POLYNOMIAL ANALOGUE OF THE KEMPNER FUNCTION

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ABSTRACT. In the integer case, the Kempner function of a positive integer \( n \) is defined to be the smallest positive integer \( k \) such that \( n \) divides the factorial \( k! \). In this paper, we first define a natural order for polynomials in \( \mathbb{F}_q[t] \) over a finite field \( \mathbb{F}_q \) and then define the Kempner function of a non-zero polynomial \( f \in \mathbb{F}_q[t] \), denoted by \( K(f) \), to be the smallest polynomial \( g \) such that \( f \) divides the Carlitz factorial of \( g \). In particular, we establish an analogue of a problem of Erdős, which implies that for almost all polynomials \( f \), \( K(f) = t^d \), where \( d \) is the maximal degree of the irreducible factors of \( f \).

1. Introduction

1.1. Motivation. In number theory, the Kempner function of a positive integer \( n \) is defined to be the smallest positive integer \( k \) such that \( n \) divides the factorial \( k! \). This function was studied by Lucas [17] for powers of primes and then by Neuberg [18] and Kempner [11] for general \( n \). In particular, Kempner [11] gave the first correct algorithm for computing this function. It is also sometimes called the Smarandache function following Smarandache’s rediscovery in 1980; see [8, 20]. This function arises here and there in number theory (for instance, see [4, 12, 16, 21]).

Clearly, the Kempner function of \( n \) is equal to the maximum of those of its prime power factors. For any integer \( n \geq 2 \), let \( P(n) \) be the largest prime factor of \( n \); and put \( P(1) = 1 \). For any \( x > 1 \), denote by \( N(x) \) the number of positive integers \( n \leq x \), whose Kempner function are not equal to that of \( P(n) \) (that is, \( P(n) \), this means \( n \nmid P(n)! \)). In 1991 Erdős [6] posed a problem answered by Kastanas [10] in 1994 that \( N(x) = o(x) \) when \( x \) goes to infinity. Later, Akbik [1] proved that

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\[ N(x) = O(x \exp(-\frac{1}{4} \sqrt{\log x})), \] and recently Ivić [9] showed that
\[ N(x) = x \exp \left( -\sqrt{2 \log x \log \log x} (1 + O(\log \log \log x / \log \log x)) \right) ; \]
see [5, 7] for some other previous results.

In this paper, we want to define and study the Kempner function for polynomials over a finite field. In particular, we want to establish an analogue of Erdős's problem.

1.2. Our consideration. Let \( \mathbb{F}_q \) be the finite field of \( q \) elements, where \( q \) is a power of a prime \( p \). Denote by \( \mathbb{A} = \mathbb{F}_q[t] \) the polynomial ring of one variable over \( \mathbb{F}_q \) and \( \mathbb{N} \) the set of non-negative integers. Let \( \mathbb{N}^* \) be the set of positive integers. For any non-zero \( g \in \mathbb{A} \), we denote by \( \text{sign}(g) \) the leading coefficient of \( g \) (which is also called the sign of \( g \)).

We write \( \mathbb{F}_q = \{ a_0 = 0, a_1 = 1, a_2, \ldots, a_{q-1} \} \) throughout the paper. For any non-zero polynomial \( f \in \mathbb{A} \) of degree \( n \), \( f \) can be uniquely written as
\[
(1.1) \quad f = a_{i_0} + a_{i_1} t + \ldots + a_{i_n} t^n, \quad a_{i_n} \neq 0, \ 0 \leq i_j \leq q - 1,
\]
then we define \( \delta(f) \) to be the integer:
\[
\delta(f) = i_0 + i_1 q + \cdots + i_n q^n ;
\]
in addition, we put \( \delta(0) = 0 \).

Clearly, \( \delta \) is a bijective map from \( \mathbb{A} \) to \( \mathbb{N} \), and for any \( m \in \mathbb{N} \),
\[
\delta^{-1}(m) = a_{i_0} + a_{i_1} t + \ldots + a_{i_k} t^k ,
\]
where \( i_0 + i_1 q + \cdots + i_k q^k \) is the \( q \)-adic expansion of \( m \).

Moreover, we define an order in \( \mathbb{A} \) based on the map \( \delta \): for any \( f, g \in \mathbb{A} \),
\[ f > g \quad \text{if and only if} \quad \delta(f) > \delta(g) ; \]
and then \( f \geq g \) if and only if \( f > g \) or \( f = g \).

With these preparations, we define a factorial in \( \mathbb{A} \).

**Definition 1.1.** For any non-zero polynomial \( f \in \mathbb{A} \), the factorial of \( f \) is defined to be
\[
f! = \prod_{g < f} (f - g).\]

Additionally, we put \( 0! = 1 \).

By definition, for any integer \( n \geq 1 \), \( t^n! \) is in fact the product of all the monic polynomials of degree \( n \).

This factorial is an analogue of the factorial of the rational integers; see [13] for another analogue. It has been used in [14, 15]. Notice that the above factorial of \( f \) is equal to the multiplication of the Carlitz
factorial of \( f \) by a constant (see the comment below Lemma 2.1). For
the Carlitz factorial, one can refer to [3, 22].

We now can define the Kempner function for polynomials in \( \mathbb{A} \). In
fact it has been used in [14, Section 4.2] for counting polynomial func-
tions in the residue class rings (see the definition of \( \lambda \) there).

**Definition 1.2.** Given a non-zero polynomial \( f \in \mathbb{A} \), the Kempner
function \( K(f) \) of \( f \) is defined to be the smallest polynomial \( g \) such that
\[
 f \mid g !; 
\]
and put \( K(0) = 0 \) by convention.

In Section 3 we establish various basic properties of the Kempner
function \( K \), such as the computation, the value set, the inverse images,
and fixed points. We emphasize that several of them haven’t be con-
sidered in the integer case, such as Proposition 3.6 on how the size of
a polynomial changes after an action of \( K \) and Proposition 3.12 on the
distance to fixed points. We then in Section 4 establish an analogue of
Erdős’s problem for \( K \) (see Theorem 4.1).

