Abstract: Employing a time-dependent mean-field-hydrodynamic model we study the generation of black solitons in a degenerate fermion-fermion mixture in a cigar-shaped geometry using variational and numerical solutions. The black soliton is found to be the first stationary vibrational excitation of the system and is considered to be a nonlinear continuation of the vibrational excitation of the harmonic oscillator state. We illustrate the stationary nature of the black soliton, by studying different perturbations on it after its formation.

Black soliton in a quasi-one-dimensional trapped fermion-fermion mixture

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1. Introduction

The one-dimensional nonlinear Schrödinger (NLS) equation in the repulsive or self-defocusing case is usually written as [1]

\[ iu_t + uu_{xx} - |u|^2u = 0, \]

where the time \( t \) and space \( x \) dependences of the wave function \( u(x, t) \) are suppressed. This equation sustains the following dark and grey solitons [2]:

\[ u(x, t) = r(x - ct) \exp\left[-i\{\phi(x - ct) - \mu t\}\right], \]

with

\[ r^2(x - ct) = \eta - 2\xi^2 \text{sech}^2[\xi(x - ct)], \]

\[ \phi(x - ct) = \tan^{-1}\left[-2\xi/c \tanh[\xi(x - ct)]\right], \]

\[ \xi = \sqrt{(2\eta - c^2)/2}, \]

where \( c \) is the velocity, \( \mu \) is the chemical potential, and \( \eta \) is related to intensity. Soliton (2) having a “notch” over a background density is grey in general. It is dark if density \( |u|^2 = 0 \) at the minimum. At zero velocity the soliton becomes a stationary dark soliton or a black soliton:

\[ |u(x, t)| = \sqrt{\eta \tanh(x\sqrt{\eta/2})}. \]

The similarity of the NLS equation (1) to the mean-field Gross-Pitaevskii (GP) equation describing a trapped zero-temperature Bose-Einstein condensate (BEC) [3] imply the possibility of a stationary dark soliton, or a black soliton, in a trapped BEC or in a degenerate trapped boson-fermion mixture (DBFM) [4]. Dark solitons represent a local minimum in a trapped BEC and have been the topic of both experimental [5] and theoretical [4,6,7] investigations. Both DBFM [8] and degenerate fermion-fermion mixture (DFFM) [9] have been experimentally observed and studied theoretically [10,11]. It has been suggested that the black soliton of a harmonically trapped BEC could be a stationary eigenstate [4] of the GP equation [3] as in

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the case of the trap-less NLS equation. We demonstrated that the black soliton in a trapped zero-temperature BEC [12] or a DBFM [13] is the first stationary vibrational excitation of the nonlinear dynamical equation for the respective systems. No such vibrational excitation of the NLS equation (1) exists in the absence of a harmonic trap.

In this paper we study the numerical simulation of a black or a stationary dark soliton in a trapped DFFM. We use a coupled time-dependent mean-field-hydrodynamic model for a DFFM and consider the formation of black solitons in a quasi-one-dimensional cigar-shaped geometry using numerical and variational solutions. The present model is inspired by the success of a similar model used recently in the investigation of collapse [14], bright [15] and dark [13] solitons in a DBFM and of mixing-demixing in a DFFM [16]. The black soliton of a DFFM is demonstrated to be a stationary vibrational excitation of the mean-field-hydrodynamic model.

We suggest a scheme for numerical simulation of a stationary dark soliton in trapped DFFM by time evolution of the linear harmonic oscillator equation starting with the analytic vibrational excitation, while the nonlinearities corresponding to the mean-field-hydrodynamic equations are slowly introduced. The simulation proceeds through successive eigenstates of the dynamical model. To illustrate the stability of our scheme we study the oscillation of the stationary dark soliton upon application of different perturbations [12].

Collective excitations in the form of solitons and vortices in trapped fermions have also been investigated recently by Damski et al. [17] and Karpiuk et al. [18]. However, these authors considered isolated ultra-cold fermions and not a realistic DFFM as in the experiments and as discussed in this paper. Also they did not demonstrate the existence of stable dark solitons with a zero at the center of the notch as in the present study. They identified grey soliton-like structure with a shallow dip in the iso-vorticity contour as in the present study. They identified dark [13] solitons in a DBFM and of mixing-demixing in a DFFM [16]. The black soliton of a DFFM is demonstrated to be a stationary vibrational excitation of the mean-field-hydrodynamic model.

