Abstract

In this paper we construct non-equivalent star products on \( \mathbb{C}P^n \) by phase space reduction. It turns out that the non-equivalent star products occur very natural in the context of phase space reduction by deforming the momentum map of the \( U(1) \)-action on \( \mathbb{C}^{n+1} \setminus \{0\} \) into a quantum momentum map and the corresponding momentum value into a quantum momentum value such that the level set, i.e., the ‘constraint surface’, of the quantum momentum map coincides with the classical one. All equivalence classes of star products on \( \mathbb{C}P^n \) are obtained by this construction.

1 Introduction

The concept of deformation quantization as introduced by Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer in \cite{BayenFlatoFronsdalLichnerowiczSternheimer} is now a well-established way to understand quantization of classical systems: the algebra of classical observables, i.e., the smooth complex-valued functions \( C^\infty(M) \) on a symplectic manifold, the phase space of the system is deformed by introducing an associative formal so-called star product \( \ast \) for \( C^\infty(M)[[\lambda]] \) depending on a formal parameter \( \lambda \) such that the zeroth order of the star product is the pointwise product and the \( \ast \)-commutator of two functions equals in first order \( i \) times the Poisson bracket. Hence the formal parameter \( \lambda \) is to be identified with Planck’s constant \( \hbar \) and the algebra of quantum observables \( C^\infty(M)[[\lambda]] \) turns out to be a deformation of the classical one in the sense of Gerstenhaber \cite{Gerstenhaber}. The existence of such star products for symplectic manifolds was shown by DeWilde and Lecomte \cite{DeWildeLecomte}, Fedosov \cite{Fedosov}, and Omori, Maeda, and Yoshioka \cite{OmoriMaedaYoshioka} and recently the existence of star products for arbitrary Poisson manifolds was stated by Kontsevich \cite{Kontsevich}. The classification of star products up to equivalence by formal power series with coefficients in the second de Rham cohomology was shown by Nest and Tsygan \cite{NestTsygan} and by Bertelson, Cahen, and Gutt \cite{BertelsonCahenGutt}.

In this paper we continue the work done together with Bordemann, Brischle and Emmrich in \cite{BordemannBrischleEmmrich1,BordemannBrischleEmmrich2} where the star product analogue of the Marsden-Weinstein phase space reduction for the
example of the $U(1)$-reduction of $\mathbb{C}^{n+1}\setminus\{0\}$ to $\mathbb{CP}^n$ was considered. Apart form a few papers the subject of reduction for star product seems not to be studied very intensively until now: In several basic examples were considered and in the example of $\mathbb{CP}^n$ and its non-compact dual was considered and the method of reduction was applied to find explicit formulas for star products on $\mathbb{CP}^n$. A generalization to complex Grassmannian manifolds is given in by Schirmer. Recently Fedosov gave a general construction for the reduction in case of a Hamiltonian group action of an arbitrary compact Lie group in . All these reductions proceed more or less the same: one starts with a suitable star product on the ‘big phase space’ such that the invariant functions form a subalgebra. Then the ideal generated by the components of the momentum map minus the momentum value is factored out and it remains to show that the quotient algebra is isomorphic to the functions on the reduced phase space endowed with a suitable star product which is hence called the reduced star product. This construction is physically reasonable and provides a way to perform a reduction in the quantum case too.

