Geometric Invariant Theory and Roth's Theorem

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Abstract

We present a new proof of Thue-Siegel-Roth's Theorem (and its more recent variants, such as those of Lang-Wirsing for number fields and that “with moving targets” of Vojta) as an application of Geometric Invariant Theory (GIT). Roth's Theorem is deduced from a general formula comparing the height of a semi-stable point and the height of its projection on the GIT quotient. In this setting, the role of the zero estimates appearing in the classical proof is played by the geometric semi-stability of the point to which we apply the formula.

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0 Introduction

In its original form, Roth’s Theorem states that given a real algebraic number \( \theta \in \mathbb{R} \) which is not rational and a real number \( \kappa > 2 \), there exist only finitely many rational numbers \( \frac{p}{q} \in \mathbb{Q} \) such that

\[
\left| \theta - \frac{p}{q} \right| < \frac{1}{|q|^\kappa}
\]

where \( p, q \) are coprime integers.

The general strategy to prove Roth’s Theorem stems back to the work of Thue. The main ingredient is the construction of an “auxiliary” polynomial in several variables \( f \) which vanishes at high order at \((\theta, \ldots, \theta)\): the crucial step is to prove that it does not vanish too much at some rational points which “approximate” \((\theta, \ldots, \theta)\).

The original argument of Roth (generalizing those of Thue, Siegel and Gel’fond) involves arithmetic considerations about the height of the rational approximations. On the other hand, in the work of Dyson — who proved an earlier version of Roth’s Theorem — the non-vanishing result (usually called “Dyson’s Lemma”) takes place over the complex numbers: being free from arithmetic constraints, it is said to be of geometric nature. The task to generalize Dyson’s Lemma from 2 to several variables was accomplished by Esnault-Viewheg [EV84]; afterwards Nakamaye [Nak99] was able to give a proof of it relying on a variant of Faltings’ Product Theorem and “elementary” concepts of intersection theory.

The advantage of having a geometric proof of Dyson’s Lemma was exploited by Bombieri in the remarkable paper [Bom82]: he showed that these methods lead to new effective results in diophantine approximation available before only through the linear forms of logarithms of Baker.

Using an arithmetic variant of the Product Theorem, Faltings and Wüstholz [FW94] gave a new proof of Schmidt’s Subspace Theorem, sensibly different from the original one. Let us remark that their Zero Lemma, as in Roth and Schmidt, is of arithmetic nature. Their proof involves a notion of semi-stability for filtered vector spaces (see also [Fal95]). The role played by semi-stability is anyway rather different from the one in the present paper: here it will collect all the geometric informations coming from Dyson’s Lemma (hence from the Product Theorem); in their paper it represents a combinatorial assumption that permits to perform an inductive step based on the Product Theorem.

The connections between Geometric Invariant Theory and Arakelov Geometry have been studied by several authors in the last twenty years (Burnol [Bur92], Bost [Bos94], Zhang [Zha94], Gasbarri [Gas00] and Chen [Che09]).

The application of these techniques to diophantine approximation was largely inspired by [Bos94], where Bost proves a lower bound for the height of cycles with semi-stable Chow point. Generalizing these arguments Gasbarri gave in [Gas00] a general a lower bound for the height of semi-stable points for a large class of representations. An explicit version of the latter has been then proved by Chen [Che09] by means of Classical Invariant Theory.

This paper is organized as follows.

In Section 1 we review some material concerning Roth’s Theorem and we state the main result of this paper (the Main Theorem, see Theorem 1.4). More precisely, we show that Roth’s Theorem with moving targets is a consequence of an effective statement (the Main Effective Lower Bound, see Theorem 1.3 and 1.5) and how this last result is obtained from the Main Theorem for a suitable choice of parameters.
In Section 2 we introduce the main tool of Geometric Invariant Theory (the Fundamental Formula, see Theorem 2.2) that we will apply to a specific “moduli problem” in order to get the Main Theorem: it is a formula relating the height of a semi-stable point with the height of its projection on the GIT quotient. In this general framework we will also state and prove a lower bound of the height on the quotient (see Theorem 2.4).

In Section 3 we introduce the situation of Geometric Invariant Theory that we are interested in. Admitting the semi-stability of the point that we introduce and some intermediate computations, we show that the Fundamental Formula translates into the Main Theorem.

These intermediate computations (upper bounds of the height and the instability measure of the point) will be developed in detail in Sections 4 and 5.

Finally, in Section 6, we show the semi-stability of the point, which is the crucial result in order to apply the Fundamental Formula. Our proof is based on the Higher Dimensional Dyson’s Lemma by Esnault-Viewheg-Nakamaye (Theorem 3.2). We will give an alternative proof in dimension 2 based on the classical constructions of wronskians.

Acknowledgements. — The results presented here are part of my doctoral thesis [Mac12] supervised by J.-B. Bost. It is a pleasure for me to thank him for his guidance and his steady encouragement. During the preparation of the present article I have been stimulated by discussions with several people: I warmly thank A. Chambert-Loir, C. Gasbarri and M. Nakamaye. This paper also benefited from the sharp advices of J. Fresán.

Conventions

We list here some convention and definitions that will be used throughout the paper.

0.0.1. — Since we will be interested on the action of \( \text{SL}_2 \) on the projective line \( \mathbb{P}^1 \), we cannot confuse the projective line and the dual one. If \( A \) is a ring and \( n \) is a positive integer we denote by \( A^n \) the dual of the \( A \)-module \( A^n \),

\[
A^n := \text{Hom}_A(A^n,A).
\]

With this notation the projective line \( \mathbb{P}^1_A \) over the ring \( A \) will be the \( A \)-scheme \( \mathbb{P}^1_A = \text{Proj}(\text{Sym} A^2) \).

0.0.2. — Let \( E \) be a finite dimensional complex vector space equipped with a hermitian norm \( \| \cdot \|_E \) and associated hermitian form \( \langle \cdot , \cdot \rangle_E \). Let \( r \) be a non-negative integer.

- On the \( r \)-th tensor power \( E^\otimes r \) we consider the hermitian norm \( \| \cdot \|_{E^\otimes r} \) associated to the hermitian form

\[
\langle v_1 \otimes \cdots \otimes v_r, w_1 \otimes \cdots \otimes w_r \rangle_{E^\otimes r} = \prod_{i=1}^r \langle v_i, w_i \rangle_E,
\]

where \( v_1, \ldots, v_r \) and \( w_1, \ldots, w_r \) are elements of \( E \).

- On the \( r \)-th symmetric power \( \text{Sym}^r E \) we consider the quotient norm \( \| \cdot \|_{\text{Sym}^r E} \) with respect to the canonical surjection \( E^{\otimes r} \to \text{Sym}^r E \). If \( e_1, \ldots, e_n \) denotes an orthonormal basis of \( E \), where \( n = \dim_C E \), for any \( n \)-tuple of non-negative integers \( (r_1, \ldots, r_n) \) such that \( r_1 + \cdots + r_n = r \) we have:

\[
\| e_1^{r_1} \cdots e_n^{r_n} \|_{\text{Sym}^r E} = \left( \frac{r}{r_1! \cdots r_n!} \right)^{-1/2}.
\]
This norm is hermitian and it is sub-multiplicative in the following sense: if \( f \in \text{Sym}^r E \) and \( g \in \text{Sym}^s E \) we have

\[
\|fg\|_{\text{Sym}^{r+s}} \leq \|f\|_{\text{Sym}^r E} \|g\|_{\text{Sym}^s E}.
\]

Let us also mention that the norm \( \| \cdot \|_{\text{Sym}^r E} \) is bigger than the sup-norm on the unit ball: for \( f \in \text{Sym}^r E \) we have

\[
\|f\|_{\sup} := \sup_{0 \neq x \in EV} \|x\|_{EV} \leq \|f\|_{\text{Sym}^r E}.
\]

- On the \( r \)-th external power \( \wedge^r E \) we consider the hermitian norm \( \| \cdot \|_{\wedge^r E} \) associated to the hermitian form

\[
\langle v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r \rangle_{\wedge^r E} = \det(\langle v_i, w_j \rangle_E : i, j = 1, \ldots, r)
\]

where \( v_1, \ldots, v_r \) and \( w_1, \ldots, w_r \) are elements of \( E \). With this notation Hadamard’s inequality reads:

\[
\|v_1 \wedge \cdots \wedge v_r\|_{\wedge^r E} \leq \prod_{i=1}^r \|v_i\|_{E}.
\]

The hermitian norm \( \| \cdot \|_{\wedge^r E} \) is not the quotient norm with respect to the canonical surjection \( E^{\otimes r} \rightarrow \wedge^r E \), but it is \( \sqrt{r!} \) times the quotient norm.

0.0.3. — Let \( K \) be a field complete with respect to a non-archimedean absolute value and let \( \sigma \) be its ring of integers. In order to do some computations it will be easier to interpret \( \sigma \)-modules as \( K \)-vector spaces endowed with a non-archimedean norm. More precisely, for any torsion free \( \sigma \)-module \( E \) let us denote by \( E := E \otimes_{\sigma} K \) its generic fiber and consider the following norm: for any \( v \in E \) we set

\[
\|v\|_E := \inf\{|\lambda| : \lambda \in K^*, v/\lambda \in E\}.
\]

The norm \( \| \cdot \|_E \) is non-archimedean and its construction is compatible with operations on \( \sigma \)-modules: for instance, if \( \varphi : E \rightarrow F \) is an injective (resp. surjective) homomorphism between torsion free \( \sigma \)-modules then the norm \( \| \cdot \|_E \) induced on \( E := E \otimes_{\sigma} K \) (resp. the norm \( \| \cdot \|_E \) induced on \( F := F \otimes_{\sigma} K \)) is the restriction of the norm \( \| \cdot \|_E \) on \( F \) (resp. is the quotient norm deduced from \( \| \cdot \|_E \) and \( \varphi \), that is, the norm defined by

\[
\|v\|_E := \inf_{\varphi(v) = w} \|v\|_E
\]

for any element \( w \) of \( F \).

It follows that, for a non-negative integer \( r \geq 0 \), the norm on symmetric powers \( \text{Sym}^r E \) (resp. on exterior powers \( \wedge^r E \)) is the norm deduced by the one on the \( r \)-th tensor power \( E^{\otimes r} \) through the canonical surjection \( E^{\otimes r} \rightarrow \text{Sym}^r E \) (resp. \( E^{\otimes r} \rightarrow \wedge^r E \)). In particular, it is sub-multiplicative (resp. Hadamard inequality holds).

0.0.4. — If \( K \) is a number field, we will denote by \( \sigma_K \) its ring of integers and by \( V_K \) the set of its places. If \( \nu \) is a place we will denote by \( K_\nu \) the completion of \( K \) with respect to \( \nu \) and by \( C_\nu \) the completion of an algebraic closure of \( K_\nu \). If \( \nu \) is an non-archimedean place extending a \( p \)-adic one, we will normalize it by

\[
|p|_\nu = p^{-[K_\nu : Q_p]}.
\]

0.0.5. — Let \( K \) be a number field, \( \sigma_K \) its ring of integers and \( V_K \) its set of places. An hermitian vector bundle \( \mathcal{E} \) is the data of a flat \( \sigma_K \)-module of finite type \( E \) and any complex embedding \( \sigma : K \rightarrow C \), a hermitian norm \( \| \cdot \|_{\mathcal{E},\sigma} \) on the complex vector space \( \mathcal{E}_\sigma := \mathcal{E} \otimes_{\sigma_K} C \). These hermitian norms are supposed to be compatible to complex conjugation. For any place \( \nu \in V_K \), we denote by \( \| \cdot \|_{\mathcal{E},\nu} \) the norm induced on the \( K_\nu \)-vector space \( \mathcal{E}_\nu := \mathcal{E} \otimes_{\sigma_K} K_\nu \).
If $L$ is a hermitian line bundle, that is a hermitian vector bundle of rank 1, we define its degree by
\[
\deg(L) := \log \#(L / s_L) - \sum_{\sigma : K \to C} \log \|s_{\sigma, L}\| = - \sum_{\nu \in V_K} \log \|s\|_{\nu, L},
\]
where $s \in L$ is non-zero. It appears clearly from the second the expression that this, according to the Product Formula, does not depend on the chosen section $s$. If $E$ is a hermitian vector bundle we define
- its degree:
\[
\deg(E) := \deg(\bigwedge^{rk(E)} E);
\]
- its slope:
\[
\theta(E) := \frac{\deg(E)}{rk(E)};
\]
- its maximal slope:
\[
\theta_{\max}(E) := \sup_{0 \neq F \subset E} \theta(F),
\]
where the supremum is taken on all non-zero sub-modules $F$ of $E$ endowed with the restriction of the hermitian metric on $E$.

With this notations the slope inequality says that for any hermitian vector bundles $E, F$ and any injective homomorphism of $K$-vector spaces $\varphi : E \otimes_{\mathcal{O}_K} K \to F \otimes_{\mathcal{O}_K} K$, we have
\[
\theta(E) \leq \theta_{\max}(F) + \sum_{\nu \in V_K} \log \|\varphi\|_{\nu, E},
\]
where for any place $\nu$ we set
\[
\|\varphi\|_{\nu, E} := \sup_{0 \neq s \in E} \frac{\|\varphi(s)\|_{\nu, F}}{\|s\|_{\nu, E}}.
\]

1 Statement of the results

1.1 Roth's Theorem with moving targets and the Main Effective Lower Bound

1.1.1. Height and distance on the projective line. — In order to state the results in their most precise way it is convenient to make the following definitions.

For a point $x = (x_0 : x_1)$ of the projective line $\mathbb{P}^1_K$ defined on a number field $K$ we consider its absolute (logarithmic) height
\[
h(x) = \frac{1}{[K : \mathbb{Q}]} \sum_{\nu \in V_K} \log \|(x_0, x_1)\|_{\nu},
\]
where $V_K$ denotes the set of places of $K$ and for any place $\nu$ we write
\[
\|(x_0, x_1)\|_{\nu} := \begin{cases} 
\max\{|x_0|_\nu, |x_1|_\nu\} & \text{if } \nu \text{ is non-archimedean} \\
\sqrt{|x_0|_\nu^2 + |x_1|_\nu^2} & \text{if } \nu \text{ is archimedean}.
\end{cases}
\]

If $K$ is a number field and $\nu \in V_K$ is a place of $K$, we consider the $\nu$-adic distance on $\mathbb{P}^1$. If $x = (x_0 : x_1)$ and $y = (y_0 : y_1)$ are $\mathbb{C}_\nu$-points of the projective line $\mathbb{P}^1$ we set
\[
d_{\nu}(x, y) := \frac{|x_0 y_1 - x_1 y_0|_\nu}{\|(x_0, x_1)\|_{\nu} \|(y_0, y_1)\|_{\nu}}.
\]
Moreover if \( x \) is a \( K \)-point of \( \mathbb{P}^1 \) and \( \theta \) is a point of \( \mathbb{P}^1 \) defined on a finite extension \( K' \) of \( K \) we set

\[
\nu_d(x, 0) := \min_{\sigma: K \to C} \nu_d(x, \sigma(\theta)).
\]

1.1.2. Roth’s Theorem with moving targets. — In this paper we are interested in the following form of Roth’s Theorem with moving targets (compare to [Voj96, Theorem 1]):

**Theorem 1.1.** Let \( K \) be a number field, \( S \subset \nu_K \) a finite subset and \( K' \) be a finite extension of \( K \) of degree \( \leq 2 \). For any real number \( \kappa > 2 \) there is no sequence \( \{(x_i, \theta_i)\}_{i \in \mathbb{N}} \) satisfying the following properties:

- for all \( i \in \mathbb{N} \), \( x_i \) is a \( K \)-rational point of \( \mathbb{P}^1 \);
- for all \( i \in \mathbb{N} \), \( \theta_i \) is a \( K' \)-rational point of \( \mathbb{P}^1 \) that is not \( K \)-rational;
- we have \( h(\theta_i) = o(h(x_i)) \) as \( i \) goes to infinity;
- for all \( i \in \mathbb{N} \) the following inequality is satisfied:

\[
-\sum_{v \in S} \nu_d(\theta_i, x_i) \geq \kappa h(x_i).
\]

Taking the points \( \theta_i \) to be all equal we find the following version of Roth’s Theorem due to Lang [BG06, Theorem 6.2.3]:

**Corollary 1.2.** Let \( K \) be a number field, \( \theta \) be an algebraic point of \( \mathbb{P}^1 \) of degree \( \geq 2 \) and \( S \subset \nu_K \) a finite subset. Then for any real number \( \kappa > 2 \), there exists only finitely many \( K \)-rational points \( x \) of \( \mathbb{P}^1 \) such that

\[
-\sum_{v \in S} \nu_d(\theta, x) \geq \kappa h(x).
\]

The original versions of Vojta and Lang of these results permit to take different algebraic points at any place \( v \in S \). An easy modification of our arguments would let us to find the same result, but we will not do it here for the sake of simplicity.

These results are not effective: one does not know an upper bound (only depending on the \( \theta_i \)'s) for the height of the rational points \( x_i \)'s well approximating the algebraic points \( \theta_i \)'s. Indeed, the proof of Roth’s Theorem (and its variants) follows a classical scheme that goes back to the work of Thue: its last step consists of an elementary absurd argument which is the principal cause of loss of effectiveness.

1.1.3. Main Effective Lower Bound. — Nevertheless there is an intermediate step in the proof of Roth’s Theorem which is effective and implies Roth’s Theorem through an elementary absurd argument that we will repeat in the next paragraph.

It is a lower bound of the height of \( K \)-rational points \( x_1, \ldots, x_n \) in terms of their \( v \)-adic distances from algebraic points \( \theta_1, \ldots, \theta_n \). Although this type of lower bounds plays a crucial role in the seminal work of Bombieri [Bom82], it is rarely stated as a stand-alone theorem.

We will call this lower bound “Main Effective Lower Bound” and the aim of this paper is to prove it by means of Geometric Invariant Theory. The statement of this result involves some auxiliary real numbers of geometric nature \( r_1, \ldots, r_n \): in the proof they will be interpreted as the multi-degree of an invertible sheaf on \( \mathbb{P}^1 \)^n.

**Theorem 1.3** (Main Effective Lower Bound). Let \( K \) be a number field and \( K' \) be a finite extension of \( K \) of degree \( \geq 2 \).

Then, there exists positive real numbers \( C_1, C_2, C_3 > 0 \) and, for any integer \( n \geq 2 \) and any real number \( 0 < \delta < 1/(2 \cdot n!) \), there exists a positive real number \( R_d(n, \delta) > 1 \) such that
• for all \(K\)'-rational points \(\theta_1, \ldots, \theta_n\) of \(\mathbb{P}^1\) generating the field \(K\),
• for all \(K\)-rational points \(x_1, \ldots, x_n\) of \(\mathbb{P}^1\)
• for any \(n\)-tuple of positive real numbers \(r = (r_1, \ldots, r_n)\) such that \(r_i/r_{i+1} > R_d(n, \delta)\) for any \(i\),

the following inequality is satisfied:

\[
t_d(n, \delta) \left( \sum_{v \in \mathcal{V}_K} \min_{i=1,\ldots,n} \left\{ -r_i \log d_v(\theta_i, x_i) \right\} \right) \leq \left( 1 + C_1 \sqrt[\delta]{2n} \right) \sum_{i=1}^n r_i h(x_i) + \frac{1}{\delta} \sum_{i=1}^n r_i (C_2 h(\theta_i) + C_3),
\]

where \(t_d(n, \delta) : [0, 1/(2 \cdot n!)] \to [0, n] \) is a continuous function such that

\[
\limsup_{\delta \to 0} \frac{n}{t(n, \delta)} = 2. \tag{1.1.1}
\]

The reason why we call this result “effective” is because the real numbers \(C_1, C_2, C_3\) can be explicitly computed. As we will see in a while, the equality (1.1.1) is where the number 2 in Roth’s Theorem comes from.

1.1.4. Deducing Roth’s Theorem from the Main Effective Lower Bound. — Let us show how the Main Effective Lower Bound (Theorem 1.3) implies Roth’s Theorem with moving targets (Theorem 1.1). Let \(K\) be a finite extension of \(K\) of degree \(d \geq 2\), \(S \subset \mathcal{V}_K\) a finite subset of places and \(\kappa > 2\) a real number.

By absurd, suppose that there exists a sequence \((x_i, \theta_i))_{i \in \mathbb{N}}\) verifying the conditions in the statement of Theorem 1.1. Replacing \(K\) by a sub-extension (different from \(K\), we may suppose that the points \(\theta_i\) generate \(K\) over \(K\).

