Abstract—In this paper we study the problem of how to optimally steer the state covariance of a continuous-time linear stochastic system over a finite time interval subject to additive “generic noise.” Optimality here means reaching a target state covariance with minimal control energy. Generic noise includes a combination of white Gaussian noise and abrupt “jump noise” that is discontinuous in time. We first establish the controllability of the state covariance for linear time-varying stochastic systems under these assumptions. We then derive the optimal control to steer the covariance subject to generic noise. The optimal control entails solving two dynamically coupled ordinary matrix differential equations (ODEs) with split boundary conditions. We show the existence and uniqueness of the solution of these coupled matrix ODEs.

Index Terms—Stochastic control, covariance steering, linear stochastic systems, additive jump noise, finite horizon.

I. INTRODUCTION

All dynamical systems are prone to disturbances whose effects persist with time. Controlling the uncertainty is critical for the overall robustness and performance of all such systems. Among the various approaches, covariance control theory, which was developed in the mid-1980s, provides a direct way to regulate the state covariance of a stochastic system subject to noise [1], [2]. By quantifying the uncertainty in a direct manner, covariance control theory has found applications in many real-life engineering problems, such as spacecraft soft landing [3] and trajectory optimization [4], vehicle path planning [5]–[8], drone delivery, multi-agent systems [9], and active cooling of stochastic oscillators [10].

Much of the earlier work dealt exclusively with controlling the stationary, or asymptotic, state covariance over an infinite horizon. Steady-state covariance control theory has addressed many problems such as the complete controllability of the state covariance [1], [2], the assignability of the state covariance, the parameterization of all covariance controllers [11]–[13], and optimal covariance control design [14]. These problems have been studied within various settings of both continuous-time and discrete-time formulations, including state feedback or output feedback cases.

Most recent work has focused on the finite-horizon optimal covariance steering problem in both continuous-time and discrete-time settings [15]–[17]. In particular, the authors in [15], [18], [19] consider the linear stochastic system with additive white Gaussian noise

$$dx(t) = A(t)x(t)\,dt + B(t)u(t)\,dt + C(t)\,dw(t), \quad (1)$$

where $(A(t),B(t))$ is a controllable matrix pair and $w(t)$ is a standard multi-dimensional Wiener process (Brownian motion). It is shown that when the noise channel coincides with the control channel, that is, $C(t) = \sqrt{\kappa}B(t)$ for some $\kappa > 0$, there exists a unique solution to (1) that steers the state covariance from any initial positive definite matrix $\Sigma_0 > 0$ to any final $\Sigma_T > 0$ over a finite time interval $[0,T]$, while minimizing a quadratic cost functional in the state and control input. The cases when one or both of the initial and final state covariances $\Sigma_0$ and $\Sigma_T$ are singular are treated in [20]. When the noise and control channels are different, and the coefficient matrices $A$, $B$, and $C$ are constant, controllability of the state covariance via state feedback is established in [18]. However, the question of existence and uniqueness of an optimal control policy that minimizes a quadratic cost subject to more general (possibly non-Gaussian) noise is still open. Interestingly, the optimal stochastic control problem, as described here, can be interpreted as a Schrödinger bridge problem and can be reduced to an optimal mass transport problem in the zero-noise limit [21]. Finite-horizon covariance steering problems with output feedback have also been explored in [22].

Interest in the finite-horizon optimal covariance steering problem for discrete-time linear stochastic systems has also grown rapidly in recent years. A variety of cost functions and inequality constraints on the terminal state distribution are investigated in [17], [23]–[26]. Additional “hard” chance constraints on the input and the state sample paths can be imposed as well [16], [27]–[29]. In the latter case, the state distribution may or may not be Gaussian. To optimally allocate the risks of chance constraint violations, an iterative algorithm is proposed in [30]. While the above-mentioned papers all deal with linear stochastic systems, covariance control theory has also been developed for nonlinear stochastic systems over both an infinite [31], [32] and a finite time horizon [33]–[37].

In the existing literature on continuous-time finite-horizon covariance steering, the noise is modeled by the “differential” of a Wiener process, namely, $dw(t)$ as in (1). Since every sample path of a Wiener processes is continuous in time, this model captures only the case of white Gaussian noise. This may be restrictive in many real-life applications where abrupt changes have to be accounted for. These abrupt changes cannot be modeled as Gaussian noise. For example, shot noise in electronic circuits [38], vehicles running on uneven ground, gusts of wind or heavy raindrops acting on drones...
[39], and abrupt faults in components of networks [40], can all be modeled by a jump noise process. In view of this, the goal of this paper is to take a first step towards bridging the gap between the conventional white Gaussian noise model assumed in the covariance control literature thus far and the emerging need for more comprehensive noise models for many covariance real control applications.

The main problem addressed in this paper is to determine the optimal control law for a continuous-time linear stochastic system subject to a generic type of noise, which may contain random jumps of any size, so that the control energy is minimized and the pre-specified mean and covariance of the terminal state are achieved over a finite time interval.

We model the noise acting on a dynamical system as the “differential” of a continuous-time martingale [41], a framework that incorporates both white Gaussian noise and random jump noise. Since such a combination captures both running noise and abrupt noise of any jump size, it is a practical model for various disturbances and event-driven uncertainties [42]. Consequently, this noise model will enable more effective control of the state distribution, improving expected system performance and safety. As the noise is, in general, no longer Gaussian, one cannot expect to target a certain terminal state performance and safety. As the noise is, in general, no longer Gaussian, we will henceforth assume that the martingale starts with $m(0) = 0$ almost surely (a.s.). We can also assume that $m(t)$ is a square integrable martingale, that is, $E[∥m(t)∥^2] < ∞$ for all $t$.

In other words, at any given time $s$, the conditional expectation of all the future values of $m(t)$ is equal to the present value $m(s)$, regardless of all its prior values. Without loss of generality, we assume that the martingale is a continuous-time martingale, that is, $E[∥m(t)∥^2] < ∞$ for all $t$.

A sequence of stopping times $τ_k$, where $k ∈ N$, is increasing and diverging if $0 ≤ τ_k < τ_{k+1}$ for all $k ∈ N$ and $\lim_{k→+∞} τ_k = +∞$ a.s.. Recall that the total variation of a function $g$ on the interval $[a, b] ⊂ R$ is

$$\sup_{P ∈ \mathcal{P}} \sum_{i=0}^{n_P-1} |g(t_{i+1}) - g(t_i)|,$$

where the supremum is taken over the set $\mathcal{P}$ of all partitions $P = \{t_0, t_1, \ldots, t_{n_P} \mid a = t_0 < t_1 < \cdots < t_{n_P} = b\}$ of the interval $[a, b]$.

Definition 1. A (multi-dimensional) martingale, denoted by $m(t)$, is a process such that $E[∥m(t)∥] < ∞$ and $E[m(t) | \{m(τ) : τ ≤ s\}] = m(s)$ for all $t$ and all $s ≤ t$.

Contributions: In this paper we first establish a controllability result for steering the state covariance from any initial positive definite matrix to any final positive definite matrix on a finite time interval for a linear time-varying stochastic system under mild assumptions. Next, a sufficient condition for optimal covariance steering in the face of generic noise is derived using a “completion of squares” argument, and is subsequently interpreted from an alternative perspective of variational calculus. The condition boils down to solving two dynamically coupled matrix differential equations with split boundary conditions. Lastly, we show the existence and uniqueness of the solution to these coupled ODEs, and hence the existence and uniqueness of an optimal control law. This result also answers the open question raised in [18] on the existence and uniqueness of an optimal control policy when the noise and control channels are different.

The rest of the paper is organized as follows. The generic noise model along with some preliminary results on stochastic processes and Itô calculus are reviewed in Section II. The main problem of interest is formulated in Section III. The controllability of the state covariance for a linear time-varying stochastic system is shown in Section IV. A sufficient condition for optimal covariance steering with generic noise is provided in Section V. This sufficient condition is analyzed and solved in Section VI. Finally, two examples are presented in Section VII to illustrate the results of this paper.

II. GENERIC NOISE MODEL AND PRELIMINARIES

In this section, we briefly review the mathematical model of the generic noise model used in this work, along with some preliminaries on stochastic processes and Itô calculus used in the rest of the paper. In the following sections, we will make use of these results to formulate the optimal covariance steering problem and to derive sufficient conditions for optimality. The following definition can be found, for example, in [43].

Definition 1. A (multi-dimensional) martingale, denoted by $m(t)$, is a process such that $E[∥m(t)∥] < ∞$ and $E[m(t) | \{m(τ) : τ ≤ s\}] = m(s)$ for all $t$ and all $s ≤ t$.

In other words, at any given time $s$, the conditional expectation of all the future values of $m(t)$ is equal to the present value $m(s)$, regardless of all its prior values. Without loss of generality, we will henceforth assume that the martingale starts with $m(0) = 0$ almost surely (a.s.). We can also assume that $m(t)$ is a square integrable martingale, that is, $E[∥m(t)∥^2] < ∞$ for all $t$.

