CLASSICAL LOGICS FOR ATTRIBUTE-VALUE LANGUAGES

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Abstract

This paper describes a classical logic for attribute-value (or feature description) languages which are used in unification grammar to describe a certain kind of linguistic object commonly called attribute-value structure (or feature structure). The algorithm which is used for deciding satisfiability of a feature description is based on a restricted deductive closure construction for sets of literals (atomic formulas and negated atomic formulas). In contrast to the Kasper/Rounds approach (cf. [Kasper/Rounds 90]), we can handle cyclicity, without the need for the introduction of complexity norms, as in [Johnson 88] and [Beierle/Pletat 88]. The deductive closure construction is the direct proof-theoretic correlate of the congruence closure algorithm (cf. [Nelson/Oppen 80]), if it were used in attribute-value languages for testing satisfiability of finite sets of literals.

1 Introduction

This paper describes a classical logic for attribute-value (or feature description) languages which are used in unification grammar to describe a certain kind of linguistic object commonly called attribute-value structure (or feature structure). From a logical point of view an attribute-value structure like e.g. the following (in matrix notation)

\[
\begin{array}{c}
PRED \, 'PROMISE'

TENSE \, PAST

SUBJ \, SUBJ \, PRED \, 'JOHN'

XCOMP \, SUBJ \, PRED \, 'COME'
\end{array}
\]

can be regarded as a graphical representation of a minimal model of a satisfiable feature description. If we assume that the attributes (in the example: PRED, TENSE, SUBJ, XCOMP) are unary partial function symbols and the values (a, 'PROMISE', PAST, 'JOHN', 'COME') are constants then the given feature structure represents graphically e.g. the minimal model of the following description:

\[
PRED \, SUBJa \approx 'JOHN' \, & \, TENSEa \approx PAST \, & \\
PREDa \approx 'PROMISE' \, & \, SUBJa \approx SUBJ \, XCOMPa \, & \\
PRED \, XCOMPa \approx 'COME'.
\]

So, in the following attribute-value languages are regarded as quantifier-free sublanguages of classical first order languages with equality whose (nonlogical) symbols are given by a set of unary partial function symbols (attributes) and a set of constants (atomic and complex values). The logical vocabulary includes all propositional connectives; negation is interpreted classically.

For quantifier-free attribute-value languages L we give an axiomatic or Hilbert type system \( H_{AV} \) which simply results from an ordinary first order system (with partial function symbols), if its language were restricted to the vocabulary of L. According to requirements of the applications, axioms for the constant-consistency, constant/complex-consistency and acyclicity can be added to force these properties for the feature structures (models).

For deciding consistency (or satisfiability) of a feature description, we assume first, that the conjunction of the formulas in the feature description is converted to disjunctive normal form. Since a formula in disjunctive normal form is consistent, if at least one of its disjuncts is consistent, we only need an algorithm for deciding consistency of finite sets of literals (atomic formulas or negated atomic formulas) \( S \). In contrast to the reduction algorithms which normalize a set \( S \) according to a complexity norm in a sequence of norm decreasing rewrite steps we use a restricted deductive closure algorithm for deciding the consistency of sets of literals. The restriction results from the fact that it is sufficient for deciding the consistency of \( S \) to consider proofs of equations from \( S \) with a certain subterm property. For the closure construction only those equations are derived from \( S \) whose terms are subterms of the terms occurring in the formulas of \( S \). This guarantees that the construction terminates with a finite set of literals. The adequacy of this subterm property restriction, which was already shown for the number theoretic calculus \( K \) in [Kreisel/Tait 61] by [Statman 74], is a necessary condition for the development of more efficient Cut-free Gentzen type systems for attribute-value languages.

1 Note that the terms are formed without using brackets. (Since all function symbols are unary, the introduction of brackets would not improve the readability essentially.) Therefore we write e.g. PRED SUBJa instead of PRED(SUBJa).

