Model equations and structures formation for the media with memory

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Abstract. We propose new types of models of the appearance of small- and large-scale structures in media with memory, including a hyperbolic modification of the Navier-Stokes equations and a class of dynamical low-dimensional models with memory effects. On the basis of computer modeling, the formation of the small-scale structures and collapses and the appearance of new chaotic solutions are demonstrated. Possibilities of the application of some proposed models to the description of the burst-type processes and collapses on the Sun are discussed.

1 Introduction

The basic equations for heat transfer and hydrodynamics are usually parabolic heat equations and the Navier-Stokes hydrodynamic equations. But these equations lose their applicability in extended media, when characteristic scales of parameters change less than correlation time and correlation length (relaxation or memory and nonlocality effects). Many examples of non-applicability were found in turbulence. Therefore, more correct equations should be applied in such cases. There are some well-established facts in the theoretical physics on the description of transport processes.

The first famous idea is the existence of hierarchy of description levels. If there are \(N \gg 1\) particles we have many levels for description: \(N\) deterministic dynamical laws of Newton for particle movement; Liouville equation for \(N\)-particle distribution function, Boltzman equation for one-particle distribution function, hydrodynamic equations for macroparameters (Navier-Stokes class equations), thermodynamic equations. The choice of the level of description depends on the measure of deviation from equilibrium.

The second important idea is the existence of many interrelating relaxation processes and many time and space scales with different relaxation times and lengths. The memory and nonlocality effects are common for all levels. Each level of description has its own specific type of chaos, autowave solutions, collapses and so on. It should be stressed that each level of description has its own model equations with typical behaviour of the solution. The problem of defining typical chaotical behaviour for a given level is especially interesting.

The turbulence is a bright example of such complex phenomena. It is well known, turbulence is encountered very often in broad classes of natural processes. It is widely recognized that turbulence is a strongly non-equilibrium phenomenon. Investigators dealing with it in the specific branches of science, as a rule, formulate what is turbulence on the intuitive level. This results in the lack of the generally accepted and unified definition of turbulence. In
concrete cases one can encounter different types of turbulence. This difference is fixed, for example, in the plasma theory (strong and weak turbulence).

Till now, strict and evident in all cases, definitions of turbulence are absent. To imagine a set of possibilities in this question we will recall a few descriptions of turbulence mainly related to hydrodynamics.

"Turbulent motion of liquid at large enough values of the Reynolds number is characterized by the extraordinarily irregular, disorderly change of speed in time in every point of stream (the developed turbulence). The velocity always pulsates near the same mean value" \[1\].

"Turbulent motion is, generally speaking, vortical" \[1\].

"So, the turbulence is the vortical flow of viscous liquid which is stochastic in the sense that it is characterized by: 1) sensitive dependence on initial conditions (this caused by exponential divergence initially closed phase trajectories); 2) that all phase trajectories are dense almost everywhere, thus, any initial nonequilibrium distributions of probability in phase space tend to the limiting equilibrium distribution; 3) mixing in phase space (and, as a result, ergodicity, by the rapid fading of temporal correlation functions and continuity of frequency spectrums). The developed turbulence possesses additionally 4) multimodes, and, as a consequence, manifests the chaos in its spatial structure at any fixed moment of time" \[2\].

"This motion is so difficult and so not studied enough, that even the question: what is turbulence? is difficult to answer. What kind of basic features of turbulent motion are there?" \[3\].

"One of the most considerable features of turbulence is co-operation of order and accident... An accident in turbulence is related to the extreme sensitiveness to disturbances. At the same time, the chosen statistics are roughly stable at disturbances" \[4\].

"What is turbulence? Some light for this difficult purpose is explained by the study of offensive turbulence or weak turbulence which is an aspect of chaos" \[5\].

"The following definitions of turbulence can be offered... A turbulent flow should be unpredictable in the sense that a small uncertainty in initial time will grow so that the strict deterministic prediction of its evolution is not possible, the second, a turbulent flow should satisfy the property of the increasing mixing" \[6\].

Still, a number of determinations is presented by the theory of plasma turbulence that, as will be argued below, it has definite interest and is hydrodynamic turbulence, so, the most common descriptive determination: "Any state of plasma with the strongly developed noises and vibrations is accepted as turbulent" \[7\].

The selection in plasma turbulence of collective degrees of freedom and construction of turbulence on their basis is important.

"Presentation which is usually inlaid in notion of turbulence substantially differs from the notion of molecular motion. Speaking, for example, about motion of any element of liquid, motion of some macroscopic volume is implied, containing the large number of molecules... In other words, hydrodynamic motion of liquid corresponds to the definite collective degrees of freedom... It is possible to define the concept of "turbulence": the turbulent motion of macroscopic bodies is such a motion in which the collective degrees of freedom are intensively excited and carried an accidental character" \[8\].

As a rule, it is assumed that turbulence in plasma and liquid are different. In \[9\] the comparison of turbulence in plasma with turbulence of incompressible liquid is given. It turns out that instability of liquid causes the excitation of whirlwinds and there are difficulties with
the flows of energy. It is considered that the indicated difficulties have a principle character and are connected with the strong turbulence in incompressible liquid. This is expressed by the following: whirlwinds generally do not have their own frequencies and the time of transmission of energy from one whirlwind to a neighbouring one is about one turn of the whirlwind.

Unlike the motion in incompressible liquids, in plasma there are a lot of collective oscillating motions which have their own definite frequencies. Time of transmission of energy of these vibrations to the neighbouring scales (or neighbouring wave numbers) can essentially exceed the characteristic period of vibrations $\omega_k^{-1}$. Because of this "elasticity" of collective motions a small parameter $\omega = \tau \omega_k^{-1} \ll 1$ can appear. It is considered that the presence of this small parameter allows to use the regular methods of decomposition on energies of turbulence and build the theory of weak turbulence [9]. The role of dispersion for plasma turbulence is underlined here.

So, in [10] it is indicated that in incompressible liquids the dispersion is absent. Therefore, in this situation we see the turbulence of whirlwinds and transmission of energy between them. Plasma possesses many excitations with dispersion, cooperation, and weak turbulence [10]. In addition, in plasma there is strong turbulence of solitons. Let us note here that the results of investigations presented in the cited work presumably shows that a sharp barrier between the types of turbulence in plasma and liquid is not present and the situation turns out to be more difficult and interesting. Plenty of determinations indicates that common determination of turbulence is absent and in every concrete process there can be one specific type of turbulence. This distinction is fixed, for example, in the theory of plasma (strong and weak turbulence).

