ESTIMATING DIFFUSION WITH COMPOUND POISSON JUMPS BASED ON SELF-NORMALIZED RESIDUALS

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Abstract. This paper considers parametric estimation problem of the continuous part of a jump diffusion model. The threshold based method was previously proposed in various papers, which enables us to distinguish whether observed increments have jumps or not, and to estimate unknown parameters. However, a data-adapted and quantitative choice of the threshold parameter is a subtle and sensitive problem, and still remains as a tough problem. In this paper, we propose a new and simple alternative based on the Jarque-Bera normality test, which makes us to attain the above two things without any sensitive fine tuning. We show that under suitable conditions the proposed estimator has a consistency property. Some numerical experiments are conducted.

1. Introduction

Suppose that we are given discrete-time but high-frequency observation \((X^n_{t_{j}})_{j=0}^{n}\) from a solution to the one-dimensional diffusion with jumps described by

\[
dX_t = a(X_t, \alpha)dw_t + b(X_t, \beta)dt + c(X_t-)dJ_t,
\]

where the ingredients are given as follows.

- \(w\) is a standard Wiener process and \(J\) a compound Poisson process associated with the Lévy measure
  \[
  \nu(dz) = \lambda F(dz)
  \]
  for some probability distribution \(F(dz)\). Throughout we assume that \(\lambda \in [0, \infty)\).
- The sampling times fulfills that
  \[
  t^n_{j} = jh_n, \quad nh^2_n \rightarrow 0
  \]
  where the terminal sampling time \(T_n := t^n_n \rightarrow \infty\); hereafter, we will largely abbreviate “\(n\)” from the notation like \(t_j = t^n_j\) and \(h = h_n\).

A well-known approach to estimate \(\theta := (\alpha, \beta) \in \Theta_\alpha \times \Theta_\beta = \Theta\) is the threshold based method independently proposed in \([5]\), \([7]\), and \([10]\). In the method, we regard that the increment

\[
\Delta_jX := X_{t_j} - X_{t_{j-1}},
\]

contains the jump component if \(|\Delta_jX| > r_n\) for a fixed jump-detection threshold \(r_n > 0\), and estimate \(\theta\) after removing such increments. It is shown that for a good \(r_n > 0\) satisfying a suitable rate, the estimator of \(\theta\) has asymptotic normality at the same rate as diffusion models. Hence the method asymptotically achieves both the estimation of \(\theta\) and the jump detection in observed data, while finite-sample performance of the threshold method strongly depends on the value of \(r_n\). Unfortunately, a data-adaptive and quantitative choice of the threshold in the jump-detection filter is a subtle and sensitive problem, and still remains as an annoying problem in practice; see \([8]\), \([9]\), as well as the references therein. Such problem can also be seen in other jump detection methods such as \([1]\).

The primary objective of this paper is to formulate an intuitively easy-to-understand strategy, which can simultaneously estimate \(\theta\) and detect jumps without any precise calibration of a jump-detection threshold. For this purpose, we utilize the approximate self-normalized residuals \([6]\) based on the Gaussian quasi maximum likelihood estimator (GQMLE), which makes a classical Jarque-Bera type test \([3]\) adapted

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to our model. More specifically, the hypothesis test whose significance level is \( \alpha \in (0, 1) \) is constructed by the following manner: let the null hypothesis be of “no jump component”: 
\[
H_0 : \nu(\mathbb{R}) = 0,
\]
against the alternative hypothesis of “non-trivial jump component”: 
\[
H_1 : \nu(\mathbb{R}) > 0.
\]
Then, if the Jarque-Bera type statistic based on the self-normalized statistics introduced later is larger than a given percentile of the chi-square distribution with the degrees of freedom being 2, we reject the null hypothesis \( H_0 \); and otherwise, we accept \( H_0 \). For such a test, we can intuitively regard that the largest increment contains at least one jump (and it will turn out to be true) when the null hypothesis is rejected. Following this inspection, our proposed method is to iteratively conduct the test with removing the largest increments at each test until \( H_0 \) is accepted, and after that, we estimate the target parameter \( \theta \) without removed increments. Our method enables us not only just to make a “pre-cleaning” of diffusion-like data sequence by removing big fluctuations which collapse the (approximate) Gaussianity of the self-normalized residuals, but also to approximately quantify jumps relative to continuous fluctuations in a natural way.

The rest of this paper is organized as follows: in Section 2, we will give a briefly summary of the GQMLE, the approximate self-normalized residuals and the Jarque-Bera test for our model. Section 3 provides the specific recipe of ours and an alternative estimator to GQMLE in order to reduce computational load. At last, we will show some numerical experiments of our method.

2. Preliminaries

In this section, we briefly review the construction of GQMLE, self-normalized residual, and Jarque-Bera statistics with its theoretical behavior. Given any function \( f \) on \( \mathbb{R} \times \Theta \), we write
\[
 f_{j-1}(\theta) = f(X_{t_{j-1}}, \theta).
\]
We denote by \( \mathbb{P}_\theta \) the image measure of \( X \) associated with the parameter value \( \theta \), and by \( \mu(dt, dz) \) the Poisson random measure associated with \( J \).

