Saturated chains in composition posets

Jan Snellman
Department of Mathematics, Stockholm University
SE-10691 Stockholm, Sweden
email: Jan.Snellman@math.su.se

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Abstract

We study some poset structures on the set of all compositions. In the first case, the covering relation consists of inserting a part of size one to the left or to the right, or increasing the size of some part by one. The resulting poset $\mathcal{N}$ was studied by the author in [5] in relation to non-commutative term orders, and then in [6], where some results about generating functions for standard paths in $\mathcal{N}$ was established. This was inspired by the work of Bergeron, Bousquet-Mélou and Dulucq [1] on standard paths in the poset $\mathcal{BBD}$, where there are additional cover relations which allows the insertion of a part of size one anywhere in the composition. Finally, following a suggestion by Richard Stanley we study a poset $\mathcal{S}$ which is an extension of $\mathcal{BBD}$. This poset is related to quasi-symmetric functions.

For these posets, we study generating functions for saturated chains of fixed width $k$. We also construct “labeled” non-commutative generating functions and their associated languages.

Keywords: Partially ordered sets, chains, enumeration, non-commutative generating functions.

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1 Introduction

To an integer partition one can associate its diagram, which is a finite subset of $\mathbb{N}^2$. Ordering the set of partitions by inclusion of diagrams, one gets a locally finite, ranked, distributive lattice $\mathcal{Y}$ which is known as Young’s lattice. The empty partition $\emptyset$ is the unique minimal element, and saturated chains in $\mathcal{Y}$ from the bottom element corresponds to an increasing sequence of diagrams, where at each step a single box is added. Such a sequence can be succinctly coded as a standard tableau on the final diagram in the chain. This well-known construction is used not only in combinatorics, but also in the representation theory of the symmetric group.
In [4] it was observed that Young’s lattice also classifies the standard term orders, i.e. admissible group orders $\leq$ on $G = \mathbb{Z}^n$ where, for a fixed choice of basis $e_1, \ldots, e_n$ of $G$ it holds that $e_1 < \cdots < e_n$. The correspondence is as follows: first, we can consider instead standard monoid orders on $\mathbb{N}^n$, which is isomorphic to the monoid of power products in $x_1, \ldots, x_n$. Secondly, we send the power product $x_i^r$ to the partition which has $r$ parts of size $i$. Third, we extend this to a monomial $x_1^{a_1} \cdots x_n^{a_n}$ in the natural way. The image of this injective map will be all partitions with parts of size $\leq n$.

This is a sublattice of Young’s lattice, and if we pull back this order to the monoid of power products in $x_1, \ldots, x_n$, we get a partial order which is the intersection of all standard term orders. Any standard term order is thus a multiplicative total extension of this poset. We can be bold and allow infinitely many indeterminates $x_1, x_2, x_3, \ldots$; the poset so obtained is isomorphic to Young’s lattice.

If we do the same for non-commutative term orders, i.e. monoid orderings of the free non-commutative monoid on $x_1, x_2, \ldots$ such that $x_1 < x_2 < \cdots$, then the resulting poset is no longer a lattice. It is natural to map a non-commutative monomial to a composition rather than a partition. If we order the set of compositions by pushing forward the order relation on non-commutative monomials via this bijection, then the resulting poset structure has the following covering relations:

1. $(1, a_1, \ldots, a_n) \succ (a_1, \ldots, a_n)$, i.e. we may insert a part of size one to the left,
2. $(a_1, \ldots, a_n, 1) \succ (a_1, \ldots, a_n)$, i.e. we may insert a part of size one to the right,
3. $(a_1, \ldots, a_i, \ldots, a_n) \succ (a_1, \ldots, a_i + 1, \ldots, a_n)$, i.e. we may increase the size of a part by one.

The “sorting map” from compositions to partitions is order-preserving, and we can regard the above poset as a non-commutative analogue of Young’s lattice.

It is, however, not the only possible such analogue! In [1], Bergeron, Bousquet-Méloü and Dulucq consider an analogous poset on the set of compositions. This poset, henceforth denoted by $\mathcal{BBD}$, is an extension of $\mathcal{R}$: there are one additional type of covering relations:

$$(a_1, \ldots, a_i, 1, \ldots, a_n) \succ (a_1, \ldots, a_n),$$

i.e. one can insert a part of size 1 anywhere in the composition. They encoded standard paths, i.e. saturated chains from the empty composition () to some composition $P$ as tableau on the diagram of $P$. This is a direct counterpart to standard Young tableaux.
Using the theory of labeled binary trees, they were able to explicitly solve the differential equation satisfied by the exponential generating function for such standard paths, and furthermore to give precise asymptotics for the number of such paths of a given length.

They also considered the simpler problem of enumerating standard paths of a fixed width \( k \), i.e. ending at a composition with \( k \) parts. Here, the generating functions turned out to be rational, given by a simple recurrence formula.

In [6] the ideas of Bergeron et al were used to give generating functions for standard paths of fixed width in \( \mathcal{R} \). In the present paper, we consider saturated chains starting from an arbitrary composition. We also introduce a non-commutative generalization, which encodes all information about the saturated chains, not only their endpoints. We’ll see that these non-commutative power series are still rational, hence recognizable and given by a finite state machine.

We also consider yet another poset, proposed by Richard Stanley. This poset, which we denote by \( S^\infty \), extends \( \mathcal{BBB} \) in such a way that a composition of \( n \) is covered by precisely \( n + 1 \) compositions. It occurs naturally in the study of the fundamental quasi-symmetric functions. We consider an infinite family of posets \( S^d \), all extending \( \mathcal{BBB} \), which have the desired poset \( S^\infty \) as their inductive limit, and introduce a compact, unifying formalism for describing these posets, together with \( \mathcal{R} \) and \( \mathcal{BBB} \).

Some related aspects of enumeration that we have not addressed are

- Enumeration of saturated chains without restriction of the width, as in [1],
- Non-saturated chains, i.e. with steps of length > 1, as in [3, 7, 8],
- Oscillating tableaux, i.e. chains going up and down in the poset, as in [2].

2 Posets of compositions

2.1 Multi-rankings on compositions

By a composition \( P \) we mean a sequence of positive integers

\[
P = (p_1, p_2, \ldots, p_k),
\]

which are the parts of \( P \). We define the width \( \ell(P) \) of \( P \) as the number of parts, and the height as the size of the largest part. The weight \( |P| = \sum_{i=1}^{k} p_k \) of \( P \) is the sum of its parts. If \( P \) has weight \( n \) then \( P \) is a composition of \( n \), and we write \( P \models n \).
Let $\mathcal{C}$ denote the set of all compositions (including the empty one). For a non-negative integer $k$, let $\mathcal{C}(k)$ denote the subset of compositions of width $k$.

The diagram of a composition $P = (p_1, \ldots, p_k)$ is the set of points $(i, j) \in \mathbb{Z}^2$ with $1 \leq j \leq p_i$. Alternatively, we can replace the node $(i, j)$ by the square with corners $(i - 1, j - 1), (i - 1, j), (i, j - 1)$ and $(i, j)$. So the composition $(1, 2, 4)$ has diagram

Thus, for a composition $P$ the height and width of $P$ is the height and width of the smallest rectangle containing its diagram.

**Definition 1.** We let $\mathbb{N}^\omega$ denote the poset of finitely supported maps $\mathbb{N}^+ \to \mathbb{N}$, with component-wise comparison. The Young lattice $\mathcal{Y}$ is the sublattice of (weakly) decreasing maps. For any positive integer $n$, $\mathbb{N}^n$ can be identified with the subposet of $\mathbb{N}^\omega$ consisting of maps with support in $\{1, 2, \ldots, n\}$.

Let $e_i$ be the $i$'th unit vector, and put

$$f_j = \sum_{i=1}^{j} e_i. \quad (2)$$

We define the multi-weight of $P$ by

$$\text{mw}(P) = \sum_{i=1}^{k} f_{p_i} \in \mathcal{Y}. \quad (3)$$

We define $\text{mw}^*(P) = \text{mw}(P)^*$, where $*$ is conjugation (which is an order-preserving involution on $\mathcal{Y}$).

Note that $\text{mw}^*(P) = (q_1, \ldots, q_k)$ is the decreasing reordering of $P = (p_1, \ldots, p_k)$.

**Definition 2.** A locally finite poset $(A, \geq)$ is said to be $\omega$-multi-ranked if there exists a map

$$\Phi : A \to \mathbb{N}^\omega \quad (4)$$

such that

$$m \succ m' \implies \Phi(m) \succ \Phi(m'). \quad (5)$$

The poset is $n$-multi-ranked if it is $\omega$-multi-ranked by $\Phi$ and $\Phi(A) \subseteq \mathbb{N}^n$. Similarly, the poset is $\mathcal{Y}$-multi-ranked if $\Phi(A) \subseteq \mathcal{Y}$.
Clearly, if $A$ is $\omega$-multiranked then it is ranked (i.e. 1-multiranked), since $\Phi$ followed by the collapsing

$$\mathbb{N}^\omega \to \mathbb{N}$$

$$\alpha = (\alpha_1, \alpha_2, \ldots) \mapsto |\alpha| = \sum_{i=1}^{\infty} \alpha_i$$

will be a ranking.

We will presently introduce the partial orders $\mathcal{N}$ and $\mathcal{BBD}$ on the set $\mathcal{C}$ of all compositions. Our main interest is the poset $\mathcal{N}$, which is related to non-commutative term orders [5]. This poset is $\mathcal{Y}$-ranked, as is the extension $\mathcal{BBD}$, studied in [1].

