Interaction solutions for supersymmetric mKdV-B equation

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The $\mathcal{N} = 1$ supersymmetric mKdV-B system is transformed to a system of coupled bosonic equations by using the bosonization approach. The bosonized supersymmetric mKdV-B (BSmKdV-B) equation includes the usual mKdV equation and a linear partial differential equation. The bosonization approach can thus effectively avoid difficulties caused by anti-commutative fermionic fields of the supersymmetric systems. The consistent tanh expansion (CTE) method is applied to the BSmKdV-B equation. A non-auto-Bäcklund (BT) theorem is obtained by using CTE method. The interaction solutions among solitons and other complicated waves including Painlevé II waves and periodic cnoidal waves are given through a non-auto-BT theorem. The features of the soliton-cnoidal interaction solutions are investigated both in analytical and graphical ways by combining the mapping and deformation method.

Keywords Supersymmetric mKdV-B equation, Bosonization approach, Consistent tanh expansion method, Interaction solutions

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I. INTRODUCTION

The study of supersymmetric integrable systems has developed great importance in recent years [1-4]. The supersymmetrization of a number of integrable equations in which the bosonic equation is independent of the fermionic variable and the system is linear in fermionic field goes by the name B-supersymmetrization. It has received a lot of attention because of their connections with string theories from the point of view of matrix models [5]. The B-supersymmetric of the Korteweg-de Vries (KdV) [5], dispersionless two boson hierarchy [6], Sawada-Kotera [7], modified KdV and Camassa-Holme quations [8] have been constructed. In the meanwhile, the methodologies involved in the study of integrable systems have been expanded to the supersymmetric framework [9-13]. The soliton solutions of supersymmetric systems have been constructed via many methodologies. However, how to find exact interaction solutions among solitons and other kinds of complicated waves is an important problem for supersymmetric integrable models. In this paper, we will use the bosonization [14, 15] and consistent tanh expansion (CTE) [17] methods on the $\mathcal{N} = 1$ supersymmetric mKdV-B (SmKdV-B) system. These interaction solutions among solitons and other types of solitary waves such as Painlevé II waves and cnoidal waves are explicitly given. All results can be directly transformed to other B-supersymmetrization systems.

The $\mathcal{N} = 1$ SmKdV-B system reads [5]

$$\Phi_t + D^6 \Phi - 6(D\Phi)^2D^2 \Phi = 0,$$

where $D = \partial_\theta + \theta \partial_x$ is the covariant derivative. The commuting space variable $x$ is extended to a doublet $(x, \theta)$, where $\theta$ is a Grassmann variable. The Taylor expansion of the superfield with
respect to $\theta$ is $\Phi(\theta, x, t) = \xi(x, t) + \theta u(x, t)$. The component version of (1) thus reads

\[
\begin{align*}
  u_t - 6u^2u_x + u_{xxx} &= 0, \\
  \xi_t - 6u^2\xi_x + \xi_{xxx} &= 0.
\end{align*}
\]

Equation (2a) is not depend on the fermionic variable, and equation (2b) is linear in the fermionic field. To avoid the difficulties in dealing with the anticommutative fermionic field of the supersymmetric equations, the component fields $\xi$ and $u$ are expanded as the following form by introducing additional two fermionic parameters

\[
\begin{align*}
  \xi(x, t) &= u_1\zeta_1 + u_2\zeta_2, \\
  u(x, t) &= u_0 + u_{12}\zeta_1\zeta_2,
\end{align*}
\]

where $\zeta_1$ and $\zeta_2$ are two Grassmann parameters, the coefficients $u_k = u_k(x, t)$ ($k = 0, 1, 2$) and $u_{12} = u_{12}(x, t)$ are four usual real or complex functions with respect to the usual space-time variables. Substituting (3) into (2) yields the bosonized SmKdV-B (BSmKdV-B) system

\[
\begin{align*}
  u_{k,t} - 6u_0^2u_{k,x} + u_{k,xxx} &= 0, \\
  u_{12,t} - 6(u_{12}u_0^2)_x + u_{12,xxx} &= 0.
\end{align*}
\]

Equation (4a) is just the usual mKdV equation which has been widely investigated. Equation (4b) is linear homogeneous in $u_{12}$. These pure bosonic systems can be easily solved in principle. This is one of the advantages of the bosonization approach.

