THE HIGGS SECTOR IN THE MINIMAL 3-3-1 MODEL WITH THE
MOST GENERAL LEPTON-NUMBER CONSERVING POTENTIAL

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Abstract

The Higgs sector of the minimal 3 - 3 - 1 model with three triplets and
one sextet is investigated in detail under the most general lepton–number
conserving potential. The mass spectra and multiplet decomposition structure
are explicitly given in a systematic order and a transparent way allowing they
to be easily checked and used in further investigations. A previously arising
problem of inconsistent signs of $f_2$ is also automatically solved.

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1 Introduction

The standard model (SM) combining the Glashow–Weinberg–Salam (GWS) model with the QCD under the gauge group $SU_C(3) \otimes SU_L(2) \otimes U_N(1)$ is one of the greatest achievements of physics in the 20th century. Many predictions of the SM have been confirmed by various experiments. However, this model works well only at the energy range below 200 GeV and gradually loses its prediction power at higher energies. Therefore, any extension of the SM to fit the theory with the higher energy phenomenology is needed. In addition, the observation of the Higgs bosons which play a central role in symmetry breaking is still an open problem. Hence the mechanism for electroweak symmetry breaking is, in some way, still a mystery.

The scalar sector has been thoroughly studied not only in the framework of the SM but also in its various extensions including the so-called 3-3-1 models based on the $SU_C(3) \otimes SU_L(3) \otimes U_N(1)$ gauge group [1–6]. The later models have the following intriguing features: firstly, the models are anomaly free only if the number of families $N$ is a multiple of three. Further, the condition of QCD asymptotic freedom valid only for the number of quark families less than five, leads to $N$ equal to 3. Secondly, the Peccei–Quinn (PQ) symmetry – a solution of the strong CP problem naturally occurs in the 3-3-1 models [8]. It is worth mentioning that the implementation of the PQ symmetry is usually possible only at a classical level (broken by quantum corrections through instanton effects) and there has been a number of attempts to find models solving the strong CP question. In these 3-3-1...
1 models the PQ symmetry following from the gauge invariant Lagrangian does not have to be imposed. The third interesting feature is that one of the quark families is treated differently from the other two. This could lead to a natural explanation for the unbalancing heavy top quarks, deviations of $A_b$ from the SM prediction, etc. Additionally, the models predict not very high new mass scales, at the order of a few TeV only.

Recently, the scalar sector of the minimal 3 - 3 - 1 model was in detail studied in [10] and [11]. There, three Higgs triplets were firstly analysed and then the sextet was added in a further consideration. It was also shown in [11] that the potential used in [10] led to inconsistent results and therefore it should be further modified or replaced by some more relevant potential. Another precise investigation on the model with a new potential is, in our opinion, really interesting and necessary. Following the previous paper [11] the present paper is devoted to such an investigation. Here, instead of the potential in [10, 11] the most general gauge–invariant postential conserving lepton numbers [12] is used. With the latter potential the scalar sector of the 3 - 3 - 1 model is investigated again at tree–level. The multiplet decomposition structure remains the same as in [11] but the masses of most of the scalars get corrections and the problem with inconsistent signs of $f_2$ is automatically solved. We emphasize that the above mentioned potential was also considered in [12, 13], but only mass matrices [12] and some their eigenvalues [13] were presented.

The paper is organized as follows. The Higgs potentials, constraint equations
and main notations are presented in section 2, while the characteristic equations are solved in section 3 where the multiplet decompositions and mass spectra of the Higgs sector are given. Our results are summarized in the last section, section 4.

2 The Higgs contents and scalar potentials

Presently, the minimal 3 - 3 - 1 models are considered with three Higgs triplets

$$\eta = \begin{pmatrix} \eta^o \\ \eta^- \\ \eta^+_i \end{pmatrix} \sim (1, 3, 0), \quad \rho = \begin{pmatrix} \rho^+ \\ \rho^o \\ \rho^{++} \end{pmatrix} \sim (1, 3, 1), \quad \chi = \begin{pmatrix} \chi^- \\ \chi^- \\ \chi^o \end{pmatrix} \sim (1, 3, -1),$$

(2.1)

and one Higgs sextet

$$S = \begin{pmatrix} \sigma^o_1 & s^+_2/\sqrt{2} & s^-_1/\sqrt{2} \\ s^+_2/\sqrt{2} & s^{++} & \sigma^o_2/\sqrt{2} \\ s^-_1/\sqrt{2} & \sigma^o_2/\sqrt{2} & s^{--} \end{pmatrix} \sim (1, 6^*, 0).$$

(2.2)

The latter is needed in order to give masses to all leptons. In [10, 11], the scalar sector of the minimal 3 - 3 - 1 models is investigated by using the potential

$$V_s(\eta, \rho, \chi, S) = V_T(\eta, \rho, \chi) + \mu_4^2 Tr(S^\dagger S) + \lambda_{10} Tr^2(S^\dagger S) + \lambda_{11} Tr[(S^\dagger S)^2] + \lambda_{12} \eta^\dagger \eta + \lambda_{13} \rho^\dagger \rho + \lambda_{14} \chi^\dagger \chi] Tr(S^\dagger S) + 2f_2 (\rho^T S \chi + h.c.),$$

with

$$V_T(\eta, \rho, \chi) = \mu_1^2 \eta^\dagger \eta + \mu_2^2 \rho^\dagger \rho + \mu_3^2 \chi^\dagger \chi + \lambda_1 (\eta^\dagger \eta)^2 + \lambda_2 (\rho^\dagger \rho)^2 + \lambda_3 (\chi^\dagger \chi)^2$$

