Notes on Vanishing Cosmological Constant
without Bose-Fermi Cancellation

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Abstract

In this article we discuss how one can systematically construct the point particle theories that realize the vanishing one-loop cosmological constant without the bose-fermi cancellation. Our construction is based on the asymmetric (or non-geometric) orbifolds of supersymmetric string vacua. Using the building blocks of their partition functions and their modular properties, we construct the theories which would be naturally identified with certain point particle theories including infinite mass spectra, but not with string vacua. They are obviously non-supersymmetric due to the mismatch of the bosonic and fermionic degrees of freedom at each mass level. Nevertheless, it is found that the one-loop cosmological constant vanishes, after removing the parameter effectively playing the role of the UV cut-off. As concrete examples we demonstrate the constructions of the models based on the toroidal asymmetric orbifolds with the Lie algebra lattices (Englert-Neveu lattices) by making use of the analysis given in [26].

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1 Introduction

String theories on the asymmetric orbifolds, which should correspond to some non-geometric backgrounds, have interesting aspects. Among others it is remarkable that the string vacua with the vanishing cosmological constant can be realized without the help of unbroken SUSY. Such studies have been initiated in type II string theories by [1–3] based on some non-abelian orbifolds, followed by studies e.g. in [4–9]. More recently, several non-SUSY vacua with this property have been constructed as asymmetric orbifolds [10] by simpler cyclic groups in [11,12].

In heterotic string theories, on the other hand, there have been many studies on the string vacua with the cosmological constant exponentially suppressed with respect to some moduli (for instance, the radii of tori of compactifications as given in [13]) in [4,14], and more recently, e.g. in [15–21] closely related with the model buildings in string phenomenology.

However, in all of these models, the vanishing (or exponentially suppressed) cosmological constant is eventually achieved by the bose-fermi cancellation, even though the space-time SUSY is broken. Toward realistic particle content, it would be more preferable to realize the vanishing (or very small) cosmological constant without the bose-fermi cancellation. In string theory this means that

$$Z(\tau) \neq 0, \quad \text{but, } \Lambda \equiv \int_F \frac{d^2 \tau}{\tau_2^2} Z(\tau) = 0 \quad (\tau \equiv \tau_1 + i\tau_2 \in \mathbb{H}^+) , \quad (1.1)$$

at the one-loop level (at least). Here, $Z(\tau)$ is the torus partition function, and $F$ denotes the familiar fundamental region of the full modular group $\Gamma(1) \equiv SL(2,\mathbb{Z})$, in other words, the moduli space of the world-sheet torus,

$$F := \left\{ \tau \in \mathbb{H}^+ : -\frac{1}{2} \leq \tau_1 < \frac{1}{2}, \ |\tau| \geq 1 \right\} . \quad (1.2)$$

Unfortunately it has been known to be very difficult to find the string vacua with this property. An elaborated mechanism has been proposed in [22] to achieve them based on some modular symmetry argument by the Atkin-Lehner involution. However, concrete constructions of such string vacua that are physically consistent along this line are still very difficult [23–25].

In this paper, instead of searching the string vacua, we discuss how one can systematically construct the particle theories with the preferable property mentioned above, that is, the vanishing one-loop cosmological constant without the bose-fermi cancellation,

$$Z_{\text{particle}}(\ell) \neq 0, \quad \text{but, } \Lambda \equiv \lim_{\varepsilon \to +0} \int_\varepsilon^\infty \frac{d\ell}{\ell} Z_{\text{particle}}(\ell) = 0 , \quad (1.3)$$

where $\ell$ denotes the Schwinger parameter (modulus of the world-line circle) and $\varepsilon$ is the UV cut-off, based on some non-geometric orbifolds of superstring vacua. Indeed, as will be demonstrated, $Z_{\text{particle}}(\ell)$ given here originates from the partition sum only of the untwisted sector.
of the relevant orbifold models rather than the total one, which is only invariant under the modular $T$-transformations $\tau \rightarrow \tau + n \ (\forall n \in \mathbb{Z})$. Consequently, the natural integration region of modulus $\tau$ should be the ‘strip region’

$$S := \left\{ \tau \in \mathbb{H}^+ : \ -\frac{1}{2} \leq \tau_1 < \frac{1}{2} \right\},$$

in place of $\mathcal{F}$. We then naturally obtain $Z_{\text{particle}}(\ell)$ by identifying $\tau_2$ with $\ell$ and integrating $\tau_1$ out, which just amounts to imposing the level-matching condition. In the main part of this article, we will clarify how to construct the appropriate non-geometric orbifold models whose untwisted sector yields the particle theories with the desired property (1.3). Of course, as working with particle theories (the integration region should be $S$ rather than $\mathcal{F}$), one would have to introduce the UV cut-off as in (1.3). We shall actually adopt a generalization of the Scherk-Schwarz type spontaneous SUSY breaking [13], in which the compactification radius turns out to effectively play the role of the UV cut-off.

This paper is organized as follows: In section 2, we discuss how one can obtain the particle theories with the property (1.3) from rather general setups of orbifolds. We propose the conditions that should be satisfied by the wanted models, and prove that they are indeed sufficient to realize (1.3). In section 3, we demonstrate a systematic construction of the concrete models satisfying these conditions based on the type II and heterotic string vacua compactified on toroidal asymmetric orbifolds. Especially, the models we focus on are the ones given in [26], with some extensions and refinements included. The main part of section 3 would look a little technical, but, we also present simple examples in order to clarify the general features of our models. In section 4, we conclude with several comments and discussion.

## 2 How to Achieve Vanishing Cosmological Constant

In this section we discuss how to obtain the particle theories from general setups of orbifolds which possess the property (1.3). Among others, we clarify the conditions that should be satisfied by the relevant orbifold models.

We start with a generic superstring vacuum with unbroken space-time SUSY, which we tentatively denote as ‘$\mathcal{M}_0$’. More precisely, we assume that $\mathcal{M}_0$ is defined as the following background,

$$\mathcal{M}_0 \cong \mathbb{R}^{D-1,1} \times \mathcal{K} \times S^1_{R},$$

where $S^1_{R}$ denotes the circle with radius $R \sqrt{\alpha'}$. The ‘internal sector’, which we also denote by $\mathcal{K}$, only has a discrete spectrum.
Let $g$ be an order $N_0$ automorphism that acts on $\mathcal{K}$ as well as the world-sheet fermions and does not commute with any space-time supercharges. We further assume the existence of the modular invariant partition function for the $\mathbb{Z}_N$-orbifold of $\mathbb{R}^{D-1,1} \times \mathcal{K} (\otimes \text{fermions})$ defined by the $g$-action, where $N$ is a certain multiple of $N_0$. This is written in the form,

$$Z_{\text{orb}}(\tau) \equiv \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} Z_{(a, b)}(\tau), \quad (2.1)$$

where $a, b \in \mathbb{Z}_N$ label the twistings along the spatial and the temporal direction respectively with respect to the orbifold action $g$ in the world-sheet torus. In other words, one may write

$$Z_{(a, b)}(\tau) \equiv \text{Tr}_{\mathcal{H}_a} \left[ g^b q^{L_0 - \frac{c}{24}} q^{-\frac{c}{24}} \right], \quad (2.2)$$

where $\mathcal{H}_a$ denotes the Hilbert space of the twisted sector associated to the $g^a$ action. $Z_{(a, b)}(\tau)$ should possess the periodicity

$$Z_{(a+N, b)}(\tau) = Z_{(a, b+N)}(\tau) = Z_{(a, b)}(\tau), \quad (2.3)$$

for consistency. Moreover, one should require the ‘modular covariance’;

$$Z_{(a, b)}(\tau) \mid_S \left( \equiv Z_{(a, b)} \left( -\frac{1}{\tau} \right) \right) = Z_{(b, -a)}(\tau), \quad (2.4)$$

$$Z_{(a, b)}(\tau) \mid_T \left( \equiv Z_{(a, b)} (\tau + 1) \right) = Z_{(a, a+b)}(\tau), \quad (2.5)$$

in the standard fashion.

Now, setting $R = N \epsilon$ with a small positive number $\epsilon$, let us consider the orbifold of $\mathcal{M}_0 \equiv \mathbb{R}^{D-1,1} \times \mathcal{K} \times S^1_{N\epsilon}$ defined by the operator

$$g := g \otimes T_{2\pi \epsilon}, \quad (2.6)$$

where $T_{2\pi \epsilon}$ denotes the translation $X \rightarrow X + 2\pi \epsilon \sqrt{\alpha'}$ along $S^1_{N\epsilon}$. One may regard this as a generalization of the Scherk-Schwarz type compactification [13]. Thus, the models we will construct are actually those with a spontaneously broken SUSY, in which the relevant SUSY is recovered when taking the ‘Scherk-Schwarz radius’ $\epsilon$ to be infinity. We note that the vacuum energy is not lifted as $\epsilon$ is varied, since it parameterizes a flat direction. We shall later take the $\epsilon \rightarrow +0$ limit. The torus partition function of the $g$-orbifold is written as

$$Z_{\text{orb}}(\tau) = \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} Z_{(a, b)}(\tau) \frac{N \epsilon}{\sqrt{\tau_2 |\eta(\tau)|^2} \sum_{w, m \in \mathbb{Z}} e^{-\frac{\pi |N|^2}{\tau_2} |(w + \frac{a}{N}) \tau + (m + \frac{b}{N})|^2}} \equiv \frac{\epsilon}{\sqrt{\tau_2 |\eta(\tau)|^2} \sum_{w, m \in \mathbb{Z}} Z_{(w, m)}(\tau) e^{-\frac{\pi |N|^2}{\tau_2} |w \tau + m|^2}, \quad (2.7)$$

$^1$We allow the cases of $N \neq N_0$, which typically happens for the asymmetric orbifolds due to the existence of non-trivial phase factors in the twisted sectors (see e.g. [11, 27, 28]). We will actually work with the orbifolds of this type in section 3.
which is manifestly modular invariant.

Then, we define the ‘theory $\mathcal{M}[\epsilon]$’ as that contains the mass spectrum read off from the level-matched sector of the following partition function

$$Z_{M[\epsilon]}(\tau) := \frac{\epsilon}{\sqrt{|\tau_2|}} \sum_{m \in \mathbb{Z}} Z_{(0,m)}(\tau) e^{-\frac{\epsilon}{2} \tau_2 m^2}. \quad (2.8)$$

This represents the contributions to $Z_{\text{orb}}(\tau)$ for the $g$-orbifold from the untwisted sector with no spatial winding number along $S^1_{N\epsilon}$. The exponential factor $e^{-\frac{\epsilon}{2} \tau_2 m^2}$ makes the infinite summation over $m \in \mathbb{Z}$ converge absolutely as long as $\epsilon > 0$. One can read off the mass spectrum from the partition function (2.8) by making the Poisson resummation for the temporal winding $m \in \mathbb{Z}$.