The mean-field dynamical equations for the system are derived from a hydrodynamic equation [15,13] following Lagrangian density [15,13]

$$\mathcal{L} = \frac{i}{2} \hbar \sum_{j=1}^{2} \left( \psi_{j} \frac{\partial \psi_{j}^{\ast}}{\partial t} - \psi_{j}^{\ast} \frac{\partial \psi_{j}}{\partial t} \right) + g_{12} n_{1} n_{2} + \left( \sum_{j=1}^{2} \left( \frac{\hbar^{2} |\nabla \psi_{j}|^{2}}{6 m_{j}} + \frac{V_{j} n_{j}}{3 \beta A_{j}} \right) \right),$$

where $n_{j}$ is the mass of component $j (j = 1, 2)$, $\psi_{j}$ a complex probability amplitude, $n_{j} = |\psi_{j}|^{2}$ the real probability density, $N_{j} \equiv \int dr n_{j}(r)$ the number of respective atoms, $A_{j} = \hbar^{2}((2\pi)^{2} 3/2) / (2m_{j})$, and $V_{j}$ is the confining trap. Here the interspecies coupling is $g_{12} = 2 \pi \hbar^{2} a_{12} / m_{R}$ with the reduced mass $m_{R} = m_{1} m_{2} / (m_{1} + m_{2})$, and $a_{12}$ is the interspecies scattering length. The interaction between intra-species fermions in spin-polarized state is highly suppressed due to the Pauli blocking term $3 \sqrt{2} \sqrt{3} / 5$ and has been neglected in Eq. (6) and will be neglected throughout. The kinetic energy terms in this equation $\hbar^{2} |\nabla \psi_{j}|^{2} / (6 m_{j})$ are derived from a hydrodynamic equation for the fermions [11] and contribute little to this problem compared to the dominating Pauli blocking term in Eq. (6). However, the inclusion of the kinetic energy terms in Eq. (6) leads to a smooth solution for the probability density everywhere [15]. To keep the algebra simple and without losing generality, in our calculation we shall take equal fermion masses: $m_{1} = m_{2} \equiv m / 3$. This simulates well the fermion mixtures in two hyperfine states of $^{6}$Li and $^{40}$K atoms observed experimentally [9]. The Lagrangian density of each fermion component in Eq. (6) is identical to that used in [13, 15].

The mean-field dynamical equations for the system are just the usual Euler-Lagrange (EL) equations [21]"
where \( x_k, k = 1, 2, 3 \), are the three space components. With Lagrangian density (6) the following EL equations of motion are derived in a straight-forward fashion [14,15]:

\[
\left[ -i \frac{\partial}{\partial \tau} - \frac{h^2 \nabla^2}{6m_j} + V_j(r) + A_j n_j^2 + g_1 n_k \right] \varphi_j = 0, \quad (8)
\]

\( j \neq k = 1, 2 \).

This is essentially a time-dependent version of a similar time-independent model recently used for fermions [22]. For a stationary state Eqs. (8) yield the same result as the formulation of Ref. [11,22]. For a system with large number of fermions both reduce to [14,22] the well-known Thomas-Fermi approximation [3,23]: \( n_j = \left[ (\mu_j - V_j)/A_j \right]^{3/2} \), with \( \mu_j \) the chemical potential.

We reduce Eq. (8) to a minimal quasi-one-dimensional form in a cigar-shaped geometry where the confining trap vanishes for any \( r \) such that \( V_j(r) = \frac{3}{2} m_j \omega^2 (r^2 + z^2) \), with \( r \) the radial vector and \( z \) the axial vector. For a cigar-shaped geometry \( \nu \ll 1 \), we consider solutions of Eq. (8) of the type \( \psi_j(r, t) = \sqrt{N_j} \phi_j(x, \tau) \psi_j^{(0)}(p)/\sqrt{L} \), where \( x = z/l, \tau = \tau_{osc}/l, L = \sqrt{\hbar/(\rho \omega m)} \), and \( \psi_j^{(0)}(p) \) is the circularly symmetric ground-state linear harmonic-oscillator wave function. In the \( \nu \ll 1 \) limit the quasi-one-dimensional functions \( \phi_j(x, \tau) \) satisfy the following dimensionless equations [16]:

\[
\left[ -i \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2} + x^2 + N_{jk} |\phi_k|^2 + N_{jj} |\phi_j|^{4/3} \right] \phi_j(x, \tau) = 0, \quad j \neq k = 1, 2, \quad (9)
\]

where \( N_{jk} = 12n_{12}(N_k/\nu)/l \), and \( N_{jj} = 9(6\pi N_j/\nu)^{1/2}/5 \). In Eq. (9), the normalization condition is given by \( \int_0^\infty |\phi_j(x, \tau)|^2 dx = 1 \).

