THE WEYL LAW OF TRANSMISSION EIGENVALUES AND THE COMPLETENESS OF GENERALIZED TRANSMISSION EIGENFUNCTIONS

HOAI-MINH NGUYEN AND QUOC-HUNG NGUYEN

Abstract. The transmission problem is a system of two second-order elliptic equations of two unknowns equipped with the Cauchy data on the boundary. After four decades of research motivated by scattering theory, the spectral properties of this problem are now known to depend on a type of contrast between coefficients near the boundary. Previously, we established the discreteness of eigenvalues for a large class of anisotropic coefficients which is related to the celebrated complementing conditions due to Agmon, Douglis, and Nirenberg. In this work, we establish the Weyl law for the eigenvalues and the completeness of the generalized eigenfunctions for this class of coefficients under an additional mild assumption on the continuity of the coefficients. The analysis is new and based on the $L^p$ regularity theory for the transmission problem established here. It also involves a subtle application of the spectral theory for the Hilbert Schmidt operators. Our work extends largely known results in the literature which are mainly devoted to the isotropic case with $C^\infty$-coefficients.

1. Introduction

The transmission eigenvalue problem plays a role in the inverse scattering theory for inhomogeneous media. This eigenvalue problem is connected to the injectivity of the relative scattering operator [12], [18]. Transmission eigenvalues are related to interrogating frequencies for which there is an incident field that is not scattered by the medium. In the acoustic setting, the transmission problem is a system of two second-order elliptic equations of two unknowns equipped with the Cauchy data on the boundary. After four decades of extensive study, the spectral properties are known to depend on a type of contrasts of the media near the boundary (i.e., a difference of some relation of the respective coefficients in each of the equations). Natural and interesting questions on the inverse scattering theory include: discreteness of the spectrum (see e.g. [7], [6], [39], [19], [42]) location of transmission eigenvalues (see [9], [22], [40], [41], and also [10] for the application in time domain), and the Weyl law of transmission eigenvalues and the completeness of the generalized eigenfunctions (see e.g. [19], [20], [5], [21], [38]). We refer the reader to [8] for a recent, and self-contained introduction to the transmission problem and its applications.

This paper concerns the Weyl law of eigenvalues and the completeness of the generalized eigenfunctions of the transmission problem in the time-harmonic acoustic setting. Let us introduce its mathematical formulation. Let $\Omega$ be a bounded, simply connected, open subset of $\mathbb{R}^d$ with $d \geq 2$. Let $A_1, A_2$ be two real, symmetric matrix-valued functions, and let $\Sigma_1, \Sigma_2$ be two bounded positive functions that are all defined in $\Omega$. Assume that $A_1$ and $A_2$ are uniformly elliptic, and $\Sigma_1$ and $\Sigma_2$ are bounded below by a positive constant in $\Omega$, i.e., for some constant $\Lambda \geq 1$, one has, for $j = 1, 2$,

$\Lambda^{-1} |\xi|^2 \leq \langle A_j(x) \xi, \xi \rangle \leq \Lambda |\xi|^2$ for all $\xi \in \mathbb{R}^d$, for a.e. $x \in \Omega$,

and

$\Lambda^{-1} \leq \Sigma_j(x) \leq \Lambda$ for a.e. $x \in \Omega$.

Here and in what follows, $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$ and $|\cdot|$ is the corresponding norm. A complex number $\lambda$ is called an eigenvalue of the transmission eigenvalue
problem associated with the pairs \((A_1, \Sigma_1)\) and \((A_2, \Sigma_2)\) in \(\Omega\) if there is a non-zero pair of functions \((u_1, u_2) \in [H^1(\Omega)]^2\) that satisfy the system

\[
\begin{align*}
\text{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 &= 0 & \text{in } \Omega, \\
\text{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 &= 0 & \text{in } \Omega, \\
u_1 &= u_2, & A_1 \nabla u_1 \cdot \nu &= A_2 \nabla u_2 \cdot \nu & \text{on } \Gamma.
\end{align*}
\] (1.3)

Here and in what follows, \(\Gamma\) denotes \(\partial \Omega\), and \(\nu\) denotes the outward, normal, unit vector on \(\Gamma\). Such a pair \((u_1, u_2)\) is then called an eigenfunction pair of (1.3).

The Weyl law of transmission eigenvalues has been investigated under various assumptions on \((A_1, \Sigma_1)\) and \((A_2, \Sigma_2)\). Robbiano [38] (see also [37]) gives the sharp order of the counting number when \(A_1 = A_2 = I\), and \(\Sigma_2 \neq \Sigma_1 = 1\) near the boundary and \(\Sigma_2\) is smooth. The analysis is based on both the microanalysis (see e.g. [15, 16]) and the regularity theory for the transmission problem. In [20], Lakshtanov and Vainberg obtained similar results when \(A_1 = I\), under certain assumptions on \(A_2\) and \(\Sigma_2\). In particular, they required that \(\Sigma_2^{-1}A_2 - I\) is positive definite or negative definite in the whole domain \(\Omega\). They also investigated the order of the counting functions for positive and negative eigenvalues under different assumptions on \(A_2\) and \(\Sigma_2\) (see also [36] [21]) via concepts on billiard trajectories. In the isotropic case, the Weyl law for the remainder was established by Petkov and Vodev [35] and Vodev [41, 42, 43] for \(C^\infty\) coefficients that satisfy the conditions (1.4) and (1.5). The case where \(A_1 = A_2\) and represent scalar functions was also investigated in their work. Their analysis is heavily based on microanalysis and required a strong smoothness condition. In addition, their work involved a delicate analysis on the Dirichlet to Neumann maps using non-standard parametrix construction initiated by Vodev [40]. It is not clear how one can improve the \(C^\infty\) condition and extend their results to the anisotropic setting using their analysis. Concerning the completeness of the generalized eigenfunctions, we want to mention the work of Robbiano [37] where \(A_1 = A_2 = I\) and \(\Sigma_2 \neq \Sigma_1 = 1\), and the work of Blästen and Päivärinta [5] where \(A_1 = A_2 = I\), and \(\Sigma_2 - \Sigma_1 = \Sigma_2 - 1 > 0\) and smooth in \(\Omega\).

In this paper, we investigate the Weyl law of eigenvalues and the completeness of the generalized eigenfunctions for transmission problem under quite general assumptions on \(A_1, A_2, \Sigma_1, \Sigma_2\). These are only imposed on the boundary of \(\partial \Omega\) except for the continuity requirement. The starting point and one of the main motivations of our work are our discreteness result established in [32]. We demonstrated the discreteness holds if \(A_1, A_2, \Sigma_1, \Sigma_2\) are continuous in a neighborhood of the boundary \(\Gamma\), and satisfy the following two conditions, with \(\nu = \nu(x)\):

\[
\begin{align*}
\langle A_2(x)\nu, \nu \rangle \langle A_2(x)\xi, \xi \rangle - \langle A_2(x)\nu, \nu \rangle^2 &\neq \langle A_1(x)\nu, \nu \rangle \langle A_1(x)\xi, \xi \rangle - \langle A_1(x)\nu, \nu \rangle^2, \\
\text{for all } x \in \Gamma \text{ and for all } \xi \in \mathbb{R}^d \setminus \{0\} \text{ with } \langle \xi, \nu \rangle = 0, \text{ and}
\end{align*}
\] (1.4)

\[
\langle A_2(x)\nu, \nu \rangle \Sigma_2(x) \neq \langle A_1(x)\nu, \nu \rangle \Sigma_1(x), \forall x \in \Gamma.
\] (1.5)

Condition (1.4) is equivalent to the celebrated complementing condition due to Agmon, Douglis, and Nirenberg [4] (see also [3]). The explicit formula given here was derived in [27].

In this paper, we establish that if conditions (1.4) and (1.5) hold then the Weyl law for transmission eigenvalues and the completeness of the generalized eigenfunctions hold as well, under the mild assumption that the coefficients are continuous in \(\Omega\). More precisely, we have

**Theorem 1.** Assume that \(A_1, A_2, \Sigma_1, \Sigma_2 \in C^0(\Omega)\), and (1.4) and (1.5) hold. Then

\[
N(t) := \#\left\{ k \in \mathbb{N} : |\lambda_k| \leq t \right\} = ct^\frac{d}{4} + o(t^\frac{d}{4}) \text{ as } t \to +\infty,
\] (1.6)

where

\[
c = \frac{1}{(2\pi)^d} \sum_{j=1}^{2} \int_{\Omega} \left| \left\{ \xi \in \mathbb{R}^d : \langle A_j(x)\xi, \xi \rangle < \Sigma_j(x) \right\} \right| dx.
\] (1.7)
Remark 2. It is worth noting that the convention of eigenvalues of the transmission problem in the work of Petkov and Vodev, and Vodev mentioned above) where \( \lambda^2 \) is used in (1.3) instead of \( \lambda \) (where \( \lambda \) is used but \( t^2 \) is considered instead of \( t \) in the formula of the counting function).
Remark 3. In [35], Petkov and Vodev considered the isotropic setting and obtained a sharper estimate for the remainder of (1.6) as in the spirit of Hörmander [17]. Other refined estimates were given in [41, 42, 43] and are obtained under the $C^\infty$ smoothness assumption. Under the continuity assumption on the smoothness of coefficients, a better estimate for the remainder of (1.6) as in [35] is implausible. Nevertheless, it is interesting to obtain better estimates for the remainder as in [35, 41, 42, 43] for sufficiently regular coefficients and/or for the anisotropic setting.

Our strategy of the analysis is to develop the approach in [32] at the level where one can apply the general spectral theory for Hilbert-Schmidt operators in Hilbert space as given in Agmon [2] (see also [1]). Two important steps are follows. One is on sharp estimates for $\|T_\lambda\|_{L^p \to W^{1,p}}$ for $p > 1$ and its consequences (see Theorem 5) for large $|\lambda|$ with an appropriate direction. This, in particular, shows that $T := T_{\hat{\mu}_1} \circ \cdots \circ T_{\hat{\mu}_{k+1}}$ with $k = [d/2]$ is a Hilbert-Schmidt operator (see Proposition 1) for an appropriate choice of $\hat{\mu}_j \in \mathbb{C}$. The analysis of this part is on the regularity theory of the transmission problems in $L^p$-scale. This is one of the cores of this paper and has its own interest. To this end, we first investigate the corresponding problems in the whole space and in a half space with constant coefficients, and then use the freezing-coefficient technique. The analysis also involves the Mikhlin-Hörmander multiplier theorem (in particular the theory of singular integrals) and Gagliardo-Nirenberg interpolation inequalities. The second step is to apply the spectral theory for Hilbert-Schmidt operators. To this end, we use the estimates for $T_\lambda$ to obtain an approximation of the trace of the kernel of the product of $T$ and its appropriate modified operator (see Proposition 3). The approximation of the trace of the kernel is then used to derive information for the Weyl law via a formula for eigenvalues established in Proposition 2. This formula is derived from the spectral theory of Hilbert-Schmidt operator and is interesting itself. The completeness of the generalized eigenfunctions follows directly from the estimates for $T_\lambda$ in Theorem 6 where we pay special attention to the possible directions of $\lambda$ where the information can be derived, after applying the spectral theory in [2].

Remark 4. We use the regularity theory and spectral theory for Hilbert-Schmidt operators to investigate the Weyl law, which was also presented by Robbiano [37]. Nevertheless, the way we derive the regularity theory in this paper is distinct from [37], which involved Carleman’s inequalities and the theory of microanalysis. The way we explore the information of Hilbert-Schmidt operators allows us to exactly obtain the first term of the Weyl Law in (1.6) instead of its magnitude order as in [37].

We propose a new approach to establish the Weyl law of eigenvalues and the completeness of the generalized eigenfunctions. This allows us to obtain new significant results and strongly weaken the smoothness assumption in various known results, that is out of reach previously. The transmission problem also appears naturally for electromagnetic waves. In this case, it is a system of two Maxwell systems equipped the Cauchy data on the boundary. The spectral theory of the transmission problem for electromagnetic waves is much less known. On this aspect, we point the reader to [11] on the discreteness, and to [16] on the completeness. More information can be found in the references therein. The analysis in this paper will be developed for the Maxwell setting in our forthcoming work.

The transmission problem has an interesting connection with the study of negative-index materials which are modeled by the Helmholtz or Maxwell equations with sign changing coefficients. In fact, our work has its roots in [27] where the stability of solutions of the Helmholtz equations with sign changing coefficients was studied. Concerning the Maxwell equations, the stability was studied in [33]. It is not coincident that the transmission problem and the Helmholtz equations with sign-changing coefficients share some common analysis. In fact, using reflections (a class of changes of variables), the Cauchy problems appear naturally in the context of the Helmholtz with sign-changing coefficients as first observed in [23] (see also [29] for the Maxwell setting).
Other properties of the Cauchy problems related to resonant (unstable) aspects and applications of negative-index materials such as cloaking and superlensing can be found in [24, 25, 26, 28, 30, 31] and the references therein.

The paper is organized as follows. In Section 2, we introduce several notations used throughout the paper. In Section 3, we establish Theorem 3, which describes the regularity theory for the transmission problem in $L^p$-scale. In Section 4, we recall some definitions, properties of Hilbert-Schmidt operators, and their finite double-norms. We then derive their applications in the context of the transmission problem. The main result of this section is Proposition 2, which is derived from Theorem 3. The Weyl law and the completeness are then established in Section 5 and in Section 6, respectively.

