Longitudinal Pochhammer - Chree Waves. Secular Equations and Spectral Analysis

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Abstract. The paper presents the spectral analysis of the Pochhammer - Cree matrix dispersion equation for longitudinal axisymmetric modes. Analytical solutions of the matrix dispersion equation describing the propagation of longitudinal harmonic waves in an infinite cylindrical rod are analyzed. Expressions for displacement fields are obtained. The dependence of displacements on the surface on the angular frequency and Poisson's ratio is analyzed. When the phase velocity is equal to the transverse wave velocity, the displacements on the surface completely disappear, which means that no axisymmetric Pochhammer - Cree waves propagate with such a phase velocity.

1. Introduction

The research related to the propagation of Pochhammer - Cree harmonic waves in rods has been conducted for over a century. These results are applied in various fields.

The first equations describing the behavior of elastic waves in an infinitely long cylindrical rod were obtained by Pochhammer [1] and Cree [2, 3]. Later in [4-10], dispersion curves (dependence of the phase velocity on frequency) were obtained numerically; in these works, bending, torsional and longitudinal axially symmetric modes were considered.

An expression for the short-wavelength limit of the phase velocity was obtained by asymptotic methods \( c_{1,lim} = c_R \). Where \( c_R \) is the speed of Rayleigh's wave [11]. And the long-wavelength limit, which is also called the core velocity and is defined by the expression \( c_{2,lim} = \sqrt{E/\rho} \). Here \( E \) is Young's modulus, \( \rho \) is the density of the rod material.

Pochhammer - Cree waves in non-circular rods were studied in [12, 13] using the finite element method.

2. Basic relations

This section mainly uses the conclusions from [14]. For a homogeneous isotropic elastic body, the equation of motion can be written in the following form

\[
\dot{c}_1^2 \nabla \text{div} \mathbf{u} - \dot{c}_2^2 \text{rot rot} \mathbf{u} = \ddot{\mathbf{u}}
\]

(1)
Where \( \mathbf{u} \) is the displacement field, \( c_1 = \sqrt{\lambda + 2\mu/\rho} \) and \( c_2 = \sqrt{\mu/\rho} \) are the volumetric velocities of longitudinal and transverse waves in an infinitely elastic medium, \( \lambda \) and \( \mu \) are the Lame constants, \( \rho \) is the density of the material.

Applying the Helmholtz expansion theorem to the displacement field, we obtain the following representation
\[
\mathbf{u} = \nabla \Phi + \text{rot} \Psi
\]  
(2)

here \( \Phi \) is scalar and \( \Psi \) is vector potentials. This cylindrical representation for the displacement components takes the form:
\[
\begin{align*}
 u_r &= \frac{\partial \Phi}{\partial r} + \frac{1}{r} \frac{\partial \Psi_z}{\partial \theta} - \frac{\partial \Psi_\theta}{\partial z} \\
u_\theta &= \frac{1}{r} \frac{\partial \Phi}{\partial \theta} + \frac{\partial \Psi_r}{\partial z} - \frac{\partial \Psi_z}{\partial r} \\
u_z &= \frac{\partial \Phi}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \Psi_\theta \right) - \frac{1}{r} \frac{\partial \Psi_r}{\partial \theta}
\end{align*}
\]  
(3)

Taking into account axial symmetry
\[
u_\theta = 0
\]  
(4)

Substituting the Helmholtz representation (2) into the equation of motion (1), we obtain 2 equations:
\[
c_1^2 \Delta \Phi = \ddot{\Phi}, \quad c_2^2 \Delta \Psi = \ddot{\Psi}
\]  
(5)

The potentials (5) for a harmonic wave propagating along the z-axis can be represented in the following form:
\[
\Phi = \Phi_0(x') e^{-i\gamma(ct-z)}, \quad \Psi = \Psi_0(x') e^{-i\gamma(ct-z)}
\]  
(6)

where \( \gamma = \omega/c \) is the wavenumber, \( c \) is the phase velocity, \( \omega \) is the circular frequency, and \( x' \) is the coordinate in the cross section.