2. Preliminaries

In this section, we gather some results which are used later on.

2.1. Some elementary results. We first compute the factorial \( f ! \) for
any \( f \in \mathbb{A} \).

**Lemma 2.1.** For any \( f \in \mathbb{A} \) of the form (1.1), then
\[
(2.1) \quad f ! = \left( \prod_{j=0}^{n} a_{ij} ! \right) \prod_{j=1}^{n} \prod_{h \in \mathbb{A}, \deg h = j, \text{sign}(h) = 1} h^j .
\]

**Proof.** Denote by \( R \) the right hand side of (2.1). We rewrite \( R \) as
\[
 R = \prod_{j=0}^{n} \prod_{k=0}^{i_j - 1} \prod_{h \in \mathbb{A}, \deg h = j, \text{sign}(h) = a_{ij} - a_k} h = \prod_{j=0}^{n} \prod_{k=0}^{i_j - 1} \prod_{h \in \mathbb{A}, \deg h = j, \text{sign}(h) = a_{ij} - a_k} (f - (f - h)) ,
\]
where one can see that \( f - h \) exactly runs over all the polynomials
\( g < f \). So, by definition we have \( R = f ! . \)

By definition and Lemma 2.1, \( f ! / (\prod_{j=0}^{n} a_{ij} !) \) is exactly the Carlitz
factorial of \( f \). So, in Definition 1.2 we can replace \( g ! \) by the Carlitz
factorial of \( g \).

The following result is a special case of Example 3 in [2]. We give a
proof here.
Lemma 2.2. Let $P \in \mathbb{A}$ be an irreducible polynomial of degree $d \geq 1$. Then, for any non-zero polynomial $f \in \mathbb{A}$ we have

$$v_P(f!) = \sum_{j \geq 1} \lfloor \delta(f) q^d j \rfloor,$$

where $v_P$ is the usual $P$-adic valuation.

Proof. Assume that $f$ is of the form (1.1). From the formula (2.1) of $f!$, we see that for any integer $j \geq 1$, if $n = \deg f \geq d j$, then the number of terms in the right hand side of (2.1) divisible by $P^j$ is exactly equal to

$$i_d + i_{d+1} q + \cdots + i_n q^{n-d}.$$

Summing up all these estimates we obtain the desired formula. \qed

Clearly, Lemma 2.2 gives the following result.

Corollary 2.3. For any $f, g \in \mathbb{A}$, if $g \leq f$, then $g! \mid f!$.

We remark that in Corollary 2.3 the converse is not true.

For the proof of Proposition 3.4, we need the following lemma, which is in fact a simple generalization of the formula of $\alpha$ in [11, page 207] (also the formula in [20, Lemma 1]).

Lemma 2.4. Fix a positive integer $n > 1$, and define a sequence \( \{b_j = \frac{n^j - 1}{n - 1} : j \in \mathbb{N}^*\} \). Then for any $e \in \mathbb{N}^*$, $e$ can be uniquely represented as

$$e = c_1 b_{j_1} + c_2 b_{j_2} + \cdots + c_k b_{j_k},$$

where $j_1 > j_2 > \cdots > j_k > 0$ and $1 \leq c_i < n, i = 1, 2, \ldots, k - 1, 1 \leq c_k \leq n$.

Proof. Obviously, $\mathbb{N}^*$ is the disjoint union of the sets $[b_j, b_{j+1}) \cap \mathbb{N}^*, j \in \mathbb{N}^*$, and $b_{j+1} = nb_j + 1$ for any $j \in \mathbb{N}^*$. So, for any $e \in \mathbb{N}^*$, there exists an unique integer $j_1 \in \mathbb{N}^*$ such that $e \in [b_{j_1}, b_{j_1+1}) \cap \mathbb{N}^*$, then by the division algorithm, we have

$$e = c_1 b_{j_1} + r_1,$$

where $1 \leq c_1 = \lfloor \frac{e}{b_{j_1}} \rfloor \leq n$ and $0 \leq r_1 < b_{j_1}$. If $r_1 = 0$, as $b_{j_1} \leq e < b_{j_1+1}$, then $k = 1, 1 \leq c_1 \leq n$ and Lemma 2.4 is proved.
If \( r_1 \neq 0 \), as \( b_{j_1} \leq e < b_{j_1+1} \), then \( 1 \leq c_1 < n \). Next procedure is the iterative process that makes use of the division algorithm in the form:

\[
\begin{align*}
    r_1 &= c_2b_{j_2} + r_2, \quad 1 \leq c_2 < n, \ 0 < r_2 < b_{j_2}, \\
    r_2 &= c_3b_{j_3} + r_3, \quad 1 \leq c_3 < n, \ 0 < r_3 < b_{j_3}, \\
    &\vdots \\
    r_{k-2} &= c_{k-1}b_{j_{k-1}} + r_{k-1}, \quad 1 \leq c_{k-1} < n, \ 0 < r_{k-1} < b_{j_{k-1}}, \\
    r_{k-1} &= c_kb_{j_k}, \quad 1 \leq c_k \leq n.
\end{align*}
\]

In the above computation the integer \( k \) is defined by the condition that \( r_{k-1} \neq 0 \) and that \( r_k = 0 \). Since \( e \geq b_{j_1} > r_1 \geq b_{j_2} > r_2 \geq \cdots \geq 0 \), such a \( k \) must exist and \( j_1 > j_2 > \cdots > j_k > 0 \).

Collecting all the equalities above, Lemma 2.4 is proved. \( \square \)

2.2. Counting polynomials. For any non-zero \( f \in A \), let \( \omega(f) \) be the number of distinct monic irreducible factors of \( f \), and let \( \tau(f) \) be the number of distinct monic factors of \( f \).

The following two results should be well-known.

**Lemma 2.5.** For any integer \( n \geq 1 \), the number of monic irreducible polynomials in \( A \) of degree at most \( n \) is at most \( q^n \).

