Semilinear automorphisms of reductive algebraic groups

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Abstract

Let $G$ be a connected reductive algebraic group over a field $k$. We study the group of semilinear automorphisms $\text{Aut}(G \to \text{Spec } k)$ consisting of algebraic automorphisms of $G$ over automorphisms of $k$. We focus on the exact sequence $1 \to \text{Aut}_G \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}_G(k) \to 1$. When $G$ is quasi-split, we show that $\text{Aut}_G(k)$ is isomorphic to $\text{Aut}_{R(G)}(k)$, where $R(G)$ denotes the scheme of based root datum of $G$. Furthermore, the exact sequence $1 \to \text{Aut}_G \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}_G(k) \to 1$ splits if and only if the exact sequence $1 \to \text{Aut} R(G) \to \text{Aut}(R(G) \to \text{Spec } k) \to \text{Aut}_{R(G)}(k) \to 1$ splits. As a corollary, we get many examples of algebraic groups $G$ over $k$ whose group of abstract automorphisms does not decompose as the semidirect product of $\text{Aut}_G$ with $\text{Aut}_G(k)$. We also study the same questions for inner forms of $\text{SL}_n$ over a local field.

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1 Introduction

The so-called abstract automorphisms of (the rational points of) a reductive algebraic group have been studied extensively in the literature. In 1955, J. Dieudonné wrote a comprehensive treatise (see [Die71] for the third edition) covering the case of classical groups, and even going beyond the world of algebraic groups (since he also considers classical groups over division algebras that are infinite dimensional over their center). Many results in this area have been subsumed in the famous article [BT73] by A. Borel and J. Tits. To wit, here is one of their results:

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Theorem ([BT73, excerpt of Corollaire 8.13]). Let $k$ be an infinite field, and let $G$ be an absolutely simple algebraic group over $k$, which is assumed to be isotropic and adjoint. Furthermore, if $k$ is of characteristic 2 or 3, assume that $k$ is not perfect. Let $\alpha$ be an automorphism of $G(k)$, considered as an abstract group. Then there exists a unique automorphism $\varphi : k \to k$ and a unique semilinear automorphism $f_\varphi : G \to G$ over $\varphi$ such that for $g \in G(k) = \Hom_{k\text{-schemes}}(\Spec k, G)$, we have $\alpha(g) = f_\varphi \circ g \circ (\Spec \varphi)^{-1}$.

By a semilinear automorphism $f_\varphi$ of a $k$-group scheme $G$ over an automorphism $\varphi : k \to k$, we mean that we have the following commutative diagram in the category of group schemes

$$
\begin{array}{c}
G \\
\downarrow \\
\Spec k
\end{array}
\xrightarrow{f_\varphi}
\begin{array}{c}
G \\
\downarrow \\
\Spec k
\end{array}
$$

where the vertical arrows are the structural morphisms of the $k$-scheme $G$. Note that if $G$ is realised as a matrix group, and if $g = (g_{ij}) \in G(k)$, then $f_\varphi \circ g \circ (\Spec \varphi)^{-1}$ is given by the matrix whose $ij$-th coefficient is $\varphi^{-1}((f_\varphi)_{ij}(g))$, so that our convention differs from [BT73] (see Remark 2.3 for a discussion of this difference).

Another way to phrase the Borel–Tits theorem is to say that the group $\Aut_{\text{abstract}}(G(k))$ of abstract automorphisms of $G(k)$ fits in the exact sequence $1 \to \Aut G \to \Aut_{\text{abstract}}(G(k)) \to \Aut(k)$. Letting $\Aut_C(k)$ denote the image of $\Aut_{\text{abstract}}(G(k))$ in $\Aut(k)$, it is natural to wonder what $\Aut_C(k)$ is, and whether the group of abstract automorphisms of $G(k)$ splits as the semi-direct product $(\Aut G) \rtimes \Aut_C(k)$. While this semi-direct product decomposition does hold when $G$ is $k$-split, it turns out that it may fail in general, even if $G$ is quasi-split. One of the aims of the present paper is to address this issue.

To illustrate some of our results in the most concrete way, here is a corollary of our results (we refer the reader to Corollary 5.14 for a proof of this statement and to Remark 5.15 for an explicit realisation of the algebraic group appearing in this corollary):

**Corollary** (of Theorem 1.3). Let $G$ be the quasi-split, absolutely simple, adjoint algebraic group of type $^2A_{n-1}$ over a field $k$ with corresponding quadratic separable extension $l$. Let $k_0$ be the field of rational numbers (respectively the field of $p$-adic numbers for some prime $p$) and assume that $k$ and $l$ are possibly infinite (respectively finite) Galois extensions of $k_0$. Then $\Aut_C(k) = \Gal(k/k_0)$ and the short exact sequence $1 \to \Aut G \to \Aut_{\text{abstract}}(G(k)) \to \Aut_C(k) \to 1$ splits if and only if the short exact sequence of abstract groups $1 \to \Gal(l/k) \to \Gal(l/k_0) \to \Gal(k/k_0) \to 1$ splits.

We now recast the problem in a more useful way for us. Let $\Aut(G \to \Spec k)$ denotes the group of semilinear automorphisms of $G$. We can then rephrase the Borel–Tits theorem as saying that under the assumptions of the theorem, the natural homomorphism $\Aut(G \to \Spec k) \to \Aut_{\text{abstract}}(G(k))$ is an isomorphism. The rest of the paper is concerned with the study of $\Aut(G \to \Spec k)$.

Given a $k$-group scheme $G$ (in this paper the letter $k$ is exclusively used to designate a field), we have a homomorphism $\Aut(G \to \Spec k) \to \Aut(k) : f_\varphi \mapsto \varphi^{-1}$. Let $\Aut_C(k)$ denotes the image of this homomorphism, and let $\Aut G$ denotes the group of $k$-algebraic automorphisms of $G$, or in other words the kernel of $\Aut(G \to \Spec k) \to \Aut(k)$. In summary, we defined the exact sequence $1 \to \Aut G \to \Aut(G \to \Spec k) \to \Aut_C(k) \to 1$.

It seems that not much is known about the subgroup $\Aut_C(k) \leq \Aut(k)$ in general (in [Die71, p. 18, last paragraph], J. Dieudonné makes a related comment, though not exactly about $\Aut_C(k)$). Actually, for $G = \text{SL}_n(D)$ with $D$ a non-commutative finite dimensional central division algebra, $\Aut_C(k)$ is isomorphic to the group of outer automorphisms of $D$ (or to a degree 2 extension of this group). But computing the latter is probably hard (for example, in [Han05] T. Hanke computes explicitly an outer automorphism of a division algebra of degree 3 over a number field, and then uses it to construct a non-crossed product division algebra over $\mathbb{Q}((t))$).

Nonetheless, in some special cases, $\Aut_C(k)$ is easily understood. For example, when $G$ is a split connected reductive group, then $\Aut_C(k) = \Aut(k)$ (this follows directly from the fact that
split connected reductive groups are defined over the prime field of $k$). The next easiest case should be to compute $\text{Aut}_G(k)$ for quasi-split groups, and this is indeed feasible.

**Theorem 1.1.** Let $G$ be a connected reductive $k$-group and let $\mathcal{R}(G)$ be its $k$-scheme of based root datum. There exists a homomorphism $\text{Aut}(G \to \text{Spec } k) \to \text{Aut}(\mathcal{R}(G) \to \text{Spec } k)$ preserving the underlying automorphism of $k$ and whose kernel is $(\text{Ad } G)(k)$. Hence $\text{Aut}_G(k) \leq \text{Aut}_{\mathcal{R}(G)}(k)$.

If $G$ is quasi-split, the corresponding exact sequence $1 \to (\text{Ad } G)(k) \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}(\mathcal{R}(G) \to \text{Spec } k)$ splits. Hence if $G$ is quasi-split $\text{Aut}_G(k) = \text{Aut}_{\mathcal{R}(G)}(k)$.

The $k$-scheme of based root datum $\mathcal{R}(G)$ appearing in this result is an object (see Definition 3.4) which can be described as a variation on the scheme of Dynkin diagram as defined in [GP11b, Exposé 24, Section 3]. In the end, the construction of the $k$-scheme of based root datum is a restatement of [GP11b, Exposé 24, Théorème 3.11]. Let us directly give a brief description of what $\mathcal{R}(G)$ is in concrete terms: for $G_0$ a split connected reductive $k$-group, $\mathcal{R}(G_0)$ is defined to be the constant object on $\text{Spec } k$ preserving $\Delta$. W e have two exact sequences $1 \to \text{Aut}(G_0 \to \text{Spec } k) \to \text{Aut}(\mathcal{R}(G_0) \to \text{Spec } k) \to H^1(k, (\text{Ad } G_0)(k_0))$ $1 \to \text{Aut}_G \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}(k) \to H^1(k, \text{Aut } G_{k_0})$.

The proof of Theorem 1.1 consists mainly of a straightforward adaptation of [GP11b, Exposé 24, Théorème 3.11] to the semilinear situation. One crucial step is to show that taking the scheme of based root datum of $G$ commutes with base change (see Lemma 3.15). We chose to prove this in concrete terms, by using cocycle computations adapted to the semilinear situation. Those cocycle computations also lead to the following Galois cohomological formulation.

**Theorem 1.2.** Let $G$ be a connected reductive $k$-group and let $\mathcal{R}(G)$ be its $k$-scheme of based root datum. We have two exact sequences $1 \to (\text{Ad } G)(k) \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}(\mathcal{R}(G) \to \text{Spec } k) \to H^1(k, (\text{Ad } G)(k_0))$ $1 \to \text{Aut}_G \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}(k) \to H^1(k, \text{Aut } G_{k_0})$.

We refer the reader to Section 4 for the definition of the coboundary maps involved in those exact sequences. After proving Theorem 1.2, we illustrate how it can be used by computing $\text{Aut}_G(k)$ when $G \cong \text{SL}_3(D)$ and $D$ is a division algebra of degree 3 over $k$. In doing so, we recover some of the results in [Han07b] without needing to introduce bycyclic algebras (see Lemma 4.17, Lemma 4.18 and Remark 4.19).

Once we have some control over $\text{Aut}_G(k)$, it is natural to wonder whether the exact sequence $1 \to \text{Aut}_G \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}_G(k) \to 1$ splits. Again, for $G$ a split connected reductive algebraic group, this exact sequence does split (a statement already made in [Thi74, Corollary 5.10]) because such a group is defined over its prime field. But somewhat surprisingly, this is not any more the case for a general quasi-split group.

**Theorem 1.3 (The bowtie theorem).** Let $G$ be a connected reductive $k$-group which is quasi-split, and let $\mathcal{R}(G)$ be its $k$-scheme of based root datum. Then the short exact sequence $1 \to \text{Aut}_G \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}_G(k) \to 1$ splits if and only if the short exact sequence $1 \to \text{Aut} \mathcal{R}(G) \to \text{Aut}(\mathcal{R}(G) \to \text{Spec } k) \to \text{Aut}_{\mathcal{R}(G)}(k) \to 1$ splits.

As it turns out, the bowtie theorem (whose name is due to the diagram appearing in its proof) is a direct corollary of Theorem 1.1 (see the end of Section 3 for the proof).
When $G$ is absolutely simple, we can identify the short exact sequence $1 \to \text{Aut} \mathcal{R}(G) \to \text{Aut}(\mathcal{R}(G) \to \text{Spec} k) \to \text{Aut}_{\mathcal{R}(G)}(k) \to 1$ as a sequence involving various automorphism groups of fields naturally associated with $G$ (see Proposition 5.10). Using this description, we can then give many explicit examples of absolutely simple, quasi-split algebraic $k$-groups $G$ for which $\text{Aut}(G \to \text{Spec} k)$ is not a split extension of $\text{Aut}_G(k)$. The corollary given at the beginning of this introduction illustrates this in the most concrete way.

In the last section of the paper, we also explore the same questions when $G$ is an inner form of $\text{SL}_n$ over a local field $K$ (see the beginning of Section 6 for a precise definition of what we mean by a local field). The first step is to get some control over $\text{Aut}_G(K)$. Actually, a direct corollary of the results of T. Hanke in [Han07b] is that in this case, $\text{Aut}_G(K) = \text{Aut}(K)$.

**Theorem 1.4** (corollary of [Han07b]). Let $K$ be a local field, let $D$ be a central division algebra over $K$ of degree $d$, and consider the algebraic group $G = \text{SL}_n(D)$. Then $\text{Aut}_G(K) = \text{Aut}(K)$.

In [Han07b], T. Hanke does not mention local fields, but he gives an algorithm over an arbitrary field to compute outer automorphisms of cyclic division algebras, and this implies the result for local fields. In fact, we only need to use the simplest version of his algorithm and for the ease of the reader we give a self contained proof of Theorem 1.4 in Corollary 6.10. Also note that in characteristic 0, this result has probably been known for a long time since it is a direct consequence of [EM48, Corollary 7.3].

Finally, we obtain an explicit characterisation for the splitting of the exact sequence $1 \to \text{Aut} G \to \text{Aut}(G \to \text{Spec} K) \to \text{Aut}_G(K) \to 1$ for $G$ an inner form of $\text{SL}_n$ over a local field.

**Theorem 1.5.** Let $K$ be a local field, let $D$ be a central division algebra over $K$ of degree $d$, and consider the algebraic $K$-group $G = \text{SL}_n(D)$. The short exact sequence $1 \to \text{Aut} G \to \text{Aut}(G \to \text{Spec} K) \to \text{Aut}_G(K) \to 1$ splits if and only if $\gcd(nd, [K : K'])$ divides $n$ for all subfields $K' \leq K$ such that $K/K'$ is finite Galois.

As we prove in Proposition 6.38, for $K = F_{p^d}(T)$, the condition “$\gcd(nd, [K : K'])$ divides $n$ for all subfields $K' \leq K$ such that $K/K'$ is finite Galois” is equivalent to requiring that $\gcd(d, p) = 1$ and that $\gcd(n, i(p^d - 1))$ divides $n$, so that this criterion is very explicit in characteristic $p$. On the other hand, in characteristic 0 note that $\mathbb{Q}_p$ is rigid (see Definition 5.11 and Lemma 5.13), so that the condition is a finite one. See also Remark 6.17 for a vivid illustration of Theorem 1.5 in characteristic 0.

The necessity of the condition is proved in Corollary 6.15. The hard part of Theorem 1.5 is to show that our explicit criterion is sufficient. Whilst in characteristic 0 this follows from Galois descent (see Theorem 6.16), no descent technique can be used in characteristic $p > 0$ since $\text{SL}_n(D)$ is not defined over the fixed field of $\text{Aut}(K)$ (which is just $F_p$) when $D$ is non commutative. Hence we have to work by hand and give the splitting explicitly. In order to achieve this, for $K = F_{p^d}(T)$, we decompose $\text{Aut}(K)$ as $(J(K) \times F_{p^d}^{	imes}) \times \text{Gal}(K/F_p(T))$ (see Definition 6.19 for the definition of $J(K)$). Fortunately, it is easy enough to find an explicit section of $\text{Aut}(G \to \text{Spec} K) \to \text{Aut}_G(K)$ for the $J(K)$ components, and the theory of Galois descent predicts when a section to $\text{Aut}(G \to \text{Spec} K) \to \text{Aut}_G(K)$ exists for the component $F_{p^d}^{	imes} \times \text{Gal}(K/F_p(T))$. It thus suffices to compute explicitly those sections predicted by Galois descent, and to check that the formulas we found on each component can be glued together.

Acknowledgements

This paper grew out from the idea that the scheme of Dynkin diagrams provides an obstruction for a semisimple algebraic group to be defined over a given field. This idea was given in a comment to my MathOverflow question [Stu15] by user grghxy (probably a close relative to nldc23), and I gratefully thank him.

Jean-Pierre Tignol suggested at an early stage of this project that it should be possible to give an explicit example in some specific cases, which gave an early form of Corollary 5.14. He also pointed out many inaccuracies and made various comments that were very helpful. In particular,
he mentioned to us the work of T. Hanke and the fact that outer automorphisms of division algebras are related to \( H^3(k, G_m) \), as explained in [EM48]. He also provided a much cleaner version and proof of Lemma 6.6. I thank him for the very interesting discussions we had on this subject.

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I am also grateful to Philippe Gille, who kindly indicated that an earlier version of this paper was reproving a special case of [GP11b, Exposé 24, Théorème 3.11]. This pointer to the literature was very useful, since in the end this result is the central one around which our paper is organised.

Finally, I warmly thank Pierre-Emmanuel Caprace for asking the question that got this paper started, for encouraging me to investigate the \( SL_n(D) \) case and for his patient teaching on mathematical exposition.

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2 Semilinear automorphisms and Galois descent

For the rest of the paper, the letter \( k \) stands for an arbitrary field. By a \( k \)-group scheme, we mean an affine group scheme of finite type over \( k \). A smooth \( k \)-group scheme is called an algebraic group. Given an object \( X \) in a category, we write \( \text{Aut} X \) for the automorphisms of \( X \) in that category. Also given a \( k \)-scheme \( X \), we denote by \( \text{Aut} X \) its \( k \)-group functor of automorphisms (i.e. for any \( k \)-algebra \( R \), \( (\text{Aut} X)(R) = \text{Aut} X_R \)). With these conventions, for \( G \) a \( k \)-group scheme, \( \text{Aut} G \) is the automorphism functor of \( G \) evaluated at \( k \), i.e. \( \text{Aut} G = (\text{Aut} G)(k) \).

Let \( G \) be a \( k \)-group scheme. We gave in the introduction the definition of a semilinear automorphism of \( G \). The vocabulary “semilinear automorphism” is already used in the literature (see for example [FSS98, Section 1.2]). It has the same meaning than our usage, except that in those references, the underlying automorphisms of the base field are assumed to fix a subfield \( k_0 \) such that \( k/k_0 \) is Galois. We do not make this assumption, and for example in Section 6, we consider the case of arbitrary automorphisms of \( k = F_p(T) \), which is a more general situation.

In the literature, the notation \( \text{SAut}(G_{\text{a}_k}) \) is used for the group of semilinear automorphisms (see for examples [BKL14, Section 3.2] and also the references therein). We prefer to use the notation \( \text{Aut}(G \to \text{Spec} k) \) so that the ground field explicitly appears in the notation.

Remark 2.1. It is tempting to define a “semilinear automorphism sheaf of \( k \)-algebras” such that \( \text{Aut}(G \to \text{Spec} k) \) would be its \( k \)-rational points. Unfortunately, this is not possible, because we do not know how to extend functorially automorphisms of \( k \) to automorphisms of an arbitrary \( k \)-algebra \( R \).

Let us continue by recalling some standard vocabulary.

Definition 2.2. Let \( \varphi : k \to l \) be a field homomorphism (if \( l \) is a field containing \( k \), we take \( \varphi \) to be the identity), let \( G, G' \) be \( k \)-group schemes and let \( H \) be an \( l \)-group scheme.

1. The group of automorphisms of \( l \) whose restriction to \( \varphi(k) \) is trivial is denoted \( \text{Aut}(l/k) \).

2. We set \( \varphi^* = \text{Spec} \varphi \). We denote the base change of \( G \) along \( \text{Spec} l \xrightarrow{\varphi^*} \text{Spec} k \) either by \( G_l \) or by \( \varphi^*G \). If \( G_l \) is isomorphic to \( H \) (as an \( l \)-group scheme), we say that \( G \) is an \( l/k \)-form of \( H \) (or just a form of \( H \) if the field extension is understood from the context). If there exists an \( l/k \)-form of \( H \), we say that \( H \) is defined over \( k \).

3. For \( f : G \to G' \) a homomorphism of \( k \)-group schemes, we denote by \( \varphi f : \varphi^* G \to \varphi^* G' \) the base change of \( f \) along \( \varphi^* \).
Remark 2.3. Having set up our notations, let us elucidate the difference between our conventions and the conventions in [BT73]. Given a \( k \)-group scheme \( G \), a \( k' \)-group scheme \( G' \) and given an abstract homomorphism \( \alpha: G(k) \to G'(k') \), A. Borel and J. Tits aim to obtain a field homomorphism \( \varphi: k \to k' \) and an isogeny \( \beta: \varphi G \to G' \) such that for \( g \in G(k) = \text{Hom}_{k\text{-schemes}}(\text{Spec} \ k, G) \), \( \alpha(g) = \beta \circ \varphi g \). The following commutative diagram summarises the situation:

\[
\begin{array}{ccc}
G' & \xrightarrow{\beta} & G \\
\downarrow{\varphi G} & & \downarrow{g} \\
\text{Spec} \ k' & \xrightarrow{\text{Spec} \varphi} & \text{Spec} \ k
\end{array}
\]

On the other hand, the present paper focuses entirely on the group of semilinear automorphisms of \( G \). To keep the Borel–Tits convention, one should define this group as \( \{ \text{Isom}_{k\text{-grp schemes}}(\varphi G, G) \mid \varphi \in \text{Aut}(k) \} \). We prefer to use the more natural definition that a semilinear automorphism over \( \varphi \in \text{Aut}(k) \) is a commutative diagram of the following kind (note that either one of the red arrows determines the other):

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G \\
\downarrow{\text{Spec} \varphi} & & \downarrow{\text{Spec} \varphi} \\
\text{Spec} \ k & \xrightarrow{\text{Spec} \varphi} & \text{Spec} \ k
\end{array}
\]

where \( f_{\varphi} \) and \( \text{Spec} \varphi \) are both automorphisms of group schemes (but they are not automorphisms of \( k \)-group schemes when \( \varphi \) is not the identity). In this setting, there are two ways (admittedly not as natural as in the Borel–Tits setting) to obtain an abstract automorphism of \( G(k) \). Either we define this abstract automorphism proceeding “from right to left”, in which case we would obtain the map \( G(k) \to G(k): g \mapsto f_{\varphi}^{-1} \circ g \circ \text{Spec} \varphi \). Or we proceed “from left to right”, in which case we obtain the map \( G(k) \to G(k): g \mapsto f_{\varphi} \circ g \circ (\text{Spec} \varphi)^{-1} \). We chose the latter option.

The following elementary observation plays a fundamental role in this work.

Lemma 2.4. Let \( k \leq l \) be a field extension of \( k \), let \( G \) be an \( l \)-group scheme and assume that \( G \) is defined over \( k \). Then there exists a homomorphism \( \text{Aut}(l/k) \to \text{Aut}(G \to \text{Spec} l) \) whose composition with \( \text{Aut}(G \to \text{Spec} l) \to \text{Aut}_G(l) \) is the identity on \( \text{Aut}(l/k) \). In particular, \( \text{Aut}_G(l) \) contains \( \text{Aut}(l/k) \).

Proof. Let \( H \) be an \( l/k \)-form of \( G \). For \( \varphi \in \text{Aut}(l/k) \), we define

\[
f_{\varphi} = \text{Id}_H \times \varphi^*: H \times_{\text{Spec} \ k} \text{Spec} l \to H \times_{\text{Spec} \ k} \text{Spec} l.
\]

The map \( \text{Aut}(l/k) \to \text{Aut}(G \to \text{Spec} l): \varphi \mapsto f_{\varphi}^{-1} \) is a homomorphism. Furthermore, its composition with \( \text{Aut}(G \to \text{Spec} l) \to \text{Aut}_G(l) \) is the identity on \( \text{Aut}(l/k) \), as wanted. \( \Box \)

In fact, if the field extension \( l/k \) appearing in Lemma 2.4 is finite Galois, then we have a converse to Lemma 2.4 by the theory of Galois descent.

Theorem 2.5 (Galois descent). Let \( k \leq l \) be a field extension of \( k \) such that \( l/k \) is a finite Galois extension and let \( G \) be an \( l \)-group scheme. If there exists a homomorphism \( \text{Gal}(l/k) \to \text{Aut}(G \to \text{Spec} l) \) whose composition with \( \text{Aut}(G \to \text{Spec} l) \to \text{Aut}_G(l) \) is the identity on \( \text{Gal}(l/k) \), then \( G \) is defined over \( k \).

Proof. This is a classical result from descent theory, see [Poo17, Section 4.4]. Note that giving such a homomorphism is the same as giving a descent datum on \( G \) by [Poo17, Proposition 4.4.2], so that the result holds by [Poo17, Corollary 4.4.6]. \( \Box \)
**Remark 2.6.** One could also treat the case of infinite Galois extensions by adding a continuity assumption as in [FSS98, Remark 1.15], but we do not need it in our work. See also [Poo17, Remark 4.4.8] for how to deal with infinite Galois extensions.

In view of the strong link between Galois descent and semilinear automorphisms, it seems natural that there should be a cocycle interpretation of semilinear automorphisms. We now take some time to set up this formalism in detail.

**Definition 2.7.**

1. Let \( k \leq l \) be a field extension of \( k \), let \( G_0 \) be an \( l \)-group scheme and let \( G \) be an \( l/k \)-form of \( G_0 \). Choose an isomorphism \( G_0 \cong G_l \), or in other words choose an exact diagram

\[
\begin{array}{ccc}
G_0 & \xrightarrow{\pi_1} & G \\
\downarrow t & & \downarrow s \\
Spec l & \xrightarrow{\pi_0} & Spec k
\end{array}
\]

For any \( \gamma \in \text{Aut}(l/k) \), by the definition of base change there exists a unique isomorphism of \( G_0 \) above \( \gamma \) such that the following diagram commutes:

\[
\begin{array}{ccc}
G_0 & \xrightarrow{\gamma \circ \pi_1} & G \\
\downarrow \gamma \circ t & & \downarrow \gamma \circ s \\
Spec l & \xrightarrow{\gamma \circ \pi_0} & Spec k
\end{array}
\]

We denote this isomorphism \( \tilde{\gamma}_G \).

