EFFECTS OF THE NOISE LEVEL ON STOCHASTIC FRACTIONAL HEAT EQUATIONS

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Abstract. We consider the nonlinear stochastic fractional heat equations of the form
\[ \partial_t u = \frac{\alpha}{2} \Delta u + \lambda \sigma(u) \dot{w}(t, x), \]
with Dirichlet boundary conditions on the interval \([0, L]\), where \(\dot{w}\) denotes the space-time white noise and \(\lambda\) is the level of the noise. For \(\gamma < \gamma_0 < 0\), we prove that the equation has a unique mild solution \(u\) in the Banach space \(B_{p,\gamma}\) and the \(p\)th moment of the solution grows exponentially for any \(\lambda > 0\). For \(\gamma > 0\), under suitable assumptions, we prove that the equation has a unique mild solution \(u\) in the Banach space \(B_{p,\gamma}\) and we show that the \(p\)th moment of \(u\) is exponentially stable if the noise intensity \(\lambda\) is small. We prove the results about some properties of the \(p\)th moment and \(p\)th energy of mild solutions and obtain the noise excitation indice of \(p\)th energy as \(\lambda \to \infty\).

1. Introduction

We consider the following stochastic semilinear fractional heat equation with Dirichlet boundary condition:
\[
\begin{align*}
\partial_t u(t, x) &= \frac{\alpha}{2} \Delta u(t, x) + \lambda \sigma(u(t, x)) \dot{w}(t, x), \quad t > 0, \ x \in (0, L), \\
u(t, 0) &= u(t, L) = 0, \quad t > 0, \\
u(0, x) &= u_0(x), \quad x \in (0, L),
\end{align*}
\]
(1.1)
where \(L\) is a positive constant, \(\lambda\) is a positive parameter, \(\dot{w}\) is the space-time white noise on \((0, \infty) \times [0, L]\), \(\sigma : \mathbb{R} \to \mathbb{R}\) is a continuous function, \(\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}\) denotes the fractional Laplacian defined by
\[
\Delta^{\alpha/2} f(x) = c(\alpha) \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{f(x + y) - f(x)}{|y|^{1+\alpha}} dy,
\]
(1.2)
where \(y \in \mathbb{R}, 1 < \alpha < 2, c(\alpha)\) is a positive constant that depends only on \(\alpha\).

The initial value \(u_0\) satisfies the following condition.

Assumption 1.1. \(u_0\) is non-random and continuous on \([0, L]\), the Lebesgue measure

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of the set \([\mu, L - \mu] \subset \text{supp}(u_0)\) is strictly positive and \(\inf_{x \in [\mu, L - \mu]} u_0(x) > 0\), where \(\text{supp}(u_0)\) denotes the support of \(u_0\) and \(\mu \in (0, L/2)\).

The function \(\sigma\) satisfies the following condition.

**Assumption 1.2.** There exist constants \(l_\sigma\) and \(L_\sigma\) such that for \(u, v \in \mathbb{R}\), \(|\sigma(u) - \sigma(v)| \leq L_\sigma |u - v|\) and \(l_\sigma |u| \leq |\sigma(u)| \leq L_\sigma |u|\).

We denote by \(\{\mathcal{F}_t\}_{t \geq 0}\) the filtration generated by the Brown sheet \(\{w(t, x); t \geq 0, x \in [0, L]\}\). For \(1 \leq p < \infty\), \(L^p[0, L]\) denotes the space of all real valued measurable functions \(u: [0, L] \to \mathbb{R}\) such that \(|u(x)|^p\) is integrable for \(x \in [0, L]\). It is a Banach space when normed by

\[
\|u\|_{L^p[0, L]} = \left( \int_0^L |u(x)|^p dx \right)^{1/p}.
\]

If \(p = \infty\), the space \(L^\infty[0, L]\) consists of all real valued measurable functions with a finite norm \(\|u\|_{L^\infty} = \text{ess sup}_{x \in [0, L]} |u(x)|\).

For each \(\gamma \in \mathbb{R}\) and \(p \geq 2\), Xie [7] define the \(p\)th energy of the solution at time \(t\) by

\[
\Phi_p(t, \lambda) = \left( \mathbb{E}[e^{\gamma t}\|u(t)\|_{L^p[0, L]}^p] \right)^{1/p}, \quad t > 0.
\]

**Definition 1.3.** The excitation index of \(u\) at time \(t\) is given by

\[
e(t) := \lim_{\lambda \to \infty} \frac{\log \log \Phi_p(t, \lambda)}{\log \lambda}.
\]

The parameter \(\lambda > 0\) in [1.1] is called the level of noise (or noise level, for short). Recently, effects of the noise level on nonlinear stochastic heat equations.
has attracted a great deal of research interest. Khoshnevisan and Kim [3] studied a stochastic heat equation of the form
\[ \frac{\partial}{\partial t} u = Lu + \lambda \sigma(u) \xi, \]
where \( \xi \) denotes space-time white noise on \( \mathbb{R}_+ \times G \), \( L \) is the generator of a Lévy process on a locally compact Hausdorff Abelian group \( G \), \( \sigma : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous, \( \lambda \) is a large parameter. They showed that if \( u \) is intermittent, the energy of the solution behaves generically as \( \exp[\text{const} \cdot \lambda^2] \) when \( G \) is discrete and \( \geq \exp[\text{const} \cdot \lambda^4] \) when \( G \) is connected. Khoshnevisan and Kim [2] studied the semilinear heat equation
\[ \partial_t u(t, x) = \partial_{xx} u(t, x) + \lambda \sigma(u(t, x)) \xi \] (1.5)
on the interval \([0, L]\), where \( \xi \) is space-time white noise, \( \sigma : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous. The authors proved when the solution of Eq. (1.5) is intermittent, the \( L^2 \)-energy of the solution grows at least \( \exp(c \lambda^2) \) and at most as \( \exp(c \lambda^4) \) as \( \lambda \to \infty \). Foondun and Joseph [13] considered Eq. (1.5), they used Gaussian estimates for Dirichlet (Neumann) heat kernel and renewal inequalities to show that the expected \( L^2 \)-energy of the mild solution is of order \( \exp[\text{const} \cdot \lambda^4] \) as \( \lambda \to \infty \). Foondun and Nualart [15] considered a stochastic heat equation on an interval
\[ \partial_t u(t, x) = \frac{1}{2} \partial_{xx} u(t, x) + \lambda \sigma(u(t, x)) \dot{w}(t, x), \] (1.6)
where \( \dot{w} \) denotes the space-time white noise. They proved the second moment of the solution grows exponentially fast if the noise intensity \( \lambda \) is large enough, the second moment decays exponentially if \( \lambda \) is small. For \( p \geq 2 \), they proved the \( p \)th moments have a similar property. Xie [7] studied Eq. (1.6) on \([0, 1]\), the author showed that for small noise level, the \( p \)th moment of \( \sup_{x \in [0,1]} |u(t, x)| \) is exponentially stable. For large noise level, the moment grows at least exponentially by an approach depending on the lower bound of the global estimate for Dirichlet heat kernel. The \( p \)th energy of the solution at time \( t \) is defined and the noise excitation index of the \( p \)th energy
of $u(t, x)$ is proved to be 4 as the noise level tends to infinity. Foondum, Tian and Liu [14] studied nonlinear parabolic stochastic equations of the form

$$\partial_t u(t, x) = \mathcal{L}u(t, x) + \lambda \sigma(u(t, x))\dot{w}(t, x)$$

on the ball $B(0, \mathbb{R})$, where $\dot{w}$ denotes white noise on $(0, \infty) \times B(0, \mathbb{R})$, $\mathcal{L}$ is the generator of an $\alpha$-stable process killed upon existing $B(0, \mathbb{R})$. The growth properties of the second moment of the solutions is obtained. Their results extend those in [13] and [2].

In this paper, we prove that there exists a $\gamma_0 < 0$ such that for any $\gamma < \gamma_0$, Eq. (1.1) has a unique mild solution $u_\lambda(t, \cdot) \in B_{p, \gamma}$ and then present an upper bound of the growth rate of the mild solution for any $\lambda > 0$. Our approach depends on the heat kernel estimates for the Dirichlet fractional Laplacian $\Delta^{\alpha/2}|_{(0,L)} := -(-\Delta)^{\alpha/2}|_{(0,L)}$. Assuming that $\lambda_1$ is the smallest eigenvalue of $(-\Delta)^{\alpha/2}$ on the interval $(0, L)$, for $p > 2/(\alpha - 1)$, $\beta \in (2/p, \alpha - 1)$ and $\gamma \in (0, (2 - \beta)\lambda_1)$, we prove that there exists $\lambda_\ell > 0$ such that for any $\lambda \in (0, \lambda_\ell)$, Eq. (1.1) has a unique mild solution $u_\lambda(t, \cdot) \in B_{p, \gamma}$. In fact, for small times, the Dirichlet fractional heat kernel behaves likes the fractional heat kernel, however, it decays exponentially for large times. Moreover, we show that for all $\lambda \in (0, \lambda_\ell)$ and $t \geq 0$, the $p$th moment of $\|u_\lambda(t)\|_{L^\infty}$ is exponentially stable. Then, we use the convolution-type inequalities and the asymptotic property of the Mittag-Leffler functions to prove that there exists a constant $T_0 > 0$ such that for any $t > T_0$, the second moment of the mild solution grows at least exponentially for large times. And we show that the second moment of the mild solution grows exponentially fast for all times. We give some properties of the $p$th moment and the $p$th energy of the mild solution and consider the nonlinear noise excitability of Eq. (1.1) for large noise level $\lambda$. We show that the noise excitation index of the solution $u_\lambda(t, x)$ with respect to $p$th energy $\Phi_p(t, \lambda)$ is $\frac{2\alpha}{\alpha - 1}$.

