On positive solutions of minimal growth for singular $p$-Laplacian with potential term

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Abstract
Let $\Omega$ be a domain in $\mathbb{R}^d$, $d \geq 2$, and $1 < p < \infty$. Fix $V \in L^\infty_{\text{loc}}(\Omega)$. Consider the functional $Q$ and its Gâteaux derivative $Q'$ given by

$$Q(u) := \frac{1}{p} \int_{\Omega} (|\nabla u|^p + V|u|^p) \, dx, \quad Q'(u) := -\nabla \cdot (|\nabla u|^{p-2}\nabla u) + V|u|^{p-2}u.$$

It is assumed that $Q \geq 0$ on $C^\infty_0(\Omega)$. In a previous paper [22] we discussed relations between the absence of weak coercivity of the functional $Q$ on $C^\infty_0(\Omega)$ and the existence of a generalized ground state. In the present paper we study further relationships between functional-analytic properties of the functional $Q$ and properties of positive solutions of the equation $Q'(u) = 0$.

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1 Introduction
Properties of positive solutions of quasilinear elliptic equations, and in particular, of equations with the $p$-Laplacian term in the principal part defined
on general domains have been extensively studied over the recent decades (see for example, [3, 4, 10, 22, 26, 31] and the references therein).

Fix \( p \in (1, \infty) \), a domain \( \Omega \subseteq \mathbb{R}^d \) and a potential \( V \in L^\infty_{\text{loc}}(\Omega) \). So, \( \Omega \) is allowed to be nonsmooth and/or unbounded, and \( V \) might blow-up near \( \partial \Omega \) or at infinity.

The \( p \)-Laplacian equation in \( \Omega \) with potential term \( V \) is the equation of the form

\[
-\Delta_p(u) + V|u|^{p-2}u = 0 \quad \text{in } \Omega, \tag{1.1}
\]

where \( \Delta_p(u) := \nabla \cdot (|\nabla u|^{p-2} \nabla u) \) is the \( p \)-Laplacian. This equation, in the semistrong sense, is a critical point equation for the functional

\[
Q(u) = Q_V(u) := \frac{1}{p} \int_{\Omega} (|\nabla u|^p + V|u|^p) \, dx \quad u \in C^\infty_0(\Omega). \tag{1.2}
\]

So, we consider solutions of (1.1) in the following weak sense.

**Definition 1.1.** A function \( v \in W^{1,p}_{\text{loc}}(\Omega) \) is a (weak) solution of the equation

\[
Q'(u) := -\Delta_p(u) + V|u|^{p-2}u = 0 \quad \text{in } \Omega, \tag{1.3}
\]

if for every \( \varphi \in C^\infty_0(\Omega) \)

\[
\int_{\Omega} (|\nabla v|^{p-2} \nabla v \cdot \nabla \varphi + V|v|^{p-2}v \varphi) \, dx = 0. \tag{1.4}
\]

We say that a real function \( v \in C^1_{\text{loc}}(\Omega) \) is a supersolution (resp. subsolution) of the equation (1.3) if for every nonnegative \( \varphi \in C^\infty_0(\Omega) \)

\[
\int_{\Omega} (|\nabla v|^{p-2} \nabla v \cdot \nabla \varphi + V|v|^{p-2}v \varphi) \, dx \geq 0 \quad \text{(resp. } \leq 0). \tag{1.5}
\]

Throughout this paper, unless otherwise stated, we assume that

\[
Q(u) \geq 0 \quad \forall u \in C^\infty_0(\Omega). \tag{1.6}
\]

The present paper continues the investigation in [22] and also in [20, 26]. These papers deal with global positivity properties of the functional \( Q \) and the set of positive solutions of the equation \( Q'(u) = 0 \) on a general domain \( \Omega \subset \mathbb{R}^d \). The existence of such global positive solutions is linked to the positivity of \( Q \) by the following Allegretto-Piepenbrink type theorem.
Theorem 1.2 ([22 Theorem 2.3]). Let $Q$ be a functional of the form (1.2). The following assertions are equivalent:

(i) The functional $Q$ is nonnegative on $C_0^\infty(\Omega)$.

(ii) Equation (1.3) admits a global positive solution.

(iii) Equation (1.3) admits a global positive supersolution.

It was established in [22] that the absence of a weak coercivity is equivalent to the existence of a (generalized) ground state. The proof hinged on the representation of $Q$ as an integral with a nonnegative Lagrangian density due to the generalized Picone identity [3, 4, 7]. In [20] a simplified equivalent (in the sense of two-sided estimates) expressions for the Lagrangian were derived, and a Liouville-type theorem was proved. In Theorem 4.5 of the present paper, we prove the equivalence of several weak coercivity properties of the functional $Q$, as well as their equivalence to the positivity of the variational $Q$-capacity of closed balls.

For $p \leq d$ it was proved in [22] that the ground state can be identified as the global positive solution of minimal growth at infinity of $\Omega$, thus extending criticality theory of positive solutions for second-order linear elliptic equations (see [19]) to the case of quasilinear equations of the form (1.3). In Section 5, we further study positive solutions of the equation $Q'(u) = 0$ of minimal growth in a neighborhood of infinity in $\Omega$, and the behavior of positive solutions near an isolated singularity. Consequently, we extend the above identification to the case $p > d$ (see Theorem 5.10).

Finally, we give conditions for a positive solution to be a positive solution of minimal growth in a neighborhood of infinity in terms of the infimum of the integral of the corresponding nonnegative Lagrangian in a neighborhood of infinity. The variational condition that we present in Section 6 for the case $p = 2$, is necessary and sufficient, while for $p \neq 2$ we give in Section 7 only a stronger sufficient condition. Weakening this condition with our methods requires the strong comparison principle, which is generally false for $p \neq 2$.

2 Preliminaries

In this section we recall local properties of solutions of (1.3) that hold in any smooth subdomain $\Omega' \subset \Omega$, where $A \Subset B$ means that $\overline{A}$ is compact in $B$. 
1. **Smoothness and Harnack inequality.** Weak solutions of (1.3) admit Hölder continuous first derivatives, and nonnegative solutions of (1.3) satisfy the Harnack inequality [24, 25, 27].

2. **Harnack convergence principle.** Let \( \{\Omega_N\}_{N=1}^{\infty} \) be an exhaustion of \( \Omega \) (i.e., a sequence of smooth, relatively compact domains such that \( \text{cl}(\Omega_N) \subset \Omega_{N+1} \), and \( \cup_{N=1}^{\infty} \Omega_N = \Omega \)). Fix a reference point \( x_0 \in \Omega_1 \). Assume also that \( V, \{V_N\}_{N=1}^{\infty} \subset L_{\text{loc}}^{\infty}(\Omega) \) satisfy \( V_N \to V \) in \( L_{\text{loc}}^{\infty}(\Omega) \). Suppose that \( \{u_N\} \) is a sequence of positive solutions of the equations

\[
Q'_N(u_N) := -\Delta_p(u_N) + V_N|u_N|^{p-2}u_N = 0 \quad \text{in} \ \Omega_N,
\]

such that \( u_N(x_0) = 1 \).

In light of the Harnack inequality, a priori interior estimates [27, Theorem 1], the Arzelà-Ascoli theorem, and a standard diagonalization argument, there exist \( 0 < \beta < 1 \), and a subsequence \( \{u_{N_k}\} \) of \( \{u_N\} \) that converges in \( C^{1,\beta}_{\text{loc}}(\Omega) \) to a positive solution \( u \) of the equation

\[
Q'(u) := -\Delta_p(u) + V|u|^{p-2}u = 0 \quad \text{in} \ \Omega.
\]

3. **Principal eigenvalue and eigenfunction.** For any smooth subdomain \( \Omega' \Subset \Omega \) consider the variational problem

\[
\lambda_{1,p}(\Omega') := \inf_{u \in W^{1,p}_0(\Omega')} \frac{\int_{\Omega'} (|\nabla u|^p + V|u|^p) \, dx}{\int_{\Omega'} |u|^p \, dx}.
\]

It is well-known that for such a subdomain, (2.2) admits (up to a multiplicative constant) a unique minimizer \( \varphi \) [8, 10]. Moreover, \( \varphi \) is a positive solution of the quasilinear eigenvalue problem

\[
\begin{cases}
Q'(\varphi) = \lambda_{1,p}(\Omega')|\varphi|^{p-2}\varphi & \text{in} \ \Omega', \\
\varphi = 0 & \text{on} \ \partial\Omega'.
\end{cases}
\]

\( \lambda_{1,p}(\Omega') \) and \( \varphi \) are called, respectively, the principal eigenvalue and eigenfunction of the operator \( Q' \) in \( \Omega' \).

It should be noted, however, that minimization statements for singular elliptic problems on unbounded or nonsmooth domains typically do not produce points of minimum, but minimizing sequences, which locally (up to subsequences) converge to solutions.

4. **Weak and strong maximum principles.**
Theorem 2.1 ([10] (see also [3, 4])). Consider a functional $Q$ of the form (1.2) such that (1.6) does not necessarily hold in $\Omega$. Let $\Omega' \subseteq \Omega$ be a bounded $C^{1+\alpha}$-subdomain, where $0 < \alpha < 1$. So, in particular, $V \in L^\infty(\Omega')$.

