Weighted Join Operators on Directed Trees

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Received: 17 July 2021 / Accepted: 20 January 2023 / Published online: 2 March 2023
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Abstract
A rooted directed tree \( T = (V, E) \) with root \( \text{root} \) can be extended to a directed graph \( T_\infty = (V_\infty, E_\infty) \) by adding a vertex \( \infty \) to \( V \) and declaring each vertex in \( V \) as a parent of \( \infty \). One may associate with the extended directed tree \( T_\infty \) a family of semigroup structures \( \sqcup_b \) with extreme ends being induced by the join operation \( \sqcup \) and the meet operation \( \sqcap \) from lattice theory (corresponding to \( b = \text{root} \) and \( b = \infty \) respectively). Each semigroup structure among these leads to a family of densely defined linear operators \( W_{\lambda u}^{(b)} \) acting on \( \ell^2(V) \), which we refer to as weighted join operators at a given base point \( b \in V_\infty \) with prescribed vertex \( u \in V \). The extreme ends of this family are weighted join operators \( W_{\lambda u}^{(\text{root})} \) and weighted meet operators \( W_{\lambda u}^{(\infty)} \). In this paper, we systematically study the weighted join operators on rooted directed trees. We also present a more involved counterpart of weighted join operators \( W_{\lambda u}^{(b)} \) on rootless directed trees \( T \). In the rooted case, these operators are either finite

Communicated by Mihai Putinar.

This article is part of the topical collection “Multivariable Operator Theory. The Jörg Eschmeier Memorial” edited by Raul Curto, Michael Hartz, Mihai Putinar and Ernst Albrecht.

Sameer Chavan and Rajeev Gupta are supported by P. K. Kelkar Fellowship and Inspire Faculty Fellowship, respectively. Kalyan B. Sinha acknowledges the support of SERB-Distinguished Fellowship as well as of INSA Senior Scientist Scheme.

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rank operators, diagonal operators or rank one perturbations of diagonal operators. In the rootless case, these operators are either possibly infinite rank operators, diagonal operators or (possibly unbounded) rank one perturbations of diagonal operators. In both cases, the class of weighted join operators overlaps with the well-studied classes of complex Jordan operators and \( n \)-symmetric operators. An important half of this paper is devoted to the study of rank one extensions \( W_{f,g}^{(b)} \) of weighted join operators \( W_{\lambda u}^{(b)} \) on rooted directed trees, where \( f \in \ell^2(V) \) and \( g : V \to \mathbb{C} \) is unspecified. Unlike weighted join operators, these operators are not necessarily closed. We provide a couple of compatibility conditions involving the weight system \( \lambda_u \) and \( g \) to ensure closedness of \( W_{f,g} \). These compatibility conditions are intimately related to whether or not an associated discrete Hilbert transform is well-defined. We discuss the role of the Gelfand-triplet in the realization of the Hilbert space adjoint of \( W_{f,g} \). Further, we describe various spectral parts of \( W_{f,g} \) in terms of the weight system and the tree data. We also provide sufficient conditions for \( W_{f,g} \) to be a sectorial operator (resp. an infinitesimal generator of a quasi-bounded strongly continuous semigroup). In case \( \mathcal{T} \) is leafless, we characterize rank one extensions \( W_{f,g} \), which admit compact resolvent. Motivated by the above graph-model, we also take a brief look into the general theory of rank one non-selfadjoint perturbations.

**Keywords** Directed tree · Join · Meet · Rank one perturbation · Discrete Hilbert transform · Commutant · Gelfand-triplet · Sectorial · Complex Jordan · \( n \)-symmetric · Form-sum

**Mathematics Subject Classification** Primary 47B37 · 47B15 · 47H06; Secondary 05C20 · 47B20

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1 Background

The present work is yet another illustration of the rich interplay between graph theory and operator theory that includes the recent developments pertaining to the weighted shifts on directed trees [18, 19, 21, 38]. This work exploits the order structure of directed trees to introduce a class of possibly unbounded linear operators to be referred to as weighted join operators $W_{\lambda u}^{(b)}$ based at the vertex $b$ and with prescribed vertex $u \in V$. We capitalize on the fact that any directed tree has a natural partial ordering induced by the notion of the directed path. This ordering satisfies all the requirements of the so-called spiral-like ordering (SLO) introduced and studied by Pruss for $p$-regular trees (see [59, Definition 6.1]; see also [17, Definition 3] for modified and extended definition). This allows us to define join and meet operations on a directed tree (refer to [31, Chapter 4] for the basics of lattice theory). These operations, in turn, induce the so-called weighted join operators $W_{\lambda u}^{(\text{root})}$ and weighted meet operators $W_{\lambda u}^{(\infty)}$ on a directed tree. The present work is devoted to a systematic study of this class. In case the directed tree is rooted, it turns out that weighted join operators $W_{\lambda u}^{(b)}$ are either (possibly unbounded) finite rank operators, diagonal operators or bounded rank one perturbations of (possibly unbounded) diagonal operators. In case the directed trees are rootless, the situation being more complex allows unbounded rank one perturbations of (possibly unbounded) diagonal operators. In particular, weighted join operators $W_{\lambda u}^{(b)}$ on rootless directed trees need not be even closable. A substantial part of this paper is devoted to the study of rank one extensions $W_{f,g}$ of weighted join operators. The so-called compatibility conditions (which control the rank one perturbation $f \otimes g$ with the help of the diagonal operator $D_{\lambda u}^{(b)}$) play an important role in the spectral theory of these operators. We also discuss the problem of determining the Hilbert space adjoint of $W_{f,g}$. Our analysis of this problem relies on the idea of the Hilbert rigging (refer to [16]). The notion of the rank one extensions of weighted join operators is partly motivated by the graph-model arising from the semigroup structures on directed trees. Interestingly, the above graph-model plays a decisive role in deriving various spectral properties of these operators.

The class of rank one perturbations of diagonal operators has been studied extensively in the context of hyperinvariant subspace problem [26–29, 37, 44, 48, 69] and spectral analysis [10–12, 24, 32, 73]. The reader is referred to [65] for a survey on the classical theory of self-adjoint rank one perturbations of self-adjoint operators (refer also to [49] for its connection with the theory of singular integral operators). The class of rank one perturbations of diagonal operators also arises naturally in a problem of
domain inclusion in the context of weighted shifts on directed trees (see [38, Theorem 4.2.2]). We find it necessary to comment upon the relationship of the present work to the existing literature. The class of weighted join operators has essentially no intersection with the existing class \((\mathcal{RO})\), as studied in [28], of bounded rank one perturbations of bounded diagonal operators. Unlike the case of operators in \((\mathcal{RO})\), commutants of weighted join operators are not necessarily abelian. It turns out that there are no non-normal hyponormal weighted join operators (refer to [41, 58] for basics of unbounded hyponormal operators). In the context of bounded rank one perturbations of bounded normal operators, similar behaviour has been observed in [44]. On the other hand, the class of weighted join operators and their rank one extensions contains bounded as well as unbounded complex Jordan operators (and \(n\)-symmetric operators) in abundance (refer to [1–4, 8, 9, 36, 45, 51] for the basic theory of Jordan operators, \(n\)-symmetric operators, and their connections with the classes of \(n\)-normal operators and \(n\)-isometries). Further, it overlaps with the class of sectorial operators, and also provides a family of examples of non-normal compact operators with large null summand in the sense of Anderson [6]. We would also like to draw attention to the works [7, 13, 53, 55, 56, 72] on the spectral theory of unbounded operator matrices on non-diagonal domains (refer also to the authoritative exposition [71] on this topic). The rank one extensions \(W_{f,g}\) of weighted join operators fit into the class of operator matrices with not necessarily of diagonal domain. Further, under some compatibility conditions, \(W_{f,g}\) is diagonally dominant in the sense of [72] with the exception that exactly one of its entries is not closable.

In Sects. 1.1 and 1.2 of this section, we collect preliminaries pertaining to the directed trees and the Hilbert space operators respectively (the reader is referred to [31, 38] for the basics of graph theory, and [64, 66] for that of Hilbert space operators). In particular, we set notations and introduce some natural and known classes of directed trees and unbounded Hilbert space operators, which are relevant to the investigations in this paper. In Sect. 1.3, we collect several simple but basic properties of bounded and unbounded rank one operators. We conclude this section with a prologue including some important aspects and the layout of the paper.

### 1.1 Directed Trees

A pair \(\mathcal{T} = (V, E)\) is said to be a directed graph if \(V\) is a non-empty set and \(E\) is a non-empty subset of \(V \times V \setminus \{(v, v) : v \in V\}\). An element of \(V\) (resp. \(E\)) is referred to as a vertex (resp. an edge) of \(\mathcal{T}\). A finite sequence \(\{v_i\}_{i=1}^n\) of distinct vertices is said to be a circuit in \(\mathcal{T}\) if \(n \geq 2\), \((v_i, v_{i+1}) \in E\) for all \(1 \leq i \leq n - 1\) and \((v_n, v_1) \in E\). Given \(u, v \in V\), by a directed path from \(u\) to \(v\) in \(\mathcal{T}\), we understand a finite sequence \(\{u_1, \ldots, u_k\}\) in \(V\) such that

\[
   u_1 = u, \quad (u_j, u_{j+1}) \in E \quad (1 \leq j \leq k - 1) \quad \text{and} \quad u_k = v.
\]

We say that two distinct vertices \(u\) and \(v\) of \(\mathcal{T}\) are connected by a path if there exists a finite sequence \(\{u_1, \ldots, u_k\}\) of distinct vertices of \(\mathcal{T}\) such that
\[ u_1 = u, \ (u_j, u_{j+1}) \text{ or } (u_{j+1}, u_j) \in E \ (1 \leq j \leq k - 1) \text{ and } u_k = v. \]

A directed graph \( T \) is said to be connected if any two distinct vertices of \( T \) can be connected by a path in \( T \). For a subset \( W \) of \( V \), define

\[ \operatorname{Chi}(W) := \bigcup_{u \in W} \{ v \in V : (u, v) \in E \}. \]

One may define inductively \( \operatorname{Chi}^{(n)}(W) \) for a non-negative integer \( n \) as follows:

\[ \operatorname{Chi}^{(n)}(W) := \begin{cases} W & \text{if } n = 0, \\ \operatorname{Chi}(\operatorname{Chi}^{(n-1)}(W)) & \text{if } n \geq 1. \end{cases} \]

Given \( v \in V \) and an integer \( n \geq 0 \), we set \( \operatorname{Chi}^{(n)}(v) := \operatorname{Chi}^{(n)}(\{v\}) \). An element of \( \operatorname{Chi}(v) \) is called a child of \( v \). For a given vertex \( v \in V \), consider the set

\[ \operatorname{Par}(v) := \{ u \in V : (u, v) \in E \}. \]

If \( \operatorname{Par}(v) \) is a singleton, then the unique vertex in \( \operatorname{Par}(v) \) is called the parent of \( v \), which we denote by \( \operatorname{par}(v) \). Define the subset \( \operatorname{Root}(T) \) of \( V \) by

\[ \operatorname{Root}(T) := \{ v \in V : \operatorname{Par}(v) = \emptyset \}. \]

An element of \( \operatorname{Root}(T) \) is called a root of \( T \). If \( \operatorname{Root}(T) \) is a singleton, then its unique element is denoted by \( \operatorname{root} \). We set \( V^\circ := V \setminus \operatorname{Root}(T) \). A directed graph \( T = (V, E) \) is called a directed tree if \( T \) has no circuits, \( T \) is connected and each vertex \( v \in V^\circ \) has a unique parent. A subgraph of a directed tree \( T \) which itself is a directed tree is said to be a subtree of \( T \). A directed tree \( T \) is said to be

(i) rooted if it has a unique root.
(ii) locally finite if \( \operatorname{card}(\operatorname{Chi}(u)) \) is finite for all \( u \in V \), where \( \operatorname{card}(X) \) stands for the cardinality of the set \( X \).
(iii) leafless if every vertex has at least one child.
(iv) narrow if there exists a positive integer \( m \) such that

\[ \operatorname{card}(\operatorname{Chi}^{(n)}(\text{root})) \leq m, \quad n \in \mathbb{N}. \quad (1.1.1) \]

The smallest positive integer \( m \) satisfying (1.1.1) will be referred to as the width of \( T \).

**Remark 1.1** Note that any narrow directed tree is necessarily locally finite. However, the converse is not true. Consider, for instance, the binary tree (see [38, Example 4.3.1]). It is worth noting that there exist narrow directed trees with \( \operatorname{card}(V^\prec) = \aleph_0 \), where \( V^\prec \) denotes the set of branching vertices of \( T \) defined by

\[ V^\prec := \{ u \in V : \operatorname{card}(\operatorname{Chi}(u)) \geq 2 \} \]
Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$. For each $u \in V$, the depth of $u$ is the unique non-negative integer $d_u$ such that $u \in \text{Chi}^{(d_u)}(\text{root})$ (see [38, Corollary 2.1.5]). We discuss here the convergence of nets associated with rooted directed trees induced by the depth. Define the relation $\leq$ on $V$ as follows:

$$v \leq w \text{ if } d_v \leq d_w,$$

where $d_v$ denotes the depth of $v$ in $\mathcal{T}$. Note that $V$ is a partially ordered set with partial order relation $\leq$, that is, $\leq$ is reflexive and transitive. Further, if $V$ is infinite, then given two vertices $v, w \in V$, there exists $u \in V$ such that $v \leq u$ and $w \leq u$ (the reader is referred to the discussion prior to [21, Remark 3.4.1] for details). In this text, we will frequently be interested in the nets $\{\mu_v\}_{v \in V}$ of complex numbers induced by the above partial order (the reader is referred to [66, Chapter 2] for the definition and elementary facts pertaining to nets).

Let $\mathcal{T} = (V, E)$ be a directed tree. For a vertex $u \in V$, we set $\text{par}^{(0)}(u) = u$. Note that the correspondence $\text{par}(\cdot) : u \mapsto \text{par}(u)$ is a partial function in $V$. For a positive integer $n$, by the partial function $\text{par}^{(n)}(\cdot)$, we understand $\text{par}(\cdot)$ composed with itself $n$-times. The descendant of a vertex $u \in V$ is defined by

$$\text{Des}(u) := \bigcup_{n=0}^{\infty} \text{Chi}^{(n)}(u) \text{ (disjoint sum)}$$

(see the discussion prior to [38, Eqn (2.1.10)]). Note that $\mathcal{T}_u = (\text{Des}(u), E_u)$ is a rooted subtree of $\mathcal{T}$ with root $u$, where

$$E_u := \{(v, w) \in E : v, w \in \text{Des}(u)\}.$$  

(1.1.3)

The ascendant or ancestor of a vertex $u \in V$ is defined by

$$\text{Asc}(u) := \{\text{par}^{(n)}(u) : n \geq 1\}.$$ 

In particular, a vertex is its descendant, while it is not its ascendant. Although this is not standard practice, we find it convenient. Note that a directed path from $u$ to $v$ in $\mathcal{T}$, denoted by $[u, v]$, is unique whenever it exists. Indeed, since there exists a path from $u$ to $v$ in $\mathcal{T}$, $v \in \text{Des}(u)$ and $d_v \geq d_u$. In this case, it is easy to see that

$$[u, v] = \{\text{par}^{(n)}(v) : n = d_v - d_u, d_v - d_u - 1, \ldots, 0\}.$$
Further, for \( u, v \in V \), we set
\[
(u, v) := \begin{cases} [u, v] \setminus \{u\} & \text{if } v \in \text{Des}(u), \\ \emptyset & \text{otherwise}. \end{cases}
\]

We also need the following subsets of \( V \): For \( u \in V \) and \( v \in \text{Des}(u) \),
\[
\begin{align*}
\text{Des}_v[u] & := \text{Des}(u) \setminus (u, v) \\
\text{Des}_v(u) & := \text{Des}(u) \setminus [u, v].
\end{align*}
\]
(1.1.4)

Note that \( \text{Des}_u[u] = \text{Des}(u) \) and \( \text{Des}_u(u) = \text{Des}(u) \setminus \{u\} \).

### 1.2 Hilbert Space Operators

For a subset \( \Omega \) of the complex plane \( \mathbb{C} \), let \( \text{int}(\Omega), \overline{\Omega} \) and \( \overline{\mathbb{C}} \setminus \Omega \) denote the interior, the closure and the complement of \( \Omega \) in \( \mathbb{C} \) respectively. We use \( \mathbb{R} \) to denote the real line, and \( \mathbb{N}_z \) and \( \mathbb{Z}_z \) denote the real and imaginary parts of a complex number \( z \) respectively. The conjugate of the complex number \( z \) will be denoted by \( \bar{z} \), while \( \text{arg}(z) \) stands for the argument of a non-zero complex number \( z \). We reserve the notation \( \mathbb{N} \) for the set of non-negative integers, while \( \mathbb{Z} \) (resp. \( \mathbb{Z}_+ \)) stands for the set of all integers (resp. all positive integers). Unless stated otherwise, all the Hilbert spaces occurring below are complex, infinite-dimensional and separable. Let \( \mathcal{H} \) be a complex, separable Hilbert space with the inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) and the corresponding norm \( \| \cdot \|_{\mathcal{H}} \). Whenever there is no ambiguity, we will suppress the suffix and simply write \( \langle x, y \rangle \) and \( \| x \| \) in place of \( \langle x, y \rangle_{\mathcal{H}} \) and \( \| x \|_{\mathcal{H}} \) respectively. By \( \text{span}\{x \in \mathcal{H} : x \in W\} \) (resp. \( \sqrt{\{x \in \mathcal{H} : x \in W\}} \)), we mean the smallest linear subspace (resp. smallest closed linear subspace) generated by the subset \( W \) of \( \mathcal{H} \). In case \( W = \{x\} \), we use the simpler notation \( [x] \) for the linear span of \( W \). The orthogonal complement of a closed subspace \( W \) of \( \mathcal{H} \) is denoted by \( \mathcal{H} \ominus W \). Sometimes \( \mathcal{H} \ominus W \) is denoted by \( W^\perp \).

Let \( S \) be a densely defined linear operator in \( \mathcal{H} \) with domain \( \mathcal{D}(S) \). The symbols \( \ker S \) and \( \text{ran} S \) will stand for the kernel of \( S \) and the range of \( S \) respectively. We use \( \sigma_p(S), \sigma_{ap}(S), \sigma(S) \) to denote the point spectrum, the approximate-point spectrum, and the spectrum of \( S \) respectively. It may be recalled that \( \sigma_p(S) \) is the set of eigenvalues of \( S \), that \( \sigma_{ap}(S) \) is the set of \( \lambda \) in \( \mathbb{C} \) for which \( S - \lambda I \) is not bounded below, and that \( \sigma(S) \) is the complement of the set of \( \lambda \) in \( \mathbb{C} \) for which \( (S - \lambda)^{-1} \) exists as a bounded linear operator on \( \mathcal{H} \). Here, by \( S - \lambda I \), we understand the linear operator \( S - \lambda I \) with \( I \) denoting the identity operator on \( \mathcal{H} \). We reserve the symbol \( B(\mathcal{H}) \) for the unital \( C^* \)-algebra of bounded linear operators on \( \mathcal{H} \). The resolvent set \( \rho(S) \) of \( S \) is defined as the complement of \( \sigma(S) \) in \( \mathbb{C} \). The resolvent function \( R_S : \rho(S) \to B(\mathcal{H}) \) is given by
\[
R_S(\lambda) := (S - \lambda)^{-1}, \quad \lambda \in \rho(S).
\]

The regularity domain \( \pi(S) \) of \( S \) is defined as the complement of \( \sigma_{ap}(S) \) in \( \mathbb{C} \). For \( \mu \in \pi(S) \), we refer to the linear subspace \( \text{ran}(S - \mu)^\perp \) of \( \mathcal{H} \) the deficiency subspace
of $S$ at $\mu$ and its dimension

$$d_S(\mu) := \dim \ker (S - \mu)^\perp,$$

(1.2.1)

the defect number of $S$ at $\mu$.

By the multiplicity function of $S$, we understand the function $m_S : \sigma_p(S) \to \mathbb{Z}_+ \cup \{\aleph_0\}$ assigning with each eigenvalue $\lambda$ of $S$, the dimension of the eigenspace $E_S(\lambda)$ of $S$ corresponding to $\lambda$. We extend $m_S$ to the entire complex plane by setting $m_S(\lambda) = 0$, $\lambda \in \mathbb{C}\setminus\sigma_p(S)$. We say that a densely defined linear operator $S$ in $\mathcal{H}$ is Fredholm if the range of $S$ is closed, $\dim \ker S$ and $\dim \ker S^*$ are finite. The essential spectrum $\sigma_e(S)$ of $S$ is the complement of the set of those $\lambda \in \mathbb{C}$ for which $S - \lambda$ is Fredholm. The Fredholm index $\text{ind}_S : \mathbb{C}\setminus\sigma_e(S) \to \mathbb{Z}$ is given by

$$\text{ind}_S(\lambda) := m_S(\lambda) - m_{S^*}(\lambda), \quad \lambda \in \mathbb{C}\setminus\sigma_e(S)$$

(the reader is referred to [46, 52, 64] for elementary properties of various spectra of unbounded linear operators).

Let $T$ be a densely defined linear operator in $\mathcal{H}$ with domain $\mathcal{D}(T)$. The closure (resp. adjoint) of $T$ is denoted by $\overline{T}$ (resp. $T^*$), whenever it exists. A subspace $\mathcal{D}$ of $\mathcal{H}$ is said to be a core of a closable linear operator $T$ if

$$\mathcal{D} \subseteq \mathcal{D}(T), \quad \overline{\mathcal{D}} = \mathcal{H}, \quad \text{and} \quad T|_\mathcal{D} = \overline{T}.$$

If $S$ is a linear operator in $\mathcal{H}$ such that

$$\mathcal{D}(S) \subseteq \mathcal{D}(T) \text{ and } Sh = Th \text{ for every } h \in \mathcal{D}(S),$$

then we say that $T$ extends $S$ (denoted by $S \subseteq T$). Note that two operators $S$ and $T$ are same if and only if $S \subseteq T$ and $T \subseteq S$. A closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be invariant for $T$ if $T(\mathcal{M} \cap \mathcal{D}(T)) \subseteq \mathcal{M}$. In this case, the restriction of $T$ to $\mathcal{M}$ is denoted by $T|_\mathcal{M}$. Note that if $T$ has invariant domain, that is, $T\mathcal{D}(T) \subseteq \mathcal{D}(T)$, then $T$ admits polynomial calculus in the sense that $p(T)$ is a well-defined linear operator with domain $\mathcal{D}(T)$ for every complex polynomial $p$ in one variable. A closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ is reducing for $T$ if there exist linear operators $T_0$ in $\mathcal{M}$ and $T_1$ in $\mathcal{M}^\perp$ such that $T = T_0 \oplus T_1$. The commutant of a linear operator $T$ is given by

$$\{T\}' := \{A \in B(\mathcal{H}) : AT \subseteq TA\}.$$

In case $T \in B(\mathcal{H})$, $\{T\}' = \{A \in B(\mathcal{H}) : AT = TA\}$. If $P_\mathcal{M}$ is an orthogonal projection of $\mathcal{H}$ onto a closed subspace $\mathcal{M}$ of $\mathcal{H}$, then $P_\mathcal{M} \in \{T\}'$ if and only if $\mathcal{M}$ is a reducing subspace for $T$ (see [64, Proposition 1.15]).

We recall definitions of some well-studied classes of unbounded linear operators, which are relevant to the present investigations (refer to [52, 57, 62, 64]). A densely defined linear operator $T$ in a complex Hilbert space $\mathcal{H}$ is said to be

- (i) **self-adjoint** if $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $T^*x = Tx$ for all $x \in \mathcal{D}(T)$. 
(ii) *normal* if $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $\|T^*x\| = \|Tx\|$ for all $x \in \mathcal{D}(T)$.

(iii) *nilpotent* if $T$ has invariant domain such that $T^n = 0$ for some positive integer $n$. The smallest positive integer with this property is referred to as the *nilpotency index* of $T$.

(iv) *complex Jordan* if there exist a normal operator $M$ and a nilpotent linear operator $N$ such that any one of the following holds:

(a) $M \in B(\mathcal{H})$, $T = M + N$ and $M \in \{N\}'$.

(b) $N \in B(\mathcal{H})$, $T = M + N$ and $N \in \{M\}'$.

1.3 Rank One Operators

We will see that the rank one (possibly unbounded) operators form building blocks in the orthogonal decomposition of weighted join operators (see Theorem 3.13). Hence we find it necessary to collect below several elementary properties of rank one operators.

Let $\mathcal{H}$ be a complex Hilbert space. For $x, y \in \mathcal{H}$, the *injective tensor product* $x \otimes y$ of $x$ and $y$ is defined by

$$x \otimes y(h) = \langle h, y \rangle x, \quad h \in \mathcal{H}.$$

Clearly, $x \otimes y$ is a rank one bounded linear operator. In fact, $\|x \otimes y\| = \|x\| \|y\|$. Conversely, every rank one bounded linear operator arises in this fashion. Indeed, if $T \in B(\mathcal{H})$ is a rank one operator with range spanned by a unit vector $y \in \mathcal{H}$, then for every $x \in \mathcal{H}$, $Tx = \alpha_x y$ for some scalar $\alpha_x \in \mathbb{C}$, and hence

$$Tx = \langle Tx, y \rangle y = \langle x, T^*y \rangle y = (y \otimes T^*y)(x), \quad x \in \mathcal{H}$$

(cf. [67, Proposition 2.1.1]). It is worth noting that a diagonal operator $D_\lambda$ on $\mathcal{H}$ with respect to an orthonormal basis $\{e_j\}_{j \in J}$ and diagonal entries $\lambda := \{\lambda_j\}_{j \in J} \subseteq \mathbb{C}$ (counted with multiplicities) can be rewritten uniquely (up to permutation) in terms of injective tensor products as follows:

$$D_\lambda = \sum_{j \in J} \lambda_j e_j \otimes e_j,$$

where $J$ is a directed set. Note that $D_\lambda$ is a bounded linear operator on $\mathcal{H}$ if and only if $\lambda$ forms a bounded subset of $\mathbb{C}$.

The analysis of weighted join operators relies on a thorough study of bounded and unbounded rank one operators. As we could not locate an appropriate reference for a number of facts essential in our investigations, we include their statements and elementary verifications.

**Lemma 1.2** Let $\mathcal{H}$ be a complex Hilbert space of dimension bigger than 1. For unit vectors $x, y, z, w \in \mathcal{H}$, we have the following statements:
(i) $x \otimes y$ is an algebraic operator:

$$p(x \otimes y) = 0 \text{ with } p(\lambda) = \lambda(\lambda - \langle x, y \rangle), \ \lambda \in \mathbb{C}. \quad (1.3.1)$$

(ii) $\sigma(x \otimes y) = \{0, \langle x, y \rangle\} = \sigma_p(x \otimes y)$. Moreover, the eigenspace $\mathcal{E}_{x \otimes y}(\mu)$ corresponding to the eigenvalue $\mu$ of $x \otimes y$ is given by

$$\mathcal{E}_{x \otimes y}(0) = [y]^\perp, \ \mathcal{E}_{x \otimes y}(\langle x, y \rangle) = [x].$$

Thus the multiplicity $m_{x \otimes y}(\mu)$ of the eigenvalue $\mu$ is given by

$$m_{x \otimes y}(0) = \aleph_0 \text{ if } \dim \mathcal{H} = \aleph_0, \ m_{x \otimes y}(\langle x, y \rangle) = 1 \text{ if } \langle x, y \rangle \neq 0.$$

(iii) The resolvent function $R_{x \otimes y}: \rho(x \otimes y) \to B(\mathcal{H})$ of $x \otimes y$ at $\mu$ is given by

$$R_{x \otimes y}(\mu) = -\frac{1}{p(\mu)} \left( (\mu - \langle x, y \rangle)P_{[x]^\perp} + x \otimes (P_{[x]^\perp}y + \tilde{\mu}x) \right),$$

where $p$ is as given in (1.3.1).

(iv) The commutant $\{x \otimes y\}'$ of $x \otimes y$ is given by

$$\{x \otimes y\}' = \{ A \in B(\mathcal{H}) : Ax = \langle Ax, x \rangle x, \ A^*y = \overline{\langle Ax, x \rangle y} \}.$$

(v) $x \otimes y$ is normal if and only if there exists a unimodular scalar $\alpha \in \mathbb{C}$ such that $x = \alpha y$.

(vi) $x \otimes y$ is self-adjoint if and only if $x = \pm y$.

**Proof** It is easy to see the following:

$$x \otimes \alpha y = \bar{\alpha} x \otimes y, \ \alpha \in \mathbb{C},$$

$$(x \otimes y)^* = y \otimes x, \ (x \otimes y)(z \otimes w) = \langle z, y \rangle x \otimes w. \quad (1.3.2)$$

To see (i), note that by (1.3.2), $x \otimes y$ satisfies

$$(x \otimes y)^2 = \langle x, y \rangle x \otimes y.$$

To see (ii), note that

$$(x \otimes y)(z) = 0 \text{ if } z \in [y]^\perp, \ (x \otimes y - \langle x, y \rangle)x = 0.$$

Further, by (i) and the spectral mapping property for polynomials [67],

$$p(\sigma(x \otimes y)) = \sigma(p(x \otimes y)) = [0],$$

where $p(z) = z(\bar{z} - \langle x, y \rangle)$. The desired conclusions in (ii) are now immediate.
To see the formula for the resolvent function $R_x \otimes y$ in (iii), let $f, g \in \mathcal{H}$ be such that $(x \otimes y - \mu) f = g$. Writing $h = P_{[x]^{\perp}}h + \langle h, x \rangle x$ for $h \in \mathcal{H}$, and comparing coefficients, we obtain

$$P_{[x]^{\perp}} f = -\frac{1}{\mu} P_{[x]^{\perp}} g, \quad \langle g, x \rangle = \langle f, y - \bar{\mu} x \rangle.$$ 

It follows that

$$\langle g, x \rangle = \langle P_{[x]^{\perp}} f + \langle f, x \rangle x, y - \bar{\mu} x \rangle$$

$$= \langle P_{[x]^{\perp}} f, y \rangle + \langle f, x \rangle (\langle x, y \rangle - \mu)$$

$$= -\frac{1}{\mu} \langle P_{[x]^{\perp}} g, y \rangle + \langle f, x \rangle (\langle x, y \rangle - \mu).$$

This yields

$$f = P_{[x]^{\perp}} f + \langle f, x \rangle x$$

$$= -\frac{1}{\mu} P_{[x]^{\perp}} g + \frac{1}{\langle x, y \rangle - \mu} \left( \langle g, x \rangle + \frac{1}{\mu} \langle P_{[x]^{\perp}} g, y \rangle \right) x.$$ 

It is now easy to see that $R_x \otimes y(\mu)$ has the desired expression.

To see (iv), note that $A$ commutes with $x \otimes y$ if and only if

$$\langle z, y \rangle Ax = \langle Az, y \rangle x, \quad z \in \mathcal{H}. \quad (1.3.3)$$

If $z \in [y]^{\perp}$, then $\langle Az, y \rangle x = 0$, and hence $Az \in [y]^{\perp}$. This shows that $A^*$ maps $[y]$ into $[y]$. Letting $z = y$ in (1.3.3), we obtain the necessity part of (iv). Conversely, if $Ax = \langle Ay, y \rangle x$, then (1.3.3) is equivalent to $\langle A(z - (z, y)y), y \rangle = 0$, which is equivalent to $A^*y = \langle Ay, y \rangle y$. To see (v), note that by (1.3.2), $(x \otimes y)^* = y \otimes x$, and apply (iv) to $A = y \otimes x$. The sufficiency part of (vi) follows immediately from (1.3.2). Assume that $(x \otimes y)^* = x \otimes y$. By (1.3.2), $x \otimes y = y \otimes x$. By (v), $x = \alpha y$ for some $\alpha \in \mathbb{C}$. Thus

$$\alpha y \otimes y = x \otimes y = y \otimes x = \bar{\alpha} y \otimes y.$$ 

It follows that $\alpha \in \mathbb{R}$. Since $x$, $y$ are unit vectors, $\alpha = \pm 1$. 

Recall that a densely defined linear operator $T$ in $\mathcal{H}$ admits a compact resolvent if there exists $\lambda \in \mathbb{C}\setminus \sigma(T)$ such that $(T - \lambda)^{-1}$ is compact. It may be concluded from Lemma 1.2(iii) that $x \otimes y$ has a compact resolvent if and only if $\dim \mathcal{H}$ is finite.

Let us discuss a class of unbounded, densely defined, but not necessarily closed (rank one) operators, which we denote as $f \otimes g$, $f \in \mathcal{H}$ and $g$ is unspecified (to be chosen later). Fix an orthonormal basis $\{e_j\}_{j \in J}$ of $\mathcal{H}$ for some directed set $J$, and let $g = \{g(j)\}_{j \in J}$. 


Define $f \odot g$ in $\mathcal{H}$ by

$$\mathcal{D}(f \odot g) = \left\{ h \in \mathcal{H} : \sum_{j \in J} h(j)\overline{g(j)} \text{ is convergent} \right\},$$

$$f \odot g(h) = \left( \sum_{j \in J} h(j)\overline{g(j)} \right) f, \quad h \in \mathcal{D}(f \odot g).$$

Note that $f \odot g$ is densely defined with span $\{e_j : j \in J\} \subseteq \mathcal{D}(f \odot g)$. Thus the Hilbert space adjoint $(f \odot g)^*$ of $f \odot g$ is well-defined. Recall that for $p \in [1, \infty]$, $\ell^p(J)$ is the Banach space of all $p$-summable complex functions $f : J \to \mathbb{C}$ endowed with the norm

$$\|f\|_p = \begin{cases} \left( \sum_{j \in J} |f(j)|^p \right)^{1/p} & \text{if } p < \infty, \\ \sup_{j \in J} |f(j)| & \text{if } p = \infty. \end{cases}$$

It turns out that $(f \odot g)^*$ is not densely defined unless $g \in \ell^2(J)$. Indeed, we have the following result:

**Lemma 1.3** Let $J$ be an infinite directed set. Fix an orthonormal basis $\{e_j : j \in J\}$ of $\mathcal{H}$, $f \in \mathcal{H}\setminus\{0\}$, and let $g = \{g(j)\}_{j \in J}$. Then the following statements are equivalent:

(i) $f \odot g$ admits a bounded linear extension to $\mathcal{H}$.

(ii) $f \odot g$ is closed.

(iii) $f \odot g$ is closable.

(iv) $g \in \ell^2(J)$.

(v) $\mathcal{D}(f \odot g) = \mathcal{H}$.

In case $g \notin \ell^2(J)$, $(f \odot g)^*$ is not densely defined, $\sigma(f \odot g) = \mathbb{C}$, and

$$\sigma_p(f \odot g) = \begin{cases} \{0, \sum_{j \in J} f(j)\overline{g(j)}\} & \text{if } f \in \mathcal{D}(f \odot g), \\ \{0\} & \text{otherwise.} \end{cases}$$

**Proof** To see the equivalence of (i)–(v), it suffices to check that (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (i). Suppose that $f \odot g$ is a closable operator with closed extension $A$. Since $A$ is a densely defined closed operator, by [52, Theorem. 1.5.15], $\mathcal{D}(A^*)$ is dense in $\mathcal{H}$ (see also [64, Theorem 1.8]). Assume that $g \notin \ell^2(J)$. Then $h \in \mathcal{D}(A^*)$ if and only if there exists a positive real number $c$ such that

$$|\langle Ax, h \rangle| \leq c\|x\|, \quad x \in \mathcal{D}(A). \quad (1.3.4)$$
However, for $x = \sum_{j \in F} g(j)e_j$ with $F \subseteq J$ and $\text{card}(F) < \infty$,

$$\langle Ax, \ h \rangle = \sum_{j \in F} |g(j)|^2 \langle f, \ h \rangle.$$ 

It follows that

$$\left( \sum_{j \in F} |g(j)|^2 \right)^{1/2} |\langle f, \ h \rangle| \leq c.$$ 

Since $g \notin \ell^2(J)$, we must have $\langle f, \ h \rangle = 0$. This shows that $\mathcal{D}(A^\ast) \subseteq \mathcal{H} \ominus [f]$. Since $f$ is non-zero, this contradicts the fact that the Hilbert space adjoint $A^\ast$ of $A$ is densely defined. This completes the verification of the implication (iii) $\Rightarrow$ (iv).

The implication (v) $\Rightarrow$ (i) may be derived from the uniform boundedness principle [23] applied to the family $\{ f \otimes g_n \}_{n \geq 0}$ of bounded linear operators, where $\{ g_n \}_{n \geq 0}$ is any finitely supported sequence converging pointwise to $g$. To see the remaining part, assume that $g \notin \ell^2(J)$. Arguing as in the preceding paragraph (with $A$ replaced by $f \otimes g$), we obtain $\mathcal{D}((f \otimes g)^\ast) \subseteq \mathcal{H} \ominus [f]$. The assertion that $\sigma(f \otimes g) = \mathbb{C}$ follows from the fact that any densely defined operator with a non-empty resolvent set is closed (see [22, Lemma 1.17]). Note that any $h \in \mathcal{D}(f \otimes g)$ such that $\sum_{j \in J} h(j)\overline{g(j)} = 0$ (there are infinitely many such vectors $h$) is an eigenvector for $f \otimes g$ corresponding to the eigenvalue 0. Suppose $f \in \mathcal{D}(f \otimes g)$. Then, as in the proof of Lemma 1.2(iii), it can be seen that $\sum_{j \in J} f(j)\overline{g(j)}$ is an eigenvalue of $f \otimes g$ corresponding to the eigenvector $f$. Since any eigenvector of $f \otimes g$ corresponding to a non-zero eigenvalue belongs to $[f]$, we conclude in this case that

$$\sigma_p(f \otimes g) = \left\{ 0, \sum_{j \in J} f(j)\overline{g(j)} \right\}.$$ 

This also shows that if $f \notin \mathcal{D}(f \otimes g)$, then $f \otimes g$ can not have a non-zero eigenvalue.

In case $f \otimes g$ is closable, the bounded extension of $f \otimes g$, as ensured by Lemma 1.3, is precisely $f \otimes g$. Otherwise, it may be concluded from (1.3.4) that $\mathcal{D}((f \otimes g)^\ast) = \mathcal{H} \ominus [f]$.

We conclude this section with a brief discussion on an interesting family of unbounded rank one operators.

**Example 1.4** Let $J$ be an infinite directed set. Let $f \in \ell^2(J) \setminus \{ 0 \}$ and let $g \in \ell^p(J)$, $1 \leq p \leq \infty$. Then $f \otimes g$ defines a densely defined rank one operator in $\ell^2(J)$ with domain $\mathcal{D}(f \otimes g) = \ell^q(J) \cap \ell^2(J)$, where $1 \leq q \leq \infty$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Indeed, since

$$\| (f \otimes g)(h) \|_2 \leq \| h \|_q \| g \|_p \| f \|_2, \quad h \in \ell^q(J),$$
f ∘ g is bounded when considered as a linear transformation from \( \ell^q(J) \) into \( \ell^2(J) \). Moreover, since \( \ell^p(J) \subseteq \ell^2(J) \) for \( 1 \leq p \leq 2 \), by Lemma 1.3, \( f \circ g \) belongs to \( B(\ell^2(J)) \) if and only if \( 1 \leq p \leq 2 \). Thus for \( g \in \ell^p(J) \setminus \ell^2(J) \) for some \( 2 < p \leq \infty \), \( f \circ g \) is a densely defined unbounded rank one operator in \( \ell^2(J) \) with domain \( \ell^q(J) \cap \ell^2(J) \).  

\[ \text{Ω} \]

Prologue

In the following discussion, we attempt to explain some important aspects of the present work with the aid of a family of rank one perturbations of weighted join operators. In the following exposition, we have tried to minimize the graph-theoretic prerequisites. In particular, we avoided the rather involved graph-theoretic definition of the class \( \{ W_{b,u} : u, b \in V \} \) of the so-called weighted join operators.

