Singular spectral curves in finite-gap integration

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Abstract. A number of examples of applications of the method of finite-gap integration of non-linear equations are considered in which singular spectral curves occur. In particular, constructions of orthogonal curvilinear coordinate systems for which a reducible spectral curve consists of rational components are discussed, along with constructions of finite-gap Frobenius manifolds, and a soliton deformation of spectral curves is demonstrated which consists in the creation and annihilation of singular points and corresponds to equations with self-consistent sources.

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4.1. Equations with self-consistent sources

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Bibliography

Introduction

This article is an expanded version of a report given by the author at the conference “Geometry, Dynamics, Integrable Systems—GDIS 2010” (Serbia, 7–13 September 2010), dedicated to the 60th birthdays of B. A. Dubrovin, V. V. Kozlov, I. M. Krichever, and A. I. Neishtadt.

In the report as well as in this text we restricted ourselves to a description of two examples studied by the author together with A. E. Mironov [1], [2] and P. G. Grinevich [3], [4] which demonstrate how singular spectral curves arise naturally in integrable problems in differential geometry.

- In the construction of finite-gap orthogonal curvilinear coordinate systems and solutions of the associativity equations [5] it is natural to consider the degenerate case when the geometric genus of a singular spectral curve is equal to zero [1], [2].

In this case the construction of the Baker–Akhiezer function and of finite-gap solutions reduces to solving linear systems, the solutions themselves are expressed in terms of elementary functions, and in particular, one obtains solutions of the associativity equations satisfying quasi-homogeneity conditions, which yields an analytic construction of infinitely many previously unknown Frobenius manifolds.

- It turned out that for soliton equations with self-consistent sources the spectral curve can be deformed, and a deformation reduces to the creation and annihilation of double points [3], [4].

We recall the main notions in § 1, we present these examples in §§ 2 and 3, respectively, and we briefly indicate in § 4 some more interesting examples of integrable problems in which singular spectral curves arise.

1. Baker–Akhiezer functions on singular spectral curves

1.1. Spectral curves. The definitions of spectral (complex) curves can be split into the following three mutually connected types.

1) For one-dimensional differential operators these curves are defined via Bloch functions, as was first done by Novikov for Schrödinger operators [6]. This procedure includes an explicit construction of Bloch functions.

2) For two-dimensional differential operators the spectral curves are implicitly defined for a given energy level via the ‘dispersion’ relation between quasi-momenta.
Such curves were introduced by Dubrovin, Krichever, and Novikov [7] in the case of the Schrödinger operator in a magnetic field:

\[ L = \partial \bar{\partial} + A \partial + U. \]  

In [7] the inverse problem of reconstructing such operators from algebro-geometric spectral data which include the spectral curve was also solved for finite-gap operators on a given energy level.

3) For problems which are integrable by using the Baker–Akhiezer functions introduced by Krichever [9], the spectral curves are Riemann surfaces on which these functions are defined. This includes the case when the spectral curve \( P(z, w) = 0 \), \( P \in \mathbb{C}[z, w] \), is defined algebraically via the Burchnall–Chaundy theorem, which asserts that commuting ordinary differential operators \( A_1 \) and \( A_2 \) satisfy an algebraic relation \( P(A_1, A_2) = 0 \).

We briefly present the main notions, referring for details to the surveys [8]–[11].

1) In finite-gap integration, spectral curves first appeared in Novikov’s initial article [6] as Riemann surfaces parametrizing the Bloch functions of the operator

\[ L = -\frac{d^2}{dx^2} + u(x) \]

in the Lax representation

\[ \frac{\partial L}{\partial t} = [L, A] \]

for the Korteweg–de Vries (KdV) equation. A **Bloch function** of an operator \( L \) with a periodic potential \( u(x) = u(x + T) \) is a joint eigenfunction of \( L \) and the operator \( \hat{T} \) of translation by \( T \): \( \hat{T}(f)(x) = f(x + T) \), that is, a solution of the system of equations

\[ L\psi = E\psi, \quad \hat{T}\psi = e^{ip(E)T}\psi. \]  

Since \( L \) has order two, for every value of \( E \) it has a two-dimensional space \( V_E \) of solutions of the equation \( L\psi = E\psi \), and this space is invariant under \( \hat{T} \). The **spectral curve** \( \Gamma \) appears as a two-sheeted cover

\[ \pi: \Gamma \rightarrow \mathbb{C}, \]

on which the Bloch function \( \psi(x, P) \) is well defined as a meromorphic function of \( P \in \Gamma \), so that \( \Gamma \) parameterizes all the Bloch functions. Since \( \det \hat{T}|_{V_E} = 1 \), the eigenvalues of \( \hat{T}|_{V_E} \) are equal to \( \pm 1 \) at the branch points \( E \) of this cover and \( \hat{T} \) is not diagonalized at these points. If there are finitely many such points, then \( L \) is called a **finite-gap** operator,\(^2\) and the Riemann surface \( \Gamma \) is completed to a (complex) algebraic curve by a branch point at infinity \( E = \infty \).

Let us recall that the KdV equation is the first equation in a hierarchy of equations which are represented in the form

\[ \frac{\partial L}{\partial t_k} = [L, A_k], \]

\(^2\)This terminology — finite-gap operators and Bloch functions — came from solid state physics. Later in English the original term ‘finite-zone’ was gradually replaced by ‘finite-gap’.
where \( A_k \) is a differential operator of order \( 2k+1 \), and the flows generated by these equations commute.

In particular, Novikov proved two fundamental results which lie at the basis of the finite-gap integration method [6]:

- if the potential \( u(x) \) satisfies an equation of the form
  \[
  [L, A_N + c_1 A_{N-1} + \cdots + c_{N-1} A_1] = 0
  \]  
  (Novikov’s equation), then the spectral curve is the Riemann surface given by an equation of the form \( w^2 = Q(E) \) with a polynomial \( Q(E) \) of degree \( 2N + 1 \) (such solutions of the KdV equation are also called finite-gap solutions);
- the spectral curve \( \Gamma \) and the quasi-momentum function \( p(E) \), which is well defined on \( \Gamma \) up to \( 2\pi k/T \), \( k \in \mathbb{Z} \), are first integrals of the KdV equation, and in particular, the roots of the equation \( Q(E) = 0 \) are first integrals for finite-gap solutions, and the classical Kruskal–Miura integrals are expressed in terms of these branch points.

For smooth real-valued solutions of the KdV equation the spectral curve is non-singular and real: the polynomial \( Q(E) \) has no multiple roots and all its roots are real.

The spectral curves for all one-dimensional (scalar and matrix) differential operators and difference operators with periodic coefficients are defined similarly. The construction of the spectral curve reduces to a solution of the ordinary differential equations to which the equation \( L\psi = E\psi \) (see (2)) reduces.

2) For two-dimensional differential operators \( L \) with periodic coefficients we define Floquet functions as solutions of the following problem:

\[
L\psi = E\psi, \quad \psi(x + T_1, y) = e^{ip_1 T_1} \psi(x, y), \quad \psi(x, y + T_2) = e^{ip_2 T_2} \psi(x, y),
\]

where \( T_1 \) and \( T_2 \) are the periods, and \( p_1 \) and \( p_2 \) are the quasi-momenta. The multipliers of the Floquet functions are defined as \( e^{ip_1 T_1} \) and \( e^{ip_2 T_2} \).

For hypoelliptic operators (say, for \( \partial \bar{\partial} + \cdots \) or \( \partial_y - \partial_x^2 + \cdots \), where the dots denote lower-order terms) one can show that the energy \( E \) and the quasi-momenta \( p_1 \) and \( p_2 \) satisfy an analytic ‘dispersion’ relation

\[
F(E, p_1, p_2) = 0.
\]  

(3)

The complex curve in \( p \)-space defined by (3) for a fixed value of \( E \) is called the spectral curve on the energy level \( E \) [7]. The existence of such a curve (possibly of infinite genus) can be established either by perturbation theory methods [12] or by using the Keldysh theorem [13].

In particular, with the help of the Weierstrass representation of tori in \( \mathbb{R}^3 \) (and in \( \mathbb{R}^4 \)) in terms of solutions of the equation \( \mathcal{D}\psi = 0 \) with \( \mathcal{D} \) a two-dimensional Dirac operator with periodic coefficients, the spectral curve of a torus immersed in \( \mathbb{R}^3 \) was introduced in [14] as the spectral curve of the operator \( \mathcal{D} \) on the zero energy level. It turned out that this spectral curve reflects the geometric properties of the surface [13] (see also [15]). In § 4.2 we discuss an example connected with spectral curves of tori.
In contrast to the one-dimensional situation, the spectral curve cannot be constructed by solution of the direct problem. However, it appears in the inverse problem data for the method of Baker–Akhiezer functions.

3) We demonstrate a solution of the inverse problem and a construction of solutions of non-linear equations by using Baker–Akhiezer functions as was done initially for the Kadomtsev–Petviashvili (KP) equation.

In developing techniques of Dubrovin and Its–Matveev for constructing finite-gap solutions of the KdV equation, Krichever defined the Baker–Akhiezer function (for the KP equation) and used it to solve the inverse problem of constructing solutions of the KP equation from the algebro-geometric spectral data as follows [16].

Let $\Gamma$ be a smooth Riemann surface of genus $g$, let $P$ be a distinguished point on it, let $k^{-1}$ be a local parameter in a neighbourhood of this point such that $k(P) = \infty$, and let $D = \gamma_1 + \cdots + \gamma_g$ be a generic effective divisor of degree $g$ ($\gamma_1, \ldots, \gamma_g \in \Gamma$). Then:

- there exists a unique function $\psi(x, y, t, \gamma)$, \( \gamma \in \Gamma \), which is meromorphic with respect to $\gamma$ on $\Gamma \setminus P$, has poles only at points in $D$ (more precisely, the pole divisor is $\leq D$) and has an asymptotic expression

$$\psi = e^{kx+k^2y+k^3t} \left( 1 + \sum_{m>0} \frac{\xi_m(x, y, t)}{k^m} \right) \text{ as } \gamma \to P;$$

the function $u(x, y, t) = -2\partial_x \xi_1$ satisfies the KP equation

$$\frac{3}{4} u_{yy} = \left( \frac{\partial}{\partial x} \left( u_t - \frac{1}{4} (6uu_x + u_{xxx}) \right) \right),$$

for all $\gamma \in \Gamma \setminus P$ we have

$$\frac{\partial \psi}{\partial y} = L \psi, \quad \text{where } L = \frac{\partial^2}{\partial x^2} + u, \quad (4)$$

and there is an explicit formula for $u(x, y, t)$ in terms of the theta function of the Riemann surface $\Gamma$,

$$u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \theta(U x + V y + W t + z_0) + c. \quad (5)$$

The tuple $(\Gamma, P, k^{-1}, D)$ gives Krichever’s ‘spectral data’ of the inverse problem for solutions of the KP equation and for the operator $\frac{\partial}{\partial y} - L$, for which, by (4), $\psi$ is a Floquet function on the zero energy level in the case when the potential $u$ is periodic.

If on $\Gamma$ there exists a meromorphic function with a unique pole which is of second order and is located at the point $P$, then there exists a two-sheeted cover $\Gamma \to \mathbb{C}P^1$ with branch point at $P$, the function $u$ does not depend on $y$, $\psi$ reduces to the Bloch function of the Schrödinger operator $L$, the formula (5) reduces to the Its–Matveev formula for finite-gap solutions of the KdV equation, and the tuple $(\Gamma, P, k^{-1}, D)$ becomes the inverse problem data for a finite-gap Schrödinger operator. These
results for the KdV equation were obtained earlier by Dubrovin, Novikov, Its, and Matveev (see the survey [8]).

The Riemann surface $\Gamma$ is called the spectral curve, and solutions obtained by this method are called finite-gap solutions. Here quasi-periodic solutions are not excluded (it is easy to see from (5) that such solutions can be obtained by this method). This terminology is applicable to all problems soluble by using Baker–Akhiezer functions (by the finite-gap integration method).

Real-valued smooth periodic solutions of the KdV, KP, and sine-Gordon equations can be constructed from smooth Riemann surfaces. The theta-function formulae are rather complicated, because the parameters in them are related by complex transcendent equations.