**Proof.** For any monic irreducible polynomial \( f \in A = \mathbb{F}_q[t] \), if \( f \) is of degree \( d \leq n \), then \( f \) corresponds to the monic polynomial \( t^r f^s \) of degree \( n \), where \( n = sd + r \) with \( 0 \leq r < d \) by the division algorithm. Note that this corresponding is injective. So the desired result follows. \( \square \)

**Lemma 2.6.** For any integer \( n \geq 1 \), we have

\[
\sum_{\text{monic } f \in A \atop \deg f = n} \tau(f) = (n + 1)q^n.
\]

**Proof.** This result has been recorded in [19, Proposition 2.5]. Here we present a different proof. It is easy to see that

\[
\sum_{\text{monic } f \in A \atop \deg f = n} \tau(f) = \sum_{\text{monic } g \in A \atop \deg g \leq n} \sum_{\text{monic } h \in A \atop \deg h = n - \deg g} 1 = \sum_{\text{monic } g \in A \atop \deg g \leq n} q^{n - \deg g} = q^n \sum_{j=0}^{n} q^{-j} \cdot q^j = (n + 1)q^n.
\]

\( \square \)
We now present some counting results for polynomials in \( A \) according to the numbers of their monic factors and their maximal monic irreducible factors. These are needed for proving Theorem 4.1.

Lemma 2.7. For any integers \( n, k \geq 1 \), let \( B = 3k \log \log q^n \) and define
\[
S_1(n, k) = \{ \text{monic } f \in A : \deg f = n, \omega(f) > B \}.
\]
Then, we have
\[
|S_1(n, k)| < \frac{3q^n}{(\log q^n)^k};
\]
if furthermore \( n \geq 3 \) and \( k \geq 2 \), we have
\[
|S_1(n, k)| < \frac{q^n}{(\log q^n)^k}.
\]
Moreover, if \( q \geq 3, n \geq 4 \) and \( k \geq 3 \), in \( S_1(n, k) \) we can choose \( B = 2k \log \log q^n \), then the estimate (2.2) still holds.

Proof. By definition, we have \( \tau(f) \geq 2 \omega(f) \) for any non-zero \( f \in A \). Using Lemma 2.6, we deduce that
\[
(n + 1)q^n = \sum_{\text{monic } f \in A, \deg f = n} \tau(f) \geq \sum_{\text{monic } f \in A, \deg f = n} 2^{\omega(f)} \geq \sum_{f \in S_1(n, k)} 2^{\omega(f)} > \sum_{f \in S_1(n, k)} 2^{3k \log \log q^n} = 2^{3k \log \log q^n} |S_1(n, k)|.
\]
So, we obtain (noticing \( q \geq 2 \))
\[
|S_1(n, k)| < \frac{(n + 1)q^n}{2^{3k \log \log q^n}} = \frac{(n + 1)q^n}{(\log q^n)^{3k \log 2}} < \frac{(n + 1)q^n}{(\log q^n)^{k+1}} < \frac{3q^n}{(\log q^n)^k}.
\]
If \( n \geq 3 \) and \( k \geq 2 \), we have
\[
|S_1(n, k)| < \frac{(n + 1)q^n}{(\log q^n)^{3k \log 2}} < \frac{(n + 1)q^n}{(\log q^n)^{2k}} < \frac{q^n}{(\log q^n)^k},
\]
where the last inequality comes from
\[
(n \log q)^k \geq (n \log q)^2 \geq (n \log 2)^2 > n + 1, \quad n \geq 3.
\]
The final part follows from
\[
|S_1(n, k)| < \frac{(n + 1)q^n}{(\log q^n)^{2k \log 2}} < \frac{q^n}{(\log q^n)^k}
\]
when \( q \geq 3, n \geq 4 \) and \( k \geq 3 \). \( \square \)
Lemma 2.8. For any integers $n, k \geq 1$, let $D = 2k \log \log q^n$ and define

$S_2(n, k) = \{ \text{monic } f \in \mathbb{A} : \deg f = n, P^2 \mid f \text{ for some irreducible polynomial } P \text{ with } \deg P > D \}.$

Then, we have

$$|S_2(n, k)| < \frac{q^n}{(\log q^n)^k}.$$  \hspace{1cm} (2.3)

Moreover, if $q \geq 3$, in $S_2(n, k)$ we can choose $D = k \log \log q^n$, then the estimate (2.3) still holds.

Proof. For any $f \in S_2(n, k)$, we can write $f = gP^2$ with $D < \deg P \leq n/2$ and $\deg g = n - 2\deg P$. So, we have

$$|S_2(n, k)| \leq \sum_{\text{monic irreducible } P \in \mathbb{A}} \sum_{\text{deg } P \leq n/2} 1 \leq \sum_{\text{monic irreducible } P \in \mathbb{A}} \sum_{\text{deg } g = n - 2\deg P} q^{n-2\deg P} < q^n \sum_{j=0}^{\infty} q^{-2j} \cdot q^j = \frac{q^n}{q^{2D} - 1} \leq \frac{q^n}{q^{2k \log \log q^n}} < \frac{q^n}{(\log q^n)^k}.$$  \hspace{1cm} (2.4)

The second part follows similarly.  \hspace{1cm} $\Box$

Lemma 2.9. For any integers $n, k \geq 1$, let $D = 2k \log \log q^n$ and define

$S_3(n, k) = \{ \text{monic } f \in \mathbb{A} : \deg f = n, P^e \mid f, e \geq D \text{ for some irreducible polynomial } P \text{ with } \deg P \leq D \}.$

