A general mathematical design method of the torque-split gear transmission with idler pinion

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Abstract
The torque-split gear transmission has been used in the transmission system of the rotorcraft, which undertaking high torque loads and requiring low weight. A universal mathematical design method of the torque-split gear transmission is proposed in this work. The teeth with the same phase of the two gears on the duplex idler are marked. And the meshing condition is defined by the whole pitch number between the two points along the pitch circle of the output gear. Then the relationship between the tooth number and gear positions is established by using this meshing condition. Unlike other existing design method, this method involves the idlers, input gear and output gear that engaged directly, and it is suitable for the design of multiple types of torque-split gear transmission. This method is validated through numerical examples of the torque-split transmission with symmetrical duplex idler. Moreover, practical applications of the torque-split transmission with planetary duplex idler, coplanar gear and concentric face gear are also studied with this method. A large number of discrete gear position solutions are observed under the same tooth number design. And different gear positions correspond to different dynamic and load sharing characteristics of the torque-split gear transmission.

Keywords: Torque-split, Design, Transmission, Duplex idler, Meshing condition

1. Introduction
The torque-split gear transmission is mainly used in the transmission system of the rotorcraft (Filler et al. and Heath et al., 2002, 2011), which uses turbines that rotate at a high speed to drive the main rotor that rotates at a lower speed. The unique characteristic of this application area is transmitting high torque through the lowest weight. The torque-split gear transmission divides the input torque into several paths, resulting in the reduction of the contact force on gear teeth, which means smaller gears can be used. The torque-split gear transmission has the advantages of compact structure, high power density and large reduction ratio, which are the main demands of the rotorcraft.

Various types of the torque-split gear transmission are patented (Yuriy, Todd et al. and Xiaolan Ai et al., 2016, 2014a, 2014b). And extensive researches have focused on the studies of design, dynamic and load sharing behavior. White (1989) is one of the first persons to have studied the torque-split gear transmission, and he has explored the advantages of the torque-split gear transmission over the traditional design. The studied topics on the dynamics of the torque-split transmission include dynamic model, natural frequencies and dynamic response (Reszuta et al. and Aydoğan et al., 2015, 2017). Moreover, because of the asymmetry of the machining and installation errors of each path of the torque-split gear transmission, the torque of each path is uneven. Gmirya (2011) developed an elastic component load sharing method. And the dynamic load sharing characteristic studies include the works in (Filler et al., Mo et al. and Fu Ai et al., 2002, 2015a, 2015b). Its application on the rotorcraft has also received attention (Gmirya and Jose et al., 2018, 2010).

The simultaneous meshing design of the torque-split gear transmission has great influence on the reliability and comfort of the rotorcraft. However, the simultaneous meshing problem, which is quite common in the field of position analysis of gear mechanisms, has not yet been fully studied with a universal mathematical design method. Vílán et al. (2010) and Abraham et al. (2012) have studied the torque-split transmission with gears in the same plane (namely coplanar gears). They defined the area formed by four coplanar gears, shown in Fig. 1, as a curvilinear quadrilateral. And a pitch difference is also defined by the sum pitches in the output and input gears minus the sum of pitches in the
idler gears at the curvilinear quadrilateral. The meshing condition is satisfied when the pitch difference coincident with a whole number. This meshing condition is effective in the design of the configuration with four coplanar gears. While for the torque-split transmission with duplex idler, the curvilinear quadrilateral, which is the basis of the meshing condition, is unclosed. Different investigations in terms of the torque-split transmission with duplex idler are done by Li Zhijun et al. (2012) and Xiangya et al. (2014). Their methods are complex and have the possibility of unsolvable.

![Fig. 1 The meshing condition proposed by Vilán and Abraham.](image)

In this paper, take the compatibility of geometrical space and simultaneous meshing of gears into account, a universal mathematical design method is developed. With this method, the torque-split transmission with duplex idler is studied numerically in two cases: the external meshing duplex idler and the internal meshing duplex idler. In each case, solutions for the simultaneous meshing problem of the torque-split gear transmission with symmetrical duplex idler are calculated. Other types of the torque-split gear transmission (planetary duplex idler, coplanar gear and concentric face gear) are also calculated. Applying this universal design method, the meshing phrasing difference between paths, which affects the dynamics and load sharing behavior of the transmission system, can be studied by adjusting the integral pitch number of the pinion contained in the closed area that formed by gears.

