The average size of Ramanujan sums over cubic number fields

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Abstract
Let $K$ be a cubic number field. In this paper, we study the Ramanujan sums $c_{\mathcal{J}}(\mathcal{I})$, where $\mathcal{I}$ and $\mathcal{J}$ are integral ideals in $\mathcal{O}_K$. The asymptotic behaviour of sums of $c_{\mathcal{J}}(\mathcal{I})$ over both $\mathcal{I}$ and $\mathcal{J}$ is investigated.

Keywords Ramanujan sum · Cubic field · Exponential sum

1 Introduction

1.1 Ramanujan sums over the rationals

For positive integers $m$ and $n$, the Ramanujan sum $c_m(n)$ is defined as

$$c_m(n) := \sum_{1 \leq j \leq m \atop \gcd(j, m) = 1} e\left(\frac{jn}{m}\right) = \sum_{d \mid \gcd(m, n)} d \mu\left(\frac{m}{d}\right),$$

(1.1)

where $e(z) = e^{2\pi i z}$ and $\mu(\cdot)$ is the Möbius function. In 2012, Chan and Kumchev [1] studied the average order of $c_m(n)$ with respect to both $m$ and $n$. They proved that

$$S_1(X, Y) = \sum_{1 \leq m \leq X} \sum_{1 \leq n \leq Y} c_m(n) = Y - \frac{3}{2\pi^2} X^2 + O(XY^{1/3} \log X) + O(X^3 Y^{-1}),$$

(1.2)
for large real numbers $Y \geq X \geq 3$, and

$$S_1(X, Y) := \begin{cases} Y, & \text{if } \delta > 2, \\ -\frac{3}{2\pi}X^2, & \text{if } 1 < \delta < 2, \end{cases} \quad (1.3)$$

if $Y \asymp X^\delta$.

Let $s$ be an arbitrary fixed positive integer. For any positive integers $m, n$ and $s \geq 2$, the sum $c_m^{(s)}(n)$ denotes a generalization of the Ramanujan sum defined by

$$c_m^{(s)}(n) := \sum_{d|m, d^s|n} ds \mu \left( \frac{m}{d} \right). \quad (1.4)$$

This sum is said to be Cohen sum or Cohen-Ramanujan sum. In the case $s = 1$, the function $c_m^{(s)}(n)$ is equal to the Ramanujan sum $c_m(n)$. Some interesting properties of (1.4) were given in detail by Kühn and Robles [11], Robles and Roy [16] and others.

More generally, for any positive integers $m, n, s$ and any arithmetic functions $f$ and $g$, define

$$s_m^{(s)}(n) := \sum_{d|m, d^s|n} f(d) g \left( \frac{m}{d} \right).$$

Kiuchi [9] considered some asymptotic formulas for weighted averages of $s_m^{(s)}(n)$.

In 2021, Kiuchi, Pillichshammer and Eddin [10] proposed a further generalization of $s_m^{(s)}(n)$ which is defined by

$$s_{f,g,h}^{(s)}(m, n) := \sum_{d|m, d^s|n} f(d) g \left( \frac{m}{d} \right) h \left( \frac{n}{d^s} \right),$$

where $s, m, n \in \mathbb{N}$ and $f, g, h$ are arithmetic functions. They derived various identities for the weighted average of the product of generalized sums $s_{f,g,h}^{(s)}(m, n)$ with weights concerning some functions.

### 1.2 Ramanujan sums in fields

Let $K$ be a number field and $O_K$ denote its ring of algebraic integers. For any nonzero integral ideal $I$ in $O_K$, the Möbius function is defined as follows: $\mu(I) = 0$ if there exists a prime ideal $P$ such that $P^2$ divides $I$, and $\mu(I) = (-1)^r$ if $I$ is a product of $r$ distinct prime ideals. For any ideal $I$, the norm of $I$ is denoted by $N(I)$. For nonzero integral ideals $I$ and $J$, the Ramanujan sum in fields is defined by

$$c_J(I) := \sum_{\mathcal{M} \in O_K} N(\mathcal{M}) \mu \left( \frac{J}{\mathcal{M}} \right), \quad (1.5)$$

which is an analogue of (1.1).

For each $n \geq 1$, let $a_K(n)$ denote the number of integral ideals in $O_K$ of norm $n$. Then

$$\sum_{n \leq x} a_K(n) = \rho_Kx + P_K(x), \quad P_K(x) = O(x^{\frac{d-1}{d+1}}), \quad (1.6)$$
where $\rho_K$ is a constant depending only on the field $K$ and $d$ is the degree of the field extension $K/\mathbb{Q}$. This is a classical result of Landau (see \cite{12}).

Let $X \geq 3$ and $Y \geq 3$ be two large real numbers. Define

$$S_K(X, Y) := \sum_{1 \leq N(J) \leq X} \sum_{1 \leq N(I) \leq Y} c_J(I),$$

which is an analogue of (1.2).

When $K$ is a quadratic number field, some authors studied the asymptotic behaviour of $S_K(X, Y)$ (see \cite{14, 18, 19}). In \cite{14}, Nowak proved

$$S_K(X, Y) \sim \rho_K Y$$

provided that $Y > X^{\delta}$ for some $\delta > \frac{1973}{820}$. In \cite{18}, Zhai improved Nowak’s results and proved that (1.8) holds provided that $Y > X^\delta$ for some $\delta > \frac{79}{34}$. Recently Zhai \cite{19} proved that (1.8) holds for $Y > X^{2+\varepsilon}$.

In this paper, we consider the asymptotic behaviour of $S_K(X, Y)$ for a cubic field $K$. We shall prove the following results.

**Theorem 1.1** Let $K$ be a cubic number field. Suppose that $Y \geq X \geq 3$ are large real numbers. Then

$$S_K(X, Y) = \rho_K Y + O(X^{\frac{8}{5}Y^{\frac{2}{5}} + X^{\frac{11}{12}Y^{\frac{1}{12}}}}),$$

provided that $Y > X^{11/4}$.

**Theorem 1.2** Let $K$ be a cubic number field. Suppose that $T \geq X \geq 3$ are two large real numbers such that $T \geq 10X$. Then

$$\int_T^{2T} | \Re_K(X, Y) |^2 dY = c(X) \int_T^{2T} Y^{\frac{2}{3}} dY + O(X^{\frac{31}{19}T^{\frac{14}{19}} + X^{\frac{26}{19}T^{\frac{29}{19}} + \varepsilon}}),$$

where

$$\Re_K(X, Y) := S_K(X, Y) - \rho_K Y$$

and $c(X)$ is defined by (4.7).

**Remark** From (4.10) we can see that $c(X) \ll X^{\frac{2}{3}+\varepsilon}$. From this estimate we get from Theorem 2 that the asymptotic formula (1.8) holds on average provided that $Y > X^{\frac{2}{3}+\varepsilon}$.