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\[ + \lambda_4(\eta^\dagger \eta)(\rho^\dagger \rho) + \lambda_5(\chi^\dagger \chi)(\eta^\dagger \eta) + \lambda_6(\rho^\dagger \rho)(\chi^\dagger \chi) + \lambda_7(\rho^\dagger \eta)(\eta^\dagger \rho) \]
\[ + \lambda_8(\chi^\dagger \eta)(\eta^\dagger \chi) + \lambda_9(\rho^\dagger \chi)(\chi^\dagger \rho) \]
\[ + \sqrt{2} f_1 \left( \varepsilon^{ijk} \eta_i \rho_j \chi_k + h.c. \right) , \quad (2.3) \]

where \( \mu \)'s are mass parameters and the coefficients \( f_1 \) and \( f_2 \) have dimensions of mass, while \( \lambda \)'s are dimensionless. Unfortunately, as shown in [11], this potential (which is not most general) leads to nonlogical results as \( f_2 \) cannot take a consistent sign. Analysing the results obtained, we conclude that we need a wider potential in order to give necessary corrections to those masses showing contradict signs of \( f_2 \). On the other hand, we suggest that the potential needed should guarantee the conservation of the lepton numbers and the continuous symmetry not higher than \( SU(3) \times U(1) \). Fortunately, there exist such potentials. The most general potential satisfying our requirements is [12]

\[ V_E(\eta, \rho, \chi, S) = V_s(\eta, \rho, \chi, S) + \lambda_{15} \eta^\dagger S^\dagger S \eta + 4 \lambda_{16} \rho^\dagger S^\dagger S \rho + 4 \lambda_{17} \chi^\dagger S^\dagger S \chi \]
\[ + 2 \sqrt{2} \lambda_{18} \rho^\dagger S^\dagger \rho \eta + 2 \sqrt{2} \lambda_{19} \chi^\dagger S^\dagger \chi \eta + \lambda_{20} S^\dagger S \eta \eta + h.c. \quad (2.4) \]

where

\[ \rho^\dagger S^\dagger \rho \eta = \varepsilon^{ijk} \rho_i S^{lti} \rho^j \eta^k , \]
\[ \chi^\dagger S^\dagger \chi \eta = \varepsilon^{ijk} \chi_i S^{lti} \chi^j \eta^k , \]
\[ S^\dagger S^\dagger \eta \eta = \varepsilon^{ijk} \varepsilon^{lmn} \eta^k \eta^l S^{nti} S^{jmi} . \]

The triplet Higgs fields \( \eta^o, \rho^o, \) and \( \chi^o \) develop VEVs \( v, u, \) and \( w, \) respectively, as follows

\[ \langle \eta \rangle = \begin{pmatrix} v/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} , \quad \langle \rho \rangle = \begin{pmatrix} 0 \\ u/\sqrt{2} \\ 0 \end{pmatrix} , \quad \langle \chi \rangle = \begin{pmatrix} 0 \\ 0 \\ w/\sqrt{2} \end{pmatrix} , \quad (2.5) \]
while for the sextet, only the $\sigma^o_2$ field develops a VEV (a nonzero VEV of $\sigma^o_1$ should provide a nonzero neutrino mass which, however, we do not consider here)

$$\langle S \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v_o/2 \\ 0 & v_o/2 & 0 \end{pmatrix}. \tag{2.6}$$

Here, all VEVs are taken to be real. (We are restrained ourselves the possibility of CP–violation arising from complex VEVs which has already been investigated in detail by D. G. Dumm [12]). The expansion of the scalar fields reads

$$\eta^o = \frac{v}{\sqrt{2}} + \xi_\eta + i\zeta_\eta, \quad \rho^o = \frac{u}{\sqrt{2}} + \xi_\rho + i\zeta_\rho, \quad \chi^o = \frac{w}{\sqrt{2}} + \xi_\chi + i\zeta_\chi,$$

$$\frac{\sigma^o_2}{\sqrt{2}} = \frac{v_o}{2} + \xi_\sigma + i\zeta_\sigma, \tag{2.7}$$

and

$$\sigma^o_1 = \xi'_{\sigma} + i\zeta'_{\sigma}. \tag{2.8}$$

Below we call a real part $\xi$ scalar and an imaginary one $\zeta$ pseudoscalar. In this case the symmetry breaking ladder is

$$SU_C(3) \otimes SU_L(3) \otimes U_N(1)$$

$$\downarrow \langle \chi \rangle$$

$$SU_C(3) \otimes SU_L(2) \otimes U_Y(1)$$

$$\downarrow \langle \rho \rangle, \langle \eta \rangle, \langle S \rangle$$

$$SU_C(3) \otimes U_Q(1)$$

where the VEVs satisfy the relation

$$v^2 + u^2 + v_o^2 \equiv v_w^2 \approx (246 \text{ GeV})^2, \tag{2.9}$$
with $v_w$ being the standard model VEV. At the first step of the symmetry breaking, $\langle \chi \rangle$ generates masses for exotic quarks and new heavy gauge bosons. At the subsequent breaking, nonzero values of $\langle \rho \rangle$, $\langle \eta \rangle$ and $\langle S \rangle$ generates masses of the familiar quarks and leptons. To keep the 3 - 3 - 1 models consistent with the low-energy phenomenology, the VEV $\langle \chi \rangle$ must be large enough in comparison with other VEV’s

$$w \gg v, u, v_\sigma.$$ 

Due to the requirement the potential to reach a minimum at the chosen VEV’s we obtain the following constraint equations in the tree–level approximation

