On trend and its derivatives estimation in repeated time series with subordinated long-range dependent errors

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Abstract

For temporal regularly spaced datasets, a lot of methods are available and the properties of these methods are extensively investigated. Less research has been performed on irregular temporal datasets subject to measurement error with complex dependence structures, while this type of datasets is widely available. In this paper, the performance of kernel smoother for trend and its derivatives is considered under dependent measurement errors and irregularly spaced sampling scheme. The error processes are assumed to be subordinated Gaussian long memory processes and have varying marginal distributions. The functional central limit theorem for the estimators of trend and its derivatives are derived and bandwidth selection problem is addressed.

Keywords: derivative estimation, unevenly spaced time series, Gaussian subordination, functional data analysis, long-range dependence.

1 Introduction

In functional data analysis (FDA) studies, it is assumed that one observes \( n \) independent random curves \( X_i(t) = \mu(t) + \sum_{l=1}^{\infty} \xi_{il} \phi_l(t) \) \((i = 1, \ldots, n)\), which come from an underlying random process \( X(t) \in L^2[0,1] \). Here \( \mu(t) \) is
the population mean, the coefficients $\xi_i$ are uncorrelated random variables with mean zero and variance $\lambda_i \left( \sum \lambda_i < \infty \right)$, and the functions $\phi_i$ build an orthonormal $L^2[0,1]$ basis. Note that $\lambda_i$ and $\phi_i$ are eigenvalues and eigenfunctions of the covariance operator $C(y) = E[(X, y)X], \ y \in L^2[0,1]$, respectively. For general overview on FDA see e.g. Ramsay and Silverman (2005), Horváth and Kokoszka (2012) and references therein.

In practice, each random curve is typically observed at discrete but not necessarily equidistant time points $t_{ij} \in [0, 1]$ ($j = 1, ..., N_i$) and observations are perturbed by random noise. That is, we observe error perturbed curves

$$Y_{ij} = \mu(t_{ij}) + \sum_{l=1}^{\infty} \xi_{il}\phi_l(t_{ij}) + \epsilon_{ij}$$

where $\epsilon_{ij}$ are mean zero random measurement errors. Existing work assumes that the random errors $\epsilon_{ij}$ are independent and Gaussian, see Cai and Yuan (2010), Peng and Paul (2009), Staniswalis and Lee (1998), and Yao et al. (2005).

However, the assumption of independence is too restrictive. For example, for certain types of EEG observations the dependence or even long-range dependence in the error processes $\epsilon_{ij}$ ($j \in \mathbb{N}$) occurs (see e.g. Bornas et al. 2013, Linkenkaer-Hansen et al. 2001, Nikulin and Brismar 2005, Watters 2000). Beran and Liu (2014, 2016) therefore consider the estimation of trend and covariance for dense functional data that are perturbed by long-range dependent Gaussian errors. For details on statistical inference for long-range dependent processes, see e.g. Beran (1994), Giraitis et al. (2012), and Beran et al. (2013) and references therein.

In addition, in many applications, the Gaussianity and equidistance assumption is unrealistic. For example, in observational studies in medicine, the patient visits reported in electronic health records are irregular and sparse. Moreover, the presence of dependent non-Gaussian measurement errors makes the analysis challenging. Another example, in the context of ecology and climate science, is palaeo-proxy irregular time series perturbed by non-Gaussian errors as has been reported (see e.g. Menéndez et al. 2010). One more example, in the context of neurobiology, is the phenomenon that subordinated long-range dependent errors exist in the optical measurements of the calcium concentration in a glomerulus of the antennal lobe of a honey bee after an olfactory stimulus as reported in Beran and Weiershauser (2010).

For single time series, Menéndez et al. (2010) and Menéndez et al. (2013) investigate the trend and its derivatives kernel estimation for observations
perturbed by subordinated Gaussian long memory errors. Ghosh, S. (2014) addresses estimation of trend and slope functions in a partial linear model when the errors are unknown time-dependent functionals of latent Gaussian processes. Beran and Weiershauser (2011) consider estimation in a spline regression model with long-range dependent errors that are generated by Gaussian subordination.

The main purpose of this paper is, for repeated time series, to investigate the influence of subordinated long-range dependent Gaussian errors on the trend and its derivatives estimation. Detailed results on the following-up analysis on covariance and principal components will be reported in a subsequent paper.

The rest of this paper is organised as follows. Section 2 explains the model in details and gives the kernel estimation of the global mean and its derivatives. In Section 3, we investigate the asymptotic bias and variance and establish the functional central limit theorem of our estimator. The proofs of the theorems appear in Appendix.

2 Definitions

2.1 The model

We consider the square-integrable random function $X(t)$ defined over $[0,1]$ with mean $\mu(t) = E[X(t)]$ and continuous covariance $C(s,t) = Cov(X(s), X(t))$. Then, by Karhunen-Loève (K.L.) expansion, $X(t)$ has the form

$$X(t) = \mu(t) + \sum_{l=1}^{\infty} \xi_l(\omega)\phi_l(t) \quad (t \in [0,1])$$

where $\{\xi_l\}$ are pairwise uncorrelated random variables and functions $\{\phi_l\}$ are continuous real-valued functions on $[0,1]$ that are pairwise orthogonal in $L^2$.