The overall factor $\epsilon$ is absorbed after this resummation. Since we have $T_{2\pi \epsilon} \equiv \exp \left[ -i 2\pi \epsilon \sqrt{\alpha'} P \right]$, where $P$ denotes the KK momentum operator along $S^1_{N\epsilon}$, the states with the $g$-eigenvalue $e^{2\pi i \frac{r}{N}}$ have to possess the KK momenta $n \equiv r \pmod{N}$, implying that these excitations at least have masses of order $\sim \frac{1}{\epsilon} M_s \left( \equiv \frac{1}{\epsilon \sqrt{\alpha'}} \right)$ except for the $r = 0$ case. In particular all the massless states in $M[\epsilon]$ are lying in $Z_0(\tau) \equiv \frac{1}{N} \sum_{b \in \mathbb{Z}_N} Z_{(0,b)}(\tau)$.

We emphasize that $Z_{M[\epsilon]}(\tau)$ is not invariant under the full modular group. This is invariant only under the modular $T$-transformations. Thus, we cannot adopt the usual fundamental region $\mathcal{F}$ of $\mathbb{H}^+ / \Gamma(1)$ given in (1.2) for the integration of the modulus $\tau$. In other words, the theory $\mathcal{M}[\epsilon]$ is not identified with some string vacuum, although we have an infinite number of mass spectrum. It is rather natural to take the ‘strip region’ $S$ defined in (1.4) as the appropriate integration region, which is identified as the fundamental region for $\mathbb{H}^+ / \langle T \rangle$. We note that the factor $e^{-\frac{\epsilon}{2} \tau_2 m^2}$ for $m \neq 0$ effectively truncates the UV region $\tau_2 \sim +0$, which makes the moduli integral well-defined, even if taking $S$ instead of $\mathcal{F}$. In other words the parameter $\epsilon$ plays the role of the UV cut-off.

Let us recall the familiar relationship between the one-loop cosmological constants of closed string theory and of point particle theory (see e.g. section 7.3 of [29]). We assume the bosonic (fermionic) mass spectrum \{m_i\}, $i \in \mathcal{H}_B$ ($i \in \mathcal{H}_F$). The one-loop cosmological constant of the particle theory on the space-time with $D$-dimensional non-compact directions is schematically expressed in terms of the summation of path-integrals over a world-line circle (with modulus $\ell$);

$$\Lambda_{\text{particle}} = \frac{1}{V_D} \left[ \sum_{i \in \mathcal{H}_B} - \sum_{i \in \mathcal{H}_F} \right] Z_{S^1}(m_i^2) \equiv \left[ \sum_{i \in \mathcal{H}_B} - \sum_{i \in \mathcal{H}_F} \right] \int_0^\infty \frac{d\ell}{\ell} \int \frac{d^Dp}{(2\pi)^D} e^{-\frac{\ell}{\epsilon} \tau_2 [p_i^2 + m_i^2]}, \quad (2.9)$$

where $p_i$ are the zero-mode momenta along $\mathbb{R}^{D-1,1}$ and $V_D$ denotes the volume factor. This is naturally compared with that of string theory. For instance, consider the type II string on the
\( \mathbb{R}^{D-1,1} \times [\text{internal sector}] \), where the ‘internal sector’ only includes the discrete spectrum. The 1-loop cosmological constant is written schematically as

\[
\Lambda_{\text{string}} = \frac{1}{V_D} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} Z(\tau),
\]

(2.10)

where

\[
Z(\tau) \equiv \frac{V_D}{\tau_2^{D-1}} \sum_{(i,\tilde{i}) \in \mathcal{H}_\perp} D(h_i, \tilde{h}_i) \, q^\frac{h_i - \frac{1}{2}}{2}q^{\tilde{h}_i - \frac{1}{2}} \quad (q = e^{2\pi i \tau}),
\]

(2.11)

denotes the modular invariant partition function on the world-sheet torus\(^2\). The coefficients of ‘degeneracy’ \(D(h_i, \tilde{h}_i)\) are positive (negative) integers for the bosonic (fermionic) states.

The correspondence between (2.9) and (2.10) is clear, if we rewrite \( \tau \equiv \theta + i\ell \) and identify the imaginary part \( \ell \) with the circle modulus (Schwinger parameter). Indeed, we obtain by formally replacing the integration region \( \mathcal{F} \) with \( \mathcal{S} \);

\[
\Lambda_{\text{string}} [\mathcal{F} \text{ replaced with } \mathcal{S}] \equiv \frac{1}{V_D} \int_{\mathcal{S}} \frac{d^2 \tau}{\tau_2} \sum_{i,\tilde{i}} D(h_i, \tilde{h}_i) \, q^\frac{h_i - \frac{1}{2}}{2}q^{\tilde{h}_i - \frac{1}{2}}
\]

\[
= \int_0^\infty d\ell \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{2\pi \alpha' \ell}{\ell} \sum_{i,\tilde{i}} D(h_i, \tilde{h}_i) \exp \left[ - \left( h_i + \tilde{h}_i - 1 \right) \ell + i\theta \left( h_i - \tilde{h}_i \right) \right],
\]

(2.12)

which easily reduces to the expression in the form as (2.9). In fact, the \( \theta \)-integral just amounts to imposing the level-matching condition \( h_i = \tilde{h}_i \), and the mass spectrum is identified as

\[
\frac{\alpha'}{4} m_i^2 = h_i - \frac{1}{2}.
\]

(2.13)

Based on these simple arguments, we propose that \( \mathcal{M}[\varepsilon] \) defined above represents a theory with an infinite number of particle spectrum, in which the one-loop cosmological constant is given as

\[
\Lambda_{\mathcal{M}[\varepsilon]} \equiv \frac{1}{V_D} \int_{\mathcal{S}} \frac{d^2 \tau}{\tau_2} \sum_{i \in \mathcal{H}[\mathcal{M}[\varepsilon]]} D(m_i) e^{-2\pi \ell \frac{\alpha'}{2} m_i^2},
\]

(2.14)

\(^2\)\((i, \tilde{i}) \in \mathcal{H}_\perp\) indicates the transverse degrees of freedom in \( \mathbb{R}^{D-2} \times [\text{internal sector}] \), and the summation is only taken over the discrete spectrum of conformal weights. The Gaussian integral of zero-mode momenta along the transverse direction \( \mathbb{R}^{D-2} \) just yields the factor \( \tau_2^{\frac{\alpha'}{2}} \), while one more factor \( \tau_2^{-1} \) due to the temporal and longitudinal zero-mode integrals are incorporated into the modular invariant measure \( d^2 \tau/\tau_2^2 \).
where the mass spectrum, which is read off from the partition function (2.8) and by imposing
the level-matching condition, is expressed as $\mathcal{H}[\mathcal{M}[\epsilon]]$. Again $D(m_i)$
denotes the degeneracy of the mass spectrum including the minus sign for fermions. Recall that the integration region
suitable for $Z_{\mathcal{M}[\epsilon]}(\tau)$ should be $\mathcal{S}$ rather than $\mathcal{F}$, as mentioned above. We can thus naturally
identify the partition function $Z_{\mathcal{M}[\epsilon]}(\tau)$ as describing a spectrum of infinite particles, rather
than a string spectrum, without making any formal replacement of the integration region of
modulus.

Under these preparations let us exhibit the fundamental requirements to be satisfied by the
models we want (some are already mentioned):

(1) The unorbifolded string vacuum $\mathcal{M}_0$ has the vanishing partition function,

$$Z_{\mathcal{M}_0}(\tau) \equiv \frac{N\epsilon}{\sqrt{2\pi |\eta(\tau)|^2}} \sum_{w,m \in \mathbb{Z}} e^{-\frac{1}{2}N^2\epsilon^2 |w\tau + m|^2} Z_{(0,0)}(\tau) = 0. \quad (2.15)$$

Needless to say, it is enough to start with any supersymmetric vacuum $\mathcal{M}_0 \equiv \mathbb{R}^{d-1,1} \times \mathcal{K} \times S_{N\epsilon}^1$.

(2) The partition function $Z_{\mathcal{M}[\epsilon]}(\tau)$ defined in (2.8) does not vanish. This may imply

$$Z_0(\tau) \equiv \frac{1}{N} \sum_{b \in \mathbb{Z}_N} Z_{(0,b)}(\tau) \neq 0.$$  

Namely, all the supercharges existing originally in $\mathcal{M}_0$ are removed by the $g$-projection.

(3) The spectrum of the level-matched states appearing in $Z_{\mathcal{M}[\epsilon]}(\tau)$ is consistent with unitarity. Alternatively, we require that $Z_0(\tau)$ possess the same property.

(4) The spectra of the level-matched states in the sectors $Z_a(\tau) \equiv \frac{1}{N} \sum_{b \in \mathbb{Z}_N} Z_{(a,b)}(\tau)$ for $^\forall a \in \mathbb{Z}_N$
do not contain tachyons. In other words, we require

$$\lim_{\tau_2 \to +\infty} e^{4\pi\tau_2 \frac{1}{\tau_1}} \left| \int_{-1/2}^{1/2} d\tau_1 Z_a(\tau) \right| < +\infty \quad (^\forall a \in \mathbb{Z}_N). \quad (2.16)$$

(We allow the existence of level-mismatched tachyons.)

(5) The orbifold partition function (2.1) vanishes.

$^3$The factor $e^{4\pi\tau_2 \frac{1}{\tau_1}}$ originates from the factor $|\eta|^{-2}$ appearing in the $S_{N\epsilon}^1$-sector. We note that the non-
tachyonic behavior for $a \neq 0$ is necessary in order to prove the statement (2.17) (see below (2.25)), even though
only the $a = 0$ sector contributes to the spectrum of the theory $\mathcal{M}[\epsilon]$. 

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Now, we state our main claim: under the requirements (1) \sim (5) given above\(^4\), the theory \(\mathcal{M}[\epsilon] (\epsilon > 0)\) yields the finite cosmological constant \(\Lambda_{\mathcal{M}[\epsilon]}\), and we further obtain

\[
\lim_{\epsilon \to +0} \Lambda_{\mathcal{M}[\epsilon]} = 0, \quad \text{for } \forall D \geq 1. \tag{2.17}
\]

[proof of the claim]

We first note that \(\Lambda_{\mathcal{M}[\epsilon]}\) defined in (2.14) is explicitly written as

\[
\Lambda_{\mathcal{M}[\epsilon]} = \frac{1}{V_D} \int_{S} \frac{d^2\tau}{\tau_2} \frac{\epsilon}{\sqrt{\tau_2} |\eta(\tau)|^2} \sum_{m \in \mathbb{Z}} \hat{Z}_{(0,m)}(\tau) e^{-\frac{\tau_0^2}{\tau_2^2} m^2}. \tag{2.18}
\]

Then, \(\Lambda_{\mathcal{M}[\epsilon]}\) should be finite for \(\forall \epsilon > 0\). Indeed, the \(m = 0\) term is absent due to the condition (1), and the potential UV-divergence around \(\tau_2 \sim 0\) is removed by the damping factor \(e^{-\frac{\tau_0^2}{\tau_2^2} m^2}\) for \(\forall m \neq 0\), as mentioned above. Moreover, the convergence in the IR-region \(\tau_2 \to +\infty\) is ensured by the condition (4).