\[
\begin{align*}
\mu_1^2 &= -\lambda_1 v^2 - \frac{\lambda_8}{2} u^2 - \frac{\lambda_5}{2} w^2 - \left( \frac{\lambda_{12}}{2} - \lambda_{20} \right) v_\sigma^2 + \frac{\lambda_{18} u^2 v_\sigma}{v} - \frac{\lambda_{19} w^2 v_\sigma}{v} - f_1 uu, \\
\mu_2^2 &= -\lambda_2 u^2 - \frac{\lambda_4}{2} v^2 - \frac{\lambda_6}{2} w^2 - \left( \frac{\lambda_{13}}{2} + \lambda_{16} \right) v_\sigma^2 + 2\lambda_{18} vv_\sigma - \frac{f_1 vu}{u} - \frac{f_2 v_\sigma w}{u}, \\
\mu_3^2 &= -\lambda_3 w^2 - \frac{\lambda_5}{2} v^2 - \frac{\lambda_6}{2} u^2 - \left( \frac{\lambda_{14}}{2} + \lambda_{17} \right) v_\sigma^2 - 2\lambda_{19} vv_\sigma - \frac{f_1 vu}{u} - \frac{f_2 v_\sigma u}{w}, \\
\mu_4^2 &= -(\lambda_{10} + \frac{\lambda_{11}}{2}) v_\sigma^2 - \left( \frac{\lambda_{12}}{2} - \lambda_{20} \right) v^2 - \left( \frac{\lambda_{13}}{2} + \lambda_{16} \right) u^2 - \left( \frac{\lambda_{14}}{2} + \lambda_{17} \right) w^2 \\
&+ 2\frac{\lambda_{18} u^2 v}{v_\sigma} - 2\frac{\lambda_{19} w^2 v}{v_\sigma} - \frac{f_2 wu}{v_\sigma},
\end{align*}
\]

which, in fact, exclude the linear terms in fields from the potential. The mass matrices, thus, can be calculated, using

$$M_{ij}^2 = \frac{\partial^2 V_e}{\partial \phi_i \partial \phi_j}$$

evaluated at the chosen minimum, where $\phi_i$’s are fields ($\xi, \zeta, \eta, \rho, \chi, s$).
3 Higgs mass spectra and physical particles

Since the $\sigma_1^\prime$ field does not develop a VEV, the associated scalar $\xi'_\sigma$ and pseudoscalar $\zeta'_\sigma$ do not mix with other fields and we have the physical field $H'_\sigma \simeq \xi'_\sigma$ with mass

$$m^2_{H'_\sigma} = \frac{f_2}{v_\sigma}uv + \frac{\lambda_{11}}{2}u^2v - \left(\frac{\lambda_{15}}{2} + \lambda_{20}\right)v^2 + \lambda_{16}u^2 + \lambda_{17}w^2 - \frac{\lambda_{18}}{v_\sigma}u^2v - \frac{\lambda_{19}}{v_\sigma}w^2v. \quad (3.1)$$

In the basis of $\xi_\eta, \xi_\rho, \xi_\sigma$ and $\xi_\chi$ the square mass matrix, after imposing the constraints (2.10), reads

$$M^2_\xi = \begin{pmatrix}
-m^2_{\xi_\eta} & -m^2_{\xi_\eta \xi_\rho} & -m^2_{\xi_\eta \xi_\sigma} & -m^2_{\xi_\eta \xi_\chi} \\
-m^2_{\xi_\rho} & -m^2_{\xi_\rho \xi_\sigma} & -m^2_{\xi_\rho \xi_\chi} \\
-m^2_{\xi_\sigma} & -m^2_{\xi_\sigma \xi_\chi} \\
-m^2_{\xi_\chi} & & & \\
\end{pmatrix} \quad (3.2)$$

where

$$m^2_{\xi_\eta} = \frac{f_1}{v}uv - 2\lambda_1v^2 - \frac{\lambda_{18}}{v}u^2v + \frac{\lambda_{19}}{v}w^2v, \quad m^2_{\xi_\eta \xi_\rho} = -f_1w - \lambda_4uv + 2\lambda_{18}uv, \quad m^2_{\xi_\eta \xi_\sigma} = -f_1w - \lambda_4uv + 2\lambda_{18}uv, \quad m^2_{\xi_\eta \xi_\chi} = -f_1w - \lambda_4uv + 2\lambda_{18}uv,$$

$$m^2_{\xi_\rho} = \frac{w}{u}(f_1 + f_2)v - 2\lambda_2v^2, \quad m^2_{\xi_\rho \xi_\sigma} = -(\lambda_{12} + 2\lambda_{20})wv + \lambda_{18}u^2 - \lambda_{19}w^2, \quad m^2_{\xi_\rho \xi_\chi} = -(f_1 + f_2)w - \lambda_2uv - 2\lambda_{19}w,$$

$$m^2_{\xi_\sigma} = -(f_1 + f_2)v - \lambda_0uv, \quad m^2_{\xi_\sigma \xi_\chi} = -(f_1 + f_2)v - \lambda_0uv - 2\lambda_{19}w,$$

and

$$m^2_{\xi_\rho \xi_\sigma} \equiv \frac{f_2}{v_\sigma}uv - (2\lambda_{10} + \lambda_{11})v^2 - \frac{\lambda_{18}}{v_\sigma}u^2v + \frac{\lambda_{19}}{v_\sigma}w^2v,$$

$$m^2_{\xi_\rho \xi_\chi} \equiv \frac{u}{v}(f_1 + f_2)v - 2\lambda_3w^2.$$

8
As in [10] we use here the approximation $|f_1|, |f_2| \sim w$ and maintain only terms of the second order in $w$ in (3.2) (using $w \gg v, u, v_\sigma$). This immediately gives us one physical field

$$H_\chi \simeq \xi_\chi$$  \hspace{1cm} (3.3)$$

with a mass

$$m^2_{H_\chi} \simeq -2\lambda_3 w^2,$$  \hspace{1cm} (3.4)$$

and a square mass matrix of $\xi_\eta, \xi_\rho, \xi_\sigma$ mixing

$$M^2_{\xi_\chi} \simeq w \begin{pmatrix} -\frac{f_1 u}{v} - \frac{\lambda_{19} w v}{v_\sigma} & f_1 & \lambda_{19} w \\ f_1 & -\frac{1}{u} (f_1 v + f_2 v_\sigma) & f_2 \\ \lambda_{19} w & f_2 & -\frac{f_2 u}{v_\sigma} - \frac{\lambda_{19} w v}{v_\sigma} \end{pmatrix}. \hspace{1cm} (3.5)$$

