ELLIPTIC OPEN BOOKS ON TORUS BUNDLES OVER THE CIRCLE

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ABSTRACT. As an application of the construction of open books on plumbed 3-manifolds, we construct elliptic open books on torus bundles over the circle. In certain cases these open books are compatible with Stein fillable contact structures and have minimal genus.

1. INTRODUCTION

Even though it has been known how to construct a contact structure starting from an open book decomposition of a closed 3–manifold since the work of Thurston and Winkelnkemper [16], only after Giroux’s fundamental work [7] on the correspondence between contact structures and open books the latter started to play an essential role in contact topology. This correspondence becomes one-to-one when certain equivalence relations are taken into account. As a consequence, given a contact structure there are infinitely many open books that correspond to it. This leads to the problem of finding the “simplest” open book compatible with a given contact structure. One way to interpret simplicity is in terms of the genus of a page. Etnyre [2] proved that whenever the contact structure is overtwisted there is a corresponding planar open book, i.e. an open book with page genus equal to 0, and he also gave nontrivial obstructions to the existence of planar open books for fillable contact structures (see [14] for other obstructions). In an earlier work Ozbagci and the author [3] described how to construct open books given a plumbing description of a 3–manifold. The open books constructed this way coincide with the ones obtained by using the methods given by Gay [4]. As an application of this construction and using the plumbing description of torus bundles over the circle [11] we explain how to construct elliptic open books, i.e. open books with page genus equal to 1, on torus bundles over the circle. We demonstrate this on many examples and obtain the following results.

**Theorem.** There exists elliptic open books compatible with Stein fillable contact structures on circle bundles over the torus with Euler number less than 5.

For negative Euler numbers the above statement was known before, and for $\mathbb{T}^3$ as a result of Etnyre’s work [2] we knew that there was no planar open book corresponding to the unique Stein fillable contact structure. The following consequence of the above theorem was previously announced by Van Horn [17].

**Corollary.** The minimal page genus among all the open books compatible with the unique Stein fillable contact structure on $\mathbb{T}^3$ is 1.

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There are exactly seven Seifert fibred spaces (besides the circle bundles over the torus) among torus bundles over the circle. The next statement is about these 3–manifolds.

**Theorem.** On the Seifert fibred spaces $M(\pm \frac{2}{3}, \frac{1}{3}, \frac{1}{3}), M(\pm \frac{1}{5}, \frac{1}{5}, \frac{1}{5}), M(\pm \frac{1}{3}, \frac{1}{5}, \frac{1}{5})$ and $M(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ there exists elliptic open books compatible with Stein fillable contact structures.

Note that on many of the 3-manifolds considered in the above statements, there is a unique Stein fillable contact structure (up to contactomorphism, and in some cases up to contact isotopy). For this observation one combines the recent result of Gay [5] stating that contact structures with positive Giroux torsion cannot be even strongly symplectically fillable with Giroux’s classification of tight contact structures on torus bundles over the circle specifically Theorem 1.3 in [6] (cf. [9]) which specifies the cases where the Giroux torsion uniquely determines the tight contact structure.

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2. **Contact structures and open books**

We will assume throughout this paper that a contact structure $\xi = \ker \alpha$ is coorientable (i.e., $\alpha$ is a global 1–form) and positive (i.e., $\alpha \wedge d\alpha > 0$). In the following we describe the compatibility of an open book decomposition with a given contact structure on a 3–manifold.

Suppose that for an oriented link $L$ in a 3–manifold $Y$ the complement $Y \setminus L$ fibers over the circle as $\pi: Y \setminus L \to S^1$ such that $\pi^{-1}(\theta) = \Sigma_{\theta}$ is the interior of a compact surface bounding $L$, for all $\theta \in S^1$. Then $(L, \pi)$ is called an open book decomposition (or just an open book) of $Y$. For each $\theta \in S^1$, the surface $\Sigma_{\theta}$ is called a page, while $L$ the binding of the open book. The monodromy of the fibration $\pi$ is defined as the diffeomorphism of a fixed page which is given by the first return map of a flow that is transverse to the pages and meridional near the binding. The isotopy class of this diffeomorphism is independent of the chosen flow and we will refer to that as the monodromy of the open book decomposition.

An open book $(L, \pi)$ on a 3–manifold $Y$ is said to be isomorphic to an open book $(L', \pi')$ on a 3–manifold $Y'$, if there is a diffeomorphism $f : (Y, L) \to (Y', L')$ such that $\pi' \circ f = \pi$ on $Y \setminus L$. In other words, an isomorphism of open books takes binding to binding and pages to pages.