2 For intuitionistic negation cf. e.g. [Dawar/Vijay-Shanker 90] and [Langholm 89].

3 Cf. e.g. [Kreisel/Tait 61], [Knuth/Bendix 70], and applied to attribute-value languages [Johnson 88], [Beierle/Pletat 88], [Smolka 89].

4 Since we allow cyclicity, unrestricted deductive closure algorithms (cf. e.g. [Kasper/Rounds 86] and [Kasper/Rounds 90]) cannot be applied.
value languages.\footnote{Cf. also [Statman 77].}

Moreover, this closure construction is the direct proof-theoretic correlate of the congruence closure algorithm (cf. [Nelson/Oppen 80]), if it were used for testing satisfiability of finite sets of literals in $H^0_{AV}$. As it is shown there, the congruence closure algorithm can be used to test consistency if the terms of the equations are represented as labeled graphs and the equations as a relation on the nodes of that graph.

On the basis of the algorithm for deciding satisfiability of finite sets of formulas we then show the completeness and decidability of $H^0_{AV}$.

2 Attribute-Value Languages

In this section we define the type of language we want to consider and introduce some additional notation.

2.1 Syntax

2.1. DEFINITION. A quantifier-free attribute-value language ($L_{AV}^0$) consists of the logical connectives $\bot$ (false), $\neg$ (negation), $\supset$ (implication), the equality symbol $\approx$ and the parentheses ($.$). The nonlogical vocabulary is given by a finite set of constants $C$ and a finite set of unary partial function symbols $r$ ($r_1 \approx r_2$).

2.2. DEFINITION. The class of terms ($T$) of $L$ is recursively defined as follows: each constant is a term; if $f$ is a function symbol and $r$ is a term, then $fr$ is a term.

2.3. DEFINITION. The set of atomic formulas of $L$ is $\{ \bot \}$.

2.4. DEFINITION. The formulas of $L$ are the atomic formulas and, whenever $\phi$ and $\psi$ are formulas, then so are $\neg \phi$, $\phi \land \psi$, $\phi \lor \psi$, $\phi \equiv \psi$.

2.5. DEFINITION. If $\alpha$ is a well-formed expression (term or formula), then $\alpha[r_1 \approx r_2]$ is used to designate an expression obtained from $\alpha$ by replacing some (possibly all or none) occurrences of $r_1$ in $\alpha$ by $r_2$.

We assume that the connectives $\lor$ (disjunction), $\land$ (conjunction) and $\equiv$ (equivalence) are introduced by their usual definitions. Furthermore, we write sometimes $r_1 \neq r_2$ instead of $\sim r_1 \approx r_2$ and drop the parentheses according to the usual conventions.\footnote{We drop the overline.}

2.2 Semantics

A model for $L$ consists of a nonempty universe $U$ and an interpretation function $\mathfrak{I}$. Since not every term denotes an element in $U$ if the function symbols are interpreted as unary partial functions, we generalize the partiality of the denotation by assuming that $\mathfrak{I}$ itself is a partial function. Thus in general not all of the constants and function symbols are interpreted by $\mathfrak{I}$. Redundancies which result from the fact that non-interpreted function symbols and function symbols interpreted as empty functions are then regarded as distinct are removed by requiring these partial functions to be nonempty. Suppose $\{X \rightarrow Y\}_{(\phi)}$ designates the set of all (partial) functions from $X$ to $Y$, then a model is defined as follows:

2.6. DEFINITION. A model for $L$ is a pair $\mathfrak{M} \mathfrak{E} (U, \mathfrak{I})$, consisting of a nonempty set $U$ and an interpretation function $\mathfrak{I} = \mathfrak{I}_C \cup \mathfrak{I}_F$, such that

(i) $\mathfrak{I}_C[C \rightarrow U]_p$,
(ii) $\mathfrak{I}_F[F_1 \rightarrow [U \rightarrow U]_p]_p$,
(iii) $\forall f \in (fctDom(\mathfrak{I}) \rightarrow \mathfrak{I}(f) \neq \emptyset)$.