A sequence of stopping times $τ_k$, where $k ∈ N$, is increasing and diverging if $0 ≤ τ_k < τ_{k+1}$ for all $k ∈ N$ and $\lim_{k→+∞} τ_k = +∞$ a.s.. Recall that the total variation of a function $g$ on the interval $[a, b] ⊂ R$ is

$$\sup_{P ∈ \mathcal{P}} \sum_{i=0}^{n_P-1} |g(t_{i+1}) - g(t_i)|,$$

where the supremum is taken over the set $\mathcal{P}$ of all partitions $P = \{t_0, t_1, \ldots, t_{n_P} \mid a = t_0 < t_1 < \cdots < t_{n_P} = b\}$ of the interval $[a, b]$.

Definition 2. A process $ℓ(t)$ is a local martingale if there exists an increasing and diverging sequence of stopping times $τ_k$, where $k ∈ N$, such that $ℓ(\min\{t, τ_k\})$ is a martingale for every $k ∈ N$. A process is of locally finite variation if there exists an increasing and diverging sequence of stopping times $τ_k$, where $k ∈ N$, such that every sample path of the process has a bounded total variation on $[0, τ_k]$ for each $k ∈ N$. A semimartingale is a càdlàg (right continuous with left limits) process that can be decomposed as the sum of a local martingale and a process of locally finite variation.

Unless otherwise stated, all processes in the sequel are semimartingales for which the Itô integral can be defined [44].

We assume “generic noise” models affecting the system, which may contain random jumps of any size. The noise signal is modeled by a martingale “differential” $dm(t)$, so the noise has a mean value of zero. Generally speaking, the noise may not be white, since a martingale may not have stationary and independent increments. Some examples of noise modeled by $dm$ are as follows.

The initial value of $m$ does not matter, as the noise is modeled by $dm$. 

[1]
1) **White Gaussian noise**: $dm_1(t) = dw(t)$, where $w(t) \in \mathbb{R}^q$ is a Wiener process, since a Wiener process is a martingale with stationary and independent increments.

2) **Noise with jumps**: $dm_2(t) = d\lambda(t)dt$, where $\lambda(t) \in \mathbb{R}^3$ is a nonhomogeneous Poisson process with (deterministic) arrival rate $\lambda(t) = [\lambda_1(t), \lambda_2(t), \ldots, \lambda_q(t)]$, 

where $I$ is the convention that the quadratic covariation $E[\int_0^t \lambda(u)du]$ is a martingale \[43\]. In general, the jump size may not be fixed but it is a random variable following a certain distribution. We may take $m_2$ to be a nonhomogeneous compound Poisson process subtracting its compensator. Thus, $dm_2$ can model random noise with any jump size.

3) **Combinations**: We can model continuous noise and noise with jumps, such as $dm_3(t) = kdw(t) + dm_2(t)$, where $k > 0$, and $dm_4(t) = dt - dw(t) - \int_0^t \lambda(u)du$, where $\lambda$ is a Lévy process and $e$ is its compensator.

For a process $v(t)$, which may be a jump process, let $v^-(t) = \lim_{s \to t} v(s)$ be the left limit of $v$ at time $t$. Then, the Itô integral $\int_0^t v^-(\tau)d\tau$ is also a martingale \[44\]. Hence, $E[\int_0^t v^-(\tau)d\tau] = 0, \forall t \geq 0$.

Clearly, $\int_0^t v^-(\tau)d\tau = \int_0^t v(\tau)d\tau$.

For a symmetric matrix $S$, let $S > 0$ and $S \succeq 0$ denote, respectively, the positive definiteness and positive semidefiniteness of $S$. By Itô calculus, $E[m(t)m^T(t)]$ is equal to the expected value of the quadratic variation of $m$ at time $t$. Thus, $E[m(t)m^T(t)]$ is increasing, in the sense that $E[m(t)m^T(t)] - E[m(s)m^T(s)] \geq 0$ for all $t > s$ \[43\]. For example, for a Wiener process $\omega(t) \in \mathbb{R}^q$, $E[\omega(t)\omega^T(t)] = tI_q$, where $I_q$ is the $q \times q$ identity matrix. For the martingale $m_2$ defined above, let $\Lambda(t) = \text{diag}[\lambda_1(t), \lambda_2(t), \ldots, \lambda_q(t)]$.

Then, $E[m_2(t)m_2^T(t)] = \int_0^t E[\lambda(u)du]$. For $m_3(t) = \alpha m(t) + \beta m_2(t)$ with $m_1$ and $m_2$ independent, it follows that $E[m_3(t)m_3^T(t)] = \alpha^2 E[m_1(t)m_1^T(t)] + \beta^2 E[m_2(t)m_2^T(t)]$. For the rest of the paper, it is assumed that $E[m(t)m^T(t)]$ is continuously differentiable with respect to $t$, with the notation $D_m(t) = E[m(t)m^T(t)]/dt \succeq 0$. This assumption is not very restrictive, as it is satisfied for $w(t)$ with $D_w(t) = I_q$ and for $m_2(t)$ with $D_{m_2}(t) = \Lambda(t)$ when the arrival rate $\lambda(t)$ is continuous. Informally, we write $E[dm(t)dm^T(t)] = dE[m(t)m^T(t)] = D_m(t)dt$.

**III. Problem Formulation**

Consider the following time-varying linear stochastic system subject to additive generic noise

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + C(t)d\lambda(t),$$

$$E[x(0)] = 0, \quad E[x(0)x^T(0)] = \Sigma_0 > 0,$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input at time $t$, $m(t) \in \mathbb{R}^q$ is a martingale independent of $x(0)$ and with $E[m(t)m^T(t)]=D(t)$, $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times p}$, and $C(t) \in \mathbb{R}^{n \times q}$ are the coefficient matrices. Let $C^k$ denote the class of $k$-times continuously differentiable functions. We assume that $A(t) \in C^{n-1}$, $B(t) \in C^n$, and $C(t), D(t) \in C^0$. Without loss of generality, assume that $A(t)$ is defined on the interval $[0, 1]$, and that the desired terminal state $x(1)$ is characterized by its mean and covariance matrix given by

$$E[x(1)] = 0, \quad E[x(1)x^T(1)] = \Sigma_1 > 0.$$  

A control input $u \in U$ is admissible if, for each $t \in [0, 1]$, it depends only on $t$ and on the past history of the states $\{x(s) : 0 \leq s \leq t\}$, and satisfies

$$J(u) \triangleq E\left[\int_0^1 u^T(t)R(t)u(t)dt\right] < \infty,$$

for some continuous $R(t) > 0$ of dimension $p \times p$ for all $t \in [0, 1]$, such that (6) with the initial condition (7) has a strong solution \[44\], and the desired terminal state mean and covariance given by (8) are achieved. The problem is to determine whether $U$ is nonempty and if so, to find the optimal control $u^* \in U$, which minimizes the quadratic cost functional (9) subject to the boundary constraints (7), (8).

**IV. Controllability of the State Covariance**

In this section, the controllability of the state covariance for (6) is established. The covariance matrix

$$\Sigma(t) \triangleq E\left[\left(x(t) - E[x(t)]\right)^T\left(x(t) - E[x(t)]\right)\right]$$

of the state $x(t)$ of (6) is said to be controllable on the time interval $[0, 1]$ if, for any given $\Sigma_0, \Sigma_1 > 0$, there exists an admissible control $u \in U$ that steers the state covariance $\Sigma(t)$ from $\Sigma(0) = \Sigma_0$ to $\Sigma(1) = \Sigma_1$, while maintaining $\Sigma(t) > 0$ for all $t \in [0, 1]$. 

Footnotes:

2 Formally, the integration by parts formula from Itô calculus states that $w = \int u^+d\tau + f(u^-) + [u, v]$, where $[u, v]$ is the quadratic covariation of $u$ and $v$ \[44\].

3 Assuming zero mean is just for convenience, otherwise the control will have an additional term; see Remark 9 in \[15\].
A. State Covariance Under State Feedback Control

Consider the state feedback control law of the form
\[ u(t) = K(t)x(t), \quad t \in [0, 1]. \tag{10} \]
Since \( \mathbb{E}[x(0)] = 0 \), we have \( \mathbb{E}[x(t)] = 0 \) for all \( t \in [0, 1] \).
Therefore, the state covariance \( \Sigma(t) = \mathbb{E}[x(t)x^T(t)] \) satisfies
\[
\Sigma(t) = \int_0^t d\Sigma(\tau) = \int_0^t d\mathbb{E}[x(\tau)x^T(\tau)] \\
= \int_0^t \mathbb{E}[-dx^Td\tau + dxx^Td\tau] \\
= \int_0^t \left( (A + BK)\Sigma + (A + BK)^T\Sigma + CDT^T \right)d\tau,
\]
where we have invoked (3) in the second last line and (6), (4), (3), and (5) in the last line. This yields
\[
\dot{\Sigma} = \left( A(t) + B(t)K(t) \right)\Sigma + \Sigma \left( A(t) + B(t)K(t) \right)^T + C(t)D(t)C^T(t). \tag{11}
\]

When the matrices \( A, B, C, \) and \( D \) are constant and the pair \((A, B)\) is controllable, (11) is known to be controllable \([18]\), and thus \( \mathcal{U} \) is nonempty, since (11) can be reduced to the form of (27) in \([18, \text{Theorem 3}]\). This shows the existence of an admissible control of the form (10). It is not difficult to check that this statement still holds when \( C(t) \) and \( D(t) \) are time-varying. It will be shown later on that when \( A(t) \) and \( B(t) \) are also time-varying, and under some mild assumptions, (11) is also controllable and therefore \( \mathcal{U} \) is nonempty.