At the same time, for some other equations to which finite-gap integration is applicable one can construct interesting solutions from singular spectral curves, including curves with geometric genus zero. In the latter case a solution can be expressed in terms of elementary functions and admits a simple qualitative investigation (see §§ 2 and 3).

1.2. Algebraic curves with singularities. We recall some of the main notions related to complex algebraic curves with singularities (singular curves; see details in [17], Chap. 4).

Let $\Gamma$ be a complex algebraic curve with singularities.

Then for a smooth algebraic curve $\Gamma_{nm}$ there exists a morphism

$$\pi: \Gamma_{nm} \to \Gamma$$

such that:

1) in $\Gamma_{nm}$ there is a finite set $S$ of points that splits into subclasses, $\pi$ maps $S$ exactly into the set $\text{Sing} = \text{Sing} \Gamma$ of singular points of $\Gamma$, and the pre-image of each point in $\text{Sing}$ is one of the subclasses of $S$;

2) the map $\pi: \Gamma_{nm} \setminus S \to \Gamma \setminus \text{Sing}$ is a smooth one-to-one projection;

3) every regular map $F: X \to \Gamma$ of a non-singular algebraic variety $X$ with dense image $F(X) \subset \Gamma$ factors through $\Gamma_{nm}$, that is, $F = \pi G$ for some regular map $G: X \to \Gamma_{nm}$.

A map $\pi$ satisfying these conditions is called the normalization of $\Gamma$ and is uniquely determined by the conditions. Sometimes the curve $\Gamma_{nm}$ itself is called the normalization.

The genus of $\Gamma_{nm}$ is called the geometric genus of $\Gamma$ and is denoted by $p_g(\Gamma)$.

The Riemann–Roch formula contains the arithmetic genus $p_a(\Gamma)$, which is the sum of the geometric genus and a positive contribution from the singularities (the points in $\text{Sing}$). For a smooth curve we have $p_a = p_g$.

A meromorphic 1-form $\omega$ on $\Gamma_{nm}$ defines a regular differential on $\Gamma$ if for every point $P \in \text{Sing}$ we have

$$\sum_{\pi^{-1}(P)} \text{Res}(f \omega) = 0$$

for all meromorphic functions $f$ on $\Gamma_{nm}$ which push forward to functions on $\Gamma$, that is, have the same value at the points in each divisor $D_i$ and do not have poles in $\pi^{-1}(P)$. Regular differentials can have poles in the pre-images of singular points. The dimension of the space of regular differentials is equal to $p_a(\Gamma)$. 
On an irreducible algebraic curve $\Gamma_{nm}$ let us take $s$ families $D_1, \ldots, D_s$ consisting of $r_1, \ldots, r_s$ points, respectively, all of which are different, and let us construct $\Gamma$ by identifying all the points in each family. Then

$$p_a(\Gamma) = p_g(\Gamma) + \sum_{i=1}^{s} (r_i - 1).$$

We formulate the Riemann–Roch theorem for algebraic curves with singularities:

- Let $L(D)$ be the space of meromorphic functions on $\Gamma$ with poles at points in $D = \sum n_P P$ of order $\leq n_P$ and let $\Omega(D)$ be the space of regular differentials on $\Gamma$ having at each point $P \in \text{Sing}$ a zero of order not less than $n_P$. Then

$$\dim L(D) - \dim \Omega(D) = \deg D + 1 - p_a(\Gamma).$$

For a generic divisor $D$ with $\deg D \geq p_a$,

$$\dim \Omega(D) = 0 \quad \text{and} \quad \dim L(D) = \deg D + 1 - p_a(\Gamma).$$

The standard scheme for proving uniqueness of the Baker–Akhiezer function is based on the Riemann–Roch theorem, and the genus $g$ of a smooth spectral curve $\Gamma$ enters into all considerations as the arithmetic genus. For the case of singular spectral curves it is enough to replace $g$ by $p_a$ in all arguments and definitions.

We present the simplest examples of one-dimensional finite-gap Schrödinger operators with singular spectral curves.

**Example 1** (a curve with a double point). Let $\Gamma_{nm} = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, with a parameter $k$. We construct the curve $\Gamma$ by identifying the points $k = \pm \lambda$ on $\Gamma_{nm}$. We have $p_a(\Gamma) = 1$, and therefore we let $D = \{k = 0\}$. From the spectral data $(\Gamma, \infty, k, D)$ we construct the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + \frac{2\lambda^2}{\sin^2(\lambda x)}$$

and the Baker–Akhiezer function

$$\psi(k, x) = \left(1 + \frac{i\lambda \cos(kx)}{k \sin(kx)}\right)e^{ikx}.$$

**Example 2** (a cuspidal curve). Let $\Gamma$ be the same as in the previous example. The curve $\Gamma$, which is homeomorphic to $\Gamma_{nm}$ but has a cuspidal singularity at $k = 0$, is obtained by restricting the class of locally holomorphic functions on $\Gamma_{nm}$: we assume that a function $f$ which is holomorphic in a neighbourhood of $k = 0$ on $\Gamma_{nm}$ is holomorphic in a neighbourhood of $k = 0$ on $\Gamma$ if and only if

$$f'|_{k=0} = 0.$$  \hspace{1cm} (6)

We have $p_a(\Gamma) = 1$. The spectral data $(\Gamma, \infty, k, D = \{k = 0\})$ determine the potential

$$u(x) = \frac{2}{x^2},$$

\text{Its form differs from (4) by the sign of } \frac{d^2}{dx^2}, \text{ and this is reflected in a change in the asymptotic behaviour of } \psi: \psi \approx e^{ikx} \text{ as } k \to \infty.
and the function
\[ \psi_1(k, x) = \left(-\frac{\partial}{\partial x} + \frac{1}{x}\right)e^{ikx} = \left(-ik + \frac{1}{x}\right)e^{ikx} \]
parameterizes all the Bloch functions and satisfies (6). After dividing it by \(-ik\) we obtain the Baker–Akhiezer function, normalized by the asymptotic behaviour
\[ \psi(k, x) = \left(1 + \frac{i}{kx}\right)e^{ikx} \quad \text{as} \quad k \to \infty. \]
This potential is obtained from the potentials described in Example 1 in the limit as \(\lambda \to 0\).

**Example 3.** The potential in Example 2 occurs in a series of rational soliton potentials:
\[ u_l(x) = \frac{l(l+1)}{x^2}. \quad (7) \]
The Baker–Akhiezer function for an operator with the potential (7) is defined on the curve \(\Gamma\) obtained from \(\Gamma_{nm} = \mathbb{C}P^1\) by assuming a condition analogous to (6):
\[ f' = f'' = \cdots = f^{(l)} = 0 \quad \text{for} \quad k = 0. \]
We have \(p_a(\Gamma) = l\) and \(D = lQ\), where \(Q = \{k = 0\}\), and the other spectral data are the same as in Example 2. The Baker–Akhiezer function is
\[ \psi_l(k, x) = \frac{1}{(-ik)^l}\left(-\frac{\partial}{\partial x} + \frac{1}{x}\right)\cdots\left(-\frac{\partial}{\partial x} + \frac{1}{x}\right)e^{ikx}. \]
The Riemann surface \(\Gamma\) is defined by the equation \(y^2 = x^{2l+1}\), and the normalization has the form
\[ \mathbb{C} \to \Gamma \setminus \{\infty\}, \quad t \to (x = t^2, \; y = t^{2n+1}). \]
The Baker–Akhiezer function is usually constructed as a function \(\psi\) on the normalization \(\Gamma_{nm}\) of the curve, and multiple points and cuspidal singularities are given by additional conditions of the form
\[ \psi(Q_1) = \psi(Q_2), \quad \psi'(Q) = \cdots = \psi^{(l)}(Q). \]
At the same time, in soliton theory one considered more general constraints of the form
\[ \sum_{j=1}^{M} \sum_{i=1}^{m_j} a_{ij}\psi^{(j)}(k, x, \ldots)|_{k=Q_i} = 0, \quad Q_1, \ldots, Q_M \in \Gamma_{nm}, \quad (8) \]
or even systems of such conditions (see, for instance, [18]). In this case \(\psi\) is a section of a certain bundle over \(\Gamma\) which we can find as follows. It is evident that generically a constraint of the form (8) imposed on rational functions on \(\Gamma_{nm}\) does not distinguish any subfield: a product of two functions satisfying such a condition does not necessarily satisfy it. At the same time, the sections of the desired bundle form a module over the field of rational functions on \(\Gamma\). Let \(f\) be such a function
and let $\psi$ be a section of the bundle given by (8). Then the minimal conditions $f$ has to satisfy in order for $f\psi$ to satisfy (8) for all sections are as follows:

$$f(Q_1) = \cdots = f(Q_M), \quad f'(Q_j) = \cdots = f^{(m_j)}(Q_j) = 0, \quad j = 1, \ldots, M,$$

so that $\Gamma$ is obtained from $\Gamma_{nm}$ by gluing together the points $Q_1, \ldots, Q_M$, and each such point with $m_j \geq 1$ is singular (see Example 3).

However, some such Baker–Akhiezer functions (apparently, all of them) can be obtained from functions on smooth spectral curves by degeneration of the curves. Let us consider the following example.

**Example 4** (A. E. Mironov). Let $\Gamma_1$ be a connected algebraic curve of genus $g$ on which the Baker–Akhiezer function $\psi$ has been constructed. For simplicity we assume that $\psi$ is constructed from spectral data for the KP equation, but also satisfies an additional condition

$$\psi(Q_1) = \alpha \psi(Q_2),$$

where $\alpha \neq 1$. By the Riemann–Roch theorem, the condition on the pole divisor of $\psi$ (it consists of $g + 1$ generic points $\gamma_1, \ldots, \gamma_{g+1}$ taken together with their multiplicities) along with the asymptotic behaviour at the essentially singular point determines $\psi$ uniquely. Let us consider a rational curve $\Gamma_2 = \mathbb{CP}^1$ with marked points $a$ and $b$ different from $\infty$, and let us identify $Q_1$ with $a$ and $Q_2$ with $b$

obtaining a reducible curve $\Gamma = \Gamma_1 \cup \Gamma_2 / \{Q_1 \sim a, Q_2 \sim b\}$ with arithmetic genus $p_a(\Gamma) = g + 1$. We construct on $\Gamma$ the Baker–Akhiezer function $\psi$ with $g + 2$ poles at the points $\gamma_1, \ldots, \gamma_{g+1} \in \Gamma_1$ and at the point $q \in \Gamma_2$ such that $(b - q)/(a - q) = \alpha$. The number of poles is greater by one than the arithmetic genus, and we impose the additional normalization condition $\psi(\infty) = 0$, where $\infty \in \Gamma_2$. Then the restriction of $\psi$ to $\Gamma_2$ is the function $\varphi = c/z - q$, $c = \text{const} \neq 0$. By the choice of $q$, we have $\varphi(a) = \alpha \varphi(b)$, and therefore the restriction of $\psi$ to $\Gamma_1$ is the desired function $\psi$.

The Riemann surface $\Gamma$ is obtained by a degeneration of the smooth surfaces upon contracting the two contours into the points $Q_1 \sim a$ and $Q_2 \sim b$, and $\psi$ is the degeneration of the Baker–Akhiezer functions on these surfaces. Since the inverse problem is solved from the asymptotic expressions for $\psi$ at points with essential singularities and there are no such singularities on the component $\Gamma_2$, the restriction of $\psi$ to $\Gamma_2$ does not play any role. At the same time, the real spectral curve on which $\tilde{\psi}$ is defined as the Baker–Akhiezer function is the reducible algebraic curve $\Gamma$ and not $\Gamma_1$.

Apparently, one can obtain by this procedure the potential Schrödinger operators in [19], which are constructed from singular spectral curves with odd arithmetic genus (for smooth spectral curves the genus is always even in this problem [12]).

In [20] Malanyuk constructed finite-gap solutions of the KPII equation for which the spectral curve is reducible and the essential singularity of the Baker–Akhiezer function lies in the rational component. These solutions are non-linear superpositions of soliton waves.

