A simple computational algorithm with inertial extrapolation for generalized split common fixed point problems

Anteneh Getachew Gebrie *, Defene Shewakena Bedane

Department of Mathematics, College of Computational and Natural Science, Debre Berhan University, Debre Berhan, Ethiopia

1. Introduction

Censor and Segal [1] introduced the following split common fixed-point problem (SCFP):

\[ x \in F(U) \text{ such that } Ax \in F(T), \]  

where \( A : H_1 \to H_2 \) is nonzero bounded linear operator, \( U : H_1 \to H_1 \) and \( T : H_2 \to H_2 \) are directed operators, \( H_1 \) and \( H_2 \) are real Hilbert spaces, and \( F(U) \) and \( F(T) \) stand for the fixed point sets \( T \) and \( U \), respectively. In particular, if \( T \) and \( U \) are projection operators, then the SCFP is reduced to the well-known split feasibility problem [2, 3], which is the problem of finding \( x \in C \) such that \( Ax \in Q \), where \( C \) and \( Q \) are nonempty closed convex subsets in \( H_1 \) and \( H_2 \), respectively. Censor and Segal [1] took the following iterative scheme, in finite dimensional spaces, solving SCFP (1):

\[ x_{n+1} = U(x_n - \gamma_n A^*(I - T)Ax_n), \]  

where \( \gamma_n = \gamma (0, 1) \) and \( \lambda \) is the largest eigenvalue of the matrix \( A^*A \). Subsequently, after the work of Censor and Segal [1], this result was extended to different class of operators, see for example [4, 5, 6, 7, 8].

For a real Hilbert space \( H \), the self-mapping \( T : H \to H \) is called \( \kappa \)-demimetric [9] if \( F(T) \neq \emptyset \) and there is \( \kappa \in (-\infty, 1) \) such that

\[ (x - s, x - Ts) \geq \frac{1 - \kappa}{2} \| x - Ts \|^2, \quad \forall (x, s) \in H \times F(T). \]

The class of \( \kappa \)-demimetric mappings in Hilbert space contains the classes of \( \kappa \)-strict pseudocontractions mappings, firmly nonexpansive mappings, nonexpansive mappings, 2-generalized hybrid mappings [10], firmly quasi-nonexpansive mappings, quasi-nonexpansive mappings, demicontractive mappings and directed mappings, see [9, 11, 12]. Moreover, several well known types of mapping arising in optimization belong to the class of demimetric mappings, see for example [9, 13] and references therein.

Generalized Split Inverse Problem (GSIP) [14] is formulated as a problem of

\[ \text{find } x^* \in X \text{ that solves IP1 such that } A_k(x^*) = y_k^* \in Y \text{ and } y_k^* \text{ solves IP2}, \quad \forall k \in \Lambda \]  

where \( \Lambda \subset R \) is an index set, IP1 and IP2 are two inverse problems installed in space \( X \) and \( Y \), respectively, \( A_k : X \to Y \) for each \( k \in \Lambda \). GSIP will be reduced to Split Inverse Problem (SIP) [15] if \( A = A_k \) for all \( k \in \Lambda \). Many models of inverse problems can be cast in this framework by choosing different inverse problems for IP1 and IP2. There is a considerable investigation in the framework of SIP, see for example [4, 16, 17, 18] and the many references therein. In this paper, we consider the type of GSIP called the generalized split common fixed point problem (in short, GSCFP) [14], formulated as finding

* Corresponding author.

E-mail address: antgetn@gmail.com (A.G. Gebrie).

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\[ \bar{x} \in N_{k=1}^N F(U_j) \text{ such that } A(\bar{x}) \in M_{j=1}^M F(T_j), \quad \forall k \in \{1, \ldots, R\}, \] (5)

where \( A_k : H_1 \to H_2 \) is a linear operator for all \( k \in \{1, \ldots, R\} \) \((R \in \mathbb{N})\), \( U_i : H_1 \to H_1 \) and \( T_j : H_2 \to H_2 \) are nonlinear operators for all \( i \in \{1, \ldots, N\} \) \((N \in \mathbb{N})\), \( j \in \{1, \ldots, M\} \) \((M \in \mathbb{N})\), \( H_1 \) and \( H_2 \) are two real Hilbert spaces. If \( A_k = A \) for all \( k \in \{1, \ldots, R\} \), GSCFP (5) will be reduced to the problem of finding

\[ \bar{x} \in N_{k=1}^N F(U_j) \text{ such that } A(\bar{x}) \in M_{j=1}^M F(T_j). \] (6)

Moreover, if \( A_k = A \) for all \( k \in \{1, \ldots, R\} \), \( U = U_j \) for all \( i \in \{1, \ldots, N\} \) and \( T = T_j \) for all \( j \in \{1, \ldots, M\} \), GSCFP (5) will be reduced to SCFP (1).

Some authors proposed methods solving (6) for a different class of mappings; for example, for directed operators [1, 19], demicontractive mappings [20], asymptotically quasi-nonexpansive mappings [21], quasi-nonexpansive operators [5, 22]. However, the implementation of the proposed algorithm requires the estimate of the operator norm \( \|A\| \), and operator norm is global invariant and is often difficult to estimate, see Theorem of Hendrickx and Olshevsky in [23]. The turning point of avoiding the estimate of \( \|A\| \) came when López et al. [24] introduced a variable step size that does not depend on the operator norm \( \|A\| \) solving SFP (1) where \( U = P_c \) and \( T = P_{\Gamma} \). Initially by López et al. [24], some authors modified (2) and established a modified step size for SCFP (1) in general so that its choice does not need any prior knowledge of \( \|A\| \), see for example in [25, 26, 27, 28].

