Self-similar Bianchi models: II. Class B models

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Abstract

In a companion article (referred to hereafter as paper I) a detailed study of the simply transitive spatially homogeneous (SH) models of class A concerning the existence of a simply transitive similarity group has been given. The present work (paper II) continues and completes the above study by considering the remaining set of class B models. Following the procedure of paper I we find all SH models of class B subjected only to the minimal geometric assumption to admit a proper homothetic vector field (HVF). The physical implications of the obtained geometric results are studied by specializing our considerations to the case of vacuum and $\gamma$-law perfect fluid models. As a result, we regain all the known exact solutions regarding vacuum and non-tilted perfect fluid models. In the case of tilted fluids, we find the general self-similar solution for the exceptional type VI$-\frac{1}{9}$ model and we identify it as an equilibrium point in the corresponding dynamical state space. It is found that this new exact solution belongs to the subclass of models $n_d^\alpha = 0$, is defined for $\gamma \in \left(\frac{2}{3}, \frac{3}{2}\right)$, and although it has a five-dimensional stable manifold there always exist two unstable modes in the restricted state space. Furthermore, the analysis of the remaining types guarantees that tilted perfect fluid models of types III, IV, V and VIIh cannot admit a proper HVF, strongly suggesting that these models either may not be asymptotically self-similar (type V) or may be extreme tilted at late times. Finally, for each Bianchi type, we give the extreme tilted equilibrium points of their state space.

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1. Introduction

In a previous article [1] (designated in the following as paper I), we presented a general study of the spatially homogeneous (SH) models of class A admitting a four-dimensional group of homotheties acting simply transitively on the spacetime manifold. We concentrated on tilted
\( \gamma \)-law perfect fluids and we have given various results concerning the existence of an admissible self-similar perfect fluid model in all the Bianchi types of class A. As a consequence, this analysis has led us to regain the type II solution [2], to show the non-existence of self-similar type VII\(_0\), VIII and IX tilted perfect fluid models [1, 3] and to prove the generality of the known perfect fluid solution for type VI\(_0\) models [4, 5].

Our intention in the present work is to address the problem regarding the existence of a proper homothetic vector field (HVF) for the remaining SH models, namely, those of class B, and to examine the physical and dynamical implications of the geometric results with the hope of providing a way of obtaining a better understanding of their past and future asymptotic dynamics.

Section 2 contains some basic elements of the two major frameworks of studying SH cosmologies: the orthonormal frame and metric approaches. In particular, and adopting the conventions and the methodology of [6], we briefly discuss the structure of the dynamical state space of the SH models, coming from the evolution equations and the algebraic restrictions for the case where the spacetime is filled with a tilted (in general) \( \gamma \)-law perfect fluid. In section 3, by exploiting the basic relations of the metric approach and the results of paper I we find all the (simply) transitively self-similar SH models, i.e., those models which admit a proper homothetic vector field (HVF) without assuming a specific matter content, filled the spacetime. From these results, we proceed in section 4 and give an analysis of their possible physical interpretation. Due to their sound dynamical significance, we apply the general geometric results only to the case of vacuum and \( \gamma \)-law perfect fluid models. For illustrative purposes, we reproduce the known vacuum and non-tilted self-similar models of types III, IV, VI\(_h\). In the case of tilted perfect fluids we find a new exact solution, representing the general self-similar tilted model of type VI\(-\frac{1}{9}\). Finally, section 5 contains a summary of all the known self-similar tilted perfect fluid solutions and a brief discussion regarding their importance in the asymptotic behaviour of generic models.

The following conventions have been used throughout: spatial frame indices are denoted by lower case Greek letters \( \alpha, \beta, \ldots = 1, 2, 3 \), lower case Latin letters denote spacetime indices \( a, b, \ldots = 0, 1, 2, 3 \) and we use geometrized units such that \( 8\pi G = c = 1 \).

2. Preliminaries

Using the so-called metric approach, one can easily derive the explicit form (in local coordinates) of the self-similar metric and the conformally mapped tilted fluid velocity. Due to the fact that the reduced field equations (FE) have purely algebraic form, the determination of a specific self-similar vacuum or \( \gamma \)-law perfect fluid model is straightforward. On the other hand, the orthonormal frame approach is used to write the FE in an autonomous form and the resulting system of decoupled first-order differential equations is studied in the dynamical state space of the corresponding model [7]. Since every (non-extreme) equilibrium point is represented by an exact self-similar solution the mutual use of the approaches is able to show the existence or not of a self-similar model and to provide the necessary tools for the description of their future asymptotic dynamics. Therefore, and in order the present paper to be self-contained as possible, we find it convenient to give the general set-up concerning the essence of both approaches which shall be used frequently in what follows (the reader is referred to [6–10] for an extensive exposition of the formalism used in this section).
2.1. The metric approach

In Bianchi models the existence of a $G_3$ Lie algebra of Killing vector fields (KVFs) $X_\alpha$ with three-dimensional spacelike orbits $C$ implies the existence of a uniquely defined unit timelike congruence $n^a$ ($n^a n_a = -1$) normal to the spatial foliations $C$:

$$n_{[a;b]} = 0 = n_{a;b} n^b \quad \Leftrightarrow \quad \frac{1}{2} \mathcal{L}_n g_{ab} \equiv n_{a;b} = \sigma_{ab} + \frac{\theta}{3} h_{ab} \quad (2.1)$$

where $\sigma_{ab}, \theta = n_{a;b} g^{ab}, h_{ab} = g_{ab} + n_a n_b$ are the kinematical quantities associated with the $n^a$ according to the standard kinematical decomposition of an arbitrary timelike congruence [11]. Because $n^a$ is irrotational and geodesic, there exists a time function $t(x^a)$ such that $n^a = \delta^a_t$, i.e., each value of $t$ essentially represents the hypersurfaces $C$. Therefore, local coordinates $(t, x^a)$ can be found in which the line element takes the following form:

$$ds^2 = -dt^2 + g_{\alpha\beta}(t) \omega^\alpha_a \omega^\beta_b dx^a dx^b \quad (2.2)$$

where we have employed a basis of invariant vector fields and 1-forms constituting of $e_\alpha$ and its dual $\omega^\alpha$.

