Quantum State Smoothing

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Smoothing is an estimation method whereby a classical state (probability distribution for classical variables) at a given time is conditioned on all-time (both past and future) observations. Here we define a smoothed quantum state for a partially monitored open quantum system, conditioned on an all-time monitoring-derived record. We calculate the smoothed distribution for a hypothetical unobserved record which, when added to the real record, would complete the monitoring, yielding a pure-state "quantum trajectory". Averaging the pure state over this smoothed distribution yields the (mixed) smoothed quantum state. We study how the choice of actual unravelling affects the purity increase over that of the conventional (filtered) state conditioned only on the past record.

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Estimation theory is used to assign values to parameters of interest, whose true values are unknown, using the available data. These parameters may evolve dynamically, and new data may arrive dynamically, through a continuous measurement process. Estimation in this instance is non-trivial because there can be noise associated with the measurement, noise affecting directly the dynamical system due to its environment, and initial uncertainty in the parameters. Optimal estimation theory can be formulated using the Bayesian approach to statistics, whereby the observer’s knowledge of the parameters is described by a conditional probability distribution. The process of determining this Bayesian state conditioned on measurements at earlier times is called filtering, while if it is conditioned on both earlier and later measurements then the process is called smoothing.

In recent decades, quantum estimation theory has become a topic of great importance [1–12]. Much has been learnt from classical estimation theory, and the analogy between the quantum state and a classical Bayesian state (probability distribution) has been fruitful even in quantum foundations [13–14]. In particular, the stochastically evolving conditioned state of an open quantum system, as introduced by physicists [16–21] and applied in quantum control [1 22–29], is now understood to be analogous to the classical filtered state [25 30–34]. However, the situation is very different regarding smoothing.

The concept of smoothing has certainly been applied in quantum estimation theory. The term “quantum smoothing” was introduced by Tsang [35 36] to mean estimating classical parameters that affect the evolution of a quantum system that is independently monitored. It has been shown to be useful to the problem of estimating a stochastically varying optical phase using the complete photocurrent record, both theoretically [37 38] and experimentally [39 40]. It has also been applied to the problem of estimating an unknown result from a measurement on a quantum system at one time, using records obtained both before and after that time, again both theoretically [41] and experimentally [42]. However, it is not possible to simply define a quantum smoothed state analogous to the classical smoothed state, because system operators do not, in general, commute with the operators for future records.

In this Letter we introduce quantum state smoothing, a method to determine the stochastic trajectory for the smoothed state of an open quantum system under partial observation. This is a major generalization of quantum trajectory theory inspired by classical estimation theory, the first since 2004 [12]. Our idea is related to that of Mølmer and co-workers [41 42], in that it involves estimating an unknown result using records in the past and future of that result. But there are two crucial differences. First, we estimate not an isolated result at a particular time, but a complete unknown record, on the basis of a complete parallel record which is known. Second, the estimation of the unknown record is just a means to the end of calculating the best reconstruction of the trajectory of the conditioned system state, given the full record over all time. We show that our approach subsumes classical state estimation by smoothing, also known as the hidden Markov model technique [43], which is applicable to quantum systems only when they are effectively classical (i.e. always diagonal in a particular basis) [44 45].

We apply our method to a genuinely quantum system (i.e. one which is not diagonal in a fixed basis) — a coherently driven two-level atom, the radiation from which is partly observed. We take the known record to be generated by homodyne detection, and the unknown record to be that corresponding to photon absorption, as this is the most intuitive picture of what happens to photons that are lost into the laboratory surroundings. These lost photons result in impurity in the standard (filtered) conditioned system state, and our smoothing technique can, on average, eliminate up to 26% of this impurity. Interestingly, however, the efficacy of the technique in recovering purity depends on which quadrature is measured in the homodyne detection. Our investigations shed light
on how well we can know the trajectory of a partially observed open quantum systems, and the relation between the quantum and classical versions of state smoothing.