Then, if $D \geq 2$ and $n \geq 2$, we have

$$|S_3(n, k)| < \frac{4q^n}{(\log q^n)^k};$$

if furthermore $n \geq 9$ and $k \geq 2$, we have

$$|S_3(n, k)| < \frac{q^n}{(\log q^n)^k}. \hspace{1cm} (2.4)$$

Moreover, if $q \geq 3, n \geq 400$ and $k \geq 3$, in $S_3(n, k)$ we can choose $D = k \log \log q^n \geq 2$, then the estimate (2.4) still holds.
Proof. Let $d = \lceil D \rceil \geq 2$. By definition, for any $f \in S_3(n, k)$, there exists a monic irreducible polynomial $P$ such that $\deg P \leq D$ and $P^d \mid f$. As in the proof of Lemma 2.8, we have

$$|S_3(n, k)| \leq \sum_{\text{monic irreducible } P \in \mathbb{A}} \sum_{1 \leq \deg P \leq D} 1$$

$$= \sum_{\text{monic irreducible } P \in \mathbb{A}} q^{n-d\deg P} < q^n \sum_{j=1}^{\infty} q^{-dj} \cdot q^j$$

$$\leq \frac{2q^n}{q^{D-1}} = \frac{2q^{n+1}}{q^{2k\log\log q^n}} = \frac{2q^{n+1}}{(\log q^n)^{2k\log q}} < \frac{4q^n}{(\log q^n)^k},$$

where we need to use the assumption $D \geq 2$ and $n \geq 2$. If moreover $n \geq 9$ and $k \geq 2$, we in fact have

$$|S_3(n, k)| < \frac{2q^{n+1}}{(\log q^n)^{2k\log q}} < \frac{q^n}{(\log q^n)^k}.$$

The final part follows similarly. \hfill \Box

3. Basic properties

3.1. Computing the Kempner function. By definition, we directly obtain two simple properties about the Kempner function.

**Proposition 3.1.** The following hold:

1. for any polynomial $f \in \mathbb{A}$ and any $a \in \mathbb{F}_{q^*}$, $K(af) = K(f)$;
2. for any non-zero polynomial $f \in \mathbb{A}$, $K(f) \leq \deg f$.

So, for computing the Kempner function, we only need to consider monic polynomials. By Definition 1.2 and Corollary 2.3, we immediately obtain the following result, which implies that we in fact only need to consider powers of monic irreducible polynomials.

**Proposition 3.2.** Suppose that $P_1, P_2, \ldots, P_k$ are distinct monic irreducible polynomials and $e_1, e_2, \ldots, e_k$ are positive integers. Then

$$K(P_1^{e_1}P_2^{e_2}\cdots P_k^{e_k}) = \max\{K(P_1^{e_1}), K(P_2^{e_2}), \ldots, K(P_k^{e_k})\}.$$ 

The case of irreducible polynomials is straightforward. We in fact can do more.

**Proposition 3.3.** Given a polynomial $f \in \mathbb{A}$ with $\deg f \geq 1$, assume that either $q \geq 3$ or $f \neq b(t+c)^2$ for any $b, c \in \mathbb{F}_q$. Then, $f$ is an irreducible polynomial if and only if $K(f) = \deg f$. 

Proof. We only need to prove the sufficiency. Assume that \( K(f) = t_{\deg f} \). Without loss of generality, we can further assume that \( f \) is monic. By Proposition 3.2, we must have \( f = P^e \) for some monic irreducible polynomial \( P \) and \( e \geq 1 \).

We first assume that \( q \geq 3 \). If \( e \geq 2 \), since \( a_2 P^{e-1} \) and \( (a_2 - 1)P^{e-1} \) are two distinct terms in the factorial \( (a_2 P^{e-1})! \) by definition, we have
\[
v_P((a_2 P^{e-1})!) \geq 2(e - 1) \geq e,
\]
and so \( K(P^e) \leq a_2 P^{e-1} < t^{de} \), which contradicts with the assumption \( K(P^e) = t^{de} \). Thus, \( f = P \) when \( q \geq 3 \).

We now assume that \( q = 2 \). By assumption, \( f \neq (t + c)^2 \) for any \( c \in \mathbb{F}_q \). So, if \( \deg P = 1 \), we must have \( e \geq 3 \), and so \( v_P(P^{e-1}) \geq e \), which implies \( K(P^e) \leq P^{e-1} < t^e \) and contradicts with the assumption \( K(P^e) = t^e \). Thus, we must have \( \deg P \geq 2 \). Let \( d = \deg P \geq 2 \). If \( e \geq 2 \), since \( tP^{e-1} \) and \( (t+1)P^{e-1} \) are two distinct terms in the factorial \( t^{d(e-1)+1}! \) by definition, we have
\[
v_P(t^{d(e-1)+1}) \geq 2(e - 1) \geq e,
\]
and so \( K(P^e) \leq t^{d(e-1)+1} < t^{de} \), which contradicts with the assumption \( K(P^e) = t^{de} \). Thus, \( f = P \). This completes the proof.

We remark that in the case \( q = 2 \), we have \( K(t^2) = K(t^2 + 1) = t^2 \).

We now handle the case of powers of irreducible polynomials by following the strategy for proving the theorem in [11, page 208] (also [20, Theorem 1]).

Proposition 3.4. Suppose that \( P \in \mathbb{A} \) is an irreducible polynomial of degree \( d \geq 1 \) and \( e \) is a positive integer. Define the sequence \( b_j = \frac{q^{j-1}}{q^d-1}, j \in \mathbb{N}^* \). Then, \( e \) is uniquely written as
\[ e = c_1b_{j_1} + c_2b_{j_2} + \cdots + c_kb_{j_k}, \]
and
\[ K(P^e) = \delta^{-1}(c_1q^{\delta_{j_1}} + c_2q^{\delta_{j_2}} + \cdots + c_kq^{\delta_{j_k}}), \]
where \( j_1 > j_2 > \cdots > j_k > 0 \) and \( 1 \leq c_i < q^d, i = 1, 2, \ldots, k - 1, 1 \leq c_k \leq q^d \).

