Asymptotic Stability of Solitons to Nonlinear Schrödinger Equations on Star Graphs

Ze Li, Lifeng Zhao

Abstract

In this paper, we prove the asymptotic stability of nonlinear Schrödinger equation on graphs, which partially solves an open problem in D. Noja [6]. The essential ingredient of our proof is the dispersive estimate for the linearized around the soliton with Kirchhoff boundary condition. In order to obtain the dispersive estimates, we use the Born’s series technique and scattering theory for the linearized operator.

**Keywords:** nonlinear Schrödinger equations on graphs; asymptotic stability

1 Introduction

In this paper we study nonlinear Schrödinger equation on star graphs, namely

\[
\begin{aligned}
  i\partial_t u^i &= -\Delta u^i + F(|u^i|^2) u^i, \\
  u^i(0, x) &= u_0^i(x)
\end{aligned}
\]  

(1.1)

where \(u^i(t, x) : [0, \infty)^2 \to \mathbb{C}\), and \(u(t, x)\) satisfies the following Kirchhoff condition on \([0, \infty)^2\),

\[
\begin{aligned}
  u^i(t, 0) &= u^j(t, 0), \forall i, j \in \{1, 2, \ldots, N\}, \\
  \sum_{i=1}^N \frac{d}{dx} u^i(t, 0) &= 0.
\end{aligned}
\]

Nonlinear Schrödinger equations (NLS) in \(\mathbb{R}^n\) and manifolds have been intensively studied in decades. Recently, NLS on graphs become an active research field in the family of dispersive equations.

Before going to mathematical settings, we describe the physical motivations. The two main fields where the NLS on graphs occurs as a nice model are the optics of nonlinear Kerr media and dynamics of Bose-Einstein condensates (BECs). These two different physical situations have
potential or actual applications to graph-like structures. In the fields of nonlinear optics such as arrays of planar self-focusing waveguides and propagation in variously shaped fibre-optic devices, Y-junction, H-junctions and more complex examples can be considered. In S. Gnutzman, U. Smilansky and S. Derevyanko [8], an example of a potential application to signal amplification in resonant scattering on networks of optical fibres is given. In the fields of BECs there has been increasing interest in one-dimensional or graph-like structures, too. In A. Tokuno, M. Oshikawa, E. Demler [13] and I. Zapata, F. Sols [15], boson liquids or condensates are treated in the presence of junctions and defects in analogy with the Tomonaga-Luttinger fermionic liquid theory, with applications to boson Andreev-like reflection, beam splitter or ring interferometers. For more concrete physical interpretations, consult [9], [10]-[12] and references therein.

The linear and cubic Schrödinger equation on simple networks with Kirchhoff conditions and special data has been studied by Cascaval R. C, and Hunter C. T [5]. The local and global well-posedness of NLS on graphs in energy space was proved by R. Adami, C. Cacciapuoti and D. Noja [1], [2]. In D. Noja [6], soliton waves were carefully studied for pure power subcritical nonlinearities, and it was proved that the soliton is orbitally stable in subcritical case.

In D. Noja [6], the asymptotic stability of solitons for NLS on graphs was raised as an open problem. Indeed, [6] conjectured that every solution starting near a standing wave is asymptotically a standing wave up to a remainder which is is a sum of a dispersive term and a tail small in time. The physical interpretation of the concept is that dispersion or radiation at infinity, provides the mechanism of stabilization or relaxation, towards the asymptotic standing wave or more generally solitons. In [6] the author stated it is very difficult to get a dispersive estimate for the linearized operator which partly makes the asymptotic stability tough.

In this paper, we try to solve this problem. However, even for NLS in Euclid space, asymptotic stability is largely open for incompletely integrable system. Because dispersive method only solves the problem for some special nonlinearities, we can not generally expect to solve the conjecture thoroughly at present time. Under some spectral assumptions, with the help of the dispersive method developed by V. S. Buslaev and G. S. Perelman [3], we obtain asymptotic stability for special nonlinearities. The most difficult part is to deduce dispersive estimates for the linearized operator. In Valeria B. and Liviu I. Ignat [14], the dispersive estimates for free Schrödinger operator on graphs was proved. However, it is more difficult to prove the same thing for the linearized operator as addressed by [6]. Inspired by the works of M. Goldberg and W. Schlag [7], we split the proof into the high energy part and low energy part. For the high energy, the further development of the method in [7] can achieve our goal with the essential ingredients of Born series and oscillatory integrations. For the low energy, we use the scattering theory developed in [3], and introduce an analogical scattering representation of the resolvent for linearized operator with Kirchhoff conditions. With the two kinds of techniques, we finally prove the desired dispersive estimates and get the asymptotic stability.
Before giving our main theorem, we introduce some notations, solitons and the linearized operator.

1.1 Preliminaries and Notations

The sole vertex of the star shape graph  is denoted by , and the  edges are denoted by  where . A function defined on means functions defined on . We say  is continuous, if . The space consists of all functions that belong to for each edge , and

Similarly, we can define as

Sobolev spaces consists all continuous functions on that belong to , for each edge, and the norm is defined as

We can also equip and with inner products, namely

and

Now we turn to introduce the Laplace operator on the graph . The details can be found in Cattaneo C. [4]. We point out is self-adjoint with domain

Furthermore, for , it holds

(1.2)
1.2 Solitons

Standing wave solutions to equation (1.1), are \( u^j = w_j(x,t,\sigma_j) \), where

\[
\begin{align*}
  w_j(t,x) &= \exp(-i\beta_j + i\frac{1}{2}v_j x)\varphi(x - b_j;\alpha), \\
  \varphi_{xx} &= \alpha^2\varphi/4 + F(\varphi^2)\varphi, \\
  \sigma_j &= (\beta_j, \omega_j, b_j, v_j), \omega_j = \frac{1}{4}(v_j^2 - \alpha^2).
\end{align*}
\]

Here \( \beta_j, \omega_j, b_j, v_j, \alpha \in \mathbb{R} \), \( \sigma_j \) is the solutions of the following equation

\[
\begin{align*}
  \beta'_j &= \omega_j, \omega'_j = 0, b'_j = v_j, v'_j = 0. \tag{1.3}
\end{align*}
\]

If \( w_j(x,t,\sigma_j) \) satisfies the Kirchhoff condition (K-condition), namely

\[
\begin{align*}
  w_j(0,t,\sigma_j) &= w_k(0,t,\sigma_k); \quad \sum_{j=1,2,\ldots,N} \frac{d}{dx}w_j(0,t,\sigma_j) = 0.
\end{align*}
\]

then we call them solitons.

We assume the following three conditions are satisfied by the nonlinearity \( F \).

(i) \( F \) is a smooth real function admitting the lower estimate

\[
F(\xi) \geq -C_1\xi^q, C_1 > 0, \xi \geq 1, q < 2.
\]

(ii) The point \( \xi = 0 \) is sufficiently strong root of \( F \):

\[
4F(\xi) = C_2\xi^p(1 + O(\xi)), p > 0.
\]

Moreover, let

\[
U(\varphi, \alpha) = -\frac{1}{8}\alpha^2\varphi^2 - \frac{1}{2}\int_0^{\varphi^2} F(\xi)d\xi,
\]

then \( U \) is negative for sufficiently small \( \varphi \) for \( \alpha \neq 0 \).

(iii) For \( \alpha \) belonging to some interval, \( \alpha \in A \subset R_+ \), the function \( \varphi \mapsto U(\varphi, \alpha) \) has a positive root, \( U(\varphi_0, \alpha) \neq 0 \), where \( \varphi_0 = \varphi_0(\alpha) \) is the smallest positive root.

**Remark 1.1** Based on (i), (ii) and (iii), we have the existence of profile \( \varphi \) and it is exponentially decay. The existence of solitons satisfying K-condition was studied in D. Noja [6] for pure power nonlinearities. For the nonlinearities satisfying (i)-(iii), it is easy to verify that (1.1) is globally well-posed in \( H^1 \). The proof is almost the same as NLS, all the ingredients needed especially Strichartz estimates are proved in [1].

Finally, caution that Einstein’s summation convention will not be used, hence the same index up and below does not mean summation.
1.3 Linearized equation

We will follow V. S. Buslaev and G. S. Perelman’s paper [3]. The linearization of (1.1) around the soliton \( \{w_j(x, t; \sigma_j)\} \) is

\[
i\partial_t \chi_j = -\Delta \chi_j + F(|w_j|^2) \chi_j + F'(|w_j|^2) w_j \chi_j + w_j \overline{\chi_j}
\]

If we denote

\[
\chi_j(x, t) = \exp(i \Phi_j f_j(y_j, t), \Phi_j = -\beta_j(t) + \frac{1}{2} v_j x, y_j = x_j - b_j(t).
\]

The function \( f_j \) satisfies the equation

\[
i\partial_t f_j = L(\alpha) f_j,
\]

where

\[
L(\alpha) f = -\Delta f + \alpha^2 f/4 + F(\varphi_j^2) f + F'(\varphi_j^2) (f + \overline{f}), \varphi_j = \varphi(y_j, \alpha).
\]

From this, we can get its complexification:

\[
i\partial_t \vec{f}_j = H(\alpha) \vec{f}_j, \vec{f}_j = (f_j, \overline{f_j})^t,
\]

\[
H(\alpha) = H_0(\alpha) + V(\alpha), H_0(\alpha) = (-\Delta + \alpha^2/4) \theta_3,
\]

\[
V(\alpha) = [F(\varphi_j^2) + F'(\varphi_j^2) \varphi_j^2] \theta_3 + i F'(\varphi_j^2) \varphi_j^2 \theta_2,
\]

where \( \theta_2 \) and \( \theta_3 \) are the matrices:

\[
\theta_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

1.4 Main Theorem

Now we give our main theorem as follows:

**Theorem 1.1.** Consider the Cauchy problem for equation (1.1) with initial data

\[
u^i(0, x) = u_0^i(x), \quad u_0^i(x) = w_j(x; \sigma_0) + \chi_0^j(x),
\]

where \( \{u_0^i(x)\} \) satisfies K-condition, \( \sigma_0 = (\vec{\beta}_0, \vec{\omega}_0, \vec{b}_0, \vec{v}_0) \) is a solution to (1.3), and \( \vec{b}_0 = (0, ..., 0), \vec{v}_0 = (0, ..., 0), \vec{\beta}_0 = (\beta(t), ..., \beta(t)), \vec{\omega}_0 = (\omega, ..., \omega). \)

Assume that the following conditions hold:

(1) The norm

\[
N = \| (1 + |x|^2) \chi_0 \|_2 + \| \chi_0' \|_2
\]
is sufficiently small.

