STALLINGS’ DECOMPOSITION THEOREM
FOR FINITELY GENERATED PRO-P GROUPS

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ABSTRACT. The main purpose of this paper is to prove a pro-p analogue of
Stallings’ decomposition theorem (cf. Thm. B), i.e., a finitely generated pro-
p group with infinitely many $\mathbb{Z}_p$-ends splits either as a non-trivial free pro-p
product with amalgamation in a finite subgroup, or as a pro-p HNN-extension
over a finite subgroup. The proof of this decomposition theorem is achieved
by characterizing finitely generated pro-p groups with a free factor isomorphic
to $\mathbb{Z}_p$ (cf. Thm. C), and the analysis of virtually free pro-p products (cf.
Thm. D).

1. Introduction

The theory of ends as established by J.R. Stallings (cf. [14]) had certainly a
major impact on combinatorial group theory. The main result of this theory states
that the number of ends $e(\Gamma)$ of a finitely generated group $\Gamma$ is either 0, 1, 2 or
$\infty$, and if $e(\Gamma) = \infty$, then $\Gamma$ splits non-trivially as an amalgamated free product or
HNN-extension over a finite group (cf. [3, Thm. 6.10]). The number of ends $e(\Gamma)$
can be expressed by the formula $1 - \text{rk}_{\mathbb{Z}}(H^0(\Gamma, \mathbb{Z}[\Gamma])) + \text{rk}_{\mathbb{Z}}(H^1(\Gamma, \mathbb{Z}[\Gamma]))$.

Following a suggestion by O.V. Mel’nikov, A.A. Korenev introduced in [6] the
number of ends $E(G)$ of a pro-$p$ group $G$ replacing $\mathbb{Z}[\Gamma]$ by $\mathbb{F}_p[[G]]$, the completed
$\mathbb{F}_p$-group algebra of $G$. He showed further that $G$ is finite if, and only if, $E(G) = 0$, and
that $G$ is virtually infinite cyclic if, and only if, $E(G) = 2$. Moreover, the only
possible values for $E(G)$ are 0, 1, 2 and $\infty$. However, he also mentioned that an
analogue of Stallings’ decomposition theorem is missing in this context. In this
paper we modify A.A. Korenev’s definition by using $\mathbb{Z}_p[[G]]$ instead of $\mathbb{F}_p[[G]]$, and
prove a pro-$p$ version of Stallings’ decomposition theorem. More precisely, for a
pro-$p$ group $G$ we define the number of $\mathbb{Z}_p$-ends $e(G)$ by

$$e(G) = 1 - \text{rk}_{\mathbb{Z}_p}(H^0_{\text{cts}}(G, \mathbb{Z}_p[[G]])) + \text{rk}_{\mathbb{Z}_p}(H^1_{\text{cts}}(G, \mathbb{Z}_p[[G]])),$$

where $H^*_{\text{cts}}(G, -)$ denotes continuous cochain cohomology as introduced by J. Tate
in [10]. One has the following properties (cf. Thm. [3,6]).

Theorem A. Let $G$ be a finitely generated pro-$p$ group. Then the following props-
erties hold.

(a) $e(G) \in \{0, 1, 2, \infty\}$.
(b) $e(G) = 0$ if, and only if, $G$ is a finite $p$-group.
(c) $e(G) = 2$ if, and only if, $G$ is infinite and virtually cyclic.
(d) $e(G) \leq E(G)$.
Moreover, one has the following analogue of Stallings decomposition theorem for finitely generated pro-$p$ groups with an infinite number of $\mathbb{Z}_p$-ends (cf. [6,4]).

**Theorem B.** Let $G$ be a finitely generated pro-$p$ group such that $e(G) = \infty$. Then either of the following holds:

(i) There exist closed subgroups $A, B, C \subseteq G$, $C \subseteq A$, $B, C$ finite, such that $G = A \ast_C B$, where $\ast$ denotes the free product with amalgamation in the category of pro-$p$ groups;

(ii) there exist closed subgroups $A, B, C \subseteq G$, $B, C$ finite and $B, C \subseteq A$, and an isomorphism $\varphi: B \to C$ such that $G = \text{HNN}_\varphi(A, t)$ is a proper HNN-extension in the category of pro-$p$ groups.

**Remark 1.1.** Just recently we heard from K. Wingberg that he independently established a virtual splitting theorem for pro-$p$ groups with infinitely many $\mathbb{F}_p$-ends. In fact one may conclude from [21, Thm. 2] that $e(G) = E(G)$ holds for any finitely generated pro-$p$ group $G$.

The proof of Theorem B is rather complicated and requires several different techniques. For a pro-$p$ group $G$ we call a sequence of homomorphisms of pro-$p$ groups $G \overset{\tau}{\twoheadrightarrow} \mathbb{Z}_p \xrightarrow{\sim} G$ a semi-direct factor isomorphic to $\mathbb{Z}_p$, if $\tau$ is surjective, and $\sigma$ is a section for $\tau$, i.e., $\tau \circ \sigma = \text{id}_{\mathbb{Z}_p}$. For a pro-$p$ group $G$ let

$$G^{ab} = G / \text{cl}([G,G]) \quad \text{and} \quad G^{ab,el} = G / \Phi(G),$$

where the direct limit is running over all open subgroups of $G$.

By construction, one has a canonical surjective homomorphism $\rho_G: G^{ab,el} \to \mathbb{T}\mathfrak{E}l_G$ of abelian pro-$p$ groups.

In his letter to J-P. Serre (cf. [13, §I, App. 1]), J. Tate introduced the abelian group

$$D_1(\mathbb{F}_p) = \lim_{\mathbb{F}_p} U_{\subseteq G} H^1(U, \mathbb{F}_p)^\vee \simeq \lim_{\mathbb{F}_p} U_{\subseteq G} U^{ab,el}$$

where the direct limit is running over all open subgroups of $G$, $\cdot^\vee$ denotes the Pontryagin dual, and the maps in the direct limits are induced by the transfer. By construction, one has a canonical map $j_{\mathfrak{E}l} : G^{ab,el} \to D_1(\mathbb{F}_p)$. Putting

$$D_1^H(\mathbb{F}_p) = \lim_{\mathbb{F}_p} U_{\subseteq G} \mathbb{T}\mathfrak{E}l_U,$$

one has a canonical map $j_{\mathbb{T}\mathfrak{E}l} : \mathbb{T}\mathfrak{E}l_G \to D_1^H(\mathbb{F}_p)$, and also a canonical surjective homomorphism of abelian groups $\rho_\text{HNN}: D_1(\mathbb{F}_p) \to D_1^H(\mathbb{F}_p)$ making the diagram

$$G^{ab,el} \xrightarrow{\rho_G} \mathbb{T}\mathfrak{E}l_G$$

commute. The first major step in the proof of Theorem B is a characterization of semi-direct factors isomorphic to $\mathbb{Z}_p$ which are free factors in case that $G$ is a

\[\text{[In the pro-$p$ case the base group $A$ does not always embed in the group $\text{HNN}_\varphi(A, t)$ (cf. [2]). If $A$ embeds in $\text{HNN}_\varphi(A, t)$, then we call $\text{HNN}_\varphi(A, t)$ proper.]}\]
finite generated pro-\( p \)-group. Indeed, a considerable part of this paper deals with establishing the following theorem (cf. Thm. 6.4).

**Theorem C.** Let \( G \) be a finitely generated pro-\( p \)-group, and let \( \delta: G \xrightarrow{\tau} \mathbb{Z}_p \xrightarrow{\sigma} G \) be a semi-direct factor isomorphic to \( \mathbb{Z}_p \), i.e., there exists a closed subgroup \( C \) of \( G \) such that \( G \cong C \amalg \text{im}(\sigma) \).

(i) \( \text{im}(\sigma) \neq 0 \) (cf. (1.6)).

(ii) \( J_G(\text{im}(\sigma)\Phi(G)) \neq 0 \) (cf. (1.6)).

For simplicity we call a semi-direct factor \( \delta: G \xrightarrow{\tau} \mathbb{Z}_p \xrightarrow{\sigma} G \) isomorphic to \( \mathbb{Z}_p \) a \( \mathbb{Z}_p \)-direction. For any \( \mathbb{Z}_p \)-direction \( \delta \) one has an associated extension of pro-\( p \) groups

\[
\{1\} \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} G/N \longrightarrow \{1\},
\]

where \( \Sigma = \text{im}(\sigma) \) and \( N = \text{cl}(\langle g\Sigma g^{-1} \mid g \in G \rangle) \). An essential step in the proof of Theorem C is to show that the extension (1.7) splits. This goal is achieved by a rather unorthodox method: We construct a complementary \( \mathbb{Z}_p[G] \)-module \( M \), show that it is isomorphic to a transitive \( \mathbb{Z}_p[G] \)-permutation module \( \mathbb{Z}_p[G/C] \), and \( C \) happens to be an \( N\)-complement in \( G \). For this purpose we use a beautiful result of A. Weiss concerning \( p \)-adic representations of finite \( p \)-groups (cf. [20]).

As a consequence of Theorem C we deduce that a finitely generated pro-\( p \)-group with infinitely many \( \mathbb{Z}_p \)-ends contains an open subgroup that splits into a non-trivial free pro-\( p \)-product. This class of pro-\( p \)-groups will be investigated in Section 4. The final task will be accomplished by proving the following theorem (cf. Thm. 5.4).

**Theorem D.** Let \( G \) be a finitely generated pro-\( p \)-group containing an open subgroup \( H \) such that

\[
H \cong F \amalg H_1 \amalg \cdots \amalg H_s,
\]

where \( H_i \neq \{1\}, i \in \{1, \ldots, s\} \), \( F \) is a free pro-\( p \)-subgroup of rank \( \text{rk}(F) = r < \infty \), and \( r + s \geq 2 \). Then \( G \) is isomorphic to the pro-\( p \)-fundamental group \( \Pi_1(G, \Gamma, v_0) \) of a finite connected graph of pro-\( p \)-groups \( (\mathcal{G}, \Gamma) \) with finite edge groups, where the graph \( \Gamma \) is not a single vertex.

2. **Profinite modules and cohomological Mackey functors**

For a prime number \( p \) we denote by \( \mathbb{F}_p \) the finite field with \( p \) elements, and by \( \mathbb{Z}_p \) the ring of \( p \)-adic integers. For a \( \mathbb{Z}_p \)-module \( M \) we denote by \( \text{gr}_a(M) \) the \( a \)-graded \( \mathbb{F}_p[t] \)-module associated with the \( p \)-adic filtration \( F_* M \), where \( F_k M = p^k M, k \geq 0 \).

### 2.1. Profinite modules for profinite groups

Let \( G \) be a profinite group. Then

\[
\mathbb{F}_p[G] = \lim_{\leftarrow \mathbb{U}_{a \in \mathbb{G}}} \mathbb{F}_p[G/U] \quad \text{and} \quad \mathbb{Z}_p[G] = \lim_{\leftarrow \mathbb{U}_{a \in \mathbb{G}}} \mathbb{Z}_p[G/U],
\]

where the inverse limits are running over all open normal subgroups of \( G \), will denote the completed \( \mathbb{F}_p \)-algebra and the completed \( \mathbb{Z}_p \)-algebra of \( G \), respectively. By \( \mathbb{F}_p[G] \text{-\text{prf}} \) we denote the abelian category of profinite left \( \mathbb{F}_p[G] \)-modules, and by \( \mathbb{Z}_p[G] \text{-\text{prf}} \) the abelian category of profinite left \( \mathbb{Z}_p[G] \)-modules. Both these categories have enough projectives. For \( M \in \text{ob}(\mathbb{Z}_p[G] \text{-\text{prf}}) \) we denote by \( \text{Ext}^k_{\mathbb{Z}_p[G]}(\_, M) \) the left derived functors of \( \text{Hom}_{\mathbb{Z}_p[G]}(\_, M) \). Then

\[
H^k_{\text{cts}}(G, M) = \text{Ext}^k_{\mathbb{Z}_p}(\mathbb{Z}_p, M)
\]
coincides with the \( k^{th} \)-continuous cochain cohomology group with coefficients in the topological \( \mathbb{Z}_p[G] \)-module \( M \). These cohomology groups were introduced by J. Tate in [16]. If \( A \subseteq G \) is a closed subgroup of \( G \), one has an exact induction functor
\[
\text{ind}^G_A(\_)_p = \mathbb{Z}_p[G] \hat{\otimes}_{\mathbb{Z}_p[A]} z_p[A]_{\text{prf}} \rightarrow \mathbb{Z}_p[G]_{\text{prf}}
\]
which is left-adjoint to the restriction functor \( \text{res}^G_\_ : z_p[G]_{\text{prf}} \rightarrow z_p[A]_{\text{prf}} \). In particular, \( \text{ind}^G_A(\_)_p \) is mapping projectives to projectives (cf. [18, Prop. 2.3.10]). Hence one has natural isomorphisms
\[
\text{Ext}^k_G(M, \text{res}^G_\_ (N)) \simeq \text{Ext}^k_G(\text{ind}^G_A(M), N)
\]
for all \( k \geq 0, M \in \text{ob}(z_p[A]_{\text{prf}}), N \in \text{ob}(z_p[G]_{\text{prf}}) \). Since \( \text{res}^G_\_ (\_)_p \) is mapping projectives to projectives (cf. [9, Cor. 5.7.2(b)]), one has a restriction map
\[
\text{res}^G_\_, A : H^k_{\text{cts}}(G, \_)_p \rightarrow H^k_{\text{cts}}(A, \_)_p.
\]
Moreover, if \( \varepsilon : \mathbb{Z}_p[G/A] \rightarrow \mathbb{Z}_p \) denotes the augmentation of the left \( \mathbb{Z}_p[G] \)-permutation module \( \mathbb{Z}_p[G/A] \), the isomorphisms (2.4) imply that this restriction mapping coincides with
\[
\text{Ext}^k_G(\varepsilon)_p : \text{Ext}^k_G(\mathbb{Z}_p, \_)_p \rightarrow \text{Ext}^k_G(\mathbb{Z}_p[G/A], \_)_p.
\]
For further details concerning profinite \( F_p[G] \)- and \( Z_p[G] \)-modules the reader may consult [11, §V.2, 13, Chap. 5] and [17].

