_i = 0 \) and will add to them if \( \hat{d}_i = 0 \), \( \hat{c}_i = 1 \).

  By a very similar reasoning, we will get a contribution of \(-\beta_i\) under exactly the same circumstances. The relation \( (\hat{c} + \hat{d})_{i-1} = (\hat{c} + \hat{d})_i \) remains the same, but the roles of \( c \) and \( d \) are reversed, giving us an overall negative sign.

  Assuming all other off-diagonal terms are zero,

  \[
  (H_L M_L H_L)_{c,d} = \alpha_i - \beta_i
  \]

\end{itemize}
if and only if \( c \) and \( d \) differ only in the \( i-1 \)th and \( i \)th positions and moreover \( d_i = 0 \) and \( c_i = 1 \). Therefore \( d < c \) in the lexicographic order and the matrix is lower triangular.

- Involution.

To complete the proof, we have to show that when \( c \) and \( d \) do not satisfy \( \hat{c} + \hat{d} = 0 \cdots 0110 \cdots 0 \) with 1s in the \( i-1 \)th and \( i \)th positions, \( (H_L M_L H_L)_{c,d} = 0 \). To do this, we introduce another sign-reversing involution just as in lemma 2. For a fixed \( c, d \) and any pair \((e, f)\) which gives a contribution of \( \alpha \) to \( (H_L M_L H_L)_{c,d} \), say, we find another pair \((e', f')\) which also gives the same contribution but such that \( c \cdot \hat{e} + f \cdot \hat{d} = c \cdot \hat{e}' + f' \cdot \hat{d} + 1 \). Notice that \((e, f)\) satisfies the condition that \( \hat{e} \) and \( \hat{f} \) differ only in the \( i \)th position and that \( f_{i-1} = 1 \).

This is easily done. Since \( c \neq d \), find the smallest value of \( j \), \( j \neq i, i-1 \), such that \( \hat{c}_j + \hat{d}_j = 1 \). Then choose \((e', f')\) as
\[
\begin{align*}
e & \mapsto e' = (e_1, \ldots, e_{L-j}, 1 - e_{L-j+1}, e_{L-j+2}, \ldots, e_L), \\
f & \mapsto f' = (f_1, \ldots, f_{L-j}, 1 - f_{L-j+1}, f_{L-j+2}, \ldots, f_L).
\end{align*}
\]

(14)

Then \( e + f = e' + f' \) and \( e \cdot (\hat{c} + \hat{d}) = e' \cdot (\hat{c} + \hat{d}) \pm 1 \) and we are done. Notice that we need \( j \neq i-1, i \) to ensure that \( e_{i-1} = e_i \) remains intact. If \( \hat{c} \) differs from \( \hat{d} \) only at these two positions, then we have already proved that there can be no such involution.

We have now done all the work necessary to write down a proof of the main result.

**Proof of theorem 1.** Since \( H_L \) is symmetric and \( H_L^2 = 1 \) by lemma 2, \( H_L M_L H_L \) has the same characteristic polynomial as \( M_L \). But the former is lower triangular. The characteristic polynomial is therefore
\[
|M_L - \lambda I_L| = \prod_{i=1}^{2^L} (-\lambda - (H_L M_L H_L)_{i,i}).
\]

(15)

Using the representation of the rows and columns using Boolean vectors of size \( L \) and using (12), we have the required result.

**Corollary 4.** The spectral gap for the system of size \( L \) is given by
\[
\min_{i=1}^{L} (\alpha_i + \beta_i).
\]

(16)

**Proof.** The proof directly follows from the explicit construction of eigenvalues in theorem 1. \( \square \)
4. Recurrence relation for the transition matrices

We will prove that there is a recursion of order one among the transition matrices. Let $M_L(\alpha_1, \ldots, \alpha_L; \beta_1, \ldots, \beta_L)$ represent the transition matrix for the system of size $L$ with parameters $\alpha_i$ and $\beta_i$.