2. The Meshing Problem

The torque-split transmission with duplex idler is shown in Fig. 2. This transmission system divides the input torque of pinion into several paths. The transmission of force is divided between several contact areas, thereby an increase of available torque. However, this also gives rise to the problem of simultaneous meshing of the gears that engaged.

![Fig. 2 The torque-split transmission with duplex idler.](image)

The input pinion 1 meshes with gears 2 and 4 respectively, and output gear 6 meshes with gears 3 and 5. The two paths are not independent of each other when transmitting torque and movement, and redundant constraints are introduced. The existence of the redundant constraints bring difficulties to the design of the torque-split gear transmission. In order to ensure continuous transmission of torque and movement, the simultaneous meshing problem of gears needs to be studied.

3. The Universal Mathematical Design Method

The gears discussed here refer to standard spur gears. Since the problem discussed is mainly about the meshing
phase and geometric compatibility, so the method is still applicable for the standard helical gears. But for nonstandard gears with modified addendum are excluded from this work. In the application of the rotorcraft, the two duplex idlers (Fig. 3) are symmetrically arranged with respect to the centerline of the input and output gears. Considering the cost of manufacture and maintenance, the two duplex idlers should be interchangeable (Li Zhijun et al. 2012). The phase between the two marked idler’s teeth on a duplex shaft can be defined as location phase. Based on the interchangeability requirement, the location phase of the two duplex idlers should be the same. If the location phase of one duplex idler is settled, the location phase of the other duplex idler is also settled. This brings redundant constraints when the two duplex idlers meshing with the input and output gears simultaneously. For analytical convenience, the location phase is defined as zero. To obtain a general mathematical design method, the location phase of the general case shown in Fig. 2 is still defined as zero.

Fig. 3 The torque-split gear transmission used in the rotorcraft.

The starting line of gear teeth is defined in Fig. 4. It is a line through the middle point of the addendum of gear tooth and gear center (Fig. 4(a)). From the top view (Fig. 4(b)) of the duplex idler, the starting lines of the two gears on the same duplex shaft coincide.

Fig. 4 The starting line of gear teeth from the view of (a) and (b).

The universal mathematical design method is derived from the general configuration of the torque-split transmission with duplex idler. And it is studied in two cases: the external meshing duplex idler and the internal meshing duplex idler.

3.1 Case 1: The External Meshing Duplex idler

The proper meshing condition of the torque-split transmission with duplex idler is illustrated as follows:

The general configuration of the external meshing duplex idler is shown in Fig. 4, the layout angles of gears are defined by $\theta_1, \theta_2, \theta_3$, and $\theta_4$. Gears 2(4) and 3(5) are connected by the duplex shaft. The two paths have the same definition of the starting line of the gear teeth.

In order to explain the meshing condition (Fig. 5), part of the gear teeth are represented by pitch circle. And the points on which the gear teeth have the same meshing position are marked with yellow dots. As two gears can mesh with the same gear correctly at any positions along the pitch circle, so it is well to assume that the pinion 1 has meshed properly with gears 2 and 4. The gear teeth on the points F and H mesh with the teeth on points E and G at the same meshing position. To ensure the correct meshing of gears 3(5) and 6, the pitch number between the two points (F and H) along the pitch circle of gear 6 must coincident with a whole number. By this meshing condition, the relationship
between the tooth number and gear positions is defined. If this relationship is not satisfied, the teeth of gears will interfere with each other.

In general, the three gears 1, 2 and 4 can mesh properly without considering gear 6. Assuming that gears 3, 5 and 6 also have been adjusted to the correct engagement. To illustrate the meshing condition of the gear system shown in Fig. 5, three moments (T0, T1 and T2) are defined. First of all, the moment shown in Fig. 5 is defined as T0. As gear 1 rotates clockwise, the point E on gear 3 will mesh with gear 6 at point F. Then, this moment is defined as T1. As gear 1 continues to rotate clockwise, the point G on gear 5 will mesh with the output gear 6 at point H. And this moment is defined as T2. The detailed status of the three moments are described as below:

**The moment T0:**
The starting line BE and the gear center line BC coincide. The starting line DG is n pitch number distance clockwise from DM. The integral pitch number of gear 1 contained in $\theta_1$ is k. And points M and N will mesh on gear center line DC.