**Types of Estimation.**—— Consider a dynamical system described by parameters $x_t$ (bold font indicates a vector of parameters) which is monitored to yield a continuous noisy output $y_t$. We denote a measurement record $Y_{\Omega} = \{y_t : t \in \Omega\}$, where $\Omega \subseteq [t_0, T]$ is typically some finite time interval. Bayesian estimation involves data processing to infer the conditional classical state

$$\varphi_{Y_\Omega}(x_r) \equiv \Pr[x_r | Y_\Omega; \varphi_0],$$

where $\varphi_0$ describes the a priori statistics of $x$ at the initial time $t_0$. It is also useful to define the unnormalized state

$$\tilde{\varphi}_{Y_\Omega}(x_r) \equiv \varphi_{Y_\Omega}(x_r) \tilde{\varphi}(Y_{\Omega}; \varphi_0) \propto \varphi(Y_{\Omega}, x_r; \varphi_0)$$

Here the $\varphi(Y_{\Omega}; \varphi_0)$ is the actual distribution for $Y_{\Omega}$ while $\varphi_{\text{post}}(Y_{\Omega}; \varphi_0)$ is an ‘ostensible’ distribution for it — it is positive and normalized, but is otherwise arbitrary and does not depend on $x_t$.

There are three types of estimation that are worth distinguishing [46, 47]: filtering, retro-filtering (as we call it), and smoothing (see Fig. 1). If — as in feedback control problems — for the time of interest $\tau$ there is only access to earlier records, $\tilde{Y} \equiv Y_{[t_0, \tau]}$ the optimal protocol is: $\varphi_{\text{F}}(x_r) \equiv \varphi_{\text{F}}(x_r)$. If instead there is access only to results obtained subsequent to $\tau$, $\tilde{Y} \equiv Y_{[\tau, t]}$ then the optimal protocol is retro-filtering: $\varphi_{\text{R}}(x_r) \equiv \varphi_{\text{R}}(x_r)$. As its name implies, this is simply the time-reverse to filtering, but starting with an uninformative final state $\varphi(X_T) \propto 1$. Finally, if the complete historical record can be accessed $\tilde{Y} \equiv Y_{[t_0, t]}$, and the time of interest $\tau$ lies in the interval it covers, then all the information can be utilised, by the technique of smoothing: $\varphi_{\text{S}}(x_r) \equiv \varphi_{\text{S}}(x_r)$. This combines filtering and retrofiltering, using unnormalized states [39]:

$$\varphi_{\text{S}}(x_r) = \frac{\tilde{\varphi}_{\text{R}}(x_r) \tilde{\varphi}_{\text{F}}(x_r)}{\int \tilde{\varphi}_{\text{F}}(x_r') \tilde{\varphi}_{\text{R}}(x_r') \, dx_r'}.$$  

(3)

Here one of the states (most conveniently the retrofiltered one) is defined using an uninformative prior, $\varphi_0 \propto 1$, to prevent double counting of the a priori information.

**Quantum analogues of (retro)filtering.**—— An extension of these results to quantum mechanics has been done partially. Quantum trajectory theory [10] is the analogue of classical state filtering. A quantum trajectory describes the path taken by the state of the quantum system over time, conditioned on the measurement result $y_t$ in each infinitesimal interval $[t, t + dt)$. This process is described by a continuum (indexed by $y_t$) of measurement operations (completely positive maps) $\mathcal{M}_{y_t}$, that evolve the state continuously forward in time:

$$\tilde{\rho}_\text{F}(t + dt) = \mathcal{M}_{y_t} \tilde{\rho}_\text{F}(t).$$

(4)

Starting with $\rho(t_0) = \rho_0$, this procedure generates the state conditioned on the whole past record: $\rho_\text{F}(\tau) = \tilde{\rho}_\text{F}(\tau)$. This is an unnormalized state (as indicated by the tilde), analogous to Eq. (2). That is, the normalized version $\rho_\text{F}(\tau)$ generates the correct filtered probability distribution $\varphi_\text{F}(x_r)$ for any system observable $X_r$, while

$$\Tr(\tilde{\rho}_\text{F}(\tau) \varphi_{\text{post}}(\tilde{Y} | \rho_0) = \varphi(\tilde{Y} | \rho_0).$$

(5)