**Notation** Let $[x]$ denote the greatest integer less or equal to $x$. The notation $U \ll V$ means that there exists a constant $C > 0$ such that $|U| \leq CV$, which is equivalent to $U = O(V)$. The notations $U \gg V$ (which implies $U \geq 0$ and $V \geq 0$), $U \asymp V$ (which means that we have both $U \ll V$ and $U \gg V$) are defined similarly. Let $\zeta(s)$ denote the Riemann zeta-function and $\tau_r(n)$ the number of ways $n$ factorized into $r$ factors. In particular, $\tau_2(n) = \tau(n)$ is the Dirichlet divisor function. At last, let $z_n$ ($n \geq 1$) denote a series of complex numbers. We set

$$\left| \sum_{N<n \leq 2N} z_n \right| := \max_{N \leq N_1 < N_2 \leq 2N} \left| \sum_{N_1 < n \leq N_2} z_n \right|.$$  

(1.10)

When we revised our manuscript, we noted that Sneha and Shivani \cite{17} established asymptotic formulas for the second moment of averages of Ramanujan sums over quadratic and cubic number fields and obtained second moment results for Ramanujan sums over some other number fields.
2 Some lemmas

In this section, we will make preparation for the proof of our theorems. From now on, we always suppose that $K$ is a cubic number field. The Dedekind zeta-function of $K$ is defined by

$$
\zeta_K(s) := \sum_{I \in \mathcal{O}_K, \ I \neq 0} \frac{1}{N^s(I)} \ (\Re s > 1).
$$

(2.1)

Then

$$
\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s} \ (\Re s > 1),
$$

(2.2)

where $a_K(n)$ is the number of integral ideals in $\mathcal{O}_K$ of norm $n$.

The function $\mu_K(n)$ is defined by

$$
\frac{1}{\zeta_K(s)} := \sum_{n=1}^{\infty} \frac{\mu_K(n)}{n^s} \ (\Re s > 1).
$$

Define

$$
M_K(x) := \sum_{n \leq x} \mu_K(n).
$$

Then there is a trivial bound

$$
M_K(x) \ll x.
$$

(2.3)

We collect the algebraic properties of cubic number fields in the following lemma.

**Lemma 2.1** (Lemma 1 in [13]) Let $K$ be a cubic number field over $\mathbb{Q}$ and $D = df^2$ ($d$ squarefree) its discriminant; then

(a) $K/\mathbb{Q}$ is a normal extension if and only if $D = f^2$. In this case

$$
\zeta_K(s) = \zeta(s)L(s, \chi_1)L(s, \overline{\chi_1}),
$$

where $\zeta(s)$ is the Riemann zeta-function and $L(s, \chi_1)$ is an ordinary Dirichlet series (over $\mathbb{Q}$) corresponding to a primitive character $\chi_1$ modulo $f$.

(b) If $K/\mathbb{Q}$ is not a normal extension, then $d \neq 1$ and

$$
\zeta_K(s) = \zeta(s)L(s, \chi_2),
$$

where $L(s, \chi_2)$ is a Dirichlet L-function over the quadratic field $F = \mathbb{Q}(\sqrt{d})$:

$$
L(s, \chi_2) = \sum_{\varrho} \chi_2(\varrho)N_F(\varrho)^{-s}, \ (\Re s > 1).
$$

Here the summation is taken over all ideals $\varrho \neq 0$ in $F$ and $N_F$ denotes the (absolute) ideal norm in $F$. 

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Remark 2.2 To describe the character $\chi_2$, let $H$ be the ideal group in $F$ according to which the normal extension $K(\sqrt{d})$ is the class field. Then $H$ divides the set $A^f$ of all ideals $\mathfrak{q} \subseteq F$ with $(\mathfrak{q}, f) = 1$ into three classes $A^f = H \cup C \cup C'$, and $(\omega = e^{2\pi i/3})$

$$\chi_2(\mathfrak{q}) = \begin{cases} 1, & \mathfrak{q} \in H, \\ \omega, & \mathfrak{q} \in C, \\ \overline{\omega}, & \mathfrak{q} \in C', \\ 0, & (\mathfrak{q}, f) \neq 1. \end{cases}$$

The substitution $\gamma = (\sqrt{d} \mapsto -\sqrt{d})$ in $F$ maps $C$ onto $C'$.

Remark 2.3 The factorization of $\zeta_K(s)$ in Lemma 2.1 gives

$$a_K(n) = \sum_{m \mid n} b(m),$$

where in the case of a normal extension $b(m) = \sum_{xy=m} \chi_1(x)\overline{\chi_1(y)}$ ($\chi_1$ is the primitive character modulo $f$). Otherwise $b(m)$ is equal to the number of ideals $\mathfrak{q} \in H$ with $N_F(\mathfrak{q}) = m$ minus two times the number of ideals $\mathfrak{q} \in C$ with $N_F(\mathfrak{q}) = m$. In both cases, $|b(m)| \ll m^{\epsilon}$.

Lemma 2.4 ((68) in [2]) Let $K$ be an algebraic number field of degree $d$. Then

$$a_K(n) \ll (\tau(n))^{d-1},$$

where $\tau(n)$ is the Dirichlet divisor function and $d = [K : \mathbb{Q}]$.

Corollary 2.5 Let $K$ be a cubic field. Then

$$a_K(n) \ll \tau^2(n).$$

Lemma 2.6 Suppose $1 \ll N \ll Y$. Then

$$P_K(Y) = \frac{Y^{1/3}}{\sqrt{3\pi}} \sum_{n \leq N} \frac{a_K(n)}{n^{2/3}} \cos(6\pi (nY)^{1/3}) + O(Y^{2/3+\epsilon}N^{-1/3}),$$

where the $O$-constant depends on $\epsilon$.

Proof This is a special case of Proposition 3.2 of Friedlander and Iwaniec [4].

Lemma 2.7 Let $T \geq 10$ be a large parameter and $y$ a real number such that $T^\epsilon \ll y \ll T$. For any $T \leq Y \leq 2T$ define

$$P_1(Y) = P_1(Y; y) := \frac{Y^{1/3}}{\sqrt{3\pi}} \sum_{n \leq y} \frac{a_K(n)}{n^{2/3}} \cos(6\pi (nY)^{1/3}),$$

$$P_2(Y) = P_2(Y; y) := P_K(Y) - P_1(Y).$$

Then we have

$$\int_T^{2T} |P_2(Y)|^2 dY \ll T^{5/3+\epsilon}y^{-1/3} \quad (y \ll T^{1/3}).$$
Proof We prove that the estimate
\[ \int_1^T |\zeta_K(7/12 + it)|^2 \, dt \ll T^{1+\epsilon} \quad (2.9) \]
holds.