But even if a rigorous definition exists, we must use some mathematical idealization in the theoretical study of this complicated phenomenon. For distributed objects we usually take the nonlinear PDEs or integro-differential equations. Then mathematical models are investigated on stochasticity connected with the unstability in systems.

The mathematical object on the hydrodynamic level is the system of partial differential equations for velocity, pressure and temperature. Such a system includes equations of motion as well as the constitutive equations, which give the expressions for the deformations versus strain dependence [11]. Instant connections between displacements and strain lead to, in the case of the incompressible fluid, the Navier-Stokes equations. The memory effects take on the form of integral constitutive equations. The simplest exponential kernels lead to the constitutive equations introduced formerly for visco-elastic media. In such cases we obtain the simplest modification of the Navier-Stokes equations, namely, the so-called equations for the Maxwell media ([12, 13, 14]).

Such modified system [12, 13, 14] is still a very complex mathematical object. In order to understand its properties, one can investigate the simpler model equations. It is well known that such a model equation for the Navier-Stokes system is the one-dimensional in space nonlinear Burgers equation [9]. This equation was introduced in 1948 empirically, and later derived within the asymptotic approach. It has been recently proposed [10] to include a second-order derivative with respect to time in the Burgers equation. The main goal of such a modification was the incorporation of memory effects. Then, we have the one-dimensional in space model equation, which is the so-called hyperbolic modification of the Burgers equation [15, 16].

Its numerical and analytical studies reveal many curious properties. One of the most
interesting properties is the existence of blow-up (collapse) solutions. However, not long ago, the two and three-dimensional model equations for visco-elastic system did not exist. Following the lines of deriving the Burgers equation we put forward multi-dimensional model equations. Analysis of the full system [12, 13, 14] and physical properties of systems with memory (fast heat processes, plasma, turbulence, statistical mechanics, visco-elasticity) allows to pick out some necessary properties that should be accounted for by model equations. Such properties are viscosity, mass transfer by convection and finite speed of disturbances propagation. Besides, the properties of such model equations should resemble in some specific cases the behaviour of original equations [12, 13, 14] and, what is more important, the behaviour of real objects (for example, the stability properties). Our analysis leads to some new model equations with such properties. The simplest two- and three-dimensional equations have the form

\[ \mu \frac{\partial^2 \vec{u}}{\partial t^2} + \frac{\partial \vec{u}}{\partial t} + u_k \frac{\partial \vec{u}}{\partial x_k} = c \Delta \vec{u}. \]

Also, the nonlocality may be incorporated in the model equations.

2 The choice of model equations

Since a deep theoretical description of mathematical models for hydrodynamic processes can be carried out by means of the methods of theoretical physics, we very briefly describe some main concepts from statistical physics relevant to modelling equations choice.

Beginning from the works of N.Bogoliubov, M.Born, M.Green, J. Kirkwood and J. Ivon, the canonical approach in the theoretical physics is as follows. Let us consider the medium constituted from \( N \) independent particles. Then, in classical physics we can describe the particle movement precisely by a system of ordinary differential equations (the Newton equations). But the statistical physics consider the ensemble of the system by introducing distribution function \( f_N(x_1, x_2, ..., x_N; t) \) for particles distribution probability at time \( t \). The function \( f_N \) is evaluated from the Liouville equation

\[ \frac{\partial f_N}{\partial t} + \{ f_N, H \} = 0, \quad (1) \]

where \( H \) is the Hamiltonian of the system and \( \{ \cdot \} \) are the Poisson brackets.

However, the function \( f_N \) is too informative for the hydrodynamic phenomena description. Usually all necessary parameters are macroparameters (for example temperature, pressure, and velocity: \( T, P \) and \( V \)). The main leading principle in such a case is the reduction of the description parameters set. The reduction procedure deals with some hierarchical levels. Firstly, by integrating on some variables in phase space, we can go to \( f_1 \)—one-particle distribution function with the BBGKI chain of equations for \( s \)-particle distribution functions. Note that for \( f_1 \) we can receive the well known Boltzman equation. These stages with distribution functions are named kinetical.

Further averaging with one-particle distribution function leads to macroparameters \( T, U, P \). Usual procedures lead to well known equations of the hydrodynamic type: the parabolic heat equation, the Navier-Stokes equations and so on. But a more correct description leads to more difficult equations with memory effects. The reason of memory effects origin under reduction processes is very well described in theoretical physics since the works of H. Mory, R. Zwanzig, R. Picirelly, D. Zubarev and many others, see review in [17, 18, 19, 20]. In this
approach we receive hydrodynamic equations for macroparameters with some constitution
equations relating to macrovariables. In general, such constitution equations have the form
of integro-differential equations [15].

Let us make some comments concerning the reduced description. Reduction of description
takes place not only at the level of the BBGKI chain. Thus, even having equations of
hydrodynamics and solving them by approximate methods, we get the chain of reduced
descriptions corresponding to different accuracy of methods. This easy to see in the Galerkins
method for the Navier-Stokes equation [21]. It is known that the construction of solutions in
the form of a series in terms of trigonometric functions leads to the system of three ordinary
differential equations (the Lorenz system) with the chaotic behaviour. If we take into account
a higher number of modes $M > 3$, then other types of chaotic or even periodic behaviour
are observed at $M > M''$.

It is clear that different Galerkins systems derived at different numbers $M$ correspond to
different levels of reduced description (with $M$ leading parameters). This can help to under-
stand the sense of such chaotic behaviour and answer the question: can the low dimensional
dynamical systems transfer the chaotic behaviour of the initial system of PDE.

Here, we can reason in the same way as in the case of the Liuvills equations. If we had
the complete phase portrait of the infinite dimensional Galerkin system of ODE, we would
pass the properties of a system of PDE completely and would know the trajectories exactly.

In order to substantiate the low dimensional dynamical system and the limiting transition
as $M \to \infty$ there is a way appearing from the classical theory of difference sch emes when
convergence and accuracy are proven as $M \to \infty$.

However, it is very difficult and in addition, presumably for any numerical schemes (at
fixed $M$), a system with such a difficult phase portrait can be found, that a numerical scheme
will not be able to correctly pass the behaviour of a solution. However, in many cases we
do not need such detailed information about the solutions, and it is important to know only
some general patterns. Here, the situation is the same as in the description of $N$ particles a
deterministic way or by means of the distribution function.

