Fate of the inert three-flavor, spin-zero color-superconducting phases

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I investigate some of the inert phases in three-flavor, spin-zero color-superconducting quark matter: the CFL phase (the analogue of the B phase in superfluid $^3$He), the A and A* phases, and the 2SC and nSC phases. I compute the pressure of these phases with and without the neutrality condition. It is shown that the 2SC phase is identical to the A* phase up to a color rotation. The CFL phase is the energetically favored phase except for a small region of intermediate densities where the 2SC/A* phase is favored.

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I. INTRODUCTION

The interaction between electrons resulting from virtual exchange of photons is attractive when the energy difference between the electrons states involved is less than the phonon energy. There is a favorable attractive channel which dominates over the repulsive screened Coulomb interactions and produces superconductivity [1]. In cold and dense quark matter, due to asymptotic freedom [2], at quark chemical potentials $\mu \gg \Lambda_{QCD}$ single-gluon exchange is the dominant interaction between quarks and it is attractive in the color-antitriplet channel. This leads to the formation of quark Cooper pairs. This kind of condensation creates a new type of superconductivity which is called color superconductivity (CSC). The difference of this type of superconductivity compared to an ordinary electronic conductor comes from the fact that quarks carry different flavors and non-Abelian color charges [3, 4]. Then, CSC can appear in different phases depending on the various colors and flavors of the quarks which participate in Cooper pairing.

It has been found that at asymptotically large baryon number densities, where the masses of the $u$, $d$, and $s$ quarks are much smaller than the chemical potential, the ground state of 3-flavor QCD is the color-flavor-locked (CFL) phase [5]. In this phase, quarks of all colors and flavors form Cooper pairs, and $SU(3)c \otimes SU(3)L \otimes SU(3)R \otimes U(1)B$ is spontaneously broken to a subgroup $SU(3)c+U(1)c+em$. This phase is a superfluid but an electromagnetic superconductor. When the strange quark does not participate in pairing, the ground state is the so-called 2SC phase [4]. The original symmetry $SU(3)c \otimes SU(2)L \otimes SU(2)R \otimes U(1)em \otimes U(1)B$ breaks down to $SU(2)c \otimes SU(2)L \otimes SU(2)R \otimes U(1)_{c+em} \otimes U(1)_{em+\beta}$. Unlike the CFL phase, the 2SC phase is not a superfluid.

In nature color superconductivity may exist in compact stars. The approximation of asymptotically large density is not valid then. Moreover, the matter in the bulk of compact stars is neutral with respect to color and electric charges. Besides, this matter should remain in $\beta$-equilibrium [6-8]. These conditions impose stress on the system. In contrast to the 2SC and the CFL phases at large densities, in this case the Fermi momenta of different quark flavors participating in pairing cannot be equal. Therefore, the ground state is neither a pure CFL nor a pure 2SC state. There might appear phases in which the Cooper pairs carry nonzero total momenta and the system exhibits a crystalline structure due to a spatially varying energy gap. This kind of a superconductor was first time studied by Larkin and Ovchinnikov [10] and Fulde and Ferrell [11], which is called LOFF superconductor. Alternately, there might be ungapped quasiparticle excitations with nonzero gap parameter for both 2SC and CFL phases, giving rise to so-called gapless phases [12, 13].

From the theoretical point of view, superfluidity is very similar to superconductivity. Both are states of interacting many-fermion systems that are distinguished from the normal matter by an order parameter. Among all possible phases of superfluid $^3$He states only the so-called inert phases have been found in experiment. These phases play a crucial role in superconductivity due to Michel’s theorem, cf. Ref. [14, 15]. It is known that the B phase is the dominant phase in a vast area of the phase diagram of the superfluid $^3$He, while the so-called A phase has a dominant pressure only in a tiny area of the diagram.

The CSC for one flavor is rather similar to superfluid $^3$He. The reason is that the order parameters are quite similar. In one-flavor CSC, the order parameter is a $3 \times 3$ matrix in color and spin space. For comparison, in superfluid $^3$He, the order parameter is a $3 \times 3$ matrix in coordinate and spin space. It is natural to expect that one-flavor CSC has inert phases which are rather similar to the ones in superfluid $^3$He. These inert phases have
already been studied in Ref. 10.
In three-flavor, spin-zero CSC the order parameter is a 3 × 3 matrix in color and flavor space. In this case, it was shown in Ref. 17 that there are 511 possible phases but using the symmetries of the system this number could be lowered. Naturally, some of these are analogues of the inert phases encountered in one-flavor CSC and superfluid 3He. The CFL phase is the analogue of the CSL phase in one-flavor CSC or the B phase in superfluid 3He. The 2SC phase is the analogue of the polar phase in one-flavor CSC or superfluid 3He. There is, however, also an analogue of the A phase of one-flavor CSC and superfluid 3He, and a closely related phase, the A* phase which I discuss here for the first time. Finally, another inert phase is the sSC phase which is the complement of the 2SC phase in the sense that strange quarks pair with up and down quarks, but up and down quarks do not pair among themselves.