The (partial) denotation function for terms $\mathfrak{T} (\mathfrak{I}_T \rightarrow U)_p$ induced by $\mathfrak{I}$ is defined as follows:\footnote{In the text following the definition we drop the overline.}

2.7. DEFINITION. For every $c \in C$ and $fr \in T$, $fr \in \mathfrak{T} (\mathfrak{I}_T \rightarrow U)_p$.

2.8. DEFINITION. The satisfaction relation between models $\mathfrak{M}$ and formulas $\phi$ ($\models \mathfrak{M} \phi$, read: $\mathfrak{M}$ satisfies $\phi$, $\phi$ is true in $\mathfrak{M}$) is defined recursively:

A formula $\phi$ is valid ($\models \phi$), if $\phi$ is true in all models. A formula $\phi$ is satisfiable, if it has at least one model. Given a set of formulas $\Gamma$, we say that $\mathfrak{M}$ satisfies $\Gamma$ ($\models \mathfrak{M} \Gamma$), if $\mathfrak{M}$ satisfies each formula in $\Gamma$. $\gamma$ is satisfiable, if there is a model that satisfies each formula in $\Gamma$. $\phi$ is logical consequence of $\Gamma$ ($\Gamma \models \phi$), if every model that satisfies $\Gamma$ is a model of $\phi$.

3 The System $H^0_{AV}$

In this section we describe an axiomatic or Hilbert type system $H^0_{AV}$ for quantifier-free attribute-value languages $L$. We give a decision procedure for the satisfiability of finite sets of formulas and show the completeness and decidability of $H^0_{AV}$ on the basis of that procedure.

3.1 Axioms and Inference Rules

If $L$ is a fixed attribute-value language, then the system consists of a traditional axiomatic propositional calculus for $L$ and two additional equality axioms. For any formulas $\phi, \psi, \chi$, terms
The system is I oms E1 and E2, cf. [Johnson 88].

3.1. THEOREM. A formula \( \psi \) is derivable from a set of formulas \( \Gamma \) (if \( \gamma \vdash \psi \)) if there is a finite sequence of formulas \( \phi_1, \ldots, \phi_n \) such that \( \phi_n = \psi \) and every \( \phi_i \) is an axiom, one of the formulas in \( \Gamma \) or follows by MP from two previous formulas of the sequence. A formula \( \phi \) is a theorem (if \( \vdash \phi \)), if \( \phi \) is derivable from the empty set. A is derivable from \( \Gamma \) (if \( \Gamma \vdash \alpha \)), if each formula of \( \alpha \) is derivable from \( \Gamma \). and \( \alpha \) are deductively equivalent (if \( \Gamma \vdash \alpha \)), if \( \Gamma \vdash \alpha \) and \( \alpha \vdash \Gamma \).

The system is sound.11

3.2. THEOREM. For every set of formulas \( \Gamma \) and every formula \( \psi \): If \( \Gamma \vdash \psi \), then \( \Gamma \models \psi \).

3.2 Satisfiability

We now prove

3.3. THEOREM. The satisfiability of a finite set of formulas \( \Gamma \) is decidable.

by providing a terminating procedure: First the conjunction of all formulas in \( \Gamma \) (denoted by \( \bigwedge \Gamma \)) is converted into disjunctive normal form (DNF) using the well-known standard techniques. Then \( \bigwedge \Gamma \) is equivalent with a DNF

\[
\bigwedge \Gamma \equiv \left( \bigwedge \phi_1 \right) \lor \left( \bigwedge \phi_2 \right) \lor \cdots \lor \left( \bigwedge \phi_n \right)
\]

where the conjuncts \( \phi_i \) (\( i = 1, \ldots, n \); \( j = 1, \ldots, k_i \)) are either atomic formulas or negations of atomic formulas, henceforth called literals. By the definition of the satisfiability we get:

3.4. LEMMA. Let \( \bigwedge S_1 \lor \bigwedge S_2 \lor \cdots \lor \bigwedge S_n \) be a DNF of \( \bigwedge \Gamma \) consisting of conjunctions \( \bigwedge S_i \) of the literals in \( S_i \), then \( \bigwedge \Gamma \) is satisfiable, iff at least one disjunct \( \bigwedge S_i \) is satisfiable.

