On a ‘universal’ class of WZW-type conformal models

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Abstract

We consider a class of sigma models that appears from a generalisation of the gauged WZW model parametrised by a constant matrix $Q$. Particular values of $Q$ correspond to the standard gauged WZW models, chiral gauged WZW models and a bosonised version of the non-abelian Thirring model. The condition of conformal invariance of the models (to one loop or $1/k$-order but exactly in $Q$) is derived and is represented as an algebraic equation on $Q$. Solving this equation we demonstrate explicitly the conformal invariance of the sigma models associated with arbitrary $G/H$ gauged and chiral gauged WZW theories as well as of the models that can be represented as WZW model perturbed by integrably marginal operators (constructed from currents of the Cartan subalgebra $H_c$ of $G$). The latter models can be also interpreted as $[G \times H]/H$ gauged WZW models and have the corresponding target space couplings (metric, antisymmetric tensor and dilaton) depending on an arbitrary constant matrix which parametrises an embedding of the abelian subgroup $H$ (isomorphic to $H_c$) into $G \times H$. We discuss the relation of our conformal invariance equation to the large $k$ form of the master equation of the affine-Virasoro construction. Our equation describes ‘reducible’ versions of some ‘irreducible’ solutions (cosets) of the master equation. We suggest a classically non-Lorentz-invariant sigma models that may correspond to other solutions of the master equation.

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1. Introduction

There exists a large class of solutions of string equations related to (gauged) WZW models that were actively discussed recently. This suggests to look for other conformal solutions based on similar WZW-type models as well as to try to understand the existence of known solutions in a systematic way. Below we shall consider a ‘universal’ model that contains other known models as special cases. Particular cases include $\sigma$-models which correspond to gauged and chiral gauged WZW theories, bosonised version [1] of the non-abelian Thirring model [2] and WZW model perturbed by integrably marginal $J\bar{J}$ operators [3].

Part of the original motivation behind this work was to explore a possibility to derive the Virasoro master equation [4][5] as a condition of conformal invariance of a standard off shell Lorentz invariant sigma model. Since the affine-Virasoro construction [4][5] (see also [6][7][8][9][10]) contains the affine-Sugawara and coset models [11][12] as special cases and is parametrised by a constant matrix $L^{ab}$ it is natural to start with a $\sigma$-model that also generalises the corresponding field theories – the WZW model [13][14] and the gauged WZW model [15][16][17].

In Section 2 we shall present the classical Lagrangian and Hamiltonian of our model which depends on a constant matrix $Q$ and explain how various known models correspond to special values of $Q$.

In Section 3 we shall put the action in the form of a $\sigma$-model one and compute the corresponding conformal anomaly coefficients (‘$\beta$-functions’) in the leading order approximation in $\alpha' = 2/k$. We shall find that the conditions of conformal invariance reduce to an algebraic equation for the basic matrix $K$ (related to $Q$) which is equal to the constant part of the target space metric in normal coordinate system. This equation can be derived from a ‘central charge action’.

1 The 2d field theory action suggested in [6] does not include the case of the coset models and is not of the standard $\sigma$-model type since it contains some extra fields (Lagrange multipliers) which are necessary for its Lorentz invariance.
In Section 4 we shall study the solutions of the conformal invariance equation and, in particular, demonstrate that the gauged and chiral gauged \( G/H \) WZW \( \sigma \)-models are conformal invariant for arbitrary simple \( G \) and \( H \). We shall also show that the \( \sigma \)-model perturbation theory gives the expected expressions for the central charges. We shall find another class of solutions with \( K \) depending on an arbitrary constant \( r \times r \) \( (r = \text{rank } G) \) matrix \( \rho \) and interpret them in terms of \([G \times H]/H\) gauged WZW models with \( H \) being an abelian group of dimension \( r \) and \( \rho \) being related to the coefficients that parametrise an embedding of \( H \) into \( G \times H \). The equation can be solved explicitly in the case of \( G = SU(2) \) when no new solutions (except the already mentioned above) are found.

In Section 5 we shall discuss the relation of our conformal invariance equation to the large \( k \) limit of the Virasoro master equation \([4][5]\). In general, the solutions of the two equations form different but intersecting sets (with the common solutions apparently been cosets only). We shall suggest that while the master equation has only ‘irreducible’ solutions (like standard WZW and cosets) our equation contains also ‘reducible’ ones (like chiral gauged WZW and its generalisation equivalent to \([G \times H_c]/H_c \) coset model). It should be possible to represent the latter (using chiral combinations of original non-chiral currents) as direct products of ‘irreducible’ solutions. Solutions of our equation correspond to field theories that are manifestly Lorentz invariant off the conformal point. At the same time, the action that corresponds to the off conformal point extension of the large \( k \) limit of the Hamiltonian of a generic affine-Virasoro construction is not Lorentz invariant \([6]\) (see also \([10]\)). We shall suggest a class of non-Lorentz-invariant \( \sigma \)-models that may have their conformal invariance conditions being related to the master equation.

2. The model

The model we shall study below can be represented as the following generalisation of the gauged WZW model \( (S_Q = kI_Q) \)

\[
I_Q(g, B) = I(g) + \frac{1}{\pi} \int d^2 z \ Tr \left[ -B\bar{\partial}gg^{-1} + g^{-1}\partial g\bar{B} + g^{-1}Bg\bar{B} + B(Q - 2I)\bar{B} \right]
\]
\[ = I(g) + \frac{1}{\pi} \int d^2z \, \text{Tr} \left[ -B^a \bar{J}_a + \bar{B}^a J_a + B^a (C_{ab} + Q_{ab} - 2\eta_{ab})\bar{B}^b \right], \quad (2.1) \]

where \( a, b, \ldots \) are the indices from the Lie algebra of a group \( G \) (which we shall assume to be simple) and \( I \)

\[ I \equiv \frac{1}{2\pi} \int d^2z \, \text{Tr} \left( \partial g^{-1} \bar{g} \right) + \frac{i}{12\pi} \int d^3z \, \text{Tr} \left( g^{-1} dg \right)^3, \quad (2.2) \]

\[ g^{-1} \partial g = J^a T_a, \quad \bar{g} g^{-1} = \bar{J}^a T_a, \quad J_a = \eta_{ab} J^b, \quad \bar{J}_a = \eta_{ab} \bar{J}^b, \quad B = B^a T_a, \quad \bar{B} = \bar{B}^a T_a, \]

\[ \text{Tr} \left( T_a T_b \right) = \eta_{ab}, \quad C_{ab} \equiv \text{Tr} \left( T_a g T_b g^{-1} \right), \quad C^T C = I, \quad (C)_{ab} = \eta^{ac} C_{cb}. \quad (2.3) \]

The model is parametrised by a constant matrix \( Q_{ab} = \text{Tr} \left( T_a Q T_b \right) \). In what follows we shall take \( Q \) to be symmetric though it would be interesting to generalise the discussion by relaxing this assumption.

We shall also assume that the vector field \((B, \bar{B})\) has the form (this will be important in the case when \( Q_{ab} \) is degenerate)

\[ B^a = Q^a_b A^b, \quad \bar{B}_a = Q^a_b \bar{A}^b, \quad Q^a_b = \eta^{ac} Q_{cb}, \quad Q^T = Q. \quad (2.4) \]

Then the \( B(...)\bar{B}\)-term in (2.1) takes the form

\[ A_a M^a_b \bar{A}^b, \quad M \equiv Q(C + Q - 2I)Q. \quad (2.5) \]

The standard WZW theory corresponds to the limit \( Q \to \infty I \) when \( B, \bar{B} \) decouple. The action of the gauged \( G/H \) WZW model (with a vector subgroup \( H \)) is recovered when

\[ \text{gauged WZW (vector)}: \quad Q = P, \quad P^2 = P, \quad (2.6) \]

\[ \text{We shall use the following conventions: } [T_a, T_b] = i f^c_{ab} T_c, \quad f_{abc} f_{abd} = c_G \eta_{cd}, \quad f_{abc} = \eta_{ad} f^d_{bc}. \quad (G) \]

\[ \text{We shall use } \eta_{ab} \text{ to raise and lower indices and to contract repeated indices. Depending on a context, both } \eta_{ab} \text{ and } \delta^a_b \text{ will be denoted by } I. \text{ We shall assume that } (ab) = \frac{1}{2} (ab + ba), [ab] = \frac{1}{2} (ab - ba), \text{ etc. We shall use the same letters } G \text{ and } H \text{ for the algebras of } G \text{ and } H. \text{ } H_c \text{ will denote the maximal abelian (Cartan) subalgebra of } G, \dim H_c = r. \]

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where \( P_{ab} \) is the projector on the Lie algebra of the subgroup \( H \). The case when \( H \) is the axial subgroup (which is not anomalous if \( H \) is abelian, see e.g. [18]) corresponds to

\[
\text{gauged WZW (axial)}: \quad Q = 3P . \quad (2.6')
\]

Another special limit of (2.1) is the chiral gauged \( G/H \) WZW model [19][18][20][21]

\[
\text{chiral gauged WZW}: \quad Q = 2P , \quad (2.7)
\]

when the matrix \( M \) in (2.5) is equal simply to \( PCP \). The model (2.1) with \( Q_{ab} = (a+1)P_{ab} \) where \( a \) is a number was discussed in [21][22].

