Connes-Lott model building on the two-sphere

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Abstract

In this work we examine generalized Connes-Lott models, with \(\mathbb{C} \oplus \mathbb{C}\) as finite algebra, over the two-sphere. The Hilbert space of the continuum spectral triple is taken as the space of sections of a twisted spinor bundle, allowing for nontrivial topological structure (magnetic monopoles). The finitely generated projective module over the full algebra is also taken as topologically non-trivial, which is possible over \(S^2\). We also construct a real spectral triple enlarging this Hilbert space to include ”particle” and ”anti-particle” fields.

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1 Introduction

In previous work \[14\], we studied the Connes-Lott program with the complex algebra $A_1 = C(S^2; \mathbb{C})$ of continuous complex-valued functions on the sphere. The Hilbert space $H_I$, on which this algebra was represented, consisted of one of the minimal left ideals $I_\pm$ of the algebra of sections of the Clifford bundle over $S^2$ with a standard scalar product. On this Hilbert space the Dirac operator was taken as $D_I = i(d - \delta)$ restricted to each of the ideals $I_\pm$. The projective modules $M$ over $A_1$ were constructed using the Bott projector $P = \frac{1}{2} (1 + \vec{n} \cdot \vec{\sigma})$, acting on the free module $A_1 \oplus A_1$. These modules are classified by the homotopy classes of the mappings $\vec{n} : S^2 \to S^2$ i.e. by $\pi_2(S^2) = \mathbb{Z}$. In Dirac’s interpretation, each integer corresponds to a magnetic monopole at the center of the sphere with magnetic charge $g$ quantised by $eg/4\pi = (n/2)\bar{\hbar}$.

In the present paper we extend the above analysis of topologically nontrivial aspects in noncommutative geometry, to the product algebra $A = A_1 \otimes A_2$, where $A_2 = \mathbb{C} \oplus \mathbb{C}$. It is clear that $A \simeq C(S^2 \times \{a, b\}; \mathbb{C})$, where $\{a, b\}$ denotes a two-point space, as in the original Connes-Lott paper \[4\]. In section \[2\] we construct the Hilbert space on which $A_1$ is represented. This Hilbert space $H(s)$ generalizes $H_I$ above, and is made of sections of what we call a Pensov spinor bundle of (integer or semi-integer) weight $s$, using a taxonomy introduced by Staruszkiewicz \[18\]. A generalized Dirac operator $D(s)$ acting on these Pensov spinors is defined and, for $s = \pm 1/2$ we recover the Kähler spinors of \[14\], while for $s = 0$ the usual Dirac spinors are obtained. These spinors may actually be identified with sections of twisted spinor bundles or, from a more physical viewpoint, as usual Dirac spinors interacting with a magnetic monopole of charge $g$ given by $eg/4\pi \bar{\hbar} = s$.

The projective modules over $A$, following Connes-Lott, are constructed in section \[3\] as $\mathcal{M} = P(A \oplus A)$ where $P = \left( P_a = 1, P_b = \frac{1}{2} (1 + \vec{n}_b \cdot \vec{\sigma}) \right)$. In Connes’ work \[1\], the smooth manifold is four-dimensional so that, taking the four-sphere $S^4$ as an example, we get mappings $\vec{n} : S^4 \to S^2$ classified by $\pi_4(S^2) = \mathbb{Z}_2$. However, the local unitary transformations acting as $P_b \to U^* P_b U$ are also classified by homotopy classes $\pi_4(U(2)) = \pi_4(S^4) = \mathbb{Z}_2$. It follows that\[2\] all Bott projectors define mod-

\[2\] We are indebted to prof. Balachandran of Syracuse University for discussions on this point.
ules isomorphic to the module obtained from $P_b = \frac{1}{2}(1 + \sigma_3)$, considered by Connes. For the two-sphere $S^2$ this does not happen since $\pi_2(S^2) = \mathbb{Z}$ and $\pi_2(U(2)) = \pi_2(SU(2)) = \pi_2(S^3) = \{1\}$. In section 4 we construct the full spectral triple obtained as the product of the Dirac-Pensov triple for the algebra $\mathcal{A}_1$ with a discrete spectral triple for the algebra $\mathcal{A}_2$. Following again Connes’ prescription à la lettre, we take the discrete Hilbert space as $H_{\text{dis}} = \mathbb{C}^{N_a} \oplus \mathbb{C}^{N_b}$ with chirality $\chi_{\text{dis}}$ given as +1 on the a-sector and -1 on the b-sector. The discrete Dirac operator $D_{\text{dis}}$ is then the most general hermitian matrix, odd with respect to the grading defined by $\chi_{\text{dis}}$. Eliminating the ”junk” in the induced representation of the universal differential envelope $\Omega^*(\mathcal{A})$, yields bounded operators $\Omega_D^*(\mathcal{A})$ in $\mathcal{H} = \mathcal{H}(s) \otimes H_{\text{dis}}$. The standard use of the Dixmier trace and of Connes’ trace theorem allows then to define a scalar product of operators in $\Omega^*_D(\mathcal{A})$. This scalar product is used in section 4.1 to construct the Yang-Mills-Higgs action. The main new features in this action, as compared with Connes’ result, are the appearance of an additional monopole potential of strength $eg/4\pi = (n/2)\bar{\hbar}$, where $n$ is the integer characterizing the homotopy class of $P_b$, and the fact that the Higgs doublet is not globally defined on $S^2$ but transforms as a Pensov field of weight $\pm n/2$. The particle sector is examined in section 4.2 and a covariant Dirac operator $D_\nabla$ acting on $H_p = \mathcal{M} \otimes_\mathcal{A} \mathcal{H}$ is defined. Here, the novelty is that, whilst the ”a-doublet” continues as a doublet of Pensov spinors of weight $s$, the ”b-singlet” metamorphoses in a Pensov spinor of weight $s+n/2$. If one should insist on a comparison with the standard electroweak model on $S^2$, this would mean that right-handed electrons see a different magnetic monopole than the left-handed and this is not really welcome. In section 5 we introduce a real Dirac-Pensov spectral triple by doubling the Hilbert space as $H_1 = \mathcal{H}(s) \oplus \mathcal{H}(-s)$. It is seen that, with the same $H_{\text{dis}}$ as before, it is not possible to define a real structure. However, a more general discrete Hilbert space $H_2 = \mathbb{C}^{N_{aa}} \oplus \mathbb{C}^{N_{ab}} \oplus \mathbb{C}^{N_{ba}} \oplus \mathbb{C}^{N_{bb}}$, as considered in [10, 17], allows for the construction of a real structure on $H_{\text{new}} = H_1 \otimes H_2$. The covariant Dirac operator on $\mathcal{M} \otimes_\mathcal{A} H_{\text{new}} \otimes_\mathcal{A} \mathcal{M}^*$ can also be defined and it is furthermore seen that, with the use of such a non trivial projective module, the abelian gauge fields are not slain, as they are when $\mathcal{M} = \mathcal{A}[21].$

Clearly this model building led us far from a toy electroweak model. The main purpose however is not to reproduce such a model on the two-sphere, but rather to examine some of the topologically nontrivial structures in model building with the simplest manifold allowing for such possibilities.
The Hilbert space of Pensov spinors on $S^2$

The standard atlas of the two-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ consists of two charts, the boreal, $H_B = \{(x, y, z) \in S^2 \mid -1 < z \leq 1\}$, and austral chart, $H_A = \{(x, y, z) \in S^2 \mid -1 \leq z < 1\}$, with coordinates:

$$
\zeta_B = \xi_B^1 + i\xi_B^2 = +\frac{x + iy}{1 + z} \text{ in } H_B,
$$

$$
\zeta_A = \xi_A^1 + i\xi_A^2 = -\frac{x - iy}{1 - z} \text{ in } H_A.
$$

In the overlap $H_B \cap H_A$, they are related by $\zeta_A \zeta_B = -1$ and the usual spherical coordinates $(\theta, \varphi)$, given by $\zeta_B = -1/\zeta_A = \tan \theta/2 \exp i\varphi$, are nonsingular. In each chart, dual coordinate bases of the complexified tangent and cotangent spaces are:

$$
\left\{ \partial = \frac{\partial}{\partial \zeta} = \frac{1}{2} \left( \frac{\partial}{\partial \xi^1} - i\frac{\partial}{\partial \xi^2} \right), \partial^* = \frac{\partial}{\partial \zeta^*} = \frac{1}{2} \left( \frac{\partial}{\partial \xi^1} + i\frac{\partial}{\partial \xi^2} \right) \right\}
$$

$$
\left\{ d\zeta = d\xi^1 + id\xi^2, d\zeta^* = d\xi^1 - id\xi^2 \right\}.
$$

In $H_B \cap H_A$ they are related by

$$
\begin{pmatrix}
\bar{\partial}_A & \bar{\partial}^*_A \\
\partial_B & \partial^*_B
\end{pmatrix} = \begin{pmatrix}
\zeta_B^2 & 0 \\
0 & \zeta_B^{*2}
\end{pmatrix} \begin{pmatrix}
\partial_A & \partial^*_A \\
\zeta_A^2 & \zeta_A^{*2}
\end{pmatrix}.
$$

The euclidean metric in $\mathbb{R}^3$ induces a metric on the sphere:

$$
g = \frac{4}{q^2} \delta_{ij} d\xi^i \otimes d\xi^j = \frac{2}{q^2} \left( d\zeta^* \otimes d\zeta + d\zeta \otimes d\zeta^* \right),
$$

where $q = 1 + |\zeta|^2$. Real and complex Zweibein fields are given by:

$$
\left\{ \theta^i = \frac{2}{q} d\xi^i ; i = 1, 2 \right\} \text{ and } \left\{ \theta = \frac{2}{q} d\zeta, \theta^* = \frac{2}{q} d\zeta^* \right\},
$$

with duals

$$
\left\{ \bar{\epsilon}^i = \frac{q}{2} \frac{\partial}{\partial \xi^i} ; i = 1, 2 \right\} \text{ and } \left\{ \bar{\epsilon} = \frac{q}{2} \frac{\partial}{\partial \zeta}, \bar{\epsilon}^* = \frac{q}{2} \frac{\partial}{\partial \zeta^*} \right\}.
$$
A rotation of the real Zweibein by an angle $\alpha$:

$$
\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \Rightarrow
\begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \end{pmatrix} =
\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}
\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix},
$$

becomes diagonal for the complex Zweibein:

$$
\begin{pmatrix} \theta \\ \theta^* \end{pmatrix} \Rightarrow
\begin{pmatrix} \tilde{\theta} \\ \tilde{\theta}^* \end{pmatrix} =
\begin{pmatrix} \exp(-i\alpha) & 0 \\ 0 & \exp(i\alpha) \end{pmatrix}
\begin{pmatrix} \theta \\ \theta^* \end{pmatrix}.
$$ (2.2)

This means that the complexified cotangent bundle $(T^*S^2)^C$ splits, in an $SO(2)$ invariant way, into the direct sum of two line bundles $(T^*S^2)'$ and $(T^*S^2)''$ with one-dimensional local bases of sections given by $\{\theta\}$ and $\{\theta^*\}$. In the overlap $H_B \cap H_A$, the Zweibein in $H_A$ and in $H_B$ are related by:

$$
\theta_A = (c_{AB})^{-1} \theta_B, \quad \theta^*_A = c_{AB} \theta^*_B,
$$ (2.3)

with the transition function $c_{AB} = \zeta_B / \zeta_B^* = \zeta_A^* / \zeta_A = \exp(2i\varphi)$, $\varphi$ being the azimuthal angle, well defined (modulo $2\pi$) in $H_B \cap H_A$.

Sections of $(T^*S^2)'$ and $(T^*S^2)''$ are written as $\Sigma' = \sigma^{(+1)} \theta$ and $\Sigma'' = \sigma^{(-1)} \theta^*$ such that, in $H_B \cap H_A$,

$$
\sigma^{(\pm 1)}|_A = (c_{AB})^{\pm 1} \sigma^{(\pm 1)}|_B.
$$

Following Staruszkiewicz [18], who refers to Pensov, we call such a field a Pensov scalar of weight $(\pm 1)$. The question is now addressed to define Pensov scalars of weight $s$ on $S^2$. In general this would require a cocycle condition on transition functions in triple overlaps. However, since the sphere is covered by only two charts, it is enough that the overlap equation

$$
\sigma^{(s)}|_A = (c_{AB})^s \sigma^{(s)}|_B
$$

be well defined. Now, $(c_{AB})^s = \exp(2is\varphi)$ is well defined when $2s$ takes integer values$^3$.

The corresponding line bundle will be denoted by $P^{(s)}$.

$^3$The integer $2s$ can be identified with the integer representing an element of the second Čech cohomology group of $S^2$ with integer values, $\tilde{H}^2(S^2, \mathbb{Z}) = \mathbb{Z}$, classifying the line bundles over the sphere $S^2$.

$^4$In the sequel, abusing the notation, we shall denote bundles and their spaces of sections by the same symbol.
A local basis of its sections in $H_B$ is denoted by $\Theta(s)_{|B}$, and in $H_A$ by $\Theta(s)_{|A}$. They are related in $H_B \cap H_A$ by the generalisation of (2.3):

$$\Theta(s)_{|A} = (c_{AB})^{-s}\Theta(s)_{|B}$$

and a local section is given by: $\Sigma(s) = \sigma(s) \Theta(s)$.

On $S^2$ with metric $g = \delta_{ij}\theta^i \otimes \theta^j$, the Levi-Civita connection reads

$$\nabla^{LC} \theta^i = -\Gamma^i_j \otimes \theta^j = -\Gamma^i_j \theta^k \otimes \theta^j,$$

where

$$\Gamma^i_j = -\frac{1}{2}\{\delta^i_k \frac{\partial q}{\partial \xi^j} - \delta^j_k \frac{\partial q}{\partial \xi^i} \delta_{kj}\}.$$

In terms of the complexified Zweibein (2.1), we write:

$$\nabla^{LC} \theta = -\Gamma \otimes \theta , \quad \nabla^{LC} \theta^* = -\Gamma^* \otimes \theta^*, \quad (2.4)$$

where

$$\Gamma = -\Gamma^* = -\frac{1}{2}\{\frac{\partial q}{\partial \zeta} - \frac{\partial q}{\partial \zeta^*} \theta^*\} = -\frac{1}{2}\{\zeta^* \theta - \zeta \theta^*\}.$$

It is easy to see that $\nabla^{LC} \Theta(s) = -s\Gamma \otimes \Theta(s)$ defines a connection in the module of Pensov $s$-scalars generalising (2.3) above. This connection maps $\mathcal{P}^{(s)}$ in $(T^*(S^2))^C \otimes \mathcal{P}^{(s)}$. Now the space of complex-valued one-forms $(T^*(S^2))^C$ is isomorphic to $\mathcal{P}^{(s+1)} \oplus \mathcal{P}^{(s-1)}$, so that $\nabla^{LC}$ is actually a mapping:

$$\nabla^{LC} : \mathcal{P}^{(s)} \mapsto \mathcal{P}^{(s+1)} \oplus \mathcal{P}^{(s-1)} ; \Sigma^{(s)} \mapsto \nabla^{LC} \Sigma^{(s)} = \left(d\sigma^{(s)} - s\Gamma \sigma^{(s)} \right) \otimes \Theta^{(s)}.$$

Projecting $\nabla^{LC} \Psi^{(s)}$ on each term in the sum $\mathcal{P}^{(s+1)} \oplus \mathcal{P}^{(s-1)}$ we obtain:

$$\nabla^{LC} \Psi^{(s)} = \frac{1}{2}\left(\dd^s \sigma^{(s)} \Theta^{(s+1)} + \dd^s \sigma^{(s)} \Theta^{(s-1)}\right),$$

where we have introduced the ”edth” operators of Newman and Penrose [15]:

$$\dd^s \sigma^{(s)} = q^{-s+1} \frac{\partial}{\partial \zeta} (q^s \sigma^{(s)}) = q \frac{\partial \sigma^{(s)}}{\partial \zeta} + s \frac{\partial q}{\partial \zeta} \sigma^{(s)}$$

and

$$\dd^s \sigma^{(s)} = q^{s+1} \frac{\partial}{\partial \zeta^*} (q^{-s} \sigma^{(s)}) = q \frac{\partial \sigma^{(s)}}{\partial \zeta^*} - s \frac{\partial q}{\partial \zeta^*} \sigma^{(s)}. \quad (2.5)$$
With respect to the scalar product of Pensov scalars:

\[
\left( \Sigma^{(s)}, T^{(s)} \right)_s = \int_{S^2} \sigma^{(s)*} \tau^{(s)} \omega,
\]

where \( \omega = \theta^1 \wedge \theta^2 = \frac{i}{2} \theta \wedge \theta^* \) is the invariant volume element on \( S^2 \), the operators \( \tilde{\delta}_s \) and \( \tilde{\delta}_s^\dagger \) are formally anti-adjoint:

\[
\left( \sigma^{(s+1)}, \tilde{\delta}_s \tau^{(s)} \right)_{s+1} = \left( -\tilde{\delta}_s^\dagger \sigma^{(s+1)}, \tau^{(s)} \right)_s.
\]

In a previous paper [14], the Dirac operator on Kähler spinors was defined as the restriction of \(-i(d - \delta)\) to the left ideals of the Clifford algebra bundle. Now, these ideals are identified with \( I^E_+ = P(0) \oplus P(1) \), with basis \( \{1 + i\omega, \theta\} \), and \( I^E_- = P(-1) \oplus P(0) \), with basis \( \{\theta^*, 1 - i\omega\} \). In these bases, the local expressions of the Dirac operators were given as:

\[
D^E_+ \left( \begin{array}{c} \sigma^{(0)} \\ \sigma^{(1)} \end{array} \right) = -i \left( \begin{array}{cc} 0 & \tilde{\delta}_0^\dagger \\ \tilde{\delta}_0 & 0 \end{array} \right) \left( \begin{array}{c} \sigma^{(0)} \\ \sigma^{(1)} \end{array} \right),
\]

\[
D^E_- \left( \begin{array}{c} \sigma^{(-1)} \\ \sigma^{(0)} \end{array} \right) = -i \left( \begin{array}{cc} 0 & \tilde{\delta}_0^\dagger \\ \tilde{\delta}_0 & 0 \end{array} \right) \left( \begin{array}{c} \sigma^{(-1)} \\ \sigma^{(0)} \end{array} \right).
\]

This suggests to define a Pensov spinor field of weight \( s \) as a section

\[
\Psi^{(s)} = \Sigma^{(s-1/2)} \oplus \Sigma^{(s+1/2)}
\]

of the Whitney sum \( \mathcal{P}^{(s-1/2)} \oplus \mathcal{P}^{(s+1/2)} \), with a Dirac operator locally expressed as:

\[
D^{(s)} \left( \begin{array}{c} \sigma^{(s-1/2)} \\ \sigma^{(s+1/2)} \end{array} \right) = -i \left( \begin{array}{cc} 0 & \tilde{\delta}_{s-1/2}^\dagger \\ \tilde{\delta}_{s-1/2} & 0 \end{array} \right) \left( \begin{array}{c} \sigma^{(s-1/2)} \\ \sigma^{(s+1/2)} \end{array} \right).
\]

The usual Dirac spinors on \( S^2 \) are recovered when \( s = 0 \).