Proof. By Lemma 2.4, we know that \( e \) is uniquely written in the form:
\[ e = c_1b_{j_1} + c_2b_{j_2} + \cdots + c_kb_{j_k}, \]
where \( j_1 > j_2 > \cdots > j_k > 0 \) and \( 1 \leq c_i < q^d, i = 1, 2, \ldots, k - 1, 1 \leq c_k \leq q^d \). Denote
\[
m = c_1q^{\delta_{j_1}} + c_2q^{\delta_{j_2}} + \cdots + c_kq^{\delta_{j_k}} = (q^d - 1)e + (c_1 + c_2 + \cdots + c_k).\]
Since $\delta$ is a bijective map from $\mathbb{A}$ to $\mathbb{N}$, we take $f = \delta^{-1}(m)$. Then, it suffices to prove $K(P^e) = f$.

By Lemma 2.2 and collecting the following equalities and inequalities
\[
\left\lfloor \frac{m}{q^d} \right\rfloor = c_1 q^{d(j_1 - 1)} + c_2 q^{d(j_2 - 1)} + \cdots + c_k q^{d(j_k - 1)},
\]
\[\vdots\]
\[
\left\lfloor \frac{m}{q^{d(j_k+1)}} \right\rfloor \geq c_1 q^{d(j_1-j_k)} + c_2 q^{d(j_2-j_k)} + \cdots + c_{k-1} q^{d(j_{k-1}-j_k)},
\]
\[\vdots\]
\[
\left\lfloor \frac{m}{q^{d(j_1)}} \right\rfloor \geq c_1,
\]
we obtain
\[
v_P(f!) = \sum_{j \geq 1} \left\lfloor \frac{m}{q^{d(j)}} \right\rfloor \geq e,
\]
which implies that $P^e \mid f!$. Actually, $v_P(f!) = e$ if and only if $c_k < q^d$.

Now, it remains to prove that for any $g \in \mathbb{A}$ and $g < f$, we have $P^e \nmid g!$. In fact, by Corollary 2.3, we only need to prove that for $g = \delta^{-1}(m - 1)$, $P^e \nmid g!$, that is, $v_P(g!) < e$. It is easy to obtain the following equalities:
\[
\left\lfloor \frac{m - 1}{q^d} \right\rfloor = c_1 q^{d(j_1 - 1)} + c_2 q^{d(j_2 - 1)} + \cdots + c_k q^{d(j_k - 1)} - 1,
\]
\[\vdots\]
\[
\left\lfloor \frac{m - 1}{q^{d(j_k)}} \right\rfloor = c_1 q^{d(j_1-j_k)} + c_2 q^{d(j_2-j_k)} + \cdots + c_{k-1} q^{d(j_{k-1}-j_k)} - 1.
\]
\[\vdots\]
\[
\left\lfloor \frac{m - 1}{q^{d(j_1)}} \right\rfloor = c_1 - 1.
\]
Then $v_P(g!) = \sum_{j \geq 1} \left\lfloor \frac{m - 1}{q^{d(j)}} \right\rfloor = e - j_1 < e$. This completes the proof. \(\square\)

By Proposition 3.4, we directly obtain the following result.

**Corollary 3.5.** Suppose that $P \in \mathbb{A}$ is an irreducible polynomial of degree $d$ and $e \leq q^d$ is a positive integer. Then
\[
K(P^e) = a_{i_0} t^d + a_{i_1} t^{d+1} + \cdots + a_{i_k} t^{d+k},
\]
where $e = \sum_{j=0}^k i_j q^j$ is the $q$-adic expansion of $e$. 
With some more efforts we can estimate how the size of a polynomial changes after an action of $K$.

**Proposition 3.6.** Given a polynomial $f \in \mathbb{A}$ with $\deg f \geq 1$, suppose that $f$ is reducible and $f \neq b(t + c)^2$ for any $b, c \in \mathbb{F}_q$. Then, we have

$$\delta(K(f)) \leq \frac{\delta(f)}{q},$$

where the equality holds if and only if $q = 2$ or $3$, $f = t^3$.

**Proof.** Without loss of generality, we can assume that $f$ is monic. When $f$ has at least two distinct monic irreducible factors, by Proposition 3.1 (2) and Proposition 3.2, we immediately have

$$\delta(K(f)) < \frac{\delta(f)}{q}.$$

So, it remains to consider the following two cases:

- $f = P^e, e \geq 2$ for a monic irreducible polynomial $P$ with $\deg P \geq 2$;
- $f = P^e, e \geq 3$ for a monic linear polynomial $P$.

Now we assume that $f = P^e, e \geq 2$ for a monic irreducible polynomial $P$ with $\deg P \geq 2$. Let $d = \deg P$, and define $b_j = \frac{q^{j-1}}{q^d-1}, j \in \mathbb{N}$. As before, $e$ can be uniquely written as

$$e = c_{j_1}b_{j_1} + c_{j_2}b_{j_2} + \cdots + c_{j_k}b_{j_k},$$

where $j_1 > j_2 > \cdots > j_k > 0$ and $1 \leq c_i < q^d, i = 1, 2, \ldots, k - 1, 1 \leq c_k \leq q^d$. By Proposition 3.4, we have

$$\delta(K(f)) = c_{j_1}q^{d_j} + c_{j_2}q^{d_j} + \cdots + c_{j_k}q^{d_j} \leq q^{d(j_1+1)}.$$

If $j_1 \geq 2$, then (noticing $d \geq 2$)

$$e \geq c_{j_1}b_{j_1} \geq b_{j_1} \geq 1 + 2^2 + \cdots + 2^{2(j_1-1)} \geq j_1 + 3,$$

and thus

$$\frac{\delta(f)}{q^d} > \frac{q^{de}}{q^d} = q^{d(e-1)} \geq q^{d(j_1+2)} > q^{d(j_1+1)} \geq \delta(K(f)).$$