Let us next prove the more non-trivial statement (2.17). Set

\[
\hat{Z}_{(w,m)}(\tau; \epsilon) := Z_{(w,m)}(\tau) \frac{\epsilon}{\sqrt{\tau_2} |\eta(\tau)|^2} e^{-\frac{\tau_0^2}{\tau_2^2} |w\tau + m|^2}, \quad (\forall w, m \in \mathbb{Z}). \tag{2.19}
\]

Because of the modular covariance

\[
\hat{Z}_{(w,m)}(\tau; \epsilon)|_S = \hat{Z}_{(m,-w)}(\tau; \epsilon), \quad \hat{Z}_{(w,m)}(\tau; \epsilon)|_T = \hat{Z}_{(w,w+m)}(\tau; \epsilon),
\]

together with the condition (1), that is, \(\hat{Z}_{(0,0)}(\tau; \epsilon) \equiv 0\), we can rewrite (2.18) as

\[
\Lambda_{\mathcal{M}[\epsilon]} \equiv \frac{1}{V_D} \int_{S} \frac{d^2\tau}{\tau_2} \sum_{m \in \mathbb{Z}} \hat{Z}_{(0,m)}(\tau; \epsilon) = \frac{1}{V_D} \int_{F} \frac{d^2\tau}{\tau_2} \sum_{w,m \in \mathbb{Z}} \hat{Z}_{(w,m)}(\tau; \epsilon), \tag{2.20}
\]

due to the arguments given in [30–32], which are commonly used in thermal string theory. Introducing the ‘Fourier transform’

\[
\hat{Z}_{(a,b)}(\tau) := \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} Z_{(a,b)}(\tau) e^{2\pi i \frac{1}{N} (ab - \beta a)}, \tag{2.21}
\]

and making use of the Poisson resummation, we can further rewrite the R.H.S of (2.20) as follows,

\[
\Lambda_{\mathcal{M}[\epsilon]} = \frac{1}{V_D} \int_{F} \frac{d^2\tau}{\tau_2} \frac{1}{N \epsilon \sqrt{\tau_2} |\eta(\tau)|^2} \sum_{w,m \in \mathbb{Z}} \hat{Z}_{(w,m)}(\tau) e^{-\frac{\tau_0^2}{\tau_2^2} N \frac{1}{N} |w\tau + m|^2}
\]

\[
= \frac{1}{V_D} \int_{F} \frac{d^2\tau}{\tau_2} \frac{1}{N \epsilon \sqrt{\tau_2} |\eta(\tau)|^2} \sum_{\substack{w,m \in \mathbb{Z} \setminus (0,0) \atop (w,m) \neq (0,0)}} \hat{Z}_{(w,m)}(\tau) e^{-\frac{\tau_0^2}{\tau_2^2} N \frac{1}{N} |w\tau + m|^2}. \tag{2.22}
\]

\(^4\)In fact, only the requirements (1), (4), (5) are sufficient to show the claim (2.17).
In the second line we made use of the fact that
\[
\hat{Z}_{(0,0)}(\tau) \equiv \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} Z_{(a,b)}(\tau) = 0,
\]
due to the condition (5).

We first estimate the non-zero winding sectors with \( w \neq 0 \) in (2.22). For each fixed \( w \neq 0 \), the summation over \( m \in \mathbb{Z} \) is evaluated by using the Poisson resummation, giving the inequality,
\[
\frac{1}{V_D N \varepsilon \sqrt{\tau_2}} \frac{1}{|\eta(\tau)|^2} \sum_{m \in \mathbb{Z}} \left| \hat{Z}_{(w,m)}(\tau) \right| e^{-\frac{\pi}{\sqrt{\tau_2}} |w_\tau + m|^2} \leq \frac{C}{\varepsilon} \frac{1}{\tau_2^{D/2}} e^{-\frac{4\pi \tau_2}{\varepsilon}(h_0 - \frac{1}{2})} e^{-\frac{\pi}{N^2 \varepsilon} |w_\tau|^2} \quad (\forall \ w \neq 0, \ \forall \tau \in \mathcal{F}),
\]
with some finite constant \( C, h_0 \). Here, \( h_0 \) is the lowest conformal weight in the discrete part of the spectrum and allowed to be tachyonic.

Therefore, replacing the integration region \( \mathcal{F} \) with the slightly larger one,
\[
\left\{ \tau \in \mathbb{H}^+: -\frac{1}{2} \leq \tau_1 < \frac{1}{2}, \ \tau_2 > \frac{\sqrt{3}}{2} \right\}, \quad (2.23)
\]
we find
\[
|\Lambda_{\varepsilon,w}| \leq \frac{C}{\varepsilon} \int_{\sqrt{3}/2}^{\infty} \frac{dt}{t^{D/2}} \frac{1}{|\eta(it)|^2} e^{-\frac{4\pi \tau_2}{\varepsilon}(h_0 - \frac{1}{2})} e^{-\frac{\pi}{N^2 \varepsilon} |w|^2} \quad (\forall \ w \neq 0).
\]
This integral is obviously finite for sufficiently small \( \varepsilon > 0 \), and we readily obtain
\[
\lim_{\varepsilon \to +0} \sum_{w \neq 0} \Lambda_{\varepsilon,w} = 0. \quad (2.24)
\]

We next focus on the \( w = 0 \) contribution. Replacing again the integration region with (2.23), we obtain the following evaluation
\[
|\Lambda_{\varepsilon,w=0}| \equiv \frac{1}{N \varepsilon V_D} \left| \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2^{D/2}} \frac{1}{\sqrt{\tau_2}} |\eta(\tau)|^2 \sum_{m \neq 0} \hat{Z}_{(0,m)}(\tau) e^{-\frac{\pi}{\tau_2 N^2 N^2}} \right| \
\leq \frac{C'}{\varepsilon} \int_{\sqrt{3}/2}^{\infty} \frac{dt}{t^{D/2}} |\eta(it)|^2 \sum_{m \neq 0} \hat{Z}_{(0,m)}(it) \left| \text{level-matched} \right| e^{-\frac{\pi}{\tau(t^2)} m^2}, \quad (2.25)
\]
with some finite constant \( C' \). Moreover, we note that
\[
e^{-\frac{\pi}{\tau} m^2} \hat{Z}_{(0,m)}(\tau) \right|_{\text{level-matched}} \equiv e^{4\pi \tau_2} \frac{1}{N} \sum_{a,b \in \mathbb{Z}_N} Z_{(a,b)}(\tau) \right|_{\text{level-matched}} e^{-2\pi \tau_2 i a},
\]

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is non-tachyonic due to the condition (4). Thus, we obtain the following evaluation with a finite constant $C''$;

$$|\Lambda_{\epsilon,w=0}| \leq \frac{C''}{\epsilon} \int_{\sqrt{3}/2}^{\infty} \frac{dt}{t^{5/2}} \frac{1}{t^{7/2}} \sum_{m=1}^{\infty} e^{-\frac{\pi}{\sqrt{2}} m^2 t}$$

$$< \frac{C''}{\epsilon} \pi^{-\frac{D+1}{2}} (N\epsilon)^{D+1} \int_{0}^{\infty} \frac{ds}{s} \frac{s^{D+1}}{s^{D+1}} \sum_{m=1}^{\infty} e^{-m^2 s}$$

$$= C'' N^{D+1} \epsilon^D \hat{\zeta}(D + 1), \quad (2.26)$$

where $\hat{\zeta}(s) \equiv \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ is the completed zeta function.

In this way, as long as $D \geq 1$, we obtain the desired result;

$$\lim_{\epsilon \to +0} \Lambda_{M[\epsilon]} \equiv \lim_{\epsilon \to +0} \left[ \Lambda_{\epsilon,w=0} + \sum_{w \neq 0} \Lambda_{\epsilon,w} \right] = 0. \quad \text{(Q.E.D)}$$

A few comments are in order:

(i) Under the requirement (2) the bosonic and fermionic mass spectra in the point particle theory $M[\epsilon]$ are obviously mismatched. Nevertheless, the 1-loop cosmological constant $\Lambda_{M[\epsilon]}$ vanishes under the $\epsilon \to +0$ limit in the cases of $D \geq 1$.

(ii) As $\epsilon$ plays the role of the UV cut-off, the integration region is truncated roughly as follows,

$$\int_{S} d^2 \tau \rightarrow \int_{\epsilon^2}^{\infty} d\tau_2 \int_{1/2}^{1} d\tau_1.$$

In other words, we have a substringy cut-off mass scale $M_\epsilon \sim \frac{1}{\epsilon \sqrt{\alpha'}} \left( \frac{1}{\epsilon} M_s \right)$, and the low energy mass spectrum $M_\epsilon$ is nearly equal to that of $Z_0(\tau) \equiv \frac{1}{N} \sum_{m \in \mathbb{Z}_N} Z_{0,m}(\tau)$, which only contains the $g$-invariant particle spectrum. On the other hand, the ‘superpartners’ that are not $g$-invariant acquire the masses greater than $M_\epsilon$. Consequently, $M_\epsilon$ is interpreted as the energy scale of the spontaneously broken SUSY in our theory $M[\epsilon]$.

It is remarkable that, while the naive dimensional estimation implies

$$\Lambda_{M[\epsilon]} \sim M_\epsilon^D \equiv \epsilon^{-D} M_s^D;$$

we indeed obtain

$$\Lambda_{M[\epsilon]} \sim \epsilon^D M_s^D \equiv \epsilon^{2D} M_\epsilon^D,$$

as was shown above. The existence of the suppression factor $\epsilon^{2D}$ is crucial in our models.
(iii) Intuitively, the counterpart of the bose-fermi cancellation in the original orbifold model is disguised by the modular transformation as other particles in the resultant particle theory $\mathcal{M}[\epsilon]$, which contribute to $\Lambda_{\mathcal{M}[\epsilon]}$ in the integration region $\mathcal{S}$ outside $\mathcal{F}$.

3 Simple Models from String Vacua

In this section we exhibit simple examples satisfying the requirements (1) $\sim$ (5) given in the previous section based on the toroidal asymmetric orbifolds of type II or heterotic superstring vacua. To this end we shall make use of the modular invariants, or modular covariant functions more generally, constructed in [26]. They are associated to Lie algebra lattices satisfying the ‘self-duality condition’ under the T-duality twist. We start with reviewing briefly the results given in [26] for our construction of the relevant models. Since we only need appropriate modular forms, slight extensions and refinements are also included below.

