Mukai flops and derived categories

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Introduction

Derived categories possibly give a new significant invariant for algebraic varieties. In particular, when the canonical line bundle is trivial, there are varieties which are not birationally equivalent, but have equivalent derived categories. The most typical example can be found in the original paper [Mu]. On the other hand, for such varieties, it is hoped that the birationally equivalence should imply the equivalence of derived categories. One of the important classes for testing this question is the class of complex symplectic manifolds (for varieties in other classes, see [B2, Ka 1, Ch]). A Mukai flop is a typical birational map between complex symplectic manifolds. In this note, we shall prove that two smooth projective varieties of dimension $2n$ connected by a Mukai flop have equivalent bounded derived categories of coherent sheaves. More precisely, let $X$ and $X^+$ be smooth projective varieties of dimension $2n$ such that there is a birational map $\phi : X \rightarrow X^+$ obtained as the Mukai flop along a subvariety $Y \subset X$ which is isomorphic to $\mathbb{P}^n$. By definition, $\phi$ is decomposed into the blowing-up $p : \bar{X} \rightarrow X$ along $Y$ and the blowing down $p^+ : \bar{X} \rightarrow X^+$ contracting the $p$-exceptional divisor $\bar{Y}$ to the subvariety $Y^+ \cong \mathbb{P}^n$ of $X^+$. On the other hand, there are birational morphisms $X \rightarrow \bar{X}$ and $X^+ \rightarrow \bar{X}$ which contract $Y$ and $Y^+$ to points respectively. We put $\hat{X} := X \times \bar{X}$ and let $q : \hat{X} \rightarrow X$ and $q^+ : \hat{X} \rightarrow X^+$ be the natural projections. Note that $\hat{X}$ is a normal crossing variety with two irreducible components $\bar{X}$ and $\mathbb{P}^n \times \mathbb{P}^n$.

Let $D(X)$ (resp. $D(X^+)$) be the bounded derived category of coherent sheaves on $X$ (resp. $X^+$).

We shall consider natural two functors

$$\Phi := R(p^+)_* \circ Lp^* : D(X) \rightarrow D(X^+)$$

$$\Psi := R(q^+)_* \circ Lq^* : D(X) \rightarrow D(X^+).$$

In this note, we first take, as $X$ and $Y$, the $\mathbb{P}^n$ bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^n} \oplus \Theta_{\mathbb{P}^n})$ over $\mathbb{P}^n$ and its negative section. In this case, $\Phi$ is not a fully-faithful functor even when $n = 2$ ($\S 2$), but $\Psi$ is an equivalence of triangulated categories for any $n \geq 2$ ($\S 3$). In $\S 4$ we prove in a general case that $\Psi$ is an equivalence of triangulated
categories (Theorem (4.4)). Theorem (4.4) holds for a Mukai flop in a more general sense. This generalization is done in §5.

Our result should be compared with elementary flops studied in Bondal-Orlov [B-O, Theorem 3.6]. For these flops, $\Phi$ becomes an equivalence of triangulated categories. But, in our case, since $\Phi \neq \Psi$, the proof there can not be directly applied. Instead, we shall use Bridgeland’s criteria [B 1, Theorem 2.3], [B-K-K, Theorem 2.4] for exact functors of triangulated categories to be fully-faithful and to be an equivalence. A Mukai flop is of particular importance when $X$ is a complex irreducible symplectic manifold; in this case, it is conjectured that the birationally equivalence implies the equivalence of bounded derived categories of coherent sheaves. This conjecture is also related with a question of Torelli type [Na, Question]. By a recent result of Wierzba and Wisniewski [W-W, Theorem 1.2] (see also [W]), one can prove that, birationally equivalent complex projective symplectic 4-folds have equivalent derived categories (Corollary (4.5)).

This is a revised version of the paper with the same title appeared in math.AG/0203287. Recently, Kawamata has independently proved the similar results in [Ka 2, §5]. In [Ka 2] the equivalence for Mukai flops is redued to the one for the elementary flops in [B-O]. But our proof here depend on direct calculations using an explicit model.

§1. (1.1) Fix an integer $n \geq 2$, and denote by $X$ the projective space bundle $P(O_P \oplus \Theta_P)$ over $P^n$ in the sense of Grothendieck (cf. [Ha, p.162]). We put $M = P^n$ and let $\pi : X \to M$ be the bundle map. Denote by $O_X(1)$ the tautological line bundles on $X$. Let $Y \subset X$ be the section of $\pi$ defined by the surjection $O_P \oplus \Theta_P \to O_P$. The normal bundle $N_{Y/X}$ is isomorphic to $\Omega_Y^1$. Let $p : \tilde{X} \to X$ be the blowing-up of $X$ along $Y$. The exceptional divisor $\tilde{Y}$ of $p$ is isomorphic to $P(\Theta_P)$. $\tilde{Y}$ has a $P^{n-1}$-bundle structure which is different from the bundle structure $Y \to Y$. One can contract $\tilde{Y} \subset \tilde{X}$ along this another ruling to $Y^+ \subset X^+$. We call this birational contraction map $p^+$. Let $\phi : X^- \to X^+$ be the birational map defined as the composite $p^+ \circ (p^{-1})$. $Y^+$ is isomorphic to $P^n$ and $N_{Y^+/X^+} \cong \Omega_{Y^+}^1$.

Lemma (1.2). $X^+$ is isomorphic to the projective space bundle $P(O_P \oplus \Theta_P)$ over $P^n$.