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4These examples were constructed at the beginning of the 1990s, and at the time Malanyuk argued similarly that their spectral curves can be obtained from smooth curves by degeneration.
It is easy to see that the construction in Example 4 generalizes to any relations of the form
\[ \alpha_1 \psi(Q_1) + \cdots + \alpha_M \psi(Q_M) = 0 \]
with generic coefficients \( \alpha_1, \ldots, \alpha_M \). Furthermore, \( \Gamma \) is obtained from \( \Gamma_1 \) by the addition of a rational curve \( \mathbb{CP}^1 \) intersecting \( \Gamma_1 \) at \( Q_1, \ldots, Q_M \).

### 2. Orthogonal curvilinear coordinate systems and Frobenius manifolds

#### 2.1. Orthogonal curvilinear coordinates and integrable systems of hydrodynamic type

Orthogonal curvilinear coordinate systems appeared in the theory of integrable systems from two directions: they arose naturally in the problem of integrability of one-dimensional Dubrovin–Novikov systems of hydrodynamic type [21]–[23]; and they also arose as a result of differential reduction as a particular case of metrics with diagonal curvature, to the explicit construction of which Zakharov applied the inverse problem method [24].

An orthogonal curvilinear coordinate system in a Riemannian manifold is defined to be a coordinate system \((u^1, \ldots, u^N)\) in which the metric tensor takes the diagonal form
\[ ds^2 = \sum_{i=1}^{N} H_i^2(u) (du^i)^2. \] (9)

We note that for \( N \geq 4 \) not every Riemannian metric locally admits orthogonal curvilinear coordinates. In fact, the existence of such coordinates is a very strong condition on a metric.

The coefficients \( H_i \) are called the Lamé coefficients, and the expressions
\[ \beta_{ij} = \frac{1}{H_j} \frac{\partial H_i}{\partial u^j} \] (10)
define the rotation coefficients.

A metric is said to be of Egorov type if locally it admits orthogonal curvilinear coordinates with symmetric rotation coefficients:
\[ \beta_{ij} = \beta_{ji}, \quad 1 \leq i, j \leq N. \]

In this case the metric is of potential type:
\[ g_{ii} = H_i^2 = \frac{\partial V}{\partial u^i}, \quad i = 1, \ldots, N, \]
for some potential \( V(u) \).

In the theory of systems of hydrodynamic type the case of Riemannian metrics is not distinguished in any way, and one considers general pseudo-Riemannian metrics for which the metric tensor in orthogonal coordinates takes the form
\[ ds^2 = \sum_{i=1}^{n} \varepsilon_i H_i^2(u) (du^i)^2, \quad \varepsilon_i = \pm 1, \]
the rotation coefficients are defined just as for Riemannian metrics, and the Egorov condition for the metric takes the form
\[ \beta_{ij} = \varepsilon_i \varepsilon_j \beta_{ji}. \]
A one-dimensional system of hydrodynamic type is an evolution system of the form
\[ u_i^t = \sum_{j=1}^{N} v_j^i(u)w_x^j, \quad i = 1, \ldots, N, \]
where \( u^1(x,t), \ldots, u^N(x,t) \) are functions of the one-dimensional spatial variable \( x \) and the time variable \( t \). Such a system is Hamiltonian if it can be represented in the form
\[ u_i^t = \{ u^i(x), \hat{H} \}, \]
where
\[ \hat{H}(u) = \int H(u) \, dx \]
and the Poisson brackets (of hydrodynamic type, or, as one now says, the Poisson–Dubrovin–Novikov brackets) have the form
\[ \{ u^i(x), u^j(y) \} = g^{ij}(u(x))\delta'(x - y) - g^{is}\Gamma_{sk}^j u_x^k \delta(x - y), \quad g^{ij} = g^{ji}. \] (11)

These notions were introduced by Dubrovin and Novikov [20], [22], who proved that:

1) the expression (11) with a non-degenerate pseudo-Riemannian metric \( g^{ij} \) (with upper indices) in the \( N \)-dimensional \( u \)-space defines Poisson brackets if and only if the metric is flat (has zero curvature) and \( \Gamma_{jk}^i \) is the corresponding Levi-Civita connection;

2) Hamiltonian systems of hydrodynamic type arise as a result of averaging integrable systems such as the KdV and sine-Gordon equations, and the non-linear Schrödinger (NS) equation.

Novikov made the conjecture that Hamiltonian systems of hydrodynamic type with a diagonal matrix \( (v_j^i) \) are integrable. For such systems the coordinates \( u^1, \ldots, u^N \) are called Riemann invariants, and the matrix \( (g^{ij}) \) is also diagonal in the \( u \)-space with orthogonal curvilinear coordinates \( u^1, \ldots, u^N \). This conjecture was proved by Tsarev, who also introduced a procedure for integration: the generalized hodograph method [25].

The interest in Egorov metrics in the theory of integrable systems is due to Dubrovin, who showed that if a flat metric \( g^{ij} \), written in terms of Riemannian invariants, is of Egorov type, then the corresponding Hamiltonian system of hydrodynamic type is superintegrable [26].

In [28] the non-local generalization
\[ \{ u^i(x), u^j(y) \} = g^{ij}(u(x))\delta'(x - y) - g^{is}\Gamma_{sk}^j u_x^k \delta(x - y) + \frac{c}{2} \text{sgn}(x - y)u_x^i u_y^j \]
of the Poisson–Dubrovin–Novikov brackets was proposed, where \( c = \text{const} \), and it was proved that this expression defines Poisson brackets if and only if \( \Gamma_{jk}^i \) is the Levi-Civita connection for a metric \( g^{ij} \) with constant sectional curvature \( c \), and that the generalized hodograph method is applicable to Hamiltonian systems with such Poisson brackets. Here the Riemann invariants give orthogonal curvilinear coordinates in the space of constant curvature \( c \).
2.2. The Lamé equations. In the 19th century the theory of orthogonal curvilinear coordinate systems was actively studied by leading geometers (Dupin, Gauss, Lamé, Bianchi, Darboux). The problem of classifying such coordinates was basically solved by the beginning of the 20th century, and the state of the theory at the time was summarized in Darboux’s book [29].

All such coordinate systems are constructed from solutions of the Lamé equations as follows.

Let us split the non-trivial components of the Riemann curvature tensor $R_{ijkl}$ into three groups:

1) the indices $i, j, k, l$ are pairwise distinct;
2) $R_{ijik}$ with pairwise distinct $i, j, k$;
3) $R_{ijij}$ with $i \neq j$.

We recall that the Riemann tensor is skew-symmetric with respect to the first and the second pairs of indices: $R_{ijkl} = -R_{jikl} = -R_{ijlk}$ and $R_{ijkl} = R_{klij}$ for all $i, j, k, l$.

For diagonalizable metrics (9) all the components of the first type vanish, and the vanishing of the components of the second type is equivalent to the system

$$\frac{\partial^2 H_i}{\partial u^j \partial u^k} = \frac{1}{H_j} \frac{\partial H_j}{\partial u^k} \frac{\partial H_i}{\partial u^j} + \frac{1}{H_k} \frac{\partial H_k}{\partial u^j} \frac{\partial H_i}{\partial u^k}, \quad i \neq j \neq k,$$

(12)

and the equations

$$\frac{\partial}{\partial u^j} \left( \frac{1}{H_j} \frac{\partial H_i}{\partial u^j} \right) + \frac{\partial}{\partial u^i} \left( \frac{1}{H_i} \frac{\partial H_j}{\partial u^i} \right) + \sum_{k \neq i, j} \frac{1}{H_k^2} \frac{\partial H_i}{\partial u^k} \frac{\partial H_j}{\partial u^k} = 0, \quad i \neq j,$$

(13)

are equivalent to the vanishing of the components of the third type: $R_{ijij} = 0$.

The systems (12) and (13) consist of $N(N-1)(N-2)/2$ and $N(N-1)/2$ equations, respectively. From counting the number of equations and the number of unknowns it is clear that the system of equations (12), (13) in the Lamé coefficients is strongly overdetermined.

The order of the system (12), (13) can be reduced by passing to the rotation coefficients. In this case the equations (12) take the form

$$\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik} \beta_{kj}, \quad i \neq j \neq k,$$

(14)

and the equations (13) can be rewritten as

$$\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{k \neq i, j} \beta_{ki} \beta_{kj} = 0, \quad i \neq j.$$

(15)

The equations (14) and (15) form the system of Lamé equations, and the equations (14) are just the compatibility condition for the equations (10). The general solution of these equations depends on $N(N-1)/2$ arbitrary functions of two variables.

For a given solution $\beta_{ij}$ of the Lamé equations, the Lamé coefficients can be found from (10) as a solution of the Cauchy problem

$$H_i(0, \ldots, 0, u^i, 0, \ldots, 0) = h_i(u^i).$$

The solution depends on the initial data consisting of $N$ functions $h_i$ of one variable.
Determination of the Euclidean coordinates $x^1, \ldots, x^N$ as functions of $u^1, \ldots, u^N$ (the immersion problem) reduces to the solution of the overdetermined system of linear equations

$$\frac{\partial^2 x^k}{\partial u^i \partial u^j} = \sum_{l=1}^{N} \Gamma^l_{ij} \frac{\partial x^k}{\partial u^l},$$

(16)

where the Christoffel symbols have the form

$$\Gamma^k_{ij} = 0, \quad i \neq j \neq k; \quad \Gamma^k_{kj} = \frac{1}{H_k} \frac{\partial H_k}{\partial u^j}; \quad \Gamma^k_{ii} = -\frac{H_i}{(H_k)^2} \frac{\partial H_k}{\partial u^k}, \quad i \neq k.$$

By (12) and (13) the system of equations (16) is compatible and determines an orthogonal curvilinear coordinate system up to motions of $\mathbb{R}^N$.

2.3. Egorov metrics with zero curvature as an integrable system: the theorem of Dubrovin. In [26] Dubrovin showed that the problem of constructing flat Egorov metrics can be integrated by methods of soliton theory. Namely, the following result holds.

**Theorem 1** [26]. 1) An Egorov (Riemannian or pseudo-Riemannian) metric is flat if and only if its rotation coefficients satisfy the equations

$$\frac{\partial^2 \beta_{ij}}{\partial u^k} = \beta_{ik} \beta_{kj}, \quad \text{where } i, j, k \text{ are pairwise distinct},$$

(17)

$$\sum_{k=1}^{N} \frac{\partial \beta_{ij}}{\partial u^k} = 0, \quad i \neq j.$$  

(18)

2) The system consisting of the equations (17) and (18) is the compatibility condition for the system of linear equations

$$\frac{\partial \psi_i}{\partial u^j} = \beta_{ij} \psi_j, \quad i \neq j,$$

$$\sum_{k=1}^{N} \frac{\partial \beta_{ij}}{\partial u^k} = \lambda \psi_i, \quad i = 1, \ldots, N,$$

where $\lambda$ is a spectral parameter.

3) For a flat Egorov metric the restriction of the rotation coefficients $\beta_{ij}(u)$ to any plane $u^i = a^i x + c^i t$ satisfies the $N$-wave equations

$$[A, B_t] - [C, B_x] = [[A, B], [C, B]],$$

where

$$A = \text{diag}(a^1, \ldots, a^N), \quad C = \text{diag}(c^1, \ldots, c^N), \quad B = (\beta_{ij}),$$

with the additional reduction

$$\text{Im } B = 0, \quad B^\top = JBJ, \quad J = \text{diag}(\varepsilon_1, \ldots, \varepsilon_N)$$

(for Riemannian metrics all the $\varepsilon_i$ are $= 1$).

This theorem can be proved by straightforward computations.
2.4. Zakharov’s method of constructing orthogonal curvilinear coordinates and metrics with diagonal curvature. Zakharov applied the inverse scattering method to the construction of a wide class of metrics with diagonal curvature [24]. Here we consider his construction.