**Theorem 5.** Let $I$ denote the identity matrix of size $2^{L-1}$. Then $M_L$ can be expressed in $2 \times 2$ block-diagonal form as

$$M_L(\alpha_1, \ldots, \alpha_L; \beta_1, \ldots, \beta_L) = \begin{pmatrix} M_{1,1} & \alpha_1 I \\ \beta_1 I & M_{2,2} \end{pmatrix},$$

where

$$M_{1,1} = M_{L-1}(\alpha_2, \alpha_3, \ldots, \alpha_L; \beta_2, \beta_3, \ldots, \beta_L) - \beta_1 I,$$

$$M_{2,2} = M_{L-1}(\beta_2, \alpha_3, \ldots, \alpha_L; \alpha_2, \beta_3, \ldots, \beta_L) - \alpha_1 I,$$

with the initial matrix for $L = 1$ given by

$$M_1(\alpha_1; \beta_1) = \begin{pmatrix} -\beta_1 & \alpha_1 \\ \beta_1 & -\alpha_1 \end{pmatrix}.$$

**Proof.** The matrix for size 1 given in (19) is easy to check. The matrix recursion (17) is proved by looking at transitions that binary vectors beginning with different two-bit vectors undergo. Let $v, v'$ denote binary vectors of length $L$ and $w, w'$ denote vectors of length $L-1$. There are then four possibilities of transitions depending on the first bits of $v$ and $v'$, namely $v = 0w$ or $1w$ and similarly for $v'$.

1. $0w \to 1w'$ (respectively $1w \to 0w'$).
   These transitions are only possible if $w = w'$ and then occur with rate $\beta_1$ (respectively $\alpha_1$). This is why the $(2, 1)$th (respectively $(1, 2)$th) block of $M_L$ is $\beta_1 I$ (respectively $\alpha_1 I$).
2. $0w \to 0w'$ (respectively $1w \to 1w'$).
   In this case, all the transitions for the system of size $L-1$ go through but with rates whose indices are one more than that of the smaller system simply because one extra site has been added to the left. For example, what occurred with rate $\alpha_2$ in the system of size $L-1$ occurs with rate $\alpha_3$ now. Because of the extra transitions in the off-diagonal blocks (described above), we must add negative terms to the diagonal so that column sums are zero. Therefore we subtract $\beta_1 I$ (respectively $\alpha_1 I$) from $M_{1,1}$ (respectively $M_{2,2}$).

$\square$
For later consideration, we also express the recursion relation using a second order recurrence relation. We express the transition matrix for the system of size $L - 1$ as

$$M_L(\alpha_1, \ldots, \alpha_L; \beta_1, \ldots, \beta_L) = \begin{pmatrix}
M_{1,1} - \frac{\alpha_2}{\beta_1 + \beta_2} & \alpha_2 & \alpha_1 & 0 \\
\beta_2 & M_{2,2} - \frac{\beta_1}{\beta_1 + \alpha_2} & 0 & \alpha_1 \\
\beta_1 & 0 & M_{1,1} - \frac{\alpha_1}{\alpha_1 + \beta_2} & \beta_2 \\
0 & \beta_1 & \alpha_2 & M_{2,2} - \frac{\beta_1}{\alpha_1 + \beta_2}
\end{pmatrix},$$  

(20)

where

$$M_{1,1} = M_{L-2}(\alpha_3, \alpha_4, \ldots, \alpha_L; \beta_3, \beta_4, \ldots, \beta_L),$$

$$M_{2,2} = M_{L-2}(\beta_3, \alpha_4, \ldots, \alpha_L; \alpha_3, \beta_4, \ldots, \beta_L).$$

(21)

**Remark 3.** Notice that the transition matrix $M_L$ is invariant under the transformation $M_L \leftrightarrow (M_L)^T$ and $\alpha \leftrightarrow \beta$.

### 5. Density in the steady state

We compute the density of 1s in the steady state at site $k$ in a system of size larger than $k$. Using remark 2, this density is completely independent of the size of the system.

Let $S_k$ be the set of subsets of $[k] = \{1, \ldots, k\}$ with an odd number of elements, and for $s \in S_k$, let $\bar{s} = S_k \setminus s$.