Nevertheless none of these approaches seems to deal with the question whether an equivalence transformation in the big phase space results in an equivalent reduced star product. In our example the possibility of non-equivalent star products has to be taken into account since the second de Rham cohomology of $\mathbb{CP}^n$ is one-dimensional. This work, originally motivated by a discussion with Bordemann, Flato, Schirmer and Sternheimer, provides a very simple example that non-equivalent star products may occur. It depends crucially on the definition of the above mentioned ideal and here one has in principle at least two reasonable possibilities: On the one hand for a fixed star product on the big phase space one can fix the value of the corresponding quantum momentum map to be the classical one which results in our example in only one star product for the quotient independent of the chosen star product on the big phase space. On the other hand one can fix the level set, i. e. the ‘constraint surface’, of the classical momentum map and hence one eventually has to modify the value of the (quantum) momentum map by ‘quantum corrections’. In this case it turns out that the resulting reduced star products are no longer equivalent in our example. Now both possibilities have their physical motivation and hence this example shows that there may occur some subtleties depending on the point of view which structure is more important: the value of the classical momentum map or the classical constraint surface. Moreover this construction shows that non-equivalent star products arise very natural in the reduction process.

The paper is organized as follows: in section we remember briefly the notation and results from . In section we describe the invariance properties and the reduction of the star products and in section we prove the non-equivalence of the star products obtained for $\mathbb{CP}^n$.

2 Preliminary results

Let us first remember the construction of star products of Wick type by phase space reduction for $\mathbb{CP}^n$ where we mainly use the notion as in . We start with the Kähler manifold $\mathbb{C}^{n+1}\setminus\{0\}$ with the usual Kähler form $\omega = \frac{i}{2}dz^k \wedge d\overline{z}^k$ where $z^0, \ldots, z^n$ are the canonical holomorphic coordinates for $\mathbb{C}^{n+1}\setminus\{0\}$ and summation over repeated indices is understood. Moreover we consider the complex projective space $\mathbb{CP}^n$ and denote by $\pi : \mathbb{C}^{n+1}\setminus\{0\} \rightarrow \mathbb{CP}^n$ the canonical (holomorphic) projection which maps $z \in \mathbb{C}^{n+1}\setminus\{0\}$ to the ray $\pi(z) \in \mathbb{CP}^n$ through $z$. On $\mathbb{C}^{n+1}\setminus\{0\}$ one has the usual $U(1)$-action $(e^{i\varphi}, z) \mapsto e^{i\varphi}z$ and the $\mathbb{C}\setminus\{0\}$-action $(\alpha, z) \mapsto \alpha z$. A function $f \in C^\infty(\mathbb{C}^{n+1}\setminus\{0\})$ is called homogeneous iff it is invariant under the $\mathbb{C}\setminus\{0\}$-action which is the case iff there exists a function $\phi \in C^\infty(\mathbb{CP}^n)$ such that $f = \pi^*\phi$. Moreover we consider the function $x : \mathbb{C}^{n+1}\setminus\{0\} \rightarrow \mathbb{R}^+$ defined by $x(z) := -\log z^k$. Then a function $R \in C^\infty(\mathbb{C}^{n+1}\setminus\{0\})$ is called radial iff there exists a function $\varrho \in C^\infty(\mathbb{R}^+)$ such that $R = \varrho \circ x$.

Let us now recall the classical Marsden-Weinstein phase space reduction procedure for $\mathbb{CP}^n$ as
e. g. in [4, p. 302] to establish our notation: the $U(1)$-action is generated by the $\text{ad}^*$-equivariant momentum map $J := -\frac{1}{2}x$ and any $\mu \in \mathbb{R}^-$ is a regular value of $J$. Moreover $J^{-1}(\{\mu\})$ is just the $2n + 1$ sphere centered at the origin with radius $\sqrt{-2}\mu$ in $\mathbb{C}^{n+1} \setminus \{0\}$. Fix now once and for all an arbitrary value $\mu \in \mathbb{R}^-$ and denote by $i_\mu : J^{-1}(\{\mu\}) \to \mathbb{C}^{n+1} \setminus \{0\}$ the inclusion and by $\pi_\mu : J^{-1}(\{\mu\}) \to J^{-1}(\{\mu\})/U(1) \cong \mathbb{CP}^n$ the projection onto the reduced phase space. Then $i_\mu^* \omega = \pi_\mu^* \omega_\mu$ determines the reduced symplectic form $\omega_\mu$ on $\mathbb{CP}^n$ and it turns out that $\omega_\mu$ is just a multiple (depending on $\mu$) of the usual Fubini-Study form. For a $U(1)$-invariant function $F \in C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ one defines the reduced function $F_\mu$ by $F_\mu([z]) := F \circ i_\mu(z)$ where $z \in J^{-1}(\{\mu\})$.