B. Sufficient Condition for Controllability

Given a time-varying matrix pair \((A(t), B(t))\) of dimensions \( n \times n \) and \( n \times p \), respectively, define
\[
\Theta_i(t) \triangleq \begin{bmatrix} \Gamma_0(t) & \Gamma_1(t) & \cdots & \Gamma_{i-1}(t) \end{bmatrix}, \quad 1 \leq i \leq n + 1, \\
\Gamma_0(t) \triangleq B(t), \\
\Gamma_k(t) \triangleq -A(t)\Gamma_{k-1}(t) + \dot{\Gamma}_{k-1}(t), \quad 1 \leq k \leq n.
\]
Then, the controllability matrix of \((A(t), B(t))\) is \( \Theta_n(t) \) \([45]\), and the pair \((A(t), B(t))\) is uniformly controllable on the time interval \([0, 1]\) if, for all \( t \in [0, 1] \), \( \text{rank} \Theta_n(t) = n \) \([45]\). Recall that the controllability property in \([15]\) requires that the controllability Gramian
\[
N(t_1, t_0) \triangleq \int_{t_0}^{t_1} \Phi_A(t_0, \tau)B(\tau)B^T(\tau)\Phi_A(t_0, \tau)^T d\tau \tag{12}
\]
is nonsingular for all \( 0 \leq t_0 < t_1 \leq 1 \), where \( \Phi_A(t, \tau) \) is the state transition matrix of \( A(t) \) (see also (24) below). This property is, in fact, equivalent to the statement that the pair \((A(t), B(t))\) is totally controllable on the time interval \([0, 1]\) \([46]\), which holds if and only if, for all \( 0 \leq t_0 < t_1 \leq 1 \), there exists \( t \in (t_0, t_1) \) such that \( \text{rank} \Theta_n(t) = n \) \([47]\). If \( A(t) \) and \( B(t) \) are analytic functions of time, then \((A(t), B(t))\) is totally controllable if and only if, for almost all \( t \in [0, 1] \), \( \text{rank} \Theta_n(t) = n \) \([45]\). Clearly, uniform controllability is a stronger property than total controllability. The pair \((A(t), B(t))\) is index invariant on the interval \([0, 1]\) if, for all \( 1 \leq i \leq n + 1 \), rank \( \Theta_i(t) \) is time-invariant on \([0, 1]\) and rank \( \Theta_n(t) = \text{rank} \Theta_{n+1}(t) \) \([48]\).

**Theorem 1.** Let the pair \((A(t), B(t))\) be uniformly controllable and index invariant on the time interval \([0, 1]\). Then, the state covariance of the linear stochastic system (6) is controllable on \([0, 1]\) and also on any subinterval of \([0, 1]\).

**Proof.** Without loss of generality, assume \( B(t) \) is of full column rank for all \( t \in [0, 1] \). Since \((A(t), B(t))\) is uniformly controllable and index invariant, there exists a time-varying coordinate transformation that brings the pair \((A(t), B(t))\) into its canonical form \([49, 50]\). That is, there exist nonsingular matrices \( P(t) \in \mathbb{R}^{n \times n} \) and \( Q(t) \in \mathbb{R}^{p \times p} \) such that the new state \( \tilde{x}(t) = P(t)x(t) \) satisfies
\[
d\tilde{x}(t) = \tilde{A}(t)\tilde{x}(t)dt + \tilde{B}u(t)dt + \tilde{C}(t)dm(t),
\]
and the new state covariance
\[
\dot{\tilde{\Sigma}}(t) \triangleq \mathbb{E}\left[ (\tilde{x}(t) - \mathbb{E}[\tilde{x}(t)]) (\tilde{x}(t) - \mathbb{E}[\tilde{x}(t)])^T \right] = P(t)\Sigma(t)P^T(t)
\]
satisfies
\[
\dot{\tilde{\Sigma}} = \left( \tilde{A}(t) + \tilde{B}\tilde{K}(t) \right)\tilde{\Sigma} + \tilde{\Sigma} \left( \tilde{A}(t) + \tilde{B}\tilde{K}(t) \right)^T + \tilde{C}(t)D(t)C^T(t),
\]
where \( \tilde{A}(t) = (P(t)A(t) + \dot{P}(t))P^{-1}(t), \tilde{B} = P(t)B(t)Q(t), \tilde{C}(t) = P(t)C(t), \tilde{K}(t) = Q^{-1}(t)K(t)P^{-1}(t), \) and \((\tilde{A}(t), \tilde{B})\) are in the canonical form
\[
\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1p} \\
A_{21} & A_{22} & \cdots & A_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
A_{p1} & A_{p2} & \cdots & A_{pp}
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\
0 & b_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b_p
\end{bmatrix}
\]
\[
\tilde{A}_{ij} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 1
\end{bmatrix}, \quad \tilde{A}_{ij} \in \mathbb{R}^{n_i \times n_j}, \quad i, j \in \{1, 2, \ldots, p\}, \quad n_1 + \cdots + n_p = n.
\]
It follows that there exists
\[
\tilde{F}(t) = -[\tilde{a}_{11}(t) \tilde{a}_{22}(t) \cdots \tilde{a}_{pp}(t)]^T \in \mathbb{R}^{p \times n},
\]
where \( \tilde{a}_{ij} \in \mathbb{R}^{1 \times n} \) is the \((i \times j)\)th row of \( \tilde{A}(t) \) for \( i \in \{1, \ldots, p\} \), such that \((\tilde{A}(t) + \tilde{B}\tilde{F}(t), \tilde{B})\) is a time-invariant matrix pair. Notice that the initial and terminal constraints (7) and (8) in the new coordinates are
\[
\mathbb{E}[\tilde{x}(0)] = 0, \quad \mathbb{E}[\tilde{x}(0)\tilde{x}(0)^T] = \tilde{\Sigma}_0 = P(0)\Sigma_0P^T(0) > 0, \\
\mathbb{E}[\tilde{x}(1)] = 0, \quad \mathbb{E}[\tilde{x}(1)\tilde{x}(1)^T] = \tilde{\Sigma}_1 = P(1)\Sigma_1P^T(1) > 0.
\]
Clearly, if \( \tilde{\Sigma}(t) > 0 \) for all \( t \in [0, 1] \), then \( \Sigma(t) > 0 \) for all \( t \in [0, 1] \). Therefore, the controllability of the state covariance...
for the time-varying system (6) follows from that of the time-invariant system \(\dot{A}(t) + B\dot{F}(t), B\) \cite{18}. Since \((A(t), B(t))\) is uniformly controllable and index invariant on \([0, 1]\), it follows that \((A(t), B(t))\) is also uniformly controllable and index invariant on any subinterval.

Remark 1. It is shown in Theorem 4 of Section VI that if the pair \((A(t), B(t))\) is totally controllable on \([0, 1]\), there exists a unique optimal control for any \(\Sigma_0, \Sigma_1 > 0\). It follows that if the pair \((A(t), B(t))\) is totally controllable on \([0, 1]\), the state covariance of (6) is controllable on \([0, 1]\) and also on any subinterval of \([0, 1]\).

V. Optimal Control of the State Covariance

In this section, a sufficient condition for optimally steering the covariance is first derived using a “completion of squares” argument \cite{15, 18, 19}. Then, the condition is interpreted from a variational calculus point of view.