The paper is organized as follows. In Section 2, we present some lemmas and preliminary facts, which will be used in the next sections. In Section 3, we obtain
some properties of the mild solutions of Eq. (1.1) in different Banach space $B_{p,\gamma}$.

In Section 4, we prove that the $p$th moment of the mild solution grows at least exponentially and obtain the excitation index of the mild solution of Eq. (1.1).

Throughout this paper, we use $c_0, c_1, c_2, \ldots$ to denote generic constants, which may change from line to line.

2. Preliminaries

Let $p(t, x, y)$ be the heat kernel of $\Delta^{\alpha/2}$ on $\mathbb{R}$. We have the following inequality (see [5])

$$0 \leq p_D(t, x, y) \leq p(t, x, y) \text{ for all } t > 0, \ x, y \in (0, L). \quad (2.1)$$

For two nonnegative functions $f_1$ and $f_2$, the notion $f_1 \asymp f_2$ means that $c_1 f_2(x) \leq f_1(x) \leq c_2 f_2(x)$, where $c_1, c_2$ are positive constants. It is well known that (see, e.g., [1, 4, 9])

$$p(t, x, y) \asymp (t^{-1/\alpha} \wedge \frac{t}{|x - y|^{1+\alpha}}),$$

that is, there exist constants $c_1, c_2$ such that for $t > 0, \ x, y \in \mathbb{R}$,

$$c_1 \left( t^{-1/\alpha} \wedge \frac{t}{|x - y|^{1+\alpha}} \right) \leq p(t, x, y) \leq c_2 \left( t^{-1/\alpha} \wedge \frac{t}{|x - y|^{1+\alpha}} \right), \quad (2.2)$$

where $c_1$ and $c_2$ are positive constants depending on $\alpha$.

Another equivalent form of (2.2) is

$$\frac{c_1 t}{(t^{1/\alpha} + |y - x|)^{1+\alpha}} \leq p(t, x, y) \leq \frac{c_2 t}{(t^{1/\alpha} + |y - x|)^{1+\alpha}}, \quad (2.3)$$

where $t > 0, \ x, y \in \mathbb{R}$, $c_1$ and $c_2$ are positive constants depending on $\alpha$.

Lemma 2.1. (Garsia-Rodemich-Rumsey Theorem, see [17, Appendix A]) Consider $f \in C([0, L], E)$, where $(E, d)$ is a complete metric space. Let $\Psi$ and $p$ be continuous strictly increasing functions on $[0, \infty)$ with $p(0) = \Psi(0) = 0$ and $\Psi(x) \to \infty$ as $x \to \infty$. Then

$$\int_0^L \int_0^L \Psi\left( \frac{f(x) - f(y)}{p(|x - y|)} \right) dx dy \leq F,$$
implies, for \( 0 \leq x < y \leq L \),

\[
d(f(x) - f(y)) \leq 8 \int_{0}^{y-x} \Psi^{-1}\left(\frac{4F}{u^2}\right) dp(u).
\]

In particular, if \( \text{osc}(f, \delta) \equiv \sup\{d(f(x), f(y)) : x, y \in [0, L], |x - y| \leq \delta\} \) denotes the modulus of continuity of \( f \), we have

\[
\text{osc}(f, \delta) \leq 8 \int_{0}^{\delta} \Psi^{-1}\left(\frac{4F}{u^2}\right) dp(u).
\]

**Lemma 2.2.** Suppose that \( \{u(x)\}_{x \in [0, L]} \) is a real valued stochastic process. If there exist \( p \geq 1 \) and positive constant \( K, \delta \) such that

\[
\mathbb{E}[|u(x) - u(y)|^p] \leq K|x - y|^{1+\delta},
\]

then \( \{u(x)\}_{x \in [0, L]} \) has a continuous modification, which will be still denoted by \( \{u(x)\}_{x \in [0, L]} \). For each \( \varepsilon \in (0, \min\{\delta, 1\}) \), there exists a positive constant \( \kappa \) depending only on \( \delta, \varepsilon \) such that

\[
|u(x) - u(y)| \leq \kappa(4B)^{1/p}|x - y|^{(\delta - \varepsilon)/p},
\]

where \( B = B(p, \varepsilon, \delta) \) is a positive random variable defined by

\[
B = \int_{0}^{L} \int_{0}^{L} \frac{|u(x) - u(y)|^p}{|x - y|^{2+\delta-\varepsilon}} dxdy.
\]

In particular, the stochastic process \( \{u(x)\}_{x \in [0, L]} \) has a \((\delta - \varepsilon)/p\)-Hölder continuous modification.

**Proof.** By Lemma 2.1, we can finish the proof, which is similar to that of Lemma 2.1 in [7], so it is omitted. \( \square \)

**Remark 2.3.** There is an inaccuracy in the formula corresponding to (4.30) in [7] (see [7], formula (2.5)). \( B \) should be changed to \( 4B \). The reason is that the Theorem A.1 (Garsia, Rodemich and Rumsey inequality) in [7] is not completely the same as Lemma 1.1 in [11].

**Lemma 2.4.** Suppose \( b \geq 0, \beta \geq 0 \) and \( a(t) \) is a nonnegative function locally on \( 0 \leq t < T \) (some \( T \leq \infty \)), and suppose \( v(t) \) is nonnegative and locally integrable on
$0 \leq t < T$ with
\[ v(t) \geq a(t) + b \int_0^t (t-s)^{\beta-1} v(s) ds, \]
on this interval; then
\[ v(t) \geq a(t) + \theta \int_0^t E'_\beta(\theta(t-s))a(s) ds, \quad 0 \leq t < T, \]
where
\[ \theta = \left(b \Gamma(\beta)\right)^{1/\beta}, \quad E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n \beta + 1)}, \quad E'_\beta(z) = \frac{d}{dz} E_\beta(z), \]
If $a(t) \equiv a$, constant, then $v(t) \geq a E_\beta(\theta t)$.

**Proof.** The proof is similar to that of Lemma 7.1.1 in [16], for the sake of completeness, we provide it in the following. Let $B \phi(t) = b \int_0^t (t-s)^{\beta-1} \phi(s) ds$, $t \geq 0$, for locally integrable functions $\phi$. Then $v \geq a + Bv$ implies that
\[ v \geq \sum_{k=0}^{n-1} B^k a + B^n v \]
and
\[ B^n v(t) = \frac{1}{\Gamma(n \beta)} \int_0^t (b \Gamma(\beta))^n (t-s)^{n \beta-1} u(s) ds \to 0 \]
as $n \to \infty$. Thus
\[ v(t) \geq a(t) + \int_0^t \left\{ \sum_{n=1}^{\infty} \frac{1}{\Gamma(n \beta)} (b \Gamma(\beta))^n (t-s)^{n \beta-1} \right\} a(s) ds \]
\[ = a(t) + \theta \int_0^t E'_\beta(\theta(t-s))a(s) ds \]
If $a(t) = a$ is a constant, then
\[ v(t) \geq a + a \theta \int_0^t E'_\beta(\theta s) ds \]
\[ \geq a + a \int_0^\theta E'_\beta(s) ds \]
\[ = a E_\beta(\theta t). \]
The proof is complete. $\Box$
3. Properties of mild solutions to Eq. (1.1) in different Banach spaces

In this section, we present the existence and uniqueness result of Eq. (1.1) in different Banach spaces $B_{p,\gamma}$. We prove that there exists $\gamma_0 < 0$, for any $\gamma < \gamma_0$, Eq. (1.1) has a unique mild solution $u_\lambda(t, \cdot) \in B_{p,\gamma}$, and then we present an upper bound of the growth rate of the solution for any $\lambda > 0$. For $\lambda$ is small, under some assumptions, we show that there exists a constant $\lambda_L$ such that for $\lambda \in (0, \lambda_L)$, Eq. (1.1) has a unique mild solution $u_\lambda(t, x)$ and the $p$th moment of $\|u_\lambda(t)\|_{L^\infty}$ is exponential stable.

**Lemma 3.1.** Let $\beta \in (0, \alpha - 1)$. There exists a positive constant $c(\alpha, \beta)$ that only depends on $\alpha$ and $\beta$ such that

$$\sup_{t \geq 0, x \in [0, L]} \int_0^t e^{\gamma s} s^{-2\beta/\alpha} \int_0^L |p_D(s, x, y)|^{2-\beta} dy ds \leq c(\alpha, \beta)|\gamma|^{(\beta+1)/\alpha}$$

for any $\gamma < 0$.

**Proof.** By (2.1), (2.3), we have

$$\int_0^t e^{\gamma s} s^{-2\beta/\alpha} \int_0^L |p_D(s, x, y)|^{2-\beta} dy ds \leq c_2^{-\beta}(\alpha) \int_0^t e^{\gamma s} s^{-2\beta/\alpha} \int_0^L \left| \frac{s^{1/\alpha} + |y-x|}{(s^{1/\alpha} + |y-x|)^{1+\alpha}} \right|^{2-\beta} dy ds$$

$$\leq c_2^{-\beta}(\alpha) \int_0^t e^{\gamma s} s^{-2\beta/\alpha} \int_0^L \frac{1}{(1 + s^{-1/\alpha}|y-x|)^{(1+\alpha)(2-\beta)}} dy ds$$

$$\leq c_2^{-\beta}(\alpha) \int_0^t e^{\gamma s} s^{-2\beta/\alpha} \int_0^L \frac{1}{(1 + |\tau|)^{(1+\alpha)(2-\beta)}} d\tau ds$$

$$\leq c_2^{-\beta}(\alpha) \int_0^t e^{\gamma s} s^{-(\beta+1)/\alpha} ds$$

$$\leq c_2^{-\beta}(\alpha) \pi \int_0^t e^{\gamma s} s^{-(\beta+1)/\alpha} ds$$

$$\leq c_2^{-\beta}(\alpha) \pi |\gamma|^{(\beta+1)/\alpha} \int_0^\infty e^{-s} s^{-(\beta+1)/\alpha} ds$$

$$\leq c_2^{-\beta}(\alpha) \pi \Gamma \left( \frac{\alpha - \beta - 1}{\alpha} \right) |\gamma|^{(\beta+1)/\alpha},$$
where $\Gamma(\cdot)$ is the Gamma function.