Let \( T = (V, E) \) be a rooted directed tree with root \( u \) and let \( b, u \in V \). Consider the closed subspace \( \ell^2(U_{b,u}) \) of \( \ell^2(V) \), where the subset \( U_{b,u} \) of \( V \) is given by

\[
U_{b,u} = \begin{cases} 
V \setminus \{ u \} & \text{if } b = u, \\
\text{Asc}(u) \cup \text{Des}_b(u) & \text{if } b \in \text{Des}_u(u), \\
\text{Des}_u(u) & \text{otherwise}
\end{cases}
\]  

(1.3.5)

(see (1.1.4)). Consider the standard orthonormal basis \( \{ e_v \}_{v \in U_{b,u}} \) of \( \ell^2(U_{b,u}) \) and let \( D_{b,u} \) denote the diagonal operator in \( \ell^2(U_{b,u}) \) defined as

\[
D_{b,u} e_v = \lambda_{u,v} e_v, \quad v \in U_{b,u},
\]

where the diagonal entries of \( D_{b,u} \) are given by

\[
\lambda_{u,v} := d_v - d_u, \quad v \in U_{b,u}.
\]  

(1.3.6)

with \( d_v \) denoting the depth of \( v \in V \) in \( T \). Consider the rank one operator \( N_{b,u} = e_u \otimes e_{A_u} \), where \( e_{A_u} := \sum_{v \in A_u} (d_v - d_u) e_v \) and the subset \( A_u \) of \( V \) is given by

\[
A_u = \begin{cases} 
\{ u, b \} & \text{if } b \in \text{Des}(u), \\
\text{Asc}(u) \cup \{ b, u \} & \text{otherwise}.
\end{cases}
\]

We also need the (possibly unbounded) rank one operator \( e_{w,x} \otimes g_x \), where \( w \in V \setminus U_{b,u} \), \( x \in \mathbb{R} \) and \( g_x : U_{b,u} \to \mathbb{C} \) is given by

\[
g_x(v) = d_v^x, \quad v \in U_{b,u}.
\]  

(1.3.7)

From the viewpoint of the spectral theory, we will be interested in the the family \( \mathcal{W} := \{ W_{w,x} : w \in V \setminus U_{b,u}, x \in \mathbb{R} \} \) of rank one extensions of weighted join operators.
defined as follows:

\[ \mathcal{D}(W_{w,x}) = \{ (h, k) : h \in \mathcal{D}(D_{u}^{(b)}) \cap \mathcal{D}(e_{w} \otimes g_{x}), k \in \ell^{2}(V \setminus U_{u}^{(b)}) \} \]

\[
W_{w,x} = \begin{bmatrix}
D_{u}^{(b)} & 0 \\
e_{w} \otimes g_{x} & N_{u}^{(b)}
\end{bmatrix}.
\]

Clearly, the linear operator \( W_{w,x} \) is densely defined in \( \ell^{2}(V) \). Further, it is not difficult to see that \( W_{w,x} \) is unbounded unless \( D_{u}^{(b)} \) belongs to \( B(\ell^{2}(U_{u}^{(b)})) \) and \( g_{x} \in \ell^{2}(U_{u}^{(b)}) \). Many conclusions can be drawn about the family \( \mathcal{W} \) of rank one extensions of weighted join operators. The analysis of Sects. 4 and 5 of this paper leads us to the following.

**Theorem 1.5** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root \( \text{root} \) and let \( b, u \in V \). Assume that \( (\text{Des}(u), E_{u}) \) is a countably infinite narrow subtree of \( \mathcal{T} \) (see (1.1.3)) and let \( U_{u}^{(b)} \) be as defined in (1.3.5). Then, for any \( w \in V \setminus U_{u}^{(b)} \) and \( x \in \mathbb{R} \), the spectrum of \( W_{w,x} \) is a proper subset of \( \mathbb{C} \) if and only if \( x < 1/2 \). In case \( x \in (-\infty, 1/2) \), we have the following statements:

(i) \( W_{w,x} \) is a closed operator with the domain being the orthogonal direct sum of \( \mathcal{D}(D_{u}^{(b)}) \) and \( \ell^{2}(V \setminus U_{u}^{(b)}) \).

(ii) \( \sigma(W_{w,x}) = \{ d_{v} - d_{u} : v \in U_{u}^{(b)} \cup \{ u \} \} = \sigma_{p}(W_{w,x}) \).

(iii) \( \sigma_{e}(W_{w,x}) \{ 0 \} = \{ d_{v} - d_{u} : v \in U_{u}^{(b)} \cup \{ u \}, \, \text{card(Chi}^{|d_{v}|}(\text{root})) = \infty \} \). Moreover, \( \text{ind}_{W_{w,x}} = 0 \) on \( \mathbb{C} \setminus \sigma_{e}(W_{w,x}) \).

(iv) \( W_{w,x} \) is a sectorial operator, which generates a strongly continuous quasi-bounded semigroup.

(v) \( W_{w,x} \) is never normal.

(vi) If, in addition, \( \mathcal{T} \) is leafless, then \( W_{w,x} \) admits a compact resolvent if and only if the set \( V_{<} \) of branching vertices of \( \mathcal{T} \) is disjoint from \( \text{Asc}(u) \).

**Remark 1.6** If \( x \geq 1/2 \), then \( \sigma(W_{w,x}) = \mathbb{C} \). Assume that \( x < 1/2 \). Then, by (1.3.5), \( \sigma(W_{w,x}) = \mathbb{N} \) if \( b \notin \text{Des}(u) \). Otherwise,

\[
\{ k \in \mathbb{N} : k \notin \{ 1, \ldots, d_{b} - d_{u} \} \}
\cup \{-1, \ldots, -d_{u} \} \subseteq \sigma(W_{w,x}) \subseteq \mathbb{N} \cup \{-1, \ldots, -d_{u} \}.
\]

The exact description of \( \sigma(W_{w,x}) \), \( x < 1/2 \) depends on the set \( V_{<} \cap \{ u, b \} \) and the leaf-structure of branches emanating from this part (see Fig. 2 for a pictorial representation of the spectra of \( W_{w,x}, x \in \mathbb{R} \), where \( x \) varies over the \( X \)-axis while the spectra \( \sigma(W_{w,x}) \) are plotted in the \( YZ \)-plane). Note that the spectral behaviour of the family \( \{ W_{w,x} \}_{x \in \mathbb{R}} \) changes at \( x = 1/2 \) (see Fig. 2).

A proof of Theorem 1.5 will be presented towards the end of Sect. 4. We remark that the conclusion of Theorem 1.5 may not hold in case \( \text{Des}(u) \) is not a narrow subtree of \( \mathcal{T} \) (see Remark 5.9). It is tempting to arrive at the conclusion that almost everything about \( W_{w,x} \) can be determined. However, this is not the case. Although, the Hilbert
space adjoint of $W_{w,x}$ is densely defined in case $-1 \leq x < 1/2$, neither we know its domain nor do we have a neat expression for $W_{w,x}^*$. Further, in case $x \geq 1/2$, we do not know whether or not $W_{w,x}$ is closable.

**Plan of the paper.** We conclude this section with the layout of the paper.

Section 2 provides the graph-theoretic framework essential in the study of weighted join operators. In particular, we introduce the notion of the extended directed tree and exploit the order structure on a rooted directed tree $\mathcal{T} = (V,E)$ to introduce a family of semigroup structures on the extended directed trees to be referred to as join operations at a base point (see Proposition 2.11). The join operation based at root (resp. $\infty$) is precisely the join (resp. meet) operation. We exhibit a pictorial illustration of the decomposition of the set $V$ of vertices into descendant, ascendant, and the rest with respect to a fixed vertex (see (2.2.1) and Fig. 4). This decomposition of $V$ helps to understand the action of join and meet operations. We conclude this section with a description of the set $M_{\lambda_u}^{(b)}(w)$ of vertices which join to a given vertex $u$ at another given vertex $w$ (see Proposition 2.14).

In Sect. 3, we see that the semigroup structures on the extended rooted directed tree $\mathcal{T} = (V,E)$, as introduced in Sect. 2, induce a family of operators referred to as weighted join operators $W_{\lambda_u}^{(b)}$. We show that for $b \neq \infty$, these operators are closed and that the linear span of the standard orthonormal basis of $\ell^2(V)$ forms a core for $W_{\lambda_u}^{(b)}$ (see Proposition 3.5). We further discuss the boundedness of weighted join operators (see Theorem 3.7). One of the main results in this section decomposes a weighted join operator with a base point in $V$ as the sum of a diagonal operator and a bounded operator of rank at most one (see Theorem 3.13). In case the base point is $\infty$, the weighted meet operator $W_{\lambda_u}^{(b)}$ turns out to be possibly an unbounded finite rank operator. Among various properties of weighted join operators, it is shown that $W_{\lambda_u}^{(b)}$ has large null summand except for at most finitely many choices of $u$ (see Corollary 3.15). It is also shown that the weighted join operator is either a complex Jordan operator of index 2 or it has a bounded Borel functional calculus (see Corollary 3.16). We also characterize compact weighted join operators and discuss an application to
the theory of commutators of compact operators (see Proposition 3.17 and Corollary 3.18). We further provide a description of the commutant of a bounded weighted join operator (see Theorem 3.19). We exhibit a concrete family of weighted join operators to conclude that the commutant of a weighted join operator is not necessarily abelian.

In Sect. 4, we capitalize on the graph-theoretic framework developed in earlier sections to introduce and study the class of rank one extensions of weighted join operators. In particular, we discuss the closedness, the structure of the Hilbert space adjoint and the spectral theory of rank one extensions of weighted join operators. We introduce two compatibility conditions which control the unbounded rank one component in the matrix representation of rank one extensions of weighted join operators and discuss their connection with certain discrete Hilbert transforms (see Proposition 4.7). Moreover, we characterize these conditions under some sparsity conditions on the weight systems of weighted join operators (see Proposition 4.10). We provide a complete spectral picture (including a description of spectra, point-spectra and essential spectra) for rank one extensions of weighted join operators (see Theorem 4.15). It turns out that the spectra of rank one extensions of weighted join operators satisfying the so-called compatibility condition I can be recovered from its point spectra. In case the compatibility condition I does not hold, either rank one extensions of weighted join operators are not closed or their domain of regularity is empty (see Corollary 4.18). We give an example of a rank one extension of a weighted join operator with spectrum properly containing the topological closure of its point spectrum (see Example 4.19). Further, under the assumption of compatibility condition I, we characterize rank one extensions of weighted join operators on leafless directed trees which admit compact resolvent (see Corollary 4.20). Given an unbounded closed subset $\sigma$ of the complex plane, we construct a non-trivial rank one extension of a weighted join operator with the spectrum same as $\sigma$ (see Corollary 4.22). Finally, we specialize Theorem 4.15 to weighted join operators and conclude that these operators are not complete unless they are complex Jordan (see Theorem 4.23 and Corollary 4.25).

In Sect. 5, we investigate various special classes of rank one extensions of weighted join operators. We exhibit a family of rank one extensions of weighted join operators, which are sectorial (see Proposition 5.1). A similar result is obtained for the generators of quasi-bounded strongly continuous semigroups (see Proposition 5.2). Further, we characterize the classes of hyponormal, cohyponormal, $n$-symmetric weighted join operators (see Propositions 5.3 and 5.5). It turns out that there are no non-normal hyponormal weighted join operators. On the other hand, if a weighted join operator is $n$-symmetric, then either $n = 1$ or $n \geq 3$. We also discuss the normality and the symmetricity of rank one extensions of weighted join operators (see Propositions 5.4 and 5.7). Towards the end of this section we present a proof of Theorem 1.5.

In Sect. 6, we discuss the counterpart of the theory of weighted join operators for rootless directed trees. The definition of join operation at a given base point becomes less obvious in view of the absence of depth of a vertex in a rootless directed tree. One of the main results in this section is a decomposition theorem analogous to Theorem 3.13 (see Theorem 6.6). It turns out that a weighted join operator $W_{\lambda b}^{(b)}$ on a rootless directed tree could be a possibly unbounded rank one perturbation of (unbounded) diagonal operator. Unlike the case of a rooted tree, the weighted join operator $W_{\lambda b}^{(b)}$, $b \neq \infty$,
need not be even closable (see Corollary 6.8). We also obtain a counterpart of Corollary 3.16 for weighted join operators on rootless directed trees (see Corollary 6.10). Finally, we briefly discuss some difficulties in the study of the rank one extensions of these operators.

In Sect. 7, we glimpse at the general theory of unbounded non-self-adjoint rank one perturbations of diagonal operators or the forms associated with diagonal operators. In particular, we discuss the sectoriality of rank one perturbations of diagonal operators and some of its immediate applications to the spectral theory. We also discuss the role of some compatibility conditions in the sectoriality of the form-sum of the form associated with a sectorial diagonal operator and a form associated with not necessarily square-summable functions $f$ and $g$.

We conclude the paper with an epilogue including several remarks and possible lines of investigations.

2 Semigroup Structures on Extended Directed Trees

In this section, we provide the graph-theoretic framework essential to introduce and study the so-called weighted join operators and their rank one extensions on rooted directed trees (see Sects. 3 and 4). In particular, we formally introduce the notion of the extended directed tree and exhibit a family of semigroup structures on it. We also present a canonical decomposition of a rooted directed tree suitable for understanding the actions of weighted join operators.

2.1 Join and Meet Operations on Extended Directed Trees

In what follows, we always assume that $\text{card}(V) = \aleph_0$. The following notion of the extended directed tree plays an important role in unifying theories of weighted join operators and weighted meet operators.

Definition 2.1 (Extended directed tree) Let $\mathcal{T} = (V, E)$ be a rooted directed tree. The extended directed tree $\mathcal{T}_\infty$ associated with $\mathcal{T}$ is the directed graph $(V_\infty, E_\infty)$ given by

$$V_\infty = V \sqcup \{\infty\}, \quad E_\infty = E \sqcup \{(u, \infty) : u \in V\}.$$ 

Remark 2.2 The element $\infty \in V_\infty$ is descendant of every vertex in $V$. Indeed, $\infty \in \text{Chi}(u)$ for every $u \in V$. In view of Friedman’s notion of the graph with boundary (see [30, Pg 490]), it’s worth pointing out that $\infty$ is a boundary point when $\mathcal{T}_\infty$ is considered as the graph with boundary.

A pictorial representation of an extended directed tree $\mathcal{T}_\infty = (V_\infty, E_\infty)$ with the vertex set $V = \{\text{root}, v_1, v_2, \ldots\}$ is given in Fig. 3.

Definition 2.3 (Join operation) Let $\mathcal{T} = (V, E)$ denote a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. 
Given $u, v \in V_\infty$, we say that $u$ joins $v$ if either $u \in \text{Des}(v)$ or $v \in \text{Des}(u)$. Further, we set

$$u \sqcup v = \begin{cases} u & \text{if } u \in \text{Des}(v), \\ v & \text{if } v \in \text{Des}(u), \\ \infty & \text{otherwise.} \end{cases}$$

By Remark 2.2, $\infty \sqcup u = \infty = u \sqcup \infty$ for any $u \in V_\infty$. Further, the join operation $\sqcup$ satisfies the following:

- (Commutativity) $u \sqcup v = v \sqcup u$ for all $u, v \in V_\infty$, 
- (Associativity) $(u \sqcup v) \sqcup w = u \sqcup (v \sqcup w)$ for all $u, v, w \in V_\infty$, 
- (Neutral element) $u \sqcup \text{root} = u = \text{root} \sqcup u$ for all $u \in V_\infty$, 
- (Absorbing element) $u \sqcup \infty = \infty = \infty \sqcup u$ for all $u \in V_\infty$.

We summarize (2.1.1) in the following lemma.

**Lemma 2.4** Let $T = (V, E)$ be a rooted directed tree with root root and let $T_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $T$. Then the pair $(V_\infty, \sqcup)$ is a commutative semigroup admitting root as a neutral element and $\infty$ as an absorbing element.

Before we define the meet operation, let us introduce the following useful notation. For $u, v \in V$,

$$\text{par}(u, v) := \{ w \in V : \text{par}^m(u) = w = \text{par}^n(v) \text{ for some } m, n \in \mathbb{N} \}.$$ 

**Definition 2.5** (Meet operation) Let $T = (V, E)$ be a rooted directed tree with root root and let $T_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $T$. Let $u, v \in V$. We say that $u$ meets $v$ if there exists a unique vertex $\omega \in V$ such that

$$\sup_{w \in \text{par}(u, v)} d_w = d_{\omega}.$$
In this case, we set \( u \cap v = \omega \). In case \( u \in V_\infty \), we set

\[
\infty \cap u = u = u \cap \infty.
\]

**Remark 2.6** Note that

\[
u \in \text{par}(v, v) \implies u \cap v = u = v \cap u. \tag{2.1.2}\]

In fact, if \( u = \text{par}(l)(v) \) for some \( l \in \mathbb{N} \), then

\[
\text{par}(u, v) = \{ w \in V : \text{par}(n)(u) = w = \text{par}(m)(v) \text{ for some } m, n \in \mathbb{N} \}
\]

\[
= \{ w \in V : \text{par}(m)(u) = w \text{ for some } m \in \mathbb{N} \}.
\]

The conclusion in (2.1.2) is now immediate.

**Lemma 2.7** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root \( \text{root} \) and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). Any two vertices \( u, v \in V \) always meet. Further, they meet in a unique vertex \( \omega \) belonging to \([\text{root}, u] \cap [\text{root}, v]\), so that \( \max_{w \in \text{par}(u, v)} d_w = d_\omega \).

**Proof** Note that the set \( \text{par}(u, v) \) is non-empty. Indeed, \( \text{root} \in \text{par}(u, v) \), since

\[
\text{par}(d_u)(u) = \text{root} = \text{par}(d_v)(v).
\]

Further, for any \( w \in \text{par}(u, v) \), there exist \( m, n \in \mathbb{N} \) such that

\[
d_w = d_u - n = d_v - m \leq \min\{d_u, d_v\} < \infty.
\]

This shows that

\[
\sup_{w \in \text{par}(u, v)} d_w < \infty. \tag{2.1.3}
\]

We claim that \( \text{par}(u, v) \) is finite. To see this, in view of (2.1.3), it suffices to check that for any two distinct vertices \( x, y \in \text{par}(u, v) \), \( d_x \neq d_y \). Note that for some integers \( n_1, n_2 \in \mathbb{N} \),

\[
\text{par}(n_1)(u) = x, \quad \text{par}(n_2)(u) = y. \tag{2.1.4}
\]

It follows that

\[
d_x = d_u - n_1, \quad d_y = d_u - n_2 \implies d_x = d_y + n_2 - n_1. \tag{2.1.5}
\]

Since \( x \neq y \), by (2.1.4), \( n_1 \neq n_2 \). Hence, by (2.1.5), \( d_x \neq d_y \). This also shows that \( \sup \text{par}(u, v) \) is attained at a unique vertex in \( \text{par}(u, v) \). The remaining part also follows from this. \( \Box \)
Let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with the rooted directed tree $\mathcal{T}$. Then the meet operation $\sqcap$ satisfies the following:

\begin{align*}
\text{(Commutativity)} & \quad u \sqcap v = v \sqcap u \quad \text{for all } u, v \in V_\infty, \\
\text{(Associativity)} & \quad (u \sqcap v) \sqcap w = u \sqcap (v \sqcap w) \quad \text{for all } u, v, w \in V_\infty, \\
\text{(Neutral element)} & \quad u \sqcap \infty = u = \infty \sqcap u \quad \text{for all } u \in V_\infty, \\
\text{(Absorbing element)} & \quad u \sqcap \text{root} = \text{root} = \text{root} \sqcap u \quad \text{for all } u \in V_\infty.
\end{align*}

We summarize (2.1.6) in the following lemma.

**Lemma 2.8** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. Then the pair $(V_\infty, \sqcap)$ is a commutative semigroup admitting $\infty$ as a neutral element and $\text{root}$ as an absorbing element.

The operations meet and join can be unified in the following manner.

**Definition 2.9** (Join operation at a base point) Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. Fix $b \in V_\infty$ and let $u, v \in V_\infty$. Define the binary operation $\sqcup_b$ on $V_\infty$ by

\[
\begin{aligned}
u \sqcup_b v &= \begin{cases} 
  u \sqcap v & \text{if } u, v \in \text{Asc}(b), \\
  u & \text{if } v = b, \\
  v & \text{if } b = u, \\
  u \sqcup v & \text{otherwise.}
\end{cases}
\end{aligned}
\]

**Remark 2.10** Clearly, $\sqcup_{\text{root}} = \sqcup$. Further, by Remark 2.2, $\sqcup_\infty = \sqcap$. Thus we have a family of countably many operations, the first of which (corresponding to $\text{root}$) is the join operation, while the farthest operation (corresponding to $\infty$) is the meet operation.

**Proposition 2.11** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. Then, for every $b \in V$, the pair $(V_\infty, \sqcup_b)$ is a commutative semigroup admitting $b$ as a neutral element and $\infty$ as an absorbing element.

**Proof** Let $b \in V$. The fact that $(V_\infty, \sqcup_b)$ is commutative and associative may be deduced from Lemmata 2.4 and 2.8. To complete the proof, note that $b$ is a neutral element, while $\infty$ is an absorbing element for $(V_\infty, \sqcup_b)$.

### 2.2 A Canonical Decomposition of an Extended Directed Tree

For a directed tree $\mathcal{T} = (V, E)$ and $u \in V$, the set $V$ of vertices can be decomposed into three disjoint subsets:

\[
V = \text{Des}(u) \sqcup \text{Asc}(u) \sqcup V_u,
\]

(2.2.1)
where \( V_u \) is the complement of \( \text{Des}(u) \sqcup \text{Asc}(u) \) in \( V \) (see Fig. 4 for a pictorial representation of this decomposition with \( u := v_4 \)). Note that if \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) is the extended directed tree associated with \( \mathcal{T} \), then \( V_\infty \) decomposes as follows:

\[
V_\infty = \text{Des}(u) \sqcup \text{Asc}(u) \sqcup V_u,
\]

(2.2.2)

where \( \infty \in \text{Des}(u) \) by the very definition of the extended directed tree.

**CAUTION** Whenever we consider the decomposition in (2.2.1), it is understood that the directed tree under consideration is \( \mathcal{T} \) (and not the extended directed graph \( \mathcal{T}_\infty \)), so that \( \infty \notin \text{Des}(u) \).

It turns out that the cardinality of \( V_u \) being infinite is intimately related to the large null summand property of weighted join operators (see Corollary 3.15). We record the following general fact for ready reference.

**Proposition 2.12** Let \( \mathcal{T} = (V, E) \) be a directed tree and let \( u \in V \). If \( V_u \) is as given in (2.2.1), then the following statements hold:

(i) If \( \mathcal{T} \) is rooted, then \( \text{card}(V_u) = \aleph_0 \) if and only if \( \text{card}(V \setminus \text{Des}(u)) = \aleph_0 \).

(ii) If \( \mathcal{T} \) is leafless, then \( \text{card}(V_u) = \aleph_0 \) if and only if there exists a branching vertex \( w \in \text{Asc}(u) \).

(iii) If \( \mathcal{T} \) is leafless, then either \( \text{card}(V_u) = 0 \) or \( \text{card}(V_u) = \aleph_0 \).

**Proof** Note that (i) follows from (2.2.1) and the fact that \( \text{card}(\text{Asc}(u)) < \infty \) for any \( u \in V \) provided \( \mathcal{T} \) is rooted. To see (ii), suppose \( \mathcal{T} \) is leafless and assume that there exists a vertex \( v \in \text{Chi}(w) \) such that \( v \notin \text{Asc}(u) \cup \text{Des}(u) \). Further, \( \text{Des}(v) \) is contained in \( V_u \). Since \( \mathcal{T} \) is leafless, \( \text{card}(\text{Des}(v)) = \aleph_0 \). This proves the sufficiency part of (ii). On the other hand, if all vertices in \( \text{Asc}(u) \) are non-branching, then \( V = \text{Des}(u) \sqcup \text{Asc}(u) \), which, by (2.2.1), implies that \( V_u = \emptyset \). This also yields (iii). \( \Box \)

To see the role of canonical decompositions (2.2.1) and (2.2.2) in determining the join and meet of two vertices, let us see an example.

**Example 2.13** Consider the rooted directed tree \( \mathcal{T} = (V, E) \) as shown in Fig. 4. To get an essential idea about the operations of meet and join, let us compute \( v_4 \sqcup v \) and \( v_4 \sqcap v \) for \( v \in V \). Note that

\[
v_4 \sqcup v = \begin{cases} 
v & \text{if } v \in \{v_4, v_7, \ldots\}, \\
v_4 & \text{if } v \in \{v_2, v_0, \text{root}\}, \\
\infty & \text{if } v \in \{v_1, v_3, \ldots\} \cup \{v_5, v_8, \ldots\}.
\end{cases}
\]

Similarly, one can see that

\[
v_4 \sqcap v = \begin{cases} 
v_4 & \text{if } v \in \{v_4, v_7, \ldots\}, \\
v & \text{if } v \in \{v_2, v_0, \text{root}\}, \\
v_0 & \text{if } v \in \{v_1, v_3, \ldots\}, \\
v_2 & \text{if } v \in \{v_5, v_8, \ldots\}.
\end{cases}
\]
Fig. 4 A rooted directed tree $\mathcal{T} = (V, E)$, where $V$ is the disjoint union of $\text{Des}(v_4) = \{v_4, v_7, \ldots\}$, $\text{Asc}(v_4) = \{v_2, v_0, \text{root}\}$, and $V_{v_4} = \{v_1, v_3, \ldots\} \cup \{v_5, v_8, \ldots\}$

Note that two vertices in $V$ can join at the vertex $\infty$, while two vertices in $V$ always meet in $V$.

Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. Fix $u \in V$. Then, for any $b \in V_\infty \setminus \{u\}$ and $v \in V$, the binary operation $\sqcup_b$ on $V_\infty$ satisfies the following:

\[
\begin{align*}
u \sqcup_b v &= \left\{ \begin{array}{ll}
u & \text{if } b \in \text{Asc}(u) \cup V_u, \ v \in \text{Asc}(u), \\
v & \text{if } b \in \text{Des}(u), \ v \in \text{Asc}(u), \\
v & \text{if } b \in \text{Asc}(u), \ v \in \text{Des}(u), \\
u & \text{if } b \in \text{Des}(u), \ v \in [u, b], \\
v & \text{if } b \in \text{Des}(u) \setminus \{\infty\}, \ v \in \text{Des}_b(u), \\
u & \text{if } b = \infty, \ v \in \text{Des}_b(u), \\
v & \text{if } b \in V_u, \ v \in \text{Des}(u),
\end{array} \right.
\end{align*}
\]

where $[u, v]$ denotes the directed path from $u$ to $v$ in a directed tree $\mathcal{T}$. The above discussion is summarized in the following table:

We conclude this section with a useful result describing the set of vertices, which join to a given vertex (with respect to a base point) at another given vertex.

**Proposition 2.14** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$ and $w \in V_\infty$, define

\[
M^{(b)}_u(w) := \{v \in V : u \sqcup_b v = w\}. \tag{2.2.3}
\]

Then the following statements hold:
Table 1 Join operation $u \sqcup b v$ at
the base point $b$

| $\frac{b \rightarrow v}{v}$ | $\text{Asc}(u)$ | $\text{Des}(u) \setminus [u, \infty]$ | $u$ | $V_u$ | $\{\infty\}$ |
|-------------------------|----------------|---------------------------------|-----|-----|---------------|
| $\text{Asc}(u)$         | $u$            | $v$                             | $v$ | $u$ | $v$           |
| $\text{Des}_b(u)$       | $v$            | $v$                             | $v$ | $u$ | $u$           |
| $[u, b]$                | $-$            | $u$                             | $u$ | $-$ | $u$           |
| $V_u \setminus \{b\}$   | $\infty$       | $\infty$                        | $v$ | $\infty$ | $u \cap v$   |
| $\{b\}$                | $u$            | $u$                             | $b$ | $u$ | $u$           |

(i) If $b = \infty$, then

$$M_u^{(b)}(w) = \begin{cases} 
\text{Des}(\text{par}^{[j]}(u)) \setminus \text{Des}(\text{par}^{[j-1]}(u)) & \text{if } w = \text{par}^{[j]}(u), \ j = 1, \ldots, d_u, \\
\text{Des}(u) & \text{if } w = u, \\
\emptyset & \text{otherwise}.
\end{cases}$$

(ii) If $b \in V$ and $u \in \text{Asc}(b)$, then

$$M_u^{(b)}(w) = \begin{cases} 
\{w\} & \text{if } w \in \text{Asc}(u) \sqcup \text{Des}_b(u), \\
[u, b] & \text{if } u \neq b \text{ and } w = u, \\
V_u & \text{if } w = \infty, \\
\emptyset & \text{otherwise}.
\end{cases}$$

(iii) If $b \in V$ and $u \not\in \text{Asc}(b)$, then

$$M_u^{(b)}(w) = \begin{cases} 
\{w\} & \text{if } b = u \text{ or } w \in \text{Des}_u(u), \\
\text{Asc}(u) \cup [u, b] & \text{if } u \neq b \text{ and } w = u, \\
V_u \setminus \{b\} & \text{if } w = \infty, \\
\emptyset & \text{otherwise}.
\end{cases}$$

(iv) If $b \in V \setminus \{u\}$, then

$$V_\infty \setminus M_u^{(b)}(\infty) = \text{Asc}(u) \cup \text{Des}(u) \cup \{b\}.$$ 

**Proof** By the definition of join operation $\sqcup_b$, $M_u^{(b)}(w)$ equals

$$\begin{cases} 
\{v \in \text{Asc}(b) : u \cap v = w\} \cup \{v \in V \setminus \text{Asc}(b) : u \sqcup v = w\} & \text{if } u \in \text{Asc}(b), w \neq u, \\
\{v \in \text{Asc}(b) : v \in \text{Des}(w)\} \cup ([b] \cap V) & \text{if } u \in \text{Asc}(b), w = u, \\
\{w\} & \text{if } b = u, \\
\{v \in V : u \sqcup v = w\} & \text{if } u \not\in \text{Asc}(b), u \neq b.
\end{cases}$$

The desired conclusions in (i)–(iii) can be easily deduced from the facts that

$$\text{Asc}(\infty) = V, \ u \cap v \in \text{par}(u, v), \ u \sqcup v \in \text{Des}(u) \cap \text{Des}(v), \ u, v \in V.$$
The parts (i)–(iii) may also be deduced from Table 1. To see (iv), let \( b \in V \setminus \{ u \} \). As seen above,

\[
M_u^{(b)}(\infty) = \begin{cases} 
\{ v \in \text{Asc}(b) : u \cap v = \infty \} \cup V_u & \text{if } u \in \text{Asc}(b), \\
\{ v \in V \setminus \{ b \} : u \sqcup v = \infty \} & \text{otherwise},
\end{cases}
\]

where \( V_u \) is as given in (2.2.1). Thus, in any case, \( M_u^{(b)}(\infty) = V_u \setminus \{ b \} \), and hence

\[
\text{Asc}(u) \cup \text{Des}(u) \cup \{ b \} \subseteq V_\infty \setminus M_u^{(b)}(\infty).
\]

To see the reverse inclusion, let \( v \in V_\infty \setminus M_u^{(b)}(\infty) \). Since \( \infty \in \text{Des}(u) \), we may assume that \( v \neq \infty \). Then \( u \sqcup b \ v \in V \), and hence we may further assume that \( v \neq b \). If \( u, v \in \text{Asc}(b) \), then \( v \in \text{Asc}(u) \cup \text{Des}(u) \). Otherwise, \( u \sqcup v = u \sqcup b \ v \in V \), and hence \( u \in \text{Des}(v) \) or \( v \in \text{Des}(u) \). In this case also, \( v \in \text{Asc}(u) \cup \text{Des}(u) \). This yields the desired equality in (iv).

The last result turns out to be crucial in decomposing the so-called weighted join operator as a direct sum of a diagonal operator and a finite rank operator.

### 3 Weighted Join Operators on Rooted Directed Trees

Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root root and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). In what follows, \( \ell^2(V) \) stands for the Hilbert space of square-summable complex functions on \( V \) equipped with the standard inner product. Note that the set \( \{ e_u \}_{u \in V} \) is an orthonormal basis of \( \ell^2(V) \), where \( e_u : V \to \mathbb{C} \) is the indicator function of \( \{ u \} \). The convention \( e_\infty = 0 \) will be used throughout this text. Note that \( \ell^2(V) \) is a reproducing kernel Hilbert space. Indeed, for every \( v \in V \), the evaluation map \( f \mapsto f(v) \) is a bounded linear functional from \( \ell^2(V) \) into \( \mathbb{C} \). For a non-empty subset \( W \) of \( V \), \( \ell^2(W) \) may be considered as a subspace of \( \ell^2(V) \). Indeed, if one extends \( f : W \to \mathbb{C} \) by setting \( F := f \) on \( W \) and 0 on \( V \setminus W \), then the mapping \( U : \ell^2(W) \to \ell^2(V) \) given by \( Uf = F \) is an isometric homomorphism. Sometimes, the orthogonal projection \( P_{\ell^2(W)} \) of \( \ell^2(V) \) onto \( \ell^2(W) \) will be denoted by \( P_w \). We say that a closed subspace \( M \) of \( \ell^2(V) \) is supported on a subset \( W \) of \( V \) if \( M = \bigvee \{ e_v : v \in W \} \). In this case, we refer to \( W \) as the support of \( M \) and we write \( \text{supp} M := W \).

**Definition 3.1** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root root and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( u \in V \) and \( b \in V_\infty \), by the weight system \( \lambda_u = \{ \lambda_{uv} \}_{v \in V_\infty} \), we understand the subset \( \{ \lambda_{uv} \}_{v \in V_\infty} \) of complex numbers such that \( \lambda_{u\infty} = 0 \).

(i) The diagonal operator \( D_{\lambda_u} \) on \( \mathcal{T} \) is given by

\[
\mathcal{D}(D_{\lambda_u}) := \left\{ f \in \ell^2(V) : \sum_{v \in V} |f(v)|^2 |\lambda_{uv}|^2 < \infty \right\}
\]
\[ D_{\lambda u} f := \sum_{v \in V} f(v) \lambda_{uv} e_v, \quad f \in \mathcal{D}(D_{\lambda u}). \]

(ii) The weighted join operator \( W_{\lambda u}^{(b)} \) on \( \mathcal{D} \) is given by

\[
\mathcal{D}(W_{\lambda u}^{(b)}) := \{ f \in \ell^2(V): A_{\lambda u}^{(b)} f \in \ell^2(V) \},
\]

\[
W_{\lambda u}^{(b)} f := A_{\lambda u}^{(b)} f, \quad f \in \mathcal{D}(W_{\lambda u}^{(b)}),
\]

where \( A_{\lambda u}^{(b)} \) is the mapping defined on complex functions \( f \) on \( V \) by

\[ (A_{\lambda u}^{(b)})(w) := \sum_{v \in M_{\lambda u}^{(b)}(w)} \lambda_{uv} f(v), \quad w \in V \]  \hspace{1cm} (3.0.1)

with the set \( M_{\lambda u}^{(b)}(w) \) given by (2.2.3).

The operator \( W_{\lambda u}^{(\infty)} \) is referred to as the weighted meet operator.

**Remark 3.2** Several remarks are in order.

(i) It is well-known that \( D_{\lambda u} \) is a densely defined closed linear operator in \( \ell^2(V) \). Its adjoint \( D_{\lambda u}^* \) is the diagonal operator with diagonal entries \( \{ \lambda_{uv} \}_{v \in V} \). Furthermore, \( D_{\lambda u} \) is normal and \( \mathcal{D}_V \) is a core for \( D_{\lambda u} \), where

\[ \mathcal{D}_U = \text{span} \{ e_v : v \in U \}, \quad U \subseteq V, \]  \hspace{1cm} (3.0.2)

(see [38, Lemma 2.2.1]).

(ii) Note that \( \mathcal{D}(W_{\lambda u}^{(b)}) \) forms a subspace of \( \ell^2(V) \). Indeed, if \( f, g \in \mathcal{D}(W_{\lambda u}^{(b)}) \), then for every \( w \in V \), the series \( (A_{\lambda u}^{(b)} f)(w) \) and \( (A_{\lambda u}^{(b)} g)(w) \) converge, and hence so does \( (A_{\lambda u}^{(b)} (f + g))(w) \). In particular, by Proposition 2.14, these series are finite sums in case \( b \neq \infty \). Also, \( A_{\lambda u}^{(b)} (f + g) \in \ell^2(V) \) if \( A_{\lambda u}^{(b)} (f) \in \ell^2(V) \) and \( A_{\lambda u}^{(b)} (g) \in \ell^2(V) \). Indeed,

\[
\| A_{\lambda u}^{(b)} (f + g) \|^2 \leq 2(\| A_{\lambda u}^{(b)} (f) \|^2 + \| A_{\lambda u}^{(b)} (g) \|^2).
\]

(iii) For every \( v \in V \), \( e_v \in \mathcal{D}(W_{\lambda u}^{(b)}) \) and

\[
(W_{\lambda u}^{(b)} e_v)(w) = \sum_{\eta \in M_{\lambda u}^{(b)}(w)} \lambda_{uv} e_v(\eta) = \lambda_{uv} e_{u \sqcup b} v(w), \quad w \in V. \]  \hspace{1cm} (3.0.3)

In particular,

\[ \mathcal{D}_V := \text{span} \{ e_v : v \in V \} \subseteq \mathcal{D}(W_{\lambda u}^{(b)}), \quad W_{\lambda u}^{(b)} \mathcal{D}_V \subseteq \mathcal{D}_V. \]  \hspace{1cm} (3.0.4)
Thus all positive integral powers of $W^{(b)}_{\lambda u}$ are densely defined and the Hilbert space adjoint $W^{(b)*}_{\lambda u}$ of $W^{(b)}_{\lambda u}$ is well-defined. To see the action of $W^{(b)*}_{\lambda u}$, let $f \in l^2(V)$ and $w \in V$. Note that

$$\langle W^{(b)}_{\lambda u} f, e_w \rangle = \sum_{v \in V} f(v) \lambda_{uv} e_{v \sqcup b} u_v = \sum_{v \in M^{(b)}_u(w)} f(v) \lambda_{uv} = \langle f, g_w \rangle,$$

where $M^{(b)}_u(w)$ is as given in (2.2.3) and $g_w = \sum_{v \in M^{(b)}_u(w)} \tilde{\lambda}_{uv} e_v$. Thus we obtain

$$\left( W^{(b)}_{\lambda u} \right)^* g = \sum_{w \in V} \sum_{v \in M^{(b)}_u(w)} g(w) \tilde{\lambda}_{uv} e_v, \quad g \in D(W^{(b)*}_{\lambda u}).$$

(iv) Finally, note that if $b = u$, then $W^{(b)}_{\lambda u}$ is the diagonal operator $D_{\lambda u}$.

Let us see two simple yet instructive examples of rooted directed trees in which the associated weighted join operators take a concrete form. Both these examples of rooted directed trees have been discussed in [38, Eqn (6.2.10)] in the context of weighted shifts on directed trees.

**Example 3.3 (With no branching vertex)** Consider the directed tree $T_1$ with the set of vertices $V = \mathbb{N}$ and root $= 0$. We further require that $\text{Chi}(n) = \{ n + 1 \}$ for all $n \in \mathbb{N}$. Let $m \in V$ and $n, b \in V_\infty$. By (3.0.3), the weighted join operator $W^{(b)}_{\lambda m}$ on $T$ is given by

$$W^{(b)}_{\lambda m} e_n = \begin{cases} 
\lambda_{mn} e_{\min\{m,n\}} & \text{if } m < b \text{ and } n < b, \\
\lambda_{mn} e_m & \text{if } n = b, \\
\lambda_{mn} e_n & \text{if } m = b \text{ and } n \in \mathbb{N}, \\
\lambda_{mn} e_{\max\{m,n\}} & \text{otherwise}, 
\end{cases}$$

where we used the assumption that $\lambda_{m\infty} = 0$ and the convention that $\max\{m, \infty\} = \infty$. In particular,

$$W^{(b)}_{\lambda m} e_n = \begin{cases} 
\lambda_{mn} e_{\max\{m,n\}} & \text{if } b = 0, \\
\lambda_{mn} e_{\min\{m,n\}} & \text{if } b = \infty. 
\end{cases}$$

The matrix representations of $W^{(0)}_{\lambda m}$ and $W^{(\infty)}_{\lambda m}$ with respect to the ordered orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ are given by
Thus $W_{\lambda m}^{(0)}$ is an at most rank one perturbation of a diagonal operator, while $W_{\lambda m}^{(\infty)}$ is a finite rank operator with a range contained in the linear span of $\{e_k : k = 0, \ldots, m\}$.

**Example 3.4 (With one branching vertex)** Consider the directed tree $T_2$ with a set of vertices

$$V = \{0\} \cup \{2j - 1, 2j : j \geq 1\}$$

and root $= 0$. We further require that $\text{Chi}(0) = \{1, 2\}$, $\text{Chi}(2j - 1) = \{2j + 1\}$ and $\text{Chi}(2j) = \{2j + 2\}$, $j \geq 1$. Let $m \in V$ and $n \in V_\infty$. By (3.0.3), the weighted join operator $W_{\lambda m}^{(b)}$ on $\mathcal{T}$ is given by

$$W_{\lambda m}^{(b)} e_n = \begin{cases} 
\lambda_{mn} e_{m \lor n} & \text{if } m, n \in \text{Asc}(b), \\
\lambda_{mn} e_m & \text{if } n = b, \\
\lambda_{mn} e_n & \text{if } m = b \text{ and } n \in \mathbb{N}, \\
\lambda_{mn} e_{m \land n} & \text{otherwise}.
\end{cases}$$

In particular, if $m$ and $n$ are positive integers, then

$$W_{\lambda m}^{(0)} e_n = \begin{cases} 
\lambda_{mn} e_{\max\{m,n\}} & \text{if } m, n \text{ are odd or } m, n \text{ are even}, \\
0 & \text{otherwise},
\end{cases}$$
Weighted Join Operators on Directed Trees

![Diagram](Image)

**Fig. 5** A pictorial representation of $T_2$ with prescribed vertex $m$

$$W_{\lambda_m}^{(\infty)} e_n = \begin{cases} 
\lambda_{mn} e_{\min(m,n)} & \text{if } m, n \text{ are odd or } m, n \text{ are even}, \\
\lambda_{mn} e_0 & \text{otherwise.}
\end{cases}$$

The matrix representations of $W_{\lambda_m}^{(0)}$ and $W_{\lambda_m}^{(\infty)}$ with respect to the ordered orthonormal basis $\{e_{2n}\}_{n \in \mathbb{N}} \cup \{e_{2n+1}\}_{n \in \mathbb{N}}$ are given by

$$W_{\lambda_m}^{(0)} = \begin{pmatrix}
0 & \cdots & \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_{m0} \lambda_{m2} \cdots \lambda_{mm} 0 & \cdots \\
0 & \cdots & 0 & \lambda_{mm+2} 0 & \cdots \\
\vdots & \cdots & \vdots & 0 & \lambda_{mm+4} 0 \\
\vdots & \cdots & \vdots & 0 & \lambda_{mm+6} \\
\end{pmatrix} \oplus 0,$$

$$W_{\lambda_m}^{(\infty)} = \begin{pmatrix}
\lambda_{m0} & 0 & \cdots & \lambda_{m1} & \lambda_{m3} & \cdots \\
0 & \lambda_{m2} & 0 & \cdots & \lambda_{m1} & \lambda_{m3} & \cdots \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 0 & \lambda_{mm-2} & 0 & \cdots & \lambda_{mm} & \lambda_{mm+2} & \cdots \\
\vdots & \ddots & \vdots & \vdots & 0 & \lambda_{mm} & \lambda_{mm+2} & \cdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}.$$ (see Fig. 5). It turns out that $W_{\lambda_m}^{(0)}$ is at most a rank one perturbation of a diagonal operator, while $W_{\lambda_m}^{(\infty)}$ is a finite rank operator with range contained in the linear span of $\{e_m, e_{m-2}, \ldots, e_0\}$. ■

The fact, as illustrated in the preceding examples, that the weighted join operator is either diagonal, a rank one perturbation of a diagonal operator or a finite rank operator holds in general (see Theorem 3.10).
3.1 Closedness and Boundedness

In this section, we discuss the closedness and boundedness of weighted join operators on rooted directed trees. Unless stated otherwise, \( b \in V_\infty \) denotes the base point of the weighted join operator \( W_{\lambda_b} \).