3.1 Asymmetric Orbifolds based on Lie Algebra Lattices

Let us consider the $r$-dimensional torus $T^r[X_r]$ associated to the Englert-Neveu lattice [33] (see also [34] for a review) for the semi-simple Lie algebra $X_r$ (rank $r$), for which we have the symmetry enhancement to the affine $X_r$-symmetry with level 1 on the string world-sheet. We simply call it the ‘Lie algebra lattice for $X_r$’ in this section. We assume that the simple parts of $X_r$ are composed only of

\[ A_1, E_7, D_n \quad (\forall n \in \mathbb{Z}_{>0}). \]

Here, we have slightly extended the argument in [26] by including $D_n$ with odd $n$. The number of ‘$D_{odd}$-components’ is supposed to be even to preserve the unitarity, as explained below. Then the asymmetric orbifold model of $T^r[X_r]$ for the chiral reflection (or the ‘T-duality transformation’)

\[ (-1)^{R}\otimes_r : (X^i_L, X^i_R) \rightarrow (X^i_L, -X^i_R) \]

is described by the following partition function [26],

\[ Z^T_{\text{orb}}[X_r](\tau) \equiv \frac{1}{16} \sum_{a,b \in \mathbb{Z}_{16}} Z^T_{(a,b)}[X_r](\tau), \]

where

\[ Z^T_{(a,b)}[X_r](\tau) \equiv \begin{cases} \varepsilon^{[\epsilon]}_{(a,b)} \left( \hat{x}^{A_1}_{(a,b)}(\tau) \right) X^X_r(\tau) & (a \in 2\mathbb{Z} + 1 \text{ or } b \in 2\mathbb{Z} + 1), \\ Z^T[X^X_r](\tau) & (a, b \in 2\mathbb{Z}), \end{cases} \]
and we introduced the notation

\[
\tilde{\chi}_{(a,b)}^{A_1}(\tau) := \left\{ \begin{array}{ll}
\sqrt{\frac{a+b}{2}} & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z}+1), \\
\sqrt{\frac{a+b}{2}} & (a \in 2\mathbb{Z}+1, \ b \in 2\mathbb{Z}), \\
\sqrt{\frac{a+b}{2}} & (a \in 2\mathbb{Z}+1, \ b \in 2\mathbb{Z}+1), \\
\end{array} \right. \tag{3.4}
\]

\[
\tilde{\epsilon}_{(a,b)}^{[r]} := \left\{ \begin{array}{ll}
e^{im_{ab}} & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z}+1), \\
e^{-im_{ab}} & (a \in 2\mathbb{Z}+1, \ b \in 2\mathbb{Z}), \\
(k_a)^r e^{-im_{ab}} & (a, \ b \in 2\mathbb{Z}+1), \\
\end{array} \right. \tag{3.5}
\]

\[
\kappa_a := e^{-\frac{i\pi}{2}(a^2-1)} \equiv \left\{ \begin{array}{ll}
+1 & (a \equiv 1, 7 \mod 8), \\
-1 & (a \equiv 3, 5 \mod 8). \\
\end{array} \right. \tag{3.6}
\]

Moreover, \(\chi^{X_r}_{(a,b)}(\tau)\) is defined as the simple product

\[
\chi^{X_r}_{(a,b)}(\tau) := \prod_i \chi^{X_r}_{(a,b)}(\tau), \quad \sum_i r_i = r,
\]

and each component is associated to the simple Lie algebra \(X_r^{(i)}\) appearing in (3.1). The explicit forms of \(Z^{T^r[X_r]}(\tau)\) and \(\chi^{X_r}_{(a,b)}(\tau)\) are summarized in appendix A.

One can derive (3.3) by first evaluating the trace,

\[
Z_{(0,1)}^{T^r[X_r]}(\tau) = \text{Tr} \left[ (-1_R)^{\otimes r} q^{La-\frac{a}{8}qLa-\frac{a}{8}} \right],
\]

with \(c = r\), and by requiring the modular covariance,

\[
Z_{(a,b)}^{T^r[X_r]}(\tau) \big|_T = Z_{(a,a+b)}^{T^r[X_r]}(\tau), \quad Z_{(a,b)}^{T^r[X_r]}(\tau) \big|_S = Z_{(b,-a)}^{T^r[X_r]}(\tau), \tag{3.7}
\]

which ensures the modular invariance of (3.2). See [26] for details. We emphasize that the non-trivial phase factor \(\tilde{\epsilon}_{(a,b)}^{[r]}\) is necessary to achieve the modular covariance (3.7), and thus, the building blocks (3.3) generically possess an order 16 periodicity, rather than the naive expectation of order 2, for the twisted sectors with \(a \neq 0\). These phases appear because the action of the chiral reflection in the target space is uplifted on the world-sheet (see e.g. [11,27,28]), and will play a crucial role in our argument given below.

We also note another interpretation of the asymmetric orbifolds given above. The Narain lattice defining the torus \(T^r[X_r]\) can be generically decomposed as

\[
\Gamma^{r}[X_r] = \sum_{\alpha_L,\alpha_R} \Lambda^{X_r}_{(\alpha_L)} \bigoplus \sum_{i=1}^r \Lambda^{A_1}_{(\alpha_R^{(i)})} \equiv \Lambda^{X_r}_{(0)} \bigoplus \left[ \Lambda^{A_1}_{(0)} \oplus \cdots \oplus \Lambda^{A_1}_{(0)} \right] + \cdots, \tag{3.8}
\]

where \(\Lambda^{X_r}_{(\alpha)}\) denotes the Lie algebra lattice of \(X_r\) associated to the conjugacy class labeled by \(\alpha\) (corresponding to a certain integrable representation of affine \(X_r\) with level 1). Especially,
$\Lambda_{(0)}^{X_r}$ is nothing but the root lattice of $X_r$ associated to the basic representation. Only the term including $[\Lambda_{(0)}^{A_1} \oplus \cdots \oplus \Lambda_{(0)}^{A_1}]$ can contribute to the trace $\text{Tr} \left[ (-1_R)^{\otimes r} \cdots \right]$ for the untwisted sector in this decomposition. If focusing on each of the right-moving momentum lattice $\Lambda_{(a)}^{A_1}$, the chiral reflection $-1_R$ is naturally described by the $SU(2)$-current algebra $\{J^a_R\}$ of level 1. Namely, one can simply identify $-1_R \equiv e^{i\pi J^1_R}$ in the basic representation (see appendix B). One can also adopt the ‘chiral half-shift’ $s_R \equiv e^{i\pi J^3_R}$ as the involutive operator in the untwisted sector to define the relevant asymmetric orbifolds, and obtains the same building blocks $Z_T^{X_r-1}((\tau))$ because of the obvious reason of the $SU(2)$ invariance.

In the following, we will mainly focus on the chiral half-shifts rather than the chiral reflections to obtain desired models, which modifies the construction in [26]. We will also maintain the world-sheet $\mathcal{N} = 1$ superconformal symmetry, since we expect that the consistency of the original superstring theories is important to ensure that of the resultant particle theories after interactions are turned on. The chiral half-shift preserves the superconformal symmetry even if trivially acting on the world-sheet fermions $\psi^i_R$, whereas we need to require

$$(-1_R)^{\otimes r} : (\psi^i_L, \psi^i_R) \rightarrow (\psi^i_L, -\psi^i_R),$$

for the chiral reflection.

### 3.2 Type II Models

Let us start with the type II string on $\mathbb{R}^{3,1} \times T^5 \times S^1_{N\epsilon}$, where $N$ is the order of the $g$-orbifold introduced below. In other words we focus on the cases with $\mathcal{K} = T^5$ in the notations given in section 2. As discussed in section 2, we consider the orbifold by $g \equiv g \otimes T_{2\pi\epsilon}$ with appropriate choices of $g$ acting on the $T^5$-sector as well as the world-sheet fermions. The modular invariant partition function of the type II string on this orbifold is written in the form as (2.7), and we shall only focus on the building block ‘$Z_{(a,b)}^{(\tau)}$’ in (2.7), which includes the contributions from $\mathbb{R}^{3,1} \times T^5$ with the $g^n(g^b)$-twist along the spatial(temporal) direction.

We consider the orbifolding by the following involutive operator with $N_0 = 2$ in the notation

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5 More precisely, the chiral half-shift operator $s_R$ (as well as the chiral reflection) has to contain the phase factor to achieve the involutive property $s_R^2 = 1$. Namely, we should adopt the definition $s_R \equiv e^{-i\pi J^3_R} e^{i\pi J^3_R}$, for the spin $\ell/2$ representation of $A_1$ with level 1 ($\ell = 0, 1$).

6 However, it may not be necessary to assume the world-sheet superconformal symmetry, since our purpose in this paper is to construct the non-SUSY particle theories with vanishing cosmological constant, rather than the non-SUSY string vacua. In other words one may be able to relax the requirements of the consistency as string vacua. We will again discuss this point in section 4.
of section 2,  
\[ g := (-1)^{F_L+F_R} \otimes s_R[k+r] \otimes s_L[k+\ell], \]  
(3.9)
where \( s_R[m] \) denotes the chiral half-shift along the \( m \) directions in \( T^5 \) and we set \( d + k + \ell + r = 5 \) \((d, k, \ell, r \geq 0)\).

We consider the case where the Narain lattice of \( T^5 \) can be decomposed as
\[ \Gamma^{5,5} = \Gamma^{d,d} \bigoplus \left[ \left( \Lambda^{A_1}_{(0)} \right)^{k+\ell} \oplus \Lambda^{X}_{(0)} \right] \bigoplus \left[ \left( \Lambda^{A_1}_{(0)} \right)^{k+r} \oplus \Lambda^{X'}_{(0)} \right] + \cdots, \]  
(3.10)
with some Lie algebra lattices for \( X_r, X'_r \) with rank \( r, \ell \) in order to make the operator \( g \) well-defined.

The standard symbol \((-1)^{F_L} \) \((-1)^{F_R} \) acts on the left(right)-moving Ramond sector as the sign-flip (‘space-time fermion number mod 2’). We assume that \( X_r, X'_r \) are composed only of \( A_1, D_n, \) and the total number of the \( D_n \) components with \( n \in 2\mathbb{Z} + 1 \) in \( X_r, X'_r \) should be even to maintain unitarity condition (2), as explained shortly.

The building block \( Z_{(a,b)}(\tau) \) is explicitly written in terms of the twisted characters given in section 3.1 (see also appendix A). For the ‘even sectors’, \( \text{i.e.} \) sectors with \( a, b \in 2\mathbb{Z} \), the building blocks are equal to the partition function of the unorbifolded model. On the other hand, those for the ‘odd sectors’, \( \text{i.e.} \) sectors with \( a \in 2\mathbb{Z} + 1 \) or \( b \in 2\mathbb{Z} + 1 \), are non-trivial. They are explicitly written as
\[ Z_{(a,b)}(\tau) = \frac{1}{\tau_2 |\eta(\tau)|^4} Z^{T^d}(\tau) \cdot |h_{(a,b)}|^2 \cdot \left| \tilde{\chi}_{(a,b)}^{A_1} \right|^{2k} \]
\[ \times \epsilon_{(a,b)}^{[r]} \left( \tilde{\chi}_{(a,b)}^{A_1} \right)^{\tau} \chi_r \cdot \epsilon_{(a,b)}^{[-\ell]} \left( \tilde{\chi}_{(a,b)}^{A_1} \right)^{\ell} \tilde{\chi}_{(a,b)}^{X'} \]
\[ = \frac{1}{\tau_2 |\eta(\tau)|^4} Z^{T^d}(\tau) \cdot |h_{(a,b)}|^2 \]
\[ \times \epsilon_{(a,b)}^{[r-\ell]} \left( \tilde{\chi}_{(a,b)}^{A_1} \right)^{\tau+k} \left( \tilde{\chi}_{(a,b)}^{A_1} \right)^{\ell+k} \tilde{\chi}_{(a,b)}^{X_r} \tilde{\chi}_{(a,b)}^{X'}. \]  
(3.11)

Here we introduced the free fermion chiral blocks twisted by \((-1)^{F_L}\),
\[ h_{(a,b)} := \left\{ \begin{array}{ll}
\left( \frac{\rho_a}{\eta} \right)^4 - \left( \frac{\rho_a}{\eta} \right)^4 + \left( \frac{\rho_a}{\eta} \right)^4 & \text{ \( (a \in 2\mathbb{Z}, b \in 2\mathbb{Z} + 1) \)}, \\
\left( \frac{\rho_a}{\eta} \right)^4 + \left( \frac{\rho_a}{\eta} \right)^4 - \left( \frac{\rho_a}{\eta} \right)^4 & \text{ \( (a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z}) \)}, \\
- \left[ \left( \frac{\rho_a}{\eta} \right)^4 + \left( \frac{\rho_a}{\eta} \right)^4 + \left( \frac{\rho_a}{\eta} \right)^4 \right] & \text{ \( (a, b \in 2\mathbb{Z} + 1) \).} 
\end{array} \right. \]  
(3.12)

One can readily confirm the modular covariance,
\[ Z_{(a,b)}(\tau) \bigg|_T = Z_{(a,a+b)}(\tau), \quad Z_{(a,b)}(\tau) \bigg|_S = Z_{(b,-a)}(\tau), \]  
(3.13)
which ensures the modular invariance of the total partition function. In fact, the first line on the R.H.S of (3.11) is written in the form that is manifestly modular covariant. We again emphasize the importance of the phase factor \( \epsilon_{(a,b)}^{[s]} \) to achieve this relation.
Now, we discuss whether the model obtained from this setup satisfies the requirements \((1) \sim (5)\) given in the previous section. First of all, it is easy to check that \((1) \sim (3)\) are satisfied.