The paper is organized as follows. In Section 2, a CTE approach is developed to the BSmKdV-B equation. It is proved that the BSmKdV-B equation is CTE solvable system. A nonauto-BT theorem is obtained with the CTE method. In section 3, some novel exact solutions of the BSmKdV-B equation are derived through a nonauto-BT theorem. The last section is a simple summary and discussion.

II. CTE METHOD FOR BS MKDV-B SYSTEM

The CTE method is developed to find interaction solutions between solitons and any other types of solitary waves. The method has been valid for lots of nonlinear integrable systems. According to the CTE method, the expansion solution has the form

\[
\begin{align*}
  u_k &= v = v_0 + v_1 \tanh(f), \\
  u_{12} &= w = w_0 + w_1 \tanh(f) + w_2 \tanh(f)^2.
\end{align*}
\]

where $v_0$, $v_1$, $w_0$, $w_1$, $w_2$ and $f$ are functions of $(x, t)$ and should be determined later. Substituting (5a) into (4a) gets

\[
\begin{align*}
  6f_xv_1(v_1^2 - f_x^2)\tanh(f)^4 &+ 6(2f_xv_1^2v_0 + f_x^2v_{1,x} + v_1f_xf_{xx} - v_1^2v_{1,x})\tanh(f)^3 - v_1(f_t + f_{xxx} - 8f_x^3 + \\
  6f_xv_1^2 - 6f_xv_0^2 + 6v_1v_{0,x} + 12v_0v_{1,x} + \frac{3(v_{1,x}f_x)}{v_1})\tanh(f)^2 + (v_{1,t} + v_{1,xxx} - 6v_0^2v_1)_x - 6f_xf_{xx}v_1 - \\
  6f_x^2v_1 - 12f_xv_1^2v_0)\tanh(f) + v_{0,t} + v_{0,xxx} + v_1(f_t + f_{xxx} - 2f_x^3) + 3(v_{1,x}f_x)_x - 6v_0^2(v_{0,x} + v_1f_x) &= 0.
\end{align*}
\]

By setting the coefficients of $\tanh(f)^4$ and $\tanh(f)^3$ in (6) to zero, one determines $v_1$ and $v_0$ as

\[
\begin{align*}
  v_1 &= -f_x, \\
  v_0 &= -\frac{f_{xx}}{2f_x}.
\end{align*}
\]
By setting the coefficient of \( \tanh(f)^2 \) in (6) to zero and using (7), one obtains the equation for \( f \)

\[
f_t = -f_{xxx} + 2f_x^3 + \frac{3f_{xx}^2}{2f_x}. \tag{8}
\]

One can verify that the coefficients of \( \tanh(f) \) and \( \tanh(f)^0 \) in (6) are identically zero by using (8). Substituting (5) and (7) into (4b) yields

\[
12f_x(f_x^2w_1 + f_xw_{2,x} - f_xw_{2})\tanh(f)^4 + 2(6f_x^2w_1 - 9f_xf_xw_1 + (2f_{xxx} - f_t + 8f_x^3 + \frac{3f_{xx}^2}{2f_x})w_2
\]

\[-3f_xw_{2,xx} \tanh(f)^3 - (3f_x^2(f_x + \frac{f_{xx}^2}{f_x^2} - \frac{f_x^2}{f_x^2}))w_2 + (f_t - 5f_{xxx} + 10f_x^3 - \frac{3f_{xx}^2}{2f_x})w_1 + 18f_xf_xw_0
\]

