EVOLUTION OF MIGRATING PLANETS UNDERGOING GAS ACCRETION

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ABSTRACT

We analyze the orbital and mass evolution of planets that undergo runaway gas accretion by means of two- and three-dimensional hydrodynamic simulations. The disk torque distribution per unit disk mass as a function of radius provides an important diagnostic for the nature of the disk-planet interactions. We present results of simulations for mass-gaining, migrating planets. For planets with an initial mass of $5 \, M_{\oplus}$, which are embedded in disks with standard parameters and which undergo runaway gas accretion to $1 \, M_{\oplus}$, the torque distributions per unit disk mass are largely unaffected by migration and accretion for a given planet mass. The migration rates for these planets are in agreement with the predictions of the standard theory for planet migration (type I and type II migration). The planet mass growth occurs through gas capture within the planet’s Bondi radius at lower planet masses, the Hill radius at intermediate planet masses, and through reduced accretion at higher planet masses due to gap formation. During runaway mass growth, a planet migrates inward by only about 20% in radius before achieving a mass of $\sim 1 \, M_{\oplus}$. For the above models, we find no evidence of fast migration driven by coorbital torques, known as type III migration. We do find evidence of type III migration for a fixed-mass planet of Saturn’s mass that is immersed in a cold and massive disk. In this case the planet migration is assumed to begin before gap formation completes. The migration is understood through a model in which the torque is due to an asymmetry in density between trapped gas on the leading side of the planet and ambient gas on the trailing side of the planet.

Subject headings: accretion, accretion disks — hydrodynamics — methods: numerical — planetary systems: formation — planetary systems: protoplanetary disks — solar system: formation

1. INTRODUCTION

In the core accretion picture of planet formation (Bodenheimer & Pollack 1986; Wuchterl 1991; Pollack et al. 1996; Hubickyj et al. 2005 and references therein), a small-mass solid core initially rapidly accretes solid material, followed by a slow evolution phase of gas and solid accretion. During this slow evolution phase, the planet is limited in its ability to accrete gas by the thermal heating caused by the impacting solids. Once the planet’s gas mass is greater than its solid mass, typically at several Earth masses, the planet undergoes “runaway” gas accretion, in which it can accrete whatever mass is provided to it. These processes have been treated by one-dimensional, spherically symmetric structure calculations in the above papers.

On the other hand, multidimensional hydrodynamical calculations of a protostellar disk interacting with the planet have revealed various flow properties of the gas, including the gap opening by tidal effects, previously anticipated by one-dimensional disk models (Lin & Papaloizou 1986). In addition, planet migration that results from disk-planet interactions has been analyzed by means of such simulations. Good agreement is often, but not always, found between the simulations and the expectations of theory (Nelson et al. 2000; Bate et al. 2003; D’Angelo et al. 2003, 2006; Nelson & Benz 2003; Li et al. 2005). These calculations typically do not include the mass evolution of the planet. Usually they apply accretion boundary conditions onto the planet as a means of modeling the runaway gas accretion process. One aim of this paper is to analyze the effects of planet mass growth on migration.

Several controversies remain on the effects of gas. The role of coorbital torques on planet migration, in the subgiant mass range, is not well understood. Masset & Papaloizou (2003, hereafter MP03) suggested on the basis of a model and simulations that a fast mode of migration (sometimes called type III migration) can occur due to strong coorbital torques. Ogilvie & Lubow (2006, hereafter OL06) found support for the concept of coorbital dominated migration under certain conditions. At higher grid resolution under the conditions specified by MP03, simulations by D’Angelo et al. (2005, hereafter DBL05) found that the migration rate was much slower.

Another subject of interest is how planet masses may be limited by a reduction in the gas accretion rate. Lin & Papaloizou (1986) proposed such a reduction by tidal torques that open a gap about the orbit of the planet. The value of the highest planet mass achieved in the presence of gap opening is somewhat controversial. Some studies (Lubow et al. 1999; Bate et al. 2003; D’Angelo et al. 2003) have suggested that the maximum planet mass is about $6-10 \, M_{\oplus}$, corresponding to the upper limit of the observed range of extrasolar planets (Marcy et al. 2005; Butler et al. 2006). This limit suggests that some other process, such as disk dispersal or other self-limiting feedback on planetary accretion, is responsible for the lower masses ($\sim 1 \, M_{\oplus}$) typically found observationally. Other studies suggest that the tidal limit is $\sim 1 \, M_{\oplus}$ and therefore no additional process is required to explain the typical masses (e.g., Dobbs-Dixon et al. 2007).

We address these and other issues in this paper by analyzing the orbital evolution of a mass-gaining planet embedded in a gas disk. In § 2 we analyze the torque distributions for planets of constant mass on fixed circular orbits. In § 3 we analyze the orbital
and mass evolution of migrating planets that undergo runaway mass accretion. Section 4 describes a model that appears to exhibit migration that is dominated by coorbital torques, i.e., type III migration. Section 5 contains the summary and discussion.

2. TORQUE DISTRIBUTION FOR A NONMIGRATING PLANET

Disk-planet gravitational torques result in planet migration (Goldreich & Tremaine 1980; Lin & Papaloizou 1993; Ward 1997). The distribution of torque with disk radius provides a means of connecting the theory with simulations. In this section we model the disk as a three-dimensional system and consider fixed-mass planets on fixed circular orbits. The torque per unit radius for a planet embedded in a disk was previously considered in Bate et al. (2003). Here we reconsider the analysis with higher resolution, especially in the coorbital region, and apply the torque distribution per unit disk mass.

2.1. Numerical Procedure

In this section we describe the torques exerted by a disk on an embedded planet with mass, $M_p$, equal to 1 $M_\odot$, 10 $M_\odot$, 0.3 $M_\odot$, and 1 $M_\odot$. For the two smallest mass planets we consider, the planet’s Hill radius is smaller than the vertical disk thickness of several percent of the distance to the star. For the two largest mass planets, the Hill radius is comparable to or larger than the disk thickness.

2.1.1. Disk Model

We use spherical polar coordinates $\{r, \theta, \phi\}$, with the origin located at the star-planet center of mass. The reference frame co-rotates with the star-planet system. The planet’s orbit lies in the plane $\theta = \pi/2$. The disk is assumed to be symmetric with respect to this plane; hence, only the disk’s northern hemisphere (i.e., the volume $\theta \leq \pi/2$) is simulated.

We assume that the material in the disk is locally isothermal and that the pressure $p$ is given by

$$p(\rho, \theta, \phi) = \rho c_s^2(r),$$

where $\rho(\rho, \theta, \phi)$ is the mass density. Quantity $c_s(r)$ is the gas sound speed, which is taken to be a function of cylindrical radius $r = R \sin \theta$. The aspect ratio of the disk, $H/r$, is taken to be constant and equal to 0.05. Therefore, the temperature distribution in the disk is only a function of the distance from the disk’s rotation axis, $r$, and decreases as $c_s^2 \propto 1/r$. Viscous forces are calculated by adopting the stress tensor for a Newtonian fluid (Mihalas & Weibel Mihalas 1999) with constant kinematic viscosity $\nu$ and zero bulk viscosity. Disk self-gravity is ignored. In Appendix C we discuss some effects of disk self-gravity and of the axisymmetric component of disk gravity on the migration rates.

2.1.2. Disk and Planet Parameters

We adopt the stellar mass $M_\ast$ as unit of mass, the orbital radius $a$ as unit of length, and $\Omega_\ast^{-1} = [G(M_\ast + M_p)/a^3]^{-1/2}$ as unit of time. In converting to dimensional units we consider $a = 5.2$ AU and $M_\ast = 1 M_\odot$.

The disk extends from 0 to $2\pi$ in azimuth around the star and, in radius, from 0.4 to either 4.0 (Jupiter-mass case) or 2.5 (lower mass cases). In the $\theta$-direction, the disk domain extends above the midplane ($\theta = \pi/2$) for $10^4$, comprising 3.5 pressure scale heights, $H$. The initial mass density distribution is independent of $\phi$, has a Gaussian profile in the $\theta$-direction, and has a radial profile proportional to $R^{-3/2}$, so that the initial (unperturbed) surface density varies as $R^{-1/2}$. We adopt a constant dimensionless kinematic viscosity $\nu$ equal to $10^{-3}$, corresponding to a turbulent viscosity parameter $\alpha = 0.004$ at the cylindrical radius $r = 1$ (5.2 AU).

As mentioned above, we perform calculations for four planet masses: $M_p = 3 \times 10^{-6}, 3 \times 10^{-5}, 3 \times 10^{-4}$, and $1 \times 10^{-3}$, which correspond, respectively, to $1 M_\Earth$, $10 M_\Earth$, 0.3 $M_\Earth$, and $1 M_\Earth$. The gravitational potential, $\Phi_p$, of the planet is smoothed over a length $\epsilon$ equal to 0.1 $R_H$ and is given by

$$\Phi_p = -\frac{GM_p}{\sqrt{S^2 + \epsilon^2}},$$

where $S$ is the distance from the planet and $R_H$ is the Hill radius of the planet.

2.1.3. Numerical Method

The mass and momentum equations that describe the evolution of the disk (e.g., DBL05) are solved numerically by means of a finite-difference scheme that applies an operator splitting procedure to perform the spatial integration of advection and source terms (Ziegler & Yorke 1997). The algorithm is second-order accurate in space and semi-second-order accurate in time. The equations are discretized over a mesh with constant grid spacing in each coordinate direction. Nested grids are used to enhance the numerical resolution in arbitrarily large regions around the planet (D’Angelo et al. 2002, 2003). This strategy allows the volume resolution to be increased by a factor of $2^n$ for each added grid level. These calculations are executed with grid systems involving five levels of grid nesting. The linear base resolution is $\Delta R = a \Delta \theta = a \Delta \phi = 0.01 a$. The linear resolution achieved in the coorbital region around the planet is approximately $9 \times 10^{-4} \Delta R$, which corresponds to $\sim 0.01 R_H$ and $\sim 0.1 R_H$ in the Jupiter-mass and Earth-mass cases, respectively. To quantify resolution effects in the Earth-mass case, we also applied a linear resolution twice as high throughout the entire grid system (base resolution of $7 \times 10^{-3} \Delta R$ and resolution in the coorbital region around the planet of $4 \times 10^{-4} \Delta R$). The torques at the two resolutions, integrated over the disk domain, differ by about 5%.

The boundary condition near the planet involves removing gas from $\sim 0.1 R_H$ of the planet at each time step. The procedure for mass removal is described in more detail in § 3.1.1. In the calculations reported in § 2.3, the removed mass is not added to the planet’s mass in order to keep it fixed. In §§ 3 and 4 (as well as in Appendices A and C) we present cases in which the planet’s mass is augmented by the mass of the gas removed from the disk.

The outer boundary of the disk domain is closed to both inflow and outflow, whereas the inner boundary allows outflow (material can flow out of the grid domain) but not inflow. Reflective and symmetry boundary conditions are applied at colatitude $\theta = \theta_{\min}$ and at the disk midplane ($\theta = \pi/2$), respectively.

Simulations are run for about 100 orbital periods. In models with 0.3 and 1 $M_\Earth$ mass planets, the initial density distribution includes a gap along the planet’s orbit to account for an approximate balance between viscous and tidal torques, which reduces the relaxation time toward steady state. In all calculations discussed here, the flow achieves a fairly steady state within $\sim 100$ orbits.

2.2. Theoretical Considerations

2.2.1. Torque Density

Consider a cylindrical coordinate system $\{r, \phi, z\}$ centered on the star-planet center of mass. The disk torque along the rotation axis per unit radius exerted on the planet is given by

$$\frac{dT}{dr}(r, t) = \left\langle r \int_0^{2\pi} d\phi \int_0^\infty dz \rho(r, t) \Phi_p(r, t) \right\rangle.$$

\[ (3) \]
where \( X(t) \) denotes the time average of \( X \) over an orbit period centered about time \( t \), \( \rho \) is the gas density, and \( \Phi_p \) is the potential due to the planet (eq. [2]).