Solving the characteristic equation for the matrix (3.5) we get one massless field $H_1$ and two physical ones ($H_2, H_3$) with masses

$$x_{2,3} = -\frac{w}{2} \left[ \frac{f_1}{uv} (u^2 + v^2) + \frac{f_2}{uu_\sigma} (v^2 + u^2) + \frac{\lambda_{19} w}{vv_\sigma} (v^2 + v_\sigma^2) \right]$$

$$\pm \frac{w}{2} \left\{ \frac{f_1}{vu} (v^2 + u^2) + \frac{f_2}{uv_\sigma} (v^2 + u^2) + \frac{\lambda_{19} w}{vv_\sigma} (v^2 + v_\sigma^2) \right\}^2$$

$$-4v^2 w \left[ \frac{f_1 f_2}{vv_\sigma} + \frac{\lambda_{19} w}{u} \left( \frac{f_1}{v_\sigma} + \frac{f_2}{v} \right) \right]^{1/2}$$

$$\equiv m^2_{H_{2,3}} \hspace{1cm} (3.6)$$

The characteristic equation corresponding to $x_{2,3}$ can be given in the following compact form

$$v \left[ F_2(n) + G_1 \right] + u \left[ F_1(n) F_2(n) - G_1 G_2 \right] + v_\sigma \left[ F_1(n) + G_2 \right] = 0, \quad n = 2, 3, \hspace{1cm} (3.7)$$
where

\[\begin{align*}
F_1(i) &= \frac{u}{v} + G_1 \frac{v^2}{v} + \frac{x_i}{f_1 w}, \\
F_2(i) &= \frac{u}{v} + G_2 \frac{v^2}{v} + \frac{x_i}{f_2 w}, \\
G_1 &= \frac{\lambda_{i9} w}{f_1}, \\
G_2 &= \frac{\lambda_{i9} w}{f_2}.
\end{align*}\]  

(3.8)

To construct physical fields we now consider the equation

\[\left( M^2 - x_i \right) H_i = 0, \quad i = 1, 2, 3, \]  

(3.9)

where \( M^2 \) is a square mass matrix and \( H_i \equiv (H_{i1}, H_{i2}, H_{i3})^T \). For \( M^2 \) we obtain a system of three equations

\[\begin{align*}
- \left( \frac{f_1 u w}{v} + \frac{\lambda_{i9} w^2 v}{v} + x_i \right) H_{i1} + f_1 w H_{i2} + \lambda_{i9} w^2 H_{i3} &= 0, \\
f_1 w H_{i1} - \left[ w \left( f_1 v + f_2 v_v \right) + x_i \right] H_{i2} + f_2 w H_{i3} &= 0, \\
\lambda_{i9} w^2 H_{i1} + f_2 w H_{i2} - \left( \frac{f_2 u w}{v} + \frac{\lambda_{i9} w^2 v}{v} + x_i \right) H_{i3} &= 0.
\end{align*}\]  

(3.10)

It is clear that this system of equations is over defined and can be reduced to two equations, say, the first and the last ones. Thus we have a freedom to suppose

\[H_{i1} = k(i),\]  

(3.11)

where \( k(i) \) will be defined by the normalization of the states. Hence,

\[H_{i2} = \frac{F_1(i) F_2(i) - G_1 G_2}{F_2(i) + G_1} k(i) \equiv \Gamma_2(i) \cdot k(i)\]  

(3.12)
and

\[ H_{i3} = \frac{F_1(i) + G_2 k(i)}{F_2(i) + G_1} \equiv \Gamma_3(i) k(i). \]  \hfill (3.13)

Then \( k(i) \) can be found

\[ k(i) = \left[ 1 + \Gamma_2^2(i) + \Gamma_3^2(i) \right]^{-1/2}, \]  \hfill (3.14)

by normalizing the states \( H_i \) written now in the form

\[
H_i = k(i) \begin{pmatrix}
1 \\
\Gamma_2(i) \\
\Gamma_3(i)
\end{pmatrix} \equiv \begin{pmatrix}
H_{i1} \\
H_{i2} \\
H_{i3}
\end{pmatrix}. \]  \hfill (3.15)

In the massless \((x_1 = 0)\) approximation \(i = 1\) we immediately find

\[ H_1 = \frac{1}{v_w} \begin{pmatrix}
v \\
u \\
v_s
\end{pmatrix}. \]  \hfill (3.16)

In the next approximation (when the \(\lambda\)'s are taken into account) the field \(H_1\) acquires a mass. Solving the characteristic equation for the exact 3 \(\times\) 3 mass matrix \(M_{3g}^2\) and the \(H_1\), namely

\[ \left( M_{3g}^2 - x_1 \right) H_1 = 0, \]  \hfill (3.17)

we obtain the following formulas for the \(H_1\) mass

\[
m_{H_1}^2 = x_1 \approx 2\lambda_1 v^2 + \lambda_4 u^2 + (\lambda_{12} - 2\lambda_{20}) v_s^2 - 2\lambda_{18} \frac{u^2 v_s}{v}, \]
\[
\approx \lambda_4 v^2 + 2\lambda_2 u^2 + (\lambda_{15} + 2\lambda_{16}) v_s^2 - 4\lambda_{18} v v_s, \]
\[
\approx (\lambda_{12} - 2\lambda_{20}) v^2 + (\lambda_{13} + 2\lambda_{16}) u^2 + (2\lambda_{19} + \lambda_{11}) v_s^2 - 2\lambda_{18} \frac{u^2 v_s}{v_s}. \]  \hfill (3.18)
The coupling constants \( \lambda \)'s must be chosen such that the Eqs. (3.18) to be compatible with the Eq. (2.9) or, in the geometric language, we need to find on the sphere (2.9) all point(s) \((v, u, v_σ)\) where all the surfaces (3.18) get together. The simplest solution of this system of equations (2.9) and (3.18) could be found if we accept the following relation among coupling constants

\[
\lambda \approx \lambda_1 \approx \frac{1}{2} \lambda_{12} - \lambda_{20} \approx \lambda_4/2 \approx \lambda_2 \approx \left( \frac{1}{2} \lambda_{13} + \lambda_{16} \right) \approx \left( \lambda_{10} + \frac{1}{2} \lambda_{11} \right). \tag{3.19}
\]