Both the gauged and chiral gauged WZW models are conformal invariant at the quantum level since their actions can be represented as combinations of independent WZW actions

\[
I(g, A) = I(\hbar^{-1}gh\bar{h}) - I(h^{-1}\hbar) , \quad I_{\text{chir}}(g, A) = I(\hbar^{-1}gh\bar{h}) - I(h^{-1}) - I(\hbar) , \quad (2.8)
\]

where \( A \) and \( \bar{A} \) have been parametrised in terms of \( h \) and \( \hbar \) which take values in \( H \), \( A = h\partial h^{-1}, \quad \bar{A} = \hbar\bar{\partial}\hbar^{-1} \). In what follows we shall find (in the one-loop approximation) the conditions on the matrix \( Q \) under which the action (2.1) describes a conformal theory.

Solving for \( A, \bar{A} \) in (2.1),(2.4) one finds the following semiclassical action for the group variable \( g \)

\[
I_Q(g) = I(g) + \frac{1}{\pi} \int d^2z \, \mathcal{M}_{ab} J^a \bar{J}^b , \quad (2.9)
\]

\[
\mathcal{M} \equiv Q M^{-1} Q = Q[Q(C + Q - 2I)Q]^{-1} Q , \quad (2.10)
\]

where the inverse is defined on the subspace on which \( Q \) is non-degenerate. Note that the matrix \( \mathcal{M} \) is a non-trivial function of \( g \) and (2.9) (as well as (2.1)) does not have a global \( G \)-invariance. The action (2.9) can be interpreted as a WZW action perturbed by an operator which is not integrably marginal [3] in general. It can be put in the form of an integrably marginal \( \mathcal{J} \mathcal{J} \)-operator in the case when \( Q \) will have the form \( P\rho P \) where \( P \) is
the projector on the Cartan subalgebra of $G$ (in agreement with the previous discussions [23][24][25][26]). Such $Q$ will be one of the solutions of the conformal invariance conditions of the model (2.9) to be derived below.

It should be stressed that our true starting point is the action (2.1) while (2.9) appears only as a semiclassical approximation. For example, to preserve conformal invariance present in (2.1) for special $Q$ (2.9) should be supplemented by the dilaton coupling originating from the integral over $A$ in (2.1). When $Q$ is non-degenerate the action (2.9) is related to the action discussed in [1] in connection with bosonisation [13][27] of the non-abelian Thirring model [2].

In this case $M$ takes the form

$$M = (C + Q - 2I)^{-1}.$$

The WZW action corresponds to $Q = \infty I$. When $Q = 2I$ (i.e. in the case of the $G/G$ chiral gauged WZW model, cf. (2.7)) eq.(2.9) takes the form of the WZW action with the sign of the first term in (2.2) reversed.\footnote{As was shown in [1] the bosonised version of the Thirring model with the coupling matrix $S$ can be represented (in the sense of equivalence of the corresponding generating functionals for currents) as the ‘deformed’ WZW model (2.9) with a non-polynomial dependence of $M$ on $S$. The relation of (in general, non-symmetric) $Q$ to $S$ in [1] is the following: in the ‘left-right decoupled’ scheme $Q - 2 = \frac{1}{2}[S/2\pi - (S/2\pi)^{-1}]$ (the transformation to other schemes is implemented by shifting $Q$ by a constant times a unit matrix, e.g. $Q \rightarrow Q + I$ in the vector scheme). Note that the duality symmetry in $S/2\pi \rightarrow -(S/2\pi)^{-1}$ is trivial in terms of $Q$. As we shall note below (see (2.19),(2.20)), there is also a non-trivial duality-type symmetry $S/2\pi \rightarrow (S/2\pi)^{-1}$ which relates two ‘dual’ models of the type (2.9) (see also [28]). In the next section we shall derive the condition of conformal invariance of this model (to the leading order in $1/k$ but to all orders in the (symmetric) coupling) as well as an ‘action’ (‘potential’) from which it follows, thus providing a natural extension of the discussion in [1]. It should be emphasized that our approach is more general than that of [1] being applicable also when $Q$ is degenerate so we are able to include systematically the cases of the $G/H$ gauged (and chiral gauged) WZW models. In these cases it is important also to take into account the dilaton coupling (see (3.1),(3.4)) that in general should be added to (2.9).}

\footnote{In the context of ref. [1] this case corresponds to the second (‘dual’) conformal point [3] of the Thirring model (see also [28]).}
It is straightforward to determine the classical Hamiltonian (or ‘00’-component of the stress tensor) that corresponds to (2.9). Starting with (2.1), (2.4) we define the momenta \( p_m \) as derivatives of the Lagrangian over \( \dot{x}^m \) where \( x^m \) are the group space coordinates. Using Minkowski notation we have for the pure WZW action

\[
I(g) = \frac{1}{4\pi} \int d^2z \left[ \frac{1}{2} G_{0mn}(x)(\dot{x}^m \dot{x}^n - x^m \dot{x}^n) + B_{0mn}(x)\dot{x}^m x^n \right],
\]

so that \( p_m = \frac{1}{4\pi}(G_{0mn}\dot{x}^n + B_{0mn}x^m) \). Assuming that \( p_m \) and \( x^m \) have standard Poisson bracket relation one finds that the currents

\[
J_a = J-a(x,p) = E_{am}(x)(\dot{x}^m - x^m) = 4\pi E^m_a(p_m - \frac{1}{4\pi}B_{0mn}x^n) - E_{am}x^m,
\]

\[
\tilde{J}_a = J+a(x,p) = T_a \tilde{E}^a_m(x)(\dot{x}^m + x^m) = 4\pi \tilde{E}^m_a(p_m - \frac{1}{4\pi}B_{0mn}x^n) + \tilde{E}_{am}x^m,
\]

form two commuting affine algebras \([13][29]\). In the case of (2.1) the expressions for momenta contain extra terms linear in \( B, \hat{B} \) or \( A, \hat{A} \). The currents that form the affine algebras are again given by the same functions of \( x^m \) and new \( p_m \) as in the WZW theory. Using the same notation \( J, \tilde{J} \) for these currents one finds for the Hamiltonian as function of independent phase space variables \( x, p, A, \dot{A} \) (cf. \([29][21]\))

\[
\mathcal{H} = \frac{1}{2} J^2 + \frac{1}{2} \tilde{J}^2 - 2J\dot{B} + 2\tilde{J}\dot{B} + 2B(Q-2I)\dot{B} + B^2 + \dot{B}^2.
\]

Elimination of \( A, \dot{A} \) defined in (2.4) gives (note that \( J^2 = \tilde{J}^2 \))

\[
\mathcal{H} = \frac{1}{2} L^{ab}(J_aJ_b + \tilde{J}_a\tilde{J}_b) + \Lambda^{ab}J_a\tilde{J}_b,
\]

\[
L = I + 2QF^{-1}Q, \quad \Lambda = 2Q(Q-2I)F^{-1}Q, \quad F \equiv Q(Q - 3I)(Q - I)Q.
\]

In the singular case of the gauged WZW model (2.6) one is to take the limit \( Q = P + \epsilon, \epsilon \to 0 \), so that the resulting Hamiltonian is finite on the subspace where \( J_H = \tilde{J}_H = 0 \) (\( J_H = PJ, \tilde{J}_H = P\tilde{J} \))

\[
\mathcal{H} = J^2 - J_H\dot{J}_H - \frac{1}{2\epsilon}(J_H - \tilde{J}_H)^2 \quad \mathcal{H} = \frac{1}{2}(J^2 - J_H^2) + \frac{1}{2}(\tilde{J}^2 - \tilde{J}_H^2).
\]

\[5\] Similar result is found in another singular limit (2.6') \( Q = 3P + \epsilon, \epsilon \to 0 \): \( \mathcal{H} = J^2 + J_H\dot{J}_H - \frac{1}{2\epsilon}(J_H + \tilde{J}_H)^2 \), i.e. \( \mathcal{H} \to \mathcal{H} = \frac{1}{2}(J^2 - J_H^2) + \frac{1}{2}(\tilde{J}^2 - \tilde{J}_H^2) \) on a subspace where \( J_H + \tilde{J}_H = 0 \).
In the chiral gauged WZW model case (2.7) \[20\] \[21\]

\[ \mathcal{H}_{\text{chir}} = \frac{1}{2}(J^2 - 2J_H^2) + \frac{1}{2}(\bar{J}^2 - 2\bar{J}_H^2) . \] (2.16)

Note that the ‘non-diagonal’ \( J\bar{J} \) term in (2.13) is non-vanishing in all other cases when \( Q^2 \neq 2Q \).