The purpose of this paper is twofold. Firstly, to present a computationally simple algorithm to approximate the solution point of GSCFP (5) for demimetric mappings \( U_i \) and \( T_j \). To be precise, the algorithm is formulated in a parallel computing platform and reduced to a simpler structure. Moreover, we constructed an extended variable step-sizes generated by the algorithms at each iteration, based on previously evaluated iterations so that the implementation of our algorithm does not need any prior information about the operator norms \( \|A_k\| \) \((k \in \{1, \ldots, R\})\). Note that iterative algorithms with a step size that does not require any prior knowledge of the operator norm are more desirable and efficient in practice. Secondly, to present inertial-type algorithm to solve the GSCFP (5) for demimetric mappings \( U_i \) and \( T_j \) and provide the strong convergence theorem for the proposed algorithm. An inertial-type algorithm is algorithm incorporating inertial extrapolation term \( a_n(x_n - x_{n-1}) \) by making use of the previous two iterates \( x_n \) and \( x_{n-1} \). The inertial extrapolation term \( a_n(x_n - x_{n-1}) \) is employed in algorithm for the purpose of speeding up the rate of convergence of the algorithm. The vector \( (x_n - x_{n-1}) \) is acting as an impulsion term and \( a_n \) is acting as a speed regulator, see [29]. Many inertial type accelerated iterative methods are proposed for different kind of problems, see for example [22, 30, 31] and references therein.

The outline of this paper is as follows. In Section 2, we present some necessary definitions, interesting properties and results. The main result of the paper is contained in Section 3 and Section 4, where in Section 3, we introduce our proposed algorithm and discuss its structure, and in Section 4, the strong convergence analysis of our proposed method is investigated. In Section 5, we give some applications that follow from our main result. Finally, in Section 6, we end the paper with numerical results about the new iteration method.

2. Preliminary

Let \( H \) be a real Hilbert space and \( C \) be a nonempty closed convex subset of \( H \). Then, the metric projection on \( C \) is a mapping \( P_C : H \to C \) defined by

\[ P_C(x) = \arg \min \{ \|y - x\| : y \in C \}. \quad x \in H. \]

Given \( x \in H \) and \( z \in C \), then \( z = P_C(x) \) if and only if \( \langle x - z, y - z \rangle \leq 0, \quad \forall y \in C. \)

**Definition 1.** The mapping \( T : H \to H \) is called

(a) firmly nonexpansive if

\[ \|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H; \]

(b) \( \kappa \)-strict pseudocontraction [32] if there exists a \( \kappa \in (0, 1) \) such that

\[ \|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H; \]

(c) directed [1] if \( F(T) \neq \emptyset \) and

\[ \|Tx - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 - \|x - T_x\|^2, \quad \forall \bar{x}, x \in H \times F(T). \]

It is clear that (3) is equivalent to the following:

\[ \|Tx - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 - \kappa \|x - T_x\|^2, \quad \forall x, \bar{x} \in H \times F(T). \]

**Lemma 1.** [9] The set of all fixed points \( F(T) \) of \( \kappa \)-demimetric mapping \( T : H \to H \) is closed and convex.

**Lemma 2.** [33] Let \( \{\Gamma_n\} \) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence \( \{\Gamma_{n,m}\}\) of \( \{\Gamma_n\} \) such that \( \Gamma_{n,m} < \Gamma_{n,m+1} \) for all \( m \geq 1 \). Moreover, consider the sequence of integers \( \{\phi(n)\}_{n \in \mathbb{N}} \) defined by \( \phi(n) = \max \{k \in \mathbb{N} : k \leq n, \Gamma_{k} \leq \Gamma_{k+1}\} \). Then \( \{\phi(n)\}_{n \in \mathbb{N}} \) is a nondecreasing sequence verifying \( \lim_{n \to \infty} \phi(n) = \infty \) and for all \( n \geq \phi(n) \), the following estimates hold: \( \Gamma_{\phi(n)} \leq \Gamma_{\phi(n+1)} \) and \( \Gamma_n \leq \Gamma_{\phi(n)+1} \).

**Lemma 3.** [34] Let \( \{c_n\} \) and \( \{\beta_n\} \) be sequences of nonnegative real numbers. \( \{\beta_n\} \) be sequences of real numbers such that

\[ c_{n+1} \leq (1 - a_n)c_n + \beta_n + \gamma_n, \quad n \geq 1, \]

where \( 0 < a_n < 1 \) and \( \sum \beta_n < \infty \).

(i) If \( \beta_n \leq \gamma_n M \) for some \( M \geq 0 \), then \( \{c_n\} \) is a bounded sequence.

(ii) If \( \sum \beta_n = \infty \) and \( \lim \sup_{n \to \infty} \frac{\gamma_n}{\beta_n} \leq 0 \), then \( c_n \to 0 \) as \( n \to \infty \).

3. Proposed method

**Lemma 4.** Suppose \( U_i : H_1 \to H_1 \) is \( \eta_i \)-demimetric mapping for all \( i \in \{1, \ldots, N\} \) and \( T_j : H_2 \to H_2 \) is \( \beta_j \)-demimetric mapping for all \( j \in \{1, \ldots, M\} \). For \( x \in H_1 \), define \( i_n \in \arg \max \{\|(I - U_i)x\| : i \in \{1, \ldots, N\}\} \)

Then, \( A_k^* (I - T_j) A_k x + (I - U_i)x = 0 \iff x \text{ solves GSCFP (5).} \)

**Proof.** Suppose \( A_k^* (I - T_j) A_k x + (I - U_i)x = 0 \). For \( p \in \Gamma \), we have

\[ 0 = \|A_k^* (I - T_j) A_k x + (I - U_i)x\| = \|x - p\| \]

\[ \geq (A_k^* (I - T_j) A_k x + (I - U_i)x, x - p) \]

\[ = (A_k^* (I - T_j) A_k x, x - p) + (I - U_i)x, x - p) \]

\[ = (I - T_j) A_k x, A_k x - A_k x, x - p) \]

\[ \geq 1 - \eta_j \| (I - T_j) A_k x \|^2 + \frac{\beta_j}{2} \| (I - U_i)x \|^2 \]

This implies \( \| (I - T_j) A_k x \|^2 \leq \| (I - U_i)x \|^2 \). Using the definition of the \( i_n \) and \( (j_n, k_n) \), we have \( \| (I - T_j) A_k x \|^2 \leq \| (I - U_i)x \|^2 \) for all \( (i, j, k) \in \{1, \ldots, N\} \times \{1, \ldots, M\} \times \{1, \ldots, R\} \). Therefore, \( x \) solves GSCFP (5).
The converse is straightforward. □

