On the existence of solutions for the Maxwell equations

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Abstract

Mathematical proofs are presented concerning the existence of solutions of the Maxwell equations with suitable boundary conditions. In particular it is stated that the well known “delayed potentials” provide effective solutions of the equations, under reasonable conditions on the sources of the fields.

1 Introduction

Any advanced test on theoretical classical electromagnetism (see for example [1], [2], [3], [4]) states that all the physical properties of the electromagnetic fields can be mathematically deduced from the Maxwell Equations, and performs such deduction obtaining formulae (the “delayed potentials”) that allow, in principle, to evaluate the fields, starting from the knowledge of the sources, i.e. charges and currents.

The problem is that the delayed potentials are obtained, as pointed out in the following section, with a method which ensures that, if solutions exist for the Maxwell Equations with suitable conditions, they must be of the provided form. Therefore the usual deduction is able to provide uniqueness theorems, but not existence theorems for the solutions of the Maxwell Equations.

This is not so surprising: also in similar areas, (e.g. the Laplace Equation for harmonic functions) the deduction of uniqueness theorems is well easier than the deduction of existence theorems.

For the above reasons, the traditional deduction of the electromagnetism leaves open the problem that solutions of the base equations could not exist. The usual way to overcome this situation is to state that “due to the physical nature of the problem, solutions must exist”, but clearly this cannot be accepted from a mathematical point of view.

The present paper presents existence theorems, i.e. states that, under suitable conditions on the sources, the Maxwell Equations admit solutions, and moreover that the delayed potentials provide an effective solution. Similar results could be obtained using the general theory of partial differential equations or distribution theory, but the presented deduction is performed without any use of such general theories, and requires no mathematical background, besides the one required for reading any theoretical test on electromagnetism.

The first section summarises the standard deduction of the delayed potentials, and points outs its defects.

The second section reports the obtained results, proving that the delayed potentials provide an effective solution.

The third section summarises the results and discusses some extensions.
2 Standard deduction of electromagnetism

The Maxwell Equations for the wide space have the form:

\[
\begin{align*}
\text{rot} E + \frac{1}{c} \frac{\partial H}{\partial t} &= 0 \\
\text{rot} H - \frac{1}{c} \frac{\partial E}{\partial t} &= 4\pi j \\
\text{div} H &= 0 \\
\text{div} E &= 4\pi \rho
\end{align*}
\] (1) (2) (3) (4)

where:

- \(E\) is the electric field
- \(H\) is the magnetic field
- \(c\) is the light speed in vacuum
- \(j\) is the current density
- \(\rho\) is the charge density

The last two quantities, named “sources of the fields”), are not independent, since they must satisfy the continuity equation

\[
\text{div} j + \frac{\partial \rho}{\partial t} = 0
\] (5)

whose physical meaning is the conservation of the total charge. The main problem of the electrodynamics is the following: given \(j\) and \(\rho\), as functions of space and time satisfying (5), find the electric and magnetic fields. The standard approach to this problem is summarised in the following subsections.

2.1 Definition of the potentials

It is trivial to prove that, given any vector function \(A\), the vector function \(H = \text{rot} A\) satisfies (3). Starting from this point, the standard developments of electrodynamics perform the claim that, in order to satisfy (3), the magnetic field \(H\) must be given by

\[
H = \text{rot} A
\] (6)

where \(A\) is a suitable vector function. A proof of this fact is not difficult, but it is not reported, since this result is not required in the following. Assuming that \(H\) is given by (6) and substituting into (1), the following is obtained:

\[
\text{rot}(E + \frac{1}{c} \frac{\partial A}{\partial t}) = 0
\] (7)

whose consequence is that the quantity in parenthesis is the gradient of some scalar function \(\phi\). Therefore the following formula is obtained:

\[
E = -\text{grad}(\phi) - \frac{1}{c} \frac{\partial A}{\partial t}
\]

