THE HELE-SHAW PROBLEM WITH SURFACE TENSION IN THE CASE OF SUBDIFFUSION

Nataliya Vasylyeva* and Vitalii Overko

Institute of Applied Mathematics and Mechanics of NAS of Ukraine,
Dobrovolskogo, 1, Slov’iansk, 84100, Ukraine

(Communicated by Bernd Kawohl)

Abstract. In this paper we analyze anomalous diffusion version of the multi-dimensional Hele-Shaw problem with a nonzero surface tension of a free boundary. We prove the one-valued solvability of this moving boundary problem in the Hölder classes. In the two-dimensional case some numerical solutions are constructed.

1. Introduction. Let \( \Omega(t) \) be a connected bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \), for each \( t \in [0, T] \) with a boundary \( \partial \Omega(t) = \Upsilon(t) \cup \Gamma_0 \), \( \Upsilon(t) \cap \Gamma_0 = \emptyset \). In the present paper, we will investigate a quasi-stationary system governed by the following set of equations:

\[
-\Delta_y p = 0 \text{ in } \Omega(t), \ t \in [0,T]; \tag{1}
\]

\[
V^\nu_{nt} = -\mu \frac{\partial p}{\partial n_t}, \quad p = \gamma \kappa(\Upsilon(t)) \text{ on } \Upsilon(t), \ \nu \in (0,1); \tag{2}
\]

\[
(1-\delta)p - \delta \frac{\partial p}{\partial n} = \varphi(y) \text{ on } \Gamma_{0T} = \Gamma_0 \times [0,T]; \tag{3}
\]

\[
\Omega(0) = \Omega, \quad \Upsilon(0) = \Upsilon \text{ are given.} \tag{4}
\]

Here \( p \) and \( \Upsilon(t) \) are a desired function and an unknown interface, correspondingly; \( \Delta_y = \nabla_y^2, \nabla_y = (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}); \mu, \gamma \) are positive constants; \( n_t \) and \( n \) are unit normals to \( \Upsilon(t) \) and \( \Gamma_0 \), correspondingly, directed outside \( \Omega(t) \); the quantity \( \kappa(\Upsilon(t)) \) defined on \( \Upsilon(t) \) denotes the mean curvature of this surface. We use the sign convention that convex hypersurfaces have positive mean curvature. \( V^\nu_{nt} \) is the fractional velocity of the boundary \( \Upsilon(t) \) in the direction of the normal \( n_t \) and is represented by (see, e.g. [39])

\[
V^\nu_{nt} = \langle D^\nu_{t} \Upsilon(t), n_t \rangle, \tag{5}
\]

where \( \langle \cdot, \cdot \rangle \) is the notation of the inner product. \( D^\nu_{t} \Upsilon(t) \) denotes the Caputo fractional derivative with respect to \( t \) and is defined by (see (2.4.6.) in [21])

\[
D^\nu_{t} w(\cdot,t) = \frac{1}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int_0^t \frac{w(\cdot,\tau) d\tau}{(t-\tau)^\nu} - \frac{w(\cdot,0)}{\Gamma(1-\nu)t^\nu}, \ \nu \in (0,1), \tag{6}
\]

2000 Mathematics Subject Classification. Primary: 35R35, 35C15; Secondary: 35B65, 35R11.

Key words and phrases. Quasistationary Stefan problem, anomalous diffusion, Caputo derivative, regularizer, coercive estimates.

Authors are supported by Grant FP7-People-2011-IRSES project number 295164 (EUMLS: EU-Ukrainian Mathematicians for Life Sciences).

* Corresponding author.
where \( \Gamma(\nu) \) is the Gamma function. Thus, the velocity of the free boundary is assumed to be nonlocal in time. Finally, \( \varphi(y) \) is a given function and the integer \( \delta \in \{0, 1\} \) is introduced to label the boundary condition on the fixed boundary \( \Gamma_0 \) (where \( \delta = 0 \) corresponds to the Dirichlet boundary condition and \( \delta = 1 \) does to the Neumann condition).

Note that the first condition in (2) implies that the motion in problem (1)-(4) is subjected by anomalous diffusion. We recall that the anomalous diffusion process is governed by non Fickian law, other words, the mean square displacement of the diffusing spaces \( \langle (\Delta y)^2 \rangle \) scales as a nonlinear power law in time, \( \langle (\Delta y)^2 \rangle \sim t^\nu \) for some real number \( \nu \) \cite{6}. In the case \( \nu \in (0, 1) \), this is referred as a subdiffusion. Fractional derivatives and fractional differential equations are used to describe mathematical models for anomalous diffusion. Note that subdiffusion or superdiffusion (\( \nu > 1 \)) arises under investigation of complex processes in physics \cite{6, 12, 19}, biology \cite{36}, medicine \cite{37}, chemistry \cite{26} and economics \cite{7}.

If \( \nu = 1 \), free boundary problem (1)-(4) is interpreted as the classical Hele-Shaw problem, which has a hydrodynamic sense and models the pressure of fluid squeezed between two parallel plates, a small distance apart \cite{18}. The motion of fluid is governed by the Darcy law stating that the velocity of fluid is proportional to the pressure gradient. Note that, if \( \gamma \neq 0 \) problem (1)-(4) describes the tumor growth in medicine \cite{15}. As it is noted in \cite{20}, there is a vast literature on the Hele-Shaw and related problems. This literature includes, in particular, well-posedness proofs of the Hele-Shaw problem in various physical settings. We do not provide here a complete survey on the results related to problem (1)-(4) for \( \nu = 1 \), but just point out some of them for \( \gamma \neq 0 \). The existence and uniqueness of a solution, for a general initial smooth or nonsmooth domain \( \Omega \), for small time interval have been proved by Chen \cite{8}, Escher and Simonett \cite{14}, Bazaliy \cite{2}, Prokert \cite{33}, Bazaliy and Friedman \cite{4}. The methods used by these authors are all different.

The mathematical model of a solute release from a polymer matrix in the quasistationary case is described by system (1)-(4) for \( \nu = 1, \gamma = 0 \) \cite{27}. If solute movement occurs in heterogeneity media (anomalous) then the process is described with (1)-(4), for \( \nu \in (0, 1) \) \cite{28}. We recall that each monolithic polymeric matrix consists of two regions: the surface zone (or liquid phase) \( y \in \Omega(t) \) and the core \( B(t) \) within the interior of \( \Upsilon(t) \). \( B(t) \cap \Omega(t) = \emptyset \) (\( \Gamma_0 \) is in the interior of \( \Upsilon(t) \)). All the solute is dissolved in the surface zone, and the core contains undissolved solute. These two zones are separated by the diffusion front (liquid-solid interface) \( \Upsilon(t) \) which moves inward as time progresses, i.e. \( \Omega(t_1) \subset \Omega(t_2), t_1 < t_2 \). Concentration of drug in the liquid phase is given by the function \( p(y, t) \) and the function \( \varphi(y) \) describes the drug concentration in the bulk liquid phase. Liu and Xu \cite{28} were the first who presented a moving boundary problem governed by time fractional derivative in drug release process. Note that if equation (1) is changed by the subdiffusion equation we will get the fractional Stefan problem, which arises under consideration of the materials with memory \cite{39} and studied by \cite{1} for the motion of planar, cylindrical and spherical domains. Problem like (1)-(4) with \( \gamma = 0, \nu \in (0, 1) \) were investigated in the case \( \Omega(t) \subset R^1 \) in \cite{28, 29, 38, 39} where explicit solutions were found for some cases and some numerical studies were performed. As for construction of mathematical theory for such problems, the classical solvability of the fractional Hele-Shaw problems for small period of time in the case of zero surface tension (\( \gamma = 0 \)) has been proved in \cite{41}, \( \Omega(t) \subset R^n, n \geq 2 \). The analogous result has been obtained for the fractional Muskat problem in \cite{43}.\n
In this paper we focus on two main questions. The first one is the one-to-one solvability of problem (1)-(4) locally in time in the Hölder classes if $\gamma > 0$. To this end we adapt the classical approach which is used for a free boundary problem in the case of normal diffusion (see, e.g. [2]) to the subdiffusion case. We reduce a moving boundary problem (1)-(4) to a nonlinear problem in a fixed domain, then linearize the problem and obtain one-valued solvability of the linearized problem. After that we use this result for the reduction of the nonlinear problem to a fixed point theorem. In this route we have to solve a lot of technically difficulties which are related to the nonlocal behavior of the free boundary velocity (i.e. with fractional derivative in time). The main one of them deals with the investigation of a nonclassical boundary value problem with the Caputo derivative in a boundary condition

$$\Delta x U = 0 \text{ in } R^2_T; \ \varphi(x',0) = 0 \text{ in } R^{n-1}; \ \frac{\partial^2 \varphi}{\partial x_i \partial x_j} = 0 \text{ on } R^{N-1}_T;$$

$$D^\nu \varphi + a_0 \frac{\partial U}{\partial x_n} = f(x',t) \text{ on } R^{n-1}_T. \quad (7)$$

Note that this is the principal model problem such that nonlinear problem (1)-(4) will inherit the main features of (7). Using Fourier and Laplace transformations, we obtain the solution of (7) as the convolutions: $U = K \ast D^2 \varphi$, $\varphi = G \ast f$ (see (74) and (75)). The kernels $G$ and $K$ can be represented only as integrals which contain the Wright functions. Thus nonlocal form of kernels is the distinguishing feature of the fractional case. As usual in the potential theory, to evaluate the functions $U$ and $\varphi$ it is necessary to describe well the properties of $K$ and $G$. Unfortunately, in virtue of nonlocal representations for these kernels, it is impossible to get the suitable local estimates as well as in the case of integer order derivative ($\nu = 1$) for $G$ and $K$ and their derivatives. We can obtain just the integral estimates of these kernels which are represented in Lemma 4.2. In this route we essential use the main properties of the Wright functions represented in Proposition 2. Generally speaking, Lemma 4.2, Propositions 2 and 4 are significant under investigation of problem (7) and their proofs contain the main difficulties caused by the presence of the fractional derivative in time.

The second issue considered in the paper is the construction of a numerical solution to problem (1)-(4) in the two-dimensional case. Here we implement like [41] an explicit boundary tracking method, which is related with nonlocal behavior of the free boundary velocity. We recall that in the case of the classical Stefan problem there exists a lot of approaches which allow to avoid explicit boundary tracking. For instance, the level set method [9] and enthalpy formulation [38] can reduce the problem with the moving boundary to the problem in the fixed domain. As it was shown in [40], the enthalpy formulation for a problem with the fractional derivative in general describes a various physical process. Level set method, in turn, demands to find the velocity $V_n$ from the first condition in (2) which is possible only for $\nu = 1$. Thus, these methods can not be applied for the free boundary problems governed by subdiffusion.

Note that this work is a continuation of the investigation of the one- or two-phase fractional Hele-Shaw problems reported in [41]– [43]. Thus we will drop here some technical calculations that are insignificantly different from similar one in the papers [41]– [43].

The paper is organized as follows. In Section 2 we describe some useful auxiliary properties related to the fractional derivatives in time: Propositions 1–3. These
results are essentially used throughout the paper. In Section 3, we reduce problem (1)-(4) with an unknown boundary \( Y(t) \) to a nonlinear problem in a fixed domain \( \Omega_T = \Omega \times (0,T) \) and formulate the main result, Theorem 3.1. In Subsection 3.2, we rewrite this nonlinear problem as \( \mathcal{A}z = F(z) \), where \( z = (u, \sigma) \) and \( \mathcal{A} \) is a linear operator, \( F \) is a nonlinear one. Section 4 is dedicated to the investigation of model problem like (7). Then in Section 5, we prove the main result of this paper, Theorem 3.1. In Subsection 5.1, adapting the technique of the regularizer Definition 2.1. The Banach space \( C \) is a contraction one. Section 6 contains numerical study of free boundary problem (1)-(4) if \( \Omega(t) \subset R^2 \), \( t \in [0,T] \). Proofs of some auxiliary assertions which are applied in Section 4 are given in Appendix.

2. Function spaces and preliminaries. In order to investigate the free boundary problem (1)-(4) we need some definitions and auxiliary results.

Let \( D \) be a given domain in \( R^n \), \( D_T = D \times (0,T) \); \( \bar{x}, x \) be any points in \( D \), \( x \neq \bar{x}; t, \tau \in [0,T] \), \( t \neq \tau \); \( \alpha, \beta \in (0,1) \). In this paper we will use the following function spaces: \( C([0,T], C^{l+\alpha}(\bar{D})), \ C^{l+\alpha, \beta}(\bar{D}_T), \ C^{l+\alpha, \frac{\nu}{l+\alpha}}(\bar{D}_T) \). The spaces \( C([0,T], C^{l+\alpha}(D)) \) are used by many authors and their definitions and properties can be found, for instance, in [31].

Denote by

\[
[w]^{(\alpha,\beta)}_{D_T} = \sup_{(x,\bar{x}) \in D, (t,\tau) \in [0,T]} \frac{|w(x,t) - w(\bar{x},t) - w(x,\tau) + w(\bar{x},\tau)|}{|x - \bar{x}|^\alpha|t - \tau|^\beta}.
\]

Next we define function spaces \( C^{l+\alpha, \beta, \alpha}(\bar{D}_T) \).

**Definition 2.1.** The Banach space \( C^{l+\alpha, \beta, \alpha}(\bar{D}_T) \) is the set of functions \( w(x,t) \) with the finite norm:

\[
\|w\|_{C^{l+\alpha, \beta, \alpha}(\bar{D}_T)} = \sum_{|j| = 0}^{l} \|D^j w\|_{C^{l+\alpha, \beta, \alpha}(\bar{D}_T)} = \sum_{|j| = 0}^{l} \{\sup_{D_T} D^j w\} + \{D^j w\}_{x, D_T} + \{D^j w\}_{t, D_T} + \{D^j w\}_{D_T^3}.
\]

where \( \langle w \rangle_{x, D_T} \) and \( \langle w \rangle_{t, D_T} \) are Hölder constants of a function \( w(x,t) \) in \( x \) and \( t \) respectively.

Note that classes \( C^{l+\alpha, \frac{\nu}{l+\alpha}}(\bar{D}_T) \), \( l = 0, 3 \), coincide with usual Hölder spaces (see (1.10)-(1.12) in Chapter 3 [24], and Section 1 in [2]) if \( \nu = 1 \).

**Definition 2.2.** We will say that the function \( w(x,t) \in C^{l+\alpha, \frac{\nu}{l+\alpha}}(\bar{D}_T) \), \( l = 0, 3 \), \( \nu \in (0,1) \), iff the following norms are finite:

\[
\|w\|_{C^{l+\alpha, \frac{\nu}{l+\alpha}}(\bar{D}_T)} = \|w\|_{C([0,T], C^{l+\alpha}(\bar{D})))} + \sum_{|j| = 0}^{l} \{D^j w\}_{x, D_T}^{\frac{\nu}{l+\alpha}} + \sum_{|j| = 1}^{3} \{D^j w\}_{t, D_T}^{\frac{\nu}{l+\alpha}} \quad \text{iff} \quad \lceil l/3 \rceil = 0,
\]

where \( \lceil l/3 \rceil \) denotes the integer part of \( l/3 \);

\[
\|w\|_{C^{l+\alpha, \frac{\nu}{l+\alpha}}(\bar{D}_T)} = \|w\|_{C([0,T], C^{l+\alpha}(\bar{D})))} + \sum_{|j| = 1}^{3} \{D^j w\}_{x, D_T}^{\frac{\nu}{l+\alpha}} + \sum_{|j| = 1}^{3} \{D^j w\}_{t, D_T}^{\frac{\nu}{l+\alpha}} + \|D^\nu w\|_{C([0,T], C^{\alpha}(\bar{D})))} + \{D^\nu w\}_{x, D_T}^{\frac{\nu}{l+\alpha}} + \{D^\nu w\}_{t, D_T}^{\frac{\nu}{l+\alpha}}.
\]
In a similar way we introduce the spaces $C^{l+\alpha,\frac{3+\nu}{2}}(\partial D_T)$ and $C^{l+\alpha,\beta,\alpha}(\partial D_T)$, where $\partial D_T = \partial D \times (0, T)$.