We now establish a self-adaptive iterative method with inertial term for solving the GSCFP (5) by making use of the following three assumptions:

(A1) The mapping \( U_i \) is \( \eta_i \)-demimetric and \( I - U_i \) is demiclosed at 0 for all \( i \in \{1, \ldots, N\} \);

(A2) The mapping \( T_j \) is \( \beta_j \)-demimetric and \( I - T_j \) is demiclosed at 0 for all \( j \in \{1, \ldots, M\} \);

(A3) The mapping \( A_k \) is nonzero bounded linear operator and \( A_k^* \) denotes the adjoint of \( A_k \) for \( k \in \{1, \ldots, R\} \);

(A4) \( \Gamma \) denotes the solution set of the GSCFP (5) and \( \Gamma \) is nonempty.

Algorithm 1 Choose \( u, x_0, x_1 \in H \). Let \( \{\alpha_n\}, \{\theta_n\} \) and \( \{\rho_n\} \) be a non-negative real sequences.

**STEP 1.** Evaluate \( z_n = x_n + \theta_n (x_n - x_{n-1}) \), where

\[
\alpha_n \in \arg \max \{ \| (I - U_i) z_n \| : i \in \{1, \ldots, N\} \} \quad \text{and} \quad \beta_n \in \arg \max \{ \| (I - T_j) A_k z_n \| : (j, k) \in \{1, \ldots, M\} \times \{1, \ldots, R\} \}.
\]

**STEP 2.** Compute \( t_n = (I - U_i) z_n \) and \( y_n = (I - T_j) A_k z_n \), where

\[
j_n, k_n \in \arg \max \{ \| (I - T_j) A_k z_n \| : (j, k) \in \{1, \ldots, M\} \times \{1, \ldots, R\} \}.
\]

If \( \| A_k^*(y_n) + t_n \| = 0 \), Stop. Otherwise, go to **STEP 3**.

**STEP 3.** Evaluate

\[
x_{n+1} = a_n u + (1 - a_n)(z_n - \rho_n \frac{\| y_n \|^2 + \| z_n \|^2}{\| z_n - y_n \|^2} (A_k^*(y_n) + t_n)).
\]

**STEP 4.** Set \( n := n + 1 \) and go to **STEP 1**.

**Lemma 5.** The stopping condition (in **STEP 2**) of Algorithm 1 is satisfied \((\| A_k^*(y_n) + t_n \| = 0 \text{ for some } n \in \mathbb{N}) \iff z_n \text{ solves GSCFP (5)}.\)

**Proof.** Straightforward using Lemma 4. □

In the next section, we analyze the strong convergence of the infinite sequence \( \{x_n\} \) generated by Algorithm 1 to the solution point of GSCFP (5) assuming that Algorithm 1 does not terminate (the stopping condition (in **STEP 2**) is not satisfied, i.e., \( \mu(z_n) = \| A_k^*(y_n) + t_n \| \neq 0 \) for all \( n \in \mathbb{N} \), or by ignoring the stopping condition (in **STEP 2**) and setting \( x_{n+1} \) in (7) (**STEP 3**) by

\[
x_{n+1} = a_n u + (1 - a_n)(z_n - \rho_n \frac{\| y_n \|^2 + \| z_n \|^2}{\| z_n - y_n \|^2} (A_k^*(y_n) + t_n)),
\]

where \( \mu(z_n) = \| A_k^*(y_n) + t_n \| \text{ if } \| A_k^*(y_n) + t_n \| \neq 0, \mu(z_n) = 1 \) otherwise.

**4. Convergence analysis**

Here in this section, we analysis the convergence of our proposed iterative method, Algorithm 1, for solving GSCFP (5).

**Theorem 1.** If the real parameters \( \{a_n\}, \{\theta_n\} \) and \( \{\rho_n\} \) in Algorithm 1 satisfy the following conditions:

(C1) \( 0 < a_n < 1, \lim_{n \to \infty} a_n = 0 \) and \( \sum_{n=1}^\infty a_n = \infty \);

(C2) \( 0 \leq \theta_n \leq \theta < 1 \) and \( \lim_{n \to \infty} \theta_n x_n - x_{n-1} \| ;

(C3) \( 0 < \rho_n < \delta \leq \min\{1 - \eta_1, \ldots, 1 - \eta_N, 1 - \beta_1, \ldots, 1 - \beta_M \} \) and \( \liminf_{n \to \infty} \rho_n (\delta - \rho_n) > 0 ;

then the sequence \( \{x_n\} \) generated by Algorithm 1 converges strongly to \( x \in \Gamma \) where \( x = P_\Gamma(u) \).

**Proof.** Let \( s_n = z_n - \rho_n \frac{|| y_n ||^2 + || z_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n) \) and let \( x = P_\Gamma(u) \).

Now, using the definition of \( s_n \), we have

\[
\| x_n - x \|^2 = \| z_n - \rho_n \frac{|| y_n ||^2 + || z_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n) - x \|^2 \\
\leq \| z_n - x \|^2 + \rho_n \frac{|| y_n ||^2 + || z_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n) - x \|^2 \\
- 2\rho_n \frac{|| y_n ||^2 + || z_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n), z_n - x).
\]

The result (8) in view of

\[
\rho_n \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n) - x \|^2 = \left( \rho_n \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} \right)^2 A_k^*(y_n) + t_n \|^2 \\
\leq \rho_n \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n) - x \|^2.
\]

and

\[
\rho_n \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n), z_n - x \|^2 = \rho_n \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n), z_n - x \|^2
\]

gives

\[
\| s_n - x \|^2 \leq \| z_n - x \|^2 + \rho_n^2 \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} - \rho_n \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n) - x \|^2 \\
\leq \| z_n - x \|^2 + \rho_n (\delta - \rho_n) \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n) - x \|^2.
\]