(II) The function $F$ is a polynomial, and the lowest degree is at least four.

(III) Discrete spectral assumption: see (Hypothesis A) in section 3.1.

(IV) The points $\pm \omega$ are not resonances.

(V) Continuous spectrum assumption: see (Hypothesis B) in section 2.

(VI) Non-degenerate assumption: see (Hypothesis C) in section 2.2.

Then there exist $\sigma_+$ and $f_+ \in L^2$ such that

$$u = w(x, \sigma_+(t)) + e^{i\eta t} e^{i\Delta t} f_+ + o(1),$$

as $t \to \infty$.

Here $\sigma_+(t)$ is the trajectory of the system (1.3) with initial data $\sigma(0) = \sigma_+$, $\eta$ is some constant and $o(1)$ assumes the $L^2$ norm. Moreover, $\sigma_+$ is sufficiently close to $\sigma_0$.

We point out that when $F(x) = |x|^p$, all the conditions except spectral assumptions seem natural, we explain this in the following three remarks.

**Remark 1.1** Although it seems strange to set $\vec{b}_0 = (0, \ldots, 0), \vec{v}_0 = (0, \ldots, 0), \vec{\beta}_0 = (\beta(t), \ldots, \beta(t)), \vec{\omega}_0 = (\omega, \ldots, \omega)$, it is the only case when the solitons satisfy K-condition for the pure power nonlinearities and $N$ odd (see D. Noja [6]).

**Remark 1.2** The polynomial assumption (II) is not essential, we use it just for simplicity. However the spectral assumptions from (IV) to (VI) are essential for dispersive estimates. Finally, we emphasize the degree restriction of $F$ prevents us from dealing with mass-subcritical pure power nonlinearities. Even for NLS in Euclid space, the asymptotic stability is largely open when the equation is not completely integrable as mentioned before.

The first step to obtain the dispersive estimates is to get an appropriate expression for the resolvent of the free linearized operator (that is the linearized operator excluding the potentials). This is done in Lemma 2.2 and Remark 2.1. The basic idea is to translate it to an ordinary differential equation with boundary conditions. The decay of the resolvent is essential for the estimates in high energy part. By scattering theory in [3], the resolvent of the linearized operator with Kirchhoff condition is constructed. This resolvent plays an important role in the estimates of low energy.

The second step aims to obtain various estimates. The $L^2$ estimate for Schrödinger operator studied in [7] is a quick corollary of the fact that the potential is real-valued. However for linearized operator considered here, the $L^2$ estimate is more involved. The other difficulty in technique, is that while applying Born’s series, the first term becomes an obstacle, because it does not enjoy enough decay. The solution is to single this term out and take advantage of the known result of dispersive estimates of free Schrödinger operator on graphs. Because of the decay of resolvent, the other terms in Born’s series can be estimated together.
The method described above can treat $L^1$, $L^2$, and weighted estimates together. Indeed by integration by parts, weighted estimates can be transformed into corresponding $L^1$ or $L^2$ estimates.

For the proof of Theorem 1.1, we begin with dispersive estimates, which will be proved for general $N$, and general nonlinearities. In fact, only the spectral assumptions $(IV)$ to $(VI)$ are required.

Different from NLS, in the process of asymptotic analysis we need consider dispersive estimates for the following operator:

$$[\mathcal{H}f]_j = H(\alpha_j) f_j.$$ 

Although in the setting of Theorem 1.1, we only need consider the case when $\alpha_j = \alpha$, but we present most proof in the case when $\alpha_j$ may be distinct for distinguished $j$. Denote the semigroup generated by $i\mathcal{H}$ by $U(t)$, then according to V. S. Buslaev and G. S. Perelman’s paper [3], in order to prove asymptotic stability, we need the following dispersive estimates:

\begin{align*}
\|U(t)Pc h\|_2 &\leq C\|h\|_2, \\
\|U(t)Pc h\|_\infty &\leq Ct^{-1/2}(\|h\|_W + \|h\|_2) \\
\|\rho U(t)Pc h\|_\infty &\leq C(1 + t)^{-3/2}(\|h\rho^{-1}\|_1 + \|h\|_{H^1}) \\
\|\rho^2 U(t)Pc h\|_2 &\leq C(1 + t)^{-3/2}\|h\rho^{-1}\|_1
\end{align*}

where $\rho(x) = (1 + |x|)^{-1}$, and $\|h\|_W = \|h\rho^{-2}\|_2$ or $\|h\rho^{-2}\|_1$.

We emphasize that in the proof of (1.5) we assume $\alpha_j = \alpha$. How we can reduce the asymptotic stability to the dispersive estimates are presented in section 3. And we point out that the dispersive estimate we get here is stronger than that of [3].

2 Dispersive estimates

It is obvious (1.7) is the corollary of (1.6). Hence, it suffices to prove (1.4), (1.5), and (1.6). First we prove (1.5). We split the proof into high energy part and low energy part. The original idea of our proof comes from M. Goldberg and W. Schlag [7].

In order to get dispersive estimates, we need a spectral assumption, namely

**Hypothesis (B)** The continuous spectrum of $\mathcal{H}$ is $\sigma_c(\mathcal{H}) = [w, \infty) \cup (-\infty, -w]$, where $w$ is some positive constant.

The base space is $\mathcal{E} = \{f \in L^2(\Gamma)\}$. $D(\mathcal{H})$ is taken as $D(\Delta_\Gamma)$ given by (1.2).

2.1 $L^1$ estimate: High energy part

For high energy part we have
Lemma 2.1. Let $\lambda_0$ be a constant to be determined, and suppose $\chi$ is a smooth cut-off such that $\chi(\lambda) = 0$ for $\lambda \leq \lambda_0$ and $\chi(\lambda) = 1$ for $\lambda \geq 2\lambda_0$. Then

$$\|e^{i\mathcal{H}t}\chi(\mathcal{H})P_{w_0}f\|_\infty \leq C|t|^{-1/2}\|\rho^{-1}f\|_1,$$

for all $t$.

Before proving Lemma 2.1, we first calculate the resolvent of the free operator $[Jf]_j = (-\Delta + w_j)\theta_3f_j$, where $w_j = \alpha_j^2/4$. Define $R_\lambda f = (\lambda - J)^{-1}f \equiv (R^{1}_{\lambda}f, R^{2}_{\lambda}f)^t$, for $f \in \mathcal{D}(\Delta\Gamma)$. Then it holds that

Lemma 2.2.

$$[R^{1}_{\lambda}f]_j = \sum_{i,l} e^{-\sqrt{w_j-\lambda}s}a_{j,i,i} \sqrt{w_i-\lambda} \int_{0}^{\infty} e^{-\sqrt{w_i-\lambda}y} f_{i,1}(y)dy + \frac{1}{2\sqrt{w_j-\lambda}} \int_{0}^{\infty} e^{-\sqrt{w_j-\lambda|x-y|}} f_{j,1}(y)dy,$$

$$[R^{2}_{\lambda}f]_j = \sum_{i,l} e^{-\sqrt{w_j+\lambda}s}b_{j,i,i} \sqrt{w_i+\lambda} \int_{0}^{\infty} e^{-\sqrt{w_i+\lambda}y} f_{i,2}(y)dy + \frac{1}{2\sqrt{w_j+\lambda}} \int_{0}^{\infty} e^{-\sqrt{w_j+\lambda|x-y|}} f_{j,2}(y)dy,$$

where $a_{j,i,i}, b_{j,i,i}$ are some constants, $m_1 = \sum_i \sqrt{w_i-\lambda}$, $m_2 = \sum_i \sqrt{w_i+\lambda}$, and $\sqrt{w_j-\lambda} (\sqrt{w_j+\lambda})$ is taken such that $\text{Re}(\sqrt{w_j-\lambda}) \geq 0$ (respectively $\text{Re}(\sqrt{w_j+\lambda}) \geq 0$).

Proof Since $JR_\lambda f = -f + \lambda R_\lambda f$, then from Duhamel principle, we have

$$[R^{1}_{\lambda}f]_j = a_{j}e^{-\sqrt{w_j-\lambda}s} + b_{j}e^{\sqrt{w_j-\lambda}s} \frac{1}{2\sqrt{w_j-\lambda}} \int_{0}^{\infty} e^{-\sqrt{w_j-\lambda|x-y|}} f_{j}(y)dy.$$

The fact $f \in L^2(\Gamma)$ implies $b_{j} = 0$. Similarly, we have the same results for $R^{2}_{\lambda}$. And from K-condition, we deduce our lemma. □

Remark 2.1. Define $a_{ij}(\lambda) = \sum_l a_{j,i,1}^l \sqrt{w_l-\lambda}$; $b_{ij}(\lambda) = \sum_l b_{j,i,1}^l \sqrt{w_l+\lambda}$, the resolvent can be written as

$$[R^{1}_{\lambda}f]_j = \sum_{i} e^{-\sqrt{w_j-\lambda}s}a_{ij} \frac{1}{\sqrt{w_i-\lambda}} \int_{0}^{\infty} e^{-\sqrt{w_i-\lambda}y} f_{i,1}(y)dy + \frac{1}{2\sqrt{w_j-\lambda}} \int_{0}^{\infty} e^{-\sqrt{w_j-\lambda|x-y|}} f_{j,1}(y)dy$$

$$[R^{2}_{\lambda}f]_j = \sum_{i} e^{-\sqrt{w_j+\lambda}s}b_{ij} \frac{1}{\sqrt{w_i+\lambda}} \int_{0}^{\infty} e^{-\sqrt{w_i+\lambda}y} f_{i,2}(y)dy + \frac{1}{2\sqrt{w_j+\lambda}} \int_{0}^{\infty} e^{-\sqrt{w_j+\lambda|x-y|}} f_{j,2}(y)dy.$$

When $k > 0$ is sufficiently large, and $\lambda = k^2 + w$, it is easily seen,

$$\sup_{\lambda = w+k^2, k \gg 1} |a_{ij}(\lambda)| + |a'_{ij}(\lambda)| \equiv a_{ij} < \infty; \quad \sup_{\lambda = w+k^2, k \gg 1} |b_{ij}(\lambda)| + |b'_{ij}(\lambda)| \equiv b_{ij} < \infty.$$
We abuse the notation $a_{ij}$ here, but it is easy to distinguish the two meanings according to the context.