Therefore, the applicability of the finite-dimensional dynamical systems as $M \to \infty$ should
be considered in the sense of characteristics of chaos by the bringing of probabilistic concep-
tions (for example, the limit of invariant measures). In addition, it is well known that the
application of probabilistic methods in the theory of cellular automatons makes the proof
easier [22].

Probabilistic considerations also allow to put forward a question about the correct trans-
mission of chaotic behaviour by means of different finite-dimensional projections. Similar is
done for the infinite - dimensional Bogolyubovs chains in the so-called lattice systems [23].

We should especially note the reasons of memory appearance and the nature of memory
in the given context. From Maxwell’s considerations of the model example, the memory
appears because of accounting the delay. In the thermodynamics of media with memory,
the memory is simply postulated in integro-differential relations for the flux of heat and
the tensor of stress with the defined kernels of integral relations. But the exact type of
relations and concrete expressions for kernels can be derived in statistical physics by means
of reduced description, when information about the ”unimportant” degrees of freedom are
taken into account by implicitly through kernels. The essence of this phenomenon was found
out in the so-called projection method of Mori Zvantzig’s type [24], and in the method of
non-equilibrium statistical operator by D. Zubarev [25, 26].
In this approach, the function of distribution $f$ is divided into two parts: $f = f_1 + f_2$. For example, as a rule, $f_1$ is chosen as the locally-maxwellian distribution depending only on the relatively slowly changing hydrodynamic parameters $T, p, u$. Part $f_2$ carries information about quickly changing micro-motions. Then, the application of $f_1$ leads to hydrodynamic equations which should contain contributions of the "rapid" part $f_2$.

In order to do this, the operator of projection $P$ is introduced so that $f_1 = Pf$, $f_2 = (1 - P)f$. Then, one can use the solution of the Liuvills equation in the form $f(t) = \exp(-itL)f(0)$, where $L$ is the Liuvill’s operator connected with the Hamiltonian system. After some manipulations \[24, 27, 26\] the expression for $f_2$ is obtained

$$f_2 = e^{-i(t-t_0)(1-P)}f_2(t_0) - i \int_{t_0}^{t-t_0} dt' e^{-i(t' - (1-P)L(1-P)Lf_1(t - t')}}$$

and equation for $f_1$

$$\frac{\partial f_1(t)}{\partial t} = -iPLf_1(t) - iPLE^{-i(t-t_0)(1-P)}f_2(t_0) - \int_{0}^{t-t_0} dt' e^{-i(t' - (1-P)L(1-P)Lf_1(t - t')}}.$$  

It easy to see that (2) and (3) have a temporal nonlocal character. This is caused by the division of function of distribution to pieces. The difficulties with the choice of the projection operator are well known, however, the form of which is not known. They were overcome in the method of nonequilibrium statistical Zubarev’s operator. According to this method, the function of distribution is assumed to depend on a certain set of parameters of reduced description $a_1, ..., a_k, f = f(a_1, ..., a_k)$, where $\{a_j\}$ are certain averages of the distribution function $f$ \[26\]. Note that, in this case, in principle, we can obtain the same relations for $f$ taking memory into account.

Concerning the hydrodynamic level of description, functions of distribution lead to the equations of hydrodynamics in the form of balance laws for momentum and energy

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V}{\partial x} = 0, \quad \frac{\partial V}{\partial t} + V_k \frac{\partial V_k}{\partial x} + \text{grad} \ p = \text{div} \ \sigma + F, \quad \frac{\partial E}{\partial t} + \frac{\partial I_E}{\partial x} = 0,$$

where $\rho$ is the density, $V$ is the velocity, $p$ is the press, $\sigma$ is the stress, $F$ is the thermal force, $E$ is the energy, $I_E$ is the energy flux.

For closing these equations, integral relations for the heat fluxes and tensors of stress are used. A short and clear presentation of these issues is given in the appendix to the book written by Day \[28\].

Let us write these relations for the one-dimensional case

$$\sigma(x,t) = \int_{-\infty}^{t} dt' \int dx' K_1 \frac{\partial u}{\partial x}(x',t'), \quad q(x,t) = \int_{-\infty}^{t} dt' \int dx' K_2 \frac{\partial T}{\partial x}(x',t').$$  

(4)

In these relations, the kernels $K_1$ and $K_2$, derived by means of uniform procedure, are connected with the correlators of variables. The correlators can be calculated with the help of the function of distribution. For example, for $K_1$ we have

$$K_1 = (q(x), q_1(x',-t)).$$  

(5)

Here, in (5) there is an assembly average.
Thus, summing up known results on statistical physics it appears that the application of a reduced description at a hydrodynamical level leads to conservation laws, which are closed by some integral relations. These integral equations can have a very difficult form including nonlinear dependence on hydrodynamic variables. The concrete type of kernels depends on a set of degrees of freedom involved in the process (this indicates the necessity of their incorporation at the reduced description).

In turn, these "necessary" degrees of freedom appear in dependence of the displacement from equilibrium. In equilibrium thermodynamics are enough to characterize the temperature of a body as whole. It is easy to see that without consideration of prehistory effects the ordinary relations for heat conductivity in the form of the Fourier law \( q = -\lambda \partial T / \partial x \) and the relations for stress of type \( \sigma = -\nu \partial u / \partial x \) are derived. These relations lead to the parabolic equation of heat conduction and to the Navier-Stokes equations accordingly. The exponential kernels of type \( \exp(-t/\tau) \) lead to the hyperbolic equation of heat conductivity and to the equations of relaxation hydrodynamics.

Exponential kernels correspond to more exact consideration of property of remoteness from equilibrium. For an adequate but rough description of the processes far from equilibrium, the hydrodynamical equations may be enough in the beginning (it is possible with nonlinear constitutive relations). But after that, a hydrodynamic level may be not valid, then we need the whole one-particle function \( f_1 \) (or even multi-particle functions of distribution) as parameters of reduced description (see [29]).

Now, let us apply the aforesaid to the problem of turbulence. Assume that each level of reduced description has the corresponding characteristic chaotic behaviour, which may be called the turbulence corresponding to the given level of reduced description. In an implicit form, this has been done in [30], and in an explicit form in [32]. A similar concept is contained in [31]. This assertion seems to allow to reconcile the seemingly irreconcilable points of view on the so-called hydrodynamic turbulence. Of course, this is a complex phenomenon which can demand the complete description with the help of function \( f_N \) (for example, for the developed isotropic turbulence). However, the levels of reduced description may be enough for weaker turbulence, and especially for the study of the initial stages of its development. Note that at different levels of description, the features of a general picture may be different. That is why, despite the disadvantages of the Navier-Stokes equations, these equations nevertheless may contain information about the turbulent behaviour. For example, in [4] the expression of Navier-Stokes turbulence appeared even. It is possible that the Navier-Stokes equations can pass the initial stages of loss of laminar flows stability.