From condition (C3) and (9), we have that

\[
\| s_n - x \|^2 \leq \| z_n - x \|^2.
\]

By (10) and the definition of \( s_{n+1} \), we get

\[
\| s_{n+1} - x \|^2 = \| (1 - a_n) s_n + a_n u - x \|^2
\]

\[
= \| (1 - a_n)(s_n - x) + a_n(u - x) \|^2
\]

\[
\leq \| 1 - a_n \| s_n - x \|^2 + a_n \| u - x \|^2
\]

\[
\leq \| 1 - a_n \| z_n - x \|^2 + a_n \| u - x \|^2
\]

\[
= \| 1 - a_n \| z_n - x \|^2 + \rho_n \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n) - x \|^2
\]

\[
\leq \| 1 - a_n \| x_n - x \|^2 + \rho_n \frac{|| y_n ||^2 + || t_n ||^2}{\mu(z_n)} (A_k^*(y_n) + t_n) - x \|^2.
\]

where \( v_n = (1 - a_n) \frac{\delta}{\mu(z_n)} \| x_n - x_{n-1} \| + \| u - x \| \). From condition (C1) and (C2), we see that \( \lim_{n \to \infty} v_n = \| u - x \| \). Hence, \( \{v_n\} \) is bounded. Let \( d > 0 \) such that \( v_n \leq d \) for all \( n \in \mathbb{N} \). Then, (11) becomes

\[
\| s_{n+1} - x \|^2 \leq \| 1 - a_n \| x_n - x \|^2 + a_n d.
\]

Thus, by Lemma 3 (i) the sequence \( \{x_n\} \) is bounded. Since \( \{x_n\} \) is bounded, there exists \( L > 0 \) such that \( \| x_n - x \| \leq L \) for all \( n \in \mathbb{N} \). Now using the definition of \( x_{n+1}, z_n \) and (9), we get
Moreover,

Now,

\[
\{ \Gamma \} = \{ \| x_n - z \| \}
\]

Since \( \alpha_n \rightarrow 0 \), we have \( \| x_n - z \| = 0 \). Thus, \( \{ \Gamma \} \) is bounded.

\[
\| x_n - z \| \leq \| x_n - x \| + \| x - z \| \leq \| x_n - x \| + \| x - z \|.
\]

Thus, \( \{ z_n \} \) is bounded.

Let us distinguish the following two cases related to the behavior of the sequence \( \{ z_n \} \) where \( z_n = \| x_n - z \| \).

Case 1. Suppose the sequence \( \{ \Gamma \} \) decreases at infinity. Thus, there exists \( n_0 \in \mathbb{N} \) such that \( \Gamma_{n+1} \leq \Gamma_n \) for \( n \geq n_0 \). Then, \( \{ \Gamma \} \) converges and \( \Gamma_n \rightarrow 0 \) as \( n \rightarrow \infty \).

From (12), we have

\[
(1 - \alpha_n) \beta_n (\delta - \rho_n) \frac{(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)} \leq (\Gamma_{n+1} - \Gamma_n) + \alpha_n \| u - z \|^2
\]

Since \( \Gamma_n \rightarrow 0 \) as \( n \rightarrow \infty \), and using (13) that

\[ (1 - \alpha_n) \beta_n (\delta - \rho_n) \frac{(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)} \rightarrow 0, \quad n \rightarrow \infty. \]

The conditions (11) and (33) together with (14) yields

\[
\frac{(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)} \rightarrow 0, \quad n \rightarrow \infty.
\]

Now, if \( \{ A^*_n\} \rightarrow \rho \neq 0 \), we have

\[
\frac{(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)} = \frac{(\| y_n \|^2 + \| t_n \|^2)}{\max\{ A_n, 1 \}} = \frac{\| y_n \|^2 + \| t_n \|^2}{\max\{ A_n, 1 \}} \leq \frac{\| y_n \|^2 + \| t_n \|^2}{\max\{ A_n, 1 \}} \leq \frac{2(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)}.
\]

Moreover, if \( \{ A^*_n\} \rightarrow \rho = 0 \), then \( \| y_n \|^2 + \| t_n \|^2 = 0 \) and thus

\[
\frac{(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)} = 0 \quad \text{and} \quad \frac{(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)} = \frac{2(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)}.
\]

Hence, by (16) and (17), for all \( n \in \mathbb{N} \) we get

\[
\frac{(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)} \leq \frac{2(\| y_n \|^2 + \| t_n \|^2)}{\mu(z_n)}.
\]

and so this together with (15) gives

\[
\lim_{n \rightarrow \infty} (\| y_n \|^2 + \| t_n \|^2) = 0 \quad \Leftrightarrow \quad \lim_{n \rightarrow \infty} \| y_n \| = \lim_{n \rightarrow \infty} \| t_n \| = 0.
\]
Therefore, in view of (27), we have
\[ \Gamma_{n+1} \leq (1 - a_n) \Gamma_n + a_n \nu_n, \] (28)
where \( \nu_n = \frac{\theta_n^2}{a_n} \| x_n - x_{n-1} \|^2 + 2 \frac{\theta_n}{a_n} \| x_n - x_{n-1} - \delta \| + 2 \theta_n L \| x_n - x_{n-1} \| + 2 \theta_n (u - \delta, x_{n+1} - \delta). \)

Case 2. Assume that \( \{ \Gamma_n \} \) does not decrease at infinity. Let \( \varphi : N \to \mathbb{N} \) be a mapping for all \( n \geq n_0 \) (for some \( n_0 \) large enough) defined by
\[ \varphi(n) = \max \{ I \in N : I \leq n, \Gamma_I \leq \Gamma_{n+1} \}. \]