2.2. The sandwich lemma. For a finite group \( G \) we will call a left \( \mathbb{Z}_p[G] \)-module \( M \) a left \( \mathbb{Z}_p[G] \)-lattice, if \( M \) is a finitely generated \( \mathbb{Z}_p \)-module. The following lemma will turn out to be useful for our purpose.

**Lemma 2.1** (Sandwich lemma). Let \( G \) be a finite \( p \)-group, and let \( K, B, Q \) three \( \mathbb{Z}_p[G] \)-lattices with the following properties:

(i) \( K \subseteq B \subseteq Q \);
(ii) \( B/K \) is torsion free;
(iii) \( B \cap pQ = K + pB \);
(iv) \( \text{soc}_G((B + pQ)/pQ) = \text{soc}_G(Q/pQ) \).

Then \( K = 0 \), and the canonical injection \( B \rightarrow Q \) is split injective.

**Proof.** Put \( C = B/K \). By (ii), one has a short exact sequence of \( \mathbb{Z}_p[G] \)-lattices
\[
0 \rightarrow K \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 0,
\]
and the canonical map \( j : C \rightarrow Q/K \) is injective. Moreover, by (iii) one has
\[
C/pC \cong B/(K + pB) = B/B \cap pQ.
\]
Thus \( j/p : C/pC \rightarrow (Q/K)/p(Q/K) \) is injective showing that \( C \) is a direct summand of \( Q/K \) in the category of finitely generated \( \mathbb{Z}_p \)-modules (cf. [19, Fact 2.2]). Hence it suffice to prove that \( K = 0 \).

Let \( F_*B \) be the integral filtration on \( B \) induced by the \( p \)-adic filtration on \( Q \), i.e., for \( k \geq 0 \) one has \( F_kB = B \cap p^kQ \). Let \( F_*K \) and \( F_*C \) denote the induced filtrations on \( K \) and \( C \), respectively, i.e., \( F_kK = K \cap p^kQ \) and \( F_kC = ((B \cap p^kQ) + K)/K \).

Since \( C \) is a direct summand of \( Q/K \)
\[
F_kC = (B \cap p^kQ) + K/K = p^kB + K/K,
\]
i.e., \( F_*C \) coincides with the \( p \)-adic filtration on \( C \). Let \( \text{gr}_*(B) \), \( \text{gr}_*(K) \) and \( \text{gr}_*(C) \) denote the graded \( F_p[t] \)-modules associated to the filtrations \( F_*B, F_*K \), etc., i.e., \( t \) is
homogeneous of degree 1 induced by multiplication with \( p \), and \( \mathrm{gr}_\bullet(B) \), \( \mathrm{gr}_\bullet(K) \) and \( \mathrm{gr}_\bullet(C) \) are left \( \mathbb{F}_p[t][G] \)-modules. By construction, one has a short exact sequence of \( \mathbb{F}_p[t][G] \)-modules

\[
(2.10) \quad 0 \longrightarrow \mathrm{gr}_\bullet(K) \longrightarrow \mathrm{gr}_\bullet(B) \longrightarrow \mathrm{gr}_\bullet(C) \longrightarrow 0.
\]

By \( (2.9) \), the maps \( t^m : \mathrm{gr}_k(C) \to \mathrm{gr}_{k+m}(C) \) are isomorphisms for all \( k, m \geq 0 \). Since \( B \) and \( K \) are torsion free \( \mathbb{Z}_p \)-modules, the homomorphisms \( t : \mathrm{gr}_k(B) \to \mathrm{gr}_{k+1}(B) \) and \( t : \mathrm{gr}_k(K) \to \mathrm{gr}_{k+1}(K) \) are injective. By hypothesis (iii), \( K \subseteq pQ \) and thus \( \mathrm{gr}_0(K) = 0 \). Hence \( \mathrm{gr}_0(\pi) \) is an isomorphism.

Suppose that there exists \( m \geq 0 \) such that \( \mathrm{gr}_m(K) \neq 0 \). As \( \mathrm{gr}_m(K) \) is a finite-dimensional \( \mathbb{F}_p[G] \)-module, this implies that \( \text{soc}_G(\mathrm{gr}_m(K)) \neq 0 \). Then \( \mathrm{gr}_m(K) \) is the kernel of the map is the composition

\[
(2.11) \quad \tau : \mathrm{gr}_m(B) \xrightarrow{\mathrm{gr}_m(\pi)} \mathrm{gr}_m(C) \xrightarrow{(t)^{-m}} \mathrm{gr}_0(C) \xrightarrow{\mathrm{gr}_0(\pi)^{-1}} \mathrm{gr}_0(B),
\]

and \( (t)^m : \mathrm{gr}_0(B) \to \mathrm{gr}_m(B) \) is a section of \( \tau \). In particular, one has an isomorphism of \( \mathbb{F}_p[G] \)-modules

\[
(2.12) \quad \mathrm{gr}_m(B) \simeq \mathrm{gr}_m(K) \oplus \mathrm{gr}_0(B).
\]

From \( (2.12) \) and hypothesis (iv) one concludes that

\[
(2.13) \quad \dim(\text{soc}_G(\mathrm{gr}_m(B))) > \dim(\text{soc}_G(\mathrm{gr}_0(B))) = \dim(\text{soc}_G(Q/pQ)) = \dim(\mathrm{gr}_m(Q)),
\]

a contradiction. This shows that \( \mathrm{gr}_\bullet(K) = 0 \) and hence \( K = 0 \). \( \square \)

### 2.3. Cohomological Mackey functors

Let \( G \) be a profinite group. As introduced in \( [19] \) \( \S 3.1 \) a \( G \)-Mackey system \( \mathcal{M} \) is a set of open subgroups of \( G \) which is closed under intersection and conjugation with elements in \( G \), e.g., the set of all open subgroups of \( G \) is a \( G \)-Mackey system which will be denoted by \( G^\circ \).

For a \( G \)-Mackey system \( \mathcal{M} \) the category \( \mathcal{CM}_{\mathcal{M}}(\mathcal{M}, \mathbb{Z}_p\text{prf}) \) of cohomological \( \mathcal{M} \)-Mackey functors with values in the abelian category \( \mathbb{Z}_p \text{prf} \) is the category of contravariant functors from the category of finitely generated \( \mathbb{Z}[G, \mathcal{M}] \)-permutation modules to the category of abelian pro-\( p \) groups with morphisms given by the natural transformations (cf. \( [19] \) \( \S 3.3 \)). In particular, \( \mathcal{CM}_{\mathcal{M}}(\mathcal{M}, \mathbb{Z}_p\text{prf}) \) is an abelian category. Any cohomological \( \mathcal{M} \)-Mackey functor \( X \) is determined uniquely by its values on \( \mathbb{Z}[G/U], U \in \mathcal{M}, \) and the homomorphisms

\[
(2.14) \quad \rho^U_g : \mathbb{Z}[G/U] \longrightarrow \mathbb{Z}[G/U], \quad \rho^U_g(xgUg^{-1}) = xgU,
\]

\[
(2.15) \quad \iota_{V,U} : \mathbb{Z}[G/V] \longrightarrow \mathbb{Z}[G/U], \quad \iota_{V,U}(xV) = xU,
\]

\[
(2.16) \quad t_{U,V} : \mathbb{Z}[G/U] \longrightarrow \mathbb{Z}[G/V], \quad t_{U,V}(xU) = \sum_{r \in R} xrV,
\]

where \( x \in G, U, V \in \mathcal{M}, V \subseteq U \) and \( R \subseteq U \) is any system of coset representatives of \( U/V \). For simplicity we put for \( X \in \mathrm{ob}(\mathcal{CM}_{\mathcal{M}}(\mathcal{M}, \mathbb{Z}_p\text{prf})) \)

\[
(2.17) \quad X_U = X(\mathbb{Z}[G/U]), \quad c^U_g = X(\rho^U_g), \quad \iota^X_U = X(\iota_{V,U}), \quad t^X_U = X(t_{U,V}),
\]

for \( U, V \in \mathcal{M}, V \subseteq U \), and \( g \in G \). On the other hand a family of abelian pro-\( p \) groups \( X_U, U \in \mathcal{M} \), with morphisms

\[
(2.18) \quad c^X_g : X_U \to X_{gU}, \quad \iota^X_U : X_U \to X_V, \quad t^X_U : X_V \to X_U
\]
for \(U, V \in \mathcal{M}, V \subseteq U, g \in G\), satisfying certain identities (cf. [19] §3.2]) defines a cohomological \(M\)-Mackey functor with values in \(z_p\text{-prf}\). Such a cohomological \(M\)-Mackey functor \(X\) will be called to be \(i\)-injective, if \(i_{U,V}^X\colon X_U \to X_V\) is injective for all \(U, V \in \mathcal{M}, V \subseteq U\). Similarly, \(X\) will be called \(t\)-surjective, if \(t_{U,V}^X\colon X_V \to X_U\) is surjective for all \(U, V \in \mathcal{M}, V \subseteq U\). For a pro-\(p\) group \(G\) the following \(G^t\)-Mackey functors will be of particular importance for our purpose:

**Example 1:** By \(Ab = Ab(G)\) we denote the cohomological \(G^t\)-Mackey functor given by \(Ab_U = U^{ab} = U/\text{cl}([U,U])\). For \(V \subseteq U\) and \(g \in G\) the mappings \(i_{g,U}^Ab\colon U^{ab} \to gU^{ab}\) and \(t_{U,V}^Ab\colon V^{ab} \to U^{ab}\) are just the canonical maps, while \(i_{U,V}^Ab : U^{ab} \to V^{ab}\) is given by the transfer.

**Example 2:** \(El = El(G) = Ab/pAb\), i.e., \(El_U = U^{ab,el} = U/\Phi(U)\) for \(U\) open in \(G\). The mappings are induced by the mappings for \(Ab\).

**Example 3:** \(TAb = TAb(G)\) is the subfunctor of \(Ab\) given by the torsion elements, i.e., \(TAb_U = \text{tor}(U^{ab})\) for \(U\) open in \(G\).

**Example 4:** \(TFab = Ab/TAb\).

**Example 5:** \(TfEl = Ab/(TAb + pAb)\).

**Example 6:** If \(M \in \text{ob}(z_p[G]\text{-prf})\) is a profinite left \(Z_p[G]\)-module, then its \(k\)-th homology functor \(h_k(M), k \geq 1\), is a cohomological \(G^t\)-Mackey functor satisfying \(h_k(M)^U = H_k(U, \text{res}^G(M))\) (cf. [19] §3.8). Here \(H_k(U, \_ ) = \text{Tor}_k^U(Z_p, \_)\) denote the homology groups as introduced by A. Brumer (cf. [19]). In particular, \(Ab = h_1(Z_p)\) and \(El = h_1(F_p)\).

**Example 7:** \(T = T(G, Z_p)\) will denote the cohomological \(G^t\)-Mackey functor given by \(T_U = Z_p, e^T_{g,U} = \text{id}_{Z_p}, t^T_{U,V} = \text{id}_{Z_p}\) and \(t^T_{U,V} = |U|/|V|\text{id}_{Z_p}\) for \(V \subseteq U\) (cf. [19] Ex. 3.1(a))). In particular, if \(G \simeq Z_p\), then \(T \simeq h_1(Z_p) = Ab\).

**Example 8:** \(Y = Y(G, Z_p)\) will denote the cohomological \(G^t\)-Mackey functor given by \(Y_U = Z_p, e^Y_{g,U} = \text{id}_{Z_p}, t^Y_{U,V} = |U|/|V|\text{id}_{Z_p}\) and \(t^Y_{U,V} = \text{id}_{Z_p}\) for \(V \subseteq U\) (cf. [19] Ex. 3.1(b))). In particular, \(Y = h_0(Z_p)\).

**2.4. Induction and restriction.** Let \(A\) be a closed subgroup of the profinite group \(G\). Then one has an **induction functor**

\[
\text{ind}^{G^t}_A(\_ ) : \text{cm}_{\mathcal{M}}^G(A^t, z_p\text{-prf}) \to \text{cm}_{\mathcal{M}}^G(G^t, z_p\text{-prf})
\]

which is the left adjoint to the **restriction functor**

\[
\text{res}^{G^t}_A(\_ ) : \text{cm}_{\mathcal{M}}^G(G^t, z_p\text{-prf}) \to \text{cm}_{\mathcal{M}}^G(A^t, z_p\text{-prf})
\]

(cf. [19] §3.5 and 3.6). Moreover, both these functors are exact, and for \(B\) open in \(A\) and \(X \in \text{ob}(\text{cm}_{\mathcal{M}}^G(G^t, z_p\text{-prf}))\) one has

\[
\text{res}^{G^t}_A(X)_B = \lim_{\downarrow B \subseteq U \subseteq A} (X_U, t^X_{V,U}),
\]

where the inverse limit is running over all open subgroups containing \(B\), and the maps are given by \(t^X_{U,V} : X_V \to X_U\) (cf. (2.18)).