We observe other types of turbulent behaviour using the levels of reduced description with the help of the complete Liouvill’s equation. Thus, the place of many works concerning the incorporation of different fluctuations is cleared up. Small-scale fluctuations can be incorporated at the level of the Boltzmann’s equation in the integral term ([31], [33] and others). Incorporation of large-scale fluctuations, which correspond to the collective motions like a transfer of liquid, are much more difficult. In this case the Reynolds’s procedure of the separation on the average flow and pulsation components does not help because it is based on the system of the Navier - Stokes equations. Therefore, since the 1960’s plenty of work was devoted to the considerations of large-scale fluctuations. These works contain the articles considering the rejection of the hypothesis of molecular chaos [34], the role of asymmetry of functions of distribution with respect to the transpositions of particles [35] and many others.

So, in [36] the role of large-scale fluctuations had been accounted for consideration of
equations of two-particle functions of distribution. The role of fluctuation is considered in book [31] as well. It would be desirable to take note of book [37] which is not devoted especially to the problem of turbulence but contains a lot of results having a direct relation to it.

Thus, in principle, it is known how to construct equations corresponding to the levels of reduced description in a general abstract form. As a rule, theoretical physics is content with the qualitative investigations of these equations. However, concrete information extracted from them is difficult because of their complication. At the same time, for hydrodynamics, it would be desirable to have the macroeffects description. In this case the study of model equations can help. At this approach the Navier-Stocks equations and equations for the Maxwell’s media can be regarded as model equations. As mentioned above, the usefulness of model equations for consideration of turbulence is confirmed by both experimental data on the turbulent flows and their derivation from the Boltsman equations. Note that, they can be obtained at the model exponential kernels of transfer from the relations [4].

It is possible also to assume that the same relations with exponential kernels will be a model in some sense for equations with the extended set of variables in comparison with the hydrodynamic one, describing the average flows. Characteristic times of relaxation determined by large-scale fluctuations can be large enough in comparison with the time of free running of molecules, as experiments demonstrate. Moreover, it is possible to make the assumption that the equations of hydrodynamics with memory are in some sense the case of ”general position” at the modelling of turbulence, while the Navier - Stocks system is a degenerate case, when the time of relaxation tends to zero.

So, it is possible to make the conclusion that, at the modelling of turbulence the equations with memory are more preferable. Therefore, in order to make some features of turbulence on a hydrodynamic stage more exact we should use the presented equations or their consequences.

It turns out that the consideration of equations with memory or their model equations lead to some unexpected results and raise problems, some of which are given in the following sections.

Also note that, some reduction procedures can be applied further for hydrodynamic equations. For example, the search of a small number of leading parameters in the dissipative structures theory or synergetic and phase transition theory also may be considered as reduction in description (see the works by I. Prigogine, H. Haken, W. Ebeling). Also remember the new investigations of reducing partial differential equations to low-dimensional ordinary differential equations and their attractors [38, 39]. The next stage of reduction consists of the transition to the pure thermodynamic description. It should be stressed that, in general, the precise abstract equations for different hierarchical levels of description are known from the theoretical physics. Usually theoretical physicists explore these very complex abstract equations (frequently qualitatively). But on a hydrodynamic level it is especially interesting to search visible macroeffects. In such a case the consideration of model equations is especially useful. Thus, as the Navier-Stokes equations and their hyperbolic counterpart and low-dimensional systems of ordinary differential equations are the model equations.

There is a great interest in the investigation of typical models equations for different hierarchical levels and their typical solutions. Note that, usually such typical solutions are the basic elementary objects for a description of complex real phenomena.
3 Collapses, elementary objects and instabilities

As was mentioned in the previous sections, various formations play an important role in turbulence. To some degree, the type of structures admitted to consideration determines possible functional spaces. Therefore, the results presented here largely join the results of the previous section.

Among elementary objects, we should first distinguish the singularities in solutions. In other words, these are collapses, modes with intensification, blow-up solutions. These solutions tend to infinity at finite time in certain points. Despite the exoticism such solutions are often used in the investigations of physical processes. It turns out that they have a long history connected with problems of turbulence. For example, vortical tubes in a liquid at the pull increase the rotation. The dissipation of energy in turbulence takes place not in whole liquid, but it is concentrated in the certain localized regions known as intermittency. Vortices can grow under the conditions of rotations of turbulent liquids.

Naturally, the phenomena of such types are represented in the mathematical models of turbulence. One class of effects is connected with the vortical motions of ideal incompressible liquid which is a good model for the study of the developed turbulence, when the Reynolds number $\text{Re} \to 0$ and the influence of viscosity is ignored. Except for pure experimental works there is a large number of investigations on the numerical modelling. According to the results of modelling it has been revealed that under the conditions of accuracy computations with the help of special numerical schemes, taking into account the character of approximating methods, there is a tendency for the increase of vorticity (see [40]). Finally, the exact analytic solutions with vorticity collapse are found.

Singularities of initial Navier-Stokes system are a much more difficult problem. Let us recall the block of ideas related to the theory of turbulence. This idea consists of the supposition that the solution or its functionals are bounded only to the finite moment of time, and then tends to infinity. After the moment of collapse the solution "becomes" finite again, next the process repeats. Note that non-uniqueness can be observed. Therefore, we should keep in mind the classical investigations on the theorems of existence and uniqueness of solutions of the Navier-Stokes equations. Remember that Leray proved the existence in the three-dimensional case of a global weak solution in the class of quadratically summable functions under the condition that the initial data has bounded energy.

However, the existence of such solutions does not guarantee the absence of singularity in the solution (i.e. the movement of a solution to infinity). It was also unknown, whether such solutions can develop from smooth initial data. The essential results in this direction are a derivation of the strong estimations above for the Hausdorff dimension. It turns out that the measure of the singularities support is equal to zero.

Recently, this problem has been developed in papers related to the multi-valued solutions. As a consequence, the concept of "concentration in the solution" appeared. This means that sequence poorly converging in $L^2$ functions can have a limiting function with the infinite values of solutions on the "emaciated" set. As an example, the sequence of functions in $R^n$ [41] is presented. The work of F.Merle is indicated in paper [41], where the behaviour of a solution after the moment of infinity is considered.