If $j_1 = 1$, then $e = c_{j_1}b_{j_1} = c_1 \leq q^d$ and $\delta(K(f)) = eq^d$, and so for $e \geq 3$

$$\frac{\delta(f)}{q^d} > q^{d(e-1)} \geq q^{2d} \geq eq^d = \delta(K(f));$$

for $e = 2$

$$\frac{\delta(f)}{q} > \frac{q^{2d}}{q} = q^{2d-1} \geq 2q^d = \delta(K(f)).$$
Finally we assume that $f = P^e, e \geq 3$ for a monic linear polynomial $P$. This means that in (3.1) and (3.2) $d = 1$. If $j_1 \geq 3$, then

$$e \geq c_{j_1} b_{j_1} \geq b_{j_1} \geq 1 + 2 + \cdots + 2^{j_1-1} \geq j_1 + 4,$$

and so

$$\frac{\delta(f)}{q} \geq q^{e-1} \geq q^{j_1+3} > q^{j_1+1} \geq \delta(K(f)).$$

If $j_1 = 2$, then for $e \geq 5$, we already have $e \geq j_1 + 3$, and so

$$\frac{\delta(f)}{q} \geq q^{e-1} \geq q^{j_1+2} > q^{j_1+1} \geq \delta(K(f));$$

for $e = 4$, we have either $q = 2, e = b_1 + b_2, \delta(K(f)) = 6$ or $q = 3, e = b_2, \delta(K(f)) = 9$, and then we still obtain

$$\frac{\delta(f)}{q} \geq q^3 > \delta(K(f));$$

for $e = 3$, we must have $q = 2, e = b_2, \delta(K(f)) = 4$, and so

$$\frac{\delta(f)}{q} \geq \frac{2^3}{2} = 4 = \delta(K(f)),$$

where the equality holds if and only if $f = t^3$. If $j_1 = 1$, then $e = c_1 b_1 = c_1 \leq q, \delta(K(f)) = eq$, and thus (noticing $e \geq 3$)

$$\frac{\delta(f)}{q} \geq q^{e-1} \geq q^2 \geq eq = \delta(K(f)),$$

where the equalities hold if and only if $q = 3, f = t^3$. This completes the proof. \[ \square \]

In the above proof, we in fact have proved the following result.

**Corollary 3.7.** For any irreducible polynomial $P \in \mathbb{A}$ with $\deg P \geq 2$ and any integer $e \geq 3$, we have

$$\delta(K(P^e)) < \frac{\delta(P^e)}{q^{\deg P}}.$$

### 3.2. Values of the Kempner function

Here we consider the value set and the inverse image sets of the Kempner function $K$.

**Proposition 3.8.** $K(\mathbb{A}) = t\mathbb{A}$.

**Proof.** By Propositions 3.2 and 3.4, it is easy to see that $K(\mathbb{A}) \subseteq t\mathbb{A}$ (note that $K(b) = 0$ for any $b \in \mathbb{F}_q$). On the other hand, for any $f = a_{i_1} t + a_{i_2} t^2 + \cdots + a_{i_k} t^k \in t\mathbb{A}$, we take $c = i_1 b_1 + i_2 b_2 + \cdots + i_k b_k$, where $b_j = \frac{q^j - 1}{q-1}, j \in \mathbb{N}^*$. Then by Proposition 3.4, we have $K(t^e) = f$, and so $t\mathbb{A} \subseteq K(\mathbb{A})$. Thus $K(\mathbb{A}) = t\mathbb{A}$. \[ \square \]
We have seen that the Kempner function $K$ is not injective; see Proposition 3.1 (1). For any non-zero polynomial $f \in \mathbb{A}$, denote by $K^{-1}(f)$ the inverse image set of $f$. We now want to determine all the powers of irreducible polynomials contained in $K^{-1}(f)$.

**Proposition 3.9.** Given a non-zero polynomial $f \in \mathbb{A}$ and an integer $d \in \mathbb{N}^*$, suppose that $q^d | \delta(f)$. Then, $\delta(f)$ is uniquely represented as

$$\delta(f) = c_1 q^{d_1} + c_2 q^{d_2} + \cdots + c_k q^{d_k},$$

with $j_1 > j_2 > \cdots > j_k > 0$, $1 \leq c_i < q^d$, $i = 1, 2, \ldots, k$; put $b_j = \frac{q^d - 1}{q^{d_j} - 1}$, $j \in \mathbb{N}^*$ and

$$e_0 = c_1 b_{j_1} + c_2 b_{j_2} + \cdots + c_k b_{j_k},$$

$K^{-1}(f)$ contains the subset

$$\{P^e : P \in \mathbb{A} \text{ is irreducible of degree } d, \ e \in [e_0 - (j_k - 1), e_0) \cap \mathbb{N}\}.$$

In particular, when exhausting all the positive integers $d$ satisfying $q^d | \delta(f)$, we obtain all the powers of irreducible polynomials contained in $K^{-1}(f)$.

**Proof.** Suppose that $P \in \mathbb{A}$ is an irreducible polynomial of degree $d$. By Proposition 3.4, we directly have $\delta(K(P^e)) = \delta(f)$, and so $K(P^e) = f$. When $j_k \geq 2$ and $e \in [e_0 - (j_k - 1), e_0) \cap \mathbb{N}$, without loss of generality, we take $e = e_0 - i$, $1 \leq i \leq j_k - 1$, then $e$ is uniquely represented in the form:

$$e = c_1 b_{j_1} + \cdots + c_{k-1} b_{j_{k-1}} + (c_k - 1) b_{j_k}$$

$$+ (q^d - 1) b_{j_{k-1}} + \cdots + (q^d - 1) b_{j_{i-1}} + q^d b_{j_{i-1}}.$$ 

By Proposition 3.4, we have $K(P^e) = f$. This in fact completes the proof. 

From Proposition 3.9, one can guess that the Kempner function $K$ is not an increasing function. We confirm this by the following result.

**Proposition 3.10.** For any irreducible polynomials $P, Q \in \mathbb{A}$ with $\deg Q > 1 + \deg P$, there exist positive integers $e_1$ and $e_2$ such that $P^{e_1} > Q^{e_2}$ but $K(P^{e_1}) < K(Q^{e_2})$.

