Higgs Bundles and Flat Connections Over Compact Sasakian Manifolds

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Abstract: Given a compact Kähler manifold $X$, there is an equivalence of categories between the completely reducible flat vector bundles on $X$ and the polystable Higgs bundles $(E, \theta)$ on $X$ with $c_1(E) = 0 = c_2(E)$ (Simpson in J Am Math Soc 1(4):867–918, 1988; Corlette in J Differ Geom 28:361–382, 1988; Uhlenbeck and Yau in Commun Pure Appl Math 39:257–293, 1986; Donaldson in Duke Math J 54(1):231–247, 1987). We extend this equivalence of categories to the context of compact Sasakian manifolds. We prove that on a compact Sasakian manifold, there is an equivalence between the category of semi-simple flat vector bundles on it and the category of polystable basic Higgs bundles on it with trivial first and second basic Chern classes. We also prove that any stable basic Higgs bundle over a compact Sasakian manifold admits a basic Hermitian metric that satisfies the Yang–Mills–Higgs equation.

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1. Introduction

Let $X$ be a compact Kähler manifold equipped with a Kähler form $\omega$, and let $E$ be a holomorphic vector bundle over $X$. Uhlenbeck and Yau proved that $E$ is polystable with respect to $\omega$ if and only if $E$ admits a Hermitian metric that satisfies the Hermite–Einstein equation defined using $\omega$ [UY]. This result was proved earlier by Donaldson under the extra assumptions that $X$ is a complex projective manifold and $\omega$ represents a rational cohomology class [Do2]. As a consequence of these theorems of Uhlenbeck–Yau and Donaldson, a holomorphic vector bundle on $X$ admits a flat unitary connection if and only if $E$ is polystable with $c_1(E) = 0 = c_2(E)$.

Hitchin proved that a Higgs bundle $(E, \theta)$ of rank two and degree zero on a compact Riemann surface $X$ admits a Hermitian metric that solves the Yang–Mills equation if and only if $(E, \theta)$ is polystable [Hi]. This implies that a polystable Higgs bundle $(E, \theta)$ of rank two and degree zero produces a completely reducible flat principal $GL(2, \mathbb{C})$-bundle (also called a principal $GL(2, \mathbb{C})$-bundle with a semi-simple flat connection) on $X$. Donaldson proved that a principal $GL(2, \mathbb{C})$-bundle on $X$ with a completely reducible flat connection admits a harmonic metric [Do3]; this produces the inverse map from the completely reducible flat $GL(2, \mathbb{C})$–bundles on $X$ to the polystable Higgs bundles of rank two and degree zero on $X$.

Simpson and Corlette extended the above bijective correspondence to the more general context of bundles over compact Kähler manifolds [Si1,Co]. More precisely, Simpson and Corlette proved that for any compact Kähler manifold $X$, there is an equivalence of categories between the completely reducible (also called semi-simple) flat vector bundles on $X$ and the polystable Higgs bundles $(E, \theta)$ on $X$ with $c_1(E) = 0 = c_2(E)$; this result is explained in detail in [Si2].

We recall that the contact manifolds constitute the odd dimensional counterpart of the symplectic manifolds. For example, while the total space of the cotangent bundle of a smooth manifold $X$ is a typical local model of a symplectic manifold, the local model of a contact manifold is the total space of the projective bundle $P(T^*M)$. In a similar vein, compact Sasakian manifolds can be thought of as the odd dimensional counterparts of the compact Kähler manifolds. A compact regular Sasakian manifold is in fact the unit circle bundle inside a holomorphic Hermitian line bundle of positive curvature on a complex projective manifold. More generally, a compact quasi-regular Sasakian manifold is the unit circle bundle inside a holomorphic Hermitian line bundle of positive curvature over a complex projective orbifold. However, the global structures of more general compact Sasakian manifolds, known as irregular Sasakian manifolds, do not admit any such explicit description.

It may be mentioned that Sasakian manifolds were introduced by Sasaki [Sa,SH], which explains the terminology. During the last twenty years there has been a very substantial increase of the interests in Sasakian manifolds, accompanied by a flurry of research activities (see [BG] and references therein). It is evident from these references that a large part of this recent investigations into Sasakian manifolds were carried out by Boyer and Galicki. Another aspect contributing to this recent activities in Sasakian manifolds is the discovery of their relevance in the string theory in theoretical physics. This was initiated in the works of Maldacena [Ma]. For further developments in this direction, see [GMSW,MSY,MS] and references therein.

Our aim here is to extend, to the context of Sasakian manifolds, the earlier mentioned equivalence of categories between the completely reducible flat vector bundles on a compact Kähler manifold and the polystable Higgs bundles $(E, \theta)$ with $c_1(E) = 0 = c_2(E)$.
The main result proved here states as follows (see Theorem 7.2).

**Theorem 1.1.** For a compact Sasakian manifold $M$, there is an equivalence between the category of semi-simple flat bundles over $M$ and the category of polystable basic Higgs bundles over $M$ with trivial first and second basic Chern classes.

In [BM1], the same question was addressed for the special case of quasi-regular Sasakian manifolds. As mentioned before, quasi-regular Sasakian manifolds are the unit circle bundles of holomorphic Hermitian line bundles of positive curvature over any complex projective orbifold. Using this fact together with a result of geometric group theory [BM1, p. 3492, Lemma 2.1], the question for quasi-regular Sasakian manifolds actually reduces to that for complex projective manifolds. In view of the known results on Higgs bundles over complex projective manifolds, the differential geometric and analytical investigations needed for the proof of Theorem 1.1 could be entirely avoided in [BM1].

To prove Theorem 1.1 we establish analogues of the theory of harmonic metrics on flat bundles and the theory of Hermitian–Yang–Mills metrics on Higgs bundles on compact Kähler manifolds. Just as Corlette and Simpson proved for compact Kähler manifolds [Co, Theorem 5.1], [Si2, Lemma 1.1], we obtain the characterization of harmonic flat bundles over compact Sasakian manifolds in terms of transversally holomorphic geometry (see Theorem 4.2). It may be mentioned that this is facilitated by a special feature of Sasakian geometry (see Theorem 3.2) which would not hold for a general transversally Kähler Geometry. This produces the functor that we are seeking in Theorem 1.1 from the semi-simple flat bundles to the basic Higgs bundles.

The proof of the opposite direction in Theorem 1.1 is inspired by a recent work of D. Baraglia and P. Hekmati [BH]. Defining the stable and polystable basic Higgs bundles over compact Sasakian manifolds, we prove for Sasakian manifolds the following analog of [Si1, Theorem 1] proved in the Kähler setting (see Theorem 5.2):

**Theorem 1.2.** For a stable basic Higgs bundle $(E, \theta)$ over a compact Sasakian manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$, there exists a basic Hermitian metric $h$ on $E$ such that

$$\Lambda R^{D^h} = 0,$$

where $R^{D^h}$ is the trace-free part of the curvature $R^{D^h}$ of the canonical connection $D^h$ associated to $h$.

Theorem 1.2 implies the following Bogomolov–Miyaoka type inequality (see Corollary 6.6):

**Corollary 1.3.** Let $(E, \theta)$ be a polystable basic Higgs bundle of rank $r$ over a compact Sasakian manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$ with $\dim M = 2n + 1$. Then

$$\int_M \left( 2c_2, B_{\gamma_i} (E) - \frac{r - 1}{r} c_1, B_{\gamma_i} (E)^2 \right) (d\eta)^{n-2} \wedge \eta \geq 0,$$

where $c_1, B_{\gamma_i} (E)$ is the $i$-th basic Chern class of $E$. If the above inequality is an equality, then $R^{D^h} = 0$.

We mention that the fundamental groups of compact Sasakian manifolds were investigated in [Ch, Ka, BM2].
2. Strongly Pseudo-convex CR Manifolds and Sasakian Manifolds

Let $M$ be a $(2n + 1)$-dimensional real smooth manifold. A CR-structure on $M$ is an $n$-dimensional complex sub-bundle $T^{1,0}$ of the complexified tangent bundle $TM = TM \otimes \mathbb{C}$ such that $T^{1,0} \cap \overline{T^{1,0}} = \{0\}$ and $T^{1,0}$ is integrable (i.e., the locally defined sections of $T^{1,0}$ are closed under the Lie bracket operation). We shall denote $\overline{T^{1,0}}$ by $T^{0,1}$. For a CR-structure $T^{1,0}$ on $M$, there is a unique sub-bundle $S$ of rank $2n$ of the real tangent bundle $TM$ together with a vector bundle homomorphism $I : S \rightarrow S$ satisfying the conditions that

1. $I^2 = -\text{Id}_S$, and
2. $T^{1,0}$ is the $\sqrt{1}$-eigenbundle of $I$.

A $(2n + 1)$-dimensional manifold $M$ equipped with a triple $(T^{1,0}, S, I)$ as above is called a CR-manifold. A contact CR-manifold is a CR-manifold $M$ with a contact 1-form $\eta$ on $M$ such that $\ker \eta = S$. Let $\xi$ denote the Reeb vector field for the contact form $\eta$. On a contact CR-manifold, the above homomorphism $I$ extends to entire $TM$ by setting $I(\xi) = 0$.

**Definition 2.1.** A contact CR-manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$ is a strongly pseudo-convex CR-manifold if the Hermitian form $L_\eta$ on $S_x$ defined by $L_\eta(X, Y) = d\eta(X, IY)$, $X, Y \in S_x$, is positive definite for every point $x \in M$.