The filtered state, on average, reproduces the Lindblad evolution [48]:

$$E_{\text{F}}[\rho_{\text{F}}(\tau)] = \exp[\mathcal{L}(\tau - t_0)]\rho_0.$$  

In general there are $K$ irreversible channels, each described by a Lindblad operator $\hat{a}_k$, with $\mathcal{L} = -i[\hat{H}, \cdot] + \sum_{k=1}^{K} D[\hat{a}_k] \cdot \cdot \cdot$, where $D[\hat{a}] = \hat{a} \cdot \hat{a}^\dagger - \frac{1}{2} (\hat{a}^\dagger \hat{a}^1 \hat{a} \cdot \cdot + \cdot \hat{a}^\dagger \hat{a} \hat{a}^1)$ and $\hat{H} = \hat{H}^1$. As an example of filtering, consider a single Lindblad operator and unit-efficiency homodyne detection with local oscillator phase $\Phi$. If we choose $\varphi_{\text{post}}(y_t)$ to be a zero-mean Gaussian of variance $1/dt$, the measurement operation is defined by $\mathcal{M}_{y_t} = M_{y_t} \cdot M_{y_t}^\dagger$, where

$$M_{y_t} = 1 - i \hat{H} dt - \frac{1}{2} \hat{a}^\dagger \hat{a} dt + e^{-i\Phi} \hat{a} y_t dt.$$  

(6)

A similar definition, $\mathcal{M}_{n_t} = M_{n_t} \cdot M_{n_t}^\dagger$, holds for the discontinuous filter equations for photon-counting. Here there are only two possible results in the interval $[t, t + dt)$: one photon or none. Choosing ostensible probabilities $\varphi_{\text{post}}(n_t := 1) = \lambda dt$ and $\varphi_{\text{post}}(n_t := 0) = 1 - \lambda dt$, gives

$$\tilde{M}_0 = \hat{a} / \sqrt{\lambda},$$  

$$\tilde{M}_1 = 1 - \left( i \hat{H} + \frac{1}{2} (\hat{a}^\dagger \hat{a} + \lambda) \right) dt.$$  

(7)

The corresponding analogue for classical state retrofiltering has been set out in [39]: it is the solution of the adjoint of equation (4),

$$\tilde{E}_{\text{R}}(t) = M_{y_t}^\dagger \tilde{E}_{\text{R}}(t + dt).$$  

(8)

In this case the effect operator evolves backwards from the final uninformative effect $\hat{E}(T) = I$ towards $\hat{E}(\tau)_{\text{R}} = \hat{E}_{\text{F}}(\tau)$, conditioned now on measurement results in the
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with the stochastic dynamics of the system.

A naive approach to construct a quantum smoothed state, given the quantum analogues of filtering $\rho_F$ and retrofiltering $E_R$, could be to combine them directly as in equation Eq. (3) so that $\rho_S(t) \propto \rho_F(t)E_R(t)$. However, as pointed out in Ref. [35], the result is not in general Hermitian or (even if symmetrized) positive semi-definite. Therefore this cannot correspond to a physical state. The deep reason for the failure of this analogy is that quantum observables at time $t$ do not commute with operator for the future record $\hat{Y}$, so there does not exist a joint distribution for $x$ and $\hat{Y}$, from which the smoothed conditional distribution $\varphi_{\rho_S}(x;\tau)$ can be found. This is the reason that Tsang gave quantum smoothing the restricted meaning of estimating an external classical parameter $x$.

Here, we explain how to generate a smoothed quantum state, in a particular situation, that avoids these difficulties. We consider an open quantum system with two groups of output channels ($b$, $c$) (Fig. 2), only the first of which can be monitored and gives us access to a complete historical record ($b \rightarrow \hat{Y}$). We then consider a hypothetical (but physically motivated) unravelling of the channel $c$, yielding an unknown record $\hat{N}$ (we take this later to be photon-counting). Although this is of quantum origin (unlike the external classical parameters considered previously by Tsang [35]), it is still classical in the sense of commuting with any future record. Thus we can apply Tsang’s quantum smoothing technique to this record to obtain $\varphi_{\rho_N}(\hat{N}) = \varphi_{\hat{N}}(\hat{N}) = P(N_{\text{true}} = \hat{N}|\hat{Y}, \rho(to))$, and thereby the smoothed quantum state,

$$\rho_S(\tau) = E_{\hat{N} | \hat{Y}}(\rho_{\rho_N}(\hat{N}, \tau)) \equiv \sum_{\hat{N}} \varphi_{\rho_N}(\hat{N})\rho_{\rho_N}(\hat{N}, \tau),$$  

(10)

Because it uses information from both the past and future $Y$ to determine a state at time $\tau$, we expect this to be a better (more pure) representation of the state than the filtered quantum state $\rho_F(t) = \rho_{\hat{Y}} = E_{\hat{N}}[\rho_{\rho_N}(\hat{N}, t)]$.

