Intersection Theory on the Moduli Space of Holomorphic Curves with Lagrangian Boundary Conditions

Jake P. Solomon *

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Abstract

We define a new family of open Gromov-Witten type invariants based on intersection theory on the moduli space of pseudoholomorphic curves of arbitrary genus with boundary in a Lagrangian submanifold. We assume the Lagrangian submanifold arises as the fixed points of an anti-symplectic involution and has dimension 2 or 3. In the strongly semi-positive genus 0 case, the new invariants coincide with Welschinger’s invariant counts of real pseudoholomorphic curves. Furthermore, we calculate the new invariant for the real quintic threefold in genus 0 and degree 1 to be 30.

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*Massachusetts Institute of Technology; email: jake@math.mit.edu.
1 Introduction

1.1 The main idea

In 1985, Gromov initiated the study of pseudoholomorphic curves in symplectic geometry with his seminal paper [7]. Motivated by applications of Gromov’s techniques in string theory, Witten developed a systematic way of organizing pseudoholomorphic curve information, later known as Gromov-Witten invariants [24, 25]. Over the following decade, mathematicians including Ruan-Tian [20], McDuff-Salamon [17], Li-Tian [15] and Fukaya-Ono [5], successfully established a rigorous foundation for Gromov-Witten invariants. Concurrently, Kontsevich [13, 12] initiated research that eventually succeeded in calculating the Gromov-Witten invariants in many situations.

We briefly recall the definition of Gromov-Witten invariants. Note that throughout the introductory portion of this paper, we assume all moduli spaces have expected dimension in order to simplify the exposition. Let \((X, \omega)\) be a symplectic manifold and denote by \(J_\omega\) the set of \(\omega\)-tame almost complex structures on \(X\). Fix a generic \(J \in J_\omega\). For \(d \in H_2(X)\), let \(\overline{M}_{g,n}(X, d)\) denote the Gromov-compactification of the moduli space of \(J\)-holomorphic maps from a surface of genus \(g\) to \(X\) representing \(d\) together with a choice of \(n\) marked points on the domain. There exist canonical evaluation maps

\[ev_i : \overline{M}_{g,n}(X, d) \rightarrow X.\]
Furthermore, denoting by \( \overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n}(\text{pt}, 0) \) the Deligne-Mumford compactification of the moduli space of genus \( g \) curves with \( n \) marked points, there exists a canonical projection

\[
\pi : \overline{\mathcal{M}}_{g,n}(X, d) \to \overline{\mathcal{M}}_{g,n}.
\]

Let \( A_i \in H^*(X) \) and \( B \in H^*(\overline{\mathcal{M}}_{g,n}) \). Choose differential forms \( \alpha_i \in \Omega^*(X) \) and \( \beta \in \Omega^*(\overline{\mathcal{M}}_{g,n}) \) such that \( [\alpha_i] = A_i \). The genus \( g \) Gromov-Witten invariant of \( X \) for cohomology classes \( A_i, B \), takes the form of the integral

\[
\int_{\overline{\mathcal{M}}_{g,n}(X, d)} \ev_1^*(\alpha_1) \wedge \ldots \wedge \ev_n^*(\alpha_n) \wedge \pi^*(\beta).
\]

It follows from Stokes’s Theorem and the fact that \( \overline{\mathcal{M}}_{g,n}(X, d) \) is a closed orbifold that this integral does not depend on the choice of the forms \( \alpha_i, \beta \), or the choice of \( J \in \mathcal{J}_\omega \). Hence, it is an invariant of the deformation class of \( \omega \) parametrized by the cohomology classes \( A_i, B \). Roughly speaking, the Gromov-Witten invariants of \( X \) count the number of \( J \)-holomorphic maps from a Riemann surface of fixed genus \( g \) to \( X \) representing \( d \) and intersecting fixed generic representatives of \( PD(A_i) \). The class \( B \) can be used to fix the conformal structure on the domain Riemann surface or the relative position of the marked points.

Now, let \( L \subset X \) be a Lagrangian submanifold, and let \( (\Sigma, \partial \Sigma) \) be a Riemann surface with boundary. For some time, physicists \([1, 14, 19, 26]\) have predicted the existence of “open” Gromov-Witten invariants counting pseudoholomorphic maps \( (\Sigma, \partial \Sigma) \to (X, L) \) satisfying certain incidence conditions. These invariants would naturally generalize classical Gromov-Witten invariants to include maps from Riemann surfaces with boundary. Note, however, that according to \([1]\) it may be necessary to specify some additional structure on \( L \) in order to uniquely determine the invariants. Katz and Liu took a first step toward the mathematical definition of open invariants given the additional structure of an \( S^1 \) action on the pair \( (X, L) \) \([10, 16]\). However, the existence of such an \( S^1 \) action is a rather restrictive condition.

Before entering a more detailed discussion, let us briefly establish some necessary notation. In the following, we denote by \( \Sigma \) a Riemann surface with boundary with fixed conformal structure. This avoids the issue of degenerations of \( \Sigma \), which the author plans to treat in another paper in the near future. For \( d \in H_2(X, L) \), let \( \mathcal{M}_{k,l}(L, \Sigma, d) \) be the moduli space of configurations of \( k \) distinct marked points in \( \partial \Sigma \), \( l \) distinct marked points in \( \Sigma \).
and \( J \)-holomorphic maps \( u : (\Sigma, \partial \Sigma) \to (X, L) \) such that \( u_*([\Sigma, \partial \Sigma]) = d \). In this moduli space, points which are equivalent by automorphisms of \( \Sigma \) are identified. We denote by \( \overline{\mathcal{M}}_{k,l}(L, \Sigma, d) \) the Gromov compactification of \( \mathcal{M}_{k,l}(L, \Sigma, d) \). Finally, we denote by \( \text{ev}_b : \overline{\mathcal{M}}_{k,l}(L, \Sigma, d) \to L, i = 1 \ldots k, \) \( \text{ev}_i : \overline{\mathcal{M}}_{k,l}(L, \Sigma, d) \to X, j = 1 \ldots l, \) the canonical evaluation maps at the marked points.

From a mathematical perspective, two main difficulties have obstructed progress on open invariants: orientation and bubbling in codimension one. Indeed, Fukaya et al. \([6]\) showed that \( \mathcal{M}_{k,l}(L, \Sigma, d) \) need not be orientable. In the same paper, they proved orientability if \( L \) is orientable and “relatively spin.” However, in many interesting examples, i.e. \((X, L) = (\mathbb{C}P^2, \mathbb{R}P^2)\), \( L \) is not orientable and neither is \( \mathcal{M}_{k,l}(L, \Sigma, d) \). In Theorem 1.1 we show that even if \( L \) is not orientable, under reasonable assumptions, the orientation bundle of \( L \) pulls-back to the orientation bundle of \( \mathcal{M}_{k,l}(L, \Sigma, d) \) under the map \( \prod_i \text{ev}_i \). This allows us to pull-back differential forms with values in the orientation bundle of \( L \), wedge and integrate.

Considerably more troublesome is the problem of bubbling in codimension one. Put differently, \( \overline{\mathcal{M}}_{k,l}(L, \Sigma, d) \) is an orbifold with corners. Intuitively, one should think of a manifold with many boundary components. The boundary consists of codimension one strata of the Gromov-compactification. This stands in contrast to the moduli space associated to a closed surface \( \overline{\mathcal{M}}_{g,n}(X, d) \), which has no boundary since all strata of the Gromov-compactification have codimension two or more. By analogy to the classical Gromov Witten invariants, we would like to define invariants parametrized by cohomology classes \( A_i \in H^*(L) \) and \( C_i \in H^*(X) \). Choose \( \alpha_i \in \Omega^*(L) \) with \([\alpha_i] = A_i\) and \( \gamma_j \in \Omega^*(X) \) with \([\gamma_j] = C_i\). The desired invariant should take the form

\[
\int_{\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)} \text{ev}_1^*(\alpha_1) \wedge \ldots \wedge \text{ev}_k^*(\alpha_k) \wedge \text{ev}_1^*(\gamma_1) \wedge \ldots \wedge \text{ev}_l^*(\gamma_l). \tag{1}
\]

However, trouble arises in trying to prove independence of the choices of \( \alpha_i, \gamma_j \). For example, suppose \( \alpha_i' \) also satisfies \([\alpha_i'] = A_i\). Then \( \alpha_i - \alpha_i' = d\delta \) for some \( \delta \) and hence

\[
\text{ev}_1^*(\alpha_1) \wedge \ldots \wedge \text{ev}_l^*(\gamma_l) - \text{ev}_1^*(\alpha_1') \wedge \ldots \wedge \text{ev}_l^*(\gamma_l) =
\]

\[
d (\text{ev}_1^*(\delta) \wedge \text{ev}_2^*(\alpha_2) \wedge \ldots \wedge \text{ev}_l^*(\gamma_l)).
\]
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We would like to integrate the right-hand side of the above equation over $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)$ to obtain zero by Stokes’s theorem. However, contributions from the integral of $ev_1^*(\delta) \wedge ev_2^*(\alpha_2) \wedge \ldots \wedge ev_l^*(\gamma_l)$ over the boundary of $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)$ may spoil this vanishing. So, the integral (1) may depend on the choice of $\alpha_i$, $\gamma_j$.

Now, let us assume there exists an anti-symplectic involution

$$\phi : X \to X, \quad \phi^*\omega = -\omega,$$

such that $L = Fix(\phi)$. We limit our discussion to the special case that $A_i \in H^{dim L}(L)$, $C_j \in H^{dim X}(X)$ and $\dim X \leq 6$. If $L$ is not orientable, we assume $\dim L \leq 4$. Consequently, we can actually prove independence of (1) from the choice of $\alpha_i$, $\gamma_j$.

We proceed to explain the idea of the proof. The extra structure $\phi$ enters the definition of the invariants through the almost complex structure. Indeed, we define

$$\mathcal{J}_{\omega, \phi} := \{ J \in \mathcal{J}_\omega | \phi^*J = -J \}.$$

In the following, we fix a generic $J \in \mathcal{J}_{\omega, \phi}$. Let $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)^{(1)}$ denote the union of the codimension one strata of $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)$, that is, strata consisting of two-component stable-maps. Think of $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)^{(1)}$ as the boundary of $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)$. Recall that it may have many connected components. We identify a subset of these components, the union of which we refer to as $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)^{(1a)}$, satisfying the following properties:

- There exists an orientation reversing involution $\tilde{\phi}_2$ of $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)^{(1a)}$ that does not preserve any single connected component. Hence the quotient $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d) := \overline{\mathcal{M}}_{k,l}(L, \Sigma, d)/\tilde{\phi}_2(x) \sim x$ carries a natural orientation. Here, we use the $\phi$ invariance of $J$.

- The forms $ev_1^*(\alpha_i)$, $ev_1^*(\gamma_j)$ and $ev_1^*(\delta)$ descend naturally to

$$\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)$$

under the assumption that the $\gamma_j$ are $\phi$ invariant.

- The differential form $ev_1^*(\delta) \wedge ev_2^*(\alpha_2) \wedge \ldots \wedge ev_l^*(\gamma_l)$ has support away from the boundary of $\overline{\mathcal{M}}_{k,l}(L, \Sigma, d)$. 

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The independence of the integral \(\int\) of the choices of \(\beta_i\) and \(\gamma_j\) follows immediately from Stokes’s theorem: Indeed, we may replace the domain of integration in \(\int\) with \(\hat{M}_{k,l}(L, \Sigma, d)\). Since \(evb_1^*(\delta) \wedge evb_2^*(\alpha_2) \wedge \cdots \wedge evi_l^*(\gamma_l)\) vanishes on the boundary of \(\hat{M}_{k,l}(L, \Sigma, d)\),

\[
\int_{\hat{M}_{k,l}(L, \Sigma, d)} d (evb_1^*(\delta) \wedge evb_2^*(\alpha_2) \wedge \cdots \wedge evi_l^*(\gamma_l)) = 0
\]
as desired. Independence of the choice of \(J \in J_{\omega,\phi}\) follows by a similar argument. Note however, that a priori the invariant so obtained should depend on the choice of \(\phi\). We may interpret the choice of \(\phi\) as the extra parameter involved in defining open invariants predicted by [1].

1.2 The definition

In the following, we denote by \((X, \omega)\) a symplectic manifold of dimension \(2n\) and by \(L \subset X\) a Lagrangian submanifold. Let \(J_{\omega}\) denote the set of \(\omega\)-tame almost complex structures on \(X\), and let \(J \in J_{\omega}\). Let \(P\) denote the set of \(J\)-anti-linear inhomogeneous perturbation terms generalizing those introduced by Ruan and Tian in [20], and let \(v \in P\). See Section 4 for more details. Fix a Riemann surface with boundary \((\Sigma, \partial \Sigma)\), let \(M_{\Sigma}\) denote the moduli space of conformal structures on \((\Sigma, \partial \Sigma)\), and fix \(j \in M_{\Sigma}\). Suppose \(\partial \Sigma = \bigsqcup_{a=1}^m (\partial \Sigma)_a\), where \((\partial \Sigma)_a \simeq S^1\). Let

\[
d = (d, d_1, \ldots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m},
\]

let \(k = (k_1, \ldots, k_m) \in \mathbb{N}^m\) and let \(l \in \mathbb{N}\). By \(M_{k,l}(L, \Sigma, d)\), we denote the moduli space of \((j, J, v)\)-holomorphic maps \(u : (\Sigma, \partial \Sigma) \to (X, L)\) with \(k_a\) marked points on \((\partial \Sigma)_a\) and \(l\) marked points on \(\Sigma\) such that \(u_*([\Sigma, \partial \Sigma]) = d\) and \(u|_{(\partial \Sigma)_a*([\partial \Sigma])} = d_a\). Let \(\overline{M}_{k,l}(L, \Sigma, d)\) denote its Gromov compactification. There exist natural evaluation maps

\[
evb_{ai} : \overline{M}_{k,l}(L, \Sigma, d) \to L, \quad i = 1 \ldots k_a, \ a = 1 \ldots m,\]

\[
evi_j : \overline{M}_{k,l}(L, \Sigma, d) \to X, \quad j = 1 \ldots l.
\]

We now digress for a moment to discuss the notion of a relatively \(Pin^\pm\) Lagrangian submanifold. Let \(V \to B\) be a vector bundle. Define the characteristic classes \(p^\pm(V) \in H^2(B, \mathbb{Z}/2\mathbb{Z})\) by

\[
p^+(V) = w_2(V), \quad p^-(V) = w_2(V) + w_1(V)^2.
\]
According to \cite{11}, $p^\pm(V)$ is the obstruction to the existence of a $Pin^\pm$ structure on $V$. See \cite{11} for a detailed discussion of the definition of the groups $Pin^\pm$ and the notion of $Pin^\pm$ structures.

Now, suppose $(X,\omega)$ is a symplectic manifold and $L \subset X$ is a Lagrangian submanifold. Note that we do not assume $L$ is the fixed points of an anti-symplectic involution yet. We say that $L$ is relatively $Pin^\pm$ if

$$p^\pm(TL) \in \text{Im} \left( i^* : H^2(X) \to H^2(L) \right).$$

and $Pin^\pm$ if $p^\pm(TL) = 0$. If $L$ is $Pin^\pm$, we define a $Pin^\pm$ structure for $L$ to be a $Pin^\pm$ structure for $TL$. If $L$ is relatively $Pin^\pm$, a relative $Pin^\pm$ structure for $L$ consists of the choice of a triangulation for the pair $(X, L)$, an oriented vector bundle $V$ over the three skeleton of $X$ such that $w_2(V) = p^\pm(TL)$ and a $Pin^\pm$ structure on $TL|_{L\cap} \oplus V|_L$. Note that the definition of a relative $Pin^\pm$ structure given here directly generalizes the notion of a relative $Spin$ structure given in \cite{6}.

By the Wu relations \cite{18}, if $n \leq 3$ then $p^-(TL) = 0$, so that $L$ is always $Pin^-$. It follows that for the applications considered in this paper, we need only consider honest $Pin^\pm$ structures. However, we state Theorem 1.1 in full generality, since that requires little extra effort.

**Theorem 1.1.** Assume $L$ is relatively $Pin^\pm$ and fix a relative $Pin^\pm$ structure on $(X, L)$. If $L$ is not orientable, assume $k_a \equiv w_1(d_a) + 1 \mod 2$. If $L$ is orientable, fix an orientation. The relative $Pin^\pm$ structure on $(X, L)$ and the orientation of $L$ if $L$ is orientable canonically determine an isomorphism

$$\det(T\overline{M}_{k,l}(L, \Sigma, d)) \sim \bigotimes_{a,i} ev_{\alpha_i}^* \det(TL).$$

**Remark 1.2.** This theorem was proved in \cite{6} in the case that $L$ is orientable.

Under the assumptions of Theorem 1.1 we define an invariant as follows. Let $H^*(L, \det(TL))$ denote the cohomology of $L$ with coefficients in the flat line bundle $\det(TL)$. Poincare duality will hold whether or not $L$ is orientable. Let $\Omega^*(L, \det(TL))$ denote differential forms on $L$ with values in $\det(TL)$, and let $\Omega^*(X)$ denote ordinary differential forms on $X$. For $a = 1, \ldots, m$, and $i = 1, \ldots, k_a$, let $\alpha_{ai} \in \Omega^*(L, \det(TL))$ represent the Poincare dual of a point in $H^n(L, \det(TL))$. Furthermore, for $j = 1, \ldots, l$, let $\gamma_j \in \Omega^{2n}(X)$ represent the Poincare dual of twice the point-class. We define

$$N_{\Sigma,d,k,l} := \int_{\overline{M}_{k,l}(L, \Sigma, d)} ev_{11}^* \alpha_{11} \wedge \ldots \wedge ev_{mk}^* \alpha_{mk} \wedge ev_{1}^* \gamma_1 \wedge \ldots \wedge ev_{m}^* \gamma_m.$$
This integral makes sense because by Theorem 1.1, the integrand is a differential form taking values in the orientation line bundle of the moduli space over which it is to be integrated. Let $\mu : H_2(X, L) \to \mathbb{Z}$ denote the Maslov index as defined in [3]. Denote by $g$ the genus of the closed Riemann surface $\Sigma \cup_{\partial \Sigma} \Sigma$ obtained by doubling $\Sigma$. Furthermore, we employ the shorthand $|k| = k_1 + \ldots + k_m$. We note that by calculating the expected dimension of $\mathcal{M}_{k,l}(L, \Sigma, d)$, it follows that unless

$$(n - 1)(|k| + 2l) = n(1 - g) + \mu(d) - \dim \text{Aut}(\Sigma) \quad (2)$$

the above integral must vanish.

Now, suppose there exists an anti-symplectic involution $\phi : X \to X$, such that $L = \text{Fix}(\phi)$. We define $\mathcal{J}_{\omega,\phi}$ to be the set of $J \in \mathcal{J}_{\omega}$ such that $\phi^*J = -J$. Define

$$\Omega^*_{\phi}(X) := \{ \gamma \in \Omega^*(X) | \phi^* \gamma = \gamma \}.$$

Furthermore, define $\tilde{h} = h \circ r$ where $h : \pi_2(X) \to H_2(X)$ is the Hurewicz homomorphism and $r : H_2(X) \to H_2(X, L)$ is the natural homomorphism.

Assume that $\dim X = 4$, and if $L$ is not orientable assume $\dim X \leq 4$. If $\dim X = 4$, assume that $k_a \cong w_1(d_a) + 1 \mod 2$. Note that these assumptions imply the hypothesis of Theorem 1.1. If $\Sigma = D^2$ and $k = 0$ assume that

$d \notin \text{Im} \left( \tilde{h} : \pi_2(X) \to H_2(X, L) \right) \quad (3)$

This is necessary to avoid a certain type of bubbling that requires taking into consideration real curves with empty real part. The author intends to treat such bubbling in another paper in the near future.

**Theorem 1.3.** The integers $N_{\Sigma, d, k, l}$ do not depend on the choice of $J \in \mathcal{J}_{\omega,\phi}$, $\nu \in \mathcal{P}$, $j \in \mathcal{M}_\Sigma$, the choice of $\alpha_{ai} \in \Omega^*(L, \det(TL))$ or the choice of $\gamma_j \in \Omega^*_\phi(X)$. That is, the numbers $N_{\Sigma, d, k, l}$ are invariants of the triple $(X, \omega, \phi)$.

**Remark 1.4.** The condition that $k_a \cong w_1(d_a) + 1 \mod 2$ when $L$ is not orientable is redundant if $g = 0$, as it can easily be derived from the dimension condition (2).

**Remark 1.5.** The definition of the integers $N_{\Sigma, d, k, l}$ does not use $\phi$ or the condition that $\dim X \leq 6$ in an essential way. The author believes that there exist far more general conditions under which similarly defined integers are invariant.
We now present an example of a non-trivial calculation of these invariants. See also Section 1.3 where we develop the relationship with Welschinger invariants, for which many interesting calculations have already been carried out.

**Example 1.6.** Let \((X, L)\) be the pair consisting of the quintic threefold and its real part. That is, \(X := \{\sum_{i=0}^{4} z_i^5 = 0\} \subset \mathbb{C}P^4\) equipped with the symplectic form coming from the restriction of the Fubini-Study metric, and \(L := X \cap \mathbb{R}P^4\). Let \(\ell \in H_2(X, L)\) denote the generator with positive symplectic area. It is not hard to see that \(\ell\) satisfies condition (3). We calculate \(N_{D^2,\ell,0,0} = 30\). This may be interpreted as the number of oriented lines in the real quintic. It is interesting to compare this with the classical computation of 2875 lines in the complex quintic.

**1.3 The relationship with real algebraic geometry**

Real algebraic geometry provides a rich source of examples of symplectic manifolds admitting anti-symplectic involutions. Indeed, given any smooth real projective algebraic variety, we can take \(X\) to be its complexification, \(\omega\) to be the pull-back of the Fubini-Study metric and \(\phi\) to be complex conjugation. For this reason, it makes sense to call triples \((X, \omega, \phi)\) real symplectic manifolds. Fix an \(\omega\)-compatible almost complex structure \(J\) such that \(\phi^*J = -J\). Let \(\Sigma\) be a Riemann surface with an anti-holomorphic involution \(c : \Sigma \to \Sigma\), and let \(\nu\) be a \(c - \phi\) equivariant inhomogeneous perturbation. We can define real \((J, \nu)\)-holomorphic curves to be \((J, \nu)\)-holomorphic maps \(u : \Sigma \to X\) such that \(\phi \circ u \circ c = u \circ a\) for some \(a \in Aut(\Sigma, \nu)\). Note that a given Riemann surface may have several different anti-holomorphic involutions. So, when we need to specify that a curve is real with respect to a particular anti-holomorphic involution, we use the terminology \(c\)-real.

Now, suppose \((\Sigma', \partial \Sigma')\) is a Riemann surface with boundary such that \(\Sigma \simeq \Sigma' \cup_{\partial \Sigma'} \Sigma\) and \(c\) acts by exchanging \(\Sigma'\) and \(\Sigma\). Any \(c\)-real \((J, \nu)\)-holomorphic curve \(u : \Sigma \to X\), must satisfy

\[
u^{-1}(Fix(\phi)) = Fix(c) \simeq \partial \Sigma'.
\]

Since, \(Fix(c)\) divides \(\Sigma\) into \(\Sigma'\) and \(\Sigma\), restricting \(u\) to either \(\Sigma'\) or \(\Sigma\) gives a \((J, \nu)\)-holomorphic curve with boundary in the Lagrangian submanifold \(L = Fix(\phi)\). Conversely, given a \((J, \nu)\)-holomorphic map \(u' : (\Sigma', \partial \Sigma') \to (X, L)\), we can construct a \(c\)-real \((J, \nu)\)-holomorphic map \(\Sigma \to X\) by gluing \(u'\) and
\( \phi \circ u' : (\overline{\Sigma'}, \partial \overline{\Sigma'}) \to (X, L) \) along their common boundary \( \partial \Sigma' \) by the Schwarz reflection principle.

Let us denote by \( \overline{\mathcal{M}}_n(X, \Sigma, d) \) the Gromov-compactification of the space of \((J, \nu)\)-holomorphic maps \( \Sigma \to X \) with \( n \) marked points and let \( \overline{\mathcal{M}}_c(X, d) \) denote its \( c \)-real locus. Let \( r : H_2(X) \to H_2(X, L) \). We have just shown there exists a canonical map

\[
\prod_{k, |k|=n, d', 2d'=r(d), \sum_a d'_a=\partial d'} \overline{\mathcal{M}}_{k,0}(L, \Sigma', d') \to \overline{\mathcal{M}}_c(X, \Sigma, d).
\]

If \( (\Sigma', \nu|_{\Sigma'}) \) is not biholomorphic to \( (\overline{\Sigma'}, \nu|_{\overline{\Sigma'}}) \), this map is 1 : 1 on the open stratum. If \( (\Sigma', \nu|_{\Sigma'}) \) is biholomorphic to \( (\overline{\Sigma'}, \nu|_{\overline{\Sigma'}}) \), then restricted to the open stratum, this map is a 2 : 1 covering map. As an immediate consequence, we have the following proposition:

**Proposition 1.7.** If \( (\Sigma', \nu|_{\Sigma'}) \) is not biholomorphic to \( (\overline{\Sigma'}, \nu|_{\overline{\Sigma'}}) \), the number of \( c \)-real \((J, \nu)\)-holomorphic maps \( \Sigma \to X \) intersecting \( n \) generic real points of \( X \) is bounded below by

\[
\sum_{k, |k|=n, d', 2d'=j_*d, \sum_a d'_a=\partial d'} N_{\Sigma', d', k, l}. \tag{4}
\]

If \( (\Sigma', \nu|_{\Sigma'}) \) is biholomorphic to \( (\overline{\Sigma'}, \nu|_{\overline{\Sigma'}}) \), then we should take one half of (4) as a lower bound instead.