A suitable starting point for the deformation quantization of this reduction is the Wick star product on $\mathbb{C}^{n+1} \setminus \{0\}$ which is given by the formal power series in $\lambda$ for $F, G \in C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ by

$$F \star G := \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \frac{\partial^r F}{\partial z_1 \cdots \partial z_r} \frac{\partial^r G}{\partial \bar{z}_1 \cdots \partial \bar{z}_r}$$

which is known to be an associative formal star product for $\mathbb{C}^{n+1} \setminus \{0\}$. For a more general treatment of this kind of star products on arbitrary Kähler manifolds see e. g. [3]. Note that in this normalization the formal parameter $\lambda$ corresponds to $2\hbar$. Moreover we define $A := C^\infty(\mathbb{C}^{n+1} \setminus \{0\})[[\lambda]]$. Remember also the definition of the bidifferential operators $M_r$ and $\tilde{M}_r$ as introduced in [4, Eqn. 5 & 23]: For $F, G \in C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ one defines

$$M_r(F, G) := \lambda^r \frac{\partial^r F}{\partial z_1 \cdots \partial z_r} \frac{\partial^r G}{\partial \bar{z}_1 \cdots \partial \bar{z}_r}$$

and since for $\phi, \psi \in C^\infty(\mathbb{CP}^n)$ clearly $M_r(\pi^* \phi, \pi^* \psi)$ is again a homogeneous function $\pi^* \tilde{M}_r(\phi, \psi) = M_r(\pi^* \phi, \pi^* \psi)$ uniquely defines a bidifferential operator $\tilde{M}_r$ on $\mathbb{CP}^n$. Crucial for the following is the observation that the formal power series with coefficients in the $U(1)$-invariant functions on $\mathbb{C}^{n+1} \setminus \{0\}$ which we shall denote by $A^0 \subset A$ build a sub-algebra with respect to the Wick product.

Let now

$$D(\lambda) := 1 + \sum_{r=1}^{\infty} \lambda^r d_r, \quad C(\lambda) := D(\lambda)^{-1} = 1 + \sum_{r=1}^{\infty} \lambda^r c_r, \quad d_r, c_r \in \mathbb{C}$$

be an arbitrary complex formal power series starting with 1 and denote by $C(\lambda)$ the inverse series. Then for any such $D$ a formal series of differential operators $S_D : A \to A$ was constructed in [3, Theorem 3.1] having the following properties: $S_D$ acts trivial on the homogeneous functions and $S_D : A^0 \to A^0$ and

$$S_D x = D \left( \frac{\lambda}{x} \right) x.$$ (4)

This operator was used to define an equivalent star product which we shall now denote by $\star^D$ to emphasise the dependence on $D$ defined by

$$F \star^D G := S_D \left( (S_D^{-1} F) \star (S_D^{-1} G) \right)$$

for $F, G \in A$. It follows that the $\star^D$-product of a radial function with an arbitrary $U(1)$-invariant function is just the pointwise product and that for homogeneous functions $f, g \in A^0$ the equation

$$f \star^D g = \sum_{r=0}^{\infty} \frac{1}{r!} \left( \lambda \frac{\partial}{\partial x} \left( \frac{\lambda}{x} \right) \right)^r \prod_{s=0}^{r} \left( 1 + s \lambda \frac{\partial}{\partial x} \left( \frac{\lambda}{x} \right) \right)^{-1} M_r(f, g)$$

for $f, g \in A^0$.
holds \[4, \text{Eqn. 4}\]. Obviously this can be rearranged such that

