Constant Factor Approximation for Balanced Cut in the PIE Model

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Abstract

We propose and study a new semi-random semi-adversarial model for Balanced Cut, a planted model with permutation-invariant random edges (PIE). Our model is much more general than planted models considered previously. Consider a set of vertices $V$ partitioned into two clusters $L$ and $R$ of equal size. Let $G$ be an arbitrary graph on $V$ with no edges between $L$ and $R$. Let $E_{random}$ be a set of edges sampled from an arbitrary permutation-invariant distribution (a distribution that is invariant under permutation of vertices in $L$ and in $R$). Then we say that $G + E_{random}$ is a graph with permutation-invariant random edges.

We present an approximation algorithm for the Balanced Cut problem that finds a balanced cut of cost $O(|E_{random}|) + n \text{polylog}(n)$ in this model. In the regime when $|E_{random}| = \Omega(n \text{polylog}(n))$, this is a constant factor approximation with respect to the cost of the planted cut.

1 Introduction

Combinatorial optimization problems arise in many areas of science and engineering. Many of them are $NP$-hard and cannot be solved exactly unless $P = NP$. What algorithms should we use to solve them? There has been a lot of research in theoretical computer science dedicated to this question. Most research has focused on designing and analyzing approximation algorithms for the worst-case model, in which we do not make any assumptions on what the input instances are. While this model is very general, algorithms for the worst-case model do not exploit properties that instances we encounter in practice have. Indeed, as empirical evidence suggests, real-life instances are usually much easier than worst-case instances, and practitioners often get much better approximation guarantees in real life than it is theoretically possible in the worst-case model. Thus it is very important to develop a model for real-life instances that will allow us to design approximation algorithms that provably work well in practice and outperform known algorithms designed for the worst case.

Several such models have been considered in the literature since the early 80’s: e.g. the random planted cut model [9, 13, 8, 19, 12, 25, 11, 10], semi-random models [14, 29, 22], and stable models [4, 3, 7, 2, 5, 6, 24].

In this paper, we propose a new very general model “planted model with permutation-invariant random edges”. We believe that this model captures well many properties of real-life instances. In particular, we argue below that our model is consistent with social network formation models studied in social sciences. We present an approximation algorithm for the Balanced Cut problem. Balanced Cut is one of the most basic and well-studied graph partitioning problems. The problem does not admit a constant factor approximation in the worst-case as was shown by Raghavendra, Steurer, and Tulsiani [27] (assuming the Small Set Expansion

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Conjecture). The best known algorithm for Balanced Cut by Arora, Rao, and Vazirani [11] gives \(O(\sqrt{\log n})\) approximation. In contrast, our algorithm gives a constant factor approximation with respect to the size of the planted cut in our model (if some conditions hold, see below).

We start with recalling the classical planted cut model of Bui, Chaudhuri, Leighton and Sipser [9] and Dyer and Frieze [13]. In this model, we generate a random graph \(F\) as follows. Let \(p\) and \(q < p\) be two numbers between 0 and 1. We take two disjoint \(G(n/2, p)\) graphs \(G_1 = (L, E_1)\) and \(G_2 = (R, E_2)\). We connect every two vertices \(x \in L\) and \(y \in R\) with probability \(q\); our random choices for all pairs of vertices \((x, y)\) are independent. We obtain a graph \(F\). We call sets \(L\) and \(R\) clusters and say that \((L, R)\) is the planted cut. We refer to the edges added at the second step as random edges. In this model, we can find the planted cut \((L, R)\) w.h.p. given the graph \(F\) (under some assumptions on \(p\) and \(q\)) [9, 13, 8, 10].

In our model, graphs \(G_1\) and \(G_2\) can be arbitrary graphs. The set of random edges is sampled from an arbitrary permutation-invariant distribution (a distribution is permutation-invariant if it is invariant under permutation of vertices in \(L\) and \(R\)). We do not make any assumptions on the distribution (aside from it being permutation-invariant). In particular, random choices for different edges may be dependent, edges may cross the cut \((L, R)\) or lie inside clusters. The set of random edges may be sampled according to a distribution that is very complex and unknown to us. For example, it may be sampled using the preferential attachment model. It can contain fairly large bicliques and dense structures that are found in many real-world networks [21, 26].

**Definition 1.1.** Consider a set of vertices \(V\) and a partition of \(V\) into two sets of equal size: \(V = L \cup R\). Let \(\Pi_{LR}\) be the set of permutations of \(V\) such that \(\pi(L) = L\) and \(\pi(R) = R\). We say that a a probability distribution \(D\) on \(\{E \subset V \times V\}\) is permutation-invariant if for every permutation \(\pi \in \Pi_{LR}\) and every set \(E \subset V \times V\), we have \(\Pr_D(\pi E) = \Pr_D(E)\).

Informally, a distribution is permutation-invariant if it “ignores” the “identities” (labels) of individual vertices; for each vertex \(u\), the distribution just “knows” whether \(u\) is in \(L\) or in \(R\).

**Definition 1.2 (Formal Definition of the Model).** Let \(V\) be a set of vertices and \(V = L \cup R\) be a partition of \(V\) into two sets of equal size. Let \(G = (V, E_G)\) be an arbitrary graph on \(V\) in which no edge crosses cut \((L, R)\). Let \(D\) be an arbitrary permutation-invariant distribution of edges. We define a probability distribution \(\Pi(L, R, E_G, D)\) of planted graphs \(F\) with permutation-invariant random edges (PIE) as follows. We sample a random set of edges \(E_R\) from \(D\) and let \(F = G + E_R\).

We give an alternative equivalent definition in Section [1, 3]. Before we state our main result, we recall the definition of the Balanced Cut problem.

**Definition 1.3.** A cut \((S, T)\) in a graph \(G = (V, E)\) is \(b\)-balanced if \(|S| \geq bn\) and \(|T| \geq bn\) (where \(b \in [0, 1/2]\) is parameter). The Balanced Cut problem is to find a \(b\)-balanced cut \((S, T)\) in a given graph \(G\) so as to minimize the number of cut edges.

We show that there is an algorithm that finds a \(\Theta(1)\)-balanced cut \((S, T)\) of cost \(O(|E_R|) + O(n \text{ polylog } n)\) w.h.p. This result is most interesting when the following conditions hold: (1) a constant fraction of edges in \(E_R\) go from \(L\) to \(R\), and (2) the number of random edges is \(\Omega(n \text{ polylog } n)\). Then, the size of the cut \((S, T)\) is at most a constant times the size of the planted cut. That is, we obtain a constant factor approximation with respect to the size of the planted cut. The algorithm does not know the graph \(G\), the distribution \(D\), and the planted cut \((L, R)\). We now formally state our main result.

**Theorem 1.4.** There is a deterministic polynomial-time algorithm that given a random graph \(F\) sampled from \(\Pi(L, R, E_G, D)\) finds a \(\Theta(1)\)-balanced cut \((S, T)\) such that

\[|E(S, T)| = O(|E_R|) + O(n \text{ polylog } n)\]

(for arbitrary sets \(L, R, E_G\), and permutation-invariant distribution \(D\), not known to the algorithm). The algorithm succeeds with probability \(1 - o(1)\) over the choice of \(F\).
model | planted graphs in L and R | random edges ER | algorithm finds a balanced cut of size (w.h.p.)
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random planted model [9, 13] | G(n/2, p) graphs | edges between L and R are sampled independently w. p. q | \(|E_R|\) The algorithm recovers the planted cut.
semi-random model [22] | arbitrary graphs | edges between L and R are sampled independently w. p. q the adversary may delete random edges | \(O(qn^2)\), equals \(O(|E_R|)\) if the adversary does not delete edges It is impossible to find the planted cut (information-theoretically).
our model | arbitrary graphs | sample \(E_R\) from an arbitrary permutation-invariant distribution (unknown to the algorithm) | \(O(|E_R|)\) It is impossible to find the planted cut (information-theoretically).

Table 1: This table compares the random planted model [9, 13], semi-random model [22], and model proposed in this paper. Algorithms for all three models succeed with high probability. In this table, we assume that \((p - q)n^2 > n\ polylog(n)\) in the first model, and \(|E_R| > n\ polylog(n)\) in the second and third models.

1.1 Comparison with other models

There is an extensive literature on the random planted model [9, 13, 8, 19, 12, 25, 11, 10] and semi-random models [14, 29, 22]. We compare our model with the random planted model of Bui, Chaudhuri, Leighton and Sipser [9] and Dyer and Frieze [13] and semi-random model from our previous work [22] (which generalizes the model of Feige and Kilian [14], see Table 1.1). In the random planted model, planted graphs in L and R are random \(G(n/2, p)\) graphs. The set of edges \(E_R\) is a random subset of all possible edges between L and R; every edge is present with the same probability \(q < p\) (which does not depend on the edge); all edges are chosen independently. The semi-random model of [22] is significantly more general. In this model, graphs inside L and R are arbitrary graphs. However, \(E_R\) is essentially the same as in the random planted model, except that we allow the adversary to delete edges between L and R. In the model we study in the current paper, not only are the graphs inside L and R arbitrary graphs, but further, \(E_R\) is sampled from an arbitrary permutation-invariant distribution (in particular, they can be random edges chosen with probability \(q\) as in the previous models [9, 13, 22]).

Bui, Chaudhuri, Leighton and Sipser [9] and Dyer and Frieze [13] showed how to find the planted cut w.h.p. in the random planted model (see also [8, 10]). This is impossible to do in our model even information-theoretically. Instead, we give an approximation algorithm that gives a constant factor approximation with respect to the size of the planted cut if conditions (1) and (2) hold.

1.2 Motivation

The random planted cut model (often referred to as the Stochastic Block Model) is widely used in statistics, machine learning, and social sciences (see e.g. [18, 30, 17, 15, 28]). The PIE model, which we study in this paper, aims to generalize it, relax its constraints, replace random choices with adversarial choices whenever possible and yet keep the model computationally tractable. In our opinion, the PIE model better

1E.g. consider the following graph \(F\): \(F[L]\) and \(F[R]\) are \(G(n/2, p)\) graphs, every edge between L and R is present independently with probability \(p\). Then \(F\) has no information about the cut \((L, R)\).
captures real-life instances than the random planted cut model. Consider two examples. The first example is clustering with noise. Suppose that we are given a set of objects \( V \). The objects are partitioned in two clusters, \( L \) and \( R \); but the clustering is not known to us. We are also given a set of “similarity” edges \( E \) on \( V \). Some edges \( E_G \subset E \) represent real similarities between objects in \( V \); these edges connect vertices within one cluster. In practice, edges in \( E_G \) are not random and our model does not impose any restrictions on them in contrast to the random planted cut model, which assumes that they are completely random. Other edges \( E_R \subset E \) are artifacts caused by measurement errors and noise. Edges in \( E_R \) are somewhat random and it is reasonable in our opinion to assume — as we do in our model — that they are sampled from a permutation-invariant distribution. Unlike the random planted model, we do not assume that edges in \( E_R \) are sampled independently.

The second example is related to social networks. There are many types of ties in social networks — there are social ties between relatives, friends, colleagues, neighbors, people with common interests and hobbies. The whole social network can be thought of as a superposition of separate networks with different types of ties. It is reasonable to assume that these networks are to large extent independent; e.g., you cannot tell much about somebody’s neighbors, if you just know his or her coauthors.

Consider a social network with several types of ties. Represent it as a graph: the vertices represent people, and edges represent social ties. Assume that people in the social network live in different geographical regions, cities, countries, etc. We divide all regions into two groups and denote the set of people who live in the regions in the first and second groups by \( L \) and \( R \), respectively. Some types of ties are usually “local” — they are ties between people living in the same region; e.g. typically friends live in the same region. Other ties are not necessarily local; e.g. coauthors, college classmates, and Twitter followers do not necessarily live in the same region. Let \( E_G \) be the set of edges representing local ties and \( E_R \) be edges representing other ties. Then the whole social network is the union of \( E_G \) and \( E_R \). The assumption that social ties of different types are independent is formalized in our model by the condition that \( E_R \) is sampled from a permutation-invariant distribution. That is, we take two social networks \( G = (V_G, E_G) \) and \( H = (V_H, E_R) \), choose a random correspondence between vertices of \( G \) and \( H \), and then identify corresponding vertices (using the notation, which we introduce in the next section, we consider the graph \( F = G \boxplus_\pi H \) for a random permutation \( \pi \)).

We believe that techniques similar to those we present in the current paper can be applied to other graph partitioning and combinatorial optimization problems. We hope that these techniques will be useful for solving real world problems on networks that we encounter in practice.

1.3 Model with Two Adversaries

We use an alternative equivalent formulation of our model in the rest of the paper. Let \( G = (V_G, E_G) \) and \( H = (V_H, E_H) \) be two graphs on \( n \) vertices, and \( \pi : V_H \to V_G \) be a bijection. Define the graph \( F = G \boxplus_\pi H \) on \( V_G \) by \( E_F = E_G \cup \pi(E_H) \). Let \( V_G = L_G \cup R_G \) and \( V_H = L_H \cup R_H \) be partitions of \( V_G \) and \( V_H \) into sets of size \( n/2 \). Define \( \Pi_{LR} \equiv \{ \pi : V_H \to V_G : \pi(L_H) = L_G \text{ and } \pi(R_H) = R_G \} \) to be the set of all bijections mapping \( L_H \) to \( L_G \) and \( R_H \) to \( R_G \).

Suppose now that one adversary chooses an arbitrary graph \( G \) with no edges between \( L_G \) and \( R_G \), and another adversary chooses an arbitrary graph \( H \) (both adversaries know the partitions \( V_G = L_G \cup R_G \) and \( V_H = L_H \cup R_H \)). Then the nature chooses a bijection \( \pi \in \Pi_{LR} \) uniformly at random. We obtain a graph \( F = G \boxplus_\pi H \).

**Theorem 1.5.** There exists a deterministic polynomial-time algorithm that given a graph \( F = G \boxplus_\pi H \) outputs a \( \Theta(1) \)-balanced partition of \( V_F = V_G \) into two sets \( L' \) and \( R' \). If there are no edges between \( L_G \) and \( R_G \) in \( G \), then the cost of the cut \( (L', R') \) is bounded by \( O(|E_H| + n \log^3 n) \) with probability \( 1 - o(1) \) over a random choice of \( \pi \in \Pi_{LR} \).
Remark: To simplify the exposition we do not attempt to optimize the constants in the \(O(\cdot)\) notation. The additive term \(n \log^2 n\) can be slightly improved.

This theorem implies Theorem \[1.3\]. Indeed, if \(|E_R|\) is a random permutation invariant set of edges, then \(E_R\) is distributed identically to \(\pi(E_R)\), where \(\pi\) is a random permutation from \(\Pi_{LR}\). Thus, graphs \((V_G, E_G \cup E_R)\) are distributed identically to graphs \(V_G \amalg \pi (V_G, E_R)\). The algorithm from Theorem \[1.5\] succeeds with probability \(1 - o(1)\) on graphs \(V_G \amalg \pi (V_G, E_R)\) for every fixed \(E_R\) and random \(\pi \in \Pi_{LR}\). Thus, it succeeds with probability \(1 - o(1)\) on graphs \((V_G, E_G \cup E_R)\).

1.4 Techniques

We present a very high-level overview of the algorithm. We are given a graph \(F = G \amalg \pi H\) and our goal is to find a balanced cut of size roughly \(O(|E_H|)\). We assume that \(|E_G| \gg |E_H|\) as otherwise any balanced cut cuts \(O(|E_H|)\) edges and we are done. We write an SDP relaxation for Balanced Cut. The relaxation is similar but slightly different from the one of Arora, Rao and Vazirani \[1\] (see Section 2 for details). The SDP solution assigns a vector \(\varphi(u)\) to every vertex \(u \in V_G\). The objective function is to minimize \(\sum_{(u,v) \in E_F} \left\| \varphi(u) - \varphi(v) \right\|^2\). The SDP constraints ensure that all vectors lie on a sphere \(S\) of radius \(\sqrt{2}/2\). Given an SDP solution, we say that an edge \((u,v)\) is \(\delta\)-short if \(\left\| \varphi(u) - \varphi(v) \right\|^2 \leq \delta\), where \(\delta\) is a fixed constant, and that it is \(\delta\)-long, otherwise.