2. Notations

We denote, for $\tau > 0$, 
\[ \Omega_\tau = \{ x \in \Omega : \text{dist}(x, \Gamma) < \tau \}. \]

For $d \geq 2$, set 
\[ \mathbb{R}_+^d = \{ x \in \mathbb{R}^d; x_d > 0 \} \quad \text{and} \quad \mathbb{R}_0^d = \{ x \in \mathbb{R}^d; x_d = 0 \}. \]

We will identify $\mathbb{R}_0^d$ with $\mathbb{R}^{d-1}$ in several places.

For $\theta \in \mathbb{R}$ and $a > 0$, denote 
\[ \mathcal{L}(\theta, a) = \{ re^{i\theta} \in \mathbb{C} : r \geq a \}. \]

3. Regularity theory for transmission problems

In this section, we establish several estimates for $T_\lambda$ for appropriate values of $\lambda$. The main results are as follows.

**Theorem 3.** Let $\varepsilon_0 > 0$ and $\Lambda \geq 1$. Assume that (1.1) and (1.2) hold, and $A_1, A_2, \Sigma_1, \Sigma_2$ are continuous in $\overline{\Omega}$. Assume that (1.4) and (1.5) hold in the following sense, with $\nu = \nu(x)$,
\[ \left| \langle A_2(x)\nu, A_2(x)\xi, \xi \rangle - \langle A_2(x)\nu, \xi, \xi \rangle - \left( \langle A_1(x)\nu, \nu \rangle \langle A_1(x)\xi, \xi \rangle - \langle A_1(x)\nu, \xi \rangle^2 \right) \right| \geq \Lambda^{-1} |\xi|^2, \]

for all $x \in \Gamma$ and for all $\xi \in \mathbb{R}^d \setminus \{0\}$ with $|\xi, \nu| = 0$, and
\[ \left| \langle A_2(x)\nu, \nu \rangle \Sigma_2(x) - \langle A_1(x)\nu, \nu \rangle \Sigma_1(x) \right| \geq \Lambda^{-1}, \quad \forall x \in \Gamma. \]

There exist two positive constants $\Lambda_0$ and $C$ depending only on $\Lambda$, $\varepsilon_0$, $\Omega$, and the continuity modulus of $A_1, A_2, \Sigma_1$, and $\Sigma_2$ in $\Omega$ such that for $\theta \in \mathbb{R}$ with $\inf_{n \in \mathbb{Z}} |\theta - n\pi| \geq \varepsilon_0$, and for $\lambda \in \mathcal{L}(\theta, \Lambda_0)$, the following fact holds: for $g = (g_1, g_2) \in [L^2(\Omega)]^2$, there exists a unique solution $u = (u_1, u_2) \in [H^1(\Omega)]^2$ of the system
\[ \begin{cases} 
\text{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 = g_1 & \text{in } \Omega, \\
\text{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 = g_2 & \text{in } \Omega, \\
u_1 - u_2 = 0, \quad (A_1 \nabla u_1 - A_2 \nabla u_2) \cdot \nu = 0 & \text{on } \Gamma.
\end{cases} \]

Moreover, for $1 < p < \infty$,
\[ \| \nabla u \|_{L^p(\Omega)} + |\lambda|^{1/2} u \|_{L^p(\Omega)} \leq C|\lambda|^{-\frac{1}{2}} \| g \|_{L^p(\Omega)}. \]

As a consequence, we have
\[ ||u||_{L^p(\Omega)} \leq C|\lambda|^{-1 + \frac{d}{2}(\frac{1}{p} - \frac{1}{q})} ||g||_{L^p(\Omega)}, \]

- for \( p > d \),
\[ ||u||_{L^\infty(\Omega)} \leq C|\lambda|^{-1 + \frac{d}{2p}} ||g||_{L^p(\Omega)}, \]

- for \( p > d \) and \( q = \frac{p}{p-1} \),
\[ ||u||_{L^q(\Omega)} \leq C|\lambda|^{-1 + \frac{d}{2} + \frac{d}{2q}} ||g||_{L^p(\Omega)}. \]

The remainder of this section contains two subsections, which are organized as follows. In the first subsection, we establish several lemmas used in the proof of Theorem 3. The proof of Theorem 3 is given in the second subsection.

3.1. Preliminaries. In this section, we establish several results used in the proof of Theorem 3, which is based on freezing coefficient technique. We begin with the corresponding settings/variants with constant coefficients in \( \mathbb{R}^d \) and in \( \mathbb{R}_+^d \). The first one is

**Lemma 1.** Let \( d \geq 2 \), \( \Lambda \geq 1 \), \( \varepsilon_0 > 0 \), \( 1 < p < \infty \), and let \( A \) and \( \Sigma \) be a symmetric matrix and a non-zero real constant, respectively. Assume that \( \inf_{n \in \mathbb{Z}} |\theta - n\pi| \geq \varepsilon_0 \) and \( \lambda \in \mathcal{C}(\theta, 1) \).
\[ \Lambda^{-1} \leq A \leq \Lambda \quad \text{and} \quad \Lambda^{-1} \leq |\Sigma| \leq \Lambda. \]

For \( g \in L^p(\mathbb{R}^d) \) and \( G \in [L^p(\mathbb{R}^d)]^d \), let \( u \in W^{1,p}(\mathbb{R}^d) \) be the unique solution of
\[ \text{div}(A\nabla u) - \lambda\Sigma u = g + \text{div}(G) \text{ in } \mathbb{R}^d. \]
We have
\[ |\lambda|^{1/2} \|\nabla u\|_{L^p(\mathbb{R}^d)} + |\lambda| \|u\|_{L^p(\mathbb{R}^d)} \leq C \left( \|g\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2} \|G\|_{L^p(\mathbb{R}^d)} \right), \]
and, if \( G = 0 \),
\[ \|\nabla^2 u\|_{L^p(\mathbb{R}^d)} \leq C \|g\|_{L^p(\mathbb{R}^d)}. \]
Here \( C \) denotes a positive constant depending only on \( p \), \( d \), \( \Lambda \), and \( \varepsilon_0 \).

Here and in what follows, for two \( d \times d \) symmetric matrices \( M_1 \) and \( M_2 \), we denote \( M_1 \geq M_2 \) (resp. \( M_1 \leq M_2 \)) if \( \langle M_1 \xi, \xi \rangle \geq \langle M_2 \xi, \xi \rangle \) (resp. \( \langle M_1 \xi, \xi \rangle \leq \langle M_2 \xi, \xi \rangle \)) for all \( \xi \in \mathbb{R}^d \).

**Proof.** For an appropriate function/vector field \( f \) defined in \( \mathbb{R}^d \), let \( \mathcal{F} f \) denote its Fourier transform. We have
\[ \mathcal{F} u(\xi) = -\mathcal{F} g(\xi) + i\xi \cdot \mathcal{F} G(\xi). \]
Set
\[ m(\xi) = \frac{1}{\langle A\xi, \xi \rangle + \lambda\Sigma}. \]
One can check that
\[ |\xi|^\ell |\mathcal{F}^\ell m(\xi)| \leq C(|\lambda|^{-1} \lambda) \text{ for } \ell \in \mathbb{N}. \]
It follows from Mikhlin-Hörmander’s multiplier theorem, see e.g. [14, Theorem 5.2.7], that
\[ ||u||_{L^p(\mathbb{R}^d)} \leq C|\lambda|^{-1} ||g||_{L^p(\mathbb{R}^d)}. \]
The other estimates in Assertion (3.5) and (3.10) can be derived in the same manner. The proof is complete.

Here is a result on a half space.
Lemma 2. Let $A_1, A_2$ be two constant, symmetric matrices, and let $\Sigma_1, \Sigma_2$ be two non-zero, real constants. Assume that, for some $\Lambda \geq 1$,

\begin{equation}
\Lambda^{-1} \leq A_1, A_2 \leq \Lambda, \quad \Lambda^{-1} \leq |\Sigma_1|, |\Sigma_2| \leq \Lambda,
\end{equation}

\begin{equation}
\begin{aligned}
&\left| \langle A_1 e_d, e_d \rangle - \langle A_1 e_d, e_d \rangle \right|^2 \geq \Lambda^{-1} |\xi|^2 \quad \forall \xi \in \mathcal{P}, \\
&\left| \langle A_2 e_d, e_d \rangle \Sigma_2 - \langle A_1 e_d, e_d \rangle \Sigma_1 \right| \geq \Lambda^{-1}.
\end{aligned}
\end{equation}

Let $\varphi \in W^{1-1/p, p}(\mathbb{R}_d^d)$, and $\varphi \in W^{1-1/p, p}(\mathbb{R}_d^d)$. There exist two positive constants $C$ and $\Lambda_0$ depending only on $\Lambda$ and $\lambda_0$ such that for $\theta \in \mathbb{R}$ with $\min_{n \in \mathbb{Z}} |\theta - n\pi| > \varepsilon_0$ and for $\lambda \in \mathcal{L}(\theta, \Lambda_0)$, there exists a unique solution $u = (u_1, u_2) \in [W^{1, p}(\mathbb{R}_d^d)]^2$ of the system

\begin{align}
&\begin{cases}
\text{div}(A_1 \nabla v_1) - \lambda \Sigma_1 v_1 = g_1 + \text{div}(G_1) \\
\text{div}(A_2 \nabla v_2) - \lambda \Sigma_2 v_2 = g_2 + \text{div}(G_2)
\end{cases} \quad \text{in } \mathbb{R}_d^d, \\
&v_1 - v_2 = \varphi, \quad (A_1 \nabla v_1 - G_1) \cdot e_d - (A_2 \nabla v_2 - G_2) \cdot e_d = 0 \quad \text{on } \mathbb{R}_0^d.
\end{align}

Moreover,

\begin{equation}
\| \nabla v \|_{L^p(\mathbb{R}_d^d)} + |\lambda|^{1/2} \| v \|_{L^p(\mathbb{R}_d^d)} \leq C \left( |\lambda|^{-1/2} \| g \|_{L^p(\mathbb{R}_d^d)} + \| G \|_{L^p(\mathbb{R}_d^d)} + \lambda^{1/2 - 1/(2p)} \| \varphi \|_{L^p(\mathbb{R}_0^d)} \right). \tag{3.15}
\end{equation}

**Proof.** We only establish (3.15). The uniqueness for $v_1, v_2$ is a consequence of (3.14). The existence of $(v_1, v_2)$ follows from the proof of (3.15) and is omitted. Let $u_j \in W^{1, p}(\mathbb{R}_d^d)$ be the unique solution of the equation

\begin{equation}
\text{div}(A_j \nabla u_j) - \lambda \Sigma_j u_j = g_j \mathbb{I}_{\mathbb{R}_d^d} + \text{div}(G_j \mathbb{I}_{\mathbb{R}_d^d}) \quad \text{in } \mathbb{R}_d^d.
\end{equation}

It follows from Lemma 4 that

\begin{equation}
\| \nabla u_j \|_{L^p(\mathbb{R}_d^d)} + |\lambda|^{1/2} \| u_j \|_{L^p(\mathbb{R}_d^d)} \leq C \left( |\lambda|^{-1/2} \| g_j \|_{L^p(\mathbb{R}_d^d)} + \| G_j \|_{L^p(\mathbb{R}_d^d)} \right),
\end{equation}

\begin{equation}
|\lambda|^{1/2 - 1/(2p)} \| u_j \|_{L^p(\mathbb{R}_d^d)} \leq C \left( \| \nabla u_j \|_{L^p(\mathbb{R}_d^d)} + |\lambda|^{1/2} \| u_j \|_{L^p(\mathbb{R}_d^d)} \right),
\end{equation}

\begin{equation}
(A_j \nabla u_j - G_j) \cdot e_d = 0 \quad \text{on } \mathbb{R}_0^d,
\end{equation}

and, by the trace theory,

\begin{equation}
\| u_j \|_{W^{1-1/p, p}(\mathbb{R}_0^d)} \leq C \| u_j \|_{W^{1, p}(\mathbb{R}_0^d)}.
\end{equation}

Therefore, without loss of generality, one might assume that $g_1 = g_2 = 0$ and $G_1 = G_2 = 0$. This will be assumed from now on.

