Universal Rational Parametrizations and Toric Varieties

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Abstract. This note proves the existence of universal rational parametrizations. The description involves homogeneous coordinates on a toric variety coming from a lattice polytope. We first describe how smooth toric varieties lead to universal rational parametrizations of certain projective varieties. We give numerous examples and then discuss what happens in the singular case. We also describe rational maps to smooth toric varieties.

1. Introduction

In geometric modeling rational curves and surfaces are widely used in the form of Bézier curves and surfaces or simple low-degree surfaces, e.g., various quadrics, torus surfaces, Dupin cyclides etc. Construction of curve arcs and patches on a given surface with the lowest possible parametrization degree is an important task. For instance this may help to solve data conversion problems which arise when translating from traditional solid modeling systems that deal with such simple surfaces to NURBS-based systems.

It follows that there is a need to understand all possible parametrizations of a given curve or surface. Is it somehow possible to find a “best” parametrization? In the case of toric surfaces (and, more generally, projective toric varieties of any dimension), this paper will offer one answer to this question, which we call a universal rational parametrization.

To illustrate what we mean by this, we give two examples of surfaces with universal rational parametrizations.

Example 1.1. Consider a quadric surface $Q$ given by the homogeneous equation $u_0u_3 = u_1u_2$ in projective space $\mathbb{P}^3$. Any rational parametrization of $Q$ can be represented by a collection of polynomials $H = (h_0, h_2, h_2, h_3)$ such that

\begin{equation}
    h_0h_3 = h_1h_2 \quad \text{and} \quad \gcd(h_0, h_2, h_2, h_3) = 1.
\end{equation}

One obvious rational parametrization is given by

\begin{equation}
    P(x_1, x_2, x_3, x_4) = (x_2x_3, x_1x_3, x_2x_4, x_1x_4)
\end{equation}

since $(x_2x_3)(x_1x_4) = (x_1x_3)(x_2x_4)$.
Now suppose that we have a collection of polynomials

\[(1.3) \quad F = (f_1, f_2, f_3, f_4), \quad \gcd(f_1, f_2) = \gcd(f_3, f_4) = 1.\]

Then let \(H = P \circ F\), i.e.,

\[H = (h_0, h_1, h_2, h_3) = (f_2f_3, f_1f_3, f_2f_4, f_1f_4).\]

It is straightforward to show that \(H\) satisfies (1.1). In other words, from the parametrization \(P\) of (1.2), we get infinitely many others by composing with any \(F\) satisfying (1.3).

But even more is true: Theorem 3.5 implies that all \(H\)'s satisfying (1.1) arise in this way. In other words, such an \(H\) is of the form \(H = P \circ F\) for some \(F\) as in (1.3). Furthermore, although \(F\) is not unique, Theorem 3.5 describes the non-uniqueness precisely: given one \(F\) with \(H = P \circ F\), then all others are of the form

\[(\lambda f_1, \lambda f_2, \lambda^{-1} f_3, \lambda^{-1} f_4)\]

for some nonzero scalar \(\lambda\).

In the language of Theorem 3.5, we say that \(P\) from (1.2) is a universal rational parametrization of the quadric \(Q\). The key property of the quadric \(Q\) is that it comes from \(\mathbb{P}^1 \times \mathbb{P}^1\). If \(x_1, x_2\) are homogeneous coordinates on the first \(\mathbb{P}^1\) and \(x_3, x_4\) are homogeneous coordinates on the second, then \(P\) induces an embedding

\[\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3\]

whose image is \(Q\).

Here is the second example of a universal rational parametrization.

**Example 1.2.** Consider the Steiner surface \(S\) in \(\mathbb{P}^3\), which is defined in homogeneous coordinates by the equation

\[u_1^2u_2^2 + u_2^2u_3^2 + u_3^2u_1^2 = u_0u_1u_2u_3.\]

Note that \(S\) is not a smooth surface—its singular locus consists of the three lines

\[(1.4) \quad u_1 = u_2 = 0, \quad u_2 = u_3 = 0, \quad u_3 = u_1 = 0.\]

A rational parametrization of \(S\) consists of polynomials \(H = (h_0, h_1, h_2, h_3)\) such that

\[(1.5) \quad h_1^2h_2^2 + h_2^2h_3^2 + h_3^2h_1^2 = h_0h_1h_2h_3 \quad \text{and} \quad \gcd(h_0, h_1, h_2, h_3) = 1.\]

One can easily show that

\[(1.6) \quad P(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2, x_1x_2, x_2x_3, x_3x_1)\]

is a parametrization of \(S\). Furthermore, given polynomials

\[(1.7) \quad F = (f_1, f_2, f_3), \quad \gcd(f_1, f_2, f_3) = 1,\]

we see that

\[H = P \circ F = (f_1^2 + f_2^2 + f_3^2, f_1f_2, f_2f_3, f_3f_1)\]

satisfies (1.5) and hence is a rational parametrization of \(S\).

In this situation, Theorem 3.5 tells us that all \(H\)'s satisfying (1.5) are of the form \(H = P \circ F\) for some \(F\) satisfying (1.7), provided the image of \(H\) does not lie in the lines (1.4). Furthermore, Theorem 3.5 implies that \(F = (f_1, f_2, f_3)\) is unique up to \(\pm 1\). \(\square\)
By Theorem 3.5, (1.6) is the universal rational parametrization of the Steiner surface \( S \). In this case, the key property of \( S \) is that it came from \( \mathbb{P}^2 \) via the map \( \mathbb{P}^2 \rightarrow S \) induced by (1.6). This map is not an embedding but is birational (i.e., is generically one-to-one). Furthermore, the three lines (1.4) are where the map fails to have an inverse.

Both \( \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \mathbb{P}^2 \) are examples of smooth toric varieties, and the coordinates \( x_1, x_2, x_3, x_4 \) for \( \mathbb{P}^1 \times \mathbb{P}^1 \) and \( x_1, x_2, x_3 \) for \( \mathbb{P}^2 \) are examples of homogeneous coordinates of toric varieties. Hence it makes sense that there should be a toric generalization of these examples. For instance, we will see that the gcd conditions (1.3) and (1.7) are dictated by the data which determines the toric variety.

The paper is organized into six sections as follows:

Section 1: Introduction
Section 2: Background and Related Work
Section 3: Universal Rational Parametrizations (Smooth Case)
Section 3: Universal Rational Parametrizations (Singular Case)
Section 5: Rational Maps to Smooth Toric Varieties
Section 6: Theoretical Justifications

In Section 2 we will describe toric varieties and homogeneous coordinates along with a summary of related work. In Section 3, we give a careful definition of rational parametrization and state Theorems 3.5, our main result about universal rational parametrizations when the toric variety involved is smooth. We also give numerous examples. Then, in Section 4, we discuss Theorem 4.3, which describes what happens when the toric variety is singular. However, in order to prove these results, we need to understand rational maps to smooth toric varieties. This is the subject of Section 5, where the main result is Theorem 5.1. Finally, Section 6 includes proofs of the results stated in Sections 3, 4 and 5.

In this paper, we will work over the complex numbers \( \mathbb{C} \) so that we can apply the tools of algebraic geometry. Let \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} = \{z \in \mathbb{C} | z \neq 0\} \).

Geometric modeling is mostly concerned with real varieties. In practice, many important real surfaces are real parts of complex toric surfaces with possibly non-standard real structures. The results of this paper hold over \( \mathbb{R} \), provided we use the standard real structure on the toric varieties involved. Our results can also be applied, with some straightforward modifications, to the case of non-standard real structures. The details about this situation and the practical issues of using universal rational parametrizations in geometric modeling will be presented elsewhere.

We would like to thank the referee for pointing out a problem in our original version of Theorem 3.5 and for suggesting the current form of Example 3.12.

2. Background Material and Related Work

The concept of universal rational parametrization was introduced at first for nonsingular quadric surfaces under the name of “generalized stereographic projection” in [5]. It was extended to more general rational surfaces in [10] and [11] (see also the recent paper [12]).

Around the same time, homogeneous coordinates for toric varieties where defined by numerous people—see [2] for a complete list. Also important were maps into toric varieties, which were first explored in [7] and [8]. This led the first author to the description of maps to smooth toric varieties given in [1].
The relation between universal rational parametrizations and toric varieties was first realized when the second author defined the toric surface patches in [9]. An account of this may also be found in [13].

2.1. Toric Varieties. In this paper, we will assume that the reader is familiar with the elementary theory of toric varieties, as explained in [4]. A toric variety $X_\Sigma$ is determined by a fan $\Sigma$, which is a collection of cones $\sigma \subset \mathbb{R}^n$ satisfying certain properties. We will assume that the union of the cones in $\Sigma$ is all of $\mathbb{R}^n$. This means that $X_\Sigma$ is a compact toric variety.

Among the cones of $\Sigma$, the edges (= one-dimensional cones) play a special role. Suppose that the edges of $\Sigma$ are $\rho_1, \ldots, \rho_r$. Then each $\rho_i$ corresponds to $x_i, n_i$ and $D_i$, where:

- The variable $x_i$ is in the homogeneous coordinate ring of $X_\Sigma$.
- The vector $n_i \in \mathbb{Z}^n$ is the first nonzero integer vector in $\rho_i$.
- The subvariety $D_i \subset X_\Sigma$ is defined by $x_i = 0$.

We think of $x_1, \ldots, x_r$ as coordinates on $\mathbb{C}^r$. We can use the $x_i$ to construct the toric variety $X_\Sigma$ as follows. Let

$$(2.1) \quad G = \{(\mu_1, \ldots, \mu_r) \in (\mathbb{C}^*)^r \mid \prod_{i=1}^r \mu_i^{(m,n_i)} = 1 \text{ for all } m \in \mathbb{Z}^n\},$$

where $\langle , \rangle$ is dot product on $\mathbb{Z}^n$. This is a subgroup of $(\mathbb{C}^*)^r$ and hence acts on $\mathbb{C}^r$ in the usual way. Also, for each cone $\sigma \in \Sigma$, let

$$x^\sigma = \prod_{n_i \not\in \sigma} x_i$$

be the product of all variables corresponding to edges not lying in $\sigma$. Finally, let the exceptional set $Z \subset \mathbb{C}^r$ be defined by the equations $x^\sigma = 0$ for all $\sigma \in \Sigma$. Then we get the quotient representation

$$(2.2) \quad X_\Sigma = (\mathbb{C}^r \setminus Z)/G.$$  

As explained in [4], this generalizes the quotient construction

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*.$$  

One consequence of (2.2) is that we have a natural map $\mathbb{C}^r \setminus Z \to X_\Sigma$. We can think of this as a rational map from $\mathbb{C}^r$ to $X_\Sigma$ which is not defined on the exceptional set $Z$. We will write this as

$$(2.3) \quad \pi : \mathbb{C}^r \dashrightarrow X_\Sigma,$$

where the broken arrow means that we have a rational map. The map (2.3) will play an important role in what follows.