**Proposition 3.5** Let \( T = (V, E) \) be a rooted directed tree with root root and let \( T_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( T \). Let \( b, u \in V \) and let \( \lambda_u = \{ \lambda_{uv} \}_{v \in V} \) be a weight system of complex numbers. Then the weighted join operator \( W_{\lambda_u} \) on \( T \) defines a densely defined closed linear operator. Moreover, \( \mathcal{D}_V := \text{span}\{ e_v : v \in V \} \) forms a core for \( W_{\lambda_u} \).

**Proof** We have already noted that \( W_{\lambda_u} \) is densely defined (see Remark 3.2). Let \( \{ f_n \}_{n \in \mathbb{N}} \) be a sequence converging to \( f \) in \( \ell^2(V) \). Suppose that \( \{ W_{\lambda_u} f_n \}_{n \in \mathbb{N}} \) converges to some \( g \in \ell^2(V) \). Since \( \ell^2(V) \) is a reproducing kernel Hilbert space, for every \( w \in V \),

\[
\lim_{n \to \infty} f_n(w) = f(w), \quad \lim_{n \to \infty} \sum_{v \in M_{\lambda_u}^{b}(w)} \lambda_{uv} f_n(v) = g(w),
\]

where \( M_{\lambda_u}^{b}(w) \) is given by (2.2.3). However, since \( b \neq \infty \), \( \text{card}(M_{\lambda_u}^{b}(w)) < \infty \) for each \( w \in V \) (see (ii) and (iii) of Proposition 2.14). It follows that \( f \in \mathcal{D}(W_{\lambda_u}) \) and \( W_{\lambda_u} f = g \). Thus \( W_{\lambda_u} \) is a closed linear operator.

To see that \( \mathcal{D}_V \) is a core for \( W_{\lambda_u} \), note that by the preceding discussion, \( W_{\lambda_u} |_{\mathcal{D}_V} \) is a closable operator such that \( W_{\lambda_u} |_{\mathcal{D}_V} \subseteq W_{\lambda_u} \). To see the reverse inclusion, let \( f = \sum_{v \in V} f(v)e_v \in \mathcal{D}(W_{\lambda_u}) \) and let

\[
f_n := \sum_{v \in V, d_v \leq n} f(v)e_v, \quad n \in \mathbb{N}.
\]

Then \( \{ f_n \}_{n \in \mathbb{N}} \subseteq \mathcal{D}_V \), \( \| f_n - f \|_{\ell^2(V)} \to 0 \) as \( n \to \infty \) and

\[
\| W_{\lambda_u} f_n - W_{\lambda_u} f \|_{\ell^2(V)}^2 = \left\| \sum_{v \in V, d_v > n} f(v)\lambda_{uv} e_{u\uparrow_b v} \right\|_{\ell^2(V)}^2
\]

\[
= \sum_{w \in V, d_w > n} \left( \sum_{u \in V, u \uparrow_b w = w} f(v)\lambda_{uv} \right)^2,
\]

which converges to 0 as \( n \to \infty \), since \( W_{\lambda_u} f \in \ell^2(V) \). It follows that \( f \in \mathcal{D}(W_{\lambda_u} |_{\mathcal{D}_V}) \) and \( W_{\lambda_u} f = W_{\lambda_u} |_{\mathcal{D}_V} f \). This yields \( W_{\lambda_u} |_{\mathcal{D}_V} = W_{\lambda_u} \). \( \square \)
Remark 3.6 This result no more holds true for the weighted meet operator \( W_{\lambda u}^{(\infty)} \). Indeed, it may be concluded from Theorem 3.7 below and Lemma 1.3 that \( W_{\lambda u}^{(\infty)} \) may not be even closable.

We discuss next the boundedness of weighted join operators \( W_{\lambda u}^{(b)} \).

**Theorem 3.7** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root \( \text{root} \) and let \( \mathcal{T}_{\infty} = (V_{\infty}, E_{\infty}) \) be the extended directed tree associated with \( \mathcal{T} \). Let \( u \in V \), \( b \in V_{\infty} \) and let \( \lambda_u = \{ \lambda_{uv} \}_{v \in V_{\infty}} \) be a weight system of complex numbers. Then the weighted join operator \( W_{\lambda u}^{(b)} \) on \( \mathcal{T} \) is bounded if and only if

\[
\lambda_u \text{ belongs to } \begin{cases} 
\ell^2(V) & \text{if } b = \infty, \\
\ell^\infty(V) & \text{if } b = u, \\
\ell^\infty(\text{Des}(u)) & \text{otherwise.} 
\end{cases} \tag{3.1.1}
\]

**Proof** Let \( f = \sum_{v \in V} f(v)e_v \in \ell^2(V) \) be of norm 1 and let \( \Lambda_{\lambda u}^{(b)} \) be as defined in (3.0.1). Recall that \( f \in \mathcal{D}(W_{\lambda u}^{(b)}) \) if and only if \( \Lambda_{\lambda u}^{(b)} f \in \ell^2(V) \). By (3.0.3),

\[
\Lambda_{\lambda u}^{(b)} f = \sum_{v \in V} f(v)\lambda_{uv} e_{u \uplus_b v}.
\]

It follows that \( f \in \mathcal{D}(W_{\lambda u}^{(b)}) \) if and only if

\[
\| \Lambda_{\lambda u}^{(b)} f \|^2 = \sum_{w \in V} \left| \sum_{v \in M_{\lambda u}^{(b)}(w)} \lambda_{uv} f(v) \right|^2 \quad \text{(3.1.2)}
\]

is finite, where \( M_{\lambda u}^{(b)}(w) \) is as given in (2.2.3). We divide the proof into the following three cases:

**Case I.** \( b = \infty \):

Let \( u_j := \text{par}^{(j)}(u) \), \( j = 0, \ldots, d_u \). By Proposition 2.14(i) and (3.1.2), we obtain

\[
\| A_{\lambda u}^{(\infty)} f \|^2 = \sum_{w \in V} \left| \sum_{v \in M_{\lambda u}^{(\infty)}(w)} \lambda_{uv} f(v) \right|^2 = \sum_{j=0}^{d_u} \left| \sum_{v \in \text{Des}(u_j) \setminus \text{Des}(u_{j-1})} \lambda_{uv} f(v) \right|^2, \tag{3.1.3}
\]

where we used the convention that \( \text{Des}(\text{par}^{(-1)}(u)) = \emptyset \). We claim that \( W_{\lambda u}^{(\infty)} \) belongs to \( B(\ell^2(V)) \) if and only if

\[
\lambda_{u_j} \in \ell^2(\text{Des}(u_j) \setminus \text{Des}(u_{j-1})) \quad \text{for } j = 0, \ldots, d_u. \tag{3.1.4}
\]
If (3.1.4) holds, then by (3.1.3) and the Cauchy–Schwarz inequality,
\[ \| \Lambda(\infty)f \|_2^2 \leq \| f \|_2^2 \sum_{j=0}^{d_u} \sum_{v \in \text{Des}(u_j) \setminus \text{Des}(u_{j-1})} |\lambda_{uv}|^2, \]
which shows that \( W_{\lambda_u}(\infty) \in B(\ell^2(V)) \). Conversely, if \( W_{\lambda_u}(\infty) \in B(\ell^2(V)) \), then by (3.1.3),
\[ \sup_{\| f \|=1} \left| \sum_{v \in \text{Des}(u_j) \setminus \text{Des}(u_{j-1})} \lambda_{uv} f(v) \right| < \infty, \quad j = 0, \ldots, d_u. \]
By the standard polar representation, the series above is indeed absolutely convergent, and hence by Riesz representation theorem [23], \( \lambda_u \in \ell^2(\text{Des}(u_j) \setminus \text{Des}(u_{j-1})) \), \( j = 0, \ldots, d_u \). Thus the claim stands verified. To complete the proof, it now suffices to check that
\[ V_\infty = \bigsqcup_{j=0}^{d_u} \left( \text{Des}(u_j) \setminus \text{Des}(u_{j-1}) \right) \quad (3.1.5) \]
(see Fig. 6). To see that, let \( v \in V_\infty \). Clearly, \( \text{Des}(u) = \text{Des}(u_0) \setminus \text{Des}(u_{-1}) \). Thus we may assume that \( v \in V_\infty \setminus \text{Des}(u) \). In view of (2.2.2), we must have \( v \in \text{Asc}(u) \sqcup V_u \).

If \( v \in \text{Asc}(u) \), then there exists \( j \in \{1, \ldots, d_u\} \) such that \( v = \text{par}^{(j)}(u) = u_j \), and hence \( v \in \text{Des}(u_j) \setminus \text{Des}(u_{j-1}) \). Hence we may further assume that \( v \in V_u \). By Lemma 2.7,
\[ u \sqcap v \in [\text{root}, u] = \{u_{d_u}, \ldots, u_0\}. \]
Thus \( u \sqcap v = u_j \) for some \( j = 0, \ldots, d_u \), and therefore \( v \in \text{Des}(u_j) \). However, by the uniqueness of the meet operation, \( v \notin \text{Des}(u_{j-1}) \). This completes the verification of (3.1.5).
**Case II.** \( b \in V \) \& \( b \in \text{Des}(u) \): 
If \( b = u \), then by Remark 3.2, \( W_{\lambda_u}^{(b)} = D_{\lambda_u} \), and hence \( W_{\lambda_u}^{(b)} \in B(\ell^2(V)) \) if and only if \( \lambda_u \in \ell^\infty(V) \). Assume that \( u \neq b \). Then, by Proposition 2.14(ii),

\[
\sum_{w \in V} \sum_{v \in M_u^{(b)}(w)} \lambda_{uv} f(v)^2 = \sum_{w \in \text{Asc}(u)} |\lambda_{uw} f(w)|^2 + \sum_{w \in \text{Des}(u)} |\lambda_{uw} f(w)|^2 + \sum_{v \in [u, b]} |\lambda_{uv} f(v)|^2.
\]

Since \( \text{Asc}(u), [u, b] \) are finite sets,

\[
\Lambda_{\mu}^{(b)} f \in \ell^2(V) \iff \sum_{w \in \text{Des}(u)} |\lambda_{uw} f(w)|^2 < \infty. \tag{3.1.6}
\]

It is now clear that \( \Lambda_{\mu}^{(b)} f \in \ell^2(V) \) for every \( f \in \ell^2(V) \) if and only if \( \lambda_u \in \ell^\infty(\text{Des}(u)) \). This shows that (3.1.1) is a necessary condition. Conversely, if (3.1.1) holds then \( \Lambda_{\mu}^{(b)} f \in \ell^2(V) \) for every \( f \in \ell^2(V) \), and hence by the closed graph theorem together with Proposition 3.5, \( W_{\lambda_u}^{(b)} \) defines a bounded linear operator on \( \ell^2(V) \).

**Case III.** \( b \in V \) \& \( b \notin \text{Des}(u) \):
By Proposition 2.14(iii),

\[
\sum_{w \in V} \sum_{v \in M_u^{(b)}(w)} \lambda_{uv} f(v)^2 = \sum_{w \in \text{Des}(u)} |\lambda_{uw} f(w)|^2 + \sum_{v \in \text{Asc}(u) \cup [u, b]} |\lambda_{uv} f(v)|^2.
\]

Once again, since \( \text{Asc}(u) \) is a finite set, we must have (3.1.6). It follows that \( \Lambda_{\mu}^{(b)} f \in \ell^2(V) \) for every \( f \in \ell^2(V) \) if and only if \( \lambda_u \in \ell^\infty(\text{Des}(u)) \). The verification of the remaining part in this case is now similar to that of Case II. \(\square\)

**Remark 3.8** Note that the weighted join operator \( W_{\lambda_u}^{(\text{root})} \) on \( T \) is bounded if and only if \( \lambda_u \in \ell^\infty(\text{Des}(u)) \). Further, the weighted meet operator \( W_{\lambda_u}^{(\infty)} \) on \( T \) is bounded if and only if \( \lambda_u \in \ell^2(V) \).

An examination of Cases II and III of the proof of Theorem 3.7 yields a neat expression for the domain of weighted join operator \( W_{\lambda_u}^{(b)} \), \( b \neq \infty \):

**Corollary 3.9** Let \( T = (V, E) \) be a rooted directed tree with root \( \text{root} \) and let \( T_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( T \). Let \( u \in V \), \( b \in V \) and let \( \lambda_u = \{\lambda_{uv}\}_{v \in V_\infty} \) be a weight system of complex numbers. For \( u \in V \), consider the weight system \( \lambda_u = \{\lambda_{uv}\}_{v \in V_\infty} \) of complex numbers and let \( D_{\lambda_u} \) be the diagonal operator with diagonal entries \( \lambda_u \). Then, for any \( b \in V \), the domain of the weighted join operator \( W_{\lambda_u}^{(b)} \) on \( T \) is given by \( D(W_{\lambda_u}^{(b)}) = D(P_{\text{Des}(u)} D_{\lambda_u}) \).

**Proof** This follows from (3.1.6), which holds for any \( b \in V \). \(\square\)
3.2 A Decomposition Theorem

One of the main results of this section shows that the weighted join operator $W^{(b)}_{\lambda u}$ on a rooted directed tree can be one of the following three types, viz. a diagonal operator, a rank one perturbation of a diagonal operator or a finite rank operator. Further, we obtain an orthogonal decomposition of $W^{(b)}_{\lambda u}$ into a diagonal operator and a rank one operator provided $b \neq u$. Among various applications, we exhibit a family of weighted join operators with large null summand. It turns out that either a weighted join operator is complex Jordan or it has bounded Borel functional calculus.

Before we state the first main result of this section, we introduce the function $e_{\mu,A}$, which appears in the decomposition of weighted join operators. For a subset $A \subseteq V$ and $\mu := \{\mu_v : v \in A\} \subseteq \mathbb{C}$, consider the function $e_{\mu,A} : V \rightarrow \mathbb{C}$ given by

$$e_{\mu,A} := \sum_{v \in A} \bar{\mu}_v e_v.$$ (3.2.1)

Note that $e_{\mu,A} \in \ell^2(V)$ if and only if $\mu \in \ell^2(A)$.

**Theorem 3.10** Let $T = (V, E)$ denote a rooted directed tree with root $r$ and let $T_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $T$. For $u \in V$, consider the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers and let $D_{\lambda u}$ be the diagonal operator with diagonal entries $\lambda_{uv}$. Then, for any $b \in V \setminus \{u\}$, the weighted join operator $W^{(b)}_{\lambda u}$ on $T$ is given by

$$W^{(b)}_{\lambda u} = \begin{cases} P_{\text{Asc}(u) \cup \text{Des}_b(u)} D_{\lambda u} + e_u \otimes e_{\lambda_{uv}}, & \text{if } b \in \text{Des}(u), \\ P_{\text{Des}(u)} D_{\lambda u} + e_u \otimes e_{\lambda_{uv}}, & \text{otherwise.} \end{cases}$$

**Remark 3.11** In case $b = u$, by Remark 3.2(iv), $W^{(b)}_{\lambda u}$ is the diagonal operator $D_{\lambda u}$. In case $b = \infty$, by (3.0.3),

$$W^{(\infty)}_{\lambda u} e_v = \lambda_{uv} e_{u \setminus v}, \quad v \in V.$$ (3.2.2)

It now follows from Lemma 2.7 that $W^{(\infty)}_{\lambda u}$ is a finite rank operator. Let us find an explicit expression for $W^{(\infty)}_{\lambda u}$. By (3.1.5), $\ell^2(V)$ admits the orthogonal decomposition

$$\ell^2(V) = \bigoplus_{j=0}^{d_u} \ell^2(\text{Des}(u)_j \setminus \text{Des}(u)_{j-1}).$$
where $\text{Des}(u) = \emptyset$ and $u_j := \text{par}^{(j)}(u)$ for $j = 0, \ldots, d_u$. By (3.2.2), with respect to the above decomposition, $W^{(\infty)}_{\lambda_u}$ decomposes as

$$
\mathcal{D}(W^{(\infty)}_{\lambda_u}) = \bigoplus_{j=0}^{d_u} \mathcal{D}(u_j \otimes e_{\lambda_u, \text{Des}(u) \setminus \text{Des}(u_{j-1})}).
$$

(3.2.3)

Thus $W^{(\infty)}_{\lambda_u}$ is an orthogonal direct sum of rank one operators

$$
e_{u_j} \otimes e_{\lambda_u, \text{Des}(u) \setminus \text{Des}(u_{j-1})}, \quad j = 0, \ldots, d_u.
$$

**Proof** Let $b \in V \setminus \{u\}$. By Corollary 3.9, $\mathcal{D}(W^{(b)}_{\lambda_u}) = \mathcal{D}(P_{\text{Des}(u)} D_{\lambda_u})$. To see the decomposition of $W^{(b)}_{\lambda_u}$, consider the subset $M^{(b)}_{\lambda_u}(\infty)$ of $V$ as given in (2.2.3). By Proposition 2.14(iv),

$$
V \setminus M^{(b)}_{\lambda_u}(\infty) = \text{Asc}(u) \cup \text{Des}(u) \cup \{b\} = (V \setminus V_u) \cup \{b\},
$$

(3.2.4)

where $V_u$ is as given in (2.2.1). Note that (3.2.4) induces the orthogonal decomposition

$$
\ell^2(V) = \begin{cases} 
\ell^2(\text{Asc}(u)) \oplus \ell^2(\text{Des}(u)) \oplus \ell^2(M^{(b)}_{\lambda_u}(\infty)) & \text{if } b \in V \setminus V_u, \\
\ell^2(\text{Asc}(u)) \oplus \ell^2(\text{Des}(u)) \oplus \ell^2(M^{(b)}_{\lambda_u}(\infty)) \oplus \ell^2(\{b\}) & \text{otherwise}.
\end{cases}
$$

(3.2.5)

It may be concluded from Table 1 that

$$
W^{(b)}_{\lambda_u}(\ell^2(\text{Asc}(u))) \subseteq \ell^2(V) \ominus \ell^2(M^{(b)}_{\lambda_u}(\infty)),
$$

$$
W^{(b)}_{\lambda_u}(\ell^2(\text{Des}(u))) \subseteq \ell^2(\text{Des}(u)),
$$

$$
W^{(b)}_{\lambda_u}(\ell^2(M^{(b)}_{\lambda_u}(\infty))) = \{0\},
$$

$$
W^{(b)}_{\lambda_u}(\ell^2(\{b\})) \subseteq \ell^2(\{u\}).
$$

With respect to the orthogonal decomposition (3.2.5) of $\ell^2(V)$, the weighted join operator $W^{(b)}_{\lambda_u}$ decomposes as follows:

$$
W^{(b)}_{\lambda_u} = \begin{cases} 
\begin{bmatrix} W^{(b)}_{11} & 0 & 0 \\
W^{(b)}_{21} & W^{(b)}_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} & \text{if } b \in V \setminus V_u, \\
\begin{bmatrix}
W^{(b)}_{11} & 0 & 0 & 0 \\
W^{(b)}_{21} & W^{(b)}_{22} & 0 & \lambda_u b e_u \otimes e_b \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} & \text{if } b \in V_u.
\end{cases}
$$

(3.2.6)
Note that for any \( v \in \text{Asc}(u) \), by Table 1,

\[
W^{(b)}_{11} e_v = P_{\text{Asc}(u)} W^{(b)}_{\lambda_u} e_v \\
= \lambda_{uv} P_{\text{Asc}(u)} e_{u \sqcup b} \\
= \begin{cases} 
0 & \text{if } b \in \text{Asc}(u) \text{ or } b \in V_u, \\
D_{\lambda_u} e_v & \text{if } b \in \text{Des}(u).
\end{cases}
\] (3.2.8)

A similar argument using Table 1 shows that for any \( v \in \text{Asc}(u) \),

\[
W^{(b)}_{21} e_v = \lambda_{uv} P_{\text{Des}(u)} e_{u \sqcup b} \\
= \begin{cases} 
\left( \sum_{w \in \text{Asc}(u)} \lambda_{uw} e_u \otimes e_w \right) e_v & \text{if } b \in \text{Asc}(u) \text{ or } b \in V_u, \\
0 & \text{if } b \in \text{Des}(u).
\end{cases}
\] (3.2.9)

Further, for any \( v \in \text{Des}(u) \), by Table 1,

\[
W^{(b)}_{22} e_v = \lambda_{uv} e_{u \sqcup b} \\
= \begin{cases} 
D_{\lambda_u} e_v & \text{if } b \in \text{Asc}(u) \text{ or } b \in V_u, \\
\left( \sum_{w \in [u, b]} \lambda_{uw} e_u \otimes e_w \right) e_v & \text{if } b \in \text{Des}(u) \text{ and } v \in [u, b], \\
D_{\lambda_u} e_v & \text{if } b \in \text{Des}(u) \text{ and } v \notin [u, b].
\end{cases}
\] (3.2.10)

It is now easy to see that

\[
W^{(b)}_{22} = \begin{cases} 
D_{\lambda_u} |_{\ell^2(\text{Des}(u))} & \text{if } b \in \text{Asc}(u) \text{ or } b \in V_u, \\
D_{\lambda_u} |_{\ell^2(\text{Des}(u))} + \sum_{w \in [u, b]} \lambda_{uw} e_u \otimes e_w & \text{if } b \in \text{Des}(u).
\end{cases}
\]

In view of (3.2.8), (3.2.9), (3.2.10), one may now deduce the desired decomposition from (3.2.6) and (3.2.7).

Theorem 3.10 together with Remark 3.11 yields the following:

**Corollary 3.12 (Dichotomy)** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root root and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( b, u \in V \), consider the weight system \( \lambda_u = \{ \lambda_{uv} \}_v \in V_\infty \) of complex numbers. Then the weighted join operator \( W^{(b)}_{\lambda_u} \) on \( \mathcal{T} \) is at most rank one perturbation of a diagonal operator, while the weighted meet operator \( W^{(\infty)}_{\lambda_u} \) on \( \mathcal{T} \) is a finite rank operator.
By Theorem 3.10, any $W_{\lambda u}^{(b)} \in B(\ell^2(V))$ can be rewritten as $C + M$, where $C$ is a diagonal operator and $M$ is a nilpotent operator of nilpotency index 2 given by

$$C = \begin{cases} P_{\text{Asc}(u) \cup \text{Des}_b(u)} D_{\lambda u} & \text{if } b \in \text{Des}(u), \\ P_{\text{Des}(u)} D_{\lambda u} & \text{otherwise}, \end{cases}$$

$$M = \begin{cases} e_u \otimes e_{\lambda u, \text{Asc}(u) \cup \{b\}} & \text{if } b \in \text{Des}(u), \\ e_u \otimes e_{\lambda u, \text{Des}_b(u)} & \text{otherwise}. \end{cases}$$

However, $W_{\lambda u}^{(b)}$ is not a complex Jordan operator unless $\lambda_{uu} = 0$. Indeed,

$$CM - MC = \lambda_{uu} M.$$

It is worth noting that $(CM - MC)^2 = 0$. Unfortunately, the above decomposition is not orthogonal. Here is a way to get such a decomposition of $W_{\lambda u}^{(b)}$.

**Theorem 3.13** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$, $b \in V \setminus \{u\}$ and the weight system $\lambda_u = \{\lambda_{uu}\}_{v \in V_\infty}$ of complex numbers, let $D_{\lambda u}$ be the diagonal operator on $\mathcal{T}$ and let $W_{\lambda u}^{(b)}$ be a weighted join operator on $\mathcal{T}$. Consider the closed subspace $\mathcal{H}_{\lambda u}^{(b)}$ of $\ell^2(V)$, given by

$$\mathcal{H}_{\lambda u}^{(b)} = \begin{cases} \ell^2(\text{Asc}(u) \cup \text{Des}_b(u)) & \text{if } b \in \text{Des}(u), \\ \ell^2(\text{Des}_u(u)) & \text{otherwise}. \end{cases}$$  (3.2.11)

Then the following statements hold:

(i) The weighted join operator $W_{\lambda u}^{(b)}$ admits the decomposition

$$W_{\lambda u}^{(b)} = D_{\lambda u}^{(b)} \oplus N_{\lambda u}^{(b)} \quad \text{on } \ell^2(V) = \mathcal{H}_{\lambda u}^{(b)} \oplus (\ell^2(V) \ominus \mathcal{H}_{\lambda u}^{(b)}),$$  (3.2.12)

where $D_{\lambda u}^{(b)}$ is a densely defined diagonal operator in $\mathcal{H}_{\lambda u}^{(b)}$ and $N_{\lambda u}^{(b)}$ is a bounded linear rank one operator on $\ell^2(V) \ominus \mathcal{H}_{\lambda u}^{(b)}$.

(ii) $D_{\lambda u}^{(b)}$ and $N_{\lambda u}^{(b)}$ are given by

$$D_{\lambda u}^{(b)} = D_{\lambda u} |_{\mathcal{H}_{\lambda u}^{(b)}}, \quad \mathcal{A}(D_{\lambda u}^{(b)}) = \left\{ f \in \mathcal{H}_{\lambda u}^{(b)} : D_{\lambda u} (f \oplus 0) \in \mathcal{H}_{\lambda u}^{(b)} \right\},$$

$$N_{\lambda u}^{(b)} = e_u \otimes e_{\lambda u, A_u},$$  (3.2.13)

where $e_{\lambda u, A_u}$ is as given in (3.2.1) and the subset $A_u$ of $V$ is given by

$$A_u = \begin{cases} \{u, b\} & \text{if } b \in \text{Des}(u), \\ \text{Asc}(u) \cup \{b, u\} & \text{otherwise}. \end{cases}$$  (3.2.14)
Proof Since the orthogonal projection $P_{\mathcal{H}_u^{(b)}}$ commutes with $D_{\lambda_u}$, one may appeal to [64, Proposition 1.15] to conclude that $\mathcal{H}_u^{(b)}$ (identified with a subspace of $\ell^2(V)$) is a reducing subspace for $D_{\lambda_u}$. The desired conclusions in (i) and (ii) now follow from Theorem 3.10. \hfill \Box

We find it convenient to denote the orthogonal decomposition (3.2.12) of $W_{\lambda_u}^{(b)}$, $b \neq u$, as ensured by Theorem 3.13, by the triple $(D_{\lambda_u}^{(b)}, N_{\lambda_u}^{(b)}, \mathcal{H}_u^{(b)})$, where $\mathcal{H}_u^{(b)}$, $D_{\lambda_u}^{(b)}$ and $N_{\lambda_u}^{(b)}$ are given by (3.2.11), (3.2.13) and (3.2.14) respectively. For the sake of convenience, we set

$$\mathcal{H}_u^{(u)} := \ell^2(V \setminus \{u\}).$$

(3.2.16)

Note that $W_{\lambda_u}^{(u)}$ admits the decomposition (3.2.12) with $N_{\lambda_u}^{(u)} = \lambda_{uu} e_u \otimes e_u$.

In what follows, we will be interested in only those vertices $u \in V$ for which $\mathcal{H}_u^{(b)}$ is of infinite dimension.

In the remaining part of this section, we present some immediate consequences of Theorem 3.13.

**Corollary 3.14** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $u$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$, $b \in V \setminus \{u\}$ and the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers, let $W_{\lambda_u}^{(b)}$ denote the weighted join operator on $\mathcal{T}$ and let $(D_{\lambda_u}^{(b)}, N_{\lambda_u}^{(b)}, \mathcal{H}_u^{(b)})$ denote the orthogonal decomposition of $W_{\lambda_u}^{(b)}$. If $\lambda_u \in \ell^\infty(Des(u))$, then $\|W_{\lambda_u}^{(b)}\| = \max \{\|D_{\lambda_u}^{(b)}\|, \|N_{\lambda_u}^{(b)}\|\}$. Further,

$$\|D_{\lambda_u}^{(b)}\| = \sup_{v \in \text{supp} \mathcal{H}_u^{(b)}} |\lambda_{uv}|, \quad \|N_{\lambda_u}^{(b)}\| = \left(\sum_{v \in A_u} |\lambda_{uv}|^2\right)^{1/2},$$

where $A_u$ is given by (3.2.15).

The following two corollaries give more insight into the structure of weighted join operators on rooted directed trees. The first of which is motivated by the work [6]. We say that a densely defined linear operator $T$ in $\mathcal{H}$ admits a large null summand if it has an infinite dimensional reducing subspace contained in its kernel.

**Corollary 3.15** Let $\mathcal{T} = (V, E)$ be a leafless, rooted directed tree with root $u$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$, $b \in V \setminus \{u\}$ and the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers, let $W_{\lambda_u}^{(b)}$ denote the weighted join operator on $\mathcal{T}$. If there exists a branching vertex $w \in Asc(u)$, then $W_{\lambda_u}^{(b)}$ has a large null summand.

**Proof** Suppose there exists a branching vertex $w \in Asc(u)$. By Theorem 3.13,

$$(\ell^2(V) \ominus \mathcal{H}_u^{(b)}) \ominus \ell^2(A_u) \subseteq \ker W_{\lambda_u}^{(b)} \cap \ker (W_{\lambda_u}^{(b)})^*,$$
where $\mathcal{A}_u$ is a finite set given by (3.2.15). In view of (2.2.1) and (3.2.11), it suffices to check that $\text{card}(V_u) = \aleph_0$. However, since $\mathcal{T}$ is leafless, this follows from Proposition 2.12(ii). □

Complex Jordan weighted join operators exist in abundance.

**Corollary 3.16** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_\infty = (V_{\infty}, E_{\infty})$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$, $b \in V\setminus\{u\}$ and the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_{\infty}}$ of complex numbers, let $W^{(b)}_{\lambda_u}$ denote the weighted join operator on $\mathcal{T}$. Then the following holds:

(i) If $\lambda_{uu} = 0$, then $W^{(b)}_{\lambda_u}$ is a complex Jordan operator of index 2.

(ii) If $\lambda_{uu} \neq 0$, then there exists a bounded homomorphism $\Phi : B_{\infty}(\sigma(W^{(b)}_{\lambda_u})) \to B(\ell^2(V))$ given by

$$\Phi(f) = f(W^{(b)}_{\lambda_u}), \quad f \in B_{\infty}(\sigma(W^{(b)}_{\lambda_u})),
$$

where $B_{\infty}(\Omega)$ denotes the algebra of bounded Borel functions from a closed subset $\Omega$ of $\mathbb{C}$ into $\mathbb{C}$. In this case, $\Phi$ extends the polynomial functional calculus.

**Proof** Let $(D^{(b)}_{\lambda_u}, N^{(b)}_{\lambda_u}, \mathcal{H}^{(b)}_{\lambda_u})$ denote the orthogonal decomposition of $W^{(b)}_{\lambda_u}$. To verify (i), assume that $\lambda_{uu} = 0$. By Theorem 3.10,

$$W^{(b)}_{\lambda_u} = P_{\mathcal{H}^{(b)}_{\lambda_u}} D_{\lambda_u} + N^{(b)}_{\lambda_u}.$$

By a routine inductive argument, we obtain

$$N^{(b)}_{\lambda_u}^k = \lambda_{uu}^{k-1} e_u \otimes e_{\lambda_u,A_u}, \quad k \geq 1. \quad (3.2.17)$$

It follows that $N^{(b)}_{\lambda_u}$ is nilpotent of nilpotency index 2. Also, it is easily seen that

$$N^{(b)}_{\lambda_u}(W^{(b)}_{\lambda_u} - N^{(b)}_{\lambda_u}) \subseteq (W^{(b)}_{\lambda_u} - N^{(b)}_{\lambda_u}) N^{(b)}_{\lambda_u} = 0,$$

which completes the verification of (i).

Assume next that $\lambda_{uu} \neq 0$. In view of (3.2.12) and [61, Theorem 13.24], it suffices to check that $N^{(b)}_{\lambda_u}$ given by (3.2.14) admits a Borel functional calculus. By (3.2.17), for any Borel measurable function $f : \sigma(W^{(b)}_{\lambda_u}) \to \mathbb{C}$, $f(N^{(b)}_{\lambda_u})$ is a well-defined bounded linear operator given by

$$f(N^{(b)}_{\lambda_u}) = \frac{f(\lambda_{uu})}{\lambda_{uu}} e_{\lambda_u,A_u} \otimes e_{\lambda_u,A_u}.$$

Further, for any bounded Borel measurable function $f$ on $\sigma(W^{(b)}_{\lambda_u})$,

$$\|f(N^{(b)}_{\lambda_u})\| \leq \frac{|f(\lambda_{uu})|}{|\lambda_{uu}|} \|e_{\lambda_u,A_u}\| \leq \|f\|_{\infty} \frac{\|e_{\lambda_u,A_u}\|}{|\lambda_{uu}|}. $$
This completes the verification of (ii). □

The following is a consequence of Theorem 3.10 and the well-known characterization of diagonal compact operators [23], once it is observed that the class of bounded finite rank operators is a subset of compact operators, dense in the operator norm (in fact, it is also dense in Schatten $p$-class in its norm for every $p \geq 1$) (cf. [38, Corollary 3.4.5]). The reader is referred to [67] for the basics of operators in Schatten classes.

**Proposition 3.17** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_{\infty} = (V_{\infty}, E_{\infty})$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$, $b \in V \setminus \{u\}$ and the weight system $\lambda_{u} = \{\lambda_{uv}\}_{v \in V_{\infty}}$ of complex numbers, let $W_{\lambda_{u}}^{(b)}$ denote the weighted join operator on $\mathcal{T}$. Then, for any $p \in [1, \infty)$, the following hold:

(i) $W_{\lambda_{u}}^{(b)}$ is compact if and only if $\lim_{v \in \text{Des}(u)} \lambda_{uv} = 0$.

(ii) $W_{\lambda_{u}}^{(b)}$ is Schatten $p$-class if and only if $\sum_{v \in \text{Des}(u)} |\lambda_{uv}|^{p} < \infty$.

Here the limit and sum are understood in a generalized sense (see (1.1.2)).

We conclude this section with an application to the theory of commutators of compact operators.

**Corollary 3.18** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_{\infty} = (V_{\infty}, E_{\infty})$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$, $b \in V \setminus \{u\}$, and the weight system $\lambda_{u} = \{\lambda_{uv}\}_{v \in V_{\infty}}$ of complex numbers, let $W_{\lambda_{u}}^{(b)}$ denote the weighted join operator on $\mathcal{T}$. Then the following statements hold:

(i) If $\lim_{v \in \text{Des}(u)} \lambda_{uv} = 0$, then there exist compact operators $K, L \in B(\ell^{2}(V))$ such that $W_{\lambda_{u}}^{(b)} = KL - LK$.

(ii) If $\sum_{v \in \text{Des}(u)} |\lambda_{uv}|^{p} < \infty$ for some $p \in [1, \infty)$, then there exist Schatten class operators $K, L \in B(\ell^{2}(V))$ such that $W_{\lambda_{u}}^{(b)} = KL - LK$.

**Proof** This follows from Proposition 3.17, Corollary 3.15 and Anderson’s Theorems [6, Theorems 1 and 3]. □

### 3.3 Commutant

The main result of this section describes commutants of weighted join operators (cf. [37, Proposition 5.4], [28, Theorem 1.8]). In general, the weighted join operators $W_{\lambda_{u}}^{(b)}$ do not belong to the class ($\mathcal{RO}$) as introduced in [28]. Indeed, in contrast with [28, Theorem 1.8], the commutant of $W_{\lambda_{u}}^{(b)}$ need not be abelian (see Corollary 3.21).

**Theorem 3.19** (Commutant) Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_{\infty} = (V_{\infty}, E_{\infty})$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$ and $b \in V \setminus \{u\}$, consider the weight system $\lambda_{u} = \{\lambda_{uv}\}_{v \in V_{\infty}}$ of complex numbers
and assume that the weighted join operator \( W_{\lambda_u}^{(b)} \) on \( \mathcal{T} \) belongs to \( B(\ell^2(V)) \). Let 
\( (D_{\lambda_u}^{(b)}, N_{\lambda_u}^{(b)}, \mathcal{H}_{\lambda_u}^{(b)}) \) denote the orthogonal decomposition of \( W_{\lambda_u}^{(b)} \), \( A_u \) be as given in (3.2.15), and let \( W_u \) be given by

\[
W_u := \left\{ v \in \text{supp} \mathcal{H}_{\lambda_u}^{(b)} : \lambda_{uuv} = \lambda_{uu} \right\}.
\]  

(3.3.1)

If \( \ker D_{\lambda_u}^{(b)} = \{0\} \) and \( N_{\lambda_u}^{(b)} \neq 0 \), then the following statements are equivalent:

(i) \( X \in B(\ell^2(V)) \) belongs to the commutant \( \{W_{\lambda_u}^{(b)}\}' \) of \( W_{\lambda_u}^{(b)} \).

(ii) \( X \in B(\ell^2(V)) \) admits the orthogonal decomposition

\[
X = \begin{bmatrix}
P & Q \\
R & S
\end{bmatrix}
\]  

on \( \ell^2(V) = \mathcal{H}_{\lambda_u}^{(b)} \oplus (\ell^2(V) \ominus \mathcal{H}_{\lambda_u}^{(b)}) \),

where \( P \) is a block diagonal operator in \( \{D_{\lambda_u}^{(b)}\}' \), \( f_0 \in \ker (D_{\lambda_u}^{(b)} - \lambda_{uu}) \),

\[
\mu_u := \{ \mu_{uuv} \} \in \text{supp} \mathcal{H}_{\lambda_u}^{(b)} \text{ belongs to } \ell^2(\text{supp} \mathcal{H}_{\lambda_u}^{(b)}), \text{ and } S \text{ is any operator in } B(\ell^2(V) \ominus \mathcal{H}_{\lambda_u}^{(b)}) \text{ such that } Se_u = (Se_u, e_u)e_u \text{ and } S^*e_{\mu_u, A_u} = (Se_u, e_u)e_{\mu_u, A_u}.
\]

**Proof** Assume that \( \ker D_{\lambda_u}^{(b)} = \{0\} \) and \( N_{\lambda_u}^{(b)} \neq 0 \). Let \( X \in B(\ell^2(V)) \) be such that \( XW_{\lambda_u}^{(b)} = W_{\lambda_u}^{(b)}X \). We decompose \( X \) as follows:

\[
X = \begin{bmatrix}
P & Q \\
R & S
\end{bmatrix}
\]  

on \( \ell^2(V) = \mathcal{H}_{\lambda_u}^{(b)} \oplus (\ell^2(V) \ominus \mathcal{H}_{\lambda_u}^{(b)}) \).

A simple calculation shows that \( XW_{\lambda_u}^{(b)} = W_{\lambda_u}^{(b)}X \) is equivalent to

\[
PD_{\lambda_u}^{(b)} = D_{\lambda_u}^{(b)}P, \quad N_{\lambda_u}^{(b)}S = SN_{\lambda_u}^{(b)}, \quad N_{\lambda_u}^{(b)}R = RD_{\lambda_u}^{(b)}, \quad DN_{\lambda_u}^{(b)} = QN_{\lambda_u}^{(b)}.
\]

(3.3.2)

We contend that \( R \) is a finite rank operator with a range contained in \( [e_u] \). To see that, let \( f \in \mathcal{H}_{\lambda_u}^{(b)} \). By (3.2.13) and (3.2.14),

\[
N_{\lambda_u}^{(b)}Rf = (Rf, e_{\mu_u, A_u})e_u, \quad RD_{\lambda_u}^{(b)}f = \sum_{v \in \text{supp} \mathcal{H}_{\lambda_u}^{(b)}} \lambda_{uuv}f(v)Re_v.
\]

Thus, by the third equation of (3.3.2), we obtain

\[
(Rf, e_{\mu_u, A_u})e_u = \sum_{v \in \text{supp} \mathcal{H}_{\lambda_u}^{(b)}} \lambda_{uuv}f(v)Re_v.
\]

(3.3.3)
Since (3.3.3) holds for arbitrary $f \in \mathcal{H}_u^{(b)}$ and $\lambda_{uv} \neq 0$ for every $v \in \text{supp} \mathcal{H}_u^{(b)}$, there exists a system $\mu_u := \{ \mu_{uv} \} \text{supp} \mathcal{H}_u^{(b)} \subseteq \mathbb{C}$ such that

$$Re_v = \mu_{uv} e_u, \quad v \in \text{supp} \mathcal{H}_u^{(b)}.$$  \hspace{1cm} (3.3.4)

This immediately yields

$$\langle Rf, e_{\lambda_{u},A_u} \rangle e_u = \sum_{v \in \text{supp} \mathcal{H}_u^{(b)}} \lambda_{uu} f(v) \mu_{uv} e_u.$$  

Combining this with (3.3.3), we obtain

$$\sum_{v \in \text{supp} \mathcal{H}_u^{(b)}} \lambda_{uu} f(v) \mu_{uv} e_u = \sum_{v \in \text{supp} \mathcal{H}_u^{(b)}} \lambda_{uv} f(v) \mu_{uv} e_u.$$  

Since $f \in \mathcal{H}_u^{(b)}$ is arbitrary, this yields that $\mu_{uu} (\lambda_{uu} - \lambda_{uv}) = 0$ for every $v \in \text{supp} \mathcal{H}_u^{(b)}$. If $v \in \text{supp} \mathcal{H}_u^{(b)} \setminus W_u$, then $\lambda_{uu} \neq \lambda_{uv}$ (see (3.3.1)), and hence $\mu_{uv} = 0$. It may be now concluded from (3.3.4) that $R = e_u \otimes e_{\lambda_u, W_u}$. Thus the claim stands verified.