The following assertions are equivalent:

(i) $Q'$ satisfies the maximum principle: If $u$ is a solution of the equation $Q'(u) = f \geq 0$ in $\Omega'$ with some $f \in L^\infty(\Omega')$, and satisfies $u \geq 0$ on $\partial\Omega'$, then $u$ is nonnegative in $\Omega'$.

(ii) $Q'$ satisfies the strong maximum principle: If $u$ is a solution of the equation $Q'(u) = f \geq 0$ in $\Omega'$ with some $f \in L^\infty(\Omega')$, and satisfies $u \geq 0$ on $\partial\Omega'$, then $u > 0$ in $\Omega'$.

(iii) $\lambda_{1,p}(\Omega') > 0$.

(iv) For some $0 \leq f \in L^\infty(\Omega')$ there exists a positive strict supersolution $v$ satisfying $Q'(v) = f$ in $\Omega'$, and $v = 0$ on $\partial\Omega'$.

(iv') There exists a positive strict supersolution $v$ satisfying $Q'(v) = f \geq 0$ in $\Omega'$, such that $v \in C^{1+\alpha}(\partial\Omega')$ and $f \in L^\infty(\Omega')$.

(v) For each nonnegative $f \in C^\alpha(\Omega') \cap L^\infty(\Omega')$ there exists a unique weak nonnegative solution of the problem $Q'(u) = f$ in $\Omega'$, and $u = 0$ on $\partial\Omega'$.

5. Weak comparison principle. We shall need also the following weak comparison principle (or WCP for brevity).

Theorem 2.2 ([10]). Consider a functional $Q$ of the form (1.2) defined on $\Omega$, such that (1.6) does not necessarily hold in $\Omega$. Let $\Omega' \subseteq \Omega$ be a bounded subdomain of class $C^{1,\alpha}$, where $0 < \alpha \leq 1$. Assume that $\lambda_{1,p}(\Omega') > 0$ and let $u_i \in W^{1,p}(\Omega') \cap L^\infty(\Omega')$ satisfying $Q'(u_i) \in L^\infty(\Omega')$, $u_i|_{\partial\Omega'} \in C^{1+\alpha}(\partial\Omega')$, where $i = 1, 2$. Suppose further that the following inequalities are satisfied

\[
\begin{cases}
Q'(u_1) \leq Q'(u_2) & \text{in } \Omega', \\
Q'(u_2) \geq 0 & \text{in } \Omega', \\
u_1 \leq u_2 & \text{on } \partial\Omega', \\
u_2 \geq 0 & \text{on } \partial\Omega'.
\end{cases}
\]

Then

\[u_1 \leq u_2 \quad \text{in } \Omega'.\]
6. Strong comparison principle.

**Definition 2.3.** We say that the *strong comparison principle* (or SCP for brevity) holds true for the functional $Q$ if the conditions of Theorem 2.2 imply that $u_1 < u_2$ in $\Omega'$ unless $u_1 = u_2$ in $\Omega'$.

**Remark 2.4.** It is well known that the SCP holds true for $p = 2$ and for $p$-harmonic functions. For sufficient conditions for the validity of the SCP see [2, 5, 6, 13, 28] and the references therein. In [5] M. Cuesta and P. Takáč present a counterexample where the WCP holds true but the SCP does not.

### 3 Picone identity and equivalent Lagrangian

Let $v \in C^1_{\text{loc}}(\Omega)$ be a positive solution (resp. subsolution) of (1.3). Using the *Picone identity* [3, 4, 7] we infer that for every $u \in C^\infty_0(\Omega)$, $u \geq 0$, we have

$$Q(u) = \frac{1}{p} \int_\Omega L(u, v) \, dx \quad \text{(resp. } Q(u) \leq \frac{1}{p} \int_\Omega L(u, v) \, dx \text{),} \quad (3.1)$$

where the Lagrangian $L$ is given by

$$L(u, v) := \frac{1}{p} \left[ |\nabla u|^p + (p - 1) \frac{u^p}{v^p} |\nabla v|^p - p \frac{u^{p-1}}{v^{p-1}} \nabla u \cdot |\nabla v|^{p-2} \nabla v \right]. \quad (3.2)$$

It can be easily verified that $L(u, v) \geq 0$ in $\Omega$.

Let now $w := u/v$, where $v$ is a positive solution of (1.3) and $u \in C^\infty_0(\Omega)$, $u \geq 0$. Then (3.1) implies that

$$Q(vw) = \frac{1}{p} \int_\Omega \left[ |w \nabla v + v \nabla w|^p - w^p |\nabla v|^p - p w^{p-1} v |\nabla v|^{p-2} \nabla v \cdot \nabla w \right] \, dx. \quad (3.3)$$

Similarly, if $v$ is a nonnegative subsolution of (1.3), then

$$Q(vw) \leq \frac{1}{p} \int_\Omega \left[ |w \nabla v + v \nabla w|^p - w^p |\nabla v|^p - p w^{p-1} v |\nabla v|^{p-2} \nabla v \cdot \nabla w \right] \, dx. \quad (3.4)$$

Therefore, a nonnegative functional $Q$ can be represented as the integral of a nonnegative Lagrangian $L$. But the expression (3.2) of $L$ contains an indefinite term, and consequently the integrand in (3.3) and (3.4) is nonnegative but with nonpositive terms. The next proposition shows that $Q$ admits a
two-sided estimate by a simplified Lagrangian containing only nonnegative terms. We call the functional associated with this simplified Lagrangian the simplified energy.

Let $f$ and $g$ be two nonnegative functions. We denote $f \asymp g$ if there exists a positive constant $C$ such that $C^{-1}g \leq f \leq Cg$.

**Proposition 3.1** ([20, Lemma 2.2]). Let $v \in C^1_{\text{loc}}(\Omega)$ be a positive solution of (1.3) and let $w \in C^1_0(\Omega)$ be a nonnegative function. Then

$$Q(vw) \asymp \int_\Omega v^2|\nabla w|^2 (w|\nabla v| + v|\nabla w|)^{p-2} \, dx. \quad (3.5)$$

In particular, for $p \geq 2$, we have

$$Q(vw) \asymp \int_\Omega (v^p|\nabla w|^p + v^2|\nabla v|^{p-2}w^{p-2}|\nabla w|^2) \, dx. \quad (3.6)$$

If $v$ is only a nonnegative subsolution of (1.3), then

$$Q(vw) \leq C \int_{\Omega \cap \{v > 0\}} v^2|\nabla w|^2 (w|\nabla v| + v|\nabla w|)^{p-2} \, dx. \quad (3.7)$$

In particular, for $p \geq 2$, we have

$$Q(vw) \leq C \int_\Omega (v^p|\nabla w|^p + v^2|\nabla v|^{p-2}w^{p-2}|\nabla w|^2) \, dx. \quad (3.8)$$

**Proof.** Let $1 < p < \infty$. The following elementary algebraic vector inequality holds true [20]

$$|a + b|^p - |a|^p - p|a|^{p-2}a \cdot b \asymp |b|^2(2|a| + |b|)^{p-2} \quad \forall a, b \in \mathbb{R}^d. \quad (3.9)$$

Set now $a := w|\nabla v|, b := v|\nabla w|$. Then we obtain (3.5) and (3.7) by applying (3.9) to (3.3) and (3.4), respectively. \square

**Remark 3.2.** It is shown in [20] that for $p > 2$ none of the two terms in the simplified energy (3.6) is dominated by the other, so that (3.6) cannot be further simplified.
4 Coercivity and ground state

**Definition 4.1.** Let $Q$ be a nonnegative functional on $C_0^\infty(\Omega)$. We say that a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ of nonnegative functions is a **null sequence** of the functional $Q$ in $\Omega$, if there exists an open set $B \subseteq \Omega$ such that $\int_B |u_k|^p \, dx = 1$, and

$$\lim_{k \to \infty} Q(u_k) = \lim_{k \to \infty} \int_\Omega (|\nabla u_k|^p + V |u_k|^p) \, dx = 0. \quad (4.1)$$

We say that a positive function $v \in C^1_{\text{loc}}(\Omega)$ is a **ground state** of the functional $Q$ in $\Omega$ if $v$ is an $L^p_{\text{loc}}(\Omega)$ limit of a null sequence of $Q$.

The functional $Q$ is **critical** in $\Omega$ if it admits a ground state in $\Omega$. If the nonnegative functional $Q$ does not admit a ground state in $\Omega$, then $Q$ is said to be **subcritical** (or strictly positive) in $\Omega$.

**Theorem 4.2** ([22, Theorem 1.6] and [17, 21] for the case $p = 2$). Suppose that the functional $Q$ is nonnegative on $C_0^\infty(\Omega)$. Then any ground state $v$ is a positive solution of (1.3). Moreover, $Q$ admits a ground state $v$ if and only if (1.3) admits a unique positive supersolution.

In this case, the following Poincaré type inequality holds: There exists a positive continuous function $W$ in $\Omega$, such that for every $\psi \in C_0^\infty(\Omega)$ satisfying $\int_\Omega \psi v \, dx \neq 0$ there exists a constant $C > 0$ such that the following inequality holds:

$$Q(u) + C \left| \int_\Omega \psi u \, dx \right|^p \geq C^{-1} \int_\Omega W |u|^p \, dx \quad \forall u \in C_0^\infty(\Omega). \quad (4.2)$$

The following result extends Theorem 2.7 of [17], which was proved for the linear case, to the case $1 < p < \infty$ (cf. [21, Theorem 4.2]).