The Riemann curvature tensor can be interpreted as the curvature operator on the space of tangent bivectors. Let $M^N$ be a Riemannian (or pseudo-Riemannian) manifold, and let $\Lambda^2 T^*M^N$ be a vector bundle over $M^N$ whose general fibre over a point is the space of tangent bivectors at this point. The metric on $M^N$ determines the natural metric on the fibres: if $e_1, \ldots, e_N$ is an orthonormal basis in the tangent space $T_xM^N$ at a point $x$, then $\{e_i \wedge e_j, i < j\}$ is an orthonormal basis in $\Lambda^2 T_xM^N$. Then the Riemann curvature tensor $R_{ijkl}$ determines the curvature operator $R$ by the formula

$$\langle R \xi, \eta \rangle = R_{AB} \xi^A \eta^B, \quad R_{AB} = R_{ijkl}, \quad A = [ij], \quad B = [kl],$$

where $\xi = \sum_A e_A \xi^A$ and $\eta = \sum_B e_B \eta^B$ are the expansions of bivectors in the basis $e_A = e_i \wedge e_j, \ i < j$. We remark that the Petrov classification (well-known in relativity theory) of the types of four-dimensional solutions of the Einstein equations is based on the classification of algebraic types of the curvature tensor.\textsuperscript{6}

In [24] a manifold with diagonal curvature is defined as a manifold such that in a neighbourhood of each point the metric is diagonalized, that is, reduces to the form (9), and in the given coordinates the quadratic form $R_{AB}$ (or the curvature operator $R$) is also diagonal.

The equations (17) exactly describe the rotation coefficients of metrics with diagonal curvature and can be written as

$$\sum_{i,j,k} \varepsilon_{ijk} \left( I_j \frac{\partial B}{\partial u^i} I_k - I_i B I_j B I_k \right) = 0,$$

(19)

where $B(u) = (\beta_{ij})$ is the $N \times N$ matrix function made up of the rotation coefficients, and $I_j = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 at the $j$th entry).

Let us consider an auxiliary function $\tilde{B} = \tilde{B}(u, s)$, with $s = u^{N+1}$ an additional variable, which satisfies the equations

$$\frac{\partial}{\partial u^i} [I_i, \tilde{B}] - \frac{\partial}{\partial u^i} [I_j, \tilde{B}] + I_i \frac{\partial \tilde{B}}{\partial s} I_j - I_j \frac{\partial \tilde{B}}{\partial s} I_i - [[I_i, \tilde{B}], [I_j, \tilde{B}]] = 0,$$

(20)

where $i, j = 1, \ldots, N + 1$ and $I_{N+1}$ is the identity matrix. If $\tilde{B}$ satisfies (20), then for any given value of $s$ the matrix-valued function $\tilde{B}$ satisfies the $N$-wave equation (19).

The system (20) admits the Lax representation

$$[L_i, L_j] = 0, \quad L_j = \frac{\partial}{\partial u^j} + I_j \frac{\partial}{\partial s} + [I_j, \tilde{B}].$$

\textsuperscript{6}By the Petrov classification people now often mean the analogous classification based on the algebraic types of the Weyl operator, which is similarly constructed from the Weyl tensor (the conformal curvature tensor).
Let us apply to it the dressing method. Namely, we consider an integral equation of Marchenko type

\[
K(s, s', u) = F(s, s', u) + \int_{s}^{\infty} K(s, q, u) F(q, s', u) \, dq,
\]

where \(F(s, s', u)\) is a matrix-valued function.

**Theorem 2** [24]. If \(F(s, s', u)\) satisfies the conditions

a) \[
\frac{\partial F}{\partial u^i} + \sum_{j} \frac{\partial F}{\partial s^j} F \frac{\partial}{\partial s^j} F I_i = 0,
\]

b) the equation (21) has a unique solution,

then the function

\[
\tilde{B}(s, u) = K(s, s, u)
\]

satisfies the equations (20), and therefore for any fixed value of \(s\) the function \(B(s) = \tilde{B}(s, u)\) satisfies (19) and determines the rotation coefficients of a metric with diagonal curvature.

The class of metrics with diagonal curvature is rather broad, and as noted by Zakharov it contains, for instance, many known solutions of the Einstein equations, including the Schwarzschild metric. The problem he posed on finding a reduction of inverse problem data that corresponds to Ricci-flat metrics remains an important open problem. By using a new and original trick, a **differential reduction**, he distinguished the class of flat metrics with diagonal curvature in the language of inverse problem data, that is, in terms of the function \(F\).

**Theorem 3** [24]. Assume the conditions of Theorem 2 and suppose that \(F\) satisfies the relation

\[
\frac{\partial F_{ij}}{\partial s'}(s, s', u) + \frac{\partial F_{ji}}{\partial s}(s', s, u) = 0.
\]

Then \(\tilde{B}\) satisfies the equations (15), and the rotation coefficients constructed from \(\tilde{B}\) correspond to flat metrics of the form (9), that is, to orthogonal curvilinear coordinate systems.

The system consisting of (22) and (24) has the following solution [24].

Let \(\Phi_{ij}(x, y)\) with \(i < j\) be \(N(N - 1)/2\) arbitrary functions of two variables, and let \(\Phi_{ii}(x, y)\) be \(N\) arbitrary skew-symmetric functions:

\[
\Phi_{ii}(x, y) = -\Phi_{ii}(y, x).
\]

Let

\[
F_{ij} = \frac{\partial \Phi_{ij}(s - u^i, s' - u^j)}{\partial s}, \quad F_{ji} = \frac{\partial \Phi_{ij}(s' - u^i, s - u^j)}{\partial s}, \quad i \neq j,
\]

\[
F_{ii} = \frac{\partial \Phi_{ii}(s - u^i, s' - u^i)}{\partial s}.
\]

Then \(F = (F_{ij})\) satisfies (22) and (24), and for any fixed value of \(s\) a solution \(K\) of the equation (21) with such a matrix \(F\) determines the rotation coefficients of an
orthogonal coordinate system: $\beta_{ij}(u) = K_{ij}(s, s, u)$. Note that we have $N(N+1)/2$ function parameters $\Phi_{ij}, i \leq j$, while the general solution depends on $N(N-1)/2$ function parameters, therefore, this method gives equivalent classes of dressings.

A general theory of reductions, including differential reductions, which can be helpful for distinguishing other classes of metrics is discussed in [30]. In particular, by a variation of the reduction (24) it is possible to describe metrics with diagonal curvature and constant sectional curvature $K \neq 0$, that is, orthogonal curvilinear coordinates in spaces of constant sectional curvature $K$ (Zakharov). In this case the system (15) is replaced by the equations

$$\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{k \neq i, j} \beta_{ki} \beta_{kj} = -KH_iH_j, \quad i \neq j.$$  

2.5. Krichever's construction of finite-gap orthogonal curvilinear coordinates. In [5] Krichever proposed for the construction of orthogonal curvilinear coordinates in Euclidean spaces a finite-gap version which, in contrast to [24], does not consist of two steps (namely, a construction of metrics with diagonal curvature, and a subsequent singling out of the flat metrics among them), but gives such coordinates directly.

Let $\Gamma$ be a smooth complex algebraic curve, that is, a compact smooth Riemann surface.

We choose three effective divisors on $\Gamma$:

$P = P_1 + \cdots + P_N, \quad D = \gamma_1 + \cdots + \gamma_{g+l-1}, \quad R = R_1 + \cdots + R_l,$

where $g$ is the genus of $\Gamma$ and $P_i, \gamma_j, R_k \in \Gamma$. In a neighbourhood of each point $P_i, i = 1, \ldots, N$, we take a local parameter $k_i^{-1}$ vanishing at the point.

The Baker–Akhiezer function corresponding to this spectral data $S = \{\Gamma, P, D, R\}$ is defined to be a function $\psi(u^1, \ldots, u^N, z), z \in \Gamma$, such that:

1) $\psi \exp(-u^k k_i)$ is analytic in a neighbourhood of $P_i, i = 1, \ldots, N$;

2) $\psi$ is meromorphic on $\Gamma \setminus \{\bigcup P_i\}$ with poles at $\gamma_j, j = 1, \ldots, g+l-1$;

3) $\psi(u, R_k) = 1, k = 1, \ldots, l$.

For a generic divisor $D$ such a function exists, is unique, and can be expressed in terms of the theta function of the curve $\Gamma$.

If $\Gamma$ is not connected, then it is assumed that the restriction of $\psi$ to every connected component satisfies such conditions.

We take an additional divisor $Q = Q_1 + \cdots + Q_N$ on $\Gamma$ such that $Q_i \in \Gamma \setminus \{P \cup D \cup R\}$ for $i = 1, \ldots, N$, and we let

$$x^j(u^1, \ldots, u^N) = \psi(u^1, \ldots, u^N, Q_j), \quad j = 1, \ldots, N.$$  

The following theorem holds.

**Theorem 4** [5]. 1) Let $\sigma: \Gamma \to \Gamma$ be a holomorphic involution on $\Gamma$ such that:

a) $\sigma$ has exactly $2m$ fixed points with $m \leq N \leq 2m$, namely, the points $P_1, \ldots, P_N$ and $2m - N$ points in $Q$;

b) $\sigma(Q) = Q$, that is, the involution either permutes the points in $Q$ or leaves them fixed, namely,

$$\sigma(Q_k) = Q_{\sigma(k)}, \quad k = 1, \ldots, N;$$
c) $\sigma(k^{-1}_i) = -k^{-1}_i$ in a neighbourhood of $P_i$, $i = 1, \ldots, N$;

d) there exists a meromorphic differential $\Omega$ on $\Gamma$ whose zero divisor and pole divisor are of the form

$$(\Omega)_0 = D + \sigma D + P, \quad (\Omega)_\infty = R + \sigma R + Q;$$

e) the quotient space $\Gamma_0 = \Gamma / \sigma$ is a smooth algebraic curve.

Then $\Omega$ is a pullback of some meromorphic differential $\Omega_0$ on $\Gamma_0$, and the following equalities hold:

$$\sum_{k,l} \eta_{kl} \frac{\partial x^k}{\partial u^i} \frac{\partial x^l}{\partial u^j} = \varepsilon_i^2 H_i^2(u) \delta_{ij},$$

where

$$H_i = \lim_{P \to P_i} (\psi e^{-u^i k_i}), \quad \eta_{kl} = \delta_{k,\sigma(l)} \text{res}_{Q_k} \Omega_0,$$

$$\Omega_0 = \frac{1}{2} \left( \varepsilon_i^2 \lambda_i + O(\lambda_i) \right) d\lambda_i, \quad \lambda_i = k_i^{-2}, \text{ at } P_i, \quad i = 1, \ldots, N.$$

2) Moreover, if there exists an antiholomorphic involution $\tau: \Gamma \to \Gamma$ such that all the fixed points of $\sigma$ are also fixed by $\tau$ and

$$\tau^* (\Omega) = \overline{\Omega},$$

then the coefficients $H_i(u)$ are real-valued for $u^1, \ldots, u^N \in \mathbb{R}$, and $u^1, \ldots, u^N$ are orthogonal curvilinear coordinates in the flat $N$-dimensional space with the metric $\eta_{kl} dx^k dx^l$.\footnote{It is easy to see that if there are points $Q_k$ and $Q_l$, $l = \sigma(k)$, which are interchanged by the involution, then the metric $\eta$ is indefinite.}

Furthermore, if all the points in $Q$ are fixed by $\sigma$ and

$$\text{res}_{Q_1} \Omega_0 = \cdots = \text{res}_{Q_N} \Omega_0 = \eta_0^2 > 0, \quad (25)$$

then the functions $x^1(u^1, \ldots, u^N), \ldots, x^N(u^1, \ldots, u^N)$ solve the immersion problem for the orthogonal curvilinear coordinates $u^1, \ldots, u^N$ and the metric $ds^2 = H_1^2(du^1)^2 + \cdots + H_N^2(du^N)^2$, where

$$H_i = \frac{\varepsilon_i h_i}{\eta_0}, \quad i = 1, \ldots, N.$$

Remark 1. The theorem still holds if instead of a) it is assumed that the functions $\psi \exp(-f^i(u^i)k_i)$ are analytic in a neighbourhood of $P_i$, where the $f^i$ are some functions of one variable which are invertible in a neighbourhood of zero, $i = 1, \ldots, N$. Therefore, we do not distinguish orthogonal coordinates which are obtained by a change of variables of the form

$$u^i \to f^i(u^i).$$
It is also possible to replace the condition \(\psi(u, R_k) = 1, \ k = 1, \ldots, l\), by
\[
\psi(u, R_k) = d_k, \quad k = 1, \ldots, l,
\] (26)
where all the constants \(d_k\) are non-zero. What is more, we can even just assume that the \(d_k\) do not vanish simultaneously:
\[
|d_1|^2 + \cdots + |d_l|^2 \neq 0,
\] (27)
and the theorem still holds.