We now show how to calculate Eq. (10) for the particular case of a single channel ($b$) yielding homodyne photocurrent $y_t$ and a single channel ($c$) yielding an unobserved photon count $n_t$. First we generate a random example of the complete record $\hat{Y}$ with the correct statistics using standard techniques [1]. With this, we calculate the filtered system state $\rho_{\rho_N}(t)$ from Eq. (4) and the corresponding retrofiltered effect operator $E_\tau(t)$ from Eq. (8) [53]. We use this to define a joint measurement operation $M_{n_t, y_t}$ such that $M_{y_t} = \sum_{n_t} \varphi_{\rho_{\rho_N}}(n_t|\hat{Y})M_{n_t, y_t}$. Using $\varphi_{\rho_{\rho_N}}(n_t|\hat{Y})$ we obtain random samples of $\hat{N}$, and thereby generate a large ensemble of pure states $\hat{\rho}_{\rho_N}(\hat{N})$ conditioned on both the observed and (hypothetical) unobserved records as in [4] but using $M_{n_t, y_t}$ [53].

Now using elementary manipulation of probabilities [19] we find $\varphi_{\rho_N}(\hat{N}) \propto \varphi_{\hat{Y}|\hat{N}, \hat{Y}}\varphi_{\hat{N}|\hat{Y}}$, where the proportionality can be determined by normalization, $\sum_{\hat{N}} \varphi_{\rho_N}(\hat{N}) = 1$. We use the equations for multiple channels corresponding to Eq. (9),

$$\varphi_{\hat{Y}|\hat{N}, \hat{Y}} = \text{Tr}[E_{\rho_N} \rho_{\rho_N}|\hat{Y}] \varphi_{\rho_N}(\hat{Y}),$$  

(11)

and to Eq. (5),

$$\text{Tr}[E_{\rho_N} \rho_{\rho_N}|\hat{N}|\hat{Y}] = \text{Tr}[E_{\rho_N} \rho_{\rho_N}|\hat{N}, \hat{Y}] \varphi_{\rho_N}(\hat{N}),$$  

(12)

to finally obtain from Eq. (10)

$$\rho_S(t) \propto \sum_{\hat{N}} \varphi_{\rho_{\rho_N}}(\hat{N}|\hat{Y}) \times \left\{ \rho_{\rho_N}(\hat{Y}, t)\text{Tr}[E_{\rho_N}(t)\rho_{\rho_N}(\hat{N}, t)] \right\}$$

$$= E_{\hat{N} \sim \varphi_{\rho_{\rho_N}}(\hat{N}|\hat{Y})} \left\{ \rho_{\rho_N}(\hat{Y}, t)\text{Tr}[E_{\rho_N}(t)\rho_{\rho_N}(\hat{N}, t)] \right\},$$  

(13)

where the notation for the expectation value (E) used in the second line is explained by the equality with the first line. But numerically we can approximate this expectation value by a finite ensemble average by generating records $\hat{N}$ according to the appropriate ostensible distribution, as outlined above. This is the method we use below to find the smoothed quantum state.

Example.— Consider a simple driven two level atom, with Hamiltonian $\hat{H} = (\Omega/2)\hat{a}^\dagger \hat{a}$ in the interaction frame, and Lindblad operator $\sqrt{\gamma} \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a}$. We take a fraction $\eta$ of the fluorescence to be observed by homodyne detection, so $\hat{b} = \sqrt{\gamma/\eta} \hat{a} \hat{a}$. The remainder is absorbed by the environment, which we model as an unobserved record of photon counts, as above, with $\hat{c} = \sqrt{\gamma(1-\eta)} \hat{a} \hat{a}$. For a fixed $\hat{Y}$ we can compare $\rho_S$ with $\rho_F$ on the interval $[0, T]$. At the final time $\rho_S(T) = \rho_F(T)$ because there is no more future record $\hat{Y}$ to give extra information to $\rho_S(T)$. Also, we take the initial state to be pure, $\rho_0 = |1\rangle \langle 1|$, which guarantees that $\rho_S(0) = \rho_F(0)$.