It would be

\[
v \approx v_σ \approx \frac{u}{\sqrt{2}} \approx \frac{v_W}{2} \approx 123 \text{ GeV}, \tag{3.20}
\]

following from

\[
\frac{u^2 v_σ}{v} \approx 2vv_σ \approx \frac{u^2 v}{v_σ} \equiv \delta^2. \tag{3.21}
\]

The assumption (3.19) is justified by examining the latest VEV’s (3.20). It is easily to see here that \( \delta^2 = \frac{1}{2} v_w^2 \). Then the mass of \( H_1 \) would take the value

\[
m_{H_1}^2 \approx 2\lambda v_w^2 - 2\delta^2 \lambda_{18} = v_w^2 (2\lambda - \lambda_{18}), \tag{3.22}
\]

while the eigenstates can be expressed, according to (3.3), (3.17) and (3.22), as follows

\[
\begin{pmatrix}
H_1 \\
H_2 \\
H_3 \\
H_χ
\end{pmatrix} \approx \begin{pmatrix}
v & u & v_σ \\
v_W & v_W & v_W \\
H_{21} & H_{22} & H_{23} \\
H_{31} & H_{32} & H_{33}
\end{pmatrix} \begin{pmatrix}
ξ_η \\
ξ_ρ \\
ξ_σ
\end{pmatrix}, \tag{3.23}
\]

\[
H_χ \approx ξ_χ. \tag{3.24}
\]
Since the matrix
\[ A_{\mu\xi} = \begin{pmatrix} \nu & \nu & \nu \\ \nu & \nu & \nu \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix} \equiv (A^{-1})^T, \quad \det A_{\mu\xi} = 1 \quad (3.25) \]
in (3.23) is an orthonormal matrix \( SO(3) \) the relation inverse to (3.23) and (3.24) can easily be found
\[ \begin{pmatrix} \xi_n \\ \xi_\rho \\ \xi_\sigma \end{pmatrix} \approx \begin{pmatrix} \nu & H_{21} & H_{31} \\ \nu & H_{22} & H_{32} \\ \nu & H_{23} & H_{33} \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}, \quad (3.26) \]
\[ \xi_\chi \approx H_\chi. \quad (3.27) \]

Similarly, in the pseudoscalar sector we obtain one physical field \( \zeta_\sigma \equiv \zeta'_\sigma \) with a mass equal to the mass of \( H'_\sigma \), and the square mass matrix of the \( \zeta_n, \zeta_\rho, \zeta_\sigma, \zeta_\chi \) mixing
\[ M^2_{\zeta_\sigma} = \begin{pmatrix} -m^2_{\zeta_\eta} & -m^2_{\zeta_\eta \zeta_\rho} & -m^2_{\zeta_\eta \zeta_\sigma} & -m^2_{\zeta_\eta \zeta_\chi} \\ -m^2_{\zeta_\rho} & -m^2_{\zeta_\rho \zeta_\sigma} & -m^2_{\zeta_\rho \zeta_\chi} \\ -m^2_{\zeta_\sigma} & -m^2_{\zeta_\sigma \zeta_\chi} \end{pmatrix}. \quad (3.28) \]
where
\[ m^2_{\zeta_\eta} = f_1uw - \frac{\lambda_{18}u^2v_\sigma}{v} + \frac{\lambda_{19}w^2v_\sigma}{v} - 2\lambda_{20}v_\sigma^2, \quad m^2_{\zeta_\eta \zeta_\rho} = f_1w, \]
\[ m^2_{\zeta_\rho} = \frac{w}{u}(f_1v + f_2v_\sigma), \quad m^2_{\zeta_\eta \zeta_\sigma} = \lambda_{18}u^2 - \lambda_{19}w^2 + 4\sqrt{2}\lambda_{19}vv_\sigma, \quad m^2_{\zeta_\rho \zeta_\sigma} = f_2w, \quad m^2_{\zeta_\eta \zeta_\chi} = f_1u, \quad m^2_{\zeta_\rho \zeta_\chi} = f_1v + f_2v_\sigma, \quad m^2_{\zeta_\sigma \zeta_\chi} = f_2u, \]
and

\[ m^2_{\zeta \sigma} = \frac{f_2 u w}{v_\sigma} - \frac{\lambda_{18} u^2 v}{v_\sigma} + \frac{\lambda_{19} w^2 v}{v_\sigma} - 2\lambda_{20} v^2, \quad m^2_{\zeta \chi} = \frac{u}{w} (f_1 v + f_2 v_\sigma). \]

In the approximation \(|f_1|, |f_2| \sim w \gg v, u, v_\sigma\) we obtain one Goldstone boson \(G_1 \approx \zeta \chi\) and the \(\zeta \eta, \zeta \rho, \zeta \sigma\) mixing

\[
M^2_{3\zeta} = w \begin{pmatrix}
-\frac{f_1 u}{v} - \frac{\lambda_{19} u v_\sigma}{v_\sigma} & -f_1 & \lambda_{19} w \\
-f_1 & -\frac{1}{w} (f_1 v + f_2 v_\sigma) & -f_2 \\
\lambda_{19} w & -f_2 & -\frac{f_1 u}{v_\sigma} - \frac{\lambda_{19} u v_\sigma}{v_\sigma} 
\end{pmatrix}. \tag{3.29}
\]

It is clear that the characteristic equation in this case gives the same roots as in the scalar sector, but a different set of the eigenstates (simply, make replacements \(H_{i2} \rightarrow -H_{i2}\))

\[
\begin{pmatrix}
A_1 \\
A_2 \\
A_3
\end{pmatrix} \approx \begin{pmatrix}
\frac{v}{v_W} & -\frac{u}{v_W} & \frac{v_\sigma}{v_W} \\
H_{21} & -H_{22} & H_{23} \\
H_{31} & -H_{32} & H_{33}
\end{pmatrix} \begin{pmatrix}
\zeta \eta \\
\zeta \rho \\
\zeta \sigma
\end{pmatrix}. \tag{3.30}
\]

or equivalently

\[
\begin{pmatrix}
\zeta \eta \\
\zeta \rho \\
\zeta \sigma
\end{pmatrix} \approx \begin{pmatrix}
\frac{v}{v_W} & H_{21} & H_{31} \\
-\frac{u}{v_W} & -H_{22} & -H_{32} \\
\frac{v_\sigma}{v_W} & H_{23} & H_{33}
\end{pmatrix} \begin{pmatrix}
A_1 \\
A_2 \\
A_3
\end{pmatrix}. \tag{3.31}
\]