For the sake of discussion, let us first make a very unrealistic assumption that the SDP solution is determined by the set of edges \(E_G\) and does not depend on the set of random edges \(E_R = \pi E_H\). Assume furthermore that all vectors \(\{\varphi(u)\}\) are distributed more-or-less uniformly on the sphere \(S\); more precisely, assume that every ball of radius \(\delta\) w.r.t. the squared Euclidean distance contains very few vectors \(\varphi(u)\). Then for every edge \(e = (u,v) \in E_H\), the probability over \(\pi\) that vectors \(\varphi(\pi u)\) and \(\varphi(\pi v)\) lie in the same ball of radius \(\delta\) is very small, and thus \(\pi e\) is a long edge with high probability. Now the total number of long edges in \(F\) is at most \(|E_R|/\delta\) since each long edge contributes at least \(\delta\) to the SDP objective function and the SDP value is at most the cost of the planted cut. This discussion suggests an approach to the problem. Let us remove all long edges in \(F\). When we do so, we decrease the number of edges in \(E_R\) by a constant factor and cut only a constant number of edges in \(E_G\) for each cut edge in \(E_R\). We repeat this step over and over until (almost) all random edges are cut. The total number of removed edges does not exceed \(O(|E_R|)\), as required.

There are several problems with this argument.

1. The SDP solution does depend on the set \(E_R\).

2. Vectors \(\varphi(u)\) are not uniformly distributed on the sphere \(S\), in general. In fact, there are only two possible values for vectors \(\varphi(u)\) in the intended integral solution.

3. We will not make any progress if we just run the same procedure over and over.

We use a Heavy Vertices Removal procedure to deal with the second and third problems. Conceptually, the procedure finds balls of radius \(\delta\) that contain many vertices and cuts them off from \(F\) so that the total number of cut edges is small. We apply this procedure more-and-more aggressively in consequent iterations.

The first problem is much more serious and most of this paper describes how to solve it. Recall that we assume that \(|E_G| \gg |E_H|\) and thus most edges in \(E_G\) are short. That means informally that short edges of \(G\) form a “skeleton” of \(G\) — edges in this skeleton are short and they locally constrain how the SDP solution may look like. The skeleton does not necessarily cover the whole graph \(G\); moreover, even if initially the skeleton covered the whole graph \(G\), it may no longer cover \(G\) after we perform a few iterations of the algorithm. We use a special Damage Control procedure to remove vertices not covered by the skeleton. This is a tricky step since the algorithm does not know which edges are in \(E_F\) and which are in \(E_R\) and consequently cannot compute the skeleton.
Now to make our argument work, we need to show that few edges in $E_R$ are short (and thus many edges in $E_R$ are long). Assume to the contrary that many edges in $E_R$ are short. Then we can also find a skeleton in the graph $H$. We prove in the Main Structural Theorem that if both graphs $G$ and $H$ have skeletons then there is a very efficient encoding of $\pi$; namely, we prove that the prefix Kolmogorov complexity $KP(\pi)$ of $\pi$ is much less than $\log_2 |\Pi_{LR}|$. The encoding consists of two parts. We identify two relatively small sets of vertices $Q_G \subset V_G$ and $Q_H \subset V_H$ and record values of $\varphi(u)$ for $u \in Q_G$ and values of $\varphi(\pi(x))$ for $x \in Q_H$ in the first part of the encoding. The first part of the encoding allows us to approximately reconstruct values of $\varphi(u)$ for all vertices $u \in V_G$ and values of $\varphi(\pi(x))$ for all vertices $x \in V_H$ using that edges in the skeletons for $G$ and $H$ are short. Note that if $u = \pi(x)$ then $\varphi(u) = \varphi(\pi(x))$. Thus if we knew the values of $\varphi(u)$ and $\varphi(\pi(x))$ exactly and all values $\varphi(u)$ were distinct, we would be able to reconstruct $\pi: \pi(x) = \varphi^{-1}(\varphi(\pi(x)))$. In fact, the encoding gives us only approximate values of $\varphi(u)$ and $\varphi(\pi(x))$ but still it tells us that $\pi(x)$ is equal to such $u$ that $\varphi(u)$ and $\varphi(\pi(x))$ are close. Given that, we can very efficiently record additional information necessary to reconstruct $\pi$ in the second part of the encoding. We show that the total length of the encoding is much less than $\log_2 |\Pi_{LR}|$ bits and thus $KP(\pi) \ll \log_2 |\Pi_{LR}|$.

Since an exponentially small fraction of permutations in $\Pi_{LR}$ has prefix Kolmogorov complexity much smaller than $\log_2 |\Pi_{LR}|$, the probability that both graphs $G$ and $H$ have skeletons is exponentially small and thus $E_R$ contains many short edges with high probability.

We note that the algorithm is quite involved and technical, and we cannot describe it accurately in the introduction. Thus the overview given above is very informal. It only gives a rough idea of how our algorithm and analysis work. In particular, we do not use the informal notion of “skeleton” in the paper.

**Technical Comparison** We use ideas introduced in papers on semi-random instances of Unique Games [20] and on semi-random instances of graph partitioning problems [22]. The very high-level approach of this paper is somewhat similar to that of our previous work [22]. As in [22], our algorithm iteratively removes long edges and uses a Heavy Vertices Removal procedure. However, overall the algorithm and analysis in this paper are very different from that of [22]. In [22], the proof of the main structural theorem relies on the fact that $H$ is a random $G(n/2, n/2, q)$ bipartite graph. That ensures that most edges in $E_R$ are long no matter what the graph $G$ is. However, that is no longer the case in the present paper. The graph $(V, E_R)$ can be a completely arbitrary graph. It does not have to be an expander or “geometric expander” (the notion we used in [22]). To prove the structural theorem, we have to analyze the skeleton formed by edges in $E_G$. As a result, the proof of the structural theorem is completely different from the proof in [22]. The algorithm is also significantly different. It needs to perform an extra Damage Control step and the Heavy Vertices Removal Step is quite different from that in [22]. There are numerous other differences between algorithms.

## 2 Preliminaries

We work with the model described in Section 1.3. We denote the number of vertices in $F = G \boxplus H$ by $n$ and let

$$d = \max\{2|E_H|/n, C \log^3 n\}$$

for sufficiently large constant $C$ ($d$ equals the average degree of vertices in the graph $H$ if the average degree is greater than $C \log^3 n$). We assume without loss of generality that $d$ is known to the algorithm (the algorithm can find $d$ using binary search). We denote the degree of a vertex $u$ in $F'$ by $\deg(u, F')$, in $G$ by $\deg(u, G)$, and in $H$ by $\deg(u, H)$.

Our algorithm performs many iterations; in each iteration, it solves an SDP relaxation for Balanced Cut on a subgraph $F'$ of $F$. The relaxation for $F'$ assigns a vector $\varphi(u) \in \mathbb{R}^n$ to every vertex $u$ of $F'$. The SDP is shown in Figure 1. The intended integral solution is $\varphi(u) = e_1/\sqrt{2}$ if $u \in L$ and $\varphi(u) = e_2/\sqrt{2}$ if $u \in R$, where $e_1$ and $e_2$ are two fixed orthogonal unit vectors. The intended solution satisfies all SDP constraints.
minimize: $\sum_{(u,v) \in E_{F'}} \|\varphi(u) - \varphi(v)\|^2$ \hspace{1cm} (1)

such that for every $u, v, w \in V_{F'}$,

$$\|\varphi(u)\|^2 = \frac{1}{2}$$ \hspace{1cm} (2)

$$\sum_{v \in V_{F'}} (1 - \|\varphi(u) - \varphi(v)\|^2) \leq n/2$$ \hspace{1cm} (3)

$$\|\varphi(u) - \varphi(v)\|^2 + \|\varphi(v) - \varphi(w)\|^2 \geq \|\varphi(u) - \varphi(w)\|^2.$$ \hspace{1cm} (4)

Figure 1: SDP relaxation for Balanced Cut

We denote the cost of a feasible SDP solution $\varphi$ for a graph $F'$ by $\text{sdp-cost}(\varphi, F')$:

$$\text{sdp-cost}(\varphi, F') = \sum_{(u,v) \in E_{F'}} \|\varphi(u) - \varphi(v)\|^2.$$  

The cost of the intended SDP solution equals the number of edges from $L$ to $R$. Since only random edges in $F$ go from $L$ to $R$, it is at most $|E_R|$. Note that the optimal SDP solution $\varphi_{opt}$ for $F$ costs at most as much as the intended solution; thus $\text{sdp-cost}(\varphi_{opt}, F) \leq |E_R| \leq dn/2$.

Our SDP relaxation for Balanced Cut is slightly different from that of Arora, Rao and Vazirani; we use different normalization in (2) and use different spreading constraints (3). However, the algorithm of Arora, Rao and Vazirani works with our SDP. We denote the approximation factor of the algorithm by $D_{ARV} = O(\sqrt{\log n})$. The algorithm given an SDP solution $\varphi$ for a subgraph $F'$ of $F$ finds a cut $(L', R')$ that cuts at most $D_{ARV}$ SDP-cost($\varphi, F'$) edges such that both sets $L'$ and $R'$ contain at most $cn$ vertices for some absolute constant $c_{ARV} \in (0, 1)$. Let $T = \lceil \log_2 D_{ARV} \rceil = O(\log \log n)$.

We say that an edge $(u, v)$ is $\delta/2$-short if $\|\varphi_t(u) - \varphi_t(v)\|^2 \leq \delta/2$; otherwise, it is $\delta/2$-long. In our algorithm, we use five parameters $K$, $\beta = 200K$, $\alpha = 50K$, $\delta = 1/12$ and $D_n = \max\{D_{ARV}, \alpha\}$. The parameter $K$ is a sufficiently large constant. Let $V_{\leq \alpha d}^G = \{u \in V_F : \text{deg}(u, G) < \alpha d\}$. It will be convenient for us to assume that $|V_{\leq \alpha d}^G| \leq n/\alpha$. If this is not the case, we run a very simple algorithm for Balanced Cut, which we present in Appendix (see Lemma [A.1]).

Our algorithm iteratively cuts edges and removes some components of the graph (a component is an arbitrary subset of vertices). We say that a vertex is removed if it lies in a removed component; otherwise, we say that the vertex is active. We distinguish between cut and removed edges. An edge $e$ is cut if the algorithm cuts it, or if $e$ belongs to the edge boundary of a removed component. An edge is removed if either it is cut or at least one of its endpoints is removed.

The algorithm we present partitions the graph into several pieces and cuts at most $O(dn) = O(|E_H| + n \log^3 n)$ edges. The size of each piece is at most $\max(c_{ARV}, 3/4)n$. We can combine all pieces into two $\max(c_{ARV}, 3/4)n$-balanced parts. The number of edges between these parts is at most $O(dn)$ as required in Theorem [1.5].
3 Algorithm

We now present the algorithm. The main steps of the algorithm are given in Figure 2. Below we describe the algorithm in more detail.

**Budget allocation:** We store a budget for every vertex \( u \). We use this budget to keep track of the number of cut edges incident on \( u \). We do that to identify vertices we need to remove at Steps 3 and 4, and also to bound the total number of cut edges. Initially, the algorithm assigns a budget to every vertex \( u \) by \( \delta \) and a budget of \( \alpha d \) if \( \deg(u, F) \geq \alpha d \); and a budget of \( \alpha d \) if \( \deg(u, F) < \alpha d \). We denote the budget of a vertex \( u \) by \( \text{budget}(u) \) and the budget of a set \( S \) by \( \text{budget}(S) = \sum_{u \in S} \text{budget}(u) \). We allocate an extra budget of \( 3nd/\delta \) units. We keep this extra budget in the variable \( \text{extra-budget} \).

**Main loop:** The algorithm works in \( T \) iterations. We let \( F_1(0) \) to be the original graph \( F \). Consider iteration \( t \). At Step 1, the algorithm solves the SDP relaxation for the graph \( F_1(t) \) and obtains an SDP solution \( \varphi_t : V_{F_1(t)} \to \mathbb{R}^n \), which is a mapping of vertices of the graph \( F_1(t) \) to \( \mathbb{R}^n \). At Step 2, the algorithm cuts all \( \delta/2 \)-long edges i.e., edges \( (u, v) \) such that \( \| \varphi_t(u) - \varphi_t(v) \|^2 \geq \delta/2 \). At Step 3, the algorithm runs the Heavy Vertices Removal procedure and at Step 3, the algorithm runs the Damage Control procedure. We describe the details of these three steps in Sections 3.5, 3.6, and 3.7. The Heavy Vertices Removal and Damage Control procedures remove some vertices from the graph. Edges on the boundary of the components removed by these procedures at iteration \( t \) are cut. We denote them by \( \Upsilon_3(t) \) and \( \Upsilon_4(t) \), respectively. We denote the set of long edges cut at Step 2 by \( \Upsilon_2(t) \). Finally, we denote the graphs obtained after Steps 2, 3, 4 by \( F_2(t) \), \( F_3(t) \) and \( F_4(t) \). At iteration \( t \), after completion of Step \( i \), the set of active vertices is \( V_{F_i(t)} \).

**Budget updates:** When we cut a long edge \( (u, v) \) at Step 2, we increase the budget of vertices \( u, v \) by 1 and decrease the extra budget by 3. When we cut an edge \( (u, v) \) at Step 3 or Step 4, we increase the budget of the active endpoint (the one we do not remove) by 1. Thus, we have the following invariant: The budget of every active vertex \( u \) always equals the initial budget of \( u \) plus the number of cut edges incident on \( u \) in the graph \( F \).

**Final partitioning:** After the last iteration of the loop is completed, we partition the graph \( F_1(T) = F_4(T - 1) \) into two balanced pieces \( L' \) and \( R' \) using the algorithm of Arora, Rao and Vazirani. We output \( L' \), \( R' \) and all components removed at Steps 3 and 4 (in all iterations).

### 3.1 Analysis

We show that the algorithm returns a solution of cost at most \( O(|E_H|) \) if the graph \( F \) satisfies Structural Properties 1–4, which we describe in Section 3.4. Then we show that the graph \( F = G \boxplus_H H \) satisfies these properties with high probability (i.e., with probability \( (1 - o(1)) \)).

Define the total budget after Step \( i \) at iteration \( t \) to be the sum of budgets of active vertices plus the extra budget:

\[
\text{total-budget} = \sum_{u \text{ is active}} \text{budget}(u) + \text{extra-budget}.
\]

We prove that at every step of the algorithm the total budget does not increase (though the budgets of some vertices do increase). Furthermore, we show that whenever we cut a set of edges \( \Upsilon_i(t) \), the total budget decreases by at least \( |\Upsilon_i(t)| \). In other words, we pay a unit of the budget for every cut edge.

**Lemma 3.1.** Let \( b_{\text{before}} \) be the total budget before executing Step \( i \) at iteration \( t \); and let \( b_{\text{after}} \) be the total budget after executing Step \( i \) at iteration \( t \). If \( F = G \boxplus_H H \) satisfies Structural Properties 1–4, then

\[
b_{\text{after}} \leq b_{\text{before}} - |\Upsilon_i(t)|.
\]
Main Algorithm

**Input:** a graph $F = G \boxplus_{\pi} H$ (graphs $G$, $H$, and the permutation $\pi$ are hidden from the algorithm).

**Output:** a partitioning of $F$ into pieces of size at most $cn$ for some $c < 1$.

- **Set the parameters:** $\beta = 200K$, $\alpha = 50\beta$, $\eta_t = 2^{-t}$ (for $t \in \mathbb{Z}^+$). Let $D_{ARV} = O(\sqrt{\log n})$ be the approximation ratio of the ARV algorithm; $D_n = \max\{D_{ARV}, \alpha\}$; $T = \lceil \log_2 D_{ARV} \rceil$.

- **Allocate budget:** For every vertex $u \in U$, set $\text{budget}(u) = \beta d$ if $\deg(u, F) \leq \alpha d$; and $\text{budget}(u) = \alpha d$ if $\deg(u, F) \geq \alpha d$.

