A PARAMETER UNIFORM FITTED MESH METHOD FOR A
WEAKLY COUPLED SYSTEM OF TWO SINGULARLY
PERTURBED CONVECTION-DIFFUSION EQUATIONS

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Abstract. In this paper, a boundary value problem for a singularly perturbed
linear system of two second order ordinary differential equations of convection-
diffusion type is considered on the interval [0, 1]. The components of the
solution of this system exhibit boundary layers at 0. A numerical method
composed of an upwind finite difference scheme applied on a piecewise uniform
Shishkin mesh is suggested to solve the problem. The method is proved to be
first order convergent in the maximum norm uniformly in the perturbation
parameters. Numerical examples are provided in support of the theory.

1. Introduction

Singular perturbation problems of convection-diffusion type arise in many areas
of applied mathematics such as fluid dynamics, chemical reactor theory, etc. Also,
linearising Navier-Stokes equations, which plays vital role in the field of science,
leads to a system of convection-diffusion equations.

For a broad introduction to singularly perturbed boundary value problems of
convection-diffusion type one can refer to [1], [2] and [3]. There, the authors sug-
gest robust computational techniques to solve them. A class of systems of singularly
perturbed reaction-diffusion equations has been examined by several authors in [4],
[5], [6] and [7].

Here, in this paper, a weakly coupled system of two singularly perturbed con-
vection - diffusion equations with distinct perturbation parameters is studied both
analytically and numerically. If the perturbation parameters are equal, then the
arguments in [3] are sufficient to show that the suggested method is parameter
uniform. But in general boundary layers of unequal width are expected for the
components of the solution because of the coupling of the components.

In the papers [8] and [9], a class of strongly coupled systems of singularly per-
turbed convection-diffusion problems is examined. A coupled system of two sin-
gularly perturbed convection-diffusion equations is considered in [10]. In [11], the
author analysed a coupled system of singularly perturbed convection-diffusion equa-
tions.

In this paper, the major assumptions $\varepsilon_1 \leq C N^{-1}$, $\varepsilon_2 \leq C N^{-1}$ in [10], are
removed. Moreover the analytical and numerical arguments are completely differ-
ent from [10] and [11] in the following sense. The decomposition of the solution

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is based on the effect of each perturbation parameter on the components of the solution. Thus, we get more information about the components of the solution and its layer pattern. Also, it is to be noted that the decomposition of the smooth component in \[10\] is given a correct definition, here in this paper.

**Notations.** For any real valued function \( y \) on \( D \), the norm of \( y \) is defined as 
\[ \| y \|_D = \sup_{x \in D} |y(x)|. \]
For any vector valued function \( \vec{z}(x) = (z_1(x), z_2(x))^T \), 
\[ |\vec{z}(x)| = (|z_1(x)|, |z_2(x)|)^T, \]
and \( \| \vec{z} \|_D = \max\{\|z_1\|_D, \|z_2\|_D\} \). Also \( \vec{z}(x) \geq \vec{0} \), if \( z_1(x) \geq 0 \) and \( z_2(x) \geq 0 \).

For any mesh function \( Y \) on \( D^n = \{x_j\}_{j=0}^N \), \( \|Y\|_{D^n} = \max_{0 \leq j \leq N} |Y(x_j)| \) and for any vector valued mesh function \( \vec{Z} = (Z_1, Z_2)^T \), 
\[ |\vec{Z}(x_j)| = (|Z_1(x_j)|, |Z_2(x_j)|)^T, \]
\[ \|\vec{Z}\|_{D^n} = \max\{|Z_1\|_{D^n}, |Z_2\|_{D^n}\}. \]

Throughout this paper, \( C \) denotes a generic positive constant which is independent of the singular perturbation and discretization parameters.

### 2. Formulation of the problem

Consider the following system of equations

\begin{equation}
L\vec{u}(x) \equiv E\vec{u}''(x) + A(x)\vec{u}'(x) - B(x)\vec{u}(x) = \vec{f}(x), \quad x \in \Omega
\end{equation}

\begin{equation}
\vec{u}(0) = \vec{l}, \quad \vec{u}(1) = \vec{r},
\end{equation}

where, \( \Omega = (0, 1) \), \( \vec{u}(x) = (u_1(x), u_2(x))^T \), \( \vec{f}(x) = (f_1(x), f_2(x))^T \),
\[ E = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}, \quad A(x) = \begin{bmatrix} a_1(x) & 0 \\ 0 & a_2(x) \end{bmatrix}, \quad B(x) = \begin{bmatrix} b_{11}(x) & -b_{12}(x) \\ -b_{21}(x) & b_{22}(x) \end{bmatrix}. \]

Here, \( \varepsilon_1 \) and \( \varepsilon_2 \) are two distinct small positive parameters and, without loss of generality, we assume that \( \varepsilon_1 < \varepsilon_2 \). The coefficient functions are taken to be sufficiently smooth on \( \Omega \) and \( a_1(x) \geq \alpha > 0 \), \( b_{ij}(x) - b_{ij}(x) \geq \beta > 0 \), \( b_{ij} > 0 \), for \( i, j = 1, 2 \) and \( i \neq j \).

The case \( a_i(x) \leq \alpha < 0 \), for \( i = 1, 2 \), is put into the form \[2.1\] by the change of independent variable from \( x \) to \( 1 - x \).

Since, the matrix \( B(x) \) is not diagonal and the matrix \( A(x) \) is diagonal, the system is weakly coupled. If the matrix \( A(x) \) is not diagonal, then the system becomes strongly coupled. If \( a_1(x) \) and \( a_2(x) \) are zero functions, then the above problem comes under the class considered in \[3\].

The reduced problem corresponding to \[2.1\]–\[2.2\] is

\begin{equation}
L_0\vec{u}_0(x) \equiv A(x)\vec{u}_0'(x) - B(x)\vec{u}_0(x) = \vec{f}(x), \quad x \in \Omega
\end{equation}

\begin{equation}
\vec{u}_0(1) = \vec{r},
\end{equation}

where, \( \vec{u}_0(x) = (u_{01}(x), u_{02}(x))^T \).

A boundary layer of width \( O(\varepsilon_2) \) is expected near \( x = 0 \) in the solution components \( u_1 \) and \( u_2 \), if \( u_2(0) \neq u_{02}(0) \) and a boundary layer of width \( O(\varepsilon_1) \) is expected near \( x = 0 \) in the solution component \( u_1 \), if \( u_1(0) \neq u_{01}(0) \). Numerical illustrations provided for each case exhibit such layer patterns.
3. Analytical Results

In this section, a maximum principle, a stability result and estimates of the derivatives of the solution of the system of equations (2.1)-(2.2) are presented.

Lemma 3.1 (Maximum Principle). Let \( \vec{\psi} \in (C^2(\Omega))^2 \) such that \( \vec{\psi}(0) \geq \vec{0}, \vec{\psi}(1) \geq \vec{0} \), \( L \vec{\psi} \leq \vec{0} \) on \((0, 1)\), then \( \vec{\psi} \geq \vec{0} \) on \([0, 1]\).