For a \(G\)-Mackey system \(\mathcal{M}\) and \(X \in \text{ob}(\text{cm}_{\mathcal{M}}^G(M, z_p\text{-prf}))\) we also put

\[
\text{tres}(X) = \lim_{\downarrow U \in \mathcal{M}} (X_U, t^X_{V,U}),
\]

\[
\text{ires}(X) = \lim_{\downarrow U \in \mathcal{M}} (X_U, i^X_{U,V}).
\]

In particular, \(\text{tres}(X)\) is a profinite left \(Z_p[G]\)-module, and \(\text{ires}(X)\) is a discrete left \(G\)-module. Moreover, if \(G \in \mathcal{M}\), then one has a canonical map \(j_X : X_G \to \text{ires}(X)\), which is a homomorphism of left \(Z_p[G]\)-modules.
3. Finitely generated pro-$p$ groups with a $\mathbb{Z}_p$-direction

In this section we consider a finitely generated pro-$p$ group $G$ with a $\mathbb{Z}_p$-direction $\delta: G \rightarrow \mathbb{Z}_p \rightarrow G$. We put $\Sigma = \text{im}(\sigma)$ and $N = N(\delta) = \text{cl}(\langle \sigma g \mid g \in G \rangle)$, i.e., $N$ is the closed normal subgroup generated by $\Sigma$. By the commutativity of the diagram (1.10), $\delta$ is also an $\mathbb{F}_p$-direction, i.e., $\mathbb{F}_p(\sigma) \Phi(G) \neq 0$. These type of semi-direct factors isomorphic to $\mathbb{Z}_p$ were investigated in [19]. Thus from [19] Thm. C one concludes the following.

**Theorem 3.1.** Let $G$ be a finitely generated pro-$p$ group with a $\mathbb{Z}_p$-direction $\delta: G \rightarrow \mathbb{Z}_p \rightarrow G$. Then $N = N(\delta)$ is a non-trivial free pro-$p$ group, and $N^{\text{ab}, \text{el}}$ is isomorphic to $\mathbb{F}_p[G/N]$ as a left $\mathbb{F}_p[G/N]$-module.

Since every $\mathbb{Z}_p$-direction is in particular an $\mathbb{F}_p$-direction, [19] Thm. 5.2 implies the following.

**Theorem 3.2.** Let $G$ be a finitely generated pro-$p$ group with a $\mathbb{Z}_p$-direction $\delta: G \rightarrow \mathbb{Z}_p \rightarrow G$ such that the extension

$$\{1\} \rightarrow N \rightarrow G \xrightarrow{\theta} G/N \rightarrow \{1\}$$

splits, i.e., there exists $\theta: G/N \rightarrow G$ such that $\pi \circ \theta = \text{id}_{G/N}$. Then $G \simeq \mathbb{Z}_p \sqcup \text{im}(\theta)$.

From Theorem 3.2 one concludes that in order to prove the inclusion (i)$\Rightarrow$(ii) in Theorem C, it remains to shows that the short exact sequence of pro-$p$ groups (3.1) splits. This goal will be achieved by showing that there exists a $\mathbb{Z}_p[G]\text{-permutation}$ module $M \simeq \mathbb{Z}_p[\Omega]$ for some profinite left $G$-set $\Omega$ satisfying $\text{res}^G_N(M) \simeq \mathbb{Z}_p[N]$. Hence any point stabilizer $G_\omega, \omega \in \Omega$, will be an $N$-complement of $G$ (cf. Thm. 5.3).

### 3.1. Pro-$p$ groups with a closed subgroup isomorphic to $\mathbb{Z}_p$.

Let $G$ be a pro-$p$, and let $\Sigma \subseteq G$ be a closed subgroup isomorphic to $\mathbb{Z}_p$. We also assume that $\Sigma$ has a distinguished generator $s \in \Sigma$. Then one has an isomorphism of cohomological $\Sigma^\sharp$-Mackey functors

$$\chi: T(\Sigma, \mathbb{Z}_p) \rightarrow \text{Ab}(\Sigma)$$

given by $\chi_\Xi(1_{\mathbb{Z}_p}) = s^p \cdot \Xi \in \Sigma^\sharp, |\Sigma : \Xi| = p^h$. This isomorphism induces an isomorphism of cohomological $\Sigma^\sharp$-Mackey functors $\chi^\sharp: T(\Sigma, \mathbb{Z}_p) \rightarrow \text{TfAb}(\Sigma)$. One has canonical isomorphisms

$$\text{tres}_{\Sigma^\sharp}(T\text{Ab}(G)) \simeq T\text{Ab}(\Sigma) = 0,$$

$$\text{tres}_{\Sigma^\sharp}(\text{Ab}(G)) \simeq \text{Ab}(\Sigma),$$

$$\text{tres}_{\Sigma^\sharp}(\text{TfAb}(G)) \simeq \text{TfAb}(\Sigma)$$

(cf. [19] Fact 3.6). Hence $\chi$ and $\chi^\sharp$ induce morphisms of cohomological $G^\sharp$-Mackey functors

$$\tilde{\chi}: \text{ind}_{\Sigma^\sharp}^{G^\sharp}(T(\Sigma, \mathbb{Z}_p)) \rightarrow \text{Ab}(G),$$

$$\chi^\sharp: \text{ind}_{\Sigma^\sharp}^{G^\sharp}(T(\Sigma, \mathbb{Z}_p)) \rightarrow \text{TfAb}(G)$$

(cf. [19] Prop. 3.4). Put $X = \text{ind}_{\Sigma^\sharp}^{G^\sharp}(T(\Sigma, \mathbb{Z}_p))$, and suppose from now on that $G$ is finitely generated. Then $X_U$ and $\text{TfAb}_U$ are finitely generated free $\mathbb{Z}_p$-modules for
all $U \in G^t$. Thus for $\underline{*} = \text{Hom}_{Z_p}(\underline{\cdot}, Z_p)$ one has an induced natural transformation

$$\chi^{tf}_U: \text{TfAb}^* \to X^*$$

(cf. [17, §3.2]). Moreover, one has a canonical isomorphism $X^* = \text{ind}^{G^t}_{G}(\Upsilon(\Sigma, Z_p))$ (cf. [19] and [17, §3.6]). In particular, $\text{tres}(X^*)$ is canonically isomorphic to $\text{ind}^{G^t}_{G}(Z_p)$. By construction, one has canonical isomorphisms

$$\text{tres}(\text{TfAb})^* \simeq \lim_{\leftarrow U \subseteq G} \text{Hom}(U^{ab}, Z_p) \simeq \lim_{\leftarrow U \subseteq G} H^1_{\text{cts}}(U, Z_p) \simeq H^1_{\text{cts}}(G, Z_p[G]).$$

One verifies easily that the diagram

$$\begin{array}{ccc}
H^1_{\text{cts}}(\Sigma, \text{res}^G_{\Sigma}(Z_p[G])) & \xrightarrow{\text{res}} & \text{ind}^G_{\Sigma}(Z_p) \\
| & & | \\
H^1_{\text{cts}}(\Sigma, Z_p[\Sigma]) \otimes_{\Sigma} Z_p[G] & \xrightarrow{\text{tres}(\chi^{tf})^*} & H^1_{\text{cts}}(\Sigma, Z_p) \otimes_{\Sigma} Z_p[G] \\
\end{array}$$

commutes. Note that the vertical maps in the diagram are isomorphisms. As $Z_p$ is an orientable Poincaré duality pro-$p$ group of cohomological $p$-dimension 1, $H^1_{\text{cts}}(\epsilon): H^1_{\text{cts}}(\Sigma, Z_p[\Sigma]) \to H^1_{\text{cts}}(\Sigma, Z_p)$ is an isomorphism (cf. [15, Thm. 4.5.6]). Hence all maps apart from $\text{res}_{G:G}$ and $\text{tres}(\chi^{tf})^*$ are isomorphisms.

### 3.2. Finitely generated pro-$p$ groups with a $Z_p$-direction

The following property will be important for our purpose.

**Proposition 3.3.** Let $G$ be a finitely generated pro-$p$ group with a $Z_p$-direction $\bar{\varnothing}: G \simeq Z_p \xrightarrow{\sigma} G$. Put $\Sigma = \text{im}(\sigma)$, and let $s = \sigma(1)$ denote a distinguished generator of $\Sigma$. Then $\bar{\chi}$ and $\chi^{tf}$ are injective, and $\chi^{tf}_U: \text{ind}^{G^t}_{G}(\Upsilon(\Sigma, Z_p)) \to \text{TfAb}_U$ is a split-injective homomorphism of finitely generated $Z_p$-modules for every $U \in G^t$.

**Proof.** Let $X = \text{ind}^{G^t}_{G}(\Upsilon(\Sigma, Z_p))$. It suffices to show that $\chi^{tf}_U: X_U \to \text{TfAb}_U$ is split-injective for every $U \in G^t$. Since $G$ is finitely generated, $\text{TfAb}_U$ is a finitely generated free $Z_p$-module for every $U \in G^t$, and the same is true for $X_U$ (cf. [19, Ex. 3.3(a)]). Hence $X$ and $\text{TfAb}$ satisfy the hypothesis (i) of [19, Lemma 4.20]. As $X$ is a Hilbert ‘90 cohomological $G^t$-Mackey functor (cf. [19, Fact 4.8]), $\text{gr}_0(X)$ is of type $H^0$ (cf. [19, Fact 4.9]), where $\text{gr}_0(\underline{\cdot})$ is defined as in the sandwich lemma (cf. [12, 2.2]). Thus hypothesis (ii) of [19, Lemma 4.20] is also satisfied. By construction, one has canonical isomorphisms

$$\text{gr}_0(\text{TfAb}) \simeq \text{TfEl}, \quad \text{res}(\text{gr}_0(\text{TfAb})) \simeq D^t_{G^t}(F_p),$$

where $\chi^{tf}_{1G} = (\text{sc}((G, G)) + \text{tor}(G^{ab})$. As $\bar{\varnothing}$ is a $Z_p$-direction, one has

$$j_{\text{TfEl}}(\text{gr}_0(\chi^{tf}_{1G})) \lesssim \Sigma 1G + pX_G = j_{\text{TfEl}}(\text{sc}((G, G)) + (\text{tor}(G^{ab}) + pG^{ab})$$

$$j_{\text{TfEl}}(\text{sc}(\Phi(G))) \neq 0$$

(cf. [16]). Since $\text{gr}_0(X_G) = F_p\Sigma 1G$, the map $j_{\text{TfEl}} \circ \text{gr}_0(\chi^{tf}_{1G})$ is injective. Hence [19, Lemma 4.20] implies that $\chi^{tf}$ is injective, and that $\chi^{tf}_U$ is split-injective for every $U \in G^t$. This yields the claim.

$\square$
As a consequence of Proposition 3.3 one obtains the following.

**Proposition 3.4.** Let $G$ be a finitely generated pro-$p$ group with a $\mathbb{Z}_p$-direction $\partial: G \rightarrow \mathbb{Z}_p \xrightarrow{\sim} G$, and let $\Sigma = \im(\sigma)$. Then the canonical map

\[
\beta_*: \Ext^1_G(\mathbb{Z}_p, \mathbb{Z}_p[G]) \rightarrow \Ext^1_G(\mathbb{Z}[\Sigma], \mathbb{Z}_p[G])
\]

is surjective.

**Proof.** Let $X = \ind_{\mathbb{F}_p}^{\mathbb{Z}_p} \left( \mathbb{T}(\Sigma, \mathbb{Z}_p) \right)$. By Proposition 3.3 $\chi^{\mathbb{Z}_p}: X \rightarrow \TfAb$ is injective, such that $\chi^{\mathbb{Z}_p}: X_U \rightarrow \TfAb_U$ is split-injective for every $U \subseteq G$. Hence the mapping $(\chi^{\mathbb{Z}_p})^*: \TfAb^* \rightarrow X^*$ is surjective. Since $\res$ is exact, this implies that $\res((\chi^{\mathbb{Z}_p})^*): H^1_{\cts}(G, \mathbb{Z}_p[G]) \rightarrow \ind_{\mathbb{Z}_p}^G(\mathbb{Z}_p)$ is surjective. From the commutativity of the diagram (3.7) one concludes the following fact. \[ \square \]

3.3. The number of $\mathbb{Z}_p$-ends. For a pro-$p$ group $G$ one has a canonical isomorphism

\[
D_1^G(\mathbb{F}_p) = \lim_{\rightarrow U \subseteq \Sigma} G((U^{ab}/\text{tor}(U^{ab}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \simeq \text{ires}(\Tf\text{El}).
\]

If $G$ is in addition finitely generated, one has a natural isomorphism

\[
H^1_{\cts}(G, \mathbb{Z}_p[G]) \simeq \lim_{\rightarrow U \subseteq \Sigma} \text{Hom}_{\text{grp}}(U, \mathbb{Z}_p);
\]

in particular, $D_1^G(\mathbb{F}_p)^{\vee} \simeq H^1_{\cts}(G, \mathbb{Z}_p[G]) \otimes \mathbb{F}_p$, where $\_^{\vee}$ denotes the Pontryagin dual. Thus for a finitely generated pro-$p$ group $G$ one has

\[
e(G) = \begin{cases} 1 + \dim(D_1^G(\mathbb{F}_p)) & \text{for } |G| = \infty, \\ 0 & \text{for } |G| < \infty \end{cases}
\]

(cf. [19 Prop. 5.3]). From (3.13) one concludes the following fact.

**Fact 3.5.** Let $G$ be a pro-$p$ group.

(a) $e(G) \leq E(G)$.

(b) $e(G) = 0$ if, and only if, $G$ is finite.

(c) $e(U) = e(G)$ for any open subgroup $U$ of $G$.

The following theorem shows that the number of $\mathbb{Z}_p$-ends of a finitely generated pro-$p$ group has properties similar to the number of ends of a finitely generated discrete group (cf. [14]).

**Theorem 3.6.** Let $G$ be a finitely generated pro-$p$ group. Then $e(G) \in \{0, 1, 2, \infty\}$. Moreover, $G$ is finite if, and only if, $e(G) = 0$; and $G$ is virtually infinite-cyclic if, and only if, $e(G) = 2$.

