Nonsingularity of the direct scattering transform for the KP II equation with real exponentially decaying at infinity potential.

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Abstract. We study the direct spectral transform for the heat equation, associated with the Kadomtsev–Petviashvili II equation. We show, that for real nonsingular exponentially decaying at infinity potentials the direct problem is nonsingular for arbitrary large potentials. Earlier this statement was proved only for potentials, satisfying the “small norm” assumption.

0 Introduction. The Kadomtsev–Petviashvili equation (KP)

\[(u_t - 6uu_x + u_{xxx})_x = -2\alpha^2 u_{yy}\]

was historically the first equation with 2 spatial variables integrated by the inverse scattering transform (IST) method (Druma, Zakharov-Shabat) [6], [12]. The auxiliary linear operator, associated with KP reads as

\[L = \alpha \partial_y - \partial_x^2 + u(x,y).\]  (1)

Usually \(\alpha^2\) is assumed to be real. Without loss of generality we can assume that \(\alpha^2 = \pm 1\) so it is sufficient to consider the following two cases:

1. \(\alpha = i\) and we have KP I or unstable KP.
2. \(\alpha = 1\) and we have KP II or stable KP.

These two cases are essentially different from the analytical point of view. The scattering transform for KP I with a decaying at infinity potential was constructed by Manakov [9] in terms of the nonlocal Riemann problem. A more regular method for constructing such transform was suggested in [1].

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The scattering transform for the KP II equation with a decaying at infinity potential was constructed by Ablowitz, Bar Yaakov and Fokas [2] in terms of the so-called \( \bar{\partial} \) problem. (Additional information and the technical details can be found in review paper [4]).

**Remark 1** In our paper we do not plan to discuss the periodic theory of KP, developed by Krichever (see [7], [8]). But we would like to point out that in the periodic theory the difference between KP I and KP II is also very essential. For example in [8] it was shown that the direct transform is well-defined for KP II and any periodic potential can be approximated by a finite-gap one, but for KPI these problems are still open.

**Remark 2** A new approach to the decaying at infinity theory, based on the resolvent technique was suggested in recent paper by Boiti, Pempinelli, Pogrebkov and Polivanov [3]. This approach can be extended to more general classes of potentials including for example, the one-dimensional solitons.

The main technical tool used in the spectral theory of operators with decaying at infinity potentials is the Fredholm theory of integral equations. Consider the following equation

\[
(\hat{1} + \hat{A}) x = b
\]

where \( \hat{A} \) is an integral (compact) operator, \( b \) is a given vector in the Hilbert space and \( x \) is an unknown vector. If the operator norm of \( \hat{A} \) is less then 1 equation (2) is uniquely solvable

\[
x = \left( \hat{1} - \hat{A}^2 - \hat{A}^3 + \ldots \right) b,
\]

but if the norm of \( \hat{A} \) is greater then 1 the situation is more complicated (see for example [11]).

Equation (2) has at least one solution if and only if the right-hand side \( b \) is orthogonal to the kernel of the adjoint equation

\[
< b, x_k^* > = 0 \quad \text{for all} \quad x_k^* \quad \text{such that} \quad (\hat{1} + \hat{A}^*) x_k^* = 0.
\]

This property is called the Fredholm alternative.
In the scattering theory the norm of the integral operator can be estimated via the norm of the potential in the direct problem or via the norm of the scattering data in the inverse problem. Hence if these norms are sufficiently small the unique solvability of the integral equations of the scattering theory can be easily proved and we essentially simplify the situation. This case was studied absolutely strictly (see [4]).

In general if the small norm assumption is not fulfilled, then the solutions of the integral equations have singularities for some special values of the spectral parameter and we have to introduce additional spectral data corresponding to these singular points. But fortunately there are some important examples such that the unique solvability of the integral equations can be proved without the “small norm” assumption.

One of such examples was found by S.P. Novikov and the author [5]. Consider the inverse problem for the two-dimensional Schrödinger operator at a fixed negative energy or the inverse problem for the operator (1) with $\alpha = 1$. The wave function is defined as a solution of the $\bar{\partial}$ equation. Assume that only the real Schrödinger operators without first order terms or only the heat operators with real potentials are considered. Then the scattering data satisfies some additional (rather simple) necessary and sufficient conditions. If these conditions are fulfilled, then the unique solvability of the $\bar{\partial}$ equation for arbitrary large data follows from the theory of the generalized analytic functions. Another important example is the solvability of the inverse problem for (1) with $\alpha = i$ and real $u(x, y)$ proved by Zhou [13].

In our paper we show that the direct scattering transform for the heat operator

$$L = \partial_y - \partial_x^2 + u(x, y).$$

with a real exponentially decaying at infinity potential is nonsingular without the “small norm” assumption. (This spectral problem corresponds to the KP II equation). It is very likely that the exponential decay rate is not too essential by it is not clear how to weaken this condition in our proof.

The plan of our paper is the following. In the first section we recall the scheme of the scattering transform for the KP II equation. In the second section we assume that equations of the direct scattering have nonzero homogeneous solutions and study the properties of these solutions. We show that these solutions have to satisfy some orthogonality conditions. In the third section we show that from these orthogonality conditions it follows
that these homogeneous solutions generate no singularities in the scattering transform.

