SYMMETRY CLASSES IN RANDOM MATRIX THEORY

by
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1. Introduction

A classification of random-matrix ensembles by symmetries was first established by Dyson, in an influential 1962 paper with the title “the threefold way: algebraic structure of symmetry groups and ensembles in quantum mechanics”. Dyson’s threefold way has since become fundamental to various areas of theoretical physics, including the statistical theory of complex many-body systems, mesoscopic physics, disordered electron systems, and the field of quantum chaos.

Over the last decade, a number of random-matrix ensembles beyond Dyson’s classification have come to the fore in physics and mathematics. On the physics side these emerged from work on the low-energy Dirac spectrum of quantum chromodynamics, and from the mesoscopic physics of low-energy quasi-particles in disordered superconductors. In the mathematical research area of number theory, the study of statistical correlations in the values of Riemann zeta and similar functions has prompted some of the same generalizations.

In this article, Dyson’s fundamental result will be reviewed from a modern perspective, and the recent extension of Dyson’s threefold way will be motivated and described. In particular, it will be explained why symmetry classes are associated with large families of symmetric spaces.

2. The framework

Random matrices have their physical origin in the quantum world, more precisely in the statistical theory of strongly interacting many-body systems such as atomic nuclei. Although random-matrix theory is nowadays understood to be of relevance to numerous areas of physics – see e.g. Guhr’s article in this volume – quantum mechanics is still where many of its applications lie. Quantum mechanics also provides a natural framework in which to classify random-matrix ensembles.

Following Dyson, the mathematical setting for classification consists of two pieces of data:

- A finite-dimensional complex vector space $\mathcal{V}$ with a Hermitian scalar product $\langle \cdot, \cdot \rangle$, called a unitary structure for short. (In physics applications, $\mathcal{V}$ will usually be the truncated Hilbert space of a family of quantum Hamiltonian systems.)
- On $\mathcal{V}$ there acts a group $G$ of unitary and antiunitary operators (the joint symmetry group of the multi-parameter family of quantum systems).

Given this setup, one is interested in the linear space of self-adjoint operators on $\mathcal{V}$ – the Hamiltonians $H$ – with the property that they commute with $G$. Such a space is reducible in general, i.e. the matrix of $H$ decomposes into blocks. The goal of classification is to enumerate the irreducible blocks that occur.

2.1. Symmetry groups. — Basic to classification is the notion of a symmetry group in quantum Hamiltonian systems, a notion that will now be explained.

In classical mechanics the symmetry group $G_0$ of a Hamiltonian system is understood to be the group of canonical transformations that commute with the phase flow of the system. An important example is the rotation group for systems in a central field.

In passing from classical to quantum mechanics, one replaces the classical phase space by a quantum mechanical Hilbert space $\mathcal{V}$ and assigns to the symmetry group $G_0$ a (projective) representation by unitary C-linear operators on $\mathcal{V}$. Beside the one-parameter continuous subgroups, whose significance is highlighted by Noether’s theorem, the components of $G_0$ not connected with the identity play an important role. A prominent example is provided by the operator for space reflection. Its eigenspaces are the subspaces of states with positive and negative parity, which reduce the matrix of any reflection-invariant Hamiltonian to two blocks.

Not all symmetries of a quantum mechanical system are of the canonical, unitary kind: the prime counterexample is the operation of inverting the time direction, called time reversal for short. In classical mechanics this operation reverses the sign of the symplectic structure of phase space; in quantum mechanics its algebraic properties reflect the fact that inverting the time direction, $t \mapsto -t$, amounts to sending $i = \sqrt{-1}$ to $-i$. Indeed, time $t$ enters in the Dirac, Pauli, or Schrödinger equation as $ih\ddt$. Therefore, time reversal is represented in the quantum theory by an anti-unitary operator $T$, which is to say that $T$ is complex anti-linear:

$$T(z\psi) = \bar{z}T\psi \quad (z \in \mathbb{C}, \psi \in \mathcal{V}),$$
and preserves the Hermitian scalar product or unitary structure up to complex conjugation:
\[
\langle T\psi_1, T\psi_2 \rangle = \overline{\langle \psi_1, \psi_2 \rangle} = \langle \psi_2, \psi_1 \rangle
\].

Another operation of this kind is charge conjugation in relativistic theories such as the Dirac equation.

By the symmetry group \( G \) of a quantum mechanical system with Hamiltonian \( H \), one then means the group of all unitary and anti-unitary transformations \( g \) of \( \mathcal{H} \) that leave the Hamiltonian invariant: \( gHg^{-1} = H \). We denote the unitary subgroup of \( G \) by \( G_0 \), and the set of anti-unitary operators in \( G \) by \( G_1 \) (not a group). If \( \mathcal{H} \) carries extra structure, as will be the case for some extensions of Dyson’s basic scheme, the action of \( G \) on \( \mathcal{H} \) has to be compatible with that structure.

The set \( G_1 \) may be empty. When it is not, the composition of any two elements of \( G_1 \) is unitary, so every \( g \in G_1 \) can be obtained from a fixed element of \( G_1 \), say \( T \), by right multiplication with some \( U \in G_0 \): \( g = TU \).

In other words, when \( G_1 \) is non-empty the coset space \( G/G_0 \) consists of exactly two elements, \( G_0 \) and \( T \cdot G_0 = G_1 \). We shall assume that \( T \) represents some inversion symmetry such as time reversal or charge conjugation. \( T \) must then be a (projective) involution, i.e. \( T^2 = z \times \text{Id} \) with \( z \) a complex number of unit modulus, so that conjugation by \( T^2 \) is the identity operation. Since \( T \) is complex anti-linear, the associative law \( T^2 \cdot T = T \cdot T^2 \) forces \( z \) to be real, and hence \( T^2 = \pm \text{Id} \).

Finding the total symmetry group of a Hamiltonian system need not always be straightforward, but this complication will not be an issue here: we take the symmetry group \( G \) and its action on the Hilbert space \( \mathcal{H} \) as fundamental and given, and then ask what is the corresponding symmetry class, meaning the linear space of Hamiltonians on \( \mathcal{H} \) that commute with \( G \).