If \( K/\mathbb{Q} \) is a normal extension, then by Lemma 2.1 we have \( \zeta_K(s) = \zeta(s)L(s, \chi_1)\overline{L(s, \chi_1)} \). From Theorem 8.4 in [7] we get that
\[ \int_1^T |\zeta(7/12 + it)|^6 \, dt \ll T^{1+\epsilon}. \quad (2.10) \]
The proof of Theorem 8.4 in [7] can be applied directly to \( L(s, \chi_1) \) to derive
\[ \int_1^T |L(7/12 + it, \chi_1)|^6 \, dt \ll T^{1+\epsilon}. \quad (2.11) \]
From (2.10), (2.11) and Hölder’s inequality we get
\[
\int_1^T |\zeta_K(7/12 + it)|^2 \, dt \\
= \int_1^T |\zeta(7/12 + it)|^2 |L(7/12 + it, \chi_1)|^4 \, dt \\
\ll \left( \int_1^T |\zeta(7/12 + it)|^6 \, dt \right)^{1/3} \left( \int_1^T |L(7/12 + it, \chi_1)|^6 \, dt \right)^{2/3} \\
\ll T^{1+\epsilon}.
\]

Now suppose that \( K/\mathbb{Q} \) is not a normal extension, then \( \zeta_K(s) = \zeta(s)L(s, \chi_2) \) from Lemma 2.1. We know that \( L(s, \chi_2) \) is an automorphic \( L \)-function of degree 2 corresponding to a cusp form \( F \) over \( \text{SL}_2(\mathbb{Z}) \) (see, for example, Fomenko [5]). So from [3, Lemma 12], which is originally proved in [8], we have
\[ \int_1^T |L(7/12 + it, \chi_2)|^3 \, dt \ll T^{1+\epsilon}. \quad (2.12) \]
By (2.10), (2.12) and Hölder’s inequality we get
\[
\int_1^T |\zeta_K(7/12 + it)|^2 \, dt \\
= \int_1^T |\zeta(7/12 + it)|^2 |L(7/12 + it, \chi_2)|^2 \, dt \\
\ll \left( \int_1^T |\zeta(7/12 + it)|^6 \, dt \right)^{1/3} \left( \int_1^T |L(7/12 + it, \chi_2)|^3 \, dt \right)^{2/3} \\
\ll T^{1+\epsilon}.
\]

Now we give a short proof of (2.8). For simplicity, we follow the proof of Theorem 1 in [3]. Take \( d = 3 \), \( a(n) = a_K(n) \), \( N = [T^{5-\epsilon}] \) and \( M = [T^{2/3}] \). From (2.9) we can take \( \sigma^* = 7/12 \). As in the proof of Theorem 1 in [3], we can write
\[ P_2(Y) = R_1^+(Y; y) + \sum_{j=2}^7 R_j(Y), \]
where
\[ R^*_1(Y; y) := \frac{Y^{1/3}}{\sqrt{3}\pi} \sum_{y < n \leq M} \frac{d_3(n)}{n^{2/3}} \cos(6\pi (nY)^{1/3}) \]
and \( R_j(Y) \) \((j = 2, 3, 4, 5, 6, 7)\) were defined in p. 2129 of [3]. Similar to (8.11) of [3], we have the estimate (noting that \( y \ll T^{1/3} \))
\[
\int_T^{2T} (R^*_1(x; y) + R_2(x))^2 \, dx \ll \sum_{y < n \leq M} \frac{d_2^2(n)}{n^{4/3}} \int_T^{2T} x^{2/3} \, dx + T^{5/3+\varepsilon} M^{-1/6} + T^{4/3+\varepsilon} M^{1/3} \ll T^{5/3+\varepsilon} y^{-1/3} + T^{14/9+\varepsilon} \ll T^{5/3+\varepsilon} y^{-1/3},
\]
which, combining (8.17) of [3], gives (2.8).
\[ \square \]

Next, we consider the following exponential sums:

\[ S_0 = \sum_{H < h \leq 2H} \sum_{N < n \leq 2N} a(h, n) \sum_{M < m \leq 2M} b(m) e\left(\frac{H^{\beta} N^{\gamma} m^\alpha}{H N M^\alpha}\right) \tag{2.13} \]

and

\[ S_1 = \sum_{H < h \leq 2H} \sum_{N < n \leq 2N} a(h, n) \left| \sum_{M < m \leq 2M} e\left(\frac{H^{\beta} N^{\gamma} m^\alpha}{H N M^\alpha}\right) \right|^*, \tag{2.14} \]

where \( H, N, M \) are positive integers, \( U \) is a real number greater than one, \( a(h, n) \) and \( b(m) \) are a complex number of modulus at most one; moreover, \( \alpha, \beta, \gamma \) are fixed real numbers such that \( \alpha(\alpha - 1)\beta\gamma \neq 0 \).

**Lemma 2.8** ([15]) We have
\[ S_0 \ll (HNM)^{1+\varepsilon} \left( \left(\frac{U}{HN M^2}\right)^{1/4} + \frac{1}{(HN)^{1/4}} + \frac{1}{M^{1/2}} + \frac{1}{U^{1/2}} \right), \tag{2.15} \]

and
\[ S_1 \ll (HNM)^{1+\varepsilon} \left( \left(\frac{U}{HN M^2}\right)^{1/4} + \frac{1}{M^{1/2}} + \frac{1}{U} \right). \tag{2.16} \]

**Lemma 2.9** (see Lemma 2.4 in [6]) Suppose that
\[ L(H) = \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j}, \]
where \( A_i, B_j, a_i, \) and \( b_j \) are positive. Assume that \( H_1 \leq H \leq H_2 \). Then there is some \( H \) with \( H_1 \leq H \leq H_2 \) and
\[ L(H) \ll \sum_{i=1}^{m} \sum_{j=1}^{n} (A_i b_j^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^{m} A_i H_{1}^{a_i} + \sum_{j=1}^{n} B_j H_{2}^{-b_j}. \]
The implied constants depend only on \( m \) and \( n \).
Lemma 2.10 (see Lemma 2.4 in [18]) Let \( l \geq 2 \) and \( q \geq 1 \) be two fixed integers. Then we have

\[
\sum_{n \leq x} \tau_l^q(n) \ll x (\log x)^{ql-1}.
\]  

(2.17)

Lemma 2.11 Let \( T \geq 2 \) be a real number. Then we have

\[
\sum_{\substack{m,n \leq T \atop m \neq n}} \frac{\tau_4^2(m) \tau_4^2(n)}{(mn)^{\frac{2}{3}} |\sqrt[3]{m} - \sqrt[3]{n}|} \ll T^{\frac{1}{3} + \varepsilon}.
\]  

(2.18)

Proof First, we write

\[
\sum_{\substack{m,n \leq T \atop m \neq n}} \frac{\tau_4^2(m) \tau_4^2(n)}{(mn)^{\frac{2}{3}} |\sqrt[3]{m} - \sqrt[3]{n}|} = S_1 + S_2,
\]

where

\[
S_1 = \sum_{\substack{m,n \leq T \atop |\sqrt[3]{m} - \sqrt[3]{n}| \geq (mn)^{1/6}/10}} \frac{\tau_4^2(m) \tau_4^2(n)}{(mn)^{\frac{2}{3}} |\sqrt[3]{m} - \sqrt[3]{n}|},
\]

\[
S_2 = \sum_{\substack{m,n \leq T \atop 0 < |\sqrt[3]{m} - \sqrt[3]{n}| < (mn)^{1/6}/10}} \frac{\tau_4^2(m) \tau_4^2(n)}{(mn)^{\frac{2}{3}} |\sqrt[3]{m} - \sqrt[3]{n}|}.
\]

Applying Lemma 2.10 with \( l = 4 \) and \( q = 2 \), we have

\[
S_1 \ll \sum_{m,n \leq T} \frac{\tau_4^2(m) \tau_4^2(n)}{(mn)^{\frac{2}{3}}} \ll T^{\frac{1}{3} + \varepsilon},
\]

where we used a summation by parts.