Let us pay attention to the work in [42] which deals with solutions with singularity. In this work, for the Burgers equations and the boundary problem, the classes of stationary solutions having singularities were studied.
Perhaps the first solution of the Navier-Stokes system with a collapse is considered in [43]. In connection with a collapse let us point out that investigations were carried out for the model equations of the hyperbolic type of the second order.

A collapse in the system of hydrodynamics with the internal degrees of freedom is considered numerically by V.A.Khrisheniuk. These results relate to strict mathematical assertions. In spite of many open questions, the solutions with a singularity of vortical type were repeatedly used in different physical investigations. So, in [44] the dynamical system consisting of a small number of vortices is considered and the conditions of chaotic motions appearance are stated. The subsequent investigations were developed toward the increase of a number of the considered vortices, rates of vortices and the distributing of vortices in media, when statistics of objects-vortices were built. We also indicate chaos built on co-operation of vortices and spiral waves.

In connection with the hydro- and gas-dynamics we mention another type of singularities related to the so-called gradient catastrophes when solutions remain bounded but the derivatives on space grow without restriction. Thus, there can be shock waves in the solution. Such a situation is especially characteristic for the compressible media. Moreover, such shock waves or solutions formed from their combinations serve the elementary solutions in some models of turbulence when a model is built on statistics of such excitations (for example, turbulence at the Burgers model).

V.E.Zaharov and co-workers dealt with collapses considering them both numerically and analytically [45, 46]. Compressed, but very capacious review is presented in [47], where the possible applications of collapses to hydrodynamic turbulence are announced. The role of collapses is especially interesting in plasma as places of dissipation (flow of energy), and also their role in Lengmyurs turbulence, when high-frequency vibrations are concluded in a diminishing in size cavity (to the cavity). These phenomena were predicted in a theory, after in their confirmation the computations were executed, and quite recently they found experimental confirmation [48]. It is important, that this process can go in retrograde on the spectrum of waves, i.e. energy is passed in a short-wave region. It differs from many models in which energy is passed on a spectrum in a long-wave region, where condensation of long-wave excitations can become even similar to the Bose-condensations in the quantum theory. In [47] the types of collapses, the volume of energy involved and the threshold effects had been considered. Thus, by basic model equations (as well as in many other works) it is the nonlinear Shredinger equation. Because of the importance of the hypotheses mentioned in [47] we will quote two extracts from skies (p.470):

"For many types of turbulence, carried out in a continuous media, the multiple development of wave collapses is characteristic in which dissipation of turbulent energy occurs. It is not set presently, whether classic turbulence of ideal incompressible liquid belongs to the given type, although for this hypothesis there are very serious grounds...however physics of plasma is the basic "user" of theory of wave collapses."

As our investigations have shown, the consideration of more general models of hydrodynamics taking into account the effects of memory, indeed give serious arguments on behalf of such a hypothesis. Note that the role of singularities in turbulence was described in [30]. Dispersion in the system is also important. In connection with collapses soliton turbulence was also considered, when modulation instability was the source of appearance of solitons as elementary objects making turbulence.

Paper [49], in which collapses in a conceptual plan are considered from a new point of
view, recently appeared (this paper intersects partially with the cited articles). It also deals with the solutions with the development of collapses on the basis of the nonlinear Shredinger equation in order to adequately describe abnormally large transfer in the turbulent flows. A model is built in such a way that the solutions with collapses do not always grow and at large amplitudes limiting factors resulting in the reduction of amplitude. This process repeats oneself by spontaneous appearance and is named the homoclinic excursion. It looks like that a model indeed passes some lines to turbulence. However, the authors of [49] note the question the applicability of the Shredinger equations to the real hydrodynamic turbulence. Let us note that equations taking into account memory allow to consider similar effects.

As numerous investigations show, which are usually described in the practically liked books on turbulence and chaos, structures usually arise as a result of display in the systems of a different sort of nonstability, thus the type of instability determines a character, size and dynamics of structures. Therefore, upon completion of this paragraph we will briefly discuss different types of nonstability, because lately definite advancements appeared in this direction. In addition, as is generally known from works on dynamic chaos, it is considered that local instability provides this phenomenon.

There is an enormous number of investigations on hydrodynamic instability. Here, we refer only to a small number on topics. The first topic concerns the problems of incorrectness in physical problems. It is known that, mostly physical problems try to formulate so that they should be correct according to Adamer. At the same time whole classes of ill-conditioned problems exist. Concerning hydrodynamics ill-conditioned problems a raise as well. We should recall, for example, the problems about explosive instability at the Burgers equations with the negative coefficient of viscosity, and the problems of the boundaries of flows (the Kelvin-Gelmgoltz instability). A non-correctness arises in the problems of motion of visco-elastic media at the change of their type from hyperbolical to elliptic. The Cauchy problems for elliptic equations arise in investigations of elliptic equations for defects. Recently, ill-conditioned problems were considered by V.P.Maslov [50]. He gave the definitions of degree of non-correctness [51].

In hydrodynamics Maslov considered the case of asymptotics on viscosity of $v \to 0$ with fast-oscillating initial data, leading to asymptotical uniqueness and the loss of causality in deterministic problems [50]. With collapses, unsteady problems were considered, when the solution grew due to resonances.

Here, we briefly describe the phenomenon of waves with negative energy in media with memory. The first such waves were considered in the problems of electronics in the 1950’s. The essence of the phenomenon consists in the distributed active media’s definite wave disturbances which are such characteristics at growth of their amplitude, that simultaneously with the energy of the system of media+wave diminishes [52]. Formally, it is expressed in the negative distribution of energy for these waves. Essentially, such effects are possible only in thermodynamically non-equilibrium media. It is also known that frequent co-operation of waves of negative energy results in collapses and that properties of such systems are determined by dispersion correlations of linearized problems.

In conclusion we will indicate another aspect, related to instability, turbulence and chaos. As already mentioned, for chaos (in the mathematical sense) in the finite-dimensional systems instability (the Oseledets theorem) is needed. However, this situation is not simple for the infinite-dimensional systems.
In addition, there is the separate complex problem related to the supervision of chaos in the infinite-dimensional systems and their interpretation. In a manner, this task is similar with the task of image regeneration, which, as is generally known, is improper. There are many examples of effects of this type and these problems deserve a deeper study. We need to note that in the problems of stability there are also intricate mathematical problems for example the study of stability in systems with singular coefficients (as in the Shredinger equation with singular potential).