\[ f *^D g = \sum_{r=0}^{\infty} \left( \frac{\lambda}{x} \right)^r K^D_r(f, g) \tag{7} \]

with some bidifferential operators \( K^D_r \) which are linear combinations of the \( M_r \) depending on the choice of \( D \). Moreover \( K^D_r(f, g) \) is clearly again homogeneous for \( f, g \) homogeneous and thus there are again uniquely determined bidifferential operators \( \tilde{K}^D_r \) on \( \mathbb{CP}^n \) such that \( \pi^* \tilde{K}^D_r(\phi, \psi) = K^D_r(\pi^*\phi, \pi^*\psi) \) for \( \phi, \psi \in C^\infty(\mathbb{CP}^n) \). By a simple computation using the associativity of \( *^D \) and the fact that the \( *^D \) product of the radial functions \( x^{-1} \) with any \( U(1) \)-invariant function is only the pointwise product one obtains that for \( \phi, \psi \in C^\infty(\mathbb{CP}^n) \)

\[ \phi *^D \mu \psi := \sum_{r=0}^{\infty} \left( \frac{\lambda}{-2\mu} \right)^r \tilde{K}^D_r(\phi, \psi) \tag{8} \]

is an associative star product for \( (\mathbb{CP}^n, \omega_\mu) \) and clearly \( (\pi^*\phi *^D \pi^*\psi)_\mu = \phi *^D \mu \psi \) \[4, \text{Theorem 4.2}\].

### 3 Invariance properties and reduction

In \[4, \text{p. 368} \] the notion of a ‘quantum moment map’ was introduced for this situation (see e. g. \[16 \] for a more general discussion) and it was shown that \( S_D J = D(\lambda/x) J \) is a quantum moment map for the \( U(1) \)-action for the star product \( *^D \), i. e. \( D(\lambda/x) J \) induces the same group action on the quantum level as \( J \) does on the classical level, i. e. for all \( F \in A \) we have

\[ F *^D S_D J - S_D J *^D F = \frac{i\lambda}{2} \{ F, J \}. \tag{9} \]

The classical observable algebra of the reduced system \( C^\infty(\mathbb{CP}^n) \) can be thought as the quotient of the \( U(1) \)-invariant functions on \( \mathbb{C}^{n+1} \setminus \{0\} \) by those which vanish on the ‘constraint surface’ \( J^{-1}(\{\mu\}) \). The later ideal is just the ideal generated by \( J - \mu \) and thus one might have the idea that the quantum version works analogously: indeed in \[4, \text{Prop. 4.1} \] it was shown that this is the case for \( D = 1 \). We shall now consider the case of arbitrary series \( D \). Then we have the already mentioned two possibilities: We can take the (left) ideal generated by \( S_D J - \mu \) in which case we can simply apply the equivalence transformation \( S_D \) to prove completely analogously to the proof in \[4 \] that the quotient space is isomorphic to the functions on \( \mathbb{CP}^n \) with the star product already obtained for \( D = 1 \). More interesting is hence the other possibility: we define the left-ideal

\[ J^D_\mu := A^0 *^D \left( S_D J - D \left( \frac{\lambda}{-2\mu} \right) \mu \right) \subset A^0 \tag{10} \]

generated by \( S_D J - D(\lambda/(-2\mu)) \mu \) which is in fact a two-sided ideal due to \( \{J^D_\mu \} \). Note that we have deformed both the classical momentum map and the momentum value in order to define the same constraint surface as in the classical case. We now shall describe the quotient \( A^0 / J^D_\mu \):

**Lemma 3.1** Denote by \( B := C^\infty(\mathbb{CP}^n)[[\lambda]] \) the vector space of the reduced observables then for any \( F \in A^0 \) we have

i.) \( F_\mu = 0 \iff F \in J^D_\mu \).

ii.) \( B \cong A^0 / J^D_\mu \) with the isomorphism \( B \ni \phi \mapsto [\pi^*\phi] \in A^0 / J^D_\mu \) and its inverse \( A^0 / J^D_\mu \ni [F] \mapsto F_\mu \in B \).
Corollary 3.2 The linear isomorphism $B \ni \phi \mapsto [\pi^* \phi] \in \mathcal{A}^0/J_\mu^D$ is an algebra isomorphism if $B$ is equipped with the star product $*^D_\mu$ as in [8] and $\mathcal{A}^0/J_\mu^D$ with the usual quotient algebra structure.