2.2. Polytopes. In Sections 3 and 4, we will consider the projective toric variety $X_\Delta$ determined by an $n$-dimensional lattice polytope $\Delta \subset \mathbb{R}^n$. The idea is that for each face of $\Delta$, we get the cone generated by the inward-pointing normals of the facets of $\Delta$ containing the face. This gives the normal fan $\Sigma_\Delta$ of $\Delta$, and the corresponding toric variety $X_{\Sigma_\Delta}$ is denoted $X_\Delta$.

Observe that edges of the normal fan correspond to facets of $\Delta$. Hence the homogeneous coordinates $x_1, \ldots, x_r$ correspond to the facets of $\Delta$. For this reason, we call $x_1, \ldots, x_r$ the facet variables of the polytope $\Delta$. 

We can use $\Delta$ to obtain some interesting monomials in the facet variables. Represent $\Delta$ as the intersection
\begin{equation}
\Delta = \bigcap_{i=1}^{r} \{ m \in \mathbb{R}^n \mid \langle m, n_i \rangle \geq -a_i \}
\end{equation}
of closed half-spaces. This gives the following monomials and polynomials.

**Definition 2.1.** For each lattice point $m \in \Delta \cap \mathbb{Z}^n$, we define the **$\Delta$-monomial** $x^m$ to be
\begin{equation}
x^m = \prod_{i=1}^{r} x_i^{\langle m, n_i \rangle + a_i}.
\end{equation}

We also define $S_\Delta$ to be the linear span of the set of $\Delta$-monomials. Thus
\[ S_\Delta = \text{Span}(x^m \mid m \in \Delta \cap \mathbb{Z}^n). \]

Since the $i$th facet is defined by $\langle m, n_i \rangle + a_i = 0$ and $\langle m, n_i \rangle + a_i \geq 0$ on $\Delta$ ($n_i$ points inward), we see that the exponent of $x_i$ measures the “distance” (in the lattice sense) from $m$ to the $i$th facet.

Here is an example of facet variables and $\Delta$-monomials.

**Example 2.2.** Consider the polytope $\Delta$

\[
\begin{array}{ccc}
x_1 & & x_4 \\
& x_5 & \\
x_2 & x_3 & \\
\end{array}
\]

with vertices $(1, 1), (-1, 1), (-1, 0), (0, -1), (1, -1)$. In terms of (2.4), we have $a_1 = \cdots = a_5 = 1$. This gives a toric surface $X_\Delta$ with variables $x_1, \ldots, x_5$ as indicated in the picture. The 8 points $m \in \Delta \cap \mathbb{Z}^2$ give the following 8 $\Delta$-monomials $x^m$:
\[
\begin{align*}
x_2 x_3 x_4^2, & \quad x_1 x_2^2 x_4^3 x_5,
\end{align*}
\]
\[
\begin{align*}
x_3 x_4^2 x_5, & \quad x_1 x_2 x_3 x_4 x_5,
\end{align*}
\]
\[
\begin{align*}
x_1 x_4 x_5^2, & \quad x_1^2 x_2^2 x_3^2.
\end{align*}
\]

We will say more about this example in Sections 3 and 5. $\square$

We should also mention that polynomials $q \in S_\Delta$ have the following important property: given $\mu = (\mu_1, \ldots, \mu_r)$ in the group $\tilde{G}$ defined in (2.1), one easily sees that
\begin{equation}
q(\mu_1 x_1, \ldots, \mu_r x_r) = \mu_\Delta q(x_1, \ldots, x_r),
\end{equation}
where
\begin{equation}
\mu_\Delta = \prod_{i=1}^{r} \mu_i^{a_i}.
\end{equation}
2.3. Rational Maps to Projective Space. Now pick a collection \( P = (p_0, \ldots, p_s) \) of \( s + 1 \) polynomials in \( S_\Delta \). This gives a rational map
\[
p : \mathbb{C}^r \longrightarrow \mathbb{P}^s.
\]
If \( X \) denotes the Zariski closure of the image, then we write \( p \) as
\[
p : \mathbb{C}^r \longrightarrow X \subset \mathbb{P}^s.
\]
We can relate \( p \) to the toric variety \( X_\Delta \) as follows.

**Proposition 2.3.** In the above situation, the map \( p \) factors \( p = \Pi \circ \pi \), where \( \pi : \mathbb{C}^r \longrightarrow X_\Delta \) is from (2.3) and
\[
\Pi : X_\Delta \longrightarrow X
\]
is a rational map.

**Proof.** Given \( a = (a_1, \ldots, a_r) \in \mathbb{C}^r \setminus \mathbb{Z} \) and \( \mu = (\mu_1, \ldots, \mu_r) \in G \), then we get \( \mu \cdot a = (\mu_1 a_1, \ldots, \mu_r a_r) \). By (2.6), we have
\[
(p_0(\mu \cdot a), \ldots, p_s(\mu \cdot a)) = \mu_\Delta (p_0(a), \ldots, p_s(a)).
\]
This shows that \( p \) induces \( \Pi : (\mathbb{C}^r \setminus \mathbb{Z})/G \longrightarrow \mathbb{P}^s \). By (2.2), we can identify the quotient with \( X_\Sigma \), and the proposition follows. \( \square \)

When \( X_\Delta \) is smooth and \( \Pi : X_\Delta \longrightarrow X \) is sufficiently nice, Theorem 3.5 asserts that \( p \) is a universal rational parametrization. In Section 3, we will use this theorem to explain Examples 1.1 and 1.2.

We will also see in Section 4 that this doesn’t quite work when \( X_\Delta \) is singular. In this case, Theorem 4.3 will show that we get a universal rational parametrization by considering a suitable resolution of singularities.

3. Universal Rational Parametrizations (Smooth Case)

In this section, we will prove the existence of universal rational parametrizations for certain projective varieties which arise naturally from smooth toric varieties associated to polytopes.

3.1. Rational Parametrizations. We first give a definition of rational parametrization which is useful in geometric modeling. Given a projective variety \( Y \subset \mathbb{P}^k \), we define its **affine cone** \( C_Y \subset \mathbb{C}^{k+1} \) to be
\[
C_Y = \pi^{-1}(Y) \cup \{0\} \subset \mathbb{C}^{k+1},
\]
where \( \pi : \mathbb{C}^{k+1} \setminus \{0\} \rightarrow \mathbb{P}^k \) is the usual map. Using this, we can make the following definition.

**Definition 3.1.** Let \( R = \mathbb{C}[y_1, \ldots, y_d] \) be the coordinate ring of \( \mathbb{C}^d \). A **rational parametrization** of a projective variety \( Y \subset \mathbb{P}^s \) consists of \( H = (h_0, \ldots, h_s) \in R^{s+1} \) such that \( \gcd(h_0, \ldots, h_s) = 1 \) and \( H(\mathbb{C}^d) \subset C_Y \).

In this paper, we use the convention that \( \gcd(0, \ldots, 0) = 0 \). Hence the gcd condition implies that the polynomials in a rational parametrization are not all zero. Then \( H(\mathbb{C}^d) \subset C_Y \) implies that \( H : \mathbb{C}^d \rightarrow \mathbb{C}^{s+1} \) induces a rational map
\[
h : \mathbb{C}^d \longrightarrow \mathbb{P}^s
\]
whose image lies in \( Y \). It is important to note that in Definition 3.1, we do not require that \( h : \mathbb{C}^d \longrightarrow Y \) be surjective or have dense image. Thus a rational
parametrization might only parametrize a proper subvariety of $Y$. Also note that the gcd condition of Definition 3.1 implies that two rational parametrizations $H$ and $H'$ give the same rational map to $\mathbb{P}^s$ if and only if $H = cH'$ for $c \neq 0$ in $\mathbb{C}$.

3.2. One Particular Parametrization. Let $\Delta$ be an $n$-dimensional lattice polytope in $\mathbb{R}^n$. This gives the toric variety $X_\Delta$ determined by the normal fan $\Sigma_\Delta$ of $\Delta$. We will assume that $X_\Delta$ is smooth.

By Definition 2.1, the facet variables $x_1, \ldots, x_r$ and the lattice points $\Delta \cap \mathbb{Z}^n$ give rise to the vector space of polynomials

$$S_\Delta = \text{Span}(x^m \mid m \in \Delta \cap \mathbb{Z}^n),$$

where $x^m$ is the $\Delta$-monomial. As in Section 2, a collection of $s + 1$ polynomials

$$(3.1) \quad P = (p_0, \ldots, p_s) \in S_\Delta^{s+1}$$

gives a rational map

$$p : \mathbb{C}^r \dashrightarrow X \subset \mathbb{P}^s$$

where $X$ is the Zariski closure of the image. Then Proposition 2.3 shows that $p = \Pi \circ \pi$, where $\pi : \mathbb{C}^r \dashrightarrow X_\Delta$ is from (2.3) and

$$\Pi : X_\Delta \dashrightarrow X \subset \mathbb{P}^s$$

is a rational map. As already mentioned, the basic idea is that $P$ is a universal rational parametrization when $\Pi$ is sufficiently nice. However, we need to make some further definitions before we can state our main result.

3.3. Sufficiently Nice. We can finally explain when $\Pi : X_\Delta \dashrightarrow X$ is sufficiently nice. Using the above notation, this means the following two things:

- First, $\Pi$ is strictly defined, which means for every $a \in \mathbb{C}^r \setminus Z$, we have $p_i(a) \neq 0$ for some $0 \leq i \leq s$. Using $X_\Delta = (\mathbb{C}^d \setminus Z)/G$ and Proposition 2.3, one can show that this condition ensures that $\Pi$ is defined everywhere. Thus we write $\Pi : X_\Delta \rightarrow X$ when $\Pi$ is strictly defined.
- Second, $\Pi$ is birational, which means that $\Pi$ induces an isomorphism between dense open subsets of $X_\Delta$ and $X$.

When we discuss projections later in the section, we will give several conditions which are equivalent to being strictly defined. However, we will see in Example 3.12 that being strictly defined is in general stronger than just assuming that the rational map $\Pi$ is defined everywhere on $X_\Delta$.

An important observation is that the $p_i$ in (3.1) are relatively prime when $\Pi$ is strictly defined. To prove this, suppose that some nonconstant polynomial $q$ divides the $p_i$. Since the exceptional set $Z \subset \mathbb{C}^r$ has codimension at least 2, we can find $a \in \mathbb{C}^r \setminus Z$ such that $q(a) = 0$. Hence $p_i(a) = 0$ for all $i$, which can’t happen when $\Pi$ is strictly defined. This proves that the $p_i$ are relatively prime. By Definition 3.1, it follows that $P$ is a rational parametrization of $X$.