**Theorem 4.3.** Suppose that the functional $Q_V$ is nonnegative on $C_0^\infty(\Omega)$. Then $Q_V$ is critical in $\Omega$ if and only if $Q_V$ admits a null sequence that converges locally uniformly in $\Omega$.

**Proof.** By Theorem 4.2 we only need to show that if $Q_V$ admits a null sequence, then it admits also a null sequence that converges locally uniformly in $\Omega$. Let $\{\Omega_N\}_{N=1}^\infty$ be an exhaustion of $\Omega$ such that $x_0 \in \Omega_1$. Pick a nonzero nonnegative function $W \in C_0^\infty(\Omega_1)$. For $t \geq 0$ and $N \geq 1$, consider the functional $Q_{V-tW}$ on $C_0^\infty(\Omega_N)$. By Propositions 4.2 and 4.4 of [22], for each $N \geq 1$ there exists a unique positive number $t_N$ such that the functional...
$Q_{V-tNW}$ admits a ground state in $\Omega_N$. Denote by $v_N$ the corresponding ground state satisfying $v_N(x_0) = 1$.

On the other hand, Proposition 4.1 of [22] implies that $t_N \to 0$. Invoking the Harnack convergence principle, it follows that there exists a subsequence $\{v_{N_k}\}$ of $\{v_N\}$ that converges as $k \to \infty$ locally uniformly to a positive solution $v$ of the equation $Q_V'(u) = 0$ in $\Omega$ such that $v(x_0) = 1$. The uniqueness of a positive solution of the equation $Q_V'(u) = 0$ in $\Omega$ satisfying $u(x_0) = 1$ (due to the criticality of $Q_V$) implies that any such subsequence converges to the same function $v$. Consequently, $v_N \to v$ locally uniformly in $\Omega$.

Since $v_N \in W^{1,p}_0(\Omega)$, we have $Q_V(v_N) = p^{-1} t_N \int_{\Omega} W|v_N|^p \, dx$. Consequently,

$$Q_V(v_N) = \frac{t_N}{p} \int_{\Omega} W|v_N|^p \, dx \to 0, \quad \text{and } \int_B |v_N|^p \, dx \asymp 1.$$ 

Therefore, $\{v_N\}$ is a null sequence of $Q_V$ and $v$ is the corresponding ground state.

A Riemannian manifold $\mathcal{M}$ is said to be $p$-parabolic if the equation

$$-\Delta_p u = 0$$

admits only trivial positive supersolutions ($p$-superharmonic functions) in $\mathcal{M}$. In [29, 30] Troyanov has established a relationship between the (variational) $p$-capacity of closed balls in a Riemannian manifold $\mathcal{M}$ and the $p$-parabolicity of $\mathcal{M}$. The following is a natural extension of the definition of $p$-capacity of compact sets.

**Definition 4.4** (cf. [11]). Suppose that the functional $Q$ is nonnegative on $C_0^\infty(\Omega)$. Let $K \subseteq \Omega$ be a compact set. The $Q$-capacity of $K$ in $\Omega$ is defined by

$$\text{Cap}_Q(K, \Omega) := \inf \{ Q(u) \mid u \in C_0^\infty(\Omega), \ u \geq 1 \text{ on } K \}.$$ 

The following theorem extends Theorem 1.6 of [22] in the spirit of [26, Proposition 3.1], as well as Troyanov’s result [29, 30] concerning the $p$-capacity of closed balls.

**Theorem 4.5.** Let $\Omega \subseteq \mathbb{R}^d$ be a domain, $V \in L^\infty_{\text{loc}}(\Omega)$, and $p \in (1, \infty)$. Suppose that the functional $Q$ is nonnegative on $C_0^\infty(\Omega)$. Then the following statements are equivalent.
(a) $Q$ does not admit a ground state in $\Omega$.

(b) There exists a continuous function $W > 0$ in $\Omega$ such that
\[ Q(u) \geq \int_\Omega W(x)|u(x)|^p \, dx \quad \forall u \in C_0^\infty(\Omega). \quad (4.3) \]

(c) There exists a continuous function $W > 0$ in $\Omega$ such that
\[ Q(u) \geq \int_\Omega W(x) (|\nabla u(x)|^p + |u(x)|^p) \, dx \quad \forall u \in C_0^\infty(\Omega). \quad (4.4) \]

(d) There exists an open set $B \Subset \Omega$ and $C_B > 0$ such that
\[ Q(u) \geq C_B \left| \int_B u(x) \, dx \right|^p \quad \forall u \in C_0^\infty(\Omega). \quad (4.5) \]

(e) The $Q$-capacity of any closed ball in $\Omega$ is positive.

Suppose further that $d > p$. Then $Q$ does not admit a ground state in $\Omega$ if and only if there exists a continuous function $W > 0$ in $\Omega$ such that
\[ Q(u) \geq \left( \int_\Omega W(x)|u(x)|^{p^*} \, dx \right)^{p/p^*} \quad \forall u \in C_0^\infty(\Omega), \quad (4.6) \]

where $p^* = pd/(d - p)$ is the critical Sobolev exponent.

Proof. By [22, Theorem 1.6], (a) $\Leftrightarrow$ (b). If (b) $\not\Rightarrow$ (c), then there exists a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ with $u_k \geq 0$, and an open set $B \Subset \Omega$ such that $\int_B |\nabla u_k|^p \, dx = 1$, while $Q(u_k) \to 0$ and $\int_B |u_k|^p \, dx \to 0$. This is false, since (3.1), Picone’s formula and Young’s inequality imply (see step 1 of the proof of [22 Lemma 3.2])
\[ \int_B |\nabla u_k|^p \, dx \leq C Q(u_k) + \int_B |u_k|^p \, dx \to 0. \]

Clearly (c) $\Rightarrow$ (d). If (d) holds and $Q$ admits a ground state in $\Omega$, then the Poincaré type inequality (4.2) implies (b) which is a contradiction. Statement (e) is immediate from Theorem 4.3

If $d > p$, then (4.6) implies (4.5). On the other hand, (4.6) is immediate from (4.4) via partition of unity and the local Sobolev inequality. \qed
Remark 4.6. The requirement in Definition 4.1 that a null sequence \( \{u_k\} \) satisfies \( \{u_k\} \subset C_0^\infty(\Omega) \) can clearly be weakened by assuming only that \( \{u_k\} \subset W^{1,p}_0(\Omega) \). Also, the requirement that \( \int_B |u_k|^p \, dx = 1 \) can be replaced by \( \int_B |u_k|^p \, dx \asymp 1 \). Moreover, by Theorem 4.5 this normalization can also be replaced by the requirement that \( \int_B u_k \, dx \asymp 1 \).

Example 4.7. Consider the functional \( Q(u) := \int_{\mathbb{R}^d} |\nabla u|^p \, dx \). It follows from [16, Theorem 2] that if \( d \leq p \), then \( Q \) admits a ground state \( \varphi = \text{constant} \) in \( \mathbb{R}^d \). On the other hand, if \( d > p \), then

\[
    u(x) := \left[1 + |x|^{p/(p-1)}\right]^{(p-d)/p}, \quad v(x) := \text{constant}
\]

are two positive supersolutions of the equation \( -\Delta_p u = 0 \) in \( \mathbb{R}^d \). Therefore, \( Q \) is strictly positive in \( \mathbb{R}^d \). For further examples see [20].

5 Solutions of minimal growth at infinity

In this section we define and study the existence of positive solutions of the equation \( Q'(u) = 0 \) of minimal growth in a neighborhood of infinity in \( \Omega \). In particular, we prove that a positive solution \( u \) of the equation \( Q'(u) = 0 \) in \( \Omega \) is a ground state if and only if \( u \) is has minimal growth in any neighborhood of infinity in \( \Omega \), a result which was proved in [22] only for \( 1 < p \leq d \).

Definition 5.1. Let \( K_0 \) be a compact set in \( \Omega \). A positive solution \( u \) of the equation \( Q'(u) = 0 \) in \( \Omega \setminus K_0 \) is said to be a positive solution of minimal growth in a neighborhood of infinity in \( \Omega \) (or \( u \in \mathcal{M}_{\Omega,K_0} \) for brevity) if for any compact set \( K \) in \( \Omega \), with a smooth boundary, such that \( K_0 \Subset \text{int}(K) \), and any positive supersolution \( v \in C((\Omega \setminus K) \cup \partial K) \) of the equation \( Q'(u) = 0 \) in \( \Omega \setminus K \), the inequality \( u \leq v \) on \( \partial K \) implies that \( u \leq v \) in \( \Omega \setminus K \).

A (global) positive solution \( u \) of the equation \( Q'(u) = 0 \) in \( \Omega \), which has minimal growth in a neighborhood of infinity in \( \Omega \) (i.e. \( u \in \mathcal{M}_{\Omega,\emptyset} \)) is called a global minimal solution of the equation \( Q'(u) = 0 \) in \( \Omega \).

