Finite-amplitude Rayleigh- Bénard convection for weakly shear thinning fluids

B Albaalbaki and Roger E Khayat

1Department of Mechanical & Materials Engineering, The University of Western Ontario, London, Ontario, Canada, N6A 5B9

Email: rkhayat@uwo.ca

Abstract. The thermo-gravitational instability in a fluid layer of a weakly shear thinning medium heated from below is investigated. A linear and a weakly nonlinear analysis are successively presented. The shear thinning fluid is modelled by means of Carreau-Bird model. The critical values for the temperature gradient and wave number corresponding to the onset of convection are determined from a linear approach. After motion has set in, particular patterns are predicted taking the form of either rolls, or hexagons. By means of a nonlinear technique, restricted to steady situations, it is determined under which specific conditions one pattern is preferred. The influence of the constitutive equation parameters is examined and discussed in detail.

1. Introduction
Our purpose is to provide a description of thermo-gravitational effects in horizontal weakly shear thinning fluid layers subject to a vertical temperature gradient. The problem is usually referred to as the Rayleigh-Bénard instability. Two main questions are asked: firstly, starting with a fluid at rest, what are the critical values of the temperature gradient and wave number when instability in the form of convective motion sets in. Above the critical threshold, one observes experimentally the occurrence of well-structured convecting cells, taking either the form of rolls, hexagons, or even more complicated polygonal patterns; an important problem is the determination of the geometrical nature of such convective planforms. The first question may be solved by a linear analysis, but the second one requires a nonlinear approach.

Although several hundreds of papers have been devoted to the problem of the Rayleigh-Bénard convection in ordinary Newtonian fluids, only a limited number of works, e.g. [1-4], deal with shear thinning fluids. Moreover, still fewer articles [4] have been devoted to nonlinear developments; moreover, they are generally restricted to two-dimensional flows. There is clearly a gap to be filled.

In rheology, one crucial problem is the formulation of the constitutive equation: the main difficulty is to ascertain the adequate constitutive equation to a given material. To circumvent this difficulty, the four parameter Carreau-Bird model [5] is adopted for expressing the behavior of shear thinning fluids, more particularly for weakly shear-thinning fluids. A major advantage of this model over other models such as power-law model is that Newton’s law of viscosity can be recovered in the limit of zero shear rates.

To whom any correspondence should be addressed.
Some of the early experiments on the thermal convection of shear thinning fluids were conducted by Pierre & Tien [1], Tsuei & Tien [2], and Liang & Acrivos [3] who found that shear thinning tends to enhance the regularity in the flow pattern. (see Shenoy and Mashelkar [6] for general review on the experiments on natural convection of shear thinning and viscoelastic fluids and Larson [7] for a general review on the instabilities in viscoelastic fluids). Nonlinear RB convection of non-Newtonian fluids was considered by Eltayeb [8], Rosenblat [9], Martinez-Mardones & Perez-Garcia [10], Martinez-Mardones et al. [11], Harder [12], and, more recently by Khayat [4,13-15], Park & Lee [16], Parmentier et al [17], Li & Khayat [18], and Zhang, Vola & Frigaard [19]. Khayat investigated nonlinear RB convection of viscoelastic [13-15] and shear-thinning [4] fluids by adopting a dynamical system constituting a generalization of the Lorenz model (Lorenz [20]) to include elastic normal stress effects. It was found that for very weakly elastic fluids, stationary convection loses its stability to oscillatory convection via a Hopf bifurcation when the Rayleigh number exceeds a critical value in the post-critical range.

An amplitude equation approach is used to assess the stability of two convection patterns, namely rolls and hexagons in the post-critical range. Readers are referred to Friedman [21], Eckhaus [22], Newell et al [23], and Cross & Hohenberg [24] for the general theory. The current derivation follows closely that developed by Parmentier et al. [17] and Li & Khayat [18].

The paper will run as follows. In Section 2, the rheological model and the governing equations of mass, momentum and energy are presented. Boussinesq's approximation is taken for granted. Section 3 is devoted to the linear approach which allows for the determination of the relevant critical parameters, like the critical temperature gradient and wave number. The study of the flow pattern observed beyond the linear instability threshold requires a weakly nonlinear approach presented in Section 4. The solution of the nonlinear stationary problem is expanded in terms of the eigenfunctions of the linear stability problem. The procedure results in a set of six amplitude equations allowing for the description of the two kinds of convective cells which are usually observed in experiments, namely: rolls, and hexagons. Conclusions and prospects are drawn in Section 5.