For any $u \in B_{p, \gamma}$, we define the stochastic convolution integral

$$Su(t, x) = \int_0^t \int_0^L p_D(t - s, x, y)u(s, y)w(dsdy).$$

**Lemma 3.2.** Suppose $p > 2/(\alpha - 1)$ and $\gamma < 0$. Then for each $\beta \in (2/p, \alpha - 1)$, there exists a positive constant $c = c(\alpha, \beta, p)$ depending only on $\alpha$, $\beta$ and $p$, such that for all $u \in B_{p, \gamma}$,

$$\|Su\|_{p, \gamma}^p \leq c\|u\|_{p, \gamma}^p (|\gamma|^{\beta + 1 - \alpha} p^{2/\alpha} + |\gamma|^{1 - \alpha} p^{2/\alpha}).$$

**Proof.** By Burkholder’s inequality, for any $p \in (0, \infty)$, there exists a positive constant $c(p)$ such that for any $x, y \in [0, L]$ and $t \geq 0$,

$$\mathbb{E}|Su(t, x) - Su(t, y)|^p \leq c(p)\mathbb{E}\left( \int_0^t \int_0^L (p_D(t - s, x, z) - p_D(t - s, y, z))^2u^2(s, z)dzds \right)^{p/2},$$

where $c(p)$ is a positive constant. By Minkowski’s integral inequality (see [18], Appendix A.1), we have for $p \geq 2$, $\beta \in (0, 2)$,

$$\mathbb{E}|Su(t, x) - Su(t, y)|^p \leq c(p)\left( \int_0^t \int_0^L |p_D(t - s, x, z) - p_D(t - s, y, z)|^\beta|p_D(t - s, x, z) - p_D(t - s, y, z)|^{2-\beta} \right. 
\left. \times (\mathbb{E}|u(s, z)|^p/\beta^{\beta/\alpha})^{2/\beta}dzds \right)^{p/2}. \quad (3.1)$$

Note that for any $x, y \in \mathbb{R}$ and $f \in C^1(\mathbb{R})$, we have

$$\int_0^1 f'(x + r(y - x))(y - x)dr = f(y) - f(x). \quad (3.2)$$

By (3.1), (3.2), we get

$$\mathbb{E}|Su(t, x) - Su(t, y)|^p \leq c(p)|x - y|^{\beta p/2} \left[ \int_0^t \int_0^L |\partial_x p_D(t - s, x + r(y - x), z)dr|^\beta|p_D(t - s, x, z) - p_D(t - s, y, z)|^{2-\beta} \right. 
\left. \times (\mathbb{E}|u(s, z)|^p/\beta^{\beta/\alpha})^{2/\beta}dzds \right]^{p/2} 
:= K_1(t, x, y)|x - y|^{\beta p/2}, \quad (3.3)$$
where

\[ K_1(t, x, y) = c(p) \left[ \int_0^t \int_0^L \left| \partial_z p_D(t - s, x + r(y - x), z) dr \right|^\beta |p_D(t - s, x, z) - p_D(t - s, y, z)|^{2-\beta} \right. \]

\[ \times \left( E|u(s, z)|^p \right)^{2/p} dz ds \right]^{p/2}. \]  

\[ (3.4) \]

From the proof of Theorem 5.1 in [10], it follows that for any \( t > 0, x, y \in (0, L) \),

\[ |\partial_z p_D(t, x, y)| \leq c_3(\alpha) t^{-1/\alpha} p(t, x, y). \]

\[ (3.5) \]

Then, (3.5) together with (3.3) yields that

\[ |\partial_z p_D(t, x, y)| \leq c_4(\alpha) t^{-2/\alpha}, \]

\[ (3.6) \]

where \( c_4(\alpha) \) is a positive constant depending only on \( \alpha \).

By (2.5), (2.6), we obtain

\[ K_1(t, x, y) \leq c_4^{\delta}(\alpha) c(p) \left[ \int_0^t \int_0^L (t - s)^{-2\beta/\alpha} |p_D(t - s, x, z) - p_D(t - s, y, z)|^{2-\beta} (E|u(s, z)|^p)^{2/p} dz ds \right]^{p/2} \]

\[ \leq c_4^{\delta}(\alpha) c(p) \left[ \int_0^t \int_0^L (t - s)^{-2\beta/\alpha} |p_D(t - s, x, z) - p_D(t - s, y, z)|^{2-\beta} (E|u(s)|^p)^{2/p} dz ds \right]^{p/2} \]

\[ \leq c_4^{\delta}(\alpha) c(p) \left[ \int_0^t \int_0^L (t - s)^{-2\beta/\alpha} |p_D(t - s, x, z) - p_D(t - s, y, z)|^{2-\beta} \right. \]

\[ \cdot (e^{-\gamma s} e^{\gamma s} E|u(s)|^p)^{2/p} dz ds \right]^{p/2} \]

\[ \leq c_4^{\delta}(\alpha) c(p) \left[ \int_0^t \int_0^L e^{-2\gamma s/p} (t - s)^{-2\beta/\alpha} |p_D(t - s, x, z) - p_D(t - s, y, z)|^{2-\beta} dz ds \right]^{p/2} \]

\[ \leq c_4^{\delta}(\alpha) c(p) e^{-\gamma t} \left[ \int_0^t \int_0^L e^{2\gamma (t - s)/p} (t - s)^{-2\beta/\alpha} |p_D(t - s, x, z) - p_D(t - s, y, z)|^{2-\beta} dz ds \right]^{p/2} \]

\[ = c_4^{\delta}(\alpha) c(p) e^{-\gamma t} \left[ \int_0^t \int_0^L e^{2\gamma s/p} (t - s)^{-2\beta/\alpha} |p_D(s, x, z) - p_D(s, y, z)|^{2-\beta} dz ds \right]^{p/2}. \]

\[ (3.7) \]

From Lemma 2.2 and (3.7), it follows that

\[ K_1(t, x, y) \leq c_5(\alpha, \beta, p) e^{-\gamma t} \left[ \int_0^t \int_0^L e^{2\gamma s/p} (t - s)^{-2\beta/\alpha} |p_D(s, x, z) - p_D(s, y, z)|^{2-\beta} dz ds \right]^{p/2}. \]

\[ = c_5(\alpha, \beta, p) e^{-\gamma t} |u|_{p, \gamma}^{\beta + 1 - \alpha/p} \]

\[ := K_2(t). \]

\[ (3.8) \]
where $c_5(\alpha, \beta, p)$ is a positive constant depending only on $\alpha, \beta, p$.

By (3.3) and (3.8), we have

$$
E |Su(t, x) - Su(t, y)|^p \leq c_5(\alpha, \beta, p)e^{-\gamma t}\|u\|_{p, \gamma}^{(\beta+1-\alpha)p/2}\|x - y\|^\beta/2
= K_2(t)|x - y|^\beta/2.
$$

(3.9)

By Lemma 2.2, we have

$$
|Su(t, x) - Su(t, y)|^p \leq \kappa(4B(t))^{1/p}|x - y|^{\beta/2 - (1+\varepsilon)/p}
$$

(3.10)

where $\kappa$ is same as that in (4.30),

$$
B(t) = \int_0^L \int_0^L \frac{|Su(t, x) - Su(t, y)|^p}{|x - y|^{1+\beta p/2 - \varepsilon}}dxdy,
$$

and $\varepsilon \in (0, \min\{\beta p/2 - 1, 1\})$.

Fix $y = y_0 \in [0, L]$. By (3.10), we get

$$
|Su(t, x)| \leq \kappa((4B(t))^{1/p}|x - y_0|^{\beta/2 - (1+\varepsilon)/p} + |Su(t, y_0)|,
$$

then for $x, y_0 \in [0, L],

$$
|Su(t, x)| \leq \kappa(4B(t))^{1/p}L^{\beta/2 - (1+\varepsilon)/p} + |Su(t, y_0)|,
$$

(3.11)

where $\beta/2 - (1 + \varepsilon)/p > 0$. Taking the $p$th moments to both sides of (3.11) and consider the inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ ($a, b \in \mathbb{R}$), we get

$$
|Su(t, x)|^p \leq 2^{p-1}\kappa^p(B(t))L^{\beta p/2 - (1+\varepsilon)} + |Su(t, y_0)|^p.
$$

(3.12)

Taking expectations of both sides of the above inequality, we have

$$
E \sup_{x \in [0, L]} |Su(t, x)|^p \leq 2^{p-1}\kappa^pL^{\beta p/2 - (1+\varepsilon)}E(B(t)) + 2^{p-1}E|Su(t, y_0)|^p.
$$

(3.13)

By (3.9), we have

$$
E(B(t)) \leq K_2(t) \int_0^L \int_0^L \frac{1}{|x - y|^{1-\varepsilon}}dxdy
\leq \frac{2L^{\varepsilon+1}}{\varepsilon(\varepsilon + 1)}K_2(t)
= \frac{2L^{\varepsilon+1}}{\varepsilon(\varepsilon + 1)}c_5(\alpha, \beta, p)e^{-\gamma t}\|u\|_{p, \gamma}^{\beta}\|\gamma|^{(\beta+1-\alpha)p/2\alpha}.
$$