Let $X_i, \ i = 1, ..., n$, be unobservable i.i.d. copies of $X$. For the $i$th subject, measurements are available at time points $t_{ij}, j = 1, ..., N_i$, and the observations at these time points are perturbed by subordinated long-range dependent noise, so the actual observations $Y_{ij}$ are:

$$Y_{ij} = X_i(t_{ij}) + \epsilon_i(j) \quad (i = 1, ..., n; j = 1, ..., N_i) \quad (1)$$

where $n$ is the number of subjects, $N_i$ is the number of sampling points
for $i$-th subject, $T_{ij} \in \mathbb{N}_+$, $T_{i1} \leq \ldots \leq T_{iN_i} \leq T_{\max} = \max_i \{T_{iN_i}\}$, and $t_{ij} = T_{ij}/T_{\max} \in [0, 1]$. Specifically,

- By K.L. expansion, the random curves $X_i(t)$ follow:

$$X_i(t) = \mu(t) + \sum_{l=1}^{\infty} \xi_{i\ell} \phi_{\ell}(t) \quad (t \in [0, 1])$$

(2)

- Let $Z(u)$ ($u \in \mathbb{R}$) be a continuous time stationary Gaussian process with $E[Z] = 0$, $\text{var}(Z) = 1$ and

$$\gamma_Z(v) = \text{cov}(Z(u), Z(u + v)) \sim c_{Z} v^{2d-1}$$

where $0 < d < \frac{1}{2}$. “$\sim$” means that the ratio of the left and right hand side tends to one. For each sample $i$, the error process $\{\epsilon_i(j)\}$ ($j \in \mathbb{N}$) has the form

$$\epsilon_i(j) = G(Z(T_{ij}), t_{ij}).$$

(3)

For each fixed $t \in [0, 1]$ the function $G(\cdot, t)$ is assumed to be in $L^2$-space of functions (on $\mathbb{R}$) with $E[(Z, t)] = 0$ and $\|G\|^2 = E[G^2(Z, t)] < \infty$. This implies a convergent $L^2$-expansion

$$G(Z, t_{ij}) = \sum_{k=q}^{\infty} \frac{c_k(t_{ij})}{k!} H_k(Z)$$

where $H_k(\cdot)$ are Hermite polynomials, $q \geq 1$ is the so-called Hermite rank, and $c_k(t) = E[G(Z, t)H_k(Z)]$.

- $\xi$ and $\epsilon$ are independent of each other.

**Remark 1** The function $G$ provides the possibility of having non-Gaussian errors with a changing marginal distribution. First, note that when $G$ is non-linear, $\epsilon$ is non-Gaussian. Therefore, $G$ can represent the departure from the Gaussian assumption which provides more flexibility for modelling. Second, a proper choice of $G$ can match any marginal distribution for $\epsilon$, which depends on $c_k(t)$. Third, from the view of robustness, $\epsilon$ can be seen as a perturbed version of $Z$ and if $\epsilon$ is close to $Z$ then $G$ is close to the identity function.
2.2 The estimation

To estimate $\mu^{(v)}(t)$, since each time series is unevenly recorded, we consider a Priestley-Chao type kernel estimator defined by

$$\hat{\mu}^{(v)}(t) = \frac{(-1)^v 1}{b^{v+1} n} \sum_{i=1}^{N_i} \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1}) K^{(v)} \left( \frac{t_{ij} - t}{b} \right) Y_{ij}$$

(4)

where $t_{i0} = 0$ and $K$ is a kernel function satisfying the following general assumptions (Gasser et al 1985, Menéndez, Ghosh and Beran 2010):

- (K1) $K \in C^{v+1}[-1, 1]$.
- (K2) $K(x) \geq 0$, $K(x) = 0$ ($|x| > 1$), $\int K(x)dx = 1$.
- (K3) $K^{(v)}(\cdot)$ is Lipschitz continuous, i.e. there exists $L \in \mathbb{R}_+$ such that, for any $x, y \in [-1, 1]$,

$$|K^{(v)}(x) - K^{(v)}(y)| \leq L|x - y|.$$

- (K4) $K(\cdot)$ is of order $(v, k)$, $v \leq k - 2$, where $k \in \mathbb{N}_+$, i.e.,

$$\int_{-1}^{1} x^j K^{(v)}(x)dx = \begin{cases} (-1)^v v!, & j = v \\
0, & j = 0, \ldots, v - 1, v + 1, \ldots, k - 1 \\
\theta, & j = k \end{cases}$$

where $\theta \neq 0$ is a constant.
- (K5) $K^{(j)}(-1) = K^{(j)}(1) = 0$ for all $j = 0, 1, \ldots, v - 1$.
- (K6) $0 \leq \kappa_{v+1} = \sup_{x \in [-1,1]} |K^{(v+1)}(x)| < \infty$.

3 Asymptotic results

3.1 Assumptions

Apart from the general assumptions on kernel function, the following technical conditions on the time points are required:

- (T1) $c_k(t)$ are continuously differentiable w.r.t. $t \in [0, 1]$. 

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• (T2) $\frac{1}{2} - \frac{1}{2d^q} < d < \frac{1}{2}$.

• (T3) For $i = 1, \ldots, n$,

$$0 \leq T_{i1} \leq \ldots \leq T_{iN} \leq T_{\text{max}} = \max_i \{T_{iN}\}, \ t_{ij} = T_{ij}/T_{\text{max}} \in [0, 1].$$

• (T4) For $i = 1, \ldots, n$,

$$\alpha_N^{-1} \leq \min_i \{t_{ij} - t_{i,j-1}\} \leq \max_i \{t_{ij} - t_{i,j-1}\} \leq \beta_N^{-1}$$

where $\alpha_N \geq \beta_N > 0$ and $\beta_N \to \infty$ as $N = \min_i \{N_i\} \to \infty$.