Furthermore, if we assume that the spacetime is filled with a perfect fluid having a linear equation of state $\tilde{p} = (\gamma - 1) \tilde{\rho}$, the Ricci tensor is written as

$$R_{ab} = \gamma \tilde{\rho} u_a u_b + \frac{(2 - \gamma) \tilde{\rho}}{2} g_{ab} \quad (2.3)$$

where $\tilde{\rho}, \tilde{p}$ are the energy density and the pressure measured by the observers comoving with the fluid velocity $u^a$. The tilted (in general) fluid velocity $u^a$ can be decomposed parallel and normal to $n^a$ as follows:

$$u = \Gamma n + \Delta a_\beta e^\beta = \Gamma n + \gamma B^a e_a \quad (2.4)$$

where $B^a(t), \Delta a(t)$ are the frame components of the spatial part of the four-velocity $u^a$ and $\Gamma$ is a smooth function of the time coordinate $t$ satisfying the constraint

$$\Gamma = (1 - B^a \Delta a)^{-\frac{1}{2}} \quad (2.5)$$

and $B^a \Delta a < 1$.

Finally, we note that under the assumption of a $H_4$ Lie algebra of homotheties, it has been shown in paper I that $\Gamma$ becomes a constant and both the frame components of the metric and the fluid velocity are determined explicitly up to integration constants.

2.2. The orthonormal frame approach

Of particular importance in the study of the asymptotic dynamics of SH models is the reformulation of the FE as an autonomous system of first-order ordinary differential equations. This has been done in [6] where, using the orthonormal frame approach, the FE are written in terms of a set of expansion-normalized variables by using the dimensionless time parameter:

$$\frac{dr}{d\tau} = \frac{1}{H}, \quad \frac{dH}{d\tau} = -(1 + q) H \quad (2.6)$$

where $q, H$ are the deceleration and expansion (Hubble) parameter, respectively, of the normal timelike congruence $n^a$.

The resulting system consists of the evolution equations for the components $\Sigma_{a\beta}$ of the shear tensor of the unit normal vector field $n^a$, the spatial curvature $A_a, n_{a\beta}$ of the orbits of the $G_3$ isometry group (due to Jacobi identities) and the spatial components of the tilted fluid velocity $v^a$.
Using the freedom of a time-dependent spatial rotation, we may choose the orthonormal tetrad to be the eigenframe of $n_{\alpha\beta}$ therefore the contracted form of Jacobi identities $n_{\alpha\beta}A^\beta = 0$ implies

$$n_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{pmatrix}, \quad A_\alpha = A_1 \delta_\alpha^1.$$  

(2.7)

We note that in types VI$_h$ and VII$_h$ the evolution equations for $n_{\alpha\beta}$ and $A^\alpha$ can be used to express the component $A_1$ in terms of the curvature variables:

$$A_1^2 = h N_2 N_3.$$  

(2.8)

In addition and following [6] we introduce the shear variables:

$$\Sigma_+ = \frac{1}{4}(\Sigma_{22} + \Sigma_{33}), \quad \Sigma_- = \frac{1}{2\sqrt{3}}(\Sigma_{22} - \Sigma_{33})$$

$$\Sigma_1 = \frac{1}{\sqrt{3}} \Sigma_{23}, \quad \Sigma_3 = \frac{1}{\sqrt{3}} \Sigma_{12}, \quad \tilde{\Sigma}_{13} = \frac{1}{\sqrt{3}} \Sigma_{13}$$  

(2.9)

(2.10)

where for the sake of simplicity we can drop the tilde.

With these identifications the expansion-normalized variables $\{\Sigma_+, \Sigma_-, \Sigma_1, \Sigma_3, N_2, N_3, \nu_\alpha\}$ satisfy the set of evolution equations (A.11)–(A.13) and (A.29) plus the algebraic constraint (A.18) given in [6]. Consequently, the state space is a compact subset $D \subset \mathbb{R}^7$ (the compactness of $D$ can be shown using the generalized Friedmann equation and the inequality $\Omega > 0$ for the energy density parameter).

We conclude this section by noting that for each Bianchi type, the evolution equations and the algebraic constraints can be determined by specializing them in the standard way:

| Type | $N_2$ | $N_3$ | Restrictions |
|------|-------|-------|--------------|
| IV   | 0     | $N_3$ | $N_1 > 0$    |
| V    | 0     | 0     | None         |
| VI$_h$ | $N_2$ | $N_3$ | $N_2 N_3 < 0$ |
| VII$_h$ | $N_2$ | $N_3$ | $N_2 N_3 > 0$ |
| VI$_{1,9}$ | $N_2$ | $N_3$ | $N_2 N_3 < 0$ |

3. Solution of the symmetry equations

In this section, we will give the general solution of the similarity equations $\mathcal{L}_H g_{\alpha\beta} = 2\psi g_{\alpha\beta}$ where $H = H_n + A^\alpha X_\alpha$ is the generator of the one-parameter group of homotheties. Using the fact that $[H, X_\alpha] = C^\beta_{\alpha\delta} X_\beta$ and $[n, X_\alpha] = 0$ it follows $H = H(t)$ and $A = A(x^\alpha)$ where $H(t), A(x^\alpha)$ are smooth functions of the spacetime manifold [1]. We note that the exact form of the self-similar SH metrics is independent of the physical assumption we made for the matter content of the spacetime. However, if we employ a perfect fluid source for the gravitational field, the well-known relation $\mathcal{L}_H R_{\alpha\beta} = \mathcal{L}_H T_{\alpha\beta} = 0$ implies [12] that the fluid velocity is mapped conformally by the HVF, i.e., $\mathcal{L}_H u_\alpha = \psi u_\alpha$.

In what follows the coordinate forms of the KVFs, the invariant basis and its dual are those which appear in [10]. We recall that in types VI$_h$ and VII$_h$ the group parameters $h_R$ and $h$ are related according to $h = -\frac{h_R + 1}{(1 - h_R)^2}$ and $h = \frac{h_R}{4 - h_R}$, respectively.
**Type III**

In contrast with the type VI$-\frac{1}{9}$ models ($h = -\frac{1}{9} \iff h_R = -2$ or $h_R = -\frac{1}{2}$) in which the ‘exceptional’ dynamical behaviour does not pass to geometry (at least regarding the determination of the corresponding self-similar metrics), type III models ($h = -1 \iff h_R = 0$) cannot be treated simultaneously within the class of type VI$_h$ metrics due to the ‘singular’ property of the value $h_R = 0$ in Jacobi identities.