\[-w_{2,t} - w_{2,xxx} + (24f_x^2 + \frac{3f_x^2}{2f_x})w_{2,x} - 3f_{xxx}w_{1,xx} + 3f_xw_{1,xx} + 6f_x^2w_{0,x})\tanh(f)^2 + (6(f_{xxx} - 2f_x^3)w_0
\]

\[+ 3f_xf_xw_{2,xxx} - 3f_x^2w_{1,xx} - 6f_xf_{xx}w_{1,xx} + 12f_xf_{xxx}w_{0,x})\tanh(f) + 3f_x(2f_x - \frac{f_{xx}^2}{f_x^2} + \frac{f_x^2}{f_x^2})w_0 + (f_t + f_{xxx} - 2f_x^3 - \frac{3f_{xx}^2}{2f_x})w_1
\]

\[+ 6f_xf_xw_2 + w_{0,t} + w_{0,xxx} + 3(f_xw_{1,xx})_x + 6f_x^2w_{2,xx} - \frac{3f_x^2w_{xx,xx}}{2f_x} = 0. \tag{9}
\]

Similarly, by setting the coefficients of \( \tanh(f)^4 \) and \( \tanh(f)^3 \) in (9) to zero, one determines \( w_1 \) and \( w_0 \) as

\[
w_1 = \frac{f_xw_2}{f_x^2} - \frac{w_{2,x}}{f_x}, \quad w_0 = (\frac{f_t}{6f_x^3} - \frac{f_{xxx}}{3f_x^3} + \frac{5f_{xx}^2}{4f_x^4} - \frac{4}{3})w_2 + \frac{w_{2,xxx}}{2f_x^2} - \frac{3f_xw_{xx,xx}}{2f_x^2}. \tag{10}
\]

We denote \( w_2 \) as \( n \) for simplicity. Setting the coefficient of \( \tanh(f)^2 \) to zero, we reduce the equation for \( n \)

\[
n_t + n_{xxx} - 6f_x^2n_x - \frac{6f_{xxx}n_{xx} + 3f_xn_{xx}n_x}{f_x} + \frac{27f_x^2n_x + 12f_xf_{xxx}n}{2f_x^2} - \frac{12f_{xx}n}{f_x} = 0. \tag{11}
\]

The coefficients of \( \tanh(f)^1 \) and \( \tanh(f)^0 \) in (9) are identically zero by using (11). While all the coefficients of powers \( \tanh(f) \) of the BSmKdV-B system can be vanished by using appropriate \( v_0, v_1, w_0, w_1, w_2 \) and \( f \), we call the expansion (5) is a CTE and the BSmKdV-B system is CTE solvable (17). In summary, we have the following a nonauto-Bäcklund (BT) theorem for the BSmKdV-B system (4).

**Nonauto-BT theorem.** The fields \( v \) and \( w \)

\[
v = -f_x \tanh(f) + \frac{f_{xx}}{2f_x}, \tag{12a}
\]

\[
w = n \tanh(f)^2 + (\frac{f_{xx}n}{f_x} - \frac{n_x}{f_x}) \tanh(f) - \frac{4}{3}n + \frac{n_{xx}}{2f_x^2} - \frac{9f_xn_{xx} + 2nf_{xxx} + nf_t}{6f_x^3} + \frac{3f_{xx}^2}{2f_x}, \tag{12b}
\]

are a solution of the BSmKdV-B system (4), while \( f \) and \( n \) satisfy (8) and (11).

### III. INTERACTION SOLUTIONS OF BSMKDV-B SYSTEM WITH A NONAUTO-BT THEOREM

Some novel solutions of the BSmKdV-B system can be found by using the above nonauto-BT theorem. Here three examples are listed in the following.

