Octonion random functions and integration of stochastic PDEs.

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Abstract

In the article random functions in modules over the octonion algebra and Cayley-Dickson algebras are investigated. For their study transition measures with values in the octonion algebra and Cayley-Dickson algebras are used. Stochastic integrals over these algebras are studied. They are applied to integration of stochastic PDEs. This approach permits subsequently to analyze and integrate PDEs of orders higher than two of different types including parabolic, elliptic and hyperbolic.

1 Introduction.

Over the complex field Feynman integrals and quasi-measures appeared to be very important in mathematics, mathematical physics, quantum mechanics, quantum field theory and partial differential equations (PDEs) (see, for example, [2, 9, 14, 16, 29]). It is worthwhile to note that in stochastic
analysis measures with values in matrix algebras or operator algebras on Hilbert spaces are frequently studied and used for solution or analysis of PDEs [2, 8, 9, 13, 14].

But there are restrictions for these approaches, because the Feynman integral works for partial differential operators (PDOs) of order not higher than two. Indeed, it is based on complex modifications of Gaussian measures. Nevertheless, if a characteristic function $\phi(t)$ of a measure has the form $\phi(t) = \exp(Q(t))$, where $Q(t)$ is a polynomial, then its degree is not higher than two according to the Marcinkiewicz theorem (Ch. II, §12 in [31]).

On the other side, hypercomplex numbers open new opportunities in these areas. For example, Dirac used the complexified quaternion algebra $H_C$ for a solution the Klein-Gordon hyperbolic PDE of the second order with constant coefficients [4]. This is important in spin quantum mechanics. It was proved in [27] that in many variants, it is possible to reduce a PDE problem to a subsequent solution of PDEs of order not higher than two with Cayley-Dickson coefficients. In general the complex field is insufficient for this purpose.

On the other hand, the Cayley-Dickson algebras $A_r$ over the real field $R$ are natural generalizations of the complex field, where $A_2 = H$ is the quaternion skew field, $A_3 = O$ denotes the octonion algebra, $A_0 = R$, $A_1 = C$. Then each subsequent algebra $A_{r+1}$ is obtained from the preceding algebra $A_r$ by the doubling procedure using the doubling generator [1, 3, 15].

They are widely applied in PDEs, non-commutative analysis, mathematical physics, quantum field theory, hydrodynamics, industrial and computational mathematics, non-commutative geometry [4, 5, 10, 11, 12, 17, 21, 22, 23, 24].

Previously measures with values in the complexified Cayley-Dickson algebra $A_{r,C}$ were studied in [18]. They appear naturally while a solution of the second order hyperbolic PDE with Cayley-Dickson coefficients. In this work the results and notation of [18] are used. This article is devoted to a realization of the plan formulated in the preceding cited work: for solution and analysis of PDEs of orders higher than two to extend Feynman integrals and quasi-measures from spaces over the complex field onto modules over the
Cayley-Dickson algebras.

In this paper random functions in modules over the complexified octonion algebra $O_C = A_{3,C}$ and the complexified Cayley-Dickson algebras $A_{r,C}$ are investigated. For their study transition measures with values in the complexified octonion algebra and the complexified Cayley-Dickson algebras are used. An existence of random functions and Markov processes in modules over the complexified Cayley-Dickson algebra $A_{r,C}$ is studied in Theorem 2.8, Corollary 2.9. Stochastic integrals of such random functions and acting on them operators are investigated in Theorems 2.14, 2.15, 2.17, 2.18. Properties of stochastic integrals over $A_{r,C}$ are described in Propositions 2.20-2.22. In Theorem 2.27 their stochastic continuity is studied. Necessary specific definitions are given. Notation is described in remarks. Lemmas 2.12, 2.13, 2.25, 2.26 are given in order to prove the theorems and propositions. These lemmas concern estimates of stochastic integrals over $A_{r,C}$. In Theorems 2.29, 2.31 and Corollary 2.30 solutions of stochastic PDEs with random functions and operators in modules over $A_{r,C}$ are scrutinized.

A formula number $(n)$ in the same subsection $m$ is referred as $(n)$, in another subsection as $m(n)$.

Main results of this work are obtained for the first time. The obtained results open new opportunities for subsequent studies of PDEs and their solutions including that of hyperbolic type and parabolic type with hyperbolic and elliptic terms of orders two or higher, related to them random functions.

2 Octonion random functions.

Definition 2.1. Suppose that $\Lambda$ is an additive group contained in $\mathbb{R}$. Suppose also that $T$ is a subset in $\Lambda$ and containing a point $t_0$. Let $X_t = X$ be a locally $\mathbb{R}$-convex space which be also a two-sided $A_{r,C}$-module for each $t \in T$, where $2 \leq r < \infty$. Put

$$(\tilde{X}_T, \tilde{U}) := \prod_{t \in T} (X_t, U_t)$$

for the product of measurable spaces, where $U_t$ is the Borel $\sigma$-algebra of $X_t$, $\tilde{U}$ is an algebra of cylindrical subsets of $\tilde{X}_T$ generated by projections
\( \pi_q : \bar{X}_T \to X^q \), where \( X^q := \prod_{t \in q} X_t \) is a left ordered direct product, \( q \subset T \) is a finite subset of \( T \), \( X^{(t)} = X_t \), \( X^{t_1, \ldots, t_{n+1}} = X_{t_{n+1}} \times (X^{t_1, \ldots, t_n}) \) for each \( t_1 < \ldots < t_{n+1} \) in \( T \).

A function \( P(t_1, x_1, t_2, A) \) with values in the complexified Cayley-Dickson algebra \( A_{r,C} \) for each \( t_1 < t_2 \in T, x_1 \in X_{t_1}, A \in U_{t_2} \) is called a transition measure if it satisfies the following conditions:

1. the set function \( \nu_{x_1,t_1,t_2}(A) := P(t_1, x_1, t_2, A) \) is a measure on \( (X_{t_2}, U_{t_2}) \);
2. the function \( \alpha_{t_1,t_2,A}(x_1) := P(t_1, x_1, t_2, A) \) of the variable \( x_1 \) is \( U_{t_1} \)-measurable, that is, \( \alpha_{t_1,t_2,A}(B(A_{r,C})) \subset U_{t_1} \);
3. \( P(t_1, x_1, t_2, A) = \int_{X_{x_1}} P(t, y, t_2, A)P(t_1, x_1, t, dy) \) for each \( t_1 < t < t_2 \in T \)
so that \( P(t, y, t_2, A) \) as the function by \( y \) is in \( L^1((X_t, U_t), \nu_{x_1,t_1,t}, A_{r,C}) \). A transition measure \( P(t_1, x_1, t_2, A) \) is called unital if

\[ P(t_1, x_1, t_2, X_{t_2}) = 1 \text{ for each } t_1 < t_2 \in T. \]

Then for each finite set \( q = (t_0, t_1, \ldots, t_{n+1}) \) of points in \( T \) such that \( t_0 < t_1 < \ldots < t_{n+1} \) there is defined a measure in \( X^q \)

\[ \mu^q_{x_0}(D) = \int_D \prod_{k=1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k), D \in U^q := \prod_{t \in q} U_t, \]
where \( g = q \setminus \{t_0\} \), variables \( x_1, \ldots, x_{n+1} \) are such that \( (x_1, \ldots, x_{n+1}) \in D, x_0 \in X_{t_0} \) is fixed.

Let the transition measure \( P(t, x_1, t_2, dx_2) \) be unital. Then for the product
\( D = D_2 \times (X_{t_j} \times D_1) \), where \( D_1 \in \prod_{i=1}^{j-1} U_{t_i}, D_2 \in \prod_{i=j+1}^{n+1} U_{t_i}, \) the equality

\[ \mu^q_{x_0}(D) = \int_{D_2 \times D_1} \prod_{k=j+1}^{n+1} P(t_{k-1}, x_{k-1}, t_k, dx_k) \]

\[ \times \left[ \int_{X_{t_j}} P(t_{j-1}, x_{j-1}, t_j, dx_j) \prod_{k=1}^{j-1} P(t_{k-1}, x_{k-1}, t_k, dx_k) \right] = \mu^q_{x_0}(D_2 \times D_1) \]
is fulfilled, where \( r = q \setminus \{t_j\} \). Equation (6) implies that

\[ [\mu^q_{x_0}]^{\pi^q_w} = \mu^v_{x_0} \]

for each \( v < q \), where finite sets are ordered by inclusion: \( v < q \) if and only if \( v \subset q \), where \( \pi^q_w : X^q \to X^w \) is the natural projection, \( g = q \setminus \{t_0\} \), \( w = v \setminus \{t_0\} \).
Denote by $\Upsilon_T$ the family of all finite linearly ordered subsets $q$ in $T$ such that $t_0 \in q \subset T$, $v \leq q \in \Upsilon_T$, $\pi_q : \tilde{X}_T \to X^q$ is the natural projection, $g = q \setminus \{t_0\}$. Hence Conditions (4), (5), (7) imply that: $\{\mu_{x_0}^q, \pi_q^*, \Upsilon_T\}$ is the consistent family of measures. It induces a cylindrical distribution $\tilde{\mu}_{x_0}$ on the measurable space $(\tilde{X}_T, \tilde{U})$ such that

$$\tilde{\mu}_{x_0}(\pi_q^{-1}(D)) = \mu_{x_0}^q(D)$$

for each $D \in U^q$.

The cylindrical distribution given by Formulas (1)-(5), (7), (8) is called the $A_{r,C}$-valued Markov distribution with time $t$ in $T$.

**Remark 2.2.** Let $X_t = X$ for each $t \in T$, $\tilde{X}_{t_0,x_0} := \{x \in \tilde{X}_T : x(t_0) = x_0\}$. Put $\bar{\pi}_q : x \mapsto x_q$ for each $x = x(t)$ in $\tilde{X}_T$, where $x_q$ is defined on $q = (t_0, \ldots, t_{n+1}) \in \Upsilon_T$ such that $x_q(t) = x(t)$ for each $t \in q$. To an arbitrary function $F : \tilde{X}_T \to A_{r,C}$ a function can be posed $(S_q F)(x) := F(x_q) = F_q(y_0, \ldots, y_n)$, where $y_j = x(t_j)$, $F_q : X^q \to A_{r,C}$, $l \in \mathbb{N}$. Put

$$F := \{F | F : \tilde{X}_T \to A_{r,C}, S_q F \text{ is } U^q \text{- measurable for each } q \in \Upsilon_T\}.$$ 
If $F \in F$, $\tau = t_0 < q$, $t_0 < t_1 < \ldots < t_{n+1}$, then the integral

$$J_q(F) = \int \chi_q(S_q F)(x_0, \ldots, x_n) \prod_{k=1}^{n+1} P(t_{k-1}, x_k, t_k, dx_k)$$

can be defined whenever it converges.

**Definition 2.3.** A function $F$ is called integrable relative to a Markov cylindrical distribution $\mu_{x_0}$ if the limit

$$\lim_{q \in \Upsilon_T} J_q(F) =: J(F)$$

along the generalized net by finite subsets $q = (t_0, \ldots, t_{n+1}) \in \Upsilon_T$ of $T$ exists.
This limit is called a functional integral relative to the Markov cylindrical distribution:

$$J(F) = \int_{\tilde{X}_{t_0,x_0}} F(x) \mu_{x_0}(dx).$$

**Remark 2.4.** Spatially homogeneous transition measure. Suppose that $P(t, A)$ is an $A_{r,C}$-valued measure on $(X, U)$ for each $t \in T$ such that $A - x \in U$ for each $A \in U$ and $x \in X$, where $A \in U$, $X$ is a locally $\mathbb{R}$-convex
space which is also a two-sided $\mathcal{A}_{r,c}$-module, $U$ is an algebra of subsets of $X$. Suppose also that $P$ is a spatially homogeneous transition measure:

\begin{equation}
(1) \quad P(t_1, x_1, t_2, A) = P(t_2 - t_1, A - x_1)
\end{equation}

for each $A \in U$, $t_1 < t_2 \in T$ and $t_2 - t_1 \in T$ and every $x_1 \in X$, where $P(t, A)$ also satisfies the following condition:

\begin{equation}
(2) \quad P(t_1 + t_2, A) = \int_X P(t_2, A - y)P(t_1, dy)
\end{equation}

for each $t_1 < t_2$ and $t_1 + t_2$ in $T$.