Let $\hat{v}_j(\xi', t)$ for $j = 1, 2$ and $\hat{\varphi}(\xi', t)$ be the Fourier transform of $v_j$ and $\varphi$ with respect to $x' \in \mathbb{R}_d^{d-1}$, i.e., for $(\xi', t) \in \mathbb{R}_d^{d-1} \times (0, +\infty)$,

\begin{equation}
\hat{v}_j(\xi', t) = \int_{\mathbb{R}_d^{d-1}} v_j(x', t) e^{-ix' \cdot \xi'} \, dx' \quad \text{for } j = 1, 2, \quad \text{and} \quad \hat{\varphi}(\xi', t) = \int_{\mathbb{R}_d^{d-1}} \varphi(x') e^{-ix' \cdot \xi'} \, dx'.
\end{equation}

Since

\begin{equation}
\text{div}(A_j \nabla v_j) - \lambda \Sigma_j v_j = 0 \quad \text{in } \mathbb{R}_d^d,
\end{equation}

it follows that

\begin{equation}
a_j v_j''(t) + 2ib_j v_j'(t) - (c_j + \lambda \Sigma_j) v_j(t) = 0 \quad \text{for } t > 0,
\end{equation}
where
\[ a_j = (A_j)_{d,d}, \quad b_j = \sum_{k=1}^{d-1} (A_j)_{d,k} \xi_k, \quad \text{and} \quad c_j = \sum_{k=1}^{d-1} \sum_{l=1}^{d-1} (A_j)_{k,l} \xi_k \xi_l. \]

Here \((A_j)_{k,l}\) denotes the \((k, l)\) component of \(A_j\) for \(j = 1, 2\) and the symmetry of \(A_j\) is used. Define, for \(j = 1, 2\),
\[ \Delta_j = -b_j^2 + a_j(c_j + \lambda \Sigma_j). \]

Denote \(\xi = (\xi', 0)\). Since \(A_j\) is symmetric and positive, it is clear that, for \(j = 1, 2\),
\[ a_j = \langle A_j e_d, e_d \rangle, \quad b_j = \langle A_j \xi, e_d \rangle, \quad c_j = \langle A_j \xi, \xi \rangle, \quad \text{and} \quad a_j c_j - b_j^2 > 0. \]

Since \(\hat{v}_j(\xi', t) \in L^2(\mathbb{R}_d^d)\), we have
\[ \hat{v}_j(\xi', t) = \alpha_j(\xi') e^{\eta_j(\xi') t}, \]
for some \(\alpha_j(\xi') \in \mathbb{C}\), where
\[ \eta_j = (-ib_j - \sqrt{\Delta_j})/a_j. \]

Here \(\sqrt{\Delta_j}\) denotes the square root of \(\Delta_j\) with positive real part. Using the fact that \(v_1 - v_2 = \varphi\) and \(A_1 \nabla v_1 \cdot e_d - A_2 \nabla v_2 \cdot e_d = 0\) on \(\mathbb{R}_0^d\), we derive that
\[ \alpha_1(\xi') - \alpha_2(\xi') = \hat{\varphi}(\xi') \quad \text{and} \quad \alpha_1(\xi') \langle i A_1 \xi + \eta_1 A_1 e_d, e_d \rangle - \alpha_2(\xi') \langle i A_2 \xi + \eta_2 A_2 e_d, e_d \rangle = 0. \]

Note that, by (3.17),
\[ \langle A_j \xi, e_d \rangle - \langle A_j e_d, e_d \rangle b_j/a_j = 0. \]

The last identity of (3.18) is equivalent to
\[ \alpha_1(\xi') \sqrt{\Delta_1} = \alpha_2(\xi') \sqrt{\Delta_2}. \]
Combining this identity and the first one of (3.18) yields
\[ \alpha_1(\xi') = \frac{\hat{\varphi}(\xi') \sqrt{\Delta_2}}{\sqrt{\Delta_2} - \sqrt{\Delta_1}} \frac{1}{\eta_j - i \xi_d}. \]

Extend \(v_1(x', t)\) by 0 for \(t < 0\). We then obtain
\[ \mathcal{F} v_1(\xi) = -\hat{\varphi}(\xi') \frac{\sqrt{\Delta_2}}{\sqrt{\Delta_2} - \sqrt{\Delta_1}} \frac{1}{\eta_j - i \xi_d}. \]

Here, \(\mathcal{F}\) is the Fourier transform in \(\mathbb{R}^d\). Set
\[ g(t) = e^{-|\lambda|^{1/2} t} 1_{t \geq 0} \text{ for } t \in \mathbb{R} \quad \text{and} \quad \Phi(x) = \varphi(x') g(x_d) \text{ for } x \in \mathbb{R}^d. \]

It follows that
\[ \mathcal{F} v_1(\xi) = \mathcal{F} \Phi(\xi) \frac{\sqrt{\Delta_2}}{\sqrt{\Delta_2} - \sqrt{\Delta_1}} \frac{|\lambda|^{1/2} + i \xi_d}{-\eta_j + i \xi_d}. \]

We have
\[ |\Delta_2 - \Delta_1|^2 \geq C(|\xi'|^4 + |\lambda|^2), \quad |\Delta_j| \leq C(|\xi'|^2 + |\lambda|), \]
and
\[ |\Re(\eta_j)| \geq C(|\xi'| + |\lambda|^{1/2}). \]

As in the proof of Lemma 1, by Mikhlin-Hörmander’s multiplier theorem, see e.g. [14] Theorem 5.2.7, one has
\[ \| v_1 \|_{L^p(\mathbb{R}^d)} \leq C \| \Phi \|_{L^p(\mathbb{R}^d)} \leq C |\lambda|^{-1/(2p)} \| \varphi \|_{L^p(\mathbb{R}^{d-1})}. \]

We next deal with \(\nabla v_1\). We have
\[ \partial_t \hat{v}_1(\xi', t) = \eta_1 \hat{v}_1(\xi', t) \text{ in } \mathbb{R}_+^d. \]
It is clear that
\[ \eta_1 = \eta_{1,1} + \eta_{1,2}, \]
where
\[ \eta_{1,1} = -\frac{\sqrt{a_1} \lambda \Sigma_1}{a_1}, \quad \text{and} \quad \eta_{1,2} = -\frac{i b_1}{a_1} - \frac{\sqrt{\Delta_1}}{a_1}. \]
As above, one can prove that
\[ \| \hat{F}^{-1}(\eta_{1,1} v_1) \|_{L^p(\mathbb{R}^d_+)} \leq C|\lambda|^{1/2}\| \Phi \|_{L^p(\mathbb{R}^d)} \leq C|\lambda|^{1/2-1/(2p)}\| \varphi \|_{L^p(\mathbb{R}^{d-1})}, \]
and, for some \( \gamma > 0 \),
\[ \| \hat{F}^{-1}(\eta_{1,2} v_1) \|_{L^p(\mathbb{R}^d_+)} \leq C\| g \|_{L^p(\mathbb{R}^d_+)}, \]
where \( \hat{F}^{-1} \) denotes the Fourier inverse with respect to \( \xi' \) in \( \mathbb{R}^{d-1} \) and
\[ \hat{g}(\xi', t) = i \xi' \cdot \hat{\varphi}(\xi') e^{-\gamma|\xi'|^2}. \]
It is clear that \( g(x) = \nabla_x v(x) \), where \( v \) is the unique solution of the system
\[ \Delta_x v + \gamma \partial_{x,a}^2 v = 0 \quad \text{in} \quad \mathbb{R}^d_+ \quad \text{and} \quad v = \varphi \quad \text{on} \quad \mathbb{R}^d_0 \]
for \( \gamma > 0 \). It follows that, see e.g. [3, Theorem 3.3], we have
\[ \| g \|_{L^p(\mathbb{R}^d_+)} \leq C\| \varphi \|_{\dot{W}^{1,p-1,p}(\mathbb{R}^{d-1})}. \]
Combining (3.22) and (3.23) yields
\[ \| \partial v_{1} \|_{L^p(\mathbb{R}^d_+)} \leq C\| \varphi \|_{\dot{W}^{1,p-1,p}(\mathbb{R}^{d-1})} + C|\lambda|^{1/2-1/(2p)}\| \varphi \|_{L^p(\mathbb{R}^{d-1})}. \]
By the same manner, we also obtain
\[ \| \nabla_x v_1 \|_{L^p(\mathbb{R}^d_+)} \leq C\| \varphi \|_{\dot{W}^{1,p-1,p}(\mathbb{R}^{d-1})}. \]
From (3.22), (3.24), and (3.25), we obtain
\[ |\lambda|^{1/2}\| v_1 \|_{L^p(\mathbb{R}^d_+)} + \| \nabla v_1 \|_{L^p(\mathbb{R}^d_+)} \leq C\| \varphi \|_{\dot{W}^{1,p-1,p}(\mathbb{R}^{d-1})} + C|\lambda|^{1/2-1/(2p)}\| \varphi \|_{L^p(\mathbb{R}^{d-1})}. \]
Similar to (3.26), we also get
\[ |\lambda|^{1/2}\| v_2 \|_{L^p(\mathbb{R}^d_+)} + \| \nabla v_2 \|_{L^p(\mathbb{R}^d_+)} \leq C\| \varphi \|_{\dot{W}^{1,p-1,p}(\mathbb{R}^{d-1})} + C|\lambda|^{1/2-1/(2p)}\| \varphi \|_{L^p(\mathbb{R}^{d-1})}. \]
The conclusion thus follows from (3.26) and (3.27). The proof is complete.

**Remark 5.** Assertion (3.14) was previously established in [32] for \( p = 2 \) (see [32, the proof of Theorem 4]). The analysis given here has its root in [27, 32]. Nevertheless, instead of using Parseval’s theorem to derive \( L^2 \)-estimates, the new ingredient involves Mikhlin-Hörmander’s multiplier theory.

We now derive consequences of Lemmas 1 and 2 via the freezing-coefficient technique. As a consequence of Lemma 1, we have

**Corollary 1.** Let \( d \geq 2, p > 1, \Lambda \geq 1, \varepsilon_0 > 0, \) and let \( A \) be a symmetric, matrix-valued function, and let \( \Sigma \) be a real function defined in \( \Omega \). Assume that \( A \) and \( \Sigma \) are continuous in \( \bar{\Omega} \),
\[ \Lambda^{-1} \leq A \leq \Lambda \quad \text{and} \quad \Lambda^{-1} \leq |\Sigma| \leq \Lambda \quad \text{in} \quad \Omega, \]
for some \( \Lambda \geq 1 \), \( \inf_{n \in \mathbb{Z}} |\theta - n \pi| \geq \varepsilon_0 \), and \( \lambda \in L(\theta, 1) \). For \( g \in L^p(\Omega) \) and \( G \in [L^p(\Omega)]^d \), let \( u \in W^{1,p}(\Omega) \) be a solution of
\[ \text{div}(A \nabla u) - \lambda \Sigma u = g + \text{div}(G) \quad \text{in} \quad \Omega. \]
We have, for $\tau > 0$,

\[
(3.29) \quad \| \nabla u \|_{L^p(\Omega, \Omega_\tau)} + |\lambda|^{1/2} \| u \|_{L^p(\Omega, \Omega_\tau)}
\leq C \left( |\lambda|^{-1/2} \| g \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} \right) + C |\lambda|^{-\frac{1}{2}} \left( \| \nabla u \|_{L^p(\Omega)} + |\lambda|^{1/2} \| u \|_{L^p(\Omega)} \right).
\]

Here $C$ denotes a positive constant depending only on $\Lambda$, $p$, $\varepsilon_0$, $\tau$, $\Omega$, and the continuity modulus of $A_1$, $A_2$, $\Sigma_1$, and $\Sigma_2$ in $\overline{\Omega}$.

**Proof.** Let $\chi$ be an arbitrary smooth function with support in $\Omega$. Set $v = \chi u$ in $\Omega$. We have

\[ \text{div}(A \nabla v) - \lambda \Sigma v = f + \text{div} F \text{ in } \Omega, \]

where

\[ f = \chi g + A \nabla u \nabla \chi - F \cdot \nabla \chi \quad \text{and} \quad F = \chi F + u A \nabla \varphi. \]

The conclusion follows from Lemma 1 by the freezing-coefficient technique and the computations above. \qed

Similarly, as a consequence of Lemma 2, we obtain

**Corollary 2.** Let $d \geq 2$, $p > 1$, $\varepsilon_0 > 0$, $\tau > 0$, and $\Lambda \geq 1$, and let $A_1$, $A_2$ be two symmetric, matrix-valued functions, and let $\Sigma_1$, $\Sigma_2$ be two real functions defined in $\Omega$. Assume that $A_1$, $A_2$, $\Sigma_1$, $\Sigma_2$ are continuous in $\overline{\Omega}_{2\tau}$, (3.28) holds, and (1.4) and (1.5) are satisfied. There exist two positive constants $\Lambda_0$ and $C$, depending only on $\Lambda$, $\varepsilon_0$, and the continuity of $A_1$, $A_2$, $\Sigma_1$, and $\Sigma_2$ in $\overline{\Omega}_{2\tau}$ such that for $\theta \in \mathbb{R}$, for $\lambda \in \mathcal{L}(\theta, \Lambda_0)$ with inf $n \in \mathbb{Z} |\theta - n\pi| \geq \varepsilon_0$, and for $g = (g_1, g_2) \in [L^p(\Omega)]^2$, and $G = (G_1, G_2) \in [L^p(\Omega)]^2 \times [L^p(\Omega)]^2$, let $u = (u_1, u_2) \in [W^{1,p}(\Omega)]^2$ be a solution of the system

\[
(3.30) \quad \left\{ \begin{array}{ll}
\text{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 = g_1 + \text{div}(G_1) & \text{in } \Omega, \\
\text{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 = g_2 + \text{div}(G_2) & \text{in } \Omega, \\
u_2 - u_1 = 0, & \text{in } \Omega, \\
u_2 - u_1 = 0, & \text{on } \Gamma.
\end{array} \right.
\]

Moreover, we have

\[
(3.31) \quad \| \nabla v \|_{L^p(\Omega, \Omega_\tau)} + |\lambda|^{1/2} \| v \|_{L^p(\Omega, \Omega_\tau)}
\leq C \left( |\lambda|^{-1/2} \| g \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} \right) + C |\lambda|^{-\frac{1}{2}} \left( \| \nabla u \|_{L^p(\Omega)} + |\lambda|^{1/2} \| u \|_{L^p(\Omega)} \right).
\]

Here $C$ denotes a positive constant depending only on $\Lambda$, $p$, $\varepsilon_0$, $\tau$, $\Omega$, and the continuity modulus of $A_1$, $A_2$, $\Sigma_1$, and $\Sigma_2$ in $\overline{\Omega}_{2\tau}$.

3.2. **Proof of Theorem 3.** We first assume the well-posedness of (3.3) and establish (3.4) - (3.7).

It is clear that (3.4) is a consequence of Corollary 1 and Corollary 2.