A. Sufficient Condition for Optimal Control

Let \(\Pi(t), t \in [0, 1],\) be a differentiable function taking values in the set of \(n \times n\) symmetric matrices. With the pre-specified boundary conditions (7) for \(x(0)\) and (8) for \(x(1)\), and for any given boundary values \(\Pi(0)\) and \(\Pi(1)\), the expected values \(\mathbb{E}[x^T(0)\Pi(0)x(0)]\) and \(\mathbb{E}[x^T(1)\Pi(1)x(1)]\) are fixed and, in particular, are independent of \(u \in \mathcal{U}\). Hence, the cost functional (9) can be equivalently written as

\[
\mathcal{J}(u) = \mathbb{E} \left[ \int_0^1 u^T(t)R(t)u(t)\,dt \right] \\
+ \mathbb{E} \left[ x^T(1)\Pi(1)x(1) \right] - \mathbb{E}[x^T(0)\Pi(0)x(0)].
\] (13)

By applying Itô calculus, we may rewrite (13) as

\[
\mathcal{J}(u) = \mathbb{E} \left[ \int_0^1 u^T(t)R(t)u(t)\,dt + \int_0^1 dt\left( x^T(t)\Pi(t)x(t) \right) \right] \\
= \mathbb{E} \left[ \int_0^1 u^T(t)Ru(t)\,dt + x^T(t)\Pi(t)x(t) \right] \\
+ \mathbb{E} \left[ \int_0^1 dx^T(t)\Pi(t)dx(t) \right] \\
= \mathbb{E} \left[ \int_0^1 u^T(t)Ru(t)\,dt + x^T(t)\Pi(t)x(t) \right] \\
+ \mathbb{E} \left[ \int_0^1 x^T(t)\Pi(t)x(t) \right] \\
+ \mathbb{E} \left[ \int_0^1 dx^T(t)\Pi(t)dx(t) \right] \\
= \mathbb{E} \left[ \int_0^1 u^T(t)Ru(t)\,dt + x^T(t)\Pi(t)x(t) \right] \\
+ \mathbb{E} \left[ \int_0^1 x^T(t)\Pi(t)x(t) \right] \\
+ \mathbb{E} \left[ \int_0^1 dx^T(t)\Pi(t)dx(t) \right].
\] (14)

To this end, choose \(\Pi(t)\) so that it satisfies the matrix Riccati equation

\[
\dot{\Pi} = -A^T(t)\Pi(t) - \Pi A(t) + \Pi B(t)R^{-1}(t)B^T(t)\Pi(t).
\] (15)

It follows that the cost functional in (14) becomes

\[
\mathcal{J}(u) = \mathbb{E} \left[ \int_0^1 \left\| R(t)u(t) + R^{-\frac{1}{2}}(t)B^T(t)\Pi(t)x(t) \right\|^2\,dt \right] \\
+ \int_0^1 \text{trace}\left( \Pi C D C^T \right)\,dt.
\] (16)

Notice now that the second term in (16) is independent of the control \(u\). Therefore, a candidate optimal control takes the form

\[
u^*(t) = -R^{-1}(t)B^T(t)\Pi(t)x(t).
\] (17)

The corresponding candidate optimal process is

\[
dx^* = \left( A(t) - B(t)R^{-1}(t)B^T(t)\Pi(t) \right)x^*\,dt + C(t)\,dm.
\] (18)

Since the initial condition is \(\mathbb{E}[x(0)] = 0\) and system (6) is subject to the state feedback control (17), it follows that \(\mathbb{E}[x(t)] = 0\) for \(t \in [0, 1]\). We still need to make sure the desired terminal state covariance \(\Sigma_1 \) is achieved.

To this end, let \(\Sigma(t) = \mathbb{E}[x^*(t)x^*(t)^T]\) be the covariance matrix of \(x^*\). Following a similar derivation as for (11), we obtain

\[
\dot{\Sigma} = \left( A(t) - B(t)R^{-1}(t)B^T(t)\Pi(t) \right)\Sigma + \Sigma \left( A(t) - B(t)R^{-1}(t)B^T(t)\Pi(t) \right)^T,
\] (19)

with boundary conditions

\[
\Sigma(0) = \Sigma_0 > 0, \quad \Sigma(1) = \Sigma_1 > 0.
\] (20)

We have thus arrived at the following result.

Theorem 2. Assume \(\Pi(t)\) and \(\Sigma(t)\) satisfy equations (15), (19), (20) for \(t \in [0, 1]\). Then, the state feedback control \(u^*\) given by (17) is optimal with respect to the cost functional (9), subject to the boundary constraints (7), (8). The corresponding optimal process is given by (18).

B. A Variational Calculus Perspective

In this subsection, we show that (15), (17), and (19) can also be obtained using a variational calculus argument. To this end, suppose we employ the state feedback control \(u(t) = K(t)x(t)\) for some \(K(t)\). Then, the state covariance \(\Sigma\) satisfies (11) with initial and final conditions given by (20). Let us treat \(\Sigma\) as the new state. Correspondingly, the new cost functional is

\[
J(K) = \mathbb{E} \left[ \int_0^1 x^T(t)K^T(t)R(t)K(t)x(t)\,dt \right] \\
\triangleq \int_0^1 \text{trace}\left( K^T(t)R(t)K(t)\Sigma(t) \right)\,dt.
\] (21)

Let

\[
\zeta(\Sigma, K) \triangleq \left( A(t) + B(t)K(t) \right)\Sigma + C(t)D(t)C^T(t) \\
+ \Sigma \left( A(t) + B(t)K(t) \right)^T.
\]
Hence, the state covariance dynamics take the form
\[ \dot{\Sigma} - \zeta(\Sigma, K) = 0, \]  
where \( K \) is the “control”. Let \( \Pi \) be the Lagrange multiplier corresponding to the dynamic constraint (22). It follows that the cost functional (21) can be written, equivalently, as
\[ J(K) = \int_0^1 \text{trace} \left( K^\top(t) R(t) K(t) \Sigma(t) \right) dt - \int_0^1 \text{trace} \left( \Pi(t) \left( \dot{\Sigma} - \zeta(\Sigma, K) \right) \right) dt. \]

Hence, the Hamiltonian is
\[ H(\Pi, \Sigma, K) = \text{trace} \left( K^\top(t) R(t) K(t) \Sigma(t) + \Pi(t) \zeta(\Sigma, K) \right). \]

Since the Hamiltonian is convex in \( K \), it follows that the unique value of \( K \) that minimizes \( H \) is given by
\[ \frac{\partial H(\Pi, \Sigma, K)}{\partial K} = 0 = 2 R(t) K(t) \Sigma(t) + 2 B^T(t) \Pi(t) \Sigma(t). \]

Since \( \Sigma(t) \succ 0 \) for all \( t \in [0,1] \) [18], the optimal feedback gain matrix is
\[ K^*(t) = - R^{-1}(t) B^T(t) \Pi(t), \quad (23) \]
which is the same as in (17). Furthermore, the differential equation for \( \Pi \) is
\[ \dot{\Pi} = \frac{\partial H(\Pi, \Sigma, K^*)}{\partial \Sigma} = -K^*(t) R(t) K^*(t) - \left( A(t) + B(t) K^*(t) \right)^\top \Pi \left( A(t) + B(t) K^*(t) \right) + A^\top(t) \Pi - \Pi A(t) + \Pi B(t) R^{-1}(t) B^T(t) \Pi, \]
which is identical to (15). Plugging (23) into (11) yields (19).

**Remark 2.** Similarly to [18], the infinite-horizon problem can be formulated as a problem seeking to determine an optimal control \( u^* \) that minimizes the power \( E[|u(t)|^2] \) for some \( R \succ 0 \) over the class of constant gain state feedback controls \( u(t) = K x(t) \), such that the linear time-invariant stochastic system
\[ dx(t) = (A + BK) x(t) dt + C dm(t), \]
with constant \( D = \text{E}[|m(t)|^2] / dt \succeq 0 \) admits a stationary state covariance \( \Sigma \succ 0 \). It is not difficult to show that a sufficient condition for optimality is \( u^*(t) = -R^{-1} B^T \Pi(t) \), provided there exists a symmetric matrix \( \Pi \) such that \( A - R B^{-1} B^T \Pi \) is Hurwitz and that
\[ (A - R B^{-1} B^T \Pi) \Sigma + \Sigma (A - R B^{-1} B^T \Pi)^\top + CDC^\top = 0. \]

**VI. SOLUTION TO THE COUPLED ODEs**

In this section, we show the existence and uniqueness of the solution to the coupled ODEs (15), (19), (20). It is assumed throughout this section that the matrix pair \((A(t), B(t))\) is totally controllable on \([0,1]\). First, it is straightforward to check that, for \( R(t) \succ 0 \), if \((A(t), B(t))\) is also totally controllable on \([0,1]\), then \((A(t), B(t) R^{-1/2}(t))\) is also totally controllable on \([0,1]\). As \( R(t) \) can always be recovered by a time-varying coordinate input transformation, for simplicity, we assume \( R(t) = I_p \) for the rest of this section. To begin with, a complete solution of \( \Pi(t) \) to (15) is presented. Then, the state transition matrix for \((A(t), B(t) R(t))\) is given. Putting them together, we obtain the solution for \( \Sigma(t) \). Next, we define a map from \( \Pi(0) \) to \( \Sigma(1) \), and show that the map is a bijection, which implies the existence and uniqueness of the solution to the coupled ODEs.

**A. Existence of Solution to Riccati Differential Equation**

Let \( \Phi_A(t,s) \) denote the state transition matrix of \( A(t) \), which satisfies
\[ \frac{\partial}{\partial t} \Phi_A(t,s) = A(t) \Phi_A(t,s), \quad \Phi_A(s,s) = I_n. \quad (24) \]

Let the matrix \( N(t,s) \) as in (12). Since \((A(t), B(t))\) is totally controllable on \([0,1]\), \( N(t,s) \succeq 0 \) for all \( 0 \leq s < t \leq 1 \).