Introducing the matrix \( K \)

\[ K \equiv \frac{Q^3 - 3Q^2}{Q^3 - Q^2} , \] (2.17)

we get

\[ L = \frac{1}{2}(K + K^{-1}) , \quad \Lambda = -\frac{1}{2}(K - K^{-1}) , \] (2.18)

so that (2.13) can be put into the form

\[ \mathcal{H} = \frac{1}{4}K^{ab}(J_a - \bar{J}_a)(J_b - \bar{J}_b) + \frac{1}{4}K^{-1ab}(J_a + \bar{J}_a)(J_b + \bar{J}_b) , \] (2.19)

which has an obvious duality-type symmetry

\[ K \rightarrow K^{-1} , \quad \bar{J} \rightarrow -\bar{J} . \] (2.20)

Note that (2.20) is not a symmetry of the action (2.9) itself, but it should relate two ‘dual’ actions which give the same generating functionals for the correlators of the corresponding currents (as in the simple case of a single coupling in \[28\]). In particular, this should be the case for \( K = 1 + P\rho P \) (with \( P \) being the projector on the Cartan subalgebra of the algebra of \( G \)) when (2.20) should be related to a particular element of the \( O(2r, 2r) \) \((r = \text{rank } G)\) duality group (see \[20\] \[23\] \[24\] \[25\] \[26\]).

\[ \text{The transformation } K \rightarrow -K^{-1} , \quad \bar{J} \rightarrow -\bar{J} \text{ is also a symmetry if it is accompanied by reversing the sign of the coefficient } k \text{ of the action (for a discussion of a similar symmetry in the quantum generating functional of the single-coupling Thirring model see \[28\]). In the case of a non-degenerate } Q \text{ the matrix } K \text{ is related to the Thirring coupling of } \] \[28\] \text{ by } Q - 2 = -\frac{k - 1}{k + 1} = \frac{1}{2}[S/2\pi - (S/2\pi)^{-1}] \text{ and so } K \rightarrow K^{-1} \text{ corresponds to } S/2\pi \rightarrow (S/2\pi)^{-1}. \]
When \( Q \) is non-degenerate we can represent the action (2.1) in the form

\[
I_Q(g, h, \bar{h}) = I(\tilde{g}) + I'(h, \tilde{h}) ,
\]

(2.21)

where

\[
\tilde{g} = h^{-1}g\bar{h} , \quad B = h\partial h^{-1} , \quad \bar{B} = \bar{h}\partial \bar{h}^{-1} .
\]

By formal manipulations in the path integral and use of the Polyakov-Wiegmann formula the action \( I' \) can be transformed into (we shall ignore the quantum shifts of \( k \), see [28])

\[
I'(h, \bar{h}) \to -I(h^{-1}) - I(\bar{h}) + \frac{1}{\pi} \int d^2 z \, \text{Tr} \left[ -B'\bar{h}\partial \bar{h}^{-1} + h\partial h^{-1}\bar{B}' + B'(Q - 2)^{-1}\bar{B}' \right] \to
\]

\[
I'' = -I(h'^{-1}) - I(\bar{h}') + I(u^{-1}) + I(\bar{u}) + \frac{1}{\pi} \int d^2 z \, \text{Tr} \left[ u\partial u^{-1}(Q - 2)^{-1}\bar{u}\partial \bar{u}^{-1} \right] ,
\]

(2.23)

\[
B' = u\partial u^{-1} , \quad \bar{B}' = \bar{u}\partial \bar{u}^{-1} , \quad h' = u^{-1}h , \quad \bar{h}' = u^{-1}\bar{h} .
\]

Integrating out \( h', \bar{h}' \) and redefining \( g \) we get the dual action \( \tilde{I}_Q'(g', u, \bar{u}) \) with \( k \to -k \) and

\[
Q' - 2 = -(Q - 2)^{-1} , \quad K' = \frac{Q' - 3}{Q' - 1} = -K^{-1} .
\]

(2.24)

3. Sigma model representation and equations of conformal invariance

As in the cases of gauged and chiral gauged WZW models (see e.g. [31][32][20][21]) one can represent the semiclassical action (2.9) or \( S_Q(g) = kI_Q(g) \) in the \( \sigma \)-model form

\[
S_Q(x) = \frac{1}{\pi\alpha'} \int d^2 z (G_{mn} + B_{mn})\partial x^m \partial x^n + \frac{1}{4\pi} \int d^2 z \sqrt{\gamma} R^{(2)}/\phi , \quad \alpha' = \frac{2}{k} ,
\]

(3.1)

\[
G_{mn} = G_{0mn} - 2M_{ac}E^{a}_{(m}\tilde{E}^{c}_{n)} = g_{ac}(x)E^{a}_{m}E^{c}_{n} ,
\]

(3.2)

\[
B_{mn} = B_{0mn} - 2M_{ac}E^{a}_{[m}\tilde{E}^{c}_{n]} = b_{ac}(x)E^{a}_{m}E^{c}_{n} ,
\]

(3.3)

\[
\phi = \phi_0 - \frac{1}{2}\ln \det M .
\]

(3.4)
We have introduced the coordinates on the group $G$ and the vielbein $E^a_M$ according to

$$g^{-1} \partial g = iT_a E^a_m(x) \partial x^m, \quad \partial g g^{-1} = iT_a \tilde{E}^a_m(x) \partial x^m, \quad \tilde{E}^a_m = C^a_b E^b_m. \quad (3.5)$$

$G_{0mn}$ and $B_{0mn}$ are the WZW couplings corresponding to the group space

$$G_{0mn} = \eta_{ac} E^a_m E^c_n, \quad B_{0mn} = b_{0ac} E^a_m E^c_n, \quad H_{0mnk} = 3 \partial_{[k} B_{0mn]} = - f_{abc} E^a_m E^b_n E^c_k. \quad (3.6)$$

We have also included the dilaton coupling that originates from the determinant of integration over $A, \bar{A}$ in passing from (2.1),(2.5) to (2.9). As in the gauged WZW case the presence of a non-trivial dilaton is related to the fact that $\det G \neq \det G_0$, i.e. the dilaton can be also represented in the form

$$\phi = \phi_0 + \frac{1}{4} \ln \det \frac{G}{G_0} = \phi_0 + \frac{1}{4} \ln \det g. \quad (3.4')$$

The matrix functions $g_{ac}$ and $b_{ac}$ in (3.2), (3.3) have the form (see (2.5),(2.10))

$$g = I - Q M^{-1} Q C - C^T Q M^{-T} Q, \quad b = b_0 - Q M^{-1} Q C + C^T Q M^{-T} Q. \quad (3.7)$$

In the gauged WZW case (2.6) when $Q^2 = Q = P$ the metric (3.2) is degenerate having dim $H$ null vectors, $G_{mn}(\tilde{E}^m_a - E^m_a) = 0$ or $g(C^T - 1)P = 0.\footnote{7}$

In what follows we shall determine the conditions on $Q$ under which the model (3.1) is conformal invariant. The one-loop equations of conformal invariance of the $\sigma$-model (3.1) have the standard form $\footnote{8}$

$$\bar{\beta}^G_{mn} = R_{mn} - \frac{1}{4} H_{mkl} H_{n}^{kl} + 2 D_m D_n \phi = 0, \quad (3.8)$$

$$\bar{\beta}^B_{mn} = - \frac{1}{2} D_l H_{mn}^{l} + H_{mn}^{l} \partial_l \phi = 0, \quad (3.9)$$

$$\bar{\beta}^\phi = \frac{1}{6} (D - C) + \alpha' \left[ - \frac{1}{2} D^2 \phi + (\partial \phi)^2 - \frac{1}{24} H_{lmmn}^2 \right] = 0, \quad (3.10)$$

\footnote{7}{We rescale the coordinates $x^m$ to make them dimensionless, absorbing the ‘radius’ of the group space into $\alpha'$.}

\footnote{8}{Similar degeneracy takes place in the axial gauging case when $Q = 3P$, $g(C^T + 1)P = 0.$}
where $D = \dim G$ and $C$ is the total central charge. Using (3.8),(3.10), i.e.

$$\tilde{\beta}^a = \beta^a - \frac{1}{4} \tilde{\beta}^G = \frac{1}{6} (D - C) + O(\alpha') = 0,$$

(3.10')

$C$ can be represented in the form

$$C = D - \frac{3}{2} \alpha' \left[R - \frac{1}{12} H_{lmn}^2 + 4D^2 \phi - 4(\partial \phi)^2\right] + O(\alpha') .$$

(3.11)

$G_{mn}$ and $B_{mn}$ depend on $x$ only through $E_a^m$ and $C_{ab}$ so it is straightforward to compute the curvature of $G_{mn}$ and $H_{mnk}$ with the help of the relations

$$\partial_m E_n^a - \partial_n E_m^a = -f_{bc}^a E_m^b E_n^c, \quad \partial_m C_{ab} = -C_{ac} f_{bd}^c E_d^m ,$$

(3.12)

i.e. $R_{nkl}^m$ and $H_{mnk}$ will be given by sums of products of $E_a^m$, $f_{bc}^a$ and matrix functions of $C_{ab}$ and $Q$. That means that the geometrical objects appearing in (3.8)–(3.10) are ‘dimensionally reducible’ (cf. [34]) in the sense that their expressions at an arbitrary point of the group space $G$ can be determined (by integrating differential equations implied by (3.12)) from their values at a particular point of $G$.