Krichever’s method was recently generalized in [31] to the case of orthogonal coordinates in spaces of constant curvature.

2.6. Coordinate systems corresponding to singular spectral curves. The Krichever construction (Theorem 4) leads to theta-function formulae which are difficult for qualitative analysis. At the same time, in the case of degeneration of a smooth spectral curve \(\Gamma\) we can obtain simpler formulae corresponding to solutions which lie on the boundary of the moduli space found by Krichever. To understand the structure of finite-gap orthogonal curvilinear systems it is also helpful to find the classical coordinate systems among them. The following theorem can serve for this.

**Theorem 5** [1]. Theorem 4 remains true for a singular algebraic curve after replacing \(g\) by the arithmetic genus \(p_a(\Gamma)\) of \(\Gamma\), and replacing the smoothness condition e) for \(\Gamma/\sigma\) by the condition that the points \(P_1, \ldots, P_N\) and the poles of \(\Omega\) are non-singular points.

Moreover, we can assume that \(\psi\) satisfies (26) and (27) instead of the equations \(\psi(u, R_k) = 1, \ k = 1, \ldots, l\).

In the case when \(\Gamma_{nm}\) is a union of smooth rational curves, that is, copies of \(\mathbb{C}P^1\), the procedure for constructing the Baker–Akhiezer function and the orthogonal coordinates reduces to simple computations with elementary functions and goes no further than solving systems of linear equations. We demonstrate this by examples.

We remark that the simplest spectral curves correspond to quite exotic coordinates, while for classical coordinate systems the spectral curves are rather complicated.

Let us recall that a regular differential \(\Omega\) on \(\Gamma\) is given by differentials \(\Omega_1, \ldots, \Omega_s\) on the irreducible components \(\Gamma_1, \ldots, \Gamma_s\). By definition, \[\sum_i \sum_{\pi^{-1}(P) \in \Gamma_i} \text{res} \Omega_i = 0,\]
where \(\pi: \Gamma_{nm} \to \Gamma\) is the normalization and the summation is taken over all singular points \(P\). The arithmetic genus \(p_a\) is the dimension of the space of regular differentials.

If the singularities are just multiple points, then these differentials can have poles at intersection points of components, and if \(P\) is an intersection point of the components \(\Gamma_{i_1}, \ldots, \Gamma_{i_r}\), then \[\sum_{k=1}^r \text{res}_P \Omega_{i_k} = 0.\] Then the arithmetic genus \(p_a\) is the dimension of the space of regular holomorphic differentials, that is, differentials such that \(\Omega_j\) can have poles only at intersection points of different components, and the sums of the residues at the intersection points are equal to zero.
Example 5. Let us take the simplest singular spectral curve $\Gamma$, consisting of two copies $\Gamma_1$ and $\Gamma_2$ of $\mathbb{CP}^1$ which intersect at two points, so that $p_a(\Gamma) = 1$: $a \sim b$, $(-a) \sim (-b)$, $\{a, -a\} \subset \Gamma_1$, and $\{b, -b\} \subset \Gamma_2$ (see Fig. 1). We consider the case

![Figure 1](image_url)

with the essential singularities lying in different irreducible components: at points $P_1 = \infty \in \Gamma_1$ and $P_2 = \infty \in \Gamma_2$. The Baker–Akhiezer function takes the form

$$
\psi_1(u^1, u^2, z_1) = e^{u^1 z_1} \left( f_0(u^1, u^2) + \frac{f_1(u^1, u^2)}{z_1 - \alpha_1} + \cdots + \frac{f_k(u^1, u^2)}{z_1 - \alpha_{s_1}} \right), \quad z_1 \in \Gamma_1,
$$

$$
\psi_2(u^1, u^2, z_2) = e^{u^2 z_2} \left( g_0(u^1, u^2) + \frac{g_1(u^1, u^2)}{z_2 - \beta_1} + \cdots + \frac{g_n(u^1, u^2)}{z_2 - \beta_{s_2}} \right), \quad z_2 \in \Gamma_2.
$$

$$
\psi_1(a) = \psi_2(b), \quad \psi_1(-a) = \psi_2(-b).
$$

The general normalization condition has the form

$$
\psi_1(R_{1,i}) = d_{1,i}, \quad \psi_2(R_{2,j}) = d_{2,j},
$$

where $R_{1,i} \in \Gamma_1$, $i = 1, \ldots, l_1$, and $R_{2,j} \in \Gamma_2$, $j = 1, \ldots, l_2$. We also have $l = l_1 + l_2 = s_1 + s_2$. Let

$$
\Omega_1 = \frac{(z_1^2 - \alpha_1^2) \cdots (z_1^2 - \alpha_{l_1}^2)}{z_1(z_1^2 - a^2)(z_1^2 - R_{1,1}^2) \cdots (z_1^2 - R_{1,l_1}^2)} \, dz_1,
$$

$$
\Omega_2 = \frac{(z_2^2 - \beta_1^2) \cdots (z_2^2 - \beta_{l_2}^2)}{z_2(z_2^2 - b^2)(z_2^2 - R_{2,1}^2) \cdots (z_2^2 - R_{2,l_2}^2)} \, dz_2,
$$

and let $Q_1 = 0 \in \Gamma_1$ and $Q_2 = 0 \in \Gamma_2$. If

$$
\text{res}_a \Omega_1 = - \text{res}_b \Omega_2, \quad \text{res}_{-a} \Omega_1 = - \text{res}_{-b} \Omega_2, \quad \text{res}_{Q_1} \Omega_1 = \text{res}_{Q_2} \Omega_2,
$$

then the differential $\Omega$ given by the differentials $\Omega_1$ and $\Omega_2$ is regular, and the condition (25) is satisfied. In this case, by Theorem 5, the coordinates $u^1$ and $u^2$ such that

$$
x^1(u) = \psi_1(u, 0), \quad x^2(u) = \psi_2(u, 0),
$$

are orthogonal.

---

8We present this simple example in detail to demonstrate the construction procedure; for details of other examples we refer to [1].
Let us consider the simplest case \( l_1 = 0 \) and \( l_2 = 1 \). Then

\[
\psi_1 = e^{u^1z_1}f_0(u^1, u^2), \quad \psi_2 = e^{u^2z_2}\left(g_0(u^1, u^2) + \frac{g_1(u^1, u^2)}{z_2 - c}\right).
\]

The gluing conditions at the intersection points and the normalization condition take the form \( \psi_1(a) = \psi_2(b), \psi_1(-a) = \psi_2(-b), \psi_2(r) = 1, r = R \in \Gamma_2 \), and this implies that

\[
\psi_1 = e^{u^1z_1}\frac{2b(c - r)e^{au^1} + (b - r)u^2}{(b + c)(b - r)e^{2bu^2} - (b + r)(b - c)e^{2au^1}},
\]

\[
\psi_2 = e^{u^2z_2}\left(\frac{e^{-ru^2}((b - c)e^{2au^1} + (b + c)e^{2bu^2})(c - r)}{(b + c)(b - r)e^{2bu^2} - (b - c)(b + r)e^{2au^1}}\right) + \frac{1}{z_2 - c}\frac{(b^2 - c^2)(r - c)e^{-ru^2}(e^{2au^1} - e^{2bu^2})}{(b + c)(r - b)e^{2bu^2} + (b - c)(b + r)e^{2au^1}}.
\]

On the components the differential \( \Omega \) is given by the differentials

\[
\Omega_1 = -\frac{dz_1}{z_1(z_1^2 - a^2)}, \quad \Omega_2 = -\frac{(z_2^2 - c^2)dz_2}{z_2(z_2^2 - b^2)(z_2^2 - r^2)},
\]

and their residues at the singular points are \( \text{res}_a \Omega_1 = \text{res}_{-a} \Omega_1 = -1/(2a^2) = -\text{res}_b \Omega_2 = -\text{res}_{-b} \Omega_2 = (b^2 - c^2)/(2b^2(b^2 - r^2)) \). The regularity condition (25) takes the form \( \text{res}_{Q_1} \Omega_1 = 1/a^2 = \text{res}_{Q_2} \Omega_2 = c^2/(r^2b^2) \), which implies that \( a = br/c \) and \( r = b/\sqrt{2 + b^2/c^2} \). By straightforward computations we get that

\[
(x^1)^2 + \left(x^2 - e^{-ru^2}\frac{b(c - r)}{c(b^2 - r^2)}\right)^2 = e^{-2ru^2}\frac{b^2(c - r)^2}{c^2(b^2 - r^2)^2}.
\]

The coordinate lines \( u^2 = \text{const} \) are circles centred on the \( x^2 \) axis. For \( b = \pm 1 \) these circles are tangent to the \( x^1 \) axis, and the second family of coordinate lines \( u^1 = \text{const} \) consists of circles centred on the \( x^1 \) axis and tangent to the \( x^2 \) axis (see Fig. 2).

\[
\text{Figure 2}
\]
The case with the same spectral curve but with the essential singularities lying on the same component is considered in [1].

**Example 6** (Euclidean coordinates). Let $\Gamma$ be a union of $n$ copies $\Gamma_1, \ldots, \Gamma_n$ of the complex projective line $\mathbb{C}P^1$, and let

$$P_j = \infty, \quad Q_j = 0, \quad R_j = -1 \in \Gamma_j, \quad \psi_j(R_j) = 1, \quad j = 1, \ldots, N.$$ 

On the components the differential $\Omega$ is given by the differentials $\Omega_j = \frac{dz_j}{z_j(z_j^2 - 1)}$, the Baker–Akhiezer function $\psi$ is $\psi_j = e^{u_j z_j} f_j(u_j), \quad j = 1, \ldots, N$, and we obtain the Euclidean coordinates in $\mathbb{R}^N$: $x^j = e^{u_j}$.

**Example 7** (polar coordinates). The curve $\Gamma$ consists of five irreducible components $\Gamma_1, \ldots, \Gamma_5$ which intersect as shown in Fig. 3. We have $p_a(\Gamma) = 1$.

**Example 8** (cylindrical coordinates). The curve $\Gamma$ is a disjoint union of the curve $\widetilde{\Gamma}$ in the previous example (polar coordinates) and a copy $\Gamma_6$ of the curve $\mathbb{C}P^1$. All the data related to $\widetilde{\Gamma}$ are the same as before. On $\Gamma_6$ we put $Q_3 = 0, \quad P_3 = \infty, \quad R_4 = -1, \quad \text{and} \quad \psi(R_4) = 1$. Then we have $\psi_6(u^3) = e^{u^3(z_0 + 1)}$ and

$$x^1 = \psi_5(Q_1) = r \cos \varphi, \quad x^2 = \psi_5(Q_2) = r \sin \varphi, \quad r = e^{u^1}, \quad \varphi = u^2.$$ 

**Example 9** (spherical coordinates in $\mathbb{R}^3$). The curve $\Gamma$ consists of 9 irreducible components which intersect as shown in Fig. 4. We have $p_a(\Gamma) = 2$. 

![Figure 3](image-url)
For certain choices of the divisors $D$, $P$, $Q$, and $R$ (see \cite{1}) these spectral data lead to the spherical coordinates

$$x^1 = \psi_5(Q_1) = r \sin \varphi, \quad x^2 = \psi_9(Q_2) = r \cos \varphi \sin \theta,$$

$$x^3 = \psi_9(Q_3) = r \cos \varphi \cos \theta, \quad r = e^{u_1}, \quad \varphi = u^2, \quad \theta = u^3.$$ 

**Example 10** (spherical coordinates in $\mathbb{R}^N$). Let $\Gamma^{(N-1)}$ be the spectral curve and let $\psi^{(N-1)}$ be the Baker–Akhiezer function for the $(N-1)$-dimensional spherical coordinates. The spectral curve $\Gamma^{(N)}$ for the $N$-dimensional spherical coordinates is obtained from $\Gamma^{(N-1)}$ and the curve in Fig. 5 by intersecting them at the points $0 \in \Gamma_{4N-7} \subset \Gamma^{(N-1)}$ and $0 \in \Gamma_{4N-6}$ (the number of irreducible components of $\Gamma^{(k)}$ equals $4k - 3$). We have $p_\alpha(\Gamma^{(N)}) = N - 1$.