In [21, 22, 23], for strongly semi-positive real symplectic manifolds, Welschinger defined invariants counting real rational \( J \)-holomorphic curves intersecting a generic \( \phi \)-invariant collection of marked points. Unlike in the usual definition of Gromov-Witten invariants, which depends on intersection theory on the moduli space of \( J \)-holomorphic curves, Welschinger defined his curve count by assigning signs to individual curves based on certain geometric-topological criteria. However, it turns out that Welschinger’s invariants admit the following intersection theoretic interpretation:

**Theorem 1.8.** Let \( X \) be a strongly semi-positive real symplectic manifold satisfying the assumptions of Theorem 1.3. Then the numbers \( N_{D^2,d,k,l} \) are twice the corresponding Welschinger invariant.
1.4 An overview of the paper

In Section 2, we define the notion of a Cauchy-Riemann Pin boundary value problem, and prove that the determinant of the associated Fredholm operator admits a natural orientation. Furthermore, we examine how this orientation is effected by certain changes in the underlying boundary value problem and determine when it is preserved by the conjugation morphism. In Section 3, we apply the results of Section 2 to the determinant bundle of the $\bar{\partial}$ operator over the moduli space of $W^{1,p}$ maps $(\Sigma, \partial \Sigma) \to (X, L)$. Section 4 explains the precise definition of the moduli space of stable maps we consider, concludes the proof of Theorem 1.1, and rigorously defines the invariants. Section 5 calculates the sign of the involution $\tilde{\phi}_2$ of Section 1.1 on the boundary of the moduli space. In Section 6, assuming that the Maslov index of all bubbles is strictly positive, we analyze the boundary of the moduli space and use the results of Section 5 to prove Theorem 1.3. Section 7 employs the Kuranishi structure developed in [5, 6] to treat the case when the Maslov index of bubbles may be 0. For the reader who is not familiar with Kuranishi structures, we summarize the relevant definitions and ideas in Appendix A. Finally, in Section 8 we introduce the notion of a short exact sequence of Cauchy-Riemann boundary value problems. We determine how the associated isomorphisms of determinant lines behave with respect to the orientation of Section 3 and apply the results to prove Theorem 1.8 and Example 1.6.

The author circulated an announcement of the results of this paper and began giving talks about it in late 2005. Late in the preparation of the full manuscript, the author learned of work of Cho [4] that addresses Theorem 1.3 in the strongly semi-positive genus 0 case when $L$ is orientable.

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2 Orienting Cauchy-Riemann operators

In this section we analyze the choices necessary to orient the determinant of a real-linear Cauchy-Riemann operator. In the following, we use the symbol $\Gamma$ to denote an appropriate Banach space completion of the smooth sections of a vector bundle. The exact choice of completion will not be important. If $V \to B$ is a vector bundle, we denote by $\mathfrak{F}(V)$ the principal $O(n)$ bundle with fiber at $x \in B$ given by the set of orthonormal frames in $V_x$. We call $\mathfrak{F}(V)$ the frame-bundle of $V$.

**Definition 2.1.** A $Pin^\pm$ structure $p = (P, p)$ on a vector bundle $V \to B$ consists of principal $Pin^\pm$ bundle $P \to B$ and a $Pin^\pm(n)\cdot O(n)$ equivariant bundle map

$$p : P \to \mathfrak{F}(V).$$

A morphism of vector bundles with $Pin$ structure $\phi : V \to V'$ is said to preserve $Pin$ structure if there exists a lifting $\tilde{\phi}$,

$$
\begin{array}{ccc}
P & \xrightarrow{\phi} & P' \\
\mathfrak{F}(V) & \xrightarrow{\phi} & \mathfrak{F}(V').
\end{array}
$$

**Definition 2.2.** A Cauchy-Riemann $Pin$ boundary value problem

$$\mathcal{D} = (\Sigma, E, F, p, D)$$

consists of

- A Riemann surface $\Sigma$ with boundary $\partial \Sigma = \bigsqcup_{a=1}^m (\partial \Sigma)_a$, $(\partial \Sigma)_a \simeq S^1$.

- A complex vector bundle $E \to \Sigma$.

- A totally real sub-bundle over the boundary

$$
\begin{array}{ccc}
F & \to & E \\
\downarrow & & \downarrow \\
\partial \Sigma & \to & \Sigma.
\end{array}
$$

- A $Pin^\pm$ structure $p$ on $F$. 

An orientation of \( F|_{(\partial\Sigma)_a} \) for each \( a \) such that \( F|_{(\partial\Sigma)_a} \) is orientable.

A differential operator

\[
D : \Gamma((\Sigma, \partial), (E, F)) \to \Gamma(\Sigma, \Omega^{0,1}(E)),
\]

satisfying, for \( \xi \in \Gamma((\Sigma, \partial), (E, F)) \) and \( f \in C^\infty(\Sigma, \mathbb{R}) \),

\[
D(f \xi) = f D \xi + (\bar{\partial} f) \xi.
\]

Such a \( D \) is known as a real-linear Cauchy-Riemann operator.

When it does not cause confusion, we will refer to such a collection by the operator alone, i.e. \( D \), leaving the domain and range implicit.

**Definition 2.3.** A morphism of Cauchy-Riemann Pin boundary value problems \( \phi : D \to D' \) consists of

- A biholomorphism \( f : \Sigma \to \Sigma' \).
- A morphism of bundles \( \phi : E \to E' \) covering \( f \) such that \( \phi|_{\partial\Sigma} \) takes \( F \) to \( F' \) and \( \phi \circ D = D' \circ \phi \).

Such a morphism is called an isomorphism if \( \phi \) is an isomorphism of vector bundles preserving Pin structure and preserving orientation if \( F, F' \), are orientable. When it causes no confusion, we may refer to such a morphism by the bundle-morphism component alone, i.e. \( \phi \).

**Definition 2.4.** We define the determinant line of a Fredholm operator \( D \) by

\[
\det(D) := \Lambda^{\max}(\ker D) \otimes \Lambda^{\max}(\text{coker } D)^*.
\]

If \( D \) is a family of Fredholm operators, then we denote by \( \det(D) \) the corresponding line bundle with the natural topology, as explained in, for example, [17, appendix A.2].

We now briefly recall the definition of the Maslov index \( \mu(E, F) \) of the vector bundle pair \( (E, F) \) appearing in the definition of a Cauchy-Riemann boundary value problem in the case that \( \partial\Sigma \neq \emptyset \). Indeed, if \( \partial\Sigma \neq \emptyset \), we may trivialize \( E \) over \( \Sigma \). Writing

\[
\partial\Sigma = \prod_{a=1}^m (\partial\Sigma)_a, \quad (\partial\Sigma)_a \simeq S^1,
\]
the restriction of \( F \) to each boundary component \( (\partial \Sigma)_a \) defines a loop of totally real subspaces of \( \mathbb{C}^n \). The Maslov index \( \mu_a \) of such a loop was defined in [2]. We define

\[
\mu(E, F) = \sum_{a=1}^{m} \mu_a.
\]

It is not hard to see that although \( \mu_a \) may depend on the choice of trivialization of \( E \), the sum \( \mu(E, F) \) does not. On the other hand, \( \mu_a \) is well defined \((\text{mod } 2)\), and coincides with the first Steifel-Whitney class \( w_1(F|_{(\partial \Sigma)_a}) \). To properly understand the results of this paper, it is useful to know the following topological classification of vector-bundle pairs \((E, F)\).

**Lemma 2.5.** Two vector bundle pairs \((E, F)\) and \((E', F')\) of the same dimension admit an isomorphism

\[
\begin{array}{ccc}
E & \overset{\sim}{\longrightarrow} & E' \\
\downarrow \phi & & \downarrow \phi' \\
F & \overset{\sim}{\longrightarrow} & F'
\end{array}
\]

if and only if

\[
\mu(E, F) = \mu(E', F'), \quad w_1(F) = w_1(F').
\]

Now, we describe a canonical orientation for the determinant line of a number of special examples of Cauchy-Riemann operators. Let \( \tau \to \mathbb{C}P^1 \) denote the tautological bundle. Let \( c' : \mathbb{C}P^1 \to \mathbb{C}P^1 \) and \( c : \tau \to \tau \) denote the automorphisms induced by complex conjugation. The fixed points of \( c' \) are simply \( \mathbb{R}P^1 \), and they divide \( \mathbb{C}P^1 \) into two copies of \( D^2 \). We define \( \tau_\mathbb{R} \) to be the fixed points of \( c' \) on \( \tau|_{\mathbb{R}P^1} \). Furthermore, let \( \mathbb{C}^n \to D^2 \) denote the trivial bundle and let \( \mathbb{R}^n \to \mathbb{C}^n \) denote the canonical trivial real sub-bundle. We define basic Cauchy-Riemann \( Pin \) boundary value problems

\[
\begin{align*}
D(-1, n) &:= (D^2, \tau|_{D^2} \oplus \mathbb{C}^{n-1}, \tau_\mathbb{R} \oplus \mathbb{R}^{n-1}, D_{-1,n}, p_{-1}), \\
D(0, n) &:= (D^2, \mathbb{C}^n, \mathbb{R}^n, D_{0,n}, p_0).
\end{align*}
\]

Here, the \( Pin^\pm \) structure \( p_0 \) is canonically induced by the splitting into line bundles. On the other hand, \( p_{-1} \) is not canonical \( per se \), but we fix one choice and remain with it for the rest of the paper. \( D_{-1,n} \) and \( D_{0,n} \) are taken to be
the standard Cauchy-Riemann operators on these bundles. Since $D_{-1,n}$ and $D_{0,n}$ are surjective, we have

$$
\det(D_{-1,n}) = \Lambda^{\text{max}}(\ker(D_{-1,n})) = \Lambda^{\text{max}}(\mathbb{R}^{n-1}),
$$

$$
\det(D_{0,n}) = \Lambda^{\text{max}}(\ker(D_{0,n})) = \Lambda^{\text{max}}(\mathbb{R}^n).
$$

So, $\det(D_{-1,n})$ and $\det(D_{0,n})$ admit canonical orientations.

Finally, we will need the following special automorphisms $Q(i)$ of the bundle pair $(\mathbb{C}^2, \mathbb{R}^2) \to (D^2, \partial D^2)$. We define the restriction of $Q(i)$ to $\mathbb{R}^2 \to \partial D^2$ to be given by a loop in $SO(2)$ with homotopy class $i \in \pi_1(O(2)) \simeq \mathbb{Z}$. We define $Q(i)$ to be an arbitrary extension of this automorphism over the inside of $D^2$ using the fact that the inclusion $SO(2) \hookrightarrow U(2)$ induces the trivial map on the fundamental group.

**Lemma 2.6.** If $n \geq 3$, automorphisms of the bundle pair

$$(\mathbb{C}^n, \mathbb{R}^n) \to (D^2, \partial D^2)$$

preserving Pin structure and orientation are all homotopic to the identity. If $n = 2$, automorphisms preserving Pin structure and orientation are homotopic to $Q(2i)$, $i \in \mathbb{Z}$. For all $n$, there are two homotopy classes of automorphisms of

$$(\tau \oplus \mathbb{C}^{n-1}, \tau_\mathbb{R} \oplus \mathbb{R}^{n-1}) \to (D^2, \partial D^2)$$

preserving Pin structure. One is homotopic to the identity and the other is homotopic to $-\text{Id}_{\tau} \oplus \text{Id}_{\mathbb{C}^n}$.

**Remark 2.7.** The assertions of Lemma 2.6 are clearly true when $n = 1$ without any reference to Pin structure. This is not surprising because all automorphisms of a real line bundle preserve Pin structure.

**Proof of Lemma 2.6.** For the trivial bundle pair $(\mathbb{C}^n, \mathbb{R}^n)$, homotopy classes of automorphisms preserving orientation are classified by $\pi_2(U(n), SO(n))$, which is easily calculated from the homotopy long exact sequence of the pair. In the case $n \geq 3$, we have

$$
\begin{array}{ccccccc}
\pi_2(U(n)) & \longrightarrow & \pi_2(U(n), SO(n)) & \overset{\sim}{\longrightarrow} & \pi_1(SO(n)) & \overset{0}{\longrightarrow} & \pi_1(U(n)) \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z}
\end{array}
$$
The automorphisms preserving \( \text{Pin} \) structure map to \( 0 \in \pi_1(SO(n)) \) so they are all homotopic to the identity. In the case \( n = 2 \), we have the same exact sequence, but \( \pi_1(SO(2)) \simeq \mathbb{Z} \) and hence \( \pi_2(U(2), SO(2)) \simeq \mathbb{Z} \). The automorphisms preserving \( \text{Pin} \) structure map to the subgroup \( 2\mathbb{Z} \subset \mathbb{Z} \simeq \pi_1(SO(2)) \). On the other hand, by definition, exactly one of the automorphisms \( Q(2i) \) maps to each element of the subgroup \( 2\mathbb{Z} \), implying the claim.

In the case of a non-trivial boundary condition, we will have to make a more explicit argument. Most of the work will be devoted to showing that the claim of the lemma is true for homotopy classes of automorphisms of the boundary condition alone. To verify this, we construct a convenient model for the boundary condition \( \tau_\mathbb{R} \oplus \mathbb{R}^{n-1} \). Indeed, let \( r_1 \in O(n) \) be the reflection that acts on \( \mathbb{R}^n \) by

\[
r(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n).
\]

We identify

\[
\tau_\mathbb{R} \oplus \mathbb{R}^{n-1} \sim \mathbb{R}^n \times [0, 1]/(x, 0) \sim (r_1(x), 1)
\]

\[
\partial D^2 \sim [0, 1]/0 \sim 1.
\]

Let \( \pi : \text{Pin}(n) \to O(n) \) denote the covering map. Letting \( e_i \) denote the standard basis vectors in \( \mathbb{R}^n \), and thinking of \( \text{Pin}(n) \) as the group generated by the unit vectors in the Clifford algebra, we have \( \pi(e_1) = r_1 \). So, we may define a \( \text{Pin} \) structure on \( \tau_\mathbb{R} \oplus \mathbb{R}^{n-1} \) by Diagram 1. Here, \( \cdot \) denotes Clifford multiplication. It follows that an automorphism of \( P \) is given by a map

\[
a : [0, 1] \to \text{Pin}(n)
\]

such that

\[
e_1 \cdot a(0) = a(1) \cdot e_1.
\]

Diagram 1

(5)
In particular, we see that $\text{Id}_{\tau \oplus \mathbb{R}^{n-1}}$ lifts to the identity automorphism of $P$, and $-\text{Id}_{\tau \oplus \mathbb{R}^{n-1}}$ lifts to the automorphism of $P$ given by $a(t) = e_1$.

We claim that up to homotopy, these are the only two possibilities. First we consider the case $n \geq 3$. Indeed, noting that $\text{Pin}(n)$ has two components, one containing $\text{Id}_{\tau \oplus \mathbb{R}^{n-1}}$ and the other containing $e_1$, it suffices to show that if automorphisms $a$ and $a'$ map to the same component of $\text{Pin}(n)$ then they are homotopic through automorphisms. Indeed, connect $a(0)$ to $a'(0)$ by an arbitrary path $b$. Connect $a(1)$ to $a'(1)$ by $\pm e_1 \cdot b \cdot e_1$, where the sign depends on whether we work in $\text{Pin}^+$ or $\text{Pin}^-$. The resulting loop is null homotopic by the simply-connectedness of each component of $\text{Pin}(n)$. Reparameterizing a null-homotopy, we obtain a family of automorphisms connecting $a$ and $a'$ as desired.

We turn to the case $n = 2$. Since $\text{Pin}(2)$ is not simply-connected, we must be more carefully. Indeed, topologically, $\text{Pin}(2) \simeq S^1 \coprod S^1$. One component consists of spinors of the form $\cos(\theta) + \sin(\theta)e_1 \cdot e_2$ and the other consist of spinors of the form $\cos(\theta)e_1 + \sin(\theta)e_2$. So, if we think of $S^1$ as the complex numbers of unit length, conjugation by $e_1$ acts by complex conjugation on each component of $\text{Pin}(2)$. So, an automorphism of $P$ is given by a path in one of the two copies of $S^1$ with complex conjugate endpoints. It suffices to show that any such path is homotopic through similar paths to the constant path at $\pm 1$ or $\pm e_1$. Indeed, $\pi(\pm 1) = \text{Id}_{\tau \oplus \mathbb{R}}$ and $\pi(\pm e_1) = -\text{Id}_{\tau \oplus \mathbb{R}}$. So, consider the covering map $\mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi ix}$. Given a path in $S^1$ with conjugate endpoints, we may lift it to path $x : [0, 1] \to \mathbb{R}$ such that $x(0) \equiv -x(1) \pmod{1}$. Since either

$$\frac{x(0) + x(1)}{2} \equiv 0 \pmod{1} \quad \text{or} \quad \frac{x(0) + x(1)}{2} \equiv \frac{1}{2} \pmod{1},$$

linear interpolation between $x$ and $\frac{x(0) + x(1)}{2}$ yields a homotopy $x_t$ such that $x_t(0) \equiv -x_t(1) \pmod{1}$, $x_0(s) = x(s)$, $x_1(s) \equiv 0$ or $\frac{1}{2} \pmod{1}$.

Then $\pi(x_t)$ yields the desired homotopy of $a$.

Now we extend our conclusion to automorphisms of the pair. Trivializing $\tau \oplus \mathbb{C}^{n-1}$ over $D^2$, we may identify an automorphism of $\tau \oplus \mathbb{C}^{n-1}$ with a map

$$A : D^2 \to U(n).$$
We are interested in $A$ that preserve $\tau \oplus \mathbb{R}^{n-1}$ over $\partial D^2$, such that the induced automorphism of $\tau \oplus \mathbb{R}^{n-1}$ preserves $Pin$ structure. By the preceding calculation, two examples are given by $\text{Id}_x \oplus \text{Id}_{\mathbb{C}^{n-1}}$ and $-\text{Id}_x \oplus \text{Id}_{\mathbb{C}^{n-1}}$. We claim that up to homotopy these are the only two examples. Indeed, given $A$, choose a lift of the induced automorphism on $\tau \oplus \mathbb{R}^{n-1}$ to $P$ and denote it by $a$. As just proved, $a$ is homotopic to either $\text{Id}_P$ in $(\mathbb{R}^n)$ or $e_1$. Denote the homotopy by $\tilde{B} : [0, 1] \to \text{Aut}(P)$, $\tilde{B}(0) = a$, $\tilde{B}(1) = \text{Id}_{Pin(n)}$ or $e_1$.

We will construct a homotopy from $A$ to $A' : D^2 \to U(n)$, where $A'$ corresponds to the automorphism

$$\text{Id}_x \oplus \text{Id}_{\mathbb{C}^{n-1}} \text{ or } -\text{Id}_x \oplus \text{Id}_{\mathbb{C}^{n-1}}$$

as $\tilde{B}(1) = \text{Id}_{Pin(n)}$ or $\tilde{B}(1) = e_1$ respectively. Indeed, composing with the given covering map, $\tilde{B}$ defines a path

$$B : [0, 1] \to \text{Aut}(\tau \oplus \mathbb{R}^{n-1}).$$

Denote by

$$i : \text{Aut}(\tau \oplus \mathbb{R}^{n-1}) \hookrightarrow \text{Aut}(\tau \oplus \mathbb{C}^{n-1}|_{\partial D^2})$$

the inclusion given by complexification. Restricting the previously mentioned trivialization of $\tau \oplus \mathbb{C}^{n-1}$ to $\partial D^2$, we may identify the path $i \circ B$ with a map

$$\hat{B} : [0, 1] \times \partial D^2 \to U(n).$$

Capping off this cylinder with the disk $A$ at one end and the disk $A'$ at the other end, we obtain a map $S^2 \to U(n)$, which is well known to be null-homotopic. Reparameterizing a null-homotopy gives the required homotopy through automorphisms from $A$ to $A'$. All these automorphisms preserve the boundary condition and its $Pin$ structure by the construction of $\hat{B}$.

**Proposition 2.8.** The determinant line of a real-linear Cauchy-Riemann $Pin$ boundary value problem $D$ admits a canonical orientation. If $\phi : D \to D'$ is an isomorphism, then the induced morphism

$$\phi : \det(D) \to \det(D')$$

preserves the canonical orientation. Furthermore, the canonical orientation varies continuously in a family of Cauchy-Riemann operators. That is, it defines a single component of the determinant line bundle over that family.
Proof. Near each boundary component \((\partial \Sigma)\) choose a closed curve \(\gamma_a\) homotopic to \((\partial \Sigma)\). Degenerate \(\Sigma\) by contracting the curves \(\gamma_a\) to points to obtain a nodal surface \(\hat{\Sigma}\). \(\hat{\Sigma}\) consists of one closed component \(\tilde{\Sigma}\), a disk \(\Delta_a\) corresponding to each boundary component \((\partial \Sigma)\) and a nodal point \(\hat{\gamma}_a\) corresponding to each curve \(\gamma_a\). There exists a continuous map \(\pi : \Sigma \to \hat{\Sigma}\) which is a smooth diffeomorphism away from the nodal points \(\hat{\gamma}_a\). So, we may define

\[
\hat{F} = (\pi|_{(\partial \Sigma)_a})^{-1} \ast F.
\]

At the same time, degenerate \(E\) to a vector bundle \(\hat{E} \to \hat{\Sigma}\) such that

\[
(\hat{E}|_{\Delta_a}, \hat{F}|_{\partial \Delta_a}) \simeq \begin{cases} 
(\tau \oplus \mathbb{C}^{n-1}, \tau_{\mathbb{R}} \oplus \mathbb{R}^{n-1}) & \text{if } w_1(F|_{(\partial \Sigma)_a}) = 1 \\
(\mathbb{C}^n, \mathbb{R}^n) & \text{if } w_1(F|_{(\partial \Sigma)_a}) = 0.
\end{cases}
\] (6)

We choose the isomorphism (6) to preserve orientation in the orientable case. Moreover, if \(n \geq 2\) we choose isomorphism (6) to preserve \(Pin\) structure. If \(n = 1\), we cannot always choose isomorphism (6) to preserve \(Pin\) structure. We compensate for this difference at the next stage of the construction.

Equip \(\hat{E}|_{\Delta_a}\) with the Cauchy-Riemann operator \(D_a\) induced by the isomorphism (6) from \(D_{-1,n}\) (resp. \(D_{0,n}\)). The isomorphism (6) induces an orientation on \(\det(D_a)\) from the canonical orientation of \(\det(D_{-1,n})\) (resp. \(\det(D_{0,n})\)). If \(n = 1\), and isomorphism (6) does not preserve \(Pin\) structure, reverse the induced orientation on \(\det(D_a)\). Choose a Cauchy-Riemann operator \(\tilde{D}\) on \(\hat{E}|_{\tilde{\Sigma}}\). Equip \(\det(\tilde{D})\) with the canonical complex orientation. Define an operator

\[
d_{\hat{\gamma}_a} : \Gamma(\hat{E}|_{\Delta_a}, \hat{F}|_{\partial \Delta_a}) \oplus \Gamma(\hat{E}|_{\tilde{\Sigma}}) \to E_{\hat{\gamma}_a}
\]

by

\[
d_{\hat{\gamma}_a}(\xi, \eta) = \xi(\hat{\gamma}_a) - \eta(\hat{\gamma}_a), \quad \xi \in \Gamma(\hat{E}|_{\Delta_a}, \hat{F}|_{\partial \Delta_a}), \quad \eta \in \Gamma(\hat{E}|_{\tilde{\Sigma}}).
\]

Gluing the \(D_a\) with \(\tilde{D}\) at \(\hat{\gamma}_a\) we obtain a Cauchy-Riemann operator \(#_a D_a \# \tilde{D}\) on \(E\) along with an isomorphism of virtual vector spaces

\[
\text{index}(\#_a D_a \# \tilde{D}) \simeq \text{index} \left( \bigoplus_a D_a \oplus \tilde{D} \oplus \bigoplus_a d_{\hat{\gamma}_a} \right),
\]

or equivalently, an isomorphism

\[
\det(\#_a D_a \# \tilde{D}) \simeq \bigotimes_a \det(D_a) \otimes \det(\tilde{D}) \otimes \bigotimes_a \det(E_{\hat{\gamma}_a}).
\] (7)
Since the space of Cauchy-Riemann operators on $E$ is contractible, choosing a one-parameter family $D_t$ with $D_0 = D$ and $D_1 = \#_a D_a \# \tilde{D}$ and trivializing the line bundle $\det(D_t)$ over the family induces an orientation on $\det(D)$.

We claim that the orientation induced on $\det(D)$ is independent of the choice of isomorphism (6), the choice of $\tilde{D}$, and the choice of $D_t$. First, we prove the independence of the choice of $D_t$. Indeed, since the space of Cauchy-Riemann operators on $E$ is contractible, given any two families $D_t$ and $D'_t$, we can construct a homotopy between them $D_{s,t}$, such that $D_{0,t} = D_t$, $D_{1,t} = D'_t$. Trivializing $\det(D_{s,t})$ over the homotopy proves that $D_t$ and $D'_t$ give the same answer.

Now we turn to proving independence of the choice of isomorphism (6) and the choice of $\tilde{D}$. Another choice of isomorphism (6) would induce a different operator $D'_a$ in place of $D_a$. Also, let $\tilde{D}'$ be another Cauchy-Riemann operator on $\hat{E}|\tilde{\Sigma}$. We prove that these new choices induce the same orientation on $D$. Choose homotopies $D_{a,t}$ and $\tilde{D}_t$ such that $D_{a,1} = D_a$, $D_{a,\frac{1}{2}} = D'_a$, $\tilde{D}_1 = \tilde{D}$, $\tilde{D}_{\frac{1}{2}} = \tilde{D}'$.