2. For \( G_0 \) a split connected reductive \( l \)-group we assume that an isomorphism with \( H_l \) has been chosen, where \( H \) is a split algebraic group over the prime field of \( l \). Now in this special situation, for \( \gamma \in \text{Aut}(l) \), instead of \( \tilde{\gamma}_H \) we use the more suggestive notation \( \text{Id}_{\gamma} \).

**Remark 2.8.**

(i) Note that when \( l/k \) is a finite Galois extension, the collection \( \{ \tilde{\gamma}_G \}_{\gamma \in \text{Gal}(l/k)} \) is nothing but a descent datum on \( G_0 \) (as defined in [Poo17, Proposition 4.4.2 (i)]) which descends to \( G \).

(ii) Note that for \( G_0 \) a split connected reductive \( l \)-group and \( \gamma \in \text{Aut}(l) \), if we choose a realisation of \( G_0 \) as a matrix group such that the realisation is defined over the prime field of \( l \), then for \( g = (g_{ij}) \in G_0(l) \) and \( \gamma \in \text{Aut}(l) \), \( \text{Id}_{\gamma} \circ g \circ (\gamma^{-1}) \in G_0(l) \) is given by the matrix whose \( ij \)-th coefficient is \( \gamma^{-1}(g_{ij}) \). This explains why we prefer to use the notation \( \text{Id}_{\gamma} \) in this situation.

We now study the behaviour of \( \tilde{\gamma}_G \) under base change.

**Lemma 2.9.** Let \( k \leq l \) be a field extension of \( k \), let \( G_0 \) be an \( l \)-group scheme and let \( G \) be a \( l/k \)-form of \( G_0 \). Fix an isomorphism \( G_l \cong G_0 \), or in other words fix an exact diagram

\[
\begin{array}{ccc}
G_0 & \xrightarrow{\pi_1} & G \\
\downarrow t & & \downarrow s \\
Spec l & \xrightarrow{\pi_0} & Spec k
\end{array}
\]
Let $\alpha \in \text{Aut}(k)$ and let $\beta \in \text{Aut}(l)$ be such that $\beta|_k = \alpha$. Further assume that $G_0$ is split connected reductive. Then there exists a unique map $\pi_\beta: G_0 \to G$ such that the following diagram commutes.

Furthermore, all squares appearing in this diagram are exact.

Proof. The existence and uniqueness of $\pi_\beta$ follows from the fact that the front square of the diagram is a base change. The fact that all squares are exact is a straightforward verification, using the fact that $\alpha^*$ and $\beta^*$ are isomorphisms.

Lemma 2.10. Keep the notations of Lemma 2.9, so that in particular we chose an isomorphism $G_0 \cong (\alpha G)_l$ via the exact diagram

$$
\begin{array}{c}
G_0 \\
\downarrow \Downarrow \\
Spec l \\
\end{array} \xrightarrow{\pi_0} \begin{array}{c}
Spec k \\
\Downarrow \\
\end{array} \xrightarrow{\alpha^*} \begin{array}{c}
Spec l \\
\Downarrow \\
\end{array}
$$

With these identifications of base change, for all $\gamma \in \text{Aut}(l/k)$ we have $\tilde{\gamma}_{\alpha G} = \text{Id}_G^{-1}(\beta^{-1}\gamma \beta)G \text{Id}_\beta$.

Proof. The proof follows from the commutativity of the following diagram

Indeed, $\tilde{\gamma}_{\alpha G}$ is defined to be the unique map such that the left hand side of the diagram commutes. But the front side and the back side of the diagram commutes by the definition of $\pi_\beta$ (see Lemma 2.9), whilst the right hand side of the diagram commutes by definition of $(\beta^{-1}\gamma \beta)_G$. Also

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note that \((\beta^{-1} \gamma \beta)^* = \beta^* \gamma^* (\beta^{-1})^*\), so that the bottom square of the diagram commutes as well. A diagram chase then shows that \(\text{Id}_\beta^{-1} (\beta^{-1} \gamma \beta) \circ \text{Id}_\beta\) satisfies the property uniquely defining \(\tilde{\gamma}_{\alpha_G}\), as was to be shown.

We can now state a clean descent formula for semilinear automorphisms in terms of cocycles. In this formula, we use the fact that for \(G_0\) a split connected reductive \(l\)-group, \(\text{Aut}(G_0 \to \text{Spec} \ l) \cong \text{Aut} G_0 \times \text{Aut}(l)\), where the splitting of the exact sequence \(1 \to \text{Aut} G_0 \to \text{Aut}(G_0 \to \text{Spec} l) \to \text{Aut}(l)\) is realised by \(\gamma \mapsto \text{Id}_\gamma^{-1}\) (see Definition 2.7 for the notation \(\text{Id}_\gamma\)). This thus defines a (left) action of \(\text{Aut}(l)\) on \(\text{Aut} G_0\) that we denote \(\gamma f\) (for \(\gamma \in \text{Aut}(l)\) and \(f \in \text{Aut} G_0\)). Explicitly, we have \(\gamma f = \text{Id}_\gamma^{-1} f \text{Id}_\gamma\).

**Lemma 2.11.** Let \(G_0\) be a split connected reductive \(l\)-group, let \(k\) be a subfield of \(l\) such that \(l/k\) is a (possibly infinite) Galois extension, and let \(G\) be a \(l/k\)-form of \(G_0\). Let \(\beta \in \text{Aut}(l)\) be such that \(\beta(k) = k\), and let \(\alpha = \beta_{\text{Gal}} \in \text{Aut}(k)\). Fix an isomorphism \(G_1 \cong G_0\) and let \(c : \text{Gal}(l/k) \to \text{Aut} G_0\) be the corresponding cocycle. Finally, let \(b \in \text{Aut} G_0\). Then \(b \text{Id}_\beta \in \text{Aut}(G_0 \to \text{Spec} l)\) descends to a semilinear automorphism of \(G\) over \(\alpha\) if and only if \(c_{\beta^{-1} \gamma \beta} \beta b \beta^{-1} c_{\gamma}^{-1} = b\) for all \(\gamma \in \text{Gal}(l/k)\).

**Proof.** Recall that a morphism of \(G_0\) over \(\beta\) is equivalent to an \(l\)-morphism from \(G_0\) to \(\beta G_0\). In this correspondence, \(b \text{Id}_\beta\) corresponds to \(\beta b \in \text{Aut} G_0\), as can be seen directly from the diagram

\[
\begin{array}{ccc}
\text{Spec} k & \xrightarrow{\beta^*} & \text{Spec} k \\
\downarrow t & & \downarrow t \\
G_0 & \xrightarrow{b \text{Id}_\beta} & G_0 \\
\downarrow \text{Id}_\gamma & & \downarrow \text{Id}_\gamma \\
\text{Spec} k & \xrightarrow{b} & \text{Spec} k
\end{array}
\]

Now by the general theory for morphisms between schemes with a descent datum, the \(l\)-morphism \(b : G_0 \to \beta G_0\) descends to a \(k\)-morphism \(G \to \alpha G\) if and only if \((\tilde{\gamma}_{\alpha_G})^{-1} (\beta b) \tilde{\gamma}_{\alpha_G} = \beta b\) for all \(\gamma \in \text{Gal}(l/k)\) (technically speaking, this is only true for finite Galois extensions, but our schemes are of finite type. Hence \(\beta b\) is defined over a finite Galois extension of \(k\) and we are just trying to descend from there). Using Lemma 2.10, we get that \(\beta b\) descends if and only if \(\text{Id}_\beta^{-1} (\beta^{-1} \gamma \beta) \circ \text{Id}_\beta (\beta b) \circ \text{Id}_\beta = \beta b\). Finally, to transfer this to a cocycle condition, recall that in the correspondence between descent datum and cocycle, we have (in our notations) \(\tilde{\gamma}_{\alpha_G} = c_{\gamma}^{-1} \text{Id}_\gamma\) (this of course relies on the fact that we used the same isomorphism \(G_0 \cong G_1\) to define \(c_\gamma\) and \(\tilde{\gamma}\)). Furthermore, by definition \(c : \text{Gal}(l/k) \to \text{Aut} G_0\) is a cocycle for the Galois action introduced before the statement of the theorem, i.e. for \(\gamma, \delta \in \text{Gal}(l/k)\) we have \(c_{\gamma \delta} = c_{\gamma} \gamma c_{\delta} = c_{\gamma} \text{Id}_{\gamma}^{-1} c_{\delta} \text{Id}_{\gamma}\). Hence, the conclusion of the theorem readily follows.

**Remark 2.12.** In our conventions, if \(\gamma \in \text{Gal}(l/k)\) appears in exponent, then it acts on the element appearing below it on the right. So if one wishes to put more parenthesis in the formula \(c_{\beta^{-1} \gamma \beta} \beta^{-1} \gamma b \beta^{-1} c_{\gamma}^{-1}\), the unique way to do so respecting this convention is by writing \(c_{\beta^{-1} \gamma \beta} (\beta^{-1} \gamma b) (\beta^{-1} c_{\gamma}^{-1})\). Note also that \(\beta \in \text{Aut}(k)\) acts by group automorphisms on \(\text{Aut} G\), so that \(\beta (c_{\gamma}^{-1}) = (\beta^{-1} c_{\gamma})^{-1}\), i.e. there is no need for any parenthesis to distinguish the two.

**Remark 2.13.** If \(\beta\) is the identity, our formula specialises to the usual condition for \(b\) to descend to a \(k\)-automorphism of \(G\), namely \(c_{\gamma} \gamma b c_{\gamma}^{-1} = b\) for all \(\gamma \in \text{Gal}(l/k)\). Also note that for \(\gamma \in \text{Gal}(l/k)\), the automorphism \(\tilde{\gamma}_{\alpha_G} = c_{\gamma}^{-1} \text{Id}_\gamma \in \text{Aut}(G_0 \to \text{Spec} l)\) satisfies the condition to descend (of course, it descends to the trivial automorphism of \(G\)).
3 Schemes of based root datum

In [GP11b, Exposé 24, section 3], the authors define what they call a “Dynkin’s scheme” of a reductive group \( G \). The strategy is to first define this Dynkin’s scheme for split reductive groups, and then to use descent. The Dynkin’s scheme is well suited to describe quasi-split semisimple groups that are adjoint or simply connected. Since there is not much more work to define a scheme of based root datum and since this allows us to treat the more general case of quasi-split reductive groups, we decided it was worth doing it.

In order to define a scheme of based root datum, we need the notion of a \( \mathbb{Z} \)-module scheme and of perfect duality between two \( \mathbb{Z} \)-module schemes. Recall that throughout the paper, the letter \( k \) stands for a field.

**Definition 3.1.** Let \( R \) be a \( k \)-algebra and let \( M \) be a \( R \)-scheme. \( M \) is called a \( \mathbb{Z} \)-module \( R \)-scheme if \( M \) is a (non necessarily affine) commutative \( R \)-group scheme.

Recall that given any set \( E \) and a \( k \)-algebra \( R \), we can consider the constant object on \( E \) which is defined to be the \( R \)-scheme \( E_R = \prod_{e \in E} \text{Spec} \, R \). This defines a fully faithfull functor from the category of Sets to the category of \( R \)-schemes, called the constant object functor. The constant object functor commutes with forming finite products (see [GP11a, Exposé 1, Section 1.8]). Hence given a \( \mathbb{Z} \)-module \( M \), the constant scheme \( M_R \) acquires the structure of a \( \mathbb{Z} \)-module \( R \)-scheme. We can now define the notion of perfect pairing for \( \mathbb{Z} \)-module \( k \)-schemes.

**Definition 3.2.** Let \( M, M' \) be two \( \mathbb{Z} \)-module \( k \)-schemes.

1. The dual of \( M \), denoted \( M^\vee \), is defined to be the functor from the category of \( k \)-algebras to the category of sets sending a \( k \)-algebra \( R \) to \( \text{Hom}_R(M, \mathbb{Z}_R) \).

2. We say that \( M \) and \( M' \) are in duality if there exists \( f \in \text{Hom}_k(M \times M' \to \mathbb{Z}_k) \). We say that the duality \( f \) is perfect if for all \( k \)-algebra \( R \), the map \( M'(R) \to M'(R): m \mapsto (f(.,m): M_R \to \mathbb{Z}_R) \) is an isomorphism (here, for any \( R \)-algebra \( R' \) and \( n \in M_R(R') \), \( f(.,m)(n) = f(n,m_{R'}) \), with \( m_{R'} \) being the image of \( m \) under \( M'(R) \to M'(R') \)).

**Remark 3.3.** As usual in this situation, one should restrict the categories under considerations to avoid set theoretic problems. One way to do so is by using universe.

Note that \( M^\vee \) is a commutative group functor, and hence \( M^\vee \) is a \( \mathbb{Z} \)-module \( k \)-scheme if \( M^\vee \) is representable. Also note that given \( f \in \text{Hom}_k(M, M') \) we can define \( f^\vee \in \text{Hom}_k(-; \mathbb{Z}_k) \) mimicking the definition for \( \mathbb{Z} \)-modules. Namely for all \( k \)-algebras \( R \), \( f^\vee(R): M^\vee(R) \to M^\vee(R): \alpha \mapsto \alpha \circ f_R \), where \( f_R: M_R \to M'_R \) denotes the base change of \( f \) to \( R \). We call \( f^\vee \) the dual of \( f \).

By a reduced based root datum \( R \), we mean a reduced root datum \( (M, M^*, \Phi, \Phi^*) \) (as defined in [GP11b, Exposé 21, Définition 1.1.1 and Définition 2.1.3]) together with a choice of simple roots \( \Delta \subseteq \Phi \). We can finally give the definition of a \( k \)-scheme of root datum.

**Definition 3.4.** 1. A \( k \)-scheme of based root datum is a 5-tuple \( \mathcal{R} = (\mathcal{M}, \mathcal{M}^*, \Psi, \Psi^*, \Gamma) \) where:

(a) \( \mathcal{M} \) and \( \mathcal{M}^* \) are \( \mathbb{Z} \)-module \( k \)-schemes in perfect duality.

(b) \( \Psi, \Psi^* \) and \( \Gamma \) are finite \( k \)-schemes, and there are closed immersions \( \Gamma \hookrightarrow \Psi \hookrightarrow \mathcal{M} \) and \( \Psi^* \hookrightarrow \mathcal{M}^* \).

(c) There is an isomorphism of \( k \)-schemes \( \Psi \cong \Psi^* \).

(d) There exists a reduced based root datum \( R = (M, M^*, \Phi, \Phi^*, \Delta) \) and a finite Galois extension \( l/k \) together with an isomorphism of \( \mathbb{Z} \)-module \( l \)-schemes \( f: M_l \to M_l \) such that \( f \) induces an isomorphism of \( l \)-schemes \( \Phi_l \cong \Psi_l, f^\vee \) induces an isomorphism of \( l \)-schemes \( \Psi^*_l \cong \Phi^*_l, f(\Delta_l) = \Gamma_l \) and the composition \( \Phi_l \cong \Psi_l \cong \Psi^*_l \cong \Phi^*_l \) induces the bijection \( \Phi \to \Phi^*: \alpha \to \alpha^* \) given in the definition of \( R \). In this case, we say that \( \mathcal{R} \) is of type \( R \).
2. Let \( R = (M, M^*, \Phi, \Phi^*, \Delta) \) be a reduced based root datum. Using the constant object functor, the 5-tuple \( R_k = (M_k, M^*_k, \Phi_k, \Phi^*_k, \Delta_k) \) has a natural structure of a \( k \)-scheme of based root datum. We call it the split k-scheme of based root datum of type \( R \). A split k-scheme of based root datum is a split k-scheme of based root datum of type \( R \) for some reduced based root datum \( R \).

3. Given \( \mathcal{R} = (M, M^*, \Psi, \Psi^*, \Gamma) \) and \( \mathcal{R}' = (M', M'^*, \Psi', \Psi'^*, \Gamma') \) two k-schemes of based root datum, a k-morphism \( \mathcal{R} \to \mathcal{R}' \) is a k-homomorphism \( f: M \to M' \) such that \( f \) induces two \( k \)-isomorphisms \( \Psi \cong \Psi' \) and \( \Gamma \cong \Gamma' \), and such that \( f^t \) induces a \( k \)-isomorphism \( \Psi^* \cong \Psi'^* \).

**Remark 3.5.** Let us stress that with this definition, if a scheme of based root datum is of type \( R \), then \( R \) is a reduced based root datum. It would be safer (but more tedious) to call these objects “k-schemes of reduced based root datum”.

**Remark 3.6.** Let \( k \) be a separable closure of \( k \), let \( R_k = (M_k, M^*_k, \Phi_k, \Phi^*_k, \Delta_k) \) be a split k-scheme of based root datum of type \( R \) and let \( E = \text{Aut} R \) (see [GP11b, Exposé 21, Définition 6.1.1] for the definition of the morphisms in the category of based root datum). Then \( \text{Aut} R_k = E \), and the action of \( \text{Gal}(k/k) \) on \( E \) is trivial. Hence elements of \( H^1(k; E) \) are continuous homomorphisms \( \text{Gal}(k/k) \to E \) up to conjugation. Also recall that \( H^1(k, E) \) classifies k-schemes of based root datum of type \( R \). Indeed, Galois descent for \( M_k \) and \( M^*_k \) is effective because they can be covered by quasi-affine open that are stable under \( \text{Gal}(k/k) \) (hence effective is ensured by [Pool87, Theorem 4.3.5]), and the other structures (the duality \( M_k \times M^*_k \to k \), the closed immersions \( \Delta_k \hookrightarrow \Phi_k \hookrightarrow M_k \), \( \Phi_k^* \hookrightarrow M^*_k \) and the \( k \)-isomorphism \( \Phi_k \cong \Phi^*_k \)) will descend as well.

**Remark 3.7.** If \( R = (M, M^*, \Phi, \Phi^*, \Delta) \) is a based root datum such that \( \Phi \) is empty, then \( \text{Aut} R \cong GL_n(k) \) where \( n \) is the rank of \( M \). Hence in this case, k-schemes of based root datum of type \( R \) classify k-tori of rank \( n \).

Note that given \( \alpha \in \text{Aut}(k) \) and a k-scheme of based root datum \( R \), we have an obvious notion of base change of \( R \) along \( \alpha \), and we denote this base change by \( \alpha R \).

**Definition 3.8.** Let \( R \) be a k-scheme of based root datum, and let \( \alpha \in \text{Aut}(k) \). A semilinear automorphism of \( R \) over \( \alpha \) is an isomorphism of k-schemes of based root datum \( f_{\alpha}: R \to \alpha R \). Given a semilinear automorphism \( f_{\alpha} \) (respectively \( f_{\beta} \)) over \( \alpha \) (respectively \( \beta \)) in \( \text{Aut}(k) \), their composition is \( (\alpha f_{\beta}) f_{\alpha} \), which is a semilinear automorphism over \( \alpha \beta \). We denote the group of semilinear automorphisms of \( R \) by \( \text{Aut}(R \to \text{Spec} k) \).

As in the case of k-group schemes, for \( R \) a k-scheme of based root datum, we have a homomorphism \( \text{Aut}(R \to \text{Spec} k) \to \text{Aut}(k); f_{\alpha} \to \alpha^{-1} \). We let \( \text{Aut}_R(k) \) be the image of this homomorphism. Furthermore denoting the \( k \)-automorphisms of the based root datum \( R \) by \( \text{Aut} R \) (or also \( (\text{Aut} R)(k) \), following the conventions discussed at the start of Section 2), we get a short exact sequence \( 1 \to \text{Aut} R \to \text{Aut}(R \to \text{Spec} k) \to \text{Aut}_R(k) \to 1 \).

We now discuss how to associate functorially a k-scheme of based root datum to a connected reductive k-group. One possible approach would be to take an inductive limit of based root datum in the split case, and then descend this canonical object to any form. This would lead to the same construction as the one we now explain.

Actually, it suffices to incorporate our definition of the k-scheme of based root datum in [GP11b, Exposé 24, Théorème 3.11], by replacing principal Galois cover of group \( E = \text{Aut} R \) with the objects over \( k \) that they classify (i.e. k-schemes of based root datum). As a corollary, we will get the definition of the k-scheme of based root datum of a connected reductive k-group. First, we recall the definition of the group scheme of exterior isomorphisms.

**Definition 3.9** ([GP11b, Exposé 24, Corollaire 1.10]). Let \( G, G' \) be two connected reductive k-group of type \( R \), for some reduced based root datum \( R \). Then \( \text{Ad} G \) acts freely (on the right) on the k-group functor \( \text{Isom}_{\text{gr}}(G, G') \). We define the k-group functor of exterior isomorphisms between \( G \) and \( G' \) to be the quotient sheaf \( \text{Ext} \text{isom}(G, G') = \text{Isom}_{\text{gr}}(G, G')/\text{Ad} G \).
Remark 3.10. Actually, [GP11b, Exposé 24, Corollaire 1.10] asserts that this quotient is representable. Since it will be useful later, here is an explicit description of $\text{Ext}_{\text{Iso}}(G, G')$ using cocycles: let $k_0$ be a separable closure of $k$, let $E = \text{Aut } R$, let $G_0$ be the (split) connected reductive $k_0$-group of type $R$ and identify $\text{Aut } G_0$ with $\text{Ad } G_0 \times E_{k_0}$ (after a choice of pinning for $G_0$, see [GP11b, Exposé 24, Théorème 1.3]). Fix an isomorphism $G_0 \cong G_{k_0}$ (respectively $G_0 \cong G'_{k_0}$), denote by $c$ (respectively $c'$) the corresponding cocycle $\text{Gal}(k_0/k) \to \text{Aut } G_0$ and let $\tilde{c}$ (respectively $\tilde{c}'$) be the composition of this cocycle with $\text{Aut } G_0 \to E$. Further assume that the pinning of $G_0$ is defined over $k$, so that the Galois action on $\text{Aut } R_{k_0} = E$ is trivial. Then $\text{Ext}_{\text{Iso}}(G, G')$ is a $k_0/k$-form of $E_{k_0}$, given by the Galois action $\gamma.f = \tilde{c}'\gamma.f\tilde{c}^{-1}$ (for all $f \in E_{k_0}$ and for all $\gamma \in \text{Gal}(k_0/k)$). This follows directly from the Galois condition for an automorphism of $G_0$ to descend to an isomorphism $G \to G'$, together with the fact that we are moding out by adjoint automorphisms.

Let us also recall the notion of a quasi-pinning.

Definition 3.11. Let $G$ be a connected reductive $k$-group. If it exists, a quasi-pinning of $G$ is:

1. A choice of a Borel subgroup $B$ containing a maximal torus $T$ of $G$. Once this is chosen, let $k_0$ be a separable closure of $k$, let $\Delta$ be the fundamental roots of $G_{k_0}$, corresponding to the pair $(T_{k_0}, B_{k_0})$ and for $\alpha \in \Delta$, let $g_\alpha$ be the corresponding one dimensional subspace of $\text{Lie}(G_{k_0})$.

2. A choice of a nontrivial element $X_\alpha \in g_\alpha$ for all $\alpha \in \Delta$ such that for all $\gamma \in \text{Gal}(k_0/k)$, $\gamma.X_\alpha = X_{\gamma(\alpha)}$.

If $G$ has a quasi-pinning, we say that $G$ is quasi-split.

The more classical definition for a connected reductive $k$-group to be quasi-split is that it possesses a Borel subgroup. It is well-known that this definition agrees with Definition 3.11 (and the equivalence is proved in a more general setting in [GP11b, Exposé 24, Proposition 3.9.1]). For the convenience of the reader, let us reprove this fact.

Lemma 3.12. Let $G$ be a connected reductive $k$-group. If $G$ has a borel subgroup, then $G$ has a quasi-pinning.

Proof. Let $B$ be a borel subgroup of $G$. Then it contains a maximal torus $T$ of $G$ (see for example [GP11b, Exposé 22, Corollaire 5.9.7]). Let $k_0, \Delta, \alpha \in \Delta$ and $g_\alpha$ be as in Definition 3.11 for the pair $(T, B)$. Let $H \leq \text{Gal}(k_0/k)$ be the stabiliser of $\alpha$ and let $k_\alpha$ be the subfield of $k_0$ fixed by $H$. Then there exists an $H$-equivariant isomorphism $k_\alpha \otimes_{k_0} k \cong g_\alpha$ (this holds because all $k_0/k_\alpha$-forms of the vector space $k_\alpha$ are equivalent by Hilbert’s 90). Set $X_\alpha = 1 \in k_\alpha \subset g_\alpha$. Now, for $\beta$ in the $\text{Gal}(k_0/k)$-orbit of $\alpha$, we set $X_\beta = \gamma.X_\alpha$ where $\gamma \in \text{Gal}(k_0/k)$ is any element such that $\gamma(\alpha) = \beta$. The point is that $X_\beta$ does not depend on a choice of $\gamma \in \text{Gal}(k_0/k)$ such that $\gamma(\alpha) = \beta$ because $H$ acts trivially on $X_\alpha$. Doing so for each orbit of $\text{Gal}(k_0/k)$ on $\Delta$ concludes the proof.

Theorem 3.13 ([GP11b, Exposé 24, Théorème 3.11]). Let $R = (M, M^*, \Phi, \Phi^*, \Delta)$ be a reduced based root datum. Consider the following categories:

1. The category $\text{BRD}$ of $k$-schemes of based root datum of type $R$. The morphisms are the isomorphisms of $k$-schemes of based root datum.