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1 The scattering transform for the KP II equation. In our paper we study the heat operator

\[ L = \partial_y - \partial_x^2 + u(x, y). \]  \hspace{1cm} (4)

with a real smooth potential \( u(x, y) \). Denote by \( r \) the distance from the origin \( r = \sqrt{x^2 + y^2} \). We shall put on \( u(x, y) \) one of the following assumptions at infinity:

D1) \( u(x, y) \) decays at infinity faster than \( r^{-2} \), i.e., there exist a constant \( \varepsilon > 0 \) and a collection of constants \( c_{mn} \) such that

\[ \left| \frac{\partial^{m+n} u(x, y)}{\partial x^m \partial y^n} \right| < \frac{c_{mn}}{(1 + r)^{2+\varepsilon}} \text{ for all } m, n \geq 0. \]

D2) \( u(x, y) \) belongs to the Schwartz class, i.e., there exist a constant \( \varepsilon > 0 \) and a collection of constants \( c_{mn}^k \) such that

\[ \left| \frac{\partial^{m+n} u(x, y)}{\partial x^m \partial y^n} \right| < \frac{c_{mn}^k}{(1 + r)^{2+k}} \text{ for all } m, n, k \geq 0. \]

D3) \( u(x, y) \) exponentially decays at infinity, i.e., there exist a constant \( \varepsilon > 0 \) and a collection of constants \( c_{mn} \) such that

\[ \left| \frac{\partial^{m+n} u(x, y)}{\partial x^m \partial y^n} \right| < c_{mn} e^{-\varepsilon r} \text{ for all } m, n \geq 0. \]

The direct scattering transform suggested in [2] (for more details see [4]) is the following:

Let \( \Psi(\lambda, x, y) \) be a solution of the heat equation

\[ L \Psi(\lambda, x, y) = 0 \]  \hspace{1cm} (5)
and
\[ \Psi(\lambda, x, y) = e^{\lambda x + \lambda^2 y} \chi(\lambda, x, y), \chi(\lambda, x, y) = 1 + o(1) \text{ as } x^2 + y^2 \to \infty, \]  
(6)
where \( \lambda \) is a complex parameter. In our text we use the following agreement: the notation \( f(\lambda), \lambda \in \mathbb{C} \) does not mean that \( f \) is holomorphic in \( \lambda \), i.e., we do not assume that \( \overline{\partial} \chi(\lambda) = 0 \).

It is convenient to introduce new variables:
\[ \lambda_R = \text{Re} \lambda, \lambda_I = \text{Im} \lambda, z = x + 2\lambda y, \bar{z} = x + 2\bar{\lambda} y, \]
\[ \partial_z = \frac{1}{4i\lambda_I} (\partial_y - 2\bar{\lambda} \partial_x), \partial_{\bar{z}} = \frac{-1}{4i\lambda_I} (\partial_y - 2\lambda \partial_x). \]  
(7)

Let
\[ L_0(\lambda) = \partial_y - \partial_z^2 - 2\lambda \partial_x = -4i\lambda_I \partial_z - (\partial_z + \partial_{\bar{z}})^2. \]
The function \( \chi(\lambda, x, y) \) satisfies the following equation:
\[ L_0(\lambda) \chi(\lambda, x, y) + u(\lambda, x, y) \chi(\lambda, x, y) = 0. \]  
(8)

Let \( G(\lambda, x, y) \) be the Green function of the operator \( L_0(\lambda) \)
\[ L_0(\lambda) G(\lambda, x, y) = \delta(x) \delta(y). \]
The function \( G(\lambda, x, y) \) reads as
\[ G(\lambda, x, y) = \frac{1}{4\pi^2} \int \int \frac{e^{i(px+qy)}}{p^2 + iq - 2i\lambda p} dpdq. \]  
(9)

Equation (8) with the boundary condition (6) is equivalent to the following integral equation
\[ \chi(\lambda, x, y) = 1 - \hat{G}(\lambda) u(\lambda, x, y) \chi(\lambda, x, y) \]  
(10)
where \( \hat{G}(\lambda) \) denotes the integral operator
\[ \hat{G}(\lambda) f(\lambda, x, y) = \int \int G(\lambda, x - x', y - y') f(\lambda, x', y') dx' dy'. \]
The function \( G(\lambda, x, y) \) is not holomorphic in \( \lambda \) but
\[ \frac{G(\lambda, x, y)}{\partial \lambda} = -\frac{i \text{ sgn } \lambda_I}{2\pi} e^{-2i\lambda_I x - 4i\lambda_I \lambda_R y}. \]
Differentiating (10) w.r.t. $\bar{\lambda}$ we get

$$\frac{\chi(\lambda, x, y)}{\partial \lambda} = -\hat{G}(\lambda)u(x, y)\frac{\chi(\lambda, x, y)}{\partial \lambda} + T(\lambda)e^{-2i\lambda_xx-4i\lambda_y\lambda Ry},$$

(11)

where

$$T(\lambda) = \frac{i\text{sgn} \lambda}{2\pi}b(\lambda), \quad b(\lambda) = \int \int u(x, y)\chi(\lambda, x, y)e^{2i\lambda_xx+4i\lambda_y\lambda Ry}dxdy.$$  

(12)

From (11) it follows that

$$\frac{\chi(\lambda, x, y)}{\partial \lambda} = T(\lambda)\chi_1(\lambda, x, y),$$

where $\chi_1(\lambda, x, y)$ is defined by the following integral equation

$$\chi_1(\lambda, x, y) = -\hat{G}(\lambda)u(x, y)\chi_1(\lambda, x, y) + e^{-2i\lambda_xx-4i\lambda_y\lambda Ry}.$$

From the symmetry properties of the Green function $G(\lambda, x, y)$ it follows that

$$\chi_1(\lambda, x, y) = e^{-2i\lambda_xx-4i\lambda_y\lambda Ry}\chi(\bar{\lambda}, x, y).$$

Finally we get

$$\frac{\chi(\lambda, x, y)}{\partial \lambda} = T(\lambda)e^{-2i\lambda_xx-4i\lambda_y\lambda Ry}\chi(\bar{\lambda}, x, y)$$

or, equivalently

$$\frac{\Psi(\lambda, x, y)}{\partial \lambda} = T(\lambda)\Psi(\bar{\lambda}, x, y)$$

The function $b(\lambda)$ is the scattering data for the KP II equation. The potential $u(x, y)$ is real if and only if

$$b(\bar{\lambda}) = \bar{b}(\lambda).$$

(13)

All calculations from this section are correct if the integral equation (11) has an unique nonsingular solution for any $\lambda \in \mathbb{C}$. The solvability of (11) is discussed below.