By a pigeonhole argument (see for example [BG06, 6.4.2]), extracting a subsequence, we may suppose that for every real number \(\varepsilon > 0\) and every place \(v \in S\), there exists a positive real number \(\lambda(\varepsilon, v)\) such that, for any \(i \in \mathbb{N}\), we have

\[
-\log d_v(\theta_i, x_i) \geq \lambda(\varepsilon, v) \left( -\sum_{v \in S} \log d_v(\theta_i, x_i) \right)
\]

and

\[
\sum_{v \in S} \lambda(\varepsilon, v) \geq 1 - \varepsilon.
\]

Take an integer \(n \geq 2\), a positive real number \(\delta\) and \(n\)-tuple of positive real numbers \(r = (r_1, \ldots, r_n)\) satisfying the conditions in the statement of Theorem 1.3. Applying it to the \(K\)'-rational points \(\theta_1, \ldots, \theta_n\) and the \(K\)-rational points \(x_1, \ldots, x_n\), we obtain:

\[
\left( 1 + C_1 \sqrt[\delta]{2n} \right) \sum_{i=1}^n r_i h(x_i) + \frac{1}{\delta} \sum_{i=1}^n r_i (C_2 h(\theta_i) + C_3) \geq t_d(n, \delta) \left( \sum_{v \in \mathcal{V}_K} \min_{i=1,\ldots,n} \left\{ -r_i \log d_v(\theta_i, x_i) \right\} \right)
\]

\[
\geq t_d(n, \delta) (1 - \varepsilon) \left( \sum_{i=1,\ldots,n} \min_{v \in S} \left\{ -r_i \cdot \sum_{v \in S} \log d_v(\theta_i, x_i) \right\} \right).
\]

By hypothesis for all \(i = 1, \ldots, n\) we have \(-\sum_{v \in S} \log d_v(\theta_i, x_i) \geq \kappa h(x_i)\), so we get

\[
\kappa t_d(n, \delta) (1 - \varepsilon) \min_{i=1,\ldots,n} \{r_i \cdot h(x_i)\} \leq \left( 1 + C_1 \sqrt[\delta]{2n} \right) \sum_{i=1}^n r_i h(x_i) + \frac{1}{\delta} \sum_{i=1}^n r_i (C_2 h(\theta_i) + C_3).
\]
The key point is that, since we have infinitely many $x_i$, extracting a subsequence we may suppose that the ratios of the heights $h(x_{i+1})/h(x_i)$ are sufficiently big (namely bigger that $R_d(n,\delta)$) so that we can take $r$ such that

$$r_j h(x_j) = r_j h(x_j),$$

for any $i, j = 1, \ldots, n$. Dividing by $r_i h(x_i) = \min(r_i h(x_i))$ the preceding inequality, we get:

$$r_i \leq \left[1 + \frac{C_1}{\sqrt{\delta}}\right] n + \frac{1}{\delta} \sum_{i=1}^{n} \frac{h(\theta_i)}{h(x_i)}.$$  

According to the hypothesis $h(\theta_i) = o(h(x_i))$ for $i \to \infty$, extracting a subsequence we may suppose that $h(\theta_i) \leq \delta \sqrt{\delta} h(x_i)$. Moreover the ratios $r_i/r_{i+1}$ and the height $h(x_i)$ can be supposed arbitrarily big. We get then

$$\kappa t_d(n, \delta)(1 - \epsilon) \leq \left[1 + (C_1 + C_2) \sqrt{\delta}\right] n$$

and letting $\delta$ and $\epsilon$ go to 0 and $n$ go to infinity, according to (1.1.1) we find

$$\kappa \leq \limsup_{\delta \to 0, n \to \infty} \frac{n}{t(n, \delta)} = 2$$

which contradicts the hypothesis $\kappa > 2$. \hfill \square

### 1.2 Statement of the Main Theorem

#### 1.2.1 Some combinatorial data. — It is convenient to fix some notation about the combinatorics that will appear in the study. Let $n \geq 1$ be a positive integer. For any $n$-tuple of positive real numbers $r = (r_1, \ldots, r_n)$ and any non-negative real number $t \geq 0$ we consider the following subsets of $\mathbb{R}^n$:

$$\square_{n,r} := \{\zeta = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n : 0 \leq \xi_i \leq r_i \text{ for every } i = 1, \ldots, n\} = \prod_{i=1}^{n} [0, r_i]$$

$$\nabla_{n,r}(t) := \{\zeta = (\xi_1, \ldots, \xi_n) \in \square_{n,r} : \frac{\xi_1}{r_1} + \cdots + \frac{\xi_n}{r_n} \geq t\}$$

$$\Delta_{n,r}(t) := \{\zeta = (\xi_1, \ldots, \xi_n) \in \square_{n,r} : \frac{\xi_1}{r_1} + \cdots + \frac{\xi_n}{r_n} < t\} = \square_{n,r} - \nabla_{n,r}(t).$$

We will often use notation in the following two ways:

- when we write $r$ we will omit $n$;
- we will just write $n$ omitting $r$ if $r = (1, \ldots, 1)$.

We will add $\mathbb{Z}$ in subscript to denote the intersection of these subsets with $\mathbb{Z}^n$ (we will write $\nabla_r(t)_{\mathbb{Z}}$ and $\Delta_r(t)_{\mathbb{Z}}$).

Let us introduce some further quantities that will appear in the statement. For any positive integer $n \geq 1$ and any $n$-tuple of positive real numbers $r = (r_1, \ldots, r_n)$ we will write

$$\epsilon_n(r) := \prod_{i=1}^{n-1} \left(1 + \max_{i+1 \leq j \leq n} \frac{r_j}{r_i} \right) (d - 1) - 1,$$

omitting the dependence on $d$. For any non-negative real number $t$, let us denote by $u_t(t)$ the unique real number in $[0, n]$ such that

$$\text{vol}\Delta_n(u_t(t)) = \min \{\max \{1 + \epsilon_n(r) - d \text{vol}\Delta_n(t), 0\}, 1\}.$$
1.2.2. Main Theorem. — Keeping the notation just introduced, the main result of the present paper is the following:

**Theorem 1.4** (Main Theorem). Let $K'$ be a finite extension of $K$ of degree $d \geq 2$. Then we have:

- for all $K'$-rational points $0_1, \ldots, 0_n$ of $P^1$ such that $K(0_i) = K'$ for any $i = 1, \ldots, n$;
- for all $K$-rational points $x_1, \ldots, x_n$ of $P^1$;
- for any $n$-tuple of positive real numbers $r = (r_1, \ldots, r_n)$;
- for any non-negative real numbers $t_0, t_x$ satisfying the condition

$$\left| \text{vol} \nabla_n(t_x) - 2 \int_{\nabla_n(t_x)} \xi_1 \, d\lambda \right| < \left| \text{vol} \nabla_n(u_r(t_0)) - 2 \int_{\nabla_n(u_r(t_0))} \xi_1 \, d\lambda \right| - \varepsilon_n(r); \quad (1.2.1)$$

the following inequality holds:

$$t_0 \cdot (1 - d \text{vol} \Delta_n(t_0)) \left\{ \sum_{i \in V_{K} = 1, \ldots, n} \min \{ -r_i \log d_v(\theta_i, x_i) \} \right\}$$

$$+ \left( \int_{\nabla_n(t_x)} \xi_1 \, d\lambda - \frac{\text{vol} \Delta_n(u_r(t_0)) + \text{vol} \nabla_n(t_x)}{2} \sum_{i = 1}^{n} \sum_{j \in V_{K}} -r_j \log d_v(\theta_i, x_i) \right)$$

$$\leq \left( \int_{\nabla_n(t_x)} \xi_1 \, d\lambda \right) \sum_{i = 1}^{n} r_i h(x_i) + d \sum_{i = 1}^{n} r_i h(\theta_i) + |r| C(t_0, t_x),$$

where

$$C(t_0, t_x) := 5(\text{vol} \Delta_n(u_r(t_0)) + \text{vol} \nabla_n(t_x)) + \log \sqrt{2d}.$$  

1.3 From the Main Theorem to the Main Effective Lower Bound

1.3.1. — We consider for any real number $\delta \in [0, 1]$ and positive integer $d \geq 1$, the unique real number $t_d(n, \delta) \in [0, n]$ such that

$$1 - d \text{vol} \Delta_n(t_d(n, \delta)) = \delta.$$  

This is the crucial concept that governs the combinatorics in Roth’s Theorem. The function defined in this way $t_d(n, \cdot) : [0, 1] \rightarrow [0, n]$ is continuous and as we already claimed in (1.1.1) we have:

$$\lim_{\delta \rightarrow 0} \sup_{n} \frac{n}{t_d(n, \delta)} = \lim_{n \rightarrow \infty} \frac{n}{t_d(n, 0)} = 2.$$  

This equality goes back to the work of Roth and it is based on an explicit version of a phenomenon of concentration of measure (see [Mil88]). Indeed, one usually proves that for any $0 \leq \varepsilon \leq 1/2$ we have (see [BG06, Lemma 6.3.5] for a proof):

$$\text{vol} \Delta_n \left( \left( \frac{1}{2} - \varepsilon \right) n \right) \leq \exp(-6n\varepsilon^2).$$  

Taking $\varepsilon := 1/2 - t_d(n)/n$ one finally deduces

$$t_d(n, 0) \geq \frac{n}{2} - \sqrt{\frac{n \log d}{6}},$$  

which in particular describes the limit behaviour of the function $n/t_d(n, 0)$ as $n$ goes to infinity.
Theorem 1.5 (Main Effective Lower Bound). Let $K'$ be a finite extension of $K$ of degree $d \geq 2$. Then,

- for any integer $n \geq 2$ and any positive real number $\delta < 1/(2\cdot n!)$;
- for all $K'$-rational points $\theta_1, \ldots, \theta_n$ of $\mathbb{P}^1$ such that $K(\theta_i) = K'$ for any $i = 1, \ldots, n$;
- for all $K$-rational points $x_1, \ldots, x_n$ of $\mathbb{P}^1$;
- for any $n$-tuple of positive real numbers $r = (r_1, \ldots, r_n)$ such that $\varepsilon_n(r) < \delta \sqrt{n}$,

the following inequality holds:

$$t_d(n, \delta) \left( \sum_{\sigma \in \mathcal{V}_K} \min_{i=1, \ldots, n} \{-r_i \log d_{\sigma}(\theta_i, x_i)\} \right) \leq \left( 1 + (4 + d) \sqrt{n} \right) \sum_{i=1}^{n} r_i \log h(\theta_i) + d \left( \frac{1}{\delta} + \sqrt{n} \right) \sum_{i=1}^{n} r_i \log h(\theta_i)$$

$$+ |r| \left( \frac{\log \sqrt{2d}}{\delta} + 5 \left( 2 + \sqrt{n} \right) \right).$$

1.3.2. — Before passing to the proof, let us remark that this is a refined version of Theorem 1.3. Indeed, one may roughly bound $\delta \sqrt{n}$ by $1/8$ and take the constants $C_1, C_2, C_3$ appearing in the statement of Theorem 1.3 as:

$$C_1 = 4 + d$$

$$C_2 = \frac{9}{8} d$$

$$C_3 = \log \sqrt{2d} + 5 \left( 2 + \frac{1}{8} \right).$$

It is probably more interesting to remark that the leading terms on the right-hand side (where $\delta$ appears in the denominator) are those that will come from the upper bound of the height of the point $[K_r(\theta, t_0)]$ (which classically corresponds to the bound for the coefficients of the auxiliary polynomial — see Section 4.2). To get rid of $\delta$ in the denominator (and hence have a sensible improvement of the Main Effective Lower Bound) one would have to find an upper bound of the height of the point $[K_r(\theta, t_0)]$ which is linear in the dimension of $K_r(\theta, t_0)$ (rather than its codimension in $\Gamma(\mathbb{P}, \mathcal{O}_\mathbb{P}(r))$).

Let us also point out that in the statement of the Theorem 1.3 the ratios of the real numbers $r_i/r_{i+1}$ are supposed all to be bigger than a real number $R_d(n, \delta)$ depending on $n$, $\delta$ and $d$. It suffices to take $R_d(n, \delta) \geq 1$ such that

$$\left( 1 + \frac{d - 1}{R_d(n, \delta)} \right)^n - 1 < \delta \sqrt{n}.$$

1.3.3. — The Main Effective Lower Bound that we stated here is deduced from Theorem 1.4 making the following choices for $t_0$ and $t_s$:

- $t_0 = t(n, \delta)$;
- Consider the unique real number $w(n, \delta) \in [n - 1, n]$ such that

$$\left| \text{vol}_{\mathcal{V}_n}(w(n, \delta)) - 2 \int_{\mathcal{V}_n(w(n, \delta))} \zeta_1 \, d\lambda \right| = \left| \text{vol}_{\mathcal{V}_n}(u(n, \delta)) - 2 \int_{\mathcal{V}_n(u(n, \delta))} \zeta_1 \, d\lambda \right| - \varepsilon_n(r),$$

where we wrote $u(n, \delta)$ instead of $u(t(n, \delta))$. We will take

$$t_s \in [w(n, \delta), n]$$

and then letting it tend to $w(n, \delta)$. 

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To verify that these choices make sense one has to check that condition (1.2.1) in Theorem 3.4 holds for $t_x > u(n, \delta)$. To begin with let us remark that by definition of $t(n, \delta)$ and $u(n, \delta)$ we have

$$\text{vol} \Delta_n(u(n, \delta)) = 1 + d \text{vol} \Delta_n(t(n, \delta)) + \varepsilon_n(r) = \delta + \varepsilon_n(r).$$

Hence, since we supposed $\delta < 1/(n+1)!$ and $\varepsilon_n(r) < \delta \sqrt{n}$ we have $u(n, \delta) \leq 1$. This entails

$$u(n, \delta) = \sqrt[n]{\delta / (\delta + \varepsilon_n(r))}.$$ 

Now to show that the condition (1.2.1) is satisfied it suffices to show $w(n, \delta) < n$, or equivalently

$$\text{vol} \Delta_n(u(n, \delta)) - 2 \int_{\Delta_n(u(n, \delta))} \zeta_1 d\lambda - \varepsilon_n(r) > 0.$$ 

Since everything here is explicit$^2$ and since this easy computation is not particularly illuminating, we leave this to the reader.

1.3.4. — Now applying Theorem 1.4 with $t_0 = t(n, \delta)$ and $t_x > w(n, \delta)$ as announced before and then letting $t_x$ tend to $w(n, \delta)$ we find:

$$t(n, \delta) \cdot (1 - d \text{vol} \Delta_n(t(n, \delta))) \left( \sum_{i \in V_k} \min_{i = 1, \ldots, n} \{-r_i \log d(v(\theta_i, x_i))\} \right)$$

$$+ \int_{\mathcal{V}} \zeta_1 d\lambda - \frac{\text{vol} \Delta_n(u(n, \delta)) + \text{vol} \nabla_n(w(n, \delta))}{2} \sum_{i = 1}^{n} \sum_{i \in V_k} -r_i \log d(v(\theta_i, x_i))$$

$$\leq \left( \int_{\mathcal{V}} \zeta_1 d\lambda \right) \sum_{i = 1}^{n} r_i h(x_i) + d \sum_{i = 1}^{n} r_i h(\theta_i) + |r| C(n, \delta),$$

where we set

$$C(n, \delta) := 5 \left( \text{vol} \Delta_n(u(n, \delta)) + \text{vol} \nabla_n(w(n, \delta)) \right) + \log 2d.$$ 

1.3.5. — By definition of $t(n, \delta)$ we have $1 - d \text{vol} \Delta_n(t(n, \delta)) = \delta$. Going back to the definition of $u(n, \delta)$ and $w(n, \delta)$ and performing some elementary estimates we find:

- $C(n, \delta) \leq 5 \delta (2 + \sqrt{d}) + \log \sqrt{2d}$
- $\int_{\mathcal{V}} \zeta_1 d\lambda - \frac{\text{vol} \Delta_n(u(n, \delta)) + \text{vol} \nabla_n(w(n, \delta))}{2} = \int_{\Delta_n(u(n, \delta))} \zeta_1 d\lambda + \frac{\varepsilon_n(r)}{2} \leq \delta \sqrt{d}$
- $\int_{\mathcal{V}} \zeta_1 d\lambda \leq \left( 1 + 4 \sqrt{d} \right) \delta$.

So we are left with

$$t(n, \delta) \left( \sum_{i \in V_k} \min_{i = 1, \ldots, n} \{-r_i \log d(v(\theta_i, x_i))\} \right) + \delta \sqrt{d} \sum_{i = 1}^{n} \sum_{i \in V_k} -r_i \log d(v(\theta_i, x_i))$$

$$\leq \delta \left( 1 + 4 \sqrt{d} \right) \sum_{i = 1}^{n} r_i h(x_i) + d \sum_{i = 1}^{n} r_i h(\theta_i) + |r| \left( 5 \delta (2 + \sqrt{d}) + \log \sqrt{2d} \right).$$

$^1$In fact for any real number $0 \leq u \leq 1$ we have

$$\text{vol} \nabla_n(u) - 2 \int_{\mathcal{V}} \zeta_1 d\lambda = 2 \int_{\Delta_n(u)} \zeta_1 d\lambda - \text{vol} \Delta_n(u) = 2 \frac{u^{n+1}}{(n+1)!} - \frac{u^n}{n!} \leq 0.$$

$^2$Indeed, since $u(n, \delta) \leq 1$ we have

$$\int_{\Delta_n(u(n, \delta))} \zeta_1 d\lambda = \frac{u(n, \delta)^{n+1}}{(n+1)!}.$$ 

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1.3.6. — This inequality is sufficient to imply the Roth-Vojta’s Theorem, but to get a nicer formulation we want to get rid of the term \( \sum_{v \in V_k} -r_i \log d_v(\theta_i, x_i) \). To do this, we will bound it by means of Liouville’s inequality (as proved, for example, in [BG06, Theorem 2.8.21]): for any \( K' \)-point \( \omega \) of \( P^1 \) generating \( K' \) and any \( K \)-point \( y \) of \( P^1 \) we have

\[
- \sum_{v \in V_k} \log d_v(\omega, y) \leq d h(y) + d h(\omega).
\]

The preceding inequality becomes

\[
t(n, \delta) \delta \left( \sum_{v \in V_k} \min_{i=1,...,n} \{-r_i \log d_v(\theta_i, x_i)\} \right) \leq \delta \left( 1 + (4+\delta) \sqrt[2d]{\delta} \right)^n \sum_{i=1}^n r_i h(\theta_i) + d \left( 1 + \delta \sqrt[2d]{\delta} \right)^n \sum_{i=1}^n r_i h(\theta_i) + \left| r \right| \left( 5\delta(2+\sqrt[2d]{\delta}) + \log \sqrt[2d]{\delta} \right)
\]

and we obtain the statement of the Main Effective Lower Bound dividing by \( \delta \).

2 Geometric Invariant Theory and Arakelov Geometry

2.1 The Fundamental Formula

Let \( K \) be a number field and \( o_K \) its ring of integers.

2.1.1. — Let \( X \) be a projective and flat \( o_K \)-scheme endowed with the action of an \( o_K \)-reductive group \( G \) and let \( L \) be a \( G \)-linearised ample invertible sheaf on \( X \). To complete the “arakelovian” data we suppose that for any embedding \( \sigma : K \rightarrow \mathbb{C} \) we are given a continuous metric \( | \cdot |_{L, \sigma} \) on \( L \) satisfying these properties :

- \( | \cdot |_{L, \sigma} \) is semi-positive: for any analytic open set \( U \subset X_{\sigma}(\mathbb{C}) \) and any section \( s \in \Gamma(U, L) \) the function \( -\log |s|_{L, \sigma} \) is plurisubharmonic;
- \( | \cdot |_{L, \sigma} \) is invariant under the action of a maximal compact subgroup of \( \sigma(\mathbb{C}) \).

Clearly we suppose that the data is compatible under complex conjugation. We denote by \( \mathcal{L} \) the resulting hermitian line bundle on \( X \).

2.1.2. — Let us consider the \( o_K \)-graded algebra of finite type

\[
\mathcal{A} := \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}\ast d).
\]

By a theorem of Seshadri [Ses77, II.4, Theorem 4], we know that the graded algebra

\[
\mathcal{A}^G := \bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}\ast d)^G
\]

3Let \( S \) be a scheme. A \( G \)-group scheme \( G \) is said to be reductive (or \( G \) is a \( S \)-reductive group) if it satisfies the following conditions:
1. \( G \) is affine, smooth and of finite type over \( S \);
2. for all \( s \in S \), the \( T \)-group scheme \( G_s := G \times_S T \) is a connected reductive algebraic group (where \( T \) is the spectrum of an algebraic closure of the residue field \( \kappa(s) \)).
of \( \mathcal{A} \)-invariants of \( \mathcal{A} \) is a \( \mathfrak{k} \)-algebra of finite type. Moreover, the projective scheme \( \mathcal{V} := \text{Proj}(\mathcal{A}^e) \) is the categorical quotient of the open subset of semi-stable points\(^4\) \( \mathcal{X}^{ss} \) of \( \mathcal{X} \) (with respect to the action of reductive group \( \mathcal{G} \) and the invertible sheaf \( \mathcal{L}^e \)). Let us denote \( \pi : \mathcal{X}^{ss} \to \mathcal{V} \) the quotient morphism.

Since \( \mathcal{V} \) is of finite type, for any sufficiently divisible integer \( D \geq 1 \), there exists an ample invertible sheaf \( \mathcal{M}_D \) on \( \mathcal{V} \) and an isomorphism of invertible sheaves

\[ \varphi_D : \pi^* \mathcal{M}_D \to \mathcal{L}^{eD} \mid \mathcal{X}^{ss} \mid \]

**2.1.3.** — For any embedding \( \sigma : K \to \mathbb{C} \) let us endow the invertible sheaf \( \mathcal{M}_D \) with a metric \( \| \|_{\mathcal{L}^e_\sigma} \) defined as follows. Take \( \in \mathcal{V}(\mathbb{C}) \) and \( t \in y^* \mathcal{M}_D \), then define

\[ \| t \|_{\mathcal{M}_D, \sigma}(y) := \sup_{\pi(s) = y} \| \varphi_D(t) \|_{\mathcal{L}^{eD}, \sigma}(x). \]

It is easy to see that this defines a metric on \( \mathcal{M}_D \) but it is not clear a priori that this metric is continuous.

**Theorem 2.1.** Under the assumptions on the metric \( \| \|_{\mathcal{L}^e_\sigma} \) made above (semi-positivity and invariance under the action of maximal compact subgroup), the metric \( \| \|_{\mathcal{M}_D, \sigma} \) is continuous.

When the metric \( \| \|_{\mathcal{L}^e_\sigma} \) is the restriction of the Fubini-Study metric on a projective space, this follows directly from the arguments of Kempf-Ness [KN79] (see also Kirwan [GIT, Chapter 8, §2], Burnol [Bur92], Schwarz [Sch00, Chapter 5] and Azad-Loeb [AL93]). The general statement can be proved replacing the convexity properties of the "special functions" of Kempf and Ness by elementary convexity properties of (pluri)subharmonic functions (namely the fact that a real function \( u : \mathbb{R} \to \mathbb{R} \) is convex if and only if \( u \circ \log |z| : \mathbb{C}^* \to \mathbb{R} \) is subharmonic — see [Mac12, Théorème II.2.18] for details).

In order to prove the Main Effective Lower Bound we will consider a metric that is a restriction of a Fubini-Study metric (see 3.1.6 below), so we will not prove this statement.