**Lemma 6.** The density of 1s at site $k$ is given by

$$\langle \eta_k \rangle = \frac{\sum_{s \in S_k} \prod_{i \in s} \beta_i \prod_{j \in \bar{s}} \alpha_j}{\prod_{j=1}^{k} (\alpha_j + \beta_j)}.$$  

(22)

**Proof.** The master equation for the density of 1s at site 1 is

$$\frac{d}{dt} \langle \eta_1 \rangle = \beta_1 (1 - \eta_1) - \alpha_1 \langle \eta_1 \rangle = 0,$$

(23)

which implies $\eta_1 = \beta_1 / ((\alpha_1 + \beta_1))$, which is consistent with (22) when $k = 1$.

For $k > 1$, the master equation is

$$\frac{d}{dt} \langle \eta_k \rangle = \beta_k \langle (1 - \eta_{k-1})(1 - \eta_k) \rangle + \alpha_k \langle \eta_{k-1}(1 - \eta_k) \rangle - \alpha_k \langle (1 - \eta_{k-1})\eta_k \rangle - \beta_k \langle \eta_{k-1}\eta_k \rangle = 0,$$

(24)

which on simplifying gives

$$\langle \eta_k \rangle = \frac{\alpha_k \langle \eta_{k-1} \rangle + \beta_k \langle 1 - \eta_{k-1} \rangle}{\alpha_k + \beta_k}.$$  

(25)
To see that (22) satisfies this recurrence, one notes that (22) implies
\[
\langle 1 - \eta_k \rangle = \frac{\sum_{s \in S_k} \prod_{i \in s} \alpha_i \prod_{j \notin s} \beta_j}{\prod_{j=1}^{k} (\alpha_j + \beta_j)}.
\]
Now partition the set $S_k$ into two subsets $S_k^{(1)}$ and $S_k^{(2)}$ such that $S_k^{(1)}$ contains the element $k$ and $S_k^{(2)}$ does not. If $s$ is an element of $S_{k-1}$, then $s \in S_k^{(2)}$ and $s \cup \{k\} \in S_k^{(1)}$. One can then clearly see that $S_k^{(2)}$ contributes to the first term in (25) and $S_k^{(1)}$ contributes to the second.

It is natural to ask whether one can find a simple formula like (22) for higher correlation functions. We have not been able to guess such a formula in general even for two-point correlation functions. We have some idea of the denominator in that case, but the main problem is that exact calculations are very time consuming. For example, for a system of size four, we get a transition matrix of size 16 with eight symbolic parameters. It takes a very long time to calculate the zero eigenvector for this matrix.

6. The normalization factor

Consider the system of size $L$. For any configuration $\eta$, we can express its steady state probability as a numerator divided by a denominator. We define the normalization factor $Z_L(\vec{\alpha}, \vec{\beta})$ as the least common multiple of the denominators of the steady state probabilities for all configurations $\eta$ such that the greatest common divisor of the corresponding numerators is 1.

The physical intuition of $Z_L$ is that it is the ratio of the probability of the most likely configuration to the least likely configuration. We have been able to find a conjecture for $Z_L$ but have not been able to prove the formula in general.

**Conjecture 1.** The normalization factor for a system of size $L$ with rates $\alpha_i$ and $\beta_i$ is given by
\[
Z_L(\vec{\alpha}, \vec{\beta}) = \prod_{i=1}^{L} (\alpha_i + \beta_i) \prod_{1 \leq i < j \leq L} (\alpha_i + \beta_i + \alpha_j + \beta_j).
\]

We obtained this conjecture by calculating the steady state probabilities exactly on a computer up to size four. In certain special cases discussed in section 7, the following conjecture does reduce correctly for all sizes. In certain special models analysed in section 7, we find that this normalization factor provably grows like $2^{L^2}$. Such growth would not be possible for a one-dimensional equilibrium model. It would be very interesting to understand why this behaviour is possible for nonequilibrium models. Further, this raises the question of the maximum possible growth of the normalization factor for (one-dimensional) nonequilibrium models.