(3.14)
By Burkholder’s inequality, Minkowski’s integral inequality and the semigroup property, we obtain

\[
E|Su(t, y_0)|^p = E \left| \int_0^t \int_0^L p_D(t - s, y_0, y)u(s, y)w(dsdy) \right|^p \\
\leq c_p E \left[ \int_0^t \int_0^L p_D^2(t - s, y_0, y)u^2(s, y)dyds \right]^{p/2} \\
\leq c_p \left[ \int_0^t \int_0^L p_D^2(t - s, y_0, y)(E|u(s, y)|^p)^{2/p}dyds \right]^{p/2} \\
\leq c_p \left[ \int_0^t \int_0^L p_D^2(t - s, y_0, y)e^{-2\gamma s/p}(e^{\gamma s}E\|u(s)\|_{L^\infty}^p)^{2/p}dyds \right]^{p/2} \\
= c_p \|u\|_{P, \gamma}^p \left[ \int_0^t e^{-2\gamma s/p} \left( \int_0^L p_D^2(t - s, y_0, y)dy \right)ds \right]^{p/2} \\
= c_p e^{-\gamma t}\|u\|_{P, \gamma}^p \left[ \int_0^t e^{2\gamma(t-s)/p}p_D(2(t-s), y_0, y_0)ds \right]^{p/2} \\
= c_p e^{-\gamma t}\|u\|_{P, \gamma}^p \left[ \int_0^t e^{2\gamma s/p}p_D(2s, y_0, y_0)ds \right]^{p/2} \\
\leq c_p e^{-\gamma t}\|u\|_{P, \gamma}^p \left( \int_0^t e^{2\gamma s/p}s^{-1/\alpha}ds \right)^{p/2} \\
\leq c_p e^{-\gamma t}\|u\|_{P, \gamma}^p \left( \int_0^t e^{2\gamma s/p}s^{-1/\alpha}ds \right)^{p/2} \\
\leq c_p e^{-\gamma t}\|u\|_{P, \gamma}^p \left( \frac{p}{2} \left(1 - \frac{1}{\alpha} \right) \right)^{p/2} e^{-\gamma t}\|u\|_{P, \gamma}^p |\gamma|^{(1-\alpha)p/2a}. \tag{3.15}
\]

Putting (3.14) and (3.15) into (3.13), we have

\[
E \sup_{x \in [0, L]} |Su(t, x)|^p \leq c_8(\alpha, \beta, p)e^{-\gamma t}\|u\|_{P, \gamma}^p |\gamma|^{(1-\alpha)p/2a} \tag{3.16}
\]

where \(c_8(\alpha, \beta, p)\) is a positive constant that only depends on \(\alpha, \beta\) and \(p\). From (3.16) and (1.3), it follows that

\[
\|Su\|_{P, \gamma}^p \leq c_8(\alpha, \beta, p)\|u\|_{P, \gamma}^p |\gamma|^{(1-\alpha)p/2a} \tag{3.17}
\]

The proof is complete. \(\square\)

**Lemma 3.3.** Suppose \(p > 2/(\alpha - 1)\) and \(\gamma < 0\). Then for each \(\beta \in (2/p, \alpha - 1)\), there exists a positive constant \(c = c(\alpha, \beta, p)\) depending only on \(\alpha, \beta\) and \(p\), such that for any \(u, v \in B_{P, \gamma}\),

\[
\|Su - Sv\|_{P, \gamma}^p \leq c\|u - v\|_{P, \gamma}^p |\gamma|^{(\beta+1-\alpha)p/2a} + |\gamma|^{(1-\alpha)p/2a}.
\]
**Proof.** The proof is similar to that of Lemma 3.2, we omit it. □

**Theorem 3.4.** Suppose \( p > 2/(\alpha - 1) \) and \( \beta \in (2/p, \alpha - 1) \). Then exists \( \gamma_0 < 0 \) such that for all \( \gamma < \gamma_0 \), Eq. (1.1) has a unique mild solution \( u_\lambda(t, \cdot) \in B_{p, \gamma} \). For all \( \gamma < \gamma_0 \), the following inequality holds

\[
\lim_{t \to \infty} \mathbb{E} \left[ \log \left| \frac{1}{t} \log \| u_\lambda(t) \|_{L^\infty} \right| \right] \leq -\gamma.
\]

That is, the growth of \( u_\lambda(t, x) \) in time \( t \) is at most in an exponential rate in the \( p \)-moment sense.

**Proof.** Define the operator \( T \) as follows

\[
Tu(t, x) = \int_0^L p_D(t, x, y)u_0(y)dy + \int_0^t \int_0^L p_D(t-s, x, y)\lambda \sigma(u(s, y))w(dsdy),
\]

where \( u \in B_{p, \gamma} \).

Since \( u_0 \) is non-random, for any \( \gamma < 0 \), we have

\[
\left| \int_0^L p_D(t, x, y)u_0(y)dy \right|^p \leq \| u_0 \|^p_{L^\infty} \sup_{t \geq 0} e^{\gamma t} \left| \int_0^L p_D(t, x, y)dy \right|^p \leq \| u_0 \|^p_{L^\infty} \sup_{t \geq 0} e^{\gamma t} \left| \int_0^L p(t, x, y)dy \right|^p \leq \| u_0 \|^p_{L^\infty}.
\]

Define the stochastic convolution \( S \sigma(u(t, x)) = \int_0^t \int_0^L p_D(t-s, x, y)\lambda \sigma(u(s, y))w(dsdy) \).

By Assumption 1.2, \( \sigma(u) \leq L_x |u| \). From Lemma 2.4, it follows that for any \( u \in B_{p, \gamma} \),

\[
\| Tu \|^p_{p, \gamma} \leq 2^{p-1} \| u_0 \|^p_{L^\infty} + 2^{p-1} \lambda^p c(\alpha, \beta, p, \kappa, L)L^p_x \| u \|^p_{p, \gamma}(|\gamma|^{(\beta+1-\alpha)p/2\alpha} + |\gamma|^{(1-\alpha)p/2\alpha}),
\]

where \( p > 2/(\alpha - 1) \), \( \beta \in (2/p, \alpha - 1) \) and \( \gamma < 0 \). Then we see that \( T \) maps \( B_{p, \gamma} \) into \( B_{p, \gamma} \) for any \( \gamma < 0 \). For any \( u, v \in B_{p, \gamma} \), by Lemma 2.5, we have

\[
\| Tu - Tv \|^p_{p, \gamma} = L^p_x \| S \sigma(u) - S \sigma(v) \|^p_{p, \gamma} \leq c(\alpha, \beta, p)\lambda^p L^p_x (|\gamma|^{(\beta+1-\alpha)p/2\alpha} + |\gamma|^{(1-\alpha)p/2\alpha}) \| u - v \|^p_{p, \gamma}, \quad (3.18)
\]
Since $2/p + 1 < \alpha < 2$, $0 < \beta < \alpha - 1$, we can choose a $\gamma_0 < 0$ such that for any $\gamma < \gamma_0$,

$$0 < c(\alpha, \beta, p)\lambda^p L_p^p(|\gamma|^{(\beta+1-\alpha)p/2\alpha} + |\gamma|^{(1-\alpha)p/2\alpha}) < 1.$$ 

From the contraction mapping theorem, it follows that $T$ has a unique fixed point on $B_{p,\gamma}$, therefore Eq.(1.1) has a unique mild solution $u_\lambda(t, \cdot) \in B_{p,\gamma}$. Since $\|u_\lambda\|_{p,\gamma} = \|Tu_\lambda\|_{p,\gamma} < \infty$, it is easy to prove that $\lim_{t \to \infty} \sup_{t} \frac{1}{t} \log \mathbb{E}\|u_\lambda(t)\|_{L^\infty} \leq -\gamma$. The proof is complete. □

**Lemma 3.5** (see [8], Theorem 1.1.) Let $D$ be a $C^{1,1}$ open subset of $\mathbb{R}^d$ with $d \geq 1$ and $\delta_D(x)$ the Euclidean distance between $x$ and $D^c$.

(i) For every $T > 0$, on $(0, T] \times D \times D$,

$$p_D(t, x, y) \asymp (1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}})(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}})(t^{-1/\alpha} \wedge \frac{t}{|x - y|^{1+\alpha}}).$$

(ii) Suppose in addition that $D$ is bounded. For every $T > 0$, there are positive constants $c_1 < c_2$ such that on $[T, \infty) \times D \times D$,

$$c_1 e^{-\lambda_0 t} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \leq p_D(t, x, y) \leq c_2 e^{-\lambda_0 t} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2},$$

where $\lambda_0 > 0$ is the smallest eigenvalue of the Dirichlet fractional Laplacian $(-\Delta)^{\alpha/2}|_D$, $p_D(t, x, y)$ is the Dirichlet fractional heat kernel. For the definition of $C^{1,1}$ open set, we refer to [8].

Throughout this paper, we assume that $\lambda_1$ is the smallest eigenvalue of the Dirichlet fractional Laplacian $(-\Delta)^{\alpha/2}|_{(0,L)}$.

**Lemma 3.6** Suppose $\beta \in (0, \alpha - 1)$ and $\gamma \in (0, (2 - \beta)\lambda_1)$. Then there exists a positive constant $c(\alpha, \beta)$, such that for any $t \geq 0$ and $x \in (0, L)$,

$$\int_0^t e^{\gamma s} s^{-2\beta/\alpha} \int_0^L |p_D(s, x, y)|^{2-\beta} dy ds \leq c(\alpha, \beta) \left(\gamma^{(\beta+1-\alpha)/\alpha} + \frac{1}{(2-\beta)\lambda_1 - \gamma}\right).$$
Proof. By (2.1), (2.3), similar to the proof of Lemma 2.3, we have
\[
\int_0^1 e^{\gamma s} s^{-(\beta+1)/\alpha} ds\leq c_2 (\alpha, \beta) (\alpha/(2 - \beta) \lambda_1 - \gamma).
\] (3.21)

By (3.20) and (3.21), the proof is complete. \(\square\)

Theorem 3.7. Suppose \( p > 2/(\alpha - 1), \beta \in (2/p, \alpha - 1) \) and \( \gamma \in (0, (2 - \beta) \lambda_1) \).
There exists \( \lambda_L > 0 \) such that for any \( \lambda \in (0, \lambda_L) \), Eq. (1.1) has a unique mild solution \( u(\lambda(t, \cdot)) \in B_{p,\gamma} \). Moreover, for all \( \lambda \in (0, \lambda_L) \) and \( x \in (0, L) \),
\[-\infty < \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[|u(\lambda(t, x))|_p^p] \leq \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[\|u(\lambda(t))\|_{L^\infty}^p] < 0.\]

Proof. Define the operator \( F \)
\[
Fu(t, x) = \int_0^L p_D(t, x, y)u_0(y)dy + \int_0^t \int_0^L p_D(t-s, x, y)\lambda \sigma(u(s, y))w(dsdy),
\] (3.22)
For any $u \in B_{p,\gamma}$.