To establish the conditions on $Q$ that follow from (3.8),(3.9) one can, therefore, use a short-cut method by expanding $E_a^m$ and $C_{ab}$ in normal coordinates near the unit element of the group (see e.g. [35])

$$E_a^m = (\frac{e^f - 1}{f})_m^a = [I + \frac{1}{2} f + \frac{1}{6} f^2 + O(x^3)]_m^a ,$$

(3.13)

$$g = \exp(i T_a x^a) , \quad (f)_b^a \equiv f_{bc}^a x^c , \quad f^T = -f , \quad E_m^a x_m = x^a ,$$

(3.14)

$$C_{ab} = (e^{-f})_b^a = [I - f + \frac{1}{2} f^2 + O(x^3)]_b^a .$$

(3.15)

Then the matrix $M$ in (3.6),(2.5) is given by

$$M = Q^3 - Q^2 - Q f Q + \frac{1}{2} Q f^2 Q + O(x^3) ,$$

and so

$$g = K + \frac{1}{2} (f K - K f) - \frac{1}{4} f K f + \frac{1}{4} K f K f + O(x^3) ,$$

(3.16)
where the constant matrix $K_{ab}$ have already appeared in (2.17), i.e.

$$K \equiv I - 2Q(Q^3 - Q^2)^{-1}Q = (Q^3 - 3Q^2)(Q^3 - Q^2)^{-1}.$$  \hspace{1cm} (3.17)

The inverse is assumed to be defined on a subspace where $Q^3 - Q^2$ is non-degenerate. When $Q$ is non-degenerate

$$K = Q - 3I, \quad Q = \frac{K - 3I}{K - I}.$$  \hspace{1cm} (3.17'')

The expansions of $G_{mn}$, $B_{mn}$ and $\phi$ have the form

$$G = K - \frac{1}{12}(Kf^2 + f^2K) + \frac{1}{4}KfKfK + O(x^3),$$  \hspace{1cm} (3.18)

$$B = B_0 + \frac{1}{2}f - \frac{1}{2}KfK + O(x^2),$$  \hspace{1cm} (3.19)

$$\phi = \phi_0 - \frac{1}{16} \text{Tr} (f^2 - KfKf) + O(x^3),$$  \hspace{1cm} (3.20)

so that one finds for the Ricci tensor, $H_{mnk}$ and $D_mD_n\phi$

$$R_{mn} = \frac{1}{4}[-f_{mkl}f_{nkl} + f_{mkl}f_{nkl} + f_{mkl}f_{nkl} + f_{kl}(m\bar{f}_{w}k_\bar{l}) - f_{kl}(m\bar{f}_{w})k\bar{l}] + O(x),$$  \hspace{1cm} (3.21)

$$H_{mnk} = \frac{1}{2}(f_{mnk} - f_{mkn} - f_{mnk} - f_{mkn}) + O(x),$$  \hspace{1cm} (3.22)

$$D_mD_n\phi = \frac{1}{8}(f_{mkl}f_{nkl} - f_{mkl}f_{nkl}) + O(x),$$  \hspace{1cm} (3.23)

where the repeated indices are contracted with $\eta_{ab}$, underlined index indicates an extra factor of $K$ and an index with a bar – a factor of $K^{-1}$, for example,

$$f_{\bar{a}bc} = K_{aa'}f_{a'bc}, \quad f_{\bar{a}bc} = K_{aa'}^{-1}f_{a'bc}.$$  \hspace{1cm} (3.24)

We have used that $f_{abc} = \eta_{ad}f^d_{bc}$ is totally antisymmetric.

The above expressions reduce to the well-known results in the group space limit when $Q = \infty I$, i.e. $K = I$,

$$R_{mn} = \frac{1}{4}f_{mkl}f_{nkl}, \quad H_{mnk} = -f_{mnk}, \quad \phi = \phi_0.$$  \hspace{1cm} (3.25)
In the case of the gauged WZW model (2.6) one should first project the metric \( G_{mn} \) and \( H_{mnk} \) on \( G/H \) to make them non-degenerate and only then expand in powers of \( x \). Then eqs. (3.13)–(3.18) still apply if all uncontracted indices are multiplied by the \( G/H \) projector \( P^\perp = I - P \) and both \( K \) and \( K^{-1} \) are replaced by \( P^\perp \)

\[
K = I - P = P^\perp, \quad K^{-1} = P^\perp, \quad P^{\perp 2} = P^\perp. \quad (3.26)
\]

In the case of the chiral gauged WZW model (2.7) one finds that \( K = K^{-1} \), i.e.

\[
K = I - 2P, \quad K^{-1} = I - 2P, \quad K^2 = I, \quad (3.27)
\]

so that it is not necessary to distinguish between the \( m \) and \( \overline{m} \) indices. If \( G \) and \( H \) are compact, \( K \) in (3.27) can be put into the form \( K_{ab} = \text{diag}(+1, \ldots, +1, -1, \ldots, -1) \).

Since the objects in eqs.(3.8)–(3.10) are expressed in terms of products of the universal quantities \( E^a_m \), \( f_{abc} \) and algebraic functions of \( C_{ab} \) they should be satisfied automatically if satisfied at \( x = 0 \). Computing the constant part of eq.(3.8) using (3.16)–(3.18) we get the following basic equation on the matrix \( K \) (or on \( Q \))

\[
(\beta^G_{mn})_{x=0} = \frac{1}{8}(f_{mkl}f_{nkl} - \frac{1}{2}f_{mkl}f_{nkl} - \frac{1}{2}f_{mkl}f_{nkl} - f_{mkl}f_{nkl}) = 0. \quad (3.28)
\]

The trace of this equation with \( K_{mn}^{-1} \) is

\[
f_{mkl}(f_{\overline{m}kl} - f_{\overline{m}kl} + \frac{1}{2}f_{\overline{m}kl} - \frac{1}{2}f_{\overline{m}kl}) = 0. \quad (3.28')
\]

The dilaton equation (3.10) implies

\[
\mathcal{C} = D - \frac{1}{16} \alpha'(f_{mkl}f_{\overline{m}kl} - 3f_{mkl}f_{\overline{m}kl} + 6f_{mkl}f_{\overline{m}kl}) + O(\alpha'^2), \quad (3.29)
\]

\[\text{Note that } K_{ab} \text{ is the constant part of the target space metric (3.18) (cf. also the Hamiltonian (2.19)) and thus the signature of it is determined by the signatures of the Killing metrics of } G \text{ and } H. \text{ For example, we get just one time-like direction if } G \text{ is compact and } H = U(1) \text{ (see also [21]).} \]
while combining (3.28') and (3.29) one gets the expression that follows from (3.11)
\[
\mathcal{C} = D - \frac{1}{32}\alpha'(-f_{mkl}f_{mnkl} - 3f_{mkl}f_{mnl} + 6f_{mkl}f_{mkl} + 6f_{mkl}f_{mkl}) + O(\alpha'^2) .
\] (3.30)

This representation is of interest since, in agreement with general expectations, one can check that \(\mathcal{C}(K)\) in (3.30) plays the role of an ‘action’ for the equation (3.28). As is well known, eqs.(3.8)–(3.10) follow from the effective action
\[
S = \int d^Dx \sqrt{G} e^{-2\phi}\tilde{\beta} \phi, \text{ or, up to a shift in the constant term in (3.10), from}
\]
\[
S = \int d^Dx \sqrt{G} e^{-2\phi} \mathcal{C} ,
\]
with \(\mathcal{C}\) given by (3.11). For the background under consideration the ‘measure factor’ \(\sqrt{G} e^{-2\phi} = \sqrt{G_0}\) is \(K\)-independent and that is why \(\partial \mathcal{C}/\partial K = 0\) is equivalent to (3.28).

Eq.(3.9) is always satisfied to the leading order. The constant term in (3.9) should be given by a sum of the products of two factors of \(f_{abc}\), one \(K^{-1}\) and several \(K\)’s but an antisymmetric tensor of such structure does not exist if \(f_{abc}\) is totally antisymmetric and \(K_{ab}\) is symmetric. The absence of an extra antisymmetric constraint coming from (3.9) is consistent with the fact that our background is parametrised by a symmetric matrix \(K_{ab}\) (which is the variable in the ‘action’ (3.30)).