**Proof of Lemma 2.1**

For $\lambda \geq w$, let $\lambda = k^2 + w, k \geq 0$, then Lemma 2.2 yields

$$
[R_1^1(\lambda \pm i0)f_j] = \sum_i e^{-s_\pm(j,k)x} a_{ij}(k) \frac{1}{s_\pm(i,k)} \int_0^\infty e^{-s_\pm(i,k)y} f_i,1(y)dy + \frac{1}{2s_\pm(j,k)} \int_0^\infty e^{-s_\pm(j,k)|x-y|} f_j,1(y)dy
$$

$$
[R_1^2(\lambda \pm i0)f_j] = \sum_i e^{-w_j+w+k^2x} b_{ij}(k) \frac{1}{s_\pm(i,k)} \int_0^\infty e^{-w_i+w+k^2y} f_i,1(y)dy
$$

where

$$
s_\pm(j,k) = \begin{cases} 
\frac{\pi i}{\sqrt{w_j-w_1-k^2}}, w_j - w_1 - k^2 \leq 0; \\
\sqrt{w_j-w_1-k^2}, w_j - w_1 - k^2 > 0.
\end{cases}
$$

Define $R_V(\lambda)f = (\lambda I - \mathcal{H})^{-1}f$, for $f \in D(\Delta_\Gamma)$. Then we have the Born series from the decay in $k$ of the free resolvent,

$$
R_V(\lambda \pm 0i) = \sum_{n=0}^\infty R_\lambda(\lambda \pm 0i)((-VR_\lambda(\lambda \pm 0i))_n,
$$

where $V$ can be viewed as a multiplying operator by $2N \times 2N$ function matrix. Precisely, from (2.8) and (2.9), for $k$ sufficiently large, we obtain

$$
\|R_\lambda(\lambda \pm 0i)f\|_\infty \leq C \frac{1}{|k|}\|f\|_1,
$$

then we get

$$
\|VR_\lambda(\lambda \pm 0i)f\|_1 \leq \frac{C}{|k|}\|f\|_1\|V\|_1,
$$

and

$$
\langle R_\lambda(\lambda \pm 0i)(VR_\lambda(\lambda \pm 0i))_n, g \rangle \leq \frac{C}{|k|^{n+1}}\|f\|_1\|g\|_1\|V\|_1^n.
$$

Thus for $k$ sufficiently large, the convergence yields

$$
\langle VR_1(\lambda \pm 0i)f, g \rangle = \sum_{n=0}^\infty \langle R_\lambda(\lambda \pm 0i)((-VR_\lambda(\lambda \pm 0i))_n, g \rangle.
$$

for $f \in L^1$ and $g \in L^1$.

Now we Introduce the truncation function $\zeta(\lambda)$ which has support in the unit ball, and equals 1 in the ball with radial 1/2, and define $\zeta_L = \zeta(\lambda/L)$. In order to prove our lemma, it suffices
to prove
\[ \sup_{L \geq 1} |\langle e^{itH} \chi_L(H) \chi(H) f, g \rangle| \leq C|t|^{-\frac{1}{2}}\|f\|_1\|g\|_1. \]

We have, for \( \lambda \geq w \),
\[ \langle E_{ac}(d\lambda)f, g \rangle = \frac{1}{2\pi i} \langle [RV(\lambda + 0i) - RV(\lambda - 0i)]f, g \rangle d\lambda. \]

Due to Hypothesis (B) and that \( \lambda_0 \) is sufficiently large, \( \chi(H)\chi_L(H)E(\lambda) = \chi(H)\chi_L(H)E_{ac}(d\lambda) \), we deduce
\[ |\langle e^{itH} \chi(H)\chi_L(H)f, g \rangle| = \left| \int_R e^{itx} \chi(x) \zeta_L(x) \langle dE(x)f, g \rangle \right|. \]

Now we only consider \( \lambda \geq w \) in the above integration, the other part namely \( \lambda \leq -w \) will be omitted since it is similar. Letting \( x = k^2 + w \), then we need estimate
\[ \frac{1}{2\pi} \int_0^\infty \left| \langle [RV(k^2 + w + 0i) - RV(k^2 + w - 0i)]f, g \rangle e^{it(k^2+w)} \chi(k^2 + w)\zeta_L(k^2 + w)kd\right| \]
\[ \leq \frac{1}{2\pi} \int_0^\infty \left| \sum_{n=1}^\infty \langle R\chi(k^2 + w - 0i)(-VR\chi(k^2 + w - 0i))n]f, g \rangle e^{it(k^2+w)} \chi(k^2 + w)\zeta_L(k^2 + w)kd\right| \]
\[ + \frac{1}{2\pi} \int_0^\infty \left| \sum_{n=1}^\infty \langle R\chi(k^2 + w - 0i)(-VR\chi(k^2 + w - 0i))n]f, g \rangle e^{it(k^2+w)} \chi(k^2 + w)\zeta_L(k^2 + w)kd\right| \]
\[ + \frac{1}{2\pi} \int_0^\infty \left| \langle [R\chi(k^2 + w - 0i) - R\chi(k^2 + w - 0i)]f, g \rangle e^{it(k^2+w)} \chi(k^2 + w)\zeta_L(k^2 + w)kd\right| . \]

Define \( \chi_L(k^2) = \chi(k^2 + w)\zeta_L(k^2 + w) \), then for the third term in above formula, it suffices to prove,
\[ \left| \int_0^\infty e^{itk^2} \chi_L(k^2)k[R\chi(k^2 + w + i0) - R\chi(k^2 + w - i0)]fkd\right| \leq Ct^{-1/2}\|f\|_1. \]

However, it is equivalent to
\[ \|e^{itJ}\chi_L(J)f\|_\infty \leq Ct^{-1/2}\|f\|_1, \]
which follows from the dispersive estimate of free Schrodinger operator on graphs in [14] and the transformation \( g \rightarrow ge^{iat} \). Now, we consider \( n \geq 1 \).

If \( k \) is large enough such that \( w_j - w_1 - k^2 \leq 0 \), define
\[ \mu(i, k) = \sqrt{w_i + w + k^2}, \quad s(i, k) = -i\sqrt{k^2 - w_j + w}, \]
then the general term for the integral expression to \( (-VR\chi(k^2 + w + 0i))n]f \) is
\[ \sum_{i_1,i_2,...,i_n} \frac{1}{\delta(k, i_1)\delta(k, i_2)...\delta(k, i_n)} \ell_{j,i_1} \ell_{i_1i_2}...\ell_{i_{n-1},i_n}. \]
\[
\int_{[0,\infty)^n} V(x)V(x_n) \cdots V(x_2)f_{i_1,\tau}(x_1) \exp\{ \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \} dx_1 dx_2 \cdots dx_n.
\]

- when \( \ell_{i_p p+1} = \frac{1}{2} \), then \( i_p = i_{p+1} \), and \( \varepsilon(k, i_p)(x_p, x_{p+1}) = s(i_{p+1}, k) |x_{p+1} - x_p| \), or \( \varepsilon(k, i_p)(x_p, x_{p+1}) = \mu(i_{p+1}, k) |x_{p+1} - x_p| \), where we arrange \( x_{n+1} = x \);
- when \( \ell_{i_p p+1} = a_{i_p p+1} \) (or \( b_{i_p p+1} \)), then \( \varepsilon(k, i_p)(x_p, x_{p+1}) = s(i_p, k) x_p + s(i_{p+1}, k) x_{p+1} \) (or \( \varepsilon(k, i_p)(x_p, x_{p+1}) = \mu(i_p, k) x_p + \mu(i_{p+1}, k) x_{p+1} \));
- and \( \delta(k, i_l) = \sqrt{w_i + w_j + k^2} \) or \( \delta(k, i_l) = -i \sqrt{k^2 - w_j + w} \), \( r = 1 \) or \( r = 2 \).

Here we have abused the notation of \( V \), regardless that they mean different potentials.

We take two special terms for explaining how to bound them. The first one is

\[
\frac{1}{(\sqrt{w_1 + w_j + k^2})^n} \sum_{i_1, i_2, \ldots, i_n} b_{j, i_n} b_{i_1, i_2} \cdots b_{i_{n-1}, i_n} \int_{[0,\infty)^n} V(x)V(x_n) \cdots V(x_2)f_{i_1, 2}(x_1) \exp\{ \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \}.
\]

In this case, the corresponding term in \( [R^{\lambda}_\chi(k^2 + w + 0i)(-VR\lambda(k^2 + w + 0i))^n f]_j \) is

\[
\frac{e^{-s(j,k)x}}{s(j,k)} a_{j n+1} \int_0^\infty e^{-s(i_{n+1}, k)x_{n+1}} \frac{1}{(\sqrt{w_1 + w_j + k^2})^n} \sum_{i_1, i_2, \ldots, i_n} b_{j, i_n} b_{i_1, i_2} \cdots b_{i_{n-1}, i_n} \int_{[0,\infty)^n} V(x_{n+1})V(x_n) \cdots V(x_2)f_{i_1, 2}(x_1) \exp\{ \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \} dx_1 \cdots dx_{n+1},
\]

\[
+ \frac{1}{2s(j, k)} \int_0^\infty e^{-s(j,k)(x-x_{n+1})} \frac{1}{(\sqrt{w_1 + w_j + k^2})^n} \sum_{i_1, i_2, \ldots, i_n} b_{j, i_n} b_{i_1, i_2} \cdots b_{i_{n-1}, i_n} \int_{[0,\infty)^n} V(x_{n+1})V(x_n) \cdots V(x_2)f_{i_1, 2}(x_1) \exp\{ \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \} dx_1 \cdots dx_{n+1}.
\]

