A Different Study on the Spaces of Generalized Fibonacci Difference $bs$ and $cs$ Spaces Sequence

Fevzi Ya¸sar * and Kuddusi Kayaduman

Department of Mathematics, Faculty of Arts and Sciences, Gaziantep University, Gaziantep 27310, Turkey; kayaduman@gantep.edu.tr
* Correspondence: fevziyasar@hotmail.com; Tel.: +9-342-360-1012

Received: 13 June 2018; Accepted: 5 July 2018; Published: 11 July 2018

Abstract: The main topic in this article is to define and examine new sequence spaces $bs(\hat{F}(s, r))$ and $cs(\hat{F}(s, r))$, where $\hat{F}(s, r)$ is generalized difference Fibonacci matrix in which $s, r \in \mathbb{R}\setminus\{0\}$. Some algebraic properties including some inclusion relations, linearly isomorphism and norms defined over them are given. In addition, it is shown that they are Banach spaces. Finally, the $\alpha$, $\beta$- and $\gamma$-duals of the spaces $bs(\hat{F}(s, r))$ and $cs(\hat{F}(s, r))$ are appointed and some matrix transformations of them are given.

Keywords: Fibonacci numbers; Fibonacci double band matrix; sequence spaces; difference matrix; matrix transformations; $\alpha$, $\beta$, $\gamma$-duals

1. Introduction

Italian mathematician Leonardo Fibonacci found the Fibonacci number sequence. The Fibonacci sequence actually originated from a rabbit problem in his first book “Liber Abaci”. This sequence is used in many fields. The Fibonacci sequence is as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

The Fibonacci sequence, which is denoted by $(f_n)$, is defined as the linear recurrence relation

$$f_n = f_{n-1} + f_{n-2}.$$  

$f_0 = 1, f_1 = 1$ and $n \geq 2$. The golden ratio is

$$\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} = \phi \text{ (Golden Ratio).}$$  

The Golden Ratio, which is also known outside the academic community, is used in many fields of science.

Let $w$ be the set of all real valued sequences. Any subspace of $w$ is called the sequence space. $c, c_0$ and $\ell_\infty$ are called as sequences space convergent, convergent to zero and bounded, respectively. In addition to these representations, $\ell_1$, $bs$ and $cs$ are sequence spaces, which are called absolutely convergent, bounded and convergent series, respectively.

Let us take a two-indexed real valued infinite matrix $A = (a_{nk})$, where $a_{nk}$ is real number and $k, n \in \mathbb{N}$. $A$ is called a matrix transformation from $X$ to $Y$ if, for every $x = (x_k) \in X$, sequence $Ax = \{A_n(x)\}$ is A transform of $x$ and in $Y$, where

$$A_n(x) = \sum_k a_{nk} x_k$$  

(1)
and Equation (1) converges for each \( n \in \mathbb{N} \).

Let \( \lambda \) be a sequence space and \( K \) be an infinite matrix. Then, the matrix domain \( \lambda_K \) is introduced by

\[
\lambda_K = \{ t = (t_k) \in w : Kt \in \lambda \}.
\]

(2)

Here, it can be seen that \( \lambda_K \) is a sequence space.

For calculation of any matrix domain of a sequence, a triangle infinite matrix is used by many authors. So many sequence spaces have been recently defined in this way. For more details, see [1–22].

Kara [23] recently introduced the \( \hat{F} \) which is derived from the Fibonacci sequence \((f_n)\) and defined the new sequence spaces \( \ell_p(\hat{F}) \) and \( \ell_\infty(\hat{F}) \) by using sequence spaces \( \ell_p \) and \( \ell_\infty \), respectively, where \( 1 \leq p < \infty \). The sequence space \( \ell_p(\hat{F}) \) has been defined as:

\[
\ell_p(\hat{F}) = \{ x \in w : \hat{F}x \in \ell_p \}, \quad (1 \leq p < \infty),
\]

where \( \hat{F} = (f_{nk}) \) defined by the sequence \((f_n)\) as follows:

\[
f_{nk} := \begin{cases} \frac{-f_{k+1}}{f_n}, & k = n - 1, \\ \frac{f_k}{f_{n+1}}, & k = n, \\ 0, & 0 \leq k < n - 1 \text{ or } k > n, \end{cases}
\]

for all \( k, n \in \mathbb{N} \). In addition, Kara et al. [24] have characterized some class of compact operators on the spaces \( \ell_p(\hat{F}) \) and \( \ell_\infty(\hat{F}) \), where \( 1 \leq p < \infty \).

Candan [25] introduced \( c(\hat{F}(s, r)) \) and \( c_0(\hat{F}(s, r)) \). Later, Candan and Kara [15] have investigated the sequence spaces \( \ell_p(\hat{F}(s, r)) \) in which \( 1 \leq p \leq \infty \).

The \( \alpha-, \beta- \) and \( \gamma- \)duals \( P^\alpha, P^\beta \) and \( P^\gamma \) of a sequence spaces \( P \) are defined, respectively, as

\[
P^\alpha = \{ a = (a_k) \in w : at = (a_k t_k) \in \ell_1 \text{ for all } t \in P \},
\]

\[
P^\beta = \{ a = (a_k) \in w : at = (a_k t_k) \in cs \text{ for all } t \in P \},
\]

\[
P^\gamma = \{ a = (a_k) \in w : at = (a_k t_k) \in bs \text{ for all } t \in P \},
\]

respectively.

In Section 2, sequence space \( bs(\hat{F}) \) and \( cs(\hat{F}) \) are defined and some algebraic properties of them are investigated. In the last section, the \( \alpha-, \beta- \) and \( \gamma- \)duals of the spaces \( bs(\hat{F}) \) and \( cs(\hat{F}) \) are found and some matrix transformations of them are given.

2. Generalized Fibonacci Difference Spaces of \( bs \) and \( cs \) Sequences

In this section, spaces \( bs(\hat{F}(s, r)) \) and \( cs(\hat{F}(s, r)) \) of generalized Fibonacci difference of sequences, which constitutes bounded and convergence series, respectively, will be defined. In addition, some algebraic properties of them are investigated.

Now, we introduce the sets \( bs(\hat{F}(s, r)) \) and \( cs(\hat{F}(s, r)) \) as the sets of all sequences whose \( \hat{F}(s, r) = \{ f_{nk}(s, r) \} \) transforms are in the sequence space \( bs \) and \( cs \),

\[
bs(\hat{F}(s, r)) = \left\{ x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left( s \frac{f_k}{f_{k+1}} x_k + r \frac{f_{k+1}}{f_k} x_{k-1} \right) \right| < \infty \right\},
\]

\[
cs(\hat{F}(s, r)) = \left\{ x = (x_k) \in w : \sum_{k=0}^{n} \left( s \frac{f_k}{f_{k+1}} x_k + r \frac{f_{k+1}}{f_k} x_{k-1} \right) \in c \right\},
\]

where \( f_{nk}(s, r) \) and \( s, r \) are positive integers. Moreover, the duals of these spaces are also introduced.
where $\hat{F}(s,r) = \{f_{nk}(s,r)\}$ is

$$f_{nk}(s,r) := \begin{cases} \frac{r}{s} f_{n+1}, & k = n - 1, \\ \frac{1}{s} \sum_{i=0}^{k} \left( \frac{r}{s} \right)^{n-k} f_{i+1}, & 0 \leq k < n - 1 \text{ or } 0 \leq k > n, \\ 0, & k = n, \end{cases}$$  \tag{3}

for all $k, n \in \mathbb{N}$ where $s, r \in \mathbb{R} \setminus \{0\}$. Actually, by using Equation (2), we can get

$$bs(\hat{F}(s,r)) = (bs)_{\hat{F}(s,r)} \text{ and } cs(\hat{F}(s,r)) = (cs)_{\hat{F}(s,r)}.$$  