Then

\begin{equation}
(3) \quad \phi(t_1, x_1, t_2, y) := \int_X P(t_1, x_1, t_2, dx) \exp(iy(x))
\end{equation}

is the characteristic functional of the transition measure $P(t_1, x_1, t_2, dx)$ for each $t_1 < t_2 \in T$ and each $x_1 \in X$, where $X^*_R$ notates the topologically dual space of all continuous $\mathbb{R}$-linear real-valued functionals $y$ on $X$, $y \in X^*_R$. Particularly for $P$ satisfying Conditions (1), (2) with $t_0 = 0$ its characteristic functional $\phi$ satisfies the equalities:

\begin{equation}
(4) \quad \phi(t_1, x_1, t_2, y) = \psi(t_2 - t_1, y)\exp(iy(x_1)),
\end{equation}

where

\begin{equation}
(5) \quad \psi(t, y) := \int_X P(t, dx) \exp(iy(x)) \quad \text{and}
\end{equation}

\begin{equation}
(6) \quad \psi(t_1 + t_2, y) = \psi(t_2, y)\psi(t_1, y)
\end{equation}

for each $t_1 < t_2 \in T$ and $t_2 - t_1 \in T$ and $t_1 + t_2 \in T$ respectively and $y \in X^*_R$, $x_1 \in X$, since $Z(\mathcal{A}_{r,c}) = \mathbb{C}$.

**Remark 2.5. Notation.** If $T$ is a $T_1 \cap T_{3.5}$ topological space, then we denote by $C^0_b(T, H)$ the Banach space of all continuous bounded functions $f : T \to H$ supplied with the norm:

\begin{equation}
(1) \quad \|f\|_{C^0} := \sup_{t \in T} \|f(t)\|_H < \infty,
\end{equation}

where $H$ is a Banach space over $\mathbb{R}$ which may be also a two-sided $\mathcal{A}_{r,c}$-module. If $T$ is compact, then $C^0_b(T, H)$ is isomorphic with the space $C^0(T, H)$ of all continuous functions $f : T \to H$. 

6
For a set $T$ and a complete locally $\mathbf{R}$-convex space $H$ which may be also a two-sided $\mathcal{A}_{r,C}$-module consider the product $\mathbf{R}$-convex space $H^T := \prod_{t \in T} H_t$ in the product topology, where $H_t := H$ for each $t \in T$.

Suppose that $\mathcal{B}$ is a separating algebra on the space either $X := X(T, H) = L^q(T, \mathcal{B}(T), \lambda, H)$ or $X := X(T, H) = C^0_b(T, H)$ or on $X = X(T, H) = H^T$, where $\lambda : \mathcal{B}(T) \to [0, \infty)$ is a $\sigma$-additive measure on the Borel $\sigma$-algebra $\mathcal{B}(T)$ on $T$, $1 \leq q \leq \infty$. Consider a random variable $\xi : \omega \mapsto \xi(t, \omega)$ with values in $(X, \mathcal{B})$, where $t \in T$, $\omega \in \Omega$. $(\Omega, \mathcal{R}, P)$ is a measure space with an $\mathcal{A}_{r,C}$-valued measure $P$, $P : \mathcal{R} \to \mathcal{A}_{r,C}$.

Events $S_1, \ldots, S_n$ are called independent in total if $P(\prod_{k=1}^n S_k) = \prod_{k=1}^n P(S_k)$. Subalgebras $\mathcal{R}^k \subset \mathcal{R}$ are said to be independent if all collections of events $S_k \in \mathcal{R}^k$ are independent in total, where $k = 1, \ldots, n$, $n \in \mathbb{N}$. To each collection of random variables $\xi_\gamma$ on $(\Omega, \mathcal{R})$ with $\gamma \in \Upsilon$ is related the minimal algebra $\mathcal{R}_T \subset \mathcal{R}$ for which all $\xi_\gamma$ are measurable, where $\Upsilon$ is a set. Collections $\{\xi_\gamma : \gamma \in \Upsilon^l\}$ are called independent if such are $\mathcal{R}_{\Upsilon^l}$, where $\Upsilon^l \subset \Upsilon$ for each $l = 1, \ldots, n$, $n \in \mathbb{N}$.

For $X = C^0_b(T, H)$ or $X = H^T$ define $X(T, H; (t_1, \ldots, t_n); (z_1, \ldots, z_n))$ as a closed submanifold in $X$ of all $f : T \to H$, $f \in \mathcal{F}$ such that $f(t_1) = z_1, \ldots, f(t_n) = z_n$, where $t_1, \ldots, t_n$ are pairwise distinct points in $T$ and $z_1, \ldots, z_n$ are points in $H$. For $n = 1$ and $t_0 \in T$ and $z_1 = 0$ we denote $X_0 := X_0(T, H) := X(T, H; t_0; 0)$.

**Definition 2.6.** Suppose that $H$ is a real Banach space which also may be a two-sided $\mathcal{A}_{r,C}$-module. Consider a random function $w(t, \omega)$ with values in the space $H$ as a random variable such that:

1. the random variable $\omega(t, \omega) - \omega(u, \omega)$ has a distribution $\mu^{F_{t,u}}$, where $\mu$ is an $\mathcal{A}_{r,C}$-valued measure on $(X(T, H), \mathcal{B})$, $\mu^g(A) := \mu(g^{-1}(A))$ for $g : X \to H$ such that $g^{-1}(\mathcal{R}_H) \subset \mathcal{B}$ and each $A \in \mathcal{R}_H$. There by $F_{t,u}$ a $\mathbf{R}$-linear operator $F_{t,u} : X \to H$ is denoted, which is prescribed by the following formula:

$$F_{t,u}(w) := w(t, \omega) - w(u, \omega)$$

for each $u < t$ in $T$, where $\mathcal{R}_H$ is a separating algebra of $H$ such that $F_{t,u}^{-1}(\mathcal{R}_H) \subset \mathcal{B}$ for each $u < t$ in $T$, where $T = [0, b]$ with $0 < b < \infty$ or
\[ T = [0, \infty), \Omega \neq \emptyset; \]

(2) the vectors \( w(t_m, \omega) - w(t_{m-1}, \omega), \ldots, w(t_1, \omega) - w(0, \omega) \) and \( w(0, \omega) \) are mutually independent for each chosen \( 0 < t_1 < \ldots < t_m \) in \( T \) and each \( m \geq 2 \), where \( \omega \in \Omega \).

Then \( \{w(t) : t \in T\} \) is called the random function with independent increments, where \( w(t) \) is the shortened notation of \( w(t, \omega) \).

It also may be put

(3) \( w(0, \omega) = 0 \).

**Remark 2.7.** The random function \( w(t, \omega) \) satisfying Conditions 2.6(1)-(3) possesses the Markovian property with the transition measure

\[ P(u, x, t, A) = \mu^{F_{t,u}}(A - x). \]

As usually it is put for the expectation

\[ E_P f = \int_{\Omega} f(\omega)P(d\omega) = P^L(f) \]

of a random variable \( f : \Omega \to \mathcal{A}_{r,C}^h \) whenever this integral exists, where \( P = P_{[r]} \) is the \( \mathcal{A}_{r,C} \)-valued measure on a measure space \((\Omega_{[r]}, [r]\mathcal{F})\) shortly denoted by \((\Omega, \mathcal{F})\), where \( f \) is \((\mathcal{F}, \mathcal{B}(\mathcal{A}_{r,C}^h))\)-measurable, \( h \in \mathbb{N}, \mathcal{B}(\mathcal{A}_{r,C}^h) \) denotes the Borel \( \sigma \)-algebra on \( \mathcal{A}_{r,C}^h \). If \( P \) is specified, it may be shortly written \( E \) instead of \( E_P \). If \( \mathcal{G} \) is a sub-\( \sigma \)-algebra in the \( \sigma \)-algebra \( \mathcal{F} \) and if there exists a random variable \( g : \Omega \to \mathcal{A}_{r,C}^h \) such that \( g \) is \((\mathcal{G}, \mathcal{B}(\mathcal{A}_{r,C}^h))\)-measurable and

\[ \int_A f(\omega)P(d\omega) = \int_A g(\omega)P(d\omega) \]

for each \( A \in \mathcal{G} \), then \( g \) is called the conditional expectation relative to \( \mathcal{G} \) and denoted by \( g = E(f | \mathcal{G}) \).

Recall that an operator \( J : \mathcal{A}_{r,C}^n \to \mathcal{A}_{r,C}^h \) is called right \( \mathcal{A}_{r,C} \)-linear in the weak sense, if

(1) \( J(xb + yc) = (Jx)b + (Jy)c \) for each \( x \) and \( y \) in \( \mathbb{R}^n \) and \( b \) and \( c \) in \( \mathcal{A}_{r,C} \), where the real field \( \mathbb{R} \) is canonically embedded into the complexified Cayley-Dickson algebra \( \mathcal{A}_{r,C} \) as \( \mathbb{R}i_0, i_0 = 1 \). Over the algebra \( \mathcal{H}_C = \mathcal{A}_{2,C} \) this gives right linear operators \( J(xb + yc) = (Jx)b + (Jy)c \) for each \( x \) and \( y \) in \( \mathcal{A}_{2,C}^n \) and \( b \) and \( c \) in \( \mathcal{A}_{2,C} \), since \( \mathcal{H}_C \) is associative. For short we omit "in
the weak sense. A set of such operators we denote by $L_r(A^u_{r,C}, A^h_{r,C})$. Then

$$
\|J\| = \sup_{z \neq 0; z \in A^u_{r,C}} \frac{\|Jz\|}{\|z\|},
$$

where $z = (z_1, ..., z_n)$, $z_j \in A_{r,C}$ for each $j \in \{1, ..., n\}$, where

$$
\|z\|^2 = \sum_{j=1}^{n} \|z_j\|^2,
$$

$\|a\|^2 = 2|b|^2 + 2|c|^2$ for each $a = b + ic$ in $A_{r,C}$ with $b$ and $c$ in $A_r$ (see also Remark 2.1 [IS]).

In particular, it is useful to consider the following case: $w = J_\xi + p$, where $\xi$ is a $\mathbb{R}^{2n}$-valued random variable on a measurable space $(\Omega_{[0]}, |_{[0]}\mathcal{F})$ and with a probability measure $P_{[0]} : |_{[0]}\mathcal{F} \to [0, 1]$, where $p \in A^u_{r,C}$, where $\mathbb{R}^{2n}$ is embedded into $A^u_{r,C}$ as $i_0\mathbb{R}^n + i_0i\mathbb{R}^n$, where $J \in L_r(A^u_{r,C}, A^h_{r,C})$. This means that $\xi$ is $(|_{[0]}\mathcal{F}, \mathcal{B}(\mathbb{R}^{2n}))$-measurable, whilst $w$ is $(|_{[r]}\mathcal{F}, \mathcal{B}(A^h_{r,C}))$-measurable, where $(\Omega_{[r]}, |_{[r]}\mathcal{F})$ is a measurable space, $P_{[r]} : |_{[r]}\mathcal{F} \to A_{r,C}$ is a measure.

Assume that there is an injection $\theta : (\Omega_{[0]}, |_{[0]}\mathcal{F}) \to (\Omega_{[r]}, |_{[r]}\mathcal{F})$ and $P_{[0]}$ has an extension $P = P^\theta_{[0]}$ on $(\Omega_{[r]}, |_{[r]}\mathcal{F})$ such that $P^\theta_{[0]}(\Omega_{[r]} \setminus \theta(\Omega_{[0]})) = 0$, $P^\theta_{[0]}(A) = P_{[0]}(\theta^{-1}(A \cap \theta(\Omega_{[0]})))$ for each $A \in |_{[r]}\mathcal{F}$ and $|P_{[r]}(\Omega_{[r]} \setminus \theta(\Omega_{[0]})) = 0$. Then it may be the case that $P$ and $P_{[r]}$ are related by Formulas 2.4(2), 2.4(3) [IS] with the help of $U = U_{[r]} = J^2$ and $U_{[0]} = I$ using the $A_{r,C}$-analytic extension. If $f = F(w)$, where $F : A^u_{r,C} \to A^h_{r,C}$ is a Borel measurable function then there exists a Borel measurable function $G : \mathbb{R}^{2n} \to A^h_{r,C}$ such that $G(\xi) = f$. Therefore if $u : A^h_{r,C} \to \mathbb{R}$ is a Borel measurable function, then using Formulas 2.4(2), 2.4(3) [IS] we put

$$
Eu(f) = \int_{\Omega_{[0]}} u(G(\xi(\omega)))P_{[0]}(d\omega).
$$

If

$$
\int_{A_{[0]}} u(G(\xi(\omega)))P_{[0]}(d\omega) = \int_{A_{[0]}} g(\theta(\omega))P_{[0]}(d\omega)
$$

for each $A \in \mathcal{G}$, where $g : \Omega_{[r]} \to \mathbb{R}$ is $(\mathcal{G}, \mathcal{B}(\mathbb{R}))$-measurable, $A_{[0]} = \theta^{-1}(A \cap \theta(\Omega_{[0]}))$, $|_{[0]}\mathcal{G} = \theta^{-1}(\mathcal{G} \cap \theta(\Omega_{[0]}))$, then $g$ will be called the conditional expectation of $u(f)$ relative to $\mathcal{G}$ and denoted by $E(u(f)|\mathcal{G}) = g$, since $P(\Omega_{[r]} \setminus \theta(\Omega_{[0]})) = 0$ and $|P_{[r]}(\Omega_{[r]} \setminus \theta(\Omega_{[0]})) = 0$, where $\mathcal{G}$ is a $\sigma$-subalgebra in $|_{[r]}\mathcal{F}$.
Henceforth this convention will be used, if some other will not be specified.