For technical reasons, we assume the group \( G_0 \) to be compact; this is an assumption that covers most (if not all) of the cases of interest in physics. The non-compact group of space translations can be incorporated, if necessary, by wrapping the system around a torus, whereby translations are turned into compact torus rotations.

While the primary objects to classify are the spaces of Hamiltonians \( H \), we shall focus for convenience on the spaces of time evolutions \( U_t = e^{-itH}/\hbar \) instead. This change of focus results in no loss, as the Hamiltonians can always be retrieved by linearizing in \( t \) at \( t = 0 \).

2.2. Symmetric spaces. — We appropriate a few basic facts from the theory of symmetric spaces.

Let \( M \) be a connected \( m \)-dimensional Riemannian manifold and \( p \) a point of \( M \). In some open subset \( N_p \) of a neighborhood of \( p \) there exists a map \( s_p : N_p \to N_p \), the geodesic inversion with respect to \( p \), which sends a point \( x \in N_p \) with normal coordinates \( (x_1, \ldots, x_m) \) to the point with normal coordinates \( (-x_1, \ldots, -x_m) \). The Riemannian manifold \( M \) is called locally symmetric if the geodesic inversion is an isometry, and is called globally symmetric if \( s_p \) extends to an isometry \( s_p : M \to M \), for all \( p \in M \). A globally symmetric Riemannian manifold is called a symmetric space for short.

The Riemann curvature tensor of a symmetric space is covariantly constant, which leads one to distinguish between three cases: the scalar curvature can be positive, zero, or negative, and the symmetric space is said to be of compact type, Euclidean type, or non-compact type, respectively. (In mesoscopic physics each type plays a role: the first provides us with the scattering matrices and time evolutions, the second with the Hamiltonians, and the third with the transfer matrices.) The focus in the current article will be on compact type, as it is this type that houses the unitary time evolution operators of quantum mechanics. The compact symmetric spaces are subdivided into two major subtypes, both of which occur naturally in the present context, as follows.

2.3. Type II. — Consider first the case where the anti-unitary component \( G_1 \) of the symmetry group is empty, so the data are \( (\mathcal{H}, G) \) with \( G = G_0 \). Let \( \mathcal{W}(\mathcal{H}) \) denote the group of all complex linear transformations that leave the structure of the vector space \( \mathcal{H} \) invariant. Thus \( \mathcal{W}(\mathcal{H}) \) is a group of unitary transformations if \( \mathcal{H} \) carries no more than the usual Hermitian scalar product; and is some subgroup of the unitary group if \( \mathcal{H} \) does have extra structure (as is the case for the Nambu space of quasi-particle excitations in a superconductor).

The symmetry group \( G_0 \), by acting on \( \mathcal{H} \) and preserving its structure, is contained as a subgroup in \( \mathcal{W}(\mathcal{H}) \).

Let now \( H \) be any Hamiltonian with the prescribed symmetries. Then the time evolution \( t \to U_t = e^{-itH}/\hbar \) generated by \( H \) is a one-parameter subgroup of \( \mathcal{W}(\mathcal{H}) \) which commutes with the \( G_0 \)-action. The total set of transformations \( U_t \) that arise in this way is called the (connected part of the) centralizer of \( G_0 \) in \( \mathcal{W}(\mathcal{H}) \), and is denoted by \( Z \). This is the “good” set of unitary time evolutions – the set compatible with the given symmetries of an ensemble of quantum systems.

The centralizer \( Z \) is obviously a group: if \( U \) and \( V \) belong to \( Z \), then so do their inverses and their product. What can one say about the structure of the group \( Z \)? In essence, this question was answered by H. Weyl in his famous treatise “The Classical Groups”. Since \( G_0 \) is compact by assumption, its group action on \( \mathcal{H} \) is completely reducible, and \( \mathcal{H} \) is guaranteed to have an orthogonal vector space decomposition
\[
\mathcal{H} = \sum_{\lambda} V_{\lambda} \simeq \sum_{\lambda} V_{\lambda} \otimes \mathbb{C}^{m_{\lambda}},
\]
where the sum is over (classes of equivalent) irreducible $G_0$-representations $\lambda$, the $V_\lambda$ are irreducible representation spaces for the $G_0$-action, and $m_\lambda$ is the multiplicity of occurrence of the representation type $\lambda$. ($G_0$ acts trivially on $\mathbb{C}^{m_\lambda}$. ) The subspaces $\mathcal{Y}_\lambda \simeq V_\lambda \otimes \mathbb{C}^{m_\lambda}$ will be called the $G_0$-isotypic components of $\mathcal{Y}$. For example, if $G_0$ is the rotation group SO(3), the $G_0$-isotypic components of $\mathcal{Y}$ are the subspaces of states with definite total angular momentum, say $\lambda = 0, 1, 2, \ldots$; $V_\lambda$ then is an SO(3)-irreducible representation space of dimension $2\lambda + 1$; and $m_\lambda$ is the number of times a multiplet of states with angular momentum $\lambda$ occurs in $\mathcal{Y}$.

Now consider any $U \in Z$. Since $U$ commutes with the $G_0$-action, it does not connect different $G_0$-isotypic components. (Indeed, in the example of SO(3)-invariant dynamics, angular momentum is conserved and transitions between different angular momentum sectors are impossible.) Thus every $G_0$-isotypic component $\mathcal{Y}_\lambda$ is an invariant subspace for the action of $Z$ on $\mathcal{Y}$, and $Z$ decomposes as $Z = \prod_\lambda Z_\lambda$ with blocks $Z_\lambda = \mathcal{Y}_\lambda \mid_{\mathcal{Y}_\lambda}$. But one can say even more: because $U$ commutes with $G_0$ and $V_\lambda$ is $G_0$-irreducible, $U$ must act like the identity on $V_\lambda$ by Schur’s lemma. Therefore, $Z_\lambda$ acts nontrivially only on the factor $\mathbb{C}^{m_\lambda}$ in $\mathcal{Y}_\lambda \simeq V_\lambda \otimes \mathbb{C}^{m_\lambda}$, so

$$Z = \prod_\lambda Z_\lambda \simeq \prod_\lambda U(m_\lambda)$$

is the direct product, if $\mathcal{Y}$ is a unitary vector space with no extra structure. In the presence of extra structure (which, by compatibility with the $G_0$-action, restricts to every subspace $\mathcal{Y}_\lambda$) the factor $Z_\lambda$ is some subgroup of $U(m_\lambda)$. In all cases, $Z$ is a direct product of connected compact Lie groups $Z_\lambda$.