**The moment T1:**
The point E on gear 3 engages with the point F on gear 6 at the pitch circle. From T0 to T1, gears 3 and 6 have rotated the same pitch number.

**The moment T2:**
The point G on gear 5 engages with the point H on gear 6 at the pitch circle. The teeth on points E and G mesh with gear 6 at the same meshing position. From T1 to T2, gears 5 and 6 have rotated the same pitch number.

The pitch number of gear 6 contained in $\theta_1$ can be expressed as

$$NUM_6 = \frac{\theta_1}{2\pi} \cdot z_6$$  \hspace{1cm} (1)

The pitch number of gear 3 contained in $\theta_1$ can be expressed as

$$NUM_3 = \frac{\theta_1}{2\pi} \cdot z_3$$  \hspace{1cm} (2)

The pitch number of gear 5 contained on the green line shown in Fig. 5 can be derived as follows:

$$NUM_5 = \frac{2\pi}{2\pi} \cdot \left[ \frac{\theta_1 - (\varphi - \Delta \gamma)}{2\pi} \right] \cdot z_5$$ \hspace{1cm} (3)

where $\varphi = n \cdot 2\pi / z_5$ is the angle of the n pitch number of gear 5, $\Delta \gamma = (\theta_4 \cdot z_4 / 2\pi - k) \cdot 2\pi / z_4$ is the angle of gear 4 that meshes with the non-integral pitch number of gear 1 contained in $\theta_1$.

In Fig. 5, the pitch numbers of gears contained in the lines that marked with the same color are equal. According to the meshing condition, the pitch number between points F and H is just an integer. Thus, the relationship between the tooth number and the gear positions is obtained as follows:

$$NUM_6 + NUM_3 - NUM_5 = Z$$ \hspace{1cm} (4)

where $Z$ is the integral pitch number of gear 6 between points F and H.

For analytical convenience, Eq. (4) can be rearranged as follows:

$$z_4z_6 \cdot \theta_1 + z_3z_5 \cdot \theta_2 - z_5z_4 \cdot \theta_3 - z_3z_6 \cdot \theta_4 = [Z \cdot z_4 + k \cdot z_5 - n \cdot z_4 - z_5z_6] \cdot 2\pi$$ \hspace{1cm} (5)

Replace the term of $Z \cdot z_4 + k \cdot z_5 - n \cdot z_4 - z_5z_6$ with an integer $N$, we obtain the following:

$$z_4z_6 \cdot \theta_1 + z_3z_5 \cdot \theta_2 - z_5z_4 \cdot \theta_3 - z_3z_6 \cdot \theta_4 = N \cdot 2\pi$$ \hspace{1cm} (6)
The relationship of the angles \((\theta_1, \theta_2, \theta_3, \theta_4)\) is obtained from the quadrilateral of ABCD as follows:

\[
\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi
\]  
(7)

Imposing the theorem of cosine to the diagonals, the following equations are derived:

\[
AB^2 + AD^2 - 2AB \cdot AD \cos \theta_1 = BC^2 + CD^2 - 2BC \cdot CD \cos \theta_2
\]  
(8)

\[
BC^2 + AB^2 - 2BC \cdot AB \cos \theta_3 = CD^2 + AD^2 - 2CD \cdot AD \cos \theta_4
\]  
(9)

For the spur gears, knowing that

\[
\text{the pitch radius of gear } i \text{ is expressed in function of the tooth number according to the following transcendental equation}
\]

\[
\frac{\pi}{2} \cdot m_i \cdot \theta_i = \text{arc} \cos \left( \frac{c - a + b \cdot \cos \theta_i}{d} \right) + w \cdot 2\pi
\]  
(13)

where

\[
a = m_i^2 (z_i + z_{ir})^2 + m_i^2 (z_i + z_{ir})^2
\]  
(14)

\[
b = 2m_i^2 (z_i + z_{ir})(z_{ir} + z_{ir})
\]  
(15)

\[
c = m_i^2 (z_i + z_{ir})^2 + m_i^2 (z_i + z_{ir})^2
\]  
(16)