However, we will consider also an extension which we denote by $\mathcal{S}$. This poset, brought to our attention by Richard Stanley, is not $\mathcal{Y}$-ranked. In order to have a concept broad enough to also encompass this poset, we define:

**Definition 3.** A locally finite poset $(a, \geq)$ is said to be almost $\mathcal{Y}$-multiranked if there exists

1. an extension $T$ of the partial order $\leq$ on $\mathcal{Y}$, and
2. a map $\Phi : A \to (\mathcal{Y}, T)$ such that
   i) $T$ is rank-preserving, i.e. $\alpha \succ \beta$ w.r.t. $T$ implies that $|\alpha| = |\beta| + 1$,
   ii) $m \succ m'$ $\implies$ $\Phi(m) \succ \Phi(m')$,
   iii) $\Phi(A) = \mathcal{Y}$,
   iv) if $\alpha \succ \beta$ w.r.t. $T$ then there are $u, v \in A$ such that $u \succ v$ and $\Phi(u) = \alpha$, $\Phi(v) = \beta$.

In particular, $A$ is ranked via $u \mapsto |\Phi(u)|$.

The definition of $\omega$-ranked posets is very natural, the definition of $\mathcal{Y}$-ranked posets somewhat less so. The definition of almost $\mathcal{Y}$-ranked posets is very *ad hoc*; it aims to capture enough of the salient features of the posets $\mathcal{S}^d$, to be defined below, so that they can be considered together with $\mathcal{N}$ and $\mathcal{BBD}$. It is a matter of aesthetics if one includes the conditions iii) and iv) or not.

3 Some (almost) $\mathcal{Y}$-multiranked posets of compositions

We will define posets $\mathcal{N}$, $\mathcal{BBD}$, $\mathcal{S}^d$, with underlying set $\mathcal{C}$.
3.1 Operations on compositions

We define the infinite alphabets

\[ A_{LR} = \{ L, R \} \]
\[ A_U = \{ U_j | j \in \mathbb{N}^+ \} \]
\[ A_{dV} = \{ V^r_i | i, r \in \mathbb{N}^+, r < d \} \]
\[ A_{LRUV} = A_{LR} \cup A_U \cup A_{dV} \] (7)

For an alphabet \( A \), we denote by \( A^* \) the free monoid on \( A \).

**Definition 4.** We define the following partially defined operations on \( C \). Let \( P = (p_1, \ldots, p_k) \) be a composition, then

1. \( L.P = (1, p_1, \ldots, p_k) \), defined for all \( P \),
2. \( R.P = (p_1, \ldots, p_k, 1) \), defined for all \( P \),
3. \( U_j.P = (p_1, \ldots, p_{j-1}, p_j + 1, p_{j+1}, \ldots, p_k) \), defined when \( j \leq k \),
4. \( V^1_i.P = (p_1, \ldots, p_{i-1}, 1, p_i, \ldots, p_k) \), defined when \( i \geq 2, p_{i-1} \geq 2 \),
5. \( V^r_i.P = (p_1, \ldots, p_{i-2}, p_{i-1} - r + 1, r, p_i, \ldots, p_k) \), defined when \( i \geq 2, p_{i-1} - r + 1 \geq 2, r \geq 2 \).

**Definition 5.** We define a partial left action on \( C \) by the free monoid \( A^*_{LRUV} \) in the following way. We define, for a word \( w = w't \), where \( t \in A_{LRUV}, P \in C \),

\[ w.P = w'.(t.P) \] (8)

if the action of \( t \) on \( P \) is defined, and if recursively the action of \( w' \) on \( t.P \) is defined.

We give \( L \) and all the \( U_j \), the highest priority, followed by the \( V^r_i \), with the convention that \( V^r_i \) has higher priority than \( V^s_j \) iff \( i < j \). The lowest priority is given to \( R \).

**Definition 6.** Let \( P \) be a composition and suppose that

\[ \mathcal{B} \subseteq A_{LRUV} \]

i) The action of one of the operations above, call it \( T \), on a composition \( P \) is admissible for \( P \) (relative to \( \mathcal{B} \)) if it is defined, and if \( T.P \neq S.P \) for all operations in \( \mathcal{B} \) with higher priority.

ii) The action of a word \( W = VT \in \mathcal{B}^* \), \( T \in \mathcal{B}, V \in \mathcal{B}^* \) is admissible for \( P \) (relative \( \mathcal{B} \)) iff the action of \( T \) on \( P \) is admissible, and recursively the action of \( V \) on \( T.P \) is admissible.
iii) We let \( \langle B; P \rangle \) be the set of words in \( B^* \) that are admissible for \( P \).

iv) We let \( \leq_B \) be the smallest poset \( \subset C \times C \) which contains
\[
\{ (Q, w.Q) \mid Q \in C, w \in \langle B^*; Q \rangle \}
\]

So \( P \leq_B Q \) if \( Q \) can be obtained from \( P \) using a sequence of admissible operations in \( B \).

3.2 A first example

Lemma 7. The poset \((C, \leq_{A_U})\) is isomorphic to the infinite direct sum

\[
\sum_{i \in \mathbb{N}} \mathbb{N}^i
\]

Proof. The map that sends the composition
\[
\alpha = (\alpha_1, \ldots, \alpha_r)
\]

to
\[
(\alpha_1 - 1, \ldots, \alpha_r - 1) \in \mathbb{N}^r
\]
is an order-preserving bijection. \( \square \)

A part of the Hasse diagram of this non locally finite poset is shown in Figure 1. The other posets that we will introduce presently are all extensions of this posets, connecting the various components and also adding links within each component.
3.2.1 Graphical representations

We have already introduced the diagram of a composition. Another graphical depiction is the so-called balls and bars representation: here, the composition \( P = (p_1, \ldots, p_r) \) is represented by \( r \) groups of balls, separated by vertical bars, the \( i \)’th group consisting of \( p_i \) balls. A third way of encoding the composition is to regard it as the “index vector” of a (non-commutative) monomial: the \( P \) above would be represented by

\[
x_{p_1} \cdots x_{p_r}
\]  

(9)

The effect of the operations \( U_j \) on \( P = (3, 4, 1, 3) \) is as follows:

| Operation | Result | Diagram | balls and bars | monomial |
|-----------|--------|---------|----------------|----------|
| \( P \)   | (3,4,1,2) | ![Diagram](image) | \( \text{o|oo|oo|o|oo} \) | \( x_3x_4x_1x_2 \) |
| \( U_1.P \) | (4,4,1,2) | ![Diagram](image) | \( \text{o|oo|oo|o|oo} \) | \( x_3^2x_1x_2 \) |
| \( U_2.P \) | (3,5,1,2) | ![Diagram](image) | \( \text{o|oo|oo|o|oo} \) | \( x_3x_5x_1x_2 \) |
| \( U_3.P \) | (3,4,2,2) | ![Diagram](image) | \( \text{o|oo|oo|o|oo} \) | \( x_3x_4x_2^2 \) |
| \( U_4.P \) | (3,4,1,3) | ![Diagram](image) | \( \text{o|oo|oo|o|oo} \) | \( x_3x_4x_1x_3 \) |

The operation \( U_j \) adds a box on top of the \( j \)’th column in the diagram, adds a ball to the \( j \)’th group of balls, and replaces the \( i \)’th variable \( x_{p_j} \) in the monomial with the variable \( x_{p_j} + 1 \).

3.3 The posets \( \mathfrak{N} \)

**Definition 8.** We define the following poset on the underlying set \( C \) of compositions:

\[
\mathfrak{N} = (C, \leq_B),
\]

where

\[
B = (A_{LR} \cup A_U).
\]
3.3.1 Graphical representations of the operations $L$ and $R$

The effect of the operations $L, R$ on $P = (3, 4, 1, 3)$ is as follows:

| Operation | Result | Diagram | balls and bars | monomial |
|-----------|--------|---------|----------------|----------|
| $P$       | (3,4,1,2) | | ooo|oooo|oo|oo | $x_3x_4x_1x_2$ |
| $L.P$     | (1,3,4,1,2) | | o|ooo|oooo|oo|oo | $x_1x_3x_4x_1x_2$ |
| $R.P$     | (3,4,1,2,1) | | ooo|ooooo|oo|oo|o | $x_3x_4x_1x_2x_1$ |

We see that $L$ adds a box to the left of the diagram, inserts a $o|$ to the left of the balls and bars, and multiplies the monomial to the left with $x_1$. Similarly, $R$ adds a box to the right of the diagram, inserts a $|o$ to the right of the boxes and bars, and multiplies the monomial to the right with $x_1$.

The poset $\mathcal{N}$ with covering relations given by the operations $L, R, U_j$ was introduced in [5] as a poset on the free monoid $X^*$, $X = \{x_1, x_2, x_3, \ldots\}$. It is the poset of all “multiplicative consequences” of the ordering $x_1 < x_2 < x_3 < x_4 < \cdots$ of the variables. For instance,

$$x_2 < x_3 \implies x_1x_2x_3^2 < x_1x_3x_2^2 = U_2(x_1x_2x_3^2).$$

Formally, it is the intersection of all standard term orders on $X^*$, where a standard term order is a total order such that

$$p < q \implies upv < uqv, \quad \forall p, q, u, v \in X^*.$$

The beginning of the Hasse diagram of $\mathcal{N}$ is shown in Figure 2.

3.4 The poset $\mathcal{BBD}$

**Definition 9.** We define $\mathcal{BBD}(\mathcal{C}, \leq _B)$ where

$$\mathcal{B} = (\{L\} \cup A_U \cup A_V^2).$$

This is the poset studied in [1]. Compared to $\mathcal{N}$, it has the additional covering relations given by $V_i^1$ which inserts a part of size one after a part of size $\geq 2$. Graphically, this looks like
Note that $V_1^4$ is not admissible for this $P$, and that adding a part of size one to the right is represented by $V_3^1$ rather than by $R$; in general, if the composition has $r$ parts, and ends with a run of $k$ parts of size 1, adding a one to right is represented by $V_1^1 r - k + 1$.