Since the star product algebra $(B, *^D_\mu)$ is isomorphic to the quotient $\mathcal{A}^0/J_\mu^D$ and since $J_\mu^D$ is the quantum analogue of the classical vanishing ideal of functions vanishing on the classical constraint surface one can indeed speak of $*^D_\mu$ as a reduced star product coming form the star product $*^D$ on $\mathbb{C}^{n+1} \setminus \{0\}$.

4 Non-equivalence of the reduced star products

Since in $\mathbb{C}^{n+1} \setminus \{0\}$ all the star products $*^D$ are equivalent this construction raises the question whether the reduced star products are still equivalent in the sense of equivalence between star products. As we shall see by comparing the star products $*^D_\mu$ for varying series $D$ this is not the case. Using the concrete formula (8) for the $*^D$-product of homogeneous functions we easily obtain the following lemma by direct computation:

Lemma 4.1 Let $D, D'$ be two formal power series starting with 1 and denote by $C, C'$ the corresponding inverse power series. Assume that $c_r = c'_r$ for $r = 1, \ldots, k - 1$ and $c_k \neq c'_k$. Then the bidifferential operators of the corresponding star products $*^D$ and $*^{D'}$ coincide $K^D_r = K^{D'}_r$ for $r = 1, \ldots, k$ and in order $k + 1$ we have

$$K^D_{k+1} - K^{D'}_{k+1} = (c_k - c'_k) \tilde{M}_1.$$  \hspace{1cm} (11)

Corollary 4.2 Under the same preconditions as in the preceding lemma we have for the bidifferential operators in $*^D_\mu$ and $*^{D'}_\mu$

$$\tilde{K}^D_r = \tilde{K}^{D'}_r \text{ for } r \leq k \quad \text{and} \quad \tilde{K}^D_{k+1} - \tilde{K}^{D'}_{k+1} = (c_k - c'_k) \tilde{M}_1.$$  \hspace{1cm} (12)

Now it is known (see e. g. [3] Prop 3.7) that if two equivalent star products on a symplectic manifold coincide up to order $k$ then the antisymmetric part of their difference in order $k + 1$ is a one-differential operator which can be written as $\Omega(X_f, X_g)$ with an exact two-form $\Omega$ where $X_f, X_g$ denote the Hamiltonian vector fields of the functions. But since in our case the antisymmetric part of $\tilde{M}_1$ is just $1/2$ times the Poisson bracket on $\mathbb{CP}^n$ which corresponds to the non-exact Fubini-Study form as two-form we immediately have the following theorem:

Theorem 4.3 Let $D, D'$ be two complex formal power series starting with 1 and let $*^D_\mu$ and $*^{D'}_\mu$ be the corresponding star product on $\mathbb{CP}^n$ according to (8). Then $*^D_\mu$ is equivalent to $*^{D'}_\mu$ iff $D = D'$ and any star product on $\mathbb{CP}^n$ is equivalent to some $*^D_\mu$.

Proof: The non-equivalence for $D \neq D'$ follows easily from corollary 4.2, the fact that the Fubini-Study form is not exact and [3] Prop 3.7. Since $H^2(\mathbb{CP}^n)$ is one-dimensional the possible equivalence classes are parametrised by $\lambda H^2(\mathbb{CP}^n)[[\lambda]]$ (for fixed Poisson bracket, see e. g. [13, 3]) which is clearly in bijection to
the antisymmetric one-differential part determined by $c_k M_1$ since $c_k$ can be chosen arbitrarily (relative to the reference star product with $D = 1$) in order $\lambda^{k+1}$.