### 2.2.2. Radial Overlap Regions

The linear theory of Lindblad resonances for disk-planet interactions demonstrates that the strongest contributing resonances have azimuthal wavenumbers \( m \sim r/H \). This estimate comes from considering the so-called torque cutoff effect that arises from Lindblad resonances that lie close to the planet (Goldreich & Tremaine 1980; Ward 1986; Artymowicz 1993). As a consequence of the resonance condition, we expect the peak torque density to be at a distance of roughly \( H \) from the planet. The torque cutoff is not sharp and there are torque contributions from resonances that lie closer than distance \( H \) from the planet, although at a decreasing level as they get closer to the planet. As seen below, the numerical results show the torque density peak to be close to distance \( H \) from the planet. However, the torque cutoff calculations assume that the orbits are such that the gas azimuthally passes by the planet, i.e., lies on circulating orbits. On the other hand, close to the planet’s orbit, this assumption breaks down and the gas flows on librating streamlines of the horseshoe orbit region. This region generally extends in the radial direction to a distance of about \( 3R_H \) from the planet’s orbital radius, where \( R_H \) is the planet’s Hill radius. But, close to the planet, the region becomes less extended radially, spanning only to approximately \( 1.2R_H \). That is, the noncoorbital (circulating) streamlines pass closest to the planet at a distance about equal to \( R_H \) (see streamline \( a \) in Lubow et al. 1999; Fig. 5 in Bate et al. 2003). In the horseshoe orbit region, the corotational resonance can play a role.

These two regions, the coorbital region (extending up to about \( 3R_H \) from the planet’s orbital radius) and Lindblad torque region (extending beyond about distance \( H \) from the planet’s orbital radius), overlap in a one-dimensional radial sense for planet-to-star mass ratios

\[
q \gtrsim \frac{1}{9} \left( \frac{H}{r} \right)^3.
\]

This condition does not necessarily imply a physical overlap in two or three dimensions. But it does affect our interpretation of the torque density reduced to one dimension, \( dT(r)/dr \). The reason is that for a given radius \( r \) such that \( R_H < |r-a| < 3R_H \), the gas lies in either the coorbital (librating) or noncoorbital (circulating) region, depending on the azimuth.

For the disk parameters considered in this section, the one-dimensional overlap occurs for planet masses greater than about 4.6 \( M_{\oplus} \), which covers all but one of the planet masses considered. For a 1 M\( \oplus \) planet, this overlap occurs out to a radius of about 1.2\( a \) or a radial distance of about 4\( H \) from planet.

The two regions physically overlap in a two- or three-dimensional sense, when the closest approach of all noncoorbital (circulating) streamlines, which occurs at a distance \( \sim R_H \) from the planet, is greater than the distance where there are maximum Lindblad torques \( \sim H \). This occurs when

\[
q \gtrsim 3 \left( \frac{H}{r} \right)^3.
\]

In this case, the usual torque cutoff condition for Lindblad resonances is questionable. This argument suggests that the torque density maximum for Lindblad resonances should occur at a radial distance from the planet

\[
|r - a| \simeq \max(R_H, H).
\]

When this condition is satisfied, the overall torque on the planet will be reduced, even if \( R_H < H \), since resonances that lie closer than distance \( H \) from the planet are suppressed.\(^2\) For the disk parameters considered in this section, this condition is satisfied for \( M_p \approx 4 \times 10^{-4} M_\odot \) (or 0.4 \( M_\odot \)).

### 2.2.3. Saturation Effects of Coorbital Torques

The flow in the coorbital region is trapped in horseshoe orbits. For a time-reversible system (e.g., no dissipation or migration), the streamlines are exactly periodic and no net torque occurs on the planet due to the disk (i.e., the torque saturates), except for possible initial transients due to initial conditions. However, turbulent viscosity introduces irreversibility that can lead to a net torque. The condition for saturation within the framework of the \( \alpha \)-disk model is that the libration timescale of the fluid in the coorbital region is shorter than the viscous radial diffusion timescale across this region. Based on scaling arguments, the saturation condition is given by (Ward 1992)

\[
\alpha \lesssim q^{3/2} \left( \frac{r}{H} \right)^{7/2}.
\]

For the parameters in this section, this constraint implies that for planets of order 10 \( M_\odot \) or greater, the corotation torques should be saturated (small). Saturation effects should be important for the larger planet masses we consider.

#### 2.3. Numerical Results

The torque per unit disk mass is defined by

\[
\frac{dT}{dM}(r, t) = \left( \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz \rho(r, \phi) \partial_t \Phi_p(r, t) \right),
\]

where \( \Sigma(r, \phi) \) is the axisymmetric disk density (i.e., the surface density averaged over the azimuth \( \phi \)) and notation \( \langle X(t) \rangle \) is defined below equation (3).

Numerically, the torque distribution per unit disk mass is determined by dividing the (three-dimensional) disk into a series of concentric shells, of radius \( R \) and thickness \( \Delta R \), centered at the origin and calculating the torque exerted by the shell and the mass of the shell. The torque per unit disk mass is obtained from the ratio of these two quantities,\(^2\) averaged over an orbit period. We use the radial grid spacing on the base grid for the value of \( \Delta R \). The torques arising from within the Hill sphere of the planet are ignored in this section but are included in later sections of this paper. We ignore such considerations here in order to compare results with the standard theory of coorbital and Lindblad torques, which does not include such contributions (Tanaka et al. 2002).

The torque per unit disk mass for four planet mass cases is shown in Figure 1. The plots are normalized such that the torque densities in the four cases would be the same, according to linear

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\(^2\) They may still partially contribute due to their finite widths.

\(^3\) There is a slight error of order \((H/r)^2\) in this procedure due to the difference between the spherical coordinate system used in the calculations and the cylindrical coordinates that apply to the definition of the torque in eq. (3).
theory, if the axisymmetric disk density gradients and gas properties (sound speeds and viscosities) were the same. That is, the torque density per unit disk mass is scaled by the square of the star-to-planet mass ratio. The 1 $M_\odot$ (solid line) and 10 $M_\odot$ (long-dashed line) cases nearly exactly overlap as predicted, while the 0.3 $M_\odot$ (dot-dashed line) and 1 $M_\odot$ (short-dashed line) cases have a smaller scaled torque density. The scaling in the plot masks the fact that the results span a large range of parameter space. In going from 1 $M_\odot$ to 1 $M_\odot$ there is a change in torque density by a large factor, 10^5, while the discrepancy is about a factor of 2.5.

The deviations in the 0.3 and 1 $M_\odot$ cases could be due to the modified torque cutoff, pressure gradients, and nonlinearities. Since $R_H \geq H$ in these cases, Lindblad resonance contributions are weakened by the modified torque cutoff, as discussed in § 2.2.2. Pressure gradients cause shifts in the resonance locations. For mild pressure gradients that change sign across the orbit of the planet (as would occur for a mild gap), the resonances shift away from the orbit of the planet (see eq. [26] of Ward 1986). The shift would then cause the torques per unit disk mass to be weaker, as seen in the figure. The situation is more complicated in the case of stronger pressure gradients, as may occur for deep gaps, and the sign of the effect on the torque depends on the detailed shape of the density profile. Nonlinearities may play a role in the 1 $M_\odot$ case, since there are shocks in the disk in that case, due to the strong forcing. But the total torque is not expected to be substantially effected by nonlinearity. For a fixed smooth background disk density distribution, resonant torques are quite insensitive to the level of nonlinearity (Yuan & Cassen 1994). For a 1 $M_\odot$ planet and a resonance with azimuthal wavenumber $m = 20 = H/a$, the nonlinearity is mild with nonlinearity parameter $f = 0.6$, as defined by Yuan & Cassen (1994). Some broadening of the torque density profile is predicted, while the total torque is reduced by only about 1%. For much stronger nonlinearity, $f = 3$, the torque reduction is only 5%. This estimate is based on considering only a single resonance. Many resonances overlap, increasing the level of nonlinearity. However, the theory does not describe overlapping resonances. So, although we cannot be definite about the importance of nonlinearities, indications for a single resonance suggest that they are not important.

The torque density per unit disk mass for the 1 $M_\odot$ planet in Figure 1 (short-dashed line) shows indications of saturation for $|r - a| < R_H$. As discussed above, this effect is suggested by theoretical considerations. The torque density peak for the 1 $M_\odot$ case is slightly displaced away from the planet relative to the smaller mass cases and lies close to a distance $R_H \approx 0.07a$ from the planet. This result is consistent with equation (6) in the 1 $M_\odot$ case, $|r - a| \approx 0.07a = 1.4H$.

Figure 2 shows that the torque in the 1 $M_\odot$ case is acquired close to the planet, well within the gap region. Most of the torque is accumulated by material with intermediate/low density interacting with an intermediate-magnitude torque per unit disk mass. About 80% of the torque is due to material within a radial distance of 0.25a from the planet.

3. MIGRATING AND GROWING PLANETS

We investigate the orbital migration of a planet that is undergoing runaway gas accretion. We consider several disk configurations, by changing the initial surface density, the pressure scale height, and the kinematic viscosity. We use disk models and numerical procedures similar to those introduced in § 2.1. Throughout this section, the disk is modeled as a three-dimensional system. The origin of the coordinate system is taken to be the star. The coordinate system rotates around the origin at a rate equal to the rotation rate of the planet. The coordinate system is then removed after the integration of the planet, under the action of disk torques and apparent forces arising from the rotation of the reference frame, as described in DBL05. The unit of length is the initial star-planet separation $a_0$ (or 5.2 AU when converting into physical units). The unit of time is the inverse of $\Omega_\star$, the initial angular speed of the planet. The unit of mass is the stellar mass $M_\star$ (1 $M_\odot$).

The grid system achieves a linear base resolution of $\Delta R = a_0/\Delta \theta = a_0/\Delta \phi = 0.014a_0$. In the coorbital region around the planet, the linear resolution is about $9 \times 10^{-4}a_0$. Nested grid levels cover extended radial regions of the disk so that the planet remains within the domain covered by the most refined grid level over the entire orbital evolution. Convergence tests were carried out with a grid system that used a volume resolution (3/2)^3 times as high throughout the whole disk domain and on all grid levels. No significant differences are observed (see § A1). To avoid depletion of the disk interior of the planet’s orbit, we apply nonreflecting
boundary conditions to the inner grid (radial) border. We test our results against possible boundary condition effects in § A2 by applying outflow boundary conditions and moving radial disk boundaries farther away from the planet’s orbit in both directions. No important effects are observed. Near the planet we apply accreting boundary conditions on the gas, as described in § 3.1.1. We consider planetary mass increases that extend over more than 2 orders of magnitude and a range of disk surface densities.

To avoid possible spurious torques exerted by material gravitationally bound to the planet, contributions from within $2 \times 10^{-4} M_0$ or $5 M_0$. However, in some applications discussed in § 4, we use an initial mass $M_p = 3 \times 10^{-4} M_0$ (about 0.3 $M_J$) in order to study the effects on migration of releasing a more massive planet in an unperturbed disk.

### 3.1. Planet Mass Growth

#### 3.1.1. Gas Accretion

In the core accretion scenario of giant planet formation, prior to the phase of runaway gas accretion, the rate at which gas is accreted is largely determined by the ability of a planetary core’s envelope to radiate away the energy delivered by gas and solids (phase of slow gas accretion; see, e.g., Hubickyj et al. 2005).3.1.2. Mass Evolution

In this section we describe the accretion rates of migrating, mass-gaining planets. Figure 3 shows the planet mass as a function of time, $M_p = M_p(t)$, for a model with initial (unperturbed) surface density at the initial orbital radius of the planet $\Sigma_p = 3 \times 10^{-4} M_0 a^{-2}$ (solid line). For a planet orbiting a solar-mass star at 5.2 AU, this density is about 100 g cm$^{-2}$, roughly corresponding to the minimum-mass solar nebula.

The mass evolution can be understood in terms of Bondi and Hill accretion. Consider a simple model in which gas is captured within some radius, $S_c$, of a planet and assume $S_c < H$. Mass is accreted with some velocity relative to the planet of order $\Omega S_c$, and so the mass accretion rate in a three-dimensional disk (where $\rho \approx \Sigma/H$) is estimated as

$$M_p \sim \frac{\Sigma}{H} \Omega S_c^3,$$

where we take $S_c$ as either the Bondi or Hill radius, with the Bondi radius given by $R_B = GM_p/c_s^2$ and the Hill radius given by $R_H = a[M_p/(3M_0)]^{1/3}$.

In the case that gas pressure prevents the gas from being bound to the planet within the Hill sphere (or, equivalently, that pressure forces dominate over gravitational three-body forces), we expect the Bondi description to be appropriate. This condition is that

$$c_s^2 \gtrsim \frac{GM_p}{R_H}$$

or

$$R_B \leq R_H.$$

Therefore, in the general case we take

$$S_c = \min(R_B, R_H).$$
It then follows that the Bondi and Hill mass growth rates, $\dot{M}_p/M_p$, of the planet are given by

$$\frac{1}{\tau_{B}} = C_B \frac{a}{H} \left( \frac{\dot{M}_p}{M_p} \right)^2,$$

$$\frac{1}{\tau_{H}} = \frac{1}{3} C_H \frac{a}{H} \left( \frac{\dot{M}_p}{M_p} \right),$$

where $C_B$ and $C_H$ are dimensionless coefficients of order unity. The overall mass growth rate is given by

$$\frac{1}{\tau_G} = \begin{cases} \frac{1}{\tau_{B}}, & M_p < M_t, \\ \frac{1}{\tau_{H}}, & M_p \geq M_t, \end{cases}$$

where

$$M_t = \frac{M_s}{\sqrt{3}} \sqrt{\frac{C_B}{C_H}} \left( \frac{H}{a} \right)^3$$

is the transition planet mass where $\tau_H = \tau_B$.