For any $\gamma \in (0, \lambda_1 p)$, by (2.1) and (ii) of Lemma 3.5, we have

\[
\| \int_0^L p_D(t, x, y)u_0(y)dy \|^p_{p,\gamma} \\
\leq \|u_0\|_{L^\infty}^p \sup_{t \geq 0} e^{\gamma t} \left( \int_0^L p_D(t, x, y)dy \right)^p \\
\leq \|u_0\|_{L^\infty}^p \left( \sup_{t \in [0,1]} e^{\gamma t} \int_0^L p_D(t, x, y)dy \right)^p + \|u_0\|_{L^\infty}^p \left( \sup_{t \in (1,\infty)} e^{\gamma t} \int_0^L p_D(t, x, y)dy \right)^p \\
\leq \|u_0\|_{L^\infty}^p \left( e^\gamma + c_2 \delta_D(x)^{\alpha p/2} \delta_D(y)^{\alpha p/2} \sup_{t \in (1,\infty)} e^{(\gamma - \lambda_1 p)t} \right) \\
\leq \|u_0\|_{L^\infty}^p (e^{\gamma} + c_2 L^\alpha p),
\]

(3.23)

which implies that for $\gamma \in (0, \lambda_1 p)$, $\int_0^L p_D(t, x, y)u_0(y)dy \in B_{p,\gamma}$.

Define the mapping $S_\lambda$

\[
S_\lambda(\sigma(u_\lambda(t, x))) = \int_0^t \int_0^L p_D(t-s, x, y)\lambda \sigma(u(s, y))w(dsdy).
\]

By Assumption 1.2, $\sigma(u) \leq L_\sigma |u|$. Similar to (3.3), we have

\[
\mathbb{E}|S_\lambda(\sigma(u_\lambda(t, x))) - S_\lambda(\sigma(u_\lambda(t, y)))|^p \\
\leq c(p) \lambda^p L_\sigma^p |x - y|^{\beta p/2} \left( \int_0^t \int_0^L |\partial_s p_D(t-s, x + r(y-x), z)dr|^\beta |p_D(t-s, x, z) - p_D(t-s, y, z)|^{2-\beta} \\
\times (\mathbb{E}|u(s, z)|^{p/2}dzds)^{p/2} \right) \\
:= K_3(t, x, y)|x - y|^{\beta p/2},
\]

(3.24)

where

\[
K_3(t, x, y) = c(p) \lambda^p L_\sigma^p \left[ \int_0^t \int_0^L |\partial_s p_D(t-s, x + r(y-x), z)dr|^\beta |p_D(t-s, x, z) - p_D(t-s, y, z)|^{2-\beta} \\
\times (\mathbb{E}|u(s, z)|^{p/2}dzds)^{p/2}. \]

(3.25)
Similar to (3.7) and by Lemma 3.6, we have

\[ K_3(t, x, y) \leq c_4^\beta c(p) \lambda^p L_\sigma^p e^{-\gamma t} \|u\|_{p, \gamma}^p \left[ \int_0^t \int_0^L e^{2\gamma s/p} s^{-2\beta/\alpha} |p_D(s, x, z) - p_D(s, y, z)|^{2-\beta} dz ds \right]^{p/2} \]

\[ \leq c_4^\beta c(p) \lambda^p L_\sigma^p e^{p/2(\alpha, \beta)} e^{-\gamma t} \|u\|_{p, \gamma}^p \left( (2\gamma/p)^{(\beta+1-\alpha)/\alpha} + \frac{1}{(2-\beta)\lambda_1 - \gamma} \right) \]

\[ := K_4(t). \quad (3.26) \]

Fix \( y = y_0 \in [0, L] \), similar to (3.13), we have

\[ \mathbb{E} \sup_{x \in [0, L]} |S_{\lambda}(u_\lambda(t, x))|^p \leq 2^p \kappa^p L^{\beta p/2 - (1+\varepsilon)} \mathbb{E}(B(t)) + 2^p \mathbb{E} |S_{\lambda}(u_\lambda(t, y_0))|^p. \quad (3.27) \]

Similar to (3.14), by (3.26), we have

\[ \mathbb{E}(B(t)) \leq K_4(t) \int_0^L \int_0^L \frac{1}{|x - y|^{1-\varepsilon}} dxdy \]

\[ \leq \frac{2L^{\varepsilon+1}}{\varepsilon(\varepsilon + 1)} K_4(t) \]

\[ = \frac{2L^{\varepsilon+1}}{\varepsilon(\varepsilon + 1)} c_4^\beta c(p) \lambda^p L_\sigma^p e^{p/2(\alpha, \beta)} e^{-\gamma t} \|u\|_{p, \gamma}^p \left( (2\gamma/p)^{(\beta+1-\alpha)/\alpha} + \frac{1}{(2-\beta)\lambda_1 - \gamma} \right). \quad (3.28) \]
Similar to the proof of (3.15), we get

\[
\mathbb{E}|S_{\lambda}(u_{\lambda}(t, y_0))|^p = \mathbb{E} \int_0^t \int_0^L p_D(t - s, y_0, y) \lambda \sigma(u(s, y)) w(ds) dy \bigg|^p
\]

\[
\leq c_p \mathbb{E} \int_0^t \int_0^L p_D^2(t - s, y_0, y) \lambda^2 \sigma^2(u(s, y)) dy ds \bigg]^p/2
\]

\[
\leq c_p \lambda^p L_\sigma^p \mathbb{E} \int_0^t \int_0^L p_D^2(t - s, y_0, y) \lambda^2 u^2(s, y) dy ds \bigg]^p/2
\]

\[
\leq c_p \lambda^p L_\sigma^p \int_0^t \int_0^L p_D^2(t - s, y_0, y) \mathbb{E}|u(s, y)|^p dy ds \bigg]^p/2
\]

\[
= c_p \lambda^p L_\sigma^p \int_0^t \int_0^L e^{-2\gamma s/p} \mathbb{E}|u(s, y)|^p_{L^\infty} dy ds \bigg]^p/2
\]

\[
= c_p \lambda^p L_\sigma^p e^{-\gamma t} \int_0^t \mathbb{E}|u|^p_{L^\gamma} dy ds \bigg]^p/2
\]

\[
= c_p \lambda^p L_\sigma^p e^{-\gamma t} \|u\|^p_{L^\gamma} \int_0^t \mathbb{E}|u|^p_{L^\infty} dy ds \bigg]^p/2.
\]

By (2.1), (2.2), we have

\[
\int_0^1 e^{2\gamma s/p} p_D(2s, y_0, y_0) ds
\]

\[
\leq c_2(\alpha) 2^{-1/\alpha} \int_0^1 e^{2\gamma s/p} s^{-1/\alpha} ds
\]

\[
\leq c_2(\alpha) (p/2)^{(\alpha - 1)/\alpha - (1 - \alpha)/\alpha} \int_0^{2\gamma/p} e^s s^{-1/\alpha} ds
\]

\[
\leq c_2(\alpha) (p/2)^{(\alpha - 1)/\alpha - (1 - \alpha)/\alpha} \int_0^{2\lambda_1} e^s s^{-1/\alpha} ds.
\]

By (ii) of Lemma 2.6, we get

\[
\int_1^\infty e^{2\gamma s/p} p_D(2s, y_0, y_0) ds
\]

\[
\leq c_2 L^\alpha \int_1^\infty e^{(2\gamma/p - 2\lambda_1)s} ds
\]

\[
= \frac{c_2 \lambda^2}{2\lambda_1 p - 2\gamma}.
\]

By (3.30) and (3.31), we have

\[
\int_0^t e^{2\gamma s/p} p_D(2s, y_0, y_0) ds \leq c(\alpha, p) \left( \gamma^{(1 - \alpha)/\alpha} + \frac{1}{\lambda_1 p - \gamma} \right),
\]
where \(c(\alpha, p, \lambda_1)\) depends on \(\alpha, p\) and \(\lambda_1\).

Putting (3.32) into (3.29), we obtain
\[
\mathbb{E}|S_{\lambda}f(u_{\lambda}(t, y_0))|^p \leq c_p \lambda^p L_\sigma^p e^{p/2}(\alpha, p)\left((\gamma^{(1-\alpha)/\alpha} + \frac{1}{\lambda_1 p - \gamma})^{p/2} e^{-\gamma t}||u||_{p, \gamma}^p. \right. \tag{3.33}
\]

Note that \(2 - \beta < 2 < p\), by (3.27), (3.28) and (3.33), for \(\gamma \in (0, (2 - \beta)\lambda_1)\), we have
\[
\sup_{\sigma \in [0, L]} \mathbb{E}|S_{\lambda}f(u_{\lambda}(t, x))|^p \leq c_1 e^{-\gamma t}||u||_{p, \gamma}^p \left((2\gamma/p)^{(\beta+1-\alpha)/\alpha} + \frac{1}{(2 - \beta)\lambda_1 - \gamma}) + c_3 e^{-\gamma t}||u||_{p, \gamma}^p \left((\gamma^{(1-\alpha)/\alpha} + \frac{1}{\lambda_1 p - \gamma})^{p/2}, \right. \tag{3.34}
\]

where \(c_1 = c(\alpha, \beta, p), c_3 = c(\alpha, p)\). This implies that for \(\gamma \in (0, (2 - \beta)\lambda_1)\),
\(S_{\lambda}f(u_{\lambda}(t, y_0)) \in B_{p, \gamma}\).