**2.1.4.** — We denote by \( \mathcal{I}_D \) the associated hermitian invertible sheaf on \( \mathcal{V} \). Since the underlying invertible sheaf \( \mathcal{M}_D \) is ample and the metrics are continuous, the height of the \( \mathcal{K} \)-points of \( \mathcal{V} \) is a priori bounded below by a real number (where \( \mathbb{K} \) is an algebraic closure of \( K \)). Let us denote it by

\[ h_{\text{min}}(\mathcal{X}, \mathcal{A})(//\mathcal{G}) := \frac{1}{D} \inf_{y \in \mathcal{V}(\mathbb{K})} \| \mathcal{I}_D(y) \| \]

(which is clearly independent of \( D \)).

**2.1.5. Instability measure.** — Let \( \nu \) be a place of \( K \). If \( \nu \) is non-archimedean we denote by \( \| \|_{\mathcal{X}^\nu} \) the continuous and bounded metric induced by the integral model \( \mathcal{L}^2 \). For a \( \mathbb{C}^\nu \)-point \( P \) of \( \mathcal{X} \) we define the instability measure \( t_\nu(P) \) as follows:

\[ t_\nu(P) = - \log \sup_{g \in \pi(G)} \frac{\| g \cdot s \|_{\mathcal{X}^\nu}(g \cdot P)}{\| s \|_{\mathcal{X}^\nu}(P)} \]

\(^4\)Namely, the open set where the rational map \( \pi : \mathcal{X} \to \text{Proj}(\mathcal{A}) \to \mathcal{V} = \text{Proj}(\mathcal{A}^e) \), given by the inclusion \( \mathcal{A}^e \subset \mathcal{A} \), is defined. Equivalently, a point \( x \in \mathcal{X} \) is semi-stable if and only if there exists, for a sufficiently big \( d \geq 1 \), a \( \mathcal{G} \)-invariant global section \( s \in \mathcal{I}(\mathcal{X}, \mathcal{L}^{ed}) \) that does not vanish at \( x \).

\(^5\)Let \( x \) be a \( \mathbb{C}^\nu \)-point of \( \mathcal{X} \). Since \( \mathcal{X} \) is proper, the \( \mathbb{C}^\nu \)-point \( x \) gives rise to a \( \mathfrak{p} \)-point \( \varepsilon_x \) of \( \mathcal{X} \), where \( \mathfrak{p} = \mathbb{Z} \) is the ring of integers of \( \mathbb{C} \). The invertible sheaf \( \varepsilon_x^* \mathcal{L} \) is a free \( \mathfrak{p} \)-module of rank \( 1 \) : choose a basis \( s \). Then any other element \( t \in x^* \mathcal{L} \) can be written in a unique way as \( t = \lambda s \) with \( \lambda \in \mathbb{C} \) and we set

\[ \| t \|_{\mathcal{X}^\nu} = |\lambda|_\nu. \]

Clearly this does not depend on the chosen basis \( s \) of \( \varepsilon_x^* \mathcal{L} \).
where $s \in P^* \mathcal{L}$ is a non-zero section. Clearly this does not depend on the chosen section $s$. The instability measure $\iota_v$ takes negative values and takes the value $-\infty$ precisely when the point $P$ is not semi-stable.

If $v$ is non-archimedean let us consider the continuous and bounded metric $\| \cdot \|_{\mathcal{O}_v,v}$ induced by the integral model $\mathcal{M}_D$. Following an argument of Burnol [Bur92, p. 122] — see also [Mac12, Corollaire II.2.21] — one can show similarly to the archimedean case that for any $Q \in \mathcal{V}(C_v)$ and $t \in Q^* \mathcal{M}_D$ we have

$$\| t \|_{\mathcal{O}_v,v}(Q) = \sup_{\pi(P) = Q} \| \pi^* t \|_{\mathcal{L}_\pi \mathcal{O}_v,v}(P).$$

However, even without the previous consideration, for any archimedean and non-archimedean place $v$ and any $P \in \mathcal{X}(C_v)$, the $\mathcal{G}$-invariance of $\pi$ gives:

$$-\frac{1}{D} \log \frac{\| t \|_{\mathcal{O}_v,v}(\pi(P))}{\| \pi^* t \|_{\mathcal{L}_\pi \mathcal{O}_v,v}(P)} \leq \iota_v(P).$$

### 2.1.6. Fundamental formula. — Summing up these considerations we obtain:

**Theorem 2.2 (Fundamental Formula).** Let $P \in \mathcal{X}(K)$ be a semi-stable point. Then for almost places $v \in V_K$ the instability measure $\iota_v(P)$ is zero and we have the inequality:

$$h_{\mathcal{F}}(P) + \frac{1}{[K : \mathbb{Q}]} \sum_{v \in V_K} \iota_v(P) \geq \frac{1}{D} h_{\mathcal{M}_D}(\pi(P)).$$

As mentioned before one can prove that this is actually an equality (that is why we name this Theorem as "Fundamental Formula" — see [Mac12, Scholie III.2.2]) but we will not need this. In practice we will use Theorem 2.2 through this Corollary:

**Corollary 2.3.** For any semi-stable point $P \in \mathcal{X}_{ss}(K)$ we have

$$h_{\mathcal{F}}(P) + \frac{1}{[K : \mathbb{Q}]} \sum_{v \in V_K} \iota_v(P) \geq h_{\min} \left( (\mathcal{X}, \mathcal{F})/\mathcal{G} \right).$$

### 2.2 Lower bound of the height on the quotient

#### 2.2.1. Statement of the lower bound. — We will need a simple lower bound for the height on the quotient in the case of a product of linear groups. Is it an explicit version of the lower bound for semi-stable points already studied by Bost [Bos94, Theorem 1] and Gasbarri [Gas00, Theorem 1]. A variant of this lower bound has been proven by means of similar techniques by Chen [Che09, Theorem 4.2].

Let $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$ be a $n$-tuple of $\mathcal{O}_K$-hermitian vector bundles of positive ranks. Suppose we are given a $\mathcal{O}_K$-hermitian vector bundle $\mathcal{F}$ and a representation

$$\rho : \text{GL}(\mathcal{E}) := \text{GL}(\mathcal{E}_1) \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \text{GL}(\mathcal{E}_n) \rightarrow \text{GL}(\mathcal{F}),$$

i.e. a morphism of $\mathcal{O}_K$-group schemes which is **unitary**, i.e., for every embedding $\sigma : K \rightarrow \mathbb{C}$, the action of the compact subgroup

$$U(\mathcal{E}_\sigma) := U(\| \cdot \|_{\mathcal{E}_1,\sigma}) \times \cdots \times U(\| \cdot \|_{\mathcal{E}_n,\sigma}) \subset \text{GL}(d_\sigma)$$

respects the hermitian norm $\| \cdot \|_{\mathcal{F},\sigma}$ (or equivalently we have $\rho(U(\mathcal{E}_\sigma)) \subset U(\| \cdot \|_{\mathcal{F},\sigma}))$.  

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Theorem 2.4. With the notation introduced above, suppose moreover that we have a homomorphism of hermitian vector bundles
\[ \varphi : \mathcal{E}_1^\otimes a_1 \otimes \cdots \otimes \mathcal{E}_n^\otimes a_n \longrightarrow \mathcal{F} \]
generically surjective and \( \text{GL}(\mathcal{E}) \)-equivariant. Then,
\[ h_{\min}\left( (\mathcal{P}(\mathcal{F}), \mathcal{O}_\mathcal{F}(1)) \# \text{SL}(\mathcal{E}) \right) \geq \sum_{i=1}^n a_i \mu(\mathcal{E}_i) - \frac{1}{2} \log \text{rk} \mathcal{E}_i, \]
where \( \mathcal{O}_\mathcal{F}(1) \) is equipped with the natural Fubiny-Study metric given by \( \mathcal{F} \).

Invariant theory for a product of linear groups

2.2.2. — Let \( k \) be a field. Let \( n \geq 1 \) be a positive integer and \( E = (E_1, \ldots, E_n) \) a \( n \)-tuple of non-zero \( k \)-vector spaces of finite dimension. We define
\[
\text{GL}(E) := \text{GL}(E_1) \times_k \cdots \times_k \text{GL}(E_n)
\]
\[
\text{SL}(E) := \text{SL}(E_1) \times_k \cdots \times_k \text{SL}(E_n).
\]

Definition 2.5. Let \( F \) be a non-zero \( k \)-vector space of finite dimension. A representation, \( \text{i.e.} \) a morphism of \( k \)-group schemes, \( \rho : \text{GL}(E) \to \text{GL}(F) \) is said to be homogeneous of weight \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) if, for every \( k \)-scheme \( S \) and all \( S \)-points \( t_1, \ldots, t_n \in G_{n}(S) \), we have
\[ \rho(t_1 \cdot \text{id}, \ldots, t_n \cdot \text{id}) = t_1^{a_1} \cdots t_n^{a_n} \cdot \text{id}_F. \]

The fact that the characters of the general linear group are powers of the determinant gives the following basic fact concerning homogeneous representations. Let \( \rho : \text{GL}(E) \to \text{GL}(F) \) be a homogeneous representation of weight \( a = (a_1, \ldots, a_n) \) and suppose that the subspace of invariant of \( F \) is non-trivial. Then:

- for any \( i = 1, \ldots, n \) the dimension \( e_i \) of \( E_i \) divides the integer \( a_i \);
- for any \( k \)-scheme \( S \), any \( S \)-point \( (g_1, \ldots, g_n) \) of \( \text{GL}(E) \) and any \( \text{GL}(E) \)-invariant element \( \nu \) of \( F \) we have:
\[
\rho(g_1, \ldots, g_n) \cdot \nu = \det(g_1)^{a_1/e_1} \cdots \det(g_n)^{a_n/e_n} \cdot \nu. \quad (2.2.1)
\]

2.2.3. — For any non-negative integer \( N \) let us denote by \( \Sigma_N \) the group of permutations on \( N \) elements
(\( \text{if} N = 0, \text{then} \Sigma_0 = \{ \text{id}_0 \} \)). If \( V \) is a \( k \)-vector space the group \( \Sigma_N \) acts on the \( N \)-th tensor product \( V^\otimes N \) permuting factors. Explicitly, if \( \sigma \in \Sigma_N \) is a permutation and \( v_1, \ldots, v_N \) are elements of \( V \) we have
\[ \sigma \cdot (v_1 \otimes \cdots \otimes v_N) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)}. \]

2.2.4. — Let \( a = (a_1, \ldots, a_n) \) be a \( n \)-tuple of integers. The \( k \)-group scheme \( \text{GL}(E) \) acts naturally on the \( k \)-vector space
\[ F_a = \text{End}_k(E_1^\otimes a_1) \otimes_k \cdots \otimes_k \text{End}_k(E_n^\otimes a_n) \]
and the associated representation \( \text{GL}(E) \to \text{GL}(F_a) \) is homogeneouse of weight \( 0 = (0, \ldots, 0) \). In particular the relation (2.2.1) says that the invariants of \( F_a \) with respect to the action of \( \text{GL}(E) \) and to the action of \( \text{SL}(E) \) are the same.

The group \( \Sigma_{a_1} \times \cdots \times \Sigma_{a_n} \) acts component-wise on the \( k \)-vector space \( E_1^\otimes a_1 \otimes \cdots \otimes E_n^\otimes a_n \). This gives a natural homomorphism of non-commutative \( k \)-algebras
\[ \varepsilon : \bigotimes_{i=1}^n k[\Sigma_{a_i}] \longrightarrow F_a = \bigotimes_{i=1}^n \text{End}_k(E_i^\otimes a_i). \]
Clearly the image of $\varepsilon$ is contained in the subspace of invariants of $F_a$. The First Main Theorem of Invariant Theory affirms that in characteristic 0 the converse inclusion holds too:

**Theorem 2.6 (First Main Theorem of Invariant Theory).** Suppose that the characteristic of $k$ is zero. Then the subspace of invariants of the $k$-vector space 

$$F_a = \bigotimes_{i=1}^{n} \text{End}_k(E \otimes a_i)$$

with respect to the natural action of $\text{SL}(E)$ (or equivalently of $\text{GL}(E)$) is the image of the natural application 

$$\varepsilon: \bigotimes_{i=1}^{n} k[\mathcal{S}_{|a_i|}] \rightarrow F_a = \bigotimes_{i=1}^{n} \text{End}_k(E \otimes a_i).$$

**2.2.5.** Suppose that the characteristic of $k$ is zero. Then, if $V$ and $W$ are linear representations of a $k$-reductive group $G$, a surjective $G$-equivariant homomorphism of $k$-vector spaces $\phi: V \rightarrow W$ the homomorphism induces a surjective homomorphism on the subspaces of $G$-invariants $\phi^*: V^G \rightarrow W^G$. This follows directly from the functoriality of the projection on the invariants.

Combining this remark with the Main First Theorem of Invariant Theory, we get immediately:

**Corollary 2.7.** Suppose that the characteristic of $k$ is zero. Let $F$ be a non-zero $k$-vector space of finite dimension and $\rho: \text{GL}(E) \rightarrow \text{GL}(F)$ be a representation. Let $a = (a_1, \ldots, a_n)$ be an $n$-tuple of integers and 

$$\phi: F_a = \bigotimes_{i=1}^{n} \text{End}_k(E \otimes a_i) \rightarrow F$$

be a surjective and $\text{GL}(E)$-equivariant homomorphism of $k$-vector spaces. The representation $\phi$ is homogeneous of weight $a = (a_1, \ldots, a_n)$ and if the subspace of $\text{SL}(E)$-invariants is non-zero we have:

- for any $i = 1, \ldots, n$ the dimension $e_i$ of $E_i$ divides the integer $a_i$;
- the subspace of $\text{SL}(E)$-invariants of $F$ is the image of the homomorphism

$$\bigotimes_{i=1}^{n} k[\mathcal{S}_{|a_i|}] \otimes \text{det}(E_i)^{a_i/e_i} \xrightarrow{\varepsilon \otimes \text{id}} \bigotimes_{i=1}^{n} \text{End}(E_i)^{a_i/e_i} \otimes \text{det}(E_i)^{a_i/e_i} \xrightarrow{\phi} F.$$

**Application to the lower bound of $h_{\min}((\mathcal{F}, \mathcal{Z})//G)$**

**2.2.6.** Let us go back to the proof of Theorem 2.4. Let us denote by $\mathcal{Y}$ the quotient of semi-stable points of $P(\mathcal{F})$ by $\text{SL}(E)$ and, for any sufficiently divisible $D$, by $\mathcal{F}_D$ the hermitian invertible sheaf on $\mathcal{Y}$ induced by $\mathcal{F}(D)$.

To prove such a statement one has to bound the (archimedean) size of a family of generators of the global sections of $\mathcal{F}_D$. This amounts to bound archimedean and non-archimedean size of a family of generators of the invariants elements of the $K$-vector space $\text{Sym}^D \mathcal{F} \otimes K$.

**2.2.7.** For any $i = 1, \ldots, n$ let us denote by $E_i$ the $K$-vector space $E_i \otimes K$ and by $F$ the $K$-vector space $\mathcal{F} \otimes K$. Since the characteristic of $K$ is zero and the homomorphism $\phi$ decreases the norms, one reduces immediately to the case where

$$\mathcal{F} = \bigotimes_{i=1}^{n} \mathcal{E}_i.$$
and \( \varphi \) is the identity. Applying the First Main Theorem of Invariant Theory (in the form given by Corollary 2.7) one finds that the subspace of invariants of \( \text{Sym}^D F \) is the image of the natural homomorphism

\[
\Phi : \bigotimes_{i=1}^{n} k[|S|D_{\sigma_i}| \otimes \det(E_i)^{a_{D_i}/e_i} \xrightarrow{\text{e} \otimes \text{id}} \bigotimes_{i=1}^{n} \operatorname{End}(E_i)^{a_{D_i}/e_i} \otimes \det(E_i)^{a_{D_i}/e_i} \xrightarrow{\phi^q} \text{Sym}^D F
\]

2.2.8. — For any \( i = 1, \ldots, n \) let us choose a non-zero element \( \delta_i \) of \( \det(E_i) \) so that a family of generators of the invariants of \( \text{Sym}^D F \) is now given by the image through \( \Phi \) of the elements of the form

\[
f_\sigma := \sigma \otimes (\delta_1^{a_{D_1}/e_1} \otimes \cdots \otimes \delta_n^{a_{D_n}/e_n})
\]

where \( \sigma = (\sigma_1, \ldots, \sigma_n) \) ranges in \( \mathcal{S}_{D[a]} := \mathcal{S}_{D[a_1]} \times \cdots \times \mathcal{S}_{D[a_n]} \). Summing up we have:

\[
h_{\min}(w(\mathcal{F}), \mathcal{O}(1)) = \inf_{Q \in \mathcal{P}(\mathcal{F})} \frac{1}{D} \sup_{0 \neq \varphi \in \mathcal{F}} \left\{ \sum_{V \in K} \log \sup_{\varphi(V)} \|\Phi(f_{\sigma})\|_{\mathcal{D}^N} \right\}.
\]

2.2.9. — By definition of the metric on \( \mathcal{D}_{\mathcal{F}} \) and the fact that the sup-norm on \( \mathcal{P}(\mathcal{F}) \) is smaller than the norm on the symmetric powers (see 0.0.2), for any place \( \nu \) and any \( \sigma = (\sigma_1, \ldots, \sigma_n) \in \mathcal{S}_{D[a]} \) one gets:

\[
\sup_{\mathcal{P}(\mathcal{F})} \|\Phi(f_{\sigma})\|_{\mathcal{D}^N} \leq \sup_{\mathcal{P}(\mathcal{F})} \|\Phi(f_{\sigma})\|_{\mathcal{D}^N} \leq \|\Phi(f_{\sigma})\|_{\text{Sym}^D \mathcal{F}, \nu}
\]

Let \( \sigma = (\sigma_1, \ldots, \sigma_n) \in \mathcal{S}_{D[a]} \) be a \( n \)-tuple of permutations and write \( \varepsilon(\sigma) = \varepsilon(\sigma_1) \otimes \cdots \otimes \varepsilon(\sigma_n) \) as an element of \( \operatorname{End}(E_1^{a_{D_1}}) \otimes \cdots \otimes \operatorname{End}(E_n^{a_{D_n}}) \). Since \( \varphi \) is a homomorphism of \( \mathfrak{g}_K \)-hermitian vector bundles, \( \nu \) decreases the norms, one has:

\[
\log \|\Phi(f_{\sigma})\|_{\text{Sym}^D \mathcal{F}, \nu} \leq \log \|\varepsilon(\sigma)\|_{\mathcal{S}_{D[a]} \operatorname{End}(\varepsilon_1^{a_{D_1}/e_1})} + \sum_{i=1}^{n} \frac{a_{D_i}}{e_i} \log \|\delta_i\|_{\det(\varepsilon_i), \nu} = \sum_{i=1}^{n} \log \|\varepsilon(\sigma_i)\|_{\operatorname{End}(\varepsilon_i^{a_{D_i}/e_i})} + \sum_{i=1}^{n} \frac{a_{D_i}}{e_i} \log \|\delta_i\|_{\det(\varepsilon_i), \nu}.
\]

**Lemma 2.8.** Let \( \mathcal{W} \) be a \( \mathfrak{g}_K \)-hermitian vector bundle and \( W := \mathcal{W} \otimes K \). Let \( N \) be an integer, \( \tau \in \mathcal{S}_N \) be a permutation and \( \epsilon_\tau \in \operatorname{End}(W^N) \) be the isomorphism on \( W^N \) obtained permuting factors by \( \tau \). Then:

- \( \epsilon_\tau \) is an isometry with respect to the norm \( \|\cdot\|_{W^N, \nu}; \)
- we endow the \( \mathfrak{g}_K \)-module \( \operatorname{End}(\mathcal{W}) \) with the natural hermitian norms on the \( \mathfrak{g}_K \)-hermitian vector bundle \( \mathcal{W} \otimes \mathcal{W} \); with this convention we have:

\[
\log \|\epsilon_\tau\|_{\operatorname{End}(W^N), \nu} = \begin{cases} \frac{|N|}{2} \log \dim_X W & \text{if } \nu \text{ is archimedean} \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof of Lemma 2.8.** First of all, thanks to the isometric isomorphism \( \operatorname{End}(\mathcal{W}^N) \to \operatorname{End}(\mathcal{W}^N) \) we may suppose that \( N \) is non-negative. The fact that \( \epsilon_\tau \) is isometric is clear so let us pass to the proof of the formula.

If the place \( \nu \) is non-archimedean one finds:

\[
\|\epsilon_\tau\|_{\operatorname{End}(W^N), \nu} = \sup_{0 \neq w \in W^N} \frac{\|\epsilon_\tau(w)\|_{W^N, \nu}}{\|w\|_{W^N, \nu}} = 1.
\]
If the place $v$ is archimedean, $n = \dim_k W$ and $w_1, \ldots, w_n$ is an orthonormal basis of $W$ one gets:

$$
\|e_i\|_{\text{End}(\mathcal{H} \otimes N_k)}^2 = \sum_{\alpha \in \{1, \ldots, n\}^N} \|e_i (w_{a_1} \otimes \cdots \otimes w_{a_n})\|_{\mathcal{H} \otimes N_k}^2
= \sum_{\alpha \in \{1, \ldots, n\}^N} \|w_{a_1} \otimes \cdots \otimes w_{a_n}\|_{\mathcal{H} \otimes N_k}^2 = \#\{1, \ldots, n\}^N = n^N,
$$

which proves the lemma.

Going back to our computation one finds:

$$
\sum_{v \in V_k} \log \|\Phi(f_0)\|_{\text{Sym}^d F, v} \leq \frac{n}{2} \sum_{i=1}^n D[a_i] \log \text{rk} \mathbb{F}_i + \sum_{v \in V_k} \frac{D[a_i]}{e_i} \log \|\delta_i\|_{\text{det}(\mathbb{F}_i), v}
= \frac{n}{2} \sum_{i=1}^n D[a_i] \log \text{rk} \mathbb{F}_i - \frac{D[a_i]}{e_i} \deg \mathbb{F}_i.
$$

According to (2.2.2), one concludes the proof changing sign and dividing by $D$.