7. Special cases: ferromagnetic and antiferromagnetic models

We now analyse two limits of this model which depend on two parameters, $\alpha_1 = \alpha$ and $\beta_1 = \beta$. Both these limits can be interpreted as nonequilibrium analogues of the ferromagnetic and antiferromagnetic Ising models respectively, hence the terminology.
The strategy for the analysis of both cases is similar. We will first use theorem 5 to write a first order recurrence for the transition matrices. We will then show that the transfer matrix ansatz [13] holds, using which we will express the steady state probabilities of the system of size \( L \) in terms of the system of size \( L - 1 \).

We first recall the definition of the transfer matrix ansatz. A family of Markov processes satisfies the transfer matrix ansatz if there exist matrices \( T_L \) for all sizes \( L \) such that

\[
M_{L+1}T_L = T_L M_L .
\]  

We also impose that this equality is nontrivial in the sense that

\[
M_{L+1}T_L \neq 0 .
\]

The rectangular transfer matrices or conjugation matrices \( T_L \) can be interpreted as a consequence of the integrability of the model.

For the system we consider here, the transition matrices are of size \( 2^L \). It is most convenient for us to take the naturally ordered basis of binary sequences of size \( L \). For example, when \( L = 2 \), the ordered list is \((00, 01, 10, 11)\).

Using the transfer matrices, we will calculate the normalization factor \( Z_L \) for both these models. Letting \(|v_1\rangle\) be the Perron–Frobenius eigenvector of the system of size one, we can define

\[
Z_L = \langle 1_L | T_{L-1} T_{L-2} \cdots T_1 | v_1 \rangle , \tag{30}
\]

where \(|1_L\rangle\) is the row vector of size \( 2^L \) composed of 1s.

For these special cases, we will explicitly calculate the density, the recurrences for the transition matrix and the transfer matrix, as well as the formula for the normalization factor. We will omit all the proofs since they are quite straightforward, if somewhat tedious. The recurrences for the transition matrix follow from the recurrence (20) and the others can be provided without too much difficulty using induction. The technique of proof is the same as was used in [13].

7.1. Ferromagnetic limit

We consider the case \( \alpha_i = 1 \) and \( \beta_i = 0 \) for \( i > 1 \). The densities are easily calculated. The only term that contributes to the sum in (22) is the subset \( \{1\} \in S_k \). Therefore,

\[
\langle \eta_k \rangle = \frac{\beta}{\alpha + \beta} \tag{31}
\]

for all \( k \). Therefore all densities are identical, which is reminiscent of the ferromagnetic limit where all spins prefer to be aligned in the same direction.

**Corollary 7.** Let \( I \) denote the identity matrix of size \( 2^{L-2} \). Suppose the transition matrix for size \( L - 1 \) is written in block-diagonal form as

\[
M_{L-1}^{(F)} = \begin{pmatrix}
M_{11}^{(F)} & \alpha I \\
\beta I & M_{22}^{(F)}
\end{pmatrix} . \tag{32}
\]
Then

\[
M^{(F)}_L = \begin{pmatrix}
M^{(F)}_{11} & \mathbb{I} & \alpha \mathbb{I} & 0 \\
0 & M^{(F)}_{22} - (1 - \alpha + \beta) \mathbb{I} & 0 & \alpha \mathbb{I} \\
\beta \mathbb{I} & 0 & M^{(F)}_{11} - (1 - \beta + \alpha) \mathbb{I} & 0 \\
0 & \beta \mathbb{I} & \mathbb{I} & M^{(F)}_{22}
\end{pmatrix},
\]

(33)

where \( M^{(F)}_L \) is written as a \( 2 \times 2 \) block matrix with each block made up of matrices of size \( 2^{L-1} \). The initial matrix for \( L = 1 \) is given by

\[
M^{(F)}_1 = \begin{pmatrix}
-\beta & \alpha \\
\beta & -\alpha
\end{pmatrix}.
\]

(34)

Let \( \sigma_L \) denote the matrix of size \( 2^L \) with 1s on the antidiagonal and zeros everywhere else. For example,

\[
\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

(35)

**Lemma 8.** Define rectangular matrices \( T^{(F)}_L \) of size \( 2^{L+1} \times 2^L \) using the following recurrence relation:

\[
T^{(F)}_1 = \begin{pmatrix}
1 - \beta + \alpha & \alpha \\
0 & \alpha \\
\beta & 0 \\
\beta & 1 - \alpha + \beta
\end{pmatrix},
\]