*Proof.* Let \( x^* \) and \( y^* \) be such that \( \psi_1(x^*) = \min_{x \in \Omega} \psi_1(x) \) and \( \psi_2(y^*) = \min_{x \in \Omega} \psi_2(x) \). Without loss of generality, we assume that \( \psi_1(x^*) \leq \psi_2(y^*) \) and suppose \( \psi_1(x^*) < 0 \); then \( x^* \notin (0, 1) \), \( \psi_1'(x^*) = 0 \) and \( \psi_1''(x^*) \geq 0 \). (\( L \vec{\psi} \)) \( \psi_1''(x^*) \geq \varepsilon_1 \psi_1''(x^*) + a_1(x^*) \psi_1'(x^*) - (b_{11}(x^*) - b_{12}(x^*)) \psi_1(x^*) > 0 \), contradiction to the assumption that \( L \vec{\psi} \leq \vec{0} \) on \((0, 1)\). Hence, \( \vec{\psi}(x) \geq 0 \), on \([0, 1]\). \( \square \)

An immediate consequence of the maximum principle is the following stability result.

Lemma 3.2 (Stability Result). Let \( \vec{\psi} \in (C^2(\Omega))^2 \), then for \( x \in \Omega \) and \( i = 1, 2 \),

\[
|\psi_i(x)| \leq \max \left\{ \|\vec{\psi}(0)\|, \|\vec{\psi}(1)\|, \frac{1}{\beta} \|L \vec{\psi}\| \right\}.
\]

Corollary 1. Let \( \vec{u} \) be the solution of (2.1)-(2.2), then

\[
|u_i(x)| \leq \max \left\{ \|\vec{u}\|, \|\vec{f}\|, \frac{1}{\beta} \|\vec{f}\| \right\}.
\]

Theorem 3.3. Let \( \vec{u} \) be the solution of (2.1)-(2.2), then for \( x \in \Omega \) and \( i = 1, 2 \),

\[
|u_i^{(k)}(x)| \leq C \varepsilon_i^{-k} \left( \|\vec{u}\| + \varepsilon_i \|\vec{f}\| \right) \text{ for } k = 1, 2
\]

\[
|u_i^{(3)}(x)| \leq C \varepsilon_i^{-3} \left( \|\vec{u}\| + \varepsilon_i \|\vec{f}\| \right) + \varepsilon_i^{-1} \|\vec{f}\|
\]

\[
|u_i^{(3)}(x)| \leq C \varepsilon_i^{-2} \varepsilon_i^{-1} \left( \|\vec{u}\| + \varepsilon_i \|\vec{f}\| \right) + \varepsilon_i^{-1} \|\vec{f}\|
\]

*Proof.* For any \( x \in [0, 1] \), there exists \( a \in [0, 1 - \varepsilon] \) such that \( x \in N_a = [a, a + \varepsilon] \). By mean value theorem, there exists \( y_i \in (a, a + \varepsilon) \) such that

\[
u_i'(y_i) = \frac{u_i(a + \varepsilon) - u_i(a)}{\varepsilon}.
\]

and hence

\[
|u_i'(y_i)| \leq C \varepsilon_i^{-1} \|\vec{u}\|.
\]

Also,

\[
u_i'(x) = u_i'(y_i) + \int_{y_i}^{x} u_i''(s)ds
\]

Substituting for \( u_i''(s) \) from (2.1) and integrating by parts, we get

\[
|u_i'(x)| \leq C \varepsilon_i^{-1} \left( \|\vec{u}\| + \varepsilon_i \|\vec{f}\| \right).
\]

Again from (2.1),

\[
|u_i''(x)| \leq C \varepsilon_i^{-2} \left( \|\vec{u}\| + \varepsilon_i \|\vec{f}\| \right).
\]

Differentiating (2.1) once and substituting the above bounds lead to

\[
|u_i^{(3)}(x)| \leq C \varepsilon_i^{-3} \left( \|\vec{u}\| + \varepsilon_i \|\vec{f}\| \right) + \varepsilon_i^{-1} \|\vec{f}\|
\]
For \( 0 \leq p \) (3.14),
\[
\|\|\| \leq C \varepsilon_2^{-2} \varepsilon_1^{-1} \left( \|\|\| + \varepsilon_2 \|f\| \right) + \varepsilon_2^{-1} \|f'\|.
\]

3.1. **Shishkin decomposition of the solution.** The solution \( \bar{u} \) of the problem \[2.1\]-\[2.2\] can be decomposed into smooth and singular components \( \bar{u} \) and \( \bar{w} \) given by
\[
\bar{u} = \bar{v} + \bar{w}
\]
where
\[
\begin{align*}
L\bar{v} &= f, \quad \bar{v}(1) = 0, \quad \bar{v}(0) \text{ suitably chosen}, \\
L\bar{w} &= 0, \quad \bar{w}(0) = \bar{v} - \bar{v}(0), \quad \bar{w}(1) = 0
\end{align*}
\]
with \( \bar{v} = (v_1, v_2)^T \) and \( \bar{w} = (w_1, w_2)^T \).

Now, \( \bar{v} \) is decomposed into \( \bar{v} = \bar{y}_0 + \varepsilon_2 \bar{y}_1 + \varepsilon_2^2 \bar{y}_2 \), where
\[
\bar{y}_0 = (y_{01}, y_{02})^T \text{ is the solution of } \[3.6\]-\[3.8\],
\]
\[
\begin{align*}
a_1(x)\bar{y}_{01}(x) - b_{11}(x)\bar{y}_{01}(x) + b_{12}(x)\bar{y}_{02}(x) &= f_1(x) \\
a_2(x)\bar{y}_{02}(x) + b_{21}(x)\bar{y}_{01}(x) - b_{22}(x)\bar{y}_{02}(x) &= f_2(x) \\
y_{01}(1) &= r_1(1), \quad y_{02}(1) = r_2(1),
\end{align*}
\]
\( \bar{y}_1 = (y_{11}, y_{12})^T \) is the solution of \[3.9\]-\[3.11\],
\[
\begin{align*}
a_1(x)\bar{y}_{11}(x) - b_{11}(x)\bar{y}_{11}(x) + b_{12}(x)\bar{y}_{12}(x) &= -\varepsilon_1 \bar{y}_{01}''(x) \\
a_2(x)\bar{y}_{12}(x) + b_{21}(x)\bar{y}_{11}(x) - b_{22}(x)\bar{y}_{12}(x) &= -\bar{y}_{02}''(x) \\
y_{11}(1) &= 0, \quad y_{12}(1) = 0
\end{align*}
\]
\( \bar{y}_2 = (y_{21}, y_{22})^T \) is the solution of \[3.12\]-\[3.14\],
\[
\begin{align*}
\varepsilon_1 \bar{y}_{21}''(x) + a_1(x)\bar{y}_{21}(x) - b_{11}(x)\bar{y}_{21}(x) + b_{12}(x)\bar{y}_{22}(x) &= -\varepsilon_1 \bar{y}_{11}''(x) \\
\varepsilon_2 \bar{y}_{22}''(x) + a_2(x)\bar{y}_{22}(x) + b_{21}(x)\bar{y}_{21}(x) - b_{22}(x)\bar{y}_{22}(x) &= -\bar{y}_{12}''(x) \\
y_{21}(0) &= p, \quad y_{22}(0) = 0, \quad y_{21}(1) = 0, \quad y_{22}(1) = 0.
\end{align*}
\]
In \[3.14\], \( p \) is a constant to be chosen such that \( |p| \leq C \).