2. Statement of the problem

We consider a thin layer of a weakly shear thinning liquid bounded from below and above by two free surfaces. The fluid layer is supposed to be of infinite horizontal extent. The plates are perfectly heat conducting at uniform temperature and assumed to remain flat, there is a heat transfer between liquid and its environment. The fluid is heated from below and is initially quiescent. When the temperature difference $\delta T$ between the lower and upper surfaces exceeds a critical value, motion sets in and convective cells taking the form of either rolls or hexagons are usually observed beyond the critical threshold. It is nowadays well recognized that two effects contribute to the onset of convection. The first one is buoyancy; this leads to Rayleigh Bénard instability and is related to the decrease of the mass density $\rho$ with temperature $T$:

$$\rho = \rho_0 \left[1 - \alpha T (T - T_0)\right]. \quad (1)$$

Here $\alpha_T$ is the coefficient of volume expansion, and $T_0$ a constant temperature, say the temperature of the upper free surface and $\rho_0$ is the mass density of the fluid at $T_0$. The second cause of instability arises from the variation of the surface tension at the upper surface with temperature. This effect is appreciable only in very thin layers, and is not considered in this work.

Consider an incompressible shear thinning fluid of viscosity $\mu$, specific heat at constant pressure $C_p$, and thermal conductivity $K$. The fluid is confined between two infinite and flat plates at $Z = 0$ and $Z = D$ with temperatures $T_0 + \delta T$ and $T_0$ respectively. If the Boussinesq's approximation, which states that the effect of compressibility is negligible everywhere in the conservation equations except in the buoyancy term, is assumed to hold, then the equations for the conservation of mass, momentum and energy, are, respectively:

$$\nabla \cdot \mathbf{V} = 0, \quad (2)$$
\[\rho_0 \left( \mathbf{V} \cdot \nabla + \mathbf{V} \cdot \nabla \mathbf{V} \right) = - \nabla P - \rho_0 \left[ 1 - \alpha_T \left( T - T_0 \right) \right] \mathbf{g} \mathbf{e}_z + \mu \Delta \mathbf{V}, \quad (3)\]

\[\rho_0 C_p \left( T - \mathbf{V} \cdot \nabla T \right) = K \Delta T + \frac{1}{2} \mu \mathbf{\dot{Y}}^2, \quad (4)\]

where \(\nabla\) is the gradient operator, \(\Delta\) is the Laplacian operator. A subscript after a comma denotes partial differentiation; \(\tau\) is the time, \(\mathbf{V}\) is the velocity vector, \(P\) is the pressure, \(\mathbf{e}_z\) is the unit vector in the direction opposite to gravity, \(g\) is the acceleration due to gravity, \(T\) is the temperature and \(\mathbf{\dot{Y}} = \nabla \mathbf{V} + (\nabla \mathbf{V})^T\) is the rate-of-strain tensor. In this work the following Carreau-Bird expression for the viscosity \(\mu\) \cite{24} is adopted:

\[\frac{\mu - \mu_\infty}{\mu_0 - \mu_\infty} = \left( 1 + \lambda^2 \mathbf{\dot{Y}}^2 \right)^{-\frac{n-1}{2}}, \quad (5)\]

where \(n\) is the power-law exponent, which for a shear thinning fluid is less than one, \(\mu_0\) is the zero-shear-rate viscosity and \(\mu_\infty\) is the infinite shear-rate-viscosity, and \(\lambda\) is the time constant of Carreau-Bird equation (5). Let \(D\), \(D^2/\kappa\) and \(\kappa D\) be, respectively, typical length, time, and velocity, and \(\mu_0 \kappa / D^2\) be the typical stress and pressure. Here \(\kappa = K \left( \rho C_p \right)^{-1}\) is the thermal diffusivity. The non-dimensional equations for the conservation of mass, momentum and energy, are respectively:

\[\nabla \cdot \mathbf{v} = 0, \quad (6)\]

\[Pr^{-1} \left( \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} \right) = - \nabla P + Pr \theta \mathbf{e}_z + \nabla \cdot \left( \eta \mathbf{\dot{Y}} \right), \quad (7)\]

\[\theta_t + \mathbf{v} \cdot \nabla \theta = \Delta \theta + \nabla \cdot \left( \eta \mathbf{e}_z + Pr Ec \frac{\eta}{2} \mathbf{\dot{Y}}^2 \right), \quad (8)\]

where \(t\) is the non-dimensional time, \(\mathbf{v}\) is the non-dimensional velocity vector, \(p = D^2 \left( \mu_0 \kappa \right)^{-1} \left( P - P_s \right)\) is the pressure deviation from the steady state pressure obeying the relation

\[\nabla P_s = -\rho_0 \left[ 1 - \alpha_T \delta T \left( 1 - \frac{Z}{D} \right) \right] \mathbf{g} \mathbf{e}_z, \quad \theta = \frac{T - T_s}{\delta T} \]

