CONSTANT CURVATURE TRANSLATION SURFACES IN
GALILEAN 3-SPACE

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Abstract. Total five different types of translation surfaces, based upon planarity of translating curves and the absolute figure, arise in a Galilean 3-space. Excepting the type in which both of translating curves are non-planar we obtain these surfaces with arbitrary constant Gaussian and mean curvature.

1. Introduction and Preliminaries

The translation surfaces, among the family of surfaces in classic differential geometry, have been commonly examined since early 1900s and for that reason an extensive literature relating to these appears. For example see [3] [5] [6], [12]-[18], [24]-[26], [31]-[36]. Such surfaces are geometrically described as translating two curves along each other up to isometries of the ambient space. As far as we know the counterparts of this notion in a Galilean space $G_3$ were firstly considered in Sipus and Divjak’s work [20] by providing translation surfaces with constant Gaussian ($K$) and mean curvature ($H$) under the restriction that the translating curves lie in orthogonal planes. Extending this restriction, which is our motivation for the present study, leads us to open fields for further investigations. More precisely, by assuming $K = const.$ and $H = const.$ we shall present the translation surfaces in $G_3$, except the ones whose both of translating curves are space curves.

A Cayley-Klein 3-space is defined as a projective 3-space $P_3(\mathbb{R})$ with certain absolute figure. Group of motions of this space are introduced by the projective transformations which leave invariant the absolute figure. Metrically arguments given up to the absolute figure are invariant under this group (cf. [24]). The Galilean 3-space $G_3$ is one of real Cayley-Klein 3-spaces with the absolute figure $\{\Gamma, l, \iota\}$, where $\Gamma$ is a plane (absolute plane) in $P_3(\mathbb{R})$, $l$ a line (absolute line) in $\Gamma$ and $\iota$ is the fixed elliptic involution of the points of $l$. For technical details, we refer the reader to [1] [2] [4], [7]-[10], [19] [21] [22] [27]-[30], [37]. Let $(x_0 : x_1 : x_2 : x_3)$ denote the homogeneous coordinates in $P_3(\mathbb{R})$. Then $\Gamma$ is characterized by $x_0 = 0$, $l$ by $x_0 = x_1 = 0$ and $\iota$ by

$$(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1 : x_3 : -x_2).$$

Passing from the homogeneous coordinates to the affine coordinates is essential to introduce the affine model of $G_3$ that is our interest field. Then, by means of the affine coordinates, the group of motions of $G_3$ is given by the transformation

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A curve given in parametric form \( \alpha = \alpha(s) = (x(s), y(s), z(s)) \) is said to be non-isotropic (or admissible) if nowhere its tangent vector is isotropic, namely \( x'(s) = \frac{dx}{ds} \neq 0 \). Otherwise the curve \( \alpha \) is said to be isotropic. If \( \alpha \) is a non-isotropic curve having unit speed (i.e. \( x'(s) = \pm 1 \)), then the curvature and torsion are given by

\[
\kappa(s) = \sqrt{[y''(s)]^2 + [z''(s)]^2}, \quad \tau(s) = \frac{\det (\alpha'(s), \alpha''(s), \alpha'''(s))}{[\kappa(s)]^2} \quad (\kappa(s) \neq 0).
\]

We call a curve planar (resp. space curve) provided \( \tau(s) = 0 \) (resp. \( \tau(s) \neq 0 \)) for all \( s \). Obviously, the space curves are non-isotropic, whereas the isotropic curves are Euclidean planar, that is, lie in a Euclidean plane.

A regular surface immersed in \( \mathbb{G}_3 \) is parameterized by the mapping

\[
r : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{G}_3, \quad (u_1, u_2) \mapsto (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).
\]

In order to specify the partial derivatives we shall notate:

\[
x_{,i} = \frac{\partial x}{\partial u_i} \quad \text{and} \quad x_{,ij} = \frac{\partial^2 x}{\partial u_i \partial u_j}, \quad 1 \leq i, j \leq 2.
\]