### 3.5 The posets $\mathcal{S}^d$

**Definition 10.** For $d$ a positive integer $> 1$ or $d = \infty$, $\mathcal{S}^d = (C, \leq_A)$, where

$$A = \{L\} \cup A_U \cup A_V^d.$$

The operations $V_j^2, V_j^3$ operate as follows on $P = (3, 4, 1, 2)$.

| Operation | Result | Diagram | balls and bars | monomial |
|-----------|--------|---------|----------------|----------|
| $P$       | (3,4,1,2) | | $\text{o00|o000|o|oo}$ | $x_3 x_4 x_1 x_2$ |
| $V^2_2.P$ | (2,2,4,1,2) | | $\text{o0|o000|o|oo}$ | $x_2^2 x_4 x_1 x_2$ |
| $V^2_3.P$ | (3,3,2,1,2) | | $\text{o00|o00|o|oo}$ | $x_3^2 x_2 x_1 x_2$ |
| $V^3_3.P$ | (3,2,3,1,2) | | $\text{o00|o000|o|oo}$ | $x_3 x_2 x_3 x_1 x_2$ |

In contrast to the other operations, the $V_i^r$’s, with $r \geq 2$, does not only involve adding an extra box to a column of the diagram, or inserting a new column; it also means taking away a box from the preceding column. This may seem unnatural and contrived, but there is another representation with respect to which these operations make perfect sense.
A compositions $\alpha = (\alpha_1, \ldots, \alpha_r)$ of $n$ can be encoded as a subset of $\{1, 2, \ldots, n - 1\}$ via the bijection

$$(\alpha_1, \ldots, \alpha_r) \mapsto S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_r - 1\}.$$  

If $\pi$ is a permutation on $\{1, 2, \ldots, n\}$ which has descent set

$$D_\pi = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_r - 1\},$$

consider all permutations on $\{0, 1, 2, \ldots, n\}$ which can be obtained by inserting a zero anywhere in the one-line representation of $\pi$. For each such permutation $\tau$ (there are of course exactly $n+1$ of them) calculate its descent set, and find the unique composition of $n + 1$ which maps to this descent set under (10). The compositions obtained are precisely the compositions which cover $\alpha$ in $\mathcal{S}$.

**Example 11.** Let $P = (3, 4, 1, 2)$ as before. This is represented as $S_P = \{3, 7, 8\} \subset \{1, 2, \ldots, 9\}$. The permutation $\pi = [1, 2, 4, 3, 5, 6, 9, 8, 7, 10]$ has descent set $S_P$. Inserting a zero at all possible places, we get 11 new permutations, 11 new descent set, and finally 11 compositions covering $P$, as shown in table 1.

| Permutation | Descent set | Composition | Operation |
|-------------|-------------|-------------|-----------|
| $[0, 1, 2, 4, 3, 5, 6, 9, 8, 7, 10]$ | $\{4, 8, 9\}$ | $(4, 4, 1, 2)$ | $U_1.P$ |
| $[1, 0, 2, 4, 3, 5, 6, 9, 8, 7, 10]$ | $\{1, 4, 8, 9\}$ | $(1, 3, 4, 1, 2)$ | $L.P$ |
| $[1, 2, 0, 4, 3, 5, 6, 9, 8, 7, 10]$ | $\{2, 4, 8, 9\}$ | $(2, 2, 4, 1, 2)$ | $V_2.P$ |
| $[1, 2, 4, 0, 3, 5, 6, 9, 8, 7, 10]$ | $\{3, 8, 9\}$ | $(3, 5, 1, 2)$ | $U_2.P$ |
| $[1, 2, 4, 3, 0, 5, 6, 9, 8, 7, 10]$ | $\{3, 4, 8, 9\}$ | $(3, 1, 4, 1, 2)$ | $V_2^1.P$ |
| $[1, 2, 4, 3, 5, 0, 6, 9, 8, 7, 10]$ | $\{3, 5, 8, 9\}$ | $(3, 2, 3, 1, 2)$ | $V_3^2.P$ |
| $[1, 2, 4, 3, 5, 6, 0, 9, 8, 7, 10]$ | $\{3, 6, 8, 9\}$ | $(3, 3, 2, 1, 2)$ | $V_3^2.P$ |
| $[1, 2, 4, 3, 5, 6, 9, 0, 8, 7, 10]$ | $\{3, 7, 9\}$ | $(3, 4, 2, 2)$ | $U_3.P$ |
| $[1, 2, 4, 3, 5, 6, 9, 8, 0, 7, 10]$ | $\{3, 7, 8\}$ | $(3, 4, 1, 3)$ | $U_4.P$ |
| $[1, 2, 4, 3, 5, 6, 9, 8, 7, 0, 10]$ | $\{3, 7, 8, 9\}$ | $(3, 4, 1, 1, 2)$ | $V_3^1.P$ |
| $[1, 2, 4, 3, 5, 6, 9, 8, 7, 10, 0]$ | $\{3, 7, 8, 10\}$ | $(3, 4, 1, 2, 1)$ | $V_4^1.P$ |

Table 1: Cover of $(3, 4, 1, 2)$ in $\mathcal{S}^\infty$

It is implicit in Stanley’s book [9] (see section 7.19, and in particular exercise 7.93), that the fundamental quasi-symmetric functions $L_\alpha$ multiply according to

$$L_1 L_\alpha = \sum_{\beta \succ \alpha} L_\beta$$  

(11)

where $\succ$ is the covering relation in $\mathcal{S}^\infty$. Here, the $L_\alpha$’s are defined by

$$L_\alpha = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$  

(12)
and the set
\[ \{ L_\alpha \mid \alpha \vdash n \} \]
is a basis for the homogeneous quasi-symmetric functions of degree \( n \).

3.6 Multi-ranking

If we identify the posets \( \mathfrak{N}, \mathfrak{BBB}, \) and \( \mathfrak{S}^d \) with their graphs, which are subsets of \( \mathcal{C} \times \mathcal{C} \), then

\[ \mathfrak{N} \subset \mathfrak{BBB} = \mathfrak{S}^2 \subset \mathfrak{S}^3 \subset \cdots \subset \cup_{d=1}^\infty \mathfrak{S}^d = \mathfrak{S}^\infty \quad (13) \]

The posets \( \mathfrak{N} \) and \( \mathfrak{BBB} \) have the same Hasse diagram up to rank 4, shown in Figure 2. For \( d > 2 \) there is an edge between (3) and (2, 2) in \( \mathfrak{S}^d \).

Lemma 12 ([5]). \( \mathfrak{N} \) and \( \mathfrak{BBB} \) are \( \mathcal{Y} \)-multiranked.

Proof. Let \( P = (p_1, \ldots, p_k) \) be a composition, and put

\[ (q_1, \ldots, q_k) = \mathfrak{mw}^*(P), \]

then \( (q_1, \ldots, q_k) \) is the decreasing reordering of \( P \). Adding a part of size 1 to \( P \) adds a part of size 1 at the end of \( \mathfrak{mw}^*(P) \), and increasing a part by one increases one part of \( \mathfrak{mw}^*(P) \) by one (a part which is strictly greater than its right neighbor). These operations are covering relations in the Young lattice, and all covering relations can be achieved. Furthermore, \( \mathfrak{mw} \) is surjective.

Lemma 13. The posets \( \mathfrak{S}^d \) are almost \( \mathcal{Y} \)-multiranked.
Proof. The operation $\text{mw}^*(P) \mapsto \text{mw}^*(V^r_i, P)$ corresponds to
\[
(q_1, \ldots, q_i, \ldots, q_k) \mapsto (q_1, \ldots, q_{i+1-r}, r, \ldots, q_k) \mapsto \text{mw}^*(q_1, \ldots, q_{i+1-r}, r, \ldots, q_k) \in \mathcal{Y},
\]
where the last step performs the necessary resorting so that the result is a partition. We can let $\mathcal{Y}^d$ be the smallest poset containing the original relations of $\mathcal{Y}$ together with these new ones. Then
\[
\text{mw}^* : \mathcal{C} \rightarrow \mathcal{Y}^d
\]
is an almost $\mathcal{Y}$-multiranking. \hfill \square

3.7 Saturated chains, standard paths, and tableaux

Now suppose that $\prec$ is a partial order on $\mathcal{C}$ such that $\text{mw}$ is an almost $\mathcal{Y}$-multiranking on $(\mathcal{C}, \prec)$.

Definition 14. If $Q$ is a composition, we define a saturated chain of length $n$, starting from $P$ and ending at $Q$ to be a sequence
\[
\gamma = (P_0 = P, P_1, P_2, \ldots, P_n = Q)
\]
of compositions such that
\[
P_0 \prec P_1 \prec P_2 \prec \cdots \prec P_n, \quad P_i \models i,
\]
Figure 4: The diagram of the saturated chain $\rho$ in $\mathcal{N}$.

i.e. $P_{i+1}$ should cover $P_i$ for all $i$.

A standard path is a saturated chain from the empty composition (\(\mathbb{0}\)).

We define the diagram, or the shape, of a saturated chain from $P$ to $Q$ to be the diagram of $Q$.

Saturated chains in $\mathcal{N}$ or in $\mathcal{BBD}$ can be coded as tableaux on the diagram of the terminal composition.

3.7.1 $\mathcal{N}$

Consider first the poset $\mathcal{N}$. With respect to this order,

$$\rho = ((1, 2), (2, 2), (1, 2, 2), (1, 2, 3), (1, 2, 4)) \quad (16)$$

is a saturated chain of length 4 from the minimal element $(1, 2)$ to the element $(1, 2, 4)$.