Second, \( 0 < |\sqrt[3]{m} - \sqrt[3]{n}| < (mn)^{1/6}/10 \) implies that \( m \asymp n \). And from the Lagrange theorem we have \( |\sqrt[3]{m} - \sqrt[3]{n}| \asymp (mn)^{-1/3} |m - n| \). By the formula \( ab \leq (a^2 + b^2)/2 \) and Lemma 2.10 with \( l = 4 \) and \( q = 4 \) we get that

\[
S_2 \ll \sum_{m,n \leq T} \frac{\tau_4^2(m) \tau_4^2(n)}{(mn)^{1/3} |m - n|}
\]

\[
\ll \sum_{m,n \leq T} \left( \frac{\tau_4^2(m)}{m^{2/3}} + \frac{\tau_4^2(n)}{n^{2/3}} \right) \frac{1}{|m - n|}
\]

\[
\ll \sum_{m \leq T} \frac{\tau_4^2(m)}{m^{2/3}} \sum_{n \leq T} \frac{1}{|m - n|} \ll T^{\frac{1}{3} + \varepsilon}.
\]

\( \square \)
3 Proof of Theorem 1.1

We begin the proof with formula (2.3) in [14], which reads

$$S_K(X, Y) = \rho_K Y + \sum_{\mathcal{M}, \mathcal{L} \in \mathcal{O}_K} N(\mathcal{M}) \mu(\mathcal{L}) P_K\left(\frac{Y}{N(\mathcal{M})}\right)$$

$$= \rho_K Y + \sum_{\mathcal{M}, \mathcal{L} \in \mathcal{O}_K} N(\mathcal{M}) \mu(\mathcal{L}) P_K\left(\frac{Y}{N(\mathcal{M})}\right). \quad (3.1)$$

Let $\mathcal{R} = \mathcal{R}_K(X, Y)$ denote the last sum in (3.1). We have

$$\mathcal{R} = \sum_{1 \leq m \leq X} ma_K(m) \mu_K(l) P_K\left(\frac{Y}{m}\right)$$

$$= \sum_{1 \leq l \leq X} \mu_K(l) \sum_{1 \leq m \leq X/l} ma_K(m) P_K\left(\frac{Y}{m}\right) \quad (3.2)$$

$$= \mathcal{R}_1^\dagger + \mathcal{R}_2^\dagger,$$

where

$$\mathcal{R}_1^\dagger := \sum_{1 \leq l \leq X^{1-\varepsilon}} \mu_K(l) \sum_{1 \leq m \leq X/l} ma_K(m) P_K\left(\frac{Y}{m}\right),$$

$$\mathcal{R}_2^\dagger := \sum_{X^{1-\varepsilon} < l \leq X} \mu_K(l) \sum_{1 \leq m \leq X/l} ma_K(m) P_K\left(\frac{Y}{m}\right).$$

First, we bound $\mathcal{R}_1^\dagger$. Müller [13] proved that $P_K(x) = O(x^{\frac{43}{96}+\varepsilon})$. So we can easily derived that

$$\mathcal{R}_1^\dagger \ll X Y^{43/96+\varepsilon}. \quad (3.3)$$

Second, we consider $\mathcal{R}_2^\dagger$. We can write

$$\mathcal{R}_2^\dagger := \sum_{1 \leq l \leq X^{1-\varepsilon}} \mu_K(l) \mathcal{R}_1(X_l, Y), \quad (3.4)$$

where

$$\mathcal{R}_1(X_l, Y) = \sum_{1 \leq m \leq X_l} ma_K(m) P_K\left(\frac{Y}{m}\right), \quad X_l = X/l. \quad (3.5)$$

Using (2.4), we can write

$$\mathcal{R}_1(X_l, Y) = \sum_{1 \leq m_1 m_2 \leq X_l} m_1 m_2 b(m_2) P_K\left(\frac{Y}{m_1 m_2}\right). \quad (3.6)$$

By a splitting argument, $\mathcal{R}_1(X_l, Y)$ can be written as a sum of the following terms

$$R(M_1, M_2) := \sum_{1 \leq m_1 m_2 \leq X_l, M_j < m_j \leq 2M_j (j=1,2)} m_1 m_2 b(m_2) P_K\left(\frac{Y}{m_1 m_2}\right). \quad (3.7)$$
Suppose that $y \ll Y/M_1 M_2$ is a parameter to be determined. By Lemma 2.6, we have

$$R(M_1, M_2) = \frac{Y^{1/3}}{\sqrt{3\pi}} \sum_{1 \leq m_1 m_2 \leq X_j} (m_1 m_2)^{2/3} b(m_2) \sum_{n \leq y} \frac{a_k(n)}{n^{2/3}} \cos \left( 6\pi \sqrt{\frac{ny}{m_1 m_2}} \right) + O((M_1 M_2)^{4/3} Y^{2/3 + \varepsilon} y^{-1/3}).$$

By a splitting argument to the sum over $n$ we get

$$R(M_1, M_2) \ll Y^{1/3} (M_1 M_2)^{2/3 + \varepsilon} N^{-2/3 + \varepsilon} |R^*(M_1, M_2, N)| + O((M_1 M_2)^{4/3} Y^{2/3 + \varepsilon} y^{-1/3}) \tag{3.8}$$

for some $1 \ll N \ll y$, where

$$R^*(M_1, M_2, N) = \sum_{1 \leq m_1 m_2 \leq X_j} \left( \frac{m_1}{M_1} \right)^{2/3} \left( \frac{m_2}{M_2} \right)^{2/3} b(m_2) \sum_{N < n \leq 2N} c(n) e \left( 6\pi \sqrt{\frac{ny}{m_1 m_2}} \right)$$

with

$$c(n) = \frac{a_k(n)}{N^\varepsilon} \left( \frac{N}{n} \right)^{2/3}.$$

Now, we give our first estimate for the sum $R^*(M_1, M_2, N)$. Obviously, we have

$$R^*(M_1, M_2, N) \ll R^\dagger(M_1, M_2, N), \tag{3.9}$$

where

$$R^\dagger(M_1, M_2, N) = \sum_{M_2 < m_2 \leq 2M_2} \sum_{N < n \leq 2N} \sum_{M_1 < m_1 \leq 2M_1} e \left( 6\pi \sqrt{\frac{ny}{m_1 m_2}} \right)^n.$$