In local coordinates $\{t, x, y, z\}$ the KVFs $\{X_\alpha\}$ and the canonical 1-forms $\{\omega^\alpha\}$ are

\[
X_1 = \partial_y, \quad X_2 = \partial_z, \quad X_3 = \partial_x + y\partial_y \tag{3.1}
\]

\[
\omega^1 = e^{-x} dy, \quad \omega^2 = dz, \quad \omega^3 = dx \tag{3.2}
\]

The non-vanishing structure constants $C_{13} = 1$ and the Jacobi identities of the homothetic Lie algebra $H_4$ imply that the remaining non-vanishing structure constants $C_{\alpha\beta4}$ are either $C_{224} = b$ or $C_{234} = c$. We consider subcases (in order to retain the generality of the results we avoid setting some group constants equal to 1):

**Case A**

\[HVF:\]

\[
H = \psi t \partial_t + D_1 \partial_x + b z \partial_z \tag{3.3}
\]

Fluid velocity:

\[
\Delta_1 = v_1 t \frac{(D_1 + \psi)}{\psi}, \quad \Delta_2 = v_2 t \frac{(\psi - b)}{\psi}, \quad \Delta_3 = v_3 t. \tag{3.4}
\]

Metric:

\[
g_{\alpha\beta} = \begin{pmatrix}
    c_{11} t^{(D_1 + \psi)/\psi} & c_{12} t^{(D_1 + 2\psi - b)/\psi} & c_{13} t^{(2\psi + D_1)/\psi} \\
    c_{12} t^{(D_1 + 2\psi - b)/\psi} & c_{22} t^{2(\psi - b)/\psi} & c_{23} t^{(2\psi - b)/\psi} \\
    c_{13} t^{(2\psi + D_1)/\psi} & c_{23} t^{(2\psi - b)/\psi} & c_{33} t^2
\end{pmatrix} \tag{3.5}
\]

**Case A**

\[HVF:\]

\[
H = \psi t \partial_t + D_1 \partial_x + D_2 e^x \partial_y + c x \partial_z \tag{3.6}
\]

Fluid velocity:

\[
\Delta_1 = v_1 t \frac{(D_1 + \psi)}{\psi}, \quad \Delta_2 = v_2 t, \quad \Delta_3 = \frac{-D_2 v_1 t (D_1 + \psi)/\psi}{D_1} - \frac{c v_2 t \ln t}{\psi} + v_3 t. \tag{3.7}
\]

Metric:

\[
g_{\alpha\beta} = \begin{pmatrix}
    c_{11} t^{(D_1 + \psi)/\psi} & c_{12} t^{(D_1 + 2\psi - b)/\psi} & g_{13} \\
    g_{12} & c_{22} t^{2(\psi - b)/\psi} & g_{23} \\
    g_{13} & g_{23} & g_{33}
\end{pmatrix} \tag{3.8}
\]

where

\[
g_{32} = t^2 \left( c_{23} - \frac{c_{22} c \ln t}{\psi} \right) - \frac{c_{12} D_2 t^{(2\psi + D_1)/\psi}}{D_1} \tag{3.9}
\]

\[
g_{33} = \frac{D_2 c_{11} t^{(2(D_1 + \psi)/\psi)}}{D_1^2} + \frac{t(D_1 + 2\psi)/\psi}{D_1} \left( \frac{2 D_2 c_{12} c \ln t}{D_1 \psi} - \frac{2 D_2 c_{13}}{D_1} \right) + \left[ \frac{c^2 c_{22} (\ln t)^2}{\psi^2} - \frac{2 c_{23} c \ln t}{\psi} + c_{33} \right] t^2. \tag{3.10}
\]
Case A

HVF:

\[ H = \psi t \partial_t + D_2 e^x \partial_y + c x \partial_z. \] (3.11)

Fluid velocity:

\[ \Delta_1 = v_1 t, \quad \Delta_2 = v_2 t, \quad \Delta_3 = t \left( v_3 - \frac{D_2 v_1 + c v_2}{\psi} \ln t \right). \] (3.12)

Metric:

\[
\begin{pmatrix}
 c_{11} t^2 & g_{12} & g_{13} \\
 c_{12} t^2 & c_{22} t^2 & g_{23} \\
 c_{13} t & c_{23} t & c_{33} t^2
\end{pmatrix}
\] (3.13)

where

\[
g_{33} = t^2 \left[ c_{33} + \frac{2 c_1 c_{13} + 2 D_2 c_{12} c + c_2^2 c_{22}}{\psi^2} (\ln t)^2 - \frac{2 (D_2 c_{13} + c_2 c_{12})}{\psi} \ln t \right].
\] (3.18)

Type IV

In type IV models, the KVFs and the dual basis have the following coordinate forms:

\[ X_1 = \partial_y, \quad X_2 = \partial_z, \quad X_3 = \partial_x + (y + z) \partial_y + z \partial_z. \] (3.14)

\[ \omega^1 = e^{-x} dy - x e^{-x} dz, \quad \omega^2 = e^{-x} dz, \quad \omega^3 = dx. \] (3.15)

Since the non-vanishing structure constants are \( C_{13}^1 = C_{23}^1 = C_{23}^2 = 1 \), a similar analysis shows that the remaining non-vanishing structure constants \( C_{\alpha \beta}^\mu \) of \( H_4 \) are \( C_{14}^1 = a, C_{24}^2 = a \). Therefore, we have the following results:

HVF:

\[ H = \psi t \partial_t + D_1 \partial_x + ay \partial_y + az \partial_z. \] (3.16)

Fluid velocity:

\[ \Delta_1 = v_1 t e^t (D_1 + \psi - a)/\psi, \quad \Delta_2 = t (D_1 + \psi - a)/\psi \left( v_2 + \frac{D_1 v_1 \ln t}{\psi} \right), \quad \Delta_3 = v_3 t. \] (3.17)

Metric:

\[
\begin{pmatrix}
 c_{11} t^{2(D_1 + \psi - a)/\psi} & g_{12} & g_{13} \\
 c_{12} t^{2(D_1 + \psi - a)/\psi} & c_{22} t^{2(D_1 + \psi - a)/\psi} & g_{23} \\
 c_{13} t^{2(D_1 + \psi - a)/\psi} & c_{23} t^{2(D_1 + \psi - a)/\psi} & c_{33} t^2
\end{pmatrix}
\] (3.18)

where

\[
g_{22} = t^{2(D_1 + \psi - a)/\psi} \left[ c_{22} + \frac{2 D_1 c_{12} \ln t}{\psi} - \frac{D_1^2 c_{11} (\ln t)^2}{\psi^2} \right].
\] (3.19)
Type V

In this case, the KVFs $\{X_\alpha\}$ and the canonical 1-forms $\{\omega^\alpha\}$ are

$$X_1 = \partial_y, \quad X_2 = \partial_z, \quad X_3 = \partial_x + y\partial_y + z\partial_z.$$  

$$\omega^1 = e^{-x} \, dy, \quad \omega^2 = e^{-x} \, dz, \quad \omega^3 = dx.$$  

It turns out that the structure of the homothetic algebra is $C_{13}^1 = C_{21}^2 = 1$, $C_{14}^2 = b$, $C_{24}^1 = a$.