We choose the family $D_t$ so that, as before, $D_0 = D$ and $D_1 = \#_a D_a \# \tilde{D}$, but we require also that $D_t = \#_a D_{a,t} \# \tilde{D}'_t$, $t \in \left[\frac{1}{2}, 1\right]$. Since this choice of $D_t$ is as good as any other, it remains only to show that the orientation on $\det(D'_a)$ induced by the isomorphism $\det(D'_a) \sim \det(D_{i,n})$, $i = -1$ or $0$, agrees with the orientation induced from $\det(D_a)$ by trivializing $\det(D_{a,t})$ over the interval $[\frac{1}{2}, 1]$. Similarly, we must show that the complex orientation on $\det(\tilde{D}')$ agrees with orientation induced from $\det(\tilde{D})$ by trivializing $\det(\tilde{D}_t)$ over the interval $[\frac{1}{2}, 1]$. The latter agreement follows from the compatibility of the topology of the determinant bundle over a family with the canonical complex orientation. To see the former agreement, note that the isomorphism (6) is determined up to an automorphism preserving $Pin$ structure of the right hand bundle pair. In the orientable case where $n \geq 3$, by
Lemma 2.6, all such automorphisms are homotopic to the identity. So, we may assume that $D_{a,t}$ is induced by a family of automorphisms. Then, it suffices to note the that the determinant bundle is tautologically trivial over a family of gauge-equivalent operators. The case $n = 2$ may be reduced to the higher dimensional case by stabilizing by a copy of the trivial bundle pair. Indeed, $Q(2i) \oplus \text{Id}_\mathbb{R}$ is homotopic to $\text{Id}_{\mathbb{R}^3}$. In the non-orientable case, we need to consider the additional possibility that the automorphism is homotopic to $-\text{Id}_\tau \oplus \text{Id}_{\mathbb{C}^{n-1}}$. But $-\text{Id}_\tau \oplus \text{Id}_{\mathbb{C}^{n-1}}$ clearly preserves the orientation of $\det(D_{-1})$, so this possibility does not affect the argument. The remaining claims of the lemma follow immediately from the construction.

**Lemma 2.9.** If the boundary condition $F|_{(\partial \Sigma)_a}$ is orientable, then reversing the orientation on $F|_{(\partial \Sigma)_a}$ will change the canonical orientation on $\det(D)$ given in Proposition 2.8.

**Proof.** This is an immediate consequence of the proof of Proposition 2.8.

In the following lemmas, we use the fact that $H^1(\mathcal{B}, \mathbb{Z}/2\mathbb{Z})$ acts naturally transitively on the set of Pin structures on a vector bundle $V \to \mathcal{B}$. See [11].

**Lemma 2.10.** Changing the Pin structure $p$ of a Cauchy-Riemann Pin boundary value problem by the action of the generator of

$$H^1((\partial \Sigma)_a, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^1(\partial \Sigma, \mathbb{Z}/2\mathbb{Z})$$

reverses the canonical orientation of Proposition 2.8.

**Proof.** By the proof of Proposition 2.8, it suffices to consider the special cases $D_{(-1,n)}$ and $D_{(0,n)}$. If $n = 1$, Lemma 2.10 is tautological. So, we assume $n \geq 2$. For the case $D_{(0,n)}$, see [6, Remark 21.6]. For the case $D_{(-1,n)}$, it suffices to show that

$$A := \text{Id}_\tau \oplus -\text{Id}_{\mathbb{C}} \oplus \text{Id}_{\mathbb{C}^{n-2}},$$

which clearly reverses the orientation of $\det(D_{-1})$, does not preserve Pin structure. We use the identification of Diagram to show that $A|_{\tau \oplus \mathbb{R}^{n-1}}$ does not lift to an automorphism of $P$. Indeed, let $r_2 \in O(n)$ be the reflection that acts on $\mathbb{R}^n$ by the formula

$$r_2(x_1, x_2, x_3, \ldots, x_n) = (x_1, -x_2, x_3, \ldots, x_n).$$
and, as before, let \( \pi : \text{Pin}(n) \to O(n) \) denote the canonical covering map. The automorphism \( A|_{\mathfrak{r} \oplus \mathbb{R}^{n-1}} \) acts on \( \mathfrak{j}(\mathfrak{r} \oplus \mathbb{R}^{n-1}) \) by the explicit formula \((o,t) \mapsto (r_2 o,t)\). If this automorphism were to lift to \( P \), it would be given by left-multiplication by \( a \in \text{Pin}(n) \) such that \( \pi(a) = r_2 \). So, thinking of \( \text{Pin}(n) \) as the group generated by the unit vectors in the Clifford algebra, we would have \( a = \pm e_2 \). But this contradicts condition \( (5) \), since

\[ e_1 \cdot e_2 = -e_2 \cdot e_1. \]

We now introduce a lemma that will play an important role in understanding the significance of relative \( \text{Pin} \) structures. Note that any real vector bundle over a Riemann surface with non-empty boundary \( \Sigma \) admits a \( \text{Pin} \) structure because \( \Sigma \) deformation retracts to a wedge of circles.

**Lemma 2.11.** Let \( V \to \Sigma \) be a real vector bundle over a Riemann surface with boundary. Consider \( D = (\Sigma, V \otimes \mathbb{C}, V|_{\partial \Sigma}, p, D) \). The canonical orientation of \( \det(D) \) is the same for any \( p \) that arises by restricting a \( \text{Pin} \) structure for \( V \) over \( \Sigma \) to \( \partial \Sigma \).

**Proof.** Let \( i : \partial \Sigma \to \Sigma \) denote the canonical inclusion. By Lemma 2.10, it suffices to show that any change of \( \text{Pin} \) structure over \( \Sigma \) would change the \( \text{Pin} \) structure over \( \partial \Sigma \) by the action of the sum of the generators of \( H^1((\partial \Sigma)_a) \) for an even number of components \( (\partial \Sigma)_a \) of \( \partial \Sigma \). Now, any two \( \text{Pin} \) structures of \( V \) over \( \Sigma \) may be related by the action of \( H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \). So we may equivalently show that for all \( \alpha \in H^1(\Sigma) \) we have \( i^* \alpha(\partial \Sigma) = 0 \) (mod 2). But this follows immediately because with \( \mathbb{Z}/2\mathbb{Z} \) coefficients \( i^* \) is the dual of \( i_* \), and tautologically \( i_*(|\partial \Sigma|) = 0 \). \( \Box \)

Now, we will calculate the sign of conjugation on the canonical orientation of the determinant line of a Cauchy-Riemann \( \text{Pin} \) boundary value problem. More precisely, given a Riemann surface \( \Sigma \), let \( \overline{\Sigma} \) denote the same topological surface with conjugate complex structure, and let

\[ t : \Sigma \to \overline{\Sigma} \]

denote the tautological anti-holomorphic map. Similarly, let \((\overline{E}, \overline{F})\) denote the same real bundle pair with the opposite complex structure on \( E \), and let

\[ T : E \to \overline{E} \]
denote the tautological anti-complex-linear bundle map. Furthermore, a Cauchy-Riemann operator \( D \) on the bundle \( E \to \Sigma \) is the same as a Cauchy-Riemann operator \( \overline{D} \) on the bundle \( \overline{E} \to \Sigma \). So, given any Cauchy-Riemann \( Pin \) boundary problem \( D \), we may construct its conjugate \( \overline{D} \). Clearly, we have a tautological map of Cauchy-Riemann \( Pin \) boundary value problems,

\[
\begin{align*}
\Gamma(\Omega^{0,1}(E)) & \xrightarrow{\tau^{-1} \otimes \tau} \Gamma(\Omega^{0,1}(\overline{E})) \\
\Gamma(E,F) & \xrightarrow{\tau} \Gamma(\overline{E},F)
\end{align*}
\]

which we denote by

\[
\tau : D \to \overline{D}.
\]

In the following proposition, we denote by \( g_0 \) the genus of \( \Sigma/\partial \Sigma \) and we write \( n = \dim_C E = \dim_R F \).

**Proposition 2.12.** The sign of the induced isomorphism

\[
T : \det(D) \to \det(\overline{D})
\]

relative to the canonical orientation is given by

\[
\begin{align*}
\mathfrak{s}_T^+(D) & := \frac{\mu(E,F)(\mu(E,F) + 1)}{2} + (1 - g_0)n + mn \\
& \quad + \sum_{a<b} w_1(F)((\partial \Sigma)_a)w_1(F)((\partial \Sigma)_b) \mod 2,
\end{align*}
\]

for a \( Pin^+ \) structure and

\[
\begin{align*}
\mathfrak{s}_T^-(D) & := \frac{\mu(E,F)(\mu(E,F) + 1)}{2} + (1 - g_0)n + mn \\
& \quad + \sum_{a<b} w_1(F)((\partial \Sigma)_a)w_1(F)((\partial \Sigma)_b) + w_1(F)(\partial \Sigma) \mod 2,
\end{align*}
\]

for a \( Pin^- \) structure.

**Remark 2.13.** When \( \Sigma = D^2 \), since \( \mu(E,F) \cong w_1(F)(\partial \Sigma) \mod 2 \), we have the relatively simple formula

\[
\mathfrak{s}_T^+(D) \cong \frac{\mu(E,F)(\mu(E,F) + 1)}{2} \mod 2.
\]
Before proving the Proposition, we will need the following lemma.

**Lemma 2.14.** The map $T : \overline{D}(-1, n) \to \overline{D}(-1, n)$ preserves orientation if $p_{-1}$ is Pin$^+$, but not if $p_{-1}$ is Pin$^-$. 

**Proof.** In this proof, and later in this paper as well, we will need to make use of the anti-holomorphic involution, 

$$c : \mathbb{CP}^1 \to \mathbb{CP}^1, \quad [z_0 : z_1] \mapsto [\bar{z}_0 : -\bar{z}_1]$$

and the natural involution $\tilde{c}$ of the tautological bundle $\tau$ covering $c$. We note that $c$ preserves the two hemispheres of $\mathbb{CP}^1$ which lie on either side of $\mathbb{RP}^1 \subset \mathbb{CP}^1$. So we may restrict $c$, $\tilde{c}$, to either of the hemispheres $D^2 \subset \mathbb{CP}^1$, and we denote the restriction as well by $c$, $\tilde{c}$. Furthermore, let $C$ denote the bundle morphism of the trivial bundle $\mathbb{C} \to \mathbb{CP}^1$ covering $c$ that acts on the fiber by complex conjugation. The lemma will follow immediately from Proposition 2.8 if we show that 

$$\tilde{c} \oplus C^{\oplus n-1} : \overline{D}(-1, n) \to \overline{D}(-1, n)$$

is an isomorphism of Cauchy-Riemann boundary value problems in the Pin$^+$ case whereas 

$$\tilde{c} \oplus -C \oplus C^{\oplus n-2} : \overline{D}(-1, n) \to \overline{D}(-1, n)$$

is an isomorphism in the Pin$^-$ case. We treat only the Pin$^-$ case since the Pin$^+$ case is very similar and not as interesting. The only property of being an isomorphism of Cauchy-Riemann boundary value problems which is not immediately evident is the preservation of Pin$^-$ structure. To verify this, we again work with the explicit model of Diagram for $\mathfrak{F}(\tau_\mathbb{R} \oplus \mathbb{R}^{n-1})$ and $P$. In this model, it is not hard to see that at the level of the frame bundle, 

$$(\tilde{c} \oplus -C \oplus C^{\oplus n-2})(o, t) = (r_1 r_2 o, 1 - t).$$

So, a lifting of this map to $P$ must act by 

$$(p, t) \sim (a \cdot p, 1 - t)$$

where $a \in Pin^-(n)$ such that $\pi(a) = r_1 r_2$. It remains to check that this lifting respects the equivalence relation defining $P$. Indeed, $(p, 0)$ and $(e_1 \cdot p, 1)$ represent the same point in $P$, so we must have 

$$(a \cdot p, 1) \sim (a \cdot e_1 \cdot p, 0).$$
But that is the same as
\[ a = e_1 \cdot a \cdot e_1. \]
Choosing, for example, \( a = e_1 \cdot e_2 \), and using the Clifford multiplication of \( \text{Pin}^- \), we verify
\[ e_1 \cdot e_1 \cdot e_1 = -e_2 \cdot e_1 = e_1 \cdot e_2. \]

**Proof of Proposition 2.12.** The strategy for the proof is to degenerate \( \Sigma \) and \( E \) as in the proof of Proposition 2.8 and then treat each irreducible component of the resulting nodal Riemann surface \( \hat{\Sigma} \) separately. Let \( \eta \) be the number of boundary components \( (\partial \Sigma)_a \) for which \( w_1(F)((\partial \Sigma)_a) \equiv 1 \) (mod 2). Then the first Chern class of the bundle \( \hat{E}|_{\hat{\Sigma}} \) from the proof of Proposition 2.8 is given by
\[
c_1 \left( \hat{E}|_{\hat{\Sigma}} \right) = \frac{\mu(E,F) + \eta}{2}. \]
Abbreviating \( \mu = \mu(E,F) \), we calculate
\[
\frac{\mu(\mu + 1)}{2} - \frac{\mu + \eta}{2} \equiv \frac{\mu^2}{2} - \frac{\eta}{2} \\
\equiv \frac{\eta^2 - \eta}{2} \\
\equiv \left( \frac{\eta}{2} \right) \\
\equiv \sum_{a < b} w_1(F)((\partial \Sigma)_a)w_1(F)((\partial \Sigma)_b) \pmod{2}. \]
Here, the second congruence uses the fact that \( \mu^2 \equiv \eta^2 \pmod{4} \). So, with \( \hat{D} \) as in the proof of Proposition 2.8, the Riemann-Roch theorem gives
\[
\text{index}_C(\hat{D}) = c_1 \left( \hat{E}|_{\hat{\Sigma}} \right) + n(1 - g_0) \\
\equiv \frac{\mu(\mu + 1)}{2} + n(1 - g_0) \\
+ \sum_{a < b} w_1(F)((\partial \Sigma)_a)w_1(F)((\partial \Sigma)_b) \pmod{2}. \tag{9}
\]
We now tally the sign of conjugation on each of the tensor factors on the right-hand side of equation (7). Note that conjugation on a complex virtual vector space leads to a sign change which is exactly its dimension (mod 2).
So, conjugation changes the orientation of $\det(\tilde{D})$ in accordance with the formula (9). Similarly, conjugation changes the orientation of $\otimes_a \det(E_{\gamma_a})$ by $hn$. The orientation change for $\det(D_a)$ was calculated in Lemma 2.14 and accounts for the difference between $s^+_T$ and $s^-_T$.

**Definition 2.15.** A short exact sequence of families of Fredholm operators

$$0 \to D' \to D \to D'' \to 0$$

consists of a parameter space $B$, short exact sequences of Banach space bundles over $B$,

$$0 \to X' \to X \to X'' \to 0, \quad 0 \to Y' \to Y \to Y'' \to 0,$$

and Fredholm Banach space bundle morphisms

$$D : X \to Y, \quad D' : X' \to Y', \quad D'' : X'' \to Y'',$$

such that the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & Y' \\
\downarrow D' & & \downarrow D \\
0 & \longrightarrow & X'
\end{array}
\begin{array}{ccc}
\longrightarrow & Y & \longrightarrow & Y'' \\
\downarrow D' & & \downarrow D'' & & \downarrow D'' \\
\longrightarrow & X' & \longrightarrow & X'' & \longrightarrow & 0
\end{array}
\]

commutes.

**Lemma 2.16.** A short exact sequence of families of Fredholm operators

$$0 \to D' \to D \to D'' \to 0$$

induces an isomorphism

$$\det(D') \otimes \det(D'') \sim \det(D).$$

### 3 Orienting moduli of open stable maps

First, we set the basic assumptions which will hold throughout this section. Let $(X, \omega)$ be a symplectic manifold with $\dim X = 2n$ and let $L \subset X$ be a Lagrangian submanifold. In the following, we assume $L$ is relatively $Pin^\pm$
and fix a relative $Pin^\pm$ structure $p$ on $L$. Furthermore, if $L$ is orientable, fix an orientation on $L$. Let $(\Sigma, \partial \Sigma)$ denote a Riemann surface with boundary and assume $\partial \Sigma = \bigsqcup_{a=1}^m (\partial \Sigma)_a$, where $(\partial \Sigma)_a \simeq S^1$. Now, for 

$$d = (d, d_1, \ldots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m},$$

define $B^{1,p}(L, \Sigma, d)$ to be the Banach manifold of $W^{1,p}$ maps 

$$u : (\Sigma, \partial \Sigma) \to (X, L)$$

such that $u_*([\Sigma, \partial \Sigma]) = d$ and $u|_{(\partial \Sigma)_a} * ([\partial \Sigma]) = d_a$. Furthermore, define 

$$B^{1,p}_{k,l}(L, \Sigma, d) := B^{1,p}(L, \Sigma, d) \times \prod_{a}(\partial \Sigma)^{k_a} \times \Sigma^l \ \Delta.$$ 

Each factor of $(\partial \Sigma)_a$ in the preceding product corresponds to a marked point on $(\partial \Sigma)_a$, and each factor of $\Sigma$ corresponds to a marked point on $\Sigma$. $\Delta$ denotes the subset of the product in which two marked points coincide. We will use $\vec{z} = (z_{a_i})$ and $\vec{w} = (w_j)$ to denote marked points in $\partial \Sigma$ and $\Sigma$ respectively, and we use $u = (u, \vec{z}, \vec{w})$ to denote elements of $B^{1,p}_{k,l}(L, \Sigma, d)$. Note that we may occasionally omit the $L, \Sigma, d$, from the preceding notation when it is clear from the context. There exist canonical evaluation maps 

$$ev_{b_{ai}} : \overline{M}_{k,l}(L, \Sigma, d) \to L, \quad i = 1 \ldots k_a, \quad a = 1 \ldots m,$$ 

$$ev_{ij} : \overline{M}_{k,l}(L, \Sigma, d) \to X, \quad j = 1 \ldots l,$$ 

given by $ev_{b_{ai}}(u) = u(z_{ai})$ and $ev_{ij}(u) = u(w_j)$. We note that the above notation is also used for the evaluation maps from the moduli spaces of holomorphic curves, which are just restrictions of the maps above.

Define the Banach space bundle $E \to B^{1,p}_{k,l}(L, \Sigma, d)$ fiberwise by 

$$E_u := L^p(\Sigma, \Omega^{0,1}(u^*TX))$$

for $u \in B^{1,p}_{k,l}(L, \Sigma, d)$. Now, fix $J \in J_\omega$, and $\nu \in \mathcal{P}$. Let 

$$\overline{\partial}_{(J, \nu)} : B^{1,p}_{k,l}(L, \Sigma, d) \to \mathcal{E}$$

denote the section of $E$ given by the $\nu$-perturbed Cauchy Riemann operator. Using the canonical identification between the vertical tangent spaces of $\mathcal{E}$ and $\mathcal{E}$ itself we define 

$$D := D\overline{\partial}_{(J, \nu)} : TB^{1,p}_{k,l}(L, \Sigma, d) \to \mathcal{E}$$
to be the vertical component of the linearization of $\bar{\partial}_{(J,\nu)}$. We will denote linearization at any given $u \in B_{k,l}^{1;p}$ by $D_u$. Finally, define $\mathcal{L} \to B_{k,l}^{1;p}(L,\Sigma,d)$ to be the determinant line bundle of the family of Fredholm operators $D$,

$$\mathcal{L} := \det(D).$$

The following proposition is a basic ingredient in the proof of Theorem \ref{thm:main}

Suppose either $L$ is orientable and provided with an orientation or $k_u \cong w_1(d_u) + 1 \mod 2$.

**Proposition 3.1.** The combination of an orientation of $L$ if $L$ is orientable and the choice of relative $Pin^\pm$ structure $p$ on $L$ canonically determine an isomorphism of line bundles

$$\mathcal{L} \cong \bigotimes_{a,i} ev_{b_i}^* \det(TL).$$

**Proof.** Clearly, the proposition will follow immediately if we succeed in providing

$$\mathcal{L}' := \mathcal{L} \otimes \bigotimes_{a,i} ev_{b_i}^* \det(TL)^*$$

with a canonical orientation depending only on the orientation of $L$ and $p$. We observe that it suffices to canonically orient the fiber $\mathcal{L}_u'$ over each $u \in B_{k,l}^{1;p}$ individually in a way that varies continuously with $u$.

Recall that the relative $Pin$ structure of $L$ specifies a triangulation of the pair $(X,L)$. Using simplicial approximation, we homotope the map $u : (\Sigma, \partial \Sigma) \to (X,L)$ to a map $\hat{u} : (\Sigma, \partial \Sigma) \to (X^{(2)},L^{(2)})$. Denote the homotopy by

$$\Phi : [0,1] \times (\Sigma, \partial \Sigma) \to (X,L).$$

We claim that the choice of $\Phi$ is unique up to homotopy. Indeed, suppose $\Phi'$ is another such homotopy. Concatenating $\Phi$ and $\Phi'$, we obtain a map

$$\Phi \# \Phi' : [-1,1] \times (\Sigma, \partial \Sigma) \to (X,L).$$

By simplicial approximation we may homotope $\Phi \# \Phi'$ to map into $(X^{(3)},L^{(3)})$. Reparameterizing this homotopy, we obtain a homotopy from $\Phi$ to $\Phi'$. We denote the homotopy from $\Phi$ to $\Phi'$ by

$$\Psi : [0,1]^2 \times (\Sigma, \partial \Sigma) \to (X,L),$$
such that,\[
\Psi(0, t) = \Phi(t), \quad \Psi(1, t) = \Phi', \quad \Psi(s, 0) = u.
\]

Now, define\[
B' := \left\{ (u, \vec{z}, \vec{w}) \in B_{k,l}^{1,p} \mid u : (\Sigma, \partial \Sigma) \to (X^{(3)}, L^{(3)}) \right\}.
\]

We now prove that the homotopy uniqueness of $\Phi$ implies that it suffices to orient $L'_{|B'}$. Indeed, think of $\Phi$ (resp. $\Psi$) as a map from $[0, 1]$ (resp. $[0, 1]^2$) to $B_{k,l}^{1,p}$. Given an orientation on $L'_{|B'}$, trivializing $\Phi^* L'$ induces an orientation of $L'_u$. This orientation agrees with the orientation induced by any other homotopy $\Phi'$, because we may trivialize $\Psi^* L'$. We note that the orientation on $L'_u$ thus induced varies continuously with $u$. Indeed, given a one parameter family $u_t \in B_{k,l}^{1,p}$ we may choose a homotopy of the one-parameter family $\Phi_t$ and trivialize $\Phi_t^* L'$.

We turn to orienting $L'_{|B'}$. The relative $Pin$ structure of $(X, L)$ provides a vector bundle $V \to X^{(3)}$ and a $Pin^\pm$ structure on $V_{|L^{(3)} \oplus TL_{|L^{(3)}}}$. We introduce the shorthand notation

\[
V_R := V_{|L^{(3)}}, \quad V_C := V \otimes \mathbb{C}.
\]

Again, it suffices to canonically orient each individual line $L'_u$ for $u \in B'$ in a way that varies continuously in families. Let $D_0$ be an arbitrary Cauchy-Riemann operator on $u^* V \otimes \mathbb{C}$. We consider the operator $D_u \oplus D_0$,

\[
TB_{k,l}^{1,p} \oplus W^{1,p}(u^* V_C, u^* V_R) \xrightarrow{D_u \oplus D_0} E \oplus L^p(\Omega^{0,1}(u^* V_C))
\]

\[
W^{1,p}(u^* (TX \oplus V_C), u|_{\partial \Sigma}^* (TL \oplus V_R)) \oplus \mathbb{R}^k \oplus \mathbb{C}^l \xrightarrow{L^p(\Omega^{0,1}(u^* (TX \oplus V_C)))}
\]

Clearly, there exists a short exact sequence of Fredholm operators

\[
0 \to D_u \to D_u \oplus D_0 \to D_0 \to 0.
\]

So, by Lemma 2.10 there exists a natural isomorphism

\[
\det(D_u) \sim \det(D_u \oplus D_0) \otimes \det(D_0)^*,
\]

and, after tensoring on both sides by $\bigotimes_{a,i} (ev_{b_{ai}}^* \det(TL)^*)_u$,

\[
L'_u = \det(D_u) \otimes \bigotimes_{a,i} (ev_{b_{ai}}^* \det(TL)^*)_u
\]

\[
\simeq \det(D_u \oplus D_0) \otimes \det(D_0)^* \otimes \bigotimes_{a,i} (ev_{b_{ai}}^* \det(TL)^*)_u.
\]
Choose a Pin structure $\tilde{p}_0$ on $u^*V \to \Sigma$ and define $p_0$ to be its restriction to $u^*V_\mathbb{R} \to \partial \Sigma$. By Lemma 2.11 the canonical orientation that the Cauchy-Riemann Pin boundary value problem

$$D_0 = (\Sigma, u^*V_\mathbb{C}, u^*V_\mathbb{R}, p_0, D_0)$$

induces on $\det(D_0)$ does not depend on the choice of $\tilde{p}_0$. So, it suffices to orient

$$\mathcal{L}'_u \otimes \det(D_0) \simeq \det(D_u \oplus D_0) \otimes \bigotimes_{a,i} (evb_{ai}^* \det(TL)^*)_u. \quad (11)$$

Note that by pull back, the relative Pin structure on $L$ gives a Pin structure on $u|_{\partial \Sigma}(TL \oplus V_\mathbb{R})$, the boundary condition for $D_u \oplus D_0$. If $L$ is orientable and given an orientation, since by definition $V$ has an orientation, we have an induced orientation on $u|_{\partial \Sigma}(TL \oplus V_\mathbb{R})$. So, by Proposition 2.8 we have a canonical orientation on $\det(D_u \oplus D_0)$. Since the orientation of $L$ is equivalent to an orientation of $\det(TL)$, we have given everything on the right-hand side of equation (11) a canonical orientation.