In case when \( N_1 = N_2 = N \), the two equations (9) become identical and reduce to

\[
\left[ -i \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2} + x^2 + 2 |\phi|^2 + |\beta \phi|^2 \right] \phi(x, \tau) = 0, \quad (10)
\]

which permits the solution \( \phi = \phi_1 = \phi_2 \), with \( \beta = N_{jj} \) and \( \gamma = N_{jk} \).

3. Variational analysis

3.1. Symmetric case \((N_1 = N_2)\)

Next we present a variational analysis of Eq. (10) based on the stationary trial wave function [24]

\[
\psi_\phi(x, \tau) = A(\tau) \exp \left[ -\frac{x^2}{2R^2(\tau)} + \frac{i}{2} b(\tau) x^2 + ic(\tau) \right], \quad (11)
\]

where \( A \) is the amplitude, \( R \) is the width, \( b \) the chirp, and \( c \) the phase. The Lagrangian density for Eq. (10) is the one-term version of Eq. (6), e.g.,

\[
\mathcal{L} = \frac{i b}{2} \left[ \phi_\phi \frac{\partial \phi_\phi}{\partial \tau} - \phi_* \frac{\partial \phi_*}{\partial \tau} \right] + \left[ \frac{\partial \phi_\phi}{\partial x} \right]^2 + \frac{1}{2} \gamma n^2 + \frac{3}{5} \beta n^3 + x^2 n, \quad (12)
\]

which is evaluated with this trial function and the effective Lagrangian \( L = \int_\infty^{-\infty} \mathcal{L} \rho d\tau \) becomes

\[
L = \frac{A^2 R^3 \sqrt{\pi}}{2} \left( \alpha \beta (R/\pi)^{1/3} + \epsilon + \frac{3R^2 \beta}{4} + \frac{3\gamma}{16R^2} A^2 R^2 + \frac{3}{2} \beta^2 R^2 + \frac{3}{2} \gamma R^2 \right), \quad (13)
\]

where \( \alpha = (54/125\sqrt{\pi})(3/5)^{1/6} \Gamma(13/6) \approx 0.242269 \) and the overhead dot denotes time derivative. The variational Lagrange equations [21]

\[
d \frac{\partial L}{\partial q} = \frac{\partial L}{\partial \dot{q}} \frac{\dot{q}}{\dot{q}}, \quad (14)
\]

where \( q \) stands for \( c, A, R \) and \( b \) can then be written as

\[
\sqrt{\frac{\pi}{2}} \frac{A^2 R^3}{4} = \text{constant} = 1. \quad (15)
\]

\[
3b = -\frac{6}{R^4} - 6b^2 + 3\gamma A^2 \frac{R^2}{2\sqrt{2}} - 6 - 20 \frac{3}{9} \alpha \beta \frac{A^4}{R^7} + 4\epsilon. \quad (16)
\]

\[
5b = -\frac{2}{R^4} - 10b^2 - 5\gamma A^2 \frac{R^2}{4\sqrt{2}} - 10 - 52 \frac{9}{9} \alpha \beta \frac{A^4}{R^7} - 4\epsilon. \quad (17)
\]

\[
\dot{R} = 2Rb. \quad (18)
\]

The constant in Eq. (15) is fixed by the normalization condition. Eliminating \( \dot{c} \) from Eqs. (16) and (18) we obtain

\[
2b = \frac{4}{R^4} - 4b^2 + \gamma \frac{A^2}{4\sqrt{2}} - 4 + \frac{8}{9} \alpha \beta \frac{A^{4/3}}{R^{13/3}}. \quad (19)
\]

The use of Eqs. (15), (17), and (19) leads to the following differential equation for the width \( R \):

\[
d^2 R \over d\tau^2 = \left( \frac{4}{R^4} + \gamma \frac{R}{2\sqrt{2} \pi R^2} - 4R + \frac{8}{9} \frac{2\epsilon \alpha \beta}{9\pi^2} \right) \quad (20)
\]

\[
= -\frac{d}{dR} \left[ \frac{2}{R^4} + \gamma \frac{1}{R^2} \frac{1}{2\sqrt{2} \pi R^2} + 2R^2 + \frac{4\epsilon \beta}{3} \left( \frac{4}{\pi R^2} \right)^{1/2} \right]. \quad (21)
\]