• (T5) $\lim_{N \to \infty} (b\alpha_N)^{1+(1-2d)q}(b\beta_N)^{-2} = 0$.

**Remark 2** Assumption (T1) implies that the residuals have a slowly changing marginal distribution. Further, (T1) also includes the necessary conditions for the weak convergence of $\mu^{(v)}(t)$. Since $\text{cov}(Z(T_{ij1}), Z(T_{ij2})) \sim |T_{ij1} - T_{ij2}|^{2d-1}$ implies $\text{cov}(\epsilon_{ij1}, \epsilon_{ij2}) \sim |T_{ij1} - T_{ij2}|^{(2d-1)q}$, the condition of (T2) implies that the long-range dependence of $Z$ is inherited by the subordinated process $\epsilon$. Assumption (T5) is a condition needed to obtain an asymptotic approximation of the variance of $\mu^{(v)}(t)$.

The trend $\mu(t)$ and the basis $\phi_l(t)$ are assumed to have the following smoothness:

• (M1) $\mu \in C^k[0, 1]$.

• (M2) $\phi_l \in C^k[0, 1]$.

### 3.2 Bias and variance of $\hat{\mu}^{(v)}$

**Theorem 1** Let $Y_{ij}$ be defined by (1)-(3). Suppose the assumptions (K1)-(K6), (T1)-(T5), (M1) and (M2) hold. Let $n \to \infty$, $N \to \infty$, $b \to 0$, $bT_{\text{max}} \to \infty$. Moreover, let

$$b^{k+1}\beta_N \to \infty. \quad (5)$$

The following holds for any $t \in (0, 1)$:
• **Bias:**

\[
E \left[ \hat{\mu}^{(v)}(t) \right] - \mu^{(v)}(t) = b^{k-v}C_{bias,v,k}(t) + o(b^{k-v}) + O \left( (b^{v+1}\beta_N)^{-1} \right) \\
= b^{k-v}C_{bias,v,k}(t) \left( 1 + o(1) \right)
\]

where

\[
C_{bias,v,k}(t) = \mu^{(k)}(t) \int K^{(v)}(x)x^k \, dx.
\]

• **Variance:**

\[
\text{var} \left( \hat{\mu}^{(v)}(t) \right) = n^{-1}C_{\text{var},v}(t) \left( 1 + O(b^{k-v}) + O \left( (b^{v+1}\beta_N)^{-1} \right) + O \left( b^{-2v}(T_{\text{max}}b)^{(2d-1)q} \right) \right)
\]

\[
= n^{-1}C_{\text{var},v}(t) \left( 1 + O(b^{k-v}) + O \left( b^{-2v}(T_{\text{max}}b)^{(2d-1)q} \right) \right)
\]

where

\[
C_{\text{var},v}(t) = \sum_{l=1}^{\infty} \lambda_l \left( \phi_l^{(v)}(t) \right)^2.
\]

**Remark 3** If (M2) is not true, say, \( \phi_l \in C^s \) with \( v < s < k \), then the order of the asymptotic variance of \( \hat{\mu}^{(v)}(t) \) has the following form:

\[
\text{var} \left( \hat{\mu}^{(v)}(t) \right) = n^{-1} \left( 1 + O(b^{s-v}) + O \left( b^{-2v}(T_{\text{max}}b)^{(2d-1)q} \right) \right).
\]

**Corollary 1** Suppose that the conditions of Theorem 7 hold. For \( v = 0 \) and \( k = 2 \), we have

\[
E \left[ \hat{\mu}(t) \right] - \mu(t) = b^2 C_{\text{bias},0,2}(t) \left( 1 + o(1) \right)
\]

and

\[
\text{var} \left( \hat{\mu}(t) \right) = n^{-1}C_{\text{bias},0}(t) \left( 1 + o(1) \right).
\]

### 3.3 Weak convergence of \( \hat{\mu}^{(v)} \)

Under additional assumptions on the sequence of bandwidth \( b \), weak convergence of \( \hat{\mu}^{(v)}(t) \) in \( C[0, 1] \) in the supremum norm sense can be obtained.
**Theorem 2** Let $Y_{ij}$ be defined by (1)-(3). Suppose the assumptions in Theorem 1 hold. Let
\[ n \to \infty, \ N \to \infty, \ b \to 0, \]
such that
\[ nb^{2(k-v)} \to 0 \] (8)
and
\[ \lim \inf b^{2(v+2)\beta^2_N(T_{\text{max}}b)^{-(2d+1)q}} > C \] (9)
for a suitable constant $C > 0$. Then
\[ Z_{n,N}(t) =: \sqrt{n} (\hat{\mu}^{(v)}(t) - \mu^{(v)}(t)) \Rightarrow \sum_{l=1}^{\infty} \sqrt{\lambda_l} \phi_l^{(v)}(t) \zeta_l \]
where “$\Rightarrow$” denotes weak convergence in $C[0,1]$ equipped with the supremum norm, and $\zeta_l \ iid \sim N(0,1)$.

**Remark 4** For observed data, the Hermite rank is unknown. Bai and Taqqu (2016) argue that a rank other than one is unstable in the sense that, when there is a slight perturbation, it typically collapses to rank one. Therefore, methods to perform valid inference of the rank are needed to be developed. As far as we know, only Beran et al (2016) performed work on designing a statistical test based on bootstrap procedures to test the Hermite rank $m = 1$ against $m > 1$.