FIG. 2: The quantum state smoothing problem: The state of a quantum system is estimated using quantum state smoothing, through the classical estimation of the results of a hypothetical unobserved noisy measurement record correlated with the stochastic dynamics of the system.
To evaluate the advantage gained by smoothing over filtering, we use the purity,

$$P[ρ_C(τ)] = \text{Tr}[ρ_C^2(τ)],$$

(14)

where $ρ_C$ could be either $ρ_F$ or $ρ_S$. If (as we can in simulations) we know the “true” unobserved record $\overline{N}^{\text{true}}$ we can also calculate the fidelity of the conditioned state to the hypothetical pure “True state” $ρ_T(τ) = ρ_{\overline{N}}^{\text{true}}(τ)$,

$$F[ρ_T(τ), ρ_C(τ)] = \text{Tr}[ρ_T(τ)ρ_C(τ)].$$

(15)

It is easy to show that these measures are related by

$$E \{P[ρ_C(τ)]\} = E \{F[ρ_T(τ), ρ_C(τ)]\}$$

(16)

where the ensemble averages here are over the actual distributions for $\overline{N} = \overline{N}^{\text{true}}$ and $\overline{Y} = \overline{Y}^{\text{true}}$.

In Fig. 3(a) we show typical trajectories, for $Y$-homodyne ($Φ = \pi/2$ in Eq. (6)) for a randomly generated true state $ρ_T(τ) = ρ_{\overline{Y}}(τ)$ featuring one jump at $τ ≈ 1.8$. We show it, and the calculated filtered $ρ_F(τ) = ρ_{\overline{Y}}(τ)$ and smoothed $ρ_S(τ) = ρ_{\overline{Y}}(τ)$ states, on a slice of the Bloch sphere in (a), and Eqs. (14)–(15) in (b) and (c) respectively. It is notable from (b) that $ρ_S$ anticipates the jump in $ρ_T$ and its uncertainty about the timing of the jump leads to a lower purity in the region of the jump than the non-anticipating $ρ_F$. Similarly, (c) shows that the fidelity of $ρ_S$ to $ρ_T$ decreases below that of $ρ_F$ prior to the jump, but is higher after the jump. In (d) we see that if there is no jump, the fidelity with $ρ_T$ is always greater for $ρ_S$.

We confirm that smoothing enables better state estimation on average by calculating the average purity, for $10^3$ realisations of $\overline{Y}$. Recall from Eq. (16) that higher purity means higher fidelity with the true state. We plot this in Fig. 4 for two different local oscillator phases: $Φ = \pi/2$ ($Y$-homodyne) in (a) and $Φ = 0$ ($X$-homodyne) in (b). Because the driving of the atom causes $δ_γ$ to oscillate at a frequencies $Ω ≫ γ$, it is harder to track the state of the atom using $Y$-homodyne detection, and the purity of the filtered state is lower than for $X$-homodyne detection [50]. It is the former case which shows the greatest improvement in purity under smoothing: about 26% of the purity lost because of the unobserved radiation is recovered in (a) compared to about 12% in (b).

One can show that our theory of quantum state smoothing includes as a special case the hidden Markov model (HMM) that applies to quantum systems that (unlike our atomic example) have no coherences and so are effectively classical. For genuinely quantum systems there are many questions about state-smoothing to explore, including: what happens if one assumes the unobserved unravelling to be different from the default one (here, photon absorption); is there a relation between quantum state smoothing and the “most probable path” formalism of Refs. [31, 32], does the HMM inevitably emerge in the classical limit, and does quantum smoothing necessarily work best in that limit; and what experiments would show uniquely quantum aspects...
of quantum state smoothing?