2. The category $\text{Red}\text{Ext}$ of connected reductive $k$-groups of type $R$. The morphisms between $G$ and $G'$ are elements of the group $\text{Ext}_{\text{Iso}}(G, G')(k)$.

3. The category $\text{QsPin}$ of connected reductive quasi-split $k$-groups of type $R$, together with a choice of quasi-pinning. The morphisms are the isomorphisms preserving the quasi-pinning.
These three categories are equivalent. More precisely, we have a diagram of functors between categories

\[
\begin{array}{ccc}
\text{BRD} & \xrightarrow{\text{qspin}} & \text{QsPin} \\
\text{brd} & \downarrow & \downarrow \\
\text{RedExt} & & \\
\end{array}
\]

such that the composition of those three functors (starting with anyone of them) is naturally isomorphic to the identity.

**Proof.** We follow the proof given in [GP11b, Exposé 24, Section 3.11], with the advantage that we can work with the more concrete Galois descent, and that the notion of pinning is simpler over fields.

Set \( E = \text{Aut} R \). Let \( k_s \) be a separable closure of \( k \) and let \( G_0 \) be the (split) connected reductive \( k_s \)-group of type \( R \). For the proof, we choose a pinning for \( G_0 \) which is defined over \( k \), i.e. we choose a pinning for the split connected reductive \( k \)-group of type \( R \) and we base change it to a pinning of \( G_0 \). In particular we choose a torus \( T \) contained in a Borel subgroup \( B_0 \) (both defined over \( k \)), and we get an identification \( \text{Aut} G_0 \cong \text{Aut} G_k \times E_{k_s} \) (where the Galois descent on \( E_{k_s} \) is trivial), and in particular an embedding \( E = E_{k_s}(k_s) \to (\text{Aut} G_0)(k_s) = \text{Aut} G_0 \).

1. The functor \( \iota \). On objects, \( \iota(G) \) is the natural inclusion whilst for \( f \in \text{Mor}_{\text{QsPin}}(G, G') \), \( \iota(f) \) is the projection of \( f \) in \( \text{Extisom}(G, G')(k) = (\text{Isom}_{k, \text{gr.}}(G, G')/\text{Ad} G)(k) \).

2. The functor \( \text{qspin} \). Let \( R \) be a \( k \)-scheme of based root datum of type \( R \). Choose an isomorphism \( R_l \cong R_l \) for some finite Galois extension \( l/k \) and let \( c: \text{Gal}(k_s/k) \to E \) be the corresponding cocycle. The quasi-split group \( \text{qspin}(R) \) is the \( k_s/k \)-form of \( G_0 \) defined by the cocycle \( c: \text{Gal}(k_s/k) \to E \to \text{Aut} G_0 \). Note that this cocycle preserves \( T_0 \) and \( B_0 \), so that \( \text{qspin}(R) \) is indeed quasi-split. We choose for quasi-pinning on \( \text{qspin}(R) \) the pair \( (T_0, B_0) \) descended to \( k \), and for \( \alpha \in \Delta \), we choose the element \( X_\alpha \in \text{Lie}(G_0) \) to be the same as the one appearing in the pinning of \( G_0 \). Since the pinning of \( G_0 \) is defined over \( k \) by assumption, this indeed constitutes a quasi-pinning of \( \text{qspin}(R) \). Finally, for a morphism \( f \in \text{Mor}_{\text{BRD}}(R, R') \), \( \text{qspin}(f) \) is defined to be the descent of \( f_{k_s} \in \text{Mor}(R_{k_s}, R'_{k_s}) \cong E \subset \text{Aut} G_0 \) to an isomorphism \( \text{qspin}(R) \to \text{qspin}(R') \).

3. The functor \( \text{brd} \). For \( G \) a connected reductive group of type \( R \), choose an isomorphism \( G_0 \cong G_{k_s} \) and let \( c: \text{Gal}(k_s/k) \to \text{Aut} G_0 \) be the corresponding cocycle. Consider \( \tilde{c} \), the composition of \( c \) with the projection \( \text{Aut} G_0 \to E \). Now \( \text{brd}(G) \) is defined to be the \( k_s/k \)-form of the split \( k_s \)-scheme of root datum \( R_{k_s} \) obtained by Galois descend using the cocycle \( \tilde{c} \). Whilst for a morphism \( f \in \text{Mor}_{\text{RedExt}}(G, G') \), \( f_{k_s} \in \text{Extisom}(G_{k_s}, G'_{k_s})(k_s) \cong \text{Aut} R_{k_s}, k_s \), and \( \text{brd}(f) \) is defined to be the descent of \( f_{k_s} \) to an isomorphism \( \text{brd}(G) \to \text{brd}(G') \).

We now check that the composition of those three functors (starting with anyone of them) is naturally isomorphic to the identity.

1. \( \text{brd} \circ \text{qspin} \cong \text{Id}_{\text{BRD}} \). Let \( c': \text{Gal}(k_s/k) \to E \) be the cocycle arising from a choice of \( R_l \cong R_l \). By definition of \( \text{qspin} \), a choice of isomorphism \( G_0 \cong \text{qspin}(R)_{k_s} \) gives rise to a cocycle \( c \) which is cohomologous to \( c' \) (as cocycles with values in \( \text{Aut} G_0 \)), hence the \( \tilde{c} \) appearing in the definition of \( \text{brd} \) is cohomologous to \( c' \) as well (as cocycles with values in \( E \)).

2. \( \text{qspin} \circ \text{brd} \circ \iota \cong \text{Id}_{\text{QsPin}} \). We need to check that given a quasi-split group \( G \) together with a choice of isomorphism \( G_0 \cong G_{k_s} \) and corresponding cocycle \( c: \text{Gal}(k_s/k) \to \text{Aut} G_0 \), then \( c \) is cohomologous to \( c \) composed with \( \text{Aut} G_0 \to E \to \text{Aut} G_0 \). The quasi-pinning on \( G \) gives a pinning of \( G_{k_s} \), which is sent by \( G_0 \cong G_{k_s} \) to a pinning of \( G_0 \). Up to conjugation by \( g \in G_0(k_s) \), which has the effect of replacing \( c \) by a cohomologous cocycle, we can assume that this pinning of \( G_0 \) is the one we chose from the outset. Because the pinning of \( G_0 \) is
defined over $k$, it is invariant under the action of $\text{Gal}(k_s/k)$. Hence, the cocycle $c$ has values in $E$, as wanted.

3. $i \circ \text{qspin} \circ \text{brd} \cong \text{Id}_{\text{RedExt}}$. Let $G$ be a connected reductive group of type $R$. We want to check that $\text{Extisom}(G, G')(k) \neq \emptyset$, where $G' = (i \circ \text{qspin} \circ \text{brd})(G)$. Let $G_0 \cong G_{k_s}$ be the chosen isomorphism to define $\text{brd}(G)$, with corresponding cocycle $c$, and let $\tilde{c}$ be the projection of $c$ under $\text{Aut}G_0 \to E$. By definition, a cocycle defining $G'$ is cohomologous to $\tilde{c}$, so we can assume that $G'$ is defined by $\tilde{c}$. Now, by Remark 3.10, the identity on $G_0$ descends to an element of $\text{Extisom}(G, G')(k)$, concluding the proof. 

In view of Theorem 3.13, one can attach in a functorial way a $k$-scheme of based root datum to any connected reductive $k$-group.

**Definition 3.14.** Let $G$ be a connected reductive $k$-group. The $k$-scheme of based root datum associated to $G$ is $\text{brd}(G)$, where $\text{brd}$ is the functor appearing in Theorem 3.13. We denote it $\mathcal{R}(G)$.

The crucial input is that taking the scheme of based root datum commutes with base change.

**Lemma 3.15.** Let $G$ be a connected reductive $k$-group and let $\alpha$ be an automorphism of $k$. Then $\mathcal{R}(\alpha G) \cong ^s\mathcal{R}(G)$, naturally in $G$.

**Proof.** Let $R$ be the type of $G$, let $k_s$ be a separable closure of $k$, and let $G_0$ be the (split) connected $k_s$-group of type $R$. Let $\beta$ be an extension of $\alpha$ to $k_s$, choose an isomorphism $G_0 \cong G_{k_s}$, and let $G_0 \cong (\alpha G)_{k_s}$ be the corresponding isomorphism defined in Lemma 2.9. Now by Lemma 2.10, if $c$ denotes the cocycle defining $G$, then the corresponding cocycle $^\alpha c$ defining $^\alpha G$ is given by $(^\alpha c)_\gamma = \text{Id}^{-1}_\beta c_{\gamma-1,\beta}\text{Id}_\beta$, for all $\gamma \in \text{Gal}(k_s/k)$. Finally, we choose a pinning of $G_0$ defined over the prime field of $k$ (so that we can identify $\text{Aut}G_0 \cong (\text{Ad}G_0)(k_s) \times \text{Aut}R$), and we let $\tilde{c}$ (respectively $^\alpha \tilde{c}$) be the projection of $c$ (respectively $^\alpha c$) under $\text{Aut}G_0 \to \text{Aut}R$. Note that since the Galois action on $\text{Aut}R$ is trivial, $(^\alpha \tilde{c})_\gamma$ is just the projection of $c_{\gamma-1,\gamma}$ onto $\text{Aut}R$.

On the other side, let $R_{k_s}$ be the split $k$-scheme of based root datum of type $R$. The (choice of) cocycle defining $\mathcal{R}(G)$ is $\tilde{c} : \text{Gal}(k_s/k) \to \text{Aut}R$. Now exactly the same computation as for algebraic groups (i.e. repeating Lemma 2.9 and Lemma 2.10 in the category of schemes of based root datum) shows that a cocycle defining $^\alpha \mathcal{R}(G)$ is given by $\gamma \mapsto \text{Id}^{-1}_\beta \tilde{c}_{\gamma-1,\gamma}\text{Id}_\beta = \tilde{c}_{\gamma-1,\gamma}$. But this is also the chosen cocycle defining $\mathcal{R}(\alpha G)$, as was to be shown. The naturality in $G$ of this isomorphism is straightforward.

**Remark 3.16.** Of course, for this whole section, we did not need the fact that the base scheme is the spectrum of a field, and for example, Lemma 3.15 should be true over any base scheme, and under any base change. The advantage of working over a field is that the notion of pinning is simpler, and that Galois descent is more concrete than fppf descent.

The proof of Theorem 1.1 now follows easily from Theorem 3.13 and Lemma 3.15.

**Proof of Theorem 1.1.** Recall that to give an automorphism $f_\alpha$ of $G$ over $\alpha \in \text{Aut}(k)$ is equivalent to give an isomorphism of $k$-group schemes $f : G \to ^\alpha G$. Hence, projecting $f$ to an element $\tilde{f} \in \text{Extisom}(G, ^\alpha G)(k)$ and using the functor $\text{brd}$ defined in Theorem 3.13, we get an isomorphism $\text{brd}(f) : \mathcal{R}(G) \to \mathcal{R}(^\alpha G) \cong ^\alpha \mathcal{R}(G)$ (where we used Lemma 3.15 for the last isomorphism). Now since $\text{brd}$ is a functor, and because the isomorphism $\mathcal{R}(^\alpha G) \cong ^\alpha \mathcal{R}(G)$ is natural in $G$, the map $f_\alpha \mapsto \text{brd}(f)$ is a group homomorphism which is natural in $G$. Furthermore, the underlying automorphism of the field is preserved by this homomorphism. To conclude the first part of the proof, note that $f_\alpha$ is in the kernel of this homomorphism if and only if $\alpha$ is trivial and $f$ is trivial in $\text{Extisom}(G, G)(k)$, which is to say that $f \in (\text{Ad}G)(k)$.

For the last assertion, assume that $G$ is quasi-split and choose a quasi-pinning of it. Define the subgroup $H = \{f_\alpha \in \text{Aut}(G) \to \text{Spec } k \mid f_\alpha \text{ preserves the quasi-pinning of } G\}$. Seeing $f_\alpha$ as an isomorphism $f$ from $G$ to $^\alpha G$, the condition for $f$ to belong to $H$ is that it preserves the quasi-pinnings (where $^\alpha G$ is endowed with the quasi-pinning on $G$ based changed to $^\alpha G$). Now the fact
that $H$ maps isomorphically onto $\text{Aut}(\mathcal{R}(G) \to \text{Spec } k)$ under $\text{Aut}(G \to \text{Spec } k) \to \text{Aut}(\mathcal{R}(G) \to \text{Spec } k)$ is a direct consequence of Lemma 3.15 and of the equivalence of categories $\text{BRD}$ and $\text{QsPin}$ in Theorem 3.13.

Remark 3.17. For $G$ a connected reductive $k$-group which is not quasi-split, the decomposition $\text{Aut } G \cong (\text{Ad } G)(k) \rtimes \text{Out } G$ as a semidirect product is usually destroyed. Similarly, one should not expect to obtain a semidirect decomposition of $\text{Aut}(G \to \text{Spec } k)$ for a general connected reductive $k$-group. Investigating a possible semidirect decomposition of the group of semilinear automorphisms of simple algebraic groups is an entirely different matter when $G$ is not quasi-split, as is illustrated by our treatment of the $\text{SL}_n(D)$ case in Section 6.

As a corollary of Theorem 1.1, we obtain a proof of Theorem 1.3.

Proof of Theorem 1.3. By Theorem 1.1, $\text{Aut}_{\mathcal{R}(G)}(k) = \text{Aut}_G(k)$. We thus obtain the following commutative diagram:

\[
\begin{array}{c c c}
1 & \rightarrow & \text{Aut}(G) \\
\downarrow & & \downarrow \\
(\text{Ad } G)(k) & \rightarrow & \text{Aut}_{\mathcal{R}(G)}(k) \\
\downarrow & & \downarrow \\
\text{Out } G & \rightarrow & \text{Aut}_{\mathcal{R}(G)} \rightarrow \text{Spec } k \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1
\end{array}
\]

where all diagonal lines and vertical lines are exact. Here, $\pi$ denotes the homomorphism provided by Theorem 1.1, and $\iota$ is a section of $\pi$ (which exists, again by Theorem 1.1). Note that in particular, $\iota$ preserves the underlying field automorphism, i.e. $p_1 \circ \iota = p_2$.

We thus conclude that the short exact sequence $1 \rightarrow \text{Aut } G \rightarrow \text{Aut}(G \to \text{Spec } k) \rightarrow \text{Aut}_G(k) \rightarrow 1$ splits if and only if the short exact sequence involving $k$-schemes of based root datum $1 \rightarrow \mathcal{R}(G) \rightarrow \text{Aut}(\mathcal{R}(G) \to \text{Spec } k) \rightarrow \text{Aut}_{\mathcal{R}(G)}(k) \rightarrow 1$ does, as was to be shown.

4 Semilinear automorphisms and Galois cohomology

We have just proved that for any connected reductive algebraic $k$-group $G$, we have a natural exact sequence $1 \rightarrow (\text{Ad } G)(k) \rightarrow \text{Aut}(G \to \text{Spec } k) \rightarrow \text{Aut}(\mathcal{R}(G) \to \text{Spec } k)$. It would be nice to be able to express the failure of surjectivity on the right using Galois cohomology. We explain in this section how to do so.

In this section, $k_0$ denotes a separable closure of $k$ with Galois group $\Gamma = \text{Gal}(k_0/k)$. $R$ is a reduced based root datum, $G_0$ is a (split) connected reductive $k_0$-group of type $R$ with a choice of pinning defined over the base field of $k$, and $R_{k_0}$ is the split $k_0$-scheme of based root datum of type $R$. We furthermore set $E = \text{Out } R$ and we let $E_{k_0}$ be the corresponding constant object over $k_0$.

Also, we again use the convention that $G_0$ comes together with a preferred split form of it over the prime field of $k$. In particular, we get a decomposition $\text{Aut}(G_0 \to \text{Spec } k_0) \cong \text{Aut } G_0 \rtimes \text{Aut}(k_0)$, where the splitting $\text{Aut}(k_0) \rightarrow \text{Aut}(G_0 \to \text{Spec } k_0)$ is given by $\beta^{-1} \mapsto \text{Id}_{\beta}$ (see Definition 2.7).
Definition 4.1. 1. Given a field extension \( l \geq k \), we set \( \text{Aut}(l \geq k) = \{ \alpha \in \text{Aut}(l) \mid \alpha(k) = k \} \)

2. We set \( \text{Aut}(G_0 \to \text{Spec} k_\alpha \geq k) = \{ f_\alpha \in \text{Aut}(G_0 \to \text{Spec} k_\alpha) \mid \text{the underlying } \alpha \in \text{Aut}(k_\alpha) \text{ belongs to } \text{Aut}(k_\alpha \geq k) \} \).

3. We denote an element of \( \text{Aut}(G_0 \to \text{Spec} k_\alpha \geq k) \cong \text{Aut}(G_0 \rtimes \text{Aut}(k_\alpha \geq k)) \) by \( b \text{Id}_b \) (where \( b \in \text{Aut}G_0 \) and \( \beta \in \text{Aut}(k_\alpha \geq k) \)).

Remark 4.2. In Definition 4.1, \( \alpha \) is required to globally preserve \( k \), but its restriction to \( k \) can be non-trivial. Also, we will use the fact that \( \text{Aut}(l/k) \) (see Definition 2.2) is a normal subgroup of \( \text{Aut}(l \geq k) \).

Definition 4.3. Let \( G \) be a connected reductive \( k \)-group of type \( R \), choose an isomorphism \( G_0 \cong G_k \), and let \( c: \Gamma \to \text{Aut}G_0 \) be the corresponding Galois cocycle. We define the \textit{semilinear Galois action corresponding to} \( c \) (also we say corresponding to \( G_0 \cong G_k \)) on \( \text{Aut}(G_0 \to \text{Spec} k_\alpha \geq k) \cong \text{Aut}(G_0 \rtimes \text{Aut}(k_\alpha \geq k)) \) as follows:

For all \( b \in \text{Aut}G_0 \), \( \beta \in \text{Aut}(k_\alpha \geq k) \) and \( \gamma \in \Gamma \),

\[
\gamma.(b \text{Id}_b) = c_{\beta^{-1}\gamma\beta b}^{\beta^{-1}\gamma\beta b} c_{\gamma b}^{-1} \text{Id}_b.
\]

Remark 4.4. In view of Lemma 2.11, an element of \( \text{Aut}(G_0 \to \text{Spec} k_\alpha \geq k) \) descends to an element of \( \text{Aut}(G \to \text{Spec} k) \) if and only if it is Galois invariant. This is the origin of Definition 4.3.

It is important to notice that in general, the \( \Gamma \)-action on \( \text{Aut}(G_0 \to \text{Spec} k_\alpha \geq k) \) does not preserve the group structure. Let us prove some elementary properties of this action.

Lemma 4.5. \textit{Keep the notations of Definition 4.3.} Let \( \gamma, \gamma_1, \gamma_2 \in \Gamma \) and let \( b \text{Id}_b, b_1 \text{Id}_{b_1}, b_2 \text{Id}_{b_2} \in \text{Aut}(G_0 \to \text{Spec} k_\alpha \geq k) \cong \text{Aut}G_0 \rtimes \text{Aut}(k_\alpha \geq k) \).

1. \( \gamma \gamma_2.(b \text{Id}_b) = \gamma_1.(\gamma_2.(b \text{Id}_b)) \)

2. \( \gamma.(b_1 \text{Id}_{b_1}, b_2 \text{Id}_{b_2}) = \left(\beta_2^{-1}\gamma\beta_2.(b_1 \text{Id}_{b_1})\right)\left(\gamma.(b_2 \text{Id}_{b_2})\right) \)

Proof.

1. \( \gamma \gamma_2.(b \text{Id}_b) = c_{\beta^{-1}\gamma\gamma_2\beta b}^{\beta^{-1}\gamma_2\beta b} c_{\gamma_2 b}^{-1} \text{Id}_b = c_{\beta^{-1}\gamma\gamma_2\beta b}^{\beta^{-1}\gamma\gamma_2\beta b} c_{\gamma_2 b}^{-1} \text{Id}_b = c_{\beta^{-1}\gamma\gamma_2\beta b}^{\beta^{-1}\gamma_2\beta b} c_{\gamma_2 b}^{-1} \text{Id}_b = \gamma_1.(c_{\beta^{-1}\gamma_2\beta b}^{\beta^{-1}\gamma_2\beta b} c_{\gamma_2 b}^{-1} \text{Id}_b) = \gamma_1.(\gamma_2.(b \text{Id}_b)) \)

2. \( \gamma.(b_1 \text{Id}_{b_1}, b_2 \text{Id}_{b_2}) = \gamma.(b_1 \beta_1^{-1}b_2 \text{Id}_{b_2 b_1 b_2}) = c_{b_1^{-1}\gamma_1\beta_1^{-1}\gamma_2\beta_1^{-1}b_2 \text{Id}_{b_2 b_1 b_2}} \text{Id}_{b_2 b_1 b_2} = c_{b_1^{-1}\gamma_1\beta_1^{-1}b_2 \text{Id}_{b_2 b_1 b_2}} \text{Id}_{b_2 b_1 b_2} = c_{b_1^{-1}\gamma_1\beta_1^{-1}b_2 \text{Id}_{b_2 b_1 b_2}} \text{Id}_{b_2 b_1 b_2} \)

Lemma 4.6. \textit{Keep the notations of Definition 4.3.} The set of elements in \( \text{Aut}(G_0 \to \text{Spec} k_\alpha \geq k) \) that are fixed by the \( \Gamma \) action is a subgroup.

Proof.

Let \( b_1 \text{Id}_{b_1}, b_2 \text{Id}_{b_2} \in \text{Aut}(G_0 \to \text{Spec} k_\alpha \geq k) \cong \text{Aut}G_0 \rtimes \text{Aut}(k_\alpha \geq k) \) be elements that are fixed by the \( \Gamma \) action. For \( \gamma \in \Gamma \), we have \( \gamma.(b_1 \text{Id}_{b_1}, b_2 \text{Id}_{b_2}) = \left(\beta_2^{-1}\gamma\beta_2.(b_1 \text{Id}_{b_1})\right)\left(\gamma.(b_2 \text{Id}_{b_2})\right) \) by Lemma 4.5. Hence \( b_1 \text{Id}_{b_1}, b_2 \text{Id}_{b_2} \) is \( \Gamma \) invariant as well.
Similarly, if \( b \cdot \text{Id}_\beta \) is \( \Gamma \) invariant, for all \( \gamma \in \Gamma \) we have \( \text{Id}_G \gamma = \gamma \cdot \text{Id}_G \gamma = (b \cdot \text{Id}_\beta)^{-1} b \cdot \text{Id}_\beta = (b^{-1} \gamma \beta \cdot ((b \cdot \text{Id}_\beta)^{-1})) \left( \gamma \cdot (b \cdot \text{Id}_\beta) \right) \). Hence, since \( b \cdot \text{Id}_\beta \) is \( \Gamma \) invariant, we get \( b^{-1} \gamma \beta \cdot ((b \cdot \text{Id}_\beta)^{-1}) = (b \cdot \text{Id}_\beta)^{-1} \) for all \( \gamma \in \Gamma \), and hence \( (b \cdot \text{Id}_\beta)^{-1} \) is \( \Gamma \) invariant as well.

**Definition 4.7.** In the notations of Definition 4.3, the subgroup of elements of \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k) \) that are fixed by \( \Gamma \) is denoted \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k)^\Gamma \).

We now aim to state that the group \( \text{Aut}(G \to \text{Spec} k) \) is the group \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k)^\Gamma \) modulo the Galois group. So we need to embed the Galois group as a normal subgroup of \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k)^\Gamma \).

**Definition 4.8.** Consider the homomorphism \( \Gamma \to \text{Aut}(G_0 \to \text{Spec} k_s \geq k)^\Gamma; \gamma \mapsto c_\gamma \cdot \text{Id}_\gamma^{-1} \). We denote the image of \( \Gamma \) by \( \bar{\Gamma} \).

**Remark 4.9.** Note that for \( \gamma \in \Gamma \), \( c_\gamma \cdot \text{Id}_\gamma^{-1} \) is an invariant element of \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k) \). Indeed, for \( \delta \in \Gamma \) we have \( \delta \cdot (c_\gamma \cdot \text{Id}_\gamma^{-1}) = c_\delta \cdot \gamma^{-1} \cdot \gamma^{-1} \gamma c_\gamma \cdot \text{Id}_\gamma^{-1} = c_\delta \cdot \gamma^{-1} \cdot \gamma c_\gamma \cdot \text{Id}_\gamma^{-1} = c_\gamma \cdot \text{Id}_\gamma^{-1} \).

**Remark 4.10.** If we denote \( \bar{\gamma} = c_\gamma \cdot \text{Id}_\gamma^{-1} \), it is unfortunate that \( \bar{\gamma} \) is the inverse of the element \( \bar{\gamma_G} = c_{\gamma^{-1}} \cdot \text{Id}_\gamma \), appearing in Definition 2.7. In the language of descent datum, \( \gamma \to \bar{\gamma_G} \) is traditionally required to be an anti-homomorphism, whereas it felt more natural to use a homomorphism in Definition 4.8, so we indulge in this inconsistency.

**Lemma 4.11.** Keeping the notations of Definition 4.3, \( \text{Aut}(G \to \text{Spec} k) \) is naturally isomorphic to \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k)^\Gamma / \bar{\Gamma} \).