In Figure 4 we plot the mass growth rate, $1/\tau_G$, for the solidline case in Figure 3. We applied equation (15) and adopted constant values of $\Sigma = \Sigma(a_0)$, at time $t = 0$, and $\Omega = \Omega_0$. The figure shows that the Bondi and Hill accretion rates in equation (15) agree with the simulation results for values of $C_B = 2.6$ and $C_H = 0.89$. The transition mass in this case evaluates to $M_t = 4.2 \times 10^{-5} M_s$. It lies between the Bondi and Hill accretion regimes in the figure, at the intersection between the two dashed line segments. For larger values of planet mass, $M_p \gtrsim 2 \times 10^{-4} M_s$, $\dot{M}_p$ is small, this simple estimate of the mass growth rate breaks down because the density is depleted near the planet due to the onset of gap formation. The density near the planet is reduced by about 40% when $M_p \approx 2 \times 10^{-4} M_s$ (see Fig. 6, right panel, below). In addition, the Hill radius becomes comparable to $H$, since $R_H = H = 0.05 a$ for $M_p = 3.75 \times 10^{-4} M_s$.

Simulations carried out in two dimensions would have different scaling behavior, since the right-hand side of equation (9) would be $\Sigma \Omega a^2$. The dependence of the mass accretion rate on planet mass and disk sound speed then artificially deviates from the three-dimensional case. In two dimensions we have that $1/\tau_G \propto (M_p/M_s)^2$ and $1/\tau_H \propto (M_p/M_s)^{3/2}$.

The maximum of the accretion rate for the solid-line case of Figure 3 is $M_p \sim 5 \times 10^{-3} \Sigma a^2 \approx 1.5 \times 10^{-3} M_j$ per orbit and occurs when $M_p \approx 0.3 M_j$. This result is consistent with the previous findings of D’Angelo et al. (2003) and Bate et al. (2003), who considered planets on fixed orbits. Also displayed in Figure 3 is the planet’s mass evolution in a disk with initial $\Sigma_p = 9 \times 10^{-4} M_s a_0^{-2}$ (dashed line) or about 300 g cm$^{-2}$ at 5.2 AU. For $M_p/M_s \lesssim 10^{-4}$, the accretion rate is a factor of 3 larger than that of the lower density disk case (solid line). Hence, equation (15) applies to the growth rate with the same coefficients $C_B$ and $C_H$ as those above. For larger planet masses, the accretion rate keeps increasing until $M_p \approx 0.7 M_j$, at which point $M_p$ starts to decline very rapidly as $M_p$ grows further. This is because effects due to gap formation are delayed. The timescale required to form a gap of half-width $\xi_{\text{diff}}$ is $\tau_{\text{gap}} \approx (a \xi^2)^{-1/3}$ (see, e.g., Bryden et al. 1999), where $\xi \approx 2$ (long-dashed line in Fig. 2). In the lower density disk model (solid line in Fig. 3), $\tau_{\text{gap}} < \tau_G$ and $\tau_{\text{gap}}$ becomes shorter than $\tau_G$ only when $M_p \gtrsim 0.7 M_j$.

In Figure 5, the mass evolution is shown for cases in which $\Sigma_p = 3 \times 10^{-4} M_s a_0^{-2} \approx 100$ g cm$^{-2}$, but with different scale heights, $H$, and kinematic viscosities, $\nu$. Near $M_p = 1 M_j$, the accretion rates of the two models with different $H/\nu$ (solid and long-dashed lines), but the same $\Sigma_p$ and $\nu$, are nearly equal, with $M_p \approx 3 \times 10^{-3} \Sigma_p a^2 \approx 9 \times 10^{-4} M_j$ per orbit. At larger planet masses, $M_p$ is smaller in the case of a colder disk (long-dashed line) because of the stronger tidal torques exerted by the planet on the disk material that produce a wider gap. When $M_p \approx 1 M_j$, the simulation with 10 times larger viscosity (short-dashed line) yields an accretion rate that is a factor of nearly 8 larger. This result is consistent with previous two-dimensional studies of planets on fixed orbits that do not gain mass. For $M_p \approx 1 M_j$ these studies showed that $M_p$ scales approximately linearly with $\nu\xi$, the overall disk accretion rate evaluated just outside the gap (Kley 1999; Lubow & D’Angelo 2006).
3.1.3. Mass within the Hill Sphere

We discuss here the relevance of torques exerted on a planet and originating within the planet’s Hill sphere. We may expect that material gravitationally bound to the planet should not be capable of exerting significantly strong torques, if resolution is appropriate (DBL05). In some situations, if the local density is large, any torque imbalance can be easily amplified by lack of numerical resolution (because torques depend on $1/S^2$, where $S$ is the distance to the planet). Artificial effects may arise when the mass within $\sim R_H$ of the planet is larger than the planet’s mass. However, not all of this material is necessarily bound to the planet. Because of the nonspherical nature of the Roche lobe, the Hill radius represents an overestimate for the size of the region where gas is bound to the planet (Paczynski 1971; Eggleton 1983). We have found that accumulated gas may be bound to the planet (Paczynski 1971; Eggleton 1983). We also consider models with initial densities larger than those discussed here (described in Appendix B). In all the cases discussed in this section, the amount of material that lies within $R_H/2$ of the planet is smaller than $M_p$, throughout the evolution, by several orders of magnitude. For models in Figure 3, as well as for those in Figure 5, the ratio of these two masses ranges from less than $\sim 10^{-3}$ to $\sim 10^{-2}$, depending mainly on the planet’s mass. We also consider models with initial densities larger than those discussed here (described in § 4). However, this mass ratio remains on the order of $10^{-2}$ or smaller. Therefore, due to the accretion boundary condition employed here at the planet location, these models do not experience a buildup of mass near the planet (with possible effects on planet migration). The accreted mass is accounted for by the increase in the planet mass.

3.2. Planet Migration

3.2.1. Theoretical Regimes of Migration

A planet that grows in mass from a few Earth masses to a few Jupiter masses is susceptible to two “classical” regimes of migration. The type I regime is expected when the planet causes small, linear disk density perturbations (e.g., Ward 1997; Tanaka et al. 2002). In the opposite limit, type II occurs when the planet mass is large enough to cause nonlinear density perturbations that result in a density gap along its orbit (Lin & Papaloizou 1986).

For the parameters we adopt (pressure scale height $H/r \sim 0.05$, kinematic viscosity of disk $\nu \sim 1 \times 10^{-5} a^2 \Omega_0$, and initial planet mass $M_p/M_\star = 1.5 \times 10^{-5}$, or $M_p = 5 M_\oplus$), it is expected that the initial evolution of the planet will follow type I migration, since the usual gap opening criteria are not satisfied. In the linear theory of Tanaka et al. (2002) the rate of migration resulting from the action of both Lindblad and (unsaturated) coorbital corotation torques is given by

$$\frac{da}{dt} = -(2.73 + 1.08s) \left( \frac{M_p}{M_\star, H} \right)^2 \frac{\Sigma_p a^2}{M_p} a^3 \Omega, \quad (17)$$

where $s$ is the slope of the unperturbed surface density. For the case of saturated (zero) corotational torques, the migration rate is given by

$$\frac{da}{dt} = -(4.68 - 0.20s) \left( \frac{M_p}{M_\star, H} \right)^2 \frac{\Sigma_p a^2}{M_p} a^3 \Omega. \quad (18)$$

The conditions for saturation are discussed in § 2.2.3. For higher planet masses that arise in the later stages of the simulations, the torques are expected to be saturated.

In the presence of a sufficiently clean density gap and for a planet whose mass is less than the local disk mass, the rate of migration follows type II theory that is dictated by disk viscous inflow

$$\frac{da}{dt} = -\zeta \frac{\nu}{a}. \quad (19)$$

Note that if there is residual material in the horseshoe orbit region, the migration rate can differ from that in equation (19). The coefficient $\zeta$ on the right-hand side of equation (19) is of order unity and also depends on the evolutionary state of the disk. For a steady state disk, the coefficient is $3/2$. But for nonsteady disks where $\nu \Sigma$ varies in radius, as in our initial states, the coefficient may differ by order unity amounts.

In the unsaturated case, some nonlinear effects of the corotation resonance can cause migration rates to differ from those predicted by equation (17) (Masset et al. 2006). For $s = \frac{1}{2}, H/r = 0.05$, these effects occur in the range of masses between $\approx 10$ and $\approx 20 M_\oplus$. However, in the models presented here, the planet grows too quickly through this mass range (taking less than a few tens of orbits) to significantly affect migration (see Fig. 23 in Appendix B).

When the amount of material in the horseshoe orbit region is larger than the planet’s mass, a regime of fast migration known as type III may occur. The origins of such a regime are not yet entirely clear. The model of MP03 suggests that it is driven by strong corotation torques originating from material that streams past the planet, while the planet is moving in the radial direction. However, an analytic model of OL06 suggests that such torques could originate from trapped librating gas. A somewhat similar model was developed by Artymowicz (2004).

3.2.2. Orbital Radius Evolution

We evaluate quantities $\Sigma_p$, $H$, and $\Omega_p$ at the planet’s orbital radius, $a$. Surface density $\Sigma_p = \Sigma_p(a)$ is evaluated according to its initial value $\Sigma_p(a) \propto (a_0/a)^2$ and so ignores evolutionary effects and tidal gap formation. The planet mass $M_p$ is regarded as a function of time that we obtain from our simulations, via piecewise polynomial fits. For the numerical models we consider, $s = -d \ln \Sigma_p/da/\ln a = 0.5$. Equations (17) and (18) are then solved numerically, providing the migration tracks $a(t)$.

In the left panel of Figure 6 we compare such tracks with outcomes from our simulations. For the first 400 orbits, while $R_H \lesssim 0.9 H$ and $M_p \lesssim 0.27 M_\oplus$, the orbital radius (i.e., semimajor axis) evolution is in good agreement with the results of type I migration. The unsaturated corotational torques appear to give a better fit than the saturated ones. But this is not always the case, as we see later when different disk parameters are considered. The right panel of Figure 6 plots the density evolution of the gas near the planet, $\Sigma_b$, computed as the ratio of the disk mass in the radial band $|r - a|/a \lesssim H/r$ to the area of the band ($\Sigma_b(a)$ is the local initial value of $\Sigma_b$). It shows that the migration rate follows the type I tracks on the left while the disk density near the planet remains close to the local initial disk value, assumed in equations (17) and (18). Up to a time of about 400 orbits, the density near the planet is reduced below its local initial value by less than 20%. At time of about 600 orbits, the density near the planet’s orbit is reduced by about a factor of 3, and we should expect the migration rates deduced from the simulation to be substantially slowed below the rates based on type I theory, in accord with the results on the left panel. After about 1000 orbits, when $M_p \gtrsim 0.9 M_\oplus$, the migration rate in the simulation becomes comparable to the (local) viscous inflow rate (long-dashed line). At this point, the disk density near the planet is depleted by a factor of about 30.
Fig. 6.—Orbital migration of a planet undergoing runaway gas accretion. *Left:* Orbital radius in units of $a_0$ (5.2 AU), as a function of time in units of the initial orbital period ($\approx 12$ yr). The initial planet mass is $5 \, M_J$. The initial surface density is $\Sigma_0 = 3 \times 10^{-4} M_0 a_0^{-2} \approx 100$ g cm$^{-2}$ at the planet’s initial orbital radius and $H/r = 0.05$. *Solid line:* Results from the three-dimensional numerical simulation of a migrating, gas-accreting planet. *Short-dashed lines:* Predictions based on type I migration theory, obtained by solving eqs. (17) and (18), for a planet that undergoes the mass growth given by the solid line in Fig. 3 and is embedded in a disk with the initial unperturbed density distribution. The upper (lower) line is for migration with unsaturated (saturated) coorbital torques. *Long-dashed line:* Consistent with type II migration, the line has slope $-1.5v/a$ and passes through $a = 0.8 a_0$ when $M_p \approx 0.9 M_J$. *Right:* Average disk density near the planet relative to the local initial value as a function of time. The density is averaged over a band of radial width $2H$ centered on the orbit of the planet (see text for details). Filled circles mark times when the mass ratio $M_p/M_0$ is equal to $5 \times 10^{-5}$ ($M_p = 16.7 \, M_J$) and when it is an integer multiple of $1 \times 10^{-4}$ ($M_p = 33.3 \, M_J$).