Proof. Let $H$ be a hyperplane of $M$. Then $\pi^{-1}(H) = P_H(O_H \oplus \Theta_M|_H)$. Let $V_H$ be the projective space subbundle of $\pi^{-1}(H) \to H$ defined by the surjection $O_H \oplus \Theta_M|_H \to O_H \oplus N_H/M$. $V_H \to H$ has a negative section and the variety obtained by contracting it to a point becomes $P^n$. Let $M^+$ be the dual projective

\footnote{Szendroi pointed out to me that a similar phenomenon to ours actually occurs in Calabi-Yau threefolds with type III contractions [Sz, Theorem 6.5].}
space of $M$. Now $\{ V_H \}_{H \in M^+}$ is a flat family of subvarieties of $X$. Let $V_H^+$ be the proper transform of $V_H$ by $\phi$. Then $\{ V_H^+ \}_{H \in M^+}$ is a flat family of subvarieties of $X^+$. Under the birational transformation $\phi$, the negative section of $V_H \to H$ is contracted to a point; hence each $V_H^+$ is isomorphic to $\mathbb{P}^n$. Moreover, one can check that $V_H^+$ and $V_{H'}^+$ are disjoint if $H \neq H'$. This implies that $X^+$ has a $\mathbb{P}^n$-bundle structure with the section $Y^+$. Let $X_\infty \subset X$ be the projective space subbundle defined by the surjection $\mathcal{O}_M \oplus \Theta_M \to \Theta_M$. $X_\infty$ is disjoint from $Y$. We see that the proper transform $X^+_\infty$ also becomes a projective space subbundle of $X^+$ disjoint from $Y^+$. Since $N_{Y^+/X^+} \cong \Omega^1_{Y^+}$, we have the result.

We denote by $\pi^+ : X^+ \to M^+$ the bundle structure introduced in (1.2) and denote by $\mathcal{O}_{X^+}(1)$ the tautological line bundle.

**Lemma (1.3).** The birational map $\phi$ induces an isomorphism

$$\phi^* : \text{Pic}(X^+) \to \text{Pic}(X)$$

such that $\phi^*(\mathcal{O}_{X^+}(1)) = \mathcal{O}_X(1)$ and $\phi^*((\pi^+)^*\mathcal{O}_M(1)) = \mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_M(-1)$.

**Proof.** Since $\phi$ is an isomorphism in codimension 1, there is a natural isomorphism $\phi^*$ between $\text{Pic}(X^+)$ and $\text{Pic}(X)$. By (1.2) we see that $\phi^*((\pi^+)^*\mathcal{O}_M(1)) = \mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_M(a)$ for some $a \in \mathbb{Z}$. Since $h^0(\mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_M(a)) = h^0((\pi^+)^*\mathcal{O}_M(1)) = n + 1$, we conclude that $a = -1$.

The birational transformation $\phi$ is symmetric with respect to $X$ and $X^+$; hence, we should have $(\phi^{-1})^*((\pi)^*\mathcal{O}_M(1)) = \mathcal{O}_{X^+}(1) \otimes (\pi^+)^*\mathcal{O}_{X^+}(-1)$ for $\phi^{-1} : \text{Pic}(X) \to \text{Pic}(X^+)$. Then we have $\phi^* \circ (\phi^{-1})^*((\pi)^*\mathcal{O}_M(1)) = \phi^*(\mathcal{O}_{X^+}(1)) \otimes \mathcal{O}_X(-1) \otimes \pi^*\mathcal{O}_M(1)$. Since $\phi^* \circ (\phi^{-1})^* = \text{id}$, $\phi^*(\mathcal{O}_{X^+}(1)) = \mathcal{O}_X(1)$.

(1.4) For $k \in \mathbb{Z}$, let $D(X)_k$ be the full subcategory of $D(X)$ whose objects are of the form $\mathcal{O}_X(k) \otimes \mathcal{L}\pi^*(F)$ with $F \in D(M)$. As a triangulated category, $D(X)$ is generated by $D(X)_{-n}, \ldots, D(X)_{-1}, D(X)_0$ by [O 1, Theorem 2.6]. Since $D(M)$ is generated by the objects $\mathcal{O}_M(-n + 1), \ldots, \mathcal{O}_M,$ and $\mathcal{O}_M(1)$, we see that $D(X)$ is generated by the set of objects $\Omega := \{ \mathcal{O}_X(j) \otimes \pi^*\mathcal{O}_M(k) \}$ with $-n \leq j \leq 0$ and $-n + 1 \leq k \leq 1$. In particular, $\Omega$ is a spanning class for $D(X)$. Here, by definition of the spanning class, if an object $a \in D(X)$ satisfies $\text{Hom}^i(\omega, a) = 0$ for all $\omega \in \Omega$ and for all $i \in \mathbb{Z}$, then $a \cong 0$. Similarly, if $\text{Hom}^i(\omega, a) = 0$ for all $\omega \in \Omega$ and for all $i \in \mathbb{Z}$, then $a \cong 0$.

(1.5) Define functors $\Phi$ and $\Phi'$ as

$$\Phi := \mathcal{R}(p^+), \mathcal{L}p^*(\cdot) : D(X) \to D(X^+).$$

$$\Phi' := \mathcal{R}p_* (\mathcal{L}(p^+)^*(\cdot) \otimes \omega_{X/X}^1) : D(X^+) \to D(X),$$

where $\omega_{X/X} := \omega_X \otimes p^*\omega_X^{-1}$. 

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Lemma (1.6). Assume that $j$ is an integer such that $-n \leq j \leq 0$.