With the complex representation of the real Clifford algebra \( \mathbb{C}^\ell(2,0) \)

\[
\gamma^1 \Rightarrow \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \gamma^2 \Rightarrow \left( \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right),
\]

\[
5 \quad \text{The real Clifford algebra } \mathbb{C}^\ell(p,q) \text{ is defined by } \gamma^k \gamma^l + \gamma^l \gamma^k = 2\eta^{kl}, \text{ where the flat metric tensor } \eta^{kl} \text{ is diagonal with } p \text{ times } +1 \text{ and } q \text{ times } -1. \text{ This entails some differences with other work using the Clifford algebra } \mathbb{C}^\ell(0,n) \text{ for Riemannian manifolds instead of } \mathbb{C}^\ell(n,0) \text{ used here.}
\]
acting on $\psi_{(s)} = \begin{pmatrix} \sigma^{(s-1/2)} \\ \sigma^{(s+1/2)} \end{pmatrix}$, the Dirac operator can also be written as:

$$D_{(s)}\psi_{(s)} = -i\gamma^k \nabla^{LC}_{k,(s)}\psi_{(s)},$$

where the covariant derivative of the spinor $\psi_{(s)}$ is given by:

$$\nabla^{LC}_{k,(s)}\psi_{(s)} = \frac{q}{2} \frac{\partial\psi_{(s)}}{\partial\xi_k} + \frac{1}{2} \Sigma^{(s)}_{ij} \Gamma^i_k \psi_{(s)},$$

Here, $\Sigma^{(s)}_{12} = i s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ reduces to $\frac{1}{4}[\gamma_1, \gamma_2]$ for $s = 0$. In terms of the “edth” operators, we may write

$$\nabla^{LC}_{(s),+} \psi_{(s)} = \frac{1}{2} \begin{pmatrix} \delta_{s-1/2} & 0 \\ 0 & \delta_{s+1/2} \end{pmatrix} \psi_{(s)},$$

$$\nabla^{LC}_{(s),-} \psi_{(s)} = \frac{1}{2} \begin{pmatrix} \delta_{s-1/2} & 0 \\ 0 & \delta_{s+1/2}^{\dagger} \end{pmatrix} \psi_{(s)}.$$

With $\gamma^{(+)} = \gamma^1 + i\gamma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and $\gamma^{(-)} = \gamma^1 - i\gamma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, the Dirac operator of (2.8) is now written as

$$D_{(s)}\psi_{(s)} = -i \left( \gamma^{(+)} \nabla^{LC}_{(s),+} + \gamma^{(-)} \nabla^{LC}_{(s),-} \right) \psi_{(s)}.$$

The transformation law for $s$-Pensov fields under a local Zweibein rotation (2.2) is related to the Spin$^c$ structure of the Pensov spinors:

$$\begin{pmatrix} \sigma^{(s-1/2)} \\ \sigma^{(s+1/2)} \end{pmatrix} \mapsto \begin{pmatrix} \sigma^{(s-1/2)} \\ \sigma^{(s+1/2)} \end{pmatrix} = \exp\{\alpha \Sigma_{12}^{(s)}\} \begin{pmatrix} \sigma^{(s-1/2)} \\ \sigma^{(s+1/2)} \end{pmatrix},$$

where

$$\exp\{\alpha \Sigma_{12}^{(s)}\} = \exp\{i s \alpha\} \begin{pmatrix} \exp(-i\alpha/2) & 0 \\ 0 & \exp(+i\alpha/2) \end{pmatrix}.$$

---

A Pensov spinor of weight $s$ can be interpreted as a usual Dirac spinor on $S^2$, interacting with a Dirac monopole of strength $s$. Indeed, in the expression of the covariant derivative, the term $i s \gamma^k \Gamma^i_k$ is the Clifford representative of the one-form (potential) $\mu_s = i s \theta \Gamma^i_k = \frac{s}{1 + |\zeta|^2} \left( \zeta^* d\zeta - \zeta d\zeta^* \right)$, which, in $\{H_B ; \cos \theta \neq +1\}$, takes the usual form $\mu_{s|B} = i s (1 - \cos \theta) d\phi$. 

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The Clifford action of $\gamma_3 = i\omega$ yields a grading on the Pensov spinors

$$\gamma_3 \begin{pmatrix} \psi(s) \\ \psi(s+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi(s-1/2) \\ \psi(s+1/2) \end{pmatrix}, \quad (\gamma_3)^2 = 1,$$

such that the Dirac operator (2.8) is odd

$$\mathcal{D}_{(s)} \gamma_3 + \gamma_3 \mathcal{D}_{(s)} = 0.$$  

According to (2.6), the scalar product of two Pensov spinors is defined as:

$$\langle \Phi_{(s)} | \Psi_{(s)} \rangle = \left( \sum^{(s-1/2)}, T^{(s-1/2)} \right)_{s-1/2} + \left( \sum^{(s+1/2)}, T^{(s+1/2)} \right)_{s+1/2}. \quad (2.11)$$

The adjointness (2.7) of $-i \tilde{\delta}_{s-1/2}$ and $-i \tilde{\delta}^\dagger_{s+1/2}$ implies that the Dirac operator is formally self-adjoint with respect to this scalar product. After completion, $\mathcal{P}^{(s-1/2)} \oplus \mathcal{P}^{(s+1/2)}$ becomes a bona fide Hilbert space $\mathcal{H}_{(s)}$ on which $\mathcal{D}_{(s)}$ acts as a self-adjoint (unbounded) operator. Its spectral resolution is completely solvable. Indeed, let $X = \left( \tilde{\partial} - \tilde{\partial}^* \right) \begin{pmatrix} X^+ \\ X^- \end{pmatrix}$ be a vector field on $S^2$, then the Lie derivatives of the Zweibein along $X$ are:

$$\mathcal{L}_X \theta = \left( \frac{\partial X^+}{\partial \xi} - \frac{1}{q} \left( \frac{\partial q}{\partial \xi^*} X^- + \frac{\partial q}{\partial \xi} X^+ \right) \right) \theta + \frac{\partial X^+}{\partial \xi^*} \theta^*,$n

$$\mathcal{L}_X \theta^* = \left( \frac{\partial X^-}{\partial \xi^*} - \frac{1}{q} \left( \frac{\partial q}{\partial \xi^*} X^- + \frac{\partial q}{\partial \xi} X^+ \right) \right) \theta^* + \frac{\partial X^-}{\partial \xi} \theta. \quad (2.12)$$

A vector field $X$ is said to be a conformal Killing vector field if $\mathcal{L}_X g = \mu g$, where $\mu$ is a scalar function on $S^2$. The expression of the Lie derivative (2.14) yields then the (anti-)holomorphic constraints:

$$\frac{\partial X^+}{\partial \xi^*} = 0, \quad \frac{\partial X^-}{\partial \xi} = 0,$n

and $\mu$ is given by:

$$\mu = q^2 \left( \frac{\partial (X^+/q^2)}{\partial \xi} + \frac{\partial (X^-/q^2)}{\partial \xi^*} \right).$$

If $X$ has to be globally defined, its Zweibein components $(2/q)X^+$ and $(2/q)X^-$ must be finite when $|\zeta| \to \infty$. For the standard metric $q = 1 + |\zeta|^2$, 

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and this implies that $X^+$, respectively $X^-$, is a quadratic polynomial in $\zeta$, respectively $\zeta^*$.
There are thus six linearly independent conformal Killing vector fields, three of which are genuinely Killing,i.e. with $\mu = 0$. They are chosen as

$$
iL_x = \frac{i}{2} \left( (\zeta^2 - 1) \bar{\partial} - (\zeta^{*2} - 1) \bar{\partial}^* \right),$$

$$
iL_y = \frac{i}{2} \left( (\zeta^2 + 1) \bar{\partial} + (\zeta^{*2} + 1) \bar{\partial}^* \right),$$

$$
iL_z = i \left( \zeta \bar{\partial} - \zeta^* \bar{\partial}^* \right).$$

The other three conformal Killing vector fields (with $\mu \neq 0$) are:

$$
iK_x = \frac{1}{2} \left( (\zeta^2 - 1) \bar{\partial} + (\zeta^{*2} - 1) \bar{\partial}^* \right),$$

$$
iK_y = \frac{i}{2} \left( -\zeta^2 - 1 \right) \bar{\partial} + (\zeta^{*2} + 1) \bar{\partial}^*,$$

$$
iK_z = \left( \zeta \bar{\partial} + \zeta^* \bar{\partial}^* \right).$$

As is well known, they form the Lie algebra $sl(2, \mathbb{C})$ with its Lie subalgebra $su(2)$ generated by $\{iL_x, iL_y, iL_z\}$. Standard angular momentum technique tells us that is easier to deal with the complex Killing vectors:

$$
\begin{align*}
L_+ &= L_x + iL_y = \zeta^2 \bar{\partial} + \bar{\partial}^*, \\
L_- &= L_x - iL_y = -\bar{\partial} - \zeta^{*2} \bar{\partial}^*, \\
L_0 &= L_z = \zeta \bar{\partial} - \zeta^* \bar{\partial}^*,
\end{align*}
$$

with commutation relations:

$$[L_0, L_{\pm}] = \pm L_{\pm}, \quad [L_+, L_-] = 2 L_0. \tag{2.13}$$

The Lie derivatives of $\theta$ with respect to these vector fields are:

$$
\mathcal{L}_+ \theta = \zeta \theta, \quad \mathcal{L}_- \theta = \zeta^* \theta, \quad \mathcal{L}_0 \theta = \theta.
$$

\footnote{Here we use the $\zeta_B$ coordinates to conform to standard conventions.}
An infinitesimal transformation of the Zweibein by Killing vectors is a rotation of the form (2.2): \[ \theta \mapsto \tilde{\theta} = \theta - i \delta \alpha \theta = \theta + \delta t L_X \theta \], and, according to (2.12), we identify
\[ -i \delta \alpha = \left( \frac{\partial X^+}{\partial \zeta} - \frac{1}{q} (\zeta X^- + \zeta^* X^+) \right) \delta t. \]
The transformation of the Pensov basis \( \Theta^{(s)} \) is then obtained as:
\[ \Theta^{(s)} \mapsto \tilde{\Theta}^{(s)} = \Theta^{(s)} - is \delta \alpha \Theta^{(s)} = \Theta^{(s)} + \delta t L_X \Theta^{(s)} , \]
with
\[ L_X \Theta^{(s)} = s \left( \frac{\partial X^+}{\partial \zeta} - \frac{1}{q} (\zeta X^- + \zeta^* X^+) \right) \Theta^{(s)}. \]
The Lie derivatives of Pensov fields \( \Sigma^{(s)} = \sigma^{(s)} \Theta^{(s)} \) along the the Killing vectors of (2.13) read:
\[ L^{(s)}_\pm \sigma^{(s)} = \left( \zeta^2 \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \zeta^*} + s \zeta \right) \sigma^{(s)} , \]
\[ L^{(s)}_0 \sigma^{(s)} = \left( \zeta \frac{\partial}{\partial \zeta} - \zeta^* \frac{\partial}{\partial \zeta^*} + s \right) \sigma^{(s)}. \]
These Lie derivatives yield a representation of \( su(2) \) on \( s \)-Pensov fields:
\[ [L^{(s)}_0, L^{(s)}_\pm] = \pm L^{(s)}_\pm , \quad [L^{(s)}_+, L^{(s)}_-] = 2 L^{(s)}_0. \]
The Casimir operator is given by:
\[ (\tilde{L}^{(s)})^2 = \frac{1}{2} \left( L^{(s)}_+ L^{(s)}_- + L^{(s)}_- L^{(s)}_+ \right) + (L^{(s)}_0)^2 = -q^2 \frac{\partial^2}{\partial \zeta \partial \zeta^*} + s q L^{(s)}_0. \]
Straightforward angular momentum algebra yields the monopole harmonics of Wu and Yang [23], solutions of
\[ (\tilde{L}^{(s)})^2 Y_{j,m}^s = j(j+1) Y_{j,m}^s , \quad L^{(s)}_0 Y_{j,m}^s = m Y_{j,m}^s , \quad (2.14) \]
where \( j = |s|, |s| + 1, \ldots \) and \( m = -j, -j + 1, \ldots, j - 1, j \).
They can be written in terms of Jacobi functions and, using some appropriate
Olinde-Rodrigues formulae, they are obtained as:

\[ Y_{j,m}^s = \frac{(-1)^{j-s}}{2^j} \sqrt{\frac{2j+1}{4\pi}} \left[ \frac{(j+m)!}{(j+s)!(j-s)!(j-m)!} \exp(i(m-s)\varphi) \right. \]

\[ (1-z)^{m-s}(1+z)^{m+s} \left( \frac{d}{dz} \right)^{j-m} \left[ (1-z)^{j-s}(1+z)^{j+s} \right] \]

\[ = \frac{(-1)^{j+m}}{2^j} \sqrt{\frac{2j+1}{4\pi}} \left[ \frac{(j-m)!}{(j+s)!(j-s)!(j+m)!} \exp(i(m-s)\varphi) \right. \]

\[ (1-z)^{m-s}(1+z)^{m+s} \left( \frac{d}{dz} \right)^{j+m} \left[ (1-z)^{j+s}(1+z)^{j-s} \right] , \quad (2.15) \]

where \( \varphi \) is the azimuthal angle and \( z = \cos(\theta) \), the cosine of the polar angle. The Lie derivative of Pensov fields along one of the vector fields \( L_{\pm}, L_0 \) "commutes" with the edth operators of (2.5) in the sense that

\[ \hat{\delta}_s \hat{\mathcal{L}}^{(s)} = \hat{\mathcal{L}}^{(s+1)} \hat{\delta}_s , \quad \hat{\delta}_{s+1} \hat{\mathcal{L}}^{(s+1)} = \hat{\mathcal{L}}^{(s)} \hat{\delta}_{s+1} . \]

With the choice of phases in (2.15), one has

\[ \hat{\delta}_s Y_{j,m}^s = -\sqrt{(j-s)(j+s+1)} \ Y_{j,m}^{s+1} , \quad \hat{\delta}_{s+1} Y_{j,m}^{s+1} = +\sqrt{(j-s)(j+s+1)} \ Y_{j,m}^s . \]

On Pensov spinors of \( \mathcal{H}_{(s)} \), one defines the "total angular momentum" as:

\[ \hat{\mathcal{L}}_{tot}^{(s)} \left( \begin{array}{c} \sigma^{(s-1/2)} \\ \sigma^{(s+1/2)} \end{array} \right) = \left( \begin{array}{cc} \hat{\mathcal{L}}^{(s-1/2)} & 0 \\ 0 & \hat{\mathcal{L}}^{(s+1/2)} \end{array} \right) \left( \begin{array}{c} \sigma^{(s-1/2)} \\ \sigma^{(s+1/2)} \end{array} \right) . \]

It commutes with the Dirac operator:

\[ \mathcal{D}_{(s)} \hat{\mathcal{L}}_{tot}^{(s)} = \hat{\mathcal{L}}_{tot}^{(s)} \mathcal{D}_{(s)} . \]

From the product of edth operators

\[ -\hat{\delta}_{s+1/2} \hat{\delta}_{s-1/2} = (\hat{\mathcal{L}}^{(s-1/2)})^2 - (s^2 - 1/4) , \]

\[ -\hat{\delta}_{s-1/2} \hat{\delta}_{s+1/2} = (\hat{\mathcal{L}}^{(s+1/2)})^2 - (s^2 - 1/4) , \]

This is quite expected since the edth operators are constructed from the Levi-Civita connection which is metric compatible and the Killing vector fields conserve this metric.
a Lichnerowicz type formula follows immediately:

\[ D(s)^2 = (\mathcal{L}_{tot}^s)^2 - (s^2 - 1/4) \mathbf{1}. \]

Using (2.14), the eigenvalues of the Dirac operator are found to be:

\[ D(s)\psi^{(\pm)}_{(s),j,m} = \pm \sqrt{(j + 1/2)^2 - s^2} \psi^{(\pm)}_{(s),j,m}, \]

with

\[ \psi^{(+)}_{(s),j,m} = \begin{pmatrix} \frac{1}{\sqrt{2}} Y_{j,m}^{s-1/2} \\ \frac{1}{\sqrt{2}} Y_{j,m}^{s+1/2} \end{pmatrix}, \quad \psi^{(-)}_{(s),j,m} = \gamma_3 \psi^{(+)}_{(s),j,m} = \begin{pmatrix} \frac{1}{\sqrt{2}} Y_{j,m}^{s-1/2} \\ -\frac{1}{\sqrt{2}} Y_{j,m}^{s+1/2} \end{pmatrix}. \]

In particular it follows that, for Dirac spinors i.e. \( s = 0 \), and only in this case, there are no zero eigenvalues. When \( s \neq 0 \), zero is an eigenvalue, \( 2|s| \) times degenerate with eigenspinors

\[
\begin{cases}
\left( Y_{s-1/2,m} \right) & \text{for positive values of } s, \\
0 & \\
\left( Y_{s+1/2} \right) & \text{if } s \text{ is negative.}
\end{cases}
\]


3 The projective modules over $\mathcal{A}$

Through the Gel’fand-Na˘ımark construction, the topology of $M = S^2 \times \{a, b\}$ is encoded in the complex $C^*$-algebra of continuous complex-valued functions on $M = S^2 \times \{a, b\}$. However, in order to get a fruitful use of a differential structure, we have to restrict this $C^*$-algebra to its dense subalgebra of smooth functions. This proviso made, let $\{f, g, \cdots\}$ denote elements of $\mathcal{A} = C(M)$ and let the value of $f$ at a point $p = \{x, \alpha\} \in M$, be written as $f(p) = f_\alpha(x)$. The vectors of the free right $\mathcal{A}$-module of rank two, identified with $\mathcal{A}^2$, are of the form $X = \sum_{i=1}^2 E_i f^i$, where $f^i \in \mathcal{A}$ and $\{E_i ; i = 1, 2\}$ is a basis of $\mathcal{A}^2$. Let $\Omega^*(\mathcal{A}) = \sum_{k=0}^\infty \Omega^{(k)}(\mathcal{A})$ denote the universal differential envelope of $\mathcal{A}$. Elements of $\Omega^{(k)}(\mathcal{A})$ can be realised, see e.g. [6], as functions on the Cartesian product of $(k + 1)$ copies of $M$, vanishing on neighbouring diagonals, i.e. $F(p_0, p_1, \cdots, p_k) = 0$ if, for some $i$, $p_i = p_{i+1}$.

The product in $\Omega^*(\mathcal{A})$ is obtained by concatenation, e.g. if $F \in \Omega^{(1)}(\mathcal{A})$ and $G \in \Omega^{(2)}(\mathcal{A})$ then their product $F \cdot G \in \Omega^{(3)}(\mathcal{A})$ is represented by

$$(F \cdot G)(p_0, p_1, p_2, p_3) = F(p_0, p_1) G(p_1, p_2, p_3).$$

The differential $d$ acts on $f \in \Omega^{(0)}(\mathcal{A})$ and on $F \in \Omega^{(1)}(\mathcal{A})$ as follows:

$$(df)(p_0, p_1) = f(p_1) - f(p_0),$$

$$(dF)(p_0, p_1, p_2) = F(p_1, p_2) - F(p_0, p_2) + F(p_0, p_1).$$

The involution $\dagger$, defined in $\mathcal{A}$ by $(f^\dagger)(p) = (f(p))^\star$, extends to $\Omega^{(k)}(\mathcal{A})$ as

$$(F^\dagger)(p_1, p_2, \cdots) = \left(F(\cdots, p_2, p_1)\right)^\star.$$ 

A (universal) connection on $\mathcal{A}^2$ is given, in the basis $\{E_i ; i = 1, 2\}$, by an $\Omega^{(1)}(\mathcal{A})$-valued $2 \times 2$ matrix $(\omega)^i_k$. It acts on $X = E_i f^i$ as:

$$\nabla_{free}(X) = E_i \otimes_{\mathcal{A}} \left(df^i + ((\omega))^i_k f^k\right). \tag{3.1}$$

\(^9\)In this section points of $S^2$ are denoted by $x, y, \cdots$, while $\alpha, \beta, \cdots$ will assume values in the two-point space $\{a, b\}$. Points of $M = S^2 \times \{a, b\}$ are thus written as $p = \{x, \alpha\}, q = \{y, \beta\}$, etc. and the value of a function $F$ at $(p, q, \cdots)$ will also be expressed as $F_{\alpha, \beta, \cdots}(x, y, \cdots)$.

