A PRIORI ESTIMATES AND BLOW-UP BEHAVIOR FOR SOLUTIONS OF $-Q_N u = V e^u$ IN BOUNDED DOMAIN IN $\mathbb{R}^N$

RULONG XIE AND HUAJUN GONG

Abstract. Let $Q_N$ be $N$-anisotropic Laplacian operator, which contains the ordinary Laplacian operator, $N$-Laplacian operator and anisotropic Laplacian operator. In this paper, we firstly obtain the properties for $Q_N$, which contain the weak maximal principle, the comparison principle and the mean value property. Then a priori estimates and blow-up analysis for solutions of $-Q_N u = V e^u$ in bounded domain in $\mathbb{R}^N$, $N \geq 2$ are established. Finally, the behavior of sole blow-up point is further considered.

1. Introduction and main results

As we know, Liouville equation was firstly studied in 1853 in [20]. This equation has many applications in geometry and physical problem, for instance, in the problem of prescribing Gaussian curvature [6,8], the mean field equation [7,10–12,19] and the Chern-Simons model [23,24,26]. About the blow-up analysis of solution of this equation, we must mention the celebrated paper of Brezis and Merle [5]. They firstly researched the blow-up behavior and uniformly estimates for solutions of Liouville equation

$$-\Delta u = V(x)e^{u(x)}$$

in bounded domain $\Omega \subset \mathbb{R}^2$ with $V(x) \in L^p(\Omega)$ and $e^u \in L^{p'}(\Omega)$ for some $1 < p \leq \infty$ and $p'$ is the Hölder conjugate index of $p$. From then, blow-up analysis about the solutions of various equation and system of equations have been extensively established (see [2,10,11,15,18,21] and their references). For example, Ren and Wei in [21] established similar results for $N$-Laplacian operators in $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Recently, Wang and Xia [27] generalized the blow-up analysis for Liouville type equation with anisotropic Laplacian (or Finsler-Laplacian) in $\Omega \subset \mathbb{R}^2$. At the same time, they also defined $N$-anisotropic Laplacian (or $N$-Finsler Laplacian) as follows:

$$-Q_N u := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(F^{N-1}(\nabla u)F_{\xi_i}(\nabla u)),$$

where $F \in C^2(\mathbb{R}^N \setminus \{0\})$ is a convex and homogeneous function (see Section 2) and $F_{\xi_i} = \frac{\partial F}{\partial \xi_i}$. The anisotropic Laplacian is closely related to a smooth, convex hypersurface in $\mathbb{R}^N$, which is called the Wulff shape. This kind of operator was

1991 Mathematics Subject Classification. 35J60, 53C60.
Key words and phrases. Blow-up behavior, $N$-anisotropic Laplacian, $N$-anisotropic Liouville equation.

The authors would like to thank Prof. Jiayu Li and Dr. Chao Xia for helpful discussion and suggestions. This work is supported by the Excellent Young Talent Foundation of Anhui Province (2013QJRL080ZD).
initiated study in Wulff’s paper (see [30]). Recently, this operator has been widely studied by many mathematicians, see [1, 3, 13, 14, 28, 29] and references therein.

It is natural to ask whether it has the similar Brezis-Merle result for Liouville type equation with $N$-anisotropic Laplacian in bounded domain in higher dimensions. This paper give the informative answer to this question. More precisely, for a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, we consider the following quasilinear equation

\begin{equation}
- Q_N u = V(x) e^{u(x)}.
\end{equation}

Equation (1.3) is also called $N$-anisotropic (or $N$-Finsler) Liouville equation.

It is well known that in the isotropic case, i.e. $F(\xi) = |\xi|$, when $N = 2$, $Q_N$ is the ordinary Laplacian operator; when $N > 2$, $Q_N$ is the $N$-Laplacian operator. In the anisotropic case, when $N = 2$, $Q_N$ is anisotropic Laplacian operator. Therefore, the results of this paper extend that of ordinary Laplacian operator, $N$-Laplacian operator and anisotropic Laplacian operator.

In this paper, we firstly obtain some properties for $Q_N$, i.e. the weak maximal principle, the comparison principle and the mean value property. Then we get an a priori estimate for solutions of $N$-anisotropic Liouville equation, i.e. Theorem 1.1 and 1.2 in the paper. Because of the nonlinearity of $Q_N$, we have no concrete Green representation formula. Therefore, to prove Theorem 1.1 and 1.2, we use the level set method in [21] and convex symmetrization technique in [1] to conquer this difficult. From this, we research the blow-up behavior about this equation in bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is the content of Theorem 1.3. In Theorem 1.4, the behavior of solo blow-up point is further studied. In the isotropic case, Li in [17] obtain this result using the method of moving plane. For the anisotropic case, one have no idea how to use this method. Here we obtain this result by analyzing the Pohozaev identity and using the expansion of Green function of $N$-anisotropic Laplacian operator.

In addition, it is worthy mentioning that the positive definiteness of $Hess(F^N)$, $N \geq 2$ is needed in proving the main results of this paper, but condition on $F$ is that $Hess(F^2)$ is positive definite. In fact, one can deduce positive definiteness of $Hess(F^N)$ from that of $Hess(F^2)$, which can be find in the proof of Theorem 3.2 in Section 3.

The main results of this paper are stated as follows.

**Theorem 1.1.** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 2$ and $u$ is a weak solution of

\begin{equation}
\begin{cases}
- Q_N u = f(x) & \text{in } \Omega, \\
u |_{\partial \Omega} = 0.
\end{cases}
\end{equation}

Then for $f \in L^1(\Omega)$ and for any $\delta \in (0, N^{-\frac{1}{N-1}}k^{-\frac{1}{N-1}}) = (0, \beta_N)$, it follows that

\begin{equation}
\int_{\Omega} \exp \left\{ \frac{(\beta_N - \delta)|u(x)|}{||f||_{L^1(\Omega)}^{1/(N-1)}} \right\} \, dx \leq \frac{\beta_N}{\delta} |\Omega|.
\end{equation}

**Theorem 1.2.** Suppose that $u$ and $v$ are the weak solution of

\begin{equation}
- Q_N u = f(x) > 0 \text{ in } \Omega
\end{equation}
and

\begin{align}
- Q_N v &= 0 \quad \text{in } \Omega, \\
v|_{\partial \Omega} &= u,
\end{align}

respectively. Then for any \( \delta \in (0, \beta_N) \), we have

\begin{equation}
\int_{\Omega} \exp \left\{ \frac{(\beta_N - \delta) d_0^{N-1} |u - v|}{\|f\|_{L^1(\Omega)}} \right\} \leq \frac{|\Omega|}{\delta},
\end{equation}

where

\[ d_0 = \inf \{ d_{X,Y} : X, Y \in \mathbb{R}^N, X \neq 0, Y \neq 0, X \neq Y \} \]

and

\[ d_{X,Y} = \frac{\langle F^{N-1}(X) F(X) - F^{N-1}(Y) F(Y), X - Y \rangle}{F^{N}(X - Y)} \]