**Lemma 1.** Given \( \Pi(s) \) for some \( s \in [0,1] \), (15) admits a unique solution \( \Pi(t) \) on \([0,1]\) if and only if
\[ N(0,s)^{-1} \prec \Pi(s) \prec N(1,s)^{-1}, \quad (25) \]
where \( N(0,0)^{-1} = -\infty \) and \( N(1,1)^{-1} = +\infty \). Moreover,
\[ \Pi(t) = \Phi_A(s,t)^\top \Pi(s) \left( I_n - N(t,s) \Pi(s) \right)^{-1} \Phi_A(s,t), \quad (26) \]
and
\[ N(0,t)^{-1} \prec \Pi(t) \prec N(1,t)^{-1}, \quad t \in [0,1]. \quad (27) \]

**Proof.** It is known that (15) admits a unique solution \( \Pi(t) \) on \([0,1]\) if and only if \( \Phi_A(t,s) - \Phi_A(t,s) N(t,s) \Pi(s) \Pi(s) \) is invertible for all \( t \in [0,1] \), and the solution \( \Pi(t) \) is given by (26) [51]. Since \( \Phi_A(t,s) \) is invertible, it suffices to show that \( I_n - N(t,s) \Pi(s) \Pi(s) \) is invertible for all \( t \in [0,1] \). When \( t = s \), we have \( I_n - N(t,s) \Pi(s) \Pi(s) = I_n \), which is invertible. When \( t \in (s,1] \), and since \( 0 \prec N(t,s) \preceq N(1,s) \), it follows from (25) that \( N(t,s)^{-1} \preceq N(1,s)^{-1} \Pi(s) \). Hence, by multiplying \( N(t,s)^{1/2} \) on both sides of the above equation, we obtain \( I_n \succeq N(t,s)^{-1} \Pi(s) N(t,s)^{1/2} \). Since \( N(t,s)^{1/2} \Pi(s) N(t,s)^{-1/2} \) is similar to \( N(t,s) \Pi(s) \), all the eigenvalues of \( N(t,s) \Pi(s) \) are strictly less than 1. Therefore, \( I_n - N(t,s) \Pi(s) \) is invertible for all \( t \in (s,1] \). The case when \( t \in [0,s] \) can be shown in an analogous way, and thus is omitted here. We have shown that (25) is necessary and sufficient for (15) to admit a unique solution \( \Pi(t) \) on \([0,1]\) and the unique solution \( \Pi(t) \) is given by (26).

\(^4\)Positive infinity of the \( n \times n \) positive semidefinite cone, written \(+\infty\), is the limit of a sequence of \( n \times n \) positive definite matrices whose eigenvalues all grow to \(+\infty\). Likewise, for \(-\infty\). Notice that as \( s \to 0^+ \), all eigenvalues of \( N(0,s)^{-1} \) go to \(-\infty\), and thus all eigenvalues of \( N(0,s)^{-1} \) go to \(+\infty\). Likewise, as \( s \to 1^- \), all eigenvalues of \( N(1,s)^{-1} \) go to \(+\infty \), and thus all eigenvalues of \( N(1,s)^{-1} \) go to \(+\infty\).
Next, we show (27). Notice that, for any fixed $r$, $N(r,t)^{-1}$ satisfies the same Riccati differential equation as $\Pi(t)$, that is,

$$\frac{\partial}{\partial t} N(r,t)^{-1} = -A^T(t)N(r,t)^{-1} - N(r,t)^{-1} A(t) + N(r,t)^{-1} B(t)B^T(t)N(r,t)^{-1}. $$

By the monotonicity of the matrix Riccati differential equation [52], it follows that $N(0,s)^{-1} < \Pi(s) < N(1,s)^{-1}$ for some $s \in [0,1]$ implies $N(0,t)^{-1} < \Pi(t) < N(1,t)^{-1}$ for all $t \in [0,1]$.

**Corollary 1.** Assume that $(A(t), B(t))$ is totally controllable for $t \in \mathbb{R}$. Let $\mathcal{I}_s \subset \mathbb{R}$ be the maximal interval of existence of the solution to (15), starting from $\Pi(s) = \Pi_s$. Then, $\mathcal{I}_s = (t_0, t_1)$, where

$$t_0 \triangleq \inf \{ t \mid t < s, \ N(t)^{-1} < \Pi_s \} ,$$

$$t_1 \triangleq \sup \{ t \mid t > s, \ N(t)^{-1} > \Pi_s \} .$$

**B. Solution to the State Covariance**

Armed with the complete solution of $\Pi(t)$, we obtain an explicit expression for the state transition matrix $\Phi_{A-BB^T}(t,s)$, which satisfies $\Phi_{A-BB^T}(t,s) = I_n$, and

$$\frac{\partial}{\partial t} \Phi_{A-BB^T}(t,s) = \left( A(t) - B(t)B^T(t) \Pi(t) \right) \Phi_{A-BB^T}(t,s).$$

**Lemma 2.** Let condition (25) hold, so that $\Pi(t)$ exists on $[0,1]$. The state transition matrix of $A(t) - B(t)B^T(t)\Pi(t)$ is given by

$$\Phi_{A-BB^T}(t,s) = \Phi_A(t,s) - \Phi_A(t,s)N(t,s)\Pi(s),$$

for $s,t \in [0,1]$.

**Proof.** One can readily check that

$$\Phi_A(s,s) - \Phi_A(s,s)N(s,s)\Pi(s) = I_n.$$

Since,

$$\frac{\partial}{\partial t} N(t,s) = \Phi_A(s,t)B(t)B^T(t)\Phi_A(s,t)^T,$$

in view of (26), we obtain that

$$\frac{\partial}{\partial t} \left( \Phi_A(t,s) - \Phi_A(t,s)N(t,s)\Pi(s) \right) = A(t)(\Phi_A(t,s) - \Phi_A(t,s)N(t,s)\Pi(s))$$

$$- B(t)B^T(t)\Phi_A(s,t)^T\Pi(s)$$

$$= \left( A(t) - B(t)B^T(t)\Pi(t) \right) \Phi_A(t,s) - \Phi_A(t,s)N(t,s)\Pi(s).$$

This completes the proof.

**Remark 3.** If (25) holds, in view of (26) and (28), we have

$$\Pi(t) = \Phi_A(s,t)^T\Pi(s)\Phi_{A-BB^T}(s,t)$$

$$= \Phi_{A-BB^T}(s,t)^T\Pi(s)\Phi_A(s,t), \quad s,t \in [0,1].$$

The next result provides an explicit solution of $\Sigma(t)$ over the interval $[0,1]$.

**Proposition 1.** Let condition (25) hold so that $\Pi(t)$ exists on $[0,1]$. Given $\Sigma(0) = \Sigma_0 \geq 0$, then, for $t \in [0,1]$:

$$\Sigma(t) = \Phi_{A-BB^T}(t,0)\Sigma_0\Phi_{A-BB^T}(t,0)^T$$

$$+ \int_0^t \Phi_{A-BB^T}(t,s)C(s)D(s)C^T(s)\Phi_{A-BB^T}(t,s)^T ds,$$

where $\Phi_{A-BB^T}(t,s)$ is given by (28) and $\Pi(t)$ is given by (26). In particular, for any $s \in [0,1]$, as $\Pi(s) \to N(1,s)^{-1}$, then $\Sigma(1) \to 0_{n \times n}$. When $\Sigma_0 > 0$, as $\Pi(s) \to N(0,s)^{-1}$, then $\Sigma(1) \to +\infty$.

**Proof.** It is not difficult to check that (29) follows directly from (19). By the continuity of $\Pi(t)$ in $\Pi(s)$ for any given $s \in [0,1]$, it follows that as $\Pi(s) \to N(1,s)^{-1}$, then $\Pi(t) \to N(1,t)^{-1}$ for each $t \in [0,1]$. By the dominated convergence theorem, it follows that, as $\Pi(t) \to N(1,t)^{-1}$ for all $t \in [0,1]$, then $\Sigma(1) \to 0_{n \times n}$. Similarly, as $\Pi(s) \to N(0,s)^{-1}$, then $\Pi(t) \to N(0,t)^{-1}$ for each $t \in [0,1]$. When $\Sigma_0 > 0$, as $\Pi(0) \to N(0,0)^{-1} = -\infty$, then $\Sigma(1) \to +\infty$.