**Example 1.** A quite trivial solution of (8) and (11) has the form

\[
f = k_1x + \omega_1t + l_1, \quad n = k_2x + \omega_2t + l_2, \tag{13}
\]
where \( k_1, k_2, l_1 \) and \( l_2 \) are arbitrary constants and \( \omega_1, \omega_2 \) are determined by the relations
\[
\omega_1 = 2k_1^3, \quad \omega_2 = 6k_2k_1^2. \tag{14}
\]
The soliton solution of BSmKdV-B system reads in the following form by using the line solution which is the equivalent Painlevé II reduction \[17\],
\[
v = -k_1 \tanh(k_1 x + 2k_1^3 t + l_1), \tag{15a}
\]
\[
w = n \tanh(k_1 x + 2k_1^3 t + l_1)^2 - \frac{k_2}{k_1} \tanh(k_1 x + 2k_1^3 t) - n. \tag{15b}
\]
Though the soliton solution \[15\] is a traveling wave in the \((x, t)\) for the boson field \( v \), it is not a traveling wave for other boson field \( w \) except for the case \( n \) being constants, i.e., \( k_2 = 0 \).

**Example II.** For the usual mKdV system, there exists a Painlevé II reduction if one uses the scaling symmetry \[26\]. The scaling group invariant solution from the nonauto-BT theorem

The interaction between solitons and Painlevé II waves of the BSmKdV-B system can be obtained

Substituting \[16\] into \( (12a) \), a second order ordinary differential equation for the field \( W_1 \) yields
\[
W_{1,\xi\xi} = 2W_1^3 + \frac{1}{3} \xi W_1 + \frac{3W_1^3}{2W_1} - c, \quad W_1 = W_\xi, \tag{17}
\]
which is the equivalent Painlevé II reduction \[17\], \( a, b \) and \( c \) are constants. The solution for other field \( n \) is obtained by solving
\[
M_{\xi\xi\xi} = \left( \frac{1}{3} \xi + 6W_1^2 \right) M_\xi + \frac{3W_1\xi M_\xi + 6W_1 M_{\xi\xi}}{W_1} - \frac{27W_1^2 M_X + 12W_{1,X} W_{1,XX} M}{2W_1^2} + \frac{12W_1^3 M_{XX}}{W_1^2}, \quad n = M(\xi), \tag{18}
\]
The interaction between solitons and Painlevé II waves of the BSmKdV-B system can be obtained from the nonauto-BT theorem
\[
v = -\frac{W_1}{(t-b)^{\frac{1}{2}}} \tanh(f) - \frac{1}{2}\frac{W_1}{(t-b)^{\frac{1}{2}}} W_1, \tag{19a}
\]
\[
w = M \tanh(f)^2 - \frac{M_{\xi} W_1 - MW_{1,\xi}}{W_1^2} \tanh(f) - \frac{4}{3} M + \frac{9M_{\xi\xi} - \xi M}{18W_1^2} + \frac{c M - 9W_{1,\xi} M_\xi - 2MW_{1,\xi\xi}}{6W_1^3} + \frac{5MW_{1,\xi}^2}{4W_1^4}. \tag{19b}
\]

**Example III.** To find the interaction solutions between solitons and cnoidal periodic waves of \[8\], the solution for the \( f \) field assume with one line solution \( k_1 x + \omega_1 t \) plus an undetermined traveling wave \( F(kx + \omega t) \)
\[
f = k_1 x + \omega_1 t + F(X), \quad X = kx + \omega t. \tag{20}
\]
Substituting \( (20) \) into \( (8) \) yields
\[
F_{1,X}^2 - 4F_1^4 - \left(C_1 k_3^3 + \frac{12k_1}{k}\right)F_1^3 - (3C_1 k_1 k_2^2 + \frac{12k_1^2}{k^2} + \frac{2\omega}{k^3})F_1^2 \tag{21}
\]
with $C_1$ is arbitrary constant. The solution of (21) can be solved out in terms of Jacobi elliptic functions (27). The solution expressed by (20) is thus the explicit interaction solutions between one soliton and cnoidal periodic waves. The field $n$ is obtained by solving the following equation

$$N_{XXX} - \frac{6k}{kF_1 + k_1} F_{1,X} N_{XX} + \left( \frac{27k^2}{2(kF_1 + k_1)^2} F_{1,X}^2 - \frac{3k}{kF_1 + k_1} F_{1,XX} - \frac{6(kF_1 + k_1)^2}{k^2} \right) N_X + \left( \frac{6k^2}{(kF_1 + k_1)^3} F_{1,XX} - \frac{2k}{kF_1 + k_1} F_{1,XX} \right) F_{1,X} N = 0, \quad n = N(X). \quad (22)$$