Let $L_{r,i}(A_{r,C}^n, A_{r,C}^b)$ denote a family of all right $A_{r,C}$-linear operators $J$ from $A_{r,C}^n$ into $A_{r,C}^b$ fulfilling the condition

$$(2)\ J(A_{r}^n) \subset A_{r}^b.$$  

**Theorem 2.8.** Suppose either $X = C_b^0(T, H)$ or $X = H^T$, where $H = A_{r,C}^n$ with $n \in \mathbb{N}$, $2 \leq r < \infty$, either $T = [0, s]$ with $0 < s \leq \infty$ or $T = [0, \infty)$. Then there exists a family $Ψ$ of pairwise inequivalent Markovian random functions with $A_{r,C}$-valued transition measures of the type $μ_{U_{t,A}}$ (see Definition 2.4 [18]) on $X$ of a cardinality $\text{card}(Ψ) = c$, where $c = 2^{R_0}$, $0 < t \in T$.

**Proof.** Naturally the algebra $A_{r,C}^n = \bigotimes_{j=1}^n A_{r,C}$, if considered as a linear space over $R$, also possesses a structure of the $R$-linear space isomorphic with $R^{2r+1,n}$. Therefore the Borel $σ$-algebra $B(A_{r,C}^n)$ of the algebra $A_{r,C}^n$ is isomorphic with $B(R^{2r+1,n})$. So put $P(t, A) = μ_{U_{t,A}}(A)$ for each $0 < t \in T$ and $A \in B(H)$, where an operator $U$ and a vector $p$ are marked satisfying conditions of Definition 2.4 and 2.3(α) [18].

Naturally an embedding of $R^n$ into $A_{r,C}^n$ exists as $i_0 R^n$, where $i_0 = 1$. If $ξ(t)$ is an $R^n$-valued random function, $J$ is a right $A_{r,C}$-linear operator $J : A_{r,C}^n \to A_{r,C}^n$ satisfying the condition $J(A_{r}^n) \subset A_{r}^n$, $v \in A_{r,C}^n$, then generally $w(t) = Jξ(t) + vt$ is an $A_{r,C}$-valued random function, where $0 \leq t \in T$, $w(t)$ is a shortened notation of $w(t, ω)$.

It is well-known, that the operators $B_j^{±1/2}$ exist (see, for example, Ch. IX, Sect. 13 in [2]), since $B_j$ is positive definite for each $j$. On the Cayley-Dickson algebra $A_r$ the function $\sqrt{a}$ exists (see §3.7 and Lemma 5.16 in [22]).

It has an extension on $A_{r,C}$ and its branch such that $\sqrt{a} > 0$ for each $a > 0$ can be specified by the following. Take an arbitrary $a = a_0 + ia_1 \in A_{r,C}$ with $a_0 \in A_r$ and $a_1 \in A_r$. Put $a_{0,0} = Re(a_0)$, $a_{1,0} = Re(a_1)$, $a_0' = a_0 - a_{0,0}$, $a_1' = a_1 - Re(a_1)$. If $a_{0,0} \neq 0$ and $a_{1,0} \neq 0$, then $a$ can be presented in the form $a = (α + iβ)(u + iv')$ with $α \in R$, $β \in R$, $u \in A_r$, $v' \in A_r$, $Re(v') = 0$. Therefore in the latter case $\sqrt{a} = \sqrt{α + iβ} \sqrt{u + iv'}$, since $C = Z(A_{r,C})$. If $a$ is such that $a_{0,0} = 0$ and $a_{1,0} \neq 0$ then for $b = ia$ there are $b_{0,0} = -a_{1,0} \neq 0$ and $b_{1,0} = 0$. On the other hand, for $a$ with $a_{1,0} = 0$ the equation $(γ + iδ)^2 = a_0 + ia_1'$ has a solution with $γ$ and $δ$ in $A_r$, since utilizing the standard basis
of the complexified Cayley-Dickson algebra this equation can be written as
the quadratic system in $2^r$ complex variables $\gamma_0 + i\delta_0, \ldots, \gamma_{2^r-1} + i\delta_{2^r-1}$. The
latter system has a solution $(\gamma, \delta)$ in $A^2_r$, since each polynomial over $C$ has
zeros in $C$ by the principal algebra theorem. Therefore the initial equation
has a solution in $A_{r,C}$. Thus the operator $U^{1/2} = \bigoplus_{j=1}^{m} a_j^{1/2} B_j^{1/2}$ exists and it
evidently belongs to $L_r(A^a_{r,C}, A^a_{r,C})$.

Particularly, $J$ can be $J = U^{1/2}$, while as $\xi(t)$ it is possible to take a
Wiener process with the zero expectation and the unit covariance operator.

If $f \in X$, then $T \ni t \mapsto f(t)$ defines a continuous $R$-linear projection
$\pi_t$ from $X$ into $H$. Therefore, $\pi_{t_n} \times (\pi_{t_{n-1}} \times \ldots \times \pi_{t_1})$ provides a continuous
$R$-linear projection $\pi_q$ from $X$ into $H^q$ for each $0 < t_1 < \ldots < t_n \in T$,
where $q = \{t_1, \ldots, t_n\}$. These projections and the Borel $\sigma$-algebras $B(H^q)$ on
$H^q$ for finite linearly ordered subsets $q$ in $T$ induce an algebra $\mathcal{R}(X)$ of $X$.
Since $H^T$ is supplied with the product Tychonoff topology, then a minimal
$\sigma$-algebra $\mathcal{R}_{\sigma}(H^T)$ generated by $\mathcal{R}(H^T)$ coincides with the Borel $\sigma$-algebra
$B(H^T)$. The topological spaces $T$ and $H$ are separable and relative to the
norm topology on $C^0_b(T, H)$ one gets also $\mathcal{R}_{\sigma}(C^0_b(T, H)) = B(C^0_b(T, H))$.

By virtue of Proposition 2.7 [18] and Formulas 2.4(2), 2.4(3) [18] a char-
pacteristic functional of $P_{U,p}(t, A) := \mu_{U,pt}$ fulfills Condition 2.4(6). It is
worth to associate with $P_{U,p}(t, A)$ a spatially homogeneous transition measure
$P_{U,p}(t_1, x_1, t_2, A)$ according to Equation 2.4(1). The representation 2.10(2)
[18] implies, that a bijective correspondence exists between $\sigma$-additive norm-
bounded $A_{r,C}$-valued measures and their characteristic functionals, since it
is valid for each real-valued addendum $\mu_{j,k}$ (see, for example, [2, 31]) and
$Z(A_{r,C}) = C$. Moreover, a characteristic functional of the ordered con-
volution $(\mu * \nu)$ of two $\sigma$-additive norm-bounded $A_{r,C}$-valued measures $\mu$
and $\nu$ is the ordered product $\hat{\mu} \cdot \hat{\nu}$ of their characteristic functionals $\hat{\mu}$ and $\hat{\nu}$
respectively. Therefore, Conditions 2.1(1)-(4) are satisfied.

Then Formulas 2.1(5), (7), (8) together with the data above describe an
$A_{r,C}$-valued Markov cylindrical distribution $P_{U,p}$ on $X$ (see Corollary 2.6 [18]
and Definition 2.1), since $t = t_2 - t_1 > 0$ for each $0 < t_1 < t_2 \in T$. The space
$H$ is Radon by Theorem I.1.2 [2], since $H$ as the metric space is separable
and complete. From Theorem 2.3 and Proposition 2.7 [18] it follows that $P_{U,p}$
is uniformly norm-bounded. In view of Theorem 2.15 and Corollary 2.17 we infer that there is a family of the cylindrical distribution has an extension to a norm-bounded measure $P_{U,p}$ on a completion $\mathcal{R}_P(X)$ of $\mathcal{R}(X)$, where $\mathcal{R}_\sigma(X) = \mathcal{B}(X)$.

Considering different operators $U$ and vectors $p$ and utilizing the Kakutani theorem (see, for example, in [2]) we infer that there is a family of the cardinality $c$ of pairwise nonequivalent and orthogonal measures of such type $P_{U,p}$ on $X$, since each $P$ has the representation 2.10(2) [18].

Let $\Omega = \Omega_{\{t\}}$ be the set of all elementary events

$$\omega := \{ f : f \in X(T, H; (t_0, t_1, \ldots, t_n); (0, x_1, \ldots, x_n)) \},$$

where $\Lambda_\omega$ is a finite subset of $\mathbb{N}$, $x_i \in H$, $(t_i : i \in \Lambda_\omega) \in \mathcal{Y}_T$ is a subset of $T \setminus \{t_0\}$ (see Remark 2.2 and Notation 2.5), where $t_0 = 0$, where $t_i < t_j$ for each $i < j$ in $\Lambda_\omega$. Hence an algebra $\hat{U}$ exists of cylindrical subsets of $X_0(T, H)$ induced by the projections $\pi_q : X_0(T, H) \to H^q$, where $q \in \mathcal{Y}_T$ is a subset in $T \setminus \{0\}$. This procedure induces the algebra $\mathcal{R}(\Omega)$ of $\Omega$. So one can consider a Markovian random function corresponding to $P_{U,p}$ (see Definition 2.6).

**Corollary 2.9.** Let $w(t, \omega)$ be a random function given by Theorem 2.8 with the transition measure $\mu_{U,p,t}$ for each $t > 0$, then

1. $E(w(t_2, \omega) - w(t_1, \omega)) = (t_2 - t_1)p$ and
2. $E((w_k(t_2, \omega) - p_k t_2)(w_h(t_1, \omega) - p_h t_1)) = (t_2 - t_1) a_j b_k - \beta_{j-1,k} - \beta_{i-1,j} \delta_{j,l}$ for each $k$ and $h$ in $\{1, \ldots, n\}$, where $0 < t_1 < t_2 \in T$, $1 + \beta_{j-1} \leq k \leq \beta_j$ and $1 + \beta_{l-1} \leq h \leq \beta_l$, $j = 1, \ldots, m$, $l = 1, \ldots, m$, where $E$ means the expectation relative to $P_{U,p}$.

**Proof.** By virtue of Theorem 2.8 the random function $w(t, \omega)$ has the transition measure

$$P(t_1, x, t_2, A) = \mu_{F_{t_2-t_1}}(A - x) = P_{U,(t_2-t_1)\Omega,(t_2-t_1)p}^L, \text{ where } x = w(t_1, \omega).$$

Therefore Formulas (1) and (2) follow from Proposition 2.8 and Theorem 2.9 [18].

**Definition 2.10.** Let $(\Omega, \mathcal{F}, P)$ be a measure space with an $\mathcal{A}_{r,c}$-valued $\sigma$-additive norm-bounded measure $P$ on a $\sigma$-algebra $\mathcal{F}$ of a set $\Omega$ with $P(\Omega) = 1$. It is said that there is a filtration $\{\mathcal{F}_t : t \in T\}$, if $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2} \subset \mathcal{F}$ for each $t_1 < t_2$ in $T$, where $\mathcal{F}_t$ is a $\sigma$-algebra for each $t \in T$, where either $T = [0, s]$ with $0 < s < \infty$ or $T = [0, \infty)$. A filtration $\{\mathcal{F}_t : t \in T\}$ is called normal, if
\{B \in \mathcal{F} : |P|(B) = 0 \} \subset \mathcal{F}_0 \quad \text{and} \quad \mathcal{F}_t = \bigcap_{T \ni t > t} \mathcal{F}_v \quad \text{for each} \quad t \in T.

Then if for each \( t \in T \) a random variable \( u(t) : \Omega \to X \) with values in a topological space \( X \) is \((\mathcal{F}_t, \mathcal{B}(X))\)-measurable, then the random function \( \{u(t) : t \in T\} \) and the filtration \( \{\mathcal{F}_t : t \in T\} \) are adapted, where \( \mathcal{B}(X) \) denotes the minimal \( \sigma \)-algebra on \( X \) containing all open subsets of \( X \) (i.e. the Borel \( \sigma \)-algebra). Let \( \mathcal{G} \) be a minimal \( \sigma \)-algebra on \( T \times \Omega \) generated by sets \( (v, t) \times A \) with \( A \in \mathcal{F}_v \), also \( \{0\} \times A \) with \( A \in \mathcal{F}_0 \). Let also \( \mu \) be a \( \sigma \)-additive measure on \((T \times \Omega, \mathcal{G})\) induced by the measure product \( \lambda \times P \), where \( \lambda \) is the Lebesgue measure on \( T \). If \( u : T \times \Omega \to X \) is \((\mathcal{G}_\mu, \mathcal{B}(X))\)-measurable, then \( u \) is called a predictable random function, where \( \mathcal{G}_\mu \) denotes the completion of \( \mathcal{G} \) by \( |\mu|\)-null sets, where \( |\mu| \) is the variation of \( \mu \) (see Definition 2.10 in [18]).