We need also the adjoint equation to (5)

$$[-\partial_y - \partial_x^2 + u(x, y)]\Psi^*(\lambda, x, y) = 0$$
where
\[ \Psi^*(\lambda, x, y) = e^{-\lambda x - \lambda^2 y} \chi^*(\lambda, x, y), \chi^*(\lambda, x, y) = 1 + o(1) \text{ as } x^2 + y^2 \to \infty. \]

Let
\[ L_0^*(\lambda) = -\partial_y - \partial_x^2 + 2\lambda \partial_x = 4i\lambda_I \partial_\tau - (\partial_z + \partial_{\bar{z}})^2. \]

The function \( \chi^*(\lambda, x, y) \) satisfies the following equation:
\[ L_0^*(\lambda)\chi^*(\lambda, x, y) + u(x, y)\chi^*(\lambda, x, y) = 0. \]

The adjoint Green function reads as
\[ G^*(\lambda, x, y) = \frac{1}{4\pi^2} \int \int \frac{e^{i(px+qy)}dpdq}{p^2 - iq + 2i\lambda p}. \]

The function \( \chi^*(\lambda, x, y) \) satisfies the following integral equation
\[ \chi^*(\lambda, x, y) = 1 - \hat{G}^*(\lambda)u(x, y)\chi^*(\lambda, x, y) \quad (14) \]

2 Homogeneous solutions of the direct scattering integral equation. If the homogeneous part of (10)
\[ \chi(\lambda, x, y) = -\hat{G}(\lambda)u(x, y)\chi(\lambda, x, y) \quad (15) \]
has no nonzero solutions for any \( \lambda \) then the direct scattering transform is well defined (see for example [4]). Assume now that equation (14) has nonzero solutions for some \( \lambda = \lambda_0 \), \( u(x, y) \) is a real smooth function, decaying at infinity faster than \( r^{-2} \) (D1 decay condition from the section [1]). In this section we study the properties of such solutions.

If \( \lambda_I = 0 \) then (10) is a Volterra type equation, i.e.,
\[ G(\lambda, x, y) = 0 \text{ for all } y < 0 \]
and (14) has no nonzero solutions. Hence without loss of generality we may assume that \( \lambda_I \neq 0 \).

It is convenient to introduce a new function
\[ \tilde{\chi}(\lambda, x, y) = e^{i\lambda_I x + 2i\lambda_I \lambda R y} \chi(\lambda, x, y). \]
The corresponding Green function reads as

\[ \tilde{G}(\lambda, x, y) = \frac{1}{4\pi^2} \int \int \frac{e^{i(px+qy)}dpdq}{p^2 - \lambda^2 + i(q - 2\lambda_Rp)}. \]

The function \( \tilde{\chi}(\lambda, x, y) \) satisfy the following integral equation

\[ \tilde{\chi}(\lambda, x, y) = e^{i\lambda_xx+2\lambda_yy} - \hat{\tilde{G}}(\lambda)u(x, y)\tilde{\chi}(\lambda, x, y). \]

The homogeneous equation reads as

\[ \tilde{\chi}(\lambda, x, y) = -\hat{\tilde{G}}(\lambda)u(x, y)\tilde{\chi}(\lambda, x, y) \quad (16) \]

The homogeneous equation (16) is real and we have the obvious one-to-one correspondence between the solutions of (15) and (16). From the Fredholm theory it follows that the spaces of these solutions are finite-dimensional. Let \( \tilde{\chi}_1(\lambda, x, y), \ldots, \tilde{\chi}_k(\lambda, x, y) \) be a real basis of solutions of (16), \( \chi_1(\lambda, x, y), \ldots, \chi_k(\lambda, x, y) \) be the corresponding solutions of (15).

**Lemma 1** The Green function \( \tilde{G}(\lambda, x, y) \) has the following asymptotic expansion as \( x^2 + y^2 \to \infty \):

\[ \tilde{G}(\lambda, x, y) = e^{i\lambda_xx+2\lambda_yy} \left\{ \frac{1}{4\lambda_I} \left( \partial_z + 2\partial_z + \partial_z^{-1}\partial_z^2 \right) \right\}^k \frac{i\text{sgn} \lambda_I}{2\pi z} + \text{Complex conjugate terms (17)} \]

All the monomials in this expansion have the form \( c_{mn} \bar{z}^m/z^n \) \( n > 2m \geq 0 \). Hence the operator \( \partial_z^{-1}\partial_z^2 \bar{z}^m/z^n \) is well defined

\[ \partial_z^{-1}\partial_z^2 \bar{z}^m/z^n = \frac{n(n+1)\bar{z}^{m+1}}{m+1\bar{z}^{n+2}}. \]

The proof of this Lemma is rather standard.

