A Short Note on some open problems in the geometry of operator ideals

1. The principle of local reflexivity for maximal Banach ideals

Throughout this Short Note, we essentially adopt notation and terminology from [15], which however cannot be explained here in detail, due to the limitation of space. Therefore, we would like to refer the interested reader to [2, 6, 15], the excellent survey article [3] and the further references therein.

Let $\mathfrak{A}, A$ be an arbitrary maximal Banach ideal. It seems to be still an open problem whether it is always possible to transfer the norm estimation in the far-reaching classical principle of local reflexivity to the ideal norm $A$. If this were not the case, we would be very interested in constructing an explicit counterexample. More precisely, we would like to know whether there exists a maximal Banach ideal $(\mathfrak{A}_0, A_0)$ which does not satisfy the following factorization property:

Definition 1.1 (cf. [9, 10]) Let $E$ and $Y$ be Banach spaces, where $E$ is finite dimensional and $F$ is a finite dimensional subspace of $Y'$. Let $(\mathfrak{A}, A)$ be a maximal Banach ideal and $\varepsilon > 0$. We say that the principle of $\mathfrak{A}$-local reflexivity (short: $\mathfrak{A}$-LRP) is satisfied, if for every $T \in \mathfrak{L}(E, Y'')$ there exists an operator $S \in \mathfrak{L}(E, Y)$ such that

(i) $A(S) \leq (1 + \varepsilon) \cdot A(T)$;

(ii) $\langle Sx, y' \rangle = \langle y', Tx \rangle$ for all $(x, y') \in E \times F$;

(iii) $j_Y Sx = Tx$ for all $x \in T^{-1}(j_Y(Y))$.

“Here, $j_Y : Y \hookrightarrow Y''$ denotes the canonical embedding from the Banach space $Y$ into its bidual $Y''$. 
Obviously, the classical principle of local reflexivity is simply the $\mathfrak{B}$-LRP, where $(\mathfrak{B}, \mathfrak{B}) := (\mathfrak{L}, \| \cdot \|)$.

**Problem 1.2** Does every maximal Banach ideal satisfy the $\mathfrak{A}$-LRP?

Due to the finite nature of maximal Banach ideals and the $\mathfrak{A}$-LRP, local versions of injectivity and surjectivity of operator ideals play a key role. Moreover, they imply interesting results for operators with infinite dimensional range (cf. [11, 12, 13]). These are the following so-called “accessibility conditions”, treated in detail in [1] [2]:

**Definition 1.3** Let $(\mathfrak{B}, \mathfrak{B})$ be a $p$-normed Banach ideal, where $0 < p \leq 1$.

(i) $(\mathfrak{B}, \mathfrak{B})$ is called right-accessible, if for any two Banach spaces $E$ and $Y$, $\text{dim}(E) < \infty$, any operator $T \in \mathfrak{L}(E, Y)$ and any $\varepsilon > 0$ there are a finite dimensional subspace $N$ of $Y$ and $S \in \mathfrak{L}(E, N)$ such that $T = J_N^Y S$ and $B(S) \leq (1 + \varepsilon)B(T)$; here $J_N^Y : N \hookrightarrow Y$ denotes the canonical embedding.

(ii) $(\mathfrak{B}, \mathfrak{B})$ is called left-accessible, if for any two Banach spaces $F$ and $X$, $\text{dim}(F) < \infty$, any operator $T \in \mathfrak{L}(X, F)$ and any $\varepsilon > 0$ there are a finite codimensional subspace $L$ of $X$ and $S \in \mathfrak{L}(X/L, F)$ such that $T = SQ_L^X$ and $B(S) \leq (1 + \varepsilon)B(T)$; here $Q_L^X : X \twoheadrightarrow X/L$ denotes the canonical quotient map.

(iii) $(\mathfrak{B}, \mathfrak{B})$ is called accessible, if it is both, right-accessible and left-accessible.

**Proposition 1.4** (cf. [13]) If $(\mathfrak{A}, \mathfrak{A})$ is a right-accessible maximal Banach ideal, then the $\mathfrak{A}$-LRP holds.

We still do not know whether the statement in Proposition [1.4] can be reversed. In 1993, Pisier constructed explicitly a maximal Banach ideal $(\mathfrak{A}_P, \mathfrak{A}_P)$ which is not right-accessible, and solved - 37 years later only - a further problem of Grothendieck’s famous Résumé (cf. [5] and [2]). Consequently, Problem [1.2] is even harder than Grothendieck’s tough accessibility problem. A maximal Banach ideal $(\mathfrak{A}, \mathfrak{A})$ which does not satisfy the $\mathfrak{A}$-LRP, necessarily cannot be right-accessible! Notice that in the investigation of those problems, Banach spaces without the approximation property (such as the Pisier space $P$) are necessarily involved (cf. [2, 9, 13]). Consequently, we arrive at

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Problem 1.5 Let \((\mathfrak{A}, \mathfrak{A})\) be a maximal Banach ideal. Are then the following statements equivalent?