Let us emphasize the central role played by the matrix \(K_{ab}\) (3.17) in the above construction. This is clear from the representation (2.19) for the Hamiltonian (2.13),(2.18) which is of course related to the following expressions for the inverse matrices to \(g_{ab}\) and \(G_{mn}\) in (3.2),(3.7)
\[
g^{-1} = \frac{1}{2}(L + C^T LC + \Lambda C + C^T \Lambda) = \frac{1}{4}[(C^T - 1)K(C - 1) + (C^T + 1)K^{-1}(C + 1)] ,
\] (3.31)
\[
G^{mn} = \frac{1}{2}L^{ab}(E^m_a E^m_b + \tilde{E}^m_a \tilde{E}^m_b) + \Lambda^{ab}E^m_a \tilde{E}^m_b
\]
\[
= \frac{1}{4}K^{ab}(E^m_a - \tilde{E}^m_a)(E^m_b - \tilde{E}^m_b) + \frac{1}{4}K^{-1ab}(E^m_a + \tilde{E}^m_a)(E^m_b + \tilde{E}^m_b) ,
\] (3.32)
which are invariant under (2.20), or \(K \rightarrow K^{-1}\), \(\tilde{E} \rightarrow -\tilde{E}\).

\[\text{Eq.(3.30) gives the ‘effective potential’ (of the structure } \ K^{-1}K^{-1}ff + KKK^{-1}ff + Kff + K^{-1}ff \text{) for the coupling }K_{ab}. \text{ A similar cubic } SSSff \text{ potential for the coupling }S \text{ of the Thirring model appeared in [36].} \]
4. Solutions of the conformal invariance condition

Let us now study possible solutions of the conformal invariance equation (3.28) or, explicitly, of

\[ f_{mkl}f_{nkl} - \frac{1}{2} f_{mkl}f_{nk\ell'} K_{k\ell'} K_{l\ell'} - \frac{1}{2} f_{mkl}f_{nk\ell'} K_{k\ell'}^{-1} K_{l\ell'}^{-1} + K_{mm'} f_{m'kl}f_{n'k\ell'} K_{k\ell'} K_{l\ell'}^{-1} - K_{mm'} f_{m'kl}f_{n'k\ell'} K_{k\ell'}^{-1} K_{l\ell'}^{-1} - K_{mm'} f_{m'kl}f_{n'k\ell'} K_{k\ell'}^{-1} K_{l\ell'}^{-1} - K_{mm'} f_{m'kl}f_{n'k\ell'} K_{k\ell'}^{-1} K_{l\ell'}^{-1} + K_{mm'} K_{nn'} f_{m'kl}f_{n'k\ell'} K_{k\ell'} K_{l\ell'}^{-1} - K_{mm'} K_{nn'} f_{m'kl}f_{n'k\ell'} K_{k\ell'} K_{l\ell'}^{-1} = 0 \]  

(4.1)

In general, this equation is not invariant under \( K \rightarrow K^{-1} \). However, the ‘coupling constant duality’ \( K \rightarrow K^{-1} \) becomes manifest in the \( \beta \)-function in the simplest possible case when \( K \) is proportional to a unit matrix

\[ K = K_0 I, \quad Q = \frac{K - 3I}{K - I} = \frac{K_0 - 3}{K_0 - 1} I. \]  

(4.2)

Then (3.28) and (3.30) take the form\(^{11}\)

\[ (\overline{\beta}_{mn}^G)_{x=0} = -\frac{1}{16} (K_0 - K_0^{-1})^2 c_G \eta_{mn} + O\left(\frac{1}{k}\right), \]  

(4.3)

\[ C = D - \frac{1}{16k} (-K_0^{-3} + 3K_0 + 6K_0^{-1}) c_G D + O\left(\frac{1}{k^2}\right). \]  

(4.4)

The only two conformal points are \( K_0 = 1 \) or \( Q = \infty I \) (WZW model) and \( K_0 = -1 \) or \( Q = 2I \) (WZW model with the reversed sign of the first term, or \( G/G \) chiral gauged WZW model). For \( K_0 = 1 \) the central charge (4.4) takes the standard form

\[ C = D - \frac{1}{2k} c_G D + O\left(\frac{1}{k^2}\right) = \frac{Dk}{k + \frac{1}{2} c_G}. \]  

(4.5)

\(^{11}\) Similar expressions (which are exact in \( K_0 \) but first order in \( 1/k \)) for the \( \beta \)-function and the central charge ‘potential’ were derived in the Thirring model context \[8][17] in \[28\]. In our approach (4.3) follows simply from the known \( \sigma \)-model \( \beta \)-function (3.8). Our result for the ‘potential’ (4.4) or (3.11) is not duality symmetric but, in general, the ‘potential’ is not defined unambiguously out of the conformal point. It should be noted also that in (4.3),(4.4) we have included the dilaton contribution (the expression for the dilaton (3.4),(3.20) is non-trivial for \( K_0^2 \neq 1 \) even in the one-coupling case) which was not considered in \[28\] where the Thirring model was the starting point.
A less trivial solution corresponds to the $G/H$ gauged WZW model (2.6),(3.26) when $K = P^\perp = P_{G/H}$. In general, if $K = K^{-1}$, eq.(4.1) reduces to

$$
(\eta_{mm'}\eta_{nn'} - K_{mm'}K_{nn'}) (\eta_{kk'}\eta_{ll'}) f_{m'kl} f_{n'k'l'} = 0 , \quad K = K^{-1} .
$$

(4.6)

Furthermore, if we formally take $K$ to be a projector, i.e. $K = K^{-1} = K^2$ then using again the notation in which the indices multiplied by $K$ are underlined, we get from (4.6) (cf.(3.28))

$$
f_{mkl} f_{nkl} - f_{mkl} f_{nkl} + f_{mkl} f_{nkl} - f_{mkl} f_{nkl} = 0 , \quad K = K^2 .
$$

(4.7)

The $KK$-projection (i.e. the product with $K_{mm'}K_{nn'}$) of this equation is satisfied automatically while the projections $KK^\perp$ and $K^\perp K^\perp$ ($K^\perp = I - K$) give the following two equations

$$
f_{\hat{m}\hat{k}l} f_{\hat{n}kl} = f_{\hat{m}kl} f_{\hat{n}kl} = c_G \eta_{\hat{m}\hat{n}} = 0 ,
$$

(4.7')

$$
f_{\hat{m}\hat{k}l} f_{\hat{n}kl} = f_{\hat{m}kl} f_{\hat{n}kl} = c_G \eta_{\hat{m}\hat{n}} ,
$$

(4.7'')

where $\hat{m}$ denotes an index projected with the help of $K^\perp$.

In the case of the gauged WZW model the gauge invariance implies that one should consider only the $KK$-projection of (4.1), i.e. (4.7'),(4.7'') are absent and thus (4.1) is satisfied.\textsuperscript{12} Then the expression for the central charge (3.29) becomes\textsuperscript{13}

$$
\mathcal{C} = D_{G/H} + \frac{1}{8} \alpha' (f_{mkl} f_{mkl} - 3 f_{mkl} f_{mkl}) + O(\alpha'^2) .
$$

(4.8)

Let the indices $r, s, t$ be from the algebra of the subgroup $H$ (i.e. from the $K^\perp$-space or the same as indices with hats) and the indices $\mu, \nu, \lambda, ...$ be from from the tangent space

\textsuperscript{12} Note that our derivation of (4.1) from (3.8) formally applies only when the metric (3.18) or $K$ is non-degenerate. The gauged WZW model case is special, having explicit gauge invariance that implies the use of appropriate projectors (or gauge fixing).

\textsuperscript{13} Because of gauge invariance of the path integral of the gauged WZW model \cite{15,16} one is also to replace the number $D$ of the degrees of freedom in (3.29) by $D_{G/H} = D_G - D_H$. 

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to $G/H$ (i.e. from the $K$-space or the same as underlined indices). Then one has $f_{\mu rs} = 0$ ($H$ is a subgroup) and
\[ f_{mkl} f_{mkl} = f_{\mu \nu \lambda} f_{\mu \nu \lambda} = f_{mkl} f_{mkl} - 3f_{\mu \nu r} f_{\mu \nu r} - f_{rst} f_{rst}, \]
\[ f_{mkl} f_{mkl} = f_{\mu kl} f_{\mu kl} = f_{\mu \nu \lambda} f_{\mu \nu \lambda} + 2f_{\mu \nu s} f_{\mu \nu s}. \]

As a result, (4.8) reproduces the first term in the $1/k$-expansion of the central charge \[12\] of the gauged WZW (or coset) model
\[ C = D_{G/H} - \frac{1}{2k} (f_{mnk} f_{mnk} - f_{rst} f_{rst}) + O(\frac{1}{k^2}) \]
\[ = D_{G/H} - \frac{1}{2k} c_G D_G - \frac{1}{2k} c_H D_H + O(\frac{1}{k^2}) = \frac{D_G k}{k + \frac{1}{2} c_G} - \frac{D_H k}{k + \frac{1}{2} c_H}, \] (4.9)

Next, let us consider the solutions with $K = K^{-1}$, $K^2 = I$. This is the case of the chiral gauged WZW model (2.7),(3.27). At the ‘self-dual’ point $K = K^{-1}$ eq.(4.1) reduces to (4.6). It is straightforward to check that this equation is satisfied if $K = I - 2P$ with $P$ being a projector on a subalgebra. Namely, one should have $f_{\mu rs} = 0$ where we set again $m = (\mu, r)$ with $r, s, t, \ldots$ corresponding to $P$ and $\mu, \nu, \lambda, \ldots$ corresponding to $P^\perp \equiv I - P$.