Having introduced the scalar potential \(\phi\) and the vector potential \(A\), the research of the fields is restricted to those given by (6) and (7). Consequently (1) and (3) are automatically satisfied, while by substituting into (2) and (4) and performing some calculations the following two equations are obtained:

\[
\begin{align*}
\Delta A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \text{grad}(\text{div} A + \frac{1}{c} \frac{\partial A}{\partial t}) &= -\frac{4\pi}{c} j \\
\Delta \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{1}{c} \frac{\partial}{\partial t}(\text{div} A + \frac{1}{c} \frac{\partial A}{\partial t}) &= -4\pi \rho
\end{align*}
\]
Normally the last equations are simplified by assuming the Gauge condition:
\[ \text{div}\mathbf{A} + \frac{1}{c} \frac{\partial\varphi}{\partial t} = 0 \] (8)

and this assumption is heuristically justified by observing that \( \varphi \) and \( \mathbf{A} \) are not uniquely defined by (6) and (7). Summarising, the restriction is done to search fields given by (6) and (7), where \( \mathbf{A} \) and \( \varphi \) satisfy (10). With the above assumptions, the equations satisfied by the potentials reduce to the form
\[ \Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j} \] (9)
\[ \Delta \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -4\pi \rho \] (10)

If solutions of the above equations, subjected to (8), can be found, the conclusion can be stated that the Maxwell equations have at least one solution, given by (6) and (7).

It is readily seen that (9) and (10) are a system of 4 independent scalar equations; moreover they are of the similar form
\[ \Delta F - \frac{1}{c^2} \frac{\partial^2 F}{\partial t^2} = -G \] (11)

2.2 Elimination of the time dependence

Equation (11) is simplified by taking its temporal Fourier transform (accepting that the involved functions can be transformed), and by assuming the possibility to exchange integrations with differentiations. By standard calculations the following equation is obtained:
\[ \Delta f + k^2 f = -g \] (12)

where

- the variable replacing \( t \) in the transformed domain is named \( \omega \) (it has a frequency meaning)
- \( k = \omega/c \), where \( c \) is the light speed in vacuum
- \( f(\omega) \) and \( g(\omega) \) are the Fourier transforms of \( F \) and \( G \), respectively.

In (12) \( k \) has simply the function of a constant parameter.

2.3 Green Theorem

Equation (12) is treated with the well known Green method, as is done for the Laplace Equation for the harmonic functions. As in this case, it is clear that the Green method provides an uniqueness theorem for the solution of (14) with assigned values on a boundary, but does not provide any existence theorem for such solution. The Green functions is chosen as \( e^{-ikd}/d \), where \( d = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \), and \( x_0, y_0, z_0 \) are the coordinates of an arbitrary point \( P_0 \) (it is easily verified that this function is everywhere regular and satisfies (12) except at \( P_0 \), therefore it is an appropriate Green function).

In such a way the Green Theorem is obtained in the form
\[ f(P_0) = -\frac{1}{4\pi} \int_D g e^{-ikd} dV + \frac{1}{4\pi} \int_S \left( f \frac{\partial e^{-ikd}}{\partial n} - e^{-ikd} \frac{\partial f}{\partial n} \right) dS \]

and provides the solution of (12) at an arbitrary point interior to a spatial domain \( D \) with boundary \( S \), using the known term \( g \) and the value of the solution and of its normal derivative taken on the boundary, in total analogy with the Green formula for the Laplace equation.
A similar formula for the solution of (11) is obtained by applying the inverse Fourier transform, and assuming interchangeability between differentiation and integration:

\[ F(P_0) = -\frac{1}{4\pi} \int_D \frac{G(t-dc)}{d} dV + \frac{1}{4\pi} \int_S \left( \frac{1}{a} \frac{\partial F}{\partial n} - \frac{1}{a} \left( \frac{1}{c} \frac{\partial F}{\partial t} + \frac{E}{a} \frac{\partial \rho}{\partial n} \right) \right) |_{t-dc} dS \]

Finally, a behavior at infinity for \( F \) is assumed that ensures the vanishing of the second integral when the domain \( G \) tends to infinity, and the following is obtained:

\[ F(P_0) = -\frac{1}{4\pi} \int \frac{G(t-dc)}{d} dV \]