Suppose that the null hypothesis \( H_0 \) is true for a moment; namely the underlying model is a diffusion process. Then, for the estimation of \( \theta = (\alpha, \beta) \), we can make use of the Gaussian quasi-(log-)likelihood
\[
 \mathcal{H}_n(\theta) := \sum_{j=1}^{n} \log \left\{ \frac{1}{\sqrt{a_{j-1}^2(\alpha)h_n}} \phi \left( \frac{\Delta_j X - b_{j-1}(\beta)h_n}{\sqrt{a_{j-1}^2(\alpha)h_n}} \right) \right\},
\]
where \( \phi \) denotes the standard normal density and
\[
 \epsilon_j(\theta) = \epsilon_{n,j}(\theta) := \frac{\Delta_j X - b_{j-1}(\beta)h_n}{\sqrt{a_{j-1}^2(\alpha)h_n}}.
\]
This quasi-likelihood is constructed based on the local-Gauss approximation of the transition probability \( \mathcal{L}(X_{t_{j}}|X_{t_{j-1}}) \) by \( N(b_{j-1}(\beta)h_n, a_{j-1}^2(\alpha)h_n) \) under \( \mathbb{P}_\theta \), and lead to the Gaussian quasi-maximum likelihood estimator (GQMLE) defined by any element
\[
 \hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n) \in \arg\max_{\theta \in \Theta} \mathcal{H}_n(\theta).
\]
It is well known that the asymptotic normality holds true \( 4 \) under suitable regularity conditions:
\[
 \left( \sqrt{n}(\hat{\alpha}_n - \alpha_0), \sqrt{n}(\hat{\beta}_n - \beta_0) \right) \overset{\mathcal{L}}{\to} N \left( 0, \text{diag}(I_1^{-1}(\alpha_0), I_2^{-1}(\beta_0)) \right),
\]
where
\[
 I_1(\alpha_0) = \frac{1}{2} \int \left( \frac{\partial_\alpha a^2(x, \alpha_0)}{a^2(x, \alpha_0)} \right)^{\otimes 2} \pi_0(dx),
\]
\[
 I_2(\theta_0) = \int \left( \frac{\partial_\beta b(x, \beta_0)}{a(x, \beta_0)} \right)^{\otimes 2} \pi_0(dx).
\]
Here \( \pi_0 \) denotes the invariant measure of \( X \).
To see whether a working model fits data well or not, diagnosis based on residual analysis is often done. Based on the GQMLE, \( \theta \) formulated Jarque-Bera normality test based on self-normalized residuals for our model. Define the self-normalized residual statistic by:

\[
\hat{N}_j = \hat{S}_n^{-1/2}(\epsilon_j(\hat{\theta}_n) - \bar{\epsilon}_n),
\]

where \( \bar{\epsilon}_n := \frac{1}{n} \sum_{j=1}^{n} \epsilon_j(\hat{\theta}_n) \) and \( \hat{S}_n := \frac{1}{n} \sum_{j=1}^{n} (\epsilon_j(\hat{\theta}_n) - \bar{\epsilon}_n)^2 \). Making use of \( \hat{N}_j \), we define Jarque-Bera type statistic \( JB_n \) by

\[
JB_n := \frac{1}{6n} \left( \sum_{j=1}^{n} (\hat{N}_j)^3 - 3 \sqrt{n} \sum_{j=1}^{n} \hat{\omega}_j b_{j-1}(\hat{\theta}_n) \right)^2 + \frac{1}{24n} \left( \sum_{j=1}^{n} ((\hat{N}_j)^4 - 3) \right)^2.
\]

Then, Jarque-bera normality test for our model is justified by the following sense:

**Theorem 2.1.** (cf. \cite{7} Theorem 3.1 and Theorem 4.1) Under the suitable regularity conditions, we have the followings:

- Under \( \mathcal{H}_0 : \nu(\mathbb{R}) = 0 \), we have \( JB_n \overset{\mathbb{P}}{\rightarrow} \chi^2(2) \);
- Under \( \mathcal{H}_1 : \nu(\mathbb{R}) > 0 \), we have \( JB_n \overset{\mathbb{P}}{\rightarrow} \infty \).

3. Proposed strategy

For brevity we write

\[
\mathbb{H}_n(\theta) = \sum_{j=1}^{n} \zeta_{n,j}(\theta)
\]

Let \( q = q_n \in (0, 1) \) be a small number. We propose the iterative jump detection procedure based on the Jarque-Bera type test below.

**Step 0.** Set \( k = k_n = 0 \), and let \( \hat{\mathcal{J}}^{0}_n = \hat{\mathcal{J}}^{0}_n \) be empty set.

**Step 1.** Calculate the modified GQMLE \( \hat{\theta}^{k}_n \) (MGQMLE, for short) by:

\[
\hat{\theta}^{k}_n \in \text{argmax} \ \hat{\mathbb{H}}^{k}_n(\theta),
\]

where \( \hat{\mathbb{H}}^{k}_n(\theta) := \sum_{j \notin \hat{\mathcal{J}}^{k}_n} \zeta_j(\theta) \). Define the following statistics:

\[
\bar{\epsilon}^{k}_n := \frac{1}{n-k} \sum_{j \notin \hat{\mathcal{J}}^{k}_n} \epsilon_j(\hat{\theta}^{k}_n), \quad \hat{S}^{k}_n := \frac{1}{n-k} \sum_{j \notin \hat{\mathcal{J}}^{k}_n} (\epsilon_j(\hat{\theta}^{k}_n) - \bar{\epsilon}^{k}_n)^2.
\]