**Theorem 1.3.** Suppose that \( \Omega \subset \mathbb{R}^N, N \geq 2 \) is a bounded domain and \( u_n \) is a sequence of weak solutions of

\begin{equation}
- Q_N u_n = V_n(x) e^{u_n} \quad \text{in } \Omega,
\end{equation}

where \( V_n(x) \geq 0, \|V_n\|_{L^q} \leq C_1 \) for some \( 1 < q \leq \infty \) and \( \|e^{u_n}\|_{L^{q'}} \leq C_2 \). Then one of the following possibilities happens (after taking subsequences):

(i) \( u_n \) is bounded in \( L^\infty_{\text{loc}}(\Omega) \);

(ii) \( u_n \to -\infty \) uniformly on any compact subset of \( \Omega \);

(iii) \( S = \{p_1, \cdots, p_m\} \) is a nonempty, finite set and \( u_n \to -\infty \) uniformly on any compact subset of \( \Omega \setminus S \). In addition, \( V_n e^{u_n} \to \sum_{i=1}^m \alpha_i \delta_{p_i} \) in the sense of measures on \( \Omega \) with \( \alpha_i \geq \left( \frac{\beta_N q}{q'} \right)^{N-1} d_0 \) for any \( i \), where the blow-up set \( S \) is defined by

\[ S := \{ x \in \Omega : \exists x_n \in \Omega \text{ such that } x_n \to x \text{ and } u_n(x_n) \to +\infty \}. \]

**Theorem 1.4.** Suppose that \( \Omega \subset \mathbb{R}^N, N \geq 2 \) is a bounded domain and \( u_n \) is a sequence of weak solutions of (1.9) with

\begin{equation}
\int_{\Omega} e^{u_n} \leq C.
\end{equation}

Let \( (V_n) \) be a sequence of Lipschitz continuous functions satisfying

\begin{equation}
V_n(x) \geq 0, V_n(x) \to V \text{ uniformly in } C_0(\overline{\Omega}), \| \nabla V_n \|_{L^\infty} \leq C.
\end{equation}

In addition, suppose that

\begin{equation}
\max_{\partial \Omega} (u_n) - \min_{\partial \Omega} (u_n) \leq C.
\end{equation}

Then if blow-up happens only at one point, the blow-up value \( \alpha = \left( \frac{N+1}{N-1} \right)^{\frac{1}{N-1}} \).
2. Preliminaries

In this section, let us recall some concepts and properties related to $F$.

Let $F : \mathbb{R}^N \to [0, \infty)$ be a convex function of $C^2(\mathbb{R}^N \setminus \{0\})$, which is even and positively homogeneous of degree 1, i.e. for any $t \in \mathbb{R}, \xi \in \mathbb{R}^N$,

$$F(t\xi) = |t|F(\xi).$$ (2.1)

We also assume that $F(\xi) > 0$ for any $\xi \neq 0$, and $Hess(F^2)$ is positive definite in $\mathbb{R}^N \setminus \{0\}$. With the help of homogeneity of $F$, there exist two constant $0 < a \leq b < \infty$, such that

$$a|\xi| \leq F(\xi) \leq b|\xi| \text{ for any } \xi \in \mathbb{R}^N.$$ (2.2)

Consider the map $\Phi : \mathbb{S}^{N-1} \to \mathbb{R}^N, \Phi(\xi) = F_\xi(\xi)$. Its image $\Phi(S^{N-1})$ is a smooth, convex hypersurface in $\mathbb{R}^N$, which is called the Wulff shape of $F$.

If one defines the support function of $F$ as $F^0(x) := \sup_{\xi \in K} < x, \xi >$, where $K := \{x \in \mathbb{R}^N : F(x) < 1\}$, then it is easy to prove that $F^0 : \mathbb{R}^N \to [0, \infty)$ is also a convex, homogeneous function and $F, F^0$ are polar to each other in the sense that

$$F^0(x) = \sup_{\xi \neq 0} \frac{< x, \xi >}{F(x)} \text{ and } F(\xi) = \sup_{x \neq 0} < x, \xi > \frac{F(\xi)}{F^0(x)}.$$ (2.3)

Let $\mathcal{W}_F := \{x \in \mathbb{R}^N : F^0(x) \leq 1\}$ and $k = k_N = |\mathcal{W}_F|$, which is the Lebesgue measure of $\mathcal{W}_F$. Also, denote $\mathcal{W}_r(x_0)$ by the Wulff ball of center at $x_0$ with radius $r$, i.e. $\mathcal{W}_r(x_0) = \{x \in \mathbb{R}^N : F^0(x - x_0) \leq r\}$.

Next, we summarize the properties on $F$ and $F^0$, which can be proved easily by the assumption on $F$. See [4, 15, 27].

**Proposition 2.1.** We have the following properties:

1. $|F(x) - F(y)| \leq F(x + y) \leq F(x) + F(y);$  
2. $|\nabla F(x)| \leq C$ for any $x \neq 0$;
3. $< \xi, \nabla F(\xi) > = F(\xi), < x, \nabla F^0(x) > = F^0(x)$ for any $x \neq 0, \xi \neq 0$;
4. $F(\nabla F^0(x)) = 1, F^0(\nabla F(\xi)) = 1$;
5. $F_{\xi_0}(t\xi) = sgn(t)F_{\xi_0}(\xi)$;
6. $F^0(x)F_{\xi}(\nabla F^0(x)) = x_i$.

Next we give a co-area formula and isoperimetric inequality in the anisotropic situation. One can refer to [1, 14].

For a bounded domain $\Omega \subseteq \mathbb{R}^N$ and a function of bounded variation $u \in BV(\Omega)$, denote the anisotropic bounded variation of $u$ with respect to $F$ by

$$\int_\Omega |\nabla u|_F = \sup \left\{ \int_\Omega u \text{div} \sigma dx : \sigma \in C^1_0(\Omega), F^0(\sigma) \leq 1 \right\},$$

and anisotropic perimeter of $E$ with respect to $F$ by

$$P_F(E) := \int_\Omega |\nabla \chi_E|_F,$$

where $E$ is a subset of $\Omega$ and $\chi_E$ is the characteristic function of $E$. The co-area formula and isoperimetric inequality can be expressed by

$$\int_\Omega |\nabla u|_F = \int_0^\infty P_F(|u| > t)dt,$$ (2.3)

and

$$P_F(E) \geq Nk^\frac{1}{N} |E|^{1-\frac{1}{N}},$$ (2.4)
respectively.

Let us review the convex symmetrization which is the generalization of the Schwarz symmetrization (see [25]). The one-dimensional decreasing rearrangement of \( u \) is

\[
u^*(t) = \sup \{ s \geq 0 : | \{ x \in \Omega : |u(x)| > s \} | > t \},
\]

for \( t \in \mathbb{R} \). The convex symmetrization of \( u \) is defined as

\[
u^*(x) = u^*(kF^0(x)^N), \quad \text{for } x \in \Omega^*,
\]

where \( \Omega^* \) is the homothetic Wulff ball centered at the origin having the same measure as \( \Omega \).