We next deal with (3.5) and (3.6). By Gagliardo-Nirenberg’s interpolation inequalities (13) (14), if $p > d$ and $u \in W^{1,p}(\Omega)$, then $u \in C(\overline{\Omega})$ and

\[
(3.32) \quad \| u \|_{L^p(\Omega)} \leq C \| u \|_{W^{1,p}(\Omega)}^{\frac{d}{p}} \| u \|_{L^p(\Omega)}^{\frac{1-d}{p}},
\]

and if $1 < p < d$, and $u \in W^{1,p}(\Omega)$, then, for $p < q < \frac{dp}{d-p}$,

\[
(3.33) \quad \| u \|_{L^p(\Omega)} \leq C \| u \|_{W^{1,p}(\Omega)}^{d(p-1)/q} \| u \|_{L^p(\Omega)}^{(1-d)(p-1)/q}.
\]

Assertions (3.5) and (3.6) now follow from (3.4), (3.32), (3.33), and Hölder’s inequality.
We finally establish (3.7). Let
\[ \mathcal{T}_\delta : [L^2(\Omega)]^2 \to [L^2(\Omega)]^2 \]
\[ g \mapsto u, \]
where \( u = (u_1, u_2) \in [H^1(\Omega)]^2 \) is the unique solution of (3.3) with \((g_1, g_2) = g\). We have, for \( q = \frac{4p}{p-1} \) and \( p > d \),
\[ \|u\|_{L^q(\Omega)} = \sup_{f \in [L^q(\Omega)]^2} \frac{|\langle u, f \rangle|}{\|f\|_{L^q(\Omega)}}, \]
and
\[ \langle u, f \rangle = \langle \mathcal{T}_\delta(g), f \rangle = \langle g, \mathcal{T}_\delta^*(f) \rangle. \]
One can check that
\[ \mathcal{T}_\delta^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
It follows that
\[ |\langle u, f \rangle| \leq \|g\|_{L^1(\Omega)} \|\mathcal{T}_\delta(f)\|_{L^\infty(\Omega)} \leq C|\lambda|^{-1+\frac{d}{2p}} \|g\|_{L^1(\Omega)} \|f\|_{L^p(\Omega)}. \]
Assertion (3.7) follows.

It remains to prove the well-posedness of (3.3). It is clear that the uniqueness of (3.3) follows from (3.4). To establish the existence for (3.3), we use the principle of limiting absorption and the Fredholm theory as in [11] the proof of Proposition 4.2 (see also [32]). We only consider the case where \( \Im(\lambda) < 0 \); the other case can be proved similarly. For \( \delta > 0 \), by the Lax-Milgram theory, there exists a unique solution \( v_\delta = (v_{1,\delta}, v_{2,\delta}) \in [H^1(\Omega)]^2 \) of the system
\[
\begin{cases}
\text{div} \left( (1 - i\delta)A_1 \nabla v_{1,\delta} \right) - \lambda \Sigma_1 v_{1,\delta} = g_1 & \text{in } \Omega, \\
\text{div} \left( (1 + i\delta)A_2 \nabla v_{2,\delta} \right) + \lambda \Sigma_2 v_{2,\delta} = g_2 & \text{in } \Omega, \\
v_{2,\delta} - v_{1,\delta} = 0, \quad (1 + i\delta)A_2 \nabla v_{2,\delta} - (1 - i\delta)A_1 \nabla v_{1,\delta} \cdot \nu = 0 & \text{on } \Gamma.
\end{cases}
\]
Moreover, by Corollaries 1 and 2 applied with \( G_1 = i\delta A_1 \nabla v_{1,\delta} \) and \( G_2 = -i\delta A_2 \nabla v_{2,\delta} \), we have, for sufficiently small \( \delta \),
\[ \|\nabla u_\delta\|_{L^2(\Omega)} + |\lambda|^{1/2} \|u_\delta\|_{L^2(\Omega)} \leq C|\lambda|^{-1/2} \|g\|_{L^2(\Omega)}. \]
By taking \( \delta \to 0_+ \), one derives the existence of a solution \( v = (v_1, v_2) \in [H^1(\Omega)]^2 \) of the system, with \( \hat{\Sigma}_2 = -\Sigma_2 
\]
\[
\begin{cases}
\text{div} \left( A_1 \nabla v_1 \right) - \lambda \Sigma_1 v_1 = g_1 & \text{in } \Omega, \\
\text{div} \left( A_2 \nabla v_2 \right) - \lambda \Sigma_2 v_2 = g_2 & \text{in } \Omega, \\
v_2 - v_1 = 0, \quad (A_2 \nabla v_2 - A_1 \nabla v_1) \cdot \nu = 0 & \text{on } \Gamma,
\end{cases}
\]
which satisfies
\[ \|\nabla v\|_{L^2(\Omega)} + |\lambda|^{1/2} \|v\|_{L^2(\Omega)} \leq C|\lambda|^{-1/2} \|g\|_{L^2(\Omega)}. \]
The uniqueness of (3.35) is again a consequence of Corollary 1 and Corollary 2.
Define
\[ \hat{\mathcal{T}} : [L^2(\Omega)]^2 \to [L^2(\Omega)]^2 \]
\[ g \mapsto v, \]
where \( v = (v_1, v_2) \in [H^1(\Omega)]^2 \) is the unique solution of (3.35). It follows from (3.36) that \( \hat{\mathcal{T}} \) is compact.
It is clear that $u = (u_1, u_2) \in [H^1(\Omega)]^2$ is a solution of (3.3) if and only if
\[
\begin{align*}
\text{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 &= g_1 \quad \text{in } \Omega, \\
\text{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 &= g_2 + 2\lambda \Sigma_2 u_2 \quad \text{in } \Omega, \\
u_1 - u_2 &= 0, \quad (A_1 \nabla u_1 - A_2 \nabla u_2) \cdot \nu = 0 \quad \text{on } \Gamma.
\end{align*}
\]
In other words,
\[
(u_1, u_2) = \hat{T}(g_1, g_2) + \hat{T}(0, 2\lambda \Sigma_2 u_2).
\]
Since this equation has at most one solution and $\hat{T}$ is compact, this equation has a unique solution by the Fredholm theory. The proof is complete. \( \square \)

**Remark 6.** Note from (3.34) that $\hat{T}_\lambda$ is not self-adjoint.

### 4. Hilbert-Schmidt operators

We now devote two subsections to the applications of Hilbert-Schmidt operators for the transmission problem. In the first subsection, we recall some basic facts on Hilbert-Schmidt operators and the finite double norms. In the second subsection, we derive their applications for the transmission problem. The main result here is Proposition 2.

#### 4.1. Some basic facts on Hilbert-Schmidt operators

In this section, we recall the definition and several properties of Hilbert-Schmidt operators. We begin with

**Definition 2.** Let $H$ be a separable Hilbert space and let $(\phi_k)_{k=1}^\infty$ be an orthogonal basis. A bounded linear operator $T : H \to H$ is Hilbert Schmidt if its finite double norm

\[
\|T\| := \left( \sum_{k=1}^\infty \|T(\phi_k)\|_H^2 \right)^{1/2} < +\infty.
\]

The trace of $T$ is then defined by

\[
\text{trace}(T) = \sum_{k=1}^\infty \langle T(\phi_k), \phi_k \rangle.
\]

**Remark 7.** The definition of $\|T\|$ and of $\text{trace}(T)$ do not depend on the choice of $(\phi_k)$, see e.g. [2, Chapter 12].

One can check, see [2, Theorem 12.12], that if $T_1$ and $T_2$ are Hilbert Schmidt then $T_1T_2$ is also Hilbert Schmidt, and

\[
|\text{trace}(T_1T_2)| \leq \|T_1\| \|T_2\|.
\]

Let $m \in \mathbb{N}$ and $T : [L^2(\Omega)]^m \to [L^2(\Omega)]^m$ be a Hilbert Schmidt operator. There exists a unique kernel $K \in [L^2(\Omega \times \Omega)]^{m \times m}$, see e.g. [2, Theorems 12.18 and 12.19], such that

\[
(Tu)(x) = \langle K(x, .), u \rangle \quad \text{for a.e. } x \in \Omega, \text{ for all } u \in [L^2(\Omega)]^m.
\]

Moreover,

\[
\|T\|^2 = \iint_{\Omega \times \Omega} |K(x, y)|^2 \, dx \, dy.
\]

Note that [2, Theorems 2.18 and 12.19] state for $m = 1$, nevertheless, the same arguments hold for $m \in \mathbb{N}$.

We have
Lemma 3. Let \( d \geq 2, m \in \mathbb{N}, \) and \( \mathbf{T} : [L^2(\Omega)]^m \to [L^2(\Omega)]^m \) be such that \( \mathbf{T}(\phi) \in C(\Omega) \) for \( \phi \in [L^2(\Omega)]^m, \) and

\[
\|\mathbf{T}(\phi)\|_{L^\infty(\Omega)} \leq M\|\phi\|_{L^2(\Omega)},
\]

for some \( M \geq 0. \) Then \( \mathbf{T} \) is a Hilbert-Schmidt operator,

\[
\|\mathbf{T}\| \leq |\Omega|^{1/2}M,
\]

and the kernel \( \mathbf{K} \) of \( \mathbf{T} \) satisfies

\[
\sup_{x \in \Omega} \left( \int \|\mathbf{K}(x, y)\|^2 dy \right)^{1/2} \leq |\Omega|^{1/2}M.
\]

Assume in addition that

\[
\|\mathbf{T}(\phi)\|_{L^\infty(\Omega)} \leq \tilde{M}\|\phi\|_{L^1(\Omega)} \quad \text{for } \phi \in [L^2(\Omega)]^m,
\]

for some \( \tilde{M} \geq 0, \) then the kernel \( \mathbf{K} \) of \( \mathbf{T} \) satisfies

\[
|\mathbf{K}(x, y)| \leq \tilde{M} \quad \forall x, y \in \Omega.
\]

Proof. The proof is quite standard as in [2]. We present the details of this proof for the convenience of the reader. Let \( (\phi_k)_{k=1}^\infty \) be an orthonormal basis of \( [L^2(\Omega)]^m \) and set \( \phi_j = \mathbf{T}(\phi_j). \) Let \( a_1, ..., a_N \in \mathbb{C} \) be arbitrary. By (4.6), we have

\[
\left| \sum_{j=1}^N a_j \phi_j(x) \right| \leq M \left( \sum_{j=1}^N |a_j|^2 \right)^{1/2} \quad \forall x \in \Omega.
\]

Choosing \( a_j = \phi_j(x) \) yields

\[
\sum_{j=1}^N |\phi_j(x)|^2 \leq M^2 \quad \forall x \in \Omega.
\]

Integrating over \( \Omega, \) we obtain

\[
\sum_{j=1}^N \|\phi_j\|^2_{L^2(\Omega)} \leq |\Omega|M^2,
\]

which implies (4.7).

Assertion (4.8) follows from (4.6) by (4.4).

It is clear that (4.10) is a consequence of (4.9) by the definition of the kernel. \( \square \)

We next recall a basic, useful property of a Hilbert-Schmidt operator, see e.g., [2, Theorem 12.21].

Lemma 4. Let \( m \in \mathbb{N} \) and let \( \mathbf{T}_1, \mathbf{T}_2 \) be two Hilbert-Schmidt operators in \( [L^2(\Omega)]^m \) with the corresponding kernels \( \mathbf{K}_1 \) and \( \mathbf{K}_2. \) Then \( \mathbf{T} := \mathbf{T}_1 \mathbf{T}_2 \) is a Hilbert-Schmidt operator with the kernel \( \mathbf{K} \) given by

\[
\mathbf{K}(x, y) = \int_{\Omega} \mathbf{K}_1(x, z)\mathbf{K}_2(z, y) dz.
\]

Moreover,

\[
\text{trace}(\mathbf{T}_1 \mathbf{T}_2) = \int_{\Omega} \text{trace} \mathbf{K}(x, x) dx.
\]

\footnote{Note that [2, Theorem 2.21] states for \( m = 1, \) nevertheless, the same arguments hold for \( m \in \mathbb{N}. \)}
Remark 8. Using (L.11), one can check that
\[
\int_{\Omega} |K(x,x)| \, dx \leq \int_{\Omega} |K_1(x,z)||K_2(z,x)| \, dz \, dx \leq \|K_1\|_{L^2(\Omega \times \Omega)} \|K_2\|_{L^2(\Omega \times \Omega)}.
\]
Hence \( K(x,x) \in [L^1(\Omega)]^{m \times m} \).

4.2. Applications of the theory of Hilbert-Schmidt operators. In this section, we apply the theory of Hilbert-Schmidt operators to the operator \( T_\lambda \) mentioned in the introduction. The main ingredient of the analysis is Theorem 3. We begin with

Definition 3. Let \( \varepsilon_0 > 0 \) and \( \Lambda \geq 1 \). Assume the assumptions of Theorem 3 hold. Let \( \Lambda_0 \) and \( C \) be the constants in Theorem 3. For \( \lambda \in \mathcal{L}(\theta, \Lambda_0) \) with \( \inf_{n \in \mathbb{Z}} |\theta - n\pi| > \varepsilon_0 \), define
\[
T_\lambda : [L^2(\Omega)]^2 \to [L^2(\Omega)]^2,
\]
where \( u = (u_1, u_2) \in [H^1(\Omega)]^2 \) is the unique solution of, with \( (f_1, f_2) = f \),
\[
\begin{aligned}
\text{div}(A_1 \nabla u_1) - \lambda \Sigma_1 u_1 &= \Sigma_1 f_1 \quad \text{in } \Omega, \\
\text{div}(A_2 \nabla u_2) - \lambda \Sigma_2 u_2 &= \Sigma_2 f_2 \quad \text{in } \Omega, \\
\end{aligned}
\]
(4.13)
where \( u_1 = u_2, \ A_1 \nabla u_1 \cdot \nu = A_2 \nabla u_2 \cdot \nu \) on \( \Gamma \).