By Lemma 2, \( \{ \varphi(n) \}_{n=n_0}^{\infty} \) is a nondecreasing sequence, \( \varphi(n) \to \infty \) as \( n \to \infty \) and
\[ \Gamma_{\varphi(n)} \leq \Gamma_{\varphi(n)+1}, \quad \forall n \geq n_0. \] (29)
In view of \( \| x_{\varphi(n)} - x \|^2 - \| x_{\varphi(n)+1} - x \|^2 \geq \Gamma_{\varphi(n)} - \Gamma_{\varphi(n)+1} \leq 0 \) for all \( n \geq n_0 \) and (27), we have for all \( n \geq n_0 \)
\[ (1 - a_{\varphi(n)}) \rho_n (\delta - \rho_{\varphi(n)}) \leq \Gamma_{\varphi(n)} - \Gamma_{\varphi(n)+1} + \frac{\theta_n^2}{a_n} \| x_n - x_{n-1} \|^2 + 2 \frac{\theta_n}{a_n} \| x_n - x_{n-1} \| + 2 \theta_n (u - \delta, x_{\varphi(n)} - \delta). \] (30)

Thus, (30) together with (C1) and (C2), we have
\[ \frac{\| y_{\varphi(n)} \|^2 + \| y_{\varphi(n)+1} \|^2}{\mu(\varphi(n))} \to 0, \quad n \to \infty. \] (31)

Using similar procedure as above in Case 1, we have
\[ \lim_{n \to \infty} \| x_{\varphi(n)} - y_{\varphi(n)} \| = \lim_{n \to \infty} \| x_{\varphi(n)+1} - x_{\varphi(n)} \| = 0. \]

Since \( \{ y_{\varphi(n)} \} \) is bounded, there exists a subsequence of \( \{ y_{\varphi(n)} \} \), still denoted by \( \{ y_{\varphi(n)} \} \) which converges weakly to \( y \). Now repeating the argument of the proof as in Case 1, we have \( \limsup_{n \to \infty} \mu(\varphi(n)) = 1 \).

From (28), we have
\[ \Gamma_{\varphi(n)+1} \leq (1 - a_{\varphi(n)}) \Gamma_{\varphi(n)} + a_{\varphi(n)} \nu_{\varphi(n)}, \] (32)
where
\[ \nu_{\varphi(n)} = \frac{\theta_n^2}{a_n} \| x_n - x_{n-1} \|^2 + 2 \frac{\theta_n}{a_n} \| x_n - x_{n-1} \| + 2 \theta_n (u - \delta, x_{\varphi(n)} - \delta). \]

Using \( \Gamma_{\varphi(n)} - \Gamma_{\varphi(n)+1} \leq 0 \) for all \( n \geq n_0 \), the inequality (32) gives
\[ 0 \leq -a_{\varphi(n)} \Gamma_{\varphi(n)} + a_{\varphi(n)} \nu_{\varphi(n)}. \]

Since \( a_{\varphi(n)} > 0 \), we obtain \( \| x_{\varphi(n)} - x \|^2 = \Gamma_{\varphi(n)} \leq \nu_{\varphi(n)} \). Moreover, since \( \limsup_{n \to \infty} \nu_{\varphi(n)} \leq 0 \), we have \( \lim_{n \to \infty} \| x_{\varphi(n)} - x \| = 0 \). Thus, \( \lim_{n \to \infty} \| x_n - x \| = 0 \) together with \( \lim_{n \to \infty} \| x_{n+1} - x \| = 0 \), gives \( \lim_{n \to \infty} \Gamma_{n+1} = 0 \). Hence, from (29), we get \( \Gamma_n \to 0 \), that is, \( x_n \to x \) as \( n \to \infty \).

This ends the proof. □

It should be noted that our iterative method, Algorithm 1, also works for approximation of solution of (6) under the assumptions (A1), (A2) and (A3).
5. Applications

5.1. Theoretical applications

As a direct consequence of our algorithm, we can have several new algorithms for different class of mappings; for example, for \( \kappa \)-strict pseudo-contractions mappings, firmly-nonexpansive mappings and directed mappings. Moreover, as applications, we can obtain several new algorithms to solve problems that can be converted to the fixed point problem of demimetric mappings. Note that \( \kappa \)-strict pseudocontractive mapping is \( \kappa \)-demimetric mapping, and firmly nonexpansive mapping is \( \kappa \)-demimetric mapping. Moreover, both \( \kappa \)-strict pseudocontractive and firmly nonexpansive mapping are demiclosed [35].

For a real Hilbert space \( H \) and maximal monotone set-valued mapping \( T : H \to 2^H \), the resolvent operator \( J_T^\lambda \) associated with \( T \) and \( \lambda > 0 \) is

\[
J_T^\lambda (x) = (I + \lambda T)^{-1}(x), \quad x \in H. \tag{35}
\]

The resolvent operator \( J_T^\lambda \) is single-valued and firmly nonexpansive. Moreover, \( \emptyset \in T(x) \) if and only if \( x \) is a fixed point of \( J_T^\lambda \) for all \( \lambda > 0 \); see [20].

Let \( H_1 \) and \( H_2 \) be real Hilbert spaces, \( A_k : H_1 \to H_2 \) be a nonzero bounded linear operator for each \( k \in \{1, \ldots, N\} \), \( T_i : H_1 \to 2^{H_2} \) and \( U_j : H_2 \to 2^{H_1} \) be maximal monotone mappings for all \( i \in \{1, \ldots, N\} \), \( j \in \{1, \ldots, M\} \). The split system of inclusion problem is to find \( x \in H_1 \) such that

\[
\begin{align*}
0 & \in T_i(x), \quad \forall i \in \{1, \ldots, N\}, \\
0 & \in U_j(A_k(x)), \quad \forall j \in \{1, \ldots, M\}, \quad \forall k \in \{1, \ldots, R\}.
\end{align*} \tag{36}
\]

Let \( X \) be the solution set of (36).

Now replacing \( U_i \) and \( T_i \) of Algorithm 1 by the resolvent operators \( J_{U_i}^\mu \) and \( J_{T_i}^\lambda \), we obtain the following algorithm, Algorithm 3, for solving (36).

**Algorithm 3** Choose \( u, x_0, x_1 \in H_1 \). Let \( \{a_n\} \), \( \{\theta_n\} \) and \( \{\rho_n\} \) be nonnegative real sequences.