\[
d = 2m_i^2 (z_i + z_{ir})(z_{ir} + z_{ir})
\]  
(17)

\[
e = m_i^2 (z_i + z_{ir})^2 + m_i^2 (z_i + z_{ir})^2
\]  
(18)

\[
f = 2m_i m_i (z_i + z_{ir})(z_{ir} + z_{ir})
\]  
(19)

\[
g = m_i^2 (z_i + z_{ir})^2 + m_i^2 (z_i + z_{ir})^2
\]  
(20)

\[
h = 2m_i m_i (z_i + z_{ir})(z_{ir} + z_{ir})
\]  
(21)

\[
u = \frac{z_i + z_{ir}}{z_i - z_{ir}}
\]  
(22)

\[
w = \frac{N + z_i z_{ir}}{z_i z_{ir} - z_i z_{ir}}
\]  
(23)

\[
x = \frac{z_i + z_{ir}}{z_i - z_{ir}}
\]  
(24)

\[
y = \frac{z_i + z_{ir}}{z_i z_{ir} - z_i z_{ir}}
\]  
(25)

\[
z = \frac{N + z_i z_{ir}}{z_i z_{ir} - z_i z_{ir}}
\]  
(26)

Newton iterative method is used to gain the root of Eq. (13). Then \(\theta_1, \theta_2\) and \(\theta_4\) are obtained from Eqs. (28)-(30).

\[
\theta_2 = \text{arc} \cos \left( \frac{c - a + b \cdot \cos \theta_1}{d} \right)
\]  
(28)

\[
\theta_1 = u \cdot \theta_1 + v \cdot \theta_1 + w \cdot 2\pi
\]  
(29)

\[
\theta_4 = x \cdot \theta_1 + y \cdot \theta_1 + z \cdot 2\pi
\]  
(30)

3.1.1 Numerical Validation 1: Symmetrical Duplex idler

The method established above is applied to the resolution of the configuration of the torque-split transmission with external meshing symmetrical duplex idlers (Fig. 6). The tooth number of gears are as follows:\(z_i = 19, z_2 = 30, z_3 = 17, z_4 = 100\). The gears are 2 module \((m_1 = m_2 = 2)\), it would be well if \(m_1 \not= m_2\).
In this configuration \((\theta_1 = \theta_4, z_2 = z_4, z_3 = z_5)\), the starting equations of Eqs. (6) and (7) become
\[
z_2z_6 \cdot \theta_1 + z_4z_5 \cdot \theta_2 - 2z_2z_3 \cdot \theta_1 = N \cdot 2\pi
\]
\[
\theta_1 + \theta_4 + 2 \cdot \theta_1 = 2\pi
\]
where \(N = Z \cdot z_2 + k \cdot z_2 - n \cdot z_3 - z_2z_3\).

Still, by the theorem of cosine to the diagonals, the Eqs. (11) and (12) are reduced to the following equation:
\[
m_z^2 (z_3 + z_4)^2 - m_z^2 (z_3 + z_4)^2 \cos \theta_1 = m_z^2 (z_3 + z_4)^2 - m_z^2 (z_3 + z_4)^2 \cos \theta_2
\]
Similarly, the transcendental equation for \(\theta_1\) is obtained as follows:
\[
z_2z_6 \cdot \theta_1 + z_4z_5 \cdot \arccos \left( \frac{b - a + a \cos \theta_1}{b} \right) - 2z_2z_3 \cdot \left[ 2\pi - \theta_1 - \arccos \left( \frac{b - a + a \cos \theta_1}{b} \right) \right] = N \cdot 2\pi
\]
where
\[
a = m_z^2 (z_3 + z_4)^2
\]
\[
b = m_z^2 (z_3 + z_4)^2
\]

Once the angle \(\theta_1\) has been determined, the other unknown parameters \(\theta_1, \theta_4\) are calculated by Eqs. (37) and (38).
\[
\theta_2 = \arccos \left( \frac{b - a + a \cos \theta_1}{b} \right)
\]
\[
\theta_4 = \frac{2\pi - \theta_1 - \theta_2}{2}
\]

Figure 7 is the representation of Eq. (34). The function value and that the root of \(N=0\) is shown. When the right side of Eq. (34) is 0, only one point on the x axis is shown in the illustration. The limit positions (shown in the first and the last pictures of Fig. 8) of gear 1 are obtained as the two endpoints of the curve intersect with the horizontal line of \(y=0\). Part of the solutions are listed in Table 1, and the corresponding configurations are shown in Fig. 8. Since there is a turning point on the equation curve, the same \(N\) may has two different solutions (for example \(N=402\)).

