A NOTE ON THE REGULARITY OF HIBI RINGS

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ABSTRACT. We compute the regularity of the Hibi ring of any finite distributive lattice in terms of its poset of join irreducible elements.

INTRODUCTION

Let $P$ be a finite poset. The set of poset ideals $L = \mathcal{I}(P)$, partially ordered by inclusion, is a distributive lattice. According to a classical result of Birkhoff any finite distributive lattice arises in this way. Now given a field $K$, there is naturally attached to $L$ the $K$-algebra $K[L]$ generated over $K$ by the elements of $L$ with defining relations $\alpha \beta - (\alpha \wedge \beta)(\alpha \vee \beta)$ with $\alpha, \beta \in L$ incomparable. This algebra was introduced by Hibi in 1987 where he showed that $K[L]$ is a Cohen–Macaulay domain with an ASL structure. He also characterized those distributive lattices for which $K[L]$ is Gorenstein. Nowadays $K[L]$ is called the Hibi ring of $L$.

By choosing for each $\alpha \in L$ an indeterminate $x_\alpha$ one obtains the presentation $K[L] \cong S/I_L$ where $S$ is the polynomial ring over $K$ in the indeterminates $x_\alpha$ and where $I_L$ is generated by the quadratic binomials $x_\alpha x_\beta - x_{\alpha \wedge \beta} x_{\alpha \vee \beta}$ with $\alpha, \beta \in L$ incomparable. Not so much is known about the graded minimal free $S$-resolution of the toric ideal $I_L$. Of course we know its projective dimension. Indeed, since $K[L]$ is Cohen-Macaulay and since $\dim K[L]$ is known to be equal to $|P| + 1$, the Auslander-Buchsbaum formula implies that $\text{proj dim } I_L = |L| - |P| - 2$. An equally important invariant of a graded module $M$ over a polynomial ring is its Castelnuovo–Mumford regularity which may be computed in terms of the shifts of the graded minimal free resolution of $M$ and which is denoted by $\text{reg } M$. As a main result of this paper we show that $\text{reg } I_L = |P| - \text{rank } P$. As a consequence we obtain the formula as given in [3] for the regularity of $I_L$ for any planar distributive lattice $L$. Our result also provides a simple proof for the classification of the distributive lattices for which $I_L$ has a linear resolution, see [3 Theorem 3.2] and [4 Corollary 10], and of those lattices for which $I_L$ is extremal Gorenstein, see [4 Theorem 3.5].

1. The regularity of $K[L]$

Let $P$ be a finite poset. A subset $\alpha \subset P$ is called a poset ideal of $P$ if whenever $p \in \alpha$ and $q \leq p$, then $q \in \alpha$. We denote by $\mathcal{I}(P)$ the set of poset ideals of $P$. Note

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Theorem 1.1. Let $v$ with $L$ elements of $L$. Then $v$ is a finite distributive lattice. Birkhoff’s fundamental theorem asserts that any finite distributive lattice $(L, \wedge, \vee)$ arises in this way. To be precise, $L \cong \mathcal{I}(P)$ where $P$ is the subposet of $L$ consisting of all join irreducible elements of $L$. Recall that $\alpha \in L$ is called join irreducible if $\alpha \neq \min L$ and whenever $\alpha = \beta \vee \gamma$, then $\alpha = \beta$ or $\alpha = \gamma$.

Due to this theorem, we may from now on assume $L = \mathcal{I}(P)$ for some poset $P$. This point of view allows us to interpret $K[L]$ as a toric ring. Indeed, let $S$ be the polynomial ring over $K$ in the variables $x_\alpha$ with $\alpha \in L$, and let $T$ be the polynomial ring over $K$ in the variables $s$ and $t_p$ with $p \in P$. We consider the $K$-algebra homomorphism $\varphi: S \to T$ with $\varphi(x_\alpha) = s \prod_{p \in \alpha} t_p$. It is shown in [8] that $I_L = \ker \varphi$. Thus we see that

$$K[L] \cong K[\{s \prod_{p \in \alpha} t_p : \alpha \in L\}] \subset T.$$ 

We henceforth identify $K[L]$ with $K[\{s \prod_{p \in \alpha} t_p : \alpha \in L\}]$. In [6, (3.3)] Hibi describes the monomial $K$-basis of $K[L]$; let $\hat{P}$ be the poset obtained from $P$ by adding the elements $-\infty$ and $\infty$ with $\infty > p$ and $-\infty < p$ for all $p \in P$, and let $\mathcal{S}(\hat{P})$ be the set of integer valued functions $v: \hat{P} \to \mathbb{N}$ with $v(\infty) = 0$ and $v(p) \leq v(q)$ for all $p \geq q$. Then the monomials