Consider the following collection of eigenfunctions of \( L_0(\lambda) \):

\[ \phi_k(\lambda, x, y) = e^{-\mu_x-\mu_y} \partial_k^\mu e^{\mu_x+\mu_y} \bigg|_{\mu=\lambda}, \quad \tilde{\phi}_k(\lambda, x, y) = e^{-\mu_x-\mu_y} \partial_k^\mu e^{\mu_x+\mu_y} \bigg|_{\mu=\lambda}, \]

\[ L_0(\lambda)\phi_k(\lambda, x, y) = L_0(\lambda)\tilde{\phi}_k(\lambda, x, y) = 0, \]
and the adjoint collection

\begin{align*}
\phi_k^*(\lambda, x, y) &= \phi_k(\lambda, -x, -y), \\
\hat{\phi}_k^*(\lambda, x, y) &= \hat{\phi}_k(\lambda, -x, -y), \\
L_0^*(\lambda)\phi_k^*(\lambda, x, y) &= L_0^*(\lambda)\hat{\phi}_k^*(\lambda, x, y) = 0, 
\end{align*}

(18)

Lemma 2

I) Let \( \tilde{\chi}_0(\lambda, x, y) \) be a real nonzero solution of the homogeneous integral equation \((16)\), \( \chi_0(\lambda, x, y) = e^{-i\lambda_l x - 2i\lambda_l \lambda_R y} \tilde{\chi}_0(\lambda, x, y) \) be the corresponding solution of \((17)\). Then there are two possibilities:

1. Either \( \tilde{\chi}_0(\lambda, x, y), \chi_0(\lambda, x, y) \) are Schwartz functions in \((x, y)\), i.e., they decay at infinity faster than any degree of \( z \).

2. Or

\( \tilde{\chi}_0(\lambda, x, y) = e^{i\lambda_l x + 2i\lambda_l \lambda_R y} \bar{c} \bar{z}^m + e^{-i\lambda_l x - 2i\lambda_l \lambda_R y} \bar{c} \bar{z}^n + o\left(\frac{1}{z^n}\right) \),

\( \chi_0(\lambda, x, y) = \frac{c}{z^n} + e^{-2i\lambda_l x - 4i\lambda_l \lambda_R y} \tilde{c} \bar{z}^m + o\left(\frac{1}{z^n}\right) \).

II) If \( \chi_0(\lambda, x, y) = O\left(\frac{1}{z^{n+1}}\right) \) then

\[
\int \int \phi_k^*(\lambda, x, y)u(x, y)\chi_0(\lambda, x, y)dxdy = \int \int \hat{\phi}_k^*(\lambda, x, y)u(x, y)\chi_0(\lambda, x, y)dxdy = 0 \quad \text{for } k = 0, \ldots, n. 
\]

(19)

Proof of Lemma 2. To get the asymptotic expansion for \( \tilde{\chi}_0(\lambda, x, y) \) as \( r \to \infty \) we substitute \((17)\) in the right hand side of \((16)\). Then we transform all monomials in the expansion of \( G(\lambda, x - x', y - y') \) to the following form

\[
\frac{(\bar{z} - \bar{z}')^m}{(z - z')^n} = \frac{z^m}{\bar{z}^n} \cdot \left(1 - \frac{\bar{z}'}{z}\right)^m \cdot \left(1 + \frac{z'}{\bar{z}} + \left(\frac{z'}{\bar{z}}\right)^2 + \left(\frac{z'}{\bar{z}}\right)^3 + \ldots\right)^n. 
\]

Integrating the right-hand side of \((16)\) in \( x', y' \) we get some formal expansion of \( \tilde{\chi}_0(\lambda, x, y) \) and the coefficients of this expansion read as

\[
\int \int P_{nm}(z, \bar{z})u(x, y)\tilde{\chi}_0(\lambda, x, y)dxdy, 
\]
where \( P_{mn} \) are some polynomials in \( z, \bar{z} \). Of course some of these integrals may diverge.

First consider the terms of this expansion decaying as \( 1/z \), then decaying as \( 1/z^2 \), then as \( 1/z^3 \) and so on. In each order of decay rate we have only finite number of monomials. It is rather easy to check that the integrals representing the lowest order nonzero coefficients in this expansion converge and these terms give us the right asymptotics of \( \tilde{\chi}_0(\lambda, x, y) \). If all coefficients are equal to zero all integrals converges and the function \( \tilde{\chi}_0(\lambda, x, y) \) decays faster than any degree of \( r \).

Assume now that at least one of the terms in the asymptotic expansion is nonzero. By definition (we use now that \( u(x, y) = o(1/z^2) \) as \( z \to \infty \)) we have

\[
L_0(\lambda) \chi_0(\lambda, x, y) = -4i\lambda \partial_z - (\partial_z + \partial_{\bar{z}})^2 \chi_0(\lambda, x, y) = o\left(\frac{1}{z^2}\right) \chi_0(\lambda, x, y).
\]

Thus the first nonvanishing coefficient in the asymptotics for \( \chi_0(\lambda, x, y) \) contains no \( \bar{z} \). This completes the proof of the first part of Lemma 2.

Let \( \kappa(\lambda, x, y) \) be one of the functions \( \phi_k^*(\lambda, x, y) \), \( \bar{\phi}_k^*(\lambda, x, y) \) \( 0 \leq k \leq n \).
By definition

\[
I = \int \int \kappa(\lambda, x, y) u(x, y) \chi_0(\lambda, x, y) dxdy =
\]

\[
= - \int \int \kappa(\lambda, x, y) L_0(\lambda) \chi_0(\lambda, x, y) dxdy.
\]

Integrating by parts we get

\[
I = \int \int \chi_0(\lambda, x, y) L_0^*(\lambda) \kappa(\lambda, x, y) dxdy + \text{Boundary terms}.
\]

If \( \chi_0(\lambda, x, y) = O\left(\frac{1}{z^{n+2}}\right) \) the boundary terms vanishes. Applying (18) we get the orthogonality property \( I = 0 \). Lemma 2 is proved.