Then \( r \) is said to satisfy admissibility criteria if nowhere it has Euclidean tangent planes, i.e., \( x_{,i} \neq 0 \) for some \( i = 1, 2 \). The first fundamental form is given by

\[
ds^2 = (g_1 du_1 + g_2 du_2)^2 + \varepsilon (h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2),
\]

where \( g_i = x_{,i}, \quad h_{ij} = y_{,i} y_{,j} + z_{,i} z_{,j}, \quad i, j = 1, 2, \) and

\[
\varepsilon = \begin{cases} 
0, & \text{if the direction } du_1 : du_2 \text{ is non-isotropic,} \\
1, & \text{if the direction } du_1 : du_2 \text{ is isotropic.}
\end{cases}
\]

Let us introduce a function \( W \) given by

\[
W = \sqrt{(x_{,1} z_{,2} - x_{,2} z_{,1})^2 + (x_{,2} y_{,1} - x_{,1} y_{,2})^2}.
\]

Then the normal vector field is defined as

\[
N = \frac{1}{W} (0, -x_{,1} z_{,2} + x_{,2} z_{,1}, x_{,1} y_{,2} - x_{,2} y_{,1})
\]
and thereafter the second fundamental form

\[ II = L_{11} du_1^2 + 2L_{12} du_1 du_2 + L_{22} du_2^2, \]

where

\[ L_{ij} = \frac{1}{g_1} \left( g_1(0, y_{i,j}, z_{i,j}) - g_{i,j}(0, y_1, z_1) \right) \cdot N, \quad g_1 \neq 0 \]

or

\[ L_{ij} = \frac{1}{g_2}(g_2(0, y_{i,j}, z_{i,j}) - g_{i,j}(0, y_2, z_2)) \cdot N, \quad g_2 \neq 0. \]

Note that the dot “.” denotes the Euclidean scalar product. Thereby, the Gaussian and mean curvature are defined as

\[ K = \frac{L_{11}L_{22} - L_{12}^2}{W} \]

and

\[ H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2W^2}. \]

A surface is said to be minimal (resp. flat) if its mean (resp. Gaussian) curvature vanishes. Recall that the minimal surfaces in \( \mathbb{G}_3 \) were classified in [29] by the result:

**Theorem 1.1.** Minimal surfaces in \( \mathbb{G}_3 \) are cones whose vertices lie on the absolute line and the ruled surfaces of type C. They are all conoidal ruled surfaces having the absolute line as the directional line in infinity.

Recall that a ruled surface of type C is of the form \( r(u, v) = (u, x(u) + vy(u), vz(u)) \).

## 2. Translation Surfaces

A translation surface in \( \mathbb{G}_3 \) is locally parameterized by

\[ r : I_1 \times I_2 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{G}_3, \quad r(x, y) = \alpha(x) + \beta(y), \]

where \( \alpha \) and \( \beta \) denote translating curves. Under the condition that \( \alpha \) and \( \beta \) are planar, the authors in [20] categorized such a surface up to the absolute figure:

- type 1: \( \alpha \) is planar non-isotropic curve and \( \beta \) isotropic curve,
- type 2: \( \alpha \) and \( \beta \) are planar non-isotropic curves.

If the planes involving translating curves are chosen to be mutually orthogonal, the surfaces of type 1 and type 2 have the parametrizations, respectively

\[ r(x, y) = (x, y, f(x) + g(y)) \quad \text{and} \quad r(x, y) = (x + y, g(y), f(x)). \]

These surfaces with \( K = \text{const.} \) and \( H = \text{const.} \) were obtained in [20]. If not, i.e. the planes are non-orthogonal, then the notion of affine translation surface naturally arises, that firstly introduced by Liu and Yu [14] as the graph surfaces of the functions

\[ z(x, y) = f(x) + g(y + ax), \quad a \neq 0. \]

By following this, the surfaces of type 1 and type 2 are generally called affine translation surfaces. We shall classify such surfaces in Section 3 with \( K = \text{const.} \) and \( H = \text{const.} \). Furthermore, the translating curves could be non-planar and hereinafter it is necessary to extend above categorization:

- type 3: \( \alpha \) is isotropic curve and \( \beta \) space curve,
- type 4: \( \alpha \) is planar non-isotropic curve and \( \beta \) space curve,
- type 5: \( \alpha \) and \( \beta \) are space curves.