As it was mentioned earlier, at asymptotically large densities, it is known 8 that the CFL phase is the energetically favored phase. At intermediate densities, the 2SC phase may be energetically preferred over the CFL phase, depending on the value of the quark masses or the diquark coupling strength \( \Lambda \). The purpose of this paper is to answer the question whether some of the other inert phases could be energetically favored over either the CFL or the 2SC phase at intermediate densities.

The paper is organized as follows. In the next section, I give a general expression for the pressure of the different phases without imposing the neutrality condition. Then, I compare the pressure of all phases to determine the ground state. In Sec.III, I calculate the pressure of the phases including the neutrality condition. In order to identify the chemical potentials that make the system color and charge neutral, I evaluate the tadpoles in each phase. At the end, I consider the symmetry breaking pattern and give a summary.

II. PRESSURE WITHOUT THE NEUTRALITY CONDITION

In this section I calculate the pressure of CSC quark matter at zero temperature without the neutrality condition. In the absence of the neutrality condition such an analysis is easy in the framework of QCD. This is because there is a single chemical potential for all quarks.

To calculate the difference between the pressure of a color-superconducting phase and the normal conducting phase, I follow the general approach of Refs. 10, 11, 13. Without going into details, here I present only the main points of the formalism.

For condensation in the even-parity channel and in the ultrarelativistic limit I use the ansatz introduced for the gap matrix in Eq. (2.24) of Ref. 10,

\[
\Phi^+(K) = \sum_{e=\pm} \phi_e(K) M_- \Lambda^e_k ,
\]

where \( \phi_e(K) \) is the gap function, \( \Lambda^e_k \equiv (1 + e\gamma_0 \gamma \cdot \hat{k})/2 \) are projectors onto the positive \( (e = +1) \) and negative \( (e = -1) \) energy states, and \( M_- \) is defined by the order parameter \( \Delta^e_k \),

\[
M_- = I_k \Delta^e_k J^h \gamma_5 ,
\]

with the following expressions,

\[
I_k \equiv (I_{ij})_k , \quad J^h \equiv (J^{fg})^h
\]

which are completely antisymmetric 3 × 3 matrices, constructed from the Levi-Civita tensor \(-i\varepsilon_{ijk}\). In spin-one CSC \( I_k \) is an antisymmetric matrix in color space and \( J^h \) is an antisymmetric matrix in spin space. The \( k \) and \( h \) indices are for the associated color and spin components respectively. \( M_- \) satisfies

\[
[M^-, \Lambda^e_k] = 0 .
\]

To proceed I introduce new quantities \( L^+ \) in terms of \( M_- \) and the Dirac matrix \( \gamma_0 \) via,

\[
L^+ \equiv \gamma_0 M^+ M_- \gamma_0 ,
\]

where \( M^+ \equiv \gamma_0 (M^-)^\dagger \gamma_0 \). The matrix \( L^+ \) can be expanded in terms of projectors \( P^r \) with eigenvalues \( \lambda_r \),

\[
L^+ = \lambda_r P^r .
\]

This imposes the following form on the projectors \( P^r \),

\[
P^r = \prod_{s \neq r} \frac{L^+ - \lambda_s}{\lambda_r - \lambda_s} ,
\]

where \( n \) is the number of different eigenvalues. Using the ansatz in Eq. (1) in the QCD gap equation, one finds the following zero-temperature result for the value of the gap \( \phi_0 \) at the Fermi surface 10

\[
\phi_0 = 2 b b' \mu \exp\left( -\frac{\pi}{2 g} \right) \left( \langle \lambda_1 \rangle^{a_1} \langle \lambda_2 \rangle^{a_2} \langle \lambda_3 \rangle^{a_3} \right)^{-1/2} .
\]

In this equation, \( a_1 \), \( a_2 \), and \( a_3 \) are positive constants defined by

\[
a_s = \frac{n_s \lambda_s}{\sum_r n_r \lambda_r} ,
\]

where \( n_s \) is the degeneracy of eigenvalue \( \lambda_s \). They obey the constraint

\[
\sum_s a_s = 1 .
\]