**Proof.** We have a homomorphism \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k)^\Gamma \to \text{Aut}(G \to \text{Spec} k) \), which maps an invariant element of \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k) \) to its descent in \( \text{Aut}(G \to \text{Spec} k) \) (see Lemma 2.11). Note that \( \bar{\Gamma} \) is the kernel of this map (since \( c_{\gamma^{-1}} \cdot \text{Id}_\gamma = \bar{\gamma_G} \) arises as a choice of base change for the identity). Now if \( b \cdot \text{Id}_\beta \) is in the kernel of this homomorphism, then \( \beta \) acts trivially on \( k \), i.e. \( \beta = \gamma \) for some \( \gamma \in \Gamma \), and \( b \cdot \text{Id}_\gamma \cdot \text{Id}_\gamma^{-1} \in \text{Aut} G_0 \) descends to the identity in \( \text{Aut} G \). But this holds if and only if \( b \cdot \text{Id}_\beta \cdot c_\gamma \cdot \text{Id}_\gamma^{-1} \) is already the identity on \( G_0 \). Since \( b \cdot \text{Id}_\beta \cdot c_\gamma \cdot \text{Id}_\gamma^{-1} = b \cdot \gamma^{-1} \cdot c_\gamma = b \cdot c_{\gamma^{-1}}^{-1} \), we conclude that \( b \cdot \text{Id}_\beta \) descends to the identity if and only if it is equal to \( c_{\gamma^{-1}} \cdot \text{Id}_\gamma \) for some \( \gamma \in \Gamma \), i.e. if and only if it belongs to \( \bar{\Gamma} \).

It remains to check that the homomorphism \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k)^\Gamma \to \text{Aut}(G \to \text{Spec} k) \) is surjective as well. But this follows from the fact that any automorphism of \( k \) can be extended to an automorphism of \( k_s \).

Note that there was nothing special about the category of algebraic \( k \)-groups, and we could as well repeat this construction for other algebraic categories over \( k \) for which descent is effective. In particular, we can repeat everything we did so far for \( k \)-schemes of based root datum. Also recall that in Theorem 3.13, still keeping the notations of Definition 4.3, the cocycle defining \( R(G) \) is obtained from \( c \) by projecting via \( \text{Aut} G_0 \to E \). Recalling that the Galois action on the split \( k_s \)-scheme of based root datum is trivial, this gives the following result.

**Lemma 4.12.** Keep the notations of Definition 4.3. Let \( \bar{c} \) be the projection of \( c \) under \( \text{Aut} G_0 \to E \). Define a seminatural Galois action on \( \text{Aut}(R_{k_s} \to \text{Spec} k_s \geq k) \cong E \times \text{Aut}(k_s \geq k) \) as follows:

For all \( b \in E \), \( \beta \in \text{Aut}(k_s \geq k) \) and \( \gamma \in \Gamma \), \( \gamma \cdot (b \cdot \text{Id}_\beta) = \bar{c}_{\beta^{-1} \gamma \beta} \cdot b \cdot \text{Id}_\gamma^{-1} \cdot \text{Id}_\beta \).

Define \( \bar{\Gamma} \leq \text{Aut}(R_{k_s} \to \text{Spec} k_s \geq k)^\Gamma \) to be the image of the homomorphism \( \Gamma \to \text{Aut}(R_{k_s} \to \text{Spec} k_s); \gamma \mapsto \bar{c}_{\gamma} \cdot \text{Id}_\gamma^{-1} \). Then \( \text{Aut}(R(G) \to \text{Spec} k) \) is naturally isomorphic to \( \text{Aut}(R_{k_s} \to \text{Spec} k_s \geq k)^\Gamma / \bar{\Gamma} \). Furthermore, the homomorphism \( \text{Aut}(G_0 \to \text{Spec} k_s) \to \text{Aut}(R_{k_s} \to \text{Spec} k_s) \) induces a homomorphism \( \text{Aut}(G_0 \to \text{Spec} k_s \geq k)^\Gamma / \bar{\Gamma} \to \text{Aut}(R_{k_s} \to \text{Spec} k_s \geq k)^\Gamma / \bar{\Gamma} \).

We can now formulate the failure of surjectivity of the map \( \text{Aut}(G \to \text{Spec} k) \to \text{Aut}(G_0 \to \text{Spec} k) \) and of the map \( \text{Aut}(G \to \text{Spec} k) \to \text{Aut}(R(G) \to \text{Spec} k) \) using a variant of Galois cohomology. We first give an approximation of this.
Proposition 4.13. Keep the notations of Lemma 4.12. Endow $\operatorname{Aut} G_0$ and $(\operatorname{Ad} G_0)(k_s)$ with the Galois action given by restricting the semilinear Galois action on $\operatorname{Aut}(G_0 \to \operatorname{Spec} k_s \geq k)$ (this is just the Galois action corresponding to the form $G_0 \cong G_{k_s}$).

1. There exists a coboundary map $\operatorname{Aut}(k_s \geq k) \xrightarrow{\partial} H^1(\Gamma, \operatorname{Aut} G_0)$ such that the sequence
   \[
   1 \to (\operatorname{Aut} G_0)^\Gamma \to \operatorname{Aut}(G_0 \to \operatorname{Spec} k_s \geq k)^\Gamma \to \operatorname{Aut}(k_s \geq k) \xrightarrow{\partial} H^1(\Gamma, \operatorname{Aut} G_0)
   \]
   is exact.

2. There exists a coboundary map $\operatorname{Aut}(R_{k_s} \to \operatorname{Spec} k_s \geq k)^\Gamma \xrightarrow{\partial} H^1(\Gamma, (\operatorname{Ad} G_0)(k_s))$ such that the sequence
   \[
   1 \to (\operatorname{Ad} G_0)(k_s)^\Gamma \to \operatorname{Aut}(G_0 \to \operatorname{Spec} k_s \geq k)^\Gamma \to \operatorname{Aut}(R_{k_s} \to \operatorname{Spec} k_s \geq k)^\Gamma \xrightarrow{\partial} H^1(\Gamma, (\operatorname{Ad} G_0)(k_s))
   \]
   is exact.

Proof. It is important not to confuse the two Galois actions we are considering on $\operatorname{Aut} G_0$. One arises from the fact that we chose a form of $G_0$ over the prime field of $k$, and the other is the Galois action arising from $G_0 \cong G_{k_s}$. For $\gamma \in \Gamma$ and $b \in \operatorname{Aut} G_0$, recall that the former is denoted $\gamma b = \operatorname{Id}_{k_s} b \operatorname{Id}_{k_s}$ whilst the latter is denoted $\gamma b = c_\gamma \gamma b c_\gamma^{-1}$.

We have a similar remark for $E = \operatorname{Aut} R_{k_s}$: for $\gamma \in \Gamma$ and $b \in E$, we denote $\gamma b = \operatorname{Id}_{k_s}^{-1} b \operatorname{Id}_{k_s} = b$ (the latter equality holds because any automorphism of $R_{k_s}$ is defined over $k$) and $\gamma b = c_\gamma \gamma b c_\gamma^{-1} = \hat{c}_\gamma b\hat{c}_\gamma^{-1}$.

1. Let $\beta^{-1} \in \operatorname{Aut}(k_s \geq k)$. As usual in this situation, we set $\partial(\beta^{-1}) : \Gamma \to \operatorname{Aut} G_0; \gamma \to \partial(\beta^{-1})_\gamma$, where $\partial(\beta^{-1})_\gamma$ is defined by the equality (in $\operatorname{Aut}(G_0 \to \operatorname{Spec} k_s \geq k)$) $\gamma \operatorname{Id}_{k_s} = \operatorname{Id}_{k_s} \gamma \partial(\beta^{-1})_\gamma$. In other words, $\partial(\beta^{-1})_\gamma = \gamma \operatorname{Id}_{k_s} \gamma^{-1} \gamma \operatorname{Id}_{k_s}$; hence $\partial(\beta^{-1})_\gamma$ clearly belongs to $\operatorname{Aut} G_0$. The fact that it is a cocycle follows directly from Lemma 4.5. Indeed, $\partial(\beta^{-1})_\gamma \gamma' = \gamma \gamma' \gamma \gamma' \gamma \operatorname{Id}_{k_s} \gamma' \operatorname{Id}_{k_s} = \gamma \gamma' \gamma \gamma' \gamma \operatorname{Id}_{k_s} \gamma' \operatorname{Id}_{k_s} = \partial(\beta^{-1})_\gamma \gamma \partial(\beta^{-1})_\gamma \gamma'$.

The sequence $1 \to \operatorname{Aut} G_0 \to \operatorname{Aut}(G_0 \to \operatorname{Spec} k_s \geq k) \to \operatorname{Aut}(k_s \geq k)$ is a $\Gamma$-equivariant exact sequence (where we endow $\operatorname{Aut}(k_s \geq k)$ with the trivial $\Gamma$ action). Hence, taking $\Gamma$-invariant elements, it remains exact. So we just need to check exactness at $\operatorname{Aut}(k_s \geq k)$.

Let $\beta^{-1} \in \operatorname{Aut}(k_s \geq k)$. Then $\partial(\beta^{-1})$ is trivial in $H^1(\Gamma, \operatorname{Aut} G_0)$ if and only if there exists $b \in \operatorname{Aut} G_0$ such that for all $\gamma \in \Gamma$, $b^{-1} \operatorname{Id}_{k_s}^{-1} \partial(\beta^{-1})_\gamma \operatorname{Id}_{k_s} = 1$. By Lemma 4.5, $b^{-1} \operatorname{Id}_{k_s}^{-1}(\gamma \gamma')(\gamma b) = (\operatorname{Id}_{k_s}^{-1} \gamma \gamma')(\operatorname{Id}_{k_s} b)$, and hence $\partial(\beta^{-1})$ is trivial if and only if there exists $b \in \operatorname{Aut} G_0$ such that $\operatorname{Id}_{k_s} b$ is a $\Gamma$-invariant element of $\operatorname{Aut}(G_0 \to \operatorname{Spec} k_s \geq k)$, which proves exactness at $\operatorname{Aut}(k_s \geq k)$.

2. Let $b \operatorname{Id}_{\beta} \in \operatorname{Aut}(R_{k_s} \to \operatorname{Spec} k_s \geq k)^\Gamma \cong (E \times \operatorname{Aut}(k_s \geq k))^\Gamma$. As usual in this situation, we set $\partial(b \operatorname{Id}_{\beta}) : \Gamma \to \operatorname{Aut} G_0; \gamma \to \partial(b \operatorname{Id}_{\beta})_\gamma$, where $\partial(b \operatorname{Id}_{\beta})_\gamma$ is defined by the equality (in $\operatorname{Aut}(G_0 \to \operatorname{Spec} k_s \geq k)$) $\gamma (b \operatorname{Id}_{\beta}) = b \operatorname{Id}_{\beta} \gamma (b \operatorname{Id}_{\beta})$. In other words, $\partial(b \operatorname{Id}_{\beta})_\gamma = (b \operatorname{Id}_{\beta})^{-1} \gamma (b \operatorname{Id}_{\beta}) (b \operatorname{Id}_{\beta})^{-1}$. Let us check that $\partial(b \operatorname{Id}_{\beta})_\gamma$ belongs to $(\operatorname{Ad} G_0)(k_s)$. For $\gamma \in \Gamma$, define $c_\gamma = c_\gamma \gamma \gamma^{-1}$. Now $\partial(b \operatorname{Id}_{\beta})_\gamma = (b \operatorname{Id}_{\beta})^{-1} c_\gamma \gamma \gamma \gamma^{-1} \gamma (b \operatorname{Id}_{\beta}) (b \operatorname{Id}_{\beta})^{-1} c_\gamma \gamma \gamma^{-1} \gamma (b \operatorname{Id}_{\beta})^{-1} c_\gamma \gamma \gamma^{-1} \gamma (b \operatorname{Id}_{\beta})$ because $b \operatorname{Id}_{\beta} \in \operatorname{Aut}(R_{k_s} \to \operatorname{Spec} k_s \geq k)^\Gamma$, hence we conclude that $\partial(b \operatorname{Id}_{\beta})_\gamma \in (\operatorname{Ad} G_0)(k_s)$ because $c_\gamma \in (\operatorname{Ad} G_0)(k_s)$ and $(\operatorname{Ad} G_0)(k_s)$ is a normal subgroup of $\operatorname{Aut}(G_0 \to \operatorname{Spec} k_s \geq k)$.

The sequence $1 \to (\operatorname{Ad} G_0)(k_s) \to \operatorname{Aut}(G_0 \to \operatorname{Spec} k_s \geq k) \to \operatorname{Aut}(R_{k_s} \to \operatorname{Spec} k_s \geq k)$ is a $\Gamma$-equivariant exact sequence. Hence, taking $\Gamma$-invariant elements, it remains exact. So we just need to check exactness at $\operatorname{Aut}(R_{k_s} \to \operatorname{Spec} k_s \geq k)^\Gamma$.

Let $b \operatorname{Id}_{\beta} \in \operatorname{Aut}(R_{k_s} \to \operatorname{Spec} k_s \geq k)^\Gamma \cong (E \times \operatorname{Aut}(k_s \geq k))^\Gamma$. Then $\partial(b \operatorname{Id}_{\beta})$ is trivial in $H^1(\Gamma, (\operatorname{Ad} G_0)(k_s))$ if and only if there exists $g \in (\operatorname{Ad} G_0)(k_s)$ such that for all $\gamma \in \Gamma$, $g^{-1} (b \operatorname{Id}_{\beta})^{-1} \gamma (b \operatorname{Id}_{\beta}) \gamma (b \operatorname{Id}_{\beta} g) = (b \operatorname{Id}_{\beta} g)^{-1} \gamma (b \operatorname{Id}_{\beta} g)$. By
Lemma 4.5, and hence \( \partial(b\text{id}_β) \) is trivial if and only if there exists \( g \in (\text{Ad } G_0)(k_α) \) such that \( b\text{id}_β \) is a \( Γ \)-invariant element of \( \text{Aut}(G_0 \to \text{Spec } k_α \geq k) \), which proves exactness at \( \text{Aut}(R_{k_α} \to \text{Spec } k_α \geq k)^Γ \).

In order to prove Theorem 1.2, it suffices now to observe that in Proposition 4.13, the image of \( \tilde{Γ} \) under the coboundary operator is trivial.

**Lemma 4.14.** Keep the notation of Proposition 4.13.

1. The image of \( Γ \subseteq \text{Aut}(k_α \geq k) \) under \( \text{Aut}(k_α \geq k) \xrightarrow{\partial} H^1(Γ, \text{Aut } G_0) \) is trivial.

2. The image of \( \tilde{Γ} \subseteq \text{Aut}(R_{k_α} \to \text{Spec } k_α \geq k)^Γ \) under the coboundary map \( \text{Aut}(R_{k_α} \to \text{Spec } k_α \geq k)^Γ \xrightarrow{\partial} H^1(Γ, (\text{Ad } G_0)(k_α)) \) is trivial.

**Proof.** The proof is just a straightforward computation, using directly the definition of the semilinear \( Γ \)-action.

1. Let \( γ \in Γ \). We want to check that the cocycle \( Γ \to \text{Aut } G_0; σ \mapsto \text{Id}_γ^{-1} \sigma \text{Id}_γ \) is trivial. By definition, \( \text{Id}_γ^{-1} \sigma \text{Id}_γ = \text{Id}_γ^{-1} c_γ^{-1}σ_γ c_γ^{-1} \text{Id}_γ = γc_γ^{-1}σ_γ c_γ^{-1} = c_γ^{-1}σ.c_γ \). Since \( c_γ \) belongs to \( \text{Aut } G_0 \), this indeed shows that \( \partial(γ^{-1}) \) is cohomologous to the trivial cocycle.

2. Let \( \tilde{c}_γ, \text{Id}_γ^{-1} \in \tilde{Γ} \subseteq \text{Aut}(R_{k_α} \to \text{Spec } k_α \geq k)^Γ \cong (E \times \text{Aut}(k_α \geq k))^Γ \). By definition, \( \partial(\tilde{c}_γ, \text{Id}_γ^{-1}) = (\tilde{c}_γ, \text{Id}_γ^{-1})^{-1}σ(\tilde{c}_γ, \text{Id}_γ^{-1}) \). Plugging the definition of the semilinear action, we get

\[
(\tilde{c}_γ, \text{Id}_γ^{-1})^{-1}σ(\tilde{c}_γ, \text{Id}_γ^{-1}) = \text{Id}_γ \tilde{c}_γ^{-1}c_γσ_γ^{-1}γσ_γ^{-1} \tilde{c}_γ γc_γ^{-1} \text{Id}_γ^{-1} = γc_γ^{-1}c_γσ_γ^{-1} \tilde{c}_γ c_γ^{-1} = γc_γ^{-1}c_γσ_γ^{-1} \tilde{c}_γ c_γ^{-1} = γc_γ^{-1}c_γσ_γ^{-1} \tilde{c}_γ c_γ^{-1} = (c_γ^{-1} \tilde{c}_γ) γ^{-1}σ(c_γ^{-1} \tilde{c}_γ) = (c_γ^{-1} \tilde{c}_γ) γ^{-1}σ(c_γ^{-1} \tilde{c}_γ)
\]

where the last equality holds because the Galois action on \( E = \text{Aut } R_{k_α} \) is trivial. Since \( c_γ^{-1} \tilde{c}_γ \) belongs to \( (\text{Ad } G_0)(k_α) \), this indeed shows that \( \partial(\tilde{c}_γ, \text{Id}_γ^{-1}) \) is cohomologous to the trivial cocycle.

**Corollary 4.15.** Keep the notations of Proposition 4.13.

1. The exact sequence

\[
1 \to (\text{Aut } G_0)^Γ \to \text{Aut}(G_0 \to \text{Spec } k_α \geq k)^Γ \to \text{Aut}(k_α \geq k) \xrightarrow{\partial} H^1(Γ, \text{Aut } G_0)
\]

induces an exact sequence

\[
1 \to (\text{Aut } G_0)^Γ \to \text{Aut}(G_0 \to \text{Spec } k_α \geq k)^Γ \xrightarrow{\partial} H^1(Γ, \text{Aut } G_0)
\]

2. The exact sequence

\[
1 \to (\text{Ad } G_0)(k_α)^Γ \to \text{Aut}(G_0 \to \text{Spec } k_α \geq k)^Γ \to \text{Aut}(R_{k_α} \to \text{Spec } k_α \geq k)^Γ \xrightarrow{\partial} H^1(Γ, (\text{Ad } G_0)(k_α))
\]

induces an exact sequence

\[
1 \to (\text{Ad } G_0)(k_α)^Γ \to \text{Aut}(G_0 \to \text{Spec } k_α \geq k)^Γ \xrightarrow{\partial} H^1(Γ, (\text{Ad } G_0)(k_α))
\]
Proof. Note that in both cases, moding out by \( \bar{\Gamma} \) does not modify exactness on the left. Hence the results follows directly from Lemma 4.14. \( \square \)

Proof of Theorem 1.2. Note that by Lemma 4.11, \( \text{Aut}(G_0 \to \text{Spec} \ k_s \geq k)^\Gamma /\bar{\Gamma} \) is naturally isomorphic to \( \text{Aut}(G \to \text{Spec} \ k) \). Similarly, by Lemma 4.12, \( \text{Aut}(R_k, G_0 \to \text{Spec} \ k_s \geq k)^\Gamma /\bar{\Gamma} \cong \text{Aut}(\mathcal{R}(G) \to \text{Spec} \ k) \). Also note that the restriction of the semilinear \( \Gamma \)-action on \( \text{Aut}(G_0 \to \text{Spec} \ k_s \geq k)^\Gamma /\bar{\Gamma} \) to \( \text{Aut} G_0 \) (respectively \( (\text{Ad} G_0)(k_s) \)) is the natural Galois action on \( \text{Aut} G_{k_s} \) (respectively \( (\text{Ad} G)(k_s) \)). In particular, the \( \Gamma \) invariant elements are the elements of \( \text{Aut} \ G \) (respectively \( (\text{Ad} \ G)(k) \)). Finally, note that \( \text{Aut}(k) \cong \text{Aut}(k_s \geq k)/\Gamma \) and that all those identifications are natural enough, we get the result. \( \square \)

We now describe how the coboundary map of the exact sequence \( 1 \to \text{Aut} G \to \text{Aut}(G \to \text{Spec} \ k) \to \text{Aut}(k) \to H^1(k, \text{Aut} G_k) \) can be used to compute \( \text{Aut}_G(k) \). To illustrate this, we set the following notations for the rest of the section:

**Definition 4.16.** 1. \( D \) denotes a central division algebra of degree 3 over \( k \) (hence by a theorem of Wedderburn, \( D \) is cyclic). We fix a maximal Galois subfield \( l \) of \( D \) so that \( \text{Gal}(l/k) \) is cyclic of order 3 (which exists because \( D \) is cyclic). We choose a generator of \( \text{Gal}(l/k) \) that we denote \( \gamma \). Choosing an element \( u \in D \) normalising \( l \) such that its action by conjugation on \( l \) generates \( \text{Gal}(l/k) \), we set \( a = u^3 \in k \). We set \( G := \text{SL}_3(D) \) to be the corresponding algebraic \( k \)-group.

2. Set \( G_0 := \text{SL}_3 \) (that we consider over \( k_s \), as in the beginning of this section). Recall that \( \text{Ad} \text{SL}_3 = \text{PGL}_3 \). We denote elements of \( \text{PGL}_3(k_s) \) as \( \left[ \begin{array}{ccc} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{array} \right] \), which is to be read as “the equivalence class corresponding to the matrix \( \left( \begin{array}{ccc} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{array} \right) \in GL_3(k_s) \)”.

3. We choose the usual pinning of \( \text{SL}_3 \) where the pair \((T, B)\) consists of diagonal matrices and of upper triangular matrices, and where we choose some generators of the corresponding “basic root groups”. Let \( R \) be the corresponding based root datum. Note that \( \text{Aut} R \) is of order 2, and that if our choice of generators for the “basic root groups” is sensible enough, the splitting of \( \text{Aut} \SL_3 \to \text{Aut} R \) is given by the automorphism \( \SL_3 \to \SL_3; g \mapsto \gamma g^{-1} \), where \( \gamma g \) denotes the anti-transposed of \( g \), i.e. “the transposed of \( g \) along the anti-diagonal”. More formally, for \( i, j \in \{1, 2, 3\} \), \( (\gamma g)_{ij} = g_{4-j;4-i} \). Note that taking anti-transpose commutes with taking inverse, so that there is no ambiguity in the notation \( \gamma g^{-1} \).

4. Consider the homomorphism \( f: \text{Gal}(l/k) \to \text{PGL}_3(l); \gamma \mapsto \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right] \) (where \( a \in k \) and \( \gamma \in \text{Gal}(l/k) \) have been defined in the first item of these definitions). We choose the cocycle \( c: \Gamma \to \text{Aut} G_0 \) defining \( G = \text{SL}_3(D) \) over \( k \) to be the composition \( \Gamma \to \text{Gal}(l/k) \xrightarrow{f} \text{PGL}_3(l) \to \text{Aut} G_0 \).

Having set those notations, we are ready to start computing. The following lemmas are two special cases of [Hau07b] that we recover without using any theory of division algebras.

**Lemma 4.17.** Keep the notations of Definition 4.16 and let \( \beta \in \text{Aut}(k_s \geq k) \) be such that \( \beta(l) = l \). Let \( \alpha \) be the restriction of \( \beta \) to \( k \). Then \( \alpha \in \text{Aut}_G(k) \) if and only if there exists \( \lambda \in l^* \) such that either \( \frac{\alpha(a)}{a} = \gamma^2 \lambda \gamma \lambda \) or \( \alpha(a)a = \gamma^2 \lambda \gamma \lambda \).

**Proof.** Note that \( \alpha \in \text{Aut}_G(k) \) if and only if \( \alpha^{-1} \in \text{Aut}_G(k) \). Using the coboundary map of Proposition 4.13, for all \( \sigma \in \Gamma \) we have \( \partial(\beta^{-1}) = \text{Id}_3^{-1} \sigma \text{Id}_3 = \text{Id}_3^{-1} c_{\beta^{-1}} \sigma c_{\beta^{-1}}^{-1} \text{Id}_3 = \beta \sigma^{-1} c_{\beta^{-1}} \sigma c_{\beta^{-1}}^{-1} \). Hence by Proposition 4.13, \( \alpha \in \text{Aut}_G(k) \) if and only if the cocycle \( \sigma \mapsto \beta \sigma^{-1} c_{\beta^{-1}} \sigma c_{\beta^{-1}} \) is trivial in \( H^1(k, \text{Aut} \SL_3) \), i.e. if and only if there exists \( g \in \text{PGL}_3(k) \) and \( c \in \text{Aut} R \leq \text{Aut} \SL_3 \) such that \( (gc)^{-1} \beta \sigma^{-1} c_{\beta^{-1}} \sigma c_{\beta^{-1}} \sigma (gc) = 1 \). But since \( \gamma \) generates \( \text{Gal}(l/k) \) and since \( c \sigma = 1 \) for all \( \sigma \) acting trivially on \( l \), this is equivalent to the existence of \( g \in \text{PGL}_3(l) \) and \( c \in \text{Aut} R \leq \text{Aut} \SL_3 \) such that
Recalling that $\gamma, (ge) = c_\gamma \gamma (ge) c_\gamma^{-1} = c_\gamma \gamma ge c_\gamma^{-1}$, this is equivalent to $\beta c_\beta^{-1} \gamma \beta c_\beta^{-1} \gamma = 1$. First assume that $\beta^{-1} \gamma \beta = \gamma$. Then $\partial (\beta^{-1})$ is trivial if and only if there exists $g \in \text{PGL}_3(l)$ such that either $\beta c_\gamma \gamma ge c_\gamma^{-1} = g$ or $\beta c_\gamma \gamma ge \gamma = g$. Letting $g = \left[ \begin{array}{ccc} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{array} \right] \in \text{PGL}_3(l)$, we have

$$\beta c_\gamma \gamma ge c_\gamma^{-1} = \left[ \begin{array}{ccc} \gamma g_{33} \beta(a) \\ \gamma g_{13} \beta(a) \\ \gamma g_{23} \beta(a) \end{array} \right].$$

If $g$ is invertible, one of $g_{13}, g_{23}$ and $g_{33}$ is non-zero. Let us for example assume that $g_{33} \neq 0$. Now $\beta c_\gamma \gamma ge c_\gamma^{-1} = g$ if and only if there exists $\lambda \in l^*$ such that $\gamma g_{33} \frac{\beta(a)}{a} = g_{11} \lambda, \gamma g_{11} = g_{22} \lambda$ and $\gamma g_{22} = g_{33} \lambda$. Hence $\gamma \left( \frac{\gamma g_{33} \frac{\beta(a)}{a}}{\gamma \lambda} \right) = g_{22} \lambda$, and furthermore $\gamma^2 \left( \frac{\gamma g_{33} \frac{\beta(a)}{a}}{\gamma \lambda} \right) = g_{33} \lambda$.