We note that \(g^2 = 1\) in the untwisted sector, and the operator \((-1)^{FL+FR}\) removes, say, all the gravitinos in the spectrum read off from \(Z_0(\tau) \equiv \frac{1}{2} \sum_{b \in \mathbb{Z}_2} Z_{(0,b)}(\tau)\). We also note that \(\epsilon_{(0,b)}^{[s]} = 1\)

\[\text{Thus } Z_0(\tau) \text{ clearly yields a unitary spectrum.}\]

We next focus on the more non-trivial requirement \((5)\). For the ‘even sectors’ with \(a, b \in 2\mathbb{Z}\), the building blocks \(Z_{(a,b)}(\tau)\) vanish as they are the same as the partition function of the original supersymmetric model. We thus need to show that the summation over the ‘odd sectors’ with \(a \in 2\mathbb{Z} + 1\) or \(b \in 2\mathbb{Z} + 1\) vanishes;

\[
\sum_{a \in 2\mathbb{Z} + 1 \text{ or } b \in 2\mathbb{Z} + 1} Z_{(a,b)}(\tau) = 0.
\]

(3.14)

To do so it is sufficient to confirm the following ‘criterion’;

\[
\sum_{b \in 2\mathbb{Z} \cap \mathbb{Z}_{16}} Z_{(a,b)}(\tau) = 0 \quad (\forall \ a \in 2\mathbb{Z} + 1).
\]

(3.15)

Indeed, if this is the case, we readily obtain (3.14) due to the modular covariance (3.13) \(^8\).

Therefore, the question is what type of the Lie algebra lattices for \(X_r, X'_\ell\) in (3.11) can satisfy the criterion (3.15). We note that each term of the theta functions appearing in \(Z_{(a,b)}(\tau)\) contains the various phase factors of the form \(e^{\pi K_{ab}}\) with \(K \equiv r - \ell \pmod{2}\). (Recall the definitions of phase \(\epsilon_{(a,b)}^{[s]}\) and the twisted characters \(\tilde{\chi}_{(a,b)}^{A_1}, \chi_{(a,b)}^{X_r}\) given in appendix A. ) Thus, it is enough that all the theta function terms contain a non-vanishing phase factor in order to satisfy (3.15). Although it seems hard to fully classify the general choices of \(X_r, X'_\ell\) satisfying (3.15), the following two simple cases are obviously sufficient\(^9\);

(i) \(\ell = 0, r \not\in 8\mathbb{Z}\) \((r = 0, \ell \not\in 8\mathbb{Z})\), and \(X_r (X'_\ell)\) is simple (i.e. made up only of a single piece of \(A_1\) (when \(r = 1\)) or \(D_r\)).

(ii) \(r - \ell \in 2\mathbb{Z} + 1\).

Finally, let us examine the requirement \((4)\). At first glance, it seems that the tachyonic behavior would appear due to the fermion chiral block \(h_{(a,b)}\) with \(a \in 2\mathbb{Z} + 1\). However, once

---

\(^7\)The assumption that ‘the number of \(D_n\)-components with \(n \in 2\mathbb{Z} + 1\) should be even’ was necessary for this statement.

\(^8\)Almost the same argument has been used in [35, 36] in a different context.

\(^9\)However, we would like to emphasize that these two are sufficient but not necessary. Namely, there are many examples that satisfy (3.15) apart from these cases (i) and (ii).
(3.15) is satisfied, by using its T-transformation we eventually obtain

$$\sum_{b \in \mathbb{Z}_{16}} Z_{(a,b)}(\tau) = 0.$$ 

Moreover, it is easy to show that we do not have any tachyonic modes in all the sectors of \( a \in 2\mathbb{Z} \) from (3.15).

### 3.3 Heterotic Models

We next try to construct the models based on the heterotic string compactified on \( T^5 \times S^1_{N\psi} \), where the left-mover is given by the 26 dimensional bosonic theory. We shall only consider the \( E_8 \times E_8 \)-cases, and the \( SO(32) \)-cases can be similarly treated.

We take the orbifold action

$$g := (-1)^{FR} \otimes s_R[k + r] \otimes s_L[k + \ell_1 + \ell_2 + \ell_3]$$

$$\equiv (-1)^{FR} \otimes s_R[k + r] \otimes s_L[k + \ell_1] \otimes s_L[\ell_2] \otimes s_L[\ell_3],$$

where all the notation is defined as in (3.9). Here, we assume \( d + k + \ell_1 + r = 5, \ell_2, \ell_3 \leq 8 \). \( s_R[k + r], s_L[k + \ell_1] \) act along the \( T^5 \)-directions, while \( s_L[\ell_2], s_L[\ell_3] \) are assigned to the two \( E_8 \)-factors. We consider the case where the total Narain lattice is decomposed as

$$\Gamma_{21,5} = \Gamma_{d,d} \bigoplus \left[ \Lambda_{A_1}^{(0)} \right]^{k+\ell_1+\ell_2+\ell_3} \bigoplus \Lambda_{X_{1}}^{(0)} \bigoplus \Lambda_{X_{8-\ell_2}}^{(0)} \bigoplus \Lambda_{X_{8-\ell_3}}^{(0)}.$$ 

(3.16)

Here, \( X_r, X_{1}^{(0)} \) are composed only of \( A_1, D_n \), and \( X_{8-\ell_2}, X_{8-\ell_3} \) can contain the pieces of \( A_1, D_n, E_7, \) or \( E_8 \). The chiral half-shifts \( s_L[*], s_R[*] \) are associated to the \( A_1 \)-pieces \( \left( \Lambda_{A_1}^{(0)} \right)^* \). We again require that the number of total \( D_n \) components with \( n \in 2\mathbb{Z} + 1 \) in the left- and right-movers should be even in order to satisfy the ‘unitarity condition’ (3).

The building blocks for the even sectors are again equal to the partition function of the unorbifolded model, and those of the odd sectors are evaluated as

$$Z_{(a,b)}(\tau) = \frac{1}{\tau_2 |\eta(\tau)|^4} Z^{T_d}(\tau) \cdot \overline{h_{(a,b)}^{1/2k}} \cdot \left( \hat{\Lambda}_{A_1}^{(a,b)} \right)^{2k} \cdot \left( \hat{X}_{(a,b)}^{(a,b)} \right)^{\ell^1_{X_{(a,b)}}} \cdot \left( \hat{X}_{(a,b)}^{(a,b)} \right)^{\ell_1_{X_{(a,b)}}} \cdot \left( \hat{X}_{(a,b)}^{(a,b)} \right)^{\ell_1_{X_{(a,b)}}}$$

$$\times \prod_{i=2,3} \left( \hat{\Lambda}_{A_1}^{(a,b)} \right)^{\ell_{X_{8-\ell_i}}} \cdot \left( \hat{X}_{(a,b)}^{(a,b)} \right)^{\ell_{X_{8-\ell_i}}} \cdot \left( \hat{X}_{(a,b)}^{(a,b)} \right)^{\ell_{X_{8-\ell_i}}} \cdot \left( \hat{X}_{(a,b)}^{(a,b)} \right)^{\ell_{X_{8-\ell_i}}} \cdot \left( \hat{X}_{(a,b)}^{(a,b)} \right)^{\ell_{X_{8-\ell_i}}} \cdot \left( \hat{X}_{(a,b)}^{(a,b)} \right)^{\ell_{X_{8-\ell_i}}} \cdot \left( \hat{X}_{(a,b)}^{(a,b)} \right)^{\ell_{X_{8-\ell_i}}},$$

(3.17)
\[
\begin{align*}
&= \frac{1}{\tau_2|\eta(\tau)|^4} \left[ Z^{T^d}(\tau) \cdot h_{(a,b)} \right] \\
&\times \epsilon_{(a,b)}^{[r-\sum \ell_i]} \left( \tilde{\chi}_{A_1}^{A_1(a,b)} \right)^{k+\sum \ell_i} \left( \chi_{D_4}^{D_4(a,b)} \chi_{A_1}^{A_1(a,b)} \right) \prod_{i=2,3} \chi_{X_8-\ell_i}^{X_8-\ell_i}.
\end{align*}
\] (3.18)

The first line of (3.18) shows its modular covariance manifestly\(^{10}\). One can likewise show that the total partition function of this model generally satisfies (1) \(\sim\) (3), and the requirements (4), (5) are achieved if the condition (3.15) is satisfied. As in the type II models, we can exhibit the two simple cases satisfying (3.15), though the general classification would be hard to describe;

(i) Only one of \(\{r, \ell_1, \ell_2, \ell_3\}\) is non-zero (mod 8), and the Lie algebra lattice corresponding to the non-vanishing integer (e.g. \(X_{8-\ell_2}\) when \(\ell_2 \not\equiv 0\) (mod 8)) is simple (i.e. made up only of a single piece of \(A_1, D_r\) or \(E_7\)).

(ii) \(r - \sum_{i=1}^3 \ell_i \in 2\mathbb{Z} + 1\).

### 3.4 Simple Examples

Here we present simple examples to demonstrate the general features given in the previous subsections.

1. **Example from type II string on** \(T^4[D_4] \times S^1 \times S^1_{N_\varepsilon}\):

The first example we consider is a particle model from the type II string on \(\mathbb{R}^{3,1} \times T^4[D_4] \times S^1 \times S^1_{N_\varepsilon}\), where \(T^4[D_4]\) is the 4-dim. torus for the Englert-Neveu lattice of \(D_4\), in other words, at the \(SO(8)\)-symmetry enhancement point. \(S^1\) is a circle with an arbitrary radius, which is not important below. We simply choose \(k = \ell = 0, r = 4\), that is, the orbifold action is defined as

\[
g = (-1)^{F_L+F_R} \otimes s_R[4],
\] (3.19)

where \(s_R[4]\) acts on \(T^4[D_4]\).