## 3 From the Fundamental Formula to the Main Theorem

### 3.1 Definition of the “moduli problem”

#### 3.1.1. Index.

Let us work on an algebraic closure $\overline{K}$ of $K$. Let $n \geq 1$ be a positive integer and $P = (\mathbb{P}^1)^n$ be the product of $n$ copies of the projective line over $\overline{K}$. For any $i = 1, \ldots, n$ let $\text{pr}_i : P \to \mathbb{P}^1$ be the projection onto the $i$-th factor.

Let $x = (x_1, \ldots, x_n)$ be a $\overline{K}$-point of $P$ and $a = (a_1, \ldots, a_n)$ be a $n$-tuple of positive real numbers. For any $i = 1, \ldots, n$ let $t_i$ be a local parameter around $x_i \in \mathbb{P}^1(\overline{K})$.

**Definition 3.1.** Let $f \in \mathcal{O}_{\mathbb{P}, x}$ be a regular function on $\mathbb{P}$ defined on an open neighborhood of $x$. The function $f$ develops into power series

$$
f = \sum_{\ell = (t_1, \ldots, t_n) \in \mathbb{N}^n} c_\ell t_1^{\ell_1} \cdots t_n^{\ell_n},
$$

with $c_\ell \in \overline{K}$. If $f$ is non-zero, then we define the *index of $f$ at $x$ with respect to the weight $a$* as the real number

$$
\text{ind}_a(f, x) := \min \{ a_1 \ell_1 + \cdots + a_n \ell_n : c_\ell \neq 0 \};
$$

if $f = 0$ we set $\text{ind}_a(0, x) := +\infty$.

If $a = (a_1, \ldots, a_n)$ is a $n$-tuple of positive real numbers we will write $1/a = (1/a_1, \ldots, 1/a_n)$. In particular the index with the respect the weight $1/a$ will be denoted by $\text{ind}_{1/a}$.

The notion of index can be naturally extended to meromorphic sections $s$ of an invertible sheaf $L$ on $\mathbb{P}$: it suffices to choose a trivializing section $s_0$ of $L$ around $x$ and set

$$
\text{ind}_a(s, x) := \text{ind}_a(s/s_0, x).
$$

The main result that we will be using concerning the index is the Higher Dimensional Dyson’s Lemma: the version stated here is due to Nakamaye [Nak99]. The original version of Esnault-Viewheg [EV84] (which has a slightly bigger error term) would work as well.

Let $r = (r_1, \ldots, r_n)$ be a $n$-tuple of positive integers. We consider the following invertible sheaf on the projective scheme $\mathbb{P}$:

$$
\mathcal{O}_P(r) := \text{pr}^*_1 \mathcal{O}_{\mathbb{P}^1}(r_1) \otimes \cdots \otimes \text{pr}^*_n \mathcal{O}_{\mathbb{P}^1}(r_n).
$$
**Theorem 3.2** (Higher dimensional Dyson’s Lemma). Let \( x_1, \ldots, x_N \) be \( \overline{K} \)-points of \( P \) and \( t_1, \ldots, t_n \) be non-negative real numbers. Suppose that

- For any \( i = 1, \ldots, n \) and any \( \sigma \neq \tau \), we have \( \text{pr}_i(x_\sigma) \neq \text{pr}_i(x_\tau) \);
- There exists a positive integer \( \alpha \geq 1 \) and a non-zero section \( f \in \Gamma(P, \mathcal{O}_P(\alpha r)) \otimes \overline{K} \) such that for any \( \sigma = 1, \ldots, N \) we have

\[
\text{ind}_{\sigma r}(f, x_\sigma) \geq \alpha r.
\]

Then the following inequality is satisfied:

\[
\sum_{\sigma=1}^{n} \text{vol} \Delta_n(t_\sigma) \leq 1 + \epsilon_{N,n}(r),
\]

where

\[
\epsilon_{N,n}(r) := \prod_{i=1}^{n-1} \left( 1 + \max_{1 \leq j \leq n} \left\{ \frac{r_j}{t_j} \right\} \max[N-2,0] \right) - 1.
\]

3.1.2. We go back to the data of the Main Effective Lower Bound. Let \( K \) be a number field, \( V_K \) its set of places and let \( K' \) be a finite extension of \( K \) of degree \( d \geq 2 \).

Let \( n \geq 1 \) be a positive integer and consider the product \( P = (P^1_{x_k})^n \) of \( n \) copies of the projective line over \( \sigma_K \). Let

- \( \theta = (\theta_1, \ldots, \theta_n) \) be a \( K' \)-point of \( P \) such that for any \( i = 1, \ldots, n \) the point \( \theta_i \) generates the field \( K' \);
- \( x = (x_1, \ldots, x_n) \) a \( K \)-rational point of \( P \).

3.1.3. Kernels of evaluation maps. — Let \( t_0, t_x \) be non-negative real numbers. We consider the following closed subschemes of the generic fiber \( P_K = P \times_{\sigma_K} K \):

- \( Z_r(0, t_0) \) : subscheme of \( P_K \) defined by the ideal sheaf of regular sections \( f \) such that \( \text{ind}_{1/r}(f, 0) \geq t_0 \);
- \( Z_r(x, t_x) \) : subscheme of \( P_K \) defined by the ideal sheaf of regular sections \( f \) such that \( \text{ind}_{1/r}(f, x) \geq t_x \).

The closed subschemes \( Z_r(0, t_0), Z_r(x, t_x) \) are respectively supported at the closed points \( 0 \) and \( x \) of \( P_K \). If \( \overline{K} \) is an algebraic closure of \( K \) we have

\[
Z_r(0, t_0) \times_{\overline{K}} \overline{K} = \bigsqcup_{\sigma : K' \rightarrow \overline{K}} Z_r(\sigma(0), t_0) \tag{3.1.1}
\]

where, for every \( \sigma : K' \rightarrow \overline{K} \), \( Z_r(\sigma(\theta), t_0) \) is the closed subscheme of \( P_{\overline{K}} = P \times_{\sigma_K} \overline{K} \) defined by ideal sheaf of regular sections \( f \) such that \( \text{ind}_{1/r}(f, \sigma(\theta)) \geq t_0 \). In an equivalent way, since we are working over \( \overline{K} \) and \( \theta \) is not \( K \)-rational, imposing an index condition to \( \theta \) imposes automatically the same condition at any conjugate of \( \theta \).

We then consider the kernels of the evaluation at these subschemes:

- \( K_r(0, t_0) \) : kernel of the evaluation map \( \Gamma(P_K, \mathcal{O}_P(r)) \rightarrow \Gamma(Z_r(0, t_0), \mathcal{O}_P(r)) \);
- \( K_r(x, t_x) \) : kernel of the evaluation map \( \Gamma(P_K, \mathcal{O}_P(r)) \rightarrow \Gamma(Z_r(x, t_x), \mathcal{O}_P(r)) \).
3.1.4. Linear actions on grassmannians. — Let \( \mathcal{E} \) be a free \( \mathcal{O}_K \)-module of finite rank. For any non-negative integer \( N \) we consider the grassmannian of subspaces of rank \( N \) of \( E \), \( \text{Grass}_N(\mathcal{E}) \), i.e. the \( \mathcal{O}_K \)-scheme representing the functor

\[
\text{Grass}_N(\mathcal{E}) : \{ \mathcal{O}_K\text{-schemes} \} \longrightarrow \{ \text{sets} \}
\]

\[
(f : X \rightarrow \text{Spec} \mathcal{O}_K) \longmapsto \{ (\mathcal{F}, \varphi) \mid \mathcal{F} = \text{locally free} \mathcal{O}_X\text{-module of rank} \, N, \quad \varphi : \mathcal{F} \rightarrow f^* \mathcal{E} \text{ injective with flat cokernel} \} \setminus \sim .
\]

Suppose that an \( \mathcal{O}_K \)-group scheme \( \mathcal{G} \) acts linearly on the \( \mathcal{O}_K \)-module \( \mathcal{E} \). Then, for any integer \( N \geq 0 \), the \( \mathcal{O}_K \)-group scheme \( \mathcal{G} \) acts naturally on the grassmannian \( \text{Grass}_N(\mathcal{E}) \) of subspaces of rank \( N \), on the projective space \( \mathbb{P}(\bigwedge^N \mathcal{E}) \) and in an equivariant way on the invertible sheaf \( \mathcal{O}_{\bigwedge^N \mathcal{E}}(1) \). Moreover, the Plücker embedding

\[
\varnothing : \text{Grass}_r(\mathcal{E}) \longrightarrow \mathbb{P}(\bigwedge^N \mathcal{E})
\]

is \( \mathcal{G} \)-equivariant.

3.1.5. — Let us denote \( k_r(t_0) \) and \( k_r(t_x) \) respectively the dimension of the \( K \)-vector spaces \( K_r(0, t_0) \) and \( K_r(x, t_x) \). In such a way, these sub-vector spaces of the global sections \( \Gamma(\mathcal{P}, \mathcal{O}_\mathcal{P}(r)) \) define the following \( K \)-points of grassmannians:

- \( [K_r(0, t_0)] \in \text{Grass}_{k_r(t_0)}(\Gamma(\mathcal{P}, \mathcal{O}_\mathcal{P}(r)))(K) \);
- \( [K_r(x, t_x)] \in \text{Grass}_{k_r(t_x)}(\Gamma(\mathcal{P}, \mathcal{O}_\mathcal{P}(r)))(K) \).

The \( \mathcal{O}_K \)-reductive group \( \text{SL}^n_{2, \mathcal{O}_K} \) acts naturally on the product \( \mathcal{P} = (\mathcal{P}_1^{\mathcal{O}_K})^n \) and we consider the natural action induced on the grassmannians mentioned above. The Plücker embeddings

- \( \text{Grass}_{k_r(t_0)}(\Gamma(\mathcal{P}, \mathcal{O}_\mathcal{P}(r))) \longrightarrow \mathbb{P}(\mathcal{F}_r(t_0)) \),
- \( \text{Grass}_{k_r(t_x)}(\Gamma(\mathcal{P}, \mathcal{O}_\mathcal{P}(r))) \longrightarrow \mathbb{P}(\mathcal{F}_r(t_x)) \),

are equivariant with respect to the action of \( \text{SL}^n_{2, \mathcal{O}_K} \), where we wrote

\[
\mathcal{F}_r(t_0) := \bigwedge^{k_r(t_0)} \Gamma(\mathcal{P}, \mathcal{O}_\mathcal{P}(r)),
\]

\[
\mathcal{F}_r(t_x) := \bigwedge^{k_r(t_x)} \Gamma(\mathcal{P}, \mathcal{O}_\mathcal{P}(r)).
\]

3.1.6. The geometric invariant theory data. — We will apply the Fundamental Formula to the following situation:

\[
\mathcal{P}_r = \{ [K_r(0, t_0)], [K_r(x, t_x)] \},
\]

\[
\mathcal{G}_r = \text{Grass}_{k_r(t_0)}(\Gamma(\mathcal{P}, \mathcal{O}_\mathcal{P}(r))) \times \text{Grass}_{k_r(t_x)}(\Gamma(\mathcal{P}, \mathcal{O}_\mathcal{P}(r))),
\]

\[
\mathcal{X}_r = \text{SL}^n_{2, \mathcal{O}_K},
\]

\[
\mathcal{L}_r = \text{polarization given by the Plücker embeddings of the grassmannians}.
\]

For any embedding \( \sigma : K \rightarrow C \), the complex vector spaces

- \( \mathcal{F}_r(t_0) \otimes_\sigma \mathbb{C} = \bigwedge^{k_r(t_0)} \bigotimes_{i=1}^n \text{Sym}^n \mathbb{C}^2 \)
are equipped with the hermitian norms obtained by tensor operations (see 0.0.2). We endow the invertible sheaf $\mathcal{L}$ with the restriction of the product of the Fubini-Study metrics on the product of projective spaces

$$P(\mathcal{F}(t_0)) \times_{K} P(\mathcal{F}(t_x)).$$

For any embedding $\sigma : K \to \mathbb{C}$ we denote by $\| \cdot \|_{\mathcal{L},\sigma}$ this metric (which is clearly invariant under the action of $SU_r^2$) and we write $\mathcal{F}$ for the associated hermitian invertible sheaf on $X$.

3.1.7. Some facts about the kernels. — We will need some estimations for the dimension of the kernels $K_r(\theta, t_0)$ and $K_r(x, t_x)$.

- **rational point:** for any $i = 1, \ldots, n$ let $T_{i0}, T_{i1}$ be a basis of $K^{2v}$ such that $T_{i1}$ vanishes at $x_i$. Then a basis of the $K$-vector space $K_r(x, t_x)$ is given by the monomials

$$\ell \in \nabla_r(t_x)_{\mathbb{Z}}. \text{ In particular we have } \dim_K K_r(x, t_x) = \# \nabla_r(t_x)_{\mathbb{Z}} \text{ so that }$$

$$\lim_{n \to \infty} \frac{\dim K_{\alpha r}(x, t_x)}{\alpha^n(r_1 \cdots r_n)} = \text{vol}_n(t_x).$$

- **algebraic point:** If we extend scalars to an algebraic closure $\overline{K}$ of $K$, it follows from (3.1.1) that the evaluation homomorphism at the index subscheme $Z_r(\theta, t_0) \times_{K} \overline{K} \subset P_{\overline{K}}$ can be written as follows:

$$\text{vol}_{\mathcal{L}(\theta, t_0)}(\mathcal{O}_P(r)) \to \text{vol}_{\mathcal{L}(\alpha(\theta), t_0)}(\mathcal{O}_P(r)) = \bigoplus_{\alpha \in \overline{K} \setminus K} \text{vol}_{\mathcal{L}(\alpha(\theta), t_0)}(\mathcal{O}_P(r)).$$

In particular, computing the dimension of $\Gamma(\mathcal{O}_P(r))$ in the same way as we did for the $K$-rational point $x$, we find:

$$\dim_k K_r(\theta, t_0) \geq \prod_{i=1}^{n} (r_i + 1) - d \# \nabla_r(t_0)_{\mathbb{Z}}.$$

Applying this estimate for any positive integer multiple of $r$, we get:

$$\lim\inf_{n \to \infty} \frac{\dim K_{\alpha r}(\theta, t_0)}{\alpha^n(r_1 \cdots r_n)} \geq 1 - d \text{vol}_{\mathcal{L}}(t_0). \quad (3.1.2)$$

In order to bound from above the dimension of $K_r(\theta, t_0)$ let us consider the unique real number $u_r(t_0) \in [0, n]$ such that

$$\text{vol}_{\mathcal{L}}(u_r(t_0)) = \min \{ 1 + \varepsilon_{d+1,n}(r) - d \text{vol}_{\mathcal{L}}(t_0), 0 \}, 1 \}.$$

This real number depends on $d$ but we will omit to write this dependence.

**Lemma 3.3.** With the notation introduced above we have

$$\lim_{\alpha \to \infty} \sup \frac{\dim K_{\alpha r}(\theta, t_0)}{\alpha^n(r_1 \cdots r_n)} = \text{vol}_{\mathcal{L}}(u_r(t_0)).$$
Let us emphasize that this upper bound is obtained as a consequence of the Higher Dimensional Dyson’s Lemma.

**Proof.** First of all, if $\text{vol} \Delta_n(u_r(t_0)) = 1$, which means $u_r(t_0) = n$, this is trivial; hence we may assume $u_r(t_0) < n$. Let $y$ be a $K$-point of $P$ such that, for any $i = 1,\ldots,n$ and any $\sigma : K' \to K$ we have

$$\text{pr}_i(y) \neq \text{pr}_i(\sigma(0)).$$

Then for any $n > t_y > u_r(t_0)$ we have $K_r(0, t_0) \cap K(y, t_y) = 0$. Indeed, if there exists a non-zero element $f$ in this intersection, applying the Higher Dimensional Dyson’s Lemma we would get

$$\sum_{\alpha : K' \to K} \text{vol} \Delta_n(t_0) + \text{vol} \Delta_n(t_y) \leq 1 + \varepsilon_{d+1,n}(r),$$

thus

$$\text{vol} \Delta_n(t_y) \leq \min \{ \max \{ 1 + \varepsilon_{d+1,n}(r) - d \text{vol} \Delta_n(t_0), 0 \}, 1 \} = \text{vol} \Delta_n(u_r(t_0))$$

which would imply $t_y \leq u_r(t_0)$ contradicting the hypothesis $t_y > u_r(t_0)$. \hfill \qedsymbol

### 3.2 Proof of the Main Theorem

#### 3.2.1. In this section we will prove Theorem 1.4 admitting some intermediate independent steps that we will prove later.

Let us begin remarking that in order to prove Theorem 1.3, by an approximation argument, we may suppose that $n$-tuple of positive real numbers $r = (r_1,\ldots,r_n)$ in the statement is made of positive rational numbers. Then, since the Main Effective Lower Bound is homogeneous in $r$, we may (and we always will) assume that $r$ is made of positive integers.

#### 3.2.2. Semi-stability conditions. — We are going to prove Theorem 1.4 applying the Fundamental Formula to the point $P_r$. In order to do this, we have to assure that under the hypothesis of Theorem 1.4 the point $P_r$ is semi-stable.

More precisely, in Section 6 we will prove the following fact:

**Theorem 3.4.** Let $n \geq 1$ be a positive integer and $r = (r_1,\ldots,r_n)$ be a $n$-tuple of positive integers. Let $t_x, t_0 \geq 0$ be non-negative real numbers. If the inequality

$$\left| \text{vol} \nabla_n(t_x) - 2 \int_{\nabla_n(t_x)} \zeta_1 \, d\lambda \right| < \left| \text{vol} \nabla_n(u_r(t_0)) - 2 \int_{\nabla_n(u_r(t_0))} \zeta_1 \, d\lambda \right| - \varepsilon_n(r)$$

is satisfied then there exists a positive integer $\alpha_0 = \alpha_0(n,d,r,t_0,t_x)$ such that, for any integer $\alpha \geq \alpha_0$, the $K$-point of the product $X_{ar} = \text{Grass}_{K(t_0)}^n \{ \Gamma(P, \mathcal{O}(ar)) \} \times \text{Grass}_{K(t_x)}^n \{ \Gamma(P, \mathcal{O}(ar)) \}$,

$$P_{ar} = ([K_{ar}(0, t_0)], [K_{ar}(x, t_x)])$$

is semi-stable under the action of $\text{SL}_d^+$ and with respect to the polarization given by the Plücker embeddings.

#### 3.2.3. Applying the Fundamental Formula. — The numerical condition appearing in the previous statement is exactly the condition (1.2.1) in Theorem 1.4. Hence according to Theorem 3.4 there exists a positive integer $\alpha_0 = \alpha_0(n,d,r,t_0,t_x)$ such that, for any integer $\alpha \geq \alpha_0$, the $K$-point

$$P_{ar} = ([K_{ar}(0, t_0)], [K_{ar}(x, t_x)])$$

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is semi-stable. The Fundamental Formula (or, better, Corollary 2.3) applied for any $\alpha \geq \alpha_0$ to the point $P_{ar}$ gives the following inequality:

$$h_{\overline{\mathcal{X}}_{ar}}(P_{ar}) + \frac{1}{[K:Q]} \sum_{v \in \mathcal{V}_K} \ell_v(P_{ar}) \geq h_{\min}\left((\mathcal{X}_{ar}, \overline{\mathcal{X}}_{ar}) / G\right).$$

Dividing by $\alpha^n(r_1 \cdots r_n)$ and letting $\alpha$ go to infinity we get

$$\limsup_{\alpha \to \infty} \frac{h_{\overline{\mathcal{X}}_{ar}}(P_{ar})}{\alpha^n(r_1 \cdots r_n)} + \frac{1}{[K:Q]} \sum_{v \in \mathcal{V}_K} \ell_v(P_{ar}) \geq \limsup_{\alpha \to \infty} \frac{h_{\min}\left((\mathcal{X}_{ar}, \overline{\mathcal{X}}_{ar}) / G\right)}{\alpha^n(r_1 \cdots r_n)}. \quad (3.2.1)$$

In order to get the inequality in Theorem 1.4 we have to bound the terms appearing in this inequality.

3.2.4. Upper bound of the height. — The Plücker embeddings give a closed isometric embedding of the product of grassmannians $\mathcal{X}_r$ into the product of projective spaces $P(\mathcal{F}_r(t_0)) \times P(\mathcal{F}_r(t_1))$. Hence we have:

$$h_{\overline{\mathcal{X}}_{r}}(P_r) = h_{\overline{\mathcal{X}}_{r}(t_0)}([K_r(x, t_1)]) + h_{\overline{\mathcal{X}}_{r}(t_0)}([K_r(0, t_0)]).$$

Now some simple estimates of Arakelov degrees (see Propositions 4.1-4.2) give:

- $h_{\overline{\mathcal{X}}_{r}(t_0)}([K_r(x, t_1)]) \leq \sum_{i=1}^n \sum_{t \in \mathcal{V}_r(t_1)/z} \ell_i h(x_i)$.  
- $h_{\overline{\mathcal{X}}_{r}(t_0)}([K_r(0, t_0)]) \leq \prod_{i=1}^n (r_i + 1) \left( \sum_{i=1}^n r_i h(\theta_i) + |r| \log \sqrt{2d}\right)$.  

Applying these estimates to any positive integer multiple of $r$ we get:

$$\limsup_{\alpha \to \infty} \frac{h_{\overline{\mathcal{X}}_{ar}}(P_{ar})}{\alpha^n(r_1 \cdots r_n)} \leq \left( \int_{\mathcal{V}_r(t_1)} \zeta_1 d\lambda \right) \sum_{i=1}^n r_i h(x_i) + \sum_{i=1}^n r_i h(\theta_i) + |r| \log \sqrt{2d}. \quad (3.2.2)$$

3.2.5. Upper bound of the instability measure. — Let $\nu$ be a place of $K$. Let us recall that the instability measure $t_{\nu}(P_r)$ of $P_r$ at $\nu$ is defined as

$$t_{\nu}(P_r) = -\log \sup_{g \in \mathcal{G}_v} \frac{\|g : \mathcal{X}_r, v\|_{\mathcal{X}_r, v}(g \cdot P_r)}{\|s\|_{\mathcal{X}_r, v}(P_r)}$$

where $s \in P^*\mathcal{X}_r$ is a non-zero section.