The focus now shifts from $Z$ to any one of the $Z_\lambda$. So we fix some $M := Z_\lambda$. Since $M$ is a group, the operation of taking the inverse, $U \mapsto U^{-1}$, makes sense for all $U \in M$. Moreover, being a compact Lie group, the manifold $M$ admits a left- and right-invariant Riemannian structure in which the inversion $U \mapsto U^{-1}$ is an isometry. By translation one gets an isometry

$$s_{U_1} : U \mapsto U_1 U^{-1} U_1$$

for every $U_1 \in M$. All these maps $s_{U_1}$ are globally defined, and the restriction of $s_{U_1}$ to some neighborhood of $U_1$ coincides with the geodesic inversion w.r.t. $U_1$. Thus $M$ is a symmetric space by the definition given above. Symmetric spaces of this kind (with group structure) are called type II.

2.4. Type I. — Consider next the case $G_1 \neq \emptyset$, where some anti-unitary symmetry $T$ is present. As before, let $Z$ be the connected component of the centralizer of $G_0$ in $\mathcal{Y}(\mathcal{Y})$. Conjugation by $T$,

$$U \mapsto \tau(U) := TUT^{-1},$$

is an automorphism of $\mathcal{Y}(\mathcal{Y})$ and, owing to $T^2 = \pm \text{Id}$, $\tau$ is involutive. Because $G_0 \subset G$ is a normal subgroup, $\tau$ restricts to an involutive automorphism (still denoted by $\tau$) of $Z$. Now recall that $T$ is complex anti-linear and the good Hamiltonians are subject to $THT^{-1} = H$.

The good time evolutions $U_t = e^{-itH/t}$ clearly satisfy $\tau(U_t) = U_{-t} = U_t^{-1}$. Thus the good set to consider is

$$\mathcal{M} := \{ U \in Z | U = \tau(U)^{-1} \}.$$ 

$\mathcal{M}$ is a manifold, but in general is not a Lie group.

Further details depend on what $\tau$ does with the factorization $Z = \prod_\lambda Z_\lambda$. If $\mathcal{Y}_\lambda$ is a $G_0$-isotypic component of $\mathcal{Y}$, then so is $T \mathcal{Y}_\lambda$, since $T$ normalizes $G_0$. Thus $T \mathcal{Y}_\lambda = \mathcal{Y}_\lambda$ for some representation type $\tilde{\lambda}$. If $\tilde{\lambda} \neq \lambda$, the involutive automorphism $\tau$ just relates $U \in Z_\lambda$ with $\tau(U) \in Z_{\tilde{\lambda}}$, whence no intrinsic constraint on $Z_\lambda$ results, and the time evolutions $(U, \tau(U)^{-1}) \in Z_{\lambda} \times Z_{\tilde{\lambda}}$ constitute a type-II symmetric space, as before.

A novel situation occurs when $\tilde{\lambda} = \lambda$, in which case $\tau$ maps the group $Z_{\lambda}$ onto itself. Let therefore $\tilde{\lambda} = \lambda$, put $K \equiv Z_{\lambda}$ for short, and consider

$$M := \{ U \in K | U = \tau(U)^{-1} \}.$$ 

Note that if two elements $p, p_0$ of $K$ are in $M$, then so is the product $p_0 p^{-1} p_0$. The group $K$ acts on $M \subset K$ by

$$k \cdot U = k U \tau(k)^{-1} \quad (k \in K),$$

and this group action is transitive, i.e. every $U \in M$ can be written as $U = k \tau(k)^{-1}$ with some $k \in K$. (Finding $k$ for a given $U$ is like taking a square root, which is possible since $\exp : \text{Lie} K \rightarrow K$ is surjective.) There exists a $K$-invariant Riemannian structure for $M$ such that for all $p_0 \in M$ the mapping $s_{p_0} : M \rightarrow M$ defined by

$$s_{p_0}(p) = p_0 p^{-1} p_0,$$

is the geodesic inversion w.r.t. $p_0 \in M$. Thus in this natural geometry $M$ is a globally symmetric Riemannian manifold and hence a symmetric space. The present kind of symmetric space is called type I. If $K_\tau$ is the set of fixed points of $\tau$ in $K$, the symmetric space $M$ is analytically diffeomorphic to the coset space $K/K_\tau$ by

$$K/K_\tau \rightarrow M \subset K, \quad U K_\tau \mapsto U \tau(U)^{-1},$$

which is called the Cartan embedding of $K/K_\tau$ into $K$.

In summary, the solution to the problem of finding the unitary time evolution operators that are compatible with a given symmetry group and structure of Hilbert space, is always a symmetric space. This is a valuable insight, as symmetric spaces are rigid objects and have been completely classified by Cartan.

If we keep the dimension of $\mathcal{Y}$ variable, the symmetric spaces that occur must be those of a large family.
3. Dyson’s threefold way

Recall the goal: given a Hilbert space $\mathcal{V}$ and a symmetry group $G$ acting on it, one wants to classify the (irreducible) spaces of time evolution operators $U$ that are “compatible” with $G$, meaning

$$U = g_0 U g_0^{-1} = g_1 U g_1^{-1} \quad (\text{for all } g_\sigma \in G_\sigma).$$

As we have seen, the spaces that arise in this way are symmetric spaces of type I or II depending on the nature of the time reversal (or other anti-unitary symmetry) $T$.

An even stronger statement can be made when more information about the Hilbert space $\mathcal{V}$ is specified. In Dyson’s classification, the Hermitian scalar product of $\mathcal{V}$ is assumed to be the only invariant structure that exists on $\mathcal{V}$. With that assumption, only three large families of symmetric spaces arise; these correspond to what we call the Wigner-Dyson symmetry classes.