The building blocks (3.11) for the odd sectors with \(a \in 2\mathbb{Z} + 1\) or \(b \in 2\mathbb{Z} + 1\) are written in this case as

\[
Z_{(a,b)}(\tau) = \frac{1}{\tau_2|\eta(\tau)|^4} \left[ Z^{S^1}(\tau) \cdot |h_{(a,b)}|^2 \cdot \epsilon_{(a,b)}^{[a]} \left( \tilde{\chi}_{A_1}^{A_1(a,b)} \right)^4 \chi_{D_4}^{D_4(a,b)} \right].
\] (3.20)

\(^{10}\)Note that \((-1)^{ab} \left( \tilde{\chi}_{A_1} \right)^8\) behaves modular covariantly up to the phase factor arising from the \(T\)-transformation of \(\eta(\tau)^{-8}\).
Especially, for the sectors with \( a \in 2\mathbb{Z} + 1, b \in 2\mathbb{Z} \), we can explicitly write it down as

\[
Z_{(a,b)}(\tau) = \frac{1}{\tau_2 |\eta(\tau)|^4} Z^{S^1}(\tau) \cdot \left| \frac{\theta_3}{\eta} \right|^4 + \left| \frac{\theta_4}{\eta} \right|^4 - \left| \frac{\theta_2}{\eta} \right|^4 \right|^2 \times (-1)^{\frac{b}{2}} \left( \frac{\theta_3 \theta_2}{\eta^2} \right)^2 \frac{1}{2} \left[ \left( \frac{\theta_3}{\eta} \right) + \left( \frac{\theta_2}{\eta} \right) \right].
\]

(3.21)

Due to the existence of the phase factor \((-1)^{\frac{b}{2}}\) the criterion (3.15) is obviously satisfied;

\[
\sum_{b: \text{even}} Z_{(a,b)}(\tau) = 0.
\]

On the other hand, the partition function of the untwisted sector \((a = 0)\) is evaluated as

\[
Z_0(\tau) = \frac{1}{2\tau_2 |\eta(\tau)|^4} Z^{S^1}(\tau) \cdot \left| \frac{\theta_3}{\eta} \right|^4 - \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right|^2 \frac{1}{2} \left[ \left| \frac{\theta_3}{\eta} \right|^8 + \left| \frac{\theta_4}{\eta} \right|^8 + \left| \frac{\theta_2}{\eta} \right|^8 \right] + \frac{1}{2\tau_2 |\eta(\tau)|^4} Z^{S^1}(\tau) \cdot \left| \frac{\theta_3}{\eta} \right|^4 - \left( \frac{\theta_4}{\eta} \right)^4 + \left( \frac{\theta_2}{\eta} \right)^4 \right|^2 \frac{1}{2} \left[ \left( \frac{\theta_3}{\eta} \right)^4 + \left( \frac{\theta_2}{\eta} \right)^4 \right],
\]

(3.22)

which does not identically vanish. In fact, this partition function describes 128 massless space-time bosons in the standard fashion, while all the space-time fermions get massive. Especially, all of the space-time fermions belonging to the massless super-multiplets in the unorbifolded theory are removed by the \(g\)-projection, which means that they acquire the masses of order \( M_\epsilon \sim \frac{1}{\epsilon \sqrt{\alpha'}} \) in the theory \( \mathcal{M}[\epsilon] \). (See the comment (ii) in section 2.) Thus, the bose-fermi cancellation is ‘maximally’ broken in \( \mathcal{M}[\epsilon] \). Nevertheless, the cosmological constant \( \Lambda_{\mathcal{M}[\epsilon]} \) vanishes under the \( \epsilon \to +0 \) limit, as we have proven in section 2.

2. **Counter example from type II string on** \( T^4[D_2 \oplus D_2] \times S^1 \times S^1_N \epsilon \)

As a digression, here we give an example which does not satisfy our criterion. We adopt almost the same background as the first one, but with the \( T^4[D_4] \) replaced with

\[
T^4[D_2 \oplus D_2] \left( \cong T^4[A_1 \oplus A_1 \oplus A_1 \oplus A_1] \right),
\]

(3.23)

and the orbifold action is again given by (3.19).

The total building blocks (3.11) now become

\[
Z_{(a,b)}(\tau) = \frac{1}{\tau_2 |\eta(\tau)|^4} Z^{S^1}(\tau) \cdot |h_{(a,b)}|^2 \cdot \epsilon_{(a,b)}^{[4]} \left( \chi_{(a,b)}^A \right)^4 \left( \chi_{(a,b)}^{D_2} \right)^2,
\]

(3.24)
and we obtain for the sectors with \( a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z} \),

\[
Z_{(a,b)}(\tau) = \frac{1}{\tau_2 |\eta(\tau)|^4} Z^{S^1}(\tau) \cdot \left( \frac{\theta_3}{\eta} \right)^4 + \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right] \\
\times (-1)^{\frac{a}{2}} \left( \frac{\theta_3 \theta_2}{\eta^2} \right)^2 \frac{1}{4} \left[ \left( \frac{\theta_3}{\eta} \right)^4 + \left( \frac{\theta_2}{\eta} \right)^4 + 2(-1)^{\frac{a}{2}} \left( \frac{\theta_3 \theta_2}{\eta^2} \right)^2 \right],
\]

(3.25)
in place of (3.21). We thus find that

\[
\sum_{b:\text{even}} Z_{(a,b)}(\tau) \propto \frac{1}{\tau_2 |\eta(\tau)|^4} Z^{S^1}(\tau) \cdot \left( \frac{\theta_3}{\eta} \right)^4 + \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right] \cdot \left( \frac{\theta_3 \theta_2}{\eta^2} \right)^4 \neq 0,
\]
because of the cancellation of the phase factors in the coefficient of the term \( \left( \frac{\theta_3 \theta_2}{\eta^2} \right)^4 \) appearing in (3.25). In this way we conclude that the criterion (3.15) is not satisfied in this case.

3. Example from heterotic string on \( T^4[D_4] \times S^1 \times S_{N^0}^1 \):

The third example is a model from the heterotic string on \( \mathbb{R}^{3,1} \times T^4[D_4] \times S^1 \times S_{N^0}^1 \), and

\[
g = (-1)^{F_D} \otimes s_{R[4]},
\]
where \( s_{R[4]} \) acts on \( T^4[D_4] \). The total building blocks (3.18) are written as

\[
Z_{(a,b)}(\tau) = \frac{1}{\tau_2 |\eta(\tau)|^4} Z^{S^1}(\tau) \cdot \eta_{(a,b)} \cdot \epsilon_{(a,b)}^{[4]} \left( \frac{\chi_{(a,b)}}{\lambda_0} \right)^2 \cdot \left( \frac{\chi_{(a,b)}}{\lambda_0} \right)^2.
\]

(3.27)

Here, \( \chi_0^{E_8} \) denotes the character of the basic representation of affine \( E_8 \) with level 1 given in (A.22). We find that the sectors with \( a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z} \) satisfy the criterion (3.15) due to the phase factor \( \epsilon_{(a,b)}^{[4]} \). We thus obtain the vanishing cosmological constant \( \lim_{\epsilon \to +0} \lambda_{M[\epsilon]} = 0 \), as in the first example.

Moreover, we obtain for the untwisted sector

\[
Z_0(\tau) = \frac{1}{2 \tau_2 |\eta(\tau)|^4} Z^{S^1}(\tau) \cdot \left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_4}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4
\]
\times \frac{1}{2} \left[ \left| \frac{\theta_3}{\eta} \right|^8 + \left| \frac{\theta_4}{\eta} \right|^8 + \left| \frac{\theta_2}{\eta} \right|^8 \right] \cdot \left( \frac{\chi_0^{E_8}}{\lambda_0} \right)^2
\]
\[+ \frac{1}{2 \tau_2 |\eta(\tau)|^4} Z^{S^1}(\tau) \cdot \left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_4}{\eta} \right)^4 + \left( \frac{\theta_2}{\eta} \right)^4
\]
\times \left( \frac{\theta_3 \theta_2}{\eta^2} \right)^2 \left[ \left( \frac{\theta_3}{\eta} \right)^4 + \left( \frac{\theta_4}{\eta} \right)^4 \right] \cdot \left( \frac{\chi_0^{E_8}}{\lambda_0} \right)^2.
\]

(3.28)
This does not vanish, meaning that we do not have the bose-fermi cancellation in the theory $\mathcal{M}[e]$ again.

As is familiar, there appear various non-abelian gauge symmetries in the heterotic string models, in contrast with the type II cases, which originate from the left-moving vertex operators with conformal weight $h = 1$. In the present case the relevant orbifold action (3.26) does not affect such bosonic massless spectrum lying in $Z_0(\tau)$ and also the $S_{N_\epsilon}$-sector omitted here. We then find the non-abelian gauge symmetry $U(1) \times U(1) \times SO(8) \times E_8 \times E_8$ (at generic points of the moduli space of the $S^1$-sector), while we always have an abelian gauge group $U(1)^6$ that originates from the right-mover. On the other hand, no massless space-time fermions appear in the manner similar to the first example.

4. Example from heterotic string on $T^4[D_4] \times S^1 \times S^1_{N_\epsilon}$ :

The fourth example is defined for the same background of example 3, $\mathbb{R}^{3,1} \times T^4[D_4] \times S^1 \times S^1_{N_\epsilon}$, but we here take

$$g = (-1)^{F_R} \otimes s_L[4],$$

instead of (3.26) as the orbifold action. Then, the total building blocks (3.18) become

$$Z_{(a,b)}(\tau) = \frac{1}{\tau_2 |\eta(\tau)|^4} Z^{S^1(\tau)} \cdot \overline{h_{(a,b)}} \cdot \epsilon^{[-4]} \left( \tilde{x}^{A_1}_{(a,b)} \right)^4 \overline{\chi^{D_4}_{(a,b)}} \cdot [\chi_{E_8}^0]^2,$$

which again satisfy the criterion (3.15).

The untwisted partition function becomes

$$Z_0(\tau) = \frac{1}{2 \tau_2 |\eta(\tau)|^4} Z^{S^1(\tau)} \cdot \left[ \frac{\theta_3}{\eta} \right]^4 \left[ \frac{\theta_4}{\eta} \right]^4 \left[ \frac{\theta_2}{\eta} \right]^4 \cdot \left[ \frac{\chi_{E_8}^0}{\eta} \right]^2$$

$$+ \frac{1}{2 \tau_2 |\eta(\tau)|^4} Z^{S^1(\tau)} \cdot \left[ \frac{\theta_3}{\eta} \right]^4 \left[ \frac{\theta_4}{\eta} \right]^4 \left[ \frac{\theta_2}{\eta} \right]^4 \cdot \left[ \frac{\chi_{E_8}^0}{\eta} \right]^2,$$

which again does not vanish.