**Remark 5** Let’s consider the estimation of the population mean, i.e. $v = 0$ under the assumption of equidistance i.e. $\beta_N = N$, $T_{\text{max}} = N$, and assume the Hermit rank $q = 1$. If the kernel function $K$ is of order 2, i.e. $k = 2$, then condition (8) reduces to $nb^4 \to \infty$ and condition (9) reduces to $\lim \inf N^{-2d+1}b^{3-2d} > C$. This situation coincides with Theorem 2 in Beran and Liu (2014) and will lead to a restriction on selecting the bandwidth (also on the relationship between $n$ and $N$), i.e. $CN^{-(1-2d)/(3-2d)} \leq b << n^{-1/4}$ for some $C > 0$. As pointed out by Beran and Liu (2014), “In order that this can be fulfilled by a sequence of bandwidths $b_N$, we need $n = n_N = o(N^{4(1-2d)/(3-2d)})$.” This means that the number of replicated time series cannot grow too fast compared to $N$, especially when $d \to 1/2$.

Under $\beta_N = N$, $T_{\text{max}} = N$, $q = 1$, and $k > 2$, condition (8) reduces to $nb^{2k} \to \infty$ and condition (9) reduces to $\lim \inf N^{-2d+1}b^{3-2d} > C$. This situation leads to a less restrictive condition compared to order 2 kernel function,
i.e. \( CN^{-(1-2d)/(3-2d)} \leq b << n^{-1/2k} \) for some \( C > 0 \). In order that this condition can be fulfilled by a sequence of bandwidths \( b_N \), we similarly need \( n = n_N = o(N^{2k(1-2d)/(3-2d)}) \) which is less restrictive, if \( k \to \infty \), even for \( d \) close to \( 1/2 \).

For irregularly designed time points, the corresponding requirements on bandwidth selection can be adapted by considering the sampling space.

**Remark 6** Now we consider the estimation of second derivative of population mean, i.e. \( v = 2 \), under the assumption of equidistance and Gaussianity of the process, i.e. \( \beta_N = N, T_{\max} = N, q = 1 \).

If the kernel function \( K \) is of order \( (2, 4) \), i.e. \( k = v + 2 \), condition \( [8] \) implies \( b << n^{-\frac{1}{4}} \), and condition \( [3] \) reduces to \( \lim \inf N^{-2d+1}b^{7-2d} > C \). These conditions result in \( CN^{-(1-2d)/(7-2d)} \leq b << n^{-1/4} \). We need \( n = n_N = o(N^{(1-2d)/(7-2d)}) \) in order the above condition to be fullfilled. This condition is more restrictive than the condition in mean estimation where \( n = n_N = o(N^{4(1-2d)/(3-2d)}) \).

If \( k > 2 \), condition \( [8] \) reduces to \( nb^{2k} \to \infty \) and condition \( [3] \) reduces to \( \lim \inf N^{-2d+1}b^{7-2d} > C \). This results in a less restrictive condition compared to order 2 kernel function, i.e. \( CN^{-(1-2d)/(7-2d)} \leq b << n^{-1/2k} \) for some \( C > 0 \). Similarly we need \( n = n_N = o(N^{2k(1-2d)/(7-2d)}) \) and this is less restrictive, if \( k \to \infty \), even for \( d \) close to \( 1/2 \).

For irregularly designed time points, the corresponding requirements on bandwidth selection can be adapted by considering the sampling space.

## 4 Final Remarks

We considered estimation of population mean and its derivatives in repeated time series with subordinated Gaussian long-range dependent errors with an FDA structure. A functional limit theorem is obtained for Priestley-Chao type kernel estimator of \( \mu^{(v)}(t) \), provided that the number of curves \( n \), the number of observations \( N_i \) on each curve and the spacings, satisify some restrictions. These restrictions provide information for selecting the bandwidth \( b_N \).

Our results allow for more flexibility in modelling repeated time series. This flexibility is needed to extract important information from temporal datasets available in health for example. Moreover, one can estimate rapid change points of \( \mu(t) \) based on our results, especially based on the second
derivative. Two more urgent questions are the covariance estimation and functional principal component analysis in this context, which are the fundamental and powerful tools to analysis functional data. These will be potential future research topics.

5 Appendix

Proof: For simplicity of presentation we consider the case with only one basis function $\phi(t)$. Therefore, let

$$Y_{ij} = \mu(t_{ij}) + \xi_i\phi(t_{ij}) + \epsilon_i(j).$$

1) The bias term is straightforward obtained and follows the standard situation. In fact,

$$E \left[ \hat{\mu}^{(v)}(t) \right] - \mu^{(v)}(t)
= \frac{(-1)^v}{b^{v+1}} \frac{1}{n} \sum_{i=1}^N \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1}) K^{(v)}(\frac{t_{ij} - t}{b}) E[Y_{ij}] - \mu^{(v)}(t)
= \frac{(-1)^v}{b^{v+1}} \frac{1}{n} \sum_{i=1}^N \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1}) K^{(v)}(\frac{t_{ij} - t}{b}) \mu(t_{ij}) - \mu^{(v)}(t)
= \frac{(-1)^v}{b^{v+1}} \int_0^1 K^{(v)}(\frac{x - t}{b}) \mu(x) dx + O(\beta_N^{-1}) - \mu^{(v)}(t)
= \frac{(-1)^v}{b^{v+1}} \int_{-1}^1 K^{(v)}(y) \mu(by + t) bdy - \mu^{(v)}(t) + O\left((b^{v+1} \beta_N)^{-1}\right)
= \frac{(-1)^v}{b^v} \int_{-1}^1 K^{(v)}(y) \sum_{l=0}^k \frac{(yb)^l}{l!} \mu^{(l)}(t) + o\left(b^k\right) dy - \mu^{(v)}(t) + O\left((b^{v+1} \beta_N)^{-1}\right)
= b^{k-v} C^{(v)}_{bias,v,k}(t) + o\left(b^{k-v}\right) + O\left((b^{v+1} \beta_N)^{-1}\right),$$

where

$$C^{(v)}_{bias,v,k}(t) = \frac{\mu^{(k)}(t)}{k!} \int K^{(v)}(x)x^k dx.$$

2) Now, we consider the asymptotic variance.