The random function given by Corollary 2.9 is called an \( \mathcal{A}_{r,C}^p \)-valued \((U, p)\)-random function or shortly \( U \)-random function for \( p = 0 \).

**Remark 2.11.** Random functions described in the proof of Theorem 2.8 are \( \mathcal{A}_{r,C} \) generalizations of the classical Brownian motion processes and of the Wiener processes.

Let \( w(t) \) be the \( \mathcal{A}_{r,C}^n \)-valued \((U, p)\)-random function provided by Theorem 2.8 and Corollary 2.9. Let a normal filtration \( \{\mathcal{F}_t : t \in T\} \) on \((\Omega, \mathcal{F}, P)\) be induces by \( w(t) \). Therefore \( w(t) \) is \((\mathcal{F}_t, \mathcal{B}(\mathcal{A}_{r,C}^n))\)-measurable for all \( t \in T \); \( w(t_1 + t_2) - w(t_1) \) is independent of any \( A \in \mathcal{F}_{t_1} \) for each \( t_1 \) and \( t_1 + t_2 \) in \( T \) with \( t_2 > 0 \). In view of Theorem 2.8 and Corollary 2.9 the conditions \( P(\Omega \setminus \theta(\Omega_{[0]})) = 0 \) and \( |P_{[0]}|(\Omega \setminus \theta(\Omega_{[0]})) = 0 \) are satisfied, where \( \Omega = \Omega_{[r]} \), \( \mathcal{F} = |r|\mathcal{F} \) (see Remark 2.7).

Suppose that \( \{S(t) : t \in T\} \) is an \( L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h) \) valued random function (that is, random operator), \( S(t) = S(t, \omega), \omega \in \Omega \) (see also the notation in Remark 2.7). It is called elementary, if a finite partition \( 0 = t_0 < t_1 < \ldots < t_k = s \) exists so that

\[
S(t) = \sum_{l=0}^{k-1} S_l \cdot ch_{(t_l, t_{l+1}]},
\]

where \( S_l : \Omega \to L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h) \) is \((\mathcal{F}_l, \mathcal{B}(L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h)))\)-measurable for each \( l = 0, \ldots, k - 1 \), where \( n \) and \( h \) are natural numbers, where \( ch_{(t_l, t_{l+1}] \} \) denotes the characteristic function of the segment \( (t_l, t_{l+1}] = \{t \in \mathbb{R} : t_l < t \leq t_{l+1}\} \), \( T = [0, s] \). A stochastic integral relative to \( w(t) \) and the elementary random
function \( S(t) \) is defined by the formula:

\[
(2) \quad \int_0^t S(\tau)dw(\tau) := \sum_{l=0}^{k-1} S_l(w(t_{l+1} \wedge t) - w(t_l \wedge t)),
\]

where \( t \wedge t' = \min(t, t') \) for each \( t \) and \( t' \) in \( T \). Similarly elementary \( L_{r,i}(A^p_{r,C}, A^b_{r,C}) \) random functions and their stochastic integrals are defined. Put

\[
(3) \quad \langle x, y \rangle := x_1y_1 + \ldots + x_ny_n \quad \text{for each } x, y \in A^b_{r,C},
\]

where \( y = (y_1, \ldots, y_n) \) with \( y_l \in A_{r,C} \) for each \( l \), \( \tilde{z} = z_0 - z' \) for each \( z = z_0 + z' \) in \( A_{r,C} \) with \( z_0 \in \mathbb{R} \) and \( z' \in A_{r,C} \), \( Re(z') = 0 \).

Denote by \( Q^* \) an adjoint operator of an \( \mathbb{R} \)-linear operator \( Q : A^u_{r,C} \to A^b_{r,C} \) such that

\[
(4) \quad \langle Qx, y \rangle = \langle x, Q^*y \rangle \quad \text{for each } x \in A^u_{r,C} \text{ and } y \in A^b_{r,C}.
\]

Then we put for \( Q = A + iB \) with \( A \) and \( B \) in \( L_{r,i}(A^u_{r,C}, A^b_{r,C}) \)

\[
(5) \quad \|Q\|_2^2 = 2Tr(AA^*) + 2Tr(BB^*).
\]

**Lemma 2.12.** Let

(i) \( S(t) \) be an elementary \( L_{r,i}(A^p_{r,C}, A^b_{r,C}) \)-valued random variable with

\[
E(\|S(t)\| ||F_a||) < \infty \quad \text{P-a.e. on } (\Omega, \mathcal{F}) \quad \text{for each } t \in [a, b] \text{ and let}
\]

(ii) \( w = w_0 + iw_1 \) be an \( A^u_{r,C} \)-valued random function with \( U_0 \) and \( U_1 \) random functions \( w_0 \) and \( w_1 \) respectively having values in \( A^u_{r,C} \) so that \( U_0 \) and \( U_1 \) belong to \( L_{r,i}(A^u_{r,C}, A^b_{r,C}) \) and the operator \( U = U_0 + iU_1 \) fulfills Conditions 2.3(a) and of Definition 2.4 [18], where \( w_0 \) and \( w_1 \) are independent; \( 0 \leq a < b < \infty \), \( [a, b] \subset T \) (see Definitions 2.10 [18], 2.10 and Remarks 2.7, 2.11 above).

Then \( E(\int_0^b S(t)dw(t)||F_a||) = 0 \quad \text{P-a.e. on } (\Omega, \mathcal{F}) \).

**Proof.** This follows from Corollary 2.9(1) and Formulas 2.10(1), 2.10(2), since \( 0 \leq (b-a)E(\sum_{l=0}^{k-1} ||S_l|| ||F_a||) < \infty \quad \text{P-a.e. and } E(w(t_2, \omega) - w(t_1, \omega)) = 0 \)

for each \( t_2 > t_1 \) in \( [a, b] \) for the \( U \)-random function \( w \).

**Lemma 2.13.** Let \( S = A + iB \), with \( A \) and \( B \) belonging to \( L_{r,i}(A^u_{r,C}, A^b_{r,C}) \),

where \( n \in \mathbb{N}, h \in \mathbb{N}, 2 \leq r < \infty \). Then

\[
(1) \quad \|S\|_2^2 = Tr[(A + iB)((A^* - iB^*))] + Tr[(A - iB)((A^* + iB^*))] < \infty \quad \text{and}
\]

\[
(2) \quad \|S\| \leq ||S||_2.
\]
Proof. Since $A$ and $B$ belong to $L_{r,i}(\mathcal{A}^n_{r,C}, A^h_{r,C})$, then
\[
\|A + iB\|_2^2 = 2Tr(AA^*) + 2Tr(BB^*) < \infty
\]
by 2.11(5), where as usually $Tr(AA^*)$ denotes the trace of the operator $AA^*$.

On the other side,
\[
[(A + iB)((A^* - iB^*)) + [(A - iB)((A^* + iB^*)] = 2(AA^* + BB^*).
\]
Since $A \in L_{r,i}(\mathcal{A}^n_{r,C}, A^h_{r,C})$, then $< Ae_k, e_l > \in \mathcal{A}_r$ for each $k = 1, \ldots, n$, $l = 1, \ldots, h$, where $\{e_k : k = 1, \ldots, m\}$ denotes the standard orthonormal base in the Euclidean space $\mathbb{R}^m$, where $m = \max(n, h)$; $\mathbb{R}^n$ is embedded into $\mathcal{A}^n_{r,C}$ as $i_0\mathbb{R}^n$. Therefore we deduce using Formulas 2.11(3) and 2.11(4) that
\[
(3) \quad Tr(AA^*) = \sum_{l,k} < e_l, Ae_k > |^2 \geq 0,
\]
since $Tr(AA^*) = \sum_t < AA^*e_t, e_t >= \sum_{l,k} < A^*e_l, e_k > < e_k, A^*e_l >$.

This implies Formula (1). From the Cauchy-Bunyakovskii-Schwarz inequality, Remark 2.7, Formulas (1) and (3) one gets Inequality (2).

**Theorem 2.14.** If $S(t)$ is an elementary random function with values in $L_{r,i}(\mathcal{A}^n_{r,C}, A^h_{r,C})$ and $w(t)$ is an $U$-random function in $\mathcal{A}^n_r$ as in Definition 2.10 with $U \in L_{r,i}(\mathcal{A}^n_{r,C}, A^n_{r,C})$, then
\[
(1) \quad E\left[< \int_a^t S(\tau) dw(\tau), \int_0^{\tau} S(\tau) dw(\tau) > \right]< \mathcal{F}_a
\]
\[
= E\left[\int_a^t Tr\{\{S(\tau)U^{1/2}\}\{U^{1/2}\}^*S^*(\tau)\}\right]d\tau < \mathcal{F}_a
\]
P-a.e. for each $0 \leq a < t$.

**Proof.** Since $Ew(t) = 0$ and $U : \mathcal{A}^n_{r,C} \to \mathcal{A}^n_{r,C}$, $U \in L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$ by the conditions of this theorem, then $a_j \in \mathcal{A}_r \setminus \{0\}$ for each $j$ and hence $U^{1/2} : \mathcal{A}^n_{r,C} \to \mathcal{A}^n_{r,C}$ and $U^{1/2} \in L_{r,i}(\mathcal{A}^n_{r,C}, \mathcal{A}^n_{r,C})$, since $U$ satisfies the conditions of Definition 2.4 and 2.3(α) \[18\] (see also Theorem 2.8). Therefore $w(t, \omega) \in \mathcal{A}^n_r$ and hence $S(t, \omega)w(t, \omega) \in \mathcal{A}^h_r$ for each $t \in T$ and $P$-almost all $\omega \in \Omega$, where $w(t)$ is a shortening of $w(t, \omega)$, while $S(t)$ is that of $S(t, \omega)$. On the other hand,
\[
(2) \quad < x, x > = |x|^2 = \sum_{j=1}^h x_j\bar{x}_j = \sum_{j=1}^h |x_j|^2
\]
for each $x \in \mathcal{A}^h_r$, where $|z|^2 = z\bar{z} = \sum_{l=0}^{2r-1} z_l^2$ for each $z$ in the Cayley-Dickson algebra $\mathcal{A}_r$, where $z = z_0i_0 + \ldots + z_{2r-1}i_{2r-1}$ with $z_l \in \mathbb{R}$ for each $l$, $\{i_0, \ldots, i_{2r-1}\}$ is the standard basis of $\mathcal{A}_r$.  

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Let $e_l \in \mathcal{A}_{r,C}^n$ and $f_l \in \mathcal{A}_{r,C}^h$, where $e_l = (\delta_{l,k} : k = 1, \ldots, n)$ and $f_l = (\delta_{l,k} : k = 1, \ldots, h)$, where $\delta_{l,k}$ is the Kronecker delta. Then for an operator $J$ in $L_{r,i}(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h)$ and each $x \in \mathcal{A}_{r,C}^n$ the representation is valid:

$$
(3) \quad Jx = \sum_{k=1}^{n} \sum_{l=1}^{h} J_{l,k} x_k f_l,
$$

where $x = x_1 e_1 + \ldots + x_n e_n$, $x_k \in \mathcal{A}_{r,C}$ and $J_{l,k} \in \mathcal{A}_r$ for each $k$ and $l$.

From the conditions imposed on $U$ (see Definition 2.4 [18]) it follows that $U$ and

$$
(4) \quad U^{1/2} = \bigoplus_{l=1}^{n} a_j^{1/2} B_j^{1/2} \quad \text{and} \quad (U^{1/2})^* = \bigoplus_{l=1}^{n} a_j^{1/2} B_j^{1/2}
$$

belong to $L_{r,i}(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^n)$, since to the positive definite operator $B_j$ the positive definite matrix $[B_j]^{1/2}$ with real matrix elements corresponds for each $j$, also $z^{1/2} \in \mathcal{A}_r$ for each $z \in \mathcal{A}_r$.

By virtue of Proposition 2.5 and Formulas 2.8(2), (3) [18] $\mu_{U^t,0}$ is the $\mathcal{A}_r$-valued measure for each $t > 0$, since the Cayley-Dickson algebra $\mathcal{A}_r$ is power-associative and $\exp(t) = \exp(z)$ for each $z \in \mathcal{A}_r$.