- **Let** $F_1(0) = F$.

  **for** $t = 0$ to $T - 1$ **do**:
  
  1. Solve the SDP on the graph $F_1(t - 1)$. Denote the SDP solution by $\varphi_t : V \rightarrow \mathbb{R}^n$.
  2. Remove $\delta/2$-long edges. Update the budgets.
  3. Run Heavy Vertices Removal procedure with $\eta_t = 2^{-t}$. Update the budgets.
  4. Run Damage Control procedure. Update the budgets.
  5. Denote the graphs obtained after Steps 2–4 by $F_2(t)$, $F_3(t)$ and $F_4(t)$. Denote the set of edges cut at these steps by $\Upsilon_2(t)$, $\Upsilon_3(t)$ and $\Upsilon_4(t)$. Let $F_1(t + 1) = F_4(t)$.

- **Partition the graph** $F_1(T)$ into two graphs $L'$ and $R'$ using the ARV algorithm.

- **Return** $L'$, $R'$ and all components removed at Steps 3 and 4.

Figure 2: Main steps of the algorithm. We present the algorithm in more detail below.

At Steps 1 and 5, we neither update the budgets of vertices, nor do we change the set of active vertices, so the total budget does not change. We consider Steps 2–4 in Lemmas 3.5, 3.6 and 3.7. In Lemma 3.5 we also show that the extra budget and hence the total budget is always non-negative (the budgets of vertices may only increase, but the extra budget may only decrease).

Structural Property 3 (see Section 3.4) guarantees that the total budget initially allocated by the algorithm is at most $3/2 \beta dn$. Hence, the total number of edges cut by the algorithm is at most $3/2 \beta dn$. We denote the set of all cut edges by $\Upsilon$:

$$\Upsilon = \bigcup_{i \in \{2, 3, 4\}, t \in \{0, \ldots, T-1\}} \Upsilon_i(t).$$

The algorithm of Arora, Rao and Vazirani partitions the graph $F_4(T)$ into two pieces of size at most $cn$ each (where $c < 1$ is an absolute constant). In Sections 3.6 and 3.7 we show that each component removed at Steps 3 and 4 has size at most $3/4 n$ (see Lemma 3.6 and Lemma 3.7). Hence, all pieces in the partition returned by the algorithm have size at most $\max(3/4, c)n$.

Now we need to verify that the size of the cut separating different pieces in the partition is at most $O(dn)$. This cut contains edges from $\Upsilon$ and edges cut by the ARV algorithm. We already know that $|\Upsilon| \leq 3/2 \beta dn = O(dn)$. It remains to prove that the ARV algorithm cuts $O(dn)$ edges. The proof follows from Theorem 3.2 which is central to our analysis.
Theorem 3.2. If the graph $F = G \boxtimes \pi H$ satisfies Structural Properties 1–4, then for every $t \in \{0, \ldots, T\}$,

$$\text{sdp-cost}(\varphi_t, F_1(t)) \leq 8K\eta_dn,$$

where $\varphi_t$ is the optimal SDP solution for $F_1(t)$, $\eta_t = 2^{-t}$, and $K$ is an absolute constant.

We also use this theorem to prove Lemma 3.6, which bounds the number of edges cut by the Heavy Vertices removal procedure. For $T = \lceil \log_2 D_{ARV} \rceil$, we get that $\text{sdp-cost}(\varphi_t, F_1(T)) \leq Kdn/D_{ARV}$. The algorithm of Arora, Rao and Vazirani outputs an integral solution of cost at most

$$D_{ARV} \times \text{sdp-cost}(\varphi_T, F_1(T)) \leq D_{ARV} \times \frac{Kdn}{D_{ARV}} = Kdn.$$

That is, the size of the cut between $L'$ and $R'$ is at most $Kdn$. This finishes the analysis of the algorithm.

### 3.2 Notation

Before proceeding to the technical part of the analysis, we set up some notation. During the execution of the algorithm, we remove some vertices and cut some edges from the graph $F$. For the purpose of analysis, we will shadow these removals in the graphs $G$ and $H$. For every $F_i(t)$ we define two graphs $G_i(t)$ and $H_i(t)$. The vertices of these graphs are the vertices of $F_i(t)$. The edges of $G_i(t)$ are edges of $F_i(t)$ that originally came from $G$; the edges of $H_i(t)$ are edges of $F_i(t)$ that originally came from $H$. Note that $G_1(0)$ equals $G$, $H_1(0)$ is isomorphic to $H$, and the isomorphism between $H$ and $H_1(0)$ equals $\pi$.

We denote by $\deg(u, F_i(t))$, $\deg(u, G_i(t))$, $\deg(u, H_i(t))$ the degree of the vertex $u$ in the graph $F_i(t)$, $G_i(t)$ and $H_i(t)$, respectively. We denote by $\deg(u, F)$, $\deg(u, G)$, $\deg(u, H)$ the degree of $u$ in the original graphs $F$, $G$, $H$. Note that strictly speaking $\deg(u, H)$ is the degree of the vertex $\pi^{-1}(u)$ in the graph $H$.

Given a graph $G$, an SDP solution $\varphi : V_G \rightarrow \mathbb{R}^n$, and a positive number $\delta > 0$, we denote by $\text{short}_{\varphi, \delta}(u, G)$ and $\text{short}_{\varphi, \delta}(u, H)$ the number of $\delta$-short edges w.r.t the SDP solution $\varphi$ leaving vertex $u$ in $G$ and $\pi H$, respectively. Finally, we denote by $N_F(u)$, $N_G(u)$ the set of neighbors of $u \in V_G$ in the graphs $F$ and $G$ and by $N_H(x)$ the set of neighbors of $x \in V_H$ in the graph $H$.

### 3.3 Overview of the Proof

The analysis of the algorithm relies on Theorem 3.2. It states that the cost of the optimal SDP solution for $F_4(t) = F_1(t + 1)$ is $O(dn/2^t)$. To prove this theorem, we construct an SDP solution of cost $O(dn/2^t)$. To this end, we first divide the graph $F_4(t)$ into two sets, the set of “undamaged” vertices $W$ and the set of “damaged” vertices $\bar{W}$. Then we further subdivide $W$ into $W \cap L$ and $W \cap R$ and get a partition of $F_4(t)$ into three pieces $W \cap L$, $W \cap R$, and $\bar{W}$. We prove that each piece contains at most $n/2$ vertices and the total number of edges cut by the partition is $O(dn/2^t)$ (we outline the proof below). The partition defines a feasible integral SDP solution that assigns the same vector to vertices in one part and orthogonal vectors to vertices in different parts. The cost of this SDP solution is $O(dn/2^t)$ as required.

Thus we need to prove that the partition into $W \cap L$, $W \cap R$ and $\bar{W}$ is balanced and cuts few edges. We first deal with the part $\bar{W}$. We run the Damage Control procedure that cuts off some components of the graph so as to ensure that $|\bar{W}| \leq n/2$ and more importantly $|\partial \bar{W}| \leq O(dn/2^t)$. We describe the procedure and prove that it cuts a small number of edges if the graph satisfies Structural Properties 2–4 in Section 3.7, and we show that a graph in the PIE model satisfies these properties w.h.p. in Sections 5.2 and 5.3.

Now consider parts $W \cap L$ and $W \cap R$. We immediately have that $|W \cap L| \leq |L| = n/2$ and $|W \cap R| \leq |R| = n/2$. There are no edges between $W \cap L$ and $W \cap R$ in $G_4(t)$ (since $(L, R)$ is the planted cut). It remains to show that there are at most $O(dn/2^t)$ edges between $W \cap L$ and $W \cap R$ in $H_4(t)$. Note that all edges in $H_4(t)$ are $\delta/2$-short w.r.t. $\varphi_t$ since we cut all $\delta/2$-long edges at Step 1. We prove in the Main
Structural Theorem (Theorem 5.1) that there are at most $O(dn/2^t)$ $\delta/2$-short edges in the induced graph $H_4(t)[W]$ and thus there are at most $O(dn/2^t)$ edges between $W \cap L$ and $W \cap R$ in $H_4(t)$.

We now sketch the proof of the Main Structural Theorem (Theorem 5.1). We present the proof in a simplified setting; most steps are somewhat different in the actual proof. We assume that all vertices in $H$ have degree $d$. Denote $\eta = 1/2^t$. All vertices in $W$ satisfy several properties — if a vertex does not satisfy these properties it is removed either by the Heavy Vertices Removal or Damage Control procedure. The Heavy Vertices Removal procedure removes all vertices $u$ such that the ball $\{v : \|\varphi_t(u) - \varphi_t(v)\|^2 \leq 3\delta\}$ has a budget of $\eta\beta dn$. We show that this implies that for every active $u$ there are at most $2\eta m$ vertices with more than $\beta d/2$ neighbors in the ball of radius $2\delta$ around $u$ (see Lemma 4.1). The Damage Control procedure removes all “damaged” vertices. We do not describe the Damage Control procedure in this overview, but we note that in particular it guarantees that $\text{short}_{\varphi_t,\delta/2}(u, G) \geq \beta d$ for all vertices $u \in W$.

For simplicity, we will assume now that $W = V_G$. Recall that $F = G \boxplus H$ in our model. We show that if $H_4(t)$ contains more than $K\eta dn\delta/2$-short edges then there is a binary encoding of $\pi$ with much fewer than $\log_2(|\Pi_{LR}|$ bits. Since any encoding needs $\log_2(|\Pi_{LR}|$ bits to encode a typical permutation in $\Pi_{LR}$, the probability that for a random $\pi \in \Pi_{LR}$ the graph $H_4(t)$ contains more than $K\eta dn$ short edges is very small.

We fix a permutation $\pi$ and assume to the contrary that $H_4(t)$ contains more than $K\eta dn\delta/2$-short edges. We are going to show that there is a short encoding of $\pi$. We sample two random subsets $Q_G \subset V_G$ and $Q_H \subset V_H$. Each vertex of $G$ and $H$ belongs to $Q_G$ and $Q_H$ (respectively) with probability $q = D_n/d$. Additionally, we choose random orderings of $Q_G$ and $Q_H$. Note that $Q_G$ and $Q_H$ are of size approximately $qn$. From now on all random events that we consider are with respect to our random choices of $Q_G$, $Q_H$ and their ordering (not the random choice of $\pi$).

For every vertex $x \in V_H$, let $x'$ be the first neighbor of $x$ in $Q_H$ w.r.t. to the random ordering of $Q_H$ if it exists. Note that the probability that $x'$ is defined for a given $x \in V_H$ is $1 - (1 - q)^d \approx 1 - e^{-D_n}$; that is, $x'$ is defined for most vertices $x$. Vertex $x'$ is uniformly distributed in $N_H(x)$. Thus the edge $(x, x')$ is short with probability $\text{short}_{\varphi_t,\delta/2}(u, H)/d$. The expected number of vertices $x$ such that $(x, x')$ is short is

$$\sum_{u \in V_H} \text{short}_{\varphi_t,\delta/2}(u, H)/d \geq K\eta nd/d = K\eta m.$$ 

If $x'$ exists and $(x, x')$ is short, define $B = \{v : \|\varphi_t(v) - \varphi_t(\pi(x'))\|^2 \leq \delta\}$. Recall that for every ball of radius $2\delta$ (or less), there are at most $2\eta m$ vertices with more than $\beta d/2$ neighbors in the ball. Thus, $|\Xi(x)| \leq 2\eta m$. Now note that $\text{short}_{\varphi_t,\delta/2}(\pi(x), G) \geq \beta d$ thus there are at least $\beta d$ vertices in $N_G(\pi(x))$ at distance at most $\delta/2 + \|\varphi_t(\pi(x)) - \varphi_t(\pi(x'))\|^2 \leq \delta$ from $\pi(x')$. That is, $|N_G(\pi(x)) \cap B| \geq \beta d$ and in expectation $Q_G \cap N_G(\pi(x)) \cap B$ contains at least $q\beta d$ vertices. Therefore, $\pi(x) \in \Xi(x)$ w.h.p.

Let $\mathcal{X}$ be the set of vertices $x$ such that $x'$ exists, the edge $(x, x')$ is $\delta/2$-short, $\pi(x) \in \Xi(x)$ and $|\Xi(x)| \leq 2\eta m$. As we showed above, $\mathcal{X}$ contains approximately $K\eta m$ vertices. We are now ready to explain how we encode the permutation $\pi$. We first record sets $Q_G$, $Q_H$ and orderings of $Q_G$ and $Q_H$ in our encoding. For each $u \in Q_G$ we record $\varphi_t(u)$; for each $x \in Q_H$ we record $\varphi_t(\pi(x))$. We record the set $\mathcal{X}$ and the restriction of $\pi$ to the complement of $\mathcal{X}$. Finally, for each $x \in \mathcal{X}$, we record the sequential number of $\pi(x)$ in the set $\Xi(x)$ w.r.t. an arbitrary fixed ordering of $V_G$ (i.e. the number of elements preceding $\pi(x)$ in $\Xi(x)$).

We show how to decode $\Pi_{LR}$ given our encoding of $\pi$. We know the value of $\pi(x)$ for $x \in \mathcal{X}$, so consider $x \notin \mathcal{X}$. First compute $x'$ and $\Xi(x)$. The encoding contains all the necessary information to do so. Now find $\pi(x)$ in $\Xi(x)$ by its sequential number in $\Xi(x)$. We showed that $\pi$ is determined by its encoding.

Now we estimate the length of the encoding. Sets $Q_G$ and $Q_H$ are of size approximately $qn$. We need $O(qn \log(1/q))$ bits to record them, $O(qn \log(qn))$ bits to record their orderings, $O(qn \log n)$ bits to record vectors $\{\varphi(u)\}_{u \in Q_G}$ and $\{\varphi(\pi(x))\}_{x \in Q_H}$ with the desired precision (that follows from the Johnson—Lindenstrauss lemma). We need $|\mathcal{X}| \log_2(1/(\eta K))$ bits to record $\mathcal{X}$ (since the size of $|\mathcal{X}|$ is approximately
We need at most \( \log_2((n/2)!((n/2 - \mathcal{X})!)) \) bits to record the restriction of \( \pi \) to \( \mathcal{X} \). Finally, we need \( \log_2|\Xi(x)| = \log_2(\eta n) + O(1) \) bits for each vertex \( u \in \mathcal{X} \) to record its position in \( \Xi(x) \). In total, we need
\[
\log_2((n/2)!((n/2 - |\mathcal{X}|)!)) + |\mathcal{X}| \log_2(n/K) + O(qn \log n)
\]
bits. In contrast, we need at least \( \log_2((n/2)!((n/2)!)) \) bits to encode a “typical” permutation in \( \Pi_{LR} \) (no matter what encoding scheme we use). That is, the encoding of \( \pi \) is shorter than the encoding of a typical permutation by at least
\[
\log_2((n/2)!((n/2)!)) - \left( \log_2((n/2)!((n/2 - |\mathcal{X}|)!)) + |\mathcal{X}| \log_2(n/K) + O(qn \log n) \right) \approx |\mathcal{X}| \log_2 K - O(qn \log n) \approx K \eta n \log_2 K - O((D_n/d) n \log n).
\]
The expression is large when \( d \gtrsim (\log^3 n) \). We conclude that a random permutation \( \pi \in \Pi_{LR} \) does not satisfy the condition of the Main Structural Theorem with small probability.

### 3.4 Structural Properties — Definitions

We now describe the Structural Properties that we use in the analysis of the algorithm. We prove that the graph \( G = G \boxplus \pi H \) satisfies these properties with probability \( 1 - o(1) \) in Section 5. We first give several definitions.

**Definition 3.3.** Consider an SDP solution \( \varphi : V_G \rightarrow \mathbb{R}^n \). We let \( \text{Ball}_\varphi(u, \delta) \) be the ball of radius \( \delta \) around \( u \) in the metric induced in \( V_G \) by the embedding \( \varphi \):
\[
\text{Ball}_\varphi(u, \delta) = \{ v \in V_G : \| \varphi(u) - \varphi(v) \|^2 \leq \delta \}.
\]
For a subset \( B \subset V_G \), we let
\[
M_\xi(B) = \sum_{v \in V_G} \min\{|N_F(v) \cap B|, \xi\}.
\]

In the proof, we need to count the number of vertices in \( F \) having at least \( \beta d \) neighbors in the \( \text{Ball}_\varphi(v, 2\delta) \). Informally, \( M_{\beta d}(\text{Ball}_\varphi(v, 2\delta)) \) is an approximation to this number scaled by \( \beta d \). We now state the Main Structural Property.