If \( K_0 \subset K_1 \) are compact sets in \( \Omega \), then clearly \( \mathcal{M}_{\Omega,K_0} \subset \mathcal{M}_{\Omega,K_1} \). On the other hand, the inverse assertion seems to depend on the SCP. More precisely, we will prove the following statement after some preparatory lemmas that are of interest in their own right.
Proposition 5.2. Suppose that the functional $Q$ is nonnegative on $C_0^\infty(\Omega)$. Assume that the strong comparison principle (SCP) holds true with respect to $Q$ in any $C^{1,\alpha}$-bounded subdomain of $\Omega$. Consider two compact sets $K_0, K_1$ in $\Omega$ such that $K_0 \Subset \text{int}(K_1)$. If $u$ is a positive solution of the equation $Q'(u) = 0$ in $\Omega \setminus K_0$ such that $u \in M_{\Omega,K_1}$, then $u \in M_{\Omega,K_0}$.

Definition 5.3. Suppose that the functional $Q$ is nonnegative on $C_0^\infty(\Omega)$. Let $\{\Omega_N\}_{N=1}^\infty$ be an exhaustion of $\Omega$. Fix $K \Subset \Omega$ with smooth boundary, and let $u$ be a positive continuous function on $\partial K$. Let $u_N$ be the solution of the following Dirichlet problem

$$
\begin{cases}
Q'(u_N) = 0 & \text{in } \Omega_N \setminus K, \\
u_N = u & \text{on } \partial K, \\
u_N = 0 & \text{on } \partial \Omega_N,
\end{cases}
$$

(5.1)

We denote:

$$u^K := \lim_{N \to \infty} u_N \text{ on } \Omega \setminus K. \quad (5.2)$$

Lemma 5.4. Suppose that the functional $Q$ is nonnegative on $C_0^\infty(\Omega)$. Let $K$ be a compact set in $\Omega$ with smooth boundary, and let $u$ be a positive continuous function on $\partial K$. Then $u^K$ is well defined (and in particular, does not depend on the exhaustion $\{\Omega_N\}_{N=1}^\infty$). Moreover, $u^K \in M_{\Omega,K}$.

Proof. We note that for $N \geq 1$ we have $\lambda_0(\Omega_N \setminus K) > 0$, and therefore the (unique) solvability of (5.1) follows from a standard sub/supersolution argument and the weak comparison principle (WCP). Using again the WCP, we see that $\{u_N\}$ is a pointwise nondecreasing sequence. Moreover, for any $K_0 \Subset \text{int}(K)$, and any positive supersolution $v$ of the equation $Q'(u) = 0$ in $\Omega \setminus K_0$ satisfying $u \leq v$ on $\partial K$, we have by the WCP that $u_N \leq v$ in $\Omega_N \setminus K$. Therefore, the limit $u^K$ exists and $0 < u^K \leq v$ for any such $v$. Consequently, $u^K$ does not depend on the exhaustion $\{\Omega_N\}_{N=1}^\infty$. Similarly, one checks that $u^K \in M_{\Omega,K}$. \qed

Lemma 5.5. Let $K_0 \Subset \Omega$, and let $u$ be a positive solution of the equation $Q'(u) = 0$ in $\Omega \setminus K_0$. Then $u \in M_{\Omega,K_0}$ if and only if for any compact set $K \Subset \Omega$ with smooth boundary, such that $K_0 \Subset \text{int}(K)$, we have $u = u^K$.

Assume further that $K_0$ has a smooth boundary, and that $u$ is positive and continuous on $(\Omega \setminus K_0) \cup \partial K_0$. Then $u^K_0 \leq u$ and equality holds if and only if $u \in M_{\Omega,K_0}$. 

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Proof. Let $K \in \Omega$ be a set as above, and let $v \in C((\Omega \setminus K) \cup \partial K)$ be a positive supersolution of the equation $Q'(u) = 0$ in $\Omega \setminus K$, satisfying the inequality $u \leq v$ on $\partial K$. Then by comparison, $u^K = \lim_{N \to \infty} u_N \leq v$ in $\Omega \setminus K$. Now if $u = u^K$, it follows that $u \leq v$. Hence $u \in M_{\Omega,K_0}$.

On the other hand, if $u \in M_{\Omega,K_0}$, then by definition $u \leq u^K$ in $\Omega \setminus K$, and since $u^K \leq u$, we have $u = u^K$ in $\Omega \setminus K$.

Assume now that $K_0$ has a smooth boundary. Suppose that $u$ is a positive solution of the equation $Q'(u) = 0$ in $\Omega \setminus K_0$ which is positive and continuous on $(\Omega \setminus K_0) \cup \partial K_0$. Then by Lemma 5.4 and its proof we infer that $u^{K_0} \leq u$, and that equality in this inequality implies that $u \in M_{\Omega,K_0}$.

On the other hand, let $u \in M_{\Omega,K_0}$. Since $u^{K_0} \leq u$, we need only to prove that $u \leq u^{K_0}$ in $\Omega \setminus K_0$. In light of the continuity of $u$ and $u^{K_0}$ and the positivity of $u^{K_0}$ we infer that for any $\varepsilon > 0$ there exists a compact smooth set $K_{\varepsilon}$ satisfying $K_0 \Subset \text{int}(K_{\varepsilon})$, and $\text{dist}(\partial K_0, \partial K_{\varepsilon}) < \varepsilon$, such that $u \leq (1 + \varepsilon)u^{K_0}$ on $\partial K_{\varepsilon}$. Since $u \in M_{\Omega,K_0}$, it follows that $u \leq (1 + \varepsilon)u^{K_0}$ in $\Omega \setminus K_{\varepsilon}$. Letting $\varepsilon \to 0$ we obtain $u \leq u^{K_0}$ in $\Omega \setminus K_0$. \hfill \Box

Remark 5.6. Let $u$ be a positive solution as in the first part of Lemma 5.5. Then $u^K$ with $K = K_0$ might be the zero solution (for smooth $K_0$ this happens if $u|_{K_0} = 0$). Therefore, without additional assumptions, the set $K$ cannot be replaced in Lemma 5.5 by $K_0$.

On the other hand, let $K_0$ be a compact set in $\Omega$ with smooth boundary, and let $u \in M_{\Omega,K_0}$ which is positive and continuous on $(\Omega \setminus K_0) \cup \partial K_0$. Then it follows from Lemma 5.5 that the comparison principle for such solutions is also valid on $\Omega \setminus K_0$ and not only in $\Omega \setminus K$ with $K_0 \Subset \text{int}(K)$ as in the definition of $M_{\Omega,K_0}$.

More precisely, under the above assumptions on $u$, for any positive supersolution $v$ of the equation $Q'(u) = 0$ in $\Omega \setminus K_0$ which is positive and continuous on $(\Omega \setminus K_0) \cup \partial K_0$ and satisfies $u \leq v$ on $\partial K_0$, we have $u \leq v$ in $\Omega \setminus K_0$.

Proof of Proposition 5.2. Let $K_0, K_1$ be compact sets in $\Omega$ satisfying $K_0 \Subset \text{int}(K_1)$, and let $u$ be a positive solution of the equation $Q'(u) = 0$ in $\Omega \setminus K_0$ such that $u \in M_{\Omega,K_1}$. Let $K', K$ be smooth compact sets in $\Omega$ satisfying

$$K_0 \Subset \text{int}(K') \subset K' \subset K_1 \Subset \text{int}(K) \Subset \Omega.$$

Since $u = u^K$ in $\Omega \setminus K$ and on $\partial(K \setminus K')$ we have $u \succ u^{K'}$, it follows by comparison and exhaustion that $u \succ u^{K'}$ in $\Omega \setminus K'$. Define

$$\varepsilon_{K'} := \max\{\varepsilon > 0 \mid \varepsilon u \leq u^{K'} \text{ in } \Omega \setminus K'\}.$$
Clearly, $0 < \varepsilon_{K'} \leq 1$. Suppose that $\varepsilon_{K'} < 1$. Since $\varepsilon_{K'} u \leq u^{K'}$ in $\Omega \setminus K'$ and $\varepsilon_{K'} u < u^{K'}$ on $\partial K'$, it follows from the SCP that $\varepsilon_{K'} u < u^{K'}$ in $\Omega_N \setminus K'$. Therefore, there exists $\delta > 0$ such that $(1 + \delta) \varepsilon_0 < 1$ and

$$(1 + \delta) \varepsilon_{K'} u(x) \leq u^{K'}(x) \quad x \in K \setminus K'.$$

Hence, by comparison and exhaustion argument on $\Omega \setminus K$, we obtain

$$(1 + \delta) \varepsilon_{K'} u(x) = (1 + \delta) \varepsilon_{K'} u_K(x) \leq u^{K'}(x) \quad x \in \Omega \setminus K.$$

Hence, $(1 + \delta) \varepsilon_{K'} u(x) \leq u^{K'}$ in $\Omega \setminus K'$ which is a contradiction to the definition of $\varepsilon_{K'}$. This implies $u \leq u^{K'}$, and therefore, $u = u^{K'}$ in $\Omega \setminus K'$. Consequently, by Lemma 5.5, $u \in \mathcal{M}_{\Omega,K_0}$. \hfill $\square$

The following two theorems extend (except for the uniqueness statement of [22, Theorem 5.4]) Theorems 5.4 and 5.5 in [22] which were proved under the assumption $1 < p \leq d$.