The quantity in the square bracket is the effective potential of the equation of motion. Small oscillation around a stable configuration is possible when there is a minimum in this potential. The variational result for width \( R \) follows by setting the right-hand-side of Eq. (20) to zero corresponding to a minimum in this effective potential, from which the variational profile for the soliton can be obtained [24] using Eq. (11).
3.2. Asymmetric Case ($N_1 \neq N_2$)

The above variational analysis can be extended to the asymmetric case. However, the algebra becomes quite involved if we take a general variational trial wave function with chirp and phase parameters. As we are interested mostly in the density profiles, we consider the following normalized stationary trial wave function for fermion component $j$ of Eq. (9)

$$\varphi_{nj} = \left(\frac{2}{R_j^2(\tau)}\right)^{1/2} x \exp\left[-\frac{x^2}{2R_j^2(\tau)}\right], \quad j = 1, 2. \quad (22)$$

Using essentially the Lagrangian density (6) in this case we obtain the following effective Lagrangian

$$L = \frac{3N_{jk}}{\sqrt{\pi}} \frac{R_j^3 R_k^3}{(R_j^2 + R_k^2)^{3/2}} + \sum_{j=1}^{2} \frac{\alpha^2}{\sqrt{\pi}} \frac{N_{jj}}{R_j^3} + \frac{3(R_j^4 + 1)}{2R_j^2}. \quad (23)$$

The variational Lagrange equations (14) for $R_1$ and $R_2$ now become

$$\frac{4}{R_1^3} - 4R_1 - \frac{4N_{1j}}{\sqrt{\pi}} \frac{2R_1 R_1^4 - 3R_1^3 R_1^2}{R_1^3 + R_2^3} + \frac{2\alpha^2}{\sqrt{\pi}} \frac{8\alpha N_{jj}}{9\pi} = 0. \quad (24)$$

Eqs. (24) can be solved for the variational widths $R_j$ and consequently the variational profile of the wave functions obtained from Eq. (22). When $N_1 = N_2$, in Eq. (24) $N_{jj} = \beta$, $N_{jk} = N_{kj} = \gamma$ and $R_1 = R_2 = R$; and in this case it is verified that Eq. (24) yields the same variational widths as from the result obtained in the symmetric case in Sec. 3.1 given by Eq. (20); e.g.,

$$\frac{4}{R^3} - 4R + \frac{\gamma}{R^2} \frac{1}{2\sqrt{2\pi}} + \frac{2\alpha^2}{\sqrt{\pi}} \frac{8\alpha}{9\pi} = 0. \quad (25)$$

3.3. Black soliton in a BEC

By a variational calculation we establish the stability of a black soliton in a trapped BEC satisfying the one-dimensional GP equation:

$$\left[-i \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2} + x^2 + n |\phi|^2\right] \phi(x, \tau) = 0. \quad (26)$$

This equation is identical with Eq. (10) with $\beta = 0$ and $\gamma = n$. Hence the variational analysis of Sec. IIIA applies to this case and the differential equation for width $R$ is given by Eq. (20) with $\beta = 0$:

$$\frac{d^2 R}{d\tau^2} = \left(\frac{4}{R^8} + \frac{n}{2\sqrt{2\pi} R^2} - 4R\right). \quad (27)$$

The variational width in this case is given by setting the right-hand-side of this equation to zero.
Fig. 2 (online color at www.lphys.org) The black soliton $|\phi(x)|$ from the numerical solution of Eq. (10) (denoted num) vs. $x$ for a DFFM with $N_1 = 10$, $N_2 = 10$, $a_{12}/l = 0.05$, while $N_{ij} = N_{22} \approx 275$, and $N_{ij} = 60$. The corresponding variational solution obtained from Eq. (27) (denoted var) is also shown.

### 4.1. Black soliton in a BEC

The results for black solitons in a BEC are obtained by solving Eqs. (26) and (27) and are presented in Fig. 1a and Fig. b for $n = 10$ and 100, respectively, where the soliton profiles are displayed. The variational profiles lie close to the numerical results and confirm the existence of a black soliton in this case. For smaller values of nonlinearity, the solitons have a probability distribution close to the ansatz given by (11). Consequently, the variational result is more accurate. For larger nonlinearities the actual probability distribution obtained from the numerical solution deviates from ansatz (11) as can be seen from Fig. 1b. By construction the variational ansatz (11) is stationary. The close agreement between the variational and numerical solutions demonstrates the stationary nature of these dark solitons. Hence, contrary to comments in the literature [7], these black solitons of a trapped BEC are stationary. Although, we illustrate the present simulation for a black soliton for a repulsive BEC, stationary dark soliton also exists for an attractive BEC [12].