Building on the MGQMLE and the above ingredients, (re-)construct the following modified self-normalized residuals \( \{\hat{N}^{k}_i\}_{i \in \{1,...,n\}} \) and Jarque-Bera type statistics \( JB^{k}_n \):

\[
\hat{N}^{k}_j := (\hat{S}^{k}_n)^{-1/2}(\epsilon_j(\hat{\theta}^{k}_n) - \bar{\epsilon}^{k}_n) \|_{j \notin \hat{\mathcal{J}}^{k}_n},
\]

\[
JB^{k}_n := \frac{1}{6(n-k)} \left( \sum_{j \notin \hat{\mathcal{J}}^{k}_n} (\hat{N}^{k}_j)^3 - 3 \sqrt{n} \sum_{j \notin \hat{\mathcal{J}}^{k}_n} \hat{\omega}_j b_{j-1}(\hat{\theta}^{k}_n) \right)^2 + \frac{1}{24(n-k)} \left( \sum_{j \notin \hat{\mathcal{J}}^{k}_n} ((\hat{N}^{k}_j)^4 - 3) \right)^2.
\]

**Step 2.** If \( JB^{k}_n > \chi^2_{q} \), then pick out the interval number

\[
j(k+1) := \text{argmax}_{j \in \{1,...,n\} \setminus \hat{\mathcal{J}}^{k}_n} |\Delta_j X|
\]

add \( j(k+1) \) to \( \hat{\mathcal{J}}^{k}_n \), and return to **Step 1**; otherwise, set the number of jumps \( k^* = k^*(\omega) = k \), and go to **Step 3**.

**Step 3.** If \( k^* = 0 \), regard that there is no jump; otherwise, the detected jumps are \( \{\Delta_{j(1)} X, \ldots, \Delta_{j(k^*)} X\} \) (they are in descending order). Finally, set \( \hat{\theta}^{k^*}_n \) as the estimator of \( \theta \).
Remark 3.1. By using its intensity parameter $\lambda$, the number of jumps of a compound Poisson process $N_t$ is expressed as $\lambda t$. Thus, as the terminal time gets larger and larger, the iteration number of our proposed methodology should also be large. In such case or the case where seemingly several jumps do exist, we could instead start from $m$-th stage for some $m \in \mathbb{N}$ which conveniently enables us to “skip” first some redundant stages.

Remark 3.2. In practice, the size of “last-removed” increment would be used as the threshold for detecting jumps for future observations: with the value $r_n(k) := |\Delta_j X|$ in hand, for future observations $(Y^n_{t})_{j=0}^n$ we regard that a jump occurred over $[t_{j-1}, t_j]$ if

$$|\Delta_j Y| > r_n(k).$$

Remark 3.3. Our method enables us to divide the set of the whole increments $(\Delta_j X)_{j=1}^n$ into the following two categories:

- “One-jump” group $\{\Delta_{(1)} X, \ldots, \Delta_{(k')} X\}$, and
- “No-jump” group $(\Delta_j X)_{j=1}^n \setminus \{\Delta_{(1)} X, \ldots, \Delta_{(k')} X\}$.

Our method conducts the estimation of the drift and diffusion part of $X$ based on continuously joined up data computed from the no-jump group pairs:

$$\{(X_{t_{j-1}}, X_{t_j}); j \notin \mathcal{J}^*_n\}.$$ 

Also, we may estimate the jump part by the members of one-jump group; namely we think that the sequence $\Delta_{(k)} X / \epsilon_{(k)}(\gamma)$ under $P_\theta$ being i.i.d. with common jump distribution of the compound Poisson process $J$.

Remark 3.4. To reduce the computational load of the calculating the GQMLE, one can alternatively use the stepwise estimator $\hat{\theta}_n := (\hat{\alpha}_n, \hat{\beta}_n)$ defined by:

$$\hat{\alpha}_n \in \operatorname{argmax}_{\alpha \in \Theta_\alpha} \sum_{j=1}^n \log \left\{ \frac{1}{\sqrt{\Delta_{j-1}^2(\alpha)h_n}} \phi \left( \frac{\Delta_j X}{\sqrt{\Delta_{j-1}^2(\alpha)h_n}} \right) \right\},$$

$$\hat{\beta}_n \in \operatorname{argmax}_{\beta \in \Theta_\beta} \sum_{j=1}^n \log \left\{ \frac{1}{\sqrt{\Delta_{j-1}^2(\hat{\alpha}_n)h_n}} \phi \left( \frac{\Delta_j X - b_{j-1}(\beta)h_n}{\sqrt{\Delta_{j-1}^2(\hat{\alpha}_n)h_n}} \right) \right\},$$

and its modified version can similarly be defined. Under the null hypothesis $H_0$ being true, the limit distribution of $\hat{\theta}_n$ is shown to be equivalent to that of $\tilde{\theta}_n$ (cf. [11]). Moreover, computation of the GQMLE and MGQMLE may become much less time-consuming one when the coefficients are of certain tractable forms: let $p_\alpha$ and $p_\beta$ be the dimension of $\Theta_\alpha$ and $\Theta_\beta$, respectively, and suppose that the diffusion coefficient $a(x)$ and the drift function $b(x)$ can be written by suitable functions $\{a^{(l)}(x)\}_{l \in \{1, \ldots, p_\alpha\}}$ and $\{b^{(k)}(x)\}_{k \in \{1, \ldots, p_\beta\}}$ as