It would be interesting to find the spectral data for other known orthogonal coordinate systems and, in particular, for the elliptic coordinates $u^1, u^2$ connected with the Euclidean coordinates $(x^1, x^2)$ by the relations

$$x^1 = \cosh u^1 \cos u^2, \quad x^2 = \sinh u^1 \sin u^2.$$ 

**2.7. A remark on discrete orthogonal coordinates.** In the area of discretization of differential geometry (now actively being developed) one means by a system of discrete coordinates in $\mathbb{R}^N$ a map

$$x : \mathbb{Z}^N \rightarrow \mathbb{R}^N$$
which is a lattice embedding. The translation operator $T_i$ along the $i$th coordinate acts on functions $F: \mathbb{Z}^N \to \mathbb{R}^k$ according to the formula

$$T_i F(u^1, \ldots, u^{i-1}, u^i, u^{i+1}, \ldots, u^N) = F(u^1, \ldots, u^{i-1}, u^i + 1, u^{i+1}, \ldots, u^N),$$

and the partial derivative operator $\Delta_i$ in the $i$th direction is then defined as

$$\Delta_i F(u) = T_i F(u) - F(u).$$

Following [32], one says that a coordinate system is orthogonal if two conditions hold:

1) (the planarity condition) the points $x(u), T_i x(u), T_j x(u), T_i T_j x(u)$ lie in one plane for any given triple $i, j, u$;

2) (the circle condition) the planar quadrangle spanned by $x(u), T_i x(u), T_j x(u)$, and $T_i T_j x(u)$ is inscribed in a circle.

In [33] there is a procedure for constructing discrete Darboux–Egorov coordinates (flat Egorov coordinates), based on a formal discretization of the Baker–Akhiezer function corresponding to the continuous Darboux–Egorov coordinates in [5]. The discrete Darboux–Egorov coordinates satisfy the planarity condition, but the circle condition is replaced by the stronger condition

3) (discrete Egorov condition) the edges $X^+_i(u) = T_i x(u) - x(u)$ and $X^-_j(u) = T_j^{-1} x(u) - x(u)$ of the lattice are orthogonal for all triples $i, j, u$ (this implies that in any quadrangle in the circle condition two opposite angles are right angles and therefore are based on a diameter of the circle in which the quadrangle is inscribed).

For discrete Darboux–Egorov coordinates there is a discrete potential: a function $\Phi$ such that $\Delta_i \Phi(u) = |T_i x(u) - x(u)|^2$.

One can similarly carry out a formal discretization (on the level of Baker–Akhiezer functions) of the coordinates obtained by Theorem 5. Since everything is very simple to compute in this case, it is possible to find explicit examples of discrete coordinates which, while satisfying the planarity condition, do not satisfy the circle condition. This demonstrates the difficulties that arise in the study of the interesting problem of finding a geometrically meaningful discretization of the Lamé equations.

3. Frobenius manifolds

3.1. The associativity equations and Frobenius manifolds. Let us consider a finite-dimensional algebra generated by $e_1, \ldots, e_n$ and with a commutative multiplication

$$e_\alpha \cdot e_\beta = c_{\alpha \beta}^\gamma e_\gamma, \quad c_{\alpha \beta \gamma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \quad c_{\alpha \beta}^\gamma = \eta^{\gamma \delta} c_{\alpha \beta \delta}, \quad \eta^\alpha \beta = \eta^\beta \alpha,$$

where $F = F(t)$ is a function of $t = (t^1, \ldots, t^N)$. The algebra is associative, that is,

$$(e_\alpha \cdot e_\beta) \cdot e_\gamma = e_\alpha \cdot (e_\beta \cdot e_\gamma) \quad \text{for all } \alpha, \beta, \gamma,$$

if and only if $F$ satisfies the associativity equations

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda \mu} \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\beta \partial t^\mu} = \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\beta \partial t^\lambda} \eta^{\lambda \mu} \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\mu}. \quad (28)$$
In fact, if these conditions hold, then we obtain a whole family of associative algebras depending on an $N$-dimensional parameter $t$.

These equations first arose in quantum field theory, and together with the conditions

1) \( c_{\alpha\beta} = \eta_{\alpha\beta}, \alpha, \beta = 1, \ldots, N, \)
2) \( \det(\eta^{\alpha\beta}) \neq 0, \eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^\alpha_\gamma, \)
3) (quasi-homogeneity condition)

\[
F(\lambda^1 t^1, \ldots, \lambda^N t^N) = \lambda^d F(t^1, \ldots, t^N) \tag{29}
\]

they form the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) system of equations \cite{34}, \cite{35}.

There are two generalizations of the quasi-homogeneity condition, namely, it is assumed that there exists a vector field \( E = (q^\alpha t^\beta + r^\alpha) \partial_\alpha \) which satisfies one of the following two conditions.

3') The equality \( E^\alpha \partial_\alpha F = d F \) holds. In the case of (29) the field \( E \) has the form \( E = d^1 t^1 \partial_1 + \cdots + d^N t^N \partial_n \). This generalization covers the case of quantum cohomology.

3'') According to \cite{35} it is only important that the correlators \( c_{ijk} \), that is, the third derivatives of \( F \), be quasi-homogeneous in the sense of (29), hence it is sufficient to require that

\[
E^\alpha \partial_\alpha F = d F + (\text{a second-order polynomial in } t^1, \ldots, t^N). \tag{30}
\]

This generalization is important for us because in our examples some of the exponents \( d_i \) are equal to \(-1\).

The notion of a Frobenius manifold, introduced by Dubrovin \cite{36}, is a geometric form of solutions of the WDVV equations: a Frobenius manifold is defined to be a domain \( U \subset \mathbb{R}^N \) endowed with a constant non-degenerate metric \( \eta_{\alpha\beta} du^\alpha du^\beta \) and with a solution \( F \) (a prepotential) of the associativity equations that satisfies the conditions 1), 2), and the most general quasi-homogeneity condition 3'').

The quasi-homogeneity condition came from physics, and besides quantum field theory Frobenius manifolds also play an important role in the theory of isomonodromic deformations \cite{37}. However, this condition is most restrictive and thus is often omitted in modern Frobenius geometry.

Before \cite{2} all the known Frobenius manifolds were given by Dubrovin’s examples of Frobenius structures on orbit spaces of Coxeter groups (here the role of a flat metric is played by the Sato metric; the solutions the WDVV equations corresponding to singularities of type \( A_n \) were found in \cite{35}) and on Hurwitz spaces, by quantum cohomology and the extended moduli spaces of complex structures on Calabi–Yau manifolds \cite{38}, and by the ‘doubles’ of Hurwitz spaces found by Shramchenko \cite{39}.

In all these examples a Frobenius manifold has a special geometric meaning. The examples in \cite{2} are obtained by analytic methods (via finite-gap integration), are algebraic in the sense that the correlators \( c_{ijk} = \frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k} \) are algebraic functions, and are not semisimple.

\[\text{For isomonodromic deformations only semisimple manifolds, that is, those without nilpotent elements, are interesting.}\]
3.2. Finite-gap Frobenius manifolds. Dubrovin discovered an important relation between solutions of the associativity equations and flat Egorov metrics [40]. It is as follows. Let

\begin{equation}
\eta_{\alpha\beta} \, dx^\alpha \, dx^\beta = \sum_{i=1}^{N} H_i^2(u) \left( du^i \right)^2
\end{equation}

be a flat Egorov metric and let \( x^1, \ldots, x^N \) be coordinates in some domain in which the coefficients \( \eta_{\alpha\beta} \) are constant. We have

\begin{equation}
\eta^{\alpha\beta} = \sum_{i=1}^{N} H_i^{-2} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^i},
\end{equation}

and the symmetry condition for the rotation coefficients and the zero-curvature condition imply the existence of a function \( F(t) \) such that

\begin{equation}
c_{\alpha\beta\gamma} = \sum_{i=1}^{N} H_i^2 \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta} \frac{\partial u^i}{\partial x^\gamma} = \frac{\partial^3 F}{\partial x^\alpha \partial x^\beta \partial x^\gamma}, \tag{31}
\end{equation}

and the associativity equations

\begin{equation}
c_{\lambda}^\alpha c^\mu_{\lambda\beta} = c^\mu_{\alpha\lambda} c^\lambda_{\beta\gamma} \quad \text{for all } \alpha, \beta, \gamma = 1, \ldots, N
\end{equation}

hold, where

\begin{equation}
c^\alpha_{\beta\gamma} = \sum_i \frac{\partial x^\alpha}{\partial u^i} \frac{\partial u^i}{\partial x^\beta} \frac{\partial u^i}{\partial x^\gamma}.
\end{equation}

If the associative algebra is semisimple, then the converse is also true: one can construct an Egorov metric satisfying (31) from a solution \( F(t) \) of the associativity equations.

Since \( u^1, \ldots, u^N \) are orthogonal curvilinear coordinates in a flat space, one can apply Krichever’s method to obtain such coordinates (see Theorem 4). In his paper [5] there are additional constraints on the spectral data which lead to Egorov metrics and therefore to solutions of the associativity equations. However, it seems that from properties of theta functions one can deduce that the explicit theta-function formulae obtained there do not give quasi-homogeneous solutions.

At the same time, one can expect that solutions expressed in terms of elementary functions and corresponding to singular spectral curves can be quasi-homogeneous and determine Frobenius manifolds. This was shown in [2].

The following theorem [2] distinguishes a special case when the construction in Theorem 5 gives flat Egorov metrics and quasi-homogeneous solutions of the associativity equations.
Theorem 6. 1) Let the assumptions of Theorem 5 hold and let $\Gamma$ be a spectral curve whose irreducible components are all rational curves (complex projective lines $\mathbb{C}P^1$).

On each component $\Gamma_i$ let a pair of points $P_i = \infty$, $Q_i = 0$ be distinguished, along with a global parameter $k_i^{-1} = z_i$, $i = 1, \ldots, N$. Assume that all singular points are double points of intersection of different components, each such intersection point $a \in \Gamma_i \cap \Gamma_j$ has the same coordinates on both components, that is,

$$z_i(a) = z_j(a),$$

and the involution $\sigma$ has on each component the form

$$\sigma(z_i) = -z_i.$$

Then the metric

$$ds^2 = \eta_{kl} dx^k dx^l = \sum_i H_i^2 (du^i)^2, \quad H_i = H_i(u^1, \ldots, u^N), \quad i = 1, \ldots, N,$$

constructed from these spectral data is a flat Egorov metric (Darboux–Egorov metric).

2) Assume also that the spectral curve is connected and the Baker–Akhiezer function is normalized at one point $r$:

$$\psi(u, r) = 1, \quad R = r \in \Gamma.$$

Then the functions

$$c_{\alpha\beta\gamma}(x) = \sum_{i=1}^n H_i^2 \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta} \frac{\partial u^i}{\partial x^\gamma}$$

are homogeneous:

$$c_{\alpha\beta\gamma}(\lambda x^1, \ldots, \lambda x^n) = \frac{1}{\lambda} c_{\alpha\beta\gamma}(x^1, \ldots, x^n).$$

To obtain a Frobenius manifold from the solution found for the associativity equations one must add to the generators $e_1, \ldots, e_N$ of the associative algebra an identity $e_0$ and a nilpotent element $e_{N+1}$. Such an extension is given by the following algebraic lemma.

Lemma 1 [2]. Let $F(t^1, \ldots, t^N)$ be a solution of the associativity equations with a constant metric $\eta_{\alpha\beta}$. Then the function

$$\widetilde{F}(t^0, t^1, \ldots, t^N, t^{N+1}) = \frac{1}{2} (\eta_{\alpha\beta} t^\alpha t^\beta t^0 + (t^0)^2 t^{N+1}) + F(t^1, \ldots, t^N)$$

satisfies the associativity equations (28) with the metric

$$\widetilde{\eta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \eta & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
and the associative algebra generated by the elements $e_0, e_1, \ldots, e_N, e_{N+1}$ with the multiplication rules

$$e_i \cdot e_j = c_{ij}^k e_k, \quad c_{ij}^k = \eta^{kl} \frac{\partial^3 F}{\partial t^l \partial t^i \partial t^j},$$

has the identity $e_0 \cdot e_k = e_k$ for all $k = 0, \ldots, n+1$, and the nilpotent element $e_{N+1}^2 = 0$. Moreover, if $F$ is quasi-homogeneous and $d_\alpha + d_\beta = c$ for all $\alpha, \beta$ such that $\eta_{\alpha\beta} \neq 0$, then $F$ is also quasi-homogeneous with $d_0 = d_F - c$, $d_{N+1} = 2c - d_F$, and the same values of $d_\alpha$, $\alpha = 1, \ldots, N$, as for $F$.