4 Low-dimensional models for two-dimensional generalized hydrodynamics

Among approaches of the investigation of hydrodynamics equations is the Galerkin method. Using this method it is easy to construct a low-dimensional dynamical system. So we shall deal with dynamical systems, obtained from nonlocal hydrodynamic models, presented above.

Since generalized models can be regarded as singular perturbations of simpler models, it is useful to carry out the comparison of the solutions of dynamical systems, squeezed out from different hydrodynamics models.

In particular, such test-systems are the well-known Lorenz system and the low-dimensional system for plane flows, investigated by C. Boldrighini and N. Franceschini in 1979 [53].

4.1 Finite-dimensional systems for 2-dimensional flows with a periodicity condition

One of the models, taking into account non-locality effects, in the case of two spatial dimensions is the Oldroyd fluid (the generalization of the Maxwell media [12, 13, 14])

\[
\begin{align*}
\frac{\partial \vec{V}}{\partial t} + V_k \frac{\partial \vec{V}}{\partial x_k} + \tau \left( \frac{\partial^2 \vec{V}}{\partial t^2} + \frac{\partial V_k}{\partial t} \frac{\partial \vec{V}}{\partial x_k} + V_k \frac{\partial^2 \vec{V}}{\partial t \partial x_k} \right) &= -\nu \Delta \vec{V} - \mu \frac{\partial \Delta \vec{V}}{\partial t} = -\left(1 + \tau \frac{\partial}{\partial t}\right) \text{grad} P + \vec{F}(x, t), \\
\text{div} \vec{V} &= 0,
\end{align*}
\]

where \( \vec{V} = \{V_1, V_2\} \) are velocity components, \( \nu \) is the viscosity coefficient, \( P \) is the pressure, \( \vec{F} = \{F_1, F_2\} \) are external forces, \( \tau \) is the relaxation time, \( \mu \) is the non-locality coefficient, \((\cdot)_k\) means summation over repeated indexes.

We considered model (6) on a plane region \( T^2 = [0, 2\pi] \times [0, 2\pi] \) with periodical boundary conditions. The geometry of flow and boundary conditions were imposed just like in [53]. In such a case we consider the flows with the velocity component \( v_z = 0 \). We assume that the flow is space-periodic in the \( (x, y) \) - plane with the periods \( 2\pi \). This implies that the flow and flow derivatives are periodic with the period \( 2\pi \). Moreover, we also assume the mean flow averages over region \( T^2 \) (see [53] and [38], pp. 103 104):

\[
\int_{T^2} \vec{V} \, dx = 0.
\]
In our case, the solution is considered as the series on harmonics $\exp(i\hat{k} \cdot \hat{x})$, where $\hat{x}$ are coordinates and $\hat{k}$ are wave vectors with integer components,

$$\hat{v}(x, t) = \sum_{k=1}^{\infty} \gamma_k(t) \exp(ikx),$$  \hspace{1cm} (7)

where $k$ is the wave vector.

Inserting (7) into (6), after long calculations we obtained the system of equations for coefficients $\gamma_k(t)$,

$$\tau \frac{\partial^2 \gamma_k}{\partial t^2} + \frac{\partial \gamma_k}{\partial t} =$$

$$-i \sum_{k_1+k_2+k} \gamma_{k_1} \gamma_{k_2} (k_1 + k_2) \left[ 1 - \frac{k_1(k_1 + k_2)}{|k|^2} \right] -$$

$$-\nu |k|^2 \gamma_k + f_1(k) \frac{k^2}{|k|^2}$$

$$-i\tau \left[ \sum_{k_1+k_2+k} \left( \gamma_{k_1} \frac{\partial \gamma_{k_2}}{\partial t} \gamma_{k_2} \frac{\partial \gamma_{k_1}}{\partial t} \right) (k_1 + k_2)(1 - k_1 - k_2) \right]$$

$$+ \tau \frac{\partial f_1}{\partial t} \left( 1 - \frac{k_1}{|k|^2} \right) - i\tau \frac{k_1}{|k|^2} \frac{\partial f_2}{\partial t},$$  \hspace{1cm} (8)

where $f_1, f_2$ correspond to Fourier expansion of function $F$. This infinite dimensional system is equivalent to the original system of PDEs, yet its investigation is a very difficult problem.

Therefore it is of common practice to ”cutoff” such a system and merely consider a finite number of its components. Truncated systems usually give some information about the original equations’ behaviour. Thus we must take $k$ in (7) from some bounded set $L$. Let us take $L = \{ k_1 = (1, 1), k_2 = (3, 0), k_3 = (2, -1), k_4 = (1, 2), k_5 = (0, 1) \}$ plus opposite in sign vectors. After reduction, in [53] the low-dimensional system was obtained ($\nu = 1.0$):

$$\dot{\gamma}_1 = 2\gamma_1 + 4\gamma_2\gamma_3 + 4\gamma_4\gamma_5,$$  \hspace{1cm} \hspace{1cm} $$\dot{\gamma}_2 = -9\gamma_2 + 3\gamma_1\gamma_3,$$

$$\dot{\gamma}_3 = -5\gamma_3 - 7\gamma_1\gamma_2 + r,$$  \hspace{1cm} \hspace{1cm} $$\dot{\gamma}_4 = -5\gamma_4 - \gamma_1\gamma_5,$$  \hspace{1cm} \hspace{1cm} $$\dot{\gamma}_5 = -\gamma_5 - 3\gamma_1\gamma_4.$$  \hspace{1cm} (9)
The memory effects lead to the ten-dimensional counterpart of (9):

\[
\begin{align*}
\frac{dx_1}{dt} &= -x_1 - 2x_6 + 4x_7x_8 + 4x_9x_{10} + 4 \left( x_2x_8 + x_7x_3 \right) + 4 \left( x_4x_{10} + x_9x_5 \right) - 2\mu x_1, \\
\frac{dx_2}{dt} &= -x_2 - 9x_7 + 3x_6x_8 + 3 \left( x_1x_8 + x_6x_3 \right) - 9\mu x_2, \\
\frac{dx_3}{dt} &= -x_3 - 5x_8 - 7x_6x_7 - 7 \left( x_1x_7 + x_6x_2 \right) - 5\mu x_3 + \frac{R}{\tau}, \\
\frac{dx_4}{dt} &= -x_4 - 5x_9 - x_6x_{10} - \left( x_1x_{10} + x_6x_5 \right) - 5\mu x_4, \\
\frac{dx_5}{dt} &= -x_5 - 3x_{10} - 3x_9 - 3 \left( x_1x_9 + x_6x_4 \right) - \mu x_5, \\
\frac{dx_6}{dt} &= x_1, \quad \frac{dx_7}{dt} = x_2, \quad \frac{dx_8}{dt} = x_3, \\
\frac{dx_9}{dt} &= x_4, \quad \frac{dx_{10}}{dt} = x_5. 
\end{align*}
\] (10)

