Relatively hyperbolic groups with semistable fundamental group at infinity

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Abstract
Suppose $G$ is a 1-ended finitely generated group that is hyperbolic relative to $\mathbf{P}$, a finite collection of 1-ended finitely generated proper subgroups of $G$. Our main theorem states that if the boundary $\partial(G, \mathbf{P})$ has no cut point, then $G$ has semistable fundamental group at $\infty$. Under mild conditions on $G$ and the members of $\mathbf{P}$, the 1-ended hypotheses and the no cut point condition can be eliminated to obtain the same semistability conclusion. We give an example that shows our main result is somewhat optimal. Finally, we improve a ‘double dagger’ result of Dahmani and Groves.

1. Introduction

In this paper, we are interested in the asymptotic behavior of relatively hyperbolic groups. We consider a property of finitely presented groups that has been well studied for over 40 years called semistable fundamental group at $\infty$. A locally finite complex $Y$ has semistable fundamental group at $\infty$ if any two proper rays $r, s : [0, \infty) \to Y$ that converge to the same end of $Y$ are properly homotopic in $Y$. A finitely presented group $G$ has semistable fundamental group at $\infty$ if for some (equivalently any) finite complex $X$ with $\pi_1(X) = G$, the universal cover of $X$ has semistable fundamental group at $\infty$. (See Section 3 for several equivalent notions of semistability.) It is unknown at this time whether all finitely presented groups have semistable fundamental group at $\infty$, but in [27] the problem is reduced to considering 1-ended groups.

The finitely presented group $G$ satisfies a weaker condition called semistable first homology at $\infty$ if and only if $H^2(G : ZG)$ is free abelian (see [18]) and it is also a long standing problem if this is always the case. Our main interest is in showing certain relatively hyperbolic groups have semistable fundamental group at $\infty$. The work of Bowditch [5] and Swarup [32] shows that if $G$ is a 1-ended word hyperbolic group, then $\partial G$, the boundary of $G$, has no (global) cut point. Bestvina and Mess [2] show (Propositions 3.2 and 3.3) the absence of cut points in $\partial G$ implies $\partial G$ is locally connected. It was pointed out by Geoghegan that $G$ has semistable fundamental group at $\infty$ if and only if $\partial G$ has the shape of a locally connected continuum (see [16] for a proof of this fact). In particular, all 1-ended word hyperbolic groups have semistable fundamental group at $\infty$.

Relatively hyperbolic groups are a much studied generalization of hyperbolic groups. Semistability only makes sense for finitely generated groups so we are only interested in finitely generated groups $G$ and finite collections $\mathbf{P}$ of finitely generated subgroups. Later in this section and again in Section 4, we say what it means for a finitely generated group to be hyperbolic relative to a finite collection of finitely generated subgroups. If a finitely generated group $G$ is hyperbolic relative to a collection of finitely generated subgroups $\mathbf{P}$, the pair $(G, \mathbf{P})$ has a well-defined compact boundary. While all 1-ended hyperbolic groups have locally connected
boundary without cut points, the boundary of the relatively hyperbolic pair \((G,P)\) may be locally connected and contain cut points. At this time, there is no known example of a finitely generated relatively hyperbolic group with a connected boundary that is not locally connected. If the boundary \(\partial(G,P)\) is connected and locally connected, then by the Hahn-Mazurkiewicz Theorem (see [34, Theorem 31.5]) it is the continuous image of the interval \([0,1]\) and so it is path connected and locally path connected, but these facts seem to be of no use in our approach to showing \(G\) has semistable fundamental group at \(\infty\). The absence of cut points is critical. Our main result follows. Note that there is no semistability hypothesis on the peripheral subgroups \(P_i\).

**Theorem 1.1.** Suppose \(G\) is a 1-ended finitely generated group that is hyperbolic relative to a collection of 1-ended finitely generated proper subgroups \(P = \{P_1,\ldots,P_n\}\). If \(\partial(G,P)\) has no cut point, then \(G\) has semistable fundamental group at \(\infty\).

Example 2.10 indicates the importance of the no cut point hypothesis in our theorem. We explain that if one could eliminate that hypothesis, then the resulting theorem would ‘nearly’ imply that all finitely presented groups have semistable fundamental group at \(\infty\).

All of our work is done in a ‘cusped’ space for \((G,P)\). When this cusped space is Gromov hyperbolic, then the pair \((G,P)\) is said to be relatively hyperbolic or that \(G\) is hyperbolic relative to \(P\). This cusped space is a locally finite 2-complex on which \(G\) acts by isometries, but not co-compactly (see §4). It follows from ([7, §6 and §9]) that the Bowditch boundary for a relatively hyperbolic pair \((G,P)\) agrees with the Gromov boundary of a corresponding cusped space. Throughout the paper, this boundary is denoted \(\partial(G,P)\) and is called the **boundary of the relatively hyperbolic pair** \((G,P)\).

One of the main tools of our proof is a ‘double dagger’ result that appears as [11, Lemma 4.2]. This result is analogous to the original double dagger lemma of [2]. The key hypothesis in both of these lemmas is that the boundary of the hyperbolic space under consideration has no cut point. To use the double dagger result of [2] effectively, it is critical that it can be iterated. Lemma 4.2 of [11] begins with a path in a certain subspace \(X_m\) of a cusped space \(X\) and ‘replaces’ it with a path that may not have image in \(X_m\). Hence the lemma cannot be iterated. We are able to improve this result (see Theorem 7.1) to provide a replacement path in \(X_m\) (a subspace of \(X\) on which \(G\) acts cocompactly by isometries) and hence iteration is now possible, just as with the [2] result. The base space \(Y\) in \(X\) is a universal cover of a finite complex with fundamental group \(G\) and our goal in Theorem 1.1 is to show that \(Y\) (and hence \(G\)) has semistable fundamental group at \(\infty\). It is straightforward to show that \(Y\) has semistable fundamental group at \(\infty\) if and only if the neighborhood \(X_m\) of \(Y\) in \(X\) has semistable fundamental group at \(\infty\) (see Theorem 6.4). This is where the specialized double dagger replacement paths play a fundamental role in our proof of the main theorem.

The remainder of the paper is organized as follows: In §2, we describe a process for generalizing our main theorem from 1-ended groups \(G\) and 1-ended peripheral subgroups to infinite ended groups. We also describe how it may be possible to extend our theorem to the situation when \(\partial(G,P)\) has cut points. These generalizations depend on classical results in the literature. We also point out limitations to our techniques. The proof of our main theorem is independent of §2. In §3, we give several equivalent definitions of semistability that are used in our proofs. In §4, we define cusped spaces and extract three results from [20]. All of our technical work takes place in cusped spaces. The Main Lemmas 5.7 and 5.9 are proved in §5. Also an elementary, but critical observation, Lemma 5.5, about ‘dead ends’ in a general finitely generated group is proved. §6 contains the proof of the main theorem and a proof that the subspaces \(X_m\) of the cusped space \(X\) have semistable fundamental group at \(\infty\) (under the hypotheses of the main theorem). §7 contains an argument that
uses the main theorem and improves the double dagger result of Dahmani–Groves. Instead of producing ‘far out’ paths in the cusped space $X$, we produce ‘far out’ paths in $X_m$. Finally in §8, a result that seems far more general than one of our key lemmas (Lemma 5.7) is proved.

2. Related results, generalizations, limitations and speculation

If $G$ is hyperbolic relative to $P$, we call the members of $P$ peripheral subgroups of $(G, P)$. Any subgroup of a conjugate of a peripheral subgroup is called a parabolic subgroup. We review several results in the literature which, in conjunction with our main theorem, imply that in many cases we can drop the 1-ended hypotheses and the no cut point hypothesis and still obtain the same semistability conclusion. Theorem 2.6 is pivotal in these applications. There seems to be no umbrella statement covering all of our applications of Theorem 1.1, but we exhibit some complimentary results that seem interesting. In Example 2.10, we explain why a conjecture that all finitely presented relatively hyperbolic groups have semistable fundamental group at $\infty$ is well beyond the current state of progress. A conjecture of this nature is very close to the speculation that all finitely presented groups have semistable fundamental group at $\infty$. The following seems more reasonable and even feasible when $G$ is finitely presented. In the case $G$ is finitely presented, this conjecture reduces to one where $G$ and each $P_i$ is 1-ended (see Theorem 2.9).

Conjecture 2.1. Suppose $G$ is a finitely generated group that is hyperbolic relative to a finite collection $\{P_1, \ldots, P_n\}$ of proper finitely generated subgroups. If each $P_i$ has semistable fundamental group at $\infty$, then $G$ has semistable fundamental group at $\infty$.

Evidence for the conjecture appears in [28] where a first homology version of it is proved. A finitely presented group $G$ has semistable first homology at $\infty$ if and only if $H^2(G; \mathbb{Z}G)$ is free abelian. At this time, there is no known example of a connected boundary of a finitely generated relatively hyperbolic group that is not locally connected.

Theorem 2.2 [28, Theorem 1.1]. Suppose a finitely presented 1-ended group $G$ is hyperbolic relative to $P = \{P_1, \ldots, P_n\}$, a set of 1-ended finitely presented subgroups (with $G \neq P_i$ for all $i$). If the boundary $\partial(G, P)$ is locally connected and for each $i$, $H^2(P_i; \mathbb{Z}P_i)$ is free abelian, then $H^2(G; \mathbb{Z}G)$ is free abelian.

Note that there is no hypothesis on the number of ends of the $P_i$ and no local connectedness hypotheses on $\partial(G, P)$ in the next result.

Corollary 2.3 [28, Corollary 1.2]. Suppose a 1-ended finitely presented group $G$ is hyperbolic relative to $P = \{P_1, \ldots, P_n\}$, a set of finitely presented subgroups (with $G \neq P_i$ for all $i$). If for each $i$, $P_i$ contains no infinite torsion subgroup and $H^2(P_i; \mathbb{Z}P_i)$ is free abelian, then $H^2(G; \mathbb{Z}G)$ is free abelian.

While the next result certainly seems relevant to our conjecture and Main Theorem, it and the techniques used to prove the last two results seem insufficient to resolve the conjecture or even be of use in proving our Main Theorem.

Theorem 2.4 [28, Theorem 1.4]. Suppose a 1-ended finitely presented group $G$ is hyperbolic relative to $P = \{P_1, \ldots, P_n\}$, a set of 1-ended finitely presented subgroups (with $G \neq P_i$ for all $i$). If the boundary $\partial(G, P)$ is locally connected, then any cusped space for $(G, P)$ has semistable fundamental group at $\infty$. 


Next we turn our attention to $\infty$-ended groups. Assume the finitely generated group $G$ is hyperbolic relative to the finite collection of proper finitely generated subgroups $\mathbf{P}$. First we consider the situation where $G$ or some of the members of $\mathbf{P}$ are $\infty$-ended. Recall that a finitely generated group is accessible if it has a finite graph of groups decomposition with each edge group finite and each vertex group either finite or 1-ended. By Dunwoody’s accessibility theorem [15], all (almost) finitely presented groups are accessible. If $G$ is finitely generated, call a graph of groups decomposition of $G$ with each edge group finite and each vertex group either finite or 1-ended, a Dunwoody decomposition. If either $G$ or some member of $\mathbf{P}$ is not accessible, we have nothing to say. If the group $G$ is finitely presented and hyperbolic relative to a finite collection of proper finitely generated subgroups $P_i$, then the $P_i$ are finitely presented as well (see [12] or for a more general result [13, Theorem 2.11]). In the next theorem, we are interested in the case where either $G$ or some peripheral subgroup is $\infty$-ended.