From Fubini theorem, in order to estimate \( \langle e^{itH} \chi(H) \zeta_L(H) \mathbf{f}, \mathbf{g} \rangle \), we need to estimate

\[
\int_{[0,\infty)^{n+2}} g(x)V(x_{n+1})V(x_n) \cdots V(x_2)f_{i_1, 2}(x_1) dx_1 \cdots dx_{n+1} dx \int_0^\infty e^{it(k^2 + w)} \chi_L(k^2 + w) \left( \sum_{i_1, i_2, \ldots, i_n} b_{i_1, i_2} \cdots b_{i_{n-1}, i_n} a_{j n+1} \frac{e^{-s(j,k)x-s(i_{n+1}, k)x_{n+1}}}{s(j,k)} \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \right) \left( \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \right)
\]

\[
+ \int_{[0,\infty)^{n+2}} g(x)V(x_{n+1})V(x_n) \cdots V(x_2)f_{i_1, 2}(x_1) dx_1 \cdots dx_{n+1} dx \int_0^\infty e^{it(k^2 + w)} \chi_L(k^2 + w)
\]

\[
\sum_{i_1, i_2, \ldots, i_n} b_{i_1, i_2} \cdots b_{i_{n-1}, i_n} b_{i_n, i_{n+1}} \frac{1}{2s(j,k)} e^{-s(j,k)(x-x_{n+1})} \left( \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \right) \left( \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \right)
\]

\[
\sum_{i_1, i_2, \ldots, i_n} b_{i_1, i_2} \cdots b_{i_{n-1}, i_n} b_{i_n, i_{n+1}} \frac{1}{(\mu(k))^n} \int_0^\infty e^{it(k^2 + w)} \chi_L(k^2 + w) dx_1 \cdots dx_{n+1} dx.
\]
Let $\vec{x} = (x_1, \ldots, x_{n+1})$ and

$$\Theta(\vec{x}, k) = \sum_{i_1, i_2, \ldots, i_n, i_{n+1}} b_{i_{n+1}, i_{n+1}}(k) b_{i_1, i_2}(k) \cdots b_{i_{n-1}, i_n}(k) a_{ji_{n+1}}(k) e^{-s(i_{n+1}, k)x_{n+1}} \exp\left\{ \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \right\} s(j, k) \frac{e^{-s(j, k)x_j}}{(\mu(k))^n}$$

we claim

$$\left| \int_0^\infty e^{it(k^2 + w)} \lambda_0^{n/2} \chi_L(k^2 + w) e^{-s(j, k)x} \Theta(\vec{x}, k) dk \right| \leq Ct^{-1/2} |\vec{x}| \left( \sum_{i,j} N_{ij} + 1 \right)^n \cdot (2.10)$$

Recall

$$(e^{-it\Delta f})(b, t) = \int_{\mathbb{R}} e^{it^2} e^{ibk} \tilde{f}(k) dk,$$

then from changing of variables, dispersive estimates of one-dimensional Schrödinger equation, the inequality $\|F(f)\|_1 \leq C\|f\|_{H^1}$, and $n \geq 1$, we deduce

$$\left| \int_0^\infty e^{it(k^2 + w)} \lambda_0^{n/2} \chi_L(k^2 + w) e^{-s(j, k)x} \Theta(\vec{x}, k) dk \right|$$

$$= \left| \int_0^\infty e^{it(k^2 + w)} \lambda_0^{n/2} \chi_L(k^2 + w) e^{i\sqrt{k^2 - w_j + w_j} x} \Theta(\vec{x}, k) dk \right|$$

$$\leq \left| \int_0^\infty e^{it(k^2 + w_j)} \lambda_0^{n/2} \chi_L(k^2 + w_j) e^{ikx} k^{(k^2 + w_j - w)}^{-1/2} \Theta(\vec{x}, \sqrt{k^2 + w_j - w}) dk \right|$$

$$\leq Ct^{-1/2} \left| \int_0^\infty e^{i\omega_j t} \lambda_0^{n/2} \chi_L(k^2 + w_j) e^{i\omega_j t} k^{(k^2 + w_j - w)}^{-1/2} \Theta(\vec{x}, \sqrt{k^2 + w_j - w}) \right|_{1}$$

$$\leq Ct^{-1/2} \left| \int_0^\infty e^{i\omega_j t} \lambda_0^{n/2} \chi_L(k^2 + w_j) e^{i\omega_j t} k^{(k^2 + w_j - w)}^{-1/2} \Theta(\vec{x}, \sqrt{k^2 + w_j - w}) \right|_{H^1}$$

$$\leq \left| \sum_{i,j} N_{ij} + 1 \right|^n \cdot (2.10)$$

The second term comes from

$$[R^2_{\chi}(k^2 + w + 0i)(-VR\chi(k^2 + w + 0i))^n f]_j,$$

namely

$$\int_{\mathbb{R}^{n+1}} g(x) V(x_{n+1}) V(x_n) \cdots V(x_2) f_{i_1, i_2}(x_1) dx_1 \cdots dx_{n+1} dx \int_0^\infty e^{it(k^2 + w)} \chi_L(k^2 + w)$$

$$\sum_{i_1, i_2, \ldots, i_n, i_{n+1}} b_{i_1, i_{n+1}} b_{i_1, i_2} \cdots b_{i_{n-1}, i_n} b_{ji_{n+1}} e^{-\sqrt{w + w_j + w} x - \sqrt{w + w_{n+1} + k^2} x_{n+1}} \exp\left\{ \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \right\} \frac{\exp\{ \sum_{p=1,2,\ldots,n} \varepsilon(k, i_p)(x_p, x_{p+1}) \}}{(\mu(k))^n} kdk.$$
Let
\[
\Omega(\vec{x}, k) = e^{itw} \chi_L (k^2 + w) \sum_{i_1, j_1, ..., i_n j_n} b_{i_n+1,j_n+1} b_{i_1,i_2} ... b_{i_{n-1},i_n} b_{j_{n+1}} e^{-\sqrt{w+w_j+k^2 x} - \sqrt{w+w_{j+1}+k^2 x_{n+1}}} \frac{\sum_{i_1,j_1,...,i_n j_n} b_{i_1,j_1} e^{-\sqrt{w+w_j+k^2 x}}}{(\mu(k))^n} \]
then from Parseval identity,
\[
\left| \int_0^\infty e^{itk^2} \Omega(\vec{x}, k) \lambda_0^{n/2} dk \right| \leq \left\| F e^{itk^2} \right\|_\infty \left\| F(\lambda_0^{n/2} \Omega(\vec{x}, k)) \right\|_1 \leq Ct^{-1/2} \left\| \lambda_0^{n/2} \Omega(\vec{x}, k) \right\|_{H^1} \leq C t^{-1/2} \left( \sum_{i,j} b_{j,i} \right)^n,
\]
where we have used
\[
\frac{k|x|}{\sqrt{w+w_j+k^2}} e^{-\sqrt{w+w_j+k^2 x}} \leq \frac{k}{w+w_j+k^2}.
\]

The other terms in \( \langle e^{it\mathcal{H}} \chi(\mathcal{H}) \zeta_L(\mathcal{H}) f, g \rangle \) can be estimated similarly. Therefore (2.21) and (2.11) give
\[
\langle e^{it\mathcal{H}} \chi(\mathcal{H}) \zeta_L(\mathcal{H}) f, g \rangle \\
\leq \sum_{n=0}^{\infty} (\sqrt{\lambda_0})^{-n} \left\| (|x| + 1)V \right\|_1^n \left\| f(|x| + 1) \right\|_1 \left\| g \right\|_1 t^{-1/2} \left( \sum_{i,j} a_{j,i} + b_{j,i} + \frac{1}{2} \right)^n \leq Ct^{-1/2} \left\| (|x| + 1)f \right\|_1 \left\| g \right\|_1.
\]

Thus Lemma 2.1 follows because \( V \) is exponentially decay and \( \lambda_0 \) is sufficiently large.

### 2.2 \( L^1 \) estimate: Low energy part

Before going to the low energy part, we recall some results in \[3\]. For convenience, we use almost the same notations. Consider the eigenvalue problem \( H(\alpha) \zeta = E \zeta \), define
\[
k = \sqrt{E - w}, \mu = \sqrt{E + w},
\]
where \( \text{Re} k \geq 0 \), and \( \text{Re} \mu \geq 0 \). Then for \( D = \{ \mu, k : \text{Re} \mu - \text{Im} k \geq \delta, \text{Im} k > -\delta \} \), where \( \delta > 0 \) is sufficiently small, it holds uniformly in \( D \) that there exists solutions \( \zeta_1 \) and \( \zeta_2 \) satisfying
\[
\zeta_1 - e^{-\mu x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = O(e^{-\gamma x}), \quad x \to \infty
\]
\[ \zeta_2 - e^{ikx} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - e^{-ix} h(k) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = O(e^{-\gamma x - imkx}), \, x \to \infty, \quad (2.12) \]

where \( h(k) = O(1 + |k|)^{-1} \). Define

\[ F_1(x, k) = (\zeta_2, \zeta_1), \quad G_2 = F_1(-x, k) \quad (2.13) \]

then the resolvent \( R(E) = (H - E)^{-1} \) has the integral kernel

\[ G(x, y, E) = \begin{cases} F_1(x, E) D^{-1}(E) G_2(y, E) \theta_3, & y \leq x; \\ G_2(x, E) D^{-1}(E) F_1(y, E) \theta_3, & y \geq x. \end{cases} \quad (2.14) \]

Meanwhile,

\[ G(x, y, E + i0) - G(x, y, E - i0) = -\frac{1}{2ik} \Lambda(x, k) \Lambda^*(y, k) \theta_3, \quad (2.15) \]

where \( E = k^2 + E_0, \, \Lambda(x, k) = (e(x, k), e(x, -k)) \), and \( e(x, k) \) has the asymptotic representation:

\[ e(x, k) = \begin{cases} s(k) \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} + O(e^{-\gamma x} (k)^{-1}); & k \geq 0 \\ e^{ikx} + r(-k)e^{-ikx} \\ 0 \end{pmatrix} + O(e^{-\gamma x} (k)^{-1}); & k \leq 0 \end{cases} \quad (2.16) \]

Moreover it was proved in Proposition 2.1.1 in [3] that there exit solutions to eigenvalue problem

\[ \mathcal{F}(x, k) = se^{ikx} \left[ e + O(e^{-\gamma x}) \right], \, x \to \infty, \]

and

\[ \mathcal{G}(x, k) = e^{-ikx} \left[ e + O(e^{-\gamma x}) \right] + r(k)e^{ikx} \left[ e + O(e^{-\gamma x}) \right], \, x \to \infty, \]

where \(|s|^2 + |r|^2 = 1, \, r\overline{s} + s\overline{r} = 0\), and \( e = (1, 0)^t \).