If $L$ is not orientable, choose an arbitrary orientation on $(evb_{ai}^*TL)_u$ for each $a$ such that $k_a \neq 0$. The complex structure on $\Sigma$ induces a natural orientation on $\Sigma$ and hence on $(\partial \Sigma)_a$ for each $a$. For each $a$ and each $i \in [2, k_a]$, trivializing $u|_{\partial \Sigma}^*TL$ along the oriented line segment in $(\partial \Sigma)_a$ from $z_{a1}$ to $z_{ai}$ induces an orientation on $(evb_{ai}^*TL)_u$. In the case that $u|_{(\partial \Sigma)_a}^*TL$ is orientable, the choice of orientation on $(evb_{ai}^*TL)_u$ induces an orientation on $u|_{(\partial \Sigma)_a}^*TL$. By Proposition 2.8 an orientation on $u|_{(\partial \Sigma)_a}^*TL$ if orientable together with the chosen orientation on $V_\mathbb{R}$ and the previously mentioned Pin structure induces a canonical orientation on $\det(D_u \oplus D_0)$ and hence the whole right-hand side of (11) is oriented after these choices.

Note that changing the orientation on $(evb_{a1}^*TL)_u$ will change all the orientations it induces. By Lemma 2.9 the condition $k_a \cong w_1(d_a) + 1 \mod 2$ now implies that changing the orientation on any given $(evb_{a1}^*TL)_u$ would make no difference because the total number of ensuing orientation changes would be even. Finally, the choice of $D_0$ is irrelevant because the space of real-linear Cauchy-Riemann operators on $u^*V$ is contractible. Since the argument for orienting $\mathcal{L}'_u$ applies word for word for a one-parameter family, we have indeed canonically oriented $\mathcal{L}'|_{B'}$.

At this point, we will modify the canonical isomorphism

$$\mathcal{L} \sim \bigotimes_{a,i} evb_{ai}^* \det(TL)$$
we constructed in the proof of Proposition 3.1. Indeed, $B_{k,l}^{1,p}(L, \Sigma, d)$ consists of many connected components, at least one for each ordering of the marked points on their respective boundary components. Let

$$\varpi = (\varpi_1, \ldots, \varpi_m)$$

where $\varpi_a$ is a permutation of the integers $1, \ldots, k_a$. Define

$$\text{sign}(\varpi) := \sum_a \text{sign}(\varpi_a).$$

Let

$$B_{k,l,\varpi}^{1,p}(L, \Sigma, d)$$

denote the component of $B_{k,l}^{1,p}(L, \Sigma, d)$ where the boundary marked points $(z_{a_i})$ are ordered within $\partial \Sigma$ by the permutations $\varpi$.

**Definition 3.2.** When $\dim L \cong 0 \pmod{2}$ we define the canonical isomorphism

$$\mathcal{L} \xrightarrow{\sim} \bigotimes_{a,i} \text{ev}_b^{*} \det(TL)$$

to be the isomorphism constructed in the proof of Proposition 3.1 twisted by $(-1)^{\text{sign}(\varpi)}$ over the component of the moduli space $B_{k,l,\varpi}^{1,p}(L, \Sigma, d)$. If $\dim L \cong 1 \pmod{2}$, then we define the canonical isomorphism to be simply the isomorphism constructed in the proof of Proposition 3.1.

Now, we move on to orienting moduli spaces of stable maps. We will restrict attention to stable maps of two components, one of which is the original Riemann surface $\Sigma$, and the other of which is a disk bubble. This will suffice for the purposes of this paper. However, it is not hard to extend the results below to stable maps of arbitrarily many components of arbitrary topological type.

We consider the case that a disk bubbles off the boundary component $(\partial \Sigma)_b$ along with $k''$ of the marked points on $(\partial \Sigma)_b$ and $l''$ of the interior marked points. Let

$$k' := k_b - k'', \quad k' := (k_1, \ldots, k', \ldots, k_m).$$

Let $l' + l'' = l$. It will turn out to be convenient to keep track of exactly which marked points bubble off. So, let $\sigma \subset [1, k_b]$ denote the subset of
boundary marked points that bubble off and let \( \sigma^c \) denote its complement. Furthermore, let \( \varrho \subset [1, l] \) denote the set of interior marked points that bubble off and let \( \varrho^c \) denote its complement. Let

\[
d' = (d', d_1, \ldots, d_b, \ldots, d_m) \in H_2(X, L) \oplus H_1(L)^{\oplus m}, \quad d'' \in H_2(X, L),
\]

satisfy

\[
d' + d'' = d, \quad d'_b + \partial d'' = d_b.
\]

We will need to add an extra marked point to each of the two irreducible components of the stable map in order to impose the condition that the two components intersect. We denote by \( z'_0 \) the extra marked point on \( \Sigma \) and by \( z''_0 \) the extra marked point on \( D^2 \). We will use the notation

\[
k' + e_b = (k_1, \ldots, k'_1 + 1, \ldots, k_m).
\]

We define the space of \( W^{1,p} \)-stable maps with this combinatorial data to be the fiber product

\[
B_{k, \sigma, l, \varrho}^{1, p}(L, \Sigma, d', d'') := B_{k' + e_b, l'}^{1, p}(L, \Sigma, d') \times_{\text{ev}_{b'_0}} B_{k'' + 1, l''}^{1, p}(L, D^2, d'').
\]

Elements \( u \in B_{k, \sigma, l, \varrho}^{1, p}(L, \Sigma, d', d'') \) take the form

\[
u = (u', u''), \quad u' \in B_{k' + e_b, l'}^{1, p}(L, \Sigma, d'), \quad u'' \in B_{k'' + 1, l''}^{1, p}(L, D^2, d''),
\]

\[
e_{b'_0}(u') = e_{b''_0}(u'').
\]

Associated to each such \( u \) there is a nodal Riemann surface with boundary

\[
\hat{\Sigma}_u := \Sigma \cup D^2 / z'_0 \sim z''_0.
\]

and a continuous map

\[
u : (\hat{\Sigma}_u, \partial \hat{\Sigma}_u) \to (X, L)
\]

given by \( u' \) on \( \Sigma \) and \( u'' \) on \( D^2 \). We denote the node of \( \hat{\Sigma}_u \) by \( z_0 \). Let

\[
p' : B_{k, \sigma, l, \varrho}^{1, p}(L, \Sigma, d', d'') \to B_{k' + e_b, l'}^{1, p}(L, \Sigma, d'),
\]

\[
p'' : B_{k, \sigma, l, \varrho}^{1, p}(L, \Sigma, d', d'') \to B_{k'' + 1, l''}^{1, p}(L, D^2, d'').
\]
denote the natural projections. Note that when various indices are clear from the context, we may abbreviate
\[ B' = B_{k'+c_k,\ell'}^{1,p}(L, \Sigma, d'), \quad B'' = B_{k''+1,\ell''}^{1,p}(L, D^2, d''), \]
\[ B^\# := B_{k,\sigma,l,\theta}^{1,p}(L, \Sigma, d', d''). \]
Define the Banach space bundle \( \mathcal{E}^\# \to B^\# \) by
\[ \mathcal{E}^\# := p_1^*\mathcal{E}' \oplus p_2^*\mathcal{E}''. \]
Fiberwise, we have
\[ \mathcal{E}_u^\# := L_p(\Sigma, \Omega^{0,1}(u^*TX) \oplus \Omega^{0,1}(u''^*TX)) \]
for \( u \in B_{k,\sigma,l,\theta}^{1,p}(L, \Sigma, d', d''). \) If \( J \in \mathcal{J}_\omega \), and \( \nu \in \mathcal{P} \), we let
\[ \overline{\partial}^\#_{(J,\nu)} : B^\# \to \mathcal{E}^\# \]
denote the section of \( \mathcal{E} \) given by the \( \nu \)-perturbed Cauchy Riemann operator. Here, the natural \( \nu \)-perturbed Cauchy-Riemann operator has a vanishing inhomogeneous term on the disk bubble. Using the canonical identification between the vertical tangent spaces of \( \mathcal{E}^\# \) and \( \mathcal{E}^\# \) itself we define
\[ D^\# := D\overline{\partial}^\#_{(J,\nu)} : TB_{k,\sigma,l,\theta}^{1,p}(L, \Sigma, d', d'') \to \mathcal{E}^\# \]
to be the vertical component of the linearization of \( \overline{\partial}^\#_{(J,\nu)} \). Finally, define \( \mathcal{L}^\# \to B_{k,l}^{1,p}(L, \Sigma, d) \) to be the determinant line bundle of the family of Fredholm operators \( D^\# \),
\[ \mathcal{L}^\# := \text{det}(D^\#). \]
Again, suppose either \( L \) is orientable and provided with an orientation or \( k_a \cong w_1(d_a) + 1 \mod 2 \).

**Proposition 3.3.** The combination of an orientation of \( L \) if \( L \) is orientable and the choice of relative Pin\(^+\) structure \( \mathfrak{p} \) on \( L \) canonically determine an isomorphism of line bundles
\[ \mathcal{L}^\# \xrightarrow{\sim} \bigotimes_{\alpha_i} ev_{b_{\alpha_i}}^* \text{det}(TL). \]
Proof. The proof is the same as the proof of Proposition 3.1 except for one extra subtlety, which is particularly important in the case that $L$ is not orientable. That is, the Riemann surface underlying the Cauchy-Riemann $Pin$ boundary problem associated to $D^#$ is singular. As in equation (11) of the proof of Proposition 3.1, it suffices to orient

$$L_u^# \otimes \det(D_0^#) \simeq \det(D_u^# + D_0^#) \otimes \bigotimes_{a,i} (evb_{\alpha_i}^* \det(TL)^*)_u$$

for $u$ such that $u : (\hat{\Sigma}_u, \partial \hat{\Sigma}_u) \rightarrow (X^{(3)}, L^{(3)})$.

To this end, we need to describe $D_0^#$ and $D^#$ in greater detail. Using the notation (10), define

$$d_{00} : W^{1,p}(u'^*V_C, u'^*V_R) \oplus W^{1,p}(u''^*V_C, u''^*V_R) \rightarrow (evb_{\alpha_0}^* V_R)_u$$

by

$$d_{00}(\xi', \xi'') = \xi'(z_0') - \xi''(z_0'').$$

Denote

$$W^{1,p}(u^*V_C, u|_{\partial \hat{\Sigma}_u}^* V_R) := \ker(d_{00}).$$

At this point, we introduce abbreviated notation

$$W'_V := W^{1,p}(u'^*V_C, u'^*V_R), \quad W''_V := W^{1,p}(u''^*V_C, u''^*V_R),$$

$$W_V := W^{1,p}(u^*V_C, u|_{\partial \hat{\Sigma}_u}^* V_R),$$

$$Y'_V := L^p(u'^*V_C), \quad Y''_V := L^p(u''^*V_C),$$

$$Y_V := L^p(u^*V_C) = L^p(u'^*V_C) + L^p(u''^*V_C).$$

Then, choose arbitrary Cauchy-Riemann operators

$$D'_0 : W'_V \rightarrow Y'_V, \quad D''_0 : W''_V \rightarrow Y''_V,$$

and define

$$D^#_0 := (D'_0 + D''_0)|_{W_V} : W_V \rightarrow Y_V.$$

Now, turning to $D^#$, define

$$d_{00} : p'^*TB_{k',t'}^{1,p} \oplus p''^*TB_{k'',t''}^{1,p} \rightarrow evb_{\alpha_0}^* TL$$
by
\[ (\xi',\xi'') \rightsquigarrow devb_0' (\xi') - devb''_0 (\xi''). \]

Note that
\[ TB_{k,a,t,q}^L (L, \Sigma, d', d'') = \ker (d_{00}). \]

So, we have a short exact sequence of families of Fredholm operators
\[
\begin{array}{c}
\mathcal{E}_u \oplus Y_V \longrightarrow p'^* \mathcal{E}'_u \oplus Y'_V \oplus p''^* \mathcal{E}''_u \oplus Y''_V \longrightarrow 0 \\
\downarrow D_u^* \oplus D_0^* \downarrow D'_u \oplus D_0' \oplus D''_u \oplus D_0'' \Downarrow d_{00} \oplus d''_{00} \end{array}
\]
\[
TB_u^* \oplus W_V \longrightarrow p'^* TB'_u \oplus W'_V \oplus p''^* TB''_u \oplus W''_V \longrightarrow evb_0^* (TL \oplus V_R)_u,
\]

and, hence, an isomorphism
\[ \det (D_u^* \oplus D_0^*) \xrightarrow{\sim} \det (D'_u \oplus D_0') \otimes \det (D''_u \oplus D_0'') \otimes evb_0^* \det (TL \oplus V_R)_u^*. \quad (13) \]

Noting isomorphism (13), if \( L \) is oriented, the whole right-hand side of equation (12) is canonically oriented by arguing as in the proof of Proposition 3.1.

If \( L \) is not orientable, choose an arbitrary orientation on \((evb_0^*TL)_u\) for each \( \alpha \) such that \( k_\alpha \neq 0 \). The complex structure on \( \Sigma \) induces a natural orientation on \( \Sigma \) and hence on \((\partial \Sigma)_a\) for each \( a \). Similarly, the complex structure on \( D^2 \) induces a natural orientation on \( \partial D^2 \). This said, any ordered pair of points \((z, z')\), \( z \neq z_0 \), in the same connected component of \( \partial \hat{\Sigma} \) can be connected by a unique oriented line segment from the first to the second. For the non-singular boundary components of \( \partial \hat{\Sigma} \), this is evident. For the singular boundary component,
\[ (\partial \hat{\Sigma})_b := (\partial \Sigma)_b \cup \partial D^2 / z_0' \sim z''_0, \]
we define the unique oriented line segment from \( z \) to \( z' \) as follows: For concreteness, assume that \( z \in (\partial \Sigma)_b \). The same exact definition applies if \( z \in \partial D^2 \). Start from \( z \) and proceed in the direction of the orientation of \((\partial \Sigma)_b\) until reaching either \( z' \) or \( z_0' \). If \( z' \) is reached first or if \( z' = z_0' \), the path ends there. If \( z_0' \) is reached first, then continue the path starting from \( z_0'' \in \partial D^2 \) and proceeding along \( \partial D^2 \) in the direction of the orientation. If \( z' \) belongs to \( \partial D^2 \) then the path ends when it reaches \( z' \). Otherwise it continues around \( \partial D^2 \) back to \( z_0'' \) and then proceeds in the direction of the orientation from \( z_0' \) along \((\partial \Sigma)_b\) until it reaches \( z' \).
For each \(a\) and each \(i \in [2, k_a]\), trivializing \(u_{\partial \Sigma}^* TL\) along the oriented line segment in \((\partial \Sigma)_a\) from \(z_{a1}\) to \(z_{ai}\) induces an orientation on \((evb_{ai}^* TL)_u\) as before. To orient the factor of \(evb_0^* det(TL \oplus V_\text{evb})_u^*\) appearing on the right-hand side of isomorphism (13), since \(V_\text{evb}\) is equipped with a chosen orientation by the definition of a relative \(Pin\) structure, it suffices to orient \((evb_0^* TL)_u\). For this purpose, we proceed as follows: If \(k_b \neq 0\) then trivializing \(u_{\partial \Sigma}^* TL\) along the oriented line segment from \(z_{b1}\) to \(z_0\) induces an orientation on \((evb_0^* TL)_u\). If \(k_b = 0\) then choose an arbitrary orientation on \((evb_0^* TL)_u\). For all \(a \neq b\), if \(u_{\partial \Sigma}^* TL\) is orientable, the choice of orientation on \((evb_{ai}^* TL)_u\) induces an orientation on \(u_{\partial \Sigma}^* TL\) since in this case \(k_a \neq 0\) by the assumption \(w_1(d_a) \cong k_a + 1 \pmod{2}\). Furthermore, we may induce an orientation on either or both of \(u_{\partial \Sigma}^* \) and \(u_{\partial \Sigma}^* TL\) if orientable, from the orientation on \((evb_0^* TL)_u\). By Proposition 2.8, the orientation on \(u_{\partial \Sigma}^* TL\) if orientable together with the chosen orientation on \(V_\text{evb}\) and the previously mentioned \(Pin\) structure induce a canonical orientation on \(det(D_u' \oplus D_0')\). Similarly, the orientation on \(u_{\partial \Sigma}^* TL\) if orientable together with the chosen orientation on \(V_\text{evb}\) and the previously mentioned \(Pin\) structure induce a canonical orientation on \(det(D_u'' \oplus D_0'')\). Hence the whole right-hand side of (13) is oriented after these choices. Since we have also chosen an orientation on \((evb_{ai}^* TL)_u\), it follows that the entire right-hand side of (12) is oriented.

Note that changing the orientation on \((evb_{ai}^* TL)_u\) will change all the orientations it induces. Similarly, in the case \(k_b = 0\), changing the chosen orientation on \((evb_0^* TL)_u\) changes any orientation it induces. By Lemma 2.9, the condition \(k_a \cong w_1(d_a) + 1 \pmod{2}\) now implies that changing the orientation on any given \((evb_{ai}^* TL)_u\) or \((evb_0^* TL)_u\) would make no difference because the total number of ensuing orientation changes would be even.

Now, we modify the canonical isomorphism

\[
\mathcal{L}^\# \sim \bigotimes_{a,i} evb_{ai}^* det(TL)
\]

constructed in the proof of Proposition 3.3 to be consistent with the isomorphism over the moduli space of irreducible stable maps defined in Definition 3.2. For this purpose, we note that there is a canonical ordering on the marked points in the boundary of the nodal curve \(\Sigma_u\). Indeed, recall from the proof of Proposition 3.3 that given any pair of points \((z, z') \subset \partial \Sigma\) such that \(z \neq z_0\), there exists a unique oriented line segment from \(z\) to \(z'\). We define \(z_{ai}\) to be ordered before \(z_{ai'}\) if and only if the segment from \(z_{a1}\) to \(z_{ai}\) lies
within the segment from $z_{a_1}$ to $z_{a_i'}$. Again, we can divide the moduli space $B^\#$ into components $B_{a_i}^\#$. By analogy with the irreducible case, we make the following definition.

**Definition 3.4.** If $\dim L = 0 \pmod{2}$, we define the canonical isomorphism

$$\mathcal{L}^\# \sim \bigotimes_{a,i} ev_b^* \det(TL)$$

to be the isomorphism constructed in the proof of Proposition 3.3 twisted by $(-1)^{\text{sign}(a_i)}$ over the component $B_{a_i}^\#$. If $\dim L = 1 \pmod{2}$, we define the canonical isomorphism to be the isomorphism constructed in the proof of Proposition 3.3.

### 4 The definition of the invariants revisited

At this point, we will rigorously define the integration carried out in defining the invariants in Section 1.2 by generalizing the techniques of Ruan and Tian [20].

First, we will carefully define the inhomogeneous perturbation to the Cauchy-Riemann equation relevant in the current situation. Let $C$ be a parameter space to be specified later. Let $\pi_i, i = 1, 2$, denote the projection from $\Sigma \times X \times C$ to the $i$th factor and let $\pi_i'$ denote the restriction of $\pi_i$ to $\partial \Sigma \times L \times C$. We define the set of inhomogeneous terms $P$ to be the set of sections

$$\nu \in \Gamma(\Sigma \times X \times C, \text{Hom}(\pi_1^* T\Sigma, \pi_2^* T\Sigma))$$

such that

(i) $\nu$ is $(j_{\Sigma}, J)$-anti-linear, i.e. $\nu \circ j_{\Sigma} = -J \circ \nu$;

(ii) $\nu|_{\partial \Sigma \times L \times C}$ carries the sub-bundle $\pi_1'^* T\partial \Sigma \subset \pi_1'^* T\Sigma$ to the sub-bundle $\pi_2'^* (JTL) \subset \pi_2'^* TX$.

For the time being, take $C = B_{k,d}^{1,p}(L, \Sigma, d)$ and define the section $\bar{\partial}_{J,\nu}$ of $E$ by

$$\bar{\partial}_{J,\nu} u := du \circ j_{\Sigma} + J \circ du - \nu(\cdot, u(\cdot), u) \in L^p\left(\Omega^{0,1}(u^* TX)\right).$$

**Lemma 4.1.** The operator $\bar{\partial}_{J,\nu}$ gives rise to an elliptic boundary value problem.
4 THE DEFINITION OF THE INVARIANTS REVISITED

Proof. Define $\tilde{\Sigma} := \Sigma \cup_{\partial \Sigma} \Sigma$. Clearly, we may extend any $\nu \in \mathcal{P}$ over $\tilde{\Sigma}$ to obtain a $\tilde{\nu}$ that continues to satisfy condition (i). Let $J_\nu$ be the automorphism of $T(\tilde{\Sigma} \times X)$ given in matrix form by

$$J_{\tilde{\nu}} := \begin{pmatrix} j_\Sigma & 0 \\ \tilde{\nu} & J \end{pmatrix}.$$  

It is not hard to check that condition (i) implies that $J_{\tilde{\nu}}$ is an almost complex structure on $\tilde{\Sigma} \times X$. Condition (ii) implies that $\partial \Sigma \times L \subset \tilde{\Sigma} \times X$ is a totally real submanifold. A map

$$u : (\Sigma, \partial \Sigma) \to (X, L)$$

satisfying $\bar{\partial}_{J_\nu} u = 0$ is equivalent to a standard $J_{\tilde{\nu}}$-holomorphic map

$$\tilde{u} : (\Sigma, \partial \Sigma) \to (\tilde{\Sigma} \times X, \partial \Sigma \times L)$$

satisfying $\pi_1 \circ \tilde{u} = \text{Id}_\Sigma$. We conclude that $\bar{\partial}_{J_\nu}$ does indeed give rise to an elliptic boundary value problem. \hfill \Box

In light of Lemma 4.1, we define

$$\tilde{\mathcal{M}}_{k,l}(L, \Sigma, d) := \bar{\partial}_{J_\nu}^{-1}(0) \subset B_{k,l}^{1,p}(L, \Sigma, d).$$

Let $\phi$ be an anti-symplectic involution of $X$ such that $L = \text{Fix}(\phi)$. We call a map

$$u : (\Sigma, \partial \Sigma) \to (X, L)$$

$\phi$-multiply covered if there does not exist $z \in \Sigma$ such that

$$du(z) \neq 0, \quad u(z) \notin u(\Sigma \setminus \{z\}), \quad u(z) \notin \text{Im} \phi \circ u.$$  

Such maps are also commonly termed not $\phi$-somewhere injective. A standard argument shows that the moduli space of $\phi$-somewhere injective maps has expected dimension for generic $J \in J_{\omega,\phi}$ even when $\nu = 0$. See [6, Section 11]. However, if we take $\nu = 0$, then the moduli space $\tilde{\mathcal{M}}_{k,l}(L, \Sigma, d)$ may be singular at $\phi$-multiply covered maps even for a generic choice of $J$. Assume for a moment that $\Sigma = D^2$ and $\mu(d) > 2$. Then by the following Lemma, the image of $\phi$-multiply covered maps under the evaluation map has codimension greater than or equal to 2.
Lemma 4.2. Suppose \( u : (D^2, \partial D^2) \rightarrow (X, L) \) is \( \phi \)-multiply covered. Then, we may factor \( u \) as a composition \( u = u' \circ \chi \), where \( u' \) is a real \( J \)-holomorphic map
\[
\chi : (D^2, \partial D^2) \rightarrow (\mathbb{C}P^1, \mathbb{R}P^1)
\]
is a holomorphic map of degree greater than or equal to 2.

Remark 4.3. Since we may extend \( \chi \) to a holomorphic map from \( \mathbb{C}P^1 \) to \( \mathbb{C}P^1 \) by the Schwarz reflection principle, \( \chi \) is actually given by polynomial. Thinking of \( D^2 \) as \( H \cup \{ \infty \} \) and thinking of \( \mathbb{C}P^1 \) as \( \mathbb{C} \cup \{ \infty \} \), the boundary condition on \( \chi \) implies that \( \chi \) is given by a real polynomial.

Proof of Lemma 4.2. Gluing together \( u \) and \( \phi \circ u \circ c' \) we obtain a real \( J \)-holomorphic map
\[
\tilde{u} : \mathbb{C}P^1 \rightarrow X, \quad \phi \circ \tilde{u} \circ c' = \tilde{u}.
\]
By a standard theorem \([17, \text{Chapter 2}] \), we can factor \( \tilde{u} = \tilde{\chi} \circ \tilde{u}' \) where \( \tilde{u}' \) is somewhere injective. Another standard result \([17, \text{Theorem E.1.2}] \) says that \( \tilde{u}' \) is injective except at a finite number of points. Since the image of \( u' \) is clearly invariant under \( \phi \), \( \phi \) induces an anti-holomorphic involution of \( \mathbb{C}P^1 \) away from a finite number of points. So, removing singularities, there exists an anti-holomorphic involution \( c'' \) on \( \mathbb{C}P^1 \) such that \( \phi \circ \tilde{u}' \circ c'' = \tilde{u}' \). Moreover, \( c'' \) fixes the image of \( \mathbb{R}P^1 \) under \( \tilde{\chi} \). It follows that \( c'' \) is conjugate to \( c' \) by some biholomorphism \( a \in PSL_2(\mathbb{C}) \). So, we may define
\[
\tilde{u}' = \tilde{u}' \circ a^{-1}, \quad \chi = a \circ \tilde{\chi}|_{\partial D^2}.
\]
Consequently, under the previously mentioned conditions, \( \phi \)-multiply covered maps are not important in the definition of intersection theory on the moduli space. So, even in the case \( \nu = 0 \), we may obtain a smooth moduli space by defining
\[
\tilde{M}_{k,i}^*(L, \Sigma, d) := \tilde{M}_{k,i}(L, \Sigma, d) \setminus \{ \phi \text{-multiply covered maps} \}.
\]
Then, to obtain an interesting intersection theory, we define
\[
M_{k,i}(L, \Sigma, d) := \tilde{M}_{k,i}^*(L, \Sigma, d)/PSL_2(\mathbb{R}), \quad (14)
\]
where $PSL_2(\mathbb{R})$ acts by
\[(u, \vec{z}, \vec{w}) \rightsquigarrow (u \circ \varphi, (\varphi^{-1})^k(\vec{z}), (\varphi^{-1})^l(\vec{w})), \quad \varphi \in PSL_2(\mathbb{R}).\]

This choice of action ensures that the evaluation maps descend to the quotient. Note that the action would not preserve the moduli space if we were to allow a general inhomogeneous term $\nu$.