**Theorem 2.5.** Suppose the finitely presented group $G$ is hyperbolic relative to a set \{\(P_1, \ldots, P_n\)\} of proper finitely generated subgroups. Let \(A\) be a Dunwoody decomposition of $G$ and \(A_i\) be a Dunwoody decomposition of $P_i$ for each $i$. Then each 1-ended vertex group $V_i$ of $A$ is hyperbolic relative to a finite collection $Q_i$ of 1-ended groups each of which is conjugate to a vertex group of some $A_j$. Furthermore, if for each 1-ended vertex group $V_i$, each member of $Q_i$ is a proper subgroup of $V_i$ and $\partial(V_i, Q_i)$ has no cut point, then $G$ has semistable fundamental group at $\infty$.

The proof of this theorem follows directly from our main theorem (Theorem 1.1) and the following list of theorems:

**Theorem 2.6.** Suppose $G$ is finitely generated and hyperbolic relative to the finitely generated groups $P_1, \ldots, P_n$. If $P_i$ is a finite graph of groups decomposition of $P_i$ such that each edge group of $P_i$ is finite, then $G$ is also hyperbolic relative to \{\(P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n\)\} $\cup$ $V(\mathcal{P}_i)$ where $V(\mathcal{P}_i)$ is the set of vertex groups of $\mathcal{P}_i$.

This last result can be derived from work of Drut¸u and Sapir [14] (Corollary 1.14) or Osin [30]. The next theorem is a result of Bowditch. A splitting of a group is relative to $\mathbf{P}$ if each element of $\mathbf{P}$ is conjugate into a vertex group of the splitting.

**Proposition 2.7** [7, Propositions 10.1-3]. Let $(G, \mathbf{P})$ be a relatively hyperbolic group. Its boundary $\partial(G, \mathbf{P})$ is disconnected if and only if $G$ splits nontrivially over a finite group relative to $\mathbf{P}$. In this case, every vertex group is hyperbolic relative to the parabolic subgroups it contains. In particular, if the parabolic subgroups of $G$ are all 1-ended, then $G$ is 1-ended if and only if $\partial(G, \mathbf{P})$ is connected.

Note that in the following combination result of Mihalik and Tschantz, there is no restriction on the number of ends of any of the groups involved.

**Theorem 2.8** [29]. Suppose $\mathcal{G}$ is a finite graph of groups decomposition of the finitely presented group $G$ where each edge group is finitely generated and each vertex group is finitely presented with semistable fundamental group at $\infty$, then $G$ has semistable fundamental group at $\infty$.

**Proof of Theorem 2.5.** By Theorem 2.6, we may replace the collection \{\(P_1, \ldots, P_n\)\} by $\mathbf{P}$, the vertex groups of the $\mathbf{A}_i$. If a peripheral subgroup is either finite or 2-ended, it may be removed from the collection of peripheral subgroups and $G$ remains hyperbolic relative
to the remaining subgroups. Hence we remove all finite and 2-ended members of $P$ and $G$ remains hyperbolic relative to the reduced collection $P$. Since each member of $P$ is 1-ended, the Dunwoody decomposition $\mathcal{A}$ of $G$ is relative to $P$. Proposition 2.7 implies that each 1-ended vertex group $V_i$ of $\mathcal{A}$ is hyperbolic relative to a finite collection $Q_i$ of 1-ended groups each of which is conjugate to a vertex group of some $A_j$. Finite groups trivially have semistable fundamental group at $\infty$. If each 1-ended $V_i$ is such that each member of $Q_i$ is a proper subgroup of $V_i$ and $\partial(V_i, Q_i)$ has no cut point, then by our Main Theorem 1.1, each 1-ended $V_i$ has semistable fundamental group at $\infty$ and so each vertex group of $\mathcal{A}$ has semistable fundamental group at $\infty$. By Theorem 2.8, $G$ has semistable fundamental group at $\infty$.

**Theorem 2.9.** If Conjecture 2.1 holds true for the case when $G$ and each $P_i$ is finitely presented and 1-ended (and each $P_i$ has semistable fundamental group at $\infty$), then the conjecture holds true in the more general setting where $G$ and each $P_i$ is finitely presented (with possibly more than 1-end), as long as the $P_i$ have semistable fundamental group at $\infty$.

**Proof.** Assume Conjecture 2.1 is true when $G$ and each $P_i$ is finitely presented and 1-ended. Consider the general case where $G$ is finitely presented. Again, if $G$ is finitely presented, then each $P_i$ is finitely presented [12]. Let $\mathcal{A}$ be a Dunwoody decomposition of $G$ and $\mathcal{A}_i$, a Dunwoody decomposition of $P_i$. Since $P_i$ has semistable fundamental group at $\infty$, each vertex group of $\mathcal{A}_i$ has semistable fundamental group at $\infty$ (see [27, Theorems 1–3]). Theorem 2.5 gives a Dunwoody decomposition $\mathcal{A}$ for $G$ such that each 1-ended vertex group $V_i$ of $\mathcal{A}$ is hyperbolic relative to a finite collection $Q_i$ of 1-ended groups, each of which is conjugate to a vertex group of some $A_j$. Hence each member of $Q_i$ is finitely presented and has semistable fundamental group at $\infty$. By our assumption, each $V_i$ has semistable fundamental group at $\infty$. By Theorem 2.8, $G$ has semistable fundamental group at $\infty$.

Next we consider a construction that illuminates how our approach breaks down if we try to show all relatively hyperbolic groups have semistable fundamental group at $\infty$.

**Example 2.10.** Suppose $B$ is a hyperbolic surface group with infinite cyclic subgroup $C$ and $A$ is an arbitrary finitely presented 1-ended group with infinite cyclic subgroup $C'$. The group $G = A *_{C' = C} B$ is a finitely presented 1-ended group. By ([10, Theorem 0.1]), $G$ is hyperbolic relative to $A$. The boundary $\partial(G, A)$ has a parabolic cut point (since $G$ splits over $C$).

It would be unreasonable to expect the theory of relatively hyperbolic groups to imply that the group $G$ of Example 2.10 has semistable fundamental group at $\infty$. The algebraic and geometric complexity of the subgroup $A$ is neutralized when one determines the hyperbolicity of the pair $(G, A)$. But no such neutralization occurs when the semistability of $G$ is at issue. In fact, it seems highly unlikely that $G = A *_{C' = C} B$ could be shown to have semistability fundamental group at $\infty$ without showing (the arbitrary group) $A$ does as well.

Using Bowditch’s results in an attempt to split $G$ in a hierarchical way, so that the final vertex groups would satisfy the hypothesis of our Main Theorem, would leave us with the vertex group $A$ which is hyperbolic relative to itself. Obviously, that is not enough to prove $A$ has semistable fundamental group at $\infty$. Under the hypothesis of Conjecture 2.1, we would assume that $A$ had semistable fundamental group at $\infty$. In that case, $G$ would have semistable fundamental group at $\infty$ by Theorem 2.8.

The proof of the next result by Hruska and Ruane is a process that begins with a group $G$ hyperbolic relative to a collection $P$. Each member of $P$ has semistable fundamental group at $\infty$ but $\partial(G, P)$ may have a cut point. The group $G$ is split in a ‘hierarchically accessible’ way so that the final vertex groups fit the hypothesis of our Main Theorem (that is, there are no
boundary cut points and all groups involved are 1-ended). This result deals with a number of splitting complexities and is not a simple application of our work here. It does support our expectations in Conjecture 2.1.

**Theorem 2.11** [21, Theorem 1.1]. Let \((G, \mathcal{P})\) be relatively hyperbolic with no noncentral element of order two. Assume each peripheral subgroup \(P \in \mathcal{P}\) is slender and coherent and all subgroups of \(P\) have semistable fundamental group at \(\infty\). Then \(G\) has semistable fundamental group at \(\infty\).

Hruska and Ruane point out that the only groups they are aware of that are slender and coherent are the virtually polycyclic groups (which have semistable fundamental group at \(\infty\) by [24]). Key ingredients in the proof of Theorem 2.11 include our Main Theorem 1.1, the combination Theorem 2.8 and a hierarchical accessibility theorem of Louder and Touikan (see [23]). As an immediate corollary they have:

**Corollary 2.12** [21, Corollary 1.3]. Let \(G\) act geometrically on a CAT(0) space \(X\) with isolated flats such that \(G\) has no noncentral element of order two. Then \(G\) has semistable fundamental group at \(\infty\).

CAT(0) groups with isolated flats are hyperbolic relative to their maximal abelian subgroups and this fact connects the corollary to the theorem. While it is known that a CAT(0) boundary of a 1-ended CAT(0) group cannot have a global cut point [31, 33], it is unknown if all CAT(0) groups are semistable at \(\infty\) (see [19]). Let \(F_2\) be the free group of rank 2. The CAT(0) group \(F_2 \times F_2\) is not hyperbolic with respect to any collection of proper subgroups (see [14]).

Hruska and Ruane discuss examples of CAT(0) groups with isolated flats in which arbitrarily deep hierarchies of peripheral splittings exist. These examples are important in understanding how our Main Theorem combines with results in the literature to produce semistability results. In particular, if cut points occur in boundaries of relatively hyperbolic groups, there are still options available to obtain semistability results.

Recall that a splitting is *relative to* \(\mathcal{P}\) if each element of \(\mathcal{P}\) is conjugate into a vertex group of the splitting.

**Definition 1** [6]. Let \((G, \mathcal{P})\) be a relatively hyperbolic group pair. A peripheral splitting of \(G\) is a representation of \(G\) as a finite bipartite graph of groups where \(\mathcal{P}\) consists precisely of the (conjugacy classes of) vertex groups of one color. A peripheral splitting is a refinement of another if there is a color-preserving folding of the first into the second.

The following organization of results of Bowditch appears in [11].

**Theorem 2.13** (B. Bowditch). Let \((G, \mathcal{P})\) be a relatively hyperbolic group pair. Assume that \(\partial(G, \mathcal{P})\) is connected.

1. [5, Theorem 0.2] If every maximal parabolic subgroup of \(G\) is (1 or 2)-ended, finitely presented, and without an infinite torsion subgroup, then, every global cut point of \(\partial(G, \mathcal{P})\) is a parabolic fixed point.
2. [4, Theorem 1.2] If there is a global cut point of \(\partial(G, \mathcal{P})\) that is a parabolic point, then \((G, \mathcal{P})\) admits a proper peripheral splitting.
3. [6, Theorem 1.2] If \((G, \mathcal{P})\) admits a proper peripheral splitting, then \(\partial(G, \mathcal{P})\) admits a global cut point.

It is established in [6, Theorem 1.3] that if \(\partial(G, \mathcal{P})\) is connected, then any nonperipheral vertex group of a peripheral splitting also has connected boundary and is hyperbolic relative
to its adjacent edge groups. The natural hyperbolic structure on vertex groups referred to in the next result might not consist solely of adjacent edge groups.

Bowditch also proves an accessibility result.

**Theorem 2.14** ([6, Theorem 1.4]). Suppose the 1-ended group $G$ is hyperbolic relative to $P$. Then $G$ splits over parabolic subgroups relative to $P$ as a finite graph of groups $\tilde{G}(G)$ with the following properties. Each vertex group $H$ inherits a natural relatively hyperbolic structure $(H, Q)$ such that $H$ does not split over a finite or parabolic subgroup relative to $Q$.

At this point, we will examine why combining Theorems 2.13 and 2.14 may not always put one in a position to apply our Main Theorem to the various vertex groups of the resulting decompositions of $G$ and then use the combination Theorem [29] to conclude that $G$ has semistable fundamental group at $\infty$.