For any \(u_{\lambda}(t, x) \in B_{p, \gamma}\), by (3.32), (3.28) and (3.31), we have
\[
\|Fu_{\lambda}\|_{p, \gamma}^p \leq 2^{p-1}(e^\gamma + c_2 L^p)\|u_0\|_{L^p}^p + 2^{p-1}c\|u_{\lambda}\|_{p, \gamma}^p \left((2\gamma/p)^{(\beta+1-\alpha)/\alpha} + \frac{1}{(2 - \beta)\lambda_1 - \gamma}) + (\gamma^{(1-\alpha)/\alpha} + \frac{1}{\lambda_1 p - \gamma})^{p/2} < \infty, \]

Then we see that for any \(\gamma \in (0, (2 - \beta)\lambda_1)\), \(F\) maps \(B_{p, \gamma}\) into \(B_{p, \gamma}\).

Similar to the proof of Lemma 2.4, we have
\[
\|Fu_{\lambda} - Fu_{\lambda}\|_{p, \gamma}^p \leq c\lambda^p L_\sigma^p \|u - v\|_{p, \gamma}^p \left((2\gamma/p)^{(\beta+1-\alpha)/\alpha} + \frac{1}{(2 - \beta)\lambda_1 - \gamma}) + c_2 (\gamma^{(1-\alpha)/\alpha} + \frac{1}{\lambda_1 p - \gamma})^{p/2}. \right. \]

We choose \(\lambda\) sufficiently small such that
\[
0 < c\lambda^p L_\sigma^p \left((2\gamma/p)^{(\beta+1-\alpha)/\alpha} + \frac{1}{(2 - \beta)\lambda_1 - \gamma}) + c_2 (\gamma^{(1-\alpha)/\alpha} + \frac{1}{\lambda_1 p - \gamma})^{p/2} < 1. \]

By the contraction mapping theorem, \(F\) has a unique fixed point on \(B_{p, \gamma}\), therefore
Eq. (1.1) has a unique mild solution \(u_{\lambda}(t, \cdot) \in B_{p, \gamma}\).

Taking the second moments to both sides of (1.1), we get
\[
\mathbb{E}|u_{\lambda}(t, x)|^2 = \left| \int_0^L p_D(t, x, y)u_0(y)dy \right|^2 + \lambda^2 \mathbb{E}\int_0^t \int_0^L p_D(t-s, x, y)\mathbb{E}|\sigma(u(s, y))|^2dyds. \tag{3.35}
\]

By (ii) of Lemma 2.6, we see that \(\int_0^L p_D(t, x, y)u_0(y)dy\) decays exponentially fast with time. By (3.35), \(\mathbb{E}|u_{\lambda}(t, x)|^2\) can not decay faster than exponential.
\[ p > 2/(\alpha - 1), \] by Jensen’s inequality, we have
\[ \mathbb{E}|u_{\lambda}(t, x)|^p = \mathbb{E}(|u_{\lambda}(t, x)|^2)^{p/2} \geq (\mathbb{E}|u_{\lambda}(t, x)|^2)^{p/2}, \tag{3.36} \]
then
\[ \lim_{t \to \infty} \sup_{1 \leq t} \frac{1}{t} \log \mathbb{E}|u_{\lambda}(t, x)|^p > -\infty. \]
Since \( \|u_{\lambda}\|_{p,\gamma} = \|Tu_{\lambda}\|_{p,\gamma} < \infty \), it is easy to prove that \( \lim_{t \to \infty} \sup_{1 \leq t} \frac{1}{t} \log \mathbb{E}\|u_{\lambda}(t)\|_{L^\infty}^p \leq -\gamma < 0 \). The proof is complete. \( \square \)

4. Noise excitation index of \( p \)th energy of the solutions to Eq. (1.1)

In this section, we first prove that the second moment of the solution to Eq. (1.1) has a lower bound on a subinterval of \((0, L)\), and the second moment grows at most exponentially on \([0, L]\). We then show that the \( p \)th absolute moment of \( u_{\lambda}(t, x) \) grows at least exponentially. At last, we prove that the excitation index of the mild solution of Eq. (1.1) is \( \frac{2\alpha}{\alpha - 1} \).

**Theorem 4.1.** Fix \( \mu \in (0, L/2) \) and suppose \( \sigma \) satisfies Assumption 1.2, then there exists constants \( \kappa_1 > 0, \kappa_2 > 0 \) and \( T_0 > 0 \) such that for \( t > T_0 \),
\[ \inf_{x \in [\mu, L-\mu]} \mathbb{E}|u_{\lambda}(t, x)|^2 \geq \kappa_1 \exp \left( \kappa_2 (\lambda^2 t_0^2 \sigma)^{\alpha/(\alpha - 1)} t \right), \]
and for \( t \in (0, T_0) \),
\[ \inf_{x \in [\mu, L-\mu]} \mathbb{E}|u_{\lambda}(t, x)|^2 \geq \kappa_1 \exp(\lambda^2 t_0^2 \kappa_2 T_0^{-1/\alpha} t), \]
where \( u_{\lambda}(t, x) \) is the unique solution of (1.1).

**Proof.** From (i) of Lemma 3.5, it follows that
\[ p_D(t, x, y) \geq c_0 \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( t^{-1/\alpha} \wedge \frac{t}{|x-y|^{1+\alpha}} \right). \]
This together with (2.2) yield
\[ p_D(t, x, y) \geq c_0 p(t, x, y), \text{ for } 0 < t \leq \mu^\alpha, \; x, y \in [\mu, L-\mu]. \tag{4.1} \]
Since $|\sigma(u)| \geq l_{\sigma}|u|$, we have
\[
\mathbb{E}\left| \int_0^t \int_0^L p_D(t-s, x, y)\lambda \sigma(u_{\lambda}(s, y))w(dsdy) \right|^2 \\
= \int_0^t \int_0^L p_D^2(t-s, x, y)\mathbb{E}|\lambda \sigma(u_{\lambda}(s, y))|^2dyds \\
\geq \lambda^2 l_{\sigma}^2 \int_0^t \int_0^L p_D^2(t-s, x, y)\mathbb{E}|u_{\lambda}(s, y)|^2dyds \tag{4.2}
\]
Set
\[a(t) = \inf_{x \in [\mu, L-\mu]} \mathbb{E}|u_{\lambda}(t, x)|^2.\]
By (4.1), we get
\[
\int_0^t \int_0^L p_D^2(t-s, x, y)\mathbb{E}|u_{\lambda}(s, y)|^2dyds \geq \int_0^t \int_{\mu}^{L-\mu} p_D^2(t-s, x, y)a(s)dyds. \tag{4.3}
\]
For $0 < t \leq \mu$, by (4.1), we obtain
\[
\int_0^t \int_{\mu}^{L-\mu} p_D^2(t-s, x, y)a(s)dyds \geq c_0 \int_0^t \int_{\mu}^{L-\mu} p_D^2(t-s, x, y)a(s)dyds. \tag{4.4}
\]
Set $A := [\mu, L-\mu] \cap \{y : |y-x| \leq (t-s)^{1/\alpha}\}$, since $0 \leq t-s \leq t < \mu$, we have $|A| \geq (t-s)^{1/\alpha}$, where $|A|$ denotes the volume of $A$. By (2.2), we have
\[
\int_{\mu}^{L-\mu} p_D^2(t-s, x, y)dy \geq c_1^2(\alpha) \int_{\mu}^{L-\mu} (t-s)^{-2/\alpha} \wedge \frac{(t-s)^2}{|x-y|^{2+2\alpha}}dy \\
\geq c_1^2(\alpha) \int_{A} ((t-s)^{-2/\alpha} \wedge \frac{(t-s)^2}{|x-y|^{2+2\alpha}}dy \\
\geq c_1^2(\alpha) \int_{A} (t-s)^{-2/\alpha}dy \\
\geq c_1^2(\alpha)(t-s)^{-1/\alpha}. \tag{4.5}
\]
By (4.3), (4.4) and (4.5), for all $0 < t \leq \mu$, we have
\[
\int_0^t \int_0^L p_D^2(t-s, x, y)\mathbb{E}|u_{\lambda}(s, y)|^2dyds \geq c_0 c_1^2(\alpha) \int_0^t (t-s)^{-1/\alpha}a(s)ds. \tag{4.6}
\]
Define
\[
(G_D u_{\lambda})(t, x) = \int_0^L p_D(t, x, y)u_0(y)dy.
\]
For $0 < t \leq \mu^\alpha$ and $x \in [\mu, L - \mu]$, we have

$$|(G_D u_\lambda)(t, x)|^2 = \int_0^L p_D(t, x, y)u_0(y)dy \geq \inf_{x \in [\mu, L - \mu]} u_0(x) \int_\mu^{L - \mu} p_D(t, x, y)dy \geq c_0 \inf_{x \in [\mu, L - \mu]} u_0(x) \int_\mu^{L - \mu} p(t, x, y)dy \geq c_0c_2 \int_\mu^{L - \mu} p(t, x, y)dy,$$

where $c_2 = \inf_{x \in [\mu, L - \mu]} u_0(x)$. By similar argument to the proof of (4.5), we have

$$(G_D u_\lambda)(t, x) \geq c_3,$$  \hspace{1cm} (4.7)

where $c_3 > 0$ is a constant.