Next we consider the equation $D_{k_u}^{(b)} Q = Q N_{\lambda_u}^{(b)}$ (see (3.3.2)). Note that by (3.2.14)

$$D_{k_u}^{(b)} Q e_u = Q N_{\lambda_u}^{(b)} e_u = \lambda_{uu} Q e_u,$$

which simplifies to $(D_{k_u}^{(b)} - \lambda_{uu}) Q e_u = 0$. Since $D_{k_u}^{(b)} - \lambda_{uu} = 0$ on $\ell^2(W_u)$ and injective on $\mathcal{H}_u^{(b)} \otimes \ell^2(W_u)$, $Q e_u \in \ell^2(W_u)$. Moreover,

$$D_{k_u}^{(b)} Q = Q (e_u \otimes e_{\lambda_u, A_u}) = (Q e_u) \otimes e_{\lambda_u, A_u}.$$  

If $Q e_u = 0$, then so is $D_{k_u}^{(b)} Q$, and hence $Q = 0$ (since, by assumption, $D_{k_u}^{(b)}$ is injective). Suppose $Q e_u \neq 0$. Then $Q : \ell^2(V) \otimes \mathcal{H}_u^{(b)} \rightarrow \mathcal{H}_u^{(b)}$ is of rank one, since so is $D_{k_u}^{(b)} Q$ and $D_{k_u}^{(b)}$ is injective. Thus $Q = f_0 \otimes g_0$ for some $f_0 \in \mathcal{H}_u^{(b)}$ and $g_0 \in \ell^2(V) \otimes \mathcal{H}_u^{(b)}$. It follows from $D_{k_u}^{(b)} Q = Q N_{\lambda_u}^{(b)}$ and (1.3.2) that

$$(D_{k_u}^{(b)} f_0) \otimes g_0 = f_0 \otimes g_0 N_{\lambda_u}^{(b)} = (f_0 \otimes g_0) (e_u \otimes e_{\lambda_u, A_u}) = g_0(u) f_0 \otimes e_{\lambda_u, A_u}.  \hspace{1cm} (3.3.5)$$

Thus for any $h \in \ell^2(V) \otimes \mathcal{H}_u^{(b)}$,

$$\langle h, g_0 \rangle D_{k_u}^{(b)} f_0 = g_0(u) \langle h, e_{\lambda_u, A_u} \rangle f_0.$$
Letting \( h = g_0 - \frac{(g_0, e_{\lambda_u} A_u)}{\|e_{\lambda_u} A_u\|^2} e_{\lambda_u} A_u \in \ell^2(V) \odot \mathcal{H}^{(b)}_\mu \), we get \( \langle h, g_0 \rangle D^{(b)}_{\lambda_u} f_0 = 0 \).

However, since \( D^{(b)}_{\lambda_u} \) is assumed to be injective and \( f_0 \neq 0 \) (since \( Q \neq 0 \)), \( \langle h, g_0 \rangle = 0 \).

It follows that \( \langle g_0, e_{\lambda_u} A_u \rangle = \|g_0\| \|e_{\lambda_u} A_u\| \). By the Cauchy–Schwarz inequality, we must have \( g_0 = \bar{\alpha} e_{\lambda_u} A_u \) for some \( \alpha \in \mathbb{C} \). Further, \( \alpha \neq 0 \), since \( g_0 \neq 0 \) (otherwise \( Q = 0 \)). This also shows that \( Q = \alpha f_0 \otimes e_{\lambda_u} A_u \). One may now infer from (3.3.5) (evaluated at \( g_0 = \bar{\alpha} e_{\lambda_u} A_u \)) that \( D^{(b)}_{\lambda_u} f_0 = \lambda_{uu} f_0 \). Further, by (3.3.2), \( P \in \{D^{(b)}_{\lambda_u}\}' \) and \( S \in \{N^{(b)}_{\lambda_u}\}' \). The fact that \( P \) is a block diagonal operator is a routine verification (see [23, Proposition 6.1, Chapter IX]). The remaining part now follows from Lemma 1.2(vi). This completes the proof of (i) \( \Rightarrow \) (ii). The reverse implication is a routine verification using (3.3.2).

\[
\square
\]

**Remark 3.20** The injectivity of \( D^{(b)}_{\lambda_u} \) can be relaxed by replacing the basis \( \{e_v\}_{v \in V} \) by \( \{e_{\alpha(v)}\}_{v \in V} \) for some permutation \( \alpha : V \to V \). We leave the details to the reader.

The following result is applicable to the case when the weight system \( \lambda_u : V \to \mathbb{C} \) of the weighted join operator \( W^{(b)}_{\lambda_u} \) is injective.

**Corollary 3.21** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root root and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( u \in V \) and \( b \in V \setminus \{u\} \), consider the weight system \( \lambda_{uu} = \{\lambda_{uv}\}_{v \in V_\infty} \) of complex numbers and assume that the weighted join operator \( W^{(b)}_{\lambda_u} \) on \( \mathcal{T} \) belongs to \( B(\ell^2(V)) \). Let \( (D^{(b)}_{\lambda_u}, N^{(b)}_{\lambda_u}, \mathcal{H}^{(b)}_{\lambda_u}) \) denote the orthogonal decomposition of \( W^{(b)}_{\lambda_u} \). Assume that \( \ker(D^{(b)}_{\lambda_u}) = 0 \) and \( N^{(b)}_{\lambda_u} \neq 0 \). If \( \lambda_{uu} \notin \sigma_P(D^{(b)}_{\lambda_u}) \), then

\[
\{W^{(b)}_{\lambda_u}\}' = \{P \oplus S : P \in \{D^{(b)}_{\lambda_u}\}', S \in \{N^{(b)}_{\lambda_u}\}'\}.
\]

**Proof** Assume that \( \lambda_{uu} \notin \sigma_P(D^{(b)}_{\lambda_u}) \) and let \( X \in \{W^{(b)}_{\lambda_u}\}' \). Thus \( X \) admits the decomposition as given in (ii) of Theorem 3.19. However, by (3.3.1), \( W_{u} \) = \( \emptyset \), and hence \( e_u \otimes e_{\mu_u} A_u = 0 \). Also, since \( f_0 \in \ker(D^{(b)}_{\lambda_u} - \lambda_{uu}) \), by our assumption, \( f_0 = 0 \). This completes the proof.

\[
\square
\]

Even under the assumptions of the preceding corollary, the commutant of a weighted meet operator can be non-abelian.

**Example 3.22** Let \( \mathcal{T} = (V, E) \) be a leafless, rooted directed tree and let \( u \in V \) be such that \( V_u = \emptyset \) (for example, take the directed tree \( \mathcal{T} \) as shown in Fig. 4 and let \( u = v_0 \)), where \( V_u \) is as given in (2.2.1). For \( b \in \text{Des}(u) \), consider the weight system \( \lambda_u = \{\lambda_{uv}\}_{v \in V_\infty} \) of distinct positive numbers and assume that the weighted join operator \( W^{(b)}_{\lambda_u} \) on \( \mathcal{T} \) belongs to \( B(\ell^2(V)) \). Let \( (D^{(b)}_{\lambda_u}, N^{(b)}_{\lambda_u}, \mathcal{H}^{(b)}_{\lambda_u}) \) denote the orthogonal decomposition of \( W^{(b)}_{\lambda_u} \). Since \( V_u = \emptyset \), by (3.2.11) and (3.2.15), we have

\[
\ell^2(V) \odot \mathcal{H}^{(b)}_{\lambda_u} = \ell^2(A_u), \quad \text{where } A_u = [u, b].
\]

(3.3.6)
We claim that \( \{ W_{\lambda u}^{(b)} \}' \) is non-abelian if and only if \( \dim (\ell^2(V) \ominus \mathcal{H}_u^{(b)}) \geq 3 \). By the preceding corollary, it suffices to check that \( \{ N_{\lambda u}^{(b)} \}' \) is non-abelian if and only if \( \dim (\ell^2(V) \ominus \mathcal{H}_u^{(b)}) \geq 3 \). We consider the following cases:

**Case 1.** \( \dim (\ell^2(V) \ominus \mathcal{H}_u^{(b)}) = 1 \):

By (3.3.6), \( b = u \), and hence \( W_{\lambda u}^{(b)} \) is the diagonal operator \( D_{\lambda u} \) with distinct diagonal entries. In this case, \( \{ W_{\lambda u}^{(b)} \}' \) is indeed maximal abelian [23].

**Case 2.** \( \dim (\ell^2(V) \ominus \mathcal{H}_u^{(b)}) = 2 \):

Consider the basis \( \{ e_u, e_{\lambda u, A_u} \} \) of \( \ell^2([u, b]) \) and let \( T \in \{ N_{\lambda u}^{(b)} \}' \). By Lemma 1.2(iv),

\[
\begin{align*}
T e_u &= (T e_u, e_u) e_u, \\
T^* e_{\lambda u, A_u} &= (T e_u, e_{\lambda u, A_u}).
\end{align*}
\]  

(3.3.7)

(3.3.8)

Let \( \alpha_T \) and \( \beta_T \) be scalars such that \( T e_{\lambda u, A_u} = \alpha_T e_u + \beta_T e_{\lambda u, A_u} \). Then

\[
\begin{align*}
\langle T e_{\lambda u, A_u}, e_{\lambda u, A_u} \rangle &= \alpha_T \lambda uu + \beta_T \| e_{\lambda u, A_u} \|^2, \\
\langle e_{\lambda u, A_u}, T^* e_{\lambda u, A_u} \rangle &= \langle T e_u, e_u \rangle \| e_{\lambda u, A_u} \|^2.
\end{align*}
\]  

(3.3.9)

It follows that

\[
\alpha_T = \frac{\| e_{\lambda u, A_u} \|^2_{\lambda uu}}{(\langle T e_u, e_u \rangle - \beta_T)}.
\]

Hence, for any \( S \in \{ N_{\lambda u}^{(b)} \}' \), we have

\[
ST e_{\lambda u, A_u} = \alpha_T S e_u + \beta_T S e_{\lambda u, A_u}
\]  

(3.3.7)

\[
(\alpha_T \langle S e_u, e_u \rangle + \beta_T \alpha_S) e_u + \beta_T \beta S e_{\lambda u, A_u}.
\]

By symmetry, \( ST e_{\lambda u, A_u} = TS e_{\lambda u, A_u} \) if and only if

\[
\alpha_T (\langle S e_u, e_u \rangle - \beta_S) = \alpha_S (\langle T e_u, e_u \rangle - \beta_T).
\]

The latter equality follows from (3.3.9). On the other hand, for any \( S \in \{ N_{\lambda u}^{(b)} \}' \), by (3.3.7), \( ST e_u = T S e_u \) always holds. This shows that \( \{ N_{\lambda u}^{(b)} \}' \) is abelian.

**Case 3.** \( \dim (\ell^2(V) \ominus \mathcal{H}_u^{(b)}) = 3 \):

Let \( f \in \ell^2(A_u) \) be orthogonal to \( \{ e_u, e_{\lambda u, A_u} \} \). Consider a bounded linear operator \( T \) on \( \ell^2(V) \ominus \mathcal{H}_u^{(b)} \) governed by

\[
T e_u = \alpha e_u, \quad T e_{\lambda u, A_u} = \beta e_u + \gamma f, \quad Tf = 0.
\]
where $\alpha, \beta, \gamma$ are complex numbers. Clearly, $TN_{\lambda_u}^{(b)}e_u = N_{\lambda_u}^{(b)}Te_u$, $TN_{\lambda_u}^{(b)}f = N_{\lambda_u}^{(b)}Tf$. Moreover, $TN_{\lambda_u}^{(b)}e_{\lambda_u} = N_{\lambda_u}^{(b)}Te_{\lambda_u}$ if and only if

$$\beta = (\|e_{\lambda_u}\|^2(Te_u, e_u))/\lambda_{\mu u}.$$

Consider another bounded linear operator $S$ on $\ell^2(V) \ominus \mathcal{H}_u^{(b)}$ governed by

$$Se_u = a e_u, \quad Se_{\lambda_u} = b e_u, \quad Sf = c e_u + d e_{\lambda_u},$$

where $a, b, c, d$ are complex numbers. A routine calculation shows that $S \in \{N_{\lambda_u}^{(b)}\}'$ if and only if

$$b\lambda_{\mu u} - a\|e_{\lambda_u}\|^2 = 0, \quad c\lambda_{\mu u} + d\|e_{\lambda_u}\|^2 = 0.$$

On the other hand, $STf = TSf$ implies that $d\gamma = 0$, and hence for non-zero choices of $\gamma$ and $d$, $S$ and $T$ do not commute. This shows that $\{N_{\lambda_u}^{(b)}\}'$ is not abelian.

Finally, in case dim $\ell^2(V) \ominus \mathcal{H}_u^{(b)} \geq 4$, $\{N_{\lambda_u}^{(b)}\}'$ contains a copy of $B(\mathbb{C}^2)$, and hence it is not abelian.

### 4 Rank One Extensions of Weighted Join Operators

In this section, we introduce and study the class of rank one extensions of weighted join operators. We introduce the so-called compatibility conditions and discuss their roles in the closedness of these operators. We also discuss the problem of determining the Hilbert space adjoint of these operators. Further, we provide a complete spectral picture for members in this class and discuss some of its applications.

**Definition 4.1** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root root and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $b, u \in V$ and the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers, let $W_{\lambda_u}^{(b)}$ be a weighted join operator on $\mathcal{T}$. Consider the orthogonal decomposition $D_{\lambda_u}^{(b)}, N_{\lambda_u}^{(b)}, \mathcal{H}_u^{(b)}$ of $W_{\lambda_u}^{(b)}$.

By a rank one extension of $W_{\lambda_u}^{(b)}$ on $\mathcal{T}$, we understand the linear operator $W_{\lambda_u}^{(b)}[f, g]$ in $\ell^2(V)$ given by

$$D(W_{\lambda_u}^{(b)}[f, g]) = \{(h, k) : h \in D(D_{\lambda_u}^{(b)}) \cap D(f \circ g), k \in \ell^2(V) \ominus \mathcal{H}_u^{(b)}\}$$

$$W_{\lambda_u}^{(b)}[f, g] = \begin{bmatrix} D_{\lambda_u}^{(b)} & 0 \\ f \circ g & N_{\lambda_u}^{(b)} \end{bmatrix},$$

(4.0.1)

where $f \in \ell^2(V) \ominus \mathcal{H}_u^{(b)}$ is non-zero and $g : \text{supp} \mathcal{H}_u^{(b)} \to \mathbb{C}$ is unspecified. For the sake of convenience, we use the simpler notation $W_{f, g}$ in place of $W_{\lambda_u}^{(b)}[f, g]$. 


Remark 4.2 Since $N_{\lambda_u}^{(b)}$ is bounded and the domains of $D_{\lambda_u}^{(b)}$ and $f \otimes g$ contains the dense subspace $\mathcal{D}_{\supp \mathcal{H}_u^{(b)}}$ of $\mathcal{H}_u^{(b)}$ (see (3.0.2)), $W_{f,g}$ is densely defined. Since the sum of a closed operator and a bounded linear operator is closed, the rank one extension $W_{f,g}$ of $W_{\lambda_u}^{(b)}$ is closed provided $g \in \mathcal{H}_u^{(b)}$. This happens in particular when $\text{Des}(u)$ has finite cardinality (see (3.2.11)). In case $g \notin \mathcal{H}_u^{(b)}$, $W_{f,g}$ need not be closed (cf. Corollary 4.18). To see this assertion, consider the situation in which $D_{\lambda_u}^{(b)}$ is bounded and $g \notin \mathcal{H}_u^{(b)}$. By Lemma 1.3, $f \otimes g$ is not closable, and hence there exists a sequence $\{h_n\}_{n \geq 0}$ in $\mathcal{H}_u^{(b)}$ such that $h_n \to 0$, $\{(f \otimes g)(h_n)\}_{n \geq 0}$ is convergent but $(f \otimes g)(h_n) \to 0$ as $n \to \infty$. Then $(h_n,0) \to (0,0)$, $\{W_{f,g}(h_n,0)\}_{n \geq 0}$ is convergent, however, $W_{f,g}(h_n,0) \to 0$, and hence $W_{f,g}$ is not even closable.

Here is a remark about the manner in which $W_{f,g}$ is defined. Certainly, one could have defined the rank one extension of $W_{\lambda_u}^{(b)}$ with the entry $f \otimes g$ appearing on the extreme upper right corner in (4.0.1). It turns out, however, that the operators defined this way are closed if and only if $g \in \ell^2(V) \ominus \mathcal{H}_u^{(b)}$. From the view point of the spectral theory, these operators are of little importance in case $g \notin \ell^2(V) \ominus \mathcal{H}_u^{(b)}$, and otherwise, these are bounded finite rank perturbations of diagonal operators. Needless to say, the latter class has been studied extensively in the literature (refer, for example, to [26–29, 37, 44, 48, 69]). Also, the way in which $W_{f,g}$ is defined (cf. [2–4, 60]), it should be referred to as rank one co-extension of $W_{\lambda_u}^{(b)}$. However, by abuse of terminology, we refer to it as a rank one extension of $W_{\lambda_u}^{(b)}$.

In what follows, we will be particularly interested in the following family of rank one extensions of weighted join operators (cf. Proposition 5.7 below).

Example 4.3 Let $W_{f,g}$ be a rank one extension of the weighted join operator $W_{\lambda_u}^{(b)}$ satisfying the intertwining relation:

$$
(f \otimes g)D_{\lambda_u}^{(b)} + N_{\lambda_u}^{(b)}(f \otimes g) = 0,
$$

where the linear operator on the left-hand side is defined on the space $\mathcal{D}_{\supp \mathcal{H}_u^{(b)}}$ (see (3.0.2)). Note that (4.0.2) is equivalent to

$$
\lambda_{uv}f + \langle f, e_{\ast, A_u}\rangle e_{\ast} = 0, \quad v \in \supp(g),
$$

where $A_u$ is given by (3.2.15) and the support $\supp(h)$ of the function $h : V \to \mathbb{C}$ is given by

$$
\supp(h) := \{v \in V : h(v) \neq 0\}.
$$

Suppose $g \neq 0$ and note that $\supp(g)$ is non-empty. We make the following observations:

(i) If $\lambda_{uv} = 0$ for some $v \in \supp(g)$, then by (4.0.3), $\langle f, e_{\ast, A_u}\rangle = 0$. Since $f \neq 0$, by another application of (4.0.3), $\lambda_{uw} = 0$ for all $w \in \supp(g)$. 


(ii) If \( \lambda_{uv} \neq 0 \) for some \( v \in \text{supp}(g) \), then by (4.0.3), \( f \in [e_u] \setminus \{0\} \) and \( \lambda_{uw} = -\lambda_{uu} \) for every \( w \in \text{supp}(g) \).

The above discussion provides the following examples of \( W_{f,g} \) satisfying (4.0.2).

(a) \( \lambda_{uv} = 0 \) for \( v \in \text{supp}(g) \), \( \text{supp } \mathcal{H}_u \setminus \text{supp}(g) \) is infinite and
\[
\sum_{w \in A_u} f(w) \lambda_{uw} = 0.
\]

(b) \( \lambda_{uv} = -\lambda_{uu} \neq 0 \) for every \( v \in \text{supp}(g) \) and \( f \in [e_u] \setminus \{0\} \).

For example, if \( \mathcal{T} \) is leafless and \( u \) is a branching vertex, then the condition that \( \text{supp } \mathcal{H}_u \setminus \text{supp}(g) \) is infinite in (a) is ensured for any \( g \) such that \( \text{supp}(g) = \text{Des}(w) \) provided
\[
 w \text{ belongs to } \left\{ \begin{array}{ll}
 \text{Chi}(b) & \text{if } b \in \text{Des}(u), \\
 \text{Chi}(u) & \text{if } b \notin \text{Des}(u).
 \end{array}ight.
\]

In case (b) holds, then the rank one extension of \( W^{(b)}_{\lambda_u} \) can be rewritten as the sum of a diagonal operator and the rank one operator \( e_u \otimes (f(u)g + e_{A_u}) \). In these cases, \( W_{f,g} \) satisfies
\[
W_{f,g}^2 = (D^{(b)}_{\lambda_u})^2 \oplus (N^{(b)}_{\lambda_u})^2 \text{ on } \mathcal{D}_{\text{supp } \mathcal{H}_u} \oplus (\ell^2(V) \ominus \mathcal{H}_u^{(b)}).
\]

Thus although \( W_{f,g} \) does not have a diagonal decomposition, the intertwining relation (4.0.2) ensures the same for its square. ■

The bounded rank one extensions of weighted join operators can be characterized easily.

**Proposition 4.4** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root root and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( u, b \in V \), consider the weight system \( \lambda_u = \{ \lambda_{uv} \}_{v \in V_\infty} \) of complex numbers and let \( W_{f,g} \) be the rank one extension of the weighted join operator \( W^{(b)}_{\lambda_u} \) on \( \mathcal{T} \). Then the following are equivalent:

(i) \( W_{f,g} \) defines a bounded linear operator on \( \ell^2(V) \).
(ii) \( g \in \mathcal{H}_u^{(b)} \) and \( D^{(b)}_{\lambda_u} \) defines a bounded linear operator on \( \mathcal{H}_u^{(b)} \).
(iii) \( g \in \mathcal{H}_u^{(b)} \) and
\[
\lambda_u \text{ belongs to } \left\{ \begin{array}{ll}
 \ell^\infty(V) & \text{if } b = u, \\
 \ell^\infty(\text{Des}(u)) & \text{otherwise}.
 \end{array}ight.
\]

**Proof** In view of Theorem 3.7, it suffices to see the equivalence of (i) and (ii). Since \( N^{(b)}_{\lambda_u} \) is a bounded linear operator on \( \ell^2(V) \ominus \mathcal{H}_u^{(b)} \), (ii) implies (i). To see the reverse
implication, assume that \( W f, g \) is bounded linear on \( \ell^2(V) \). Thus \( \mathcal{D}(D_{\lambda u}^{(b)}) \cap \mathcal{D}(f \circ g) = \mathcal{H}_{u}^{(b)} \). By the closed graph theorem, \( D_{\lambda u}^{(b)} \), being a closed operator defined on \( \mathcal{H}_{u}^{(b)} \), is a bounded linear operator on \( \mathcal{H}_{u}^{(b)} \). Further, for any \( h \in \mathcal{D}(D_{\lambda u}^{(b)}) \cap \mathcal{D}(f \circ g) \),

\[
\| W f, g(h, 0) \|_2 = \left\| \left( D_{\lambda u}^{(b)} h, \left( \sum_{v \in \text{supp } \mathcal{H}_{u}^{(b)}} h(v)g(v) \right) f \right) \right\|_2
\]

\[
= \| D_{\lambda u}^{(b)} h \|_2^2 + \| (f \circ g)(h) \|_2^2.
\]

Since \( f \neq 0 \) and \( W f, g \) is bounded, \( f \circ g \) must be bounded linear, and hence by Lemma 1.3, \( g \in \mathcal{H}_{u}^{(b)} \). This completes the proof.

\[\square\]

### 4.1 Compatibility Conditions and Discrete Hilbert Transforms

It turns out that the inclusion \( \mathcal{D}(D_{\lambda u}^{(b)}) \subseteq \mathcal{D}(f \circ g) \) of domains of \( D_{\lambda u}^{(b)} \) and \( f \circ g \) plays a central role in deciding whether or not a given rank one extension \( W f, g \) of a weighted join operator is closed (cf. Remark 4.2). Indeed, we will see that the so-called compatibility conditions on \( W f, g \) always ensure the above inclusion as well as the closedness of \( W f, g \). We formally introduce these conditions below (cf. [37, Proposition 2.4(iv)], [48, Proposition 4.1]).

**Definition 4.5** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root root and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( u, b \in V \), consider the weight system \( \lambda_u = {\lambda_{uv}}_{v \in V_\infty} \) of complex numbers and let \( W f, g \) be the rank one extension of the weighted join operator \( W_{\lambda u}^{(b)} \) on \( \mathcal{T} \), where \( f \in \ell^2(V) \otimes \mathcal{H}_{u}^{(b)} \) is non-zero and \( g : \text{supp } \mathcal{H}_{u}^{(b)} \to \mathbb{C} \) is given. Set

\[
\text{dist}(\mu, \lambda_u) = \inf \{|\mu - \lambda_{uv}| : v \in \text{supp } \mathcal{H}_{u}^{(b)}\}, \quad \mu \in \mathbb{C},
\]

\[\Gamma_{\lambda u} = \{\mu \in \mathbb{C} : \text{dist}(\mu, \lambda_u) > 0\}. \tag{4.1.1}\]

(i) We say that \( W f, g \) satisfies compatibility condition I if there exists \( \mu_0 \in \Gamma_{\lambda u} \) such that \( g_{\lambda u, \mu_0} \in \mathcal{H}_{u}^{(b)} \), where

\[
g_{\lambda u, \mu_0}(v) := \frac{g(v)}{\lambda_{uv} - \mu_0}, \quad v \in \text{supp } \mathcal{H}_{u}^{(b)}. \tag{4.1.2}\]

(ii) We say that \( W f, g \) satisfies compatibility condition II if the function \( g \) satisfies

\[
\sum_{v \in \text{supp } \mathcal{H}_{u}^{(b)}} \frac{|g(v)|^2}{|\lambda_{uv}|^2 + 1} < \infty. \tag{4.1.3}\]

If one of the above conditions holds, then we say that \( W f, g \) satisfies a compatibility condition.
Remark 4.6  It turns out that $g_{\lambda u, \mu_0} \in \mathcal{H}_u^{(b)}$ for some $\mu_0 \in \Gamma_{\lambda u}$, then $g_{\lambda u, \mu} \in \mathcal{H}_u^{(b)}$ for every $\mu \in \Gamma_{\lambda u}$. This may be derived from

$$|\lambda_{uv} - \mu_0|^2 \leq 2(|\lambda_{uv} - \mu|^2 + |\mu_0 - \mu|^2), \quad v \in \text{supp } \mathcal{H}_u^{(b)}, \quad \mu \in \Gamma_{\lambda u}\setminus\{\mu_0\},$$

and the fact that $|\mu_0 - \mu| \leq c \text{ dist}(\mu, \lambda u)$ for some $c > 0$ (see (4.1.1)).

Here we discuss the relationships between the above compatibility conditions and the domain inclusion $\mathcal{D}(D_{\lambda u}^{(b)}) \subseteq \mathcal{D}(f \circ g)$.

Proposition 4.7  Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root root and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u, b \in V$, consider the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers and let $W_{f,g}$ be the rank one extension of the weighted join operator $W_{\lambda_u}^{(b)}$ on $\mathcal{T}$, where $f \in \ell^2(V) \ominus \mathcal{H}_u^{(b)}$ is non-zero and $g : \text{supp } \mathcal{H}_u^{(b)} \to \mathbb{C}$ is given. If $W_{f,g}$ satisfies a compatibility condition, then we have the domain inclusion $\mathcal{D}(D_{\lambda u}^{(b)}) \subseteq \mathcal{D}(f \circ g)$. Moreover, if $\Gamma_{\lambda u}$ is non-empty, then the following statements are equivalent:

(i) $\mathcal{D}(D_{\lambda u}^{(b)}) \subseteq \mathcal{D}(f \circ g)$.
(ii) $W_{f,g}$ satisfies compatibility condition I.
(iii) The discrete Hilbert transform $H_{\lambda u,g}(h)$ given by

$$H_{\lambda u,g}(h) = \sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} \frac{h(v)\overline{g(v)}}{\mu - \lambda_{uv}}$$

is well-defined for every $\mu \in \Gamma_{\lambda u}$ and every $h \in \mathcal{H}_u^{(b)}$.
(iv) For every $\mu \in \Gamma_{\lambda u}$, the linear operator $L_{\lambda u, \mu} := (f \circ g)(D_{\lambda u}^{(b)} - \mu)^{-1}$ defines a bounded linear transformation from $\mathcal{H}_u^{(b)}$ into $\ell^2(V) \ominus \mathcal{H}_u^{(b)}$.

Remark 4.8  Note that $\sigma(D_{\lambda u}^{(b)}) = \mathbb{C}\setminus\Gamma_{\lambda u}$ (see [64, Example 3.8]). This clarifies the expression $(D_{\lambda u}^{(b)} - \mu)^{-1}$ appearing in (iv). It is worth noting that a discrete Hilbert transform appears in [37, Corollary 2.5], which characterizes the set of eigenvalues of a bounded rank one perturbation of a diagonal operator (see also [37, Corollary 2.6]). Also, the operator $L_{\lambda u, \mu}$, as appearing in Proposition 4.7(iv), is precisely the operator $G(\mu)$, as appearing in the Frobenius-Schur-type factorization in [7, Equation (1.6)].

Proof  Let $h \in \mathcal{D}(D_{\lambda u}^{(b)})$. We divide the verification of the domain inclusion $\mathcal{D}(D_{\lambda u}^{(b)}) \subseteq \mathcal{D}(f \circ g)$ into the following cases:

Assume that $W_{f,g}$ satisfies the compatibility condition I. Then, for some $\mu_0 \in \Gamma_{\lambda u}$, by the Cauchy–Schwarz inequality,

$$\left| \sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} h(v)\overline{g(v)} \right| \leq \|(D_{\lambda u}^{(b)} - \mu_0)h\| \|g_{\lambda u, \mu_0}\|.$$
where we used (4.1.2) and the fact that \( \mathcal{D}(D_{\lambda u}^{(b)} - \mu_0) = \mathcal{D}(D_{\lambda u}^{(b)}) \). This shows that \( h \in \mathcal{D}(f \otimes g) \).

Next assume that \( W_{f,g} \) satisfies the compatibility condition II. Since \( \mathcal{D}(D_{\lambda u}^{(b)}) = \mathcal{D}((D_{\lambda u}^{(b)})^* D_{\lambda u}^{(b)} + I)^{1/2} \), by the Cauchy–Schwarz inequality,

\[
\left| \sum_{v \in \text{supp} \mathcal{H}_u^{(b)}} h(v) \overline{g(v)} \right|^2 \leq \| (D_{\lambda u}^{(b)})^* D_{\lambda u}^{(b)} + I \|^{1/2} \sum_{v \in \text{supp} \mathcal{H}_u^{(b)}} |g(v)|^2 / |\lambda_{uv}|^2 + 1.
\]

Thus \( h \in \mathcal{D}(f \otimes g) \) in this case, as well.

The preceding discussion also yields the implication (ii) \( \Rightarrow \) (i). To see the equivalence of (i)–(iv), assume that \( \Gamma_{\lambda u} \) is non-empty.

(i) \( \Rightarrow \) (ii): Let \( \mu \in \Gamma_{\lambda u} \). For \( h \in \mathcal{H}_u^{(b)} \), consider the function \( k_h : \text{supp} \mathcal{H}_u^{(b)} \to \mathbb{C} \) defined by

\[
k_h(v) = \frac{h(v)}{\lambda_{uv} - \mu}, \quad v \in \text{supp} \mathcal{H}_u^{(b)}.
\]

Clearly, \( k_h \) belongs to \( \mathcal{D}(D_{\lambda u}^{(b)}) \) for every \( h \in \mathcal{H}_u^{(b)} \). By assumption, \( \mathcal{D}(D_{\lambda u}^{(b)}) \subseteq \mathcal{D}(f \otimes g) \), and hence the linear functional \( \phi_h : \mathcal{H}_u^{(b)} \to \mathbb{C} \) given by

\[
\phi_h(h) = \sum_{v \in \text{supp} \mathcal{H}_u^{(b)}} k_h(v) g(v), \quad h \in \mathcal{H}_u^{(b)}
\]

is well-defined. By the standard polar representation, the series \( \sum_{v \in \text{supp} \mathcal{H}_u^{(b)}} k_h(v) g(v) \) is absolutely convergent for every \( h \in \mathcal{H}_u^{(b)} \). One may now apply the uniform boundedness principle [66] to the family of linear functionals \( \phi_F,g(h) = \sum_{v \in F} h(v) \frac{g(v)}{\lambda_{uv} - \mu} \), \( F \) is a finite subset of \( \text{supp} \mathcal{H}_u^{(b)} \) to derive the boundedness of \( \phi_g \). By (4.1.4) and (4.1.5), the boundedness of \( \phi_g \) in turn is equivalent to \( g_{\lambda u, \mu} \in \mathcal{H}_u^{(b)} \).

(iii) \( \Rightarrow \) (ii): This may be derived from the uniform boundedness principle (see the verification of (i) \( \Rightarrow \) (ii)).

(ii) \( \Rightarrow \) (iii): In view of the Cauchy–Schwarz inequality, it suffices to check that \( g_{\lambda u, \mu_0} \in \mathcal{H}_u^{(b)} \) for some \( \mu_0 \in \Gamma_{\lambda u} \), then \( g_{\lambda u, \mu} \in \mathcal{H}_u^{(b)} \) for every \( \mu \in \Gamma_{\lambda u} \). This is observed in Remark 4.6.

(ii) \( \Rightarrow \) (iv): By the Cauchy–Schwarz inequality, for any \( h \in \mathcal{H}_u^{(b)} \),

\[
\| L_{\lambda u, \mu} h \| = \| (f \otimes g)(D_{\lambda u}^{(b)} - \mu)^{-1} h \| = \left| \sum_{v \in \text{supp} \mathcal{H}_u^{(b)}} h(v) \frac{g(v)}{\lambda_{uv} - \mu} \| f \| \leq \| h \| \| g_{\lambda u, \mu} \|.
\]

Since \( g_{\lambda u, \mu} \in \mathcal{H}_u^{(b)} \), this shows that \( L_{\lambda u, \mu} \) is bounded linear.

(iv) \( \Rightarrow \) (iii): This is straightforward. \( \square \)
Here is an instance in which the compatibility condition II implies the compatibility condition I.

**Corollary 4.9** Under the hypotheses of Proposition 4.7 and the assumption that \( \Gamma_{\lambda_u} \neq \emptyset \), if \( W_{f,g} \) satisfies the compatibility condition II, then it satisfies the compatibility condition I.

**Proof** If \( W_{f,g} \) satisfies the compatibility condition II, then by the first half of Proposition 4.7, we obtain \( \mathcal{D}(D_{\lambda_u}(b)) \subseteq \mathcal{D}(f \circ g) \). The desired conclusion now follows from the implication (i) \( \Rightarrow \) (ii) of Proposition 4.7. \( \square \)

Following [15], for any \( g : \text{supp} \, \mathcal{H}_u(b) \to \mathbb{C} \setminus \{0\} \), we set
\[
(\Gamma, g)^* = \{ \mu \in \mathbb{C} : g_{\lambda_u, \mu} \in \mathcal{H}_u(b) \}.
\]

The following has been motivated by the discussion from [15, Pg 2] on discrete Hilbert transforms in a slightly different context. Note that the compatibility condition II is nothing but the existence of admissible weight sequence in the sense of [15].

**Proposition 4.10** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root \( \text{root} \) and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( u, b \in V \), consider the weight system \( \lambda_u = \{ \lambda_{uv} \}_{v \in V_\infty} \) of complex numbers and let \( W_{f,g} \) be the rank one extension of the weighted join operator \( W_{\lambda_u}(b) \) on \( \mathcal{T} \), where \( f \in \ell^2(V) \otimes \mathcal{H}_u(b) \) is non-zero and \( g : \text{supp} \, \mathcal{H}_u(b) \to \mathbb{C} \setminus \{0\} \) be given. Then the following statements are true:

(i) If \( \{ \lambda_{uv} : v \in \text{supp} \, \mathcal{H}_u(b) \} \) is closed, then \( W_{f,g} \) satisfies the compatibility condition I if and only if
\[
\Gamma_{\lambda_u} = \mathbb{C} \setminus \{ \lambda_{uv} : v \in \text{supp} \, \mathcal{H}_u(b) \} = (\Gamma, g)^* \quad (\text{see (4.1.1) and (4.1.6)}).
\]

(ii) If \( \{ \lambda_{uv} : v \in \text{supp} \, \mathcal{H}_u(b) \} \) has an accumulation point only at \( \infty \) with each of its entries appearing finitely many times, then \( W_{f,g} \) satisfies compatibility condition II if and only if \( \Gamma_{\lambda_u} = (\Gamma, g)^* \).

**Proof** To see (i), assume that \( \{ \lambda_{uv} : v \in \text{supp} \, \mathcal{H}_u(b) \} \) is closed. Clearly,
\[
\Gamma_{\lambda_u} = \mathbb{C} \setminus \{ \lambda_{uv} : v \in \text{supp} \, \mathcal{H}_u(b) \}.
\]

Since \( g \) is nowhere vanishing,
\[
(\Gamma, g)^* \subseteq \mathbb{C} \setminus \{ \lambda_{uv} : v \in \text{supp} \, \mathcal{H}_u(b) \}.
\]

Note further that if \( W_{f,g} \) satisfies the compatibility condition I, then \( g_{\lambda_u, \mu} \) belongs to \( \mathcal{H}_u(b) \) for every \( \mu \in \Gamma_{\lambda_u} \) (see Remark 4.6). In this case, \( \Gamma_{\lambda_u} \subseteq (\Gamma, g)^* \), and hence the
necessity part in (i) follows from (4.1.7) and (4.1.8). Since $C \{ \lambda u^v : v \in \text{supp } \mathcal{H}_u^{(b)} \}$ is always a non-empty set (by the assumption that $\text{card}(V) = \aleph_0$, $\text{supp } \mathcal{H}_u^{(b)}$ is always countable), the sufficiency part of (i) follows from (4.1.6).

To see (ii), assume that $\{ \lambda u^v : v \in \text{supp } \mathcal{H}_u^{(b)} \}$ has an accumulation point only at $\infty$ with each of its entries appearing finitely many times. The necessity part follows from (i), (4.1.7) and Corollary 4.9. To see the sufficiency part of (ii), suppose that $\langle \Gamma, g \rangle^* = \Gamma \lambda_u$. By (i), for some $\mu_0 \in \mathbb{C}$, we must have

$$\sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} \frac{|g(v)|^2}{|\mu_0 - \lambda u^v|^2} < \infty.$$  \hspace{1cm} (4.1.9)

However, since $\infty$ is the only accumulation point for $\{ \lambda u^v : v \in \text{supp } \mathcal{H}_u^{(b)} \}$, there exists (sufficiently large) $M > 0$ such that $|\mu_0 - \lambda u^v|^2 \leq M(\lambda u^v)^2 + 1$ for every $v \in \text{supp } \mathcal{H}_u^{(b)}$. It now follows from (4.1.9) that $W_{f,g}$ satisfies the compatibility condition II. $\Box$

4.2 Closedness and Relative Boundedness

In this section, we show that any rank one extension $W_{f,g}$ of weighted join operator satisfying a compatibility condition is closed (cf. [7, Theorem 1.1], [72, Theorems 2.5 and 2.6]). This is achieved by decomposing $W_{f,g}$ as $A + B$, where $A$ is closed and $B$ is $A$-bounded. We begin recalling some definitions from [46]. Given densely defined linear operators $A$ and $B$ in $\mathcal{H}$, we say that $B$ is $A$-bounded if $D(B) \supseteq D(A)$ and there exist non-negative real numbers $a$ and $b$ such that

$$\|Bx\|^2 \leq a\|Ax\|^2 + b\|x\|^2, \quad x \in D(A).$$

The infimum of all $a \geq 0$ for which there exists a number $b \geq 0$ such that the above inequality holds is called the $A$-bound of $B$. Note that $B$ is $A$-bounded if and only if $D(B) \supseteq D(A)$ and there exist non-negative real numbers $a$ and $b$ such that

$$\|Bx\| \leq a\|Ax\| + b\|x\|, \quad x \in D(S).$$ \hspace{1cm} (4.2.1)

For basic facts pertaining to $A$-bounded operators, the reader is referred to [46, 64, 68]. For the sake of convenience, we recall here the statement of the Kato-Rellich theorem from [46]. Suppose that $A$ is a closed operator in $\mathcal{H}$. Let $B$ be a linear operator such that $D(A) \subseteq D(B)$ and there exist $a \in (0, 1)$ and $b \in (0, \infty)$ with the property (4.2.1), then the linear operator $A + B$ with the domain $D(A)$ is a closed operator in $\mathcal{H}$ (see [46, Theorem 1.1, Chapter IV]).