### 4.2. Black soliton in a DFFM

The numerical results for black solitons in a DFFM are obtained by solving Eqs. (10) and (9) for the symmetric ($N_1 = N_2$) and asymmetric ($N_1 \neq N_2$) cases, respectively. The corresponding variational results are obtained from Eqs. (25) and (24), respectively. First we present the numerical and variational results for the symmetric situation with $a_{12}/l = 0.05$, $\nu = 0.1$, and $N_1 = N_2 = 10$ in Fig. 2. The nonlinearities here are quite large, e.g., $N_{ij} \approx 275$ and $N_{ij} = 60$, $j \neq k = 1, 2$. The variational and numerical results in this case compare favorably as can be seen in Fig. 2.

Next we present the numerical and variational results for the asymmetric case with $N_1 = 10$ and $N_2 = 20$.

Fig. 3 (online color at www.lphys.org) The black solitons (a) $\phi_1(x)$ and (b) $\phi_2(x)$ from the numerical solution of Eqs. (9) (denoted num) for a DFFM with $N_1 = 10$, $N_2 = 20$, $a_{12}/l = 0.05$, while $N_{11} \approx 275$, $N_{22} \approx 436$, $N_{12} = 120$, and $N_{21} = 60$. The corresponding variational solutions (denoted var) obtained from Eqs. (24) are also shown.

Fig. 4 (online color at www.lphys.org) The dark soliton profile for the black soliton of a DFFM of Fig. 2 with $N_1 = N_2 = 10$ at times $\tau = 0$, $\pi/2$, $3\pi/2$, $2\pi$, $5\pi/2$, and $3\pi$ after the soliton has been set into oscillation by displacing the trap through a distance of $2 \mu$m.

$a_{12}/l = 0.05$ and $\nu = 0.1$ corresponding to even larger nonlinearities $N_{11} \approx 275$, $N_{22} \approx 436$, $N_{12} = 120$, and $N_{21} = 60$. The variational and numerical results in this case for black solitons for components 1 and 2 are exhibited in Fig. 3a and Fig. b.

From Figs. 1–3 we see that although the variational result is a good quantitative approximation to the exact
The stability of the dark soliton for $N_1 = N_2 = 10$ under this sudden change in the trap frequency is exhibited in Fig. 6 where we plot the soliton profile at different times. A poor initial variational harmonic-oscillator ansatz for the soliton is responsible for the imprecise variational results for large nonlinearities.

To demonstrate the robustness of the black solitons, after their formation we increased suddenly the strength of the trap by 10%: $x^2 \rightarrow 1.1x^2$. The soliton executes reasonably stable breathing oscillation. To study the breathing oscillation in detail we calculate the root-mean-square (rms) size $\langle s \rangle$ of the soliton at different times from a numerical solution of the mean-field equation and the result is plotted in Fig. 7b. The amplitude and frequency of the variational oscillation are in good qualitative agreement with the precise numerical results. However, there is quantitative disagreement between the two. After multiplying the variational size by 0.89, the results in Fig. 7b are in good agreement with the numerical result in Fig. 7a. The period of breathing oscillation from Fig. 7a is 7.3 ms and that from Fig. 7b is 7.6 ms. The fair agreement between the variational and numerical results of breathing oscillation illustrates that the variational description of the stationary dark soliton is on the right track and confirms the stationary nature of the black soliton.
5. Summary

In conclusion, we used a coupled mean-field-hydrodynamic model for a DFFM to study the formation of black solitons in a quasi-one-dimensional geometry by numerical and variational methods. The existence of stationary dark solitons is predicted by variational study and confirmed by detailed numerical analysis. The stationary nature of the present solitons is demonstrated numerically through their (a) sustained oscillation initiated by a sudden displacement of the harmonic trap, as well as (b) breathing oscillation initiated by a sudden increase of the strength of the harmonic trap by 10%: $x^2 \rightarrow 1.1x^2$. Here we used a set of mean-field equations for the DFFM. A proper treatment of a degenerate Fermi gas or DFFM should be done using a fully antisymmetrized many-body Slater determinant wave function [10] as in the case of scattering involving many electrons [26]. However, in view of the success of a fermionic mean-field model in studies of collapse [14], bright [15] and dark [13] soliton in a DBFM and of mixing and demixing in a DFFM [16], we do not believe that the present study on bright solitons in a DFFM to be so peculiar as to have no general validity.

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