$$\hat{\mu}^{(v)}(t) - E \left[ \hat{\mu}^{(v)}(t) \right] = \frac{(-1)^v}{b^{v+1}} \frac{1}{n} \sum_{i=1}^N \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1}) K^{(v)}(\frac{t_{ij} - t}{b}) [\xi_i\phi(t_{ij}) + \epsilon_i(j)].$$
Since $\xi$ and $\epsilon$ are independent, we have
\[
\text{var} \left( \hat{\mu}^{(v)}(t) \right) = n^{-2}b^{-2(v+1)}(A_{n,N}(t) + B_{n,N}(t))
\]
where
\[
A_{n,N}(t) = \text{var} \left( \sum_{i=1}^{n} \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1}) K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \xi_i \phi(t_{ij}) \right),
\]
\[
B_{n,N}(t) = \text{var} \left( \sum_{i=1}^{n} \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1}) K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \epsilon_i (j) \right).
\]

i) For the item $A_{n,N}(t)$, since $\phi \in C^k[0,1]$ and $\xi_i$ are independent, we have
\[
A_{n,N}(t) = \text{var} \left( \sum_{i=1}^{n} \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1}) K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \phi(t_{ij}) \xi_i \right)
\]
\[
= n\lambda \left( \int_{0}^{1} K^{(v)} \left( \frac{x - t}{b} \right) \phi(x) dx + O (\beta_N^{-1}) \right)^2
\]
\[
= n\lambda \left( \int_{-1}^{1} K^{(v)}(y) \phi(by + t)bdy + O (\beta_N^{-1}) \right)^2
\]
\[
= n\lambda \left( b \int_{-1}^{1} K^{(v)}(y) \left[ \sum_{l=0}^{k} \frac{(yb)^l}{l!} \phi^{(l)}(t) + o (b^k) \right] dy + O (\beta_N^{-1}) \right)^2
\]
\[
= nb^{2(v+1)} \lambda \left( \phi^{(v)}(t) \right)^2 \left( 1 + O (b^{k-v}) + O \left( (b^{v+1} \beta_N)^{-1} \right) \right).
\]
ii) For the item $B_{n,N}(t)$, we have

$$B_{n,N}(t) = \sum_{i=1}^{n} \text{var} \left( \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1}) K(v) \left( \frac{t_{ij} - t}{b} \right) \epsilon_i(j) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1})^2 \left( K(v) \left( \frac{t_{ij} - t}{b} \right) \right)^2 V_{i,j}$$

$$+ \sum_{i=1}^{n} \sum_{j_1 < j_2} (t_{ij_1} - t_{i,j_1-1})(t_{ij_2} - t_{i,j_2-1}) K(v) \left( \frac{t_{ij_1} - t}{b} \right) K(v) \left( \frac{t_{ij_2} - t}{b} \right) V_{i,j_1,j_2}$$

$$+ \sum_{i=1}^{n} \sum_{j_1 > j_2} (t_{ij_1} - t_{i,j_1-1})(t_{ij_2} - t_{i,j_2-1}) K(v) \left( \frac{t_{ij_1} - t}{b} \right) K(v) \left( \frac{t_{ij_2} - t}{b} \right) V_{i,j_1,j_2}$$

$$=: B_{n,N,1}(t) + B_{n,N,2}(t) + B_{n,N,3}(t)$$

where

$$V_{i,j} = \text{cov}(\epsilon_i(j), \epsilon_i(j)) = \sum_{l=q}^{\infty} \frac{c_l^2(t_{ij})}{l!}$$

$$V_{i,j_1,j_2} = \text{cov}(\epsilon_i(j_1), \epsilon_i(j_2)) = \sum_{l=q}^{\infty} \frac{c_l(t_{ij_1})c_l(t_{ij_2})}{l!} \gamma_Z(T_{ij_1} - T_{ij_2})$$

and

$$\gamma_Z(T_{ij_1} - T_{ij_2}) \sim c_Z |T_{ij_1} - T_{ij_2}|^{2d-1}.$$
For $B_{n, N, 1}(t)$ we have,

$$|B_{n, N, 1}(t)| = \sum_{i=1}^{n} \sum_{j=1}^{N_i} (t_{ij} - t_{ij-1})^2 \left( K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \right)^2 V_{i,j}$$

$$\leq C_1 \sum_{i=1}^{n} \sum_{j=1}^{N_i} (t_{ij} - t_{ij-1})^2 \left( K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \right)^2$$

$$= b^2 C_1 \sum_{i=1}^{n} \sum_{j=1}^{N_i} \left( \frac{t_{ij} - t_{ij-1}}{b} \right)^2 \left( K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \right)^2$$

$$\leq b^2 C_1 \sum_{i=1}^{n} \sum_{j=1}^{N_i} \left( \frac{t_{ij} - t_{ij-1}}{b} \right) \frac{1}{b\beta_N} \left( K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \right)^2$$