From \[3.6\]-\[3.11\], it is not hard to see that, for \( 0 \leq k \leq 3 \),
\[
\|\bar{y}_0^{(k)}\| \leq C, \quad \|\bar{y}_1^{(k)}\| \leq C.
\]

Now, consider the equations \[3.12\]-\[3.14\] and using Lemma 3.2
\[
\|\bar{y}_2\| \leq C.
\]
Using the estimate \[3.1\] from Theorem 3.3 we get,
\[
|\bar{y}_2^{(k)}(x)| \leq C \varepsilon_2^{-k} \text{ for } k = 1, 2
\]
From \[3.12\],
\[
\varepsilon_1 \bar{y}_{21}'' + a_1(x)\bar{y}_{21}'(x) - b_{11}(x)\bar{y}_{21}(x) = -\varepsilon_1 \bar{y}_{11}''(x) - b_{12}(x)\bar{y}_{22}(x).
\]
Decompose $y_{21}$ as $y_{21}(x) = z_0(x) + \varepsilon_1 z_1(x) + \varepsilon_1^2 z_2(x)$ with

\begin{align}
(3.19) & \quad a_1(x) z_0'(x) - b_{11}(x) z_0(x) = -\frac{\varepsilon_1}{\varepsilon_2} y_{11}'(x) - b_{12}(x) y_{22}(x), \quad z_0(1) = 0, \\
(3.20) & \quad a_1(x) z_1'(x) - b_{11}(x) z_1(x) = -z_0''(x), \quad z_1(1) = 0, \\
(3.21) & \quad \varepsilon_1 z_2''(x) + a_1(x) z_2'(x) - b_{11}(x) z_2(x) = -z_1''(x), \quad z_2(0) = 0, z_2(1) = 0.
\end{align}

Estimating $z_0$ and $z_1$ from (3.19) and (3.20) and using Chapter 8 of [1] for the problem (3.21), the following estimates hold for $0 \leq k \leq 3$,

\begin{align*}
|z_0^{(k)}| & < C(1 + \varepsilon_2^{1-k}), \quad |z_1^{(k)}| < C(1 + \varepsilon_2^{-2} \varepsilon_1^{-k}), \quad |z_2^{(k)}| < C(1 + \varepsilon_2^{-2} \varepsilon_1^{-k}).
\end{align*}

Then $p = z_0(0) + \varepsilon_1 z_1(0)$ and for $k = 0, 1, 2$,

\begin{align}
(3.22) & \quad |y_{21}^{(k)}(x)| \leq C \varepsilon_2^{-2}, \quad |y_{21}^{(3)}(x)| \leq C \varepsilon_2^{-1} \varepsilon_2^{-2}.
\end{align}

Differentiating (3.13) once and using (3.17) and (3.22),

\begin{align}
(3.23) & \quad |y_{22}^{(3)}(x)| \leq C \varepsilon_2^{-3}.
\end{align}

Hence, from (3.15) - (3.17) and (3.22) - (3.23), the estimates of the components $v_1 = y_{01} + \varepsilon_2 y_{11} + \varepsilon_2^2 y_{21}$ and $v_2 = y_{02} + \varepsilon_2 y_{12} + \varepsilon_2^2 y_{22}$ of $\vec{v}$ are as follows.

\begin{align}
(3.24) & \quad |v_1^{(k)}(x)| \leq C, \quad |v_2^{(k)}(x)| \leq C \quad \text{for} \quad 0 \leq k \leq 2, \\
(3.25) & \quad |v_1^{(3)}(x)| \leq C \varepsilon_2^{-1}, \quad |v_2^{(3)}(x)| \leq C \varepsilon_2^{-1}.
\end{align}

**Theorem 3.4.** Let $\vec{w}(x)$ be the solution of (3.4), then for $x \in \Omega$, the following estimates hold.

\begin{align}
(3.26) & \quad |w_1(x)| \leq C \exp{\frac{-\alpha x}{\varepsilon_2}}, \quad |w_2(x)| \leq C \exp{\frac{-\alpha x}{\varepsilon_2}}, \\
(3.27) & \quad |w_1^{(k)}(x)| \leq C \left( \varepsilon_2^{-k} \exp{\frac{-\alpha x}{\varepsilon_1}} + \varepsilon_2^{-k} \exp{\frac{-\alpha x}{\varepsilon_2}} \right), \quad \text{for} \quad k = 1, 2, 3, \\
(3.28) & \quad |w_2^{(k)}(x)| \leq C \varepsilon_2^{-k} \exp{\frac{-\alpha x}{\varepsilon_2}}, \quad \text{for} \quad k = 1, 2, \\
(3.29) & \quad |w_2^{(3)}(x)| \leq C \varepsilon_2^{-1} \left( \varepsilon_1^{-1} \exp{\frac{-\alpha x}{\varepsilon_1}} + \varepsilon_2^{-2} \exp{\frac{-\alpha x}{\varepsilon_2}} \right).
\end{align}

**Proof.** Estimates (3.26) - (3.28) follow from Lemma 4 of [10]. From (3.35), we have

\begin{align*}
\varepsilon_2 w_2''(x) + a_2(x) w_2'(x) + b_{21}(x) w_1(x) - b_{22}(x) w_2(x) = 0.
\end{align*}

Differentiating the above equation once,

\begin{align*}
|\varepsilon_2 w_2^{(3)}(x)| \leq C (|w_2''(x)| + |w_1'(x)|)
\end{align*}

and hence,

\begin{align*}
|w_2^{(3)}(x)| \leq C \varepsilon_2^{-1} \left( \varepsilon_1^{-1} \exp{\frac{-\alpha x}{\varepsilon_1}} + \varepsilon_2^{-2} \exp{\frac{-\alpha x}{\varepsilon_2}} \right).
\end{align*}
3.2. Improved estimates for the bounds of the singular components. Let $B_1(x)$ and $B_2(x)$ be the layer functions defined on $[0, 1]$ as follows

$$B_1(x) = \exp \frac{-\alpha x}{\varepsilon_1}, \quad B_2(x) = \exp \frac{-\alpha x}{\varepsilon_2}.$$  

Using the arguments similar to those used in Lemma 5 of [6], it is not hard to see that there exists point $x_s \in (0, \frac{1}{2})$ such that

$$B_1(x_s) = \frac{B_2(x_s)}{\varepsilon_2^s}, \quad s = 1, 2, 3$$  

and

$$B_1(x) > \frac{B_2(x)}{\varepsilon_2^s}, \text{ for } x \in [0, x_s), \quad B_1(x) < \frac{B_2(x)}{\varepsilon_2^s}, \text{ for } x \in (x_s, 1].$$  