From now on, we fix the constant \( \Lambda_0 \) as required in Definition 3 for a given \( \varepsilon_0 \) and set
(4.14)
\[
\lambda_0 = \Lambda_0 e^{\varepsilon_0/2}.
\]

Remark 9. Let \( T_\lambda^* \) be the adjoint operator of \( T_\lambda \), i.e., \( \langle T_\lambda(f), g \rangle = \langle f, T_\lambda^*(g) \rangle \) for any \( f, g \in [L^2(\Omega)]^2 \). Integrating by parts, one has
\[
\begin{aligned}
\left( \int_{\Omega} \text{div}(A_1 \nabla u_1) v_1 - \text{div}(A_2 \nabla u_2) v_2 \right) - \left( \int_{\Omega} u_1 \text{div}(A_1 \nabla v_1) - u_2 \text{div}(A_2 \nabla v_2) \right) \\
= \left( \int_{\Gamma} v_1 A_1 \nabla u_1 \cdot \nu - v_2 A_2 \nabla u_2 \cdot \nu \right) - \left( \int_{\Gamma} u_1 A_1 \nabla v_1 \cdot \nu - u_2 A_2 \nabla v_2 \cdot \nu \right) \\
\end{aligned}
\]
(4.15)
This implies
\[
T_\lambda^* = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & -\Sigma_2 \end{pmatrix} T_\pi \begin{pmatrix} 1/\Sigma_1 & 0 \\ 0 & -1/\Sigma_2 \end{pmatrix}.
\]
Thus \( T_\lambda \) is not self-adjoint.

For the operator \( T_\lambda \) defined above, the following estimates hold:

Proposition 1. We have
(4.16)
\[
\|T_\lambda\|_{L^p \to L^2} \leq C|\lambda|^{-1 + \frac{d}{p}} \quad \text{if } p > d,
\]
(4.17)
\[
\|T_\lambda\|_{L^p \to L^q} \leq C|\lambda|^{-1 + \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \quad \text{if } 1 < p < d, \quad p \leq q < \frac{dp}{d-p},
\]
(4.18)
\[
\|T_\lambda\|_{L^1 \to L^q} \leq C|\lambda|^{-1 + \frac{d}{2} - \frac{d}{q}} \quad \text{if } 1 < q < \frac{d}{d-1}.
\]
Assume that \( \lambda_1, ..., \lambda_{k+1} \) satisfy the assumption of Theorem 3 with \( k = k_d = [\frac{d}{2}] \), and \( |\lambda_1| \sim |\lambda_2| \sim ... \sim |\lambda_{k+1}| = t \). Then operator \( \prod_{j=1}^{k+1} T_{\lambda_j} = T_{\lambda_{k+1}} \circ T_{\lambda_k} \circ ... \circ T_{\lambda_1} \) is Hilbert-Schmidt, and
(4.19)
\[
\| \prod_{j=1}^{k+1} T_{\lambda_j} \| \leq C t^{\frac{d}{2} - 1 - k}.
\]
Proof. Clearly, (4.16) - (4.18) follow from (3.5) - (3.7). Fix $p_1 = 2 < p_2 < \ldots < p_k < p_{k+1} < +\infty$ with $p_{k+1} > d$ and $p_j < \frac{dp_{j-1}}{d-p_{j-1}}$. By (4.16) and (4.17), we obtain

$$
\left\| \prod_{j=1}^{k-1} T_{\lambda_j} \right\|_{L^2 \rightarrow L^\infty} \leq \| T_{\lambda_1} \|_{L^{p_1} \rightarrow L^{p_2}} \ldots \| T_{\lambda_k} \|_{L^{p_{k-1}} \rightarrow L^{p_k}} \| T_{\lambda_{k+1}} \|_{L^{p_{k+1}} \rightarrow L^{\infty}}
$$

$$
\leq C \left( \prod_{j=1}^{k} |\lambda|^{-1+\frac{1}{q}} \left( \frac{1}{p_j - \frac{1}{p_{j+1}}} \right) \right) |\lambda|^{-1+\frac{d}{2(p_{k+1})}} = C |\lambda|^{\frac{d}{2}(k+1)}.
$$

The conclusions now follow from Lemma 3. \(\square\)

The following is the main result of this section and plays a crucial role in our analysis.

**Proposition 2.** Let $k = k_d = \left[ \frac{d}{2} \right]$ and denote

$$
\theta_j = \left( \frac{1}{4} + 2(j-1) \right) \frac{\pi}{k+1} \quad \text{and} \quad \theta_{k+1+j} = \left( \frac{5}{4} + 2(j-1) \right) \frac{\pi}{k+1}
$$

for $1 \leq j \leq k+1$. Let $t > 10\Lambda_0$ and set $\mu_j = \lambda_0 + t z_j$ with $z_j = \overline{e^{i\theta_j}}$ for $j = 1, \ldots, 2(k+1)$. We have

$$
\text{trace} \left( T_{\mu_{2(k+1)}} \circ T_{\mu_{2k+1}} \circ \ldots \circ T_{\mu_{1}} \right) = \sum_{j} \frac{1}{\lambda_j^{2(k+1)} - it(2(k+1))},
$$

where each characteristic value $\overline{\lambda}_j$ of $T_{\lambda_0}$ is repeated a number of times equal to its multiplicity.

**Remark 10.** In Proposition 2, $\Lambda_0$ is chosen large and corresponds with $\varepsilon_0 = \frac{\pi}{8(k+1)}$.

Proof. It is clear that $z_1, \ldots, z_{k+1}$ are the solutions of $z^{k+1} - e^{i\frac{\pi}{4}} = 0$ in $\mathbb{C}$ and $z_{k+2}, \ldots, z_{2(k+1)}$ are the solutions of $z^{k+1} - e^{i\frac{\pi}{4}} = 0$ in $\mathbb{C}$. One then has, for $z \in \mathbb{C}$,

$$
\prod_{j=1}^{k+1} (z - z_j) = z^{k+1} - e^{i\frac{\pi}{4}}, \quad \prod_{j=1}^{k+1} (1 - z_j z) = 1 - e^{i\frac{\pi}{4}} z^{k+1}
$$

and

$$
\prod_{j=k+2}^{2(k+1)} (z - z_j) = z^{k+1} - e^{i\frac{\pi}{4}}, \quad \prod_{j=k+2}^{2(k+1)} (1 - z_j z) = 1 - e^{i\frac{\pi}{4}} z^{k+1}.
$$

Note that, if $T_{\lambda}$ and $T_{\lambda+s}$ exist, and $T_{\lambda}$ is compact, then $s$ is not a characteristic value of $T_{\lambda}$, and

$$
T_{\lambda+s} = T_{\lambda}(I - sT_{\lambda})^{-1} = (I - sT_{\lambda})^{-1}T_{\lambda}.
$$

Indeed, if $T_{\lambda}$ and $T_{\lambda+s}$ exist, one can check that $I - sT_{\lambda}$ is injective, and therefore subjective since $T_{\lambda}$ is compact. One can then show that (4.23) holds.

As a consequence of (4.23), $T_{\lambda+s}$ is the modified operator of $T_{\lambda}$ with respect to $s$. Set

$$
T = T_{\mu_{2(k+1)}} \circ \ldots \circ T_{\mu_{k+2}}.
$$

It follows from Proposition 3 that $T$ is Hilbert-Schmidt, and

$$
\|T\| \leq C t^{\frac{d}{4} - 1 - k}, \quad \text{and} \quad \|T\|_{L^2 \rightarrow L^2} \leq C t^{-k-1}.
$$
Let \( s_1, s_2, \ldots \) be the characteristic values of \( T \) repeated a number of times equal to their multiplicities. Thanks to \([2]\) Theorem 12.17, one has, for a non-characteristic value \( \lambda \) of \( T \),

\[
\text{trace} \ (T \circ (T)_\lambda) = \sum_j \frac{1}{s_j(s_j - \lambda)} + c_t,
\]

where \((T)_\lambda\) is the modified operator associated with \( T \) and \( \lambda \), i.e., \((T)_\lambda := T(I - \lambda T)^{-1}\), for some \( c_t \in \mathbb{C} \).

By applying (4.24) with \( \lambda = 2e^{\frac{2\pi i}{t}} t^{k+1} \), it suffices to establish

\[
(T)_\lambda = T_{\mu_{k+1}} \circ \ldots \circ T_{\mu_1} \quad \text{for} \; \lambda = 2e^{\frac{2\pi i}{t}} t^{k+1},
\]

(4.26)

for some \( \ell \), and

(4.27) the multiplicity of \( s_j \) is equal to the sum of the multiplicity of \( \lambda_{\ell} \) such that (4.26) holds, and

(4.28) \( c_t = 0 \).

This will be done in the next three steps.

Step 1: Proof of (4.25). Since \( \mu_j - \lambda_0 = z_j t \), it follows from the second identity in (4.21) that

\[
\prod_{l=1}^{k+1} (1 - (\mu_l - \lambda_0)z) = 1 - e^{i\pi t} t^{k+1} z^{k+1}.
\]

(4.29)

One has

\[
T_{\mu_{k+1}} \circ \ldots \circ T_{\mu_1} = T_{\lambda_0} (I - (\mu_{k+1} - \lambda_0)T_{\lambda_0})^{-1} \circ \ldots \circ T_{\lambda_0} (I - (\mu_1 - \lambda_0)T_{\lambda_0})^{-1}
\]

\[
= T_{\lambda_0}^{k+1} \prod_{l=1}^{k+1} (I - (\mu_l - \lambda_0)T_{\lambda_0})^{-1} = T_{\lambda_0}^{k+1} (I - e^{i\pi t} t^{k+1} T_{\lambda_0}^{k+1})^{-1}
\]

(4.30)

In other words, we have

\[
T_{\mu_{k+1}} \circ \ldots \circ T_{\mu_1} = (T_{\lambda_0}^{k+1}) e^{i\pi t} t^{k+1}.
\]

(4.31)

Similarly, we obtain

\[
T = T_{\mu_{2(k+1)}} \circ \ldots \circ T_{\mu_{k+2}} = (T_{\lambda_0}^{k+1}) e^{i\pi t} t^{k+1}.
\]

(4.32)

Using the property

\[
\left( \begin{array}{c} T_{\lambda_0}^{k+1} \\ \gamma_1 \end{array} \right)_{\gamma_2} = \left( \begin{array}{c} T_{\lambda_0}^{k+1} \\ \gamma_1 + \gamma_2 \end{array} \right),
\]

for \( \gamma_1 \) and \( \gamma_1 + \gamma_2 \) non-characteristic values of \( T_{\lambda_0}^{k+1} \), we derive from (4.31) and (4.32) that

\[
(T)_{\lambda} = (T_{\lambda_0}^{k+1}) e^{i\pi t} t^{k+1} = T_{\mu_{k+1}} \circ \ldots \circ T_{\mu_1},
\]

and (4.25) follows.

Step 2: Proof of (4.26) and (4.27). Since \( T = (T_{\lambda_0}^{k+1}) e^{i\pi t} t^{k+1} \), it follows, see e.g. \([2]\) Theorem 12.4, that \( s_j^{-1} \) is an eigenvalue of \( T \) that is not equal to \(-e^{-i5\pi/4}t^{-(k+1)}\) if and only if \( \frac{s_j^{-1}}{1 + s_j^{-1} e^{i5\pi/4} t^{k+1}} = \frac{1}{s_j + e^{i5\pi/4} t^{k+1}} \) is an eigenvalue of \( T_{\lambda_0}^{k+1} \) (or equivalently \( s_j + e^{i5\pi/4} t^{k+1} \) is a characteristic value of \( T_{\lambda_0}^{k+1} \),
and they have the same multiplicity. One can check that $-e^{-i\frac{5\pi}{4}l-(k+1)}$ is not an eigenvalue of $T$. Assertions (4.26) and (4.27) follow.

**Step 3:** Proof of (4.28). For $z \in \mathcal{L}(\theta, 1)$ with $\inf_{n \in \mathbb{Z}}|\theta - n\pi| > \varepsilon_0$ and $|z|$ large enough, let $\tau_1, \cdots , \tau_{k+1}^0$ be the $k + 1$ distinct roots in $\mathbb{C}$ of the equation $x^{k+1} = z$. Set $\eta_l = \lambda_0 + \tau_l$ for $1 \leq l \leq k + 1$.

As in the proof of (4.30), one has

$$ T_{\eta_{k+1}} \circ \cdots \circ T_{\eta_1} = T_{\lambda_0}^{k+1}\left(I - zT_{\lambda_0}^{k+1}\right)^{-1}. $$

It follows that

$$ T_{\eta_{k+1}} \circ \cdots \circ T_{\eta_1} = \left(T_{\lambda_0}^{k+1}\right)_z. $$

Consider $\lambda$ defined by $e^{\frac{i\pi}{4}l^{k+1} + \lambda} = z$. We have, for large $|z|$,

$$ |\text{trace } (T \circ (T)\lambda)| \leq \|T\|\|T\lambda\| \leq C_t |z|^{\frac{d-1-k}{2}} \to 0 \text{ as } |z| \to +\infty, $$

and

$$ \left|\sum_j \frac{1}{s_j(s_j - \lambda)}\right| \leq \left(\sum_j |s_j|^{-2}\right)^{1/2} \left(\sum_j |s_j - \lambda|^{-2}\right)^{1/2}. $$

Applying [2, Theorem 12.14], we have

$$ \sum_j |s_j|^{-2} \leq \|T\| \leq C_t, $$

and applying [2, Theorems 12.4 and 12.14], we obtain

$$ \sum_j |s_j - \lambda|^{-2} \leq \|T\lambda\| \leq C_t |z|^{\frac{d-1-k}{2}} \to 0 \text{ as } |z| \to +\infty. $$

We derive from (4.34), (4.35), and (4.36) that

$$ \sum_j \frac{1}{s_j(s_j - \lambda)} \to 0 \text{ as } |z| \to +\infty. $$

Combining (4.33) and (4.37) yields $c_t = 0$.