With a basic calculation, we can find the inverse matrix of $\hat{F}(s,r) = \{f_{nk}(s,r)\}$. The inverse matrix of $\hat{F}(s,r) = \{f_{nk}(s,r)\}$ is $\hat{F}^{-1}(s,r) = (f_{nk}^{-1}(s,r))$ such that

$$f_{nk}^{-1}(s,r) = \begin{cases} \frac{1}{s} \sum_{i=0}^{k} \left( \frac{r}{s} \right)^{n-k} f_{i+1}, & 0 \leq k < n, \\ 0, & k > n, \end{cases} \tag{4}

for all $k, n \in \mathbb{N}$. If $y = (y_n)$ is $\hat{F}(s,r)$-transform of a sequence $x = (x_n)$, then the below equality is justified:

$$y_n = (\hat{F}(s,r)x)_n = \begin{cases} sx_0, & n = 0, \\ \frac{s}{f_k} x_n + r \frac{f_{k+1}}{f_k} x_{n-1}, & n \geq 1, \end{cases} \tag{5}

for all $n \in \mathbb{N}$. In this situation, we see that $x_n = \hat{F}^{-1}(s,r)y_n$, i.e.,

$$x_n = \sum_{k=0}^{n} \frac{1}{s} \left( \frac{r}{s} \right)^{n-k} \frac{f_{k+1}}{f_k} y_k \tag{6}

for all $n \in \mathbb{N}$.

**Theorem 1.** $bs(\hat{F}(s,r))$ is the linear space with the co-ordinatewise addition and scalar multiplication.

**Proof.** We omit the proof because it is clear and easy. \(\square\)

**Theorem 2.** $cs(\hat{F}(s,r))$ is the linear space with the co-ordinatewise addition and scalar multiplication.

**Proof.** We omit the proof because it is clear and easy. \(\square\)

**Theorem 3.** The space $bs(\hat{F}(s,r))$ is a normed space with

$$\|x\|_{bs(\hat{F}(s,r))} = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left( \frac{s}{f_k} x_k + r \frac{f_{k+1}}{f_k} x_{k-1} \right) \right|. \tag{7}

**Proof.** It is clear that space $bs(\hat{F}(s,r))$ ensures normed space conditions. \(\square\)

**Theorem 4.** The space $cs(\hat{F}(s,r))$ is a normed space with norm Equation (7).

**Proof.** It is clear that normed space conditions are ensured by space $cs(\hat{F}(s,r))$. \(\square\)

**Theorem 5.** $bs(\hat{F}(s,r))$ is linearly isomorphic as isometric to the space $bs$, that is, $bs(\hat{F}(s,r)) \cong bs$.

**Proof.** For proof, we must demonstrate that bijection and linearly transformation $T$ exist between the space $bs(\hat{F}(s,r))$ and $bs$. Let us take the transformation $T : bs(\hat{F}(s,r)) \rightarrow bs$ mentioned above with the
Theorem 8. The space \( cs \) is linearly isomorphic as isometric to the space \( cs \), that is, \( cs(\hat{F}(s,r)) \cong cs \).

Proof. If we write \( cs \) instead of \( bs \) and \( cs(\hat{F}(s,r)) \) instead of \( bs(\hat{F}(s,r)) \) in Theorem 5, the proof will be demonstrated. \( \square \)

Theorem 7. The space \( bs(\hat{F}(s,r)) \) is a Banach space with the norm, which is given in Equation (7).

Proof. We can easily see that norm conditions are ensured. Let us take that \( x^i = (x_k^i) \) is a Cauchy sequence in \( bs(\hat{F}(s,r)) \) for all \( i \in \mathbb{N} \). By using Equation (5), we have

\[
y_k^i = s \frac{f_k}{f_{k+1}} x_k^i + r \frac{f_{k+1}}{f_k} x_{k-1}^i
\]

for all \( i, k \in \mathbb{N} \). Since \( x^i = (x_k^i) \) is a Cauchy sequence, for every \( \varepsilon > 0 \), there exists \( n_0 = n_0(\varepsilon) \) such that

\[
\|x^i - x^m\|_{bs(\hat{F}(s,r))} = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left( s \frac{f_k}{f_{k+1}} (x_k^i - x_k^m) + r \frac{f_{k+1}}{f_k} (x_{k-1}^i - x_{k-1}^m) \right) \right| < \varepsilon
\]

for all \( i, m \geq n_0 \). Since \( bs \) is complete, \( y^i \to y \) \( (i \to \infty) \) such that \( y \in bs \) exist and since the sequence spaces \( bs(\hat{F}(s,r)) \) and \( bs \) are linearly isomorphic as isometric \( bs(\hat{F}(s,r)) \) is complete. Consequently, \( bs(\hat{F}(s,r)) \) is a Banach space. \( \square \)

Theorem 8. The space \( cs(\hat{F}(s,r)) \) is a Banach space with the norm, which is given in Equation (7).

Proof. We can easily see that norm conditions are ensured. Let us take that \( x^i = (x_k^i) \) is a Cauchy sequence in \( cs(\hat{F}(s,r)) \) for all \( i \in \mathbb{N} \). By using Equation (5), we have

\[
y_k^i = s \frac{f_k}{f_{k+1}} x_k^i + r \frac{f_{k+1}}{f_k} x_{k-1}^i
\]
for all $i, k \in \mathbb{N}$. Since $x^i = (x^i_k)$ is a Cauchy sequence, for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that
\[
\|x^i - x^m\|_{cs(F(s,r))} = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left( s \frac{f_k}{f_{k+1}} (x^i_k - x^m_k) + r \frac{f_{k+1}}{f_k} (x^i_{k+1} - x^m_{k+1}) \right) \right| < \varepsilon
\]
for all $i, m \geq n_0$. Since $cs$ is complete, $y^i \to y (i \to \infty)$ such that $y \in cs$ exists and since the sequence spaces $cs(\hat{F}(s,r))$ and $cs$ are linearly isomorphic as isometric $cs(\hat{F}(s,r))$ is complete. Consequently, $cs(\hat{F}(s,r))$ is a Banach space. \(\square\)

Now, let $A = (a_{nk})$ be an arbitrary infinite matrix and list the following:
\[
\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty, \tag{8}
\]
\[
\lim_{k} a_{nk} = 0 \text{ for each } n \in \mathbb{N}, \tag{9}
\]
\[
\sup_{m} \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{n,k+1}) \right| < \infty, \tag{10}
\]
\[
\lim_{n} \sum_k a_{nk} = \alpha \text{ for each } k \in \mathbb{N}, \alpha \in \mathbb{C}, \tag{11}
\]
\[
\sup_{n} \sum_k |a_{nk} - a_{n,k+1}| < \infty, \tag{12}
\]
\[
\lim_{n} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \alpha_k \in \mathbb{C}, \tag{13}
\]
\[
\sup_{N,K \in F} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k+1}) \right| < \infty, \tag{14}
\]
\[
\sup_{N,K \in F} \left| \sum_{n \in N} \sum_{k \in K} (a_{nk} - a_{n,k-1}) \right| < \infty, \tag{15}
\]
\[
\lim_{n} (a_{nk} - a_{n,k+1}) = \alpha \text{ for each } k \in \mathbb{N}, \alpha \in \mathbb{C}, \tag{16}
\]
\[
\lim_{n \to \infty} \sum_k |a_{nk} - a_{n,k+1}| = \sum_k \lim_{n \to \infty} |a_{nk} - a_{n,k+1}|, \tag{17}
\]
\[
\sup_{n} \left| \lim_{k} a_{nk} \right| < \infty, \tag{18}
\]
\[
\lim_{n \to \infty} \sum_k |a_{nk} - a_{n,k+1}| = 0 \text{ uniformly in } n, \tag{19}
\]
\[
\lim_{m} \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{n,k+1}) \right| = 0, \tag{20}
\]
\[
\lim_{m} \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{n,k+1}) \right| = \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{n,k+1}) \right|, \tag{21}
\]
\[
\sup_{N,K \in F} \left| \sum_{n \in N} \sum_{k \in K} [(a_{nk} - a_{n,k+1}) - (a_{n,k} - a_{n-1,k+1})] \right| < \infty, \tag{22}
\]
\[
\sup_{m \in \mathbb{N}} \left| \lim_{k} \sum_{n=0}^{m} a_{nk} \right| < \infty. \tag{23}
\]
\[
\exists \alpha_k \in \mathbb{C} \ni \sum_{n} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}, \quad (24)
\]

\[
\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} \left( a_{nk} - a_{n-1,k} \right) - \left( a_{n,k-1} - a_{n-1,k-1} \right) \right| < \infty, \quad (25)
\]

where \( \mathcal{F} \) denote the collection of all finite subsets of \( \mathbb{N} \).