is the departure from steady state temperature, \(T_s = -\frac{Z}{D} \delta T + T_0 + \delta T\), \(\mathbf{\dot{Y}} = \nabla \mathbf{V} + (\nabla \mathbf{V})^T\) is the non-dimensional rate-of-strain tensor, \(\eta = \mu / \mu_0\) is the non-dimensional viscosity and \(\nu = \mu_0 / \rho_0\) is the kinematic viscosity. The following non-dimensional groups have been introduced, namely the Rayleigh number, Prandtl number and Eckert number:

\[Pr = \frac{v}{\kappa}, \quad Ra = \frac{\delta T g \alpha_T D^3}{v \kappa}, \quad Ec = \frac{\kappa^2}{C_v \delta T D^2}. \quad (9)\]

The relevant boundary conditions at the lower and upper free surfaces are:

\[\mathbf{v} \cdot \mathbf{e}_z = 0, \quad (10)\]

\[\mathbf{v} \cdot \mathbf{e}_z = 0, \quad \theta = 0.\]

The non-dimensional Carreau-Bird equation for viscosity is given below:

\[\eta = s + \left( 1 - s \right) \left[ 1 + E^2 \lambda^2 \right]^{-\frac{n-1}{2}}, \quad (11)\]

where \(s\) is the ratio of zero- to infinite-shear-rate viscosities, and \(E\), the elasticity number, is the ratio of time constant \(\lambda\) of the fluid to thermal diffusion time.
\[ E = \frac{\lambda \kappa}{D^2}. \] (12)

Thus in the limit \( E \to 0 \), the Newtonian viscosity \( \eta = 1 \) is recovered, namely, the viscosity is equal to the zero-shear-rate value.

3. Linear approach

The linear approximation is the starting point for treating the more realistic nonlinear problem. Since the latter is the main objective of the present work, we shall briefly sketch the linear analysis, by emphasizing the most important points.

The basic equations are obtained by linearizing the set (6) – (8) and by seeking solutions of the form:

\[ \phi(x, y, z, t) = \varphi(x, y)F(z)\exp(st), \] (13)

where \( \varphi \) stands for anyone of the unknowns: \( v, p, \) and \( \theta \); \( \varphi(x, y) \) describes the horizontal dependence of the disturbance and is solution of:

\[ \varphi_{xx} + \varphi_{yy} = k^2 \varphi, \] (14)

where \( k \) designates the magnitude of the horizontal wave number \( k \), the complex quantity \( s = s_r + is_i \) dictates the time evolution of the perturbation and \( F = (F_u, F_v, F_w, F_p, F_\theta) \) are the amplitudes of the disturbances that depend only on the vertical coordinate \( z \). After substitution of equation (13) in the set (6)-(8), following set of linear coupled partial differential equations are obtained:

\[ Pr^{-1} sv = -\nabla p + Ra \theta \varepsilon_z + \Delta v, \] (15)
\[ \nabla \cdot v = 0, \] (16)
\[ s\theta = \Delta \theta + v \cdot e_z. \] (17)

where the eigenvalue \( s \) obeys the following characteristic equation:

\[ s^2 + \left( k^2 + m^2 \pi^2 \right)(Pr + 1)s + \frac{k^2 Pr}{k^2 + m^2 \pi^2} \left( \frac{(k^2 + m^2 \pi^2)^3}{k^2} - Ra \right) = 0. \] (18)

It is not difficult to establish, as in the case of a Newtonian fluid, that the value of critical Rayleigh number for the onset of stationary convection is equal to \( \left( \pi^2 + k^2 \right)^3 k^{-2} \) for the most dominant mode. The critical (smallest) Rayleigh number and the corresponding wavenumber are equal to \( Ra_c = 657.51 \) and \( k_c = 2.221 \), respectively.

4. Weakly nonlinear analysis

4.1 The amplitude equations

The linear analysis developed in the previous section does not allow determining the pattern of the flow beyond the critical threshold. Our aim is to answer the following questions:

(i) What is the geometrical nature of the planform observed beyond the critical threshold?

(ii) In which temperature intervals are these structures stable?

The present analysis is classified as a weakly nonlinear approach in the sense that the deviation:

\[ \varepsilon = \frac{Ra - Ra_c}{Ra_c}, \] (19)
with respect to the critical rate of heating remains small, say smaller than one. As it was shown in section 3 that overstability does not present, we shall here restrict our nonlinear approach to the stationary situation.