The remaining constants in Eq. (8) are as follows,

\[
b = 256\pi^4 \left( \frac{2}{N_f g^2} \right)^{5/2} , \quad b' \equiv \exp\left( -\frac{\pi^2 + 4}{8} \right) .
\]
where \( N_f \) accounts for the number of flavors that occur, \( g \) is the strong coupling constant, and

\[
b \equiv \tilde{b} \exp(-d) .
\] (12)

The constant \( d \) originates from subleading contributions to the gap equation. For spin-zero condensates, due to an accidental cancellation of some of the subleading terms arising from static electric and non-static magnetic gluon exchange, \( d \) is zero. In the spin-one cases, this cancellation does not occur and \( d \neq 0 \), cf. Ref. [15].

Finally, at zero temperature and without color and electric charge neutrality conditions, the difference between the pressure of the color-superconducting phase and the normal conducting phase is given by [16]

\[
\Delta p = \frac{\mu^2}{16 \pi^2} \phi_0^2 \text{Tr}[L^+] .
\] (13)

In three-flavor, spin-zero color superconductivity the order parameter \( \Delta_h \), cf. Eq. [4], is a 3 \times 3 matrix in color (\( k = 1, 2, 3 \)) and flavor (\( h = 1, 2, 3 \)) space. Different order parameters \( \Delta_h \) lead to different gap matrices \( \Phi^+ \), and thus to different physical states. As we see from Eq. (13), the value of the pressure is given in terms of \( \text{Tr}[L^+] \) and the value of the gap. Then, different order parameters also produce in general different values for the pressure. In the following, after introducing the order parameter of each phase I calculate their pressure.

### A. A phase

The order parameter of the so-called A phase has the following form, cf. Refs. [15, 16],

\[
\Delta_h^k = \delta^{k3}(\delta_{h1} + i \delta_{h2}) .
\] (14)

where the upper index stands for color and the lower index for flavor. Using this equation the matrix \( \mathcal{M}^- \) becomes

\[
\mathcal{M}^- = I_3(J^1 + i J^2)\gamma_5 .
\] (15)

Inserting this expression in Eq. (3) leads to

\[
[L^+]^{fg}_{ij} = (\delta_{ij} - \delta_{i3} \delta_{j3})(2 \delta^{fg} - \delta^{f1} \delta^{g1} - \delta^{f2} \delta^{g2} - i \delta^{f1} \delta^{g2}) .
\] (16)

One should notice that because of the summation rule in Eq. (2), the role of the indices is interchanged, so that the upper index stands for flavor and the lower index for color. After some straightforward calculation, one finds that,

\[
[L^+]^n = 2^{n-1} [L^+] .
\] (17)

The results for the eigenvalues \( \lambda_r \) come from the roots of the following equation,

\[
det(\lambda - L^+) = 0 ,
\] (18)

The left-hand side of this equation can be rewritten in the form

\[
det(\lambda - L^+) = \exp \{ \text{Tr}[\ln(\lambda - L^+)] \} .
\] (19)

which, after expanding the logarithm and making use of Eq. (17), gives

\[
\lambda^5(\lambda - 2)^4 = 0 .
\] (20)

This equation yields two different eigenvalues for the A phase,

\[
\begin{cases}
\lambda_1 = 2 & (4 \text{- fold}) \rightarrow a_1 = 1 , \\
\lambda_2 = 0 & (5 \text{- fold}) \rightarrow a_2 = 0 .
\end{cases}
\] (21)

From Eqs. (8) and (13) I find the value of the pressure for this phase,

\[
\Delta p_A = 4 \alpha ,
\] (22)

where \( \alpha \) is defined as

\[
\alpha = \frac{\mu^2}{4 \pi^2} b^2 b_0^2 \exp \left( -\frac{\pi}{g} \right) .
\] (23)

### B. A* phase

Here I define a new phase motivated by the order parameter of the A phase and I call it A* phase. This phase is not included in Ref. [16] but was introduced in Ref. [15]. The order parameter of this phase is a transposed form of that in the A phase, i.e., the roles of the color and flavor indices are interchanged,

\[
\Delta_h^k = (\delta^{k1} + i \delta^{k2})\delta_{h3} .
\] (24)

By the same argument which led to this phase one realizes that the corresponding \( L^+ \) matrix can be derived by interchanging the color and flavor indices of the matrix \( L^+ \) in the A phase,

\[
[L^+]^{fg}_{ij} = (2 \delta_{ij} - \delta_{i1} \delta_{j1} - \delta_{i2} \delta_{j2} - i \delta_{i2} \delta_{j1} + i \delta_{i1} \delta_{j2})(\delta^{fg} - \delta^{f3} \delta^{g3}) .
\] (25)

Therefore, one has the same expression for \( [L^+]^n \) and the same eigenvalues \( \lambda_{1,2} \) given for the A phase, cf. Eqs. (17) and (21) respectively. Considering Eqs. (8) and (13), the pressure of this phase is given by

\[
\Delta p_{A*} = 4 \alpha .
\] (26)

which is equal to the pressure of the A phase.