Fig. 6 The torque-split transmission with external meshing symmetrical duplex idler.

Fig. 7 The representation of Eq. (34) for \(N=0\).
Table 1  Solutions of Eqs. (34), (37) and (38) in sexagesimal degrees

| N(x₀) | -510(1) | -80(1) | 50(1) | 190(1) | 323(1) | 380(1) | 402(1) | 402(6) | 323(6) |
|-------|---------|--------|-------|--------|--------|--------|--------|--------|--------|
| 01    | 0       | 25.95  | 32.99 | 39.38  | 43     | 40.67  | 36.74  | 27.74  | 0      |
| 02    | 0       | 76.50  | 103.02| 136.57 | 180    | 213.25 | 239.32 | 277.27 | 360    |
| 03    | 180     | 128.78 | 112.00| 92.02  | 68.50  | 53.04  | 41.97  | 27.50  | 0      |
| 04    | 180     | 128.78 | 112.00| 92.02  | 68.50  | 53.04  | 41.97  | 27.50  | 0      |

Fig. 8 Solutions for the torque-split transmission with external meshing symmetrical duplex idler.

It is noteworthy that the integer \( n \), which indicates the position of the starting line of gear 4, determines the installation phase of gear tooth of the duplex idler. Between the two limit positions, there are numerous of the gear positions (corresponding to \( N \) and the initial value of \( x₀ \)). Among those gear positions, only the gear positions with an integer pitch number of input pinion contained in \( \theta_2 \) have no meshing phase difference between the first stage of the two paths. And the meshing phase difference has great influence on the dynamic and load sharing behavior of the torque-split gear transmission.

3.2 Case 2 : The Internal Meshing Duplex idler

The general configurations of the torque-split transmission with internal meshing duplex idler are shown in Fig. 9. It has two types: noncrossed gear center line (Fig. 9(a)) and crossed gear center line (Fig. 9(b)).

Fig. 9 General configurations of the torque-split transmission with internal meshing duplex idler: (a) noncrossed gear center line; (b) crossed gear center line.

3.2.1 Type 1: Noncrossed Gear Center Line

In this configuration (Fig. 9(a)), the starting equations of Eqs. (6) and (7) become

\[
\begin{align*}
2z_i z_4 \cdot \theta_1 - z_i z_6 \cdot \theta_2 - z_4 z_5 \cdot (\pi - \theta_4) + z_i z_5 \cdot (\pi + \theta_4) &= N \cdot 2\pi \\
\theta_1 + \theta_2 + \theta_4 + \theta_4 &= 2\pi
\end{align*}
\]

(39) \hspace{1cm} (40)

where \( N = Z \cdot z_4 - k \cdot z_4 - n \cdot z_4 \).

Once again, by the theorem of cosine to the diagonals, the following equations are derived.

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\[ m_1^2(z_a - z_i)^2 + m_1^2(z_a - z_i)^2 - 2m_1^2(z_a - z_i)(z_a - z_i)\cos \theta_1 = m_1^2(z_i + z_i)^2 + m_1^2(z_i + z_i)^2 - 2m_1^2(z_i + z_i)(z_i + z_i)\cos \theta_1 \quad (41) \]
\[ m_2^2(z_a + z_i)^2 + m_2^2(z_a - z_i)^2 - 2m_2m_1(z_i + z_i)(z_a - z_i)\cos \theta_1 = m_2^2(z_i + z_i)^2 + m_2^2(z_i + z_i)^2 - 2m_2m_1(z_i + z_i)(z_i + z_i)(z_a - z_i)\cos \theta_1 \quad (42) \]

From the four equations above, the transcendental equation for \( \theta_1 \) and the equations for \( \theta_2, \theta_3, \theta_4 \) are the same with Eqs. (13) and (28)-(30). The differences are the following expressions.