In the singly charged sector the mixing occurs in the set of \(\eta_1^+, \rho^+, s_1^+\) and in the set of \(\eta_2^+, \chi^+, s_2^+\) with the following square mass matrices

\[
M^2_{+1} = \begin{pmatrix}
-m^2_{\eta_1} & -m^2_{\rho^+ \eta_1} & -m^2_{s_1^+ \eta_1} \\
-m^2_{\rho^+ \rho^+} & -m^2_{s_1^+ \rho^+} & -m^2_{s_1^+ s_1^+}
\end{pmatrix}. \tag{3.32}
\]
where

\[
m_1^2 = \frac{f_1 w}{v} - \frac{\lambda_s w^2}{2} - \left(\frac{\lambda_{15}}{4} + \lambda_{20}\right)v^2 - \frac{\lambda_{18} w^2 v}{v} + \frac{\lambda_{19} w^2 v}{v},
\]

\[
m_{\rho^+ \eta_1^-}^2 = f_1 w - \frac{\lambda_s w}{2} - \lambda_{18} w v, \quad m_{\rho^+ \rho^-}^2 = \frac{w}{u} (f_1 v + f_2 v) - \frac{\lambda_s v^2}{2} + \lambda_{16} v^2 - 2 \lambda_{18} w v,
\]

\[
m_{s_1^+ s_1^-}^2 = \frac{\lambda_{15} v v}{4} + \lambda_{19} w^2 - \lambda_{20} v^2, \quad m_{s_1^+ s_1^-}^2 = -f_2 w - \lambda_{16} w v + \lambda_{18} w,
\]

and

\[
M_{\pi^+}^2 = \begin{pmatrix}
-m_{\eta_2}^2 - f_1 u + \frac{\lambda_s w v}{2} - \lambda_{18} w v - \left(\frac{\lambda_{15}}{4} + \lambda_{20}\right)\left(\frac{f_1 v + f_2 v}{v}\right) - \frac{\lambda_{18} w^2 v}{v} + \frac{\lambda_{19} w^2 v}{v} \\
-m_{\chi^+}^2 - f_2 u + \lambda_{17} w v + \lambda_{19} w v \\
-m_{s_1^+ s_1^-}^2
\end{pmatrix}.
\]

where

\[
m_{\eta_2}^2 = \frac{f_1 w}{v} - \frac{\lambda_s w^2}{2} - \left(\frac{\lambda_{15}}{4} + \lambda_{20}\right)v^2 - \frac{\lambda_{18} w^2 v}{v} + \frac{\lambda_{19} w^2 v}{v},
\]

\[
m_{\chi^+}^2 = \frac{u}{w} (f_1 v + f_2 v) - \frac{\lambda_s v^2}{2} + \lambda_{17} v^2 + 2 \lambda_{19} v v,
\]

\[
m_{s_2^+ s_2^-}^2 = \frac{f_2 w}{v} + \frac{\lambda_{17} v}{4} - \left(\frac{\lambda_{15}}{4} + \lambda_{20}\right)v^2 + \lambda_{17} w^2 - \frac{\lambda_{18} w v}{v} + \frac{\lambda_{19} w v}{v}.
\]

Applying the approximation for \(M_{\pi^+}^2\) we obtain one Goldstone boson \(G_2^+ \approx \chi^+\) and two physical fields associated with \(\eta_2^+\) and \(s_2^+\) with masses

\[
m_{\eta_2}^2 = \frac{f_1 w}{v} - \frac{\lambda_s w^2}{2} + \frac{\lambda_{18} w^2 v}{v}, \quad m_{s_2^+ s_2^-}^2 = \frac{f_2 w}{v} + \lambda_{17} w^2 + \frac{\lambda_{19} w^2 v}{v}. \quad (3.34)
\]

For the \(\eta_1^+, \rho^+, s_1^+\) mixing, we have

\[
M_{\pi^+}^2 = w \begin{pmatrix}
-f_1 u - \frac{\lambda_{19} w v}{v} & -f_1 & -\lambda_{19} w \\
-f_1 & -\frac{1}{u} (f_1 v + f_2 v) & f_2 \\
-\lambda_{19} w & f_2 & -\frac{f_2 u}{v} - \frac{\lambda_{19} w v}{v}
\end{pmatrix}.
\]

(3.35)
As before, the characteristic equation of (2.41) has the same roots, but the eigenstates are different and are given by (necessary replacements: $H_{i2} \to -H_{i2}, H_{i3} \to -H_{i3}$)

\[
\begin{pmatrix}
  h_1^+ \\
  h_2^+ \\
  h_3^+
\end{pmatrix} \approx \begin{pmatrix}
  \frac{u}{v_w} & -\frac{u}{v_w} & -\frac{v_w}{v_w} \\
  H_{21} & -H_{22} & -H_{23} \\
  H_{31} & -H_{32} & -H_{33}
\end{pmatrix} \begin{pmatrix}
  \eta_1^+ \\
  \rho^+ \\
  s_1^+
\end{pmatrix}
\] (3.36)

or equivalently

\[
\begin{pmatrix}
  \eta_1^+ \\
  \rho^+ \\
  s_1^+
\end{pmatrix} \approx \begin{pmatrix}
  \frac{v}{v_w} & H_{21} & H_{31} \\
  -\frac{u}{v_w} & -H_{22} & -H_{32} \\
  -\frac{v_w}{v_w} & -H_{23} & -H_{33}
\end{pmatrix} \begin{pmatrix}
  h_1^+ \\
  h_2^+ \\
  h_3^+
\end{pmatrix}.
\] (3.37)