When substituted (6) into (5), we obtain the Helmholtz equations for the corresponding potentials
\[
\Delta \Phi_0 + \left( \frac{c_1^2}{c_1^2} - 1 \right) \gamma^2 \Phi_0 = 0, \quad \Delta \Psi_0 + \left( \frac{c_2^2}{c_2^2} - 1 \right) \gamma^2 \Psi_0 = 0
\]  
(7)

When passing the scalar potential (7) to cylindrical coordinates, taking into account the axial symmetry (\( \partial \Phi_0 / \partial \theta = 0 \)), we obtain the Bessel equation:
\[
1 \frac{d}{dr} r \frac{d}{dr} \Phi_0(r) + \left( \frac{c_1^2}{c_1^2} - 1 \right) \gamma^2 \Phi_0(r) = 0
\]  
(8)

Its solution is expressed in terms of the Bessel functions of the first and second kind.
\[
\Phi_0(r) = C_1 J_0(q_1 r) + C_2 Y_0(q_1 r)
\]  
(9)

where \( C_1, C_2 \) are unknown complex coefficients, \( q_1^2 = \left( c_1^2/c_1^2 - 1 \right) \gamma^2 \)

The axial symmetry condition for the vector potential has the form
\[
\partial \Psi_r / \partial \theta = \partial \Psi_\theta / \partial \theta = \partial \Psi_z / \partial \theta = 0
\]  
(10)
When passing to cylindrical coordinates for the vector potential, taking into account the condition (10), we obtain the Bessel equations (for each of the components)

\[
\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \Psi_r (r) + \left( \frac{c^2}{c^2} - 1 \right) \gamma^2 - \frac{1}{r^2} \Psi_r (r) = 0
\]

\[
\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \Psi_\theta (r) + \left( \frac{c^2}{c^2} - 1 \right) \gamma^2 - \frac{1}{r^2} \Psi_\theta (r) = 0
\]

\[
\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \Psi_z (r) + \left( \frac{c^2}{c^2} - 1 \right) \gamma^2 \Psi_z (r) = 0
\]

The solutions to these equations have the form

\[ \Psi_\theta (r) = C_3 J_1 (q_2 r) + C_4 Y_1 (q_2 r) \]

\[ \Psi_r (r) = C_5 J_1 (q_2 r) + C_6 Y_1 (q_2 r) \]

\[ \Psi_z (r) = C_7 J_0 (q_2 r) + C_8 Y_0 (q_2 r) \]

where \( C_3, \ldots, C_8 \) are unknown complex coefficients, \( q_2^2 = \left( \frac{c^2}{c^2} - 1 \right) \gamma^2 \)

The condition of axial symmetry of the vector potential imposes one more restriction [6]:

\[ \Psi_r = \Psi_z = 0 \]

Considering (3), (4), (9), (12), (13), the vector field of displacements is represented in the form [8]

\[ u_r = - \left( q_1 (C_1 J_1 (q_1 r) + C_2 Y_1 (q_1 r)) + i \gamma (C_3 J_1 (q_2 r) + C_4 Y_1 (q_2 r)) \right) e^{-i \gamma (ct - z)} \]

\[ u_\theta = 0 \]

\[ u_z = \left( i \gamma (C_1 J_0 (q_1 r) + C_2 Y_0 (q_1 r)) + q_2 (C_3 J_0 (q_2 r) + C_4 Y_0 (q_2 r)) \right) e^{-i \gamma (ct - z)} \]

Taking into account that the Bessel functions of the second kind are not bounded for \( r = 0 \), we obtain

\[ u_r = - \left( q_1 C_1 J_1 (q_1 r) + i \gamma C_2 J_1 (q_2 r) \right) e^{-i \gamma (ct - z)} \]

\[ u_\theta = 0 \]

\[ u_z = \left( i \gamma C_1 J_0 (q_1 r) + q_2 C_2 J_0 (q_2 r) \right) e^{-i \gamma (ct - z)} \]

In expressions (15), a constant \( C_3 \) is indicated by \( C_2 \).