When $K = K^{-1}$ the central charge (3.29),(3.30) takes the form similar to (4.8)
\[ C = D + \frac{1}{8} \alpha'(K_{mn'} K_{nn'} K_{ll'} f_{mnl} f_{m'n'l'} - 3K_{mn'} f_{mkl} f_{m'k'l'}) + O(\alpha'^2). \] (4.10)

For $K = I - 2P$ and $f_{\mu st} = 0$ we get (here $D = D_G$)
\[ C = D + \frac{1}{8} \alpha' (-2f_{mnk} f_{mnk} + 4f_{rst} f_{rst} + 12f_{\mu st} f_{\mu st}) + O(\alpha'^2) \] (4.11)
\[ = D_G - \frac{1}{2k} c_G D_G + \frac{1}{k} c_H D_H + O(\frac{1}{k^2}) = \frac{D_G k}{k + \frac{1}{2} c_G} + \frac{c_H D_H}{k + \frac{1}{2} c_H}, \] (4.12)

Eq.(4.10) thus reproduces the $O(1/k)$ term in the expansion of the central charge of the chiral gauged WZW model as given in [21]. Another solution is found by reversing the sign of $K$, i.e. $K = 2P - 1 = 1 - 2P^\perp$ (then the sign of the $O(1/k)$ term in (4.10) is also
reversed). In the case when \( K = K^{-1} \) the expressions (2.18), (2.19), (3.31), (3.32) simplify, in particular, the Hamiltonian (2.13), (2.19) takes the form

\[
\mathcal{H} = \frac{1}{2} K^{ab}(J_a J_b + \bar{J}_a \bar{J}_b) , \quad K = K^{-1} .
\] (4.13)

Let us now relax the condition \( K = K^{-1} \) and look for other solutions of (4.1). One possible ansatz is a generalisation of \( K = I - 2P \), namely,

\[
K = \gamma I + P \rho P , \quad P^2 = P ,
\] (4.14)

where the constant \( \gamma \) and the constant matrix \( \rho_{ab} \) are to be determined. One can show (e.g. by taking the \( PP, P^\perp P, P^\perp P^\perp \)-projections of (4.1)) that the solution exists only if \((I - P)_{mn} P_{k^\prime l^\prime} f_{m^\prime k^\prime l^\prime} = 0\), i.e. if \( P \) is a projector on a subalgebra. If the subalgebra is non-abelian then the only solution is the chiral gauged WZW one (3.27), i.e. \(|\gamma| = 1 , \ \rho = -2\gamma I\). If, however, \( P \) is a projector on an abelian subalgebra (any subalgebra of Cartan algebra \( H_c \)), i.e. \( f_{mnk} P_{nn} P_{kk^\prime} = 0 \), then \( \gamma^2 = 1 \) but the matrix \( \rho \) can be arbitrary, i.e.

\[
K = I + P \rho P , \quad Q = \frac{K - 3I}{K - I} = P(1 - 2\rho^{-1})P ,
\] (4.14’)

\[
K = I + \bar{\rho} , \quad \bar{\rho} \equiv P \rho P , \quad \bar{\rho} H \subseteq H_c .
\]

The reason why one finds a conformal model for an arbitrary \( \rho \) can be understood in the following way (for a related discussion see [26][21]). Consider the gauged \([G \times H]/H\) WZW model with \( H \) isomorphic to (a subalgebra of) the Cartan subalgebra of \( G \). Its action can be represented as the sum of the action of the gauged \( G/H \) WZW model and the gauged action corresponding to \( H \),

\[
I_H = \frac{1}{2\pi} \int d^2 z (\partial y_s + \lambda_{st} B_t)(\bar{\partial} y_s + \lambda_{st^\prime} \bar{B}_{t^\prime}) ,
\]

where \( y_s (s = 1, ..., r) \) are the variables of the WZW theory for \( H \) and \( \lambda_{st} \) are constants that parametrise the embedding of \( H \) into \( G \times H \). In the gauge \( y_s = 0 \) this action is equivalent to (2.1) if \( Q \) is given by (4.14’) with \( \rho_{st} = -4\lambda_{st}^2 \). Since the gauged \([G \times H]/H\) WZW
theory is conformal, the model (2.1), (4.14′) should, of course, also be conformal. The chiral gauged WZW model corresponds to the particular case of \( \rho = -2I \).

The models (2.9), (4.14′) with different values of \( \rho_{rs} \) can be generated from the pure WZW theory by the \( O(2r, 2r) \) duality transformations corresponding to the isometries along the Cartan algebra directions \([23][24][25][26]\) and thus are also related by the duality. The second term in (2.9) in this case can be put into the form of an integrably marginal deformation \([23][24][25][26]\).

The Hamiltonian corresponding to the solution (4.14′) is given by (2.13),(2.18)

\[
\mathcal{H} = \frac{1}{2} [I + \frac{1}{2} \bar{\rho}(P + \rho)^{-1}\rho](J_a J_b + \bar{J}_a \bar{J}_b) - \frac{1}{2} \bar{\rho}(P + \rho)^{-1}(2P + \rho)J_a \bar{J}_b ,
\]

i.e. contains the \( J \bar{J} \)-term. The central charge (3.29) for \( K \) in (4.14′) is \( \rho \)-independent and is the same as for the \([G \times H_c]/H_c \) coset or simply \( G \) affine-Sugawara model, i.e. is given by (4.5).

One may look for other solutions of (4.1) representing the symmetric matrix \( K \) in the ‘diagonal’ form (as in \([7][9]\))

\[
K_{ab} = \sum_c p_c \Omega_{ac} \Omega_{bc} , \quad \Omega^T \Omega = I .
\] (4.15)

Here \( \Omega \) is an element of \( SO(D_G) \). Then the basic equation (4.1) reduces to

\[
\sum_{k,l} \left[ \frac{(p_k - p_l)^2}{p_k p_l} - \frac{(p_k p_l - 1)^2}{p_k p_l} \right] \hat{f}_{mkl} \hat{f}_{nkl} = 0 ,
\] (4.16)

\[
\hat{f}_{mkl} \equiv \Omega_{mm'} \Omega_{kk'} \Omega_{ll'} \hat{f}_{m'k'l'} .
\]

The solutions of (4.16) correspond to extrema of the central charge ‘action’ (3.30) which in the case of (4.15) is given by

\[
\mathcal{C} = D - \frac{1}{32} \alpha' \sum_{m,k,l} (-p_m^{-1} p_k^{-1} p_l^{-1} - 3p_m p_k p_l^{-1} + 6p_m + 6p_m^{-1}) \hat{f}_{mkl} \hat{f}_{mkl} + O(\alpha'^2) .
\] (4.17)

14 This is a reflection of the general equivalence \([21]\) of the \( G/H \) chiral gauged WZW model with an abelian \( H \) to the \([G \times H]/H \) gauged WZW model with a special value of the embedding parameter of the axial subgroup \( H \) into the group \( G \times H \).
A special solution of (4.16) is equivalent to (4.14'). If $K^2 = I$ we have $p_n^2 = 1$ and (4.16) becomes

$$ (p_m p_n - 1) \sum_{k,l} (p_k p_l - 1) \hat{f}_{mkl} \hat{f}_{nkl} = 0 \ . \quad (4.18) $$

The non-zero components of (4.18) correspond to $m \neq n$ and $p_m = -p_n = 1$; then

$$ \sum_{k,l} (p_k p_l - 1) \hat{f}_{mkl} \hat{f}_{nkl} = 0. $$

The previously discussed chiral gauged WZW solution (3.27) is reproduced as a particular case.

It is easy to show that no additional solutions of (4.16) are found if $G = SU(2)$ or $SL(2, R)$. Here $\hat{f}_{mkl} = f_{mkl} = \epsilon_{mkl}$ and we get from (4.16)

$$ p_1^2 (p_2 - p_3)^2 = (p_2 p_3 - 1)^2 , \quad p_2^2 (p_1 - p_3)^2 = (p_1 p_3 - 1)^2 , \quad p_3^2 (p_2 - p_1)^2 = (p_2 p_1 - 1)^2 , \quad (4.19) $$

with the only non-trivial solution (up to permutations and replacements of $+1$ by $-1$) being (4.14'), i.e. $p_1 = 1, p_2 = 1, p_3 = 1 + \rho =$ arbitrary. For $\rho = -2$ we have the solution (3.27). For $G = SL(2, R)$ the resulting model (2.1),(3.1)–(3.4),(4.14') is equivalent to the ‘charged black string’ or $[SL(2, R) \times R]/R$ gauged WZW model [38] which, in fact, is conformally invariant for an arbitrary value of one free parameter (charge) related to $\rho$.

5. Relation to Virasoro master equation

An obvious question is how eq.(4.1) is related to the Virasoro master equation of refs.[4][5]

$$ L^{ab} = L^{ac} L^{cb} + \frac{1}{2k} \left( f_{cd} f_{d'b'} L^{bb'} L^{dd'} + f_{cd} f_{d'd'} L^{aa'} L^{dd'} - f_{cd} f_{d'd'} L^{cc'} L^{dd'} \right) . \quad (5.1) $$

We have rescaled $L$ of [4][5] by $2k$. Using the same representation for $L$ as for $K$ in (4.15) one can put (5.1) into the form [4][5]

$$ \lambda_a (1 - \lambda_a) \eta_{ab} = \frac{1}{2k} \sum_{c,d} \lambda_c (\lambda_a + \lambda_b - \lambda_d) \hat{f}_{acd} \hat{f}_{bcd} , \quad L^{ab} = \sum_c \lambda_c \Omega^{ac} \Omega^{bc} \ . \quad (5.2) $$
In the large $k$ limit the system (5.2) reduces to

$$L = L^2, \quad \lambda_a(1 - \lambda_a) = 0, \quad (5.3)$$

$$\sum_{c,d} \lambda_c(\lambda_a + \lambda_b - \lambda_d)\hat{f}_{acd}\hat{f}_{bcd} = 0, \quad a \neq b, \quad (5.4)$$

where $\hat{f}$ in (5.4) depends on the leading-order form of the ‘angular’ variables $\Omega$.