**Proof.** By Fact 3.5(b), we may assume that $G$ is infinite and $e(G) > 1$. Then there exists an open subgroup $U$ of $G$ such that the map $\tilde{j}_{\TfAb_U}: \TfAb_U \rightarrow D_1^U(\mathbb{F}_p)$ is non-trivial (cf. [3.13]). Choose a $\mathbb{Z}_p$-submodule $C$ of $\TfAb_U$ and an element $x \in \TfAb_U$ such that $\TfAb_U = \mathbb{Z}_p x \oplus C$ and $\tilde{j}_{\TfAb_U}(x) \neq 0$. Let $\tau: U \rightarrow \mathbb{Z}_p$ be the homomorphism induced by the map $\tilde{\tau}: \TfAb_U \rightarrow \mathbb{Z}_p$, $\tilde{\tau}|_C = 0$, $\tilde{\tau}(x) = 1_{\mathbb{Z}_p}$, and let $y \in U$ be such that $(ycl(U, U)) + \text{tor}(U^{ab}) = x$. Then for $\sigma: \mathbb{Z}_p \rightarrow U$, $\sigma(1_{\mathbb{Z}_p}) = y$, $\tilde{\sigma}: U \rightarrow \mathbb{Z}_p \xrightarrow{\sim} U$ is a $\mathbb{Z}_p$-direction. Let $\Sigma = \im(\sigma)$. One may distinguish two cases:

**Case 1:** $|U : \Sigma| < \infty$. Then $G$ is virtually infinite-cyclic. As $H^1_{\cts}(\Sigma, \mathbb{Z}_p[\Sigma]) \simeq \mathbb{Z}_p$ (cf. [15, Thm. 4.5.6]), one has $e(G) = e(U) = 2$ (cf. Fact 3.5(c)).
**Case 2:** \([U : \Sigma] = \infty\). As \(\text{ind}_{\Sigma}^U(\mathbb{Z}_p)\) is a homomorphic image of \(H^1_{\text{cts}}(U, \mathbb{Z}_p[U])\) (cf. (3.7)) and Prop. 3.4, one has that \(\text{rk}(H^1_{\text{cts}}(U, \mathbb{Z}_p[U])) = \infty\). Thus, by Fact 3.5(c), \(e(G) = e(U) = \infty\) (cf. \([19]\) Prop. 5.3). \(\square\)

### 3.4. Complementary modules and short exact sequences of a pro-p group associated with a \(\mathbb{Z}_p\)-direction.

Let \(\varphi: G \rightarrow \mathbb{Z}_p \rightarrow G\) be a pro-p group with a \(\mathbb{Z}_p\)-direction. Then one has canonical mappings \(\mathbb{Z}_p[G] \xrightarrow{\alpha} \mathbb{Z}_p[G/\Sigma] \xrightarrow{\beta} \mathbb{Z}_p\) of profinite left \(\mathbb{Z}_p\)-modules. We say that the profinite left \(\mathbb{Z}_p\)-module \(M\) is a complementary module for \(\varphi\), if there exist mappings \(\eta: M \rightarrow \mathbb{Z}_p\), \(\xi: \mathbb{Z}_p[G] \rightarrow M\), and \(j: \ker(\beta) \rightarrow \mathbb{Z}_p[G]\) of profinite \(\mathbb{Z}_p\)-modules such that the diagram

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{\iota} & \ker(\beta) & \xrightarrow{j} & \mathbb{Z}_p[G] & \xrightarrow{\alpha} & M & \xrightarrow{\eta} & 0 \\
& & \downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{\varphi} & & \\
0 & \xrightarrow{\iota} & \ker(\beta) & \xrightarrow{\iota} & \mathbb{Z}_p[G/\Sigma] & \xrightarrow{\beta} & \mathbb{Z}_p & \xrightarrow{\iota} & 0 \\
\end{array}
\]

commutes and has exact rows. The main goal of the following subsection is to establish the existence of such a module in case that \(G\) is finitely generated and to study its properties. One consequence of the existence of complementary modules is the following: As \(\alpha\) is surjective, the snake lemma implies that \(\eta\) is surjective. Moreover, the induced map \(\xi: \ker(\alpha) \rightarrow \ker(\beta)\) is an isomorphism. Hence the diagram (3.14) can be completed to a commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{\iota} & \ker(\beta) & \xrightarrow{j} & \mathbb{Z}_p[G] & \xrightarrow{\alpha} & M & \xrightarrow{\eta} & 0 \\
& & \downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{\varphi} & & \\
0 & \xrightarrow{\iota} & \ker(\beta) & \xrightarrow{\iota} & \mathbb{Z}_p[G/\Sigma] & \xrightarrow{\beta} & \mathbb{Z}_p & \xrightarrow{\iota} & 0 \\
\end{array}
\]

where \(\cdot(s - 1)\) denotes right multiplication by the element \(s - 1 \in \mathbb{Z}_p[\Sigma]\). Thus the existence of a complementary \(\mathbb{Z}_p[G]\)-module \(M\) gives rise to a short exact sequence

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{\iota} & \mathbb{Z}_p[G] & \xrightarrow{\omega} & M & \xrightarrow{\eta} & \mathbb{Z}_p & \xrightarrow{\iota} & 0 \\
\end{array}
\]

which will be called a complementary short exact sequence associated to the complementary profinite \(\mathbb{Z}_p[G]\)-module \(M\).

**Lemma 3.7.** Let \(G\) be a pro-p group, and let \(\Sigma = \text{cl}(\langle s \rangle) \subseteq G\) be a subgroup isomorphic to \(\mathbb{Z}_p\) such that \(\beta_s: \text{Ext}^1_G(\mathbb{Z}, \mathbb{Z}_p[G]) \rightarrow \text{Ext}^1_G(\mathbb{Z}_p[G/\Sigma], \mathbb{Z}_p[G])\) is surjective.
Then there exists an injective mapping \( j : \ker(\beta) \to \mathbb{Z}_p[G] \) making the diagram
\[
\begin{array}{ccc}
\mathbb{Z}_p[G] & \xrightarrow{\beta} & \mathbb{Z}_p \\
\downarrow{\alpha} & & \downarrow{0} \\
\ker(\beta) & \xrightarrow{\iota} & \mathbb{Z}_p[G]/\Sigma \\
\end{array}
\]
commute.

**Proof.** From the long exact sequence for \( \text{Ext}^\bullet_G(\_ , \_ ) \) one obtains a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_G(\ker(\beta), \mathbb{Z}_p[G]) & \xrightarrow{\alpha \circ} & \text{End}_G(\mathbb{Z}_p[G]/\Sigma) \\
\downarrow{\beta} & & \downarrow{0} \\
\text{Ext}^1_G(\mathbb{Z}_p, \mathbb{Z}_p[G]) & \xrightarrow{\delta} & \text{Ext}^1_G(\mathbb{Z}_p[G]/\Sigma, \mathbb{Z}_p[G]) \\
\end{array}
\]
with exact rows and columns. In particular, \( \delta(\iota) = 0 \). Hence there exists \( j \in \text{Hom}_G(\ker(\beta), \mathbb{Z}_p[G]) \) such that \( \iota = \alpha \circ j \). As \( \iota \) is injective, \( j \) is injective. This yields the claim. \( \square \)

From the diagram (3.17) one obtains for \( M = \text{coker}(j) \) a commutative diagram
\[
\begin{array}{ccc}
0 & \xrightarrow{\iota} & \mathbb{Z}_p[G] \\
\downarrow{\alpha} & & \downarrow{0} \\
\ker(\beta) & \xrightarrow{\iota} & \mathbb{Z}_p[G]/\Sigma \\
\end{array}
\]
with exact rows. Here \( \xi : \mathbb{Z}_p[G] \to M \) and \( \eta : M \to \mathbb{Z}_p \) are the canonical maps. Hence from (3.19) and Lemma 3.7 one concludes the following.

**Proposition 3.8.** Let \( G \) be a finitely generated pro-\( p \) group with a \( \mathbb{Z}_p \)-direction \( \sigma : G \xrightarrow{\tau} \mathbb{Z}_p \xrightarrow{\rho} G \). Then \( \sigma \) has a complementary module.

**3.5. Elementary properties of complementary modules.** Complementary modules have the following elementary properties.

**Proposition 3.9.** Let \( G \) be a finitely generated pro-\( p \) group with a \( \mathbb{Z}_p \)-direction \( \sigma : G \xrightarrow{\tau} \mathbb{Z}_p \xrightarrow{\rho} G \), and let \( M \) be a complementary left \( \mathbb{Z}_p[G] \)-module with complementary short exact sequence \( 0 \to \mathbb{Z}_p[G] \xrightarrow{\omega} M \xrightarrow{\eta} \mathbb{Z}_p \to 0 \).

(a) One has \( \text{res}^G_N(M) \simeq \mathbb{Z}_p[N] \).
(b) \( M_G \simeq \mathbb{Z}_p \).
(c) The connecting homomorphism \( \delta'_1 : G^{ab} \to \mathbb{Z}_p[G]_G \) is surjective.

**Proof.** Put \( \bar{G} = G/N \).
(a) Applying $\Tor^G_p(F_p[G], -)$ to the commutative diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & Z_p[G] & \xrightarrow{(\alpha-1)} & Z_p[G] & \longrightarrow & 0 \\
\downarrow & & & & & \downarrow & \\
0 & \longrightarrow & Z_p[G] & \xrightarrow{\delta} & M & \xrightarrow{\eta} & Z_p & \longrightarrow & 0
\end{array}$$

with exact rows one obtains

$$\begin{array}{cccccc}
0 & \longrightarrow & F_p[G] & \cong & F_p[G] & \longrightarrow & F_p[G] & \longrightarrow & F_p[G] & \longrightarrow & 0 \\
\downarrow & & & & & & & & & & \\
0 & \longrightarrow & F_p[G] & \xrightarrow{\omega_0} & F_p[G] & \cong & F_p[G] \otimes_G M & \longrightarrow & F_p & \longrightarrow & 0
\end{array}$$

with $\omega_0$ injective. From the commutativity of the left handside square one concludes that $\delta_1$ is surjective, and $\omega_0 = 0$. By Theorem 3.1 $\delta_1$ is an isomorphism. Thus $\Tor_1^G(F_p, \res^G_N(M)) \simeq \Tor_1^G(F_p[F_p], M) = 0$, and $\res^G_N(M)$ is a projective, profinite $Z_p[N]$-module (cf. [1] Cor. 3.2). As $\omega_0 = 0$, one has $F_p \otimes_N \res^G_N(M) \simeq F_p[G] \otimes_G M \simeq F_p$. This shows that $\res^G_N(M) \simeq Z_p[N]$.

For (b) and (c) applying $\Tor^G_p(Z_p, -)$ to the commutative diagram \[3.20\] yields a commutative diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & Z_p & \longrightarrow & Z_p & \longrightarrow & Z_p & \longrightarrow & Z_p & \longrightarrow & 0 \\
\downarrow & & & & & & & & & & \\
0 & \longrightarrow & Z_p & \xrightarrow{\omega'_0} & Z_p & \xrightarrow{\delta'} & Z_p & \cong & Z_p & \longrightarrow & 0
\end{array}$$

Hence $\omega'_0 = 0$, $\delta'$ is surjective and $M_G \simeq Z_p$. \[\square\]

4. Subfunctors of El and Ab

The main purpose of this section is to show that for any finitely generated pro-$p$ group $G$ with a $Z_p$-direction $\delta: G \rightarrow Z_p \rightarrow G$ the coinvariants $M_U$ of a complementary profinite $Z_p[G]$-module $M$ are torsion free (cf. Cor. 4.11). In this section all cohomological Mackey functors we are considering will have values in $Z_p$.prf.

4.1. Normal subgroups. Let $N$ be a closed normal subgroup of the pro-$p$ group $G$. For $U \in G^\delta$ define

(4.1) $\Ab(N, G)_U = (U \cap N) / \cl([U, U]) / \cl([-U, U]) \subseteq \Ab_U$ and

(4.2) $\El(N, G)_U = (U \cap N) \Phi(U) / \Phi(U) \subseteq \El_U$.

Then for $U, V \in G^\delta$, $V \subseteq U$, $i_{U, V}^{Ab}$ maps $\Ab(N, G)_V$ to $\Ab(N, G)_U$, and $i_{U, V}^{Ab}$ maps $\Ab(N, G)_V \rightarrow \Ab(N, G)_U$, and the same applies to $\El(N, G)$. Hence $\Ab(N, G)$ is a subfunctor of $\Ab$, and $\El(N, G)$ is a subfunctor of $\El$. Moreover, one has the following properties.

Fact 4.1. Let $G$ be a pro-$p$ group, let $N$ be a closed normal subgroup of $G$ and put $\tilde{G} = G/N$. Then $\res^G_{G\tilde{N}}(\Ab(N, G))$ and $\res^G_{G\tilde{N}}(\El(N, G))$ are $t$-surjective. Moreover,
one has canonical short exact sequences

\[
\begin{array}{cccccc}
0 & \xrightarrow{\text{res}} & \text{res}_G^G(\text{Ab}(N, G)) & \xrightarrow{\text{res}} & \text{res}_G^G(\text{Ab}(G)) & \xrightarrow{} \text{Ab}(G) & 0 \\
0 & \xrightarrow{\text{res}} & \text{res}_G^G(\text{El}(N, G)) & \xrightarrow{\text{res}} & \text{res}_G^G(\text{El}(G)) & \xrightarrow{} \text{El}(G) & 0
\end{array}
\]

of cohomological \(G\)-Mackey functors.

For the definition of the restriction functor \(\text{res}_G^G(\_\_\_\_\_)\) see \cite{[19]} §3.4.