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[1] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, England, 2010).
[2] C. W. Helstrom, *Quantum Detection and Estimation Theory*, vol. 123 of *Mathematics in Science and Engineering* (Academic Press, New York, 1976).
[3] V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).
[4] V. Giovannetti, S. Lloyd, and L. Maccone, Nature Photonics 5, 222 (2011).
[5] A. S. Holevo, *Quantum Systems, Channels, Information* (de Gruyter, Berlin, 2012).
[6] V. B. Braginsky and F. Y. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).
[7] M. Tsang, H. Wiseman, and C. Caves, Phys. Rev. Lett. 106, 090401 (2011).
[8] M. Tsang and R. Nair, Phys. Rev. A 86, 042115 (2012).
[9] M. Tsang, New Journal of Physics 15, 073005 (2013).
[10] M. Tsang, Phys. Rev. Lett. 102, 253601 (2009).
[11] M. A. Taylor, J. Janousek, V. Daria, J. Knittel, B. Hage, H. A. Bachor, and W. P. Bowen, Nature Photonics 7, 229 (2013).
[12] J. Bollinger, W. Itano, D. Wineland, and D. Heinzen, Phys. Rev. A 54, R4649 (1996).
[13] C. W. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics* (Springer, Berlin, 2004).
[14] C. M. Caves, C. A. Fuchs, and R. Schack, Phys. Rev. A 65, 022305 (2002).
[15] C. A. Fuchs, N. D. Mermin, and R. Schack, American Journal of Physics 82, 749 (2014).
[16] H. J. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer, Berlin, 1993).
[17] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. 68, 580 (1992).
[18] C. W. Gardiner, A. S. Parkins, and P. Zoller, Phys. Rev. A 46, 4363 (1992).
[19] H. M. Wiseman and G. J. Milburn, Phys. Rev. A 47, 1652 (1993).
[20] A. Barchielli, Int. J. Theor. Phys. 32, 2221 (1993).
[21] A. N. Korotkov, Phys. Rev. B 60, 5737 (1999).
[22] H. M. Wiseman and G. J. Milburn, Phys. Rev. Lett. 70, 548 (1993).
[23] H. M. Wiseman, Phys. Rev. Lett. 75, 4587 (1995).
[24] D. B. Horoshko and S. Y. Kilin, Phys. Rev. Lett. 78, 840 (1997).
[25] A. C. Doherty, S. Habib, K. Jacobs, H. Mabuchi, and S. M. Tan, Phys. Rev. A 62, 012105 (2000).
[26] W. P. Smith, J. E. Reiner, L. A. Orozco, S. Kuhr, and H. M. Wiseman, Phys. Rev. Lett. 89, 133601 (2002).
[27] H. M. Wiseman and A. C. Doherty, Phys. Rev. Lett. 94, 070405 (2005).
[28] P. Bushev, D. Rotter, A. Wilson, F. Dubin, C. Becher, J. Eschner, R. Blatt, V. Steixner, P. Rabl, and P. Zoller, Phys. Rev. Lett. 96, 043003 (2006).
[29] R. Vijay, C. Macklin, D. H. Slichter, S. J. Weber, K. W. Murch, R. Naik, A. N. Korotkov, and I. Siddiqi, Nature 490, 77 (2012).
[30] V. P. Belavkin, in *Lecture Notes in Control and Information Sciences* (Springer, Berlin, 1988), p. 245.
[31] V. P. Belavkin, Rep. Math. Phys. 43, A405 (1999).
[32] M. R. James, Phys. Rev. A 69, 032108 (2004).
[33] H. Hofmann, G. Mahler, and O. Hess, Phys. Rev. A 57, 4877 (1998).
[34] A. N. Korotkov, Phys. Rev. B 63, 115403 (2001).
[35] M. Tsang, Phys. Rev. Lett. 102, 250403 (2009).
[36] M. Tsang, Phys. Rev. A 80, 033840 (2009).
[37] M. Tsang, J. Shapiro, and S. Lloyd, Phys. Rev. A 79, 053843 (2009).
[38] D. Berry, M. Hall, and H. Wiseman, Phys. Rev. Lett. 111, 113601 (2013).
[39] T. Wheatley, D. Berry, H. Yonezawa, D. Nakane, H. Arao, D. Pope, T. Ralph, H. Wiseman, A. Furusawa, and E. Huntington, Phys. Rev. Lett. 104, 093601 (2010).
[40] H. Yonezawa, D. Nakane, T. A. Wheatley, K. Iwasawa, S. Takeda, H. Arao, K. Okhi, K. Tsumura, D. W. Berry, T. C. Ralph, et al., Science 337, 1514 (2012).
[41] S. Gammelmark, B. Jalsgaard, and K. Mølmer, Phys. Rev. Lett. 111, 160401 (2013).
[42] D. Tan, S. J. Weber, I. Siddiqi, K. Mølmer, and K. W. Murch, Phys. Rev. Lett. 114, 090403 (2015).
[43] W. H. Press, A. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipies* The art of Scientific Computing (Cambridge University Press, New York, 2010).
[44] M. A. Armen, A. E. Miller, and H. Mabuchi, Phys. Rev. Lett. 103, 173601 (2009).
[45] S. Gammelmark, K. Mølmer, W. Alt, T. Kampschulte, and D. Meschede, Phys. Rev. A 89, 043839 (2014).
[46] E. Pardoux, Stochastics 6, 38 (1981).
[47] J. L. Junkins, *Optical Estimation of Dynamic Systems* (Chapman and Hall, 2011).
[48] G. Lindblad, Commun. Math. Phys. 48, 11 (1976).
[49] See Supplemental Material at [URL] for more detailed analytical calculations.
[50] D. J. Atkins, Z. Brady, K., and H. M. Wiseman, Europhys. Lett. 69, 163 (2005).
[51] A. Chantasri, J. Dressel, and A. N. Jordan, Phys. Rev. A 88, 042110 (2013).
[52] S. J. Weber, A. Chantasri, J. Dressel, A. N. Jordan, K. W. Murch, and I. Siddiqi, Nature 511, 570 (2014).
[53] We tested this numerically for the two-level atom example below by verifying the \(t\)-independence of \(\varphi(\overline{Y}) = \text{Tr}[\overline{E}(\overline{t})\rho_\varphi(t)]\).
[54] We tested this numerically for the example below by verifying that the ensemble average of the calculated ‘doubly’ filtered states over the random sample of \(\overline{Y}\) coincides with the filtered states only conditioned on \(\overline{Y}\) i.e., \(E_\overline{Y}[\rho_\varphi(y)(t)] = \rho_\varphi(t), \forall t \in [t_0, T]\).
Quantum State Smoothing: Supplemental material