(i) \((\mathfrak{A}, \mathfrak{A})\) is right-accessible.

(ii) \((\mathfrak{A}, \mathfrak{A})\) satisfies the \(\mathfrak{A}\)-LRP.

Problem 1.6 Does \((\mathfrak{A}_p, \mathfrak{A}_p)\) satisfy the \(\mathfrak{A}_p\)-LRP?

Let \((\mathcal{B}, \mathcal{B})\) be a \(p\)-normed Banach ideal, where \(0 < p \leq 1\). In 1971, Pietsch defined the adjoint operator ideal \((\mathcal{B}^*, \mathcal{B}^*)\) to uncover the structure of maximal Banach ideals. In 1973, Gordon-Lewis-Retherford introduced a related smaller “conjugate” operator ideal \((\mathcal{B}^\triangle, \mathcal{B}^\triangle)\), which - in the contrary - is somehow “skew” under the point of view of trace duality but nevertheless quite useful (cf. \([4, 9]\):

Definition 1.7 Let \(X, Y\) two Banach spaces and \(\mathcal{B}^\triangle(X, Y)\) be the set of all those operators \(T \in \mathcal{L}(X, Y)\) satisfying

\[
\mathcal{B}^\triangle(T) := \sup \{ |\text{tr}(TL)| : L \in \mathcal{F}(Y, X), \mathcal{B}(L) \leq 1 \} < \infty.
\]

Then \((\mathcal{B}^\triangle, \mathcal{B}^\triangle)\) is a Banach ideal.\(^\dagger\) It is called the conjugate ideal of \((\mathcal{B}, \mathcal{B})\).

It is easy to see that the various approximation properties of Banach spaces and the accessibility of operator ideals are intrinsically related to a “good behaviour” of trace duality. If \(\mathfrak{A}\) is a maximal Banach ideal, then \((\mathfrak{A}^\triangle, \mathfrak{A}^\triangle)\) is right-accessible (cf. \([9, 10]\)). However, we do not know whether \(\mathfrak{A}^\triangle\) is left-accessible, since:

Theorem 1.8 (cf. \([9, 10]\)) Let \((\mathfrak{A}, \mathfrak{A})\) be a maximal Banach ideal. Then the following statements are equivalent:

(i) \(\mathfrak{A}^\triangle\) is left–accessible.

(ii) \(\mathfrak{A}(E, Y^{\prime\prime}) \cong \mathfrak{A}(E, Y)''\) for all finite dimensional Banach spaces \(E\) and arbitrary Banach spaces \(Y\).

(iii) The \(\mathfrak{A}\)-LRP holds.

\(^\dagger\)Here \(\text{tr} : \mathcal{F}(Y, Y) \to \mathbb{K}\) denotes the trace on the operator ideal component of all finite rank operators on \(Y\).
2. Property (I) and normed products of Banach ideals

After extending finite rank operators in suitable quasi-Banach ideals \( \mathcal{B} \neq \mathcal{L} \), the \( \mathcal{B} \)-LRP and the calculation of conjugate ideal-norms allow us to ignore the structure of the range space and to leave the finite dimensional case. There are sufficient conditions on \( \mathcal{B} \) which imply that for any two Banach spaces \( X \) and \( Y \), any Banach space \( Z \) which contains \( X \) isometrically, any finite rank operator \( L \in \mathcal{F}(X, Y) \) and any \( \varepsilon > 0 \) there exists a finite rank extension \( \tilde{L} \in \mathcal{F}(Z, Y) \) such that \( B(\tilde{L}) \leq (1 + \varepsilon) \cdot B(L) \) (cf. [13]).

Consequently, we have to look for a suitable class of product ideals \( \mathfrak{A} \circ \mathcal{B} \) which satisfy the following factorization property: Given \( \varepsilon > 0 \), any finite rank operator \( L \in \mathfrak{A} \circ \mathcal{B} \) can be factorized as \( L = AB \) such that \( A(A) \cdot B(B) \leq (1 + \varepsilon) \cdot A \circ B(L) \) and \( A \) respectively \( B \) has finite dimensional range. Product ideals \( \mathfrak{A} \circ \mathcal{B} \) of this type are said to have property (I) respectively property (S). They had been introduced in [7] to prepare a detailed investigation of trace ideals.

In view of looking for a counterexample of a maximal Banach ideal \((\mathfrak{A}_0, \mathfrak{A}_0)\) which does not satisfy the \( \mathfrak{A}_0 \)-LRP, property (I) of product ideals of type \( \mathfrak{A}^* \circ \mathcal{L}_\infty \) seemingly plays a key role.

**Theorem 2.1** (cf. [13]) Let \((\mathfrak{A}, \mathfrak{A})\) be a maximal Banach ideal such that \( \mathfrak{A}^* \circ \mathcal{L}_\infty \) has property (I). If space(\( \mathfrak{A} \)) contains a Banach space \( X_0 \) such that \( X_0 \) has the bounded approximation property but \( X_0'' \) has not, then the \( \mathfrak{A}^*-\text{LRP} \) is not satisfied.