The random function $S(t)w(t)$ is obtained from the standard Wiener process $\xi$ in $\mathbb{R}^n$ with the zero expectation and the unit covariance operator with the help of the operator $U^{1/2}$ as

$$
(5) \quad S(t)w(t) = S(t)U^{1/2}\xi(t)
$$

according to Theorem 2.8. Therefore, from the Ito isometry theorem (see it, for example, in [2, 8]), Formulas (2)-(5) above and Remarks 2.7, 2.11 the statement of this theorem follows.

**Theorem 2.15.** Suppose that

(i) $S(t)$ is an elementary $L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^n)$ valued random function and

(ii) $w = w_0 + iw_1$ is an $\mathcal{A}_{r,C}^n$-valued random function satisfying Condition 2.12(ii), then

$$
(1) \quad E \left[ \left\| \int_a^t S(\tau)dw(\tau) \right\|^2 \bigg| \mathcal{F}_a \right] \leq \max(\|U_0^{1/2}\|^2_2, \|U_1^{1/2}\|^2_2) E \left[ \int_a^t \|S(\tau)\|^2_2 d\tau \bigg| \mathcal{F}_a \right]
$$

$\mu$-a.e. for each $0 \leq a < t \in T$.

**Proof.** We consider the following representation $S(x + iy) = (S_{0,0} x) + (S_{0,1} y) + i(S_{1,0} x) + i(S_{1,1} y)$ of $S$ with $S_{l,k} \in L_{r,i}(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^n)$ for every $l, k \in \{0, 1\}$ and $z = x + iy \in \mathcal{A}_{r,C}^n$ with $x$ and $y$ in $\mathcal{A}_{r,C}^n$. For each $z = x + iy \in \mathcal{A}_{r,C}^n$ we have $|Sz|^2 = |(S_{0,0} x) + (S_{0,1} y)|^2 + |(S_{1,0} x) + (S_{1,1} y)|^2$ (see Remark 2.1
and Formula 2.14(2) above. On the other hand, $|v|^2 = \langle v, v \rangle$ for each $v \in \mathcal{A}_b^h$. For two operators $G$ and $H$ in $L_{r,\alpha}(\mathcal{A}^a_C, \mathcal{A}^b_C)$ the inequality is valid $|\text{Tr}(GH^*)|^2 \leq |\text{Tr}(GG^*)| \cdot |\text{Tr}(HH^*)|$ due to the representation 2.14(3).

Applying Theorem 2.14 and Lemma 2.13 (see also Remarks 2.7, 2.11) to $S_{0,0}w_0 + S_{0,1}w_1 = (S_{0,0} \oplus S_{0,1})\eta$ and $S_{1,0}w_0 + S_{1,1}w_1 = (S_{1,0} \oplus S_{1,1})\eta$, where $\eta = w_0 \oplus w_1$ and $U = U_0 \oplus U_1$, we infer that

$$E\left[\left\| \int_0^t S(\tau) dw(\tau) \right\|^2 \right| \mathcal{F}_\alpha] =$$

$$2E\left[ \int_0^t \left( \sum_{l,k=0}^1 \text{Tr}(\{S_{l,k}(\tau)U^{1/2}_k\})\{(U^{1/2}_k)^*S_{l,k}(\tau)\}) \right) d\tau \right| \mathcal{F}_\alpha]$$

$$\leq \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2) E\left[ \int_0^t \|S(\tau)\|_2^2 d\tau \right| \mathcal{F}_\alpha]$$

P-a.e. for each $0 \leq a < t \in T$, since $|\text{Tr}(GH^*)| = |\text{Tr}(HG^*)|$ and $|a + b| \leq |a| + |b|$ for each $a$ and $b$ in $\mathcal{A}_b^h$.

**Lemma 2.16.** If conditions 2.15(i), 2.12(ii) are satisfied, then

$$\mathcal{P}\left\{ \left\| \int_a^b S(t) dw(t) \right\| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2) \right\} \leq$$

$$\alpha \beta^{-2} + \mathcal{P}\left\{ \int_a^b \|S(t)\|^2_2 dt > \alpha \right\}$$

for each $\alpha > 0$, $\beta > 0$, $[a, b] \subset T$, $0 \leq a < b < \infty$.

**Proof.** According to Formula 2.10(1) $S(t) = S(t_i)$ for each $t_i < t \leq t_{i+1}$, where $a = t_0 < t_1 < \ldots < t_k = b$. Since $S(t)$ is $(\mathcal{F}_{t_i}, \mathcal{B}(L_r(\mathcal{A}^a_C, \mathcal{A}^b_C)))$-measurable for each $t \in (t_i, t_{i+1}]$, then $\int_a^{t+1} \|S(t)\|^2_2 dt$ is $(\mathcal{F}_{t_i}, \mathcal{B}([0, \infty]))$-measurable. We consider a modified elementary random function $S_a(t)$ such that $S_a(t) = S(t)$ for each $t \leq t_i$ if $\int_a^{t+1} \|S(t)\|^2_2 dt \leq \alpha$; otherwise $S_a(t) = 0$ for each $t \in (t_i, b]$ if $\int_a^b \|S(t)\|^2_2 dt \leq \alpha < \int_a^{t+1} \|S(t)\|^2_2 dt$ for some $l$. Therefore $\int_a^b \|S_a(t)\|^2_2 dt \leq \alpha$ for each $t \in [a, b]$ and hence

$$\mathcal{P}\left\{ \sup_{t \in [a, b]} \|S_a(t) - S(t)\|_2 > 0 \right\} = \mathcal{P}\left\{ \int_a^b \|S(t)\|^2_2 dt > \alpha \right\}.$$

Then we deduce that

$$\mathcal{P}\left\{ \left\| \int_a^b S(t) dw(t) \right\| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2) \right\} =$$

$$\mathcal{P}\left\{ \left\| \int_a^b S_a(t) dw(t) + \int_a^b (S(t) - S_a(t)) dw(t) \right\| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2) \right\} \leq$$

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\[ P\left\{ \left\| \int_a^b S_\alpha(t) dw(t) \right\| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2) \right\} + P\left\{ \left\| \int_a^b (S(t) - S_\alpha(t)) dw(t) \right\| > 0 \right\} \leq \frac{E\left[ \left\| \int_a^b S_\alpha(t) dw(t) \right\|^2 \right]}{\beta^2 \max(\|U_0^{1/2}\|_2^2, \|U_1^{1/2}\|_2^2)} + P\left\{ \int_a^b \|S(t)\|^2 dt > \alpha \right\} \]

by Chebyshev inequality (see it, for example, in Sect. II.6 [31]), Equality (2) above, Formulas 2.10(1), (2) in [18]. By virtue of Theorem 2.15

\[ E\left[ \left\| \int_a^b S_\alpha(t) dw(t) \right\|^2 \right] \leq \max(\|U_0^{1/2}\|_2^2, \|U_1^{1/2}\|_2^2) E\left[ \int_a^b \|S(t)\|^2 dt \right], \]

since \( E[E(\zeta | F_a)] = E\zeta \) for a random variable \( \zeta : \Omega \rightarrow [0, \infty] \) which is \( (F_a, B([0, \infty])) \)-measurable (Sect. II.7 [31]). This implies inequality (1).

**Theorem 2.17.** If \( w \) is a \( U \)-random function and \( \{S(t) : t \in T\} \) is an \( L_{r,i}(\mathcal{A}_{r,C}^a, \mathcal{A}_{r,C}^h) \)-valued predictable random function satisfying the condition

\[(1) \quad E\left[ \int_a^t \text{Tr}(\{S(\tau)U^{1/2}\}\{(U^{1/2})^*S^*(\tau)\}) d\tau \right] < \infty \]

for each \( 0 \leq a < t \) in \( T \), where the operator \( U \) is specified in Definition 2.4 [18] such that \( U \in L_{r,i}(\mathcal{A}_{r,C}^a, \mathcal{A}_{r,C}^h) \), then a sequence \( \{S_\kappa(t) : \kappa \in \mathbb{N}\} \) of elementary random functions exists with \( t \in T \) such that

\[(2) \quad \lim_{\kappa \rightarrow \infty} E\left[ \int_a^t \text{Tr}(\{(S(\tau) - S_\kappa(\tau))U^{1/2}\}\{(U^{1/2})^*(S^*(\tau) - S_\kappa(\tau))\}) d\tau \right] = 0 \]

for each \( 0 \leq a < t \) in \( T \).

**Proof.** Notice that \( \text{Tr}(\{S(\tau)U^{1/2}\}\{(U^{1/2})^*S^*(\tau)\}) \geq 0 \) for each \( \tau \in T \), since \( U \in L_{r,i}(\mathcal{A}_{r,C}^a, \mathcal{A}_{r,C}^h) \) implying \( a_j \in \mathcal{A}_r \) and hence \( a_j^{1/2} \in \mathcal{A}_r \) for each \( j \). In view of Formulas 2.14(1), (3) the random function \( S(\tau)U^{1/2} \) having values in \( L_{r,i}(\mathcal{A}_r^a, \mathcal{A}_r^h) \) has the decomposition into a finite \( \mathbb{R} \)-linear combination

\[(3) \quad S(t)U^{1/2} = \sum_{l=1}^{n} \sum_{k=1}^{h} \sum_{j=0}^{2^r-1} \eta_{l,k,j} e_l \otimes f_k i_j \]

of real random functions \( \eta_{l,k,j} \) using vectors \( e_l, f_k \) and the standard basis \( \{i_0, i_1, ..., i_{2^r-1}\} \) of the Cayley-Dickson algebra \( \mathcal{A}_r \) over \( \mathbb{R} \). For each real-valued random function the condition

\[(4) \quad E\left[ \int_a^t \eta_{l,k,j}^2 d\tau \right] < \infty \]

is fulfilled for each \( 0 \leq a < t \) in \( T \), hence a sequence of real-valued random functions \( \eta_{l,k,j;\kappa} \) exists such that

\[(5) \quad \lim_{\kappa \rightarrow \infty} E\left[ \int_a^t (\eta_{l,k,j} - \eta_{l,k,j;\kappa})^2 d\tau \right] = 0 \]
for each $t \in T$. Thus Formulas (3) and (5) imply (2).

**Theorem 2.18.** If $w$ fulfills Condition 2.12(ii) and $S(t)$ is a $L_r(\mathcal{A}^a_{r,C}, \mathcal{A}^b_{r,C})$-valued predictable random function satisfying the following condition

$$
\int_a^b F(S; U_0, U_1)(\tau) d\tau < \infty
$$

for each $0 \leq a < b$ in $T$, where

$$
F(S; U_0, U_1)(t) = \sum_{l,k=0}^1 Tr(\{S_{l,k}(t)U_k^{1/2}\} \{(U_k^{1/2})^*S_{l,k}(t)\}),
$$

then a sequence $\{S_\kappa(t) : \kappa \in \mathbb{N}\}$ of elementary random functions exists with $t \in T$ such that

$$
\lim_{\kappa \to \infty} E\left[ \int_a^b F((S(\tau) - S_\kappa(\tau)); U_0, U_1)(\tau) d\tau \right] = 0
$$

for every $0 \leq a < b$ in $T$.

The proof is analogous to that of Theorem 2.17 with the help of Formula 2.15(2), since $E(E(\xi | F_a)) = E\xi$ with $\xi = \int_a^b F(S; U_0, U_1)(\tau) d\tau$, $\xi \geq 0$ $\mathbb{P}$-a.e.

**Definition 2.19.** It will be said that a sequence $\{S_\kappa(t) : \kappa \in \mathbb{N}\}$ of elementary $L_r(\mathcal{A}^a_{r,C}, \mathcal{A}^b_{r,C})$-valued random functions with $t \in T$ is mean absolute square convergent to a predictable $L_r(\mathcal{A}^a_{r,C}, \mathcal{A}^b_{r,C})$-valued random function $\{S(t) : t \in T\}$, where $w$ satisfies Condition 2.12(ii), if Condition 2.18(3) is fulfilled. The corresponding mean absolute square limit is induced by Formulas 2.15(2), 2.18(3) and is denoted by $l.i.m.$ The family of all predictable $L_r(\mathcal{A}^a_{r,C}, \mathcal{A}^b_{r,C})$-valued random functions $\{S(t) : t \in T\}$ satisfying condition 2.18(1) will be denoted by $V_{2,1}(U_0, U_1, a, b, n, h)$

A stochastic integral of $S \in V_{2,1}(U_0, U_1, a, b, n, h)$ is:

$$
(1) \quad \int_0^t S(\tau)d\tau := \lim_{\kappa \to \infty} \int_0^t S_\kappa(\tau)d\tau,
$$

where (2) $w = w_0 + iw_1$ is an $\mathcal{A}^a_{r,C}$-valued random function with $U_0$ and $U_1$ random functions $w_0$ and $w_1$ respectively having values in $\mathcal{A}^a_r$, where $0 \leq a \leq t \leq b$ in $T$, where $w$ satisfies Condition 2.12(ii).