\(^{10}\)Note that $d(f^\dagger) = -(df)^\dagger, f \in \mathcal{A}$ and, more generally, if $F \in \Omega^{(k)}(\mathcal{A})$, then $d(F^\dagger) = (-1)^{k+1}(dF)^\dagger$. 

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Let $X = E_i f^i$ and $Y = E_i g^i$ be two vectors of $A^2$, then the standard hermitian product with values in $A$ is given by:

$$h(X, Y) = \sum_{i,j} (f^i)\dagger \delta_{ij} g^j = \sum_i (f^i)\dagger g^i$$

It extends as $\Omega^*(A)$-valued on $\left(A^2 \otimes_A \Omega^*(A)\right) \times \left(A^2 \otimes_A \Omega^*(A)\right)$ as

$$h(X \otimes F, Y \otimes G) = F\dagger h(X, Y) G.$$ 

The connection is hermitian if $\mathbf{d}\left(h(X, Y)\right) = h(X, \nabla_{\text{free}} Y) - h(\nabla_{\text{free}} X, Y)$. For the product above, this yields

$$((\omega))^{i,j} = \delta^{i,k} ((\omega))^{k,\ell} \delta_{\ell,j}.$$

The representation of $\omega$ by functions on $M \times M$ is given by:

$$((\omega)) \Rightarrow \begin{pmatrix} K_{\alpha\beta}(x,y) & S_{\alpha\beta}(x,y) \\ T_{\alpha\beta}(x,y) & L_{\alpha\beta}(x,y) \end{pmatrix}$$

and the hermiticity condition reads:

$$K_{\alpha\beta}(x,y) = \left(K_{\beta\alpha}(y,x)\right)^*,$$
$$L_{\alpha\beta}(x,y) = \left(L_{\beta\alpha}(y,x)\right)^*,$$
$$T_{\alpha\beta}(x,y) = \left(S_{\beta\alpha}(y,x)\right)^*.$$ (3.2)

The action of the connection (3.1) is represented by:

$$\left(\nabla_{\text{free}} X\right)^{i,\alpha}_{\beta}(x,y) = f^i_{\beta}(y) - f^i_{\alpha}(x) + ((\omega))^{i,k}_{,\alpha\beta}(x,y) f^k_{\beta}(y).$$

A projective module is defined by an endomorphism $\mathbf{P}$ of $A^2$ which is idempotent, $\mathbf{P}^2 = \mathbf{P}$, and hermitian, $\mathbf{P}^\dagger = \mathbf{P}$, where the adjoint $A^\dagger$ of an endomorphism $A$ is defined by $h(X, A^\dagger Y) = h(A X, Y)$. In the basis $\{E_i\}$, the projector is given by a $2 \times 2$ matrix $(P)^{i,j}_{\alpha\beta}$ with entries in $A$ and is represented by $(P)^{i,j}_{\alpha\beta}(x)$. The projective module $\mathcal{M}$ is defined as the image of $\mathbf{P}$:

$$\mathcal{M} = \left\{ \mathbf{P} X \mid X \in A^2 \right\} = \left\{ X \in A^2 \mid \mathbf{P} X = X \right\}.$$ 

The hermiticity of the projector guarantees that $\mathbf{h}$, restricted to $\mathcal{M}$, defines a hermitian product in $\mathcal{M}$.

In the Connes-Lott model [4], the projectors are of the form

$$((P))^{i,j}_{\alpha\beta}(x) = \delta^i_j$$

and

$$((P))^{i,j}_{\alpha\beta}(x) = \frac{1}{2} \left(1 + \vec{n}(x) \sigma^i\sigma^j\right).$$
where \( \vec{\sigma} \) are the Pauli matrices and \( \vec{n}(x) \) is a real unit vector, mapping \( S^2 \mapsto S^2 \) so that the projectors are classified by \( \pi_2(S^2) = \mathbb{Z} \). Furthermore, since \( \pi_2(U(2)) = \pi_2(SU(2)) = \pi_2(S^3) = \{1\} \), projectors, belonging to different homotopy classes, cannot be unitarily equivalent.

The target sphere \( S^2 \) also has two coordinate charts \( H^\text{target}_B \) and \( H^\text{target}_A \). In these charts, the projector \( \langle P_b \rangle \) can be written as \( \langle P^B_B \rangle = |\nu_B\rangle \langle \nu_B| \), respectively \( \langle P^A_b \rangle = |\nu_A\rangle \langle \nu_A| \), where \( \nu_B \), respectively \( \nu_A \), is the complex coordinate of \( \vec{n} \) in \( H^\text{target}_B \), respectively \( H^\text{target}_A \). We have used the Dirac ket-and bra-notation:

\[
|\nu_B\rangle = \frac{1}{\sqrt{1 + |\nu_B|^2}} \begin{pmatrix} 1 \\ \nu_B \end{pmatrix}, \quad \langle \nu_B| = \frac{1}{\sqrt{1 + |\nu_B|^2}} \begin{pmatrix} 1 \\ \nu_B^* \end{pmatrix},
\]

\[
|\nu_A\rangle = \frac{1}{\sqrt{1 + |\nu_A|^2}} \begin{pmatrix} -\nu_A \\ 1 \end{pmatrix}, \quad \langle \nu_A| = \frac{1}{\sqrt{1 + |\nu_A|^2}} \begin{pmatrix} -\nu_A^* \\ 1 \end{pmatrix}.
\]

An element \( X \) of \( A^2 \) is represented by the column matrix \( X \Rightarrow \begin{pmatrix} |f_a(x)\rangle \\ |f_b(x)\rangle \end{pmatrix} \),

where \( |f_a(x)\rangle = \begin{pmatrix} f^1_a(x) \\ f^2_a(x) \end{pmatrix} \) and \( |f_b(x)\rangle = \begin{pmatrix} f^1_b(x) \\ f^2_b(x) \end{pmatrix} \). It belongs to \( \mathcal{M} \) if \( PX = X \), which yields no restriction on \( |f_a(x)\rangle \) but there is one on \( |f_b(x)\rangle \).

In \( H^\text{target}_B \) it is expressed as \( |f_b\rangle = |\nu_B\rangle f^B_b \), where

\[
f^B_b = \langle \nu_B| f_b \rangle = \frac{1}{\sqrt{1 + |\nu_B|^2}} (f^1_b + \nu_B^* f^2_b).
\]

In the same way, in \( H^\text{target}_A \) one has \( |f_b\rangle = |\nu_A\rangle f^A_b \), where

\[
f^A_b = \langle \nu_A| f_b \rangle = \frac{1}{\sqrt{1 + |\nu_A|^2}} (-\nu_A^* f^1_b + f^2_b).
\]

As representatives of the homotopy class \( [n] \in \pi_2(S^2) \equiv \mathbb{Z} \), we choose a mapping \( \vec{n} \) transforming \( H_B \), respectively \( H_A \) of the range \( S^2 \), into \( H^\text{target}_B \), respectively \( H^\text{target}_A \) of the target \( S^2 \). Such a choice is

\[
\nu_B(x) = \left( \frac{\zeta_B}{\zeta_B^2} \right)^{n-1} \zeta_B, \quad \nu_A(x) = \left( \frac{\zeta_A}{\zeta_A^2} \right)^{n-1} \zeta_A \quad n \in \mathbb{Z},
\]

\[\text{(3.3)}\]

\[\text{Note that this choice is different from the one in previous work}^{[14]}\.]
where $\zeta_B$, $\zeta_A$ are the complex coordinates of $x \in S^2$.

In the overlap $H_A \cap H_B$, with transition function $c_{AB}$ given by (2.3), we have:

$$f_b^A(x) = (c_{AB}(x))^{n/2} f_b^B(x)$$

and this tells us that $f_b(x)$ is a Pensov scalar of weight $n/2$.

In the rest of this paper we shall omit the $A$ and $B$ labels except when relating quantities in $H_A$ with those in $H_B$ in the overlap $H_A \cap H_B$.

An element of $\mathcal{M}$ is thus represented by:

$$X \Rightarrow (|f_a\rangle |\nu\rangle) f_b^*$$

and its scalar product with $Y \Rightarrow (|g_a\rangle |\nu\rangle) g_b$ is given by:

$$(h_P(X,Y))_a = (f^*_1(x)) g_a^1(x) + (f^*_2(x)) g_a^2(x),$$

$$(h_P(X,Y))_b = (f_b(x))^* g_b(x).$$

An active gauge transformation in the free module $A^2$ is given by a unitary $2 \times 2$ matrix $U$ with values in $A$. It restricts to a gauge transformation in $M$ when it commutes with $P$: $PU = UP$.

An element $X \in \mathcal{M}$ transforms as $X \Rightarrow UX$ given by:

$$|\left(UX\right)_a(x)\rangle = (|U_a(x)\rangle |f_a\rangle),$$

$$|\left(UX\right)_b(x)\rangle = |\nu(x)\rangle u_b(x) f_b(x),$$

(3.4)

where $(|U_a(x)\rangle) \in U(2)$ and $u_b(x) \in U(1)$.

The connection in the free module (3.1) induces a connection in the projective module $\mathcal{M}$ given by $\nabla X = P \nabla_{\text{free}} X$ where $X \in \mathcal{M}$.

It is represented by

$$|(\nabla X)_{\alpha\beta}(x,y)\rangle = ((P_\alpha(x))(f_\beta(y)) - |f_\alpha(x)\rangle + (A_{\alpha\beta}(x,y)) |f_\beta(y)\rangle.$$  (3.5)

The $2 \times 2$ matrices $(A_{\alpha\beta}(x,y))$ are given by:

$$((A_{aa}(x,y))) = (\omega_{aa}(x,y)),$$

$$((A_{ab}(x,y))) = |\Phi_{ab}(x,y)\rangle \langle \nu(y)|,$$

$$((A_{ba}(x,y))) = |\nu(x)\rangle \langle \Phi_{ba}(x,y)|,$$

$$((A_{bb}(x,y))) = |\nu(x)\rangle \omega_b(x,y) |\nu(y)\rangle.$$  (3.6)
where we have introduced the $\Omega^{(1)}(\mathcal{A})$-valued ket- and bra- vectors:

$$
|\Phi_{ab}(x, y)\rangle = \langle \omega_{ab}(x, y) \rangle |\nu(y)\rangle,
$$

and the universal one-form:

$$
\omega_{b}(x, y) = \langle \nu(x) |\omega_{bb}(x, y)\rangle |\nu(y)\rangle.
$$

The hermiticity condition (3.2) yields:

$$
\langle \omega_{aa}(x, y) \rangle^+ = \langle \omega_{aa}(y, x) \rangle,
$$

$$
\omega_{b}(x, y)^\ast = \omega_{b}(y, x),
$$

and the universal one-form:

$$
\omega_{b}(x, y) = \langle \nu(x) |\omega_{bb}(x, y)\rangle |\nu(y)\rangle.
$$

The hermiticity condition (3.2) yields:

$$
(\omega_{aa}(x, y))^+ = (\omega_{aa}(y, x))^\ast,\quad \omega_{b}(x, y)^\ast = \omega_{b}(y, x),
$$

and the universal one-form:

$$
\omega_{b}(x, y) = \langle \nu(x) |\omega_{bb}(x, y)\rangle |\nu(y)\rangle.
$$

In $H_B \cap H_A$:

$$
|\Phi_{ab}^A(x, y)\rangle = |\Phi_{ab}^B(x, y)\rangle \left( c_{AB}(y) \right)^{-n/2},
$$

$$
\langle \Phi_{ba}^A(x, y) \rangle = \left( c_{AB}(x) \right)^{n/2} \langle \Phi_{ba}^B(x, y) \rangle,
$$

$$
\omega_{b}^A(x, y) = \left( c_{AB}(x) \right)^{n/2} \omega_{b}^B(x, y) \left( c_{AB}(y) \right)^{-n/2}.
$$

The action of $\nabla$ on $X \in \mathcal{M}$ is obtained using (3.6), (3.7) and (3.8):

$$
|\nabla X_{ab}(x, y)\rangle = |f_a(y)\rangle - |f_a(x)\rangle + ((\omega_{aa}(x, y)) |f_a(y)\rangle,
$$

$$
|\nabla X_{ab}(x, y)\rangle = |H_{ab}(x, y)\rangle f_b(y) - |f_a(x)\rangle,
$$

$$
|\nabla X_{ba}(x, y)\rangle = |\nu(x)\rangle \left[ \langle H_{ba}(x, y) |f_a(y)\rangle - f_b(x) \right],
$$

$$
|\nabla X_{bb}(x, y)\rangle = |\nu(x)\rangle \left[ f_b(y) - f_b(x) + (\omega_b(x, y) + m_b(x, y)) f_b(y) \right],
$$

where

$$
|H_{ab}(x, y)\rangle = |\Phi_{ab}(x, y)\rangle + |\nu(y)\rangle,
$$

$$
\langle H_{ba}(x, y) \rangle = \langle \Phi_{ba}(x, y) \rangle + \langle \nu(x) \rangle.
$$

and the ”monopole” connection $m_b(x, y)$ appears as:

$$
m_b(x, y) = \langle \nu(x) |\nu(y)\rangle - 1.
$$
As seen from (3.10) and (3.12), the off-diagonal connections $|H_{ab}(x, y)\rangle$, $\langle H_{ba}(x, y)\rangle$ and also $\omega_b(x, y)$ transform homogeneously from $H_B$ to $H_A$ but $m_b(x, y)$ transforms with the expected inhomogeneous term:

$$m_b^A(x, y) = (c_{AB}(x))^{n/2} m_b^B(x, y)(c_{AB}(y))^{-n/2} + (c_{AB}(x))^{n/2}[(c_{AB}(y))^{-n/2} - (c_{AB}(x))^{-n/2}].$$

In terms of abstract universal differential one forms, (3.13) and (3.14) read:

$$m_b = \frac{1}{\sqrt{1 + |\nu|^2}}(\nu^* d\nu - \sqrt{1 + |\nu|^2} d\sqrt{1 + |\nu|^2}),$$

$$m_b^A = (c_{AB})^{n/2} m_b^B (c_{AB})^{-n/2} + (c_{AB})^{n/2} d(c_{AB})^{-n/2}.$$  

The curvature of the connection is defined by:

$$\langle \nabla^2 X \rangle = (\langle R \rangle) \langle X \rangle.$$  

It is a right-module homomorphism $\mathcal{M} \to \mathcal{M} \otimes \Omega(2)(\mathcal{A})$ given in the basis $\{E_i\}$ by the $2 \times 2$ matrix with values in $\Omega(2)(\mathcal{A})$:

$$\langle (R) \rangle = \langle (P) \rangle \left( d\langle (A) \rangle \right) \langle (P) \rangle + \langle (A) \rangle^2 + \langle (P) \rangle \left( d\langle (P) \rangle \right) \left( d\langle (P) \rangle \right) \langle (P) \rangle,$$

or, within the used realisation, by

$$\langle (R) \rangle = \langle (P) \rangle \langle (A) \rangle \langle (P) \rangle + \langle (A) \rangle^2 + \langle (P) \rangle \left( d\langle (P) \rangle \right) \left( d\langle (P) \rangle \right) \langle (P) \rangle.$$  

A connection $\nabla$ compatible with the hermitian structure in $\mathcal{M}$ implies in a self-adjoint curvature:

$$R^i_j = \delta^{i\ell}(R^k_\ell)^{+} \delta_{kj}.$$  

Let the connection $\nabla$ be extended to $\mathcal{M} \otimes \Omega^\bullet(\mathcal{A})$ by

$$\nabla(X \otimes_A F) = (\nabla X) F + X \otimes_A dF,$$

then $\nabla^2$ becomes an endomorphism of the right $\Omega^\bullet(\mathcal{A})$-module $\mathcal{M} \otimes \Omega(\mathcal{A})$. The active gauge transformation (3.4) acts on the right on the space of
connections as $\nabla \mapsto \nabla^U = U^{-1} \circ \nabla \circ U$. The action of $\nabla^U$ on $X$ is given by a similar expression as in (3.5) with the matrices $(A)$ replaced by $(A^U) = (U)^{-1}(A)(U) + (P)(U^{-1})(d(U))(P)$ and (3.11) becomes

$$\langle (\nabla^U X)_{aa}(x,y) \rangle = \langle f_a(y) - f_a(x) + (\omega_{aa}^U(x,y)) | f_a(y) \rangle,$$

$$\langle (\nabla^U X)_{ab}(x,y) \rangle = \langle H_{ab}^U(x,y) f_b(y) - f_a(x) \rangle,$$

$$\langle (\nabla^U X)_{ba}(x,y) \rangle = \langle \nu(x) \left[ (H_{ba}^U(x,y) | f_a(y) - f_b(x) \right] \rangle,$$

$$\langle (\nabla^U X)_{bb}(x,y) \rangle = \langle \nu(x) \left[ f_b(y) - f_b(x) + (\omega_{bb}^U(x,y) + m_{bb}^U(x,y)) f_b(y) \right] \rangle.$$

with

$$\langle (\omega_{aa}^U(x,y)) \rangle = \langle (U_a(x))^{-1}(\omega_{aa}(x,y))((U_a(y))$$

$$+ \langle (U_a(x))^{-1}(\omega_{aa}(x,y)) - (U_a(x)) \rangle,$$

$$\langle H_{ab}^U(x,y) \rangle = \langle (U_a(x))^{-1} H_{ab}(x,y) \rangle u_b(y),$$

$$\langle H_{ba}^U(x,y) \rangle = \langle (U_b(x))^{-1} H_{ba}(x,y) \rangle ((U_a(y)),$$

$$m_{bb}^U(x,y) = \langle (U_b(x))^{-1} m_{bb}(x,y) u_b(y)$$

$$\omega_{bb}^U(x,y) = \langle (U_b(x))^{-1} \omega_{bb}(x,y) u_b(y)$$

$$+ (U_b(x))^{-1} (u_b(y) - u_b(x)).$$

(3.17)

It is thus seen that $\langle H_{ab}(x,y) \rangle, \langle H_{ba}(x,y) \rangle$ and the monopole connection (3.13) $m_{bb}(x,y)$ transform homogeneously under an active gauge transformation, while $\langle (\omega_{aa}(x,y)) \rangle$ and $\omega_b(x,y)$ have the expected inhomogeneous terms $U^{-1}du, u^{-1}du$. 

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4 The spectral triple \( \{ \mathcal{A}, \mathcal{H}, D, \Gamma \} \)

An (even) spectral triple, \( \{ \mathcal{A}, \mathcal{H}, D, \chi \} \), as defined by Connes [1], is given by a \( C^\ast \) algebra \( \mathcal{A} \) and a Hilbert space \( \mathcal{H} \), graded by \( \chi \), with \( \pi : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H}) : f \mapsto \pi(f) \), a faithful, \( \text{Ker}(\pi) = 0 \), and even, \( [\chi, \pi(f)] = 0 \), \( \ast \)-representation, \( \pi(f^*) = (\pi(f))^\ast \), of \( \mathcal{A} \) acts as bounded operators.

Furthermore, on \( \mathcal{H} \), there is a self-adjoint Dirac operator \( D \), which is odd, \( D\chi + \chi D = 0 \), and such that \( (D - \lambda)^{-1} \) is compact for \( \lambda \notin \mathbb{R} \). It should be a first order operator in the sense that \( [D, \pi(f)] , [D, \pi(g)] = 0 \), \( f,g \in \mathcal{A} \).

Here the algebra is \( \mathcal{A} = C(S^2 \times \{ a,b \}; \mathbb{C}) \) and as Hilbert space we take
\[
\mathcal{H} = \mathcal{H}_{(s)} \otimes \mathcal{H}_{\text{dis}},
\]
the tensor product of the Hilbert space \( \mathcal{H}_{(s)} \) of Pensov spinors with a finite Hilbert space \( \mathcal{H}_{\text{dis}} = (\mathbb{C}^{N_a} \oplus \mathbb{C}^{N_b}) \) where \( N_a, N_b \) are natural numbers giving the number of generations in each chirality sector.