The proof is complete. \hfill \Box

**Remark 11.** Let $\lambda_j$ be an eigenvalue of the transmission problem. Then $\lambda_j - \lambda$ and $\lambda_j - \hat{\lambda}$ are the characteristic values of $T_\lambda$ and $T_{\hat{\lambda}}$ respectively, provided that $T_\lambda$ and $T_{\hat{\lambda}}$ exist. Using (4.23) and applying [2, Theorem 12.4], one can show that the multiplicity of $\lambda_j - \lambda$ and the multiplicity of $\lambda_j - \hat{\lambda}$ are the same.

## 5. The Weyl Law for Eigenvalues of the Transmission Problem - Proof of Theorem 1

**5.1. Approximation of the Trace of the Kernel and Their Applications.** For $\lambda \in \mathcal{L}(\theta, 1)$ with $\theta \neq n\pi$ for all $n \in \mathbb{Z}$, and $x_0 \in \Omega$, set

$$ S_{j,\lambda,x_0} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) $$

$$ f_j \mapsto v_j, $$

(5.1)
where \(v_j \in H^1(\mathbb{R}^d)\) is the unique solution of
\[
\text{div}(A_j(x_0) \nabla v_j) - \lambda \Sigma_j(x_0)v_j = \Sigma_j(x_0)f_j \quad \text{in } \mathbb{R}^d.
\]
We also define
\[
S_{\lambda,x_0} : [L^2(\mathbb{R}^d)]^2 \to [L^2(\mathbb{R}^d)]^2
\]
\[
(f_1,f_2) \mapsto (S_{1,\lambda,x_0}f_1, S_{2,\lambda,x_0}f_2).
\]
One then has
\[
S_{j,\lambda,x_0}f(x) = \int_{\mathbb{R}^d} F_{j,\lambda}(x_0,x-y)f_j(y)dy,
\]
where
\[
F_{j,\lambda}(x_0,z) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{i\xi} \lambda^{-\frac{1}{2}}}{\xi} d\xi \quad \text{for } z \in \mathbb{R}^d.
\]

By Lemma 1, we get, for \(1 < p < +\infty\),
\[
\|\nabla^2 S_{j,\lambda,x_0}f_j\|_{L^p(\mathbb{R}^d)} + \|\lambda\|^{1/2} \|\nabla S_{j,\lambda,x_0}f_j\|_{L^p(\mathbb{R}^d)} + \|\lambda\| \|S_{j,\lambda,x_0}f_j\|_{L^p(\mathbb{R}^d)} \leq C\|f_j\|_{L^p(\mathbb{R}^d)}.
\]
As in the proof of Theorem 2, we obtain from the interpolation inequalities (3.32) and (3.33) that
\[
\|S_{\lambda,x_0}\|_{L^p \to L^\infty} \leq C|\lambda|^{-1+\frac{d}{2p}} \quad \text{if } p > d,
\]
\[
\|S_{\lambda,x_0}\|_{L^p \to L^q} \leq C|\lambda|^{-1+\frac{d}{2q}} \left( \frac{d}{d-1} \right) \quad \text{if } 1 < q < \frac{d}{d-1}.
\]

Let \(t > 10\Lambda_0\) and let \(\mu_1, \ldots, \mu_{2(k+1)}\) be defined in Proposition 2. Set, for \(z \in \mathbb{R}^d\),
\[
F_{j,t}(x_0,z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{i\xi} \lambda^{-\frac{1}{2}}}{\xi} d\xi,
\]
and define
\[
S_{j,t,x_0} = \prod_{l=1}^{2(k+1)} S_{j,\mu_l,x_0}.
\]
Then
\[
S_{j,t,x_0}f_j(x) = \int_{\mathbb{R}^d} F_{j,t}(x_0,x-y)f_j(y)dy.
\]
Since, by the definition of \(\mu_l\) and \(z_l\),
\[
\prod_{l=1}^{2(k+1)} \left( \Sigma_j(x_0)^{-1}\langle A_j(x_0)\xi,\xi\rangle + \mu_l \right) = \prod_{l=1}^{2(k+1)} \left( \Sigma_j(x_0)^{-1}\langle A_j(x_0)\xi,\xi\rangle + \lambda_0 + tz_l \right)
\]
\[
= \left( \Sigma_j(x_0)^{-1}\langle A_j(x_0)\xi,\xi\rangle + \lambda_0 \right)^{2(k+1)} - it^{2(k+1)},
\]
it follows from (5.7) that
\[
F_{j,t}(x_0,z) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{i\xi} \lambda^{-\frac{1}{2}}}{\xi} d\xi.
\]
As a consequence of (5.8), we obtain, by a change of variables,

\[(5.9) \quad \mathcal{F}_{j,t}(x_0,0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{(\Sigma_j^{-1}(x_0)\langle A_j(x_0)\xi, \xi \rangle + \lambda_0)^{2(k+1)} - it^{2(k+1)}} d\xi \]

\[= \frac{t^{\frac{d}{2} - 2(k+1)}}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{(\Sigma_j^{-1}(x_0)\langle A_j(x_0)\xi, \xi \rangle + t^{-1}\lambda_0)^{2(k+1)} - i} d\xi.\]

This implies, by the dominated convergence theorem,

\[(5.10) \quad \mathcal{F}_{j,t}(x_0,0) = \frac{t^{\frac{d}{2} - 2(k+1)}}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{(\Sigma_j^{-1}(x_0)\langle A_j(x_0)\xi, \xi \rangle)^{2(k+1)} - i} d\xi + O(t^{\frac{d}{2} - 2(k+1)-1}).\]

We next introduce \(S_{t,x_0} : [L^2(\mathbb{R}^d)]^2 \rightarrow [L^2(\mathbb{R}^d)]^2\) by

\[S_{t,x_0} = S_{\mu_2(k+1),x_0} \circ \cdots \circ S_{\mu_{1},x_0}, \quad \text{where} \quad S_{\mu_{1},x_0} = \begin{pmatrix} S_{1,\mu_{1},x_0} & 0 \\ 0 & S_{2,\mu_{1},x_0} \end{pmatrix}.\]

Set

\[F_{t}(x_0,\cdot) = \begin{pmatrix} \mathcal{F}_{1,t}(x_0,\cdot) & 0 \\ 0 & \mathcal{F}_{2,t}(x_0,\cdot) \end{pmatrix}.\]

We then have

\[S_{t,x_0} f(x) = \int_{\mathbb{R}^d} F_{t}(x_0,x-y)f(y)dy.\]

Let \(\mathcal{K}_t\) denote the kernel corresponding to \(\prod_{\mu_{1}}^{2(k+1)} T_{\mu_{1}} = T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_{1}}\). Here is the main result of this section.

**Proposition 3.** We have

\[(5.11) \quad \int_{\Omega} \text{trace } \mathcal{K}_t(x,x)dx = \hat{\epsilon} t^{\frac{d}{2} - 2(k+1)} + o(t^{\frac{d}{2} - 2(k+1)}) \quad \text{as} \quad t \to \infty,\]

where

\[(5.12) \quad \hat{\epsilon} = \frac{1}{(2\pi)^d} \sum_{j=1}^{2} \int_{\Omega} \int_{\mathbb{R}^d} \frac{1}{(\Sigma_j^{-1}(x)\langle A_j(x)\xi, \xi \rangle)^{2(k+1)} - i} d\xi dx.\]

**Proof.** We claim that

\[(5.13) \quad \int_{\Omega} \text{trace } \mathcal{K}_t(x,x)dx = \int_{\Omega} \text{trace } F_{t}(x,0)dx + o(t^{\frac{d}{2} - 2(k+1)}) \quad \text{as} \quad t \to \infty.\]

The conclusion then follows from (5.10).

The main point of the proof is to establish (5.13). Let \(\varphi\) be a function in \(C^\infty(\mathbb{R}^d)\) such that \(0 \leq \varphi \leq 1\), \(\varphi = 1\) in \(B_1^\frac{d}{2}\) and \(\text{supp } \varphi \subset B_2^\frac{d}{2} \setminus \overline{B_1^\frac{d}{2}}\). Let \(\delta_0 > 0\) and \(x_0 \in \Omega\) be such that \(\text{dist}(x_0,\partial\Omega) > \delta_0\).

For \(\delta \in (0,10^{-2}\delta_0)\), set \(\varphi_{\delta}(x) = \varphi(\delta^{-1}(x-x_0))\) and

\[\Phi(\delta,x_0) = \sup_{B_{10\delta}(x_0)} \sum_{j=1}^{2} \left| A_j(x) - A_j(x_0) \right| + \left| \Sigma_j(x) - \Sigma_j(x_0) \right|.\]

The essential ingredient of the analysis is the following estimate, for \(t > \delta^{-4}:

\[(5.14) \quad \|\varphi_{2\delta} \left( T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_{1}} - S_{t,x_0} \right) \varphi_{\delta} \|_{L^1 \to L^\infty} \leq C_{\delta_0} \left( \Phi(\delta,x_0) + \delta^{-1} t^{-1/2} \right) t^{\frac{d}{2} - 2(k+1)}.\]
We first assume (5.13) and continue the proof. We have

\[(\varphi_{2\delta} \circ (T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_1} - S_{t,x_0}) \varphi_{\delta}) (f)(x) = \varphi_{2\delta}(x) \int_{\Omega} (K_t(x,y) - F_t(x_0,x-y)) \varphi_{\delta}(y) f(y) dy.\]

It follows from (5.14) that, for \(x,y \in \Omega\) and for \(t > \delta^{-4}\),

\[|\varphi_{2\delta}(x) \varphi_{\delta}(y) (K_t(x,y) - F_t(x_0,x-y))| \leq C_{\delta_0} \left( \Phi(\delta,x_0) + \delta^{-1/2} t^{-1/2} \right) \frac{\delta^2}{t^{2k+1}}.\]

This implies that, for \(t > \delta^{-4}\),

\[|\text{trace } K_t(x_0,x) - \text{trace } F_t(x_0,0)| \leq C_{\delta_0} \left( \Phi(\delta,x_0) + \delta^{-1/2} t^{-1/2} \right) \frac{\delta^2}{t^{2k+1}}.\]

Here we used the fact that \(\varphi_{2\delta}(x_0) = \varphi_{\delta}(x_0) = 1\). Using Proposition 4 and (4.9) - (4.10), we have

\[(5.16) \quad |K_t(x,x)| \leq Ct^{d/2-2k+1} \text{ for } x \in \Omega.\]

By (5.10), we obtain

\[(5.17) \quad |F_t(x,0)| \leq Ct^{d/2-2k+1} \text{ for } x \in \Omega.\]

Assertion (5.13) now follows from (5.15), (5.16), and (5.17) by noting that \(\sup_{x \in \Omega} \Phi(\delta,x) \to 0\) as \(\delta \to 0\).

It remains to prove (5.14). We have

\[(5.18) \quad \varphi_{2\delta} \circ (T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_1} - S_{t,x_0}) \varphi_{\delta} \]

\[= \sum_{l=1}^{2k+1} \varphi_{2\delta} \circ (T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_{l+1}} \circ (T_{\mu_l} - S_{\mu_1,x_0}) \circ S_{\mu_{l-1},x_0} \cdots \circ S_{\mu_1,x_0}) \varphi_{\delta}.\]

Fix \(\beta_0 = 1 < \beta_1 < \cdots < \beta_{2k+1} = 2\) with \(\beta_{l+1} - \beta_l > 1/(10d)\). Set \(S_{\mu_{l-1},x_0} = \varphi_{\beta_l} \delta S_{\mu_l,x_0}\) and \(S_{\mu_{l-1},x_0} = (1 - \varphi_{\beta_l} \delta) S_{\mu_l,x_0}\).