(1) When $-n+1 \leq k \leq 0$, $\Phi(\mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(k)) = \mathcal{O}_{X^+}(j+k) \otimes \pi^* \mathcal{O}_{M^+}(-k)$.

When $k = 1$, $\Phi(\mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(1)) = \mathcal{O}_{X^+}(j+1) \otimes \pi^* \mathcal{O}_{M^+}(-1) \otimes I_{Y^+}$, where $I_{Y^+}$ is the ideal sheaf of $Y^+$ in $X^+$.

(2) When $-n+1 \leq k \leq 0$, $\Phi' \circ \Phi(\mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(k)) = \mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(k)$.

Proof. (1): At first, we have $\mathcal{O}_X(1)|_Y \cong \mathcal{O}_Y$, hence, for $y^+ \in Y^+$, $(\nu^*(\mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(k)))|_{\nu^{-1}(y^+)}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(k)$. Therefore, $\nu^*(\mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(k))$ is $\nu^+$-acyclic for $k \geq -n+1$. If $k \leq 0$, then $\nu^*(\mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(k))$ is a line bundle on $X^+$. If $k = 1$, then $\nu^*(\mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(1))$ is a line bundle outside $Y^+$ but not a line bundle on $Y^+$. Finally, by (1.3) we conclude that, if $k \leq 0$, then $\Phi(\mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(k)) = \mathcal{O}_{X^+}(j+k) \otimes \pi^* \mathcal{O}_{M^+}(-k)$, and that if $k = 1$, then $\Phi(\mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(1)) = \mathcal{O}_{X^+}(j+1) \otimes \pi^* \mathcal{O}_{M^+}(-1) \otimes I_{Y^+}$, where $I_{Y^+}$ is the ideal sheaf of $Y^+$ in $X^+$.

(2): For $y \in Y$, $(\nu^+)^*(\mathcal{O}_{X^+}(j+k) \otimes (\pi^+)^* \mathcal{O}_{M^+}(-k)) \otimes (\pi_+)^* \mathcal{O}_{X^+}(-1) \otimes (\pi^+)^* \mathcal{O}_{M^+}(-1) \cong \mathcal{O}_{\mathbb{P}^{n-1}(-k-n+1)}$ because $\omega_{X^+)/\nu^{-1}(y)} \cong \mathcal{O}_{\mathbb{P}^{n-1}(-k-n+1)}$. By the same argument as (1), we conclude that

$$\Phi'(\mathcal{O}_{X^+}(j+k) \otimes (\pi^+)^* \mathcal{O}_{M^+}(-k)) = \mathcal{O}_X(j) \otimes \pi^* \mathcal{O}_M(k).$$

(1.7) We shall construct a locally free resolution of the ideal sheaf $I_{Y^+}$.

Put $E := \mathcal{O}_{M^+} \oplus \Theta_{M^+}$. Let

$$0 \to \mathcal{O}_{X^+}(-1) \to (\pi^+)^* E^* \to Q \to 0$$

be the universal exact sequence, where $E^*$ is the dual of $E$. Let us consider the fiber product $X^+ \times_{M^+} X^+$ and denote by $p_1 : X^+ \times_{M^+} X^+ \to X^+$ and $p_2 : X^+ \times_{M^+} X^+ \to X^+$ the first projection and the second projection respectively. One has an isomorphism ([O 1, p.136])

$$H^0(X^+ \times_{M^+} X^+, p_1^* \mathcal{O}_{X^+}(1) \otimes p_2^* Q) \cong H^0(M^+, E \otimes E^*).$$

Corresponding to the identity map $id : E \to E$, one has a section $s \in H^0(X^+ \times_{M^+} X^+, p_1^* \mathcal{O}_{X^+}(1) \otimes p_2^* Q)$. Then the set of zeros of $s$ coincides with the diagonal $\Delta \subset X^+ \times_{M^+} X^+$ (cf. [O 1, p.136]). Hence, we have the following Koszul complex which becomes a resolution of the $\mathcal{O}_{X^+ \times_{M^+} X^+}$ module $\mathcal{O}_\Delta$:

$$0 \to \wedge^n (p_1^* \mathcal{O}_{X^+}(-1) \otimes p_2^* Q^*) \to \wedge^{n-1} (p_1^* \mathcal{O}_{X^+}(-1) \otimes p_2^* Q^*) \to \ldots \to$$

$$p_1^* \mathcal{O}_{X^+}(-1) \otimes p_2^* Q^* \to \mathcal{O}_{X^+ \times_{M^+} X^+} \to \mathcal{O}_\Delta \to 0.$$

By the projection $p_2 : X^+ \times_{M^+} X^+ \to X^+$, each term becomes a flat $\mathcal{O}_{X^+}$ module. Thus, by taking the tensor product of the complex with the $\mathcal{O}_{X^+}$ module $\mathcal{O}_{Y^+}$, we get the exact sequence

$$0 \to \wedge^n (\mathcal{O}_{X^+}(-1) \otimes (\pi^+)^* \Theta_{M^+}) \to \wedge^{n-1} (\mathcal{O}_{X^+}(-1) \otimes (\pi^+)^* \Theta_{M^+}) \to \ldots \to$$

$$\mathcal{O}_{Y^+} \to 0.$$
$\mathcal{O}_{X}(-1) \otimes (\pi^+)^*\Theta_{M^+} \to I_{Y^+} \to 0$. 
Here we have used the fact that $\text{Ker}[\mathcal{O}_{X^+} \to \mathcal{O}_{Y^+}] = I_{Y^+}$ and $Q^*|_{Y^+} \cong \Theta_{Y^+}$.

