Difference Ramsey Numbers and Issai Numbers

Aaron Robertson
Department of Mathematics, Temple University
Philadelphia, PA 19122
e-mail: aaron@math.temple.edu
Classification: 05D10, 05D05

Abstract

We present a recursive algorithm for finding good lower bounds for the classical Ramsey numbers. Using notions from this algorithm we then give some results for generalized Schur numbers, which we call Issai numbers.

Introduction

We present two new ideas in this paper. The first will be dealing with classical Ramsey numbers. In this part we give a recursive algorithm for finding so-called difference Ramsey numbers. Using the ideas from this first part we then define Issai numbers, a generalization of the Schur numbers. We give some easy results, values, and bounds for these Issai numbers.

Recall that $N = R(k_1, k_2, \ldots, k_r)$ is the minimal integer with the following property:

**Ramsey Property:** If we $r$-color the edges of the complete graph on $N$ vertices, then there exist $j$, $1 \leq j \leq r$, such that a monochromatic $j$-colored complete graph on $k_j$ vertices is a subgraph of the $r$-colored $K_N$.

To find a lower bound, $L$, for one of these Ramsey numbers, it suffices to find an edgewise coloring of $K_L$ which avoids the Ramsey property. To this end, we will restrict our search to the subclass of difference graphs. After presenting some results, we will show that the Issai numbers are a natural consequence of the difference Ramsey numbers, and a natural extension of the Schur numbers.

The Difference Ramsey Numbers part of this article is accompanied by the Maple package **AUTORAMSEY**. It has also been translated into Fortran77 and is available as DF.f at the author’s website. The Issai Numbers part of this article is accompanied by the Maple package **ISSAI**. All computer packages are available for download at the author’s website.

1 webpage: www.math.temple.edu/~aaron/
This paper is part of the author’s Ph.D. thesis under the direction of Doron Zeilberger.
This paper was supported in part by the NSF under the PI-ship of Doron Zeilberger.
Difference Ramsey Numbers

Our goal here is to find good lower bounds for the classical Ramsey numbers. Hence, we wish to find edge-wise colorings of complete graphs which avoid the Ramsey Property. Our approach is to construct a recursive algorithm to find the best possible colorings among those colorings we search. Since searching all possible colorings of a complete graph on any nontrivial number of vertices is not feasible by today’s computing standards, we must restrict the class of colored graphs to be searched. The class of graphs we will search will be the class of *difference graphs*.

**Definition:** *Difference Graph:*
Consider the complete graph on $n$ vertices, $K_n$. Number the vertices 1 through $n$. Let $i < j$ be two vertices of $K_n$. Let $B_n$ be a set of arbitrary integers between 1 and $n - 1$. Call $B_n$ the set of blue differences on $n$ vertices. We now color the edges of $K_n$ as follows: if $j - i \in B_n$ then color the edge connecting $i$ and $j$ blue, otherwise color the edge red. The resulting colored graph will be called a *difference graph*.

Given $k$ and $l$, a difference graph with the maximal number of vertices which avoids both a blue $K_k$ and a red $K_l$ will be called a *maximal difference Ramsey graph*. Let the number of vertices of a maximal difference Ramsey graph be $V$. Then we will define the *difference Ramsey number*, denoted $D(k, l)$, to be $V + 1$. Further, since the class of difference graphs is a subclass of all two-colored complete graphs, we have that $D(k, l) \leq R(k, l)$. Hence, by finding the difference Ramsey numbers, we are finding lower bounds for the classical Ramsey numbers.

Before we present the computational aspect of these difference Ramsey numbers, we establish an easy result: $D(k, l) \leq D(k - 1, l) + D(k, l - 1)$, which is analogous to the upper bound derived from Ramsey’s proof [GRS p. 3], does not follow from Ramsey’s proof.