In order to make the problem more tractable, the binomial expansion in equation (11) is carried out, and $E$ is taken small enough to guarantee the convergence. The present formulation and subsequent numerical results are thus limited to flows at moderately small shear rate or relaxation time. This assumption is not unreasonable since, although large strains may be present during thermal convection, only small shear rates are usually involved. Thus, keeping terms to $O(E^2)$, the binomial expansion of equation (11) leads to:

$$\eta = 1 + \alpha E^2,$$

where $\alpha$ is the shear thinning parameter:

$$\alpha = \frac{n-1}{2} E^2.$$

The basic set of equations given by relations (6)-(8) and the boundary conditions (10) can be written in a compact form as:

$$L_c(\Phi) + L_\Delta(\Phi) - M \frac{\partial \Phi}{\partial t} = N(\Phi),$$

where $\Phi$ designates the following set of unknowns:

$$\Phi = (v, p, \theta, \gamma),$$

and where the explicit expressions of operators $L_c$ and $L_\Delta$ are given by:

$$L_c = \begin{bmatrix} \Delta & -\nabla & Ra e_z \\ \nabla & 0 & 0 \\ e_z & 0 & \Delta \end{bmatrix},$$

$$L_\Delta = \begin{bmatrix} 0 & 0 & \varepsilon Ra e_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$}

The quantity $N(\Phi)$ and $M$ in the equation (22) represents the nonlinear and time-rate contributions and can be written as:

$$N(\Phi) = \begin{bmatrix} Pr^{-1} v \cdot \nabla v \\ 0 \\ v \cdot \nabla \theta \end{bmatrix} - \alpha \begin{bmatrix} \nabla \cdot (\gamma \gamma) \\ 0 \\ 0 \end{bmatrix},$$

$$M = \begin{bmatrix} Pr^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$}

The method of solution consists of expanding the solution $\Phi$ of the nonlinear problem in terms of eigenfunctions of the linear problem. To be explicit, $\Phi$ can be written in the form:

$$\Phi(x, y, z, t) = \sum_{n=1}^{\infty} \sum_q A_n^q(t) f_n^q(x, y, z),$$
The summation \( \sum_{q=1}^{\infty} \) extends on the whole set of allowable wave vectors \( k_q \), which in the case of an infinite horizontal extent can take all possible directions, while the summation \( \sum_{n=1}^{\infty} \) runs over the eigenvalues \( s_0, s_1, ..., s_n, ... \) pertaining to a given \( k_q \); \( A_n^q(t) \) designates the amplitude of the mode \( f_n^q \) and must satisfy:

\[
A\( s_n, k_q \) = \overline{A}\( \overline{s}_n, -k_q \),
\]

(29)
in order that \( \Phi \) be real, a bar over a symbol means complex conjugate; the eigen-functions \( f_n^q \) are solutions of the linear problems:

\[
L_c f_n^q = s_n^q M f_n^q ,
\]

(30)
where \( L_c \) and \( M \) are given by expressions (24) and (27). The eigen-values of linear problem are ordered in such a way that \( \text{Re}\( s_1^q \) > \text{Re}\( s_2^q \) > ... > \text{Re}\( s_p^q \) where \( \text{Re}(s) \) stands for the real part of \( s \).

Solutions \( f_n^q(x,y,z) \) are sought in the form:

\[
f_n^q(x,y,z) = \exp\left[ jk_q \cdot \left( xe_x + ye_y \right) \right] F_n^q(z),
\]

(31)
where \( F_n^q(z) \) stands for amplitudes \( F_n^q = \{ F_u(z;n,q), F_v(z;n,q), F_w(z;n,q), F_p(z;n,q), F_q(z;n,q) \} \) and is determined after substitution of equation (31) in equation (30). Now, the solution \( \Phi(x,y,z,t) \) of the nonlinear problem (22) is expressed in terms of the eigenfunctions \( f_n^q \):

\[
\sum_{n,q=1}^{\infty} \left( A_n^q(t) L_c \left( f_n^q \right) + \epsilon A_n^q L_\Delta \left( f_n^q \right) - M f_n^q \frac{dA_n^q}{dt} \right) = N(\Phi).
\]

(32)
By applying relation (30) in (32), the following equation is obtained:

\[
\sum_{n,q=1}^{\infty} \left( A_n^q(t) s_n^q M f_n^q + \epsilon A_n^q L_\Delta \left( f_n^q \right) - M f_n^q \frac{dA_n^q}{dt} \right) = N(\Phi).
\]

(33)
The next step of the procedure consists of projecting the non-linear equations (22) on the eigenfunctions \( f_m^p \) of the linearized adjoint problem:

\[
L_c^* f_m^p = s_m^p M f_m^p ,
\]

(34)
where \( s_m^p \) and \( f_m^p \) are respectively, the adjoint eigenvalue and eigenfunction of the linearized adjoint operator \( L_c^* \), defined by the following relation:

\[
\left< L_c f, f^* \right> = \left< f, L_c^* f^* \right> .
\]