**Proof.** For simplicity, denote $d_1 = \deg P$ and $d_2 = \deg Q$, and put $b_j = \frac{d_1 - 1}{d_2 - 1}$, $j \in \mathbb{N}^*$. Since $1 \leq d_1 < d_2$, by the division algorithm, there exist $k, r \in \mathbb{N}$ such that

$$d_2 = kd_1 + r,$$

where $k \geq 1$ and $0 \leq r < d_1$. By assumption, we have $d_2 - d_1 \geq 2$. 


We first assume \( r \neq 0 \). Take \( e_1 = b_1 + q^{r-1}b_k \) and \( e_2 = 1 \), then \( e_1 \geq k + 1 \). So, using Proposition 3.4 we have

\[
\delta(P^{e_1}) \geq q^{d_1e_1} \geq q^{d_1(k+1)} = q^{d_2+d_1-r} \geq q^{d_2+1} > \delta(Q^{e_2})
\]

and

\[
\delta(K(P^{e_1})) = q^{d_1} + q^{r-1}q^{kd_1} = q^{d_1} + q^{d_2-1} < q^{d_2} = \delta(K(Q^{e_2})).
\]

Hence, \( P^{e_1} > Q^{e_2} \) but \( K(P^{e_1}) < K(Q^{e_2}) \).

We now assume \( r = 0 \). Then \( k \geq 2 \). We take \( e_1 = b_1 + b_{k-1} + b_k \) and \( e_2 = 2 \), then \( e_1 \geq 2k + 1 \). So, using Proposition 3.4 we deduce that

\[
\delta(P^{e_1}) \geq q^{d_1e_1} \geq q^{d_1(2k+1)} = q^{2d_2+d_1} > q^{2d_2+1} > \delta(Q^{e_2})
\]

and

\[
\delta(K(P^{e_1})) = q^{d_1} + q^{d_1(k-1)} + q^{kd_1} = q^{d_1} + q^{d_2-d_1} + q^{d_2} < 2q^{d_2} = \delta(K(Q^{e_2})).
\]

Hence, \( P^{e_1} > Q^{e_2} \) but \( K(P^{e_1}) < K(Q^{e_2}) \). This completes the proof. \( \square \)

We remark that by Proposition 3.4, for any irreducible polynomials \( P, Q \in \mathbb{A} \) and any positive integer \( e \), if \( \deg P = \deg Q \), then \( K(P^{e}) = K(Q^{e}) \).

3.3. Fixed points. For any \( f \in \mathbb{A} \), if \( K(f) = f \), then we call \( f \) a fixed point of \( K \). We first determine the fixed points of the Kempner function \( K \).

**Proposition 3.11.** Given a non-zero polynomial \( f \in \mathbb{A} \), \( f \) is a fixed point of the Kempner function \( K \) if and only if

\[
f = \begin{cases} 
  t, & \text{if } q > 2, \\
  t \text{ or } t^2, & \text{if } q = 2.
\end{cases}
\]

**Proof.** If \( f \) is a fixed point, then \( K(f) = f \), and by Proposition 3.1 (1), (2) and Proposition 3.2, we must have \( f = t^e, e \in \mathbb{N}^* \). So, by definition we obtain the desired result. Indeed, by the definition of factorial (Definition 1.1), we have \( t^e | t^{e-1}! \) if \( e > 2 \); and if \( q > 2 \), then \( t^2 | (a_2t)! \). \( \square \)

We remark that in the integer case all the prime numbers are fixed points of the Kempner function.

For any integer \( n \geq 1 \), let \( K^{(n)} \) be the \( n \)-th iteration of \( K \). It is easy to see that for any \( f \in \mathbb{A} \) with \( \deg f \geq 1 \) there exists some integer \( n \) such that \( K^{(n)}(f) \) is a fixed point of \( K \). We now want to estimate the number of iterations, which can be viewed as the distance to fixed points.

**Proposition 3.12.** For any \( f \in \mathbb{A} \) with \( \deg f \geq 1 \), there exists a positive integer \( n \leq 2 + \deg f \) such that \( K^{(n)}(f) \) is a fixed point of \( K \).
Proof. We first note that by Proposition 3.8, for any polynomial \( g \in \mathbb{A} \) with \( \deg g \geq 1 \), we have \( K(g) \in t\mathbb{A} \), and so, if \( \deg K(g) \geq 2 \), then \( K(g) \) must be a reducible polynomial.

Now, given \( f \in \mathbb{A} \) with \( \deg f \geq 1 \), if \( \deg K(f) \geq 3 \), then \( K(f) \) satisfies the condition in Proposition 3.6, and so

\[
\delta(K^{(2)}(f)) \leq \frac{\delta(K(f))}{q};
\]

if again \( \deg K^{(2)}(f) \geq 3 \), we have

\[
\delta(K^{(3)}(f)) \leq \frac{\delta(K^{(2)}(f))}{q} \leq \frac{\delta(K(f))}{q^2};
\]

this process stops when we reach \( \deg K^{(j)}(f) \leq 2 \) for some integer \( j \). So, this integer \( j \) satisfies

\[
\delta(K^{(j)}(f)) < q^3.
\]

This automatically holds if

\[
\frac{\delta(K(f))}{q^{j-1}} < q^3,
\]

which, together with \( \delta(K(f)) \leq q^{\deg f} \) by Proposition 3.1 (2), implies in

\[
q^{\deg f - j + 1} < q^3.
\]

This gives \( j \geq \deg f - 1 \). Note that \( K^{(j+3)}(f) \) must be a fixed point of \( K \). Here, the term 3 comes from the fact that when \( q \geq 3 \),

\[
K^{(3)}(g) = K^{(2)}(t^2) = K(a_2t) = t
\]

for any irreducible polynomial \( g \in \mathbb{A} \) of degree 2. Hence, for \( n = 2 + \deg f \), \( K^{(n)}(f) \) must be a fixed point of \( K \).

4. Analogue of Erdős’s problem

In this section, we want to establish an analogue of Erdős’s problem.