We complete the proof of Theorem 3.3 by an algorithm that converts a finite set of literals \( S_i \) into a deductively equivalent set of literals in normal form \( S' \) which is satisfiable iff it is not equal to \( \{ \perp \} \).

3.2.1 A Normal Form for Sets of Literals

The normal form is constructed by closing \( S \) deductively by those equations whose terms are subterms of the terms occurring in \( S \). For the construction we use the following derived rules:

- \( \sigma \models \tau' \lor \tau \) Subterm Reflexivity
- \( \tau \models \tau' \land \phi \models \phi[r/\tau'] \) Substitutivity
- \( \tau \models \tau' \land \tau' \models \tau \) Symmetry.

We get R1 and R2 from E1 and E2 by the deduction theorem. R3 is derivable from R1 and R2, since we get from \( \tau \models \tau' \) first \( \tau \models \tau \) by R1 and then \( \tau \models \tau \) by R2.

If \( T_\Sigma \) denotes the set of terms occurring in the formulas of \( S \) \((T_\Sigma = \{ r, \tau' \models (\neg \alpha \models r') \}) \), and SUB(\( T_\Sigma \)) denotes the set of all subterms of the terms in \( T_\Sigma \)

\[
\text{SUB}(T_\Sigma) = \{ r | \sigma \models \tau, \text{with } \sigma F_\phi \}
\]

then the normal form is constructed according to the following inductive definition.

3.5. DEFINITION. For a given set of literals \( S \) we define a sequence of sets \( S_i \) (\( i \geq 0 \)) by induction:

With \( S_0 = S \lor \{ r' \models \tau \lor \tau' \models S \} \)

\[
S_0 = \begin{cases}
\{ \perp \} & \text{if } \perp \models S; \\
S_0 \lor \{ r \models \sigma \models \tau \models \sigma F_\phi \} & \text{otherwise}
\end{cases}
\]

\[
S_{i+1} = \begin{cases}
\{ \perp \} & \text{if } \exists \phi \models S_i \models \neg \theta; \text{otherwise}
\end{cases}
\]

where \( S_i \subseteq S_{i+1}, \) for \( S_i+1 \neq \{ \perp \} \), the construction terminates on the basis of the subterm condition either with a finite set of literals or with \( \{ \perp \} \). If each term of the equations in \( S_{i+1} \) is a subterm of the terms in \( T_\Sigma \), no term of the equations in \( S_{i+1} \) can be longer than the longest term in \( T_\Sigma \).

EXAMPLE 1. Assume that \( L \) consists of the constants \( a, b, c, e \) and the function symbols \( f, g, h, m, n, p \). Then, for the set of literals

\[
S = \{ ge \models pm, c \models me, nb \models ngef sc, c \models a, \}
\]

the following sequence of sets is constructed. We represent the equations of a set \( S \) by the system of sets of equivalent terms induced by \( S \). I.e.: If \( \Theta \) is a set of terms under \( S \) and

\[12\]
r, r' ∈ Θ, then r ≈ r' ∈ S_0. Furthermore, we mark by an arrow that a set under S_i is also induced (without modifications) by the equations in S_{i+1}.

Furthermore, we mark by an arrow that a set under S_i is also induced (without modifications) by the equations in S_{i+1}.