Now we are going to describe some properties of system (10). It is easily seen that stationary points for original variables \((x_6, ..., x_{10})\) coincide with the stationary points for 5-dimensional systems from [53] (because in steady states for (10) \(x_i \equiv 0, i = 1, ..., 5\) must be fulfilled). This is because the complement terms in (6) with coefficients \(\tau, \mu\) have time derivatives. Thus, the difference in stationary points between the usual and generalized cases consist in stability properties. Full investigation of ODE systems for generalized hydrodynamics is a forthcoming problem (especially in cases of higher-dimensional systems), so now we present only some numerical results.

Numerical calculations of the dynamical system (10) had been performed. It demonstrates complicated new irregular behaviour of the solution. This type of phase portrait (called the butterfly) corresponds to the Lorenz-type chaos. Behaviour is much more complicated with memory accounting. It is characterized by the large value of the maximal Lyapunov exponent (which is larger than 1) and the broad power spectrum. The projections of trajectories have a cross-like shape. Let us note that, such forms were found earlier in the 7-dimension system for the Navier-Stokes equations [54]. In analogy with [54] we may suppose that such a pattern is created when the eigenvalue of the Poincare map for the periodic orbits crosses the unit circle in point +1 and when nonsteady hyperbolic-type orbits are created.

According to the numerical integration, if \(R\) is increased, the amplitude of the limit cycle grows till stability loss, because the period double bifurcation occurs. Subsequent development of the periodic regime was studied with the help of the Poincare bifurcation diagram. We can distinguish several period double bifurcations attached to the chaotic region in the diagram. It is possible to make from the conclusion form of the chaotic region that, two different types of chaotic attractors interact in the phase space of dynamical system (10).

Thus it follows from obtained results that memory effects can cause more complicated behaviour of solutions of the hydrodynamical model, while nonlocality can play the stabilizing role (at least in considered cases).
5 Low-dimensional model for a three-dimensional case

Low-dimensional dynamical systems were constructed for three dimensional flows. Among results for such system it is necessary to point out two new features. Firstly, the flow with external force demonstrates the intermittency-II-type behaviour under certain conditions. Secondly, autooscillations were observed even without the external forces.

Continuing investigations of model (6), let us take the boundary conditions for the three-dimensional system (6) in the following form

$$\bar{V}|_{\partial \Omega} = 0,$$

where $\Omega \in \mathbb{R}^3$ is a region, bounded by surface $\partial \Omega$, which is the rigid boundary of the fluid volume. Relation (11), known as "stick" boundary condition, means that velocity $\bar{V}$ is vanishing on the rigid boundary (nonslip boundary conditions). Physically, it means that fluid doesn’t move near boundary $\bar{V}|_{\partial \Omega} = 0$.

According to the Galerkin method the solutions of model (6) are looked for as series expansions

$$V(x,t) = \sum_k z_k(t) \Phi_k(x),$$

where $\{\Phi_k(x)\}$ form the full system of orthogonal eigenfunctions for the linear eigenvalue problem for eigenvalues $\mu_k$ and eigenfunctions $\Phi_k$

$$\nu \Delta \Phi_k = -\mu_k \Phi_k + \text{grad } p_k, \quad \text{div } \Phi_k = 0,$$

$$\Phi_k|_{\partial \Omega} = 0, \quad \int_\Omega \Phi_k(x) dx = 1.$$ (13)

After some lengthy computations we get system for amplitudes $z_k$, $k \in \mathbb{N}$

$$\tau \frac{d^2 z_\ell}{dt^2} + \frac{dz_\ell}{dt} + \sum_k \sum_m c_{kml} z_k z_m + \mu_\ell z_\ell$$

$$+ \tau \sum_k \sum_m \left\{ z_m \frac{dz_k}{dt} + z_k \frac{dz_m}{dt} + \nu \frac{dz_\ell}{dt} \right\} c_{kml} = f_\ell,$$ (14)

where $\mu_\ell$, $\Phi_\ell$ are given by (13),

$$f_\ell = \int_\Omega \bar{F} \Phi_\ell dx, \quad c_{kml} = \int_\Omega (\Phi, \nabla) \Phi_m \Phi_k dx, \ell \in \mathbb{N}.$$ (15)

Values $f_\ell$ and $c_{kml}$ coincide with the analogous value from [55]. It is seen that for $\tau = \mu = 0$ system (14) coincides with that obtained in [39, 55] for the Navier-Stokes equations. To investigate the influence of memory and non-locality on truncated Galerkin approximations we take the system of ODEs from [39]:

$$\frac{dx_1}{dt} = -\eta_1 x_1 + A x_2 x_3 + F_1,$$

$$\frac{dx_2}{dt} = -\eta_2 x_2 + B x_1 x_3 + F_2,$$

$$\frac{dx_3}{dt} = -\eta_3 x_3 + C x_1 x_2 + F_3.$$ (16)
where \(A, B, C, F\) are constant and an additional relation is fulfilled \(A + B + C = 0\). The last condition is derived in [55] from consideration of non-viscous fluid kinetic energy with nullifying mass forces \((\nu = 0, \bar{F} = 0)\). Let us fix parameters

\[
A = 1, B = -2, C = 1, \\
F_1 = -\frac{5+\sqrt{76}}{2}, F_2 = -20 - \sqrt{76}, F_3 = \frac{34+5\sqrt{76}}{4}, \\
\eta_i = \nu, \quad i = 1, 2, 3.
\] (17)

for illustration. Then the low-dimensional system in this case is

\[
\frac{dx_1}{dt} = \frac{-x_1 - x_3x_6 - \nu x_4 + F_1}{\tau} - (x_3x_5 + x_6x_2) - \\
- \nu \text{sgn} |\mu| x_1, \\
\frac{dx_2}{dt} = \frac{-x_2 + 2x_4x_6 - \nu x_5 + F_2}{\tau} + 2 (x_1x_6 + x_3x_4) + \\
+ \nu \text{sgn} |\mu| x_2, \\
\frac{dx_3}{dt} = \frac{-x_3 - x_4x_5 - \nu x_6 + F_3}{\tau} - (x_2x_3 + x_1x_5) - \\
- \nu \text{sgn} |\mu| x_3, \\
\frac{dx_4}{dt} = x_1, \frac{dx_5}{dt} = x_2, \frac{dx_6}{dt} = x_3. \quad (18)
\]

It was shown in [55] that in the case of Navier-Stokes system there is a Hopf bifurcation, when \(\nu = 0\). Value \(\nu_{cr} = 2\) separates steady state and the region with periodic orbits. There is only a stable equilibrium for \(\bar{F} = 0\). Now, let us present the results of the numerical integration of system (18) performed with the help of the Runge-Kutta method. It is seen that growth of amplitudes is observed. Then we have decaying oscillations and almost constant solution on time length about five time units. However, near time 9.0 there is a strong burst of the solution. After time of 12.0 the burst decays. Then after time 14.0 the burst takes place again. This is similar with type-II intermittency.