3.1. Class A. — Recall that in Dyson’s case, the connected part of the centralizer of $G_0$ in $\mathcal{A}(\mathcal{V})$ is a direct product of unitary groups, each factor being associated with one $G_0$-isotypic component $\mathcal{V}_\lambda$ of $\mathcal{V}$. The type-II situation occurs when the set $G_1$ of anti-unitary symmetries is empty or else exchanges different $\mathcal{V}_\lambda$. In both cases, the set of good time evolution operators restricted to one $G_0$-isotypic component $\mathcal{V}_\lambda$ is a unitary group $U(m_\lambda)$, with $m_\lambda$ being the multiplicity of the irreducible $G_0$-representation $\lambda$ in $\mathcal{V}_\lambda$.

The unitary groups $U(N = m_\lambda)$ or to be precise, their simple parts $SU(N)$, are called type-II symmetric spaces of the $A$ family or $A$ series – hence the name class A. The Hamiltonians $H$, the generators of time evolutions $U_t = e^{-iHt/\hbar}$, in this class are represented by complex Hermitian $N \times N$ matrices. By putting a $U(N)$-invariant Gaussian probability measure

$$\exp\left(\frac{-\text{Tr} H^2}{2\sigma^2}\right) d\mathcal{H} \quad (\sigma \in \mathbb{R})$$

on that space, one gets what is called the GUE – the Gaussian Unitary Ensemble – which defines the Wigner-Dyson universality class of unitary symmetry.

3.2. Classes AI and AII. — Consider next the case $G_1 \neq \emptyset$, with anti-unitary generator $T$. Let $\mathcal{V}_\lambda$ be any $G_0$-isotypic component which is invariant under $T$ (the type-I situation). The mapping $U \mapsto TUT^{-1} = \tau(U)$ then is an automorphism of the groups $U(\mathcal{V}_\lambda)$, $G_0$ and $K = Z_\lambda \simeq U(m_\lambda)$. If $K_\tau$ is the subgroup of fixed points of $\tau$ in $K$, the space of good time evolutions can be identified with the symmetric space $K/K_\tau$ by the Cartan embedding. The task is to determine $K_\tau$. As was emphasized by Dyson, the answer for $K_\tau$ does not follow from any single piece of data, but is determined by the combination of three anti-unitary involutions on $\mathcal{V}_\lambda$.

The first of these is the standard operation of taking the complex conjugate (w.r.t. the complex structure of $\mathcal{V}_\lambda$), denoted by $\psi \mapsto \bar{\psi}$ as usual. Recall that the unitary vector space $\mathcal{V}_\lambda$ decomposes as an orthogonal sum of $m_\lambda$ identical copies of an irreducible representation space $V_\lambda$ for the compact group $G_0$:

$$\mathcal{V}_\lambda = V_\lambda^{(1)} \oplus V_\lambda^{(2)} \oplus \ldots \oplus V_\lambda^{(m_\lambda)} \simeq V_\lambda \otimes \mathbb{C}^{m_\lambda}.$$

From the multitude of such decompositions, we select one that is invariant under complex conjugation: if $\psi$ lies in a subspace $V_\lambda^{(j)}$, so does $\bar{\psi}$.

Next we look at the operation $g_0 \mapsto \bar{g}_0$ on any one of the $G_0$-irreducible subspaces of $\mathcal{V}_\lambda$, say $V_\lambda \equiv V_\lambda^{(1)}$. Since $\lambda = \bar{\lambda}$ by assumption, the $G_0$-action on $V_\lambda$ is unitarily equivalent to its complex conjugate. Thus there exists some unitary transformation $s \in U(V_\lambda)$ such that

$$\bar{g}_0 = s^{-1} g_0 s$$

holds for every $g_0 \in G_0$. Given $s$, one defines an anti-unitary operator $S$ on $\mathcal{V}_\lambda$ by

$$S(v^{(1)} + \ldots + v^{(m_\lambda)}) = s v^{(1)} + \ldots + s v^{(m_\lambda)},$$

where the decomposition (1) is invoked. By construction, $S$ commutes with the action of $G_0$. Since the $G_0$-action on $V_\lambda$ is irreducible, Schür’s lemma applied to $S^2$ forces $S$ to be a projective involution, and associativity $(S^2 \cdot S = S \cdot S^2)$ results in $S^2 = \pm 1$.

Recall next that $g_0 \mapsto T g_0 T^{-1}$ is an automorphism of $G_0$, and remains so when the $G_0$-action is restricted to $\mathcal{V}_\lambda$. Since all of the $G_0$-representations in the isotypic component $\mathcal{V}_\lambda$ are equivalent, there exists a unitary transformation $R \in U(\mathcal{V}_\lambda)$ such that

$$T g_0 T^{-1} = R^{-1} g_0 R \quad (\text{for all } g_0 \in G_0)$$

holds as an operator identity on $\mathcal{V}_\lambda$. Note that the composition $RT$ intertwines $G_0$-actions: $RT g_0 = g_0 RT$, but changes the complex structure of $\mathcal{V}_\lambda$ (by anti-linearity of $T$). A better object to consider is $RT \circ S$ which, being composed of two anti-unitary operators, is unitary. $RTS$ commutes with the $G_0$-action and thus lies in the centralizer $K$. Using it, one defines another anti-unitary operator $T'$ on $\mathcal{V}_\lambda$ and an automorphism $\tau'$ of $U(\mathcal{V}_\lambda)$ by

$$T' \psi = RT S \psi, \quad \tau'(k) = T' k T'^{-1}.$$

$T'$ determines a complex bilinear form $Q$ on $\mathcal{V}_\lambda$ by

$$Q(\psi_1, \psi_2) = \langle T' \psi_1, \psi_2 \rangle_{\mathcal{V}_\lambda} \quad (\text{for all } \psi_1, \psi_2 \in \mathcal{V}_\lambda).$$

The remaining steps toward identifying $K_\tau$ depend on the nature of $R$. Consider first the easy case where the automorphism $\tau$ of $G_0$ is inner, i.e. $R \in G_0$. Then $k = \tau(k)$ for $k \in K$ is equivalent to $k = RT k (RT)^{-1}$, which in turn amounts to $k = RT S \bar{k} (RT)^{-1} = \tau(k)$.
Hence another description of $K_t$ is to say that its elements $k$ are the unitary transformations of $Y_\lambda$ that centralize $G_0$ and leave the pairing $Q$ invariant (the latter follows from $T'k = kT'$ and invariance of $\langle \cdot, \cdot \rangle_{Y_\lambda}$).