$$= b^2 \frac{1}{b\beta_N} C_1 \sum_{i=1}^{n} \left( \int_{\frac{t_{iN}}{b}}^{t_{iN}} \left( K^{(v)} \left( x - \frac{t}{b} \right) \right)^2 dx + O \left( (b\beta_N)^{-1} \right) \right)$$

$$= b^2 \frac{n}{b\beta_N} C_1 \left( \int_{-1}^{1} (K^{(v)}(u))^2 du + O \left( (b\beta_N)^{-1} \right) \right)$$

where

$$C_1 = \frac{\sup_{t \geq q} \sup_{t \in [0, 1]} C^2(t)}{e}$$

For $B_{n, N, 2}(t)$, notice assumption (T2)

$$-1 < (2d - 1)q < 0,$$

we have

$$V_{i,j_1,j_2} \sim \frac{C^2_q(t)}{q!} \gamma^q_Z(T_{j_1} - T_{j_2}) \sim |T_{ij_1} - T_{ij_2}|^{(2d-1)q}$$

for $j_1, j_2 \in U_b(t)$ with $U_b(t) = \{ k \in \mathbb{N} : |t - t_{ik}| \leq b \}$. Moreover since
\[ K(x) = 0 \text{ for } |x| > 1, \text{ we have} \]

\[ B_{n,N,2}(t) = \sum_{i=1}^{n} \sum_{j_1 < j_2} (t_{ij_1} - t_{ij_1-1})(t_{ij_2} - t_{ij_2-1})K^{(v)} \left( \frac{t_{ij_1} - t}{b} \right) K^{(v)} \left( \frac{t_{ij_2} - t}{b} \right) (T_{ij_1} - T_{ij_2})^{(2d-1)q} \]

\[ = b^2(T_{\text{max}} b)^{(2d-1)q} \sum_{i=1}^{n} \sum_{j_1 \in A_{i,j_1}} K^{(v)} \left( \frac{t_{ij_1} - t}{b} \right) \frac{t_{ij_1} - t_{ij_1-1}}{b} \times \sum_{j_2 \in A_{i,j_2}} K^{(v)} \left( \frac{t_{ij_2} - t}{b} \right) \frac{t_{ij_2} - t_{ij_2-1}}{b} \]

where

\[ A_{i,j_1} = \{ j_1 \in \mathbb{N} : |T_{ij_1} - tT_{\text{max}}| \leq bT_{\text{max}} \} \]

and

\[ A_{i,j_2} = \{ j_2 \in \mathbb{N} : 1 \leq j_2 \leq j_1 - 1, |T_{ij_2} - tT_{\text{max}}| \leq bT_{\text{max}} \}. \]

Setting

\[ h_{N_i}(x) = K^{(v)} \left( x - \frac{t}{b} \right) \left( \frac{t_{ij_1}}{b} - x \right)^{(2d-1)q} \]

we have

\[ \sum_{j_2 \in A_{i,j_2}} K^{(v)} \left( \frac{t_{ij_2} - t}{b} \right) \left( \frac{t_{ij_1} - t_{ij_2}}{b} \right)^{(2d-1)q} t_{ij_2} - t_{ij_2-1} \]

\[ = \int_{t_{ij_1-1}^1}^{t_{ij_1-1}^2} h_{N_i}(x) dx + \sum_{j_2 \in A_{i,j_2}} h'_{N_i}(x_{j_2}) \left( \frac{t_{ij_2} - t_{ij_2-1}}{b} \right)^2 \]

\[ = \int_{t_{ij_1-1}^1}^{t_{ij_1-1}^2} h_{N_i}(x) dx + r_{N_i,j_1} \]

where \( \frac{t_{ij_2-1}}{b} \leq x_{ij_2} \leq \frac{t_{ij_2}}{b} \) and \( h'_{N_i}(x) = g_{N_i,1}(x) + g_{N_i,2}(x) \) with

\[ g_{N_i,1}(x) = K^{(v+1)} \left( x - \frac{t}{b} \right) \left( \frac{t_{ij_1}}{b} - x \right)^{(2d-1)q} \]

and

\[ g_{N_i,2}(x) = K^{(v)} \left( x - \frac{t}{b} \right) \left( \frac{t_{ij_1}}{b} - x \right)^{(2d-1)q-1} (2d - 1)q. \]
Notice that

\((T2):\) \(-1 < (2d - 1)q < 0,\)
\((T4):\) \(\alpha_N^{-1} \leq \min_i \{t_{ij} - t_{i,j-1}\} \leq \max_i \{t_{ij} - t_{i,j-1}\} \leq \beta_N^{-1},\)
\((T5):\) \(\lim_{N \to \infty} (b\alpha_N)^{1+(1-2d)q}(b\beta_N)^{-2} = 0,\)
\((K6):\) \(0 \leq \kappa_{v+1} = \sup_{x \in [-1,1]} |K^{(v+1)}(x)| < \infty,\)

moreover
\(b\beta_N \to \infty\) which implies \(b\alpha_N \to \infty,\)

using the notation \(k_1 = \lceil \alpha_N(t-b) \rceil\) and \(k_2 = \lceil \alpha_N(t+b) \rceil,\) an upper bound can be given by