It is clear what meant by saying that the boxes in the diagram of $P_n$ should be labeled “in the order that they appear in the path”, if we (to avoid ambiguity) use the convention that whenever $P_i$ consists of $i$ ones and $P_{i+1}$ consists of $i+1$ ones, the extra one is considered to have been added to the left. This in accordance with the above notion of priority of operations, since $L$ has the highest priority. Thus, the only possible tableau for standard paths ending at the composition $(1, 1, 1, 1)$ is $\begin{array}{cccc} 4 & 3 & 2 & 1 \end{array}$.

As another example, the path $\rho$ corresponds to the tableau in Figure 4

3.7.2 $\mathcal{BBD}$

Now consider the poset $\mathcal{BBD}$, where there is the additional possibility of $P < V_j^1.P$. If $P = (p_1, p_2, \ldots, p_\ell, 1, \ldots, 1, p_s, \ldots, p_n)$, with $p_\ell > 1$, then if $\ell < i < j < s$ we have that $V_j^1.P = V_j^1.P$. However, the priority ordering and the rules for admissibility gives that only $i = \ell + 1$ is admissible. In other words, parts of size one can be inserted either to the left, or immediately after a part of size $> 1$. As an illustration, consider the following standard path (taken from [1]):

14
Figure 5: The diagram of the standard path $\gamma$ in the poset $\mathcal{BBD}$.

\[ \gamma: (\emptyset) \prec (1) \prec (1,1) \prec (1,2) \prec (1,2,1) \prec (1,3,1) \prec (1,3,1,1) \prec (1,3,1,2) \prec (2,3,1,2) \prec (2,3,1,3) \prec (2,3,1,4) \prec (2,3,1,5) \prec (2,3,1,1,5) \]

The diagram of $\gamma$ is shown in Figure 5.

It is clear that for these two posets, given a tableau we can reconstruct the saturated chain.

Nota bene: the step $(3,1,1,1) \leadsto (3,1,1)$ is considered as adding a part at the right in $\mathcal{N}$, but as adding a part after 3 in $\mathcal{BBD}$.

3.7.3 The shadow of a tableau

Since the mapping $\text{mw}^*: \mathcal{C} \to \mathcal{Y}$ is a multiranking for $\mathcal{N}$ and $\mathcal{BBD}$, every saturated chain $\gamma$, as in, in $\mathcal{N}$ or in $\mathcal{BBD}$ “lies over” the saturated chain

\[ \text{mw}^*(\gamma) = (\text{mw}^*(P_0), \text{mw}^*(P_1), \text{mw}^*(P_2), \ldots, \text{mw}^*(P_n)) \] (17)

in $\mathcal{Y}$. We call $\text{mw}^*(\gamma)$ the shadow of $\gamma$. If $T$ is a tableau representing $\gamma$, then we let $\gamma(T)$ be the standard skew-tableau of shape $\text{mw}^*(P_n)/\text{mw}^*(P_0)$ which encodes the way boxes are added to $\text{mw}^*(P_0)$ to obtain $\text{mw}^*(P_n)$; we call this the shadow of $T$. If $S$ is a standard skew-tableau of shape $\lambda/\mu$, then we define its multiplicity (w.r.t. $\mathcal{N}$ or $\mathcal{BBD}$) to be the number of saturated chains in $\mathcal{N}$ or in $\mathcal{BBD}$ having $\lambda/\mu$ as its shadow.

Example 15. The shadow of the tableaux in Figure 4 is shown in Figure 6. The shadow has multiplicity 8. The eight tableaux in $\mathcal{N}$ lying over the shadow is shown in Table 2.

4 Enumeration of saturated chains of fixed width

4.1 Definitions

Let $\mathcal{Q}$ be one of the posets on compositions considered above. For a saturated chain $\gamma$ of shape $(p_1, p_2, \ldots, p_k)$ we set

\[ v(\gamma) = x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k} \] (18)
Figure 6: The shadow of the saturated chain $\rho$ in $\mathcal{R}$.

Table 2: The eight tableaux in $\mathcal{R}$ that lie over the tableaux $S$.

Note that this is a commutative monomial, different from the representation used in (9).

We define the generating function

$$f_{\alpha}^k(x_1, \ldots, x_k) = f_{\alpha}^k[Q](x_1, \ldots, x_k) = \sum_{\gamma \text{ saturated chain of width } k \text{ starting from } \alpha} v(\gamma) \quad (19)$$

If $\alpha$ is the empty composition, then we omit the superscript.

**Definition 16.** If $f$ is a series in $x_1, x_2, x_3, \ldots$ and $d, i, j$ are positive integers, then

$$\Lambda_j(f)(x_1, x_2, x_3, \ldots) = f(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, x_{j+2}, \ldots)$$

$$\Delta_i^d(f) = \frac{x_i^d}{d!} \frac{\partial^d f}{\partial x_i^d}(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots) \quad (20)$$

We put $\Lambda = \Lambda_1$.

**4.2 Recurrence relations for the generating functions**

**Lemma 17.** Let $P = (p_1, \ldots, p_{k-1})$ be a composition. Then

(i) $v(L.P) = x_1 \Lambda(v(P))$,

(ii) $v(R.P) = x_k v(P)$,
(iii) \( v(U_j.P) = x_j v(P), \) for \( j < k, \)

(iv) \( v(V_i^1.P) = x_i \Lambda_i \left( v(P) - \Delta_{i-1}^1(v(P)) \right), \) \( 2 \leq i \leq k, \)

(v) \( v(V_i^d.P) = \frac{x^d}{x_{i-1}} \Lambda_i \left( v(P) - \sum_{j=1}^d \Delta_{j-1}^j(v(P)) \right), \) \( 2 \leq i \leq k, d \geq 2. \)

if the respective operations are admissible (otherwise the RHS is zero).

Proof. \( v(P) = x_1^{p_1} \cdots x_{k-1}^{p_{k-1}}, \) so

\[
v(L.P) = v((1, p_1, \ldots, p_{k-1})) \\
= x_1 x_2^{p_1} \cdots x_k^{p_{k-1}} \\
= x_1 \Lambda \left( x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} \right) \\
= x_1 \Lambda(v(P))
\]

\[
v(R.P) = v((p_1, \ldots, p_{k-1}, 1)) \\
= x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} x_k \\
= x_k v(P)
\]

\[
v(U_j.P) = v(p_1, \ldots, p_{j-1}, p_j + 1, p_{j+1}, \ldots, p_{k-1}) \\
= x_1^{p_1} \cdots x_{j-1}^{p_{j-1}} x_j^{p_j + 1} x_{j+1}^{p_{j+1}} \cdots x_{k-1}^{p_{k-1}} \\
= x_j x_1^{p_1} \cdots x_{j-1}^{p_{j-1}} x_j x_{j+1}^{p_{j+1}} \cdots x_{k-1}^{p_{k-1}} \\
= x_j v(P)
\]

If \( p_{i-1} \geq 2 \) then

\[
v(V_i^1.P) = v(p_1, \ldots, p_{i-2}, p_{i-1}, 1, p_i, \ldots, p_k) \\
= x_1^{p_1} \cdots x_{i-2}^{p_{i-2}} x_{i-1}^{p_{i-1}} x_i^{p_i} x_{i+1}^{p_{i+1}} \cdots x_{k-1}^{p_{k-1}} \\
= x_i x_1^{p_1} \cdots x_{i-2}^{p_{i-2}} x_{i-1}^{p_{i-1}} x_i^{p_i} x_{i+1}^{p_{i+1}} \cdots x_{k+1}^{p_{k+1}} \\
= x_i \Lambda_i(v(P)) \\
= x_i \Lambda_i \left( (v(P) - \Delta_{i-1}^1(v(P))) \right)
\] (22)

where the last equality follows from

\[
\Delta_{i-1}^1(v(P)) = \Delta_{i-1}^1(mx_{i-1}^{p_{i-1}} m') = m0m' = 0
\] (23)

On the other hand, if \( p_{i-1} < 2 \), i.e. if \( p_{i-1} = 1 \), then

\[
\Delta_{i-1}^1(v(P)) = \Delta_{i-1}^1(mx_{i-1} m') = x_{i-1} mm' = v(P)
\] (24)

so

\[
x_i \Lambda_i \left( (v(P) - \Delta_{i-1}^1(v(P))) \right) = x_i \Lambda_i (v(P) - v(P)) = 0
\] (25)

which is consistent with the fact that \( V_i^1 \) is not admissible for \( P \).
Similarly, to show that the action of $V_i^d$ on $P$ corresponds to

$$\frac{d_i^d}{x_i^{d-1}} \Lambda_i(v(P)) - \sum_{j=1}^{d} \Delta_{i-1}^j(v(P))$$

we want to show that

$$\sum_{j=1}^{d} \Delta_{i-1}^j(v(P)) = \begin{cases} 0 & \text{if } p_i - d + 1 \geq 2 \\ v(P) & \text{otherwise} \end{cases} \quad (26)$$

Suppose first that $t = p_i - 1 \geq d + 1$. Write $v(P) = mx_i^t$. Then, for all $1 \leq j \leq d$,

$$\frac{\partial^j}{\partial x_i^j} mx_i^t = mj! x_i^{t-j}$$

is divisible by $x_i$, hence

$$\sum_{j=1}^{d} \Delta_{i-1}^j(mx_i^t) = 0.$$ 

Suppose now that $t = p_i - 1 \leq d$. Then

$$\frac{\partial^j}{\partial x_i^j} mx_i^t = \begin{cases} 0 & j < t \\ j!m & j = t \\ 0 & j > t \end{cases} \quad (27)$$

hence

$$\Delta_{i-1}^j(mx_i^t) = \begin{cases} 0 & j < t \\ mx_i^t & j = t \\ 0 & j > t \end{cases} \quad (28)$$

This proves the assertion. \hfill \Box

The above result gives recurrence relations for $f_k^\alpha[Q]$:

Lemma 18 ([6]). Let $Q = \mathfrak{N}$, let $\alpha$ be a composition with $r$ parts, and let $f_k = f_k^\alpha[\mathfrak{N}](x_1, \ldots, x_k)$ be the generating function for saturated chains, starting from $\alpha$, of width $k$. Then $f_k = 0$ for $k < r$, and

$$f_r = v(\alpha) + (x_1 + \cdots + x_r)f_r.$$ 

Furthermore, for $k > r$, $f_k$ satisfies the following recurrence relation

$$f_k = x_1 \Lambda(f_{k-1}) + x_k f_{k-1} + (x_1 + \cdots + x_k)f_k \quad (29)$$

if $\alpha$ is not all-ones, and

$$f_k = x_1 \Lambda(f_{k-1}) + x_k f_{k-1} + (x_1 + \cdots + x_k)f_k - x_1 \cdots x_k \quad (30)$$

if $\alpha$ is all-ones.
Proof. This follows from Lemma 17 since $Q \in \mathcal{C}_k$ can be obtained from $P \in \mathcal{C}_{k-1}$ either as $Q = L.P$ or $Q = R.P$, and from $W \in \mathcal{C}_k$ as $Q = U_i.W$.