By taking $(H, N, M) = (M_2, N, M_1)$ and $U = \sqrt[3]{NY}/\sqrt[3]{M_1 M_2}$ in Lemma 2.8, we get that

$$R^\dagger(M_1, M_2, N) Y^{-\varepsilon} \ll N^{\varepsilon} \frac{1}{\pi} M_1 \frac{5}{3} M_2^2 + N M_1 \frac{2}{3} M_2 + N^{2/3} Y^{-\frac{1}{3}} (M_1 M_2)^{\frac{4}{3}},$$

which combining (3.9) gives

$$R^*(M_1, M_2, N) Y^{-\varepsilon} \ll N^{\varepsilon} \frac{1}{\pi} M_1 \frac{5}{3} M_2^2 + N M_1 \frac{2}{3} M_2 + N^{2/3} Y^{-\frac{1}{3}} (M_1 M_2)^{\frac{4}{3}}, \tag{3.10}$$

$$= N^{\varepsilon} \frac{1}{\pi} (M_1 M_2)^{\frac{5}{3}} M_2^2 + N (M_1 M_2)^{\frac{1}{3}} M_2^2 + N^{2/3} Y^{-\frac{1}{3}} (M_1 M_2)^{\frac{4}{3}}.$$

Next, we give another estimate for $R^*(M_1, M_2, N)$. Clearly we have

$$R^*(M_1, M_2, N) \ll R^\ddagger(M_1, M_2, N), \tag{3.11}$$

where

$$R^\ddagger(M_1, M_2, N) = \sum_{M_1 < m_1 \leq 2M_1} \sum_{N < n \leq 2N} \sum_{M_2 < m_2 \leq 2M_2} e \left( 6\pi \sqrt{\frac{ny}{m_1 m_2}} \right).$$
By taking \((H, N, M) = (M_1, N, M_2)\) and \(U = \sqrt[3]{NY}/\sqrt{M_1M_2}\) in Lemma 2.8, we get that
\[
R^*(M_1, M_2, N)Y^{-\varepsilon} \ll N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_1^{\frac{1}{4}} + N^3(M_1M_2)^{\frac{3}{8}}M_2^{\frac{3}{8}}
+ N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}} + N^5Y^{-\frac{1}{6}}(M_1M_2)^{\frac{7}{6}}.
\]

So
\[
R^*(M_1, M_2, N)Y^{-\varepsilon} \ll N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_1^{\frac{1}{4}} + N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}} + N^3(M_1M_2)^{\frac{3}{8}}M_2^{\frac{3}{8}} + N^5Y^{-\frac{1}{6}}(M_1M_2)^{\frac{7}{6}}. 
\]

From (3.10) and (3.12), we get
\[
R^*(M_1, M_2, N)Y^{-\varepsilon} \ll J_1 + J_2 + J_3 + J_4 + N^5Y^{-\frac{1}{6}}(M_1M_2)^{\frac{7}{6}} + N^3(M_1M_2)^{\frac{3}{8}}M_2^{\frac{3}{8}},
\]
where
\[
J_1 = \min \left( N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}}, N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_1^{\frac{1}{4}} \right),
\]
\[
J_2 = \min \left( N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_2^{\frac{1}{4}}, N^3Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_1^{\frac{1}{4}} \right),
\]
\[
J_3 = \min \left( N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}}, N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}} \right),
\]
\[
J_4 = \min \left( N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_2^{\frac{1}{4}}, N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}} \right).
\]

Noticing the fact that \(\min(X_1, \ldots, X_k) \leq X_1^{a_1} \ldots X_k^{a_k}\), where \(X_1, \ldots, X_k > 0, a_1, \ldots, a_k \geq 0\) satisfies \(a_1 + \cdots + a_k = 1\), we have
\[
J_1 \leq \left( N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}} \right)^{\frac{1}{3}} \left( N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_1^{\frac{1}{4}} \right)^{\frac{2}{3}} \leq N^8Y^{\frac{1}{12}}(M_1M_2)^{\frac{11}{12}},
\]
\[
J_2 \leq \left( N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_2^{\frac{1}{4}} \right)^{\frac{1}{3}} \left( N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_1^{\frac{1}{4}} \right)^{\frac{2}{3}} \leq N^8Y^{\frac{1}{12}}(M_1M_2)^{\frac{13}{12}},
\]
\[
J_3 \leq \left( N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}} \right)^{\frac{1}{3}} \left( N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}} \right)^{\frac{2}{3}} \leq N(M_1M_2)^{\frac{3}{4}},
\]
\[
J_4 \leq \left( N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{5}{12}}M_2^{\frac{1}{4}} \right)^{\frac{1}{3}} \left( N(M_1M_2)^{\frac{1}{2}}M_i^{\frac{1}{2}} \right)^{\frac{2}{3}} \leq N^8Y^{\frac{1}{12}}(M_1M_2)^{\frac{11}{12}}.
\]

It now follows that
\[
R^*(M_1, M_2, N)Y^{-\varepsilon} \ll N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{11}{12}} + N(M_1M_2)^{\frac{3}{4}} + N^5Y^{\frac{1}{12}}(M_1M_2)^{\frac{13}{12}} + N^5Y^{-\frac{1}{6}}(M_1M_2)^{\frac{7}{6}} + N^3(M_1M_2)^{\frac{3}{8}}M_2^{\frac{3}{8}},
\]

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Combining (3.8) with (3.13), we get (recalling $N \ll y$)

$$R(M_1, M_2) Y^{-\varepsilon}$$

$$\ll N^{\frac{2}{7}} Y^{\frac{5}{3}} (M_1 M_2)^{\frac{23}{28}} + N^{\frac{1}{4}} Y^{\frac{1}{3}} (M_1 M_2)^{\frac{17}{22}} + N^{\frac{1}{6}} Y^{\frac{5}{3}} (M_1 M_2)^{\frac{29}{22}}$$

$$+ N^{\frac{1}{6}} Y^{\frac{1}{3}} (M_1 M_2)^{\frac{11}{10}} + N^{\frac{1}{7}} Y^{\frac{1}{3}} (M_1 M_2)^{\frac{1}{2}} + Y^{\frac{2}{3}} (M_1 M_2)^{\frac{4}{3}} y^{-\frac{1}{3}} + (M_1 M_2)^{2}$$

(3.14)

$$\ll Y^{\frac{7}{7}} Y^{\frac{5}{9}} (M_1 M_2)^{\frac{23}{28}} + Y^{\frac{1}{7}} Y^{\frac{7}{3}} (M_1 M_2)^{\frac{17}{22}} + Y^{\frac{5}{7}} Y^{\frac{1}{6}} (M_1 M_2)^{\frac{29}{22}}$$

$$+ Y^{\frac{1}{6}} Y^{\frac{1}{6}} (M_1 M_2)^{\frac{11}{10}} + Y^{\frac{1}{6}} Y^{\frac{1}{6}} (M_1 M_2)^{\frac{5}{3}} + Y^{\frac{2}{3}} (M_1 M_2)^{\frac{3}{2}} y^{-\frac{1}{3}} + (M_1 M_2)^{2}.