The analysis of the symmetry equations suggests that we must again consider subcases according to the vanishing of the parameter $a$:

**Case A$_1$ ($a \neq 0$)**

**HVF:**

$$H = \psi t \partial_t + D_1 \partial_x + b z \partial_y + a y \partial_z.$$  

Fluid velocity:

$$\Delta_1 = v_1 \exp \left( \frac{(D_1 - \sqrt{ab} + \psi) \ln t}{\psi} \right) + \frac{2a}{v_2 \exp \left( \frac{(D_1 + \sqrt{ab} + \psi) \ln t}{\psi} \right)},$$  

$$\Delta_2 = \frac{-\psi t (\Delta_1)_{,t} - (D_1 + \psi) \Delta_1}{a}, \quad \Delta_3 = v_3 t.$$  

Metric:

$$g_{11} = c_{12} \exp \left( \frac{2(D_1 - \sqrt{ab} + \psi) \ln t}{\psi} \right) + c_{21} \exp \left( \frac{2(D_1 + \sqrt{ab} + \psi) \ln t}{\psi} \right) + c_{23} t^{2/2(1+ab)},$$  

$$g_{12} = -\frac{\psi t (g_{11})_{,t} - 2(D_1 + \psi) g_{11}}{2a},$$  

$$g_{13} = c_{13} \exp \left( \frac{(D_1 - \sqrt{ab} + 2\psi) \ln t}{\psi} \right) + c_{31} \exp \left( \frac{(D_1 + \sqrt{ab} + 2\psi) \ln t}{\psi} \right),$$  

$$g_{22} = \frac{-\psi t^2 (g_{11})_{,tt} - \psi (4D_1 + 3\psi)(g_{11})_{,t} + 2(D_1^2 + 4D_1 \psi - ab + 2\psi^2) g_{11}}{2a^2},$$  

$$g_{23} = \frac{-\psi t (g_{13})_{,t} - (D_1 + 2\psi) g_{13}}{a}, \quad g_{33} = c_{33} t^2.$$  

**Case A$_2$**

**HVF:**

$$H = \psi t \partial_t + D_1 \partial_x + b z \partial_y.$$  

Fluid velocity:

$$\Delta_1 = v_1 \frac{c_{13} t^{2/2(1+ab)}}{\psi}, \quad \Delta_2 = t \frac{c_{13} t^{2/2(1+ab)}}{\psi}, \quad \Delta_3 = v_3 t.$$  

Metric:

$$g_{\alpha\beta} = \begin{pmatrix} 
\frac{c_{11} t^{2/2(1+ab)}}{\psi} & \frac{c_{12} - b c_{11} \ln t}{\psi} & \frac{g_{21}}{\psi} \\
\frac{c_{22} - 2 b c_{11} \ln t}{\psi} & \frac{c_{23} - b c_{11} \ln t}{\psi} & \frac{g_{31}}{\psi} \\
\frac{c_{13}}{t^{2/2(1+ab)}} & \frac{c_{13} t^{2/2(1+ab)}}{\psi} & \frac{g_{32}}{\psi} 
\end{pmatrix}.$$  

$$c_{11} t^{2/2(1+ab)} - c_{13} t^{2/2(1+ab)}.$$
The KVFs \( \{ X_\alpha \} \) and the canonical 1-forms \( \{ \omega^\alpha \} \) are
\[
X_1 = \partial_y, \quad X_2 = \partial_z, \quad X_3 = \partial_x + y\partial_y + h_R z\partial_z, \quad (3.32)
\]
\[
\omega^1 = e^{-x} dy, \quad \omega^2 = e^{-h_R x} dz, \quad \omega^3 = dx, \quad (3.33)
\]
where \( h_R \neq 0 \) and \( C_{13}^1 = h_R^{-1} C_{23}^2 = 1 \). Hence, the remaining non-vanishing structure constants \( C^\alpha_{\beta\gamma} \) are \( C_{14}^1 = a, C_{24}^2 = b \). Using these results we get
HVF:
\[
H = \psi t \partial_t + D_1 \partial_x + ay \partial_y + b z \partial_z, \quad (3.34)
\]
Fluid velocity:
\[
\Delta_1 = v_1 t \left( D_1 + \psi - a \right) / \psi, \quad \Delta_2 = v_2 t \left( D_1 + h_R \psi - b \right) / \psi, \quad \Delta_3 = v_3 t. \quad (3.35)
\]
Metric:
\[
g_{\alpha\beta} = \begin{pmatrix}
c_{11} t^{2(D_1 + \psi - a) / \psi} & c_{12} t^{2(D_1 + \psi - a) / \psi} & c_{13} t^{2(D_1 + \psi - a) / \psi} \\
c_{12} t^{2(D_1 + \psi - a) / \psi} & c_{22} t^{2(D_1 + \psi - b) / \psi} & c_{23} t^{2(D_1 + \psi - b) / \psi} \\
c_{13} t^{2(D_1 + \psi - a) / \psi} & c_{23} t^{2(D_1 + \psi - b) / \psi} & c_{33} t^2 \\
\end{pmatrix}, \quad (3.36)
\]
Type VII
Finally, in type VII models we have
\[
X_1 = \partial_y, \quad X_2 = \partial_z, \quad X_3 = \partial_x - z \partial_y + \left( y + h_R z \right) \partial_z, \quad (3.37)
\]
\[
\omega^1 = \left( A_1 - \frac{h_R}{2} A_2 \right) dy - A_2 dz, \quad \omega^2 = A_2 dy + \left( A_1 + \frac{h_R}{2} A_2 \right) dz, \quad \omega^3 = dx \quad (3.38)
\]
where
\[
A_1 = \exp \left( -\frac{h_R}{2} x \right) \cos wx, \quad A_2 = \frac{1}{w} \exp \left( -\frac{h_R}{2} x \right) \sin wx, \quad w = \sqrt{4 - h_R^2 / 4}, \quad (3.39)
\]
The non-vanishing structure constants are \(( h_R^2 < 4 \) \) \( C_{13}^1 = -C_{23}^1 = h_R^{-1} C_{23}^2 = 1 \) leading to the following expressions:
\[
C_{14}^1 = a, \quad C_{24}^2 = a. \quad (3.40)
\]
HVF:
\[
H = \psi t \partial_t + D_1 \partial_x + ay \partial_y + az \partial_z, \quad (3.41)
\]
Fluid velocity:
\[
\Delta_1 = \frac{v_1}{\psi^{1/2}} \left[ v_{12} \cos \left( \frac{D_1 w}{\psi} \ln t \right) + v_{21} \sin \left( \frac{D_1 w}{\psi} \ln t \right) \right], \quad (3.42)
\]
\[
\Delta_2 = \frac{\psi t (\Delta_1)}{D_1} + (a - \psi) \Delta_1, \quad \Delta_3 = v_3 t. \quad (3.43)
\]
Metric:

