Stochastic Acceleration in Strong Random Fields

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Abstract

Diffusion of particles in velocity space undergoing turbulent field was extensively studied in the problem of warm beam relaxation. Under low field intensities the diffusion is described by the Fokker-Planck equation with the diffusion coefficient given by quasilinear theory. This diffusion coefficient is calculated on the free particle propagator and for weak fields its renormalization due to orbit diffusion is not necessary. To study effects which should be taken into account when the intensity of the turbulent field is increased a numerical simulation of particle motion in the external field of Langmuir waves with given \( k \)-spectrum and random phases is done. For strong fields mean square velocity evolution shows that ballistic regime in the very beginning is changed for oscillatory one in the intermediate stage and later for diffusion. Asymptotically it behaves like fractional power of elapsed time with the exponent dependent on the particular field spectrum. Such evolution in the whole temporal interval of simulation is recovered from the numerical solution of generalized Fokker-Planck equation with time dependent diffusion coefficient obtained from the microscopic approach. The analytical approximation for this solution is also given.

1 Introduction

Diffusion of particles in random external fields could be considered in relation to a general problem of transport in plasmas. The test particle approach, which is simpler and more controllable than selfconsistent one, helps to analyze some particular aspects of turbulent transport.

One of not clarified issues reported in papers \cite{1,2} concerns the enhancement of a diffusion coefficient in velocity space as compared to its quasilinear value. It was supposed \cite{1} that the enhancement was caused by peaks in spectrum being formed in a course of evolution. The tendency of formation of nonuniform structures in plasma-beam system was pointed out in the works \cite{3}. Here we are interested in what effects are to be taken into account when the spectrum of external field become stronger and narrower, i.e. more peaked.

We have made a direct simulation of test particle motion in prescribed random fields for a variety of spectra, and found from obtained data the evolution of average velocity and dispersion. Then the generalization of the Fokker-Planck equation were considered in order the solutions would be consistent with simulation in wide range of variation of external field spectra.

As it was expected for low intensity and broad spectrum (small Kubo number) solutions of Fokker-Planck equation with quasilinear diffusion coefficient agree with the results of simulation. When the form of spectrum is taken peaked (Kubo number becomes larger than the unit) the velocity dispersion grows very fast at a time less than the field correlation time, and such jump of dispersion on the very early (prekinetic) stage could give a substantial contribution to overall dispersion. In this case the distribution function is governed by a Fokker-Planck equation with time dependent diffusion coefficient. Solutions of the Fokker-Planck equation with quasilinear and time dependent diffusion coefficients were found numerically, and an analytical approximation for them was proposed as well.

The numerical experiment gives the power law behavior of the dispersion at simulation times, and solutions of the Fokker-Planck equation shows it for extended interval. However the
exponent is not unique for all spectra of the same type, as this follows from scaling consideration [4], but depends on the particular spectrum.

2 Numerical model

We consider the motion of noninteracting particles in an external random electric field. The potential of the field is taken as a superposition of \( M \) waves [5]

\[
\varphi(x,t) = \sum_{i=1}^{M} \varphi_i \cos (\omega t - k_i x + \alpha_i)
\]

(1)

with fixed frequency \( \omega \) and wave numbers from the interval \((k_0 - 2.5\Delta k, k_0 + 2.5\Delta k)\). The total intensity of the field \( \varphi^2_0 \) is distributed between the partial waves according to the Gaussian

\[
\varphi^2_i = \frac{2}{\sqrt{\pi}} \varphi^2_0 \frac{\delta k}{\Delta k} \exp \left( \frac{(k_i - k_0)}{\Delta k} \right)^2,
\]

(2)

where \( k_0 \) is the central wave number, \( \Delta k \) is the width of the spectrum, \( \delta k = k_{i+1} - k_i = 5\Delta k/M \).