**Property 1** (Main Structural Property). There exists a constant \( K > 0 \) (note that \( \alpha \), \( \beta \) and \( D_n \) depend on \( K \); see Section 2) such that for every feasible SDP solution \( \varphi : V_G \rightarrow \mathbb{R}^n \) and \( \eta = 2^{-t} (t \leq T) \), there are at most \( K \eta n d \) edges \((u, v) \in E_F\) satisfying the following conditions:

1. \((u, v)\) is a \( \delta/2\)-short edge in \( \pi(H) \) i.e., \( \| \varphi(u) - \varphi(v) \|^2 \leq \delta/2 \) and \((u, v) \in \pi E_H\).
2. \( M_{\beta d}(\text{Ball}_\varphi(v, 2\delta)) \leq \eta \beta d n \).
3. \( \text{short}_{\delta/2}(u, G) \geq \max\{ \beta d, \deg(u, H)/D_n \} \) i.e., there are at least \( \max\{ \beta d, \deg(u, H)/D_n \} \) edges of length \( \delta/2 \) leaving \( u \) in the graph \( G \).

In some sense, this is the main property that we need for the proof of Theorem 3.2. Roughly speaking, we show that condition 2 is satisfied if \( u \) is not a “heavy vertex”, and condition 3 is satisfied if \( u \) is not a “damaged vertex”. Hence, after removing short edges, heavy vertices, and damaged vertices, we obtain a graph \( (F_4(t)) \) which does not have more than \( K \eta n d \) edges from \( H \). This implies Theorem 3.2. Unfortunately, the Damage Control procedure does not remove all damaged vertices — it just controls the number of such vertices. We need Properties 2–4 to show that the edge boundary of the set of the remaining damaged vertices is small.

The following property is an analog of the Main Structural Property with graphs \( G \) and \( H \) switched around. Notice that it has an extra condition (4) on edges \((u, v)\) that are counted.
Property 2. For every feasible SDP solution $\varphi : V \to \mathbb{R}^n$ and $\eta = 2^{-t}$ ($t \leq T$) there are at most $K\eta dn$ edges $(u, v) \in E_F$ satisfying the following conditions:

1. $(u, v)$ is a $\delta/2$-short edge in $G$, i.e., $\|\varphi(u) - \varphi(v)\|^2 \leq \delta/2$ and $(u, v) \in E_G$.
2. $M_{\beta d}(\text{Ball}_{\varphi}(v, 2\delta)) \leq \eta \beta dn$.
3. $\text{short}_{\varphi, \delta/2}(u, H) \geq \beta d$, i.e., there are at least $\beta d$ edges of length $\delta/2$ leaving $u$ in the graph $H$.
4. $\deg(u, G) \leq \alpha d$.

Let

$$V_G^{\leq \alpha d} = \{u \in V_G : \deg(u, G) \leq \alpha d\}$$

be the set of vertices in $G$ of degree at most $\alpha d$; and let

$$V_H^{\geq \beta d} = \{u \in V_G : \deg(u, H) \geq \beta d\}$$

be the set of vertices in $H$ of degree at least $\beta d$. As we assumed in Section 2, $|V_G^{\leq \alpha d}| \leq n/\alpha$ (otherwise, we use an alternative simple algorithm). We now state this assumption as Structural Property 3.

Property 3. There are at most $n/\alpha$ vertices of degree less than $\alpha d$ in $F$. In other words, $|V_F^{\geq \alpha d}| \leq n/\alpha$. Consequently, there are at most $n/\alpha$ vertices of degree less than $\alpha d$ in $G$.

We use this property in several places, particularly to get a bound on the initial total budget: Since we give a budget of $\beta d$ to vertices with $\deg(u, F) \geq \alpha d$, and $\alpha d$ to vertices with $\deg(u, F) \leq \alpha d$, the initial budget allocated to vertices is at most $\beta d \times n + (\alpha - \beta) d \times n/\alpha \leq (\beta d + 1)n$. The initial total budget is bounded by

$$(\beta d + 1)n + 3nd/\delta \leq 3/2 \beta dn.$$

Finally, we describe the last structural property. This property is rather technical. Roughly speaking, it says that every vertex $u$ has much more neighbors in $V_H^{\geq \beta d} \setminus V_G^{\leq \alpha d}$ than in $V_H^{\geq \beta d} \cap V_G^{\leq \alpha d}$. This happens because $V_H^{\geq \beta d}$ is the image of the set $\{x \in V_H : \deg(x, H) \geq \beta d\}$ under $\pi$. Every element in $\{x \in V_H : \deg(x, H) \geq \beta d\}$ is much more likely to be mapped to $V_G \setminus V_G^{\leq \alpha d}$ than to $V_G^{\leq \alpha d}$ just because the set $V_G^{\leq \alpha d}$ is very small.

Property 4. For every vertex $u \in V_F$,

$$\sum_{v : (u, v) \in \pi E_H \atop v \in V_H^{\geq \beta d} \cap V_G^{\leq \alpha d}} \frac{\beta d}{\deg(v, H)} \leq \frac{8}{\alpha} \sum_{v : (u, v) \in \pi E_H \atop v \in V_H^{\geq \beta d} \setminus V_G^{\leq \alpha d}} \frac{\beta d}{\deg(v, H)} + 4 \log n.$$

We prove that the graph $F$ satisfies these Structural Properties w.h.p in Section 5. Now we proceed with the analysis of the algorithm.

### 3.5 Long Edges Removal

We say that an edge $(u, v)$ is $\delta$-long with respect to the SDP solution $\varphi_t$ if $\|\varphi_t(u) - \varphi_t(v)\|^2 \geq \delta$. At Step 2 of the main loop of the algorithm, we cut all $\delta/2$-long edges in the graph $F_1(t)$. For every $\delta/2$-long edge $(u, v)$ we cut, we increase the budgets of the endpoints of the edge, vertices $u$ and $v$, by 1 (each) and decrease the extra budget (the variable extra-budget) by 3. This way the total budget decreases by the number of edges cut at this step. We need to verify that the extra budget is always non-negative. To do so, we bound the total number of $\delta/2$-long edges cut during the execution of the algorithm.
**Lemma 3.4.** The total number of $δ/2$-long edges cut by the algorithm is at most $δ^{-1}dn$.

*Proof.* At iteration $t$ the algorithm cuts a set $Υ_2(t)$ of $δ/2$-long edges. Each edge contributes at least $δ/2$ to $\text{sdp-cost}(ϕ_t, F_1(t))$. Once we cut edges in the set $Υ_2(t)$ the SDP value decreases by at least $δ/2|Υ_2(t)|$, i.e., $\text{sdp-cost}(ϕ_t, F_1(t)) - \text{sdp-cost}(ϕ_t, F_1(t+1)) \leq \frac{δ}{2} |Υ_2(t)|$. Observe that $ϕ_t$ restricted to $V_{F_1(t+1)}$ is a feasible (but possibly suboptimal) solution for the graph $F_1(t+1)$. Hence,

$$\text{sdp-cost}(ϕ_{t+1}, F_1(t+1)) \leq \text{sdp-cost}(ϕ_t, F_1(t+1)) \leq \text{sdp-cost}(ϕ_t, F_2(t)) - \frac{δ}{2} |Υ_2(t)|.$$ 

Thus, $|Υ_2(t)| \leq \frac{2}{δ} \cdot (\text{sdp-cost}(ϕ_t, F_1(t)) - \text{sdp-cost}(ϕ_{t+1}, F_1(t+1)))$, and

$$\sum_{t=0}^{T-1} |Υ_2(t)| \leq \frac{2}{δ} \cdot \sum_{t=0}^{T-1} \text{sdp-cost}(ϕ_t, F_1(t)) - \text{sdp-cost}(ϕ_{t+1}, F_1(t+1)) \leq \frac{2}{δ} \cdot \text{sdp-cost}(ϕ_t, F(t)) \leq δ^{-1}dn,$$

since the cost of the optimal bisection in graph $F(t)$ is at most $|E_H| \leq dn/2$, and hence $\text{sdp-cost}(ϕ_t, F(t)) \leq dn/2$. 

As a corollary we get the following lemma.

**Lemma 3.5.** I. Denote by $b_{\text{before}}$ the total budget before removing $δ/2$-long edges; denote by $b_{\text{after}}$ the total budget after removing $δ/2$-long edges. Then,

$$b_{\text{after}} \leq b_{\text{before}} - |Υ_2(t)|.$$

II. The total budget is always non-negative.

*Proof.* I. Whenever we cut a long edge we increase the budgets of the endpoints by 1 and decrease the extra budget by 3.

II. We never decrease budgets of individual vertices, so their budgets remain positive all the time (note: the total budget of all active vertices may decrease, because the set of active vertices may decrease). By Lemma 3.4 the number of long edges cut is at most $δ^{-1}dn$, hence the extra budget may decrease by at most $3δ^{-1}dn$ (the algorithm uses the extra budget only to pay for cut long edges). Hence the extra budget is always non-negative. 

### 3.6 Heavy Vertices Removal

We say that a vertex $u \in V_{F_2(t)}$ is $η$-heavy if the vertices in the ball of radius $3δ$ around $u$ have budget at least $βηnd$:

$$\text{budget}\{\{v : |ϕ_t(u) - ϕ_t(v)|^2 \leq 3δ\}\} \geq βηnd.$$ 

The Heavy Vertices Removal procedure sequentially picks vertices $u$ in $V_{F_2(t)}$. If $u$ is active (i.e., it was not removed at the current step, or previous steps) and it is an $η$-heavy vertex, then we find the radius $r \in [3δ, 4δ]$ that minimizes the edge boundary $Ȝ B_u$ of the ball

$$B_u = \{v : |ϕ_t(u) - ϕ_t(v)|^2 \leq r\}.$$ 

We remove the set $B_u$ from the graph. Thus, the Heavy Vertices Removal Step removes a collection of balls $B_u$. The set of cut edges $Υ_3(t)$ is the union of the corresponding $Ȝ B_u$.

We need to prove that the procedure satisfies the invariant of the loop: the total budget decreases by at least $|Υ_3(t)|$. The Heavy Vertices Removal procedure may remove several components from the graph $F_3(t)$. We verify the invariant for each of them independently.
Lemma 3.6. Consider one of the removed components $B_u$. Let $\partial B_u$ be the edge boundary of the set $B_u$.

I. Denote by $b_{\text{before}}$ the total budget before removing the set $B_u$; and denote by $b_{\text{after}}$ the total budget after removing the set $B_u$. Then,

$$b_{\text{after}} \leq b_{\text{before}} - |\partial B_u|.$$ 

II. The size of the set $B_u$ is at most $3n/4$.

Proof. I. The set $B_u$ contains the ball of radius $3\delta$ around $u$. The budget of vertices in this ball is at least $\beta \eta n d$, because $u$ is a heavy vertex. Hence, the budget of $B_u$ is also at least $\beta \eta n d$. After we remove the set $B_u$, the vertices in $B_u$ are no longer active, so we decrease the total budget by at least $\beta \eta n d$.

We now need to bound the size of the edge boundary $\partial B$. To do so, we use the bound on the cost of the SDP solution. By Theorem 3.2,

$$\text{sdp-cost}(\varphi_t, F_3(t)) \leq \text{sdp-cost}(\varphi_t, F_1(t)) \leq 8K\eta d n.$$ 

Since we pick the radius $r$ in the range $[3\delta, 4\delta]$, we get by the standard ball growing argument, that the size of the edge boundary $|\partial B|$ is at most $8K\eta d n / \delta \leq 2 \eta d n$.

After removing the set $B_u$, the total budget decreases by

$$\text{budget}(B_u) - |\partial B_u| \geq \beta \eta n d - \beta / 2 \cdot \eta d n = \beta / 2 \cdot \eta d n \geq |\partial B_u|.$$ 

Above, we subtract $|\partial B_u|$ from $\text{budget}(B_u)$, because for every cut edge $(v', v'') \in \partial B_u$, $v' \in B_u$, $v'' \notin B_u$, the algorithm increased the budget of $v''$ by 1.

II. We upper bound the size of the Ball $\varphi_t(u, 4\delta)$ containing the set $B_u$. We apply the SDP spreading constraint (3) for vertex $u$:

$$\sum_{v \in \text{Ball}_{\varphi_t}(u, 4\delta)} (1 - 4\delta) \leq \sum_{v \in V_{F_2(t)}} (1 - \|\varphi_t(u) - \varphi_t(v)\|^2) \leq \frac{n}{2}.$$ 

Using that $\delta = 1/12$ and $(1 - 4\delta) = 2/3$, we get the bound

$$|\text{Ball}_{\varphi_t}(u, 4\delta)| \leq \frac{3}{4} n.$$ 

\[\square\]

3.7 Damage Control

The Damage Control procedure removes components with a small edge boundary and large budget. We find a set of vertices $Y \subset V_{F_2(t)}$ to maximize

$$\Delta(Y) \equiv \text{budget}(Y) - 2|E_{F_3(t)}(Y, \bar{Y})| - 2\beta d|Y|.$$ 

(8)

To find the set $Y$ we solve a maximum flow problem on the graph $F_3(t)$ with two extra vertices – the source and the sink. We connect every vertex $u$ in $F_3(t)$ to the source with an edge of capacity $\text{budget}(u)$ and to the sink with an edge of capacity $2\beta d$. We set the capacity of every edge in $F_3(t)$ to 2. Then we find the minimum cut between the source and the sink. The set $Y$ is the set of vertices lying in the same part of the cut as the source. It is easy to check that $Y$ minimizes (8). We give the details in Appendix B.

If $\Delta(Y) > 0$ we remove the set $Y$ from the graph $F_3(t)$. We denote the edge boundary of $Y$ by $\Upsilon_4(t)$; we denote the obtained graph by $F_4(t) = F_3(t) - Y$. Observe that when we remove the set $Y$ we cut only edges in $\Upsilon_4(t)$. For every edge $(u, v) \in \Upsilon_4(t)$, we increase the budget of the endpoint $u$ that we do not remove (i.e., $u \notin Y$) by 1. If $\Delta(Y) < 0$, then we do nothing: We let $F_4(t) = F_3(t)$ and $\Upsilon_4(t) = \varnothing$.

We need to show that for every edge removed from $F_3(t)$ the Damage Control procedure decreases the total budget by at least 1, and that the size of the set $Y$ is at most $3n/4$. 

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Lemma 3.7. Let $b_{before}$ be the total budget before applying the Damage Control procedure at step $t$; and let $b_{after}$ be the total budget after applying the Damage Control procedure at step $t$. Then,

$$b_{after} \leq b_{before} - |\mathcal{Y}_t|.$$ 

II. The size of the set $Y$ is at most $3n/4$.

Proof. If $\Delta(Y) \leq 0$, then the Damage Control procedure does not do anything and thus the statements I and II are trivial, so we assume $\Delta(Y) \geq 0$.

I. The procedure decreases the total budget by $\text{budget}(Y) - |\mathcal{Y}_t|$: it removes the set $Y$, which decreases the total budget by $\text{budget}(Y)$; however, for every removed edge $(u, v) \in E_{F_t}(Y, \bar{Y})$, $u \in Y$, $v \notin Y$, it increases the budget of $v$ by 1, which increases the total budget by $E_{F_t}(Y, \bar{Y})$. Since $\Delta(Y) \geq 0$, we have

"the change in the budget" = $\text{budget}(Y) - |E_{F_t}(Y, \bar{Y})| \geq |E_{F_t}(Y, \bar{Y})| = |\mathcal{Y}_t|.$ 

II. Since $\Delta(Y) \geq 0$, we have $\text{budget}(Y) \geq 2\beta d|Y|$. The budget of the set $Y$ is at most the total budget. Initially, the total budget is at most $3/2 \beta dn$, and during the execution of the algorithm it may only decrease (by Lemma 3.1), so $\text{budget}(Y) \leq 3/2 \beta dn$. Hence, $|Y| \leq 3/4 n$. 