**Theorem 4.11** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root root and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u, b \in V$, consider the weight system $\lambda_u = \{ \lambda uv \}_{v \in V_\infty}$ of complex numbers and let $W_{f,g}$ be the rank one
extension of the weighted join operator $W_{\lambda_u}^{(b)}$ on $\mathcal{T}$, where $f \in \ell^2(V) \ominus \mathcal{H}_u^{(b)}$ is non-zero and $g : \text{supp} \mathcal{H}_u^{(b)} \to \mathbb{C}$ is given. Suppose that $W_{f,g}$ satisfies a compatibility condition. Then $W_{f,g}$ defines a closed linear operator with the domain given by

$$\mathcal{D}(W_{f,g}) = \{(h,k) : h \in \mathcal{D}(D_{\lambda_u}^{(b)}), k \in \ell^2(V) \ominus \mathcal{H}_u^{(b)}\}. \quad (4.2.2)$$

Moreover, $\mathcal{D}_V$, as given by (3.0.2), forms a core for $W_{f,g}$.

**Proof** Suppose that $W_{f,g}$ satisfies the compatibility condition I for some $\mu \in \Gamma_{\lambda_u}$. Let $a$ be a positive real number less than 1. Since $g_{\lambda_u,\mu} \in \mathcal{H}_u^{(b)}$, there exists a finite subset $F$ of $\text{supp} \mathcal{H}_u^{(b)}$ such that

$$\sum_{\text{supp} \mathcal{H}_u^{(b)} \setminus F} \frac{|g(v)|^2}{|\lambda_{uv} - \mu|^2} \leq \frac{a}{4 \|f\|^2}. \quad (4.2.3)$$

Define closed linear operators $N_F$ and $D_F$ in $\mathcal{H}_u^{(b)}$ by

$$N_F = \sum_{v \in F} \lambda_{uv} e_v \otimes e_v, \quad D_F = D_{\lambda_u}^{(b)} - N_F.$$ 

Further let $g_F = \sum_{v \in \text{supp} \mathcal{H}_u^{(b)} \setminus F} g(v) e_v$. We rewrite $W_{f,g}$ as $A + B + C$, where $A$, $B$, $C$ (with their natural domains) are densely defined operators in $\ell^2(V)$ given by

$$A = \begin{bmatrix} D_F & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ f \otimes g_F & 0 \end{bmatrix}, \quad C = \begin{bmatrix} N_F & 0 \\ f \otimes (g - g_F) N_{\lambda_u}^{(b)} \end{bmatrix}.$$ 

Note that $\mathcal{D}(D_F) = \mathcal{D}(D_{\lambda_u}^{(b)})$ and $\mathcal{D}(f \otimes g_F) = \mathcal{D}(f \otimes g)$. Furthermore, $A$ is a closed linear operator in $\ell^2(V)$ and $C$ is a bounded linear operator on $\ell^2(V)$. Moreover, $\mathcal{D}(D_F) \subseteq \mathcal{D}(f \otimes g_F)$ (cf. Proposition 4.7). Indeed, by the Cauchy–Schwarz inequality and (4.2.3), for any $h \in \mathcal{D}(D_F)$,

$$\left| \sum_{v \in \text{supp} \mathcal{H}_u^{(b)} \setminus F} h(v) g(v) \right| \leq \frac{\sqrt{a}}{2} \frac{\|D_F - \mu\| h}{\|f\|}. \quad (4.2.4)$$

Since $C$ is a bounded operator, we obtain that $\mathcal{D}(A) \subseteq \mathcal{D}(B + C)$. We claim that for all $h \in \mathcal{D}(A)$,

$$\|B + C\| h \|^2 \leq a \|Ah\|^2 + b(\mu) \|h\|^2, \quad (4.2.5)$$

where $b(\mu) = a|\mu|^2 + 2\|C\|^2$. To see the claim, let $h = (h_1, h_2) \in \mathcal{D}(A)$. By repeated applications of $|\alpha + \beta|^2 \leq 2(|\alpha|^2 + |\beta|^2)$, $\alpha, \beta \in \mathbb{C}$, we obtain

$$\|B + C\| h \|^2 \leq 2\|f \otimes g_F\| h_1 \|^2 + 2\|C\|^2 \|h\|^2.$$
This completes the verification of (4.2.5). Thus \( B + C \) is \( A \)-bounded with \( A \)-bound less than 1. Hence, by the Kato-Rellich Theorem, \( W_{f, g} \) is a closed operator with the domain given by (4.2.2).

Next suppose that \( W_{f, g} \) satisfies the compatibility condition II. Let

\[
G_m := \sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} \frac{|g(v)|^2}{|\lambda_{uv}|^2 + m^2}, \quad m \geq 1. \tag{4.2.6}
\]

Note that

\[
0 \leq G_m \leq G_1 < \infty, \quad m \geq 1. \tag{4.2.7}
\]

We rewrite \( W_{f, g} \) as \( A + B + C \), where \( A, B, C \) (with their natural domains) are densely defined operators in \( \ell^2(V) \) given by

\[
A = \begin{bmatrix} D^{(b)}_h & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ f \otimes g & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & N^{(b)}_h \end{bmatrix}. \tag{4.2.8}
\]

Note that \( A \) is a closed linear operator in \( \ell^2(V) \) and \( C \) is a bounded linear operator on \( \ell^2(V) \). Given a positive integer \( m \), consider the inner-product space \( \mathcal{D}(A) \) endowed with the inner-product

\[
\langle x, y \rangle_{A, m} = \langle Ax, Ay \rangle + m^2 \langle x, y \rangle, \quad x, y \in \mathcal{D}(A).
\]

Since \( A \) is a closed linear operator, \( \mathcal{H}_{A, m} = (\mathcal{D}(A), \langle \cdot, \cdot \rangle_{A, m}) \) is a Hilbert space. Further, by Kato’s second representation theorem [46, Theorem 2.23, Chapter VI],

\[
\mathcal{D}(A) = \mathcal{D}((A^* A + m^2 I)^{1/2}), \langle x, y \rangle_{A, m} = \langle (A^* A + m^2 I)^{1/2} x, (A^* A + m^2 I)^{1/2} y \rangle, \quad x, y \in \mathcal{D}(A). \tag{4.2.9}
\]

By Proposition 4.7, \( \mathcal{D}(A) \subseteq \mathcal{D}(B) \). Moreover, if \( h \in \mathcal{D}(A) \), then by (4.2.9), we have

\[
\|h\|_{A, m}^2 = \sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} (|\lambda_{uv}|^2 + m^2)|h(v)|^2 < \infty,
\]

and hence by (4.1.3) and (4.2.7),

\[
\left| \sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} h(v)g(v) \right|^2 \leq \|h\|_{A, m}^2 G_m \leq G_1 \|h\|_{A, m}^2. \tag{4.2.10}
\]
Since \(C\) is a bounded linear operator, \(\mathcal{D}(A) \subseteq \mathcal{D}(B + C)\). We claim that for any (arbitrarily small) \(a > 0\), there exists (large enough) \(b\) such that
\[
\|(B + C)h\|^2 \leq a\|Ah\|^2 + b\|h\|^2, \quad h \in \mathcal{D}(A).
\] (4.2.11)

To see the claim, let \(h = (h_1, h_2) \in \mathcal{D}(A)\). By repeated applications of \(|\alpha + \beta|^2 \leq 2(|\alpha|^2 + |\beta|^2), \alpha, \beta \in \mathbb{C}\), we obtain
\[
\|(B + C)h\|^2 \leq 2\|f \odot g\|_{m}^2 + 2\|C\|^2\|h_2\|^2 \quad (4.2.10)
\]
\[
\leq 2\|f\|^2\|h\|^2_{\Lambda} + 2\|C\|^2\|h_2\|^2
\]
\[
\leq 2\|f\|^2\|Ah\|^2_{\Lambda} + (2m^2\|f\|^2\|G_m + 2\|C\|^2\|h\|^2.
\]

However, an application of Lebesgue dominated convergence theorem together with (4.2.7) shows that \(G_m \to 0\) as \(m \to \infty\). This completes the proof of the claim. Another application of the Kato-Rellich Theorem shows that \(W_{f,g}\) is closed.

To prove that \(D_V\) forms a core for \(W_{f,g}\), it suffices to check that \(W_{f,g} \subseteq \overline{W_{f,g}|_{\mathcal{D}(f \odot g)}}\). To see that, let \((h, k) \in \mathcal{D}(W_{f,g})\). By (4.2.2), \(h \in \mathcal{D}(D_{\lambda_n}^{(b)})\) and \(k \in \ell^2(V) \odot \mathcal{H}_\mu^{(b)}\).

Since \(\mathcal{D}\supp \mathcal{H}_\mu^{(b)}\) is a core for \(D_{\lambda_n}^{(b)}\), there exists a sequence \(\{h_n\}_{n \in \mathbb{N}} \subseteq \mathcal{D}\supp \mathcal{H}_\mu^{(b)}\) such that \(h_n \to h\) and \(D_{\lambda_n}^{(b)}h_n \to D_{\lambda_n}^{(b)}h\) as \(n \to \infty\). Let \(\{k_n\}_{n \in \mathbb{N}}\) be any sequence in \(\mathcal{D}\supp \mathcal{H}_\mu^{(b)}\) converging to \(k\). Note that by Proposition 4.7, \(h \in \mathcal{D}(f \odot g)\). Further, since \(g_{\lambda_n, \mu} \in \mathcal{H}_\mu^{(b)}\) and \((D_{\lambda_n}^{(b)} - \mu)h_n \to (D_{\lambda_n}^{(b)} - \mu)h\) as \(n \to \infty\), by the Cauchy-Schwarz inequality, \(f \odot g(h_n) \to (f \odot g)(h)\) as \(n \to \infty\). It is now easy to see that

\[(h_n, k_n) \to (h, k), \quad W_{f,g}(h_n, k_n) \to W_{f,g}(h, k)\] as \(n \to \infty\).

Thus \((h, k) \in \mathcal{D}(\overline{W_{f,g}|_{\mathcal{D}(f \odot g)}})\) and \(W_{f,g}(h, k) = \overline{W_{f,g}|_{\mathcal{D}(f \odot g)}}(h, k)\), as desired. \(\square\)

The following is a consequence of the proof of Theorem 4.11.

**Corollary 4.12** Let \(\mathcal{T} = (V, E)\) be a rooted directed tree with root root and let \(\mathcal{T}_\infty = (V_\infty, E_\infty)\) be the extended directed tree associated with \(\mathcal{T}\). For \(u, b \in V\), consider the weight system \(\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}\) of complex numbers and let \(W_{f,g}\) be the rank one extension of the weighted join operator \(W_{\lambda_n}^{(b)}\) on \(\mathcal{T}\), where \(f \in \ell^2(V) \odot \mathcal{H}_\mu^{(b)}\) is non-zero and \(g : \supp \mathcal{H}_\mu^{(b)} \to \mathbb{C}\) is given. Suppose that \(W_{f,g}\) satisfies the compatibility condition II. Then \(W_{f,g}\) decomposes as \(A + B + C\), where \(A, B, C\) are densely defined operators given by (4.2.8) such that \(B + C\) is A-bounded with A-bound equal to 0.

### 4.3 Adjoint and Gelfand-Triplets

In this section, we try to unravel the structure of the Hilbert space adjoint of the rank one extension \(W_{f,g}\) of weighted join operators. In particular, we discuss the question of determining the action of the Hilbert space adjoint of \(W_{f,g}\). For an interesting
discussion on the relationship between the Hilbert space adjoint and the formal adjoint of an unbounded operator matrix, the reader is referred to [53]. Unfortunately, the situation in our context suggests that there is no obvious way in which one can identify the Hilbert space adjoint of \( W_{f,g} \) with its formal adjoint (that is, the transpose of the matrix formed after taking entry-wise adjoint) unless \( g \in \mathcal{H}_u^{(b)} \). We conclude this section with a brief discussion on the role of a Gelfand-triplet naturally associated with \( W_{f,g} \) in the realization of \( W_{f,g}^* \).

We begin with the following proposition which shows that the adjoint of \( W_{f,g} \) coincides with the adjoint of the associated weighted join operator \( W_{\lambda_u}^{(b)} \) on a possibly non-dense subspace of \( \mathcal{D}(W_{f,g}^*) \).

**Proposition 4.13** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root \( \text{root} \) and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( u, b \in V \), consider the weight system \( \lambda_u = \{\lambda_{uv}\}_{v \in V_\infty} \) of complex numbers and let \( W_{f,g} \) be the rank one extension of the weighted join operator \( W_{\lambda_u}^{(b)} \) on \( \mathcal{T} \), where \( f \in \ell^2(V) \cap \mathcal{H}_u^{(b)} \) is non-zero and \( g : \text{supp} \mathcal{H}_u^{(b)} \to \mathbb{C} \) is given. Then \( W_{f,g}^* \) is a closed operator. Moreover,

\[
\mathcal{D} := \{k = (k_1, k_2) \in \mathcal{D}(D_{\lambda_u}^{(b)}) \oplus (\ell^2(V) \oplus \mathcal{H}_u^{(b)}) : \langle k_2, f \rangle = 0\} \subseteq \mathcal{D}(W_{f,g}^*),
\]

\[
W_{f,g}^* k = (D_{\lambda_u}^{(b)})^* k_1 + (N_{\lambda_u}^{(b)})^* k_2, \quad k = (k_1, k_2) \in \mathcal{D}.
\]

Further, if \( W_{f,g} \) satisfies a compatibility condition, then \( W_{f,g}^* \) is densely defined.

**Proof** Since \( W_{f,g} \) is densely defined in \( \ell^2(V) \), the Hilbert space adjoint \( W_{f,g}^* \) is a closed linear operator [64]. Let \( k = (k_1, k_2) \in \mathcal{D} \). Then, for any \( h = (h_1, h_2) \in \mathcal{D}(W_{f,g}) \),

\[
\langle W_{f,g} h, k \rangle = \langle D_{\lambda_u}^{(b)} h_1, k_1 \rangle + \langle (f \otimes g) h_1, k_2 \rangle + \langle N_{\lambda_u}^{(b)} h_2, k_2 \rangle
\]

\[
= \langle h_1, (D_{\lambda_u}^{(b)})^* k_1 \rangle + \langle h_2, (N_{\lambda_u}^{(b)})^* k_2 \rangle,
\]

where we used the fact that \( \mathcal{D}((D_{\lambda_u}^{(b)})^*) = \mathcal{D}(D_{\lambda_u}^{(b)}) \). It follows that \( k \in \mathcal{D}(W_{f,g}^*) \) and

\[
W_{f,g}^* k = (D_{\lambda_u}^{(b)})^* k_1 + (N_{\lambda_u}^{(b)})^* k_2.
\]

Finally, if \( W_{f,g} \) satisfies a compatibility condition, then by Theorem 4.11, \( W_{f,g} \) is closed, and hence, by [64, Theorem 1.8(i)], \( W_{f,g}^* \) is densely defined. This completes the proof. \( \square \)

**Remark 4.14** Note that for any \( k \in \ell^2(V) \cap \mathcal{H}_u^{(b)}, (0, k) \notin \mathcal{D}(W_{f,g}^*) \). Otherwise,

\[
\phi(h) = \langle W_{f,g} h, (0, k) \rangle = \langle (f \otimes g) h_1, k \rangle + \langle N_{\lambda_u}^{(b)} h_2, k \rangle, \quad h = (h_1, h_2) \in \mathcal{D}(W_{f,g})
\]

extends as a bounded linear functional, which is not possible since \( g \notin \mathcal{H}_u^{(b)} \) (see Lemma 1.3). In particular, \( (0, f) \notin \mathcal{D}(W_{f,g}^*) \).
Note that the closability of $W_{f,g}$ is equivalent to the density of the domain $\mathcal{D}(W^*_{f,g})$ of $W^*_{f,g}$ (see [64, Theorem 1.8]). In particular, it would be interesting to obtain conditions ensuring the density of $\mathcal{D}(W^*_{f,g})$ (with or without a compatibility condition). In view of the decomposition $W_{f,g} = A + B + C$ as given in (4.2.8), it is tempting to ask whether $W^*_{f,g}$ can be decomposed as $A^* + B^* + C^*$. It may be concluded from [14, Proposition] that if $W_{f,g}$ is Fredholm such that $B$ is $A$-compact and $B^*$ is $A^*$-compact, then $W^*_{f,g} = A^* + B^* + C^*$ (see [54, Theorem 2.2] for a variant). We will see in the proof of Theorem 4.15(iv) that under the assumption of compatibility condition I, $A$-compactness of $B$ can be ensured (Recall that $B$ is $A$-compact if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $B$ maps $\{h \in D(A) : \|h\| + \|Ah\| \leq 1\}$ into a pre-compact set).

Another natural problem which arises in finding $W^*_{f,g}$ is whether it is possible to have a matrix decomposition of $W^*_{f,g}$ similar to the one we have it for $W_{f,g}$. One possible candidate for $W^*_{f,g}$ is its formal adjoint, that is, the transpose of the operator matrix $W^\times_{f,g}$ obtained by taking the Hilbert space adjoint of each entry of $W_{f,g}$. A direct application of [53, Theorem 6.1] shows the following:

(i) $W^\times_{f,g}$ is a closable operator such that $W^\times_{f,g} \subseteq W^*_{f,g}$.

(ii) If $W^\times_{f,g}$ is densely defined, then $W_{f,g}$ is closable.

It is evident that there is no natural way to recover $W^*_{f,g}$ from $W^\times_{f,g}$. One such way has been shown in [53, Proposition 6.3], which provides a sufficient condition for the equality of the Hilbert space adjoint and the formal adjoint of an unbounded operator matrix. Unfortunately, this result is not applicable to $W_{f,g}$ unless all its entries are closable operators. Recall that $f \otimes g$ is not even closable in case $g \notin \mathcal{H}^{(b)}_\mu$ (see Lemma 1.3). That’s why to understand the action of $W^\times_{f,g}$, we need to replace $\mathcal{H}^{(b)}_\mu$ by a larger Hilbert space. We will see below that the notion of Hilbert rigging turns out to be handy in this context.

Consider the inner-product space $\mathcal{D}(D^{(b)}_{k_u})$ endowed with the inner-product

$$\langle x, y \rangle_\circ := \langle D^{(b)}_{k_u} x, D^{(b)}_{k_u} y \rangle + \langle x, y \rangle, \quad x, y \in \mathcal{D}(D^{(b)}_{k_u}).$$

Note that $\mathcal{H}_\circ := \mathcal{D}(D^{(b)}_{k_u})$ endowed with the inner-product $\langle \cdot, \cdot \rangle_\circ$ is a Hilbert space. Clearly, the inclusion map $i : \mathcal{H}_\circ \hookrightarrow \mathcal{H}^{(b)}_\mu$ is contractive. Consider further the topological dual $\mathcal{H}^*_\circ$ of $\mathcal{H}_\circ$, which we denote by $\mathcal{H}^\circ$. We claim that any element $h \in \mathcal{H}^{(b)}_\mu$ can be realized as a bounded conjugate-linear functional in $\mathcal{H}^\circ$. To see this, consider the mapping $j : \mathcal{H}^{(b)}_\mu \to \mathcal{H}^\circ$ given by $j(h) = \phi_h$, where

$$\phi_h(k) = \sum_{v \in \text{supp} \mathcal{H}^{(b)}_\mu} k(v) h(v), \quad k \in \mathcal{H}_\circ.$$
The contractivity of \( j \) follows from the Cauchy–Schwarz inequality and
\[
\| \phi_h \| = \sup_{\| k \|_\circ = 1} \left| \sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} \overline{k(v)} h(v) \right| = \left( \sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} \frac{|h(v)|^2}{1 + | \lambda_{uv} |^2} \right)^{1/2} = \|(D_{\lambda_u}^{(b)})^* D_{\lambda_u}^{(b)} + I)^{-1/2} h\|, \tag{4.3.1}
\]
and hence the claim stands verified. This also shows that \( \mathcal{H}^\circ \) can be identified with the completion of \( \mathcal{H}_u^{(b)} \) endowed with the inner-product
\[
\langle x, y \rangle^\circ = \langle (D_{\lambda_u}^{(b)})^* D_{\lambda_u}^{(b)} + I)^{-1} x, y \rangle, \quad x, y \in \mathcal{H}_u^{(b)}.
\]
Thus we have the following chain of Hilbert spaces:
\[
\mathcal{H}_o \subsetneq \mathcal{H}_u^{(b)} \subsetneq \mathcal{H}^\circ,
\]
where \( \mathcal{H}_o \) is dense in \( \mathcal{H}_u^{(b)} \) and \( \mathcal{H}_u^{(b)} \) is dense in \( \mathcal{H}^\circ \). One may refer to this chain of Hilbert spaces as the Hilbert rigging of \( \mathcal{H}_u^{(b)} \) by \( \mathcal{H}_o \) and \( \mathcal{H}^\circ \). The triplet \( (\mathcal{H}_o, \mathcal{H}_u^{(b)}, \mathcal{H}^\circ) \) is known as the Gelfand-triplet (refer to [16, Chapter 14] for an abstract theory of rigged spaces). If \( \{ | \lambda_{uv} |^2 + 1 \}^{-1/2} : v \in \text{supp } \mathcal{H}_u^{(b)} \} \) is square-summable, then by (4.3.1), the above Hilbert rigging is quasi-nuclear in the sense that the inclusion \( j : \mathcal{H}_u^{(b)} \to \mathcal{H}^\circ \) is Hilbert-Schmidt (see [16, Pg 121]).

Suppose that \( W_{f,g} \) satisfies the compatibility condition II. Define \( \phi_g : \mathcal{H}_o \to \mathbb{C} \) by
\[
\phi_g(k) = \sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} \overline{k(v)} g(v), \quad k \in \mathcal{H}_o.
\]
It may be concluded from (4.1.3) that \( \phi_g \in \mathcal{H}^\circ \). This allows us to introduce the bounded linear transformation \( B : \mathcal{H}_o \to \ell^2(V) \otimes \mathcal{H}_u^{(b)} \) by setting
\[
Bk = \phi_g(k) f, \quad k \in \mathcal{H}_o,
\]
Note that for any \( l \in \ell^2(V) \otimes \mathcal{H}_u^{(b)} \) and \( k \in \mathcal{H}_o \),
\[
(B^* l)(k) = \langle l, Bk \rangle = \phi_g(k) \langle l, f \rangle = (\phi_g \otimes f)(l)(k).
\]
Thus \( B^* \) can be identified with \( g \otimes f \). In particular, the Hilbert space adjoint of \( W_{f,g} \) can be identified with the formal adjoint of \( W_{f,g} \) after replacing the Hilbert space \( \mathcal{H}_u^{(b)} \) by the larger Hilbert space \( \mathcal{H}^\circ \).
4.4 Spectral Analysis

We now turn our attention to the spectral properties of rank one extensions of weighted join operators. The main result of this section provides a complete spectral picture for rank one extensions $W_{f,g}$ of weighted join operators (cf. [70, Theorem 1], [37, Theorem 2.3], [45, Theorem 2.3], [56, Corollary 2.8], [7, Theorem 2.2]). It turns out that $W_{f,g}$ has non-empty resolvent set if and only if it satisfies the compatibility condition I. Among various applications, we characterize rank one extensions of weighted join operators on leafless directed trees which admit compact resolvent.

**Theorem 4.15** (Spectral picture) Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root root and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u, b \in V$, consider the weight system $\lambda_u = \{\lambda_{uv} v \in V_\infty\}$ of complex numbers and let $W_{f,g}$ be the rank one extension of the weighted join operator $W_{b,\lambda_u}$ on $\mathcal{T}$, where $f \in \ell^2(V) \ominus \mathcal{H}_b$ is non-zero and $g : \text{supp} \mathcal{H}_b \rightarrow \mathbb{C}$ is given (see (4.0.1)). Then, we have the following statements:

(i) The point spectrum $\sigma_p(W_{f,g})$ of $W_{f,g}$ is given by

$$\sigma_p(W_{f,g}) = \begin{cases} \{\lambda_{uv} : v \in V\} & \text{if } b = u, \\ \{\lambda_{uv} : v \in \text{Asc}(u) \cup \text{Des}_b(u) \cup \{0\}\} & \text{if } b \in \text{Des}_u(u), \\ \{\lambda_{uv} : v \in \text{Des}(u) \cup \{0\}\} & \text{otherwise}. \end{cases}$$

(ii) The spectrum $\sigma(W_{f,g})$ of $W_{f,g}$ is given by

$$\sigma(W_{f,g}) = \begin{cases} \sigma_p(W_{f,g}) & \text{if } W_{f,g} \text{ satisfies the compatibility condition I}, \\ \mathbb{C} & \text{otherwise}. \end{cases}$$

If, in addition, $W_{f,g}$ satisfies the compatibility condition I, then we have the following:

(iii) For every $\mu \in \mathbb{C} \setminus \sigma_p(W_{f,g})$, the resolvent of $W_{f,g}$ at $\mu$ is given by

$$(W_{f,g} - \mu)^{-1} = \begin{bmatrix} (D_{\lambda_u}^{(b)} - \mu)^{-1} & 0 \\ - (N_{\lambda_u}^{(b)} - \mu)^{-1} L_{\lambda_u,\mu} (N_{\lambda_u}^{(b)} - \mu)^{-1} \end{bmatrix}, \quad (4.4.1)$$

where the linear transformation $L_{\lambda_u,\mu} := (f \otimes g)(D_{\lambda_u}^{(b)} - \mu)^{-1}$ defines a Hilbert-Schmidt integral operator from $\mathcal{H}_u^{(b)}$ into $\ell^2(V) \ominus \mathcal{H}_u^{(b)}$.

(iv) The essential spectrum $\sigma_e(W_{f,g})$ of $W_{f,g}$ is given by

$$\sigma_e(W_{f,g}) = \begin{cases} \sigma_e(D_{\lambda_u}^{(b)}) & \text{if } \dim (\ell^2(V) \ominus \mathcal{H}_u^{(b)}) < \infty, \\ \sigma_e(D_{\lambda_u}^{(b)}) \cup \{0\} & \text{otherwise}. \end{cases}$$

Moreover, ind$_{W_{f,g}} = 0$ on $\mathbb{C} \setminus \sigma_e(W_{f,g})$. 

Remark 4.16 By [48, Theorem 7.1], for any diagonal operator $D$ with simple point spectrum and perfect spectrum, there exists a bounded rank one perturbation $h \otimes k$ of an arbitrarily small positive norm such that $D + h \otimes k$ has no point spectrum. This situation does not appear in the context of rank one extensions of weighted join operators, where the point spectrum is always non-empty.

Proof Let $\mu$ be a complex number and $(h, k) \in \mathcal{D}(W_{f,g})$ be a non-zero vector such that $W_{f,g}(h, k) = \mu(h, k).$ By (4.0.1), $h \in \mathcal{D}(D^{(b)}_{\lambda u}) \cap \mathcal{D}(f \otimes g),$ $k \in \ell^2(V) \ominus \mathcal{H}^{(b)}_u$ and

$$D^{(b)}_{\lambda u} h = \mu h, \quad \left( \sum_{v \in \text{supp} \mathcal{H}^{(b)}_u} h(v)g(v) \right) f + \langle k, e_{\lambda u, u} \rangle e_u = \mu k. \quad (4.4.2)$$

Case I. $h = 0$: In this case, $(k, e_{\lambda u, u}) e_u = \mu k.$ Accordingly, any one of the following possibilities occur:

1. $e_u$ (considered as the vector $(0, e_u)$) is an eigenvector of $W_{f,g}$ corresponding to the eigenvalue $\mu = \lambda_{uu}.$
2. $k$ is an eigenvector of $W_{f,g}$ corresponding to the eigenvalue $\mu = 0$ provided $b \neq u,$ where $k \in \left( \ell^2(V) \ominus \mathcal{H}^{(b)}_u \right) \ominus [e_{\lambda u, u}].$

Here, in the second assertion, we used the facts that $\dim \ell^2(A_{uu}) \geq 2$ (since $b \neq u$) and

$$\dim \left( \ell^2(V) \ominus \mathcal{H}^{(b)}_u \right) \geq \dim \ell^2(A_{uu}) \quad (see \ (3.2.15)).$$

Case II. $h \neq 0$: In this case, $\mu \in \sigma_p(D^{(b)}_{\lambda u}),$ and hence $\mu = \lambda_{uw}$ for some $w \in \text{supp} \mathcal{H}^{(b)}_u$ and

$$h \in \mathcal{E}_{D^{(b)}_{\lambda u}}(\mu) = \ell^2(W_w),$$

where $W_w$ is as given in (3.3.1). It follows from (4.4.2) that

$$\left( \sum_{v \in W_w} h(v)g(v) \right) f + \langle k, e_{\lambda u, u} \rangle e_u = \lambda_{uw} k. \quad (4.4.3)$$

Taking inner-product with $e_{\lambda u, u}$ on both sides, we get

$$\left( \sum_{v \in W_w} h(v)g(v) \right) \langle f, e_{\lambda u, u} \rangle = (\lambda_{uw} - \lambda_{uu}) \langle k, e_{\lambda u, u} \rangle. \quad (4.4.4)$$

Accordingly, any one of the following possibilities occur:
(1) \( \lambda_{uw} \) is a non-zero number equal to \( \lambda_{uu} \): In this case,

\[
\langle f, e_{\lambda_{uu}} \rangle = 0 \text{ or } \sum_{v \in W_w} h(v)g(v) = 0.
\]

Thus \( k \) belongs either to \([f]\) or to \([e_u]\).

(2) \( \lambda_{uw} \) is a non-zero number not equal to \( \lambda_{uu} \): By (4.4.3) and (4.4.4), \( k \) takes the form

\[
k = \frac{\sum_{v \in W_w} h(v)g(v)}{\lambda_{uw}} \left( f + \frac{\langle f, e_{\lambda_{uu}} \rangle}{\lambda_{uw} - \lambda_{uu}} e_u \right).
\]

In this case, \( k \) belongs to the span of \( f + \frac{\langle f, e_{\lambda_{uu}} \rangle}{\lambda_{uw} - \lambda_{uu}} e_u \).

(3) \( \lambda_{uw} = 0 \): In this case, any non-zero vector \((h, k)\) with \( h \in \ell^2(W_w), k \in \ell^2(V) \cap \mathcal{H}_u(b) \) satisfying the following identity will be an eigenvector of \( W_{f,g} \) corresponding to the eigenvalue 0:

\[
\left( \sum_{v \in W_w} h(v)g(v) \right) f + \langle k, e_{\lambda_{uu}} \rangle e_u = 0.
\]

In particular, the cases above show that

\[
\sigma_p(W_{f,g}) = \begin{cases} 
\sigma_p(D_{\lambda_u}^{(b)}) \cup \{\lambda_{uu}\} & \text{if } b = u, \\
\sigma_p(D_{\lambda_u}^{(b)}) \cup \{\lambda_{uu}, 0\} & \text{otherwise.}
\end{cases}
\]

The conclusion in (i) is now clear from the fact that \( \sigma_p(D_{\lambda_u}^{(b)}) = \{\lambda_{uv} : v \in \text{supp } \mathcal{H}_u^{(b)}\} \), (3.2.11) and (3.2.16).

To see (ii), let \( \mu \in \mathbb{C}\setminus\sigma_p(W_{f,g}) \), that is, \( \mu \) is a non-zero number such that \( \mu \neq \lambda_{uu} \) and

\[
\text{dist}(\lambda_u, \mu) = \inf_{v \in \text{supp } \mathcal{H}_u^{(b)}} |\lambda_{uv} - \mu| > 0.
\]

By (4.0.1), \((k_1, k_2) \in \text{ran}(W_{f,g} - \mu)\) if and only if there exists \((h_1, h_2) \in \mathcal{D}(W_{f,g})\) such that

\[
(D_{\lambda_u}^{(b)} - \mu)h_1 = k_1, \quad \left( \sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} h_1(v)g(v) \right) f + \langle h_2, e_{\lambda_{uu}} \rangle e_u - \mu h_2 = k_2.
\]

(4.4.7)
We claim that
\[ W_{f,g} - \mu \text{ is surjective if and only if } \mathcal{D}(D^{(b)}_{\lambda u}) \subseteq \mathcal{D}(f \circ g). \quad (4.4.8) \]

To see the claim, suppose that \( W_{f,g} - \mu \) is surjective and let \( h' \in \mathcal{D}(D^{(b)}_{\lambda u}) \). Letting \( k_1 = (D^{(b)}_{\lambda u} - \mu)h' \) and \( k_2 = 0 \), by surjectivity of \( W_{f,g} - \mu \), we get \( (h_1, h_2) \in \mathcal{D}(W_{f,g}) \) such that \( (4.4.7) \) holds. However, since \( D^{(b)}_{\lambda u} - \mu \) is injective, \( h_1' = h_1 \), and hence \( h_1' \in \mathcal{D}(f \circ g) \). To see the reverse implication, assume that \( \mathcal{D}(D^{(b)}_{\lambda u}) \subseteq \mathcal{D}(f \circ g) \), and let \( h_1 \in \mathcal{H}_u^{(b)} \) and \( h_2 \in \ell^2(V) \ominus \mathcal{H}_u^{(b)} \). By \( (4.4.6) \), \( D^{(b)}_{\lambda u} - \mu \) is invertible, and hence there exists \( h_1 \in \mathcal{D}(D^{(b)}_{\lambda u}) \) such that \( (D^{(b)}_{\lambda u} - \mu)h_1 = k_1 \). By assumption, \( h_1 \in \mathcal{D}(f \circ g) \). Since \( \mu \neq \lambda_{uu} \), the following equation can be uniquely solved for \( \langle h_2, e_{\lambda_{uu},Au} \rangle \):

\[ \langle h_2, e_{\lambda_{uu},Au} \rangle (\lambda_{uu} - \mu) = \langle k_2, e_{\lambda_{uu},Au} \rangle - \sum_{\upsilon \in \text{supp } \mathcal{H}_u^{(b)}} h_1(\upsilon)g(\upsilon) \langle f, e_{\lambda_{uu},Au} \rangle. \]

Since \( \mu \neq 0 \), substituting the above value of \( \langle h_2, e_{\lambda_{uu},Au} \rangle \) in \( (4.4.7) \) determines \( h_2 \in \ell^2(V) \ominus \mathcal{H}_u^{(b)} \) uniquely. This completes the verification of \( (4.4.8) \). The first part in (ii) now follows from Proposition 4.7.

To see the remaining part in (ii), suppose that \( W_{f,g} \) does not satisfy the compatibility condition I. Let \( \Gamma_{\lambda u} \) be as given in \( (4.1.1) \). Thus there are two possibilities:

**Case 1.** \( \Gamma_{\lambda u} = \emptyset \):

In this case, \( \sigma_p(D^{(b)}_{\lambda u}) \) is necessarily dense in \( \mathbb{C} \), and hence by (i), \( \sigma_p(W_{f,g}) \) is also dense in \( \mathbb{C} \). If possible, then assume that \( \sigma(W_{f,g}) \) is not equal to \( \mathbb{C} \). Then, by [22, Lemma 1.17], \( W_{f,g} \) is closed. However, the spectrum of a closed operator is always closed (see [64, Proposition 2.6]). This is not possible since \( \sigma_p(W_{f,g}) \) is a dense and proper subset of \( \mathbb{C} \), and hence we must have \( \sigma(W_{f,g}) = \mathbb{C} \).

**Case 2.** \( \Gamma_{\lambda u} \neq \emptyset \):

If possible, then suppose that \( \sigma(W_{f,g}) \subseteq \mathbb{C} \). Thus there exists \( \mu \in \mathbb{C} \setminus \sigma(W_{f,g}) \) and a linear operator \( R(\mu) \in \mathcal{B}(\ell^2(V)) \) such that

\[ (W_{f,g} - \mu)R(\mu)h = h, \quad h \in \ell^2(V). \quad (4.4.9) \]

Consider the following decomposition of \( R(\mu) \):

\[ R(\mu) = \begin{bmatrix} A(\mu) & B(\mu) \\ C(\mu) & D(\mu) \end{bmatrix} \text{ on } \ell^2(V) = \mathcal{H}_u^{(b)} \oplus (\ell^2(V) \ominus \mathcal{H}_u^{(b)}). \]

Note that for any \( k \in \ell^2(V) \ominus \mathcal{H}_u^{(b)} \),

\[ \begin{bmatrix} 0 \\ k \end{bmatrix} \overset{(4.4.9)}{=} (W_{f,g} - \mu)R(\mu) \begin{bmatrix} 0 \\ k \end{bmatrix} \]
\[
= \begin{pmatrix}
(D_{\mathcal{H}}^{(b)} - \mu)B(\mu)k \\
f \otimes g(B(\mu)k) + (N_{\mathcal{H}}^{(b)} - \mu)(D(\mu)k)
\end{pmatrix},
\]

which yields \((D_{\mathcal{H}}^{(b)} - \mu)B(\mu)k = 0\). However, \(\mu \notin \sigma(D_{\mathcal{H}}^{(b)})\), and consequently, \(B(\mu) = 0\). Also, since for any \(h \in \mathcal{H}_{\mathcal{H}}^{(b)}\),

\[
R(\mu) \begin{bmatrix} h \\ 0 \end{bmatrix} \in \mathcal{D}(W_{f,g} - \mu) = \mathcal{D}(W_{f,g}),
\]

by the definition of the domain of \(W_{f,g}\), we must have

\[
A(\mu)h \in \mathcal{D}(f \otimes g), \quad (D^{(b)}_{\mathcal{H}} - \mu)A(\mu) = I \quad (4.4.10)
\]

(see \((4.4.9)\)). It follows that \(A(\mu) = (D^{(b)}_{\mathcal{H}} - \mu)^{-1}\) and hence any arbitrary vector in \(\mathcal{D}(D^{(b)}_{\mathcal{H}})\) is of the form \(A(\mu)h\) for some \(h \in \mathcal{H}_{\mathcal{H}}^{(b)}\). This together with \((4.4.10)\) yields the inclusion \(\mathcal{D}(D_{\mathcal{H}}^{(b)}) \subseteq \mathcal{D}(f \otimes g)\). An application of Proposition 4.7, however, shows that \(W_{f,g}\) satisfies the compatibility condition I, which is contrary to our assumption. This completes the proof of (ii).

To see (iii) and (iv), assume that \(W_{f,g}\) satisfies the compatibility condition I. Let \(\mu \in \mathbb{C} \setminus \sigma_p(W_{f,g})\). By Proposition 4.7, \(L_{\mathcal{H},\mathcal{H}}\) is bounded. A routine verification shows that \(L_{\mathcal{H},\mathcal{H}}\) is an integral operator with kernel \(K_{\mathcal{H},\mathcal{H}}\) given by

\[
K_{\mathcal{H},\mathcal{H}}(w, v) := \frac{g(v)f(w)}{\lambda_{uv} - \mu}, \quad w \in V \setminus \text{supp } \mathcal{H}_{\mathcal{H}}^{(b)}, \quad v \in \text{supp } \mathcal{H}_{\mathcal{H}}^{(b)}.
\]

By the compatibility condition I and the assumption that \(f \in \ell^2(V) \subseteq \mathcal{H}_{\mathcal{H}}^{(b)}\), \(K_{\mathcal{H},\mathcal{H}}\) belongs to \(\ell^2((V \setminus \text{supp } \mathcal{H}_{\mathcal{H}}^{(b)}) \times \text{supp } \mathcal{H}_{\mathcal{H}}^{(b)})\). By [67, Theorem 3.8.5], \(L_{\mathcal{H},\mathcal{H}}\) is a Hilbert-Schmidt operator. We leave it to the reader to verify that the expression given by \((4.4.1)\) defines the resolvent of \(W_{f,g}\) at \(\mu\).