Let $d$ is a smooth surface and $d_T = d \times [0, T]$.

**Definition 2.3.** We will say $w \in P^{l+\alpha}(d_T)$ iff $w \in C([0, T], C^\alpha(\bar{d}))$ and $D_x w \in C^{l+\alpha,\frac{3+\nu}{2}}(\partial D_T)$:

$$\|w\|_{P^{l+\alpha}(d_T)} = \|w\|_{C([0, T], C^\alpha(\bar{d}))} + \|D_x w\|_{C^{l+\alpha,\frac{3+\nu}{2}}(\partial D_T)}.$$

We define classes $C([0, T], C^{l+\alpha}(\bar{D}))$ and $C^{k+\alpha,\frac{3+\nu}{2}}(\partial D_T), k = 0, 1, 2$, as the subspaces of $C([0, T], C^{l+\alpha}(\bar{D}))$ and $C^{k+\alpha,\frac{3+\nu}{2}}(\partial D_T)$ respectively such that $D_x^k w(x, 0) = 0$, $|j| = 0, 1, 2$. In the same way we define $P_0^{l+\alpha}(\bar{d_T})$.

Moreover, we will use the usual H"older spaces $C^{l+\alpha}(\bar{D})$ and $C^{l+\alpha}(\partial D)$, their definitions can be found, for instance, in [25].

Throughout the paper we will need in interpolation inequality (see Corollary 1.2.18 [31]):

$$\|w\|_{C^{l+\alpha(2-2\gamma)}(\bar{D})} \leq C\|w\|_{C^{l+\alpha}(\bar{D})}\|w\|_{C^{l-\alpha}(\bar{D})},$$

where $\partial D \in C^{l_2}, a \in (0, 1), 0 \leq l_1 < l_2$.

Next, we define the fractional Riemann-Liouville integral and derivative of a function $w(\cdot, t)$ with respect to $t$ as (see, e.g., (2.1.1) and (2.1.8) [21]):

$$I^\theta w(\cdot, t) := \frac{1}{\Gamma(\theta)} \int_0^t \frac{w(\cdot, \tau)}{(t-\tau)^{1-\theta}} d\tau, \quad \theta > 0, \ t > 0;$$

$$\partial^\theta_t w(\cdot, t) := \frac{1}{\Gamma(1-\theta)} \frac{\partial}{\partial t} \int_0^t \frac{w(\cdot, \tau)}{(t-\tau)^{\theta}} d\tau, \quad \theta \in (0, 1).$$

The following results which are well known in the case of an integer order derivative will be essentially applied in Sections 4 and 7.1.

**Proposition 1.** Let $\alpha, \nu \in (0, 1), \psi, \varphi \in C([0, T], C^\alpha(\bar{D})), \varphi(x, t), \varphi_1(x, t) \in C([0, T], L_\infty(D)), \partial_t^\nu \varphi(x, t) \in L^1(D_T).$ Then

i: 

$$|\psi(x, t) - \psi(x, t_2)| \leq C|t_1 - t_2|^{\nu}\sup_{\partial D_T} |\partial_t^\nu \psi|;$$

$$\langle \psi(x, t_1) - \psi(x, t_2) \rangle^{(\alpha)}_{x, D_T} \leq C(\partial_t^\alpha \psi(x, t_1, t_2) \rangle \sup_{\partial D_T} |\partial_t^\nu \psi| \cdot \sup_{\partial D_T} \|\psi\|_{L_\infty(D_T)} \forall t_1, t_2 \in [0, T].$$

ii: 

$$D_t^\nu \varphi(x, t) = \partial_t^\nu \varphi(x, t), \quad \text{if} \ \varphi(x, 0) = 0, \ \text{where} \ \partial_t^\nu \varphi(x, t) \ \text{is given by (10)};$$

iii: 

$$\partial_t^\nu \int_0^t \varphi_1(x, t - \tau) \varphi(x, \tau) d\tau = \int_0^t \varphi(x, t - \tau) \partial_t^\nu \varphi_1(x, \tau) d\tau + \varphi(x, t) \lim_{z \to t} I_z^\nu \varphi_1(x, z), \ \forall t \in [0, T].$$

The results of Proposition 1 have been obtained in [43] (see Propositions 2.1 and 2.2 there).

Note that the Wright functions (see, e.g. [13] or [32]):

$$W(z; b, d) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(bk + d)}, \quad z, \ b, \ d \in \mathbb{C},$$
play fundamental roles in various applications of the fractional calculus. Here, we formulate the main properties of this function: asymptotic representations, estimates, formula for fractional differentiation and integration, which will be essentially used to investigate the model problem in Section 4 and to obtain the corresponding coercive estimates of its solution. The proofs of the following properties for the Wright functions are represented in [22,34,35] and [23].

**Proposition 2.** Let $C_1, C_2, A, b, c$ be positive constants, $d \in \mathbb{R}^1, z \in \mathbb{C}$, then there are the following:

1. 
   \[ W(-z; -b, d - 1) + (d - 1)W(-z; -b, d) = bzW(-z; -b; d - b); \]  
   \[ W(z; -b, d) = W(z; -b, d - b); \]  
   \[ 0 < W(-z; -b, d) \leq C_2 \exp\{-C_3|z|^\frac{1}{\alpha}\}, \forall z > 0, d \geq 0, \]  
   where $C_3 := C_3(b)$ is a bounded constant which can be chosen positive if $b \in (0, 1) \text{ and either } d \geq 0 \text{ or } 0 < b \leq d < 1; \]

2. 
   \[ W(-z; -b, 1 - b) \leq C_4(b)z^{\frac{b-1}{2}}\exp\{-C_5(b)|z|^\frac{1}{\alpha}\}, \]
   where $C_4(b) = \frac{1}{\sqrt{2\pi(1-b)}}, C_5(b) = \frac{1}{b} - 1; \]

3. 
   \[ \int_0^{+\infty} x^{b-1}W(-x; -b, d)dx = \frac{\Gamma(l)}{\Gamma(bl + d)}, \forall l > 0 \text{ and } b > 0, \]  
   \[ W(y, 0) = \Gamma(\Omega_T) = \Gamma_0 \times [0, T]; \]

4. 
   \[ W = g_2(y, t) \text{ on } \bar{\Gamma}_T = \bar{\Gamma}_0 \times [0, T]; \]

5. 
   \[ W(0, 0) = g_0(y, t) \text{ in } \Omega_T, \]

6. 
   \[ W(0, 0) = g_1(y, 0) \text{ on } \bar{\Gamma}_T = \bar{\Gamma}_0 \times [0, T]; \]

7. 
   \[ W = g_2(y, t) \text{ on } \bar{\Gamma}_T = \bar{\Gamma}_0 \times [0, T], \]

where $\Delta_y W = g_0(y, t)$ in $\Omega_T = \Omega \times (0, T)$; $W(y, 0) = 0, y \in \Omega$; 

\[ (1 - \delta)W - \frac{\partial W}{\partial n} = g_1(y, t) \text{ on } \Gamma_0 \times [0, T]; \]

\[ W = g_2(y, t) \text{ on } \bar{\Gamma}_T = \bar{\Gamma}_0 \times [0, T], \]

where the integer $\delta \in \{0, 1\}, n$ is the outward normal to $\Omega$.

**Proposition 3.** Let $\alpha, \beta, \nu \in (0, 1), \Gamma_0 \in C^{k+\alpha}, k \geq 2; g_0(y, 0), g_1(y, 0), \]

\[ g_2(y, 0) = 0. \]

1. 
   \[ \mathbb{C}[0, T], C^{k+\alpha}(\bar{\Omega}), \]

2. 
   \[ g_1 \in C([0, T], C^{k-\delta+\alpha}(\Gamma_0)), g_2 \in C([0, T], C^{k+\alpha}(\bar{\Omega})), \]

then there exists a unique solution of problem (23):

\[ \|W\|_{C([0, T], C^{k+\alpha}(\bar{\Omega}))} \leq C[\|g_0\|_{C([0, T], C^{k+\alpha}(\bar{\Omega}))} + \|g_1\|_{C([0, T], C^{k-\delta+\alpha}(\Gamma_0))} + \|g_2\|_{C([0, T], C^{k+\alpha}(\bar{\Omega}))}; \]
ii: If \( g_0 \in C^{k-2+\alpha,\beta,\alpha}(\bar{\Omega}_T) \), \( g_1 \in C^{k-\delta,\beta,\alpha}(\Gamma_0 T) \), \( g_2 \in C^{k+\alpha,\beta,\alpha}(\bar{\Omega}_T) \), then there exists a unique solution of problem (23):
\[
\|W\|_{C^{k+\alpha,\beta,\alpha}(\bar{\Omega}_T)} \leq C[\|g_0\|_{C^{k-2+\alpha,\beta,\alpha}(\bar{\Omega}_T)} + \|g_1\|_{C^{k-\delta,\beta,\alpha}(\Gamma_0 T)} + \|g_2\|_{C^{k+\alpha,\beta,\alpha}(\bar{\Omega}_T)}],
\]
\[(25)\]

iii: If \( g_0, g_1 \equiv 0, g_2 = \sum_{i,j=1}^{n-1} c_{ij} \frac{\partial^2 \bar{g}_2}{\partial x_i \partial x_j}, \bar{g}_2 \in H^{1+\alpha}(\bar{\Omega}_T) \), \( c_{ij} \) are given constants, there is a unique solution of problem (23), \( W \in C^{2+\alpha,\frac{2\alpha}{1+\alpha}}(\bar{\Omega}_T) \):
\[
\|W\|_{C^{2+\alpha,\frac{2\alpha}{1+\alpha}}(\bar{\Omega}_T)} \leq C\|\bar{g}_2\|_{H^{1+\alpha}(\bar{\Omega}_T)},
\]
\[
\max_{\bar{\Omega}} |W| \leq C T^\alpha/2 [\|\bar{g}_2\|_{L^2(0,T),C^4(\bar{T})}] + \|D^\alpha_{\bar{\tau}} \bar{g}_2\|_{L^2(0,T),C(\bar{T})}.
\]
\[(26)\]
\[
\|(\bar{W}(t_1) - \bar{W}(t_2))\|_{C^k(\bar{\Omega})} \leq C T^\frac{\alpha}{2} / \|\bar{g}_2\|_{L^2(0,T),C^4(\bar{T})} \|\bar{W}(t_1) - \bar{W}(t_2)\|_{C^{k+\alpha}(\bar{\Omega})}
\]
\[
\leq C T^{\alpha/2} / \|\bar{g}_2\|_{L^2(0,T),C^4(\bar{T})} \|\bar{W}(t_1) - \bar{W}(t_2)\|_{C^{k+\alpha}(\bar{\Omega})}.
\]
\[(27)\]

Proof. Statements (i), (ii) are simple consequences of classical theory for elliptic equations (see, e.g., Chapter 3 [25]). Let us prove statement (iii). Note that estimate (27) follows from the maximum principle and interpolation inequality (8) which is applied to the function \( D^2_{\bar{\tau}} \bar{g}_2 \).

As it follows from inequality (24), in order to get (27) we should to evaluate the Hölder constant of \( D^k_{\bar{\tau}} W \), \( k = \frac{1}{2} \), with respect to time \( t \). The maximum principle together with properties of the function \( \bar{g}_2 \) allow us to obtain inequality
\[
\langle W \rangle_{t,\Omega_T} \leq C \langle D^2_{\bar{\tau}} \bar{g}_2 \rangle_{t,\Omega_T}.
\]
\[(28)\]

Moreover, the principle maximum and interpolation inequality (8) applying to the function \( D^2_{\bar{\tau}} \bar{g}_2 \) lead to estimate (27). At last, from (8) and (28) we have
\[
\|W(\cdot, t_1) - W(\cdot, t_2)\|_{C^{k+\alpha}(\bar{\Omega})} \leq C T^{\alpha/2} / \|\bar{g}_2\|_{L^2(0,T),C^4(\bar{T})} \|W(\cdot, t_1) - W(\cdot, t_2)\|_{C^{k+\alpha}(\bar{\Omega})}.
\]
\[(29)\]

Then, inequalities (28), (29) together with (24) lead to estimate (26).

\[\square\]

Remark 1. The statements of Proposition 3 are true if \( \Omega \equiv R^n_+ \), \( \Gamma_0 = \emptyset \), \( g_1 \equiv 0 \), \( g_0 \) and \( g_2 \) are finite functions which satisfy conditions of Proposition 3.

3. The nonlinear mapping. Main results.

3.1. Reduction of problem (1)-(4) to a problem in the fixed domain. We assume that
\[
\Upsilon \in C^{k+\alpha}, \; \Gamma_0 \in C^{k-2+\alpha}, \; k \geq 7.
\]
\[(30)\]

To prove the solvability of problem (1)-(4), it is convenient to reduce the one to a problem in a fixed domain. To this end, we use the Hanzawa method [17]. Let \( \omega = (\omega_1, \ldots, \omega_{n-1}) \) be some coordinates on \( \Upsilon \). We represent \( \Upsilon \) in the form \( y = \bar{m}(\omega) \) and denote by \( \bar{\eta}(\omega) \) the normal to \( \Upsilon \) directed outside \( \Omega \).