Lemma 19 ([1]). Let $Q = \mathcal{BBD}$, and let $f_k = f_k^\alpha[\mathcal{BBD}](x_1, \ldots, x_k)$ be the generating function for saturated chains, from $\alpha$ and of width $k$. Then $f_k = 0$ for $k < r$, where $r$ is the number of parts in $\alpha$, and

$$f_r = v(\alpha) + (x_1 + \cdots + x_r)f_r.$$  

Furthermore, for $k > r$, $f_k$ satisfies the following recurrence relation

$$f_k = x_1\Lambda(f_{k-1}) + (x_1 + \cdots + x_k)f_k + \sum_{i=2}^{k} x_i\Lambda_i \left( f_{k-1} - \Delta_{i-1}^1(f_{k-1}) \right) \quad (31)$$

For the posets $\mathcal{G}^d$, the recurrence is as follows:

Lemma 20. Let $Q = \mathcal{G}^d$, where $d = \infty$ or $d > 2$ is a positive integer, and let $f_k = f_k^\alpha[\mathcal{G}^d](x_1, \ldots, x_k)$ be the generating function for saturated chains, from $\alpha$ and of width $k$. Then $f_k = 0$ for $k < r$, where $r$ is the number of parts in $\alpha$, and

$$f_r = v(\alpha) + (x_1 + \cdots + x_r)f_r.$$  

Furthermore, for $k > r$, $f_k$ satisfies the following recurrence relation

$$f_k = x_1\Lambda(f_{k-1}) + (x_1 + \cdots + x_k)f_k +$$

$$+ \sum_{i=2}^{k} \sum_{1 \leq v < d} \frac{x_i^v}{x_i^{v-1}}\Lambda_i \left( f_{k-1} - \sum_{j=1}^{v} \Delta_{i-1}^j(f_{k-1}) \right) \quad (32)$$

Definition 21. Let $Q$ be one of the posets above, and let $a_{n,k}^\alpha$ denote the number of saturated chains of width $k$ and length $n - |\alpha|$, starting from $\alpha$. Define

$$L_k^\alpha[Q](t) = L_k^\alpha(t) = \sum_{n \geq 0} a_{n,k}^\alpha t^n = f_k^\alpha(t, \ldots, t) \quad (33)$$

Note that $L_k^0 = L_k^{(1)}$ for $k > 0$, so we may assume that $\alpha$ has a positive number of parts.
4.3 Enumeration of saturated chains of fixed width in the poset $\mathcal{N}$

The generating functions $f^\mathcal{N}_{k}[\mathcal{N}] = f^\alpha_k$ are displayed below for some small $k, \alpha$. Note that $f^{(1)}_k = f^{(1)}_k$ for $k > 0$.

\begin{align*}
  f^{(1)}_1 &= \frac{x_1}{1 - x_1} \\
  f^{(1)}_2 &= \frac{x_1 x_2 (1 - x_1 x_2)}{(1 - x_1) (1 - x_2) (1 - x_1 - x_2)} \\
  f^{(1,1)}_2 &= \frac{x_1 x_2}{1 - x_1 - x_2} \\
  f^{(1,1)}_3 &= \frac{x_1 x_2 x_3 (1 - x_1 x_2 - x_1 x_3 - x_2^2 - x_2 x_3)}{(1 - x_1 - x_2) (1 - x_2 - x_3) (1 - x_1 - x_2 - x_3)} \\
  f^{(1,2)}_3 &= \frac{x_1 x_2 x_3 (1 - 2 x_2 x_3 + x_2 - x_1 x_3 + x_3)}{(1 - x_1 - x_2) (1 - x_2 - x_3) (1 - x_1 - x_2 - x_3)}
\end{align*}

Using the recurrence relation (29) we can prove by induction

**Theorem 22 (\[6\]).** For each $k$,

\begin{equation}
  f^{(1)}_k(x_1, \ldots, x_k) = \frac{x_1 \cdots x_k}{\prod_{i=1}^k \prod_{j=i}^k (1 - x_i - x_{i+1} - \ldots - x_j)} f_k(x_1, \ldots, x_k) \tag{35}
\end{equation}

where $\tilde{f}_k$ is a polynomial.

The corresponding result for $f^\alpha_k$ is as follows:

**Theorem 23.** Let $\alpha$ be a composition with $r > 0$ parts. For each $k \geq r$,

\begin{equation}
  f^\alpha_k(x_1, \ldots, x_k) = \frac{x_1 \cdots x_k}{\prod_{i=1}^k \prod_{j=i+r-1}^k (1 - x_i - x_{i+1} - \ldots - x_j)} \tilde{f}^\alpha_k(x_1, \ldots, x_k) \tag{36}
\end{equation}

where $\tilde{f}^\alpha_k$ is a polynomial.

**Proof.** When $k = r$ we have that $f^\alpha_k = v(\alpha) (1 - x_1 - x_2 - \cdots - x_k)^{-1}$, which has the desired form. For $k > r$, assume that $f^\alpha_k$ has the above form.
If $\alpha$ is all-ones, then by the recurrence relation (39) it follows that

$$f_k^\alpha(1-x_1-\cdots-x_k) = x_1f_k^\alpha(x_2,\ldots,x_k) + x_kf_k^\alpha(x_1,\ldots,x_{k-1}) - x_1\cdots x_k$$

$$= x_1x_2\cdots x_kf_k^\alpha(x_2,\ldots,x_k) \prod_{i=2}^k \prod_{j=i+r-1}^k (1-x_i-\cdots-x_j)^{-1}$$

$$+ x_1x_2\cdots x_k\tilde{f}_{k-1}^\alpha(x_1,\ldots,x_{k-1}) \prod_{i=1}^{k-1} \prod_{j=i+r-1}^{k-1} (1-x_i-\cdots-x_j)^{-1} - x_1\cdots x_k$$

$$= x_1\cdots x_k \left[ \tilde{f}_{k-1}^\alpha(x_1,\ldots,x_{k-1}) \prod_{i=1}^{k-1} \prod_{j=i+r-1}^{k-1} (1-x_i-\cdots-x_j)^{-1} \right]$$

(37)

hence

$$\tilde{f}_k^\alpha = \frac{f_k(1-x_1-\cdots-x_k)\prod_{i=1}^k \prod_{j=i+r-1}^k (1-x_i-\cdots-x_j)}{x_1\cdots x_k}$$

(38)

is a polynomial.

If $\alpha = (1,\ldots,1)$ is all-ones and has $r$ parts, then by the recurrence relation (29) it follows that

$$f_k^\alpha(1-x_1-\cdots-x_k) = x_1f_k^\alpha(x_2,\ldots,x_k) + x_kf_k^\alpha(x_1,\ldots,x_{k-1})$$

$$= x_1x_2\cdots x_kf_k^\alpha(x_2,\ldots,x_k) \prod_{i=2}^k \prod_{j=i+r-1}^k (1-x_i-\cdots-x_j)^{-1}$$

$$+ x_1x_2\cdots x_k\tilde{f}_{k-1}^\alpha(x_1,\ldots,x_{k-1}) \prod_{i=1}^{k-1} \prod_{j=i+r-1}^{k-1} (1-x_i-\cdots-x_j)^{-1}$$

$$= x_1\cdots x_k \left[ \tilde{f}_{k-1}^\alpha(x_1,\ldots,x_{k-1}) \prod_{i=1}^{k-1} \prod_{j=i+r-1}^{k-1} (1-x_i-\cdots-x_j)^{-1} \right]$$

(39)

hence

$$\tilde{f}_k^\alpha = \frac{f_k(1-x_1-\cdots-x_k)\prod_{i=1}^k \prod_{j=i+r-1}^k (1-x_i-\cdots-x_j)}{x_1\cdots x_k}$$

(40)

is a polynomial.
The generating functions
\[ L^\alpha_k[N](t) = L^\alpha_k(t) = F^\alpha_k(t, \ldots, t) \]
are clearly rational functions. We have that
\[ L^1(1) = \frac{t}{1-t} \]
\[ L^2(1) = \frac{(t+1)t^2}{(1-2t)(1-t)} \]
\[ L^3(1) = \frac{t^3(-2t^2+5t+1)}{(1-3t)(1-2t)(1-t)} \]
\[ L^4(1) = \frac{(6t^3-15t^2+16t+1)t^4}{(1-4t)(1-3t)(1-2t)(1-t)} \]
\[ L_{5}^{(1,1)} = \frac{(-24t^3+38t^2-27t-1)t^5}{(1-5t)(1-4t)(1-3t)(1-2t)} \]
\[ L_{5}^{(2,3)} = 8\frac{t^8}{(1-5t)(1-4t)(1-3t)(1-2t)} \]