We also note that being strictly defined implies that $\Pi$ and hence $p$ are onto, i.e., $X$ is the image of $p : \mathbb{C}^r \dashrightarrow \mathbb{P}^s$. This follows because $X_\Delta$ is compact.

Finally, note that when $\Pi$ is birational, there is a nonempty Zariski open subset

$$U \subset X$$

on which $\Pi^{-1}$ is defined. We may assume that $U$ is the maximal such open set.
3.4. ΣΔ-Irreducible Polynomials. The rough idea of a universal rational parametrization is that any rational parametrization $H$ should arise from $P$ by composition with a polynomial map $\mathbb{C}^d \to \mathbb{C}^r$. But if the image of $\mathbb{C}^d \to \mathbb{C}^r$ lies in the exceptional set $Z$, then the composition doesn’t make sense since $p$ is not defined on $Z \subset \mathbb{C}^r$. It follows that we need to exclude certain polynomial maps. The precise definition is as follows. Let $R = \mathbb{C}[y_1, \ldots, y_d]$.

**Definition 3.2.** We say that $F = (f_1, \ldots, f_r) \in R^r$ is $\Sigma\Delta$-irreducible if $\gcd(f_{i_1}, \ldots, f_{i_k}) = 1$ whenever no cone of $\Sigma\Delta$ contains $\rho_{i_1}, \ldots, \rho_{i_k}$.

Because of our gcd convention, Definition 3.2 implies in particular that the edges $\rho_i$ such that $f_i = 0$ all lie in some cone of $\Sigma\Delta$. In the discussion which follows, we will identify $F$ with the polynomial function $\mathbb{C}^d \to \mathbb{C}^r$ it induces.

**Example 3.3.** Consider the toric variety $X_\Delta = \mathbb{P}^2$ coming from the polytope $\Delta$ with vertices $(0, 0), (2, 0), (0, 2)$.

The polytope $\Delta$ is on the left with facet variables $x_1, x_2, x_3$ and the normal fan is on the right with edges $\rho_1, \rho_2, \rho_3$. The only choice for $\rho_{i_1}, \ldots, \rho_{i_k}$ in Definition 3.2 is $\rho_1, \rho_2, \rho_3$, so that $F = (f_1, f_2, f_3)$ is $\Sigma$-irreducible if $\gcd(f_1, f_2, f_3) = 1$. This is the gcd condition (1.7) in Example 1.2.

**Example 3.4.** Let $\Delta$ be the unit square in the plane with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$. This gives $X_\Delta = \mathbb{P}^1 \times \mathbb{P}^1$.

As in the previous example, $\Delta$ and the facet variables $x_1, x_2, x_3, x_4$ are on the left and $\Sigma\Delta$ and the edges are $\rho_1, \rho_2, \rho_3, \rho_4$ are on the right. The minimal choices for $\rho_{i_1}, \ldots, \rho_{i_k}$ in Definition 3.2 are $\rho_1, \rho_2$ and $\rho_3, \rho_4$ (you should check this). Thus $F = (f_1, f_2, f_3, f_4)$ is $\Sigma$-irreducible if $\gcd(f_1, f_2) = \gcd(f_3, f_4) = 1$. This is the gcd condition (1.3) in Example 1.1.

3.5. The Main Result. Before stating our main result, we need to introduce some notation. As above, let $R = \mathbb{C}[y_1, \ldots, y_d]$ be the coordinate ring of $\mathbb{C}^d$. Also, given polynomials $F = (f_1, \ldots, f_r) \in R^r$ and $m \in \Delta \cap \mathbb{Z}^n$, we set

$$f^m = \prod_{i=1}^r f_i^{(m, n_i) + a_i}.$$
Recall the group $G$ from (2.1) and that $\mu \in G$ gives $\mu_\Delta \in \mathbb{C}^*$ defined in (2.7). The map $\mu \mapsto \mu_\Delta$ is a group homomorphism $G \to \mathbb{C}^*$. Let $G_\Delta$ be the kernel of this map. This group will measure the lack of uniqueness in Theorem 3.5. Finally, let $\sum_m \cdot$ denote summation over all $m \in \Delta \cap \mathbb{Z}^n$.

Here is our precise result.

**Theorem 3.5.** Let $P = (p_0, \ldots, p_s) = (\sum_m a_{0m}x^m, \ldots, \sum_m a_{sm}x^m) \in S_\Delta^{s+1}$, where $X_\Delta$ is smooth and $\Pi : X_\Delta \to X$ is strictly defined and birational, with an inverse defined on $U \subset X$ which we assume to be maximal. Then $P$ is a universal rational parametrization of $X \subset \mathbb{P}^s$ in the following sense:

1. If $F = (f_1, \ldots, f_r) \in R^r$ is $\Sigma_\Delta$-irreducible, then
   \[ P \circ F = (\sum_m a_{0m}f^m, \ldots, \sum_m a_{sm}f^m) \in R^{s+1} \]

   is a rational parametrization of $X \subset \mathbb{P}^s$.

2. Conversely, given any rational parametrization $H \in R^{s+1}$ of $X$ whose image meets $U \subset X$, there is a $\Sigma_\Delta$-irreducible $F = (f_1, \ldots, f_r) \in R^r$ such that $H = P \circ F$.

3. If $F$ and $F'$ are $\Sigma_\Delta$-irreducible, then $P \circ F = P \circ F'$ as rational parametrizations if and only if $F' = \mu \cdot F$ for some $\mu \in G_\Delta$.

The proof will be given in Section 6. Here is a corollary of Theorem 3.5.

**Corollary 3.6.** Assume the same hypothesis as Theorem 3.5 and suppose that $H' = (h'_0, \ldots, h'_r) \in R^{s+1}$ gives a rational map $\mathbb{C}^d \dashrightarrow \mathbb{P}^s$ whose image lies in $X$ and meets $U$. Then there is a polynomial $q \in R$ and a $\Sigma_\Delta$-irreducible $F = (f_1, \ldots, f_r) \in R^r$ such that

\[ H' = q \cdot P \circ F. \]

**Proof.** Write $H' = q \cdot H$, where the entries of $H$ are relatively prime. Since $H$ is a rational parametrization, we are done by Theorem 3.5. \hfill \Box

### 3.6. Embeddings.

In order for Theorem 3.5 to be useful, we need to have a good supply of parametrizations $P = (p_0, \ldots, p_s) \in S_\Delta^{s+1}$ which satisfy the hypotheses of the theorem. The first crucial observation is that since $X_\Delta$ is a smooth toric variety, it is a standard result that the collection of all $\Delta$-monomials gives a projective embedding (see [6, Sec. 3.4]).

This means the following. Suppose that $\Delta \cap \mathbb{Z}^n = \{m_0, \ldots, m_r\}$ and let

\[ P_\Delta = (x^{m_0}, \ldots, x^{m_r}) \]

Then $P_\Delta$ induces $p_\Delta : \mathbb{C}^r \dashrightarrow \mathbb{P}^d$, and in the factorization $p_\Delta = \pi \circ \Pi_\Delta$ of Proposition 2.3, the map $\Pi : X_\Delta \to \mathbb{P}^d$ is an embedding. Hence we can write $X_\Delta \subset \mathbb{P}^d$.

All of the hypotheses of Theorem 3.5 are satisfied in this situation, and the open set $U$ is all of $X = X_\Delta$. Thus $P_\Delta$ is a universal rational parametrization in the strong sense that every rational parametrization is of the form $P_\Delta \circ F$ for some $\Sigma_\Delta$-irreducible $F$.

Here are two examples of this result.
Example 3.7. Let \( \Delta \) be the unit square from Example 3.4.

The labeling of \( x_1, x_2, x_3, x_4 \) is consistent with Example 3.4. In terms of (2.4), \( a_1 = a_3 = 0 \) and \( a_2 = a_4 = 1 \). Then

\[
P_\Delta = (x_2x_3, x_1x_3, x_2x_4, x_1x_4) \in S^4_\Delta
\]
gives a universal rational parametrization of its image in \( \mathbb{P}^3 \), which is the quadric \( Q \) of Example 1.1. This means that any parametrization of \( Q \) is of the form \( P_\Delta \circ F \), where \( F = (f_1, f_2, f_3, f_4) \) satisfies the gcd condition worked out in Example 3.4.

To study uniqueness, we need to compute

\[
G = \{ (\mu_1, \mu_2, \mu_3, \mu_4) \in (\mathbb{C}^*)^4 \mid \prod_{i=1}^4 \mu_i^{(m, n_i)} = 1 \text{ for all } m \in \mathbb{Z}^4 \}.
\]

Since it suffices to use \( m = (1, 0) \) and \( (0, 1) \), we see that

\[
G = \{ (\mu_1, \mu_2, \mu_3, \mu_4) \in (\mathbb{C}^*)^4 \mid \mu_1\mu_2^{-1} = \mu_3\mu_4^{-1} = 1 \},
\]

and it follows that

\[
G_\Delta = \{ (\lambda, \lambda, \lambda^{-1}, \lambda^{-1}) \mid \lambda \in \mathbb{C}^* \}.
\]

Thus, when we write a rational parametrization of \( Q \) as \( P_\Delta \circ F \), we see that \( F = (f_1, f_2, f_3, f_4) \) is unique up to

\[
(\lambda f_1, \lambda f_2, \lambda^{-1} f_3, \lambda^{-1} f_4).
\]

for some nonzero scalar \( \lambda \). It follows that we obtain precisely the description given in Example 1.1.

Here is how Theorem 3.5 applies to one of our earlier examples.

Example 3.8. Consider the toric variety \( X_\Delta \) of the polytope \( \Delta \) with vertices \( (1, 1), (-1, 1), (-1, 0), (0, -1), (1, -1) \) from Example 2.2.
As usual, the polytope is on the left and the normal fan of $\Delta$ is on the right. One can show that $X_\Delta$ is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point.

In terms of (2.4) and the above labeling, we have $a_1 = \cdots = a_5 = 1$. The 8 points of $\Delta \cap \mathbb{Z}^2$ give an embedding of $X_\Delta$ into $\mathbb{P}^7$. It follows that

$$P_\Delta = (x_2x_3^2x_4^5, x_1x_2^2x_3^2x_4, x_1x_2^2x_3^2x_5, x_1x_2x_3x_4x_5, x_1x_2x_3x_5x_6, x_1x_2x_4x_5, x_1x_2^2x_3^2, x_1x_2x_2x_2) \in S_\Delta^8,$$

is the universal rational parametrization of $X_\Delta$ by Theorem 3.5.