Since our equation (4.1) was derived in the leading order approximation in $\alpha' = 2/k$, it should be compared with (5.3),(5.4). While all $k \to \infty$ solutions the master equation correspond to $L$ being a projector our equation (4.1) has also solutions with $K \neq K^2$. As it is clear from the structure of the Hamiltonian of our model (2.13),(2.18), a correspondence should be possible only in the special case when our matrix $K$ satisfies $K^2 = K = K^{-1}$, i.e. is a projector on a subspace. Then we can identify $L$ with $K$ (i.e. $p_a$ with $\lambda_a$). The equation that follows from (4.1) when $K^2 = K$ is (4.7). If one uses the representation (4.15) this equation is the same as (4.18) (now with $p_a^2 = p_a$). Though (4.18) looks similar to (5.4) the two equations are not equivalent in general (but for $SU(2)$ the solutions are the same).

That (5.1) with $L^2 = L$ is different from (4.1) with $K^2 = K$ (i.e. from (4.7)) is easy to see also without using the ‘diagonal’ representation. In terms of the same notation as in (4.7’),(4.7’’$$(f_{\hat{m}kl} = L^{\perp}_{mm'}f_{m'kl}, \quad f_{nkl} = L_{nn'}f_{n'kl})$$) we can represent the $LL^\perp$ and $L^\perp L^\perp$ projections of (5.1) (equivalent to (5.4)) in the following way

$$f_{\hat{m}kl}f_{nkl} - f_{\hat{m}kl}f_{nkl} = f_{\hat{m}kl}f_{\hat{n}kl} = 0, \quad (5.5)$$

$$f_{\hat{m}kl}f_{\hat{n}kl} = 0. \quad (5.6)$$

The system (5.5),(5.6) is obviously different from (and is much less restrictive than) the system (4.7’),(4.7’’) that follows from (4.1),(4.7).

The only obvious common solution of (4.1) and the $k \to \infty$ limit of (5.1) is the coset one ($L = K = I - P$). The $k \to \infty$ master equation, i.e. (5.3),(5.5),(5.6) has many other
solutions with $L = L^2$. At the same time, our equation (4.1) has also other solutions with $K^2 \neq K$, namely, (4.14') and, in particular, the chiral gauged one (3.27). The sets of solutions of (4.1) and (5.3),(5.5),(5.6) thus intersect but do not coincide.

The relation between eq.(4.1) and the $k \to \infty$ limit of the Virasoro master equation (5.1) is the following. The master equation describes only ‘irreducible’ solutions while (4.1) contains ‘reducible’ solutions corresponding to some of the solutions (cosets) of the master equation. The ‘reducible’ solutions can be understood as some ‘twisted’ products of ‘irreducible’ WZW models with ‘twisting’ being due to mixing of group variables and reducing of the configuration space by integrating out some of the degrees of freedom (group variables that parametrise the 2d gauge field).

For example, it is clear from (2.8) that before one integrates out the 2d gauge field the action of the chiral gauged WZW model is just a sum of the three WZW actions so that the corresponding stress tensor is given by the three independent affine-Sugawara terms each of which is of course a solution of the master equation. At the same time, the Hamiltonian (2.16) one finds upon elimination of the gauge field is not of the standard coset model type. The reason why it still corresponds to a conformal theory (i.e. represents a solution of (4.1)) is that the currents that appear in (2.13),(2.16) are not chiral (as it is assumed in the affine–Virasoro construction). In fact, if the $J\bar{J}$-term in (2.9) is not treated just as a perturbation of the WZW theory the currents are no longer (anti)holomorphic on the equations of motion. As was mentioned in [21], it should be possible to define the new (anti)holomorphic currents (combinations of $J$, $\bar{J}$, $J_H$ and $\bar{J}_H$) in terms of which the Hamiltonian will take again its standard Sugawara-like form. Similar remark applies to the general case of the models (4.14') since they are equivalent to the $[G \times H_c]/H_c$ coset models.$^{15}$

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$^{15}$ Since the basis of the affine-Virasoro construction is the current algebra, the starting point in [6] is the classical Hamiltonian which does not contain $J\bar{J}$-terms. This is natural since if the currents are (anti)holomorphic then such a structure is implied by conformal invariance. At the same time, the currents that appear in the Hamiltonian of our model (2.13) are not, in general, (anti)holomorphic on shell (since the equations of motion that follow from (2.1),(2.9) are different from the standard equations of the WZW model). As a result, we got conformal solutions (4.14') (with $K \neq K^{-1}$) for which there is a $J\bar{J}$-term in the Hamiltonian.
In general, our model (2.1),(4.1) should not be expected to describe ‘reducible’ solutions corresponding to ‘irreducible’ solutions of the master equation other than cosets. In fact, since $K$ is the constant part of the metric (3.18) the derivation of (4.1) from the $\bar{\beta}$-function equations (3.8) formally applies only when $K$ is non-degenerate. The case of the gauged WZW model when $K = K^2$, i.e. is singular, is a special one; it can still be treated in a consistent way because of the explicit gauge invariance of the action (2.1) when $K = I - P$. It is clear, however, that other solutions of the master equation with $K = L = L^2$ should fall outside of our class of models (2.1). In fact, it is known that an action [6] that reproduces the off conformal point extension of the (large $k$) Hamiltonian of a generic non-coset affine–Virasoro construction is not Lorentz invariant [6][10]. When $L$ is subject to the master equation the action of [6] is Lorentz invariant provided one does not ignore its dependence on extra degrees of freedom (Lagrange multipliers) $v_{zz}$ and $\bar{v}_{\bar{z}\bar{z}}$.

To summarise, the solutions of equation (4.1) are only the ‘reducible’ counterparts of such solutions of the master equation (cosets) that can be described by field theories that are manifestly Lorentz invariant off the conformal point. A natural problem then is to find an analog of (4.1) which will contain more general solutions of the master equation (as well as their ‘reducible’ counterparts) by starting from a non-Lorentz-invariant field-theoretic model and imposing the conditions of conformal/Lorentz invariance at the quantum level. An existence of such generalised $\sigma$-models that are not Lorentz invariant at the classical level but become invariant at the conformal point was conjectured in [39] (where the $\sigma$-models with doubled number of coordinates were introduced in order to make the target space duality symmetry manifest at the string world sheet action level).

One possible starting point is the group space action of [6] (in the gauge $v = \bar{v} = 0$). We would like, however, to suggest what seems to be a natural alternative approach which is based on doubling of the number of group space variables (but not of the degrees of freedom). The idea is that in trying to construct an off shell extension, one may represent the chiral currents $J, \bar{J}$ of the affine-Virasoro construction either in terms of one group field
consider the following analog of (2.1) gauge symmetries. Such actions with Lagrange multipliers may be related to the action in [6].

to deal with extra (Lagrange multiplier) degrees of freedom and preserve [46] the corresponding for chiral scalars appeared in [45]. The latter approach is manifestly Loren tz invariant but one has

one replaces the field \( g(z, \bar{z}) \) by the two ‘chiral’ fields \( g_-(z, \bar{z}) \) and \( g_+(z, \bar{z}) \) described by the Floreanini-Jackiw type [40] WZW Lagrangians [41] (cf. (2.2))

\[
I_\pm(g_\pm) = \pm \frac{1}{8\pi} \int d^2z \, \text{Tr} (\partial_1 g_\pm^{-1} \partial_\mp g_\pm) + \frac{1}{12\pi} \int d^3z \, \text{Tr} (g_\pm^{-1} dg_\pm)^3
\]

\[
= \frac{1}{8\pi} \int d^2z \left[ G_{0mn}(x_\pm)(\pm \dot{x}_\pm^m x_\pm^m - x_\pm^m x_\pm^m) + B_{0mn}(x_\pm) \dot{x}_\pm^m x_\pm^m \right], \quad (5.7)
\]

or

\[
I_\pm(g_\pm) = I'_\pm(g_\pm) - \frac{1}{8\pi} \int d^2z G_{0mn}(x_\pm)x_\pm^m x_\pm^m,
\]

\[
I'_\pm(g_\pm) \equiv \frac{1}{8\pi} \int d^2z (\pm G_{0mn} + B_{0mn})(x_\pm) \dot{x}_\pm^m x_\pm^m. \quad (5.8)
\]

These models are Lorentz invariant only on the equations of motion. In the absence of interaction between \( g_-(z, \bar{z}) \) and \( g_+(z, \bar{z}) \) it is possible to integrate out the ‘ratio’ of \( g_- \) and \( g_+ \) explicitly, ending up with the standard WZW action for \( g = g_- g_+ \) [14] [39]. Let us consider the following analog of (2.1)

\[
I_L(g_\pm, A_\pm) = I_+(g_+) + I_-(g_-)
\]

\[
+ \frac{1}{2\pi} \int d^2z \, \text{Tr} \left[ -A_-(L-I)J_+ + J_-(L-I)A_+ - A_-(L-I)A_- - A_+(L-I)A_+ \right], \quad (5.9)
\]

\[
J_+ = g_+^{-1} \partial_1 g_+ = iT_a \bar{E}_m^a x^m, \quad J_- = \partial_1 g_- g_-^{-1} = iT_a \bar{E}_m^a x^m, \quad (5.10)
\]

where \( L^{ab} \) is a constant symmetrix matrix. Integration over \( A_\pm \) gives (cf. (2.9))

\[
I_L(g_\pm) = I_L(g_+) + I_L(g_-) = I_+(g_+) + I_-(g_-) + \frac{1}{8\pi} \int d^2z \, \text{Tr} \left[ J_-(L-I)J_- + J_+(L-I)J_+ \right]
\]

(5.11)

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16 These actions can be also obtained from the manifestly Lorentz invariant actions in the Siegel’s approach to chiral scalars [42] by gauge-fixing the Lagrange multiplier [43].