**Theorem 5.7.** Suppose that $1 < p < \infty$, and $Q$ is nonnegative on $C_0^\infty(\Omega)$. Then for any $x_0 \in \Omega$ the equation $Q'(u) = 0$ has a positive solution $u \in \mathcal{M}_{\Omega\setminus\{x_0\}}$.

**Proof.** Consider an exhaustion $\{\Omega_N\}_{N=1}^\infty$ of $\Omega$ such that $x_0 \in \Omega_1$. Let $\{f_N\}$ be a sequence of nonzero nonnegative smooth functions such that $f_N$ is compactly supported in $B(x_0, 2/N) \setminus B(x_0, 1/N)$.

Fix $N \geq 1$, and denote $A_N := \Omega_N \setminus \overline{B(x_0, 1/N)}$. There exists (with a suitable $c_N > 0$) a unique positive solution of the Dirichlet problem

$$\begin{cases}
Q'(u_N) = c_N f_N & \text{in } A_N, \\
u_N = 0 & \text{on } \partial A_N, \\
u_N(x_1) = 1,
\end{cases}$$

(5.3)

where $x_1 \in A_1$ is a fixed reference point. Note that $u_N$ is a positive solution of the homogeneous equation in $\Omega_N \setminus \overline{B(x_0, 2/N)}$, and $u_N(x_1) = 1$. By the Harnack convergence principle, $\{u_N\}$ admits a subsequence which converges locally uniformly in $\Omega \setminus \{x_0\}$ to a positive solution $u$ of the equation $Q'(u) = 0$ in $\Omega \setminus \{x_0\}$.

Let $K \subseteq \Omega$ be a compact set with a smooth boundary such that $x_0 \in \text{int}(K)$, and let $v \in C(\Omega \setminus \text{int}(K))$ be a positive supersolution of the equation $Q'(u) = 0$ in $\Omega \setminus K$ such that the inequality $u \leq v$ holds on $\partial K$. 

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For $N \geq N_K$ we have that $\text{supp} f_N \subset B(x_0, 2/N) \Subset K$. Fix $\delta > 1$ and $x \in \Omega \setminus K$. Applying the WCP (Theorem 2.2) in $\Omega_N \setminus K$, we obtain that $u_N(x) \leq (1 + \delta)v(x)$ for all $N$ sufficiently large. By letting $N \to \infty$, and then $\delta \to 0$, we obtain that $u \leq v$ in $\Omega \setminus K$. Hence, $u \in \mathcal{M}_{\Omega, \{x_0\}}$.

**Theorem 5.8.** The functional $Q_V$ is strictly positive in $\Omega$ if and only if the equation $Q'_V(u) = 0$ does not admit a global minimal solution in $\Omega$. In particular, $u$ is ground state of the equation $Q'_V(u) = 0$ in $\Omega$ if and only if $u$ is a global minimal solution of this equation.

**Proof.** 1. **Necessity.** Assume that there exists a global minimal solution $u > 0$ of the equation $Q'_V(u) = 0$ in $\Omega$, and suppose that $Q_V$ is strictly positive. It follows from Theorem 4.5 and Theorem 1.2 (see also [22, Proposition 4.4]) that there exists a nonzero nonnegative function $V_1 \in C_0^\infty(\Omega)$ with $\text{supp} V_1 \subset B(x_0, \delta)$ for some $\delta > 0$, such that $Q_{V-V_1}$ is strictly positive in $\Omega$. Therefore, due to Theorem 1.2 there exists a positive solution $v$ of the equation $Q'_{V-V_1}(u) = 0$ in $\Omega$.

In particular, $v$ is a positive supersolution of the equation $Q'_V(u) = 0$ in $\Omega$ which is not a solution. On the other hand, $u$ is a positive solution of the equation $Q'_V(u) = 0$ in $\Omega$ which has minimal growth in a neighborhood of infinity in $\Omega$. Therefore, there exists $\varepsilon > 0$ such that $\varepsilon u \leq v$ in $\Omega$. Define

$$\varepsilon_0 := \max\{\varepsilon > 0 \mid \varepsilon u \leq v \text{ in } \Omega\}.$$ 

Clearly $\varepsilon_0 u \smallsetminus v$ in $\Omega$. Consequently, there exist $\delta_1, \delta_2 > 0$ and $x_1 \in \Omega$ such that

$$(1 + \delta_1)\varepsilon_0 u(x) \leq v(x) \quad x \in B(x_1, \delta_2).$$

Hence, by the definition of minimal growth, we have

$$(1 + \delta_1)\varepsilon_0 u(x) \leq v(x) \quad x \in \Omega \setminus B(x_1, \delta_2),$$

and thus $(1 + \delta_1)\varepsilon_0 u \leq v$ in $\Omega$, which is a contradiction to the definition of $\varepsilon_0$.

2. **Sufficiency.** Fix $x_0 \in \Omega_1$. Assume that $Q$ is not strictly positive. Then $Q$ admits a (unique) ground state $u$ in $\Omega$ satisfying $u(x_1) = 1$, where $x_1 \in \Omega \setminus \Omega_1$ is another fixed reference point. It suffices to prove that $u$ is a global minimal solution of the equation $Q'(u) = 0$ in $\Omega$.

Fix $n \in \mathbb{N}$, and let $f_n \in C_0^\infty(B(x_0, 1/n))$ be a nonzero nonnegative function. For $N \geq 1$, let $v_{N,n}$ be the unique positive solution of the Dirichlet
problem
\[
\left\{ \begin{array}{ll}
Q'(v_{N,n}) = f_n & \text{in } \Omega_N, \\
v_{N,n} = 0 & \text{on } \partial\Omega_N.
\end{array} \right.
\] (5.4)

By the WCP, \(\{v_{N,n}\}_{N \geq 1}\) is a nondecreasing sequence. Recall from [22, Theorem 1.6] that if \(Q\) admits a ground state \(u\) in \(\Omega\), then any positive supersolution for the equation \(Q'(u) = 0\) in \(\Omega\) equals to \(cu\) with some \(c > 0\). On the other hand, if \(\{v_{N,n}(x_1)\}\) is bounded, then \(v_{N,n} \to v_n\), where \(v_n\) satisfies the equation \(Q'(v_n) = f_n \geq 0\) in \(\Omega\). Therefore, \(v_{N,n}(x_1) \to \infty\).

Consider now the sequence of functions \(u_{N,n}(x) := v_{N,n}(x)/v_{N,n}(x_1), N \geq 1\). Then \(u_{N,n}\) solves the Dirichlet problem
\[
\left\{ \begin{array}{ll}
Q'(u_{N,n}) = \frac{f_n(x)}{v_{N,n}(x_1)^{p-1}} & \text{in } \Omega_N, \\
u_{N,n} = 0 & \text{on } \partial\Omega_N, \\
u_{N,n}(x_1) = 1.
\end{array} \right.
\] (5.5)

By the Harnack convergence principle, we may extract a subsequence of \(\{u_{N,n}\}\) that converges as \(N \to \infty\) to a positive solution \(u_n\) of the equation \(Q'(u) = 0\) in \(\Omega\). By the uniqueness of the ground state, we have \(u_n = u\).

Let \(K \Subset \Omega\) be a compact set with a smooth boundary, and let \(v \in C(\Omega \setminus \text{int}(K))\) be a positive supersolution of the equation \(Q'(u) = 0\) in \(\Omega \setminus K\) such that the inequality \(u \leq v\) holds on \(\partial K\). Without loss of generality, we may assume that \(x_0 \in \text{int}(K)\).

Let \(n \in \mathbb{N}\) be sufficiently large number such that \(\text{supp } f_n \Subset K\). By comparison it follows (as in the first part of the proof) that \(u = u_n \leq v\) in \(\Omega \setminus K\). Since \(K \Subset \Omega\) is an arbitrary smooth compact set, it follows that the ground state \(u\) is a global minimal solution of the equation \(Q'(u) = 0\) in \(\Omega\). \(\square\)

Consider a positive solution \(u\) of the equation \(Q'(u) = 0\) in a punctured neighborhood of \(x_0\) which has a nonremovable singularity at \(x_0 \in \mathbb{R}^d\). Without loss of generality, we may assume that \(x_0 = 0\). If \(1 < p \leq d\), then the behavior of \(u\) near an isolated singularity is well understood. Indeed, due to a result of L. Véron (see [22, Lemma 5.1]), we have that
\[
u(x) \sim \begin{cases} 
|x|^{\alpha(d,p)} & p < d, \\
-\log|x| & p = d,
\end{cases} \quad \text{as } x \to 0,
\] (5.6)
where \( \alpha(d, p) := (p - d)/(p - 1) \), and \( f \sim g \) means that
\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = C
\]
for some positive constant \( C \). In particular, \( \lim_{x \to 0} u(x) = \infty \).

Assume now that \( p > d \). A general question is whether in this case, any positive solution of the equation \( Q'(u) = 0 \) in a punctured ball centered at \( x_0 \) can be continuously extended at \( x_0 \) (see [14] for partial results).

We answer below this question under the assumption that \( u \approx 1 \) near the isolated singular point.

**Lemma 5.9.** Assume that \( p > d \), and let \( v \) be a positive solution of the equation \( Q'(u) = 0 \) in a punctured neighborhood of \( x_0 \) satisfying \( v \approx 1 \) near \( x_0 \). Then \( v \) can be continuously extended at \( x_0 \).

