BV-Norm Convergence of Interpolation Approximations
for Frobenius-Perron Operators

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Abstract. Let $S$ be a chaotic transformation from an interval into itself and let $P$ be the corresponding Frobenius-Perron operator associated with $S$. In this paper we prove the convergence under the variation norm for a piecewise linear interpolation method that was recently proposed by the authors for computing a stationary density of $P$.

1. Introduction
Since the pioneering work [5] of Ulam concerning Ulam's conjecture on the convergence of his piecewise constant approximation method [6] for computing absolutely continuous invariant measures of chaotic interval transformations, Ulam's method and its various higher order extensions have attracted researchers in the field of numerical dynamical systems and many applied areas such as, statistical physics, computational molecular dynamics, and electric engineering. Because of wide applications of Frobenius-Perron operators in nonlinear analysis [4], the efficient computation of their fixed densities is very important when one attempts to study complicated behavior of dynamical systems with statistical and stochastic approaches.

In [3] a piecewise linear interpolation method was proposed for computing absolutely continuous invariant measures of chaotic deterministic dynamical systems. Although the resulting finite dimensional approximation operators are no longer Markov ones as the approximated Frobenius-Perron operator, they keep not only the positivity of the Frobenius-Perron operator but also piecewise linear functions unchanged. The numerical experiments have shown that the new piecewise linear interpolation method converges faster than the piecewise linear Markov approximations method in [1] without significantly increasing the computational cost.

This paper is devoted to proving the convergence of the piecewise linear interpolation method under the variation norm, called the BV-norm, which is stronger than the $L^1$-norm. In the following section we briefly outline this method and present some of its basic properties. The convergence under

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the variation norm will be proven in Section 3. We conclude in Section 4 with some open problems for further research.

2. The Interpolation Method

Let $S$ be a nonsingular map from the unit interval $[0, 1]$ into itself in the sense that $m(A) = 0$ implies that $m(S^{-1}(A)) = 0$, where $m$ is the Lebesgue measure on the line. Let $L'(0,1)$ be the Banach space of all Lebesgue integrable functions with the $L'$-norm $\| f \| = \int_0^1 | f | \, dm$. A nonnegative function $f \in L'(0,1)$ with $\| f \| = 1$ is called a density. The set of all $L'$-functions of bounded variation is denoted as $BV(0,1)$ which is a Banach space under the $BV$-norm $\| f \|_BV = \| f \| + V_1^0 f$, where $V_1^0 f$ is the variation of $f$ over $[0,1]$. By Helly's Theorem, a bounded closed subset of $BV(0,1)$ is compact in $L'(0,1)$, a fact which will be used in Section 3.

The Frobenius-Perron operator $P : L'(0,1) \to L'(0,1)$ associated with $S$ is defined by

$$\int_A Pf \, dm = \int_{S^{-1}(A)} f \, dm, \quad \forall \text{ measurable subsets } A \text{ of } [0,1]. \quad (1)$$

It is well known [4] that a density function $f^*$ is a fixed point of $P$, called a stationary density of $P$ if and only if the absolutely continuous probability measure $\mu$, defined by $\mu(A) = \int_A f^* \, dm$, is $S$-invariant, that is, $\mu(S^{-1}(A)) = \mu(A)$ for all measurable subsets $A$ of $[0,1]$. Absolutely continuous invariant measures describe the statistical properties of the dynamics of the transformation $S$ for almost all initial states.

Divide $[0,1]$ into $n$ equal subintervals with $x_i = i h, i = 0, 1, \ldots, n$, where $h = 1/n$, and denote by $\Delta_n$ the subspace of $L'(0,1)$ consisting of all continuous piecewise linear functions defined on $[0,1]$ corresponding to the partition. The “generalized” Lagrange interpolation operator $L_n : L'(0,1) \to \Delta_n$ is defined as

$$L_n f(x) = \sum_{i=0}^n f_i e_i(x), \quad (2)$$

where $f_i = \lim_{t \to 0} (2t)^{-1} \int_{x_i-t}^{x_i+t} f(x) \, dx$ is the “trace” value of $f$ at $x_i$ and the functions

$$e_i(x) = \omega \left( \frac{x-x_i}{h} \right), \quad i = 1, 2, \ldots, n$$

constitute the canonical basis for $\Delta_n$ constructed from the usual tent function

$$\omega(x) = (1 - |x|)_{[-1,1]}(x).$$

It is clear that $f_i$ is the average value of $f$ around the node $x_i$ and

$$L_n f(x_i) = f_i \quad \text{for} \quad 0 \leq i \leq n.$$
Clearly \( \lim_{n \to \infty} \| f - L_n f \| = 0. \) Let \( P_n = L_n P. \) Then \( P_n : \Delta_n \to \Delta_n \) is a finite dimensional positive operator, so the spectral radius \( r(P_n) \) of \( P_n \) is a dominant eigenvalue of \( P_n. \) In the piecewise linear interpolation method we calculate a density eigenvector \( f_n \) associated with the eigenvalue \( \lambda_n \equiv r(P_n), \) which is an approximation to a stationary density of \( f^* \) of \( P. \)

Let \( f = \sum_{i=0}^{n} v_i e_i \in \Delta_n. \) Then, from

\[
P_n f = \sum_{i=0}^{n} v_i P_n e_i = \sum_{i=0}^{n} v_i \lim_{t \to \infty} \frac{1}{2t} \int_{t}^{t+1} P e_i \ dm \ e_j = \sum_{j=0}^{n} \left( \sum_{i=0}^{n} v_i \lim_{t \to \infty} \frac{1}{2t} \int_{t}^{t+1} P e_i \ dm \right) e_j,
\]

the eigenvector equality

\[
P_n f_n = \lambda_n f_n
\]

holds if and only if \( \lambda_n v = v \overline{P}_n, \) where \( v = (v_0, v_1, \ldots, v_n) \) and

\[
\overline{P}_n = \left[ \lim_{t \to \infty} \frac{1}{2t} \int_{t}^{t+1} e_i \ dm \right]_{i=0}^{n}.
\]

The following properties of \( L_n \) have been proved in [3].