\[ g_{11} = -\frac{t^{p_1/\psi} \left[ 16D_1E_1w^3p_1^2 \cos \left( \frac{2D_1w\ln t}{\psi} \right) + 4E_2w^2p_1 \sin \left( \frac{2D_1w\ln t}{\psi} \right) \right]}{64D_1^5w^5p_1^2} \]

\[ g_{12} = \frac{\psi t(g_{11}, t) + 2(\alpha - \psi)g_{11}}{2D_1} \]

\[ g_{13} = t^{(p_1+2\psi)/2\psi} \left[ c_{13} \cos \left( \frac{D_1w\ln t}{\psi} \right) + c_{31} \sin \left( \frac{D_1w\ln t}{\psi} \right) \right] \]

\[ g_{22} = \frac{\psi^2t^2(g_{11}, t) - \psi t(g_{11}, t)(p_1 + \psi - 2\alpha) + 2\left[D_1^2 + (\psi - \alpha)p_1\right]g_{11}}{2D_1^2} \]

\[ g_{23} = \frac{\psi t(g_{11}, t) + (\alpha - 2\psi)g_{13}}{D_1}, \quad g_{33} = c_{33}t^2 \]

where \( E_1, E_2 \) are constants and we have set \( p_1 = D_1h_R + 2(\psi - \alpha) \).

4. Exact solutions

The well-known feature of self-similarity to reduce the FE to a purely algebraic form makes the determination of vacuum and perfect fluid models straightforward. Following the methodology of paper I and for illustrative purposes, we reproduce all the known vacuum and non-tilted perfect fluid solutions. In addition, we find the general tilted perfect fluid model for the exceptional class \( VI_{-1/9} \). For completeness we also give, for each Bianchi type, the extreme tilted equilibrium points of their state space.

Vacuum plane wave models of type III

This exact vacuum solution can be found from the case \( A_1 \) of section 2 and has been given in [13]. The HVF is \( H = \psi t\partial_t + D_1\partial_x + b\partial_z \) and the constants \( c_{\alpha\beta}, \psi, b \) are

\[ c_{13} = c_{23} = 0, \quad c_{11} = c_{22} = 0, \quad c_{33} = \frac{b^2}{(D_1 + b)^2}, \quad \psi = b \]

where without loss of generality we can set \( b = 1 \).

We note that the above spacetime is algebraically special (Petrov type N) since it admits the gradient null KVF \( \partial^\nu = e^{-b_1/(D_1 + b)}\left( \partial_t + \frac{b}{D_1 + b}\partial_x \right) \) which essentially represents the repeated principal null direction of the Weyl tensor.

Non-tilted perfect fluid models of type III

It is a special case of the non-tilted perfect fluid solution found by Collins [14] for \( h = -1 \). The HVF is \( H = \psi t\partial_t + b\partial_x \) and the constants \( c_{\alpha\beta}, \psi, b \) are given by

\[ c_{13} = c_{23} = c_{12} = 0, \quad c_{11} = c_{22} = 1, \quad c_{33} = \frac{\gamma^2}{(2 - \gamma)(3\gamma - 2)}, \quad \mu = \frac{4(1 - \gamma)}{\gamma^2}, \quad \psi = \frac{by}{3\gamma - 2} \]

\(^1\) We recall that an extreme tilted equilibrium point satisfies \( v^\mu v_\mu = 1 \).
Tilted perfect fluid models of type III

Using the results of section 2 we can show that there is no physically acceptable tilted perfect solution of type III. We note that this conclusion can also be confirmed from the set of the tilted equilibrium points of the dynamical state space given in [15].

Vacuum plane wave models of type IV

From the self-similar metrics (3.18) we can find the general type IV vacuum solution, first given in [16]. In this case, the HVF is
\[ \mathbf{H} = \psi t \partial_t + D_1 \partial_x + ay \partial_y + az \partial_z, \]
or, by performing a suitable change of the basis of the homothetic Lie algebra,
\[ \mathbf{H} = \psi t \partial_t + (D_1 - a) \partial_x - az \partial_y. \]
In addition, the constants \( c_{\alpha\beta}, \psi \) are given by
\[ c_{13} = c_{23} = 0, \quad c_{22} = \frac{D_1(c_{11}^2 - 4c_{12}^2) + 4ac_{12}^2}{4c_{11}(a - D_1)}, \quad c_{33} = \frac{a^2}{D_1^2}, \quad \psi = a. \] (4.3)

It can be verified that the above one-parameter family of models is also algebraically special and admits the gradient null KVF
\[ l^a = e^{-ax/D_1} \left( \partial_t + D_1 \frac{a}{D_1} \partial_x \right). \]

Non-tilted and tilted perfect fluid models of type IV

It is found that no orthogonal or tilted perfect fluid self-similar models of type IV exist. Nevertheless, there exists the following extreme tilted equilibrium point [17]:
\[ N_2 = 0, \quad N_3 = N_3, \quad v^\alpha v_\alpha = 1, \]
\[ \Sigma^2 = N_1^2 + 3q^2, \quad \Sigma_3 = \Sigma_1 = \Sigma_2 = 0, \quad \Sigma_+ = \frac{q}{2} \]
\[ \Sigma_1 = \frac{\sqrt{3}N_1}{6}, \quad v_1 = 1, \quad A_1 = \frac{2 - q}{2} \]
\[ \Omega = \frac{3q(2 - q) - N_2^2}{6}, \quad 0 < \gamma < 2. \]