The torque per unit disk mass as a function of distance from the planet for the case in Fig. 6 is plotted in Fig. 7. The plot shows very similar behavior to the case of a stationary, non-growing planet seen in Fig. 1. Therefore, there is no evidence that planet migration or growth substantially affects the disk-planet torques for these model parameters. In particular, there is no evidence for strong coorbital torques.

The results obtained from a model with $H/r = 0.04$ (i.e., with a lower disk temperature compared to the model in Fig. 6) are shown in the left panel of Fig. 8. As in the case of the warmer disk, the type I migration tracks (short-dashed lines) reproduce reasonably well the radial migration from the simulation (solid line) while $M_p \lesssim 0.14 M_J$ (see long-dashed line in Fig. 5) or $R_h \lesssim 0.9 H$. As before, the right panel of Fig. 8 shows that the migration rate follows the type I tracks while the disk density near the planet remains close to the local unperturbed value. Again, when $M_p \gtrsim 0.75 M_J$, $\Sigma_0/\Sigma_0^0 \lesssim 0.03$ and $|\dot{a}/\dot{a}|$ is on the order of the viscous inflow velocity (long-dashed line).

Fig. 7.—Torque per unit disk mass on the planet as a function of normalized distance from the migrating and growing planet plotted in Fig. 6 ($\Sigma_0 = 3 \times 10^{-4} M_0 a_0^{-2} \approx 100$ g cm$^{-2}$ at the planet’s initial orbital radius and $H/r = 0.05$). The vertical scale is in units of $G M_p (M_0/M_J)^2/a$, where $a = \alpha t_0$. The solid, long-dashed, dot-dashed, and short-dashed lines refer to times when $M_p = 6.0 \, M_J$, 9.3 $M_J$, 0.36 $M_J$, and 1.0 $M_J$, respectively.

The dependence of migration on viscosity was investigated by running a simulation with kinematic viscosity $\nu = 1 \times 10^{-3} a_0^2 \Omega_0$ ($\alpha = 0.04$), 10 times the value in Fig. 6 with all other parameters being the same. The results are shown in Fig. 9. The left panel shows the orbital migration from the simulation as a solid line and the type I migration based on equations (17) and (18) as dashed lines. In this case, the relation $M_p = M_0(t)$ represented by a short-dashed line in Fig. 5 is used in equations (17) and (18). The long-dashed line indicates a migration at a constant rate of $|\dot{a}| \approx 0.7 v/a$, with $\alpha \approx 0.9 a_0$. The long-dashed line passes through a range of masses that spans from $0.2 \approx 1.2 M_J$. However, at $M_p \approx 1 M_J (t \approx 600$ orbits), the density gap along the planet’s orbit has not yet fully formed. This can be observed in the right panel of Fig. 9, which displays the averaged disk density near the planet normalized to the local unperturbed (initial) disk value. There is a drop of only a factor of 2.5 in the disk density near the planet by the time $M_p \approx 1 M_J$. The reason is that one of the conditions for steady state gap formation, $M_p/M_0 > 40 v/(a^2 \Omega_0) \sim 4 \times 10^{-3}$ (Lin & Papaloizou 1993), is not fulfilled in this higher viscosity case until $M_p \approx 4 M_J$. At about 780 orbits, $\Sigma_0/\Sigma_0^0 \approx 0.1$, but the planet mass has reached beyond 2 $M_J$ and is therefore more massive than the local disk mass. At those stages of the orbital evolution, inertia effects and further gap clearing are likely playing an important role in reducing the migration rate, as demonstrated in the next paragraph.

Figure 10 displays a comparison of the orbital radius evolution from two calculations. The solid line is the same as that in the left panel of Fig. 9. The dotted line with filled circles is the outcome of a three-dimensional simulation in which the planet mass is fixed at $M_p = 1 M_J$. Material is removed from the vicinity of the planet according to the usual procedure we apply (see § 3.1.1), but in this case it is not added to the mass of the planet. The planet’s orbit is held fixed for the first 100 orbital periods, after which time it is allowed to evolve under the action of disk torques. The plot shows that there is general agreement, while $M_p \approx 1 M_J$, with the variable mass model and that the effect of adding mass to the planet in this regime is to slow its migration rate.

The local viscous timescale, $t_\nu = r^2/\nu$, in the models presented in Figures 9 and 10 is about 1600 orbital periods at $r = \sigma_0$. Therefore, one might wonder whether the viscous evolution of the disk
at radii larger than the outer grid boundary has any significant impact on the orbital evolution of the planet. We address this issue in Appendix B and show that extending the disk farther out at larger radii does not affect the migration tracks shown in Figures 9 and 10. In Appendix B we also present results for cases with viscosity parameter $\alpha = 0.2$ (kinematic viscosity $\nu = 5 \times 10^{-4} a_0^2 \Omega_0$) that have $t_p \approx 320$ orbits at $r = a_0$. This case also leads to inward migration that can be interpreted as a type I regime, partially modified by the perturbed surface density of the disk.

4. TYPE III MIGRATION

Figures 6 and 8 indicate that a growing planet undergoes type I migration, as long as the disk density near the planet remains undepressed. At higher planet masses where the gap opening sets in, there is a smooth transition toward type II migration with migration speeds that are on the order of the viscous inflow velocity. There is no evidence for another form of migration, since the torque distributions are essentially the same in the migrating and nonmigrating cases explored thus far (compare Figs. 1 and 7). Type III migration was suggested to involve coorbital material that provides a fast form of migration (MP03). In this section we discuss planet migration for several variants on the models of §3 that should be favorable for a type III regime of migration. We describe a case that appears to exhibit type III migration.

4.1. Higher Disk Mass

Cooorbital torques are stronger for higher mass disks. Masses in the coorbital region are on the order of $8\pi R_{\text{H}}a\Sigma(a)$. For the model presented in Figure 6, involving disks of relatively low density, the coorbital disk mass is approximately equal to the planet mass when $M_p \approx 0.2 M_1$. We describe here results of three-dimensional calculations with initial surface densities $\Sigma_p = 9 \times 10^{-4} M_0 a_0^{-2} \approx 300$ g cm$^{-2}$ and $\Sigma_{p3} = 1.5 \times 10^{-3} M_0 a_0^{-2} \approx 500$ g cm$^{-2}$ at the planet’s initial orbital radius of $a_0 = 5.2$ AU. The mass evolution in the former case is plotted as the dashed line in Figure 3. The mass evolution in the latter case is similar, but the growth proceeds very rapidly, reaching about $1 M_1$ within 130 orbital periods. The resulting orbital radius evolution for both simulations is plotted in Figure 11 (left panel) along with the average disk density near the planet normalized to the local unperturbed value (right panel). For both cases presented in the figure, at earlier times ($t \leq 170$ and $\leq 100$ initial orbits, respectively), the simulated
migration rates are comparable to the type I rates. During that stage of the evolution, the coorbital region is more massive than the planet. In the model with initial $\Sigma p \approx 300 \, \text{g cm}^{-2}$ at 5.2 AU (upper migration track in Fig. 11), for times $t \lesssim 170$ orbits ($M_p \approx 0.3 \, \text{M}_J$) the ratio of coorbital region mass to planet mass is larger than 2. In the model with initial $\Sigma p \approx 500 \, \text{g cm}^{-2}$ at 5.2 AU (lower migration track in Fig. 11), for times $t \approx 100$ orbits ($M_p \approx 0.4 \, \text{M}_J$) the ratio of coorbital region mass to planet mass is larger than 3. However, during those stages, the results are generally consistent with the type I migration and some slowing at later times, with no indication of another form of migration.

To examine the situation in more detail, we plot the torque per unit disk mass as a function of distance from the planet in Figure 12. The plot shows very similar behavior to the case of a nonmigrating, non-growing planet seen in Figure 1, as well as to the case of a migrating, growing planet within a lower density disk presented in Figure 7. Again, there is no evidence that planet migration or growth substantially affects the disk-planet torques for the parameters adopted in these models. Furthermore, there is no evidence for strong coorbital torques dominating planet’s migration.

In carrying out calculations at higher disk masses, we have introduced a possible inconsistency between the orbital motion of the disk and the planet. The orbital motion of the planet is affected by the axisymmetric gravitational force of the disk. On the other hand, the motion of the disk near the planet is not affected by this force, since disk self-gravity is ignored. This difference in rotation rates can lead to an artificial increase in the planet migration rate (Pierens & Hure 2005; Baruteau & Masset 2008). This issue has some quantitative effect on our results in this section. But, the qualitative results (approximately following the expectations of standard type I and II theory) remain. We examine this issue further in Appendix C.

4.2. Higher Initial Planet Mass

We have shown that if a low-mass protoplanet is allowed to rapidly grow in mass while it migrates, the orbital radius evolution begins at the type I rate (eqs. [17] and [18]) and approaches the type II migration rate as a clean gap develops. Since the evolving planet gains mass at the fastest possible rate, the runaway accretion rate, the time available for gap clearing is relatively short. Such conditions should be favorable for migration dominated by coorbital torques. But as we saw in Figure 11, such situations only reveal type I and II migration. In this section we explore a more extreme situation for providing coorbital material. We consider the case that a planet of higher initial mass (higher than the 5 $\text{M}_J$ considered thus far) is suddenly immersed in a smooth disk. Gap clearing is then not initially present for the higher mass planets. More coorbital gas is available for affecting migration.

We consider a planet with initial mass $M_p = 0.3 \, \text{M}_J$ ($M_p/M_J = 3 \times 10^{-4}$) that is allowed to grow and migrate in a three-dimensional disk with initial density $\Sigma p \approx 300 \, \text{g cm}^{-2}$ at $a_0 = 5.2$ AU (same as the lower initial density disk in Fig. 11). $H/r = 0.05$, and $\nu = 1 \times 10^{-2} a_0^{-1} \Omega_0$. Its orbital radius evolution is plotted as a dotted line with filled circles in Figure 13, together with the migration track of the model that starts with $M_p = 5 \, \text{M}_J$ at $t = 0$ (plotted as a solid line). For purposes of comparison, the initial orbital radius $a_0$ for the dotted-line case is chosen to be the $a$-value of the solid-line case when its planet mass is also 0.3 $\text{M}_J$. Given the large initial mass of the planet for the dotted-line case, the mass growth is very rapid: the planet gains about 0.7 $\text{M}_J$ over the first ~50 orbits.
of evolution. The figure shows that, with these disk conditions, migration rates differ only for a brief period of time, but they soon converge to values compatible with orbital migration in the more relaxed disk (compare slopes of solid and dotted lines). The dashed lines in the figure show that the planet initially migrates at the type I rate. But it later slows to nearly the same rate as the solid-line case. Again, there is no indication of type III migration.

4.3. Nongrowing Planets in a Colder Disk

We consider the case of a nongrowing 0.3 $M_\oplus$ planet by removing gas mass near the planet without adding the mass of this material to the planet’s mass. This situation may mimic the effects of an efficient disk wind. These models differ from those in MP03 and DBL05, who considered nonaccreting planets, only with respect to the accretion boundary conditions near the planet and the time of planet release.

Unlike the mass removal case, the nonaccreting case may introduce a complication because of the buildup of gas within the planet’s Hill sphere, which can become more massive than the planet. It has been argued that inertia effects from material close to the planet could introduce complications in self-consistently analyzing the dynamics of the system (Papaloizou et al. 2007). Appendix D describes some effects of the nonaccreting boundary condition. To avoid this potential problem, we remove gas near the planet and ignore torques exerted by the gas on the planet within the inner half of the Hill sphere (by radius), where most of the bound gas resides (see Fig. 29 below).

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Fig. 12.—Torque per unit disk mass on the planet as a function of normalized distance for the migrating and growing planets plotted in Fig. 11. Left: Case with initial surface density at the initial orbit of the planet equal to $\Sigma_0 \approx 300 \, \text{g cm}^{-2}$. Right: Case with initial surface density at the initial orbit of the planet equal to $\Sigma_0 \approx 500 \, \text{g cm}^{-2}$. The vertical scale is in units of $GM_\ast(m/M_p)^2a$, where $a = a(t)$. The solid, long-dashed, dot-dashed, and short-dashed lines refer to times when $M_p = 6.0 M_\oplus$, $9.3 M_\oplus$, $0.36 M_\oplus$, and $1.0 M_\oplus$, respectively.