§2.

In this section, we observe that the functor $\Phi$ is not fully faithful.

**Lemma (2.1).** Assume that $n = 2$, that is, $\dim X = \dim X^+ = 4$. Then $\text{Ext}^2(\Phi^*\mathcal{O}_M(1)), \Phi^*\mathcal{O}_M(1)) \neq 0$.

**Proof.** By (1.6) we have

\[
\text{Ext}^2(\Phi^*\mathcal{O}_M(1)), \Phi^*\mathcal{O}_M(1)) = 
\text{Ext}^2(\mathcal{O}_{X^+}(1) \otimes \pi^*\mathcal{O}_{M^+}(-1) \otimes I_{Y^+}, \mathcal{O}_{X^+}(1) \otimes \pi^*\mathcal{O}_{M^+}(-1) \otimes I_{Y^+}) = 
\text{Ext}^2(I_{Y^+}, I_{Y^+}).
\]

By (*) in §1, we have an exact sequence

\[
0 \to \wedge^2(\mathcal{O}_{X^+}(-1) \otimes (\pi^+)^*\Theta_{M^+}) \to \mathcal{O}_{X^+}(-1) \otimes (\pi^+)^*\Theta_{M^+} \to I_{Y^+} \to 0.
\]

This exact sequence yields the exact sequence

\[
\text{Ext}^1(\mathcal{O}_{X^+}(-1) \otimes (\pi^+)^*\Theta_{M^+}, I_{Y^+}) \to \text{Ext}^1(\wedge^2(\mathcal{O}_{X^+}(-1) \otimes (\pi^+)^*\Theta_{M^+}, I_{Y^+}) 
\to \text{Ext}^2(I_{Y^+}, I_{Y^+}) \to \text{Ext}^2(\wedge^2(\mathcal{O}_{X^+}(-1) \otimes (\pi^+)^*\Theta_{M^+}, I_{Y^+}).
\]

Note that, since $\mathcal{O}_{X^+}(-1) \otimes (\pi^+)^*\Theta_{M^+}$ and $\wedge^2(\mathcal{O}_{X^+}(-1) \otimes (\pi^+)^*\Theta_{M^+}$ are both locally free sheaves, the first, second and fourth terms are identified with suitable cohomology groups. Then, by a direct calculation one can check that the first and fourth term vanish but the second one is not zero.

**Corollary (2.2).** When $n = 2$, the functor $\Phi : D(X) \to D(X^+)$ is not fully faithful.

**Proof.** One has

\[
\text{Hom}(\pi^*\mathcal{O}_M(1), \pi^*\mathcal{O}_M(1)[2]) = \text{Ext}^2(\pi^*\mathcal{O}_M(1), \pi^*\mathcal{O}_M(1)) = 
H^2(X, \mathcal{O}_X) = 0.
\]

On the other hand, by (2.1)

\[
\text{Hom}(\Phi^*\mathcal{O}_M(1)[2]) = \text{Hom}(\Phi^*\mathcal{O}_M(1)[2]) = 
\text{Ext}^2(\Phi^*\mathcal{O}_M(1)[2]) \neq 0.
\]

We shall give a more intrinsic proof to Corollary (2.2).

For this we need a lemma.

**Lemma (2.3).** Notation being the same as §1. Then, $\text{Ext}^i(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) = \mathcal{C}$ if $i$ is an even integer with $0 \leq i \leq 2n$, and otherwise $\text{Ext}^i(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) = 0$.

**Proof.** By the exact sequence

\[
0 \to I_{Y^+} \to \mathcal{O}_{X^+} \to \mathcal{O}_{Y^+} \to 0
\]
one has the exact sequence
\[ \text{Hom}(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) \to \text{Hom}(\mathcal{O}_{X^+}, \mathcal{O}_{Y^+}) \to \text{Hom}(I_{Y^+}, \mathcal{O}_{Y^+}) \to \text{Ext}^1(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) \to 0. \]

Since the first map is an isomorphism, \( \text{Hom}(I_{Y^+}, \mathcal{O}_{Y^+}) \cong \text{Ext}^1(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) \).

Here the left hand side is the normal bundle \( N_{Y^+/X^+} \) which is isomorphic to \( \Omega_{Y^+}^1 \). Hence, we have
\[ \text{Ext}^1(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) \cong \Omega_{Y^+}^1. \]

Now, by taking \( \text{Hom}(\cdot, \mathcal{O}_{Y^+}) \) of the Koszul complex \((\ast)\) in (1.7) we see that
\[ \text{Ext}^1(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) \cong \wedge^1 \text{Ext}^1(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) \cong \Omega_{Y^+}^1. \]

Therefore, in the spectral sequence
\[ E_2^{p,q} := H^p(X^+, \text{Ext}^q(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+})) \Rightarrow \text{Ext}^{p+q}(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}), \]
\[ E_2^{p,q} = 0 \] if \( p \neq q \). This implies that this spectral sequence degenerates at \( E_2 \)-terms; hence we have the result.