- If the place $\nu$ is non-archimedean we find:

$$t_{\nu}(P_r) \leq k_r(t_0) \left( t_0 \max_{i=1,\ldots, n} \{ r_i \log d_v(\theta_i, x_i) \} \right) + \sum_{i=1}^n \left( \sum_{t \in \mathcal{V}_r(t_1)/z} \ell_i - \frac{k_r(t_1) + k_r(t_0)}{2} r_i \right) \log d_v(\theta_i, x_i).$$

- If the place $\nu$ is archimedean we find:

$$t_{\nu}(P_r) \leq k_r(t_0) \left( t_0 \max_{i=1,\ldots, n} \{ r_i \log d_v(\theta_i, x_i) \} \right) + \sum_{i=1}^n \left( \sum_{t \in \mathcal{V}_r(t_1)/z} \ell_i - \frac{k_r(t_1) + k_r(t_0)}{2} r_i \right) \log d_v(\theta_i, x_i) + 2 (k_r(t_1) + k_r(t_0)) |r|.$$

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These bounds will be proved in Section 5. Applying these estimates to any positive integer multiple of \( r \) we get:

- if \( v \) is non-archimedean:
  \[
  \limsup_{a \to -\infty} \frac{1}{a^n(r_1 \cdots r_n)} \leq \left(1 - d \vol_{\Delta_n(t_0)} t_0 \max_{i=1, \ldots, n} \left\{ r_i \log d_{\nu}(\theta_i, x_i) \right\} \right) + \left( \int_{\nu_n(t_0)} \frac{\zeta_1}{2} \, d\lambda - \frac{\vol_{\Delta_n(u_r(t_0))} + \vol \nabla_n(t_2)}{2} \right) \sum_{i=1}^{n} r_i \log d_{\nu}(\theta_i, x_i) \\
  \leq \left(1 - d \vol_{\Delta_n(t_0)} t_0 \max_{i=1, \ldots, n} \left\{ r_i \log d_{\nu}(\theta_i, x_i) \right\} \right) + \left( \int_{\nu_n(t_0)} \frac{\zeta_1}{2} \, d\lambda - \frac{\vol_{\Delta_n(u_r(t_0))} + \vol \nabla_n(t_2)}{2} \right) \sum_{i=1}^{n} r_i \log d_{\nu}(\theta_i, x_i) \\
  \leq \left(1 - d \vol_{\Delta_n(t_0)} t_0 \max_{i=1, \ldots, n} \left\{ r_i \log d_{\nu}(\theta_i, x_i) \right\} \right) + \left( \int_{\nu_n(t_0)} \frac{\zeta_1}{2} \, d\lambda - \frac{\vol_{\Delta_n(u_r(t_0))} + \vol \nabla_n(t_2)}{2} \right) \sum_{i=1}^{n} r_i \log d_{\nu}(\theta_i, x_i) + 2|\nu| (\vol_{\Delta_n(u_r(t_0))} + \vol \nabla_n(t_2)).
  \]  
(3.2.3)

- if \( v \) is archimedean:
  \[
  \limsup_{a \to -\infty} \frac{1}{a^n(r_1 \cdots r_n)} \leq \left(1 - d \vol_{\Delta_n(t_0)} t_0 \max_{i=1, \ldots, n} \left\{ r_i \log d_{\nu}(\theta_i, x_i) \right\} \right) + \left( \int_{\nu_n(t_0)} \frac{\zeta_1}{2} \, d\lambda - \frac{\vol_{\Delta_n(u_r(t_0))} + \vol \nabla_n(t_2)}{2} \right) \sum_{i=1}^{n} r_i \log d_{\nu}(\theta_i, x_i) \\
  + 2|\nu| (\vol_{\Delta_n(u_r(t_0))} + \vol \nabla_n(t_2)).
  \]  
(3.2.4)

### 3.2.6. Lower bound for the height on the quotient.

— Combining the Plücker and the Segre embeddings we get a \( \mathcal{G} \)-equivariant closed embedding \( \mathcal{E}_r \to \text{P}(\mathcal{F}_r(t_0) \otimes \mathcal{F}_r(t_2)) \)

Applying the lower bound given by Theorem 2.4 to the representation \( \mathcal{G} \to \text{GL}(\mathcal{F}_r(t_0) \otimes \mathcal{F}_r(t_2)) \) and to the natural surjection

\[
\varphi : \bigotimes_{i=1}^{n} (\mathfrak{a}_K^{2v})^{\otimes r_i (k_r(t_0) + k_r(t_2))} \longrightarrow \mathcal{F}_r(t_0) \otimes \mathcal{F}_r(t_2)
\]

(hence with \( \mathfrak{a}_i = \mathfrak{a}_K^2 \) and with \( a_i = -r_i (k_r(t_0) + k_r(t_2)) \) for any \( i = 1, \ldots, n \) we get

\[
h_{\min} \left( (\mathcal{E}_r, \mathcal{E}_r) || \mathcal{G} \right) \geq h_{\min} \left( (\text{P}(\mathcal{F}_r(t_0) \otimes \mathcal{F}_r(t_2)), \mathcal{G}(1)) || \mathcal{G} \right) \geq - (k_r(t_0) + k_r(t_2))|n| \log \sqrt{2} - \frac{1}{2} (\log k_r(t_0)! + \log k_r(t_2)!)
\]

where the term \( -1/2 (\log k_r(t_0)! + \log k_r(t_2)!) \) is due to the ratio between the hermitian norm on the alternating product and the quotient norm with the respect to surjection \( \varphi \) (see 0.0.2). Thanks to Stirling’s approximation one gets

\[
h_{\min} \left( (\mathcal{E}_r, \mathcal{E}_r) || \mathcal{G} \right) \geq -3 (k_r(t_0) + k_r(t_2))|n|.
\]

Applying this formula to any positive integer multiple of \( r \) we conclude:

\[
\liminf_{a \to -\infty} \frac{h_{\min} \left( (\mathcal{E}_r, \mathcal{E}_r) || \mathcal{G} \right)}{a^n(r_1 \cdots r_n)} \geq -3 (\vol_{\Delta_n(u_r(t_0))} + \vol \nabla_n(t_2))|r|.
\]  
(3.2.5)

Now to conclude the proof one has the asymptotic terms in (3.2.1) with the corresponding bounds given by the inequalities (3.2.2)-(3.2.5).

### 4 Upper bound of the height

We go back to the notation introduced in the paragraphs 3.1.3 and 3.1.5.
4.1 Rational point

**Proposition 4.1.** With the notation introduced above, we have

\[ h_{\mathcal{F}(t_{\ell})}([K_{r}(x_{i}, t_{\ell})]) \leq \sum_{i=1}^{n} \sum_{\ell \in \mathcal{V}(r, t_{\ell}, Z)} \ell_{i} h(x_{i}). \]

**Proof.** Let \( T_{0}, T_{1} \) be the canonical basis of \( K^{\mathcal{V}} \). For any \( i = 1, \ldots, n \) let \( (x_{i0}, x_{i1}) \) be a non-zero element of \( K^{2} \) representing the point \( x_{i} \in \mathbb{P}^{1}(K) \). We may suppose that \( x_{i0} \) is non-zero. For any \( n \)-tuple of non-negative integers \( \ell \in \triangle_{r} \), define

\[ T(\ell) := \bigotimes_{i=1}^{n} T_{0}^{\ell_{i}} T_{x_{i}}^{\ell_{i}} \]

where \( T_{x_{i}} = x_{0} T_{1} - x_{1} T_{0} \). A basis of the \( K \)-vector space \( K_{r}(x_{i}, t_{\ell}) \) is given by the elements \( T(\ell) \) while \( \ell \) ranges in \( \mathcal{V}_{r}(t_{\ell}) \).

Let \( \nu \) be a place of \( K \). The Hadamard inequality gives

\[ \log \| T(\ell) \|_{\mathcal{F}(t_{\ell}), \nu} \leq \sum_{\ell \in \mathcal{V}(r, t_{\ell}, Z)} \log \| T(\ell) \|_{\mathcal{F}(t_{\ell}, \mathcal{O}_{P}(r)), \nu}. \]

For any \( n \)-tuple of non-negative integers \( \ell \in \triangle_{r} \), the sub-multiplicativity of the norm on symmetric powers gives

\[ \log \| T(\ell) \|_{\mathcal{F}(t_{\ell}, \mathcal{O}_{P}(r)), \nu} = \sum_{i=1}^{n} \log \| T_{0}^{\ell_{i}} T_{x_{i}}^{\ell_{i}} \nu \leq \sum_{i=1}^{n} (r_{i} - \ell_{i}) \log \| T_{0} \nu + \ell_{i} \log \| T_{x_{i}} \nu. \]

Thanks to the Hadamard inequality, we conclude taking the sum over all places. \( \square \)

4.2 Algebraic point

**Proposition 4.2.** With the notation introduced above, we have

\[ h_{\mathcal{F}(t_{0})}([K_{r}(0, t_{0})]) \leq \prod_{i=1}^{n} \left( r_{i} \right) \left( \sum_{i=1}^{n} r_{i} h(0_{i}) + |r| \log \sqrt{2d} \right). \]

The rest of this section is devoted to the proof of this upper bound.

4.2.1. — We begin equipping the \( \mathcal{O}_{K} \)-module \( \Gamma(P, \mathcal{O}_{P}(r)) \) with the natural hermitian metric induced by the identification

\[ \Gamma(P, \mathcal{O}_{P}(r)) = \bigotimes_{i=1}^{n} \text{Sym}^{r_{i}} \left( \mathcal{O}_{K}^{2 \mathcal{V}} \right). \]

We denote by \( T(P, \mathcal{O}_{P}(r)) \) the resulting \( \mathcal{O}_{K} \)-hermitian vector bundle. We remark that the \( \mathcal{O}_{K} \)-hermitian vector bundle \( T(P, \mathcal{O}_{P}(r)) \) is not trivial since the basis of \( \Gamma(P, \mathcal{O}_{P}(r)) \) given by the elements

\[ T(\ell) = \bigotimes_{i=1}^{n} T_{0}^{\ell_{i}} T_{1}^{\ell_{i}} \]

is orthogonal but not orthonormal. Anyway, for any place \( \nu \), the sub-multiplicativity of the norm on symmetric powers gives

\[ \log \| T(\ell) \|_{\mathcal{F}(t_{\ell}, \mathcal{O}_{P}(r)), \nu} \leq \sum_{i=1}^{n} (r_{i} - \ell_{i}) \log \| T_{0} \nu + \ell_{i} \log \| T_{1} \nu = 0. \]
In particular we have
\[ \tilde{\mu}(\Gamma(P, \mathcal{E}_P(r))) \geq - \sum_{v \in V_K} \sum_{\ell \in \Omega^0} \log \|\Gamma(\ell\|_{\mathcal{E}_P(\mathcal{E}_P(r)), v}) \geq 0. \] (4.2.1)

4.2.2. — We endow the K-vector space \( \mathcal{K}_r(\theta, t_0) \) with the structure of \( \sigma_K \)-hermitian vector bundle induced by the one of \( \Gamma(P, \mathcal{E}_P(r)) \). Namely, we consider the \( \sigma_K \)-module
\[ \mathcal{K}_r(\theta, t_0) = \Gamma(P, \mathcal{E}_P(r)) \cap \mathcal{K}_r(\theta, t_0) \]
equipped with the restriction of the hermitian norms on \( \Gamma(P, \mathcal{E}_P(r)) \). Let us then consider
\[ \mathcal{E} = \Gamma(P, \mathcal{E}_P(r))/\mathcal{K}_r(\theta, t_0) \]
and endow it with quotient norms with respect to the surjection \( \Gamma(P, \mathcal{E}_P(r)) \to \mathcal{E} \). We denote by \( \overline{\mathcal{E}} \) the \( \sigma_K \)-hermitian vector bundle obtained in this way. With these choices and according to (4.2.1) we have
\[ h_{\mathcal{E}, t_0}(\mathcal{O}_K(\theta, t_0)) = - \deg \mathcal{E}_r(\theta, t_0) = \deg \overline{\mathcal{E}} - \deg \Gamma(P, \mathcal{E}_P(r)) \leq \deg \overline{\mathcal{E}}. \]

4.2.3. — Let us denote by \( E \) the K-vector space \( \Gamma(Z_r(\theta, t_0), \mathcal{E}_P(r)) \) and let \( \Omega \) be a Galois closure of \( K' \) over \( K \). Then let us endow the \( \Omega \)-vector space \( E \otimes_K \Omega \) with a structure of \( \sigma_\Omega \)-hermitian vector bundle as follows. As mentioned before (see 3.1.3), we have
\[ Z_r(\theta, t_0) \times_K \Omega = \bigcup_{\sigma: K \to \Omega} Z^\Omega_r(\sigma(\theta), t_0) \]
(where, for every \( \sigma: K' \to \Omega \), \( Z^\Omega_r(\sigma(\theta), t_0) \) is the closed subscheme of \( \mathcal{P}_\Omega = \mathcal{P} \times_{\sigma_K} \Omega \) defined by ideal sheaf of regular sections \( f \) such that \( \text{ind}_{1/\Omega}(f, \sigma(\theta)) \geq t_0 \)). Hence
\[ E \otimes_K \Omega = \bigoplus_{\sigma: K \to \Omega} E_\sigma := \Gamma \left( Z^\Omega_r(\sigma(\theta), t_0), \mathcal{E}_{P_\Omega}(r) \right) \] (4.2.2)

For any \( i = 1, \ldots, n \) let \( \hat{\theta}_i = (\theta_{i0}, \theta_{i1}) \) be a \( K' \)-point of \( \mathbb{A}^2 \) representing the point \( \theta_i \). Since \( \theta_i \) is not K-rational, we may (and we will) assume \( \theta_{i0} = 1 \).

Fix \( \sigma: K' \to \Omega \) an embedding. A basis of the \( \Omega \)-vector space \( E_\sigma \) is given by the elements
\[ T_{\sigma(\theta_i)}(\ell) = \prod_{i=1}^n T_{\sigma(\theta_i)}(\ell) \]
where \( T_{\sigma(\theta_i)} = T_1 - \theta_{i1} T_0 \) and \( \ell = (\ell_1, \ldots, \ell_n) \) ranges in the elements of \( \Delta_{\tau}(t_0)_Z \). We consider the \( \sigma_\Omega \)-submodule \( \mathcal{E}_\sigma \subset E_\sigma \) generated by the elements \( \Gamma(\ell) \)'s and we equip it with the hermitian norm having the elements \( T(\ell) \)'s as an orthonormal basis. We denote by \( \overline{\mathcal{E}}_\sigma \) the associated \( \sigma_\Omega \)-hermitian vector bundle.

Finally, according to (4.2.2), we endow \( \Omega \)-vector space \( E \otimes_K \Omega \) with the structure of \( \sigma_\Omega \)-hermitian vector bundle given by the orthogonal direct sum of the \( \sigma_\Omega \)-hermitian vector bundles \( \overline{\mathcal{E}}_\sigma \)'s.

4.2.4. — The evaluation homomorphism \( \eta: \Gamma(P_K, \mathcal{E}_P(r)) \to E = \Gamma(Z_r(\theta, t_0), \mathcal{E}_P(r)) \) factors through an injection \( \varepsilon: \mathcal{E} \otimes K \to E \). Extending the scalars to \( \Omega \) and applying the slope inequality, one gets
\[ h_{\mathcal{E}, t_0}(\mathcal{O}_K(\theta, t_0)) \leq \deg \overline{\mathcal{E}} \leq \text{rk} \mathcal{E} \left( \mu_{\max}(\overline{\mathcal{E}}_\Omega) + \sum_{v \in V_\Omega} \log \|\varepsilon\|_{\sup, v} \right) \] (4.2.3)

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where for any place $v \in V_E$ we denoted by $\|e\|_{sup,v}$ the $v$-adic operator norm of the injection $e$,

$$
\|e\|_{sup,v} := \sup_{0 \neq f \in \mathcal{O}^*_{\Omega}} \frac{\|e(f)\|_{\ell,v}}{\|f\|_{\ell,v}}.
$$

Clearly this coincides with the operator norm $\|\eta\|_{sup,v}$ of the evaluation morphism $\eta$. Let us also remark that, by definition, the $\mathcal{O}_{\Omega}$-hermitian vector bundle $\overline{\mathcal{E}}_{\Omega}$ is trivial hence $\|\eta\|_{sup,v} = 0$.

4.2.5. — So we are left with bounding the $v$-adic size of the evaluation homomorphism $\eta$. For any embedding $\sigma : K' \to \Omega$ let us consider $\etao : \Gamma(P_{\Omega}, \mathcal{E}_{P(r)}) \to E_o$ the composition of the homomorphism $\eta$ with the canonical projection $E \to E_o$. Let us also denote by $\|\etao\|_{sup,v}$ the operator norm of $\etao$. With this notation we have:

- $v$ non-archimedean: $\|\eta\|_{sup,v} = \max_{\sigma : K' \to \Omega} \|\etao\|_{sup,v}$;
- $v$ archimedean: $\|\eta\|_{sup,v} \leq \sqrt{d} \max_{\sigma : K' \to \Omega} \|\etao\|_{sup,v}$.

For any $\sigma : K' \to \Omega$ and any $i = 1, \ldots, n$ let us consider the automorphism $\varphi_{oi}$ of the $\Omega$-vector space $\Omega^{2^i}$ defined by

$$
\varphi_{oi} : T_0 \to T_0,
T_1 \mapsto T_{\sigma([0])} - T_1 - \theta_1 T_0
$$

and $\Phi_{oi}$ the corresponding automorphism of $P_{\Omega}$. Let us denote by $\Phi : P_{\Omega} \to P_{\Omega}$ the automorphism that operates as $\Phi_{oi}$ on the $i$-th factor. With this notation we have

$$
\Phi_{oi}^{-1}(Z_{\Omega,v}(\sigma([0]), t_0)) = Z_{r}(1 : 0), t_0 \times K \Omega.
$$

In particular the homomorphism $\etao \circ \Phi_{oi} : \Gamma(P_{\Omega}, \mathcal{E}_{P(r)}) \to E_o$ coincides with the evaluation morphism at the closed subscheme $Z_{r}(1 : 0), t_0 \times K \Omega$, i.e. it is described as follows

$$
T(\ell) = \bigotimes_{i=1}^{n} T_{0_{T_{\ell_i}}}^{-\epsilon} T_{t_{ij}} \leftrightarrow \begin{cases} T_{\sigma([0])} & \text{if } \ell \in \Delta_r(t_0) \mathcal{Z} \\ 0 & \text{otherwise.} \end{cases}
$$

The elements $T_{\sigma([0])} \ell$'s form an orthonormal basis of the trivial $\mathcal{O}_{\Omega}$-hermitian vector bundle $\overline{\mathcal{E}}_{\Omega}$. Thus we have $\|\etao \circ \Phi_{oi} \|_{sup,v} \leq 1$ and we deduce $\|\etao\|_{sup,v} \leq \|\Phi_{oi}^{-1}\|_{sup,v}$. The pullback by the morphism $\Phi_{oi}^{-1}$ induces on the global sections of $\mathcal{E}_{P(r)}$ the homomorphism

$$
\text{Sym}^n \varphi_{oi}^{-1} \otimes \cdots \otimes \text{Sym}^n \varphi_{oi}^{-1}.
$$

Recalling that for an endomorphism $\psi$ of a $\mathcal{O}_k$-hermitian vector bundle $\overline{\mathcal{V}}$ the sup-norm of $\psi$ is smaller than its norm as an element of $\overline{\mathcal{V}} \otimes \overline{\mathcal{V}}$, we have the following inequalities:

$$
\|\Phi_{oi}^{-1}\|_{sup,v} = \log \|\text{Sym}^n \varphi_{oi}^{-1} \otimes \cdots \otimes \text{Sym}^n \varphi_{oi}^{-1}\|_{sup,v} 
\leq \log \|\text{Sym}^n \varphi_{oi}^{-1} \otimes \cdots \otimes \text{Sym}^n \varphi_{oi}^{-1}\|_{\text{End}(\mathcal{V}, \mathcal{E}_{P(r)}), v} 
\leq \sum_{i=1}^{n} r_i \log \|\varphi_{oi}^{-1}\|_{\text{End}(\mathcal{O}_{\Omega}), v}
$$

Now one has to treat separately the archimedean and the non-archimedean case.
• \( v \) non-archimedean:

\[
\log \| \varphi_{\alpha T}^{-1} \|_v = \log \max \{ \| \varphi_{\alpha T}^{-1}(T_0) \|_v, \| \varphi_{\alpha T}^{-1}(T_1) \|_v \}
\]

\[
= \log \max \{ \| (1, 0) \|_v, \| (\sigma(\theta_i), 1) \|_v \} = \log \| \sigma(\theta_i) \|_v.
\]

• \( v \) archimedean:

\[
\log \| \varphi_{\alpha T}^{-1} \|_v = \log \sqrt{\| \varphi_{\alpha T}^{-1}(T_0) \|_v^2 + \| \varphi_{\alpha T}^{-1}(T_1) \|_v^2}
\]

\[
= \log \sqrt{\| (1, 0) \|_v^2 + \| (\sigma(\theta_i), 1) \|_v^2}
\]

\[
= \log \sqrt{1 + \| \sigma(\theta_i) \|_v^2} \leq \log \| \sigma(\theta_i) \|_v + \log \sqrt{2}.
\]

Taking the sum over all the places of \( \Omega \) we finally get

\[
\sum_{v \in V_{\Omega}} \log \| \eta \|_{\sup, v} \leq \sum_{v \in V_{\Omega}} \max_{\alpha K' \subset \Omega} \left\{ \sum_{i=1}^{n} r_i \log \| \sigma(\theta_i) \|_v \right\} + \log \sqrt{2^d} = \sum_{i=1}^{n} r_i h(\theta_i) + \log \sqrt{2^d}.
\]

According to (4.2.3) we finish multiplying by \( \text{rk} \mathcal{C} \) and bounding it by \( \text{rk} \Gamma(P, \mathcal{O}_P(r)) = \prod_{i=1}^{n} (r_i + 1) \).