Notice that all the asymptotic relations above can be differentiated by \( \xi \) and \( x \).

Now we are ready to give the integral kernel for our resolvent \( R_V \).

**Lemma 2.3.** We have solutions to eigenvalue problem \( \mathcal{F} \) and \( \mathcal{G} \) such that

\[ \mathcal{F}(x, k) = se^{ikx} \left[ e + O(e^{-\gamma x}) \right], \, x \to \infty, \]

and

\[ \mathcal{G}(x, k) = e^{-ikx} \left[ e + O(e^{-\gamma x}) \right] + r(k)e^{ikx} \left[ e + O(e^{-\gamma x}) \right], \, x \to \infty, \]

where \(|s|^2 + |r|^2 = 1, \, r\overline{s} + s\overline{r} = 0\), and \( e = (1, 0)^t \).

**Proof** Set \( \mathcal{F} = \mathcal{F}, \, \mathcal{G} = \mathcal{G} - \frac{r}{s}\mathcal{F} \), then the lemma follows.
Lemma 2.4. Define $\tilde{\mathfrak{g}}_j$ as the $\mathfrak{g}$ when $E = \frac{1}{4} \alpha_j^2$, and similarly define $\mathfrak{g}_j$. Then

$$[R_V(k^2 + w + i0)f]_j = c_j\mathfrak{g}_j(x,k) + e_j\tilde{\mathfrak{g}}_j + \int_0^\infty G_j(x,y,k)f_j(y)dy. \quad (2.17)$$

$$[R_V(k^2 + w - i0)f]_j = d_j\mathfrak{g}_j(x,k) + h_j\tilde{\mathfrak{g}}_j + \int_0^\infty G_j(x,y,k)f_j(y)dy. \quad (2.18)$$

where

$$c_j = \frac{N_j,l(k)}{W(k)} \int_0^\infty G_l(0,y,k)f_i(y)dy + \frac{M_j,l(k)}{W(k)} \int_0^\infty \partial_y G_l(0,y,k)f_i(y)dy$$

$$e_j = \frac{N_j,l(k)}{W(k)} \int_0^\infty G_l(0,y,k)f_i(y)dy + \frac{M_j,l(k)}{W(k)} \int_0^\infty \partial_y G_l(0,y,k)f_i(y)dy$$

$$d_j = \frac{\tilde{N}_j,l(k)}{W(k)} \int_0^\infty G_l(0,y,k)f_i(y)dy + \frac{M_j,l(k)}{W(k)} \int_0^\infty \partial_y G_l(0,y,k)f_i(y)dy$$

$$h_j = \frac{\tilde{N}_j,l(k)}{W(k)} \int_0^\infty G_l(0,y,k)f_i(y)dy + \frac{\tilde{N}_j,l(k)}{W(k)} \int_0^\infty \partial_y G_l(0,y,k)f_i(y)dy.$$ 

Proof Generally, we have

$$[R_V(\lambda)(f)]_j = c_j\mathfrak{g}_j + e_j\tilde{\mathfrak{g}}_j + d_{j,1}\mathfrak{g}_j + d_{j,2}\tilde{\mathfrak{g}}_j - \int_0^\infty G_j(x,y,E)f_j(y)dy.$$ 

For $\lambda = k^2 + w + i\varepsilon, \varepsilon > 0$, then $L^2$ condition makes $d_{j,i} = 0$.

Considering the K-condition, denote $c = (c_1, e_1, e_2, e_2, ..., c_N, e_N)$, then $c$ solves

$$Ac = Y,$$

where

$$A = \begin{pmatrix}
\mathfrak{g}_1(0,k) & \tilde{\mathfrak{g}}_1(0,k) & -\mathfrak{g}_2(0,k) & -\tilde{\mathfrak{g}}_2(0,k) & 0 \\
\mathfrak{g}_2(0,k) & \tilde{\mathfrak{g}}_2(0,k) & -\mathfrak{g}_3(0,k) & -\tilde{\mathfrak{g}}_3(0,k) & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\partial_y \mathfrak{g}_1(0,k) & \partial_y \tilde{\mathfrak{g}}_1(0,k) & \partial_y \mathfrak{g}_2(0,k) & \partial_y \tilde{\mathfrak{g}}_2(0,k) & \cdots \\
\partial_y \mathfrak{g}_2(0,k) & \partial_y \tilde{\mathfrak{g}}_2(0,k) & \partial_y \mathfrak{g}_3(0,k) & \partial_y \tilde{\mathfrak{g}}_3(0,k) & \cdots
\end{pmatrix}$$

and

$$Y = \left(\int_0^\infty G_2(0,y,k)dy - \int_0^\infty G_1(0,y,k)dy, \cdots, \sum_j \int_0^\infty \partial_y G_j(0,y,k)\right)^t.$$ 

Denote $W(k) = det(A)$, then we get (2.17), (2.18) is similar.

Next, we assume
Hypothesis (C')

\[
\begin{align*}
\frac{N_{j,l}(k)}{W(k)}, & \quad \frac{M_{j,l}(k)}{W(k)}, & \quad \frac{\overline{N}_{j,l}(k)}{W(k)}, & \quad \frac{\overline{M}_{j,l}(k)}{W(k)}, \\
\frac{\widetilde{N}_{j,l}(k)}{W(k)}, & \quad \frac{\widetilde{M}_{j,l}(k)}{W(k)}, & \quad \frac{\widetilde{N}_{j,l}(k)}{W(k)}, & \quad \frac{\widetilde{N}_{j,l}(k)}{W(k)}, \\
\end{align*}
\]

are analytic near 0.

In the setting of Theorem 1.1, namely \(\alpha_j = \alpha\), Hypothesis (C') reduces to

**Hypothesis C** When \(k = 0\), we have \(\det(\mathcal{F}(0, k)) \neq 0\), \(\det(\partial_x \mathcal{F}(0, k), \partial_x \mathcal{F}(0, k)) \neq 0\).

**Lemma 2.5.** Define a truncation function \(\psi(x)\) which equals 1 in the ball of radial \(2\lambda_0\), and vanishes outside \(3\lambda_0\), then

\[
\|e^{itH}\psi(H)P_{ac}f\|_\infty \leq Ct^{-1/2} (\|f\|_2 + \|f\|_W).
\]

**Proof** From Lemma 2.4 and (2.15), for \(\lambda = k^2 + w\), we deduce

\[
[E_{ac}(d\lambda)]_j = \frac{1}{2\pi i} \left[ c_j \mathcal{F}_j(x, k) + e_j \mathcal{G}_j(x, k) - d_j \mathcal{G}_j(x, k) - h_j \mathcal{G}_j(x, k) \right] dk \\
+ \frac{1}{2i} \Lambda_j(x, k) \Lambda^*_j(y, k) \theta_3 dk,
\]

where \(\Lambda_j\) is \(\Lambda\) when \(E = \frac{1}{4} \alpha_j^2\). Thus

\[
\left[ e^{itH} \psi(H)P_{ac}f \right]_j \\
= \left[ \int e^{it\lambda} \psi(\lambda) E_{ac}(d\lambda)f \right]_j \\
= \frac{1}{2\pi i} \int_0^\infty \left[ e^{it(k^2+w)} \psi(k) [c_j \mathcal{F}_j(x, k) + e_j \mathcal{G}_j(x, k) - d_j \mathcal{G}_j(x, k) - h_j \mathcal{G}_j(x, k)] dk \\
+ \frac{1}{2i} \int_0^\infty e^{it(k^2+w)} \psi(k) \Lambda(x, k) \Lambda^*(y, k) \theta_3 f_j(y) dk \right]
\]

(2.19)

(2.20) has been dealt with in [3]. It suffices to prove (2.19). In fact, we only need to estimate

\[
\int_0^\infty e^{itw+itk^2} \psi(k) c_j \mathcal{F}_j(x, k) dk,
\]

since the other terms are similar. For this term, from Parseval identity, we obtain

\[
\int_0^\infty e^{itw+itk^2} \psi(k) c_j \mathcal{F}_j(x, k) dk \\
\leq \left\| F_k(e^{itw+itk^2}) \right\|_\infty \left\| F_k[\psi(k)c_j \mathcal{F}_j(x, k)] \right\|_1
\]

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For $I$, by (2.14), (2.12), (2.13), Lemma 2.3 and Hypothesis C, it is easily seen

\[
\left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} G_j(0, y, k) \psi(k) \tilde{\mathcal{F}}_j(x, k) \right) \right\|_1 \\
\leq \left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} \begin{pmatrix} 1 & 0 \\ h(k) & 1 \end{pmatrix} D^{-t}(k) \begin{pmatrix} e^{iky} & 0 \\ 0 & 0 \end{pmatrix} \psi(k) \begin{pmatrix} s(k) e^{ikx} \\ 0 \end{pmatrix} \right) \right\|_1 \\
+ \left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} \begin{pmatrix} 1 & 0 \\ h(k) & 1 \end{pmatrix} D^{-t}(k) \begin{pmatrix} e^{iky} & 0 \\ 0 & 0 \end{pmatrix} \psi(k) O((k)^{-1} e^{-\gamma x}) \right) \right\|_1 \\
+ \left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} \psi(k) \begin{pmatrix} s(k) e^{ikx} \\ 0 \end{pmatrix} O((k)^{-1} e^{-\gamma'}) \right) \right\|_1 \\
\leq \left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} \begin{pmatrix} 1 & 0 \\ h(k) & 1 \end{pmatrix} D^{-t}(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} s(k) \psi(k) \right) (\xi - x - y) \right\|_1 \\
+ \left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} \begin{pmatrix} 1 & 0 \\ h(k) & 1 \end{pmatrix} D^{-t}(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi(k) O((k)^{-1} e^{-\gamma x}) \right) (\xi - y) \right\|_1 \\
+ \left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} \psi(k) \begin{pmatrix} s(k) \\ 0 \end{pmatrix} O((k)^{-1} e^{-\gamma'}) \right) (\xi - x) \right\|_1 \\
\leq \left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} \begin{pmatrix} 1 & 0 \\ h(k) & 1 \end{pmatrix} D^{-t}(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} s(k) \psi(k) \right) \right\|_1 \\
+ \left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} \begin{pmatrix} 1 & 0 \\ h(k) & 1 \end{pmatrix} D^{-t}(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi(k) O((k)^{-1} e^{-\gamma x}) \right) \right\|_1 \\
+ \left\| F_k \left( \frac{N_{i,j}(k)}{W(k)} \psi(k) \begin{pmatrix} s(k) \\ 0 \end{pmatrix} O((k)^{-1} e^{-\gamma'}) \right) \right\|_1 \\
\leq \left\| \frac{N_{i,j}(k)}{W(k)} \begin{pmatrix} 1 & 0 \\ h(k) & 1 \end{pmatrix} D^{-t}(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} s(k) \psi(k) \right\|_{H^1}
\]
\[\begin{align*}
&\frac{N_{ij}(k)}{W(k)} \left( \begin{array}{cc}
1 & 0 \\
h(k) & 1
\end{array} \right) D^{-i}(k) \left( \begin{array}{cc}
1 & 0 \\
0 & 0
\end{array} \right) \psi(k) O(\langle k \rangle^{-1} e^{-\gamma x}) \\
&\frac{N_{ij}(k)}{W(k)} \psi(k) \left( \begin{array}{c}
s(k) \\
0
\end{array} \right) O(\langle k \rangle^{-1} e^{-\gamma y}) \\
&\leq C
\end{align*}\]

\[\Pi\] is almost the same. For \(\lambda = -k^2 - w\), the proof is similar and we omit it. Hence, the Lemma follows.