### C. Planar or sSC phase

Another experimentally observed phase in superfluid $^3$He is the so-called planar phase which has the following form for the order parameter

\[
\Delta_h^k = \delta_{h1}^k - \delta_{h3}^k \delta_{h3} ,
\] (27)
normal conducting matter is found to be

\[ L^+ | f g \]_{ij} = 2 \delta_{ij} \delta^{fg} - \delta^{fg} (\delta_{ij} \delta_{j1} + \delta_{j2} \delta_{j2}) - \delta_{ij} (\delta_1 \delta_1^{fg} + \delta_2 \delta_2^{fg}) + (\delta_{ij} \delta_{j1} + \delta_{j2} \delta_{j2}) (\delta_{ij} \delta_{j1}^{fg} + \delta_{j2} \delta_{j2}^{fg}), \quad (28)\]

which gives the following result

\[ [L^+]^n = 2^{n-1} L^+ + (2^{n-1} - 1) \ell, \quad (29)\]

where \( \ell \) is

\[ \ell | f g \]_{ij} = 2 (\delta_{ij} \delta_{j1}^{fg} - \delta_{j2} \delta_{j2}^{fg}) (\delta_{ij} \delta_{j1} + \delta_{j2} \delta_{j2}) - \delta_{ij} (\delta_1 \delta_1^{fg} + \delta_2 \delta_2^{fg}). \quad (30)\]

Then following the method introduced for calculating the eigenvalues of the A phase I derive

\[ \begin{align*} 
\lambda_1 &= 2 \quad \text{(2 - fold)} \rightarrow a_1 = 1/2, \\
\lambda_2 &= 1 \quad \text{(4 - fold)} \rightarrow a_2 = 1/2, \\
\lambda_3 &= 0 \quad \text{(3 - fold)} \rightarrow a_3 = 0.
\end{align*} \quad (31)\]

The difference between the pressure of the sSC phase and normal conducting matter is found to be

\[ \Delta p_{\text{sSC}} = 8 \frac{2^{1/2}}{\alpha}. \quad (32)\]

D. Polar or 2SC phase

Analogous to the previous Sec. [110], the phase called the polar phase for superfluid \(^3\)He is analogous to the 2SC phase in CSC,

\[ \Delta^k_h = \delta^{k3} \delta_{h3}, \quad (33)\]

which gives a zero value for the gaps \( \Delta^1_1 \) and \( \Delta^2_2 \). The \( L^+ \) matrix of this phase is

\[ [L^+]^f g |_{ij} = (\delta_{ij} - \delta_{h3} \delta_{h3}) (\delta^{fg} - \delta^{f3} \delta^{g3}), \quad (34)\]

and yields

\[ [L^+]^n = L^+, \quad (35)\]

so that with the same methods one arrives at

\[ \begin{align*} 
\lambda_1 &= 1 \quad \text{(4 - fold)} \rightarrow a_1 = 1, \\
\lambda_2 &= 0 \quad \text{(5 - fold)} \rightarrow a_2 = 0,
\end{align*} \quad (36)\]

with the pressure difference equal to

\[ \Delta p_{\text{2SC}} = 4 \alpha. \quad (37)\]

E. CFL phase

To make a detailed comparison with the previous results I copy the results given for the CFL phase from Ref. [10]. The order parameter of the CFL phase is

\[ \Delta^k = \delta^k_h, \quad (38)\]

and the \( L^+ \) matrix has the following form,

\[ [L^+]^f g |_{ij} = \delta^{f1} \delta^{g2} + \delta^{f2} \delta^{g1}, \quad (39)\]

with the following quantities,

\[ \begin{align*} 
\lambda_1 &= 4 \quad \text{(1 - fold)} \rightarrow a_1 = 1/3, \\
\lambda_2 &= 1 \quad \text{(8 - fold)} \rightarrow a_2 = 2/3,
\end{align*} \quad (40)\]

which are sufficient to find the pressure of this phase,

\[ \Delta P_{\text{CFL}} = \frac{12}{\alpha}. \quad (41)\]

Using all results, one can compare the pressure of the inert phases

\[ P_{\text{CFL}} > P_{\text{sSC}} > P_{\text{A}} = P_{\text{A}*} = P_{\text{2SC}}. \quad (42)\]

As we see, the pressure of the CFL phase is larger than the pressure of the other phases, i.e., the CFL phase is the dominant phase. Another interesting result is a larger value for the pressure of the sSC phase without the neutrality condition than that for the 2SC phase.