For sufficiently small \( \gamma_0 > 0 \), \( \omega \)-surfaces: \( \bar{m}(\omega) + \bar{\eta}(\omega), \, |\eta| < 2\gamma_0 \), do not intersect each other and \( \Gamma_0 \). On the set \( N = \{ y \in R^n : \; dist(y, \Upsilon) < 3\gamma_0/2 \} \) we introduce the local coordinates \( (\omega, \eta) \) by
\[
y = (y_1, \ldots, y_n) = \bar{m}(\omega) + \bar{\eta}(\omega), \; \bar{m}(\omega) \subset \Upsilon.
\]

We represent the free boundary in problem (1)-(4) as
\[
\Upsilon(t) = \{(y, t) : y(\omega, t) = \bar{m}(\omega) + \rho(\omega, t)\bar{\eta}(\omega), \; t \in [0, T]\},
\]
\[(31)\]
where \( \rho(\omega, t) \) is an unknown function, and
\[
|\rho(\omega, t)| < \gamma_0 C_0, \quad 0 < C_0 < 1, \quad \rho(\omega, 0) = 0. \tag{32}
\]
That means the surface \( \Upsilon(t) \) in the local variables is given by
\[
\Phi_\rho(y, t) = \eta(y) - \rho(\omega(y), t) = 0. \tag{33}
\]
Using (5), (31) and (33), we can rewrite boundary conditions in (2) as
\[
\mathbf{D}_\mu^\gamma \rho < \nabla_y \Phi_\rho(y, 0), \nabla_y \Phi_\rho(y, t) > \quad \text{such that the transform}
\]
\[
\mathbf{D}_\mu^\gamma \rho < \nabla_y \Phi_\rho(y, 0), \nabla_y \Phi_\rho(y, t) > = -\mu < \nabla_y p, \nabla_y \Phi_\rho(y, t) >, \tag{34}
\]
\[
p = \gamma \nabla_y \left( \frac{\nabla_y \Phi_\rho(y, t)}{\nabla_y \Phi_\rho(y, t)} \right) \bigg|_{\Upsilon(t)}. \tag{35}
\]
Let \( \chi(\lambda) \in C_0^\infty(R^1) \), \( \chi(\lambda) = 1 \) if \(|\lambda| < \gamma_0 / 4 \) and \( \chi(\lambda) = 0 \) if \(|\lambda| > 3\gamma_0 / 4 \), \(|\chi^{(k)}| \leq C_1 / (\gamma_0^k), k = 1, 3 \). We choose constant \( C_0 \) in (32) such that \( C_0 < \frac{1}{2C_1} \), then \( 1 + \chi(\lambda) \mu \geq 1 / 2 \) if \(|\mu| \leq \gamma_0 C_0 \). We will use the coordinates \((\omega, \eta)\) to define the diffeomorphism
\[
e_\rho: (x, t) \to (y, t)
\]
from \( X_T = R^n \times [0, T] \) onto \( Y_T = R^n \times [0, T] \) by setting
\[
\begin{align*}
\{ & y = x, \text{ if } \text{dist}(x, \Upsilon) > 3\gamma_0 / 4, \\
& \omega(y) = \omega(x), \eta(y) = \lambda(x) + \chi(\lambda(x))\rho(\omega(x), t) \text{ otherwise},
\}
\end{align*}
\]
\[
\text{such that the transform } e_\rho^{-1} \text{ maps } \Omega(t) \text{ onto } \Omega \times (0, T) = \Omega_T, \text{ and } \Upsilon(t) \text{ onto } \Upsilon \times [0, T] = \Upsilon_T; \text{ the free boundary is given by}
\]
\[
e_\rho(\{\lambda(x) = 0\}), \tag{36}
\]
and \( \omega(x), \lambda(x) \) are the coordinates in \( X_T \) similar to the coordinates \( \omega(y), \eta(y) \) in \( Y_T \). After the change of variables (36), we have a new desired function
\[
v(x_1, \ldots, x_n, t) = p(y_1, \ldots, y_n, t) \circ e_\rho(x, t). \tag{37}
\]
Denote by \( \nabla_\rho = (E_\rho^*)^{-1} \nabla_x \) where \( E_\rho \) is the Jacobi matrix of the mapping \( y = e_\rho(x, t) \) so that
\[
\nabla_y = \nabla_\rho \text{ and } \Delta_y = \nabla^2_\rho.
\]
Taking into account that \( y = x \) near \( \Gamma_{\partial T} \), we can reduce free boundary problem (1)-(4) to the nonlinear problem in the fixed domain:
\[
\nabla^2_\rho v(x, t) = 0 \text{ in } \Omega_T; \tag{39}
\]
\[
v(x, t) = \gamma \sum_{i=1}^{n-1} A_{ij}(\omega, \rho, \nabla_\omega \rho) \frac{\partial^2 \rho}{\partial \omega_i \partial \omega_j} + A_0(\omega, \rho, \nabla_\omega \rho) \quad \text{on } \Upsilon_T; \tag{40}
\]
\[
\mathbf{D}_\mu^\gamma \rho = -\mu S(\omega, \rho, \nabla_\omega \rho) \frac{\partial v}{\partial \lambda} - \mu \sum_{i=1}^{n-1} S_i(\omega, \rho, \nabla_\omega \rho) \frac{\partial v}{\partial \omega_i} \quad \text{on } \Upsilon_T; \tag{41}
\]
\[
(1 - \delta)v - \delta \frac{\partial v}{\partial \lambda} = \varphi(x) \text{ on } \Gamma_{\partial T}, \quad \delta \in \{0, 1\}; \tag{42}
\]
\[
\rho(\omega, 0) = 0, \quad \omega \in \Upsilon, \tag{43}
\]
where \( S(\omega, \rho, \nabla_\omega \rho), S_i(\omega, \rho, \nabla_\omega \rho), i = 1, n - 1, A_{ij}(\omega, \rho, \nabla_\omega \rho), i, j = 1, n - 1, A_0(\omega, \rho, \nabla_\omega \rho) \) are some specific smooth functions (their representations and properties can be found in [24,41] and [10]) such that
\[
S(\omega, 0, 0) = 1, \quad \frac{\partial S}{\partial \rho_i}(\omega, 0, 0) = 0, \quad S_i(\omega, 0, 0) = 0, \quad i = 1, n - 1,
\]
\[
\det \{ A_{ij}(\omega, 0, 0) \} \geq \varepsilon_0 > 0. 
\] (44)

Moreover, one can easily check that
\[
\nabla^2 \rho|_{t=0} = \Delta_x. 
\] (45)

Let us define the function \( v_0(x) \) as a solution of the boundary-value problem:
\[
\Delta_x v_0 = 0 \text{ in } \Omega; \quad v_0 = \gamma \kappa(\Upsilon) \text{ on } \Upsilon; \quad (1 - \delta) v_0 - \delta \frac{\partial v_0}{\partial n} = \varphi(x) \text{ on } \Gamma_0. 
\] (46)

We assume that condition (30) hold, and
\[
\varphi(x) \in C^{5-\delta+\alpha}, \quad \delta = 0 \text{ or } 1. 
\] (47)

According to the results of Chapter 3 in [25], there exists a unique solution \( v_0(x) \) of problem (46), and
\[
\| v_0 \|_{C^{5+\alpha}(\bar{\Omega})} \leq C(\| \varphi \|_{C^{5-\delta+\alpha}(\Upsilon)} + \| \kappa(\Upsilon) \|_{C^{5+\alpha}(\Upsilon)}), \quad v_0(x) = v(x, 0) \text{ in } \bar{\Omega}, 
\] (48)

where \( C \) is a positive constant.

Henceforward the letter \( C \) will be used to denote different constants encountered in our formulae. The main result of our paper is the following.

**Theorem 3.1.** Let \( \alpha, \nu \in (0, 1); \Gamma_0 \) and \( \Upsilon \) satisfy assumptions mentioned in Section 1 and conditions (30), (47) hold. Then for some small \( T \), there exists a unique solution \( (v(x,t), \rho(\omega,t)) \) of nonlinear problem (39)-(43) for \( t \in [0, T] \) such that
\[
v(x,t) \in C^{2+\alpha, \frac{\alpha}{\nu}}(\bar{\Omega}_T), \quad \rho(\omega,t) \in C^{4+\alpha, \frac{\alpha}{\nu}}(\bar{\Upsilon}_T), \quad D^\nu_t \rho(\omega,t) \in C^{1+\alpha, \frac{\alpha}{\nu}}(\bar{\Upsilon}_T),
\] and \( v(x,0) \) is given with (46) and (48).

**3.2. Linearization of system (39)-(43).** Here we linearize nonlinear problem (39)-(43) on the initial data and rewrite it as a system \( Az = F(z) \), where \( A \) is a linear operator and \( F \) is a nonlinear perturbation.

As follows from (41), (44) and (48),
\[
D^\nu_t \rho(\omega, 0) = -\mu \frac{\partial v_0}{\partial \lambda} \bigr|_{\Upsilon}. 
\] (49)

Let a function \( s(\omega,t) \) be such that
\[
s(\omega, 0) = 0, \quad D^\nu_t s(\omega, 0) = D^\nu_t \rho(\omega, 0) \text{ on } \Upsilon. 
\] (50)

Due to \( D^\nu_t \frac{\mu^\nu}{(1+\nu)} = 1 \), we may choose the function \( s(\omega,t) \) as
\[
s(\omega, t) = -\mu \frac{\mu^\nu}{(1+\nu)} \frac{\partial v_0}{\partial \lambda} \bigr|_{\Upsilon}. 
\] (51)

Using properties (48) and representation (51), we can conclude the following:

**Corollary 1.** The function \( s(\omega,t) \) given by (51) satisfies (50) and
\[
\| s \|_{C([0,T], C^{4+\alpha}(\Upsilon))} + \| D^\nu_t s \|_{C([0,T], C^{4+\alpha}(\Upsilon))} + \sum_{|i|=0}^4 \| D^\nu_t D^\nu_{i,Y_\nu} s \| \leq C(\| \varphi \|_{C^{5-\delta+\alpha}(\Upsilon)} + \| \kappa(\Upsilon) \|_{C^{5+\alpha}(\Upsilon)}), \quad \beta \in (0, 1), \quad \delta = 0 \text{ or } 1.
\]
Next, we introduce the new unknown functions \( u(x,t) \) and \( \sigma(\omega,t) \) in the following way
\[
\sigma(\omega,t) = \rho(\omega,t) - s(\omega,t), \quad u(x,t) = v(x,t) - v_0(x).
\] (52)
In terms of the functions \( \sigma(\omega,t) \) and \( u(x,t) \) system (39)-(43) can be rewritten as follows:
\[
\Delta_x u = F_0(u,\sigma) \quad \text{in } \Omega_T;
\]
\[
u - \sum_{ij=1}^{n-1} a_{ij}(\omega,t) \frac{\partial^2 \sigma}{\partial \omega_i \partial \omega_j} + \sum_{i=1}^{n-1} a_{i1}(\omega,t) \frac{\partial \sigma}{\partial \omega_i} = F_1(\sigma) \quad \text{on } \Upsilon_T;
\]
\[
D_{\sigma}^{l} \sigma + b_0(\omega) \frac{\partial u}{\partial \tilde{\Omega}(\omega)} = F_2(u,\sigma) \quad \text{on } \Upsilon_T;
\]
\[
(1 - \delta) u - \delta \frac{\partial u}{\partial \bar{\Omega}} = 0 \quad \text{on } \Gamma_{\text{GR}}; \quad \sigma(\omega,0) = 0, \quad \omega \in \Upsilon,
\] (53)
where the representations of the functions \( F_i(u,\sigma), i = 0,2, F_1(\sigma), a_{ij}, a_{i1}, l,j = \bar{1},n-\bar{1}, b_0 \) can be found in \([2,41]\).

Thus system (39)-(43) can be written briefly in the form
\[
Az = F(z), \quad \text{where } \ z = (u,\sigma),
\] (54)
where a linear operator \( A \) is defined by the left-hand side of (53), and \( F \) is a nonlinear operator, \( F(z) = \{F_0(z), F_1(z), F_2(z)\} \). Based on representations of \( F_i, i = 0,2, a_{ij}(\omega,t), a_{i1}(\omega,t), l,j = \bar{1},n-\bar{1}, b_0(\omega), \) properties (44), (48) and Corollary 1, we can deduce the following results.

**Corollary 2.** The functions \( F_i(u,\sigma), i = 0,2, F_1(\sigma) \) contain the higher derivatives of \( u(x,t) \) and \( \sigma(\omega,t) \) with the coefficients that tend to zero as \( t \to 0 \), the "quadratic" terms of minor differential orders of unknown functions. Moreover,
\[
F_i(u,\sigma)|_{t=0} = 0, \quad i = 0,2, \quad F_1(\sigma)|_{t=0} = 0;
\] (55)
\[
a_{ij}(\omega,t), \ a_{i1}(\omega,t) \in C^{3+\alpha,\nu,\alpha}(\Upsilon_T); \quad b_0(\omega) \in C^{6+\alpha}(\Upsilon_T), \quad l,j = \bar{1},n-\bar{1};
\] (56)
\[
\text{matrix}\{a_{ij}\}^{i,j=1}_{i,j=1} \text{ is positive, and } \beta_0|\xi|^2 \leq a_{ij}\xi_i \xi_j \leq \beta_0^{-1}|\xi|^2, \quad \xi \in \mathbb{R}^{n-1},
\]
\[
\beta_0 \text{ is a positive constant; } b_0(\omega) > 0.
\] (57)

Note that equality (55) together with conditions (53) provide
\[
u(x,0) = 0 \quad \text{if } x \in \bar{\Omega}.
\] (58)
In a next step of our investigation we prove the boundedness of the linear operator \( A \) in the corresponding function spaces. To this end, we freeze the functional arguments in the functions \( F_i(u,\sigma), i = 0,2, F_1(\sigma) \). Then system (54) will be a linear system with variable coefficients, which will be studied in detail in Subsection 5.1. This investigation will be based on the properties of solutions to the corresponding model problems.

4. Model problem with fractional temporal derivative in a boundary condition. In this section we formulate and study the principal model problem such that nonlinear problem (54) will inherit the main feature of this model problem.

Let \( a_0 \) and \( a_{ij}, i,j = \bar{1},n-\bar{1}, \) be some given positive constants,
\[
R^n_+ = \{(x',x_n): x' \in \mathbb{R}^{n-1}, \ x_n > 0\}, \quad x' = (x_1,...,x_{n-1});
\]
\[
R^n_+ \times (0,T), \quad R^n_{-1} = \mathbb{R}^{n-1} \times (0,T).
\]
We look for a solution \((U(x, t), \psi(x', t))\) bounded at infinity and
\[
\Delta_x U = f_0(x, t) \text{ in } \mathbb{R}^n_+;
\]
\[
U - \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} = f_1(x', t) \text{ on } \mathbb{R}^n_T - 1;
\]
\[
\mathcal{D}_t \psi + a_0 \frac{\partial U}{\partial x_n} = f(x', t) \text{ on } \mathbb{R}^n_T - 1;
\]
\[
\psi(x', 0) = 0 \text{ in } \mathbb{R}^n - 1,
\]
where \(f_0, f_1, f\) are given functions:
\[
f_0, f_1, f \equiv 0 \text{ if either } t = 0 \text{ or } |x| > R_0 \quad (63)
\]
for a positive number \(R_0\).

Due to the quadratic form connected with the Laplace equation is positive, we can choose the coordinate \(x'\) such that the form \(\sum_{i,j=1}^{n-1} a_{ij} \xi_i \xi_j\) is reduced to the diagonal view. Thus, we believe that \(a_{ij} = 0\) if \(i \neq j\).

**Theorem 4.1.** Let \(\alpha, \nu \in (0, 1)\), condition (63) hold.

i: If \(f_0 \in C([0, T], C^\alpha(\bar{R}_n^+)), f_1 \in C([0, T], C^{2+\alpha}(\mathbb{R}^n - 1)), f \in C([0, T], C^{1+\alpha}(\mathbb{R}^n - 1))\). Then there exists a unique solution \((U, \psi)\) of problem (59)-(62): \(U \in C([0, T], C^{2+\alpha}(\bar{R}_n^+)), \psi \in C([0, T], C^{1+\alpha}(\mathbb{R}^n - 1))\), and the following estimate holds:
\[
\|U\|_{C([0, T], C^{2+\alpha}(\bar{R}_n^+))} + \|
ψ\|_{C([0, T], C^{1+\alpha}(\mathbb{R}^n - 1))} + \|\mathcal{D}_t \psi\|_{C([0, T], C^{1+\alpha}(\mathbb{R}^n - 1))} 
\leq C_1\|f_0\|_{C([0, T], C^\alpha(\bar{R}_n^+))} + \|f_1\|_{C([0, T], C^{2+\alpha}(\mathbb{R}^n - 1))} + \|f\|_{C([0, T], C^{1+\alpha}(\mathbb{R}^n - 1))}. \quad (64)
\]

ii: If \(f_0 \in C^{\alpha, \alpha/3, \alpha}(\bar{R}_n^+), f_1 \in C^{2+\alpha, \alpha/3, \alpha}(\mathbb{R}^n - 1), f \in C^{1+\alpha, \alpha/3, \alpha}(\mathbb{R}^n - 1)\). Then there exists a unique solution \((U, \psi)\) of problem (59)-(62):
\[
\|U\|_{C^{2+\alpha, \alpha/3, \alpha}(\bar{R}_n^+)} + \|
ψ\|_{C^{1+\alpha, \alpha/3, \alpha}(\mathbb{R}^n - 1)} + \|\mathcal{D}_t \psi\|_{C^{1+\alpha, \alpha/3, \alpha}(\mathbb{R}^n - 1)} 
\leq C_2\|f_0\|_{C^{\alpha, \alpha/3, \alpha}(\bar{R}_n^+)} + \|f_1\|_{C^{2+\alpha, \alpha/3, \alpha}(\mathbb{R}^n - 1)} + \|f\|_{C^{1+\alpha, \alpha/3, \alpha}(\mathbb{R}^n - 1)}. \quad (65)
\]

iii: If \(f \in C^{1+\alpha, \frac{1+\alpha}{6}\nu}(\mathbb{R}^n - 1)\) and
\[
f_0, f_1 \equiv 0, \quad (66)
\]
then there is a unique solution \((U, \psi)\): \(U \in C^{2+\alpha, \frac{2+\alpha}{3}\nu}(\bar{R}_n^+), \psi \in C^{1+\alpha, \frac{1+\alpha}{3}\nu}(\mathbb{R}^n - 1)\), and the following estimate holds:
\[
\|U\|_{C^{2+\alpha, \frac{2+\alpha}{3}\nu}(\bar{R}_n^+)} + \|
ψ\|_{C^{1+\alpha, \frac{1+\alpha}{3}\nu}(\mathbb{R}^n - 1)} \leq C_3\|f\|_{C^{1+\alpha, \frac{1+\alpha}{3}\nu}(\bar{R}_n^+)}. \quad (67)
\]
where \(C_i, i = 1, 3,\) are positive constant independent of the right-hand side.