Each realization of the field (1) is characterized by unique set of random phases \( \{\alpha_i\} \). For a given realization of the potential (1) the equations of particle motion,

\[
\begin{align*}
\dot{x} &= v, \\
\dot{v} &= \frac{e}{m} E(x,t), \\
E(x,t) &= -\frac{\partial}{\partial x} \varphi(x,t),
\end{align*}
\]

(3)

with the initial conditions \( x(0) = 0, v(0) = v_0 \) are integrated numerically to obtain the particle trajectory in velocity space \( v(t) \). Average particle velocity \( \bar{v}(t) \) and velocity dispersion \( \langle \Delta v^2 \rangle_t \) are found by averaging over \( N \) realization

\[
\bar{v}(t) = \frac{1}{N} \sum_{j=1}^{N} v_j(t),
\]

\[
\langle \Delta v^2 \rangle_t = \frac{1}{N} \sum_{j=1}^{N} (v_j(t) - \bar{v}(t))^2.
\]

(4)

In simulations length is normalized to \( 2\pi/k_0 \), time to \( 2\pi/\omega \), dimensionless potential and spectrum width are

\[
\sigma = \frac{e \varphi^2}{m \omega^2} k_0^2 \quad \text{and} \quad d = \frac{\Delta k}{k_0}.
\]

Kubo number, \( Q \), which is the ratio of the correlation time to characteristic period of particle oscillations for this model is

\[
Q = \sqrt{\sigma/d}.
\]

(5)

Overlap parameter

\[
A_j = 4\pi^2 \frac{e}{m} \varphi_j \frac{k_j^4}{\delta k^2 \omega^2}
\]

much exceeds the unit for most harmonics \( j \) except of few at the wings of Gaussian distribution (2), and particle motion can be treated as stochastic. Note, that the random phase ensemble we used here does not provide stochastization by itself, but gives the explicit way for calculation
of Euler correlation function of fields. According to Eqs. (1), (2) the correlation function for the potential \( \langle \varphi^2 \rangle_{xt} \) is of the form
\[
\langle \varphi^2 \rangle_{xt} = \varphi_0^2 \exp - \frac{(\Delta k x)^2}{4} \cos(\omega t - k_0 x).
\]
Obtained in simulation \( \bar{v}(t) \) and \( \langle \Delta v^2 \rangle_t \) will be compared in Section 4 with numerical and approximate analytical solutions of the Fokker-Planck equation.

3 Equation for distribution function and approximate analytical solution

Introduce here the particle distribution function \( f(v, t) \) as microscopic distribution function averaged over random phase ensemble and integrated over the spatial variable \( x \). As far as the averaging over the ensemble of random phase does not imply the averaging over any small but finite time scale, the distribution function is defined at all time scales, as well for \( t < \tau_{cor} \), i.e. in prekinetic stage. Generalized Fokker-Planck equation for \( f(v, t) \) in external fields could be obtained from Ref. 6 in the form
\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} D(v, t) \frac{\partial}{\partial v} f(v, t) \quad (7)
\]
with a time dependent diffusion coefficient
\[
D(v, t) = \left( \frac{e}{m} \right)^2 \int_0^t \langle E^2 \rangle_{v\tau} d\tau. \quad (8)
\]

For correlation function (6) it takes the form
\[
D(v, t) = \frac{1}{2} \left( \frac{e}{m} \varphi_0 \Delta k \right)^2 \int_0^t d\tau \exp - \frac{1}{2} (\Delta kv\tau)^2 \left( 1 + 2 \left( \frac{k_0}{\Delta k} \right)^2 - \frac{1}{2} (\Delta kv\tau)^2 \right) \cos(\omega - k_0 v)\tau + 2k_0 v\tau \sin(\omega - k_0 v)\tau. \quad (9)
\]

It will be shown that for moderate Kubo number the agreement with simulation is recovered by more accurate treatment of distribution function evolution on early stage \( t < \tau_{cor} \). Here the use of a time dependent diffusion coefficient is required.

The asymptotic value of \( D(v, t) \) at large times \( t \gg \tau_{cor} \) gives the well known quasilinear diffusion coefficient \( D_{ql}(v) \)
\[
D_{ql}(v) = D(v, t \to \infty). \quad (10)
\]

When the correlation function is given by Eq. (6) the quasilinear diffusion coefficient takes the form
\[
D_{ql}(v) = \left( \frac{e}{m} \varphi_0 \right)^2 \sqrt{\frac{\pi}{\Delta k|v|^3}} \exp - (\frac{\omega - k_0 v}{\Delta k v})^2. \quad (11)
\]

In the following section it will be shown that in the cases of narrow and/or high intensive spectrum it is important to retain the dependence of diffusion coefficient on time.