Equivalently, instead of quotienting by $PSL_2(\mathbb{R})$, we could consider an appropriate section of the group action. Fortunately, this approach, as observed in [20], generalizes to the situation where we allow a generic $\nu$ and so provides a definition of the moduli space that works even when $(X, L)$ may admit holomorphic disks of Maslov index 0 or when $\Sigma \neq D^2$.

So, we construct a section of the reparameterization group action in the following manner. Let
\[\pi_j : \tilde{M}_{k,l}(L, \Sigma, d) \to \Sigma\]
be the projection sending $(u, \vec{z}, \vec{w}) \rightsquigarrow w_j$. First, suppose that $\Sigma \simeq D^2$. Choose an interior point $s_0 \in \Sigma$ and a line $\ell \subset D^2$ connecting $s_0$ to $\partial \Sigma$ such that for any pair of points $(w, w') \in \Sigma$ there exists a unique $\varphi \in Aut(\Sigma) \simeq PSL_2(\mathbb{R})$ that satisfies
\[\varphi(w) = s_0, \quad \varphi(w') \in \ell.\]

For the time being, assume $l \geq 2$. We require that the dependence of $\nu$ on $u \in B_{k,l}$ factors through $\pi_1 \times \pi_2$. Moreover, letting $d(\cdot, \cdot)$ denote the distance function on $D^2$, we impose on $\nu$ the condition
\[\nu(\cdot, \cdot, (w_1, w_2)) = \frac{d(w_1, w_2)}{d(w'_1, w'_2)} \nu(\cdot, \cdot, (w'_1, w'_2)). \quad (15)\]

In particular, $\nu$ vanishes uniformly in the limit $w_2 \to w_1$. We define
\[\mathcal{M}_{k,l}(L, \Sigma, d) := (\pi_1 \times \pi_2)^{-1}(s_0 \times \ell) \subset \tilde{M}_{k,l}(L, \Sigma, d). \quad (16)\]

Standard arguments show that for a generic choice of $\nu$ satisfying (15), $\mathcal{M}_{k,l}(L, \Sigma, d)$ will be a smooth manifold of expected dimension. We digress briefly to explain the significance of condition (15). To prove the invariance of $N_{\Sigma, d, k,l}$, we need to argue that stable maps in the Gromov compactification of $\mathcal{M}_{k,l}(L, \Sigma, d)$ involving sphere bubbles occur only in codimension two. A sequence $u^i \in \mathcal{M}_{k,l}(L, \Sigma, d)$ such that $w^i_2 \to w^i_1$ will Gromov converge to
a stable map consisting of one disk component and one sphere component with both \( w_1 \) and \( w_2 \) on the sphere component. The nodal point where the sphere and disk are attached is fixed at \( s_0 \). However, there are no other fixed marked points on the disk component to compensate for the remaining \( T^1 \) symmetry. Condition \([15]\) implies that the limit inhomogeneous term on the disk component is zero. So, we can simply quotient by the residual \( T^1 \) action. Although \( \phi \)-multiply covered maps can arise, we may disregard them because by a standard argument, their image under the evaluation map has high codimension.

If, \( \Sigma \cong S^1 \times I \), we choose a line \( \ell \subset \Sigma \) connecting the two boundary components of \( \Sigma \) such that for any \( w \in \Sigma \) there exists a unique \( \varphi \in Aut(\Sigma) \cong T^1 \) such that \( \varphi(w) \in \ell \). We choose \( \nu \) to be entirely independent of \( u \). Assuming for the time being that \( l \geq 1 \), define

\[
\mathcal{M}_{k,l}(L, \Sigma, d) := \pi_{-1}^{-1}(\ell) \subset \tilde{\mathcal{M}}_{k,l}(L, \Sigma, d).
\]

Again, for a generic choice of \( \nu \), standard arguments show that \( \mathcal{M}_{k,l}(L, \Sigma, d) \) will be a smooth manifold of expected dimension.

Now we turn to the case when \( \Sigma \cong D^2 \) and \( l = 0 \). This case is of particular interest because it arises when \( X \) is a Calabi-Yau manifold and \( L \) is a special Lagrangian submanifold. The cases \( \Sigma \cong D^2, l = 1 \), and \( \Sigma \cong S^1 \times I, l = 0 \), use a very similar argument.

We start with the moduli space \( \mathcal{M}_{k,2}(L, \Sigma, d) \) constructed above and proceed as follows. Choose smooth manifolds \( A, B, \) and maps

\[
f : A \to X, \quad g : B \to X
\]

such that \((A, f)\) and \((B, g)\) define pseudo-cycles representing the Poincare-dual of any non-trivial \( \phi \)-anti-invariant 2nd cohomology class. The symplectic form \( \omega \) provides at least one example, and for simplicity, we will continue with this example.

**Lemma 4.4.** We can choose \((A, f)\) and \((B, f)\) that are \( \phi \)-anti-invariant and do not intersect \( L \).

**Proof.** A straightforward transversality argument shows that we may assume \((A, f)\) is transversal to \( L \). If necessary, replacing \( A \) by \( \frac{1}{2}A \bigsqcup -\frac{1}{2}A \) and \( f \) by \( f \bigsqcup \phi \circ f \), we may assume that the pseudo-cycle \((A, f)\) is \( \phi \)-anti-invariant just like \( \omega \). Now, we consider a local model for \((A, f)\) near \( L \) and show how
to modify \((A,f)\) near \(L\) to avoid intersecting \(L\). Choose \(\phi\)-invariant local symplectic coordinates \(\Theta : X \to \mathbb{C}^n\) near \(L\) such that \(\Theta(L) = \mathbb{R}^n\). We may assume that the image of \(f\) is given by the union of the vanishing sets of conjugate real-linear maps

\[
\ell, \bar{\ell} : \mathbb{C}^n \to \mathbb{R}^2.
\]

More precisely, choose complex coordinates \(z = (z_1, \ldots, z_n)\) on \(\mathbb{C}^n\). Taking \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\), we write \(z = x + iy\). So, we can decompose

\[
\ell(z) = \ell^x(x) + \ell^y(y), \quad \bar{\ell}(z) = \ell^x(x) - \ell^y(y).
\]

Letting \(\ell_i, i = 1, 2\), denote the \(i\)th component of \(\ell\), the equations for the image of \(f\) may be written as

\[
0 = \ell_i \cdot \bar{\ell}_i = \ell_i^x - \ell_i^y, \quad i = 1, 2.
\]

So, choosing small constants \(\epsilon_i > 0, i = 1, 2\), we modify \((A, f)\) so that locally the image of \(f\) satisfies equations

\[
\ell_i^x - \ell_i^y = -\epsilon_i, \quad i = 1, 2.
\]

Clearly, these equations have no real solutions. The same applies for \(B\).

Furthermore, we may assume that \((A \times B, f \times g)\) is transversal to \(evi_1 \times evi_2\) so that we may define

\[
\mathcal{M}_{k,0}(L, \Sigma, d) := \frac{1}{\omega(d)^2} \mathcal{M}_{k,2}(L, \Sigma, d) \times_{X \times X} (A \times B).
\]

The factor \(\frac{1}{\omega(d)^2}\) in front of the fiber-product means that each point in the moduli space should be counted with weight \(\frac{1}{\omega(d)^2}\). This correction is designed to cancel the contribution of the divisors \((A, f)\) and \((B, g)\) as predicted by the divisor axiom of formal Gromov-Witten theory. Indeed a map \(u : (\Sigma, \partial\Sigma) \to (X, L)\) representing the class \(d \in H_2(X, L)\) should intersect a pseudo-cycle Poincare dual to \(\omega\) exactly \(\omega(d)\) times.

Remark 4.5. Note that even when \(l > 0\), we are free to fix the group action by adding divisor constraints just as in the case when \(l = 0\).
Proof of Theorem 1.1. Given this definition of $\mathcal{M}_{k,l}(L, \Sigma, d)$, Proposition 3.1 immediately implies Theorem 1.1. We choose the isomorphism according to Definition 3.2.

Now, for a sufficiently generic choice of points $\vec{x} = (x_{a_i}), x_{a_i} \in L$, and pairs of points $\vec{y} = (y_j)$,

$$y_j : \{0, 1\} \to X, \quad y_j(1) = \phi \circ y_j(0),$$

the total evaluation map

$$\text{ev} := \prod_{a,i} \text{ev}_{b_{ai}} \times \prod_j \text{ev}_{i_j} : \mathcal{M}_{k,l}(L, \Sigma, d) \to L^{k|} \times X^l$$

will be transverse to

$$\prod_{a,i} x_{a_i} \times \prod_j y_j : \{0, 1\}^l \to L^{k|} \times X^l.$$ 

So, assuming the dimension condition (2) is satisfied, we may define

$$N_{\Sigma,d,k,l} := \#\text{ev}^{-1}(\vec{x}, \vec{y}).$$

Here, $\#$ denotes the signed count with the sign of a given point $v \in \text{ev}^{-1}(\vec{x}, \vec{y})$ depending on whether or not the isomorphism

$$d\text{ev}_v : \text{det}(T\mathcal{M}_{k,l}(L, \Sigma, d))_v \cong \text{ev}^* \text{det}
\left( T \left( L^{k|} \times X^l \right) \right)_v$$

agrees with the isomorphism of Theorem 1.1 up to the action of the multiplicative group of positive real numbers. If the dimension condition (2) is not satisfied, we define $N_{\Sigma,d,k,l} := 0$.

5 The sign of conjugation on the moduli space

Now, suppose $\Sigma$ is biholomorphic to $\overline{\Sigma}$. It follows that there exists a complex conjugation $c : \Sigma \to \Sigma$. Let $\phi$ be an anti-symplectic involution of $X$ such that $\text{Fix}(\phi) = L$. Fix $J \in \mathcal{J}_{\omega, \phi}$, define $\mathcal{P}_{\phi,c}$ to be the set of $\nu \in \mathcal{P}$ such that $d\phi \circ \nu \circ dc = \nu$ and let $\nu \in \mathcal{P}_{\phi,c}$. We define an involution

$$\phi_B : B_{k,l}^{1,\nu}(L, \Sigma, d) \to B_{k,l}^{1,\nu}(L, \Sigma, d)$$
by

\[(u, \vec{z}, \vec{w}) \sim (\phi \circ u \circ c, (c|_{\partial D^2})^k(\vec{z}), c'(\vec{w})).\]

Furthermore, define an involution \(\phi_E : \mathcal{E} \to \mathcal{E}\) covering \(\phi_B\) by sending \(\eta \in \mathcal{E}_u\) to \(d\phi \circ \eta \circ dc \in \mathcal{E}_{\phi_B(u)}\). It is easy to see that \(\overline{J,\nu}\) is \(\phi_B - \phi_E\) equivariant, and hence \(\phi_E\) induces an involution \(\phi_L : \mathcal{L} \to \mathcal{L}\) covering \(\phi_B\). Now, the trivial bundle morphism of

\[\bigotimes_{a,i} \text{ev}_{b_i}^* \det(TL)\]

covers \(\phi_B\). So, the involution \(\phi_L\) induces an involution \(\phi'_L\) of the bundle

\[\mathcal{L}' := \text{Hom} \left( \bigotimes_{a,i} \text{ev}_{b_i}^* \det(TL), \mathcal{L} \right) \simeq \mathcal{L} \otimes \bigotimes_{a,i} \text{ev}_{b_i}^* \det(TL).\]

covering \(\phi_B\). Now, the bundle \(\mathcal{L}'\) has a canonical orientation corresponding to the canonical isomorphism of Definition 3.2. So, the involution \(\phi'_L\) may either preserve the orientation component of the complement of the zero section of \(\mathcal{L}'\), exchange it with the opposite component or some combination of the two over different connected components of the base. If \(\phi'_L\) preserves the orientation of \(\mathcal{L}'\), we say it has sign 0. If \(\phi'_L\) does not preserve the orientation of \(\mathcal{L}'\) anywhere, we say it has sign 1.

We would like to show the sign of \(\phi_L\) is well defined and calculate it. To properly formulate the result, we define a degree 0 homomorphism

\[\psi : H_*(X, L; \mathbb{Z}/2\mathbb{Z}) \to H_*(X; \mathbb{Z}/2\mathbb{Z})\]

on the level of singular chains as follows: We implicitly use \(\mathbb{Z}/2\mathbb{Z}\) coefficients everywhere. Suppose

\[\sigma \in C_*(X, L) := C_*(X)/C_*(L)\]

is a relative singular chain. Let \(\tilde{\sigma} \in C_*(X)\) represent \(\sigma\). Define

\[\widehat{\psi}(\sigma) := (\text{Id} + \phi_*)\tilde{\sigma}.\]

\(\widehat{\psi}\) is well defined because if \(\sigma = 0\) then \(\tilde{\sigma} \in C_*(L)\) and hence

\[(\text{Id} + \phi_*)\tilde{\sigma} = 2\tilde{\sigma} \equiv 0 \mod 2.\]

Since \(\phi_*\) commutes with the boundary operator, so does \(\widehat{\psi}\), so we can define

\[\psi := H(\widehat{\psi}).\]

Now, let \(g_0\) denote the genus of \(\Sigma/\partial \Sigma\).
Proposition 5.1. Let $n = \dim L$. If $p$ is a relative Pin$^-$ structure then the involution $\phi'_L$ has sign

$$s^-(d, k, l) \cong \frac{\mu(d)(\mu(d) + 1)}{2} + (1 - g_0)n + mn + |k| + l$$

$$+ w_2(V)(\psi(d)) + w_1(\partial d) + \sum_{a < b} w_1(d_a)w_1(d_b)$$

$$+ \sum_a w_1(d_a)(k_a - 1) + (n + 1)\sum_a \frac{(k_a - 1)(k_a - 2)}{2} \mod 2.$$ 

If $p$ is a relative Pin$^+$ structure, $\phi'_L$ has sign

$$s^+(d, k, l) \cong \frac{\mu(d)(\mu(d) + 1)}{2} + (1 - g_0)n + mn + |k| + l$$

$$+ w_2(V)(\psi(d)) + \sum_{a < b} w_1(d_a)w_1(d_b)$$

$$+ \sum_a w_1(d_a)(k_a - 1) + (n + 1)\sum_a \frac{(k_a - 1)(k_a - 2)}{2} \mod 2.$$ 

Remark 5.2. Suppose $\dim L \leq 3$. Then $L$ is Pin$^-$ by the Wu relations, so we can take $p$ to be a standard Pin$^-$ structure. In particular, $w_2(V) = 0$. Since $w_1(\partial d) \cong \mu(d) \mod 2$, we have

$$s^-(d, k, l) \cong \frac{\mu(d)(\mu(d) - 1)}{2} + (1 - g_0)n + mn + |k| + l$$

$$+ \sum_{a < b} w_1(d_a)w_1(d_b) + \sum_a w_1(d_a)(k_a - 1)$$

$$+ (n + 1)\sum_a \frac{(k_a - 1)(k_a - 2)}{2} \mod 2.$$ 

In the particularly simple case that $\Sigma \simeq D^2$, we have

$$s^-(d, k, l) \cong \frac{\mu(d)(\mu(d) - 1)}{2} + k + l + \mu(d)(k - 1)$$

$$+ (n + 1)\frac{(k - 1)(k - 2)}{2} \mod 2.$$ 

We omit the proof of the preceding proposition as it is very similar to the proof of the next proposition and we do not actually use it.
5 THE SIGN OF CONJUGATION ON THE MODULI SPACE

Now, recall from Section 3 that

\[ B_{k,p,k',\sigma,l,\rho}(L,\Sigma,d',d'') := B_{k'+e_{b'},\rho}(L,\Sigma,d') \times B_{k'+1,\rho}(L, D^2, d''). \]

Note that \( D^2 \simeq \overline{D^2} \) as demonstrated by the standard conjugation \( c : D^2 \to D^2 \).

So, we have an involution \( \phi_{B''} \) of the second factor of the fiber product. Since \( L \subset \text{Fix}(\phi) \), the involution \( \phi_{B''} \) of the second factor induces an involution \( \phi_{B'} \) of the whole fiber product. Similarly, the involution \( \phi_{E''} \) of the bundle \( E'' \) over the second factor of the fiber product induces an involution \( \phi_{E'} \) on the bundle \( E' \to B' \). Recall that the natural inhomogeneous perturbation \( \nu \) for stable maps vanishes on bubble components. In particular, it is \( \phi \)-invariant. So, \( \bar{\partial}_{J,\nu} \) defines a \( \phi_{B''} \) invariant section of \( E'' \to B'' \). Consequently, \( \phi_{B''} \) and \( \phi_{E''} \) induce an involution \( \phi_{L'} \) of the determinant bundle \( L' \to B' \).

As before, \( \phi_{L'} \) induces an involution \( \phi'_{L'} \) of

\[ L'' := \text{Hom} \left( \bigotimes_{a,i} \text{ev}_{b_i}^* \det(TL), L' \right) \simeq L' \otimes \bigotimes_{a,i} \text{ev}_{b_i}^* \det(TL). \]

Proving that the sign of \( \phi'_{L'} \) is well defined and calculating it will play a crucial role in the proof of the invariance of \( N_{\Sigma,d,k,l} \). Before writing down the formula for the sign of \( \phi'_{L'} \), let us introduce some new notation. We define

\[ \Upsilon'(d'', k'') \equiv \mu(d'')k'' \equiv w_1(\partial d'')k'' \pmod{2} \]

and

\[ \Upsilon''(d'_b, d'', k', k'') \equiv \begin{cases} 0, & w_1(d'_b) = w_1(\partial d'') = 0 \\ k', & w_1(d'_b) = w_1(\partial d'') = 1 \\ k'' - 1, & w_1(d'_b) = 1, \ w_1(\partial d'') = 0 \\ k_a - 1 = k'' + k' - 1, & w_1(d'_b) = 0, \ w_1(\partial d'') = 1. \end{cases} \]

**Proposition 5.3.** Let \( n = \dim L \). Suppose the marked point \( z_1 \) does not bubble off, i.e. \( 1 \notin \sigma \). Then the involution \( \phi'_{L'} \) of the line-bundle

\[ L'' \to B_{k,p,k',\sigma,l,\rho}(L,\Sigma,d',d'') \]
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has sign

\[
\mathcal{s}'_\pm(d'', k'', l'') \coloneq \frac{\mu(d'')(\mu(d'') \pm 1)}{2} + w_2(\psi(d'')) + k'' + 1 + l''
\]
\[
\quad + \Upsilon'(d'', k'') + (n + 1)\frac{k''(k'' - 1)}{2} \mod 2,
\]
(17)

with + in the Pin\(^+\) and − in the Pin\(^-\) case. On the other hand, suppose now that the marked point \(z_1\) does bubble off, i.e. \(1 \in \sigma\). Then the involution \(\phi_L\) has sign

\[
\mathcal{s}''_\pm(d_b', d'', k', k'', l'') \coloneq \frac{\mu(d'')(\mu(d'') \pm 1)}{2} + w_2(\psi(d''))
\]
\[
\quad + k'' + 1 + l'' + \Upsilon''(d_b', d'', k', k'') + w_1(d_b')w_1(\partial d'')
\]
\[
\quad + (n + 1)\left(\frac{(k'' - 1)(k'' - 2)}{2} + k_bk'' + k_b\right) \mod 2,
\]
(18)

with + in the Pin\(^+\) and − in the Pin\(^-\) case.

Remark 5.4. Suppose \(L\) is orientable and \(\dim L\) is odd. Then, using the fact that \(\mu(d'')\) is even if \(L\) is orientable, we have

\[
\mathcal{s}' = \mathcal{s}''' = \frac{\mu(d'')(\mu(d'') \pm 1)}{2} + w_2(\psi(d'')) + k'' + 1 + l''.
\]
(19)

Proof of Proposition 5.3: The first term in \(\mathcal{s}'\) comes from the formula (8). This accounts for the sign of conjugation on the moduli space of unmarked disks. The terms \(k'' + 1 + l''\) account for conjugation on the configuration space of the marked points, adding one extra point for the incidence condition. Recall from the proof of Proposition 3.3 that the unique oriented path from \(z \neq z_0\) to \(z'\) in the boundary of \(\partial \Sigma_u\) played an important role in determining the canonical orientation of \(L\). This path depended on the orientation of \(\partial \Sigma\), which is reversed under conjugation. The terms \(\Upsilon'\) (resp. \(\Upsilon''\)) account for this dependence. The remaining terms account for the reordering of the marked points under conjugation, which plays a role only in even dimensions according to Definition 3.4.

We now explain in more detail how the unique oriented path from \(z \neq 0\) to \(z'\) changes under conjugation, and how that effects the orientation of \(L\)'.
First, suppose $1 \notin \sigma$. The path from $z_{b_1}$ to $z_{bi}$ for $i \in \sigma$ will change when the orientation of the boundary of the bubble $\partial D^2$ changes under conjugation. If $w_1(\partial d'') = 1$, this change of path changes the orientation of $(evb^*_u TL)_u$ for each $i \in \sigma$. Since $|\sigma| = k''$, we obtain the total change of orientation given by $\Upsilon'$. The explanation of $\Upsilon''$ is similar.

The additional orientation change $w_1(d'_b)w_1(\partial d'')$ when $1 \in \sigma$ enters because then the path from $z_{b_1}$ to $z_0$ changes under conjugation. This path effects the orientation of $(evb^*_u TL)_u$ when $w_1(\partial d'') = 1$. The orientation of $(evb^*_u TL)_u$ enters twice into the orientation of $\mathcal{L}'#$ when $w_1(\partial d'_b) = 0$. Indeed, it determines the orientation of $evb^*_u det(TL \oplus V_R)_u$, and it determines the orientation of $u_{|_{(\partial \Sigma)_{b}}^*} TL$, which in turn determines the orientation of $det(D^*_u \oplus D^*_0)$. Both of these determinants appear on the right-hand side of isomorphism (13). However, when $w_1(d'_b) = 1$ we cannot orient $u_{|_{(\partial \Sigma)_{b}}^*} TL$.

So, the orientation of $(evb^*_u TL)_u$ enters only once into the orientation of $\mathcal{L}'#$. So, there is an extra contribution to the orientation change of $\mathcal{L}'#$ exactly when

$$w_1(d'_b)w_1(\partial d'') = 1.$$ 

\[ \square \]

6 Proof of invariance

Proof of Theorem 1.3 In order to prove independence from various choices, we construct cobordisms from parameterized moduli spaces. Complications arise in compactifying these cobordisms. For concreteness, we focus on independence of a variation of the constraints on marked points. The proof of independence of a variation of $J, \Sigma$, or $A, B$, is very similar.

Recall that the definition of $N_{\Sigma, d, k, l}$ depends on the choice of points $\vec{x} = (x_{ai}), x_{ai} \in L$, and pairs of points, $\vec{y} = (y_j), y_j : \{0, 1\} \rightarrow X, y_j(1) = \phi(y_j(0))$.

Suppose we choose different points $\vec{x}'$ and $\vec{y}'$ satisfying the same conditions. This corresponds to changing the forms $\alpha_{ai}$ and $\gamma_j$ mentioned in Theorem 1.3 Let

$$\begin{align*}
x : [0, 1] \rightarrow L^{|k|}, & \quad x(0) = \vec{x}, \quad x(1) = \vec{x}', \\
y : [0, 1] \times \{0, 1\}^l \rightarrow X^l, & \quad y(t, 1) = \phi(y(t, 0)), \\
y(0, *) = \vec{y}(*) & \quad y(1, *) = \vec{y}'(*)
\end{align*}$$
If we choose \( x \) and \( y \) generically, they will be transverse to the total evaluation map
\[
ev : \mathcal{M}_{k,l}(L, \Sigma, d) \to L^{|k|} \times X^l.
\]
So,
\[
\mathcal{W} := \mathcal{W}(x, y) := \mathcal{M}_{k,l}(\Sigma, L, d)_{\ev \times (x \times y) \circ \Delta} ([0, 1] \times \{0, 1\}^l)
\]
gives a smooth oriented cobordism between
\[
\ev^{-1}(\vec{x}, \vec{y}) \quad \text{and} \quad \ev^{-1}(\vec{x}', \vec{y}').
\]
However, \( \mathcal{W} \) is generally not compact. So, in order to show the invariance of \( N^{\Sigma, d, k, l} \) we must study the non-trivial stable maps arising in the Gromov-compactification of \( \mathcal{W} \), which we denote by \( \partial_G \mathcal{W} \).

We now digress for a moment to describe \( \partial_G \mathcal{W} \) more explicitly. We define
\[
\tilde{\mathcal{M}}_{k,\sigma, l,\varrho}(L, \Sigma, d', d'') := \bar{\partial}_{J, \nu}^{-1} \subset \mathcal{M}_{k,\sigma, l,\varrho}(L, \Sigma, d', d'').
\]
Recall that the inhomogeneous perturbation \( \nu \) vanishes on the bubble \( D^2 \subset \hat{\Sigma} \). This means that the moduli space \( \tilde{\mathcal{M}}_{k,\sigma, l,\varrho}(L, \Sigma, d', d'') \) may be singular at \( \phi \)-multiply covered maps. As long as \( \mu(d'') > 0 \), this does not present a problem because Lemma 4.1 then shows that the image under the evaluation map of stable maps with \( \phi \)-multiply covered bubbled components has codimension at least two. Also, constant holomorphic disks have expected dimension. On the other hand, in the case that \( (X, L) \) admits holomorphic disks of positive energy of Maslov index zero, we must take \( \phi \)-multiply covered maps into consideration. We postpone the argument in this case to Section 7.