To see (iv), let \(A, B, C\) be as given by \((4.2.8)\) and note that \(A + B + C = W_{f,g}\). Since \(A = D_{\mathcal{H}}^{(b)} \oplus 0\) on \(\mathcal{H}_{\mathcal{H}}^{(b)} \oplus (\ell^2(V) \subseteq \mathcal{H}_{\mathcal{H}}^{(b)})\), it suffices to check that \(\sigma_e(W_{f,g}) = \sigma_e(A)\) and \(\text{ind } W_{f,g} = \text{ind } A\). In view of [46, Theorems 5.26 and 5.35, Chapter IV], it is sufficient to verify that \(B + C\) is \(A\)-compact (see also the foot note 1 on [46, Pg 244]). Let \(\{h_n\}\) be a bounded sequence in \(\mathcal{D}(A) \subseteq \mathcal{D}(B)\) such that \(\{Ah_n\}\) is bounded. Since \(C\) is a finite rank operator, it suffices to check that \(\{Bh_n\}\) has a convergent subsequence. By part (iii), \(L_{\mathcal{H},\mathcal{H}}\) is a compact operator, and hence so is \((A - \mu)^{-1}\). However, \((A - \mu)h_n\) is bounded, and hence \(\{Bh_n\}\) admits a convergent subsequence. The remaining part follows from the fact that the index function for a diagonal operator is identically 0. \(\square\)

**Remark 4.17** In this remark, we describe eigenspaces of \(W_{f,g}\) (under the hypotheses of Theorem 4.15). To see that, we need some notations. Given a subspace \(\ell^2(W)\) of
\( \ell^2(V) \) and a rank one operator \( h \otimes k \) from \( \ell^2(V) \) into \( \ell^2(U) \), we introduce the linear transformation \( h \otimes k|_{\ell^2(W)} \) from \( \ell^2(W) \) into \( \ell^2(U) \) as follows:

\[
\mathcal{D}(h \otimes k|_{\ell^2(W)}) = \mathcal{D}(h \otimes k) \cap \ell^2(W), \quad h \otimes k|_{\ell^2(W)}(l) = h \otimes k(l), \quad l \in \mathcal{D}(h \otimes k|_{\ell^2(W)}).
\]

For \( \mu \in \mathbb{C} \), let \( W_\mu \) be given by

\[
W_\mu = \{ w \in \text{supp} \mathcal{H}^{(b)}_u : \lambda_{uw} = \mu \}. \tag{4.4.11}
\]

In case \( \mu = \lambda_{uv} \) for some \( v \in V \), we denote \( W_\mu \) by the simpler notation \( W_v \). Further, we reserve the notation \( \text{graph}(T) \) for the graph of a linear operator \( T \) in \( \mathcal{H} \). If \( \mathcal{E}_{W_{f,g}}(\mu) \) denotes the eigenspace corresponding to the eigenvalue \( \mu \) of \( W_{f,g} \), then we have the following statements:

(a) If \( \lambda_{uv} \neq 0 \) for every \( v \in \text{supp} \mathcal{H}^{(b)}_u \), then

\[
\mathcal{E}_{W_{f,g}}(0) = \{0\} \oplus \ker(N^{(b)}_u).
\]

(b) If \( \lambda_{uv} = 0 \) for some \( v \in \text{supp} \mathcal{H}^{(b)}_u \), then

\[
\mathcal{E}_{W_{f,g}}(0) = \begin{cases} 
\ker(f \otimes \tilde{g}) & \text{if } f \in [e_u], \\
\ker(f \otimes g|_{\ell^2(W_v)}) \oplus \ker(N^{(b)}_u) & \text{otherwise},
\end{cases}
\]

where \( \tilde{g} : W_v \cup (V \setminus \text{supp} \mathcal{H}^{(b)}_u) \rightarrow \mathbb{C} \) is given by

\[
\tilde{g}(w) = \begin{cases} 
\bar{f}(w) g(w) & \text{if } w \in W_v, \\
e_{\lambda_{uv}}(w) & \text{otherwise}.
\end{cases}
\]

(c) If \( \mu = \lambda_{uu} \) is non-zero, then

\[
\mathcal{E}_{W_{f,g}}(\mu) = \begin{cases} 
\text{graph}(\tilde{f} \otimes g|_{\ell^2(W_u)}) \oplus [e_u] & \text{if } f \in \ker(N^{(b)}_u), \\
\ker(f \otimes g|_{\ell^2(W_u)}) \oplus [e_u] & \text{otherwise},
\end{cases}
\]

where \( \tilde{f} = f/\lambda_{uu} \).

(d) If \( \mu = \lambda_{uv} \) for some \( v \in \text{supp} \mathcal{H}^{(b)}_u \) and \( \mu \notin \{0, \lambda_{uu}\} \), then

\[
\mathcal{E}_{W_{f,g}}(\mu) = \text{graph}(\tilde{f} \otimes g|_{\ell^2(W_u)}),
\]

where \( \tilde{f} = \frac{1}{\lambda_{uv}} \left( f + \frac{\langle f, e_{\lambda_{uv}} \rangle}{\lambda_{uv} - \lambda_{uu}} e_u \right) \).
To see the above statements, suppose that \((h, k) \in \mathcal{E}_{W_{f,g}}(\mu)\). Since \(b \neq u\), by Case I(2) of the proof of Theorem 4.15,

\[
(\ell^2(V) \ominus \mathcal{K}_u^{(b)}) \ominus [e_{\lambda_u A_u}] = \ker(N_{\lambda_u}^{(b)}) \subseteq \mathcal{E}_{W_{f,g}}(0).
\] (4.4.12)

The desired conclusion in (a) now follows from (4.4.2) and the assumption that \(\lambda_{uv} \neq 0\) for every \(v \in \text{supp} \mathcal{K}_u^{(b)}\). To see (b), assume that \(\lambda_{uv} = 0\) for some \(v \in \text{supp} \mathcal{K}_u^{(b)}\), and suppose that \(f = c e_u\) for some non-zero scalar \(c\). By (4.4.3) and (4.4.12), \((h, k) \in \mathcal{E}_{W_{f,g}}(0)\) if and only if

\[
c \left( \sum_{v \in W_u} h(v) g(v) \right) + \langle k, e_{\lambda_u A_u} \rangle = 0,
\]

which is equivalent to \((h, k) \in \ker(f \otimes \tilde{g})\). This yields the first part in (b). If \(f\) and \(e_u\) are linearly independent, then \(\sum_{v \in W_u} h(v) g(v) = 0\) and \(\langle k, e_{\lambda_u A_u} \rangle = 0\). Thus the other part in (b) follows at once from (4.4.12). To see (c), suppose that \(f = \lambda_{uu}\) is non-zero. By Case I(1) and (4.4.3), \(k = \alpha f + \beta e_u\) for some \(\alpha, \beta \in \mathbb{C}\). Combining this with (4.4.3) yields

\[
\left( \sum_{v \in W_u} h(v) g(v) - \alpha \lambda_{uu} \right) f + \alpha \langle f, e_{\lambda_u A_u} \rangle e_u = 0
\]

If \(f \in \ker(N_{\lambda_u}^{(b)})\), then \(\langle f, e_{\lambda_u A_u} \rangle = 0\), and the above equation determines \(\alpha\) uniquely, whereas \(\beta\) can be chosen arbitrarily to get the conclusion in the first part of (c). If \(f \notin \ker(N_{\lambda_u}^{(b)})\), then by (4.4.4), \(\sum_{v \in W_u} h(v) g(v) = 0\), that is, \(h \in \ker(f \otimes g|_{\ell^2(W_u)})\).

However, in this case, \(k \in [e_u]\), which yields the remaining part of (c). To see (d), assume that \(\mu = \lambda_{uu} \notin \{0, \lambda_{uu}\}\). Once again, by (4.4.5), \(k = f \otimes g(h)\). This completes the verification of (d).

The following sheds more light into the spectral picture of rank one extensions of weighted join operators.

**Corollary 4.18** Let \(T = (V, E)\) be a rooted directed tree with root \(r\) and let \(T_\infty = (V_\infty, E_\infty)\) be the extended directed tree associated with \(T\). For \(u, b \in V\), consider the weight system \(\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}\) of complex numbers and let \(W_{f,g}\) be the rank one extension of the weighted join operator \(W_{\lambda_u}^{(b)}\) on \(T\), where \(f \in \ell^2(V) \ominus \mathcal{K}_u^{(b)}\) is non-zero and \(g : \text{supp} \mathcal{K}_u^{(b)} \rightarrow \mathbb{C}\) is given. Then \(\sigma(W_{f,g})\) is a proper closed subset of \(\mathbb{C}\) if and only if \(W_{f,g}\) satisfies the compatibility condition I. Further, we have the following:

(a) In case \(W_{f,g}\) satisfies the compatibility condition I, \(W_{f,g}\) defines a closed linear operator such that the following hold:

(a1) \(\sigma(W_{f,g}) = \tilde{\sigma}_p(W_{f,g})\).
(a2) \( \pi(W_{f,g}) = \mathbb{C}\setminus\sigma_p(W_{f,g}) \).

(b) In case \( W_{f,g} \) does not satisfy the compatibility condition I, the following hold:

(b1) \( \sigma(W_{f,g}) = \mathbb{C} \).

(b2) Either \( W_{f,g} \) is not closed or \( \pi(W_{f,g}) = \emptyset \).

**Proof** Suppose that \( \sigma(W_{f,g}) \) is a proper closed subset of \( \mathbb{C} \). Thus there exists \( \mu \in \mathbb{C}\setminus\sigma_p(W_{f,g}) \) such that \( W_{f,g} - \mu \) is surjective. By (4.4.8), we obtain the domain inclusion \( \mathcal{D}(D^{(b)}_{\lambda u}) \subseteq \mathcal{D}(f \otimes g) \), and hence by Proposition 4.7, \( W_{f,g} \) satisfies the compatibility condition I. Conversely, if \( W_{f,g} \) satisfies the compatibility condition I, then \( \mathcal{I}_{\lambda u} \) is non-empty (see (4.1.1)). It may now be concluded from (i) and (ii) of Theorem 4.15 that \( \sigma(W_{f,g}) \) is a proper closed subset of \( \mathbb{C} \).

To see (a), assume that \( W_{f,g} \) satisfies the compatibility condition I. Since \( \sigma(W_{f,g}) \) is a proper subset of \( \mathbb{C} \), \( W_{f,g} \) is closed (see [22, Lemma 1.17]). Further, (a1) follows from Theorem 4.15(ii). Since the complement of the regularity domain of a densely defined closed operator is a closed subset of the spectrum that contains the point spectrum (see [64, Proposition 2.1]), the conclusion in (a2) follows.

To see (b), assume that \( W_{f,g} \) does not satisfy the compatibility condition I. Clearly, (b1) follows from the first part of this corollary. To see (b2), assume that \( W_{f,g} \) is closed. By [64, Proposition 2.6],

\[
\{ \mu \in \pi(W_{f,g}) : d_{W_{f,g}}(\mu) = 0 \} = \mathbb{C}\setminus\sigma(W_{f,g}) \quad \text{(b1)}
\]

(see (1.2.1)). Let \( \mu \in \pi(W_{f,g}) \). Since \( \pi(W_{f,g}) \subseteq \mathbb{C}\setminus\sigma_p(W_{f,g}) \), \( \mu \notin \sigma_p(W_{f,g}) \).

Then, by the proof of Theorem 4.15(i), \( \mu \notin \sigma_p(D^{(b)}_{\lambda u}) \cup \sigma(N^{(b)}_{\lambda u}) \). It follows that for every \( v \in \text{supp} \mathcal{H}^{(b)}_u \),

\[
\begin{bmatrix}
D^{(b)}_{\lambda u} - \mu & 0 \\
f \otimes g & N^{(b)}_{\lambda u} - \mu
\end{bmatrix}
\begin{bmatrix}
(\lambda_{uv} - \mu)^{-1}e_v \\
-(\lambda_{uv} - \mu)^{-1}g(v)(N^{(b)}_{\lambda u} - \mu)^{-1}(f)
\end{bmatrix}
= \begin{bmatrix}
e_v \\
0
\end{bmatrix},
\]

which implies that \( (W_{f,g} - \mu)\mathcal{D}(W_{f,g}) \) is dense in \( \mathcal{H}^{(b)}_u \). Also,

\[
(W_{f,g} - \mu)(\ell^2(V) \ominus \mathcal{H}^{(b)}_u) = \ell^2(V) \ominus \mathcal{H}^{(b)}_u,
\]

which implies that \( d_{W_{f,g}}(\mu) = 0 \). This together with (4.4.13) shows that \( \pi(W_{f,g}) = \emptyset \) completing the proof.

In general, the spectrum of \( W_{f,g} \) may not be the topological closure of its point spectrum.

**Example 4.19** Let \( g : \text{supp} \mathcal{H}^{(b)}_u \rightarrow \mathbb{C} \) be such that \( \sum_{v \in \text{supp} \mathcal{H}^{(b)}_u} |g(v)|^2 = \infty \) and \( \text{card}(\text{supp}(g)) = \aleph_0 = \text{card}(\text{supp} \mathcal{H}^{(b)}_u \setminus \text{supp}(g)) \).
(for instance, one may let $u=v_2$ and $\text{supp}(g) = \text{Des}(v_5)$ in the rooted directed tree as given in Fig. 4). Let $\lambda_u$ be a weight system such that

$$\{\lambda_{uv} : v \in \text{supp}(g)\} \subseteq \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}, \quad (4.4.14)$$

$$\{\lambda_{uv} : v \in \text{supp} \mathcal{H}_u^{(b)}\} \neq \mathbb{C}. \quad (4.4.15)$$

It is easy to see using (4.4.14) that $W_{f,g}$ does not satisfy the compatibility condition. Hence, by Corollary 4.18, $\sigma(W_{f,g}) = \mathbb{C}$. Further, since $\sigma_p(W_{f,g})$ is not dense in $\mathbb{C}$ (see (4.4.15)), we must have $\sigma_p(W_{f,g}) \subsetneq \sigma(W_{f,g})$. Thus the spectral picture of a rank one extension $W_{f,g}$ of a weighted join operator can be summarized as follows:

(i) If $g$ satisfies the compatibility condition I, then $\sigma(W_{f,g}) = \sigma_p(W_{f,g})$ is a proper subset of $\mathbb{C}$.

(ii) If $g$ does not satisfy the compatibility condition I, then $\sigma(W_{f,g}) = \mathbb{C}$ and $\sigma_p(W_{f,g})$ may be a proper subset of $\mathbb{C}$.

In the last case, either $W_{f,g}$ is not closed or $\pi(W_{f,g}) = \emptyset$.

As an application to Theorem 4.15, we characterize those rank one extensions of weighted join operators on leafless directed trees, which admit compact resolvent.

**Corollary 4.20** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $\text{root}$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$ and $b \in V \setminus \{u\}$, consider the weight system $\lambda_u = \{\lambda_{uv} : v \in V_\infty\}$ of complex numbers and let $W_{f,g}$ be the rank one extension of the weighted join operator $W_{\lambda_u}^{(b)}$ on $\mathcal{T}$, where $f \in \ell^2(V) \ominus \mathcal{H}_u^{(b)}$ is non-zero and $g : \text{supp} \mathcal{H}_u^{(b)} \rightarrow \mathbb{C}$ is given. Suppose that $W_{f,g}$ satisfies the compatibility condition I. If $\mathcal{T}$ is leafless, then the following are equivalent:

(i) The rank one extension $W_{f,g}$ of the weighted join operator $W_{\lambda_u}^{(b)}$ on $\mathcal{T}$ admits a compact resolvent.

(ii) The set $\{\lambda_{uv} : v \in \text{supp} \mathcal{H}_u^{(b)}\}$ has an accumulation point only at $\infty$ with each of its entries appearing finitely many times and the set $V_\prec$ of branching vertices of $\mathcal{T}$ is disjoint from $\text{Asc}(u)$.

**Proof** We need a couple of general facts in this proof.

(a) The diagonal operator $D_\lambda$ has compact resolvent if and only if the weight system $\lambda$ has an accumulation point only at $\infty$ with each of its entries appearing finitely many times.

(b) A finite block matrix with operator entries being bounded linear is compact if and only if all of its entries are compact.

To see the equivalence of (i) and (ii), assume that $\mathcal{T}$ is leafless. In view of (b), the formula (4.4.1) and Theorem 4.15(iii), $W_{f,g}$ has compact resolvent if and only if $D_{\lambda_u}^{(b)}$ and $N_{\lambda_u}^{(b)}$ have compact resolvents. On the other hand, by Proposition 2.12(ii) and (3.2.11), $\ell^2(V) \ominus \mathcal{H}_u^{(b)}$ is finite dimensional if and only if $V_\prec \cap \text{Asc}(u) = \emptyset$. In view of Lemma 1.2(iii), this is equivalent to the assertion that $N_{\lambda_u}^{(b)}$ has compact resolvent. The desired equivalence now follows from (a).
Remark 4.21 Assume that $\mathcal{T}$ is leafless and $b = u$. Then, by (3.2.16), $\ell^2(V) \ominus \mathcal{H}_u^{(b)}$ is one-dimensional. One may now argue as the proof of Corollary 4.20 to show that $W_{f,g}$ has compact resolvent if and only if the set $\{\lambda_{uv} : v \in V\}$ has an accumulation point only at $\infty$ with each of its entries appearing finitely many times.

It is well-known that given any closed subset $\sigma$ of the complex plane, there exists a diagonal operator $D_\lambda$ on $\ell^2(\mathbb{N})$ such that $\sigma(D_\lambda) = \sigma$. Here is a variant of this fact for rank one extensions of weighted join operators.

Corollary 4.22 Let $\mathcal{T} = (V, E)$ be a rooted directed tree and let $b \in V$. Let $u \in V$ be such that $\mathcal{U}_u = (\operatorname{Des}(u), E_u)$ is an infinite directed subtree of $\mathcal{T}$. Then, for any closed, unbounded proper subset $\sigma$ of the complex plane, there exists a rank one extension $W_{f,g}$ of a weighted join operator $W^{(b)}_{\lambda_u}$ on $\mathcal{T}$ such that the following hold:

(i) $g \notin \mathcal{H}_u^{(b)}$,

(ii) $W_{f,g}$ satisfies the compatibility condition I, and

(iii) $\sigma(W_{f,g}) = \sigma$.

**Proof** Let $f = eu$ and let $z_0 \in \mathbb{C} \setminus \sigma$. Let $\{\mu_n\}_{n \geq 1}$ be a countable dense subset of $\sigma$ and let $\{v_n\}_{n \geq 1}$ be a subset of $\sigma$ such that

$$|v_n - z_0| \geq 2^{n/2}$$

for every integer $n \geq 1$ (4.4.16)

(which exists since $\sigma$ is unbounded). Consider the countable dense subset $\{\lambda_n\}_{n \geq 1}$ of $\sigma$ defined by

$$\lambda_n = \begin{cases} 
\mu_k & \text{if } n = 2k, \ k \geq 1, \\
v_k & \text{if } n = 2k - 1, \ k \geq 1.
\end{cases}$$

By axiom of choice [66, Pg 11], there exists a sequence $\{v_n\}_{n \geq 1} \subseteq \operatorname{supp} \mathcal{H}_u^{(b)}$ such that $d_{v_n} = n$ for every integer $n \geq 1$. Set

$$\lambda_{uv} = \begin{cases} 
\lambda_n & \text{if } v = v_n, \\
0 & \text{otherwise.}
\end{cases}$$

Define $g : \operatorname{supp} \mathcal{H}_u^{(b)} \to \mathbb{C}$ by

$$g(v) = \begin{cases} 
\lambda_n - z_0 & \text{if } v = v_n, \\
0 & \text{otherwise.}
\end{cases}$$

Then, by the choice of vertices $\{v_n\}_{n \geq 1}$, the weight system $\lambda_u$ and $g$,

$$\sum_{u \in \operatorname{supp} \mathcal{H}_u^{(b)}} \frac{|g(v)|^2}{|\lambda_{uv} - z_0|^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$
and hence the rank one extension \( W_{f,g} \) of the weighted join operator \( W^{(b)}_{\lambda_u} \) satisfies the compatibility condition \( I \). This combined with Theorem 4.15(ii) yields (iii). On the other hand, by (4.4.16) and the definition of \( \{\lambda_n\}_{n \geq 1} \),

\[
\sum_{v \in \text{supp } H_u^{(b)}} |g(v)|^2 \geq \sum_{k=1}^{\infty} \frac{|v_k - z_0|^2}{|2k - 1|^2} \geq \sum_{k=1}^{\infty} \frac{2^k}{|2k - 1|^2},
\]

which shows that \( g \notin H_u^{(b)} \). This completes the proof. \( \square \)

It is worth noting that the conclusion of Corollary 4.22 does not hold in case \( \sigma \) is a bounded subset of \( \mathbb{C} \). Indeed, the boundedness of the spectrum of a rank one extension \( W_{f,g} \) of a weighted join operator \( W^{(b)}_{\lambda_u} \) implies that the diagonal operator \( D^{(b)}_{\lambda_u} \) is bounded. Then, by Remark 4.2, \( W_{f,g} \) is not even closable. So, by Theorem 4.11, \( W_{f,g} \) can not satisfy a compatibility condition.

A special case of Theorem 4.15 (the case of \( g = 0 \)) provides a complete spectral picture for weighted join operators. This together with some additional properties is summarized in the next result. We first introduce some notations and definitions.

Let \( T \) be a densely defined linear operator in \( \mathcal{H} \). A complex number \( \mu \) is said to be a generalized eigenvalue of \( T \) if there exists a positive integer \( k \) and a non-zero vector \( f \in D(T^k) \) (to be referred to as generalized eigenvector corresponding to \( \mu \)) such that \((T - \mu)^k f = 0\). The rootspace \( \mathcal{R}_T(\mu) \) of \( T \) corresponding to the generalized eigenvalue \( \mu \) is defined as the closed space spanned by the corresponding generalized eigenvectors of \( T \). Any vector in the rootspace of \( T \) is referred to as root vector for \( T \).

We say that \( T \) is complete if it has a complete set of root vectors (refer to [32] for the basics of completeness of root systems and to [11] for completeness of root systems for the class of rank one perturbations of self-adjoint operators).

**Theorem 4.23** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root root and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( u \in V \) and \( b \in V \setminus \{u\} \), consider the weight system \( \lambda_u = \{\lambda_{uv}\}_{v \in V_\infty} \) of complex numbers and let \( W^{(b)}_{\lambda_u} \) be the weighted join operator on \( \mathcal{T} \). Then, we have the following statements:

(i) The point spectrum \( \sigma_p(W^{(b)}_{\lambda_u}) \) of \( W^{(b)}_{\lambda_u} \) is given by

\[
\sigma_p(W^{(b)}_{\lambda_u}) = \begin{cases} 
\{\lambda_{uv} : v \in V\} & \text{if } b = u, \\
\{\lambda_{uv} : v \in \text{Asc}(u) \cup \text{Des}_b[u] \} \cup \{0\} & \text{if } b \in \text{Des}_u(u), \\
\{\lambda_{uv} : v \in \text{Des}(u) \} \cup \{0\} & \text{otherwise}.
\end{cases}
\]

(ii) The spectrum \( \sigma(W^{(b)}_{\lambda_u}) \) of \( W^{(b)}_{\lambda_u} \) is the topological closure of \( \sigma_p(W^{(b)}_{\lambda_u}) \).
(iii) If $E_{W_{k_u}^{(b)}}(\mu)$ denotes the eigenspace corresponding to the eigenvalue $\mu$ of $W_{k_u}^{(b)}$, then

$$E_{W_{k_u}^{(b)}}(\mu) = \begin{cases} \ell^2(W_{\mu}) & \text{if } \mu \neq 0, \mu \neq \lambda_{uu}, \\ \ell^2(W_{\mu}) \oplus [e_u] & \text{if } \mu \neq 0, \mu = \lambda_{uu}, \\ \ell^2(W_0) \oplus (\ell^2(V) \ominus \mathcal{H}_{u}^{(b)}) \oplus [e_{k_u,A_u}] & \text{if } \mu = 0, \end{cases}$$

where $W_{\mu}$ is given by (4.4.11), $\mathcal{H}_{u}^{(b)}$ is given by (3.2.11) and $A_u$ is given by (3.2.15).

(iv) The multiplicity function $m_{W_{k_u}^{(b)}} : \sigma_p(W_{k_u}^{(b)}) \to \mathbb{Z}_+ \cup \{\mathbb{N}_0\}$ is given by

$$m_{W_{k_u}^{(b)}}(\mu) = \text{card } \{v \in \{u\} \cup \text{supp } \mathcal{H}_{u}^{(b)} : \lambda_{uv} = \mu\} \text{ if } \mu \neq 0.$$  

In addition, if $\mathcal{T}$ is leafless and if there exists a branching vertex $w \in \text{Asc}(u)$, then $m_{W_{k_u}^{(b)}}(0) = \mathbb{N}_0$.  

(v) The rootspace $R_{W_{k_u}^{(b)}}^{(b)}(\mu)$ of $W_{k_u}^{(b)}$ corresponding to the generalized eigenvalue $\mu$ is given by

$$R_{W_{k_u}^{(b)}}^{(b)}(\mu) = \begin{cases} E_{W_{k_u}^{(b)}}(\mu) & \text{if } \mu \in \sigma_p(W_{k_u}^{(b)}) \setminus \{0\}, \\ E_{W_{k_u}^{(b)}}(0) & \text{if } \mu = 0, \lambda_{uu} \neq 0, \\ E_{W_{k_u}^{(b)}}(0) \oplus [e_{k_u,A_u}] & \text{if } \mu = 0, \lambda_{uu} = 0. \end{cases}$$

Remark 4.24 In case $b = u$, the spectral picture of $W_{k_u}^{(b)}$ coincides with that of the diagonal operator $D_{k_u}$ (see Remark 3.11). We discuss here the spectral picture of $W_{k_u}^{(\infty)}$. By (3.2.3), $W_{k_u}^{(\infty)}$ is an orthogonal direct sum of rank one operators

$$e_{u_j} \otimes e_{k_u,\text{Des}(u_j) \setminus \text{Des}(u_{j-1})}, \quad j = 0, \ldots, d_u,$$

where Des$^{-1}(u_j) = \emptyset$ and $u_j := \text{par}(j)(u)$ for $j = 0, \ldots, d_u$. If $\lambda_{u} \in \ell^2(V)$, then it follows that $W_{k_u}^{(\infty)} \subseteq B(\ell^2(V))$, and hence by Lemma 1.2(iii),

$$\sigma(W_{k_u}^{(\infty)}) = \{0\} \cup \{\lambda_{uu} : j = 0, \ldots, d_u\} = \sigma_p(W_{k_u}^{(\infty)}).$$

Assume now that $\lambda_{u} \notin \ell^2(V)$. By Lemma 1.3, $\sigma_p(W_{k_u}^{(\infty)})$ is given by the same formula as above. Further, another application of Lemma 1.3 shows that $W_{k_u}^{(\infty)}$ is not closed, and hence $\sigma(W_{k_u}^{(\infty)}) = \mathbb{C}$.

Proof The conclusions in (i) and (ii) follow from (i) and (iii) of Theorem 4.15. Since $\ell^2(A_u) \subseteq \ell^2(V) \ominus \mathcal{H}_{u}^{(b)}$ and card$(A_u) \geq 2$ (see (3.2.15)), by Lemma 1.2(iii) and (3.2.14), we obtain
\[\sigma_p(N^{(b)}_{\lambda_u}) = \{0, \lambda_{uu}\} = \sigma(N^{(b)}_{\lambda_u}),\]

\[m_{N^{(b)}_{\lambda_u}}(0) = \mathbb{N}_0 \text{ if } \dim(\ell^2(V) \ominus \mathcal{H}^{(b)}_{\lambda_u}) = \mathbb{N}_0, \quad m_{N^{(b)}_{\lambda_u}}(\lambda_{uu}) = 1 \text{ if } \lambda_{uu} \neq 0,\]

\[\lambda_{uu} \neq 0 \implies \mathcal{E}^{(b)}_{N^{(b)}_{\lambda_u}}(\mu) = \begin{cases} [e_{\lambda_u, \lambda_u}]^\perp & \text{if } \mu = 0, \\ [e_u] & \text{if } \mu = \lambda_{uu}.\end{cases}\]

In view of the fact that \(\dim(\ell^2(V) \ominus \mathcal{H}^{(b)}_{\lambda_u}) = \mathbb{N}_0\) if and only if \(\text{card}(V_u) = \mathbb{N}_0\), the conclusions in (iii) and (iv) pertaining to the eigenspaces and multiplicities now follow from Proposition 2.12.

To see (v), let \(k\) be a positive integer and let \(\mu \in \mathbb{C}\backslash\{0\}\). By (3.2.17),

\[(N^{(b)}_{\lambda_u} - \mu)^k = \sum_{l=0}^k (-\mu)^{k-l} \binom{k}{l} N^{(b)}_{\lambda_u}^l = (-\mu)^k I + \sum_{l=1}^k (-\mu)^{k-l} \binom{k}{l} \lambda_{uu}^{-l-1} e_u \otimes e_{\lambda_u, \lambda_u} - e_{\lambda_u, \lambda_u}.\]

It is not difficult to see that for any \(h \in \ell^2(V) \ominus \mathcal{H}^{(b)}_{\lambda_u}\) such that \((N^{(b)}_{\lambda_u} - \mu)^k h = 0\), we must have \(h = h(u)e_u\) and \(\mu = \lambda_{uu}\). However, \(e_u\) is an eigenvector, and hence a root vector. This shows that the rootspace of \(N^{(b)}_{\lambda_u}\) corresponding to \(\mu\) is spanned by \(e_u\). Since the generalized eigenvalues and eigenvalues coincide for a diagonal operator, the desired conclusion in (v) follows provided \(\lambda_{uu} \neq 0\). In case \(\lambda_{uu} = 0\), by Corollary 3.16, \(N^{(b)}_{\lambda_u}\) is a nilpotent operator of nilpotency index 2, and hence any \(h \in \ell^2(V) \ominus \mathcal{H}^{(b)}_{\lambda_u}\) is a root vector. This completes the verification of (v). \(\square\)

A result of Wermer says that the spectral synthesis holds for all normal compact operators [73]. As shown by Hamburger [33], this no longer holds for compact operators. Interestingly, the following result can be used to construct examples of compact non-normal operators which are not even complete.

**Corollary 4.25** Let \(\mathcal{T} = (V, E)\) be a rooted directed tree with root root and let \(\mathcal{T}_\infty = (V_\infty, E_\infty)\) be the extended directed tree associated with \(\mathcal{T}\). For \(u \in V\) and \(b \in V \setminus \{u\}\), consider the weight system \(\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}\) of complex numbers and let \(W^{(b)}_{\lambda_u}\) be the weighted join operator on \(\mathcal{T}\). Then \(W^{(b)}_{\lambda_u}\) is complete if and only if \(\lambda_{uu} = 0\) or \(\lambda_{uv} = 0\) for every \(v \in A_u \setminus \{u\}\).

**Proof** If \(\lambda_{uu} = 0\), then by (iii) and (v) of Theorem 4.23, the root vectors for \(W^{(b)}_{\lambda_u}\) forms a complete set. If \(\lambda_{uv} = 0\) for every \(v \in A_u \setminus \{u\}\), then by Theorem 3.13, \(W^{(b)}_{\lambda_u}\) is a diagonal operator, and hence we get the sufficiency part. To see the necessity part, suppose that \(\lambda_{uu} \neq 0\) and \(\lambda_{uu} \neq 0\) for some \(v \in A_u \setminus \{u\}\). Thus \(e_{\lambda_u, \lambda_u}\) is a non-zero vector in \(\ell^2(V)\), and by (iv) and (v) of Theorem 4.23,

\[\bigvee_{\mu \in \mathbb{C}} \mathcal{R}^{(b)}_{W^{(b)}_{\lambda_u}}(\mu) \subseteq \ell^2(V) \ominus [e_{\lambda_u, \lambda_u}].\]

This shows that \(W^{(b)}_{\lambda_u}\) is not complete. \(\square\)
5 Special Classes

In this section, we discuss some special classes of weighted join operators and their rank one extensions. In particular, we exhibit families of sectorial operators and infinitesimal generators of quasi-bounded strongly continuous semigroups within these classes. Further, we characterize hyponormal operators and $n$-symmetric operators within the class of weighted join operators on rooted directed trees. We also investigate the classes of hyponormal and $n$-symmetric rank one extensions $W_{f,g}$ of weighted join operators. The complete characterizations of these classes seem to be beyond reach at present, particularly, in view of the fact that structures of positive integral powers and the Hilbert space adjoint of $W_{f,g}$ are complicated.

5.1 Sectoriality

A densely defined linear operator $T$ in $\mathcal{H}$ is sectorial if there exist $a \in \mathbb{R}$, $M \in (0, \infty)$ and $\theta \in (0, \pi/2)$ such that

$$\lambda \in \rho(T) \text{ and } \|RT(\lambda)\| \leq \frac{M}{|\lambda - a|} \text{ whenever } \lambda \in \mathbb{C} \text{ and } |\arg(\lambda - a)| \geq \theta.$$  \hspace{1cm} (5.1.1)

Sometimes we say that $T$ is sectorial with angle $\theta$ and vertex $a$. Note that there is considerable divergence of terminology in the literature, for example, Kato [46] calls it $m$-sectorial, while we call it just sectorial, the correspondence being a minus sign between the two. For the basic theory of sectorial operators, the reader is referred to [5, 16, 22, 34, 46, 52, 64, 67, 68]). The following result yields a family of sectorial rank one extensions of weighted join operators (cf. [41, Proposition 3]).

**Proposition 5.1** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root root and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u, b \in V$, consider the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers and let $W_{f,g}$ be the rank one extension of the weighted join operator $W_{\lambda_u}$ on $\mathcal{T}$, where $f \in \ell^2(V) \ominus \mathcal{H}(b_u)$ is non-zero and $g : \text{supp} \mathcal{H}(b_u) \to \mathbb{C}$ is given. Suppose that $W_{f,g}$ satisfies the compatibility condition II. If $\{\lambda_{uv} : v \in \text{supp} \mathcal{H}_u^{(b)}\}$ is contained in the sector $\mathbb{S}_{\theta,a} := \{z \in \mathbb{C} : |\arg(z - \alpha)| < \theta\}$ for some $\theta \in (0, \pi/2)$ and $\alpha \in \mathbb{R}$, then $W_{f,g}$ is a sectorial operator.

**Proof** Assume that $\{\lambda_{uv} : v \in \text{supp} \mathcal{H}_u^{(b)}\}$ is contained in $\mathbb{S}_{\theta,a}$ for some $\theta \in (0, \pi/2)$. After replacing $\theta$ by $\theta + \epsilon \in (0, \pi/2)$ for some $\epsilon > 0$, we may assume without loss of generality that $\sigma(D_{\lambda_u}^{(b)}) \subseteq \mathbb{S}_{\theta',\alpha}$ for some $\theta' \in (0, \theta)$. It is well-known that the diagonal operator $D_{\lambda_u}^{(b)}$ satisfies the estimate (5.1.1) (see, for instance, [34, Chapter 2, Section 2.2.1] and [52, Example 4.5.2]). Indeed, for any $\mu \in \mathbb{C} \setminus \overline{\mathbb{S}_\theta}$,

$$\|(D_{\lambda_u}^{(b)} - \mu)^{-1}\| = \sup_{v \in \text{supp} \mathcal{H}_u^{(b)}} |\lambda_{uv} - \mu|^{-1}$$
\[ \frac{1}{|\mu - \alpha|} \leq \inf_{\lambda u : v \in \text{supp } H(b)} |\lambda uv - \mu| \]

\[ \leq \frac{1}{\inf_{t \in \mathbb{R}^t, \alpha} |t - \mu|} \]

\[ \leq \begin{cases} \frac{1}{|\mu - \alpha|} & \text{if } |\arg(\mu - \alpha)| \geq \theta + \pi/2, \\ \frac{1}{|\mu - \alpha| \sin(\arg(\mu - \alpha) - \theta')} & \text{otherwise} \end{cases} \]

(see Fig. 7). Thus (5.1.1) holds with \( M = \frac{1}{\sin(\theta - \theta')} \). By Corollary 4.12, \( W_{f,g} = A + B + C \), where \( B + C \) is \( A \)-bounded with \( A \)-bound equal to 0 (see (4.2.8) and (4.2.11)). Since \( A \) is sectorial (since so is \( D(b) \)), an application of [52, Theorem 4.5.7] shows that \( W_{f,g} \) is sectorial. \( \square \)

For all relevant definitions and basic theory of strongly continuous quasi-bounded semigroups, the reader is referred to [46, 52, 68]. A result similar to the following has been obtained in [55, Proposition 3.1] for a family of upper triangular operator matrices on non-diagonal domains (cf. [13, Theorem 7.11]).

**Proposition 5.2** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root \( \text{root} \) and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( u, b \in V \), consider the weight system \( \lambda_u = \{\lambda_{uv} : v \in V_\infty \} \) of complex numbers and let \( W_{f,g} \) be the rank one extension of the weighted join operator \( W_{\lambda_u}(b) \) on \( \mathcal{T} \), where \( f \in \ell^2(V) \ominus \mathcal{H}_u(b) \) is non-zero and \( g : \text{supp } \mathcal{H}_u(b) \rightarrow \mathbb{C} \) is given. Suppose that \( W_{f,g} \) satisfies the compatibility condition II. If \( \{\lambda_{uv} : v \in \text{supp } \mathcal{H}_u(b) \} \) is contained in the right half plane \( \mathbb{H}_\alpha = \{z \in \mathbb{C} : \Re z \geq \alpha \} \) for some \( \alpha \in \mathbb{R} \), then \( W_{f,g} \) is the generator of a strongly continuous semigroup \( \{Q(t)\}_{t \geq 0} \) satisfying \( \|Q(t)\| \leq Me^{-\alpha t} \) for \( t \geq 0 \).
Proposition 5.3 Let \( \lambda_{uv} : v \in \text{supp } \mathcal{H}_u^{(b)} \) be contained in the right half plane \( \mathbb{H}_\alpha = \{ z \in \mathbb{C} : \Re z \geq \alpha \} \) for some \( \alpha \in \mathbb{R} \). Thus \( \sigma(D^{(b)}_{\lambda_u}) \subseteq \mathbb{H}_\alpha \). It is easy to see that

\[
\| (D^{(b)}_{\lambda_u} - \mu)^{-n} \| \leq \frac{1}{|\alpha - \mu|^n}, \quad \mu \in (-\infty, \alpha), \; n = 1, 2, \ldots.
\]

Hence, by the Hille-Yoshida Theorem (see [52, Theorem 4.3.5], [68, Theorem 2.3.3]), \( D^{(b)}_{\lambda_u} \) is the generator of a strongly continuous semigroup \( \{ S(t) \}_{t \geq 0} \) satisfying the quasi-boundedness condition \( \| S(t) \| \leq M e^{-\alpha t}, \; t \geq 0 \). By Corollary 4.12, \( W_{f,g} = A + B + C \), where \( B + C \) is \( A \)-bounded with \( A \)-bound equal to 0 (see (4.2.8) and (4.2.11)). Note that \( A \) is the generator of the strongly continuous semigroup \( \{ S(t) \oplus I \}_{t \geq 0} \). The desired conclusion may now be derived from [46, Corollary 2.5, Chapter IX]. \( \square \)

If \( W_{f,g} \) is as in the preceding result, one can define fractional powers of \( W_{f,g} \) (refer to [52, Chapter 6]). Further, one may obtain a counterpart of [41, Corollary 2] ensuring \( H^\infty \)-functional calculus for rank one extensions of weighted join operators (cf. [20, Proposition 3.4]). We refer the reader to [34] for more details on this topic.

5.2 Normality

A densely defined linear operator \( T \) in \( \mathcal{H} \) is said to be \textit{hyponormal} if \( \mathcal{D}(T) \subseteq \mathcal{D}(T^*) \) and \( \| T^* x \| \leq \| Tx \| \) for all \( x \in \mathcal{D}(T) \). We say that \( T \) is \textit{cohyponormal} if \( T \) is closed and \( T^* \) is hyponormal. There has been significant literature on the classes of hyponormal and cohyponormal operators (refer to [25, 38–44, 58]).

We begin with a rigidity result stating that no weighted join operator can be hyponormal unless it is diagonal. A variant of this fact in the context of bounded operators has been obtained in [44, Theorem 2.3] (cf. [37, Proposition 3.1]).

Proposition 5.3 Let \( \mathcal{I} = (V, E) \) be a rooted directed tree with root \( r \) and let \( \mathcal{I}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{I} \). For \( u, b \in V \), consider the weight system \( \lambda_u = \{ \lambda_{uv} \}_{v \in V_\infty} \) of complex numbers and let \( W^{(b)}_{\lambda_u} \) be the weighted join operator on \( \mathcal{I} \). Then the following statements are equivalent:

(i) \( W^{(b)}_{\lambda_u} \) is normal.

(ii) \( W^{(b)}_{\lambda_u} \) is hyponormal.

(iii) \( W^{(b)}_{\lambda_u} \) is cohyponormal.

(iv) \( W^{(b)}_{\lambda_u} \) is diagonal with respect to the orthonormal basis \( \{ e_v \}_{v \in V} \).

(v) \( b = u \) or \( \lambda_{uv} = 0 \) for every \( v \in A_u \setminus \{ u \} \), where \( A_u \) is given by (3.2.15).

Proof By Remark 3.11, \( W^{(b)}_{\lambda_u} \) is a diagonal operator if \( b = u \) or \( \lambda_{uv} = 0 \) for every \( v \in A_u \setminus \{ u \} \), and hence (v) implies (i)–(iv). By Theorem 3.13, \( W^{(b)}_{\lambda_u} \) admits the decomposition \( (D^{(b)}_{\lambda_u}, N^{(b)}_{\lambda_u}, \mathcal{H}^{(b)}_{\lambda_u}) \). Thus the Hilbert space adjoint of \( W^{(b)}_{\lambda_u} \) is given by

\[
\mathcal{D}(W^{(b)}_{\lambda_u}^*) = \mathcal{D}(W^{(b)}_{\lambda_u}),
\]
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Since $D_{\lambda_u}$ is normal, $W_{\lambda_u}$ is hyponormal (resp. cohyponormal) if and only if so is $N^{(b)}_{\lambda_u}$. Note that by (1.3.2), $N_{\lambda_u}^{(b)*} = e_{\lambda_u} \otimes e_u$. Also, by (1.3.2), for $x, y \in \ell^2(V)$,

$$
[x \otimes y, (x \otimes y)^*] = (x \otimes y)(y \otimes x) - (y \otimes x)(x \otimes y) = \|y\|^2 x \otimes x - \|x\|^2 y \otimes y.
$$

It follows that

$$
[N_{\lambda_u}^{(b)*}, N_{\lambda_u}^{(b)}] = \begin{cases} 
    e_{\lambda_u, [u, b]} \otimes e_{\lambda_u, [u, b]} - \|e_{\lambda_u, [u, b]}\|^2 e_u \otimes e_u & \text{if } b \in \text{Des}(u), \\
    e_{\lambda_u, \text{Asc}(u), [u, b]} \otimes e_{\lambda_u, \text{Asc}(u), [u, b]} - \|e_{\lambda_u, \text{Asc}(u), [u, b]}\|^2 e_u \otimes e_u & \text{otherwise.}
\end{cases}
$$

This yields

$$
\langle [N_{\lambda_u}^{(b)*}, N_{\lambda_u}^{(b)}]e_u, e_u \rangle = \begin{cases} 
    -\|e_{\lambda_u, [u, b]}\|^2 & \text{if } b \in \text{Des}(u), \\
    -\|e_{\lambda_u, \text{Asc}(u), [u, b]}\|^2 & \text{otherwise,}
\end{cases}
$$

which is always negative provided $b \neq u$ and $\lambda_{uv} \neq 0$ for some $v \in A_u \setminus \{u\}$. Further, in this case,

$$
\langle [N_{\lambda_u}^{(b)*}, N_{\lambda_u}^{(b)}]e_v, e_v \rangle = |\lambda_{uv}|^2, \quad v \in A_u \setminus \{u\},
$$

and hence $N_{\lambda_u}^{(b)}$ is not cohyponormal. This completes the proof. 