We have established that Step 4 of the algorithm does not violate the invariants of the loop. We now show that after applying the Damage Control procedure, the boundary of every set $Y' \subset V_{F_t}$ is not too large.

Lemma 3.8. After Step 4, for every $Y' \subset V_{F_t}$,

$$\text{budget}(Y') \leq 2|E_{F_t}(Y', \bar{Y'})| + 2\beta d|Y'|.$$ 

(9)

Proof. Suppose that at Step 4, the algorithm removed a set $Y$ of vertices, and a set $\mathcal{Y}_t$ of edges from $F_t$ (then $\mathcal{Y}_t$ is the edge boundary of $Y$). Note that the sets $Y$ and $\mathcal{Y}_t$ can possibly be empty. Assume to the contrary that for some set $Y'$ the inequality (9) is violated. We argue that in this case, $\Delta(Y \cup Y')$ would be greater than $\Delta(Y)$ and hence the Damage Control procedure would remove the set $(Y \cup Y')$ instead of $Y$ from $F_t$. This easily follows from the following observation: The Damage Control procedure has increased the budget of $Y'$ by the size of the edge boundary between $Y$ and $Y'$ i.e., by $|E_{F_t}(Y, Y')|$. The edge boundary of $Y'$ has decreased by $|E_{F_t}(Y, Y')|$. Hence, before applying the Damage Control procedure, we had

$$(\text{budget}(Y') + |E_{F_t}(Y, Y')|) > (2|E_{F_t}(Y', \bar{Y'})| + 2|E_{F_t}(Y, Y')|) - 2\beta d|Y'|.$$ 

Thus, $\Delta(Y') > 0$, and $\Delta(Y \cup Y') \geq \Delta(Y) + \Delta(Y') > \Delta(Y)$. We get a contradiction with the assumption that inequality (9) is violated. 

4 Bounding the Cost of the SDP

In this section, we upper bound the cost of the SDP solution $\text{sdp-cost}(\varphi_t, F_1(t))$. We have a trivial upper bound of $OPT \leq dn/2$ for $t = 0$ (as $F_1(0) = F$), so we consider the case $t > 0$. In fact, we upper bound $\text{sdp-cost}(\varphi, F_1(t))$ for the optimal $\varphi$, which equals $\text{sdp-cost}(\varphi_{t+1}, F_1(t+1))$. To this end, we show in Lemma 4.2 that $F_1(t)$ can be partitioned into 3 balanced pieces with edge boundary at most $4K\eta nd$. As we see in Section 4.2, this immediately gives us an upper bound on the cost of the SDP solution: $\text{sdp-cost}(\varphi, F_1(t)) \leq 4K\eta nd$, and $\text{sdp-cost}(\varphi_t, F_1(t+1)) \leq 8K\eta_{t+1}nd$. 

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4.1 Partitioning into Three Balanced Sets

Define a new feasible SDP solution $\varphi'_t$ for $F$ i.e., a mapping of $V \rightarrow \mathbb{R}^n$ satisfying the SDP constraints. Let

$$\varphi'_t(u) = \begin{cases} \varphi_t(u), & \text{if } u \in V_{F_3(t)}; \\ e_u, & \text{otherwise}; \end{cases}$$

(10)

where $e_u$ is a vector of length $\sqrt{2}/2$ orthogonal to all other vectors in the SDP solution (including other $e_v$’s). This SDP solution coincides with $\varphi_t$ on the set of active vertices. Note that all edges in $F_3(t)$ are $\delta/2$-short w.r.t. $\varphi'_t$, and any edge connecting an active vertex and a removed vertex has length 1.

We now show that all active vertices $u \in V_{F_3(t)}$ satisfy the second condition on edges $(u, v)$ in Property 1 and in Property 2 for the SDP solution $\varphi'_t$.

Lemma 4.1. For all vertices $u \in V_{F_3(t)}$, $M_{\beta d}(\text{Ball}_{\varphi'_t}(u, 2\delta)) \leq \eta \beta d n$, where $\varphi'_t$ is defined in (10).

Proof. Fix a vertex $u \in V_{F_3(t)}$. Let $B_{2\delta} = \text{Ball}_{\varphi'_t}(u, 2\delta)$ and

$$B_{3\delta} = \text{Ball}_{\varphi'_t}(u, 3\delta) \cap V_{F_3(t)} = \{ v \in V_{F_3(t)} : \|\varphi_t(v) - \varphi'_t(u)\|^2 \leq 3\delta \}.$$

That is, $B_{2\delta}$ is the ball of radius $2\delta$ around $u$ in the graph $F_3(t)$; $B_{3\delta}$ is the ball of radius $3\delta$ in the graph $F_3(t)$ w.r.t $\varphi_t$ (and not $\varphi'_t$). Note that $B_{2\delta} \subset B_{3\delta}$. Write the definition of $M_{\beta d}(\text{Ball}_{\varphi'_t}(u, 2\delta)) = M_{\beta d}(B_{2\delta})$:

$$M_{\beta d}(B_{2\delta}) = \sum_{v \in V_G} \min\{|N_F(v) \cap B_{2\delta}|, \beta d\}$$

$$= \sum_{v \in B_{3\delta}} \min\{|N_F(v) \cap B_{2\delta}|, \beta d\}$$

(11)

$$+ \sum_{v \in V_G \setminus B_{3\delta}} \min\{|N_F(v) \cap B_{2\delta}|, \beta d\}.$$

(12)

Observe that all vertices in $B_{2\delta}$ are active, since the distance to already removed vertices equals 1 (see (10)). We separately bound the first and second sums above. We bound the size of the first sum by $|B_{3\delta}| \beta d$. To bound the size of the second sum, consider a vertex $v \in V_G \setminus B_{3\delta}$. There are two options:

1. $v \notin V_{F_3(t)}$ i.e. $v$ was removed at one of the previous iterations or at Step 3. In this case, all edges going from $v$ to $B_{2\delta}$ were cut at one of the previous iterations or at Step 3.

2. $\|\varphi_t(u) - \varphi_t(v)\|^2 \geq 3\delta$ (but $v \in V_{F_3(t)}$). In this case, all edges going from $v$ to $B_{2\delta}$ have length at least $\delta$, and thus they were cut at Step 2 at one of the iterations.

In any case, all edges between $v$ and vertices in $B_{2\delta}$ have been cut by the algorithm at Steps 2, 3 of the current iteration, or at any step of one of the previous iterations. Note that none of these edges were cut at Step 4 of the current iteration. Let $\rho$ be the number of edges going from $V_G \setminus B_{3\delta}$ to $B_{2\delta}$. Then, $\rho$ is an upper bound on the second sum in (12). We have

$$M_{\beta d}(B_{2\delta}) \leq |B_{3\delta}| \beta d + \rho.$$

We know that every cut edge has increased the budget of the endpoint lying in $B_{2\delta}$ by 1. Initially, the algorithm assigned a budget of at least $\beta d$ to each vertex in $B_{3\delta}$, hence

$$\text{budget}(B_{3\delta}) \geq \beta d \cdot |B_{3\delta}| + \rho \geq M_{\beta d}(B_{2\delta}).$$
In the equation above, we compute the budget of $B_{3\delta}$ after Step 3. That is, we ignore the changes of the budgets that occurred at Step 4.

We now use that the Heavy Vertices Removal procedure has removed all balls of radius $3\delta$ having a budget of $\eta_d \beta dn$ or more. Thus, $\text{budget}(B) \leq \beta \eta_d dn$ (again, here we compute the budget after Step 3). We conclude that $M_{\beta d}(B_{2\delta}) \leq \eta_d \beta dn$. \hfill \Box

We now state the main technical result of this section.

**Lemma 4.2.** Let $(L, R)$ be the planted partition in the graph $G$. For every $t \in \{0, \ldots, T - 1\}$, the graph $F_4(t)$ can be partitioned into two sets $W, \bar{W}$ such that the sizes of the sets $L \cap W, R \cap W$ and $W$ are at most $n/2$ each; and the size of the edge boundary between $L \cap W, R \cap W$ and $W$ is at most $4K\eta_d dn$.

**Proof.** Define the set $W$ as follows:

$$W = \{u \in V_{F_4(t)} : \deg(u, G_4(t)) \geq \max\{\beta d, \deg(u, H)/D_n\}\}.$$  

Let $\varphi'_t$ be the SDP solution defined in (10). We claim that all edges $(u, v)$ in $E_{H_4(t)}$ with $u \in W$ satisfy conditions 1–3 of the Main Structural Property. Indeed, all edges $(u, v) \in E_{F_4(t)}$ are $\delta/2$-short, otherwise they would be removed by Step 2. By Lemma 4.1, all active vertices satisfy the second condition. Finally, by the definition of $W$, the degree of every $u \in W$ in the graph $G_4(t)$ is at least $\max\{\beta d, \deg(u, H)/D_n\}$, and since all uncut edges are $\delta/2$-short, short $\varphi'_t, \beta_d (u, G) \geq \max\{\beta d, \deg(u, H)/D_n\}$. Therefore, by the Main Structural Property, there are at most $K\eta_d dn$ edges $(u, v) \in E_{H_4(t)}$ with $u \in W$.

The edge boundary between the sets $L \cap W, R \cap W$ and $W = V_{F_4(t)} \setminus \bar{W}$ is the union of the sets $E_{G_{4}(t)}(L \cap W, R \cap W), E_{G_{4}(t)}(W, \bar{W}), E_{H_4(t)}(L \cap W, R \cap W)$ and $E_{H_4(t)}(W, \bar{W})$. Observe, that $E_{G_{4}(t)}(L \cap W, R \cap W)$ is the planted cut in $G$. We already have an upper bound

$$|E_{H_4(t)}(L \cap W, R \cap W)| + |E_{H_4(t)}(W, \bar{W})| \leq K\eta_d dn$$

(13) since all edges in $E_{H_4(t)}(L \cap W, R \cap W)$ and $E_{H_4(t)}(W, \bar{W})$ are incident on the set $W$. We now bound the size of the set $E_{G_{4}(t)}(W, \bar{W})$. Let

$$X = \{u \in V_{F_4(t)} : \beta d \leq \deg(u, G_4(t)) \leq \deg(u, H)/D_n\};$$

$$Y = \{u \in V_{F_4(t)} : \deg(u, G_4(t)) \leq \beta d; \text{budget}(u) \geq (\alpha - \beta)d\};$$

$$Z = \{u \in V_{F_4(t)} : \deg(u, G_4(t)) \leq \beta d; \text{budget}(u) < (\alpha - \beta)d\}.$$  

The budgets in the expressions above are computed in the end of the $t$-th iteration. We will need several bounds on the degrees of vertices $u$ in $Z$.

**Claim 4.3.** For every $u \in Z$,

1. $\deg(u, H_4(t)) \geq \max\{\beta d, 1/6 \deg(u, H)\};$

2. $\deg(u, G) \leq (\alpha - \beta)d.$

**Proof.** 1. Denote by $\rho = \deg(u, F) - \deg(u, G_4(t))$ the number of cut edges incident on $u$. By the definition of $Z$, $\text{budget}(u) < (\alpha - \beta)d$. Therefore, $\deg(u, F) \geq (\alpha + \beta)d$ (otherwise, $u$ would receive a budget of $\alpha d$ at the initialization step); and the initial budget of $u$ is $\beta d$. Hence, the current budget of $u$ equals $\text{budget}(u) = \beta d + \rho$. We get $\rho = \text{budget}(u) - \beta d \leq (\alpha - 2\beta)d$. Thus, $\deg(u, F_4(t)) = \deg(u, F) - \rho \geq 2\beta d$. Then, $\deg(u, H_4(t)) \geq \deg(u, F_4(t)) - \deg(u, G_4(t)) \geq \beta d$.  

By Lemma 4.8 (applied with $Y' = \{u\}$),

$$\text{budget}(u) \leq 2\deg(u, F_4(t)) + 2\beta d \leq 2\deg(u, H_4(t)) + 4\beta d.$$
Hence, \( \rho \leq 2 \deg(u, H_4(t)) + 3\beta d \), and
\[
\deg(u, H) \leq \deg(u, H_4(t)) + \rho \leq 3 \deg(u, H_4(t)) + 3\beta d \leq 6 \deg(u, H_4(t)).
\]

II. We have \( \deg(u, G) \leq \deg(u, G_4(t)) + \rho \leq \beta d + (\alpha - 2\beta)d \leq (\alpha - \beta)d \).

As an immediate corollary we get that \( Z \subset V_G^{\leq \alpha d} \cap V_H^{\geq \beta d} \) (see (6) and (7) in Section 3.4 for the definitions of \( V_G^{\leq \alpha d} \) and \( V_H^{\geq \beta d} \)).

**Corollary 4.4.** \( Z \subset V_G^{\leq \alpha d} \cap V_H^{\geq \beta d} \).

We will also need an upper bound on the size of \( X \).

**Claim 4.5.** \(|X| \leq n/(\beta D_n)\).

**Proof.** For all \( u \in X \), \( \deg(u, H) \geq \beta d \). The average degree of vertices in \( H \) is at most \( d \). Hence, by Markov’s inequality, \( |X| \leq n/(\beta d) \).

We return to the proof of Lemma 4.2. Write
\[
|E_{G_4(t)}(W, \bar{W})| = |E_{G_4(t)}(Y, W)| + |E_{G_4(t)}(Z, W)|.
\]

Observe that every edge \((u, v)\) in \( E_{G_4(t)}(Z, W) \) (\( u \in Z, v \in W \)) satisfies conditions 1–4 of Structural Property 2: Each edge \((u, v)\) is \( \delta/2 \)-short. Then, \( M_{\beta d}(\text{Ball}_\delta(v, 2\delta)) \leq \eta n \) by Lemma 4.1 (note that \( v \) is active, because \((u, v)\) is \( \delta/2 \)-short); short \( \varphi_{\delta/2}(u, G) \geq \deg(u, G_4(t)) \geq \beta d \) by Claim 4.3 and \( \deg(u, G) \leq \alpha d \) by Claim 4.3. Hence,
\[
|E_{G_4(t)}(Z, W)| \leq K\eta nd.
\]

Claim 4.8 shows that \( |E_{G_4(t)}(Y, W)| \leq K\eta nd \), and Claim 4.6 shows that \( |E_{G_4(t)}(X, \bar{W})| \leq K\eta nd \). Hence, the total size of the edge boundary is at most \( 4K\eta nd \). Before proving Claims 4.8 and 4.6 we verify that the sizes of the sets \( L \cap W, R \cap W \) and \( \bar{W} \) are bounded by \( n/2 \). The sizes of the sets \( L \cap W, R \cap W \) and \( \bar{W} \) are bounded by \( |L| = |R| = n/2 \). The size of the set \( X \) is bounded by \( n/(\beta d) \leq n/6 \) (since the budget of every vertex in \( Y \) is at least \( (\alpha - \beta)\alpha d \); the size of \( Z \) is bounded by \( n/\beta \leq n/6 \) (since the average degree in \( H \) is at most \( d \); the degrees of all vertices in \( Y \) are at least \( \beta d \)). Thus, \( |\bar{W}| = |X| + |Y| + |Z| \leq n/2 \).

**Claim 4.6.** The size of the edge boundary between \( W \) and \( X \) in the graph \( G_4(t) \) is at most \( \eta \alpha d \):
\[
|E_{G_4(t)}(W, X)| \leq \eta \alpha d.
\]

**Proof.** We count the number of edges incident on the vertices of \( X \) in the graph \( G_4(t) \). By the definition of \( X \), \( \deg(x, G_4(t)) \leq \deg(x, H)/D_n \). Thus,
\[
\sum_{u \in X} \deg(u, G_4(t)) \leq \sum_{u \in X} \frac{\deg(u, H)}{D_n} \leq \frac{2|E_H|}{D_n} \leq \frac{dn}{D_n} \leq \eta \alpha d;
\]
since \( \eta \geq 1/D_n \).