Given any strongly pseudo-convex CR-manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$, there is a canonical Riemann metric $g_\eta$ on $M$, called the Webster metric, which is defined to be

$$g_\eta(X, Y) := L_\eta(X, Y) + \eta(X)\eta(Y), \quad X, Y \in T_x M, \ x \in M.$$ 

**Proposition 2.2** [Ta, We]. For a strongly pseudo-convex CR-manifold

$$(M, (T^{1,0}, S, I), (\eta, \xi))$$

there exists a unique affine connection $\nabla^{TW}$ on $TM$ such that the following hold:

1. $\nabla^{TW}(C^\infty(S)) \subset A^1(M, S)$, where $A^k(M, S)$ is the space of differential $k$-forms on $M$ with values in the vector bundle $S$.
2. $\nabla^{TW}\xi = 0, \nabla^{TW}I = 0, \nabla^{TW}d\eta = 0, \nabla^{TW}\eta = 0$ and $\nabla^{TW}g_\eta = 0$.
3. The torsion $T^{TW}$ of the affine connection $\nabla^{TW}$ satisfies the equation

$$T^{TW}(X, Y) = -d\eta(X, Y)\xi$$

for all $X, Y \in S_x$ and $x \in M$.

The affine connection $\nabla^{TW}$ in Proposition 2.2 is called the Tanaka–Webster connection.

**Definition 2.3.** A Sasakian manifold is a strongly pseudo-convex CR-manifold

$$(M, (T^{1,0}, S, I), (\eta, \xi))$$

satisfying any (all) of the following equivalent conditions:

1. $[\xi, A^0(M, S)] \subset A^0(M, S)$.
2. $\mathcal{L}_\xi\eta = 0$ and $\mathcal{L}_\xi I = 0$. 


3. $T^{TW}(\xi, v) = 0$ for all $v \in TM$.

For a Sasakian manifold, the curvature $R^{TW}$ of the Tanaka–Webster connection $\nabla^{TW}$ satisfies the equation

$$R^{TW}(\xi, v) = 0$$

for all $v \in TM$. See [BG] for Sasakian manifolds.

3. Harmonic Metrics on Sasakian Manifolds

Let $M$ be a compact Riemannian manifold and $E$ a flat complex vector bundle over $M$ equipped with a flat connection $\nabla^E$. For any Hermitian metric $h$ on $E$, we have a unique decomposition

$$\nabla^E = \nabla^h + \phi^h$$

such that $\nabla^h$ is a unitary connection and $\phi^h$ is a 1-form on $M$ with values in the self-adjoint endomorphisms of $E$ with respect to $h$.

**Theorem 3.1** [Co]. If a flat complex vector bundle $(E, \nabla^E)$ is semi-simple (meaning, direct sum of irreducible flat connections), then there exists a Hermitian metric (called the harmonic metric) $h$ on $E$ such that

$$(\nabla^h)^* \phi^h = 0,$$

where $(\nabla^h)^*$ is the formal adjoint operator of $\nabla^h$. If the connection $\nabla^E$ is irreducible, then the harmonic metric is unique up to multiplication by a constant scalar.

**Theorem 3.2.** Let $M$ be a compact Sasakian manifold with a Reeb vector field $\xi$, and let $(E, \nabla^E)$ be a semi-simple flat complex vector bundle over $M$. Then,

$$\phi^h(\xi) = 0$$

for any harmonic metric $h$ on $E$ for the flat connection $\nabla^E$.

**Proof.** This theorem is proved by modifying the proof of [Pe, Theorem 4.1]. Consider the vector bundle $\bigwedge T^*M \otimes \text{End}(E)$ over $M$ equipped with the connection

$$\tilde{\nabla} = \nabla^{TW} \otimes \text{Id}_{\text{End}(E)} + \text{Id}_{T^*M} \otimes \nabla^h.$$

Denote by $Cl(M)$ the Clifford bundle of $M$ associated with the Sasakian metric $g_\eta$. Then, using the canonical isomorphism $Cl(M) \cong \bigwedge T^*M \otimes \text{End}(E)$, consider $\bigwedge T^*M \otimes \text{End}(E)$ as a Dirac bundle. Define the Dirac operator corresponding to $\tilde{\nabla}$ to be

$$D = \sum_i e_i \cdot \tilde{\nabla}_{e_i},$$

where $\{e_i\}$ is a local orthonormal frame for $TM$ and “$\cdot$” denotes the Clifford multiplication. Then this $D$ is in fact a formal self-adjoint operator (see [Pe, Proposition 2.1]).

We have the following formula (cf. [Pe, Lemma 3.1]).

**Lemma 3.3.** $D \phi^h = -d \eta \otimes \phi^h(\xi)$.
Proof. By the flatness of the connection $\nabla^E$, we have $(\nabla^h + \phi_h)^2 = 0$, and this implies that

$$R^h = -\frac{1}{2} [\phi_h, \phi_h] \quad (3.2)$$

and $\nabla^h \phi_h = 0$, where $R^h$ is the curvature of the Hermitian connection $\nabla^h$ on $E$ associated to $h$ (see (3.1)). By a computation as in the proof of [Pe, Lemma 3.1], we conclude that $D \phi_h - (\nabla^h)^* \phi_h$ is the anti-symmetrization of covariant derivative on $\bigwedge T^*M \otimes \text{End}(E)$, and we have

$$(D \phi_h - (\nabla^h)^* \phi_h)(X, Y) = (\nabla^h \phi_h)(X, Y) - \phi_h(T^{TW}(X, Y)).$$

By the harmonicity of $h$,

$$(\nabla^h)^* \phi_h = 0 = \nabla^h \phi_h$$

as above. Now the lemma follows from Proposition 2.2. \hfill $\square$

Consider the following formula

$$\frac{1}{2} [D^2, (\xi \cdot )] = \frac{1}{2} (D^2 (\xi \cdot ) - \xi \cdot D^2) = -2 \xi \cdot \mathcal{R}_\xi$$

(see [Pe, Corollary 2.1]), where $\mathcal{R}_\xi$ is the endomorphism defined by

$$\mathcal{R}_\xi = -\frac{1}{2} \sum_i \xi \cdot e_i \cdot \tilde{R}(\xi, e_i)$$

with $\tilde{R}$ being the curvature of the connection $\tilde{\nabla} = \nabla^{TW} \otimes \text{Id}_{\text{End}(E)} + \text{Id}_{T^*M \otimes \nabla^h}$ on $T^*M \otimes \text{End}(E)$. For convenience, we take $\{e_i\}$ to be a local orthonormal frame of $TM$ such that $e_0 = \xi$ and $e_1, \ldots, e_{2n}$ is a local orthonormal frame of $S$ associated with $L_\eta$. We have

$$\frac{1}{2} \int_M \langle [D^2, (\xi \cdot )] \phi_h, \xi \cdot \phi_h \rangle = -2 \int_M \langle \xi \cdot \mathcal{R}_\xi \phi_h, \xi \cdot \phi_h \rangle = -2 \int_M \langle \mathcal{R}_\xi \phi_h, \phi_h \rangle$$

$$= -\sum_{i=0}^{2n} \int_M \langle \xi \cdot e_i \cdot R(\xi, e_i) \phi_h, \phi_h \rangle = \sum_{i=0}^{2n} \int_M \langle e_i \cdot R(\xi, e_i) \phi_h, \xi \cdot \phi_h \rangle.$$

By $R^{TW}(\xi, -) = 0$ [see (2.1)], we have that

$$-\sum_{i=0}^{2n} \int_M \langle e_i \cdot R(\xi, e_i) \phi_h, \xi \cdot \phi_h \rangle$$

$$= -\sum_{i=1}^{2n} \int_M \left( \langle (R^h(\xi, e_i) \phi_h)(e_i), \phi_h(\xi) \rangle - \langle (R^h(\xi, e_i) \phi_h)(\xi), \phi_h(e_i) \rangle \right)$$

(see [Pe, formula (17)]). Using $R^h = -\frac{1}{2} [\phi_h, \phi_h]$ [see (3.2)], this is equal to

$$-\frac{1}{2} \sum_{i=1}^{2n} \int_M \langle [\phi_h(\xi), \phi_h(e_i)], \phi_h(e_i) \rangle - \langle [\phi_h(\xi), \phi_h(e_i)], \phi_h(\xi) \rangle \rangle.$$


Since $\phi_h$ is a 1-form with values in the self-adjoint endomorphisms of $E$, this is equal to

$$-\sum_{i=1}^{2n} \int_M \langle [\phi_h(\xi), \phi_h(e_i)], [\phi_h(\xi), \phi_h(e_i)] \rangle.$$ 

Thus we obtain the inequality

$$\frac{1}{2} \int_M \langle [D^2, (\xi \cdot)] \phi_h, \xi \cdot \phi_h \rangle \leq 0. \quad (3.3)$$

On the other hand, we can directly compute that

$$\frac{1}{2} \int_M \langle [D^2, (\xi \cdot)] \phi_h, \xi \cdot \phi_h \rangle = 4 \int_M \left( \langle \xi \cdot D(\phi_h), \tilde{\nabla}_\xi \phi_h \rangle + \langle \tilde{\nabla}_\xi \phi_h, \tilde{\nabla}_\xi \phi_h \rangle \right)$$

as done in [Pe, p. 594]. Using $(\xi \cdot) = (\eta \wedge) - i_\xi$ and $D\phi_h = -d\eta \otimes \phi_h(\xi)$, where $i_\xi$ denotes the interior product, together with the fact that $i_\xi d\eta = 0$ [this means that the form $d\eta$ is basic; see (4.1)], we conclude that $\langle \xi \cdot D(\phi_h), \tilde{\nabla}_\xi \phi_h \rangle = 0$. Thus, we have the inequality

$$\frac{1}{2} \int_M \langle [D^2, (\xi \cdot)] \phi_h, \xi \cdot \phi_h \rangle = \int_M \langle \tilde{\nabla}_\xi \phi_h, \tilde{\nabla}_\xi \phi_h \rangle \geq 0. \quad (3.4)$$

Now from (3.3) and (3.4) it follows that $\tilde{\nabla}_\xi \phi_h = 0$.

By the same argument as in [Pe, p. 598], we have that $\phi_h(\xi) = 0$.  \qed

### 4. Basic Vector Bundles

Let $M$ be a compact manifold equipped with a nonsingular foliation $\mathcal{F}$. Then, a differential form $\omega$ on $M$ is called **basic** if for every vector field $X$ on $M$ which is tangent to the leaves of $\mathcal{F}$, the equations

$$i_X \omega = 0 = \mathcal{L}_X \omega \quad (4.1)$$

hold.