Now the singular components $w_1(x)$ and $w_2(x)$ are decomposed as follows

$$w_1(x) = w_{11}(x) + w_{12}(x), \quad w_2(x) = w_{21}(x) + w_{22}(x),$$

where, $w_{11}, w_{12}, w_{21}$ and $w_{22}$ are defined by

$$w_{11}(x) = \begin{cases} \sum_{k=0}^{3} \frac{(x - x_3)^k}{k!} w_1^{(k)}(x_3), & \text{for } x \in [0, x_3) \\ w_1(x), & \text{for } x \in [x_3, 1] \end{cases}$$

$$w_{12}(x) = w_1(x) - w_{11}(x)$$

$$w_{21}(x) = \begin{cases} \sum_{k=0}^{3} \frac{(x - x_1)^k}{k!} w_2^{(k)}(x_1), & \text{for } x \in [0, x_1) \\ w_2(x), & \text{for } x \in [x_1, 1] \end{cases}$$

$$w_{22}(x) = w_2(x) - w_{21}(x).$$

Lemma 3.5. Let $w_{11}, w_{12}, w_{21}$ and $w_{22}$ are as defined in (3.33)-(3.36), then for $x \in \Omega$, the following estimates hold.

$$|w_{11}^{(3)}(x)| \leq C\varepsilon_2^{-3} B_2(x), \quad |w_{12}^{(3)}(x)| \leq C\varepsilon_1^{-2} B_1(x),$$

$$|w_{21}^{(3)}(x)| \leq C\varepsilon_2^{-3} B_2(x), \quad |w_{22}^{(3)}(x)| \leq C\varepsilon_2^{-3} B_1(x).$$

Proof. For $x \in [0, x_3)$, by the definition of $w_{11}(x)$ and using (3.27) and (3.30),

$$|w_{11}^{(3)}(x)| = |w_1^{(3)}(x_3)| \leq C\varepsilon_3^{-3} B_2(x_3) \leq C\varepsilon_2^{-3} B_2(x).$$

For $x \in [x_3, 1]$, by the definition of $w_{11}(x)$ and using (3.27) and (3.31),

$$|w_{11}^{(3)}(x)| = |w_1^{(3)}(x)| \leq C\varepsilon_2^{-3} B_2(x).$$

Hence,

$$|w_{11}^{(3)}(x)| \leq C\varepsilon_2^{-3} B_2(x), \text{ on } \Omega.$$  

Similar arguments lead to,

$$|w_{21}^{(3)}(x)| \leq C\varepsilon_2^{-3} B_2(x), \text{ on } \Omega.$$  

Using (3.34), (3.27), (3.41) and (3.31), it is not hard to see that, for $x \in [0, x_3)$,

$$|w_{12}^{(3)}(x)| \leq |w_1^{(3)}(x)| + |w_{11}^{(3)}(x)| \leq C\varepsilon_1^{-3} B_1(x).$$
Since \( w''_{12}(1) = 0 \), it follows that for any \( x \in [0, 1] \),
\[
|w''_{12}(x)| = \left| \int_x^1 w^{(3)}_{12}(t) dt \right| \leq C \int_x^1 \varepsilon_1^{-3} B_1(t) dt \leq C \varepsilon_1^{-2} B_1(x).
\]
Hence,
\[
(3.41) \quad |w''_{12}(x)| \leq C \varepsilon_1^{-2} B_1(x), \text{ on } \Omega.
\]
Similar arguments lead to,
\[
(3.42) \quad |w''_{22}(x)| \leq C \varepsilon_2^{-2} B_1(x), \text{ on } \Omega.
\]

Now consider the alternate decomposition of the singular component \( w_1(x) \) as below.
\[
(3.43) \quad w_1(x) = w_{11}(x) + w_{12}(x),
\]
where \( w_{11} \) and \( w_{12} \) are defined by
\[
(3.44) \quad w_{11}(x) = \begin{cases} 
\sum_{k=0}^{2} ((x - x_2)^k/k!) w_1^{(k)}(x_2), & \text{for } x \in [0, x_2) \\
\frac{w_1(x)}{x_2 - x_1}, & \text{for } x \in [x_2, 1]
\end{cases}
\]
\[
(3.45) \quad w_{12}(x) = w_1(x) - w_{11}(x).
\]
Then, arguments similar to Lemma 3.5 lead to
\[
(3.46) \quad |w_{11}''(x)| \leq C \varepsilon_2^{-2} B_2(x), \quad |w_{12}'(x)| \leq C \varepsilon_1^{-1} B_1(x).
\]

4. Numerical Method

A piecewise uniform Shishkin mesh \( \Omega_N \) is defined on \([0, 1]\), so as to resolve the layers in the neighbourhood of \( x = 0 \). Let \( N \) denote the number of mesh elements which is taken to be a multiple of 4. The interval \([0, 1]\) is divided into three subintervals \([0, \tau_1]\), \([\tau_1, \tau_2]\) and \([\tau_2, 1]\), where \( \tau_1 \) and \( \tau_2 \) are the transition parameters given by,
\[
\tau_2 = \min \left\{ \frac{1}{2}, \frac{2\varepsilon_2}{\alpha} \ln N \right\}, \quad \tau_1 = \min \left\{ \frac{\tau_2}{2}, \frac{2\varepsilon_1}{\alpha} \ln N \right\}.
\]
In each of the intervals \([0, \tau_1]\), \([\tau_1, \tau_2]\), \(N/4\) mesh elements are placed and \(N/2\) mesh elements are placed in the interval \([\tau_2, 1]\) so that the mesh is piecewise uniform. The mesh becomes uniform when \( \tau_2 = 1/2 \) and \( \tau_1 = \tau_2/2 \).

Let \( H_1, H_2 \) and \( H_3 \) denote the step sizes in the intervals \([0, \tau_1]\), \([\tau_1, \tau_2]\) and \([\tau_2, 1]\) respectively. Thus,
\[
H_1 = \frac{4\tau_1}{N}, \quad H_2 = \frac{4(\tau_2 - \tau_1)}{N} \quad \text{and} \quad H_3 = \frac{2(1 - \tau_2)}{N}.
\]
Therefore the possible four Shishkin meshes are represented by $\Omega^N = \{x_j\}^N_{j=0}$, where,

$$x_j = \begin{cases} 
  jH_1, & \text{if } 0 \leq j \leq \frac{N}{4} \\
  \tau_1 + (j - \frac{N}{4})H_2, & \text{if } \frac{N}{4} \leq j \leq \frac{N}{2} \\
  \tau_2 + (j - \frac{N}{2})H_3, & \text{if } \frac{N}{2} \leq j \leq N.
\end{cases}$$

To resolve the layers, the mesh is constructed in such a way that it condenses at the inner regions where the layers are exhibited and is coarse in the outer region, away from the layers.

To solve the BVP (2.1)-(2.2) numerically the following upwind classical finite difference scheme is applied on the mesh $\Omega^N$.

(4.1) \quad $L^N \vec{U}(x_j) \equiv E\delta^2 \vec{U}(x_j) + A(x_j)D^+ \vec{U}(x_j) - B(x_j)\vec{U}(x_j) = \vec{f}(x_j)$,

(4.2) \quad $\vec{U}(x_0) = \vec{l}$, $\vec{U}(x_N) = \vec{r}$,

where, $\vec{U}(x_j) = (U_1(x_j), U_2(x_j))^T$ and for $1 \leq j \leq N - 1$,

$$D^+ U(x_j) = \frac{U(x_{j+1}) - U(x_j)}{h_{j+1}}, \quad D^- U(x_j) = \frac{U(x_j) - U(x_{j-1})}{h_j},$$

$$\delta^2 U(x_j) = \frac{1}{h_j} \left( D^+ U(x_j) - D^- U(x_j) \right),$$

with

$$h_j = x_j - x_{j-1}, \quad \overline{h_j} = \frac{(h_j + h_{j+1})}{2}.$$

5. Error Analysis

In this section a discrete maximum principle, a discrete stability result and the first order convergence of the proposed numerical method are established.