(13)

where presently the integral is extended to the whole space. (13) provides an explicit solution of (11), if the assumption is made that a solution exists satisfying all the outlined conditions. Coming back to the equations for the potentials, (13) provides:

\[ A(P_0) = -\frac{1}{c} \int \frac{J(x,y,z,t-dc)}{d} dV \]

(14)

\[ \varphi(P_0) = -\int \frac{\rho(x,y,z,t-dc)}{d} dV \]

(15)

The above formulae are known as “delayed potentials”, and are believed to provide, with (6) and (7), a general solution to the electrodynamics problem.

2.4 Criticism

As pointed out by the above review of the standard deduction, the fields obtained applying (6) and (7) to (14) and (15) provide a solution to the Maxwell equation assuming that the following conditions are verified:

- Existence of a solution which can be derived by the potentials according to (6) and (7) (this would not be difficult to prove, if the assumption is made of existence of a solution of (1), (2), (3), (4))
- Existence of a solution with suitable conditions at infinity
- As a minor point, validity of various hypothesis concerning the possibility of exchanging integrals (singular and with infinite domain) and derivatives
- Existence of the integrals appearing in (14) and (15)
- Validity, for the potentials given by (14) and (15), of the condition (8), with is necessary to conclude that (9) and (10) provide solutions for the potentials.

In conclusion, the summarised deduction provides only an heuristic feeling to have found possible solutions of the Maxwell equations.

3 Solubility Theorems

In order to overcome the summarised limits, the chosen strategy does not attempt to verify the validity of the various conditions pointed out at the end of the preceding section, but attacks directly the problem to prove that, under suitable conditions on the field sources \( \rho \) and \( \mathbf{j} \), (14) and (15) provide functions that satisfy (8), (9) and (10), from which it is straightforward to prove that the fields obtained by (6) and (7) satisfy the Maxwell equations (1), (2), (3), (4).

The assumed conditions on \( \rho \) and \( \mathbf{j} \) are the following:
• Regularity, i.e. existence and continuity up to the first derivatives

• Concerning the space variables, vanishing outside of a finite regular domain \( D \), where “regular” means that its boundary admits everywhere a tangent plane

• Concerning the time dependence, sinusoidal behaviour, or existence of the Fourier transform, vanishing outside of a finite interval for the transformed variable \( \omega \) (in practice, finitely extended frequency spectrum)

These conditions, in particular the second and the third, are surely redundant, even if acceptable from a physical point of view. The deduction will show some possibility of extensions. The proofs are reported for the following cases, of increasing complexity:

• Electrostatic case
• Magnetostatic case
• Monochromatic case
• General case.

3.1 Electrostatic case

This is the case in which all the charges are fixed, i.e. the density \( \rho \) does not depend on the time, and the current density \( j \) vanishes (note that condition (5) is trivially satisfied). In this case the proposed solutions (16) and (17) assume the form

\[
\begin{align*}
A(P_0) &= 0 \\
\varphi(P_0) &= -\int_D \frac{\rho(x,y,z)}{d}dV 
\end{align*}
\]  

where \( D \) is the finite domain in which \( \rho \) does not vanish, while the equations to be verified ((8), (9) and (10)) reduce to

\[
\Delta \varphi = -4\pi \rho
\]  

If a point \( P_0 \) is external to \( D \), the integral appearing in (15) is not singular and can be differentiated under the sign, providing trivially \( \Delta \varphi = -4\pi \rho \), as required. If the point \( P_0 \) is internal to \( D \), a transformation to polar coordinates \( u, \alpha, \beta \) centered on \( P_0 \) is performed, giving

\[
\varphi(P_0) = -\int_0^{2\pi} d\beta \int_{-\pi/2}^{\pi/2} \sin^2 \alpha \int_0^{g(P_0,\alpha,\beta)} \rho(P_0 + \mathbf{v})u du 
\]  

in which the vector \( \mathbf{v} = (u \sin \alpha \cos \beta, u \sin \alpha \sin \beta, u \cos \beta) \) has been introduced, and \( g(P_0,\alpha,\beta) \) is the distance between \( P_0 \) and the point on the boundary of \( D \) with polar angles \( \alpha, \beta \), which is a regular function.