Comment. In the expression (15) at \( r = 0 \), the displacement component \( u_r = 0 \) disappears. This is due to \( J_1 (0) = 0 \). At the same time \( J_0 (0) = 1 \), the expression \( u_z \) takes the following form

\[ u_z = \left( i \gamma C_1 + q_2 C_2 \right) e^{-i \gamma (ct - z)} \]

This expression gives the following condition for the disappearance of the displacement components \( u_z \) at \( r = 0 \)

\[ \gamma C_1 = i q_2 C_2 \]

The condition that the forces on the lateral surface of the cylinder are equal to zero can be represented as
\[ \mathbf{T}_v \equiv \left( \lambda (\text{tr} \epsilon) \mathbf{v} + 2 \mu \epsilon \cdot \mathbf{v} \right)_{r=R} = 0 \]  
\[(18)\]

here \( \mathbf{v} \) is the unit vector of the outward normal to the lateral surface

Substituting (18) into the previously obtained expressions (15), we obtain the boundary conditions (up to an exponential factor \( e^{-i\gamma (ct-z)} \))

\[ t_{rr} \equiv \lambda I_e + 2 \mu \epsilon_{rr} = - \left[ \begin{array}{c} \lambda \left( q_1^2 + \gamma^2 \right) J_0(q_1r) C_1 + \\ + \frac{2\mu}{r} \left[ q_1 C_1 \left( q_1 r J_0(q_1r) - J_1(q_1r) \right) + \\ + i\gamma C_2 \left( q_2 r J_0(q_2r) - J_1(q_2r) \right) \right]_{r=R} \end{array} \right] = 0 \]
\[(19)\]

\[ t_{rz} = 2 \mu \epsilon_{rz} = - \mu \left[ \left[ \gamma \left[ i q_1 C_1 J_1(q_1r) - \gamma C_2 J_1(q_2r) \right] + \\ + \left[ i \gamma q_1 C_1 J_1(q_1r) + q_2^2 C_2 J_1(q_2r) \right] \right]_{r=R} \right] = 0 \]

Using this identity \( \lambda/\mu = c_1^2/c_2^2 - 2 \), the conditions (19) result in the dispersion equation

\[ \det \mathbf{A} = 0 \]
\[(20)\]

The matrix elements \( A \) are the following expressions

\[ A_{11} = - \lambda \left( q_1^2 + \gamma^2 \right) J_0(q_1R) - \frac{2\mu}{R} q_1 \left( q_1 R J_0(q_1R) - J_1(q_1R) \right) \]
\[ A_{12} = - \frac{2i\mu\gamma}{R} \left( q_2 R J_0(q_2R) - J_1(q_2R) \right) \]
\[ A_{21} = - 2i\mu \gamma q_1 J_1(q_1R) \]
\[ A_{22} = - \mu \left( q_2^2 - \gamma^2 \right) J_1(q_2R) \]
\[(21)\]

The eigenvectors of this matrix dispersion equation, which are advised to be zero eigenvalues, determine the polarization of the waves.

3. Spectral analysis of the matrix \( A \)

Spectral analysis is an analysis of the eigenvalues and eigenvectors of a differential equation describing the propagation of Pochhammer - Cree waves.

To search for eigenvalues, a characteristic equation is used, which can be represented in the following form (here \( \mathbf{I} \) is the identity matrix, \( \lambda \) is eigenvalues)

\[ \det (\mathbf{A} - \lambda \mathbf{I}) = 0, \]
\[(22)\]

With the following replacement

\[ s = \frac{A_{11} + A_{22}}{2}, \quad d = \sqrt{f^2 + A_{12} A_{21}}, \quad f = \frac{A_{11} - A_{22}}{2}, \]
\[(23)\]

We obtain the characteristic equation, which is investigated in the work

\[ \lambda^2 - 2s\lambda + s^2 - d^2 = 0 \]
\[(24)\]

The spectral analysis of the matrix \( \mathbf{A} \) is divided into 2 cases.
3.1. The matrix $\mathbf{A}$ is simple (semisimple)

The matrix $\mathbf{A}$ does not contain Jordan blocks and has two different eigenvectors. In spectral decomposition, each eigenvalue corresponds to an eigenvector $\bar{\alpha}_1 \leftrightarrow \lambda_1$, $\bar{\alpha}_2 \leftrightarrow \lambda_2$. Here $\bar{\alpha}_{1,2}$ are eigenvectors.