(35)
Here an angle bracket denotes the average integral of the scalar product defined by:

\[
\langle a, b \rangle = \lim_{\ell \to \infty} \frac{1}{4\ell} \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} P_v(a, b) dz dy dx ,
\]

(36)
where \( P_v(a, b) \) is scalar product defined on the volume \( v \) of the fluid between two infinite flat plates at \( z = 0 \) and \( z = 1 \). The explicit equations for the adjoint problem are given by the following set:
\[ Pr^{-1} s^* \mathbf{v}^* = -\nabla p^* + \Delta \mathbf{v}^* + w e_z, \]
\[ \nabla \cdot \mathbf{v}^* = 0, \]
\[ s^* \theta^* = \Delta \theta^* + \text{Raw}^*. \]

Now, non-linear equations (22) are projected on the eigenfunctions \( \mathbf{f}_m^p \) of the linearized adjoint problem:
\[
\sum_{n,q=1}^{\infty} \left( A_n^q(t) s_n^q \left< \mathbf{Mf}_n^q, \mathbf{f}_m^p \right> + \varepsilon A_n^q \left< \mathbf{L}_\Delta \left( \mathbf{f}_n^q \right), \mathbf{f}_m^p \right> - \left< \mathbf{Mf}_n^q, \mathbf{f}_m^p \right> \frac{dA_n^q}{dt} \right) = \left< \mathbf{N}(\Phi), \mathbf{f}_m^p \right>. \tag{40}
\]

The following orthogonality identities can be established for any linear operator \( \mathbf{L} \):
\[
\left< \mathbf{L} \left( \mathbf{f}_n^q \right), \mathbf{f}_m^p \right> = \delta_{pq} \left< \mathbf{L} \left( \mathbf{f}_n^q \right), \mathbf{f}_m^p \right>, \tag{41}
\]
\[
\left< \mathbf{Mf}_n^p, \mathbf{f}_m^p \right> = \delta_{mm} \left< \mathbf{Mf}_n^p, \mathbf{f}_m^p \right>. \tag{42}
\]

The set of equations (40) stands for an infinite number of ordinary differential equations for the unknown amplitude \( A_n^q(t) \). After integration by parts, and applying the relations (41) and (42), infinite sequence of amplitude equations are obtained:
\[
A_m^p(t) s_m^p C_0(m,p) + \sum_{n=1}^{\infty} \varepsilon A_n^p C_1(m,n,p) - \frac{dA_m^p}{dt} C_0(m,p) = \\
\sum_{n,L=1}^{\infty} \sum_{q,r=1}^{\infty} A_n^q A_L^r \delta(\mathbf{k}_q + \mathbf{k}_r - \mathbf{k}_p) C_2(m,n,L,p,q,r) + \\
- \alpha \sum_{n,L,a=1}^{\infty} \sum_{q,r,b=1}^{\infty} A_n^q A_L^a A_L^b \delta(\mathbf{k}_q + \mathbf{k}_r + \mathbf{k}_b - \mathbf{k}_p) C_3(m,n,L,a,p,q,r,b), \tag{43}
\]

where \( C_0, C_1, C_2, \) and \( C_3 \) are coefficients of the amplitude equations. The \( \delta \) functions are defined by:
\[
\delta(\mathbf{k}_q + \mathbf{k}_r - \mathbf{k}_p) = \begin{cases} 
0, & \mathbf{k}_q + \mathbf{k}_r - \mathbf{k}_p \neq 0 \\
1, & \mathbf{k}_q + \mathbf{k}_r - \mathbf{k}_p = 0
\end{cases}, \tag{44}
\]
\[
\delta(\mathbf{k}_q + \mathbf{k}_r + \mathbf{k}_b - \mathbf{k}_p) = \begin{cases} 
0, & \mathbf{k}_q + \mathbf{k}_r + \mathbf{k}_b - \mathbf{k}_p \neq 0 \\
1, & \mathbf{k}_q + \mathbf{k}_r + \mathbf{k}_b - \mathbf{k}_p = 0
\end{cases}. \tag{45}
\]

It is, of course, highly desirable to reduce the infinite number of nonlinear coupled ordinary differential equations equation (43) to a finite set of equations. This will be achieved by separating the set \( K \) of eigenmodes \( \mathbf{f}_n^q \) in two subsets \( K_C \) and \( K_S \) so that:
\[
K = K_C \cup K_S. \tag{46}
\]

The subset \( K_C \) contains the critical eigen-modes with a zero growth rate \( \text{Re} \left( s_m^p \right) = 0 \) while the set \( K_S \) is formed by the ensemble of stable modes characterized by a negative growth rate \( \text{Re} \left( s_m^p \right) < 0 \).