Note that statement (iii) of Theorem 4.1 can not be extended into arbitrary right-hand sides \(f_0, f_1,\) due to the spaces \(C^{k+\alpha, \frac{1+\alpha}{3}\nu}\) are related with subdiffusion equation. Nevertheless, this result is essentially used to prove solvability of the homogenous linear problem (109)-(112) (see Section 5.1). As for statements (i) and (ii) of Theorem 4.1, they are necessary to obtain one-valued solvability of the linear problem (109)-(112) in the case of arbitrary right-hand sides.

We remark that conditions of Theorem 4.1 together with (63) lead to
\[
U(x, 0) = 0 \text{ in } \mathbb{R}_n^+.
\]
In virtue of Proposition 3 and Remark 1, it is enough to prove Theorem 4.1 in the case of (66).

4.1. **Integral representation of the solution to problem (59)-(63) in the case of (66).** Let \( \zeta = (\zeta_1, \ldots, \zeta_{n-1}) \) and \( |\zeta| = \left( \sum_{i=1}^{n-1} \zeta_i^2 \right)^{1/2} \). We denote by \( \hat{w}(\zeta, x_n, t) \) the Fourier transform of \( w(x', x_n, t) \) and by \( \hat{w}(\cdot, p) \) the Laplace transform of \( w(\cdot, t) \), and use the notation “\( * \)” instead of \( \hat{\cdot} \).

Due to condition (63), we can extend the function \( f(x', t) \) by 0 for \( t \in (-\infty, 0) \) and then apply the Fourier and Laplace transformations to problem (59)-(62).

After some simple calculations, we have:

\[
\varphi^*(\zeta, p) = \frac{f^*(\zeta, p)}{p^\nu + 8\pi^3 a_0|\zeta| < a\zeta, \zeta^\nu}, \quad a = \{a_{11}, a_{22}, \ldots, a_{n-1n-1}\},
\]

\[
U^*(\zeta, x_n, p) = -\frac{4\pi^2}{p^\nu + 8\pi^3 a_0|\zeta| < a\zeta, \zeta^\nu} e^{-2\pi|\zeta|x_n} f^*(\zeta, p).
\]

One can easily check that

\[
(p^\nu + 8\pi^3 a_0|\zeta| < a\zeta, \zeta^\nu)^{-1} = \int_0^{+\infty} e^{z} \exp\{-zp^\nu - 8\pi^3 a_0 z|\zeta| < a\zeta, \zeta^\nu\} \, dz,
\]

if \( \text{Re } p^\nu > 0 \).

Let

\[
L(y', z) := \int_{R^{n-1}} \exp\{-8\pi^3 a_0 z|\zeta| < a\zeta, \zeta^\nu + 2\pi i < \zeta, y' >\} \, d\zeta,
\]

\[
y' = (y_1, \ldots, y_{n-1});
\]

\[
G(y', \tau) := \int_0^{\infty} \tau^{-1} W(-z\tau^{-\nu}; -\nu, 0)L(y', z) \, dz;
\]

\[
K(y', x_n) = \int_{R^{n-1}} \exp\{-2\pi|\zeta|x_n + 2\pi i < \zeta, y' >\} \, d\zeta,
\]

where \( W(z; b, d) \) is the Wright function (see (14)).

Then, applying the inverse Laplace and Fourier transformations in (69), (70), we obtain

\[
\varphi(x', t) = \int_0^t d\tau \int_{R^{n-1}} f(y', \tau)G(x' - y', t - \tau) \, dy',
\]

\[
U(x', x_n, t) = \sum_{i=1}^{n-1} a_{ii} \int_{R^{n-1}} K(y', x_n) \frac{\partial^2 \varphi}{\partial y'^2}(x' - y', t) \, dy'.
\]

Remark that by virtue of the smoothness properties of the function \( f(x', t) \) and its behavior at infinity all above performed operations are justified.

4.2. **Estimates of the constructed solution** \( (U(x', x_n, t), \varphi(x', t)) \). As usual in the potential theory, to estimate functions \( \varphi(x', t) \) and \( U(x', x_n, t) \) we need to describe well the properties of kernels \( G(y', t), L(y', z) \) and \( K(y', x_n) \). Note that, the behavior of function \( K(y', x_n) \) was described in Lemma 3.1 [41]. Thus, it is necessary to evaluate the functions \( G(y', t) \) and \( L(y', z) \).

**Lemma 4.2.** Let \( \alpha, \nu \in (0, 1); \; y' \in R^{n-1}, \; y'' \in R^{n-2}, \; z \in (0, +\infty), \; t \in [0, T], \; \varepsilon \) and \( B \) be positive numbers. Then functions \( L(y', z) \) and \( G(y', t) \) which are given by (71), (72) satisfy the following inequalities:
\[ |D_y^\alpha L(y', z)| \leq C \exp\left(-\frac{B}{z} \sum_{i=1}^{n-1} |y_i| / z^{1/3} \right), \quad |m| = \sum_{i=1}^{n-1} m_i, \quad |m| = 0, 1, 2, \ldots; \quad (76) \]

\[ \left| \frac{\partial L(y', z)}{\partial z} \right| \leq C \exp\left(-\frac{B}{z} \sum_{i=1}^{n-1} |y_i| / z^{1/3} \right); \quad (77) \]

ii:

\[ \int_{R^{n-1}} L(y', z) dy' = 1; \quad (78) \]

\[ \int_{R^{n-1}} D_{y'}^{2k+1} L(y', z) dy' = 0, \quad k = 0, 1, 2, \ldots; \quad (79) \]

\[ \int_{R^{n-1}} G(y', t) dy' = \frac{\nu^{-1}}{\Gamma(\nu)}; \quad (80) \]

where \( \delta(y) \) is the Dirac delta function;

iv:

\[ \int_0^t d\tau \int_{R^{n-1}_+} y_i^\alpha |D_y^\alpha G(y', \tau)| dy' \leq C t^{3\alpha} \quad , \quad i = 1, n-1; \quad (82) \]

v:

\[ \int_0^t d\tau \int_{R^{n-2}_+} dy'' \int_{-\infty}^{t} y_i^\alpha |D_y^\alpha G(y', \tau)| dy_i \leq C \varepsilon^\alpha, \quad i = 1, n-1; \quad (83) \]

\[ \int_0^t d\tau \int_{R^{n-2}_+} dy'' \int_{-\infty}^{t} y_i^\alpha |D_y^\alpha G(y', \tau)| dy_i \leq C (t + t^2), \quad i = 1, n-1; \quad (84) \]

\[ \int_0^t d\tau \int_{R^{n-2}_+} dy'' \int_{-\infty}^{t} y_i^\alpha |D_y^\alpha G(y', \tau)| dy_i \leq C \varepsilon^{\alpha-1}, \quad i = 1, n-1. \quad (85) \]

Proof. From (71), one can easily obtain that

\[ D_y^\alpha L(y', z) = \frac{\text{const.}}{z^{\frac{n-1}{3}} (\alpha + 1)} \int_{R^{n-1}} \exp\left\{ -8\pi^3 a_0 |\mu| (\alpha, \mu) + 2\pi i (\alpha, y'/z^{1/3}) \right\} \]

\[ \times p_{\mu_1^{m_1} \cdots \mu_{n-1}^{m_{n-1}}} \frac{d\mu}{|\mu|^{n-1}}, \quad \mu_k = \zeta_k z^{1/3}, \quad k = 1, n-1. \quad (86) \]

Since the function \( \mu_1^{m_1} \cdots \mu_{n-1}^{m_{n-1}} \exp\left\{ -8\pi^3 a_0 |\mu| (\alpha, \mu) + 2\pi i (\alpha, y'/z^{1/3}) \right\} \) vanishes if \( \mu_k \to \infty \), for every \( k \); and does not have any poles, we can calculate the integrals in (86) along the shifted contours \( \mu_k = \lambda_k \pm iB, k = 1, n-1, B \) is a positive constant. The sign " + " is chosen if \( y_k > 0 \), and " - " if \( y_k < 0 \). Thus we deduce

\[ D_y^\alpha L(y', z) = C\exp\left( -\frac{B}{z} \sum_{k=1}^{n-1} |y_k| / z^{1/3} \right) \int_{R^{n-1}} (\lambda_1 \pm iB)^{m_1} \cdots (\lambda_{n-1} \pm iB)^{m_{n-1}} \]

\[ \times \exp\left\{ -8\pi^3 a_0 |\lambda \pm iB| (\alpha (\lambda \pm iB), \lambda \pm iB) + 2\pi i \left\langle \lambda, y' / z^{1/3} \right\rangle \right\} d\lambda, \quad B = 2\pi i B. \quad (87) \]

It is not hard to check the uniformly boundedness of the module of the integrals in (87). Thus we obtain (76). As for inequality (77), it is proved with the same way.

Based on the following relation from Lemma 5.3 [3]:

\[ \Re \int_0^\infty e^{iy} dy = \pi \delta(-x), \quad (88) \]
where $\delta(x)$ is the Dirac delta function, it is easy to get equalities (78), (79) and (81). To obtain (80), we use equality (78) and statement (i) from Proposition 2 and deduce

$$
\int_{R^{n-1}} G(y', t) dy' = \nu t^{\nu-1} \int_0^{+\infty} \eta W(-\eta; -\nu, 1-\nu) d\eta,
$$

(89)

$$
\eta = z t^{-\nu}.
$$

(90)

After that it is enough to apply (22) to the integral in the right-hand side in (89).

To prove statement (iv), we use (76) and conclude:

$$
I_{1} := \int_0^t d\tau \int_{R^{n-1}_+} y_i^{\alpha} |D^3_y G(y', \tau)| dy' \\
\leq \int_0^t d\tau \int_{R^{n-1}_+} y_i^{\alpha} \int_0^{+\infty} \exp \left\{ - B \sum_{k=1}^{n-1} y_k / z^{1/3} \right\} W(-z\tau^{-\nu}; -\nu, 0) d\eta \int_{R^{n-1}_+} y_i^{\alpha-1} \int_0^{+\infty} e^{\eta^{2/3} \tau^{1/3} - \nu^{1/3} \eta^{1/3}} W(-\eta; -\nu, 1-\nu) d\eta
$$

(91)

Then, performing the consecutive change of variables:

$$
y_k = z^{1/3} p_k, \quad k = 1, n-1,
$$

(92)

and (90) in the right-hand sides of (91), we deduce

$$
I_{1} \leq \text{const.} \int_0^t d\tau \int_{R^{n-1}_+} d\eta \int_0^{+\infty} e^{\eta^{2/3} \tau^{1/3} - \nu^{1/3} \eta^{1/3}} W(-\eta; -\nu, 1-\nu) d\eta
$$

(93)

Note that the last inequality in (93) follows from statements (i) and (vii) of Proposition 2. As for estimate (83), we apply again inequality (76) and change of variables (92) with $k \neq i$ and (90). Thus, we get

$$
I_{2} := \int_0^t d\tau \int_{R^{n-2}_+} d\eta \int_0^{+\infty} e^{\eta^{2/3} \tau^{1/3} - \nu^{1/3} \eta^{1/3}} W(-\eta; -\nu, 1-\nu) d\eta
$$

The change of variable

$$
r = y_i \tau^{-\nu/3} \eta^{-1/3}
$$

in the last integral leads to

$$
I_{2} \leq \text{const.} \int_0^{+\infty} d\eta \int_0^{+\infty} dr e^{-B r} \int_0^{+\infty} W(-\eta; -\nu, 1-\nu) d\eta.
$$

At last, applying equality (22) allows us to obtain estimate (83). As for estimate (85), it is obtained in the same way.
Let us get estimate (84). As before we can deduce from (16), (76) and (20) that
\[
\int_0^t d\tau \int_{R_{R_{T_n}^{-1}}^{\nu}} dy' \int_{\varepsilon}^{+\infty} |D_y^3 G(y', \tau)| dy_i
\]
\[
\leq \text{const.} \int_0^t d\tau \int_{0}^{+\infty} e^{-\frac{\tau}{\tau^2 + \eta^2 + \nu^2 \eta}} \left| W(-\eta^{-\nu}; -\nu, -1) \right| + \int_{\tau}^{+\infty} \frac{1}{\tau^{1+\nu}} d\eta
\]
\[
\leq \text{const.} \int_0^t d\tau \left[ \int_{\varepsilon}^{+\infty} e^{-\frac{\tau}{\tau^2 + \eta^2 + \nu^2 \eta}} [1 + \tau] d\eta + \int_{1}^{+\infty} [\eta^{-2-\frac{\nu}{2}} + \eta^{-1-\frac{\nu}{2}}] d\eta \right]
\]
\[
\leq \text{const.} [t + t^2]. \tag{95}
\]
Note that the constants in (95) are independent of \( \varepsilon \). Thus, the last inequality in (95) ensures (84). That completes the proof of Lemma 4.2. \( \square \)

**Lemma 4.3.** Let \( \alpha, \nu \in (0, 1) \), conditions (63) and (66) hold and \( f \in C([0, T], C^{1+\alpha}(R_{R_{T_n}^{-1}}^{\nu})) \). Then there is the following inequality:
\[
\|\varphi\|_{C([0, T], C^{1+\alpha}(R_{R_{T_n}^{-1}}^{\nu}))} \leq C \|f\|_{C([0, T], C^{1+\alpha}(R_{R_{T_n}^{-1}}^{\nu}))}. \tag{96}
\]

**Proof.** First of all, we evaluate \( \max_{\overline{R_{T_n}^{\nu}}} |\varphi| \). To this end, we combine inequality (76) with change of variables (90) and (92) and obtain
\[
|\varphi(x', t)| \leq \text{const.} \max_{\overline{R_{T_n}^{\nu}}} |f| \int_0^t d\tau \int_{R_{R_{T_n}^{-1}}^{\nu}} [D_y^2 f(y', \tau) - D_y^2 f(x', \tau)] D_y^3 G(x' - y', t - \tau) dy'. \tag{97}
\]
After that, we evaluate \( \|D_x^4 \varphi\|_{C([0, T], C^{1+\alpha}(R_{R_{T_n}^{-1}}^{\nu}))} \). Due to (78) and (79), we can represent \( D_x^4 \varphi(x', t) \) as
\[
D_x^4 \varphi(x', t) = \int_0^t d\tau \int_{R_{R_{T_n}^{-1}}^{\nu}} [D_y f(y', \tau) - D_y f(x', \tau)] D_y^3 G(x' - y', t - \tau) dy'. \tag{98}
\]
Based on representation (98) and inequality (82), we conclude that
\[
\max_{\overline{R_{T_n}^{\nu}}} D_x^4 \varphi \leq CT^{\nu/3} \langle D_x^4 f \rangle_{x', R_{R_{T_n}^{-1}}^{\nu}}^{(\alpha)}. \tag{99}
\]
To evaluate the H"older seminorm \( \langle D_x^4 \varphi \rangle_{x', R_{R_{T_n}^{-1}}^{\nu}}^{(\alpha)} \), we use the representation from Chapter 4 [24] for the difference \( [D_x^4 \varphi(x', t) - D_x^4 \varphi(\bar{x}', t)] \) together with statement (v) from Lemma 4.2; and obtain
\[
\langle D_x^4 \varphi \rangle_{x', R_{R_{T_n}^{-1}}^{\nu}}^{(\alpha)} \leq C \|D_x^4 f\|_{C([0, T], C^{1+\alpha}(R_{R_{T_n}^{-1}}^{\nu}))}. \tag{100}
\]
Thus, estimates (97), (99), (100) together with interpolation inequality (8) lead to (96). \( \square \)

We will use the following results to evaluate the functions \( D_t^\nu \varphi(x', t) \) and \( U(x', x_n, t) \) given with (75) and (74).