(2.4) (Another proof of (2.2)): For simplicity, put \( L := \pi^* \mathcal{O}_M(1) \) and \( L^+ := \mathcal{O}_{X^+}(1) \otimes (\pi^+)^* \mathcal{O}_M(-1) \). Then, \( \Phi(L) = L^+ \otimes I_{Y^+} \). We have \( \text{Ext}^2(L, L) = H^2(X, \mathcal{O}_X) \). On the other hand, \( \text{Ext}^2(L^+ \otimes I_{Y^+}, L^+ \otimes I_{Y^+}) = \text{Ext}^2(I_{Y^+}, I_{Y^+}) \). Take \( \text{Hom}(\cdot, I_{Y^+}) \) of the exact sequence \( 0 \to I_{Y^+} \to \mathcal{O}_{X^+} \to \mathcal{O}_{Y^+} \to 0 \). Here \( \text{Ext}^2(I_{Y^+}, I_{Y^+}) = 0 \) because, in the exact sequence
\[ \text{Ext}^1(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) \to \text{Ext}^2(\mathcal{O}_{Y^+}, I_{Y^+}) \to \text{Ext}^2(\mathcal{O}_{Y^+}, \mathcal{O}_{X^+}) = 0, \]
the first term vanish by Lemma (2.3). Moreover, we calculate
\[ \text{Ext}^3(\mathcal{O}_{X^+}, I_{Y^+}) = H^3(X^+, I_{Y^+}) \cong H^3(X^+, \mathcal{O}_{X^+}) = 0. \]
Therefore, we have an exact sequence
\[ 0 \to \text{Ext}^2(\mathcal{O}_{X^+}, I_{Y^+}) \to \text{Ext}^2(I_{Y^+}, I_{Y^+}) \to \text{Ext}^3(\mathcal{O}_{Y^+}, I_{Y^+}) \to 0. \]

The first term is isomorphic to \( H^2(X^+, I_{Y^+}) \cong H^2(X^+, \mathcal{O}_{X^+}) \). Note that, \( h^2(X^+, \mathcal{O}_{X^+}) = h^2(X, \mathcal{O}_X) \). The third term is isomorphic to \( \text{Ext}^2(\mathcal{O}_{Y^+}, \mathcal{O}_{Y^+}) \). By Lemma (2.3) this is a one dimensional \( \mathbb{C} \)-vector space. Therefore,
\[ \dim \text{Ext}^2(L, L) = \dim \text{Ext}^2(\Phi(L), \Phi(L)) - 1. \]

\[ \S3. \]

There are birational morphisms \( X \to \bar{X} \) and \( X^+ \to \bar{X} \) which contract \( Y \) and \( Y^+ \) to points respectively. We put \( \bar{X} := X \times_{\bar{X}} X^+ \) and let \( q : \bar{X} \to X \) and
$q^+ : \hat{X} \to X^+$ be the natural projections. Note that $\hat{X}$ is a normal crossing variety with two irreducible components $X$ and $Y \times Y^+$. Define a functor $\Psi$ as

$$\Psi := R(q^+)_* \circ Lq^* : D(X) \to D(X^+).$$

**Theorem (3.1).** The functor $\Psi$ is an equivalence of triangulated categories.

The remainder consists of the proof of this theorem.

**Outline of the Proof:** We define a suitable spanning class $\Omega'$ for $D(X)$. We shall prove that, for any $a, b \in \Omega$, $\text{Hom}^i(a, b) \cong \text{Hom}^i(\Psi(a), \Psi(b))$ for all $i \in \mathbb{Z}$. Then, by [B 1, Theorem 2.3] we see that $\Psi$ is fully-faithful.

Finally, apply Theorem 2.4. in [B-K-R] to $\Psi$ and the spanning class $\Omega'$ to conclude that $\Psi$ is an equivalence.

**Remark:** For $f : Y \times Y^+ \to Y$ and $f^+ : Y \times Y^+ \to Y^+$ be the natural projections, respectively. Note that $Y$ is a subvariety of $Y \times Y^+$. For an $\mathcal{O}_Y$ module $F$ and for an $\mathcal{O}_{Y^+}$ module $G$, write $F \otimes G$ for $f^* F \otimes (f^+)^* G$. We write $F \otimes_Y G$ for $f^* F \otimes (f^+)^* G|_Y$.

We have an exact sequence (cf. [Fr])

$$0 \to \mathcal{O}_Y \to \mathcal{O}_X \oplus \mathcal{O}_{Y \times Y^+} \to \mathcal{O}_Y \to 0.$$
\[ \pi^*\mathcal{O}_M(k) = \mathcal{O}_X(j + k) \otimes (\pi^+)^*\mathcal{O}_{M^+}(−k) \] because \( k \leq 0 \). As a consequence, we have
\[
\Psi(\mathcal{O}_X(j) \otimes \pi^*\mathcal{O}_M(k)) \cong \mathcal{O}_X(j + k) \otimes (\pi^+)^*\mathcal{O}_{M^+}(−k).
\]

**Lemma (3.4)** Let \( \pi : X \to M \) be the same as \( \S 1 \). Let \( l \) and \( m \) be integers such that \( −n \leq l \leq n \) and \( −n \leq m \leq n \). Then we have:

1. \( H^i(X, \mathcal{O}_X(l) \otimes \mathcal{O}_M(m)) = 0 \) for \( i > 0 \).
2. \( H^i(X, \mathcal{O}_X(l + m) \otimes \mathcal{O}_M(−m)) = 0 \) for \( i > 0 \).

**Proof.** (1): Since \( l \geq −n \), \( H^i(X, \mathcal{O}_X(l) \otimes \mathcal{O}_M(m)) = H^i(M, \pi_*\mathcal{O}_X(l) \otimes \mathcal{O}_M(m)) \). We only have to consider the case where \( l \geq 0 \). In this case \( \pi_*\mathcal{O}_X(l) \otimes \mathcal{O}_M(m) \cong \text{Sym}^m(\mathcal{O}_M \oplus \Theta_M) \otimes \mathcal{O}_M(m) \). Now (1) follows from the fact that \( m \geq −n \).