In the doubly charged sector the mixing occurs up all states $\rho^{++}, s_2^{++}, \chi^{++}, s_1^{++},$

and the square mass matrix is given

\[
M^2_{4++} = \begin{pmatrix}
  -m_{\rho^{++}\rho^{--}}^2 & -m_{s_2^{++}\rho^{--}}^2 & -m_{\chi^{++}\rho^{--}}^2 & -m_{s_1^{++}\rho^{--}}^2 \\
  -m_{s_2^{++}\rho^{--}}^2 & -m_{\chi^{++}\rho^{--}}^2 & -m_{s_2^{++}\rho^{--}}^2 & -m_{s_1^{++}\rho^{--}}^2 \\
  -m_{\chi^{++}\rho^{--}}^2 & -m_{s_1^{++}\rho^{--}}^2 & -m_{\chi^{++}\rho^{--}}^2
\end{pmatrix},
\] (3.38)

where

\[
m_{\rho^{++}\rho^{--}}^2 = \frac{w}{u}(f_1 v + f_2 v_s) - \frac{\lambda_{18} w^2}{2} - 4\lambda_{18} vv_s, \\
m_{s_2^{++}\rho^{--}}^2 = -\sqrt{2}(f_2 w + \lambda_{16} uv_s - \lambda_{18} uv), \\
m_{\chi^{++}\rho^{--}}^2 = f_1 v + f_2 v_s - \sqrt{2} \lambda_{18} uv, \\
m_{s_1^{++}\rho^{--}}^2 = \sqrt{2}(-\lambda_{17} uv_s + \lambda_{19} uv), \\
m_{s_2^{++}\rho^{--}}^2 = -\sqrt{2}(f_2 u + \lambda_{17} uv_s + \lambda_{19} uv), \\
m_{s_1^{++}\rho^{--}}^2 = -\lambda_{20} v^2, \\
m_{s_1^{++}\rho^{--}}^2 = -\sqrt{2}(f_2 u + \lambda_{17} uv_s + \lambda_{19} uv),
\]
and
\[ m^2_{s^+_1 s^-_2} = \frac{f_2 uu}{v_o} + \lambda_{16} u^2 - \lambda_{17} w^2 - \frac{\lambda_{18} u^2 v}{v_o} + \frac{\lambda_{19} w^2 v}{v_o} - \lambda_{20} v^2, \]
\[ m^2_{\chi^++\chi^-} \equiv \frac{u}{w}(f_1 v + f_2 v_o) - \frac{\lambda_9 u^2}{2} + 4\lambda_{19} v v_o, \]
\[ m^2_{s^+_1 s^-_1} \equiv \frac{f_2 uu}{v_o} - \lambda_{16} u^2 + \lambda_{17} w^2 - \frac{\lambda_{18} u^2 v}{v_o} + \frac{\lambda_{19} w^2 v}{v_o} - \lambda_{20} v^2. \]

By the same way as considered above we obtain one Goldstone boson \( G^+_3 \approx \chi^+ \)
and one physical field \( s^+_1 \) with mass
\[ m^2_{s^+_1 s^-_1} = \frac{f_2 uu}{v_o} + \lambda_{16} u^2 + \lambda_{17} w^2 - \frac{\lambda_{18} u^2 v}{v_o} + \frac{\lambda_{19} w^2 v}{v_o} - \lambda_{20} v^2. \]

and a matrix of \( \rho^{++}, s^{++}_2 \) mixing
\[ M_{2++}^2 = w \begin{pmatrix} -\frac{1}{u}(f_1 v + f_2 v_o) + \frac{\lambda_9 u}{2} & \sqrt{2}f_2 \\ \sqrt{2}f_2 & -f_2 v_o + \lambda_{17} w - \frac{\lambda_{19} w v}{v_o} \end{pmatrix}. \]

Solving the characteristic equation we get two square masses
\[ x_{4,5} = \frac{w}{2} \left[ \lambda_{17} w - \frac{\lambda_{19} w v}{v_o} - f_2 \left( \frac{u}{v_o} + \frac{v_o}{u} \right) - \frac{f_1 v}{v_o} \right] \]
\[ \pm \frac{w}{2} \left[ \left( \lambda_{17} w - \frac{\lambda_{19} w v}{v_o} - f_2 \left( \frac{u}{v_o} + \frac{v_o}{u} \right) - \frac{f_1 v}{v_o} \right)^2 \right. \]
\[ \left. + \frac{2\sqrt{2}}{u v_o} \left( f_1 v + f_2 v_o \right) \left( f_2 u - \frac{\lambda_{17} w v}{\sqrt{2}} + \lambda_{19} w v \right) - f_2^2 \right]^{1/2} \]
\[ \equiv m^2_{d_1 = d_2}. \]

for two physical fields
\[ \begin{pmatrix} d^{++}_1 \\ d^{++}_2 \end{pmatrix} = \begin{pmatrix} n_4 \frac{1}{\sqrt{2}} \left( \frac{u}{v_o} + \frac{x_4}{f_2 u} - \frac{\lambda_{17} w}{f_2} + \frac{\lambda_{19} w v}{f_2 v_o} \right) n_4 \\ n_5 \frac{1}{\sqrt{2}} \left( \frac{u}{v_o} + \frac{x_5}{f_2 u} - \frac{\lambda_{17} w}{f_2} + \frac{\lambda_{19} w v}{f_2 v_o} \right) n_5 \end{pmatrix} \begin{pmatrix} \rho^{++} \\ s^{++}_2 \end{pmatrix}. \]
corresponding to (3.41) in [11] where

\[
n_i = \left[ 1 + \frac{1}{2} \left( \frac{u}{v} + \frac{x_i}{f_2 w} - \frac{\lambda_{12} w + \lambda_{13} w v}{f_2 v} \right)^2 \right]^{-\frac{1}{2}}, \quad (i = 4, 5), \quad (3.43)
\]

is found by normalizing states. Here, using a shorter notation

\[
M^2_{2++} = \begin{pmatrix} a_{11} & a \\ a & a_{22} \end{pmatrix}, \quad (3.44)
\]

of the matrix (3.40) we, however, can rewrite (3.42) in another way

\[
\begin{pmatrix} d_{1+}^+ \\ d_{2+}^+ \end{pmatrix} = \begin{pmatrix} aN_4 & X_4 N_4 \\ aN_5 & X_5 N_5 \end{pmatrix} \begin{pmatrix} \rho^{++} \\ s_{2+}^{++} \end{pmatrix}, \quad (3.45)
\]

where

\[
N_i = \left( a^2 + X_i^2 \right)^{-\frac{1}{2}} \quad (3.46)
\]

and

\[
X_i = \frac{1}{2} \left[ -a_{11} + a_{22} \pm \sqrt{\left( a_{11} - a_{22} \right)^2 + 4a^2} \right], \quad (i = 4, 5). \quad (3.47)
\]