### 2.3 \(L^2\) estimates

**Lemma 2.6.** For the \(\chi\) in Lemma 2.1, we have

\[\| e^{it\mathcal{H}} \chi(\mathcal{H}) P_{ac} f \|_2 \leq C \| f \|_2.\]

**Proof** We use Born’s series again. Notice that \(n = 0\) is trivial. Indeed, in this case, it reduces to the dispersive estimates for the free operator \(e^{itJ}\). For \(e^{itJ}\), consider

\[i\partial_t u^i = (-\Delta u^i + w_i)\theta_3 u^i,\]  
\[(2.21)\]

and \(\{u^i\}\) satisfies Kirchhoff condition, where \(w_i = \frac{1}{4}\alpha_i^2\). Then \(L^2\) estimate comes immediately from multiplying \((2.21)\) with \(\overline{w^i}\) and taking inner products.

From now on, we suppose \(n \geq 1\). We pick up a term in \(e^{it\mathcal{H}} \chi(\mathcal{H}) P_{ac} f\) to illustrate the ideas, namely

\[\int_{[0,\infty)^{n+1}} V(x_{n+1})V(x_n) \cdots V(x_2)f_{i_1,2}(x_1)dx_1 \cdots dx_{n+1} \int_0^\infty e^{it(k^2 + w)} \chi_L(k^2 + w) \sum_{i_1,i_2,\ldots,i_n} \frac{b_{i_1,i_2,\ldots,i_{n-1}},i_n a_{j_{i_{n+1}}} e^{-s(j,k)x - s(i_{n+1},k) x_{n+1}}}{s(j,k)} \exp\{ \sum_{p=1,2,\ldots,n} \varepsilon(k,i_p)(x_p, x_{p+1}) \} \sum_{i_1,i_2,\ldots,i_n} \frac{b_{i_1,i_2,\ldots,i_{n-1}},i_n a_{j_{i_{n+1}}} e^{-s(i_{n+1},k)x_{n+1}}}{s(j,k)} \exp\{ \sum_{p=1,2,\ldots,n} \varepsilon(k,i_p)(x_p, x_{p+1}) \} \sum_{i_1,i_2,\ldots,i_n} \frac{b_{i_1,i_2,\ldots,i_{n-1}},i_n a_{j_{i_{n+1}}} e^{-s(i_{n+1},k)x_{n+1}}}{s(j,k)} \exp\{ \sum_{p=1,2,\ldots,n} \varepsilon(k,i_p)(x_p, x_{p+1}) \} \frac{1}{(\mu(k))^n} dk.\]

Let \(\vec{x}_1 = (x_2, x_3, \ldots, x_{n+1})\), and

\[\Xi(k, \vec{x}_1) = \int_0^\infty e^{-\mu(k)x^1} f_{i_1,2}(x_1)dx_1 e^{it(k^2 + w)} \chi_L(k^2 + w) \sum_{i_1,i_2,\ldots,i_n} \frac{b_{i_1,i_2,\ldots,i_{n-1}},i_n a_{j_{i_{n+1}}} e^{-s(i_{n+1},k)x_{n+1}}}{s(j,k)} \exp\{ \sum_{p=1,2,\ldots,n} \varepsilon(k,i_p)(x_p, x_{p+1}) \} \frac{1}{(\mu(k))^n} dk.\]
Then by change of variables, Parseval identity and Hölder inequality, we have

\[
\left\| \int_0^\infty e^{-s(j,k)x} \Xi(x,1,k) \lambda_0^{n/2} dk \right\|_{L^2(dx)} = \left\| \int_0^\infty e^{-i\sqrt{k^2-w_j+w} \lambda_0^{n/2} \Xi(x,1,k) dk \right\|_{L^2(dx)} \\
\leq \left\| \int_0^\infty e^{-ikx \lambda_0^{n/2} \Xi(x,1,k) dk} \right\|_{L^2(dx)} \\
\leq \left\| \lambda_0^{n/2} \Xi(x,1,k) \right\|_{L^2(dx)} \\
C \left\| \int_0^\infty e^{-\mu(k)x} f_{i_1,i_2}(x) dx \right\|_{L^2(dx)} \left( \sum_{i,j=1}^n a_{i,j} + b_{i,j} + \frac{1}{2} \right)^n \leq C \left\| f \right\|_2 \left( \sum_{i,j=1}^n a_{i,j} + b_{i,j} + \frac{1}{2} \right)^n,
\]

where we have used \( \left\| e^{-\mu(k)x} \right\|_{L^2(dx)} \leq C(\lambda_0) \).

Besides this type, we illustrate the following one, which is another typical representative in all terms of \( e^{it\mathcal{H}} \chi(\mathcal{H})P_{ac} f \):

\[
\int_{[0,\infty)^{n+1}} V(x_{n+1})V(x_n) \cdots V(x_2)f_{i_1,i_2}(x_1)dx_1 \cdots dx_{n+1} \int_0^\infty e^{-it(k^2+w)} \chi_L(k^2+w) \\
\sum_{i_1,i_2,\ldots,i_n} b_{i_1,i_2}\cdots b_{i_{n-1},i_n}a_{ji_{n+1}} e^{-\sqrt{k^2+w+jx-s(i_{n+1},k)x_{n+1}}} \exp\left\{ \sum_{p=1,2,\ldots,n} \varepsilon(k,i_p)(x_p,x_{p+1}) \right\} \left( \mu(k) \right)^n \frac{1}{s(j,k)} \right\|_{L^2(dx)} \right|_{L^2(dx)} \\
\leq \left\| f \right\|_2 \left( \sum_{i,j=1}^n a_{i,j} + b_{i,j} + \frac{1}{2} \right)^n \leq C(\lambda_0) \left( \sum_{i,j=1}^n a_{i,j} + b_{i,j} + \frac{1}{2} \right)^n \left\| f \right\|_2.
\]

The other terms in \( e^{it\mathcal{H}} \chi(\mathcal{H})P_{ac} f \) can be treated similarly. Thus we have proved our result.
Lemma 2.7. For $\psi$ in Lemma 2.5, it holds
\[ \|e^{it\mathcal{H}}\psi(\mathcal{H})P_{ac}f\|_2 \leq C\|f\|_2. \]

Proof From the integral expression of resolvent $R_V$ in Lemma 2.5, it suffices to prove
\[ \left\| \int_0^\infty e^{it^2+itw} \psi(k)c_j(k)k\mathfrak{F}_j(x,k)dk \right\|_2 \leq C\|f\|_2, \quad (2.22) \]
since the $\Lambda$ term has been proved in [3], and the other terms are similar. For (2.22), from the asymptotic representation of $\mathfrak{F}_j$, we have
\[
\left\| \int_0^\infty e^{it^2+itw} \psi(k)c_j(k)k\mathfrak{F}_j(x,k)dk \right\|_2 \\
\leq \left\| \int_0^\infty e^{it^2+itw} \psi(k)c_j(k)ks_j(k)e^{ixk}dk \right\|_2 + \left\| \int_0^\infty e^{it^2+itw} \psi(k)c_j(k)kO(e^{-\gamma x})dk \right\|_2 \\
\leq C\|c_j(k)ks_j(k)\psi(k)\|_2 + C\|c_j(k)k\psi(k)\|_2 \\
\leq C\|c_j(k)\psi(k)\|_2.
\]

We write
\[
c_j(k) = \frac{N_{i,j}(k)}{W(k)} \int_0^\infty G_i(0,y,k)f_i dy + \frac{M_{i,j}(k)}{W(k)} \int_0^\infty \partial_y G_i(0,y,k)f_i dy \\
\equiv I + II.
\]

From the asymptotic relations, we have
\[
I = \frac{N_{i,j}(k)}{W(k)} \int_0^\infty \begin{pmatrix} 1 & 0 \\ h(k) & 1 \end{pmatrix} \begin{pmatrix} e^{iky} & 0 \\ 0 & 0 \end{pmatrix} \partial_y f_i dy + \frac{N_{i,j}(k)}{W(k)} \int_0^\infty O(e^{-\gamma y})f_i dy.
\]

By Parseval identity, we deduce
\[
I \leq C\|f\|_2.
\]

$II$ can be estimated similarly. Hence
\[
\|c_j(k)\psi(k)\|_2 \leq \|f\|_2.
\]

Thus we finish the proof of Lemma 2.7. Combined with Lemma 2.6, we have proved (1.4).
2.4 Weighted estimates

**Lemma 2.8.** For $\chi$ in Lemma 2.1, we have
\[
\| \rho(x)e^{it\mathcal{H}\chi(H)}P_{ac}f \|_\infty \leq Ct^{-3/2} \| \rho(x)^{-1}f \|_1.
\]