(36)

and define \( T^{(F)}_L \) by the following recursion. If

\[
T^{(F)}_L = \begin{pmatrix}
T^{(F)}_{11} & T^{(F)}_{12} \\
0 & T^{(F)}_{22} \\
T^{(F)}_{31} & 0 \\
T^{(F)}_{41} & T^{(F)}_{42}
\end{pmatrix}
\]

(37)

then

\[
T^{(F)}_L = \begin{pmatrix}
2T^{(F)}_{11} & T^{(F)}_{11} & 2T^{(F)}_{12} & T^{(F)}_{12} \\
0 & \sigma_{L-2} T^{(F)}_{11} \sigma_{L-2} & 0 & T^{(F)}_{22} \\
0 & 0 & T^{(F)}_{12} & 0 \\
2T^{(F)}_{31} & T^{(F)}_{31} & 0 & 0 \\
0 & 0 & T^{(F)}_{41} & 0 \\
T^{(F)}_{31} & 0 & \sigma_{L-2} T^{(F)}_{42} \sigma_{L-2} & 0 \\
T^{(F)}_{41} & T^{(F)}_{41} & 2T^{(F)}_{42} & 2T^{(F)}_{42}
\end{pmatrix},
\]

(38)

Then the family of Markov processes given by the transition matrices \( \{ M^{(F)}_L \} \) satisfy the transfer matrix ansatz with \( T^{(F)}_L \) defined above, i.e. \( M^{(F)}_L T^{(F)}_{L-1} = T^{(F)}_{L-1} M^{(F)}_{L-1} \).
Using lemma 8, we can compute the normalization factor $Z^{(F)}_L(\alpha, \beta)$ using (30) by setting

$$|v_1\rangle = \left( \begin{array}{c} \beta \\ \alpha \end{array} \right).$$

(39)

For example, $Z^{(F)}_1 = \alpha + \beta$. We find a remarkable property of the normalization factor of the system, namely its super-extensive growth with the size of the system.

**Corollary 9.** The normalization factor of the system of size $L$ is given by

$$Z^{(F)}_L = 2 \left( \frac{L - 1}{2} \right) (\alpha + \beta)(1 + \alpha + \beta)^{L-1}.$$  

(40)

### 7.2. Antiferromagnetic limit

We consider the case $\alpha_i = 0$ and $\beta_i = 1$ for $i > 1$. Now the only terms that can contribute to the sum in (22) are the subsets $\bar{s} = \phi, \{1\} \in S_k$. Therefore, the density depends on the parity of the site. If $k$ is odd (respectively even), $s = [k]$ (respectively $s = \{2, \ldots, k\}$) is the only contribution,

$$\langle \eta_k \rangle = \begin{cases} \frac{\beta}{\alpha + \beta}, & k \text{ odd,} \\ \frac{\alpha}{\alpha + \beta}, & k \text{ even.} \end{cases}$$

(41)

This is reminiscent of the antiferromagnetic Ising model in which neighbouring sites prefer to be aligned opposite to one another.

**Corollary 10.** Let $\mathbb{I}$ denote the identity matrix of size $2^{L-2}$. Suppose the transition matrix for size $L - 1$ is written in block-diagonal form as

$$M^{(A)}_{L-1} = \begin{pmatrix} M^{(A)}_{11} & \alpha \mathbb{I} \\ \beta \mathbb{I} & M^{(A)}_{22} \end{pmatrix}.$$

(42)

Then

$$M^{(A)}_L = \begin{pmatrix} M^{(A)}_{11} - \mathbb{I} & 0 & \alpha \mathbb{I} & 0 \\ \mathbb{I} & M^{(A)}_{22} - (\alpha - \beta) \mathbb{I} & 0 & \alpha \mathbb{I} \\ \beta \mathbb{I} & 0 & M^{(A)}_{11} - (\beta - \alpha) \mathbb{I} & \mathbb{I} \\ 0 & \beta \mathbb{I} & 0 & M^{(A)}_{22} - \mathbb{I} \end{pmatrix},$$