Models of type V

The analysis has shown that no self-similar vacuum or perfect fluid models exist for type V. This result can also be proved by using the set of evolution equations given in [6] (see also [15]). However, there exists the following extreme tilted equilibrium point [18, 19]:
\[ N_2 = N_3 = 0, \quad v^\alpha v_\alpha = 1, \]
\[ \Sigma^2 = (A_1 - 1)^2, \quad \Sigma_3 = \Sigma_1 = \Sigma_2 = 0, \quad \Sigma_+ = \frac{q}{2} \]
\[ v_1 = 1, \quad A_1 = \frac{2 - q}{2}, \quad \Omega = 2A_1(1 - A_1), \quad 0 < \gamma < 2. \]

Vacuum plane wave models of type VI_{h_R}

In this case, the HVF is
\[ \mathbf{H} = \psi t \partial_t + D_1 \partial_x + ay \partial_y + b \partial_z, \]
and the constants \( c_{\alpha\beta}, \psi \) are given by [13]
\[ c_{13} = c_{23} = 0, \quad c_{11} = \frac{c_{12}^2(h_R + 1)[D_1(h_R - 1) - a - b]}{2c_{22}D_1(h_R + 1) - a - bh_R}, \]
\[ c_{33} = \frac{(ah_R - b)^2}{[D_1(h_R - 1) + a - b]^2}, \quad \psi = \frac{ah_R - b}{h_R - 1}. \] (4.4)
The gradient null KVF is
\[ l^a = \exp \left( - \frac{ah_R - b}{D_1(h_R - 1) + a - b} \right) \left[ \partial_t + \frac{D_1(h_R - 1) + a - b}{(ah_R - b)^2} \partial_x \right]. \]
As we have mentioned in section 2, in order to determine the corresponding self-similar solution for the exceptional type VI_{-1/9} models we can use the geometric results (3.34)–(3.36) and solve the FE by setting \( h_R = -2 \) or \( h_R = -1/2 \) (for these values of the group parameter the 01 or 02 component of (2.3) vanishes and essentially represents the ‘exceptional’ behaviour of the VI_{-1/9} models in terms of the FE). The resulting solution was first given by Robinson and Trautman (RT) [20] (see also [13]) and has the following form \((c_{11} \neq 0)\):

\[
\begin{align*}
  c_{12} &= c_{23} = 0, & c_{33} &= \frac{75}{8}, & c_{11} &= \frac{24c_{13}^2}{125}, \\
  D_1 &= -\frac{b}{3}, & \psi &= \frac{5b}{6}, & a &= \frac{b}{3}.
\end{align*}
\]  

Non-tilted perfect fluid models of type VI_{h_R}

The family of non-tilted perfect fluid models has been found by Collins [14]. The HVF is given by equation (3.34) and the constants \( c_{\alpha\beta}, \psi, a, b \) are

\[
\begin{align*}
  c_{13} &= c_{23} = c_{12} = 0, & c_{33} &= c_{22} = 1, & c_{11} &= \frac{\gamma^2(h_R - 1)^2}{(2 - \gamma)(3\gamma - 2)}, \\
  \mu &= -\frac{4(h_R^2(\gamma - 1) + h_R\gamma + \gamma - 1)}{\gamma^2(h_R - 1)^2}, & \psi &= \frac{\gamma(D_1 - a)(h_R - 1)}{h_R(2 - 3\gamma)}, \\
  b &= \frac{D_1(h_R + 1) - a}{h_R}.
\end{align*}
\]

The exceptional type VI_{-1/9} non-tilted perfect fluid solution can be found similarly by setting \( h_R = -2 \) (or \( h_R = -1/2 \)) and has been given in [21]:

\[
\begin{align*}
  c_{12} &= c_{23} = 0, & c_{33} &= \frac{75}{8}, & \psi &= \frac{5b}{2}, & D_1 = 0, & a = 2b, \\
  \mu &= \frac{9(25c_{11} - 24c_{13}^2)}{25(75c_{11} - 8c_{13}^2)t^2}, & \gamma &= \frac{10}{9}.
\end{align*}
\]

Tilted perfect fluid models of type VI_{h_R}

It is convenient to employ the constants \( p_1, p_2 \):

\[
D_1 = a + p_1\psi - 2\psi, \quad a = bh_R - \psi(p_1 + p_2 - 4).
\]

The frame components of the metric and the fluid velocity become

\[
\begin{align*}
  g_{\alpha\beta} &= \begin{pmatrix}
    c_{11}t^{2p_1} & g_{21}c_{12}t^{2(p_1 + 1)} & g_{31}c_{13}t^{2p_1} \\
    g_{21}c_{12}(b(h_R^2 - 1) + h_R\psi(2 - p_2 + p_1)\psi)/\psi & g_{22}t^{2(h_R^2 - 1) + h_R\psi(2 - p_2 + 2\psi)/\psi} & g_{23}t^{2h_R^2 - 1 + h_R\psi(2 - p_2 + 2\psi)/\psi} \\
    g_{31}c_{13}t^{2p_1} & g_{32}c_{23}t^{2h_R^2 - 1 + h_R\psi(2 - p_2 + 2\psi)/\psi} & g_{33}c_{33}t^{2p_1}
  \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
  \Delta_1 &= v_1t^{p_1 - 1}, & \Delta_2 &= v_2t^{b(h_R^2 - 1) + h_R\psi(2 - p_2 + 2\psi)/\psi}, & \Delta_3 &= v_3t.
\end{align*}
\]

whereas the HVF assumes the form

\[
H = \psi t \partial_t + [bh_R - \psi(p_2 - 2)] \partial_x + [bh_R - \psi(p_1 + p_2 - 4)] y \partial_y + b z \partial_z.
\]

For the case \( h_R \neq -2 \) (or \( h_R \neq -1/2 \)), the general self-similar solution has been found recently [15]. Here, we give the corresponding general solution for the type VI_{-1/9} models.
We define the parameter $s$ according to
\[ \gamma = \frac{2}{2s + 1}. \]  
(4.12)

Then, the 00-conservation equation implies that the constant $p_1$ is given by
\[ p_1 = \frac{2(6s + 1) - p_2}{4}. \]  
(4.13)