The critical eigenmodes \( \mathbf{f}_n^q \) are the ones corresponding to \( |\mathbf{k}_q| = k_c \). Since the intention is the study of the stability of hexagonal and roll planforms (which are the most usual patterns observed experimentally), only the six wave vectors \( \pm \mathbf{k}_q \) \( (q = 1, 2, 3) \) distributed on a circumference of radius
and making an angle of \(60^\circ\) between each other are considered. Solely considering one of the wave-vectors and its negative one, corresponds to the roll planform with axis perpendicular to the selected wave-vector. Including all the wave-vectors represented in figure 1 corresponds to the hexagonal planform. It follows that the ensemble \(K_C\) is formed by the six eigenmodes (see figure 1).

Considering each of wave-vectors represented in figure 1. The set of critical wave vectors 

The subset \(K_S\) is constituted by all the stable eigenmodes \(\mathbf{f}_n^q\), \(q \in \{1, 2, 3, \ldots\}\), \(\mathbf{k}_q \in \mathbb{R}^2\). In a weakly nonlinear regime, these stable eigen-modes are rapidly relaxing from which it follows that the amplitudes in equation (43) corresponding to these stable modes can be notably simplified. It is justified to neglect the terms in equation (43) containing the time derivative because these modes are quickly damped so that, finally, we are left with the only nonlinear terms in the critical amplitudes. Under the above assumptions, the following relation between the stable and critical amplitudes is obtained:

\[
A_{m,p}^P C_0(m,p) = -\varepsilon \left( A_{1}^P C_1(m,1,p) + \right.
\]

\[+ \beta \sum_{q,r=1}^{\infty} A_q^A_r^C \delta(\mathbf{k}_q - \mathbf{k}_r - \mathbf{k}_p) C_2(m,1,1,p,q,r) + \]

\[- \alpha \sum_{q,r,b=1}^{\infty} A_q^A_r^A_b^C \delta(\mathbf{k}_q + \mathbf{k}_r + \mathbf{k}_b - \mathbf{k}_p) C_3(m,1,1,p,q,r,b) \]

where \(\mathbf{k}_p \in K_S\). The wave vectors of ensemble \(K_S\) must satisfy relations \(\mathbf{k}_p \in K_C\), \(\mathbf{k}_p = \mathbf{k}_q + \mathbf{k}_r\), or \(\mathbf{k}_p = \mathbf{k}_q + \mathbf{k}_r + \mathbf{k}_b\) in order to allow the right-hand side of equation (48) has a nonzero value:

\[
K_S = K_C \cup \{ \mathbf{k}_p | \mathbf{k}_p = \mathbf{k}_q + \mathbf{k}_r, \mathbf{k}_q, \mathbf{k}_r \in K_C \} \cup \{ \mathbf{k}_p | \mathbf{k}_p = \mathbf{k}_q + \mathbf{k}_r + \mathbf{k}_b, \mathbf{k}_q, \mathbf{k}_r, \mathbf{k}_b \in K_C \}. \quad (48)
\]

It follows from the above considerations that the infinite number of ordinary differential equations (43) reduces to a finite number of differential equations taking the form:
After substituting relation (47) in equation (49) and omitting terms of order higher than four (this is justified as one remains in the weakly nonlinear regime), the following set of equations for the amplitude are obtained, after use has been made of property (29) expressing that the solutions \( q^n_f \) are real:

\[
\frac{dA_i}{dt} = a_1 A_1 + a_2 \bar{A}_2 \bar{A}_3 + a_3 A_1 |A_1|^2 + a_4 \left( A_1 |A_2|^2 + A_1 |A_3|^2 \right) + a_5 \left( A_1 \right)^2 \bar{A}_2 A_3 + a_6 |A_2|^2 \bar{A}_2 A_3 + a_7 \left( |A_2|^2 \bar{A}_2 A_3 + |A_3|^2 \bar{A}_2 A_3 \right) 
\]

\[
\frac{dA_2}{dt} = a_1 A_2 + a_2 \bar{A}_3 A_1 + a_3 A_2 |A_2|^2 + a_4 \left( A_2 |A_3|^2 + A_2 |A_1|^2 \right) + a_5 \left( A_2 \right)^2 A_3 A_1 + a_6 |A_3|^2 \bar{A}_3 A_1 + a_7 \left( |A_3|^2 \bar{A}_3 A_1 + |A_1|^2 \bar{A}_3 A_1 \right) . 
\]

\[
\frac{dA_3}{dt} = a_1 A_3 + a_2 \bar{A}_1 A_2 + a_3 A_3 |A_3|^2 + a_4 \left( A_3 |A_1|^2 + A_3 |A_2|^2 \right) + a_5 \left( A_3 \right)^2 A_1 A_2 + a_6 |A_1|^2 \bar{A}_1 A_2 + a_7 \left( |A_1|^2 \bar{A}_1 A_2 + |A_2|^2 \bar{A}_1 A_2 \right) . 
\]