To see this consider the difference Ramsey number $D(3, 3) = 6$. Let the set of red differences be $R_6 = \{1, 2, 4\}$ (and thus the set of blue differences is $B_6 = \{3, 5\}$). Call this difference graph $D_6$. In Ramsey’s proof, a vertex $v$ is isolated. The next step is to notice that, regardless of the choice of $v$, the number of red edges from $v$ to $D_6 \setminus \{v\} \geq D(2, 3) = 3$. Call the graph which has each vertex connected to the vertex $v$ by a red edge $G$. If $v \in 1, 6$ then $G$ has 3 vertices, otherwise it has 4 vertices. Either way, the number of vertices of $G$ is at least $D(2, 3) = 3$.

In order for Ramsey’s argument to work in the difference graph situation, we must show that $G$ is isomorphic to a difference graph. Assume there exists an isomorphism, $\phi : \{1, 2, 3, 4, 5, 6\} \rightarrow \{1, 2, 3, 4, 5, 6\}$, such that the vertex set of $G$, is mapped onto $\{1, 2, 3\}$ or $\{1, 2, 3, 4\}$ (depending on the number of vertices of $G$), and the edge coloring is preserved. Then $\phi(G)$ would be a difference graph. Notice now that $\phi(\{v\}) \in \{4, 5, 6\}$. For any choice of $\phi(\{v\})$ we obtain the contradiction that the difference 1 must be both red and blue (for different edges). Hence, no such isomorphism can exist. Hence we cannot use the difference Ramsey number property to conclude that the inequality holds. 

2
However, the difference Ramsey numbers seem to be, for small values, quite close to the Ramsey numbers. This may just be a case of the Law of Small Numbers, but numerical evidence from this paper leads us to make the following

**Conjecture 1:** $D(k, l) \leq D(k - 1, l) + D(k, l - 1)$

The set of difference graphs is a superclass of the often searched circular (or cyclic) graphs (see the survey [CG] by Chung and Grinstead), which are similarly defined. The distinction is that, using the notation above, for a graph to be circular we require that if $b \in B_n$, then we must have $n - b \in B_n$. By removing this circular condition, we remove from the coloring the dependence on $n$ (the number of vertices), and can thereby construct a recursive algorithm to find the set of maximal difference Ramsey graphs:

The recursive step in the algorithm is described as follows. A difference graph on $n$ vertices consists of $B_n$, the set of blue differences, and $R_n$, the set of red differences. Thus $B_n \cup R_n = \{1, 2, 3, \ldots, n-1\}$. To obtain a difference graph on $n+1$ vertices, we consider the difference $d = n$. If $B_n \cup \{d\}$ avoids a red clique, then we have a difference graph on $n+1$ vertices where $B_{n+1} = B_n \cup \{d\}$ and $R_{n+1} = R_n$. (Note that now $B_{n+1} \cup R_{n+1} = \{1, 2, 3, \ldots, n\}$.) Likewise, if $R_n \cup \{d\}$ avoids a blue clique, then we have a different difference graph on $n+1$ vertices with $B_{n+1} = B_n$ and $R_{n+1} = R_n \cup \{d\}$. Hence, we have a simple recursion which is not possible with circular graphs. (By increasing the number of vertices from $n$ to $n+1$, a circular graph goes from being circular, to being completely noncircular (if $b \in B_n$, then $n - b \notin B_n$).)

We can now use our recursive algorithm to find automatically (and we must note theoretically due to time and memory constraints, but much less time and memory than would be required to search all graphs) all maximal difference Ramsey graphs for any given $k$ and $l$.

**About the Maple Package AUTORAMSEY**

AUTORAMSEY is a Maple package that automatically computes the difference graph(s) with the maximum number of vertices that avoids both a blue $K_k$ and a red $K_l$. Hence, this package automatically finds lower bounds for the Ramsey number $R(k, l)$. In the spirit of automation, and to take another step towards AI, AUTORAMSEY can create a verification Maple program tailored to the maximal graph(s) calculated in AUTORAMSEY (that can be run at your leisure) and can write a $\LaTeX$ paper giving the lower bound for the Ramsey number $R(k, l)$ along with a maximal difference graph that avoids both a blue $K_k$ and a red $K_l$.