For the special case when $\Sigma_0 > 0$ and $C(t)D(t)C^T(t) = B(t)B^T(t)$ for all $t \in [0,1]$, we have that

$$\int_0^1 \Phi_{A-BB^T}(1,\tau)B(\tau)B^T(\tau)\Phi_{A-BB^T}(1,\tau)^T d\tau$$

$$= \int_0^1 \left( \Phi_A(1,\tau) - \Phi_A(1,\tau)N(1,\tau)\Pi(\tau) \right)B(\tau)B^T(\tau)$$

$$\times \left( \Phi_A(1,\tau)^T - \Pi(\tau)N(1,\tau)\Phi_A(1,\tau)^T \right) d\tau$$

$$= -N(0,1)+\Phi_A(1,t)N(1,t)\Pi(t)N(1,t)\Phi_A(1,t)^T \int_0^1$$

$$= -N(0,1)-\Phi_A(1,0)N(1,0)\Pi(0)N(1,0)\Phi_A(1,0)^T. $$

For notational simplicity, in the sequel let $N(1,0)$ be denoted by $N_{10}$, and let $\Pi(0)$ be denoted by $\Pi_0$. Then,

$$\Phi_A(0,1)\Sigma(1)\Phi_A(0,1)^T = N_{10}\left( N_{10}^{-1} - \Pi_0 \right)N_{10}$$

$$+ N_{10}\left( N_{10}^{-1} - \Pi_0 \right)\Sigma_0\left( N_{10}^{-1} - \Pi_0 \right)N_{10}$$

$$= N_{10}\Sigma_0^{-\frac{1}{2}}\left( \Sigma_0^\frac{1}{2}\left( N_{10}^{-1} - \Pi_0 \right)\Sigma_0^\frac{1}{2} \right)\Sigma_0^{-\frac{1}{2}}N_{10}$$

$$+ \Sigma_0^\frac{1}{2}\left( N_{10}^{-1} - \Pi_0 \right)\Sigma_0^\frac{1}{2}N_{10}$$

$$= N_{10}\Sigma_0^{-\frac{1}{2}}\left( \left[ \Sigma_0^\frac{1}{2}\left( N_{10}^{-1} - \Pi_0 \right)\Sigma_0^\frac{1}{2} \right]^2 - \frac{I}{2} \right) - \frac{I}{4} \right)\Sigma_0^{-\frac{1}{2}}N_{10}.$$
This is the same solution reported in [15].

Since $\Phi_{A-BB^T}\Pi(1, s) = \Phi_{A-BB^T}\Pi(1, 0)\Phi_{A-BB^T}(0, s)$, Proposition 1 provides an explicit map from $\Pi(0)$ to $\Sigma(1)$. Specifically, define $f : \{\Pi_0 \in \mathbb{R}^{n \times n} | \Pi_0 = \Pi_0^T \prec N_1^{-1}\} \to \{\Sigma_1 \in \mathbb{R}^{n \times n} | \Sigma_1 = \Sigma_1^T > 0\}$ such that

$$f(\Pi_0) = \Phi_{A10}(I - N_0 \Pi_0)\left[\Sigma_0 + \int_0^1 (I - N_0 \Pi_0)^{-1} \Phi_{A0s} C_s D_s C_s^T \Phi_{A0s} (I - \Pi_0 N_0)^{-1} ds\right] \times \left(I - \Pi_0 N_0\right) \Phi_{A10},$$

(30)

where $\Phi_{Ats} \triangleq \Phi_A(t, s), N_{ts} \triangleq N(t, s), C_s \triangleq C(s)$, and $D_s \triangleq D(s)$. Proposition 1 leads naturally to another sufficient condition for optimality stated below.

**Theorem 3.** If there exists an $n \times n$ symmetric matrix $\Pi_0 \prec N_1^{-1}$ such that the desired terminal state covariance $\Sigma_1$ can be written as $\Sigma_1 = f(\Pi_0)$, then, the optimal control is

$$u^*(t) = -B^T(t)\Pi(t)x(t),$$

where $\Pi(t)$ is the unique solution to (15) with $\Pi(0) = \Pi_0$.

C. Existence and Uniqueness of the Solution

We can compute the Jacobian of the map $f$ defined by (30). To this end, let us denote the integrand in (30) by

$$P_s \triangleq (I - N_0 \Pi_0)^{-1} \Phi_{A0s} C_s D_s C_s^T \Phi_{A0s} (I - \Pi_0 N_0)^{-1} \geq 0.$$

Let $\Pi_0$ denote a small increment of $\Pi_0$. Then, it follows from [53] that

$$\left(I - N_0 \Pi_0 - N_0 \Pi_0^T\right)^{-1} = \left(I - N_0 \Pi_0\right)^{-1} + \left(I - N_0 \Pi_0\right)^{-1} N_0 \Pi_0 \left(I - N_0 \Pi_0\right)^{-1} + O(\Pi_0^3).$$

Hence, by collecting all the first order terms of $\Pi_0$, we have

$$f(\Pi_0 + \Pi_0) - f(\Pi_0) = O(\Pi_0^3)$$

$$- \Phi_{A10} N_0 \Pi_0 \left[\Sigma_0 + \int_0^1 P_s ds\right] \left(I - \Pi_0 N_0\right) \Phi_{A10}$$

$$- \Phi_{A10} \left(I - N_0 \Pi_0\right) \left[\Sigma_0 + \int_0^1 P_s ds\right] \Pi_0 N_0 \Phi_{A10}$$

$$+ \Phi_{A10} \left(I - N_0 \Pi_0\right) \left[I - N_0 \Pi_0\right]^{-1} N_0 \Pi_0 P_s$$

$$+ P_s N_0 \Pi_0 \left(I - \Pi_0 N_0\right)^{-1} \left(I - \Pi_0 N_0\right) \Phi_{A10}.$$

For notational simplicity, let

$$\Phi_{II0} \triangleq \Phi_{A-BB^T}\Pi(1, 0) = \Phi_{A10} \left(I - N_0 \Pi_0\right),$$

and

$$T_{s0} \triangleq \left\{\begin{array}{ll}
N_0^{-1} - \Pi_0^{-1} > 0, & s \in (0, 1],
0_{n \times n}, & s = 0.
\end{array}\right.$$ 

Then, we can write

$$f(\Pi_0 + \Pi_0) - f(\Pi_0) = O(\Pi_0^3)$$

$$- \Phi_{II0} \left[T_{10} \Pi_0 \Sigma_0 + \Sigma_0 \Pi_0 T_{10}\right]$$

$$+ \int_0^1 \left(T_{10} - T_{s0}\right) \Pi_0 P_s + P_s \Pi_0 \left(T_{10} - T_{s0}\right) ds \left[I - \Pi_0 N_0\right] \Phi_{A10}.$$

In the following, the vectorization $\text{vec}(H)$ of an $n \times n$ matrix $H = [h_{ij}]$ is

$$\text{vec}(H) \triangleq [h_{11} \ldots h_{n1} h_{12} \ldots h_{n2} \ldots h_{1n} \ldots h_{nn}]^T,$$

and $\otimes$ denotes the Kronecker product. Vectorizing both sides of (31) yields

$$\text{vec}\left(f(\Pi_0 + \Pi_0) - f(\Pi_0)\right) = J_f(\Pi_0) \text{vec}(\Pi_0) + O(\|\Pi_0\|^2),$$

where,

$$J_f(\Pi_0) = -\Phi_{II10} \otimes \Phi_{II10} \left[\Sigma_0 \otimes T_{10} + T_{10} \otimes \Sigma_0\right]$$

$$+ \int_0^1 P_s \otimes \left(T_{10} - T_{s0}\right) + \left(T_{10} - T_{s0}\right) \otimes P_s ds.$$

(32)

It follows that $J_f(\Pi_0)$ is the Jacobian of the map $f$ at $\Pi_0$.

**Lemma 3.** For any given $\Sigma_0 \succ 0$, the map $f$ defined by (30) is a homeomorphism. Thus, for any $\Sigma_1 \succ 0$, there exists a unique $\Pi_0 \prec N_1^{-1}$ such that $\Sigma_1 = f(\Pi_0)$.

**Proof.** Since $T_{s0}$ is continuous in $\Pi_0$ and $s \in [0, 1]$, $J_f(\Pi_0)$ is continuous in $\Pi_0$. First, we show that $J_f(\Pi_0)$ is nonsingular at each $\Pi_0 \prec N_1^{-1}$. Since $\Phi_{II10}$ is nonsingular, $\Phi_{II10} \otimes \Phi_{II10}$ is nonsingular as well. It suffices to show that the term in the square brackets of (32), that is,

$$S \triangleq \Sigma_0 \otimes T_{10} + T_{10} \otimes \Sigma_0$$

$$+ \int_0^1 P_s \otimes \left(T_{10} - T_{s0}\right) + \left(T_{10} - T_{s0}\right) \otimes P_s ds,$$

is nonsingular. Notice that $S$ is symmetric, since $\Sigma_0, T_{10} \succ 0$ and $P_s, T_{10} - T_{s0} \geq 0$ are all symmetric. Let $X \neq 0$ be an $n \times n$ matrix, not necessarily symmetric. Then,

$$\text{vec}(X)^T S \text{vec}(X) = \text{vec}(X)^T \text{vec} \left(T_{10} X \Sigma_0 + \Sigma_0 X T_{10}\right)$$

$$+ \int_0^1 \left(T_{10} - T_{s0}\right) X P_s + P_s X \left(T_{10} - T_{s0}\right) ds$$

$$= \text{trace} \left(X^T T_{10} \Sigma_0 + \Sigma_0 X T_{10}\right)$$

$$+ \int_0^1 X^T \left(T_{10} - T_{s0}\right) X P_s + P_s X \left(T_{10} - T_{s0}\right) ds$$

$$\geq \text{trace} \left(T_{10}^2 \Sigma_0 X^T T_{10}^2 + T_{10}^2 X^T \Sigma_0 X T_{10}^2\right) > 0.$$

Thus, $S \succ 0$, which implies that $J_f(\Pi_0)$ is nonsingular at each $\Pi_0 \prec N_1^{-1}$. 