4.2. Subfunctors of \(\text{El}\) of finite codimension. Let \(G\) be a pro-\(p\) group, and let \(X \subseteq \text{El}\) be a \(G\)-subfunctor of the cohomological \(G\)-Mackey functor \(\text{El}\) of finite codimension, i.e., for every open, normal subgroup \(U\) of \(G\) the map \(j_U: X_U \to U^{\text{ab,el}}\) is given by inclusion, and \(\dim(\ker(j_U)) < \infty\). Put

\[
\Upsilon(U) = \{ u \in U \mid u \Phi(U) \subseteq \ker(j_U) \}
\]

Then \(\Upsilon(U)\) is an open, normal subgroup of \(G\) which is contained in \(U\). We define a family \((G_k)_{k \geq 0}\) of open, normal subgroups of \(G\) associated to \(X\) inductively by \(G_0 = G\) and \(G_k = \Upsilon(G_{k-1})\) for \(k \geq 1\). Then

\[
\mathcal{O}^X(G) = \bigcap_{k \geq 0} G_k
\]

is a closed normal subgroup of \(G\) which will be called the \(X\)-residue of \(G\). The family \((G_k)_{k \geq 0}\) of open, normal subgroups of \(G\) has the following properties.

**Proposition 4.2.** Let \(G\) be a pro-\(p\) group, and let \(X \subseteq \text{El}\) be a \(G\)-subfunctor of the cohomological \(G\)-Mackey functor \(\text{El}\) of finite codimension. Let \(N = \mathcal{O}^X(G)\), \(\bar{G} = G/N\), and let \(\mathcal{M} = \{ G_k \mid k \geq 0 \}\) denote the family of open normal subgroups associated to \(X\). Put \(\bar{G}_k = G_k/N\) and \(\bar{G}_k = \bar{G}_{k-1}\) for \(k \geq 1\). Then for all \(k \geq 0\) one has the following:

(a) \(G_{k+1} = N\Phi(G_k)\).
(b) \(\bar{G}_{k+1} = \Phi(G_k)\).
(c) \(G_k^{\text{ab,el}} \cong \bar{G}_k^{\text{ab,el}}/\ker(j_{G_k})\).

Moreover, one has

\[
\text{res}_G^G(\text{El}(N, G)) \subseteq \text{res}_G^G(X) \quad \text{and} \quad \text{res}_G^G(\text{El}(N, G)) = \text{res}_G^G(X).
\]

In particular, if \(\text{res}_G^G(X)\) is \(t\)-surjective, one has

\[
\text{res}_G^G(\text{El}(N, G)) = \text{res}_G^G(X).
\]

**Proof.** (a) By construction, one has \(\Phi(G_k) \subseteq G_{k+1} \subseteq G_k\). This shows that \(G_{k+1}\) contains \(N\Phi(G_k)\). By definition, the canonical map

\[
t_{\bar{G}_{k+1}, \bar{G}_k}: \bar{G}_k^{\text{ab,el}} \to \bar{G}_{k+1}^{\text{ab,el}}
\]

is the trivial map. Hence \(\bar{G}_{k+1} \subseteq \Phi(\bar{G}_k)\) showing that \(G_{k+1} \subseteq N\Phi(G_k)\).

(b) is a direct consequence of (a).

(c) By construction, \(G_{k+1}/\Phi(G_k) \cong \ker(j_{G_k})\). Hence the third isomorphism theorem yields the claim.

Note that (a) implies that \(\text{res}_G^G(\text{El}(N, G)) = \text{res}_G^G(X)\). Let \(U\) be an open subgroup of \(G\) containing \(N\). By construction, there exists \(k \geq 0\) such that \(G_k \subseteq U\). As \(\text{res}_G^G(\text{El}(N, G))\) is \(t\)-surjective and \(X_{G_k} = N\Phi(G_k)/\Phi(G_k)\), one has \(N\Phi(U)/\Phi(U) = \ker(t_{G_k, U}) \subseteq X_U\). This shows \(\text{[1.6]}\) and \(\text{[4.7]}\). \(\square\)
From the definition of the closed normal subgroup $\mathcal{O}^X(G)$ associated to a $G'$-subfunctor $X$ of $\text{El}$ of finite codimension one concludes the following fact:

**Fact 4.3.** Let $G$ be a pro-$p$ group, and let $X \subseteq \text{El}$ be a $G'$-subfunctor of the cohomological $G'$-Mackey functor $\text{El}$ of finite codimension. Then for an open normal subgroup $U$ of $G$, the closed subgroup $\mathcal{O}^X(U) = \text{res}^G_{U}^{G'}(X)(U)$, is normal in $G$.

**Remark 4.4.** (a) If $G$ is a finitely generated pro-$p$ group, one has $\dim(U_{ab,\text{el}}) < \infty$ for all $U \in G'$. Hence every subfunctor $X \subseteq \text{El}$ is closed and of finite codimension.

(b) Let $X \subseteq \text{El}$ be a smooth subfunctor of finite codimension. Then by part (b) of Proposition 4.2, $\overline{G} = G/\mathcal{O}^X(G)$ is necessarily a finitely generated pro-$p$ group; part (a) implies that for $Y = \text{El}(\mathcal{O}^X(G), G)$ one has $\mathcal{O}^Y(G) = \mathcal{O}^X(G)$.

4.3 Smooth subfunctors of $\text{El}$. Let $G$ be a pro-$p$ group, and let $X \subseteq \text{El}$ be a $G'$-subfunctor of $\text{El}$. Then $X$ will be called smooth, if it is of finite codimension, and if for $\overline{G} = G/\mathcal{O}^X(G)$ the cohomological $\overline{G}$-Mackey functor $\text{res}^{\overline{G}}_{G}(X)$ is $t$-surjective.

**Fact 4.5.** Let $G$ be a pro-$p$ group, and let $N$ be a closed, normal subgroup of $G$ such that $G/N$ is finitely generated. Then $\text{El}(N, G) \subseteq \text{El}$ is smooth.

Let $G$ be a pro-$p$ group, and let $Z \subseteq \text{El}$ be a smooth $G'$-subfunctor, i.e., for $\overline{G} = G/\mathcal{O}^Z(G)$ one has $\text{res}^{G}_{G}(Z) = \text{res}^{G}_{G}(\text{El}(\mathcal{O}^{Z}(G), G))$. We call $U \in G'$ $Z$-unramified, if $\mathcal{O}^Z(G) \subseteq U$. For $Z$-unramified open subgroups one has the following.

**Fact 4.6.** Let $G$ be a pro-$p$ group, let $Z \subseteq \text{El}$ be a smooth $G'$-subfunctor, and let $U, V \in G'$, $U$ $Z$-unramified, $\Phi(U) \subseteq V \subseteq U$. Then $V$ is $Z$-unramified if, and only if, $Z_U \subseteq V/\Phi(U)$, i.e., $\mathcal{O}^Z(G)\Phi(U)$ is the minimal $Z$-unramified, open subgroup of $U$ containing $\Phi(U)$.

The concept of ‘ramification’ can be used to show that $\mathcal{O}$ is order-preserving, i.e., one has the following.

**Proposition 4.7.** Let $G$ be a pro-$p$ group, and let $X, Y \subseteq \text{El}$ be a $G'$-subfunctor of the cohomological $G'$-Mackey functor $\text{El}$ of finite codimension satisfying $X \subseteq Y$. Then $\mathcal{O}^X(G) \subseteq \mathcal{O}^Y(G)$.

**Proof.** Let $(G_k)_{k \geq 0}$ be the family of open normal subgroups of $G$ associated with $Y$, i.e., $G_k = \mathcal{O}^Y(G)\Phi(G_k)$ (cf. Prop. 4.2(a)). Let $Z = \text{El}(\mathcal{O}^{X}(G), G)$. By Fact 4.4 $Z$ is smooth. If $U \in G'$ is $Z$-unramified, one has $\mathcal{O}^Z(G) \subseteq U$, and, by (4.6), also $Z_U \subseteq X_U$. Since $\bigcap_{k \geq 0} G_k = \mathcal{O}^Y(G)$, it suffices to show that $G_k$ is $Z$-unramified for all $k \geq 0$ (cf. Rem. 4.4(b)). We proceed by induction on $k$. For $k = 0$ there is nothing to prove. Suppose that $G_k$ is $Z$-unramified. Then $\Phi(G_k) \subseteq G_{k+1} \subseteq G_k$ and

$$G_{k+1}/\Phi(G_k) = Y_{G_k} \supseteq X_{G_k} \supseteq Z_{G_k}. \tag{4.9}$$

By Fact 4.6 $G_{k+1}$ is $Z$-unramified. This yields the claim. $\square$

From (cf. 19 Cor. B) one concludes the following property.

**Proposition 4.8.** Let $G$ be a pro-$p$ group, and let $X \subseteq \text{El}$ be a $G'$-subfunctor of the cohomological $G'$-Mackey functor $\text{El}$ of finite codimension such that $\text{El}/X$ is $i$-injective. Then $\overline{G} = G/\mathcal{O}^X(G)$ is a free pro-$p$ group of rank $\dim(\text{coker}(j_G))$ and $X$ is smooth.
Proof. Let \((G_k)_{k \geq 0}\) denote the family of open, normal subgroups of \(G\) associated to \(X\), and let \(\tilde{G}_k = G_k/\mathcal{O}(G)\). Then \(\mathcal{M} = \{ \tilde{G}_k \mid k \geq 0 \}\) is a base of neighbourhoods of 1 consisting of open, normal subgroups of \(\tilde{G} = G/\mathcal{O}(G)\). By Fact 4.1 and Proposition 4.2, the cohomological \(\tilde{G}\)-Mackey functor \(\mathcal{E}(\tilde{G})\) restricted to \(\mathcal{M}\) is \(i\)-injective. Hence \(\tilde{G}\) is a free group (cf. [19 Cor. B]) of rank \(\dim(\ker(j_G))\) (cf. Prop. 4.2(c)).

Let \(\tilde{X} = \text{res}^G_{\tilde{G}_k}(X)\), \(Y = \text{res}^G_{\tilde{G}_k}(\mathcal{E}(\mathcal{O}(G), G)\) and \(Z = \text{res}^G_{\tilde{G}_k}(\mathcal{E}(\tilde{G}/\mathcal{E}(\mathcal{O}(G), G)\) which is canonically isomorphic to \(\mathcal{E}(\tilde{G})\) (cf. Fact 4.1). Then, by (4.10), \(\tilde{X}/\tilde{Y}\) is a \(\tilde{G}\)-subfunctor of \(\tilde{Z}\) satisfying \(\tilde{X}/\tilde{Y}\) for all \(k \geq 0\). Moreover, as \(\tilde{G}\) is a free pro-\(p\) group, \(Z = \mathcal{E}(\tilde{G})\) is \(i\)-injective (cf. [19 Cor. B]). Hence \(\tilde{X}/\tilde{Y}\) is \(\tilde{G}\)-injective. Let \(U \subseteq \tilde{G}\). Then as \(\mathcal{M}\) is a Mackey basis - there exists \(k \geq 0\) such that \(\tilde{G}_k \subseteq U\). As \(\tilde{X}/\tilde{Y}: (\tilde{X}/\tilde{Y})_{\tilde{U}} \rightarrow (\tilde{X}/\tilde{Y})_{\tilde{G}_k}\) is injective, one has \((\tilde{X}/\tilde{Y})_{\tilde{U}} = 0\). Hence \(\tilde{X}/\tilde{Y} = 0\), and Fact 4.5 yields the claim. □

4.4. Subfunctors of \(\text{Ab}\). For subfunctors of \(\text{Ab}\) one has the following.

**Proposition 4.9.** Let \(G\) be a finitely generated pro-\(p\) group, and let \(Y \subseteq \text{Ab}\) be a \(\tilde{G}\)-subfunctor of the cohomological \(\tilde{G}\)-Mackey functor \(\text{Ab}\). Let \(X = Y/pY\) and suppose that in the induced commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\gamma} & \text{Ab} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\gamma} & \mathcal{E}
\end{array}
\]

the mapping \(\gamma\) is injective. Let \(\mathcal{M} = \{ G_k \mid k \geq 0 \}\) denote the family of open normal subgroups associated to \(\gamma(X) \subseteq \mathcal{E}\), \(N = \mathcal{O}(X)\) and \(\tilde{G} = G/N\). Put \(\tilde{G}_k = G_k/N\) and \(\mathcal{M} = \{ G_k \mid k \geq 0 \}\). Then for all \(k \geq 0\) one has

\[
Y_{G_k} = N \text{cl}([G_k, G_k]) / \text{cl}([G_k, G_k]),
\]

and

\[
\text{res}^G_{\tilde{G}_k}(\text{Ab}(N, G)) \subseteq \text{res}^G_{\tilde{G}_k}(Y) \quad \text{and} \quad \text{res}^G_{\tilde{G}_k}(\text{Ab}(N, G)) = \text{res}^G_{\tilde{G}_k}(Y).
\]

Moreover, if \(\text{res}^G_{\tilde{G}_k}(Y)\) (resp. \(\text{res}^G_{\tilde{G}_k}(X)\)) is \(t\)-surjective one has

\[
\text{res}^G_{\tilde{G}_k}(\text{Ab}(N, G)) = \text{res}^G_{\tilde{G}_k}(Y).
\]

**Proof.** Since \(\text{res}^G_{\tilde{G}_k}(X)\) is \(t\)-surjective, \(\tilde{Y} = \text{res}^G_{\tilde{G}_k}(Y)\) is \(t\)-surjective (cf. [19 Fact 2.1]). Let \(C_k = \text{cl}([G_k, G_k])\), and let \(V_k\) be the closed normal subgroup of \(G\) satisfying \(V_k/C_k = Y_{G_k}\). From the \(t\)-surjectivity of \(\tilde{Y}\) one concludes that \(V_k = V_{k+1}C_k\). Put \(V = \bigcap_{k \geq 0} V_k\). In particular, as \(V_k \Phi(G_k) = G_k\), one has \(V \subseteq N\).