Next, let us give some lemmas which will be important in proving main results of this paper.

**Lemma 2.2.** (see [1]) Let \( u \in W^{1,p}_0(\Omega) \) for \( p \geq 1 \). Then \( u^* \in W^{1,p}_0(\Omega^*) \) and

\[
\int_{\Omega^*} F^p(\nabla u^*) dx \leq \int_{\Omega} F^p(\nabla u) dx.
\]

**Lemma 2.3.** (see [1]) Let \( u, v \in W^{1,2}_0(\Omega) \) be the weak solutions of the following equations

\[
\begin{aligned}
\begin{cases}
-\text{div}(a(x,u,\nabla u)) = f(x) \quad \text{in } \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
\begin{cases}
-Q_N v = f^*(x) \quad \text{in } \Omega^*, \\
v|_{\partial \Omega^*} = 0,
\end{cases}
\end{aligned}
\]

respectively, where \( f \in L^{\frac{2N}{N-2}}(\Omega) \) if \( N \geq 3 \) or \( f \in L^p(\Omega), p > 1 \) if \( N = 2 \). Suppose that \( a(x,\eta,\xi) \) is vector-value Carathéodory function satisfying

\[
\langle a(x,\eta,\xi),\xi \rangle \geq F^2(\xi) \quad \text{a.e. } x \in \Omega, \ \eta \in \mathbb{R}, \ \xi \in \mathbb{R}^N.
\]

Then it follows that

\[
u^* \leq v \quad \text{in } \Omega^*.
\]

**Lemma 2.4.** (see [27]) Assume \( u \) satisfies \(-Q_N u = 0 \) in \( \Omega \setminus \{0\} \) such that \( u(x)/\Gamma(x) \) remains bounded in some neighborhood of 0. Then there exists a real number \( \gamma \) and \( h \in C^0(\Omega) \) such that

\[
\begin{aligned}
\begin{cases}
\Gamma(x) = \frac{1}{(Nk)^{\frac{1}{N}}} \log F^0(x).
\end{cases}
\end{aligned}
\]

Moreover, when \( \gamma \neq 0 \), the following relation holds

\[
\lim_{x \to 0} F^0(x) \nabla h(x) = 0
\]

and \( u \) satisfies \(-Q_N u = \gamma \delta_0 \) in the sense of measures in \( \Omega \), where

\[
\Gamma(x) = \frac{1}{(Nk)^{\frac{1}{N}}} \log F^0(x).
\]
Thus $\Omega$ (3.1)

Proof. Let $\Omega$ in $\Omega$ using the integration by parts, it follows that

Moreover, $G = \Gamma + h$ with $h \in C^0(\Omega)$ satisfying (2.8).

3. The properties of $Q_N$

In this section, we will give the weak maximum principle, weak comparison principle and mean value property for the $N$-anisotropic Laplacian operator $Q_N$.

Theorem 3.1. (Weak maximum principle) Suppose that $-Q_Nu \leq 0$ in $\Omega$ and $\Omega^+$ has measures zero and $u \leq M$ a.e. in $\Omega$.

Proof. Let $\Omega^+ = \{ x \in \Omega : u(x) > M \}$. Multiplying $-Q_Nu \leq 0$ by $(u - M)^+$ and using the integration by parts, it follows that

Thus $\Omega^+$ has measures zero and $u(x) \leq M$ a.e. in $\Omega$.

Theorem 3.2. (Comparison principle) Suppose that $-Q_Nu \leq -Q_Nv$ in $\Omega$ and $u \leq v$ on $\partial \Omega$, then $u \leq v$ a.e. in $\Omega$.

Proof. Let $\Omega^+ = \{ x \in \Omega : u(x) > v(x) \}$. Assume that $\Omega^+$ has positive measure. Multiplying $-Q_Nu \leq -Q_Nv$ by $(u - v)^+$, it follows that

Since

and

where $[A]^T$ denotes the transpose of matrix $A$. Then from the positive definiteness of $Hess(F^2)$, we obtain that $Hess(F^N)$, $N \geq 2$ is also positive definite. Thus we have that $\nabla u = \nabla v$ a.e. in $\Omega^+$. Since $u = v$ on $\partial \Omega^+$, it is easy to see that $\Omega^+$ has measures zero and $u \leq v$ a.e. in $\Omega$. 

Theorem 3.3. (Mean value property) Suppose that $F$ and $F^0$ satisfy
\begin{align}
<F^0(x), F^0_\xi(y)> &= \frac{<x, y>}{F^{N-1}(x)F^0(y)}
\end{align}
for all $x, y \in \mathbb{R}^N$. If $Q_Nu = 0$ in $\Omega$ and $\mathcal{W}_\rho(0) = \{x \in \mathbb{R}^N : F^0(x) < \rho\} \subset \Omega$, then for every ball of radius $r \in (0, \rho)$, $u$ satisfies the mean value property on spheres,
\begin{align}
u(0) &= \frac{1}{|\partial \mathcal{W}_r(0)|} \int_{\partial \mathcal{W}_r(0)} u(x)ds
\end{align}
and the corresponding mean value property on balls
\begin{align}
u(0) &= \frac{1}{kr^N} \int_{\mathcal{W}_r(0)} u(x)dx.
\end{align}

Proof. Let
\begin{align}
\phi(r) &:= \frac{1}{|\partial \mathcal{W}_r(0)|} \int_{\partial \mathcal{W}_r(0)} u(x)ds(x)
\end{align}
\begin{align}
&= \frac{1}{|\partial \mathcal{W}_r(0)|} \int_{\partial \mathcal{W}_r(0)} u(rz)ds(z).
\end{align}
Then
\begin{align}
\phi'(r) &= \frac{1}{|\partial \mathcal{W}_r(0)|} \int_{\partial \mathcal{W}_r(0)} <\nabla u(rz), z>ds(z)
\end{align}
\begin{align}
&= \frac{1}{|\partial \mathcal{W}_r(0)|} \int_{\partial \mathcal{W}_r(0)} <\nabla u(x), \frac{x}{r}>ds(x).
\end{align}
By the condition (3.3), it follows that
\begin{align}
<\nabla u, x> &= F^{N-1}(\nabla u) < F_\xi(\nabla u), F^0_\xi(x) > F^0(x).
\end{align}
Since $F^0 = r$ and $\nu = F^0_\xi(x)$ on $\partial \mathcal{W}_r(0)$, thus by integration by parts we have
\begin{align}
\phi'(r) &= \int_{\partial \mathcal{W}_r(0)} \sum_{i=1}^N F^{N-1}(\nabla u) F_\xi(\nabla u) \nu_i ds = \int_{\mathcal{W}_r(0)} Q_N u ds = 0.
\end{align}
Thus the proof of the mean value property on spheres is completed. Integrating with respect to $r$, one can get the mean value property on ball. \hfill \Box

4. A Priori Estimates

In this section, with the method of level set and convex symmetrization, we firstly prove Brezis-Merle type concentration-compactness formula, i.e. Theorem 1.1 and Theorem 1.2. At the same time, we obtain the $L^\infty$ estimates for a single solution and uniform $L^\infty$ bounds for solutions of $-Q_Nu = Ve^u$.