Based on ideas developed by Haulmark, Hruska and Osin, it should be possible to construct examples of finitely presented groups that decompose according to Bowditch’s accessibility result (Theorem 2.14) with vertex groups that are finitely generated and not finitely presented. A brief outlining of their idea follows. For any (finitely generated but not finitely presented) recursively presented group $Q$ and finitely presented group $P$ containing a subgroup isomorphic to $Q$, one can use [1, Theorem 1.1] to construct a finitely generated relatively hyperbolic group $A$ which contains $Q$ as a maximal parabolic subgroup. Since $Q$ is not finitely presented neither is $A$ [12]. Their idea is to then use [3, Corollary 1.5] to show $G = A * Q P$ is finitely presented and hyperbolic relative to $P$.

In this type of situation, the finitely generated version of our main theorem might come into play in an attempt to show that the finitely presented group $G$ has semistable fundamental group at $\infty$. It could also be the case that $Q$ is not accessible and our techniques would break down even if $P$ had semistable fundamental group at $\infty$.

If every vertex group in the splitting of Theorem 2.14 has only proper 1-ended peripheral subgroups, then those vertex groups satisfy the hypotheses of our Main Theorem by [6, Proposition 5.2] and $G$ has semistable fundamental group at $\infty$ (by Theorem 2.8). Some care must be taken here. Examples of relatively hyperbolic groups $(G, P)$ are described in [21] where a vertex group $V$ given by Theorem 2.14 does not satisfy the hypotheses of our Main Theorem. The ‘natural’ relative hyperbolic structure $Q$ on $V$ ensured by Theorem 2.14 contains a 2-ended peripheral subgroup even though all of the members of $P$ are 1-ended. While $\partial(V, Q)$ contains no cut point, if $Q'$ is obtained from $Q$ by removing the 2-ended peripheral subgroup, then $\partial(V, Q')$ has a cut point. So in either situation, our Main Theorem cannot be applied. Even so, reapplying Theorem 2.14 to vertex groups eventually (hierarchically) leads to vertex groups covered by our Main Theorem and so Theorem 2.8 implies the original groups have semistable fundamental group at $\infty$.

### 3. Semistability background

The best reference for the notion of semistable fundamental group at $\infty$ is [17] and we use this book as a general reference throughout this section. While semistability makes sense for multiple ended spaces, for the most part, we are only interested in 1-ended spaces in this article. Suppose $K$ is a locally finite connected CW complex. A ray in $K$ is a continuous map $r : [0, \infty) \to K$. A continuous map $f : X \to Y$ is proper if for each compact set $C$ in $Y$, $f^{-1}(C)$ is compact in $X$. Proper rays $r, s : [0, \infty) \to K$ converge to the same end if for any compact set $C$ in $K$, there is an integer $k(C)$ such that $r([k, \infty))$ and $s([k, \infty))$ belong to the same component of $K - C$. The space $K$ has semistable fundamental group at $\infty$ if any
two proper rays in $K$ that converge to the same end are properly homotopic. Note that when $K$ is 1-ended, this means that $K$ has semistable fundamental group at $\infty$ if any two proper rays in $K$ are properly homotopic. Suppose $C_0, C_1, \ldots$ is a collection of compact subsets of a locally finite 1-ended complex $K$ such that $C_i$ is a subset of the interior of $C_{i+1}$ and $\cup_{i=0}^{\infty} C_i = K$, and $r : [0, \infty) \to K$ is proper, then $\pi_1^\infty(K, r)$ is the inverse limit of the inverse system of groups:

$$\pi_1(K - C_0, r) \leftarrow \pi_1(K - C_1, r) \leftarrow \cdots$$

This inverse system is pro-isomorphic to an inverse system of groups with epimorphic bonding maps if and only if $K$ has semistable fundamental group at $\infty$ (see [24, Theorem 2.1] or [17, Theorem 16.1.2]). It is an elementary exercise to see that semistable fundamental group at $\infty$ is an invariant of proper homotopy type and Brick [8] proved that semistability is a quasi-isometry invariant. When $K$ is 1-ended with semistable fundamental group at $\infty$, $\pi_1^\infty(K, r)$ is independent of proper base ray $r$ (in direct analogy with the fundamental group of a path connected space being independent of base point). Theorem 2.1 of [24] and [25, Lemma 9] provide several equivalent notions of semistability. The space considered in [24] is simply connected, but simple connectivity is not important in that argument. A slight modification of proofs gives the following result. Condition 2 is the semistability criterion used in the proof of our main theorem.

**Theorem 3.1** [9, Theorem 3.2]. Suppose $K$ is a connected 1-ended locally finite CW-complex. Then the following are equivalent.

1. $K$ has semistable fundamental group at $\infty$.
2. Suppose $r : [0, \infty) \to K$ is a proper base ray. Then for any compact set $C$, there is a compact set $D(C, r)$ such that for any third compact set $E$ and loop $\alpha$ based on $r$ and with image in $K - D$, $\alpha$ is homotopic rel $\{r\}$ to a loop in $K - E$, by a homotopy with image in $K - C$.
3. For any compact set $C$, there is a compact set $D(C)$ such that if $r$ and $s$ are proper rays based at $v$ and with image in $K - D$, then $r$ and $s$ are properly homotopic rel $\{v\}$, by a proper homotopy in $K - C$.

If $G$ is a finitely presented group and $X$ is a finite connected complex with $\pi_1(X) = G$, then $G$ has **semistable fundamental group at $\infty$** if the universal cover of $X$ has semistable fundamental group at $\infty$. This definition only depends on $G$ (see the proof of [22, Theorem 3] or the opening paragraph of [17, section 16.5]) and it is unknown if all finitely presented groups have semistable fundamental group at $\infty$.

In order to prove certain finitely presented groups have semistable fundamental group at $\infty$, the notion of semistable fundamental group at $\infty$ for a finitely generated group was introduced in [26]. Our main interest here is also finitely presented groups, but we might begin with a finitely presented group $G$, apply a decomposition theorem and end up with finitely generated vertex groups that are not finitely presented. In order to show that $G$ has semistable fundamental group at $\infty$, we might need to show some finitely generated group has semistable fundamental group at $\infty$. Now we say what it means for a finitely generated group to have semistable fundamental group at $\infty$. Suppose $S$ is a finite generating set for a finitely generated group $G$. Let $\Gamma(G, S)$ be the Cayley graph of $(G, S)$. If there is a finite set of relations $R$ of $G$ such that the space resulting from attaching 2-cells to $\Gamma$ (one at each vertex for each $R \in R$) produces a space that has semistable fundamental group at $\infty$, then $G$ is said to have **semistable fundamental group at $\infty$**. We first prove our main theorem in the finitely presented case. An elementary adjustment provides the finitely generated version.
4. Cusped spaces and relatively hyperbolic groups

Given a finitely generated group $G$ and a collection of finitely generated subgroups $P$ of $G$, there are a number of equivalent definitions of what it means for the pair $(G, P)$ to be relatively hyperbolic or $G$ to be relatively hyperbolic with respect to $P$. Theorem 4.2 enables us to say the pair $(G, P)$ is relatively hyperbolic if a certain cusped space is Gromov hyperbolic, so we take this as our definition. The Gromov boundary of this cusped space is the boundary of the pair $(G, P)$ and is denoted $\partial(G, P)$. As noted in Section 1, this boundary agrees with the Bowditch boundary of the pair $(G, P)$.

Groves and Manning [20] investigate a locally finite space $X$ derived from a finitely generated group $G$ and a collection $P$ of finitely generated subgroups. The following definitions are directly from [20].

**Definition 2.** Let $\Gamma$ be any 1-complex. The combinatorial horoball based on $\Gamma$, denoted $\mathcal{H}(\Gamma)$, is the 2-complex formed as follows.

(A) $\mathcal{H}(0) = \Gamma(0) \times \{0\} \cup \mathbb{N}$.

(B) $\mathcal{H}(1)$ contains the following three types of edges. The first two types are called horizontal, and the last type is called vertical.

(B1) If $e$ is an edge of $\Gamma$ joining $v$ to $w$, then there is a corresponding edge $\bar{e}$ connecting $(v,0)$ to $(w,0)$.

(B2) If $k > 0$ and $0 < d_\Gamma(v,w) \leq 2^k$, then there is a single edge connecting $(v,k)$ to $(w,k)$.

(B3) If $k \geq 0$ and $v \in \Gamma(0)$, there is an edge joining $(v,k)$ to $(v,k+1)$.

(C) $\mathcal{H}(2)$ contains three kinds of 2-cells.

(C1) If $\gamma \subset \mathcal{H}(1)$ is a circuit composed of three horizontal edges, then there is a 2-cell (a horizontal triangle) attached along $\gamma$.

(C2) If $\gamma \subset \mathcal{H}(1)$ is a circuit composed of two horizontal edges and two vertical edges, then there is a 2-cell (a vertical square) attached along $\gamma$.

(C3) If $\gamma \subset \mathcal{H}(1)$ is a circuit composed of three horizontal edges and two vertical ones, then there is a 2-cell (a vertical pentagon) attached along $\gamma$, unless $\gamma$ is the boundary of the union of a vertical square and a horizontal triangle.

**Definition 3.** Let $\Gamma$ be a graph and $\mathcal{H}(\Gamma)$ the associated combinatorial horoball. Define a depth function

$$D : \mathcal{H}(\Gamma) \to [0, \infty),$$

which satisfies:

1. $D(x) = 0$ if $x \in \Gamma$;
2. $D(x) = k$ if $x$ is a vertex $(v,k)$; and
3. $D$ restricts to an affine function on each 1-cell and on each 2-cell.

**Definition 4.** Let $\Gamma$ be a graph and $\mathcal{H} = \mathcal{H}(\Gamma)$ the associated combinatorial horoball. For $n \geq 0$, let $\mathcal{H}_n \subset \mathcal{H}$ be the full sub-graph with vertex set $\Gamma(0) \times \{0, \ldots, N\}$, so that $\mathcal{H}_n = D^{-1}[0,n]$. Let $\mathcal{H}' = D^{-1}[n, \infty)$ and $N(n) = D^{-1}(n)$. The set $\mathcal{H}(n)$ is often called a horosphere or $n$th level horosphere.

**Lemma 4.1 [20, Lemma 3.10].** Let $\mathcal{H}(\Gamma)$ be a combinatorial horoball. Suppose that $x, y \in \mathcal{H}(\Gamma)$ are distinct vertices. Then there is a geodesic $\gamma(x,y) = \gamma(y,x)$ between $x$ and $y$ which consists of at most two vertical segments and a single horizontal segment of length at most 3.
Moreover, any other geodesic between \( x \) and \( y \) is Hausdorff distance at most 4 from this geodesic.

**Definition 5.** Let \( G \) be a finitely generated group, let \( \mathbf{P} = \{P_1, \ldots, P_n\} \) be a \( (\text{finite}) \) family of finitely generated subgroups of \( G \) and let \( S \) be a generating set for \( G \) containing generators for each of the \( P_i \). For each \( i \in \{1, \ldots, n\} \), let \( T_i \) be a left transversal for \( P_i \) (that is, a collection of representatives for left cosets of \( P_i \) in \( G \) which contains exactly one element of each left coset).