As the well known exact Jacobi elliptic functions solutions of (21), we try to build the mapping and deformation relationship (15) between the solution for $F_1$ and $N$ by using (22). In order to get the mapping and deformation relationship, the variable transformation is introduced (15, 16)

$$N(X) = N'(F_1(X)). \quad (23)$$

Substituting the transformation (23) into (22) and vanishing $F_{1,X}$ via (21), equation (22) becomes

$$L(F_1 + \frac{k_1}{k}) \frac{3}{d^2F_1^3} \frac{d^3N'}{dF_1^3} - \frac{3}{2} M(F_1 + \frac{k_1}{k}) \frac{2}{d^2N'} \frac{d^2N'}{dF_1^2} + 3M(F_1 + \frac{k_1}{k}) \frac{dN'}{dF_1} - 3MN' = 0, \quad (24)$$

where

$$L = 4k^4 F_1^4 + k^3(C_1 k^4 + 8k_1) F_1^2 + 2k(C_1 k_1 k^5 + 2k k_1 + \omega) F_1 + C_1 k_1^2 k^5 + k \omega_1 + k_1 \omega,$$

$$M = k^3(C_1 k^4 - 4k_1) F_1^2 + 2k(C_1 k_1 k^5 - 4k k_1 + 2\omega) F_1 + C_1 k_1^2 k^5 - 4k k_1^3 + 3k \omega_1 + k_1 \omega,$$

for simplicity. The solution for (24) can be directly obtained by using Maple since (24) is the linear ordinary differential equation. The mapping and deformation relation is thus constructed by solving (24)

$$N = N' = C_2 k F_1^2 + (C_2 k_1 + C_3 k) F_1 + C_3 k_1, \quad (25)$$

with $C_2$ and $C_3$ are arbitrary constants.

Substituting (20), (23) and (25) into (12), the corresponding solution of the BS-KdV-B system reads

$$v = -(kF_1 + k_1) \tanh(f) - \frac{k^2}{2} \frac{F_{1,X}}{kF_1 + k_1}, \quad (26a)$$

$$w = (kF_1 + k_1)(C_2 F_1 + C_3) \tanh(f) - C_2 k F_{1,X} \tanh(f) + \frac{k^2(4C_2 k F_1 + 3C_2 k_1 + C_3 k)}{6(kF_1 + k_1)^2} F_{1,XX} \quad (26b)$$

$$- \frac{k^3(3C_2 k F_1 + 2C_2 k_1 + C_3 k)}{4(F_1 + k_1)^3} F_{1,X}^2 - \frac{(C_2 F_1 + C_3)(8k^3 F_1^3 + 24k k_1^2 F_1^2 + (24k k_1^2 - \omega) F_1 + 8k^3 _1 - \omega_1)}{6(F_1 + k_1)^2}.$$ 

To show more clearly of this kind of solution, we give two special cases.