\[
\left| \sum_{j_2 \in A_i,j_2} g_{N,i,j_2}(x_{j_2}) \left( \frac{t_{ij_2} - t_{i,j_2-1}}{b} \right)^2 \right|
\]
\[= \left| \sum_{j_2 \in A_i,j_2} K^{(v+1)} \left( x_{ij_2} - \frac{t}{b} \right) \left( \frac{t_{ij_1} - t_{ij_2}}{b} - x_{ij_2} \right)^{(2d-1)q} \left( \frac{t_{ij_2} - t_{i,j_2-1}}{b} \right)^2 \right|
\]
\[\leq \kappa_{v+1}(b\beta_N)^{-2} \left| \sum_{j_2=k_1}^{k_2} \left( \frac{t_{ij_1}}{b} - \frac{t_{ij_2}}{b} \right)^{(2d-1)q} \right|
\]
\[\leq \kappa_{v+1}(b\beta_N)^{-2} \left| \sum_{j_2=1}^{[2b\alpha_N]} \left( \frac{j}{b\alpha_N} \right)^{(2d-1)q} \right|
\]
\[\leq \kappa_{v+1}b\alpha_N(b\beta_N)^{-2} \int_0^2 x^{(2d-1)q}dx.
\]

Notice that \((2d-1)q > -1,\) \(b\alpha_N \to \infty,\) and \(\lim_{N \to \infty} (b\alpha_N)^{1+(2d-1)q}(b\beta_N)^{-2} = 0\) imply \(\lim_{N \to \infty} (b\alpha_N)(b\beta_N)^{-2} = 0.\) Moreover, \(\int_0^2 x^{(2d-1)q}dx < \infty\) since \((2d-1)q > -1.\) Thus, there is a uniform upper bound on \(r_{N,j_1}.

Analogous arguments apply to \(B_{n,N,3}(t),\) and at last, we obtain

\[
B_{n,N}(t) = nb^2(T_{max,b})^{(2d-1)q}I_q(t)(1 + o(1))
\]

where
\[
I_q(t) = \frac{c_q^2(t)}{q!} c_z^q \int \int K(x)K(y)|x - y|^{(2d-1)q}dxdy.
\]
Therefore,
\[
\text{var} \left( \hat{\mu}^{(v)}(t) \right) = n^{-2}b^{-2(v+1)}(A_{n,N}(t) + B_{n,N}(t)) \\
= n^{-1}\lambda \left( \phi^{(v)}(t) \right)^2 \left( 1 + O \left( b^{k-v} \right) + O \left( (b^{v+1} \beta_N)^{-1} \right) \right) \\
+ n^{-1}b^{-2v}(T_{\text{max}}b)^{(2d-1)q}I_q(t)(1 + o(1)).
\]

**Proof:** Condition (8) is only required to make the bias of order \( o(n^{-1/2}) \) since
\[
b^{2(k-v)} \to 0
\]
together with Theorem 7 implies
\[
\lim_{n \to \infty} \sqrt{n} \sup_{t \in [0,1]} |E \left[ \hat{\mu}^{(v)}(t) \right] - \mu^{(v)}(t) | = 0.
\]
Therefore, it is sufficient to consider the process
\[
Z^{0}_{n,N}(t) := \sqrt{n} \left( \hat{\mu}^{(v)}(t) - E \left[ \hat{\mu}^{(v)}(t) \right] \right).
\]
For simplicity of presentation, as in the proof of Theorem 7, we consider the case with only one basis function \( \phi(t) \). Therefore,
\[
Y_{ij} = \mu(t_j) + \xi_i \phi(t_j) + \epsilon_i(j).
\]

Denote
\[
c_{iN}(t) = \frac{(-1)^v}{b^{v+1}} \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1})K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \phi(t_{ij}),
\]
\[
u_n = n^{-1/2} \lambda^{-1/2} \sum_{i=1}^{n} \xi_i
\]
and
\[
e_n(j) = n^{-1/2} \sum_{i=1}^{n} \epsilon_i(j),
\]
we can write
\[
Z^{0}_{n,N}(t) = S_{n,1}(t) + S_{n,2}(t)
\]
where
\[
S_{n,1}(t) = \sqrt{\lambda}n^{-1/2} \lambda^{-1/2} \sum_{i=1}^{n} \xi_i c_{iN}
\]
and
\[ S_{n,N,2}(t) = \frac{(-1)^v}{b^{v+1}} n^{-1/2} \sum_{i=1}^{n} \sum_{j=1}^{N} (t_{ij} - t_{ij-1}) K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \epsilon_i(j) \]
are independent.

Now, we consider \( S_{n,N,1}(t) \). Notice that, since \( \xi_i \overset{iid}{\sim} N(0, \lambda) \), we have \( u_n \sim N(0,1) \) for all \( n \). For \( c_N(t) \), we have
\[ |c_iN(s) - c_iN(t)| \leq |\phi^{(v)}(s) - \phi^{(v)}(t)| + C b^{k-v} + C^* (b^{v+1} \beta_N)^{-1}, \]
where \( C \) and \( C^* \) are suitable constants. Thus
\[ \omega_{S_{n,N,1}}(\Delta) \leq \sqrt{\lambda} \left( \omega_{\phi^{(v)}}(\Delta) + C b^{k-v} + C^* (b^{v+1} \beta_N)^{-1} \right) |u_n|, \]
where
\[ \omega_f(\Delta) = \sup_{|s-t| \leq \Delta} |f(t) - f(s)| \]
is the modulus of continuity of function \( f(t) \). Let \( \tau > 0 \) and \( \Delta > 0 \) be small. Then, we have
\[ P(\omega_{S_{n,N,1}}(\Delta) > \tau) \leq P \left( \sqrt{\lambda} \left( \omega_{\phi^{(v)}}(\Delta) + C b^{k-v} + C^* (b^{v+1} \beta_N)^{-1} \right) |u_n| > \tau \right). \]
Taking the \( \limsup \) over \( n \) and \( N \) such that conditions of Theorem 2 hold, we have
\[ \limsup P(\omega_{S_{n,N,1}}(\Delta) > \tau) \leq 2 \left[ \left| 1 - \Phi \left( \tau \omega_{\phi^{(v)}}^{-1}(\Delta) \lambda^{-1/2} \right) \right] \right], \]
where \( \Phi \) denotes the cumulative distribution function of standard normal random variable. Since \( \phi^{(v)} \) is uniformly continuous on \([0,1]\), we have
\[ \lim_{\Delta \to 0} \limsup P(\omega_{S_{n,N,1}}(\Delta) > \tau) = 0. \]
Therefore,
\[ S_{n,N,1}(t) \Rightarrow \sqrt{\lambda} \phi^{(v)}(t) \zeta. \]