So S_0 $\subseteq$ S_1 = S_2 = S_ν.

\[ngffα \neq e\rightarrow
\begin{array}{l}
\{e, me\} \rightarrow \\
\{a, ffa\} \rightarrow \\
\{f c\} \rightarrow \\
\{fb\} \rightarrow \\
\{eb, gffα\} \rightarrow \\
\{gα, ha\} \rightarrow
\end{array}
\]

Then r ≈ r' is in S_ν, iff the nodes which represent the terms r and r' in the graph constructed for S are congruent. Moreover, for unary partial functions the algorithm is simpler, since the arity does not have to be controlled.

3.9. Lemma. The set of all equations in S_ν is closed under subterm reflexivity, symmetry and transitivity.

Proof. For S_ν = {⊥} trivial. If S_ν ≠ {⊥}, then S_ν is closed under subterm reflexivity and symmetry, since these properties are inherited from S_0 to its successor sets. S_ν is closed under transitivity, since we first get r_2 ∈ SUB(T_2) from r_1 ≈ r_2, r_2 ≈ r_3 ∈ S_0 and then according to the construction also r_1 ≈ r_2[r_2/r_3] ∈ S_{ν+1} = S_ν, with r_2[r_2/r_3] = r_3.

3.2.2 Satisfiability of Sets of Literals

For the proof that the satisfiability of a finite set of literals is decidable we first show that a set of literals in normal form is satisfiable, if the set is not equal to {⊥}. For S_ν = {⊥} we get trivially:

3.10. Lemma. S_ν = {⊥} $\rightarrow$ ¬3M(∃M S_ν).

Otherwise we can show the satisfiability of S_ν by the construction of a canonical model, which satisfies S_ν.

Let E_ν be the set of all (nonnegated) equations in S_ν, T_ν the set of terms occurring in E_ν and ≈_{E_ν} the relation induced by E_ν on T_ν. Then, we choose as the universe of the canonical model M_ν = (U_ν, =_{M_ν}) the set of all equivalence classes of ≈_{E_ν} on T_ν, if T_ν ≠ ∅. By Lemma 3.9 this set exists. If S_ν contains no (unnegated) equation, we set U_ν = {ν}, since the universe has to be nonempty.

3.11. Definition. For a set of literals S_ν in normal form, the canonical term model for S_ν is given by the pair M_ν = (U_ν, =_{M_ν}), consisting of the universe

\[U_ν = \begin{cases}
T_ν / ≈_{E_ν} & \text{if } T_ν \neq ∅ \\
\emptyset & \text{otherwise}
\end{cases}
\]

and the interpretation function =_{M_ν}, which is defined for c ∈ C, f ∈ F and [r] ∈ T_ν by:

\[=_{M_ν}(c) = \begin{cases}
[c] & \text{if } c \in T_ν \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

\[=_{M_ν}(f)([r]) = \begin{cases}
[f(r)] & \text{if } r' ∈ [r] \text{ and } f'(r) \in T_ν \\
\text{undefined} & \text{otherwise}
\end{cases}
\]

It follows from the definition that =_{M_ν} is a partial function. Suppose further for Θ F_1 (f) that [r_1] ∈ [r_2] and that Θ F_1 (f)([r_1]) is defined. Then

\[Θ F_1 (f)([r_1]) = Θ F_1 (f)([r_2]).\]

For this, suppose Θ F_1 (f)([r_1]) = [f'(r)], with r' ∈ [r_1]. Since ≈_{E_ν} is an equivalence relation we get r' ∈ [r_2] and thus Θ F_1 (f)([r_2]) = [f'(r)].

14 Cf. [Wedekind 90].
15 We drop the ≈_{E_ν}-index of the equivalence classes.
EXAMPLE 3. The canonical model for $S$ of Example 1 which is constructed using $S_2 = S_3$ is given by:

$$\mathcal{U}_c = \{ [c, me], [b], [a], [mefafa], [ge, pnb], [mb, ncfgfc, ngffen], [f, fc, [affc, gffa, ga, ha] \}$$

For each term $r$ in $T_{E_3}$ it follows from the definition of $\mathcal{Q}_c$ and $\mathcal{Q}_f$ that $\mathcal{Q}_c\{c\} = [c]$.