For any non-constant polynomial \( f \in \mathbb{A} \), let \( \mathcal{P}(f) \) be the maximal monic irreducible factor of \( f \). Following Erdős’s problem, for any integer \( n \geq 1 \), we define the subset of \( \mathbb{A} \):

\[
\mathcal{S}(n) = \{ \text{monic } f \in \mathbb{A} : \deg f = n, K(f) \neq t^{\deg \mathcal{P}(f)} \},
\]

where one should note that \( K(\mathcal{P}(f)) = t^{\deg \mathcal{P}(f)} \).

Using the strategy in [1], which is in fact a classical approach by considering the number of distinct prime factors and the maximal prime factor, we establish the following analogue of Erdős’s problem.
Theorem 4.1. For the sets $S(n)$, we have

$$|S(n)| < \begin{cases} 4^n \exp(-\sqrt{n}/3), & \text{if } n \geq (9 \log \log q^n)^2 \text{ and } n \geq 1600, \\ 4^n \exp(-\sqrt{n}/2), & \text{if } q \geq 3, n \geq (6 \log \log q^n)^2 \text{ and } n \geq 30. \end{cases}$$

Theorem 4.1 implies that for almost all polynomials $f \in A$, $K(f) = t^d$, where $d$ is the maximal degree of the irreducible factors of $f$.

Recall the sets $S_1(n, k), S_2(n, k), S_3(n, k)$ defined in Section 2.2. To prove Theorem 4.1, we need one more preparation.

Lemma 4.2. For any integers $n, k \geq 1$, define

$$S_4(n, k) = S(n) \setminus (S_1(n, k) \cup S_2(n, k) \cup S_3(n, k)).$$

Then, for any $f \in S_4(n, k)$ we have

$$\deg P(f) < D + \frac{\log D}{\log q}.$$ 

Proof. For any $f \in S_4(n, k)$, we have $K(f) \neq t^{\deg P(f)}$, which implies that $f \nmid t^{\deg P(f)}$. So, there exists a monic irreducible polynomial $P$ such that $P \mid f$ and $v_P(f) > v_P(t^{\deg P(f)})$. Note that $v_P(t^{\deg P(f)}) \geq 1$ by definition. So, we have $v_P(f) \geq 2$. Then, in view of $f \notin S_2(n, k)$ and $f \notin S_3(n, k)$, we must have $\deg P \leq D$ and $v_P(f) < D$.

Hence, using Lemma 2.2 we obtain

$$D > v_P(f) > v_P(t^{\deg P(f)}) \geq \frac{q^{\deg P(f)}}{q^{\deg P}} \geq \frac{q^{\deg P(f)}}{q^D},$$

which gives the desired result. \qed

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. For any $f \in S_4(n, k)$, we have $f \notin S_1(n, k) \cup S_2(n, k) \cup S_3(n, k)$. So, $\omega(f) \leq B = 3k \log \log q^n$, and also, if $P^e, e \geq 1$, is any positive power of a monic irreducible polynomial $P$ such that $P^e \mid f$, then we only have two cases:

(i) $\deg P \leq D, e < D$,

(ii) $\deg P > D, e = 1$,

where $D = 2k \log \log q^n$. Case (i) yields at most $Dq^D$ positive powers of monic irreducible polynomials (using Lemma 2.5). For Case (ii), since $\deg P \leq \deg P(f) < D + \log D/\log q$ by Lemma 4.2, it also gives at most $Dq^D$ positive powers of monic irreducible polynomials. Hence, the number of possible powers of monic irreducible polynomials which divides an $f \in S_4(n, k)$ is at most $2Dq^D$. However, such an $f$
is the product of at most \(B = 3k \log \log q^n\) distinct powers of monic irreducible polynomials. Hence, we have

\[
|S_4(n, k)| \leq (2Dq^D)^{3k \log \log q^n} \\
= (4k \log \log q^n)^{3k \log \log q^n} \cdot q^{6k^2(\log \log q^n)^2} \\
\leq q^{7k^2(\log \log q^n)^2}
\]

when \(k \geq 3\) and \(\log \log q^n \geq 7\). Then, using Lemmas 2.7, 2.8 and 2.9 (assuming moreover \(n \geq 9\)), we obtain

\[
|S(n)| \leq |S_1(n, k)| + |S_2(n, k)| + |S_3(n, k)| + |S_4(n, k)| \\
\leq \frac{3q^n}{(\log q^n)^k} + q^{7k^2(\log \log q^n)^2}.
\]

Now, choosing

\[
k = \frac{\sqrt{n}}{3 \log \log q^n},
\]

we obtain

\[
|S(n)| < 4q^n \exp(-\sqrt{n}/3),
\]

where \(n \geq (9 \log \log q^n)^2\) and \(n \geq 1600\) (due to \(k \geq 3\) and \(\log \log q^n \geq 7\)).

Finally we assume \(q \geq 3\). In this case, using Lemmas 2.7, 2.8 and 2.9 we can choose \(B = 2k \log \log q^n\) and \(D = k \log \log q^n\), and then (4.1) becomes

\[
|S_4(n, k)| \leq (2Dq^D)^{2k \log \log q^n} \\
= (2k \log \log q^n)^{2k \log \log q^n} \cdot q^{2k^2(\log \log q^n)^2} \\
\leq q^{3k^2(\log \log q^n)^2}
\]

when \(k \geq 3\) and \(\log \log q^n \geq 3\). So, (4.2) becomes

\[
|S(n)| < \frac{3q^n}{(\log q^n)^k} + q^{3k^2(\log \log q^n)^2}.
\]

Now, choosing

\[
k = \frac{\sqrt{n}}{2 \log \log q^n},
\]

we obtain

\[
|S(n)| < 4q^n \exp(-\sqrt{n}/2),
\]

where \(n \geq (6 \log \log q^n)^2\) and \(n \geq 30\) (due to \(k \geq 3\) and \(\log \log q^n \geq 3\)). This completes the proof. □
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