**Lemma 3** Let \( \Psi(x, y) \) be a global real solution of the heat equation

\[
(\partial_y - \partial_x^2 + u(x, y)) \Psi(x, y) = 0 \tag{20}
\]

\( (u(x, y) \) is assumed to be a real smooth function defined in the whole plane).

Denote by \( D \) the open set of points \( (x, y) \) such that \( \Psi(x, y) \neq 0 \). Denote by \( D_j \) the arcwise connected components of \( D \).
Assume that the component $D_j$ is locally bounded in the $x$-direction, i.e., for any $y_1$, $y_2$ the intersection of the set $D_j$ with the strip $y_1 < y < y_2$ is bounded.

Then the set $D_j$ is unbounded from below in the $y$-direction, i.e., there exists a path $[0, \infty) \to D_j$, $t \to (x(t), y(t))$ such that

$$\frac{\partial y(t)}{\partial t} < 0, \ y(t) \to -\infty \text{ as } t \to +\infty.$$  

**Remark 3** In Lemma 3 the functions $\Psi(x, y)$ and $u(x, y)$ are assumed to be real, smooth and defined for all $(x, y)$, but they may grow arbitrary fast as $r \to \infty$.

**Proof of Lemma 3.** Consider the following function

$$J(y) = \int_{\mathcal{L}(y)} \frac{1}{2} \Psi^2(x, y) dx$$

where $\mathcal{L}(y)$ denotes the intersection of $D_j$ with the line $y = \text{const}$. The function $\Psi^2(x, y)$ vanishes on the boundary of $D_j$. Hence

$$\frac{\partial J(y)}{\partial y} = \int_{\mathcal{L}(y)} \Psi(x, y) \frac{\partial}{\partial y} \Psi(x, y) dx$$

(the terms arousing from differentiation by the position of the boundaries of $\mathcal{L}(y)$ are equal to 0). From (20) it follows that

$$\frac{\partial J(y)}{\partial y} = - \int_{\mathcal{L}(y)} u(x, y) \Psi^2(x, y) dx + \int_{\mathcal{L}(y)} \Psi(x, y) \frac{\partial^2}{\partial x} \Psi(x, y) dx.$$

The function $\Psi(x, y)$ vanishes on the boundary of $D_j$. Hence integrating by parts we get

$$\frac{\partial J(y)}{\partial y} = - \int_{\mathcal{L}(y)} u(x, y) \Psi^2(x, y) dx - \int_{\mathcal{L}(y)} (\partial_x \Psi(x, y))^2 dx < 2J(y) \max_{x \in \mathcal{L}(y)} |u(x, y)|.$$

For any $y < y_0$ such that $J(y_0) > 0$ we have the following estimate

$$J(y) > e^{c_{y, y_0}(y-y_0)}J(y_0) > 0, \text{ where } c_{y, y_0} = 2 \max_{x \in \mathcal{L}(\tilde{y}), \tilde{y} \leq y \leq y_0} |u(x, \tilde{y})|.$$
Thus if $\mathcal{L}(y_0)$ is not empty then $\mathcal{L}(y)$ is not empty for any $y < y_0$. This completes the proof.

Now we are ready to prove:

**Theorem 1** Let $\chi_0(\lambda, x, y)$ be a real nonzero solution of the homogeneous integral equation (15) where $u(x, y)$ is a real smooth function decaying at infinity faster then $r^{-2}$ (D1 decay condition). Then

1. $\chi_0(\lambda, x, y)$ decays at infinity faster than any degree of $r$, i.e., belongs to the Schwartz class.

2. The function $u(x, y)\chi_0(\lambda, x, y)$ is orthogonal to the functions $\phi_k^*(\lambda, x, y)$, $\tilde{\phi}_k^*(\lambda, x, y)$ for all $k \geq 0$

$$
\int \int \phi_k^*(\lambda, x, y)u(x, y)\chi_0(\lambda, x, y)dxdy = 0
$$

$$
= \int \int \tilde{\phi}_k^*(\lambda, x, y)u(x, y)\chi_0(\lambda, x, y)dxdy = 0. \tag{21}
$$

**Proof of Theorem 1.** Without loss of generality we can assume that the functions

$$
\tilde{\chi}_0(\lambda, x, y) = e^{i\lambda_1 x + 2i\lambda_1 \lambda_2 y} \chi_0(\lambda, x, y), \quad \Psi_0(\lambda, x, y) = e^{\lambda_1 x + 2\lambda_1 \lambda_2 y} \chi_0(\lambda, x, y)
$$

are real. The zeroes of $\tilde{\chi}_0(\lambda, x, y)$ coincide with the zeroes of $\Psi_0(\lambda, x, y)$, $\Psi_0(\lambda, x, y)$ is a global solution of (20). Thus the zeroes of $\tilde{\chi}_0(\lambda, x, y)$ have the properties formulated in Lemma 3.