Next, we show that the map $f$ is proper, that is, for any compact subset $K \subset \{ \Sigma_1 \in \mathbb{R}^{n \times n} | \Sigma_1 = \Sigma_1' > 0 \}$, the inverse image $f^{-1}(K) \subset \{ \Pi_0 \in \mathbb{R}^{n \times n} | \Pi_0 = \Pi_0' < N_{10}^{-1} \}$ is compact. Since $f$ is continuous, the inverse image of a closed set is closed. Since $K$ is bounded, in view of (30), the set
\[
\left\{ \Phi_{A_10} \left( I - N_{10} \Pi_0 \right) \Sigma_0 \left( I - \Pi_0 N_{10} \right) \Phi_{A_10} \right| \Pi_0 \in f^{-1}(K) \right\}
\]
is bounded. It follows that $f^{-1}(K)$ is bounded. Hence, $f^{-1}(K)$ is compact, and $f$ is proper.

Since the set of positive definite matrices is convex, it is simply connected. By Hadamard’s global inverse function theorem, $f$ is a homeomorphism, thus a bijection.

A direct consequence of Lemma 3 and Theorem 3 is summarized below.

**Theorem 4.** For any given $\Sigma_0, \Sigma_1 > 0$, the unique optimal control solving the covariance steering problem is given by (17), where $\Pi(t)$ is the unique solution to (15), (19), (20).

**D. One-Dimensional Case**

When $n > 1$, the map $f$ from $\Pi(0)$ to $\Sigma(1)$, defined by (30), is not monotone in the Loewner order. However, when $n = 1$, $f$ is monotone decreasing. In this subsection, we exploit the monotonicity of $f$ when $n = 1$, and also explain how to optimally steer the state covariance when the initial covariance is zero, that is, the initial state is deterministic. Without loss of generality, assume $p = q = 1$. With $n = 1$, the property of total controllability reduces to the property that, for all $0 \leq t_0 < t_1 \leq 1$, there exists $t \in (t_0, t_1)$ such that $b(t) \neq 0$, which is assumed throughout this section.

When $n = 1$, the coupled ODEs (15), (19), (20) reduce to
\[
\dot{\pi} = -2a(t)\pi + r^{-1}(t)b^2(t)\pi^2,
\]
\[
\dot{\sigma} = 2 \left( a(t) - r^{-1}(t)b^2(t)\pi(t) \right) \sigma + d(t)c^2(t),
\]
\[
\sigma(0) = \sigma_0, \quad \sigma(1) = \sigma_1.
\]

Let $\pi(t; \sigma_0)$ and $\sigma(t; \sigma_0)$ respectively denote the solution to (33) and (34) starting from $\pi(0) = \sigma_0 < N(1, 0)^{-1}$ and $\sigma(0) = \sigma_0 \geq 0$. Since for any fixed $t \in [0, 1]$, $\pi(t; \sigma_0)$ is continuous and monotonically increasing in $\sigma_0$ [52], it follows that for any given $\sigma_0 \geq 0$ and any fixed $t \in [0, 1]$, $\sigma(t; \sigma_0)$ is continuous and monotonically decreasing in $\sigma_0$. In particular, $\sigma(1; \pi_0)$ is continuous and monotonically decreasing in $\pi_0$. In view of Proposition 1, as $\pi_0 \to -\infty$, $\sigma(1; \pi_0)$ $\to +\infty$, and as $\pi_0 \to N(1, 0)^{-1}$, $\sigma(1; \pi_0) \to 0^+$. This observation suggests that when $\sigma(0) = 0$, there may also exist a unique solution to the ODEs (33), (34), (35). The following lemma makes this statement more precise.

**Lemma 4.** Let $\sigma_0 = 0$. For any given $\sigma_1 \in (0, \eta)$, where
\[
\eta \triangleq \int_0^1 \left( \int_0^1 e^{\int_1^t 2a(s)ds}b^2(s)ds \right)^2 e^{-\int_1^t 2a(s)ds}d(t)c^2(t)dt,
\]
there exists a unique solution to (33), (34), (35). As $\pi(0) \to -\infty$, $\sigma(1) \to \eta^-$. Moreover, if $c(0)d(0) \neq 0$, $\eta = +\infty$.

**Proof.** By continuity and monotonicity of $\pi(t; \sigma_0)$ in $\sigma_0$ for all $t \in [0, 1]$, it follows that as $\sigma_0 \to -\infty$, $\pi(t; \sigma_0) \to N(0, t)^{-1}$ monotonically. By (29) and the monotone convergence theorem, it follows that
\[
\lim_{\sigma_0 \to -\infty} \sigma(1; \sigma_0) = \eta^-,
\]
for $\sigma_0 = 0$.

Let now $c(0)d(0) \neq 0$. By continuity, $d(t)c^2(t)$ is bounded below from zero on some interval $[0, \varepsilon]$ for $\varepsilon \in (0, 1]$. Let $Z > 0$ satisfy
\[
e^{-\int_1^t 2a(s)ds}d(t)c^2(t) \geq Z > 0, \quad \forall t \in [0, \varepsilon],
\]
and let
\[
\rho \triangleq \int_0^1 e^{\int_1^t 2a(s)ds}b^2(s)ds \geq 0.
\]
Let $G \in \mathbb{R}$ and $\Omega > 0$ satisfy
\[
2a(t) \leq G, \quad r^{-1}(t)b^2(t) \leq \Omega, \quad \forall t \in [0, \varepsilon].
\]
Without loss of generality, assume $\varepsilon$ is sufficiently small, such that, for all $t \in [0, \varepsilon]$, $e^{Gt} - 1 = Gt + O(t^2)$. Then, for all $t \in [0, \varepsilon]$,
\[
\int_0^t e^{\int_1^x 2a(s)ds}b^2(s)ds \leq \int_0^t e^{G(t-x)}\Omega dx
\]
\[
= \frac{\Omega (e^{Gt} - 1)}{G} \approx \Omega t.
\]
Therefore,
\[
\eta \geq \int_0^\varepsilon \left( \frac{\rho}{\Omega t} \right)^2 Z dt = +\infty.
\]
This completes the proof.

An immediate result of Lemma 4 and Theorem 2 is summarized below.

**Theorem 5.** Assume $c(0)d(0) \neq 0$. Then, for any $\sigma_0 \geq 0$, $\sigma_1 > 0$, the unique optimal control solving the covariance steering problem is given by $u^*(t) = -r^{-1}(t)b(t)\pi(t)x(t)$, where $\pi(t)$ is the unique solution to (33), (34), (35).

**VII. Computation of the Optimal Control**

In this section, we summarize the numerical methods of computing the optimal control for the covariance steering problem, and provide two numerical examples to demonstrate the results.

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5Loewner order is a partial order on the set of $n \times n$ symmetric matrices in that, for two such matrices $M_1$ and $M_2$, $M_1 \geq M_2$ if $M_1 - M_2$ is positive semidefinite, and $M_1 > M_2$ if $M_1 - M_2$.

6If $c(0)d(0) = 0$, the value of $\eta$ depends on the asymptotic behavior of $d(t)c^2(t)$ as $t \to 0^+$. 
A. Numerical Methods

In this subsection, we briefly discuss how to compute the solution to the coupled ODEs (15), (19), (20), and to the optimal covariance steering problem.

In the special case when
\[ C(t)D(t)C^T(t) = \kappa B(t)R^{-1}(t)B^T(t), \quad t \in [0, 1], \]
for some \( \kappa > 0 \), given any boundary conditions \( \Sigma_0, \Sigma_1 > 0 \), a closed-form expression of \( \Pi(0) \) is provided in [15], [19].

In general, it is not straightforward to get an analytic expression of the inverse map \( f \) from \( \Sigma(1) \) to \( \Pi(0) \). Instead, we have to resort to numerical methods. When \( n = 1 \), we can find \( \pi(0) \) based on the monotonicity of \( \sigma(1) \) in \( \pi(0) \) using, for example, bisection. When \( C(t)D(t)C^T(t) \) is close to a scalar multiple of \( B(t)R^{-1}(t)B^T(t) \) in the sense that
\[
C(t)D(t)C^T(t) - \kappa B(t)R^{-1}(t)B^T(t) = O(\varepsilon), \quad t \in [0, 1],
\]
for some \( \kappa > 0 \) and small \( \varepsilon \), we can adopt perturbation methods to find an approximate solution to the coupled ODEs. Since we know the map \( f \) from \( \Pi(0) \) to \( \Sigma(1) \), given by (30), and its Jacobian in (32), we can use root-finding algorithms to find \( \Pi(0) \) for a given \( \Sigma(1) \). Lastly, as explained in [18], the general optimal covariance steering problem can always be solved numerically by recasting it as a semidefinite program.