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SMOOTHING PROBABILITY

Elementary manipulation of probabilities is necessary to find the smoothing probability $\varphi_S(\hat{N}) = \varphi_T(\hat{N}) = \text{Pr}[\hat{N}_\text{true} = \hat{N} | \hat{Y}, \rho(t_0)]$. In this section we will show this manipulation in detail. Starting from the probability definition

$$\varphi_S(\hat{N}) = \frac{\varphi(\hat{N} | \hat{Y}, \hat{Y})}{\varphi(\hat{Y}, \hat{Y})},$$

we can use Baye’s rule to rewrite the expression as:

$$\varphi_S(\hat{N}) = \frac{\varphi(\hat{Y} | \hat{N}, \hat{Y}) \varphi(\hat{N}, \hat{Y})}{\varphi(\hat{Y}, \hat{Y})},$$

so that, as $\varphi(A, B) = \varphi(A|B) \varphi(B)$ and $\varphi(A) = \sum_B \varphi(A|B) \varphi(B)$, we finally get,

$$\varphi_S(\hat{N}) = \sum_{\hat{Y}} \frac{\varphi(\hat{Y} | \hat{N}, \hat{Y}) \varphi(\hat{N}| \hat{Y})}{\varphi(\hat{Y}| \hat{Y})}. \quad (1)$$

AVERAGE PURITY AND AVERAGE FIDELITY

Smoothed quantum state’s purity and fidelity with the true state are appropriate measures of the model success. We will demonstrate in this section that for the case of average values of these two quantities, they are totally equivalent for filtering and smoothing.

In Eq. (10) the quantum smoothed state is defined as an ensemble average over realizations of the unknown record $\hat{N}$. If we define a general condition $C$, filtered and smoothed states can be described similarly,

$$\rho_C(\tau) = E_{\hat{N} | C}[\rho_{\hat{Y}, \hat{N}}(\tau)] = \sum_{\hat{N}} \varphi_C(\hat{N}) \rho_{\hat{Y}, \hat{N}}(\tau), \quad (2)$$

where $C$ can be taken as $\hat{Y}$ for smoothing or $\hat{Y}$ for filtering. From the definition for purity in Eq.(14)

$$P[\rho_C(\tau)] = \sum_{\hat{N}} \varphi_C(\hat{N}) \text{Tr}[\rho_{\hat{Y}, \hat{N}}(\tau) \rho_C(\tau)]. \quad (3)$$