A further interesting class is given by the family of all operator ideals which contain \((\mathcal{L}_2, \mathcal{L}_2)\) as a factor (i.e., the maximal and injective Banach ideal of all operators which factor through a Hilbert space). This class does not only show surprising connections with the principle of local reflexivity for operator ideals. There are also links to the existence of an ideal-norm on certain product ideals and the property (I), reflected e.g. in the following two results (cf. [14]):

**Theorem 2.2** Let \((\mathfrak{A}, \mathfrak{A})\) be an arbitrary maximal Banach ideal such that \( \mathfrak{A} \circ \mathcal{L}_2 \) is normed. If the \((\mathfrak{A} \circ \mathcal{L}_2)^*\)-LRP is satisfied, then \((\mathfrak{A} \circ \mathcal{L}_2)^* \circ \mathcal{L}_\infty \) has both, property (I) and property (S).

\(^1\)Recall that there is no Hahn-Banach extension theorem for finite rank operators, viewed as elements of the Banach ideal \((\mathcal{L}, \| \cdot \|))\).
Theorem 2.3 Let \((\mathcal{A}, \mathcal{A})\) be an arbitrary maximal Banach ideal such that \(\mathcal{A} \circ \mathcal{L}_2\) is normed. If \(\mathcal{L}_2 \subseteq \mathcal{A}^*\), then the \((\mathcal{A} \circ \mathcal{L}_2)^{**}\)-LRP cannot be satisfied.

Here is a nice application of Theorem 2.3: Put \(\mathcal{A} := (\mathcal{L}_1 \circ \mathcal{L}_\infty)^{**}\). Due to Grothendieck’s inequality in operator form, it follows that \(\mathcal{L}_2 \subseteq \mathcal{A}^*\) (cf. [2, 12]). Hence, in view of Theorem 2.3, a natural question appears:

Problem 2.4 Is the product ideal \((\mathcal{L}_1 \circ \mathcal{L}_\infty)^{**} \circ \mathcal{L}_2\) normed?

However, we do not know criteria which are sufficient for the existence of an ideal-norm on a given product of quasi-Banach ideals. Here, one has to study very carefully the structure of \(F\)-spaces which are not locally convex, such as \(L^p([0,1])\) and the Hardy space \(H^p\) on the unit disk, for some \(0 < p < 1\) (cf. [8]). It seems to be much easier to provide arguments which imply the non-existence of such an ideal-norm (by using trace ideals) (cf. [14]).

Theorem 2.5 (cf. [12, 13]) Let \((\mathcal{A}, \mathcal{A})\) be a maximal Banach ideal such that \(\mathcal{A}^* \circ \mathcal{L}_\infty\) is right-accessible and has property (I). Let \(X\) and \(Y\) be Banach spaces such that \(X'\) and \(Y\) are of cotype 2. If the \(\mathcal{A}^*-LRP\) is satisfied, then

\[
\text{inj}(X, Y) \subseteq \mathcal{L}_2(X, Y)
\]

and

\[
\mathcal{L}_2(T) \leq (2C_2(X') \cdot C_2(Y))^{\frac{3}{7}} \cdot \text{inj}(T)
\]

for all operators \(T \in \text{inj}(X, Y)\).

Corollary 2.6 Let \((\mathfrak{B}, \mathfrak{B})\) be a maximal Banach ideal. Let \(X_0\) and \(Y_0\) be Banach spaces such that \(X_0'\) and \(Y_0\) are of cotype 2 and \(\mathfrak{B}^*(X_0, Y_0) \nsubseteq \mathcal{L}_2(X_0, Y_0)\). If \(\mathfrak{B} \circ \mathcal{L}_\infty\) is right-accessible and has property (I), then the \(\mathfrak{B}\)-LRP is not satisfied.

Consequently, an application of Proposition 1.4 immediately implies the following surprising result:

Corollary 2.7 Let \((\mathfrak{B}, \mathfrak{B})\) be a maximal Banach ideal. Let \(X_0\) and \(Y_0\) be Banach spaces such that \(X_0'\) and \(Y_0\) are of cotype 2 and \(\mathfrak{B}^*(X_0, Y_0) \nsubseteq \mathcal{L}_2(X_0, Y_0)\). If \(\mathfrak{B}\) is right-accessible, then \(\mathfrak{B} \circ \mathcal{L}_\infty\) does not have property (I).

Problem 2.8 Let \((\mathfrak{C}_2, \mathfrak{C}_2)\) denote the maximal injective Banach ideal of all cotype 2 operators. Does \(\mathfrak{C}_2^* \circ \mathcal{L}_\infty\) have property (I)?
If Problem 2.8 had a positive answer, Corollary 2.7 would imply that \((C_2, C_2)\) cannot be left-accessible; and a further open problem of Defant and Floret could be solved (see [2], 21.2., p. 277).

**Problem 2.9** Can we drop the assumption “\(B \circ L_\infty\) is right-accessible” in Corollary 2.6?

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