(43)

where $M^{(A)}_L$ is written as a $2 \times 2$ block matrix with each block made up of matrices of size $2^{L-1}$. The initial matrix for $L = 1$ is given by

$$M^{(A)}_1 = \begin{pmatrix} -\beta & \alpha \\ \beta & -\alpha \end{pmatrix}.$$  

(44)
Lemma 11. Define rectangular matrices $T_L^{(A)}$ of size $2^{L+1} \times 2^L$ using the following recurrence relation:

$$T_1^{(A)} = \begin{pmatrix} 0 & \alpha \\ 1 - \beta + \alpha & \alpha \\ \beta & 1 - \alpha + \beta \\ 0 \end{pmatrix},$$

and if

$$T_{L-1}^{(A)} = \begin{pmatrix} 0 & T_1^{(A)} \\ T_2^{(A)} & T_3^{(A)} \\ T_4^{(A)} & 0 \end{pmatrix},$$

then

$$T_L^{(A)} = \begin{pmatrix} 0 & 0 & 0 & T_1^{(A)} \\ 0 & 0 & 2T_2^{(A)} & T_3^{(A)} \\ \sigma T_2^{(A)} & 2\sigma T_1^{(A)} & T_4^{(A)} & 2T_4^{(A)} \\ 0 & T_3^{(A)} & 0 & T_5^{(A)} \\ 2T_4^{(A)} & T_5^{(A)} & 2\sigma T_3^{(A)} & \sigma T_4^{(A)} \\ T_5^{(A)} & 2T_4^{(A)} & 0 & 0 \\ T_4^{(A)} & 0 & 0 & 0 \end{pmatrix}.$$  (47)

Then the family of Markov processes given by the transition matrices $\{M_L^{(A)}\}$ satisfy the transfer matrix ansatz with $T_L^{(A)}$ defined above, i.e. $M_L^{(A)} T_{L-1}^{(A)} = T_{L-1}^{(A)} M_L^{(A)}$.

Using lemma 8, we can compute the normalization factor $Z_L^{(A)}(\alpha, \beta)$ using (30) by using the same formula for $|v_1\rangle$ as in the ferromagnetic case, (39). We find that the normalization factor is exactly the same as for the ferromagnetic model.

Corollary 12. The normalization factor of the system of size $L$ is given by

$$Z_L^{(A)} = 2 \left( \frac{L-1}{2} \right) (\alpha + \beta)(1 + \alpha + \beta)^{L-1}.$$  (48)

7.3. Relation between these special cases

The transition matrix $M_L^{(X)}$ for $X = F, A$ is invariant, in addition to the transposition symmetry, under the transformation

$$(M_L^{(X)})_{i,j} \leftrightarrow (M_L^{(X)})_{2L+1-i,2L+1-j} \quad \text{and} \quad \alpha \leftrightarrow \beta$$  (49)

in the usual matrix coordinate notation where $i, j$ vary from 1 to $2^L$. Similarly, the transfer matrix $T_L^{(X)}$ is invariant under the transformation

$$(T_L^{(X)})_{i,j} \leftrightarrow (T_L^{(X)})_{2L+1-i,2L+1-j} \quad \text{and} \quad \alpha \leftrightarrow \beta$$  (50)

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where this time \( i \) varies from 1 to \( 2^{L+1} \) and \( j \) varies from 1 to \( 2^L \). The normalization factors for the two systems are exactly the same, \( Z_L^{(F)} = Z_L^{(A)} \). One also sees that the recurrence relations for the transition matrices and the transfer matrices are similar.

The reason that there are so many similarities between the two models is that one can establish an exact correspondence between the ferromagnetic and antiferromagnetic limits defined previously using remark 1. In this case, we use the second part of the remark. As a consequence of this coupling, time-dependent correlation functions in the ferromagnetic model are related to those in the antiferromagnetic model.

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Note added in proof. After this article was accepted, an article appeared on the asymmetric Glauber dynamics of a one-dimensional Ising chain with periodic boundary conditions [15]. The dynamics there is partially asymmetric as opposed to the total asymmetry here. It would certainly be interesting to extend that study to systems with open boundaries.

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