Now, we can give some matrix transformations in the following Lemma for the next step that we will need in the inclusion Theorems.

**Lemma 1.** Let \( A = (a_{nk}) \) be an arbitrary transformations infinite matrix. Then,

1. \( A = (a_{nk}) \in (bs, \ell_\infty) \iff \text{Equations (9) and (12) hold (Stieglitz and Tietz [26])} \),
2. \( A = (a_{nk}) \in (bs, c) \iff \text{Equations (12) and (13) hold (Wilansky [27])} \),
3. \( A = (a_{nk}) \in (bs, \ell_1) \iff \text{Equations (9) and (14) hold (K.-G. Grosse-Erdman [28])} \),
4. \( A = (a_{nk}) \in (cs, \ell_1) \iff \text{Equation (15) holds (Stieglitz and Tietz [26])} \),
5. \( A = (a_{nk}) \in (bs, c) \iff \text{Equations (9), (16) and (17) hold (K.-G. Grosse-Erdman [28])} \),
6. \( A = (a_{nk}) \in (cs, \ell_\infty) \iff \text{Equations (12) and (18) hold (Stieglitz and Tietz [26])} \),
7. \( A = (a_{nk}) \in (bs, c_0) \iff \text{Equations (9) and (20) hold (Zeller [29])} \),
8. \( A = (a_{nk}) \in (bs, c) \iff \text{Equations (9) and (21) hold (Zeller [29])} \),
9. \( A = (a_{nk}) \in (bs, c_0) \iff \text{Equations (9) and (22) hold (Zeller [29])} \),
10. \( A = (a_{nk}) \in (bs, bs) \iff \text{Equations (9) and (10) hold (Zeller [29])} \),
11. \( A = (a_{nk}) \in (cs, c) \iff \text{Equations (10) and (11) hold (Hill, [30])} \),
12. \( A = (a_{nk}) \in (bs, b_0) \iff \text{Equations (12), (19) and (22) hold (Stieglitz and Tietz [26])} \),
13. \( A = (a_{nk}) \in (cs, c_0) \iff \text{Equation (12) holds and Equation (13) also holds with } \alpha_k = 0 \text{ for all } k \in \mathbb{N} \) (Dienes [31]),
14. \( A = (a_{nk}) \in (cs, c_0) \iff \text{Equation (10) holds and Equation (24) also holds with } \alpha_k = 0 \text{ for all } k \in \mathbb{N} \) (Zeller [29]).
15. \( A = (a_{nk}) \in (cs, bs) \iff \text{Equations (10) and (23) hold (Zeller [29])} \),
16. \( A = (a_{nk}) \in (cs, c) \iff \text{Equation (10) holds and Equation (13) also holds with } \alpha_k = 0 \text{ for all } k \in \mathbb{N} \) (Zeller [29]).
17. \( A = (a_{nk}) \in (cs, bs) \iff \text{Equation (25) holds (Zeller [29])} \),
18. \( A = (a_{nk}) \in (cs, b_0) \iff \text{Equation (25) holds and Equation (13) also holds with } \alpha_k = 0 \text{ for all } k \in \mathbb{N} \) (Stieglitz and Tietz [26]).

**Theorem 9.** The inclusion \( bs \subset bs(\hat{F}(s,r)) \) is valid.

**Proof.** Let \( x \in bs \). We must demonstrate that \( x \in bs(\hat{F}(s,r)) \). It means that \( \hat{F}(s,r) \in (bs,bs) \). For \( \hat{F}(s,r) \in (bs,bs) \), \( \hat{F}(s,r) \) must ensure to the conditions of (11) of Lemma 1. We see that

\[
\lim_{k} f_{nk}(s,r) = 0 \text{ for each } n \in \mathbb{N}.
\]

The other condition also holds as follows:

\[
\sup_{m} \sum_{k} \left| \sum_{n=0}^{m} (f_{nk}(s,r) - f_{n,k+1}(s,r)) \right| = \sup_{m} \lim_{p} \left( \frac{|s + r|}{f_{1,f_{2}}} + \frac{|s + r|}{f_{2,f_{3}}} + \ldots + \frac{|s + r|}{f_{p+1,f_{p+2}}} \right)
\]

\[
= \frac{17}{10} |s + r| < \infty.
\]

Consequently, the conditions of (11) of Lemma 1 hold. The proof is complete. \( \square \)

**Theorem 10.** If \( |r/s| < 1/4 \), then \( bs(\hat{F}(s,r)) \subset \ell_\infty \) is valid.

**Proof.** Let \( x \in bs(\hat{F}(s,r)) \). Then, \( y = \hat{F}(s,r)x \in bs \). We must demonstrate that \( x = \hat{F}^{-1}(s,r)y \in \ell_\infty \). That is, \( \hat{F}^{-1}(s,r) \in (bs,\ell_\infty) \). For \( \hat{F}^{-1}(s,r) \in (bs,\ell_\infty), \) \( \hat{F}^{-1}(s,r) \) must satisfy the conditions of (1) of Lemma 1. It is clear that

\[
\lim_{k} f_{nk}^{-1}(s,r) = 0 \text{ for each } n \in \mathbb{N}.
\]
The other condition is also holds as follows:

\[
\sup_n \frac{1}{k} \left( |f_{nk}^{-1}(s, r) - f_{nk+1}^{-1}(s, r)| \right) \leq 2 \sup_n \frac{1}{k} \left| \left( f_{nk}^{-1}(s, r) - \frac{r}{s} \right) \right| \\
\leq \frac{4}{s} \sum_k \left( \frac{4r}{s} \right)^k < \infty.
\]

(Theorem 11. The inclusion \( cs \subset cs(\hat{F}(s, r)) \) is valid.

Proof. Let \( x \in cs \). We must demonstrate that \( x \in cs(\hat{F}(s, r)) \). It means that \( \hat{F}(s, r) \in (cs, cs) \).
For \( \hat{F}(s, r) \in (cs, cs) \), \( \hat{F}(s, r) \) must satisfy the conditions of (12) of Lemma 1. Equation (10) has been satisfied in Theorem 9. Now, we must demonstrate Equation (11). For every \( k \in \mathbb{N} \),

\[
\lim_{n \to \infty} \sum \chi f_{nk}(s, r) = \lim_n \frac{f_{n+1}}{s} + \frac{f_{n+1}}{f_n} = \frac{s}{\varphi} + r \varphi = \ell
\]

such that \( \ell \in \mathbb{C} \) exist. Consequently, the conditions of (12) of Lemma 1 hold. The proof is complete. \( \square \)

(Theorem 12. If \(|r/s| < 1/4\), then \( cs(\hat{F}(s, r)) \subset c \) is valid.

Proof. Let \( x \in cs(\hat{F}(s, r)) \). Then, \( y = \hat{F}(s, r)x \in cs \). We must demonstrate that \( x = \hat{F}^{-1}(s, r)y \in c \).
That is, \( \hat{F}^{-1}(s, r) \in (cs, c) \). For \( \hat{F}^{-1}(s, r) \in (cs, c) \), \( \hat{F}^{-1}(s, r) \) must satisfy the conditions of (2) of Lemma 1. Equation (12) has been satisfied in Theorem 10. Now, we must demonstrate Equation (13). For each \( k \in \mathbb{N} \),

\[
\lim_n f_{nk}^{-1}(s, r) \leq \lim_n \left| f_{nk}^{-1}(s, r) \right| = \lim_n \left| \frac{f_{n+1}}{s} \right| \left( -\frac{r}{s} \right)^{n-k} \left| \frac{f_{n+1}}{f_n} \right| = \lim_n \left| \frac{f_{n+1}}{s} \right| \left( \frac{4r}{s} \right)^{n-k} = \frac{\varphi}{|s|} 0 = 0.
\]

Thus, Equation (13) is also satisfied. \( \square \)

(Theorem 13. The inclusion \( cs(\hat{F}(s, r)) \subset bs(\hat{F}(s, r)) \) is valid.