We denote by $A^*_B(\mathcal{F})(M)$ the subspace of basic forms in the de Rham complex $A^*(M)$. It is straight-forward to check that $A^*_B(\mathcal{F})(M)$ is a sub-complex of the de Rham complex $A^*(M)$. Denote by

$$H^*_B(\mathcal{F})(M) = \bigoplus_{i \geq 0} H^i_B(\mathcal{F})(M) \quad (4.2)$$

the cohomology of the basic de Rham complex $A^*_B(\mathcal{F})(M)$. Note that there is a natural homomorphism from $H^i_B(\mathcal{F})(M)$ to the $i$-th de Rham cohomology of $M$.

Let $E$ be a complex $C^\infty$ vector bundle over $M$ of rank $r$. This $E$ is said to be **basic** if it has local trivializations with respect to an open covering $M = \bigcup_{\alpha} U_\alpha$ satisfying the condition that each transition function $f_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{C})$ is basic on $U_\alpha \cap U_\beta$, meaning it is constant on the leaves of the foliation $\mathcal{F}$. This condition is equivalent to the condition that $E$ has a flat partial connection in the direction of $\mathcal{F}$.
For a basic vector bundle $E$ over $M$, a differential form $\omega \in A^*(M, E)$ with values in $E$ is called basic if $\omega$ is basic on every $U_\alpha$, meaning $\omega|_{U_\alpha} \in A^*_{B,\pi}(U_\alpha) \otimes \mathbb{C}^r$ for every $\alpha$. Let

$$A^*_{B,\pi}(M, E) \subset A^*(M, E)$$

denote the subspace of basic forms in the space $A^*(M, E)$ of differential forms with values in $E$.

We shall consider any flat vector bundle $(E, \nabla^E)$ over $M$ as a basic vector bundle by local flat frames. Then, $A^*_{B,\pi}(M, E)$ is a sub-complex of the de Rham complex $A^*(M, E)$ equipped with the differential $d_E$ associated to the flat connection $\nabla^E$.

Let $(E, \nabla^E)$ be a flat vector bundle over $M$. For a Hermitian metric $h$ on $E$, consider the canonical decomposition $\nabla^E = \nabla^h + \phi^h$ in (3.1).

**Proposition 4.1.** The following two conditions are equivalent:

- $\phi^h(X) = 0$ for all $X \in T_x\mathcal{F}$ and $x \in M$.
- The Hermitian structure $h$ is basic, meaning $h \in A^0_{B,\pi}(M, E^* \otimes \bar{E}^*)$.

These conditions imply that $\phi^h \in A^1_{B,\pi}(M, \text{End}(E))$.

**Proof.** For local flat frames of $E$ with respect to an open covering $M = \bigcup \alpha U_\alpha$, we can write

$$\phi^h = -\frac{1}{2} df_\alpha f_\alpha^{-1}$$

on each $U_\alpha$ for certain functions $f_\alpha$ on $U_\alpha$ with values in the positive definite Hermitian matrices $Herm^+_r$ with respect to $h$. Then, $\phi^h(X) = 0$ for any $X \in T_x\mathcal{F}$ if and only if each $f_\alpha$ is basic. Thus the proposition follows. \(\square\)

Let $(M, (T^{1,0}, S, I), (\eta, \xi))$ be a compact Sasakian manifold. Then the Reeb vector field $\xi$ defines a 1-dimensional foliation $\mathcal{F}_\xi$ on $M$. It is known that the map $I : TM \rightarrow TM$ associated with the CR-structure $T^{1,0}$ defines a transversely complex structure on the foliated manifold $(M, \mathcal{F}_\xi)$. Furthermore, the closed basic 2-form $d\eta$ is a transversely Kähler structure with respect to this transversely complex structure. Corresponding to the decomposition $S_C = T^{1,0} \oplus T^{0,1}$, we have the bigrading

$$A^r_{B,\mathcal{F}_\xi}(M)_C = \bigoplus_{p+q=r} A^{p,q}(M)$$

as well as the decomposition of the exterior differential $d|_{A^r_{B,\mathcal{F}_\xi}(M)_C} = \partial_\xi + \bar{\partial}_\xi$ on $A^r_{B,\mathcal{F}_\xi}(M)_C$, so that

$$\partial_\xi : A^{p,q}_B(M) \rightarrow A^{p+1,q}_B(M) \quad \text{and} \quad \bar{\partial}_\xi : A^{p,q}_B(M) \rightarrow A^{p,q+1}_B(M).$$
We shall now use transverse Hodge theory \([KT,Ka]\). Consider the usual Hodge star operator

\[
* : A^r(M) \longrightarrow A^{2n+1-r}(M)
\]

associated to the Sasakian metric \(g_\eta\) and the formal adjoint operator

\[
\delta = -*d* : A^r(M) \longrightarrow A^{r-1}(M).
\]

We define the homomorphism

\[
*_\xi : A^r_{B_{\mathcal{F}_\xi}}(M) \longrightarrow A^{2n-r}_{B_{\mathcal{F}_\xi}}(M)
\]

to be \(*_\xi \omega = *(\eta \wedge \omega)\) for \(\omega \in A^r_{B_{\mathcal{F}_\xi}}(M)\). Also define the operators

\[
\delta_\xi = -*_\xi d*_\xi : A^r_{B_{\mathcal{F}_\xi}}(M) \longrightarrow A^{r-1}_{B_{\mathcal{F}_\xi}}(M),
\]

\[
\partial_\xi^* = -*_\xi \overline{\partial}_\xi*_\xi : A^{p,q}_{B_{\mathcal{F}}}(M) \longrightarrow A^{p-1,q}_{B_{\mathcal{F}}}(M),
\]

\[
\overline{\partial}_\xi^* = -*_\xi \partial_\xi*_\xi : A^{p,q}_{B_{\mathcal{F}}}(M) \longrightarrow A^{p,q-1}_{B_{\mathcal{F}}}(M)
\]

and \(\Lambda = -*_\xi (d\eta \wedge *)\). They are the formal adjoints of \(d, \partial_\xi, \overline{\partial}_\xi\) and \((d\eta \wedge)\) respectively for the pairing

\[
A^0_{B_{\mathcal{F}_\xi}}(M) \times A^r_{B_{\mathcal{F}_\xi}}(M) \ni (\alpha, \beta) \longmapsto \int_M \eta \wedge \alpha \wedge *_\xi \beta.
\]

Define the Laplacian operators

\[
\Delta : A^r(M) \longrightarrow A^r(M) \quad \text{and} \quad \Delta_\xi : A^r_{B_{\mathcal{F}_\xi}}(M) \longrightarrow A^r_{B_{\mathcal{F}_\xi}}(M)
\]

by

\[
\Delta = d\delta + \delta d \quad \text{and} \quad \Delta_\xi = d\delta_\xi + \delta d
\]

respectively. For \(\omega \in A^r_{B_{\mathcal{F}_\xi}}(M)\), since the relation \(*\omega = (*_\xi \omega) \wedge \eta\) holds, we have the relation

\[
\delta \omega = \delta_\xi \omega + *(d\eta \wedge *_\xi \omega).
\]

Thus, for \(\omega \in A^1_{B_{\mathcal{F}_\xi}}(M)\), the equality \(\delta_\xi \omega = \delta \omega\) holds, and hence for \(f \in A^0_{B_{\mathcal{F}_\xi}}(M)\), we have that \(\Delta_\xi f = \Delta f\). The usual Kähler identities

\[
[A, \partial_\xi] = -\sqrt{-1} \partial_\xi^* \quad \text{and} \quad [A, \overline{\partial}_\xi] = \sqrt{-1} \overline{\partial}_\xi^*
\]

hold, and these imply that

\[
\Delta_\xi = 2\Delta_\xi' = 2\Delta_\xi''
\]

where \(\Delta_\xi' = \partial_\xi \partial_\xi^* + \partial_\xi^* \partial_\xi\) and \(\Delta_\xi'' = \overline{\partial}_\xi \overline{\partial}_\xi^* + \overline{\partial}_\xi^* \overline{\partial}_\xi\).

Let \(E\) be a complex \(C^\infty\) vector bundle over \(M\). We say that \(E\) is transversely holomorphic if it admits local trivializations with respect to an open covering \(M = \bigcup_{\alpha} U_\alpha\)
such that each transition function $f_{\alpha \beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{C})$ is basic and holomorphic (i.e., $\overline{\partial}_s f_{\alpha \beta} = 0$). For a transversely holomorphic vector bundle $E$ over $M$, define the canonical Dolbeault operator

$$\overline{\partial}_E : A^{p, q}_{B_F}(M, E) \longrightarrow A^{p, q+1}_{B_F}(M, E)$$

satisfying the following two conditions:

$$\overline{\partial}_E (\omega \wedge s) = (\overline{\partial}_s \omega) \wedge s + (-1)^{p+q} \omega \wedge \overline{\partial}_E s$$

and $\overline{\partial}_E \overline{\partial}_E = 0$. Conversely, on a basic complex vector bundle $E$, if we have an operator $\overline{\partial}_E : A^{p, q}_{B_F}(M, E) \longrightarrow A^{p, q+1}_{B_F}(M, E)$ satisfying the above two conditions, then the differential operator $\overline{\partial}_E$ defines a canonical transversely holomorphic structure on $E$ by the Frobenius theorem [Ko, p. 9, Proposition 3.7].

Let $(E, \nabla^E)$ be a flat complex vector bundle over $M$ such that $E$ is equipped with a Hermitian metric $h$. Let

$$\nabla^E = \nabla^h + \phi_h$$

be the canonical decomposition of the connection $\nabla^E$ [see (3.1)]. Then, using the pairing $A^r(M, E) \times A^{2n+1-r}(M, E) \rightarrow A^{2n+1}(M)$ associated with $h$, define the Hodge star operator

$$*_h : A^r(M, E) \rightarrow A^{2n+1-r}(M, E)$$

as well as the formal adjoint operator $(\nabla^h)^* = -*_h \nabla^h *_h$.