**Proof.** We use Véron’s method (see [22, Lemma 5.1]). Without loss of generality, we may assume that \( x_0 = 0 \). For \( 0 < r \leq r_0 \) denote
\[
m(r) := \min_{|x| = r} v(x), \quad M(r) := \max_{|x| = r} v(x), \quad M := \limsup_{r \to 0} M(r).
\]

By our assumption \( 0 < M < \infty \). Let \( \{x_n\} \) be a sequence such that \( x_n \to 0 \), and \( M = \lim_{n \to \infty} v(x_n) \). Define \( v_n(x) := v(|x_n|x) \).

Since \( u \approx 1 \) near \( x_0 \), it follows that there exists \( C > 0 \) such that in an arbitrarily large punctured ball
\[
C^{-1} \leq v_n(x) \leq C
\]
for all \( n \) large enough. Moreover, in such a ball, \( v_n \) is a positive solution of the quasilinear elliptic equation
\[
-\Delta_p v_n(x) + |x_n|^p V(|x_n|x)v_n^{p-1}(x) = 0. \quad (5.7)
\]
Since \( \{v_n\} \) is locally bounded and bounded away from zero in any punctured ball, the Harnack convergence principle implies that there is a subsequence of \( \{v_n\} \) that converges locally uniformly in \( \mathbb{R}^d \setminus \{0\} \) to a positive bounded solution \( U \) of the limiting equation \( -\Delta_p U = 0 \) in the punctured space. Recall that by Example [17, 17] if \( p \geq d \), then the constant function is a ground state of the \( p \)-Laplacian [20], and in particular, it is the unique positive \( p \)-(super)harmonic function in \( \mathbb{R}^d \). Therefore, Theorem [17, 17] implies
that the origin has zero $p$-capacity. By Theorem 7.36 in [11], the singularity of $U$ at the origin is removable. Hence, $U$ is an entire positive $p$-harmonic function in $\mathbb{R}^d$, and consequently, $U = \text{constant} = \alpha$.

This implies that
\[
\lim_{n \to \infty} \|v(x) - \alpha\|_{L^\infty(|x|=|x_n|)} = 0.
\tag{5.8}
\]
In other words, $v$ approximates $\alpha$ uniformly over concentric spheres whose radii converge to 0.

We need to prove that $v$ approximates $\alpha$ uniformly over the concentric annuli $A_n := \{|x_n| \leq |x| \leq |x_{n+1}|\}$. Let $\alpha_-(x) := \alpha - \delta|x|^a$ and $\alpha_+(x) := \alpha + \delta|x|^a$ (for some $a > 0$ and $\delta > 0$ sufficiently small). It turns out that $\alpha_-$ (resp., $\alpha_+$) is a radial positive subsolution (resp., supersolution) of the equation $Q'(u) = 0$ near the origin, and therefore using the comparison principle in the annulus $A_n$, $n \in \mathbb{N}$, and (5.8), it follows that
\[
\lim_{r \to 0} \|v(x) - \alpha\|_{L^\infty(|x|=r)} = 0.
\]

Let $u \in M_{\Omega \setminus \{0\}}$ (without loss of generality, we assume that $x_0 = 0 \in \Omega$). If $u$ has a removable singularity at 0, then by definition, $u$ is a global minimal solution. Let us show that the converse is also true.

Suppose that $u$ has a nonremovable singularity at 0, and set
\[
m := \liminf_{r \to 0} m(r) = \liminf_{r \to 0} \min u(x), \quad M := \limsup_{r \to 0} M(r) = \limsup_{r \to 0} \max u(x).
\]

By Harnack inequality, for any positive solution $v$ of the equation $Q'(u) = 0$ in a punctured neighborhood of 0 there exists $\tilde{C} > 0$ such that
\[
\tilde{C}^{-1} M(r) \leq m(r) \leq M(r) \quad 0 < r \leq r_0.
\tag{5.9}
\]
If $m = 0$, then by comparing $u$ with any positive global (super)solution and using Harnack inequality (5.9), we infer that $u$ must be identically zero, which is a contradiction.

On the other hand, if $M = \infty$, then the equation $Q'(u) = 0$ in $\Omega$ does not admit a global minimal solution due to the Harnack inequality (5.9) and the weak comparison principle. Hence, Theorem 5.8 implies that $Q$ is strictly positive.

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Assume now that $0 < m \leq M < \infty$. Then by Lemma 5.9, $u$ can be continuously extended at the origin. If the equation $Q'(u) = 0$ in $\Omega$ admits a global minimal solution $v$, then by comparison, $u = cv$, where $c$ is a positive constant, and thus $u$ has a removable singularity at $0$, a contradiction.

Thus, we proved the following result which extends the second part of [22 Theorem 5.4], where the case $1 < p \leq d$ is considered.

**Theorem 5.10.** Let $x_0 \in \Omega$, and let $u \in M_{\Omega,\{x_0\}}$. Then $Q$ is strictly positive in $\Omega$ if and only if $u$ has a nonremovable singularity at $x_0$.

## 6 Variational principle for solutions of minimal growth in the linear case

Throughout this section we restrict our consideration to the linear case ($p = 2$). In fact, as in [21], we can actually consider in the linear case the following somewhat more general functional than $Q_V$ of the form (1.2).

We assume that $A : \Omega \to \mathbb{R}^{d^2}$ is a measurable symmetric matrix valued function such that for every compact set $K \subset \Omega$ there exists $\mu_K > 1$ so that

$$\mu_K^{-1} I_d \leq A(x) \leq \mu_K I_d \quad \forall x \in K,$$

where $I_d$ is the $d$-dimensional identity matrix, and the matrix inequality $A \leq B$ means that $B - A$ is a nonnegative matrix on $\mathbb{R}^d$. Let $V \in L^q_{\text{loc}}(\Omega; \mathbb{R})$, where $q > d/2$. We consider the quadratic form

$$a_{A,V}[u] := \frac{1}{2} \int_{\Omega} (A \nabla u \cdot \nabla u + V |u|^2) \, dx$$

(6.2)

on $C^\infty_0(\Omega)$ associated with the Schrödinger equation

$$Pu := (-\nabla \cdot (A \nabla) + V)u = 0 \quad \text{in } \Omega.$$ (6.3)

We say that $a_{A,V}$ is nonnegative on $C^\infty_0(\Omega)$, if $a_{A,V}[u] \geq 0$ for all $u \in C^\infty_0(\Omega)$.

Let $v$ be a positive solution of the equation $Pu = 0$ in $\Omega$. Then by [21 Lemma 2.4] we have the following analog of (3.1). For any nonnegative $w \in C^\infty_0(\Omega)$ we have

$$a_{A,V}[w] = \frac{1}{2} \int_{\Omega} v^2 |A \nabla (w/v) \cdot \nabla (w/v)| \, dx.$$ (6.4)
Moreover, it follows from [21, 22] that all the results mentioned in the present paper concerning criticality/subcriticality of the functional \( Q \) are also valid for the form \( a_{A,V} \).

The aim of this section is to characterize positive solutions of minimal growth in a neighborhood of infinity in \( \Omega \) in terms of a modified null sequence of the form \( a_{A,V} \).

**Theorem 6.1.** Suppose that \( a_{A,V} \) is nonnegative on \( C^\infty_0(\Omega) \). Let \( \Omega_1 \Subset \Omega \) be an open set, and let \( u \in C(\Omega \setminus \Omega_1) \) be a positive solution of the equation \( Pu = 0 \) in \( \Omega \setminus \Omega_1 \).

Then \( u \in \mathcal{M}_{\Omega_1} \) if and only if for every smooth open set \( \Omega_2 \) satisfying \( \Omega_1 \Subset \Omega_2 \Subset \Omega \), and an open set \( B \Subset (\Omega \setminus \Omega_2) \) there exists a sequence \( \{u_k\} \subset C^\infty_0(\Omega), \ u_k \geq 0 \), such that for all \( k \in \mathbb{N} \), \( \int_B |u_k|^2 \, dx = 1 \), and

\[
\lim_{k \to \infty} \int_{\Omega \setminus \Omega_2} u^2 A \nabla (u_k/u) \cdot \nabla (u_k/u) \, dx = 0. \tag{6.5}
\]

**Proof.** 1. Sufficiency. Let \( u \in C(\Omega \setminus \Omega_1) \) be a positive solution of the equation \( Pu = 0 \) in \( \Omega \setminus \Omega_1 \). Let \( \Omega_2 \) be an open set with smooth boundary such that \( \Omega_1 \Subset \Omega_2 \Subset \Omega \), and let \( B \) be an open set so that \( B \Subset (\Omega \setminus \Omega_2) \).

Suppose that \( \{u_k\} \subset C^\infty_0(\Omega) \) is a sequence of nonnegative functions such that \( \int_B |u_k|^2 \, dx = 1 \) for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \int_{\Omega \setminus \Omega_2} u^2 A \nabla (u_k/u) \cdot \nabla (u_k/u) \, dx = 0. \tag{6.6}
\]

It follows (cf. [21] Lemma 2.5) that \( u_k \to cu \) in \( W^{1,2}_{\text{loc}}(\Omega \setminus \overline{\Omega_2}) \), where \( c > 0 \).