$$a(x, \alpha) = \sqrt{\sum_{l=1}^{p_\alpha} (\alpha^{(l)} a^{(l)}(x)), \quad b(x, \beta) = \sum_{k=1}^{p_\beta} \beta^{(k)} b^{(k)}(x),}$$

where $x^{(l)}$ denotes the $l$-th element of $x$ for every vector $x$. Then the stepwise estimator $\hat{\theta}_n$ is given by

$$\hat{\alpha}_n = \frac{1}{h_n} \left( \sum_{j=1}^n \mathcal{A}_{j-1} \mathcal{A}_{j-1}^\top \right)^{-1} \sum_{j=1}^n (\Delta_j X)^2 \mathcal{A}_{j-1},$$

$$\hat{\beta}_n = \frac{1}{h_n} \left( \sum_{j=1}^n \mathcal{A}_{j-1}^{-2}(\hat{\alpha}_n) \mathcal{B}_{j-1} \mathcal{B}_{j-1}^\top \right)^{-1} \sum_{j=1}^n \mathcal{A}_{j-1}^{-2}(\hat{\alpha}_n) \Delta_j X \mathcal{B}_{j-1},$$

where $\mathcal{A}(x) = (a^{(1)}(x), \ldots, a^{(p_\alpha)}(x))^\top$ and $\mathcal{B}(x) = (b^{(1)}(x), \ldots, b^{(p_\beta)}(x))^\top$. What is important from these expressions is that the modified version of $\hat{\theta}_n$ can be calculated simply by removing the corresponding indices from the sum without repetitive numerical optimizations, thus reducing the computational time to a large extent.
### Table 1. The performance of the GQMLE and MGQMLE: the mean is given with the standard deviation in parenthesis.

|                | $\hat{\alpha}_n$ | $\hat{\beta}_n$ |
|----------------|-------------------|------------------|
| before jump removal | 0.31 (0.06) | 2.00 (0.49) |
| after jump removal   | 0.30 (0.01) | 1.00 (0.01) |

### 4. Asymptotic property of the MGQMLE

In this section, we look at the asymptotic properties of the MGQMLE for the following toy model:

\[ dX_t = \beta dt + \sqrt{\alpha} dw_t + dJ_t, \]

where \( J \) is a compound Poisson process expressed as

\[ J_t := \sum_{i=1}^{N} \xi_i. \]

In this expression, \( N \) and \( \{\xi_i\}_{i \in \mathbb{N}} \) denote a Poisson process whose intensity parameter is \( \lambda \) and i.i.d random variables, respectively. Recall that the observations \((X_{t_0}, \ldots, X_{t_n})\) are obtained in \( t_j = t_j^n := j h_n, T_n := nh_n \to \infty \). To deduce the asymptotic properties of the MGQMLE, we introduce some assumptions below.

**Assumption 4.1.**

- \( E[|\xi|^2] < \infty \), and there exists positive deterministic sequence \((a_n)\) satisfying the conditions

\[ \max_{1 \leq j \leq \lambda + 1} |\xi_j| = O_p(a_n), \]

\[ a_n^3 \sqrt{h_n \log n} \vee nh_n^2 a_n^2 \log n = o(1). \]

- For any \( K > 0 \), we have

\[ P(|\xi|^2 \leq h_n \log n) = o(T_n), \]

\[ P(|\xi|^4 \leq K a_n^4 \sqrt{h_n \log n}) = o(T_n). \]

- the number of jump removal \( k_n = o\left( \frac{n \sqrt{h_n \log n}}{h_n} \right) \).

The following theorem ensures a consistency property of the MGQMLE:

**Theorem 4.2.** If Assumption [4,4] holds, then we have

\[ P(|\hat{\theta}^{k_n} - \theta_0| > \epsilon) \cap \{JB^{k_n} \leq \chi^2_{q}(2)\} \to 0 \]

for each \( \epsilon > 0 \) and \( q \in (0,1) \).

### 5. Numerical experiments

We consider the following SDE model:

\[ dX_t = \alpha(1 + X_t^2)^{-1/2} dw_t - \beta dt + dN_t, \]

where \( \Theta = \Theta_\alpha \times \Theta_\beta = (0.01,100) \times (0.01,100) \). Here we set the true values as \( \alpha_0 = 0.3 \) and \( \beta_0 = 1 \).

Under the conditions where:

- \( n = 10^4, h_n = 5n^{-2/3}; \)
- \( q = 10^{-4}; \)
- \( ^3MC = 10^3; \)
- \( ^3\text{jump times} = 10, \) and jump size = \( U(0.5,5) \) or \( U(-5,-0.5) \) selected with equal probabilities.