The computer generated program is a straightforward program that can be used to (double) check that the results obtained in AUTORAMSEY do indeed avoid both a blue $K_k$ and a red $K_l$. Further, this program can be easily altered (with instructions on how to do so) to search two-colored complete graphs for $k$-cliques and $l$-anticliques.
AUTORAMSEY has also been translated into Fortran77 as DF.f to speed up the algorithm implementation. The code for the translated programs (dependent upon the clique sizes we are trying to avoid) is available for download at my webpage.

The Algorithm

Below we will give the pseudocode which finds the maximal difference Ramsey graph(s). Hence, it also will find the exact value of the difference Ramsey numbers $D(k,l)$. Because the number of difference graphs is of order $2^n$ as compared to $2^{n^2/2}$ for all colored graphs, the algorithm can feasibly work on larger Ramsey numbers.

Let $D_n$ be the class of difference graphs on $n$ vertices. Let $\text{GoodSet}$ be the set of difference graphs that avoid both a blue $K_k$ and a red $K_l$.

Let $m = \min(k,l)$

Find $D_{m-1}$, our starting point.

Set $\text{GoodSet} = D_{m-1}$.

Set $j = m - 1$

WHILE $\text{flag} \neq 0$ do

FOR $i$ from 1 to $|\text{GoodSet}|$ do

Take $T \in \text{GoodSet}$, where $T$ is of the form $T = [B_j, R_j]$ where $B_j$ and $R_j$ are the blue and red difference sets on $j$ vertices

Consider $S_B = [B_j \cup \{j\}, R_j]$ and $S_R = [B_j, R_j \cup \{j\}]$

If $S_B$ avoids both a blue $K_k$ and a red $K_l$ then

$\text{NewGoodSet} := \text{NewGoodSet} \cup S_B$

If $S_R$ avoids both a blue $K_k$ and a red $K_l$ then

$\text{NewGoodSet} := \text{NewGoodSet} \cup S_R$

Repeat FOR loop with a new $T$

If $|\text{NewGoodSet}| = 0$ then RETURN $\text{GoodSet}$ and set $\text{flag} = 0$

Otherwise, set $\text{GoodSet} = \text{NewGoodSet}$, $\text{NewGoodSet} = \{\}$, and $j = j + 1$

Repeat WHILE loop

For this algorithm to be efficient we must have the subroutine which checks whether or not a monochromatic clique is avoided be very quick. We use the following lemma to achieve quick results in the Fortran77 code. (The Maple code is mainly for separately checking (with a different, much slower, but more straightforward, algorithm) the Fortran77 code for small cases.)

**Lemma 1:** Define the binary operation $*$ to be $x * y = | x - y |$. Let $D$ be a set of differences. If $D$ contains a $k$-clique, then there exists $K \subset D$, with $|K| = k - 1$, such that for all $x, y \in K$, $x * y \in D$.

**Proof:** We will prove the contrapositive. Let $K = \{d_1, d_2, \ldots, d_{k-1}\}$. Order and rename the elements of $K$ so that $d_1 < d_2 < \ldots < d_{k-1}$. Let $v_0 < v_1 < \ldots < v_{k-1}$ be
the vertices of a $k$-clique where $d_i = v_i - v_0$. By supposition, there exists $I < J$ such that $d_J * d_I = d_J - d_I \notin D$. This is the edge connecting $v_J$ with $v_I$. Since this edge is not in $D$, $D$ contains no $k$-clique.

By using this lemma we need only check pairs of elements in a $k$-set, rather than constructing all possible colorings using the $k$-set. Further, we need not worry about the ordering of the pairs; the operation $*$ is commutative.

Some Results

It is easy to find lower bounds for $R(k, l)$, so we must show that the algorithm gives “good” lower bounds. Below are two tables of the difference Ramsey number results obtained so far. The first table is of the difference Ramsey number values. The second table is of the number of maximal difference Ramsey graphs. If we are considering the diagonal Ramsey number $R(k, k)$, then the number of maximal difference graphs takes into account the symmetry of colors; i.e. we do not count a reversal of colors as a different difference graph. Where lower bounds are listed we have made constraints on the size of the set $\text{GoodSet}$ in the algorithm due to memory and/or (self-imposed) time restrictions.