Assume the Hermitian structure $h$ to be basic (equivalently, $\phi_h(\xi) = 0$ by Proposition 4.1). Since $\nabla^E_\xi = \nabla^h_\xi$, it follows that the unitary connection $\nabla^h$ restricts to a homomorphism

$$\nabla^h : A^r(M, E) \rightarrow A^{r+1}(M, E).$$

Now, on $A^{p, q}_{B_F}(M, E)$, decompose

$$\nabla^h = \partial_{h, \xi} + \overline{\partial}_{h, \xi}$$

such that $\partial_{h, \xi} : A^{p, q}_{B_F}(M, E) \rightarrow A^{p+1, q}_{B_F}(M, E)$ and $\overline{\partial}_{h, \xi} : A^{p, q}_{B_F}(M, E) \rightarrow A^{p, q+1}_{B_F}(M, E)$. Since $\nabla^h$ is a unitary connection, we have that

$$\overline{\partial}_s h(s_1, s_2) = h(\overline{\partial}_{h, \xi} s_1, s_2) + h(s_1, \overline{\partial}_{h, \xi} s_2)$$

for $s_1, s_2 \in A^{0, 0}_{B_F}(M, E)$. We define the operator

$$*_h, \xi : A^r(M, E) \rightarrow A^{2n-r}_{B_F}(M, E)$$

and the formal adjoint operators

$$(\nabla^h)^*_{\xi} = -*_h, \xi \nabla^h *_{h, \xi} : A^r_{B_F}(M, E) \rightarrow A^{r-1}_{B_F}(M, E),$$
\[ \partial_{\xi}^{\ast} h_{\xi} = - \ast h_{\xi} \partial_{h_{\xi}} h_{\xi} : A_{B_{M}}^{p,q} (M, E) \rightarrow A_{B_{M}}^{p-1,q} (M, E) \] and

\[ \overline{\partial}_{h_{\xi}}^{\ast} = - \ast h_{\xi} \partial_{h_{\xi}} h_{\xi} : A_{B_{M}}^{p,q} (M, E) \rightarrow A_{B_{M}}^{p,q-1} (M, E), \]
as well we \[ \Lambda_{h} := - \ast h_{\xi} (d \eta_{\xi}) \ast h_{\xi} \] in the same way as above. We now have the Kähler identities

\[ [\Lambda_{h}, \partial_{h_{\xi}}] = -\sqrt{-1 \partial_{h_{\xi}}^{\ast}} \] and \[ [\Lambda_{h}, \overline{\partial}_{h_{\xi}}] = \sqrt{-1 \overline{\partial}_{h_{\xi}}^{\ast}}. \]

**Theorem 4.2.** Let \( (M, (T^{1,0}, S, I), (\eta, \xi)) \) be a compact Sasakian manifold and \( (E, \nabla^{E}) \) a flat complex vector bundle over \( M \) with a Hermitian metric \( h \). Then the following two conditions are equivalent:

- The Hermitian structure \( h \) is harmonic, i.e., \( (\nabla^{h})^{\ast} \phi_{h} = 0 \).
- The Hermitian structure \( h \) is basic \( \iff \phi_{h}(\xi) = 0 \) and implying that \( \phi_{h} \in A_{B_{M}}^{1,0} (M, \text{End}(E)) \)

by Proposition 4.1, and for the decomposition

\[ \phi_{h} = \theta_{h_{\xi}}^{1,0} + \theta_{h_{\xi}}^{0,1} \]

with \( \theta_{h_{\xi}}^{1,0} \in A_{B_{M}}^{1,0} (M, \text{End}(E)) \) and \( \theta_{h_{\xi}}^{0,1} \in A_{B_{M}}^{0,1} (M, \text{End}(E)) \). \[ \overline{\partial}_{h_{\xi}} \overline{\partial}_{h_{\xi}} = 0, \quad [\theta_{h_{\xi}}^{1,0}, \theta_{h_{\xi}}^{1,0}] = 0 \quad \text{and} \quad \overline{\partial}_{h_{\xi}} \overline{\partial}_{h_{\xi}}^{1,0} = 0. \]

**Proof.** First suppose that \( h \) is basic. Then by the relation similar to the relation between \( \delta \) and \( \delta_{\xi} \), we have that

\[ (\nabla^{h})^{\ast} \phi_{h} = (\nabla^{h})^{\ast} \phi_{h}. \]

By the same computation as in [Co, p. 376], we have that

\[ (\nabla^{h})^{\ast} \phi_{h} = \sqrt{-1} \Lambda_{h} (\partial_{h_{\xi}} - \overline{\partial}_{h_{\xi}}) \phi_{h} = 2 \partial_{h_{\xi}}^{\ast} \theta_{h_{\xi}}^{1,0}. \]

Thus, \( h \) is harmonic if and only if \( \overline{\partial}_{h_{\xi}} \theta_{h_{\xi}}^{1,0} \) is primitive, meaning \( \Lambda_{h} \overline{\partial}_{h_{\xi}} \theta_{h_{\xi}}^{1,0} = 0 \). From this it follows that the second condition in the theorem implies the first one.

To prove the converse, suppose that \( h \) is harmonic. Then, by Theorem 3.2, \( h \) is basic. By the flatness of the connection \( \nabla^{E} \), we have that \( (\nabla^{h})^{2} = -\frac{1}{2} \left[ \phi_{h}, \phi_{h} \right] \) and \( \nabla^{h} \phi_{h} = 0 \). Now using \( (\nabla^{h})^{2} = -\frac{1}{2} \left[ \phi_{h}, \phi_{h} \right] \) it follows that

\[ \partial_{h_{\xi}} \overline{\partial}_{h_{\xi}} = -[\theta_{h_{\xi}}^{1,0}, \theta_{h_{\xi}}^{1,0}] \quad \text{and} \quad \overline{\partial}_{h_{\xi}} \overline{\partial}_{h_{\xi}} = -[\theta_{h_{\xi}}^{0,1}, \theta_{h_{\xi}}^{0,1}]. \]

Now \( \nabla^{h} \phi_{h} = 0 \) implies that

\[ \partial_{h_{\xi}} \theta_{h_{\xi}}^{1,0}, \overline{\partial}_{h_{\xi}} \theta_{h_{\xi}}^{0,1} \quad \text{and} \quad \overline{\partial}_{h_{\xi}} \theta_{h_{\xi}}^{1,0} + \partial_{h_{\xi}} \theta_{h_{\xi}}^{0,1} = 0. \]

We have (cf. the proof of [Co, Theorem 5.1])

\[ \partial_{\xi} \overline{\partial}_{\xi} h(\theta_{h_{\xi}}^{1,0}, \theta_{h_{\xi}}^{1,0}) = h (\partial_{\xi} \overline{\partial}_{\xi} \theta_{h_{\xi}}^{1,0}, \theta_{h_{\xi}}^{1,0}) + h (\overline{\partial}_{h_{\xi}} \theta_{h_{\xi}}^{1,0}, \overline{\partial}_{\xi} \theta_{h_{\xi}}^{1,0}) \]
\[ -h(\partial_{h,\xi}\partial_{h,\xi}\theta^{0,1}_{h,\xi}, \theta^{1,0}_{h,\xi}) + h(\partial_{h,\xi}\theta^{1,0}_{h,\xi}, \partial_{h,\xi}\theta^{1,0}_{h,\xi}) \]
\[ = h \left( [\theta^{1,0}_{h,\xi}, \theta^{1,0}_{h,\xi}], [\theta^{0,1}_{h,\xi}, \theta^{1,0}_{h,\xi}] \right) + h \left( \partial_{h,\xi}\theta^{1,0}_{h,\xi}, \partial_{h,\xi}\theta^{1,0}_{h,\xi} \right) \]
\[ = -h \left( [\theta^{0,1}_{h,\xi}, \theta^{1,0}_{h,\xi}], [\theta^{1,0}_{h,\xi}, \theta^{0,1}_{h,\xi}] \right) + h \left( \partial_{h,\xi}\theta^{1,0}_{h,\xi}, \partial_{h,\xi}\theta^{1,0}_{h,\xi} \right). \]

Since \( \partial_{h,\xi}\theta^{1,0}_{h,\xi} \) is primitive by the above argument, using the Lefschetz decomposition of basic forms corresponding to the transversely Kähler form \( d\eta \), and integrating the wedge product of this equation and \( (d\eta)^{n-2} \wedge \eta \), from the Stokes theorem, we obtain that
\[ 0 = -C_1 \int_M [\partial_{h,\xi}\theta^{1,0}_{h,\xi}] - C_2 \int_M [[\theta^{1,0}_{h,\xi}, \theta^{1,0}_{h,\xi}]]. \]
for some positive constant \( C_1 \) and \( C_2 \). Consequently, we have
\[ [\theta^{1,0}_{h,\xi}, \theta^{1,0}_{h,\xi}] = 0 \quad \text{and} \quad \partial_{h,\xi}\theta^{1,0}_{h,\xi} = 0. \]

Using \( \partial_{h,\xi}\partial_{h,\xi} = -[\theta^{1,0}_{h,\xi}, \theta^{1,0}_{h,\xi}] \) it follows that \( \partial_{h,\xi}\partial_{h,\xi} = 0 \) and hence the theorem is proved. \( \square \)

5. Basic Higgs Bundles

Let \((M, (T_{1,0}, S, I), (\eta, \xi))\) be a compact Sasakian manifold. As in (4.2), \( H_{B_{\mathcal{F}_\xi}}^*(M) \) is the cohomology of the basic de Rham complex \( A_{B_{\mathcal{F}_\xi}}^*(M) \). By the Kähler identities on \( A_{B_{\mathcal{F}_\xi}}^{p,q} (M) \), as in the usual Kähler case, we have the canonical Hodge decomposition
\[ H_{B_{\mathcal{F}_\xi}}^r (M) \otimes \mathbb{C} = \bigoplus_{p+q=r} H_{B_{\mathcal{F}_\xi}}^{p,q} (M). \]