Let's work out what this means. According to Definition 3.2, $F = (f_1, \ldots, f_5)$ is $\Sigma_\Delta$-irreducible provided that

$$\gcd(f_1, f_3) = \gcd(f_1, f_4) = \gcd(f_2, f_4) = \gcd(f_2, f_5) = \gcd(f_3, f_5) = 1.$$

Then any rational parametrization of $X_\Sigma$ is of the form $P_\Delta \circ F$ for some $F$ satisfying this gcd condition. To determine the lack of uniqueness, we need to compute the group $G$. Using the methods of Example 3.7, one obtains

$$G = \{(\lambda, \mu, \nu, \lambda\mu, \mu\nu) \mid \lambda, \mu, \nu \in \mathbb{C}^*\},$$

and then $a_1 = \cdots = a_5 = 1$ imply that

$$G_\Delta = \{(\lambda, \mu, \nu, \lambda\mu, \mu\nu) \mid \lambda, \mu, \nu \in \mathbb{C}^*, \lambda^2\mu^3\nu^2 = 1\}.$$

Thus rational parametrizations of $X_\Sigma$ are all of the form $P_\Delta \circ F$, where $F$ is unique up to $(\lambda, \mu, \nu, \lambda\mu, \mu\nu) \cdot F$ for $\lambda^2\mu^3\nu^2 = 1$.  

3.7. Projections. Although $P_\Delta = (x^{m_0}, \ldots, x^{m_e})$ from (3.2) always gives a universal rational parametrization, it is rarely useful in practice since it usually gives an embedding into a projective space of high dimension. An important observation is that we can think of the general case

$$P = (p_0, \ldots, p_s) = (\sum_{i=0}^\ell a_{0i}x^{m_i}, \ldots, \sum_{i=0}^\ell a_{si}x^{m_i}) \in S_\Delta^{s+1}$$

in terms of projections. Let $\mathbb{P}^\ell$ be a projective space with homogeneous coordinates $z_0, \ldots, z_\ell$. Then the $s + 1$ linear forms $\sum_{i=0}^\ell a_{ji}z_i$ define a projection $\mathbb{P}^\ell \dashrightarrow \mathbb{P}^s$ defined by

$$(z_0, \ldots, z_\ell) \mapsto (\sum_{i=0}^\ell a_{0i}z_i, \ldots, \sum_{i=0}^\ell a_{si}z_i).$$

The center $L \subset \mathbb{P}^\ell$ of this projection is defined by $\sum_{i=0}^\ell a_{ji}z_i = 0$ for $j = 0, \ldots, s$. This tells us where the projection is not defined.

If we compose this projection with $p_\Delta : \mathbb{C}^r \dashrightarrow \mathbb{P}^\ell$, then we get the rational map $p : \mathbb{C}^r \dashrightarrow \mathbb{P}^s$ induced by $P$. Furthermore, since the image of $p_\Delta$ is $X_\Delta \subset \mathbb{P}^\ell$, it follows that the variety $X$ parametrized by $P$ is the image of $X_\Delta$ under the projection.

From this point of view, we can think of $\Pi$ as a projection. It is then straightforward to check that $\Pi$ is strictly defined if and only if $X_\Delta$ is disjoint from the center $L$ of the projection. (For more sophisticated readers, we point out that being strictly defined is equivalent to the assertion that the linear system on $X_\Delta$ spanned by the $p_i$ has no base points. One can also show that $X_\Delta$ is the normalization of $X$ when $\Pi$ is strictly defined and birational.)

Let's give an example from geometric modeling which involves the projection of a toric variety.
EXAMPLE 3.9. Consider the toric variety $X_\Delta = \mathbb{P}^2$, where $\Delta$ is the polytope from Example 3.3. In terms of (2.4), we have $a_1 = a_2 = 0$ and $a_3 = 2$. The 6 points of $\Delta \cap \mathbb{Z}^2$ define

$$P_\Delta = (x_1^2, x_2^2, x_3^2, x_1 x_2, x_2 x_3, x_3 x_1).$$

This gives the usual Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$.

The composition of this map with the projection $\mathbb{P}^5 \dasharrow \mathbb{P}^3$ defined by

$$(3.3) \quad (z_0, z_1, z_2, z_3, z_4, z_5) \mapsto (z_0 + z_1, z_2, z_3, z_4, z_5)$$

gives a rational parametrization

$$(3.4) \quad P = (x_1^2 + x_2^2 + x_3^2, x_1 x_2, x_2 x_3, x_3 x_1)$$

of the Steiner surface $S \subset \mathbb{P}^3$ defined by

$$u_1^2 u_2 - u_2 u_3 + u_3^2 u_1 = u_0 u_1 u_2 u_3,$$

where $u_0, u_1, u_2, u_3$ are homogeneous coordinates on $\mathbb{P}^3$. We saw this equation earlier in Example 1.2.

It is easy to check that the center of the projection (3.3) is disjoint from $X_\Delta$. Thus the map $\Pi : X_\Delta \to S$ is strictly defined. Furthermore, since

$$x_1^2 = \frac{(x_1 x_2)(x_3 x_1)}{x_2 x_3} = \frac{u_1 u_3}{u_2},$$

one easily sees that $\Pi : X_\Delta \to S$ is birational with inverse

$$\Pi^{-1}(u_0, u_1, u_2, u_3) = \left(\frac{u_1 u_3}{u_2}, \frac{u_1 u_2}{u_3}, \frac{u_2 u_3}{u_1}, u_1, u_2, u_3\right)
= (u_1^2 u_2^3, u_2^2 u_3^2, u_2^2 u_3^2, u_1^2 u_2 u_3, u_1 u_2 u_3^2, u_1 u_2 u_3^2).$$

Also notice that $\Pi^{-1}$ is defined on the complement of the three lines $u_1 = u_2 = 0, u_2 = u_3 = 0, u_3 = u_1 = 0$.

By Theorem 3.5, (3.4) is the universal rational parametrization of the Steiner surface $S$. It follows that if $H$ is a rational parametrization of $S$ whose image is not contained in any of the above three lines, then $H = P \circ F$, where $F = (f_1, f_2, f_3)$. Furthermore, we know that $F$ is $\Sigma_\Delta$-irreducible, which by Example 3.3 means $\gcd(f_1, f_2, f_3) = 1$.

Finally, we know that $G = \{ (\lambda, \lambda, \lambda) \mid \lambda \in \mathbb{C}^* \} \simeq \mathbb{C}^*$ in this case. Since $a_1 = a_2 = 0$ and $a_3 = 2$, we see that $\mu_\Delta = \lambda^2$ when $\mu = (\lambda, \lambda, \lambda)$. It follows that the kernel of $\mu \mapsto \mu_\Delta$ is $\pm(1, 1, 1)$, so that in $H = P \circ F$, $F$ is unique up to $\pm 1$. Hence we recover the description of the rational parametrizations of the Steiner surface given in Example 1.2.

Observe that $F = (f_1, f_2, f_3)$ may fail to exist when the image of $H$ is contained in one of the three lines $z_1 = z_2 = 0, z_2 = z_3 = 0, z_3 = z_1 = 0$. For example, $H = (u, 0, 0, v)$ is a rational parametrization from $\mathbb{C}^2$ to $S$ which is not of the form $P \circ F$ for any $F \in \mathbb{C}[u, v]^3$. Notice also that the union of these lines is the singular locus of $S$. $\square$

We next describe one important class of projections which always lead to universal rational parametrizations. Suppose that the smooth $n$-dimensional toric variety $X_\Delta$ is embedded into $\mathbb{P}^2$ via $P_\Delta$. Then let

$$P = (p_0, \ldots, p_{n+1}) \in S^{n+2}_\Delta$$

be chosen generically. Then $X \subset \mathbb{P}^{n+1}$ is the image of $X_\Delta$ under a generic projection. It is well-known that in this situation, $X_\Delta$ is disjoint from the center of
the projection and the restriction of the projection to $X_\Delta$ is birational. Hence $P$ is a universal rational parametrization in this generic case. Notice that $X$ has dimension $n$ and hence is a hypersurface in $\mathbb{P}^{n+1}$.

In particular, when $X_\Delta$ is a smooth toric surface, it follows that

$$P = (p_0, \ldots, p_3) \in S_\Delta^4$$

is a universal rational parametrization whenever the $p_i$ are chosen generically. Here, we parametrize a surface in $\mathbb{P}^3$, which is the case of greatest interest in geometric modeling.

However, we should also mention that there are some non-generic projections which also work nicely. Here is another class of projections which are guaranteed to give universal rational parametrizations.

**Proposition 3.10.** Let $X_\Delta$ be the smooth toric variety of a polytope $\Delta$ and let $A = \{ \tilde{m}_0, \ldots, \tilde{m}_k \} \subset \Delta \cap \mathbb{Z}^n$. Assume that $\Delta$ is the convex hull of $A$ and that $A$ generates $\mathbb{Z}^n$ affinely over $\mathbb{Z}$ (meaning that $\mathbb{Z}^n$ is the $\mathbb{Z}$-span of $\{ m - m' \mid m, m' \in A \}$). Then $P_A = (x^{\tilde{m}_0}, \ldots, x^{\tilde{m}_k}) \in S_{\Delta}^{k+1}$ induces an everywhere defined birational map

$$\Pi : X_\Delta \rightarrow X_A \subset \mathbb{P}^k$$

and $P_A$ is the universal rational parametrization of $X_A$.

**Proof.** Proposition 5.3 of [3] implies that the map $\Pi : X_\Delta \rightarrow X_A$ is the normalization map. (In [3], Proposition 5.3 does not assume that $A$ generates $\mathbb{Z}^n$ affinely, but this is necessary since the proposition depends on Proposition 5.2, which does assume that $A$ generates the lattice affinely.) It follows immediately that $\Pi$ is a birational morphism. One can also show that $\Pi$ is strictly defined in this case. Then the final assertion follows immediately from Theorem 3.5. This completes the proof.

Here is an example of this proposition.

**Example 3.11.** In the situation of Example 3.8, let $A \subset \Delta \cap \mathbb{Z}^2$ be the five vertices of $\Delta$. Since $A$ satisfies all of the conditions of Proposition 3.10, it follows that

$$P_A = (x_2x_4^2x_5^2, x_1^2x_2^2x_3^2, x_3x_4^2x_5, x_1x_4x_5^2, x_1^2x_2x_5^3) \in S_{\Delta}^5$$

is the universal rational parametrization of $X_A \subset \mathbb{P}^4$. Also note that $X_A$ is the image of $X_\Delta \subset \mathbb{P}^7$ under the projection $\mathbb{P}^7 \longrightarrow \mathbb{P}^4$ determined by $A$.

One final comment about Proposition 3.10 is that $X_A$ is itself a toric variety (possibly non-normal). In contrast, the image of $X_\Delta$ under a generic projection may fail to be a toric variety.

### 3.8. Further Examples

Here are two examples which show what happens when we violate one of the hypotheses of Theorem 3.5.
Example 3.12. Consider the quadrilateral $\Delta$ with vertices $(1,0), (0,1), (-1,1)$ and $(-1,0)$:

![Diagram of quadrilateral]

The lattice points $\Delta \cap \mathbb{Z}^2$ give the five monomials

$$x_1 x_2, x_1 x_4, x_2 x_3 x_4, x_3 x_4^2,$$

which in turn give an embedding $X_\Delta \subset \mathbb{P}^4$.