The solution to the equation (24) is the eigenvalues

$$\lambda_{1,2} = s \pm d$$

(25)

The corresponding (normalized) eigenvectors are determined by the expression

$$\bar{\alpha}_{1,2} = \frac{1}{\sqrt{|A_{21}|^2 + |f| d^2}} \begin{pmatrix} f \\ A_{21} \end{pmatrix}. \quad (26)$$

Analyzing expressions (25), (26) we can formulate the following.

Suggestions 1:

a) A necessary and sufficient condition for the simplicity of the matrix is the condition

$$d \neq 0. \quad (27)$$

b) The condition for the degeneration of the matrix $\mathbf{A}$ has the form

$$A_{11} A_{22} = A_{12} A_{21}. \quad (28)$$

Evidence:

a) Analyzing the expression (25), we see that a necessary and sufficient condition for the simplicity of the matrix $\mathbf{A}$ is the condition (27), since under this condition, we obtain two different eigenvectors.

b) According to (25), the degeneracy condition has the form

$$s^2 = d^2, \quad (29)$$

since in this case, one of the eigenvalues vanishes. Given the substitution (23), the expression (29) results in the equation (28).

Remark 1. Note that the condition (28) is equivalent to the dispersion equation (20).

3.2. The matrix $\mathbf{A}$ is not semisimple

This is the case when the matrix $\mathbf{A}$ contains a Jordan block. The expression (25) implies the non-semisimplicity condition of the matrix

$$d = 0. \quad (30)$$

The spectral decomposition of the matrix $\mathbf{A}$ taking into account (30) has the form

$$\frac{1}{\sqrt{|A_{21}|^2 + |f|^2}} \begin{pmatrix} f \\ A_{21} \end{pmatrix} \leftrightarrow \lambda_{1,2} = s. \quad (31)$$

Taking into account (30), the matrix $\mathbf{A}$ becomes not only a complex matrix, but also non-semisimple, since it has only one eigenvector (31).

Under the conditions (30) and (25), the double degeneracy of the matrix $\mathbf{A}$ corresponds to

$$s = 0. \quad (32)$$

Taking into account substitutions (23), the following suggestions follow

Suggestions 2:

a) The necessary and sufficient condition for the non-semisimplicity of the matrix $\mathbf{A}$ has the form

$$f^2 = -A_{21} A_{22}. \quad (33)$$

b) The double degeneration of the matrix $\mathbf{A}$ taking into account (33) takes the form

$$A_{11} = -A_{22}. \quad (34)$$
Evidence:

a) Suggestion 1. a guarantees that with (30), the matrix $A$ becomes nonsemisimple, since its two eigenvalues coincide. But, under the condition (30), given the expressions for the corresponding eigenvectors (26), both eigenvectors coincide, so $A$ becomes non-semisimple. Then the substitution of (23) into (30) leads to the condition (33).

b) Degeneration of the matrix $A$ considering (25) and (30) is equivalent to the following condition

$$s = 0.$$ (35)

But the condition (35) is equivalent to (34).

Comment 2. For the considered case of degeneration of a nonsemisimple matrix, the corresponding dispersion equation takes the form

$$(q_2^2 - s^2) \left( J_1(q_2 R) \right)^2 - \gamma^2 q_1 q_2 J_0(q_2 R) J_1(q_1 R) + \frac{\gamma^2 q_1}{R} J_1(q_2 R) J_1(q_1 R) = 0.$$ (36)

4. Conclusions

Analytical solutions of the linear Pochhammer - Cree equation for propagating harmonic axisymmetric longitudinal waves $L(0, m)$ in a cylindrical rod are analyzed.

Spectral analysis of the matrix dispersion equation for longitudinal axisymmetric modes (L(0, m), m > 0) of Pochhammer - Cree waves is carried out, which showed that no longitudinal modes can propagate with a phase velocity equal to the volume velocity of transverse waves in an infinitely elastic medium ($c_v$).

The change in the polarization of the wave on the free surface on the angular frequency and Poisson's ratio is analyzed. It was found that there is a phase velocity dependent on Poisson’s ratio $c_k(v)$ at which the longitudinal component of the fundamental longitudinal mode $U_z$ disappears.

5. References

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