\[ a = m_1^2(z_a - z_i)^2 + m_1^2(z_a - z_i)^2 \quad (43) \]
\[ b = 2m_1^2(z_a - z_i)(z_a - z_i) \quad (44) \]
\[ c = m_1^2(z_i + z_i)^2 + m_1^2(z_i + z_i)^2 \quad (45) \]
\[ d = 2m_1^2(z_i + z_i)(z_i + z_i) \quad (46) \]
\[ e = m_2^2(z_i + z_i)^2 + m_2^2(z_i - z_i)^2 \quad (47) \]
\[ f = 2m_2m_1(z_i + z_i)(z_i - z_i) \quad (48) \]
\[ g = m_2^2(z_i + z_i)^2 + m_2^2(z_i - z_i)^2 \quad (49) \]
\[ h = 2m_2m_1(z_i + z_i)(z_i - z_i) \quad (50) \]
\[ u = \frac{z_a - z_i}{z_a - z_i} \quad (51) \]
\[ v = \frac{z_a z_i - z_i z_i}{z_a z_i - z_i z_i} \quad (52) \]
\[ w = \pi + \frac{N - z_a z_i}{z_a z_i - z_i z_i} \cdot 2\pi \quad (53) \]
\[ x = -\frac{z_i - z_i}{z_i - z_i} \quad (54) \]
\[ y = \frac{z_i z_i + z_i z_i}{z_i z_i - z_i z_i} \quad (55) \]
\[ z = -\pi - \frac{N - z_i z_i}{z_i z_i - z_i z_i} \cdot 2\pi \quad (56) \]

### 3.2.2 Type 2: Crossed Gear Center Line

In the configuration shown in Fig. 9(b), Eqs. (39) and (40) become

\[ z_a z_i \cdot \theta_i - z_i z_i \cdot \theta_i - z_i z_i \cdot (\pi - \theta_i) + z_i z_i \cdot (\pi - \theta_i) = N \cdot 2\pi \quad (57) \]
\[ \theta_1 - \theta_3 = 0 \quad (58) \]

With the quadrilateral cosine theorem, the equations obtained are the same with Eqs. (41) and (42). From the equations above, the same equations for \( \theta_1, \theta_2, \theta_3, \theta_4 \) are obtained with the following expressions:

\[ u = \frac{z_a - z_i}{z_a - z_i} \quad (59) \]
\[ v = \frac{z_a z_i - z_i z_i}{z_a z_i - z_i z_i} \quad (60) \]
\[ w = \pi + \frac{N}{z_a z_i - z_i z_i} \cdot 2\pi \quad (61) \]
\[ x = \frac{z_i - z_i}{z_i - z_i} \quad (62) \]
\[ y = \frac{z_i z_i - z_i z_i}{z_i z_i - z_i z_i} \quad (63) \]
\[ z = \pi + \frac{N}{z_i z_i - z_i z_i} \cdot 2\pi \quad (64) \]

### 3.2.3 Numerical Validation 2: Symmetrical Duplex idler

Likewise, the method established in this section is applied to the resolution of the configuration of the torque-split transmission with external meshing symmetrical duplex idler (Fig. 10) which is a specific instance. The tooth number of gears are also as follows: \( z_1 = 19, z_2 = 30, z_3 = 17, z_6 = 100 \). The gears are 2 module \( m_1 = m_2 = 2 \), it would be well if \( m_1 \neq m_2 \).
The torque-split transmission with internal meshing symmetrical duplex idler is a type of noncrossed gear center line. In this configuration, \( \theta_1 = \theta_2 \), \( z_2 = z_1 \), \( \theta_3 = z_3 \). Eqs. (39) and (40) become

\[
 z_2z_6 \cdot \theta_1 - z_1z_5 \cdot \theta_2 + 2z_2z_3 \cdot \theta_1 = N \cdot 2\pi \\
 \theta_1 + \theta_2 + 2\theta_3 = 2\pi
\]  

(65)

(66)

Equations (41) and (42) are reduced to the following equation:

\[
m_2^2 (z_a - z_b)^2 - m_1^2 (z_a - z_b)^2 \cos \theta_1 = m_2^2 (z_1 + z_2)^2 - m_1^2 (z_1 + z_2)^2 \cos \theta_2
\]  

(67)