Then

\[(S_{\mu_{l-1},x_0} \circ \cdots \circ S_{\mu_1,x_0}) \varphi_{\delta} = (S_{\mu_{l-1},x_0} + S_{\mu_{l-1},x_0} \varphi_{\delta}) \varphi_{\beta_l} \delta \circ \cdots \circ (S_{\mu_1,x_0} \varphi_{\beta_l} \delta) \circ (S_{\mu_1,x_0} \varphi_{\beta_l} \delta) \varphi_{\delta}.\]

Since \(\varphi_{\beta_l} \delta \circ \varphi_{\beta_l} \delta\), it follows from (5.18) that

\[(5.19) \quad \varphi_{2\delta} \circ (T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_1} - S_{t,x_0}) \varphi_{\delta} \]

\[= \sum_{l=1}^{2k+1} \varphi_{2\delta} \circ (T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_{l+1}}) \circ (T_{\mu_l} - S_{\mu_1,x_0}) \circ (S_{\mu_{l-1},x_0} \circ \cdots \circ S_{\mu_1,x_0} \varphi_{\delta}) + \]

\[\sum_{l=1}^{2k+1} \varphi_{2\delta} \circ (T_{\mu_{2(k+1)}} \circ \cdots \circ T_{\mu_{l+1}}) \circ (T_{\mu_l} - S_{\mu_1,x_0}) \circ (S_{\mu_{l-1},x_0} \circ \cdots \circ S_{\mu_1,x_0} \circ S_{\mu_1,x_0} \varphi_{\delta}).\]

Let \(p_1 = 1 < p_2 < \cdots < p_{2k+1} < d \leq p_{2k+1} = +\infty\) be such that \(p_{l+1} < p_l/(d-p_l)\) for \(1 \leq l \leq 2k+1\). Using the exponential decay property: for \(\gamma > 1, r > 0, y \in \mathbb{R}^d\), and for \(f \text{ with supp } f \subset B_r\), it holds, for \(t > r^{-3/2}\)

\[(5.20) \quad \|S_{\mu_l} f\|_{L^\infty(B_{r+y}(y))} \leq C_{\gamma} e^{-\gamma r t} \|f\|_{L^r(B_r(y))},\]

one has for \(l = 2, \ldots, 2k+1 + 1\),

\[(5.21) \quad \| (S_{\mu_{l-1},x_0} \cdots \circ S_{\mu_1,x_0} \varphi_{\delta} - (S_{\mu_{l-1},x_0} \cdots \circ S_{\mu_1,x_0} \varphi_{\delta}) \|_{L^1 \to L^{p_{l+1}}} \leq C e^{-\delta t}.\]
Combining (5.19)-(5.21), and using (4.16)-(4.18) for $T_{\mu}$, and (5.4)-(5.6) for $S_{\mu}$, it suffices to prove that
\begin{equation}
\|\varphi_{2\delta}(T_{\mu_{2(k+1)}^2} - S_{\mu_{2(k+1)},x_0})\|_{L^p(\mathbb{R}^{2k+1})} \leq C_{\delta_0}(\Phi(\delta, x_0) + t^{-1/2}\delta^{-1}) t^{-1+\frac{1}{2} \gamma} \|\varphi_{2\delta}(T_{\mu_{2(k+1)}^2})\|_{L^p(\mathbb{R}^{2k+1})},
\end{equation}
and for $l = 1, 2, ..., 2k + 1$,
\begin{equation}
\|\varphi_{\beta\gamma}(T_{\mu_{l}} - S_{\mu_{l},x_0})\|_{L^{p_l}\rightarrow L^{p_{l+1}}} \leq C_{\delta_0}(\Phi(\delta, x_0) + \delta^{-1/2} t^{-1/2}) t^{-1+\frac{1}{2} \gamma} \left(\frac{1}{p_l} - \frac{1}{p_{l+1}}\right).
\end{equation}

Step 1: Proof of (5.22). We will prove the following stronger result, which will be used in the proof of (5.23): for $\lambda \in \mathcal{L}(\theta, \Lambda_0)$ and $\sup_{n \in \mathbb{Z}} |\theta - n\pi| > \varepsilon_0$, $\beta \in [1, 2]$; and for $1 < p < d$, $p < q < \frac{PD}{d-p}$ or for $d > p$ and $q = +\infty$:
\begin{equation}
\|\varphi_{\beta\delta}(T_{\lambda} - S_{\lambda,x_0})\|_{L^p\rightarrow L^q} \leq C_{\delta_0,p}(\Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}) |\lambda|^{-1+\frac{1}{2} \gamma} \left(\frac{1}{p} - \frac{1}{q}\right).
\end{equation}

Denote
\[u = T_\lambda(f) \quad \text{and} \quad v = S_{\lambda,x_0}f.\]

Set
\[u_{j,\delta} = \varphi_{\beta\delta} u_j \quad \text{and} \quad v_{j,\delta} = \varphi_{\beta\delta} v_j.\]

Since, in $\Omega$,
\[\text{div}(A_j \nabla u_j) - \lambda \Sigma_j u_j = \Sigma_j f_j,\]
and
\[\text{div}(A_j(x_0) \nabla v_j) - \lambda \Sigma_j(x_0) v_j = \Sigma_j(x_0) f_j,\]
we have, in $\Omega$,
\[\text{div}(A_j(x_0) \nabla u_{j,\delta}) - \lambda \Sigma_j(x_0) u_{j,\delta} = f_{j,\delta} \quad \text{and} \quad \text{div}(A_j(x_0) \nabla v_{j,\delta}) - \lambda \Sigma_j(x_0) u_{j,\delta} = g_{j,\delta},\]
where
\[f_{j,\delta} = \tilde{f}_{j,\delta} + \text{div} \tilde{F}_{j,\delta} \quad \text{and} \quad g_{j,\delta} = \tilde{g}_{j,\delta} + \text{div} \tilde{G}_{j,\delta},\]
with
\[\tilde{f}_{j,\delta} = \varphi_{\beta\delta} \Sigma_j f_j + A_j \nabla u_j \nabla \varphi_{\beta\delta} - \lambda (\Sigma_j(x_0) - \Sigma_j(x)) u_{j,\delta},\]
\[\tilde{F}_{j,\delta} = u_j A_j \nabla \varphi_{\beta\delta} + (A_j(x) - A_j(x)) \nabla u_{j,\delta},\]
\[\tilde{g}_{j,\delta} = \varphi_{\beta\delta} \Sigma_j(x_0) f_j + A_j(x_0) \nabla v_j \nabla \varphi_{\beta\delta}, \quad \text{and} \quad \tilde{G}_{j,\delta} = v_j A_j(x_0) \nabla \varphi_{\beta\delta}.\]

By Theorems 3 and 3.3, we have for $1 < p < +\infty$,
\begin{equation}
\|\nabla u\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2} \left(\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\mathbb{R}^d)}\right) \leq C |\lambda|^{-\frac{1}{2}} \|f\|_{L^p(\Omega)}.
\end{equation}

By Lemma 1, we obtain for $1 < p < +\infty$,
\begin{equation}
|\lambda|^{1/2} \|\nabla (u_{j,\delta} - v_{j,\delta})\|_{L^p(\mathbb{R}^d)} + |\lambda| \|u_{j,\delta} - v_{j,\delta}\|_{L^p(\mathbb{R}^d)}
\leq C \left(\|\tilde{f}_{j,\delta} - \tilde{g}_{j,\delta}\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2} \|\tilde{F}_{j,\delta} - \tilde{G}_{j,\delta}\|_{L^p(\mathbb{R}^d)}\right).
\end{equation}

Using (5.25), we derive, for $1 < p < +\infty$, that, with $u_{\delta} = (u_{1,\delta}, u_{2,\delta})$ and $v_{\delta} = (v_{1,\delta}, v_{2,\delta})$,
\[\|\nabla (u_{\delta} - v_{\delta})\|_{L^p(\mathbb{R}^d)} + |\lambda|^{1/2} \|u_{\delta} - v_{\delta}\|_{L^p(\mathbb{R}^d)} \leq C_{\delta_0}(\Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}) |\lambda|^{-1/2} \|f\|_{L^p(\Omega)}.
\]

By Gagliardo-Nirenberg’s interpolation inequalities, one gets that for $1 < p < d$ and $p < q < \frac{PD}{d-p}$
\[\|u_{\delta} - v_{\delta}\|_{L^q(\mathbb{R}^d)} \leq C_{\delta_0}(\Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}) |\lambda|^{-1+\frac{1}{2} \gamma} \left(\frac{1}{p} - \frac{1}{q}\right) \|f\|_{L^p(\Omega)},\]
and for \( p > d \),
\[
\|u_\delta - v_\delta\|_{L^p(\Omega)} \leq C_{\delta_0} \left( \Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}\right)|\lambda|^{-1 + \frac{d}{n}}\|f\|_{L^p(\Omega)},
\]
and assertion (5.24) follows.

Step 2: Proof of (5.23). By (4.15),
\[
T_{\lambda}^* - S_{\lambda,x_0}^* = \left( \frac{1}{\Sigma_1} 0 \right) (T_{\lambda}^* - S_{\lambda,x_0}^*) \left( \frac{1}{\Sigma_1} 0 \right),
\]
it follows that by (5.24), for \( \lambda \in \mathcal{L}(\theta, \Lambda_0) \) with \( \sup_{n \in \mathbb{Z}} |\theta - n\pi| > \varepsilon_0, \beta \in [1,2] \); and for \( 1 < p < d \),
\[
\|\varphi_{\beta\delta} (T_{\lambda}^* - S_{\lambda,x_0}^*)\|_{L^p \to L^p} \leq C_{\delta_0, \varepsilon_0} \left( \Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}\right)|\lambda|^{-1 + \frac{d}{n}}\left( \frac{1}{p} - \frac{1}{q} \right),
\]
which implies
\[
(5.27) \quad \| (T_{\lambda} - S_{\lambda,x_0}) \varphi_{\beta\delta}\|_{L^p \to L^p} \leq C_{\delta_0, \varepsilon_0} \left( \Phi(\delta, x_0) + |\lambda|^{-1/2}\delta^{-1}\right)|\lambda|^{-1 + \frac{d}{n}}\left( \frac{1}{p} - \frac{1}{q} \right).
\]
This gives (5.23). The proof is complete.

As a consequence of Proposition 2 and Proposition 3, we obtain

**Corollary 3.** We have
\[
\sum_j \frac{1}{|\lambda_j|^{2(k+1)} - \ell 2(k+1)} = \hat{c} t^{\frac{d}{2} - 2(k+1)} + o(t^{\frac{d}{2} - 2(k+1)}) \quad \text{as} \quad t \to \infty,
\]
where each characteristic value \( \tilde{\lambda}_j \) of \( T_{\lambda_0} \) is repeated a number of times equal to its multiplicity, and \( \hat{c} \) is defined by (5.12).

**Proof.** By Propositions 2 and 3, and (4.12) in Lemma 4, we have
\[
\sum_j \frac{1}{|\lambda_j|^{2(k+1)} - \ell 2(k+1)} = \hat{c} t^{\frac{d}{2} - 2(k+1)} + o(t^{\frac{d}{2} - 2(k+1)}) \quad \text{as} \quad t \to \infty.
\]
For \( j_0 \in \mathbb{N} \) large and for \( t \geq 2|\tilde{\lambda}_{j_0}| \), we have
\[
(5.28) \quad \left| \sum_{j=1}^{j_0} \frac{1}{|\lambda_j|^{2(k+1)} - \ell 2(k+1)} - \sum_{j=1}^{\infty} \frac{1}{|\lambda_j|^{2(k+1)} - \ell 2(k+1)} \right| 
\]
\[
\leq \sum_{j=1}^{j_0} \frac{1}{|\lambda_j|^{2(k+1)} - \ell 2(k+1)} + \frac{1}{\left|\lambda_j\right|^{2(k+1)} - \ell 2(k+1)} 
\]
\[
+ \sum_{j=j_0+1}^{\infty} \frac{C|\lambda_j|^{2(k+1)|3\lambda_j|}}{|\lambda_j|^{2(k+1)} - \ell 2(k+1)} \left| \lambda_j \right|^{2(k+1)} \ell 2(k+1) 
\]
\[
\leq 2 j_0 t^{2(k+1)} + \left( \sup_{j \geq j_0} \left| \frac{3\lambda_j}{|\lambda_j|} \right| \right) \sum_{j=j_0+1}^{\infty} \frac{1}{|\lambda_j|^{2(k+1)}}.
\]
By [2] Theorem 12.14] and Theorem 3, we have

\begin{equation}
\sum_{j \geq j_0} \left( |\bar{\lambda}_j| + |t| \right)^{-2(k+1)} \leq C \|T_{-it}^{k+1}\|^2,
\end{equation}

and by Proposition 1, we obtain

\begin{equation}
\|T_{-it}\|^2 \leq Ct^{d-2(k+1)}.
\end{equation}

Since, by Theorem 3,

\begin{equation}
\left( \sup_{j \geq j_0} \left| \Im \bar{\lambda}_j \right| \right) \to 0 \text{ as } j_0 \to +\infty,
\end{equation}

it follows from (5.28), (5.29), and (5.30) that

\begin{equation}
\left| \sum_{j=1}^{\infty} \frac{1}{|\bar{\lambda}_j|^{2(k+1)} - t^2(2(k+1))} - \sum_{j=1}^{\infty} \frac{1}{|\bar{\lambda}_j|^{2(k+1)} - t^2(2(k+1))} \right| = o(1)t^{d/2-2(k+1)} \text{ as } t \to \infty.
\end{equation}

The proof is complete. \hfill \Box

5.2. Proof of Theorem 1. Before giving the proof of Theorem 1, we recall a Tauberian theorem of Hardy and Littlewood, see e.g. [44, Theorem 2a] or [2, Theorem 14.5].