### Difference Ramsey Numbers

| $k$ | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 3   | 6   | 9   | 14  | 17  | 22  | 27  | 36  | 39  | 46  |
| 4   |     | 18  | 25  | 34  | 47  | $\geq 53$ | $\geq 62$ |     |     |
| 5   |     |     |     | 42  |     | $\geq 57$ |     |     |     |

### Number of Maximal Difference Ramsey Graphs

| $k$ | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 3   | 1   | 2   | 3   | 7   | 13  | 13  | 4   | 21  | 6   |
| 4   |     | 1   | 6   | 24  | 21  | n/a | n/a |     |     |
| 5   |     |     |     |     |     |     |     | 11  | n/a |

When we compare our test results to the well known maximal Ramsey graphs for $R(3, 3), R(3, 4), R(3, 5), R(4, 4)$ [GG], and $R(4, 5)$ [MR], we find that the program has found the critical colorings for all of these numbers. The classical coloring in [GRS] for $R(3, 4)$ is not a difference graph, and hence is not found by the program. More importantly, however, is that it does find a difference graph on 8 vertices that avoids both a blue $K_3$ and a red $K_4$. Hence, for the Ramsey numbers found by Gleason and Greenwood [GG], and for $R(4, 5)$ found by McKay and Radziszowski [MR] we have found critical Ramsey graphs which are also difference graphs.
The algorithm presented above can be trivially extended to search difference graphs with more than two colors. The progress made so far in this direction follows.

**Multicolored Difference Ramsey Numbers**

The algorithm presented here can be applied to an arbitrary number of colors. The recursive step in the algorithm simply becomes the addition of the next difference to each of the three color set $B_n$, $R_n$, and $G_n$ ($G$ for green). Everything else remains the same. Hence, the alteration of the program to any number of colors is a simple one. The main hurdle encountered while searching difference graphs of more than two colors is that the size of the set $\text{GoodSet}$ in the algorithm grows very quickly. In fact, the system’s memory while fully searching all difference graphs was consumed within seconds for most multicolored difference Ramsey numbers.

\[D(3, 3, 3) = 15\]
\[D(3, 3, 4) = 30\]
\[D(3, 3, 5) = 42\]
\[D(3, 3, 6) \geq 60\]

We note here that $D(3, 3, 6) \geq 60$ implies that $R(3, 3, 6) \geq 60$, which is a new result. The previous best lower bound was 54 [SLZL]. The coloring on 59 vertices is cyclic, hence we need only list the differences up to 29:

**Color 1:** 5,12,13,14,16,20,22
**Color 2:** 10,15,19,24,26,27
**Color 3:** 1,2,3,4,6,7,8,9,11,17,18,21,23,25,28,29

**Future Directions**

Currently the algorithm which searches for the maximal difference Ramsey graphs is a straightforward search. In other words, if the memory requirement exceeds the space in the computer, the algorithm will only return a lower bound. In the future this algorithm should be adapted to backtrack searches or network searching. For a backtrack search we would note for which difference the memory barrier is reached and then start splitting up the searches. This would create a tree-like structure. We then check all leaves on this tree and choose the maximal graph. For network searching, the same type of backtrack algorithm would be used except that difference branches of the tree would be sent to different computers. This would be much quicker, but of course would cost much more in computer facilities.
**Issai Numbers**

Issai Schur proved in 1916 the following theorem which is considered the first Ramsey theorem to spark activity in Ramsey Theory.

*Schur’s Theorem: Given* \( r \), there exists an integer \( N = N(r) \) such that any \( r \)-coloring of the integers 1 through \( N \) must admit a monochromatic solution to \( x + y = z \).

We may extend this to the following theorem:

**Theorem 1:** Given \( r \) and \( k \), there exists an integer \( N = N(r, k) \) such that any \( r \)-coloring of the integers 1 through \( N \) must admit a monochromatic solution to

\[
\sum_{i=1}^{k-1} x_i = x_k.
\]

This is not a new theorem. In fact it is a special case of Rado’s Theorem [GRS p. 56]. We will, however, present a simple proof which relies only on the notions already presented in this paper.