**Case I.** One solution of (21) take as

$$F_1 = a_0 + a_1 \text{sn}(a_2 X, m), \quad (27)$$
where $a_0$, $a_1$ and $a_2$ are constants, $\text{sn}(a_2 X, m)$, $\text{dn}(a_2 X, m)$ and $\text{cn}(a_2 X, m)$ are the usual Jacobi elliptic functions with the modulus $m$. $\hat{S}$, $\hat{D}$ and $\hat{C}$ stand for $\text{sn}(a_2 X, m)$, $\text{dn}(a_2 X, m)$ and $\text{cn}(a_2 X, m)$ respectively for simplicity. Substituting (27) into (21) and vanishing all the coefficients of different powers of $\hat{S}$, a nontrivial solution is obtained for constants

\[
\begin{align*}
a_0 &= -\frac{C_1 k^3}{4} - \frac{3a_2}{2}, \\
a_1 &= -\frac{a_2 m}{2}, \\
\omega &= \frac{k^3 a_2^2 (m^2 - 5)}{2}, \\
\omega_1 &= \frac{k^3 a_2^2 (32a_2 + 5C_1 k^3 - C_1 k^3 m^2)}{8}.
\end{align*}
\]

In this case, the interaction solution of of the BSmKdV-B system becomes by using (26)

\[
\begin{align*}
v &= -\frac{1}{2} k a_2 (1 + m\hat{S}) \tan(f) - \frac{a_2 k m}{2(1 + m\hat{S})} \hat{C} \hat{D}, \\
w &= \frac{1}{8} a_2 k (1 + m\hat{S})(2C_2 a_2 m\hat{S} + 4C_3 - 6C_2 a_2 - C_1 C_2 k^3) \tan(f)^2 + \frac{1}{2} C_2 a_2^2 k m \hat{C} \hat{D} \tan(f) \\
&\quad + \frac{1}{8} a_2 k (C_1 C_2 k^3 - 4C_3)(1 + m\hat{S}) + \frac{3}{4} a_2^2 C_2 k m \hat{S} + \frac{a_2^2 C_2 k (m^3 \hat{S}^3 - 2m^2 + 3)}{4(1 + m\hat{S})}, \\
&\quad \left( f = \frac{ka_2}{2} x + \frac{a_2^3 k^3 (3m^2 + 1)t}{4} - \frac{1}{2} \ln(\hat{D} - m\hat{C}) \right).
\end{align*}
\]

The figure 1 plots the interaction solutions between solitons and cnoidal waves for the field $v$ and $w$ respectively. The parameters are $k = 1$, $m = 0.3$, $a_2 = 1$, $k_1 = 1$, $C_1 = 1$, $C_2 = 1$, $C_3 = 1$.

**Case II.** Another special solution of (21) reads

\[
F_1 = -b_1 b_2 + \frac{b_1 b_2}{1 - n \text{sn}(b_1 X, m)^2},
\]

where $b_1, b_2, n$ and $m$ are constants, $\text{sn}(b_1 X, m)$, $\text{dn}(b_1 X, m)$ and $\text{cn}(b_1 X, m)$ are the usual Jacobi elliptic functions. Here, $\hat{S}$, $\hat{D}$ and $\hat{C}$ stand for $\text{sn}(b_1 X, m)$, $\text{dn}(b_1 X, m)$ and $\text{cn}(b_1 X, m)$ respectively for similarity. Substituting (30) into (21) and vanishing all the coefficients of different powers of $\hat{S}$ yields constraints of constants

\[
\begin{align*}
b_1 &= \frac{C_1 k^3 \sqrt{n(1 - n)(m^2 - n)}}{4(m^2 n - 2m^2 + n)}, \\
b_2 &= \sqrt{\frac{(1 - n)(m^2 - n)}{n}}, \\
k_1 &= \frac{C_1 k^4 (1 - n)(m^2 - n)}{4(m^2 n - 2m^2 + n)}.
\end{align*}
\]
Fig. 2: (a) Plot of a kink soliton on the cnoidal wave background expressed by (31a). (b) Plot of a second special soliton-cnoidal wave interaction solution by (31b). The parameters are k = 1, m = 0.8, n = 0.2, C1 = 3, C2 = 1, C3 = 1.

\[
\omega = \frac{C_1^2 k^9 (1-n)(n-m^2)(m^2n-3m^2+n)}{8(m^2n-2m^2+n)^2}, \quad \omega_1 = \frac{C_1^3 k^{12} (m^2n-m^2-n^2+n)^2 (m^2-m^2n-n)}{32(m^2n-2m^2+n)^3}.
\]