Eqs. (7), (9) or (7), (11) which determine the evolution of \( f(v, t) \) with time dependent or, respectively, quasilinear diffusion coefficient are solved numerically. Initial condition for
The average velocity and dispersion are calculated as 

$$\bar{v}(t) = \int dv \, v \, f(v, t),$$  \hspace{1cm} (12) 

$$\langle \Delta v^2 \rangle_t = \int dv (v - \bar{v}(t))^2 f(v, t).$$  \hspace{1cm} (13) 

They are compared in the following section with $\bar{v}(t)$ and $\langle \Delta v^2 \rangle_t$ obtained in simulation.

In Fig. 1 the time dependent diffusion coefficient $D(v, t)$ for narrow spectrum of external field along with its profiles are shown.

**Fig. 1.** Time dependent diffusion coefficient $D(v, t)$ for times less than the correlation time $\tau_{\text{cor}}$ (left), sections of $D(v, t)$ at $t = 1$, 3 and 100 (right), $\sigma = 0.01$, $d = 0.04$.

Time dependent diffusion coefficient (9) evolve from very broad distribution through oscillating regime to its asymptotic quasilinear value (11).

Approximate WKB-type solution of the Fokker-Planck equation (7) with velocity dependent diffusion coefficient $D_{\text{ql}}(v)$ for the distribution function $f(v, v_0, t)$ with initial conditions $f(v, t = 0) = \delta(v - v_0)$ may be given in the form

$$f(v, v_0, t) = C(t) \exp \left( -\frac{Y^2(v, v_0)}{4t} \right),$$  \hspace{1cm} (14) 

with

$$Y(v, v_0) = \int_{v_0}^{v} \frac{du}{\sqrt{D(u)}},$$  \hspace{1cm} (15) 

and $C(t)$ to be taken from the condition of normalization

$$C^{-1} = \int dv \exp \left( -\frac{Y^2(v, v_0)}{4t} \right).$$  \hspace{1cm} (16) 

This approximation was proposed in Ref. 4, however with other $C(t)$, which does not gives a proper time scaling of dispersion.

For a time dependent diffusion coefficient the approximate WKB-type solution could be generalized as

$$f(v, v_0, t) = C(t) \exp \left( -\frac{1}{4} Z^2(v, v_0, t) \right),$$  \hspace{1cm} (17) 

where
\[ Z(v, v_0, t) = \int_{v_0}^{v} \frac{du}{\sqrt{\int_{0}^{t} D(u, \tau) d\tau}}. \]  

(18)

Here, similarly to the previous case, \( C(t) \) should be defined from the condition of normalization.

4 Comparison of simulation with numerical and analytical solutions of the Fokker-Planck equation

In this section results of simulations are compared with numerical and analytical solutions of the Fokker-Planck equation. For small Kubo numbers the solutions of Fokker-Planck equation with asymptotic quasilinear diffusion coefficient give the similar evolution of velocity dispersion and average velocity as the solutions with time dependent diffusion coefficient. In addition, for very small Kubo numbers at the beginning of evolution the velocity dispersion grows almost linearly. Whether Kubo number increase the deviation from the linear law due to dependence of diffusion coefficient on velocity becomes evident, however the solutions with \( D_{ql}(v) \) and \( D(v, t) \) are still rather close. For a moderate Kubo numbers of the order of the unit, the difference between solutions with time dependent and asymptotic diffusion coefficient becomes noticeable.

In Fig. 2 the curves obtained in simulation for a wide spectrum and moderate field are compared with the numerical solution of Fokker-Planck equation and the WKB solution (17), (18).

![Fig. 2. Dispersion \( \langle \Delta v^2 \rangle \) and average velocity \( \bar{v} \) for a wide spectrum and moderate field, \( d = 0.4, \sigma = 0.01 \) and \( v_0 = 1 \). Kubo number \( Q = 0.25 \). Simulation and solution of Fokker-Planck equation with diffusion coefficient \( D(v, t) \) are compared with WKB solution (17), (18).]

In Fig. 3 is shown how WKB solution reproduces the early evolution of \( \langle \Delta v^2 \rangle \) in the case of a narrow spectrum and moderate field with initial jump of dispersion.