Another interesting result is an oscillating regime without mass forces. These oscillations take place in nonzero initial values. This case can be interpreted as an evolution of flow without external influence. The presented behaviour illustrates one of the possible ways of explanation of turbulence in flows without restoring to negative viscosity. Our computations also confirm the stabilizing role of nonlocality.

Let us summarize the numerical and analytical results of the investigations of systems (16), (18), (9) and (10).

For the six-dimensional system (18) we found the emergence of periodical solutions. Such solutions were also found without external forces \((F = 0)\) with nonzero initial conditions. We also found, in some cases, phenomena similar to ”intermittency” (bursts in solutions). The ten-dimensional system (10) was investigated for some parameters. We changed the values of relaxation time and the initial conditions. In the case with no memory \((\tau = 0)\) the projections of the phase portrait on two-dimensional planes have a ”butterfly” type of attractor similar to the diagrams for the Lorenz chaotic attractor. In the case with \(\tau \neq 0\), there is complex behaviour of a new type. Visually, the trajectory densely fills a bounded volume (named ”container”). Trajectory projections on the planes have a broken form.
in many points. Locally, the projection of phase portrait looks like the ball of thread or "patience". Visually, the behaviour is similar to two-dimensional mappings with homoclinic tangency and quasiattractors which was described in [56]. Similar pictures were found in systems with the so-called "fat-fractals" with larger fractal dimension than the Cantor-type set. In many cases the results of numerical calculations look like the projections of motion on the torus. The first Liapunov characteristic exponent stayed positive for a long time (but with diminishing value). Numerical investigation of (10) with different values of \( R \) displayed the still excitation of complex behaviour. Standard one-dimensional bifurcation diagram is entirely different from the usual period doubling scenario of transition to chaos. We also made some numerical investigations of bifurcation for the 10-dimensional system. We found that in case of vanishing external forces \( (R=0) \) there was a unique stationary point with zero co-ordinates. The Jacobian of the right part of (10) had a pair of pure imaginary eigenvalues. Some further bifurcation was received by increasing the number of stationary points to ten with increasing the values of \( R \). At this processes the pairs of complex conjugate eigenvalues cross the imaginary axis from left to right. Further investigations of the systems above look very promising in the case \( \tau \ll 1 \). This is the singular perturbation of the usual systems of ODEs with chaos. The evaluation of bounds for the attractor dimension in the case \( \tau \to 0 \) is interesting (especially in the limit \( N \) or/and \( R \) tending to infinity). Note that (14) consists of ODEs of the second order in time. Hence, such a system recalls the collection of oscillators. Each \( z_k \) corresponds to a wave number of harmonics \( k \). So, we may anticipate the properties like transmission of energy on the spectrum of harmonics, the existence of harmonics clusters, resonances and so on. Also, some ideas of memory effects in turbulence may be reconsidered. There are many space and time scales in fluid flows. This implies the existence of many types of chaotic behaviour in fluids. The account of memory leads presumably to a new type of chaos similar to the chaos in media constructed from oscillators. Let us also note that, the above complex behaviour may serve as a prototype of a new possible type of chaos in media with finite speed of propagation and with gauge symmetry.

6 Applications of model equations to physical processes

Proposed model are rather new and are useful objects for further mathematical investigations. But just now some interpretations of the solutions may be proposed for real physical processes. Some applications to heat and mass transfer processes had been proposed and tested experimentally earlier: hyperbolic heat conduction equation, ignition theory by heat explosion, heat and mass transfer in turbulent and heterogeneous media (see [14]). New peculiarities of solutions founded in new model equations allows considering some aspects of very important recent physical and technical problems where strong non equilibrium properties are essential. Wide range of applications supplies the processes in astrophysics. The first subclass of problems is the description of the phenomena in the sparse plasma, including also MHD. The second relevant field of application is the investigation of the processes in the near-Earth space where a great number of different structures have been found experimentally. The third field of the investigation is the processes at the Sun, especially forming of hot and cool spots, arcs, bursts, cellular lattices on the surface etc. Many theories have been proposed before but the problems are still open. The most interesting consequences from proposed modal equations for such problems considerations are presumable fragmentation
of flow with diminishing of scales and increasing of vorticites and of solutions amplitudes for fast flows. Other important problem is the building of thermonuclear power plants. One of the main problems is such processes and apparatus control and suppression of wide spectrum of possible instabilities. Usually classical MHD equations are used. But proposed considerations lead to the conclusion of new more accurate equations needs far from equilibrium in such processes. Finally proposed model equations with memory and non-locality effects can be used for extending the classical synergetic problems to processes much far from equilibrium the in usual synergetic. Note for example that in classical synergetic classical Kuramoto-Sivashinsky equation is used for considering many kinds of instabilities. So, for the case of far from equilibrium processes the new equations for instabilities investigation may be proposed. Such equations may be extending of Kuramoto-Sivashinsky equation but with accounting memory and non-locality effects. For example one of such generalized equations (for one space dimension case) for structures formation has the form:

$$\tau \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial^3 u}{\partial x^3} + \beta \frac{\partial^4 u}{\partial x^4} = 0,$$

where $\alpha, \beta, \mu, \tau, \nu$ parameters. Note that the ordinary differential equation for the form of travelling wave solution coincides with the same in classical case ($\tau = 0$) but the stability conditions for such solutions are different from the classical case.

7 Conclusions

Thus, in this paper we propose to extend the studies of memory effects on some distributed systems which have classical description by partial differential and ordinary equations. This allows posing new mathematical problems on investigation of these new objects and their solutions. The first studies allow to find new properties and to propose a lot of new research problems.

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