Proof. Let \( x \in cs(\hat{F}(s, r)) \). Then, \( y = \hat{F}(s, r)x \in cs \). Hence, \( \sum_k \hat{F}(s, r)x \in c \subset \ell_\infty \), so it becomes \( \sum_k \hat{F}(s, r)x \in \ell_\infty \). That is, \( \hat{F}(s, r)x \in bs \). Hence, \( x \in bs(\hat{F}(s, r)) \). Consequently, \( cs(\hat{F}(s, r)) \subset bs(\hat{F}(s, r)) \).

Before giving the corollary about the Schauder basis for the space \( cs(\hat{F}(r, s)) \), let us define the Schauder basis which was introduced by J. Schauder in 1927. Let \( (X, ||\cdot||) \) be a normed space and be a sequence \( (a_k) \in X \). There exists a unique sequence \( (\hat{a}_k) \) of scalars such that \( x = \sum_{k=0}^{\infty} \lambda_k a_k \), and

\[
\lim_{n \to \infty} \left| x - \sum_{k=0}^{n} \lambda_k a_k \right| = 0.
\]

Then, \((a_k)\) is called a Schauder basis for \( X \). \( \square \)

Now, we can give the corollary about Schauder basis.
Corollary 1. Let us sequence $b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}}$ defined in the $cs(\hat{F}(s,r))$ such that

$$b_n^{(k)} = \begin{cases} \frac{1}{2}(-\frac{r}{s})^{n-k} \frac{F_{k+1}}{n!} , & n > k, \\ \frac{1}{2} \frac{F_{k+1}}{n!} , & n = k, \\ 0, & n < k. \end{cases}$$

Then, sequence $\{b^{(k)}\}_{n \in \mathbb{N}}$ is a basis of $cs(\hat{F}(s,r))$ and every sequence $x \in cs(\hat{F}(s,r))$ has a unique representation $x = \sum_k y_k b^k$, where $y_k = (\hat{F}(s,r)x)_k$.

3. The $\alpha$, $\beta$- and $\gamma$-Duals of the Spaces $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ and Some Matrix Transformations

In this section, the alpha-, beta-, gamma-duals of the spaces $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ are determined and characterized the classes of infinite matrices from the space $bs(\hat{F}(s,r))$ and $cs(\hat{F}(s,r))$ to some other sequence spaces.

Now, we give the two lemmas to prove the theorems that will be given in the next stage.

Lemma 2. Suppose that $a = (a_n) \in w$ and the infinite matrix $B = (b_{nk})$ is defined by $B_n = a_n(\delta^{-1}(s,r))_n$, that is,

$$b_{nk} = \begin{cases} a_nf_{nk}^{-1}(s,r), & 0 \leq k < n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$, $\delta \in \{bs, cs\}$. Then, $a \in \{\delta(\hat{F}(s,r))\}^a$ iff $B \in (\delta, \ell_1)$. 

Proof. Let $a = (a_n)$ and $x = (x_n)$ be an arbitrary subset of $w$. $y = (y_n)$ such that $y = \hat{F}(s,r)x$, which is defined by Equation (5). Then,

$$a_nx_n = a_n(\delta^{-1}(s,r)y)_n = (By)_n$$

for all $n \in \mathbb{N}$. Hence, we obtain by Equation (5) that $ax = (a_nx_n) \in \ell_1$ with $x = (x_n) \in \delta(\hat{F}(s,r))$ iff $By \in \ell_1$ with $y \in \delta$. That is, $B \in (\delta, \ell_1)$. □

Lemma 3. Let $[32] C = (c_{nk})$ be defined via a sequence $a = (a_k) \in w$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $U = (u_{nk})$ by

$$c_{nk} = \begin{cases} \sum_{j=k}^{n-1} a_{nj} v_{jk}, & 0 \leq k < n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, for any sequence space $\lambda$,

$$\lambda_U^\gamma = \{a = (a_k) \in w : C \in (\lambda, \ell_\infty)\},$$

$$\lambda_U^\beta = \{a = (a_k) \in w : C \in (\lambda, c)\}.$$ 

If we consider Lemmas 1–3 together, the following is obtained.

Corollary 1. Let $B = (b_{nk})$ and $C = (c_{nk})$ such that

$$b_{nk} = \begin{cases} a_nf_{nk}^{-1}(s,r), & 0 \leq k < n, \\ 0, & k > n, \end{cases}$$

and $c_{nk} = \sum_{j=k}^{n} \frac{1}{2}(-\frac{r}{s})^{-j-k} \frac{F_{k+1}}{F_{j}F_{j+1}} a_j,$

then $\lambda_U^\gamma = \{a = (a_k) \in w : C \in (\lambda, \ell_\infty)\}$ and $\lambda_U^\beta = \{a = (a_k) \in w : C \in (\lambda, c)\}$. 

If we take $t_1, t_2, t_3, t_4, t_5, t_6, t_7$ and $t_8$ as follows:

\[
\begin{align*}
  t_1 &= \left\{ a = (a_k) \in w : \sup_{N, K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (b_{nk} - b_{n,k+1}) \right| < \infty \right\}, \\
  t_2 &= \left\{ a = (a_k) \in w : \sup_{N, K \in \mathcal{F}} \left| \sum_{n \in N} \sum_{k \in K} (b_{nk} - b_{n,k-1}) \right| < \infty \right\}, \\
  t_3 &= \left\{ a = (a_k) \in w : \lim_{k \to \infty} c_{nk} = 0 \right\}, \\
  t_4 &= \left\{ a = (a_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_{n \to \infty} (c_{nk} - c_{n,k+1}) = \alpha \right\}, \\
  t_5 &= \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_k |c_{nk} - c_{n,k+1}| = \sum_k \lim_{n \to \infty} |c_{nk} - c_{n,k+1}| \right\}, \\
  t_6 &= \left\{ a = (a_k) \in w : \exists \alpha \in \mathbb{C} \ni \lim_{n \to \infty} c_{nk} = \alpha, \text{ for all } k \in \mathbb{N} \right\}, \\
  t_7 &= \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_k |c_{nk} - c_{n,k+1}| < \infty \right\}, \\
  t_8 &= \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \lim_{k} |c_{nk}| < \infty \right\}.
\end{align*}
\]

Then, the following statements hold:

1. $\{bs(\tilde{f}(s, r))\}^a = t_1$,
2. $\{cs(\tilde{f}(s, r))\}^a = t_2$,
3. $\{bs(\tilde{f}(s, r))\}^b = t_3 \cap t_4 \cap t_5$,
4. $\{cs(\tilde{f}(s, r))\}^b = t_6 \cap t_7$,
5. $\{bs(\tilde{f}(s, r))\}^c = t_3 \cap t_7$,
6. $\{cs(\tilde{f}(s, r))\}^c = t_7 \cap t_8$.

**Theorem 14.** Let $\lambda \in \{bs, cs\}$ and $\mu \subset w$. Then, $A = (a_{nk}) \in (\lambda(\tilde{f}(s, r)), \mu)$ iff

\[D^m = (d_{nk}^{(m)}) \in (\lambda, c) \text{ for all } n \in \mathbb{N},\]

\[D = (d_{nk}) \in (\lambda, \mu),\]

where

\[d_{nk}^{(m)} = \begin{cases}
    \sum_{j=k}^{m} \frac{1}{s} (-\frac{r}{s})^j - k \frac{f_{j+1}^2}{f_{j+1}} a_{nj}, & 0 \leq k < m, \\
    0, & k > m,
\end{cases}\]

and

\[d_{nk} = \sum_{j=k}^{\infty} \frac{1}{s} (-\frac{r}{s})^j - k \frac{f_{j+1}^2}{f_{j+1}} a_{nj}\]

for all $k, m, n \in \mathbb{N}$.