For \( S_{n,N,2}(t) \), first, we show convergence of finite-dimensional distributions. Notice that
\[ \text{var}(S_{n,N,2}(t)) = b^{-2v}(T_{\text{max}} b)^{(2d-1)q} I_q(t) + r_N(t). \]
where
\[
\lim_{T_{\max} \to \infty} b^{2v}(T_{\max} b)^{(1-2d)q} \sup_{t \in [0,1]} |r_N(t)| = 0.
\]

Thus,
\[
\sup_{t \in [0,1]} \text{var}(S_{n,N,2}(t)) \leq C b^{-2v}(T_{\max} b)^{(2d-1)q} (n \geq n_0, \ N \geq N_0)
\]
for \( n_0 \) and \( N_0 \) large enough. Thus, for all \( p \in \mathbb{N}, t_1, \ldots, t_p \in [0,1] \),
\[
(S_{n,N,2}(t_1), \ldots, S_{n,N,2}(t_p))^T \overset{d,b}{\Rightarrow} (0, \ldots, 0)^T.
\]

Now we have to show the tightness of \( S_{n,N,2}(t) \). Therefore,
\[
E \left[ (S_{n,N,2}(t) - S_{n,N,2}(s))^2 \right]
\]
\[
= \frac{1}{b^{2(v+1)n}} E \left[ \left( \sum_{i=1}^{n} \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1})K^{(v)} \left( \frac{t_{ij} - t}{b} \right) \epsilon_i(j) \right)^2 \right]
\]
\[
- \sum_{i=1}^{n} \sum_{j=1}^{N_i} (t_{ij} - t_{i,j-1})K^{(v)} \left( \frac{t_{ij} - s}{b} \right) \epsilon_i(j))^2
\]
\[
\leq b^{-2(v+1)} \kappa_{v+1}^2 \left( \frac{t - s}{b} \right)^2 \beta_N^{-2} n^{-1} \sum_{i=1}^{n} \sum_{j_1,j_2=1}^{2T_{\max} b} V_{i,j_1,j_2}.
\]

where
\[
V_{i,j_1,j_2} = \text{Cov}(\epsilon_i(j_1), \epsilon_i(j_2)) = \sum_{l=0}^{\infty} c_l(t_{ij_1}) c_l(t_{ij_2}) \gamma_l(T_{ij_1} - T_{ij_2}).
\]

Recalling that
\[
\gamma_l(T_{ij_1} - T_{ij_2}) \sim c_Z |T_{ij_1} - T_{ij_2}|^{2d-1}
\]
and the condition (T2), i.e. \(-1 < (2d-1)q < 0\), we have
\[
V_{i,j_1,j_2} \sim \frac{c_2^2(t)}{q!} \gamma_2(T_{ij_1} - T_{ij_2}).
\]

Then, notice that \( |T_{ij_1} - T_{ij_2}| \geq 1 \) and assumption (T1), we have
\[
E \left[ (S_{n,N,2}(t) - S_{n,N,2}(s))^2 \right]
\]
\[
\leq b^{-2(v+1)} \kappa_{v+1}^2 (t - s)^2 b^{-2} \beta_N^{-2} n^{-1} \sum_{i=1}^{n} \sum_{j_1,j_2=1}^{2T_{\max} b} C_V |T_{ij_1} - T_{ij_2}|^{(2d-1)q}
\]
\[
\leq C b^{-2(v+1)} b^{-2} \beta_N^{-2} (T_{\max} b)^{(2d+1)q} (t - s)^2
\]
where
\[ \kappa_{v+1} = \sup_{t \in [-1,1]} |K^{(v+1)}(x)|. \]

Therefore, assumption (9) \( S_{n,N} \to 0 \) implies that there is a finite constant \( C^* \) such that
\[ E \left[ (S_{n,N,2}(t) - S_{n,N,2}(s))^2 \right] \leq C(t - s)^2. \]

Therefore, tightness of \( S_{n,N,2}(t) \ (t \in [0,1]) \) and the weak convergence of \( S_{n,N,2}(t) \) to 0 in the Skorohod topology can be obtained from Billingsley (1999). Therefore
\[ Z_{0}^{n,N}(t) \Rightarrow \sqrt{\lambda} \phi^{(v)}(t) \zeta \]
and further
\[ Z_{n,N}(t) \Rightarrow \sqrt{\lambda} \phi^{(v)}(t) \zeta. \]

The above proof can be extended to the general case with an arbitrary number of basis functions \( \phi_l \).

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