In this case, an interaction solution for the BSmKdV-B system is obtained as

\[
v = \frac{C_1 k^4 (n-1)(m^2-n)}{4(nS^2-1)(m^2n-2m^2+n)} \tanh(f) - \frac{nC_1 k^4 \sqrt{(m^2-n)(1-n)}}{4(m^2n-2m^2+n)(nS^2-1)} \tilde{S} \tilde{C} \tilde{D}, \tag{32a}
\]

\[
w = \left( \frac{C_1 k^4 (n-1)(m^2-n)}{4(nS^2-1)(m^2n-2m^2+n)} + \frac{nC_1^2 k^7 (n-1)^2 (m^2-n)^2}{16(nS^2-1)^2 (m^2n-2m^2+n)^2} \tilde{S} \tilde{D} \tanh(f)^2 \right) + \frac{C_1 C_2 k^7 n (n-1)(m^2-n)}{8(nS^2-1)^2 (m^2n-2m^2+n)^2} \tilde{S} \tilde{C} \tilde{D} \tanh(f) + \frac{C_1 C_2 k^7 (n-1)(m^2-n)}{4(nS^2-1)(m^2n-2m^2+n)} + \frac{C_1 C_2 k^7 n (n-1)(m^2-n)(m^2n^2 \tilde{S} \tilde{C} - 3 \tilde{S} ^4 m^2n + \tilde{S} ^2 m^2n + \tilde{S} ^2 m \tilde{S} ^2 n - n)}{16(nS^2-1)^2 (m^2n-2m^2+n)^2}, \tag{32b}
\]

\[
(f = \frac{C_3 m^2 k^{12} (1-n)(m^2-n)^2 t}{16(m^2n-2m^2+n)^3} - \sqrt{(1-n)(m^2-n) n} E_{\pi}(\tilde{S}, n, m))
\]

where \(E_{\pi}\) is the third type of incomplete elliptic integral. The figure 2 plots the interaction solutions between solitons and cnoidal waves expressed by (32) with the parameters \(k = 1, m = 0.8, n = 0.2, C_1 = 3, C_2 = 1, C_3 = 1\). There are some typical nonlinear waves such as interaction solutions between solitary waves and cnoidal periodic waves in the ocean \([27, 29]\). Solutions (29) and (32) may be useful for studying these types of ocean waves \([28, 29]\).

**Remark.** For a given solution \(v\) and \(w\) of BSmKdV-B system, the interaction solutions for component fields \(\xi\) and \(\nu\) of SmKdV-B can be constructed by using (8). The bosonization approach can thus effectively avoid difficulties caused by anticommutative fermionic fields for supersymmetric nonlinear systems.

**IV. CONCLUSIONS**

In summary, the \(N = 1\) SmKdV-B system is mapped to a system of coupled bosonic equations by means of the bosonization approach. The BSmKdV-B system is just the usual mKdV equation together with a linear differential equation. Then, the CTE approach is applied to the BSmKdV-B equation. It is proved that the BSmKdV-B equation is CTE solvable system. A nonauto-BT
theorem is obtained with the CTE method. Various explicit solutions of the BS\(\text{mKdV-B}\) system such as soliton-Painlevé II waves and soliton-cnoidal waves are obtained by using the nonauto-BT theorem. For the interaction soliton-cnoidal waves, two cases are given both in analytical and graphical ways via combining the mapping and deformation method. This kind of interaction solutions may be useful in real physical phenomena.

For the bosonization approach, we can also introduce \(N \geq 2\) fermionic parameters \(\zeta_i (i = 2, 3, \ldots, N)\) to study the B-supersymmetrization of nonlinear evolution system. On the other hand, lots of B-supersymmetrization systems are introduced from the usual action principle \[8\. The interaction solutions among solitons and other complicated waves for these B-supersymmetrization systems are worth studying.

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