We continue now with the description of \( \partial_G \mathcal{W} \). Since \( \nu = 0 \) on bubble components, we have an action of \( PSL_2(\mathbb{R}) \) on \( \tilde{\mathcal{M}}_{k,\sigma, l,\varrho}(L, \Sigma, d', d'') \) given by
\[
(u', (u'', \vec{z}'', \vec{w}'')) \sim \left( u', (u'' \circ \varphi, (\varphi^{-1})''(\vec{z}''), (\varphi^{-1})''(\vec{w}'')) \right), \quad \varphi \in PSL_2(\mathbb{R})
\]
On the other hand, a generic perturbation term \( \nu \) will break the \( Aut(\Sigma) \) invariance of \( \bar{\partial}_{J, \nu}^{-1} \). So, we construct a section of the \( Aut(\Sigma) \) action that would exist if \( \nu \) vanished in the following manner. Let
\[
\pi_j : \tilde{\mathcal{M}}_{k,\sigma, l,\varrho}(L, \Sigma, d', d'') \to \Sigma
\]
be the projection sending \( (u, \vec{z}, \vec{w}) \sim w_j \). In addition, we define
\[
\pi'_0 : \tilde{\mathcal{M}}_{k,\sigma, l,\varrho}(L, \Sigma, d', d'') \to \Sigma
\]
by \( \pi'_0(\mathbf{u}) = z'_0 \), the point where the bubble attaches.

First, suppose that \( \Sigma \simeq D^2 \) and \( l \geq 2 \). Recall from the construction of \( \mathcal{M}_{k,l}(L, \Sigma, \mathbf{d}) \) in Section 4 that we chose an interior point \( s_0 \in \Sigma \) and a line \( \ell \subset D^2 \) connecting \( s_0 \) to \( \partial \Sigma \). We imposed the conditions \( w_1 = s_0 \) and \( w_2 \in \ell \). Since \( s_0 \) is an interior point, \( w_1 \) cannot possibly bubble off onto a disk bubble, i.e. \( 1 \notin \mathcal{g} \). However, \( w_2 \) could bubble onto a disk bubble that bubbles off at \( \ell \cap \partial \Sigma \). So, we consider the following two cases. If \( w_2 \) does not bubble off, i.e. \( 2 \notin \mathcal{g} \), define

\[
\mathcal{M}_{k,\sigma, l, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') := \left( \pi_1 \times \pi_2 \right)^{-1}(s_0 \times \ell) / \text{PSL}_2(\mathbb{R})
\]

\[
\subset \tilde{\mathcal{M}}_{k,\sigma, l, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') / \text{PSL}_2(\mathbb{R}).
\]

If \( w_2 \) does bubble off, that is, \( 2 \in \mathcal{g} \), we define

\[
\mathcal{M}_{k,\sigma, l, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') := \left( \pi_1 \times \pi_0' \right)^{-1}(s_0 \times (\ell \cap \partial \Sigma)) / \text{PSL}_2(\mathbb{R}).
\]

Now, suppose \( \Sigma \simeq S^1 \times I \) and \( l \geq 1 \). Recall from the construction of \( \mathcal{M}_{k,l}(L, \Sigma, \mathbf{d}) \) that we chose a line \( \ell \subset \Sigma \) connecting the two boundary components of \( \Sigma \) and imposed the condition \( w_1 \in \ell \). So, \( w_1 \) could bubble off at a disk connecting to \( \ell \cap \partial \Sigma \). So, we consider the following two cases. If \( w_1 \) does not bubble off, i.e. \( 1 \notin \mathcal{g} \), define

\[
\mathcal{M}_{k,\sigma, l, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') := \pi_1^{-1}(\ell) / \text{PSL}_2(\mathbb{R}) \subset \tilde{\mathcal{M}}_{k,\sigma, l, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') / \text{PSL}_2(\mathbb{R}).
\]

If \( w_1 \) does bubble off, i.e. \( 1 \in \mathcal{g} \), define

\[
\mathcal{M}_{k,\sigma, l, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') := \pi_0^{-1}(\ell \cap \partial \Sigma) / \text{PSL}_2(\mathbb{R}).
\]

Now we turn to the case when \( \Sigma \simeq D^2 \) and \( l = 0 \). The cases \( \Sigma \simeq D^2, l = 1 \), and \( \Sigma \simeq S^1 \times I, l = 0 \) use a very similar argument, which we omit. Recall from the construction of \( \mathcal{M}_{k,l}(L, \Sigma, \mathbf{d}) \) that we chose \( \phi \)-anti-invariant pseudo-cycles \((A, f)\) and \((B, g)\) representing the Poincare dual of \( \omega \) and satisfying various transversality conditions. Taking \( \mathcal{M}_{k,\sigma,2, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') \) as defined above, we may perturb \((A, f)\) and \((B, g)\) slightly so that the evaluation map

\[
ev_1 \times ev_2 : \mathcal{M}_{k,\sigma, 2, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') \to X \times X
\]

is transversal to \((A \times B, f \times g)\). So, we may define

\[
\mathcal{M}_{k,\sigma,0, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') := \frac{1}{\omega(\mathbf{d})^2} \mathcal{M}_{k,\sigma, 2, \mathcal{g}}(L, \Sigma, \mathbf{d}', d'') \times _{X \times X} (A \times B).
\]
Perturbing $x$ and $y$ slightly assures they are transverse to the total evaluation map
\[ \text{ev} : \mathcal{M}_{k,\sigma,l,\varrho}(L, \Sigma, d', d'') \to X^l \times L^{|k|}. \]
Each moduli space $\mathcal{M}_{k,\sigma,l,\varrho}$ contributes a boundary stratum of the cobordism $\mathcal{W}$, which we denote by
\[ \partial_G \mathcal{W}_{\sigma,\varrho} := \mathcal{M}_{k,\sigma,l,\varrho}(\Sigma, L, d', d'') \text{ev} \times_{(x \times y) \circ \Delta} ([0, 1] \times \{0, 1\}). \]
In total, the boundary of the Gromov compactification of $\mathcal{W}$ takes the form
\[ \partial_G \mathcal{W} = \bigcup_{\alpha \in [1,m], \beta \subset [1,k_\alpha]} \partial_G \mathcal{W}_{\sigma,\varrho}. \]

In general, one might expect an extra term in $\partial_G \mathcal{W}$ coming from sphere bubbles attached to a constant disk. If there are no marked points on the disk, this may happen in codimension 1. However, assumption \[3\] precludes this possibility.

Recall that $\mathbb{Z}/2\mathbb{Z}$ acts on $B_{k,\sigma,l,\varrho}(L, \Sigma, d', d'')$ by the involution $\phi_{B^\#}$ that exchanges a disk bubble with its conjugate. Now, the boundary strata $\mathcal{W}_{\sigma,\varrho}$ are constructed from $B_{k,\sigma,l,\varrho}(L, \Sigma, d', d''), X$ and $L$, by considering the vanishing set of a $\mathbb{Z}/2\mathbb{Z}$ equivariant section and then taking various fiber products with respect to $\mathbb{Z}/2\mathbb{Z}$ equivariant maps. So, each stratum admits a canonical $\mathbb{Z}/2\mathbb{Z}$ action by an involution that we denote by $\phi_{\partial \mathcal{W}}$. We claim this action is fixed point free and orientation reversing. In other words,
\[ \#\partial_G \mathcal{W} = 0 \]
so that
\[ 0 = \#\partial \mathcal{W} = \#(\text{ev}^{-1}(\vec{x}', \vec{y}')) - \text{ev}^{-1}(\vec{x}, \vec{y}) + \partial_G \mathcal{W} \]
\[ = \#\text{ev}^{-1}(\vec{x}', \vec{y}') - \#\text{ev}^{-1}(\vec{x}, \vec{y}) \]
(21)
and $N_{\Sigma,d,k,l}$ does not depend on the choice of $\vec{x}, \vec{y}$.

First, we show that $\phi_{\partial \mathcal{W}}$ is fixed point free. Indeed, as noted above, we are presently considering the case where we may assume there are no $\phi$-multiply covered disks of positive energy. By definition, a $\phi$-somewhere injective disk cannot be a fixed point of $\phi_{\partial \mathcal{W}}$. So, $\phi_{\partial \mathcal{W}}$ could only have a fixed point if a zero energy disk bubbled off. That would correspond to an interior marked
point moving to the boundary. Clearly, marked points that are constrained
away from \( L \) cannot possibly move to the boundary. This is where we use
the fact that by Lemma 4.4 we have chosen \((A, f)\) and \((B, g)\) not to intersect
\( L \).

The following calculations show that \( \phi_{\partial W} \) reverses orientation. First, we
consider the case that \( \dim X = 6 \) and \( L \) is orientable. Since \( \dim L = 3 \), by
Wu’s relations, \( w_2(TL) = 0 \). So, by formula (19), the sign of \( \phi'_{L^\#} \) is given by
\[
\sigma^\# \cong \frac{\mu(d'')}{2} + k'' + l'' + 1. \tag{22}
\]
Furthermore, \( \mathbb{Z}/2\mathbb{Z} \) acts on \( M_{k,\sigma,l,\emptyset}(L, \Sigma, d', d'') \) with this same sign. Indeed, in the case \( l \leq 2 \), we need to add marked points and fiber product
with the divisors along the corresponding evaluation map. One of the extra
marked points could be on the bubble component, so that \( \mathbb{Z}/2\mathbb{Z} \) acts on it
non-trivially. However, since the divisors are chosen to be \( \phi \) anti-invariant,
the total sign change induced on \( M_{k,\sigma,l,\emptyset} \) from the extra marked point will
be zero. Also, we note at this point that the sign of the conjugation auto-
morphism of \( PSL_2(\mathbb{R}) \) is zero.

Note that the sign of \( \phi_{\partial W} \) is independent of \( l'' \). Indeed, the action of \( \phi \)
on \( X \) reverses orientation because \( \phi^* \omega^3 = -\omega^3 \). So, the sign of the \( \mathbb{Z}/2\mathbb{Z} \) action on the fiber product of \( M_{k,\sigma,l,\emptyset}(L, \Sigma, d', d'') \) with \([0, 1] \times \{0, 1\}^l \) over
\( X^l \), where \( \phi \) acts non-trivially on \( l'' \) of the factors of \( X \), does not depend on
\( l'' \). On the other hand, a straightforward virtual dimension calculation shows
that \( \partial G W_{\sigma,\emptyset} \) must be empty unless
\[
\mu(d'') = 2k'' + 4l''.
\]
This in turn implies that \( \mu(d'')/2 \cong k'' \pmod{2} \), or equivalently,
\[
\frac{\mu(d'')}{2} + k'' + 1 \cong 1 \pmod{2}.
\]
So, by equation (22), \( \phi_{\partial W} \) reverses orientation.

Now we turn to the more difficult situation where \( \dim X = 4 \) and \( L \) may
not be orientable. By the Wu relations, \( L \) is \( Pin^- \). So, we assume that \( p \) is
given by a standard \( Pin^- \) structure. By the same argument as above, we
conclude that \( \mathbb{Z}/2\mathbb{Z} \) acts on \( M_{k,\sigma,l,\emptyset}(L, \Sigma, d', d'') \) with sign given by formulas
(17) or (18) depending on whether or not \( 1 \in \sigma \). Note that \( \phi \) preserves the
orientation of \( X \) because \( \phi^* \omega^2 = \omega^2 \). This implies that (17, 18), also give
the right signs for the action of $\phi_{W_W}$. Next, observe that by virtual dimension counting, the stratum $\partial_G \mathcal{W}_{o, \ell}$ will be empty unless

$$\mu(d'') + r = k'' + 2l''$$

for $r = 0$ or $-1$. We claim that this restriction implies that the signs (17) and (18) always simplify to exactly 1.

We first consider the case $1 \notin \sigma$. Using the restriction (23), we calculate

$$\frac{k''(k'' - 1)}{2} = \frac{(\mu(d'') + r - 2l'')(\mu(d'') + r - 2l'' - 1)}{2}$$

$$= \frac{\mu(d'')^2 + 2r\mu(d'') + r^2 - 4l''(\mu(d'') + r) + 4l''^2 - \mu(d'') - r - 2l''}{2}$$

$$\equiv \frac{\mu(d'')(\mu(d'') - 1)}{2} + \frac{r(r - 1)}{2} + l'' + r\mu(d''). \quad (\text{mod } 2) \quad (24)$$

Again using the restriction (23), we calculate

$$\Upsilon'(d'', k'') \equiv \mu(d'')^k'' \cong \mu(d'')^2 + r\mu(d'') + 2l''\mu(d'')$$

$$\cong \mu(d'') + r\mu(d'') \quad (\text{mod } 2). \quad (25)$$

Substituting equations (24) and (25) into (17), eliminating the remaining $k''$ by (23) and canceling expressions which occur in pairs yields

$$s^\#(d'', k'', l'') \equiv \frac{r(r + 1)}{2} + 1 \quad (\text{mod } 2).$$

This is always exactly 1 since $r = 0$ or $-1$.

We turn now to the case $1 \in \sigma$. Using the restriction (23), we calculate

$$\frac{(k'' - 1)(k'' - 2)}{2} = \frac{(\mu(d'') + r - 2l'')(\mu(d'') + r - 2l'' - 2)}{2}$$

$$= \frac{1}{2} \left[ \mu(d'')^2 + 2r\mu(d'') + r^2 - 3\mu(d'') - 3r$$

$$+ 2 + 4l''^2 + 4l''(\mu(d'') + r) + 6l'' \right]$$

$$\equiv \frac{\mu(d'')(\mu(d'') + 1)}{2} + \frac{r(r + 1)}{2}$$

$$+ r\mu(d'') + l'' + 1 \quad (\text{mod } 2). \quad (26)$$

Furthermore, using the condition $w_1(d_b) \cong k_b + 1$ and (23),

$$k_b k'' + k_b \cong (w_1(d_b) + 1)(\mu(d'') + r + 1) \quad (\text{mod } 2). \quad (27)$$
Substitute calculations (26) and (27) in sign formula (18) and use restriction (23). Cancelling pairs of similar terms, we obtain

\[ s_2'' \equiv \frac{r(r+1)}{2} + r\mu(d'') + r 
+ (w_1(d_b) + 1)(\mu(d'') + r + 1) + \Upsilon'' + w_1(d_b')w_1(d'') \]
\[ \cong r\mu(d'') + r + (w_1(d_b) + 1)(\mu(d'') + r + 1) + \Upsilon'' + w_1(d_b')w_1(d''). \quad (28) \]

Here, the second congruence follows from the fact \( r = 0 \) or \( -1 \).

We now expand \( \Upsilon'' \) to further analyze \( s_2'' \). Using the fact that \( w_1(d_b') = w_1(d_b') + w_1(\partial d'') \), it is easy to verify that

\[ \Upsilon'' = w_1(d_b)(k'' - 1) + w_1(\partial d'')k'. \quad (29) \]

By repeatedly applying restriction (23), the condition that \( w_1(d_a) = 1 + k_a \) and the fact that \( \mu(d'') \equiv w_1(\partial d'') \mod 2 \), we calculate,

\[ w_1(\partial d'')k' \equiv w_1(\partial d'')(k_b - k'') \]
\[ \cong w_1(\partial d'')(w_1(d_b) + 1 + \mu(d'') + r) \]
\[ \cong w_1(\partial d'')(w_1(d_b') + w_1(\partial d'') + 1 + \mu(d'') + r) \]
\[ \cong w_1(\partial d'')w_1(d_b') + \mu(d'')(1 + r) \mod 2. \]

Substituting this calculation in formula (29), and using restriction (23) again, we obtain

\[ \Upsilon'' \cong w_1(d_b)(\mu(d'') + r + 1) + w_1(\partial d'')w_1(d_b') + \mu(d'')(1 + r) \mod 2. \]

Substituting this expression for \( \Upsilon'' \) in formula (28) and cancelling all repeated terms leaves

\[ s_2'' \equiv r\mu(d'') + r + (1)(\mu(d'') + r + 1) + \mu(d'')(1 + r) \]
\[ \cong r\mu(d'') + r + \mu(d'') + r + 1 + \mu(d'') + \mu(d'')r \]
\[ \cong 1 \mod 2, \]

as desired.
7 An equivariant Kuranishi structure

In this section, we complete the proof of invariance of $N_{\Sigma; d, k, l}$ in the case when $(X, L)$ may admit holomorphic disks of Maslov index 0. If $\dim L = 2$, the expected dimension of holomorphic disks with Maslov index 0 is negative. For generic $J$, by Lemma 4.2, such disks don’t exist. So, we consider the case $\dim L = 3$. By assumption, $L$ is orientable. The main tool we use to prove invariance in this case is the notion of a Kuranishi structure, introduced in [5] and extended in [6]. For a summary of relevant information on Kuranishi structures, see Appendix A.

Suppose $(X, K)$ is a space with Kuranishi structure $K = (V_p, E_p, \Gamma_p, s_p, \psi_p)$.

Let $\iota$ be an involution of $X$.

**Definition 7.1.** An extension $\tilde{\iota}$ of an involution $\iota$ to an involution of $K$ consists of $\Gamma_p$-equivariant maps $\iota_p : V_p \to V_{\iota(p)}$ and $\hat{\iota}_p : E_p \to E_{\iota(p)}$ covering $\iota_p$ such that

1. $\iota_{\iota(p)} \circ \iota_p = \text{Id}_{V_p}$.
2. $s_{\iota(p)} \circ \iota_p = \hat{\iota}_p \circ s_p$.
3. $\psi_{\iota(p)} \circ \iota_p|_{s_p^{-1}(0)} = \iota \circ \psi_p$.
4. $\iota_q$ maps $V_{pq} \subset V_q$ to $V_{\iota(p)\iota(q)} \subset V_{\iota(q)}$.
5. $\iota_p \circ \varphi_{pq} = \varphi_{\iota(p)\iota(q)} \circ \iota_q$ and $\hat{\iota}_p \circ \hat{\varphi}_{pq} = \hat{\varphi}_{\iota(p)\iota(q)} \circ \hat{\iota}_q$.

Note that $\iota_p, \hat{\iota}_p$, induce bundle morphisms

$$
\iota_p^T : \det(TV_p) \otimes \det(E_p) \to \det(TV_{\iota(p)}) \otimes \det(E_{\iota(p)})
$$

covering $\iota_p$. Now, suppose that $(X, K)$ has a tangent bundle given by $\Phi_{pq}$. We say that $\tilde{\iota}$ acts smoothly on $(X, K)$ if

$$
\Phi_{\iota(p)\iota(q)} \circ \hat{\iota}_q = \hat{\iota}_p \circ \Phi_{pq}.
$$

If $\iota$ acts smoothly and $(X, K)$ is oriented, the bundle morphisms $\iota_p^T$ may either preserve or reverse the orientation of $K$ over each connected component of $X$. Furthermore, let $G = (P, V'_p, E'_p, s'_p, \psi'_p)$ be a good coordinate system for...
\( K \). We say that \( \mathcal{G} \) is compatible with \( \iota \) if \( \iota(P) = P \) and \( t_p(V'_p) = V'_p \). Finally, a collection \( \{t_p\}_{p \in P} \) of multisections of \( E'_p \) is said to be \( t_p - \iota_p \) equivariant if \( t_{i(p)} \circ \iota_p = \iota_p \circ t_p \) for all \( p \in P \).

Now, we give the main idea of the proof. As we will explain, it is possible to construct an oriented Kuranishi structure with tangent bundle \( K = (V_p, E'_p, s_p, \psi_p) \) on \( \mathcal{W} \). Moreover, we may construct \( K \) so that the induced Kuranishi structure on \( \partial_G \mathcal{W} \) admits a smooth orientation reversing involution \( \bar{\phi}_{\mathcal{W}} \) extending \( \phi_{\mathcal{W}} \). There exists a good coordinate system \( \mathcal{G} = (P, V'_p, E'_p, s'_p, \psi'_p) \) for \( K \) such that its restriction to \( \partial_G \mathcal{W} \) is compatible with \( \bar{\phi}_{\mathcal{W}} \). Finally, we may choose \( \phi_{\mathcal{W}} = \bar{\phi}_{\mathcal{W}} \) equivariant generic transverse multisections \( s'_{p,n} \) satisfying conditions [P1]-[P4] of Theorem [A.3] and coinciding exactly with \( s'_p \) away from \( \psi_p^{-1}(\partial_G \mathcal{W}) \).

Note that the charts of the induced Kuranishi structure on \( \partial \mathcal{W} \) are just \( \partial V_p \). We define 
\[
\partial_G V_p := \psi_p^{-1}(\partial_G \mathcal{W}) \subset \partial V_p.
\]

The same applies for the charts of the good coordinate system and we use the analogous notation. The vanishing sets of the \( s'_{p,n} \) define a 1-dimensional simplicial complex with boundary contained in the \( \partial V'_p \). The boundary is simply a collection of points with signed rational weights. By the same reasoning as in equation (21), it suffices to show that the part of this boundary contained in \( \partial_G V'_p \) has total cardinality zero. Again, we use the sign reversing involutions \( \phi_{\mathcal{W}} \) to cancel the points in pairs. The only slight complication arises because \( \phi_{\mathcal{W}} \) may have fixed points. However, this is easily resolved by the observation that a point \( x \) of the vanishing set of \( s'_{p,n} \) fixed by \( \phi_{\mathcal{W}} \) must have weight zero. Indeed, by definition of transversality for multisections, each branch \( s''_{p,n} \) of \( s'_{p,n} \) is transverse to zero. So, if \( s''_{p,n} \) vanishes at \( x \), the differential
\[
d s''_{p,n} : T_x \partial V'_p \to (E'_p)_x
\]
defines a non-zero element \( \omega \in \det(TV'_p) \otimes \det(E'_p) \). Since \( s'_{p,n} \) is \( \phi_{\mathcal{W}} \) equivariant,
\[
\bar{\phi}_{\mathcal{W}} \circ s''_{p,n} \circ \phi_{\mathcal{W}}
\]
must also be a branch of \( s'_{p,n} \); defining an element \( \omega' \in \det(TV'_p) \otimes \det(E'_p) \). But since \( \bar{\phi}_{\mathcal{W}} \) is orientation reversing, we know that \( \omega \) and \( \omega' \) belong to opposite components of \( \det(TV'_p) \otimes \det(E'_p) \setminus \{0\} \). So, the branches of \( s'_{p,n} \) that vanish at \( x \) come in pairs that induce opposite orientations on \( x \). Therefore, the total weight of \( x \) is zero.
We turn our attention to the construction of $K$. Recall from (20) that

$$W = \mathcal{M}_{k,l}(\Sigma, L, d)_{ev \times (x \times y) \in \Delta} \left( \{0, 1\} \times \{0, 1\} \right).$$

Let $\partial \mathcal{M}_{k,l}(\Sigma, L, d)$ denote the union of all the strata of the Gromov compactification of $\mathcal{M}_{k,l}$ except $\mathcal{M}_{k,l}$ itself. Let $\phi_{\partial \mathcal{M}}$ denote the involution of $\partial \mathcal{M}_{k,l}$ induced by $\phi_{B^\#}$. As explained in [6, Appendix 2], a weakly submersive Kuranishi structure $\mathcal{K}_{\mathcal{M}}$ on $\overline{\mathcal{M}}_{k,l}(\Sigma, L, d)$ naturally induces a Kuranishi structure $\mathcal{K}$ on the fiber product $W$. If we let $\mathbb{Z}/2\mathbb{Z}$ act on $X$ by $\phi$ and on $[0, 1] \times \{0, 1\}$ by exchanging 0 and 1, $x \times y$ is clearly $\mathbb{Z}/2\mathbb{Z}$ equivariant. So, if $\mathcal{K}_{\mathcal{M}|\partial \mathcal{M}_{k,l}}$ admits an extension of $\phi_{\partial \mathcal{M}}$, then $K|_{\partial \mathcal{W}}$ will admit an extension of $\phi_{\partial \mathcal{W}}$. Consequently, we focus on the construction of $\mathcal{K}_{\mathcal{M}}$.

We assume $\Sigma = D^2$ and $l \geq 2$. The other cases are similar. Interpreting (16) as a fiber product, we have

$$\mathcal{M}_{k,l}(\Sigma, L, d) = \tilde{\mathcal{M}}_{k,l}(\Sigma, L, d) \times_{\Sigma} (s_0 \times \ell).$$

Let $\partial \tilde{\mathcal{M}}_{k,l}(\Sigma, L, d)$ denote the union of all strata of the Gromov compactification of $\mathcal{M}_{k,l}$ except $\tilde{\mathcal{M}}_{k,l}$ itself. Let $\phi_{\partial \tilde{\mathcal{M}}}$ denote the involution of $\partial \tilde{\mathcal{M}}_{k,l}$ induced by $\phi_{B^\#}$. Again, we reduce to constructing a weakly submersive Kuranishi structure $\mathcal{K}_{\mathcal{M}}$ on the Gromov compactification of $\tilde{\mathcal{M}}_{k,l}(\Sigma, L, d)$ such that its restriction to $\partial \tilde{\mathcal{M}}_{k,l}(\Sigma, L, d)$ admits an extension of $\phi_{\partial \tilde{\mathcal{M}}}$.

In [6], Fukaya et al. constructed a Kuranishi structure on the moduli space of $J$-holomorphic disks with Lagrangian boundary conditions. Their construction generalizes immediately to the higher genus fixed conformal structure situation considered in this paper. Away from $\partial \tilde{\mathcal{M}}_{k,l}$, we use their construction without further discussion. Near $\partial \tilde{\mathcal{M}}_{k,l}$, we must impose additional conditions on several choices made in the construction to make sure we can find an extension of $\phi_{\partial \tilde{\mathcal{M}}}$. So, we briefly recount the idea of the construction of the Kuranishi neighborhood of a point $p \in \partial \tilde{\mathcal{M}}_{k,l}$ given in [6].