**Proposition 2.20.** Let the conditions of Theorem 2.18 be satisfied and let $S \in V_{2,1}(U_0, U_1, a, b, n, h)$, $0 \leq a < c < b \in T$, then there exists $\int_a^b S(t)d\tau$ for each $a \leq \beta \leq \gamma \leq b$ and

$$
(1) \quad \int_a^b S(t)d\tau = \int_a^c S(t)d\tau + \int_c^b S(t)d\tau.
$$
**Proof.** In view of Theorem 2.18, Definitions 2.10, 2.19 and Remark 2.11 there exists $f_{\beta}^\gamma S(t)dw(t)$ for each $a \leq \beta \leq \gamma \leq b$. Formula (1) for elementary random functions $S_\kappa$ for each $\kappa \in \mathbb{N}$ follows from 2.11(2). Hence taking $l.i.m._{k \to \infty}$ we infer Equality (1) for $S \in V_{2,1}(U_0, U_1, a, b, n, h)$ by Theorem 2.18.

**Proposition 2.21.** If $S \in V_{2,1}(U_0, U_1, a, b, n, h)$, $S_\kappa \in V_{2,1}(U_0, U_1, a, b, n, h)$ for each $\kappa \in \mathbb{N}$, $w$ satisfies Condition 2.12(ii), and

\[ l.i.m._{\kappa \to \infty} \int_a^b S_\kappa(t)dw(t) = \int_a^b S(t)dw(t). \]

**Proof.** In view of Proposition 2.20 stochastic integrals $\int_a^b S(t)dw(t)$ and $\int_a^b S_\kappa(t)dw(t)$ exist for each $\kappa \in \mathbb{N}$. From Theorem 2.18 and Definition 2.19 it follows that

\[ l.i.m._{\kappa \to \infty} \int_a^b S_\kappa(t)dw(t) = \int_a^b S(t)dw(t). \]

**Proposition 2.22.** If $S \in V_{2,1}(U_0, U_1, a, b, n, h)$, and if $w$ satisfies Condition 2.12(ii), where $0 \leq a < b \in T$, then

\[ E \left[ \int_a^b S(t)dw(t) \right] = 0 \quad \text{P-a.e.} \]

and

\[ E \left[ \left\| \int_a^t S(\tau)d\tau \right\|^2 \right] \leq \max(\|U_0^{1/2}\|_2^2, \|U_1^{1/2}\|_2^2) E \left[ \int_a^t \|S(\tau)\|_2^2 d\tau \right] \]

P-a.e. for each $0 \leq a < t \in T$.

**Proof.** From Lemmas 2.12, 2.13, Proposition 2.20 the identity (1) follows. Then Theorem 2.15 and Proposition 2.20 imply Inequality (2), since $E(E(\zeta | \mathcal{F}_a)) = E\zeta$ with $\zeta = \int_a^b F(S; U_0, U_1)(t)dt$ and since

\[ P \left\{ \omega \in \Omega : E \left[ \int_a^b F(S; U_0, U_1)(t)dt \right] (\omega) = \infty \right\} = 0; \quad \zeta \geq 0 \quad \text{P-a.e.} \]

**Remark 2.23.** Let $ch_{[0, \infty)}(t) = 1$ for each $t \geq 0$, and $ch_{[0, \infty)}(t) = 0$ for each $t < 0$, be a characteristic function of $[0, \infty)$, $[0, \infty) \subset \mathbb{R}$. Then $G(\tau) \in$
V_{2,1}(U_0,U_1,a,b,n,h) for each \( t \in [a,b] \), if \( S(\tau) \in V_{2,1}(U_0,U_1,a,b,n,h) \), where \( G(\tau) := S(\tau)ch_{[0,\infty)}(\tau-t) \). It is put

\[
(1) \quad \eta(t) = \int_a^t S(\tau)dw(\tau) := \int_a^b S(\tau)ch_{[0,\infty)}(t-\tau)dw(\tau)
\]

for each \( t \in [a,b] \). From Proposition 2.22 it follows that \( \eta(t) \) is defined \( \mathbb{P} \)-a.e. By virtue of Theorem IV.2.1 in [8] \( \eta(t) \) is the separable random function up to the stochastic equivalence, since \((\mathcal{A}_{r,C}^h,|\cdot|)\) is the metric space. Therefore \( \eta(t) \) will be considered as the separable random function.

**Definition 2.24.** Let \( \zeta(t) \), \( t \in T \), be a \( L_{r,C}^h \)-valued random function adapted to the filtration \( \{\mathcal{F}_t : t \in T\} \) of \( \sigma \)-algebras \( \mathcal{F}_t \) and let \( E|\zeta(t)| < \infty \) for each \( t \in T \). If \( E(\zeta(t)|\mathcal{F}_s) = \zeta(s) \) for each \( s < t \) in \( T \), then the family \( \{\zeta(t),\mathcal{F}_t : t \in T\} \) is called a martingale. If \( \zeta(t) \in \mathbb{R} \) for each \( t \in T \) and \( E(\zeta(t)|\mathcal{F}_s) \geq \zeta(s) \) for each \( s < t \) in \( T \), then \( \{\zeta(t),\mathcal{F}_t : t \in T\} \) is called a sub-martingale.

**Lemma 2.25.** Assume that \( S(t) \in V_{2,1}(U_0,U_1,a,b,n,h) \) and \( w \) satisfies Condition 2.12(ii), \( 0 \leq a < b \leq \infty \), \( [a,b] \subset T \) and

\[
(1) \quad E\left[ \int_a^b F(S;U_0,U_1)(t)dt \bigg| \mathcal{F}_a \right] < \infty
\]

and \( \eta(t) \) is provided by Formula 2.23(1), then \( \{\eta(t),\mathcal{F}_t : t \in [a,b]\} \) is a martingale and \( \{|\eta(t)|^2,\mathcal{F}_t : t \in [a,b]\} \) is the sub-martingale.

**Proof.** By virtue of Proposition 2.22 \( \eta(t) \) is \((\mathcal{F}_t,\mathcal{B}(\mathcal{A}_{r,C}^h))\)-measurable and \( E(\eta(t_2)-\eta(t_1)|\mathcal{F}_{t_1}) = E\left[ \int_{t_1}^{t_2} S(\tau)d\tau \bigg| \mathcal{F}_{t_1} \right] = 0 \) for each \( a \leq t_1 < t_2 \leq b \). Hence \( \{\eta(t),\mathcal{F}_t : t \in [a,b]\} \) is the martingale.

The random function \( \eta(t) \) has the decomposition:

\[
(2) \quad \eta(t) = \sum_{k \in \{0,1\}; \ j \in \{0,1,\ldots,2^r-1\}; \ l \in \{1,\ldots,h\}} \eta_{k,j,l}(t) i_j^k e_l
\]

with \( \eta_{k,j,l}(t) \in \mathbb{R} \), for each \( k, j, l \), where \( \{e_l : l = 1, \ldots, h\} \) is the standard orthonormal basis of the Euclidean space \( \mathbb{R}^h \), where \( \mathbb{R}^h \) is embedded into \( \mathcal{A}^h_{r,C} \) as \( i_0^0 \mathbb{R}^h \). Therefore each random function \( \eta_{k,j,l}(t) \) is the martingale. Then

\[
(3) \quad |\eta(t)|^2 = \sum_{k \in \{0,1\}; \ j \in \{0,1,\ldots,2^r-1\}; \ l \in \{1,\ldots,h\}} |\eta_{k,j,l}(t)|^2.
\]

By virtue of Theorem 1 and Corollary 2 in Ch. III, Sect. 1 [8] and Formula 2.22(2) above \( \{|\eta_{k,j,l}(t)|^2,\mathcal{F}_t : t \in [a,b]\} \) is the sub-martingale for each \( k, j, \)
l, consequently, \(|\eta(t)|^2, \mathcal{F}_t : t \in [a, b]\) is the sub-martingale by Formulas (2) and (3).

**Lemma 2.26.** Let \(S(t) \in V_{2,1}(U_0, U_1, a, b, n, h)\) and \(w\) satisfy Condition 2.12(ii) such that

\[
(1) \quad E\left[ \int_a^b F(S; U_0, U_1)(t) dt \bigg| \mathcal{F}_a \right] < \infty,
\]

then

\[
(2) \quad \mathbb{P}\{ \sup_{t \in [a,b]} |\int_a^t S(\tau) dw(\tau)| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2) | \mathcal{F}_a \} \leq \beta^{-2} E\left[ \int_a^b F(S; U_0, U_1)(t) dt \bigg| \mathcal{F}_a \right],
\]

\[
(3) \quad \mathbb{P}\{ \sup_{t \in [a,b]} |\int_a^t S(\tau) dw(\tau)| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2) \} \leq \beta^{-2} E\left[ \int_a^b F(S; U_0, U_1)(t) dt \right].
\]

**Proof.** From (2) it follows (3). Therefore, it remains to prove (2). We take an arbitrary partition \(a = t_0 < t_1 < ... < t_n = b\) of \([a, b]\). Then we consider \(\eta_k := \int_{t_{k-1}}^{t_k} S(\tau) dw(\tau)\). In view of Lemma 2.25 \(\{\eta_l, \mathcal{F}_t : l = 1, ..., n\}\) is the martingale and \(\{||\eta_l||^2, \mathcal{F}_t : l = 1, ..., n\}\) is the sub-martingale.

Therefore, from Theorem 5 in Ch. III, Sect. 1 [3] and Formulas 2.10(1), (2) [13] we deduce that

\[
\mathbb{P}\{ \sup_{0 \leq l \leq n} |\eta_l| > \beta(\max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2)| \mathcal{F}_a \} \leq \beta^{-2} E(|\eta_n|^2| \mathcal{F}_a)
\]

(see also Remark 2.11). Together with Proposition 2.22 above and the Fubini theorem (II.6.8 [31]) this implies that

\[
(4) \quad \mathbb{P}\{ \sup_{0 \leq l \leq n} |\int_{t_{l-1}}^{t_l} S(\tau) dw(\tau)| > \beta \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2) | \mathcal{F}_a \} \leq \beta^{-2} E\left[ \int_a^b F(S; U_0, U_1)(t) dt | \mathcal{F}_a \right].
\]

The random function \(\int_a^b S(\tau) dw(\tau)\) is separable (see Remark 2.23), hence from (4) it follows (2).

**Theorem 2.27.** Let \(S \in V_{2,1}(U_0, U_1, a, b, n, h)\) be a predictable \(L_r(\mathcal{A}^n_{r,C}, \mathcal{A}^b_{r,C})\)-valued random function, let \(w\) satisfy Condition 2.12(ii), \([a, b] \subset T\). Then
the random function \( \eta(t) = \int_a^t S(\tau)dw(\tau) \) is stochastically continuous, where \( t \in [a, b] \).

**Proof.** If \( S_\kappa(\tau) \) is an elementary \( L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h) \)-valued random function, then \( \eta_\kappa(t) = \int_a^t S_\kappa(\tau)dw(\tau) \) is stochastically continuous by Formula 2.10(2), since \( w(t) \) is stochastically continuous.

For each \( S \in V_{2,1}(U_0, U_1, a, b, n, h) \) according to Definition 2.19 and the Fubini theorem \( \int_a^b E(F(S; U_0, U_1))(t)dt < \infty \). By virtue of Theorem 2.18 there exists a sequence \( \{S_\kappa(t) : \kappa \in \mathbb{N}\} \) of elementary \( L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h) \)-valued random functions such that 2.18(3) is satisfied. From Lemma 2.26 and the Fubini theorem we infer that

\[
P\{ \sup_{t \in [a,b]} | \int_a^t S(\tau)dw(\tau) - \int_a^t S_\kappa(\tau)dw(\tau) | > \epsilon \max(\|U_0^{1/2}\|_2, \|U_1^{1/2}\|_2) \} \leq \epsilon^{-2} \int_a^b E(F(S - S_\kappa; U_0, U_1))(t)dt.
\]

Therefore, there exists a sequence \( \{\epsilon_\kappa : \kappa \in \mathbb{N}\} \) with \( \lim_{\kappa \to \infty} \epsilon_\kappa = 0 \) and a sequence \( \{n_k \in \mathbb{N} : k \in \mathbb{N}\} \) such that

\[
\sum_{k=1}^{\infty} \epsilon_\kappa^{-2} \int_a^b E(F(S - S_{n_k}; U_0, U_1))(t)dt < \infty,
\]

consequently,

\[
\sum_{k=1}^{\infty} P\{ \sup_{t \in [a,b]} | \int_a^t S(\tau)dw(\tau) - \int_a^t S_{n_k}(\tau)dw(\tau) | > \epsilon_k \} < \infty.
\]

In view of the Borel-Cantelli lemma (see, for example, Ch. II, Sect. 10 [31]) a natural number \( k_0 \in \mathbb{N} \) exists such that

\[
P\{ \sup_{t \in [a,b]} | \int_a^b S(\tau)dw(\tau) - \int_a^b S_{n_k}(\tau)dw(\tau) | > \epsilon_k \} = 1
\]

for each \( k \geq k_0 \). Hence \( \int_a^t S(\tau)dw(\tau) \) is stochastically continuous, since \( \int_a^t S_{n_k}(\tau)dw(\tau) \) is stochastically continuous for each \( k \in \mathbb{N} \).