**Lemma 5.1.** (Discrete Maximum Principle) Assume that the vector valued mesh function $\vec{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j))^T$ satisfies $\vec{\psi}(x_0) \geq \vec{0}$ and $\vec{\psi}(x_N) \geq \vec{0}$. Then $L^N \vec{\psi}(x_j) \leq \vec{0}$ for $1 \leq j \leq N - 1$ implies that $\vec{\psi}(x_j) \geq \vec{0}$ for $0 \leq j \leq N$.

**Proof.** Let $k_1$ and $k_2$ be such that $\psi_1(x_{k_1}) = \min_{x_j} \psi_1(x_j)$ and $\psi_2(x_{k_2}) = \min_{x_j} \psi_2(x_j)$.

Without loss of generality, we assume that $\psi_1(x_{k_1}) \leq \psi_2(x_{k_2})$ and suppose $\psi_1(x_{k_1}) < 0$. Then, $k_1 \notin \{0, N\}$, $\psi_1(x_{k_1+1}) - \psi_1(x_{k_1}) \geq 0$ and $\psi_1(x_{k_1}) - \psi(x_{k_1-1}) \leq 0$, implies that $(L^N \vec{\psi})_1(x_{k_1}) > 0$, a contradiction. Therefore, $\psi_1(x_{k_1}) \geq 0$ and hence, $\vec{\psi}(x_j) \geq \vec{0}$ for $0 \leq j \leq N$. \(\square\)

An immediate consequence of the above discrete maximum principle is the following discrete stability result.

**Lemma 5.2.** (Discrete Stability Result) If $\vec{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j))^T$ is any vector valued mesh function defined on $\Omega^N$, then for $i = 1, 2$ and $0 \leq j \leq N$,

$$|\psi_i(x_j)| \leq \max \left\{ ||\vec{\psi}(x_0)||, ||\vec{\psi}(x_N)||, \frac{1}{\beta} L^N ||\vec{\psi}||_{\Omega^N} \right\}.$$
5.1. Error Estimate. Analogous to the continuous case, the discrete solution $\bar{U}$ can be decomposed into $\bar{V}$ and $\bar{W}$ as defined below.

(5.1) \[ L^N \bar{V}(x_j) = \bar{f}(x_j), \text{ for } 0 < j < N, \quad \bar{V}(x_0) = \bar{v}(x_0), \quad \bar{V}(x_N) = \bar{v}(x_N) \]
(5.2) \[ L^N \bar{W}(x_j) = \bar{0}, \text{ for } 0 < j < N, \quad \bar{W}(x_0) = \bar{w}(x_0), \quad \bar{W}(x_N) = \bar{w}(x_N) \]

Lemma 5.3. Let $\bar{v}$ be the solution of (5.4) and $\bar{V}$ be the solution of (5.1), then

$$\|\bar{V} - \bar{v}\|_{\Omega^N} \leq CN^{-1}. \quad (5.3)$$

Proof. For $1 \leq j \leq N-1$,

$$L^N(\bar{V} - \bar{v})(x_j) = \bar{f}(x_j) - L^N\bar{v}(x_j)$$
$$= (L - L^N)\bar{v}(x_j)$$
$$= \left( \frac{d^2}{dx^2} - \delta^2 \right)E\bar{v}(x_j) + \left( \frac{d}{dx} - D^+ \right)A(x_j)\bar{v}(x_j)$$
$$= \left( \varepsilon_1 \frac{d^2}{dx^2} - \delta^2 \right)v_1(x_j) + \left( \varepsilon_2 \frac{d^2}{dx^2} - \delta^2 \right)v_2(x_j) = \left( \varepsilon_1 \frac{d}{dx} - \delta \right)v_1(x_j) + \left( \varepsilon_2 \frac{d}{dx} - \delta \right)v_2(x_j). \quad (3)$$

By the standard local truncation used in the Taylor expansions,

$$|\varepsilon_1 \frac{d^2}{dx^2} - \delta^2 |v_1(x_j) + a_1(x_j)\left( \frac{d}{dx} - D^+ \right)v_1(x_j) | \leq C(x_{j+1} - x_{j-1})(\varepsilon_1 v_1^{(3)} + v_1^{(2)}), \quad (3)$$

$$|\varepsilon_2 \frac{d^2}{dx^2} - \delta^2 |v_2(x_j) + a_2(x_j)\left( \frac{d}{dx} - D^+ \right)v_2(x_j) | \leq C(x_{j+1} - x_{j-1})(\varepsilon_2 v_2^{(3)} + v_2^{(2)}). \quad (2)$$

Since $(x_{j+1} - x_{j-1}) \leq CN^{-1}$, using (3.24) and (3.25),

$$\|L^N(\bar{V} - \bar{v})\|_{\Omega^N} \leq CN^{-1}. \quad (3)$$

Using Lemma 5.2

$$\|\bar{V} - \bar{v}\|_{\Omega^N} \leq CN^{-1}. \quad (3)$$

To estimate the error in the singular components, we consider the mesh functions $B_1^N(x_j)$ and $B_2^N(x_j)$ on $\Omega^N$ defined by

$$B_1^N(x_j) = \prod_{i=1}^{j}(1 + \frac{\alpha h_i}{2\varepsilon_1})^{-1} \quad \text{and} \quad B_2^N(x_j) = \prod_{i=1}^{j}(1 + \frac{\alpha h_i}{2\varepsilon_2})^{-1}$$

with $B_1^N(x_0) = B_2^N(x_0) = 1$.

It is to be observed that $B_1^N$ and $B_2^N$ are monotonically decreasing.

Lemma 5.4. The layer components $W_1$ and $W_2$ satisfy the following bounds on $\Omega^N$.

$$|W_1(x_j)| \leq CB_2^N(x_j) \quad \text{and} \quad |W_2(x_j)| \leq CB_2^N(x_j).$$

Proof. Consider the following vector valued mesh functions on $\Omega^N$,

$$\psi^+(x_j) = C(B_2^N(x_j), B_2^N(x_j))^T \pm \bar{W}(x_j).$$

□
Then for sufficiently large $C$, $\psi^\pm(x_0) \geq 0$, $\psi^\pm(x_N) \geq 0$ and

$$L^N \psi^\pm(x_j) = CL^N \left( \frac{B^N_2(x_j)}{B^N_2(x_j)} \right) \leq 0.$$ 

Using discrete maximum principle, we have $\bar{\psi}^\pm(x_j) \geq 0$ on $\Omega^N$, which implies that $|W_1(x_j)| \leq CB^N_2(x_j)$ and $|W_2(x_j)| \leq CB^N_2(x_j)$.

\[ \square \]

**Lemma 5.5.** Let $\bar{w}$ be the solution of (5.3) and $\tilde{W}$ be the solution of (5.2), then

$$\|\tilde{W} - \bar{w}\|_{\Omega^N} \leq CN^{-1} \ln N.$$ 

**Proof.** By the standard local truncation used in the Taylor expansions,

$$|\varepsilon_1 \left( \frac{d^2}{dx^2} - \delta^2 \right) w_1(x_j) + a_1(x_j) \left( \frac{d}{dx} - D^+ \right) w_1(x_j)| \leq C(x_{j+1} - x_{j-1}) (\varepsilon_1 \|w^{(3)}_1\| + \|w^{(2)}_1\|)$$

$$|\varepsilon_2 \left( \frac{d^2}{dx^2} - \delta^2 \right) w_2(x_j) + a_2(x_j) \left( \frac{d}{dx} - D^+ \right) w_2(x_j)| \leq C(x_{j+1} - x_{j-1}) (\varepsilon_2 \|w^{(3)}_2\| + \|w^{(2)}_2\|)$$

where the norm is taken over the interval $[x_{j-1}, x_{j+1}]$.