We shall also provide the surfaces of type 3 and type 4 in next sections with \( K = \text{const.} \) and \( H = \text{const.} \).
3. Constant Curvature Affine Translation Surfaces

Assume that $A = (a_{ij})$ is a regular real matrix, $i, j = 1, 2$, and $w = \det A \neq 0$. Let us consider the following planar curves:

$$
\begin{align*}
\alpha &= \alpha(u) = \left( \frac{a_{22}}{w} u - \frac{-a_{21}}{w} u, f(u) \right), \quad P_\alpha : a_{21} x + a_{22} y = 0, \\
\beta &= \beta(v) = \left( \frac{-a_{12}}{w} v, \frac{a_{11}}{w} v, g(v) \right), \quad P_\beta : a_{11} x + a_{12} y = 0,
\end{align*}
$$

where $P_\alpha$ and $P_\beta$ denotes the planes involving the curves. It is easily seen that $P_\alpha$ is orthogonal to $P_\beta$ provided $A$ is an orthogonal matrix. If $a_{12} = 0$ (resp. $a_{22} = 0$) in (3.1) then $\beta$ (resp. $\alpha$) becomes an isotropic curve. Otherwise both of them are non-isotropic curves. Therefore, by a translation of $\alpha$ and $\beta$, we derive the following admissible surface

$$
r(u, v) = \left( \frac{a_{22}}{w} u - \frac{a_{12}}{w} v, \frac{a_{11}}{w} v, g(v) \right).
$$

By changing the coordinates $u = a_{11} x + a_{12} y, \ v = a_{21} x + a_{22} y$, (3.2) turns to the standard parametrization of affine translation surface given by

$$
r(x, y) = (x, y, f(a_{11} x + a_{12} y) + g(a_{21} x + a_{22} y)).
$$

This one represents the surfaces of both type 1 and type 2 as well as a natural generalization of the surfaces given by (2.1). Throughout this section, we shall only distinguish the cases relating to $f$ due to the fact that the roles of $f$ and $g$ are symmetric. After a calculation, we have the Gaussian curvature:

$$
K = \frac{w^2 f'' g''}{\left[ 1 + (a_{12} f' + a_{22} g')^2 \right]^2},
$$

where $f' = \frac{df}{dx}$ and $g' = \frac{dg}{dx}$, etc.

**Theorem 3.1.** If an affine translation surface given by (3.3) has constant Gaussian curvature $K_0$ in $\mathbb{G}_3$, then it is either

1. a generalized cylinder with isotropic or non-isotropic rulings ($K_0 = 0$);
2. or a certain surface parameterized by, up to suitable translations and constants,

$$
r(x, s) = \left( x, c_1 x + \frac{K_0 c_2}{c_2} s^2, c_3 x^2 + \frac{1}{2} s \sqrt{1 - \frac{K_0 c_2}{c_2} s^2} + \frac{c_2}{16 K_0} \arcsin \left( \sqrt{\frac{4 K_0 c_2}{c_2} s} \right) \right),
$$

where $c_1, c_2, c_3 \in \mathbb{R} - \{0\}$ and $s$ is the arc-length parameter of $\beta$.

**Proof.** Assume that $K_0 = 0$. Then (3.4) leads $f$ to be a linear function and thus the surface becomes a generalized cylinder (so-called cylindrical surface, see [11], p. 439). Otherwise, i.e. $K_0 \neq 0$, by (3.4) we get $f'' g'' \neq 0$. Taking the partial derivative of (3.4) with respect to $u$ gives

$$
4 K_0 [1 + (a_{12} f' + a_{22} g')^2] [a_{12} f'' + a_{22} g''] [a_{12} f'''] = w^2 f''' g''.
$$

To solve (3.5), we have two cases:

**Case (A)** $a_{12} = 0$. (3.5) follows that $f''' = c_1 \neq 0$. Then by (3.4) we get

$$
\frac{K_0}{a_{11} a_{22}} = \frac{a_{22} g''}{\left[ 1 + (a_{22} g')^2 \right]^2},
$$

where $g'' = \frac{dg}{dx}$. 


where \(a_{11}a_{22} \neq 0\) since \(w \neq 0\). We treat the method used in [11] in order to solve (3.6). Since \(a_{12} = 0\), \(\beta\) is an isotropic curve and its reparametrization having unit speed is given by

\[
\beta(s) = (0, p(s), q(s)), \quad (p')^2 + (q')^2 = 1,
\]

where the prime denotes the derivative with respect to the arc-length parameter. In this case (3.4) turns to

\[
K_0 = f''q''.
\]