We present two of the simplest examples of Frobenius manifolds given by Theorem 6.

We note that statement 1) of this theorem gives the solutions of the associativity equations, while statement 2) distinguishes among them a certain subclass of quasi-homogeneous solutions: there are quasi-homogeneous solutions whose spectral data do not satisfy the sufficient conditions in 2). Example 11 below is covered by 2), but Example 12 is not.

**Example 11.** Let $\Gamma$ be formed by two complex projective lines $\Gamma_1$ and $\Gamma_2$ which intersect in a pair of points (see Fig. 6): \{a, -a $\in \Gamma_1\} \sim \{a, -a \in \Gamma_2\}.

![Figure 6](image)

Let $N = 2$ and $l = 1$, and let the Baker–Akhiezer function $\psi$ be normalized by the condition $\psi_2(r) = 1$ at a point $r \in \Gamma_2$. The prepotential takes the form

$$F_{a,c}(x_1^1, x_2^2) = \frac{1}{4ac} \left( 2x_2^2 \sqrt{(a^2 - c^2)x_1^2 + c^2 x_2^2} + 2cx_1^2 \log \left( -\frac{cx_2}{x_1} + \sqrt{(a^2 - c^2)x_1^2 + c^2 x_2^2} \right) - \sqrt{2c^2 - a^2} (x_1^2 + x_2^2) \right.$$

$$\left. \times \log \left( c^2(x_1^2 - 3x_2^2) + a^2(x_2^2 - x_1^2) - 2x_2 \sqrt{2c^2 - a^2} \sqrt{(a^2 - c^2)x_1^2 + c^2 x_2^2} \right) \right)$$

and satisfies the associativity equations for $\eta_{\alpha\beta} = \delta_{\alpha\beta}$. It depends on two additional parameters $a$ and $c$, and for $a = 1$ and $c = 2/\sqrt{7}$ the formulae for the coordinates
and for the correlators are rather simple:

\[ x^1 = \frac{4(7 - \sqrt{7})e^{u_1-u^2}}{(21 - 6\sqrt{7})e^{2u_1} + (7 + 2\sqrt{7})e^{2u_2}}, \]
\[ x^2 = \frac{e^{-2u_2}(3(\sqrt{7} - 3)e^{2u_1} + (5 + \sqrt{7})e^{2u_2})}{3(\sqrt{7} - 2)e^{2u_1} + (2 + \sqrt{7})e^{2u_2}}, \]

\[ c_{111} = \frac{-9x_1^8 + 51x_1^6x_2^2 + 88x_1^4x_2^4 + (2x_1^2x_2^3 + 4x_2^5)\sqrt{(3x_1^2 + 4x_2^2)^3} + 48x_1^2x_2^6}{2x_1(3x_1^4 + 7x_1^2x_2^2 + 4x_2^4)^2}, \]
\[ c_{112} = \frac{9x_1^6x_2 + 15x_1^4x_2^3 - 8x_1^2x_2^5 + (2x_1^3x_2^2 + 4x_1^2x_2^3 + 4x_2^4)\sqrt{(3x_1^2 + 4x_2^2)^3} - 16x_2^7}{2(3x_1^4 + 7x_1^2x_2^2 + 4x_2^4)^2}, \]
\[ c_{122} = \frac{-9x_1^7 + 15x_1^5x_2^2 - 8x_1^3x_2^4 + (2x_1^3x_2^2 + 4x_1^2x_2^3 + 4x_2^4)\sqrt{(3x_1^2 + 4x_2^2)^3} - 16x_1x_2^6}{2(3x_1^4 + 7x_1^2x_2^2 + 4x_2^4)^2}, \]
\[ c_{222} = \frac{-27x_1^6x_2 + 16x_2^7 - 72x_1^2x_2^3 + (4x_1^2x_2^2 + 2x_1^3)\sqrt{(3x_1^2 + 4x_2^2)^3} - 81x_1^4x_2^3}{2(3x_1^4 + 7x_1^2x_2^2 + 4x_2^4)^2}. \]

**Example 12.** Let \( \Gamma \) be the same complex curve as in Example 11. Contrary to Example 11, we assume here that

\[ P_1 = \infty \in \Gamma_1, \quad P_2 = 0 \in \Gamma_1, \quad Q_1 = \infty \in \Gamma_2, \quad Q_2 = 0 \in \Gamma_2, \]

the normalization point \( R = r \) lies in \( \Gamma_1 \), and the pole divisor \( D = c \) lies in \( \Gamma_2 \) (see Fig. 7). However, we do not assume that the intersection points have the same coordinates (the assumptions of the statement 2) in Theorem 6 are not satisfied):

\[ a \sim b, \quad -a \sim -b, \quad \pm a \in \Gamma_1, \quad \pm b \in \Gamma_2, \quad a \neq b. \]

![Figure 7](image)

The prepotential \( F(x^1, x^2) \) is

\[ F(x^1, x^2) = -\frac{1}{8} \left( (x^1)^2 + (x^2)^2 \right) \log((x^1)^2 + (x^2)^2) \]

and is included in a linear family of quasi-homogeneous functions

\[ F_q(x^1, x^2) = q((x^1)^2 + (x^2)^2) \arctan \left( \frac{x^1}{x^2} \right) - \frac{1}{8} \left( (x^1)^2 + (x^2)^2 \right) \log((x^1)^2 + (x^2)^2), \quad q \in \mathbb{R}, \]

which satisfy the associativity equations for \( \eta_{\alpha\beta} = \delta_{\alpha\beta} \).
The correlators for $F$ (that is, $q = 0$) have the simplest form:

\[
\begin{align*}
c_{111} &= -\frac{3}{2} \frac{x^1}{(x^1)^2 + (x^2)^2} + \frac{(x^1)^3}{((x^1)^2 + (x^2)^2)^2}, \\
c_{112} &= -\frac{1}{2} \frac{x^2}{(x^1)^2 + (x^2)^2} + \frac{(x^1)^2 x^2}{((x^1)^2 + (x^2)^2)^2},
\end{align*}
\]

and the formulae for $c_{122}$ and $c_{222}$ are obtained from these by the interchange of indices $1 \leftrightarrow 2$.

Examples 11 and 12 describe quasi-homogeneous deformations of the cohomology ring of $\mathbb{C}P^2 \# \mathbb{C}P^2$. Indeed, we have the standard generators $e_0, \ldots, e_3$ in $H^*(\mathbb{C}P^2 \# \mathbb{C}P^2; \mathbb{C})$:

\[
e_0 \in H^0, \quad e_1, e_2 \in H^2, \quad e_3 \in H^4, \quad e_1^2 = e_2^2 = e_3, \quad e_1 e_2 = 0.
\]

The identity $d_i = (\deg e_i)/2$ holds. These deformations change the multiplication rules for two-dimensional classes by adding two-dimensional terms:

\[
e_i e_j = e_3 + c_{ij}^k(t)e_k, \quad 1 \leq i, j \leq 2.
\]

4. Soliton equations with self-consistent sources and the corresponding deformations of the spectral curves

4.1. Equations with self-consistent sources. Contrary to $(1+1)$-dimensional soliton equations, which have the Lax representation

\[
L_t = [L, A],
\]

equations with self-consistent sources are represented in the form

\[
L_t = [L, A] + C,
\]

where $C$ is an expression in terms of solutions $\psi_k$ of the linear problems $L \psi_k = \lambda_k \psi_k$, $k = 1, \ldots, l$. The inverse problem method is applicable to equations with self-consistent sources, and they were apparently first derived from this formally algebraic point of view by Mel’nikov [41]. In the physics literature they first appeared in the paper [42] by Zakharov and Kuznetsov, who, in particular, described the physical meaning of the Korteweg–de Vries and Kadomtsev–Petviashvili equations

\[
3u_{yy} - \frac{\partial}{\partial x}(u_t + 6uu_x + u_{xxx}) = 3 \frac{\partial^2}{\partial x^2} |\psi|^2, \quad i\psi_t = \psi_{xx} + u\psi
\]

with self-consistent sources.

We note two important facts:

1) each such equation has as its predecessor a soliton equation $L_t = [L, A]$;

2) to correctly pose the problem it is necessary to normalize the functions $\psi$ (they have to correspond to certain fixed spectral characteristics) and to have an expression $C$ which falls in the class of solutions under consideration — for instance, to be rapidly decreasing or periodic.
Solutions of these equations have interesting qualitative properties: for example, creation and annihilation of solitons become possible [43]. We discuss this in § 4.3.

Corresponding to the operator $L$ in the periodic case is a spectral curve which, as it turns out, can be deformed under the action of an equation with self-consistent sources, and such a deformation consists in the appearance or disappearance of singular points. This was first observed in [3] for a conformal flow which arises in the differential geometry of surfaces, and later was studied for the KP equation with self-consistent sources [4]. These two facts are presented below in § 4.2 and § 4.3, respectively.

4.2. Spectral curves of immersed tori and the conformal flow. In the theory of the Weierstrass representation of immersed surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ (see the survey [13]) surfaces are described in terms of solutions of the equation $D\psi = 0$ for surfaces in $\mathbb{R}^3$ and of the equations $D\psi = D^\vee \varphi = 0$ for surfaces in $\mathbb{R}^4$, where

$$
D = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}, \quad D^\vee = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \bar{U} & 0 \\ 0 & U \end{pmatrix}.
$$

For surfaces in $\mathbb{R}^3$ the potential $U$ is real-valued and $D = D^\vee$.

For tori these operators are doubly periodic, and it is natural to consider their spectral curves on the zero energy level. It turned out that these spectral curves contain information about the values of the Willmore functional, which is defined as

$$W(M) = \int_M |H|^2 d\mu,$$

where $H$ is the mean curvature vector and $d\mu$ is the volume form induced by the immersion on the surface $M$. This functional is conformally invariant in the sense that

$$W(M) = W(f(M)),$$

where $M$ is a closed surface and $f: S^k \to S^k$ is a conformal transformation of $S^k = \mathbb{R}^k \cup \{\infty\}$, $k \geq 3$, such that $M \subset \mathbb{R}^k$ and $f(M) \subset \mathbb{R}^k$.

For tori of revolution the potential $U$ depends on one variable, and we obtain a reduction of the problem $D\psi = 0$ to the Zakharov–Shabat spectral problem for the one-dimensional Dirac operator. The Willmore functional $W$ is the first Kruskal–Miura integral connected with this operator for the modified KdV (mKdV) hierarchy. Other Kruskal–Miura integrals of the mKdV hierarchy can be generalized to (non-local) conservation laws for the hierarchies connected with the two-dimensional operator $D$ (the hierarchies of the modified Novikov–Veselov equations and Davy–Stewartson equations).

This leads us to an observation about the existence of a relation between the geometry of a surface and the spectral properties of the operator $D$ in the Weierstrass representation of the surface [15].

Our proposed approach to proving the Willmore conjecture, based on this relation, has so far not met with success.

Another conjecture of ours, namely, that the higher two-dimensional generalizations of the Kruskal–Miura integrals, and together with them the whole spectral curve, are conformally invariant, was confirmed in [44]. The conformal flow introduced by Grinevich was used for this.
Since for \( k \geq 3 \) the conformal group of \( \mathbb{R}^k \cup \{\infty\} \) is generated by translations, dilations, and inversions, and since the potential \( U \) is invariant under translations and dilations, it is enough to prove the invariance of spectral curve under inversions. To do this, let us consider the generator of inversions in the conformal group and express its action on the potential. It has the form

\[
U_t = |\psi_2|^2 - |\psi_1|^2,
\]

where a torus \( \mathbb{R}^3 \) is given by a solution \( \psi = (\psi_1, \psi_2)^T \) of the equation \( \mathcal{D} \psi = 0 \).

**Proposition 1** [44]. All the Floquet multipliers of the operator \( \mathcal{D} \) are preserved by the flow (32) corresponding to an inversion of a torus in \( \mathbb{R}^4 \), and therefore they are preserved by inversions.