By choosing a best $y$ with Lemma 2.6 (recalling that $X_l = X/l$), we get that

$$R(M_1, M_2) Y^{-\varepsilon} \ll Y^{\frac{7}{7}} (M_1 M_2)^{\frac{17}{20}} + Y^{\frac{1}{7}} (M_1 M_2)^{\frac{1}{2}} + Y^{\frac{7}{7}} (M_1 M_2)^{\frac{3}{2}}$$

$$+ Y^{\frac{1}{6}} (M_1 M_2)^{\frac{1}{2}} + (M_1 M_2)^{2}$$

(3.15)

$$\ll Y^{\frac{7}{7}} X_l^{\frac{13}{22}} + Y^{\frac{1}{7}} X_l^{\frac{11}{10}} + Y^{\frac{7}{7}} X_l^{\frac{3}{5}} + Y^{\frac{3}{5}} X_l^{\frac{5}{3}} + Y^{\frac{2}{3}} X_l^{\frac{5}{3}} + X^2.$$

From (3.4)-(3.7) and (3.15), we get

$$\mathcal{R} Y^{-\varepsilon} \ll X^{\frac{17}{20}} Y^{\frac{1}{2}} + X^{\frac{11}{10}} Y^{\frac{1}{2}} + X^{\frac{3}{5}} Y^{\frac{1}{2}} + X^{\frac{3}{5}} Y^{\frac{2}{3}} + X^2$$

$$\ll X^{\frac{17}{20}} Y^{\frac{1}{2}} + X^{\frac{8}{10}} Y^{\frac{2}{3}}$$

by noting that $Y \geq X$. This together with (3.2) and (3.3) yields

$$\mathcal{R} Y^{-\varepsilon} \ll X^{\frac{17}{20}} Y^{\frac{1}{2}} + X^{\frac{8}{10}} Y^{\frac{2}{3}} + X Y^{\frac{43}{56}} \ll X^{\frac{17}{20}} Y^{\frac{1}{2}} + X^{\frac{8}{10}} Y^{\frac{2}{3}}.$$

This completes the proof of Theorem 1.

4 The proof of Theorem 1.2

We begin with the first expression of $\mathcal{R}$ in (3.2)

$$\mathcal{R} = \sum_{1 \leq m \leq X} ma_K(m) \mu_K(l) P_K\left(\frac{Y}{m}\right)$$

$$= \sum_{1 \leq m \leq X} ma_K(m)M_K\left(\frac{X}{m}\right) P_K\left(\frac{Y}{m}\right)$$

(4.1)

$$= \mathcal{R}_1 + \mathcal{R}_2,$$

where

$$\mathcal{R}_1 = \frac{Y^{1/3}}{\sqrt{3\pi}} \sum_{m \leq X} m^{2/3} a_K(m)M_K\left(\frac{X}{m}\right) \sum_{n \leq y} a_K(n) n^{2/3} \cos\left(6\pi \sqrt{n Y/m}\right),$$

$$\mathcal{R}_2 = \sum_{1 \leq m \leq X} ma_K(m)M_K\left(\frac{X}{m}\right) P_2\left(\frac{Y}{m}\right).$$
A. Evaluation of $\int_T^{2T} \mathfrak{H}_2^2 \, dY$

Suppose that $0 < y < \left( \frac{T}{X} \right)^{1/3}$, it is not hard to find that

$$\mathfrak{H}_2 \ll \sum_{m \sim M} ma_K(m)M_K\left(\frac{X}{m}\right)P_2\left(\frac{Y}{m}\right) \log X$$

$$\ll X \sum_{m \sim M} a_K(m)P_2\left(\frac{Y}{m}\right) \log X$$

for some $1 \ll M \ll X$ and $M_K(t) \ll t$. By Cauchy’s inequality we get

$$\mathfrak{H}_2^2 \ll X^2 \sum_{m \sim M} a_K(m) \sum_{m \sim M} a_K(m)P_2^2\left(\frac{Y}{m}\right) \log^2 X$$

$$\ll X^2 M \sum_{m \sim M} a_K(m)P_2^2\left(\frac{Y}{m}\right) \log^2 X,$$

which together with $Xy^3 \ll T$ implies that

$$\int_T^{2T} \mathfrak{H}_2^2 \, dY \ll X^2 M \sum_{m \sim M} a_K(m) \log^2 X \int_T^{2T} P_2^2\left(\frac{Y}{m}\right) \, dY$$

$$\ll X^2 M \sum_{m \sim M} a_K(m) \log^2 X \int_T^{2T} P_2^2\left(\frac{Y}{m}\right) \, d\left(\frac{Y}{m}\right)$$

$$\ll X^2 M \sum_{m \sim M} a_K(m) m \left(\frac{T}{m}\right)^{\frac{5}{3}+\varepsilon} y^{-\frac{1}{3}} \log^2 X$$

$$\ll X^2 M^{\frac{4}{3}} T^{\frac{5}{3}+\varepsilon} y^{-\frac{1}{3}}$$

$$\ll X^{\frac{10}{3}} T^{\frac{5}{3}+\varepsilon} y^{-\frac{1}{3}}. \tag{4.2}$$

B. Evaluation of $\int_T^{2T} \mathfrak{H}_1^2 \, dY$

Noting that

$$\mathfrak{H}_1^2 = \frac{Y^\frac{2}{3}}{3\pi^2} \sum_{1 \leq m_1, m_2 \leq X} (m_1m_2)^\frac{2}{3} a_K(m_1)a_K(m_2)M_K\left(\frac{X}{m_1}\right)M_K\left(\frac{X}{m_2}\right)$$

$$\times \sum_{n_1, n_2 \leq y} \frac{a_K(n_1)}{n_1^{2/3}} \frac{a_K(n_2)}{n_2^{2/3}} \cos\left(6\pi \sqrt{\frac{n_1 Y}{m_1}}\right) \cos\left(6\pi \sqrt{\frac{n_2 Y}{m_2}}\right)$$

and using the elementary formula $\cos \alpha \cos \beta = \frac{1}{2} \left( \cos(\alpha - \beta) + \cos(\alpha + \beta) \right)$, we get

$$\mathfrak{H}_1^2 = Q_1(Y) + Q_2(Y) + Q_3(Y), \tag{4.3}$$
where

\[
Q_1(Y) := \frac{Y^\frac{3}{2}}{6\pi^2} \sum_{m_1, n_1, m_2, n_2 \leq Y} (m_1 m_2)^\frac{3}{2} a_K(m_1) a_K(m_2) \\
\times M_K\left(\frac{X}{m_1}\right) M_K\left(\frac{X}{m_2}\right) a_K(n_1) a_K(n_2) \frac{1}{n_1^{2/3} n_2^{2/3}},
\]