Proof of Theorem 1.1. Assume that $R > 0$ is the constant such that $|\Omega| = kR^N$. Let $v$ be the unique solution of the convex symmetrized Dirichlet problem (2.2). According to Lemma 2.3, it follows that
\begin{align}
u^* \leq v,
\end{align}
where \( u^* \) is the convex symmetric decreasing rearrangement of \( u \). In addition, \( v(x) = v(F^0(x)) = v(r) \) is convex symmetric with respect to \( F \) and satisfies the following equation:

\[
\begin{cases}
(-v'' + \frac{N-1}{r}v' + f^*(r)) = 0, \\
v'(0) = 0, v(R) = 0.
\end{cases}
\]

(4.1)

It follows that

\[
-v'(r) = \left( \int_0^r s^{N-1} f^*(s) ds \right) \frac{1}{r} \leq \frac{1}{(Nk) \frac{1}{N-\epsilon}} \frac{1}{r} \| f^* \|_{L^1(\Omega^r)}^\bullet = \frac{1}{(Nk) \frac{1}{N-\epsilon}} \frac{1}{r} \| f \|_{L^1(\Omega)}^\bullet.
\]

Then

\[
v(r) = -\int_r^R v'(t) dt \leq \int_r^R \frac{1}{(Nk)^{\frac{1}{N-\epsilon}}} \frac{1}{r} \| f \|_{L^1(\Omega)}^\bullet dt = \frac{1}{(Nk)^{\frac{1}{N-\epsilon}}} \| f \|_{L^1(\Omega)}^\bullet \log \frac{R}{r}.
\]

Therefore

\[
\int_\Omega \exp \left( \frac{(N-\epsilon)(Nk)^{\frac{1}{N-\epsilon}} |u(x)|}{\| f \|_{L^1(\Omega)}^\bullet} \right) dx = \int_{\Omega^\star} \exp \left( \frac{(N-\epsilon)(Nk)^{\frac{1}{N-\epsilon}} u^*(x)}{\| f \|_{L^1(\Omega)}^\bullet} \right) dx
\]

\[
\leq \int_{\Omega^\star} \exp \left( \frac{(N-\epsilon)(Nk)^{\frac{1}{N-\epsilon}} v(x)}{\| f \|_{L^1(\Omega)}^\bullet} \right) dx = \int_0^R Nkr^{N-1} \exp \left( \frac{R}{r} \right)^{N-\epsilon} dr = \frac{Nk}{\epsilon} R^N.
\]

(4.2)

Let \( \epsilon(Nk)^{\frac{1}{N-\epsilon}} = \delta \), one have

\[
\int_\Omega \exp \left( \frac{(\beta_N - \delta) |u(x)|}{\| f \|_{L^1(\Omega)}^\bullet} \right) dx \leq \frac{\beta_N}{\delta} kR^N = \frac{\beta_N}{\delta} |\Omega|.
\]

(4.3)

Corollary 4.1. Suppose that \( u \) is a weak solution of (1.4) with \( f \in L^1(\Omega) \). Then for any constant \( s > 0 \), we have \( \exp(s|u|) \in L^1(\Omega) \).

Proof. Let \( 0 < \epsilon < \frac{1}{N-1} \), we split \( f \) as \( f = f_1 + f_2 \) with \( \| f_1 \|_{L^1(\Omega)} < \epsilon \) and \( f_2 \in L^\infty(\Omega) \). Assume that \( u_1 \) is the solution of

\[
\begin{cases}
-QNu_1 = f_1(x) \quad \text{in } \Omega, \\
u_1|_{\partial \Omega} = 0.
\end{cases}
\]

(4.4)

Choosing \( \delta = \beta_N - 1 \) in Theorem 1.1, it follows that \( \int_\Omega \exp \left[ \frac{|u_1|}{\| f_1 \|_{L^1(\Omega)}^\bullet} \right] < \infty \) and thus \( \int_\Omega \exp(s|u_1|) < \infty \). In addition, by the Fundamental Theorem of Calculus, we have

\[
f_2 = -(QN - QNu_1) = -Q_N(u - u_1).
\]
where
\[ \widetilde{Q}_N(u - u_1) = \sum_{i,j} \partial_x_i \left[ \int_0^1 \left( \frac{1}{N} F_{\xi,\xi_i}(t \nabla u + (1-t) \nabla u_1) \right) dt \partial_x_j (u - u_1) \right]. \]

As in the proof of Theorem 3.2, from \( \text{Hess}(F^2) \) is positive definite, we get that \( \text{Hess}(F^N), N \geq 2 \) is also positive definite. It is easy to see that \( \widetilde{Q}_N \) is an elliptic operator. From standard elliptic theory, we easily obtain \( ||u - u_1||_{L^\infty} \leq C ||f_2||_{L^\infty}. \) The desired conclusion is followed.

**Corollary 4.2.** Suppose that \( u \) is a weak solution of
\[
\begin{aligned}
- Q_N u & = V(x)e^u \quad \text{in } \Omega, \\
u|_{\partial \Omega} & = 0,
\end{aligned}
\]
where \( V(x) \in L^q(\Omega) \) and \( e^u \in L^{q'}(\Omega) \) for some \( 1 < q \leq \infty. \) Then \( u \in L^\infty(\Omega). \)

**Proof.** By Corollary 4.1 and the standard elliptic theory for quasilinear equation, one can obtain the desired result.

**Corollary 4.3.** Suppose that \( u_n \) is a weak solution of (1.9) with \( u_n = 0 \) on \( \partial \Omega, \)
where
\[
||V_n||_{L^s(\Omega)} \text{ for some } 1 < s \leq \infty,
\]
and
\[
\int_\Omega |V_n| e^{u_n} \leq \epsilon_0 < \frac{\beta_N}{q'}.
\]
Then
\[
||u_n||_{L^\infty(\Omega)} \leq C.
\]

**Proof.** Choosing \( \delta > 0 \) such that \( \beta_N - \delta > \epsilon_0(q' + \delta). \) By Theorem 1.1, we have \( e^{u_n} \) is bounded in \( L^{q'+\delta}(\Omega) \) and \( V_n e^{u_n} \) is bounded in \( L^p(\Omega) \) for some \( p > 1. \) Then the conclusion can be obtained by the standard Morse iteration method.