4 Conclusion

We have just considered the Higgs sector of the minimal 3 - 3 - 1 model under the most general gauge–invariant potential conserving lepton numbers. In comparison with the previous paper [11] the content of the particles and their multiplet decomposition structure remain the same but most of the masses get corrections:
– in the neutral scalar sector, physical fields are: $H_1, H_2, H_3, H'_\sigma$ and $H_\chi$

$$m^2_{H_1} \approx 2\lambda v^2_w - 2\delta^2 \lambda_{18} = v^2_w (2\lambda - \lambda_{18}), \quad m^2_{H_2} = x_2, \quad m^2_{H_3} = x_3,$$

$$m^2_{H'_\sigma} = \frac{f_2 u w}{v_\sigma} + \frac{\lambda_{11}}{2} v^2_\sigma - \left(\frac{\lambda_{15}}{2} + \lambda_{20}\right) v^2 + \lambda_{16} u^2 + \lambda_{17} w^2 - \frac{\lambda_{18} w^2 v}{v_\sigma} + \frac{\lambda_{19} w^2 v}{v_\sigma},$$

$$\approx \sqrt{2} f_2 w + 4 \left(\frac{\lambda_{11}}{2} - \frac{\lambda_{15}}{2} + 2\lambda_{16} - 2\lambda_{18} - \lambda_{20}\right) v^2_w + (\lambda_{17} + \lambda_{19}) w^2,$$

$$m^2_{H_\chi} \approx -2\lambda_3 w^2,$$  \hspace{1cm} (4.1)

where the relation (3.20) is used,

– in the neutral pseudoscalar sector, physical fields are: $A_2, A_3, A_\sigma$ and two Goldstone bosons $G_1 \approx \zeta_\chi$ and $G_2$ (corresponding to the massless $A_4$)

$$m^2_{A_2} = x_2, \quad m^2_{A_3} = x_3, \quad m^2_{A_\sigma} = m^2_{H'_\sigma},$$  \hspace{1cm} (4.2)

– in the singly charged sector, there are two Goldstone bosons $G_3 = h^+_1, G^+_4 \approx \chi^+$ and three physical fields: $h^+_2, h^+_3, \eta^+_2, s^+_2$ with masses:

$$m^2_{h^+_2} = m^2_{H_2}, \quad m^2_{h^+_3} = m^2_{H_3},$$

$$m^2_{\eta_2} \approx \frac{f_1 u w}{v} - \frac{\lambda_3 w^2}{2} + \frac{\lambda_{19} w^2 v^2_\sigma}{2 v^2}, \quad m^2_{s^+_2} \approx \frac{f_2 u w}{v_\sigma} + \lambda_{17} w^2 + \frac{\lambda_{19} w^2 v}{v_\sigma},$$  \hspace{1cm} (4.3)

– in the doubly charged sector, there are one Goldstone ($G^{++}_5 \approx \chi^{++}$) and three physical fields $d^{++}_1, d^{++}_2, s^{++}_1$ with masses:

$$m^2_{d^{++}_1} = x_4, \quad m^2_{d^{++}_2} = x_5, \quad m^2_{s^{++}_1} = \frac{f_2 u w}{v_\sigma} + \lambda_{17} w^2 + \frac{\lambda_{19} w^2 v}{v_\sigma} \equiv m^2_{s^{++}_1}.$$  \hspace{1cm} (4.4)

Eqs. (4.1) – (4.4) show that $f_1$ and $f_2$ can take a definite consistent sign and there are three degenerate states $H_2, A_2$ and $h^+_2$ in mass $x_2$, three degenerate states $H_3, A_3$ and $h^+_3$ in mass $x_3$ and two degenerate states $H'_\sigma, A'_\sigma$ in mass $m^2_{H'_\sigma}$. 

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Combining the assumption (3.19) and the positiveness of the mass squares we get the following bounds for coupling constants

$$\lambda \approx \lambda_1 \approx \frac{1}{2} \lambda_{12} - \lambda_{20} \approx \lambda_4/2 \approx \lambda_2 \approx (\frac{1}{2} \lambda_{13} + \lambda_{16}) \approx (\lambda_{10} + \frac{1}{2} \lambda_{11}) \gtrsim 0,$$

\[ \lambda_3 \lesssim 0. \tag{4.5} \]

Note that new coupling constants $\lambda_{15}$, $\lambda_{17}$, $\lambda_{18}$ and $\lambda_{19}$ remain unconstrained by (3.19) and (4.5). It is worth mentioning that the system of Eqs. (2.9) and (3.18) may admit more general solutions with other coupling constants rather than those constrained by (3.19). This question deserves to be furthermore investigated.

In conclusion, the present paper is an extension of previous investigations [10, 11] on the Higgs sector of the minimal 3 - 3 - 1 model with three triplets and one sextet. Under the most general lepton–number conserving potential the mass spectra and the multiplet decomposition structure of this sector are investigated in detail at tree-level. Due to the fact that most of the scalar masses get corrections the problem with inconsistent signs of $f_2$ arising in the previous case [10, 11] is solved. The results of this paper are exposed in a systematic order and a transparent way allowing them to be easily checked and used in further studies.

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