\[
Q_2(Y) := \frac{Y^\frac{3}{2}}{6\pi^2} \sum_{m_1, n_1, m_2, n_2 \leq Y} (m_1 m_2)^\frac{3}{2} a_K(m_1) a_K(m_2) M_K\left(\frac{X}{m_1}\right) M_K\left(\frac{X}{m_2}\right) \\
\times \frac{a_K(n_1) a_K(n_2)}{n_1^{2/3} n_2^{2/3}} \cos \left(\frac{6\pi}{\sqrt{Y}} \left(\frac{n_1}{m_1} - \frac{n_2}{m_2}\right)\right),
\]

\[
Q_3(Y) := \frac{Y^\frac{3}{2}}{6\pi^2} \sum_{m_1, n_1, m_2, n_2 \leq Y} (m_1 m_2)^\frac{3}{2} a_K(m_1) a_K(m_2) M_K\left(\frac{X}{m_1}\right) M_K\left(\frac{X}{m_2}\right) \\
\times \frac{a_K(n_1) a_K(n_2)}{n_1^{2/3} n_2^{2/3}} \cos \left(\frac{6\pi}{\sqrt{Y}} \left(\frac{n_1}{m_1} + \frac{n_2}{m_2}\right)\right).
\]

Firstly, we consider \(Q_3(Y)\). By using the first derivative test, (2.3) and the elementary formula \(a + b \geq 2\sqrt{ab}\) \((a > 0, b > 0)\), we get

\[
\int_T^{2T} Q_3(Y) \, dY \ll T^\frac{1}{2} \sum_{m_1, n_1, m_2, n_2 \leq Y} (m_1 m_2)^\frac{3}{2} a_K(m_1) a_K(m_2) M_K\left(\frac{X}{m_1}\right) M_K\left(\frac{X}{m_2}\right) \\
\times \frac{a_K(n_1) a_K(n_2)}{n_1^{2/3} n_2^{2/3}} \times \frac{1}{\sqrt{\frac{n_1}{m_1} + \frac{n_2}{m_2}}} \tag{4.4}
\]

\[
\ll X^2 T^{\frac{3}{2}} \sum_{m_1, n_1, m_2, n_2 \leq Y} \frac{a_K(m_1) a_K(m_2)}{(m_1 m_2)^{1/6}} \sum_{n_1, n_2 \leq Y} \frac{a_K(n_1) a_K(n_2)}{n_1^{5/6} n_2^{5/6}} \\
\ll X^{\frac{3}{2}} T^{\frac{3}{2}} \sqrt{\frac{Y}{X^2}}
\]

where in the last step we used (1.6) and a summation by parts.

Secondly, we consider \(Q_2(Y)\). By the first derivative test and (2.3) again we get with the help of Lemma 2.11 that

\[
\int_T^{2T} Q_2(Y) \, dY \\
\ll T^\frac{1}{2} \sum_{m_1, n_1, m_2, n_2 \leq Y} (m_1 m_2)^\frac{3}{2} a_K(m_1) a_K(m_2) M_K\left(\frac{X}{m_1}\right) M_K\left(\frac{X}{m_2}\right) \\
\times \frac{a_K(n_1) a_K(n_2)}{n_1^{2/3} n_2^{2/3}} \times \frac{1}{\sqrt{\frac{n_1}{m_1} - \frac{n_2}{m_2}}} \\
\ll X^2 T^{\frac{3}{2}} \sum_{m_1, n_1, m_2, n_2 \leq Y} \frac{a_K(m_1) a_K(m_2) a_K(n_1) a_K(n_2)}{(n_1 n_2)^{2/3} |\sqrt{n_1 m_2} - \frac{3n_2 m_1}{n_1}|}
\]
Noting that 

\[ a_K(n) = \frac{1}{\sqrt{m_1m_2}} \sum_{n_1m_2 \neq n_2m_1} \frac{a_K(m_1)a_K(m_2)a_K(n_1)a_K(n_2)}{(m_1m_2)^{2/3}(n_1n_2)^{2/3} | \sqrt{n_1m_2} - \sqrt{n_2m_1}|} \]

we obtain

\[ \ll X^{1/2} T^{1/2} \\sum_{m_1,m_2 \leq X, n_1,n_2 \leq \sqrt{X}} \tau_2^2(l_1) \tau_2^2(l_2) \frac{n_1}{l_1^{2/3} l_2^{2/3} (\sqrt{T_1} - \sqrt{T_2})} \]

\[ \ll T^{1/2} X(Y) \frac{1}{X} + \epsilon \]

\[ \ll X^{1/2} T^{1/2} (XY)^{1/2} + \epsilon \]

where we used the estimate \( a_K(m)a_K(n) \leq \tau^2(m) \tau^2(n) \leq \tau^2(mn) \).

Finally, we consider \( Q_1(Y) \). Let \( m = (m_1, m_2) \). Write \( m_1 = mm'_1, m_2 = mm'_2 \) such that \( (m'_1, m'_2) = 1 \). If \( n_1m_2 = n_2m_1 \), we immediately get that \( n_1 = mm'_1, n_2 = mm'_2 \) for some positive integer \( n \). It follows that

\[ Q_1(Y) = \frac{Y^2}{6\pi^2} \sum_{mm_1,mm_2 \leq X} \sum_{\gcd(m_1,m_2) = 1} m^{4/3} a_K(mm_1)a_K(mm_2)M_K \left( \frac{X}{mm_1} \right) M_K \left( \frac{X}{mm_2} \right) \]

\[ \times \sum_{n \leq \min \left( \frac{X}{mm_1}, \frac{X}{mm_2} \right)} \frac{a_K(nm_1)a_K(nm_2)}{n^{4/3}} \tag{4.6} \]

\[ = c(X) Y^2 + E(Y), \]

where

\[ c(X) = \frac{1}{6\pi^2} \sum_{mm_1,mm_2 \leq X} \sum_{\gcd(m_1,m_2) = 1} m^{4/3} a_K(mm_1)a_K(mm_2)M_K \left( \frac{X}{mm_1} \right) M_K \left( \frac{X}{mm_2} \right) \]

\[ \times \sum_{n=1}^{\infty} \frac{a_K(nm_1)a_K(nm_2)}{n^{4/3}}, \tag{4.7} \]

\[ E(Y) = \frac{Y^2}{6\pi^2} \sum_{mm_1,mm_2 \leq X} \sum_{\gcd(m_1,m_2) = 1} m^{4/3} a_K(mm_1)a_K(mm_2)M_K \left( \frac{X}{mm_1} \right) M_K \left( \frac{X}{mm_2} \right) \]

\[ \times \sum_{n > \min \left( \frac{X}{mm_1}, \frac{X}{mm_2} \right)} \frac{a_K(nm_1)a_K(nm_2)}{n^{4/3}}. \]