For the case $\tau_2 = 1/2$ and $\tau_1 = 1/4$, the mesh is uniform, $h = N^{-1}$, $\varepsilon_1^{-1} \leq C \ln N$ and $\varepsilon_2^{-1} \leq C \ln N$ and thus we obtain,

$$L^N (\tilde{W} - \bar{w})(x_j) \leq \left( \frac{CN^{-1} (\varepsilon_1^{-2} B_1(x_{j-1}) + \varepsilon_2^{-2} B_2(x_{j-1}))}{CN^{-1} (\varepsilon_1^{-1} \varepsilon_2^{-1} B_1(x_{j-1}) + \varepsilon_2^{-1} B_2(x_{j-1}))} \right).$$

Consider the following barrier function $\phi$ given by

$$\phi_1(x_j) = \frac{CN^{-1}}{\gamma (\alpha - \gamma)} \left( \exp(2\gamma h/\varepsilon_1) \varepsilon_1^{-1} Y_j + \exp(2\gamma h/\varepsilon_2) \varepsilon_2^{-1} Z_j \right)$$

$$\phi_2(x_j) = \frac{CN^{-1}}{\gamma (\alpha - \gamma)} \left( \exp(2\gamma h/\varepsilon_2) \varepsilon_1^{-1} Z_j \right)$$

where $\gamma$ is a constant such that $0 < \gamma < \alpha$,

$$Y_j = \frac{\Lambda^{N-j} - 1}{\Lambda^{N} - 1} \text{ with } \Lambda = 1 + \frac{\gamma h}{\varepsilon_1}$$

and

$$Z_j = \frac{\Lambda^{N-j} - 1}{\Lambda^{N} - 1} \text{ with } \Lambda = 1 + \frac{\gamma h}{\varepsilon_2}.$$ 

It is not hard to see that

$$0 \leq Y_j, Z_j \leq 1,$$

$$(\varepsilon_1 \delta^2 + \gamma D^+) Y_j = 0, \quad (\varepsilon_2 \delta^2 + \gamma D^+) Z_j = 0,$$

$$D^+ Y_j \leq -\frac{\gamma}{\varepsilon_1} \exp(-\gamma x_{j+1}/\varepsilon_1), \quad D^+ Z_j \leq -\frac{\gamma}{\varepsilon_2} \exp(-\gamma x_{j+1}/\varepsilon_2).$$

Hence,

$$\left( L^N \phi \right)(x_j) \leq \frac{CN^{-1}}{\gamma (\alpha - \gamma)} \left( \frac{\varepsilon_1^{-1} \exp(2\gamma h/\varepsilon_1) D^+ Y_j + \varepsilon_2^{-1} \exp(2\gamma h/\varepsilon_2) D^+ Z_j}{\varepsilon_1^{-1} \exp(2\gamma h/\varepsilon_2) (a_2 - \gamma) D^+ Z_j} \right)$$

$$\leq -CN^{-1} \left( \frac{\varepsilon_1^{-2} B_1(x_{j-1}) + \varepsilon_2^{-2} B_2(x_{j-1})}{\varepsilon_1^{-1} \varepsilon_2^{-1} B_1(x_{j-1})} \right).$$

(5.5)
Consider the discrete functions
\[ \tilde{\psi}^\pm(x_j) = \tilde{\phi}(x_j) \pm (\tilde{W} - \tilde{w})(x_j), x_j \in \tilde{\Omega}^N. \]
Then for sufficiently large C, using (5.4) and (5.5), \( \tilde{\psi}^\pm(x_0) > 0, \) \( \tilde{\psi}^\pm(x_N) = 0 \) and \( L^N \tilde{\psi}^\pm(x_j) \leq 0 \) on \( \tilde{\Omega}^N. \)

Using discrete maximum principle, \( \tilde{\psi}^\pm(x_j) \geq 0 \) on \( \tilde{\Omega}^N. \) Hence,
\[ ||(\tilde{W} - \tilde{w})(x_j)|| \leq \left( C N^{-1} \left( \frac{\varepsilon_1^{-1} + \varepsilon_2^{-1}}{N} \right) \right) \leq \left( C N^{-1} \ln N \right) \]
implies that
\[ (5.6) \quad ||(\tilde{W} - \tilde{w})||_{\tilde{\Omega}^N} \leq CN^{-1} \ln N. \]

For other choices of \( \tau_1 \) and \( \tau_2, \) estimate of \( ||(\tilde{W} - \tilde{w})||_{\tilde{\Omega}^N} \) is as follows.

Let \( \tilde{\Omega}^N_1 = \{ x_j \}_{j=0}^{N/2}^N \) and \( \tilde{\Omega}^N_2 = \{ x_j \}_{j=N/2}^{N}, \) then for \( x_j \in \tilde{\Omega}^N_2, \) using Lemma 5.4 and Theorem 3.3
\[ ||(W_1 - w_1)(x_j)|| \leq |W_1(x_j)| + |w_1(x_j)| \leq CB^N_2(x_j) + CB_2(x_j) \]
\[ \leq CB^N_2(\tau_2) + CB_2(\tau_2). \]
\[ B_2(\tau_2) = \exp \left( \frac{-\alpha \tau_2}{\varepsilon_2} \right) \leq \exp ( - \ln N ) \leq N^{-1}. \]
\[ B^N_2(\tau_2) = \prod_{i=1}^{j} \left( 1 + \frac{\alpha h_i}{2\varepsilon_2} \right)^{N/2} \]
\[ = \left( 1 + \frac{\alpha h_1}{2\varepsilon_2} \right)^{N/2} \left( 1 + \frac{\alpha h_2}{2\varepsilon_2} \right)^{N/2} \]
\[ = \left( 1 + \frac{2\alpha \tau_1}{N\varepsilon_2} \right)^{N/2} \left( 1 + \frac{2\alpha (\tau_2 - \tau_1)}{N\varepsilon_2} \right)^{N/2} \]
\[ \leq \left( 1 + \frac{2\alpha \tau_2}{N\varepsilon_2} \right)^{N/2}. \]

Hence, \( ||(W_1 - w_1)(x_j)|| \leq CN^{-1}. \)

Similarly, it is true that \( ||(W_2 - w_2)(x_j)|| \leq CN^{-1} \) and hence,
\[ (5.7) \quad ||(\tilde{W} - \tilde{w})||_{\tilde{\Omega}^N_{2}} \leq CN^{-1}. \]