For each \( i \) and each \( t \in T_i \), let \( \Gamma_{i,t} \) be the full subgraph of the Cayley graph \( \Gamma(G, S) \) which contains \( tP_i \). Each \( \Gamma_{i,t} \) is isomorphic to the Cayley graph of \( P_i \) with respect to the generators \( P_i \cap S \). Then define

\[
X(G, \mathbf{P}, S) = \Gamma(G, S) \cup \bigcup \{ \mathcal{H}(\Gamma_{i,t})^{(1)} \, | \, 1 \leq i \leq n, t \in T_i \},
\]

where the graphs \( \Gamma_{i,t} \subset \Gamma(G, S) \) and \( \Gamma_{i,t} \subset \mathcal{H}(\Gamma_{i,t}) \) are identified in the obvious way.

The space \( X(G, \mathbf{P}, S) \) is called the cusped space for \( G, \mathbf{P} \) and \( S \). If \( G \) has a finite presentation \( \mathcal{A} = (S; R) \), we add 2-cells to \( \Gamma(G, S) \) to form the Cayley 2-complex of this presentation. The resulting expansion of \( X(G, \mathbf{P}, S) \) is called the cusped space for \( G, \mathbf{P} \) and \( \mathcal{A} \) and is denoted \( X(G, \mathbf{P}, \mathcal{A}) \). The next result shows cusped spaces are fundamentally important spaces. We prove our results in cusped spaces.

**Theorem 4.2** [20, Theorem 3.25]. Suppose that \( G \) is a finitely generated group and \( \mathbf{P} = \{P_1, \ldots, P_n\} \) is a \( (\text{finite}) \) collection of finitely generated subgroups of \( G \). Let \( S \) be a finite-generating set for \( G \) containing generating sets for the \( P_i \). A cusped space \( X(G, \mathbf{P}, S) \) is hyperbolic if and only if \( G \) is hyperbolic with respect to \( \mathbf{P} \).

Assume \( G \) is finitely presented and hyperbolic with respect to the subgroups \( \mathbf{P} = \{P_1, \ldots, P_n\} \) and \( S \) is a finite-generating set for \( G \) containing generating sets for the \( P_i \). For a finite presentation \( \mathcal{A} \) of \( G \) with respect to \( S \), let \( Y(\mathcal{A}) \) be the Cayley 2-complex for \( \mathcal{A} \). So \( Y \) is simply connected with 1-skeleton \( \Gamma(G, S) \), and the quotient space \( G/Y \) has fundamental group \( G \). The cusped space \( X(G, \mathbf{P}, S) \) is quasi-isometric to the cusped space \( X(G, \mathbf{P}, \mathcal{A}) \) and so one is hyperbolic if and only if the other is hyperbolic, and these two spaces have the same boundary. For \( g \in G \) and \( i \in \{1, \ldots, n\} \), we call \( gP_i \) a peripheral coset in a cusped space. The depth functions on the horoballs over the peripheral cosets extend from \( X(G, \mathbf{P}, S) \) to \( X(G, \mathbf{P}, \mathcal{A}) \). So that

\[
\mathcal{D} : X(G, \mathbf{P}, \mathcal{A}) \to [0, \infty),
\]

where \( \mathcal{D}^{-1}(0) = Y \) and for each horoball \( H \) (over a peripheral coset), we have \( H \cap \mathcal{D}^{-1}(m) = H(m), H \cap \mathcal{D}^{-1}[0, m) = H_m \) and \( H \cap \mathcal{D}^{-1}[m, \infty) = H^m \). We call each \( H^m \) an \( m \)-horoball.

**Lemma 4.3** [20, Lemma 3.26]. If a cusped space \( X \) is \( \delta \)-hyperbolic, then the \( m \)-horoballs of \( X \) are convex for all \( m \geq \delta \).

Given two points \( x \) and \( y \) in a horoball \( H \), there is a shortest path in \( H \) from \( x \) to \( y \) of the form \((\alpha, \tau, \beta)\) where \( \alpha \) and \( \beta \) are vertical and \( \tau \) is horizontal of length \( \leq 3 \). Note that if \( \alpha \) is nontrivial and ascending and \( \beta \) is nontrivial and descending, then \( \tau \) has length either 2 or 3.

If \( Y(\mathcal{A}) \) is the Cayley 2-complex for the finite presentation \( \mathcal{A} \) of the group \( G \), then the isometric action of \( G \) on \( Y \) extends to an isometric action of \( G \) on \( X(G, \mathbf{P}, \mathcal{A}) \). This action is height preserving.
LEMMA 4.4. Let $X(G, P, A)$ be a cusped space. For $i \in \{1, 2\}$, let $z_i$ be vertices of the horosphere $H_i(m)$ where $H_i$ is a horoball over the peripheral coset $g_iP_j$. Then there is an element $g \in G$ such that $gz_1 = z_2$. Furthermore, $gH_1 = H_2$.

Proof. For $j \in \{1, 2\}$, let $\alpha_j$ be the vertical path of length $m$ in $X$ from $z_j$ to the level-0 point $y_j \in g_j(P_j) \subset Y(A)$. The element of $g \in G$ that takes the vertex $y_1$ to $y_2$ maps $\alpha_1$ to $\alpha_2$ and $g_1P_i$ to $g_2P_i$. Hence $gz_1 = gz_2$ and $gH_1 = H_2$. \hfill \Box

5. The main lemmas

In this section, we prove the two lemmas (5.7 and 5.9) that provide the combinatorial base of our semistability proof.

Again assume that $G$ is a 1-ended finitely presented group, hyperbolic relative to $P$, a finite collection of proper 1-ended subgroups of $G$. Let $A = \langle S; R \rangle$ be a finite presentation for $G$ where $S$ contains generators for each member of $P$. Let $Y$ be the Cayley 2-complex for $A$, $\ast$ the identity vertex of $Y \subset X \equiv X(G, P, A)$ and $d$ the edge path distance in $X$. If $H$ is a horoball, and $x, y$ elements of the horosphere $H(m)$, then let $d^m(x, y)$ be the length of the shortest edge path from $x$ to $y$ in $H(m)$. For $z \in H(m)$, let $\hat{B}_n(z)$ be the set of all vertices $v$ of $H(m)$ such that $d^m(v, z) \leq n$.

LEMMA 5.1. Suppose $t_1$ and $t_2$ are vertices of depth $d \geq \delta$ in a horoball $H$ of $X$. Then for each $i \in \{1, 2\}$, there is a geodesic $\gamma_i$ from $\ast$ to $t_i$ such that $\gamma_i$ has the form $(\eta_i, \alpha_i, \tau_i, \beta_i)$, where the end point $x_i$ of $\eta_i$ is the first point of $\gamma_i$ in the horosphere $H(d)$, $\alpha_i$ and $\beta_i$ are vertical and of the same length in $H^d$ and $\tau_i$ is horizontal of length $\leq 3$. Furthermore $d(x_1, x_2) \leq 2\delta + 1$.

Proof. The proof of the first part follows directly from Lemma 4.3. For the last part, simply consider the geodesic triangle formed by the geodesics $\eta_1, \eta_2$ and the geodesic between $x_1$ and $x_2$. The two vertical segments of this last geodesic cannot be longer than $\delta - 1$ since the triangle is $\delta$-thin. \hfill \Box

We call a path of the form $(\eta_i, \alpha_i, \tau_i, \beta_i)$ a $D(t_i)$-standard geodesic from $\ast$ to $t_i$. The next lemma basically says if $D(t) = d \geq \delta$, $H$ is a horoball containing $t$ and $z$ is a closest point of $H(d)$ to $\ast$, then the $(X)$ distance $d(t, \ast)$ is approximately the distance from $\ast$ to $z$ plus the distance (in $H^d$) from $z$ to $t$.

LEMMA 5.2. Suppose $t$ is a vertex of a horoball $H$ of $X$ such that $D(t) = d \geq \delta$. Let $z$ be a point of the horosphere $H(d)$ closest to $\ast$. Then

$$d(\ast, t) \leq d(\ast, z) + d(z, t) \leq 2\delta + 1 + d(\ast, t).$$

Proof. The first inequality follows from the triangle inequality. Let $(\eta, \alpha, \tau, \beta)$ be a $d$-standard geodesic from $\ast$ to $t$. Let $p$ be the end point of $\eta$. By Lemma 5.1, $d(z, p) \leq 2\delta + 1$ and so by the triangle equality

$$d(z, t) \leq d(z, p) + d(p, t) \leq 2\delta + 1 + d(p, t).$$

Also, $d(\ast, z) \leq d(\ast, p)$. Adding inequalities,

$$d(\ast, z) + d(z, t) \leq d(\ast, p) + 2\delta + 1 + d(p, t) = 2\delta + 1 + d(\ast, t).$$

The next two results follow directly from the definition of a cusped space.
Lemma 5.3. Suppose $z$ and $t$ are vertices of a horosphere $H(m)$ ($m \geq \delta$) of $X$, and $d^m(z,t) \leq 2n$. Then $d(z,t) \leq 2n$, and so $B_{2n}(z) \subset B_{2n}(z)$.

Lemma 5.4. Suppose $z$ and $t$ are vertices of a horosphere $H(m)$ ($m \geq \delta$), and $d(z,t) \leq 2n + 3$ (respectively; $2n + 2$). Then $d^m(z,t) \leq 3(2^n)$ (respectively, $2^{n+1}$), and so $B_{2n+3}(z) \cap H(m) \subset B_{3(2^n)}(z)$ (respectively, $B_{2n+2}(z) \cap H(m) \subset B_{2^{n+1}}(z)$). In particular, for any integer $k > 0$, $B_k(z) \cap H(m) \subset B_{2^{k+1}}(z)$.

Lemma 5.5. Suppose $G$ is a finitely generated group with finite generating set $S$. Let $\Gamma$ be the Cayley graph of $G$ with respect to $S$. Then for any integer $r > 0$ and any point $x \in \Gamma - B_{2r}(*),$ there is a geodesic ray at $x$ which avoids $B_r(*)$.

Proof. Let $L$ be a geodesic line in $\Gamma$. By translation, we may assume $x$ is a vertex of $L$. Now consider the two geodesic rays $q^+$ and $q^-$ in opposite directions on $L$, both with initial point $x$. If both $q^+$ and $q^-$ intersect $B_r(*)$ at $x^+$ and $x^-$, respectively, then since $x \notin B_{2r}(*)$, both of $x^+$ and $x^-$ are of distance at least $r + 1$ from $x$. This means the distance from $x^+$ to $x^-$ is at least $2r + 2$. But, since both belong to $B_r(*)$, they are at most distance $2r$ from one another — the desired contradiction.

Suppose $x, y \in X$ are points of a horosphere $H(k)$. Any path (without backtracking) in the $k$-horoball $H^k$ between $x$ and $y$ can be written as

$$\psi = (\tau_0, \alpha_1, \tau_1, \alpha_2, \tau_2, \ldots, \alpha_n, \tau_n),$$

where for each $i$, $\alpha_i$ is nontrivial and vertical and $\tau_i$ is horizontal. With the possible exception of $\tau_0$ and $\tau_n$, $\tau_i$ is nontrivial. If $\tau_1$ is the edge path $(e_1,\ldots,e_m)$, let $\alpha_{1,j}$ be the vertical path from the end point of $e_j$ to the horosphere $H(k) \subset H^k$. Also let $\alpha^{-1}_i = \alpha_{i,0}$. Next let $\gamma_{1,j}$ be a shortest edge path in $H(k)$ from the end point of $\alpha_{1,j-1}$ to the end point of $\alpha_{1,j}$ for all $j \in \{1,\ldots,m\}$. Let $\gamma_1 = (\gamma_{1,1},\ldots,\gamma_{1,m})$ and note that for each vertex $v$ of $\gamma_1$ the vertical edge path of length $|\alpha_1|$ is within 1 (horizontal) unit of a vertex of one of the $e_i$. For each $\tau_i$, similarly construct $\gamma_i$. The path $\gamma = (\gamma_1,\ldots,\gamma_n)$ is a projection of $\psi$ into $H(k)$. Thus we have:

Lemma 5.6. Suppose $H$ is a horoball in $X$, $x$ and $y$ are points in the horosphere $H(k)$ and

$$\psi = (\alpha_1, \tau_1, \alpha_2, \tau_2, \ldots, \alpha_n, \tau_n, \alpha_{n+1})$$

is a path in $H^k$ between $x$ and $y$, where for each $i$, $\alpha_i$ is nontrivial and vertical and $\tau_i$ is nontrivial and horizontal. If $\gamma$ is a projection of $\psi$ into $H(k)$, then each vertical line at a vertex of $\gamma$ passes within 1 horizontal unit of a vertex of one of the $\tau_i$.