Noting that \( a_K(mn) \leq \tau^2(mn) \leq \tau^2(m) \tau^2(n) \), we get that

\[ E(Y) \ll Y^2 \sum_{mm_1,mm_2 \leq X} \sum_{\gcd(m_1,m_2) = 1} m^{4/3} a_K(mm_1)a_K(mm_2)M_K \left( \frac{X}{mm_1} \right) M_K \left( \frac{X}{mm_2} \right) \]

\[ \times \sum_{n > \min \left( \frac{X}{mm_1}, \frac{X}{mm_2} \right)} \frac{a_K(nm_1)a_K(nm_2)}{n^{4/3}} \]

\[ \ll X^2 Y^2 \sum_{m \leq X} \frac{\tau^4(m)}{m^{2/3}} \sum_{m_1 \leq \frac{X}{m}, m_2 \leq \frac{X}{m}, \gcd(m_1,m_2) = 1} \frac{\tau^4(m_1)\tau^4(m_2)}{m_1m_2} \sum_{n > \min \left( \frac{X}{mm_1}, \frac{X}{mm_2} \right)} \frac{\tau^4(n)}{n^{4/3}} \]

\[ \ll T^{1/2} X(Y)^{1/2} + \epsilon, \]
This together with (4.6) yields
\[
\int_T^{2T} Q_1(Y) \, dY = c(X) \int_T^{2T} Y^{\frac{7}{3}} \, dY + O(X^{\frac{7}{3}} T^{\frac{4}{3}} + e^{-\frac{1}{3}} + X^{\frac{5}{3}} T^{\frac{2}{3}} + e^{-\frac{1}{3}} + X^{\frac{2}{3}} T^{\frac{1}{3}} + e^{-\frac{1}{3}} + X^{\frac{1}{3}} T^{\frac{0}{3}} + e^{-\frac{1}{3}}).
\] (4.9)

Similar to (4.8), we obtain the estimate
\[
c(X) \ll X^{\frac{7}{3} + e}.
\] (4.10)

From (4.3)–(4.5) and (4.9), we get
\[
\int_T^{2T} \mathcal{R}_1^2 \, dY = c(X) \int_T^{2T} Y^{\frac{2}{3}} \, dY + O(X^\frac{7}{3} T^{\frac{5}{3}} + e^{-\frac{1}{3}} + X^{\frac{5}{3}} T^{\frac{2}{3}} + e^{-\frac{1}{3}} + X^{\frac{2}{3}} T^{\frac{1}{3}} + e^{-\frac{1}{3}}.
\] (4.11)

**C. Evaluation of \(\int_T^{2T} \mathcal{R}_2^2 \, dY\)**

From (4.2), (4.10), (4.11) and Cauchy’s inequality, we get
\[
\int_T^{2T} \mathcal{R}_1 \mathcal{R}_2 \, dY \ll X^{\frac{17}{3}} T^{\frac{5}{3} + e} y^{-\frac{1}{6}} + X^{\frac{7}{3}} T^{\frac{2}{3} + e}.
\] (4.12)

Combining (4.1), (4.2) and (4.11), we finally get
\[
\int_T^{2T} \mathcal{R}_2^2 \, dY = c(X) \int_T^{2T} Y^{\frac{2}{3}} \, dY + O(X^{\frac{11}{3}} T^{\frac{4}{3} + e} y^{\frac{1}{3}} + X^{\frac{7}{3}} T^{\frac{2}{3} + e} y^{-\frac{1}{3}} + X^{\frac{17}{3}} T^{\frac{5}{3} + e} y^{-\frac{1}{6}} + X^{\frac{7}{3}} T^{\frac{2}{3} + e}.
\] (4.13)

By choosing a best \(y \in (1, (T/X)^{1/3})\) via Lemma 2.9, we get
\[
\int_T^{2T} |\mathcal{R}_K(X, Y)|^2 \, dY = c(X) \int_T^{2T} Y^{\frac{2}{3}} \, dY + O(X^{\frac{11}{3}} T^{\frac{4}{3} + e} + X^{\frac{26}{3}} T^{\frac{29}{3} + e}),
\]
where \(c(X)\) is defined by (4.7). This completes the proof of Theorem 2.

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**References**

1. T.H. Chan, A.V. Kumchev, On sums of Ramanujan sums. Acta Arith. **152**, 1–10 (2012)
2. K. Chandrasekharan, R. Narasimhan, The approximate functional equation for a class of zeta-functions. Math. Ann. **152**, 30–64 (1963)
3. X. Cao, Y. Tanigawa, W. Zhai, Tong-type identity and the mean square of the error term for an extended Selberg class. Sci. China Math. **59**, 2103–2144 (2016)
4. J.B. Friedlander, H. Iwaniec, Summation formulae for coefficients of $L$-functions. Can. J. Math. **57**, 494–505 (2005)
5. O.M. Fomenko, Mean values connected with the Dedekind zeta function. J. Math. Sci. **150**, 2114–2122 (2008)
6. S.W. Graham, G. Kolesnik, *Van Der Corput’s Method of Exponential Sums* (Cambridge University Press, Cambridge, 1991)
7. A. Ivić, *The Riemann Zeta-Function* (John Wiley & Sons, New York, 1985)
8. A. Ivić, On zeta-functions associated with Fourier coefficients of cusp forms, in *Proceedings of the Amalfi Conference on Analytic Number Theory*. ed. by E. Bombieri et al. (Università di Salerno, Salerno, 1992), pp.231–246
9. I. Kiuchi, Sums of averages of generalized Ramanujan sums. J. Number Theory **180**, 310–348 (2017)
10. I. Kiuchi, F. Pillichshammer, S.S. Eddin, On the multivariable generalization of Anderson–Apostol sums. [arXiv:1811.06022 [math.NT]]
11. P. Kühn, N. Robles, Explicit formulas of a generalized Ramanujan sum. Int. J. Number Theory **12**, 383–408 (2016)
12. E. Landau, *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale*, 2nd edn. (New York, 1949)
13. W. Müller, On the distribution of ideals in cubic number fields. Monatsh. Math. **106**, 211–219 (1988)
14. W.G. Nowak, The average size of Ramanujan sums over quadratic number fields. Arch. Math. **99**, 433–442 (2012)
15. O. Robert, P. Sargos, Three-dimensional exponential sums with monomials. J. Reine Angew. Math. **591**, 1–20 (2006)
16. N. Robles, A. Roy, Moments of averages of generalized Ramanujan sums. Monatsh. Math. **182**, 433–461 (2017)
17. C. Sneha, G. Shivani, On the distribution of Ramanujan sums over number fields. [arXiv:2109.09398 [math.NT]]
18. W. Zhai, The average size of Ramanujan sums over quadratic number fields. Ramanujan J. **56**, 953–969 (2021)
19. W. Zhai, The average size of Ramanujan sums over quadratic number fields (II) (submitted)

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