In the next section I calculate the pressure including the neutrality condition. As it was mentioned above, in this case the pressure for some of these inert phases (2SC, sSC, and CFL) were considered in the literature, cf. Ref. [21, 22]. Therefore, in the following I calculate the pressure of only those phases which are still left out, the A and A* phases.

III. PRESSURE INCLUDING THE NEUTRALITY CONDITION

In this section I impose the neutrality condition on the system. For this one has to know the relevant chemical potentials of each phase. Thus one has several parameters to be evaluated and using QCD to calculate the pressure becomes more complicated. Therefore, I will use the Nambu-Jona-Lasinio model (NJL), cf. [24].

To find the pressure of the A and the A* phase under the neutrality condition using the NJL model one has to know the relevant chemical potentials for these phases. Since a nonvanishing tadpole leads to the violation of the neutrality of the system, one has to introduce a chemical potential which makes the tadpole vanish, cf. [24, 25]. The sum of these chemical potentials together with the quark chemical potential is the relevant chemical potential for the system under color and electric charge neutrality condition. Thus, before I proceed to evaluate the pressure I compute tadpoles of the system. Afterwards, I go further to find the pressure of the systems.
A. Calculating the tadpoles

In order to compute the tadpoles of a system I use Eq. (19) of Ref. [24],
\[ T^a = -\frac{g}{2} \int \frac{d^4Q}{i(2\pi)^4} \text{Tr}_{D,c.f.}[\Gamma_0^a G^+(Q) + \Gamma_0^a G^-(Q)] \, . \tag{43} \]
Here \( \Gamma_0^a = \gamma_\alpha T^a \), \( \Gamma_0^0 = -\gamma_0(T^a)^T \) with \( T^a = \lambda^a/2 \) for \( a = 1, \ldots, 8 \) where \( \lambda^a \) are the Gell-Mann matrices in flavor space, and \( G^\pm \) are the fermion propagators for quasiparticles and charge-conjugate quasiparticles,
\[ G^\pm(Q) \equiv (G_0^\pm)^{-1} - \Sigma^\pm^{-1} , \quad \Sigma^\pm \equiv M^T G_0^\pm M^\pm \, . \tag{44} \]
In these equations, \( \Sigma^\pm \) is the quark self-energy generated by exchanging particles or charge conjugate particles with the condensate. The role of different phases appears in the quark self-energy via the matrix \( M^\pm \), cf. Eq. (2). In the next subsections, I insert the order parameter of each phase into \( M^\pm \) to find the tadpoles.

B. Tadpoles in the A phase

In Eq. (43) I first evaluate the trace over color and flavor space, and afterwards the trace over the Dirac space. The inverse free fermion propagator for quarks \((G_0^\pm)^{-1})_{ij}\) has the following color and flavor structure,
\[ (G_0^\pm)^{-1})_{ij} = (\gamma^\mu K_\mu \pm \mu \gamma_0 - M)\delta^{fg} \delta_{ij} \, . \tag{45} \]
Using Eq. (44) yields
\[ (\Sigma^\pm)_{ij} = (M^T G_0^\mp M^\pm)_{ij} = \gamma_5 \frac{\Delta^2}{\gamma^\mu K_\mu \pm \mu \gamma_0 - \lambda M} \times \gamma_5 (\delta^{fg} - \delta^{f3} \delta^{g3}) (2 \delta_{ij} - \delta_{i1} \delta_{j1} - \delta_{i2} \delta_{j2}) \pm i \delta_{i1} \delta_{j2} + i \delta_{i2} \delta_{j1} \, , \tag{46} \]
where \( \Delta \) is the value of the gap for this phase. After some algebraic calculation, one can find the color and flavor structure of the propagator \( G^\pm(Q) \) as,
\[ [G^\pm]_{ij} = ((G_0^\pm)^{-1} - 2\gamma_5 G^T \gamma_5 \Delta^2)^{-1}(\delta^{fg} - \delta^{f3} \delta^{g3}) \delta_{ij} \]
\[ + [G_0^\mp]_{ij} \delta^{fg} \delta_{ij} - ((G_0^\pm)^{-1} - 2\gamma_5 G^T \gamma_5 \Delta^2)^{-1} \]
\[ \times (\gamma_5 G^T \gamma_5 \Delta^2)(G_0^\pm)(\delta_{ij} - \delta^{f3} \delta^{g3}) \]
\[ \times (\delta_{i1} \delta_{j1} + \delta_{i2} \delta_{j2} + i \delta_{i1} \delta_{j2} + i \delta_{i2} \delta_{j1}) \, . \tag{47} \]
Now one has to put these equations back into Eq. (43) and evaluate the traces. By doing so, it is revealed that all components of \( T^a \) are zero except for \( a = 8 \). This forces us to introduce a chemical potential \( \mu_a \) for the system to make the tadpole vanish and achieve color and electric charge neutrality.