After solving (3.8), up to suitable translations and constants, we deduce

\[
q = \frac{K_0}{c_1}s^2.
\]

Considering it into (3.7) leads to

\[
p(s) = \frac{1}{2}s\sqrt{1 - \frac{K_0}{c_1}s^2} + \frac{1}{4}\sqrt{\frac{c_1}{K_0}}\arcsin\left(\frac{2\sqrt{\frac{K_0}{c_1}}s}{K_0}\right),
\]

which proves the second statement of the theorem.

**Case (B) \(a_{12} \neq 0\).** By the symmetry we have \(a_{22} \neq 0\) and then (3.5) can be rewritten as

\[
1 + \frac{3(a_{12}f' + a_{22}g')^2}{a_{12}f' + a_{22}g' + (a_{12}f' + a_{22}g')^2} = \frac{g''}{a_{22}(g'')^2}.
\]

The partial derivative of (3.9) with respect to \(v\) yields

\[
1 + 3\frac{(a_{12}f' + a_{22}g')^2}{a_{12}f' + a_{22}g' + (a_{12}f' + a_{22}g')^2} = \frac{g''}{a_{22}(g'')^2}.
\]

Again taking the partial derivative of (3.10) with respect to \(u\) gives the following polynomial equation on \((a_{12}f' + a_{22}g')\):

\[
1 + 3(a_{12}f' + a_{22}g')^4 = 0,
\]

which is a contradiction and completes the proof.

For the mean curvature, we have

\[
H = \frac{a_{12}f'' + a_{22}g''}{\left[1 + (a_{12}f' + a_{22}g')^2\right]^2},
\]

**Theorem 3.2.** Let an affine translation surface given by (3.3) have constant mean curvature \(H_0\) in \(\mathbb{G}_3\). Then:

1. If \(H_0 = 0\), it is either
   1.1 a non-cylindrical ruled surface of type C whose the base curve is a parabolic circle.
   1.2 a generalized cylinder with isotropic rulings, or
   1.3 an isotropic plane.
2. Otherwise \((H_0 \neq 0)\); it is either
   2.1 a certain surface given by

\[
r(x, y) = \left(x, y, f(a_{11}x) - \frac{1}{H_0}\left(1 - \left(\frac{H_0}{a_{22}}v\right)^2\right)^2\right), \quad a_{22} \neq 0,
\]
or a generalized cylinder with non-isotropic rulings given by

\[ r(x, y) = \left( x, y, \frac{c_1 w}{a_{22}} x - \frac{1}{H_0} \sqrt{1 - \left( \frac{H_0}{a_{22}} v \right)^2} \right), \quad a_{22} \neq 0, \]

where \( v = a_{21} x + a_{22} y \).

Proof. We divide the proof into two cases:

Case (A) \( H_0 = 0 \). Then (3.11) reduces to

(3.12) \[ a_{12}^2 f'' + a_{22}^2 g''' = 0. \]

We have again cases:

Case (A.i) \( f'' = 0 = g''' \) is a solution for (3.12). This leads the surface to be an isotropic plane, which implies the statement (1.1) of the theorem.

Case (A.ii) \( a_{12} = 0 \). Since \( w \neq 0 \), we get \( a_{22} \neq 0 \). Thus (3.12) immediately implies \( g''' = 0 \), which proves the statement (2) of the theorem.

Case (A.iii) \( a_{12} \neq 0 \). The symmetry implies \( a_{22} \neq 0 \). Solving (3.12) gives, up to suitable translations and constants,

\[ f(u) = \frac{c_1}{2a_{12}} u^2, \quad g(v) = -\frac{c_1}{2a_{22}} v^2. \]

Substituting this into (3.3) gives

\[ r(x, y) = \left( x, 0, \frac{c_1}{2} \left( \frac{a_{11}}{a_{12}} \right)^2 - \left( \frac{a_{21}}{a_{22}} \right)^2 \right) \frac{x^2}{2}, + y \left( 0, 1, 2x \left( \frac{a_{11}}{a_{12}} - \frac{a_{21}}{a_{22}} \right) \right), \]

which parametrizes the non-cylindrical ruled surface whose the base curve is a parabolic circle and the rulings are isotropic.