The various integration constants are
\[ c_{12} = c_{13} = v_1 = 0, \quad \psi = -\frac{2b}{p_2 - 2}. \]  
(4.14)

\[ c_{22} = \frac{c_{23}^2(p_2 - 2)^2[2(3s - 1) - p_2][−9p_2^2 + 4p_2(3 - 2s) + 624s^2 - 208s + 12]}{96s[25p_2^2 - 20p_2(6s + 1) + 4(6s + 1)^2]} \]
\[ c_{33} = \frac{48[3p_2^2 + 4p_2(8s - 3) - 4(12s^2 + 8s - 3)]}{(p_2 - 2)^2[9p_2^2 + 4p_2(2s - 3) - 4(156s^2 - 52s + 3)]} \]
\[ \Gamma^2 = \frac{s(5p_2 - 12s - 2)(2s^2 - 1) - p_2][−9p_2^2 + 4p_2(3 - 2s) + 624s^2 - 208s + 12]}{2[p_2 + 2(2s - 1)][p_2^2(31s - 6) + 4p_2s(78s - 17) - 4(1476s^3 - 660s^2 + 101s - 6)]} \]
\[ v_3 = \frac{9p_2^3 + 4p_2(4s - 15) - 12p_2(52s^2 - 16s - 1) + 8(156s^2 - 52s + 3)}{36(p_2 + 4s - 2)[p_2 + 2(2s - 1)]} \]
\[ v_2 = \frac{3c_{23}[p_2^2 + 4p_2(s - 1) - 4(2s - 1)]}{4[2(6s + 1) - 5p_2]}. \]  
(4.15)

The constant $p_2$ is related to the ‘state parameter’ $s$ according to
\[ p_2 = \frac{2[−6\sqrt{36s^4 - 204s^3 + 133s^2 - 22s + 1}(4s - 1)] + s(336s^3 - 214s + 19)}{160s^2 - 91s + 66}, \]  
(4.16)

and the energy density of the model is
\[ \bar{\mu} = \frac{(2s + 1)[p_2^2(31s - 6) + 4p_2s(78s - 17) - 4(1476s^3 - 660s^2 + 101s - 6)]}{144s(p_2 - 12s + 2)\gamma^2}. \]  
(4.17)

In order to ensure the positivity of the energy density and the real values of the constant $p_2$ (together with the overall signature of the metric) this new exact solution is only defined when
\[ s \in \left(\frac{1}{3}, \frac{1}{2}\right) \quad \leftrightarrow \quad \gamma \in \left(\frac{1}{3}, \frac{2}{3}\right). \]  
(4.18)

It is interesting to note that, similar to the types VI₀ and VI₀, the above rotating cosmological model admits the hypersurface orthogonal KVΦ $X_2 = \partial_\gamma$ and belongs to the subclass $n_2^* = 0$. In fact, the present solution arises as an equilibrium point of the tilted perfect fluid Bianchi type VI-1,9 dynamical state space which we denote by $E$ and has the following kinematical and dynamical quantities:
\[ N_1 = \frac{3[q(3\gamma - 6) + 7\gamma - 6][q^2(4\gamma^2 - 4\gamma - 12) - 2q(3\gamma^2 - 22\gamma + 12) - 37\gamma^2 + 48\gamma - 12]}{16\gamma^2[q(\gamma - 2) + 2(\gamma - 1)]} \]
\[ N_2 = -N_1, \quad \Sigma_+ = 0, \quad \Sigma_\gamma = -\frac{q}{2}, \quad v_2 = -v_3. \]
We also note that there exists the following extreme tilted equilibrium point [19]:

\[ \Sigma_1 = -\frac{\sqrt{3}[q(5\gamma - 12) + 2(7\gamma - 6)]}{6\gamma}, \]
\[ \Sigma_3 = -\frac{\sqrt{6(2 - \gamma)}[2q^2(\gamma - 3) + 3q(3\gamma - 4) + 7\gamma - 6]}{12\gamma\sqrt{(q + 1)[q(\gamma - 2) + 2(\gamma - 1)]}}, \]
\[ \Sigma_{13} = \frac{\sqrt{6(2 - \gamma)}[2q^3(\gamma - 3) + q^2(11\gamma - 18) + 2q(8\gamma - 9) + 7\gamma - 6]}{12\gamma\sqrt{(q + 1)^3[q(\gamma - 2) + 2(\gamma - 1)]}}, \]
\[ v_1 = \frac{3(2q - 3\gamma + 2)}{2N_2\gamma}, \]
\[ v_3 = -\frac{3\sqrt{2(\gamma - 3)(q + 1)(2q - 3\gamma + 2)}}{8N_3\gamma\sqrt{q(\gamma - 2) + 2(\gamma - 1)}}, \]
\[ v^2 = \frac{9(2q - 3\gamma + 2)^2[3q(\gamma - 2) + 7\gamma - 6]}{16N_3^2\gamma^2[q(\gamma - 2) + 2(\gamma - 1)]}, \]
\[ \Omega = \frac{[q(3\gamma - 6) + 7\gamma - 6][2q(\gamma - 3) + 7\gamma - 6][4q^2(4 - \gamma) + 2q(8 - 15\gamma) + 27\gamma^2 - 26\gamma]}{12\gamma^2(2q - 3\gamma + 2)[q(\gamma - 2) + 2(\gamma - 1)]}, \]
\[ \Sigma_2 = \frac{[3q(\gamma - 2) + 7\gamma - 6][4q^2(2\gamma^2 - 8\gamma + 9) + 4q(9\gamma^2 - 29\gamma + 18) + 49\gamma^2 - 84\gamma + 36]}{12\gamma^2[q(\gamma - 2) + 2(\gamma - 1)]}, \]
\[ q = \gamma\sqrt{(\gamma - 1)(73\gamma^3 - 253\gamma^2 + 240\gamma - 36)[3\gamma^2 - 4\gamma - 2(\gamma - 1)(33\gamma^3 - 179\gamma^2 + 294\gamma - 144)]} \]
\[ 6(\gamma - 1)(3\gamma^2 - 19\gamma + 24). \]