Here, we set number of jumps fixed just for numerical comparison purpose. Then the performance of our method is given in the table [1]. The transition of our estimators and the logarithmic values of the JB statistic in the last iteration are shown in Figures [1][2]. As can be seen from Table [1] the estimation accuracy is drastically improved by our method. In this example, we set the jump distribution symmetric, thus the improvement of the estimation of the drift parameter is small compared with that
Lemma 6.1. Let \( \{\tau_i\}_{i \in \{1, \ldots, \}} \) denote jump times of \( N \). Then we have
\[
P(2i, j \text{ s.t. } \tau_i, \tau_{i+1} \in [t_{j-1}, t_j]) \sim \sum_{i=2}^{\infty} P(2j \in \{2, \ldots, i\} \text{ s.t. } \tau_j - \tau_{j-1} < h_n) P(N_{T_n} = i) \leq (1 - e^{-\lambda h_n}) \sum_{i=2}^{\infty} \frac{(AT_n)^i}{(i-1)!} e^{-\lambda T_n} \lesssim nh_n \to 0.
\]

For convenience, we hereafter write
\[
B_{k_n,n} = \left\{ 2i, j \text{ s.t. } \tau_i, \tau_{i+1} \in [t_{j-1}, t_j] \right\}.
\]

Thanks to Lemma 6.1 in proving Theorem 4.2 we may and do focus on the set \( B_{k_n,n} \).

Lemma 6.2. We have
\[
P(\left\{ 2i, j \text{ s.t. } \tau_i \in [t_{j-1}, t_j] \text{ and } \Delta_j X \notin \hat{J}_n \right\} \cap \{1 \leq N_{T_n} \leq k_n \} \cap B_{k_n,n}) \to 0, \quad n \to \infty.
\]

Proof. Hereafter we use the following notations:
\[
D_n = \{j : \exists \tau_i \in [t_{j-1}, t_j]\}, \quad C_n = \{1, \ldots, n\} \setminus D_n.
\]

For every \( j \), we write \( \eta_j = \frac{\Delta_{j-w}}{\sqrt{\lambda_n}} \). Then we have
\[
P(\left\{ 2i, j \text{ s.t. } \tau_i \in [t_{j-1}, t_j] \text{ and } \Delta_j X \notin \hat{J}_n \right\} \cap \{1 \leq N_{T_n} \leq k_n \} \cap B_{k_n,n}) \leq P(\{2j' \in D_n, j'' \in C_n \text{ s.t. } |\Delta_j X| < |\Delta_j' X| \} \cap \{1 \leq N_{T_n} \leq k_n \} \cap B_{k_n,n}) \leq P(\{2j' \in D_n, j'' \in C_n \text{ s.t. } |\Delta_j' X| - |\beta_0 h_n + \alpha_0 \Delta_j' w| < |\beta_0 h_n + \alpha_0 \Delta_j' w| \} \cap \{1 \leq N_{T_n} \leq k_n \} \cap B_{k_n,n}) \leq P\left( \left\{ \min_{1 \leq j \leq n} |\Delta_j J|^2 \geq 0 \right\} \frac{4\beta_0^2 h_n + 4\alpha_0^2 \max_{1 \leq j \leq n} |\eta_j|^2}{h_n} \right) \cap \{1 \leq N_{T_n} \leq k_n \} \cap B_{k_n,n} \right) \cdot
From extreme value theory (cf. [2] Table 3.4.4), we have
\[
\max_{1 \leq j \leq n} |\eta_j|^2 - \left( \log n - \frac{1}{2} \log \log n - \log \Gamma \left( \frac{1}{2} \right) \right) = O_p(1).
\]

Therefore it suffices to show that
\[
P \left( \min_{1 \leq j \leq n} (|\Delta_j|^2 \lor 0) < 1 + r_n \right) \cap \{1 \leq N_{T_n} \leq k_n \} \cap B_{k_n} \to 0,
\]
where
\[
r_n := \frac{4 \log h_n + 4 \log \left\{ \max_{1 \leq j \leq n} |\eta_j|^2 - \left( \log n - \frac{1}{2} \log \log n - \log \Gamma \left( \frac{1}{2} \right) \right) \right\}}{\log \left( \log n - \frac{1}{2} \log \log n - \log \Gamma \left( \frac{1}{2} \right) \right)} = o_p(1).
\]

Assumption 4.1 implies that
\[
P(\{\exists \tau_i \in [t_{j-1}, t_j) \ s.t. \ \Delta_j X \notin \tilde{J}_n^k \} \cap \{1 \leq N_{T_n} \leq k_n \} \cap B_{k_n})
\leq P \left( \min_{1 \leq j \leq n} (|\Delta_j|^2 \lor 0) < \frac{3}{2} \right) \cap \{1 \leq N_{T_n} \leq k_n \} \cap B_{k_n}) + P \left( |r_n| \geq \frac{1}{2} \right)
\leq P \left( \sum_{j=1}^{k_n} (\lambda T_n)^j e^{-\lambda T_n} + P \left( |r_n| \geq \frac{1}{2} \right) \right) \to 0,
\]
hence the claim follows.

\[\square\]

Lemma 6.2 implies that all increments containing jumps are correctly picked up as long as \(1 \leq N_{T_n} \leq k_n\). Similarly, we can derive
\[
P \left( \{\exists j \in \tilde{J}_n^k \cap C_n \} \cap \{N_{T_n} \geq k_n + 1\} \right) \to 0.
\]