Lemma 5. Let \( \sigma(s) \) be a non-decreasing function for \( s > 0 \), let \( a \in (0,1) \) and \( P \geq 0 \). Then, as \( t \to \infty \),

\begin{equation}
\int_0^\infty \frac{d\sigma(s)}{s+t} = P t^{a-1} + o(t^{a-1}),
\end{equation}

if and only, as \( s \to \infty \),

\begin{equation}
\sigma(s) = \frac{P}{a} \int_0^\infty t^{a-1} (1+t)^{-1} dt + o(s^a).
\end{equation}

We are ready to give

Proof of Theorem 1. We have, by Corollary 3:

\begin{equation}
\sum_{j=1}^{\infty} \frac{1}{|\bar{\lambda}_j|^{2(k+1)} - t^2(2(k+1))} = \hat{c} t^{d/2-2(k+1)} + o(t^{d/2-2(k+1)}) \text{ as } t \to \infty,
\end{equation}

where \( \hat{c} \) is given by (5.12):

\begin{equation}
\hat{c} = \frac{1}{(2\pi)^d} \int_{\Omega} \sum_{j=1,2} \int_{\mathbb{R}^d} \frac{1}{(\Sigma_j(x) - \langle A_j(x)\xi, \xi \rangle t^2(2(k+1)) - t} d\xi dx.
\end{equation}

Considering the imaginary part yields,

\begin{equation}
\sum_{j=1}^{\infty} \frac{t^{2(k+1)}}{|\bar{\lambda}_j|^{2(k+1)} + t^4(k+1)} = \hat{c}_1 t^{d/2-2(k+1)} + o(t^{d/2-2(k+1)}) \text{ as } t \to +\infty,
\end{equation}

where

\begin{equation}
\hat{c}_1 = \Im(\hat{c}) = \frac{1}{(2\pi)^d} \int_{\Omega} \sum_{j=1,2} \int_{\mathbb{R}^d} \frac{1}{(\Sigma_j(x) - \langle A_j(x)\xi, \xi \rangle t^4(k+1) + 1} d\xi dx.
\end{equation}

This implies, by replacing \( t^4(k+1) \) by \( t \),

\begin{equation}
\sum_{j=1}^{\infty} \frac{1}{|\bar{\lambda}_j|^{4(k+1)} + t} = \hat{c}_1 t^{d/(2(k+1))-1} + o(t^{d/(2(k+1))-1}) \text{ as } t \to +\infty.
\end{equation}
Since $\tilde{\lambda}_j = \lambda_j - \lambda_0$, one obtains
\[
\sum_{j=1}^{\infty} \frac{1}{|\lambda_j|^{k+1}} = \hat{c}_1 t^{\frac{d}{8(k+1)-1}} + o(t^{\frac{d}{8(k+1)-1}}) \quad \text{as} \ t \to +\infty.
\]
We can write this identity under the form
\[
\int_0^{\infty} dN(s^{\frac{d}{8(k+1)}}) = \hat{c}_1 t^{\frac{d}{8(k+1)-1}} + o(t^{\frac{d}{8(k+1)-1}}) \quad \text{as} \ t \to \infty.
\]
By Lemma 5, one has
\[
(5.32) \quad N(t) = ct^{\frac{d}{2}} + o(t^{\frac{d}{2}}),
\]
where
\[
c = \frac{\hat{c}_1}{d(k+1) t^{\frac{d}{8(k+1)-1}(1 + t)^{-1}} dt}.
\]
We have, by Fubini’s theorem,
\[
\int_{\mathbb{R}^d} \frac{1}{(\Sigma_j(x))^{-1}(\langle A_j(x)\xi, \xi \rangle)^{4(k+1)} + 1} d\xi = \int_{0}^{\infty} \left| \left\{ \xi : (\Sigma_j(x))^{-1}(\langle A_j(x)\xi, \xi \rangle)^{4(k+1)} < t \right\} \right| dt \frac{d}{(t + 1)^{2}}.
\]
Since
\[
\left| \left\{ \xi : (\Sigma_j(x))^{-1}(\langle A_j(x)\xi, \xi \rangle)^{4(k+1)} < t \right\} \right| = t^{\frac{d}{8(k+1)}} \left| \left\{ \xi : \langle A_j(x)\xi, \xi \rangle < \Sigma_j(x) \right\} \right|,
\]
it follows that
\[
\int_{\mathbb{R}^d} \frac{1}{(\Sigma_j(x))^{-1}(\langle A_j(x)\xi, \xi \rangle)^{4(k+1)} + 1} d\xi = \left| \left\{ \xi : \langle A_j(x)\xi, \xi \rangle < \Sigma_j(x) \right\} \right| \int_{0}^{\infty} t^{\frac{d}{8(k+1)}} dt \frac{d}{(t + 1)^{2}}
\]
\[
= \left| \left\{ \xi : \langle A_j(x)\xi, \xi \rangle < \Sigma_j(x) \right\} \right| \frac{d}{8(k+1)} \int_{0}^{\infty} t^{\frac{d}{8(k+1)-1}(1 + t)^{-1}} dt.
\]
Here in the last identity, an integration by parts is used. We therefore have
\[
c = \frac{1}{(2\pi)^d} \sum_{j=1,2} \int_{\Omega} \left| \left\{ \xi : \langle A_j(x)\xi, \xi \rangle < \Sigma_j(x) \right\} \right| |dx|.
\]
The proof is complete. □

6. Completeness of Generalized Eigenfunctions of the Transmission Problem -
Proof of Theorem 2

Fix $\varepsilon_0 > 0$. For $z \in \mathcal{L}(\theta, 1)$ with $\inf_{n \in \mathbb{Z}} |\theta - n\pi| > \varepsilon_0$ and $|z|$ large enough, let $\tau_1, \cdots, \tau_{k+1}$ with $k = k_d = [d/2]$ be the $k + 1$ distinct roots in $\mathbb{C}$ of the equation $x^{k+1} = z$. Set
\[
\eta_l = \lambda_0 + \tau_l \quad \text{for} \ 1 \leq l \leq k + 1.
\]
As in the proof of (4.30), one has
\[
T_{\eta_{k+1}} \circ \cdots \circ T_{\eta_1} = T_{\lambda_0}^{k+1} \left( I - zT_{\lambda_0}^{k+1} \right)^{-1}.
\]
It follows that
\[
T_{\eta_{k+1}} \circ \cdots \circ T_{\eta_1} = \left( T_{\lambda_0}^{k+1} \right)^z.
\]
Since $T_{\lambda_0}^{k+1}$ is a Hilbert-Schmidt operator, it follows from [2, Theorem 16.4] that:
1) the space spanned by the general eigenfunctions of $T_{\lambda_0}^{k+1}$ is equal to $\mathcal{R}(T_{\lambda_0}^{k+1})$, the closure of the range of $T_{\lambda_0}^{k+1}$ with respect to the $L^2$-topology.
On the other hand, we have
2) the range $\mathbf{R}(T_{\lambda_0}^{k+1})$ of $T_{\lambda_0}^{k+1}$ is dense in $[L^2(\Omega)]^2$, since $\mathbf{R}(T_{\lambda_0})$ is dense in $[L^2(\Omega)]^2$ and $T_{\lambda_0}$ is continuous,

3) the space spanned by the general eigenfunctions of $T_{\lambda_0}^{k+1}$ is equal to the space spanned by the general eigenfunctions of $T_{\lambda_0}$.

The conclusion now follows from 1), 2), and 3). □

References

[1] Shmuel Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. 15 (1962), 119–147. MR 147774

[2] _________, *Lectures on elliptic boundary value problems*, Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965. MR 0178246

[3] Shmuel Agmon, Avron Douglis, and Louis Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I.*, Comm. Pure Appl. Math. 12 (1959), 623–727. MR 125307

[4] _________, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II.*, Comm. Pure Appl. Math. 17 (1964), 35–92.

[5] Eemeli Blästen and Lassi Päivärinta, *Completeness of generalized transmission eigenstates*, Inverse Problems 29 (2013), no. 10, 104002, 22. MR 3116197

[6] Anne-Sophie Bonnet-Ben Dhia, Lucas Chesnel, and Houssem Haddar, *On the use of $T$-coercivity to study the interior transmission eigenvalue problem*, C. R. Math. Acad. Sci. Paris 349 (2011), no. 11-12, 647–651. MR 2817384

[7] Fioralba Cakoni, David Colton, and Drossos Gintides, *The interior transmission eigenvalue problem*, SIAM J. Math. Anal. 42 (2010), no. 6, 2912–2921. MR 2745796

[8] Fioralba Cakoni, David Colton, and Houssem Haddar, *Inverse scattering theory and transmission eigenvalues*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 88, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016. MR 3601119

[9] Fioralba Cakoni, Drossos Gintides, and Houssem Haddar, *The existence of an infinite discrete set of transmission eigenvalues*, SIAM J. Math. Anal. 42 (2010), no. 1, 237–255. MR 2596553

[10] Fioralba Cakoni and Peter Monk and Virginia Selgas, *Analysis of the linear sampling method for imaging penetrable obstacles in the time domain*, Anal. & PDE 2020, to appear.

[11] Fioralba Cakoni and Hoai-Minh Nguyen, *On the Discreteness of Transmission Eigenvalues for the Maxwell Equations*, 2020, submitted, https://arxiv.org/abs/2004.14670.

[12] David Colton and Peter Monk, *The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium*, Quart. J. Mech. Appl. Math. 41 (1988), no. 1, 97–125. MR 934695

[13] Emilio Gagliardo, *Ulteriori proprietà di alcune classi di funzioni in più variabili*, Ricerche Mat. 8 (1959), 24–51. MR 109295

[14] Loukas Grafakos, *Classical Fourier analysis*, second ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2008. MR 2445437

[15] Alain Grigis and Johannes Sjöstrand, *Microlocal analysis for differential operators*, London Mathematical Society Lecture Note Series, vol. 196, Cambridge University Press, Cambridge, 1994. An introduction. MR 1299107

[16] Houssem Haddar and Shixiu Meng, *The spectral analysis of the interior transmission eigenvalue problem for Maxwell’s equations*, J. Math. Pures Appl. (9) 120 (2018), 1–32. MR 3906154

[17] Lars Hörmander, *The spectral function of an elliptic operator*, Acta Math. 121 (1968), 193–218. MR 609014

[18] Andreas Kirsch, *The denseness of the far field patterns for the transmission problem*, IMA J. Appl. Math. 37 (1986), no. 3, 213–225. MR 83987

[19] Evgeny Lakshtanov and Boris Vainberg, *Ellipticity in the interior transmission problem in anisotropic media*, SIAM J. Math. Anal. 44 (2012), no. 2, 1165–1174. MR 2914264

[20] _________, *Remarks on interior transmission eigenvalues*, Weyl formula and branching billiards, J. Phys. A 45 (2012), no. 12, 125202, 10. MR 2902328

[21] _________, *Sharp Weyl law for signed counting function of positive interior transmission eigenvalues*, SIAM J. Math. Anal. 47 (2015), no. 4, 3212–3234. MR 3390149

[22] Yuk-J Leung and David Colton, *Complex transmission eigenvalues for spherically stratified media*, Inverse Problems 28 (2012), no. 7, 075005, 9. MR 2944956
[23] Hoai-Minh Nguyen, *Asymptotic behavior of solutions to the Helmholtz equations with sign changing coefficients*, Trans. Amer. Math. Soc. **367** (2015), no. 9, 6581–6595. MR 3356948

[24] ———, *Superlensing using complementary media*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), no. 2, 471–484. MR 3325246

[25] ———, *Cloaking via anomalous localized resonance for doubly complementary media in the quasistatic regime*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 6, 1327–1365. MR 3353803

[26] ———, *Cloaking using complementary media in the quasistatic regime*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 6, 1509–1518. MR 3569240

[27] ———, *Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients*, J. Math. Pures Appl. (9) **106** (2016), no. 2, 342–374. MR 3515306

[28] ———, *Cloaking an arbitrary object via anomalous localized resonance: the cloak is independent of the object*, SIAM J. Math. Anal. **49** (2017), no. 4, 3208–3232. MR 3689138

[29] ———, *Superlensing using complementary media and reflecting complementary media for electromagnetic waves*, Adv. Nonlinear Anal. **7** (2018), no. 4, 449–467. MR 3871415

[30] ———, *Cloaking using complementary media for electromagnetic waves*, ESAIM Control Optim. Calc. Var. **25** (2019), Art. 29, 19. MR 3990650

[31] ———, *Cloaking property of a plasmonic structure in doubly complementary media and three-sphere inequalities with partial data*, preprint, https://arxiv.org/abs/1912.00908.

[32] Hoai-Minh Nguyen and Quoc-Hung Nguyen, *Discreteness of interior transmission eigenvalues revisited*, Calc. Var. Partial Differential Equations **56** (2017), no. 2, Paper No. 51, 38. MR 3626617

[33] Hoai-Minh Nguyen and Swarnendu Sil, *Limiting absorption principle and well-posedness for the Maxwell equations with anisotropic sign changing coefficients*, Comm. Math. Phys. (2020), https://doi.org/10.1007/s00220-020-03805-1.

[34] Louis Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **13** (1959), 115–162. MR 109940

[35] Vesselin Petkov and Georgi Vodev, *Asymptotics of the number of the interior transmission eigenvalues*, J. Spectr. Theory **7** (2017), no. 1, 1–31. MR 3629406

[36] Ha Pham and Plamen Stefanov, *Weyl asymptotics of the transmission eigenvalues for a constant index of refraction*, Inverse Probl. Imaging **8** (2014), no. 3, 795–810. MR 3295945

[37] Luc Robbiano, *Spectral analysis of the interior transmission eigenvalue problem*, Inverse Problems **29** (2013), no. 10, 104001, 28. MR 3116196

[38] ———, *Counting function for interior transmission eigenvalues*, Math. Control Relat. Fields **6** (2016), no. 1, 167–183. MR 3448675

[39] John Sylvester, *Discreteness of transmission eigenvalues via upper triangular compact operators*, SIAM J. Math. Anal. **44** (2012), no. 1, 341–354. MR 2888291

[40] Georgi Vodev, *Transmission eigenvalue-free regions*, Comm. Math. Phys. **336** (2015), no. 3, 1141–1166. MR 3324140

[41] ———, *High-frequency approximation of the interior Dirichlet-to-Neumann map and applications to the transmission eigenvalues*, Anal. PDE **11** (2018), no. 1, 213–236. MR 3707296

[42] ———, *Parabolic transmission eigenvalue-free regions in the degenerate isotropic case*, Asymptot. Anal. **106** (2018), no. 3-4, 147–168. MR 3764153

[43] ———, *Improved parametrization in the glancing region for the interior Dirichlet-to-Neumann map*, Comm. Partial Differential Equations **44** (2019), no. 5, 367–396. MR 3948385

[44] David Vernon Widder, *The Laplace Transform*, Princeton Mathematical Series, v. 6, Princeton University Press, Princeton, N. J., 1941. MR 0005923

[45] Maciej Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, vol. 138, American Mathematical Society, Providence, RI, 2012. MR 2952218

(Hoai-Minh Nguyen) École Polytechnique Fédérale de Lausanne, EPFL, CAMA, Station 8, CH-1015 Lausanne, Switzerland.

E-mail address: hoai-minh.nguyen@epfl.ch

(Quoc-Hung Nguyen) Institute of Mathematical Sciences, ShanghaiTech University, 393 Middle Huaxia Road, Pudong, Shanghai, 201210, China.

E-mail address: qhnguyen@shanghaitech.edu.cn