**Proposition 4.** Let conditions of Lemma 4.3 hold. Then the functions \( \varphi(x', t) \) and \( U(x', x_n, t) \) which are represented by (74) and (75) satisfy boundary conditions (60), (61) with \( f_1 \equiv 0 \). Moreover, there are the following representations:
\[
D_t^\nu \varphi(x', t) = f(x', t) + \int_0^t d\tau \int_{R_{R_{T_n}^{-1}}^{\nu}} [f(x' - y', t - \tau) - f(x', t - \tau)] \partial_y^2 G(y', \tau) dy', \tag{101}
\]
Let conditions of Lemma 4.3 hold, then functions \( \varphi(x', y', t) \) and \( \psi(x', t) \) given by \((74)\) and \((75)\) satisfy to inequalities:

\[
\|U\|_{C([0,T], C^{2+\alpha}(\mathbb{R}^n_+))} \leq C\|f\|_{C([0,T], C^{1+\alpha}(\mathbb{R}^{n-1}))},
\]
\[
\|D_{x'}\varphi\|_{C([0,T], C^{1+\alpha}(\mathbb{R}^{n-1}))} \leq C\|f\|_{C([0,T], C^{1+\alpha}(\mathbb{R}^{n-1}))}.
\]

In addition, if \( f \in C^{1+\alpha, 3\alpha/3}(\mathbb{R}^{n-1}_T) \), then

\[
\sum_{|l|=0}^{4} |D_{x'}^l \varphi|_{R^2_1}^{(\alpha, \alpha/v/3)} + \sum_{|l|=0}^{1} |D_{x'}^l D_{x''} \varphi|_{R^2_1}^{(\alpha, \alpha/v/3)} \leq C\|f\|_{C^{1+\alpha, 3\alpha/3}(\mathbb{R}^{n-1}_T)}.
\]

Next step of our investigation is a proof of the corresponding estimates to the functions \( \varphi(x', t), D_{x'}^l \varphi(x', t) \) and \( U(x', x_n, t) \) with respect to time.

**Proposition 5.** Let conditions of Lemma 4.3 be satisfied and \( f \in C^{1+\alpha, 3\alpha/3}(\mathbb{R}^{n-1}_T) \). Then the following estimates hold:

\[
\sum_{|l|=0}^{4} |D_{x'}^l \varphi|_{t, R^2_1}^{(4+\alpha, -|l|)} \leq \|f\|_{C^{1+\alpha, 3\alpha/3}(\mathbb{R}^{n-1}_T)};
\]
\[
\sum_{|l|=0}^{2} |D_{x'}^l U|_{t, R^2_1}^{(2+\alpha, -|l|)} \leq \|f\|_{C^{1+\alpha, 3\alpha/3}(\mathbb{R}^{n-1}_T)};
\]
\[
\sum_{|l|=0}^{1} |D_{x'}^l D_{x''} \varphi|_{t, R^2_1}^{(4+\alpha, -|l|)} \leq \|f\|_{C^{1+\alpha, 3\alpha/3}(\mathbb{R}^{n-1}_T)}.
\]

**Proof.** As for estimate \((106)\), it follows from interpolation inequality \((8)\), Propositions 1 and 4, and Lemma 4.3. Note that, boundary condition \((60)\) together with inequality \((106)\) lead to estimate of \( \langle U \rangle_{t, R^2_1}^{(2+\alpha, -|l|)} \). To evaluate \( \langle D_{x'}^l U \rangle_{t, R^2_1}^{(2+\alpha, -|l|)} \), it is enough to use representation \((102)\) and estimates \((3.30), (3.31)\) from [41]. After that interpolation inequality \((8)\) together with corresponding estimates of \( \langle U \rangle_{t, R^2_1}^{(2+\alpha, -|l|)} \) and \( \langle D_{x'}^l U \rangle_{t, R^2_1}^{(2+\alpha, -|l|)} \) allow us to obtain \((107)\).

At last, using Proposition 4, boundary condition \((61)\) and inequality \((107)\), we deduce estimate \((108)\). That completes Proposition 6. \(\square\)
4.3. Proof of Theorem 4.1. Note that, estimates (64) and (67) follow immediately from Lemma 4.3 and Propositions 5 and 6. To obtain inequality (65), it is enough to use Remark 1 together with Proposition 4 and estimate (105).

The uniqueness of the constructed solution \((U(x', x_n, t), \varphi(x', t))\) follows from (64). Moreover, representation (75) and estimate (96) mean that \(U = K * D^2\varphi\), where \(\varphi_{x, x_j} \in C^{2+\alpha}(R^{n-1})\) for \(\forall t \in [0, T]\). Thus arguments from Chapter 3 [25] allow us to show that \(U(x', x_n, t)\) satisfies equation (59) with \(f_0 \equiv 0\). Inequality (97) ensures condition (62). Hence, all the written above prove statements (ii) and (iii) of Theorem 4.1. As for validity of statement (i) of Theorem 4.1, it is necessary to show continuity of the function \(\varphi(x', t)\) and \(U(x', x_n, t)\) together with their derivatives with respect to time \(t\). To this end, we repeat the arguments from Subsection 3.3 in [41] and apply statement (ii) from Theorem 4.1. Thus, the proof of Theorem 4.1 is finished.

5. Proof of the main result: Theorem 3.1. Our proof splits into two steps. In the first stage we show the boundedness of the linear operator \(A\) (see (54)) in the spaces \(C^{k+\alpha, \beta, \alpha}\). After that we prove that the nonlinear operator \(A^{-1}F\) is a contraction one in the corresponding classes.

5.1. Linear problem corresponding to (54). Here we analyze the following linear problem which is obtained from (54) with the fixed right-hand sides:

\[
\Delta_x u = F_0(x, t) \text{ in } \Omega_T; \tag{109}
\]

\[
u - \sum_{i,j=1}^{n-1} a_{ij}(\omega, t) \frac{\partial^2 \sigma}{\partial \omega_i \partial \omega_j} + \sum_{i=1}^{n-1} a_{i1}(\omega, t) \frac{\partial \sigma}{\partial \omega_i} = F_1(x, t) \text{ on } \Upsilon_T; \tag{110}
\]

\[
D^l \sigma + b_0(\omega) \frac{\partial u}{\partial n(\omega)} = F_2(x, t) \text{ on } \Upsilon_T; \tag{111}
\]

\[(1 - \delta)u - \delta \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\sigma T}; \tag{112}
\]

Here \(\delta\) is integer \(\delta \in \{0, 1\}\), \(F_1, l = \bar{\Omega}\); \(a_{ij}(\omega, t), a_{i1}(\omega, t), i, j = \bar{\Gamma}, n - 1; b_0(\omega)\) are some given functions and

\[
F_0 \in C^{\alpha, \frac{\alpha}{\alpha}, \alpha}(\bar{\Omega}_T), F_1 \in C^{2+\alpha, \frac{\alpha}{\alpha}, \alpha}(\Upsilon_T), F_2 \in C^{1+\alpha, \frac{\alpha}{\alpha}, \alpha}(\Upsilon_T), F_1(x, 0) = 0, l = \bar{\Omega}\; \text{ on } \Omega_T; \tag{113}
\]

\[
\frac{1}{C} |\xi|^2 \leq \sum_{i,j=1}^{n-1} a_{ij}(\omega, t) \xi_i \xi_j \leq C |\xi|^2, (\omega, t) \in \Upsilon_T, \xi \in R^{n-1}, b_0(\omega) > 0; \tag{114}
\]

\[
|b_0| \in C^{1+\alpha}(\bar{\Upsilon}), a_{ij}, a_{i1} \in C^{2+\alpha, \alpha, \alpha}(\bar{\Upsilon}_T). \tag{115}
\]

Theorem 5.1. Let \(\alpha, \nu \in (0, 1), \Upsilon, \Gamma_0 \in C^{3+\alpha}\); conditions (113)-(115) hold. Then for some small \(T\), there exists a unique solution \((u, \sigma)\) of problem (109)-(112), such that

\[
\|u\|_{C^{2+\alpha, \frac{\alpha}{\alpha}, \alpha}(\bar{\Upsilon}_T)} + \|\sigma\|_{C^{1+\alpha, \frac{\alpha}{\alpha}, \alpha}(\bar{\Upsilon}_T)} + \|D^l \sigma\|_{C^{1+\alpha, \frac{\alpha}{\alpha}, \alpha}(\Upsilon_T)} \leq C\|F_0\|_{C^{\alpha, \frac{\alpha}{\alpha}, \alpha}(\bar{\Omega}_T)} + \|F_1\|_{C^{2+\alpha, \frac{\alpha}{\alpha}, \alpha}(\Upsilon_T)} + \|F_2\|_{C^{1+\alpha, \frac{\alpha}{\alpha}, \alpha}(\Upsilon_T)}, \tag{116}
\]

where \(C\) is a positive constant independent of the right-hand sides of (109)-(112) and depends on the measure of \(\Omega, \Upsilon, \Gamma_0\); and \(\|b_0\|_{C^{1+\alpha}(\bar{\Upsilon})}, \|a_{ij}\|_{C^{2+\alpha, \alpha, \alpha}(\bar{\Upsilon}_T)}, \|a_{i1}\|_{C^{2+\alpha, \alpha, \alpha}(\bar{\Upsilon}_T)}\).
Note that due to statement (i) from Proposition 3, it is is enough to prove Theorem 5.1 in the case of homogenous right-hand sides in (109) and (110).

First of all we assume that
\[ F_0, \ F_1 \equiv 0; \ F_2 \in C^{1+\alpha, \frac{1+\alpha}{3}}(Y_T), \]
and prove existence of solution (109)-(112) in the smoother classes.

For the sake of simplicity, we consider \( \delta = 0 \) in (112). Note that the case of \( \delta = 1 \) will be proved in the same way.

One can easily deduce from equalities (109)-(112) and condition (113) that
\[ u(x, 0) = 0 \text{ if } x \in \bar{\Omega}. \]

To prove the solvability of (109)-(112) under conditions (117) and (118), we construct the regularizer (see §7 Chapter iv [24] or Section 4 [5]). To this end, we reduce system (109)-(112) to the nonlocal equation
\[ L\sigma = F_2 \text{ on } Y_T, \quad \sigma |_{t=0} = 0 \text{ on } Y, \]
where the operator \( L \) is built as follows. Let \( \sigma \) be given in equation (110). Then equations (109), (110) and (112) formulate the Dirichlet problem \( (\delta = 0) \) for the functions \( u \). The solution of this Dirichlet problem can be used in equation (111). Thus, \( L\sigma \) is given by the left-hand side of (111).

**Definition 5.2.** An operator
\[ R : C^{1+\alpha, \frac{1+\alpha}{3}}(Y_T) \to P^{4+\alpha}(Y_T) \]
we call the right regularizer of the operator \( L \), if \( R \) is bounded and there is the following equality for each function \( F_2 \in C^{1+\alpha, \frac{1+\alpha}{3}}(Y_T) \)
\[ LR F_2 = F_2 + TF_2. \]

Here \( T \) is a bounded operator in \( C^{1+\alpha, \frac{1+\alpha}{3}}(Y_T) \):
\[ ||T||_{C^{1+\alpha, \frac{1+\alpha}{3}}(Y_T)} \leq \frac{1}{2} ||F_2||_{C^{1+\alpha, \frac{1+\alpha}{3}}(Y_T)}, \]
for enough small \( T \).

As it follows from the principle of contracting maps, inequality (122) ensures the existence of \( (I + T)^{-1} \) where \( I \) is the identity operator. Then one can conclude from (121) that the operator \( L^{-1} := R(I + T)^{-1} \) is the right inverse operator of \( L \) and
\[ \sigma = L^{-1}F_2. \]

A nonlocal character of system (120) causes difficulties requiring technical tricks related to a suitable localization. As suggested in either Chapter 4 from [24] or Section 4 from [5], we introduce the two collections of open sets: \{\( \theta^m \)\} and \{\( \Theta^m \)\}, such that
\[ \theta^m \subset \Theta^m \subset \Omega, \quad \cup_m \theta^m = \cup_m \Theta^m = \bar{\Omega}, \]
\[ \theta^m = B_{r/2}(x^m) \cap \bar{\Omega}; \quad \Theta^m = B_r(x^m) \cap \bar{\Omega} \]
with \( m = 1, ..., N_0; \) and \( B_r(x^m), \ B_{r/2}(x^m) \) are balls with the center in \( x^m \) and the radii of \( r \) and \( r/2, \) correspondingly. Denote by \( \Gamma^m = \Gamma \cap B_r(x^m), \) and \( \Gamma_0^m = \Gamma_0 \cap B_r(x^m). \) The index \( m \) belongs to one of two sets:
\[ m \in M \text{ if } \Theta^m \cap \Gamma = \emptyset \text{ and } m \in N \text{ if } \Theta^m \cap \Gamma \neq \emptyset. \]
The covering \{θ^m\} and \{Θ^m\} define a partition of unity for the domain Ω. Let ξ^m : Ω → [0, 1] be a smooth function such that
\[ ξ^m = 1, \text{ if } x ∈ θ^m, \quad ξ^m = 0, \text{ if } x ∈ Ω \setminus Θ^m, \quad ξ^m ∈ (0, 1), \text{ if } x ∈ Θ^m \setminus θ^m, \]
and \(|∇^i|ξ^m| ≤ const. r^{-|i|}, 1 ≤ ξ^m ≤ M_0\). Using the functions ξ^m, we define the function
\[ η^m = \frac{ξ^m}{∑_j(ξ^j)^2}. \]
From the properties of the functions ξ^m it follows that the functions η^m vanish for \(x ∈ Ω \setminus Θ^m\); and in addition, \(|∇^i|η^m| ≤ const. r^{-|i|} \).

The functions \(η^mξ^m\) define the partition of unity by the following formula
\[ ∑_m η^mξ^m = 1. \]

For each \(m ∈ N\), we pick out one point \(x^m ∈ θ^m ∩ Γ\) which will be the origin of the local coordinate system. The description of this system can be found in Chapter 4 [24].

