Pre-inflationary primordial perturbations

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Abstract

The large-scale power deficit in the CMB fluctuations might be relevant with the physics of pre-inflation, a bounce or a superinflationary phase preceding slow-roll inflation, which can provide a singular-free realization of inflation. We investigate the primordial perturbations from such pre-inflationary evolutions, which generally may consist of multiple phases with different background dynamics, and give a universal formula for the power spectrum of primordial perturbations in terms of the recursive Bogoliubov coefficients. We also apply our formula to corresponding cases, and show how the intensity of large-scale power suppression is affected by the pre-inflationary physics.

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I. INTRODUCTION

As the paradigm of the early universe, inflation has been generally regarded as a possible solution of the horizon, flatness, entropy, homogeneity, isotropy and primordial monopole problems [1], [2], [3], [4]. But maybe far more attractive is that inflation can generate the primordial perturbations, which have grown into all the structures observed in our universe today. The observations of CMB by Planck and WMAP has provided us with more and more information of the early universe, which shows that the single field slow-roll inflationary model is more likely to be the right one.

However, a large-scale (or low-\(l\)) power deficit in CMB TT-mode spectrum observed by WMAP [5] and recently confirmed by the Planck Collaboration [6], [7] with higher precision is not concordant with the standard slow-roll inflation. It is hard to attribute this power deficit to the foreground as it has been observed by experiments with higher and higher statistical significance. Though the cosmic variance could be a source of this deficit, it is still very likely that the large-scale anomalies are induced by the physics preceding inflation, as the larger is the scales of the perturbations, the earlier is the time corresponding to their horizon exiting.

The slow-roll inflation might last for just the minimal number of efolds, i.e. just enough [8], and thus the power deficit on large scale may be attributed to the pre-inflationary non-slow-roll evolution. In this case, the Planck best-fit single-field inflationary model only actually provides a fit for the intermediate and small angular scales. After the WMAP1 data were released, some studies have been done in Refs. [9], [10], [11], [12], [13], [14], [15] along this line, and also recent [16], [17], [18], [19], [20], [21], [22], [23], see Ref. [24] for a review.

However, it might be more interesting that the large-scale anomalies could be relevant with the physics solving the initial singularity problem of inflation [11], [19]. In the bouncing model, see e.g. [25], [26] for reviews, initially the universe is in a contracting phase, and then it bounces into an expanding phase, which results in a solution to the cosmological singularity problem. In Refs. [11], [14], [15], [16], [17], [21], it was noticed that if the universe is initially in a contracting phase and after the bounce it begins to inflate, the power spectrum of primordial perturbations will get a large-scale cutoff, which may naturally explain the power deficit of CMB TT-mode spectrum at low-\(l\), see e.g. [16] for the details. In addition, it was also observed in [27] that if the pre-inflationary bounce actually occurs, the BB-mode correlation
at low-$l$ is also suppressed, while the TB and EB-mode correlations on corresponding scale may be enhanced.

The superinflationary phase before slow-roll inflation may also provide a singular-free realization of inflation, in the meantime explains the anomalies of CMB power spectrum \cite{18,28}, see also \cite{29,30} for an almost flat pre-inflationary universe.

In above pre-inflationary scenarios, e.g.\cite{11,16,18}, initially the primordial perturbation is deep inside the horizon, which naturally set itself in Bunch-Davies(BD) vacuum. This implies that we can calculate large-scale power spectrum without any assumption for the initial state of primordial perturbations. However, it is possible that the pre-inflationary era might consist of multiple phases with different background evolution, e.g.\cite{24}. In certain sense, the introducing of multiple pre-inflationary phases may be better simulate the physics of the pre-inflationary era, since due to the complexity of pre-inflationary physics, sometimes a single phase can hardly reflect the drastic change of the background parameters, e.g.\cite{20}. Therefore, it is interesting to have a quantitative estimate for the power spectra of primordial perturbations from arbitrary pre-inflationary era, involving the multiple phases with different background dynamics.

This paper is organized as follows. In Sec.IIA, we introduce the evolution of pre-inflationary background, which we will focus on, consisting of multiple phases. We require that the primordial perturbations can be produced in these phases. In Sec.IIB, we perform a model-independent calculation for the primordial perturbations, and give a universal formula for the power spectra of primordial perturbations in terms of the recursive Bogoliubov coefficients. In Sec.III, we apply our formula to the bounce inflation and the superinflation preceding slow-roll inflation, and show how the intensity of large-scale power suppression of primordial spectrum is affected by the pre-inflationary physics. We will see a large-scale suppressed primordial spectrum may result in the power deficit at low-$l$ in the CMB TT-mode spectrum, however, the intensity of the power suppression is model-dependent. Sec.IV is the discussion.