**Lemma 24.** Let \( r \) denote the number of parts of \( \alpha, \alpha \models N \). Then the following recurrence relation holds:
\[ L^\alpha_r = \frac{t^N(1-rt)^{-1}}{1-kt}, \quad k > r, \alpha \text{ all-ones} \]
\[ L^\alpha_k = \frac{2tL^\alpha_{k-1} - t^k}{1-kt}, \quad k > r, \alpha \text{ not all-ones} \]

**Proof.** Specialize (29) and (30).

We get by induction:

**Lemma 25.** Suppose that \( \alpha \) is not all-ones. Then
\[ L^\alpha_k = 2^{k-r}t^{N+k-r} \prod_{i=r}^{k}(1-it)^{-1} \]

Since
\[ \prod_{j=r}^{k}(1-jt)^{-1} = \frac{k^{k-r}}{(k-r)!}(1-kt)^{-1} + l.o.t, \]
we get that, when \( \alpha \) is a composition of \( N \) with \( r < N \) parts,
\[ a^\alpha_{n,k} \sim \frac{2^{k-r}}{k^N(k-r)!}k^n \quad \text{as } n \to \infty \]

Now suppose that \( \alpha \) is all-ones, i.e. \( N = r \).
Proposition 26. Suppose that \( \alpha \) is all-ones and has \( r > 0 \) parts. Then

\[
L_k^\alpha(t) = \frac{t^k D_k^\alpha(t)}{\prod_{i=r}^{k}(1-it)}
\]

(46)

where \( D_k^\alpha(t) \) is a polynomial satisfying the recurrence

\[
D_k^\alpha(t) = 2D_{k-1}^\alpha(t) - \prod_{i=r}^{k-1}(1-it).
\]

(47)

with initial conditions \( D_r^\alpha(t) = 1 \).

Proof. This is true for \( k = r \). The assertion follows by induction, the induction step being

\[
L_k = \frac{t^k D_k}{\prod_{i=r}^{k}(1-it)} = \frac{2tL_{k-1} - t^k}{1 - kt} = \frac{2t^{k-1}D_{k-1} - t^k}{\prod_{i=r}^{k-1}(1-it)} = \frac{2t^kD_{k-1} - t^k\prod_{i=r}^{k-1}(1-it)}{\prod_{i=r}^{k}(1-it)}
\]

(48)

from which (47) follows.

The following proposition is a generalization of a result in [6] for \( \alpha = () \).

Proposition 27. Suppose that \( \alpha \) is all-ones and has \( r > 0 \) parts. Then the polynomial \( D_k^\alpha(t) \) is 1 for \( k = r \), and for \( k > r \) this polynomial has

- degree \( k - r \),
- constant term 1,
- leading coefficient \( (-1)^{k-r+1}(k-1)!/(r-1)! \).

As in (45) we have that when \( \alpha \) is a composition consisting of \( r \) ones,

\[
a_{n,k}^\alpha \sim \frac{1}{k^k} D_k^\alpha(1/k) \frac{k^{k-r}}{(k-r)!} k^n = \frac{D_k^\alpha(1/k)}{(k-r)!} k^{n-r} \quad \text{as } n \to \infty
\]

(49)

Remark 28. This was stated incorrectly in [6, Corollary 4]; the numerator was evaluated at 1 rather than at \( 1/k \).

Remark 29. We have not been able to determine a formulae for the value of \( D_k^\alpha(1/k) \).
Although the poles of the rational function $L_k$ is of greater interest than the zeroes (since the pole of smallest modulus, namely $1/k$, determines the asymptotic growth of the Taylor coefficients), we could still ask where the zeroes are located. By (46), the zeroes of $L_k$ are 0 together with the zeroes of $D_k$. We make the following conjecture:

**Conjecture 30.** There is some $R \ni c \approx 8$ and a curve $C \subset \mathbb{C}$ such that, when $k$ is large, the zeroes of $D_k^{(1)}(x/k)$ are either close to the set $\{ k/m | k \in \mathbb{N}^+ \} \cap [c, k]$ or lie interspersed close to the curve $C$.

Thus the zeroes of $D_k^{(1)}(x)$ are either close to $\{ 1/m | m \in \mathbb{N}^+ \} \cap [c/k, 1]$ or lie interspersed close to the curve $k^{-1}C$.

The zeroes of $D_k^{(1)}(x/k)$ is shown in Figure 7 and those zeroes that approach the curve $C$ is shown in greater detail in Figure 8.

### 4.4 Enumeration of saturated chains of fixed width in the posets $\mathcal{S}^d$

The generating functions $f_k[SBD] = f_k^{(1)}[SBD]$ were studied in [1]. The authors derived an explicit formula for the coefficient

$$[x_1^{a_1} \cdots x_k^{a_k}] f_k^{(1)}[SBD](x_1, \ldots, x_k)$$

Recall that $SBD = \mathcal{S}^2$ and that $\mathcal{S}^d$ is an increasing family of posets on $\mathcal{C}$, with union $\mathcal{S}^\infty$. Since the generating functions $f_k^a[\mathcal{S}^d]$, for $d < \infty$, satisfies the recurrence relation $\mathcal{S}^d$, we’ll be able to give some simple results about these functions. We note for instance that $f_k^a[\mathcal{S}^d]$ are rational...
functions for $d < \infty$. Furthermore, for fixed $k, \alpha$ it holds that
\[
\lim_{d \to \infty} f_{\alpha}^d(S^d) = f_{\alpha}^\infty(S^\infty)
\]
in the natural formal topology on $\mathbb{Z}[[x_1, \ldots, x_k]]$.

Similarly to Theorem 23 one can show

**Theorem 31.** Let $\alpha$ be a composition with $r$ parts, and let $k, d$ be positive integers, such that $r \leq k$. Then the following hold:

1. The denominator of $f_{\alpha}^d(S^d)$ is of the form
\[
\prod_{\emptyset \neq S \subseteq \{1, \ldots, k\}} \left(1 - \sum_{i \in S} x_i\right)^{e(\alpha, k, d, S)}
\]
where $e(\alpha, k, d, S)$ are non-negative integers, with
\[
eq e(\alpha, k, d, \{1, 2, \ldots, k\}) = 1.
\]

2. The denominator of
\[
L_{\alpha}^d(S^d)(t) = f_{\alpha}^d(S^d)(t, \ldots, t)
\]
is of the form
\[
\prod_{i=1}^{k} (1 - it)^{c(\alpha, k, d, i)}
\]
with $c(\alpha, k, d, k) = 1$. 

Figure 8: The non-sporadic zeroes of $D_{45}^{(1)}(x/k)$. 

Table 3: the numbers $e(1, 3, d, S)$

3. The coefficient of $t^n$ in $L_k^\alpha[\mathbb{G}^d](t)$, i.e. the number of saturated chains of length $n$, starting from $\alpha$, grow as (some constant times) $k^n$ with $n$.

Example 32. Let us look at some small examples, for $\alpha = (1)$. We have that

$$f_1[\mathbb{G}^d] = \frac{x_1}{1 - x_1}$$

for all $d \geq 2$, hence that

$$f_1[\mathbb{G}^\infty] = \frac{x_1}{1 - x_1}.$$

Furthermore,

$$f_1[\mathbb{G}^\infty] = \frac{x_1}{1 - x_1}$$

$$f_2[\mathbb{G}^2] = \frac{x_1x_2(1 - x_1x_2)}{(1 - x_1 - x_2)(1 - x_2)(1 - x_1)}$$

$$f_2[\mathbb{G}^3] = \frac{x_2^2x_1(1 - x_1x_2^2)}{(1 - x_1 - x_2)(1 - x_2)(1 - x_1)}$$

$$f_2[\mathbb{G}^4] = \frac{x_2x_1^2(1 - x_1x_2^3)}{(1 - x_1 - x_2)(1 - x_2)(1 - x_1)}$$

$$f_2[\mathbb{G}^\infty] = \frac{x_2x_1}{(1 - x_1 + x_2)(1 - x_2)(1 - x_1)}$$

(52)

From the above example, it might look like $f_k[\mathbb{G}^\infty]$ should be rational for all $k$. This is in fact not the case. Already for $k = 3$ the denominators fail to stabilize: the numbers $e(1, 3, d, S)$ are shown in Table 3.

We see that for large $d$ the denominator of $f_3[\mathbb{G}^d]$ is of the form

$$(1 - x_2)(1 - x_1 - x_2)(1 - x_1 - x_3)(1 - x_2 - x_3) \times$$

$$(1 - x_1 - x_2 - x_3)(1 - x_1)^d(1 - x_3)^d$$

This means that

$$f_3[\mathbb{G}^\infty] = \lim_{d \to \infty} f_3[\mathbb{G}^d]$$

is not a rational function. Similarly, the specialization $L_3[\mathbb{G}^\infty]$ is not rational, since $L_3[\mathbb{G}^d]$ has a denominator of the form $(1 - t)^{d+1}(1 - 2t)(1 - 3t)$.
### Table 4: the numbers $e(\alpha, 3, 2, S)$

| $\alpha$ | 1 2 12 3 13 23 123 |
|-----------|---------------------|
| (1)       | 2 1 1 2 1 1 1       |
| (2)       | 2 1 1 2 1 1 1       |
| (3)       | 2 1 1 2 1 1 1       |
| (1,1)     | 1 0 1 1 1 1 1       |
| (2,1)     | 1 0 1 0 1 1 1       |
| (2,2)     | 0 0 1 0 1 1 1       |
| (3,2)     | 0 0 1 0 1 1 1       |
| (4,4)     | 0 0 1 0 1 1 1       |
| (1,1,1)   | 0 0 0 0 0 0 1       |

4.4.1 The poset $\mathcal{BBB}$

For $d = 2$, i.e. for the $\mathcal{BBB}$ poset, the numbers $e(\alpha, k, 2, S)$, for $k = 3, 4$, are shown in Table 4 and Table 5.