C. Tadpoles in the A* phase

Following the same procedure for calculating the tadpoles of the A phase, the color and flavor structure of the propagator \( G^\pm (Q) \) is given by
\[ [G^\pm]_{ij} = ((G_0^\pm)^{-1} - 2\gamma_5 G^T \gamma_5 \Delta^2)^{-1}(\delta_{ij} - \delta_{i3} \delta_{j3}) \delta^{fg} \]
\[ + [G_0^\mp]_{ij} \delta_{i3} \delta_{j3} \delta^{fg} - ((G_0^\pm)^{-1} - 2\gamma_5 G^T \gamma_5 \Delta^2)^{-1} \]
\[ \times (\gamma_5 G^T \gamma_5 \Delta^2)(G_0^\pm)(\delta_{ij} - \delta_{i3} \delta_{j3}) \]
\[ \times (\delta^{f1} \delta^{g1} + \delta^{f2} \delta^{g2} + i \delta^{f1} \delta^{g2} + i \delta^{f2} \delta^{g1}) \, . \tag{48} \]
This form of the full fermion propagator yields a nonzero value for the \( a = 2 \) and \( a = 8 \) tadpoles. Hence, for this phase the chemical potential has to contain \( \mu_2 \) and \( \mu_8 \) to provide neutrality.

Now one is able to calculate the pressure.

D. Pressure

The grand partition function is given by
\[ Z = e^{-\Omega V/T} = \int D\bar{\psi} D\psi e^{i S_{N\uparrow} (\bar{\psi} + \mu \gamma_5 \psi)} \, , \tag{49} \]
where \( \Omega \) is the thermodynamic potential density, \( V \) is the volume, \( \bar{\mu} \) is the matrix of the quark chemical potentials and \( L \) is the Lagrangian density for three-flavor quark matter for which a local NJL-type interaction is given by
\[ \mathcal{L} = \bar{\psi} (i\gamma \cdot \hat{\mu} - \hat{m} ) \psi + G_S \sum_{a=0}^8 \left[ (\bar{\psi} \lambda_a \lambda_a ) \right] \psi \]
\[ + G_D \sum_{k,h} \left[ \bar{\psi} \gamma_5 \psi \delta^{kj} \right] \left[ (\bar{\lambda}_C)^f \gamma_5 \psi \right] \psi \]
\[ - \bar{K} \{ \bar{\psi} (1 + \gamma_5) \psi \} + \det \left[ \bar{\psi} (1 - \gamma_5) \psi \right] \} \, , \tag{50} \]
where the quark spinor field \( \psi_i^f \) carries color \( (i = r, g, b) \) and flavor \( (f = u, d) \) indices. The matrix of quark current masses is given by \( \hat{m} = \text{diag}(m_u, m_d, m_s) \) and \( \lambda_0 = \sqrt{2\mathcal{F}} \mathds{1}_F \). The charge-conjugate spinors are defined as \( \psi_C = C \psi^T \) and \( \bar{\psi}_C = \psi^T C \), where \( \psi = \psi^\dagger \gamma^0 \) is the Dirac-conjugate spinor and \( C = i\gamma^2 \gamma^0 \) is the charge conjugation matrix. Note that I include the 't Hooft interaction whose strength is determined by the coupling constant \( K \). This term breaks the \( U(1) \) axial symmetry.