The transcendental equation for \( \theta_1 \) is obtained as follows:

\[
z_2z_6 \cdot \theta_1 - z_1z_5 \cdot \theta_1 \arccos \left( \frac{b - a + a\cos \theta_1}{b} \right) + z_2z_3 \cdot \left[ 2\pi - \theta_1 - \arccos \left( \frac{b - a + a\cos \theta_1}{b} \right) \right] = N \cdot 2\pi
\]  

(68)

where

\[
a = m_2^2 (z_a - z_b)^2
\]  

(69)

\[
b = m_1^2 (z_1 + z_2)^2
\]  

(70)

Once N is given, \( \theta_1 \) is solved through the Newton iterative method. The angles \( \theta_2, \theta_3 \) are calculated by the following equations:

\[
 \theta_2 = \arccos \left( \frac{b - a + a\cos \theta_1}{b} \right)
\]  

(71)

\[
 \theta_3 = \frac{2\pi - \theta_1 - \theta_2}{2}
\]  

(72)

Figure 11 is the representation of Eq. (68). The function value and that the root of N=0 is shown. When the right side of Eq. (68) is 0, only one point on the x axis is shown in the illustration. When the two endpoints of the curve intersect with the horizontal line of y=0, the limit positions (shown in the first and the last pictures of Fig. 12) of gear 1 are obtained. Nine different solutions are listed in Table 2, and the corresponding configuration are shown in Fig. 12. Since there is a turning point on the curve of Eq. (68), the same N may has two different solutions (for example N=640).
Table 2  Solutions of Eqs. (68), (71) and (72) in sexagesimal degrees

| N(x₀) | 510(1) | 640(6) | 650(1) | 640(6) | 593(6) | 530(6) | 400(6) | 200(6) | -323(6) |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|---------|
| 01    | 0      | 47.76  | 53.40  | 70.15  | 72     | 70.73  | 62.94  | 47.11  | 0       |
| 02    | 0      | 86.59  | 99.13  | 153.52 | 180    | 202.77 | 235.68 | 274.79 | 360     |
| 03    | 180    | 112.82 | 103.73 | 68.17  | 54     | 43.25  | 30.69  | 19.05  | 0       |
| 04    | 180    | 112.82 | 103.73 | 68.17  | 54     | 43.25  | 30.69  | 19.05  | 0       |

Fig. 12 Solutions for the torque-split transmission with internal meshing symmetrical duplex idler.

4. Applications with Simplified Mathematical Method

4.1 The Planetary Duplex idler

![Image of planetary duplex idler](image)

The planetary duplex idler (Fig. 13) is studied as a particular case of the torque-split transmission with internal meshing symmetrical duplex idler. In this case, the axes of gear 1 and gear 6 coincide, and the following equations are obtained.

\[ \theta_1 = \theta_2 = \frac{2\pi}{q} \]  \hspace{1cm} \text{(73)}

\[ \theta_3 = \theta_4 = 0 \]  \hspace{1cm} \text{(74)}

where \( q \) is the number of the duplex idler.

Equation (65) becomes

\[ z_1z_3 - z_2z_6 = q \cdot N \]  \hspace{1cm} \text{(75)}

Considering the geometrical constraints of the planetary wheel, the following equation is obtained.

\[ m_1 (z_i + z_j) = m_2 (z_k - z_l) \]  \hspace{1cm} \text{(76)}

The tooth numbers of gears 2 and 3 can be solved by the following equations.
\[
\begin{align*}
\frac{z_2}{z_1} &= \frac{m_2 \cdot z_2 - m_1 \cdot z_1}{m_2 z_1 + m_1 z_2} \\
\frac{z_3}{z_1} &= \frac{z_2 - m_2 \cdot (z_1 + z_2)}{m_2}
\end{align*}
\]

(77)  
(78)

### 4.2 The Configurations with Coplanar Gears

Given the same tooth number of the gears on the duplex idler shafts in Sec. 3, namely \( z_2 = z_3, \quad z_1 = z_0 \), the configuration discussed here becomes the configuration of torque-split transmission with coplanar gears. This configuration has been fully studied with a different method (Vilán et al. and Abraham et al., 2010, 2012). So only the meshing condition, which is different from their works, is studied in this section.