\( \mathcal{H}_{\text{dis}} \) is endowed with a grading operator:
\[
\chi_{\text{dis}} = \begin{pmatrix}
1_a & 0 \\
0 & -1_b
\end{pmatrix},
\tag{4.1}
\]
where \( 1_\alpha , \alpha = a,b \) is the \( N_\alpha \times N_\alpha \) unit matrix. On \( \mathcal{H}_{\text{dis}} \) there a finite Dirac operator \( D_{\text{dis}} \) represented by a hermitian matrix odd with respect to \( \chi_{\text{dis}} \) :
\[
D_{\text{dis}} = \begin{pmatrix}
0 & M^+ \\
M & 0
\end{pmatrix},
\tag{4.2}
\]
where \( M \) is a \( N_b \times N_a \) matrix describing the phenomenology of the masses.

The total grading in \( \mathcal{H} \) is given by
\[
\chi = \gamma_3 \otimes \chi_{\text{dis}} = \begin{pmatrix}
\gamma_3 \otimes 1_a & 0 \\
0 & -\gamma_3 \otimes 1_b
\end{pmatrix},
\tag{4.3}
\]

The Dirac operator in \( \mathcal{H} \) is obtained from (2.10) and (4.2) as :
\[
D = D_{(s)} \otimes 1_{a+b} + \gamma_3 \otimes D_{\text{dis}}.
\tag{4.4}
\]

It is odd with respect to the total grading \( \chi \). The grading operator \( \gamma_3 \) has been introduced in (1.4) so that the square of the total Dirac operator reads
\[
D^2 = D_{(s)}^2 \otimes 1_{a+b} + 1 \otimes D_{\text{dis}}^2.
\]
An element $\Psi$ of $\mathcal{H}$ is represented as
\[
\begin{pmatrix}
\psi(s),a(x) \\
\psi(s),b(x)
\end{pmatrix},
\] (4.5)

where each $\psi(s),\alpha(x)$, ($\alpha = a, b$), is a Pensov spinor with $N_\alpha$ "generation" indices. The scalar product is the obvious extension of (2.11):
\[
(\Psi; \Phi) = \int_{S^2} \left[ (\psi_a)^+ \phi_a + (\psi_b)^+ \phi_b \right] \omega.
\] (4.6)

An element $f \in A$ is represented as:
\[
\pi(f) \begin{pmatrix}
\psi_a(x) \\
\psi_b(x)
\end{pmatrix} = \begin{pmatrix}
f_a(x) \mathbf{1}_c \otimes \mathbf{1}_a & 0 \\
0 & f_b(x) \mathbf{1}_c \otimes \mathbf{1}_b
\end{pmatrix} \begin{pmatrix}
\psi_a(x) \\
\psi_b(x)
\end{pmatrix},
\] (4.7)

where $\mathbf{1}_c$ is the $2 \times 2$ unit matrix in the Clifford algebra. Acting on vectors of the form (4.5), (4.4) becomes
\[
\mathcal{D} = \begin{pmatrix}
\mathcal{D}(s) \otimes \mathbf{1}_a & \gamma_3 \otimes M^+ \\
\gamma_3 \otimes M & \mathcal{D}(s) \otimes \mathbf{1}_b
\end{pmatrix}.
\]

Its commutator with $\pi(f)$ is:
\[
[D, \pi(f)] = \begin{pmatrix}
-i \mathbf{c}(d f_a) \otimes \mathbf{1}_a & (f_b - f_a) \gamma_3 \otimes M^+ \\
(f_a - f_b) \gamma_3 \otimes M & -i \mathbf{c}(d f_b) \otimes \mathbf{1}_b
\end{pmatrix},
\] (4.8)

where the de Rham exterior differential is $d f_a = \left( \tilde{e}_k(f_a) \right) \theta^k$ and where $\mathbf{c}(\sigma^{(k)})$ denotes the Clifford representation of the k-form $\sigma^{(k)}$:
\[
\mathbf{c}(\sigma_{i_1 \ldots i_k} \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}) = \sigma_{i_1 \ldots i_k} \gamma^{i_1} \ldots \gamma^{i_k}.
\]

The representation $\pi$ of (4.7) extends to a $\star$-representation of $\Omega^\bullet(A)$ by:
\[
\pi(f_0 d f_1 \cdots d f_k) = \pi(f_0) [D, \pi(f_1)] \cdots [D, \pi(f_k)].
\]

From (4.7) and (4.8) it follows that the element $f d g \in \Omega^{(1)}(A)$ is represented by
\[
\pi(f d g) = \begin{pmatrix}
f_a - i \mathbf{c}(d g_a) \otimes \mathbf{1}_a & f_a(g_b - g_a) \gamma_3 \otimes M^+ \\
(\overline{f}_b(g_a - g_b) \gamma_3 \otimes M & f_b - i \mathbf{c}(d g_b) \otimes \mathbf{1}_b
\end{pmatrix},
\] (4.9)

\[\text{These "generation" indices are not written down explicitely and the \(s\) subscript, fixed once for all, will also be omitted in this section.}\]
A general element $F \in \Omega^{(1)}(\mathcal{E})$, given by $F_{\alpha\beta}(x,y)$, is then represented as an operator on $\mathcal{H}$ by:

$$
\pi(F) = \begin{pmatrix}
-i \mathbf{c}(\sigma^1_a) \otimes \mathbf{1}_a & \sigma^0_{ab} \gamma_3 \otimes M^+
\end{pmatrix},
$$

where the $\sigma^{(k)}$'s are differential k-forms given by:

$$
\sigma^1_a(x) = \left(\tilde{e}_{k,y} F_{aa}(x,y)\right)_{y=x} \theta_x^k, \quad \sigma^1_b(x) = \left(\tilde{e}_{k,y} F_{bb}(x,y)\right)_{y=x} \theta_x^k.
$$

The representative of a universal 2-form $\mathbf{fdfgh}$ will be given by the product of the matrix (4.9) with

$$
\begin{pmatrix}
-i \mathbf{c}(dh_a) \otimes \mathbf{1}_a & (h_b - h_a) \gamma_3 \otimes M^+
(h_a - h_b) \gamma_3 \otimes M & -i \mathbf{c}(dh_b) \otimes \mathbf{1}_b
\end{pmatrix}.
$$

The result is:

$$
\pi(\mathbf{fdfgh}) = \begin{pmatrix}
\pi(\mathbf{fdfgh})_{[aa]} & \pi(\mathbf{fdfgh})_{[ab]}
\pi(\mathbf{fdfgh})_{[ba]} & \pi(\mathbf{fdfgh})_{[bb]}
\end{pmatrix},
$$

where

$$
\pi(\mathbf{fdfgh})_{[aa]} = -f_a \mathbf{c}(dg_a) \mathbf{c}(dh_a) \otimes \mathbf{1}_a + f_a(g_b - g_a)(h_a - h_b) \otimes M^+ M,
$$

$$
\pi(\mathbf{fdfgh})_{[ab]} = -i \left(f_a \mathbf{c}(dg_a)(h_b - h_a) - f_a(g_b - g_a) \mathbf{c}(dh_b)\right) \gamma_3 \otimes M^+,
$$

$$
\pi(\mathbf{fdfgh})_{[ba]} = -i \left(-f_b(g_a - g_b) \mathbf{c}(dh_a) + f_b \mathbf{c}(dg_b)(h_a - h_b)\right) \gamma_3 \otimes M,
$$

$$
\pi(\mathbf{fdfgh})_{[bb]} = -f_b \mathbf{c}(dg_b) \mathbf{c}(dh_b) \otimes \mathbf{1}_b + f_b(g_a - g_b)(h_b - h_a) \otimes MM^+.
$$

A generic universal two-form $G$ is represented by

$$
\pi(G) = \begin{pmatrix}
\pi(G)_{[aa]} & \pi(G)_{[ab]}
\pi(G)_{[ba]} & \pi(G)_{[bb]}
\end{pmatrix},
$$

(4.12)
with

\[ \pi(G)_{aa} = -c(\rho_{aaa}^{(2+0)}) \otimes 1_a + \rho_{aba}^{(0)} \otimes M^+ M, \]
\[ \pi(G)_{ab} = -i \, c(\rho_{ab}^{(1)}) \gamma_3 \otimes M^+, \]
\[ \pi(G)_{ba} = -i \, c(\rho_{ba}^{(1)}) \gamma_3 \otimes M, \]
\[ \pi(G)_{bb} = -c(\rho_{bbb}^{(2+0)}) \otimes 1_b + \rho_{bab}^{(0)} \otimes MM^+, \]

where the differential forms \( \rho^{(k)}(x) \) are given by:

\[ \rho_{aaa}^{(2+0)}(x) = \rho_{aaa}^{(2)} + \rho_{aaa}^{(0)}, \]
\[ \rho_{bbb}^{(2+0)}(x) = \rho_{bbb}^{(2)} + \rho_{bbb}^{(0)}, \]
\[ \rho_{aaa}^{(2)}(x) = \left( \frac{1}{2} [\vec{e}_{k,y} \, \vec{e}_{\ell,z} - \vec{e}_{\ell,y} \, \vec{e}_{k,z}] G_{aaa}(x,y,z) \right)_{|y=x,z=x} \theta^k \wedge \theta^\ell, \]
\[ \rho_{bbb}^{(2)}(x) = \left( \frac{1}{2} [\vec{e}_{k,y} \, \vec{e}_{\ell,z} - \vec{e}_{\ell,y} \, \vec{e}_{k,z}] G_{bbb}(x,y,z) \right)_{|y=x,z=x} \theta^k \wedge \theta^\ell, \]
\[ \rho_{aaa}^{(0)}(x) = \left( G_{aba}(x,y,z) \right)_{|y=x,z=x}, \]
\[ \rho_{bbb}^{(0)}(x) = \left( G_{bab}(x,y,z) \right)_{|y=x,z=x}, \]
\[ \rho_{ab}^{(1)}(x) = \left( \vec{e}_{k,y} G_{aab}(x,y,z) - \vec{e}_{k,z} G_{abb}(x,y,z) \right)_{|y=x,z=x} \theta^k, \]
\[ \rho_{ba}^{(1)}(x) = \left( \vec{e}_{k,y} G_{bba}(x,y,z) - \vec{e}_{k,z} G_{bab}(x,y,z) \right)_{|y=x,z=x} \theta^k. \]

The representation \( \pi \) of \( \Omega^*(A) \) is a star-representation but it is not a differential representation of \( \Omega^*(A) \) in the sense that \( G \in Ker(\pi) = \mathcal{J}_0 \) does not imply \( \pi(\mathbf{d}G) = 0 \). To obtain a graded differential algebra of operators in \( \mathcal{H} \), it is necessary to take the quotient of \( \Omega^*(A) \) by the graded differential ideal \( \mathcal{J} = \mathcal{J}_0 + \mathbf{d} \mathcal{J}_0 \) with canonical projection

\[ \pi_D : \Omega^*(A) \mapsto \Omega^*_D(A) = \frac{\Omega^*(A)}{\mathcal{J}} = \bigoplus_{k=0}^{\infty} \Omega^*_D(k)(A). \]

\(^{13}\)Here we have used \( c(\theta^k)c(\theta^\ell) = c(\theta^k \wedge \theta^\ell) + \delta^{k\ell}. \)
Using the homomorphism theorem for algebras, it is easily seen that
\[
\Omega_D^{(k)}(A) = \frac{\pi(\Omega(k)(A))}{\pi(dJ_0^{(k-1)})}.
\]
Since \( \pi \) is a faithful representation of \( A \), its kernel \( J_0^{(0)} \) has to be zero and \( \Omega_D^{(1)}(A) \cong \pi(\Omega^{(1)}(A)) \). So \( F \in \Omega^{(1)}(A) \) has in \( \Omega_D^{(1)}(A) \) the same representative as given by (4.10).

To compute \( \Omega_D^{(2)}(A) \), we need \( dJ_0^{(1)} \). According to (4.10), \( J_0^{(1)} \) is given by:
\[
\left\{ F \in \Omega^{(1)}(A) \parallel F_{ab}(x,x) = 0 = F_{ba}(x,x); \bar{e}_{k,y} F_{\alpha\alpha}(x,y)_{y=x} = 0, \alpha = a, b \right\}
\]
so that (4.12) with \( G = dF \) and \( F \in J_0^{(1)} \), yield a \( \pi(dF) \) of the form
\[
\begin{pmatrix}
  j_a^{(0)} \otimes 1_a & 0 \\
  0 & j_b^{(0)} \otimes 1_b
\end{pmatrix},
\]
where \( j_a^{(0)}(x) = -\delta^{k\ell}(\bar{e}_{k,y} \bar{e}_{\ell,z} F_{\alpha\alpha}(y,z))_{y=x,z=x} \), with \( \alpha = a, b \).

In the space \( \pi(\Omega^{(k)}(A)) \), whose elements are bounded operators in \( \mathcal{H} \), the scalar product of \( \pi(G_1) \) and \( \pi(G_2) \) is defined by
\[
\langle \pi(G_1); \pi(G_2) \rangle_k = \text{Tr}_{Dix} \left\{ \pi(G_1)^+ \pi(G_2) | D |^{-d} \right\},
\]
where \( \text{Tr}_{Dix} \) is the Dixmier trace and \( d \) (here \( d = 2 \)) is the dimension of the spectral triple as defined in Connes’ book [1].

With respect to this scalar product, \( \pi(\Omega^{(k)}(A)) \) can be completed to a Hilbert space \( \mathcal{H}^{(k)} \) and its quotient by \( \pi(dJ_0^{(k-1)}) \), i.e. \( \Omega_D^{(k)}(A) \), will be a dense subspace of \( \pi(dJ_0^{(k-1)})^\perp \), the orthogonal complement of \( \pi(dJ_0^{(k-1)}) \).

Let \( P^{(k)} \) be the projector on \( \mathcal{H}^{(k)}_D = \pi(dJ_0^{(k-1)})^\perp \), then a scalar product in \( \Omega_D^{(k)}(A) \) is defined by:
\[
\langle \pi_D(G_1); \pi_D(G_2) \rangle_{k,D} = \langle P^{(k)}(\pi(G_1)); P^{(k)}(\pi(G_2)) \rangle_k.
\]

It can also be shown that \( \mathcal{H}^{(k)}_D \) is a Hilbert \( A \)-bimodule with a two-sided representation of the unitaries \( \mathcal{U}(A) = \{ u \in A | uu^+ = u^+ u = 1 \} \). Indeed, if
$G \in \Omega^{(k)}(\mathcal{A})$ then $uG$ and $Gu$ also belong to $\Omega^{(k)}(\mathcal{A})$ so that $\mathcal{H}^{(k)}$ is a Hilbert \mathcal{A}-bimodule. Furthermore $\pi(u)\pi_D(G) = \pi_D(uG)$ and $\pi_D(G)\pi(u) = \pi_D(Gu)$ show that $\mathcal{P}^{(k)} : \mathcal{H}^{(k)} \to \mathcal{H}^{(k)}_D$ is a bimodule homomorphism and the Dixmier trace properties guarantee that

$$\langle \pi_D(G_1); \pi_D(G_2) \rangle_{k,D} = \langle \pi(u)\pi_D(G_1); \pi_D(G_2) \rangle_{k,D} = \langle \pi_D(G_1)\pi(u); \pi_D(G_2)\pi(u) \rangle_{k,D} .$$

(4.16)

The following trace theorems of Connes [1] will be needed in the sequel of the calculations.

1) If $A_s \otimes B$ is a bounded operator in $\mathcal{H} = \mathcal{H}_{(s)} \otimes \mathbb{C}^N$, then

$$\text{Tr}_{\text{Dix}}\left\{ (A_s \otimes B) \mid D \mid^{-2} \right\} = \text{Tr}_{\text{Dix}}\left\{ A_s \mid D_{(s)} \mid^{-2} \right\} \text{tr}\{B\} ,$$

where $\text{tr}$ is the ordinary trace on $N \times N$ matrices.

2) Let $A_s$ be a section of the Clifford bundle over a compact $d$-dimensional manifold $M = S^2$ with its action on the (Pensov) spinors, then

$$\text{Tr}_{\text{Dix}}\left\{ A_s \mid D_{(s)} \mid^{-d} \right\} = \left( \frac{1}{4\pi} \right)^{d/2} \frac{1}{\Gamma(d/2 + 1)} \int_M \text{tr}_c\{A_s\} \omega ,$$

(4.18)

where $\text{tr}_c$ is the trace on the representation of the Clifford algebra.

An element $\pi(G)$ of $\Omega_D^{(2)}(\mathcal{A}) \cong (\pi(\mathcal{dJ}_0^{(1)}))^\perp$ is of the form given in (4.12) and has to obey :

$$\forall j^{(0)}_\alpha : \text{Tr}_{\text{Dix}}\left\{ \begin{pmatrix} j^{(0)}_a(x) & 1_a \\ 0 & j^{(0)}_b(x) \end{pmatrix} \pi(G) \mid D \mid^{-2} \right\} = 0 .$$

Using the trace properties (4.17) and (4.18), we obtain the orthogonality condition :

$$\rho^{(0)}_{aaa} - \frac{1}{N_a} \rho^{(0)}_{aba} \text{tr}\{M^+M\} = 0 ; \quad \rho^{(0)}_{bbb} - \frac{1}{N_b} \rho^{(0)}_{bab} \text{tr}\{MM^+\} = 0 .$$

(4.19)

Subtracting these equalities from the diagonals of (4.12) yields the following representative of $\pi_D(G) \in \Omega_D^{(2)}(\mathcal{A})$ :

$$\pi_D(G) = \begin{pmatrix} \pi_D(G)_{aa} & \pi_D(G)_{ab} \\ \pi_D(G)_{ba} & \pi_D(G)_{bb} \end{pmatrix} ,$$

(4.20)
where
\[
\begin{align*}
\pi_D(G)_{[aa]} &= -c(\rho_{aaa}^{(2)}) \otimes 1_a + \rho_{aba}^{(0)} \otimes [M+M]_{NT}, \\
\pi_D(G)_{[ab]} &= -i c(\rho_{ab}^{(1)}) \gamma_3 \otimes M^+, \\
\pi_D(G)_{[ba]} &= -i c(\rho_{ba}^{(1)}) \gamma_3 \otimes M, \\
\pi_D(G)_{[bb]} &= -c(\rho_{bbb}^{(2)}) \otimes 1_b + \rho_{bab}^{(0)} \otimes [MM^+]_{NT}.
\end{align*}
\]

with the traceless matrices\(^{14}\):
\[
\begin{align*}
[M+M]_{NT} &= M+M - \frac{1}{N_a} \text{tr}\{M+M\}, \\
[MM^+]_{NT} &= MM^+ - \frac{1}{N_b} \text{tr}\{MM^+\}.
\end{align*}
\]

The scalar product (4.15) in \(\Omega^2_D(A)\) is calculated using, besides the trace theorems, the identities:
\[
\frac{1}{2d/2} \text{tr}\left\{(\mathbf{c}(\rho^{(k)})^*)^a \mathbf{c}(\rho^{(k)})^b\right\} = k! (\rho^{(k)}_{i_1...i_k})^a (\rho^{(k)}_{j_1...j_k})^b \delta^{i_1j_1}...\delta^{i_kj_k} \\
= g^{-1}(\rho^{(k)*}; \rho^{(k)}).
\]

In terms of the Hodge dual \(*\), defined by
\[
g^{-1}(\rho^{(k)*}; \rho^{(k)}) \omega = (\rho^{(k)})^* \wedge \rho^{(k)},
\]
the scalar product reads:
\[
\begin{align*}
\langle \pi_D(G); \pi_D(G') \rangle_{2,D} &= \\
&= \frac{1}{2\pi} \left\{ N_a \int_{S^2} \rho_{aaa}^{(2)*} \wedge *\rho_{aaa}^{(2)} + N_b \int_{S^2} \rho_{bbb}^{(2)*} \wedge *\rho_{bbb}^{(2)} \\
&+ \text{tr}\{MM^+\} \int_{S^2} \rho_{ab}^{(1)*} \wedge *\rho_{ab}^{(1)} + \text{tr}\{M+M\} \int_{S^2} \rho_{ba}^{(1)*} \wedge *\rho_{ba}^{(1)} \\
&+ \text{tr}\{[M+M]^2\}_{NT} \int_{S^2} \rho_{aba}^{(0)*} \wedge *\rho_{aba}^{(0)} + \text{tr}\{[MM^+]^2\}_{NT} \int_{S^2} \rho_{bab}^{(0)*} \wedge *\rho_{bab}^{(0)} \right\}. \\
&= (4.21)
\end{align*}
\]