The term in the second line of Eq. (49) describes a scalar diquark interaction in the color-antitriplet and flavor-antitriplet channel. For symmetry reasons there should also be a pseudoscalar diquark interaction with the same coupling constant but for the sake of simplicity I do not consider it here.
I use the following set of model parameters \[26]\:

\[ m_{u,d} = 5.5\, \text{MeV}, \quad m_s = 140.7\, \text{MeV}, \quad G_S \Lambda^2 = 1.835, \quad K \Lambda^2 = 12.36, \quad \Lambda = 602.3\, \text{MeV}. \]

In general, it is expected that the diquark coupling \( G_D \) is of the same order as the quark-antiquark coupling \( G_S \) and in this paper, I study the regime of intermediate coupling strength with \( G_D = \frac{4}{3} G_S \).

All quarks carry baryon charge 1/3 and thus have a diagonal contribution \( \mu \delta g \delta_{ij} \) to their matrix of chemical potentials. By definition \( \mu = \mu_B/3 \) with \( \mu_B \) being the baryon chemical potential. In order to fulfill the neutrality condition one has to take the chemical potentials introduced in the previous sections, cf.\[16\] and \[17\] into account. For the \( A \) phase one has

\[ \mu_{ij}^f = \left( \mu \delta g + \mu_Q Q_F g \right) \delta_{ij} + \mu_s (T_s)_{ij} \delta g, \]

and for the \( A^* \) phase

\[ \mu_{ij}^f = \left( \mu \delta g + \mu_Q Q_F g \right) \delta_{ij} + \left( \mu_2 (T_2)_{ij} + \mu_s (T_s)_{ij} \right) \delta g, \]

where \( \mu_Q \) is the chemical potential of the electric charge and \( Q_F \) is the electric charge matrix \( Q_F = \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \).

To calculate the mean-field thermodynamic potential at temperature \( T \), one has to linearize the interaction in the presence of the diquark condensates \( \Delta_k^\pm \sim (\bar{\psi}_C)\mathbf{i} \gamma_5 \varepsilon^{ijk} \varepsilon_{fgb} \psi^g_{ij} \) and the quark-antiquark condensates \( \sigma_\alpha \sim \bar{\psi}_C \psi_\alpha \) (no sum over \( \alpha \)). Then, integrating out the quark fields and neglecting the fluctuations of composite order parameters gives the following expression for the thermodynamic potential:

\[ \Omega = \Omega_L + \frac{1}{4 G_D} \sum_{k,h=1}^3 |\Delta_k^h|^2 + 2 G_S \sum_{\alpha=1}^3 \sigma_\alpha^2 \]

\[ - 4 K \sigma_u \sigma_d \sigma_s - \frac{T}{2V} \sum_K \ln \det \frac{S^{-1}}{T}, \]

where the number of the non-degenerate eigenvalues \( \epsilon_i \) decreases again to 18. With \( p \equiv -\Omega \) I find

\[ p = \frac{1}{2 \pi^2} \sum_{i=1}^{18} \int_0^\Lambda \frac{dk}{k^2} \left[ \epsilon_i + 2 T \ln \left( 1 + e^{-\frac{|\epsilon_i|}{T}} \right) \right] \]

\[ + \frac{4 K \sigma_u \sigma_d \sigma_s - \frac{1}{4 G_D} \sum_{h,k=1}^3 |\Delta_h^k|^2 - 2 G_S \sum_{\alpha=1}^3 \sigma_\alpha^2}{\frac{T}{2V} \sum_{i=\epsilon,\mu,\mu} \sum_{\epsilon=\mu} \int_0^\infty \frac{dk}{k^2} \ln \left( 1 + e^{-\frac{K \sigma u - \mu_i}{T}} \right)}, \]

where the contribution of electrons and muons with masses \( m_e \approx 0.511\, \text{MeV} \) and \( m_\mu \approx 105.66\, \text{MeV} \) are included. The expression for the pressure in Eq. \[32\] has a physical meaning only when the chiral and color-superconducting order parameters, \( \sigma_\alpha \) and \( \Delta_h^k \), satisfy the following set of equations:

\[ \frac{\partial p}{\partial \sigma_\alpha} = 0, \]

\[ \frac{\partial p}{\partial \Delta_h^k} = 0. \]

In order to enforce the condition of local charge neutrality in dense matter, for the \( A \) phase one has to require that

\[ n_Q \equiv \frac{\partial p}{\partial \mu_Q} = 0, \]

\[ n_s \equiv \frac{\partial p}{\partial \mu_s} = 0. \]
three-flavor, spin-zero color superconductivity is a 3 parameter of these phases and find a unitary transformation $U$? To answer that, one has to consider the order parameter of the A* phase. This was not the case for superfluid $^3$He. Hence, one has to find a unitary matrix $U$ which changes the last expression to that in the 2SC phase, 

$$U = e^{i\pi(\lambda_5 + \lambda_8)/2\sqrt{2}} ,$$

by which one has

$$U^T (\lambda_5 + i\lambda_7) U \rightarrow i\sqrt{2}\lambda_2 .$$

and the following results are derived,

$$\mu_2^{A^*} = -\frac{\sqrt{3}}{2} \mu_2^{2SC} ,$$

$$\mu_8^{A^*} = -\frac{1}{2} \mu_8^{2SC} .$$

I conclude that the A* phase is the same as the 2SC phase. This was not the case for superfluid $^3$He, cf. Ref. [15].