Since the computation is similar if $g_{13} \neq 0$ or $g_{23} \neq 0$ instead, we conclude that if such a $g$ exists, then there exists $\lambda \in l$ such that $\frac{\beta(a)}{a} = \gamma^2 \lambda^2 \lambda \lambda$ (because $\gamma$ acts trivially on $\alpha \in k$ and $\gamma^3$ acts trivially on $g_{33} \in l$). Conversely, if there exists $\lambda \in l^*$ such that $\frac{\beta(a)}{a} = \gamma^2 \lambda^2 \lambda \lambda$, then $g = \left[ \begin{array}{ccc} \gamma^2 \lambda^2 \lambda^2 & 0 & 0 \\ 0 & \gamma \lambda & 0 \\ 0 & 0 & 1 \end{array} \right]$ is such that $\beta c_\gamma \gamma ge c_\gamma^{-1} = g$. A similar computation shows that there exists $g \in \text{PGL}_3(l)$ such that $\beta c_\gamma \gamma ge \gamma = g$ if and only if there exists $\lambda \in l^*$ such that $\beta(a) a = \gamma^2 \lambda^2 \lambda \lambda$.

Now assuming that $\beta^{-1} \gamma \beta = \gamma^{-1}, \partial (\beta^{-1})$ is trivial if and only if there exists $g \in \text{PGL}_3(l)$ such that either $\beta c_\gamma^{-1} \gamma ge c_\gamma^{-1} = g$ or $\beta c_\gamma^{-1} \gamma ge \gamma = g$. Or equivalently if and only if there exists $g \in \text{PGL}_3(l)$ such that either $\beta c_\gamma \gamma^{-1} ge c_\gamma^{-1} = g$ or $\beta c_\gamma^{-1} \gamma ge c_\gamma^{-1} = g$. Hence we get the same setting as for the case $\beta^{-1} \gamma \beta = \gamma$ up to replacing $\gamma g_{ij}$ with $\gamma^{-1} g_{ij}$, which leaves the conclusion unchanged.

**Lemma 4.18.** Keep the notations of Definition 4.16 and let $\beta \in \text{Aut} (k, \geq k)$ be such that $\beta(l) = l^* \neq 1$. Let $\delta = \beta \gamma \beta$ (a generator of $\text{Gal} (l'/k)$), let $K$ be the compositum $l l^* \leq k_\delta$ and let $\text{Gal} (K/k) = \text{Gal}(l/k) \times \text{Gal}(l'/k)$ be the corresponding decomposition of $\text{Gal}(K/k)$. Let $\alpha$ be the restriction of $\beta$ to $K$. Then $\alpha \in \text{Aut} G(k)$ if and only if there exist $\lambda, \mu \in K^*$ such that

1. either $\gamma^2 \lambda^2 \lambda^2 = \frac{1}{a}, \delta^2 \mu \delta \mu = \beta(a)$ and $\frac{\lambda}{\mu} = \frac{a}{c},$

2. or $\gamma^2 \lambda^2 \lambda^2 = a, \delta^2 \mu \delta \mu = \beta(a)$ and $\frac{\lambda}{\mu} = \frac{b}{c}.$

**Proof.** Arguing as in the beginning of the proof of Lemma 4.17, $\alpha \in \text{Aut} G(k)$ if and only if there exists $g \in \text{PGL}_3(k)$ and $\epsilon \in \text{Aut} R \leq \text{Aut} SL_3$ such that

$$\beta c_\beta^{-1} \sigma \beta c_\beta^{-1} \sigma^{-1} \epsilon = g$$

for all $\sigma \in \text{Gal}(K/k)$ (1)

We first do the case $\epsilon = 1$. Taking $\sigma = \gamma$ in Equation 1, we get $\gamma ge c_\gamma^{-1} = g$. On the other hand, taking $\sigma = \delta$ in Equation 1, we get $\beta c_\gamma \gamma \delta g = g$. Hence Equation 1 implies that

$$\left[ \begin{array}{ccc} g_{51} & g_{52} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \end{array} \right] = \left[ \begin{array}{ccc} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{array} \right] \cdot \left[ \begin{array}{ccc} \delta g_{51} & \delta g_{52} & \delta g_{53} \\ \delta g_{21} & \delta g_{22} & \delta g_{23} \\ \delta g_{31} & \delta g_{32} & \delta g_{33} \end{array} \right].$$

To fix ideas, assume $g_{11} \neq 0$ (the computation is similar if $g_{21} \neq 0$ or $g_{31} \neq 0$ instead). So we can further assume that $g_{11} = 1$. Hence there exists $\lambda, \mu \in K^*$ such that
can easily be adapted for any finite-dimensional central division algebra $D$ and for two finite-dimensional central division algebras $D_\alpha$ and $D_\beta$, there exists a non-trivial automorphism of $D$ which does not commute with any other automorphism, such that $D_\alpha \sim D_\beta$. Finally, by looking at the central coefficient, we see that $D_\alpha \sim D_\beta$. In the case where $D_\alpha$ and $D_\beta$ are not isomorphic, the classifying field of $D_\alpha$ up to conjugation.

Remark 4.19. The kind of computations we perform in Lemma 4.17 can easily be adapted for any cyclic division algebra, and the computations in Lemma 4.18 can easily be adapted to any cyclic division algebras of prime degrees. When the degree is not prime, there exists a non-trivial automorphism of $R$, one uses the fact that $c_\gamma e^{-1} = c_\gamma^{-1}$ and then imitates the above computation to get the second condition on $\lambda, \mu$ (to carry out this computation most easily, assume that $g_{11} = 1$ and for the last condition, consider the 23 coefficient).

Remark 4.20. In light of the results in [Han07b], we proved that $\gamma \in \operatorname{Aut}_G(k)$ if and only if $\gamma^2 \lambda^\gamma = 1$ and $\gamma^2 \mu^\gamma = 1$. Conversely, if there exists $\lambda, \mu \in K$ such that $\lambda^2 \gamma^\lambda = 1$ and $\mu^2 \gamma^\mu = 1$, one can check that

$$g = \begin{pmatrix} \lambda^{-1} & \gamma^\lambda \mu^{-1} \\ \gamma^\mu \lambda^{-1} & \lambda^{-1} \gamma^{-1} \mu^{-1} \end{pmatrix}$$

satisfies Equation 1.

In the case where $e$ is the non-trivial automorphism of $R$, one uses the fact that $e^2 = 1$ and then imitates the above computation to get the second condition on $\lambda, \mu$ (to carry out this computation most easily, assume that $g_{11} = 1$ and for the last condition, consider the 23 coefficient).

5 Semilinear automorphisms of based root datum

We aim to give an explicit description of the short exact sequence $1 \to \operatorname{Aut}(\mathcal{R}(G)) \to \operatorname{Aut}(\mathcal{R}(G) \to \operatorname{Spec} k) \to \operatorname{Aut}_\mathcal{R}(G)(k) \to 1$. We base this computation on Lemma 4.12.

Recall that for $R_k$, a split $k$-scheme of based root datum of type $R$, the $\operatorname{Gal}(k_s/k)$-action on $\operatorname{Aut} R_k$ is trivial, so that $H^1(k_s/k, \operatorname{Aut} R_k)$ is isomorphic to the set of continuous homomorphisms $\operatorname{Hom}(\operatorname{Gal}(k_s/k), \operatorname{Aut} R)$ up to conjugation.

Definition 5.1. Let $\mathcal{R}$ be a $k$-scheme of based root datum of type $R$.

1. Fix an isomorphism $\mathcal{R}_k \cong R_k$, and let $\tilde{c} : \operatorname{Gal}(k_s/k) \to \operatorname{Aut} R$ be the corresponding cocycle. Let $N \subseteq \operatorname{Gal}(k_s/k)$ be the kernel of the homomorphism $\tilde{c}$, and let $l$ be the Galois extension of $k$ fixed by $N$. We call $l$ the classifying field of $\mathcal{R}$. Once a separable closure of $k$ has been fixed, the classifying field of $\mathcal{R}$ is uniquely determined by $\mathcal{R}$.

2. We say that $\mathcal{R}$ (or $R$) is semisimple (respectively simply connected, respectively adjoint, respectively simple) if the split connected reductive group of type $R$ is semisimple (respectively simply connected, respectively adjoint, respectively simple).

Remark 5.2. We use the following terminology: a connected reductive $k$-group is simple if it is non-abelian and has no non-trivial connected closed normal subgroup (some authors prefer to call such groups quasi-simple).
Lemma 5.3. Let \( R \) be a simple reduced based root datum and let \( k_s \) be a separable closure of \( k \). The map which associates to a \( k_s/k \)-form of \( R_{k_s} \) its classifying field is a bijection between \( k \)-schemes of based root datum of type \( R \) (up to \( k \)-isomorphism) and subfields \( l \leq k_s \) such that \( l \) is Galois over \( k \) and \( \text{Gal}(l/k) \) is isomorphic to a subgroup of \( \text{Aut} R \).

Proof. Let \( D \) be the Dynkin diagram associated to \( R \). Since \( R \) is semisimple and reduced, \( \text{Aut} R \leq \text{Aut} D \). Furthermore, since \( R \) is simple \( \text{Aut} D \) is either trivial, \( \mathbb{Z}/2\mathbb{Z} \) or \( S_3 \). Hence, if two subgroups of \( \text{Aut} R \) are isomorphic, they are actually conjugate. The result follows from the fact that the Galois action on \( \text{Aut} R \) is trivial, and hence \( H^1(k_s/k, \text{Aut} R_k) \) is isomorphic to the set of continuous homomorphisms \( \text{Hom}(\text{Gal}(k_s/k), \text{Aut} R) \) modulo conjugation. \( \square \)

Remark 5.4. In the notations of the proof of Lemma 5.3, one might wonder when the inclusion \( \text{Aut} R \leq \text{Aut} D \) is an equality. This is always the case, except possibly when \( R \) is not simply connected or adjoint and is of type \( D_{2n} \). See [Conl14, Proposition 1.5.1] for a precise statement.

Definition 5.5. Let \( R \) be a simple \( k \)-scheme of based root datum, and let \( k_s \) be a separable closure of \( k \). We define the **Tits index of** \( R \) to be \( g X_{n,l} \) where

1. \( l \leq k_s \) is the classifying field of \( R \) (hence \( l \) is a finite Galois extension of \( k \)).
2. \( X_n \) is the label of the Dynkin diagram associated to \( R \).
3. \( g \) is the order of the Galois group \( \text{Gal}(l/k) \).

Lemma 5.6. Let \( R \) be a simple \( k \)-scheme of based root datum of type \( R \) with index \( g X_{n,l} \).

1. \( g \in \{1, 2, 3, 6\} \).
2. If \( g = 1 \), \( \text{Aut}(R \to \text{Spec} k) \cong \text{Aut} R \times \text{Aut}(k) \), and this isomorphism restricts to \( \text{Aut} R \cong \text{Aut} l \).
3. If \( g = 2 \) or \( g = 3 \), \( \text{Aut}(R \to \text{Spec} k) \cong \text{Aut}(l \geq k) \), and this isomorphism restricts to \( \text{Aut} R \cong \text{Gal}(l/k) \).
4. If \( g = 6 \), \( \text{Aut}(R \to \text{Spec} k) \cong \text{Aut}(l_3 \geq k) \), where \( l_3 \) is any non-normal cubic subextension of \( l/k \). Furthermore, \( \text{Aut} R \) is trivial.

Proof. 1. Let \( D \) be the Dynkin diagram associated to \( R \). Since \( R \) is semisimple and reduced, \( \text{Aut} R \leq \text{Aut} D \). Furthermore, since \( R \) is simple \( \text{Aut} D \) is either trivial, \( \mathbb{Z}/2\mathbb{Z} \) or \( S_3 \). It follows that \( g \in \{1, 2, 3, 6\} \).

2. The case \( g = 1 \) means that \( R \) is a split \( k \)-scheme of based root datum. Hence, \( \text{Aut} R \cong \text{Aut} l \) (because the functor of constant objects is fully faithfull). Furthermore, the short exact sequence \( 1 \to \text{Aut} R \to \text{Aut}(R \to \text{Spec} k) \to \text{Aut}(k) \to 1 \) splits. Also note that \( \text{Aut}(k) \) acts trivially on \( \text{Aut} R \), so that the result follows.

3. Recall that by Lemma 4.12, \( \text{Aut}(R \to \text{Spec} k) \cong (\text{Aut} R \times \text{Aut}(k_s \geq k))^{\tilde{\Gamma}} / \tilde{\Gamma} \), where \( \text{Aut} R \times \text{Aut}(k_s \geq k) \) is endowed with a semilinear \( \Gamma \)-action arising from a choice of cocycle \( \tilde{c} : \Gamma \to \text{Aut} R \) defining \( R \). For \( \beta \in \text{Aut}(l \geq k) \), let \( \tilde{\beta} \) denote an extension of \( \beta \) to an element of \( \text{Aut}(k_s \geq k) \). Let also \( s \in \text{Aut} R \) be an element of order 2 (note that the only case where the existence of such an element is not clear is in type \( 3D_4 \), in which case it follows from Lemma 5.7). We define a map

\[
\Phi : \text{Aut}(l \geq k) \to (\text{Aut} R \times \text{Aut}(k_s \geq k))^{\tilde{\Gamma}} / \tilde{\Gamma}
\]

\[
\beta \mapsto \begin{cases} 
\text{Id}^{\gamma} & \text{if } \beta \gamma \beta^{-1} = \gamma \text{ for all } \gamma \in \text{Gal}(l/k). \\
\ast \text{Id}^{-1} & \text{if } \beta \gamma \beta^{-1} \neq \gamma \text{ for } \gamma \text{ a generator of } \text{Gal}(l/k).
\end{cases}
\]
We first check that $\Phi$ is well-defined. If $\bar{\beta}$ and $\bar{\beta}'$ are two extensions of $\beta$, we have to check that $\Id_{\bar{\beta}} \Id_{\bar{\beta}}^{-1}$ belongs to $\Gamma$. But $(\bar{\beta}')^{-1}\bar{\beta}$ acts trivially on $l$, hence the image of $(\bar{\beta}')^{-1}\bar{\beta}$ under $\Gamma \to (\text{Aut } R \times \text{Aut}(k_s \geq k))^\Gamma; \delta \mapsto \tilde{c}_\delta \Id_{\bar{\beta}}^{-1}$ is indeed equal to $\Id_{(\bar{\beta}')^{-1}\bar{\beta}} = \Id_{\bar{\beta}} \Id_{\bar{\beta}}^{-1}$.

We now check that the image of $\Phi$ is $\Gamma$-invariant. Let $\delta \mapsto \tilde{\delta}$ denotes the projection $\text{Aut}(k_s/k) \to \text{Aut}(l/k)$. When $\beta \gamma^{-1} = \gamma$ for all $\gamma \in \text{Gal}(l/k)$, $\delta, \Id_{\bar{\beta}}^{-1} = \tilde{c}_{\beta \delta \bar{\gamma}}^{-1} \Id_{\bar{\beta}}^{-1} = \tilde{c}_{\bar{\delta}} \Id_{\bar{\beta}}^{-1} = \Id_{\bar{\beta}}^{-1}$ for all $\delta \in \text{Gal}(k_s/k)$. On the other hand, when $\beta \gamma^{-1} \neq \gamma$ for $\gamma$ a generator of $\text{Gal}(l/k)$, we have

$$\delta, (s \Id_{\bar{\beta}}^{-1}) = \tilde{c}_{\beta \delta \bar{\gamma}}^{-1} \Id_{\bar{\beta}}^{-1} = \Id_{\bar{\beta}}^{-1} = s \Id_{\bar{\beta}}^{-1}$$

for all $\delta \in \text{Gal}(k_s/k)$.

It is readily checked that $\Phi$ is a homomorphism, so it remains to check that $\Phi$ is bijective. If $\Phi(\delta)$ is trivial, then $\Id_{\bar{\beta}}^{-1}$ or $s \Id_{\bar{\beta}}^{-1}$ belongs to $\Gamma$, i.e. there exists $\delta \in \Gamma$ such that either $\Id_{\bar{\beta}}^{-1}$ or $s \Id_{\bar{\beta}}^{-1}$ is equal to $\tilde{c}_\delta \Id_{\bar{\beta}}^{-1}$. This implies that $\tilde{c}_\delta$ is trivial, so that $\delta$ acts trivially on $l$, and hence $\beta$ is trivial.

Finally, we check that $\Phi$ is surjective. Let $b \Id_{\bar{\beta}} \in (\text{Aut } R \times \text{Aut}(k_s \geq k))^\Gamma$. We claim that $\bar{\beta}$ preserves $l$. Indeed, for all $\delta \in \Gamma$, $\delta, (b \Id_{\bar{\beta}}) = \tilde{c}_{\beta \delta \bar{\gamma}}^{-1} \Id_{\bar{\beta}}$. Hence $b \Id_{\bar{\beta}}$ is $\Gamma$-invariant if and only if $bc_{\delta} = \tilde{c}_{\beta \delta \bar{\gamma}}^{-1} \Id_{\bar{\beta}}$. But if $\bar{\beta}$ does not preserve $l$, there exists $\delta \in \Gamma$ such that $\tilde{c}_\delta = 1 \neq \tilde{c}_{\beta \delta}$, a contradiction. Hence the claim is proved. To conclude, note that if $b \Id_{\bar{\beta}}$ is $\Gamma$-invariant, $\bar{\beta}$ preserves $l$ and hence up to an element in the image of $\Phi$, we can assume that $\bar{\beta}$ acts trivially on $l$. Hence, since $b \Id_{\bar{\beta}}$ is $\Gamma$-invariant, either $b$ is trivial, or $b$ commutes with the image of $\bar{\gamma}$, and hence belongs to the image of $\bar{c}$. In the first case, $b \Id_{\bar{\beta}} = \Id_{\bar{\beta}}$ is in $\Gamma$. In the second case, $b \Id_{\bar{\beta}} = \tilde{c}_{\beta} \Id_{\bar{\beta}}^{-1} \Id_{\bar{\beta}}$ for some $\gamma \in \text{Gal}(l/k)$. But this is in the image of $\Phi$ because $\tilde{c}_{\beta} \Id_{\bar{\gamma}}^{-1}$ and $\Id_{\bar{\beta}}$ are in $\Gamma$, whilst $\Id_{\bar{\gamma}} = \Phi(\gamma^{-1})$.

For the last statement, note that under the isomorphism $\text{Aut}(l \geq k) \cong \text{Aut}(R \to \text{Spec } k)$, the algebraic automorphisms are the one acting trivially on $k$, i.e. we have $\text{Aut } R \cong \text{Aut}(l/k)$.

4. We begin by proving the following claim.

**Claim 1.** Any automorphism $\beta \in \text{Aut}(l_3 \geq k)$ has a unique extension $\beta_0 \in \text{Aut}(l \geq k)$ such that for all $\gamma \in \text{Gal}(l/k)$, $\beta_0^{-1} \gamma \beta_0 = \gamma$.

**Proof of the claim:** Let $\beta \in \text{Aut}(l_3 \geq k)$ and let $\tilde{\beta} \in \text{Aut}(k_s/k)$ be an extension to $k_s$. Since $\beta$ preserves $l_3$ and since $l$ is the normal closure of $l_3$, $\tilde{\beta}$ preserves $l$. Let $l'_3$ and $l'_3$ be the two other degree 3 extension of $k$ contained in $l$. Either $\tilde{\beta}(l'_3) = l'_3'$, or $\tilde{\beta}(l'_3) = l''_3$. In the latter case, replace $\tilde{\beta}$ by $\tilde{\beta} \gamma_0$, where $\gamma_0 \in \text{Gal}(l/k)$ acts trivially on $l_3$ and exchanges $l'_3$ and $l''_3$. Hence we can assume that $\tilde{\beta}$ preserves $l_3$, $l'_3$ and $l''_3$. But now the restriction of $\beta$ to $l$ has the desired property. For uniqueness, note that if $\beta'_0$ is another such extension, then $\beta'_0^{-1}$ is an element of $\text{Gal}(l/k)$ preserving $l_3$, $l'_3$ and $l''_3$, hence $\beta'_0 \beta_0^{-1} = 1$.

For $\beta \in \text{Aut}(l_3 \geq k)$, we denote by $\beta_0$ the unique extension of $\beta$ to an element of $\text{Aut}(l \geq k)$ provided by Claim 1, and by $\beta_0$ an extension of $\beta_0$ to $\text{Aut}(k_s/k)$. Now the proof follows the same line as the previous proof of the previous item, and we discuss it more briefly. We define a map

$$\Phi: \text{Aut}(l_3 \geq k) \to (\text{Aut } R \times \text{Aut}(k_s \geq k))^\Gamma; \beta \mapsto \Id_{\beta_0}^{-1}$$

The proof that $\Phi(\beta)$ does not depend on a lift of $\beta_0$ and that $\Id_{\beta_0}^{-1}$ is $\Gamma$-invariant follows the same line as in the previous item. Furthermore, $\Phi$ is clearly a homomorphism.
Assume now that $\Phi(\beta)$ is trivial. Hence there exists $\delta \in \Gamma$ such that $\Id^{-1}_{\beta_0} = \tilde{e}_3 \Id^{-1}_g$. Hence $\tilde{e}_3$ is trivial, which implies that $\delta$ acts trivially on $l$, so that $\beta$ was trivial. Hence $\Phi$ is injective. Let us now prove surjectivity. Let $b \Id_{\tilde{\beta}} \in (\Aut R \times \Aut(k_3 \geq k))^2$. Since $\tilde{c} : \Gal(k_3/k) \to \Aut R$ is surjective and since we are working modulo $\tilde{\Gamma}$, we can assume that $b = 1$. We claim that $\tilde{\beta}$ preserves $l$ and that $\tilde{\beta}^{-1} \gamma \tilde{\beta} = \gamma$ for all $\gamma \in \Aut(l/k)$. Indeed, for all $\delta \in \Gamma$, $\delta \Id_{\beta} = \tilde{e}_{\beta^{-1} \delta} \tilde{e}_3^{-1} \Id_{\beta}$. Hence $\Id_{\tilde{\beta}}$ is $\Gamma$-invariant if and only if $\tilde{e}_3 = \tilde{e}_{\beta^{-1} \delta}$ for all $\delta \in \Gamma$. But if $\tilde{\beta}$ does not preserve $l$, there exists $\delta \in \Gamma$ such that $\tilde{e}_3 = 1 \neq \tilde{e}_{\beta^{-1} \delta}$, a contradiction. The fact that $\tilde{\beta}^{-1} \gamma \tilde{\beta} = \gamma$ for all $\gamma \in \Aut(l/k)$ also follows directly, and the claim is proved.

To conclude, note that the claim implies that $\tilde{\beta}$ preserves $l_3$, and hence up to an element in the image of $\Phi$, we can assume that $\tilde{\beta}$ acts trivially on $l$, so that $\Id_{\tilde{\beta}}$ is trivial modulo $\tilde{\Gamma}$, as wanted. \hfill $\blacksquare$

In the proof of Lemma 5.6, we needed the following lemma.

**Lemma 5.7.** Let $\mathcal{R}$ be a simple based root datum of type $R$ with Tits index $3D_{4,1}$. Then $\mathcal{R}$ is simply connected or adjoint, and hence $\Aut R$ contains an element of order $2$.

**Proof.** If $R$ is neither simply connected nor adjoint, the corresponding split connected reductive group is the split $SO_8$ (there are actually three proper subgroups in the center of the split $Spin_8$, but the corresponding intermediate quotients are all isomorphic). But the split $SO_8$ does not have an outer automorphism of order $3$, contradicting the fact that the Tits index of $\mathcal{R}$ is $3D_{4,1}$. The last part of the lemma follows from the fact that if $R$ is simply connected or adjoint, $\Aut R = \Aut D_4$ (see [Con14, Proposition 1.5.1]) and the fact that $\Aut D_4 = S_3$. \hfill $\blacksquare$

**Corollary 5.8.** Let $\mathcal{R}$ be a simple $k$-scheme of based root datum with classifying field $l$. If $\Aut(l/k) \not\cong S_3$, then $\Aut_{\mathcal{R}}(k) \cong \{ \alpha \in \Aut(k) \mid \text{there exists } \tilde{\alpha} \in \Aut(l) \text{ extending } \alpha \}$. While if $\Aut(l/k) \cong S_3$, then $\Aut_{\mathcal{R}}(k) \cong \{ \alpha \in \Aut(k) \mid \text{there exists } \bar{\alpha} \in \Aut(l_3) \text{ extending } \alpha \}$, where $l_3$ is a chosen non-normal cubic subextension of $l/k$.