In (17) every singularity has disappeared, and this ensures the existence and regularity of \( \varphi(P_0) \), and the existence of the vector field given by

\[ \mathbf{E} = -\nabla \varphi \]

The problem is if \( \mathbf{E} \) can be calculated by taking the derivatives respect to \( P_0 \) under the sign of integral appearing in (19). The response is certainly ”yes” if the integrals obtained by formal derivation under the sign are uniformly convergent. Performing the formal derivation, one obtains

\[
\frac{\partial \varphi}{\partial x_0} = \int_D \rho \frac{x_0 - x}{d^3}dV
\]
where the interrogation mark means that the result must be justified. Performing the same transformation to polar coordinates just used, the last integral transforms to

\[ \varphi(P_0) = ^2 - \int_0^{2\pi} d\beta \int_{-\pi/2}^{\pi/2} \sin^2 \alpha \int_0^{\rho(P_0,\alpha,\beta)} \rho(P_0 + \mathbf{v}) d\alpha \]

which is without singularities. Therefore it represents a continuous function of \( P_0 \), and the uniform convergence is ensured, justifying the formal derivation. It is important to note that a similar reasoning could not be applied to the second derivatives, vanishing any attempt to verify (20) with a direct calculation. Having justified first order derivatives under the sign of \( \varphi(P_0) \), the following result is obtained

\[ \mathbf{E}(P_0) = \int_D \rho(P) \frac{r_0 - r}{|r_0 - r|} dV \]

with uniform convergence of the integral at its unique singular point \( P_0 \). Now let \( V \) be an arbitrary regular domain internal to \( D \), with boundary \( S \). The aim is to calculate the flux of \( \mathbf{E} \) out of \( V \). With some straightforward calculations, performing an integration exchange justified by the uniform convergence already established, one finds:

\[
\int_S \mathbf{E} \cdot \mathbf{n} dS_0 = \int_V \rho(P) dV \int_S \frac{(r_0 - r) \cdot \mathbf{n}}{|r_0 - r|^2} dS_0 + \int_{D - V} \rho(P) dV \int_S \frac{(r_0 - r) \cdot \mathbf{n}}{|r_0 - r|^3} dS_0
\]

(19)

where \( \mathbf{n} \) is the vector of modulus 1 normal to \( S \). In (19) the second integral does not contain singularities, since \( P_0 \) is on \( S \), while \( P \) is external to \( S \). Since a trivial calculation gives

\[
\text{div} \left( \frac{r_0 - r}{|r_0 - r|^3} \right) = 0
\]

when \( P_0 \neq P \), a standard application of Gauss Theorem proves that the second integral vanishes. By the same reasoning, in the first integral the boundary \( S \) can be modified to a sphere centered on \( P \), and totally included in \( D \), without modifying its value. In this conditions, a direct calculation shows that

\[
\int_S \mathbf{E} \cdot \mathbf{n} dS_0 = 4\pi \int_V \rho(P) dV
\]

and using again Gauss Theorem

\[
\int_V \text{div} \mathbf{E} dV = 4\pi \int_V \rho(P) dV
\]

Finally, keeping into account that the domain \( V \) is arbitrary and that the functions under integral are continuous, one obtains

\[
\text{div} \mathbf{E} = 4\pi \rho
\]

which, inserting the definition of \( \mathbf{E} \), is equivalent to the equation to be verified (16). The focal point in the reported prove is that a vector function like (18) has flux zero from any closed surface not including singularities of the integral; from this it follows that the flux from an arbitrary surface can be calculated by modifying the surface up to a sphere concentrated on the unique singular point, for which the calculation is trivial. Similar methods are applied to the
calculation of the residues of an analytical function, providing Cauchy formula, whose aspect is similar to (18). This localisation property allows also to understand how the restriction on the finiteness of the domain $D$ could be lowered: it is sufficient to separate $D$ into a finite part surrounding the point $P$ and a remaining infinite part. For the finite part the reported reasoning applies, while the remaining one gives no contribution. Obviously some conditions must be added, which authorise the performed interchange of integrals; a sufficient condition is the existence of the integrals appearing in (19) as multiple integrals, by Fubini Theorem.