**STEP 1.** Evaluate \( z_0 = x_0 + \theta_0(x_0 - x_1) \).

**STEP 2.** Compute \( z_n = (I - J_{U_i}^\mu)z_{n-1} \) and \( y_n = (I - J_{T_i}^\lambda)z_n \) where

\[
i_n = \arg \max \{ \|I - J_{U_i}^\mu z_n\| : i \in \{1, \ldots, N\}\}
\]

and \( (i_n, k_n) = \arg \max \{ \|I - J_{T_i}^\lambda z_n\| : (i, k) \in \{1, \ldots, N\} \times \{1, \ldots, M\} \} \).

**STEP 3.** Evaluate

\[
x_{n+1} = a_n x_n + (1 - a_n) \left( z_n - \rho_n \left\| x_n - x_{n-1} \right\| z_n + \rho_n \mu(z_n) \right),
\]

where \( \mu(z_n) = \|A_k^* (y_n) + t_n\| \text{ if } \|A_k^* (y_n) + t_n\| \neq 0, \mu(z_n) = 1 \text{ otherwise.} \)

**STEP 4.** Set \( n := n+1 \) and go to **STEP 1**.

**Corollary 3.** If the real parameters \( \{a_n\} \), \( \{\theta_n\} \) and \( \{\rho_n\} \) in Algorithm 3 satisfy the following conditions:

\[
\begin{align*}
&0 < a_n < 1, \quad \lim_{n \to \infty} a_n = 0 \text{ and } \sum_{n=1}^{\infty} a_n = \infty; \quad \text{(C1)} \\
&0 < \theta_n \leq \theta < 1 \text{ and } \lim_{n \to \infty} \frac{\theta_n a_n}{a_{n+1} + \rho_n} = 0; \quad \text{(C2)} \\
&0 < \rho_n < 2 \text{ and } \lim_{n \to \infty} \frac{\rho_n}{a_n} = 0; \quad \text{(C3)}
\end{align*}
\]

then the sequence \( \{x_n\} \) generated by Algorithm 3 converges strongly to the solution point \( x \) problem (36) where \( x = P_X(u) \).

Let \( H \) be a real Hilbert space. Note that for a proper, lower semicontinuous convex function \( f : H \to \mathbb{R} \cup \{+\infty\} \) the subdifferential of \( f \) (denoted by \( \partial f \)) is a maximal monotone operator; \( f \) at \( x \in H \), we denote by \( \partial f(x) \), which is given by \( \partial f(x) = \{ y \in H : f(z) \geq f(x) + \langle y, z - x \rangle, \quad \forall z \in H \} \). Notice that a point \( x \in H \) minimizes \( f \) if and only if \( 0 \in \partial f(x) \). On top of that, \( \text{prox}_{IJ}(x) = (I + \lambda f_j)^{-1}, \) i.e., the point \( x \) minimizes \( f \) if and only if \( \text{prox}_{IJ}(x) = x \), see [36].

If \( f_j : H_1 \to \mathbb{R} \cup \{+\infty\} \) and \( g_j : H_2 \to \mathbb{R} \cup \{+\infty\} \) are proper, lower semicontinuous convex functions for \( i \in \{1, \ldots, N\} \), \( j \in \{1, \ldots, M\} \), then taking \( T_i = \text{prox}_{f_i} \) and \( U_j = \text{prox}_{g_j} \) in (5), Algorithm 3 yields strong convergence result solving

\[
\hat{x} \in \bigcap_{i=1}^{N}(\arg \min f_i) \text{ such that } A_k(\hat{x}) \in \bigcap_{j=1}^{M}(\arg \min g_j), \quad \forall k \in \{1, \ldots, R\}. \tag{37}
\]

The multiple-set split feasibility problem (MSFPF) is the problem of finding

\[
\hat{x} \in \bigcap_{i=1}^{N} C_i \text{ such that } A_k(\hat{x}) \in \bigcap_{j=1}^{M} Q_j, \quad \forall k \in \{1, \ldots, R\}, \tag{38}
\]

where \( C_i (i \in \{1, \ldots, N\}) \) and \( Q_j (j \in \{1, \ldots, M\}) \) nonempty closed convex subsets of \( H_1 \) and \( H_2 \), respectively. The MSFPF (38) is a special case of (37), i.e., take \( f_i = \delta_{C_i} \) and \( g_j = \delta_{Q_j} \) (the indicator functions) in (37).

**Remark 3.** The results mentioned as an application extend and improve the results in literature, for example, [5, 8, 16, 17, 18, 19, 20, 21, 37] and the reference therein.

5.2. Real world application

Several problems in the fields of applications that can be cast into a MSSF (38) and as we illustrated above MSSF (38) is particular case of GSCFP (5). The following are exhibits of the potential application of our proposed method presented as particular case of GSCFP (5).

i. Sparse Binary Tomography: Digital image processing plays an important role in medical and astronomical imaging, file restoration, image and video coding, and other applications. Some approaches have been suggested for the image reconstruction processing analysis (see, Gibali and Petra [38]) are presented, for example \( l_0 \)-superiorization [38] and \( l_\infty \)-minimization, see [39] and reference therein. Consider \( l_0 \) constraint approach of the image reconstruction by Gibali and Petra [38]:

\[
\min \frac{1}{2} \|Ax - b\|^2_2 \text{ subject to, } \quad x \in [0, 1]^p \text{ and } \|x\|_1 \leq t. \tag{39}
\]

where \( b \in \mathbb{R}^t \), \( A \in \mathbb{R} \times p \), \( [0, 1]^p \) is the \( [0, 1] \) cube in \( \mathbb{R}^t \). The problem (39) takes MSSF (38) given by

\[
\min \frac{1}{2} \|Ax - b\|^2_2 \text{ subject to, } \quad x \in \mathbb{R}^t \text{ and } \|x\|_1 \leq t. \tag{40}
\]

where \( b \in \mathbb{R}^t \), \( A \in \mathbb{R} \times p \), \( C_1 \) is cube in \( \mathbb{R}^t \) and \( C_2 = \{ x \in \mathbb{R}^t : \|x\|_1 \leq t \} \) and \( Q = \{b\} \).

ii. Radiotherapy treatment is very important in the field of medicine, specially in the dose calculation process in Radiation therapy treatment planning (RTTP) [40] and intensity-modulated radiation therapy (IMRT) [31, 42]. The RTTP considered in [40] and IMRT considered in [41, 42] are examples that can be translated into MSSF (38).