By Lemma 4 if the function $\tilde{\chi}_0(\lambda, x, y)$ is not from the Schwartz class then the leading term of $\tilde{\chi}_0(\lambda, x, y)$ reads as

$$
\tilde{\chi}_0(\lambda, x, y) \sim e^{i\lambda_1 x + 2i\lambda_1 \lambda_2 y} \frac{c}{z^n} + e^{-i\lambda_1 x - 2i\lambda_1 \lambda_2 y} \frac{c}{z^n}. \tag{22}
$$

It is easy to check that (22) approximate $\tilde{\chi}_0(\lambda, x, y)$ with the first derivative. The zeroes of the leading term are non-degenerate hence they approximate the zeroes of $\tilde{\chi}_0(\lambda, x, y)$ for sufficiently large $z$. Let $C$ be a sufficiently big circle.
Let us draw the zeroes of $\tilde{\chi}_0(\lambda, x, y)$ outside $C$. Consider the set of domains $D_j$ intersecting the circle $C$ (the domains $D_j$ were defined in Lemma 3). Let $N$ be the number of such domains coming from the direction $y = -\infty$. Elementary analysis of (22) shows that the number of domains coming from $y = +\infty$ and intersecting $C$ is equal to $N + 2n$. Hence at least one of the domains coming from $y = +\infty$ vanishes as $y \to -\infty$. But by Lemma 3 it is impossible. Thus assuming that $\chi_0(\lambda, x, y)$ has nonvanishing terms in the asymptotic expansion we get a contradiction. Comparing this result with Lemma 2 we complete the proof.

3 Regularity of the direct spectral transform. Assume now that $u(x, y)$ is a real smooth potential decaying at infinity faster then any degree of $r$ (D2 decay condition).

Let us call a point $\lambda_0$ regular if for $\lambda = \lambda_0$ the homogeneous integral equation (15) has no nonzero solutions. Otherwise the point $\lambda_0$ will be called irregular. The aim of this section is to study the properties of the wave function $\Psi(\lambda, x, y)$ at irregular points.

In section 2 we defined the functions $\phi_k(\lambda, x, y)$, $\tilde{\phi}_k(\lambda, x, y)$ for $k \geq 0$. Let us define the following formal series

$$\phi_k(\lambda, x, y) = \left\{1 + \sum_{k=1}^{\infty} \left[ \frac{i}{4\lambda_I} \left( \partial_z + 2\partial_x + \partial_{\bar{v}}^{-1}\partial_{\bar{v}}^2 \right) \right]^k \right\} z^k,$$

$$\tilde{\phi}_k(\lambda, x, y) = e^{-2i\lambda_I x - 4i\lambda_I \lambda_R y} \tilde{\phi}_k(\lambda, x, y),$$

$$\phi^*_k(\lambda, x, y) = \phi_k(\lambda, -x, -y), \quad \tilde{\phi}^*_k(\lambda, x, y) = \tilde{\phi}_k(\lambda, -x, -y). \quad (23)$$
For $k \geq 0$ these series contain only finite number of nonzero terms and coincide with (18).

**Theorem 2**  Let $u(x, y)$ be a real smooth potential, decaying at infinity faster than any degree of $r$, $a_0, \ldots, a_n, b_0, \ldots, b_n$ be arbitrary constants. Then for any $\lambda$ the following integral equation

$$
\chi(\lambda, x, y) = \sum_{k=0}^{n} \left( a_k \phi_k(\lambda, x, y) + b_k \tilde{\phi}_k(\lambda, x, y) \right) - \hat{G}(\lambda) u(x, y) \chi(\lambda, x, y) \tag{24}
$$

has at least one nonsingular solution with the following asymptotics as $z \to \infty$:

$$
\chi(\lambda, x, y) = \sum_{k=\infty}^{n} \left( a_k \phi_k(\lambda, x, y) + b_k \tilde{\phi}_k(\lambda, x, y) \right).
$$

The coefficients $a_k, b_k$ with $k < 0$ are uniquely defined by the coefficients $a_k$, $b_k$ with $k \geq 0$.

**Proof of Theorem 2.** If $\lambda$ is a regular point then this statement is absolutely standard (see for example [4]). If $\lambda$ is an irregular point then (24) has at least one nonsingular solution if and only if the non-homogeneous part is orthogonal to the kernel of the adjoint equation (see for example [11]). By definition the adjoint equation to the integral equation (15) reads as

$$
f(\lambda, x, y) = -u(x, y) \hat{G}(\lambda) f(\lambda, x, y) \tag{25}
$$

(we use here $(AB)^* = B^*A^*$). The basis of solutions of (25) is formed by the functions $u(x, y)\chi_1^*(\lambda, x, y), \ldots, u(x, y)\chi_k^*(\lambda, x, y)$ where $\chi_1^*(\lambda, x, y), \ldots, \chi_k^*(\lambda, x, y)$ is some basis of solutions of the homogeneous part of (14)

$$
\chi^*(\lambda, x, y) = -\hat{G}^*(\lambda) u(x, y) \chi^*(\lambda, x, y).
$$

Thus we have the following orthogonality conditions:

$$
\int \int \phi_k(\lambda, x, y) u(x, y) \chi_k^*(\lambda, x, y) dxdy =
$$

$$
= \int \int \tilde{\phi}_k(\lambda, x, y) u(x, y) \chi_k^*(\lambda, x, y) dxdy = 0. \tag{26}
$$

Conditions (26) are adjoint to the orthogonality conditions (19) from Theorem [1] and can be proved absolutely in the same way.
The function $u(x, y)$ vanishes at infinity faster than any degree of $r$. Then asymptotically for any $N > 0$ we have

$$L_0(\lambda)\chi(\lambda, x, y) = o(z^{-N}).$$

(27)

Using the same approach as Lemma 2 we can construct asymptotic expansion for $\chi(\lambda, x, y)$. Substituting this expansion in (27) we obtain that the asymptotics is formed by linear combinations of $\phi_k(\lambda, x, y), \tilde{\phi}_k(\lambda, x, y)$.