\(^{14}\)Note that when \(N_a = N_b = N\) and \(M\) is a scalar matrix, these traceless matrices \([M+M]_{NT}\) and \([MM^+]_{NT}\) vanish and there is no \(\rho_{aa}^{(0)}\) term in \(\pi_D(G)\). Physically this implies that, in order to have a Higgs mechanism, a nontrivial mass spectrum is necessary!
4.1 The Yang-Mills-Higgs action

The universal connection in $\mathcal{M}$, given by the matrices $(\langle A_{\alpha\beta}(x, y) \rangle)$ of (3.10), is represented in $\Omega_B^1(\mathcal{A})$ by an operator of the form (4.11) where the differential forms $\sigma^{(k)}$ are matrix-valued.

\[
\begin{align*}
\left(\left(\sigma_a^{(1)}(x)\right)\right) &= \left(\left(\alpha_a(x)\right)\right) \equiv \left(\tilde{e}_{k,y}(\omega_{aa}(x, y))\right)_{y=x} \theta^k_x, \\
\left(\left(\sigma_b^{(1)}(x)\right)\right) &= \alpha_b(x)\langle P_b(x)\rangle \equiv \left(\tilde{e}_{k,y}\omega_{bb}(x, y)\right)_{y=x} \theta^k_x \langle P_b(x)\rangle, \\
\left(\left(\sigma_{ab}^{(0)}(x)\right)\right) &= \left(\langle \Phi_{ab}(x, x)\rangle \langle \nu(x)\rangle\right), \\
\left(\left(\sigma_{ba}^{(0)}(x)\right)\right) &= \left(\langle \nu(x)\rangle \langle \Phi_{ba}(x, x)\rangle\right). \\
\end{align*}
\] (4.22)

The monopole connection (3.13) also implements a differential one-form:

\[
\mu_b(x) = (\tilde{e}_{k,b} m_b(x, y))_{y=x} \theta^k_x = \langle \nu(x) \rangle \left(\frac{1}{2} \theta^k_x \right) = \frac{1}{|\nu(x)|^2} \left(\langle \nu(x) \rangle^* \langle d \nu(x) \rangle - \langle \nu(x) \rangle \langle d \nu(x) \rangle^* \right). \\
\] (4.23)

It is also convenient to introduce the Higgs field doublets:

\[
\begin{align*}
|\eta_{ab}(x)\rangle &= |H_{ab}(x, x)\rangle = |\Phi_{ab}(x, x)\rangle + |\nu(x)\rangle, \\
\langle \eta_{ba}(x)| &= \langle H_{ab}(x, x)| = \langle \Phi_{ba}(x, x)| + \langle \nu(x)|. \\
\end{align*}
\] (4.24)

The hermiticity of the connection (3.9) yields:

\[
\left(\left(\alpha_a\right)\right)^* = -\left(\left(\alpha_a\right)\right), \quad \langle \alpha_a^* \rangle = -\langle \alpha_a \rangle, \quad \langle \mu_b^* \rangle = -\langle \mu_b \rangle, \quad |\eta_{ab}\rangle = |\eta_{ba}\rangle^+. \\
\] (4.25)

From (3.17) it follows that, under an active gauge transformation, the differential forms (4.22) and (4.23) behave as:

\[
\begin{align*}
\langle U_a \rangle^{\dagger} = \langle U_a \rangle^{-1} \langle \alpha_a \rangle \langle U_a \rangle + \langle U_a \rangle^{-1} d \langle U_a \rangle, \\
\alpha_b^U &= (u_b)^{-1} \alpha_b (u_b) + (u_b)^{-1} d u_b = \alpha_b + (u_b)^{-1} d u_b, \\
\mu_b^U &= (u_b)^{-1} \mu_b (u_b) = \mu_b, \\
|\eta_{ab}^U| &= \langle U_a \rangle^{-1} |\eta_{ab}| u_b, \quad \langle \eta_{ba}^U | = (u_b)^{-1} \langle \eta_{ba} | \langle U_a \rangle. \\
\end{align*}
\]

On the other hand, under a passive gauge transformation $H_B \rightarrow H_A$, according to (3.10) and (3.14), they transform as:

\[
\begin{align*}
\langle \alpha_a^A \rangle &= \langle \alpha_a^B \rangle, \quad \alpha_b^A = \alpha_b^B, \\
|\eta_{ab}^A| &= |\eta_{ab}^B| (c_{AB})^{-n/2}, \quad \langle \eta_{ba}^A | = (c_{AB})^{+n/2} \langle \eta_{ba}^B |, \\
\mu_b^A &= \mu_b^B + (c_{AB})^{+n/2} d (c_{AB})^{-n/2} = \mu_b^B - (n/2) (c_{AB})^{-1} d c_{AB}. \\
\end{align*}
\]

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This means that the Higgs fields \( \{|\eta_{ab}\rangle, \langle\eta_{ba}|\} \) are actually Pensov scalars of weight \(-n/2; +n/2\) and that the monopole potential cannot be represented by a globally defined one-form on the sphere, but acquires the inhomogeneous term \(- (n/2)(c_{AB})^{-1}d_{CAB}\) in \(H_B \cap H_A\).

The canonical projection (4.14) induces a \(\Omega^{(1)}_D(\mathcal{A})\)-valued connection in \(\mathcal{M}\):

\[
\nabla_D : \mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{A}} \Omega^{(1)}_D(\mathcal{A}) : X \mapsto \nabla_D X,
\]

defined by \(\nabla_D = (1_{\mathcal{M}} \otimes \pi_D) \circ \nabla\).

From (3.11) and (4.22) it follows that:

\[
(\nabla_D X)_{aa} = -i c (df_a + (\langle\alpha_a\rangle)|f_a\rangle),
\]

\[
(\nabla_D X)_{ab} = (|\eta_{ab}\rangle f_b - |f_a\rangle) \gamma_3 M^+,
\]

\[
(\nabla_D X)_{ba} = |\nu\rangle (\langle\eta_{ba}|f_a\rangle - f_b) \gamma_3 M,
\]

\[
(\nabla_D X)_{bb} = |\nu\rangle - i c (df_b + (\alpha_b + \mu_b)f_b).
\]

The curvature (3.13) is represented in \(\Omega^{(2)}_D(\mathcal{A})\) by \(\pi_D(R)\) of the form (4.20), where the differential forms \(\rho^{(k)}\)'s are now \(2 \times 2\)-matrix valued.

The diagonal elements of \(\pi_D(R)\) are given by:

\[
\begin{align*}
(\rho^{(2)}_{aaa}) &= \langle F_a \rangle = d(\langle\alpha_a\rangle) + (\langle\alpha_a\rangle) \wedge (\langle\alpha_a\rangle), \\
(\rho^{(2)}_{bbb}) &= F_b\langle P_b \rangle = (d\alpha_b + d\mu_b)\langle P_b \rangle, \\
(\rho^{(0)}_{a\alpha b}) &= |\eta_{ab}\rangle \langle\eta_{ba}| - \langle\text{Id}\rangle, \\
(\rho^{(0)}_{b\beta a}) &= \langle\eta_{ba}|\eta_{ab}\rangle - 1 \langle\text{Id}\rangle.
\end{align*}
\]

The off-diagonal elements are given in terms of the covariant differentials of the Higgs fields (4.24):

\[
\begin{align*}
|\nabla\eta_{ab}\rangle &= d |\eta_{ab}\rangle + (\langle\alpha_a\rangle)|\eta_{ab}\rangle - |\eta_{ab}\rangle (\alpha_b + \mu_b), \\
\langle\nabla\eta_{ba}| &= d \langle\eta_{ba}| - \langle\eta_{ba}|(\langle\alpha_a\rangle) + (\alpha_b + \mu_b)\langle\eta_{ba}|.
\end{align*}
\]

They read

\[
\begin{align*}
(\rho^{(1)}_{ab}) &= |\nabla\eta_{ab}\rangle \langle\nu|, \\
(\rho^{(1)}_{ba}) &= |\nu\rangle \langle\nabla\eta_{ba}|.
\end{align*}
\]
The Yang-Mills-Higgs action is constructed as:

$$S_{YMH}(\nabla D) = \lambda \text{tr}_{\text{matrix}}\left\{ \langle \pi_D(R); \pi_D(R) \rangle_{2,D} \right\}$$

$$= \lambda \text{tr}_{\text{matrix}}\left\{ \text{TrDix}\{ \pi_D(R)^\dagger \pi_D(R) |D|^2 \} \right\}$$

where $\lambda$ is a coupling constant and $\text{tr}_{\text{matrix}}$ is the trace of the $2 \times 2$ matrices, product of matrices $(\rho^{(k)})^+$ with $(\rho^{(k)})$. Since the curvature transforms as $R \rightarrow R U = U^{-1} R U$, the gauge invariance of the action follows from the obvious extension of the representation (4.16) of the unitaries in $\mathcal{H}_D^{(2)}$. With the scalar product given by (4.21), the action (4.26) reads:

$$S_{YMH}(\nabla D) = \frac{\lambda}{2\pi} \left\{ N_a \int_{S^2} \text{tr}_{\text{matrix}}\left\{ (F_a)^+ \wedge *F_a \right\} \right. \right.$$

$$+ N_b \int_{S^2} (F_b)^* \wedge *F_b$$

$$+ 2 \text{tr}\{MM^+\} \int_{S^2} \langle \nabla \eta_{ba} \rangle \wedge * \langle \nabla \eta_{ab} \rangle$$

$$+ \text{tr}\left\{ [M^+ M]_{NT}^2 \right\} \int_{S^2} *(\langle \eta_{ba} \eta_{ab} \rangle - 1)^2 + 1$$

$$+ \text{tr}\left\{ [M^+ M]_{NT}^2 \right\} \int_{S^2} *(\langle \eta_{ba} \eta_{ab} \rangle - 1)^2 \right\}.$$ (4.28)

### 4.2 The Hilbert space of particle states and the covariant Dirac operator

The tensor product over $A$ of the right $A$-module $M$ with the (left-module) Hilbert space $\mathcal{H}$ is itself a Hilbert space $\mathcal{H}_p = M \otimes_A \mathcal{H}$, with scalar product induced by the scalar product (4.9) in $\mathcal{H}$ and the hermitian structure $h$ in the module $M$:

$$(X \otimes_A \Psi; Y \otimes_A \Phi) = (\Psi; \pi(h(X,Y))\Phi).$$

A generic element of $\mathcal{H}_p$ can be written as $\|\Psi_p\| = E_i \otimes_A \Psi^i$, where $\Psi^i \in \mathcal{H}$ obeys $\pi(P^i_j) \Psi^j = \Psi^i$. In the model considered here, $\mathcal{H} = \mathcal{H}_{(a)} \otimes (C^{N_a} \oplus C^{N_b})$ and the projective module is $M = P A^2$, with $P$ defined by the homotopy class $[g]$ in (3.3). A state $\|\Psi_p\|$ describing particles, is thus represented by:
1. A pair of Pensov spinors of $\mathcal{H}_{(s)}$, given by:

$$|\psi_a(x)\rangle = \left(\begin{array}{c} \psi^1_a(x) \\ \psi^2_a(x) \end{array}\right)$$

each with $N_a$ values of the generation index.

2. A single Pensov spinor $\psi_b(x)$ of $\mathcal{H}_{(s+n/2)}$, with a $N_b$-valued generation index, such that

$$|\psi_b(x)\rangle = \left(\begin{array}{c} \psi^1_b(x) \\ \psi^2_b(x) \end{array}\right) = |\nu(x)\rangle \psi_b(x) \text{ in } H_B.$$

The $*$-representation $\pi$ of $\Omega^*(A)$ in $\mathcal{H}$ induces a mapping

$$\pi_1: M \otimes A \Omega^*(A) \mapsto \mathcal{B}(\mathcal{H}, \mathcal{H}_p): X \otimes A F \mapsto \pi_1(X \otimes A F), \quad (4.29)$$

where $\mathcal{B}(\mathcal{H}, \mathcal{H}_p)$ are the bounded linear operators from $\mathcal{H}$ to $\mathcal{H}_p$. It is defined by $\pi_1(X \otimes A F)\Psi = X \otimes A \pi(F)\Psi$.

Furthermore, there is a mapping

$$\pi_2: \text{HOM}_{A}(M, M \otimes A \Omega^*) \mapsto \mathcal{B}(\mathcal{H}_p): T \mapsto \pi_2(T), \quad (4.30)$$

defined by $\pi_2(T)(X \otimes A \Psi) = \pi_1(TX)\Psi$.

The covariant Dirac operator in $\mathcal{H}_p$ is defined, using (4.29), as

$$D_{\nabla}(X \otimes A \Psi) = X \otimes A D\Psi + \pi_1(\nabla X)\Psi. \quad (4.31)$$

It is easy to check that $D_{\nabla}(X f \otimes A \Psi) = D_{\nabla}(X \otimes A \pi(f)\Psi)$ so that $D_{\nabla}$ is well defined in $\mathcal{H}_p$.

A grading in $\mathcal{H}_p$ is defined by

$$\Gamma_p: X \otimes A \Psi \mapsto X \otimes A \Gamma\Psi \quad (4.32)$$

and the covariant Dirac operator is odd with respect to this grading:

$$D_{\nabla}\Gamma_p + \Gamma_p D_{\nabla} = 0 \quad (4.33)$$

With $\|\Psi_p\|$ as above, $D_{\nabla}$ is calculated as follows.

$$D_{\nabla}\|\Psi_p\| = E_i \otimes A \left(D_{\nabla}\|\Psi_p\|\right)^i,$$

where $\left(D_{\nabla}\|\Psi_p\|\right)^i = \left(\pi(A)D + \pi(A)\right)_{ij}^i\Psi^j$ is represented by

$$\left(\begin{array}{cc} (D_{\nabla})_{aa} & (D_{\nabla})_{ab} \\ (D_{\nabla})_{ba} & (D_{\nabla})_{bb} \end{array}\right) \left(\begin{array}{c} |\psi_a\rangle \\ |\nu\rangle |\psi_b\rangle \end{array}\right), \quad (4.34)$$
\[
\begin{align*}
(D_{\nabla})_{aa} &= D_{(s)} \otimes 1_a - i \, c(\alpha_a) \otimes 1_a, \\
(D_{\nabla})_{ab} &= (|\eta_{ab}\rangle \langle \nu|) \otimes \gamma_3 M^+, \\
(D_{\nabla})_{ba} &= (|\nu \rangle \langle \eta_{ba}|) \otimes \gamma_3 M, \\
(D_{\nabla})_{bb} &= ((P_b)D_{(s)}(P_b)) \otimes 1_b - i \, c(\alpha_b) \langle (P_b) .
\end{align*}
\]

Now, \( \langle \nu | D_{(s)} | \nu \rangle \psi_b = D_{(s)} \psi_b - i \, c(m_b) \psi_b \) and with our choice (3.3) of the representative of the homotopy class \([n] \in \mathbb{Z}\), we obtain

\[
\langle (P_b)D_{(s)}| \nu \rangle \psi_b = |\nu \rangle D_{(s+n/2)} \psi_b. \quad (4.35)
\]

Substituting (4.35) and (4.34) in (4.31) yields finally:

\[
\begin{align*}
(D_{\nabla} || \Psi_p ||)_a &= \left( D_{(s)} - i \, c(\alpha_a) \right) |\psi_a\rangle + |\eta_{ab}\rangle \gamma_3 M^+ \psi_b, \\
(D_{\nabla} || \Psi_p ||)_b &= |\nu \rangle \left[ \left( D_{(s+n/2)} - i \, c(\alpha_b) \right) \psi_b + \gamma_3 M \langle \eta_{ba}| \psi_a \rangle \right]. \quad (4.36)
\end{align*}
\]

The matter action functional is then constructed as:

\[
S_{\text{Mat}} (|| \Psi_p ||, \nabla_D) = \left( || \Psi_p ||; D_{\nabla} || \Psi_p || \right) = \int_{S^2} \star \left\{ \langle \psi_a | \left( D_{(s)} - i \, c(\alpha_a) \right) | \psi_a \rangle \\
+ \langle \psi_a | \eta_{ab} \rangle \gamma_3 M^+ \psi_b + \langle \psi_b \rangle^+ \gamma_3 M \langle \eta_{ba}| \psi_a \rangle \\
+ \langle \psi_b \rangle^+ \left( D_{(s+n/2)} - i \, c(\alpha_b) \right) \psi_b \right\}. \quad (4.37)
\]

The hermiticity condition (4.25) guarantees that the action is real. If an Euclidean chiral theory is aimed for, then \( \Gamma_p || \Psi_p || = || \Psi_p || \) implies that the action (4.37) vanishes identically due to the oddness of the Dirac operator (4.33). A proposed way out, as in [22], is just to make an easy switch going to an indefinite Minkowski type metric changing the \( \psi^* \) to a \( \tilde{\psi} = \psi^* \gamma^0 \) so that the presence of the \( \gamma^0 \) provides an extra factor minus one and the action does not vanish. Such a ”usual incantation” [12] appears highly unaesthetic and rather unsatisfactory. It seems necessary to double the Hilbert space in order to deal with this issue. This can be achieved introducing a Hilbert space.

\[\text{Note that with a different choice in (3.3), a globally defined differential one-form would be added to } D_{(s+n/2)} \text{ and this can always be absorbed in } \alpha_b.\]
\( \mathcal{H}_p \) of "anti-particle" states. The need of doubling the Fermion fields also arises in the usual Euclidean quantum field theory, where the fermion fields are operator valued in Fock space, in order to cure inconsistent hermiticity properties of the propagators [16, 5]. Alternative proposals were made by [13] and more recently by [20]. Related comments by [11] in a non-commutative geometric setting, should also be mentioned.

Here, however, we choose to remain with the primary interpretation of \(|\Psi_p\rangle\rangle\) as a state in the Hilbert space \(\mathcal{H}_p\) represented by Euclidean wave functions. This means that in this work we endeavour an Euclidean one-particle (plus one would-be anti-particle) field theory, which, in a path integral formalism, may hopefully lead to a proper quantum theory.
5  Real spectral triples

5.1  The real Pensov-Dirac spectral triple

The complex conjugation $K$ transforms a Pensov field of weight $s$, $\sigma^{(s)}$, into a Pensov field $\sigma^{(s)*}$ of weight $-s$. Besides the Hilbert space of Pensov spinors $\psi_{(+)}$ of weight $s$, denoted here as $\mathcal{H}_{1(+)}$, we also introduce $\mathcal{H}_{1(-)}$, with spinors $\psi_{(-)}$ of weight $-s$. A real structure will be induced by a pair of anti-linear mappings $J_{1(\pm)}: \mathcal{H}_{1(\pm)} \to \mathcal{H}_{1(\mp)}$, which are required to preserve the real Clifford-algebra module structure of the spinor spaces:

$$
J_{1(\pm)} \lambda \psi_{(\pm)} = \bar{\lambda} J_{1(\pm)} \psi_{(\pm)}
$$

$$
J_{1(\pm)} \left( \gamma^k \psi_{(\pm)} \right) = \alpha \gamma^k \left( J_{1(\pm)} \psi_{(\pm)} \right),
$$

(5.1)

where we allow for $\alpha$ to be a sign factor $\pm 1$.