In Appendix A, we will investigate the Wronskian constrain for primordial perturbations, and argue that in certain sense the scenario with pre-inflationary era is equivalent to the inflation scenario with the non-Bunch-Davis initial state. In Appendix B, we will approximately estimate the spectra index of primordial perturbations produced in each phase.
II. PRE-INFLATIONARY PRIMORDIAL PERTURBATIONS

A. The pre-inflationary background

The pre-inflationary era may consist of multiple phases. We define the inflation as the phase 0, and the latest pre-inflationary phase as the phase 1, and so on.

The cosmological evolution of the phase $i$ is

$$a_i \sim \eta_i^{\frac{2}{1+3\omega_i}}, \quad \text{or} \quad (-\eta_i)^{\frac{2}{1+3\omega_i}},$$

(1)

where $\omega_i = p_i / \rho_i$ is a constant, and $\omega_i \neq -1/3$, and $\eta = \int dt / a$ is the conformal time. The phase $i$ may be expanding or contracting.

However, noting the continuities of $a$ and $|H|$, the evolutions of the inflation and the phase $i$ can be rewritten as

$$a_0(\eta) = \frac{a_0(\eta_0)}{1 - \mathcal{H}_{inf} \cdot (\eta - \eta_0)},$$

(2)

$$a_i(\eta) = a_i(\eta_{i-1}) \left[1 + \frac{1 + 3\omega_i}{2} \mathcal{H}_i(\eta_{i-1}) \cdot (\eta - \eta_{i-1})\right]^{\frac{2}{1+3\omega_i}},$$

(3)

respectively, where $\eta_0$ is that at the onset of inflation and $\mathcal{H}_{inf} = \mathcal{H}_0(\eta_0)$ is the conformal Hubble parameter during slow-roll inflation, and $\eta_{i-1}$ is the matching time between phase $i$ and phase $i-1$, which signals the onset of phase $i-1$.

It seems that $\omega_i$ is arbitrary. However, if we require that initially all perturbation modes are in BD state, which is right only if its wavelength $\lambda \ll 1/\mathcal{H}$, $\omega_i$ will be constrained. That the perturbation mode with $\lambda \sim 1/k$ extends outside the horizon in phase $i$ marks the primordial perturbation is produced in this phase, which requires $\mathcal{H}_i(\eta_i) < k < \mathcal{H}_i(\eta_{i-1})$.

Thus $|\mathcal{H}_i(\eta)|$ must increase with time, which implies

$$\omega_i < -1/3 \quad \text{for the expanding phase,}$$

or

$$\omega_i > -1/3 \quad \text{for the contracting phase.}$$

(4)

The slow-roll inflation is the evolution with $\omega \simeq -1$, which satisfies (4).

The pre-inflationary universe with $\omega > -1/3$ must be contracting, or such a phase hardly produces the primordial perturbation. After the bounce, the universe begins to inflate. The bounce may be implemented by applying higher-order derivative field [31,32,33,34], which is ghost-free, and also viscous fluid [35], modified gravity [36,37,38,39,40].
FIG. 1: The green line denotes the evolution of the comoving Hubble radius $\frac{1}{a|\dot{H}|}$, and $\eta_i$ signals the onset of the phase $i$. The blue dashed lines denote the evolutions of the primordial perturbations.

The expansion with $\omega < -1$ is superinflation. The primordial perturbations generated during the superinflation have been studied earlier in Refs. [41, 42]. The case with $\omega \ll -1$ corresponds to the slow expansion scenario, which has been proposed in Ref. [44], see also [45, 46] for genesis scenario, and investigated in details in Ref. [47, 48]. It is showed in Ref. [48] that there is no ghost instability during superinflation. Thus the pre-inflationary universe may also be superinflating. It is interesting to notice that both bounce and super-inflation preceding slow-roll inflation may provide a singular-free realization of inflation.

However, if the pre-inflationary era is the expanding phase with $\omega = 1$ characterized by fast-rolling dominance e.g. [9, 49, 50], except the initial fast-roll inflation e.g. [51], or the expanding phase with radiation dominance [10, 13], initially the perturbation mode should be outside the horizon, so it is not clear how to set its initial condition. It is possible that its initial value is set in a phase preceding fast-rolling dominance, however, this phase is still required to satisfy [1], e.g. a higher energy inflation preceding fast-rolling phase [52], see [24] for comments, see also [23] for a domain wall or cosmic string phase. Here, we will not involve this issue, and will assume that all pre-inflationary phases satisfy [1], which assures initially all perturbation modes are naturally set in BD vacuum and each phase may contribute to the production of primordial perturbations with the certain range of wavenumber.