Fig. 13.—Migration with different initial conditions. Solid line: Orbital radius evolution of a planet with initial mass $M_p = 5 M_\oplus$ that interacts with a three-dimensional disk having initial surface density at the planet's initial radial position $\Sigma_0 \approx 300 \, \text{g cm}^{-2}$ at $a_0 = 5.2 \, \text{AU}$ (same as the upper migration track plotted in Fig. 11). It has mass $M_p = 0.3 M_\oplus$ at a time of about 165 orbits (see Fig. 3, dashed line), when $a \approx 0.92a_0$. Dotted line with filled circles: Orbital radius evolution of a planet with initial mass $M_p = 0.3 M_\oplus$ that interacts with the same initial unperturbed disk density distribution as the solid-line case has at time $t = 0$. The planet starts at the same radius ($a \approx 0.92a_0$) as the solid line where that planet has acquired a mass of 0.3 $M_\oplus$. The difference in the two cases is that the solid-line case has a partially cleared gap when $M_p = 0.3 M_\oplus$ (see Fig. 11, right panel), while the dotted-line case starts in a smooth unperturbed disk. Dashed lines: Orbital radius evolution of a planet according to type I theory (eqs. [17] and [18]) for a planet of fixed mass $M_p = 0.3 M_\oplus$ (lower line of pair for saturated coorbital torques) and disk density at $r = 0.92a_0$ for the unperturbed initial disk.

Fig. 14.—Orbital migration of a Saturn-mass planet of fixed mass ($M_p = 0.3 M_\oplus$) in a cold ($H/r = 0.03$) and high-mass disk ($\Sigma_0 \approx 670 \, \text{g cm}^{-2}$ at the planet’s initial position). Mass is removed from the disk near the planet to prevent a mass buildup there. The planet is embedded in a two-dimensional disk and held on a fixed orbit for $t_{rf} = 50$ (dotted lines), 100 (solid line), and 200 (long-dashed line) initial orbital periods. The dotted line with filled circles plots the migration track from a three-dimensional disk model with $t_{rf} = 50$ orbits. The orbital radius is in units of $a_0$. For $a_0 = 5.2 \, \text{AU}$, the unit of time is $\approx 12 \, \text{yr}$. The predicted type I migration tracks, assuming that the planet does not open a gap, are plotted for a two-dimensional (short-dashed line) and a three-dimensional (short-dashed line with filled circles) disk.
We are interested in seeing whether these situations could give rise to strong torques outside the Hill sphere that cannot be accounted for by type I or type II theory in the case of a planet of fixed mass. As we show below, there are conditions under which such strong torques occur in the coorbital region.

4.3.1. Simulation Setup

We consider a Saturn-mass (0.3 $M_J$) planet in a disk having $H/r = 0.03$. The initial (unperturbed) disk surface density varies as $\Sigma = \Sigma_0(a_0)/(a_0/r)^{3/2}$, with $\Sigma_0(a_0) = 2 \times 10^{-3} M_\odot \alpha_0^{-2} \approx 670 \text{ g cm}^{-2}$ and kinematic viscosity $\nu = 1 \times 10^{-5} \alpha_0^2 \Omega_0$. Most of these calculations are carried out in two dimensions since $R_{\text{Hill}} \approx 1.5H$. However, we checked that results from three-dimensional models are in general agreement with those from two-dimensional models (see dotted lines in Fig. 14). Simulations in three dimensions use the grid system outlined in §3. As in the three-dimensional case, the two-dimensional grid has a linear base resolution of 0.014$a_0$. In the coorbital region around the planet, the linear resolution is 0.02$R_H$. Since we intend to study some global properties of flow dynamics in the coorbital region, three grid levels extend $2\pi$ in azimuth around the star. Convergence tests at these grid resolutions are presented in Appendix D.

As anticipated above, here we assume that some process removes gas from the disk, according to the procedure detailed in §3.1.1, but that the planet mass remains constant. The migration rate of the fixed-mass planet is not substantially affected by the assumption that the gas is removed. In Appendix D we show that configurations with a nonaccreting planet result in similar migration tracks. Hence, our conclusions would apply to nonaccreting planets as well.

In MP03 and DBL05, the planet’s orbital radius was initially fixed for over 470 orbits, so that a steady-state disk gap would form before it underwent migration. Here we reconsider that configuration but examine cases where the planet’s initial orbital radius is fixed for a shorter time, only 100 orbits. This case is somewhat like that of §4.2, which has more gas in the coorbital region (lower line case in Fig. 11), but instead has a fixed-mass planet in a cooler disk. The following factors applied here should help increase the torques from the coorbital region: cooler disk, fixed planet mass, higher disk density, and reduced time on initially fixed orbit.

Figure 14 shows that the migration timescale of the planet is quite short and that it lengthens as the release time increases (and the gap deepens). We focus on the case with $t_{\text{release}} = 100$ (solid line), which has a migration timescale of order 100 initial orbital periods. Although short, this migration timescale is longer than the type I migration timescale that would be predicted if the planet did not open a gap (lower short-dashed line of pair). The planet does open a partial gap, as seen in Figure 15. So it might appear that a weakened form of type I migration, due to partial gap opening, could explain the simulated migration rate. However, we demonstrate below that the migration cannot be explained by the usual type I theory.

Figure 16 shows the torque distribution per unit disk mass exerted on the planet, as defined by equation (8). The torque density distribution at the time of release of the planet, 100 orbits after the start of the simulation, reveals a curve characteristic of type I torques. The distribution is similar to the cases plotted in Figure 1, although it is somewhat larger in magnitude, as expected by the lower sound speed of the gas and the two (rather than three) dimensions of the simulation. However, at later times the torque distribution changes character, with much larger values in the coorbital zone, within radial distances of about $2R_H \approx 0.1 a(t)$ from the orbit of the planet. In particular, there is substantial torque occurring in the radial band $|r - a| < R_H$, where $R_H = 0.046a$. We argued in §2.2.2 that this region involves only coorbital torques (not Lindblad torques). We have verified that this torque is not originating from within the planet’s Hill sphere (see Fig. 27 below). The contribution from within the Hill sphere is about 20% of the net torque at release time and generally less than about 10% at later times.

Figure 15 shows that the planet is migrating on a shorter timescale than that of gap opening. At release, the planet is fairly symmetrically positioned in the gap. Later, the planet lies much closer to the inner edge of the gap than to the outer edge, and the gap is less deep. Such a situation would be expected to lead to slower or even outward migration according to type I theory of Lindblad resonances, since the inner resonances (which provide outward migration) are more strongly activated than the outer

5 Note that, for a constant-mass planet and $\Sigma_0 \propto a^{-3/2}$, it follows that $\dot{a}_1$ is a constant (see eq. [17]).
resonances (which provide inward migration) due to the asymmetric density distribution near the planet. If fact, the slowing/stalling of inward migration due to the feedback from the inward disk density of a migrating planet was envisioned by Hourigan & Ward (1984) and Ward & Hourigan (1989) in their consideration of the inertial limit to planet migration. To quantify the effects of standard type I torques, we apply the type I torque distribution taken at the time of planet release in Figure 16. In doing so, we are ignoring pressure effects on $dT/dM$ due to the changing gap shape. We determine the type I torques at the later times by integrating this torque distribution (appropriately shifted to the instantaneous position of the planet) over the disk mass distributions in Figure 15. The torque is then given by

$$T_I(t) = 2\pi \int \frac{dT}{dM}(x, r_0) \Sigma(r, t) r dr,$$

where $x = (r - a)/a$ and $a = a(t)$. We find that the resulting type I migration rates at times of 10 and 20 orbits after release are outward and equal to $\dot{a} = 2 \times 10^{-4} a_0 \Omega_0$ and $4 \times 10^{-5} a_0 \Omega_0$, respectively. Clearly, results from the simulation are not consistent with the expectations of the usual type I migration theory. Instead, we claim that the effects of the corotation resonances are critical for migration here.

In the model by OL06, fast migration is due to torques caused by a density asymmetry in the coorbital region between gas on the leading and trailing sides of the planet. The gas on the leading side of the planet is trapped and contains gas acquired at other radii, while the trailing side contains ambient gas near the planet. The contrast between the trapped and ambient gas is limited by viscous diffusion. The trapped gas is in a quasi-steady advective-diffusive equilibrium. The density asymmetry and thus the torque are caused by the motion of the planet.

To test this model, we analyzed streamlines in the coorbital region in the frame comoving with the planet. We determine the streamlines in the simulations by following the motion of tracer particles that move with the velocity of the gas (see §D1). In Figure 17 we plot coorbital streamlines near before the planet is released, i.e., while the planet was on a fixed stationary orbit. The figure shows good agreement between the simulation and theory. The streamlines are symmetric between the leading ($\phi > \phi_p$) and trailing ($\phi < \phi_p$) sides of the planet. Figure 18 shows the streamlines after the planet is released, while the planet is migrating. Strictly speaking, these are not streamlines in the simulation case but trajectories, since the flow is not in a strict steady state in the comoving frame of the planet because the planet is migrating at a variable rate. The theoretical streamlines depend on the planet-to-star mass ratio and the migration rate of the planet. They are
calculated assuming a steady state and constant migration rate by means of the linear perturbation model of OL06. The theoretical streamlines were calculated by using intermediate parameter values from the simulation during the interval of planet migration: \( \dot{a} = -0.002a_0\Omega_0 \) and \( r = 0.85a_0 \). The simulated and theoretical streamlines in Figure 18 are in approximate agreement. They show closed streamlines on the leading side of the planet’s azimuthal motion. They contain the trapped gas described above. The open streamlines on the trailing side of the planet involve ambient gas that streams outward past the planet. The smaller closed streamlines are centered at about the same azimuth in the two plots, about 0.2\( \pi \) ahead of the planet. Figure 19 shows that the gas density asymmetry in the coorbital region between the leading and trailing sides of the planet increases with time. The unperturbed background density increases with time as the planet encounters higher density gas in its inward migration. Notice that the density increase is higher on the trailing side of the planet than on the leading side. This result suggests that the trapped gas approximately retains its initial density as the planet migrates. The gas on the trailing side more fully reflects the local density. The density asymmetry then gives rise to the dominant torque on the planet.

The OL06 model does not determine the value of coorbital corotational torque for a migrating planet. It does provide a detailed analysis for the noncoorbital corotational torque. In that case, the effect of migration is to amplify the standard coorbital torque for a nonmigrating planet (Goldreich & Tremaine 1979). By analogy, one might expect similar behavior in the coorbital case. The theoretical streamlines on the trailing side of the planet involve ambient gas that streams outward past the planet. The smaller closed streamlines are centered at about the same azimuth in the two plots, about 0.2\( \pi \) ahead of the planet. Figure 19 shows that the gas density asymmetry in the coorbital region between the leading and trailing sides of the planet increases with time. The unperturbed background density increases with time as the planet encounters higher density gas in its inward migration. Notice that the density increase is higher on the trailing side of the planet than on the leading side. This result suggests that the trapped gas approximately retains its initial density as the planet migrates. The gas on the trailing side more fully reflects the local density. The density asymmetry then gives rise to the dominant torque on the planet.

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4.4. Conditions for Type III Migration

The results in § 4.3.2 provide evidence for migration dominated by coorbital torques, or type III migration. Within the framework of the OL06 model we generalize the results of the simulations and describe some conditions that are favorable for type III migration.

As in § 4.3.2, we consider a planet of fixed mass. For a planet whose mass is large enough to open a gap, we apply the initial condition that the planet undergoes migration before steady gap formation completes, as was the case in § 4.3.2.

We require that the planet does not strongly deplete the gas in the coorbital region as it migrates. This requirement implies that the migration timescale across the coorbital region be shorter than the timescale to clear a gap over that region. Figure 15 demonstrates that this condition holds for the model in § 4.3.2.

To derive a crude estimate for this condition, we assume that the torques exerted by the planet lead to a local change in disk angular momentum over a region whose size is comparable to the coorbital region. Each one-sided torque (interior and exterior to the orbit of the planet) on the gas is capable of clearing a gap, while the net effect of both interior and exterior torques results in migration. The condition that the gap clearing timescale is longer than the migration timescale becomes

\[
M_c \gtrsim \frac{M_p}{A},
\]

where \( M_c \) is the mass of the coorbital region and \( A \) is the dimensionless torque asymmetry

\[
A = \frac{|T_i| - |T_e|}{|T_c|},
\]

where \( T_i \) and \( T_e \) denote the torques interior and exterior to the planet’s orbit, respectively.