To prove Claim 4.8 we need to bound \( \beta d |Z| \).

**Claim 4.7.** We have
\[
\beta d |Z| \leq \frac{\text{budget}(\bar{W})}{10} + 7|E_{H_4(t)}(\bar{W}, W)|.
\]
Proof. Consider the following mental experiment: We give \( \beta d \) blue tokens to every \( z \in Z \), and \( \beta d \) red tokens to every \( y \in X \cup Y \). Then, every vertex \( x \in \bar{W} \) sends \( \beta d / \deg(x, H) \) tokens to each neighbor of \( x \) in the graph \( H \). Let us write that the total number of blue tokens, \( \beta d | Z | \), equals the number of tokens sent from vertices in \( Z \):

\[
\beta d | Z | = \sum_{y \in Z} \deg(y, H) \times \frac{\beta d}{\deg(y, H)}.
\]

Using the bound \( \deg(z, H_4(t)) \geq 1/6 \deg(z, H) \) from Claim 4.3, we get

\[
\beta d | Z | \leq 6 \sum_{y \in Z} \frac{\deg(z, H_4(t))}{\deg(y, H)} = 6 \sum_{y \in Z} \sum_{x : (x, y) \in E_{H_4(t)}} \frac{\deg(y, H)}{\deg(y, H)}.
\]

Now, in the right hand side, we count the number of blue tokens sent along edges in \( H_4(t) \). Observe that every vertex \( y \) sends at most one blue token to each of its neighbors — simply because \( \deg(y, H) \geq \beta d \) (see Claim 4.3). Hence, the number of tokens sent to \( W \) from all \( y \)’s in \( Z \) is bounded by the size of the edge boundary between \( Z \) and \( W \). We have

\[
\beta d | Z | \leq 6 \sum_{y \in Z} \sum_{x : (x, y) \in E_{H_4(t)}} \frac{\deg(y, H)}{\deg(y, H)} + 6 | E_{H_4(t)}(Z, W) |\]

\[
= 6 \sum_{x \in W} \left( \sum_{y \in Z} \frac{\deg(y, H)}{\deg(y, H)} \right) + 6 | E_{H_4(t)}(Z, W) |.
\]

The expression in the brackets above is the number of blue tokens a vertex \( x \in \bar{W} \) receives. We compare it with the number of red tokens received by the same vertex. Using Structural Property 4, we get (keep in mind that \( Z \subset V_{G}^{\leq \alpha d} \cap V_{H}^{\geq \beta d} \), see Corollary 4.4)

\[
\sum_{y : (x, y) \in \pi E_{H}} \frac{\beta d}{\deg(y, H)} \leq \sum_{y : (x, y) \in \pi E_{H}} \frac{\beta d}{\deg(y, H)} \leq \frac{\beta d}{\deg(y, H)} + 4 \log n \leq \frac{\beta d}{\deg(y, H)} + 4 \log n.
\]

We cover the domain \( \{ y : (x, y) \in \pi E_{H} \text{ and } y \in V_{H}^{\geq \beta d} \setminus Z \} \) with three sets \( S_1 = \{ y : (x, y) \in \pi E_{H} \setminus E_{H_4(t)} \text{ and } y \in V_{H}^{\geq \beta d} \} \), \( S_2 = \{ y : (x, y) \in E_{H_4(t)} \text{ and } y \in W \} \), and \( S_3 = \{ y : (x, y) \in E_{H_4(t)} \text{ and } y \in X \cup Y \} \). The size of \( S_1 \) is at most budget\((x) \) – \( \beta d \), since all edges from \( S_1 \) have been cut and hence budget\((x) \) ≥ \( \beta d + | S_1 | \). The set \( S_2 \) equals \( E_{H_4(t)}(\{ x \}, W) \). Therefore, using that \( \beta d / \deg(y, H) \leq 1 \), we get

\[
\sum_{y : (x, y) \in \pi E_{H}} \frac{\beta d}{\deg(y, H)} \leq \frac{\beta d}{\deg(y, H)} + | S_2 | + \sum_{y \in S_3} \frac{\beta d}{\deg(y, H)} + 4 \log n
\]

\[
\leq \frac{\beta d}{\deg(y, H)} + | E_{H_4(t)}(\{ x \}, W) | + \text{budget}(x) + 4 \log n.
\]
Plugging this inequality in (15), we get

$$\beta d |Z| \leq \frac{48}{\alpha} \sum_{x \in W} \left( \sum_{y : (x, y) \in E_{H_4(t)}} \frac{\beta d}{\deg(y, H)} + \text{budget}(x) + E_{H_4(t)}(\{x\}, W) \right)$$

+ \left( 4|\bar{W}| \log n + 6|E_{H_4(t)}(\bar{W}, W)| \right)

$$\leq \frac{48}{\alpha} \left( \beta d |X| + \beta d |Y| + \text{budget}(W) + |E_{H_4(t)}(\bar{W}, W)| \right) + 4|\bar{W}| \log n + 6|E_{H_4(t)}(\bar{W}, W)|.$$

Using that $\alpha = 50\beta$, $\beta = 200K$, $d > \beta \log n$, and $\text{budget}(X) \geq \beta d |X|$, $\text{budget}(Y) \geq (\alpha - \beta)d |Y|$, $\text{budget}(Z) \geq \beta d |Z|$, we get the following bounds

- $48\beta d |X|/\alpha + 48 \text{budget}(X)/\alpha + 4 |X| \log n \leq \text{budget}(X)/10$;
- $48\beta d |Y|/\alpha + 48 \text{budget}(Y)/\alpha + 4 |Y| \log n \leq \text{budget}(Y)/10$; and
- $48 \text{budget}(Z)/\alpha + 4 |Z| \log n \leq \text{budget}(Z)/10$.

Therefore,

$$\beta d |Z| \leq \frac{\text{budget}(\bar{W})}{10} + 7|E_{H_4(t)}(\bar{W}, W)|.$$  

We now use the upper bound on $\beta d |Z|$ to get an upper bound on $\zeta \equiv |E_{G_4(t)}(W, Y)|$.

**Claim 4.8.** The size of the edge boundary between $W$ and $Y$ in the graph $G_4(t)$ is at most $K \eta n d$:

$$\zeta \equiv |E_{G_4(t)}(W, Y)| \leq K \eta n d.$$

**Proof.** The Damage Control procedure ensures that for $Y' = \bar{W}$ (see Lemma 3.8), we have

$$\text{budget}(\bar{W}) \leq 2|E_{F_4(t)}(\bar{W}, W)| + 2\beta d |\bar{W}| = 2|E_{F_4(t)}(\bar{W}, W)| + 2\beta d |X| + 2\beta d |Y| + 2\beta d |Z|.$$  

We bound the term $\beta d |X|$ using Claim 4.5 and the term $\beta d |Z|$ using Claim 4.7. We get the following bound.

$$\text{budget}(\bar{W}) \leq 2|E_{F_4(t)}(\bar{W}, W)| + 2\beta d |Y| + \frac{\text{budget}(\bar{W})}{5} + 14|E_{H_4(t)}(\bar{W}, W)| + \frac{2dn}{D_n}.$$  

Hence,

$$\text{budget}(\bar{W}) \leq \frac{5}{2} \beta d |Y| + \frac{5}{2} |E_{F_4(t)}(\bar{W}, W)| + 18|E_{H_4(t)}(\bar{W}, W)| + \frac{3dn}{D_n}.$$  

We replace $|E_{F_4(t)}(\bar{W}, W)|$ with $|E_{G_4(t)}(\bar{W}, W)| + |E_{H_4(t)}(\bar{W}, W)|$,

$$\text{budget}(\bar{W}) \leq \frac{5}{2} \beta d |Y| + \frac{5}{2} |E_{G_4(t)}(\bar{W}, W)| + 21|E_{H_4(t)}(\bar{W}, W)| + \frac{3dn}{D_n}.$$  

Recall, that every vertex in $Y$ has a budget of at least $(\alpha - \beta)d$ (by the definition of $Y$). Thus, $\text{budget}(\bar{W}) \geq \text{budget}(Y) \geq (\alpha - \beta)d |Y|$. We get

$$(\alpha - \frac{7}{2}\beta)d |Y| \leq \frac{5}{2} |E_{G_4(t)}(\bar{W}, W)| + 21|E_{H_4(t)}(\bar{W}, W)| + \frac{3dn}{D_n}.$$  

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The degree of every vertex $u$ in $Y$ in the graph $G_4(t)$ is at most $\beta d$ (by the definition of $Y$). Hence, $\zeta \equiv |E_{G_4(t)}(Y, W)| \leq \beta d |Y|$, and

$$\zeta \leq \frac{\beta}{\alpha - 7/2} \times \left[ \frac{5/2 |E_{G_4(t)}(Y, W)|}{\alpha - 7/2} + 21 |E_{H(t)}(Y, W)| + \frac{3dn}{D_n} \right].$$

Finally, we use the inequalities $|E_{H_4(t)}(Y, W)| \leq K \eta dn$ (see (13)); $3dn/D_n \leq 3 \eta dn$; and

$$|E_{G_4(t)}(Y, W)| = |E_{G_4(t)}(X, W)| + |E_{G_4(t)}(Y, W)| + |E_{G_4(t)}(Z, W)| \leq \eta dn + \zeta + K \eta dn$$

(see Claim 4.6 and Equation (14)) to obtain the bound:

$$\zeta \leq \frac{\beta}{\alpha - 7/2} \times \left[ \frac{5/2 \zeta + 24K \eta dn}{\alpha - 7/2} \right],$$

which implies $\zeta < K \eta dn$. □

4.2 Proof of Theorem 3.2

Proof of Theorem 3.2 Let $L \cap W$, $R \cap W$ and $\bar{W}$ be the partitioning from Lemma 4.2. We pick three orthogonal vectors $e_L$, $e_R$ and $e_{\bar{W}}$ of lengths $\sqrt{2}/2$. We define a new SDP solution $\varphi : V_{F_4(t)} \rightarrow \mathbb{R}^n$ as follows. Let $\varphi(u) = e_L$ for $u \in L \cap W$; $\varphi(u) = e_R$ for $u \in R \cap W$ and $\varphi(u) = e_{\bar{W}}$ for $u \in \bar{W}$. It is easy to check that this SDP solution is feasible: it trivially satisfies the $\ell^2$-triangle inequalities (since it is a 0-1 metric); and it satisfies the spreading constraints since the sets $L \cap W$, $R \cap W$ and $\bar{W}$ are balanced. The cost of the solution, sdp-cost$(\varphi, F_4(t))$ exactly equals the number of edges cut by the partition, which is bounded by $4K \eta dn = 8K \eta_{t+1} dn$:

$$\text{sdp-cost}(\varphi_{t+1}, F_1(t + 1)) \leq \text{sdp-cost}(\varphi, F_1(t + 1)) = \text{sdp-cost}(\varphi, F_4(t)) \leq 8K \eta_{t+1} dn.$$ □

5 Structural Properties — Proofs

In this section we show that $F = G \boxplus_H H$ satisfies Structural Properties 1–4 with high probability (see Section 3.4 for definitions). The main technically interesting and conceptually important part of our proof is the Main Structural Theorem.

Theorem 5.1 (Main Structural Theorem). There exist a constant $K$, such that for every $\beta > 1$, $D_n > \sqrt{\log n}$, and $d \geq D_{ARV} D_n \log^2 n$, every graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ on vertex sets $V_G = L_G \cup R_G$ and $V_H = L_H \cup R_H$ with $|E_H| \leq dn^2/2$ and $|L_G| = |R_G| = |L_H| = |R_H| = n/2$, the following statement holds with probability $(1 - 1/n^2)$ for $\pi$ chosen uniformly at random from $\Pi_{LR}$. For every feasible SDP solution $\varphi : V_G \rightarrow \mathbb{R}^n$ and $\eta = 2^{-t} (t \leq T = \log_2 D_{ARV} = O(\log \log n))$, there are at least $K \eta dn$ elements in the set $S$ defined as follows: the elements of $S$ are ordered pairs; a pair $(u, v) \in V_G \times V_G$ belongs to $S$, if

1. $(u, v)$ is a $\delta/2$-short edge in $\pi(H)$ i.e., $\|\varphi(u) - \varphi(v)\|^2 \leq \delta/2$ and $(u, v) \in \pi E_H$.
2. $M_{\beta d}(\text{Ball}(\varphi(u), 2\delta)) \leq \eta \beta d n$.
3. $\text{short}_{\varphi, \delta/2}(u, G) \geq \max\{\beta d, \deg(u, H)/D_n\}$ i.e., there are at least $\max\{\beta d, \deg(u, H)/D_n\}$ edges of length $\delta/2$ leaving $u$ in the graph $G$.  

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Conditions 1–3 in the statement of the theorem are the same as in the Main Structural Property. Note that we set $\beta$ to $200K$ in the algorithm, and thus $\beta$ depends on $K$. In this theorem, we prove that a universal constant $K$ exists that works for every $\beta > 1$ and $D_n > \sqrt{\log n}$, particularly, for $\beta = 200K$, $\alpha = 50\beta$, and $D_n = \max\{D_{ARV}, \alpha\}$. We do not assume that $G$ has a planted cut i.e., some edges in $G$ may cross the cut $(L_G, R_G)$.

In the proof, we use the notion of prefix Kolmogorov complexity. We denote the complexity of the bijection $\pi \in \Pi_{LR}$ i.e., $KP(\pi)$ is the number of bits required to store $\pi$.

5.1 Proof of the Main Structural Theorem

Proof. Fix $\eta = 2^{-t}$ and an SDP solution $\varphi$. Let $\gamma = |S|/(dn)$. We show that if $\gamma > K\eta$ (for some constant $K$), then the permutation $\pi$ can be encoded with a binary string of length less than $\log_2 |\Pi_{LR}| - 2 \log_2 n$. In other words, the prefix Kolmogorov complexity of $\pi$ is at most $\log_2 |\Pi| - 2 \log_2 n$. This is an unlikely event for a random $\pi$ sampled from $\Pi_{LR}$ uniformly. So we conclude that $|S| \geq K\eta dn$ with small probability.

To construct the encoding we need to identify a set of vertices $x \in \mathcal{X}$ for which the description of $\pi(x)$ is short. Denote the set of vertices whose degrees are in the range $[2^i, 2^{i+1})$ in $H$ by $U_i$:

$$\mathcal{U}_i = \{x \in V_H : \deg(x, H) \in [2^i, 2^{i+1} - 1]\}.$$ 

Let $\mathcal{E}_i = \{(x, y) \in E_H : x \in \mathcal{U}_i\}$ and let $\mathcal{S}_i = \{(x, y) \in \mathcal{S} : x \in \mathcal{U}_i\}$. The pairs $(x, y)$ in $\mathcal{E}_i$ and in $\mathcal{S}_i$ are ordered pairs. Let $\lambda_i = |\mathcal{E}_i|/(dn)$, and $\gamma_i = |\mathcal{S}_i|/|\mathcal{E}_i|$. Then,

$$\gamma = \sum_i \lambda_i \gamma_i. \tag{16}$$

Note that $|E_H| \leq dn/2$ (by the definition of $d$), thus $\sum \lambda_i \leq 1$. Consider the set of indices $I = \{i : \lambda_i \gamma_i \geq \gamma/(2 \log_2 n)\}$. We have

$$\sum_{i \in I} \lambda_i \gamma_i \geq \gamma - \sum_{i : \lambda_i \gamma_i \leq \gamma/(2 \log_2 n)} \lambda_i \gamma_i \geq \gamma/2.$$ 

We pick one $i \in I$ with $\gamma_i \geq \gamma/2$.