With

$$
J_{1(\pm)} \psi_{(\pm)} = a_{(\pm s)} \mathcal{C}_{1,\alpha} K \psi_{(\pm)},
$$

(5.2)

where $a_{(\pm s)}$ is an arbitrary complex number, we should have

$$
\mathcal{C}_{1,\alpha} \left( \gamma^k \right)^* \mathcal{C}_{1,\alpha}^{-1} = \alpha \gamma^k.
$$

In the chiral representation $\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\gamma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, we may choose $\mathcal{C}_{1,\alpha} = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}$.

The adjoint of $J_{1(\pm)}$ is given by:

$$
\left( J_{1(\pm)} \right)^+ \psi_{(\mp)} = a_{(\pm s)} \mathcal{C}_{1,\alpha}^t K \psi_{(\mp)}.
$$

(5.3)

Demanding that

$$
J_{1(\pm)} \left( J_{1(\pm)} \right)^+ = 1_{(\mp)}; \left( J_{1(\pm)} \right)^+ J_{1(\pm)} = 1_{(\pm)},
$$

(5.4)

restricts $a_{(\pm s)}$ be a phase factor. On $\mathcal{H}_1 = \mathcal{H}_{1(+)} \oplus \mathcal{H}_{1(-)}$, the antilinear isometry defined by

$$
J_1 = \begin{pmatrix} 0 & J_{1(-)} \\ J_{1(+)} & 0 \end{pmatrix}
$$

(5.5)
obeys

\[
J_1 \begin{pmatrix} \gamma^k & 0 \\ 0 & \gamma^k \end{pmatrix} = \alpha \begin{pmatrix} \gamma^k & 0 \\ 0 & \gamma^k \end{pmatrix} J_1.
\]

If we require

\[J_1^2 = \epsilon_1 \mathbf{1}_1, \quad \text{with } \epsilon_1 = \pm 1, \quad (5.6)\]

the phases \(a_{(\pm s)}\) are related by \(a_{(-s)} = \alpha \epsilon_1 a_{(+s)}\) and \(J_{1(-)} = \epsilon_1 (J_{1(+)})^\dagger\).

The antilinear mappings \(J_{1(\pm)}\) intertwine with the Dirac operators \(D_{(\pm s)}\) as:

\[
J_{1(\pm)} D_{(\pm s)} = -\alpha D_{(\mp s)} J_{1(\pm)}.
\] (5.7)

On \(H_{1(+)\dagger}\), the Dirac operator is chosen as \(D_{1(+)\dagger} = D_{(s)}\), but on \(H_{1(-)}\) we may choose \(D_{(-s)}\) up to a sign. Let \(\epsilon'_1\) be another arbitrary sign factor, then the choice

\[
D_{1(-)} = -\alpha \epsilon'_1 D_{(-s)},
\] (5.8)

yields a Dirac operator \(D_1 = \begin{pmatrix} D_{1(+)\dagger} & 0 \\ 0 & D_{1(-)} \end{pmatrix}\) intertwining with \(J_1\) as:

\[
J_1 D_1 = \epsilon'_1 D_1 J_1.
\] (5.9)

The representation of \(A_1 = C(S^2; C)\) in \(H_1\) is obtained by taking two copies of the representation in \(H_{(s)}\):

\[
\pi_1(f) \begin{pmatrix} \psi_{(+)} \\ \psi_{(-)} \end{pmatrix}(x) = \begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} \begin{pmatrix} \psi_{(+)}(x) \\ \psi_{(-)}(x) \end{pmatrix}.
\] (5.10)

In general, a real structure \(J_1\) induces a representation of the opposite algebra \(A_1^o\) by:

\[
\pi_1^o(f) = J_1 (\pi_1(f))^\dagger J_1^+, \quad \forall f \in A_1^o.
\]

so that the Hilbert space \(H_1\) becomes an \(A_1\) bimodule. Here \(A_1\) is abelian, and with the representation \(\pi_1\) above, we have \(\pi_1^o(f) = \pi_1(f)\).

Since \([D_1, \pi_1(f)] = \begin{pmatrix} -i\kappa_{(+)} c(df) & 0 \\ 0 & -i\kappa_{(-)} c(df) \end{pmatrix}\), the first-order condition

\[
[[D_1, \pi_1(f)], \pi_1^o(g)] = 0,
\] (5.11)

\[\text{For notational convenience, we define } \kappa(+) = +1 \text{ and } \kappa(-) = -\alpha \epsilon'_1 \text{ so that } D_{1(\pm)} = \kappa(\pm) D_{(\pm s)}.\]
which is needed to define a connection in bimodules \([\square]\), is satisfied. Since \(J^{1(\pm)}\gamma_3 = -\gamma_3 J^{1(\pm)}\), the chirality in \(H_{1(+)} \oplus H_{1(-)}\) will be taken as

\[
\chi_1 = \begin{pmatrix} \gamma_3 & 0 \\ 0 & \gamma_3 \end{pmatrix}.
\]

With this choice:

\[
J_1 \chi_1 = \epsilon''_1 \chi_1 J_1 , \text{ with } \epsilon''_1 = -1.
\]

The \(\epsilon\)-sign table of Connes \([\square]\), corresponding to \(n = 2\), can be satisfied if we choose \(\epsilon_1 = -1\) and \(\epsilon'_1 = +1\), but for the moment we shall leave these choices open.

The spectral triple \(T_1 = \{A_1, H_1, D_1, \chi_1, J_1\}\) is actually a 0-sphere real spectral triple as defined in \([\square]\). For our pragmatic purposes, an \(S^0\)-real spectral triple may be defined as a real spectral triple with an hermitian involution \(\sigma^0\) commuting with \(\pi(A_1), D_1, \chi_1\) and anticommuting with \(J_1\).

It is implemented by the decomposition \(H_1 = H_{1(+)} \oplus H_{1(-)}\), which it is given in, by:

\[
\sigma^0 = \begin{pmatrix} 1_{1(+)} & 0 \\ 0 & -1_{1(-)} \end{pmatrix}.
\]

The doubling of the Hilbert space is justified if we interpret the Penso spinors of \(H_{(s)}\) as usual (Euclidean!) Dirac spinors interacting with a magnetic monopole of strenght \(s\). It seems then natural to consider the (Euclidean!) anti-particle fields as Dirac spinors “seeing” a monopole of strenght \(-s\) i.e. as Penso spinors of \(H_{(-s)}\).

### 5.2 The real discrete spectral triple

Proceeding further, as in section \([\square]\), we have to compose the above \(S^0\)-real ”Dirac-Pensov" spectral triple \(T_1\) with a real discrete spectral triple \(T_2 = \{A_2, H_2, D_2, \chi_2, J_2\}\) over the algebra \(A_2 = \mathbb{C} \oplus \mathbb{C}\). The most general finite Hilbert space allowing a \(A_2\)-bimodule structure\(^{17}\) is given by the direct sum

\[
H_2 = \bigoplus_{\alpha,\beta} \mathbb{C}^{N_{\alpha\beta}},
\]

\(^{17}\) For a general discussion on real discrete spectral triples, we refer to \(([\square], [\square])\).
where \( \alpha \) and \( \beta \) vary over \{a,b\} and where \( N_{\alpha\beta} \) are integers. Its elements are of the form

\[
(\xi) = \begin{pmatrix}
\xi_{aa} \\
\xi_{ab} \\
\xi_{ba} \\
\xi_{bb}
\end{pmatrix},
\]

where each \( \xi_{\alpha\beta} \) is a column vector with \( N_{\alpha\beta} \) rows. An element \( \lambda = (\lambda_a, \lambda_b) \) of \( A_1 \) acts on the left on \( H_2 \) by:

\[
\pi_2(\lambda) = \begin{pmatrix}
\lambda_a & 0 & 0 & 0 \\
0 & \lambda_a & 0 & 0 \\
0 & 0 & \lambda_b & 0 \\
0 & 0 & 0 & \lambda_b
\end{pmatrix},
\]

and on the right by:

\[
\pi_2^o(\lambda) = \begin{pmatrix}
\lambda_a & 0 & 0 & 0 \\
0 & \lambda_b & 0 & 0 \\
0 & 0 & \lambda_a & 0 \\
0 & 0 & 0 & \lambda_b
\end{pmatrix}. \tag{5.17}
\]

Although \( A_2 \) is an abelian algebra and, as such, isomorphic to its opposite algebra, it is not a simple algebra so that, in general, \( \pi_2(\lambda) \neq \pi_2^o(\lambda) \). The discrete real structure, given by \( J_2 = C_2 K \), relates both by \( \pi^o(\lambda) = J_2 \pi(\lambda)^+ J_2^{-1} \) so that \( C_2 \) is an intertwining operator for the two representations:

\[
\pi_2^o(\lambda) = C_2 \pi_2(\lambda) C_2^{-1}. \tag{5.18}
\]

This requires that \( N_{ab} = N_{ba} = N \) and, since we require \( J_2 \) to be anti-unitary, the basis in \( H_2 \) may be chosen such that:

\[
C_2 = \begin{pmatrix}
1_{N_{aa}} & 0 & 0 & 0 \\
0 & 0 & 1_N & 0 \\
0 & 1_N & 0 & 0 \\
0 & 0 & 0 & 1_{N_{bb}}
\end{pmatrix}. \tag{5.19}
\]

This implies that:

\[
J_2^2 = \epsilon_2 1_2, \text{ with } \epsilon_2 = +1. \tag{5.20}
\]

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The chirality, $\chi_2$, defining the orientation of the spectral triple is the image of a Hochschild 0-cycle, i.e. an element of $A_2 \otimes A_2^*$. This implies that $\chi_2$ is diagonal and $\chi_{\alpha\beta} = \pm 1$ on each subspace $C^{N_{\alpha\beta}}$.

Furthermore, demanding that

$$ J_2 \chi_2 = \epsilon'' \chi_2 J_2 , \text{ with } \epsilon'' = +1 , \quad (5.21) $$

requires $\chi_{ab} = \chi_{ba} = \chi'$ so that the chirality in $H_2$ reads:

$$ \chi_2 = \begin{pmatrix} 
\chi_{aa} & 1_{N_{aa}} & 0 & 0 & 0 \\
0 & \chi' & 1_N & 0 & 0 \\
0 & 0 & \chi' & 1_N & 0 \\
0 & 0 & 0 & \chi_{bb} & 1_{N_{bb}} 
\end{pmatrix} , \quad (5.22) $$

We consider the following three possibilities leading to a non trivial hermitian Dirac operator, odd with respect to this chirality :

2.a $+\chi_{aa} = +\chi' = -\chi_{bb} = \pm 1$

2.b $-\chi_{aa} = +\chi' = +\chi_{bb} = \pm 1$

2.c $-\chi_{aa} = +\chi' = -\chi_{bb} = \pm 1$

The corresponding Dirac operators have the form

2.a,2.b

$$ D_{2,a} = \begin{pmatrix} 0 & 0 & 0 & K^+ \\
0 & 0 & 0 & A^+ \\
0 & 0 & 0 & B^+ \\
K & A & B & 0 
\end{pmatrix} ; \quad D_{2,b} = \begin{pmatrix} 0 & B^+ & A^+ & K^+ \\
B & 0 & 0 & 0 \\
A & 0 & 0 & 0 \\
K & 0 & 0 & 0 
\end{pmatrix} $$

2.c

$$ D_{2,c} = \begin{pmatrix} 0 & B'^+ & A'^+ & 0 \\
B' & 0 & 0 & A^+ \\
A' & 0 & 0 & B^+ \\
0 & A & B & 0 
\end{pmatrix} . $$

\footnote{If we should require that $\epsilon'' = -1$, then $N_{aa} = N_{bb} = 0$ and $\chi_{ab} = -\chi_{ba}$ and the corresponding odd Dirac operator would not satisfy the first order condition.}
The first-order condition \[ [D_2, \pi_2(\lambda), \pi_2(\mu)] = 0, \] satisfied in case 2.c), implies that in case 2.a) and 2.b) \( K \) must vanish.

If we assume

\[ J_2 D_2 = \epsilon_2' D_2 J_2, \text{ with } \epsilon_2' = +1, \] (5.23)

then

\[ B = A^*; B' = A'^*. \] (5.24)

It should be stressed that, in order to have a non trivial Dirac operator, necessarily \( N \neq 0 \). This confirms that the discrete Hilbert space \( H_{dis} \) used in 4 does not allow for a real structure in the above sense. At last, it can be shown[10, 17] that noncommutative Poincaré duality, in the discrete case, amounts to the non degeneracy of the intersection matrix with elements \( \chi_{\alpha\beta} = \chi_{\alpha\beta} N_{\alpha\beta} \). This non degeneracy condition in case 2.a) and 2.b) reads \( N_{aa} N_{bb} + N^2 \neq 0 \) and is always satisfied. In case 2.c) it is required that \( N_{aa} N_{bb} - N^2 \neq 0 \) and if all \( N \)'s should be equal, this would not be satisfied.

If we insist on equal \( N \)'s, which is not strictly necessary, we are limited to the models 2.a) and 2.b) with representations given by (5.16), (5.17), Dirac operators and chirality assignments by:

\[
D_{2,a} = \begin{pmatrix}0 & 0 & 0 & 0 \\ 0 & 0 & A^+ & 0 \\ 0 & 0 & B^+ & 0 \\ 0 & A & B & 0 \end{pmatrix}; \chi_{2,a} = \chi' \begin{pmatrix}1_N & 0 & 0 & 0 \\ 0 & 1_N & 0 & 0 \\ 0 & 0 & 1_N & 0 \\ 0 & 0 & 0 & -1_N \end{pmatrix}. \] (5.25)

\[
D_{2,b} = \begin{pmatrix}0 & B^+ & A^+ & 0 \\ B & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \chi_{2,b} = \chi' \begin{pmatrix}-1_N & 0 & 0 & 0 \\ 0 & 1_N & 0 & 0 \\ 0 & 0 & 1_N & 0 \\ 0 & 0 & 0 & 1_N \end{pmatrix}. \] (5.26)

5.3 The product

The product of \( T_1 \) with \( T_2 \) yields the total triple \( T = \{A, \mathcal{H}, D, \chi, J\} \) with algebra \( A = A_1 \otimes A_2 \). Since \( T_1 \) is \( S^0 \)-real, the product is also \( S^0 \)-real with hermitian involution \( \Sigma^0 = \sigma^0 \otimes 1_2 \). The total Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) is decomposed as

\[
\mathcal{H} = \mathcal{H}_{(+)} \oplus \mathcal{H}_{(-)}, \] (5.27)
where $\mathcal{H}(\pm) = \mathcal{H}_1(\pm) \otimes \mathcal{H}_2$. Elements of $\mathcal{H}(\pm)$ are represented by column matrices of the form

$$\Psi(\pm)(x) = \begin{pmatrix} \psi_{\pm}^{a\alpha}(x) \\ \psi_{\pm}^{b\beta}(x) \\ \psi_{\pm}^{a\beta}(x) \\ \psi_{\pm}^{b\alpha}(x) \end{pmatrix},$$

(5.28)

where each $\psi_{\pm}^{a\alpha}(x)$ is a Pensov spinor of $\mathcal{H}_1(\pm)$ with $N$ "internal" indices (not explicitly written down). The total Dirac operator is

$$\mathcal{D} = \mathcal{D}_1 \otimes \mathcal{1}_2 + \chi_1 \otimes \mathcal{D}_2,$$

(5.29)

and the chirality is given by

$$\chi = \chi_1 \otimes \chi_2.$$  

(5.30)

The continuum spectral triple $\mathcal{T}_1$ of 5.1 is of dimension two and its real structure $J_1$ obeys

$$J_1^2 = \epsilon_1 \mathcal{1}_1, \quad J_1 \mathcal{D}_1 = \epsilon'_1 \mathcal{D}_1 J_1, \quad J_1 \chi_1 = \epsilon''_1 \chi_1 J_1,$$

where $\epsilon''_1 = -1$ was fixed but $\epsilon_1$ and $\epsilon'_1$ were independent free $\pm 1$ factors. On the other hand the discrete triple $\mathcal{T}_2$ of 5.2 is of zero dimension and $J_2$ obeys

$$J_2^2 = \epsilon_2 \mathcal{1}_2, \quad J_2 \mathcal{D}_2 = \epsilon'_2 \mathcal{D}_2 J_2, \quad J_2 \chi_2 = \epsilon''_2 \chi_2 J_2,$$

with $\epsilon_2 = \epsilon'_2 = \epsilon''_2 = +1$.

The real structure $J$ of the product triple should obey

$$J^2 = \epsilon \mathcal{1}, \quad J \mathcal{D} = \epsilon' \mathcal{D} J, \quad J \chi = \epsilon'' \chi J.$$  

(5.31)

If we require that Connes' sign table be satisfied, i.e.

- for $\mathcal{T}_1$, $n_1 = 2$ and $\epsilon_1 = -1$, $\epsilon'_1 = +1$, $\epsilon''_1 = -1$,
- for $\mathcal{T}_2$, $n_2 = 0$ and $\epsilon_2 = \epsilon'_2 = \epsilon''_2 = +1$ for $n_2 = 0$,
- for the product $\mathcal{T}$, $n = 2$ and $\epsilon = -1$, $\epsilon' = +1$, $\epsilon'' = -1,$
it is seen that, with \( J = J_1 \otimes J_2 \), the sign table for the product is not obeyed, since such a \( J \) implies the consistency conditions

\[
\begin{align*}
\epsilon &= \epsilon_1 \epsilon_2 , \\
\epsilon' &= \epsilon'_1 = \epsilon''_1 \epsilon'_2 , \\
\epsilon'' &= \epsilon''_1 \epsilon''_2 ,
\end{align*}
\]

and the second condition is not satisfied. If we keep the same Dirac operator \( (5.29) \), it is the definition of \( J \) that should be changed to

\[
J = J_1 \otimes (J_2 \chi_2) ,
\]

and with this \( J \) the consistency conditions become

\[
\begin{align*}
\epsilon &= \epsilon_1 \epsilon_2 \epsilon''_2 , \\
\epsilon' &= \epsilon'_1 = -\epsilon''_1 \epsilon'_2 , \\
\epsilon'' &= \epsilon''_1 \epsilon''_2 ,
\end{align*}
\]

and these are satisfied. In the rest of this section, we shall assume that these choices are made. Also, in order to simplify the forthcoming formulae, we take \( \alpha = -1 \) so that \( C_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( a_{(+s)} = a_{(-s)} = +1 \) which imply that \( J_{1(-)} = -J_{1(+)}^+ = J_{1(+)} \) and \( D_{1(\pm)} = D_{(\pm s)} \). The change of \( J_2 \) to \( J_2 \chi_2 \) will not change the representation \( \pi^o_2 \) since \( \pi_2 \) is even with respect to \( \chi_2 \).