3.2 Magnetostatic case

This is the case in which both the density $\rho$ and the current density $j$ do not depend on time. In this case it will be necessary to use condition (5), which becomes

$$\text{div } j = 0$$

(20)

In this case the proposed solutions (14) and (15) assume the form

$$A(P_0) = -\frac{1}{c} \int_D \frac{j(x,y,z)}{d} dV$$

(21)

$$\varphi(P_0) = - \int_D \frac{\rho(x,y,z)}{d} dV$$

(22)

where $D$ is the finite domain in which $\rho$ and $j$ do not vanish, while the equations to be verified ((8), (9) and (10)) reduce to

$$\text{div } A = 0$$

(23)

$$\Delta A = -\frac{4\pi}{c} j$$

(24)

$$\Delta \varphi = -4\pi \rho$$

(25)

(25) has been already proved in the preceding section. The same reasoning, applied to the individual components of $A$ and $j$, proves (24). Therefore all what is required is to prove (23). To this aim, the first point is to consider two vectors $r_0$ and $r$ and to evaluate the flux of the vector $\frac{j}{|r_0-r|}$ from the boundary $S$ of $D$. Considering firstly the case in which $P_0$ is external to $D$, which avoids any singularity, and using (20), (21), (22) and Gauss Theorem, one obtains

$$\int_S \frac{\mathbf{j} \cdot \mathbf{n}}{|r_0-r|} dS = \int_D \text{div} \frac{\mathbf{j}}{|r_0-r|} dV = \int_D \frac{1}{|r_0-r|} \text{div} \mathbf{j} dV + \int_D \mathbf{j} \cdot \text{grad} \frac{1}{|r_0-r|} dV = \int_D \frac{j(|r_0-r|)}{|r_0-r|^3} dV = \text{div}_0 A(r_0)$$

(in the last passage a derivative under the sign has been taken, justified as in the preceding section).

Therefore in this case the proof of (23) is obtained, if the flux appearing at the left hand side is zero. A sufficient condition for that is that at each point of $S$ $j$ is tangent to $S$, or in particular null. To prove this, suppose that exists a point $P$ on $S$ at which $j$ is directed e.g. towards the extern. Then it should be possible to find an neighborhood of $P$ satisfying the same condition (remember that $j$ is assumed continuous), with the consequence that the flux of $j$ from such neighborhood, completed to a closed surface, should be strictly positive, in contradiction with (20). Considering now the case in which $P_0$ is internal to $D$, the integral appearing in (21) becomes singular, and the above passages are not valid. But they can be applied excluding
from the integral the contribution of a sphere \( D_\varepsilon \) of ray \( \varepsilon \) centered on \( r_0 \) and with boundary \( S_\varepsilon \), finding

\[
0 = \int_{S_\varepsilon} \frac{j \cdot n}{|r_0 - r|} \, dS = \int_{S} + \int_{\varepsilon} - \int_{S_\varepsilon} \frac{j \cdot n}{|r_0 - r|} \, dS = \int_{D-D_\varepsilon} \text{div} \frac{j}{|r_0 - r|} \, dV + \int_{S_\varepsilon} \frac{j \cdot n}{|r_0 - r|} \, dS
\]

The first integral is not singular, and can be treated as in the first case; for the second one the mean value theorem is applied, obtaining

\[
\int_{D-D_\varepsilon} \frac{j(r_0 - r)}{|r_0 - r|} \, dV + K^* \varepsilon = 0
\]

where \( K^* \) is a suitable value comprised between the minimum and the maximum of \( j \cdot n \) on \( S \), and therefore bounded. Taking the limit for \( \varepsilon \to 0 \), the desired result \( \text{div}_0 A(r_0) = 0 \) is obtained.