The general pattern seems to be quite involved, even if we concentrate on $\alpha = (1)$, i.e. on standard paths. However, in [1] an explicit, though intricate formula for the coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ in $f_k[\mathcal{BBB}] = f_k^{(1)}[\mathcal{BBB}]$ is given.

The specializations $L_k^{(1)}[\mathcal{BBB}](t)$ looks like follows:

\[
L_1^{(1)} = \frac{t}{1-t}, \quad L_2^{(1)} = \frac{(t+1)t^2}{(1-t)(1-2t)}, \quad L_3^{(1)} = \frac{t^3(3t^2 - 4t - 1)}{(1-t)^2(1-2t)(1-3t)}, \quad L_4^{(1)} = \frac{(12t^4 - 19t^3 - 19t^2 + 13t + 1)t^4}{(1-4t)(1-t)^2(1-2t)^2(1-3t)}
\]

Unfortunately, the recurrence relation (31) for $f_k[\mathcal{BBB}]$ does not specialize to a recurrence relation for $L_k[\mathcal{BBB}]$ in the way that the recurrence relation (29) for $f_k[\mathcal{N}]$ does, so even if one should be able to guess the general form of $L_k[\mathcal{BBB}]$ it would not be trivial to prove it.

4.5 Enumeration of shadow skew tableaux in $\mathcal{N}$ and $\mathcal{BBB}$

Let $Q$ denote either the poset $\mathcal{N}$ or the poset $\mathcal{BBB}$. For these two $\mathcal{V}$-graded posets, we have defined (in subsection 3.7.3) the shadow of a tableau encoding a saturated chain: this is a skew tableau encoding a saturated chain in the Young lattice. Conversely, for a saturated chain in the Young
Table 5: The numbers \( e(\alpha, 4, 2, S) \). Here, the columns, corresponding to \( \emptyset \neq S \subset 1, 2, 3, 4 \), are coded in binary, e.g. \( 5 = 2^2 + 2^0 \) correspond to \( \{1, 3\} \).

| \( \alpha \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
|------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| (1)        | 2  | 2  | 2  | 2  | 1  | 1  | 1  | 2  | 2  | 1  | 1  | 2  | 1  | 1  | 1  |
| (2)        | 2  | 2  | 2  | 2  | 1  | 1  | 1  | 2  | 2  | 1  | 1  | 2  | 1  | 1  | 1  |
| (3)        | 2  | 2  | 2  | 2  | 1  | 1  | 1  | 2  | 2  | 1  | 1  | 2  | 1  | 1  | 1  |
| (1,1)      | 1  | 1  | 2  | 1  | 1  | 1  | 1  | 2  | 1  | 1  | 2  | 1  | 1  | 1  | 1  |
| (1,2)      | 0  | 0  | 2  | 1  | 1  | 1  | 1  | 1  | 2  | 1  | 1  | 2  | 1  | 1  | 1  |
| (1,3)      | 0  | 0  | 2  | 1  | 1  | 1  | 1  | 1  | 2  | 1  | 1  | 2  | 1  | 1  | 1  |
| (1,4)      | 0  | 0  | 2  | 1  | 1  | 1  | 1  | 1  | 2  | 1  | 1  | 2  | 1  | 1  | 1  |
| (2,2)      | 0  | 0  | 2  | 0  | 1  | 1  | 1  | 0  | 2  | 1  | 1  | 2  | 1  | 1  | 1  |
| (2,3)      | 0  | 0  | 2  | 0  | 1  | 1  | 1  | 0  | 2  | 1  | 1  | 2  | 1  | 1  | 1  |
| (5,9)      | 0  | 0  | 2  | 0  | 1  | 1  | 1  | 0  | 2  | 1  | 1  | 2  | 1  | 1  | 1  |
| (1,1,1)    | 0  | 0  | 1  | 0  | 0  | 0  | 1  | 0  | 1  | 0  | 1  | 1  | 1  | 1  | 1  |
| (1,1,2)    | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 1  | 0  | 1  | 1  | 1  | 1  | 1  |
| (1,2,2)    | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 0  | 0  | 1  | 1  | 1  | 1  | 1  |
| (5,5,5)    | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 0  | 0  | 0  | 1  | 0  | 1  | 1  | 1  |
| (1,1,1,1)  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  |

lattice, we have defined its multiplicity as the number of saturated chains in \( Q \) having the \( Y \)-chain as its shadow.

Let \( \mathcal{R} : \mathbb{C}[[x_1, \ldots, x_k]] \to \mathbb{C}[[x_1, \ldots, x_k]] \) be the continuous, \( \mathbb{C} \)-linear map defined on monomials by

\[
\mathcal{R}(x^\alpha) = x^\beta,
\]

where \( \beta \in \mathbb{N}^k \) is the dominant weight associated to \( \alpha \), i.e. the entries in \( \alpha \) sorted in decreasing order.

Now suppose that \( \beta \in Y \) has all parts equal, so that there is only one composition which has the same parts. Then it is clear that the generating functions for saturated chains in \( Y \), starting from \( \beta \), and counted with multiplicity \( m(\gamma) \), is given by

\[
\tilde{f}_k^\beta[Q](x_1, \ldots, x_k) := \sum_{\gamma \text{ saturated chain in } Y \text{ of width } k \text{ starting from } \beta} m(\gamma)v(\gamma)
\]

\[
= \mathcal{R}(f_k^\beta[Q](x_1, \ldots, x_k))
\]

Example 33. Let \( Q = \mathcal{N} \), \( \alpha = () \). Then

\[
\tilde{f}_2 = \mathcal{R}(f_2) = \mathcal{R}\left(\frac{x_1x_2(1-x_1x_2)}{(1-x_1)(1-x_2)(1-x_1-x_2)}\right)
\]

\[
= \mathcal{R}\left(x_1x_2 + 2x^2_1x_2 + 2x_1x^2_2 + 3x^3_1x_2 + 4x^2_1x^2_2 + 3x_1x^2_2 + \ldots\right)
\]

\[
= x_1x_2 + x^2_1\left(4x_2 + 4x^2_2\right) + x^3_1\left(6x_2 + 14x^3_2 + 14x^2_2\right) + \ldots
\]
Table 6: Standard paths ending in a composition with shadow (2, 1).

Figure 9: The Hasse diagram of $\mathcal{N}$. Edges are labeled according to the type of the covering relation.

That the coefficient of $x_1^2x_2$ is 4 is consistent with the fact that there are 4 standard paths in $\mathcal{N}$ that ends either in (2, 1) or in (1, 2), namely the standard paths which has diagrams shown in Table 6.

We conjecture that the series $\hat{f}_k[\mathcal{Q}]$ are non-rational in all non-degenerate cases.

5 Labeled enumeration of saturated chains of fixed width

5.1 Labeling the edges of the Hasse diagram

Let $\mathcal{Q}$ be one of the posets $\mathfrak{P}$, $\mathfrak{T}$ or $\mathfrak{S}$. We label the edges in the Hasse diagram of $\mathcal{Q}$ with $L, R, U_j, V_i^j$, according to the type of covering relation. Saturated chains are labeled with the sequence of labels occurring along the edges. Admissible words for a composition $\alpha$ now correspond bijectively to saturated chains starting from $\alpha$. In Figure 9 we show the the labeling of the edges of $\mathcal{N}$. 
If $\alpha \in C$ and $W = W_r W_{r-1} \ldots W_1$ is a word which is admissible for $\alpha$, then we give the corresponding chain $\gamma = (\alpha, W_1, \alpha, W_2, W_1, \alpha, \ldots, W, \alpha)$ non-commutative weight

$$V(\gamma) = v(\gamma)W$$

were $v(\gamma)$ is as in (18). The non-commutative generalization of (19) is

$$F^\alpha_k[Q] = F^\alpha_k = \sum_{\gamma} V(\gamma)$$

were the sum is over all saturated chains $\gamma$ of width $k$ that starts from $\alpha$. Here, the $x_i$’s commute with each other and with the variables $R, L, U_j$, but the latter variables do not commute with each other. Note that $F^\alpha_k$ only involves finitely many variables.

One observes that the coefficient in $F^\alpha_k$ of a non-commutative monomial $W$ is a single monomial in $x_1, \ldots, x_k$, namely the monomial encoding the endpoint of the path encoded by $W$. Similarly the coefficient in $F^\alpha_k$ of a commutative monomial $x^a$ is a non-commutative polynomial in $L, R, U_j, V^j_i$ with non-negative coefficients, encoding all paths (from the starting composition) that ends at $a$. As an example, for $Q = \mathfrak{N}$ the coefficient of $x_1 x_2$ in $F^\alpha_2$ is $U_2 L^2 + LU_1 L$.