By Theorem 1 if $\lambda$ is an irregular point then all homogeneous solutions decay at infinity faster than any degree of $z$; hence they do not affect the asymptotic expansion. Theorem 2 is proved.

**Remark 4** Theorem 2 says that for any $\lambda$ we have at least one solution of (24). But we have not proved that these solutions are continuous in $\lambda$. This question needs a more detailed analysis.

Let $\lambda_0$ be an irregular point. By Theorem 2 the direct scattering equation (5) has at least one solution with the asymptotics (6) at the point $\lambda_0$. Our next step is to construct a formal solution of (5) in the neighborhood of $\lambda_0$. Consider the following formal series in $(\lambda - \lambda_0), (\bar{\lambda} - \bar{\lambda}_0)$

$$\Psi(\lambda, x, y) = e^{\lambda_0 x + \lambda_0^2 y} \sum_{m,n \geq 0} \frac{(\lambda - \lambda_0)^m (\bar{\lambda} - \bar{\lambda}_0)^n}{m! n!} \psi_{mn}(x, y),$$

(28)

where the functions $\psi_{mn}(x, y)$ are defined by the following properties:

1. All $\psi_{mn}(x, y)$ satisfy (8) with $\lambda = \lambda_0$.
2. $\psi_{mn}(x, y)$ are smooth in $x, y$ and

$$\psi_{mn}(x, y) = \sum_{k=-\infty}^{\max(m,n)} c_{mn}^{k} \phi_k(\lambda_0, x, y) + \sum_{k=-\infty}^{\max(m,n)} d_{mn}^{k} \tilde{\phi}_k(\lambda_0, x, y)$$

(29)

as $z \to \infty$.

3. In all orders of perturbation theory in $(\lambda - \lambda_0), (\bar{\lambda} - \bar{\lambda}_0)$ the asymptotic condition (6) is fulfilled.
The last property needs some explanations. Substituting (29) to (28) we get
\[
\Psi(\lambda, x, y) = e^{\lambda_0 x + \lambda_0^2 y} \Phi_1(\lambda, x, y) + e^{\bar{\lambda}_0 x + \bar{\lambda}_0^2 y} \Phi_2(\lambda, x, y)
\]
where
\[
\Phi_1(\lambda, x, y) = \sum_{m,n \geq 0} \sum_{k = -\infty}^{\max(m,n)} c_k^{mn} \frac{(\lambda - \lambda_0)^m (\bar{\lambda} - \bar{\lambda}_0)^n}{m!n!} \phi_k(\lambda_0, x, y)
\]
\[
\Phi_2(\lambda, x, y) = \sum_{m,n \geq 0} \sum_{k = -\infty}^{\max(m,n)} d_k^{mn} \frac{(\lambda - \lambda_0)^m (\bar{\lambda} - \bar{\lambda}_0)^n}{m!n!} \bar{\phi}_k(\lambda_0, x, y)
\]
Let
\[
\Xi(\lambda, x, y) = e^{(\lambda_0 - \lambda)x + (\lambda_0^2 - \lambda^2)y} \Phi_1(\lambda, x, y),
\]
\[
\hat{\Xi}(\lambda, x, y) = e^{(\bar{\lambda}_0 - \bar{\lambda})x + (\bar{\lambda}_0^2 - \bar{\lambda}^2)y} \Phi_2(\lambda, x, y)
\]
Expanding the exponents in \((\lambda - \lambda_0), (\bar{\lambda} - \bar{\lambda}_0)\) we get some formal series
\[
\Xi(\lambda, x, y) = \sum_{m,n \geq 0} \frac{(\lambda - \lambda_0)^m (\bar{\lambda} - \bar{\lambda}_0)^n}{m!n!} \xi_{mn}(x, y)
\]
\[
\hat{\Xi}(\lambda, x, y) = \sum_{m,n \geq 0} \frac{(\lambda - \lambda_0)^m (\bar{\lambda} - \bar{\lambda}_0)^n}{m!n!} \hat{\xi}_{mn}(x, y)
\]
The functions \(\xi_{mn}(x, y), \hat{\xi}_{mn}(x, y)\) are some asymptotic Laurent series in \(z, \bar{z}\). The property 3) means the following:
\[
\xi_{mn}(x, y) = \delta_{m0}\delta_{n0} + o(1), \hat{\xi}_{mn}(x, y) = o(1) \text{ as } z \to \infty.
\] (30)

**Theorem 3** Let \(u(x, y)\) be a real smooth potential, decaying at infinity faster then any degree of \(r\), \(\lambda = \lambda_0\) be an irregular point. Then

1. The equation of direct scattering (7) has at least one formal solution (28) with the properties 1)–3).

2. The functions \(\psi_{mn}(x, y)\) are defined uniquely up to adding arbitrary solutions of the homogeneous equation (15).
3. The constants $c_{mn}^k$, $d_{mn}^k$ are uniquely determined by the properties 1)–3).

4. The “scattering data” $b(\lambda)$ is uniquely defined as a formal Taylor series in $(\lambda - \lambda_0)$, $(\bar{\lambda} - \bar{\lambda}_0)$.