By iterating $k = \tau^2(k) = \tau^2(k)$ one infers that $T'^{2}$ commutes with the $K_t$-action on $Y_\lambda$. But $T'^{2}$ also commutes with $T'$ and with the $G_0$-action, which implies $T'^{2} = \varepsilon \times 1_{Y_\lambda}$ with $\varepsilon = \pm 1$, by standard reasoning. From

$$Q(\psi_1, \psi_2) = \langle T'^{2} \psi_1, \psi_2 \rangle_{Y_\lambda} = \varepsilon Q(\psi_2, \psi_1)$$

one sees that the pairing $Q$ is symmetric for $\varepsilon = +1$, and skew for $\varepsilon = -1$. The Lie group $K_t$ is now easily identified. Since the unitary operator $RTS$ is an element of $K \simeq U(m_\lambda)$, $Q$ restricts to a pairing on the factor $\mathbb{C}^{m_\lambda}$ in the decomposition $Y_\lambda \simeq V_\lambda \otimes \mathbb{C}^{m_\lambda}$. Thus for $\varepsilon = +1$, $K_t$ can be viewed as a subgroup of $U(m_\lambda)$ that preserves a symmetric pairing (or orthogonal structure) on $\mathbb{C}^{m_\lambda}$; consequently $K_t \simeq O(m_\lambda)$. For $\varepsilon = -1$, the multiplicity $m_\lambda$ must be even, and $K_t$ preserves a skew pairing (or symplectic structure); in that case $K_t \simeq USp(m_\lambda)$, the unitary symplectic group.

In the general case ($R \notin G_0$) drawing these conclusions is more difficult, and one must exploit the rigidity of the symmetric space $K/K_t$ under deformations of the automorphism $\tau(U) = TUT^{-1}$. Actually, the method used by Dyson to handle the general case is quite different: Dyson chooses to regard $Y_\lambda$ as a real vector space (with complex structure) and expresses the group action of $G$ by orthogonal matrices. In this real setup he then exploits a deep theorem of Weyl on the structure of group algebras and their commutator algebras, which leads him to the conclusion that the above two possibilities are in fact the only ones that can occur in the present type-I situation. Thus there is a dichotomy for the sets of good time evolutions $M \simeq K/K_t$:

- Class AI: $K/K_t \simeq U(N)/O(N)$ \hspace{1cm} ($N = m_\lambda$),
- Class AII: $K/K_t \simeq U(2N)/USp(2N)$ \hspace{1cm} ($2N = m_\lambda$).

Again we are referring to symmetric spaces by the names they – or rather their simple parts $SU(N)/SO(N)$ and $SU(2N)/USp(2N)$ – have in the Cartan classification. Be warned that Weyl’s theorem by itself does not allow to decide between the alternatives AI or AII for a given data set $(Y_\lambda, G_0, T)$ (rather, to do so you must determine the “Wigner type” of the $G$-representation on $Y_\lambda$).

In the case where the $G_0$-automorphism $\tau$ is inner (which actually covers most of the known examples of physical interest) Dyson’s reasoning is basically identical to the one reviewed above. There, as we have seen, the dichotomy is ruled by the number $\varepsilon$ computed from $(T')^2 = R S R T S = \varepsilon \times 1_{Y_\lambda}$. Important examples are provided by physical systems with spin-rotation symmetry, $G_0 = SU(2)$, and time-reversal symmetry. The physical operation of time reversal, $T$, commutes with spin rotations, so $\tau$ is inner here with $R = 1$. On states with spin $|S|$, one has $T^2 = (-1)^{2|S|}$ and $S^2 = (-1)^{|S|}$, which gives $T'^{2} = +1$ in all cases. Thus time-reversal invariant systems with no symmetries other than energy and spin are always class AI. By breaking spin-rotation symmetry ($G_0 = \{1\}$), so $S^2 = 1$ while maintaining $T$-symmetry for states with half-integer spin (say single electrons, which carry spin $|S| = 1/2$), one gets $T'^{2} = T^2 = +1$, thereby realizing class AII.

The Hamiltonians $H$, obtained by passing to the tangent space of $K/K_t$ at unity, are represented by Hermitian matrices with entries that are real numbers (class AI) or real quaternions (class AII). If you put $K_t$-invariant Gaussian probability measures on these spaces, you get the Wigner-Dyson universality classes of orthogonal resp. symplectic symmetry. In mesoscopic physics these are realized in disordered metals with time-reversal invariance (absence of magnetic fields and magnetic impurities). Spin-rotation symmetry is broken by strong spin-orbit scatterers such as gold impurities.

### 4. Disordered superconductors

When Dirac first wrote down his famous equation in 1928, he assumed that he was writing an equation for the wavefunction of the electron. Later, because of the instability caused by negative-energy solutions, the Dirac equation was reinterpreted (via second quantization) as an equation for the fermionic field operators of a quantum field theory. A similar change of viewpoint is carried out in reverse in the Hartree-Fock-Bogoliubov mean field description of quasi-particle excitations in superconductors. There, one starts from the equations of motion for linear superpositions of the electron creation and annihilation operators, and reinterprets them as a unitary quantum dynamics for what might be called the quasi-particle “wavefunction”.

In both cases – the Dirac equation and the quasi-particle dynamics of a superconductor – there enters a structure not present in the standard quantum mechanics underlying Dyson’s classification: the field operators for fermionic particles are subject to a set of requirements called the canonical anti-commutation relations, and these are preserved by the quantum dynamics. Therefore, whenever second quantization is undone (assuming it can be undone) to return from field operators to wavefunctions, the wavefunction dynamics is required to preserve some extra structure. This puts a linear constraint on the allowed Hamiltonians $H$. For our
purposes, the best viewpoint to take is to attribute the extra invariant structure to the Hilbert space $\mathcal{V}$, thereby turning it into a Nambu space.