Let \( E \) be a complex basic vector bundle over \( M \). Consider a connection operator
\[ \nabla : A_{B_{\mathcal{F}_\xi}}^* (M, E) \rightarrow A_{B_{\mathcal{F}_\xi}}^{*+1} (M, E) \]
satisfying the equation
\[ \nabla(\omega s) = (d\omega)s + (-1)^{r} \omega \wedge \nabla s \]
for \( \omega \in A_{B_{\mathcal{F}_\xi}}^r (M) \) and \( s \in A_{B_{\mathcal{F}_\xi}}^0 (M, E) \). Let
\[ R^\nabla = \nabla^2 \in A_{B_{\mathcal{F}_\xi}}^2 (M, \text{End}(E)) \]
be the curvature of \( \nabla \). For any \( 1 \leq i \leq n \), Define \( c_{i, B_{\mathcal{F}_\xi}} (E, \nabla) \in A_{B_{\mathcal{F}_\xi}}^{2i} (M) \) by
\[ \det \left( I - \frac{R^\nabla}{2\pi \sqrt{-1}} \right) = 1 + \sum_{i=1}^{n} c_{i, B_{\mathcal{F}_\xi}} (E, \nabla). \]

Then, as the case of usual Chern–Weil theory, the cohomology class
\[ c_{i, B_{\mathcal{F}_\xi}} (E) \in H_{B_{\mathcal{F}_\xi}}^{2i} (M) \]
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of each \(c_{i, B_{F_{\xi}}}(E, \nabla)\) is independent of the choice of the connection \(\nabla\) taking \(A^*_{B_{F_{\xi}}}(M, E)\) to \(A^{*+1}_{B_{F_{\xi}}}(M, E)\). If \(E\) is a transversely holomorphic vector bundle, just as in the case of Chern classes of holomorphic vector bundles over compact Kähler manifolds, we have that \(c_{i, B_{F_{\xi}}}(E) \in H^{i}_{B_{F_{\xi}}}(M)\).

A basic Higgs bundle over \((M, (T^{1,0}, S, I), (\eta, \xi))\) is a pair \((E, \theta)\) consisting of a transversely holomorphic vector bundle \(E\) and a section \(\theta \in A^{2,0}_{B_{F_{\xi}}}(M, \text{End}(E))\) satisfying the following two conditions:

\[\overline{\partial}_E \theta = 0 \quad \text{and} \quad \theta \wedge \theta = 0.\]

This section \(\theta\) is called a Higgs field on \(E\). Note that \(\theta \wedge \theta \in A^{2,0}_{B_{F_{\xi}}}(M, \text{End}(E))\).

For a basic Higgs bundle \((E, \theta)\), we define the operator

\[D''_{E, \theta} = \overline{\partial}_E + \theta : A^*_{B_{F_{\xi}}}(M, E) \longrightarrow A^{*+1}_{B_{F_{\xi}}}(M, E).\]

Let \(h\) be a Hermitian metric on \(E\). Assume that \(h\) is basic. Define \(\overline{\partial}_h \in A^{0,1}_{B_{F_{\xi}}}(M, \text{End}(E))\) by

\[(\theta(e_1), e_2) = (e_1, \overline{\partial}_h(e_2)) \quad (5.1)\]

for \(e_1, e_2 \in E\).

Let \(\nabla\) be a unitary connection on \(E\) preserving \(h\) associated to \(h\) such that \(\nabla\) actually restricts as \(\nabla : A^*_{B_{F_{\xi}}}(M, E) \longrightarrow A^{*+1}_{B_{F_{\xi}}}(M, E)\). Define the connection

\[D = \nabla + \theta + \overline{\partial}_h,\]

and consider the curvature \(R^D = D^2\) of \(D\). Then, by the assumptions on \(\theta\) and \(h\), we have that

\[R^D \in A^{2}_{B_{F_{\xi}}}(M, \text{End}(E)).\]

Define the degree of \(E\) to be

\[\deg(E) = \frac{\sqrt{-1}}{2\pi} \int_M \text{Tr}(\Lambda R^D).\]

Note that we have

\[\deg(E) = \int_M c_{1, B_{F_{\xi}}}(E) \wedge d\eta \wedge \eta,\]

and hence \(\deg(E)\) depends only on \(E\).

We define the canonical (Chern) connection \(\nabla^h\) on the transversely holomorphic Hermitian bundle \((E, h)\) in the following way. Take local basic holomorphic frames
$e^\alpha_1, \ldots, e^\alpha_n$ of $E$ with respect to an open covering $M = \bigcup_\alpha U_\alpha$. For the Hermitian matrices $H_\alpha = (h^\alpha_{i,j})$ with $h^\alpha_{i,j} = h(e^\alpha_i, e^\alpha_j)$, we define

$$\nabla^h = d + H^{-1}_\alpha \partial_{\xi} H_\alpha$$  \hfill (5.2)$$
on each $U_\alpha$.

Let us consider the canonical connection $D^h = \nabla^h + \theta + \bar{\theta}_h$ on $E$. We also define the operator $\partial E, h : A^{p,q}_{B,F,\xi}(M,E) \to A^{p+1,q}_{B,F,\xi}(M,E)$ such that

$$\partial E, h = \partial_{\xi} + H^{-1}_\alpha \partial_{\xi} H_\alpha$$
on each $U_\alpha$; so $\partial E, h$ is the $(1,0)$-component of $\nabla^h$.

We now define the stable Higgs bundles (cf. [BH,BS]). Denote by $O_{B,F,\xi}$ the sheaf of basic holomorphic functions on $M$, and for a transversely holomorphic vector bundle $E$ on $M$, denote by $O_{B,F,\xi}(E)$ the sheaf of basic holomorphic sections of $E$. Consider $O_{B,F,\xi}(E)$ as a coherent $O_{B,F,\xi}$-sheaf.

For a basic Higgs bundle $(E, \theta)$, a sub-Higgs sheaf of $(E, \theta)$ is a coherent $O_{B,F,\xi}$-subsheaf $V$ of $O_{B,F,\xi}(E)$ such that $\theta(V) \subset V \otimes \Omega_{B,F,\xi}$, where $\Omega_{B,F,\xi}$ is the sheaf of basic holomorphic 1-forms on $M$. By [BH, Proposition 3.21], if $rk(V) < rk(E)$ and $O_{B,F,\xi}(E)/V$ is torsion-free, then there is a transversely analytic sub-variety $S \subset M$ of complex co-dimension at least 2 such that $V$ is given by a transversely holomorphic sub-bundle $V \subset E$; the degree $deg(V)$ can be defined by integrating on this complement $M \setminus S$.

**Definition 5.1.** We say that a basic Higgs bundle $(E, \theta)$ is stable if $E$ admits a basic Hermitian metric and for every sub-Higgs sheaf $V$ of $(E, \theta)$ such that $rk(V) < rk(E)$ and $O_{B,F,\xi}(E)/V$ is torsion-free, the inequality

$$\frac{deg(V)}{rk(V)} < \frac{deg(E)}{rk(E)}$$

holds.

A basic Higgs bundle $(E, \theta)$ is called polystable if

$$(E, \theta) = \bigoplus_{i=1}^k (E_i, \theta_i),$$

where each $(E_i, \theta_i)$ is a stable basic Higgs bundle with

$$\frac{deg(E_i)}{rk(E_i)} = \frac{deg(E)}{rk(E)}.$$  

**Theorem 5.2.** For a stable basic Higgs bundle $(E, \theta)$ over a compact Sasakian manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$, there exists a basic Hermitian metric $h$ on $E$ such that

$$\Lambda R^{Dh} = 0,$$

where $R^{Dh}$ is the trace-free part of the curvature $R^{Dh}$ of the canonical connection $D^h$ associated to $h$.  

Theorem 5.2 will be proved in the next section.

**Proposition 5.3.** Let \((E, \theta)\) be a basic Higgs bundle over a compact Sasakian manifold \((M, (T^{1,0}, S, I), (\eta, \xi))\). Suppose that \(h\) is a basic Hermitian metric on \(E\) such that \(\Lambda R^{D_h} = 0\). If \(c_{1, B_{F_\xi}}(E) = 0\) and \(c_{2, B_{F_\xi}}(E) = 0\), then the connection \(D^h\) is flat.

**Proof.** The arguments in the proof of [Si1, Proposition 3.4] for the usual case go through after using the Riemann bilinear relations for basic forms on \(M\). \(\square\)

### 6. Proof of Theorem 5.2

#### 6.1. Preliminaries

The proof of Theorem 5.2 will closely follow the proof of [Si1, Theorem 1].