This time, the unbroken gauge symmetry originating from the left-mover is given as $U(1) \times U(1) \times SO(4) \times SO(4) \times E_8 \times E_8$, that is, the $SO(8)$ gauge symmetry is broken to $SO(4) \times SO(4)$ by the relevant orbifolding (3.29). On the other hand, as opposed to the third example, we have
massless space-time fermions which belong to the vector representation for both factors of the
$SO(4)$ gauge groups\textsuperscript{11}.

4 Discussions and Comments

In this paper we have demonstrated how one can systematically construct the point particle
theories that realize the vanishing one-loop cosmological constant without the bose-fermi can-
cellation, namely, the theories with the property (1.3). The main idea to reach the desired
theories is to utilize the building blocks of certain asymmetric orbifolds of supersymmetric
string vacua, in which all the space-time supercharges are removed at least in the untwisted
sectors. We have interpreted the partition function for these untwisted sectors, which is not
modular invariant, as that of the particle theories satisfying (1.3). We have clarified the several
conditions to be satisfied by the relevant orbifold models, and assumed the consistency of the
superstring theories on these orbifolds. Especially, we required the world-sheet superconformal
symmetry in the RNS formalism.

However, if one focuses only on the spectrum, it is possible to relax the consistency conditions
of the original superstring theories, or even not to start from superstring theories. In such cases,
the possibility of the construction is largely enhanced, since appropriate modular forms can be
used as building blocks, regardlessly of the consistency at intermediate stages.

An important issue here is the consistency after interactions are turned on. If the resultant
particle theories are still consistent, we may adopt them, as our purpose is to construct the non-
SUSY particle theories rather than the non-SUSY string vacua. We have, however, considered
the particle theories which descend from consistent superstring theories in this paper, since the
consistency of the latter may be inherited by the former.

At one loop, for example, the multi-particle scattering amplitudes must be compatible with
the requirement of unitarity. It would be possible to reduce this issue to that in superstring
theory. Indeed, the superstring multi-particle amplitudes are given by the integral of the con-
tributions from each spin structure over the torus modulus (see e.g. [37,38]). Each contribution
is invariant under the modular subgroup $\Gamma(2)$, which preserves the spin structures. Further-
more, it is decomposed into the factors coming from the partition functions and from the vertex
operators, and those factors are separately modular covariant. One may thus expect that the

\textsuperscript{11} The simplest way to observe these aspects is as follows; The $T^4[D_4]$ sector is fermionized in the standard
manner, that is, described by the 8 pairs of chiral fermions $(\lambda^i_L, \lambda^i_R) \ (i = 1, \ldots, 8)$, and $s_L[4]$ is just regarded as
the sign-flip of the 4 left-moving fermions, say, $\lambda^i_L, i = 1, \ldots, 4$. Then, one can easily understand the massless
spectrum mentioned here.
consistency of the relevant amplitudes in the particle theories is deduced from that of superstring by an argument similar to the one given in section 2, with the help of the modular property of the amplitudes. It would be also possible to confirm the UV-finiteness of the multi-particle amplitudes in our particle theories, even though the integration region of the modulus (or Schwinger parameter) is $S$, rather than $F$, as for the cosmological constant.

Once establishing the consistency of multi-particle amplitudes at one-loop, one would be able to evaluate the higher loop corrections to cosmological constant by the ‘cutting and sewing procedure’ of Riemann surfaces. We expect that the modular arguments presented in section 2 still work (especially, the natural extensions of (2.20) to higher genera). If this is the case, the higher loop cosmological constant would also be shown to behave as

$$
\Lambda_{\text{higher loop}} \sim e^\lambda \times \text{[power of coupling constant]} \times M_D^{\epsilon},
$$

similarly to the one-loop case, where $\lambda$ is a positive number depending on the loop number. We would like to discuss these issues in more detail elsewhere.

Let us further add several comments:

(i) We have demonstrated the two types of constructions based on type II and heterotic strings by utilizing the building blocks studied in [26]. They might look quite similar. However, we have a crucial difference between them.

In the type II cases, the spectra read off from $Z_0(\tau)$ (and thus, the theory $\mathcal{M}[\epsilon]$) do not include massless space-time fermions, while the heterotic cases can do. This aspect originates from the fact that the chiral half-shift only acts on the bosonic coordinates. In fact, the possible massless fermions for the type II cases should be of the form

$$
|\text{R-vacuum}\rangle_L \otimes |\psi_i^j \rangle_R \text{ or } |\psi_i^j \rangle_L \otimes |\text{R-vacuum}\rangle_L.
$$

However, both of them are projected out by the orbifold action (3.9), since the chiral half-shifts $s_L[\ast], s_R[\ast]$ assign +1 to any NS massless states in addition to $(-1)^{F_L+F_R} = -1$.

On the other hand, in the heterotic models, one can obtain the massless fermionic states of the form

$$
|p_L\rangle_L \otimes |\text{R-vacuum}\rangle_R,
$$

where $|p_L\rangle_L$ is any massless Fock vacuum (i.e. $h = 1$) with $s_L[\ast] = -1$. (See example 4 presented in subsection 3.4.)
(ii) As mentioned above, the models we constructed satisfy the unitarity condition \((3)\), that is, the unitarity in the untwisted sector with \(a = 0\) in \(Z_{(a,b)}\), which is easily confirmed. We need not impose the same condition in the twisted sectors with \(a \neq 0\) at least at the level of the spectrum, though the condition for \(a \neq 0\) is also satisfied in our examples.

(iii) We here only treated the twist for the world-sheet fermions by \((-1)^{F_L + F_R}\) in the type II cases and by \((-1)^{F_R}\) in the heterotic cases in order to break the space-time SUSY, in other words, to achieve the condition \((2)\), \(Z_0(\tau) \neq 0\). As another possibility to break SUSY, one can also make use of the chiral reflections

\[
(X_L, X_R) \rightarrow (-X_L, X_R), \quad (\psi_L, \psi_R) \rightarrow (-\psi_L, \psi_R),
\]

or those for the right-mover. The fermionic building blocks for the chiral reflection acting on the \(2p\) \((p = 1, ..., 4)\) components \((-1)^{1L} \otimes 2p\) are explicitly given in appendix A and denoted as \(g_{(a,b)}^{[p]}(\tau)\). They do not identically vanish for \(p \neq 2\), which means the absence of the bose-fermi cancellation.

One can construct more elaborated models satisfying the requirements \((1) \sim (5)\) by combining the chiral half-shifts and the chiral reflections. The crucial point is that the total building blocks \(Z_{(a,b)}(\tau)\) for the odd sectors should include the non-trivial phase factors such as \(e^{\frac{2\pi i}{K} K_{ab}}\), especially to satisfy the condition \((5)\). To this aim the inclusions of the chiral half-shifts are useful, since the phases that originate from the chiral reflections tend to be canceled out after combining the bosonic and fermionic building blocks. (Compare \((3.5)\) and \((A.25)\).) We also note that the massless fermions can appear in these models even in the type II cases. This is in a sharp contrast with those constructed only with the chiral half-shifts and \((-1)^{F_L + F_R}\) mentioned in the comment (i).

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Appendix A: Summary of Notations

In appendix A, we summarize the notations used in this paper, and the building blocks to compose the modular invariants for the asymmetric orbifolds given in section 3. We set \( q := e^{2\pi i \tau}, \ y := e^{2\pi iz} \ (\forall \ \tau \in \mathbb{H}^+, \ \forall \ z \in \mathbb{C}). \)

1. Theta Functions

\[
\begin{align*}
\theta_1(\tau, z) &:= i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} = 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - y q^m)(1 - y^{-1} q^m), \\
\theta_2(\tau, z) &:= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} = 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + y q^m)(1 + y^{-1} q^m), \\
\theta_3(\tau, z) &:= \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n = \prod_{m=1}^{\infty} (1 - q^m)(1 + y q^{m-1/2})(1 + y^{-1} q^{m-1/2}), \\
\theta_4(\tau, z) &:= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n = \prod_{m=1}^{\infty} (1 - q^m)(1 - y q^{m-1/2})(1 - y^{-1} q^{m-1/2}). \\
\Theta_{m,k}(\tau, z) &:= \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2\tau})^2} y^{k(n+\frac{m}{2\tau})}, \\
\widetilde{\Theta}_{m,k}(\tau, z) &:= \sum_{n=-\infty}^{\infty} (-1)^n q^{k(n+\frac{m}{2\tau})^2} y^{k(n+\frac{m}{2\tau})}, \\
\eta(\tau) &:= q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\end{align*}
\]

We often use abbreviations, \( \theta_i \equiv \theta_i(\tau) \equiv \theta_i(\tau, 0) \) (\( \theta_1 \equiv \theta_1(\tau) \equiv 0 \)), \( \Theta_{m,k} \equiv \Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau, 0) \), and \( \widetilde{\Theta}_{m,k} \equiv \widetilde{\Theta}_{m,k}(\tau) \equiv \widetilde{\Theta}_{m,k}(\tau, 0) \).

2. Bosonic Building Blocks

We next exhibit the bosonic building blocks for the ‘odd sectors’ with \( a \in 2\mathbb{Z} + 1 \) or \( b \in 2\mathbb{Z} + 1 \) in the relevant asymmetric orbifolds repeatedly appearing in section 3.

• \( A_1 \)-type :

We first describe the building blocks twisted by the ‘chiral reflection’ \( e^{i\pi J^0} \) in the current
algebra $\hat{A}_1$;

$$\tilde{\chi}_{(a,b)}^{A_1}(\tau) := \begin{cases} 
\sqrt{\frac{\theta_3 \theta_4}{\eta^2}} & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z} + 1), \\
\sqrt{\frac{\theta_3 \theta_2}{\eta^2}} & (a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z}), \\
\sqrt{\frac{\theta_4 \theta_2}{\eta^2}} & (a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z} + 1). 
\end{cases}$$  (A.8)

We note that the same functions can be rewritten in the expressions which look more natural for the 'chiral half-shift' $e^{i\pi J^3}$:

$$\tilde{\chi}_{(a,b)}^{A_1}(\tau) \equiv \begin{cases} 
\tilde{\Theta}_{0,1} & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z} + 1), \\
\sqrt{2} \tilde{\Theta}_{0,1} & (a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z}), \\
\sqrt{2} \tilde{\Theta}_{0,1} & (a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z} + 1). 
\end{cases}$$  (A.9)

We also introduce the functions

$$\chi_{(a,b)}^{A_1}(\tau) := \begin{cases} 
\frac{1}{2} \left\{ \chi_+^{A_1} + e^{-2\pi i a b} \chi_-^{A_1} \right\} & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z} + 1), \\
\frac{1}{\sqrt{2}} \left\{ \chi_0^{A_1} + e^{4\pi i a b} \chi_1^{A_1} \right\} & (a \in 2\mathbb{Z} + 1). 
\end{cases}$$  (A.10)

where

$$\chi^{A_1}_\ell(\tau) := \frac{\Theta_{\ell,1}}{\eta},$$  (A.11)

denotes the spin $\ell/2$ character of $\hat{A}_1$ with level 1 ($\ell = 0, 1$), and we set

$$\chi^{A_1}_\pm(\tau) := \chi_0^{A_1} \pm \chi_1^{A_1}. $$  (A.12)