For \( N/4 \leq j < N/2, \) if \( \varepsilon_2/2 \leq \varepsilon_1 \leq \varepsilon_2, \) then \( \tau_2 \leq (4\varepsilon_1/\alpha) \ln N \) implies that
\[ (5.8) \quad |L^N(\tilde{W} - \tilde{w})(x_j)| \leq CN^{-1} \ln N \left( \varepsilon_1^{-1} B_1(x_{j-1}) + \varepsilon_2^{-1} B_2(x_{j-1}) \right). \]

On the other hand, if \( \varepsilon_2 > 2\varepsilon_1, \) then using (3.32),
\[ \begin{align*}
&\left( |\varepsilon_1 (\frac{d^2}{dx^2} - \delta^2) w_1 (x_j)| \right) + \left( |\varepsilon_1 (\frac{d^2}{dx^2} - \delta^2) w_{11} (x_j)| \right) \leq \left( |\varepsilon_1 (\frac{d^2}{dx^2} - \delta^2) w_{12} (x_j)| \right) + \left( |\varepsilon_1 (\frac{d^2}{dx^2} - \delta^2) w_{21} (x_j)| \right) \right).
\end{align*} \]
Also, by the standard local truncation used in the Taylor expansions and using Lemma 3.5

\[
\begin{align*}
\left( \varepsilon_1 \left( \frac{d^2}{dx^2} - \delta^2 \right) w_{11}(x_j) \right) & \leq \left( C \varepsilon_1 (x_{j+1} - x_{j-1}) \| w'''_{11} \| \right) \\
\left( \varepsilon_2 \left( \frac{d^2}{dx^2} - \delta^2 \right) w_{21}(x_j) \right) & \leq \left( C \varepsilon_2 (x_{j+1} - x_{j-1}) \| w'''_{21} \| \right)
\end{align*}
\]

\[
\leq C \varepsilon_2^{-1} N^{-1} \ln N \left( \frac{B_2(x_{j-1})}{B_2(x_{j-1})} \right),
\]

\[
\begin{align*}
\left( \varepsilon_1 \left( \frac{d^2}{dx^2} - \delta^2 \right) w_{12}(x_j) \right) & \leq C \left( \varepsilon_1 \| w''_{12} \| [x_{j-1}, x_{j+1}] \right) \\
\left( \varepsilon_2 \left( \frac{d^2}{dx^2} - \delta^2 \right) w_{22}(x_j) \right) & \leq C \left( \varepsilon_2 \| w''_{22} \| [x_{j-1}, x_{j+1}] \right)
\end{align*}
\]

\[
\leq C \left( \varepsilon_1^{-1} B_1(x_{j-1}) \right) \left( \varepsilon_2^{-1} B_1(x_{j-1}) \right)
\]

Thus, for \( N/4 \leq j < N/2, \)

\[
(5.9) \quad \left( \varepsilon_1 \left( \frac{d^2}{dx^2} - \delta^2 \right) w_1(x_j) \right) \leq \left( C \varepsilon_2^{-1} N^{-1} \ln N B_2(x_{j-1}) + C \varepsilon_1^{-1} B_1(x_{j-1}) \right) \left( C \varepsilon_2^{-1} N^{-1} \ln N B_2(x_{j-1}) + C \varepsilon_1^{-1} B_1(x_{j-1}) \right).
\]

Using the alternate decomposition of \( w_1(x) \) given in (3.43) and the arguments similar to the above, it is not hard to verify that for \( N/4 \leq j < N/2, \)

\[
(5.10) \quad \left( \left( \frac{d}{dx} - D^+ \right) w_1(x_j) \right) \leq \left( C \varepsilon_2^{-1} N^{-1} \ln N B_2(x_{j-1}) + C \varepsilon_1^{-1} B_1(x_{j-1}) \right) \left( C \varepsilon_2^{-1} N^{-1} \ln N B_2(x_{j-1}) + C \varepsilon_1^{-1} B_1(x_{j-1}) \right).
\]

Hence, for \( N/4 \leq j < N/2, \) expressions (5.9) & (5.10) yield

\[
(5.11) \quad \| L^N(\bar{W} - \bar{w})(x_j) \| \leq \left( C \varepsilon_2^{-1} N^{-1} \ln N B_2(x_{j-1}) + C \varepsilon_1^{-1} B_1(x_{j-1}) \right) \left( C \varepsilon_2^{-1} N^{-1} \ln N B_2(x_{j-1}) + C \varepsilon_1^{-1} B_1(x_{j-1}) \right).
\]

For \( 0 < j < N/4, \) \( \tau_1 \leq (\varepsilon_1/\alpha) \ln N \) and hence

\[
(5.12) \quad \| L^N(\bar{W} - \bar{w})(x_j) \| \leq C N^{-1} \ln N \left( \varepsilon_1^{-1} B_1(x_{j-1}) + \varepsilon_2^{-1} B_2(x_{j-1}) \right).
\]

Consider the following barrier functions for \( 0 < j < N/4 \)

\[
(5.13) \quad \phi_1(x_j) = C N^{-1} \ln N \left( \exp(2\alpha H_1/\varepsilon_1) B_1^N(x_j) + \exp(2\alpha H_1/\varepsilon_2) B_2^N(x_j) \right)
\]

\[
(5.14) \quad \phi_2(x_j) = C N^{-1} \ln N \exp(2\alpha H_1/\varepsilon_2) B_2^N(x_j)
\]

and for \( N/4 \leq j \leq N/2, \)

\[
(5.15) \quad \phi_1(x_j) = C N^{-1} \ln N \exp(2\alpha H_2/\varepsilon_2) B_2^N(x_j) + C B_1^N(x_j)
\]

\[
(5.16) \quad \phi_2(x_j) = C N^{-1} \ln N \exp(2\alpha H_2/\varepsilon_2) B_2^N(x_j) + C N^{-1} \left( (\tau_2 - x_j) \varepsilon_2^{-1} + 1 \right)
\]

Let \( \vec{\phi} = (\phi_1, \phi_2)^T \) and consider the following vector valued mesh functions, for \( 0 \leq j \leq N/2, \)

\[
\vec{\psi}^b(x_j) = \vec{\phi}(x_j) \pm (\bar{W} - \bar{w})(x_j).
\]

For sufficiently large \( C, \)

\[
\vec{\psi}^b(x_0) \geq \vec{0}, \quad \vec{\psi}^b(x_{N/2}) \geq \vec{0} \quad \text{and} \quad L^N \vec{\psi}^b(x_j) \leq \vec{0}, \quad \text{for } 0 < j < N/2.
\]

Then by Lemma 5.1, \( \vec{\psi}^b(x_j) \geq \vec{0} \) for \( 0 \leq j \leq N/2. \) Hence,

\[
(5.17) \quad \| (\bar{W} - \bar{w}) \|_{\mathbb{H}^N} \leq C N^{-1} \ln N.
\]
Therefore, for any choice of $\tau_1$ and $\tau_2$,

$$
\|(\tilde{W} - \tilde{w})\|_{T^N} \leq CN^{-1} \ln N.
$$

\[\square\]

**Theorem 5.6.** Let $\tilde{u}$ be the solution of the problem (2.1)-(2.2) and $\tilde{U}$ be the solution of the problem (4.1)-(4.2), then,

$$
\|(\tilde{u} - \tilde{U})\|_{T^N} \leq CN^{-1} \ln N.
$$

**Proof.** The result follows by using triangle inequality, (5.3) and (5.18). \[\square\]

### 6. Numerical Illustrations

**Example 6.1.** Consider the boundary value problem for the system of convection diffusion equations on $(0,1)$

- (6.1) $\varepsilon_1 u''_1(x) + (1 + x^2)u'_1(x) - (4 + \sin x)u_1(x) + 2u_2(x) = -e^x$,
- (6.2) $\varepsilon_2 u''_2(x) + (2 + x)u'_2(x) + u_1(x) - (2 + \cos x)u_2(x) = -x^2$,
- (6.3) with $u_1(0) = 3, u_2(0) = 3, u_1(1) = 1, u_2(1) = 1$.