To encode $\pi$, we need to store the embedding $\varphi$. However, we cannot afford to store the whole embedding, so we only encode the embeddings of two subsets $Q_G \subset V_G$ and $Q_H \subset V_H$ of size at most $3qn$, where $q = D_n/2^t$. For every $x \in \mathcal{U}_i$, let $x'$ be the first element in $Q_H$ according to the order in $Q_H$ that is a neighbor of $x$ in the graph $H$ i.e. $(x, x') \in E_H$. If $x'$ exists, then we define two sets $\Xi'(x)$ and $\Xi''(x)$ as follows:

$$\Xi'(x) = \{u : |Q_G \cap N_G(u) \cap \{v : \|\varphi(v) - \varphi(\pi(x'))\| \leq \delta\}| \geq q\beta d/2\};$$

$$\Xi''(x) = \{u : |Q_G \cap N_G(u) \cap \{v : \|\varphi(v) - \varphi(\pi(x'))\| \leq 2\delta\}| \geq q\beta d/2\}.$$ 

The only difference in the definitions of $\Xi'(x)$ and $\Xi''(x)$ is that the radius of the ball around $\varphi(\pi(x'))$ is $\delta$ for $\Xi'(x)$ and $2\delta$ for $\Xi''(x)$. Note that $\Xi'(x) \subset \Xi''(x)$. Consider the set $\mathcal{X}$ of vertices $x$ for which the following conditions hold:

1. $x'$ is defined and $(x, x')$ is a $\delta/2$-short edge w.r.t. $\varphi$;

2. $\pi(x) \in \Xi'(x);$  

\(^2\)In the standard notation our $KP(x)$ is $KP(x \mid G, H)$. 

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3. $|\Xi''(x)| \leq 4\eta n$.

We show that for the right choice of $Q_G$ and $Q_H$, the set $\mathcal{X}'$ is sufficiently large.

**Lemma 5.2.** There exist sets $Q_G \subset V_G$ and $Q_H \subset V_H$ such that $|\mathcal{X}'| \geq c\gamma_i|U_i|$ (for some absolute constant $c$).

**Proof.** Let $Q_G$ and $Q_H$ be random subsets of $V_G$ and $V_H$ such that every $u \in V_G$ belongs to $Q_G$, and every $x \in V_H$ belongs to $Q_H$ with probability $q$. Then, by Chernoff’s bound, the probability that $|Q_G|, |Q_H| \leq 3qn$ is at least

$$1 - 2e^{-qn} \geq 1 - 2e^{-D_n} \geq 1 - 2e^{-\sqrt{\log n}}.$$

We now estimate the expected size of $\mathcal{X}'$.

The degrees of all vertices in $\mathcal{E}_i$ are at least $2i$. Hence, the probability that a vertex $x \in U_i$ does not have neighbors in $Q_H$ is at most $(1 - D_n/2^i)^2 \ll 1/4$. Consequently, for every $x \in U_i$, $x'$ is defined with probability at least $3/4$. The edge $(x, x')$ belongs to $S_i$ with probability $|N_H(x) \cap S_i|/|N_H(x)| \geq |N_H(x) \cap S_i|/2^{i+1}$. Note that the event $(x, x') \in S_i$ depends only on the set $Q_H$. Let us condition on $Q_H$ and assume that $(x, x') \in S_i$.

Let $B(x)$ be the set of neighbors of $\pi(x)$ in $G$ connected to $\pi(x)$ via $\delta/2$-short edges. Since $(x, x') \in S$, the set $B(x)$ has at least $\min\{\beta d, 2i/D_n\}$ vertices (by condition 3 of the definition of $S$). The size of $B(x) \cap Q_H$ is distributed as a Binomial distribution with parameters $|B(x)|$ and $q$. The median of the distribution is at least $[qB(x)]$. Note that $[qB(x)] \geq 1$, since $|B(x)| \geq 2^i/D_n$. Hence, $[qB(x)] \geq q\beta d/2$. Therefore, $\Pr(|B(x) \cap Q_H| \geq q\beta d/2) \geq 1/2$. The distance from $\pi(x)$ to $\pi(x')$ is at most $\delta/2$, hence, $B(x) \subset \{v \in V_G : ||\varphi(v) - \varphi(\pi'(x'))||^2 \leq \delta\}$. Thus, if $|B(x) \cap Q_G(x)| \geq q\beta d/2$, then $\pi(x) \in \Xi'(x)$. We get

$$\Pr(\pi(x) \in \Xi'(x) \mid (x, x') \in S_i) \geq \frac{1}{2} \quad \text{(17)}$$

We now estimate $\Pr(|\Xi''(x)| \leq 8\eta n \mid (x, x') \in S_i)$. By Markov’s inequality the probability that $u \in \Xi''(x)$ (over a random $Q_G$ and fixed $x'$) is bounded by

$$\Pr(u \in \Xi''(x)) \leq \min \left\{ \frac{|N_G(u) \cap \text{Ball}_\varphi(\pi'(x'), 2\delta)|}{\beta d/2}, 1 \right\}.$$

The expected size of $\Xi''(x)$ is bounded by

$$\mathbb{E} |\Xi''(x)| \leq \frac{1}{\beta d} \sum_{u \in V_G} \min \left\{ 2|N_G(u) \cap \text{Ball}_\varphi(\pi'(x'), 2\delta)|, \beta d \right\}.$$

Now if $(x, x') \in S_i$, then by the definition of $S$, $M_{\beta d}(\text{Ball}_\varphi(\pi'(x'), 2\delta)) \leq \eta n$. Thus,

$$\sum_{u \in V_G} \min \left\{ 2|N_G(u) \cap \text{Ball}_\varphi(\pi'(x'), 2\delta)|, \beta d \right\} \leq 2 \sum_{u \in V_G} \min \left\{ |N_F(u) \cap \text{Ball}_\varphi(\pi'(x'), 2\delta)|, \beta d \right\} \equiv 2M_{\beta d}(\text{Ball}_\varphi(\pi'(x'), 2\delta)) \leq 2\eta \beta dn.$$

We obtain the bound $\mathbb{E} |\Xi''(x)| \leq \eta n$. Applying Markov’s inequality, we get

$$\Pr(|\Xi''(x)| \geq 4\eta n \mid (x, x') \in S) \leq \frac{1}{4}.$$

Combining this inequality with (17), we obtain the following bound:

$$\Pr(\pi(x) \in \Xi'(x) \text{ and } |\Xi''(x)| \leq 8\eta n \mid (x, x') \in S) \geq \frac{1}{4},$$

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which is equivalent to
\[
\Pr(\pi(x) \in \Xi'(x); |\Xi''(x)| \leq 8\eta n; (x, x') \in S) \geq \frac{\Pr((x, x') \in S)}{4}.
\]

We conclude that the expected size of $\mathcal{X}$ is lower bounded by
\[
\mathbb{E} |\mathcal{X}| \geq \sum_{x \in \mathcal{U}} \frac{\Pr((x, x') \in S)}{4} \geq \sum_{x \in \mathcal{U}} \frac{3|N_{H}(x) \cap S|}{16 \cdot 2^{\log n + 1}} = \frac{|S|}{11 \cdot 2^{\log n}}.
\]

Since $|\mathcal{X}|$ never exceeds $|\mathcal{U}|$, we get $\Pr(|\mathcal{X}| \geq (\gamma_i/22)|\mathcal{U}|) \geq \gamma_i/22 \geq 1/(44D_{ARV})$. This finishes the proof of the lemma.

We now continue the proof of the Main Structural Theorem. We fix sets $Q_{G}$ and $Q_{H}$ satisfying the conditions of Lemma 5.2. We embed all vectors $\varphi(u)$ and $\varphi(\pi(x))$ for $u \in Q_{G}$ and $x \in Q_{H}$ in a low dimensional space using the Johnson—Lindenstrauss transform. We pick the dimension and scaling in a such a distance function inequalities). In other words, where $d(u, x)$ are the distances between the embedded vectors. In Lemma C.1 in Appendix, we show that such a distance function $d$ can be encoded using $O(qn \log n)$ bits (the function $d$ may not satisfy triangle inequalities). In other words, $KP(d(\cdot, \cdot)) = O(qn \log n)$. We define the set $\Xi(x)$ as follows:
\[
\Xi(x) = \{u \in V_{G} : |N_{G}(u) \cap \{v \in Q_{G} : d(v, x') \leq \delta\}| \geq q\delta d/2\}.
\]

By our choice of $d$, we have $\Xi'(x) \subset \Xi(x) \subset \Xi''(x)$. Particularly, for $x \in \mathcal{X}$, $\pi(x) \in \Xi(x)$ and $|\Xi(x)| \leq 4\eta n$. Note that $\Xi(x)$ depends only on the graphs $G$, $H$, the sets $Q_{G}$, $Q_{H}$ and the distance function $d$. It does not depend on the permutation $\pi$.

We now show how to encode the pair $(\mathcal{X}, \pi|_{\mathcal{X}})$ (here $\pi|_{\mathcal{X}}$ is the restriction of $\pi$ to $\mathcal{X}$). We first encode $i$ using $[\log_{2} \log_{2} n]$ bits, then we encode $|\mathcal{X}|$ using $[\log_{2} n]$ bit. We encode $\mathcal{X} \subset \mathcal{U}_{i}$ using $[\log_{2} (|\mathcal{U}_{i}|)] \leq \log_{2} (|\mathcal{U}_{i}|)] + 1$ bits. We can do so, since there are $(|\mathcal{U}_{i}|)/|\mathcal{X}|$ subsets of $\mathcal{U}_{i}$ of size $|\mathcal{X}|$. Note, that
\[
\log_{2} \left(\frac{|\mathcal{U}_{i}|}{|\mathcal{X}|}\right) + 1 \leq \log_{2} \left(\frac{c|\mathcal{U}_{i}|}{|\mathcal{X}|}\right)^{|\mathcal{X}|} + 1 < |\mathcal{X}| \log_{2} \left(\frac{|\mathcal{U}_{i}|}{|\mathcal{X}|}\right) + 2|\mathcal{X}|.
\]

We denote $\gamma' = |\mathcal{X}|/|\mathcal{U}_{i}|$ By Lemma 5.2 $\gamma' \geq c\gamma i \geq c\gamma/2$. We encode $Q_{G}$ and $Q_{H}$ using $O(qn \log n)$ bits. Finally, for every $x \in \mathcal{X}$, we encode the index of $\pi(x)$ in $\Xi(x)$ using $[\log_{2} \Xi(x)] \leq \log_{2} (16\eta m)$ bits. Altogether we use at most
\[
KP(\mathcal{X}, \pi|_{\mathcal{X}}) \leq \log_{2} \log_{2} n + |\log_{2} n| + (|\mathcal{X}| \log_{2} (1/\gamma') + 2|\mathcal{X}|) + O(qn \log n) + |\mathcal{X}| \cdot \log_{2} (16\eta m) \\
\leq |\mathcal{X}| \log_{2} (16\eta m/\gamma') + 2|\mathcal{X}| + 2[\log_{2} n] + O(qn \log n)
\]
bits. Lemma 5.4 which we prove below, shows that we can extend the encoding of $(\mathcal{X}, \pi|_{\mathcal{X}})$ to the encoding of $\pi$ using extra $\log_{2} |\Pi_{LR}| - |\mathcal{X}| \log_{2} n + 3|\mathcal{X}|$ bits. Hence, the total number of bits we need is
\[
KP(\pi) \leq KP(\mathcal{X}, \pi|_{\mathcal{X}}) + KP((\mathcal{X}, \pi|_{\mathcal{X}}) + O(1) \\
\leq \log_{2} |\Pi_{LR}| - |\mathcal{X}| \log_{2} n + 3|\mathcal{X}| + (|\mathcal{X}| \log_{2} (16\eta m/\gamma') + 2|\mathcal{X}| + 2[\log_{2} n] + O(qn \log n)) \\
\leq \log_{2} |\Pi_{LR}| + |\mathcal{X}| \log_{2} (16\eta m/\gamma') + 5|\mathcal{X}| + 2[\log_{2} n] + O(qn \log n) \\
\leq \log_{2} |\Pi_{LR}| - |\mathcal{X}| \log_{2} (c'\gamma/\eta) + O(qn \log n) + 2[\log_{2} n].
\]
for some constant $c'$. Here we used that $\gamma' \geq c\gamma_i \geq c\gamma/2$. To finish the proof of the Main Structural Theorem, we need to show that $|X| \geq \Omega(\max\{qn \log n, \log n\})$ (see Claim 5.3). This would imply that for a sufficiently large constant $K$, if $\gamma/\eta > K$, then

$$|X| \log_2(c'/\eta) - O(qn \log n) - 2[\log_2 n] > |X| \log_2(c'K) - O(qn \log n) - 2[\log_2 n] > 2\log_2 n.$$ 

Hence, if $\gamma/\eta > K$, then

$$KP(\pi) \leq \log_2 |\Pi_{LR}| - 2\log_2 n.$$ 

Therefore, $\gamma/\eta > K$ with probability at most $n^{-2}$ (since the number of $\pi$’s with prefix Kolmogorov complexity smaller than $\log_2 |\Pi_{LR}| - 2\log_2 n$ is at most $2^{\log_2 |\Pi_{LR}| - 2\log_2 n} \leq |\Pi_{LR}|/n^2$).

Claim 5.3. The following bound holds.

$$|X| \geq \Omega(\max\{qn \log n, \log n\}).$$

Proof. We lower bound the size of $X$ as follows. By Lemma 5.2, $|X| = \Omega(\gamma'|U_i|) \geq \Omega(\gamma_i|U_i|)$. Then,

$$\gamma_i|U_i| \geq \frac{\gamma_i|E_i|}{2^{i+1}} = \gamma_i\lambda_idn \geq \frac{2^{-i+1}2\log_2 n}{2^{i+2}2\log_2 n}.$$ 

Here we used that $i \in I$, and thus $\lambda_i \gamma_i \geq \gamma/(2\log_2 n)$. Since $2^i \leq n$, $d = \Omega(\log n)$ and $\gamma \geq \eta = \Omega(1/D_{ARV}) = \Omega(1/(\sqrt{\log n}))$, we have $\gamma_i|U_i| \geq \Omega(\log n)$. Again using the bounds on $d$ and $\gamma$, we get $\gamma d/\log_2 n \geq D_n \log_2 n$ and

$$\gamma_i|U_i| \geq \frac{2^{i+1}2\log_2 n}{2^{i+2}2\log_2 n} \geq \left(\frac{D_n}{2^i}\right)n \log_2 n = \Omega(qn \log n).$$

\[ \square \]

Lemma 5.4. Let $\pi \in \Pi_{LR}$ be a bijection from $V_H$ to $V_G$ mapping $L_H$ to $L_G$ and $R_H$ to $R_G$. Consider a subset $X \subset V_H$. Then,

$$KP(\pi) \leq KP((\pi|_X, X')) + \log_2 |\Pi_{LR}| - |X| \log_2 n + 3|X| + O(1).$$ 

That is, if the pair $(\pi|_X, X')$ can be encoded using $KP((\pi|_X, X'))$ bits, then $\pi$ can be encoded using $\log_2 |\Pi_{LR}| - |X| \log_2 n + 3|X| + O(1)$ bits.

Proof. We first encode $X'$ and $\pi|_X$ using $KP((\pi|_X, X'))$ bits. Then, we encode the restriction $\pi|_{V_H \setminus X}$. To do so, we split the set $V_H \setminus X$ into two subsets $X_L = L_H \setminus X$ and $X_R = R_H \setminus X$. Let $m_L = |X_L|$ and $m_R = |X_R|$. The restrictions of $\pi$ to $X_L$ and to $X_R$ are bijections from $X_L$ to $L_G \setminus \pi(V_H)$ and from $X_R$ to $R_G \setminus \pi(V_H)$ respectively. Hence, we can encode $\pi|_{X_L}$ and $\pi|_{X_R}$ using $\lceil \log_2 m_L \rceil$ and $\lceil \log_2 m_R \rceil$ bits (given $X'$ and $\pi(X)$). In other words,

$$KP(\pi) \leq KP((\pi|_X, X')) + KP(\pi|_{X_L} | X, \pi(X)) + KP(\pi|_{X_R} | X, \pi(X)) + O(1)$$

$$\leq KP((\pi|_X, X')) + \log_2 (m_L!) + \log_2 (m_R!) + O(1).$$
Thus, we now show how to derive Property 2 from the Main Structural Theorem. Structural Property 3 immediately follows from the assumption we made in Section 2, and thus it is always satisfied. We estimate $\log_2 m_L! + \log_2 m_R!$ using Stirling’s approximation:

\[
\log_2 m_L! = \log_2((n/2)! - \sum_{i=m_L+1}^{n/2} \log_2 i \leq \log_2(n/2)! - \int_{m_L}^{n/2} \log_2 x \, dx
\]

\[
= \log_2((n/2)! - \frac{n}{2} \log_2 \frac{n}{2} + m_L \log_2 m_L + \log_2 e \cdot (\frac{n}{2} - m_L)
\]

\[
\leq \log_2((n/2)! - (\frac{n}{2} - m_L)(\log_2 \frac{n}{2} - 3/2).
\]

Thus,

\[
\log_2(m_L!) + \log_2(m_R!) \leq 2\log_2((n/2)! - (n - m_L - m_R)(\log_2 \frac{n}{2} - 3/2) = \log_2 |\Pi_{LR}| - |\mathcal{X}|(\log_2 n - 5/2).
\]

\[
\square
\]

### 5.2 Proof of the Structural Properties 2 and 3

Structural Property 3 immediately follows from the assumption we made in Section 2 and thus it is always satisfied. We now show how to derive Property 2 from the Main Structural Theorem.