The set of equations (50), usually is referred to as the Landau equations. The coefficients \( a_1 \) to \( a_8 \) are dependent generally on the rheological properties and \( \varepsilon \):

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7 \\
a_8
\end{bmatrix} = \begin{bmatrix}
0 & 0 & b_{13} & 0 \\
0 & b_{21} & b_{22} & b_{23} & b_{24} \\
0 & b_{31} & b_{32} & b_{33} & b_{34} \\
0 & b_{41} & b_{42} & b_{43} & b_{44} \\
0 & b_{51} & b_{52} & 0 & b_{54} \\
0 & b_{61} & b_{62} & 0 & b_{64} \\
0 & b_{71} & b_{72} & 0 & b_{74} \\
\end{bmatrix} \begin{bmatrix}
1 \\
\alpha \\
\varepsilon \\
\alpha \varepsilon
\end{bmatrix} .
\]

where \( \alpha \) is the shear thinning parameter defined in (21). The coefficients \( b_{i,j} \), \( i=\{1,2,...,7\} \), \( j=\{1,2,3,4\} \) are dependent on the Prandtl number.

### 4.2 Hexagons and rolls patterns

The amplitude equations (50) can be rewritten in the compact form:

\[
\frac{dA_i}{dt} = g(A_i) .
\]

whose steady solutions are obtained by setting \( g(A_i) = 0 \), their stability is determined by the sign of the eigenvalues \( \sigma_i \) of the \( 6\times6 \) Jacobian matrix \( J (= \partial g/\partial A_i) \):

\[
\begin{bmatrix}
1 \\
\alpha \\
\varepsilon \\
\alpha \varepsilon
\end{bmatrix} .
\]
\[ J A_i = \sigma_i A_i. \]  

(53)

Now the stability of the most usually observed patterns, i.e. rolls, and hexagons are discussed.

### 4.2.1 Rolls

A solution of equation (50) corresponding to steady rolls is:

\[ A_1 \neq 0, A_2 = A_3 = 0. \]  

(54)

This follows to:

\[ |A_1| = \sqrt{-\frac{a_1}{a_3}} = \sqrt{-\frac{\varepsilon}{b_{3,1} + b_{3,2}a + b_{3,3}e + b_{3,4}ae}}. \]  

(55)

To determine the stability of the solutions, one calculates the 6 eigenvalues which must be negative for stable rolls. It is found that:

\[ \sigma_1 = 0, \]

\[ \sigma_2 = -2a_1, \]

\[ \sigma_{3,4} = \frac{a_3\left(a_1a_3 - a_4a_1\right) - \left(a_3a_2 - a_1a_7\right)\sqrt{-a_3a_1}}{a_3^2}, \]  

(56)

\[ \sigma_{5,6} = \frac{a_3\left(a_1a_3 - a_4a_1\right) + \left(a_3a_2 - a_1a_7\right)\sqrt{-a_3a_1}}{a_3^2}. \]

For stability of rolls the following condition is obtained:

\[ a_3^2 - a_3a_4 + a_7\sqrt{-a_3a_1} < 0. \]  

(57)

### 4.2.2 Hexagons

The solutions representing hexagons are of the form

\[ A_1 = A_2 = A_3, \]  

(58)

This follows to a third order equation for determination of \( A_1 \):

\[ 0 = \left(a_5 + a_6 + 2a_7\right)A_1^3 + \left(a_3 + 2a_4\right)A_1^2 + a_2A_1 + a_1. \]  

(59)

To determine the stability of the solutions, one calculates the 6 eigenvalues which must be negative for stable hexagons. This procedure is the same as calculating eigen values of rolls and is not presented here.

### 4.3 Domain of stability

The current calculations are based on the free–free boundary conditions only. The stability picture is best illustrated in the \((\varepsilon, \alpha)\)-plane. Figure 2 shows typical regions of existence of roll and hexagonal patterns for a fluid with \(Pr = 7\). In the figure, H, and R indicate stable hexagonal and roll regions, respectively. Also, H/R indicates regions where the both roll and hexagon patterns are stable. It can be observed that by increasing the degree of shear thinning, rolls will be replaced by hexagons. Regions indicated by ‘Not valid’ representing regions which shear rates grow unlimitedly and therefore contradict the main assumption of equation (20).
5 Conclusion

The objective of the present work is to present a mixed, numerical and analytical, 3-D description of the coupled Rayleigh-Bénard instability problem for some classes of shear thinning fluids. The analysis covers linear and weakly nonlinear situations. The Carreau-Bird rheological model is used in order to illustrate the behaviour of shear thinning fluids. In the present work, only instabilities in fluid layers with an upper and lower flat interface have been studied. It would, of course, be interesting to relax these two restrictions in the future.