Put \(Z = \text{Ab}/Y\), i.e., \(Z_{G_k} = G_k/V_k\). By hypothesis, one has a canonical isomorphism \(\mathcal{E}(Z)/\text{im}(\gamma) \simeq Z/pZ\). Since \(t_{G_k+1}^\mathcal{E}/\text{im}(\gamma)\) is the 0-map (cf. Prop. 4.2(b)), one has

\[
G_{k+1}V_k/V_k = \text{im}(t_{G_k+1}^Z_{G_k+1, G_k}) \subseteq p(G_k/V_k).
\]

Thus by induction and (4.14), one concludes that for \(m \geq 0\) one has

\[
G_{k+m}V_k/V_k = \text{im}(t_{G_k+m, G_k}^Z) \subseteq p^m(G_k/V_k).
\]
Proof. One has a commutative diagram of cohomological δ subgroup.

Note that the cofiniteness of The connected short exact sequence Theorem 4.10. (cf. Prop. 4.2(c)).

Let $s$ be a short exact sequence of profinite left $\mathbb{Z}$-modules. By definition, the long exact sequence in homology yields an exact sequence of cohomological $G^t$-Mackey functors

\begin{equation}
\cdots \longrightarrow h_1(M) \xrightarrow{h_1(\tau)} \text{El} \xrightarrow{\delta} h_0(Q) \xrightarrow{h_0(\rho)} h_0(M) \xrightarrow{h_0(\gamma)} h_0(\mathbb{F}_p) \longrightarrow 0.
\end{equation}

The connected short exact sequence $s$ will be called cofinite, if $\text{dim}(\text{im}((\delta_U))) < \infty$ for all $U \in G^t$. Note that if $G$ is finitely generated, every connected short exact sequence is cofinite. We define the canonical quotient associated to the cofinite, connected short exact sequence $s$ by $p_B : G \to G(s)$, where $G(s) = G/\text{im}(h_t(\tau))(G)$. Note that the cofiniteness of $s$ implies that $G(s)$ is a finitely generated pro-$p$ group (cf. Prop. 4.2(c)).

4.6. Modules with torsion free coinvariants. The following theorem will be important for our purpose.

**Theorem 4.10.** Let $G$ be a finitely generated pro-$p$ group, $r > 0$, and let

\begin{equation}
0 \longrightarrow \mathbb{Z}_p[\mathbb{Z}] \xrightarrow{\gamma} M \xrightarrow{\tau} \mathbb{Z}_p \longrightarrow 0
\end{equation}

be a short exact sequence of profinite left $\mathbb{Z}_p[\mathbb{Z}]$-modules such that the induced map $\delta_G : G^\text{ab} \to (\mathbb{Z}_p[\mathbb{Z}]^r)_G$ is surjective. Then $M_U$ is torsion free for every open, normal subgroup $U$ of $G$.

**Proof.** One has a commutative diagram of cohomological $G^t$-Mackey functors

\begin{equation}
\begin{array}{ccccccc}
0 & \longrightarrow & h_1(M) & \xrightarrow{\alpha} & \text{Ab} & \xrightarrow{\delta} & h_0(\mathbb{Z}_p[G]^r) & \xrightarrow{\gamma} & h_0(M) & \longrightarrow & \cdots \\
0 & \xrightarrow{0} & h_1(M/p) & \xrightarrow{\tilde{\alpha}} & \text{El} & \xrightarrow{\tilde{\delta}} & h_0(\mathbb{F}_p[G]^r) & \xrightarrow{\tilde{\gamma}} & h_0(M/p) & \longrightarrow & \cdots
\end{array}
\end{equation}

with exact rows, and $\xi, \phi$ and $\psi$ are surjective. Put $X = \text{im}(\tilde{\alpha} \circ \mu) \subseteq \text{El}$, $Y = \text{im}(\alpha)$ and $Z = \text{im}(\tilde{\alpha})$. Note that the long exact sequence in homology implies that $X = Y/pY$. By definition, $j : X \to \text{El}$ is injective. Moreover, one has $X \subseteq Z \subseteq \text{El}$. Let $Q = h_0(\mathbb{Z}_p[G]^r)$, $\tilde{Q} = h_0(\mathbb{F}_p[G]^r)$, $B = \text{im}(\tilde{\delta})$ and $\tilde{B} = \text{im}(\tilde{\delta})$. By hypothesis, $B_G = Q_G$, and - since $\phi_G$ and $\delta_G$ are surjective - $\tilde{B}_G = \tilde{Q}_G$. By construction, one
has the following commutative diagram

\[
\begin{array}{c}
\text{X} \rightarrow \text{Z} \quad \text{0} \\
\downarrow \quad \downarrow \\
\text{B} \quad \text{B} \quad \text{0}
\end{array}
\]

Let \( U \) be an open, normal subgroup of \( G \). In particular, \( O^X(U) \) and \( O^Z(U) \) are normal in \( G \) (cf. Fact 4.3), and \( O^X(U) \subseteq O^Z(U) \) (cf. Prop. 4.1). Let \( V = U/O^X(U) \) and \( W = U/O^Z(U) \).

As \( B \) is a subfunctor of \( \overline{Q_G} \), \( B \) is \( i \)-injective (cf. [19, Fact 4.3]). Hence \( W \) is a free pro-p group (cf. Prop. 4.8), and \( W_{ab} \) is torsion free.

By Proposition 4.9, one has a canonical isomorphism \( \tau: V_{ab} \rightarrow B_U \). Moreover, for \( K = \tau(O^Z(U) \text{ cl}(U,U))/O^X(U) \text{ cl}(U,U)) \), one has \( B_U/K \simeq W_{ab} \).

Consider the triple \((K, B_U, Q_U)\) of \( Z_p[G/U] \)-lattices. The previously mentioned arguments show that it satisfies hypothesis (i) and (ii) of the sandwich lemma (cf. Lemma 2.1). By (4.18) and (4.19), one has \( \overline{B_U} \simeq B_U/B_U \cap pQ_U \) and

\[
(4.20) \quad \overline{B_U} \simeq W_{ab,el} \simeq B_U/(pB_U + K).
\]

Hence \( B_U \cap pQ_U = pB_U + K \) and the triple satisfies also hypothesis (iii) of Lemma 2.1. Moreover,

\[
(4.21) \quad \text{im}(i_{G, U}) = \text{im}(i_{B, U}) = \text{soc}_{G/U}(\overline{Q_U}).
\]

Hence the triple \((K, B_U, Q_U)\) satisfies the hypothesis (i)-(iv) of the sandwich lemma. Thus \( K = 0 \) and the canonical injection \( B_U \rightarrow Q_U \) is split injective (cf. Lemma 2.1). In particular, \( M_U \) is torsion free.

From Theorem 4.10 one concludes the following.

**Corollary 4.11.** Let \( G \) be a finitely generated pro-p group with a \( Z_p \)-direction \( \partial: G \rightarrow Z_p \rightarrow G \), and let \( M \) be a profinite complementary left \( Z_p[G] \)-module with complementary short exact sequence

\[
(4.22) \quad 0 \longrightarrow \overline{Z_p[G]} \overset{\omega}{\longrightarrow} M \overset{\xi}{\longrightarrow} Z_p \longrightarrow 0.
\]

Then \( M_U \) is torsion free for every open normal subgroup \( U \) of \( G \).

**Proof.** By Proposition 3.9(c), the edge homomorphism \( \overline{\partial}_1: G^{ab} \rightarrow \overline{Z_p[G]} \) is surjective. Hence the claim is a direct consequence of Theorem 4.10. \( \square \)

## 5. Transitive permutation modules for pro-p groups

The scope of this section is to establish a criterion ensuring that a given profinite left \( Z_p[G] \)-module \( M \) for a pro-p group \( G \) is a transitive \( Z_p[G] \)-permutation module (cf. Thm. 5.3), i.e., there exists a closed subgroup \( C \) of \( G \) such that \( M \simeq Z_p[G/C] \).
5.1. Permutation modules of finite $p$-groups. For a finite $p$-group $G$ it is in general quite difficult to give a necessary and sufficient criterion ensuring that a given left $\mathbb{Z}_p[G]$-lattice $M$ is isomorphic to a left $\mathbb{Z}_p[G]$-permutation module. The following sufficient criterion has been established by A. Weiss (cf. [5 Chap. 8, Thm. 2.6], [20 Thm. 2]).

**Theorem 5.1** (A. Weiss). Let $G$ be a finite $p$-group, let $N$ be a normal subgroup of $G$, and let $M$ be a left $\mathbb{Z}_p[G]$-lattice. Assume further that

(i) $\text{res}_N^G(M)$ is a projective left $\mathbb{Z}_p[N]$-module, and

(ii) $M^N$ is a $\mathbb{Z}_p[G/N]$-permutation module.

Then $M$ is a $\mathbb{Z}_p[G]$-permutation module.

5.2. Transitive permutation modules for pro-$p$ groups. For pro-$p$ groups one has the following property.

**Proposition 5.2.** Let $G$ be a pro-$p$ group, and let $B = \{U_i \mid i \in I\}$ be a basis of neighbourhoods of $1 \in G$ consisting of open normal subgroups of $G$. Let $M$ be a profinite left $\mathbb{Z}_p[G]$-module with the property that $M_{U_i}$, $i \in I$, is a transitive left $\mathbb{Z}_p[G/U_i]$-permutation module. Then $M$ is a transitive left $\mathbb{Z}_p[G]$-permutation module, i.e., there exists a closed subgroup $C$ of $G$ such that $M \simeq \mathbb{Z}_p[G/C]$.

**Proof.** Let $M_i = M_{U_i}$, and let $\tau_i : M \rightarrow M$ and $\pi_i : M_i \rightarrow M_i/pM_i$, $i \in I$, denote the canonical projections. Let $d_i = \text{rk}(M_i)$, and let $P_{d_i}(M_i)$ be the set of all subsets of $M_i$ of cardinality $d_i$. We define $X_i \subset P_{d_i}(M_i)$ to be the set of all subsets $S \subset P_{d_i}(M_i)$ such that $S$ is $G$-invariant, consists of a single $G$-orbit, and $\pi_i(S) = \{ \pi_i(s) \mid s \in S \}$ is an $\mathbb{F}_p$-basis of $M_i/pM_i$. Then $X_i$ is a profinite set, and by hypothesis - $X = \lim_{\longleftarrow} X_i$ is a non-empty profinite set. By construction, $\Omega \in X$ is a $G$-set. Moreover, as $\tau_i(\Omega)$ is a transitive left $G$-set for all $i \in I$, $M = \mathbb{Z}_p[\Omega]$ is a transitive left $\mathbb{Z}_p[G]$-permutation module. \qed

From Proposition 5.2 one concludes the following pro-$p$ version of A. Weiss theorem.

**Theorem 5.3.** Let $G$ be a pro-$p$ group, let $N$ be a closed normal subgroup of $G$, and let $M$ be a profinite left $\mathbb{Z}_p[G]$-module with the following properties:

(i) $M_U$ is a torsionfree abelian pro-$p$ group for every open, normal subgroup $U$ of $G$; and

(ii) $\text{res}_N^G(M) \simeq \mathbb{Z}_p[N]$.

Then $M$ is a transitive $\mathbb{Z}_p[G]$-permutation module. In particular, there exists a closed subgroup $C$ of $G$ which is an $N$-complement, i.e., $G = C.N$ and $C \cap N = \{1\}$.

**Proof.** Let $B$ be a basis of neighbourhoods of $1 \in G$ consisting of open, normal subgroups of $G$. For $U \in B$, hypothesis (ii) implies that $M_{N \cup U}$ is a free $\mathbb{Z}_p$-module of rank $|N : N \cup U|$. As $M_{N \cup U} = \lim_{\longleftarrow} \bigcup_{U \subseteq V \subseteq G} M_{(N \cup U)V}$, and all mapping in the inverse limits are surjective, one concludes that there exists an element $V(U) \in B$, $V(U) \subseteq U$, such that the canonical map $M_{N \cup U} \rightarrow M_{(N \cup U)V(U)}$ is an isomorphism. Put $M(U) = M_{(N \cup U)V(U)}$. Note that $\{(N \cup U)V(U) \mid U \in B\}$ is again a basis of neighbourhoods of $1 \in G$. By construction, one has

$$N \cap ((N \cup U)V(U)) = (N \cup V(U)).(N \cap U) = N \cap U. \tag{5.1}$$

Put $\bar{G}_U = G/(N \cap U)V(U)$, and

$$\bar{B}_U = NV(U)/(N \cap U)V(U) \simeq N/(N \cap ((N \cup U)V(U)) \simeq N/N \cap U. \tag{5.2}$$
As \( \text{res}^{G/N_U}_N(M_{NCU}) \simeq \text{res}^{G_U}_N(M(U)) \) is an isomorphism of left \( Z_p[N/N_U] \)-modules, one has \( \text{res}^{G_U}_N(M(U)) \simeq Z_p[N_U] \). The map

\[
N_{\bar{B}U} : M(U)_{\bar{B}U} \rightarrow M(U)^{\bar{B}U}, \quad N_{\bar{B}U}(m + \omega_{\bar{B}U} M) = \sum_{b \in \bar{B}U} b m,
\]

where \( \omega_{\bar{B}U} M \) is the augmentation ideal of \( Z_p[\bar{B}U] \), is an isomorphism of left \( Z_p[\bar{G}_U] \)-modules. In particular, as \( M(U)^{\bar{B}U} \simeq Z_p \) and \( M(U)_{\bar{G}_U} \simeq Z_p \) by hypothesis, \( M(U)^{\bar{B}U} \) is a trivial left \( Z_p[\bar{G}_U] \)-module. Hence \( M(U)^{\bar{G}_U} \simeq Z_p \). Thus by A. Weiss’ theorem (cf. Thm. 5.1), \( M(U) \) is a transitive \( Z_p[G_U] \)-module. Applying Proposition 5.2 to the base \{\((N \cap U)V(U) \mid U \in \mathcal{B}\)\} shows that \( M \) is a transitive \( Z_p[G] \)-module, i.e., there exists a closed subgroup \( C \) of \( G \) such that \( M \simeq Z_p[G/C] \). As \( M_N \simeq Z_p \), \( N \) must act transitively on \( G/C \). Since \( \text{res}^G_N(Z_p[G/C]) \simeq Z_p[N] \), \( N \) is also acting fixed point freely on \( G/C \). Hence \( G = NC \) and \( C \cap N = \{1\} \). This yields the claim. \( \square \)

5.3. Free factors isomorphic to \( Z_p \). Let \( G \) be a pro-\( p \) group with a semi-direct factor \( \bar{G} : G \rightarrow Z_p \rightarrow G \). Then \( \bar{G} \) is called a free factor isomorphic to \( Z_p \) if there exists a pro-\( p \) group \( A \) and homomorphisms of pro-\( p \) groups \( A \rightarrow G \rightarrow A \) such that \( A \rightarrow G \rightarrow Z_p \) is a free pro-\( p \) product, i.e., for every pro-\( p \) group \( H \) and homomorphisms of pro-\( p \) groups \( \chi : A \rightarrow H \), \( \gamma : Z_p \rightarrow H \), there exists a unique homomorphism of pro-\( p \) groups \( \theta = \theta(\chi, \gamma) : G \rightarrow H \) making the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\beta} & G \\
\downarrow{\chi} & & \downarrow{\gamma} \\
 & \xrightarrow{\theta} & H
\end{array}
\]

commute. Moreover, one has \( \alpha = \theta(\text{id}_A, 1_{Z_p}, A) \) and \( \tau = \theta(1_{A, Z_p}, \text{id}_{Z_p}) \), where \( 1_{Z_p, A} : Z_p \rightarrow A \) and \( 1_{A, Z_p} : A \rightarrow Z_p \) denote the trivial homomorphisms.