The next lemma combines with Lemma 5.2 to show that if $\psi$ is an edge path in $X - B_r(*)$, has image in $H^d (d \geq \delta)$ for some horoball $H$, begins and ends in $H(\overline{d})$ and the image of $\psi$ is ‘far’ from a vertex of $H(\overline{d})$ closest to $*$, then $\psi$ projects to a path in $H(\overline{d})$ that avoids $B_{r-2\delta}(*)$. The subsequent Lemma 5.9 shows that if $\psi$ has a vertex ‘close’ to a vertex of $H(\overline{d})$ closest to $*$, then $\psi$ can be replaced by a path in $H(\overline{d})$ that avoids $B_{r-2\delta}(*)_\delta$. These are the two main lemmas of the paper. In the proof of our main theorem, we only need these lemmas with $d = \delta$. In the proof of Theorem 7.1, we will need the general version of these lemmas. Lemma 5.7 has a variation (Lemma 8.1) that is in some sense stronger and of separate interest. We only need the version that immediately follows in order to prove our main theorem, and so we delay the introduction of Lemma 8.1 to §8.
Lemma 5.7. Suppose $H$ is a horoball, $\bar{d}$ is an integer $\geq \delta$, $x \neq y$ are vertices of the horosphere $H(\bar{d})$, $z$ is a closest vertex of $H(\bar{d})$ to $*$, $\psi$ is a path in $H^\delta - B_r(*)$ between $x$ and $y$ that only intersects $H(\bar{d})$ at $x$ and $y$ and $L = |\psi|$.

If $\gamma$ is a projection of $\psi$ to $H(\bar{d})$ and each vertex $v$ of $\psi$ is such that $d(v, z) > L + 2$, then each vertex of $\gamma$ is at distance greater than $L + 2$ from $z$. Furthermore, the image of $\gamma$ avoids $B_r - (2\delta + 1)(*)$.

Proof. By Lemma 4.3, $H^\delta$ is convex. Let $p$ be a vertex of the projection $\gamma$ and $(\alpha_p, \tau_p, \beta_p)$ a geodesic (in $H^\delta$) from $z$ to $p$ where $\alpha_p$ and $\beta_p$ are vertical of the same length and $|\tau_p| \leq 3$. Let $y$ be the end point of $\tau_p$. By Lemma 5.6, there is a vertical segment that begins at $p$ and ends at a vertex $w$ which is at most 1 (horizontal) unit from a vertex $v$ of $\psi$ (and $D(v) \geq \bar{d} + 1$). The vertical line at $p$ contains $y$ and $w$. There are two cases. If $y$ is closer to $p$ than is $w$, (Figure 1(1)), then there is a path $\rho$ from $v$ to $z$ that begins with a horizontal edge from $v$ to $w$ on the vertical segment, followed by a vertical segment from $w$ to $y$, followed by $\tau_p^{-1}$, followed by $\alpha_p^{-1}$.

Since $\psi$ begins and ends in $H(\bar{d})$ has image in $H^\delta$ and has length $L$, at most $\frac{L-1}{2}$ vertical edges are required to go from a vertex of $\psi$ down to $H(\bar{d})$. In particular, the sum of the lengths of the two vertical segments of $\rho$ is less than or equal to $\frac{L-1}{2}$ (since $v$ is on $\psi$), and so

$$L + 2 < d(v, z) \leq \frac{L - 1}{2} + 4 = \frac{L}{2} + \frac{7}{2}$$

implying

$$2L + 4 < L + 7 \text{ and } L < 3,$$

which is impossible since $x \neq y$.

Instead, the vertical line segment from $p$ to $y$ contains a vertex $w$ (other than $y$) within 1 horizontal unit of a vertex $v$ of $\psi$ (Figure 1(2)). Now

$$d(z, p) = d(p, w) + d(w, y) + |\tau_p| + |\alpha_p| \text{ and }$$

$$L + 2 < d(v, z) \leq 1 + d(w, y) + |\tau_p| + |\alpha_p|.$$ 

Since $d(p, w) \geq 1$, $d(z, p) > L + 2$, completing the first part of the lemma.
Since the depth of $v$ (and hence the depth of $w$) is at least $\tilde{d} + 1$,
\[ d(p, z) \geq 1 + d(w, z) \geq d(v, z). \]
Combining this inequality with Lemma 5.2 and the triangle inequality, we have for each vertex $p$ of $\gamma$:
\[ d(p, \ast) + 2\delta + 1 \geq d(p, z) + d(z, \ast) \geq d(v, z) + d(z, \ast) \geq d(v, \ast) > r \]
So $d(p, \ast) > r - 2\delta - 1$.

Remark 5.8. Suppose $H$ is a horoball, $\tilde{d} \geq \delta$ is an integer and $\psi$ is a path in $H^{\tilde{d}} - B_r(\ast)$ connecting distinct points $x$ and $y$ of the horosphere $H(\tilde{d})$ such that $\psi$ only meets $H(\tilde{d})$ at its initial and end point. Let $L = |\psi|$ and $z$ be a closest vertex of $H(\tilde{d})$ to $\ast$. If each vertex $v$ of $\psi$ is such that $d(v, z) > L + 2$, then $\psi$ satisfies the hypothesis of Lemma 5.7. Otherwise, some vertex $v$ of $\psi$ is such that $d(v, z) \leq L + 2$ and $\psi$ satisfies the hypothesis of Lemma 5.9. In any case, each such $\psi$ satisfies the hypothesis (and hence the conclusion) of either Lemma 5.7 or 5.9.

Our proof of Lemma 5.9 heavily depends on the integer $k = r - d(z, \ast)$ (so that $r = d(z, \ast) + k$). Note that for any $v \in B_{L+2}(z)$:
\[ d(v, \ast) \leq d(v, z) + d(z, \ast) \leq L + 2 + d(z, \ast) = L + 2 + r - k \]
So if $k \geq L + 2$, then $d(v, \ast) \leq r$ and $B_{L+2}(z) \subseteq B_r(\ast)$. Since $\psi$ has image in $X - B_r(\ast)$, each vertex $v$ of $\psi$ is such that $d(v, z) > L + 2$. Hence if some vertex $v$ of $\psi$ is within $L + 2$ of $z$, (the case considered in Lemma 5.9) it must be that $k < L + 2$.

Lemma 5.9. Let $L > 0$ and $\tilde{d} \geq \delta$ be fixed integers. Then there is an integer $F_{5.9}(L, \tilde{d})$ satisfying the following. Suppose:

1. $x$ and $y$ are vertices of $H(\tilde{d}) - B_r(\ast)$ for some horoball $H$ of $X$;
2. $\psi$ is an edge path of length $\leq L$ in $H^{\tilde{d}} - B_r(\ast)$ between $x$ and $y$ whose image intersects the horosphere $H(\tilde{d})$ only at $x$ and $y$; and
3. some vertex $v$ of $\psi$ is such that $d(v, z) \leq L + 2$ where $z$ is a closest point of $H(\tilde{d})$ to $\ast$;

then there is an edge path in $H(\tilde{d})$ of length $\leq F_{5.9}(L, \tilde{d})$ from $x$ to $y$ that avoids $B_{r-(2\delta+5)}(\ast)$.

Proof. The proof is somewhat technical and an outline is helpful. It is critical that our peripheral subgroups are 1-ended in this lemma. First observe that by the discussion following Remark 5.8, $k = r - d(z, \ast) < L + 2$. When $k < 0$, it is elementary to show there are paths of length $\leq 2^{L-1}$ satisfying the conclusion of the lemma. The more difficult case occurs when $k \in \{0, 1, \ldots, L + 1\}$. Note that in this case, only $L + 2$ integers $r = k + d(z, \ast)$ occur for any particular horosphere. For a fixed horosphere $H(\tilde{d})$ satisfying the hypothesis of our lemma, and each $k \in \{0, 1, \ldots, L + 1\}$, we produce constants $F_k$ so that the conclusion of the lemma is satisfied (for this $H(\tilde{d})$) with $F_{5.9}(L, \tilde{d})$ equal to $F$, the largest of the $F_k$. Next we show that for any $G$-translate of $H(\tilde{d})$, the same constant $F$ satisfies the conclusion of the lemma. This is not a simple matter. The action of $G$ on the cusped space $X$ can certainly be used to translate certain horospheres $H(\tilde{d})$ to one another, but this action moves the base point $\ast$ and the conclusion of our lemma requires paths in the compliment of a ball based at $\ast$. Fortunately, the hyperbolic structure on $X$ gives the leverage needed to overcome this issue by utilizing the fact that geodesics from $\ast$ to a point of any horosphere must pass near a closest point of the horosphere to $\ast$. Lemmas 5.3 and 5.4 allow us to translate metric ball information based at $\ast$ to metric ball information based at closest points of a horosphere to $\ast$. The group action translating a closest point (to $\ast$) of one horosphere to a closest point (to $\ast$) of another
horosphere is then used to show that the constant $F$ satisfying the conclusion of our lemma for $H(\hat{d})$ will also satisfy the conclusion of our lemma for all translates of $H(d)$ in $X$. Since there are only finitely many level $\bar{d}$ horospheres up to translation (one for each peripheral subgroup), we can simply pick $F_{3,\bar{d}}(L, \bar{d})$ to be the largest of a finite set of integers.

We begin with the inequality $k = r - d(z, *) < L + 2$. Since $|\psi| = L$ and $d(v, z) \leq L + 2$, both $d(x, z)$ and $d(y, z)$ are $\leq 2L + 2$. By Lemma 5.4,

$$x, y \in \hat{B}_{2L+2}(z)(\subset H(\bar{d}))$$

If $k = r - d(z, *) < 0$, then (since $z$ is a closest point of $H(\bar{d})$ to $*$) no point of $H(\bar{d})$ belongs to $B_1(*)$. Hence any path in $H(\bar{d})$ between $x$ and $y$ will avoid $B_1(*)$. In particular, a projection of $\psi$ to $H(\bar{d})$ will have length $\leq 2^{\frac{k-1}{2}}$ and avoid $B_1(*)$. Let $F_{-1}(L) = 2^{\frac{k-1}{2}}$.