**PROOF.** (By induction on the length of $r$.) Suppose first that $\mathcal{Q}_c$ is defined for $r$, then $\mathcal{Q}_c(r) = [r]$, with $r \in T_{E_3}$.

By the following lemma we show in addition that the domain of $\mathcal{Q}_c$ restricted to $T_{E_3}$ is equal to $T_{E_3}$.

3.12. LEMMA. For each term $r$ in $T_{E_3}$: If $\mathcal{Q}_c\{c\} = [r]$, then $\mathcal{Q}_c(r) = [r]$, with $r \in T_{E_3}$.

**PROOF.** (By induction on the length of $r$.) Suppose first that $\mathcal{Q}_c\{c\}$ is defined for $r$. For every constant $c$ it follows from the definition of $\mathcal{Q}_c$ that $\mathcal{Q}_c\{c\} = [c]$, with $r \in T_{E_3}$. Assume for $r'$ by induction hypothesis $\mathcal{Q}_c\{c\}(r') = [r']$, with $r' \in T_{E_3}$, then it follows from the inductive hypothesis $\mathcal{Q}_c\{c\}(r') = [r']$, with $r' \in T_{E_3}$, that $\mathcal{Q}_c\{c\}(f)([r]) = [f[r]]$, with $f \in T_{E_3}$. Since $f$ is a subterm of $r'$, we first get $f([r]) = [f[r]]$ and by Lemma 3.9 $f \in T_{E_3}$. Because of $f \in T_{E_3}$, it follows that $r \approx r'$. Then also $r \approx r'$. So, $r \approx r'$. Next we show for the model $M_{S_3}$:

3.13. LEMMA. $S_3 \not\equiv \{1\} \rightarrow \models_{M_{S_3}} S_3$.

**PROOF.** (We prove $\models_{M_{S_3}} \phi$, for every $\phi$ in $S_3$ by induction on the structure of $\phi$.)

$\bot$ is not element of $S_3$. If $\bot$ were in $S_3$, we would get by the definition of $S_3$ that $\mathcal{S}_3 = \{1\}$ which contradicts our assumption.

For $\phi = \bot, \models_{M_{S_3}} \bot$ holds trivially.

Suppose $\phi = \tau \approx \tau'$, then $\tau, \tau'$ are in $T_{E_3}$, $\mathcal{Q}_c\{c\}$ is defined for $\tau, \tau'$, and $\mathcal{Q}_c\{c\}(r) = [r]$, $\mathcal{Q}_c\{c\}(r') = [r']$. Because of $\tau \approx \tau'$, we first get $\mathcal{Q}_c\{c\}(r) = [r]$ and by Lemma 3.9 $\mathcal{Q}_c\{c\}(r') = [r']$. From $\mathcal{Q}_c\{c\}(f)([r]) = [f[r]]$, which is derivable from $\mathcal{Q}_c\{c\}(f)([r]) = [f[r]]$, we get $\mathcal{Q}_c\{c\}(f)([r]) = [f[r]]$. From $\mathcal{Q}_c\{c\}(f)([r]) = [f[r]]$, we get $\mathcal{Q}_c\{c\}(r) \approx \mathcal{Q}_c\{c\}(r')$.

Assume that $\phi = \tau \approx \tau'$, then $\tau, \tau'$ were satisfied by $M_{S_3}$, $\mathcal{Q}_c\{c\}(r) = [r]$ and $\mathcal{Q}_c\{c\}(r') = [r']$. From $\mathcal{Q}_c\{c\}(r) = [r]$ and $\mathcal{Q}_c\{c\}(r') = [r']$, and by the deduction theorem first $\models_{M_{S_3}} \Mathcal{Q}_c\{c\}(r) \approx \tau \approx \tau' \models_{M_{S_3}}$.