**Proposition 6.1.** Let \((M, (T^{1,0}, S, I), (\eta, \xi))\) be a compact Sasakian manifold. Let \(B \in \mathbb{R}_{>0}\) be a positive number. There exist positive constants \(C_1(B), C_2(B)\) and \(C_3(B)\) depending on \(B\) and an increasing function \(a: [0, +\infty) \rightarrow [0, +\infty)\) with \(a(0) = 0, a(x) = x\) for \(x > 1\), such that for any positive basic function \(f \in A^0_{B_{F_\xi}}(M)\) on \(M\) satisfying \(\Delta f \leq B\), the following two inequalities hold:

\[
\sup_M f \leq C_1(B) + C_2(B) \int_M f
\]

and

\[
\sup_M f \leq C_3(B) a \left( \int_M f \right).
\]

**Proof.** In view of \(\Delta f = \Delta f\) for \(f \in A^0_{B_{F_\xi}}(M)\), it suffices to prove the two inequalities for \(f \in A^0_{B_{F_\xi}}(M)\) satisfying \(\Delta f \leq B\). We next note that this is already proved in [Si1, Proposition 2.1] and [Do2]. \(\square\)

Let \((E, \theta)\) be a basic Higgs bundle, equipped with a basic Hermitian metric \(h\), over a compact Sasakian manifold \((M, (T^{1,0}, S, I), (\eta, \xi))\). We define the operator

\[
D^{t,h}_{E,\theta} = \partial_{E,h} + \overline{\theta}_h : A^*_0(\mathcal{B}_{F_\xi}) \rightarrow A^{*+1}_0(\mathcal{B}_{F_\xi}),
\]

where \(\partial_{E,h}\) and \(\overline{\theta}_h\) are defined as in (5.1) and (5.2) respectively. As in [Si1, Lemma 3.1], we have the following formulas:

1. \[
\sqrt{-1}[\lambda, D'^{t,h}_{E,\theta}] = (D'^{t,h}_{E,\theta})^* = -(D'^{t,h}_{E,\theta})^*,
\]

where \((D'^{t,h}_{E,\theta})^* = -*_{h,\xi} \overline{\theta}_h *_{h,\xi} + *_{h,\xi} \theta *_{h,\xi}\) and

\[
(D''^{t,h}_{E,\theta})^* = -*_{h,\xi} \partial_{E,h} *_{h,\xi} + *_{h,\xi} \overline{\theta}_h *_{h,\xi} + \theta *_{h,\xi} \overline{\theta}_h *_{h,\xi};
\]

note that they are the formal adjoints of \(D'^{t,h}_{E,\theta}\) and \(D''^{t,h}_{E,\theta}\) respectively for the \(L^2\) inner product \(A^*_{B_{F_\xi}}(M, E) \times A^*_0(\mathcal{B}_{F_\xi}) \ni (\alpha, \beta) \mapsto \int_M \langle \alpha, \beta \rangle\).
2. For self-adjoint basic sections \( \sigma, \tau \in A^0_{B^g_{\xi}}(M, \text{end}(E)) \),

\[
|D''_{E,\theta}(\sigma)\tau|_h^2 = -\sqrt{-1}\text{Tr}(D''_{E,\theta}(\sigma)\tau^2 D'_{E,\theta}(\sigma)).
\]

3. If \( k = h\sigma \) for a basic positive self-adjoint section \( \sigma \in A^0_{B^g_{\xi}}(M, \text{end}(E)) \), then

\[
D'_{E,\theta} = D'_{E,\theta} + \sigma^{-1} D'_{E,\theta}(\sigma)
\]

and

\[
\Delta'_h(\sigma) = h\sqrt{-1}(\Lambda R^{Dk} - \Lambda R^{Db}) + \sqrt{-1}\Lambda D''_{E,\theta}(\sigma)\sigma^{-1} D'_{E,\theta}(\sigma),
\]

where \( \Delta'_h = (D'_{E,\theta})^* D'_{E,\theta} = \sqrt{-1}\Lambda D''_{E,\theta} D'_{E,\theta} \).

4. Also,

\[
\Delta_\xi \log \text{Tr}(\sigma) \leq 2(|\Lambda R^{Dk}|_h + |\Lambda R^{Db}|_h).
\]

(6.1)

6.2. Donaldson’s functional. Let \((E, \theta)\) be a basic Higgs vector bundle over the Sasakian manifold \((M, (T^{1,0}, S, I), (\eta, \xi))\). Fix a basic Hermitian metric \( h \) on \( E \). Denote by \( S(E) \) the smooth vector bundle of self-adjoint endomorphisms of \( E \) corresponding to \( h \). For smooth functions \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) and \( \Psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), we define the maps \( \phi : S(E) \rightarrow S(E) \) and \( \Psi : S(E) \rightarrow S(\text{End}E) \) as in [Si1, p. 880]. Consider the \( L^p \)-completion \( L^p_{B^g_{\xi}}(S(E)) \) of the space of basic sections of \( S(E) \), and also the Sobolev space \( L^{p,1}_{B^g_{\xi}}(S(E)) \) with the norm \( \|s\|_{L^{p,1}_{B^g_{\xi}}(S(E))} = \|s\|_{L^p} + \|D'_{E,\theta} s\|_{L^p} \) for any section \( s \) of \( S(E) \). As done in [Si1, Proposition 4.1], we can extend \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) and \( \Psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) to continuous maps

\[
\phi : L^p_{B^g_{\xi}}(S(E)) \rightarrow L^p_{B^g_{\xi}}(S(E)),
\]

\[
\Psi : L^p_{B^g_{\xi}}(S(E)) \rightarrow \text{Hom}(L^p_{B^g_{\xi}}(\text{End}E), L^q_{B^g_{\xi}}(\text{End}E))
\]

and \( \phi : L^{p,1}_{B^g_{\xi}}(S(E)) \rightarrow L^{q,1}_{B^g_{\xi}}(S(E)) \) for \( q < p \).

Denote by \( \mathcal{P} \) the space of all basic Hermitian metrics on \( E \). We parametrize \( \mathcal{P} \) by \( A^0_{B^g_{\xi}}(S(E)) \) as \( A^0_{B^g_{\xi}}(S(E)) \ni \sigma \mapsto h \exp(\sigma) \in \mathcal{P} \). For \( h, h' \in \mathcal{P} \) with \( h' = h \exp(s) \), define

\[
M(h, h') = \sqrt{-1} \int_M \text{Tr}(s \Lambda R^{Db}) + \int_M \left\langle \Psi_1(s)(D'_{E,\theta}s), D''_{E,\theta}s \right\rangle_h,
\]

where \( \Psi_1 : S(E) \rightarrow S(\text{End}E) \) is defined by the function

\[
\Psi_1(\lambda_1, \lambda_2) = \frac{e^{\lambda_2-\lambda_1} - (\lambda_2 - \lambda_1) - 1}{(\lambda_2 - \lambda_1)^2}.
\]

By the same proof as of [Si1, Proposition 5.1], for all \( h, h', h'' \in \mathcal{P} \), the identity

\[
M(h, h') + M(h', h'') = M(h, h'')
\]

is obtained.

We have the following important estimate as in [Si1, Proposition 5.3].
Proposition 6.2. Fix a positive number $B$. Suppose a basic Hermitian metric $h$ on $E$ satisfies the inequality $\sup_M |\Delta R^{Dh}| \leq B$. If $(E, \theta)$ is a stable basic Higgs bundle, then there are positive constants $C_1$ and $C_2$ such that
\[
\sup_M |s| \leq C_1 + C_2 M(h, h \exp(s))
\]
for every $s \in A_B^0(S(E))$ with $\text{Tr}(s) = 0$ and $\sup_M |\Delta R^{Dh_{\exp(s)}}| \leq B$.

Proposition 6.2 will be proved after Theorem 6.4.

Definition 6.3. Let $E$ be a transversely holomorphic vector bundle, over a compact Sasakian manifold $M$, equipped with a basic Hermitian metric $h$. A transversely weakly holomorphic subbundle of $E$ is $\Pi \in L^{2,1}_{\overline{B}}(S(E))$ satisfying the following two conditions
\[
\Pi = \Pi^2 \quad \text{and} \quad (\text{Id}_E - \Pi)\overline{\partial}_{\text{End}(E)}(\Pi) = 0.
\]

In [BH, Theorem 5.7], the following result is proved.

Theorem 6.4. [BH] Let $\Pi$ be a transversely weakly holomorphic subbundle of $E$. Then there is a transversely coherent sheaf $V$ and a transverse analytic subset $S \subset M$ such that the following three hold:
1. The complex codimension of $S$ in $M$ is at least two.
2. The restriction of $\Pi$ to $M \setminus S$ is smooth and defines a transversely holomorphic subbundle $V \subset E|_{M \setminus S}$.
3. The restriction of $V$ to $M \setminus S$ is the sheaf of basic holomorphic sections of $V$. Moreover these properties imply that $O_{\overline{\mathcal{F}}}(E)/V$ is torsion-free.

Now we can prove Proposition 6.2 in the same way as done for the proof of [Si1, Proposition 5.3].

Proof of Proposition 6.2. For a section $s$ as in the statement of the proposition, by the inequality
\[
\Delta_\xi \log \text{Tr}(\sigma) \leq 2(|\Delta R^{Dh}|_k + |\Delta R^{Dh}|_h)
\]
in (6.1) we have that $\Delta_\xi |s| \leq 4B$, and hence using Proposition 6.1 it follows that $\sup_M |s| \leq C_1 + C_2 \|s\|_{L^1}$.

Assume that the estimate in the statement of the proposition does not hold in the following sense. We can find a sequence $s_i$ of basic sections of $S(E)$ with $\text{Tr}(s_i) = 0$ such that $\lim_{i \to \infty} \|s_i\|_{L^1} = +\infty$ and $\|s_i\|_{L^1} \geq M(h, h \exp(s_i))$.

Then, by the same arguments given between [Si1, Lemma 5.4] and [Si1, Lemma 5.7], we have a transversely weakly holomorphic subbundle $\Pi$ of $E$ such that $(\text{Id}_E - \Pi)\theta(\Pi) = 0$, and the inequality
\[
\frac{\text{deg}(\Pi)}{\text{Tr}(\Pi)} \geq \frac{\text{deg}(E)}{\text{rk}(E)}
\]
holds, where
\[
\text{deg}(\Pi) = \sqrt{-1} \int_M \text{Tr}(\Pi \Delta R^{Dh}) - \int_M |D_{E,\theta}^\pi \Pi|^2.
\]
We have a sub-Higgs sheaf $V$ of $(E, \theta)$ satisfying the properties in Theorem 6.4. We conclude that $\text{deg}(\Pi) = \text{deg}(V)$ as in [Si1, Lemma 3.2], and this contradicts the given stability condition. This completes the proof of Proposition 6.2 \[\Box\]
6.3. The heat equation on Kähler cone. For a compact Sasakian manifold

\[(M, (T^{1,0}, S, I), (\eta, \xi)),\]

consider the cone \(C(M) = M \times \mathbb{R}^{>0}\) and define the real 2-form \(\omega \in A^2(C(M))\) to be

\[\omega = 2rd\eta \wedge \eta + r^2d\eta;\]

also, define the bundle homomorphism \(J : TC(M) \rightarrow TC(M)\) by

- \(J(X) = I(X)\) for \(X \in S\),
- \(J\left(r \frac{\partial}{\partial r}\right) = -\xi\), and
- \(J(\xi) = r \frac{\partial}{\partial r}\), where \(r\) is the parameter of \(\mathbb{R}^{>0}\).