The \( S^0 \)-real structure implies that \( \pi \) and \( \pi^o \), \( D \) and \( \chi \) are block diagonal in the decomposition \( (5.27) \) of \( \mathcal{H} \) and we obtain

the representations : let \( f \in \mathcal{C}(S^2, \mathbb{C} \oplus \mathbb{C}) \), then

\[
\pi(f) = \pi_{(\pm)}(f) \oplus \pi_{(\mp)}(f) ; \quad \pi^o(f) = \pi^o_{(+)}(f) \oplus \pi^o_{(-)}(f) ,
\]

\[
\pi_{(\pm)}(f(x)) = \begin{pmatrix}
 f_a(x)1_N & 0 & 0 & 0 \\
 0 & f_a(x)1_N & 0 & 0 \\
 0 & 0 & f_b(x)1_N & 0 \\
 0 & 0 & 0 & f_b(x)1_N
\end{pmatrix} ,
\]

\[19\text{A general examination of the \( \epsilon \) sign table and the product of two real spectral triples is done in [19].}\]
\[ \pi_{(\pm)}'(f(x)) = \begin{pmatrix} f_a(x)1_N & 0 & 0 & 0 \\ 0 & f_b(x)1_N & 0 & 0 \\ 0 & 0 & f_a(x)1_N & 0 \\ 0 & 0 & 0 & f_b(x)1_N \end{pmatrix}. \] (5.36)

the Dirac operator: \( \mathcal{D} = \mathcal{D}_{(+)} \oplus \mathcal{D}_{(-)} \) is given by

- in case 2.a):
\[
\mathcal{D}_{a(\pm)} = \begin{pmatrix} \mathcal{D}_{1(\pm)}1_N & 0 & 0 & 0 \\ 0 & \mathcal{D}_{1(\pm)}1_N & 0 & \gamma_3 A^+ \\ 0 & 0 & \mathcal{D}_{1(\pm)}1_N & \gamma_3 B^+ \\ 0 & \gamma_3 A & \gamma_3 B & \mathcal{D}_{1(\pm)}1_N \end{pmatrix}, \quad (5.37)
\]

- in case 2.b):
\[
\mathcal{D}_{b(\pm)} = \begin{pmatrix} \mathcal{D}_{1(\pm)}1_N & \gamma_3 B^+ & \gamma_3 A^+ & 0 \\ \gamma_3 B & \mathcal{D}_{1(\pm)}1_N & 0 & 0 \\ \gamma_3 A & 0 & \mathcal{D}_{1(\pm)}1_N & 0 \\ 0 & 0 & 0 & \mathcal{D}_{1(\pm)}1_N \end{pmatrix}. \quad (5.38)
\]

the chirality: \( \chi = \chi_{(+)} \oplus \chi_{(-)} \), is given by

- in case 2.a):
\[
\chi_{(\pm)} = \chi' \begin{pmatrix} \gamma_3 1_N & 0 & 0 & 0 \\ 0 & \gamma_3 1_N & 0 & 0 \\ 0 & 0 & \gamma_3 1_N & 0 \\ 0 & 0 & 0 & -\gamma_3 1_N \end{pmatrix}, \quad (5.39)
\]

- in case 2.b):
\[
\chi_{(\pm)} = \chi' \begin{pmatrix} -\gamma_3 1_N & 0 & 0 & 0 \\ 0 & \gamma_3 1_N & 0 & 0 \\ 0 & 0 & \gamma_3 1_N & 0 \\ 0 & 0 & 0 & \gamma_3 1_N \end{pmatrix}, \quad (5.40)
\]
the real structure: \( \mathbf{J} = \begin{pmatrix} 0 & \mathbf{J}^{(-)} \\ \mathbf{J}^{(+)} & 0 \end{pmatrix} \), exchanges \( \mathcal{H}^{(+)} \) and \( \mathcal{H}^{(-)} \) and \( \mathbf{J}^{(\pm)} = \mathcal{C}_1 \otimes \mathcal{C}_2 \chi_2 \) yields

- in case 2.a):
  \[
  \mathbf{J}_{a(\pm)} = \chi' \begin{pmatrix}
  C_11_N & 0 & 0 & 0 \\
  0 & 0 & C_11_N & 0 \\
  0 & C_11_N & 0 & 0 \\
  0 & 0 & 0 & \!-C_11_N
  \end{pmatrix} \mathcal{K}, \tag{5.41}
  \]

- in case 2.b):
  \[
  \mathbf{J}_{b(\pm)} = \chi' \begin{pmatrix}
  \!-C_11_N & 0 & 0 & 0 \\
  0 & 0 & C_11_N & 0 \\
  0 & C_11_N & 0 & 0 \\
  0 & 0 & 0 & C_11_N
  \end{pmatrix} \mathcal{K}, \tag{5.42}
  \]

5.4 The “Real” Yang-Mills-Higgs action

The representations \( \pi^{(\pm)} \) of (5.35) and \( \pi_{(\pm)}^{(0)} \) of (5.36), with the Dirac operators of (5.37) and (5.38), induce representations of \( \Omega^*(\mathcal{A}) \).

Let \( F \in \Omega^{(1)}(\mathcal{A}) \), then, using the same techniques which led to (4.10), we obtain in case 2.a):

\[
\pi_{(\pm)}^{(\pm)}(F) = \begin{pmatrix}
  -i \mathbf{c}(\sigma_a^{(1)})1_N & 0 & 0 & 0 \\
  0 & -i \mathbf{c}(\sigma_a^{(1)})1_N & 0 & \sigma_{ab}^{(0)} \gamma_3 A^+ \\
  0 & 0 & -i \mathbf{c}(\sigma_b^{(1)})1_N & 0 \\
  0 & \sigma_{ba}^{(0)} \gamma_3 A & 0 & -i \mathbf{c}(\sigma_b^{(1)})1_N
  \end{pmatrix},
\]

and in case 2.b):
\[
\pi_{(\pm)}^{(\pm)}(F) = \begin{pmatrix}
  -i \mathbf{c}(\sigma_a^{(1)})1_N & 0 & \sigma_{ab}^{(0)} \gamma_3 A^+ & 0 \\
  0 & -i \mathbf{c}(\sigma_b^{(1)})1_N & 0 & 0 \\
  \sigma_{ba}^{(0)} \gamma_3 A & 0 & -i \mathbf{c}(\sigma_b^{(1)})1_N & 0 \\
  0 & 0 & 0 & -i \mathbf{c}(\sigma_b^{(1)})1_N
  \end{pmatrix},
\]

43
where the differential forms $\sigma^{(k)}$’s are given in (4.11).

Introducing the $2N \times 2N$ matrix in case 2.a) as $M_{(2,a)} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and in case 2.b) as $M_{2b}$, we may also write:

$$\pi(\pm)(F) = \begin{pmatrix} -i c(\sigma_{a}^{(1)})_{12N} & \sigma_{ba}^{(0)} \gamma_{3}M + i c(\sigma_{b}^{(1)})_{12N} \\ \sigma_{ba}^{(0)} \gamma_{3}M & -i c(\sigma_{b}^{(1)})_{12N} \end{pmatrix}. \quad (5.43)$$

A universal two-form $G \in \Omega^{(2)}(A)$ has representations $\pi_{(\pm)}(G)$ given by a similar expression as in (4.12). The unwanted differential ideal $J$ is removed using the orthonality condition analogous to (4.19)

$$\rho_{aaa}^{(0)} - \frac{1}{2N}\rho_{aba} \text{ tr}\{M^{+}M\} = 0 \quad ; \quad \rho_{bbb}^{(0)} - \frac{1}{2N}\rho_{bab} \text{ tr}\{MM^{+}\} = 0. \quad (5.44)$$

The representative of $G$ in $\Omega^{(2)}_{D}(A)$, as in (4.20), is given by :

$$\pi_{D}(\pm)(G) = \begin{pmatrix} \pi_{D}(\pm)(G)_{[aa]} & \pi_{D}(\pm)(G)_{[ab]} \\ \pi_{D}(\pm)(G)_{[ba]} & \pi_{D}(\pm)(G)_{[bb]} \end{pmatrix}, \quad (5.45)$$

with

$$\pi_{D}(\pm)(G)_{[aa]} = -c(\rho_{aaa}^{(2)}) \otimes 1_{2N} + \rho_{aba}^{(0)} \otimes \left[ M^{+}M \right]_{NT},$$

$$\pi_{D}(\pm)(G)_{[ab]} = -i c(\rho_{ab}^{(1)}) \gamma_{3}(\pm) \otimes M^{+},$$

$$\pi_{D}(\pm)(G)_{[ba]} = -i c(\rho_{ba}^{(1)}) \gamma_{3}(\pm) \otimes M,$$

$$\pi_{D}(\pm)(G)_{[bb]} = -c(\rho_{bbb}^{(2)}) \otimes 1_{2N} + \rho_{bab}^{(0)} \otimes \left[ MM^{+} \right]_{NT},$$

where the differential forms $\rho^{(k)}$ are given in (1.13) and

$$\left[ M^{+}M \right]_{NT} = M^{+}M - \frac{1}{2N}\text{ tr}\{M^{+}M\},$$

$$\left[ MM^{+} \right]_{NT} = MM^{+} - \frac{1}{2N}\text{ tr}\{MM^{+}\}.$$

The scalar product in $\Omega^{(2)}_{D}(A)$ is the same for representatives in $\mathcal{H}_{(+)}$ or $\mathcal{H}_{(-)}$ and is given by a similar expression as (4.21) :

$$\langle \pi_{D}(\pm)(G); \pi_{D}(\pm)(G') \rangle_{2,D} =$$
\[
\frac{1}{2\pi} \left\{ 2N \int_{S^2} \left( \rho^{(2)*}_{aaa} \wedge * \rho^{(2)}_{aaa} + \rho^{(2)*}_{bbb} \wedge * \rho^{(2)}_{bbb} \right) \\
+ \text{tr}\{A^+ A\} \int_{S^2} \left( \rho^{(1)*}_{ab} \wedge * \rho^{(1)}_{ab} + \rho^{(1)*}_{ba} \wedge * \rho^{(1)}_{ba} \right) \\
+ \left[ \text{tr}\{(A^+ A)^2\} - \frac{1}{2N} (\text{tr}\{A^+ A\})^2 \right] \int_{S^2} \left( \rho^{(0)*}_{aba} \wedge * \rho^{(0)}_{aba} + \rho^{(0)*}_{bab} \wedge * \rho^{(0)}_{bab} \right) \right\}.
\]

(5.46)

From this expression of the scalar product, the Yang-Mills-Higgs action is essentially twice the action (4.28) obtained in section 4.1:

\[
S_{YMH}(\nabla_D) = \frac{\lambda}{\pi} \left\{ 2N \int_{S^2} \left( \text{tr}_{\text{matrix}} \left\{ \langle (F_a)^+ \wedge * (F_a) \rangle + F_b^* \wedge * F_b \right\} \\
+ 2 \text{tr}\{A^+ A\} \int_{S^2} \langle \nabla \eta_{ba} \rangle \wedge * \langle \nabla \eta_{ab} \rangle \\
+ \left[ \text{tr}\{(A^+ A)^2\} - \frac{1}{2N} (\text{tr}\{A^+ A\})^2 \right] \int_{S^2} \left( 2(\langle \eta_{ba}, \eta_{ab} \rangle - 1)^2 + 1 \right) \right\}.
\]

(5.47)

5.5 The "Real" covariant Dirac operator

Matter in this "real spectral triple" approach is represented by states of the covariant Hilbert space \( \mathcal{H}_{\text{cov}} = \mathcal{M} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{M}^* \). For the \( S^0 \)-real spectral triple \( \mathcal{H} = \mathcal{H}_{(+)} \oplus \mathcal{H}_{(-)} \) so that also \( \mathcal{H}_{\text{cov}} \) splits in a sum of “particle” and “antiparticle” Hilbert spaces \( \mathcal{H}_{\text{cov}} = \mathcal{H}_{(\pm p)} \oplus \mathcal{H}_{(\mp p)} \) where each \( \mathcal{H}_{(\pm p)} = \mathcal{M} \otimes_{\mathcal{A}} \mathcal{H}_{(\pm)} \otimes_{\mathcal{A}} \mathcal{M}^* \) has typical elements \( \| \Psi_{(\pm p)} \| \). The projective module \( \mathcal{M} = \mathcal{P}_A^2 \) and its dual \( \mathcal{M}^* \) were examined in section 3 and in appendix A. In bases \( \{ E_i \} \) and \( \{ E^j \} \) of the free modules \( A^2 \) and \( A^{2*} \), we may represent a state of \( A^2 \otimes_{\mathcal{A}} \mathcal{H}_{(\pm)} \otimes_{\mathcal{A}} A^{2*} \) as \( E_i \otimes_{\mathcal{A}} \Psi_{(\pm)}^i \otimes_{\mathcal{A}} E^j \). It is a state \( \| \Psi_{(\pm p)} \| \) of \( \mathcal{H}_{(\pm p)} \) if \( \Psi_{(\pm)}^i \in \mathcal{H}_{(\pm)} \) obeys

\[
\Psi_{(\pm)}^i = \pi_{(\pm)}(P^i_k) \pi_{(\pm)}^k (P^j_j) \Psi_{(\pm)}^k.
\]

(5.48)

\(^{20}\)In this section we consider case 2.a) only. At the end the final result for the covariant Dirac operator in case 2.b) will also be given.

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Since \( P_{k,a}(x) = \delta_k^i \) and \( P_{k,b}(x) = |\nu(x)|^2 \langle \nu(x) |_k \) (cf. section 3) the vector \( \Psi^{i}_j \in \mathcal{H}(\pm) \) is represented by the column vector

\[
\Psi^{i}_j = \begin{pmatrix}
\langle (\psi^{(\pm)ab})_{\alpha} \rangle_j^i \\
|\psi^{(\pm)ab} \rangle^i \langle \nu |_j \\
|\nu \rangle^i \langle \psi^{(\pm)ba} |_j \\
|\nu \rangle^i \langle \psi^{(\pm)bb} |_j \\
\end{pmatrix},
\]

where \( \langle (\psi^{(\pm)aa}) \rangle \) is a quadruplet and \( \psi^{(\pm)\bar{b}} = \langle \nu | (\psi^{(\pm)\bar{b}}) | \nu \rangle \) a singlet of Pensov spinor fields of spin weight \( \pm s \), while \( |\psi^{(\pm)ab} \rangle = \langle (\psi^{(\pm)ab}) | \nu \rangle \), respectively \( \langle \psi^{(\pm)ba} \rangle = \langle (\psi^{(\pm)ba}) | \nu \rangle \), are doublets of Pensov spinors of weight \( \pm s - n/2 \), respectively \( \pm s + n/2 \).

The covariant real structure \( J_{Cov} \), as defined in appendix 3, acts on \( \langle \Psi^{i}_j \rangle \) as

\[
(J_{Cov} \langle \Psi^{i}_j \rangle )^{i}_j = \chi' \begin{pmatrix}
\delta^{i\ell} \langle C_{1} \mathcal{K} (\psi^{(\pm)aa})_{\ell}^k \rangle \delta_{k\bar{j}} \\
\delta^{i\ell} \langle C_{1} \mathcal{K} (\psi^{(\pm)ba})_{\ell}^k \rangle \langle \nu |_j \\
|\nu \rangle^i \langle C_{1} \mathcal{K} (\psi^{(\pm)ab})_{\ell}^k \rangle \delta_{k\bar{j}} \\
-|\nu \rangle^i \langle C_{1} \mathcal{K} (\psi^{(\pm)bb})_{\ell}^k \rangle \langle \nu |_j \\
\end{pmatrix}.
\]

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The covariant Dirac operator, defined in (B.10), is also block diagonal:

\[ D = D_0 \oplus D_{-0} \]

and is given by:

\[
D_{\pm 0} \left\{ (E_k P^i_k) \otimes A \psi^{\pm \ell}_k \otimes A (P^j_\ell E_j) \right\} = \\
E_k \otimes A \pi_{\pm 0} \left( (A)^{i}_k \right) \psi^{\pm \ell}_k \otimes A E^j \\\n+ E_k \otimes A \pi_{\pm 0} (P^i_k \pi_{\pm 0} (P^j_\ell) D_{\pm 0} \psi^{\pm \ell}_k \otimes A E^j \\\n+ \epsilon E_k \otimes A \pi_{\pm 0} \left( (A)^{i}_j \right) \psi^{\pm \ell}_k \otimes A E^j ,
\]

(5.51)

where \((A)\) is the \(2 \times 2\) matrix of universal one-forms given in (B.6). It is represented by the matrix valued differential one- and zero-forms given in terms of \((\alpha_a), \alpha_b, |\Phi_{ab}\rangle\) and \(\langle \Phi_{ba}|\), defined in (4.22) by:

\[
\begin{align*}
\langle A_a \rangle &= -i \gamma^r \langle \alpha_{a,r} \rangle , \\
\langle A_b \rangle &= -i \gamma^r \alpha_{b,r} |\nu\rangle \langle \nu| , \\
\langle A_{ab} \rangle &= |\Phi_{ab}\rangle |\nu\rangle \gamma_3 , \\
\langle A_{ba} \rangle &= |\nu\rangle \langle \Phi_{ba}| \gamma_3 .
\end{align*}
\]

(5.52)

We obtain:

\[
\pi_{\pm 0}(\langle A \rangle) = \begin{pmatrix}
\langle A_a \rangle 1_N & 0 & 0 & 0 \\
0 & \langle A_b \rangle 1_N & 0 & (\langle A_{ab} \rangle) A^+ \\
0 & 0 & \langle A_b \rangle 1_N & 0 \\
0 & \langle A_{ba} \rangle A & 0 & \langle A_b \rangle 1_N \\
\end{pmatrix} .
\]

(5.53)

The \(\pi_{\pm 0}^0\) representative of \(\langle A \rangle\) is computed as:

\[
\epsilon \pi_{\pm 0}^0(\langle A \rangle) = \begin{pmatrix}
\langle A_a \rangle 1_N & 0 & 0 & 0 \\
0 & \langle A_b \rangle 1_N & 0 & 0 \\
0 & 0 & \langle A_a \rangle 1_N & -\langle A_{ab} \rangle B^+ \\
0 & 0 & -\langle A_{ab} \rangle B & \langle A_b \rangle 1_N \\
\end{pmatrix} .
\]

(5.54)
Substituting \((5.53)\) and \((5.54)\) in \((5.51)\), we obtain:

\[
\langle D\nabla \Psi_{Cov}(\pm)aa \rangle = D_{1(\pm)}(\psi(\pm)aa) - i \gamma^r (\alpha_{a,r}) \langle (\pm a)aa \rangle ,
\]

\[
\langle D\nabla \Psi_{Cov}(\pm)ab \rangle = D_{1(\pm)}(\psi(\pm)ab) \pm |\eta_{ab}| \gamma_3 A^+ \psi(\pm)bb
\]

\[
-i \gamma^r (\alpha_{a,r}) \langle (\pm a)ab \rangle - |\psi(\pm)ab| \alpha_{b,r} \rangle ;
\]

\[
\langle D\nabla \Psi_{Cov}(\pm)ba \rangle = D_{1(\pm)}(\psi(\pm)ba) + \gamma_3 A \langle \eta_{ba} \psi(\pm)bb \rangle
\]

\[
-i \gamma^r (\alpha_{b,r} \psi(\pm)ba - |\psi(\pm)ba| (\alpha_{a,r})) \rangle ;
\]

\[
\langle D\nabla \Psi_{Cov}(\pm)bb \rangle = D_{1(\pm)}(\psi(\pm)bb) + \gamma_3 A \langle \eta_{ba} \psi(\pm)ab \rangle + \gamma_3 B \langle \psi(\pm)ba \rangle .
\]

In case 2.b) a similar result is obtained:

\[
\langle D\nabla \Psi_{Cov}(\pm)aa \rangle = D_{1(\pm)}(\psi(\pm)aa) - i \gamma^r (\alpha_{a,r}) \langle (\pm a)aa \rangle
\]

\[
+ \gamma_3 A^+ |\eta_{ab}| \langle \psi(\pm)ab \rangle + \gamma_3 B^+ |\psi(\pm)ab| \langle \eta_{ba} \rangle ;
\]

\[
\langle D\nabla \Psi_{Cov}(\pm)ab \rangle = D_{1(\pm)}(\psi(\pm)ab) + \gamma_3 B \langle (\pm a)aa \rangle |\eta_{ab} \rangle
\]

\[
-i \gamma^r (\alpha_{a,r}) \langle (\pm a)ab \rangle - |\psi(\pm)ab| \alpha_{b,r} \rangle ;
\]

\[
\langle D\nabla \Psi_{Cov}(\pm)ba \rangle = D_{1(\pm)}(\psi(\pm)ba) + \gamma_3 A \langle \eta_{ba} \psi(\pm)aa \rangle
\]

\[
-i \gamma^r (\alpha_{b,r} \psi(\pm)ba - |\psi(\pm)ba| (\alpha_{a,r})) \rangle ;
\]

\[
\langle D\nabla \Psi_{Cov}(\pm)bb \rangle = D_{1(\pm)}(\psi(\pm)bb) .
\]

The difference is that here the Higgs field interact with the quadruplet \((\psi(\pm)aa)\), while in case 2.a) it interacts with the singlet \(\psi(\pm)bb\) and it is this interaction that gives masses to the particles.