**Proof** The proof is almost the same as the proof of Lemma 2.1 except for the first step. We use the following example to show how an integration by parts leads to the $t^{-3/2}$ decay:
\[
\int_0^\infty e^{it(k^2+w)}k\chi L(k^2+w) \sum_{i_1,i_2,\ldots,i_n} b_{i_1,i_2} \cdots b_{i_{n-1},i_n} a_{j_{i_{n+1}}} e^{-\sqrt{k^2+w+x-s(i_{n+1},k)x_{n+1}}} \frac{e^{-\sqrt{k^2+w+x-s(j,k)x_n}}}{s(j,k)}
\exp\left\{ \sum_{p=1,2,\ldots,n} \varepsilon(k,i_p)(x_p,x_{p+1}) \right\}
\frac{\mu(k)^n}{(\mu(k))^n} d\Gamma(k,\bar{x}).
\]

Define
\[
\Gamma(k,x,\bar{x}) = \chi L(k^2+w) \sum_{i_1,i_2,\ldots,i_n} b_{i_1,i_2} \cdots b_{i_{n-1},i_n} a_{j_{i_{n+1}}} e^{-\sqrt{k^2+w+x-s(i_{n+1},k)x_{n+1}}} \frac{e^{-\sqrt{k^2+w+x-s(j,k)x_n}}}{s(j,k)}
\exp\left\{ \sum_{p=1,2,\ldots,n} \varepsilon(k,i_p)(x_p,x_{p+1}) \right\}
\frac{\mu(k)^n}{(\mu(k))^n},
\]
then
\[
\int_0^\infty \Gamma(k,x,\bar{x})ke^{it(k^2+w)} dk
\leq C \frac{1}{t} \int_0^\infty \Gamma(k,x,\bar{x}) \frac{d}{dk}e^{it(k^2+w)} dk
\leq C \frac{1}{t} \int_0^\infty \frac{d}{dk} \Gamma(k,x,\bar{x})e^{it(k^2+w)} dk.
\]

Then same arguments as Lemma 2.1 imply our desired result. The other terms are similar, thus we have proved our Lemma.

For low energy part, we use the same technique.

**Lemma 2.9.** For $\psi$ in Lemma 2.5, then under the Hypothesis (C), it holds
\[
\| \langle x \rangle^{-1} e^{it\mathcal{H}} \psi(H) P_{ac} f \|_\infty \leq Ct^{-3/2} \| \langle x \rangle f \|_1.
\]

Since the weighted dispersive estimates we give here is stronger than \[3\], we have to deal with
A term differently. By noticing $\Lambda(x,0) = 0$, and it is analytic with respect to $k$ (see [3]), we have

$$\int_0^\infty e^{itk^2 + itw} \psi(k) \Lambda_j(x,k) \Lambda^*_j(y,k) \theta_3 f_j(y) dydk$$

$$= \frac{1}{2it} \int_0^\infty \frac{d}{dk} \left( e^{itk^2 + itw} \right) \frac{1}{k} \psi(k) \Lambda_j(x,k) \Lambda^*_j(y,k) \theta_3 f_j(y) dydk$$

$$= -\frac{1}{2it} \int_0^\infty e^{itk^2 + itw} \frac{d}{dk} \left( \frac{1}{k} \Lambda_j(x,k) \Lambda^*_j(y,k) \psi(k) \right) \theta_3 f_j(y) dydk$$

$$= \frac{1}{2it} \int_0^\infty e^{itk^2 + itw} \frac{1}{k^2} \psi(k) \Lambda_j(x,k) \Lambda^*_j(y,k) \theta_3 f_j(y) dydk$$

$$- \frac{1}{2it} \int_0^\infty e^{itk^2 + itw} \frac{1}{k} \left( \Lambda_j(x,k) \Lambda^*_j(y,k) \psi(k) \right)^' \theta_3 f_j(y) dydk$$

From the asymptotic representation in (2.16), we can deduce our lemma as what we have done in the proof of Lemma 2.5. In fact, roughly speaking,

$$\Lambda_j(x,k) = O(|x|).$$

The $\mathfrak{F}$ and $\mathfrak{G}$ terms are similar, we omit them. Therefore, we have proved all the dispersive estimates.

3 Scattering for the linearized operator

**Lemma 3.1.** If $\alpha_j = \alpha$, then for any function $f \in L^2$ satisfying $\|\rho^2 U(t)f\|_2 \leq Ct^{-3/2}$, there exists $f_+ \in L^2$ such that

$$\lim_{t \to \infty} \left\| e^{-iHt} f - e^{i\omega t} e^{i\Delta t} f_+ \right\|_2 = 0.$$

**Proof** First, we prove there exists $h \in L^2$ such that

$$\lim_{t \to \infty} \left\| e^{-iHt} f - e^{-i\mathcal{H}t} h \right\|_2 = 0.$$

Define $g(t,x) = e^{i\mathcal{H}t} e^{-i\mathcal{H}t} f$, since $e^{i\mathcal{H}t}$ keeps the $L^2$ norm, it suffices to prove

$$\frac{d}{dt} g(t,x) \in L^1([1, \infty); L^2(dx)).$$

Direct calculation shows

$$\left\| \frac{d}{dt} e^{i\mathcal{H}t} e^{-i\mathcal{H}t} f \right\|_2 = \left\| e^{i\mathcal{H}t} (J - \mathcal{H}) e^{-i\mathcal{H}t} f \right\|_2 \leq \left\| Ve^{-i\mathcal{H}t} f \right\|_2 \leq C \left\| \rho^2 U(t) f \right\|_2 \leq Ct^{-3/2},$$

which combined with the transformation $g \to ge^{i\omega t}$, gives Lemma 3.1.
4 Proof of theorem 1.1

Although, the following sketch is a repetition of the arguments in V. S. Buslaev, G. S. Perelman [3], we present it here for the reader’s convenience. Some differences are addressed.

4.1 Generalized eigenfunctions

In $L^2(\mathbb{R})$ setting without boundary conditions, we know that there exists at least four generalized eigenfunctions, and the root space is exactly four dimensional for subcritical pure power nonlinearity. The explicit expressions for them are:

$$
\xi_1 = \left( \begin{array}{c} u_1 \\ \bar{u}_1 \end{array} \right), \xi_3 = \left( \begin{array}{c} u_3 \\ \bar{u}_3 \end{array} \right), \xi_2 = \left( \begin{array}{c} u_2 \\ \bar{u}_2 \end{array} \right), \xi_4 = \left( \begin{array}{c} u_4 \\ \bar{u}_4 \end{array} \right).
$$

where $u_1 = -i\varphi(y, \alpha), u_3 = -\varphi_y(y, \alpha), u_2 = -\frac{2}{\alpha}\varphi_\alpha(y, \alpha), u_4 = \frac{i}{2}y\varphi(y, \alpha)$. They satisfies the relations

$$
H\xi_1 = H\xi_3 = 0, \quad H\xi_2 = i\xi_1, \quad H\xi_4 = i\xi_3.
$$

Combining them together. With the continuity condition, we get four generalized “eigenfunctions” for zero to $\mathcal{H}$, namely

$$
\vec{\xi}_j = (\xi_j(\alpha), ..., \xi_j(\alpha)), \quad j = 1, 2, 3, 4;
$$

and we also have

$$
\mathcal{H}\vec{\xi}_1 = \mathcal{H}\vec{\xi}_3 = 0, \quad \mathcal{H}\vec{\xi}_2 = i\vec{\xi}_1, \quad \mathcal{H}\vec{\xi}_4 = i\vec{\xi}_3.
$$

Since K-condition is added to the spectral problem, we need check whether the four generalized eigenfunctions are “real”.

In the pure power case, namely $F(x) = |x|^{\mu}$, we have the explicit expression for $\varphi$, namely

$$
\varphi(x; \sigma, \omega) = e^{i\sigma[(\mu + 1)\omega]^{1/2}} \sec^{1/\mu}(\mu\sqrt{\omega}x).
$$

It is direct to check only $\vec{\xi}_1$ and $\vec{\xi}_2$ satisfy K-condition, we conjecture that the dimension of root space for $\mathcal{H}$ is two in subcritical pure power case.

Thus we assume

**Hypothesis A:** Zero is the only discrete spectrum for $\mathcal{H}(\alpha)$, the dimension for its root space is two, and it is spanned by $\vec{\xi}_1$ and $\vec{\xi}_2$, where

$$
\xi_1 = \left( \begin{array}{c} u_1 \\ \bar{u}_1 \end{array} \right), \xi_2 = \left( \begin{array}{c} u_2 \\ \bar{u}_2 \end{array} \right), \quad u_1 = -i\varphi(y, \alpha), u_3 = -\varphi_y(y, \alpha)
$$

$$
\vec{\xi}_1 = (\xi_1(\alpha), ..., \xi_1(\alpha)), \quad \vec{\xi}_2 = (\xi_2(\alpha), ..., \xi_2(\alpha)).
$$
4.2 Orthogonality conditions.

We write the solution $u$ of equation (1.1) in the form of a sum

$$
\begin{align*}
\quad \quad u(x, t) &= w_j(x, \sigma_j(t)) + \chi_j(x, t) \\
\quad \quad w_j(x, \sigma_j(t)) &= \exp(i\Phi_j) \varphi(y, \alpha_j(t)), \Phi = -\beta_j(t) + \frac{1}{2}v_j(t)x \\
\quad \quad y &= x - b_j(t),
\end{align*}
$$

(4.23)

here $\sigma_j(t) = (\beta_j(t), \omega_j(t), b_j(t), v_j(t))$ may not be solutions to (1.3), but we assume

$$
\beta_j(t) = \beta(t), \quad \omega_j(t) = \omega(t), \quad b_j(t) = v_j(t) = 0,
$$

(4.24)

Hence $w_j(x, \sigma_j(t))$ satisfies K-condition, and thus the same holds for $\{\chi_j\}$. Let $\chi_j(x, t) = e^{i\Phi}f(x, t)$, $\Phi = -\beta(t)$. And $\{f_j\}$ is imposed by the following orthogonal conditions:

$$
\sum_{j=1}^{N} (f_j(t), \theta_3 \xi_{ji}(t)) = 0,
$$

(4.25)

where $\{\xi_{ji}(t)\}$ are the functions in the root space, namely $\xi_{j1} = \xi_1$, and $\xi_{j2} = \xi_2$.

There exists $\sigma_j(t)$ such that (4.25) holds, in fact we have the following lemma:

**Lemma 4.1.** If $\chi_j(t, x)$ is sufficiently small in $L^2$ norm, then there exists a unique representation (4.23), in which (4.24) and (4.25) hold.