We qualitatively plot the evolutions of background and perturbation modes, which we will focus on, in Fig. 1.
B. The power spectrum of pre-inflationary perturbations

The equation of the primordial perturbation $\mathcal{R}$ is \[53, 54\]

$$u'' + \left( c_s^2 k^2 - \frac{z''}{z} \right) u = 0, \quad (5)$$

where $u \equiv z\mathcal{R}$, the prime is the derivative with respect to conformal time, $z \equiv a \sqrt{2M_p^2 \epsilon / c_s}$ \[55\], and $z \simeq a \sqrt{2M_p^2 |\epsilon| / c_s}$ \[48\] for the superinflationary phase, and the definition of $\epsilon$ is $\epsilon \equiv -\dot{H}/H^2 = \frac{3(1+\omega)}{2}$. We assume $c_s^2 = 1$ for simplicity.

In slow-roll inflationary phase, we have

$$\frac{z''}{z_0} \simeq \frac{2H_{\text{inf}}^2}{[1 - H_{\text{inf}}(\eta - \eta_0)]^2}, \quad (6)$$

the solution of Eq.(5) is

$$u_0 = \sqrt{-k\eta_{\text{eff}} f_0 \left[ C_{0,1} H_{3/2}^{(1)} (-k\eta_{\text{eff}} f_0) + C_{0,2} H_{3/2}^{(2)} (-k\eta_{\text{eff}} f_0) \right]}, \quad (7)$$

where $H_{3/2}^{(1)}$ and $H_{3/2}^{(2)}$ are the 3/2-th order Hankel functions of the first and second kind, respectively, and $\eta_{\text{eff}} f_0 = \eta - \eta_0 - \frac{1}{H_{\text{inf}}}$. In pre-inflationary phase $i$, we have

$$\frac{z''}{z_i} \simeq \frac{(1 - 3\omega_i)H_{\text{inf}}^2(\eta_{i-1})}{2 \left[ 1 + \frac{1 + 3\omega_i}{2} H_i(\eta_{i-1}) : (\eta - \eta_{i-1}) \right]^2}, \quad (8)$$

the solution of Eq.(5) is

$$u_i = \sqrt{-k\eta_{\text{eff}} f_i \left[ C_{i,1} H_{v_i}^{(1)} (-k\eta_{\text{eff}} f_i) + C_{i,2} H_{v_i}^{(2)} (-k\eta_{\text{eff}} f_i) \right]}, \quad (9)$$

where $v_i = \frac{3}{2} \left| \frac{1 - \omega_i}{1 + 3\omega_i} \right|$. $H_{v_i}^{(1)}$ and $H_{v_i}^{(2)}$ are the $v_i$-th order Hankel functions of the first and second kind, respectively, and $\eta_{\text{eff}} f_i = \eta - \eta_{i-1} + \frac{2}{(1 + 3\omega_i)H_i(\eta_{i-1})}$. In fact, Eq.(7) can be obtained from Eq.(9) while $i = 0$, but $\eta_{i-1}$ should be replaced by $\eta_0$.

Here, we require that around and at the matching surface besides there is not the ghost instability, there is also not gradient instability, i.e. $c_s^2 > 0$. In this case, the perturbation can continuously pass through the matching surface between two adjacent phases, and its spectrum is insensitive to the physical detail around the matching surface, see e.g. \[56\] for the bounce. The coefficients $C_i$ in Eq.(7) and Eq.(9) are determined by requiring the continuity of $u$ and $u'$ at the matching surface. We can write the coefficients $C_{i,1}$ and $C_{i,2}$ of phase $i$...
recursively as
\[
\begin{pmatrix}
C_{i,1} \\
C_{i,2}
\end{pmatrix}
= \mathcal{M}^{(i,i+1)} \times 
\begin{pmatrix}
C_{i+1,1} \\
C_{i+1,2}
\end{pmatrix}
= \mathcal{M}^{(i,i+1)} \times \mathcal{M}^{(i+1,i+2)} \times \cdots \times \mathcal{M}^{(i_{\text{max}}-1,i_{\text{max}})} \times 
\begin{pmatrix}
C_{i_{\text{max}},1} \\
C_{i_{\text{max}},2}
\end{pmatrix},
\]
where the earliest phase is defined as the phase \(i_{\text{max}}\), and \(\mathcal{M}^{(i,i+1)} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}\) is the recursive matrix, which is given by