**Theorem 5.5.** Structural Property 2 holds with probability $1 - o(1)$.

**Proof.** Consider a subset of edges $E'_G$ of the graph $G$. An edge $(u, v)$ belongs to $E'_G$ if one of the endpoints, $u$ or $v$, belongs to $V_G^{\leq \alpha d}$ (see (5) for the definition of $V_G^{\leq \alpha d}$) i.e.,

\[
E'_G = \{(u, v) \in E_G : u \in V_G^{\leq \alpha d}\}.
\]

The degrees of vertices in $V_G^{\leq \alpha d}$ are upper bounded by $\alpha d$. The size of $V_G^{\leq \alpha d}$ is bounded by $n/\alpha$. Thus, the set $E'_G$ has at most $dn$ edges. Let $G' = (V_G, E'_G)$ be the graph with edges $E'_G$. We now apply the Main Structural Theorem to the graph $F' = H \boxplus_{\pi^{-1}} G'$ i.e., we switch around $G$ and $H$. This is possible, since the graph $G'$ has at most $dn$ edges. The theorem implies that for every $\varphi$ and $\eta = 2^{-i}$, there are at most $K_\eta dn$ edges satisfying the conditions below:

1. $(u, v)$ is a $\delta/2$-short edge in $E'_G$
2. $M^F_{\beta d}(\text{Ball}_\varphi(u, 2\delta)) \leq \eta m$, here $M^F_{\beta d}(\cdot)$ is defined as $M_{\beta d}(\cdot)$ in (5) but only for the graph $F'$.
3. $\text{short}_{\varphi, \delta/2}(u, H) \geq \max\{\beta d, \deg(u, H)/D_n\}$.

The first condition is equivalent to conditions 1 and 4 of Property 2. The second condition is less restrictive than the second condition of Property 2, because $M^F_{\beta d}(\text{Ball}_\varphi(u, 2\delta)) \leq M^{F'}_{\beta d}(\text{Ball}_\varphi(u, 2\delta))$ (since all edges of $\pi(F')$ are also edges of $F$). Finally, the third condition above is equivalent to the third condition of Property 2, because

\[
\max\{\beta d, \deg(u, H)/D_n\} \leq \max\{\beta d, \alpha d/D_n\} = \beta D_n.
\]

\[
\square
\]
5.3 Proof of the Structural Property 4

We first prove a simple technical lemma.

Lemma 5.6. Consider a vertex weighted graph $H = (V_H, E_H)$ with weights $c_y : V_H \to [0, 1]$. Let $T \subset V_H$ be a subset of $V_H$ of size $n' = n/2$. Fix an integer $k \leq n'/12$. Let $S$ be a random subset of $T$ of size $k$. Then,

$$
\Pr \left\{ \forall x \in V_H, \sum_{(x,y) \in E_H} c_y \leq \frac{4k}{n'} \sum_{y \in V_H \setminus S} c_y + 2 \log n \right\} \geq 1 - o(1).
$$

Proof. We first prove a simple technical lemma.

We first bound the probability that

$$
\sum_{(x,y) \in E_H} c_y \leq \frac{4k}{n'} \sum_{y \in V_H \setminus S} c_y + 2 \log n.
$$

(18)

for a fixed vertex $x \in V_H$. Let

$$
\tilde{c}_y = \begin{cases} 
  c_y, & \text{if } (x, y) \in E_H; \\
  0, & \text{otherwise.}
\end{cases}
$$

Write the inequality (18) in terms of $\tilde{c}_y$:

$$
\sum_{y \in S} \tilde{c}_y \geq \frac{4k}{n'} \sum_{y \in V_H \setminus S} \tilde{c}_y + 2 \log n \geq \frac{4k}{n'} \sum_{y \in S} \tilde{c}_y + 2 \log n,
$$

or, equivalently,

$$
(1 + \frac{4k}{n'}) \sum_{y \in S} \tilde{c}_y \geq \frac{4k}{n'} \sum_{y \in \tilde{T} \setminus S} \tilde{c}_y + 2 \log n.
$$

We denote $\mu = \frac{k}{n'} \sum_{y \in T} \tilde{c}_y$. Then,

$$
\Pr \left( \text{(18) holds} \right) \leq \Pr \left( \sum_{y \in S} \tilde{c}_y \geq \frac{n'}{n'} + 4k (4 \mu + 2 \log n) \right) \leq \Pr \left( \sum_{y \in S} \tilde{c}_y \geq 3 \mu + 3/2 \log n \right)
$$

$$
\leq \Pr \left( \sum_{y \in S} \tilde{c}_y - \mu \geq 2 \mu + 3/2 \log n \right)
$$

here we used that $k/n' \leq 1/12$ and $n'/12 (n' + 4k) \geq 3/4$. Let $S'$ be a random multiset sampled from $T$ with replacement such that each $y$ belongs to $S'$ with probability $k/n'$. Hoeffding [16] Theorem 4] showed that

$$
\Pr \left( \sum_{y \in S} \tilde{c}_y - \mu \geq 2 \mu + 3/2 \log n \right) \leq \Pr \left( \sum_{y \in S'} \tilde{c}_y - \mu \geq 2 \mu + 3/2 \log n \right).
$$

Observe that $\sum_{y \in S'} \tilde{c}_y = \mu$; $\text{Var} \left[ \sum_{y \in S'} \tilde{c}_y \right] \leq \mu$; and $c_y \in [0, 1]$ for all $y \in T$. Thus, by Bernstein’s inequality,

$$
\Pr \left( \sum_{y \in S'} \tilde{c}_y - \mu \geq 2 \mu + 3/2 \log n \right) \leq \exp \left( -\frac{(2 \mu + \frac{3}{2} \log n)^2}{2 \mu + \frac{3}{4} (2 \mu + \frac{3}{2} \log n)} \right)
$$

$$
\leq \exp \left( -\frac{(2 \mu + \frac{3}{2} \log n)^2}{4 \mu + \log n} \right)
$$

$$
\leq \exp(-2 \log n) = n^{-2}.
$$

By the union bound, the probability that the inequality (18) holds for some $x \in V_H$ is at most $1/n$. This concludes the proof. \qed
This inequality implies (21) since

\[ \sum_{v : (x,y) \in \pi E_H \cap V_G^{\leq \alpha d}} \frac{\beta d}{\deg(v, H)} \leq \frac{8}{\alpha} \sum_{v : (x,y) \in \pi E_H \cap V_G^{\leq \alpha d}} \frac{\beta d}{\deg(v, H)} + 4 \log n. \]  

(19)

Proof. Consider two sets \( S = \pi^{-1}(V_G^{\leq \alpha d}) \) and \( U = \pi^{-1}(V_H^{\geq \beta d}) = \{ y \in V_H : \deg(y, H) \geq \beta d \} \). The vertices of \( S \) and \( U \) belong to the graph \( H \) and not to the graph \( F \). We slightly abuse notation to denote by \( \deg(y, H) \) the degree of \( y \) in \( H \) (previously we used this notation for \( u \in V_F \)). Note that the set \( S \) is a random though non-completely uniform subset of \( V_H \), but \( U \) is not a random subset and does not depend on \( \pi \). We rewrite (19) as follows: for every \( x \in V_H \),

\[ \sum_{y : (x,y) \in E_H \cap U \cap S} \frac{\beta d}{\deg(y, H)} \leq \frac{8}{\alpha} \sum_{y : (x,y) \in E_H \cap U \cap S} \frac{\beta d}{\deg(v, H)} + 4 \log n. \]  

(20)

We split the set \( S \) into two sets \( S_L = S \cap L_H \) and \( S_R = S \cap R_H \). We show that for every \( x \in V_H \), the following two inequalities hold:

\[ \sum_{v : (x,y) \in \pi E_H \cap U \cap S_L} \frac{\beta d}{\deg(v, H)} \leq \frac{8}{\alpha} \sum_{v : (x,y) \in \pi E_H \cap U \cap S_L} \frac{\beta d}{\deg(v, H)} + 2 \log n; \]  

(21)

\[ \sum_{v : (x,y) \in \pi E_H \cap U \cap S_R} \frac{\beta d}{\deg(v, H)} \leq \frac{8}{\alpha} \sum_{v : (x,y) \in \pi E_H \cap U \cap S_R} \frac{\beta d}{\deg(v, H)} + 2 \log n, \]  

(22)

which together imply (20) and (19). These inequalities are the same up to renaming of \( L \) and \( R \). So we consider only the first inequality. We set the weight of each vertex \( y \in L_H \) to be

\[ c_y = \begin{cases} \frac{\beta d}{\deg(y, H)}, & \text{if } y \in U; \\ 0, & \text{otherwise}. \end{cases} \]

Note that for all \( y \in V_H \), we have \( c_y \in [0, 1] \) and \( |S_L| = |V_G^{\leq \alpha d} \cap L_H| \leq |V_G^{\leq \alpha d}| \leq n/\alpha \leq n/24 \). The set \( S_L \) is a random subset of \( L_H \) of size \( |V_G^{\leq \alpha d} \cap L| \). Hence, by Lemma 5.6

\[ \Pr \left\{ \forall x \in V_H, \sum_{(x,y) \in E_H \cap U \cap S_L} c_y = \frac{8|V_G^{\leq \alpha d} \cap L|}{n} \sum_{(x,y) \in E_H \cap U \cap S_L} c_y + 2 \log n \right\} \geq 1 - o(1). \]

This inequality implies (21) since \( 8|S_L|/n \leq 8/\alpha \). \( \Box \)

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A Proof of Lemma A.1

Lemma A.1. Suppose that $|V^\leq d_G| \geq n/\alpha$. Consider the following algorithm for Balanced Cut: sort all vertices according to their degree in $F$, let $L'$ be the $\lceil n/(3\alpha) \rceil$ vertices with least degrees and $R' = V_F \setminus L'$, return the cut $(L', R')$. The algorithm return a $\Theta(1)$-balanced cut of cost $O(dn)$ with high probability.

Proof. The cut $(L', R')$ is $1/(3\alpha)$ balanced, as required. We show that its cost is $\Theta(dn)$ w.h.p. Note that at least half of all vertices in $H$ have degree at most $2d$ by Markov's inequality. The permutation $\pi$ maps at least a $|V^\leq d_G|/n$ fraction of them to $V^\leq d_G$. In expectation. Thus the fraction of vertices in $F$ with degree at most $(\alpha + 2)d$ is at least $(1/2) \cdot (|V^\leq d_G|/n) \geq 1/(2\alpha)$ in expectation. With high probability, there are at least $n/(3\alpha)$ vertices in $F$ of degree at most $(\alpha + 2)d$. Then all vertices in $L$ have degrees at most $(\alpha + 2)d$. Thus the cost of the cut $(L', R')$ is at most $(\alpha + 2)d \times |L'| = (\alpha + 2)d \cdot \lceil n/(3\alpha) \rceil = O(dn)$.

B Min Cut in Damage Control

In the Damage Control procedure, we solve a minimum cut problem in order to find $Y$ that maximizes $\sum_{i,j} c_{ij}$. Let us verify that the solution we obtain indeed maximizes $Y$. Consider an arbitrary cut

$$(\{\text{“source”}\} \cup Y, \{\text{“sink”}\} \cup \bar{Y}).$$
This cut cuts all edges going from $Y$ to $\bar{Y}$. The capacity of these edges is $2|E_{F_3(t)}(Y, \bar{Y})|$. Then, it cuts all edges going from the source to $\bar{Y}$. The capacity of these edges equals budget($\bar{Y}$). Finally, it cuts all edges going from $Y$ to the sink. The capacity of these edges equals $2\beta d|Y|$. Thus, the total size of the cut equals

$$2|E_{F_3(t)}(Y, \bar{Y})| + \text{budget}(\bar{Y}) + 2\beta d|Y| = 2|E_{F_3(t)}(Y, \bar{Y})| + \text{budget}(V_{F_3(t)}) - \text{budget}(Y) + 2\beta d|Y|.$$  

The term budget($V_{F_3(t)}$) does not depend on the cut. Hence, the cut is minimized, when the expression (8) is maximized.

C Proof of Lemma C.1

We show that there exists a distance function $d : Q_G \times Q_H \rightarrow \mathbb{R}^+$ of small complexity that approximately preserves balls of radius $\delta$.

Lemma C.1. There exists a function $d : Q_G \times Q_H \rightarrow \mathbb{R}^+$ such that

$$ KP(d \mid Q_G, Q_H) = O(\max\{|Q_G|, |Q_H|\} \log n) $$

and for every $x \in Q_H$,

$$ \{u \in Q_G : \|\varphi(u) - \varphi(\pi(x))\|^2 \leq \delta\} \subset \{u \in Q_G : d(u, x) \leq \delta\} \subset \{u \in Q_G : \|\varphi(u) - \varphi(\pi(x))\|^2 \leq 2\delta\}. $$

Proof. The proof of the lemma is very standard. We embed all vectors in $\varphi(Q_G)$ and $\varphi(Q_H)$ in a lower dimensional space via the Johnson—Lindenstrauss transform and then replace the embedded vectors with vectors in sufficiently dense low dimensional epsilon net. Instead of presenting the details we use a lemma from our previous work [23].

Lemma C.2 (Lemma 2.7 in [23]). For every $m$ and $\varepsilon \in (0, 1)$, there exists a set of matrices $A$ of size at most $|A| \leq \exp(O(\frac{m \log m}{2\varepsilon^2}))$ such that: for every collection of vectors $L(1), \ldots, L(m), R(1) \ldots R(m)$ with $\|L(i)\| \leq 1, \|R(j)\| \leq 1$ and $\langle L(i), R(j) \rangle \geq 0$, there exists $A \in A$ satisfying for every $u$ and $x$:

$$ a(u, x) \leq \langle L(u), R(x) \rangle \leq a(u, x) + \gamma; $$

$$ a(u, x) \in [0, 1]. $$

Let $m = \max\{|Q_G|, |Q_H|\}$ and $\varepsilon = \delta/2$. We pick $A$ as in the lemma above. The set $A$ depends only on $m$ and $\varepsilon$. We find a matrix $a \in A$ such that $a(u, x) \leq \langle \varphi(u), \varphi(\pi(x)) \rangle \leq a(u, x) + \varepsilon$ and let $d(u, x) = (1 - 2a(u, x))/2$. The complexity $KP(d \mid Q_G, Q_H)$ is at most $\lceil \log_2 |A| \rceil = O(m \log m)$ since $d$ can be reconstructed from $a$, and $a$ is chosen among $\exp(O(\frac{m \log m}{2\varepsilon^2}))$ possible matrices. If $\|\varphi(u) - \varphi(\pi(x))\|^2 \leq \delta$, then $\langle \varphi(u), \varphi(\pi(x)) \rangle \leq (1 - \delta)/2$. Hence, $a(u, x) \geq (1 - 2\delta)/2$ and $d(u, x) \leq \delta$. If $\|\varphi(u) - \varphi(\pi(x))\|^2 > 2\delta$, then $a(u, x) \leq \langle \varphi(u), \varphi(\pi(x)) \rangle < (1 - 2\delta)/2$, and $d(u, x) > \delta$. □