B. Numerical Examples

We illustrate the results of the theory using two examples. The first example is a simple problem of controlled population growth. Assume the population is susceptible to environmental uncertainties such as epidemics. Let \( x \geq 0 \) be the population under control. The effects of epidemics on the population can be modeled by the negative of a nonhomogeneous Poisson process \( h(t) \) with arrival rate \( \lambda(t) \) [55]. The controlled population subject to environmental uncertainties is modeled by the linear stochastic differential equation
\[
dx(t) = a(t)x(t) \, dt + u(t) \, dt - 4 \, dh(t) + 2 \, dw(t), \quad t \in [0, 1].
\]
The equation can be written, equivalently, as
\[
dx(t) = a(t)x(t) \, dt - 4\lambda(t) \, dt + u(t) \, dt + dm(t), \quad t \in [0, 1],
\]
where \( dm(t) = -4 (dh(t) - \lambda(t) \, dt) + 2 \, dw(t) \). Assume that \( a(t) = 0.8 - 0.1 \, t \) and \( \lambda(t) = 2 + t \). The goal is to control the population over the time interval \([0, 1]\) from an initial distribution with mean \( \mu_0 = 50 \) and variance \( \sigma_0 = 6 \) to a target distribution with mean \( \mu_1 = 60 \) and variance \( \sigma_1 = 2 \) using the least control energy \( \mathbb{E} \left[ \int_0^1 u^2(t) \, dt \right] \). The optimal control \( u^* \) is given by
\[
u(t) = \phi(1, t) \left( \int_0^1 \phi(1, \tau)^2 \, d\tau \right)^{-1} (\mu_1 - \phi(1, 0) \mu_0),
\]
\[
\dot{\phi}(1, t) = (\pi(t) - a(t)) \phi(1, t), \quad \phi(1, 1) = 1.
\]
Assuming \( x(0) \sim \mathcal{N}(\mu_0, \sigma_0) \), ten sample paths of \( x(t) \) using the optimal control \( u^* \) are plotted in Figure 1. The transparent blue region is the “three standard deviation” interval between \( \mathbb{E} [x(t)] - 3\sqrt{\sigma(t)} \) and \( \mathbb{E} [x(t)] + 3\sqrt{\sigma(t)} \), \( t \in [0, 1] \).

In the second example, we consider a drone flying in heavy rain, which is modeled by a kinematic point subject to Poisson and Gaussian noise [56]. We only care about its motion in the vertical direction. Let \( x(t) \) and \( v(t) \) denote the vertical position and velocity, respectively. The effect of rain on the drone is modeled by a nonhomogeneous compound Poisson process \( h(t) \). The vertical dynamics of the drone is therefore given by
\[
dx(t) = v(t) \, dt,
\]
\[
dv(t) = u(t) \, dt + 0.2 \, dw(t) + dh(t),
\]
where \( h(t) \) has arrival rate \( \lambda(t) = 5 - t \) and i.i.d. jump size \( \chi \sim \mathcal{N}(−0.5, 0.1^2) \). Our goal is to control the covariance of the drone hovering at a certain height from an initial covariance of \( \Sigma_0 = \text{diag}[0.6, 0.6] \) to a target covariance of \( \Sigma_1 = \text{diag}[0.2, 0.1] \) with the least effort \( \mathbb{E} \left[ \int_0^1 u^2(t) \, dt \right] \).
Assume \( [x(0) \, v(0)]^T \sim \mathcal{N}(0, 0, \Sigma_0) \). Figure 2 illustrates ten controlled sample paths in the phase space as a function of time. The “three standard deviation” tolerance interval for \( t \in [0, 1] \) is depicted as the transparent blue tube.
REFERENCES

[1] A. F. Hotz and R. E. Skelton, “A covariance control theory,” in Proc. IEEE Conf. Decision Control, Lauderdaile, FL, 1985, pp. 552–557.
[2] E. Collins and R. Skelton, “Covariance control of discrete systems,” in Proc. IEEE Conf. Decision Control, Lauderdaile, FL, 1985, pp. 542–547.
[3] J. Ridderhof and P. Tsiotras, “Uncertainty quantification and control during Mars powered descent and landing using covariance steering,” in AIAA Guidance, Navigation, Control Conf., Kissimmee, FL, 2018.
[4] J. Ridderhof, J. Pilipovsky, and P. Tsiotras, “Chance-constrained covariance control for low-thrust minimum-fuel trajectory optimization,” in AAAIA/ AIAA Astrodynamics Specialist Conf., South Lake Tahoe, CA, 2020.
[5] K. Okamoto and P. Tsiotras, “Optimal stochastic vehicle path planning using covariance steering,” IEEE Robot. Autom. Lett., vol. 4, no. 3, pp. 2276–2281, 2019.
[6] K. Okamoto and P. Tsiotras, “Stochastic model predictive control for constrained linear systems using optimal covariance steering,” arXiv:1905.12396, 2019.
[7] D. Zheng, J. Ridderhof, P. Tsiotras, and A.-a. Agha-mohammadi, “Belief space planning: A covariance steering approach,” arXiv:2105.11092, 2021.
[8] J. Yin, Z. Zhang, E. Theodorou, and P. Tsiotras, “Trajectory distribution control for model predictive path integral control using covariance steering,” arXiv:2109.12147, 2022.
[9] A. D. Saravanos, A. G. Tsolovikos, E. Bakolas, and E. A. Theodorou, “Distributed covariance steering with consensus ADMM for stochastic multi-agent systems,” in Robot.: Sci. Syst., 2021.
[10] Y. Chen, T. T. Georgiou, and M. Pavon, “Optimal transport in systems subject to additive generic noise, which can be regarded as a combination of continuous white Gaussian noise and abrupt noise of any jump size. The results extend the applications of covariance steering to a larger class of problems of interest to engineering applications. Interesting potential extensions include the development of efficient algorithms for solving coupled ODEs (15), (19), (20), optimal steering of the state covariance with partially fixed terminal covariance and/or free final time with generic noise, and covariance steering with output feedback subject to non-Gaussian noise.

VIII. CONCLUDING REMARKS

In this paper, the optimal control law is established for steering the state covariance of a linear time-varying stochastic system subject to additive generic noise, which can be regarded as a combination of continuous white Gaussian noise and abrupt noise of any jump size. The results extend the applications of covariance steering to a larger class of problems of interest to engineering applications. Interesting potential extensions include the development of efficient algorithms for solving coupled ODEs (15), (19), (20), optimal steering of the state covariance with partially fixed terminal covariance and/or free final time with generic noise, and covariance steering with output feedback subject to non-Gaussian noise.
[45] L. M. Silverman and H. Meadows, “Controllability and observability in
time-variable linear systems,” *SIAM J. Control*, vol. 5, no. 1, pp. 64–73,
1967.
[46] E. Kreindler and P. Sarachik, “On the concepts of controllability and
observability of linear systems,” *IEEE Trans. Autom. Control*, vol. 9,
no. 2, pp. 129–136, 1964.
[47] A. Stubberud, “A controllability criterion for a class of linear systems,”
*IEEE Trans. Ind. Appl.*, vol. 83, no. 75, pp. 411–413, 1964.
[48] A. Morse and L. Silverman, “Structure of index-invariant systems,”
*SIAM J. Control*, vol. 11, no. 2, pp. 215–225, 1973.
[49] W. Wolovich, “On the stabilization of controllable systems,” *IEEE Trans.
Autom. Control*, vol. 13, no. 5, pp. 569–572, 1968.
[50] C. Seal and A. Stubberud, “Canonical forms for multiple-input time-
variable systems,” *IEEE Trans. Autom. Control*, vol. 14, no. 6, pp. 704–
707, 1969.
[51] S. Kilicaslan and S. P. Banks, “Existence of solutions of Riccati
differential equations for linear time varying systems,” in *Proc. Amer.
Control Conf.*, Baltimore, MD, 2010, pp. 1586–1590.
[52] G. Freiling, G. Jank, and H. Abou-Kandil, “Generalized Riccati differ-
ence and differential equations,” *Linear Algebra Its Appl.*, vol. 241, pp.
291–303, 1996.
[53] H. V. Henderson and S. R. Searle, “On deriving the inverse of a sum
of matrices,” *SIAM Rev.*, vol. 23, no. 1, pp. 53–60, 1981.
[54] S. G. Krantz and H. R. Parks, *The Implicit Function Theorem: History,
Theory, and Applications*. Springer, 2002.
[55] M. O’Driscoll, C. Harry, C. A. Donnelly, A. Cori, and I. Dorigatti, “A
comparative analysis of statistical methods to estimate the reproduction
number in emerging epidemics, with implications for the current coron-
avirus disease 2019 (COVID-19) pandemic,” *Clin. Infect. Dis.*, vol. 73,
no. 1, pp. e215–e223, 2021.
[56] P. Cowpertwait, V. Isham, and C. Onof, “Point process models of
rainfall: developments for fine-scale structure,” *Proc. R. Soc. A: Math.
Phys. Eng. Sci.*, vol. 463, no. 2086, pp. 2569–2587, 2007.