Another condition is that the migration rate be large enough that there is a strong asymmetry in streamlines between the leading and trailing sides of the planet, as seen in Figure 18. According to OL06, the asymmetry is strong for migration rates greater than \( |\dot{a}_d| = 1.45\Omega aM_p/M_c \). Since the migration begins as type I migration (see short-dashed line in Fig. 16), the condition is that \( |\dot{a}_i| \gtrsim |\dot{a}_d| \). This condition is approximately

\[
\Sigma_p \gtrsim \frac{M_c}{a^2} \left( \frac{H}{a} \right)^2.
\]

Notice that the condition is independent of planet mass, since both \( \dot{a}_i \) and \( \dot{a}_d \) are linear in the planet mass.

We now apply these conditions to the model simulated in § 4.3.2. The asymmetry parameter is estimated as \( A \approx 0.3 \), from the initial torque distribution in Figure 16. The condition given by equation (21) is then satisfied for this model, since \( M_c \approx 7M_p \). The condition given by equation (23) is also (marginally) satisfied, since the initial density is \( \Sigma_p a^2/M_\star = 2 \times 10^{-3} \) and \( (H/a)^2 = 9 \times 10^{-3} \). None of the other models discussed in this paper satisfy both conditions.

It also appears that the condition that the planet mass is fixed (or slowly increasing) is important. The dashed line in the right panel of Figure 27 in Appendix D suggests that a planet that grows in mass at the runaway rate would not undergo rapid migration long enough to move very far. The slow-down is partly due to gap opening that reduces the torques.

The picture is then that a fixed-mass planet, initially undergoing sufficiently fast type I migration, develops strong coorbital torques due to asymmetric trapped gas. The situation is not simple, however. We saw in § 4.3.2 that the disk’s feedback to the planet’s motion might slow or halt type I migration, but the coorbital torques allow the inward migration to continue. We found that the rate of the resulting migration is actually slower than type I migration for a smooth disk. So the conditions given by equations (21) and (23) are suggestive only at this point and require further testing.

---

Fig. 19.—Radial average of the surface density within the coorbital region [defined as having radial extent \( 2R_g \leq \rho(\tau) \) of the planet’s orbit] as a function of azimuth at three different times: the time of planet release \( t_{\text{ini}} = 100 \) initial orbital periods (short-dashed line), \( t_{\text{ini}} + 10 \) initial orbital periods (long-dashed line), and \( t_{\text{ini}} + 20 \) initial orbital periods (solid line).
5. SUMMARY AND DISCUSSION

We have analyzed the evolution of migrating planets that undergo runaway gas accretion by means of multidimensional numerical simulations. The results agree with the predictions of type I and II migration (see Figs. 6–9) for a planet of time-varying mass that we obtain from simulations. The set of simulations includes cases with disk densities as low as the minimum-mass solar nebula value and as high as 5 times that value, viscosities $\alpha \approx 0.004$ and 10 times that value (also 50 times that value, $\alpha \approx 0.2$, in a test reported in Appendix B), disk temperatures corresponding to $H/r = 0.05$, and a colder case of $H/r = 0.04$. The mass accretion rates onto the planet (see Figs. 3 and 5) are in general agreement with previous determinations based on fixed-mass planets on fixed circular orbits, when comparing cases with the same planet mass and disk properties for which the growth timescale is longer than the gap opening timescale. Planet mass growth rates can be understood in terms of accretion within the Bondi radius at lower planet masses, accretion within the Hill radius at intermediate masses, and accretion through the gap at higher planet mass (see Fig. 4). Mass growth rates typically peak at the intermediate planet masses, a few tenths of a Jupiter mass.

An important diagnostic for the nature of the disk-planet torques is the scaled torque density distribution per unit disk mass as a function of the scaled radial distance from the planet. In the linear regime of the standard theory of disk-planet resonances, for a fixed form of the disk density distribution $\Delta \ln \Sigma / \Delta \ln r$ and gas properties (sound speed and viscosity), this scaled torque distribution should be universal, independent of planet mass and disk density value. We verified this universality for low-mass planets, although the distribution varies somewhat with planet mass for larger mass planets that open gaps (compare low-mass cases in Figs. 1, 7, and 12).

There is no fundamental distinction between the torques involved in type I and II migration. This follows from the near independence of the scaled torque density distribution with planet mass. Previous concepts of type II suggested that a planet in a clean gap would migrate inward like a test particle that follows the disk accretion. Here we describe a view for cases where the gap is not completely clear of material. The difference in type I and II rates is due to the mass density distribution of the gas that multiplies torque density in determining the torque on the planet.

In type II migration, the density distribution adjusts so that the net torque on the planet causes it to migrate at approximately the viscous evolution rate of the disk. The transition between the two forms of migration is quite smooth. To illustrate this point, we plot in Figure 20 the orbital evolution of the planet through the phase where gap formation sets in. The figure shows that the migration rate can be accounted for by standard type I theory, corrected for the gas depletion in the gap region, although there is no unique prescription to do this. In the figure, the theoretical curves (dashed lines) are determined by type I migration theory, equations (17) and (18), with the density $\Sigma$ taken to be the average value in a radial band of half-width 0.15$r_0$ (a typical gap width) centered on the planet.

For a given planet mass, the torque density diagnostic reveals that the distribution is not strongly affected by migration or accretion (Fig. 1 is quite similar to Figs. 7 and 12 for similar planet masses). In particular, there is no evidence for strong coorbital torques. However, in a certain case, we do find evidence for strong coorbital torques, or type III migration. This case has a planet of fixed mass, 0.3 $M_J$, that is immersed into a cold, smooth disk. The planet is held at a fixed orbit for relatively short time ($\sim 100$ orbits) before being released, so that gap clearing is incomplete. The torque distribution at the time the planet is released follows the expectations of the standard theory for nonmigrating planets. This can be seen by comparing the line for the 0.3 $M_J$ case in Figure 1 (dot-dashed line) with the short-dashed line (time $t = t_{db}$) in Figure 16. (There are differences in the magnitude of the distributions because of differences in gas sound speeds and dimensionalities of the calculations, but the forms of the distributions are similar.) At later times, Figure 16 reveals a transition to a completely different torque distribution, where coorbital torques play a critical role.

The strong coorbital torque can be understood in terms of an asymmetry in the streamlines between the leading and trailing sides of the planet, in accord with the analytic model of OL06 (see Figs. 17 and 18) and also along the lines of Artymowicz (2004). This asymmetry causes trapped material to persist on the leading side of the planet, which has a different density from the ambient gas that flows on the trailing side (see Fig. 19). The asymmetry gives rise to the coorbital torque. We suggest some criteria for this form of migration (see § 4.4). More exploration is needed to test them.

Although we find evidence for type III migration, the conditions required appear somewhat artificial, i.e., incomplete gap clearing (nonequilibrium gap) of a cool disk with a planet of fixed mass. It is not clear whether and/or how conditions for type III could arise in a more plausible evolution scenario.

We have generally assumed that the planet is able to accrete almost all gas the disk is able to provide (so-called runaway gas accretion). For a disk viscosity $\nu \geq 1 \times 10^{-2} a_0^2 \Omega_0$, the mass accretion rates are large. The time to build a 1 $M_J$ planet starting with a 5 $M_{\odot}$ planet in a minimum mass solar nebula is shorter than $\sim 10^5$ yr, substantially less than the observationally determined disk lifetimes of $\sim 10^6$ yr (Haisch et al. 2001; Flaherty & Muzerolle 2008). Over this $10^5$ yr time interval, the planet has radially migrated inward by only $\sim 20\%$ of its initial radius. In other words, for these models, the mass doubling timescale for a $M_p \leq 1 M_J$ planet is short compared to the migration timescale and the disk lifetime. This situation stands in strong contrast to the earlier phases of planet formation where the migration timescales are shorter.
than the planet mass doubling timescales and disk lifetimes (e.g., Ward 1997; Hubickyj et al. 2005).

The runaway accretion rates pose some challenges for explaining the mass distribution of planets (Butler et al. 2006). Typical accretion rates in T Tauri stars are \(\sim 1 \times 10^{-8} M_\odot \text{ yr}^{-1}\) (Hartmann et al. 1998). For a steady state unperturbed disk (without a planet), the accretion rate is given by \(3\pi \nu \Sigma\). The initial disks considered in this paper are not in a steady state, but this accretion rate provides a reasonable estimate. For the minimum-mass nebula model (Fig. 3), the kinematic turbulent viscosity we typically adopt, \(\nu \sim 10^{-5} a_0^2 \Omega_0\), was chosen so that the accretion rate evaluates to about this same value, \(\sim 1 \times 10^{-8} M_\odot \text{ yr}^{-1}\). In the case of a planet embedded in a disk, if there is a comparable accretion rate onto a planet of mass \(M_p \lesssim M_f\) (as found by Lubow & D’Angelo 2006), then the mass doubling timescale for a Jupiter-mass planet is about \(10^5\) yr, consistent with what we found in the simulations in this paper. But then it is not at all clear why planets would not almost always achieve masses higher than \(\sim 1 M_f\) in contradiction with the observed mass distribution of extrasolar planets and the case of Saturn. Special timing for disk dispersal could be invoked but may be artificial.

There are a few possible explanations. A colder disk \((H/r < 0.05)\) would experience stronger tidal truncation effects from a Jupiter-mass planet, as \(H\) becomes significantly smaller than \(R_0 \sim 0.07 a\). This effect could certainly reduce the accretion rate by a large enough factor (say, 10) so that the mass doubling timescale for a \(\sim 1 M_f\) planet would become of order \(10^5\) yr, consistent with the suggestion by Dobbs-Dixon et al. (2007). The issue then is how cold the disk would need to be. The model in this paper with \(H/r = 0.04\) (Fig. 5, long-dashed line) has the same unperturbed overall disk accretion rate as the \(H/r = 0.05\) case (Figs. 3 and 5, solid line), since \(\nu\) and the initial surface density \(\Sigma(r)\) are the same. The accretion rate onto the planet from the colder disk \((H/r = 0.04)\) differs from the case with \(H/r = 0.05\) by only 10%. Model \(b\) in Lubow & D’Angelo (2006) for a nonmigrating planet of fixed mass, having \(H/r = 0.03\), has an accretion rate that is \(~25\%\) less than model \(b\), which has \(H/r = 0.05\) and the same unperturbed overall disk accretion rate. Consequently, in this disk thickness range \((0.03 \leq H/r \leq 0.05)\), we find that the reduction in the accretion rate onto the planet due to cooler disks is not significant. For higher mass planets \((5\sim 10 M_f)\), we expect that the accretion rate will be reduced by tidal truncation effects to a level where the planet mass doubling timescale is comparable to the disk lifetime, as found in previous studies of planets on fixed orbits (Lubow et al. 1999; Bate et al. 2003; D’Angelo et al. 2003). This effect may set the upper limit to planet masses.

Another possibility is that there is a feedback effect that limits the gas accretion rate. Perhaps the heating of the protoplanet envelope by impacting solids continues to later times than is assumed in the standard core accretion model. Depletion of disk solids near the planet occurs in the standard core accretion model, when planet migration is not included. With migration, it is possible that continued accretion of disk solids would occur (e.g., Alibert et al. 2005), resulting in continued heating that could limit the gas accretion rate further. It is not clear how well this possibility works, since planetesimals will get trapped into resonances as the planet migrates (Zhou & Lin 2007). Having a higher mass solid core is problematic in the case of Jupiter, whose solid core mass is thought to be a small fraction of the total mass (see Guillot 2005; see also discussion in Lissauer & Stevenson 2007). It is also possible that winds emanating from the circumpalnetary disk within the planet’s Hill sphere could reduce the accretion rate onto the planet. Magetically driven winds are believed to play an important role in the case of young stars (Blandford & Payne 1982; Podritz & Norman 1986; Shu et al. 1994). There are likely differences in the flow properties from the stellar outflow case (Fendt 2003). However, it is not clear that the winds would be able to expel a large enough fraction (say, 90%) of the accreting gas to sufficiently reduce the accretion rate onto the planet.

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APPENDIX A

NUMERICAL SENSITIVITY STUDY

We conducted several tests to assess the sensitivity of the results presented in § 3 to the choice of various numerical parameters. Since the main objective of that section is the mass and orbital evolution of an embedded planet, we present here quantitative comparisons of planetary masses and orbital radii as a function of time.