**Proof** First we prove it for $t = 0$. In the view of (4.24), we aim to find $\beta$ and $\alpha$ such that

$$
\begin{align*}
\sum_{j=1}^{N} \text{im} \left( [u_j(0, x) - e^{-i\beta} \varphi(y, \alpha)], e^{-i\beta} \varphi(y, \alpha) \right) &= 0 \\
\sum_{j=1}^{N} \text{im} \left( [u_j(0, x) - e^{-i\beta} \varphi(y, \alpha)], e^{-i\beta} \varphi(y, \alpha) \right) &= 0.
\end{align*}
$$

The solvability is the consequence of the nonsingular of the main term to the corresponding Jacobian:

$$
\begin{pmatrix}
0 & N \\
\frac{N}{2} e & 0
\end{pmatrix}
$$

where $e = \frac{d}{d\alpha} \|\varphi(y, \alpha)\|_2^2$. Then the existence of $\{\sigma_j(t)\}$ follows in the same way as Proposition 1.3.1 and “important remark” there in [3].
4.3 Reduction to a spectral problem.

Define \( \beta(t) = \int_0^t \omega(\tau)d\tau + \gamma(t) \). Differentiate (4.25), we obtain the equations for \( \beta(t) \), namely

\[
\gamma(t)\frac{d}{d\alpha} \|\varphi\|_2^2 = [\gamma' + \omega'(t)]O_1(f, \varphi) + O_2(f, \varphi),
\]

\[
\frac{1}{\alpha} \omega'(t) \frac{d}{d\alpha} \|\varphi\|_2^2 = [\gamma' + \omega'(t)]O_1(f, \varphi) + O_2(f, \varphi),
\]

(4.26)

where \( O_1(f, \varphi) \) is the linear term of \( \{f_j\} \), and \( O_2(f, \varphi) \) is at least quadratic for \( \{f_j\} \), moreover they satisfy the following estimates:

\[
|O_1(f, \varphi)| \leq \|\varphi\|_2^2; \quad |O_2(f, \varphi)| \leq \|\varphi\|_2^2.
\]

(4.27)

Fixed a \( t_1 > 0 \), suppose the solution to (4.26) at time \( t_1 \) is \( \sigma_{j,1}(t) = (\beta_1, w_1, 0, 0) \);

and let \( \beta_1 = w_1 t_1 + \gamma_1 \),

\[
\chi_j(x, t) = \exp(i\Phi_1)g_j(x, t), \quad \Phi_1 = -\omega t - \gamma_1.
\]

(4.28)

Since \( \chi_j(x, t) \) satisfies K-condition, we infer that \( \{g_j\} \) satisfies K-condition by the special form of the transformation. Furthermore \( g \) satisfies,

\[
i\partial_t g = \mathcal{H}g + D.
\]

where \( D = D_0 + D_1 + D_2 + D_3 + D_4 \),

\[
D_0 = -e^{-\Omega}[\gamma' \varphi(x, \alpha) + \frac{2i}{\alpha} \omega' \varphi_\alpha(y; \alpha)], \quad \Omega = \Phi_1 - \Phi;
\]

\[
D_1 = F'(\varphi^2(x, \alpha))\varphi^2(x, \alpha)[\exp(-2i\Omega) - 1]\tilde{g};
\]

\[
D_2 = [F(\varphi^2(x, \alpha)) + F'(\varphi(x, \alpha))\varphi^2(x, \alpha)
- F(\varphi^2(x, \alpha_1)) - F'(\varphi(x, \alpha_1))\varphi^2(x, \alpha_1)]g;
\]

\[
D_3 = [F'(\varphi^2(x, \alpha))\varphi^2(x, \alpha) - F'(\varphi(x, \alpha_1))\varphi^2(x, \alpha_1)]\tilde{g};
\]

\[
D_4 = e^{-\Omega}N(\varphi(x, \alpha), e^{i\Omega}g),
\]

where \( -\frac{1}{4} \alpha(t)^2 = \omega(t) \) as before, and \( N \) is at least quadratic to \( \{f_j\} \). In order to determine the asymptotic behavior of \( g \), we split it into continuous part and discrete spectral part as follows:

\[
\tilde{g}_j = k_1(-i\varphi(x, \alpha), i\varphi(x, \alpha)) + k_2(\varphi_\alpha(x, \alpha), \varphi_\alpha(x, \alpha)) + \bar{h}_j(x, t).
\]
Then the orthogonal condition (4.25) reduces to
\[
\begin{cases}
\sum_{j=1}^{N} \sum_{i=1}^{2} \sum_{k} k_{i}(\Lambda \xi_{i}(\alpha_{1}), \theta_{3} \xi_{1}(\alpha)) + \sum_{j=1}^{N} (A_{j} \bar{h}_{j}, \theta_{3} \xi_{1}(\alpha)) = 0, \\
\sum_{j=1}^{N} \sum_{i=1}^{2} k_{i}(\Lambda \xi_{i}(\alpha_{1}), \theta_{3} \xi_{2}(\alpha)) + \sum_{j=1}^{N} (A_{j} \bar{h}_{j}, \theta_{3} \xi_{2}(\alpha)) = 0.
\end{cases}
\]
(4.29)

4.4 Nonlinear estimates

Define
\[M_0(t) = |\alpha^2 - \alpha_0^2|, \quad M_1(t) = \|k\|, \quad M_2(t) = \|\rho^2 h\|_2, \quad M_3 = \|g\|_{\infty}, \quad M_0 = \sup_{\tau \leq t} M_0(\tau), \text{ and} \]
\[\mathcal{M}_1(t) = \sup_{\tau \leq t} (1 + \tau)^{3/2} M_1(\tau), \quad \mathcal{M}_2(t) = \sup_{\tau \leq t} (1 + \tau)^{3/2} M_2(\tau), \quad \mathcal{M}_3(t) = \sup_{\tau \leq t} (1 + \tau)^{1/2} M_3(\tau).\]

(4.26) and (4.27) imply
\[
\|\gamma\| + \|\omega\| \leq \frac{1}{1 - c \|\rho^2 f\|_2} |O_2| \leq \frac{C \|\rho^2 f\|_2^2}{1 - c \|\rho^2 f\|_2}.\]

Hence
\[
\|\gamma\| + \|\omega\| \leq W(\mathcal{M})(1 + t)^{-3}(\mathcal{M}_1 + \mathcal{M}_2)^2, \quad (4.30)
\]
where \(W(\mathcal{M})\) is a function of \(\mathcal{M}_0\) to \(\mathcal{M}_3\) that is bounded near 0. Then we have
\[
|\Omega| \leq W(\mathcal{M})(\mathcal{M}_1 + \mathcal{M}_2)^2. \quad (4.31)
\]

Combing (4.31) and (4.29), we get
\[
\mathcal{M}_1 \leq W(\mathcal{M})(\mathcal{M}_1 + \mathcal{M}_2)^3. \quad (4.32)
\]

As §1.3 in [3], using dispersive estimates, we can prove
\[
\mathcal{M}_1 + \mathcal{M}_2, \mathcal{M}_3 \leq W(\mathcal{M})[N + (\mathcal{M}_1 + \mathcal{M}_2)^2 + (\mathcal{M}_1 + \mathcal{M}_2)^3 + \mathcal{M}_3^2 + \mathcal{M}_3^{2p-1}].
\]

Thus from continuity method, we can prove all \(\mathcal{M}_j\) are bounded, if \(N\) is sufficiently small.

4.5 The limit soliton

Since all \(\mathcal{M}_j\) are bounded, by (4.30), we obtain
\[
\|\gamma\| + \|\omega\| \leq C(1 + t)^{-3}. \quad (4.33)
\]
Then $\gamma, \omega$ have limits $\gamma_\infty$ and $\omega_\infty$. Thus we can introduce the limit trajectory:

$$
\beta_+ = \omega_+ + \gamma_+, \quad \omega_+ = \omega_\infty, \quad \gamma_+ = \gamma_\infty + \int_0^\infty (\omega(\tau) - \omega_\infty) d\tau.
$$

Obviously, $\sigma(t) - \sigma_+(t) = O(t^{-1})$, and then

$$
w(x; \sigma(t)) - w(x; \sigma_+(t)) = O(t^{-1}), \quad (4.33)
$$
in $L^2 \cap L^\infty$.

### 4.6 End of the proof

Let $\chi_j$ in decomposition (4.28) be $\chi_j = e^{i\Phi_j} g_j(x, t)$, $\Phi_\infty = -\beta_+(t)$, taking $t_1 = \infty$, splitting $g$ into continuous part $h$ and discrete part $k$ corresponding to $H(\alpha_+)$, and repeating the same procedure, we can prove

$$
\| h \rho^2 \|_2 \leq C t^{-3/2},
$$

and

$$
\| k \|_{L^2 \cap L^\infty} \leq C t^{-3/2}.
$$

Recall that $h$ satisfies

$$
h = e^{-i\mathcal{H}t} P_c(\mathcal{H}) h_0 - i \int_0^t e^{-i\mathcal{H}(t-\tau)} P_c(\mathcal{H}) D d\tau.
$$

Let $h = e^{-i\mathcal{H}t} h_\infty + R$, where

$$
h_\infty = P_c(h_0 + h_1), \quad h_1 = -i \int_0^\infty e^{i\mathcal{H}t} D d\tau, \quad R = i \int_t^\infty [-i\mathcal{H}(t-\tau)] P_c D d\tau.
$$

We have $R = O(t^{-1/2})$ in $L^2 \cap L^\infty$, and

$$
\| \rho^2 U(t) h_\infty \|_2 = O(t^{-3/2}). \quad (4.34)
$$

Thus we can state the following result:

$$
u(t) = w(x; \sigma_+(t)) + e^{-i\beta_+(t)} e^{-i\mathcal{H}t} h_\infty + \chi,
$$

where $\| \chi \|_{L^2 \cap L^\infty} \leq C t^{-1/2}$. From Lemma 3.1, because of (4.33), there exists $f_+ \in L^2$ such that

$$
\lim_{t \to \infty} \| e^{-i\mathcal{H}t} h_\infty - e^{i\omega_+ t} e^{i\Delta t} f_+ \|_2 = 0.
$$
Hence, Theorem 1.1 has been proved.

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