(2): When \( l + m \geq −n \), \( H^i(X, \mathcal{O}_X(l + m) \otimes \mathcal{O}_M(−m)) = H^i(M, \pi_*\mathcal{O}_X(l + m) \otimes \mathcal{O}_M(−m)) \). Since \( m \geq −n \), this is true for \( i \neq 2n \). When \( i = 2n \), this is proved by a case-by-case checking. For example, when \( l + m = −2n \), we must have \( l = −n \) and \( m = −n \). But, then \( \pi_*\mathcal{O}_X(n − 1) \otimes \mathcal{O}_M(−n) = \text{Sym}^{n−1}(\mathcal{O}_M \oplus \Theta_M) \otimes \mathcal{O}_M(−n) \) has no global sections.

**Proposition (3.5)** Let \( a \) and \( b \) be elements of \( \Omega' \). Then, for all \( i \in \mathbb{Z} \), \( \text{Hom}^i(a, b) \cong \text{Hom}^i(\Psi(a), \Psi(b)) \).

**Proof.** By Lemma (3.4), (1), we have \( \text{Hom}^i(a, b) = 0 \) for \( i > 0 \). By (3.3) and Lemma (3.4), (2), we have \( \text{Hom}^i(\Psi(a), \Psi(b)) = 0 \) for \( i > 0 \). Let \( a \) and \( b \) be represented by the line bundles \( L \) and \( L' \) on \( X \). Then \( \Psi(a) \) and \( \Psi(b) \) are represented by the proper transform \( \phi(L) \) and \( \phi(L') \) where \( \phi : X \to X^+ \) is the Mukai flop. Therefore, \( \text{Hom}^0(a, b) \cong \text{Hom}^0(\Psi(a), \Psi(b)) \).

(3.6). By (3.5), for any \( a, b \in \Omega' \), and for all \( i \in \mathbb{Z} \), \( \text{Hom}^i(a, b) \cong \text{Hom}^i(\Psi(a), \Psi(b)) \). \( \Psi \) has a left adjoint and a right adjoint; in fact, let \( p_1 : X \times X^+ \to X \) and \( p_2 : X \times X^+ \to X^+ \) be the first and second projections. Then
\[
\Psi(\cdot) = \mathbf{R}(p_2)_*(\mathcal{O}_X \otimes \mathcal{O}_{X \times X^+} \otimes p_2^*(\cdot)).
\]

Now, put \( Q := \mathbf{R}\text{Hom}_{D(X \times X^+)}(\mathcal{O}_X, \mathcal{O}_{X \times X^+}) \otimes (p_2)^*\omega_{X^+}[2n] \). Then,
\[
\Psi(\cdot) := \mathbf{R}(p_1)_*(Q \otimes \mathcal{O}_{X} \otimes (p_2)^*(\cdot)) : D(X^+) \to D(X)
\]
becomes the left adjoint of \( \Psi \) (cf. [B-K-R, \S 6, Step 1]). Let \( S_X : D(X) \to D(X) \) be a Serre functor defined as \( S_X(\cdot) = \cdot \otimes \omega_X[2n] \). Similarly, we define \( S_{X^+} : D(X^+) \to D(X^+) \) as...
\(D(X^+) \to D(X^+)\). Then \(S_X \circ \Psi' \circ S_{X^+}^{-1}\) becomes a right adjoint of \(\Psi\). Since \(\Omega\) is a spanning class, we conclude that \(\Psi\) is fully-faithful by [B 1, Theorem 2.3]. By [B-K-R, Theorem 2.4], in order to prove that \(\Psi\) is an equivalence of triangulated categories, we have to check that \(\Psi \circ S_X(c) = S_{X^+} \circ \Psi(c)\) for all \(c \in \Omega\). Since \(\omega_X \cong O_X(-n - 1)\), \(\Psi \circ S_X(c) = \Psi(c) \otimes O_X, (-n - 1)[2n]\). This coincides with \(S_{X^+} \circ \Psi(c)\) because \(\omega_{X^+} \cong O_X, (-n - 1)\).

\(\S 4.\)

The result in the previous section was concerned with a special example. In this section we prove that the same holds in a general situation.

(4.1) Let \(Z\) and \(Z^+\) be birationally equivalent smooth projective varieties of dimension \(2n\) with \(n \geq 2\). Assume that there are subvarieties \(W \subset Z\) and \(W^+ \subset Z^+\) which are isomorphic to \(\mathbb{P}^n\). Assume that \(N_W/Z \cong \Omega_{\mathbb{P}^n}, N_{W^+/Z^+} \cong \Omega_{\mathbb{P}^n}\) and that \(Z^+\) (resp. \(Z\)) is the Mukai flop of \(Z\) (resp. \(Z^+\)) along \(W\) (resp. \(W^+)\). Let \(Z \to \overline{Z}\) and \(Z^+ \to \overline{Z^+}\) be the birational morphisms which contract \(W\) and \(W^+\) to points respectively. We put \(\overline{Z} := Z \times \overline{Z^+}\), and denote by \(u : \overline{Z} \to Z\) and \(u^+ : \overline{Z} \to Z^+\) the natural projections. We define the functor
\[
\Psi_Z := R(u^*) \circ L u^* : D(Z) \to D(Z^+).
\]