**Proof.** To prove the necessary part of the theorem, let us suppose that $A = (a_{nk}) \in (\lambda(\tilde{f}(s, r), \mu)$ and $x = (x_k) \in \lambda(\tilde{f}(s, r))$. By using Equation (6), we find

\[\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} a_{nk} \sum_{j=0}^{k} \frac{1}{s} (-\frac{r}{s})^j - k \frac{f_{j+1}^2}{f_{j+1}} y_j\]

\[= \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{1}{s} (-\frac{r}{s})^j - k \frac{f_{j+1}^2}{f_{j+1}} a_{nj} y_k = \sum_{k=0}^{m} d_{nk}^{(m)} y_k = D_{n}^{(m)}(y)\]
for all $m, n \in \mathbb{N}$. For each $m \in \mathbb{N}$ and $x = (x_k) \in \lambda(\hat{F}(s, r))$, $A_n(x)$ exists and also lies in $c$. Then, $D^{(m)}_c$ also lies in $c$ for each $m \in \mathbb{N}$. Hence, $D^{(m)} \in (\lambda, c)$. Now, from Equation (32), we consider for $m \to \infty$, and then $Ax = Dy$. Consequently, we obtain $D = (d_{nk}) \in (\lambda, \mu)$.

If we want to prove the sufficient part of the theorem, then let us assume that Equations (28) and (29) are satisfied and $x = (x_k) \in \lambda(\hat{F}(s, r))$. By using Corollary 1 and Equations (28) and (32), we obtain $y = \hat{F}(s, r)x \in \lambda$ and $D^{(m)}(y) = \sum_{k=0}^{m} d^{(m)}_{nk} y_k = \sum_{k=0}^{m} a_{nk} x_k = A^{(m)}(x) \in c$. Hence, $A = (a_{nk})_{k \in \mathbb{N}}$ exists. In addition, in Equation (32), if we consider $m \to \infty$. Then, $Ax = Dy$. Consequently, we obtain $A = (a_{nk}) \in (\lambda(\hat{F}(s, r)), \mu)$.

In Theorem 14, we take $\lambda(\hat{F}(s, r))$ instead of $\mu$ and $\mu$ instead of $\lambda(\hat{F}(s, r))$, and then we get the following theorem. □

**Theorem 15.** Let $\lambda \in \{bs, cs\}$ and $\mu$ be an arbitrary subset of $w$ and $A = (a_{nk})$ and $B = (b_{nk})$ be infinite matrices. If we take

$$b_{nk} := r \frac{f_{n+1}}{f_n} a_{n-1,k} + s \frac{f_n}{f_{n+1}} a_{nk}$$

for all $k, n \in \mathbb{N}$, then $A \in (\mu, \lambda(\hat{F}(s, r)))$ iff $B \in (\mu, \lambda)$.

**Proof.** Let us suppose that $A \in (\mu, \lambda(\hat{F}(s, r)))$ and Equation (33) exist. For $z = (z_k) \in \mu$, we obtain $Az \in \lambda(\hat{F}(s, r))$ from $A \in (\mu, \lambda(\hat{F}(s, r)))$. Hence, $\hat{F}(s, r)(Az) \in \lambda$. On the other hand, we have

$$\sum_{k=0}^{m} b_{nk} z_k = \sum_{k=0}^{m} \left( r \frac{f_{n+1}}{f_n} a_{n-1,k} + s \frac{f_n}{f_{n+1}} a_{nk} \right) z_k$$

for all $m, n \in \mathbb{N}$. If we carry out $m \to \infty$ to Equation (34), we obtain that

$$(Bz)_n = \left( (\hat{F}(s, r)A)z \right)_n = \left( \hat{F}(s, r)(Az) \right)_n.$$  (35)

Since $\hat{F}(s, r)(Az) \in \lambda$, we find $Bz = (Bz)_n \in \lambda$ for $z = (z_k) \in \mu$ from Equation (35). Hence, we obtain that $B \in (\mu, \lambda).$ This is the desired result. □

At this stage, let us consider almost convergent sequences spaces, which were given by Lorentz [33]. This is because they will help in calculating some of the results of Theorems 14 and 15. Let a sequence $x = (x_k) \in \ell_{\infty}$. $x$ is said to be almost convergent to the generalized limit $\ell$ iff $\lim_{m \to \infty} \frac{\sum_{k=0}^{m} x_{n+k}}{m+1} = \ell$ uniformly in $n$ and is denoted by $f - \lim x = \ell$. By $f$ and $f_0$, we indicate the space of all almost convergent and almost null sequences, respectively. However, in this article, we use $\hat{c}$ and $\hat{c}_0$ instead of $f$ and $f_0$, respectively, in order to avoid any confusion. This is because the Fibonacci sequence is also denoted by $f$. In addition, by $\hat{c}s$, we indicate the space of sequences, which is composed of all almost convergent series. The sequences spaces $\hat{c}$ and $\hat{c}_0$ are

$$\hat{c}_0 = \left\{ x = (x_k) \in \ell_{\infty} : \lim_{m \to \infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n \right\},$$

$$\hat{c} = \left\{ x = (x_k) \in \ell_{\infty} : \exists \ell \in \mathbb{C} \ni \lim_{m \to \infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = \ell \text{ uniformly in } n \right\}.$$

Now, let $A = (a_{nk})$ be an arbitrary infinite matrix and list the following conditions:

$$\exists a_k \in \mathbb{C} \ni f - \lim a_{nk} = a_k \text{ for each } k \in \mathbb{N},$$  (36)

$$\lim_{q \to \infty} \sum_{k=0}^{q} \frac{1}{q+1} \sum_{j=0}^{q} \left| \sum_{i=j}^{n+i} (a_{jk} - a_k) \right| = 0 \text{ uniformly in } n,$$  (37)
\[
\sup_{n \in \mathbb{N}} \sum_{k} \left| \Delta \left[ \sum_{j=0}^{n} a_{jk} \right] \right| < \infty, \tag{38}
\]

\[
\exists \alpha_k \in \mathbb{C} \ni \lim_{n} \sum_{j=0}^{n} a_{jk} = \alpha_k \text{ for each } k \in \mathbb{N}, \tag{39}
\]

\[
\sup_{n \in \mathbb{N}} \sum_{k} \left| \sum_{j=0}^{n} a_{jk} \right| < \infty, \tag{40}
\]

\[
\exists \alpha_k \in \mathbb{C} \ni \sum_{n} a_{nk} = \alpha_k \text{ for all } k \in \mathbb{N}, \tag{41}
\]

\[
\lim_{n} \sum_{k} \left| \Delta \left[ \sum_{j=0}^{n} (a_{jk} - \alpha_k) \right] \right| = 0, \tag{42}
\]

\[
\sup_{n \in \mathbb{N}} \sum_{k} \left| \sum_{j=0}^{n} a_{jk} \right|^q < \infty, \quad q = \frac{p}{p-1}, \tag{43}
\]

\[
\sup_{m, n \in \mathbb{N}} \left| \sum_{j=0}^{n} a_{nk} \right| < \infty, \tag{44}
\]

\[
\sup_{m, l \in \mathbb{N}} \left| \sum_{n=0}^{m} \sum_{l=0}^{n} a_{nk} \right| < \infty, \tag{45}
\]

\[
\sup_{m, l \in \mathbb{N}} \left| \sum_{n=0}^{m} \sum_{l=0}^{n} a_{nk} \right| < \infty, \tag{46}
\]

\[
\lim_{m} \sum_{k} \left| \sum_{n=m}^{\infty} a_{nk} \right| = 0, \tag{47}
\]

\[
\sum_{n} a_{nk}, \text{ convergent}, \tag{48}
\]

\[
\lim_{m \to \infty} \sum_{n=0}^{m} (a_{nk} - a_{n,k+1}) = \alpha, \text{ for each } k \in \mathbb{N}, \alpha \in \mathbb{C}. \tag{49}
\]

Let us give some matrix transformations in the following Lemma for use in the next step.