Then, the pair \((\omega, J)\) is a Kähler structure on the complex manifold \(C(M)\). Consider the real 1-dimensional foliation \(\overline{\mathcal{F}}_\xi\) on \(C(M)\) generated by \(\xi\). Denote by \(A\) the 1-parameter group of automorphisms of the Kähler manifold \(C(M)\) corresponding to \(\overline{\mathcal{F}}_\xi\). Then the space

\[A^\ast(C(M))^A \subset A^\ast(C(M))\]

of \(A\)-invariant differential forms on \(C(M)\) contains the basic de Rham complex \(A^\ast_{\overline{\mathcal{F}}_\xi}(C(M))\). In particular, we have \(A^0(C(M))^A = A^0_{\overline{\mathcal{F}}_\xi}(C(M))\). For a complex basic vector bundle \(E\) over \(C(M)\), we can naturally define the \(A\)-action on \(A^\ast(C(M), E)\) so that for \(a \in A\), \(\omega \in A^r(C(M))\) and \(s \in A^0_{\overline{\mathcal{F}}_\xi}(C(M), E)\), we have \(a(\omega \otimes s) = (a^\ast \omega) \otimes s\).

Let \((E, \theta)\) be a basic Higgs bundle over \((M, (T^{1,0}, S, I), (\eta, \xi))\). Consider the pair \((\tilde{E}, \tilde{\theta})\) defined by the pull back of \((E, \theta)\) using the projection

\[\pi_0 : C(M) \rightarrow M\]

defined by \((x, t) \mapsto x\). Then \((\tilde{E}, \tilde{\theta})\) is a usual Higgs bundle over the Kähler manifold \((C(M), \omega, J)\). Consider the operator

\[\tilde{D}'' = \overline{\alpha}_E + \theta : A^\ast(C(M), \tilde{E}) \rightarrow A^{\ast+1}(C(M), \tilde{E}),\]

where \(\overline{\alpha}_E\) is the usual Dolbeault operator for the holomorphic vector bundle \(E\). Let \(\tilde{h}\) be a Hermitian metric on \(\tilde{E}\). Assume that \(\tilde{h}\) is basic, i.e., \(\tilde{h} \in A^0_{\overline{\mathcal{F}}_\xi}(C(M), \tilde{E}^* \otimes \tilde{E}^*)\).

Then, we have the operator

\[\partial_{E, \tilde{h}} : A^{p,q}(C(M), \tilde{E}) \rightarrow A^{p,q+1}(C(M), \tilde{E})\]

such that \(\overline{\alpha}_E + \partial_{E, \tilde{h}}\) is the canonical Chern connection for the Hermitian holomorphic vector bundle \((\tilde{E}, \tilde{h})\). Define \(\overline{\partial}_{\tilde{h}} \in A^{0,1}(C(M), \text{End}(\tilde{E}))\) by

\[h(\overline{\partial}_1 e_1, e_2) = \tilde{h}(e_1, \overline{\partial}_{\tilde{h}} e_2)\]

and also define the operator \(\tilde{D}''_{\tilde{h}} = \partial_{E, \tilde{h}} + \overline{\partial}_{\tilde{h}} : A^\ast(C(M), \tilde{E}) \rightarrow A^{\ast+1}(C(M), \tilde{E})\). The curvature of the connection \(\tilde{D}'' + \tilde{D}''_{\tilde{h}}\) will be denoted by \(\tilde{R}_{\tilde{h}}\).
Denote by $\tilde{\Lambda}$ the formal adjoint of the Lefschetz operator associated with the Kähler form $\omega$ on $C(M)$. Consider the heat equation

$$
\tilde{h}^{-1}_t \frac{d\tilde{h}_t}{dt} = -\sqrt{-1} \Lambda \tilde{R} \tilde{h}_t\perp,
$$

where $\tilde{R} \tilde{h}_t\perp$ is the trace-free part of the curvature $\tilde{R} \tilde{h}_t$.

Fix a basic Hermitian metric $h_0$ on the basic holomorphic bundle $E$ over the Sasakian manifold $M$. Let $\tilde{h}_0 = \pi_0^* h_0$ be the pull-back of $h_0$ to a Hermitian metric on $\tilde{E} := \pi_0^* E$, where $\pi_0$ is the projection in (6.2).

Write $\tilde{h}_0^{-1} \tilde{h}_t = \sigma_t$. Then the equation in (6.3) becomes

$$
\left( \frac{d}{dt} + \Delta'_{h_0} \right) \sigma_t = -\sqrt{-1} \sigma_t \Lambda \tilde{R} \sigma_t\perp + \sqrt{-1} \Lambda \tilde{D}''(\sigma_t) \sigma_t^{-1} \tilde{D}'_{h_0}(\sigma_t),
$$

where $\Delta'_{h_0}$ is the Laplacian operator of $\tilde{D}'_{h_0}$. Recall that $A$ is a group of automorphisms of the Kähler manifold $C(M)$. Since the Hermitian metric $h_0$ on $E$ is basic, the action of $A$ on $A^0(C(M), \text{End}(\tilde{E}))$ commutes with the operators $\tilde{D}'$, $\tilde{D}'_{h_0}$, $\Lambda$ and $\Delta'_{h_0}$, and we have that

$$
\tilde{\Lambda} \tilde{R} \tilde{h}_0 \in A^0_{\bar{g}_{\tilde{\mathcal{F}}}}(C(M), \text{End}(\tilde{E})).
$$

Thus, the set of solutions of the heat equation (6.3) is invariant under the action of $A$.

The positive function $r^2$ on $C(M)$ is plurisubharmonic, because $\sqrt{-1} \Lambda \partial \bar{\partial} r^2 = \omega$. Just as the results in [Si1, Section 6] are derived using the arguments in [Ha,Do1], we have the following.

**Theorem 6.5.** Let $\epsilon$ be a positive real number. There exists a unique solution $\tilde{h}$, defined for all time $(0, +\infty)$, of the heat equation

$$
\tilde{h}^{-1}_t \frac{d\tilde{h}_t}{dt} = -\sqrt{-1} \tilde{R} \tilde{h}_t\perp
$$

on the compact manifold $M \times [1, 1 + \epsilon]$ with boundary satisfying

$$
\det(\tilde{h}_0) = \det(\tilde{h}_t), \quad \tilde{h}_t|_{\partial} = \tilde{h}_0
$$

together with the Neumann boundary condition $\frac{\partial \tilde{h}}{\partial r}|_{r=1,1+\epsilon} = 0$.

6.4. **Proof of Theorem 5.2.** Consider the solution $\tilde{h}_t$ in Theorem 6.5. It was observed above that the set of solutions of the heat equation is $A$-invariant. Therefore, we conclude that $\tilde{h}_t$ is basic by the uniqueness property. Define the Hermitian metrics $h_t$ on $E$ by the pull-backs of $\tilde{h}_t$ for the embedding $M \rightarrow M \times [1, 1 + \epsilon]$ determined by $x \mapsto (x, 1)$. In view of the Neumann boundary condition $\frac{\partial h}{\partial r}|_{r=1,1+\epsilon} = 0$ in Theorem 6.5, we conclude that the pull-back, by this embedding, of the canonical Chern connection $\nabla_{\tilde{E}} + \partial_{E,\tilde{h}_t}$, on $(\tilde{E}, \tilde{h}_t)$, is identified with the canonical Chern connection $\nabla_{h_t}$ on $E$, and moreover the pull-back of $(\tilde{\Lambda} \tilde{R} \tilde{h}_t\perp)$ is identified with $\Lambda R^{b_t\perp}$. Thus, $h_t$ satisfies the basic heat equation

$$
h_t^{-1} \frac{d h_t}{dt} = -\sqrt{-1} \Lambda R^{b_t\perp}.
By the formulas in Sect. 6.1, for $h_{t_0}^{-1} h_t = \sigma_t$, this equation can be written as

$$
\left( \frac{d}{dt} + \Delta_{h_0} \right) \sigma_t = -\sqrt{-1} \sigma_t \Lambda R^{h_0 \perp} + \sqrt{-1} \Lambda D''(\sigma_t) \sigma_t^{-1} D'_{h_0}(\sigma_t).
$$

By the same proof of [Si1, Lemma 7.1], we have the formula

$$
\frac{d}{dt} M(h_0, h_t) = -\int_M |\Lambda R^{D_{h_t \perp}}|^2_{h_t}.
$$

Now we assume that the Higgs bundle $(E, \theta)$ is stable. Applying Proposition 6.2, as done in [Si1, p. 895], we can take a sequence $\{t_i\}$ of time instances, with $t_i \to +\infty$, such that

$$
\lim_{i \to +\infty} \int_M |\Lambda R^{D_{h_{t_i} \perp}}|^2_{h_{t_i}} = 0
$$

and $h_{t_i} \to h_\infty$ weakly in $L_1$. By the basic Sobolev embedding theorem [BH, Theorem 2.6], this $h_{t_i}$ is a Cauchy sequence in $L^1$. For a positive number $B$ such that $|\Lambda R^{D_{h_0}}| \leq B$, by the inequality $\Delta_{\xi} \log \text{Tr}(\sigma) \leq 2(|\Lambda R^{D_{h_0}}|_k + |\Lambda R^{D_{h_0}}|_h)$ in (6.1), we have that

$$
\Delta_{\xi} \log \text{Tr}(h_{t_i}^{-1} h_{t_j}) \leq 2B,
$$

and hence Proposition 6.1 implies that $\log \text{Tr}(h_{t_i} h_{t_j}) \to 0$ in $C^0$. Thus the convergence $h_{t_i}^{-1} \to h_\infty$ is in $C^0$. By the $C^0$-convergence and the uniform boundedness of $|\Lambda R^{D_{h_t \perp}}|_{h_t}$, as done in the arguments in the proof of [Do1, Lemma 19] (also [Si1, Lemma 6.4]), we conclude that $h_{t_i}$ is actually bounded in $C^1$.