**Definition 2.28.** The generalized Cauchy problem over the complexified Cayley-Dickson algebra \( \mathcal{A}_{r,C} \). Let

1. \( H : T \times \mathcal{A}_{r,C}^h \to L_r(\mathcal{A}_{r,C}^n, \mathcal{A}_{r,C}^h) \),
2. \( G : T \times \mathcal{A}_{r,C}^h \to \mathcal{A}_{r,C}^h \) and
3. \( w = w_0 + iw_1 \) be a random function in \( \mathcal{A}_{r,C}^n \) satisfying Condition 2.12(ii), where \( n \) and \( h \) are natural numbers.
A stochastic Cauchy problem over $\mathcal{A}_{r,C}$ is:

\begin{equation}
Y(t) = \zeta + \int_{a}^{t} G(\tau, Y(\tau)) d\tau + \int_{a}^{t} H(\tau, Y(\tau)) d\tau, \quad t \in [a, b] \subset T.
\end{equation}

Then $Y(t)$ is called a solution, if it satisfies the following conditions (6)-(8):

6. $Y(t)$ is predictable,
7. $\forall t \in [a, b] \quad \mathbb{P}\{Y(t) : \int_{a}^{t} \|G(\tau, Y(\tau))\| d\tau = \infty\} = 0$ and
8. $\zeta + \int_{a}^{t} G(\tau, Y(\tau)) d\tau + \int_{a}^{t} H(\tau, Y(\tau)) d\tau = 0$,

where $Y(t)$ is a shortening of $Y(t, \omega)$.

**Theorem 2.29.** Let $G(t, y)$ and $H(t, y)$ be Borel functions, let $w$ satisfy Condition 2.12(ii), let $K = \text{const} > 0$ be such that

1. $\|G(t, x) - G(t, y)\| + \|H(t, x) - H(t, y)\| \leq K\|x - y\|$ and
2. $\|G(t, y)\|^2 + \|H(t, y)\|^2 \leq K^2(1 + \|y\|^2)$ for each $x$ and $y$ in $\mathcal{A}_{r,C}$,
3. $t \in [a, b] = T$, where $0 \leq a < b < \infty$,
4. $E[\|\zeta\|^2] < \infty$.

Then a solution $Y$ of Equation 2.28(5) exists; and if $Y$ and $Y_1$ are two stochastically continuous solutions, then

1. $\mathbb{P}\{\sup_{t \in [a, b]} \|Y(t) - Y_1(t)\| > 0\} = 0$.

**Proof.** We consider a Banach space $B_{2,\infty} = B_{2,\infty}[a, b]$ consisting of all predictable random functions $X : [a, b] \times \Omega \to \mathcal{A}_{r,C}$ such that $X(t)$ is $(\mathcal{F}_t, \mathcal{B}(\mathcal{A}_{r,C}))$-measurable for each $t \in [a, b]$ and $\sup_{t \in [a, b]} E[\|X(t)\|^2] < \infty$ with the norm

\begin{equation}
\|X\|_{B_{2,\infty}} = (\sup_{t \in [a, b]} E[\|X(t)\|^2])^{1/2}.
\end{equation}

In view of Proposition 2.21 there exists and operator $Q$ on $B_{2,\infty}$ such that

\begin{equation}
QX(t) = \zeta + \int_{a}^{t} G(\tau, X(\tau)) d\tau + \int_{a}^{t} H(\tau, X(\tau)) d\tau
\end{equation}

for each $t \in [a, b]$, since $G$ and $H$ satisfy Condition (ii). Then $QX(t)$ is $(\mathcal{F}_t, \mathcal{B}(\mathcal{A}_{r,C}))$-measurable for each $t \in [a, b]$, since $G$ and $H$ are Borel functions and $X \in B_{2,\infty}$. By virtue of Proposition 2.22, using the inequality $(\alpha + \beta +
\( \gamma )^2 \leq 3(\alpha^2 + \beta^2 + \gamma^2) \) for each \( \alpha, \beta \) and \( \gamma \) in \( \mathbb{R} \), the Cauchy-Bunyakovsky-Schwarz inequality, and Condition \((ii)\) of this theorem, we infer that

\[
E[\|QX(t)\|^2] \leq 3E[\|\gamma\|^2] + 3(b-a) \int_a^t K^2(1+\|X(\tau)\|^2)d\tau + 3E \int_a^t K^2(1+\|X(\tau)\|^2)d\tau
\]

\[
\leq 3E[\|\gamma\|^2] + 3K^2[2(b-a) + 1]E \int_a^b (1+\|X(\tau)\|^2)d\tau
\]

\[
\leq 3E[\|\gamma\|^2] + 3K^2(b-a)[(b-a) + 1](1+\|X\|^2_{B_{2,\infty}}).
\]

Thus \( Q : B_{2,\infty} \to B_{2,\infty} \). Then using the Cauchy-Bunyakovsky-Schwarz inequality, 2.3(12) \[18\], Proposition 2.22, Condition \((i)\) of this theorem, and the inequality \((\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)\) for each \( \alpha \) and \( \beta \) in \( \mathbb{R} \), we deduce that

\[
E[\|QX(t) - X_1(t)\|^2] \leq 2(b-a) \int_a^t E[\|G(\tau, X(\tau)) - G(\tau, X_1(\tau))\|^2]d\tau
\]

\[
+ 2E[\int_a^t \{H(\tau, X(\tau)) - H(\tau, X_1(\tau))\}dw(\tau)\|^2]
\]

\[
\leq C_1 \int_a^t E[\|X(\tau) - X_1(\tau)\|^2]d\tau \leq C_1(t-a)\|X - X_1\|_{B_{2,\infty}}^2
\]

for each \( X \) and \( X_1 \) in \( B_{2,\infty} \), \( t \in [a, b] \), where \( C_1 = 2K^2(b-a+1) \). Therefore, the operator \( Q : B_{2,\infty} \to B_{2,\infty} \) is continuous. Then we infer that

\[
E[\|Q^mX(t) - Q^mX_1(t)\|^2] \leq C_1 \int_a^t E[\|Q^{m-1}X(\tau) - Q^{m-1}X_1(\tau)\|^2]d\tau
\]

\[
\leq \ldots \leq C_1^m \int_{a<t_1<\ldots<t_m<t} E[\|X(t_m) - X_1(t_m)\|^2]dt_1\ldots dt_m
\]

\[
\leq C_1^m \|X - X_1\|_{B_{2,\infty}}^2 (b-a)^m/m!
\]

for each \( X \) and \( X_1 \) in \( B_{2,\infty} \), \( m = 1, 2, 3, \ldots \). Therefore,

\[
\|Q^{m+1}X - Q^mX\|_{B_{2,\infty}} \leq C_1^m (b-a)^m \|QX - X\|^2_{B_{2,\infty}}/m! \text{ for each } m = 1, 2, 3, \ldots .
\]

Hence the series \( \sum_{m=1}^\infty \|Q^{m+1}X - Q^mX\|_{B_{2,\infty}} \) converges. Thus the following limit exists \( \lim_{m \to \infty} Q^mX(t) =: Y(t) \) in \( B_{2,\infty} \). From the continuity of \( Q \) it follows that \( \lim_{m \to \infty} Q(Q^mX) = QY \), hence \( QY = Y \). Thus

\[
\|QY - Y\|_{B_{2,\infty}} = 0, \text{ consequently, } P\{Y(t) = QY(t)\} = 1 \text{ for each } t \in [a, b].
\]

This means that \( Y(t) \) is the solution of 2.28(5). In view of Theorem 2.27 and Condition \((ii)\) the solution \( Y(t) \) is stochastically continuous up to the stochastic equivalence.

Let now \( Y \) and \( Y_1 \) be two stochastically continuous solutions of Equation 2.28(5). We consider a random function \( q_N(t) \) such that \( q_N(t) = 1 \) if
$\|Y(\tau)\| \leq N$ and $\|Y_1(\tau)\| \leq N$ for each $\tau \in [a, t]$, $q_N(t) = 0$ in the contrary case, where $t \in [a, b]$, $N > 0$. Therefore $q_N(t)q_N(\tau) = q_N(t)$ for each $\tau < t$ in $[a, b]$, consequently,

$$q_N(t)[Y(t) - Y_1(t)] = q_N(t)[\int_a^t q_N(\tau)[G(\tau, Y(\tau)) - G(\tau, Y_1(\tau))]d\tau$$

$$+ \int_a^t q_N(\tau)[H(\tau, Y(\tau)) - H(\tau, Y_1(\tau))]dw(\tau)].$$

On the other hand,

$$q_N(\tau)[\|G(\tau, Y(\tau)) - G(\tau, Y_1(\tau))\| + \|H(\tau, Y(\tau)) - H(\tau, Y_1(\tau))\|]$$

$$\leq Kq_N(\tau)\|Y(\tau) - Y_1(\tau)\| \leq 2KN$$

by Condition (i). This implies that $E[q_N(t)\|Y(t) - Y_1(t)\|^2] < \infty$. Then using the Fubini theorem, 2.3(12) [13], Proposition 2.22, Lemma 2.26, we deduce that

$$E[q_N(t)\|Y(t) - Y_1(t)\|^2] \leq 2E[q_N(t)\|\int_a^t q_N(\tau)[G(\tau, Y(\tau)) - G(\tau, Y_1(\tau))]d\tau\|^2] +$$

$$2E[q_N(t)\|\int_a^t q_N(\tau)[H(\tau, Y(\tau)) - H(\tau, Y_1(\tau))]dw(\tau)\|^2]$$

$$\leq 2(b - a)\int_a^t E[q_N(\tau)\|G(\tau, Y(\tau)) - G(\tau, Y_1(\tau))\|^2]d\tau$$

$$+ 4\int_a^t E[q_N(\tau)F(H(\tau, Y(\tau)) - H(\tau, Y_1(\tau)); U_0, U_1)]d\tau$$

$$\leq 2K^2[b - a + \max(\|U_0^{1/2}\|_2^2, \|U_1^{1/2}\|_2^2)]\int_a^t E[q_N(\tau)\|Y(\tau) - Y_1(\tau)\|^2]d\tau.$$

Thus a constant $C_2 > 0$ exists such that

$$E[q_N(t)\|Y(t) - Y_1(t)\|^2] \leq C_2\int_a^t E[q_N(\tau)\|Y(\tau) - Y_1(\tau)\|^2]d\tau.$$

The Gronwall inequality [8, 9] implies that $E[q_N(t)\|Y(t) - Y_1(t)\|^2] = 0$, consequently,

$$P\{Y(t) \neq Y_1(t)\} \leq P\{\sup_{t \in [a, b]} \|Y(t)\| > N\} + P\{\sup_{t \in [a, b]} \|Y_1(t)\| > N\}.$$

The random functions $Y(t)$ and $Y_1(t)$ are stochastically continuous, hence stochastically bounded, consequently, $\lim_{N \to \infty} P\{\sup_{t \in [a, b]} \|Y(t)\| > N\} = 0$ and $\lim_{N \to \infty} P\{\sup_{t \in [a, b]} \|Y_1(t)\| > N\} = 0$. Therefore, the random functions $Y(t)$ and $Y_1(t)$ are stochastically equivalent. Thus $P\{\sup_{t \in [a, b]} \|Y(t) - Y_1(t)\| > 0\} = 0$.

**Corollary 2.30.** Let operators $G$ and $H$ be $G \in L_r(A_{eC}^b) = L_r(A_{eC}^b)$ and $H \in L_r(A_{eC}^a, A_{eC}^b)$ such that $G$ be a generator of a semigroup $\{S(t) : t \in [0, \infty]\}$. 26
Let also \( w(t) \) be a random function fulfilling Condition 2.12(ii). Then the Cauchy problem

\[
(1) \quad Y(t) = \zeta + \int_0^t GY(\tau) d\tau + \int_0^t Hdw(\tau),
\]

where \( t \in T, E[||\zeta||^2] < \infty \), has a solution

\[
(2) \quad Y(t) = S(t)\zeta + \int_0^t S(t-\tau)Hdw(\tau)
\]

for each \( 0 \leq t \in T \).