We define local coordinate systems connected with each point \(x^m, m ∈ N\). Let Ψ be described with \(y_n = Ψ^m(y_1, ..., y_{n−1})\) in a small vicinity of every point \(x^m, m ∈ N\), and
\[ y = B^{(m)}(x − x^m), \quad \frac{∂Ψ^m}{∂y_i} ≤ const. r, \quad i = 1, n − 1, \quad (123) \]
where \(B^{(m)} = (β_{ij}^{(m)})_{i,j=1,n}\) is an orthogonal matrix with elements \(β_{ij}^{(m)}\) and \((β_{ij}^{(m)})^{-1}\) is an element of the inverse matrix to \(B^{(m)}\). To obtain the local "flatness" of the boundary, we change the variables as
\[ z_i = y_i, \quad z_n = y_n − Ψ^m(y_1, ..., y_{n−1}), \quad i = 1, n − 1. \quad (124) \]
Thus, the variables \((x_1, ..., x_n)\) are connected with \((z_1, ..., z_n)\) by the mappings \(Z_m\) (see (123), (124)), such that
\[ x = Z_m(z), \quad \text{and } z = Z_m^{-1}(x), \quad m ∈ N. \]

Denote by
\[ B\left(x, \frac{∂}{∂x}\right) := b_0(ω(x))\frac{∂}{∂n(ω)} = \sum_{i=1}^n b_0(ω(x))n_i(x)\frac{∂}{∂x_i}, \]
\[ B^{(m)}\left(y, \frac{∂}{∂y}\right) = \sum_{i=1}^n b_0(ω(y))n_i^{(m)}(y)\frac{∂}{∂y_i}, \quad (125) \]
\[ n_i^{(m)}(y) = \sum_{k=1}^n n_k(y)β_{ik}^{(m)}, \quad \left. n_i^{(m)}(z) = \sum_{k=1}^n n_k(x)β_{ik}^{(m)}\right|_{x = Z_m(z)}, \quad (126) \]
\[ a_{ij}^{(m)} = \sum_{k,s=1}^{n−1} a_{ks}(x^m, 0)β_{ik}^{(m)}β_{js}^{(m)}, \quad b_{0i}^{(m)} = b_0(ω(x^m))n_i^{(m)}(x^m). \quad (127) \]
It is easily to check that
\[ B^{(m)}\left(x^m, \frac{∂}{∂z}\right) = b_{0m}^{(m)}\frac{∂}{∂z_n}. \quad (128) \]
Let us introduce the function $F_2^{(m)}(z, t)$ as
\[ F_2^{(m)}(z, t) = Z_m^{-1}(\xi^m(x) F_2(x, t)), \quad m \in \mathbb{N}, \] (129)
and construct the operator $\mathcal{R}$.

We define the functions $u^{(m)}(z, t), \sigma^{(m)}(z', t), m \in \mathbb{N}$, as a solution of problem
\[ \Delta_z u^{(m)} = 0, \quad (z, t) \in \mathbb{R}^n_T, \quad u^{(m)} - \sum_{ij=1}^{n-1} a_{ij}^{(m)} \frac{\partial^2 \sigma^{(m)}}{\partial z_i \partial z_j} = 0 \text{ on } \mathbb{R}^{n-1}_T, \]
and $\sigma^{(m)}(z', t) = 0, z' \in \mathbb{R}^{(n-1)}$.

After that we can define the operator $\mathcal{R}$ and the function $u$ as
\[ \mathcal{R} F_2 = \sum_{m \in \mathbb{N}} \eta^m(x) \partial \alpha^m(x, t), \quad \sigma := \mathcal{R} F_2 = \sum_{m \in \mathbb{N}} \eta^m(x) \sigma^{(m)}(x, t) = \sum_{m \in \mathbb{N}} \eta^m(x) \mathcal{R} (\mathcal{R}^{(m)}(Z_m^{-1}[\xi^m F_2])) (129)

Next we introduce the function $v(x, t)$ as a solution of the following Dirichlet problem
\[ \Delta_z v = 0 \text{ in } \Omega_T, \quad v = \sum_{ij=1}^{n-1} a_{ij} \eta^m(x) \frac{\partial^2 \mathcal{R} F_2}{\partial \omega_i \partial \omega_j} \text{ on } \Gamma_T, \quad v = 0 \text{ on } \Gamma_\partial T. \]

The definition of the operator $\mathcal{L}$ (see (120)) gives
\[ \mathcal{L} F_2 = \mathcal{D}_\nu^r \mathcal{R} F_2 + B \left( x, \frac{\partial}{\partial x} \right) v. \]

Then, taking into account (129)-(134), we rewrite (127) as
\[ \mathcal{L} F_2 = F_2 + \sum_{m \in \mathbb{N}} \left[ B \left( x, \frac{\partial}{\partial x} \right) \eta^m \hat{u}^{(m)} - \eta^m B \left( x, \frac{\partial}{\partial x} \right) \partial \alpha^m \right] + \sum_{m \in \mathbb{N}} \left( B \left( x, \frac{\partial}{\partial x} \right) Z_m[u^{(m)}] - Z_m \left[ B^{(m)} \left( x^{(m)}, \frac{\partial}{\partial z} \right) u^{(m)} \right] \right) + B \left( x, \frac{\partial}{\partial x} \right) (v - u), \]
or
\[ \mathcal{L} F_2 = F_2 + T_1 F_2 + T_2 F_2 + T_3 F_2. \]
Lemma 5.3. Let \( \alpha, \nu \) and \( \Gamma_0, \Upsilon \) satisfy conditions of Theorem 5.1; (117) and (118) hold. Then
\[
\sum_{i=1}^{3} \| T F_2 \|_{C^{1+\alpha, \frac{2+\alpha}{\nu}}(\Upsilon_T)} \leq C(r, T) \| F_2 \|_{C^{1+\alpha, \frac{2+\alpha}{\nu}}(\Upsilon_T)},
\]
where
\[
C(r, T) = C[r^{1-\alpha} + r^\alpha + T^{\frac{\alpha}{2}} + T^{\frac{\alpha(1-\alpha)}{2}} + T^{\frac{\alpha}{2} r^{-1}}] < 1/2
\]
for small quantities \( r \) and \( T \).

Since the proof of Proposition 7 is standard (see, e.g., Section 4 [5]) and technically tedious, we represent it in Appendix 7.2.

Results of Proposition 7 provide existence of an element \( \sigma \) which satisfies (120) and \( \sigma \in P^4+\alpha(\Upsilon_T) \). Then the Schauder approach together with estimates (131), (119) and (28), (29) allow us to obtain the corresponding coercive estimate for \( \sigma \in P^4+\alpha(\Upsilon_T) \) and \( u \in C^{2+\alpha, \frac{2+\alpha}{\nu}}(\overline{\Omega_T}) \).

Thus, we assert the following result.

Lemma 5.3. Let \( \alpha, \nu \in (0, 1), \Upsilon \) and \( \Gamma_0 \in C^{3+\alpha} \), conditions (114), (115) and (117), (118) hold, and \( F_2(x, 0) = 0, x \in \Upsilon \). Then for some small \( T \) there exists a unique solution \((u, \sigma)\) of problem (109)-(112): \( u \in C^{2+\alpha, \frac{2+\alpha}{\nu}}(\overline{\Omega_T}), \sigma \in P^4+\alpha(\Upsilon_T) \), such that
\[
\|u\|_{C^{2+\alpha, \frac{2+\alpha}{\nu}}(\Upsilon_T)} + \|\sigma\|_{P^4+\alpha(\Upsilon_T)} \leq C \| F_2 \|_{C^{1+\alpha, \frac{2+\alpha}{\nu}}(\Upsilon_T)},
\]
where \( C \) is a positive constant depends only on the measure of \( \Omega, \Upsilon, \Gamma_0, \) and \( \|b_0\|_{C^{1+\alpha}(\Upsilon)}, \|a_{ij}\|_{C^{2+\alpha, \nu/\alpha}(\Upsilon_T)} \).

Note that results of Lemma 5.3 hold in the case of \( a^1_j \neq 0, a^1_i \in C^{2+\alpha, \nu/\alpha}(\Upsilon_T) \). To show this it is enough to apply the contraction mapping theorem to (109)-(112) together with Lemma 5.3 and interpolation inequalities.

One can check that the results of Lemma 5.3 can not be extended into linear problem (109)-(112) with arbitrary nonzero \( F_0 \in C^{\alpha, \nu/3}(\overline{\Omega_T}) \) and \( F_1 \in C^{2+\alpha, \frac{2+\alpha}{\nu}}(\overline{\Omega_T}) \). It is explained that the spaces \( C^{k+\alpha, \frac{2+\alpha}{\nu}} \) are related with the subdiffusion equation like \( D_t^\nu u - \Delta_x u = F_0 \) in \( \Omega_T \), but we have only elliptic equation \( \Delta_x u = F_0 \) in \( \Omega_T \).

It is not hard to exam the following embedding: \( C^{k+\alpha, \frac{2+\alpha}{\nu}}(\overline{\Omega_T}) \subset C([0, T], C^{k+\alpha}(\Omega)) \) and \( C^{k+\alpha, \nu/3+\alpha}(\overline{\Omega_T}) \subset C([0, T], C^{k+\alpha}(\Omega)) \). However we don’t have the same embedding between the spaces \( C^{k+\alpha, \frac{2+\alpha}{\nu}} \) and \( C^{k+\alpha, \nu/3+\alpha} \). Thus, based on the results of Lemma 5.3, we will show one-valued solvability of problem (109)-(112) in the spaces \( C([0, T], C^{k+\alpha}(\Omega)) \).

After that, using Schauder technique together with statements (ii) from Theorem 4.1 and Propositions 1 and 3, we deduce that the obtained unique solution \((u, \sigma)\): \( u \in C([0, T], C^{2+\alpha}(\Omega)), \sigma \in C([0, T], C^{4+\alpha}(\Upsilon)) \), belongs to spaces \( C^{k+\alpha, \nu/3+\alpha} \) with the corresponding \( k \).

Let us consider the sequence of “smooth” functions \( F_{2k} \in C^{1+\alpha, \frac{2+\alpha}{\nu}}(\Upsilon_T): F_{2k} \to F_2 \in C([0, T], C^{1+\alpha}(\Upsilon)) \), and apply Lemma 5.3 to the following problems:
\[
\Delta_x u_k = 0 \text{ in } \Omega_T; \quad (1 - \delta) u_k - \delta \frac{\partial u_k}{\partial n} = 0 \text{ on } \Gamma_0 T; \quad \sigma_k(\omega, 0) = 0, \omega \in \Upsilon;
\]
\[
u \sum_{i=1}^{n} a_{ij}(\omega, t) \frac{\partial^2 \sigma_k}{\partial \omega_i \partial \omega_j} + \sum_{i=1}^{n} a_{i}^1(\omega, t) \frac{\partial \sigma_k}{\partial \omega_i} = 0 \text{ on } \Upsilon_T;
\]
Thus, we obtain the sequence of solutions \((u_k, \sigma_k)\): \(u_k \in P_{\frac{\nu}{4} + \alpha}(\bar{\Omega})\), \(u_k \in C^{2+\alpha, \frac{\nu}{2} + \alpha}(\bar{\Omega})\). Next we apply Schauder approach together with inequalities (11), (12), (24), (27) and (64) and deduce next estimate
\[
\|u_k\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))} + \|\sigma_k\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))} + \|D^\nu_T \sigma_k\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))} \leq C\|F_{2k}\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))},
\]
where the positive constant \(C\) is independent of \(k\).

Then, using the completeness of spaces \(C([0,T], C^{2+\alpha}(\bar{\Omega}))\), \(C([0,T], C^{2+\alpha}(\bar{\Omega}))\) and passing to the limit in (142) and (143), we conclude
\[
\sigma_k \to \sigma \in C([0,T], C^{2+\alpha}(\bar{\Omega})); \quad D^\nu_T \sigma_k \to D^\nu_T \sigma \in C([0,T], C^{1+\alpha}(\bar{\Omega}));
\]
\[
u_k \to u \in C([0,T], C^{2+\alpha}(\bar{\Omega})),
\]
where \(u\) and \(\sigma\) satisfy inequality (143) with the right-hand side \(\|F_{2}\|_{C([0,T], C^{1+\alpha}(\bar{\Omega}))}\).

Therefore, we get a unique solution \((u, \sigma)\) of problem (109)-(112) where \(F_0, F_1 \equiv 0\): \(u \in C([0,T], C^{2+\alpha}(\bar{\Omega})); \quad \sigma \in C([0,T], C^{2+\alpha}(\bar{\Omega})); \quad D^\nu_T \sigma \in C([0,T], C^{1+\alpha}(\bar{\Omega})).\)

Finally, we apply the standard Schauder technique together with Proposition 1, statement (ii) from Theorem 4.1 and interpolation inequality (8) and get
\[
\|u\|_{C^{2+\alpha, \frac{\nu}{2} + \alpha}(\bar{\Omega})} + \|\sigma\|_{C^{2+\alpha, \frac{\nu}{2} + \alpha}(\bar{\Omega})} + \|D^\nu_T \sigma\|_{C^{1+\alpha}(\bar{\Omega})} \leq C\|F_2\|_{C^{2+\alpha, \frac{\nu}{2} + \alpha}(\bar{\Omega})} + \langle u \rangle_{L^2_{(t, \bar{\Omega})}}.
\]

As for estimate of \(\langle u \rangle_{L^2_{(t, \bar{\Omega})}}\), we employ the maximum principle to elliptic boundary-value problem like (23), where \(w := u(x,t) - u(x,\tau)\), \(g_0 := 0\), \(g_1 := 0\), \(g_2 := \sum_{i=j=1}^{n-1} a_{ij}(\omega, \tau) \frac{\partial^2 \sigma(\omega, \tau)}{\partial \omega_i \partial \omega_j} + \sum_{i=1}^{n-1} a_1(\omega, \tau) \left[ \frac{\partial \sigma(\omega, \tau)}{\partial \omega_i} - \frac{\partial \sigma(\omega, \tau)}{\partial \omega_j} \right] - \sum_{i=1}^{n-1} [a_i(\omega, \tau) - 1] \frac{\partial \sigma(\omega, \tau)}{\partial \omega_i} \right] \]
Hence we have
\[
\langle u \rangle_{L^2_{(t, \bar{\Omega})}} \leq |t-\tau|^{\frac{\nu}{2}} \|\sigma\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))} \left[ \sum_{i=1}^{n-1} \|a_{ij}\|_{C^{2+\alpha, \frac{\nu}{2} + \alpha}(\bar{\Omega})} + \sum_{i=1}^{n-1} \|a_1\|_{C^{2+\alpha, \frac{\nu}{2} + \alpha}(\bar{\Omega})} \right]
\]
\[
+ |t-\tau|^{\frac{\nu}{2}} \sum_{i=1}^{n-1} \|a_{1i}\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))} \left[ \frac{\partial^2 \sigma(\omega, \tau)}{\partial \omega_i \partial \omega_j} - \frac{\partial^2 \sigma(\omega, \tau)}{\partial \omega_i \partial \omega_j} \right]_{C([0,T], C^{1+\alpha}(\bar{\Omega}))} + |t-\tau|^{\frac{\nu}{2}} \sum_{i=1}^{n-1} \|a_{1i}\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))} \left[ \frac{\partial \sigma(\omega, \tau)}{\partial \omega_i} - \frac{\partial \sigma(\omega, \tau)}{\partial \omega_i} \right]_{C([0,T], C^{1+\alpha}(\bar{\Omega}))}.\]

Here we used properties (115) of the functions \(a_{ij}\) and \(a_{1i}\). Based on inequality (11), one can easily check that
\[
|t-\tau|^{\frac{\nu}{2}} \sum_{i=1}^{n-1} \|a_{1i}\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))} \left[ \frac{\partial \sigma(\omega, \tau)}{\partial \omega_i} - \frac{\partial \sigma(\omega, \tau)}{\partial \omega_i} \right]_{C([0,T], C^{1+\alpha}(\bar{\Omega}))} \leq C|t-\tau|^{\frac{\nu}{2}} \sum_{i=1}^{n-1} \|a_{1i}\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))} \]
\[
\times \|D^\nu_T \sigma\|_{C([0,T], C^{1+\alpha}(\bar{\Omega}))} \sum_{i=1}^{n-1} \|a_{1i}\|_{C([0,T], C^{2+\alpha}(\bar{\Omega}))}.\]
At last, interpolation inequality together with (11) lead to
\[\max_{\gamma} |D_2^2 \sigma(\omega, t) - D_2^2 \sigma(\omega, \tau)| \leq \varepsilon |t - \tau|^{\frac{2}{\nu'}} \left( \frac{T^{2+\nu'}}{2\varepsilon} |t - \tau|^{2+\nu'} (D_2^2 \sigma)_{t, \tau} + \frac{T^{2+\nu'}}{2\varepsilon} \right). \tag{148}\]
Choosing \(\varepsilon < 1/4\) and \(T^{2+\nu'} < 1/4\) and taking into account (146)-(148), we deduce that
\[\langle u \rangle_{t, \tau} \leq C(T, \varepsilon) \left[ \|\sigma\|_{C^{1,\alpha}, \alpha}^{\|\sigma\|_{H^{1,\alpha}}} + \|D_2^2 \sigma\|_{C^{1,\alpha}, \alpha}^{\|\sigma\|_{H^{1,\alpha}}} \right], \tag{149}\]
where \(C(T, \varepsilon) \ll 1\).
Thus, estimate (116) follows from (145) and (149). That completes the proof of Theorem 5.1.