Let $\mathcal{A} = \{(-1,1), (-1,0), (0,0)\} \subset \Delta \cap \mathbb{Z}^2$. This gives

$$P_\mathcal{A} = (x_1 x_2, x_2 x_3, x_2 x_3 x_4)$$

and the rational map

$$\Pi : X_\Delta \longrightarrow X_\mathcal{A} \subset \mathbb{P}^2$$

defined by

$$(x_1 x_2, x_1 x_4, x_2 x_3 x_4, x_3 x_4^2) \mapsto (x_1 x_2, x_2 x_3, x_2 x_3 x_4).$$

The center of this projection is the line $L = \{(0,u,0,v) \mid (u,v) \neq (0,0)\}$. One can check that $L$ is entirely contained in $X_\Delta$ and corresponds to those points where $x_2 = 0$. Thus $\Pi$ is not strictly defined. The surprise is that $\Pi$ is nevertheless defined everywhere on $X_\Delta$. At first glance, this seems impossible, since $x_2 = 0$ corresponds to points

$$(0, x_1 x_4, 0, 0, x_3 x_4^2) \in X_\Delta$$

which project to $(0,0,0)$. We get around this difficulty by letting $x_2 = \varepsilon$, where $\varepsilon \in \mathbb{C}$ is nonzero but close to zero. Then (3.5) becomes

$$(x_1 \varepsilon, x_1 x_4, \varepsilon^2 x_3, \varepsilon x_3 x_4, x_3 x_4^2) \mapsto (x_1 \varepsilon, x_1 x_4, \varepsilon x_3 x_4) = (x_1, x_3 x_4) \in \mathbb{P}^2$$

since $\varepsilon \neq 0$. Letting $\varepsilon \to 0$, this suggests that

$$\Pi(0, x_1 x_4, 0, 0, x_3 x_4^2) = (x_1, 0, x_3 x_4) \in \mathbb{P}^2.$$

In fact, one can prove rigorously that $\Pi$ is defined everywhere on $X_\Delta$.

We also note that $X_\Delta = \mathbb{P}^2$ and that $\Pi$ is birational. This follows from

$$\Pi^{-1}(u_0, u_1, u_2) = (u_0 u_1, u_0 u_2, u_1^2, u_1 u_2, u_2^2),$$

where $u_0, u_1, u_2$ are homogeneous coordinates on $\mathbb{P}^2$. This is the inverse of $\Pi$ on the open subset of $\mathbb{P}^2$ where $u_0 u_1 u_2 \neq 0$. So $\Pi$ is defined everywhere and is birational. However, Theorem 3.5 fails in this case. For example, $P_\mathcal{A}$ is not a rational parametrization since $x_2$ divides the polynomials of $P_\mathcal{A}$. Yet the definition of rational parametrization requires relatively prime polynomials. Hence $P_\mathcal{A}$ has no chance of being a universal rational parametrization. \qed

Our final example concerns a singular polygon.
Example 3.13. Consider the triangle $\Delta$ with vertices $(1,0)$, $(0,1)$, and $(-1,0)$:

The lattice points $\Delta \cap \mathbb{Z}^2$ give the four monomials

$$x^2 \quad z \quad xy \quad y^2$$

which in turn give an embedding $X_\Delta \subset \mathbb{P}^3$ as the singular quadric surface $u_2^2 = u_1 u_3$. One can show that $X_\Delta$ is the weighted projective space $\mathbb{P}(1,1,2)$.

Even though $X_\Delta$ is singular, it is easy to see that $P_\Delta = (z,x^2,xy,y^2)$ satisfies the other hypotheses of Theorem 3.5. So how close is $P_\Delta$ to being a universal rational parametrization?

For an example of what can go wrong, consider $H = (v,u,u,u)$. This is a rational parametrization of $X_\Delta$, yet $H$ is not of the form $P_\Delta \circ F$ for any $F = (f_1,f_2,f_3) \in \mathbb{C}[u,v]^3$ since $u$ is not a square. So Theorem 3.5 fails in this case.

However, it is true that $H = P_\Delta \circ \tilde{F}$, where

$$\tilde{F} = (\sqrt{u},\sqrt{u},v).$$

So it may be that for singular toric varieties, square roots and other radicals appear naturally in considering what a universal parametrization means. But in the next section, we will learn a better method which uses resolution of singularities.\hfill $\square$

4. Universal Rational Parametrizations (Singular Case)

So far, we have always assumed that $X_\Delta$ is smooth, and we saw in Example 3.13 how things can go wrong when $X_\Delta$ has singularities. We will use a toric resolution of singularities to show that we still have universal rational parametrizations, where $\Delta$ is now allowed to be any $n$-dimensional lattice polytope in $\mathbb{Z}^n$.

As in Section 3, assume that we have

$$P = (p_0,\ldots,p_s) \in S_{\Delta}^{s+1},$$

which induces a strictly defined birational map

$$\Pi : X_\Delta \to X \subset \mathbb{P}^s.$$

Our goal is to describe a universal rational parametrization of $X \subset \mathbb{P}^s$. Our main tool will be a toric resolution of singularities. As shown in [6], the normal fan $\Sigma_\Delta$ of $\Delta$ has a refinement $\Sigma$ such that $X_\Sigma$ is smooth. It follows that the induced toric morphism

$$\varphi : X_\Sigma \to X_\Delta$$

is a resolution of singularities. We may assume that $\varphi^{-1}$ is defined on the smooth part of $X_\Delta$. 
Let \( x_1, \ldots, x_r \) be the homogeneous coordinates of \( X_\Sigma \) and let \( \tilde{n}_i \) generate the edge of \( \Sigma \) corresponding to \( x_i \). Some of the \( \tilde{n}_i \)'s will be inner normals to facets of \( \Delta \), while others will be new vectors added in the process of resolving singularities. We will regard the new \( \tilde{n}_i \)'s as inner normals to "virtual facet hyperplanes" of \( \Delta \) in the following way.

Given \( \tilde{n}_i \), we know that it lies in some cone \( \sigma \in \Sigma_\Delta \). We pick the smallest such cone. Its generators are facet normals of \( \Delta \), and the intersection of the corresponding facets is a face \( \Delta_\sigma \) of \( \Delta \). Using the support functions defined in [6], one can prove that there is a unique integer \( \tilde{a}_i \) such that
\[
\{ m \in \mathbb{R}^n \mid \langle m, \tilde{n}_i \rangle + \tilde{a}_i = 0 \} \cap \Delta = \Delta_\sigma.
\]
We call \( \{ m \in \mathbb{R}^n \mid \langle m, \tilde{n}_i \rangle + \tilde{a}_i = 0 \} \) the virtual facet hyperplane of \( \Delta \) with \( \tilde{n}_i \) as inner normal. When \( \tilde{n}_i \) is the inner normal of a facet of \( \Delta \), then one easily sees that the virtual facet hyperplane is the facet hyperplane \( \langle m, \tilde{n}_i \rangle + \tilde{a}_i = 0 \) containing the corresponding facet of \( \Delta \).

Let's illustrate what this looks like in one of our previous examples.

**Example 4.1.** Consider the triangle \( \Delta \) of Example 3.13 and let \( \Sigma \) be the following refinement of its normal fan:

\[
\begin{array}{c}
\bullet \quad x_1 \\
\bullet \quad x_2 \\
\bullet \quad x_3 \\
\bullet \quad x_4
\end{array}
\]

(\text{So the refinement is given by adding the edge corresponding to} \ x_3. \text{)} \ Let \( \tilde{n}_i \) generate the edge corresponding to \( x_i \). Thus \( \tilde{n}_1, \tilde{n}_2 \) and \( \tilde{n}_4 \) are inner normals of facets of the triangle \( \Delta \) of Example 3.13, while \( \tilde{n}_3 \) was added to make \( X_\Sigma \) smooth. Then we can draw the virtual facet hyperplanes (= lines in this case) and their corresponding variables as follows:

\[
\begin{array}{c}
\bullet \quad x_3 \\
\bullet \quad x_2 \\
\bullet \quad x_4 \\
\bullet \quad x_1
\end{array}
\]

The facet hyperplanes are solid lines, while the one virtual facet hyperplane is the dashed line corresponding to \( x_3 \). Note also that
\[ \tilde{a}_1 = 0 \quad \text{and} \quad \tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 1. \]

We will return to this example shortly. \( \Box \)
Given this setup, a lattice point \( m \in \Delta \cap \mathbb{Z}^n \) gives the monomial

\[
x^m = \prod_{i=1}^{\tilde{r}} x_i^{(m, \tilde{a}_i) + \tilde{a}_i}.
\]

We call \( x^m \) a \( \Delta \)-monomial of the toric variety \( X_{\Sigma} \). Note that the exponent of \( x_i \) in \( x^m \) measures the lattice distance from \( m \) to the corresponding virtual facet hyperplane. Here is an example.

**Example 4.2.** In the situation of Example 4.1, the lattice points of \( \Delta \cap \mathbb{Z}^2 \) give the \( \Delta \)-monomials

\[
x_1^2 x_3 \quad x_2 x_3 x_4 \quad x_3 x_4^2
\]

in the homogeneous coordinates of the toric variety \( X_{\Sigma} \) which resolves the singularities of \( X_{\Delta} \).

One useful observation is that when dealing with lattice polygons, the only places we need to add virtual facet hyperplanes are at vertices whose adjacent inner normals do not form a basis of \( \mathbb{Z}^2 \) over \( \mathbb{Z} \). Furthermore, in this situation, there is a unique minimal resolution of singularities which can be computed algorithmically—see [6, Sec. 2.6]. Thus there is an algorithm for finding the virtual inner normals that need to be added at these vertices.