Taking the second moment to (1.4), we have

$$E|u_\lambda(t, x)|^2 = |(G_D u_\lambda)(t, x)|^2 + E\left| \int_0^t \int_0^L p_D(t - s, x, y)\lambda \sigma(u_\lambda(s, y))w(dsdy) \right|^2.$$

By (4.2), (4.6) and (4.7), for $t \in (0, \mu^\alpha]$, we get

$$E|u_\lambda(t, x)|^2 \geq c_3^2 + \lambda^2 l_\sigma^2 c_0 c_3^2 \int_0^t (t - s)^{-1/\alpha} a(s)ds. \hspace{1cm} (4.8)$$

For any fixed $t, T > 0$, we get

$$E|u_\lambda(T + t, x)|^2 = |(G_D u_\lambda)(T + t, x)|^2 + \lambda^2 \int_0^{T+t} \int_0^L p_D^2(T + t - s)E[\sigma(u_\lambda(s, y))]^2 dyds$$

$$= |(G_D u_\lambda)(T + t, x)|^2 + \lambda^2 \int_0^{T} \int_0^L p_D^2(T + t - s)E[\sigma(u_\lambda(s, y))]^2 dyds$$

$$+ \lambda^2 \int_0^t \int_0^L p_D^2(t - s)E[\sigma(u_\lambda(T + s, y))]^2 dyds. \hspace{1cm} (4.9)$$

Then

$$E|u_\lambda(T + t, x)|^2 \geq |(G_D u_\lambda)(T + t, x)|^2 + \lambda^2 l_\sigma^2 \int_0^t \int_0^L p_D^2(t - s)E[u_\lambda(T + s, y)]^2 dyds. \hspace{1cm} (4.10)$$

Then we see that for any $t > 0$, (4.8) holds. From the definition of $a(t)$, it follows that

$$a(t) \geq c_3^2 + \lambda^2 l_\sigma^2 c_4 \int_0^t (t - s)^{-1/\alpha} a(s)ds, \hspace{1cm} t > 0, \hspace{1cm} (4.11)$$
where $c_4 = c_0 c_1^2(\alpha)$.

By Lemma 2.4, we have

$$a(t) \geq c_3^2 E_{1-1/\alpha}(\lambda^2 l_c^2 c_4 \Gamma(1-1/\alpha))^{\alpha/(\alpha-1)} t). \quad (4.12)$$

Recall the definition of the Mittag-Leffler functions (see formula (1.66) in [20]),

$$F_{\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta + 1)}, \beta > 0, \ z \in \mathbb{C}, \quad (4.13)$$

where $\mathbb{C}$ denotes the complex plane. We see that $E_{\beta}(z) = F_{\beta}(z^\beta)$ for any $\beta > 0$.

Then by (4.12), we have

$$a(t) \geq c_3^2 F_{1-1/\alpha}(\lambda^2 l_c^2 c_4 \Gamma(1-1/\alpha))^{\alpha/(\alpha-1)/\alpha}$$

$$= c_3^2 F_{1-1/\alpha}(\lambda^2 l_c^2 c_4 \Gamma(1-1/\alpha))^{(\alpha-1)/\alpha}. \quad (4.14)$$

The Mittag-Leffler functions has the following asymptotic property as $z \to \infty$ (see Theorem 1.3 in [20]). If $0 < \tau < 2$, $\mu$ is an arbitrary real number such that $\pi \tau/2 < \mu < \min\{\pi, \pi \beta\}$, then for arbitrary integer $p \geq 1$, the following expansion holds:

$$F_{\tau}(z) = \frac{1}{\tau} \exp(z^{1/\tau}) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(1-\tau k)} + O(|z|^{-1-p}), \quad (4.15)$$

$z \to \infty$, arg($z$) $\leq \mu$, where arg($z$) denotes the principal value of the argument of $z$.

Choosing $p = 1$, by (4.15), we get

$$F_{\tau}(z) = \frac{1}{\tau} \exp(z^{1/\tau}) - \frac{z^{-1}}{\Gamma(1-\tau)} + O(|z|^{-2}), \quad (4.16)$$

$z \to \infty$, arg($z$) $\leq \mu$.

By (4.14) and (4.16), we have

$$a(t) \geq \frac{\alpha}{\alpha - 1} c_3^2 \exp(\lambda^2 l_c^2 c_4 \Gamma(1-1/\alpha))^{\alpha/(\alpha-1)} t) - \frac{t^{(1-\alpha)/\alpha}}{\Gamma(1/\alpha)\Gamma(1-1/\alpha)} \lambda^2 l_c^2 c_4$$

$$+ O(t^{2(1-\alpha)/\alpha}), \quad (4.17)$$

as $t \to \infty$.

For $\alpha \in (1, 2)$, note that $\Gamma(1-1/\alpha) > 0$, therefore we can choose $T_0 > 0$ is sufficiently
large such that for all $t > T_0$,

$$a(t) \geq c_5 \exp((\lambda^2 l_0^2 c_4 \Gamma(1 - 1/\alpha))^{\alpha/(\alpha-1)})t),$$  \hspace{1cm} (4.18)

where $c_5 > 0$ ia a constant.

On the other hand, for $0 < t < T_0$, by (2.1), we have

$$a(t) \geq c_3^2 + \lambda^2 l_0^2 c_4 T_0^{-1/\alpha} \int_0^t a(s)ds.$$  \hspace{1cm} (4.19)

By Lemma 2.4,

$$a(t) \geq c_3^2 \exp(\lambda^2 l_0^2 c_4 T_0^{-1/\alpha} t).$$

The proof is complete. \hspace{1cm} \square

**Theorem 4.2.** There exists constants $\kappa_6 > 0, \kappa_7 > 0$ such that for all $t > 0$,

$$\sup_{x \in [0, L]} \mathbb{E}|u_\lambda(t, x)|^2 \leq \kappa_3 \exp(\kappa_4(\lambda^2 L_0^2)^{\alpha/(\alpha-1)}t),$$

where $u_\lambda(t, x)$ is the unique mild solution of (1.1).

**Proof.** The proof is similar to that of Theorem 4.1, and we note that Lemma 7.1.1 in [16] and it is known that (see [21], formula (2.9)) the Mittag-Leffler type function

$$F_\tau(\omega t) \leq c \exp(\omega^{1/\tau} t), \hspace{1cm} t \geq 0, \hspace{1cm} \tau \in (0, 2),$$  \hspace{1cm} (4.20)

it is easy to prove the theorem. The proof is omitted. \hspace{1cm} \square

**Theorem 4.3.** Under the assumptions in Theorem 4.1 and Theorem 3.4, then for $x \in [\mu, l - \mu]$,

$$\lim_{t \to \infty} \inf_{x \in [\mu, l - \mu]} \frac{1}{t} \log \mathbb{E}|u_\lambda(t, x)|^p \leq \lim_{t \to \infty} \sup_{x \in [\mu, l - \mu]} \frac{1}{t} \log \mathbb{E}|u_\lambda(t, x)|^p_{L^\infty} < \infty,$$  \hspace{1cm} (4.21)

where $\kappa_2$ is the same constant as that in Theorem 4.1.

**Proof.** By Theorem 4.1, we have

$$\log(\inf_{x \in [\mu, l - \mu]} \mathbb{E}|u_\lambda(t, x)|^p) \geq \log(\kappa_1 \exp(\kappa_2(\lambda^2 l_0^2)^{\alpha/(\alpha-1)}t)), \hspace{1cm} t > T_0.$$
Then
\[
\lim_{t \to \infty} \inf \frac{1}{t} \log \left( \inf_{x \in [\mu, L-\mu]} \mathbb{E}|u_\lambda(t, x)|^2 \right) \geq \kappa_2 (\lambda^2 l^2_\sigma)^{\alpha/(\alpha-1)}.
\] (4.22)

By Jensen’s inequality, for \( p > 2/(\alpha - 1) > 2 \), we have
\[
(\mathbb{E}|u_\lambda(t, x)|^2)^{1/2} \leq (\mathbb{E}|u_\lambda(t, x)|^p)^{1/p}.
\]

This together with Theorem 3.4 yield that the inequality (4.21) holds. □

Under the assumptions in Theorem 2.2, for \( \lambda \in (0, \lambda_L) \), the \( p \)th energy of the solution of (1.1) satisfies the following inequality
\[
-\infty < \lim_{t \to \infty} \sup_{t} \frac{1}{t} \log \Phi_p(t, \lambda) < 0.
\] (4.23)

Since for \( p \geq 2 \), \( \mathbb{E}\|u_\lambda(t)\|_{L^p}^p \leq \mathbb{E}\|u_\lambda(t)\|_{L^\infty}^p \), by Theorem 3.4, it is easy to see that (4.23) holds.

**Corollary 4.4.** Under the assumptions in Theorem 4.1 and Theorem 3.4, then the \( p \)th energy of the solution of (1.1) satisfies
\[
\frac{1}{2} (\kappa_2 \lambda^2 l^2_\sigma)^{\alpha/(\alpha-1)} < \lim_{t \to \infty} \inf_{t} \frac{1}{t} \log \Phi_p(t, \lambda) < \infty,
\] (4.24)

where \( \kappa_2 \) is the same constant as that in Theorem 4.1.