**Theorem 2.1** (Alexander [1]). Every closed and oriented 3–manifold admits an open book decomposition.

In fact, every closed oriented 3–manifold admits a planar open book, which means that a page is a disk $D^2$ with holes [15]. On the other hand, every closed oriented 3–manifold admits a contact structure [13]. So it seems natural to strengthen the contact condition $\alpha \wedge d\alpha > 0$ in the presence of an open book decomposition on $Y$ by requiring that $\alpha > 0$ on the binding and $d\alpha > 0$ on the pages.

An open book decomposition of a 3–manifold $Y$ and a contact structure $\xi$ on $Y$ are called compatible if $\xi$ can be represented by a contact form $\alpha$ such that the binding is a transverse link, $d\alpha$ is a symplectic form on every page and the orientation of the transverse binding induced by $\alpha$ agrees with the boundary orientation of the pages.

**Theorem 2.2** (Giroux [7]). Every contact 3–manifold admits a compatible open book. Moreover two contact structures compatible with the same open book are isotopic.
Under the light of this correspondence a natural question that comes to mind is the following:

**Question.** Given a contact structure, what is the minimal (page) genus of all the compatible open books?

As Etnyre proved in [2], if the contact structure is overtwisted, there is always a compatible planar open book. He also proved that for fillable contact structures, there is an obstruction for having a compatible planar open book (also see [14] for other obstructions).

**Theorem 2.3** (Etnyre [2]). If $X$ is a symplectic filling of a contact 3–manifold $(M, \xi)$ carried by a planar open book, then $b_2^+(X) = b_2^-(X) = 0$.

At this point we should also note that the monodromy of a compatible open book may be a good indicator of the fillability and tightness of the contact structure.

**Theorem 2.4** (Giroux [7]). A contact structure is Stein fillable if and only if it is compatible with an open book whose monodromy can be written as a product of right-handed Dehn twists.

**Theorem 2.5** (Honda-Kazez-Matić [10]). A contact structure is tight if and only if the monodromy of any compatible open book is right-veering.

An important consequence of Theorem 2.5 which applies to some examples we construct in the later sections is that if the monodromy of an open book can be written as a product of Dehn twists along disjoint curves and at least one of these Dehn twists is left-handed and parallel to a boundary component, then the monodromy is not right-veering and hence the open book is compatible with an overtwisted contact structure.

3. **Kirby Diagrams and Plumbing Graphs of Torus Bundles Over the Circle**

Let $Y$ be a 3-manifold which fibers over the circle with torus fibers. It is uniquely determined by the monodromy of the fibration. On the other hand, the automorphisms of the torus can be viewed as (conjugacy classes of) matrices in $SL(2, \mathbb{Z})$ by considering the induced automorphisms on the first homology of the torus. In the appendix to [11], Kirby and Melvin explain in detail
how to obtain a Kirby diagram on \( Y \) given its monodromy \( A \) as a matrix in \( SL(2, \mathbb{Z}) \). If \( A = S T^{a_1} S T^{a_2} \cdots S T^{a_k} S \) for integers \( a_1, a_2, \ldots, a_k \), where

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

are matrices that generate \( SL(2, \mathbb{Z}) \) with relations \( S^4 = I \) and \( S^2 = (ST)^3 \), then a Kirby diagram is given in Figure 1 (left). As a consequence \( Y \) has a plumbing description given by the plumbing graph in Figure 1 (right). Note that the unknots in the Kirby diagram are oriented and according to these orientations the edges in the plumbing graph all have negative signs. If we had a tree as a plumbing graph the signs on the edges would be irrelevant as we could have changed them as we wish by changing the orientation of the base sphere of the circle bundle that corresponds to one of the two adjacent vertices. In our case we obviously don’t have a tree and, even though we can change the signs on the edges in certain ways, we cannot, for example, make all the signs + without changing the 3–manifold \( Y \).

4. FROM PLUMBING DIAGRAMS TO OPEN BOOKS

In [3], Ozbagci and the author explain how to obtain open books on 3–manifolds from a plumbing description of the manifold. Even though the main focus of that paper is horizontal open books, a general procedure is described. Apparently, the open books obtained this way coincide with the ones constructed by Gay [4]. Here we explain this construction on an example (this is Example 10 in [3]).

Consider the 3–manifold \( Y \) obtained by plumbing circle bundles over tori according to the graph in Figure 2.

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**Figure 2.** The integer weights at each vertex are the Euler number and the genus of the base of the corresponding circle bundle, respectively. Each edge is assumed to be positively signed.