Case (B) \( H_0 \neq 0 \). We have two cases:

Case (B.i) \( a_{12} = 0 \). Then (3.11) reduces to

(3.13) \[ H_0 = \frac{a_{22}^2 g''}{\left[ 1 + (a_{22} g')^2 \right]^2}. \]

After solving (3.13), up to suitable translations and constants, we deduce

\[ g(a_{22} y) = -\frac{1}{H_0} \sqrt{1 - (H_0 y)^2}, \]

where \( a_{22} \neq 0 \) since \( w \neq 0 \). This proves the statement (2.1) of the theorem.

Case (B.ii) \( a_{12} \neq 0 \). Taking partial derivative of (3.11) with respect to \( u \) gives

(3.14) \[ 3H_0 [1 + (a_{12} f' + a_{22} g')^2]^{\frac{3}{2}} [a_{12} f' + a_{22} g'] [a_{12} f'''] = a_{12}^2 f''''. \]

We have again two cases:

Case (B.ii.1) \( f'' = 0 \). Then from (3.11), we have

(3.15) \[ \frac{H_0}{a_{22}} = \frac{a_{22} g''}{\left[ 1 + (a_{22} c_1 + a_{22} g')^2 \right]^2}. \]
where $f' = c_1$. By solving (3.15), up to suitable translations and constants, we obtain

$$g(v) = -\frac{1}{H_0} \sqrt{1 - \left(\frac{H_0}{a_{22}}\right)^2 - \frac{c_1 a_{12}}{a_{22}} v},$$

which gives the statement (2.2) of the theorem.

**Case (B.ii.2) $f'' \neq 0$.** Then (3.14) can be rewritten as

$$[1 + (a_{12} f' + a_{22} g')^2]^\frac{3}{2} [a_{12} f' + a_{22} g'] = \frac{a_{12} f'''}{3H_0 f''}.$$

The partial derivative of (3.16) with respect to $v$ gives

$$1 + 2 (a_{12} f' + a_{22} g')^2 = 0,$$

which is a contradiction. This completes the proof. □

### 4. Constant Curvature Surfaces of Type 3

Let one translating curve be the space curve given by $\alpha = \alpha(u) = (u, f_1(u), f_2(u))$ and another one the unit speed isotropic curve by

$$\begin{align*}
\beta &= \beta(v) = (0, g_1(v), g_2(v)), \\
(g_1')^2 + (g_2')^2 &= 1, \quad g_i' = \frac{dg_i}{dv}, \quad i = 1, 2,
\end{align*}$$

where we may assume $g_1' \neq 0$ without loss of generality. The last equality yields

$$g_1' g_2'' - f_1'' g_2' = 0.$$

Further, since the torsion of $\alpha$ is different from zero, we get

$$f_1' f_2'' - f_1''' f_2' = 0,$$

where $\frac{df_i}{du} = f_i'$, etc. $i = 1, 2$. Thereby the obtained translation surface belongs to type 3 and is given by

$$r(u, v) = (u, f_1(u) + g_1(v), f_2(u) + g_2(v)).$$

By a calculation, the Gaussian curvature is

$$K = -\frac{g_2''}{g_1'} (f_1'' g_2' - f_2'' g_1').$$

**Theorem 4.1.** If the surface given by (4.1) has constant Gaussian curvature $K_0$ in $\mathbb{G}_3$, then it is a generalized cylinder with isotropic rulings ($K_0 = 0$).

**Proof.** If $K_0$ vanishes then either $g_2'' = 0$ or $f_1' g_2'' - f_2'' g_1' = 0$ in (4.4). The second possibility is eliminated due to (4.2) and thus $\beta$ becomes an isotropic line. Otherwise, $K_0 \neq 0$, we have $g_2'' \neq 0$. Then by taking partial derivative of (4.4) with respect to $u$, we get

$$0 = f_1''' g_1' - f_2''' g_2'.$$

From (4.2) at least one of $f_1'''$ and $f_2'''$ is different from zero. Thus (4.5) implies $g_2' = c g_1'$, $c \in \mathbb{R} - \{0\}$. Considering it into (4.1) yields a contradiction, which proves the theorem. □

**Theorem 4.2.** If the surface given by (4.1) has constant mean curvature $H_0$ in $\mathbb{G}_3$ then either
(1) it is either a generalized cylinder with isotropic rulings \((H_0 = 0)\); or
(2) the translating isotropic curve is a Euclidean circle with radius \(|H_0|\) \((H_0 \neq 0)\).