Let us now investigate the class of normal rank one extensions $W_{f, g}$ of weighted join operators.

**Proposition 5.4** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root root and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u, b \in V$, consider the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers and let $W_{f, g}$ be the rank one extension of the weighted join operator $W_{\lambda_u}^{(b)}$ on $\mathcal{T}$, where $f \in \ell^2(V) \otimes \mathcal{H}^{(b)}_u$ is non-zero and $g : \text{supp } \mathcal{H}^{(b)}_u \rightarrow \mathbb{C}$ is given. Suppose that $W_{f, g}$ satisfies the compatibility condition $I$. Then the following statements are equivalent:

(i) $W_{f, g}$ is normal.
(ii) $W_{f, g}$ is diagonal with respect to the orthonormal basis $\{e_v\}_{v \in V}$.
(iii) $g = 0$ and

$$
either b = u or \lambda_{uv} = 0 for every v \in A_u \setminus \{u\}.\tag{5.2.1}
$$

**Proof** Assume that $W_{f, g}$ is normal. Note that $N_{\lambda_u}^{(b)}$, being the restriction of $W_{f, g}$ to $\ell^2(V) \otimes \mathcal{H}^{(b)}_u$, is hyponormal. By Proposition 5.3, (iii) holds. Thus $N_{\lambda_u}^{(b)} = \lambda_{uu} e_u \otimes e_u$. 

Since $W_{f,g}$ satisfies the compatibility condition I, by Corollary 4.18, $\sigma(W_{f,g})$ is a proper closed subset of $\mathbb{C}$. Let $\mu \in \mathbb{C}\setminus \sigma(W_{f,g})$. By Theorem 4.15(iii),

$$(W_{f,g} - \mu)^{-1} = \begin{bmatrix} (\mathcal{D}_{\lambda_u}^{(b)} - \mu)^{-1} & 0 \\ - (\mathcal{N}_{\lambda_u}^{(b)} - \mu)^{-1} L_{\lambda_u,\mu} & (\mathcal{N}_{\lambda_u}^{(b)} - \mu)^{-1} \end{bmatrix}, \quad (5.2.2)$$

where the linear transformation $L_{\lambda_u,\mu}$ is given by $L_{\lambda_u,\mu} = \frac{f}{obslash g}(\mathcal{D}_{\lambda_u}^{(b)} - \mu)^{-1}$.

On the other hand, by [64, Proposition 3.26(v)], $(W_{f,g} - \mu)^{-1}$ is normal. Let $A = (\mathcal{D}_{\lambda_u}^{(b)} - \mu)^{-1}$, $B = -(\mathcal{N}_{\lambda_u}^{(b)} - \mu)^{-1} L_{\lambda_u,\mu}$ and $C = (\mathcal{N}_{\lambda_u}^{(b)} - \mu)^{-1}$. Since $A$ and $C$ are normal, it may be concluded from (5.2.2) that

$$[((W_{f,g} - \mu)^{-1})^*, (W_{f,g} - \mu)^{-1}] = \begin{bmatrix} B^* B & B^* C - AB^* \\ C^* B - BA^* & BB^* \end{bmatrix}.$$  

Since $W_{f,g}$ is normal, $B = 0$. It follows that $L_{\lambda_u,\mu} = 0$, and hence $f \odot g = 0$ on $\mathcal{D}(\mathcal{D}_{\lambda_u}^{(b)})$. Since $f$ is non-zero, we must have $g = 0$. The remaining implications follow from Proposition 5.3.

The methods of proofs of Propositions 5.3 and 5.4 are different. In particular, the unavailability of a formula for the Hilbert space adjoint of $W_{f,g}$ necessitated us to characterize the normality of $W_{f,g}$ with the help of the resolvent function. An inspection of the proof of Proposition 5.4 shows that (5.2.1) is a necessary condition for $W_{f,g}$ to be a hyponormal operator.

### 5.3 Symmetry

A densely defined linear operator $T$ in $\mathcal{H}$ is said to be $n$-symmetric if

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \langle T^j x, T^{n-j} y \rangle = 0, \quad x, y \in \mathcal{D}(T^n),$$

where $n$ is a positive integer. We refer to 1-symmetric operator as symmetric operator. We say that $T$ is strictly $n$-symmetric if it is $n$-symmetric, but not $(n-1)$-symmetric, where $n$ is a positive integer bigger than 1. For the basic properties of $n$-symmetric operators and its connection with the theory of differential equations, the reader is referred to [2–4, 8, 9, 35, 36, 62, 63].

The following proposition describes all $n$-symmetric weighted join operators. It reveals the curious fact that there is no strictly 2-symmetric weighted join operator. On the other hand, strictly 3-symmetric weighted join operators exist in abundance.

**Proposition 5.5** Let $\mathcal{T} = (V, E)$ be a rooted directed tree with root $root$ and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $b, u \in V$ and the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers, let $W_{\lambda_u}^{(b)}$ denote the weighted join operator on $\mathcal{T}$. Then, for any positive integer $n$, the following are equivalent:
(i) $W_{\lambda_u}^{(b)}$ is $n$-symmetric.

(ii) The weight system $\lambda_u$ satisfies

$$\lambda_{uv} \in \mathbb{R}, \quad v \in \begin{cases} \text{Asc}(u) \cup \text{Des}_b[u] & \text{if } b \in \text{Des}(u), \\ \text{Des}(u) & \text{otherwise}. \end{cases}$$

and one of the following holds:

(a) $\lambda_{uu} = 0$ and $n \geq 3$.
(b) $\lambda_{uv} = 0$ for $v \in A_u \setminus \{u\}$, where $A_u$ is as given in (3.2.15).

(iii) One of the following holds:

(a) $\lambda_{uu} = 0$ and $n \geq 3$.
(b) The weighted join operator $W_{\lambda_u}^{(b)}$ is symmetric.

**Proof** By Theorem 3.13, the weighted join operator $W_{\lambda_u}^{(b)}$ admits the decomposition $(D_{\lambda_u}^{(b)}, N_{\lambda_u}^{(b)}, \mathcal{H}_u^{(b)})$. Thus $W_{\lambda_u}^{(b)}$ is $n$-symmetric if and only if $D_{\lambda_u}^{(b)}$ and $N_{\lambda_u}^{(b)}$ are $n$-symmetric. It is easy to see that a diagonal operator is $n$-symmetric if and only if it is symmetric, which in turn is equivalent to the assertion that its diagonal entries are real. Assume that $N_{\lambda_u}^{(b)}$ is $n$-symmetric and let $v, w \in A_u$. Note that by (3.2.17),

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \langle N_{\lambda_u}^{(b)}^j e_v, N_{\lambda_u}^{(b)^n-j} e_w \rangle =$$

$$= \begin{cases} (\lambda_{uu} - \bar{\lambda}_{uu})^n & \text{if } v = u, \ w = u, \\ (-1)^n \bar{\lambda}_{uw} \bar{\lambda}_{uu}^{n-1} + \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \bar{\lambda}_{uw} \bar{\lambda}_{uu}^{j-1} \bar{\lambda}_{uu}^{n-j-1} & \text{if } v = u, \ w \neq u, \\ \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \lambda_{uw} \tilde{\lambda}_{uu} \lambda_{uu}^{j-1} \lambda_{uu}^{n-j-1} & \text{if } v \neq u, \ w \neq u. \end{cases}$$

(5.3.1)

Thus, by the first identity, $\lambda_{uu}$ is real, and hence, by the second identity, for every $w \in A_u \setminus \{u\}$,

$$\lambda_{uu}^{n-1} \bar{\lambda}_{uw} \sum_{j=0}^{n-1} (-1)^{n-j} \binom{n}{j} = 0.$$

Thus

$$\lambda_{uu} = 0 \text{ or } \lambda_{uw} = 0 \text{ for every } w \in A_u \setminus \{u\}. \quad (5.3.2)$$
Suppose \( \lambda_{uv} \neq 0 \) for some \( w \in A_u \setminus \{u\} \) and \( n \leq 2 \). Then, by (5.3.2), \( \lambda_{uu} = 0 \), and hence, by the third identity in (5.3.1),

\[
\sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \lambda_{uv} \lambda_{uu}^{n-2} = 0, \quad v \in A_u \setminus \{u\}.
\]

If \( n = 2 \), then \(-2\lambda_{uv} \lambda_{uu} = 0\), which is not possible for \( v = w \) (since by assumption \( \lambda_{uv} \neq 0 \)). This proves the implication (i) \( \Rightarrow \) (ii). Also, since condition (b) of (ii) is equivalent to the assertion that \( N_{\lambda_u}^{(b)} \) is equal to the normal rank one operator \( \lambda_{uu} e_u \otimes e_u \) (see (3.2.14)), we also obtain the equivalence of (ii) and (iii). Finally, the implication (ii) \( \Rightarrow \) (i) may be easily deduced from (5.3.1) and (3.0.4).

\[\square\]

**Remark 5.6** The weighted join operator \( W_{\lambda_u}^{(b)} \) is either symmetric or strictly 3-symmetric. In particular, \( W_{\lambda_u}^{(b)} \) is never strictly 2-symmetric.

We capitalize on the last proposition to exhibit a family of \( n \)-symmetric rank one extensions of weighted join operators.

**Proposition 5.7** Let \( T = (V, E) \) be a rooted directed tree with root root and let \( T_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( T \). For \( u, b \in V \), consider the weight system \( \lambda_u = \{\lambda_{uv}\}_{v \in V_\infty} \) of complex numbers and let \( W_{f,g} \) be the rank one extension of the weighted join operator \( W_{\lambda_u}^{(b)} \) on \( T \), where \( f, g \in \ell^2(V) \odot H_{\lambda_u}^{(b)} \) is non-zero and \( g : \text{supp } H_u^{(b)} \to \mathbb{C} \) is a non-zero function. Assume that \( \langle f, e_{\lambda_u} \rangle = 0 \). Then the following statements hold:

(i) If \( n \geq 2 \) and \( \text{supp}(g) \cap \{v \in \text{supp } H_u^{(b)} : \lambda_{uv} \neq 0\} = \emptyset \), then \( W_{f,g} \) is \( n \)-symmetric if and only if \( \{\lambda_{uv} : v \in \text{supp } H_u^{(b)}\} \) is contained in \( \mathbb{R} \) and either \( f(u) = 0 \) or (5.3.2) holds.

(ii) If \( \text{supp}(g) \cap \{v \in \text{supp } H_u^{(b)} : \lambda_{uv} \in \mathbb{R} \setminus \{0\}\} = \emptyset \), then \( W_{f,g} \) is never \( n \)-symmetric.

**Proof** By (4.0.1), \( W_{f,g}^n \) can be decomposed as

\[
W_{f,g}^n = \begin{bmatrix}
(D_{\lambda_u}^{(b)})^n & 0 \\
L_n & (N_{\lambda_u}^{(b)})^n
\end{bmatrix},
\]

where \( L_n, n \geq 0 \) is defined inductively as follows:

\[
L_0 = 0, \quad L_1 = f \otimes g, \quad L_n = L_{n-1} D_{\lambda_u}^{(b)} + (N_{\lambda_u}^{(b)})^{n-1} f \otimes g, \quad n \geq 2.
\]

Recall from (3.2.17) that \( N_{\lambda_u}^{(b)k} = \lambda_{uu}^{k-1} N_{\lambda_u}^{(b)} \), \( k \geq 1 \). An inductive argument now shows that

\[
L_k = f \otimes g (D_{\lambda_u}^{(b)})^{k-1} + \sum_{j=1}^{k-1} \lambda_{uu}^{j-1} N_{\lambda_u}^{(b)} (f \otimes g) (D_{\lambda_u}^{(b)})^{k-j-1}, \quad k \geq 1. \quad (5.3.3)
\]
Note that $W_{f,g}$ is $n$-symmetric if and only if $N_{\lambda_n}^{(b)}$ is $n$-symmetric and for every $(h_1, h_2), (k_1, k_2)$ in $\mathcal{P}(W^n_{f,g})$,

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \langle (D^{(b)}_{\lambda_n})^j h_1, (D^{(b)}_{\lambda_n})^{n-j} k_1 \rangle$$

$$+ \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \langle L_j h_1, L_{n-j} k_1 \rangle = 0, \quad (5.3.4)$$

$$\sum_{j=0}^{n-1} (-1)^{n-j} \binom{n}{j} \langle (N^{(b)}_{\lambda_n})^j h_2, L_{n-j} k_1 \rangle = 0. \quad (5.3.5)$$

Assume now that $n \geq 2$ and $\langle f, e_{\lambda_n} \rangle = 0$. By (5.3.3),

$$L_k = f \otimes g (D^{(b)}_{\lambda_n})^{k-1}, \quad k \geq 1. \quad (5.3.6)$$

It follows that for any $v, w \in \text{supp} \mathcal{H}^{(b)}_u$,

$$\sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \langle L_j e_v, L_{n-j} e_w \rangle = \frac{g(v)g(w)}{g(v)} \|f\|^2 \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \lambda_{uv}^{j-1} n^{n-j-1}. \quad (5.3.7)$$

Hence (5.3.4) holds if and only if for any $v, w \in \text{supp} \mathcal{H}^{(b)}_u$,

$$(\lambda_{uv} - \bar{\lambda}_{uv})^n + |g(v)|^2 \|f\|^2 \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \lambda_{uv}^{j-1} n^{n-j-1} = 0,$$

$$\frac{g(v)g(w)}{g(v)} \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \lambda_{uv}^{j-1} n^{n-j-1} = 0, \quad v \neq w. \quad (5.3.7)$$

Further, (5.3.5) holds if and only if for every $v \in V \setminus \text{supp} \mathcal{H}^{(b)}_u$ and $w \in \text{supp} \mathcal{H}^{(b)}_u$,

$$(-1)^n \lambda_{uw}^{n-1} g(w) f(v) + \frac{f(u)g(w)}{f(v)} e_v, e_{\lambda_n} = \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \lambda_{uu}^{j-1} n^{n-j-1} = 0. \quad (5.3.8)$$

Assume that $\text{supp}(g) \cap \{v \in \text{supp} \mathcal{H}^{(b)}_u : \lambda_{uv} \neq 0\} = \emptyset$. In this case, $(f \otimes g) D^{(b)}_{\lambda_n} = 0$. Hence, by (5.3.6), $L_k = 0$ for $k \geq 2$. Let $v \in \text{supp}(g)$, $h_1 = e_v = k_1$ in (5.3.4). Then

$$\langle L_1 e_v, L_{n-1} e_v \rangle = 0, \quad n \geq 2,$$
which is possible only if \( n \geq 3 \). In this case, (5.3.4) holds if and only if

\[
\{ \lambda_{uv} : v \in \text{supp} \mathcal{H}_u^{(b)} \} \subseteq \mathbb{R}.
\]

Further, (5.3.5) holds if and only if

\[
\langle (N_{\lambda_u}^{(b)})^{n-1} e_w, L_1 e_v \rangle = 0, \ v \in \text{supp} \mathcal{H}_u^{(b)}, \ w \in V \setminus \text{supp} \mathcal{H}_u^{(b)},
\]

which is possible if and only if either \( f(u) = 0 \) or \( \lambda_{uu} = 0 \). This completes the verification of (i).

Assume next that \( \text{supp}(g) \cap \{ v \in \text{supp} \mathcal{H}_u^{(b)} : \lambda_{uv} \in \mathbb{R} \setminus \{0\} \} \neq \emptyset \). In this case, there exists \( \eta \in \text{supp}(g) \) such that \( \lambda_{u\eta} \in \mathbb{R} \setminus \{0\} \). If possible, then assume that \( W_{f,g} \) is \( n \)-symmetric. By (5.3.7) with \( v = \eta \),

\[
(\lambda_{u\eta} - \bar{\lambda}_{u\eta})^n + \frac{|g(\eta)|^2}{|\lambda_{u\eta}|^2} \| f \|^2 (\lambda_{u\eta} - \bar{\lambda}_{u\eta})^n - (\lambda_{u\eta} - \bar{\lambda}_{u\eta}) = 0.
\]

However, \( \lambda_{u\eta} \in \mathbb{R} \setminus \{0\} \), and hence \( n \) is necessarily an odd integer. Further, since \( N_{\lambda_u}^{(b)} \) is also \( n \)-symmetric, by (5.3.1) and (5.3.2), \( \lambda_{uu} \in \mathbb{R} \) and

\[
\lambda_{uu} = 0 \text{ or } \lambda_{uw} = 0 \text{ for every } w \in A_u \setminus \{u\}.
\]

(5.3.9)

By (5.3.8) with \( w = \eta \), for any \( v \in V \setminus (\text{supp} \mathcal{H}_u^{(b)} \cup A_u) \), \( f(v) = 0 \). Further, if \( \lambda_{uu} \neq 0 \), then by (5.3.9), \( e_{\lambda_{uu}u} = \lambda_{uu}e_u \), which by assumption is orthogonal to \( f \). This forces \( f(u) \) to be equal to 0. In that case, by (5.3.8) with \( w = \eta \), \( f(v) = 0 \) for all \( v \in A_u \). This is not possible since \( f \neq 0 \). Thus \( \lambda_{uu} = 0 \) and \( f(u) \neq 0 \). Once again, by (5.3.8) with \( v = u \) and \( w = \eta \),

\[
(-1)^n \lambda_{uu}^{n-1} + \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} \lambda_{uu}^j = 0.
\]

It follows that \( (\lambda_{uu} - \lambda_{u\eta})^n = \lambda_{uu}^n \). This is not possible, since \( n \) is an odd integer and \( \lambda_{uu}, \lambda_{u\eta} \in \mathbb{R} \setminus \{0\} \). Thus we arrive at a contradiction to the assumption that \( W_{f,g} \) is \( n \)-symmetric. \( \square \)

Remark 5.8 Let \( g : \text{supp} \mathcal{H}_u^{(b)} \to \mathbb{C} \) be given. Then \( W_{f,g} \) is symmetric if and only if \( \{ \lambda_{uv} : v \in \text{supp} \mathcal{H}_u^{(b)} \} \) is contained in \( \mathbb{R} \), \( g = 0 \) and \( \lambda_{uw} = 0 \) for \( v \in A_u \setminus \{u\} \), where \( A_u \) is as given in (3.2.15). This may be concluded from (5.3.4), (5.3.5) (with \( n = 1 \)) and Proposition 5.5.

We are now in a position to present a proof of Theorem 1.5 (recall the notations \( U_u^{(b)}, g_x, W_{w,x} \) as introduced in the Prologue).
**Proof of Theorem 1.5** Note that $\mathcal{H}_u^{(b)}$ is nothing but $\mathcal{H}_u^{(b)}$, while $W_{w,x}$ is a rank one extension of a weighted join operator in $\ell^2(V)$. Clearly, $W_{w,x}$ is densely defined with $\{e_v : v \in V\}$ contained in $\mathcal{D}(W_{w,x})$. By (1.3.6) and Theorem 4.15(i),

$$\sigma_p(W_{w,x}) = \{d_v - d_u : v \in U_u^{(b)} \cup \{u\}\}.$$  

Let us find conditions on $x \in \mathbb{R}$ which ensure that $W_{w,x}$ satisfies the compatibility condition I. To see that, note first that $\sigma_p(W_{w,x})$ is a closed subset of $\mathbb{C}$ and $\mu_0 = -d_u - 1 \notin \sigma_p(W_{w,x})$. By assumption, $(\text{Des}(u), E_u)$ is a narrow tree of width $m$, and hence by (1.3.6) and (1.3.7), we have the estimate

$$\sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} \frac{|g_0(v)|^2}{|\lambda_{uv} - \mu_0|^2} \leq m \sum_{n=0}^{\infty} \frac{n^{2x}}{(n + 1)^2}.$$  \hspace{1cm} (5.3.10)

Since $\text{card}(\text{Des}(u)) = \aleph_0$, we must have

$$\sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} \frac{|g_0(v)|^2}{|\lambda_{uv} - \mu_0|^2} \geq \sum_{n=0}^{\infty} \frac{n^{2x}}{(n + 1)^2}.$$  

It follows from the last two estimates that $W_{w,x}$ satisfies the compatibility condition I if and only if $x < 1/2$. It now follows from Corollary 4.18 that $\sigma(W_{w,x})$ is a proper closed subset of $\mathbb{C}$ if and only if $x < 1/2$. The conclusion in (i), (ii) and (iii) now follow from Theorem 4.11 and Corollary 4.18. Since the spectrum of $W_{w,x}$ is a subset of $\{-d_u + k : k \in \mathbb{N}\}$, parts (iv) and (v) may be deduced from Propositions 5.1, 5.2 and 5.4. Finally, part (vi) may be deduced from Corollary 4.20. \hfill $\square$

**Remark 5.9** Note that the conclusion of Theorem 1.5 may not hold in case $(\text{Des}(u), E_u)$ is not narrow. For instance, if $\mathcal{T}$ is the binary tree, then an examination of (5.3.10) shows that

$$\sum_{v \in \text{supp } \mathcal{H}_u^{(b)}} \frac{|g_0(v)|^2}{|\lambda_{uv} - \mu_0|^2} = \infty.$$  

In this case, $\sigma(W_{w,0}) = \mathbb{C}$ (see Corollary 4.18). Finally, note that by Proposition 5.5, the weighted join operator $W_{\lambda u}^{(b)}$ with weight system given by (1.3.6) is strictly 3-symmetric, while by Proposition 5.7(ii), its rank one extension $W_{w,x}, x \in \mathbb{R}$ is never $n$-symmetric.

### 6 Weighted Join Operators on Rootless Directed Trees

In this section, we extend the notion of join operation at a given base point to rootless directed trees and study the associated weighted join operators. This is achieved by introducing a partial order relation on a rootless directed tree.
6.1 Semigroup Structures on Extended Rootless Directed Trees

Let $\mathcal{T} = (V, E)$ be a rootless directed tree and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. Fix $u, v \in V_\infty$. Define

$$u \leq v \text{ if there exists a directed path } [u, v] \text{ from } u \text{ to } v.$$ 

Note that $\leq$ defines a partial order on $V_\infty$. Further, $\leq$ is anti-symmetric, since $\mathcal{T}$ is a directed tree. Moreover, since $v \leq \infty$ for every $v \in V_\infty$, $\infty$ can be considered as a maximal element of $\mathcal{T}_\infty$, whereas $\mathcal{T}_\infty$, being rootless, has no minimal element. One may now define the join operation $u \sqcup v$ on $\mathcal{T}_\infty$ by setting

$$u \sqcup v = \begin{cases} u & \text{if } v \leq u, \\ v & \text{if } u \leq v, \\ \infty & \text{otherwise.} \end{cases}$$

As in the case of rooted directed trees (see Lemma 2.4), $(V_\infty, \sqcup)$ is a commutative semigroup admitting $\infty$ as an absorbing element.

Let us now define the meet operation. Let $u \in V$ and $v \in V_\infty$. Note that by [38, Proposition 2.1.4] and the definition of the extended directed tree, there exists $w_0 \in V$ such that $\{u, v\} \subseteq \text{Des}(w_0)$. Thus $\text{par}^{(n)}(u) = w_0 = \text{par}^{(m)}(v)$ for some $m, n \in \mathbb{N}$. In particular, the set $\text{par}(u, v)$ given by

$$\text{par}(u, v) := \{w \in V : \text{par}^{(n)}(u) = w = \text{par}^{(m)}(v) \text{ for some } m, n \in \mathbb{N}\}$$

is non-empty. Further, $w \leq u$ for every $w \in \text{par}(u, v)$. Moreover, it is totally ordered, that is, for any $w_1, w_2 \in \text{par}(u, v)$, $w_1 \leq w_2$ or $w_2 \leq w_1$. This follows since each vertex in $V$ has a unique parent. One may now define $u \sqcap v$ by

$$u \sqcap v = \max ([w_0, u] \cap [w_0, v]), \quad (6.1.1)$$

where $w_0$ is any element in $\text{par}(u, v)$. Note that

$$u \sqcap v \in \text{Asc}(u) \cap \text{Asc}(v). \quad (6.1.2)$$

Since $\text{par}(u, v)$ is totally ordered, $u \sqcap v$ is independent of the choice of $w_0$. Further, we set $\infty \sqcap \infty = \infty$. Once again, $(V_\infty, \sqcap)$ is a commutative semigroup admitting identity element as $\infty$ (cf. Lemma 2.8). We leave the verification to the reader.

Before we define the join operation at a given base point, we present a decomposition of the extended directed tree.

**Lemma 6.1** Let $\mathcal{T} = (V, E)$ be a rootless directed tree and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. Then, for any $u \in V$, we have

$$V_\infty = \bigsqcup_{j=0}^{\infty} (\text{Des}^{(j)}(u) \setminus \text{Des}^{(j-1)}(u)).$$
where we used the convention that \( \text{Des}(\text{par}^{(-1)}(u)) = \emptyset \).

**Proof** Let \( v \in V \). By (6.1.1), there exists a non-negative integer \( j \) such that \( u \cap v = \text{par}^{(j)}(u) \). It now follows from (6.1.2) and the uniqueness of \( u \cap v \) that

\[
v \in \text{Des}(\text{par}^{(j)}(u)) \setminus \text{Des}(\text{par}^{(j-1)}(u)).
\]

Consequently, we get the inclusion

\[
V \subseteq \bigcup_{j=0}^{\infty} \text{Des}(\text{par}^{(j)}(u)) \setminus \text{Des}(\text{par}^{(j-1)}(u)).
\]

Since \( \infty \in \text{Des}(u) \), we get the desired equality. \( \square \)

**Definition 6.2 (Join operation at a base point)** Let \( \mathcal{T} = (V, E) \) be a rootless directed tree and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). Fix \( b \in V_\infty \) and let \( u, v \in V_\infty \). Define the binary operation \( \sqcup_b \) on \( V_\infty \) by

\[
u \sqcup_b v = \begin{cases} 
  u \cap v & \text{if } u, v \in \text{Asc}(b), \\
  u & \text{if } v = b, \\
  v & \text{if } b = u, \\
  u \sqcup v & \text{otherwise}.
\end{cases}
\]

Note that \( (V_\infty, \sqcup_b) \) is a commutative semigroup admitting identity element as \( b \). Further, \( \sqcup_\infty = \sqcap \). The table for join operation \( u \sqcup_b v \) at the base point \( b \) for a rootless directed tree is identical with Table 1.

**Definition 6.3** Let \( \mathcal{T} = (V, E) \) be a rootless directed tree and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). Fix \( u, b \in V_\infty \) and the weight system \( \lambda_u = \{\lambda_{uv}\}_{v \in V_\infty} \) of complex numbers, we define the **weighted join operator** \( W_{\lambda_u}^{(b)} \) \( \text{(based at } b) \) on \( \mathcal{T} \) by

\[
\mathcal{D}(W_{\lambda_u}^{(b)}) := \{ f \in \ell^2(V) : A_u^{(b)} f \in \ell^2(V) \},
\]

\[
W_{\lambda_u}^{(b)} f := A_u^{(b)} f, \quad f \in \mathcal{D}(W_{\lambda_u}^{(b)}),
\]

where \( A_u^{(b)} \) is the mapping defined on complex functions \( f \) on \( V \) by

\[
(A_u^{(b)} f)(w) := \sum_{v \in M_u^{(b)}(w)} \lambda_{uv} f(v), \quad w \in V
\]

with \( M_u^{(b)}(w) \) given by

\[
M_u^{(b)}(w) := \{ v \in V : u \sqcup_b v = w \}.
\]
Remark 6.4 As in the rooted case, it can be seen that $\mathcal{D}(W_{\lambda_0}^{(b)})$ forms a subspace of $\ell^2(V)$. Clearly, $e_v \in \mathcal{D}(W_{\lambda_0}^{(b)})$ and $(W_{\lambda_0}^{(b)} e_v)(w) = \lambda_{uv} e_{u \downarrow v}(w)$, $w \in V$. Thus

$$\mathcal{D}_V := \text{span} \{e_v : v \in V\} \subseteq \mathcal{D}(W_{\lambda_0}^{(b)}), \quad W_{\lambda_0}^{(b)} \mathcal{D}_V \subseteq \mathcal{D}_V.$$ 

Thus all positive integral powers of $W_{\lambda_0}^{(b)}$ are densely defined and the Hilbert space adjoint $W_{\lambda_0}^{(b)*}$ of $W_{\lambda_0}^{(b)}$ is defined.

To get an idea about the structure of weighted join operators on rootless directed trees, let us discuss one example.

Example 6.5 (With one branching vertex) Let $\mathcal{T}_3$ denote the directed tree as shown in Fig. 8 (see [38, Eqn (6.2.10)]). Consider the ordered orthonormal basis

$$\{e_{3n} : n \in \mathbb{N}\} \cup \{e_{3n+2} : n \in \mathbb{N}\} \cup \{e_{3n+1} : n \in \mathbb{N}\}$$

of $\ell^2(V)$. The matrix representation of the weighted join operator $W_{\lambda_0}^{(m)}$ and weighted meet operator $W_{\lambda_m}^{(\infty)}$ on $\mathcal{T}_3$ are given by

$$W_{\lambda_0}^{(m)} = \begin{pmatrix}
\ldots & 0 & \ldots \\
\vdots & & \vdots \\
\ldots & 0 & \lambda_{m3} & \lambda_{m0} & \lambda_{m2} & \ldots & \lambda_{mm} & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \ldots & 0 & \lambda_{mm+3} & 0 & \ldots \\
\vdots & & \vdots & & 0 & \lambda_{mm+6} & 0 \\
\vdots & & \vdots & & 0 & \lambda_{mm+9} & \ddots \\
\end{pmatrix} \oplus \mathbf{0},$$

Fig. 8 The rootless directed tree $\mathcal{T}_3$ with prescribed vertex $m$
\[
W_{\lambda}^{(\infty)} = \begin{pmatrix}
\ldots & \lambda_{m3} & & & \\
& \lambda_{m0} & 0 & \cdots & \cdots & 0 & \lambda_{m1} & \lambda_{m4} & \cdots \\
& 0 & \lambda_{m2} & 0 & \cdots & \\
& \vdots & 0 & \cdots & 0 & \cdots \\
& \vdots & \lambda_{mm-3} & 0 & \cdots & 0 & \lambda_{mm} & \lambda_{mm+3} & \cdots \\
& \vdots & 0 & 0 & \cdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\]

These expressions should be compared with the matrix representations of weighted join operator and weighted meet operator on \( T_2 \) discussed in Example 3.4.

6.2 A Decomposition Theorem and Spectral Analysis

Note that \( W_{\lambda}^{(\infty)} = D_{\lambda} \), the diagonal operator with diagonal entries \( \lambda \). The structure of the weighted join operator \( W_{\lambda}^{(\infty)} \), \( u \neq b \) turns out to be quite involved in case of rootless directed trees. We present below a counterpart of Theorem 3.13 for weighted join operators on rootless directed trees.

**Theorem 6.6** Let \( T = (V, E) \) be a rootless directed tree and let \( T_{\infty} = (V_{\infty}, E_{\infty}) \) be the extended directed tree associated with \( T \). For \( b \in V_{\infty} \), \( u \in V \setminus \{b\} \) and the weight system \( \lambda_{uv} \in \mathbb{V}_{\infty} \) of complex numbers, let \( D_{\lambda} \) be the diagonal operator on \( T \) and let \( W_{\lambda}^{(\infty)} \) be the weighted join operator on \( T \). Then the following hold:

(i) Assume that \( b \in V \). Consider the subspace \( \mathcal{H}_{\lambda}^{(b)} \) of \( \ell^2(V) \) given by

\[
\mathcal{H}_{\lambda}^{(b)} = \begin{cases} 
\ell^2(\text{Asc}(u) \cup \text{Des}_b(u)) & \text{if } b \in \text{Des}(u), \\
\ell^2(\text{Des}_u(u)) & \text{otherwise}. 
\end{cases} \tag{6.2.1}
\]

Then the weighted join operator \( W_{\lambda}^{(b)} \) admits the decomposition

\[
W_{\lambda}^{(b)} = D_{\lambda}^{(b)} \oplus N_{\lambda}^{(b)} \text{ on } \ell^2(V) = \mathcal{H}_{\lambda}^{(b)} \oplus (\ell^2(V) \ominus \mathcal{H}_{\lambda}^{(b)}), \tag{6.2.2}
\]

where \( D_{\lambda}^{(b)} \) is a densely defined diagonal operator in \( \mathcal{H}_{\lambda}^{(b)} \) and \( N_{\lambda}^{(b)} \) is a rank one densely defined linear operator on \( \ell^2(V) \ominus \mathcal{H}_{\lambda}^{(b)} \) with invariant domain. Further, \( D_{\lambda}^{(b)} \) and \( N_{\lambda}^{(b)} \) are given by

\[
D_{\lambda}^{(b)} = D_{\lambda}\big|_{\mathcal{H}_{\lambda}^{(b)}}, \quad \mathcal{D}(D_{\lambda}^{(b)}) = \{ f \in \mathcal{H}_{\lambda}^{(b)} : D_{\lambda} f \in \mathcal{H}_{\lambda}^{(b)} \}, \tag{6.2.3}
\]

\[
N_{\lambda}^{(b)} = e_u \otimes e_{\lambda, A_u}, \tag{6.2.4}
\]
where the subset $A_u$ of $V$ is given by

$$A_u = \begin{cases} [u, b] & \text{if } b \in \text{Des}(u), \\ \text{Asc}(u) \cup \{b, u\} & \text{otherwise.} \end{cases}$$

(ii) Assume that $b = \infty$. Consider the orthogonal decomposition of $\ell^2(V)$ (as ensured by Lemma 6.1) given by

$$\ell^2(V) = \bigoplus_{j=0}^{\infty} \ell^2(\text{Des}(u_j) \setminus \text{Des}(u_{j-1})), $$

where $\text{Des}(u_{-1}) = \emptyset$ and $u_j := \text{par}^{(j)}(u)$, $j \in \mathbb{N}$. Further, with respect to the above decomposition, $W^{(b)}_{\lambda_u}$ decomposes as

$$\mathcal{D}(W^{(b)}_{\lambda_u}) = \bigoplus_{j=0}^{\infty} \mathcal{D}(e_{u_j} \otimes e_{\lambda_u, \text{Des}(u_j) \setminus \text{Des}(u_{j-1})}),$$

$$W^{(b)}_{\lambda_u} = \bigoplus_{j=0}^{\infty} e_{u_j} \otimes e_{\lambda_u, \text{Des}(u_j) \setminus \text{Des}(u_{j-1})}. \quad (6.2.5)$$

**Proof** Let $V_u$ be the complement of $\text{Des}(u) \sqcup \text{Asc}(u)$ in $V$. We divide the proof into three cases.

**Case I.** $b \notin \text{Des}(u)$:
Consider the following decomposition of $V$:

$$V = \text{Des}_u(u) \sqcup A_u \sqcup \left(V_u \setminus \{b\}\right).$$

Thus

$$\ell^2(V) = \ell^2(\text{Des}_u(u)) \oplus \ell^2(A_u) \oplus \ell^2(V_u \setminus \{b\}).$$

Note that $\ell^2(\text{Des}_u(u))$, $\ell^2(A_u)$ and $\ell^2(V_u \setminus \{b\})$ are invariant subspaces of $W^{(b)}_{\lambda_u}$. We claim that the weighted join operator $W^{(b)}_{\lambda_u}$ is given by

$$\mathcal{D}(W^{(b)}_{\lambda_u}) = \mathcal{D}(D_{\lambda_u} |_{\ell^2(\text{Des}_u(u))}) \oplus \mathcal{D}(e_u \otimes e_{\lambda_u, A_u}) \oplus \ell^2(V_u \setminus \{b\}),$$

$$W^{(b)}_{\lambda_u} = D_{\lambda_u} |_{\ell^2(\text{Des}_u(u))} \oplus e_u \otimes e_{\lambda_u, A_u} \oplus 0.$$

To see the above decomposition, let $f \in \ell^2(V)$ be of the form $f_1 \oplus f_2 \oplus f_3$ with $f_1 \in \ell^2(\text{Des}_u(u))$, $f_2 \in \ell^2(A_u)$, $f_3 \in \ell^2(V_u \setminus \{b\})$. Note that $f \in \mathcal{D}(W^{(b)}_{\lambda_u})$ if and only if $W^{(b)}_{\lambda_u} f \in \ell^2(V)$, where $W^{(b)}_{\lambda_u} f$ takes the form
\[
\sum_{v \in \text{Des}_u(u)} f_1(v) \lambda_{u,v} e_{u \downarrow_b v} \oplus \sum_{v \in A_u} f_2(v) \lambda_{u,v} e_{u \downarrow_b v} \oplus \sum_{v \in V_u \setminus \{b\}} f_3(v) \lambda_{u,v} e_{u \downarrow_b v}
\]

It follows that \( f \in \mathcal{D}(W^{(b)}_{\lambda_u}) \) if and only if
\[
f_1 \in \mathcal{D}(D_{\lambda_u \mid _{\ell^2(\text{Asc}(u))}}), \quad f_2 \in \mathcal{D}(e_u \ominus e_{\lambda_u,\lambda_u}), \quad f_3 \in \ell^2(V_u \setminus \{b\}) .
\]

This yields the desired orthogonal decomposition of \( W^{(b)}_{\lambda_u} \).

**Case II.** \( b \in \text{Des}(u) \setminus \{\infty\} \):

Consider the following decomposition of \( V \):
\[
V = (\text{Asc}(u) \cup \text{Des}_b(u)) \sqcup A_u \sqcup V_u .
\]

Thus
\[
\ell^2(V) = \ell^2(\text{Asc}(u) \cup \text{Des}_b(u)) \oplus \ell^2(A_u) \oplus \ell^2(V_u).
\]

Note that \( \ell^2(\text{Asc}(u) \cup \text{Des}_b(u)) \), \( \ell^2(A_u) \) and \( \ell^2(V_u) \) are invariant subspaces of \( W^{(b)}_{\lambda_u} \).

As in the previous case, one can verify that the weighted join operator \( W^{(b)}_{\lambda_u} \) is given by
\[
\mathcal{D}(W^{(b)}_{\lambda_u}) = \mathcal{D}(D_{\lambda_u \mid _{\ell^2(\text{Asc}(u))}}) \oplus \mathcal{D}(e_u \ominus e_{\lambda_u,\lambda_u}) \oplus \ell^2(V_u),
\]
\[
W^{(b)}_{\lambda_u} = D_{\lambda_u \mid _{\ell^2(\text{Asc}(u))}} \oplus e_u \ominus e_{\lambda_u,\lambda_u} \oplus 0.
\]

**Case III.** \( b = \infty \):

The decomposition (6.2.5) follows from
\[
W^{(b)}_{\lambda_u} e_v = \lambda_{u,v} e_{u \downarrow v} = \lambda_{u,v} e_{u_j}, \quad v \in \text{Des}(u_j) \setminus \text{Des}(u_{j-1}).
\]

This completes the proof. \(\square\)

Here are some immediate consequences of Theorem 6.6.

**Corollary 6.7 (Dichotomy)** Let \( \mathcal{T} = (V,E) \) be a rootless directed tree and let \( \mathcal{T}_\infty = (V_\infty,E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( b,u \in V \) and the weight system \( \lambda_u = \{ \lambda_{u,v} \}_{v \in V_\infty} \) of complex numbers, the weighted join operator \( W^{(b)}_{\lambda_u} \) on \( \mathcal{T} \) is at most rank one (possibly unbounded) perturbation of a diagonal operator, while the weighted meet operator \( W^{(\infty)}_{\lambda_u} \) on \( \mathcal{T} \) is an infinite rank operator provided \( \lambda_u \subseteq \mathbb{C} \setminus \{0\} \).
The orthogonal decomposition (6.2.2) of $W_{\lambda_u}^{(b)}$, as ensured by Theorem 6.6, is given by the triple $(D_{\lambda_u}^{(b)}, N_{\lambda_u}^{(b)}, \mathcal{H}_u^{(b)})$, where $\mathcal{H}_u^{(b)}$, $D_{\lambda_u}^{(b)}$ and $N_{\lambda_u}^{(b)}$ are given by (6.2.1), (6.2.3) and (6.2.4) respectively.

**Corollary 6.8** Let $\mathcal{T} = (V, E)$ be a rootless directed tree and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $b \in V_\infty$, $u \in V \setminus \{b\}$ and the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers, let $D_{\lambda_u}$ be the diagonal operator on $\mathcal{T}$ and let $W_{\lambda_u}^{(b)}$ be a weighted join operator on $\mathcal{T}$. Then the following holds true:

(i) If $b \notin \text{Des}(u)$, then $W_{\lambda_u}^{(b)}$ is bounded if and only if

$$\lambda_u \in \ell^\infty(\text{Des}(u)) \text{ and } \sum_{v \in \text{Asc}(u)} |\lambda_{uv}|^2 < \infty.$$  

(ii) If $b \in \text{Des}(u) \setminus \{\infty\}$, then $W_{\lambda_u}^{(b)}$ is bounded if and only if $\lambda_u \in \ell^\infty(\text{Asc}(u) \cup \text{Des}_b(u))$.