First of all, we construct an open book for each vertex, i.e. circle bundle over a surface (these are shown in the upper part of Figure 3). In general for each vertex we obtain an open book with a page of genus and punctures given by the base genus and the absolute value of the Euler number, respectively, of the circle bundle which corresponds to the vertex and the monodromy is the product of Dehn twists around the punctures (right-handed if the Euler number is negative and left-handed otherwise). Note that, whenever necessary, we can add two boundary components to a page of an open book without changing the underlying 3–manifold by adding opposite-handed Dehn twists around these punctures. Going back to our example, we then “join” the pages of these open books according to the plumbing graph and keeping in mind that the edges are all positive (see the lower part of Figure 3). In general, if there is a negative edge, to join the pages of the open books that correspond to the adjacent vertices we use boundary components that have left-handed Dehn twists around them.
Combining the procedures described in the previous two sections, one can obtain an elliptic open book on any 3–manifold which fibers over the circle with torus fibers. Since we are interested in realizing the minimal genus for a contact structure and since it is already known that the minimal genus of any overtwisted contact structures is zero [2], we should focus on tight contact structures. On the other hand, even though we can detect the tightness of a contact structure by considering open books compatible with it [10], at least for the moment, we have to consider all the compatible open books. This is why we try to construct open books compatible with Stein fillable contact structures which are much easier to detect by considering the monodromy of a single compatible open book. On the other hand, obtaining open books compatible with Stein fillable contact structures requires some care. In what follows we focus on two special classes among torus bundles over the circle: Seifert fibred spaces and circle bundles over the torus.

5.1. Open books on some Seifert fibred spaces. As a first example, let us consider the small Seifert fibred space $M(-2/3, 1/3, 1/3)$. Using the fact that this manifold fibers over the circle with monodromy

$$
\begin{pmatrix}
0 & 1 \\
-1 & -1
\end{pmatrix}
= ST^1ST^0S
$$

(see, for example, the table on page 89 in [9]), we provide an open book decomposition of this manifold and the corresponding plumbing graph in the first row of Table I. Figure 4 demonstrates how we obtain this plumbing graph from the initial plumbing graph via blow-ups.

This open book is compatible with a Stein fillable contact structure since its monodromy can be written as a product of right-handed Dehn twists. To see this, let us denote the right-handed Dehn twists around the punctures by $\delta_i$ and the right-handed Dehn twists around the essential curves by $\alpha_i$ as in Figure 5. So the monodromy is $\phi = \delta_1 \delta_2 \alpha_1^{-3} \alpha_2^{-3}$ (note that since the curves are pairwise disjoint, all these Dehn twists commute with each other). We have the relation $\delta_1 \delta_2 = (\alpha_1 \alpha_2 \beta)^4$, where $\beta$ is the right-handed Dehn twists around the essential curve disjoint from the $\delta_i$’s and intersecting each $\alpha_i$ geometrically once (see [12] for this and other such relations on punctured tori). It is crucial that we have no more than 4 parallel left-handed Dehn twists on a twice-punctured

\[ \text{Figure 3. A page and the monodromy of the open book constructed from the plumbing above.} \]
Figure 4. From the first graph to the second a single blow-up on the edge connecting 0-vertices and from the second to the third one blow-up on each side edge are performed. Every edge has a minus sign and every blow-up is with a +1-framed unknot.

Figure 5.

torus in order to be able to use this relation to write the monodromy in terms of right-handed Dehn twists.

Theorem 5.1. On the Seifert fibred space $M(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ there is an elliptic open book compatible with a Stein fillable contact structure.

In fact, using the same methods it is not difficult to do the same for the other six Seifert fibred spaces which are torus bundles over the circle. Examples of such open books are given in Tables 1 and 2.

Theorem 5.2. On the Seifert fibred spaces $M(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}), M(\pm \frac{1}{2}, \mp \frac{1}{4}, \mp \frac{1}{4}), M(\pm \frac{1}{2}, \mp \frac{1}{3}, \mp \frac{1}{6}), and M(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ there are elliptic open books compatible with Stein fillable contact structures.

5.2. Open books on circle bundles over the torus. In this subsection, we will give examples of open books on some circle bundles over the torus. $Y_k$ denotes the circle bundle with Euler number $k$. Since $Y_k$ is also a torus bundle over the circle, we can use the previous techniques to obtain open books. As in the previous subsection, we are especially interested in open books compatible with Stein fillable contact structures.