$$s^{v(\infty)} \prod_{p \in P} t_p^{v(p)}, \quad v \in \mathcal{S}(\hat{P})$$

form a $K$-basis of $K[L]$. Note that $K[L]$ is standard graded with

$$\deg(s^{v(\infty)} \prod_{p \in P} t_p^{v(p)}) = v(\infty).$$

Let $\omega_L$ be the canonical ideal of $K[L]$. By using a result of Stanley [8, pg. 82], Hibi shows in [6, (3.3)] that the monomials

$$s^{v(\infty)} \prod_{p \in P} t_p^{v(p)}, \quad v \in \mathcal{T}(\hat{P})$$

form a $K$-basis of $\omega_L$, where $\mathcal{T}(\hat{P})$ is the set of integer valued functions $v: \hat{P} \to \mathbb{N}$ with $v(\infty) = 0$ and $v(p) < v(q)$ for all $p > q$.

Based on these facts, we are now ready to prove the following

**Theorem 1.1.** Let $L$ be a finite distributive lattice and $P$ the poset of join irreducible elements of $L$. Then $\reg I_L = |P| - \rank P$.

**Proof.** Let $H_{K[L]}(t)$ be the Hilbert series of $K[L]$. Then

$$H_{K[L]}(t) = \frac{Q(t)}{(1-t)^d},$$

where $Q(t) = \sum_i h_i t^i$ is a polynomial and where $d = |P| + 1$ is the Krull dimension of $K[L]$. Since $K[L]$ is Cohen-Macaulay, it follows that $\reg K[L] = \deg Q(t)$. 


The $a$-invariant $a(K[L])$ of $K[L]$ is defined to be the degree of the Hilbert series of $K[L]$ (see \[2, \text{Def. 4.4.4}\]) which by definition is equal to $\deg Q(t) - d$. Thus we see that

\[(4) \quad \reg I_L = \reg K[L] + 1 = a(K[L]) + |P| + 2.\]

On the other hand, following Goto and Watanabe \[5\], who introduced the $a$-invariant, we have

\[a(K[L]) = -\min\{i: (\omega_L)_i \neq 0\},\]

see \[2, \text{Def. 3.6.13}\]. Thus, since $\hat{\reg} P = \hat{\reg} K[L] + 2$, the desired formula for the regularity of $K[L]$ follows from \[4\] once we have shown that $\min\{i: (\omega_L)_i \neq 0\} = \hat{\reg} \hat{P}$.

Let $v \in \mathcal{T}(\hat{P})$ and let $-\infty < p_1 < \cdots < p_r < \infty$ be a maximal chain in $\hat{P}$ with $r = \hat{\reg} P + 1$. Then

\[0 < v(p_r) < v(p_{r-1}) < \cdots < v(p_1) < v(-\infty).\]

It follows that $v(-\infty) \geq \hat{\reg} \hat{P}$, and hence \[3\] implies that $\min\{i: (\omega_L)_i \neq 0\} \geq \hat{\reg} \hat{P}$. In order to prove equality, we consider the depth function $\delta: \hat{P} \to \mathbb{N}$ which for $p \in \hat{P}$ is defined to be the supremum of the lengths of chains ascending from $p$. Obviously, $\delta \in \mathcal{T}(\hat{P})$ and $\delta(-\infty) = \hat{\reg} \hat{P}$. This concludes the proof of the theorem. $\square$

Recall that $L = \mathcal{I}(P)$ is called simple if there is no $p \in P$ with the property that for every $q \in P$ either $q \leq p$ or $q \geq p$. In the further discussions we may assume without any restrictions that $L$ is simple, because if we consider the subposet $P'$ of $P$ which is obtained by removing a vertex $p \in P$ which is comparable with any other vertex of $P$ and let $L' = \mathcal{I}(P')$, then $I_L$ and $I_{L'}$ have the same regularity. Indeed, $|P'| = |P| - 1$, and since any maximal chain of $P$ passes through $p$, it also follows that $\hat{\reg} P' = \hat{\reg} P - 1$. Thus the assertion follows from Theorem \[\square\].