A1. A RESOLUTION TEST

For purposes of a resolution study, we performed a three-dimensional calculation in which the grid resolution is raised by a factor of 3/2 over the standard resolution (see § 3) in each coordinate direction, throughout the entire disk domain, and on all grid levels. Note that such an increase implies an overall refinement gain of a factor of \((3/2)^3 \approx 3.4\), in terms of volume resolution of the system or number of grid elements. Nested grids cover extended disk regions, so that the planet always remains in the domain described by the most refined grid during the calculation. We focus on the disk model with initial surface density at the planet’s initial position \(\Sigma_p = 9 \times 10^{-4} M_\odot a_0^2\) (300 g cm\(^{-2}\)), \(H/r = 0.05\), and \(\nu = 1 \times 10^{-5} a_0^2 \Omega_0\). The planet’s mass and orbital radius evolution, obtained at standard grid resolution, are shown in Figure 3 (dashed line) and Figure 11 (topmost solid line), respectively. In order to carry out a quantitative comparison, results (for both \(M_p\) and \(a\)) from the two calculations, which will be labeled as 1 and 2, are averaged over time intervals of half of the (initial) orbital period, and then relative differences are computed as

\[
\frac{\Delta X}{X} = 2 \left( \frac{X_1 - X_2}{X_1 + X_2} \right),
\]  

(A1)
where $X$ is either $M_p$ or $a$. In order to give time-averaged estimates of the relative differences, over the course of the calculations, we perform a running-time average of quantity $\Delta X/\bar{X}$, which is defined by

$$
\left\langle \frac{\Delta X}{\bar{X}} \right\rangle = \frac{1}{t} \int_0^t \frac{\Delta X}{\bar{X}} \, dt'.
$$

(A2)

Results are shown in the left panels of Figure 21. The largest differences are observed in the results for the mass evolution (top left panel), after the onset of the rapid accretion phase (see dashed line in Fig. 3). The average difference, over the entire evolution, stays within 10%–15%. The average relative difference between the evolution results of the orbital radii (bottom left panel) is much smaller and remains well within 1%.

In order to test whether the orbital evolution is consistent with type I migration theory of Tanaka et al. (2002) at our standard resolution, we set up a three-dimensional disk model with a planet that grows at a prescribed rate in a three-dimensional disk whose initial (unperturbed) surface density has slope $s = -\ln \Sigma_0 / \ln a = 3/2$. Results from a calculation (solid line) are compared to predictions of type I theory (dashed line; see § 3.2.1). The top panel shows the orbital radius evolution, whereas the bottom panel shows the migration rate as a function of the planet mass.

![Figure 21](https://example.com/figure21.png)

**Fig. 21.**—Left: Running-time average, defined by eq. (A2), of the relative differences between two three-dimensional calculations, whose numerical resolutions differ by a factor of 3.4 in terms of grid elements (see text for details). Most of the difference in the mass growth of the planet (top) is accumulated between about 100 and 150 orbits, during the early phases of the rapid mass growth, when $M_p \sim 1 \times 10^{-4}$ (see dashed line in Fig. 3). Overall, the running-time average of $\Delta M_p/M_p$ is in the range 10%–15%. Relative differences between the evolution of the orbital radii (bottom) remain negligible over the course of the calculations, and the running-time average is contained within 1%. Right: Orbital migration of a planet that grows at a prescribed rate in a three-dimensional disk whose initial (unperturbed) surface density has slope $s = -\ln \Sigma_0 / \ln a = 3/2$. Results from a calculation (solid line) are compared to predictions of type I theory (dashed line; see § 3.2.1). The top panel shows the orbital radius evolution, whereas the bottom panel shows the migration rate as a function of the planet mass.

**A2. BOUNDARY CONDITION EFFECTS**

Boundary conditions may play some role and affect the late stages of the system’s evolution, especially when $M_p$ becomes on the order of a Jupiter mass. In our situation, the major concerns that may arise are related to the positions of the grid radial boundaries. The finite extent of the inner radius of the grid ($R_{\text{min}}$) may lead to an augmented depletion of the disk within the orbit of the planet. At the outer radial boundary ($R_{\text{max}}$), reflection of waves may affect torques exerted on the planet. The first effect can be mitigated by reducing the inner radius of the grid or adopting nonreflective boundary conditions (e.g., Godon 1996, 1997), whereas the second can be largely suppressed by choosing $R_{\text{max}} \gg a$. However, while increasing the outer grid radius possibly lengthens the computing time only because a larger number of grid elements in the radial direction is required (for a given value of $\Delta R$), decreasing the inner grid radius directly affects the time step of the calculation, which is proportional to $R_{\text{min}}^{3/2}$ because of the stability criterion imposed by the Courant-Friedrichs-Lewy condition.

The left panels of Figure 22 show a comparison of results obtained from a three-dimensional calculation ($\Sigma_0 = 9 \times 10^{-4} M_J a_0^{-2}$ at $t = 0$, $H/r = 0.05$, and $\nu = 1 \times 10^{-7} a_0^{-1}$) with nonreflective boundary conditions at $R_{\text{min}} = 0.4 a_0$ and one with $R_{\text{min}} = 0.19 a_0$ but outflow boundary conditions, as outlined in § 2.1.3. The position of the inner grid boundary affects the density distribution interior to the planet’s orbit. The effect on the planet’s mass is at the 3% level, on average over the entire course of the simulation (top left panel). The orbital radius evolution (bottom left panel) displays average relative differences much smaller than 1%. In the right panels of Figure 22,
results from a three-dimensional calculation with outer grid radius at \( R_{\text{max}} = 2.5a_0 \) are compared to those from a calculation in which \( R_{\text{max}} = 4.9a_0 \). The evolution of both planet mass (top right panel) and orbital radius (bottom right panel) is hardly affected by the position of the outer grid boundary.

### A3. EFFECTS OF EXCLUDED TORQUES

The region of space in which material is gravitationally bound to the planet depends on several disk parameters (including \( H/r \) and \( \nu \)) and on the planet’s mass. Calculations with fixed-mass and fixed-orbit planets indicate that, for \( H/r \approx 0.05 \) and \( \nu \approx 1 \times 10^{-5}a_0^2\Omega_0 \), only material within about \( 0.3r_H \) is gravitationally bound to the planet when \( 5 M_p \leq M_f \leq 40 M_p \) (Hubickyj et al. 2007). Analytical and numerical models of disk formation around a Jupiter-mass planet suggest that such disks may extend over a distance of about \( R_H/4 \) (or less) around the planet. In the calculations presented in §§ 3 and 4, torques originating within \( R_H/2 \) of the planet are not taken into account, which may include nonzero net torques exerted by unbound material. In order to estimate how this choice affects the evolution of the orbital radius, we also considered cases in which only torques from within \( 0.3r_H \) are neglected. The models with initial surface density \( \Sigma_0 = 3 \times 10^{-3}M_0a_0^{-2} \) and \( 9 \times 10^{-4}M_0a_0^{-2} \) at \( r = a_0 \) (\( H/r = 0.05 \), \( \nu = 1 \times 10^{-5}a_0^2\Omega_0 \)), discussed in §§ 3.1.2 and 3.2.2, are restarted at regular time intervals, covering entirely their respective mass range. The evolution is then integrated for time periods of \( \geq 100 \) orbits at each restart. We compare the evolution of orbital radii by measuring the relative differences \( \Delta a/a \) and find that for none of the cases considered does \( |\Delta a/a| \) exceed 1%. We therefore conclude that excluded torques from unbound material have only marginal effects on the results presented in §§ 3.2 and 4.1.

### APPENDIX B

### MIGRATION IN HIGH-VISCOSITY DISKS

Results presented in § 3.2 indicate that the orbital radius evolution of a growing planet can be described reasonably well in terms of standard type I and II regimes of migration (as long as the local disk mass is comparable to or larger than the planet’s mass). This conclusion holds when the disk’s kinematic viscosity is \( \nu \approx 10^{-5}a_0^2\Omega_0 \) (see Fig. 6), as well as when \( \nu \approx 10^{-4}a_0^2\Omega_0 \) (see Fig. 9), which brackets a range of \( \alpha \)-parameters between \( 4 \times 10^{-3} \) and \( 4 \times 10^{-2} \) at the location of the planet. Here we present further analysis of a case with \( \nu = 1 \times 10^{-4}a_0^2\Omega_0 \) and examine cases with \( \nu = 5 \times 10^{-4}a_0^2\Omega_0 \) (\( \alpha \approx 0.2 \)), which still result in a form of type I migration modified by the perturbed surface density of the disk.

#### B1. AN ADDITIONAL MODEL WITH \( \nu = 1 \times 10^{-4}a_0^2\Omega_0 \)

As anticipated in § 3.2.2, some concern may arise when the viscous timescale, \( t_v = r^2/\nu \), at \( R_{\text{max}} \) becomes comparable to the length of time over which the orbital evolution of the planet is calculated. However, it is unlikely that the disk’s viscous evolution at \( R > R_{\text{max}} \) has...
a large impact on the results displayed in Figure 9 since \( t_\nu \) at \( R = R_{\text{max}} \) is more than 6 times as long as the viscous timescale at \( a \), about \( 10^4 \) (initial) orbital periods of the planet. Therefore, at this viscosity level, we may experience some effects only over simulations covering timescales longer than \( 10^4 \) orbits (note that inward migration will increase even further this timescale).

In order to address this issue more in detail, we set up a three-dimensional model with \( \nu = 1 \times 10^{-4} a_0^2 \Omega_0 \) and \( H/r = 0.05 \) at time \( t = 0 \) (thinner solid line) and at times when \( M_p = M_p(t) \equiv 0.1 M_1 \) (short-dashed line), \( 0.3 M_1 \) (long-dashed line), and \( 1 M_1 \) (thicker solid line). Right: Migration rate \( (da/dt) \) as a function of the planet mass \( (M_p) \) over the first 330 orbital periods of the simulation, during which time the planet’s orbit is held fixed. The planet mass growth is prescribed. Migration rates are evaluated by means of Gaussian perturbation equations. The upper and lower dashed lines indicate type I migration rates predicted by eqs. (17) and (18), respectively.

![Diagram](image)

**Fig. 23.** Left: Azimuthally averaged surface density of a three-dimensional disk with \( \nu = 1 \times 10^{-4} a_0^2 \Omega_0 \) and \( H/r = 0.05 \) at time \( t = 0 \) (thinner solid line) and at times when \( M_p = M_p(t) \equiv 0.1 M_1 \) (short-dashed line), \( 0.3 M_1 \) (long-dashed line), and \( 1 M_1 \) (thicker solid line). Right: Migration rate \( (da/dt) \) as a function of the planet mass \( (M_p) \) over the first 330 orbital periods of the simulation, during which time the planet’s orbit is held fixed. The planet mass growth is prescribed. Migration rates are evaluated by means of Gaussian perturbation equations. The upper and lower dashed lines indicate type I migration rates predicted by eqs. (17) and (18), respectively.

B2. A MODEL WITH \( \nu = 5 \times 10^{-4} a_0^2 \Omega_0 \)

We wish to examine here whether the migration trend observed in raising the viscosity from \( \nu \sim 10^{-5} a_0^2 \Omega_0 \) to \( \sim 10^{-4} a_0^2 \Omega_0 \) persists at larger viscosity. As shown in the left panel of Figure 23 (solid line), when \( \nu = 1 \times 10^{-4} a_0^2 \Omega_0 \), a Jupiter-mass planet is able to open only a
shallow gap along its orbit (there is a drop in density of about a factor of 3 relative to the value just outside the gap). This is because gap opening conditions are not satisfied (see § 3.2.2). At larger disk viscosity we therefore expect an even shallower gap.

We perform two three-dimensional simulations with the same setup as that outlined above but with $\nu = 5 \times 10^{-4} a_0^2 \Omega_0$ ($\alpha = 0.2$ at $r = a_0$) and outer radial boundaries at $R_{\text{max}} = 6.7 a_0$ and $13 a_0$, respectively. The linear base resolution is $\Delta R = a_0 \Delta \theta = a_0 \Delta \phi = 0.014 a_0$, while the resolution in the coorbital region around the planet is $9 \times 10^{-4} a_0$. The viscous diffusion timescale, $t_v$, at $r = a_0$ is approximately 320 (initial) orbital periods, whereas $t_v$ at $R_{\text{max}} = 6.7 a_0$ is over $1.4 \times 10^4$ orbits (and $5.4 \times 10^4$ orbits at $R_{\text{max}} = 13 a_0$). We also consider a two-dimensional version of such models, having the same grid structure and resolution in the $r$-$\phi$ plane and outer grid boundary located at $r_{\text{max}} = 13 a_0$ (nearly 68 AU from the central star). The planet mass grows, at a prescribed rate, from $M_p \approx 3 M_\oplus$ to $1 M_J$ over about 330 periods (which is similar to $t_u$ at the planet position), while the planet’s orbit is held fixed. The left panel of Figure 25 shows the initial surface density (thin solid line) and the azimuthally averaged density profile when $M_p = 1 M_J$ for the three-dimensional model with $R_{\text{max}} = 6.7 a_0$ (thick solid line) and the two-dimensional model (dotted line with filled circles). As a reference, the azimuthally averaged density for the case with viscosity $\nu = 1 \times 10^{-4} a_0^2 \Omega_0$ and $M_p = 1 M_J$ is also plotted as a dashed line (same as the thick solid line in the left panel of Fig. 24). In the overlapping disk region, two- and three-dimensional calculations give consistent results. No significant deviations from the initial density distribution are observed at $r \gg a$.