**Lemma 2.1.**

(i) \( V_1^n L_n f \leq V_1^n f, \quad \forall \ n, \quad \forall \ f \in BV(0,1). \)

(ii) \( V_1^n (f - L_n f) \leq h \max_{x \in [0,1]} |f^*(x)|, \quad \forall \ f \in C^2[0,1]. \)

3. **BV-Convergence Analysis**

We assume that \( P \) satisfies the following Lasota-Yorke inequality [4]

\[
V_0^1 Pf \leq a V_0^1 f + b \| f \|, \quad \forall \ f \in BV(0,1),
\]

where \( 0 < a < 1 \) and \( b > 0 \) are two constants. It follows from the definition of the BV-norm and (i) of Lemma 2.1 that

\[
\| P_n f \|_{BV} \leq a \| f \|_{BV} + b \| f \|, \quad \forall \ f \in BV(0,1), \quad \forall \ n.
\]

It is well known (see, e.g., [2]) that the Lasota-Yorke inequality (4) implies that the Frobenius-Perron operator \( P :BV(0,1) \to BV(0,1) \) is quasi-compact, so that the spectral radius 1 of \( P, \) which is the dominant eigenvalue of \( P \) is an isolated eigenvalue of \( P. \) Fix a circle \( \Gamma \) centered at 1 with radius \( \varepsilon \) and a positive number \( \delta \) such that \( 1 - \varepsilon > a + \delta \) and \( \Gamma \) does not enclose any spectral point of \( P \) except for 1. Let \( f_n \) be a density function eigenvector of \( P_n \) corresponding to the dominant eigenvalue \( \lambda_n \) of \( P_n \) for any \( n. \) The following lemma has been established in [3].

**Lemma 3.1.**
For $n$ large enough,

$$\lambda_n > 1 - \varepsilon, \quad V^1_n f_n \leq \frac{b}{\delta}$$

and

$$\lambda_n - 1 \leq \frac{2a}{n} V^1_n f_n + \frac{2b}{n}.$$  \hspace{1cm} (7)

Now we can the following result based on the Lasota-Yorke inequality.

**Lemma 3.2.**

$$\| f_n - f^* \|_{BV} \leq \frac{1}{1 - a} (b \| f_n - f^* \| + \| \lambda_n - 1 \| f_n - f^* \|_{BV} + \| f^* - Q_n f^* \|_{BV}).$$  \hspace{1cm} (8)

**Proof.** Since $P_n f_n = \lambda_n f_n$,

$$f^* - f_n = P_n (f^* - f_n) + (\lambda_n - 1)f_n + f^* - Q_n f^*.$$  Then, using (5), we obtain

$$\| f_n - f^* \|_{BV} \leq \| P_n (f^* - f_n) \|_{BV} + \| (\lambda_n - 1)f_n \|_{BV} + \| f^* - Q_n f^* \|_{BV} \leq a \| f_n - f^* \|_{BV} + b \| f_n - f^* \| + \| \lambda_n - 1 \| f_n \|_{BV} + \| f^* - Q_n f^* \|_{BV}.$$  Thus (8) follows. Q.E.D.

With the help of the above three lemmas, we are ready to prove the main convergence result.

**Theorem 3.1.**

Suppose that the Frobenius-Perron operator $P$ has a unique fixed density $f^*$ such that $f^* \in C^2[0,1]$. Let $f_n$ be a sequence of piecewise linear approximate densities of the Frobenius-Perron operator obtained from the interpolation method. Then

$$\lim_{n \to \infty} \| f_n - f^* \|_{BV} = 0.$$  \hspace{1cm} (9)

**Proof.** Lemma 2.1 (ii) shows that $\| f^* - Q_n f^* \|_{BV} = O(h)$ if $f^*$ is smooth enough, and together with Lemma 3.1, we know that $\| \lambda_n - 1 \| f_n \|_{BV} = O(h)$. Therefore, Lemma 3.2 tells us that if $f^* \in C^2[0,1]$, then

$$\| f_n - f^* \|_{BV} = O(\| f_n - f^* \|) + O\left(\frac{1}{n}\right).$$  \hspace{1cm} (10)

It has been proved in [3] that $\lim_{n \to \infty} \| f_n - f^* \| = 0$, so (9) holds. Q.E.D.

**4. Conclusions**

In this paper, using the Lasota-Yorke inequality, we proved the convergence of the piecewise linear interpolation method under the BV-norm for a class of nonsingular transformations of the unit interval, which is a much stronger convergence than the usual $L^1$-norm one. This explains fast convergence observed from the numerical experiment result in [3]. In the future we shall estimate the convergence rate of the interpolation method under both the BV-norm and the $L^1$-norm. A preliminary analysis
based on the concept of strongly stable approximations of linear operators makes us believe the validity of the following convergence rate equality under the BV-norm:

$$\| f_n - f^* \|_{BV} \leq C \| f^* - L_n f^* \|_{BV} = O\left(\frac{1}{n}\right)$$

for some constant $C$ uniformly for all $n$, when the unique fixed density function $f^*$ of the Frobenius-Perron operator is regular enough, such as $f^* \in C^2[0, 1]$. The detailed convergence rate analysis under both the $L^1$-norm and the BV-norm will be given in a forthcoming paper.

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