*Proof:* Consider the \( r \)-colored difference Ramsey number \( N = D(k, k, \ldots, k) \). Then any \( r \)-coloring of \( K_N \) must have a monochromatic \( K_k \) subgraph. Let the vertices of this subgraph be \( \{v_0, v_1, \ldots, v_{k-1}\} \), with the differences \( d_i = v_i - v_0 \). By ordering and renaming we may assume that \( d_1 < d_2 < \ldots < d_{k-1} \). Since \( K_k \) is monochromatic, we have that the edges \( \overline{v_i v_j} \), \( i = 1, 2, \ldots, k-1 \), and \( \overline{v_{k-1} v_0} \) must all be the same color. Since the \( r \)-colored \( K_N \) is a difference graph we have that \( (d_{i+1} - d_i), i = 1, 2, \ldots, k-1, d_1, \) and \( d_{k-1} \) must all be assigned the same color. Hence we have the monochromatic solution \( d_1 + \sum_{i=1}^{k-2} (d_{i+1} - d_i) = d_{k-1} \).

Using this theorem we will define *Issai numbers*. But first, another definition is in order.

**Definition:** *Schur k-tuple*

We will call a \( k \)-tuple, \( (x_1, x_2, \ldots, x_k) \), a *Schur k-tuple* if \( \sum_{i=1}^{k-1} x_i = x_k \).

In the case where \( k = 3 \), the 3-tuple \( (x, y, x+y) \) is called a Schur triple. In Schur’s theorem the only parameter is \( r \), the number of colors. Hence, a Schur number is defined to be the minimal integer \( S = S(r) \) such that any \( r \)-coloring of the integers 1 through \( S \) must contain a monochromatic Schur triple. It is known that \( S(2) = 5 \), \( S(3) = 14 \), and \( S(4) = 45 \). The Schur numbers have been generalized in [BB] and [S] in directions different from what will be presented here. We will extend the Schur numbers in the same fashion as the Ramsey numbers were extended from \( R(k, k) \) to \( R(k, l) \).

**Definition:** *Issai Number*

Let \( S = S(k_1, k_2, \ldots, k_r) \) be the minimal integer such that any \( r \)-coloring of the integers from 1 to \( S \) must have a monochromatic Schur \( k_i \)-tuple, for some \( i \in \{1, 2, \ldots, r\} \). \( S \) will be called an *Issai number*. The existence of these Issai numbers is trivially im-
plied by the existence of the difference Ramsey numbers \( D(k_1, k_2, \ldots, k_r) \). In fact, we have the following result:

**Lemma 2**: \( S(k_1, k_2, \ldots, k_r) \leq D(k_1, k_2, \ldots, k_r) - 1 \)

**Proof**: By definition, there exists a minimal integer \( N = D(k_1, k_2, \ldots, k_r) \) such that any \( r \)-coloring of \( K_N \) must contain a monochromatic \( K_{k_i} \), for some \( i \in \{1, 2, \ldots, r\} \). Using the same reasoning as in the proof of Theorem 1 and the fact that the differences in the difference graph are \( 1, 2, \ldots, N - 1 \), we have the stated inequality.

Using this new definition and notation, it is already known that \( S(3, 3) = 5 \), \( S(3, 3, 3) = 14 \), and \( S(3, 3, 3, 3) = 45 \). We note here that since \( D(3, 3, 3) = 15 \) we immediately have \( S(3, 3, 3) \leq 14 \), whereas before, since \( R(3, 3, 3) = 17 \), we had only that \( S(3, 3, 3) \leq 16 \).

Attempts to find a general bound for \( S(k, l) \) have been unsuccessful. The values below lead me to make the following seemingly trivial conjecture:

**Conjecture 2**: \( S(k - 1, l) \leq S(k, l) \)

The difficulty here is that a monochromatic Schur \( k \)-tuple in no way implies the existence of a monochromatic Schur \((k - 1)\)-tuple. To see this, consider the following coloring of \( \{1, 2, \ldots, 9\} \). Color \( \{1, 3, 5, 9\} \) red, and the other integers blue. Then we have the red Schur 4-tuple \((1, 3, 5, 9)\). However no red Schur triple exists in this coloring.