**Proof.** The condition \( G \in L_r(\mathcal{A}_{r,C}^h, \mathcal{A}_{r,C}^h) \) implies that

\[
||G|| = \sup_{x \in \mathcal{A}_{r,C}^h, ||x||=1} ||Gx|| < \infty,
\]

where \( ||x||^2 = ||x_1||^2 + \ldots + ||x_h||^2 \), \( x = (x_1, \ldots, x_h) \in \mathcal{A}_{r,C}^h \), \( x_k \in \mathcal{A}_{r,C} \) for each \( k \). As a realization of the semigroup \( S(t) \) it is possible to take \( \{S(t) = \exp(tG) : t \geq 0\} \), since \( G \) is a bounded operator and \( ||\exp(tG)|| \leq \exp(||G||t) \) for each \( t \geq 0 \) by Formulas 2.1(9) and 2.3(12) [18]. Therefore from Theorem 2.29 the assertion of this corollary follows.

**Theorem 2.31.** Let \( G, H \) and \( w \) satisfy conditions of Theorem 2.29, let \( Y_{t,z}(t) \) be an \( \mathcal{A}_{r,C}^h \)-valued random function satisfying the following equation:

\[
(1) \quad Y_{t,z}(t_1) = z + \int_t^{t_1} G(\tau, Y_{t,z}(\tau)) d\tau + \int_t^{t_1} H(\tau, Y_{t,z}(\tau)) dw(\tau),
\]

where \( z \in \mathcal{A}_{r,C}^h, t < t_1 \) in \([a, b] \subset T, 0 \leq a < b < \infty \). Then the random function \( Y \) satisfying Equation 2.28(5) is Markovian with the transition measure

\[
(2) \quad P(t, z, t_1, A) = P\{Y_{t,z}(t_1) \in A\} \text{ for each } A \in \mathcal{B}(\mathcal{A}_{r,C}^h).
\]

**Proof.** The random function \( Y(t) \) is \( (\mathcal{F}_t, \mathcal{B}(\mathcal{A}_{r,C}^h)) \)-measurable for each \( t \in [a, b] \). On the other hand, \( Y_{t,z}(t_1) \) is induced by the random function \( w(t_1) - w(t) \) for each \( t_1 \in (t, b] \), where \( w(t_1) - w(t) \) is independent of \( \mathcal{F}_t \). Therefore \( Y_{t,z}(t_1) \) is independent of \( Y(t) \) and each \( A \in \mathcal{F}_t \). By virtue of Theorem 2.29 \( Y(t_1) \) is unique (up to stochastic equivalence) solution of the equation

\[
(3) \quad Y(t_1) = Y(t) + \int_t^{t_1} G(\tau, Y(\tau)) d\tau + \int_t^{t_1} H(\tau, Y(\tau)) dw(\tau)
\]

and \( Y_{t,Y(0)}(t_1) \) also is its solution, consequently, \( P\{Y(t_1) = Y_{t,Y(0)}(t_1)\} = 1 \).
Let \( f \in C_b^0(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C}) \), where \( C_b^0(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C}) \) denotes the family of all bounded continuous functions from \( \mathcal{A}^h_{r,C} \) into \( \mathcal{A}_{r,C} \). Let \( g \in R_b(\Omega, \mathcal{A}_{r,C}) \), where \( R_b(\Omega, \mathcal{A}_{r,C}) \) denotes the family of all random variables \( g : \Omega \to \mathcal{A}_{r,C} \) such that there exists \( C_g = \text{const} > 0 \) for which
\[
P\{\|g\| < C_g\} = 1,
\]
where \( C_g \) may depend on \( g \). We put
\[
(4) \quad q(z, \omega) = f(Y_{t,z}(t_1, \omega)), \quad \text{hence } f(Y(t_1, \omega)) = q(Y(t_1), \omega),
\]
where \( Y(t) \) is a shortening of \( Y(t, \omega) \) as above, \( \omega \in \Omega \). Assume at first that
\( q \) has the following decomposition:
\[
(5) \quad q(z, \omega) = \sum_{k=1}^m q_k(z)u_k(\omega),
\]
where \( q_k : \mathcal{A}^h_{r,C} \to \mathcal{A}_{r,C}, \quad u_k : \Omega \to \mathcal{A}_{r,C}, \quad m \in \mathbb{N} \). This implies that \( u_k(\omega) \) is independent of \( \mathcal{F}_t \) for each \( k \). Therefore we deduce that
\[
E[\sum_{k=1}^m q_k(Y(t))u_k(\omega)] = \sum_{k=1}^m E[q_k(Y(t))]E[u_k(\omega)]
\]
and
\[
E[\sum_{k=1}^m q_k(Y(t))u_k(\omega)]Y(t) = \sum_{k=1}^m q_k(Y(t))E[u_k(\omega)],
\]
consequently,
\[
(6) \quad Egf(Y(t_1)) = EgE[f(Y(t_1))]Y(t)
\]
for \( q \) of the form (5). This implies that
\[
(7) \quad E[f(Y(t_1))|\mathcal{F}_t] = v(Y(t)),
\]
where \( v(z) = Ef(Y_{t,z}(t_1)) \).

Then
\[
E[\|q(Y(\tau), \omega)\|^2] \leq C_g^2\|f\|_{C}^2
\]
for each \( \tau \in [a, b] \) by 2.3(12) [15], since \( g \) and \( f \) are bounded, where \( \|f\|_{C} := \sup_{z \in \mathcal{A}^h_{r,C}} \|f(z)\| < \infty \). Therefore for each \( \epsilon > 0 \) there exists \( f_\epsilon \in C_b^0(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C}) \) for which \( q_\epsilon(z, \omega) = f_\epsilon(Y_{t,z}(t_1, \omega)) \) has the decomposition of type (5) and such that \( E[\|q_\epsilon(Y(t), \omega) - q(Y(t), \omega)\|^2] < \epsilon/C_g^2 \). Taking \( \epsilon \downarrow 0 \) one gets that Formulas (6) and (7) are accomplished for each \( f \in C_b^0(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C}) \). Therefore \( P\{Y(t_1) \in A|\mathcal{F}_t\} = P\{Y(t_1) \in A|Y(t)\} \) for each \( A \in \mathcal{B}(\mathcal{A}^h_{r,C}), \) \( t < t_1 \) in \( [a, b] \), since the families \( R_b(\Omega, \mathcal{A}_{r,C}) \) and \( C_b^0(\mathcal{A}^h_{r,C}, \mathcal{A}_{r,C}) \) of all such \( g \) and \( f \) separate points in \( \mathcal{A}^h_{r,C} \). This implies that \( P\{Y(t_1) \in A|\mathcal{F}_t\} = P_{t,Y(t)}(t_1, A) \) for each \( A \in \mathcal{B}(\mathcal{A}^h_{r,C}), \) where \( P_{t,Y(t)}(t_1, A) = P\{Y_{t,z}(t_1) \in A\} \).

**Conclusion 2.32.** The obtained in this paper results open new opportunities for integration of PDEs of order higher than two. Indeed, a solution of a stochastic PDE with real or complex coefficients of order higher than
two can be decomposed into a solution of a sequence of PDEs of order one or two with $A_{r,C}$ coefficients \cite{27, 28}. They can be used for further studies of random functions and integration of stochastic differential equations over octonions and the complexified Cayley-Dickson algebra $A_{r,C}$. It is worth to mention that equations of the type 2.28(5) are related with generalized diffusion PDEs of the second order. For example, this approach can be applied to PDEs describing unsteady heat conduction in solids \cite{30}, fourth order Schrödinger or Klein-Gordon type PDEs.

Another application of obtained results is for the implementation of the plan described in \cite{18}. It is related with investigations of analogs of Feynman integrals over the complexified Cayley-Dickson algebra $A_{r,C}$ for solutions of PDEs of order higher than two.

References

[1] Baez, J.C.: The octonions. Bull. Am. Math. Soc. 39 (2), 145-205 (2002)

[2] Dalecky, Yu.L.; Fomin, S.V.: Measures and differential equations in infinite-dimensional space. Kluwer, Dordrecht (1991)

[3] Dickson L.E.: The collected mathematical papers, vol. 1-5. Chelsea Publishing Co., New York (1975)

[4] Dirac, P.A.M.: Die Prinzipen der Quantenmechanik. Hirzel, Leipzig (1930)

[5] Emch, G.: Mécanique quantique quaternionienne et Relativité restreinte. Helv. Phys. Acta. 36, 739-788 (1963)

[6] Freidlin, M.: Functional integration and partial differential equations. Princeton University Press, Princeton (1985)

[7] Gantmacher, F.R.: Theory of matrices Nauka, Moscow (1988)

[8] Gilman, I.I.; Skorohod, A.V.: The theory of stochastic processes. Springer-Verlag, New York (1975); or Nauka, Moscow (1977)
[9] Gulisashvili, A.; van Casteren J.A.: Non-autonomous Kato classes and Feynman-Kac propagators. World Scientific, New Jersey (2006)

[10] Gürlebeck, K.; Sprössig W.: Quaternionic and Clifford calculus for physicists and engineers. John Wiley and Sons, Chichester (1997)

[11] Gürlebeck, K.; Sprössig, W.: Quaternionic analysis and elliptic boundary value problem. Birkhäuser, Basel (1990)

[12] Gürsey, F.; Tze, C.-H.: On the role of division, Jordan and related algebras in particle physics. World Scientific Publ. Co., Singapore (1996)

[13] Pap. E.: Handbook of measure theory. vol. 1-2. Elsevier, Amsterdam (2002)

[14] Johnson, G.W.; Lapidus, M.L.: The Feynman integral and Feynman’s operational calculus. Oxford Univ. Press, Clarendon Press, New York (2000)

[15] Kantor, I.L.; Solodovnikov A.S.: Hypercomplex numbers. Springer, New York (1989)

[16] Kim, B.J.; Kim B.S.: Integration by parts formulas for analytic Feynman integrals of unbounded functionals. Integr. Transforms and Spec. Funct. 20 (1), 45-57 (2009)

[17] Lawson, H.B.; Michelson, M.-L.: Spin geometry Princeton University Press, Princeton (1989)

[18] Ludkowski, S.V.: Octonion measures for solutions of PDEs. Adv. Appl. Clifford Algebras. 30, 1-23 (2020); doi: 10.1007/s00006-020-01062-y.

[19] Ludkovsky, S.V.: Feynman integration over octonions with application to quantum mechanics. Math. Methods Appl. Sci. 33 (9), 1148-1173 (2010)

[20] Ludkovsky, S.V.: Noncommutative quasi-conformal integral transforms over quaternions and octonions. J. Math. Sci., N.Y. 157 (2), 199-251 (2009)
[21] Ludkovsky, S.V.; van Oystaeyen, F.: Differentiable functions of quaternion variables. Bull. Sci. Math. (Paris). Ser. 2. 127, 755-796 (2003)

[22] Ludkovsky, S.V.: Differentiable functions of Cayley-Dickson numbers and line integration. J. Math. Sci., N.Y. 141 (3), 1231-1298 (2007)

[23] Ludkovsky, S.V.: Functions of several Cayley-Dickson variables and manifolds over them. J. Math. Sci., N.Y. 141 (3), 1299-1330 (2007)

[24] Ludkovsky, S.V.: Geometric loop groups and diffeomorphism groups of manifolds, stochastic processes on them, associated unitary representations. In: Ying, L.M. (ed.) Focus on Groups Theory Research, pp. 59-136. Nova Science Publishers, Inc., New York (2006)

[25] Ludkovsky, S.V.: Two-sided Laplace transform over Cayley-Dickson algebras and its applications. J. Math. Sci., N.Y. 151 (5), 3372-3430 (2008)

[26] Ludkovsky, S.V.: Residues of functions of octonion variables. Far East J. of Math. Sci. (FJMS) 39 (1), 65-104 (2010)

[27] Ludkowski, S.V.: Decompositions of PDE over Cayley-Dickson algebras. Rendic. dell‘Ist. di Math. dell‘Università di Trieste. Nuova Ser. 46, 1-23 (2014)

[28] Ludkovsky, S.V.: Line integration of Dirac operators over octonions and Cayley-Dickson algebras. Comput. Methods and Function Theory 12 (1), 279-306 (2012)

[29] Nicola, F.: Convergence in $L^p$ for Feynman path integrals. Adv. Math. 294, 384-409 (2016)

[30] Ozherelkova, L.M., Savin, E.S.: The temperature dependence of unsteady heat conduct in solids. Russ. Technol. J. 7 (2), 49-60 (2019).

[31] Shiryaev, A.N.: Probability. MTsNMO, Moscow (2011)