**Proof of Theorem 1.2.** Let \( \Omega_t = \{ x \in \Omega : |u - v| > t \} \) and \( \mu(t) = |\Omega_t|. \) It follows that
\[
- (Q_N(u) - Q_N(v)) = - \widetilde{Q}_N(u - v)
\]
\[
= - \sum_{i,j} \partial_x_i \left[ \int_0^1 \left( \frac{1}{N} F_{\xi,\xi_i}((1-t)\nabla u + t\nabla v) \right) dt \partial_x_j (u - v) \right] = f > 0.
\]
Since this equation is uniformly elliptic, applying Hopf’s boundary lemma, one can deduce that
\[ \partial_n (u - v) < 0, \quad \nabla u - \nabla v \neq 0 \text{ on } \partial \Omega_t. \]

Then
\[
\int_{\Omega_t} f(x) dx = \int_{\Omega_t} -(Q_N(u) - Q_N(v)) dx
\]
\[
= \int_{\partial \Omega_t} \left< F^{N-1}(\nabla u)F_{\xi}(\nabla u) - F^{N-1}(\nabla v)F_{\xi}(\nabla v), \frac{\nabla u - \nabla v}{|\nabla u - \nabla v|} \right>
\]
\[
\geq d_0 \int_{\partial \Omega_t} \frac{F^N(\nabla u - \nabla v)}{|\nabla u - \nabla v|}.
\]
By the isoperimetric inequality, the co-area formula and the Hölder’s inequality, it follows that
\[
N_k \mu(t) \frac{N-1}{N} \leq P_F(\Omega_t) = -\frac{d}{dt} \int_{\Omega_t} F(\nabla u - \nabla v)dx
\]
\[
= \int_{\partial \Omega_t} \frac{F(\nabla u - \nabla v)}{|\nabla u - \nabla v|} \leq \left( \int_{\partial \Omega_t} \frac{F^N(\nabla u - \nabla v)}{|\nabla u - \nabla v|} \right)^{\frac{1}{N}} \left( \int_{\partial \Omega_t} \frac{1}{|\nabla u - \nabla v|} \right)^{\frac{N-1}{N}}
\]
\[
= \left( \int_{\partial \Omega_t} \frac{F^N(\nabla u - \nabla v)}{|\nabla u - \nabla v|} \right)^{\frac{1}{N}} \left(-\mu'(t)\right)^{\frac{N-1}{N}}.
\]

It follows that
\[
-\mu'(t) \geq \frac{d_0^{\frac{1}{N}} N^{\frac{N}{N-1}} k^{\frac{1}{N-1}} \mu(t)}{\left(\int_{\Omega_t} f(x)dx \right)^{\frac{1}{N-1}}}.
\]

Therefore
\[
\frac{-d}{d\mu} \leq \frac{||f||^\frac{1}{N} L^1(\Omega)}{d_0^{\frac{1}{N}} N^{\frac{N}{N-1}} k^{\frac{1}{N-1}} \mu(t)}.
\]

Integrating (4.9) over \((\mu, |\Omega|)\), we obtain
\[
t(\mu) \leq \frac{||f||^\frac{1}{N} L^1(\Omega)}{d_0^{\frac{1}{N}} N^{\frac{N}{N-1}} k^{\frac{1}{N-1}}} \log\left(\frac{|\Omega|}{\mu}\right).
\]

It is easy to see that
\[
\exp\left( (1-\epsilon)d_0^{\frac{1}{N}} N^{\frac{N}{N-1}} k^{\frac{1}{N-1}} t(\mu) \right) \leq \left( \frac{|\Omega|}{\mu} \right)^{1-\epsilon}.
\]

Thus one can get
\[
\int_{\Omega} \exp\left( (1-\epsilon)d_0^{\frac{1}{N}} N^{\frac{N}{N-1}} k^{\frac{1}{N-1}} |u - v| \right) \frac{d\mu}{||f||^\frac{1}{N} L^1(\Omega)} dx
\]
\[
= \int_{0}^{\infty} \exp\left( (1-\epsilon)d_0^{\frac{1}{N}} N^{\frac{N}{N-1}} k^{\frac{1}{N-1}} t(\mu) \right) (-\mu'(t)) dt
\]
\[
= \int_{0}^{\Omega} \exp\left( (1-\epsilon)d_0^{\frac{1}{N}} N^{\frac{N}{N-1}} k^{\frac{1}{N-1}} t(\mu) \right) \frac{d\mu}{||f||^\frac{1}{N} L^1(\Omega)} \leq \frac{|\Omega|}{\epsilon}.
\]

This means that
\[
\int_{\Omega} \exp\left( (1-\epsilon)d_0^{\frac{1}{N}} N^{\frac{N}{N-1}} k^{\frac{1}{N-1}} |u - v| \right) \frac{d\mu}{||f||^\frac{1}{N} L^1(\Omega)} dx \leq \frac{|\Omega|}{\epsilon}.
\]

Choosing \(\delta = \epsilon N^{\frac{N}{N-1}} k^{\frac{1}{N-1}}\), one can easily obtain the desired result. \(\square\)
Corollary 4.4. Suppose that \( u_n \) is a weak solution of (1.9), where \( V_n \geq 0 \) in \( \Omega \). For \( B_R \subset \Omega, \|V_n\|_{L^q(B_R)} \leq C_1 \) for some \( 1 < q \leq \infty \), \( \|u_n^+\|_{L^q(B_R)} \leq C_2 \) and

\[
\int_{B_R} V_n e^{u_n} \leq \epsilon_0 < \left(\frac{\beta N}{q'}\right)^{-1} d_0.
\]

Then there exists a positive constant \( C \) such that
\[
\|u_n^+\|_{L^\infty(\Omega)} \leq C.
\]

Proof. Consider the problem

\[
\begin{cases}
-Q_N v_n = 0 & \text{in } B_R, \\
v_n = u_n^+ & \text{on } \partial B_R.
\end{cases}
\]

Then, by the weak comparison principle Theorem 3.2, one implies that \( v_n \leq u_n \) in \( B_R \), that is, \( v_n^+ \leq u_n^+ \). Therefore,

\[
\|v_n^+\|_{L^\infty(B_R)} \leq \|u_n^+\|_{L^\infty(B_R)} \leq C_2.
\]

By Serrin’s local a priori estimates (see \[22\]), we obtain that \( \|v_n^+\|_{L^\infty(B_R)} \leq C \).