We assume that $p$ is a stable map of two components. The construction for more components is similar. By definition, $p$ is an equivalence class of quadruples $u = (\Sigma, u, \vec{z}, \vec{w}) \in B_{k,\sigma,l,\ell}^p(L, \Sigma, d', d'')$ such that $\bar{\partial}_{\epsilon}^u \epsilon = 0$. The equivalence relation in the case of two components equates reparameterizations of the bubble component. When there are more than two components, the equivalence relation also takes into consideration automorphisms of the underlying tree of the stable map. We choose some $u$ such that $[u] = p$. Locally trivializing $\mathcal{E}^\#$ by parallel translation and projection to $\Lambda^{0,1}(TX)$,
we define
\[ D_u^\# = D_u \tilde{\partial}_{J,\nu}^\# : T_u \mathcal{B}^{1,p}_{k,\sigma,t,\varrho} (L, \Sigma, d', d'') \rightarrow \mathcal{E}_u^\# \]
to be the linearization of \( \tilde{\partial}_{J,\nu}^\# \) at \( u \). Since the bubble component \( u'' \) of \( u \) may be \( \phi \)-multiply covered, \( D_u^\# \) may not be surjective even for generic \( J \) and \( \nu \). However, since \( D_u^\# \) is Fredholm, we may choose a finite dimensional subspace \( E_u \subset \mathcal{E}_u^\# \) such that if \( \pi : \mathcal{E}_u^\# \rightarrow \mathcal{E}_u^\#/E_u \) denotes the natural projection, then \( \pi \circ D_u^\# \) is surjective. Possibly enlarging \( E_u \), we may assume that the evaluation maps from \( \ker \pi \circ D_u^\# \) to \( T_{u(z_i)}L \) and \( T_{u(w_i)}X \) are surjective. This is necessary for the Kuranishi structure we are describing to be weakly submersive. By elliptic regularity, we may choose \( E_u \) to consist of smooth sections of \( \Lambda^{0,1}(u^*TX) \). By the unique continuation theorem, we may assume these sections are compactly supported away from the singular point \( z_0 \).

Let \( u = (\bar{\Sigma}_e, u_e, \bar{z}_e, \bar{w}_e) \) be sufficiently close to \( u \). More precisely, choose a small \( \delta > 0 \). We allow \( \bar{\Sigma}_e \) to differ from \( \bar{\Sigma} \) in a \( \delta \) neighborhood \( N_\delta \) of the singular point \( z_0 \). Either \( z_0 \) may move slightly, or small neighborhoods of \( z_0 \) in each component of \( \bar{\Sigma} \) may be removed and their boundaries glued to smooth the singularity. In particular, outside \( N_\delta \), there exists a canonical identification of \( \bar{\Sigma} \) with \( \bar{\Sigma}_e \). The pre-gluing construction explained in detail in [5] [17] gives a smooth map \( \tilde{u} : (\bar{\Sigma}', \partial \bar{\Sigma}') \rightarrow (X, L) \) that agrees with \( u \) outside \( N_\delta \) and stays \( \delta \)-close to \( u(z_0) \) within \( N_\delta \). We assume that \( u_e \) is \( \delta \) close to \( \tilde{u} \) in the \( W^{1,p} \) norm. Also, we assume that \( \bar{z}_e, \bar{w}_e \) are \( \delta \)-close to \( \bar{z}, \bar{w} \). Then, for \( \delta \) sufficiently small, there exist unique shortest geodesics from \( u(z) \) to \( u_e(z) \) for each \( z \in \bar{\Sigma} \setminus N_\delta \). For \( \delta \) sufficiently small, we may assume \( N_\delta \) is disjoint from the support of the sections of \( \Lambda^{0,1}(u^*TX) \) constituting \( E_u \).

So, we may parallel translate \( E_u \) along length minimizing geodesics and then project to \( \Lambda^{0,1}(TX) \) to obtain a subspace of \( \mathcal{E}_{u_e}^{(\#)} \). Here, we parenthesize \( \# \) because \( u \) may be an irreducible \( W^{1,p} \) stable map. If \( \delta \) is sufficiently small this subspace has constant dimension. We let \( E_{u_e} \) denote the sub-bundle of \( \mathcal{E}_{u_e}^{(\#)} \) so obtained. Similarly, we let \( \bar{\pi} : \mathcal{E}^{(\#)} \rightarrow \mathcal{E}^{(\#)}/E_{u_e} \) define the projection to the quotient bundle.

According to [5] [15], we may essentially define \( V_p \) to be the set of \( u \) as above such that \( \bar{\pi} \circ \tilde{\partial}_{J,\nu}^{(\#)} u_e = 0 \). Then, we define \( E_p = E_{u_e}|_V \) and \( s_p = \tilde{\partial}_{J,\nu}^{(\#)} \). The definition of \( \psi_p \) is tautological. A great deal of hard analysis then shows that \( V_p \) is actually a smooth manifold with boundary modelled on
\[ \ker (\pi \circ D_u^\#) \times (R, \infty). \]
However, we do not need to know the details of this analysis at all for our
purposes. Also, we note that it is necessary to further enlarge $E_u$ in order to construct the maps $\varphi_{pq}, \hat{\varphi}_{pq}$. However, this step is essentially formal, and it is not hard to make it $\varphi$-equivariant. So, we do not discuss it here and instead refer the reader to [6, end of Section 18], or for more detail [5, Section 15].

Finally, we note that canonical orientation of det($D^\#$) induces an orientation of det($TV_p \otimes det(E_p)$). Indeed, note that there exist natural isomorphisms $\ker DS_p \simeq \ker D\partial_{J,\nu}$ and $\coker DS_p \simeq \coker D\partial_{J,\nu}$. On the other hand, the exact sequence

$$0 \to \ker DS_p \to TV_p \to E_p \to \coker DS_p \to 0$$

induces a natural isomorphism

$$\text{det}(TV_p) \otimes \text{det}(E_p) \simeq \text{det}(\ker DS_p) \otimes \text{det}(\coker DS_p).$$

We now detail additional conditions on the choices made in the above construction, which ensure that $K_{\tilde{M}} |_{\partial \tilde{M}}$ admits an extension of $\phi_{\partial \tilde{M}}$. Clearly, we must choose a $\varphi$-invariant metric on $X$ for measuring all distances, and constructing geodesics and parallel transport. We consider two cases: First suppose $p_\varphi := \varphi_{\partial \tilde{M}}(p) \neq p$. We simultaneously construct Kuranishi neighborhoods of $p$ and $p_\varphi$ as well as the extension of $\phi_{\partial \tilde{M}}$. Indeed, given a representative $u$ of $p$, we choose the representative $u_\varphi := \varphi_{B^\#}(u)$ of $p_\varphi$. Furthermore, we choose $E_{u_\varphi} = \phi_{E^\#}(E_u)$. This is compatible with the construction above because of the $\varphi$-invariance of the metric. Again, by $\varphi$-invariance of the metric, it follows that $\phi_{E^\#}$ maps $E_{u_\varphi} |_{B^\#}$ to $E_{u_\varphi} |_{B^\#}$. Since $\partial_{J,\nu}^\#$ is $\phi_{B^\#} - \phi_{E^\#}$ equivariant, it follows that $\phi_{B^\#}$ maps $\partial V_p$ to $\partial V_p$. So, we define the extension of $\phi_{\partial \tilde{M}}$ by

$$\tilde{\phi}_{\partial \tilde{M}} := \phi_{B^\#} |_{\partial V_p}, \quad \hat{\phi}_{\partial \tilde{M}} := \phi_{E^\#} |_{E_p}.$$ 

On the other hand, suppose that $\phi_{\partial \tilde{M}}(p) = p$. This may happen when $u''$ is a $\varphi$-multiply covered disk of even multiplicity. It is not hard to see that we may choose $u$ representing $p$ such that $\phi_{B^\#}(u) = u$. Indeed, this follows from the fact that all anti-holomorphic involutions of $D^2$ are conjugate under the action of $PSL_2(\mathbb{R})$. Furthermore, we choose $E_u$ to be $\phi_{E^\#}$-invariant. This said, we may define the extension of $\phi_{\partial \tilde{M}}$ exactly as above. This completes the construction of $K_{\tilde{M}}$.

Now, a minor adaptation of the proof of [4, Lemma 6.3] gives the good cover $G$. To obtain generic transverse multisections $s'_{p,n}$ such that $s'_{p,n} |_{G V_p}$ is $\tilde{\phi}_{GW}$ equivariant, we use an argument from [3, Section 11]. That is, since
\( \mathcal{K}|_{\partial \mathcal{G} \mathcal{W}} \) admits an extension of \( \phi|_{\partial \mathcal{G} \mathcal{W}} \), it descends naturally to a Kuranishi structure on \( \mathcal{W}/(\mathbb{Z}/2\mathbb{Z}) \). Similarly, \( \mathcal{G}|_{\partial \mathcal{G} \mathcal{W}} \) descends to a good coordinate system \( \hat{\mathcal{G}} = (\hat{P}', \hat{V}', \hat{E}'_p, \hat{s}'_p, \hat{\psi}'_p) \) on \( \partial \mathcal{G} \mathcal{W}/(\mathbb{Z}/2\mathbb{Z}) \). We denote by 
\[
\Pi : \partial \mathcal{W} \to \partial \mathcal{W}/(\mathbb{Z}/2\mathbb{Z})
\]
the quotient map. We can apply the standard machinery of Kuranishi structures developed in [5, Chapter 1], reviewed in Theorem A.4, to obtain generic transversal multisectons \( \bar{s}'_{p,n} \) perturbing \( \bar{s}_p \). Pulling back \( \bar{s}'_{p,n} \) under \( \Pi \), we obtain \( \phi|_{\partial \mathcal{V}_p'} - \hat{\phi}|_{\partial \mathcal{V}_p'} \) equivariant transverse multisectons over \( \partial \mathcal{G} \mathcal{V}_p' \). Since \( \mathcal{V}_p' \) is a manifold with corners, it is not hard to extend a transverse section from \( \partial \mathcal{G} \mathcal{V}_p' \) to a neighborhood of \( \partial \mathcal{G} \mathcal{V}_p' \). In fact, away from corners, a neighborhood of \( \partial \mathcal{G} \mathcal{V}_p' \) is diffeomorphic to \([0, 1) \times \partial \mathcal{G} \mathcal{V}_p' \) and we can extend sections transverse to zero as constants over \([0, 1) \times \partial \mathcal{G} \mathcal{V}_p' \). Since \( \dim K = 1 \), a transverse section cannot have zeros at corners of \( \mathcal{V}_p' \), so we can extend near corners arbitrarily. Note that by choosing a generic inhomogeneous perturbation \( \nu \) we have already made \( s'_p \) transverse away from \( \partial \mathcal{G} \mathcal{V}_p' \). So, we may patch the \( s'_p \) with the extensions we have just constructed to obtain the desired \( s'_{p,n} \), maintaining transversality everywhere. This completes the proof of Theorem 1.3. \( \blacksquare \)

8 Calculations

In this section we prove Theorem 1.8 and Example 1.6. The main tool of the proofs is the notion of a short exact sequence of Cauchy-Riemann \( \text{Pin} \) boundary value problems. The first step in understanding short exact sequences of \( \text{Pin} \) boundary problems is to understand short exact sequences of bundles with \( \text{Pin} \) structure. Suppose
\[
0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0
\]
is a short exact sequence of real vector bundles over a base \( B \).

**Lemma 8.1.** Assume that at least one of \( V' \) and \( V'' \) is orientable. Then a \( \text{Pin} \) structure on any two of \( V, V', V'' \) naturally induces a \( \text{Pin} \) structure on the third.

**Proof.** For the proof of this Lemma, we write \( \dim V' = n \) and \( \dim V'' = m \). Choosing a metric on \( V \), we may identify \( V \simeq V' \oplus V'' \). By symmetry of the direct sum, we may assume that \( V' \) is orientable. We use the orientation of
$V'$ to reduce its structure group to $SO(n)$. So, the Lemma follows from the existence of the commutative square of group homomorphisms

$$
\begin{array}{ccc}
Spin(n) \times Pin(m) & \longrightarrow & Pin(n+m) \\
\downarrow & & \downarrow \\
SO(n) \times O(n) & \longrightarrow & O(n+m).
\end{array}
$$

Indeed, we work on the level of transition functions which satisfy the co-cycle condition. The commutativity of the two factors of the product of groups ensures that the direct sum does not effect the cocyle condition. \hfill \square

**Lemma 8.2.** Let $V \to B$ be a real vector bundle. If $B$ is a one dimensional manifold, then $V$ automatically carries a $Pin$ structure. On the other hand, if $V$ is one dimensional and admits a $Pin$ structure, the $Pin$ structure on $V$ may be chosen canonically.

**Proof.** The first claim of the Lemma follows because the obstruction to the existence of a $Pin$ structure is a second cohomology class. The second claim follows when $B = \mathbb{R}P^1$ because, as noted in Remark 2.7, all automorphisms of a line bundle over $\mathbb{R}P^1$ preserve $Pin$ structure and so we can induce a $Pin$ structure canonically from a previously chosen one on $\tau_\mathbb{R} \to \mathbb{R}P^1$ or $\mathbb{R} \to \mathbb{R}P^1$. This extends to general $B$ because a $Pin$ structure on $V \to B$, if it exists, is determined by its restriction to loops in $B$. \hfill \square

Now, let $E, E', E''$, be complex vector bundles over a Riemann surface with boundary $\Sigma$ and let $F, F', F''$, be totally real subbundles of $E, E', E''$, respectively, over $\partial \Sigma$. Suppose further that we have an exact sequence

$$
0 \to E' \to E \to E'' \to 0
$$

that restricts to an exact sequence

$$
0 \to F' \to F \to F'' \to 0. \quad (30)
$$

We refer to such a short exact sequence as a short exact sequence of pairs of vector bundles. Let $p, p', p''$, be $Pin$-structures on $F, F', F''$, respectively. We say that $p$ is compatible with the short exact sequence (30) if $p$ agrees with the $Pin$ structure naturally induced on $F'$ by $p'$ and $p''$ by Lemma 8.1. If $F'$ or $F''$ is one dimensional, even if it does not come equipped with a $Pin$ structure, we extend the notion of compatibility by equipping it with the canonical $Pin$ structure of Lemma 8.2.
Definition 8.3. A short exact sequence of Cauchy-Riemann Pin boundary value problems

\[ 0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0 \]

consists of

- An exact sequence of pairs of vector bundles

\[ 0 \longrightarrow (E', F') \longrightarrow (E, F) \longrightarrow (E'', F'') \longrightarrow 0 \quad (31) \]

such that at least one of \( F' \) and \( F'' \) is orientable.

- Orientations on each of \( F, F', F'' \), that is orientable. If all three are orientable, we assume the orientation of \( F \) agrees with the orientation induced from \( F' \) and \( F'' \).

- Pin structures \( p, p', p'' \), on \( F, F', F'' \), respectively. If \( F' \) (resp. \( F'' \)) has dimension 1, we do not require \( p' \) (resp. \( p'' \)) as part of the definition, since it may be chosen canonically by Lemma 8.2.

- Cauchy-Riemann operators

\[
D : \Gamma(E, F) \longrightarrow \Gamma(\Omega^{0,1}(E)), \quad D' : \Gamma(E', F') \longrightarrow \Gamma(\Omega^{0,1}(E')), \\
D'' : \Gamma(E'', F'') \longrightarrow \Gamma(\Omega^{0,1}(E'')),
\]

such that the diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\Omega^{0,1}(E')) & \longrightarrow & \Gamma(\Omega^{0,1}(E)) & \longrightarrow & \Gamma(\Omega^{0,1}(E'')) & \longrightarrow & 0 \\
& & D' \downarrow & & D \downarrow & & D'' \downarrow & & \\
0 & \longrightarrow & \Gamma(E', F') & \longrightarrow & \Gamma(E, F) & \longrightarrow & \Gamma(E'', F'') & \longrightarrow & 0
\end{array}
\]

commutes.

Note that a short exact sequence of Cauchy-Riemann Pin boundary value problems is an example of a short exact sequence of Fredholm operators. See Definition 2.15.

Proposition 8.4. Let

\[ 0 \longrightarrow D' \longrightarrow D'' \longrightarrow D' \longrightarrow 0 \]
be a short exact sequence of Cauchy-Riemann Pin boundary value problems. Up to a universal sign, the isomorphism
\[ \det(D') \otimes \det(D'') \xrightarrow{\sim} \det(D) \]
given by Lemma 2.16 respects the canonical orientations of Proposition 2.8 if and only if \( p \) is compatible with the short exact sequence. The universal sign depends on the dimension of \( E, E', E'' \), the topology of \( \Sigma \) and the orientability of \( F, F', F'' \), restricted to each boundary component of \( \Sigma \).

**Proof.** By a deformation argument, similar to the proof of Proposition 2.8, we would like to reduce the problem to a standard short exact sequence. Throughout the proof, we assume that \( p \) is compatible with the short exact sequence. The other case follows from Lemma 2.10. As in the proof of Proposition 2.8, degenerate \( \gamma \) to a nodal Riemann surface \( \hat{\Sigma} \) with nodal points \( \hat{\gamma}_a \), and irreducible components \( \Delta_a \simeq D^2 \) and \( \hat{\Sigma} \simeq \Sigma/\partial \Sigma \). Simultaneously, degenerate \( E, E', E'' \), to complex vector bundles \( \hat{E}, \hat{E}', \hat{E}'' \), over \( \hat{\Sigma} \) that all satisfy condition (6) for appropriate \( n \). Now, by degenerate, we mean identify the fibers of \( E|_{\gamma_a} \) (resp. \( E'|_{\gamma_a}, E''|_{\gamma_a} \)) with the single fiber \( \hat{E}|_{\hat{\gamma}_a} \) (resp. \( \hat{E}'|_{\hat{\gamma}_a}, \hat{E}''|_{\hat{\gamma}_a} \)). Such a degeneration satisfying condition (6) is unique up to homotopy. Furthermore, we may choose the degeneration of \( E \) to extend the degeneration of \( E' \). These two degenerations induce a degeneration of \( E'' \) via the short exact sequence (31). So, we may assume that there exists a natural induced short exact sequence
\[ 0 \rightarrow (\hat{E}', \hat{F}') \rightarrow (\hat{E}, \hat{F}) \rightarrow (\hat{E}'', \hat{F}'') \rightarrow 0. \] (32)

Choose a particular isomorphism (6) for \((\hat{E}'|_{\Delta_a}, \hat{F}'|_{\partial \Delta_a})\). Extend it to an isomorphism (6) for \((\hat{E}|_{\Delta_a}, \hat{F}|_{\partial \Delta_a})\). Denote the canonical bundle pairs over \((D^2, \partial D^2)\) by
\[
(E_{i,n}, F_{i,n}) := \begin{cases} (\tau \oplus \mathbb{C}^{n-1}, \tau \oplus \mathbb{R}^{n-1}), & i = -1 \\ (\mathbb{C}^n, \mathbb{R}^n), & i = 0. \end{cases}
\]
The following diagram shows that we have a naturally induced isomorphism (6) for \((\hat{E}'|_{\Delta_a}, \hat{F}'|_{\partial \Delta_a})\) :

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (E_{i,n}, F_{i,n}) & \rightarrow & (E_{i+j,n+m}, F_{i+j,n+m}) & \rightarrow & (E_{j,m}, F_{j,m}) & \rightarrow & 0 \\
0 & \rightarrow & (\hat{E}'|_{\Delta_a}, \hat{F}'|_{\partial \Delta_a}) & \rightarrow & (\hat{E}|_{\Delta_a}, \hat{F}|_{\partial \Delta_a}) & \rightarrow & (\hat{E}'|_{\Delta_a}, \hat{F}'|_{\partial \Delta_a}) & \rightarrow & 0.
\end{array}
\]
Here, the top row makes sense because by assumption, either $i$ or $j$ or both are 0. So, the top row is tautological. The bottom row is just a restriction of short exact sequence \[ (32) \]. Since the morphisms in the top row commute with the canonical Cauchy-Riemann operators on the bundles $E_{i,n}$, the isomorphisms \[ (6) \] just chosen induce Cauchy-Riemann operators $D_a, D'_a$ and $D''_a$, on $\hat{E}|_{\Delta_a}, \hat{E}'|_{\Delta_a}$ and $\hat{E}''|_{\Delta_a}$, respectively, that commute with the morphisms of the short exact sequence \[ (32) \]. We claim that if $n + m \geq 3$, the preceding construction is unique up to homotopy. Indeed, choosing a metric on $\hat{E}|_{\Delta_a}$ induces a splitting of the bottom row of the above diagram. Since the space of metrics is contractible, this choice is unique up to homotopy. Then, the middle vertical morphism determines both of the other vertical morphisms. On the other hand, when $n + m \geq 3$, the middle vertical morphism is unique up to homotopy by Lemma 2.6. For $n + m = 2$, we use a stabilization argument as in the proof of Proposition 2.8 to reduce to the case $n + m = 3$.

Choose operators $\tilde{D}, \tilde{D}'$ and $\tilde{D}''$, on $\hat{E}|_{\tilde{\Sigma}}, \hat{E}'|_{\tilde{\Sigma}}$ and $\hat{E}''|_{\tilde{\Sigma}}$, compatible with the short exact sequence \[ (32) \]. Note that the induced isomorphism

$$\det(\tilde{D}') \otimes \det(\tilde{D}'') \sim \det(\tilde{D})$$

always preserves the canonical complex orientations of each side. Finally, choose homotopies of operators $D_t, D'_t$ and $D''_t$, on $E, E'$ and $E''$, respectively, compatible with the short exact sequence \[ (32) \], such that

$$D_0 = D, \quad D_1 = \# a D_a \# \tilde{D}$$

and similarly for $D'_t$ and $D''_t$. Applying Lemma 2.16 to the short exact sequence of families of Fredholm operators,

$$0 \rightarrow D'_t \rightarrow D_t \rightarrow D''_t \rightarrow 0$$

proves that the sign is universal, as claimed. \qed

We now prove a technical lemma that will be useful in the proof of Theorem 1.8. The idea of the proof is taken from \[ [17, proof of Theorem C.1.10(iii)] \], which in turn follows the work of Hofer-Lizan-Sikorav. \[ 8. \]

**Lemma 8.5.** Let $(E, F) \rightarrow (D^2, \partial D^2)$ be a vector bundle pair with $\dim_{\mathbb{C}} E = 1$, and denote its Maslov index by $\mu = \mu(E, F) \geq -1$. Let $D$ be a real
Cauchy-Riemann operator on $E$. Let $z_1, \ldots, z_k \in \partial D^2$, and $w_1, \ldots, w_l \in D^2$ be distinct marked points. Assume $l + 2k = \mu + 1$. Denote by

$$ev : \ker(D) \to \mathbb{R}^k \oplus \mathbb{C}^l$$

the evaluation map defined by

$$\xi \mapsto (\xi(z_1), \ldots, \xi(z_k), \xi(w_1), \ldots, \xi(w_l)), \quad \xi \in \ker D.$$  

Then $ev$ is always surjective.

**Proof.** The Fredholm index of $D$ is well known to be $\mu + 1$. It follows that the Fredholm index of $D \oplus ev$ is 0. So, if $ev$ is not surjective, there must exist some non-zero $\xi \in \ker(D)$ such that $\xi(z_i) = 0$, $i = 1, \ldots, k$, and $\xi(w_j) = 0$, $j = 1, \ldots, l$. According to [17, proof of Theorem C.1.10(iii)], there exists a complex linear Cauchy-Riemann operator $D'$ on $E$ and a function $u \in W^{1,p}(D^2, \mathbb{C})$, such that $D'(u\xi) = 0$. This leads to a contradiction since a holomorphic section of $(E, F)$ may have at most $\mu$ zeros where interior zeros are counted twice.

**Remark 8.6.** Note that it is crucial for this argument that the underlying Riemann surface is a disk. Otherwise, the Fredholm index of $D$ is not $\mu + 1$. This explains, at least in part, why Welschinger’s counting scheme does not immediately extend to curves of higher genus.

**Proof of Theorem 1.8** First, we treat the case $\dim L = 2$. Welschinger’s invariants are defined only in the strongly semipositive case when $\Sigma = D^2$. So, we take $\nu = 0$. As explained in Section 4, we could consider the moduli space defined by quotienting by the action of $PSL_2(\mathbb{R})$, but for this proof it seems more natural to take a section of the $PSL_2(\mathbb{R})$ action. In particular, we add two marked points constrained to divisors, even when $l > 0$, as explained in Remark 4.5. As in Section 4, we denote by $(A, f)$ and $(B, g)$ the divisor constraint pseudo-cycles. In this proof, we will refer to the extra added marked points as $z_{-1}$ and $z_{-2}$. As in Section 4 we fix $z_{-1}$ to be at a point $s_0$ and we fix $z_{-2}$ to lie on a line $\ell$.