First of all, in case $k < 0$ (resp. $k > 0$), as it was explained in [3], one can easily obtain an open book $ob_k$ with $|k|$-times punctured torus page and the product of one right-handed (resp. left-handed) Dehn twist around each boundary component as monodromy on $Y_k$. Note that when $k < 0$, since the monodromy of $ob_k$ is the product of right-handed Dehn twists, $ob_k$ is compatible with a Stein fillable contact structure on $Y_k$ (see Figure 6).
Table 1. Elliptic open books on some Seifert fibred spaces. The edges in the plumbing graphs have negative sign. Punctured tori (black dots on the tori are the punctures) represent a page, the monodromy is the product of Dehn twists along with the curves drawn on this page and the handedness of these twists are given by the sign next to each curve: + indicates a right-handed (positive) Dehn twist and − indicates a left-handed (negative) Dehn twist.

| 3–manifold | monodromy | plumbing | open book |
|------------|-----------|----------|-----------|
| $M\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$ | \[
\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}
\] | ![Plumbing Graph](image) | ![Open Book](image) |
| $M\left(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ | \[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\] | ![Plumbing Graph](image) | ![Open Book](image) |
| $M\left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right)$ | \[
\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}
\] | ![Plumbing Graph](image) | ![Open Book](image) |
| $M\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ | \[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\] | ![Plumbing Graph](image) | ![Open Book](image) |

One obvious way to obtain an elliptic open book on $Y_0 = \mathbb{T}^3$ is to let a page be the twice punctured torus and the monodromy be the product of opposite-handed Dehn twists around the punctures. But, like $\text{ob}_k$ for $k > 0$, this open book is compatible with an overtwisted contact structure since the monodromy is not right veering [10].

The total space $Y_k$ of the circle bundle of Euler number $k$ over the torus is also the total space of the torus bundle over the circle with monodromy $T^k = S^2T^kS^2 = ST^kST^kST^0S$. Therefore $Y_k$ can be given by the Kirby diagram and the corresponding plumbing graph in Figure [7].
Using the recipe in [3] to get an open book on $Y_k$ from this plumbing description we obtain an open book $\text{ob}_k'$ with $6 + |k - 2|$-times punctured torus page and a monodromy given by the product of Dehn twists around the curves indicated in Figure 8 for $k > 2$. Note that in case $k > 2$ we get left-handed Dehn twists around $k - 2$ boundary components (and the other Dehn twists are around disjoint curves) which implies that the monodromy is not right-veering hence the open book $\text{ob}_k'$ is compatible with an overtwisted contact structure [10]. On the other hand, when $k \leq 2$, we have right-handed Dehn twists around boundary components. In particular, when $k = 0$ or 1, by using certain relations [12] in the mapping class group of torus with 8 and 7 punctures, respectively, the monodromy can be written as a product of right-handed Dehn twists implying that $\text{ob}_k'$ is compatible with a Stein fillable contact structure. Elliptic open books compatible with
the unique Stein fillable contact structure on $\mathbb{T}^3$ are especially interesting as minimal genus open books compatible with this contact structure since there is no planar open book compatible with it: If there were a planar open book compatible with the unique Stein fillable contact structure on $\mathbb{T}^3$, then by Theorem 4.1 in [2] any such filling would have vanishing $b_2^+$ and $b_2^0$ while on the other hand $T^2 \times D^2$ has $b_2^0 \neq 0$ and it is a Stein filling of $\mathbb{T}^3$.

**Theorem 5.3.** The minimal page genus among all the open books compatible with the unique Stein fillable contact structure on $\mathbb{T}^3$ is 1.

For $k = 2$ case we weren’t able to verify that the monodromy of $ob'_2$ can be written as a product of right-handed Dehn twists, but obtained another elliptic open book on $Y_2$ which is compatible with a Stein fillable contact structure. To see this, take the plumbing diagram for $Y_2$ in Figure 7. After “blowing-up” this diagram four times we get the plumbing diagram and the corresponding open book in the third row of Table 3. Using a relation [12] in the mapping class group of the torus with four punctures one can see that the monodromy of this new open book can be written as a product of right-handed Dehn twists. In fact, all of the open books given in Table 3 are obtained this way, i.e. by blowing-up the plumbing diagram for $Y_k$ in Figure 7 in a way that the monodromy of the resulting open book can be written in terms of right-handed Dehn twists by using the relations in [12]. We should note that the elliptic open book given in the first row of Table 3 is compatible...


**Table 3. Examples of elliptic open books on some circle bundles over the torus**

| 3–manifold | plumbing | open book |
|------------|----------|-----------|
| $Y_0 = \mathbb{T}^3$ | ![Plumbing Diagram](image1) | ![Open Book Diagram](image2) |
| $Y_1$ | ![Plumbing Diagram](image3) | ![Open Book Diagram](image4) |
| $Y_2$ | ![Plumbing Diagram](image5) | ![Open Book Diagram](image6) |
| $Y_3$ | ![Plumbing Diagram](image7) | ![Open Book Diagram](image8) |
| $Y_4$ | ![Plumbing Diagram](image9) | ![Open Book Diagram](image10) |
with the unique Stein fillable contact structure on \( Y_0 = \mathbb{T}^3 \) and it was first constructed by Van Horn [17] using different techniques.

**Theorem 5.4.** For \( k \leq 4 \) there exists elliptic open books on \( Y_k \) compatible with Stein fillable contact structures.

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