Clearly, specializing all non-commutative variables in $F^\alpha_k$ to one gives $f^\alpha_k$. On the other hand, specializing all commutative variables to one gives a formal power series in non-commuting variables, all whose occurring coefficients are one. If $\mathcal{F}$ denotes the free monoid on the relevant non-commuting variables, then this power series is the generating function of the language

$$\langle \mathcal{F}; \alpha \rangle \subset \mathcal{F}.$$  

5.2 Labeled enumeration in the poset $\mathfrak{N}$

The poset $\mathfrak{N}$ has covering relations given by the partial action of the free monoid $(\mathcal{A}_L \cup \mathcal{A}_U)^*$. The generating function $F^\alpha_k[\mathfrak{N}] = F^\alpha_k$ has commuting variables $x_1, \ldots, x_k$ and non-commuting variables in $\{L, R\} \cup \mathcal{A}_U$. In fact, no $U_j$ with $j > k$ will occur in $F^\alpha_k$, hence we regard $F^\alpha_k$ as having non-commuting variables in

$$\{L, R, U_1, U_2, \ldots, U_k\}$$

Theorem 34. The non-commutative generating function for labeled saturated chains in $\mathfrak{N}$, starting from the composition $\alpha = (a_1, \ldots, a_s)$, satisfies the recurrence

$$F^\alpha_k = F^\alpha_k(x_1, \ldots, x_k; L, R, U_1, U_2, \ldots, U_k)$$

$$= \begin{cases} 
0 & \text{if } k < s \\
A + v(\alpha) & \text{if } k = s \\
A + B + C & \text{if } k > s \text{ and } \alpha \text{ not all-ones} \\
A + B + C - D & \text{if } k > s \text{ and } \alpha \text{ all-ones}
\end{cases}$$

(60)
where
\[
A = (x_1U_1 + \cdots + x_kU_k)F_{k}^{\alpha}
\]
\[
B = x_1L \cdot \Lambda(F_{k-1}^{\alpha})
\]
\[
C = x_kR \cdot F_{k-1}^{\alpha}
\]
\[
D = RL^{k-1}v(\alpha)
\]

Proof. This follows from Lemma 17 in the same way that Lemma 18 follows. \qed

For \(\alpha = (2)\), we get that
\[
F_0^{(2)} = 0
\]
\[
F_1^{(2)} = (1 - x_1U_1)^{-1}x_1^2
\]
\[
F_2^{(2)} = (1 - x_1U_1 - x_2U_2)^{-1} \times
\]
\[
[x_1L((1 - x_2U_1)^{-1}x_2^2 + x_2R(1 - x_1U_1)^{-1}x_1^2)]
\]

It is known that non-commutative rational series in finitely many variables are recognizable, so that the coefficients correspond to the labels of walks from a start node to an end node in a certain labeled digraph. As an example,
\[
F_1^{(2)} = (1 - x_1U_1)^{-1}x_1^2 = x_1^2 + x_1^3U_1 + x_1^4U_1^2 + \cdots
\]
corresponds to paths from • to ○ in the following digraph:

An immediate consequence of (60) is the following:

**Theorem 35.** Let \(\alpha\) be a composition with \(r\) parts. Denote the language defined by \(F_{k}^{\alpha}(1, \ldots, 1, L, R, U_1, \ldots, U_K)\) by \(\mathcal{L}_{k}^{\alpha}\). This is a regular language, and abusing notation by equating a regular language to some regular expression that defines it, we can write
\[
\mathcal{L}_{r}^{\alpha} = (U_1 + U_2 + \cdots + U_r)^*
\]
(A) If $\alpha$ is not all-ones, then a digraph for $F^\alpha_k$, which enumerates saturated chains of width $k$ in $\mathcal{N}$, starting from $\alpha$, by walks from $\bullet$ to $\circ$, is obtained from the one for $F^\alpha_{k-1}$ by

\[ \Lambda(F^\alpha_{k-1}) \]

Here, $F^\alpha_{k-1}$ denotes the digraph yields $F^\alpha_{k-1}$, and $\Lambda(F^\alpha_{k-1})$ denotes the digraph which yields $\Lambda(F^\alpha_{k-1})$; this digraph is obtained from the former by transforming each label using $\Lambda$.

It follows that

\[ \mathcal{L}^\alpha_k = (U_1 + U_2 + \cdots + U_k)^*(L + R)\mathcal{L}^\alpha_{k-1} \quad \text{for } k \geq r \quad (66) \]

so that

\[ \mathcal{L}_k = \left( \sum_{i=1}^{k} U_i \right)^* (L + R) \left( \sum_{i=1}^{k-1} U_i \right)^* (L + R) \cdots \]

\[ \cdots \left( \sum_{i=1}^{r+1} U_i \right)^* (L + R) \left( \sum_{i=1}^{r} U_i \right)^* \quad (67) \]

(B) If $\alpha$ is all-ones, then a digraph for $F^\alpha_k$, which enumerates saturated chains of width $k$ in $\mathcal{N}$, starting from $\alpha$, by walks from $\bullet$ to $\circ$, is obtained
from the one for $F_{k-1}^\alpha$ by

$$L_k^\alpha = (U_1 + U_2 + \cdots + U_k)^* (L + (U_1 + U_2 + \cdots + U_k)R) L_{k-1}^\alpha$$

$$= (U_1 + U_2 + \cdots + U_k)^* (L + R) L_{k-1}^\alpha - R L^{k-1} \tag{69}$$

**Example 36.** Since

$$F_1^{(2)} = \begin{array}{c}
\cdots \\
\circ \leftarrow x_1 U_1 \\
\uparrow \\
\circ \rightarrow x_1^2 \\
\end{array}$$

and

$$\Lambda(F_1^{(2)}) = \begin{array}{c}
\cdots \\
\circ \leftarrow x_1 U_1 \\
\uparrow \\
\circ \rightarrow x_1^2 \\
\end{array},$$

a digraph for $F_2^{(2)}$ is

$$F_2^{(2)} = \begin{array}{c}
\cdots \\
\circ \leftarrow x_1 U_1 \\
\uparrow \\
\circ \rightarrow x_2 U_1 \\
\end{array}$$

$$\circ \rightarrow x_1^2 \\
\downarrow \\
\circ \rightarrow x_2^2$$
and
\[ L_2^{(2)} = (U_1 + U_2)^*(L + R)U_1^* \] (70)

Note that specializing \( x_1 = x_2 = 1 \) in (62) gives
\[ (1 - U_1 - U_2)^{-1} \left[ L(1 - U_1)^{-1} + R(1 - U_1)^{-1} \right] = (1 - U_1 - U_2)^{-1}(L + R)(1 - U_1)^{-1} \] (71)

which correspond precisely to (70).

Example 37. A digraph for \( F_3^{(2)} \) is

\[ L_3^{(2)} = (U_1 + U_2 + U_3)^*(L + R)(U_1 + U_2)^*(L + R)U_1^* \] (72)

5.3 Labeled enumeration in the poset \( \mathbb{BBD} \)

In the poset \( \mathbb{BBD} \), we label the edges of the Hasse diagram with \( L, U_j \) or \( V_1 \). Defining \( F_k^\alpha = F_k^\alpha[\mathbb{BBD}] \) as before, we get:

**Theorem 38.** The non-commutative generating function for labeled saturated chains in \( \mathbb{BBD} \), starting from the composition \( \alpha = (a_1, \ldots, a_s) \), satisfies the recurrence
\[
F_k^\alpha = F_k^\alpha[\mathbb{BBD}](x_1, \ldots, x_k; L, U_1, U_2, \ldots, U_k, V_2, \ldots, V_k)
= \begin{cases} 
0 & \text{if } k < s \\
A + v(\alpha) & \text{if } k = s \\
A + B + \sum_{i=2}^{k} C_i & \text{if } k > s
\end{cases}
\] (73)
where

\[ A = (x_1 U_1 + \cdots + x_k U_k) F_k^\alpha \]
\[ B = x_1 L \Lambda (F_{k-1}^\alpha) \]
\[ C_i = x_i V_i \Lambda_i (F_{k-1} - \Delta_{i-1}^1 (F_{k-1}^\alpha)) \]

**(Example 39.)** We have that

\[
F_1^{(1)} = x_1 (1 - x_1 U_1)^{-1} \\
= x_1 + x_1^2 U_1 + x_1^3 U_1^2 + x_1^4 U_1^3 + \ldots \\
F_2^{(1)} = [1 - x_1 U_1 - x_2 U_2]^{-1} \left( x_1 [x_2 (1 - x_2 U_1)^{-1}] + x_2 V_2 (x_1^2 U_1 (1 - x_1 U_1)^{-1}) \right)
\]

**(Theorem 40.)** Let \( \alpha \) be a composition with \( r \) parts. Put

\[ F_k^\alpha = F_k^\alpha [\mathcal{BBD}] (1, \ldots, 1, L, V_2, \ldots, V_k, U_1, \ldots, U_k) \]

Then a digraph for \( F_k^\alpha \), which enumerates saturated chains of width \( k \) in \( \mathcal{N} \), starting from \( \alpha \), by walks from \( \bullet \) to \( \circ \), is obtained from the one for \( F_{k-1}^\alpha \) by

\[
\text{where } \boxed{B} \text{ is the digraph for } \Lambda (F_{k-1}^\alpha) \text{ and } \boxed{C_i} \text{ is the digraph for } \\
\Lambda_i (F_{k-1} - \Delta_{i-1}^1 (F_{k-1}^\alpha))
\]

This theorem is less informative than Theorem 35 since it is somewhat complicated to construct the digraph generating \( F_k^\alpha - \Delta_{i-1}^1 (F_{k-1}^\alpha) \) given the digraph generating \( F_{k-1}^\alpha \). The process may require the addition of extra edges, and is somewhat irregular.
Example 41. We have that

\[ F_1^{(1)} = \]

and that

\[ F_1^{(1)} - \Delta_1^{(1)}(F_1^{(1)}) = \]

so a digraph for \( F_2^{(1)} \) is

\[ F_2^{(1)} = \]

We furthermore see that

\[ F_2^{(1)} - \Delta_1^{(1)}(F_2^{(1)}) = \]

but that

\[ F_2^{(1)} - \Delta_2^{(1)}(F_2^{(1)}) = \]
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