Recall from Section 4 that, by definition,

$$N_{D^2,d,k,l} := \#ev^{-1}(\vec{x}, \vec{y}).$$
So, we are counting holomorphic curves through a generic collection of marked points, just like Welschinger. The sign of a given point \( u = (u, \vec{z}, \vec{w}) \in \text{ev}^{-1}(\vec{x}, \vec{y}) \) depends on whether or not the isomorphism

\[
\text{dev}_u : \text{det}(T\mathcal{M}_{k,l}(L, \Sigma, d))_u \xrightarrow{\sim} \text{ev}^* \text{det} \left( T \left( L^{[k]} \times X^l \right) \right)_u
\]

agrees with the isomorphism of Theorem 1.1 up to the action of the multiplicative group of positive real numbers. We would like to reduce this sign to the Welschinger sign associated with the curve \( u \), up to a universal correction factor. For this purpose, we apply Proposition 8.4 to a particular short exact sequence of Cauchy-Riemann boundary value problems. Indeed, we take the underlying short exact sequence of vector bundle pairs

\[
0 \rightarrow (T\partial D^2, T\partial D^2) \xrightarrow{du} (u^*TX, u^*TL) \rightarrow (N_{u,X}^X, N_{u,L}^L) \rightarrow 0.
\]

Pulling back the \( \text{Pin} \) structure on \( TL \) induces a \( \text{Pin} \) structure \( p_u \) on \( u^*TL \). We equip \( (u^*TX, u^*TL) \) with the linearized \( \bar{\partial}_J \) operator \( D_u = D\bar{\partial}_J|_u \) and we denote by \( D_u' \) and \( D_u'' \) the natural operators it induces on the other terms of the short exact sequence. Note that \( T\partial D^2 \) is orientable and has a natural orientation, so we are in the situation of Definition 8.3. The natural orientation on \( T\partial D^2 \) also induces an orientation on \( \text{ker} D_u' \).

Before continuing, we introduce some notation for configuration space. Define the configuration space of \( k \) boundary points and \( l \) interior points of the disk to be

\[
\mathcal{C}_{k,l} := (\partial D^2)^k \times (D^2)^l \setminus \Delta.
\]

Thinking of \( \mathcal{M}_{k,l} \) as the fiber product

\[
\mathcal{M}_{k,l} := \widetilde{\mathcal{M}}_{k,l+2} \times X^2 \times \mathcal{C}_{0,2} (A \times B \times s_0 \times \ell),
\]

we obtain Diagram 2. The central column of Diagram 2 is the short-exact sequence for the tangent space of the fiber product. The main content of the central row is the short exact sequence of solutions of the short exact sequence of Cauchy-Riemann boundary value problems \( \text{34} \). This short exact sequence exists because by assumption the Cauchy-Riemann operator at each term of the sequence is surjective and we can apply the snake lemma from homological algebra. Diagram 2 shows how to construct a natural isomorphism

\[
T\mathcal{M}_{k,l} \simeq T_{\vec{z}, \vec{w}}\mathcal{C}_{k,l} \oplus \text{ker} D_u''.
\]
First, suppose $u^*TL$ is orientable. Choose an orientation $\Omega_0$ on $u^*TL$. Then, $\Omega_0$ together with the complex orientation of $T\partial D^2$ induces an orientation on $N^L_u$. Furthermore, as in the proof of Proposition 3.1, $\Omega_0$ induces an orientation on $\Theta^* \det (T (L^{[k]} \times X^l))$. If $u^*TL$ is not orientable, as in the proof of Proposition 3.1, $\Theta^* \det (T (L^{[k]} \times X^l))$ admits a canonical orientation. Since $D_u$ is surjective by assumption, by Proposition 2.8, $p_u$ and $\Omega_0$ induce an orientation on $\ker(D_u)$ and hence on $T_M$ by isomorphism (35). By Proposition 8.4, the $\Omega''$ orientation on $T_M$ agrees with the $\Omega$ orientation on $T_M$ if and only if $p_u$ is compatible with the short exact sequence (34).

Recall that Welschinger’s invariant counts curves with sign determined by the parity of the isolated real double points. By the adjunction formula, a rational curve $u$ of degree $d$ in a symplectic four manifold has a topologically determined total number of double points

$$\delta(u) = \frac{d \cdot d - c_1(d) - 2}{2}.$$ 

Complex double points come in pairs. So, the parity of the real non-isolated double points is determined by the parity of the real isolated double points. On the other hand, the parity of the real non-isolated double points determines the parity of the number of twists of the real part of the curve.
about itself. This exactly determines when the Pin structure \( p_u \) is compatible with the short exact sequence (34).

Finally, we claim that the map (33) always preserves orientation if we consider the \( \Omega'' \) orientation on \( TM_{k,l} \) together with the \( \Omega_0 \) orientation on \( \text{ev}^* \det \left( T (L^k \times X^l) \right) \). Indeed, by Lemma 8.5 if we vary \( D'' \) to the standard complex linear Cauchy-Riemann operator on \( N_u^X \) and vary \( N_l^L \) to a standard boundary condition keeping the marked points distinct, the evaluation map will remain surjective during the whole variation. So, the sign is standard for a given ordering of the marked points. At this point, we use Definition 3.2 which twists the orientation of \( TM_{k,l} \) by \( \text{sign}(\varpi) \). Indeed, switching the order of the marked points induces a change of sign of the evaluation map (33) since each marked point leads to a codimension \( n-1 \) condition and \( n = 2 \) is even. The twisting cancels this sign change.

The case \( n = 3 \) is very similar. The same exact sequence (34) again plays a central role. Since \( N_u^X \) is now two dimensional, we need to define a Pin structure on \( N_u^L \) with which \( p_u \) may or may not be compatible. In [22], Welschinger does exactly that using the splitting of \( N_u^X \) into holomorphic line bundles. It is important in that paper that \( X \) be convex so that \( J \) may be taken to be integrable. Then the Cauchy-Riemann operator \( D'' \) is the standard one, so the evaluation map is also standard and has a standard sign. If \( D'' \) were not standard, we could not apply Lemma 8.5 as before since \( N_u^X \) is no longer one dimensional. In [23], Welschinger modifies the definition of spinor states to take into account possible changes of orientation arising from walls where the evaluation map is not surjective. This allows him to extend the definition of his invariants to general strongly-semipositive real symplectic manifolds \( X \).

Now we turn to the proof of the calculation in Example 1.6. We generalize Kontsevich’s idea for calculating the closed Gromov-Witten invariants of the quintic threefold [12] to deal with the open case as well. Extending previous notation in the case \( n = 1 \), we denote by \( \tau \) the tautological line bundle of \( \mathbb{C}P^n \) equipped with its canonical complex structure and we denote by \( \tau_{\mathbb{R}} \) the tautological line bundle of \( \mathbb{R}P^n \). Furthermore, we let

\[
\tilde{c} : \mathbb{C}P^n \to \mathbb{C}P^n, \quad \bar{c} : \tau \to \tau,
\]

denote complex conjugation and the bundle-map of \( \tau \) covering complex conjugation respectively. Throughout the following discussion, we use \( \Gamma \) to de-
note holomorphic sections or holomorphic sections satisfying a real boundary condition.

Let \( s \in \Gamma(\tau_{\mathbb{C}P^4}^{*\otimes 5}) \) be a real section, i.e.
\[
\tilde{c}' \circ s \circ c' = s.
\]

We take
\[
X_s = \{ s = 0 \} \subset \mathbb{C}P^4, \quad \omega_s = \omega_{FS}|_{X}, \quad \phi_s = c'|_{X_s}, \quad L_s = Fix(\phi_s) \subset X_s.
\]
That is, \( X_s \) is a quintic threefold equipped with the symplectic form induced by the restriction of the Fubini-Study form of \( \mathbb{C}P^4 \). Moreover, \( X_s \) is equipped with an anti-symplectic involution \( \phi_s \) and \( L_s = Fix(\phi_s) \) is the corresponding Lagrangian submanifold. Choosing \( s \) generically, we may assume that \( X_s \) is a smooth manifold. When it does not lead to confusion, we may drop the subscript \( s \).

We define a bundle \( F_d \) over the moduli space \( M_{0,0}(\mathbb{R}P^4, D^2, d) \) of disks in \( \mathbb{C}P^4 \) with boundary in \( \mathbb{R}P^4 \) by specifying its fibers,
\[
F_{d_u} := \Gamma(u^*\tau^{*\otimes 5}, u^*\tau_{\mathbb{R}}^{*\otimes 5}), \quad u \in M_{0,0}(\mathbb{R}P^4, D^2, d).
\]
By restricting to the image of each curve, \( s \) induces a section \( \hat{s} \) of \( F_d \) that vanishes exactly on those curves entirely contained in \( X_s \).

**Lemma 8.7.** The total space of \( F_d \) is orientable for each \( d \). For \( d \) odd, \( F_d \) is an orientable vector bundle.

**Proof.** Let \( u \in M_{0,0}(\mathbb{R}P^4, D^2, d) \) and \( \xi \in F_u \). After choosing a connection on \( F_d \), there exists a canonical isomorphism
\[
T_{(u,\xi)}F_d \simeq \Gamma(u^*\tau^{*\otimes 5}, u^*\tau_{\mathbb{R}}^{*\otimes 5}) \oplus \Gamma(u^*T\mathbb{C}P^4, u^*T\mathbb{R}P^4)
\]
\[
\simeq \Gamma(u^*(\tau^{*\otimes 5} \oplus T\mathbb{C}P^4), u^*(\tau_{\mathbb{R}}^{*\otimes 5} \oplus T\mathbb{R}P^4)).
\]
Since \( \tau_{\mathbb{R}}^{*\otimes 5} \oplus T\mathbb{R}P^4 \) is orientable, after choosing an orientation, Proposition \ref{prop:orientability} gives a canonical orientation on each tangent space. It is not hard to see that this orientation varies continuously with \( u \) and \( \xi \).

When \( d \) is odd, we have
\[
F_{d_u} := \Gamma(u^*\tau^{*\otimes 5}, u^*\tau_{\mathbb{R}}^{*\otimes 5}) \simeq \Gamma(\tau^{*\otimes 5d}|_{D^2}, \tau_{\mathbb{R}}^{*\otimes 5d}),
\]
where we think of \( D^2 \) as one hemisphere of \( \mathbb{C}P^1 \) with boundary \( \partial D^2 = \mathbb{R}P^1 \). Since \( 5d \) is odd, again Proposition \ref{prop:orientability} gives each \( F_u \) a canonical orientation that varies continuously with \( u \). \( \square \)
Let $V \to B$ be an orientable real vector bundle. We denote by $e(V)$ the Euler class of $V$.

**Proposition 8.8.** Suppose $\hat{s}$ is transverse to the zero section of $F$. Let $d$ be odd. Then

$$N_{D^2,d,0,0} = e(F_d)$$

**Remark 8.9.** This proposition should still hold true when $\hat{s}$ is not transverse to the zero section of $F$. However, the proof will be slightly more complicated. The author plans to address this issue in a future paper that will calculate the invariants in higher degrees as well.

**Remark 8.10.** If $d$ is even, when the details of the necessary corrections from real curves with empty real part are worked out, an argument similar to the proof of Proposition 8.8 should show that $N_{D^2,d,0,0}$ is zero. Indeed, $N_{D^2,d,0,0}$ should be given by the self-intersection number of the zero section of $F_d$. Since $\dim \mathcal{M}_{0,0}(\mathbb{R}P^4, D^2, d) = 5d + 1$, which is odd when $d$ is even, the self intersection number should be zero.

**Proof of Proposition 8.8.** Let $N_{L_s}$ denote the normal bundle of $L_s$ in $\mathbb{R}P^4$. By the adjunction formula, it is isomorphic to $\tau_0 \otimes \tau^* |_{L_s}$. Since $N_{L_s}$ is one dimensional, by Lemma 8.2 we may choose its $Pin^+$ structure canonically. Equip $\tau_0 \otimes \tau^*$ with a $Pin^+$ structure corresponding to the $Pin^+$ structure of $N_{L_s}$ under the isomorphism of the adjunction formula. Choose $Pin^+$ structures on $T_{L_s}$ and $T\mathbb{R}P^4$ compatible with the short exact sequence

$$0 \to TL_s \to T\mathbb{R}P^4 \to N_{L_s} \to 0.$$ 

As in isomorphism (35) of the proof of Theorem 1.8 we may identify

$$T_u M_{0,0}(L_s, D^2, d) \simeq \ker D''_u,$$ (36)

where $D''_u$ is the operator that $D_u$ induces on the bundle pair $(N^{X_s}_u, N^{L_s}_u)$. Equip $N^{L_s}_u$ with the $Pin^+$ structure induced by the short exact sequence

$$0 \to T\partial D^2 \to u^*TL_s \to N^{L_s}_u \to 0.$$ 

Then, by Diagram 2 and Proposition 8.4 we may assume that isomorphism (36) is orientation preserving when $\ker D''_u$ is given the canonical orientation of Proposition 2.8. From Diagram 3 along with its conjugation invariant
part, we deduce a short exact sequence of Cauchy-Riemann boundary value problems

$$0 \rightarrow (N_u^X, N_u^L) \rightarrow (N_u^{CP^4}, N_u^{RP^4}) \rightarrow (u^* N_X, u^* N_L) \rightarrow 0.$$  \hspace{1cm} (37)

The Cauchy-Riemann operators at each term of the sequence are induced by the rows of Diagram 3. The conjugation invariant parts of the rows of Diagram 3 induce $Pin^+$ structures on each of the boundary conditions in short exact sequence (37). The induced $Pin^+$ structures are compatible with the exact sequence because the already chosen $Pin^+$ structures on the conjugation invariant parts of the first two columns of Diagram 3 are compatible. Recall from the proof of the adjunction formula that the isomorphism $N_{X_s} \simeq \tau^{* \otimes 5}|_{X_s}$ is given by the differential $ds$. So, we have a diagram

$$0 \rightarrow \Gamma(N_u^X, N_u^L) \rightarrow \Gamma(N_u^{CP^4}, N_u^{RP^4}) \rightarrow \Gamma(u^* N_X, u^* N_L) \rightarrow 0.$$  \hspace{1cm} (37)

Since $CP^4$ is convex, $\Gamma(N_u^{CP^4}, N_u^{RP^4})$ has expected dimension. Since $\tau^{* \otimes 5}$ is a line bundle, its sections always have expected dimension. By assumption, $d\hat{s}$ is an isomorphism. So, by the snake lemma, $\Gamma(N_u^X, N_u^L)$ must have expected dimension, i.e., 0. So, its orientation is just a sign. Since all $Pin^+$ structures in the above diagram have been chosen compatibly, the sign is given exactly by the sign of $d\hat{s}$, as claimed. \hfill $\square$
Proof of Example 8.8. The section $s_F \in \Gamma(\tau^* \otimes 5)$ defining the Fermat quintic does not satisfy the assumptions of Proposition 8.8 even in degree 1. However, an elementary transversality argument shows that we may choose a nearby section $s$ which does, in degree 1. Note that if $s$ is not sufficiently close to $s_F$, the topology of $L_s$ could be different from that of $L_{s_F}$. However, by Moser’s method, if $s$ is sufficiently close to $s_F$, we know that $(X_s, \omega_s)$ is symplectomorphic to $(X_{s_F}, \omega_{s_F})$ by a $\phi_s - \phi_{s_F}$ equivariant symplectomorphism. In particular, this symplectomorphism sends $L_s = Fix(\phi_s)$ to $L_{s_F} = Fix(\phi_{s_F})$. So, fixing some $s$ sufficiently close to $s_F$, we may think of the deformation of $s_F$ to $s$ as a deformation of $\phi_{s_F}$-invariant complex structure on $X_{s_F}$, which leaves the invariants unchanged. By Proposition 8.8 it suffices to calculate $e(\mathcal{F}_1)$. Let $G(k, n)$ denote the Grassmannian of real oriented $k$ planes in $n$ space and let $\tau_G$ denote its tautological bundle. It is not hard to see that

$$\mathcal{M}_{0,0}(\mathbb{R}P^4, D^2, 1) \simeq G(2, 5), \quad \mathcal{F}_1 \simeq Sym^5(\tau_G).$$

Applying the splitting principle, we calculate the Pontryagin class

$$p_3(Sym^5(\tau_G)) = 225p_1(\tau_G)^3.$$ (38)

Then, taking square roots, we have

$$e(Sym^5(\tau_G)) = 15e(\tau_G)^3 = 15e(\tau_G^3).$$

Here, we have to include $G(2, 5)$ into $G(2, n)$ for $n$ sufficiently large so that both sides of equation (38) are not just zero. We use the unique factorization property of the polynomial ring $H^*(G(2, \infty))$ to justify taking square roots on both sides.

Finally, since there are two oriented 2-planes in the intersection of three generic hyperplanes in $\mathbb{R}^5$, we know that

$$\int_{G(2,5)} e(\tau_G^{\oplus 3}) = 2 \text{ or } 0.$$ (39)

To show the integral is actually 2, we proceed as follows. Let $\hat{G}(k, n)$ denote the Grassmannian of unoriented $k$-planes in $n$-space and let $\hat{\pi}_G$ denote its tautological bundle. Note that $\pi : G(2, 5) \rightarrow \hat{G}(2, 5)$ is the orientation cover. Moreover, $\pi^*\hat{\tau}_G \simeq \tau_G$ and $w_1(\hat{\tau}_G) = w_1(T\hat{G}(2, 5))$. So, both points count the same and integral (39) is 2 as desired. \qed
A Kuranishi structures

In this appendix, we briefly review the definition of a space with Kuranishi structure and perturbations thereof, as introduced in [5] and extended in [6]. We essentially follow the conventions of [6, Appendix 2]. In the following discussion, we take $X$ to be a compact metrizable space.

**Definition A.1.** A Kuranishi structure with corners on $X$ of dimension $d$ consists of the following data:

1. For each point $p \in X$,
   1.1. A smooth manifold with corners $V_p$ and a smooth vector bundle $E_p \to V_p$ such that $\dim V_p - \text{rank } E_p = d$.
   1.2. A finite group $\Gamma_p$ acting on $E_p \to V_p$.
   1.3. An $\Gamma_p$-equivariant smooth section $s_p$ of $E_p$.
   1.4. A homeomorphism $\psi_p$ from $s_p^{-1}(0)/\Gamma_p$ to a neighborhood of $p$ in $X$.

2. For each $p \in X$ and for each $q \in \text{Im } \psi_p$,
   2.1. An open subset $V_{pq} \subset V_q$ containing $\psi_q^{-1}(q)$.
   2.2. A homomorphism $h_{pq} : \Gamma_q \to \Gamma_p$.
   2.3. An $h_{pq}$-equivariant embedding $\varphi_{pq} : V_{pq} \to V_p$ and an injective $h_{pq}$-equivariant bundle map $\hat{\varphi}_{pq} : E_q|_{V_{pq}} \to E_p$ covering $\varphi_{pq}$.

Furthermore, the above data should satisfy the following compatibility conditions:

(C1) $\hat{\varphi}_{pq} \circ s_q = s_p \circ \varphi_{pq}$.

(C2) $\psi_q = \psi_p \circ \varphi_{pq}$.

(C3) If $r \in \psi_q \left( s_q^{-1}(0) \cap V_q/\Gamma_q \right)$, then in a sufficiently small neighborhood of $r$,

\[ \varphi_{pq} \circ \varphi_{qr} = \varphi_{pr}, \quad \hat{\varphi}_{pq} \circ \hat{\varphi}_{qr} = \hat{\varphi}_{pr}. \]
We note that occasionally we may regard \( \psi_p \) as a map from \( s_p^{-1}(0) \) to \( X \) by composing with the quotient map \( V_p \to V_p/\Gamma_p \).

A crucial ingredient in the construction of a perturbation of a space with Kuranishi structure is the notion of the tangent bundle of a Kuranishi structure. We take the following definition from [5, Section 5].

**Definition A.2.** A tangent bundle for a Kuranishi structure consists of a collection of vector bundle isomorphisms

\[
\Phi_{pq} : N_{U_p U_q} \sim E_p|V_{pq}/E_q|V_{pq}
\]

covering the embeddings \( \varphi_{pq} \). Furthermore, if

\[
q \in \text{Im} \psi_p, \quad r \in \psi_q \left( s_q^{-1}(0) \cap V_{pq} \right),
\]

then in a sufficiently small neighborhood of \( r \), we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & N_{V_{q} V_{r}} & \to & N_{V_{p} V_{r}} & \to & N_{V_{p} V_{q}} & \to & 0 \\
& & \Phi_{qr} & \downarrow & \Phi_{pr} & \downarrow & \Phi_{pq} & & \\
0 & \to & E_q/E_r & \to & E_p/E_r & \to & E_p/E_q & \to & 0.
\end{array}
\]

We also need the notion of an orientation for a Kuranishi structure, which again comes from [5, Section 5].

**Definition A.3.** An orientation of a Kuranishi structure with tangent bundle consists of a family of trivializations of \( \det(TV_p) \otimes \det(E_p) \) compatible with the isomorphisms

\[
\det(TV_q) \otimes \det(E_q)|V_{pq} \sim \det(TV_p) \otimes \det(E_p)|V_{pq}
\]

induced by \( \Phi_{pq} \).

Without providing full detail, we remind the reader of certain definitions relating to multisections that are used in Section 7 of this paper. For details, see [5, Section 3]. In the following, for \( Z \) a space, we denote by \( S^\ell(Z) \) its \( \ell \)th symmetric power. That is

\[
S^\ell(Z) := Z^{\ell}/S_\ell,
\]

where \( S_\ell \) is the group of permutations of \( \ell \) objects acting on \( Z^{\ell} \) by permuting the factors. Let \( E \to V \) be a vector bundle. If \( U \subset V \) is sufficiently small,
then $E|_U$ is trivial. So, if rank $E = r$, a section of $E$ over $U$ may be specified by a map $U \to \mathbb{R}^r$. Analogously, a multisection $s$ of $E$ over $U$ may be specified by a map $s_U : U \to S^\ell(\mathbb{R}^r)$. Note that globally, the multiplicity $\ell$ can change. By definition, the multisection $s$ is said to be smooth if after possibly shrinking $U$, there exists a smooth lifting of $s_U$ to the Cartesian product,

$$\tilde{s}_U : U \to (\mathbb{R}^k)^\ell.$$  

The components of this lifting locally define $\ell$ sections $s_U^i$ of $E$. We call the $s_U^i$ branches of $s$ over $U$. If $E \to V$ is a $\Gamma$-equivariant vector bundle, then there is a natural notion of a $\Gamma$-equivariant multisection coming from the induced action on the symmetric power. Note that for $\Gamma$-equivariant smooth sections, we do not require the local lifts $\tilde{s}_U$ to be $\Gamma$-equivariant. We call a smooth multisection transverse if each branch of each local lifting is transverse. The vanishing set of a multisection $s$ is defined locally by

$$s^{-1}(0) \cap U = \bigcup_i (s_U^i)^{-1}(0).$$  

If $s$ is transverse and sufficiently generic, then $s^{-1}(0)$ admits a smooth triangulation. If we fix a trivialization of $\det(E) \otimes \det(V)$, then the vanishing set of any smooth section is oriented. So, $s^{-1}(0)$ actually defines a rational singular chain by weighting each simplex of its triangulation by the signed count of branches $s_U^i$ that vanish on it, divided by $\ell$.

Finally, we need to recall the notion of a good coordinate system introduced in [5, Section 6]. Fix a Kuranishi structure on $X$. We denote the various parts of the Kuranishi structure by the same symbols as in Definition A.1. For $V'_p \subset V_p$, we denote by $E'_p, \varphi'_p, s'_p$, etc. the restrictions of all the related parts of the Kuranishi structure. A good coordinate system specifies a finite ordered set $P \subset X$ and $V'_p \subset V_p$ for each $p \in P$ such that

$$X \subset \bigcup_{p \in P} \text{Im } \psi'_p.$$  

Furthermore, for $q, p \in P$ such that $q < p$, it specifies a neighborhood

$$V''_p \supset \psi^{-1}_q(\text{Im } \psi'_p),$$

an embedding $\varphi'_p : V''_p \hookrightarrow V'_p$ and an injective bundle map

$$\hat{\varphi}_{pq} : E'_q|_{V''_p} \to E'_p.$$
covering $\varphi'_{pq}$. Of course, we must require $s'_p \circ \varphi'_{pq} = \hat{\varphi}'_{pq} \circ s'_q$. Also, $\varphi'_{pq}$ (resp. $\varphi'_{pq}$) must respect the actions of $\Gamma_q$ and $\Gamma_p$ in such a way as to define a map of the quotient orbifolds (resp. orbi-bundles). A few additional technical conditions ensure requisite compatibility.

The following is a restatement of [5, Theorem 6.4].

**Theorem A.4.** Let $(P, V'_P, \psi'_P, s'_P, \varphi'_P, \hat{\varphi}'_{pq})$ be a good coordinate system on a space with Kuranishi structure $(X, K)$. Suppose that $K$ has a tangent bundle in the sense of Definition A.2. Then, for each $p \in P$, there exists a sequence of smooth $\Gamma_p$-equivariant multisections $s'_{p,n}$ of $E_p$ such that

1. $s'_{p,n} \circ \varphi'_{pq} = \hat{\varphi}'_{pq} \circ s'_{q,n}$.
2. $\lim_{n \to \infty} s'_{p,n} = s'_p$.
3. $s'_{p,n}$ is transversal to 0.
4. The restriction to $\text{Im} \varphi'_{pq}$ of the differential of the composition of any branch of $s'_{p,n}$ and the projection $E'_p \to E'_p/E'_q$ coincides with the isomorphism $\Phi'_{pq} : N_{V'_p} V'_q \cong E'_p/E'_q$.

Fix $n$ sufficiently large. The rational chain given by the vanishing sets of the $s'_{p,n}$ constructed in Theorem A.4 constitutes a transverse perturbation of the space with Kuranishi structure $(X, K)$.

For the reader’s convenience, we outline the proof of Theorem A.4. We use induction on the ordered set $P$. Write $P = \{p_1, p_2, \ldots\}$. Assume the existence of perturbations $s'_{p_i,n}$ satisfying conditions (P1)-(P4) for $i < j$. The embeddings $\varphi'_{p_j p_i}$, the bundle maps $\varphi'_{p_j p_i}$, and the isomorphisms $\Phi_{p_j p_i}$, allow the extension of $s'_{p_i,n}$ to a neighborhood of

$$\bigcup_{i<j} \text{Im} \varphi'_{p_j p_i} \subset V_{p_j}.$$

A small perturbation of the extension produces $s'_{p_j,n}$ as desired. Full detail is given in [5, Section 6].

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