Finally, in the case of \( P \) on the boundary of \( D \), it is sufficient to use the already obtained results and the continuity of \( \text{div}A \), whose prove is obtained by the method of the preceding section.

### 3.3 Monocromatic case

This is the case in which the density \( \rho \) and the current density \( j \) have a sinusoidal time dependence. Using the complex exponential notation, which simplifies some arguments, they are given by

\[
\begin{align*}
\rho(x, y, z, t) &= \rho_a(x, y, z) \exp(-i\omega t) \\
j(x, y, z, t) &= j_a(x, y, z) \exp(-i\omega t)
\end{align*}
\]

where \( \rho_a(x, y, z) \) and \( j_a(x, y, z) \) depend only on the spatial variables, and \( \omega \) is a fixed parameter. In this case the proposed solutions (14) and (15) assume the form

\[
\begin{align*}
A(P_0) &= -\frac{1}{c} \int_D \frac{j_a(x, y, z) \exp(i\omega d)}{d} \, dV \cdot \exp(-i\omega t) \\
\varphi(P_0) &= -\int_D \rho_a(x, y, z) \exp(i\omega d) \, dV \cdot \exp(-i\omega t)
\end{align*}
\]

where \( D \) is a finite domain outside of which \( \rho \) and \( j \) vanish. Posing

\[
A = A_a \exp(-i\omega t)
\]

\[
\varphi = \varphi_a \exp(-i\omega t)
\]

equations (28) and (29) give

\[
\begin{align*}
A_a(P_0) &= -\frac{1}{c} \int_D \frac{j_a(x, y, z) \exp(i\omega d)}{d} \, dV \\
\varphi_a(P_0) &= -\int_D \rho_a(x, y, z) \exp(i\omega d) \, dV
\end{align*}
\]

while the equations to be verified ((8), (9) and (10)) reduce to
\[
\begin{align*}
\text{div}\mathbf{A}_a - ik\varphi_a &= 0 \quad (32) \\
\Delta\varphi_a + k^2\varphi_a &= -4\pi\varphi_a \quad (33) \\
\Delta\mathbf{A}_a + k^2\mathbf{A}_a &= -\frac{4\pi}{c}\mathbf{j}_a \quad (34)
\end{align*}
\]

where \( k = \omega/c \).

Finally the continuity condition (5) becomes

\[
\text{div}\mathbf{j}_a - i\omega\rho_a = 0 \quad (35)
\]

The first step is to verify (32). The detailed steps are not reported, since they are identical to those used in the preceding section, with the only difference that the starting point is the calculation of the flux of \( \frac{\mathbf{j}_a}{|r_0 - r|} \) which vanishes due to the tangent condition on \( \mathbf{j}_a \).

Coming to the proof of (33), one defines

\[
\mathbf{E}_a = -\text{grad}\varphi_a + ik\mathbf{A}_a \quad (36)
\]

and, using (32), obtains

\[
\text{div}_0\mathbf{E}_a = -\text{div}_0 \int_D \rho_a(ik|r_0 - r| - 1) \exp(ik|r_0 - r|) \frac{r_0 - r}{|r_0 - r|^3} dV + k^2\varphi_a
\]

Using the fact that an integral with first order singularity in \( |r_0 - r| \) can be differentiated under the sign and developing some calculations, the following is obtained:

\[
\text{div}_0\mathbf{E}_a = -ik \int_D \rho_a(ik|r_0 - r| - 1) \exp(ik|r_0 - r|) \frac{1}{|r_0 - r|^2} dV + \text{div}_0\mathbf{C} \quad (37)
\]

having defined

\[
\mathbf{C} = \int_D \rho_a \exp(ik|r_0 - r|) \frac{r_0 - r}{|r_0 - r|^3} dV
\]