4.1. Nambu space. — Starting from the standard formalism of second quantization, consider a set of single-particle creation and annihilation operators $c^\dagger_\alpha$ and $c_\alpha$, where $\alpha = 1, \ldots, N$ labels single-particle states that are orthogonal to each other. Such operators are subject to the canonical anti-commutation relations

\begin{align}
    c^\dagger_\alpha c_\beta + c_\beta c^\dagger_\alpha &= \delta_{\alpha\beta}, \\
    c^\dagger_\alpha c^\dagger_\beta + c_\beta c_\alpha &= 0 = c_\alpha c^\dagger_\beta + c^\dagger_\beta c_\alpha.
\end{align}

When written in terms of $c_\alpha + c^\dagger_\alpha$ and $i(c_\alpha - c^\dagger_\alpha)$, these become the defining relations of a Clifford algebra. Field operators are linear combinations $\psi = \sum_\alpha (u_\alpha c^\dagger_\alpha + v_\alpha c_\alpha)$ with complex coefficients $u_\alpha$ and $v_\alpha$.

Now take $H$ to be some Hamiltonian which is quadratic in the creation and annihilation operators:

$$
    H = \sum_{\alpha\beta} A_{\alpha\beta} c^\dagger_\alpha c_\beta + \frac{1}{2} \sum_{\alpha\beta} \left( B_{\alpha\beta} c^\dagger_\alpha c^\dagger_\beta + \bar{B}_{\alpha\beta} c_\beta c_\alpha \right),
$$

and let $H$ act on field operators $\psi$ by the commutator: $H \cdot \psi \equiv [H, \psi]$. The time evolution of $\psi$ is then determined by the equation

$$
    \frac{d\psi}{dt} = -\frac{i}{\hbar} H \cdot \psi,
$$

which integrates to $\psi(t) = e^{-itH/\hbar} \cdot \psi(0)$, and is easily verified to preserve the relations (2).

The dynamical equation (3) is equivalent to a system of linear differential equations for the amplitudes $u_\alpha$ and $v_\alpha$. If these are assembled into vectors, and the $A_{\alpha\beta}$ and $B_{\alpha\beta}$ into matrices, equation (3) becomes

$$
    \frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = -\frac{i}{\hbar} \begin{pmatrix} A & B \\ -\bar{B} & -A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
$$

The Hamiltonian matrix on the right-hand side has some special properties due to $B_{\alpha\beta} = -\bar{B}_{\beta\alpha}$ (from $c^\dagger_\alpha c_\beta = -c_\beta c^\dagger_\alpha$) and $A_{\alpha\beta} = \bar{A}_{\beta\alpha}$ (from $H$ being self-adjoint as an operator in Fock space). To keep track of these properties while imposing some unitary and anti-unitary symmetries, it is best to put everything in invariant form.

Let $V$ be the complex vector space of annihilation operators $v = \sum_\alpha v_\alpha c^\dagger_\alpha$, and view the creation operators $u = \sum_\alpha u_\alpha c^\dagger_\alpha$ as lying in the dual vector space $V^*$. The field operators $\psi = u + v$ then are elements of the direct sum $V^* \oplus V = \mathcal{V}$, called Nambu space. On $\mathcal{V}$ there exists a canonical unitary structure expressed by

$$
    \langle u_1 + v_1, u_2 + v_2 \rangle = \sum_\alpha (\bar{u}_1 u_2 + \bar{v}_1 v_2).
$$

A second canonical structure on $\mathcal{V}$ is given by the symmetric $C$-bilinear form

$$
    \{u_1 + v_1, u_2 + v_2\} = \sum_\alpha (u_1 v_{2\alpha} + u_{2\alpha} v_1)\,.
$$

Note that $\{\psi_1, \psi_2\}$ agrees with the anti-commutator of the field operators, $\psi_1 \psi_2 + \psi_2 \psi_1$, by the relations (2).

Now recall that the quantum dynamics is determined by a Hamiltonian $H$ that acts on $\psi$ by the commutator $H \cdot \psi = [H, \psi]$. The one-parameter groups $t \mapsto e^{-itH/\hbar}$ generated by this action (the time evolutions) preserve the symmetric pairing:

$$
    \{\psi_1, \psi_2\} = \left\{ e^{-itH/\hbar} \cdot \psi_1, e^{-itH/\hbar} \cdot \psi_2 \right\},
$$

since the anti-commutation relations (2) do not change with time. They also preserve the unitary structure,

$$
    \langle \psi_1, \psi_2 \rangle = \left\langle e^{-itH/\hbar} \cdot \psi_1, e^{-itH/\hbar} \cdot \psi_2 \right\rangle,
$$

because probability in Nambu space is preserved. (Physically speaking, this holds true as long as $H$ is quadratic, i.e. many-body interactions are negligible.)

One can now pose Dyson’s question again: given Nambu space $\mathcal{V}$ and a symmetry group $G$ acting on it, what is the set of time evolution operators that preserve the structure of $\mathcal{V}$ and are compatible with $G$? From Section 2 we know the answer to be some symmetric space, but which are the symmetric spaces that occur?

4.2. Class $D$. — Consider a superconductor with no symmetries in its quasi-particle dynamics, so $G = \{1\}$. (A concrete example would be a disordered spin-triplet superconductor in the vortex phase). The time evolutions $U = e^{-itH/\hbar}$ are then constrained only by invariance of the unitary structure and the symmetric pairing $\langle \cdot, \cdot \rangle$ of Nambu space. These two structures are consistent; they are related by particle-hole conjugation $C$:

$$
    \{\psi_1, \psi_2\} = \langle C\psi_1, \psi_2 \rangle,
$$

which is an anti-unitary operator with square $C^2 = +\text{Id}$. The condition $\{\psi_1, \psi_2\} = \{U\psi_1, U\psi_2\}$ (invariance of an orthogonal structure) selects a complex orthogonal group, and imposing unitarity yields a real subgroup $\text{SO}(\mathcal{V}) \simeq \text{SO}(4N)$ — a symmetric space of the $D$ family.