#### 4.2.1 The External Meshing Coplanar Gears

The torque-split transmission with external meshing coplanar gears is shown in Fig. 14. With the same method in Sec. 3, the following equation is derived.

\[
\theta_1 + \theta_2 - \theta_3 - \theta_4 = (Z + k - z_3) \cdot 2\pi \tag{79}
\]

Replace the term of \( Z + k - z_3 \) with an integral number (N), the final form of Eq. (79) is as follows:

\[
\theta_1 + \theta_2 - \theta_3 - \theta_4 = N \cdot 2\pi \tag{80}
\]

When \( \theta_2 = 0 \), the integral numbers (Z and k) are equal to zero, and \( N = -z_3 \) is obtained. When \( \theta_2 = 2\pi \), the integral numbers are equal to \( z_3 \), and \( N = z_3 \) is obtained.

Fig. 14 The torque-split transmission with external meshing coplanar gears.

#### 4.2.2 The Internal Meshing Coplanar Gears

![Fig. 15 The torque-split transmission with internal meshing coplanar gears: (a) noncrossed gear center line; (b) crossed gear center line.](image)

The torque-split transmission with internal meshing coplanar gears is shown in Fig. 15. Eq. (79) becomes
\[ z_4 \theta_4 - z_1 \theta_1 - z_2 \left( \pi - \theta_2 \right) + z_3 \left( \pi \pm \theta_3 \right) = (Z - k) \cdot 2\pi \]  
(81)

where \( \pm (+, -) \) correspond to Fig. 15(a) and Fig. 15(b).

Replace the term of \( Z-k \) with an integral number (N), the final form of Eq. (81) is:

\[ z_4 \theta_4 - z_1 \theta_1 - z_2 \left( \pi - \theta_2 \right) + z_3 \left( \pi \pm \theta_3 \right) = N \cdot 2\pi \]  
(82)

4.3 The Concentric Face Gear

The torque-split transmission with concentric face gear is shown in Fig. 16(a). The two face gears arranged face to face one over the other. This arrangement has significant weight reduction and large reduction ratio. Its meshing condition is illustrated in Fig. 16(b), with \( z_2 = z_4, \ z_3 = z_5, \ \theta_2 = \pi \).

The meshing condition can be expressed as follows:

\[ \frac{\theta_1}{2\pi} Z + n = z_1 \]  
(83)

where \( Z \) is the integral pitch number between points F and H along the pitch circle of gear 4, and \( n \) is the integral pitch number of gear 3 from the first gear tooth (clockwise from \( O_1D \)) in clockwise direction.

4.3.1 Specific Instance: Equal Spaced Star Gears

For the configuration with cylindrical gears arranged around the face gear equally, Fig. 16(a), the following equation is derived.

\[ \theta_1 = 2\pi / q \]  
(84)

where \( q \) is the number of the cylindrical gears that are arranged around the face gear.

Imposing Eq. (84), the Eq. (83) becomes

\[ \frac{z_1}{q} = Z - n \]  
(85)

A meshing phase difference of \( \pi / z_1 \) between the upper and lower meshes of gear 1 is observed as the tooth number of the cylindrical gear is odd. While if the tooth number of the cylindrical gear is even, the meshing phase difference is zero. The meshing phase difference leads to the unsynchronized meshing stiffness of paths, and has influence on the dynamics and load sharing behaviors of the transmission system.

5. Conclusions

The torque-split gear transmission is mainly developed for the rotorcraft to achieve high power density and large reduction ratio. It reduces the contact force of gear teeth by dividing the force between several contact areas. This results in an increase of available torque and smaller gears. In this work, a universal mathematical design method of the torque-split gear transmission is proposed. Unlike other existing methods, this method is proved to be suitable for multiple types of torque-split gear transmission. This method is applied to the calculation of the torque-split transmission with duplex idler, planetary duplex gear, coplanar gear and concentric face gear. Numerous discrete solutions of the gear positions are observed. And these solutions of gear position correspond to different meshing phase difference between paths. The meshing phase difference, which has great influence on the dynamics and load sharing behaviors of the torque-split gear transmission, can be changed by adjusting the pitch number of pinion contained in the area formed by gears. Moreover, an even tooth number of the cylindrical gear of the concentric face gear transmission is recommended for the reason of minimizing the meshing phase difference.
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