5 Upper bound of the instability measure

5.1 General notation

5.1.1. — We go back to the notation introduced in the paragraphs 3.1.3 and 3.1.5 and we fix now some supplementary notation that we will keep during all this section. We start choosing

- \( \mathcal{R}_v(x, t) \) a non-zero element of \( \mathcal{F}_v(t) \otimes \mathcal{K}_v \) representing the point \([K_v(x, t)]\),
- \( \mathcal{R}_v(\theta, t_0) \) a non-zero element of \( \mathcal{F}_v(t_0) \otimes \mathcal{K}_v \) representing the point \([K_v(\theta, t_0)]\),

so that, by the very definition of instability measure (paragraph 3.2.5), we have:

\[
\tau_v(P_r) = \log \inf_{g \in \text{SL}_2(C_v)} \frac{\| g \cdot (\mathcal{R}_v(x, t_0) \otimes \mathcal{R}_v(\theta, t_0)) \|_{\mathcal{F}_v(t_0) \otimes \mathcal{F}_v(\theta, t_0), v}}{\| \mathcal{R}_v(x, t_0) \otimes \mathcal{R}_v(\theta, t_0) \|_{\mathcal{F}_v(t_0) \otimes \mathcal{F}_v(\theta, t_0), v}}.
\]

Since we are simply interested in a upper bound for the instability measure, what we will actually do is bounding the instability with the size of some special points in the orbit of \( P_r \). More precisely, for every embedding \( \sigma : K' \to C_v \), we will define an element \( \tilde{\mathcal{R}}^\sigma \) of the group \( \text{SL}_2(C_v) \), and in such a way we will have:

\[
\tau_v(P_r) \leq \inf_{\sigma : K' \to C_v} \left\{ \tau_v(\tilde{\mathcal{R}}^\sigma, [K_v(x, t_0)]) + \tau_v(\tilde{\mathcal{R}}^\sigma, [K_v(\theta, t_0)]) \right\} \tag{5.1.1}
\]

where

\[
\tau_v(\tilde{\mathcal{R}}^\sigma, [K_v(x, t_0)]) = \log \frac{\| \tilde{\mathcal{R}}^\sigma \cdot \mathcal{R}_v(x, t_0) \|_{\mathcal{F}_v(t_0), v}}{\| \mathcal{R}_v(x, t_0) \|_{\mathcal{F}_v(t_0), v}};
\]

\[
\tau_v(\tilde{\mathcal{R}}^\sigma, [K_v(\theta, t_0)]) = \log \frac{\| \tilde{\mathcal{R}}^\sigma \cdot \mathcal{R}_v(\theta, t_0) \|_{\mathcal{F}_v(t_0), v}}{\| \mathcal{R}_v(\theta, t_0) \|_{\mathcal{F}_v(t_0), v}}.
\]

According to (5.1.1), the upper bounds that we stated in the paragraph 3.2.5 are obtained summing the upper bounds of \( \tau_v(\tilde{\mathcal{R}}^\sigma, [K_v(x, t_0)]) \) and \( \tau_v(\tilde{\mathcal{R}}^\sigma, [K_v(\theta, t_0)]) \) that will be respectively proved in Propositions 5.1 and 5.2.
5.2 Instability measure at the rational point

Let us pass to the definition of the elements $\delta^\sigma_i$ that we alluded before. Let $\sigma : K \to C_\nu$ be an embedding and for any $i = 1, \ldots, n$ let:

- $\delta^\sigma_i = (\theta_i^\sigma, \theta_i^{\sigma^*}) \in A^2(C_\nu)$ be a representative of $\sigma(\theta_i) \in P^1(C_\nu)$ such that $\|\delta^\sigma_i\|_\nu = 1$;
- $\bar{x}_i = (x_{i0}, x_{i1}) \in A^2(K_\nu)$ be a representative of $x_i \in P^1(K)$ such that $\|\bar{x}_i\|_\nu = 1$;

We consider the automorphism $g^\sigma_i$ of $C_\nu^2$ such that the automorphism induced on the dual vector space $C_\nu^{2^*}$ is given by the following matrix (with respect to the canonical basis $T_0, T_1$):

$$
\left[\begin{array}{cc}
\theta_i^\sigma & \theta_i^{\sigma^*} \\
x_{i0} & x_{i1}
\end{array}\right].
$$

Explicitly $g^\sigma_i$ is given by the inverse and transpose matrix of the preceding one:

$$
g^\sigma_i := \left[\begin{array}{cc}
\theta_i^\sigma & \theta_i^{\sigma^*} \\
x_{i0} & x_{i1}
\end{array}\right]^{-1} = \frac{1}{\theta_i^{\sigma x_{i1}} - \theta_i^{\sigma^* x_{i0}}} \left[\begin{array}{cc}
x_{i1} - x_{i0} & -x_{i0} \\
-x_{i1} & -\theta_i^{\sigma^*}
\end{array}\right] \in GL_2(C_\nu).
$$

Let us point out some straightforward properties of $g^\sigma_i$ that we will need in the computation:

- The key property of the element $g^\sigma_i$ is that it measures distance of a point $y \in P^1(C_\nu)$ and the points $x_i$ and $\sigma(\theta_i)$. In fact, let $y = (y_0, y_1) \in A^2(C_\nu)$ be a non-zero representant of the point $y$ and suppose that $\|y_0\|_\nu = 1$. We have:

$$
\|g^\sigma_i \cdot (y_0 T_1 - y_1 T_0)\|_\nu = \begin{cases} 
\max\{d_\nu(\sigma(\theta_i), y_1), d_\nu(x_i, y_1)\} & \text{if } \nu \text{ is non-archimedean} \\
\sqrt{d_\nu(\sigma(\theta_i), y_1)^2 + d_\nu(x_i, y_1)^2} & \text{if } \nu \text{ archimedean}
\end{cases} \quad (5.1.2)
$$

- Since the points $\delta^\sigma_i$ and $\bar{x}_i$ are of norm 1 we have $d_\nu(\sigma(\theta_i), x_i) = |\delta_i^\sigma x_{i1} - \theta_i^{\sigma^* x_{i0}}|$. Hence:

$$
|\det g^\sigma_i|_\nu = |\delta_i^\sigma_{i1} - \theta_i^{\sigma^*_{i0}}| = d_\nu(\sigma(\theta_i), x_i). \quad (5.1.3)
$$

Finally we choose a square root $\delta_i \in C_\nu$ of $\det g^\sigma_i$ so that the automorphism $g^\sigma_i := g^\sigma_i / \delta_i^\sigma$ is of determinant 1. The element $g^\sigma_i$ is the $n$-tuple of the elements $g^\sigma_i$'s,

$$
g^\sigma := (g^\sigma_1, \ldots, g^\sigma_n).
$$

5.2 Instability measure at the rational point

5.2.1. — In this section we will prove the following upper bound for $t_\nu \left(\delta^\sigma, [K, r(x, t_3)]\right)$.

**Proposition 5.1.** With the notation introduced above, we have :

- if $\nu$ is non-archimedean:

$$
t_\nu \left(\delta^\sigma, [K, r(x, t_3)]\right) \leq \sum_{i=1}^n \left( \sum_{l \in \mathbb{N} \cap \{r_i(x)\}} \frac{\ell_l - k_i(t_3)}{2} r_l \right) \log d_\nu(\sigma(\theta_i), x_i);
$$

- if $\nu$ is archimedean:

$$
t_\nu \left(\delta^\sigma, [K, r(x, t_3)]\right) \leq \sum_{i=1}^n \left( \sum_{l \in \mathbb{N} \cap \{r_i(x)\}} \frac{\ell_l - k_i(t_3)}{2} r_l \right) \log d_\nu(\sigma(\theta_i), x_i) + 2 k_i(t_3) |r|.
$$
5.2.2. — To begin with, let us remark that, since the representation $\mathbf{GL}_n^+ \rightarrow \mathbf{GL}(\mathcal{F}_\tau(t_x))$ is homogeneous of weight $k_\tau(t_x)\tau$, writing $g^\sigma = ((\delta^\sigma_1)^{-k_\tau(t_x)\tau}g_1, \ldots, (\delta^\sigma_n)^{-k_\tau(t_x)\tau}g_n)$ we have

$$t_\nu(g^\sigma, [K_\tau(x, t_x)]) = t_\nu(g^\sigma, [K_\tau(x, t_x)]) - k_\tau(t_x) \left( \sum_{i=1}^n r_i \log |\delta_i| \right)$$

$$= t_\nu(g^\sigma, [K_\tau(x, t_x)]) - \frac{k_\tau(t_x)}{2} \left( \sum_{i=1}^n r_i \log d_\nu(\sigma(\theta_i), x_i) \right), \quad (5.2.1)$$

where

$$t_\nu(g^\sigma, [K_\tau(x, t_x)]) = \log \frac{\|g^\sigma \cdot \tilde{K}_\nu(x, t_x)\|_{\mathcal{F}_\tau(t_x), \nu}}{\|K_\tau(x, t_x)\|_{\mathcal{F}_\tau(t_x), \nu}}.$$

5.2.3. — Let us consider for any $i = 1, \ldots, n$ the following elements of $K^2v$.

$$T_{i0} = x_{i0}T_0 + x_{i1}T_{i1}$$

$$T_{i1} = -x_{i1}T_0 + x_{i0}T_{i0}.$$ 

Since the point $\tilde{x}_i \in K^2$ is of norm 1, the elements $T_{i0}$ and $T_{i1}$ are of norm 1. If $\nu$ is non-archimedean, and $o_\nu$ denotes ring of integers of $K_\nu$, they form a basis of the $o_\nu$-module $o_\nu^{1/2}$; if $\nu$ is archimedean, they are orthogonal so they form an orthonormal basis of $K^{2v}$.

A basis of the K-vector space $K_\tau(x, t_x)$ is now given by the elements of the form

$$T(\ell) = \bigotimes_{i=1}^n T_{i0}^{r_{i0} - \ell_i} T_{i1}^{r_{i1}}$$

where $\ell = (\ell_1, \ldots, \ell_n)$ ranges in the elements of $\mathcal{V}_\tau(t_x)\mathbb{Z}$.

- If $\nu$ is non-archimedean the elements $T(\ell)$’s form a basis of the $o_\nu$-module $(K_\tau(x, t_x) \otimes K_\nu) \cap (T'P, \mathcal{O}_P(r)) \otimes o_\nu$). Hence

$$\log \|\tilde{K}_\nu(x, t_x)\|_{\mathcal{F}_\tau(t_x), \nu} = 0.$$

- If $\nu$ is archimedean the elements $T(\ell)$’s are orthogonal but they are not of norm 1. Precisely we have

$$\log \|\tilde{K}_\nu(x, t_x)\|_{\mathcal{F}_\tau(t_x), \nu} = \log \left( \bigwedge_{\ell \in \mathcal{V}_\tau(t_x)\mathbb{Z}} T(\ell) \bigg|_{\mathcal{F}_\tau(t_x), \nu} \right) = \sum_{\ell \in \mathcal{V}_\tau(t_x)\mathbb{Z}} \log \|T(\ell)\|_{\mathcal{F}_\tau(t_x), \nu}$$

$$= -\frac{1}{2} \sum_{\ell \in \mathcal{V}_\tau(t_x)\mathbb{Z}} \sum_{i=1}^n \log \left( \ell_i \right).$$

Bounding the binomial $\binom{n}{r_i}$ with $\binom{r_i}{r_i/2}$ and using Stirling approximation we obtain

$$\log \|\tilde{K}_\nu(x, t_x)\|_{\mathcal{F}_\tau(t_x), \nu} = -\frac{1}{2} \sum_{\ell \in \mathcal{V}_\tau(t_x)\mathbb{Z}} \sum_{i=1}^n \log \left( \ell_i \right) \approx -\frac{1}{2} \sum_{\ell \in \mathcal{V}_\tau(t_x)\mathbb{Z}} \sum_{i=1}^n 2r_i = -k_\tau(t_x)\tau.$$
norm on symmetric powers, we have:

\[
\log \| g^\sigma \cdot \tilde{K}_r(x, t_x)\|_{\mathcal{F}_r(t_x), v} = \log \left\| g^\sigma \cdot \left( \bigwedge_{(\ell \in \mathbb{V}_r(t_x)) \mathbb{Z}} T(\ell) \right) \right\|_{\mathcal{F}_r(t_x), v} \\
\leq \sum_{\ell \in \mathbb{V}_r(t_x) \mathbb{Z}} \log \| g^\sigma \cdot T(\ell) \|_{v} + \ell_i \log \| g^\sigma \cdot T_{\tilde{I}_1} \|_{v}
\]

The property (5.1.3) of \( g_i^\sigma \) is the one that justifies its definition. In fact we have:

- \( \| g^\sigma \cdot T_{\tilde{I}_0} \|_v \leq \begin{cases} 1 & v \text{ non-archimedean} \\ \sqrt{2} & v \text{ archimedean} \end{cases} \)
- \( \| g^\sigma \cdot T_{\tilde{I}_1} \|_v = d_v(\sigma(\theta), x_i) \).

Employing this in the previous upper bound, if \( v \) is non-archimedean, we obtain:

\[
t_v(g^\sigma, [K_r(x, t_x)]) = \log \| g^\sigma \cdot \tilde{K}_r(x, t_x)\|_{\mathcal{F}_r(t_x), v} \leq \sum_{i=1}^n \left( \sum_{\ell \in \mathbb{V}_r(t_x) \mathbb{Z}} \ell_i \right) d_v(\sigma(\theta_i), x_i).
\]

Otherwise, if \( v \) archimedean, we have:

\[
t_v(g^\sigma, [K_r(x, t_x)]) \leq \log \| g^\sigma \cdot \tilde{K}_r(x, t_x)\|_{\mathcal{F}_r(t_x), v} + k_r(t_x) |r| \\
\leq \sum_{i=1}^n \left( \sum_{\ell \in \mathbb{V}_r(t_x) \mathbb{Z}} \ell_i \right) d_v(\sigma(\theta_i), x_i) + \left( \log \sqrt{2} + 1 \right) k_r(t_x) |r|.
\]

So we conclude the proof according to (5.2.1) and, in the archimedean case, bounding \( \log \sqrt{2} + 1 \) by 2.

\[\square\]

5.3 Instability measure at the algebraic point

5.3.1. — In this section we will prove the following upper bound for \( t_v\left(g^\sigma, [K_r(\theta, t_0)]\right)\).

Proposition 5.2. With the notation introduced above, we have:

- if \( v \) is non-archimedean:

\[
t_v\left(g^\sigma, [K_r(\theta, t_0)]\right) \leq k_r(t_0) \left( t_0 \max_{i=1, \ldots, n} \{ r_i \log d_v(\sigma(\theta_i), x_i) \} - \frac{1}{2} \sum_{i=1}^n r_i \log d_v(\sigma(\theta_i), x_i) \right);
\]

- if \( v \) is archimedean:

\[
t_v\left(g^\sigma, [K_r(\theta, t_0)]\right) \leq k_r(t_0) \left( t_0 \max_{i=1, \ldots, n} \{ r_i \log d_v(\sigma(\theta_i), x_i) \} - \frac{1}{2} \sum_{i=1}^n r_i \log d_v(\sigma(\theta_i), x_i) \right) + 2k_r(t_0) |r|.
\]

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5.3.2. — As in the proof of the Proposition 5.1 let us begin remarking that, since the representation $GL_n^\sigma \rightarrow GL(\mathcal{F}_r(t_0))$ is homogeneous of weight $k_r(t_0) \cdot r$, we have

$$
\tau_v(g^\sigma, [K_r(0, t_0)]) = \tau_v(g^\sigma, [K_r(0, t_0)]) - k_r(t_0) \left( \sum_{i=1}^{n} r_i \log |\delta_i| \right)
$$

\[= \tau_v(g^\sigma, [K_r(0, t_0)]) - \frac{k_r(t_0)}{2} \left( \sum_{i=1}^{n} r_i \log d_v(\sigma(\theta_i), x_i) \right), \quad (5.3.1) \]

where

$$
\tau_v(g^\sigma, [K_r(0, t_0)]) = \log \frac{\| g^\sigma \cdot \tilde{K}_r(0, t_0) \|_{\mathcal{F}_r(t_0), v}}{\| K_r(0, t_0) \|_{\mathcal{F}_r(t_0), v}}.
$$

5.3.3. — Let us consider a basis $f_1, \ldots, f_{k_r(t_0)}$ of the $C_v$-vector space $K(\theta, t_0) \otimes C_v$ such that:

- if $v$ is non-archimedean and $\tilde{\sigma}_v$ denotes the ring of integers of $C_v$, the elements $f_{a}$'s are a basis of the $\tilde{\sigma}_v$-module $(K_r(\theta, t_0) \otimes C_v) \cap \{ \Gamma(p, \mathcal{O}_p(r)) \otimes G_v \}$.

- if $v$ is archimedean, the elements $f_{a}$'s are an orthonormal basis.

Then, by Hadamard's inequality, we have:

$$
\tau_v(g^\sigma, [K_r(0, t_0)]) = \log \left\| g^\sigma \cdot \left( \bigwedge_{\alpha=1}^{k_r(t_0)} f_{a} \right) \right\|_{\mathcal{F}_r(t_0), v} \leq \sum_{a=1}^{k_r(t_0)} \log \| g^\sigma \cdot f_a \|_{\Gamma(p, \mathcal{O}_p(r)), v},
$$

so we are left with the proof of the following lemma:

**Lemma 5.3.** Let $f$ be a non-zero element of $K_r(\theta, t_0)$. With the notation introduced above, we have

- if $v$ is non-archimedean:

$$
\log \frac{\| g^\sigma \cdot f \|_{\Gamma(p, \mathcal{O}_p(r)), v}}{\| f \|_{\Gamma(p, \mathcal{O}_p(r)), v}} \leq t_0 \max_{i=1, \ldots, n} \{ r_i \log d_v(\sigma(\theta_i), x_i) \};
$$

- if $v$ is archimedean:

$$
\log \frac{\| g^\sigma \cdot f \|_{\Gamma(p, \mathcal{O}_p(r)), v}}{\| f \|_{\Gamma(p, \mathcal{O}_p(r)), v}} \leq t_0 \max_{i=1, \ldots, n} \{ r_i \log d_v(\sigma(\theta_i), x_i) \} + 2|r|.
$$

5.3.4. — To begin with, let us consider the following elements of $C_v^{32}$:

$$
T_{10} := \theta_t^0 T_1 + \theta_t^0 T_1
$$

$$
T_{11} := \theta_t^0 T_1 - \theta_t^0 T_0
$$

Since the points $\tilde{\theta}_t^0$ are of norm 1, the elements $T_{10}$ and $T_{11}$ are of norm 1. If the place $v$ is non-archimedean they form a basis of the $\tilde{\sigma}_v$-module $\tilde{C}_v^{32}$ (where $\tilde{\sigma}_v$ is the ring of integers of $C_v$); if the place $v$ is archimedean they are orthogonal and so they form an orthonormal basis.

For any $n$-tuple of integers $\ell = (\ell_1, \ldots, \ell_n) \in \square_n$, let us define

$$
T(\ell) = \bigotimes_{i=1}^{n} T_{10}^{\ell_i} T_{11}^{\ell_i}.
$$
The monomials \( T(\ell) \)'s form a basis of the \( C_\nu \)-vector space \( \Gamma(\mathbb{P}, O_\mathbb{P}(r)) \otimes C_\nu \). If \( \nu \) is non-archimedean the elements \( T(\ell) \)'s form a basis of the \( \mathfrak{a}_\nu \)-module \( K_r(x, t_3) \cap \Gamma(\mathbb{P}, O_\mathbb{P}(r)) \); if \( \nu \) is archimedean the elements \( T(\ell) \)'s are orthogonal and

\[
\|T(\ell)\|_{\Gamma(\mathbb{P}, O_\mathbb{P}(r)), \nu} = \left( \sum_{i=1}^{n} r_i \right)^{-1/2} = \prod_{i=1}^{n} \left( \frac{r_i}{\ell_i} \right)^{-1/2}.
\]

5.3.5. — The same computation of that in proof of the Proposition 5.1, that is using the sub-multiplicativity of the norm on symmetric powers, gives:

\[
\log \|g^\sigma \cdot T(\ell)\|_{\Gamma(\mathbb{P}, O_\mathbb{P}(r)), \nu} \leq \sum_{i=1}^{n} (r_i - \ell_i) \log \|g^\sigma_i \cdot T_{i0}\|_{\nu} + \ell_i \log \|g^\sigma_i \cdot T_{i1}\|_{\nu}.
\]

Now, using the key property (5.1.3) of \( g^\sigma \) we have:

- \( \|g^\sigma \cdot T_{i0}\|_{\nu} = \left\{ \begin{array}{ll} 1 & \text{if } \nu \text{ non-archimedean} \\
\sqrt{2} & \text{if } \nu \text{ archimedean} \end{array} \right. \)

- \( \|g^\sigma \cdot T_{i1}\|_{\nu} = \nu(\sigma(\theta_1), x_1) \).

So that we obtain the following upper bounds:

- if \( \nu \) non-archimedean:

\[
\log \|g^\sigma \cdot T(\ell)\|_{\Gamma(\mathbb{P}, O_\mathbb{P}(r)), \nu} \leq \sum_{i=1}^{n} \ell_i \log \nu(\sigma(\theta_1), x_1);
\]

- if \( \nu \) archimedean:

\[
\log \|g^\sigma \cdot T(\ell)\|_{\Gamma(\mathbb{P}, O_\mathbb{P}(r)), \nu} \leq \sum_{i=1}^{n} \ell_i \log \nu(\sigma(\theta_1), x_1) + \log \sqrt{2} \nu(t_0)|r|.
\]

5.3.6. — We are now ready for the proof of the Proposition 5.2. Let us develop the global section \( f \in K_r(\theta, t_0) \) with respect to the basis \( \{ T(\ell) \} \):

\[
f = \sum_{\ell \in \Delta, \nu} c_\ell T(\ell)
\]

where \( c_\ell \in C_\nu \). Moreover,

- if \( \nu \) is non-archimedean: \( \|f\|_{\Gamma(\mathbb{P}, O_\mathbb{P}(r)), \nu} = \max \{|c_\ell|_\nu : \ell \in \Delta, \nu\} \).