**Lemma 4.** Let \( A = (a_{nk}) \) be an infinite matrix for all \( k, n \in \mathbb{N} \). Then,

1. \( A = (a_{nk}) \in \left( \hat{c}, cs \right) \) iff Equations (24) and (40)–(42) hold (Başar [34]).
2. \( A = (a_{nk}) \in \left( cs, \hat{c} \right) \) iff Equations (12) and (36) hold (Başar and Çolak [35]).
3. \( A = (a_{nk}) \in \left( bs, \hat{c} \right) \) iff Equations (9), (12), (36) and (37) hold (Başar and Solak [36]).
4. \( A = (a_{nk}) \in \left( bs, cs \right) \) iff Equations (9) and (37)–(39) hold (Başar and Solak [36]).
5. \( A = (a_{nk}) \in \left( \hat{c}, cs \right) \) iff Equations (38) and (39) hold (Başar and Çolak [35]).
6. \( A = (a_{nk}) \in \left( \ell_{\infty}, bs \right) = (c, bs) = (c_0, bs) \) iff Equation (40) holds (Zeller [29]).
7. \( A = (a_{nk}) \in \left( \ell_{p}, bs \right) \) iff Equation (43) holds (Jakimovski and Russell [37]).
8. \( A = (a_{nk}) \in \left( \ell, bs \right) \) iff Equation (44) holds (Zeller [29]).
9. \( A = (a_{nk}) \in \left( bv, bs \right) \) iff Equation (45) holds (Zeller [29]).
10. \( A = (a_{nk}) \in \left( \ell_{p,0}, bs \right) \) iff Equation (46) holds (Jakimovski and Russell [37]).
11. \( A = (a_{nk}) \in \left( \ell_{\infty, cs} \right) \) iff Equation (47) holds (Zeller [29]).
12. \( A = (a_{nk}) \in \left( c, cs \right) \) iff Equations (11), (40) and (48) hold (Zeller [29]).
13. \( A = (a_{nk}) \in \left( c_{0}, cs \right) \) iff Equations (10) and (49) hold (Zeller [29]).
14. \( A = (a_{nk}) \in \left( \ell_{p, cs} \right) \) iff Equations (11) and (43) hold (Jakimovski and Russell [37]).
15. \( A = (a_{nk}) \in \left( \ell, cs \right) \) iff Equations (11) and (44) hold (Jakimovski and Russell [37]).
16. \( A = (a_{nk}) \in \left( bv, cs \right) \) iff Equations (11), (44) and (46) hold (Zeller [29]).
Now, let us list the following condition, where $d_{nk}$ and $d_{nk}^{(m)}$ are taken as in Equations (30) and (31):

\[
\lim_{k \to \infty} d_{nk}^{(m)} = 0 \text{ for all } n \in \mathbb{N},
\]

(50)

\[
\exists d_{nk} \in \mathbb{C} \ni \lim_{n \to \infty} (d_{nk}^{(m)} - d_{nk+1}^{(m)}) = d_{nk} \text{ for all } k, n \in \mathbb{N},
\]

(51)

\[
\lim_{n \to \infty} \sum_{k} |d_{nk}^{(m)} - d_{nk+1}^{(m)}| < \infty \text{ uniformly in } n,
\]

(52)

\[
\lim_{k} d_{nk} = 0 \text{ for all } n \in \mathbb{N},
\]

(53)

\[
\sup_{n} \sum_{k} |d_{nk} - d_{n,k+1}| < \infty,
\]

(54)

\[
\exists d_{k} \in \mathbb{C} \ni \lim_{n \to \infty} (d_{nk} - d_{n,k+1}) = d_{k} \text{ for all } k, n \in \mathbb{N},
\]

(55)

\[
\exists \alpha \in \mathbb{C} \ni \lim_{n \to \infty} \sum_{k} |d_{nk} - d_{n,k+1}| = \alpha \text{ uniformly in } n,
\]

(56)

\[
\sup_{m \in \mathbb{N}} \left| \sum_{n=0}^{m} (d_{nk} - d_{n,k+1}) \right| < \infty,
\]

(57)

\[
\lim_{m} \sum_{k} \left| \sum_{n=0}^{m} (d_{nk} - d_{n,k+1}) \right| = \sum_{k} \left| \sum_{n} (d_{nk} - d_{n,k+1}) \right|,
\]

(58)

\[
\lim_{m} \sum_{k} \left| \sum_{n=0}^{m} (d_{nk} - d_{n,k+1}) \right| = 0,
\]

(59)

\[
\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{K} (d_{nk} - d_{n,k+1}) \right| < \infty,
\]

(60)

\[
\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{K} (d_{nk} - d_{n,k+1}) - (d_{n-1,k} - d_{n-1,k+1}) \right| < \infty,
\]

(61)

\[
\sup_{n} \sum_{k} |d_{nk}^{(m)} - d_{nk+1}^{(m)}| < \infty,
\]

(62)

\[
\exists d_{k} \in \mathbb{C} \ni \lim_{n \to \infty} d_{nk}^{(m)} = d_{k} \text{ for all } k, n \in \mathbb{N},
\]

(63)

\[
\sup_{n \in \mathbb{N}} \left| \lim_{k} d_{nk} \right| < \infty,
\]

(64)

\[
\exists d_{k} \in \mathbb{C} \ni \lim_{n \to \infty} d_{nk} = d_{k} \text{ for all } k, n \in \mathbb{N},
\]

(65)

\[
\sup_{m \in \mathbb{N}} \left| \lim_{k} \sum_{n=0}^{m} d_{nk} \right| < \infty,
\]

(66)

\[
\sup_{m \in \mathbb{N}} \sum_{k} \left| \sum_{n=0}^{m} (d_{nk} - d_{n,k-1}) \right| < \infty,
\]

(67)

\[
\exists d_{k} \in \mathbb{C} \ni \sum_{n} d_{nk} = d_{k} \text{ for each } k \in \mathbb{N},
\]

(68)

\[
\sup_{N,K \in \mathcal{F}, n \in \mathbb{N}} \left| \sum_{k \in \mathbb{K}} (d_{nk} - d_{n,k-1}) \right| < \infty,
\]

(69)
\[
\exists d_k \in \mathbb{C} \ni f - \lim_{n \to \infty} d_{nk} = d_k \text{ for each } k \in \mathbb{N}, \tag{70}
\]

\[
\sup_{N,K \in \mathcal{F}} \left| \sum_{n \in N, k \in K} (d_{nk} - d_{n-1,k}) - (d_{n,k-1} - d_{n-1,k-1}) \right| < \infty, \tag{71}
\]

\[
\lim \sum_{n=0}^{\infty} \frac{1}{n} \sum_{i=0}^{n} \left| \Delta \left[ \sum_{j=0}^{n} (d_{jk} - a_k) \right] \right| = 0 \text{ uniformly in } n, \tag{72}
\]

\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=0}^{n} d_{jk} \right| < \infty, \tag{73}
\]

\[
\exists d_k \in \mathbb{C} \ni \sum_{n} d_{nk} = d_k \text{ for all } k \in \mathbb{N}, \tag{74}
\]

\[
\lim_{n \to \infty} \sum_{k} \left| \Delta \left[ \sum_{j=0}^{n} (d_{jk} - a_k) \right] \right| = 0, \tag{75}
\]

\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=0}^{n} d_{jk} \right| < \infty, \tag{76}
\]

\[
\exists d_k \in \mathbb{C} \ni f - \lim_{n \to \infty} \sum_{j=0}^{n} d_{jk} = d_k \text{ for each } k \in \mathbb{N}, \tag{77}
\]

Now, we can give several conclusions of Theorems 14 and 15, and Lemmas 1 and 4.