Since the time evolutions are real orthogonal, there exists a basis of $\mathcal{V}$ (called Majorana fermions in physics) in which the matrix of $iH \in \text{so}(\mathcal{V})$ is real skew, and that of $H$ imaginary skew. The simplest random matrix model for class $D$, the SO-invariant Gaussian ensemble of imaginary skew matrices, is analyzed in the second edition of Mehta’s book. From the expressions given by Mehta it is seen that the level correlation functions at high energy coincide with those of the Wigner-Dyson universality class of unitary symmetry. The level
4.3. Class DIII. — Let now magnetic fields and magnetic impurities be absent, so that time reversal $T$ is a symmetry of the quasi-particle system: $G = \{\text{Id}, T\}$. Following Section 2, the set of good time evolutions is $M \simeq K/K_T$ with $K = \text{SO}(\mathcal{V})$ and $K_T$ the set of fixed points of $U \mapsto \tau(U) = TUT^{-1}$ in $K$. What is $K_T$?

The time-reversal operator has square $T^2 = -\text{Id}$ (for particles with spin 1/2), and commutes with particle-hole conjugation $C$, which makes $Q := iCT$ a useful operator to consider. Since $C$ by definition commutes with the action of $K$, and hence also with that of $K_T$, the subgroup $K_T$ has an equivalent description as

$$K_T = \{k \in U(\mathcal{V}) \mid k = QkQ^{-1} = \tau(k)\}.$$

The operator $Q$ is easily seen to have the following properties: (i) $Q$ is unitary, (ii) $Q^2 = \text{Id}$, and (iii) $\text{Tr}_\mathcal{V} Q = 0$. Consequently $Q$ possesses two eigenspaces $\mathcal{V}_{\pm}$ of equal dimension, and the condition $k = QkQ^{-1}$ fixes a subgroup $U(\mathcal{V}_+) \times U(\mathcal{V}_-) \subset U(\mathcal{V})$. Since $Q$ contains a factor $i = \sqrt{-1}$ in its definition, it anti-commutes with the anti-linear operator $T$. Therefore the automorphism $\tau$ exchanges $U(\mathcal{V}_+)$ with $U(\mathcal{V}_-)$, and the fixed point set $K_T$ is the same as $U(\mathcal{V}_+) \simeq U(2N)$. Thus

$$M \simeq K/K_T \simeq \text{SO}(4N)/U(2N),$$

a symmetric space in the DIII family. Note that for particles with spin 1/2 the dimension of $\mathcal{V}_\pm$ has to be even.

By realizing the algebra of involutions $C, T$ as $C\psi = (i\sigma_\chi \otimes 1_{2N})\bar{\psi}$ and $T\psi = (i\sigma_\chi \otimes 1_{2N})\bar{\psi}$, the Hamiltonians $H$ in class DIII are brought into the standard form

$$H = \begin{pmatrix} 0 & \bar{Z} \\ -\bar{Z} & 0 \end{pmatrix},$$

where the $2N \times 2N$ matrix $Z$ is complex and skew.

4.4. Class C. — Next let the spin of the quasi-particles be conserved, as is the case for a spin-singlet superconductor with no spin-orbit scatterers present, and let time-reversal invariance be broken by a magnetic field. The symmetry group of the quasi-particle system then is the spin-rotation group: $G = G_0 = \text{Spin}(3) = SU(2)$.

Nambu space $\mathcal{V}$ can be arranged to be a tensor product $\mathcal{V} = \mathcal{W} \otimes \mathbb{C}^2$ so that $G_0$ acts trivially on $\mathcal{W}$ and by the spinor representation on the spinor space $\mathbb{C}^2$. Since two spinors combine to give a scalar, the latter comes with a skew-symmetric form $\epsilon : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$. In a suitable basis, the anti-commutation relations $\{\cdot, \cdot\}$ factor on particle-hole and spin indices. The symmetric bilinear form $\{\cdot, \cdot\}$ of $\mathcal{V}$ correspondingly factors under the tensor product decomposition $\mathcal{V} = \mathcal{W} \otimes \mathbb{C}^2$ as

$$\{w_1 \otimes s_1, w_2 \otimes s_2\} = [w_1, w_2] \times \epsilon(s_1, s_2),$$

where $[\cdot, \cdot]$ is a skew bilinear form on $\mathcal{W}$, giving $\mathcal{W}$ the structure of a symplectic vector space.

The good set $M$ now consists of the time evolutions that, in addition to preserving the structure of Nambu space, commute with the spin-rotation group $SU(2)$:

$$M = \{U \in U(\mathcal{W}) \mid UC = CU, \forall R \in SU(2) : RU = UR\}.$$

By the last condition, all time evolutions act trivially on the factor $\mathbb{C}^2$. The condition $UC = CU$, which expresses invariance of the orthogonal structure of $\mathcal{V}$, then implies that time evolutions preserve the symplectic pairing of $\mathcal{W}$. Time evolutions therefore are unitary symplectic transformations of $\mathcal{W}$, hence $M = \text{USp}(\mathcal{W}) \simeq \text{USp}(2N) - \text{a symmetric space of the C family}$. The Hamiltonian matrices in class C have the standard form

$$H = \begin{pmatrix} A & B \\ \bar{B} & -\bar{A} \end{pmatrix},$$

with $A$ being Hermitian and $B$ complex and symmetric.

4.5. Class CI. — The next class is obtained by taking the time reversal $T$ as well as the spin rotations $R \in SU(2)$ to be symmetries of the quasi-particle system.