- if \( \nu \) is archimedean: \( \|f\|^2_{\Gamma(\mathbb{P}, O_\mathbb{P}(r)), \nu} = \sum_{\ell \in \Delta, \nu} |c_\ell|_\nu^2 \left( \frac{r}{\ell} \right)^{-1} \).

Since the distance on the projective line is bounded by 1, for every \( \ell \in \Delta, \nu \) we have:

\[
\sum_{i=1}^{n} \ell_i \log \nu(\sigma(\theta_1), x_1) \leq \sum_{i=1}^{n} \frac{\ell_i}{r_i} \max \{ r_i \log \nu(\sigma(\theta_1), x_1) \}
\]

\[
\leq t_0 \max \{ r_i \log \nu(\sigma(\theta_1), x_1) \},
\]

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By definition the global section $f$ satisfy $\text{ind}_1(f, \sigma(0)) \geq t_0$, i.e. we have $c_\ell = 0$ for every $\ell \in \Delta_\sigma(t_0)\mathbb{Z}$. Applying the previous considerations we obtain for $n$ non-archimedean:

$$\log \| g^\sigma \cdot f \|_{\Gamma(P, \mathcal{O}_P(r))}, v \leq \max_{\ell \in V_\ell(\mathbb{Z})} \left\{ \log |c_\ell|_v + \log \| g^\sigma \cdot T(\ell) \|_{\Gamma(P, \mathcal{O}_P(r)), v} \right\} \leq t_0 \max_{i=1, \ldots, n} \left\{ r_i \log d_{\sigma}(\sigma(0)), x_i \right\} + \log \| f \|_{\Gamma(P, \mathcal{O}_P(r)), v},$$

which actually concludes the proof in the non-archimedean case. Let us then suppose that $v$ is archimedean. Applying the triangle inequality we have:

$$\| g^\sigma \cdot f \|_{\Gamma(P, \mathcal{O}_P(r))} \leq \sum_{\ell \in V_\ell(\mathbb{Z})} |c_\ell|_v \| g^\sigma \cdot T(\ell) \|_{\Gamma(P, \mathcal{O}_P(r)), v} \leq \max_{i=1, \ldots, n} \left\{ d_{\sigma}(\sigma(0)), x_i \right\} \left\{ \sum_{\ell \in V_\ell(\mathbb{Z})} |c_\ell| \prod_{i=1}^n 2^{r_i - \ell_i} \right\} \left( \prod_{i=1}^n (r_i + 1) \right).$$

Comparing $\ell^1$ and $\ell^2$ norms on $\Gamma(P, \mathcal{O}_P(r))$ thanks to Jensen’s inequality we have:

$$\sum_{\ell \in V_\ell(\mathbb{Z})} |c_\ell|_v \prod_{i=1}^n 2^{r_i - \ell_i} \leq \max_{i=1, \ldots, n} \left\{ d_{\sigma}(\sigma(0)), x_i \right\} \left\{ \sum_{\ell \in V_\ell(\mathbb{Z})} |c_\ell| \prod_{i=1}^n 2^{r_i - \ell_i} \right\} \left( \prod_{i=1}^n (r_i + 1) \right) \leq \max_{i=1, \ldots, n} \left\{ d_{\sigma}(\sigma(0)), x_i \right\} \left\{ \prod_{i=1}^n (r_i + 1) \right\} \| f \|_{\Gamma(P, \mathcal{O}_P(r)), v}^2.$$

Finally, using Stirling approximation we find

$$\sum_{i=1}^n \log (r_i + 1) + \max_{\ell \in V_\ell(\mathbb{Z})} \left\{ \log \binom{r}{\ell} + \sum_{i=1}^n (r_i - \ell_i) \log 2 \right\} \leq \frac{5}{2} |r| \leq 4 |r|,$$

so we conclude according to (5.3.2).

\[\square\]

6 Semi-stability

6.1 Basic facts about the semi-stability of subspaces

6.1.1. Instability coefficient. — Let $G$ a K-reductive group acting on a proper K-scheme equipped with a G-equivariant invertible sheaf $L$. Let $x$ be a K-point of $X$. Let $\lambda : G_m \to G$ be a one-parameter subgroup of $G$ and consider the morphism $\lambda_x : G_m \to X$ given by

$$\lambda_x(t) := \lambda(t) \cdot x.$$

By properness of $X$, the morphism $\lambda_x$ extends in a unique way to a morphism $\overline{\lambda}_x : A^1 \to X$. We denote by $x_0$ the K-point $\overline{\lambda}_x(0)$. Since it is a fixed point under the action of $G_m$, then $G_m$ acts on the K-vector space $x_0^*L$ through a character

$$t \mapsto t^{|l|_1(x,x)},$$

with $|l|_1(\lambda, x) \in \mathbb{Z}$. We call it the instability coefficient of $x$ with respect to the one-parameter subgroup $\lambda$ and the invertible sheaf $L$. Let us remark that the sign in this definition is opposite with respect to the convention adopted in [GIT, Definition 2.2].

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Theorem 6.1 (Hilbert-Mumford-Kempf-Rousseau, [Kem78, Theorem 4.2]). With the notation introduced above suppose that \( L \) is ample and \( K \) is perfect. Then the point \( x \) is semi-stable if and only if
\[
\mu_L(\lambda, x) \leq 0
\]
for every one-parameter subgroup \( \lambda : \mathbb{G}_m \to X \).

6.1.2. Linear action on grassmannians. — Let \( V \) be a finite dimension \( \mathbb{K} \)-vector space and \( r \) be a non-negative integer. We consider the grassmannian of \( r \)-dimensional subspaces \( \text{Grass}_r(V) \) and its Plücker embedding \( \tilde{\varphi} : \text{Grass}_r(V) \to \mathbb{P}(\bigwedge^r V) \).

Suppose that a \( \mathbb{K} \)-reductive group \( G \) acts linearly on \( V \). Then it acts on the grassmannian \( \text{Grass}_r(V) \), on the projective space \( \mathbb{P}(\bigwedge^r V) \) and in an equivariant way on the invertible sheaf \( \mathcal{O}(1) \) on \( \mathbb{P}(\bigwedge^r V) \).

Since the Plücker embedding \( \tilde{\varphi} \) is \( G \)-equivariant with respect to this action, the ample invertible sheaf \( \tilde{\varphi}^* \mathcal{O}(1) \) on \( \text{Grass}_r(V) \) is naturally endowed with a \( G \)-equivariant action.

Definition 6.2. For any subspace \( W \subset V \) of dimension \( r \) we will denote simply by \( \mu(\lambda, [W]) \) the integer
\[
\mu_{\tilde{\varphi}^* \mathcal{O}(1)}(\lambda, [W]) = \mu_{\mathcal{O}(1)}(\lambda, \tilde{\varphi}([W]))
\]
omitting the polarisation \( \tilde{\varphi}^* \mathcal{O}(1) \).

6.1.3. Formula for the instability coefficient. — Keeping the notation introduced here above, let \( \lambda : \mathbb{G}_m \to G \) be a one-parameter subgroup. For any integer \( a \) let us consider the subspace \( V_a := \{ v \in V : \lambda(t) \cdot v = t^a v \} \).

For a sub-space \( W \subset V \) we define \( W_a := W \cap V_a \) and
\[
W[a] := \bigoplus_{b \leq a} W_b.
\]
The subspaces \( W[a] \) define a decreasing filtration of \( W \) and we have
\[
\mu(\lambda, [W]) = \sum_{a \in \mathbb{Z}} a(\dim_{\mathbb{K}} W[a] - \dim_{\mathbb{K}} W[a + 1])
= a_{\text{min}} \dim_{\mathbb{K}} W + \sum_{a = a_{\text{min}} + 1}^{a_{\text{max}}} \dim_{\mathbb{K}} W[a],
\]
where \( a_{\text{min}} \) (resp. \( a_{\text{max}} \)) denotes the smallest (resp. the biggest) integer \( a \) such that \( V_a \) is non-zero. Let us draw some easy consequences of this formula.

- **Inclusion formula**: Let \( W' \subset W \) be subvector spaces of \( V \). Then we have:
\[
\mu(\lambda, [W']) \leq \mu(\lambda, [W]) + a_{\text{min}}(\dim_{\mathbb{K}} W - \dim_{\mathbb{K}} W') \tag{6.1.1}
\]

- **Grassmann formula**: let \( W_1 \) and \( W_2 \) be sub-vector spaces of \( V \). Then we have:
\[
\mu(\lambda, [W_1]) + \mu(\lambda, [W_2]) \leq \mu(\lambda, [W_1 + W_2]) + \mu(\lambda, [W_1 \cap W_2]). \tag{6.1.2}
\]

In fact for any integer \( a \in \mathbb{Z} \) the usual Grassmann formula for dimensions gives
\[
\dim_{\mathbb{K}} W_1[a] + \dim_{\mathbb{K}} W_2[a] = \dim_{\mathbb{K}} (W_1[a] + W_2[a]) + \dim_{\mathbb{K}} (W_1[a] \cap W_2[a])
\]
and we conclude noticing that \( W_1[a] + W_2[a] \subset (W_1 + W_2)[a] \).
In the following will be also useful to remark the following fact.

Let $W \subset V$ be a sub-vector space of $V$ and $w_1, \ldots, w_r$ be a basis for $W$ where $r$ is the dimension of $W$. For any $i = 1, \ldots, r$ let us consider the instability coefficient $\mu(\lambda, [w_i])$ of the point $[w_i] \in P(V)$. Then the vector $w_i$ writes as

$$
\lambda(\tau) \cdot w_i = \tau^{i(\lambda, [w_i])} w_{i,\text{min}} + \text{(terms of higher order in } \tau)$$

with $w_{i,\text{min}} \in V$. Suppose that the basis $[w_i]$ is such that the elements $w_{1,\text{min}}, \ldots, w_{r,\text{min}} \in V$ are linearly independent; then

$$
\mu(\lambda, [W]) = \sum_{i=1}^r \mu(\lambda, [w_i]).
$$

### 6.2 Asymptotic semi-stability of $P_r$

6.2.1. — We go back to the notation introduced in the paragraphs 3.1.3 and 3.1.5. Since the integral models do not play any role here, we will silently work over $K$ and not over $\sigma_K$ (for example $P$ will denote the product of $n$ copies of the projective $P_K^n$ over $K$).

**Theorem 6.3.** Let $n \geq 1$ be a positive integer and $r = (r_1, \ldots, r_n)$ be a $n$-tuple of positive integers. Let $t_x, t_0 \geq 0$ be non-negative real numbers. If the inequality

$$
\left| \text{vol} \nabla_n(t_x) - 2 \int_{\nabla_n(t_x)} \xi_1 \ d\lambda \right| < \left| \text{vol} \nabla_n(t_0) - 2 \int_{\nabla_n(t_0)} \xi_1 \ d\lambda \right| - \epsilon(r)
$$

is satisfied then there exists a positive integer $\alpha_0 = \alpha_0(n, d, r, t_0, t_x)$ such that, for any integer $\alpha \geq \alpha_0$, the $K$-point of the product $X_{\alpha r} = \text{Grass}_{\alpha r}(t_0) \left( \Gamma(P, \mathcal{O}_P(\alpha r)) \right) \times \text{Grass}_{\alpha r}(t_x) \left( \Gamma(P, \mathcal{O}_P(\alpha r)) \right)$,

$$
P_{\alpha r} = ([K_{\alpha r}(0, t_0), [K_{\alpha r}(t_x, t_x)])
$$

is semi-stable under the action of $\text{SL}_r^n$ and with respect to the polarization given by the Plücker embeddings.

6.2.2. — To begin let us remark that the condition (6.2.1) implies that there exists positive real numbers $\delta = \delta(n, d, r, t_0, t_x), \rho = \rho(n, d, r, t_0, t_x)$ and a positive integer $\alpha_0 = \alpha_0(n, d, r, t_0, t_x)$ such that, for any integer $\alpha \geq \alpha_0$ and any $u$ such that $|u - u_r(t_0)| < \rho$ we have:

$$
\left| \# \nabla(t_x) - 2 \sum_{\ell \in \nabla(t_x)} \ell \right| < \left| \# \nabla(u) - 2 \sum_{\ell \in \nabla(u)} \ell \right| - (\epsilon(r) + \delta)(\alpha r).
$$

We will need $\alpha_0$ and $\rho$ to satisfy other three technical conditions on the approximation of some cardinalities by volumes — up to increase $\alpha_0$ and decrease $\rho$ they can always be obtained:

- Since we have

$$
\lim_{\alpha \to \infty} \frac{\dim K \Gamma(P, \mathcal{O}_P(\alpha r))}{\alpha^n(r_1 \cdots r_n)} = 1,
$$

we will suppose that for any $\alpha \geq \alpha_0$ we have

$$
\left| \frac{\dim K \Gamma(P, \mathcal{O}_P(\alpha r))}{\alpha^n(r_1 \cdots r_n)} - 1 \right| < \frac{\delta}{3}.
$$

(6.2.3)
• We recall that the dimension $k_r(t_0)$ of $K_r(\theta, t_0)$ is asymptotically given by (see (3.1.2)):

$$\liminf_{a \to \infty} \frac{k_{ar}(t_0)}{a^n(r_1 \cdots r_n)} \geq 1 - d \Delta_n(t_0).$$

Hence, we will suppose that for any integer $\alpha \geq \alpha_0$ and any $u$ such that $|u - u_r(t_0)| < \rho$ we have:

$$\frac{k_{ar}(t_0)}{a^n(r_1 \cdots r_n)} > (1 - d \Delta_n(t_0)) - \frac{\delta}{3} \quad (6.2.4)$$

• Let us remark here that the semi-stability condition (6.2.1) is never satisfied for $u_r(t_0) = 0, n$. Hence, by the very definition of $u_r(t_0)$, we have

$$\text{vol} \Delta_n(u_r(t_0)) = 1 - d \text{vol} \Delta_r(t_0) + \epsilon_n(r).$$

In particular,

$$\lim_{a \to \infty} \frac{\#\nabla_u(u_r(t_0))]_Z}{a^n(r_1 \cdots r_n)} = \text{vol} \nabla_u(u_r(t_0)) = 1 - \text{vol} \Delta_n(u_r(t_0)) = d \text{vol} \Delta_n(t_0) - \epsilon_n(r).$$

We will finally suppose that for any integer $\alpha \geq \alpha_0$ and any $u$ such that $|u - u_r(t_0)| < \rho$ we have:

$$\left| \frac{\#\nabla_u(u)]_Z}{a^n(r_1 \cdots r_n)} - (d \text{vol} \Delta_n(t_0) - \epsilon_n(r)) \right| < \frac{\delta}{3}. \quad (6.2.5)$$

In the remainder of the proof we fix $\alpha = \alpha_0$ and we prove that $P_{ar}$ is semi-stable.

**6.2.3.** — We will prove Theorem 6.3 thanks to the Hilbert-Mumford criterion in its $K$-rational version proved by Kempf and Rousseau (Theorem 6.1).

Let $\lambda : G_m \to \text{SL}_2$ be a $K$-rational one-parameter subgroup. For any $i = 1, \ldots, n$ let $\lambda_i : G_m \to \text{SL}_2$ be the one-parameter subgroup induced by the projection onto the $i$-th factor. There exists a basis $T_{i0}, T_{i1}$ of $K^{2v}$ and a *non-negative* integer $m_i \geq 0$ such that

$$\lambda_i(\tau) \cdot T_{i0} = \tau^{m_i} T_{i0}$$

$$\lambda_i(\tau) \cdot T_{i1} = \tau^{-m_i} T_{i1}.$$  

**6.2.4.** — In general the instability coefficient is additive on a product. Hence, in this situation, we have:

$$\mu(\lambda, P_{ar}) = \mu(\lambda, [K_{ar}(x, t_x)]) + \mu(\lambda, [K_{ar}(\theta, t_0)]).$$

We will treat separately the instability coefficient at the rational point and at the algebraic one. Let us postpone the proof of the results that we state here in order to show how they permit to conclude the proof.

• **rational point:** an easy computation based on the explicit knowledge of a basis of $K_r(x, t_x)$ gives (see paragraph 6.2.5 below):

$$\mu(\lambda, [K_{ar}(x, t_x)]) \leq \sum_{i=1}^n m_i \left| k_{ar}(t_x) \cdot r_i - 2 \sum_{\ell \in \nabla_u(t_x)_Z} \ell_i \right|. \quad (6.2.6)$$

• **algebraic point:** an application of Higher Dimensional Dyson's Lemma will show that for any $u_r(t_0) < u < u_r(t_0) + \rho$ we have (see paragraph 6.2.6 here below):

$$\mu(\lambda, [K_{ar}(\theta, t_0)]) \leq \alpha(r \cdot m) (\epsilon_n(r) + \delta) - \sum_{i=1}^n m_i \left| \#\nabla_u(u)_Z - 2 \sum_{\ell \in \nabla_u(u)_Z} \ell_i \right|. \quad (6.2.7)$$
According to the additivity of the instability coefficient, summing up the previous facts, we conclude that the instability coefficient \( \mu(\lambda, P_{\alpha r}) \) is non-positive if the following condition is satisfied:

\[
\sum_{i=1}^{n} m_i \left| k_{\alpha r}(t_i) \cdot r_i - 2 \sum_{\ell \in V_{\alpha r}(t_i) Z} \ell_i \right| \leq \sum_{i=1}^{n} m_i \left( \|v_{\alpha r}(u) Z\| - 2 \sum_{\ell \in V_{\alpha r}(u) Z} \ell_i + \alpha r_i (\epsilon_n(r) + \delta) \right).
\]

Since we supposed the integers \( m_i \) to be non-negative, this is assured by (6.2.2).

**6.2.5. Instability coefficient of \([K_\tau(x, t)]\).** — For any \( i = 1, \ldots, n \) let us consider \( \chi_i(x) \in \{0, 1\} \) defined as follows

\[
\chi_i(x) := \begin{cases} 
1 & \text{if } T_{i0} \text{ vanishes at } x_i \\
0 & \text{otherwise.}
\end{cases}
\]

**Lemma 6.4.** With the notation introduced above we have:

\[
\mu(\lambda, [K_\tau(x, t)]) = \sum_{i=1}^{n} (\sum_{\ell \in V_{\tau_i}(t_i) Z} \ell_i) (-\chi_i(x)) m_i (r_i - 2 \ell_i)
\]

\[
= \sum_{i=1}^{n} (-\chi_i(x)) m_i \left( k_{\tau_i}(t_i) \cdot r_i - 2 \sum_{\ell \in V_{\tau_i}(t_i) Z} \ell_i \right).
\]

**Proof.** Let \( i = 1, \ldots, n \) be such that \( T_{i0} \) does not vanish at the point \( x_i \), i.e. \( \chi_i(x) = 0 \), so that there exists \( \xi_\ell \in K \) such that \( T_{\ell 1} - \xi_\ell T_{\ell 0} \) vanishes at \( x_i \). A basis of the K-vector space \( K \cdot (x, t) \) is given by elements of the form

\[
T(\ell) := \left( \bigotimes_{\chi_i(x) = 0} T_{i0}^{\ell_i - \ell} (T_{i1} - \xi_\ell T_{\ell 0})^{\ell_i} \right) \otimes \left( \bigotimes_{\chi_i(x) = 1} T_{i0}^{\ell_i} T_{i1}^{\ell_i - \ell} \right),
\]

where \( \ell = (\ell_1, \ldots, \ell_n) \) is integer \( n \)-tuples belonging to \( V_{\tau_i}(t_i) \). A straightforward computation gives

\[
\lambda(\tau) \cdot T(\ell) = \tau^{\mu_\ell} \left( \bigotimes_{\chi_i(x) = 0} T_{i0}^{\ell_i - \ell} (T_{i1} - \xi_\ell T_{\ell 0})^{\ell_i} \right) \otimes \left( \bigotimes_{\chi_i(x) = 1} T_{i0}^{\ell_i} T_{i1}^{\ell_i - \ell} \right),
\]

where

\[
\mu_\ell := \sum_{i=1}^{n} (-\chi_i(x)) m_i (r_i - 2 \ell_i). \tag{6.2.8}
\]

Since the integers \( m_i \geq 0 \) are supposed to be non-negative, the component of \( \lambda(\tau) \cdot T(\ell) \) of minimal weight (i.e. the polynomial multiplied by \( \tau^{\mu_\ell} \)) is

\[
T(\ell)_{\text{min}} = \left( \bigotimes_{\chi_i(x) = 0} T_{i0}^{\ell_i} (T_{i1} - \xi_\ell T_{\ell 0})^{\ell_i} \right) \otimes \left( \bigotimes_{\chi_i(x) = 1} T_{i0}^{\ell_i} T_{i1}^{\ell_i - \ell} \right).
\]

In particular the elements \( \{T(\ell) : \ell \in V_{\tau_i}(t_i) Z\} \) are linearly independent, which entails

\[
\mu(\lambda, [K_\tau(x, t)]) = \sum_{\ell \in V_{\tau_i}(t_i) Z} \mu_\ell.
\]

According to (6.2.8) this conclude the proof. \( \Box \)

where for any \( \ell \in V_{\tau_i}(t_i) Z \) we denoted by \( \mu(\lambda, [T(\ell)]) \) the instability coefficient of \( T(\ell) \) as point of the projective space \( \mathbf{P}(\Gamma(\mathbf{P}, \partial \mathbf{P}(\tau))) \). Since we supposed the integer \( m_i \) to be non-negative, the smallest exponent of \( \tau \) appearing in the expansion of \( \lambda(\tau) \cdot T(\ell) \) is \( \mu_\ell \), hence we have \( \mu(\lambda, [T(\ell)]) = \mu_\ell \).
Finally, since we supposed the integers $m_i$ to be non-negative, we recover (6.2.6) simply by

$$
\sum_{i=1}^{n} (-1)^{\chi_i} m_i \left( k_r(t_2) \cdot r_i - 2 \sum_{\ell \in V_r(t_2)_{\mathbb{Z}}} \ell_i \right) \leq \sum_{i=1}^{n} m_i \left| k_r(t_2) \cdot r_i - 2 \sum_{\ell \in V_r(t_2)_{\mathbb{Z}}} \ell_i \right|.
$$

### 6.2.6. Instability coefficient of $\{K_r(\emptyset, t_0)\}$. — Let $y$ be a $K$-point of $P^1$ and $t_y$ be a non-negative real number. We consider the subscheme $Z_r(y, t_y)$ of $P$ defined by the regular functions $f$ such that $\text{ind}_f(t_y, y) \geq t_y$.