**Corollary 2.** Let \( A = (a_{nk}) \) be an infinite matrix for all \( k, n \in \mathbb{N} \). Then,

1. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), c_0) \) iff Equations (50)–(53) hold and Equation (56) also holds with \( \alpha = 0 \).
2. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), c_0) \) iff Equations (50)–(53) and (59) hold.
3. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), c) \) iff Equations (50)–(53), (55) and (56) hold.
4. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), cs) \) iff Equations (50)–(53) and (58) hold.
5. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), \ell_\infty) \) iff Equations (50)–(54) hold.
6. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), bs) \) iff Equations (50)–(53) and (57) hold.
7. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), \ell_1) \) iff Equations (50)–(53) and (60) hold.
8. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), bv) \) iff Equations (50)–(53) and (61) hold.
9. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), bv_0) \) iff Equations (50)–(52), (54) and (61) hold and Equation (56) also holds with \( \alpha = 0 \).

**Corollary 3.** Let \( A = (a_{nk}) \) be an infinite matrix for all \( k, n \in \mathbb{N} \). Then,

1. \( A = (a_{nk}) \in (cs(\hat{F}(s,r), c_0) \) iff Equations (54), (62) and (63) hold and Equation (65) also holds with \( d_k = 0 \) for all \( k \in \mathbb{N} \).
2. \( A = (a_{nk}) \in (cs(\hat{F}(s,r), c_0) \) iff Equations (57), (62) and (63) hold and Equation (68) also holds with \( d_k = 0 \) for all \( k \in \mathbb{N} \).
3. \( A = (a_{nk}) \in (cs(\hat{F}(s,r), c) \) iff Equations (54), (62), (63) and (65) hold.
4. \( A = (a_{nk}) \in (cs(\hat{F}(s,r), cs) \) iff Equations (62), (63), (67) and (68) hold.
5. \( A = (a_{nk}) \in (cs(\hat{F}(s,r), \ell_\infty) \) iff Equations (54) and (62)–(64) hold.
6. \( A = (a_{nk}) \in (cs(\hat{F}(s,r), bs) \) iff Equations (57), (62), (63) and (66) hold.
7. \( A = (a_{nk}) \in (cs(\hat{F}(s,r), \ell_1) \) iff Equations (62), (63) and (69) hold.
8. \( A = (a_{nk}) \in (cs(\hat{F}(s,r), bv) \) iff Equations (62), (63) and (71) hold.
9. \( A = (a_{nk}) \in (cs(\hat{F}(s,r), bv_0) \) iff Equations (62), (63) and (65) hold and Equation (71) also holds with \( d_k = 0 \) for all \( k \in \mathbb{N} \).

**Corollary 4.** Let \( A = (a_{nk}) \) be an infinite matrix for all \( k, n \in \mathbb{N} \). Then,

1. \( A = (a_{nk}) \in (bs(\hat{F}(s,r), \ell) \) iff Equations (50)–(54), (70) and (72) hold.
The Fibonacci difference matrix \( \hat{F} \) is derived from the Fibonacci sequence \( F \), which is introduced for the first time in the literature by Kizmaz [38].

The Fibonacci difference matrix \( \hat{F} \), which is derived from the Fibonacci sequence \( F \), was recently introduced by Kara [23] in 2013 and defined the new sequence spaces \( \ell_p(\hat{F}) \) and \( \ell_\infty(\hat{F}) \), which are derived by the matrix domain of \( \hat{F} \) from the sequence spaces \( \ell_p \) and \( \ell_\infty \), respectively, where \( 1 \leq p < \infty \).

Corollary 5. Let \( A = (a_{nk}) \) be an infinite matrix for all \( k, n \in \mathbb{N} \). Then,

\[
(1) \quad A = (a_{nk}) \in (\ell_\infty, bs(\hat{F}, s)) = (c, bs) = (c_0, bs) \text{ iff Equation (40) holds with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by (33)).}
\]

\[
(2) \quad A = (a_{nk}) \in (\ell_p, bs(\hat{F}, s)) \text{ iff Equation (43) holds with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by (33)).}
\]

\[
(3) \quad A = (a_{nk}) \in (\ell, bs(\hat{F}, s)) \text{ iff Equation (44) holds with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

\[
(4) \quad A = (a_{nk}) \in (\ell, bs(\hat{F}, s)) \text{ iff Equation (45) holds with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

\[
(5) \quad A = (a_{nk}) \in (\ell, bs(\hat{F}, s)) \text{ iff Equation (46) holds with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

\[
(6) \quad A = (a_{nk}) \in (\ell_\infty, cs(\hat{F}, s)) \text{ iff Equation (47) holds with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

\[
(7) \quad A = (a_{nk}) \in (\ell_\infty, cs(\hat{F}, s)) \text{ iff Equation (11), (40) and (48) hold with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

\[
(8) \quad A = (a_{nk}) \in (\ell_\infty, cs(\hat{F}, s)) \text{ iff Equation (11) and (43) hold with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

\[
(9) \quad A = (a_{nk}) \in (\ell_\infty, cs(\hat{F}, s)) \text{ iff Equation (11) and (43) hold with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

\[
(10) \quad A = (a_{nk}) \in (\ell_\infty, cs(\hat{F}, s)) \text{ iff Equation (11) and (43) hold with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

\[
(11) \quad A = (a_{nk}) \in (\ell_\infty, cs(\hat{F}, s)) \text{ iff Equation (11) and (43) hold with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

\[
(12) \quad A = (a_{nk}) \in (\ell_\infty, cs(\hat{F}, s)) \text{ iff Equation (11) and (43) hold with } b_{nk} \text{ instead of } a_{nk}, \text{ where } b_{nk} \text{ is defined by Equation (33)).}
\]

4. Discussion

The difference sequence operator was introduced for the first time in the literature by Kizmaz [38]. Kirişçi and Başar [4] have characterized and investigated generalized difference sequence spaces. The Fibonacci difference matrix \( \hat{F} \), which is derived from the Fibonacci sequence \( F \), was recently introduced by Kara [23] in 2013 and defined the new sequence spaces \( \ell_p(\hat{F}) \) and \( \ell_\infty(\hat{F}) \), which are derived by the matrix domain of \( \hat{F} \) from the sequence spaces \( \ell_p \) and \( \ell_\infty \), respectively, where \( 1 \leq p < \infty \).

Candan [25] in 2015 introduced the sequence spaces \( c(\hat{F}, s) \) and \( \ell_p(\hat{F}, s) \). Later, Candan and Kara [15] studied the sequence spaces \( \ell_p(\hat{F}, s) \) in which \( 1 \leq p \leq \infty \). In addition, Kara et al. [24] have characterized some class of compact operators in the spaces \( \ell_p(\hat{F}) \) and \( \ell_\infty(\hat{F}) \), where \( 1 \leq p < \infty \).

The study is concerned with matrix domain on the sequences space of a triangle infinite matrix. In this article, we defined spaces \( bs(\hat{F}, s) \) and \( cs(\hat{F}, s) \) of Generalized Fibonacci difference of sequences, which constituted bounded and convergence series, respectively. We have demonstrated the sets of \( bs(\hat{F}, s) \) and \( cs(\hat{F}, s) \), which are the linear spaces, and both spaces have the same norm

\[
\|x\| = \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \left( s \frac{f_k}{f_{k+1}} x_k + r \frac{f_{k+1}}{f_k} x_{k-1} \right) \right|
\]
where \( x \in bs(\hat{F}(s, r)) \) or \( x \in cs(\hat{F}(s, r)) \). In addition, it was shown that they are normed space and Banach spaces. It was found that \( bs(\hat{F}(s, r)) \) and \( bs \) are linearly isomorphic as isometric. At the same time, \( cs(\hat{F}(s, r)) \) and \( cs \) are linearly isomorphic as isometric. Some inclusions’ theorems were given with respect to \( bs(\hat{F}(s, r)) \) and \( cs(\hat{F}(s, r)) \). According to this, inclusions \( bs \subset bs(\hat{F}(s, r)) \), \( cs \subset cs(\hat{F}(s, r)) \) are valid. In addition, if \( |r/s| < 1/4 \), then \( bs(\hat{F}(s, r)) \subset \ell_\infty \) and \( cs(\hat{F}(s, r)) \subset c \) are valid. It was concluded that \( cs(\hat{F}(s, r)) \) has a Schauder basis.