In the next section I briefly mention the result derived for the symmetry pattern of the A* and 2SC phases and then investigate the generators of the symmetry groups for the A* phase in terms of those in the 2SC phase.

IV. PATTERN OF SYMMETRY BREAKING

The study of the order parameter of the A* phase reveals that just like the 2SC phase the strange quark does not participate in pairing. Thus, the A* phase is a two-flavor CSC. Without including the neutrality condition and in the case that all the quark masses are zero, the following results are derived,
initial symmetry group for the A\(^\ast\) phase is the same as the 2SC phase

\[
G = SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \\
\otimes U(1)_B \otimes U(1)_{em} ,
\tag{81}
\]

where \(SU(3)_c\) is the color gauge group and \(SU(2)_L\) and \(SU(2)_R\) are the representations of the flavor group. \(U(1)_B\) and \(U(1)_{em}\), respectively, are accounting for baryon number conservation symmetry and the electromagnetic gauge group. The order parameter \(\Delta\) is an element of a representation of \(G\). After pairing, the group \(G\) is spontaneously broken to a residual subgroup \(H \subseteq G\) so that any transformation \(g \in H\) leaves the order parameter invariant,

\[
g(\Delta) = \Delta .
\tag{82}\]

To find all possible order parameters and the corresponding residual groups \(H\) one has to satisfy this invariance condition. Here I restrict the calculations to those which lead to the residual group of the A\(^\ast\) phase. Using the method given in Ref. [15, 18] I find that

\[
H_{A\ast} = SU(2)_c \otimes SU(2)_L \otimes SU(2)_R \\
\otimes \tilde{U}(1)_B \otimes \tilde{U}(1)_{em} ,
\tag{83}\]

which is exactly the same residual group as for the 2SC phase. This result confirms the equivalence of the A\(^\ast\) phase with the 2SC phase from this point of view. To complete this subsection, utilizing the results of the previous section and knowing the generators for the residual group of the 2SC phase, I want to find the corresponding generators for the A\(^\ast\) phase. In the 2SC phase the generator of baryon number conservation is

\[
\tilde{B} = B - \frac{2}{\sqrt{3}}T_8 ,
\tag{84}\]

and that for an unbroken \(\tilde{U}(1)_{em}\) is

\[
\tilde{Q} = Q - \frac{1}{\sqrt{3}}T_8 .
\tag{85}\]

Under the same color transformation for which the A\(^\ast\) phase goes to the 2SC phase, Eq. (77), one finds the generators for the residual group of the A\(^\ast\) phase,

\[
\tilde{B}' = B + \left( T_2 + \frac{1}{\sqrt{3}}T_8 \right) ,
\tag{86}\]

\[
\tilde{Q}' = Q + \frac{1}{2}\left( T_2 + \frac{1}{\sqrt{3}}T_8 \right) ,
\tag{87}\]

which is a linear combination of generators of the 2SC phase.

V. CONCLUSIONS

I investigated the inert phases in three-flavor CSC, the A, A\(^\ast\), 2SC, sSC, CFL phases. Without the neutrality condition, calculating the pressure of the phases I found the CFL phase to be the dominant phase. Besides, in this case I found that the pressure of the sSC phase is larger than that for the 2SC phase. Including the neutrality condition, the CFL is again the dominant phase and the dominance of the sSC phase over the 2SC phase ceases. On the other hand, I found that for all values of the chemical potential the pressure of the A\(^\ast\) and the 2SC phases are equal. This led me to show that the A\(^\ast\) phase is different from the 2SC phase only by a color rotation. At the end I found the generators of the A\(^\ast\) phase in terms of those for the 2SC phase.

Although none of the newly investigated phases are favored over the 2SC or CFL phases, this does not preclude that they could not exist if the external conditions are changed. For instance, at zero temperature the A phase in superfluid \(^3\)He appears only if a sufficient external magnetic field is applied. The same also could happen in neutron stars which have strong magnetic fields. This could be an interesting subject for further studies.

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