6. Numerical result

In this section, we will present two preliminary numerical experiments to show the performance of our proposed iterative algorithm, Algorithm 1, and to show our algorithm converges faster than the algorithm in [1, 19]. All the code is written in MATLAB and is performed on HP laptop with Intel(R) Core(TM) i5-7200U CPU @ 2.70 GHz and RAM 4.00 GB.
Problem 1. Consider GSCFP (5) for $H_1 = H_2 = l_2$ with usual norm, where

$$U_i : x = (x^{(1)}, x^{(2)}, x^{(3)}, \ldots) \mapsto (a_{i,1}x^{(1)}, a_{i,2}x^{(2)}, \ldots, a_{i,p}x^{(p)}, 0, 0, 0, \ldots),$$

$$T_j : y = (y^{(1)}, y^{(2)}, y^{(3)}, \ldots) \mapsto (b_{j,1}y^{(1)}, b_{j,2}y^{(2)}, b_{j,3}y^{(3)}, \ldots)$$

and

$$A_k : x = (x^{(1)}, x^{(2)}, x^{(3)}, \ldots) \mapsto (x^{(1)}, x^{(2)}, x^{(3)}, \ldots),$$

where $a_{i,j} \leq -1$ for all $(i, t) \in \{1, \ldots, N\} \times \{1, \ldots, p\}$, and $b_{j, s} \leq -1$ for all $j \in \{1, \ldots, M\}$ and $x = (1, 2, 3, \ldots)$ is given by

$$z^{(i)} = \begin{cases} 0, & \text{if } s \leq k, \\ x^{(s)}, & \text{if } s = k + m, \end{cases}$$

for $k \in \{1, \ldots, R\}$. It is not hard to show that $U_i$ is $q_i$-strict pseudocontraction mapping ($q_i$-demimetric mapping) where $0 \leq q_i \leq q_i < 1$ for $\xi_i = \max \left\{ \frac{a_{i,j}^{(s-1)}}{q_i a_{i,j}^{(s-2)}} : i \in \{1, \ldots, n\} \right\}$, and $T_j$ is $\beta_j$-strict pseudocontraction mapping ($\beta_j$-demimetric mapping) where $0 \leq \beta_j \leq \beta_j < 1$ for $\xi_j = \frac{b_{j,1}^{(s-1)}}{q_j b_{j,1}^{(s-2)}}$. This implies, $I - U_i$ and $I - T_j$ are demiclosed.

We study our numerical results for our algorithm for different step-sizes, $u = 0$ and for different starting points $x_0$ and $x_1$, and different $N$, $M$, and $p$. The step size $\theta_n$ is set $\theta_n = \theta_0$, where $\theta_0$ is defined as in Remark 2, i.e., by taking $\theta = 0.8$ and $\epsilon_n$ with $\epsilon_n = \alpha(\theta_n)$ and (34). The numerical results are shown in Table 1, 2 and 3. The tables illustrate the numerical behavior of our algorithm in terms of number of iterations (Iter(n)) and CPU time in seconds (cput), where

$$\max_{i \in \{1, \ldots, N\}} \max_{(x_1, x_2, \ldots, x_N) \in C} \left| \frac{1}{\|x_1 - x_2\|} \left| x_1^{(s)} \right| \right| < 0.001$$

as the stopping criterion. Based on the numerical results in Table 1 and 2 we see that the CPU time and a number of iterations of the algorithm to decrease linearly for the choice of the term $a_n$ close to 1 or $\rho_n$ close to 1. In addition, from Table 1, 2 and 3, we observe that the number of iterations and CPU time of Algorithm 1 depends on initial points and $N$ and $M$, and our iteration method can successively find high accuracy approximations to the solution GSCFP (5).

We next consider an example of GSCFP (5) in a finite-dimensional Hilbert space to compare our method, with the results in [1, 19].

Problem 2. Consider GSCFP (5) for $H_1 = \mathbb{R}^p$ and $H_2 = \mathbb{R}^p$, with $A_k = G^{(1)p}\rho\rho^*$.

$$U_i : x = (x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(p)}) \mapsto (a_{i,1}x^{(1)}, a_{i,2}x^{(2)}, \ldots, a_{i,p}x^{(p)})$$

and

$$T_j : y = (y^{(1)}, y^{(2)}, y^{(3)}, \ldots, y^{(p)}) \mapsto (b_{j,1}y^{(1)}, b_{j,2}y^{(2)}, \ldots, b_{j,p}y^{(p)}).$$
where \(0 \leq a_{i,j} < 1\) for all \((i,t) \in \{1, \ldots, N\} \times \{1, \ldots, p\}\) and \(0 \leq b_{j,r} < 1\) for all \((i,t) \in \{1, \ldots, M\} \times \{1, \ldots, q\}\) and \(G_{ij}^{(k)}\) is \(q \times p\) matrix for \(k \in \{1, \ldots, R\}\). Obviously, for this case \(U_i\), \(\eta_i\)-demimetric mapping where \(\xi_i \leq \eta_i \leq 1\) for \(\xi_i = \max\left\{ a_{i,j}^{\beta_{ij}^{-1}} : r \in \{1, \ldots, p\}\right\}\), and \(T_j\) is \(\beta_j\)-demimetric \(\zeta_j \leq \beta_j \leq 1\) for \(\zeta_j = \max\left\{ b_{j,r}^{\gamma_{jr}^{-1}} : r \in \{1, \ldots, q\}\right\}\). This implies, each \(U_i\) and \(T_j\) are directed mapping. The mappings \(I - U_i\) and \(I - T_j\) are demimetric from the fact that \(U_i\) and \(T_j\) are also nonexpansive mappings.