![Figure 24](image1.png)

**Fig. 24.** Left: Orbital radius evolution of a 1 $M_J$ planet in a disk whose kinematic viscosity is $\nu = 1 \times 10^{-4} a_0^2 \Omega_0$ (H/$r = 0.05$). The solid line represents the model discussed in this appendix, while the dotted line with filled circles is the same as Fig. 10. Note that the two models produce very similar migration tracks (see inset), although they use grids with different outer radial boundaries, at $6.7 a_0$ and $2.5 a_0$, respectively, and although the surface density profiles at $r \leq a$ are different. In both calculations, the planet’s orbit is held fixed over the first $t_u$ (initial) orbital periods (see text for details). The dashed (straight) line has a slope about equal to $-7 \times 10^{-4} a_0^2 \Omega_0$. Right: Cumulative torque at time $t = t_u + 600$ initial orbital periods (thicker line), in units of $GM_p a / a$, where $a = a(t)$. See text for an explanation of the thinner line. Nearly all torque is exerted by material within a radial distance of $0.25 a$ from the planet’s orbit.

![Figure 25](image2.png)

**Fig. 25.** Left: Azimuthally averaged surface density of simulations with disk kinematic viscosity $\nu = 5 \times 10^{-4} a_0^2 \Omega_0$ and $H/r = 0.05$. The planet mass grows from $M_p \approx 3 M_\oplus$ to $1 M_J$ at a prescribed rate, over roughly 330 initial orbital periods. The planet’s orbit is fixed during this time interval. Thin solid line: Initial surface density distribution (same as the thin solid line in the left panel of Fig. 23). Thick solid line: Averaged surface density at time when $M_p = 1 M_J$ from a three-dimensional model whose outer radial boundary is $R_{\text{max}} = 6.7 a_0$. Dotted line: Surface density for the case with viscosity $\nu = 1 \times 10^{-4} a_0^2 \Omega_0$ and $M_p = 1 M_J$. Right: Orbital radius evolution after release, $t_u = 340$ initial orbits, obtained from the three-dimensional model with $R_{\text{max}} = 6.7 a_0$ (solid line) and the two-dimensional model (dotted line with filled circles). Note that the orbital evolution covers more than 1.8 local viscous timescales. The inset shows migration tracks obtained from the three-dimensional calculations over $t_u$/2 viscous timescales at the initial orbital radius of the planet. The filled diamonds represent data from the case with $R_{\text{max}} = 13 a_0$. The solid line represents the same model as in the main panel.
The viscosity condition for gap opening requires that $M_p/M_* \gtrsim 0.02$ (see § 3.2.2), or $M_p \gtrsim 20 M_*$ in order for gravitational torques to overcome viscous torques. In fact, the density distribution in the left panel of Figure 25 (thick solid line and dotted line with filled circles) shows a form of rather shallow gap. Hence, type II migration should not be expected.

The planet is released from its fixed orbit at time $t_{\text{fhi}} = 340$ ($M_p = 1 M_*$ for $t \geq t_{\text{fhi}}$), and the orbit is integrated for 1.8$t_0$ viscous timescales at $r = a_0$. The right panel of Figure 25 displays the orbital radius evolution for the two-dimensional (dotted line with filled circles) and three-dimensional (thick solid line) simulations. The two migration tracks in the main panel closely follow one another. A comparison between the results obtained from the three-dimensional models, over $t_0/2$ at the initial orbital radius of the planet, is shown in the inset. Again, there is no indication that the disk’s evolution at $r \gg a$ has a significant influence on the planet’s migration. As for the case discussed above, nearly all the torque is accumulated by material within a radial band $|r - a| \leq 0.25a$ centered on the planet’s orbit.

The rate of migration after release time is approximately $-8 \times 10^{-5} a_0^2 \Omega_0$, which is similar to that of the solid line in the left panel of Figure 24. This near equality is expected for type I migration, since it is independent of the level of disk viscosity. We use the torque per unit disk mass, $dT/dM$, at $t = t_{\text{fhi}}$ from the model with $\nu = 1 \times 10^{-4} a_0^2 \Omega_0$ discussed above in § B1 and the averaged surface density profile in the left panel of Figure 25 (thick solid line). By applying equation (20), we estimate the total torque expected under the assumption that $dT/dM$ has similar shapes in the two models. There will be some dependence of $dT/dM$ on viscosity, since viscosity affects the resonance widths. This estimate yields a migration rate that agrees within a factor of 1.7 with the value stated above, indicating that the intrinsic character of the torque per unit disk mass is roughly similar in these two cases.

APPENDIX C

CORRECTIONS FOR DISK GRAVITY

As discussed in § 4.1, there is a possible artificial torque that can act on a planet surrounded by a massive disk, when the planet responds to the gravity of the disk but the disk self-gravity is not included (Pierens & Huré 2005). This torque is a consequence of the disk’s axisymmetric gravitational force in changing the planet’s orbital rotation rate, but not changing the disk’s rotation rate (since the disk is not self-gravitating). This artificial difference leads to a shift in disk resonances that in turn leads to an artificial increase in the planet’s inward migration. It can largely be remedied by forcing the planet to rotate at the local Keplerian rate, i.e., at the same speed as the gas rotates apart from effects of gas pressure. In this prescription, the planet responds to the nonaxisymmetric forces of the disk that result in migration, while undergoing orbital motion at the Keplerian rate. This scheme is in reasonable accord with simulations that include the full effects of disk self-gravity (Baruteau & Masset 2008). The full effects of self-gravity cause a slightly faster migration rate than this approximation suggests. We have carried out three-dimensional simulations with such imposed Keplerian planetary orbits for various disk mass cases discussed in §§ 3 and 4. For mass distributions and disk thicknesses, as those applied in § 3, migration tracks show negligible differences, over the entire planet mass range. The only cases that produce changes beyond a few percent in migration rates are those in § 4. In Figure 26 we plot the resulting migration for the same disk models as in Figure 11. The migration rates are slower, as found by Baruteau & Masset (2008). However, they are still in approximate agreement with the predictions of migration theory.

APPENDIX D

ADDITIONAL TESTS ON FAST MIGRATION

Simulations of the orbital evolution of a fixed Saturn-mass planet ($M_p = 0.3 M_*$) in a cold ($H/r = 0.03$) and massive disk ($\Sigma_0 = 2 \times 10^{-3} M_0 a_0^{-2} \approx 670$ g cm$^{-2}$ at the planet’s initial orbital radius) can lead to a buildup of gas within the planet’s Hill sphere, which is eventually halted when a sufficiently large pressure gradient is established. The mass of material that accumulates around the planet can
exceed the planet’s mass, with possible effects on migration rates. In order to prevent the accumulation of gas within the Hill sphere, in the models presented in §4.3 we applied accreting boundary conditions near the planet, without adding the gas mass to the planet mass. In this appendix we wish to reconsider the nonaccreting configuration (as in MP03 and DBL05). The nonaccreting approach may be considered to be crudely simulating a case where some process prevents the planet from gaining further mass.

In the left panel of Figure 27, migration tracks from a two-dimensional model with an accreting planet (solid lines) are compared to those obtained from a two-dimensional model with a nonaccreting planet (dotted lines). The gas masses within the Hill spheres are drastically different: $0.07 M_\text{p}$ and $1.6 M_\text{p}$ in the accreting and nonaccreting planet cases, respectively. In the nonaccreting case, the mass of the gas ($1 M_\text{p}$) within the bound region (see Fig. 29, right panel) should be added to the inertial mass of the planet, thereby slowing migration (Papaloizou et al. 2007). This effect is indeed seen at early times (less than 10 orbits after release). The migration of the nonaccreting planet with bound gas (dotted line) is slowed by about a factor of 2 relative to the accreting case (solid line). At later times, the migration rates are closer, although it is not clear why. The reason may be related to our determination that the mass of “bound” gas decreases at later times in the nonaccreting case (see also right panel of Fig. 29).

The left panel of Figure 27 also plots the orbital evolution when torques from within the Hill sphere are not taken into account (solid and dotted lines with filled circles). The similarity of these migration tracks to the others plotted in the figure indicates that torques from gas in this region do not dominate the migration rates in this particular case.

In the nonaccreting case, dense gas that accumulates around the planet could be thought of as forming an envelope, once it becomes bound to the planet. A massive envelope would then participate in both the gravitational and inertial mass of the planet. We set up a three-dimensional model, with the same disk properties mentioned above (see §4.3.1 for details) and planet mass $M_\text{p} = M_\text{c} + M_\text{e}$, where $M_\text{c} = 0.3 M_\text{J}$ is a “core” mass and $M_\text{e} = M_\text{e}(t)$ is the mass of the gas within $R_H/4$ of the planet. Given the large initial mass in the
coorbital region ($\sim 2 M_J$), the planet rapidly gains mass, growing beyond $1 M_J$ in less than 25 initial orbits. The planet is released in a smooth disk after a few orbits. The orbital radius evolution is shown as a dashed line in the right panel of Figure 27, together with those from three-dimensional models with a fixed-mass planet and accreting (solid line) and nonaccreting (dotted line) boundary conditions near the planet. The initial migration rates are similar in all three configurations, but migration starts to rapidly slow down when the planet mass, in the dashed-line case, grows beyond $M_p \approx 0.8 M_J$. This behavior resembles that seen in Figure 13 (dotted line with filled circles).

Figure 28 displays numerical convergence tests for the accreting (left panel) and nonaccreting (right panel) planet models presented in the left panel of Figure 27. The two simulations in the left panel have coarsest (linear) resolutions in the coorbital region that differ by a factor of 4, in both radial and azimuthal directions. Calculations in the right panel have resolutions in the coorbital region around the planet that differ by a factor of 2 in each direction.

D1. GAS BOUND TO THE PLANET

In the calculations with an accreting planet, torque contributions from within $R_H/2$ of the planet are ignored. By following fluid paths, here we show that most of this material is captured and eventually accreted by the planet.

In a nonstationary flow, streamlines can be used as a proxy for fluid trajectories only over short distances and periods of time. Therefore, we track trajectories of fluid parcels by deploying tracer (massless) particles in the flow and then following their motion. This procedure allows us to obtain a reliable determination of fluid paths regardless of whether the flow is close to or far from steady state.

The equations of motion of each particle are integrated every hydrodynamical time step by interpolating the velocity field at the particle’s location and by advancing its position in time via a second-order Runge-Kutta method. Both spatial and temporal interpolations are performed by using the velocity field with the highest resolution available, i.e., that belonging to the most refined grid level in which the particle resides. The spatial interpolation is based on a monotonized harmonic mean (van Leer 1977), which is second-order accurate and capable of handling discontinuities and shock conditions. Hence, trajectories are formally second-order accurate in both space and time.

Here we employ tracer particles to estimate the size to the region occupied by gas bound to nonmigrating planets. Tracers are deployed in the disk within about $2 R_H$ of a Saturn-mass planet ($M_p = 0.3 M_J$). We use both models with and without accreting boundary conditions near the planet discussed in this appendix (Fig. 27, left panel) and in § 4.3. Figure 29 shows the distance from the planet, $S$, of particles as a function of time. Tracers are deployed at a time $t_d$, when the mass within the Hill sphere has reached a nearly steady value.

The distance along the trajectories is normalized to the Hill radius, $R_H$. The left panel refers to the accreting planet case, whereas the right panel refers to the nonaccreting planet case. Thin lines mark distances of trajectories that are not captured and thus return to the disk. Thick lines mark distances of trajectories that are captured in the planet’s gravitational potential (left panel) or otherwise remain within $R_H/2$ of the planet, over the simulated evolution (right panel). In the accreting case, bound trajectories rapidly decay toward the planet and it is therefore possible to make a clear distinction between bound and unbound trajectories. In the nonaccreting case, the distinction is less clear and may apply only over a given amount of time.

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