**Proof.** Assume that the surface given by (4.1) has constant mean curvature \(H_0\). Then we have the relation

\[
H_0 = \frac{g''_2}{g'_1},
\]

which immediately implies that \(H_0\) vanishes provided \(\beta\) is an isotropic line. If \(H_0 \neq 0\), then we have

\[
g''_2 = H_0 g'_1.
\]

Considering (4.7) into (4.1) gives

\[
g''_1 = -H_0 g''_2.
\]

We may formulate the equations (4.7) and (4.8) as follows:

\[
\begin{aligned}
g'''_1 + H_0^2 g'_1 &= 0, \\
g'''_2 + H_0^2 g'_2 &= 0.
\end{aligned}
\]

After solving (4.9) we obtain, up to suitable constants,

\[
\begin{aligned}
g_1 &= \frac{c_1}{|H_0|} \sin(|H_0|u) + \frac{c_2}{|H_0|} \cos(|H_0|u), \\
g_2 &= \frac{d}{du} \sin(|H_0|v) + \frac{e}{|H_0|} \cos(|H_0|v).
\end{aligned}
\]

Since \((g'_1)^2 + (g'_2)^2 = 1\), we have \((c_1)^2 + (c_2)^2 = 1, (c_2)^2 + (c_4)^2 = 1\) and \(c_1c_2 + c_3c_4 = 0\). This means that \(\beta\) is a Euclidean circle with radius \(|H_0|\). \(\square\)

5. **Constant Curvature Surfaces of Type 4**

In last section, we are interested in the surfaces generated by translating a space curve \(\alpha = \alpha(u) = (u, f_1(u), f_2(u))\) and a planar non-isotropic curve \(\beta = \beta(v) = (v, g(v), av), a \in \mathbb{R}\). Since the torsion of \(\alpha\) is different from zero, we have

\[
f''_1 f'''_2 - f'''_1 f''_2 \neq 0,
\]

where \(\frac{df_i}{du} = f'_i\) and so on, \(i = 1, 2\). Therefore the obtained translation surface is of the form

\[
r(u, v) = (u + v, f_1(u) + g(v), f_2(u) + av).
\]

By a calculation, the Gaussian curvature turns to

\[
K = \frac{g'' \left[f''_1 (f'_2 - a)^2 - f''_2 (f'_2 - a)(f'_1 - g')\right]}{\left[(f'_2 - a)^2 + (f'_1 - g')^2\right]^2}.
\]

**Theorem 5.1.** If the surface given by (5.2) has constant Gaussian curvature \(K_0\) in \(\mathbb{R}^3\), then it is a generalized cylinder with non-isotropic rulings \((K_0 = 0)\).

**Proof.** We divide the proof into two cases:
Case (A) $K_0 = 0$. From (5.3), we conclude either $g'' = 0$, namely the surface is generalized cylinder with non-isotropic rulings, or

\[(5.4) \quad f''_1(f_2' - a) - f''_2(f_1' - g') = 0.\]

Taking partial derivative of (5.4) with respect to $v$, we get $f''_2 = 0$, which is not possible due to (5.1).

Case (B) $K_0 \neq 0$. By taking twice partial derivative of (5.3) with respect to $v$, we deduce

\[(5.5) \quad -4K_0 \left[3(f_1' - g')^2 + (f_2' - a)^2 \right] = \frac{1}{g'} \left( \frac{\zeta'}{g''} \right)' \left[ f''_1(f_2' - a)^2 - f''_2(f_2' - a)(f_1' - g') \right] + 2\frac{g'''}{g'}(f_2' - a)f''_2,\]

where $g'' \neq 0$ due to our assumption. Put $\frac{\zeta'}{g''} = \frac{1}{g'(g'' - 1)}$ into (5.5). After taking partial derivative of (5.5) with respect to $v$, we conclude