5.2. Solvability of nonlinear problem (54). Let us introduce function spaces \(H_1\) and \(H_2\), such that \(z \in H_1\) and \(F(z) \in H_2\):
\[H_1 = C^{2+\alpha, \frac{\alpha}{2+\alpha}}(\Omega_T) \times C^{4+\alpha, \frac{2\alpha}{2+\alpha}}(Y_T) \times C^{1+\alpha, \frac{\alpha}{2+\alpha}}(Y_T);\]
\[H_2 = C^{\alpha, \frac{\alpha}{2}}(\Omega_T) \times C^{2+\alpha, \frac{\alpha}{2+\alpha}}(Y_T) \times C^{1+\alpha, \frac{\alpha}{2+\alpha}}(Y_T) \times C^{2+\alpha, \frac{\alpha}{2+\alpha}}(\Gamma_0_T),\]
where \(\delta = 0\) or \(\delta = 1\), and
\[\|z\|_{H_1} = \|u, \sigma\|_{H_1} := \|u\|_{C^{2+\alpha, \frac{\alpha}{2+\alpha}}(\Omega_T)} + \|\sigma\|_{C^{4+\alpha, \frac{2\alpha}{2+\alpha}}(\Omega_T)} + \|D_2^2 \sigma\|_{C^{1+\alpha, \frac{\alpha}{2+\alpha}}(\Omega_T)};\]
\[\|F(z)\|_{H_2} = \|F_0(z), F_1(z), F_2(z), 0\|_{H_2} := \|F_0\|_{C^{\alpha, \frac{\alpha}{2}}(\Omega_T)} + \|F_1\|_{C^{2+\alpha, \frac{\alpha}{2+\alpha}}(\Omega_T)};\]
\[\|F_2\|_{C^{1+\alpha, \frac{\alpha}{2+\alpha}}(\Omega_T)} + \|F_2\|_{C^{1+\alpha, \frac{\alpha}{2+\alpha}}(\Omega_T)}.\]
It is easy to see that the linear operator \(A\) obtained in (54) acts from \(H_1\) into \(H_2\).
In virtue of Theorem 5.1, we rewrite equation (54) in the form
\[z = A^{-1} F(z) \equiv \mathbb{R}(z), \tag{150}\]
\(\mathbb{R}(z)\) is a nonlinear operator. We will show that \(\mathbb{R}\) is a contraction operator.
Let \(B_d(z) = \{z \in H_1 : \|z\|_{H_1} \leq d, z(x, 0) = 0\}\) be a ball of the radius \(d\) in the space \(H_1\) centered at the origin.

**Lemma 5.4.** The following estimates hold for the right-hand side of problem (54):
\[\|F(0)\|_{H_2} \leq C_1(T), \|F(z_1) - F(z_2)\|_{H_2} \leq C_2(T, d)\|z_1 - z_2\|_{H_1}, \ z_1, z_2 \in B_d,\]
where \(C_1(T) \to 0, C_2(T, d) \to 0\) as \(d, T \to 0\).
Details of the proof of this lemma are similar to those from Lemma 5.1 in [43] and use results of Theorem 5.1, Proposition 1, interpolation inequality (8) together with Corollaries 1 and 2. So we omit these arguments here.
From Lemma 5.4 it follows that for sufficiently small \(T\) and \(d\), the nonlinear operator \(\mathbb{R}\) in (150) satisfies the conditions of the fixed point theorem for the contraction operator. This proves Theorem 3.1.
Note that Theorem 3.1 gives one-to-one solvability locally in time of the fractional Hele-Shaw problem with surface tension in the Hölder classes \(C^{k+\alpha, \beta, \alpha}\) with \(\beta := \nu\alpha/3\). Thus, the obtained results in the nonlocal case (\(\nu \in (0, 1)\)) represent the marked difference with the local case (see [2] and [4]) where the exponent \(\beta := \frac{\alpha}{4}\) is greater.
6. Numerical solution of (1)-(4). In this section we employ the numerical technique from [41] to get some numerical solution of problem (1)-(4) if \(\Omega(t) \subset \mathbb{R}^2, t \in [0, T] \). To this end we separate (1)-(4) into two problems. The first problem is the elliptic one defined on \(\Omega(t)\):

\[
\Delta p = 0 \text{ in } \Omega(t), t \in [0, T]; \quad (1-\delta)p - \delta \frac{\partial p}{\partial n} = \varphi(y) \text{ on } \Gamma_{\partial T}, \quad \delta = 0 \text{ or } 1; \quad p = \gamma \kappa(\Upsilon(t)), \quad \text{ (151)}
\]

while the second problem is the initial value problem on \(\Upsilon(t)\):

\[
D^p(\omega, t) = -\mu \frac{\partial p}{\langle n(\omega), n_t \rangle_{\Upsilon(t)}} \quad \rho(\omega, 0) = 0, \quad \nu \in (0, 1), \quad \omega \in [0, 1]. \quad \text{ (152)}
\]

Note that problem (152) is obtained from the first boundary condition in (2) and representations (5) and (32) for the "fractional" velocity of the free boundary and the unknown interface \(\Upsilon(t)\).

Problems (151) and (152) are connected through the reciprocal usage of their solutions. That is why, the numerical approach for solving quasistationary fractional free boundary problem (1)-(4) may be constructed as the splitting scheme. Other words, at each time point \(t_i\), problems (151) and (152) are solved separately with the mutual exchange of their solutions on the next time step.

Elliptic problem (151) is solved by the finite element method implemented using the finite element library DOLFIN [30]. At every time point \(t_i\), we construct a new mesh in accordance with solution of (152) which provides \(\Upsilon(t_i)\). The curvature \(\kappa(\Upsilon(t_i))\) is given by

\[
\kappa(\Upsilon(t_i)) = \frac{\partial y_1(\omega, t) \partial^2 y_2(\omega, t) - \partial^2 y_1(\omega, t) \partial y_2(\omega, t)}{\left[\left(\frac{\partial y_1(\omega, t)}{\partial \omega}\right)^2 + \left(\frac{\partial y_2(\omega, t)}{\partial \omega}\right)^2\right]^{3/2}} \bigg|_{t=t_i},
\]

where (see (31))

\[
y_i(\omega, t) = m_i(\omega) + \rho(\omega, t)n_i(\omega), \quad \omega \in [0, 1], \quad i = 1, 2.
\]

The predictor-corrector scheme from [11] is employed to solve (152) where on the predictor stage we use an analytical approximation like (52): 

\[
\rho^{pr}(\omega, t_i) = -\frac{\mu t_i^{\nu}}{\Gamma(\nu + 1)} \frac{\partial p_{0}}{\partial \bar{n}(\omega)} \bigg|_{\Upsilon(t)}. \quad \text{ (153)}
\]

Thus the solving algorithm for the coupled problems (151) and (152) looks like the algorithm from Subsection 6.1 [41] and can be represented as:

**Algorithm**

Given \(\Omega, \Upsilon, \mu, \phi(\omega), \bar{n}(\omega), \delta, \gamma\), the temporal grid \(t_i = i\Delta t; i \in \overline{0, N}, \Delta t = T/N\). First, we solve problem (151) and compute \(\frac{\partial p_{0}}{\partial \bar{n}(\omega)}\) 

Next for each \(i = 1, N\) we perform

- compute \(\rho^{pr}(\omega, t_i)\) from (153)
- update \(\Upsilon\) (and \(\Omega\)) according to (31) using \(\rho^{pr}(\omega, t_i)\)
- solve problem (151) and compute \(\frac{\partial p_{pr}}{\partial \bar{n}_{t_i}}\)
- compute corrected \(\rho(\omega, t_i)\) as

\[
\rho(\omega, t_i) = -\frac{\mu}{(\bar{n}(\omega), n_{t_i}^{pr})} \frac{(\Delta t)^{\nu}}{\Gamma(2 + \nu)} \left( \sum_{k=0}^{i-1} c_{k, t} \frac{\partial p_{k}}{\partial \bar{n}_{t_k}} + \frac{\partial p_{pr}^{pr}}{\partial \bar{n}_{t_i}} \right), \quad \text{ (154)}
\]
\( c_{k,i} = \begin{cases} (1 + \nu)\nu - i^{1+\nu} + (i - 1)^{1+\nu}, & \text{if } k = 0; \\
(i - k + 1)^{1+\nu} - 2(i - k - 1)^{1+\nu} + (i - k - 1)^{1+\nu}, & \text{if } 0 < k < i - 1; \\
1, & \text{if } k = i; \end{cases} \) \tag{155}

- update \( \Upsilon \) (and \( \Omega \)) according to (31) using \( \rho(\omega, t_i) \)
- solve for the problem (151) and compute \( \frac{\partial \rho_i}{\partial n_i} \).

Next we represent some example of this simulations with different \( \nu \in (0, 1) \) and the same initial spatial domain \( \Omega \). Let \( \Omega \) be restricted by two elliptic boundaries: \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 1 \), where \( a := a_0, b := b_0 \) for the internal fixed boundary \( \Gamma_0 \), and \( a := a_0^{\prime}, b := b_0^{\prime} \) for the external initial shape of the free boundary \( \Upsilon(0) = \Upsilon \). We assume that \( \mu, \gamma = 1, \delta = 0, \varphi(y) = 1; a_0 = 1.25, b_0 = 2.5; a_0^{\prime} = 2.5, b_0^{\prime} = 5.0 \). Due to the symmetry of the problem, it is solved on one quarter of \( \Omega(t) \) corresponding to the first quadrant.

Numerical experiments are performed for three different values of \( \nu \), namely, \( \nu_1 = 0.25, \nu_2 = 0.5, \nu_3 = 0.75 \). We analyze the dynamics of the free boundary for different values of \( \nu \). As it follows from Fig. 1, smaller values of \( \nu \) require longer time to change the boundaries.

Moreover, Fig. 2 demonstrates that \( \nu \) controls dynamics of the moving boundary \( \Upsilon(t) \) in the following way. There is a critical time \( T^* := T^*(\nu) \), such that for \( t < T^* \) the free boundary moves rapid if \( \nu \) decreases. However, the effect is opposite for \( t > T^* \), i.e. the free boundary moves slowly for small \( \nu \) and much faster for \( \nu \) approaching 1. As it follows from Fig. 2, \( T^* \) is located in the neighborhood \( T = 0.75 \). Drawing the shape of free boundaries for \( t \) near \( T = 0.75 \) and \( \nu = \nu_i \), \( i = 1, 2, 3 \), (see Fig.3 in the case of \( \nu_2 = 0.5, \nu_3 = 0.75 \)) we find critical time:

- \( T^* = 0.75, \nu_1 = 0.25, \nu_2 = 0.5 \),
- \( T^* = 0.83, \nu_1 = 0.25, \nu_2 = 0.75 \),
- \( T^* = 0.90, \nu_1 = 0.5, \nu_2 = 0.75 \).

7. Appendix.

7.1. Proof of Proposition 4. To get representation (101), we repeat the arguments from Section 7 [41] and use the results from Lemma 4.2 and Proposition 1. Then taking into account relation (3.32) from [41], we can deduce from (75)

\[
\frac{\partial^2 U}{\partial x_i \partial x_j} = \begin{cases} \sum_{i=1}^{n-1} a_{ii} \int_{R_{n-1}} \left[ \frac{\partial^3 \psi(x' - y', t)}{\partial y_i \partial y_l} - \frac{\partial^2 \psi(x', t)}{\partial x_i \partial x_l} \right] \frac{\partial K}{\partial y_j}(y', x_n) dy', j \neq l; \\
\sum_{i=1}^{n-1} a_{ii} \int_{R_{n-1}} \left[ \frac{\partial^3 \psi(x' - y', t)}{\partial y_j \partial y_l} - \frac{\partial^2 \psi(x', t)}{\partial x_i \partial x_l} \right] \frac{\partial K}{\partial y_i}(y', x_n) dy', l \neq n, j = n; \\
\sum_{i=1}^{n-1} a_{ii} \int_{R_{n-1}} \left[ \frac{\partial^2 \psi(x' - y', t)}{\partial y_j^2} - \frac{\partial \psi(x', t)}{\partial x_i^2} \right] \frac{\partial^2 K}{\partial y_i^2}(y', x_n) dy', j = l = n, \end{cases}\tag{156}
\]

for \( l, j \neq 1, n \). Hence, representation (102) is an easy consequent of representation (156) if \( l \neq n \). As for case \( l = j = n \), one can check with simple calculations that

\[
\frac{\partial^2 K(y', x_n)}{\partial x_n^2} = \sum_{m=1}^{n-1} \frac{\partial^2}{\partial y_m^2} K(y', x_n). \tag{157}
\]

After that we integrate by parts in (156), \( l = j = n \), and taking into account relation (157) together with estimate (3.30) [41] to get representation (102).
Next, we apply equality (88) to representation (73) and get condition (60). Thus, to finish the proof of Proposition 4, it is necessary to show that the functions $U(x', x_n, t)$ and $\varphi(x', t)$ represented by (74) and (75) satisfy condition (61).
Figure 2. Comparison of dynamics of the free boundary for (a) \( \nu_1 = 0.25, \nu_2 = 0.5 \), (b) \( \nu_1 = 0.25, \nu_2 = 0.75 \), (c) \( \nu_2 = 0.5, \nu_3 = 0.75 \).

First we calculate \( \left. \frac{\partial U}{\partial x_n} \right|_{x_n=0} \):

\[
\left. \frac{\partial U}{\partial x_n} \right|_{x_n=0} = -\frac{1}{a_0} \int_0^t d\tau \int_{R^{n-1}} dy' \left[ f(y', t-\tau) - f(x', t-\tau) \right] \int_0^{\tau^{-1}} \int_{-\nu}^{0} W(-z^\nu; -\nu, 0) d\zeta \langle a\zeta, \zeta \rangle |\zeta| \exp \left\{ -8\pi^2 a_0 |\zeta| \langle a\zeta, \zeta \rangle \right\} \int_{R^{n-1}} e^{2\pi i \langle \zeta, y' \rangle} dy'.
\]

(158)
Figure 3. Behaviour of the free boundary for $\nu_2 = 0.5$, $\nu_3 = 0.75$, and for time near $T^* = 0.90$, (a) the case $t < T^*$, (b) the case $t = T^*$, (c) the case $t > T^*$. 
After that we apply (88) to the second term in (158) and obtain
\[ \frac{\partial U}{\partial x_n} \bigg|_{x_n=0} = -\frac{\nu}{\alpha_0} \int_0^t dr \int_{R^{n-1}} dy' [f(y', t-\tau) - f(x', t-\tau)] \int_0^{+\infty} W(z\tau^{-\nu}; -\nu, 1-\nu) \times \alpha \tau^{-1-\nu} \frac{\partial L}{\partial z} (y', z) dz. \] (159)

Then, we return to representation (101) and calculate \( \partial_x T \). Proposition 2, equality (81) together with integrating by parts lead to the relation
\[ \partial_x T(y', \tau) = \nu \int_0^{+\infty} W(z\tau^{-\nu}; -\nu, 1-\nu) \alpha \tau^{-1-\nu} \frac{\partial L}{\partial z} (y', z) dz + \nu \tau^{-\nu-1} \sum_{j=1}^{\gamma} \delta(-y_j) \times \lim_{z \rightarrow 0} z W(z\tau^{-\nu}; -\nu, 1-\nu). \] (160)

Therefore, condition (61) follows from equalities (101), (159) and (160). That completes the proof of Proposition 4.