By Theorem 1.2, we know

\[
\int_{B_{R/2}} \exp \left\{ \frac{(\beta N - \delta) d_0^{-1} (u_n - v_n)}{||V_n e^{u_n}||_{L^1(B_R)}} \right\} \leq \frac{|B_R|}{\delta}.
\]

Note that \( v_n \leq u_n \) implies \( u_n^+ - v_n^+ \leq u_n - v_n \), then

\[
\int_{B_{R/2}} \exp \left\{ \frac{(\beta N - \delta) d_0^{-1} (u_n^+ - v_n^+)}{||V_n e^{u_n}||_{L^1(B_R)}} \right\} \leq \frac{|B_R|}{\delta}.
\]

Combining this with \( \|v_n^+\|_{L^\infty(B_R)} \leq C \) and the smallness condition \[4.11\], we know

\[
\int_{B_{R/2}} \exp \left\{ \frac{(\beta N - \delta) d_0^{-1} u_n^+}{\epsilon_0} \right\} \leq \int_{B_{R/2}} \exp \left\{ \frac{(\beta N - \delta) d_0^{-1} u_n^+}{||V_n e^{u_n}||_{L^1(B_R)}} \right\} \leq \frac{|B_R|}{\delta}.
\]

Now, choosing \( \delta \) such that \( (\beta N - \delta) d_0^{-1} \epsilon_0^{-1} > \epsilon_0^{-1} (q' + \delta) \), we deduce that \( e^{u_n^+} \) is bounded in \( L^{q' + \delta}(B_{R/2}) \), which implies that \( V_n e^{u_n^+} \) is bounded in \( L^p(B_{R/2}) \) for some \( p > 1 \). If we take the problem

\[
\begin{cases}
-Q_N w_n = V_n e^{u_n^+} & \text{in } B_{R/2}, \\
w_n = u_n^+ & \text{on } \partial B_{R/2}.
\end{cases}
\]

Using Serrin’s local a priori estimate again, we obtain \( \|w_n\|_{L^\infty(B_{R/2})} \leq C \). Note that \( u_n^+ \leq w_n \) by the weak comparison principle, then \( \|u_n^+\|_{L^\infty(B_{R/2})} \leq C \).

5. Proof of Blow-up theorem

Using the results of Section 4, let us prove the Blow-up Theorem, i.e. Theorem 1.3.
Proof of Theorem 1.3. Since $V_n e^{u_n}$ is bounded in $L^1(\Omega)$, then there exists a non-negative bounded measures $\mu$ such that for a subsequence (still denoted by $V_n e^{u_n}$),
\[ \int V_n e^{u_n} \psi \to \int \psi d\mu \]
in the sense of measures on $\Omega$ for any $\psi \in C_c(\Omega)$. One call that a point $x_0 \in \Omega$ is a $\gamma$ regular point if for some $\gamma > 0$, there is a function $\psi \in C_c(\Omega)$, $0 \leq \psi \leq 1$ with $\psi = 1$ in some neighborhood of $x_0$, such that
\[ \int \psi d\mu < \gamma. \]

Let $\Sigma$ be the set of non $\gamma$ regular point in $\Omega$. It is easy to see that $x_0 \in \Sigma \iff \mu(\{x_0\}) \geq \gamma$.

In addition, since $\mu$ is a bounded measure, one know that $\Sigma$ is finite.

We now split the proof into three steps.

Step 1. $S = \Sigma$. If $x_0$ is a $(\beta N)q^{-1}d_0$ regular point, then by Corollary 4.4, there is $R_0 > 0$ such that $u_n^+_{x_0}$ is bounded in $L^\infty(B_{R_0}(x_0))$. In other words, if $x_0 \notin \Sigma$, then $x_0 \notin S$, i.e. $S \subset \Sigma$. Conversely, with the similar method in [5], one can get $\Sigma \subset S$. Here we omit the details.

Step 2. $S = \emptyset$ implies (i) or (ii) holds. $S = \emptyset$ means that $u_n$ is bounded in $L^\infty_{\text{loc}}(\Omega)$ and $V_n e^{u_n}$ is bounded in $L^q_{\text{loc}}(\Omega)$. This implies that $\mu \in L^1(\Omega) \cap L^q_{\text{loc}}(\Omega)$.

Assume that $v_n$ is the weak solution of
\[
\begin{cases}
-Q_N v_n = V_n e^{u_n} & \text{in } \Omega, \\
v_n = 0 & \text{on } \partial \Omega.
\end{cases}
\] (5.1)

Obviously, $v_n \to v$ uniformly on any compact subset of $\Omega$, where $v$ is the weak solution of
\[
\begin{cases}
-Q_N v = \mu & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\] (5.2)

Let $w_n = u_n - v_n$, then $-\tilde{Q}_N w_n = 0$ in $\Omega$ and $w_n^+$ is bounded in $L^\infty_{\text{loc}}(\Omega)$, where $\tilde{Q}_N$ is defined as in Corollary 4.1. Using the Harnack’s inequality (see [22]), one obtains the following two possibility:

(a) a subsequence $(w_{n_k})$ is bounded in $L^\infty_{\text{loc}}(\Omega)$; or

(b) $(w_{n})$ converges uniformly to $-\infty$ on compact subset of $\Omega$.

It is easy to see that (a) corresponds to Case (i) and (b) to Case (ii).

Step 3. $S \neq \emptyset$ implies (iii) holds. Similar with Step 2, by Harnack’s inequality [22], one gets the following two possibility:

(c) a subsequence $(w_{n_k})$ is bounded in $L^\infty_{\text{loc}}(\Omega \setminus S)$;

(d) $(w_{n})$ converges uniformly to $-\infty$ on compact subset of $\Omega \setminus S$.

With the similar method in [5], we can exclude the possibility (c). It is important to note that we use the Green function $G(x) = \frac{1}{(Nk)^{N/2}} \log \frac{1}{(x-x_0)^n}$ of $Q_N$ in $W_R(x_0)$ instead of the ordinary Laplacian operator in a ball.

Combing the above argument, the proof of Theorem 1.3 is completed. \qed
6. Proof of Theorem 1.4

In this section, we prove the Theorem 1.4 by analyzing the Pohozaev identity and using the expansion of Green function of \(N\)-anisotropic Laplacian operator.

Proof of Theorem 1.4. Without loss of generality, we assume that \(u_n\) blow up at 0. Let \(v_n\) be the weak solution of

\[
\begin{cases}
-Q_N v_n = V_n e^{u_n} & \text{in } \Omega, \\
v_n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

and \(z_n = u_n - \min_{\partial \Omega} u_n - v_n\). One easily obtains that

\[
\begin{cases}
-Q_N z_n = 0 & \text{in } \Omega, \\
z_n = u_n - \min_{\partial \Omega} u_n & \text{on } \partial \Omega.
\end{cases}
\]

By the standard quasilinear uniformly elliptic equation theory and condition (1.11), it follows that

\[
\|z_n\|_{L^\infty(\Omega)} \leq \|z_n\|_{L^\infty(\partial \Omega)} \leq C \quad \text{and} \quad \|\nabla z_n\|_{L^\infty(\Omega)} \leq C.
\]

In addition, we may assume that \(z_n \to z\) uniformly in \(C^0(\overline{\Omega}) \cap C^1_{\text{loc}}(\Omega)\). Let \(Z_n = V_n \exp\{z_n + \min_{\partial \Omega} u_n\}\), we get

\[
-Q_N v_n = V_n e^{u_n} = Z_n e^{v_n}.
\]