To complete the proof, we use the basic elliptic estimate and regularity explained in [BH, Section 2]. For the transversely elliptic operator $\Delta'_{h_0}$, as in [BH, Remark 2.2], we can non-canonically extend $\Delta'_{h_0}$ to a second order differential operator

$$
L : A^0(S(E)) \to A^0(\text{End}(E)).
$$

For the linear operator

$$
\nabla_{\xi}^h : A^0(S(E)) \to A^0(S(E))
$$

associated with the connection $\nabla^h$, and its formal adjoint $(\nabla_{\xi}^h)^*$, define the second order differential operator

$$
\square := (\nabla_{\xi}^h)^* \nabla_{\xi}^h + L : A^0(S(E)) \to A^0(\text{End}(E)).
$$

Then, by the transverse ellipticity of $\Delta'_{h_0}$, the differential operator $\square$ is elliptic. We have $
abla = \Delta'_{h_0}$ on $A^0_{B_\xi}(S(E))$. Therefore, by the

- elliptic estimate of the elliptic operator $\square$,
- the $C^1$-boundedness of $h_{t_i}$, and
- the uniform boundedness of $|\Lambda R^{D_{h_t \perp}}|_{h_t}$,
we conclude that $h_t$ is bounded in $L^{p,2}$, and hence the convergence $h_t \to h_\infty$ is weakly in $L^{p,2}$. Thus, $R^{D_{h_\infty}}$ is defined, and $\Lambda R^{D_{h_\infty}} = 0$.

We shall prove that $h_\infty$ is a smooth basic Hermitian metric. For that it is sufficient to show that $\sigma_\infty = h^{-1}_0 h_\infty \in A^0_{\mathcal{B}_\mathcal{F}_\mathcal{E}} (S(E))$. We consider the Sobolev space $L^{p,k}(S(E))$.

Then, the basic Sobolev space $L^{p,1}_{\mathcal{B}_\mathcal{F}_\mathcal{E}} (S(E))$ is the closure of $A^0_{\mathcal{B}_\mathcal{F}_\mathcal{E}} (S(E))$ in $L^{p,1}(S(E))$. By the elliptic regularity for the elliptic operator $\square$ (see [BH, Lemma 2.8]), we conclude that

$$\sigma_\infty \in L^{p,1}_{\mathcal{B}_\mathcal{F}_\mathcal{E}} (S(E)) \cap A^0(S(E)).$$

We have $A^0_{\mathcal{B}_\mathcal{F}_\mathcal{E}} (S(E)) = \text{kernel}(\nabla^h)$ for the linear differential operator $\nabla^h : A^0(S(E)) \to A^0(S(E))$. Extend $(\nabla^h)_{p,1} : L^{p,1}(S(E)) \to L^{p,0}(S(E))$ continuously. Then we have $L^{p,1}_{\mathcal{B}_\mathcal{F}_\mathcal{E}} (S(E)) \subset \text{kernel}((\nabla^h)_{p,1})$. From the commutative diagram

$$
\begin{array}{ccc}
A^0(S(E)) & \longrightarrow & L^{p,1}(S(E)) \\
\nabla^h \downarrow & & \downarrow (\nabla^h)_{p,1} \\
A^0(S(E)) & \longrightarrow & L^{p,0}(S(E))
\end{array}
$$

where $A^0(S(E)) \to L^{p,1}(S(E))$ and $A^0(S(E)) \to L^{p,0}(S(E))$ are the natural inclusions, it follows that $\text{kernel}((\nabla^h)_{p,1}) \cap A^0(S(E)) \subset \text{kernel}(\nabla^h)$. Thus we have $\sigma_\infty \in A^0_{\mathcal{B}_\mathcal{F}_\mathcal{E}} (S(E))$. This completes the proof of Theorem 5.2.

6.5. Bogomolov–Miyaoka inequality. The following Bogomolov–Miyaoka type inequality is derived from Theorem 5.2 just as [Si1, pp. 878–879, Proposition 3.4] is derived from [Si1, p. 878, Theorem 1].

**Corollary 6.6.** Let $(E, \theta)$ be a polystable basic Higgs bundle of rank $r$ over a compact Sasakian manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$ with $\dim M = 2n + 1$. Then

$$
\int_M \left( 2c_{2,b_{\mathcal{F}_\mathcal{E}}} (E) - \frac{r - 1}{r} c_{1,b_{\mathcal{F}_\mathcal{E}}} (E)^2 \right) (d\eta)^{n-2} \wedge \eta \geq 0,
$$

where $c_{i,b_{\mathcal{F}_\mathcal{E}}} (E)$ is the $i$-th basic Chern class of $E$. If the above inequality is an equality, then $R^{D_{h_\infty}} = 0$.

7. Correspondence Between Flat Bundles and Higgs Bundles

**Proposition 7.1** (See [BH, Theorem 4.7]). Let $(E, \theta)$ be a basic Higgs bundle over a compact Sasakian manifold $(M, (T^{1,0}, S, I), (\eta, \xi))$ with $\deg(E) = 0$. Suppose that $h$ is a basic Hermitian metric on $E$ with $\Lambda R^{D_{h}} = 0$. Then $(E, \theta)$ is a direct sum of stable basic Higgs bundles of degree zero.
Proof. Assume that \((E, \theta)\) is not stable. Then there exists a sub-Higgs sheaf \(\mathcal{V}\) of \((E, \theta)\) such that

- \(\text{rk}(\mathcal{V}) < \text{rk}(E)\) with \(\mathcal{O}_{B_{F_{\xi}}} (E) / \mathcal{V}\) is torsion-free, and
- the inequality
  \[
  \deg(\mathcal{V}) \geq 0
  \]
  holds.

Let \(\pi\) be the projection to the transversely holomorphic subbundle \(V \subset E\) defined almost everywhere using \(\mathcal{V}\), constructed as in [BH, Proposition 3.21] associated with the Hermitian metric \(h\). Then, \(\pi \in L^{2,1}_{B}(S(E))\), and we have the Chern–Weil formula [Si1, Lemma 3.2]

\[
\deg(\mathcal{V}) = \sqrt{-1} \int_{M} \text{Tr}(\pi \Lambda R^{Dh}) - \int_{M} |D''_{E,\theta} \pi|^{2}.
\]

By \(\Lambda R^{Dh} = 0\), we have that \(\deg(\mathcal{V}) = 0\) and \(D''_{E,\theta} \pi = 0\). By \(\pi \in L^{2,1}_{B}(S(E))\), we also have that \(D'_{E,\theta} \pi = 0\). Using the elliptic regularity it follows that \(\pi \in L^{p,1}_{B_{F_{\xi}}}(S(E)) \cap A^{0}(S(E))\), and hence \(\pi \in A^{0}_{B_{F_{\xi}}}(S(E))\) by the same argument as in the last part of the proof of Theorem 5.2.

Using \(D''_{E,\theta} \pi = 0\) and \(D'_{E,\theta} \pi = 0\), it can be seen that \(\pi\) is the projection to a globally defined basic Higgs sub-bundle \(V \subset E\) of \((E, \theta)\). Thus have the direct sum decomposition \(E = V \oplus V^\perp\) of basic Higgs bundles that satisfies the condition that \(\deg(V) = 0 = \deg(V^\perp)\). Restricting \(h\) to \(V\) and \(V^\perp\), and repeating the arguments the proposition can now be proved inductively. \(\square\)

In view of Theorems 4.2, 5.2, Propositions 7.1 and 5.3, on a compact Sasakian manifold \((M, (T^{1,0}, S, I), (\eta, \xi))\), we have a bijective correspondence between

- the semi-simple flat bundles \((E, \nabla_{E})\), and
- the polystable basic Higgs bundles \((E, \theta)\) over \(M\) with \(c_{1,B_{F_{\xi}}}(E) = 0 = c_{2,B_{F_{\xi}}}(E)\). via harmonic metrics \(h\). It should be clarified that the transversely holomorphic structure of the vector bundle underlying \((E, \nabla_{E})\) is in general different from the transversely holomorphic structure of the vector bundle underlying the corresponding basic Higgs bundle \((E, \theta)\); however the underlying \(C^{\infty}\) vector bundles coincide.

In the above correspondence, considering the basic de Rham complex \((A^{*}_{B_{F_{\xi}}}(M, E), d_{E})\) with values in a semi-simple flat bundle \((E, \nabla_{E})\), we have the decomposition

\[
d_{E} = D''_{E,\theta} + D'_{E,\theta}^{\prime \prime}.
\]

Denote by

\[
H_{dR,B_{F_{\xi}}}(M, E) \quad \text{and} \quad H_{Dol,B_{F_{\xi}}}(M, E)
\]

the cohomologies of complexes \((A^{*}_{B_{F_{\xi}}}(M, E), d_{E})\) and \((A^{*}_{B_{F_{\xi}}}(M, E), D''_{E,\theta})\) respectively. By the Kähler identities, we have an isomorphism

\[
H^{*}_{dR,B_{F_{\xi}}}(M, E) \cong H^{*}_{Dol,B_{F_{\xi}}}(M, E)
\]
by the transverse Hodge theory of usual basic cohomology (see \cite{Ka}). In particular, a
section \( \varphi \in A^*_{BF}(M, E) \) is flat (i.e., \( \nabla^E \varphi = 0 \)) if and only if \( D''_{E, \theta} \varphi = 0 \).

As done in \cite[Corollary 1.3]{Si2}, we now obtain the following result.

**Theorem 7.2.** Let \((M, (T^{1,0}, S, I), (\eta, \xi))\) a compact Sasakian manifold. Then there
is an equivalence between the category of semi-simple flat vector bundles over \( M \) and
the category of polystable basic Higgs bundles over \( M \) with trivial first and second basic
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