Finally, the \( \alpha \), \( \beta \) - and \( \gamma \)-duals of the both spaces are calculated and some matrix transformations of them were given.

5. Conclusions

In this article, we have defined spaces \( bs(\hat{F}(s, r)) \) and \( cs(\hat{F}(s, r)) \) of Generalized Fibonacci difference of sequences, which constituted bounded and convergence series, respectively. We have demonstrated that the sets of \( bs(\hat{F}(s, r)) \) and \( cs(\hat{F}(s, r)) \) are the linear spaces and both spaces have the same norm. In addition, it was shown that they are Banach spaces. Some inclusions theorems were given with respect to \( bs(\hat{F}(s, r)) \) and \( cs(\hat{F}(s, r)) \). It was concluded that \( cs(\hat{F}(s, r)) \) has a Schauder basis. Finally, the \( \alpha \), \( \beta \) - and \( \gamma \)-duals of the both spaces were calculated and some matrix transformations of them were given.

Author Contributions: Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

Funding: This research received no external funding

Acknowledgments: The authors are grateful to the responsible editor and the anonymous reviewers for their valuable comments and suggestions, which have greatly improved this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

- iff: if and only if

References

1. Altay, B.; Başar, F. On some Euler sequence spaces of nonabsolute type. *Ukranian Math. J.* 2005, 57, 3–17. [CrossRef]
2. Malkowsky, E.; Savaş, E. Matrix transformations between sequence spaces of generalized weighted mean. *Appl. Math. Comput.* 2004, 147, 333–345.
3. Aydın, C.; Başar, F. On the new sequence spaces which include the spaces \( c_0 \) and \( c \) *Hokkaido Math. J.* 2004, 33, 1–16. [CrossRef]
4. Kirişçi, M.; Başar, F. Some new sequence spaces derived by the domain of generalized difference matrix. *Comput. Math. Appl.* 2010, 60, 1299–1309. [CrossRef]
5. Şengönül, M.; Başar, F. Some new Cesáro sequence spaces of non-absolute type which include the spaces \( c_0 \) and \( c \) *Soochow J. Math.* 2005, 31, 107–119.
6. Altay, B.; Başar, F. Some paranormed Riesz sequence spaces of non-absolute type. *Southeast Asian Bull. Math.* 2006, 30, 591–608.
7. Mursaleen, M.; Noman, A.K. On the spaces of \( \lambda \)-convergent and bounded sequences. *Thai J. Math.* 2010, 8, 311–329.
8. Candan, M. Domain of the double sequential band matrix in the classical sequence spaces. *J. Inequal. Appl.* 2012, 2012, 281. [CrossRef]
9. Candan, M.; Kayaduman, K. Almost convergent sequence space derived by generalized Fibonacci matrix and Fibonacci core. *Br. J. Math. Comput. Sci.* 2015, 7, 150–167. [CrossRef]
10. Candan, M. Almost convergence and double sequential band matrix. Acta Math. Sci. 2014, 34, 354–366. [CrossRef]
11. Candan, M. A new sequence space isomorphic to the space $\ell(p)$ and compact operators. J. Math. Comput. Sci. 2014, 4, 306–334.
12. Candan, M. Domain of the double sequential band matrix in the spaces of convergent and null sequences. Adv. Differ. Equ. 2014, 2014, 163. [CrossRef]
13. Candan, M.; Güneş, A. Paranormed sequence space of non-absolute type founded using generalized difference matrix. Proc. Nat. Acad. Sci. India Sect. A 2015, 85, 269–276. [CrossRef]
14. Candan, M. Some new sequence spaces derived from the spaces of bounded, convergent and null sequences. Int. J. Mod. Math. Sci. 2014, 12, 74–87.
15. Candan, M.; Kara, E.E. A study on topological and geometrical characteristics of new Banach sequence spaces. Gulf J. Math. 2015, 3, 67–84.
16. Candan, M.; Kılınç, G. A different look for paranormed Riesz sequence space derived by Fibonacci matrix. Konuralp J. Math. 2015, 3, 62–76.
17. Şengönül, M.; Kayaduman, K. On the Riesz almost convergent sequence space. Abstr. Appl. Anal. 2012, 2012, 18. [CrossRef]
18. Kayaduman, K.; Şengönül, M. The spaces of Cesàro almost convergent sequences and core theorems. Acta Math. Sci. 2012, 32, 2265–2278. [CrossRef]
19. Çakan, C.; Çoşkun, H. Some new inequalities related to the invariant means and uniformly bounded function sequences. Appl. Math. Lett. 2007, 20, 605–609. [CrossRef]
20. Çoşkun, H.; Çakan, C. A class of statistical and $\sigma$-conservative matrices. Czechoslov. Math. J. 2005, 55, 791–801. [CrossRef]
21. Çoşkun, H.; Çakan, C.M. On the statistical and $\sigma$-cores. Stud. Math. 2003, 154, 29–35. [CrossRef]
22. Kayaduman, K.; Furkan, H. Infinite matrices and $\sigma^{(A)}$-core. Demonstr. Math. 2006, 39, 531–538. [CrossRef]
23. Kara, E.E. Some topological and geometrical properties of new Banach sequence spaces. J. Inequal. Appl. 2013, 2013, 38. [CrossRef]
24. Kara, E.E.; Başarır, M.; Mursaleen, M. Compact operators on the Fibonacci difference sequence spaces $\ell_p^{(\wedge)}$ and $\ell_\infty^{(\wedge)}$. In Proceedings of the 1st International Eurasian Conference on Mathematical Sciences and Applications, Pristhine, Kosovo, 3–7 September 2012.
25. Candan, M. A new approach on the spaces of generalized Fibonacci difference null and convergent sequences. Math. Aeterna 2015, 5, 191–210.
26. Stieglitz, M.; Tietz, H. Matrix transformationen von folgenräumen eine ergebnisübersicht. Math. Z. 1997, 154, 1–16. [CrossRef]
27. Wilansky, A. Summability through Functional Analysis, North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 1984; Volume 85.
28. Grosse-Erdman, K.-G. Matrix transformations between the sequence space of Maddox. J. Math Anal. Appl. 1993, 180, 223–238. [CrossRef]
29. Zeller, K. Allgemeine Eigenschaften von Limitierungsverfahren die auf Matrixtransformationen beruhen Wissenschaftliche Abhandlung. Math. Z. 1951, 53, 463–487. [CrossRef]
30. Hill, J.D. On the space ($\gamma$) of convergent series. Tohoku Math. J. 1939. Available online: https://www.jstage.jst.go.jp/article/tmj1911/45/0/45_0_332/_article/-char/ja/.
31. Dienes, P. An Introduction to the Theory of Functions of a Complex Variable; Taylor Series; Clarendon Press: Oxford, UK; Dover, NY, USA, 1957.
32. Altay, B.; Başar, F. Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space. J. Math. Anal. Appl. 2007, 336, 632–645. [CrossRef]
33. Lorentz, G.G. A contribution to the theory of divergent sequences. Acta Math. 1948, 80, 167–190. [CrossRef]
34. Başar, F. Strongly-conservative sequence-to-series matrix transformations. Ercline Üniversitesi Fen Bilimleri Dergisi 1989, 5, 888–893.
35. Başar, F.; Çolak, R. Almost-conservative matrix transformations. Turkish J. Math. 1989, 13, 91–100.
36. Başar, F.; Solak, I. Almost-coercive matrix transformations. Rend. Mat. Appl. 1991, 11, 249–256.
37. Jakimovski, A.; Russell, D.C. Matrix mapping between BK-spaces. *Lond. Math. Soc.* **1972**, *4*, 345–353. [CrossRef]

38. Kızmaz, H. On certain sequence spaces. *Can. Math. Bull.* **1981**, *24*, 169–176. [CrossRef]