On the other hand, since 0 is the blow-up point, we obtain that when \(x_n \to 0\), \(u_n(x_n) = \max\{x_n \to \infty\}\). Let \(\delta_n = \exp\{-u_n(x_n)/N\}\) and \(\tilde{u}(x) = u_n(\delta_n x + x_n) + N \log \delta_n\). Clearly, \(\tilde{u} \to \tilde{u}\) locally in \(C^1(\mathbb{R}^N)\), where \(\tilde{u}\) is the solution of

\[
\begin{cases}
-Q_N \tilde{u} = V(0) e^{\tilde{u}} & \text{in } \mathbb{R}^N, \\
\tilde{u}(0) = 0, \\
\int_{\mathbb{R}^N} e^{\tilde{u}} < +\infty.
\end{cases}
\]

If \(V(0) = 0\), then \(\tilde{u}\) must be a constant by Liouville type theorem, which contradicts \(\int_{\mathbb{R}^N} e^{\tilde{u}} < +\infty\). Thus it follows that \(V(0) > 0\), which deduces that \(V_0\) has positive lower bound near the origin. Since \(\nabla \log Z_n = \nabla \log V_n + \nabla z_n\), by using (6.3) and the condition (1.10), one can get

\[
\|\nabla \log Z_n\|_{L^\infty(B_r)} \leq C
\]

for some \(r > 0\).

From Theorem 1.3, we have for any \(\psi \in C_0^\infty(\Omega),\)

\[
\int_{\Omega} -Q_N v_n \psi = \int_{\Omega} V_n e^{u_n} \psi \to \alpha \psi(0).
\]

Let \(\Omega_1 = \{x : a|\nabla v_n(x)| \leq 1\}, \Omega_2 = \{x : a|\nabla v_n(x)| > 1\}\). By (2.2) and Proposition 2.1, we obtain

\[
\|\nabla v_n\|_{L^q(\Omega)} \leq \|\nabla v_n\|_{L^q(\Omega_1)} + \|\nabla v_n\|_{L^q(\Omega_2)} \leq C + \sup\{\int_{\Omega_2} \nabla v_n \nabla \psi : ||\psi||_{W^{1,q}_0} = 1\}\]

\[
\leq C + C \sup\{\int_{\Omega_2} F^{N-1}(\nabla v_n) F_\xi (\nabla v_n) \nabla \psi : ||\psi||_{W^{1,q}_0} = 1\}.
\]
It is easy to see that \(||\psi||_{L^\infty(\Omega)} \leq C\) by the Sobolev embedding. Thus
\[
\int_{\Omega_2} F^{N-1}(\nabla v_n) F^\xi(\nabla v_n) \nabla \psi = \int_{\Omega_2} -Q_N v_n \psi \leq ||V_n e^{u_n}||_{L^1(\Omega)} ||\psi||_{L^\infty(\Omega)} \leq C.
\]
Therefore \(||\nabla v_n||_{L^\infty(\Omega)} \leq C\) for any \(1 < q < \frac{N}{N-1}\).

By Lemma 2.4 and 2.5, we have a unique Green function of
\[
\begin{cases}
-Q_N G(\cdot, 0) = \alpha \delta_0 \text{ in } \Omega, \\
G(\cdot, 0) = 0 \text{ on } \partial \Omega,
\end{cases}
\]
and \(G\) has a decomposition
\[
G(x) = -\frac{\alpha}{(Nk)^{\frac{N-1}{N}}} \log F^0(x) + h(x),
\]
with
\[
h(x) \in C^0(\Omega) \text{ and } \lim_{x \to 0} h(x) \text{ exists}, \lim_{x \to 0} |x| \nabla h(x) = 0.
\]
One easily obtains that \(v_n \to G\) weakly in \(W^{1,q}(\Omega)\).

By Corollary 4.4, from \(\int_{\Omega} V_n e^{u_n} \to 0\) for any \(\overline{\Omega} \subset \subset \Omega \setminus 0\), we obtain \(||v_n^+||_{L^\infty(\overline{\Omega})} \leq C\). Thus \(||v_n^+||_{C^{1,\beta}(\overline{\Omega})} \leq C\) for some \(0 < \beta < 1\) by equation (6.4). Hence \(v_n \to G\) strongly in \(C^{1,\beta}(\Omega)\).

Multiplying (6.4) by \(<x, \nabla v_n>\) and integrating by parts, one can get the Pohozaev identity as follows:
\[
\int_{\partial W_\epsilon} -F^{N-1}(\nabla v_n) < F^\xi(\nabla v_n, \nu) > <x, \nabla v_n>
+ \frac{1}{N} F^N(\nabla v_n) <x, \nu>
= \int_{\partial W_\epsilon} Z_n e^{u_n} <x, \nu> - \int_{W_\epsilon} N Z_n e^{u_n} + <x, \nabla \log Z_n> Z_n e^{u_n},
\]
where \(\nu = \frac{\nabla F^a}{|\nabla F^a|}\) is the unit outward normal.

Letting \(n \to \infty\), the left-hand side of (6.11) converges to
\[
I := \int_{\partial W_\epsilon} -F^{N-1}(\nabla G) < F^\xi(\nabla G), \nu > <x, \nabla G>
+ \frac{1}{N} F^N(\nabla G) <x, \nu>.
\]

By (6.9), (6.10) and Proposition 2.1, we obtain that on \(\partial W_\epsilon\),
\[
F^{N-1}(\nabla G) = F^{N-1} \left( -\frac{\alpha}{(Nk)^{\frac{N-1}{N}}} \frac{\nabla F^0}{F^0} + o\left(\frac{1}{F^0}\right) \right)
\leq \left( \frac{\alpha}{\epsilon (Nk)^{\frac{N-1}{N}}} + o\left(\frac{1}{\epsilon}\right) \right)^{N-1},
\]

\[
F^{N-1}(\nabla G) = \left( -\frac{\alpha}{(Nk)^{\frac{N-1}{N}}} \frac{\nabla F^0}{F^0} + o\left(\frac{1}{F^0}\right) \right)^{N-1}.
\]
\( < F_\xi(\nabla G), \nu > = < F_\xi(\nabla G), \frac{\nabla F^0}{|\nabla F^0|} > \)

\[
= \left\langle F_\xi(\nabla G), \left( \frac{(Nk)^{\frac{1}{Nk}} F^0}{\alpha} \right) \nabla G - o\left(\frac{1}{F^0}\right) \right\rangle \\
= - \epsilon \frac{(Nk)^{\frac{1}{Nk}}}{\alpha} \left( \frac{F(\nabla G)}{|\nabla F^0|} - \frac{o(1)}{|\nabla F^0|} \right) \\
= -(1 + o(1)) \frac{1}{|\nabla F^0|},
\]

(6.14)

\( < x, \nabla G > = < x, -\frac{\alpha}{(Nk)^{\frac{1}{Nk}}} \frac{\nabla F^0}{F^0} + o\left(\frac{1}{F^0}\right) > \)