We take \(a_{i,j} = \frac{1}{i+1}\) and \(b_{j,r} = \frac{1}{j+r}\) for all \((i,t) \in \{1, \ldots, N\} \times \{1, \ldots, p\}\) and \(G_{ij}^{(k)} = A = A = G_{ij}^{(k)}\) for all \(k \in \{1, \ldots, R\}\), we compare our iterative algorithm, Algorithm 1 (Alg. 1), with Algorithm 17 of Censor and Segal [1] (CS-Alg. 17) and Algorithm 1 of Tang and Liu [19] (TL-Alg. 1) to solve Problem 2 for randomly generated \(q \times p\) matrix \(G_{ij}^{(k)}\). We take the following step sizes:

**Alg. 1:** \(a_n = \frac{1}{n+1}, \quad \rho_n = 1, \quad \theta_n\) defined as in Remark 2 taking \(e_n = \frac{1}{\log(1+\nu)}\);

**CS-Alg. 17:** \(\gamma = \frac{1}{M}, \quad a_i = \frac{1}{1+i+n}, \quad \beta_j = \frac{1}{1+j+n}, \quad (i \in \{1, \ldots, N\}, j \in \{1, \ldots, M\})\), see [1];

**TL-Alg. 1:** \(\alpha_n = \frac{1}{n+1}, \quad \omega_n = \frac{1}{n+1}, \quad \eta_j = \frac{1}{1+j+n}, \quad (i \in \{1, \ldots, N\}, j \in \{1, \ldots, M\})\), see [19];

In Table 3 and 4, and Fig. 1 present the corresponding numerical results, where Fig. 1 demonstrates error \((\text{err}) = ||x_n - x||\) versus the number of iterations, and Table 3 shows the CPU time exclusion (cput) and a number of iterations (Iter(o)) of Alg. 1, CS-Alg. 17 and TL-Alg. 1 for the stopping criteria \(\text{err} = \frac{\|x_{n} - x\|}{\|x_{0} - x\|} < 0.001\).

From Fig. 1, and Table 4 and 5, we see that our proposed iterative method outperform Algorithm 17 of Censor and Segal [1] and Algorithm 1 of Tang and Liu [19]. Our method requires most likely fewer iterations and CPU time than that of the two compared algorithms as seen from Table 4 and 5. Furthermore, the numerical results further confirm the effectiveness of our proposed method.

### 7. Conclusions

In this paper, we introduced a novel algorithm involving an inertial term and a step size independent of the operator norm for approximating a solution of a generalized split common fixed point problem (5) for demimetric mappings in a real Hilbert space. Under mild assumptions, we proved the convergence of the proposed iterative methods. We applied our result to solve problems that can be studied as fixed points of demimetric mappings. The numerical experiments reported show the efficiency of the proposed iterative method.

### Declarations

**Author contribution statement**

Anteneh Getachew Gebrie, Dejene Shewakana Bedane: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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### References

[1] Y. Censor, A. Segal, *The split common fixed point problem for directed operators*, J. Convex Anal. 16 (2) (2009) 587–600.
[2] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Probl. 18 (2) (2002) 41.
[3] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms 8 (2) (1994) 221–239.
[4] C. Byrne, Y. Censor, A. Gibali, S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. 13 (4) (2012) 795–775.
[5] A. Cegielski, *General method for solving the split common fixed point problem*, J. Optim. Theory Appl. 165 (2) (2015) 385–404.
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[26] F. Wang, A new method for split common fixed-point problem without priori knowledge of matrix norms, Inverse Probl. 28 (8) (2012) 085004.

[27] H.-K. Xu, Self-adaptive algorithms for the split problem of the demicontractive operators, Optimization 66 (1) (2002) 240–256.

[28] H.-K. Xu, A self-adaptive iterative algorithm for the split common fixed point problem, Numer. Algorithms 66 (1) (2004) 240–256.

[29] P.-E. Maingé, Inertial iterative process for fixed points of certain quasi-nonexpansive mappings, Set-Valued Anal. 15 (1) (2007) 67–79.

[30] M.A. Glona, T.O. Alakoya, O.-I.O. Aib-semi, O.T. Mewomo, Inertial shrinking projection algorithm with self-adaptive step size for split generalized equilibrium and fixed point problems for a countable family of nonexpansive multivalued mappings, Demonstr. Math. 54 (1) (2021) 47–67.

[31] G. Ogwo, C. Iuchukwu, O. Mewomo, Inertial methods for finding minimum-norm solutions of the split variational inequality problem beyond monotonicity, Numer. Algorithms (2021) 1–28.

[32] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (2) (1967) 197–228.

[33] P.-E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (7–8) (2008) 899–912.

[34] H.-K. Xu, Iterative algorithms for nonlinear operators, J. Lond. Math. Soc. 66 (1) (2002) 240–256.

[35] G. Marino, H.-K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (1) (2007) 336–346.

[36] H.H. Bauschke, P.L. Combettes, et al., Convex Analysis and Monotone Operator Theory in Hilbert Spaces, vol. 408, Springer, 2011.

[37] J. Zhaoa, Q. Yang, Several acceleration schemes for solving the multiple-sets split feasibility problem, Linear Algebra Appl. 437 (7) (2012) 1648–1657.

[38] A. Gibali, S. Petra, Dc-programming versus l0-regularization for discrete tomography, An. Ştiinţ. Univ. ‘Ovidius’ Constanţa, Ser. Mat. (2017).

[39] A. Moudafi, A. Gibali, l0-l1 regularization of split feasibility problems, Numer. Algorithms 78 (3) (2018) 739–757.

[40] M. Brooke, Y. Censor, A. Gibali, Dynamic string-averaging cQ-methods for the split feasibility problem with percentage violation constraints arising in radiation therapy treatment planning, Int. Trans. Oper. Res. (2020).

[41] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, Phys. Med. Biol. 51 (10) (2006) 2233.

[42] C. Nutting, D. Dearnaley, S. Webb, Intensity modulated radiation therapy: a clinical review, Br. J. Radiol. 73 (869) (2000) 459–469.