7.2. Proof of Proposition 7. It is easy to check that results of Lemmas 4.1-4.4 from Chapter 4 [24] can be extended into the case of spaces \( C^{k+\alpha, \frac{k\alpha}{3}} \), \( k = 0, 3 \). Based on these results and inequalities (119), (131), one can obtain with the technique of §7 Chapter 4 [24], Section 4 [5]:
\[ \|T_1 F_2\|_{C^{k+\alpha, \frac{k\alpha}{3}}(\Omega_T)} + \|T_2 F_2\|_{C^{k+\alpha, \frac{k\alpha}{3}}(\Omega_T)} \leq C_1 [r^{\alpha} + T^{\alpha/3} + \omega]\|F_2\|_{C^{k+\alpha, \frac{k\alpha}{3}}(\Omega_T)}, \]
\[ \omega = \frac{T^{\alpha/3}}{r} \ll 1, \] (161)

if \( r \) and \( T \) are enough small.

To evaluate the term \( T_3 F_2 = B \left( x \frac{\partial}{\partial x} \right) (v - u) \) in (138), we have to describe well the properties of the function \( w := v - u \). The simple calculations give (if one takes into account definitions of functions \( v \) and \( u \) and interpolation inequality (8))
\[ \|Bw\|_{C^{k+\alpha, \frac{k\alpha}{3}}(\Omega_T)} \leq C \left( \sum_{i,j=1}^n \|w_{x_i x_j}\|_{C^{k\alpha/3}} + \|w_{x_i}\|_{C^{k\alpha/3}} \right) \leq C \left( \sum_{i=1}^n \|w_{x_i}\|_{C^{k\alpha/3}} \right). \] (162)

Moreover, one can check that the function \( w \) satisfies to the following Dirichlet problem
\[ \Delta_x w = G(x, t), \quad (x, t) \in \Omega_T; \quad w = 0, \quad (x, t) \in \Gamma_0; \quad w = g(x, t), \quad (x, t) \in \Omega_T, \]
(163)
where
\[ G(x, t) = \Delta_x \left( \sum_{m \in N} \eta^m(x) \hat{u}^{(m)}(x, t) \right), \] (164)
\[ g(x, t) = \sum_{m \in N} \left\{ \sum_{i,j=1}^{n-1} a_{ij}(x, t) - a_{ij}^{(m)} \right\} \eta^m \frac{\partial^2 \hat{u}^{(m)}}{\partial \omega_i \partial \omega_j} + \sum_{i,j=1}^{n-1} a_{ij}(x, t) \frac{\partial^2 \eta^m}{\partial \omega_i \partial \omega_j} \hat{u}^{(m)} + 2 \sum_{i,j=1}^{n-1} a_{ij}(x, t) \frac{\partial \eta^m}{\partial \omega_i} \frac{\partial \hat{u}^{(m)}}{\partial \omega_j} + \sum_{i,j=1}^{n-1} \eta^m \frac{\partial^2 \eta^m}{\partial \omega_i \partial \omega_j} Z_m [a_{ij}^{(m)} \sigma^{(m)}(z, t)] \]

THE HELE-SHAW PROBLEM IN THE CASE OF SUBDIFFUSION 1969
\[ -Z_m \left[ a_{ij}^{(m)} \frac{\partial^2 \sigma^{(m)}}{\partial z_i \partial z_j} \right]. \] (165)

As it follows from [25]

\[ \sum_{i,j=1}^n \langle w_{x_i x_j} \rangle_{x, T_T}^{(a)} \leq \| w \|_{C([0, T]; C^{2+a}(\Omega))] \leq C(\| g \|_{C([0, T]; C^{2+a}(\Omega))] + \| g \|_{C([0, T]; C^{2+a}(\Omega)])} \]

\[ \leq C \left( \| G \|_{C^{\alpha}}(\Omega_T) + \| g \|_{C^{2+a, 2\alpha/\nu}(\Omega_T)} \right). \] (166)

As for the estimate of the right-hand side in (166), we get

\[ \| G \|_{C^{\alpha}}(\Omega_T) \leq c_2 [r + T^{\frac{2\alpha}{\nu}} + \alpha] \| F_2 \|_{C^{1+a, \frac{2\alpha}{\nu}}(\Omega_T)}. \] (167)

The proof of (167) is based on tedious calculations by using of properties of \( a_{ij}(\omega, t), \)
\( b_0(\omega), \) inequalities (119), (123), (130) and results like Lemmas 4.1-4.3 from Chapter 4 [24]. Here we represent only the proof of (167) for \( \| D_\omega u^{m} a_{ij} \|_{C^{\alpha, \alpha/3}(\Omega_T)} \)
which is the “worst” term in \( \| g \|_{C^{2+a, 2\alpha/\nu}(\Omega_T)}. \)

It is easily to check that

\[ \sup_{T_T} \| D_\omega u^{m} a_{ij} \|_{C^{\alpha, \alpha/3}(\Omega_T)} \leq C \frac{T^{1+\alpha/\nu}}{r} \| a_{ij} \|_{C([0, T], C(\Omega))} \langle D_\omega^3 \phi \rangle_{t, T_T}, \] (168)

\[ (D_\omega u^{m} a_{ij} \|_{C^{\alpha, \alpha/3}(\Omega_T)} \leq \frac{c_2^2}{r} \langle a_{ij} \rangle_{t, T_T} T^{\alpha/2} \langle D_\omega^3 \phi \rangle_{t, T_T} \]

\[ + |a_{ij}|_{C([0, T], C(\Omega))} T^2 \langle D_\omega^3 \phi \rangle_{t, T_T}, \] (169)

\[ \langle D_\omega u^{m} a_{ij} \|_{C^{\alpha, \alpha/3}(\Omega_T)} \leq C \frac{1}{r} \langle a_{ij} \rangle_{t, T_T} T^{\alpha/2} \langle D_\omega^3 \phi \rangle_{t, T_T}, \] (170)

where the constant \( c_2 \) is independent of \( m, r, T. \)

Thus, we deduce from inequalities (168)-(170), (131) that

\[ \sup_m \| D_\omega u^{m} a_{ij} \|_{C^{\alpha, \alpha/3}(\Omega_T)} \leq C \omega \left[ 1 + T^{\alpha/3} + T^{\nu(1-\alpha)/3} \right] \| a_{ij} \|_{C^{2+a, \nu}(\Omega_T)} \]

\[ + \| F_2 \|_{C^{1+a, \frac{2\alpha}{\nu}}(\Omega_T)}. \] (171)

Note that the rest terms in the left-hand side of (167) are estimated with the same way.

Hence, inequalities (166) and (167) ensures the corresponding estimate of the first term in the right-hand side of (162)

\[ \sum_{i,j=1}^n \langle w_{x_i x_j} \rangle_{x, T_T}^{(a)} \leq c_2 [r + \alpha + T^{\frac{2\alpha}{\nu}}] \| F_2 \|_{C^{1+a, \frac{2\alpha}{\nu}}(\Omega_T)}. \] (172)

To evaluate the second term in the right-hand side of (162), we use interpolation inequality (8) and deduce

\[ \sum_{i,j=1}^n \langle w_{x_i x_j} \rangle_{t, T_T}^{(\alpha/3)} \leq \sup_{t, t' \in [0, T]} \left| w(\cdot, t) - w(\cdot, t') \right|_{C^2(\bar{\Omega})} \leq C \left| \| w \|_{C([0, T], C^{2+a}(\bar{\Omega})]} \right|^{\frac{1}{1-a}} \]

\[ \times \left[ \| w \|_{t, T_T}^{(1+a/\nu)} + \sum_{i,j=1}^n \langle w_{x_i} \rangle_{t, T_T}^{(1+a/\nu)} \right]^{\frac{a}{1+a}}. \] (173)
It is easy to see
\[
\|u\|_{C([0,T], C^{2+\alpha}(\Omega))} \leq C(\sum_{i,j=1}^{n} (w_{x_i x_j}, \alpha)_{x,\Omega_T} + \sup_{\Omega_T} |u|)
\]
\[
\leq C(|r + \alpha + T^{\frac{2+\alpha}{\nu}}|) \|F_x\|_{C^{1+\alpha, \frac{1+\alpha}{\nu}}(\Omega_T)},
\]
(174)
Hence, to complete the proof of estimate (162), we have to evaluate the term
\[
\langle w_{x_i, x_j} \rangle_{x, \Omega_T}.
\]
First of all, we represent the function \(G\) in (164) as
\[
G = G_1 + d\nu G_2, \quad G_1 \in C([0,T], C(\Omega)), \quad G_2, i \in C([0,T], C^m(\Omega)), \quad i = \overline{1,n}.
\]
(175)
After simple calculations we rewrite the function \(G\) as
\[
G(x, t) = \sum_{m \in N} \left[ - \hat{u}^{(m)} \Delta_x \eta^m + 2 \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial \eta^m}{\partial x_i} \hat{u}^{(m)} \right) + \sum_{j=1}^{n} \frac{\partial \eta^m}{\partial x_j} Z_{m}[(\beta_{m})^{-1}]
\]
\[
x \times \sum_{i=1}^{n-1} (\Psi_{1z}^{(m)} \hat{u}_{z_{n}}^{(m)} - \Psi_{1z}^{(m)} u^{(m)} - 2 \Psi_{1z}^{(m)} u^{(m)}]) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (\eta^m Z_{m}[(\beta_{m})^{-1}])
\]
\[
\times \sum_{i=1}^{n-1} (\Psi_{1z}^{(m)} \hat{u}_{z_{n}}^{(m)} - \Psi_{1z}^{(m)} u^{(m)} - 2 \Psi_{1z}^{(m)} u^{(m)}]), \quad x \in \text{supp} \eta^m(x), \quad m \in N.
\]
(176)
Here we take into account that \(\Delta_x u^{(m)}(z, t) = 0\).
It is not hard to exam that
\[
\frac{\partial \eta^m}{\partial x_j} Z_{m}[(\beta_{m})^{-1} \Psi_{1z}^{(m)} u^{(m)}] = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial \eta^m}{\partial x_j} Z_{m}[(\beta_{m})^{-1} \Psi_{1z}^{(m)} u^{(m)}] \right) - \sum_{i=1}^{n} \frac{\partial^2 \eta^m}{\partial x_i \partial x_j}
\]
\[
x \times Z_{m}[(\beta_{m})^{-1} \Psi_{1z}^{(m)} u^{(m)}] - \frac{\partial \eta^m}{\partial x_i} Z_{m} \left[ \frac{\partial}{\partial z_i} \left( \frac{\partial \eta^m}{\partial z_i} \right) \right] u^{(m)},
\]
(177)
where \(d_{m}^{(m)} := (\beta_{m})^{-1} (\beta_{m})^{-1}, \quad l = \overline{1,n}.
\)
Thus, we deduce from (176), (177) that
\[
G_1 = \sum_{m \in N} G_1^m, \quad G_1^m = - \Delta_x \eta^m \hat{u}^{(m)} - \sum_{j=1}^{n} \frac{\partial \eta^m}{\partial x_j} Z_{m}[(\beta_{m})^{-1} \sum_{i=1}^{n-1} (\Psi_{1z}^{(m)} \hat{u}_{z_{n}}^{(m)} - \Psi_{1z}^{(m)} u^{(m)}])
\]
\[
+ 2 \sum_{j=1}^{n} \frac{\partial \eta^m}{\partial x_j} Z_{m} \left[ \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i} \left( \frac{\partial \eta^m}{\partial z_i} \right) u^{(m)} \right]
\]
\[- 2 \sum_{j=1}^{n} \frac{\partial^2 \eta^m}{\partial x_j \partial x_j} Z_{m} \left[ \sum_{i=1}^{n-1} \frac{\partial d_{m}^{(m)}}{\partial x_{1}} \hat{u}_{z_{n}}^{(m)} \right],
\]
(178)
\[
G_2 = \sum_{m \in N} G_2^m, \quad G_2^m = \frac{\partial \eta^m}{\partial x_j} \hat{u}^{(m)} + \eta^m Z_{m}[(\beta_{m})^{-1} \sum_{i=1}^{n-1} (\Psi_{1z}^{(m)} \hat{u}_{z_{n}}^{(m)} - \Psi_{1z}^{(m)} u^{(m)}])
\]
\[- 2 \sum_{j=1}^{n} \frac{\partial \eta^m}{\partial x_j} Z_{m} \left[ \sum_{i=1}^{n-1} \frac{\partial d_{m}^{(m)}}{\partial x_{1}} \hat{u}_{z_{n}}^{(m)} \right].
\]
(179)
After that, we apply Theorems 8.33 and 8.34 [16] to the solution \( w \) of problem (163) and obtain
\[
\sup_{t,t'\in[0,T]} \|w(\cdot, t') - w(\cdot, t)\|_{C^{1+\alpha}(\Omega)} \leq \text{const.} \sup_{t,t'\in[0,T]} \|G_1(\cdot, t') - G_1(\cdot, t)\|_{C^{1+\alpha}(\Omega)} + \sup_{t,t'\in[0,T]} \|G_2(\cdot, t') - G_2(\cdot, t)\|_{C^{1+\alpha}(\Omega)} + \sup_{t,t'\in[0,T]} \|g(\cdot, t') - g(\cdot, t)\|_{C^{1+\alpha}(\Omega)}.
\]

(180)

Then we deduce from interpolation inequalities and estimate (131) that
\[
\sup_{t,t'\in[0,T]} \langle \tilde{u}^{(m)}(\cdot, t') - \tilde{u}^{(m)}(\cdot, t) \rangle_{x,\frac{1}{3}} \leq cT^{(1-\alpha)\nu} \|F_2^{(m)}\|_{C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\Gamma)}
\]

(181)

Inequalities (180), (181), (131) together with representation (178) and (179) provide the following estimate
\[
\langle w\rangle_{t,\frac{1}{3}\nu} + \sum_{i,j=1}^{n} \langle w_{ij}\rangle_{t,\frac{1}{3}\nu} \leq c_3 [\bar{x} + r^{1-\alpha} + T^{\frac{1-\alpha}{3}} + T^{\frac{\nu}{3}}] \|F_2\|_{C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\Gamma)}.
\]

(182)

Next, we combine estimates (167), (172), (174) and (182) and conclude
\[
\|T_3 F_2\|_{C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\Gamma)} \leq c_4 [\bar{x} + r^{1-\alpha} + T^{\frac{1-\alpha}{3}} + T^{\frac{\nu}{3}}] \|F_2\|_{C^{1+\alpha,\frac{1+\alpha}{3}\nu}(\Gamma)}.
\]

(183)

At last, we choose \( r \) and \( T \) such that
\[
\bar{x} + r^{1-\alpha} + T^{\frac{1-\alpha}{3}} + T^{\frac{\nu}{3}} + c_1 [r^{\alpha} + T^{\nu/3} + \bar{x}] \leq \tilde{C} [\bar{x} + r^{1-\alpha} + T^{\frac{1-\alpha}{3}} + T^{\frac{\nu}{3}}] < 1/2,
\]

(184)

and get estimates (139) and (140) from (161) and (184). That completes the proof of Proposition 7.

Acknowledgments. This work was prepared during the visit to the Institute of Mathematics, Lübeck University. The authors are very grateful to Professor Jürgen Prestin for his hospitality and to Dr. Lyudmyla Vynnytska for useful discussion.

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Received July 2015; revised February 2016.

E-mail address: nataliy_v@yahoo.com

E-mail address: vitaliioverko@gmail.com