At this point the flux of \( \mathbf{C} \) through a surface \( S \) contained in \( V \) is evaluated by steps similar to those reported in the section on electrostatics. The result is

\[
\int_V \text{div}\mathbf{C} dV_0 = \int_S dS_0 \int_D \rho_a(P) \frac{r_0 - r}{|r_0 - r|^3} \exp(ik|r_0 - r|) dV + \int_D \rho_a(P) dV + \int_S \frac{r_0 - r}{|r_0 - r|^3} \exp(ik|r_0 - r|) dS_0
\]

In the above equation the second integral is regular, and its value is

\[
\int_D \rho_a(P) dV \int_S \frac{(r_0 - r)_n}{|r_0 - r|^3} \exp(ik|r_0 - r|) dS_0 = \int_D \rho_a(P) dV \int_D \frac{ik}{|r_0 - r|^3} \exp(ik|r_0 - r|) dV
\]

On the contrary, in the first integral the surface is moved up to the already used sphere of radius \( \varepsilon \), and the following is obtained

\[
\int_V \text{div}\mathbf{C} dV_0 = \int_D \rho_a dV - \int_{V - V_\varepsilon} \rho_a \frac{ik \exp(ik|r_0 - r|)}{|r_0 - r|^2} dV + 4\pi \int_D \rho_a dV
\]
Taking the limit $\varepsilon \to 0$ and using the fact that $V$ is arbitrary, the final result is obtained

$$\text{div} \mathbf{C} = ik \int_V \rho_a \frac{\exp(ik|r_0 - r|)}{|r_0 - r|^2} dV + 4\pi \varphi_a$$

which inserted into (37) provides the desired result (33).

The final step is the verification of (34). At a first sight, it could appear sufficient to apply the last obtained result to the three components of (34). However, there is a problem, due to the fact that to obtain (33) from (31), essential use has been made of (32), which depends on the fact that a-priori $\rho_a$ is not arbitrary, but tied to a certain vector $\mathbf{j}_a$ by (35). Therefore the proof is complete if the fact can be proved that to each component $j_{a'}$ of a vector $\mathbf{j}_a$ another vector $\mathbf{j}_a^*$ can be associated, satisfying the equation

$$\text{div} \mathbf{j}_a^* - i\omega j_{a'} = 0$$

(38)

and moreover tangent, as $\mathbf{j}_a$, on the boundary of the domain $D$.

This is consequence of the following property:

The equation in $\mathbf{j}$

$$\text{div} \mathbf{j} = f(x, y, z)$$

(39)

has solutions, even if $\mathbf{j}$ is constrained to be tangent to an assigned surface $S$. In fact it is easy to verify that (39) (but in general not the constraint) is satisfied by the vector

$$\mathbf{j}^{**} = (0, 0, \int f(x, y, z)dz)$$

and therefore also by

$$\mathbf{j} = \mathbf{j}^{**} + \text{rot} \mathbf{P}$$

where $\mathbf{P}$ is an arbitrary vector. Therefore all is reduced to choice $\mathbf{P}$ in such a way that on $S$:

$$\text{rot} \mathbf{P} \cdot \mathbf{n} = -\mathbf{j}^{**} \cdot \mathbf{n}$$

i.e. to prove that the equation in $\mathbf{P}$

$$\text{rot} \mathbf{P} \cdot \mathbf{n} = g(x, y, z)$$

(40)

has solutions on $S$.

In fact this is possible even by imposing $P_x = P_y = 0$, and reducing (40) to the form

$$\frac{\partial P_z}{\partial y} n_x - \frac{\partial P_z}{\partial x} n_y = g(x, y, z)$$
which is a first order partial differential linear equation and can be solved through a method which transform it into an ordinary differential equation (see any text of Mathematical Analysis, e.g. [5]).