(4.2) Let \(X, X^+\) and others be the same as the previous sections. Define a set of objects \(\Omega' := \{O_x\}_{x \in X}\), where \(O_x\) is the structure sheaf of a point \(x\). Then \(\Omega'\) becomes a spanning class for \(D(X)\). Since the functor \(\Psi : D(X) \to D(X^+)\) is an equivalence of triangulated categories, for any points \(x, x' \in X\),
\[
\text{Hom}^i(O_x, O_{x'}) = \text{Hom}^i_i(D(X^+))(O_x, O_{x'}),
\]
and
\[
\text{Hom}^i(\Psi(O_x), \Psi(O_{x'})) = \text{Hom}^i\left(D(X^+), \Psi(O_x), \Psi(O_{x'})\right).
\]

(4.3) Let \(Z_W\) (resp. \(Z_{W^+}\)) be the formal completion of \(Z\) (resp. \(Z^+\)) along \(W\) (resp. \(W^+)\). We have \(Z_W \cong X_Y\) and \(Z_{W^+}^+ \cong X_{Y^+}^+\). By these identifications, we regard a point \(x \in Y\) as a point on \(W\). Now, by (4.2), if \(x \in W\) and \(x' \in W\), then we have
\[
\text{Hom}^i(O_x, O_{x'}) = \text{Hom}^i(\Psi_Z(O_x), \Psi_Z(O_{x'})).
\]
When one of \(x \in Z\) and \(x' \in Z\) is not in \(W\), it is clear that both spaces are isomorphic. Therefore, by [B 1, Theorem 2.3] \(\Psi_Z\) is fully-faithful. Let \(\Omega_Z\) be the spanning class for \(D(Z)\) consisting of the objects \(O_x\) with \(x \in Z\), and let \(S_Z\) (resp. \(S_{Z^+}\)) be the Serre functor defined in (3.10). Since \(\omega_{Z^+}\) is trivial along
W^+, it is easily checked that \( \Psi \circ S(c) = S_{Z^+} \circ \Psi_Z(c) \) for all \( c \in \Omega^r_{W} \). Hence, by [B-K-R, Theorem 2.4], \( \Psi_Z \) is an equivalence of triangulated categories. As a consequence, we have proved that

**Theorem (4.4).** Let \( Z, Z^+ \) and \( \Psi_Z \) be the same as (4.1). Then

\[ \Psi_Z : D(Z) \to D(Z^+) \]

is an equivalence of triangulated categories.

**Corollary (4.5).** Let \( Z \) and \( Z' \) be birationally equivalent, complex projective symplectic 4-folds. Then \( D(Z) \) and \( D(Z') \) are equivalent.

**Proof.** By Wierzba and Wisniewski [W-W, Theorem 1.2](see also [W]), \( Z \) and \( Z' \) are connected by a finite sequence of Mukai flops. Theorem (4.4) together with this implies that \( D(Z) \) and \( D(Z') \) are equivalent.

§5.

Theorem (4.4) holds for a Mukai flop in a more general sense. Namely, let \( Z \) be a smooth projective variety of dim \( 2n \). Let \( f : Z \to \bar{Z} \) be a projective birational morphism which contracts a smooth subvariety \( W \subset Z \) of dim \( n + r \) \((0 \leq r \leq n - 2)\) to a smooth subvariety \( S \subset \bar{Z} \) of dim \( 2r \). Assume that \( W \to S \) is a \( \mathbb{P}^{n-r} \) bundle and assume that, for all \( s \in S \), \( N_{W/Z} \vert_{W_s} \cong \Omega^r_{W_s} \). Then we can perform Mukai flops in a family to get a new variety \( Z^+ \). Let us consider \( Z \times_{\bar{Z}} Z^+ \) and let \( \Psi_Z : D(Z) \to D(Z^+) \) be the corresponding functor. Then we have the following.

**Theorem (5.1).** \( \Psi_Z \) is an equivalence of triangulated categories.

(5.2) Let \( X \) and \( Y \) be smooth projective varieties. For an object \( \mathcal{E} \in D(X \times Y) \), let \( F_{X \to Y, \mathcal{E}} : D(X) \to D(Y) \) be the Fourier-Mukai functor defined as \( F_X(\cdot) := \mathbb{R}(p_Y)_+(L(p_X)^\ast(\cdot) \otimes L \mathcal{E}) \). Here \( p_X \) (resp. \( p_Y \)) is the first projection (resp. the second projection) of \( X \times Y \). If there is an object \( \mathcal{G} \in D(Y \times X) \) such that \( F_Y \circ F_{X \to Y, \mathcal{E}} \cong \text{id}_X \) and \( F_X \circ F_{X \to Y, \mathcal{E}} \cong \text{id}_Y \), then \( F_{X \to Y, \mathcal{E}} \) is called an equivalence as a Fourier-Mukai transform. Note that the functor \( \Psi \) in §4 is an equivalence as a Fourier-Mukai transform.

Let \( S \) be a (not necessarily projective) smooth algebraic variety.

Let \( p_{X \times Y \times S} : X \times Y \times S \to X \times Y \) be the projection to the first and second factors (similarly, we define projections \( p_{X \times Y \times S \to X \times S} \) and \( p_{X \times Y \times S \to Y \times S} \)).

Put \( \mathcal{E} \otimes S := L(p_{X \times Y \times S \to X \times Y})^\ast \mathcal{E} \in D(X \times Y \times S) \). Define a functor

\[ F_{X \times S \\to Y \times S, \mathcal{E} \otimes S} : D(X \times S) \to D(Y \times S) \]

as

\[ \mathbb{R}(p_{X \times Y \times S \to Y \times S})_+(L(p_{X \times Y \times S \to X \times S})^\ast(\cdot) \otimes (\mathcal{E} \otimes S)) \).