Now we need only consider the cases where $k \in \{0, 1, \ldots, L + 1\}$. For simplicity, assume $k$ is a positive even number. If $d(t, z) \leq k = r - d(*, z)$, then $r \geq d(*, z) + d(z, t) \geq d(*, t)$, and so $t \in B_1(*)$. In particular, $B_k(z) \subset B_1(*)$. By Lemma 5.3,

$$\hat{B}_{\frac{k}{2}^d}(z) \subset B_k(z) \subset B_1(*)$$

so that

$$x, y \in \hat{B}_{2L+2}(z) - \hat{B}_{\frac{k}{2}^d}(z).$$

This is an initial step to transfer metric information based at $*$ to metric information based at $z$ (and opens the door for group action arguments). For every horoball $H$ in $X$, $H(\bar{d})$ is a Cayley graph for some peripheral subgroup and hence is 1-ended. This implies $H(d) - \hat{B}_{2L+1}^L(z)$ has exactly one unbounded component $Q$. Since $x, y \in H(\bar{d}) - \hat{B}_{\frac{k}{2}^d}(z)$, Lemma 5.5 provides rays at $x$ and $y$ that must be in $Q$. In particular, $x$ and $y$ are in $Q$. Hence there is a path $\gamma(x, y)$ from $x$ to $y$, with image in $Q \subset H(d) - \hat{B}_{\frac{k}{2}^d}(z)$. In order to show that the $\gamma$ paths satisfy the conclusion of the lemma, it is enough to prove the following claim.

**Claim 5.10.** $B_{r-(2\delta+5)}(*) \cap H(\bar{d}) \subset \hat{B}_{\frac{k}{2}^d}(z)$.

**Proof.** By Lemma 5.4, $B_{m}(z) \cap H(\bar{d}) \subset \hat{B}_{\frac{k}{2}^d}(z)$ and so

$$B_{k-4}(z) \cap H(\bar{d}) \subset \hat{B}_{\frac{k}{2}^d}(z).$$

Say $t \in B_1(*) \cap H(\bar{d})$ and let $p$ be the first vertex of $H(\bar{d})$ on a geodesic from $* \to t$. By Lemma 5.1, $d(z, p) \leq 2\delta + 1$. Then $r \geq d(t, *) = d(*, p) + d(p, t)$ so that $k = r - d(*, z) \geq r - d(*, p) \geq d(p, t)$, and

$$d(t, z) \leq d(z, p) + d(p, t) \leq 2\delta + 1 + k$$

In particular, $t \in B_{2\delta+1+k}(z) \cap H(\bar{d})$ and

$$B_r(*) \cap H(\bar{d}) \subset B_{2\delta+1+k}(z) \cap H(\bar{d}) = B_{2\delta+1+r-d(*, z)}(z) \cap H(\bar{d}).$$

This last formula is true for all $r$, and replacing $r$ by $r - 2\delta - 5$ gives

$$B_{r-2\delta-5}(*) \cap H(\bar{d}) \subset B_{r-4-d(*, z)}(z) \cap H(\bar{d}) = B_{k-4}(z) \cap H(\bar{d}) \subset \hat{B}_{\frac{k}{2}^d}(z).$$

This concludes the proof of the claim. \qed

Now, for vertices $x, y \in \hat{B}_{2L+2}(z) - \hat{B}_{\frac{k}{2}^d}(z)$, $\gamma(x, y)$ is an edge path from $x$ to $y$ with image in $H(\bar{d}) - \hat{B}_{\frac{k}{2}^d}(z)$ and so $\gamma(x, y)$ avoids $B_{r-2\delta-5}(*).$ Since $B_{2L+2}(z) - \hat{B}_{\frac{k}{2}^d}(z)$ is a finite set, the set $\{\gamma(x, y) : x, y \in \hat{B}_{2L+2}(z) - \hat{B}_{\frac{k}{2}^d}(z)\}$ is a finite set of integers. Let $F_k(L, \bar{d})$ be the
The proof of Theorem 1.1 is completed in this section. The two papers finitely many peripheral subgroups. Finally, let $H$ the largest integer in this set. Certainly $M. MIHALIK AND E. SWENSON

Again by Remark 5.8, we only need to consider the $L + 2$ integers $r$ such that $k = r - d(z', \ast) \in \{0, 1, \ldots, L + 1\}$. By Lemma 4.4, there is $g \in G$ such that $g^z = z'$ (so $gH(\bar{d}) = H'(\bar{d})$). Note that $g$ maps $B_k(z)$ to $B_k(z')$ for any integer $s$. So if $x', y' \in B_i(z) - \bar{B}_{\bar{z}},$ then the points $x = g^{-1}x'$ and $y = g^{-1}y'$ of $B_{2L+2}(z) - \bar{B}_{\bar{z}}$ are connected by a path $\gamma$ of length $\leq F_k$ in $H(\bar{d}) - \bar{B}_{\bar{z}}(z)$.

The path $g^\gamma$ (of length $\leq F_k$) in $H'(\bar{d}) - \bar{B}_{\bar{z}}(z')$ connects $x'$ and $y'$. By Claim 5.10, $B_{(2\delta + 5)}(\ast) \cap H'(\bar{d}) \subset \bar{B}_{\bar{z}}(z')$, so $g^\gamma$ avoids $B_{(2\delta + 5)}(\ast) \cap H'(\bar{d})$. Of course the $F_i$ depend on the peripheral subgroup that corresponds to the horoball $H$, but there are only finitely many peripheral subgroups. Finally, let $F_{\delta, \gamma}(L, \bar{d})$ be the largest of (the finitely many) $F_i$ that show up this way.

6. Proof of the main theorem

The proof of Theorem 1.1 is completed in this section. The two papers [11, 20] along with many of Bowditch’s results are used. We list a few results that we will need for our proof. Theorem 3.33 of [20] insures that there are geodesic lines in $X$ that are axes for elements of $G$. Our next result is a simple application of Theorem 3.33.

**THEOREM 6.1.** There is an infinite order element $g \in G$ so that if $\rho$ is a geodesic from $\ast$ to $g\ast$, then the line $(\ldots, g^{-1}\rho, \rho, gp, g^2\rho, \ldots)$ is a bi-infinite geodesic that has image in $D^{-1}([0, 19\delta])$.

**Proof.** By [20, Theorem 3.33], there is a geodesic line $\ell$ in $D^{-1}([0, 19\delta])$ and an infinite-order element $g_1 \in G$ such that $g_1 \ell = \ell$. Certainly the image of $\ell$ is not a subset of a horoball and so $\ell$ must contain a vertex $v = h\ast$ (for $h \in G$) of $Y$. The element $g = h^{-1}g_1h$ stabilizes the geodesic line $h^{-1}\ell$ (containing $\ast$). If $\rho$ is the geodesic from $\ast$ to $g\ast$, then $h^{-1}\ell = (\ldots, g^{-1}\rho, \rho, gp, g^2\rho, \ldots)$. Since $G$ is height preserving, $h^{-1}\ell$ has image in $D^{-1}([0, 19\delta])$. □

We need a result of [11], but some background first. In [11, §2.3] several constants are defined. There $\delta$ is any positive integer such that $X$ is $\delta$-hyperbolic and $C = 3\delta$. Following [11], define

$$M(\delta) = 6(C + 45\delta) + 2\delta + 3 = 6(45\delta) + 20\delta + 3$$

and $K(\delta) = 2M(\delta)$.

Two conditions on points in $X$ are defined. First:

Given $\epsilon \geq 0$, and two points $x, y \in X$, say that $x$ and $y$ satisfy $\ast_{\epsilon}$ if

$$\ast_{\epsilon} : |d(\ast, x) - d(\ast, y)| \leq \epsilon \text{ and } d(x, y) \leq M(\delta).$$

(†) Given an integer $N$, say $x, y \in X$ satisfying $\ast_{\epsilon}$, satisfy condition $\frac{1}{2}((\epsilon, N)(x, y)$ if there is a path of length at most $N$ from $x$ to $y$ in $X - B_{m-48\delta}(\ast)$ where $m = \min\{d(x, x), d(y, y)\}$.

**LEMMA 6.2 [11, Lemma 4.2].** If the boundary $\partial X$ is connected, and has no global cut point, then there is an integer $N$ such that $\frac{1}{2}(10\delta, N)(x, y)$ holds for all $x, y \in X_{K(\delta)} = D^{-1}([0, K(\delta)])$ satisfying $\ast_{10\delta}$. \hfill □
Note that the path between $x$ and $y$ of this last lemma has image in $X$ (not $X_K$). Later we improve this result (see Theorem 7.1) to get a path in $X_K$. The improved result is not used in our proof of the main theorem.

**Lemma 6.3.** For any vertex $v$ of $Y$, there is a geodesic ray $r_v$ in $X$ such that $r_v(0) = \ast$, there is $t_v \in [0, \infty)$ such that $d(r_v(t_v), v) \leq \delta$ and $r_v|_{[t_v, \infty)}$ has image in $D^{-1}([0, 21\delta])$.

**Proof.** Let $\ell$ be the line of Theorem 6.1. By definition, the vertices of $Y$ are the elements of $G$, so that $v\ell$ is a geodesic line through $v$ and with image in $D^{-1}[0, 19\delta]$. Write the consecutive vertices of $v\ell$ as $\{\ldots, v_{i-1}, v_i, v_{i+1}, \ldots\}$ where $v_0 = v$. Let $\rho_i$ be a geodesic from $\ast$ to $v_i$ for $i \geq 1$. Infinitely many of the $\rho_i$ have the same first edge $e_1$. Of these, infinitely many have the same second edge $e_2$. Continuing, the geodesic ray $r_v = (e_1, e_2, \ldots)$ converges to $v\ell^+\infty$. Similarly, the geodesic ray $s_v$ is constructed converging to the other end $v\ell^-\infty$ of $v\ell$. By construction, for any integer $n \in [0, \infty)$, $r_v([0, n])$ can be extended to some $\rho_i$ (ending at the vertex $v_i$ of $v\ell$). The same is true for $s_v$. Let $k > d(v, \ast) + \delta$ be an integer. Extend $r_v([0, k])$ to a geodesic $\alpha$ ending on $v\ell([0, \infty))$ and extend $s_v([0, k])$ to a geodesic $\beta$ ending on $v\ell([-\infty, 0])$. Let $\gamma$ be the geodesic segment of $v\ell$ connecting the endpoints of $\alpha$ and $\beta$ and note that $v$ is a vertex of $\gamma$. Consider the geodesic triangle formed by $\alpha$, $\beta$ and $\gamma$. Without loss assume that $v$ is within $\delta$ of $\alpha$. Then since $\alpha$ extends $r_v([0, k])$ and $k = d(v, \ast) + \delta$, $v$ is within $\delta$ of $r_v([0, k])$. (In slightly different terms, the ideal geodesic triangle formed by $r_v$, $s_v$ and $v\ell$ is $\delta$-thin.) In any case, $v$ is within $\delta$ of some vertex of $r_v$ or $s_v$. Assume there is $t_v \in [0, \infty)$ such that $d(r_v(t_v), v) \leq \delta$. Since for any $n \geq t_v$, $r_v([0, n])$ can be extended to a geodesic ending on $v\ell$, $d(r_v(n), v\ell) \leq 2\delta$. As $v\ell$ has image in $D^{-1}([0, 19\delta])$, $r_v|_{[t_v, \infty)}$ has image in $D^{-1}([0, 21\delta])$. \hfill $\square$

**Proof of the main theorem.** First assume that $G$ is finitely presented, so that the Cayley complex $Y$ is simply connected. All of our compact sets will be finite subcomplexes of $X$ and all paths will be edge paths. Given a compact set $C$ in $Y$, our goal is to find a compact set $D$ in $Y$ with the following semistability property (see Theorem 3.1(2)): For any third compact set $E \subset Y$ and $Y$-loop $\beta$ based on (the base ray) $\hat{r}$ and with image in $Y - D$, show that $\beta$ is homotopic rel$\{\hat{r}\}$ to a loop in $Y - E$ by a homotopy in $Y - C$. We will define a set of integers $N_0$, $N_1$, $N_2$, $N_3$ and $N_4$. Our compact set $D$ will be $B_{N_4}(C) \cap Y$, the ball in $X$ of radius $N_4$ about $C$ intersected with $Y$.