We are almost ready to state our main result. As above, \( \Delta \) is an \( n \)-dimensional lattice polytope in \( \mathbb{R}^n \) and \( \varphi : X_{\Sigma} \rightarrow X_{\Delta} \) is a toric resolution. The lattice points in \( \Delta \cap \mathbb{Z}^n \) determine

\[
S_\Delta = \text{Span}(x^m \mid m \in \Delta \cap \mathbb{Z}^n)
\]

where \( x^m \) is now the \( \Delta \)-monomial (4.2) in the homogeneous coordinates \( x_1, \ldots, x_\tilde{r} \) of \( X_{\Sigma} \). Now let

\[
P = (p_0, \ldots, p_s) = \left( \sum_m a_{0m} x^m, \ldots, \sum_m a_{sm} x^m \right) \in S_{\Delta}^{s+1}.
\]

Then \( P \) induces a rational map \( p : \mathbb{C}^\tilde{r} \dashrightarrow \mathbb{P}^s \), and similar to Proposition 2.3, one can show that \( p \) factors as

\[
\mathbb{C}^\tilde{r} \dashrightarrow X_{\Sigma} \xrightarrow{\varphi} X_{\Delta} \dashrightarrow X \subset \mathbb{P}^s,
\]

where as usual, \( X \) is the Zariski closure of the image of \( p \). The map from \( X_{\Delta} \) to \( X \) will be denoted \( \Pi \), and as in Theorem 3.5, we will assume that \( \Pi \) is strictly defined and birational. Let \( U \subset X \) be the maximal open set on which the inverse of \( \Pi : X_{\Delta} \rightarrow X \) is defined, and then set

\[
\tilde{U} = U \cap \left( X \setminus \Pi(\{ x \in X_{\Delta} \mid x \text{ is a singular point of } X_{\Delta} \}) \right).
\]

Finally, we have \( G \subset (\mathbb{C}^*)^\tilde{r} \). Then \( \mu = (\mu_1, \ldots, \mu_\tilde{r}) \in G \) gives \( \mu_{\Delta, \Sigma} = \prod_{i=1}^{\tilde{r}} \mu_i^{\tilde{a}_i} \).

Let \( G_{\Delta, \Sigma} \) be the kernel of the homomorphism \( \mu \mapsto \mu_{\Delta, \Sigma} \). We use this notation because \( \mu_{\Delta, \Sigma} \) and \( G_{\Delta, \Sigma} \) depend not only on the polytope \( \Delta \) but also on the fan \( \Sigma \).

We now show that \( P \) is a universal rational parametrization of \( X \).
THEOREM 4.3. Let $\Delta$, $\Sigma$, $P = (p_0, \ldots, p_s)$, $X$ and $\bar{U}$ be as above. Then $P$ is a universal rational parametrization of $X \subset \mathbb{P}^s$ in the following sense:

1. If $F = (f_1, \ldots, f_r) \in R^r = \mathbb{C}[y_1, \ldots, y_d]^r$ is $\Sigma$-irreducible, then
   \[ P \circ F = (\sum_m a_{m,f}^m, \ldots, \sum_m a_{m,f}^m) \in \mathbb{R}^{r+1} \]
   is a rational parametrization of $X \subset \mathbb{P}^s$.

2. Conversely, given any rational parametrization $H \in \mathbb{R}^{r+1}$ of $X$ whose image meets the open set $\bar{U} \subset X$, there is a $\Sigma$-irreducible $F = (f_1, \ldots, f_r) \in R^r$ such that $H = P \circ F$.

3. If $F$ and $F'$ are $\Sigma$-irreducible, then $P \circ F = P \circ F'$ as rational parametrizations if and only if $F' = \mu \cdot F$ for some $\mu \in G_{\Delta, \Sigma}$.

The proof will be given in Section 6. Note that the theorem uses the concept of $\Sigma$-irreducible. This uses the obvious modification of Definition 3.2 which applies to any fan $\Sigma$.

Let’s apply Theorem 4.3 to the singular example we’ve been studying.

EXAMPLE 4.4. Let $\Delta$ be the triangle of Examples 3.13, 4.1 and 4.2. This gives the singular toric variety $X_\Delta$. The fan $\Sigma$ from Example 4.1 gives a resolution of singularities, and the $\Delta$-monomials $x^m$ for $m \in \Delta \cap \mathbb{Z}^2$ are given in (4.3). Let

\[ P = (x_1, x_2^2 x_3, x_2 x_3 x_4, x_3 x_4^2) \]

Since $\Pi : X_\Delta \rightarrow X \subset \mathbb{P}^3$ is an isomorphism in this case, Theorem 4.3 implies that $P$ is a universal rational parametrization of $X$.

This means the following. If $u_0, u_1, u_2, u_3$ are coordinates on $\mathbb{P}^3$, then $X$ is defined by $u_2^2 = u_1 u_3$. Hence, if $H = (h_0, h_1, h_2, h_3)$ is a rational parametrization whose image is not the singular point $(1, 0, 0, 0) \in X$, then there is $F = (f_1, f_2, f_3, f_4)$ such that

\[ H = P \circ F = (f_1, f_2^2 f_3, f_2 f_3 f_4, f_3 f_4^2) \]

Furthermore, one can show that

- $F$ is $\Sigma$-irreducible if and only if $\gcd(f_1, f_3) = \gcd(f_2, f_4) = 1$.
- $F = (f_1, f_2, f_3, f_4)$ is unique up to $(f_1, \lambda f_2, \lambda^{-1} f_3, \lambda f_4)$ for $\lambda \in \mathbb{C}^*$.

For $H = (v, u, u, u) \in \mathbb{C}[u, v]^4$ as in Example 3.13, one easily sees that $H = P \circ F$ for $F = (v, 1, u, 1)$ in this case. So unlike Example 3.13, we don’t need square roots.

In the smooth case, we analyzed $P$ in terms of the embedding given by $P_\Delta$ followed by a projection. In the singular case, the analog of $P_\Delta$ need not give an embedding. However, when $\Delta$ is a toric surface, then it is. Hence the discussion of embeddings and projections given in Section 3 applies to any toric surface.

Finally, we remark that while toric resolutions are in general not unique, in the surface case one can always find a minimal resolution which is unique up to isomorphism. It follows that we have a canonical choice of universal rational parametrization when $\Delta$ is a lattice polygon.

5. Rational Maps to Smooth Toric Varieties

In order to prove the results of Sections 3 and 4, we need to study rational maps to an abstract toric variety. So in this section we will assume that $X_\Sigma$ is a compact toric variety, possibly non-projective.
In algebraic geometry, there is a well-defined notion of a rational map between irreducible varieties, regardless of whether they are affine, projective or defined abstractly like $X_\Sigma$. Our goal here is to describe all rational maps
\[ \mathbb{C}^d \longrightarrow X_\Sigma \]
when $X_\Sigma$ is a smooth toric variety. Recall that this means that the generators of every $n$-dimension cone of $\Sigma$ are a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

The natural candidate for the universal rational map to $X_\Sigma$ is the rational map
\[ \pi : \mathbb{C}^r \longrightarrow X_\Sigma \]
of (2.3). So we need to explain what universal means in this context.

Given a polynomial map $F : \mathbb{C}^d \rightarrow \mathbb{C}^r$ such that $F$ is $\Sigma$-irreducible, we will show in Section 6 that the composition
\[ \pi \circ F : \mathbb{C}^d \longrightarrow X_\Sigma. \]
is a well-defined rational map. One of the key assertions of Theorem 5.1 below is that this gives all rational maps from $\mathbb{C}^d$ to $X_\Sigma$.

However, the map $F$ in (5.1) is not unique. Recall from (2.1) that we have the subgroup $G \subset (\mathbb{C}^*)^r$ which is used in the quotient representation of $X_\Sigma$. If $F = (f_1, \ldots, f_r) \in R^r$ is $\Sigma$-irreducible and $\mu = (\mu_1, \ldots, \mu_r) \in G$, then
\[ \mu : F = (\mu_1 f_1, \ldots, \mu_r f_r) \]
is also $\Sigma$-irreducible and gives the same rational map as $F$ when composed with $\pi$ (because of the quotient (2.2)). Another key assertion of Theorem 5.1 is that this is the only way that two $\Sigma$-irreducible $F$’s can give the same $\pi \circ F$. Thus we have complete control of the lack of uniqueness.

We can now state the main result of this section. Let $R = \mathbb{C}[y_1, \ldots, y_d]$.

**Theorem 5.1.** Let $X_\Sigma$ be a smooth compact toric variety. Then:

1. If $F = (f_1, \ldots, f_r) \in R^r$ is $\Sigma$-irreducible, then $\pi \circ F$ gives a well-defined rational map $\pi \circ F : \mathbb{C}^d \longrightarrow X_\Sigma$.

2. If $F$ and $F'$ are $\Sigma$-irreducible, then $\pi \circ F = \pi \circ F'$ as rational maps if and only if $F' = \mu \cdot F$ for some $\mu \in G$.

3. Finally, every rational map $f : \mathbb{C}^d \longrightarrow X_\Sigma$ is of the form $\pi \circ F$ for some $\Sigma$-irreducible $F \in R^r$.

Hence rational maps $f : \mathbb{C}^d \longrightarrow X_\Sigma$ correspond bijectively to $G$-equivalence classes of $\Sigma$-irreducible $(f_1, \ldots, f_r) \in R^r$.

The proof will be given in Section 6. Here is an example of Theorem 5.1.

**Example 5.2.** Let $X_\Sigma$ be the toric variety of Example 3.8. There, we saw that $F = (f_1, \ldots, f_5)$ is $\Sigma$-irreducible provided
\[ \gcd(f_1, f_3) = \gcd(f_1, f_4) = \gcd(f_2, f_4) = \gcd(f_2, f_5) = \gcd(f_3, f_5) = 1 \]
and that
\[ G = \{(\lambda, \mu, \nu, \lambda \mu, \mu \nu) \mid \lambda, \mu, \nu \in \mathbb{C}^* \}. \]

By Theorem 5.1, it follows that rational maps to $X_\Sigma$ are all of the form $\pi \circ F$, where $F$ is unique up to $(\lambda, \mu, \nu, \lambda \mu, \mu \nu) \cdot F$.

Let’s look at the specific example of the map $F' : \mathbb{C}^2 \rightarrow \mathbb{C}^5$ defined by
\[ F'(u, v) = (uv, 1, u, v, 1) \]
This induces a rational map \( \pi \circ F' : C^2 \longrightarrow X_\Sigma \). However, \( F' \) is not \( \Sigma \)-irreducible. To get a \( \Sigma \)-irreducible representation, observe that

\[
F' = (uv, 1, u, v, 1) = (uv, u^{-1}, u, v, 1) \cdot (1, u, 1, 1, 1).
\]

Since \( (uv, u^{-1}, u, v, 1) \in G \) for \( u, v \neq 0 \), we see that \( F' \) and \( F = (1, u, 1, 1, 1) \) give the same rational map. Since \( F \) is \( \Sigma \)-irreducible, this is the representation given by Theorem 5.1.

Notice that even though \( (5.2) \) is given by polynomials, \( (5.3) \) shows that it is not a polynomial multiple of the \( \Sigma \)-irreducible representation \( F = (1, u, 1, 1, 1) \).

We can also look at \( (5.2) \) from the point of view of Theorem 3.5 and Corollary 3.6. If we compose \( (5.2) \) with \( P_\Delta \) from Example 3.8, we obtain

\[
H' = (u^2v^2, u^3v^2, uv^2, u^2v^2, u^3v^2, uv^2, u^2v^2),
\]

which satisfies the hypothesis of Corollary 3.6. Factoring out the gcd \( uv^2 \), we can write this as

\[
H' = uv^2(u, u^2, u^3, 1, u, u^2, 1, u) = uv^2H.
\]

Furthermore, one easily computes that \( H = P_\Delta(1, u, 1, 1, 1) \). Thus

\[
H' = uv^2P_\Delta(1, u, 1, 1, 1).
\]

Notice also that unlike \( (5.3) \), the representation given by Corollary 3.6 involves only polynomials.