They yield the modular covariant blocks (3.3) for the $T^1[A_1]$-case with the phase factors $\epsilon_{(a,b)}^{[1]}$ defined in (3.5), that is,

$$Z_{(a,b)}^{T^1[A_1]}(\tau) \equiv \epsilon_{(a,b)}^{[1]} \tilde{\chi}_{(a,b)}^{A_1}(\tau) \chi_{(a,b)}^{A_1}(\tau) \quad (a \in 2\mathbb{Z} + 1, \text{ or } b \in 2\mathbb{Z} + 1), $$  (A.13)

satisfies

$$Z_{(a,b)}^{T^1[A_1]}(\tau) \big|_T = Z_{(a,a+b)}^{T^1[A_1]}(\tau), \quad Z_{(a,b)}^{T^1[A_1]}(\tau) \big|_S = Z_{(b,-a)}^{T^1[A_1]}(\tau).$$
• $E_7$-type:

The relevant building blocks for the $E_7$-type are written as

$$
\chi_{E_7}^{(a,b)}(\tau) := \begin{cases}
\frac{1}{2} \left\{ \chi_+^{E_7} + e^{\frac{i\pi}{2}ab}\chi_-^{E_7} \right\} & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z} + 1), \\
\frac{1}{\sqrt{2}} \left\{ \chi_0^{E_7} + e^{-\frac{i\pi}{2}ab}\chi_1^{E_7} \right\} & (a \in 2\mathbb{Z} + 1),
\end{cases}
$$

(A.14)

where

$$
\chi_0^{E_7}(\tau) := \frac{1}{2\eta^r} \left[ \Theta_{0,1} \left( \theta_3^6 + \theta_4^6 \right) + \Theta_{1,1} \theta_2^6 \right],
$$

$$
\chi_1^{E_7}(\tau) := \frac{1}{2\eta^r} \left[ \Theta_{1,1} \left( \theta_3^6 - \theta_4^6 \right) + \Theta_{0,1} \theta_2^6 \right],
$$

(A.15)

denotes the characters of the basic and the fundamental representation of $\hat{E}_7$ with level 1, which contain the states with dimension $h = 0$ and $h = \frac{3}{4}$ respectively, and we set

$$
\chi_\pm^{E_7}(\tau) := \chi_0^{E_7} \pm \chi_1^{E_7}.
$$

(A.16)

(A.14) yields the modular covariant blocks (3.3) for the $T^r[E_7]$-case together with $(\tilde{\chi}^{A_1})^7$ and $e^{[7]}_{(a,b)}$.

• $D_r$-type:

The relevant building blocks for the $D_r$-type are written as

$$
\chi_{D_r}^{(a,b)}(\tau) := \begin{cases}
\frac{1}{2\eta^r} \left\{ \theta_3^r + e^{-\frac{i\pi}{2}ab}\theta_4^r \right\} & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z} + 1), \\
\frac{1}{2\eta^r} \left\{ \theta_3^r + e^{\frac{i\pi}{2}ab}\theta_2^r \right\} & (a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z}), \\
\frac{1}{2\eta^r} \left\{ \theta_4^r + e^{\frac{i\pi}{2}ab}\theta_2^r \right\} & (a, b \in 2\mathbb{Z} + 1).
\end{cases}
$$

(A.17)

We note that the modular covariant blocks for the $T^r[D_r]$-case are likewise composed from the functions (A.17) as well as $(\tilde{\chi}^{A_1})^r$, but with the phase factors slightly different from $e^{[r]}_{(a,b)}$:

$$
Z_{\chi_{D_r}}^{(a,b)}(\tau) := \tilde{e}^{[r]}_{(a,b)} \left( \tilde{\chi}^{A_1}_{(a,b)}(\tau) \right)^r \chi_{D_r}^{(a,b)}(\tau),
$$

(A.18)

where we define

$$
\tilde{e}^{[r]}_{(a,b)} := \begin{cases}
(k_b)^r e^{[r]}_{(a,b)} & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z} + 1), \\
(k_a)^r e^{[r]}_{(a,b)} & (a \in 2\mathbb{Z} + 1).
\end{cases}
$$

(A.19)
The distinction between \( r \) and \( s \) affects only in the cases of \( r \in 2\mathbb{Z} + 1 \), and thus one may simply adopt \( r \) in (3.3) as long as the number of \( D_{odd} \)-pieces in the total Lie algebra lattice \( X \) is even. The building blocks for odd \( r \) extend those for even \( r \) in [26].

The building blocks \( Z^{T[X]} \) for the ‘even sectors’ with \( a, b \in 2\mathbb{Z} \) are given by the diagonal modular invariants of the characters of the associated affine Lie algebras. Therefore,

\[
Z^{T[A_1]}(\tau) = |\chi^{A_1}_0|^2 + |\chi^{A_1}_1|^2, \quad Z^{T[E_7]}(\tau) = |\chi^{E_7}_0|^2 + |\chi^{E_7}_1|^2,
\]

where \( \chi^{A_1}_e \) and \( \chi^{E_7}_c \) are given in (A.11), (A.15), and for the \( D_r \)-cases,

\[
\begin{align*}
\chi^{D_r}_0 &= \frac{1}{2\eta^r} (\theta_3^r + \theta_4^r) \quad \text{(basic rep.),} \\
\chi^{D_r}_v &= \frac{1}{2\eta^r} (\theta_3^r - \theta_4^r) \quad \text{(vector rep.),} \\
\chi^{D_r}_s &= \chi^{D_r}_c = \frac{1}{2\eta^r} \theta_2^r \quad \text{(spinor or cospinor rep.).}
\end{align*}
\]

We also note the familiar formula of the character of the basic representation of affine \( E_8 \) with level 1,

\[
\chi^{E_8}_0(\tau) = \frac{1}{2} \left[ \left( \frac{\theta_3}{\eta} \right)^8 + \left( \frac{\theta_4}{\eta} \right)^8 + \left( \frac{\theta_2}{\eta} \right)^8 \right].
\]

3. Fermionic Building Blocks

Here we summarize the modular covariant chiral blocks of the world-sheet fermions appearing in section 3. We first describe those twisted by \((-1)^{F_L} \), where \( F_L \) denotes the space-time fermion number operator,

\[
h_{(a,b)}(\tau) = \begin{cases} 
\left( \frac{\theta_3}{\eta} \right)^4 + \left( \frac{\theta_4}{\eta} \right)^4 \quad & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z} + 1), \\
\left( \frac{\theta_3}{\eta} \right)^4 + \left( \frac{\theta_4}{\eta} \right)^4 - \frac{\theta_2}{\eta} \quad & (a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z}), \\
- \left( \frac{\theta_3}{\eta} \right)^4 + \left( \frac{\theta_4}{\eta} \right)^4 + \frac{\theta_2}{\eta} \quad & (a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z} + 1), \\
\left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_4}{\eta} \right)^4 \quad & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z}).
\end{cases}
\]

The block for the ‘even sectors’ with \( a, b \in 2\mathbb{Z} \) is the supersymmetric one that is identically zero. The modular covariance of \( h_{(a,b)}(\tau) \) means that

\[
h_{(a,b)}(\tau)|_T = -e^{-2\pi i \frac{a}{6} r + \frac{a}{6}} h_{(a,a+b)}(\tau), \quad h_{(a,b)}(\tau)|_S = h_{(b,-a)}(\tau).
\]

\[\text{In [26], only the cases of } r \in 2\mathbb{Z} \text{ are treated among the } D_r \text{-type Lie algebra lattices, in which the phases } (\kappa_a)^r \text{ or } (\kappa_b)^r \text{ are absent.}\]
We next consider the chiral reflection acting on the $2p (p = 1, \ldots, 4)$ components of the world-sheet fermions $(-1)^F L^\otimes 2p$. The relevant blocks are written as

\[
g^{[p]}_{(a,b)}(\tau) \equiv \begin{cases} 
 e^{-i\pi ab} \left\{ \left( \frac{\alpha}{\eta} \right)^{4-p} \left( \frac{\beta}{\eta} \right)^p - e^{i\pi ab} \left( \frac{\alpha}{\eta} \right)^{4-p} \left( \frac{\beta}{\eta} \right)^p \right\} & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z} + 1), \\
 e^{i\pi ab} \left\{ \left( \frac{\alpha}{\eta} \right)^{4-p} \left( \frac{\beta}{\eta} \right)^p - e^{-i\pi ab} \left( \frac{\alpha}{\eta} \right)^{4-p} \left( \frac{\beta}{\eta} \right)^p \right\} & (a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z}), \\
 -e^{i\pi ab} \left\{ \left( \frac{\alpha}{\eta} \right)^{4-p} \left( \frac{\beta}{\eta} \right)^p + e^{-i\pi ab} \left( \frac{\alpha}{\eta} \right)^{4-p} \left( \frac{\beta}{\eta} \right)^p \right\} & (a \in 2\mathbb{Z} + 1, \ b \in 2\mathbb{Z} + 1), \\
 \left( \frac{\alpha}{\eta} \right)^4 - \left( \frac{\beta}{\eta} \right)^4 & (a \in 2\mathbb{Z}, \ b \in 2\mathbb{Z}).
\end{cases}
\]

(A.25)

The blocks with $p = 2$ are supersymmetric and identically vanish. They satisfy the relation of the modular covariance as in (A.24), that is,

\[
g^{[p]}_{(a,b)}(\tau) |_{T} = -e^{-2\pi i b} g^{[p]}_{(a,a+b)}(\tau), \quad g^{[p]}_{(a,b)}(\tau) |_{S} = g^{[p]}_{(b,-a)}(\tau).
\]

(A.26)

We note that the non-trivial phase factors are necessary to achieve the correct modular covariance in contrast to the previous ones $h_{(a,b)}(\tau)$.

**Appendix B: Free Compact Boson with Self-Dual Radius**

In appendix B, we summarize the basic fact about the $\hat{A}_1$-symmetry with level 1 realized by the free boson compactified on the circle with the self-dual radius $R = \sqrt{\alpha'}$. Assuming the standard normalization of free boson $X_L(z)X_L(0) \sim -\frac{\alpha'}{2} \ln z$, the chiral part of the $A_1$-current algebra with level 1 is given by the currents,

\[
J^3_L(z) = \frac{1}{\sqrt{\alpha'}} i \partial X_L(z), \quad J^+_{L}(z) \equiv J^1_L(z) \pm i J^2_L(z) = e^{\pm i \frac{\pi}{\sqrt{2\alpha'}} X_L(z)}.
\]

(B.1)

We note that $e^{i\pi J^3_{L,0}}$ acts as the chiral reflection,

\[
e^{i\pi J^3_{L,0}} : X_L \longrightarrow -X_L,
\]

(B.2)

while $e^{i\pi J^3_{L,0}}$ acts as the chiral half-shift:\footnote{Combining the left and right movers, $e^{i\pi J^3_{L,0}} \otimes e^{i\pi J^3_{R,0}}$ acts on the free boson $X \equiv X_L + X_R$ as $X \rightarrow X + \pi \sqrt{\alpha'}$, which is identified as the half-shift.}

\[
e^{i\pi J^3_{L,0}} : X_L \longrightarrow X_L + \frac{\pi}{2} \sqrt{\alpha'}.
\]

(B.3)
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