The above problem is solved using the suggested numerical method and plot of the approximate solution for $N = 1024, \varepsilon_1 = 5^{-4}, \varepsilon_2 = 2^{-7}$ is shown in Figure 1. Parameter uniform error and order of convergence of the numerical method are shown in Table 1 which are computed using two mesh algorithm, a variant of the one suggested in [3].

![Figure 1. Approximate solution of Example 6.1](image)

From Table 1 it is to be noted that the error decreases as number of mesh elements $N$ increases. Also for each $N$, the error stabilizes as $\varepsilon_1$ and $\varepsilon_2$ tends to zero.
Consider the boundary value problem for the system of convection diffusion equations on \((0,1)\)

\begin{align}
\varepsilon_1 u''_1(x) + u'_1(x) - 2u_1(x) + u_2(x) &= -3(x - 1), \\
\varepsilon_2 u''_2(x) + (1 + x)u'_2(x) + xu_1(x) - (2x + 1)u_2(x) &= -2x,
\end{align}

with \(u_1(0) = 0\), \(u_2(0) = 3\), \(u_1(1) = 2\), \(u_2(1) = 2\).

The reduced problem corresponding to (6.4) - (6.6) is

\begin{align}
u''_{01}(x) - 2u_{01}(x) + u_{02}(x) &= -3(x - 1), \\
(1 + x)u''_{01}(x) + xu_{01}(x) - (2x + 1)u_{02}(x) &= -2x,
\end{align}

with \(u_{01}(0) = 2\), \(u_{02}(1) = 2\).

Solution of the reduced problem is \((u_{01}(x), u_{02}(x))^T = (2x, x+1)^T\). Even though \(u_{01}(x)\) coincides with \(u_1(x)\) at the boundary points, \(u_{02}(0) \neq u_2(0)\) implies that \(\varepsilon_2\)-layer may occur at \(x = 0\) in both the solution components \(u_1\) and \(u_2\). For \(N = 1024\), \(\varepsilon_1 = 5^{-6}\), \(\varepsilon_2 = 2^{-6}\), the plots of the approximate solution components of (6.4) - (6.6) shown in Figures 2 and 3 ensure the foresaid layer patterns.

**Example 6.3.** Consider the boundary value problem for the system of convection diffusion equations on \((0,1)\)

\begin{align}
\varepsilon_1 u''_1(x) + u'_1(x) - 2u_1(x) + u_2(x) &= -3(x - 1), \\
\varepsilon_2 u''_2(x) + (1 + x)u'_2(x) + xu_1(x) - (2x + 1)u_2(x) &= -2x,
\end{align}

with \(u_1(0) = 1\), \(u_2(0) = 1\), \(u_1(1) = 2\), \(u_2(1) = 2\).

Solution of the reduced problem is \((u_{01}(x), u_{02}(x))^T = (2x, x+1)^T\). Since, \(u_{01}(0) \neq u_1(0)\) and \(u_{02}(0) = u_2(0)\), \(\varepsilon_1\)-layer is expected near \(x = 0\) only in the

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\(\varepsilon_1\) & \(\varepsilon_2\) & \multicolumn{4}{c|}{Number of mesh elements \(N\)} \\
\hline
& & 128 & 256 & 512 & 1024 & 2048 \\
\hline
5^{-4} & 2^{-7} & 4.725E - 02 & 2.887E - 02 & 1.775E - 02 & 1.019E - 02 & 5.779E - 03 \\
5^{-5} & 2^{-8} & 4.789E - 02 & 2.919E - 02 & 1.792E - 02 & 1.028E - 02 & 5.827E - 03 \\
5^{-6} & 2^{-9} & 6.282E - 02 & 4.456E - 02 & 2.644E - 02 & 1.535E - 02 & 8.425E - 03 \\
5^{-7} & 2^{-10} & 7.146E - 02 & 5.089E - 02 & 3.212E - 02 & 1.914E - 02 & 1.095E - 02 \\
5^{-8} & 2^{-11} & 7.393E - 02 & 5.243E - 02 & 3.365E - 02 & 2.009E - 02 & 1.159E - 02 \\
5^{-9} & 2^{-12} & 7.470E - 02 & 5.321E - 02 & 3.437E - 02 & 2.033E - 02 & 1.174E - 02 \\
5^{-10} & 2^{-13} & 7.497E - 02 & 5.355E - 02 & 3.462E - 02 & 2.040E - 02 & 1.177E - 02 \\
5^{-11} & 2^{-14} & 7.508E - 02 & 5.367E - 02 & 3.471E - 02 & 2.042E - 02 & 1.179E - 02 \\
5^{-12} & 2^{-15} & 7.512E - 02 & 5.372E - 02 & 3.475E - 02 & 2.042E - 02 & 1.180E - 02 \\
5^{-13} & 2^{-16} & 7.513E - 02 & 5.374E - 02 & 3.477E - 02 & 2.043E - 02 & 1.181E - 02 \\
5^{-14} & 2^{-17} & 7.514E - 02 & 5.375E - 02 & 3.478E - 02 & 2.043E - 02 & 1.181E - 02 \\
5^{-15} & 2^{-18} & 7.514E - 02 & 5.375E - 02 & 3.478E - 02 & 2.044E - 02 & 1.181E - 02 \\
5^{-16} & 2^{-19} & 7.515E - 02 & 5.376E - 02 & 3.478E - 02 & 2.044E - 02 & 1.181E - 02 \\
5^{-17} & 2^{-20} & 7.515E - 02 & 5.376E - 02 & 3.478E - 02 & 2.044E - 02 & 1.181E - 02 \\
5^{-18} & 2^{-21} & 7.515E - 02 & 5.376E - 02 & 3.478E - 02 & 2.044E - 02 & 1.181E - 02 \\
\hline
\end{tabular}
\end{table}

Computed order of \((\varepsilon_1, \varepsilon_2)\)-uniform convergence, \(p^* = 0.4833\).

Computed \((\varepsilon_1, \varepsilon_2)\)-uniform error constant, \(C_{p^*} = 2.7546\).
solution component $u_1$. For $N = 1024$, $\varepsilon_1 = 5^{-4}$, $\varepsilon_2 = 2^{-4}$, the plots of the approximate solution components of (6.10) - (6.12) shown in Figure 4 ensures the foresaid layer patterns.
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