Remarks: In our example the choice $D = 1$ with the classical value $\mu$ is clearly preferred since only for this choice the corresponding star product $*^D$ is strong $U(1)$-invariant, i.e. the quantum momentum map coincides with the classical one. Nevertheless one can think of more general situation where no strong invariant star products are available. Here one might weaken the strong invariance to the existence of a quantum momentum map and hence there might occur some subtilities in the choice of the star product and the choice of the quantum momentum value leading to non-equivalent star products for the quotient. Hence it would be very interesting to examine these aspects of reduction in more general situations. A good starting point for this programme should perhaps be Fedosov’s reduction scheme [10].

Acknowledgements

We would like to thank Martin Bordemann, Moshe Flato, Joachim Schirmer and Daniel Sternheimer for a motivating discussion and Martin Bordemann for carefully reading the manuscript and useful comments.

References

[1] Abraham R., Marsden, J. E.: Foundations of Mechanics. 2nd edition, Addison Wesley Publishing Company, Inc., Reading Mass. 1985.
[2] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation Theory and Quantization. Ann. Phys. 111 (1978) part I: 61–110, part II: 111–151.
[3] Bertelson, M., Cahen, M., Gutt, S.: Equivalence of star products. Class. Quantum Grav. 14 (1997) A93–A107.
[4] Bordemann, M., Brischle, M., Emmrich, C., Waldmann, S.: Phase Space Reduction for Star-Products: An Explicit Construction for $\mathbb{C}P^n$. Lett. Math. Phys. 36 (1996) 357–371.
[5] Bordemann, M., Brischle, M., Emmrich, C., Waldmann, S.: Subalgebras with converging star products in deformation quantization: An algebraic construction for $\mathbb{C}P^n$. J. Math. Phys. 37 (1996) 6311–6323.
[6] Bordemann, M., Waldmann, S.: A Fedosov Star Product of the Wick Type for Kähler Manifolds. Lett. Math. Phys. 41 (1997) 243–253.
[7] DeWilde, M., Lecomte, P. B. A.: Existence of star-products and of formal deformations of the Poisson Lie Algebra of arbitrary symplectic manifolds. Lett. Math. Phys. 7 (1983) 487–496.
[8] B. Fedosov: A Simple Geometrical Construction of Deformation Quantization. J. Diff. Geom. 40 (1994) 213–238.
[9] Fedosov, B.: Deformation Quantization and Index Theory. Akademie Verlag, Berlin 1996.
[10] Fedosov, B.: Non-Abelian Reduction in Deformation Quantization. Preprint 1997.
[11] Gerstenhaber, M., Schack, S.: Algebraic Cohomology and Deformation Theory. in: Hazewinkel, M., Gerstenhaber, M. (eds): Deformation Theory of Algebras and Structures and Applications. Kluwer, Dordrecht 1988.
[12] Kontsevich, M.: *Deformation Quantization of Poisson Manifolds*. Preprint, September 1997, [arXiv:q-alg/9709040](http://arxiv.org/abs/q-alg/9709040).

[13] Nest, R., Tsygan, B.: *Algebraic Index Theorem*. Commun. Math. Phys. **172** (1995) 223–262.

[14] Omori, H., Maeda, Y., Yoshioka, A.: *Weyl manifolds and deformation quantization*. Adv. Math. **85** (1991) 224–255.

[15] Schirmer, J.: *A Star Product for Complex Grassmann Manifolds*. Preprint Freiburg, September 1997, [arXiv:q-alg/9709021](http://arxiv.org/abs/q-alg/9709021).

[16] Xu, P.: *Fedosov *-Products and Quantum Momentum Maps*. Preprint 1996, [arXiv:q-alg/9608006](http://arxiv.org/abs/q-alg/9608006).