Finally, we observe that the smoothness assumption in Theorem 5.1 is necessary, as shown by the following example.

**Example 5.3.** Consider the weighted projective plane \( \mathbb{P}(1, 1, 2) \). Here, \( (2.2) \) represents this toric variety as the quotient of \( \mathbb{C}^3 \setminus \{0\} \) under the action of \( \mathbb{C}^* \) given by \( \lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda^2 z) \). Then consider the rational map

\[
C^2 \longrightarrow \mathbb{P}(1, 1, 2)
\]

defined by

\[
(u, v) \mapsto (\sqrt{u}, \sqrt{u}, v).
\]

This looks crazy, but notice that

\[
(-1) \cdot (\sqrt{u}, \sqrt{u}, v) = ((-1)\sqrt{u}, (-1)\sqrt{u}, (-1)^2 v) = (-\sqrt{u}, -\sqrt{u}, v).
\]

In fact, one can prove that \( (5.4) \) gives a well-defined rational map whose image is the curve \( x = y \) in \( \mathbb{P}(1, 1, 2) \). Since this map cannot be written in the form \( \pi \circ F \) where \( F \) consists of polynomials, we see that Theorem 5.1 fails in this case.

We should also mention that this example is a version of Examples 3.13 and 4.4 in disguise. In fact, the “crazy” rational map \( (5.4) \) is exactly the map we used in Example 3.13 to show that Theorem 3.5 fails for singular toric varieties. Recall also that we gave a purely polynomial version of this map by using the toric resolution described in Example 4.4.

6. Theoretical Justification

The purpose of this section is to prove the three main results of this paper, Theorems 4.3, 3.5 and 5.1. We begin with Theorem 5.1 since it will be used to prove the other two theorems.
Proof of Theorem 5.1. Suppose that \( F = (f_1, \ldots, f_r) \) is \( \Sigma \)-irreducible. The discussion following Definition 3.2 implies that there is some \( \sigma \in \Sigma \) such that \( f_1 \neq 0 \) for all \( \rho \not\subset \sigma \). Thus \( x^\sigma = \Pi_{\rho \not\subset \sigma} x_i \) does not vanish on the image of \( F \), so that the image of \( F \) is not contained in the exceptional set \( Z = V(x^\sigma \mid \sigma \in \Sigma) \). This shows that \( V = F^{-1}(Z) \) is a proper subvariety of \( C^d \). It follows that \( F \) induces a map
\[
\mathbb{C}^d \setminus V \xrightarrow{F} \mathbb{C}^r \setminus Z \xrightarrow{\pi} (\mathbb{C}^r \setminus Z)/G = X_\Sigma.
\]
Since \( \mathbb{C}^d \setminus V \) is a nonempty Zariski open subset of \( \mathbb{C}^d \), we get a well-defined rational map \( \pi \circ F : \mathbb{C}^d \dashrightarrow X_\Sigma \). This proves assertion (1) of the theorem.

Before proving (2), we need to describe \( Z \). Since \( V(x^\sigma) \) is a union of coordinate hyperplanes, their intersection \( Z \) is a union of coordinate subspaces \( W_1 \cup \cdots \cup W_k \). Let one of these be \( W_j = V(x_{i_1}, \ldots, x_{i_k}) \). Suppose that \( \rho_{i_1}, \ldots, \rho_{i_k} \subset \sigma \) for some \( \sigma \in \Sigma \). Then the point \( a \in C^r \) with \( a_{i_1} = \cdots = a_{i_k} = 0 \) and \( a_i = 1 \) otherwise lies in \( V(x_{i_1}, \ldots, x_{i_k}) \subset Z \), yet \( x_\sigma \) is nonvanishing at \( a \). This contradiction shows that no cone of \( \Sigma \) contains \( \rho_{i_1}, \ldots, \rho_{i_k} \).

Note that \( V = F^{-1}(Z) = F^{-1}(W_1) \cup \cdots \cup F^{-1}(W_k) \), and if we write \( W_j = V(x_{i_1}, \ldots, x_{i_k}) \) as above, then
\[
F^{-1}(W_j) = V(f_{i_1}, \ldots, f_{i_k}).
\]
The above paragraph and Definition 3.2 imply that \( \gcd(f_{i_1}, \ldots, f_{i_k}) = 1 \). This shows that each \( F^{-1}(W_j) \) has codimension at least 2 in \( \mathbb{C}^d \), so that the same is true for their union \( V \).

Now we can prove uniqueness. Suppose that another \( \Sigma \)-irreducible \( F' = (f_1', \ldots, f_r') \) gives the same rational map \( f \). This means the following. Let \( V' = (F')^{-1}(Z) \). Then as above \( V' \) has codimension at least 2 and the induced rational map \( f' \) is defined on \( \mathbb{C}^d \setminus V' \). Then \( f = f' \) as rational maps implies that \( f = f' \) as functions on \( U = \mathbb{C}^d \setminus (V \cup V') \). Since \( X_\Sigma = (\mathbb{C}^r \setminus Z)/G \), this means that for each \( y \in U \), there is \( \mu(y) \in G \) such that \( \mu(y) \cdot F(y) = F'(y) \). Since \( X_\Sigma \) is smooth, the quotient map \( \mathbb{C}^r \setminus Z \to X_\Sigma \) is a smooth fibration with fibers isomorphic to \( G \). This implies that the map \( y \mapsto \mu(y) \) is an algebraic map \( U \to G \).

Using \( G \subset (\mathbb{C}^r)^* \), \( \mu \) gives maps \( \mu_i : U \to \mathbb{C}^* \) such that \( f_i(y) = \mu_i(y) f'_i(y) \) for all \( y \in U \). Now comes the key point: since \( V \cup V' \) has codimension at least 2, \( \mu_i \) must be constant. (To see this, write \( \mu_i = A/B \), where \( A, B \) are relatively prime polynomials. Then \( A \) nonconstant \( \Rightarrow \) the zeros of \( \mu_i \) have codimension 1 and \( B \) nonconstant \( \Rightarrow \) the poles of \( \mu_i \) have codimension 1. But \( \mu_i \) is defined and nonzero outside of codimension at least 2.)

Hence the \( \mu_i \) are constant. It follows \( \mu \in G \) is also constant, and then \( \mu \) gives the desired equivalence between \( (f_1, \ldots, f_r) \) and \( (f_1', \ldots, f_r') \). This completes the proof of (2).
$X_\Sigma$ is smooth, $D_i \subset X_\Sigma$ is locally defined by a single equation, say $h = 0$, and then $f^{-1}(D_i) \subset U$ is defined locally by $h \circ f = 0$. It follows that every irreducible component of $f^{-1}(D_i)$ in $U$ has codimension 1, although the components may have multiplicities. Now, using $U \subset \mathbb{C}^d$, we get the Zariski closure $Z_i \subset \mathbb{C}^d$ of $f^{-1}(D_i) \subset U$. The irreducible components of $Z_i$ also have codimension 1, with the same multiplicities. It follows that there is $f_i \in \mathbb{R}$, unique up to a constant, such that $V(f_i) = Z_i$ with the same multiplicities.

We claim that $(f_1, \ldots, f_r)$ is $\Sigma$-irreducible. Suppose that $\rho_i, \ldots, \rho_k$ are contained in no cone of $\Sigma$. Then the relation between cones and divisors implies that $D_i \cap \cdots \cap D_k = \emptyset$ in $X_\Sigma$. Thus, in $U$, we have

$$f^{-1}(D_i) \cap \cdots \cap f^{-1}(D_k) = \emptyset.$$

Since $V(f_i) \cap U = f^{-1}(D_i)$ for all $i$, it follows that

$$(6.1) \quad V(f_1, \ldots, f_k) \cap U = \emptyset.$$  

Hence $V(f_1, \ldots, f_k) \subset \mathbb{C}^d \setminus U$. Since $\mathbb{C}^d \setminus U$ has codimension at least 2, this implies that $\gcd(f_1, \ldots, f_k) = 1$. Thus $(f_1, \ldots, f_r)$ is $\Sigma$-irreducible. It follows that $(c_1 f_1, \ldots, c_r f_r)$ is $\Sigma$-irreducible whenever $c_i \in \mathbb{C}^*$. This will be useful below.

We next claim that there are $c_i \in \mathbb{C}^*$ such that $(c_1 f_1, \ldots, c_r f_r)$ gives the rational map $f$. Let $f'$ be the rational map determined by $(f_1, \ldots, f_r)$. Using $(6.1)$ and our earlier description of $F^{-1}(Z)$, one easily shows that $f'$ is defined on $U$. Furthermore, the $f_i$ were defined so that in $U$, we have

$$(6.2) \quad (f')^{-1}(D_i) = f^{-1}(D_i)$$

for all $i$. This equality also gives the correct multiplicities.

Now take a $n$-dimensional cone $\sigma \in \Sigma$. This gives the affine toric variety $U_\sigma \subset X_\Sigma$, and one easily sees that $U_\sigma = X_\Sigma \setminus \bigcup_{\rho_i \not\in \sigma} D_i$. Then $(6.2)$ implies that $(f')^{-1}(U_\sigma) = f^{-1}(U_\sigma)$. Call this $U'_\sigma$ and note that $U'_\sigma \neq \emptyset$ since $f(U) \cap (\mathbb{C}^*)^n \neq \emptyset$. Thus $f$ and $f'$ give maps $U'_\sigma \to U_\sigma$. But since $X_\Sigma$ is smooth, we have $U'_\sigma \simeq \mathbb{C}^n$.