3.4 General case

This is the case in which the density \( \rho \) and the current density \( j \) have arbitrary time dependence, but with the restriction to admit a Fourier transform, producing functions vanishing outside of a finite domain. Under these conditions, \( \rho \) and \( j \) are of the following form

\[
\rho(r, t) = \frac{1}{\sqrt{2\pi}} \int_B \rho_a(r, \omega) \exp(-i\omega t) d\omega
\]

\[
j(r, t) = \frac{1}{\sqrt{2\pi}} \int_B j_a(r, \omega) \exp(-i\omega t) d\omega
\]

where \( B \) is a finite domain. Inserting (41) and (42) into (14) and (15) and interchanging bounded integrations, the proposed solutions take the form

\[
A(P_0, t) = -\frac{1}{\sqrt{2\pi c}} \int_B \exp(-i\omega t) d\omega \int_D j_a(r, \omega) \exp(i\omega d/c) dV = \frac{1}{\sqrt{2\pi}} \int_B j_a(r, \omega) \exp(-i\omega t) d\omega
\]

\[
\varphi(P_0, t) = -\frac{1}{\sqrt{2\pi c}} \int_B \exp(-i\omega t) d\omega \int_D \rho_a(r, \omega) \exp(i\omega d/c) dV = \frac{1}{\sqrt{2\pi}} \int_B \varphi_a(r, \omega) \exp(-i\omega t) d\omega
\]

where \( A_a \) and \( \varphi_a \) are given by (28) and (29).

Similarly, taking derivatives under the sign of the regular range bounded integrals (41) and (42) and using inversion properties for the Fourier transform, the continuity equation (5) provides (32) for \( A_a \) and \( \varphi_a \). Using the results of the preceding section, it can be concluded that \( A_a \) and \( \varphi_a \) satisfy (32), (33), (34). Using this fact, one obtains, performing derivatives under bounded regular integrals:

\[
\text{div}A + \frac{1}{c} \frac{\partial \varphi}{\partial t} = \frac{1}{\sqrt{2\pi}} \int_B (\text{div}A_a - \frac{i\omega}{c} \varphi_a) \exp(-i\omega t) d\omega = 0
\]

\[
\Delta A_a - \frac{\omega^2}{c^2} A_a = \frac{1}{\sqrt{2\pi}} \int_D (\Delta A_a + \frac{\omega^2}{c^2} A_a) \exp(-i\omega t) d\omega = -4\pi \frac{1}{\sqrt{2\pi}} \int_D j_a \exp(-i\omega t) d\omega = -4\pi \frac{1}{\sqrt{2\pi}} j
\]

\[
\Delta \varphi_a - \frac{\omega^2}{c^2} \varphi_a = \frac{1}{\sqrt{2\pi}} \int_D (\Delta \varphi_a + \frac{\omega^2}{c^2} \varphi_a) \exp(-i\omega t) d\omega = -4\pi \frac{1}{\sqrt{2\pi}} \int_D \varphi_a \exp(-i\omega t) d\omega = -4\pi \rho
\]

i.e. (8), (9) and (10) have been proved.

The conditions of finitely extended Fourier transform on \( \rho \) and \( j \) have been imposed since they are the simplest way to justify the last passages of interchanging integrations and performing derivatives under integration. However, the same justifications are ensured more generally by conditions of existence and uniform absolute convergence of multiple integrals of the form

\[
\int_B \int_D \frac{\rho_a(r, \omega)}{d} \omega^2 d\omega dV
\]

\[
\int_B \int_D \frac{j_a(r, \omega)}{d} \omega^2 d\omega dV
\]

having considered that the factor \( \exp(-i\omega t) \), of modulus 1, does not disturb such absolute convergence.
4 Conclusions

A proof has been given that the fields calculated applying (6) and (7) to the delayed potentials (14) and (15) provide effectively a solution to the Maxwell equations (1), (2), (3), (4). The complete details of the proof have been reported using the assumption that the sources $\rho$ and $j$ are regular functions up to the first order derivatives, that they are vanishing outside some finite spatial domain, and that their temporal Fourier transform is vanishing outside some finite frequency interval. It has also been shown that the last two conditions can be lowered, assuming only the multiple summability of certain space/frequency functions constructed from the sources.

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