Let \( Z \) be a smooth projective variety. Let \( \mathcal{G} \) be an object of \( D(Y \times Z) \). By a similar argument as [Mu, Proposition 1.3], we have

\[ F_{Y \times S \to Z \times S, \mathcal{G} \otimes S} \circ F_{X \times S \to Y \times S, \mathcal{E} \otimes S} \]
\[ \cong F_{X \times S \to Z \times S; \langle \mathcal{G}, \mathcal{E} \rangle \circ \mathcal{S}}. \]

Here
\[ \langle \mathcal{G}, \mathcal{E} \rangle := R(p_{X \times Y \times Z \to X \times Z})_* (L(p_{X \times Y \times Z \to X \times Y})^* \mathcal{E} \otimes L(p_{X \times Y \times Z \to Y \times Z})^* \mathcal{G}). \]

**Proposition (5.3).** Assume that \( F_{X \to Y; \mathcal{E}} : D(X) \to D(Y) \) is an equivalence as a Fourier-Mukai transform. Then \( F_{X \times S \to Y \times S; \mathcal{E} \circ \mathcal{S}} : D(X \times S) \to D(Y \times S) \) is also an equivalence.

**Proof of (5.3).** Since \( F_{X \to Y; \mathcal{E}} \) is an equivalence as a Fourier-Mukai transform, one can find \( \mathcal{G} \in D(Y \times X) \) such that \( F_{Y \to X; \mathcal{G}} \circ F_{X \to Y; \mathcal{E}} \cong id_X \) and \( F_{X \to Y; \mathcal{E}} \circ F_{X \to Y; \mathcal{G}} \cong id_Y \). This means that \( F_{X \to X; \langle \mathcal{G}, \mathcal{E} \rangle} \cong F_{X \to Y, O_{\Delta X}} \) and \( F_{Y \to X; \langle \mathcal{E}, \mathcal{G} \rangle} \cong F_{Y \to Y, O_{\Delta Y}} \). By a theorem of Orlov [O 2, Theorem 2.2], we conclude that \( \langle \mathcal{G}, \mathcal{E} \rangle \cong O_{\Delta X} \) in \( D(X \times X) \) and \( \langle \mathcal{E}, \mathcal{G} \rangle \cong O_{\Delta Y} \) in \( D(Y \times Y) \).

Therefore, we have
\[
F_{X \times S \to X \times S; \langle \mathcal{G}, \mathcal{E} \rangle \circ \mathcal{S}} \circ F_{X \times S \to Y \times S; \mathcal{E} \circ \mathcal{S}} = F_{X \times S \to X \times S; \langle \mathcal{G}, \mathcal{E} \rangle \circ \mathcal{S}} \cong id_X.
\]

Similarly, we have
\[
F_{X \times S \to Y \times S; \mathcal{E} \circ \mathcal{S}} \circ F_{Y \times S \to X \times S; \mathcal{G} \circ \mathcal{S}} \cong id_Y.
\]

(5.4) (Proof of (5.1)): It is sufficient to show that, for \( z, z' \in Z \) and for \( i \in \mathbb{Z} \), \( \text{Hom}^i(\mathcal{O}_z, \mathcal{O}_{z'}) \cong \text{Hom}^i(\Psi_{\mathcal{Z}}(\mathcal{O}_z), \Psi_{\mathcal{Z}}(\mathcal{O}_{z'})) \). Let \( g : W \to S \) be the natural map induced by \( f : Z \to \mathbb{Z} \). We only have to consider the case where \( g(z) = g(z') \). We put \( s := g(z) \). Let \( s \in U(s) \) be an open set of \( S \) such that \( g^{-1}(U(s)) \cong P^r \times U(s) \). Under this identification, let \( (y, s) \) and \( (y', s) \) be the points which correspond to \( z \) and \( z' \) respectively. Put \( X := P(\mathcal{O}_{P^r} \oplus \Theta_{P^r}) \).

Then \( P^r \times U(s) \) can be identified with the (negative) section of \( X \times U(s) \to P^r \times U(s) \). Now, the Mukai flop \( X \times - \to X^+ \) induces a birational map \( X \times U(s) \to X^+ \times U(s) \). Put \( \hat{X} := X \times_X X^+ \) as in §1. Then we have an equivalence \( F_{\mathcal{O}_{\hat{X} \circ U(s)}} : D(X \times U(s)) \to D(X^+ \times U(s)) \) (cf. (5.3)). Note that, the formal completion of \( Z \) along \( g^{-1}(U(s)) \) is isomorphic to the formal completion of \( X \times U(s) \) along \( P^r \times U(s) \). By the same argument as (4.2) and (4.3), the homomorphism
\[
\text{Hom}^i_{D(Z)}(\mathcal{O}_z, \mathcal{O}_{z'}) \to \text{Hom}^i_{D(Z^+)}(\Psi_{\mathcal{Z}}(\mathcal{O}_z), \Psi_{\mathcal{Z}}(\mathcal{O}_{z'}))
\]
can be identified with
\[
\text{Hom}^i_{D(X \times U(s))}(\mathcal{O}_{(y,s)}, \mathcal{O}_{(y',s)}) \to \text{Hom}^i_{D(X^+ \times U(s))}(F_{\mathcal{O}_{\hat{X} \circ U(s)}}(\mathcal{O}_{(y,s)}), F_{\mathcal{O}_{\hat{X} \circ U(s)}}(\mathcal{O}_{(y',s)})).
\]

This homomorphism is an isomorphism by (5.3).

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