**Proof.** This follows from the surjectivity of $\Aut(\mathcal{R} \to \Spec k) \to \Aut_{\mathcal{R}}(k)$ and from the description of $\Aut(\mathcal{R} \to \Spec k)$ contained in Lemma 5.6. \hfill $\blacksquare$

In view of Corollary 5.8, it is useful to introduce the following notation.

**Definition 5.9.** Let $l \geq k$ be a field extension of $k$. We denote by $\Aut_{\mathcal{R}}(k)$ the group of automorphisms of $k$ which extend to an automorphism of $l$, i.e. $\Aut_{\mathcal{R}}(k) = \{ \alpha \in \Aut(k) \mid \text{there exists } \tilde{\alpha} \in \Aut(l) \text{ extending } \alpha \}$. Using the identifications we made in Lemma 5.6 and Corollary 5.8, we can rewrite in a very explicit form the short exact sequence $1 \to \Aut \mathcal{R}(G) \to \Aut(\mathcal{R}(G) \to \Spec k) \to \Aut_{\mathcal{R}}(G)(k) \to 1$.

**Proposition 5.10.** Let $\mathcal{R}$ be a simple $k$-scheme of based root datum of type $R$ with Tits index $^gX_{n,1}$.

1. If $g = 1$, the short exact sequence $1 \to \Aut \mathcal{R}(G) \to \Aut(\mathcal{R}(G) \to \Spec k) \to \Aut_{\mathcal{R}}(k) \to 1$ is isomorphic to the short exact sequence $1 \to \Aut R \to \Aut R \times \Aut(\mathcal{R}(G) \to \Spec k) \to \Aut_{\mathcal{R}}(G)(k) \to 1$. In particular, it always splits.

2. If $g = 2$ or $g = 3$, the short exact sequence $1 \to \Aut \mathcal{R} \to \Aut(\mathcal{R}(G) \to \Spec k) \to \Aut_{\mathcal{R}}(k) \to 1$ is isomorphic to $1 \to \Aut(l/k) \to \Aut(l \geq k) \to \Aut_{\mathcal{R}}(k) \to 1$.

3. If $g = 6$, let $l_3$ be a non-normal cubic subextension of $l/k$. The short exact sequence $1 \to \Aut \mathcal{R} \to \Aut(\mathcal{R}(G) \to \Spec k) \to \Aut_{\mathcal{R}}(k) \to 1$ is isomorphic to $1 \to 1 \to \Aut(l_3 \geq k) \to \Aut_{\mathcal{R}}(k) \to 1$. In particular, it always splits.
Proof. This is a direct consequence of Lemma 5.6 and Corollary 5.8. Note that in each case, the map $\text{Aut}(l \geq k) \to \text{Aut}(k)$ is given by restriction to $k$. Also note that when $g = 6$, and since $l_3$ is a non normal cubic extension of $k$, the group $\text{Aut}(l_3/k)$ is trivial, and $\text{Aut}(l_3 \geq k) \cong \text{Aut}_{l_3}(k)$. □

We end this discussion with examples where the short exact sequence $1 \to \text{Aut} R \to \text{Aut}(R \to \text{Spec } k) \to \text{Aut}_{\text{R}}(k) \to 1$ does not split.

Definition 5.11. The field $k$ is called rigid if for any finite Galois extension $k'$ of $k$ such that $k'$ is not algebraically closed, every automorphism of $k'$ fixes $k$ pointwise.

Definition 5.12. A prime field is either the field of rational numbers of a finite field of order $p$ for some prime $p$.

Examples of rigid fields include prime fields and $\mathbb{Q}_p$ (the field of $p$-adic numbers) for any prime $p$. Let us give a reference for this latter assertion.

Lemma 5.13. Let $p$ be a prime number and let $\mathbb{Q}_p$ be the field of $p$-adic numbers. Let $\mathbb{Q}_p \leq k'$ be a finite Galois extension. Then every automorphism of $k'$ fixes pointwise $\mathbb{Q}_p$.

Proof. The field $k'$ is complete and non algebraically closed. Hence by [Sch33], all complete norms on $k'$ are equivalent. Hence an automorphism of $k'$ has to preserve the norm, which is to say that it has to be continuous. But since any automorphism acts trivially on $\mathbb{Q}$, by continuity it also has to act trivially on $\mathbb{Q}_p$. □

Corollary 5.14. Assume that $k$ is a finite (respectively possibly infinite) Galois extension of a rigid (respectively prime) field $k_0$. Let $G$ be a connected reductive $k$-group which is quasi-split and absolutely simple. Assume that $R(G)$ has Tits index $\{X_{n,1}\}$, with $q = 2$ or $q = 3$. Further assume that $l$ is a Galois extension of $k_0$. Then $\text{Aut}_G(k) = \text{Aut}(k)$ and the short exact sequence $1 \to \text{Aut} G \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}_G(k) \to 1$ splits if and only if $1 \to \text{Gal}(l/k) \to \text{Gal}(l/k_0) \to \text{Gal}(k/k_0) \to 1$ splits.

Proof. In view of Theorem 1.3 and Proposition 5.10, the short exact sequence $1 \to \text{Aut} G \to \text{Aut}(G \to \text{Spec } k) \to \text{Aut}_G(k) \to 1$ splits if and only if the short exact sequence $1 \to \text{Gal}(l/k) \to \text{Aut}(l \geq k) \to \text{Aut}_l(k) \to 1$ splits. Since $k_0$ is rigid (or even prime if $k$ is an infinite Galois extension) and $k$ is a normal extension, $\text{Aut}(l \geq k) = \text{Gal}(l/k_0)$. Furthermore, $\text{Aut}(k) = \text{Gal}(k/k_0)$, and since $l/k_0$ is Galois, every element of $\text{Gal}(k/k_0)$ extends to $\text{Gal}(l/k_0)$. Hence $\text{Aut}_l(k) = \text{Gal}(k/k_0)$, as wanted. □

Remark 5.15. Corollary 5.14 directly implies the corollary stated at the beginning of the introduction of this paper. Indeed, $\mathbb{Q}$ is a prime field and $\mathbb{Q}_p$ is rigid by Lemma 5.13. Furthermore, $\text{Aut}_{\text{abstract}}(G(k)) = \text{Aut}(G \to \text{Spec } k)$ by the Borel–Tits theorem that we stated at the very beginning of the introduction. For the ease of non-expert readers, let us also give an explicit realisation of the quasi-split, absolutely simple, adjoint algebraic $k$-group of type $2A_{n-1}$ with corresponding quadratic separable extension $l$: denote the Galois conjugation on $l$ by $x \mapsto \bar{x}$, and for $g \in \text{PGL}_n(l)$, set $(\bar{g}^{-1}g)_{ij} = \delta_{n+1-j,n+1-i}$ (i.e. the anti-transposed conjugated matrix). We define $\text{PGU}_n(k) = \{g \in \text{PGL}_n(l) \mid \bar{g}^{-1}g = 1\}$. This is easily interpreted as the $k$-rational points of an algebraic $k$-group, and one readily sees that this algebraic $k$-group is the quasi-split, absolutely simple, adjoint algebraic $k$-group of type $2A_{n-1}$ with corresponding quadratic separable extension $l$ (because the corresponding cocycle is $g \mapsto \bar{g}^{-1}$, which is an outer automorphism of $\text{PGL}_n$ preserving its Borel subgroup consisting of upper triangular matrices).

6 The SL$_n(D)$ case over a local field

6.1 Outer automorphisms of finite dimensional central simple algebras over local fields

We now explore the same question for algebraic groups of the form SL$_n(D)$. First, we need to be a bit more precise and make a distinction between the algebraic $k$-group and its group of $k$-rational
points.

**Definition 6.1.** Let $A$ be a finite dimensional central simple $k$-algebra. Following the notation of [KMRT98], we denote the corresponding algebraic $k$-group of “reduced norm 1 elements” by $\text{SL}_n(A)$. The $k$-rational points of $\text{SL}_n(A)$ are the elements of $A$ of reduced norm 1, and we denote this group by $\text{SL}_n(A)$. When $A = M_n(A')$ for some finite dimensional central simple $k$-algebra $A'$, we also denote $\text{SL}_n(A)$ (respectively $\text{SL}_n(A')$) by $\text{SL}_n(A)$ (respectively $\text{SL}_n(A')$).

**Remark 6.2.** Note that for $A$ a finite dimensional central simple $k$-algebra and $\alpha \in \text{Aut}(k)$, $\alpha \text{SL}_n(A)$ is naturally isomorphic (as an algebraic $k$-group) to $\text{SL}_n(\alpha A)$. Hence by (a slightly enhanced version of) [KMRT98, Remark 26.11], $\alpha \in \text{Aut}_{\text{SL}_n(A)}(k)$ if and only if $A \cong \alpha A$ or $A^{op} \cong \alpha A$ (as $k$-algebras).

We will restrict ourselves to working over a local field. For us, a **local field** is a non-archimedean non-discrete topological field which is locally compact (or equivalently, a field isomorphic to $\mathbb{F}_p((T))$ or a finite extension of $\mathbb{Q}_p$ for some prime number $p$). For the rest of the paper, the letter $K$ exclusively stands for a local field. Let us begin by recalling the classification of central simple algebras over local fields.

**Definition 6.3.** Let $k$ be a field and let $l/k$ be a finite cyclic extension of degree $d$. Let $\sigma \in \text{Gal}(l/k)$ be a generator of the cyclic group $\text{Gal}(l/k)$, let $a \in k$ and let $u$ be an abstract symbol. The cyclic algebra $A(l/k, \sigma, a, u)$ is defined as follows: as a $k$-vector space, $A(l/k, \sigma, a, u) \cong \bigoplus_{i=0}^{d-1} u^i l$, and the multiplication is defined by using the relations $u^d = a$ and $u^{-1} xu = \sigma(x)$ for all $x \in l$. We also denote it $A(l/k, \sigma, a)$.

We recall that the algebra $A(l/k, \sigma, a, u)$ of Definition 6.3 is always central simple over $k$, and that it is isomorphic to the $k$-algebra $M_n(k)$ if and only if $a$ is the norm of an element in $l$.

**Definition 6.4.** Let $K$ be a local field and let $d, r \in \mathbb{N}$ with $d \geq 1$. Let $K_d$ be the unramified extension of $K$ of degree $d$, let $\sigma \in \text{Gal}(K_d/K)$ be the Frobenius automorphism (i.e. the automorphism inducing the Frobenius automorphism on $\text{Gal}(\overline{K_d}/K)$), and let $\pi$ be a uniformiser of $K$. We define $A(d, r)$ to be the cyclic algebra $A(K_d/K, \sigma, \pi^r)$.

Note that up to isomorphism, $A(d, r)$ does not depend on the choice of $\pi$. In fact, given two uniformisers $\pi$ and $\tilde{\pi}$, an explicit isomorphism $(K_d/K, \sigma, \pi^r) \cong (K_d/K, \sigma, \tilde{\pi}^r)$ having the same form as the one appearing in Lemma 6.5 can be given.

**Lemma 6.5.** Let $K$ be a local field. Let $A = A(d, r)$ and $K_d, \sigma, \pi$ be as in Definition 6.4. Let $\alpha$ be an automorphism of $K_d$ such that $\alpha(K) = K$, and assume that there exists an element $x$ in $K_d$ such that $N_{K_d/K}(x) = \frac{\alpha(x)}{x}$. Then the map $\phi(\alpha, x): A \to A$: $\sum_{i=0}^{d-1} u^i a_i \mapsto \sum_{i=0}^{d-1} (ux)^i \alpha(a_i)$ is a ring automorphism of $A$.

**Proof.** We view $A$ as a quotient of the twisted polynomial ring $K_d[u; \sigma]$ (see [Jac96, Section 1.1] for the definition of a twisted polynomial ring) modulo the relation $u^d = \pi^r$. Given an automorphism $\alpha$ in $\text{Aut}(K_d)$, we can define a map $f_\alpha: K_d[u; \sigma] \to K_d[u; \sigma]$: $u \mapsto ux$, $a \mapsto \alpha(a)$ for all $a \in K_d$. By [Jac96, Proposition 4.6.20], $f_\alpha$ is a ring automorphism as soon as $\alpha \sigma = \sigma \alpha$. Recall that by assumption, $\alpha(K) = K$. Hence $\sigma^{-1} \alpha \sigma^{-1}$ belongs to $\text{Gal}(K_d/K)$, and its induced automorphism on the residue field $\overline{K_d}$ is a commutator in $\text{Aut}(\overline{K_d})$, thus trivial (note that since every automorphism of a local field is continuous, it always induces an automorphism of the residue field). We conclude that $\sigma^{-1} \alpha \sigma^{-1}$ itself was trivial by [Ser79, Chapter III, §5, Theorem 3]. Hence, $f_\alpha$ is indeed a ring automorphism.

Furthermore, if it passes to the quotient, $f_\alpha$ induces the automorphism $\phi(\alpha, x)$. Hence it suffices to check that $f_\alpha$ preserves the relation. But we have $f_\alpha(u^d - \pi^r) = (ux)^d - \alpha(\pi^r) = u^d N_{K_d/K}(x) - \alpha(\pi^r) = (u^d - \pi^r) \frac{\alpha(x)}{x}$, as wanted. \qed
For $\alpha$ an automorphism of a (non-necessarily commutative) ring $R$, we denote by $\bar{\alpha}$ the corresponding automorphism of $M_n(R)$ (the algebra of $n \times n$ matrices with coefficient in $R$) obtained by applying $\alpha$ coefficient by coefficient. Also, for $A$ a finite dimensional central simple algebra over a field $k$, we denote by $\text{Nrd}: A \to k$ its reduced norm.

**Lemma 6.6.** Let $k$ be a field and let $A$ be a central simple $k$-algebra. For every ring automorphism $\alpha$ of $A$ and $x \in A$,

$$\text{Nrd}(\alpha(x)) = \alpha(\text{Nrd}(x)).$$

**Proof.** Let $k_s$ be a separable closure of $k$. Every automorphism $\alpha$ of $A$ preserves the center $k$; the restriction $\alpha_k$ extends to an automorphism $\beta$ of $k_s$, and we may consider the tensor product

$$\alpha \otimes \beta: A \otimes_k k_s \to A \otimes_k k_s.$$

Since $k_s$ splits $A$, we may also consider an isomorphism of $k_s$-algebras $f: A \otimes_k k_s \to M_d(k_s)$. The ring automorphism $f \circ (\alpha \otimes \beta) \circ f^{-1}$ of $M_d(k_s)$ restricts to $\beta$ on the center $k_s$, hence $f \circ (\alpha \otimes \beta) \circ f^{-1} \circ \beta^{-1}$ is the identity on $k_s$. Since every $k_s$-automorphism of $M_d(k_s)$ is inner, we may find $g \in \text{GL}_d(k_s)$ such that

$$f \circ (\alpha \otimes \beta) \circ f^{-1} \circ \beta^{-1} = \text{int}(g).$$

The following diagram then commutes:

$$\begin{array}{ccc}
A \otimes_k k_s & \xrightarrow{f} & M_d(k_s) & \xrightarrow{\det} & k_s^\times \\
\downarrow{\alpha \otimes \beta} & & \downarrow{\text{int}(g) \circ \beta} & & \\
A \otimes_k k_s & \xrightarrow{f} & M_d(k_s) & \xrightarrow{\det} & k_s^\times 
\end{array}$$

Since $\text{Nrd} = \det \circ f$, the lemma follows. \hfill \square

We set some notations that we use for the rest of the paper.

**Definition 6.7.** Let $K$ be a local field. Let $A(d, r)$ and $K_d, \sigma, \pi$ be as in Definition 6.4. Let $\alpha$ be an automorphism of $K_d$ such that $\alpha(K) = K$, and assume that there exists an element $x$ in $K_d$ such that $N_{K_d/K}(x) = \frac{\alpha(x)}{x}$. The map $\bar{\phi}(\alpha, x): M_n(A) \to M_n(A)$ corresponding to the automorphism $\phi(\alpha, x): A \to A$ from Lemma 6.5 preserves elements of reduced norm 1 by Lemma 6.6. We again denote its restriction to $\text{SL}_n(A)$ by $\bar{\phi}(\alpha, x)$.

**Remark 6.8.** In the notations of Definition 6.7, $\bar{\phi}(\alpha, x)$ is an isomorphism of $k$-algebras $M_n(A) \cong \alpha^{-1} M_n(A)$. Hence, by [KMRT98, Theorem 26.9], $\bar{\phi}(\alpha, x)$ corresponds to a unique $k$-isomorphism of algebraic groups $\text{SL}_n(A) \cong \alpha^{-1} \text{SL}_n(A)$. More concretely, this can also be seen by using a representation of $A$ in $M_d(K)$ (where $d$ is the degree of $A$).

The following observation explains in part why the local field case is so much simpler than say the global field case (see also the end of Remark 6.11).

**Lemma 6.9.** Let $K$ be a local field and let $K_d$ be a finite dimensional unramified extension of $K$. Any automorphism of $K$ extends to an automorphism of $K_d$.

**Proof.** Let $\alpha \in \text{Aut}(K)$. There exists an extension $\beta$ of $\alpha$ to the separable closure of $K$. Note that if $K_d \cong K[X]/(f)$, then $\beta(K_d) \cong K[X]/(\alpha f)$, where $\alpha f$ is the polynomial obtained from $f$ by applying $\alpha$ to its coefficients. But $\alpha$ is continuous, and an extension is unramified if and only if it is isomorphic to $K[X]/(g)$ for some polynomial $g$ whose coefficients are all of valuation 0. Hence by uniqueness of unramified extensions of a given degree, $\beta$ preserves $K_d$. \hfill \square

**Corollary 6.10.** Let $A$ be a finite dimensional central simple algebra over a local field $K$. Every automorphism of $K$ extends to an automorphism of $A$. Hence, $\text{Aut}_{\text{SL}_n(A)}(K) = \text{Aut}(K)$.
**Proof.** By Theorem A.1, the central simple algebra $A$ is an algebra of the form $A(d,r)$, i.e. a cyclic algebra of the form $(K_d/K, \sigma, \pi^t)$ with $K_d, \sigma, \pi$ as in Definition 6.4.

Let $\alpha \in \text{Aut}(K)$. By Lemma 6.9, there exists $\beta \in \text{Aut}(K_d)$ extending $\alpha$. Also, by [Ser79, Chapter V.3.2, Corollary], $N_{K_d/K}$ is surjective on $O_K^\times$. Furthermore, any automorphism of a local field preserves the valuation. Hence there exists $x \in K_d$ such that $N_{K_d/K}(x) = \frac{a(x)}{x}$. Then the automorphism $\phi(\beta, x)$ defined in Lemma 6.5 is an extension of $\alpha$ to $A$. Finally, $\phi(\beta, x)$ from Definition 6.7 is defined over $\alpha^{-1}$, so that the last claim follows from Remark 6.8. 

**Remark 6.11.** If $\alpha \in \text{Aut}(K)$ is of finite order, the result in Corollary 6.10 asserting that $\alpha$ extends to an automorphism of $A$ is an old result. Indeed, using Lemma A.4, it is a direct consequence of [EM48, Corollary 7.3] (see also [Han07a, Theorem 5.6]) and the fact that $A = M_n(D)$ for some division algebra $D$. This already settles the question in characteristic 0. In particular, Theorem 6.10 can be seen as a direct corollary of the results in [Han07b].

Note that the fact that any extension of $\alpha \in \text{Aut}(K)$ to the separable closure of $K$ preserves $K_d$ simplifies matters (compare with Lemma 4.18 when the extension $\beta$ does not preserve the chosen maximal subfield $l$).

### 6.2 Sufficient condition for the exact sequence not to split

We turn to the splitting question for the exact sequence $1 \to \text{Aut}G \to \text{Aut}(G \to \text{Spec}k) \to \text{Aut}_G(k) \to 1$, still assuming $G = \text{SL}_n(A)$ over a local field. Let us introduce another notation for a subgroup of the group of semilinear automorphisms, which allow us to introduce a “ground field”.

**Definition 6.12.** Let $G$ be a $k$-group scheme. Let $k'$ be a subfield of $k$. We denote by $\text{Aut}(G \to \text{Spec}k/k')$ the subgroup of $\text{Aut}(G \to \text{Spec}k)$ consisting of semilinear automorphisms over an automorphism $\alpha$ belonging to $\text{Aut}(k/k')$. Furthermore, we denote by $\text{Aut}_G(k/k')$ the image of $\text{Aut}(G \to \text{Spec}k/k')$ under the map $\text{Aut}(G \to \text{Spec}k) \to \text{Aut}_G(k)$.

**Theorem 6.13.** Let $D$ be a central division algebra of degree $d$ over a local field $K$ and let $G = \text{SL}_n(D)$. Let $K'$ be a subfield of $K$ such that $K/K'$ is a finite Galois extension. Then the short exact sequence $1 \to \text{Aut}G \to \text{Aut}(G \to \text{Spec}K/K') \to \text{Aut}_G(K/K') \to 1$ splits if and only if $\gcd(nd, [K : K'])$ divides $n$.

**Proof.** By Corollary 6.10, $\text{Aut}_G(K) = \text{Aut}(K)$. Hence, since $\text{Gal}(K/K')$ is contained in $\text{Aut}_G(K)$, the short exact sequence splits if and only if $G$ is defined over $K'$ (see Theorem 2.5). Let $H$ be this hypothetical form of $G$ over $K'$. The case $d = 1$ being obviously true, let us assume that $d \geq 2$. Now, by the classification of simple groups over local fields (see [Tit79, Section 4.2 and 4.3]), the Tits index of $H$ is of the form $1A^{(d)}$ or $2A^{(1)}$, since these are the only groups of type $A$ over local fields. Note that a distinguished orbit has to remain distinguished after scalar extension, because a non-trivial root remains non-trivial after scalar extension. Hence $H$ cannot be of type $2A^{(1)}$, because groups of type $2A^{(1)}$ have extremal roots that are distinguished, whereas $G$ has undistinguished extremal roots when $d \geq 2$. But the only groups of type $1A^{(d)}$ are groups of the form $\text{SL}_{n'}(D')$ where $n' \geq 1$ and $D'$ is a division algebra over $K'$. So we conclude that $H$ is of this form.

We use the notation $\text{inv}$ for the map classifying division algebras over local fields (see Theorem A.1 for a precise definition of inv). Let $d'$ be the degree of $D'$ over $K'$, and let $r'$ be such that $[\frac{d'}{n'}] = \text{inv}([D'])$ in $Q/Z$. Also, let $a = \gcd(d', [K : K'])$. The base change of $\text{SL}_{n'}(D')$ from $K'$ to $K$ is the algebraic group $\text{SL}_{n'}(A(\frac{d'}{a}, \frac{dK/K'}{a}r'))$ by Proposition A.5. Since $H$ is isomorphic to $G$ over $K$, $an' = n$ and $ad = d'$. Hence, $a = \gcd(ad, [K : K'])$, which implies that $\gcd(adn', [K : K'])$ divides $an'$. Now, the equation $an' = n$ already proves that if $H$ exists, then $\gcd(nd, [K : K'])$ divides $n$.

Conversely, let $a = \gcd(nd, [K : K'])$, and assume that $a$ divides $n$. We then set $n' = \frac{n}{a}$, $d' = ad$ and $r'$ such that $\frac{[K:K']}{a}r' \in \mathbb{Z}$ (such an $r'$ exists because $\frac{[K:K']}{a}$ is prime to $d$). With these parameters, the algebraic group $\text{SL}_{n'}(A(d', r'))$ is a form of $G$ over $K'$, as wanted. 

\[ \square \]
Remark 6.14. The condition that gcd$(nd, [K : K'])$ divides $n$ is equivalent to require that for all primes $p$ dividing $d$, the $p$-adic valuation of $[K : K']$ is less than or equal to the $p$-adic valuation of $n$.

Corollary 6.15. Let $D$ be a central division algebra of degree $d$ over a local field $K$ and let $G = SL_n(D)$. The short exact sequence 1 $\rightarrow$ Aut$(G)$ $\rightarrow$ Aut$(G \rightarrow \text{Spec } K)$ $\rightarrow$ Aut$_G(K) \rightarrow$ 1 does not split if there exists a subfield $K' \leq K$ such that $K/K'$ is finite Galois and gcd$(nd, [K : K'])$ does not divide $n$.

Proof. 1 $\rightarrow$ Aut$(G)$ $\rightarrow$ Aut$(G \rightarrow \text{Spec } K')$ $\rightarrow$ Aut$_G(K/K') \rightarrow$ 1 does not split by Theorem 6.13, hence neither does 1 $\rightarrow$ Aut$(G)$ $\rightarrow$ Aut$(G \rightarrow \text{Spec } K)$ $\rightarrow$ Aut$_G(K) \rightarrow$ 1.

6.3 Sufficient condition for the exact sequence to split

In characteristic 0, it is actually straightforward to prove the converse of Corollary 6.15.