By arguments that should be familiar by now, the set of good time evolutions is a symmetric space $M \simeq K/K_T$ with $K = \text{USp}(\mathcal{W})$ and $K_T$ the set of fixed points of $\tau$ in $K$. Once again, the question to be answered is: what’s $K_T$? The situation here is very similar to the one for class DIII, with $\mathcal{W}$ and $\text{USp}(\mathcal{W})$ taking the roles of $\mathcal{V}$ and $\text{SO}(\mathcal{V})$. By adapting the previous argument to the present case, one shows that $K_T$ is the same as $U(\mathcal{W}_+) \simeq U(N)$, where $\mathcal{W}_+$ is the positive eigenspace of $Q = iCT$ viewed as a unitary operator on $\mathcal{W}$. Thus

$$M \simeq K/K_T \simeq \text{USp}(2N)/U(N).$$

The standard form of the Hamiltonian matrices here is

$$H = \begin{pmatrix} 0 & Z \\ \bar{Z} & 0 \end{pmatrix},$$

with the $N \times N$ matrix $Z$ being complex and symmetric.

5. Dirac fermions: the chiral classes

Three large families of symmetric spaces remain to be implemented. Although these, too, occur in mesoscopic physics, their most natural realization is by 4d Dirac fermions in a random gauge field background.
Consider the Lagrangian $L$ for the Euclidean space-time version of quantum chromodynamics with $N_c \geq 3$ colors of quarks coupled to an SU($N_c$) gauge field $A_\mu$:

$$L = i\bar{\psi} \gamma^\mu (\partial_\mu - A_\mu) \psi + im\bar{\psi} \psi .$$

The massless Dirac operator $D = i\gamma^\mu (\partial_\mu - A_\mu)$ anticommutes with $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$. Therefore, in a basis of eigenstates of $\gamma^5$ the matrix of $D$ takes the form

$$D = \begin{pmatrix} 0 & Z \\ -Z^* & 0 \end{pmatrix} .$$

If the gauge field carries topological charge $\nu \in \mathbb{Z}$, the Dirac operator $D$ has at least $|\nu|$ zero modes by the index theorem. To make a simple model of the challenging situation where $A_\mu$ is distributed according to Yang-Mills measure, one takes the matrices $Z$ to be complex rectangular, of size $p \times q$ with $p - q = \nu$, and puts a Gauss measure on that space. This random-matrix model for $D$ captures the universal features of the QCD Dirac spectrum in the massless limit.

The exponential of the truncated Dirac operator, $e^{itD}$ (where $t$ is not the time), lies in a space equivalent to $U(p+q)/U(p) \times U(q)$ — a symmetric space of the AIII family. We therefore say that the universal behavior of the QCD Dirac spectrum is that of symmetry class AIII.

But hold on! Why are we entitled to speak of a symmetry class here? By definition, symmetries always commute with the Hamiltonian, never do they anti-commute! (The relation $D = -\gamma^5 D \gamma^5$ is not a symmetry in the sense of Dyson, nor is it a symmetry in our sense.)

5.1. Class AIII. — To incorporate the massless QCD Dirac operator into the present classification scheme, we adapt it to the Nambu space setting. This is done by reorganizing the 4-component Dirac spinor $\psi$, $\bar{\psi}$ as an 8-component Majorana spinor $\Psi$, to write

$$L_{\text{m=0}} = \frac{i}{2} \bar{\Psi} \Gamma^\mu (\partial_\mu - A_\mu) \Psi .$$

The $8 \times 8$ matrices $\Gamma^\mu$ are real symmetric besides satisfying the Clifford relations $\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2\delta^{\mu\nu}$. A possible tensor-product realization is

$$\Gamma^0 = 1 \otimes \sigma_3 \otimes 1 , \quad \Gamma^1 = \sigma_x \otimes \sigma_y \otimes \sigma_y , \quad \Gamma^2 = \sigma_y \otimes \sigma_3 \otimes 1 , \quad \Gamma^3 = \sigma_3 \otimes \sigma_y \otimes \sigma_y .$$

The gauge field in this Majorana representation is $A_\mu = 1 \otimes \sigma_3 \otimes 1 \otimes (\tilde{A}_\mu^(-) - \tilde{A}_\mu^+ \sigma_y)$ where $\tilde{A}_\mu^{(\pm)} = \frac{1}{2} (A_\mu^{(\pm)} \pm A_\mu^{(T)})$ are the symmetric and skew parts of $A_\mu \in \text{su}(N_c)$.

The operator $H = i\Gamma^\mu (\partial_\mu - \tilde{A}_\mu)$ is imaginary skew, therefore $e^{\tilde{H}}$ is real orthogonal. This means that there exists a Nambu space $\mathcal{Y}$ with unitary structure $\langle \cdot, \cdot \rangle$ and symmetric pairing $\{ \cdot, \cdot \}$, both of which are preserved by the action of $e^{\tilde{H}}$. No change of physical meaning or interpretation is implied by the identical rewriting from Dirac $D$ to Majorana $H$. The fact that Dirac fermions are not truly Majorana is encoded in a U(1)-symmetry $He^{i\tilde{Q} \tilde{H}} = e^{i\tilde{Q} \tilde{H}} H$ generated by $Q = 1 \otimes 1 \otimes \sigma_y$.

Now comes the essential point: since $H$ obeys $\tilde{H} = -H$, the chiral “symmetry” $H = -\Gamma_5 \tilde{H} \Gamma_5$ with $\Gamma_5 = 1 \otimes \sigma_z \otimes 1$ can be recast as a true symmetry:

$$H = +\Gamma_5 \tilde{H} \Gamma_5 = THT^{-1} ,$$

with anti-linear $T : \Psi \mapsto \Gamma_5 \bar{\Psi}$. Thus the massless QCD Dirac operator is indeed associated with a symmetry class in the present, post-Dyson sense: that’s class AIII, realized by self-adjoint operators on Nambu space with Dirac U(1)-symmetry and an anti-unitary symmetry $T$.

5.2. Classes BDI and CII. — Consider Hamiltonians $D$ still of the form (4) but now with matrix entries taken from either the real numbers or the real quaternions. Their one-parameter groups $e^{itD}$ belong to two further families of symmetric spaces:

- Class BDI: $SO(p+q)/SO(p) \times SO(q)$,
- Class CII: $USp(2p+2q)/USp(2p) \times USp(2q)$.

These large families are known to be realized as symmetry classes by the massless Dirac operator with gauge group SU(2) (for BDI), or with fermions in the adjoint representation (for CII). For the details we must refer to Verbaarschot’s paper, as there is no space left here.

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December 31, 2003

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