We may assume that the edges of $\Sigma$ are labeled so that $\rho_1, \ldots, \rho_n$ are the edges of $\sigma$. Then write

$$f|_{U'_\sigma} = (h_1, \ldots, h_n) : U'_\sigma \to \mathbb{C}^n$$

$$f'|_{U'_\sigma} = (h'_1, \ldots, h'_n) : U'_\sigma \to \mathbb{C}^n.$$

We have set things up so that $D_i \cap U_\sigma$ is defined by the vanishing of the $i$th coordinate. Since $(6.2)$ respects multiplicities, we see that $h'_i/h_i = \beta_i$ is a nonvanishing function on $U'_\sigma$. Thus

$$\beta_\sigma = (\beta_1, \ldots, \beta_n) : U'_\sigma \to (\mathbb{C}^*)^n$$

is an algebraic map which satisfies

$$\beta_\sigma(y) \cdot f(y) = f'(y)$$

for all $y \in U'_\sigma$. If $\tau$ is another $n$-dimensional cone, then we get $\beta_\tau$ defined on $U'_\tau$ with similar properties. However, for any $y$ in the nonempty open subset $f^{-1}((\mathbb{C}^*)^n) \subset U'_\sigma \cap U'_\tau$, there is a unique element of $(\mathbb{C}^*)^n$ which takes $f(y)$ to $f'(y)$. It follows easily that $\beta_\sigma = \beta_\tau$ on $U'_\sigma \cap U'_\tau$. Furthermore, the $U'_\sigma$ cover $U$ since the $U_\sigma$ cover $X$. It follows that we get an algebraic map

$$\beta : U \to (\mathbb{C}^*)^n$$
with the property that
\[ \beta(y) \cdot f(y) = f'(y) \]
for all \( y \in U \). But arguing as above, we see that \( \beta \) must be constant since \( \mathbb{C}^d \setminus U \) has codimension at least 2. Thus there is \( \beta \in (\mathbb{C}^*)^n \) such that \( \beta \cdot f(y) = f'(y) \) for all \( y \in U \).

Since \( (\mathbb{C}^*)^r/G = (\mathbb{C}^*)^n \), we can pick \( (c_1, \ldots, c_r) \in (\mathbb{C}^*)^r \) which maps to \( \beta \in (\mathbb{C}^*)^n \). We conclude that \( (c_1f_1, \ldots, c_rf_r) \) is \( \Sigma \)-irreducible and gives \( f \).

Finally, we need to discuss what happens when our rational map \( f \) satisfies \( f(U) \cap (\mathbb{C}^*)^n = \emptyset \). Here, the idea is that there is a smallest torus orbit which meets \( f(U) \). The Zariski closure of this orbit will be \( D_{i_1} \cap \cdots \cap D_{i_t} \), where \( \rho_{i_1}, \ldots, \rho_{i_t} \) are the edges of some \( \sigma_0 \in \Sigma \). Let \( \text{orb}(\sigma_0) \) denote this orbit. Then make the following changes in the above proof:

1. First, let \( f_{i_1} = \cdots = f_{i_t} = 0 \).
2. Second, replace \( (\mathbb{C}^*)^n \) with \( \text{orb}(\sigma_0) \).
3. Third, for \( \rho_i \not\subset \sigma_0 \), pick \( f_i \) so that \( V(f_i) \cap U = f^{-1}(D_i) \) (with the same multiplicities).
4. Fourth, use \( n \)-dimensional cones \( \sigma \) which contain \( \sigma_0 \) as a face.

With these changes, the above argument shows that \( f \) comes from a \( \Sigma \)-irreducible element of \( R^r \). We omit the details. \( \square \)

**Remark 6.1.** In the existence part of the above proof, notice that the set \( U \) was the maximal open subset of \( \mathbb{C}^d \) on which \( f \) was defined. Yet the \( f' \) we constructed was defined on a potentially bigger set, namely \( \mathbb{C}^d \setminus F^{-1}(Z) \). Once we prove \( \beta \cdot f = f' \), it follows that \( U = \mathbb{C}^d \setminus F^{-1}(Z) \). Using this, we obtain the following corollary.

**Corollary 6.2.** Let \( f : \mathbb{C}^d \dashrightarrow X_\Sigma \) be induced by a \( \Sigma \)-irreducible \( F = (f_1, \ldots, f_r) \in R^r \). Then the maximal open subset of \( \mathbb{C}^d \) on which \( f \) is defined is given by \( U = \mathbb{C}^d \setminus F^{-1}(Z) \).

We now turn to the proof of the existence of universal rational parametrizations for smooth toric projective toric varieties.

**Proof of Theorem 3.5.** To prove (1), first assume that \( F = (f_1, \ldots, f_r) \) is \( \Sigma_\Delta \)-irreducible. We need to prove that the polynomials \( \sum_m a_m f_m \) are relatively prime. So suppose that an irreducible polynomial \( q \in R \) divides all of them.

We will use the interpretation of \( \Pi : X_\Delta \to X \) as the composition of the embedding \( X_\Delta \subset \mathbb{P}^d \) given by \( \Pi_\Delta \) followed by a projection. In particular, if \( L \) is the center of the projection, then \( X_\Delta \cap L = \emptyset \) since \( \Pi \) is strictly defined on \( X_\Delta \).

Suppose that we have a \( a \in \mathbb{C}^d \) such that \( q(a) = 0 \). If \( F(a) \in \mathbb{C}^d \setminus Z \), then \( p(F(a)) \) gives a point in \( X_\Delta \cap L \), which is empty by assumption. It follows that \( F(V(q)) \subset Z \). Since \( q \) is irreducible, \( F(V(q)) \) must lie in some irreducible component of \( Z \). By the proof of Theorem 5.1, it follows that \( F(V(q)) \subset V(x_{i_1}, \ldots, x_{i_k}) \), where no cone of \( \Sigma \) contains \( \rho_{i_1}, \ldots, \rho_{i_k} \). Thus \( f_{i_j} \) vanishes on \( V(q) \), so that \( q \) divides \( f_{i_j} \). But this is impossible since \( F \) is \( \Sigma_\Delta \)-irreducible. This completes the proof of (1).

Next suppose that \( H = (h_0, \ldots, h_s) \) is a rational parametrization of \( X \) whose image meets \( U \subset X \). This gives a rational map denoted \( h : \mathbb{C}^d \dashrightarrow X \). Since \( \Pi : X_\Delta \to X \) is birational and \( \Pi^{-1} \) is defined on \( U \), we get a rational map
\[ f = \Pi^{-1} \circ h : \mathbb{C}^d \dashrightarrow X_\Delta. \]
By Theorem 5.1, \( f \) is induced by a \( \Sigma_\Delta \)-irreducible \( F = (f_1, \ldots, f_r) \in \mathbb{R}^r \). It follows that \( H \) and \( P \circ F \) give the same rational map \( \mathbb{C}^d \to \mathbb{P}^s \). Since both satisfy the gcd condition of Definition 3.1, we see that \( H = c P \circ F \) for some constant \( c \neq 0 \).

We claim that there is \( \mu \in G \) such that \( H = P \circ (\mu \cdot F) \). Recall from (2.6) that if \( \mu = (\mu_1, \ldots, \mu_r) \in G \), then
\[
(6.3) \quad P(\mu \cdot (x_1, \ldots, x_r)) = \mu_\Delta P(x_1, \ldots, x_r),
\]
where
\[
\mu_\Delta = \prod_{i=1}^{r} \mu_i^{a_i}.
\]
Assume for the moment that the map
\[
(6.4) \quad G \to \mathbb{C}^*
\]
defined by \( \mu \mapsto \mu_\Delta \) is surjective. Then we can find \( \mu \in G \) such that \( \mu_\Delta = c \). It follows that
\[
H = c P \circ F = \mu_\Delta P \circ F = P \circ (\mu \cdot F),
\]
as claimed. Since \( G_\Delta \) is the kernel of (6.4), the uniqueness assertion of Theorem 5.1 easily implies that \( \mu \cdot F \) is unique up to \( G_\Delta \)-equivalence.

We still need to prove that (6.4) is surjective. Since this map is a character, its image is either finite or all of \( \mathbb{C}^* \). Furthermore, it is well-known that \( G \) is connected since \( X_\Delta \) is smooth. Hence the image is either the identity or \( \mathbb{C}^* \). So all we need to prove is that (6.4) is nonconstant.

If the map is constant, then \( \mu_\Delta = 1 \) for all \( \mu \in G \). We claim this implies the existence of \( m \in M \) such that
\[
(6.5) \quad \langle m, n_i \rangle = a_i \quad \text{for all} \; i = 1, \ldots, r.
\]
We prove this as follows. As explained in [2], the inclusion \( G \subset (\mathbb{C}^*)^r \) induces an exact sequence
\[
1 \to G \to (\mathbb{C}^*)^r \xrightarrow{\phi} (\mathbb{C}^*)^n \to 1.
\]
The map \( \mu \to \mu_\Delta = \prod_{i=1}^{r} \mu_i^{a_i} \) extends to the character \( (\mathbb{C}^*)^r \to \mathbb{C}^* \) corresponding to \( (a_1, \ldots, a_r) \in \mathbb{Z}^r \). If \( \mu \to \mu_\Delta \) is constant on \( G \), then above exact sequence shows that it induces a character \( \chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^* \). Since the map \( \phi \) is dual to the inclusion \( \mathbb{Z}^n \to \mathbb{Z}^r \) which sends \( m \) to \( \langle \langle m, n_1 \rangle, \ldots, \langle m, n_r \rangle \rangle \), it follows that \( (a_1, \ldots, a_r) = \langle \langle m, n_1 \rangle, \ldots, \langle m, n_r \rangle \rangle \), as claimed. Thus (6.5) is proved.

However, if we compare (6.5) to (2.4), we see that \( -m \) lies in every facet of \( \Delta \), which is clearly impossible. This contradiction shows that (6.4) must be nonconstant, and we are done. \( \square \)

Finally, we prove the existence of universal rational parametrizations for arbitrary projective toric varieties.

**Proof of Theorem 4.3.** Recall that \( P \) induces a rational map \( p : \mathbb{C}^r \dashrightarrow \mathbb{P}^s \) which factors
\[
\mathbb{C}^r \dashrightarrow X_\Sigma \xrightarrow{\xi} X_\Delta \xrightarrow{\Pi} X \subset \mathbb{P}^s.
\]
Furthermore, the argument preceding the statement of Theorem 3.5 shows that \( p \) is a rational parametrization of \( X \). From here, the proof of (1) is identical to the proof of the first part of Theorem 3.5.
As for (2), observe that the composition
\[ X_{\Sigma} \xrightarrow{\varphi} X_{\Delta} \xrightarrow{\Pi} X \]
is birational. Furthermore, since \( \Pi^{-1} \) is defined on \( U \) and \( \varphi^{-1} \) is defined on the smooth part of \( X_{\Delta} \), it follows that
\[ \tilde{\Pi} : X_{\Sigma} \rightarrow X \]
is a birational morphism whose inverse is defined on \( \tilde{U} \). Hence, if a rational parametrization \( H \) induces a rational map \( h : \mathbb{C}^d \dashrightarrow X \subset \mathbb{P}^s \) whose image meets \( \tilde{U} \), then
\[ f = \tilde{\Pi}^{-1} \circ h : \mathbb{C}^d \dashrightarrow X_{\Sigma} \]
is a rational map. As in the proof of Theorem 3.5, Theorem 5.1 implies that \( f \) is given by a \( \Sigma \)-irreducible \( F \). This means that \( H \) and \( P \circ F \) agree up to a constant.

For the third assertion of the theorem, the proof follows from what we did in the proof of Theorem 3.5. This completes the proof of the theorem. □

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