(6.15)

and

\( < x, \nu > = < x, \frac{\nabla F^0}{|\nabla F^0|} > = \epsilon \frac{1}{|\nabla F^0|}. \)

Substituting the above formula (6.13)-(6.16) into (6.12), one obtains

\[
I = \int_{\partial W_0} \left( \frac{\alpha}{\epsilon (Nk) \frac{1}{Nk}} + o\left(\frac{1}{\epsilon}\right) \right)^{N-1} (1 + o(1)) \\
\times \frac{1}{|\nabla F^0|} \left( - \frac{\alpha}{(Nk) \frac{1}{Nk}} + o(1) \right) \\
+ \frac{1}{N} \left( \frac{\alpha}{\epsilon (Nk) \frac{1}{Nk}} + o\left(\frac{1}{\epsilon}\right) \right)^N \epsilon \frac{1}{|\nabla F^0|} \\
= - \frac{N - 1}{N} \left\{ \left( \frac{\alpha}{(Nk) \frac{1}{Nk}} \right)^N + o(1) \right\} \frac{1}{\epsilon^{N-1}} \int_{\partial W_0} \frac{1}{|\nabla F^0|} \\
= - (N - 1)k \left( \frac{\alpha}{(Nk) \frac{1}{Nk}} \right)^N + o(1).
\]

(6.17)

Letting \( n \to \infty \) and \( \epsilon \to 0 \), one deduce that the right-hand side of (6.11) converges to \( -Na \) by (6.6) and \( V_n e^{\alpha_n} \to \alpha_0 \delta_0 \). Then we easily get \( \alpha = \left( \frac{N^{N+1} k^{\frac{1}{Nk-1}}}{N^{\frac{1}{Nk-1}}} \right)^{\frac{1}{Nk-1}} \).

Thus the proof of Theorem 1.4 is completed. \( \square \)
References

[1] A. Alvino, V. Ferone, G. Trombetti, P.-L. Lions. Convex symmetrization and applications. Ann. Inst. H. Poincaré Anal. Non Linéaire, 14 (1997) 275-293.
[2] D. Bartolucci, G. Tarantello. Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory. Comm. Math. Phys, 229 (2002) 3-47.
[3] M. Belloni, V. Ferone, B. Kawohl. Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators. Z. Angew. Math. Phys, 54 (2003) 771-783.
[4] G. Bellettini, M. Paolini. Anisotropic motion by mean curvature in the context of Finsler geometry. Hokkaido Math. J. 25 (1996) 537-566.
[5] H. Brezis, F. Merle. Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. Comm. Partial Differential Equations, 16 (1991) 1223-1253.
[6] S. Chang, P. Yang. Prescribing Gaussian curvature on $S_2$. Acta Math., 159 (1987) 215-259.
[7] C. Chen, C. Lin. Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. Comm. Pure Appl. Math., 55 (2002) 728-771.
[8] W. Chen, C. Li. Prescribing Gaussian curvatures on surfaces with conical singularities. J. Differ. Equal., 1 (4) (1991) 359-372.
[9] L. Damascelli. Comparison theorems for some quasilinier degenerate elliptic operators and applications to symmetry and monotonicity results. Ann. Inst. H. Poincare Anal. Non Lineaire, 15 (1998) 493-516.
[10] W. Ding, J. Jost, J. Li, G. Wang. The differential equation $\Delta u = 8\pi - 8\pi e^u$ on a compact Riemann surface. Asian J. Math., 1 (1997) 230-248.
[11] W. Ding, J. Jost, J. Li, G. Wang. Existence results for mean field equations. Ann. Inst. H. Poincare Anal. Non Lineaire, 16 (5) (1999) 653-666.
[12] Z. Djadli. Existence result for the mean field problem on Riemann surfaces of all genera. Commun. Contemp. Math., 10 (2008) 205-220.
[13] V. Ferone, B. Kawohl. Remarks on a Finsler-Laplacian. Proc. Amer. Math. Soc., 137 (2009) 247-253.
[14] I. Fonseca, S. Muller. A uniqueness proof for the Wulff theorem. Proc. Roy. Soc. Edinburgh Sect., A 119 (1991) 125-136.
[15] J. Jost, G. Wang. Analytic aspects of the Toda system. I. A Moser-Trudinger inequality. Comm. Pure Appl. Math., 54 (2001) 1289-1319.
[16] J. Li, Y. Li. Solutions for Toda systems on Riemann surfaces. Annali della Scuola Normale Superiore di Pisa Classe di Scienze Serie V, 4 (2005) 703-728.
[17] Y. Li. Harnack inequality: the method of moving planes. Comm. Math. Phys., 200 (1999) 421-444.
[18] Y. Li, I. Shafrir. Blow-up analysis for solutions of $-\Delta u = V(x)e^u$ in dimension two. Indiana Univ. Math. J., 43 (1994) 1255-1270.
[19] C. Lin. An expository survey of the recent development of mean field equations. Discrete Contin. Dyn. Syst., 19 (2007) 387-410.
[20] J. Liouville. Sur l'équation aux derivees partielles $\frac{\partial^2 \log \lambda}{\partial u \partial v} \pm \frac{\lambda}{2\pi^2} = 0$. J. Math. Pures Appl., 18 (1853) 71-72.
[21] X. Ren, J. Wei. Counting peaks of solutions to some quasilinear elliptic equations with large exponents. J. Differ. Equations, 117 (1995) 28-55.
[22] J. Serrin. Local behavior of solutions of quasi-linear equations. Acta Math., 111 (1964) 247-302.
[23] J. Spruck, Y. Yang. On multivortices in the electroweak theory I: Existence of periodic solutions. Comm. Math. Phys., 144 (1992) 1-16.
[24] M. Struwe, G. Tarantello. On the multivortex solutions in the Chern-Simons gauge theory. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat., 1 (1998) 109-121.
[25] G. Talenti. Elliptic equations and rearrangements. Ann. Sc. Norm. Super. Pisa Cl. Sci., 3 (1976) 697-718.
[26] G. Tarantello. Multiple condensate solutions for the Chern-Simons-Higgs theory. J. Math. Phys., 37 (1996) 3769-3796.
[27] G. Wang, C. Xia. Blow-up analysis of a Finsler-Liouville equation in two dimensions. J. Differ. Equations, 252 (2012) 1668-1700.
[28] G. Wang, C. Xia. A characterization of the Wulff shape by an overdetermined anisotropic PDE. Arch. Ration. Mech. Anal., 99 (2011) 99-115.
[29] G. Wang, C. Xia. A Brunn-Minkowski inequality for a Finsler-Laplacian. Analysis (Munich), 31 (2011) 103-115.
[30] G. Wulff. Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Kristallflächen. Z. Krist., 34 (1901) 449-530.

RULONG XIE. SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI 230026, CHINA
E-mail address: xierl@mail.ustc.edu.cn;

HUAJUN GONG (CORRESPONDING AUTHOR). SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI 230026, CHINA
E-mail address: huajun84@hotmail.com.