\[(5.6) \quad 24K_0(f_1' - g') = \frac{\zeta'}{g''} \left[ f''_1(f_2' - a)^2 - f''_2(f_2' - a)(f_1' - g') \right] + 3\zeta(f_2' - a)f''_2,\]

where $\zeta' = \frac{d\zeta}{dv}$. The partial derivative of (5.6) with respect to $v$ implies

\[(5.7) \quad -\frac{24}{f''_2(f_2' - a)} = \frac{1}{g''} \left( \frac{\zeta'}{g''} \right)' \left[ f''_1(f_2' - a) \frac{f''_2(f_2' - a)}{f''_2} - (f_1' - g') \right] + 3\frac{\zeta'}{g''},\]

After again taking partial derivative of (5.7) with respect to $u$ and $v$, we deduce

\[(5.8) \quad 0 = \left( \frac{1}{g''} \left( \frac{\zeta'}{g''} \right)' \right)' \left[ \left( \frac{f''_1(f_2' - a)}{f''_2} \right)' - f''_2 \right].\]

We have two cases to solve (5.8):

Case (B.i) $\left( \frac{\zeta'}{g''} \right)' = c_3g''$. Up to suitable constant, we have $\frac{\zeta'}{g''} = c_3g''$. Substituting these into (5.7) gives

\[-\frac{24}{f''_2(f_2' - a)} = c_3 \frac{f''_1(f_2' - a)}{f''_2} - c_3 f'_1 + 4c_3g',\]

which implies $c_3 = 0$ and thus $\zeta' = 0$. Considering it into (5.6) leads to

\[(5.9) \quad 24K_0(f_1' - g') = 3c_4(f_2' - a)f''_2,\]

where $\zeta = c_4$. (5.9) yields a contradiction due to $K_0 \neq 0$.

Case (B.ii) $\left( \frac{f''_2(f_2' - a)}{f''_2} \right)' - f''_2 = 0$. Up to suitable constant, we have

\[(5.10) \quad \frac{f''_2}{f''_2} = \frac{f''_2}{f''_2 - a}.\]

After solving (5.10) we obtain $f''_1 = c_5f''_2$ which is a contradiction due to (5.1). Therefore the proof is completed.\[\Box\]
By a calculation, the mean curvature turns to
\[
H = \frac{(f_2' - a)g'' + (f_2' - a)f_2'' - (f_1' - g')f_2''}{\left[(f_2' - a)^2 + (f_1' - g')^2\right]^{\frac{1}{2}}}.
\]
(5.11)

First we investigate the minimality case:

**Theorem 5.2.** There does not exist a minimal translation surface given by (5.2) in \(\mathbb{G}_3\).

**Proof.** Let us assume the contrary situation. Then (5.11) reduces to
\[
(f_2' - a)\left(g'' + f_2'' - (f_1' - g')f_2''\right) = 0.
\]
(5.12)

The partial derivative of (5.12) with respect to \(v\) yields
\[
(f_1' - c_1)g'' + f_2''g'' = 0.
\]
(5.13)

We have two cases:

**Case (A)** \(g' = c_1, \ c_1 \in \mathbb{R}\). Then (5.13) turns to
\[
\frac{f_1''}{f_1' - c_1} = \frac{f_2''}{f_2' - a}
\]
and solving (5.14) yields \(f_1'' = c_2f_2'', \ c_2 \in \mathbb{R} - \{0\}\). This leads to a contradiction due to (5.1).

**Case (B)** \(g'' \neq 0\). Then (5.13) can be rewritten as
\[
\frac{g'''}{g''} = c_3 = \frac{-f_2''}{f_2' - a}, \ c_3 \in \mathbb{R} - \{0\}.
\]
(5.15)

which implies \(g'' = c_3g'\), up to suitable constant. Substituting these into (5.12) gives
\[
f_1'' + c_3f_1' = 0.
\]
(5.16)

From (5.15) and (5.16) we derive
\[
f_2''' = -c_3f_2'' \quad \text{and} \quad f_1''' = -c_3f_1'',
\]
which is no possible due to (5.1). Therefore the proof is completed. \(\square\)