**Proof.** For \( p > 2/(\alpha - 1) > 2 \), by Jensen’s inequality and Theorem 4.1,
\[
\mathbb{E} \int_0^L |u_\lambda(t, x)|^p dx \geq \int_0^1 (\mathbb{E}|u_\lambda(t, x)|^2)^{p/2} dx
\]
\[
\geq \int_{\mu}^{L-\mu} (\inf_{x \in [\mu, L-\mu]} \mathbb{E}|u_\lambda(t, x)|^2)^{p/2}
\]
\[
\geq (L - 2\mu) (\inf_{x \in [\mu, L-\mu]} \mathbb{E}|u_\lambda(t, x)|^2)^{p/2}
\]
\[
\geq (L - 2\mu) (\kappa_1 \exp (\kappa_2 (\lambda^2 l^2_\sigma)^{\alpha/(\alpha-1)} t) - \kappa_3 t^{(1-\alpha)/\alpha})^{p/2}.
\]

Then
\[
\log \Phi_p(t, \lambda) = \frac{1}{p} \log(\mathbb{E}\|u_\lambda(t)\|_{L^p}^p)
\]
\[
= \frac{1}{p} \log \left( \mathbb{E} \int_0^L |u_\lambda(t, x)|^p dx \right)
\]
\[
\geq \frac{1}{p} \log(L - 2\mu) + \frac{1}{2} \log \left( \kappa_1 \exp(\kappa_2 (\lambda^2 l^2_\sigma)^{\alpha/(\alpha-1)} t) - \kappa_3 t^{(1-\alpha)/\alpha} \right). \] (4.25)
Therefore, we get
\[
\lim_{t \to \infty} \inf \frac{1}{t} \log \Phi_p(t, \lambda) \geq \frac{1}{2}(\kappa_2 \lambda^2 t^2)^{\alpha/(\alpha-1)}.
\] (4.26)

This together with Theorem 3.4 yield that (4.24). □

**Lemma 4.5.** (Minkowski’s Inequality) Let \( X \) and \( Y \) be random variables. Then, for \( 1 \leq p < \infty \),
\[
(\mathbb{E}|X + Y|^p)^{1/p} \leq (\mathbb{E}|X|^p)^{1/p} + (\mathbb{E}|Y|^p)^{1/p}.
\]

**Theorem 4.6.** Under the assumptions in Theorem 4.1, there exists a constant \( c_p > 0 \) such that for \( t > T_0 \),
\[
c_p^2 \lambda^{2/(\alpha-1)} t \leq \lim_{\lambda \to \infty} \inf \frac{1}{\lambda} \log \left( \inf_{x \in [\mu, L]} \mathbb{E}|u_\lambda(t, x)|^p \right)
\leq \lim_{\lambda \to \infty} \sup \lambda^{2/(\alpha-1)} \log \left( \sup_{x \in [0, L]} \mathbb{E}|u_\lambda(t, x)|^p \right),
\] (4.27)

where \( T_0 \) is the same constant as that in Theorem 4.1, and for \( t > 0 \),
\[
\lim_{\lambda \to \infty} \sup \lambda^{2/(\alpha-1)} \log \left( \sup_{x \in [0, L]} \mathbb{E}|u_\lambda(t, x)|^p \right) \leq c_p^{-1} L^{2/(\alpha-1)} t.
\] (4.28)

**Proof.** First, we prove (4.28). For all \( t > 0 \), since \( u_0 \) is continuous on \([0, L]\),
\[
\sup_{x \in [0, L]} |(G_D u_\lambda)(t, x)| = \sup_{x \in [0, L]} \int_0^L |p_D(t, x, y)u_0(y)|dy \leq c_1.
\]

By similar arguments as that in Lemma 3.2, we have
\[
(\mathbb{E} \int_0^t \int_0^L p_D(t - s, x, y)\lambda \sigma(u_\lambda(s, y))w(dsdy)|^p)^{2/p} \leq c_2 \lambda^2 L^2 \int_0^t \int_0^L p^2_D(t - s, x, y)(\mathbb{E}|u_\lambda(s, y)|^p)^{2/p} dy ds
\]
\[
\leq c_2 \lambda^2 L^2 \int_0^t \int_0^L p^2_D(t - s, x, y)(\sup_{y \in [0, L]} \mathbb{E}|u_\lambda(s, y)|^p)^{2/p} dy ds
\]
\[
\leq c_2 \lambda^2 L^2 L \int_0^t p^2_D(t - s, x, y)(\sup_{y \in [0, L]} \mathbb{E}|u_\lambda(s, y)|^p)^{2/p} ds
\]
\[
\leq c_2 \lambda^2 L^2 L \int_0^t p_D(2(t - s), x, x)(\sup_{y \in [0, L]} \mathbb{E}|u_\lambda(s, y)|^p)^{2/p} ds
\]
\[
\leq c_2 \lambda^2 L^2 L \int_0^t (t - s)^{-1/\alpha}(\sup_{y \in [0, L]} \mathbb{E}|u_\lambda(s, y)|^p)^{2/p}.
\] (4.29)
By Lemma 3.13,
\[
(\sup_{x \in [0,L]} \mathbb{E}|u_\lambda(t,y)|^p)^{2/p} \leq 2(\mathbb{E} \sup_{x \in [0,L]} |(G_Du_\lambda)(t,x)|^p)^{2/p} \\
+ 2(\mathbb{E} \int_0^t \int_0^L p_D(t-s,x,y)\lambda \sigma(u_\lambda(s,y))w(dsdy)|^p)^{2/p} \\
\leq c_4 + c_5 \lambda^2 L^2 \int_0^t (t-s)^{-1/\alpha} (\mathbb{E} \sup_{x \in [0,L]} |u_\lambda(s,x)|^p)^{2/p} ds.
\]
(4.30)

where \(c > 0\) is a constant.

By Lemma 7.1.1 in [16] and (4.20), for any \(t > 0\),
\[
(\sup_{x \in [0,L]} \mathbb{E}|u_\lambda(s,x)|^p)^{2/p} \leq c_4 E_1 - 1/\alpha ((c_5 \lambda^2 L^2 \Gamma(1 - 1/\alpha))^{\alpha/(\alpha - 1)} t) \\
= c_4 F_{1-\alpha} (c_5 \lambda^2 L^2 \Gamma(1 - 1/\alpha) t^{(\alpha - 1)/\alpha}) \\
\leq c_6 \exp[(c_5 \lambda^2 L^2 \Gamma(1 - 1/\alpha))^{\alpha/(\alpha - 1)} t].
\]
(4.31)

Therefore, we obtain (4.28).

By Theorem 4.1, for \(t > T_0\),
\[
\log(\inf_{x \in [\mu,L-\mu]} \mathbb{E}|u_\lambda(t,x)|^2) \geq \log \kappa_1 + \kappa_2 (\lambda^2 L^2)^{\alpha/(\alpha - 1)} t.
\]
(4.32)

By Jensen’s inequality and (4.32), we see that (4.27) holds. □

Set \(f_{p,t}(\lambda) := \sup_{x \in [0,L]} \mathbb{E}|u(t,x)|^p\).

**Lemma 4.7.** (See Proposition 2.3 in [14]) For any \(t > 0\), we have
\[
\lim_{\eta \to \infty} \sup_{\lambda \to \infty} \frac{\log \log F_{\beta}(\eta t)}{\log \eta} \leq \frac{1}{\beta^2},
\]
and
\[
\lim_{\eta \to \infty} \inf_{\lambda \to \infty} \frac{\log \log F_{\beta}(\eta t)}{\log \eta} \geq \frac{1}{\beta^2},
\]
where \(F_{\beta}(\cdot)\) is the Mittag-Leffler function, see (4.13).

**Lemma 4.8.** Fix \(t > 0\), then
\[
\lim_{\lambda \to \infty} \frac{\log \log f_{p,t}(\lambda)}{\log \lambda} \leq \frac{2\alpha}{\alpha - 1}.
\]
Proof. By (4.31),
\[
\sup_{x \in [0,L]} \mathbb{E}|u_\lambda(s, x)|^p \leq \left[ c_4 F_{1-1/\alpha}(c_5 \lambda^2 L_o^2 L \Gamma(1 - 1/\alpha)) \right]^{p/2}.
\]
Then by Lemma 4.7, we get the conclusion. □

For any \( \mu \in (0, L/2) \), set
\[
I_{p, \mu, t}(\lambda) = \inf_{x \in [\mu, L-\mu]} \mathbb{E}|u_\lambda(t, x)|^p.
\]

Lemma 4.9. For any \( \mu \in (0, L/2) \), fix \( t > 0 \),
\[
\lim_{\lambda \to \infty} \frac{\log \log I_{p, \mu, t}(\lambda)}{\log \lambda} \geq \frac{2\alpha}{\alpha - 1}.
\]
Proof. By (4.14), we have
\[
a(t) \geq c_2^2 F_{1-1/\alpha}(\lambda^2 L_o^2 c_4 \Gamma(1 - 1/\alpha))^{1/\alpha},
\]
where \( a(t) = \inf_{x \in [\mu, L-\mu]} \mathbb{E}|u_\lambda(t, x)|^2 \). By Jensen’s inequality, for \( p > 2/(\alpha - 1) > 2 \),
\[
(\mathbb{E}|u_\lambda(t, x)|^2)^{1/2} \leq (\mathbb{E}|u_\lambda(t, x)|^p)^{1/p}.
\]
By Lemma 4.7, (4.33) and (4.34), we obtain the result. □

Theorem 4.10. Fix \( \mu \in (0, L/2) \), then for any \( t > 0 \) and \( x \in [\mu, L - \mu] \),
\[
\lim_{\lambda \to \infty} \frac{\log \log \mathbb{E}|u_\lambda(t, x)|^p}{\log \lambda} = \frac{2\alpha}{\alpha - 1}.
\]
Proof. By Lemma 4.8 and Lemma 4.9, we get the result. □

Corollary 4.11. The excitation index of the solution of (1.1) is \( \frac{2\alpha}{\alpha - 1} \).

Proof. Since the \( p \)th energy is defined by
\[
\Phi_p(t, \lambda) = (\mathbb{E}\|u_\lambda(t)\|_{L^p}^p)^{1/p} = \left( \int_0^L \mathbb{E}|u_\lambda(t, x)|^pdx \right)^{1/p}, \quad t > 0.
\]
By Theorem 4.10 and the following inequalities
\[
\int_0^L \mathbb{E}|u_\lambda(t, x)|^pdx \leq L \sup_{x \in [0,L]} \mathbb{E}|u_\lambda(t, x)|^p,
\]
\[
\int_0^L \mathbb{E}|u_\lambda(t, x)|^pdx \geq (L - 2\mu) \inf_{x \in [\mu, L-\mu]} \mathbb{E}|u_\lambda(t, x)|^p,
\]
we obtain the result. □
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