Averaging over an ensemble of measurement records $C$,

$$E_C[P[\rho_C(\tau)]] = \sum_{\hat{N}} \sum_C \varphi(\hat{N}|C) \varphi(C) \text{Tr}[\rho_{\hat{Y}, \hat{N}}(\tau) \rho_C(\tau)]$$

$$= \sum_{C, \hat{N}} \varphi(C, \hat{N}) \text{Tr}[\rho_{\hat{Y}, \hat{N}}(\tau) \rho_C(\tau)]$$

$$= E_{C, \hat{N}} \{ F[\rho_{\hat{Y}, \hat{N}}(\tau), \rho_C(\tau)] \}$$

$$= E_{C, \hat{N}} \{ F[\rho_T(\tau), \rho_C(\tau)] \}, \quad (4)$$

where here we can think of the ensemble average as being over all possible “true” records $\hat{N}$ and $\hat{Y}$.
RELATION TO CLASSICAL HIDDEN MARKOV MODEL PREVIOUS RESULTS

In the paper we mention that the results in [1, 2] and applied in [3] can be easily obtained as a restriction of the quantum state smoothing model. This comes from the fact that in the case of state transitions like quantum jumps, the state of the system is perfectly defined and a classical variable is enough to describe it. In this section we will show in detail the connection between these two models.

For this purpose let us consider a conditional state

\[
\tilde{\rho}_{\bar{N}, \bar{Y}}(\tau) \tilde{\rho}_{\text{post}}(\bar{Y}, \bar{N}) = \Pi_{x(\bar{N})} \varphi(\bar{Y}, \bar{N}),
\]

such that the classical hidden variable \( x \) depending on an hypothetical monitoring record \( \bar{N} \) is a classical hidden Markov model: \( x(\bar{N}) \). Replacing this state in Eq.(12) and the expression for \( \varphi_{\text{post}} \) in (1),

\[
\varphi_{\text{post}}(\bar{Y}, \bar{N}) \propto \frac{\text{Tr}[\hat{E}_{\bar{Y}}(\tau) \hat{\Pi}_{x(\bar{N})}] \varphi(\bar{Y}, \bar{N})}{\varphi_{\text{post}}(\bar{Y}, \bar{N})} = \varphi(\bar{Y}|x(\bar{N})) \varphi(\bar{Y}, \bar{N}).
\]

Rewriting the quantum state of the system after smoothing:

\[
\rho_{\text{S}} \propto \sum_{\bar{N}} \varphi(\bar{Y}|x(\bar{N})) \varphi(\bar{Y}, \bar{N}) \hat{\Pi}_{x(\bar{N})}.
\]

Using the completeness condition in the space of the classical hidden variable \( x \):

\[
\sum_{x'} \delta_{x', x(\bar{N})} = \sum_{x'} \text{Pr}[x' = x(\bar{N})|\bar{Y}, \bar{N}] = 1,
\]

and in the space of the hypothetical measurement record \( \bar{N} \) and considering a fixed record \( \bar{Y} \)

\[
\rho_{\text{S}} \propto \sum_{x'} \varphi(\bar{Y}|x') \hat{\Pi}_{x'} \sum_{\bar{N}} \varphi(x'|\bar{Y}, \bar{N}) \varphi(\bar{Y}, \bar{N}) \frac{\varphi(\bar{Y}|x(\bar{N}))}{\varphi_{\text{post}}(\bar{Y}, \bar{N})}
\]

\[
\propto \sum_{x'} \varphi(\bar{Y}|x') \varphi(x'|\bar{Y}) \hat{\Pi}_{x'}.
\]

This expression coincides with the hidden Markov model reproduced by Gammelmark et al in [1, 2] where the measurement operators are restricted to those like quantum jumps, e.g. photon-detection.

[1] S. Gammelmark, B. Julsgaard, and K. Mølmer, Phys. Rev. Lett. 111, 160401 (2013).
[2] S. Gammelmark, K. Mølmer, W. Alt, T. Kampuschulte, and D. Meschede, Phys. Rev. A 89, 043839 (2014).
[3] M. A. Armen, A. E. Miller, and H. Mabuchi, Phys. Rev. Lett. 103, 173601 (2009).