**Proof of Theorem 3.** Suppose $c_{mn}^k$, $d_{mn}^k$ are arbitrary constants such that $c_{mn}^k = d_{mn}^k = 0$ for $k > \max(m,n)$. Then the formal series $\Xi(\lambda, x, y)$, $\tilde{\Xi}(\lambda, x, y)$ are well-defined in all orders of $(\lambda - \lambda_0)$, $(\bar{\lambda} - \bar{\lambda}_0)$ and satisfy

$$[L_0(\lambda_0) - 2(\lambda - \lambda_0)\partial_x]\Xi(\lambda, x, y) = 0, \quad [L_0(\bar{\lambda}_0) - 2(\bar{\lambda} - \bar{\lambda}_0)\partial_x]\tilde{\Xi}(\lambda, x, y) = 0.$$  \hspace{1em} (31)

By Theorem 4 for any $m, n$ and arbitrary $c_{mn}^k$, $d_{mn}^k$ we have at least one solution of (24). This solutions is defined up to adding arbitrary solutions of the homogeneous equation (15), the constants $c_{mn}^k$, $d_{mn}^k$ with $k < 0$ are uniquely defined. Hence to prove the theorem it is sufficient to show the existence of an unique collection of $c_{mn}^k$, $d_{mn}^k$, $k \geq 0$ such that the property 3) is fulfilled.

We calculate these constants by induction. Putting $\lambda = \lambda_0$ we get $c_{00}^{00} = 1$, $d_{00}^{00} = 0$. Assume that for some $l$ we know all coefficients $c_{mn}^k$, $d_{mn}^k$ with $m + n < l$. Consider the function $\xi_{mn}^{lm}(x, y)$, $m + n = l$. By (31)

$$L_0(\lambda_0)\xi_{mn}^{lm}(x, y) - 2m\partial_x\xi_{m-1}^{m-n}(x, y) = 0.$$  \hspace{1em} (32)

By (30)

$$\partial_x\xi_{m-N}^{m-N}(x, y) = o(1) \text{ as } z \to \infty.$$  \hspace{1em} (33)

Hence

$$L_0(\lambda_0)\xi_{mn}^{lm}(x, y) = o(1), \quad \xi_{mn}^{lm}(x, y) = \sum_{k=0}^{\max(m,n)} v_{mn}^k \phi_k(\lambda_0, x, y) + o(1).$$  \hspace{1em} (34)

From the definition of $\Xi(\lambda, x, y)$ it follows that

$$\xi_{mn}^{lm}(x, y) = F(\lambda, x, y, c_{mn}^k, d_{mn}^k) + \sum_{k=0}^{\max(m,n)} c_{mn}^k \phi_k(\lambda_0, x, y) + o(1),$$

where $F(\lambda, x, y, c_{mn}^k, d_{mn}^k)$ is a linear function of $c_{mn}^k, d_{mn}^k$ with $m + n < l$. 


Thus there exists an unique collection of $c_{mn}^k$, $0 \leq k \leq \max(m, n)$ such that $\xi_{mn}^{x,y} = o(1)$. Similar analysis of $\hat{\Xi}(\lambda, x, y)$ gives us the coefficients $d_{mn}^k$, $m + n = 1$.

To define $b(\lambda)$ consider the asymptotic expansion of $\Psi(\lambda, x, y)$ for large $x, y$

$$\Psi(\lambda, x, y) = e^{\lambda x + \lambda^2 y} \left[ 1 + \frac{c_{-1}}{z} + \ldots \right] + e^{\lambda x + \lambda^2 y} \left[ \frac{d_{-1}}{z} + \ldots \right].$$

The function $b(\lambda)$ is defined by (12). Integrating (12) by parts we get

$$T(\lambda) = d_{-1}(\lambda), b(\lambda) = 2\pi i \text{sgn} \lambda d_{-1}(\lambda).$$

At the previous step of the proof we have defined the function $d_{-1}(\lambda)$ as a formal series in $(\lambda - \lambda_0), (\bar{\lambda} - \bar{\lambda}_0)$. Thus $b(\lambda)$ is uniquely defined as a formal series. This completes the proof.

Unfortunately the results of Theorem 3 are not sufficient to prove that the scattering data $b(\lambda)$ is a nonsingular function because the series of perturbation theory may diverge. Assume now that the potential $u(x, y)$ decays exponentially at infinity (D3 condition). Then the scattering data $b(\lambda)$ is a ratio of two analytic functions of two real variables $\lambda_R, \lambda_I$. For the Schrödinger operator the proof of analyticity can be found in [10], the scheme of this proof can also be applied to the heat operator without serious modifications. Thus if $u(x, y)$ exponentially decays at infinity then the series of perturbation theory for $b(\lambda)$ converges and from Theorem 3 it follows that the functions $b(\lambda)$ and $\Psi(\lambda, x, y)$ are regular in $\lambda$. Let us recall that for nonsingular spectral data with the reality constraint (13) the corresponding potential and the wave function $\Psi(\lambda, x, y)$ are uniquely defined by $(\lambda)$ without the small norm assumption. Hence we have proved the main result of the paper:

**Theorem 4** Let $u(x, y)$ be a real smooth potential, exponentially decaying at infinity (D3 condition), $L$ be the heat operator (4) associated with the KP II equation. Then the direct spectral transform for $L$ described in section 2 is nonsingular for arbitrary large $u(x, y)$.

**Remark 5** The scattering transform for the heat operator can be obtained by an appropriate limiting procedure from the scattering transform for the two-dimensional Schrödinger operator at a fixed negative energy studied in
These two scattering problems look rather similar. But now is is clear that these two problems are essentially different at least in one aspect. In the theory of the Schrödinger operator with real decaying at infinity potential if our fixed energy level is located above the ground state then we always have singularities in the spectral transform. It is not clear now how to pose the inverse scattering problem for the singular scattering data (some preliminary results can be found in [5]). Thus in some sense the scattering transform for the heat operator is simpler (at least for sufficiently fast decaying potentials).

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