**Theorem 5.4.** Let \( G \) be a finitely generated pro-\( p \) group, and let \( \bar{G} : G \rightarrow Z_p \rightarrow G \) be a semi-direct factor isomorphic to \( Z_p \). Then \( \bar{G} \) is a free factor isomorphic to \( Z_p \) if, and only if, \( \bar{G} \) is a \( Z_p \)-direction.

**Proof.** Let \( \bar{G} : G \rightarrow Z_p \rightarrow G \) be a \( Z_p \)-direction, and let

\[
0 \rightarrow Z_p[G] \xrightarrow{\bar{\iota}} M \xrightarrow{\bar{\xi}} Z_p \rightarrow 0
\]

be a complementary exact sequence associated with \( \bar{G} \) (cf. Prop. 3.8 and 3.10). Put \( \Sigma = \text{im}(\sigma) \) and \( N = \text{cl}(\langle \Sigma \mid g \in G \rangle) \). Then, by Theorem 3.4 \( N \) is a free pro-\( p \) subgroup, and \( N^\text{ab,el} \simeq F_p[N] \) as left \( F_p[N] \)-module.

Since \( M \) is a complementary \( Z_p[G] \)-module, one has \( \text{res}^G_N(M) \simeq Z_p[N] \) (cf. Prop. 3.3(2)). By Corollary 4.11 \( M_U \) is torsion free for every open, normal subgroup \( U \) of \( G \). Thus, by Theorem 5.3, \( M \simeq Z_p[G/C] \) is a transitive left \( Z_p[G] \)-module, and \( C \) is an \( N \)-complement in \( G \), i.e., \( G = CN \) and \( C \cap N = \{1\} \). In particular, the extension of pro-\( p \) groups

\[
\begin{array}{cccc}
\{1\} & \rightarrow N & \rightarrow G & \xrightarrow{\alpha} G/N \rightarrow \{1\}
\end{array}
\]

splits, and \( \beta = (\alpha|C)^{-1} \) is a section for \( \alpha \). Moreover, \( G \simeq Z_p \# G/N \) is a free pro-\( p \) product (cf. Thm. 3.2). and \( \bar{G} \) is a free factor isomorphic to \( Z_p \).
Suppose that $\bar{\sigma}$ is a free factor isomorphic to $\mathbb{Z}_p$, and let $A \xrightarrow{\beta} G \xrightarrow{\sigma} A$ denote the associated homomorphisms of pro-$p$ groups. As before put $\Sigma = \text{im}(\sigma)$. Then $G$ acts on a pro-$p$ tree $T$ with two orbits on vertices and transitive on the profinite set of geometric edges (cf. [19, Thm. 4.1]). In particular, one has a short exact sequence of left $\mathbb{Z}_p[G]$-modules (cf. [22, (1.15)])

$$0 \longrightarrow \mathbb{Z}_p[G] \longrightarrow \mathbb{Z}_p[G/A] \oplus \mathbb{Z}_p[G/\Sigma] \xrightarrow{\varepsilon_{A+\Sigma}} \mathbb{Z}_p \longrightarrow 0. \tag{5.7}$$

Let $X = h_1(Z_p[G/\Sigma]) \in \text{ob}(\text{cM}_{G}(G^2, z_p\text{prf}))$ (cf. [19, §3.8]), i.e., for $U \in G^2$ one has

$$X_U = \text{Tor}^G_1(Z_p[U\setminus G], Z_p[G/\Sigma]) \simeq \text{Tor}^G_1(Z_p[Z_p[U\setminus G]), Z_p) \tag{5.8}$$

$$\simeq \text{Ext}^G_0(Z_p[G/U], Z_p) \simeq \text{Hom}_G(Z_p[G/U], Z_p), \tag{5.9}$$

where $\text{coind}^G_\Sigma(Z_p)[\Sigma] \text{dis} \longrightarrow Z_p[G] \text{dis}$ denotes coinduction (cf. [13, §1.2.5]). In particular, one has an isomorphism of cohomological $G^2$-Mackey functors

$$X \simeq h^0(\text{coind}^G_\Sigma(Z_p)) \simeq \text{ind}^G_\Sigma(T(\Sigma, Z_p)), \tag{5.10}$$

(cf. [19, Ex. 3.5(c)]), i.e., $X_U$ is torsion free for all $U \in G^2$, and $X$ is Hilbert '90 (cf. [19, Fact 3.8]). From these properties one concludes that $X_p = X/pX$ is of type $H^0$ (cf. [19, Fact 4.9]), and the canonical map $j_{X_p}: (X_p)_G \to \text{ires}(X_p)$ is injective.

Since $h_1(Z_p[G]) = 0$ and $h_0(Z_p[G])_U$ is torsion free for all $U \in G^2$, the maps $h_1(\varepsilon_{\Sigma}): X_U \to \text{Ab}_U$ are split injective for all $U \in G^2$. Let

$$h_1(\varepsilon_{\Sigma})_{tf}: X \longrightarrow \text{TFAb} \tag{5.11}$$

denote the induced natural transformation (cf. [23, Ex. 4]). Then $h_1(\varepsilon_{\Sigma})_{tf}$ is injective, and $h_1(\varepsilon_{\Sigma})_{tf}: X_U \to U^{ab,tf}$ is split injective for all $U \in G^2$. In particular, the induced natural transformation $h_1(\varepsilon_{\Sigma})_{tf}/p: X_p \to \text{TFEl}$ is injective. Hence $\text{ires}(h_1(\varepsilon_{\Sigma})_{tf}/p): \text{ires}(X_p) \to D^f_p(F_p)$ is injective, and $j_G(\sigma(1)\Phi(G)) \neq 0$, i.e., $\bar{\sigma}$ is a $\mathbb{Z}_p$-direction.

\section{6. Virtually free pro-$p$ products}

\subsection{6.1. The fundamental pro-$p$ group}

Let $(G, \Gamma, v_0)$ be a finite graph of finitely generated pro-$p$ groups. The \emph{fundamental pro-$p$ group} $G = \Pi_1(G, \Gamma, v_0)$ of it will be the pro-$p$ completion of the usual fundamental group $\pi_1(G, \Gamma, v_0)$ (cf. [12, §5.1]). In general, $\pi_1(G, \Gamma, v_0)$ does not have to be residually $p$, but this will be the case in all of our considerations. In particular, edge and vertex groups will be subgroups of $\Pi_1(G, \Gamma, v_0)$.

Let $(G, \Gamma, v_0)$ be a finite graph of finite $p$-groups. By [22, Thm. 3.10], every finite subgroup of $G = \Pi_1(G, \Gamma, v_0)$ is conjugate to a subgroup of a vertex group of $(G, \Gamma, v_0)$. Hence $G$ has only finitely many finite subgroups up to conjugation. In particular, every maximal finite subgroup of $G$ is $G$-conjugate to a vertex group of $(G, \Gamma, v_0)$. In order to make the converse also true one has to modify the graph of
groups (without changing the fundamental group) by collapsing superfluous edges in the graph \( \Gamma \), i.e., geometric edges \( \{e, \bar{e}\} \) which are not loops and for which either \( \alpha_e : \mathcal{G}(e) \to \mathcal{G}(t(e)) \) or \( \alpha_{\bar{e}} : \mathcal{G}(e) \to \mathcal{G}(o(e)) \) is an isomorphism. Such an edge can be collapsed without changing the fundamental group, i.e., \( \{e, \bar{e}\} \) will be removed from the edge set of \( \Gamma \), and \( o(e) \) will be identified with \( t(e) \) in a point \( y \). For the induced graph of groups \( \mathcal{G}' \) one defines \( \mathcal{G}'(y) = \mathcal{G}(o(e)) \) if \( \alpha_e \) is an isomorphism, and \( \mathcal{G}'(y) = \mathcal{G}(t(e)) \) if \( \alpha_{\bar{e}} \) is not an isomorphism. This procedure can be continued until \( \alpha_e \) is not surjective for all edges not defining loops and does not change the fundamental group \( \Pi_1(\mathcal{G}, \Gamma, v_0) \). For short we call a graph of groups with the just mentioned property reduced. For such a graph of groups every vertex group becomes a maximal finite subgroup of \( \mathcal{G} = \Pi_1(\mathcal{G}, \Gamma, v_0) \).

6.2. Virtually free pro-\( p \) groups. The following properties are well known.

**Proposition 6.1.** Let \( H = \bigsqcup_{i \in I} H_i \bigsqcup F \) be a free pro-\( p \) product of finite \( p \)-groups \( H_i \) and a free pro-\( p \) group \( F \):

(a) Any finite subgroup of \( H \) is conjugate to a subgroup of \( H_i \) for some \( i \in I \);

(b) \( H_i \cap H_j^p = 1 \) if either \( i \neq j \) or \( h \not\in H_i \);

(c) For \( K \subseteq H_i, K \neq \{1\} \), one has \( N_H(K) \subseteq H_i \); in particular, \( N_H(K) \) is finite.

**Proof.** For (a) and (b) see Theorems 4.2 (a) and 4.3 (a) in [10]. In order to prove (c) take \( h \in N_G(K) \). Then \( K \subseteq H_i \cap H_i^p \), and, by (b), one has \( h \in H_i \). \( \square \)

From Proposition 6.1 one concludes the following properties for virtual free pro-\( p \) products.

**Proposition 6.2.** Let \( (\mathcal{G}, \Gamma) \) be a reduced finite graph of finite \( p \)-groups, and suppose that \( G = \Pi_1(\mathcal{G}, \Gamma, v_0) \) contains an open, normal subgroup \( H = F \bigsqcup H_1 \bigsqcup \cdots \bigsqcup H_s \), \( H_i \neq \{1\} \), for some free pro-\( p \) subgroup \( F \) of rank \( r \), \( 0 \leq r < \infty \), such that \( r+s \geq 2 \). Then one has the following.

(a) For any edge \( e \) of \( \Gamma \) one has \( \mathcal{G}(e) \cap H = \{1\} \); in particular, \( |\mathcal{G}(e)| \leq |G : H| \).

(b) \( |E(\Gamma)| \leq 2(r+s) - 1 \) and \( |V(\Gamma)| \leq 2(r+s) \), where \( V(\Gamma) \) is the set of vertices of \( \Gamma \), and \( E(\Gamma) \) is the set of geometric edges of \( \Gamma \).

**Proof.** Put \( X = \pi_1(\mathcal{G}, \Gamma, v_0) \) and \( Y = X \cap H \). Hence \( G \) and \( H \) are the pro-\( p \) completions of \( X \) and \( Y \), respectively. Moreover, \( |X : Y| = |G : H| \).

(a) Suppose that \( \mathcal{G}(e) \cap H \neq \{1\} \). Since \( H \) is normal in \( G \), \( N_G(\mathcal{G}(e)) \) normalizes \( \mathcal{G}(e) \cap H \). We claim that \( N_G(\mathcal{G}(e)) \) is infinite. Indeed one has to distinguish two cases: Case 1: \( \{e, \bar{e}\} \) is not a loop. In this case \( N_G(\mathcal{G}(e)) \) contains the infinite group \( \langle N_{\mathcal{G}(v)}(\mathcal{G}(e)), N_{\mathcal{G}(w)}(\mathcal{G}(e)) \rangle \), where \( v = o(e), w = t(e) \). Case 2: \( \{e, \bar{e}\} \) is a loop. Let \( v = t(e) = o(e) \), and let \( z_e \in G \) be the stable letter associated with \( e \). If \( \mathcal{G}(e) = \mathcal{G}(v) \), then \( N_G(\mathcal{G}(e)) \) contains the infinite group \( \langle z_e \rangle \). Otherwise \( N_G(\mathcal{G}(e)) \) contains the infinite group \( \langle N_{\mathcal{G}(v)}(\mathcal{G}(e)), z_e, N_{\mathcal{G}(v)}(\mathcal{G}(e))z_e^{-1} \rangle \).