A conclusion to be stressed and drawn from the linear analysis is that the critical Rayleigh number and wave number remain unchanged for shear thinning fluids with respect to Newtonian fluids. Another important result, although not new, is that overstability is absent at the instability of the onset of conduction.

The 3-D weakly nonlinear description is essentially based on the amplitudes method. Our nonlinear analysis was restricted to steady convection, for the sake of simplicity. Work relating to nonlinear description of oscillatory instabilities is presently under progress. We have discussed in detail the stability of the two patterns which are usually displayed in experimental observations, namely rolls and hexagons. The most relevant results of present investigation can be summarized as follows:

i) At low degree of shear thinning behavior (defined by the shear thinning parameter in (21)) the rolls are the stable pattern at the onset of stationary convection.

ii) When the level of shear thinning behavior exceeds a certain value only hexagonal pattern are stable at the onset of stationary convection.

The later result is in contrast to the Newtonian case, where only rolls are predicted to be stable.

References

[1] Pierre C St and Tien C 1963 Experimental investigation of natural convection heat transfer in confined space for non-Newtonian fluid Can J. Chem. Eng. 41 122
[2] Tsuei H S and Tien C 1973 Free convection heat transfer in a horizontal layer of non-Newtonian fluid Can J. Chem. Eng. 51 249
[3] Liang S F and Acrivos A 1970 Experiments on buoyancy driven convection in non-Newtonian fluids Rheol. Acta 9 447
[4] Khayat R E 1996 Chaos in the thermal convection of weakly shear thinning fluids J. Non-Newtonian Fluid Mech. 63 153
[5] Bird R B, Armstrong R C and Hassager O 1987 Dynamics of polymeric liquids Vol. 1 2nd edn. *John Wiley & Sons*
[6] Shenoy A V and Mashelkar R A 1982 Thermal convection in non-Newtonian fluids, Advances in heat transfer Vol. 15 143
[7] Larson R G 1992 Instabilities in viscoelastic flows *Rheol. Acta* 31 213
[8] Eltayeb I A 1977 Nonlinear thermal convection in an viscoelastic layer heated from below *Proc. R. Soc. Lond A* 356 161
[9] Rosenblat S 1986 Thermal convection of a viscoelastic fluid *J. Non-Newtonian Fluid Mech.* 21 201
[10] Martinez-Mardones J and Perez-Garcia C 1992 Bifurcation analysis and amplitude equations for viscoelastic convective fluids *Il Nuovo Cimento* 14 961
[11] Martinez-Mardones J, Tienmann R and Walgraef D 1996 Amplitude equations and pattern selection in viscoelastic convection *Phys. Rev. E* 54 1478
[12] Harder H 1991 Numerical simulation of thermal convection with Maxwellian viscoelasticity *J. Non-Newtonian Fluid Mech.* 36 67
[13] Khayat R E 1994 Chaos and overstability in the thermal convection of viscoelastic fluids *J. Non-Newtonian Fluid Mech.* 53 227
[14] Khayat R E 1995 Nonlinear overstability in the thermal convection of viscoelastic fluids *J. Non-Newtonian Fluid Mech.* 58 331
[15] Khayat R E 1995 Fluid elasticity and the onset of chaos in thermal convection *Phys. Rev. E* 51 380
[16] Park H M, Lee H S 1996 Hopf bifurcation of viscoelastic fluids heated from below *J. Non-Newtonian Fluid Mech.* 66 1
[17] Parmentier P, Lebon G and Reginer V 2000 Weakly nonlinear analysis of Bénard-Marangoni instability in viscoelastic fluids *J. Non-Newtonian Fluid Mech* 89 63
[18] Li Z and Khayat R E 2005 Finite-amplitude Rayleigh-Bénard convection and pattern selection for viscoelastic fluids *J. Fluid Mech.* 529 221
[19] Zhang J, Vola D and Frigaard I A 2006 Yield stress effects on Rayleigh-Bénard convection *J. Fluid Mech.* 556 389
[20] Lorenz E N 1963 Deterministic nonperiodic flows *J. Atoms Sci* 20 130
[21] Friedman B 1956 Principles and techniques of applied mathematics *John Wiley & Sons*
[22] Eckhaus W 1965 Studies in nonlinear stability theory *Springer*
[23] Newell A C, Passot T, Lega J 1993 Order parameter equations for patterns *Anni. Rev. Fluid Mech.* 25 399
[24] Cross M C, Hohenberg P C 1993 Pattern formation outside equilibrium *Rev. Mod. Phys.* 65 851