The Dirac operators, \(D_{1(\pm)}^{(-n/2)} = D_{1(\pm)}^{(+n/2)}\) and \(D_{1(\pm)}^{(-n/2)} = D_{1(\pm)}^{(+n/2)}\), acting on Penso spinor fields of spin weight \(\pm s + n/2\) or \(\pm s - n/2\), arise from

\[
\langle \psi(\pm)ab \rangle |\nu \rangle = D_{1(\pm)}^{(-n/2)} |\psi(\pm)ab \rangle ,
\]

\[
\langle \psi(\pm)ba \rangle |\nu \rangle = D_{1(\pm)}^{(+n/2)} |\psi(\pm)ba \rangle ,
\]

where \((P_b) = |\nu \rangle \langle \nu |\) is the representative, chosen in \((3.3)\), of the homotopy class \([n] \in \pi_2(S^2)\). The induced contribution of the “magnetic monopole” is
hidden in this modification of the Dirac operator.
The Higgs doublets of Pensov fields of weight \( \mp n/2 \), \(|\eta_{ab}\rangle\) and \( \langle\eta_{ba}|\) were defined in (1.24).

A suitable action of the matter field would be

\[
S_{\text{Mat}}(\|\Psi_{\text{Cov}}\rangle, \nabla D) = \left(\|\Psi_{\text{Cov}}\rangle ; D\nabla\|\Psi_{\text{Cov}}\rangle\right).
\]

(5.57)

But, if we aim for a theory admitting only chiral matter, i.e. if we restrict the Hilbert space to those vectors obeying say

\[
\Gamma\|\Psi_{\text{Cov}}\rangle = +\|\Psi_{\text{Cov}}\rangle,
\]

(5.58)

then the above action vanishes identically. Another choice for the action would be

\[
S_{\text{Chiral}}(\|\Psi_{\text{Cov}}\rangle, \nabla D) = \left(J_{\text{Cov}}\|\Psi_{\text{Cov}}\rangle ; D\nabla\|\Psi_{\text{Cov}}\rangle\right).
\]

(5.59)

It is easy to show that this action does not vanish identically if

\[
\epsilon'' = -1,
\]

(5.60)

\[
\epsilon' = +1.
\]

(5.61)

In two dimensions Connes’ sign table obeys the first but not the second condition\(^{21}\). It should however be stressed that Connes’ sign table, with its modulo eight periodicity, comes from representation theory of the real Clifford algebras and if we restrict our (generalized) spinors to Weyl spinors, we lose the Clifford algebra representation and the sign tables cease to be mandatory. We could then go back to (5.3) and

- with \( J = J_1 \otimes J_2 \) require (5.32) to hold with \( \epsilon'_1 = -1\epsilon_1 \) or,

- with \( J = J_1 \otimes (J_2\chi_2) \) require (5.34) to hold with \( \epsilon'_1 = +1 = \epsilon_1 \).

\(^{21}\)In four dimensions it is the first condition that is not satisfied.
6 Conclusions and Outlook

The Connes-Lott model over the two-sphere, with $\mathbb{C} \oplus \mathbb{C}$ as discrete algebra, has been generalized such as to allow for a nontrivial topological structure. The basic Hilbert space $H_1(s)$ was made of Pensov spinors which can be interpreted as usual spinors interacting with a Dirac monopole "inside" the sphere. Covariantisation of the Hilbert space $H_1(s) \otimes H_{dis}$ with a nontrivial projective module $\mathcal{M}$ induced a "spin" change in certain matter fields so that we obtained singlets and doublets of different spin content. The Higgs fields also acquired a nontrivial topology since they are no longer ordinary functions on the sphere, but rather Pensov scalars i.e. sections of nontrivial line bundles over the sphere.

A real spectral triple has also been constructed essentially through the doubling of the Pensov spinors so that the Hilbert space of the continuum spectral triple became $\mathcal{H}_1 = \mathcal{H}_{(s)} \oplus \mathcal{H}_{(-s)}$. The discrete spectral triple had also to be extended in order that the first order condition could be met. In contrast with the standard noncommutative geometry model of the standard model, in our model the continuum spectral triple has an $S^0$-real structure while the discrete spectral triple has not. Some physical plausibility arguments for this were given in section 5.1. It was also shown that the covariantisation of the real spectral triple with the nontrivial $\mathcal{M}$ allows the abelian gauge fields to survive, while they are slain if covariantisation is done with a trivial module. Finally a possibility of solution to the the problem of a non vanishing action of chiral matter has been indicated, paying the price of using a complex action.

If we address the quantisation problem in a path integral formalism, let us first recall that we have Higgs fields which are Pensov scalars of a $\mathcal{P}^{(s_1)}$ and matter fields Pensov spinors of type $\mathcal{H}_{(s_2)}$. It is thus tempting to assume that the Higgs fields should be even Grassmann variables if $s_1$ is integer valued and odd Grassmann variables if it is half-integer. In the same vein the matter fields should be odd or even Grassmann variables if $s_2$ is integer or half integer valued. A thorough examination of this issue is however beyond the scope of this work.
A On modules and connections

Let $\mathcal{M}$ be a right module over the $\star$-algebra $\mathcal{A}$. The set of endomorphisms $\mathcal{B} = \text{END}_{\mathcal{A}}(\mathcal{M})$ has an obvious algebra structure and the left action of $\mathcal{A} \in \mathcal{B}$ on $X \in \mathcal{M}$ commutes with the right action of $a \in \mathcal{A}$ : $(AX)a = A(Xa)$ so that $\mathcal{M}$ acquires a canonical $\mathcal{B} - \mathcal{A}$ bimodule structure. The dual module $\mathcal{M}^* = \text{HOM}_{\mathcal{A}}(\mathcal{M}, \mathcal{A})$ is a left $\mathcal{A}$-module and $\mathcal{A} \in \mathcal{B}$ acts on the right on $\xi \in \mathcal{M}^*$ by $(\xi A, X) = (\xi, AX)$ so that $\mathcal{M}^*$ is a $\mathcal{A} - \mathcal{B}$ bimodule.

When $\mathcal{M}$ is endowed with a sesquilinear, hermitian and non-degenerate form $h : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$ : $(X, Y) \rightarrow h(X, Y)$, there is a canonical bijective mapping $h^\# : \mathcal{M} \rightarrow \mathcal{M}^*$ : $X \rightarrow h^\#(X)$,

$$h^\#: \mathcal{M} \rightarrow \mathcal{M}^*: X \rightarrow h^\#(X) + \left( h^\#(X), Y \right) = h(X, Y) \ . \quad (A.1)$$

Since $h^\#(Xa) = a^*h^\#(X)$, $h^\#$ is an anti-isomorphism with inverse

$$(h^\#)^{-1} = h^\#: \xi \rightarrow h^\#(\xi) \ ,$$

and $h^\#(a\xi) = h^\#(\xi)a^*$. The inverse form $h^{-1}$ is defined by :

$$h^{-1} : \mathcal{M}^* \times \mathcal{M}^* \rightarrow \mathcal{A} : (\xi, \eta) \rightarrow h^{-1}(\xi, \eta) = h\left(h^\#(\xi), h^\#(\eta)\right) \ .$$

The hermitian conjugate of $\mathcal{A} \in \mathcal{B}$ is defined by $h(X, A^*Y) = h(AX, Y)$.

Let $\Omega^\star(\mathcal{A}) = \bigoplus_{k \in \mathbb{Z}} \Omega^k(\mathcal{A})$ be a graded differential $\star$-envelope of $\mathcal{A}$, then $h$ can be extended to $\mathcal{M}^\star = \mathcal{M} \otimes_{\mathcal{A}} \Omega^\star(\mathcal{A})$ by :

$$h(X \otimes_A F, Y \otimes_A G) = F^*h(X, Y)G \ , \quad (A.2)$$

where $X, Y \in \mathcal{M}$ and $F, G \in \Omega^\star(\mathcal{A})$.

Defining $\mathcal{M}^{\star\star} = \Omega^\star(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}^*$, $h^\#$ can be extended as a mapping $\mathcal{M}^* \rightarrow \mathcal{M}^{\star\star}$ by $(X \otimes_A F)^\dagger = F^+ \otimes_A X^\dagger$, where $h^\#(\ )$ is written as $(\ )^\dagger$.

A connection $\nabla$ in $\mathcal{M}$ is an additive map

$$\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathcal{A}} \Omega^{(1)}(\mathcal{A}) : X \rightarrow \nabla X \ , \quad (A.3)$$

which is additive and obeys the Leibniz rule

$$\nabla(Xa) = (\nabla X)a + X \otimes_A da \ .$$

It defines an associate dual connection $\nabla^\star$ in $\mathcal{M}^*$ by :

$$(\nabla^\star \xi, X) + (\xi, \nabla X) = d (\xi, X) \ . \quad (A.4)$$
It obeys $\nabla^* (a \xi) = da \otimes_A \xi + a(\nabla^* \xi)$.

The extension of $\nabla$ to $\mathcal{M}^\bullet$ by $\nabla (X \otimes_A F) = (\nabla X) F + X \otimes_A dF$, allows to define the curvature as $\nabla^2 : \mathcal{M}^\bullet \rightarrow \mathcal{M}^\bullet$ and it is seen that $\nabla^2$ is in fact an endomorphism of $\mathcal{M}^\bullet$ considered as a right $\Omega^\bullet$-module:

$$\nabla^2 \left( (X \otimes_A F) G \right) = \left( \nabla^2 (X \otimes_A F) \right) G . \quad (A.5)$$

Similarly, $\nabla^*$ is extended to $\mathcal{M}^{\bullet\bullet}$ and

$$\left( \nabla^{*2} (G \otimes_A \xi), X \otimes_A F \right) = \left( G \otimes_A \xi, \nabla^2 (X \otimes_A F) \right) .$$

When $\mathcal{M}$ has an hermitian structure, the connection is said to be compatible with this hermitian structure, if

$$d \left( h(X, Y) \right) = -h(\nabla X, Y) + h(X, \nabla Y) \quad (A.6)$$

or equivalently if

$$d \left( h^{-1}(\xi, \eta) \right) = h^{-1}(\nabla^* \xi, \eta) - h^{-1}(\xi, \nabla^* \eta) .$$

The mapping $h^\sharp$ relates both connections through

$$\nabla^* h^\sharp(X) = -h^\sharp(\nabla X) \text{ or } \nabla^* X^\dagger = - (\nabla X)^\dagger . \quad (A.7)$$

The curvature of a compatible connection is hermitian:

$$h(\nabla^2 X, Y) = h(X, \nabla^2 Y) . \quad (A.8)$$

The algebras $\mathcal{A}$ and $\mathcal{B} = END_A(\mathcal{M})$ are said to be Morita equivalent in the sense that there exists a $B - A$ bimodule $\mathcal{M}$ and a $A - B$ bimodule $\mathcal{M}^*$ such that $\mathcal{M} \otimes_A \mathcal{M}^* \simeq \mathcal{B}$, with the identification $(X \otimes_A \xi) Y = X (\xi, Y)$ and $\eta (X \otimes_A \xi) = (\eta, X) \xi$, and $\mathcal{M}^* \otimes_B \mathcal{M} \simeq \mathcal{A}$ with $Y (\xi \otimes_B X) = Y (\xi, X)$ and $(\xi \otimes_B X) \eta = (\xi, X) \eta$.

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22Here we have chosen $d(a^*) = -(da)^*$.
B  Real Covariant Spectral Triples

Let \{A, H, D, \chi, J\} be a real spectral triple with a faithful \(*\)-representation in the bounded operators of \(H\), \(\pi : A \to \mathcal{B}(H) : a \to \pi(a)\). The Dirac operator \(D\) allows to extend this \(*\)-representation to \(\Omega^\bullet(A)\) by

\[
\pi : A \to \mathcal{B}(H) : a \to \pi(a_0)[D, \pi(a_1)] \cdots [D, \pi(a_k)].
\] (B.1)

The real structure \(J\) defines a right action of \(A\) on \(H\) or equivalently a representation of the opposite algebra \(A^0\) by

\[
\pi^o : A^0 \to \mathcal{B}(H) : a \to \pi^o(a) = J(\pi(a))^+ J^+.
\] (B.2)

It is assumed that \([\pi(a), \pi^o(b)] = 0\) so that the Hilbert space \(H\) is provided with an \(A\)-bimodule structure and the tensor product

\[
\mathcal{H}_{Cov} = \mathcal{M} \otimes_A H \otimes_A \mathcal{M}^*
\] (B.3)
is well defined. The vectors \(\langle \cdot \rangle\) of \(\mathcal{H}_{Cov}\) are generated by elements of the form \(X \otimes_A |\Psi\rangle \otimes_A \eta\). A scalar product \(\langle \cdot \rangle \cdot \langle \cdot \rangle\) in \(\mathcal{H}_{Cov}\) is defined in terms of the scalar product \(\langle \cdot ; \cdot \rangle\) in \(H\), by

\[
\left( X \otimes_A |\psi\rangle \otimes_A \eta \big| R \otimes_A |\phi\rangle \otimes_A \sigma \right) = \left( |\psi\rangle ; \pi(h(X, R))\pi^o(h^{-1}(\sigma, \eta))|\phi\rangle \right),
\] (B.4)

where \(X, R \in \mathcal{M}, \eta, \sigma \in \mathcal{M}^*,\) and \(|\psi\rangle, |\phi\rangle \in H\).

It is well defined and, as usual, completion of \(\mathcal{H}_{Cov}\) defines the covariant Hilbert space also denoted by \(\mathcal{H}_{Cov}\).

The representation \(\pi\) defines a mapping \(\tilde{\pi}\) from \(\mathcal{M}^*\) to the bounded operators \(\mathcal{B}(\mathcal{H}, \mathcal{M} \otimes_A \mathcal{H})\) by:

\[
\tilde{\pi}(X \otimes_A F) |\psi\rangle = X \otimes_A \pi(F) |\psi\rangle.
\] (B.5)

\footnote{This definition reads \(\pi^o(a_0 \cdots a_k) = [\pi^o(a_k), D'] \cdots [\pi^o(a_1), D'] \pi^o(a_0)\), where \(D' = JDJ^+ = \epsilon D\).}

\footnote{Elements of \(\mathcal{H}_{Cov}\) could also be represented by \(X \otimes_A |\Psi\rangle \otimes_A Y^\dagger\), if we choose to represent \(\eta \in \mathcal{M}^*\) as \(Y^\dagger\).}
It obeys \( \tilde{\pi}(X \otimes_A FG) = \tilde{\pi}(X \otimes_A F) \circ \pi(G) \).

In the same way, \( \pi^o \) defines a mapping \( \tilde{\pi}^o \) from \( \mathcal{M}^{**} \) to the bounded operators \( \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes_A \mathcal{M}^*) \) by:

\[
\tilde{\pi}^o(G \otimes_A \eta) |\phi\rangle = \pi^o(G) |\phi\rangle \otimes_A \eta, \tag{B.6}
\]

and \( \tilde{\pi}^o(FG \otimes_A \eta) = \tilde{\pi}^o(G \otimes_A \eta) \circ \pi^o(F) \).

It is further assumed that \( \mathcal{D} \) is a first-order operator, i.e. \( \forall a, b \in \mathcal{A}, \)

\[
\left[ [\mathcal{D}, \pi(a)], \pi^o(b) \right] = 0. \tag{B.7}
\]

Since \( [\pi(a), \pi^o(b)] = 0 \), it follows also that \( [ [\mathcal{D}, \pi^o(b)], \pi(a) ] = 0 \) and more generally that:

\[
[\pi(F), \pi^o(b)] = 0; \quad [\pi^o(G), \pi(a)] = 0. \tag{B.8}
\]

It follows that

\[
\tilde{\pi}(x \otimes_A F)|\psi\rangle \otimes_A b\eta = \tilde{\pi}(X \otimes_A F)\left(\pi^o(b)|\psi\rangle\right) \otimes_A \eta
\]

\[
(Xa) \otimes_A \tilde{\pi}^o(G \otimes_A \eta)|\psi\rangle = X \otimes_A \tilde{\pi}^o(G \otimes \eta)\left(\pi(a)|\psi\rangle\right). \tag{B.9}
\]

The covariant Dirac operator in \( \mathcal{H}_{Cov} \) is defined by:

\[
\mathcal{D}_\nabla \left(X \otimes_A |\psi\rangle \otimes_A \eta\right) = \tilde{\pi}(\nabla X)|\psi\rangle \otimes_A \eta + X \otimes_A \mathcal{D}|\psi\rangle \otimes_A \eta - \epsilon' X \otimes_A \tilde{\pi}^o(\nabla^* \eta)|\psi\rangle. \tag{B.10}
\]

As can be checked using (B.8) and (B.9), this operator is well defined on the tensor product over \( \mathcal{A} \) defining \( \mathcal{H}_{Cov} \).

The chirality in \( \mathcal{H}_{Cov} \) is defined as:

\[
\Gamma(X \otimes_A |\psi\rangle \otimes_A \eta) = X \otimes_A \chi|\psi\rangle \otimes_A \eta. \tag{B.11}
\]

Since \( \chi \) commutes with \( \pi(a) \) and \( \pi^o(a) \) and anticommutes with \( \mathcal{D} \), it follows that

\[
\Gamma \pi(F) = \pi(\alpha(F)) \Gamma; \quad \Gamma \pi^o(F) = \pi^o(\alpha(F)) \Gamma,
\]

\(^{25}\)The \( \epsilon' \) sign appears due to our definition (B.3).
where $\alpha$ is the main automorphism in the graded algebra $\Omega^\bullet(A)$. It is then easy to show that
\[
\Gamma D_{\nabla} + D_{\nabla} \Gamma = 0 .
\] (B.12)

The real structure in $\mathcal{H}_{Cov}$ is defined by:
\[
J_h \left( X \otimes_A |\psi\rangle \otimes_A \eta \right) = h^\flat(\eta) \otimes_A J|\psi\rangle \otimes_A h^\sharp(X) .
\] (B.13)

Obviously, if $J^2 = \epsilon 1$, then also $J_h^2 = \epsilon 1$ and, if $J \chi = \epsilon'' \chi J$, also $J_h \Gamma = \epsilon'' \Gamma J_h$ holds.

Using
\[
J_h \left( X \otimes_A \tilde{\pi}(\nabla^* \eta)|\psi\rangle \right) = -\tilde{\pi}(\nabla h^\flat(\eta))(J|\psi\rangle) \otimes_A h^\sharp(X) ,
\]
\[
J_h \left( \tilde{\pi}(\nabla X)|\psi\rangle \otimes_A \eta \right) = -h^\flat(\eta) \otimes_A \tilde{\pi}^\circ(h^\sharp(\nabla X))|\psi\rangle ,
\]
it is seen that $JD = \epsilon'DJ$ implies
\[
J_h D_{\nabla} = \epsilon' D_{\nabla} J_h .
\] (B.14)

Just as $\mathcal{H}$ is a Hilbert $A$-bimodule, $\mathcal{H}_{Cov}$ is a Hilbert $B$-bimodule with the action of $A, B \in B$ given by:
\[
X \otimes_A |\psi\rangle \otimes_A \eta \to (AX) \otimes_A |\psi\rangle \otimes_A (\eta B) .
\]

The spectral data $\{A, \mathcal{H}, D, \chi, J\}$, also called (non-commutative) geometry in [21], is said to be Morita equivalent to the geometry defined by $\{B, \mathcal{H}_{Cov}, D_{\nabla}, \Gamma, J_h\}$.

Not only $\mathcal{H}_{Cov} = \mathcal{M} \otimes_A \mathcal{H} \otimes_A \mathcal{M}^*$, but also $\mathcal{H} = \mathcal{M}^* \otimes_B \mathcal{H}_{Cov} \otimes_B \mathcal{M}$!
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