As an immediate consequence of Theorem \[\square\] we get the following characterization of simple distributive lattices whose Hibi rings have linear resolutions, previously obtained in \[3\] and \[4\].

**Corollary 1.2.** Let $L$ be a finite simple distributive lattice and $P$ the poset of join irreducible elements of $L$. Then $I_L$ has a linear resolution if and only if $P$ is the sum of a chain and an isolated element.

**Proof.** The ideal $I_L$ has a linear resolution if and only if $\reg I_L = 2$. By Theorem \[\square\] this is the case if and only if $|P| - \hat{\reg} P = 2$. Say, $\hat{\reg} P = r$, and let $C = p_0 < p_1 < \cdots < p_r$ be a maximal chain in $P$. Thus $|P| - \hat{\reg} P = 2$, if and only if there exists a unique $q \in P$ not belonging to $C$. Suppose $q$ is comparable with some $p_i$. Then $p_i$ is comparable with any other element of $P$, contradiction the assumption that $L$ is simple. Thus if $L$ is simple, then $|P| - \hat{\reg} P = 2$ if and only if $P$ is the sum of the chain $C$ and the isolated element $q$. $\square$

The preceding corollary implies that a finite simple distributive lattice is planar if $I_L$ has a linear resolution. Now let $L$ be any simple planar lattice and $P$ the poset
of join irreducible elements of $L$. Then there exist two chains $C_1$ and $C_2$ such that $P$ as a set is the disjoint union of them. We may assume that $|C_1| \geq |C_2|$. It follows from Theorem 1.1 that $\text{reg} I_L = |C_2| + 1$. This result may also be obtained with the characterization given in [4, Theorem 4].

We would like to remark that, given a number $k$, Theorem 1.1 allows us to determine in a finite number of steps all finite simple distributive lattices $L$ with $\text{reg} I_L = k$. As an example, we consider the case $k = 3$. Let $P$ be the poset of join irreducible poset of $L$. By Theorem 1.1 it is enough to find all finite posets $P$ with $|P| - \text{rank } P = 3$. Let $C$ be a maximal chain in $P$. Since $|P| = \text{rank } P + 3$, it follows that there exist precisely two elements $q, q' \in P$ which do not belong to $C$. The only posets satisfying $|P| = \text{rank } P + 3$ for which $L = \mathcal{I}(P)$ is simple are displayed in Figure 1.

![Figure 1](image1.png)

The Gorenstein ideals $I_L$ with $\text{reg} I_L = 3$ are called extremal Gorenstein. Hibi showed in [6, pg. 105, d) Corollary] that for any distributive lattice $L$, the ideal $I_L$ is Gorenstein if and only if the poset of join irreducible elements of $L$ is pure. Combining this fact with the above consideration, we recover the result of [3, Theorem 3.5] which says that for a simple distributive lattice $L$, the ideal $I_L$ is extremal Gorenstein if and only if $L$ is one of the lattices shown in Figure 2.

![Figure 2](image2.png)

References

[1] G. Birkhoff, *Lattice Theory* (3rd ed.), Amer. Math. Soc. Colloq. Publ. No. 25. Providence, R. I.: Amer. Math. Soc.
[2] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Revised Ed., Cambridge University Press, 1998.
[3] V. Ene, J. Herzog, T. Hibi, *Linearly related polyominoes*, preprint, arXiv: 1403.4349v1.
[4] V. Ene, A. A. Qureshi, A. Rauf, Regularity of join-meet ideals of distributive lattices, Electron. J. Combin. 20 (3) (2013), #P20.
[5] S. Goto, K.-i. Watanabe, On graded rings, I, J. Math. Soc. Japan 30(2) (1978), 179–213.
[6] T. Hibi, Distributive lattices, affine semigroup rings and algebras with straightening laws, In: “Commutative Algebra and Combinatorics” (M. Nagata and H. Matsumura, Eds.), Adv. Stud. Pure Math. 11, North–Holland, Amsterdam, (1987), 93–109.
[7] P. Schenzel, Über die freien Auflösungen extremaler Cohen-Macaulay Ringe, J. Algebra 64 (1980), 93–101.
[8] R. P. Stanley, Hilbert functions of graded algebras, Adv. Math. 28 (1978), 57–83.

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