Theorem 6.16. Let $D$ be a central division algebra of degree $d$ over a local field $K$ of characteristic 0 and let $G = SL_n(D)$. The short exact sequence 1 $\rightarrow$ Aut$(G)$ $\rightarrow$ Aut$(G \rightarrow \text{Spec } K)$ $\rightarrow$ Aut$_G(K) \rightarrow$ 1 does not split only if there exists a subfield $K' \leq K$ such that $K/K'$ is finite Galois and gcd$(nd, [K : K'])$ does not divide $n$.

Proof. By Corollary 6.10, Aut$_G(K) = \text{Aut}(K)$. Since $K$ is of characteristic 0, it is a finite extension of $\mathbb{Q}_p$ for some prime $p$. But every automorphism of $K$ acts trivially on $\mathbb{Q}_p$ by Lemma 5.13. Hence, by Galois theory, Aut$(K)$ is a finite group. Furthermore, letting $K^\text{Aut}(K)$ be the subfield of $K$ fixed by Aut$(K)$, the extension $K/K^\text{Aut}(K)$ is Galois with Galois group Aut$(K)$.

Let $a = \gcd(nd, [K : K^\text{Aut}(K)])$. Assuming that there does not exist a subfield $K' \leq K$ such that $K/K'$ is finite Galois and such that gcd$(nd, [K : K'])$ does not divide $n$, we have in particular that $a$ divides $n$. Also, let $r \in \mathbb{N}$ be such that $[\mathbb{Z}^r] = \text{inv}(D)$. Since $K_{K^\text{Aut}(K)}^a$ is prime to $d$, there exists $r' \in \mathbb{N}$ such that $[K_{K^\text{Aut}(K)}^a]^{-1} \cdot r' - r \in d\mathbb{Z}$. Hence, by Proposition A.5, the algebraic group $\text{SL}_n(A(ad, r'))$ is a form of $G$ over $K^\text{Aut}(K)$, because gcd$(ad, [K : K^\text{Aut}(K)]) = a$. But in view of Lemma 2.4, this implies that the homomorphism Aut$(G \rightarrow \text{Spec } K)$ $\rightarrow$ Aut$(K) = \text{Gal}(K/K^\text{Aut}(K))$ has a section, as wanted.

Remark 6.17. Putting Corollary 6.15 and Theorem 6.16 together already proves Theorem 1.5 in characteristic 0. In particular, the sequence always splits for $K = \mathbb{Q}_p$ (this actually directly follows from the rigidity of $\mathbb{Q}_p$, which was used in the proof of Theorem 6.16). For a more interesting example, if $K$ is a Galois extension of $\mathbb{Q}_p$ of degree $p^i$ for some prime $p$ and some $i \in \mathbb{N}$, then Theorem 1.5 asserts that the following are equivalent:

1. The sequence 1 $\rightarrow$ Aut$\text{SL}_n(D)$ $\rightarrow$ Aut$(\text{SL}_n(D) \rightarrow \text{Spec } K)$ $\rightarrow$ Aut$\text{SL}_n(D)(K) \rightarrow$ 1 splits.

2. If $n$ is not divisible by $p^i$, the degree of $D$ is not divisible by $p$.

We now aim to prove an analogue of Theorem 6.16 but in positive characteristic. When $K$ is of positive characteristic, the fixed field $K^\text{Aut}(K)$ is finite and $K/K^\text{Aut}(K)$ is not Galois. Thus we cannot use the same method than in characteristic 0.

Instead, the strategy goes as follows: we decompose Aut$(K)$ in various pieces, we give a section of Aut$(\text{SL}_n(D) \rightarrow \text{Spec } K)$ $\rightarrow$ Aut$(K)$ separately for each pieces and then we check that everything can be glued. Let us begin by decomposing Aut$(K)$.

Lemma 6.18. Let $K = F_{p^i}(T)$. Since $F_{p^i}$ is the algebraic closure in $K$ of the prime field of $K$, $F_{p^i}$ is preserved by any automorphism of $K$. Let $N(K) = \{\alpha \in \text{Aut}(K) \mid \alpha \text{ acts trivially on } F_{p^i}\}$. We have Aut$(K) \cong N(K) \rtimes \text{Gal}(K/F_{p^i}(T))$.

Proof. We want to show that the short exact sequence 1 $\rightarrow$ $N(K)$ $\rightarrow$ Aut$(K)$ $\rightarrow$ Gal$(K/F_{p^i}(T))$ $\rightarrow$ 1 splits. But by [Ser79, Chapter III, §5, Theorem 3], $f$ maps Gal$(K/F_{p^i}(T))$ isomorphically onto Gal$(F_{p^i}/F_p)$, hence the result.\]
We furthermore decompose the group \( N(K) \). Since automorphisms of \( K \) are continuous, an element \( \alpha \) of \( N(K) \) is therefore determined by its action on \( T \), and we have \( \alpha(T) = \sum_{j=2}^{\infty} a_j T^j \), where \( a_1 \in F_p^\times \) and \( a_j \in F_p \) for all \( j \geq 2 \).

**Definition 6.19.** Let \( J(K) = \{ \alpha \in N(K) \mid \alpha(T) = T + \sum_{j=2}^{\infty} a_j T^j, \quad a_j \in F_p^\prime \} \) and let \( C_{p^\prime - 1} = \{ \alpha \in N(K) \mid \alpha(T) = aT, \quad a \in F_p^\times \} \). With those notations, the group \( N(K) \) is isomorphic to \( J(K) \times F_p^\times \). For \( x \in F_p^\times \), we denote by \( \text{ev}(xT) \) the corresponding element of \( \text{Aut}(K) \).

In summary, we have decomposed \( \text{Aut}(K) \) as the group \( (J(K) \times F_p^\times) \rtimes \text{Gal}(K/F_p((T))) \). We go on by giving a section to \( \text{Aut}(SL_n(D)) \rightarrow \text{Spec} \mathcal{O}(K) \rightarrow \text{Aut}(K) \) for each component of \( \text{Aut}(K) \), one at a time. In doing so, we will at the same time take care that the given section glues well with the other sections (though each are studied separately). Hence, a given formula for a section on one component of \( \text{Aut}(K) \) will at times be slightly more complicated than a formula one would naturally consider if one was not aiming for a global section. In each case, we write a remark to explain how the given formula could be simplified if not aiming for a global splitting.

We need to set a few notations.

**Definition 6.20.**
1. Let \( F_p^\text{alg} \) be an algebraic closure of \( F_p \). We denote by \( F_p \) the Frobenius automorphism of \( F_p^\text{alg}(T) \) (i.e. the automorphism of \( F_p^\text{alg}(T) \) fixing \( F_p((T)) \)) and inducing the Frobenius automorphism on \( F_p^\text{alg} \). For any finite extension extension \( L \) of \( F_p \), we also denote by \( F \) the restriction of \( F \) to \( L \) and to \( L((T)) \).

2. We fix \( p \) a prime number, \( i, n, d, r \in \mathbb{N} \) such that \( \text{gcd}(d, r) = 1 \) and two symbols \( u, T \). We set \( K = F_p((T)), \quad E = F_{p^r}(T) \) and \( D = (E/K, F^a, T^r, u) \) (a cyclic division algebra of degree \( d \) over \( K \) with symbol \( u \) as in Definition 6.4). Furthermore, we let \( G \) be the algebraic \( K \)-group \( SL_n(D) \).

3. For \( \alpha \in N(K) \) we define its extension \( \alpha_E \) to \( \text{Aut}(E) \) as follows: \( \alpha_E \) acts trivially on the residue field, while \( \alpha_E(T) = \alpha(T) \). We thus get an injective homomorphism \( N(K) \rightarrow \text{N}(E); \alpha \mapsto \alpha_E \). Abusing notations, we again denote \( \alpha_E \) by \( \alpha \).

4. We fix \( a, b \in \mathbb{N} \) such that \( ab = p^i - 1 \), \( \text{gcd}(d^i - 1, p^i - 1) = \text{gcd}(d^i, b) = b \) and \( \text{gcd}(d, a) = 1 \).

5. We fix \( a', b' \in \mathbb{N} \) such that \( a'b' = i \), \( \text{gcd}(d^i, i) = \text{gcd}(d^i, b') = b' \) and \( \text{gcd}(d, a') = 1 \).

6. We choose a generator \( \zeta \) of the multiplicative group \( F_p^\times \).

7. For \( g \in \text{PGL}_n(D) \), we denote by \( \text{int}(g) \) the automorphism by conjugation of \( g \) on \( G \), i.e. \( \text{int}(g); G \rightarrow G; h \mapsto ghg^{-1} \).

**Remark 6.21.** The natural numbers \( a, b, a', b' \) are uniquely determined by their definition. Note that we have in particular \( \text{gcd}(a, b) = 1 = \text{gcd}(a', b') \). Also note that by Definition 6.4, \( u^{-1} x u = F^g(x) \) for all \( x \in E \).

We will further make use of the following notation: if \( l, m \in \mathbb{N} \) and \( A_1, \ldots, A_l \) are \( m \times m \) matrices, we denote \( \text{Diag}(A_1, \ldots, A_l) \) the corresponding block diagonal \( lm \times lm \) matrix. Furthermore, the \( m \times m \) identity matrix is denoted \( \text{Id}_m \). We will denote the cyclic group of order \( m \) by \( C_m \).

**Proposition 6.22.** Keep the notations of Definition 6.20. Assume that \( \text{gcd}(p, d) = 1 \) and that \( bb' \) divides \( n \). For \( \alpha \in J(K) \), there exists a unique \( x_\alpha \in 1 + TF_p'[T] \) such that \( x_\alpha^{bb'} = x_\alpha^{bb' - 1} \). Let \( M = \text{Diag}(\text{Id}_b, x_\alpha \text{Id}_b, x_\alpha^2 \text{Id}_b, \ldots, x_\alpha^{b-1} \text{Id}_b) \) (so that \( M \) is a \( bb' \times bb' \) matrix which is block diagonal) and let \( X_\alpha = \text{Diag}(M, \ldots, M) \) where we have \( \frac{n}{bb'} \) terms (so that \( X_\alpha \) is a \( n \times n \) matrix which is
block diagonal with coefficients in $1 + TF_p[[T]]$). Recalling the notation introduced in Remark 6.8, the map

$$f_{J(K)} : J(K) \to \text{Aut}(G \to \text{Spec } K)$$

$$\alpha \mapsto \text{int}(X_\alpha) \phi(\alpha, x_\alpha^{b'})$$

is a homomorphism whose composition with the map $\text{Aut}(G \to \text{Spec } K) \to \text{Aut}_G(K)$ is the identity on $J(K)$.

Proof. Note that $\gcd(p, d) = 1$ and $\gcd(d^{b'}, b') = b'$ implies $\gcd(p, b') = 1$, so that $\gcd(p, db') = 1$. Hence, for $\alpha \in J(K)$ the existence and uniqueness of $x_\alpha$ in $1 + TF_p[[T]]$ such that $x_\alpha^{b'} = \frac{\alpha(T')}{T}$ follows directly from Hensel's lemma. We claim that for $\alpha, \beta \in J(K)$, $x_{\beta\circ\alpha} = x_\beta \beta(x_\alpha)$. By uniqueness, this equation holds if and only if $(\frac{2\alpha(x)}{p}) = (\frac{x_\beta\beta(x_\alpha)}{p})^{db'}$. But the right hand side is equal to $\frac{\beta(T')}{T} \beta(\alpha(T'))$, which is indeed equal to $(\frac{2\alpha(x)}{p})$. Checking that $f_{J(K)}$ is a homomorphism is now straightforward:

$$\text{int}(X_\beta) \phi(\beta, x_\beta^{b'}) \circ \text{int}(X_\alpha) \phi(\alpha, x_\alpha^{b'}) = \text{int}(X_\beta \beta(X_\alpha)) \phi(\beta \circ \alpha, x_\beta^{b'} \beta(x_\alpha^{b'})) = \text{int}(X_{\beta\circ\alpha}) \phi(\beta \circ \alpha, x_\beta^{b'}).$$

Note that if we were not aiming to define a global section of $\text{Aut}(G \to \text{Spec } K) \to \text{Aut}_G(K)$, we could just as well get rid of the factor $\text{int}(X_\alpha)$ and hence we would not need the assumption that $bb'$ divides $n$. In light of this, the next proposition really is a converse to Proposition 6.22.

Proposition 6.23. Keep the notations of Definition 6.20. If $\gcd(p, d) \neq 1$, there does not exist a homomorphism $J(K) \to \text{Aut}(G \to \text{Spec } K)$ whose composition with $\text{Aut}(G \to \text{Spec } K) \to \text{Aut}_G(K)$ is the identity on $J(K)$.

Proof. By Theorem 6.13, it suffices to prove that there exists $K' \leq K$ such that $K/K'$ is finite Galois with $\text{Gal}(K/K') \leq J(K)$ and such that $\gcd(nd_i, [K : K'])$ does not divide $n$. Let $H$ be a group of order $p^n$. By [Cam97, Theorem 3], there exists an injective homomorphism $H \to J(\mathbb{F}_p[[T]])$. Also note that $J(\mathbb{F}_p[[T]])$ can be seen as a subgroup of $J(K)$ in a natural way, so that $J(K)$ has a subgroup of order $p^n$, that we again denote by $H$. Now, let $K' = K^H = \{ x \in K \mid \alpha(x) = x \text{ for all } \alpha \in H \}$. Hence, $K/K'$ is a Galois extension with $\text{Gal}(K/K') = H \leq J(K)$ and $\gcd(nd_i, [K : K']) = \gcd(nd_i, p^n)$ does not divide $n$ because $\gcd(p, d) \neq 1$, as wanted.

We now construct a section of $\text{Aut}(G \to \text{Spec } K) \to \text{Aut}(K)$ for $\mathbb{F}_p^\times$. In fact, using the same line of argument as for Theorem 6.13, we know that a section for $\mathbb{F}_p^\times$ exists if and only if $\gcd(nd_i, p^i - 1)$ divides $n$ (where $d$ and $n$ appear in the form of $G = \text{SL}_n(D)$, $d$ denoting as usual the degree of $D$). But we need to have an explicit formula, since we want to ensure that it glues well with the map $f_{J(K)}$ constructed in Proposition 6.22. We found those explicit formulas by working out by hand some low degree examples for which we could follow explicitly what the theory was predicting, and then by generalising our findings to any degree. Again, we give a section which is going to be slightly more complicated than necessary, because we aim to define a global section in the end.

We first need a $(db')$-th root of $\zeta^{br}$.

Lemma 6.24. Keep the notations of Definition 6.20 and let $C_a$ be the group generated by $\zeta^b$ in $\mathbb{F}_p^\times$. There exists a unique $z \in C_a$ such that $z^{db'} = \zeta^{br}$.

Proof. Note that $\gcd(d, a) = 1$ and $\gcd(db', b') = b'$ implies $\gcd(b', a) = 1$, so that $\gcd(db', a) = 1$. Hence the result follows from the fact that $db'$ is invertible in the cyclic group of order $a$.

Proposition 6.25. Keep the notations of Definition 6.20 and of Lemma 6.24, so that $z^{db'} = \zeta^{br}$. Assume that $bb'$ divides $n$. Let $M = \text{Diag}(I_d, z I_d, z^2 I_d, \ldots, z^{b'-1} I_d)$ (so that $M$ is a $bb' \times bb'$ matrix which is block diagonal) and let $Z = \text{Diag}(M, \ldots, M)$ where we have $\frac{db'}{bb'}$ terms (so that $Z$ is
a $n \times n$ matrix which is block diagonal with coefficients in $F_p)$. Recalling the notation introduced in Remark 6.8, the map

$$f_{C_a}: C_a \to \text{Aut}(G \to \text{Spec} K)$$

$$\text{ev}(\zeta^j) \mapsto \text{int}(Z^j) \tilde{\phi}(\text{ev}(\zeta^j), z^b)$$

is a homomorphism whose composition with the map $\text{Aut}(G \to \text{Spec} K) \to \text{Aut}_G(K)$ is the identity on $C_a$.

Proof. Note that $\text{ev}(\zeta^j) = \zeta^b$ and $z^b \in \text{Aut}(G)$, so that we can indeed use Definition 6.7. With these definitions, for all $j, j' \in \mathbb{N}$, we have

$$\text{int}(Z^j) \tilde{\phi}(\text{ev}(\zeta^j), z^b) \circ \text{int}(Z^{j'}) \tilde{\phi}(\text{ev}(\zeta^j), z^b) = \text{int}(Z^j Z^{j'}) \tilde{\phi}(\text{ev}(\zeta^j) \circ \text{ev}(\zeta^j), z^{b-j') = \text{int}(Z^{j+j'}) \tilde{\phi}(\text{ev}(\zeta^{j+j'}), z^{b+j')}.$$}

Hence the fact that $f_{C_a}$ is well-defined follows from $z^a = 1$ (which holds because $z^a$ is the unique $(db')$-th root in $C_a$ of $\zeta^{abr} = 1$).

Remark 6.26. In the proof of Proposition 6.25, we needed to show that $f_{C_a}(\text{ev}(\zeta^j))\circ = a$ is a trivial element of $\text{Aut}((G \to \text{Spec} K)$. We proved it by showing that this algebraic automorphism of SL$_n(D)$ induces a trivial automorphism of SL$_n(D)$, hence it is itself trivial by the density of rational points for $G$. This will be used repeatedly to show that $K$-automorphisms of $G$ are trivial. In using that argument, it is also important to notice that a semilinear automorphism of the form $\text{int}(Y) \tilde{\phi}(\alpha, x)$ is algebraic if and only if $\alpha$ acts trivially on $K$.

Remark 6.27. Note that in Proposition 6.25, the factor $\text{int}(Z^j)$ is unnecessary if one is just interested in a section defined on $C_a$ alone. Hence, a section of $\text{Aut}(G \to \text{Spec} K) \to \text{Aut}_G(K)$ only defined on $C_a$ always exists (i.e. one does not need to assume that $bb'$ divides $n$).

Proposition 6.28. Keep the notations of Definition 6.20 and assume that $b$ divides $n$. Let $C_b$ be the group generated by $\zeta^b$ in $F_p$. There exists an element $y \in F_{p, a}$ such that $\frac{F^{\tilde{a}}(y)}{y} = \zeta^a$. Choosing a $F_{p, a}$-basis of $F_{p, a}$, we obtain an embedding $\varphi: F_{p, a} \to M_b(F_{p, a})$. Let $y = \varphi(y^{-1})$ and let $Y = \text{Diag}(g, \ldots, g)$ where we have $\frac{y}{y}$ terms (so that $Y$ is a $n \times n$ matrix which is block diagonal with coefficients in $F_{p, a}$). Recalling the notation introduced in Remark 6.8, the map

$$f_{C_b}: C_b \to \text{Aut}(G \to \text{Spec} K)$$

$$\text{ev}(\zeta)^{a} \mapsto \text{int}(Y^j) \tilde{\phi}(\text{ev}(\zeta^j), (\frac{F^{i}(y)}{y})^j)$$

is a homomorphism whose composition with the map $\text{Aut}(G \to \text{Spec} K) \to \text{Aut}_G(K)$ is the identity on $C_b$.

Proof. For the existence of $y \in F_{p, a}$ such that $\frac{F^{\tilde{a}}(y)}{y} = \zeta^a$, note that $N_{F_{p, a}/F_{p, a}}(\zeta^a) = \zeta^{abr} = 1$. Also note that the extension $F_{p, a}/F_{p, a}$ is Galois cyclic, and that $F_{p, a}$ generates its Galois group. Hence, by Hilbert’s Theorem 90, there indeed exists $y \in F_{p, a}$ such that $\frac{F^{\tilde{a}}(y)}{y} = \zeta^a$. For the rest of the proof, we choose such an $y$.

From $\frac{F^{\tilde{a}}(y)}{y} = \zeta^a$, it readily follows that $F^{\tilde{a}}(y)y^{-1}$ and $y^b$ belong to $F_{p, a}$, since they are both invariant under $F_{p, a}$. Note that $\frac{F^{\tilde{a}}(y)y^{-1}}{y} = (F^{i}(y))^{i} = N_{E/K}(F^{i}(y))$, so that we can indeed use Definition 6.7.

It remains to check that $f_{C_b}$ is well-defined and is a homomorphism. Note that for all $j, j' \in \mathbb{N}$, we have

$$\text{int}(Y^j) \tilde{\phi}(\text{ev}(\zeta^j), (\frac{F^{i}(y)}{y})^j) \circ \text{int}(Y^{j'}) \tilde{\phi}(\text{ev}(\zeta^{j'}), (\frac{F^{i}(y)}{y})^{j'})$$

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By Theorem 

Proof. Aut is generated by the automorphism of $K$. Keep the notations of Definition Proposition 6.32.

Those two elements of Aut($Y$) belong to Aut($F$) by definition of $Y$. Hence, $uY^bu = F^a(Y)^b$ (see Remark 6.21). Hence int$(Y^b)\tilde{\phi}(1, (F^a(y)^{-1})^b)(u, Id_n) = Y^b(u(F^ay)^{-1}b)Id_nY^b = uF^ayY^{-1}b = u, Id_n$. Since int$(Y^b)\tilde{\phi}(1, (F^a(y)^{-1})^b)$ also acts trivially on $SL_n(E) \leq SL_n(D)$, this concludes the proof. 

Remark 6.29. When trying to find a section of Aut($G \rightarrow Spec K$) → Aut$_G(K)$ only defined on $C$, this is the only formula we could come up with. Otherwise stated, the complicatedness of the formula defining $f_{C_b}$ does not come from the need to adjust it to other partial sections of Aut($G \rightarrow Spec K$) → Aut$_G(K)$.

Though not needed, we check that the automorphism int$(Y^j)\tilde{\phi}(ev(\zeta^nT), (F^i(y)^{-1})^j)$ appearing in Proposition 6.28 does not depend on the choice of $y$.

Lemma 6.30. Keep the notations of Proposition 6.28. Let $y' \in F_{p,a}$ such that $F^{id}(y') = \zeta^{ar}$. Let $g' = \phi(y'^{-1})$ and let $Y' = \text{Diag}(g', \ldots, g')$ where we have $\frac{n}{r}$ terms. Then $f_{C_b}(ev(\zeta^nT)) = \text{int}(Y^j)\tilde{\phi}(ev(\zeta^nT), (F^{id}(y'))^j)$.

Proof. Those two elements of Aut($G \rightarrow Spec K$) differ by int$(Y^jY'^{-j})\tilde{\phi}(1, (F^{id}(y'))^j)$. Let $x = y'y'^{-1}$. Note that $x$ belongs to $F_{p,a}$ because $x$ is invariant under $F^{id}$, and hence $Y'Y'^{-j} = \text{Diag}(gg'^{-1}, \ldots, gg'^{-1})$ is actually the diagonal matrix $x^j.Id_n$.

Hence int$(Y^jY'^{-j})\tilde{\phi}(1, (F^{id}(y'))^j) = \text{int}(x, Id_n)\tilde{\phi}(1, (\frac{F^{id}(x)}{x})^j)$. But this automorphism is trivial, since int$(x, Id_n)\tilde{\phi}(1, (\frac{F^{id}(x)}{x})^j)(u, Id_n) = x^j.u(\frac{F^{id}(x)}{x})^j.x^{-j}Id_n = u, Id_n$. 

As before, we can also prove a converse to Proposition 6.28.

Proposition 6.31. Keep the notations of Proposition 6.28. If $b$ does not divide $n$, there does not exist a homomorphism $C_b \rightarrow \text{Aut}(G \rightarrow Spec K)$ whose composition with $\text{Aut}(G \rightarrow Spec K) \rightarrow \text{Aut}_G(K)$ is the identity on $C_b$.

Proof. By Theorem 6.13, it suffices to prove that there exists $K' \leq K$ such that $K/K'$ is finite Galois with $\text{Gal}(K/K') \leq C_b$ and such that $\text{gcd}(nd, [K : K'])$ does not divide $n$. Recall (see Definition 6.20) that $a$ is prime to $b$ with $ab = p^d - 1$ so that $\zeta^n$ is a $b$-th primitive root of unity of $K$. Hence, $K' = F_{p,a}(T^b)$ is such that $K/K'$ is Galois of degree $b$, and $\text{Gal}(K/K')$ is generated by the automorphism of $K$ sending $T$ to $\zeta^nT$, so that $\text{Gal}(K/K') = C_b$. Finally, $\text{gcd}(nd, [K : K']) = \text{gcd}(nd, b)$ does not divide $n$ because by definition $b = \text{gcd}(d^b)$. 

Finally, we construct a section to $\text{Aut}(G \rightarrow Spec K) \rightarrow \text{Aut}(K)$ for $\text{Gal}(F_{p,a}(T)/F_{p,a}(T))$.

Proposition 6.32. Keep the notations of Definition 6.20. Let $C_{a'}$ be the group generated by $F^{id}$ in $\text{Gal}(K/F_{p,a}(T))$. Let $c \in \mathbb{N}$ be such that $ca' + 1 \in d\mathbb{Z}$ (which exists because $\text{gcd}(a', d) = 1$). Recalling the notation introduced in Remark 6.8, the map $f_{C_{a'}}: C_{a'} \rightarrow \text{Aut}(G \rightarrow Spec K)$ $F^{id} \rightarrow \tilde{\phi}(F^{id(ca'+b)}, 1)$ is a homomorphism. Furthermore, its composition with the map $\text{Aut}(G \rightarrow Spec K) \rightarrow \text{Aut}(K)$ is the identity on $C_{a'}$. 