It [3] the analogous problem was considered for tori in \( \mathbb{R}^4 \), and it was shown that to the generator of inversions

\[
\partial_t x^1 = 2x^1 x^3,
\]
\[
\partial_t x^2 = 2x^2 x^3,
\]
\[
\partial_t x^3 = (x^3)^2 - (x^1)^2 - (x^2)^2 - (x^4)^2,
\]
\[
\partial_t x^4 = 2x^4 x^3
\]

there corresponds a deformation of \( U \) of the form

\[
\partial_t U = \varphi_1 \overline{\psi}_1 - \varphi_2 \overline{\psi}_2, \quad \partial_t \overline{U} = \overline{\varphi}_1 \psi_1 - \varphi_2 \overline{\psi}_2, \quad \mathcal{D} \psi = \mathcal{D}^\gamma \varphi = 0,
\]

where \( \psi = (\psi_1, \psi_2)^T \) and \( \varphi = (\varphi_1, \varphi_2)^T \) define a torus in \( \mathbb{R}^4 \) via the Weierstrass representation. The following result was shown in a way similar to that in the three-dimensional case.

**Proposition 2** [3]. All the Floquet multipliers of \( \mathcal{D} \) are preserved by a transformation of \( U \) of the form (33), where \( \psi \) and \( \varphi \) satisfy the equations \( \mathcal{D} \psi = \mathcal{D}^\gamma \varphi = 0 \) and the quadratic expressions \( \psi_i^2 \) and \( \varphi_j^2 \) (\( i, j = 1, 2 \)) are doubly periodic.

We note that (32) and (33) are equations with self-consistent sources which in the absence of these sources reduce to the stationary equation \( U_t = 0 \).

At the same time, there are explicitly computed spectral curves of the Clifford torus

\[
x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2},
\]

in the three-sphere \( S^3 \subset \mathbb{R}^4 \) and of its stereographic projection into \( \mathbb{R}^3 \). The Willmore conjecture asserts that on the Clifford torus the Willmore functional attains its minimal value (for all immersed tori), which is equal to \( 2\pi^2 \).

For the Clifford torus in \( S^3 \subset \mathbb{R}^4 \) the spectral curve is the complex projective line, \( \Gamma = \mathbb{C}P^1 \), while for the projection into \( \mathbb{R}^3 \) it is a rational curve with two double points [45]. A detailed analysis of the flow (33) led us to the following result.

**Theorem 7** [3]. The evolution flow (33) acting on doubly periodic potentials preserves the Floquet multipliers of the operator \( \mathcal{D} \), but can deform its spectral curve (on the zero energy level), with the deformation reducing to the gluing together or splitting apart of double points.

Since the higher conservation laws for the modified Novikov–Veselov hierarchy and the Davy–Stewartson hierarchy are expressed in terms of the asymptotic behaviour of multipliers, they are also conservation laws for the flow (33).
4.3. Finite-gap solutions of KdV and KP equations with self-consistent sources. Let \( \Gamma \) be a Riemann surface of genus \( g \) (the spectral curve) with a marked point \( P \) and with a local parameter \( k^{-1} \) near this point, \( k(P) = \infty \), and let \( D = \gamma_1 + \cdots + \gamma_g, \gamma_i \in \Gamma, \ i = 1, \ldots, N, \) be a generic divisor on it. Assume also that on \( \Gamma \) there are \( 2N \) pairwise distinct marked points \( R^\pm_l, \ l = 1, \ldots, N, \) which also differ from \( P \) and from all the points in \( D \).

By the theory of Baker–Akhiezer functions there is a unique function \( \psi(\gamma, x, y, t, \tau) \) with \( \tau = (\tau_1, \ldots, \tau_N) \) and \( \gamma \in \Gamma \) such that:

1) \( \psi \) is meromorphic with respect to \( \gamma \) on \( \Gamma \setminus P \) and has \( g + N \) simple poles at the points \( \gamma_1, \ldots, \gamma_g, R_1^+, \ldots, R_N^+; \)

2) \( \text{res} \psi(\lambda, x, y, t, \tau)|_{\lambda = R^+_l} = \tau_l \psi(R^-_l, x, y, t, \tau), \ l = 1, \ldots, N; \)

3) \( \psi \) has an essential singularity at \( P \), and

\[
\psi(\gamma, x, y, t, \tau) = e^{kx + k^2y + k^3t} \left( 1 + \sum_{m>0} \frac{\xi_m(x, y, t, \tau)}{k^m} \right) \quad \text{for} \quad \gamma \to P.
\]

Moreover, there exists a unique adjoint Baker–Akhiezer 1-form\(^{10} \) \( \psi^*(\gamma, x, y, t, \tau) \) with the following properties:

1\(^*\) \( \psi^* \) is meromorphic with respect to \( \gamma \) on \( \Gamma \setminus P \) and has \( g + N \) simple zeros at the points \( \gamma_1, \ldots, \gamma_g, R^-_1, \ldots, R^-_N; \)

2\(^*\) \( \text{res} \psi^*(\lambda, x, y, t, \tau)|_{\lambda = R^-_l} = -\tau_l \frac{\psi^*(\lambda, x, y, t, \tau)}{d\lambda} \left|_{\lambda = R_l^+} \right., \ l = 1, \ldots, N; \)

3\(^*\) \( \psi^* \) has an essential singularity at \( P \), and

\[
\psi^*(\gamma, x, y, t, \tau) = e^{-kx - k^2y - k^3t} \left( 1 + o(1) \right) dk \quad \text{for} \quad \gamma \to P.
\]

Let

\[
u(x, y, t) = 2 \frac{\partial}{\partial x} \xi_1(x, y, t, \tau), \quad \tau_l = \alpha_l - \beta_l t, \quad l = 1, \ldots, N. \tag{34}
\]

The functions \( \psi(\gamma, x, y, t) \) and the forms \( \psi^*(\gamma, x, y, t) \) are constructed similarly from \( \psi(\gamma, x, y, t, \tau) \) and \( \psi^*(\gamma, x, y, t, \tau) \).

We have the following theorem.

**Theorem 8** [4]. The function \( \nu(x, y, t) \) of the form \( (34) \) satisfies the KP equation with \( N \) self-consistent sources:

\[
u_t = KP[\nu] + 2 \frac{\partial}{\partial x} \sum_{l=1}^N \beta_l \psi(R^-_l, x, y, t, \tau) \psi^*(\lambda, x, y, t)|_{\lambda = R^+_l}, \tag{35}
\]

where \( \nu_t = KP[\nu] \) is the KP flow.

If there exists a two-sheeted cover \( \pi: \Gamma \to \mathbb{C}P^1 \) that is branched at the point \( P, \pi(P) = \infty, \pi(k) = \pi(-k), \) and \( \pi(R^+_l) = \pi(R^-_l) \) for \( l = 1, \ldots, N, \) then the formula \( (35) \) gives a solution \( \nu(x, t) \) of the KdV equation with self-consistent sources (in \( (35) \) one must replace \( KP[\nu] \) by \( \text{KdV}[\nu] \), where \( u_t = \text{KdV}[\nu] \) is the Korteweg–de Vries equation).

\(^{10}\)The adjoint Baker–Akhiezer 1-forms were introduced in [46]. The proof of their existence and uniqueness is analogous to proofs of the same facts for ordinary Baker–Akhiezer functions.
Similarly, one can construct finite-gap solutions of all the equations in the KdV and KP hierarchies with self-consistent sources [4]. The proof of this theorem is essentially based on the theory of Cauchy–Baker–Akhiezer kernels introduced by Grinevich and Orlov [47].

**Remark 2.** The Baker–Akhiezer function $\psi$ in this theorem is defined on the spectral curve with double points $R_l$ which is obtained from $\Gamma$ by pairwise identifications of the points $R_l^+$ and $R_l^-$, $l = 1, \ldots, N$. A double point $R_l$ splits apart if and only if $\tau_l = 0$. If even for the initial potential $u(x, y, t)$ at $t = 0$ the spectral curve is regular, then the equations with self-consistent sources immediately lead to the creation of double points on the spectral curve (for almost all times their number is equal to the number $N$ of sources), and these singularities are preserved for almost all times.

**Example 13.** Let $\Gamma = \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$, with a parameter $k$, $N = 1$, and $R^\pm = \pm \kappa$. After simple computations we obtain the formula

$$u(x, t, \tau) = -\frac{16\kappa^3}{\tau e^{-(\kappa x + \kappa^3 t)} + 2\kappa e^{\kappa x + \kappa^3 t}},$$

which gives a (regular) soliton for $\tau > 0$, a zero solution for $\tau = 0$, and a singular soliton for $\tau < 0$. From this we conclude that the function $u(x, t) = u(x, t, \alpha + \beta t)$ satisfies the KdV equation with a self-consistent source

$$u_t = \frac{1}{4} u_{xxx} - \frac{3}{2} u u_x + 2\beta \partial_x \psi^2(-\kappa, x, t), \quad (36)$$

where

$$\psi(-\kappa, x, t) = \left(1 - \frac{\tau}{\tau + 2\kappa e^{2(\kappa x + \kappa^3 t)}}\right) e^{-\kappa x - \kappa^3 t}, \quad \tau = \alpha + \beta t.$$

There are the following qualitative effects, first noted by Mel’nikov [43]:

1) starting from a small initial value $c = c(0)$, we attain $c = 0$ in a finite time (annihilation of a soliton);
2) upon inversion of the flow (36) with the initial data $c(0) = 0$ we immediately obtain a soliton for $t > 0$ (creation of a soliton).

For rapidly decreasing potentials these two effects are the analogues of an annihilation and a creation of a double point for the spectral curves of doubly periodic potentials.

### 5. Some other examples of integrable problems with singular spectral curves

We consider another pair of interesting examples of integrable physical problems with singular spectral curves:

1) Recently Grinevich, Mironov, and Novikov found a class of algebro-geometric spectral data from which, by using the two-point Baker–Akhiezer functions introduced in [7], one can construct operators of the form

$$L = (\partial + A) \tilde{\partial}$$

where

$$\partial = \frac{\partial}{\partial x}, \quad \tilde{\partial} = \frac{\partial}{\partial \tau}, \quad A = \frac{\partial^2}{\partial x^2}.$$
(a magnetic two-dimensional Pauli operator) [48]. The spectral curve \( \Gamma \) consists of two smooth components \( \Gamma' \) and \( \Gamma'' \) from which it is obtained by identifying \( k + 1 \) pairs of points \( Q_1' \sim Q_1'', \ldots, Q_{k+1}' \sim Q_{k+1}'', \) where \( Q_i' \in \Gamma' \) and \( Q_j'' \in \Gamma'' \) for \( i, j = 1, \ldots, k + 1 \). Moreover, there exists an antiholomorphic involution \( \sigma : \Gamma \to \Gamma \), \( \sigma^2(\gamma) = \gamma, \gamma \in \Gamma \), which permutes the irreducible components:

\[
\sigma(\Gamma') = \Gamma'', \quad \sigma(Q_{k}') = Q_{\sigma(k)''}.
\]

This implies that \( \Gamma' \) and \( \Gamma'' \) have the same genus \( g \), and \( p_a(\Gamma) = 2g + k \). Physically interesting examples arise already in the case when \( \Gamma' \) and \( \Gamma'' \) are complex projective lines.

There is an integrable two-dimensional generalization of the Burgers equation connected with another real reduction of \( L \),

\[
\tilde{L} = (\partial_x + A) \partial_y,
\]

which is constructed from the same singular spectral curves. [49].

2) At the beginning of the 1980s Krichever applied algebro-geometric methods to a study of solutions of the Yang–Baxter equation for \( 4 \times 4 \) matrices [50]. The construction of these solutions does not use Baker–Akhiezer functions, but the general ideology is taken from finite-gap integration, and the role of the spectral curve \( \Gamma \) is played by a smooth elliptic curve. The solutions are classified by their rank, which takes the values \( l = 1, 2 \), and as shown in [50], all the solutions of rank one are gauge equivalent to the Baxter solution or are obtained from it by transformations corresponding to simple symmetries. Dragovich considered degenerate cases of this construction and showed that if \( \Gamma \) splits into two rational components intersecting in two points, then we have the well-known Yang solution [51], while if \( \Gamma \) is a rational curve with a double point, then we get Cherednik’s solution [52] (in both cases, up to gauge equivalence).

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