In the cases of broad spectrum and low intensity particles for long time slowly diffuse on small distance on \( v \), which is less than halfwidth of \( D_{ql}(v) \); and this time is enough for \( D(v, t) \) to evolve to its asymptotic value. In the opposite case, corresponding to Fig. 3, particles on times substantially less then \( \tau_{cor} \), while \( D(v, t) \) is a broad in \( v \) (c.f. Fig. 1), diffuse at large distance which is more than halfwidth of \( D_{ql}(v) \).
Fig. 3. Dispersion $\langle \Delta v^2 \rangle$ for a narrow spectrum and moderate field, $d = 0.04$, $\sigma = 0.01$, $v_0 = 1.0$. Kubo number $Q = 2.5$. Simulation and solution of Fokker-Planck equation with $D(v, t)$ are compared with WKB solution (17), (18) in a small time scale.

The above examples were given for particles which initial velocity are not so far from the phase velocity of the central harmonic. Such particles in each instant are in resonance with some harmonic of considerable intensity and diffusion prevail over oscillations. For particles which initial velocities are far from resonance with intensive harmonics the diffusivity is small and oscillations become more distinct. In Fig. 4 dispersion are given for nonresonant particle. The curves obtained in simulation, as numerical solution of the Fokker-Planck equation with time dependent diffusion coefficient and in WKB approximation are shown.

Fig. 4. Dispersion of nonresonant particles $\langle \Delta v^2 \rangle$ for a narrow spectrum and weak field, $d = 0.04$, $\sigma = 0.0001$, $v_0 = 1.2$. Kubo number $Q = 0.25$. Simulation and solution of Fokker-Planck equation with diffusion coefficient $D(v, t)$ is compared with WKB solution (17), (18) on a small time scales.

5 Power law dispersion

Simulation shows the velocity dispersion obeys a power law dependence on time. Numerical solutions of Fokker-Planck equation give the same power law, as simulation, and it easily could be calculated for much longer time. Such power law is also recovered from WKB approximation, and in this case it could be related to power law dependence of normalizing constants $C(t)$. The example with $Q = 0.079$ is given in Ref. 7. Here, in Fig. 5 the plot is given for the different
Kubo number, $Q = 0.25$, for numerical and WKB solutions of Fokker-Planck equation along with time dependence of $G = C(t) t^q$ (note that $q = p/2$).

![Graph showing power law dispersion](image)

**Fig. 5.** $U = \langle \Delta v^2 \rangle / t^p$ against $t$. Power law dispersion for a wide spectrum, $p = 0.29$ (left), and $G = C(t) t^q$ against $t$, $q = p/2 = 0.145$ (right); $d = 0.4$, $\sigma = 0.01$, $v_0 = 1$, Kubo number $Q = 0.25$.

The velocity dispersion shows power law time dependence, with exponent dependent on particular spectrum. In terms of WKB solution it is related to power law dependence of normalizing constant $C(t)$.

**Conclusions**

For small field intensity and wide spectrum (Kubo number less than the unit) the solution of the Fokker-Planck equation with quasilinear diffusion coefficient gives good agreement with results of numerical experiment. To have a consistency for high intensity and/or narrow spectrum (Kubo number of the order or larger than the unit) the generalization of the Fokker-Planck equation is to be done by introducing a time dependent diffusion coefficient. An analytical approximation for such solutions is proposed. Velocity dispersion manifests power law time dependence and the exponent is dependent on the spectrum.

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[1] I Doxas, J R Cary *Phys. Plasmas* 4 2508 (1997)
[2] J R Cary, I Doxas, D F Escande, A D Verga *Phys. Fluids* 4 2062 (1992)
[3] A S Bakaj *Dokl. Acad. Nauk SSR* 237, 1069 (1977); A S Bakaj, Y S Sigov *ibid*, 1326
[4] E Vanden Eijnden *Phys. Plasmas* 4 1486 (1997)
[5] F Doveil, D Gresillon *Phys. Fluids* 25 1396 (1982)
[6] S A Orszag, R H Kraichman. *Phys. Fluids* 10 1720 (1967)
[7] A Zagorodny, V Zasenko, J Weiland 23th EPS Conference on Contr. Fusion and Plasma Phys., St. Petersburg, 7-11 July 2003 ECA Vol. 27A, P-2.3