(iii) If $b = \infty$, then $W_{\lambda_u}^{(b)}$ is bounded if and only if

$$\sup_{j \geq 0} \sum_{v \in \text{Des}(u_j) \setminus \text{Des}(u_{j-1})} |\lambda_{uv}|^2 < \infty,$$

where $\text{Des}(u_{-1}) = \emptyset$ and $u_j := \text{par}(j)(u)$ for $j \in \mathbb{N}$.

(iv) If $b \notin \text{Des}(u)$ and $\sum v \in \text{Asc}(u) |\lambda_{uv}|^2 = \infty$, then $W_{\lambda_u}^{(b)}$ is not closable.

**Proof** The desired conclusions in (i)–(iii) follow from Theorem 6.2.2, while (iv) follows from (i) and Lemma 1.3. \qed

**Remark 6.9** Let us briefly discuss the spectral picture for a weighted join operator $W_{\lambda_u}^{(b)}$ on the rootless directed tree $\mathcal{T}$. Consider the orthogonal decomposition $(D_{\lambda_u}^{(b)}, N_{\lambda_u}^{(b)}, \mathcal{H}_u^{(b)})$ of $W_{\lambda_u}^{(b)}$. In case $b \in \text{Des}(u) \setminus \{\infty\}$, the operator $N_{\lambda_u}^{(b)}$ in the decomposition of $W_{\lambda_u}^{(b)}$ is bounded. Hence the spectral picture of $W_{\lambda_u}^{(b)}$ can be described as in the rooted case (see Theorem 4.15). We leave the details to the reader. In case $b \notin \text{Des}(u)$ and $\sum v \in \text{Asc}(u) |\lambda_{uv}|^2 < \infty$, $N_{\lambda_u}^{(b)}$ is bounded and the same remark as above is applicable. Suppose now that $b \notin \text{Des}(u)$ and $\sum v \in \text{Asc}(u) |\lambda_{uv}|^2 = \infty$. Then, by Corollary 6.8, $N_{\lambda_u}^{(b)}$ is unbounded. Hence, by Lemma 1.3,

$$\sigma(W_{\lambda_u}^{(b)}) = \mathbb{C}, \quad \sigma_p(W_{\lambda_u}^{(b)}) = \sigma_p(D_{\lambda_u}^{(b)}) \cup \{0, \lambda_{uu}\}.$$  

The verification of the following is similar to that of Corollary 3.16, and hence we skip its verification.

**Corollary 6.10** Let $\mathcal{T} = (V, E)$ be a rootless directed tree and let $\mathcal{T}_\infty = (V_\infty, E_\infty)$ be the extended directed tree associated with $\mathcal{T}$. For $u \in V$ and $b \in V \setminus \{u\}$, consider the weight system $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty}$ of complex numbers and let $W_{\lambda_u}^{(b)}$ denote the weighted
join operator on $\mathcal{T}$. Consider the orthogonal decomposition $(D^{(b)}_{\lambda u}, N^{(b)}_{\lambda u}, H^{(b)}_{\lambda u})$ of $W^{(b)}_{\lambda u}$ as ensured by Theorem 6.6. Then the following statements hold:

(i) If $\lambda_{uu} = 0$, then $W^{(b)}_{\lambda u}$ is a complex Jordan operator of index 2 provided $D^{(b)}_{\lambda u}$ belongs to $B(H^{(b)}_{\lambda u})$ or $N^{(b)}_{\lambda u}$ belongs to $B(\ell^2(V) \ominus H^{(b)}_{\lambda u})$.

(ii) If $\lambda_{uu} \neq 0$ and $N^{(b)}_{\lambda u} \in B(\ell^2(V) \ominus H^{(b)}_{\lambda u})$, then $W^{(b)}_{\lambda u}$ admits a bounded Borel functional calculus.

As in the case of rooted directed trees (see Definition 4.1), one may introduce the rank one extension $W^{(b)}_{f,g}$ of the weighted join operator $W^{(b)}_{\lambda u}$ on a rootless directed tree in a similar fashion. In case the operator $N^{(b)}_{\lambda u}$ appearing in the decomposition of $W^{(b)}_{\lambda u}$ is unbounded, it turns out (due to the fact that $D^{(b)}_{\lambda u}$ has no “good influence” on $N^{(b)}_{\lambda u}$) that $W^{(b)}_{f,g}$ is not even closable. On the other hand, in case $N^{(b)}_{\lambda u}$ is bounded, one can obtain counterparts of Theorems 4.11, 4.15 and Propositions 5.1, 5.2 for rank one extensions of weighted join operators on rootless directed trees along similar lines. We leave the details to the reader.

7 Rank One Perturbations

The considerations in Sect. 4 around the notion of rank one extensions of weighted join operators were mainly motivated by the graph-model developed in earlier sections. Some of these can be replicated in a general set-up simply by replacing the vertex set of the underlying rooted directed tree by a countably infinite directed set. The results in this section give a few glimpses of this general scenario. In particular, we discuss the role of some compatibility conditions (differing from compatibility conditions I and II as introduced in Sect. 4) in the sectoriality of rank one perturbations of diagonal operators. We also discuss the sectoriality of the form-sum of the form associated with a sectorial diagonal operator and a form associated with not necessarily square-summable functions $f$ and $g$.

7.1 Operator-Sum

Throughout this section, $J$ denotes a countably infinite directed set and let $\{e_j : j \in J\}$ be the standard orthonormal basis of $\ell^2(J)$. Let $D_\lambda$ stand for the diagonal operator in $\ell^2(J)$ with diagonal entries $\lambda = \{\lambda_j : j \in J\}$ given by

$$D_\lambda e_j = \lambda_j e_j, \quad j \in J.$$

The following main result of this section shows that a compatibility condition ensures the sectoriality of the operator-sum of a sectorial diagonal operator and an unbounded rank one operator.
Theorem 7.1 Let $D_\lambda$ be a sectorial operator in $\ell^2(J)$ and let $f \in \ell^2(J)$. Let $g : J \to \mathbb{C}$ be such that for some $z_0 \in \rho(D_\lambda)$,
\[
\sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j - z_0|^2} < \infty.
\]
(7.1.1)
Then $D_\lambda + f \circ g$ defines a sectorial operator in $\ell^2(J)$ with the domain $\mathcal{D}(D_\lambda)$.

Clearly, Theorem 7.1 generalizes Proposition 5.1. In its proof, we need a couple of observations of independent interests. The first of which characterizes the $B$-boundedness of $A$ in terms of the strict contractivity of $B(A - z)^{-1}$ for some $z \in \rho(A)$, where $A$ is a normal operator satisfying certain growth condition.

Proposition 7.2 Let $A$ be a normal operator in $\mathcal{H}$ and let $B$ be a linear operator in $\mathcal{H}$ with $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. If there exists $z \in \rho(A)$ such that $\|B(A - z)^{-1}\| < 1$, then
\[
\|Bx\| \leq a\|Ax\| + b\|x\|, \quad x \in \mathcal{D}(A),
\]
(7.1.2)
where $a = \|B(A - z)^{-1}\|$ and $b \in (0, \infty)$. Conversely, if there exist $a \in (0, 1)$ and $b \in (0, \infty)$ such that (7.1.2) holds and if for some $\theta \in \mathbb{R}$,
\[
\max\{|\mu|, n\} \leq |\mu - e^{i\theta}n|, \quad n \in \mathbb{N}, \mu \in \sigma(A),
\]
(7.1.3)
then $\|B(A - z)^{-1}\| < 1$ for some $z \in \rho(A)$.

Remark 7.3 There are two particular instances in which (7.1.3) can be ensured.

(i) If $A$ is self-adjoint, then by [64, Corollary 3.14], $\sigma(A) \subseteq \mathbb{R}$, and hence (7.1.3) holds with $\theta = \pm \pi/2$.

(ii) If $A$ is sectorial with vertex at 0, then (7.1.3) holds with $\theta = \pi$.

Proof Note that if $z \in \rho(A)$ is such that $\|B(A - z)^{-1}\| < 1$, then for any $x \in \mathcal{D}(A)$
\[
\|Bx\| = \|B(A - z)^{-1}(A - z)x\| \leq a\|Ax\| + b\|x\|,
\]
where $a = \|B(A - z)^{-1}\|$ and $b = \|z\|\|B(A - z)^{-1}\|$. This yields (7.1.2).

To see the converse, suppose that (7.1.2) and (7.1.3) hold. For any $z \in \rho(A)$, $(A - z)^{-1}$ is a bounded operator on $\mathcal{H}$ with range equal to $\mathcal{D}(A)$. Thus (7.1.2) becomes
\[
\|B(A - z)^{-1}y\| \leq a\|A(A - z)^{-1}y\| + b\|(A - z)^{-1}y\|, \quad z \in \rho(A), \quad y \in \mathcal{H}.
\]
(7.1.4)
Let $E(\cdot)$ denote the spectral measure of $A$ and let $n \in \mathbb{N}$. Clearly, by (7.1.3), $e^{i\theta}n \in \rho(A)$. It follows from the spectral theorem [61, Theorem 13.24] and (7.1.4), that for any $y \in \mathcal{H}$,
\[ \| B(A - e^{i\theta}n)^{-1}y \| \leq a \sqrt{\int_{\sigma(A)} \left| \frac{\mu}{\mu - e^{i\theta}n} \right|^2 \| E(d\mu)(y) \|^2 + b \| (A - e^{i\theta}n)^{-1}y \| } \]

\[ \leq a \| y \| + \frac{b}{n} \| y \|. \]

(7.1.3)

Thus, for sufficiently large integer \( n \),

\[ \| B(A - e^{i\theta}n)^{-1} \| \leq a + \frac{b}{n} < 1. \]

This completes the proof. \( \square \)

We need one more fact in the proof of Theorem 7.1 (cf. Theorem 4.15(iii)).

**Proposition 7.4** Let \( D_\lambda \) be a sectorial operator in \( \ell^2(J) \) and let \( f \in \ell^2(J) \). Let \( g : J \to \mathbb{C} \) be such that for some \( z_0 \in \rho(D_\lambda) \), (7.1.1) holds. Then, for any \( z \in \rho(D_\lambda) \), \( G_z := f \otimes g(D_\lambda - z)^{-1} \) is a Hilbert-Schmidt integral operator with square-summable kernel

\[ K_z(j, k) := \frac{f(j) \overline{g(k)}}{\lambda_k - z}, \quad j, k \in J. \]

Moreover, there exists a sequence \( \{z_n\}_{n \in \mathbb{N}} \subseteq \rho(D_\lambda) \) such that

\[ \lim_{n \to \infty} \| G_{z_n} \|_2 = 0, \]

where \( \| \cdot \|_2 \) denotes the Hilbert-Schmidt norm.

**Proof** Let \( z \in \rho(D_\lambda) \). Then, as in the proof of Theorem 4.15, it can be seen that \( G_z \) is a Hilbert-Schmidt integral operator with kernel \( K_z \in \ell^2(J \times J) \). Moreover,

\[ \| G_z \|_2^2 = \| f \|_2^2 \sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j - z|^2}. \]

(7.1.5)

On the other hand, it is easily seen that there exists a sequence \( \{z_n\}_{n \in \mathbb{N}} \subseteq \rho(D_\lambda) \) with the only accumulation point at \( \infty \) such that

\[ |\lambda_j - z_0| \leq |\lambda_j - z_n|, \quad n \in \mathbb{N}, \quad j \in J. \]

Using Lebesgue dominated convergence theorem, we see that

\[ \sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j - z_n|^2} \to 0 \quad \text{as} \quad n \to \infty. \]

Hence, by (7.1.5), we obtain the remaining part. \( \square \)
**Proof of Theorem 7.1** As in the proof of Proposition 4.7, it is easily seen that

\[ \mathcal{D}(D_\lambda) \subseteq \mathcal{D}(f \otimes g). \]  

(7.1.6)

Also, by Proposition 7.4, for any \( a \in (0, 1) \), there exists \( z \in \rho(D_\lambda) \) such that the Hilbert-Schmidt norm of \( f \otimes g(D_\lambda - z)^{-1} \) is less than \( a \). Since the operator norm of any Hilbert-Schmidt operator is less than or equal to its Hilbert-Schmidt norm, it follows from Proposition 7.2 that \( f \otimes g \) is \( D_\lambda \)-bounded with \( D_\lambda \)-bound equal to 0. The desired conclusion now follows from [52, Theorem 4.5.7].

The following provides a variant of Corollary 4.12. Since the bounded component \( N^{(b)}_{\lambda_u} \) in the rank one extension \( W_{f,g} \) has no effect in the \( D^{(b)}_{\lambda_u} \)-boundedness of \( f \otimes g \), this variant may be obtained by imitating the proof of Theorem 7.1.

**Corollary 7.5** Let \( \mathcal{T} = (V, E) \) be a rooted directed tree with root \( \text{root} \) and let \( \mathcal{T}_\infty = (V_\infty, E_\infty) \) be the extended directed tree associated with \( \mathcal{T} \). For \( u, b \in V \), consider the weight system \( \lambda_u = \{\lambda_{uv}\}_{v \in V_\infty} \) of complex numbers and let \( W_{f,g} \) be the rank one extension of the weighted join operator \( W^{(b)}_{\lambda_u} \) on \( \mathcal{T} \), where \( f \in \ell^2(V) \otimes \mathcal{H}^{(b)}_u \) is non-zero and \( g : \text{supp} \mathcal{H}^{(b)}_u \rightarrow \mathbb{C} \) is given. Suppose that \( W_{f,g} \) satisfies the compatibility condition \( I \). Then \( W_{f,g} \) decomposes as \( A + B + C \), where \( A, B, C \) are densely defined operators given by (4.2.8) such that \( B + C \) is A-bounded with A-bound equal to 0.

We conclude this section with a brief discussion on some spectral properties of rank one perturbations of the diagonal operator \( D_\lambda \). Assume that there exists \( z_0 \in \rho(D_\lambda) \) such that \( g : J \rightarrow \mathbb{C} \) satisfies (7.1.1). By (7.1.6), \( D_\lambda + f \otimes g \) is a densely defined operator in \( \ell^2(J) \) with the domain \( \mathcal{D}(D_\lambda) \). Let \( \mu \in \mathbb{C} \setminus \lambda \) be an eigenvalue of \( D_\lambda + f \otimes g \). Thus there exists a non-zero vector \( h \) in \( \ell^2(J) \) such that for every \( j \in J \),

\[
\lambda_j h(j) + \left( \sum_{k \in J} h(k) \overline{g(k)} \right) f(j) = \mu h(j)
\]

\[
\implies (\mu - \lambda_j) h(j) = \left( \sum_{k \in J} h(k) \overline{g(k)} \right) f(j)
\]

\[
\implies h(j) = a \frac{f(j)}{\mu - \lambda_j},
\]

where \( a = \sum_{k \in J} h(k) \overline{g(k)} \) is non-zero. Therefore, we have

\[
\sum_{j \in J} \frac{f(j) \overline{g(j)}}{\mu - \lambda_j} = 1.
\]

(7.1.7)

Notice that expression in (7.1.7) is an analytic function in \( \mu \) outside the spectrum of \( D_\lambda \). Also, for \( \mu \in \mathbb{C} \setminus \lambda \) to be an eigenvalue for \( D_\lambda + f \otimes g \), it has to satisfy (7.1.7). Therefore, the set of all eigenvalues of \( D_\lambda + f \otimes g \) outside the set \( \sigma(D_\lambda) \) has to be discrete (cf. [37, Corollary 2.5]). One may argue now as in the proof of Theorem 4.15(iv) using Weyl’s theorem to conclude that \( \sigma_e(D_\lambda + f \otimes g) = \sigma_e(D_\lambda) \).
7.2 Form-Sum

Consider a sectorial diagonal operator \( D_\lambda \) in \( \ell^2(J) \). By [46, Theorem 3.35, Chapter V] \( D_\lambda \) (and hence \( D_\lambda^* \) as well) has a unique square root; let us denote it by \( R_\lambda \) and note that \( R_\lambda \) is a sectorial operator with \( \mathcal{D}(R_\lambda) = \mathcal{D}(R_\lambda^*) \). Consider the form \( Q_R \) given by

\[
Q_R(h, k) := \langle R_\lambda h, R_\lambda^* k \rangle, \quad h, k \in \mathcal{D}(R_\lambda).
\] (7.2.1)

Then having the unbounded form-perturbation of \( Q_R \) by \( Q_f, g \) is to find conditions on \( f, g \) so that the form

\[
Q_{f, g}(h, k) := \sum_{j \in J} h(j) \overline{g(j)} \sum_{j \in J} f(j) \overline{k(j)}
\] (7.2.2)

is well defined for all \( h, k \in \mathcal{D}(R_\lambda) \) and to ensure that the perturbation by \( Q_{f, g} \) is small, (so that an application of [46, Theorems 1.33 and 2.1, Chapter VI] can be made through a choice of large enough \( z \) in the appropriate sector). In such a case, the form-sum \( Q_R + Q_{f, g} \) is closed and defines a sectorial operator with domain contained in the domain of \( Q_R \) (the reader is referred to [46, Chapter VI] for all definitions pertaining to sesquilinear forms in Hilbert spaces). This is made precise in the following theorem.

**Theorem 7.6** Let \( D_\lambda \) be a sectorial diagonal operator in \( \ell^2(J) \) with angle \( \theta \in (0, \pi/2) \) and vertex 0. Let \( f : J \to \mathbb{C} \) and \( g : J \to \mathbb{C} \) be such that for some \( z_0 \in (-\infty, 0) \),

\[
\sum_{j \in J} \frac{|g(j)|^2}{|\sqrt{\lambda_j} - z_0|^2} < \infty, \quad \sum_{j \in J} \frac{|f(j)|^2}{|\sqrt{\lambda_j} - z_0|^2} < \infty,
\] (7.2.3)

where the square root is obtained by the branch cut at the non-positive real axis. Let \( Q_R \) and \( Q_{f, g} \) be as given by (7.2.1) and (7.2.2). Then \( Q_{f, g}(h, k) \) is defined for all \( h, k \in \mathcal{D}(R_\lambda) \). Moreover, the form \( Q_R + Q_{f, g} \) is sectorial and there exists a unique sectorial operator \( T \) in \( \ell^2(J) \) with domain contained in the domain of \( Q_R \) such that

\[
Q_R(h, k) + Q_{f, g}(h, k) = \langle Th, k \rangle, \quad h \in \mathcal{D}(T), \ k \in \mathcal{D}(Q_R).
\]

**Proof** Note that \( z_0 \in \rho(R_\lambda) \), and hence by (7.2.3) and the definition of \( \mathcal{D}(R_\lambda) \), \( Q_{f, g}(h, k) \) is well-defined for all \( h, k \in \mathcal{D}(R_\lambda) \). To see the remaining part, we make some general observations. Notice first that

\[
\Re Q_R(h, h) = \sum_{j \in J} \Re \lambda_j |h(j)|^2, \quad \Im Q_R(h, h) = \sum_{j \in J} \Im \lambda_j |h(j)|^2.
\] (7.2.4)

Further, since \( R_\lambda \) is normal,

\[
\|R_\lambda h\|^2 \geq |Q_R(h, h)|^2, \quad h \in \mathcal{D}(R_\lambda).
\] (7.2.5)
Furthermore, since $D_\lambda$ is a sectorial operator with angle $\theta$,

\[
\begin{align*}
|\Re \lambda_j| & \leq \tan \theta \Re \lambda_j, \quad j \in J, \\
|\Im Q_R(h, h)| & \leq \tan \theta \Re Q_R(h, h), \quad h \in \mathcal{D}(R_\lambda).
\end{align*}
\]  

(7.2.6)

We claim that $Q_R$ is a closed form. It suffices to check that $\Re Q_R$ is closed (see [46, Pg 336]). Let $h \in \ell^2(J)$, $\{h_n\}_{n \in \mathbb{N}} \subseteq \mathcal{D}(Q_R)$ be such that $h_n \to h$ as $n \to \infty$ and $\Re Q_R(h_n - h_m, h_n - h_m) \to 0$ as $n$ and $m$ tend to $\infty$. It follows that

\[
\Re Q_R(h_n - h_m, h_n - h_m) \overset{(7.2.4)}{=} \sum_{j \in J} \Re \lambda_j |h_n(j) - h_m(j)|^2 \\
\overset{(7.2.6)}{=} \frac{1}{\sqrt{1 + \tan^2 \theta}} \sum_{j \in J} |\lambda_j||h_n(j) - h_m(j)|^2 \\
= \frac{1}{\sqrt{1 + \tan^2 \theta}} \sum_{j \in J} |\sqrt{\lambda_j} h_n(j) - \sqrt{\lambda_j} h_m(j)|^2 \\
= \frac{1}{\sqrt{1 + \tan^2 \theta}} \|R_\lambda(h_n - h_m)\|^2.
\]

This shows that $\{R_\lambda(h_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\ell^2(J)$. Thus there exists $g \in \ell^2(J)$ such that $R_\lambda(h_n) \to g$ as $n \to \infty$. Since $R_\lambda$ is closed, $h \in \mathcal{D}(R_\lambda) = \mathcal{D}(Q_R)$ and $g = R_\lambda h$. By (7.2.5), $Q_R(h_n - h, h_n - h) \to 0$ as $n \to \infty$. This completes the verification of the claim.

We next show that $Q_{f,g}$ is $Q_R$-bounded with $Q_R$-bound less than 1. To see this, let $z \leq z_0$ be a negative real number. Note that $|w - z_0| \leq |w - z|$ for any $w \in \mathbb{C}$ such that $|\arg w| < \theta$. It follows from (7.2.3) that

\[
\begin{align*}
\sum_{j \in J} \frac{|f(j)|^2}{|\lambda_j|^{1/2} - z|^2} & \leq \sum_{j \in J} \frac{|f(j)|^2}{|\lambda_j|^{1/2} - z_0|^2} < \infty, \\
\sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j|^{1/2} - z|^2} & \leq \sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j|^{1/2} - z_0|^2} < \infty.
\end{align*}
\]  

(7.2.7)

For any $k \in \mathcal{D}(R_\lambda)$, note that

\[
\left| \sum_{j \in J} f(j) k(j) \right|^2 \leq \sum_{j \in J} \frac{|f(j)|^2}{|\lambda_j|^{1/2} - z|^2} \sum_{j \in J} |\lambda_j|^{1/2} - z|^2 |k(j)|^2 \\
\leq 2 \sum_{j \in J} \frac{|f(j)|^2}{|\lambda_j|^{1/2} - z|^2} \left( \sum_{j \in J} |\lambda_j||k(j)|^2 + \sum_{j \in J} |z|^2 |k(j)|^2 \right) \\
\overset{(7.2.6)}{\leq} 2 \sum_{j \in J} \frac{|f(j)|^2}{|\lambda_j|^{1/2} - z|^2} \left( \frac{1}{\sqrt{1 + \tan^2 \theta}} |Q_R(k, k)| + |z|^2 \|k\|^2 \right).
\]

(7.2.8)
Similarly, we can conclude that for any $h \in \mathcal{D}(R_\lambda)$,

$$
\left| \sum_{j \in J} h(j) \overline{g(j)} \right|^2 \leq 2 \sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j^{1/2} - z|^2} \left( \frac{1}{\sqrt{1 + \tan^2 \theta}} |Q_R(h, h)| + |z|^2 \|h\|^2 \right).
$$

(7.2.9)

Combining (7.2.8) and (7.2.9) together, for any $h \in \mathcal{D}(R_\lambda)$, we obtain

$$
|Q_{f,g}(h, h)| \leq 2 \left( \frac{|Q_R(h, h)|}{\sqrt{1 + \tan^2 \theta}} + |z|^2 \|h\|^2 \right) \left[ \sum_{j \in J} \frac{|f(j)|^2}{|\lambda_j^{1/2} - z|^2} \right] \left[ \sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j^{1/2} - z|^2} \right].
$$

(7.2.10)

Also, note that for any $j \in J$,

$$
\lim_{z \to -\infty} \frac{f(j)}{\lambda_j^{1/2} - z} = 0, \quad \lim_{z \to -\infty} \frac{g(j)}{\lambda_j^{1/2} - z} = 0.
$$

We now conclude from (7.2.7) and the Lebesgue dominated convergence theorem that

$$
\lim_{z \to -\infty} \sum_{j \in J} \frac{|f(j)|^2}{|\lambda_j^{1/2} - z|^2} = 0, \quad \lim_{z \to -\infty} \sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j^{1/2} - z|^2} = 0.
$$

Let $h \in \mathcal{D}(R_\lambda)$. Then, for any $a \in (0, 1)$, there exists $z \leq z_0$ such that

$$
\sum_{j \in J} \frac{|f(j)|^2}{|\lambda_j^{1/2} - z|^2} < \frac{a \sqrt{1 + \tan^2 \theta}}{2}, \quad \sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j^{1/2} - z|^2} < \frac{a \sqrt{1 + \tan^2 \theta}}{2},
$$

and hence by (7.2.10), we conclude that for some $b > 0$,

$$
|Q_{f,g}(h, h)| \leq a |Q_R(h, h)| + b \|h\|^2.
$$

(7.2.11)

Since $Q_R$ is a sectorial form, (7.2.11) together with [46, Theorem 1.33, Chapter VI] implies that $Q_R + Q_{f,g}$ is a sectorial form. The remaining assertion about the existence of $T$ follows from [46, Theorem 2.1, Chapter VI].

We next present a variant of Theorem 7.6, where we discuss the sectoriality of form-sum with $Q_R$ replaced by the form $Q$ defined as

$$
Q(h, k) := \langle |D_\lambda|^{1/2} h, |D_\lambda|^{1/2} k \rangle, \quad h, k \in \mathcal{D}(|D_\lambda|^{1/2}),
$$

(7.2.12)

where $|A|$ denotes the modulus of a densely defined closed operator $A$. Needless to say, we alter compatibility conditions as per the requirement.
Theorem 7.7 Let $D_\lambda$ be a sectorial diagonal operator in $\ell^2(J)$ with angle $\theta$ and vertex 0. Let $f : J \to \mathbb{C}$ and $g : J \to \mathbb{C}$ be such that for some $\beta_0 > 0$,

$$
\sum_{j \in J} \frac{|f(j)|^2}{(|\lambda_j|^{1/2} + \beta_0)^2} < \infty, \quad \sum_{j \in J} \frac{|g(j)|^2}{(|\lambda_j|^{1/2} + \beta_0)^2} < \infty. \quad (7.2.13)
$$

Let $Q$ and $Q_{f,g}$ be as given by (7.2.12) and (7.2.2). Then $Q_{f,g}(h, k)$ is defined for all $h, k \in \mathcal{D}(|D_\lambda|^{1/2})$. Moreover, the form $Q + Q_{f,g}$ is sectorial and there exists a unique sectorial operator $T$ in $\ell^2(J)$ with domain contained in the domain of $Q$ such that

$$
Q(h, k) + Q_{f,g}(h, k) = (Th, k), \quad h \in \mathcal{D}(T), \quad k \in \mathcal{D}(Q).
$$

Proof For $h, k \in \mathcal{D}(|D_\lambda|^{1/2})$, note that

$$
Q_{f,g}(h, k) = \sum_{p, q \in J} K_\beta(p, q)\tilde{h}(p)\tilde{k}(q), \quad (7.2.14)
$$

where, for $p, q \in J$ and $\beta \in [\beta_0, \infty)$,

$$
K_\beta(p, q) := \frac{f(p)}{|\lambda_p|^{1/2} + \beta} \frac{g(q)}{|\lambda_q|^{1/2} + \beta},
$$

$$
\tilde{h}(p) := (|\lambda_p|^{1/2} + \beta)h(p), \quad \tilde{k}(p) := (|\lambda_p|^{1/2} + \beta)k(p).
$$

Let $G_\beta$ denote the integral operator with kernel $K_\beta$ and notice that

$$
\|G_\beta\|_2^2 = \sum_{p \in J} \frac{|f(p)|^2}{(|\lambda_p|^{1/2} + \beta)^2} \sum_{q \in J} \frac{|g(q)|^2}{(|\lambda_q|^{1/2} + \beta)^2}.
$$

From (7.2.13), it is clear that $G_\beta$ is a Hilbert-Schmidt operator for any $\beta \geq \beta_0$. One may argue as in Proposition 7.4, using Lebesgue dominated convergence theorem, to conclude that $\|G_\beta\|_2 \to 0$ as $\beta \to \infty$. Now one may use (7.2.14) to see that for any $h \in \mathcal{D}(|D_\lambda|^{1/2})$,

$$
|Q_{f,g}(h, h)| \leq \sum_{q \in J} |\tilde{h}(q)||g(q)||\sum_{p \in J} |f(p)||k(p)|
$$

$$
\leq \|G_\beta\|_2 \sum_{p \in J} (|\lambda_p|^{1/2} + \beta)^2 |h(p)|^2
$$

$$
\leq 2\|G_\beta\|_2(\sum_{p \in J} |\lambda_p||h(p)|^2 + \beta^2 \|h\|^2)
$$

$$
= 2\|G_\beta\|_2(Q(h, h) + \beta^2 \|h\|^2).
$$
Since \( \lim_{\beta \to \infty} \| G_\beta \|_2 = 0 \), with an arbitrarily small \( a > 0 \), we obtain for some \( b > 0 \),
\[
|Q_{f,g}(h,h)| \leq a \, Q(h,h) + b \| h \|_2^2, \quad h \in \mathcal{D}(\mathcal{D}_\lambda)^{1/2}).
\]
Since \( Q \) defines a sectorial form, one may now argue as in the proof of Theorem 7.6 to complete the proof.

In general, the form \( Q_{f,g} \) is far from being closable. In fact, since \( Q_{f,g} \) is associated
with the rank one operator \( f \otimes g \), it may be derived from Kato’s first representation
theorem [46, Theorem 2.1, Chapter VI] and Lemma 1.3 that \( Q_{f,g} \) is closable if and
only if \( g \in \ell^2(J) \). Under suitable compatibility conditions, Theorems 7.6 and 7.7
above ensure that the form-sums \( Q_R + Q_{f,g} \) and \( Q + Q_{f,g} \) are indeed closed. Finally,
we note that as in the case of operator-sum, under the compatibility condition (7.1.1),
it can be seen that the eigenvalues of the sectorial operator \( T \) associated with \( Q + Q_{f,g} \)
outside \( \sigma(D_\lambda) \) is discrete and the essential spectrum of \( T \) coincides with that of \( D_\lambda \).

**Epilogue**

The present paper capitalizes on the order structure of directed trees to introduce and
study the classes of weighted join operators and their rank one extensions. In particular,
we discuss the issue of the closedness, unravel the structure of the Hilbert space
adjoint and identify various spectral parts of members of these classes. Certain discrete
Hilbert transforms arise naturally in the spectral theory of rank one extensions of
weighted join operators. The assumption that the underlying directed trees are rooted or
rootless brings several prominent differences in the structures of these classes. Further,
these classes overlap with the well-studied classes of complex Jordan operators, \( n \)-
symmetric operators and sectorial operators. This work also takes a brief look into
the general theory of rank one perturbations. As a natural outgrowth of this work, the
study of finite rank extensions of weighted join operators would be desirable. In this
regard, we would like to draw attention to the very recent work [50] on finite rank
(self-adjoint) perturbations of self-adjoint operators.

In the remaining part of this section, we discuss some problems pertaining to the
theory of weighted join operators and their rank one extensions, which arise naturally
in our efforts to understand these operators. In what follows, let \( W^{(b)}_{\lambda u} \) be a weighted join
operator on a rooted directed tree \( \mathcal{T} = (V,E) \) and let \( W_{f,g} \) be its rank one extension.
Here \( u \in V, b \in V, f \in \ell^2(V) \otimes \mathcal{H}^{(b)}_u \) is non-zero and \( g : \text{supp } \mathcal{H}^{(b)}_u \to \mathbb{C} \) is given.

**Numerical range and Friedrichs extension**

Let \( (D_{\lambda u}^{(b)}, N_{\lambda u}^{(b)}, \mathcal{H}^{(b)}_u) \) denote the orthogonal decomposition of \( W^{(b)}_{\lambda u} \). Recall that the
**numerical range** \( \Theta(T) \) of a densely defined linear operator \( T \) is given by
\[
\Theta(T) := \{ \langle T f, f \rangle : f \in \mathcal{D}(T), \| f \| = 1 \}.
\]
Since the numerical range of a diagonal operator is contained in the closed convex hull of its diagonal entries, we obtain \( \Theta(D_{\lambda u}^{(b)}) \subseteq \text{conv}\{ \lambda_{uv} : v \in \text{supp } \mathcal{H}^{(b)}_u \} \). where
conv(A) denotes the closed convex hull of A. Further, for any \( f \in \ell^2(V) \otimes H^{(b)}_u \) of unit norm, we have

\[
\langle e_u \otimes e_{\lambda_u A_u} f, f \rangle = \langle f, e_{\lambda_u A_u} \rangle \frac{f(u)}{f(v)} = \frac{f(u)}{\sum_{v \in A_u} \lambda_{uv} f(v)},
\]

where \( A_u \) is as given in (3.2.15). Thus the numerical range \( \Theta(N^{(b)}_{\lambda_u}) \) of \( N^{(b)}_{\lambda_u} \) satisfies

\[
\Theta(N^{(b)}_{\lambda_u}) = \left\{ \frac{f(u)}{\sum_{v \in A_u} \lambda_{uv} f(v)} : \|f\|_{\ell^2(V)} = 1 \right\}
\]

\[
\subseteq \left\{ z \in \mathbb{C} : |z|^2 \leq \sum_{v \in A_u} |\lambda_{uv}|^2 \right\}.
\]

The numerical range of the rank one extension \( W_{f,g} \) of \( W^{(b)}_{\lambda_u} \) is given by

\[
\left\{ \langle D^{(b)}_{\lambda_u} h, h \rangle + ((f \otimes g) h, k) + \langle N^{(b)}_{\lambda_u} k, k \rangle : \langle h, k \rangle \in D(W_{f,g}), \|h\|^2 + \|k\|^2 = 1 \right\}.
\]

Recall that if the numerical range is a proper subset of the complex plane, then the underlying operator is closable (see [46, Theorem 3.4, Chapter V]). It would be interesting otherwise also to find conditions on \( g \) (different from the compatibility conditions) so that the numerical range of \( W_{f,g} \) is a proper subset of the complex plane or is contained in a sector.

Let us now discuss the so-called Friedrichs extensions of weighted join operators and their rank one extensions [52]. Suppose, for some \( r \in \mathbb{R} \) and \( M \in (0, \infty) \), we have

\[
|\Im \langle W^{(b)}_{\lambda_u} h, h \rangle| \leq M \Re \langle (W^{(b)}_{\lambda_u} - r) h, h \rangle, \quad h \in D(W^{(b)}_{\lambda_u}). \quad (7.2.15)
\]

By [52, Theorem 2.12.1], there exist a subspace \( \Gamma \) of \( \ell^2(V) \), an inner product \( \langle \cdot, \cdot \rangle_{\Gamma} \) on \( \Gamma \) with the corresponding norm \( \| \cdot \|_{\Gamma} \), and a sectorial sesquilinear form \( \mathcal{F} \) on \( \Gamma \) such that the following assertions hold:

(a) \( D(W^{(b)}_{\lambda_u}) \) is a dense subspace of \( \Gamma \) (in \( \| \cdot \|_{\Gamma} \)).

(b) \( \langle h, k \rangle_{\Gamma} = (1/2) \left( \langle W^{(b)}_{\lambda_u} h, k \rangle + \langle h, W^{(b)}_{\lambda_u} k \rangle \right) + (1 - r) \langle h, k \rangle, \ h, k \in D(W^{(b)}_{\lambda_u}). \)

(c) \( \mathcal{F}(h, k) = \langle W^{(b)}_{\lambda_u} h, k \rangle \) for all \( h, k \in D(W^{(b)}_{\lambda_u}) \).

It turns out that the linear operator \( \mathcal{A} \) associated with the sectorial sesquilinear form \( \mathcal{F} \), referred to as the Friedrichs extension of \( W^{(b)}_{\lambda_u} \), turns out to be \( W^{(b)}_{\lambda_u} \) itself. Since \( W^{(b)}_{\lambda_u} \) is a closed linear operator (Proposition 3.5), this fact may be deduced from [52, Lemma 1.6.14] and Theorem 4.15 (see also [46, Theorem 2.9, Chapter VI]).

It would be desirable to find conditions on \( g \) (similar to compatibility conditions) so that (7.2.15) is ensured for the rank one extension \( W_{f,g} \) of \( W^{(b)}_{\lambda_u} \). In this case, \( W_{f,g} \)
admits a Friedrichs extension. In case $W_{f,g}$ is closed, then it can be seen once again that this extension coincides with $W_{f,g}$ itself. The Friedrichs extension would be of interest, particularly, in case either $W_{f,g}$ is not closed or $\sigma(W_{f,g})$ is the entire complex plane. One may be keen to know whether or not there exists a rank one extension $W_{f,g}$, which is closable but not closed.

**Hyponormality and $n$-Symmetricity** It has been seen in Proposition 5.3 that the notions of the hyponormality and the normality coincide in the context of rank one extensions of weighted join operators. We do not know whether or not there exists any non-normal hyponormal rank one extension $W_{f,g}$ of a weighted join operator. The essential difficulty in this problem is unavailability of an explicit expression for the Hilbert space adjoint of $W_{f,g}$. As evident from Propositions 5.5 and 5.7, $n$-symmetric rank one extensions of weighted join operators exist in abundance. The problem of classifying all $n$-symmetric rank one extensions of weighted join operators remains unsolved.

**$C^*$-algebras** Let $\mathcal{T}$ be a leafless rooted directed tree and let $W^{(b)}_{\lambda_u}$ be a weighted join operator on $\mathcal{T}$. Assume that $W^{(b)}_{\lambda_u}$ is bounded and $V_u \neq \emptyset$ (see (2.2.1)). Recall that the essential spectrum of an orthogonal direct sum of two bounded operators $A, B \in B(H)$ is a union of essential spectra of $A$ and $B$. Also, since the essential spectrum is invariant under compact perturbations [23], Theorem 3.13 together with Proposition 2.12(iii) implies that

$$
\sigma_e(W^{(b)}_{\lambda_u}) = \begin{cases} 
\sigma_e(D_{\lambda_u} |_{L^2(\text{Des}(u))}) \cup \{0\} & \text{if } b \in \text{Des}(u) \setminus \{\infty\}, \\
\{0\} & \text{if } b = \infty, \\
\sigma_e(D_{\lambda_u} |_{L^2(\text{Des}(u))}) \cup \{0\} & \text{otherwise}
\end{cases}
$$

On the other hand, the essential spectrum of a normal operator is the complement of isolated eigenvalues of finite multiplicity in its spectrum [23]. Thus the weight system $\lambda_u$ completely determines the essential spectra of bounded weighted join operators. Let $C^*(W^{(b)}_{\lambda_u})$ denote the $C^*$-algebra generated by $W^{(b)}_{\lambda_u}$. Since $W^{(b)}_{\lambda_u}$ is essentially normal (see Theorem 3.13), the quotient $C^*$-algebra $C^*(W^{(b)}_{\lambda_u})/\mathcal{K}$ can be identified with $C(\sigma_e(W^{(b)}_{\lambda_u}))$, where $\mathcal{K}$ is the $C^*$-algebra of compact operators and $C(X)$ denotes the $C^*$-algebra of continuous functions on a compact Hausdorff space $X$ endowed with sup norm.

We conclude this paper with another possible line of investigation. For $b \in V$, consider the family $\mathcal{F}_b := \{W^{(b)}_{\lambda_u}\}_{u \in V}$ of bounded linear weighted join operators $W^{(b)}_{\lambda_u}$ on a directed tree $\mathcal{T} = (V, E)$. A routine verification shows that $W^{(b)}_{\lambda_u}W^{(b)}_{\lambda_v} = W^{(b)}_{\lambda_v}W^{(b)}_{\lambda_u}$ if and only if

$$
\lambda_{uv}\lambda_{uv\downarrow \downarrow w} = \lambda_{uw}\lambda_{uv\downarrow \downarrow w}, \quad w \in V.
$$

The latter condition holds, in particular, for the constant weight systems $\lambda_u$, $u \in V$ with value 1. Assume that the family $\mathcal{F}_b$ is commuting. By Theorems 3.13 and 6.6, the family $\mathcal{F}_b$ is essentially normal. Motivated by [47, Theorem 2.11], one may ask...
whether the \( \mathcal{C}^* \)-algebra \( \mathcal{C}^*(\mathcal{F}_b) \) generated by \( \mathcal{F}_b \) is completely determined by the directed tree \( \mathcal{T} \) and weight systems \( \lambda_u, u \in V \)? In case the answer is no, what are the complete invariants which determine \( \mathcal{C}^*(\mathcal{F}_b) \)?

Acknowledgements  The authors would like to thank Shailesh Trivedi and Soumitra Ghara for some stimulating conversations related to the subject of this paper. We also convey our sincere thanks to Sumit Mohanty for drawing our attention to the Refs. [30, 59], where respectively the notions of the graph with boundary and the spiral-like ordering, relevant to the present investigations, were introduced.

Data Availability Statement  No new data were created or analyzed in this study.

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