**Some Issai Values and Colorings**

We used the Maple package ISSAI to calculate the exact values as well as an exceptional coloring given below. ISSAI is written for two colors, but can easily be extended to any number of colors. The value \( S(3, 3) = 5 \) has been known since before Schur proved his theorem. The value \( S(4, 4) = 11 \) follows from Beutelspacher and Brestovansky in [BB], who more generally show that \( S(k, k) = k^2 - k - 1 \). The remaining values are new.

| Issai Numbers |
|---------------|
| \( k \) | \( l \) | 3 | 4 | 5 | 6 | 7 |
| 3 | 5 | 7 | 11 | 13 | \( \geq 17 \) |
| 4 | 11 | 14 |

The exceptional colorings found by ISSAI are as follows. Let \( S(k, l) \) denote the minimal number such that and 2-coloring of the integers from 1 to \( S(k, l) \) must contain
either a red Schur $k$-tuple or a blue Schur $l$-tuple. It is enough to list only those integers colored red:

\[
\begin{align*}
S(3,4) > 6: & \quad \text{Red: 1,6} \\
S(3,5) > 10: & \quad \text{Red: 1,3,8,10} \\
S(4,4) > 10: & \quad \text{Red: 1,2,9,10} \\
S(3,6) > 12: & \quad \text{Red: 1,3,10,12} \\
S(4,5) > 13: & \quad \text{Red: 1,2,12,13} \\
S(3,7) > 16: & \quad \text{Red: 1,3,5,12,14,16}
\end{align*}
\]

Acknowledgment

I would like to thank my advisor, Doron Zeilberger, for his guidance, his support, and for sharing his mathematical philosophies. I would also like to thank Hans Johnston for his expertise and help with my Fortran code. Further, I would like to thank Daniel Schaal for his help with some references.

References

[BB] A. Beutelspacher and W. Brestovansky, Generalized Schur Numbers, Lecture Notes in Mathematics (Springer), 969, 1982, 30-38.

[C] F. Chung, On the Ramsey Numbers $N(3, 3, \ldots, 3)$, Discrete Mathematics, 5, 1973, 317-321.

[CG] F.R.K. Chung and C.M. Grinstead, A Survey of Bounds for Classical Ramsey Numbers, Journal of Graph Theory, 7, 1983, 25-37.

[E] G. Exoo, On Two Classical Ramsey Numbers of the Form $R(3, n)$, SIAM Journal of Discrete Mathematics, 2, 1989, 5-11.

[GG] A. Gleason and R. Greenwood, Combinatorial Relations and Chromatic Graphs, Canadian Journal of Mathematics, 7, 1955, 1-7.

[GR] C. Grinstead and S. Roberts, On the Ramsey Numbers $R(3, 8)$ and $R(3, 9)$, Journal of Combinatorial Theory, Series B, 33, 1982, 27-51.

[GRS] R. Graham, B. Rothschild, and J. Spencer, Ramsey Theory, John Wiley and Sons, 1980, 74-76.

[GY] J.E. Graver and J. Yackel, Some Graph Theoretic Results Associated with Ramsey's Theorem, Journal of Combinatorial Theory, 4, 1968, 125-175.

[K] J. G. Kalbfleisch, Chromatic Graphs and Ramsey's Theorem, Ph.D. Thesis, University of Waterloo, 1966.

[Rad] S. Radziszowski, Small Ramsey Numbers, Electronic Journal of Combinatorics, Dynamic Survey DS1, 1994, 28pp.
[RK] S. Radziszowski and D. L. Kreher, *On R(3, k) Ramsey Graphs: Theoretical and Computational Results*, Journal of Combinatorial Mathematics and Combinatorial Computing, 4, 1988, 207-212.

[S] D. Schaal, *On Generalized Schur Numbers*, Cong. Numer., 98, 1993, 178-187.

[SLZL] Su Wenlong, Luo Haipeng, Zhang Zhengyou, and Li Guiqing, *New Lower Bounds of Fifteen Classical Ramsey Numbers*, to appear in Australasian Journal of Combinatorics.