Let \( E : W^{2,2}_{\text{loc}}(\Omega \setminus \Omega_2) \to W^{2,2}_{\text{loc}}(\Omega) \) be a bounded linear extension operator (cf. [9] Section 5.4), and in particular, Remark (i) therein). Note that the operator \( E \) in [9] extends a given function to a bounded open set outside a smooth compact boundary, so the construction from [9] applies also to our situation, combined with a straightforward use of partition of unity. For completeness, we outline the construction of \( E \) below.

In suitable neighborhoods \( U_1, \ldots, U_m \) covering \( \partial \Omega_2 \) there exist local diffeomorphisms \( \psi_j \) such that for each \( 1 \leq j \leq m \) the diffeomorphism \( \psi_j \) maps the open set \( U_j \cap (\Omega \setminus \overline{\Omega_2}) \) into a portion of the half-space \( \mathbb{R}^d_+ = \{(x', x_d) \in \mathbb{R}^d \mid x_d > 0\} \), and \( \psi_j(U_j \cap (\partial \Omega_2)) \subset \{(x', 0) \in \mathbb{R}^d\} \). We may
assume that \( w \in C^\infty(\Omega \setminus \Omega_2) \) and define in \( U_1, \ldots, U_m \) up to the diffeomorphism

\[
w_j(x', x_d) := \begin{cases} w(x', x_d) & \text{if } x_d \geq 0, \\ 4w(x', -x_d/2) - 3w(x', -x_d) & \text{if } x_d < 0. \end{cases}
\]

Let \( U_0 \) and \( U_\infty \) be open sets, \( U_0 \Subset \Omega_2 \) and \( U_\infty \subset \Omega \setminus \Omega_2 \), such that together with \( U_1, \ldots, U_m \), they form an open covering of \( \Omega \). Set \( w_0(x) := 1 \) for \( x \in U_0 \), and \( w_\infty(x) := w(x) \) if \( x \in U_\infty \). Let \( \{\chi_j\}_{j=0, \ldots, m, \infty} \) be a partition of unity subordinated to \( U_0, \ldots, U_m, U_\infty \), and define the extension operator as

\[
(Ew)(x) := w_\infty(x)\chi_\infty(x) + \sum_{j=0}^m w_j(x)\chi_j(x).
\]

Let \( \tilde{u} := Eu \), and \( \tilde{v}_k := E(u_k/u) \), and \( \tilde{u}_k := \tilde{u}\tilde{v}_k \). Note that, since \( u > 0 \) and \( \partial\Omega_2 \) is compact, one can always choose the neighborhoods \( U_1, \ldots, U_m \) sufficiently small so that \( \tilde{u} > 0 \). Clearly, \( Ew|_{\Omega \setminus \Omega_2} = w \). Let \( f := P\tilde{u} \), and define \( W := f/\tilde{u} \). Notice that \( W \) has a compact support in \( \Omega \). Moreover, by elliptic regularity, \( W \in L^q(\Omega) \). It follows that \( \tilde{u} \) is a positive solution of the equation \((P - W)u = 0\) in \( \Omega \). Moreover, by the continuity of \( E \) and the continuity of the form \( a \) due (6.4), it follows from (6.6) that

\[
\lim_{k \to \infty} a_{A,V - W}[\tilde{u}_k] = \lim_{k \to \infty} \int_{\Omega} \tilde{u}_k^2 A\nabla(\tilde{u}_k/\tilde{u}) \cdot \nabla(\tilde{u}_k/\tilde{u}) \, dx = 0.
\]

On the other hand, \( \int_B |\tilde{u}_k|^2 \, dx = \int_B |u_k|^2 \, dx = 1 \). Therefore, Corollary 1.6 in [21] implies that \( \tilde{u}_k \to c\tilde{u} \), and \( \tilde{u} \in M_{\Omega,0} \) (Corollary 1.6 of [21] is analogous to Theorem 5.8, but note that the terminology in [21] is different from the terminology of the present paper). Hence \( u \in M_{\Omega,0} \). Since \( \Omega_2 \) is an arbitrary smooth open set satisfying \( \Omega_1 \Subset \Omega_2 \Subset \Omega \), we have \( u \in M_{\Omega,0} \).

2. Necessity. Suppose that \( u \in C(\Omega \setminus \Omega_1) \cap M_{\Omega,0} \). Let \( \Omega_2 \) be any open set with smooth boundary such that \( \Omega_1 \Subset \Omega_2 \Subset \Omega \).

Let \( \tilde{u} \) be a positive function in \( W_{1,2}^{loc}(\Omega) \) such that \( \tilde{u}|_{\Omega \setminus \Omega_2} = u \). Let \( f := P\tilde{u} \), and define \( W := f/\tilde{u} \). Recall that \( W \) has a compact support in \( \Omega \), and that the SCP holds in the linear case. Therefore, Proposition 5.2 implies that \( \tilde{u} \) is a global minimal solution of the equation \((P - W)u = 0\) in \( \Omega \). Consequently, it follows from [21, Corollary 1.6] that \( \tilde{u} \) is a ground state of the equation
\((P - W)u = 0\) in \(\Omega\). Let \(\{u_k\} \subset C_0^\infty(\Omega)\) be a null sequence for the form \(a_{A,V - W}\). So, for some open set \(B \Subset (\Omega \setminus \overline{\Omega}_2)\) we have \(\int_B |u_k|^2 \, dx = 1\), and

\[
\lim_{k \to \infty} \int_\Omega \tilde{u}^2 A \nabla (u_k/\tilde{u}) \cdot \nabla (u_k/\tilde{u}) \, dx = \lim_{k \to \infty} a_{A,V - W}[u_k] = 0.
\]

Thus,

\[
\lim_{k \to \infty} \int_{\Omega \setminus \overline{\Omega}_2} u^2 A \nabla (u_k/u) \cdot \nabla (u_k/u) \, dx = 0. \tag{6.8}
\]

Finally, we prove a sub-supersolution comparison principle for our singular elliptic equation. This general Phragmén-Lindelöf type principle, which seems to be new even in the linear case, holds in unbounded or nonsmooth domains, and for irregular potential \(V\), provided the subsolution satisfies a certain decay property of variational type (cf. \([1, 12, 15, 23]\)).

**Theorem 6.2 (Comparison Principle).** Let \(P\) be a nonnegative Schrödinger operator of the form (6.3). Fix smooth open sets \(\Omega_1 \Subset \Omega_2 \Subset \Omega\). Let \(u, v \in W^{1,2}_\text{loc}(\Omega \setminus \overline{\Omega}_1) \cap C(\Omega \setminus \overline{\Omega}_1)\) be, respectively, a positive subsolution and a supersolution of the equation \(Pw = 0\) in \(\Omega \setminus \overline{\Omega}_1\) such that \(u \leq v\) on \(\partial \Omega_2\).

Assume further that \(Pu \in L^q_{\text{loc}}(\Omega \setminus \overline{\Omega}_1)\), where \(q > d/2\), and that there exist an open set \(B \Subset (\Omega \setminus \overline{\Omega}_2)\) and a sequence \(\{u_k\} \subset C_0^\infty(\Omega), u_k \geq 0\), such that

\[
\int_B |u_k|^2 \, dx = 1 \quad \forall k \geq 1, \quad \text{and} \quad \lim_{k \to \infty} \int_{\Omega \setminus \overline{\Omega}_1} u^2 A \nabla (u_k/u) \cdot \nabla (u_k/u) \, dx = 0. \tag{6.9}
\]

Then \(u \leq v\) on \(\Omega \setminus \Omega_2\).

**Proof.** Let \(\Omega_1 \Subset \Omega_2 \Subset \Omega\), and let \(0 < \tilde{u} \in W^{1,2}_{\text{loc}}(\Omega)\) be the extension of \(u\) provided in the proof of Theorem 6.1 and define analogously \(f := P\tilde{u}\) and \(W := f/\tilde{u}\). Clearly, \(W \leq 0\) in \(\Omega \setminus \Omega_2\). Theorem 6.1 and Proposition 5.2 imply that \(\tilde{u}\) is a positive solution of the equation \((P - W)u = 0\) in \(\Omega\) which is a global minimal solution. On the other hand, \(v\) is a positive supersolution of the equation \((P - W)u = 0\) in \(\Omega \setminus \Omega_2\), therefore \(u \leq v\) on \(\Omega \setminus \Omega_2\).

**Remark 6.3.** In Theorem 6.2 we have assumed that the subsolution \(u\) is strictly positive. It would be useful to prove the above comparison principle under the assumption that \(u \geq 0\) (cf. \([12]\)).
Remark 6.4. Let $K$ be a compact set in $\Omega$, and $\phi$ be a positive solution of minimal growth in a neighborhood of infinity in $\Omega$ of the equation $\dot{P}u := -\nabla \cdot (\dot{A}\nabla u) + \dot{V}u = 0$ in $\Omega$ for some $\dot{A}$ satisfying (6.1), and $\dot{V} \in L^q_{\text{loc}}(\Omega; \mathbb{R})$, with $q > d/2$. If $u \in W^{1,2}_{\text{loc}}(\Omega \setminus \Omega_1) \cap C(\Omega \setminus \Omega_1)$ is a positive subsolution of the equation $Pw = 0$ in $\Omega \setminus \Omega_1$ such that

$$u^2(x)A(x) \leq \phi^2(x)\tilde{A}(x)$$

in $\Omega \setminus K$, then Condition (6.9) is satisfied (cf. [18]).