Next we construct the base ray $\hat{r}$ for our semistability criterion. Let $g$ and $\rho$ be as in Lemma 6.1 and say $\rho$ is the edge path $(e_1, \ldots, e_{L_0})$ so that $|\rho| = L_0$. Say the consecutive vertices of $\rho$ are $\ast = v_0, \ldots, v_{L_0}$. If $v_i \in Y$, then let $y_i = v_i$. Otherwise, let $y_i \in Y$ be the end point of a vertical path (of length $\leq 19\delta$) from $v_i$ to $Y$. Let $\hat{r}_i$ be a shortest path in $Y$ from $y_{i-1}$ to $y_i$.

Let $\hat{\rho}$ be the $Y$-path $\hat{\rho} = (\hat{r}_1, \ldots, \hat{r}_{L_0})$. Then our base ray $\hat{r} = (\hat{\rho}, g\hat{\rho}, g^2\hat{\rho}, \ldots)$ is a proper edge path ray in $Y$ beginning at $\ast$. Note that the $X$-geodesic ray $r = (\rho, g\rho, g^2\rho, \ldots)$ tracks $\hat{r}$ and for any vertex $v$ of $r$, there is a vertical path of length $\leq 19\delta$ from $v$ to a point of $\hat{r}$. Furthermore, $r$ and $\hat{r}$ share the vertices $\ast, g, g^2, \ldots$ (where $g^n = g^{\ast}$). (See Figure 2.)
Suppose \( \beta \), with consecutive vertices \( b_0 = r(t_0), b_1, b_2, \ldots, b_{q-1}, b_q = b_0 \), is a loop based at a common vertex \( b_0 \) of \( \hat{r} \) and \( r \). Let \( r = r_0 = r_Q \). By Lemma 6.3, for each \( i \in \{1, \ldots, q-1\} \), there is a geodesic ray \( r_i \) at \( * \) such that for some integer \( t_i \in [0, \infty) \), \( d(r_i(t_i), b_i) \leq \delta \) and \( r_i(\delta(t_i, \infty)) \) has image in \( D^{-1}([0,21\delta]) \). Define \( \tau_i \) to be a path from \( b_i \) to \( r_i(t_i) \) of length \( \leq \delta \). (See Figure 3.) Since \( b_0 = r_0(t_0) \), \( \tau_0 \) is the trivial path. Note that \( d(r_{i-1}(t_{i-1}), r_i(t_i)) \leq 2\delta + 1 \). Let \( w_i = r_i(t_i + 3(45\delta) + 9\delta) \) for all \( i \). Then

\[
d(w_{i-1}, w_i) \leq 2[3(45\delta) + 9\delta] + 2\delta + 1 = 6(45\delta) + 20\delta + 1 < M.
\]

Since \( d(r_{i-1}(t_{i-1}), r(t_i)) \leq 2\delta + 1 \) and \( d(r_i(t_i), w_i) \) is the same for all \( i \), \( |d(w_{i-1}, *) - d(w_i, *)| \leq 2\delta + 1 < 10\delta \). As \( w_i \in D^{-1}(\{0,21\delta\}) \) and \( 21\delta < K \), Lemma 6.2 implies \( w_{i-1} \) and \( w_i \) satisfy \( \frac{1}{2}(10\delta, N_0)(w_{i-1}, w_i) \) for some positive integer \( N_0 \). That means there is a path \( \mu_i \) in \( X \) from \( w_{i-1} \) to \( w_i \) of length \( \leq N_0 \) in the compliment of the ball \( B_{m-48\delta}(*) \), where \( m = \min\{d(*, w_{i-1}), d(*, w_i)\} \). As \( D(w_i) \leq 21\delta \), there is a vertical path \( \alpha_i \) of length \( \leq 20\delta \) from a point \( z_i \in D^{-1}(\delta) \) to \( w_i \). (By the definition of \( \hat{r} \), the vertical path from \( w_0 \) to \( Y \) ends at a point in the image of \( \hat{r} \) and has length \( \leq 19\delta \).) Hence the vertical path from \( w_0 \) to the point \( z_0 \in D^{-1}(\delta) \) is of length \( \leq 18\delta \). Let \( \alpha_0 \) be the vertical path from \( z_0 \) to \( w_0 \).

The path \( \psi_i = (\alpha_{i-1}, \mu_i, \alpha_i^{-1}) \) begins at \( z_{i-1} \in D^{-1}(\delta) \) and ends at \( z_i \in D^{-1}(\delta) \). As \( |\mu_i| \leq N_0 \) and each \( |\alpha_j| \leq 20\delta \), \( |\psi_i| \leq N_0 + 40\delta \). Since \( \mu_i \) avoids \( B_{m-48\delta} \) and for each \( j \), \( |\alpha_j| \leq 20\delta \), \( \psi_i = (\alpha_{i-1}, \mu_i, \alpha_i^{-1}) \) has image in \( X - B_{m-68\delta}(*) \). The path \( \psi_i \) can be decomposed into subpaths where each either has image in \( D^{-1}(\{0,\delta\}) \), or satisfies the conclusion of either Lemma 5.7 or Lemma 5.9 (with \( d = \delta \)). Let \( \lambda \) be one of these subpaths.

If \( \lambda \) satisfies the conclusion of Lemma 5.7, then it has a projection \( \lambda' \) (to \( D^{-1}(\delta) \)) with image in \( X - B_{m-68\delta-(2\delta+1)}(*) \). The length of \( \lambda \), and in particular the length of \( \lambda' \), is bounded by a constant only depending on \( |\psi_i| \leq N_0 + 40\delta \).
If λ satisfies the conclusion of Lemma 5.9, then there is path λ′ (in \( D^{-1}(\delta) \)) connecting the end points of λ, with image in \( X - B_{m-68-25+5}(*). \) and again the length of λ′ is bounded by a constant only depending on the length of λ, and in particular, by a constant depending only on \(|w_1| \leq N_0 + 40\delta.\) Replacing each of the λ subpaths by the corresponding λ′ paths in \( D^{-1}(\delta) \) gives us a path \( \psi''_i \) in \( D^{-1}([0, \delta]) \) from \( z_{i-1} \) to \( z_i, \) of length \( \leq N_1 \) (a constant depending only on the number \( N_0 + 40\delta. \)) Then \(|\psi''_i| \leq N_1 \) and the image of \( \psi''_i \) is in \( X - B_{m-68-25+5}(**). \)

Let \( \alpha'_i \) be the vertical path of length \( \delta \) from \( z'_i \in Y \) to \( z_i. \) Note that \( z'_i \) is in the image of \( \hat{r}. \) Let \( \psi''_i \) be a projection of \( \psi''_i \) to Y. Again, the length of \( \psi''_i \) is \( \leq N_2 \) (a constant depending only on the number \( N_0 + 40\delta. \) By Lemma 5.6, each vertex of \( \psi''_i \) is within \( \delta + 1 \) of a vertex of \( \psi'_i. \) Hence, \(|\psi''_i| \leq N_2 \) and the image of \( \psi''_i \) is in \( X - B_{m-68-25+5-\delta+1}(**) = X - B_{m-71\delta-6}(**). \)

Our goal is to show that the edges \([b_{i-1}, b_i] \) are compatibly homotopic to the \( \psi''_i, \) and that each vertex of \( \psi''_i \) is further from * than either of \( b_{i-1} \) or \( b_i. \) We must also show the homotopies that exchange the edges of \( \beta \) for the \( \psi''_i \) avoid the compact set \( C \) (but first we must decide how large the yet to be defined \( N_4 \) must be). Once this is accomplished, then repeating the process moves the original path (rel \( \{ \hat{r} \} \)) to a path outside of any preassigned compact set \( E \) (by a homotopy avoiding \( C \)) to finish the proof.

First we show each vertex \( v \) of \( \psi''_i \) is further from * than either \( b_i \) or \( b_{i-1}. \) We have

\[
d(v, *) \geq m - 71\delta - 6, \quad d(v, *), d(*, w_i) = \min\{d(*, w_{i-1}), d(*, w_i) \} =
\]

Then,

\[
d(v, *) \geq \min\{d(*, r_{i-1}(t_{i-1})), d(*, r_i(t_i)) \} + 3(45\delta) + 9\delta.
\]

Since \( d(r_i(t_i), b_i) \leq \delta \) for all \( i: \)

\[
d(v, *) \geq \min\{d(b_{i-1}, *), d(b_i, *) \} + 72\delta - 6.
\]

Since \( d(b_{i-1}, b_i) \leq 1, \) \( v \) is at least \( 72\delta - 7 \) further from * than either \( b_{i-1} \) or \( b_i \) is from *. We have shown:

Each vertex of \( \psi''_i \) is at least \( 72\delta - 7 \) further from * than either \( b_{i-1} \) or \( b_i \) is from *.

Recall that \(|\tau_i| \leq \delta, \quad d(r(t_i), u_i) = 3(45\delta) + 9\delta \) and \(|(\alpha'_i, \alpha_i)| \leq 218. \) Consider the X path \((\tau_i, [r_i(t_i), u_i], \alpha_i^{-1}, \alpha_i^{-1}) \) from \( b_i \) to \( z'_i \) (both vertices in Y) of length \( \leq 3(45\delta) + 31\delta = L_1. \)

There is a positive integer \( N_3 \) only depending on \( L_1 \) so that for any edge path \( \lambda \) in the Y of length \( \leq L_1, \) that begins and ends in Y, there is an edge path \( \lambda' \) of length \( \leq N_3 \) in Y connecting the end points of \( \lambda. \) In particular, there is a Y-edge path \( \lambda'_i \) from \( b_i \) to \( z_i \) of length \( \leq N_3. \)

When \( i = 0 \) or \( i = n, \) we want \( \lambda'_i = \lambda_i \) to be the subpath of \( \hat{r} \) from \( b_0 = b_n \) to \( z_0 = z_n. \) Since \( b_0 = r_0(t_0) \) and \( d(r_0(t_0), u_0) = 3(45\delta) + 9\delta, \) we can assume \( N_3 \) is larger than the length of the subpath of \( \hat{r} \) from \( b_0 = r_0(t_0) \) is \( im(\hat{r}) \) to \( z_0 = z_n. \)

Given any vertices \( v_1 \) and \( v_2 \) of Y, there is an element of \( g \in G \) such that \( gv_1 = v_2. \) Hence if \( J \) is any positive integer, there are only finitely many edge loops paths in Y of length \( \leq J, \) up to translation by the action of \( G \) on Y. Since Y is simply connected, there is an integer \( I(J) \) such that any edge path loop in Y of length \( \leq J \) is homotopically trivial in \( Y(J) \) (the ball in Y of radius \( I(J) \)) for any vertex \( v \) of the loop.

The loops \( \ell_i = ([b_{i-1}, b_i], \psi_i', \psi_i'') \) have image in Y and are of length \( \leq 1 + 2N_3 + N_2. \) Let \( N_4 = I(1 + 2N_3 + N_2) \) so that any loop of length \( \leq 1 + 2N_3 + N_2 \) in Y is homotopically trivial by a homotopy in (the Y-ball) \( B_{N_4}(v) \) for any vertex v of the loop. Define the compact set \( D \) to be \( B_{N_4}(C) \) and \( Y \) (the ball in X of radius \( N_4 \) about C, intersected with Y). Assume that \( \beta \) is a loop in Y based at a vertex \( r \) common to \( \hat{r} \) and with image in Y - D. The homotopies killing the \( \ell_i \) move \( v \) (with image in \( Y(J) \)) for any vertex \( v \) of \( \ell_i \) move each edge \([b_{i-1}, b_i] \) to a path \( \psi''_i \) by a homotopy in Y whose image avoids the compact set \( C. \) These homotopies
string together compatibly to move \( \beta \) further out along \( \tilde{r} \). Continuing, \( \beta \) is eventually moved outside of any compact subset \( E \) of \( Y \). We have shown that \( Y \), and hence \( G \) has semistable fundamental group at \( \infty \).