**Lemma 6.5.** Suppose $0 < u_r(t_0) < n$. With the notation introduced above, for any $t_y > u_r(t_0)$ the intersection of the subspaces $K_r(\emptyset, t_0)$ and $K_r(y, t_y)$ is zero and we have

$$
\mu(\lambda, |K_r(\emptyset, t_0)|) + \mu(\lambda, |K_r(y, t_y)|) \leq (r \cdot m)(\dim(\Gamma(P, \mathcal{O}_P(r))) - k_r(t_0) - k_r(t_y)).
$$

**Proof.** We begin showing that the intersection of the subspaces $K_r(\emptyset, t_0)$ and $K_r(y, t_y)$ is zero. By absurd suppose that there exists a non-zero global section

$$
f \in K_r(\emptyset, t_0) \cap K_r(y, t_y).
$$

Let us pass to an algebraic closure $\overline{K}$ of $K$. Applying the Higher Dimensional Dynson’s Lemma to the $\overline{K}$-points $[\sigma(\emptyset) : \sigma : K' \to \overline{K}]$ and $y$ we get

$$
\sum_{\sigma : K' \to \overline{K}} \text{vol} \Delta_n(t_0) + \text{vol} \Delta_n(t_y) \leq 1 + \varepsilon_n(r),
$$

so that

$$
\text{vol} \Delta_n(t_y) \leq 1 + \varepsilon_n(r) - d \text{vol} \Delta_n(t_0) = \text{vol} \Delta_n(u_r(t_0)).
$$

Since the function $\text{vol} \Delta$ is strictly increasing on $[0, n]$ this contradicts $t_y > u_r(t_0)$.

In order to prove the upper bound in the statement we will apply the Grassmann formula for instability coefficients (6.1.2) to the subspaces $K_r(\emptyset, t_0)$ and $K_r(y, t_y)$. Since their intersection is zero we have:

$$
\mu(\lambda, |K_r(\emptyset, t_0)|) + \mu(\lambda, |K_r(y, t_y)|) \leq \mu(\lambda, |K_r(\emptyset, t_0) + K_r(y, t_y)|)
$$

Let us remark that, going back to the notation of paragraph 6.1.3, here the smallest weight $a$ such that $\Gamma(P, \mathcal{O}_P(r))_a \neq 0$ is

$$
a_{\text{min}} = -(r \cdot m)
$$

(which occurs only for the monomial $t_0^r \cdots \otimes t_n^r$). Hence, applying the inequality (6.1.1) to the inclusion $K_r(\emptyset, t_0) + K_r(y, t_y) \subset \Gamma(P, \mathcal{O}_P(r))$ we find

$$
\mu(\lambda, |K_r(\emptyset, t_0) + K_r(y, t_y)|) \leq -(r \cdot m)(k_r(t_0) + k_r(t_y) - \dim(\Gamma(P, \mathcal{O}_P(r))))
$$

which achieves the proof.

The upper bound (6.2.7) for the instability coefficient of the point $[K_{ar}(\emptyset, t_0)]$ is now obtained choosing a $K$-rational point $y$ in a convenient way. Let us take $u_r(t_0) < t_y < u_r(t_0) + \rho$ and consider, for any $i = 1, \ldots, n$, the $K$-point $y_i$ of $P$ defined by

$$
y_i := \begin{cases} [T_{i0} = 0] & \text{if } \#\mathcal{V}_{ar}(t_y)_{\mathbb{Z}} - 2 \sum_{\ell \in V_{ar}(t_y)_{\mathbb{Z}}} \ell_i \leq 0 \\ [T_{i1} = 0] & \text{otherwise} \end{cases}
$$

\text{Let us remark that the condition (6.2.1) is never satisfied with } u_r(t_0) = 0, n \text{ because in these cases we have }

$$
\text{vol} \mathcal{V}_{ar}(u_r(t_0)) - 2 \int_{\mathcal{V}_{ar}(u_r(t_0))} \zeta_i \, d\lambda = 0.
$$

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Applying the computation for the rational point $x$ to the K-rational point $y = (y_1, \ldots, y_n)$ we get:

$$\mu(\lambda, K_r(y, t_y)) = \sum_{i=1}^{n} m_i \left| \#V_{ar(t_y)}Z - 2 \sum_{\xi \in V_{ar(t_y)}Z} \xi_i \right|.$$  

We now use the technical conditions (6.2.3)-(6.2.5). Recalling that $k_{ar}(t_y) = \#V_r(t_y)Z$ we have:

- $\frac{\text{dim}_K \Gamma(P, O_P(r))}{\alpha^n(r_1 \cdots r_n)} - 1 < \frac{\delta}{3};$
- $\frac{k_{ar}(t_0)}{\alpha^n(r_1 \cdots r_n)} - (1 - d \text{vol} \Delta_{n}(t_0)) < \frac{\delta}{3};$
- $\frac{k_{ar}(u)}{\alpha^n(r_1 \cdots r_n)} - (d \text{vol} \Delta_{n}(t_0) - \varepsilon_n(r)) < \frac{\delta}{3}.$

Taking the sum of these three inequalities we find

$$\text{dim} \Gamma(P, O_P(ar)) - k_{ar}(t_0) - k_{ar}(t_y) \leq \alpha^n(r \cdot m)(\varepsilon_n(r) + \delta),$$

so that, applying Lemma 6.5 we have:

$$\mu(\lambda, [K_{ar}(\theta, t_0)]) \leq (r \cdot m)\left( \text{dim} \Gamma(P, O_P(r)) - k_{ar}(t_0) - k_{ar}(t_y) \right) - \mu(\lambda, [K_{ar}(y, t_y)])$$

$$\leq \alpha^n(r \cdot m)(\varepsilon_n(r) + \delta) - \sum_{i=1}^{n} m_i \left| \#V_{ar(t_y)}Z - 2 \sum_{\xi \in V_{ar(t_y)}Z} \xi_i \right|.$$

This is just the upper bound (6.2.7) that we were looking for, where we wrote $t_y$ instead of $u$.  

## 6.3 Semi-stability in the case $n = 2$.

In this section we are going to prove semi-stability in the case $n = 2$ in a slightly different way. By definition in this case we have

$$\epsilon_2(r) = (d - 1)\frac{r_2}{r_1}.$$  

### 6.3.1. — The semi-stability statement we will prove is the following one:

**Theorem 6.6.** Let $r = (r_1, r_2)$ be un couple of positive integers. Let $t_x, t_0 \geq 0$ be non-negative real numbers. If the inequality

$$2 \int_{V_2(t_x)} \xi_i d\lambda - \text{vol} V_2(t_x) < \left(1 - d \text{vol} \Delta_2(t_0)\right) \left(1 - 2(1 - d \text{vol} \Delta_2(t_0) + \epsilon_2(r))\right)$$

(6.3.1)

is satisfied then there exists a positive integer $\alpha_0 = \alpha_0(d, r, t_0, t_x)$ such that, for any integer $\alpha \geq \alpha_0$, the K-point of the product $X_{ar} = \text{Grass}_{\alpha_0(t_0)}(\Gamma(P, O_P(\alpha r))) \times \text{Grass}_{\alpha_0(t_x)}(\Gamma(P, O_P(\alpha r)))$,

$$P_{ar} = ([K_{ar}(0, t_0), [K_{ar}(x, t_x)])$$

is semi-stable under the action of $\text{SL}_2$ and with respect to the polarization given by the Plücker embeddings.

In particular, given $0 < \delta < 1$ we can apply it with
• $t_0 = t(2, \delta) = \frac{2}{d}(1 - \delta)$;
• $t_\delta = 2 - \sqrt{2\delta}$,

and this is enough to derive the Main Effective Lower Bound in the case $n = 2$.

The rest of this section is devoted to the proof of Theorem 6.6.

6.3.2. — To begin let us remark that the condition (6.3.1) implies that there exists positive real numbers $\delta = \delta(d, r, t_0, t_\delta) < 1$ and a positive integer $\alpha_0 = \alpha_0(d, r, t_0, t_\delta)$ such that, for any integer $\alpha \geq \alpha_0$ we have:

$$\left(2 \sum_{\ell \in \mathbb{V}_{\lambda_2}(t_\delta)} \ell_i \Delta \ell\right) < \alpha^3 r_1 r_2 \left(1 - d \text{vol} \Delta_2(t_0) - \delta \right) \left(1 - 2 \left(1 - d \text{vol} \Delta_2(t_0) + \varepsilon_2(r)\right)\right)$$

(6.3.2)

Up to increase $\alpha_0$ we will suppose that the following condition is also satisfied. We recall that the dimension $k_\ell(t_0)$ of $K_\ell(0, t_0)$ is asymptotically given by (see (3.1.2)):

$$\liminf_{\alpha \to \infty} \frac{k_{ar}(t_0)}{\alpha^n(r_1 \cdots r_n)} \geq 1 - d \text{vol} \Delta_n(t_0).$$

Hence, we will suppose that for any integer $\alpha \geq \alpha_0$ and any $u$ such that $|u - u_\ell(t_0)| < \rho$ we have:

$$\frac{k_{ar}(t_0)}{\alpha^n(r_1 \cdots r_n)} > \left(1 - d \text{vol} \Delta_n(t_0)\right) - \delta.$$

(6.3.3)

In the remainder of the proof we fix $\alpha \geq \alpha_0$ and we prove that $P_{ar}$ is semi-stable.

6.3.3. — As for Theorem 6.3 we will prove Theorem 6.6 thanks to the Hilbert-Mumford criterion in its K-rational version proved by Kempf and Rousseau (Theorem 6.1).

Let $\lambda : \mathbb{G}_m \to \mathbb{SL}_2$ be a K-rational one-parameter subgroup. For any $i = 1, 2$ let $\lambda_i : \mathbb{G}_m \to \mathbb{SL}_2$ be the one-parameter subgroup induced by the projection onto the $i$-th factor. There exists a basis $T_{i0}, T_{i1}$ of $K^{2\nu}$ and a non-negative integer $m_i \geq 0$ such that

$$\lambda_i(t) \cdot T_{i0} = \tau^{m_i} T_{i0}$$
$$\lambda_i(t) \cdot T_{i1} = \tau^{-m_i} T_{i1}.$$

6.3.4. — In general the instability coefficient is additive on a product. Hence, in this situation, we have:

$$\mu(\lambda, P_{ar}) = \mu(\lambda, [K_{ar}(x, t_\delta)]) + \mu(\lambda, [K_{ar}(0, t_0)]).$$

We will treat separately the instability coefficient at the rational point and at the algebraic one. Let us postpone the proof of the results that we state here in order to show how they permit to conclude the proof.

• rational point: we already proved in paragraph 6.2.5 that:

$$\mu(\lambda, [K_{ar}(x, t_\delta)]) \leq \sum_{i=1}^2 m_i \left| k_{ar}(t_\delta \cdot r_i - 2 \sum_{\ell \in \mathbb{V}_{\lambda_2}(t_\delta)} \ell_i \right|.$$

• algebraic point: in what follows we are going to prove the following upper bound:

$$\mu(\lambda, [K_{ar}(0, t_0)]) \leq \alpha^3 r_1 r_2 (r \cdot m) \left(1 - d \text{vol} \Delta(t_0) - \delta \right) \left(2 \left(1 - d \text{vol} \Delta(t_0) + \varepsilon_2(r)\right) - 1\right)$$

(6.3.4)
According to the additivity of the instability coefficient, summing up the previous facts, we conclude that the instability coefficient \( \mu(\lambda, P_{arr}) \) is non-positive if the following condition is satisfied:

\[
\sum_{i=1}^{m_i} m_i \left| k_{arr}(t_i) \cdot r_i - 2 \sum_{\ell \in V_{arr}(t_i) \mathbb{Z}} \ell \right| \leq \alpha \cdot r_1 r_2 (r \cdot m) \left( 1 - d \cdot \text{vol} \Delta(t_0) - \delta \right) \left( 1 - 2(1 - d \cdot \text{vol} \Delta_2(t_0) + \varepsilon_2(r)) \right).
\]

Since we supposed the integers \( m_i \) to be non-negative, this is assured by (6.3.2).

**6.3.5.** — The remainder of this section is devoted to the proof of the upper bound (6.3.4). To simplify notation we will drop the subscript \( i \) from \( T_{j_0}, T_{j_1} \) and we will denote by \( \infty \) the point of \( \mathbb{P}^1 \) where \( T_0 \) vanishes. Moreover we will identify \( \mathbb{A}^2 \) with the open set

\[
[T_0 \neq 0] \times [T_0 \neq 0] = (\mathbb{P}^1 - [\infty]) \times (\mathbb{P}^1 - [\infty]) \subset \mathbb{P}.
\]

What takes the place of the Higher Dimensional's Dyson Lemma here is the following fact (that actually follows the classical proof of Dyson [Dys47, Bom82]):

**Lemma 6.7.** Let \( f \) be a non-zero element of \( K_{arr}(\theta, t_0) \). Then

\[
\text{ind}_{\mathbb{P}}(f, \Delta(\infty)) \leq \alpha \max\{m_1 r_1, m_2 r_2\} (1 - d \cdot \text{vol} \Delta(t_0) + \varepsilon_2(r)).
\]

Before proving the lemma let us show how this gives the upper bound for the instability coefficient. On one side let us notice that going back to the formulas given in paragraph 6.1.3 we see easily that

\[
\mu(\lambda, [K_{arr}(\theta, t_0)]) \leq \overline{\alpha} \cdot \text{dim}_K K_{arr}(\theta, t_0)
\]

where here \( \overline{\alpha} \) denotes the biggest integer such that \( K_{arr}(\theta, t_0)[a] \) is non-zero. On the other side the crucial remark is:

\[
f \in \Gamma(\mathbb{P}, \mathcal{O}_\mathbb{P}(ar))[a] \iff \text{ind}_{\mathbb{P}}(f, \Delta(\infty)) \geq \frac{a + ar \cdot m}{2}.
\]

With this notation the previous lemma implies that

\[
\overline{\alpha} \leq 2\alpha \max\{m_1 r_1, m_2 r_2\} (1 - d \cdot \text{vol} \Delta(t_0) + \varepsilon_2(r)) - ar \cdot m
\]

\[
\leq a(r \cdot m) \left( 2(1 - d \cdot \text{vol} \Delta(t_0) + \varepsilon_2(r)) - 1 \right)
\]

and since the right-hand side is negative we have

\[
\mu(\lambda, [K_{arr}(\theta, t_0)]) \leq \overline{\alpha} \cdot \text{dim}_K K_{arr}(\theta, t_0)
\]

\[
\leq \alpha^2 r_1 r_2 (1 - d \cdot \text{vol} \Delta(t_0) - \delta)(r \cdot m) \left( 2(1 - d \cdot \text{vol} \Delta(t_0) + \varepsilon_2(r)) - 1 \right)
\]

that is what we wanted to prove.

**6.3.6.** — Let us pass to the proof of the Lemma 6.7. Since the result is homogeneous in \( r \) we may suppose \( \alpha = 1 \). To begin with, let us identify \( f \) with a polynomial \( f \in K[X_1, X_2] \) on \( \mathbb{A}^2 \) of multi-degree \( \leq (r_1, r_2) \). Then there exists a decomposition

\[
f(X_1, X_2) = \sum_{k=0}^{s} A_{1k}(X_1) A_{2k}(X_2)
\]

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with \( s \leq r_2 \) and such that \( A_{i0}, \ldots, A_{is} \) are linearly independent for \( i = 1, 2 \). By Wronsky's criterion of linear independence, for \( i = 1, 2 \) the determinant

\[
W_i(X_i) = \det \left( \frac{\partial^{\ell}A_{im}}{\partial X_i^\ell} \right)_{\ell,m=0,\ldots,s}
\]

is a non-zero polynomial in \( K[X_i] \). Hence their product

\[
W(X_1, X_2) = W_1(X_1) \cdot W_2(X_2) = \det \left( \frac{\partial^{\ell_1+\ell_2}f}{\partial X_1^{\ell_1} \partial X_2^{\ell_2}} \right)_{\ell_1, \ell_2=0,\ldots,s}
\]

is also non-zero.

6.3.7. — We can make the following elementary observations about the Wronskian \( W \):

- We have \( \deg_{X_i} W_i \leq (s + 1)(r_i - s) \) for \( i = 1, 2 \).

  **Proof.** In fact there is a basis \( A'_{i0}, \ldots, A'_{is} \) of the vector space spanned by the polynomials \( A_{i0}, \ldots, A_{is} \) such that

  \[
  0 \leq \deg_{X_i} A'_{i0} < \deg_{X_i} A'_{i1} < \cdots < \deg A_{is} \leq r_i.
  \]

  Since the wronskian of the polynomials \( A'_{i0}, \ldots, A'_{is} \) is proportional to \( W \) we may assume that

  \[
  \deg_{X_i} A_{it} \geq r_i - \ell.
  \]

  Hence we can conclude

  \[
  \deg_{X_i} W_i \leq \max_{\pi \in \mathcal{E}_1} \left\{ \sum_{\ell=0}^{s} \deg_{X_i} A_{it} - \pi(\ell) \right\} \leq \max_{\pi \in \mathcal{E}_1} \left\{ \sum_{\ell=0}^{s} r_i - \ell - \pi(\ell) \right\} \leq (s + 1)(r_i - s). \quad \square
  \]

- For any \( \sigma : \mathbb{K}(0) \to \mathbb{K} \) we have \( \text{ind}_{1/\sigma}(W, \sigma(0)) \geq (s + 1) \left( \text{vol} \Delta(t_0) - \frac{s}{r_1} \right) \).

  **Proof.** In fact for any \( \ell_1, \ell_2 = 0, \ldots, s \) we have

  \[
  \text{ind}_{1/\sigma} \left( \frac{\partial^{\ell_1+\ell_2}f}{\partial X_1^{\ell_1} \partial X_2^{\ell_2}}, \sigma(0) \right) \geq \max \left\{ 0, \text{ind}_{1/\sigma} (f, \sigma(0)) - \left( \frac{\ell_1}{r_1} + \frac{\ell_2}{r_2} \right) \right\} \geq \max \left\{ 0, t_0 - \frac{\ell_2}{r_2} - \frac{s}{r_1} \right\}.
  \]

  Since the index is valuation we have

  \[
  \text{ind}_{1/\sigma} (W, \sigma(0)) \geq \min_{\pi \in \mathcal{E}_{s+1}} \left\{ \sum_{\ell=0}^{s} \text{ind}_{1/\sigma} \left( \frac{\partial^{\pi(\ell)+\ell}f}{\partial X_1^{\ell_1} \partial X_2^{\ell_2}}, \sigma(0) \right) \right\}
  \]

  \[
  \geq \min_{\pi \in \mathcal{E}_{s+1}} \left\{ \sum_{\ell=0}^{s} \max \left\{ 0, t_0 - \frac{\ell}{r_2} - \frac{s}{r_1} \right\} = \sum_{\ell=0}^{s} \max \left\{ 0, t_0 - \frac{\ell}{r_2} - (s + 1) \frac{s}{r_1} \right\} \right\}
  \]

  \[
  \geq (s + 1) \frac{1}{2} \min \left\{ t_0, \frac{r_0^2}{r_1} \right\} - (s + 1) \frac{s}{r_1} = (s + 1) \left( \text{vol} \Delta(t_0) - \frac{s}{r_1} \right),
  \]

  the last equality coming from \( t_0 \leq 1 \). \quad \square

- Since deriving on \( A^2 \) does not affect the index at \( \Delta(\infty) \) of \( f \) we have

  \[
  \text{ind}_m (W, \Delta(\infty)) \geq (s + 1) \text{ind}_m (f, \Delta(\infty)).
  \]
6.3.8. — Let us collect the informations we proved here above. Let us identify $W$ with a global section of the invertible sheaf $\mathcal{O}_{\mathbb{P}}(r_1 - s, r_2 - s)^{\oplus(s+1)}$. We have

$$\text{ind}_{1/r}(W, \Delta(\infty)) = \frac{2}{r_i} \left( s + 1 \right) \left( r_i - s \right) - \sum_{i=1}^{2} \deg_{K_i} W_i$$

$$\leq (s + 1) \left( \sum_{i=1}^{2} \left( 1 - \frac{s}{r_i} \right) - \frac{\deg_{K_i} W_i}{r_i} \right)$$

$$= (s + 1) \left( \sum_{i=1}^{2} \left( 1 - \frac{s}{r_i} \right) - \frac{\text{mult}(W_i, \sigma(0))}{r_i} \right)$$

$$\leq (s + 1) \left( \sum_{i=1}^{2} \left( 1 - \frac{s}{r_i} \right) - \text{ind}_{1/r}(W, \sigma(0)) \right)$$

$$\leq (s + 1) \left( \sum_{i=1}^{2} \left( 1 - \frac{s}{r_i} \right) - \text{vol}(\Delta(\theta)) - \frac{s}{r_i} \right)$$

$$\leq (s + 1) \left( 2 - \frac{2}{r_2} - d \text{vol}(\Delta(\theta)) + (d - 1)\varepsilon(r) \right)$$

Let us remark that we have

$$\text{ind}_{m}(W, \Delta(\infty)) = m_1 \text{mult}(W_1, \infty) + m_2 \text{mult}(W_2, \infty) \leq \max_{i=1,2} \{r_i m_i\} \text{ind}_{1/r}(W, \Delta(\infty)).$$

Hence

$$\text{ind}_{m}(f, \Delta(\infty)) \leq \max_{i=1,2} \left\{ 2 - \frac{s}{r_2} - d \text{vol}(\Delta(\theta)) + (d - 1)\varepsilon(r) \right\}.$$ 

Now a simple argument shows that, taking powers of $f$, we can take $s/r_2$ arbitrarily close to 1 (see [Bom82, II.2 Lemma 2] for a proof). This terminates the proof.

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