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The only assertion that requires a justification is that the map is well-defined, i.e. we have

\[ f_{C_{\nu^i}} \colon C_{\nu^i} \to \text{Aut}(G \to \text{Spec} K) \to \text{Aut}_{C}(K) \text{ on } \text{Gal}(K/\mathbb{F}_{p^i}(T)). \]

**Definition 6.33.** With the notations of Definition 6.20, let \( 0_b \) be the zero \( b \times b \) matrix, and let \( w \) be the following \( bb' \times bb' \) matrix:

\[
w = \begin{pmatrix}
0_b & 0_b & 0_b & \ldots & 0_b & u \text{Id}_b \\
\text{Id}_b & 0_b & 0_b & \ldots & 0_b & 0_b \\
0_b & \text{Id}_b & 0_b & \ldots & 0_b & 0_b \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0_b & 0_b & 0_b & \ldots & \text{Id}_b & 0_b
\end{pmatrix}
\]

**Proposition 6.34.** Keep the notations of Definition 6.20. Assume that \( bb' \) divides \( n \). Let \( C_{\nu'} \) be the group generated by \( F^{\nu'} \) in \( \text{Gal}(K/\mathbb{F}_{p^i}(T)) \). With the notations of Definition 6.33, let \( W = \text{Ding}(w, \ldots, w) \) where we have \( \frac{bb'}{n} \) terms (so that \( W \) is a \( n \times n \) matrix which is block diagonal with coefficients in the set \( \{0, 1, u\} \)). Recalling the notation introduced in Remark 6.8, the map

\[
f_{C_{\nu'}} : C_{\nu'} \to \text{Aut}(G \to \text{Spec} K)
\]

\[
F^{\nu'}j \mapsto \text{int}(Wj)\phi(F^{\nu'}j, 1)
\]

is a homomorphism. Furthermore, its composition with the map \( \text{Aut}(G \to \text{Spec} K) \to \text{Aut}_{C}(K) \) is the identity on \( C_{\nu'} \).

**Proof.** The only assertion that requires a justification is that the map is well-defined, i.e. we have to check that \( (\text{int}(W)\phi(F^{\nu'}, 1))^{bb'} \) is the identity on \( \text{SL}_n(D) \). By definition, \( W^{bb'} \) is the scalar matrix \( u. \text{Id}_n \). Hence \( (\text{int}(W)\phi(F^{\nu'}, 1))^{bb'} = \text{int}(W^{bb'})\phi(F^{\nu'}, 1) = \text{int}(u. \text{Id}_n)\phi(F^{\nu'}, 1) \), which indeed acts as the identity on \( \text{SL}_n(D) \) because for all \( x \in E \), \( uwxu^{-1} = F^{-i}(x) \) (see Remark 6.21).  

**Remark 6.35.** Note that if one is just interested in a section defined on \( C_{\nu'} \) alone, one can take \( b = 1 \) in Proposition 6.34. Hence, a section of \( \text{Aut}(G \to \text{Spec} K) \to \text{Aut}_{C}(K) \) only defined on \( C_{\nu'} \) exists if and only if \( b' \) divides \( n \) (i.e. the stronger assumption that \( bb' \) divides \( n \) is there to ensure that we can glue \( f_{C_{\nu'}} \) with \( f_{C_{\nu}} \)).

As before, we have a converse to Proposition 6.34.

**Proposition 6.36.** Keep the notations of Proposition 6.34. If \( b' \) does not divide \( n \), there does not exist a homomorphism \( C_{\nu'} \to \text{Aut}(G \to \text{Spec} K) \) whose composition with \( \text{Aut}(G \to \text{Spec} K) \to \text{Aut}_{C}(K) \) is the identity on \( C_{\nu'} \).

**Proof.** By Theorem 6.13, it suffices to prove that there exists \( K' \leq K \) such that \( K/K' \) is finite Galois, \( \text{Gal}(K/K') \leq C_{\nu'} \) and \( \gcd(nd, [K : K']) \) does not divide \( n \). But \( K' = \mathbb{F}_{p^{i'}}(T) \) is such a subfield.  

We can finally glue all the previous constructions to obtain a global splitting of the initial short exact sequence.

**Theorem 6.37.** Keep the notations of Definition 6.20. Assume that for all subfields \( K' \leq K \) such that \( K/K' \) is finite Galois, \( \gcd(nd, [K : K']) \) divides \( n \). Then the short exact sequence

\[
1 \to \text{Aut} G \to \text{Aut}(G \to \text{Spec} K) \to \text{Aut}_{C}(K) \to 1
\]

splits.
Proof. In view of Proposition 6.38, the hypotheses imply that \( \gcd(d, p) = 1 \) and \( \gcd(nd, i(p^i - 1)) \) divides \( n \). Hence \( bb' \) divides \( n \) and we can apply Propositions 6.22, 6.25, 6.28, 6.32 and 6.34. For the rest of the proof, we strictly adhere to the notations that are introduced in the statements of those propositions.

Recall that \( \text{Aut}_G(K) = \text{Aut}(K) \) (Corollary 6.10). Also recall that we decomposed \( \text{Aut}(K) \) as \( (J(K) \rtimes (C_a \times C_b)) \times (C_{a'} \times C_{b'}) \), where

(i) \( C_a \leq F_p^\times \) is generated by \( \zeta^b \).

(ii) \( C_b \leq F_p^\times \) is generated by \( \zeta^a \).

(iii) \( C_{a'} \leq \text{Gal}(K/F_p((T))) \) is generated by \( F^{b'} \) restricted to \( K \).

(iv) \( C_{b'} \leq \text{Gal}(K/F_p((T))) \) is generated by \( F^{a'} \) restricted to \( K \).

We define a map

\[
\begin{align*}
\varphi : (J(K) \rtimes (C_a \times C_b)) \times (C_{a'} \times C_{b'}) & \to \text{Aut}(G \to \text{Spec} K) \\
(g_1, g_2, g_3, g_4, g_5) & \mapsto \varphi_{J(K)}(g_1) \varphi_{C_a}(g_2) \varphi_{C_b}(g_3) \varphi_{C_{a'}}(g_4) \varphi_{C_{b'}}(g_5)
\end{align*}
\]

We claim that \( \varphi \) is a homomorphism. To prove this claim, it suffices to compute various commutators in \( \text{Aut}(G \to \text{Spec} K) \). To carry the computation, we pick \( j, j' \in \mathbb{N} \).

1. The images of \( f_{C_a} \) and \( f_{C_b} \) commute. Indeed, \( \text{int}(Z^j) \tilde{\phi}(ev(\zeta^b T), z^{b'}) \) readily commutes with \( \text{int}(Y^{j'}) \tilde{\phi}(ev(\zeta^{b'} T), (\frac{E(q)}{g})^{j'}) \) (note that \( Y \) and \( Z \) are both \( b \times b \) block diagonal matrices, and that the blocks defining \( Z \) are scalars).

2. Let \( \alpha \in J(K) \). We compute \( f_{C_a}(ev(\zeta^b T)) f_{J(K)}(\alpha) f_{C_a}(ev(\zeta^{-b} T)) \):

\[
\begin{align*}
\text{int}(Z^j) \tilde{\phi}(ev(\zeta^b T), z^{b'}) & \circ \text{int}(X_{\alpha}) \tilde{\phi}(\alpha, z^{b'}) \circ \text{int}(Z^{-j}) \tilde{\phi}(ev(\zeta^{-b} T), z^{-b'}) \\
= \text{int}(Z^j) ev(\zeta^{b'} T)(X_{\alpha}) \alpha(Z^{-j}) & \tilde{\phi}(ev(\zeta^b T), \alpha \circ ev(\zeta^{-b} T), z^{b'}) ev(\zeta^{b'} T)(x_{\alpha}) \alpha(z^{-b'}) \\
= \text{int}(ev(\zeta^{b'} T)(X_{\alpha})) & \tilde{\phi}(ev(\zeta^b T), \alpha \circ ev(\zeta^{-b} T), (ev(\zeta^{b'} T)(x_{\alpha}))^{b'})
\end{align*}
\]

where the last equality follows from the fact that \( Z \) and \( X_{\alpha} \) commute because they are block diagonal matrices, together with the equality \( \alpha(Z^{-j}) = Z^{-j} \) which holds because \( Z \) has coefficients in \( F_p \).

But \( ev(\zeta^{b'} T)(x_{\alpha}) = x_{ev(\zeta^{b'} T) \circ ev(\zeta^{-b} T)} \), because \( ev(\zeta^{b'} T)(x_{\alpha}) \) belongs to \( 1 + TF_p[[T]] \), and

\[
ev(\zeta^b T)(x_{\alpha})^{b'} = ev(\zeta^b T)(\frac{\alpha(T^r)}{T^r}) = \zeta^{-b} \frac{ev(\zeta^b T) \alpha(T^r)}{T^r} = \frac{(ev(\zeta^b T) \circ \alpha \circ ev(\zeta^{-b} T))(T^r)}{T^r}
\]

Hence \( f_{C_a}(ev(\zeta^b T)) f_{J(K)}(\alpha) f_{C_a}(ev(\zeta^{-b} T)) = f_{J(K)}(ev(\zeta^b T) \circ \alpha \circ ev(\zeta^{-b} T)) \).

3. The equality \( f_{C_a}(ev(\zeta^a T)) f_{J(K)}(\alpha) f_{C_a}(ev(\zeta^{-a} T)) = f_{J(K)}(ev(\zeta^a T) \circ \alpha \circ ev(\zeta^{-a} T)) \) is proved by doing a similar computation than in the previous item.

4. The images of \( f_{C_{a'}} \) and \( f_{C_{b'}} \) commute. Indeed, \( \tilde{\phi}(F^{b'+c}, 1) \) readily commutes with \( \text{int}(W^{j'}) \tilde{\phi}(F^{a', j'}, 1) \) (recall that \( W \) has coefficients in \( \{0, 1, u\} \)).
5. We check that $f_{C_\varphi}(F^{b'})f_{C_{\varphi}}(\text{ev}(\varphi^j T))f_{C_{\varphi}}(F^{-b'} j) = f_{C_\varphi}(F^{b'} j \circ \text{ev}(\varphi^j T) \circ F^{-b'} j)$. We have
\[
\tilde{\phi}(F^{j(c+b')}, 1) \circ \text{int}(Y^j) \tilde{\phi}(\text{ev}(\varphi^j T), (\frac{F^{j}(y)}{y})^j) \circ \tilde{\phi}(F^{-j(c+b')}, 1)
\]
\[
= \text{int}(F^{j(c+b')}(Y^j)) \tilde{\phi}(F^{j(c+b')}, \text{ev}(\varphi^j T) \circ F^{-j(c+b')}, F_j(c+b'))((\frac{F^{j}(y)}{y})^j)
\]
Noting that $F^{j(c+b')} \circ \text{ev}(\varphi^j T) \circ F^{-j(c+b')} = \text{ev}(F^{j(c+b')}(\varphi^j T))$, the desired equality follows from the fact that the Frobenius automorphism on $\mathbf{F}_{p^{ad}}$ is just elevating to the power $p$.

6. One readily check that $f_{C_\varphi}(F^{b'})f_{C_{\varphi}}(\text{ev}(\varphi^j T))f_{C_{\varphi}}(F^{-b'} j) = f_{C_\varphi}(F^{b'} j \circ \text{ev}(\varphi^j T) \circ F^{-b'} j)$.

7. We have $f_{C_\varphi}(F^{b'})f_{J(K),(\alpha)}f_{C_{\varphi}}(F^{-b'} j) = f_{J(K)}(F^{b'} j \circ \alpha \circ F^{-b'} j)$. Indeed, $F^{j(c+b')}(x_\alpha)$ belongs to $1 + TF_{p^{ad}}[T]$, and $F^{j(c+b')}(x_\alpha)^{a'd} = F^{j(c+b')}(\alpha(T^a)) = F^{j(c+b')}(F^{-j(c+b')})(T^a)$ because $F$ acts trivially on $T$. Hence
\[
\tilde{\phi}(F^{j(c+b')}, 1) \text{int}(X_\alpha) \tilde{\phi}(\alpha, x_{\alpha'}) \tilde{\phi}(F^{-j(c+b')}, 1)
\]
\[
= \text{int}(F^{j(c+b')}(x_\alpha)) \tilde{\phi}(F^{j(c+b')}(\alpha F^{-j(c+b')}), F^{j(c+b')}(x_{\alpha'}))
\]
\[
= \text{int}(X_{F^{j(c+b')}\alpha F^{-j(c+b')}}) \tilde{\phi}(F^{j(c+b')}(\alpha F^{-j(c+b')}), x_{F^{j(c+b')}\alpha F^{-j(c+b')}})
\]
as wanted.

8. We check that $f_{C_{\varphi}}(F^{a'} j)f_{C_{\varphi}}(\text{ev}(\varphi^j T))f_{C_{\varphi}}(F^{-a'} j) = f_{C_\varphi}(F^{a'} j \circ \text{ev}(\varphi^j T) \circ F^{-a'} j)$. It is obviously enough to check this when $j = j' = 1$. Then
\[
\text{int}(W) \tilde{\phi}(F^{a'} j, 1) \circ \text{int}(Y) \tilde{\phi}(\text{ev}(\varphi^j T), (\frac{F^{j}(y)}{y})) \circ \text{int}(W^{-1}) \tilde{\phi}(F^{-a'} j, 1) =
\]
\[
\text{int}(WF^{a'}(Y) \tilde{\phi}(\text{ev}(\varphi^j T), (\frac{F^{j}(y)}{y})) \tilde{\phi}(F^{a'} j \circ \text{ev}(\varphi^j T) \circ F^{-a'} j, \frac{F^{j}(y)}{y}))
\]
Again, since $F^{a'} \circ \text{ev}(\varphi^j T) \circ F^{-a'} = \text{ev}(F^{a'}(\varphi^j T))$, it suffices to show that the matrix appearing in the argument of int() is equal to $F^{a'}(Y)$. Since $W$ is made up of $bb' \times bb'$ block matrices, whilst $Y$ is made up of $b \times b$ block matrices, we can work with one block at a time. Otherwise stated, we may assume that $bb' = n$. Now $\tilde{\phi}(\text{ev}(\varphi^j T), (\frac{F^{j}(y)}{y})) \tilde{\phi}(F^{a'}(Y), (\frac{F^{j}(y)}{y}))$ is the matrix
\[
\begin{pmatrix}
0_b & \text{Id}_b & 0_b & 0_b & \ldots & 0_b \\
0_b & 0_b & \text{Id}_b & 0_b & \ldots & 0_b \\
0_b & 0_b & 0_b & \text{Id}_b & \ldots & 0_b \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0_b & 0_b & 0_b & \ldots & \text{Id}_b & 0_b \\
(u^{F^{j}(y)} y)^{-1} & \text{Id}_b & 0_b & 0_b & \ldots & 0_b
\end{pmatrix}
\]
so that the matrix appearing in the argument of int() is the $n \times n$ block diagonal matrix $\text{Diag}(u^{F^{a'}(g)}(u^{F^{i}(y)} y)^{-1}, F^{a'}(g), \ldots, F^{a'}(g))$. But $u^{F^{a'}(g)}(u^{F^{i}(y)} y)^{-1} = F^{a'}(u^{F^{i}(y)} y)^{-1}$. Recalling the embedding $\varphi: \mathbf{F}_{p^{ad}} \rightarrow M_b(\mathbf{F}_{p^{ad}})$ of Proposition 6.28, we have
\[
u \varphi(u^{F^{i}(y)} y)^{-1} = u \varphi(y^{-1}) \varphi(y) F^{i}(\varphi(y)^{-1}) u^{-1}
\]
\[
= F^{-i}(F^{i}(g)) = g
\]
We conclude that the argument of int() is indeed $\text{Diag}(F^{a'}(g), F^{a'}(g), \ldots, F^{a'}(g)) = F^{a'}(Y)$, as wanted.
9. Let us check that $f_{C_{ab}}(F'_{ij})f_{C_{ab}}(\text{ev}(\zeta b^j T))f_{C_{ab}}(F^{-a'-j}) = f_{C_{ab}}(F'_{ij} \circ \text{ev}(\zeta b^j T) \circ F^{-a'-j})$. It is obviously enough to check this when $j = j' = 1$. Then

$$
\begin{align*}
\text{int}(W) \hat{\phi}(F', 1) \text{int}(Z) \circ \hat{\phi}(\text{ev}(\zeta b^j T), z b^j) \text{int}(W^{-1}) \hat{\phi}(F^{-a'}, 1) &= \\
\text{int}(WF'^a(Z) \hat{\phi}(\text{ev}(\zeta b^j T), z b^j)(W^{-1})) \hat{\phi}(F' \circ \text{ev}(\zeta b^j T) \circ F^{-a'}, F'(z b^j))
\end{align*}
$$

Again, since $F' \circ \text{ev}(\zeta b^j T) \circ F^{-a'} = ev(F' \circ (\zeta b^j) T)$, it suffices to show that the matrix appearing in the argument of int() is equal to $F'_{ij}(Z)$. As in the previous item, we can assume that $bb' = n$, and doing the same kind of computation as in the previous item, we find that the matrix in the argument of int() is $F'_{ij}(\text{Diag}(uz^{-1}u^{-1} \text{Idp}, \text{Idp}, \text{Idp}, \ldots, z b^{\alpha - 2} \text{Idp}))$. Since $z \in F_p$, $uz^{-1}u^{-1} = z^{-1}$, and we conclude that multiplication by the scalar matrix $z \text{Id}_n$ (which belongs to the center of $GL_n(D)$ because $z \in F_p$), the argument appearing in int() is equal to $F'_{ij}(Z)$, as wanted.

10. Checking that $f_{C_{ab}}(F'_{ij})f_{J(K)}(\alpha)f_{C_{ab}}(F^{-a'-j}) = f_{J(K)}(F'_{ij} \circ \alpha \circ F^{-a'-j})$ is a similar computation than in the previous item.

We conclude that $f$ is indeed a homomorphism. The fact that $f$ is a splitting of the short exact sequence in the statement of the proposition follows from the fact that the restriction of $f$ to each component is locally a section of $\text{Aut}(G \to \text{Spec} K) \to \text{Aut}_G(K)$.

The first step in the proof of Theorem 6.37 is to translate the existence of Galois subfields of some degree into some divisibility relations between $p, d, i$ and $p' - 1$. We now prove the ad hoc proposition. We warn the reader that the notations of Definition 6.20 (and of the subsequent propositions) are not in use any more.

**Proposition 6.38.** Let $K = F_p^q((T))$, let $q$ be a prime number and let $a \in \mathbb{N}$. There exists a subfield $K'$ such that $K/K'$ is finite Galois and $q^a$ divides $[K : K']$ if and only if $q = p$ or $q^a$ divides $i(p' - 1)$.

**Proof.** First assume that such a $K'$ exists. Since $K/K'$ is Galois and $q^a$ divides $[K : K']$, there exists $K$ such that $K/K$ is Galois and $[K : K] = q^a$. Up to replacing $K'$ by $K$, we can thus assume that $[K : K'] = q^a$. Let also $K'_u$ be the maximal unramified extension of $K'$ inside $K$.

Note that $K'$ and $K'_u$ are local fields, so that in particular $K' \cong F_p^q((T))$ and $K'_u \cong F_p^q((T))$. Since $[K'_u : K']$ divides $q^a$, there exists $a_1$ such that $q^{a_1} = \frac{p}{2}$. Letting $a_2 = a - a_1$, we have that $K/K'_u$ is a totally ramified extension of degree $q^{a_2}$.

If $p = q$, the proposition is proved, hence there just remains to investigate the case $p \neq q$. In this case, $K$ is a tamely totally ramified extension of $K'_u$. Thus, $K$ is isomorphic to $K'_u[X]/(X^{q^{a_2}} - \pi)$ for some uniformiser $\pi \in F_p^q((T))$. But $K$ is a Galois extension, and hence this implies that $F_p^q((T))$ has a primitive $q^{a_2}$-th root of unity, so that $q^{a_2}$ divides $p' - 1$, as wanted.

To prove the converse, we use a classical fact from local class field theory: there exists an extension $K_\pi$ of $K$ which is Galois and totally ramified, and such that $\text{Gal}(K_\pi/K)$ is isomorphic to the group of invertible elements $F_p^q[T]^\times$ of $F_p^q[T]$ (see for example [Iwa86, Section 5.3]). Note that the degree of $F_p^q[T]^{\times} + T^{p+1}F_p^q[T]$ in $F_p^q[T]^\times$ is equal to $p^2$. Let $L_1$ be the Galois extension of $K$ corresponding to $F_p^q[T]^{\times} + T^{p+1}F_p^q[T]$. Let also $L_2$ be the splitting field of $X^{q^{a_2}} - 1$ over $F_p^q(T)$. For $j = 1$ or 2, $L_j$ is totally ramified of finite degree over $K$, so that there exists an isomorphism $\phi_j : K \to L_j$. Hence $K_1 = \phi_1^{-1}(K)$ (respectively $K_2 = \phi_2^{-1}(F_p^q(T))$) is such that $K/K_1$ (respectively $K/K_2$) is Galois, and $[K : K_1] = p^2$ (respectively $[K : K_2] = i(p' - 1)$), which concludes the proof.

## A Base change of the algebraic group $SL_n(D)$

We begin by recalling some classical facts about finite dimensional central simple algebras over local fields.
Theorem A.1. Let $K$ be a local field. Every central simple algebra over $K$ is isomorphic to an algebra of the form $A(d,r)$ as in Definition 6.4. Furthermore, the map $\text{inv}: Br(K) \rightarrow \mathbb{Q}/\mathbb{Z}$: $[A(d,r)] \mapsto \left[\frac{d}{r}\right]$ is an isomorphism of groups.

Proof. See for example [Mor97, Theorem 8] for the first assertion, while the second is precisely the content of [Pie82, Chapter 17, §10, Theorem].

Corollary A.2. Let $K$ be a local field and let $d, r \in \mathbb{N}$ with $d \geq 1$. Let $a = \gcd(d, r)$. Then $A(d, r)$ is a division algebra if and only if $a = 1$, and $A(d, r) \cong M_n(A(d, r/a))$.

Proof. The central simple algebra $A(d, r)$ is a division algebra if and only if all central simple algebras over $K$ in the same Brauer class have a higher degree. In view of Theorem A.1, it readily implies that $A(d, r)$ is a division algebra if and only if $a = 1$. Furthermore, by Wedderburn’s theorem, $A(d, r)$ is isomorphic to $M_n(D)$ for some division algebra $D$ and some $1 \leq n \in \mathbb{N}$, and by definition of the Brauer group, $[D] = [A(d, r)]$. Hence, using the first part of the Theorem, $D \cong A(d, r/a)$. Now, comparing degrees readily imply that $n = a$, and the result is proved.

We now study the base change of the algebraic group $SL_n(A)$.

Lemma A.3. Let $A$ be a central simple algebra over a field $k$, and let $SL_1(A)$ be the corresponding algebraic $k$-group (see Definition 6.1). For $k'$ a field extension of $k$, $SL_1(A)_{k'} = SL_1(A \otimes_k k')$.

Proof. Let $k'$ be an algebraic closure of $k'$. Since $k'$ splits $A$, the reduced norm is the map $f: A \rightarrow A \otimes_k k' \cong M_n(k') \overset{\text{det}}{\rightarrow} k'$. Let $\varphi$ denote the isomorphism $A \otimes_k k' \cong M_n(k')$. If we take a $k$-basis of $A$ to get coordinates on $A \otimes_k k'$, the map $\varphi$ is actually a polynomial map on $A \otimes_k k'$ with coefficients in $k$, by [Bon73, Chapitre VIII, §12, Proposition 11]. Hence, $f_{k'} = \text{det} \circ \varphi$. This implies that $f_{k'}: A \otimes_k k' \rightarrow k'$ is just the composition $A \otimes_k k' \rightarrow A \otimes_k k' \cong M_n(k') \overset{\text{det}}{\rightarrow} k'$, i.e. $f_{k'}$ is the reduced norm map of the algebra $A \otimes_k k'$, as wanted.

Before giving the formula for the base change of $SL_n(A)$, we recall the effect of extending scalars for central simple algebras over local fields.

Lemma A.4. Let $K$ be a local field and let $A(d, r)$ be the central simple algebra over $K$ defined in Definition 6.4. Let $L$ be a finite extension of $K$. Then $A(d, r) \otimes_K L \cong A(d, r[L : K])$.

Proof. By Wedderburn’s theorem, a central simple algebra over a field is uniquely determined by its degree and its Brauer class. By [Pie82, Chapter 17, Section 17.10, Proposition], we have $\text{inv}([A(d, r) \otimes_K L]) = [L : K]. \text{inv}([A(d, r)])$. Hence $A(d, r[L : K])$ and $A(d, r) \otimes_K L$ are in the same Brauer class. Since they have the same degree as well, this concludes the proof.

Proposition A.5. Let $A(d', r')$ be a division algebra over a local field $K'$ as in Definition 6.4. Let $L/K'$ be a finite field extension and let $a = \gcd(d', [K, K'])$. Then the base change of $SL_{a'}(A(d', r'))$ to $K$ is isomorphic to $SL_{a'}(A(d, r))$.

Proof. The base change of $SL_{a'}(A(d', r')) = SL_1(M_{a'}(A(d', r'))) \otimes_{K'} K \cong SL_{a'}(A(d', r') \otimes_K K')$ by Lemma A.3. But by Corollary A.2 and Lemma A.4, $A(d', r') \otimes_K K \cong M_a(A(d, r/a))$. To conclude, note that for any central simple algebra $A$, $SL_{a'}(M_a(A)) \cong SL_{a'}(A)$.
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