Note that in order to show that \( Y \) has semistable fundamental group at \( \infty \), the only homotopies we need are those that kill loops of length \( \leq 1 + 2N_3 + N_2 \) in \( Y \). If \( G \) is only finitely generated (as opposed to finitely presented), then let \( Y \) be obtained from \( \Gamma(G,S) \) (the Cayley graph of \( G \) with respect to the generating set \( S \)) by attaching 2-cells that bound every edge path loop of length \( \leq 1 + 2N_3 + N_2 \) in \( \Gamma(G,S) \). Since each edge of \( \Gamma(G,S) \) is directed and labeled by an element of \( S \), each such loop spells out a relator of \( G \) of length \( \leq 1 + 2N_3 + N_2 \) (and so only a finite collection of relators). The same proof now shows that \( Y \) has semistable fundamental group at \( \infty \). According to our definition of semistability for a finitely generated group, \( G \) has semistable fundamental group at \( \infty \).

**Theorem 6.4.** Suppose \( G \) is a 1-ended finitely generated group that is hyperbolic relative to the finitely generated proper subgroups \( P_1, \ldots, P_n \). Let \( X \) be a corresponding cusped space. If \( Y \) has semistable fundamental group at \( \infty \), then \( X_k \) has semistable fundamental group at \( \infty \) for all \( k \geq 0 \).

**Proof.** Let \( r \) and \( s \) be proper rays at \(*\) in \( X_k \) for \( k \geq 0 \). Let \( r' \) and \( s' \) be projections of \( r \) and \( s \) to \( Y \). By the product structure on horoballs, \( r \) is properly homotopic to \( r' \) and \( s \) to \( s' \). Since \( r' \) and \( s' \) are properly homotopic (by the semistability of \( Y \)), a combination of proper homotopies shows \( r \) is properly homotopic to \( s \) in \( X_k \). \( \Box \)

7. An improved \( \dagger \) result (finding paths in the spaces \( X_{K(\delta)} \))

As pointed out in the introduction, 1-ended word hyperbolic groups have semistable fundamental group at \( \infty \). This result follows by combining work of Swarup, Bowditch and Bestvina–Mess with a shape theoretic observation by Geoghegan. It is also an elementary fact that the Bestvina–Mess \( \dagger \) result (in conjunction with the fact that the boundary of a 1-ended word hyperbolic group does not have a cut point) allows one to prove word hyperbolic groups have semistable fundamental group at \( \infty \). The proof is a simple matter of starting with a \( \text{‘far out’} \) loop \( \ell_1 \) in the appropriate space, using \( \dagger \) to move it further out to a loop \( \ell_2 \) and then iteratively applying \( \dagger \) again and again to (properly) move successive loops as far out as desired. It is critical that this process can be iterated.

The \( \dagger \) result of [11] (Lemma 6.2) cannot be used in a similar way to easily prove our main theorem. While the path it produces begins and ends in \( X_{K(\delta)} \), it can only be guaranteed to have image in \( X - B_{m-48\delta}(\ast) \) and so the process cannot be iterated. If instead, this path were guaranteed to have image in say \( X_{K(\delta)} - B_{m-(50\delta+5)}(\ast) \) (as in our Theorem 7.1), then one could iterate this result and model a proof of our main result along the lines of the one that shows 1-ended word hyperbolic groups have semistable fundamental group at \( \infty \).

Throughout this section, \( G \) is a finitely generated group that is hyperbolic relative to a finite collection \( P \) of proper 1-ended finitely generated subgroups. Assume again that \( X \) is a cusped space for \((G,P)\). Say \( X \) is \( \delta_0 \) hyperbolic and that \( \delta \geq \delta_0 \) is an integer. Lemmas 5.7 and 5.9 allow us to improve Lemma 6.2 from a partial result about \( X_{K(\delta)} \) to one completely about \( X_{K(\delta)} \). More precisely, Lemma 6.2 guarantees a path in \( X - B_{m-48\delta}(\ast) \) and we want to replace it by a path in \( X_{K(\delta)} - B_{m-(50\delta+5)}(\ast) \).

First some observations. As Lemma 6.2 gives a separate result for each \( \delta \geq \delta_0 \), the integers \( K, M \) and \( N \) will in fact be \( K(\delta), M(\delta) \) and \( N(\delta) \). Consider \( \delta \geq \delta_0 \). Say \( x, y \in X_{K(\delta)} \) satisfy \( *_{10\delta} \) and \( \psi \) is a path in \( X \) of length \( \leq N(\delta) \) joining \( x \) and \( y \) in \( X - B_{m-48\delta}(\ast) \) where \( m = \min\{d(\ast,x),d(\ast,y)\} \). Lemmas 5.7 and 5.9 allow us to replace the segments of \( \psi \) that leave
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X_{K(\delta)} by paths in H(K(\delta)) (for some horoball H). The number of such segments and the
lengths of the replacement paths are bounded in terms of N(\delta) and the image of each such
replacement path is in X_{K(\delta)} - B_{m - (48\delta + 2\delta + 5)}(\ast).

Recall the \ast_\epsilon and \dagger condition of [11], where for any \delta \geq \delta_0, we define

M(\delta) = 6(45\delta) + 20\delta + 3 and K(\delta) = 2M(\delta).

Given \epsilon \geq 0, and two points x, y \in X, say that x and y satisfy \ast_\epsilon if

\ast_\epsilon : |d(\ast, x) - d(\ast, y)| \leq \epsilon and d(x, y) \leq M.

Note that if \epsilon' \leq \epsilon and x, y \in X satisfy \ast_\epsilon', then x, y satisfy \ast_\epsilon.

Given an integer N, say x, y \in X satisfying \ast_\epsilon, satisfy condition \dagger(\epsilon, N)(x, y) if there is a path
of length at most N from x to y in X - B_{m - 48\delta + 5}(\ast) where m = \min\{d(\ast, x), d(\ast, y)\}.

This induces the following definition.

\dagger(\epsilon, N)(x, y) if there is a path of length at most N from x to y in X_{K(\delta)} - B_{m - 50\delta + 5}(\ast) where m = \min\{d(\ast, x), d(\ast, y)\}.

Lemma 6.2 states that if the boundary \partial X is connected, and has no global cut point, then
there is an integer N(\delta) such that \dagger(10\delta, N(\delta))(x, y) holds for all x, y \in X_{K(\delta)} = D^{-1}([0, K(\delta)])

satisfying \ast_{10\delta}. The path of length \leq N(\delta) connecting x and y and guaranteed by Lemma 6.2
has image in X. Lemmas 5.7 and 5.9 enhance this result to give a path in X_{K(\delta)} that will become
larger and the image of the path may come 2\delta + 5 units closer to \ast than the one
given by \dagger(10\delta, N(\delta))(x, y).

Combining these observations we have:

THEOREM 7.1. Suppose G is a finitely generated group that is relatively hyperbolic with
respect to a finite collection P of finitely generated 1-ended proper subgroups, X is a cusped
space for (G, P) that is \delta_0-hyperbolic and \delta is an integer \geq \delta_0. If the boundary \partial X is connected,
and has no global cut point, then there is an integer N(\delta) such that \dagger(10\delta, N(\delta))(x, y) holds
for all x, y \in X_{K(\delta)} = D^{-1}([0, K(\delta)])
satisfying \ast_{10\delta}.

8. A variant of Lemma 5.7 (finding paths that only depend on depth)

Lemmas 5.7 and 5.9 are the most important technical facts in this paper. If X is a cusped
space for a relatively hyperbolic group, these lemmas provide the leverage to move certain ‘far out’
paths that begin and end in the horosphere X(\delta) (\delta \geq \delta) to far out paths in X(\delta) by a
homotopy that fixes the end points and has ‘far out’ image. Lemmas of this sort are invaluable
in studying the asymptotic structure of groups with “boundaries” (like hyperbolic, relatively
hyperbolic and CAT(0) groups). While we see no immediate application of the next lemma,
we find it intriguing.

The following lemma seems more general than Lemma 5.7, since there is no restriction on
the length of the path \psi considered here, only a restriction on the depth of the path. The proof
parallels that of Lemma 5.7 and the second half is exactly the same.

LEMMA 8.1. Suppose H is a horoball, \bar{d} an integer \geq \delta, x \neq y vertices of H(\bar{d}) and \psi is a
path in H^{\bar{d}} - B_{\bar{d}}(\ast) between x and y that only intersects H(\bar{d}) at x and y. Let L be such that
D(\psi) = [d, L] (the depth of \psi).

Suppose \gamma is a projection of \psi to H(\bar{d}). If z is a closest vertex of H(\bar{d}) to \ast and each vertex v
of \psi is such that d(v, z) > \bar{d} + 4, then each vertex of \gamma is at distance greater than L - \bar{d} + 4
from z. Furthermore, the image of \gamma avoids B_{\bar{d} - (2\delta + 1)}(\ast).
Proof. By Lemma 4.3, $H^i$ is convex. Let $p$ be a vertex of $\gamma$ and $(\alpha_p, \tau_p, \beta_p)$ a geodesic (in $H^i$) from $z$ to $p$ where $\alpha_p$ and $\beta_p$ are vertical of the same length and $|\tau_p| \leq 3$. Let $y$ be the end point of $\tau_p$. By Lemma 5.6, there is a vertical segment that begins at $p$ and ends at most 1 (horizontal) unit from a vertex $v$ of $\psi$ (and $D(v) \geq \bar{d} + 1$). If $y$ is on that vertical line segment (Figure 1.1), then there is a path $\rho$ from $v$ to $z$ that begins with a horizontal edge from $v$ to a vertex $w$ on the vertical segment, followed by a vertical segment from $w$ to $y$, followed by $\tau_p^{-1}$, followed by $\alpha_p^{-1}$.

The sum of the lengths of the two vertical segments of $\rho$ is less than or equal to $L - \bar{d}$ (since $v$ is on $\psi$), and so

$$L - \bar{d} + 4 < d(v, z) \leq L - \bar{d} + 4,$$

which is impossible.

Instead, $y$ is on the vertical line at $p$ and the vertical line segment from $p$ to $y$ contains a vertex $w$ (other than $y$) within 1 horizontal unit of a vertex $v$ of $\psi$ (Figure 1.2). Now

$$d(z, p) = d(p, w) + d(w, y) + |\tau_p| + |\alpha_p| \quad \text{and} \quad L - \bar{d} + 4 < d(v, z) \leq 1 + d(w, y) + |\tau_p| + |\alpha_p|.$$ 

Since $d(p, w) \geq 1$, $d(z, p) > L - \bar{d} + 4$, completing the first part of the lemma.

Since the depth of $v$ (and hence the depth of $w$) is at least $\bar{d} + 1$, $d(p, z) \geq 1 + d(w, z) \geq d(v, z)$

Combining this inequality with Lemma 5.2 and the triangle inequality, we have for each vertex $p$ of $\gamma$:

$$d(p, *) + 2\delta + 1 \geq d(p, z) + d(z, *) \geq d(v, z) + d(z, *) \geq d(v, *) > r$$

So $d(p, *) > r - 2\delta - 1$. 

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