Since \( |G : H| < \infty \), the fact that \( N_G(\mathcal{G}(e)) \) is infinite implies that \( N_H(\mathcal{G}(e) \cap H) = N_G(\mathcal{G}(e) \cap H) \cap H \) is infinite as well contradicting Proposition 6.1 (c). Hence one has \( \mathcal{G}(e) \cap H = \{1\} \) as required.
(b) It suffices to show the first inequality. By \([12, §2.6, \text{Ex. 3}]\), one has

\[
-\chi_X = \sum_{e \in E(\Gamma)} \frac{1}{|G(e)|} - \sum_{v \in V(\Gamma)} \frac{1}{|G(v)|}
\]

\[
(6.1)
\]

\[
= \frac{1}{|X : Y|} \cdot \chi_Y
\]

\[
= \frac{1}{|X : Y|} \cdot \left( r + s - 1 - \sum_{1 \leq i \leq s} \frac{1}{|H_i|} \right).
\]

Thus one obtains

\[
(6.2)
\]

\[
 r + s - 1 \geq |X : Y| \left( \sum_{e \in E(\Gamma)} \frac{1}{|G(e)|} - \sum_{v \in V(\Gamma)} \frac{1}{|G(v)|} \right)
\]

As \(\Gamma\) is reduced, for every edge \(e\) in a maximal subtree \(T\) of \(\Gamma\) the edge group \(G(e)\) is isomorphic to a proper subgroup of \(G(t(e))\). Hence, \(|G(t(e))| \geq 2|G(e)|\). Taking into account that \(|E(T)| = |V(\Gamma)| - 1\), one concludes from (a) that

\[
(6.3)
\]

\[
 r + s - 1 \geq \frac{1}{2} \cdot \sum_{e \in E(\Gamma) \setminus \{f\}} \frac{|X : Y|}{|G(e)|} \geq \frac{1}{2} \cdot (|E(\Gamma)| - 1).
\]

Here \(f\) is any (geometric) edge in the maximal subtree \(T\). This yields the claim. \(\Box\)

From Proposition 6.2 one concludes the following straightforward fact.

**Corollary 6.3.** Let \((G, \Gamma)\) be a reduced finite graph of finite \(p\)-groups, and suppose that \(G = \Pi_1(G, \Gamma, v_0)\) contains a free open subgroup \(H\) of rank \(r \geq 2\). Then there are finitely many reduced finite graphs of finite \(p\)-groups \((G', \Gamma')\) such that \(G \simeq \Pi_1(G', \Gamma', w_0)\).

The proof of the structure theorem for virtual free \(pro\)-\(p\) products (cf. Thm. 6.5) in the subsequent subsection is based on the following result due to W.N. Herfort and the second author.

**Theorem 6.4.** (cf. [4, Thm. 1.1]) Let \(G\) be a finitely generated \(pro\)-\(p\) group with a free open subgroup \(F\). Then \(G\) is the fundamental group of a finite graph of \(pro\)-\(p\) groups of order bounded by \(|G : F|\).

6.3. **Virtual free \(pro\)-\(p\) products.** A \(pro\)-\(p\) group \(G\) will be called a free \(pro\)-\(p\) product if there exist non-trivial closed subgroups \(A\) and \(B\) such that \(G = A \amalg B\). The following theorem gives a description of the structure of virtual free \(pro\)-\(p\) products.

**Theorem 6.5.** Let \(G\) be a finitely generated \(pro\)-\(p\) group containing an open subgroup \(H = F \amalg H_1 \amalg \cdots \amalg H_s\) for some free \(pro\)-\(p\) subgroup \(F\) of rank \(r < \infty\) and non-trivial subgroups \(H_i\) such that \(r + s \geq 2\). Then \(G\) is isomorphic to the \(pro\)-\(p\) fundamental group of a finite graph of \(pro\)-\(p\) groups with finite edge stabilizers.

**Proof.** Note that by hypothesis the subgroups \(H_i, 1 \leq i \leq s\), are finitely generated. By replacing \(H\) by its core and applying the Kurosh subgroup theorem for open subgroups (cf. [9, Thm. 9.1.9]), we may assume that \(H\) is normal in \(G\).
**Step 1:** Let \( \mathcal{B} \) be a basis of neighbourhoods of \( 1_G \in G \) consisting of open normal subgroups \( U \) of \( G \) which are contained in \( H \) with \( H \) \( \not\subseteq U \) for every \( i = 1, \ldots, s \). For \( U \in \mathcal{B} \) put
\[
\tilde{U} = \text{cl}(\{ U \cap H_i^g \mid g \in G, \ 1 \leq i \leq s \}).
\]
Then \( \tilde{U} \) is a closed, normal subgroup of \( G \), and \( G/\tilde{U} \) contains the open normal subgroup
\[
H/\tilde{U} = F \sqcup H_1 \tilde{U}/\tilde{U} \sqcup \cdots \sqcup H_s \tilde{U}/\tilde{U}
\]
(cf. [7, Prop. 1.18]). In particular, \( H/\tilde{U} \) is a finitely generated, virtually free pro-\( p \) group, and thus \( G/\tilde{U} \) is a finitely generated, virtually free pro-\( p \) group.

**Step 2:** By Theorem [6.2] \( G/\tilde{U} \) is isomorphic to the fundamental group \( \Pi_1(\mathcal{G}_U, \Gamma_U, v_U) \) of a finite graph of finite \( p \)-groups. Using the procedure described in subsection 6.1 we may assume that \( \langle \mathcal{G}_U, \Gamma_U, v_U \rangle \) is reduced. Hence from now on we may assume that for every \( U \in \mathcal{B} \) the vertex groups of \( G/\tilde{U} = \Pi_1(\mathcal{G}_U, \Gamma_U, v_U) \) are representatives of the \( G/\tilde{U} \)-conjugacy classes of maximal finite subgroups. Note that by Proposition [6.2] a, one has \( \mathcal{G}_U(e) \cap H/\tilde{U} = 1 \).

**Step 3:** By [11] Prop. 1.10], for \( V \subseteq U \) both open and normal in \( G \) the decomposition \( G/V = \Pi_1(\mathcal{G}_V, \Gamma_V, v_V) \) gives rise to a natural decomposition of \( G/\tilde{U} \) as the fundamental group \( G/\tilde{U} = \Pi_1(\mathcal{G}_{V, U}, \Gamma_V, v_V) \) of a graph of groups \( (\mathcal{G}_{V, U}, \Gamma_V) \) as follows: \( \mathcal{G}_{V, U}(x) = \mathcal{G}_V(x)\tilde{U}/\tilde{V} \sqcup \tilde{V}/\tilde{U} \) for every \( x \in \Gamma_V \). Thus we have a morphism \( \eta: (\mathcal{G}_V, \Gamma_V) \rightarrow (\mathcal{G}_{V, U}, \Gamma_V) \) of graph of groups such that the induced homomorphism on the pro-\( p \) fundamental groups coincides with the canonical projection \( \varphi_{UV}: G/\tilde{V} \rightarrow G/\tilde{U} \).

**Claim.** Let \( U, V \in \mathcal{B}, V \subseteq U \). If \( (\mathcal{G}_V, \Gamma_V) \) is reduced, then \( (\mathcal{G}_{V, U}, \Gamma_V) \) is reduced.

**Proof.** Suppose that there exists an edge \( e \in \Gamma_V \) which is not a loop such that for \( v = \ell(e) \) one has \( \mathcal{G}_V(v)\tilde{U}/\tilde{V} = \mathcal{G}_V(e)\tilde{U}/\tilde{V} \subseteq G/\tilde{V} \sqcup \tilde{V}/\tilde{U} \). Then by the second isomorphism theorem
\[
\mathcal{G}_V(v) = \mathcal{G}_V(e)(\tilde{U}/\tilde{V}) \cap \mathcal{G}_V(v).
\]
By the definition of \( \mathcal{B} \), one has \( \{1\} \neq (\tilde{U}/\tilde{V}) \cap \mathcal{G}_V(v) \subseteq H/\tilde{V} \). Since \( H/\tilde{V} = F \sqcup H_1 \tilde{V}/\tilde{U} \sqcup \cdots \sqcup H_s \tilde{V}/\tilde{U} \), Proposition [6.1] a implies that \( (\tilde{U}/\tilde{V}) \cap \mathcal{G}_V(v) \) is contained in some \( \tilde{g}H_i \tilde{V}/\tilde{V} \). Note that \( \mathcal{G}_V(e) \subseteq \tilde{N}_{G/\tilde{V}}((\tilde{U}/\tilde{V}) \cap \mathcal{G}_V(v)) \). So for \( y \in \mathcal{G}_V(e) \), one has \( (\tilde{g}H_i \tilde{V}/\tilde{V}) \cap (\tilde{g}H_i \tilde{V}/\tilde{V}) = \{1\} \). Hence by Proposition [6.1] b, \( \tilde{g}H_i \tilde{V}/\tilde{V} = \tilde{g}H_i \tilde{V}/\tilde{V} \) and thus \( \mathcal{G}_V(e) \subseteq \tilde{N}_{G/\tilde{V}}(\tilde{g}H_i \tilde{V}/\tilde{V}) \). In particular, \( \mathcal{G}_V(e)\tilde{g}H_i \tilde{V}/\tilde{V} \) is a finite subgroup of \( G/\tilde{V} \) containing \( \mathcal{G}_V(v) \) (cf. (6.6)). Hence, by the maximality of \( \mathcal{G}_V(v) \), one has
\[
\mathcal{G}_V(e)((\tilde{U}/\tilde{V}) \cap \mathcal{G}_V(v)) = \mathcal{G}_V(v) = \mathcal{G}_V(e)\tilde{g}H_i \tilde{V}/\tilde{V}.
\]
Since \( (\tilde{U}/\tilde{V}) \cap \mathcal{G}_V(v) \subseteq \tilde{g}H_i \tilde{V}/\tilde{V} \), and as \( \mathcal{G}_V(e) \cap \tilde{g}H_i \tilde{V}/\tilde{V} = \{1\} \) by Proposition [6.2] a, one concludes that \( \tilde{g}H_i \tilde{V}/\tilde{V} \subseteq \tilde{U}/\tilde{V} \), contradicting the choice of \( \mathcal{B} \). \(\square\)

**Step 4:** By Proposition [6.2] the number \( |V(\Gamma_U)| + |E(\Gamma_U)| \) is bounded by \( 4(r + s) - 1 \). Therefore, by passing to a cofinal system \( \mathcal{C} \) of \( \mathcal{B} \) if necessary, we may assume that \( \Gamma_U = \Gamma \) for each \( U \in \mathcal{C} \). Then, by Corollary [6.3] the number of isomorphism classes of finite reduced graphs of finite \( p \)-groups \( (\mathcal{G}_U, \Gamma) \) which are based on \( \Gamma \) and
satisfy $G/\tilde{U} \cong \Pi_1(G', \Gamma, v_0)$ is finite. Suppose that $\Omega_U$ is a set containing a copy of every such isomorphism class. For $V \in \mathcal{C}$, $V \subseteq U$, one has a map $\omega_{V,U}: \Omega_V \to \Omega_U$ (cf. Claim in Step 2). Hence $\Omega = \lim_{\to \in C} \Omega_U$ is non-empty. Let $(G'_U, \Gamma) \in C \subseteq \Omega$. Then $(G'_U, \Gamma)$ given by $G'_U(x) = \lim_{\to \in C} G'_U(x)$ if $x$ is either a vertex or an edge of $\Gamma$, is a reduced finite graph of finitely generated pro-$p$ groups satisfying $G \cong \Pi_1(G', \Gamma, v_0)$. By Proposition 6.2(a), $G'(e)$ is finite for every edge $e$ of $\Gamma$. This yields the claim. □

From Theorem 5.1 one concludes the following.

Corollary 6.6. Let $G$ be a finitely generated torsion free pro-$p$ group which is a virtual free pro-$p$ product. Then $G$ is a free pro-$p$ product.

6.4. The final conclusion.

Proof of Theorem B. Let $G$ be a finitely generated pro-$p$ group with $e(G) = \infty$. Hence, by [3, 13], there exists an open subgroup $U$ of $G$ such that $j_{\text{TFEl}}: \text{TFEl}_U \to D^1_{\text{TFEl}}(p_\mathfrak{p})$ is non-trivial. Let $x \in \text{TFEl}_{U}$ be such that $j_{\text{TFEl}}(x + p \text{TFEl}_{U}) \neq 0$. By construction, $\mathbb{Z}_p x \subseteq \text{TFEl}_{U}$ is a direct summand of $\text{TFEl}_{U}$, i.e., there exists $D \subseteq \text{TFEl}_{U}$ such that $\text{TFEl}_{U} = \mathbb{Z}_p x \oplus D$. Let $\tilde{D} \subseteq G$ be the canonical preimage of $D$ in $U$, let $\hat{x}$ be a preimage of $x$ in $U$, and let $\tau: U \to \mathbb{Z}_p$ be the surjective homomorphism given by $\tau(\hat{x}) = 1$ and $\tilde{D} = \ker(\tau)$. Then for $\sigma: \mathbb{Z}_p \to U$, $\sigma(1) = \hat{x}$, the semi-direct factor $U \overset{\tau}{\to} \mathbb{Z}_p \overset{\tau}{\to} U$ is a $\mathbb{Z}_p$-direction. In particular, by Theorem 6.4 there exists a closed subgroup $C$ of $U$ such that $U = \mathbb{Z}_p \overset{\tau}{\to} C$. As $G$ is not virtually cyclic (cf. Theorem A), $C \neq \{1\}$. Hence $U$ is a free pro-$p$ product. By Theorem 6.5, $G \cong \Pi_1(G, \Gamma, v_0)$ for some finite reduced graph of finitely generated pro-$p$ groups and all edge groups $G(e)$ are finite $p$-groups. Removing a (geometric) edge $e$ from the graph $\Gamma$ we are left with two cases for the resulting subgraph $\Gamma_e$: (i) $\Gamma_e$ is disconnected in which case $G$ splits as an amalgamated free pro-$p$ product over the edge group $G(e)$; or (ii) $\Gamma_e$ is connected in which case $G$ splits as an HNN-extension over $G(e)$. □

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