17 A model of this type was considered in [33]. Similar actions in the Siegel’s formulation [42] for chiral scalars appeared in [35]. The latter approach is manifestly Lorentz invariant but one has to deal with extra (Lagrange multiplier) degrees of freedom and preserve [16] the corresponding gauge symmetries. Such actions with Lagrange multipliers may be related to the action in [3].
\[ I_+^\prime(g_+) + I_-^\prime(g_-) + \frac{1}{8\pi} \int d^2z \ Tr \ (J_-LJ_- + J_+LJ_+) , \]  

(5.12)

or

\[ I_L = \frac{1}{8\pi} \int d^2z \left[ (G_{0mn} + B_{0mn})(x_+) \dot{x}_+^m x_+^m - (G_{0mn} - B_{0mn})(x_-) \dot{x}_-^m x_-^m \right. \]
\[ - L^{ab} E_{am} E_{bn} x_+^m x_+^n - L^{ab} \tilde{E}_{am} \tilde{E}_{bn} x_-^m x_-^n \]  

(5.13)

Since the actions \( I_\pm' \) are linear in time derivatives, the corresponding Hamiltonian is proportional to the third term in (5.12). In fact, one can consider each of \( x_+^m \) and \( x_-^m \) as phase space coordinates, i.e. as a mixture of ‘true’ coordinates and momenta [1]. If the Lagrangian of a mechanical system is \( L = a_i(q) \dot{q}_i - V(q) \), then the Hamiltonian is just \( H = V \) and the Poisson bracket of functions on the phase space is

\[ \{X_1(q), X_2(q)\} = F^{-1ij} \partial_i X_1 \partial_j X_2, \quad F_{ij} \equiv \partial_i a_j - \partial_j a_i \]  

[13, 47]. It is possible to check that if one starts with the action (5.11) or, equivalently, with \( I_\pm(g_\pm) \) or \( I_\pm^\prime(g_\pm) \), then the brackets one gets are such that the currents (5.10) form the standard affine algebras [41].

The Hamiltonian for the pure ‘kinetic’ action \( I_\pm' \) is zero, for the chiral WZW action \( I_\pm \) is given by the ‘potential’ \( \partial_1 g_\pm^{-1} \partial_1 g_\pm \) term or \( H_\pm = \frac{1}{2} \eta^{ab} J_\pm a J_\pm b \), while for (5.12) is obviously

\[ H = \frac{1}{2} L^{ab}(J_{+-} J_{+-} + J_{-+} J_{-+}) . \]  

(5.14)

We conclude that the action (5.11) may be considered as an alternative Lagrangian realisation of the Hamiltonian of the affine-Virasoro construction [4, 5]. While in [6] the current algebra was represented in terms of one set of group coordinates of the standard WZW action, we got a simpler action by using two ‘chiral’ sets \( x_+^m \) and \( x_-^m \). In the case when \( L \) satisfies (the large \( k \) form (5.3) of) the master equation the Hamiltonian system

\[ \text{A suggestion that a different ‘Thirring-like’ action for two interacting WZW fields} \ g_L \text{ and} \ g_R \text{ may correspond to the solutions of the master equation was made in}[48]. \text{However, we believe the approach of}[48] \text{is not consistent since the fields} \ g_L \text{ and} \ g_R \text{ where assumed to be ‘non-chiral’ having manifestly Lorentz invariant action (i.e. describing doubled number of degrees of freedom) while the chirality constraint was imposed later ‘by hands’. This does not seem to go beyond a trivial rephrasing of the original affine–Virasoro construction.} \]
has invariance implied by the conjugation invariance \( L' = I - L \) as in the corresponding constraints can be added (with Lagrange multipliers \( v, \bar{v} \)) to the action (5.12) by replacing \( L \) by \( L + v(I - L) \) in the ‘+’ part and \( L \) by \( L + \bar{v}(I - L) \) in the ‘−’ part. The classical equations that follow from (5.12) can be represented as two separate equations for the currents \( J_+ \) and \( J_- \) (symbolically, \( \dot{J}_+ = fLJ_+J_+ + LJ'_+ \), etc) and are nothing but the Hamiltonian equations \( \dot{J}_\pm = \{H, J_\pm\} \) corresponding to (5.13) if \( J_\pm \) there form commuting affine algebras.

An advantage of the action (5.11) (5.9) is that it naturally incorporates the case of the coset solution \( L = I - P_H \) where \( P_H \) is a projector on a subalgebra \( H \). If we assume that \( A, \bar{A} \) take values in \( H \) eq.(5.9) becomes

\[
I_{G/H}(g_\pm, A_\pm) = I_+(g_+) + I_-(g_-) + \frac{1}{2\pi} \int d^2z \; \text{Tr} \; (-A_-J_+ + J_-A_+ - A_-A_- - A_+A_+) .
\]

(5.15)

This is just the sum of the analogs of the gauged WZW action in the ‘chiral’ case. In fact, consider, e.g.,

\[
I_{G/H}(g_+, A_-) = I_+(g_+) + \frac{1}{2\pi} \int d^2z \; \text{Tr} \; (-A_-J_+ - A_-A_-) = I_+(hg_+) - I(h) ,
\]

(5.16)

where \( A_- = \partial_-hh^{-1} \) (\( h \) is from \( H \)) and \( I(h) \) is the usual WZW action (see also \( [H] \)). The action (5.16) is invariant under \( g_+ \to fg_+, \; h \to hf^{-1}, \; f = f(x_-) \).

The action (5.12) is not, in general, Lorentz invariant at the classical level (even on the equations of motion). The coset case \( L = I - P_H \) (including \( L = I \)) is special since here the action (5.12) \is Lorentz invariant on the mass shell. This is not surprising since in the coset case one can integrate out the ‘ratio’ of \( g_+ \) and \( g_- \) explicitly, getting a local,

\[\text{\footnote{To have a non-chiral gauge invariance one needs to add an extra term } A_-A_+. \text{ This corresponds to a ‘vector’ regularisation scheme. In the present setting the ‘left-right decoupled’ scheme seems more natural. If } A_-A_+ \text{-term is added to (5.9) the action (5.11) and the Hamiltonian (5.14) become more complicated and, in particular, contain } J_+J_- \text{-term. The absence of the } A_-A_+ \text{-term is the reason why by integrating out the ‘ratio’ of } g_+ \text{ and } g_- \text{ one gets not a standard gauged WZW action but a chiral gauged WZW action for } g = g_+g_- .}\]

25
manifestly Lorentz invariant action for $g = g_+ g_-$. In fact, a combination of (5.16) with a similar action for $g_-, A_+$ leads to a (chiral) gauged WZW action for $g = g_+ g_-$. Since $I \rightarrow I_\text{chir}$,

$$I_+(g_+) + I_-(g_-) \rightarrow I(g), \quad g = g_+ g_-,$$  \hspace{1cm} (5.17)

we get (cf. (2.8))

$$I_+(g_+, A_-) + I_-(g_-, A_+) \rightarrow I_{chir}(g, A_+, A_-).$$  \hspace{1cm} (5.18)

It may be possible to do a similar integration for some other special values of $L$ returning back to the Lorentz invariant model (2.1) for $g = g_- g_+$. One may expect that the condition of (one-loop) conformal invariance of the non-Lorentz-invariant models (5.9),(5.12) can be put into correspondence with the master equation (5.3),(5.4). Then (5.9) would describe ‘reducible’ counterparts of both the coset and non-coset ‘irreducible’ solutions of the master equation.

In conclusion, let us mention that eqs.(3.2)–(3.3) give the universal expressions for the basic target space fields for the models of the class (2.1) making it possible to study the corresponding geometries in a systematic way. Another open problem is to find if there are other solutions of the conformal invariance equations (4.1),(4.16) in addition to gauged WZW, non-abelian chiral gauged WZW and (4.14').

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20 To prove (5.17) one changes the variables $g_+ = g^{1/2} f$, $g_- = f^{-1} g^{1/2}$. Then the sum of the actions in (5.17) becomes equal to $I(g)$ plus an additional term of the structure $\int (B + T(g))^2$, $B \equiv \partial_1 f f^{-1}$. The integral over $f$ gives a trivial contribution since one can replace $f$-integral by the integral over $B$-variable (the Jacobian is trivial since $\partial_1^{-1} = \theta(z_1 - z'_1)$).
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