It can be easily shown that $M_{S_3}$ is a unique (up to isomorphism) minimal model for $S_3$. Strictly speaking, if $M$ is a model for $S_3$, homomorphic to $M_{S_3}$, then every minimal submodel of $M$ that satisfies $S_3$ is isomorphic to $M_{S_3}$.

From the two lemmata above it follows first that the satisfiability of sets of formulas in normal form is decidable:

$$S_3 \not\equiv \{1\} \rightarrow \exists M(\models_{M_{S_3}} S_3).$$

Since $S_3$ and $S$ are deductively equivalent, we can establish by the following lemma that the satisfiability of arbitrary finite sets of literals $S$ is decidable.

3.14. LEMMA. $S \not\equiv \{1\} \rightarrow \exists M(\models_{M_{S_3}} S)$. 

**PROOF.** (If) If $S \not\equiv \{1\}$, we know by Lemma 3.13 that $M_{S_3}$ is a model for $S_3$. Then, by the soundness $S_3 \models S \rightarrow \forall M(\models_{M_{S_3}} S_3 \models S)$. Since $S$ is derivable from $S_3$, it follows $\models_{M_{S_3}} S$ and thus $S \not\equiv \{1\} \rightarrow \exists M(\models_{M_{S_3}} S)$. (If) If $S \not\equiv \{1\}$, then for each model $M \models_{M_{S_3}} S_3$. From the soundness we get $S_3 \models S \rightarrow \forall M(\models_{M_{S_3}} S \rightarrow \models_{M_{S_3}} S_3)$. Since $S_3$ is derivable from $S_3$, it follows $\forall M(\models_{M_{S_3}} S \rightarrow \models_{M_{S_3}} S_3)$ and hence $S_3 = \{1\} \rightarrow \models_{M_{S_3}} S$.

3.3 Completeness and Decidability

Using the procedure for deciding satisfiability we can easily show the completeness and decidability of $H_{S_3}^v$.

3.15. THEOREM. For every finite set of formulas $\Gamma$, and for each formula $\phi$: $\Gamma \models \phi$, then $\Gamma \models \phi$.

**PROOF.** By definition $\phi$ is a logical consequence of $\Gamma$, iff $\Gamma \cup \{\phi\}$ is unsatisfiable. Using the equivalences of Theorem 3.3, we first get:

$$\Gamma \cup \{\phi\} \models \{\lambda \cup \{\phi\}\}.$$ Suppose, that $\Gamma \cup \{\phi\} \models \{\lambda \cup \{\phi\}\}$ is a DNF of $\{\lambda \cup \{\phi\}\}$, then $\Gamma \cup \{\phi\} \models \{\lambda \cup \{\phi\}\}$ and by the decision procedure

$$\models_{\Gamma \cup \{\phi\}} \models \mathcal{S}_{S} = \{1\} \wedge \ldots \wedge \mathcal{S}_{S} = \{1\}.$$ If $\Gamma \cup \{\phi\} \models \mathcal{S}_{S}$ is unsatisfiable, it follows that $\Gamma \cup \{\phi\} \models \{\lambda \cup \{\phi\}\}$, since each $\mathcal{S}_{S}$ is deductively equivalent with $\{1\}$. From $\mathcal{S}_{\Gamma} \models \{\phi\}$ it follows by the deduction theorem first $\Gamma \models \{\phi\}$ and thus $\Gamma \models \{\phi\}$. From $\Gamma \models \{\phi\}$ and $\mathcal{S}_{\Gamma} \models \mathcal{S}_{S} \models \mathcal{S}_{S} \models \{1\}$.

3.16. COROLLARY. For every finite set of formulas $\Gamma$ and each formula $\phi$, $\Gamma \models \phi$ is decidable.

**PROOF.** By the completeness and soundness we know $\Gamma \models \phi$ iff $\Gamma \cup \{\phi\}$.

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