7 Variational principle for solutions of minimal growth for the quasilinear case

In this section we extend the results of the previous section to the case $1 < p < \infty$. Since the SCP does not always hold, we obtain weaker results.

**Theorem 7.1.** Suppose that $1 < p < \infty$, and let $Q_V$ be nonnegative on $C^\infty_0(\Omega)$. Let $\Omega_1 \subseteq \Omega$ be an open set, and let $u \in C(\Omega \setminus \Omega_1)$ be a positive solution of the equation $Q'_V(u) = 0$ in $\Omega \setminus \overline{\Omega_1}$ satisfying $|\nabla u| \neq 0$ in $\Omega \setminus \overline{\Omega_1}$.

Then $u \in \mathcal{M}_{\Omega_1}$ if for every smooth open set $\Omega_2$ satisfying $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, and an open set $B \subseteq (\Omega \setminus \overline{\Omega_2})$ there exists a sequence $\{u_k\} \subset C^\infty_0(\Omega)$, $u_k \geq 0$, such that for all $k \in \mathbb{N}$, $\int_B |u_k|^p \, dx = 1$, and

$$\lim_{k \to \infty} \int_{\Omega \setminus \overline{\Omega_2}} L(u_k, u) \, dx = 0,$$

(7.1)

where $L$ is the Lagrangian given by (3.2).

**Proof.** Let $u$ be a positive solution of the equation $Q'_V(u) = 0$ in $\Omega \setminus \overline{\Omega_1}$ satisfying the theorem’s assumptions. Let $\Omega_2$ be an open set with smooth boundary such that $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, and let $B$ an open set so that $B \subseteq (\Omega \setminus \overline{\Omega_2})$.

Suppose that $\{u_k\} \subset C^\infty_0(\Omega)$ is a sequence of nonnegative functions such that $\int_B |u_k|^p \, dx = 1$ for all $k \in \mathbb{N}$, and

$$\lim_{k \to \infty} \int_{\Omega \setminus \overline{\Omega_2}} L(u_k, u) \, dx = 0.$$

(7.2)

As in the proof of [22] Lemma 3.2], it follows that $u_k \to cu$ in $W^{1,p}_{\text{loc}}(\Omega \setminus \overline{\Omega_1.5})$, where $c > 0$. 

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Let $E : W^{2,p}_{\text{loc}}(\Omega \setminus \Omega_2) \to W^{2,p}_{\text{loc}}(\Omega)$ be the bounded extension operator, constructed in the proof of Theorem 6.1 (referring here, as in [9], to the general case $1 < p < \infty$):

$$(Ew)(x) := w_{\infty}(x)\chi_{\infty}(x) + \sum_{i=0}^{m} w_i(x)\chi_i(x).$$

Let $\tilde{u} = Eu$, $\tilde{u}_k = Eu_k$ and note that, since $u > 0$, one can always choose the neighborhoods $U_1, \ldots, U_m$ sufficiently small so that $\tilde{u} > 0$.

Let $f := Q'_V(\tilde{u})$, and define $W := f\tilde{u}^{1-p}$. Clearly, $W$ has a compact support. Since $|\nabla u| \neq 0$ in $\Omega \setminus \Omega_1$, a standard elliptic regularity argument implies that $W \in L^\infty_{\text{loc}}(\Omega)$. It follows that $\tilde{u}$ is a positive solution of the equation $Q'_V(\tilde{u}) = 0$ in $\Omega$. Moreover, by the continuity of $E$ and the continuity of $Q'_V - W$ due (3.5), it follows from (7.2) that

$$\lim_{k \to \infty} Q'_V(\tilde{u}_k) = \lim_{k \to \infty} Q'_{V-W}(\tilde{u}_k) = \lim_{k \to \infty} \int_{\Omega} L(|\tilde{u}_k|, \tilde{u}) \, dx = 0. \quad (7.3)$$

Therefore, [22, Theorem 1.6] implies that $|\tilde{u}_k| \to c\tilde{u}$, and $\tilde{u}$ is a ground state of the functional $Q'_{V-W}$ in $\Omega$. Consequently, Theorem 5.8 implies that $\tilde{u} \in \mathcal{M}_{\Omega, \emptyset}$, and therefore $u \in \mathcal{M}_{\Omega_1, \Omega_2}$. Since $\Omega_2$ is an arbitrary smooth open set satisfying $\Omega_1 \subset \Omega_2 \subset \Omega$, we have $u \in \mathcal{M}_{\Omega_1, \Omega_2}$.

**Remark 7.2.** Suppose that for all $V \in L^\infty_{\text{loc}}(\Omega)$, any positive solution of the equation $Q'_V(u) = 0$ in $\Omega$ satisfying $u \in \mathcal{M}_{\Omega_1, \Omega_2}$ for some smooth open set $\Omega_1 \subset \Omega_2 \subset \Omega$ is a global minimal solution (this assumption seems to depend on the SCP, cf. Proposition 5.2). Then the condition of Theorem 7.1 is also necessary.

Indeed, suppose that $u \in C(\Omega \setminus \Omega_1) \cap \mathcal{M}_{\Omega_1, \Omega_2}$ satisfying $|\nabla u| \neq 0$ in $\Omega \setminus \Omega_1$. Let $\Omega_2$ be any open set with smooth boundary such that $\Omega_1 \subset \Omega_2 \subset \Omega$. Let $\tilde{u}$ be a positive smooth function in $\Omega$ such that $\tilde{u}|_{\Omega \setminus \Omega_2} = u$. Let $f := Q'_V(\tilde{u})$, and define $W := f\tilde{u}^{1-p}$. Recall that $W$ has a compact support in $\Omega$, and by our assumption $\tilde{u}$ is a global minimal solution of the equation $Q'_{V-W}(u) = 0$ in $\Omega$. So, by Theorem 5.7, $\tilde{u}$ is a ground state of the functional $Q'_{V-W}$. Let $\{u_k\} \subset C^\infty(\Omega)$ be a null sequence for $Q'_{V-W}$. So, $\lim_{k \to \infty} Q'_{V-W}(u_k) = 0$, and for an open set $B \subset (\Omega \setminus \Omega_2)$, we have $\int_B |u_k|^p \, dx = 1$ for all $k \in \mathbb{N}$. By Picone identity,

$$\lim_{k \to \infty} \int_{\Omega} L(u_k, \tilde{u}) \, dx = \lim_{k \to \infty} Q'_{V-W}(u_k) = 0.$$
Since \( L(u_k, \bar{u}) \geq 0 \) in \( \Omega \), it follows that
\[
\lim_{k \to \infty} \int_{\Omega \setminus \overline{\Omega}_2} L(u_k, u) \, dx = 0. \tag{7.4}
\]

Finally, we formulate a sub-supersolution comparison principle for our singular elliptic equation.

**Theorem 7.3** (Comparison Principle). Suppose that \( 1 < p < \infty \), and let \( Q_V \) be nonnegative on \( C_0^\infty(\Omega) \). Fix smooth open sets \( \Omega_1 \Subset \Omega_2 \Subset \Omega \). Let \( u, v \in W^{1,p}_{\text{loc}}(\Omega \setminus \Omega_1) \cap C(\Omega \setminus \Omega_1) \) be, respectively, a positive subsolution and a supersolution of the equation \( Q'_V(w) = 0 \) in \( \Omega \setminus \overline{\Omega}_1 \) such that \( u \leq v \) on \( \partial \Omega_2 \).

Assume further that \( Q'_V(u) \in L^\infty_{\text{loc}}(\Omega \setminus \Omega_1) \), \( |\nabla u| \neq 0 \) in \( \Omega \setminus \overline{\Omega}_1 \), and that there exist an open set \( B \Subset (\Omega \setminus \overline{\Omega}_2) \) and a sequence \( \{u_k\} \subset C_0^\infty(\Omega) \), \( u_k \geq 0 \), such that
\[
\int_B |u_k|^p \, dx = 1 \quad \forall k \geq 1, \quad \text{and} \quad \lim_{k \to \infty} \int_{\Omega \setminus \Omega_1} L(u_k, u) \, dx = 0. \tag{7.5}
\]

Then \( u \leq v \) on \( \Omega \setminus \Omega_2 \).

**Proof.** As in the proof of Theorem 7.1 let \( 0 < \bar{u} \in W^{1,p}_{\text{loc}}(\Omega) \) be an extension of \( u \) such that \( \bar{u}|_{\Omega \setminus \Omega_2} = u \). Let \( f := Q'_V(\bar{u}) \), and define \( W := f\bar{u}^{1-p} \). Clearly, \( W \leq 0 \) in \( \Omega \setminus \Omega_2 \). By the proof of Theorem 7.1 \( \bar{u} \) is a positive solution of the equation \( Q'_{V-W}(u) = 0 \) in \( \Omega \) which is a global minimal solution. On the other hand, \( v \) is a positive supersolution of the equation \( Q'_{V-W}(w) = 0 \) in \( \Omega \setminus \Omega_2 \), therefore \( u \leq v \) on \( \Omega \setminus \Omega_2 \). \( \square \)

**Remark 7.4.** Following Remark 4.6 the normalization condition
\[
\int_B |u_k|^p \, dx = 1
\]
in Theorems 6.1, 6.2, 7.1 and 7.3 can be replaced by the condition
\[
\int_B u_k \, dx = 1.
\]

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