**Theorem 5.3.** If the surface given by (5.2) has nonzero constant mean curvature \(H_0\) in \(\mathbb{G}_3\), then it is a generalized cylinder with non-isotropic rulings whose the base curve satisfies the equation
\[
f_1 = cu + H_0\left\{\frac{1}{2} (f_2' - au)^2 \zeta(\sigma) - \frac{1}{2} \int (f_2' - au)^2 \frac{d\zeta(\sigma)}{du} \right\},
\]
where \(c \in \mathbb{R}\) and
\[
\zeta(\sigma) = \int (f_2' - au) d\sigma \quad \text{for} \quad \sigma = \frac{f_1' - c_1}{f_2' - a}.
\]

**Proof.** The partial derivative of (5.11) with respect to \(v\) gives
\[
3H_0 \left[(f_2' - a)^2 + (f_1' - g')^2\right]^{\frac{1}{2}} (f_1' - g')g'' = (f_2' - a)g''' + f_2''g''.
\]
(5.17)

To solve (5.17), we have two cases:
Case (A) $g' = c_1$, $c_1 \in \mathbb{R}$. (5.11) turns to

\begin{equation}
H_0 = \frac{(f'_2 - a) f''_1 - (f'_1 - c_1) f''_2}{\left[ (f'_2 - a)^2 + (f'_1 - c_1)^2 \right]^\frac{3}{2}}.
\end{equation}

(5.18)

Put $\sigma = \frac{f'_1 - c_1}{f'_2 - a}$ into (5.18). Then we get

\begin{equation}
H_0 (f'_2 - a) = \frac{d\sigma}{du} \frac{1}{(1 + \sigma^2)^\frac{3}{2}}.
\end{equation}

(5.19)

Up to suitable constant, an integration of (5.19) with respect to $u$ gives

\begin{equation}
H_0 (f'_2 - au) = \frac{\sigma}{(1 + \sigma^2)^\frac{3}{2}}.
\end{equation}

(5.20)

Again an integration of (5.20) with respect to $\sigma$, we conclude

\begin{equation}
H_0 \int (f'_2 - au) d\sigma = \sqrt{1 + \sigma^2}.
\end{equation}

(5.21)

Substituting (5.21) into (5.20) yields

\begin{equation}
H_0^2 (f'_2 - au) \int (f'_2 - au) d\sigma = \sigma,
\end{equation}

or

\begin{equation}
f'_1 - c_1 = H_0^2 (f'_2 - au) (f'_2 - a) \zeta(\sigma),
\end{equation}

(5.22)

where $\zeta(\sigma) = \int (f'_2 - au) d\sigma$. The partial integration of (5.22) with respect to $u$ gives

\[ f_1 = c_1 u + H_0^2 \left\{ \frac{1}{2} (f'_2 - au)^2 \zeta(\sigma) - \frac{1}{2} \int (f'_2 - au)^2 \frac{d\zeta(\sigma)}{du} du \right\}. \]

Case (B) $g'' \neq 0$. (5.17) can be rewritten as

\begin{equation}
3H_0 \left[ (f'_2 - a)^2 + (f'_1 - g')^2 \right]^\frac{3}{2} (f'_1 - g') = (f'_2 - a) \frac{g''}{g'''} + f''_2.
\end{equation}

(5.23)

The partial derivative of (5.23) with respect to $v$ gives

\begin{equation}
\frac{2 (f'_1 - g')^2 + f'_2 - a}{\left[ (f'_2 - a)^2 + (f'_1 - g')^2 \right]^\frac{3}{2}} = -\frac{1}{3H_0g''} \left( \frac{g'''}{g''} \right)'.
\end{equation}

(5.26)

By again taking partial derivative of (5.26) with respect to $u$ we derive a polynomial equation on $(f'_1 - g')$. In that equation, the coefficient of the term of highest degree is $(f'_2 - a) f''_2$. This one cannot vanish due to (5.1) and therefore we achieve a contradiction which completes the proof.

$\Box$
6. Conclusions

This study is devoted to obtain the translation surfaces in $G_3$ with $K = \text{const.}$ and $H = \text{const.}$ when at least one of the translating curves is planar. In this sense, to classify the surfaces in $G_3$ whose both of translating curves are non-planar is still an open problem, that is not easy to solve. However, it is obvious that such a surface can be neither flat nor minimal (see Theorem 1.1). Consequently, the known results can be summarized as in Table 1:

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