We also note that there exists the following extreme tilted equilibrium point [19]:

\[ N_2 = N_2, \quad N_3 = N_3, \quad v^a v_a = 1, \]
\[ \Sigma_2 = \frac{3(N_3 - N_2)^2 - 4(N_2 N_3 + 6\sqrt{-N_2 N_3} - 6)}{36}, \quad \Sigma_3 = \Sigma_{13} = \Sigma_- = 0, \]
\[ \Sigma_1 = \frac{\sqrt{3}(N_3 - N_2)}{6}, \quad \Sigma_+ = -\frac{q}{2}, \quad q = \frac{2(3 - \sqrt{-N_2 N_3})}{3}, \quad v_1 = 1, \]
\[ \Omega = -\frac{3(N_3 - N_2)^2 - 4(N_2 N_3 + 3\sqrt{-N_2 N_3})}{18}, \quad 0 < \gamma < 2. \]

Models of type VII_{h,k}

The question of the existence of tilted perfect type VII_{h,k} models is notoriously difficult to answer due to the complexity of the self-similar metrics (3.44)–(3.48). In fact, the resulting FE are difficult to handle analytically even for the case of expressing the known VII_{h,k} vacuum solution (see, e.g., [7] p 192 for an elegant form of this solution and [22] for a complete study of its stability properties against vorticity, shear and Weyl curvature perturbations). Nevertheless, using the set of evolution equations we can show that the assumption of the existence of a proper HVF in incompatible with tilted perfect fluid models and that type VII_{h,k} models possess the following extreme tilted equilibrium point (this equilibrium point has also been given in [17]):

\[ N_2 = N_2, \quad N_3 = N_3, \quad v^a v_a = 1, \]
\[ \Sigma_2 = \frac{(N_3 - N_2)^2 + 12(h N_2 N_3 - 2\sqrt{h} N_2 N_3 + 1)}{12}, \quad \Sigma_3 = \Sigma_{13} = \Sigma_- = 0, \]
Table 1. This table contains all the self-similar tilted perfect fluid SH models and the relevant range of the state parameter $\gamma$.

| Type | Self-similar model | Reference |
|------|-------------------|-----------|
| I    | $\mathbb{R}$     | [23]      |
| II   | $\gamma \in \left(\tfrac{3}{4}, 2\right)$ | [2]       |
| VI0  | $\gamma \in \left[\tfrac{5}{7}, \tfrac{3}{2}\right)$ | [4, 5]    |
| VII0 | $\mathbb{R}$     | [1]       |
| III  | $\mathbb{R}$     | [3]       |
| IV   | $\mathbb{R}$     | [3]       |
| V    | $\mathbb{R}$     | Paper II  |
| VI   | $\gamma \in \left(\tfrac{1}{3}, \tfrac{1}{2}\right)$ | [15]      |
| VII  | $\mathbb{R}$     | Paper II  |
| VI_{1/9} | $\gamma \in \left(\tfrac{1}{4}, \tfrac{1}{2}\right)$ | Paper II  |

\[
\Sigma_1 = \frac{\sqrt{3}(N_3 - N_2)}{6}, \quad \Sigma_\tau = -\frac{q}{2}, \quad q = 2(1 - \sqrt{hN_2N_3}), \quad v_1 = 1, \\
\Omega = -\frac{(N_3 - N_2)^2 + 12(hN_2N_3 - \sqrt{hN_2N_3})}{6}, \quad 0 < \gamma < 2.
\]

5. Discussion

The main concern of papers I and II can be regarded as having two main branches:

(a) a geometric nature concerning the determination of all the SH geometries restricted only by the requirement of admitting a proper HVF and

(b) the physical implications of the general geometric results by finding, whenever they exist, the corresponding vacuum or perfect fluid models.

The underlying importance of (a) is that we have not incorporated a specific form for the matter fields filling the spacetime, therefore we can associate the self-similar metrics with more general matter configurations with a view to analysing their physical significance. On the other hand, the well established aspect of the SH equilibrium points as past or future attractors for general vacuum or perfect fluid models necessitates the knowledge of all the self-similar equilibrium points in order to gain deeper insight into their asymptotic dynamics. This was accomplished in the series of articles I, II, [3, 4, 15] in which the complete set of tilted perfect fluid self-similar models has been found. For convenience, we collect them in table 1 together with the corresponding references.

As an immediate application of the geometric results of section 3, we reproduced the known exact vacuum and non-tilted solutions for the Bianchi types of class B and we have found the general form of the self-similar tilted type VI_{1/9} models. As we have seen in section 4 this new exact solution arises as an equilibrium point in the VI_{1/9} state space and shares many of the common properties of type VI0 and VI_h ($h \neq -1/9$) models and important differences as well. For example, the stability analysis of vacuum and non-tilted equilibrium
points [25] has shown that the Collins VI $-1/9$ solution is stable for $\gamma \in \left(\frac{2}{3}, \frac{10}{7}\right)$. At the arc of equilibrium points $\gamma = \frac{10}{7}$ [24] there is an exchange of stability with the RT vacuum solution $[8, 20]$ which is stable whenever $\gamma \in \left(\frac{10}{7}, \frac{4}{3}\right)$. As a consequence, we expect that tilted models will be future dominated against the RT model when $\gamma > \frac{4}{3}$. Indeed, the equilibrium point $E$ is found to have a five-dimensional stable manifold for $\gamma \in \left(\frac{4}{3}, \frac{3}{2}\right)$. However, it can be shown that, at least for the case where $n_\alpha^\alpha = 0$, there always exist two unstable modes in the restricted state space. One can make a step further and conjecture that for the type VI$_h$ and VI $-1/9$ models the values $\gamma = \frac{2(3+\sqrt{-h})}{5+3\sqrt{h}}$ and $\gamma = \frac{4}{3}$, respectively, represent line bifurcations similarly to the type VI$_0$ models with $\gamma = \frac{5}{2}$. However, no such bifurcations are found in either models.

Furthermore, for the remaining types, the mutual use of the metric and orthonormal frame approaches guarantees that tilted perfect fluid models of types III, IV, V and VII$_h$ cannot admit a proper HVF indicating that these models either may not be asymptotically self-similar (a preliminary analysis suggests that this behaviour occurs in type V models) or may be extreme tilted at late times (see [17] for a thorough discussion on the asymptotic dynamics of type IV models).

We conclude by noting that an interesting aspect of the analysis presented in papers I and II is that vacuum and tilted perfect fluid SH self-similar models exhibit further geometric constraints coming from the existence of covariantly constant null vector fields or a hypersurface orthogonal KVF. Therefore, we believe that it will be interesting to study how the assumptions of the self-similarity and a specific dynamic description lead to extra restrictions on the geometry of the SH models.

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