**Proof of Theorem 4.2**

We introduce the following events:
\[
A_{k_n,n,\epsilon} := \left\{ \{\hat{\eta}_n^{k_n} - \theta_0| > \epsilon \} \cap \{ J_{B_n}^{k_n} \leq \lambda^2(2) \} \right\},
\]
\[
C_{k_n,n} := \left\{ \exists j, j \text{ s.t. } \tau_i \in [t_{j-1}, t_j) \text{ and } \Delta_j X \notin \tilde{J}_n^k \} \right\}^c
\]
\[
= \left\{ \forall i, j, \tau_i \notin [t_{j-1}, t_j) \text{ or } \Delta_j X \in \tilde{J}_n^k \right\}.
\]

Taking the lemmas into consideration, we can split \(P(A_{k_n,n,\epsilon})\) as
\[
P(A_{k_n,n,\epsilon}) = P_{1,n} + P_{2,n} + P_{3,n} + o(1),
\]
where
\[
P_{1,n} := P(A_{k_n,n,\epsilon} \cap \{N_{T_n} = 0\}),
\]
\[
P_{2,n} := P(A_{k_n,n,\epsilon} \cap B_{k_n,n} \cap C_{k_n,n} \cap \{1 \leq N_{T_n} \leq k_n\}),
\]
\[
P_{3,n} := P \left( A_{k_n,n,\epsilon} \cap B_{k_n,n} \cap \{\exists j \in \tilde{J}_n^k \cap C_n \} \right) \cap \{N_{T_n} \geq k_n + 1\}.
\]

Since \(P_{1,n} \leq P(N_{T_n} = 0) = e^{-\lambda T_n} \to 0\), it suffices to show \(P_{2,n} \to 0\) and \(P_{3,n} \to 0\). From now on, for an event \(A\) we denote by \(\mathbb{I}_A = \mathbb{I}_A(\omega)\) the indicator function of \(A\):
\[
\mathbb{I}_A = \mathbb{I}_A(\omega) := \begin{cases} 1 & \omega \in A, \\ 0 & \omega \notin A. \end{cases}
\]

First we focus on the estimate of \(P_{2,n}\). By virtue of the foregoing discussion, we have the following expression:
\[
\hat{\eta}_n^{k_n} \mathbb{I}_{A_{k_n,n,\epsilon} \cap B_{k_n,n} \cap C_{k_n,n} \cap \{1 \leq N_{T_n} \leq k_n\}}
= \frac{1}{(n-k_n)h_n} \sum_{j \notin \tilde{J}_n^k} (\Delta_j X)^2 \mathbb{I}_{A_{k_n,n,\epsilon} \cap B_{k_n,n} \cap C_{k_n,n} \cap \{1 \leq N_{T_n} \leq k_n\}}
= \frac{1}{(n-k_n)h_n} \sum_{j \notin \tilde{J}_n^k} (\beta_0 h_n + \alpha_0 \Delta_j w)^2 \mathbb{I}_{A_{k_n,n,\epsilon} \cap B_{k_n,n} \cap C_{k_n,n} \cap \{1 \leq N_{T_n} \leq k_n\}},
\]
Since

For abbreviation, we write

Thus

Hence it follows that

\[ P_{2,n} \leq P \left( \frac{1}{n-k_n} \sum_{j \notin \mathcal{J}_n^{k_n}} (\beta_0 h_n + \alpha_0 \Delta_j w)^2 - \alpha_0 \left| \frac{1}{n-k_n} \sum_{j \notin \mathcal{J}_n^{k_n}} (\beta_0 h_n + \alpha_0 \Delta_j w) - \beta_0 \right| > \epsilon \right). \]

The law of large numbers for triangular sequences implies that

\[ \frac{1}{n-k_n} \sum_{j=1}^{n} (\beta_0 h_n + \alpha_0 \Delta_j w)^2 \xrightarrow{p} \alpha_0, \]

\[ \frac{1}{n-k_n} \sum_{j=1}^{n} (\beta_0 h_n + \alpha_0 \Delta_j w) \xrightarrow{p} \beta_0. \]

Again applying (6.1), we have

\[ \frac{1}{n-k_n} \sum_{j \notin \mathcal{J}_n^{k_n}} (\beta_0 h_n + \alpha_0 \Delta_j w)^2 \leq \frac{k_n}{n-k_n} \left( 2\beta_0^2 h_n + 2\alpha_0^2 \max_{1 \leq j \leq n} \eta_j^2 \right) = o_p(1), \]

\[ \left| \frac{1}{n-k_n} \sum_{j \notin \mathcal{J}_n^{k_n}} (\beta_0 h_n + \alpha_0 \Delta_j w) \right| \leq \frac{k_n}{n} + \frac{k_n}{n\sqrt{h_n}} \max_{1 \leq j \leq n} |\eta_j| = o_p(1). \]

Thus \( P_{2,n} \to 0. \)

Let us now move on to the estimates of \( P_{3,n}. \) From the representation \( N_{T_n} = N_{t_n} + (N_{t_1} - N_{t_0}) + \cdots + (N_{T_n} - N_{t_{n-1}}) \) and the central limit theorem, it follows that

\[ \frac{N_{T_n} - \lambda T_n}{\sqrt{\lambda T_n}} \to N(0,1). \]

Hence if \( \frac{k_n - \lambda T_n}{\sqrt{\lambda T_n}} \to \infty, \) we have

\[ P_{3,n} \leq P \left( \frac{N_{T_n} - \lambda T_n}{\sqrt{\lambda T_n}} \geq \frac{k_n - \lambda T_n}{\sqrt{\lambda T_n}} \right) \to 0. \]

Next we consider the case where \( \lim \sup_n \frac{k_n - \lambda T_n}{\sqrt{\lambda T_n}} < \infty. \) Again using the central limit argument, we have

\[ P_{3,n} = P \left( \{ JB_{n}^{k_n} \leq X^2 \} \cap B_{k_n,n} \cap \left\{ \exists j \in \mathcal{J}_n^{k_n} \cap C_n \right\}^c \cap \{ k_n + 1 \leq N_{T_n} \leq [\lambda + 1)T_n \} \right) + o(1). \]

For abbreviation, we write

\[ D_{k_n,n} = B_{k_n,n} \cap \left\{ \exists j \in \mathcal{J}_n^{k_n} \cap C_n \right\}^c \cap \{ k_n + 1 \leq N_{T_n} \leq [\lambda + 1)T_n \}, \]

\[ \zeta_j = \frac{1}{\sqrt{h_n}} \left( \Delta_j X - \frac{1}{n-k_n} \sum_{j \notin \mathcal{J}_n^{k_n}} \Delta_j X \right). \]

Since \( \frac{1}{n-k_n} \sum_{j \notin \mathcal{J}_n^{k_n}} \zeta_j = 0, \) we have

\[ \epsilon_j (\hat{\theta}_n^{k_n}) = \frac{\zeta_j}{n-k_n} \sum_{j \notin \mathcal{J}_n^{k_n}} (\Delta_j X)^2; \]

\[ N_j^{k_n} = \left\{ \frac{1}{n-k_n} \sum_{j \notin \mathcal{J}_n^{k_n}} (\zeta_j)^2 \right\}^{-1/2} \zeta_j \mathcal{J}_n^{k_n}. \]
We decompose $\zeta_j$ as

$$\zeta_j = \frac{1}{\sqrt{h_n}} \left\{ \beta_0 h_n + \alpha_0 \Delta_j w + \Delta_j J - \frac{1}{n - k_n} \sum_{j \notin J_n^h} (\beta_0 h_n + \alpha_0 \Delta_j w + \Delta_j J) \right\} = \zeta_{1,j} + \zeta_2 + \zeta_3,$$

where

$$\zeta_{1,j} := \alpha_0 \eta_j + \frac{1}{\sqrt{h_n}} \Delta_j J,$$

$$\zeta_2 := -\frac{\alpha_0}{n - k_n} \sum_{j=1}^n \eta_j - \frac{1}{n - k_n} \sum_{j=1}^n \frac{1}{\sqrt{h_n}} \Delta_j J,$$

$$\zeta_3 := \frac{1}{n - k_n} \sum_{j \notin J_n^h} \left( \alpha_0 \eta_j + \frac{1}{\sqrt{h_n}} \Delta_j J \right).$$

Then, the following estimate holds:

$$\frac{1}{n - k_n} \sum_{j \notin J_n^h} \zeta_j^2 \mathbb{1}_{D_{k,n}} = \frac{1}{n - k_n} \sum_{j=1}^n \left( \alpha_0 \eta_j^2 + \frac{2 \alpha_0 \eta_j \Delta_j J}{\sqrt{h_n}} + \frac{(\Delta_j J)^2}{h_n} \right) + R_n \mathbb{1}_{D_{k,n}},$$

where

$$R_n := \frac{1}{n - k_n} \sum_{j \notin J_n^h} \{ \zeta_j^2 - \zeta_{1,j}^2 \} = - (\zeta_2 + \zeta_3)^2,$$

$$\bar{R}_n := R_n - \frac{1}{n - k_n} \sum_{j \notin J_n^h} \alpha_0 \eta_j^2 - \frac{1}{n - k_n} \sum_{j \notin J_n^h} 2 \alpha_0 \eta_j \Delta_j J \sqrt{h_n}.$$

Since $\zeta_2$ can be written as

$$\zeta_2 = -\frac{\alpha_0}{n - k_n} \sum_{j=1}^n \eta_j - \sqrt{h_n} \frac{1}{T_n} \sum_{j=1}^{N_{\tau_n}} \xi_j,$$

the law of large numbers implies that

$$\left| \zeta_2 \mathbb{1}_{D_{k,n}} \right| \leq \left| \frac{1}{n} \sum_{j=1}^n \eta_j \right| + \sqrt{h_n} \frac{1}{T_n} \sum_{j=1}^{N_{\tau_n}} |\xi_j| = O_p(\sqrt{h_n}).$$

It also follows that

$$\left| \zeta_3 \mathbb{1}_{D_{k,n}} \right| \leq \frac{k_n}{n - k_n} \left( \max_{1 \leq j \leq n} |\eta_j| + \frac{1}{\sqrt{h_n}} \sum_{1 \leq j \leq \lambda + 1} |\xi_j| \right) = O_p \left( h_n \sqrt{\log n} \vee a_n \sqrt{h_n} \right).$$

Thus we have

$$\left| \bar{R}_n \mathbb{1}_{D_{k,n}} \right| \leq \left( \zeta_2^2 + \zeta_3^2 + \frac{1}{k_n} \max_{1 \leq j \leq n} \eta_j^2 + \frac{N_{\tau_n} - k_n}{(n - k_n) \sqrt{h_n}} \max_{1 \leq j \leq N_{\tau_n}} |\eta_j| \max_{1 \leq j \leq N_{\tau_n}} |\xi_j| \right) \mathbb{1}_{D_{k,n}},$$

$$= O_p(\log h_n n \vee a_n \sqrt{h_n}) = o_p(1).$$

Since $E \left[ \frac{1}{n - k_n} \sum_{j=1}^n \left( \frac{\Delta_j J}{h_n} \right)^2 \mathbb{1}_{D_{k,n}} \right] = E[\zeta_j^2] + o(1)$, we get

$$0 \leq \frac{1}{n - k_n} \sum_{j \notin J_n^h} \frac{(\Delta_j J)^2}{h_n} \mathbb{1}_{D_{k,n}} \leq \frac{1}{n - k_n} \sum_{j=1}^n \frac{(\Delta_j J)^2}{h_n} = O_p(1),$$

and

$$\frac{1}{n - k_n} \sum_{j \notin J_n^h} \frac{(\Delta_j J)^2}{h_n} \mathbb{1}_{D_{k,n}} = E[\zeta_j^2] + o(1).$$
so that for any $\epsilon > 0$, we can choose sufficiently small $\delta$ and large $M$ satisfying
\[
P \left( \left\{ \frac{1}{n-k_n} \sum_{j \notin J_{k_n}} \zeta_j^2 < \delta \right\} \cap D_{k_n,n} \right) + P \left( \left\{ \frac{1}{n-k_n} \sum_{j \notin J_{k_n}} \zeta_j^2 > M \right\} \cap D_{k_n,n} \right) < \epsilon.
\]
For $m = 1, \ldots, 4$ and $m' = 4 - m$, we have
\[
\left( \sum_{j \notin J_{k_n}} \left( \frac{\Delta_j u}{\sqrt{h_n}} \right)^m \right)^m \leq (N_{T_n} - k_n) h_n^{-\frac{m}{2}} \max_{1 \leq j \leq n} \max_{1 \leq j' \leq \lambda(1) T_n} \left| \zeta_j \right|^m.
\]
By the definition, we have
\[
P \left( (J'B_{k_n}^n) \subseteq \left( \sum_{j \notin J_{k_n}} \left( (\hat{N}_{j_n}^k)^4 - 3 \right) \right) \cap D_{k_n,n} \right) \leq P \left( \left\{ \frac{1}{\sqrt{24(n-k_n)}} \sum_{j \notin J_{k_n}} \left( (\hat{N}_{j_n}^k)^4 - 3 \right) \right\} \cap D_{k_n,n} \right).
\]
For abbreviation, we write $V_n = \left( \frac{1}{n-k_n} \sum_{j \notin J_{k_n}} \zeta_j^2 \right)^{-2}$. By definition of $\hat{N}_{j_n}^k$, the following estimate holds:
\[
\frac{1}{\sqrt{24(n-k_n)}} \sum_{j \notin J_{k_n}} \left( (\hat{N}_{j_n}^k)^4 - 3 \right) = \frac{V_n}{\sqrt{24(n-k_n)}} \left[ \sum_{j \notin J_{k_n}} \left( \frac{(\Delta_j u)}{\sqrt{h_n}} \right)^4 + 4 \left( \frac{(\Delta_j u)}{\sqrt{h_n}} \right) \left( \frac{(\Delta_j J)}{\sqrt{h_n}} \right)^3 \right] + \tilde{R}_n,
\]
where
\[
\tilde{R}_n := \sum_{j \notin J_{k_n}} \left\{ \zeta_j^4 - \left( \frac{(\Delta_j J)}{\sqrt{h_n}} \right)^4 - 4 \left( \frac{(\Delta_j u)}{\sqrt{h_n}} \right) \left( \frac{(\Delta_j J)}{\sqrt{h_n}} \right)^3 - \frac{3}{V_n} \right\}.
\]
Therefore we obtain the following estimate:
\[
\frac{1}{\sqrt{24(n-k_n)}} \sum_{j \notin J_{k_n}} \left( (\hat{N}_{j_n}^k)^4 - 3 \right) \ll D_{k_n,n}.
\]
Utilizing (6.1) and (6.2), for any positive constants $\delta < M$, we can derive from Assumption 4.1 that
\[
\sqrt{h_n} \max_{1 \leq j \leq n} \left| \zeta_j \right| = O_p \left( n^{3/2} \sqrt{\log n} \right),
\]
\[
h_n^2 \tilde{R}_n \ll \Delta_{k_n,n} \sup_{\delta \leq \lambda(1) T_n} = O_p \left( n^{3/2} \sqrt{\log n} \right) = o_p(1).
\]
We also have
\[
P \left( \min_{1 \leq j \leq \lambda(1) T_n} \left| \zeta_j \right|^4 > K a_n^3 \sqrt{\log n} \right) = \left( 1 - P \left( \left| \zeta_1 \right|^4 \leq K a_n^3 \sqrt{\log n} \right) \right)^{\lambda(1) T_n} \to 1.
\]
Hence we get
\[
P \left( \left\{ \frac{1}{\sqrt{24(n-k_n)}} \sum_{j \notin J_{k_n}} \left( (\hat{N}_{j_n}^k)^4 - 3 \right) \leq \sqrt{\chi^2_\delta} \cap D_{k_n,n} \right\} \right) \to 0,
\]
and the desired result follows.

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