\[
\mathcal{M}_{11} = \frac{i \pi \sqrt{xy}}{8} \left\{ \left[ -H_{-1+v_{i+1}}^{(1)}(y) + H_{1+v_{i+1}}^{(1)}(y) \right] H_{1+v_i}^{(2)}(x) + \left[ H_{-1+v_i}^{(2)}(x) - H_{1+v_i}^{(2)}(x) \right] H_{v_{i+1}}^{(1)}(y) + (x^{-1} - y^{-1}) H_{v_{i+1}}^{(2)}(x) H_{v_{i+1}}^{(1)}(y) \right\},
\]

\[
\mathcal{M}_{12} = \frac{i \pi \sqrt{xy}}{8} \left\{ \left[ -H_{-1+v_{i+1}}^{(2)}(y) + H_{1+v_{i+1}}^{(2)}(y) \right] H_{1+v_i}^{(2)}(x) + \left[ H_{-1+v_i}^{(2)}(x) - H_{1+v_i}^{(2)}(x) \right] H_{v_{i+1}}^{(2)}(y) + (x^{-1} - y^{-1}) H_{v_{i+1}}^{(2)}(x) H_{v_{i+1}}^{(2)}(y) \right\},
\]

\[
\mathcal{M}_{21} = \frac{i \pi \sqrt{xy}}{8} \left\{ \left[ H_{-1+v_{i+1}}^{(1)}(y) - H_{1+v_{i+1}}^{(1)}(y) \right] H_{1+v_i}^{(1)}(x) - \left[ H_{-1+v_i}^{(1)}(x) - H_{1+v_i}^{(1)}(x) \right] H_{v_{i+1}}^{(1)}(y) - (x^{-1} - y^{-1}) H_{v_{i+1}}^{(1)}(x) H_{v_{i+1}}^{(1)}(y) \right\},
\]

\[
\mathcal{M}_{22} = \frac{i \pi \sqrt{xy}}{8} \left\{ \left[ H_{-1+v_{i+1}}^{(2)}(y) - H_{1+v_{i+1}}^{(2)}(y) \right] H_{1+v_i}^{(1)}(x) - \left[ H_{-1+v_i}^{(1)}(x) - H_{1+v_i}^{(1)}(x) \right] H_{v_{i+1}}^{(2)}(y) - (x^{-1} - y^{-1}) H_{v_{i+1}}^{(1)}(x) H_{v_{i+1}}^{(1)}(y) \right\},
\]

where \(x = -\frac{2k}{(1+3\omega_i)\eta_i} \) and \(y = -\frac{2k}{(1+3\omega_{i+1})\eta_{i+1}}\). When \(i = 0\), \(\eta_{i-1}\) should be replaced with \(\eta_0\).

In the earliest phase, the coefficients \(C_{i_{\text{max}},1}\) and \(C_{i_{\text{max}},2}\) are determined by the initial condition. When \(k^2 \gg \frac{z''_{\text{max}}}{z_{\text{max}}}\), the perturbation mode is deep inside the horizon, which is set in BD vacuum,

\[
u \sim \frac{1}{\sqrt{2k}} e^{-ik\eta}.
\]

When \(k^2 \gg \frac{z''_{\text{max}}}{z_{\text{max}}}\), \(u_{i_{\text{max}}}\) given in Eq. (9) should approximate to the form in Eq. (12). Thus we get

\[
C_{i_{\text{max}},1} = \frac{\sqrt{\pi}}{2\sqrt{k}}, \quad C_{i_{\text{max}},2} = 0.
\]
Obviously, $C_{i,\text{max},1}$ and $C_{i,\text{max},2}$ satisfy the so-called Wronskian (or canonical normalization) constraint
\[\frac{4k}{\pi} (|C_{i,\text{max},1}|^2 - |C_{i,\text{max},2}|^2) = 1. \tag{14}\]

And actually, for following phase $i$, $C_{i,1}$ and $C_{i,2}$ will always satisfy the Wronskian constraint, which will be proved in Appendix A. $C_{i,1}$ and $C_{i,2}$ are related to the so-called Bogoliubov coefficients by Eq.\,(25).

The power spectrum of $\mathcal{R}$ is
\[\mathcal{P}_{\mathcal{R}} = \frac{k^3}{2\pi^2} \left| \frac{u_0}{z_0} \right|^2. \tag{15}\]

After substituting Eq.\,(7) into Eq.\,(15) and requiring $\left| k(\eta - \eta_0) - \frac{k}{\frac{H_{\text{inf}}}{2\pi}} \right| \ll 1$, we get a universal formula
\[\mathcal{P}_{\mathcal{R}} = \mathcal{P}_{\mathcal{R}}^{\text{inf}} \frac{4}{\pi} k |C_{0,1} - C_{0,2}|^2, \tag{16}\]
where $\mathcal{P}_{\mathcal{R}}^{\text{inf}} = \frac{1}{2M_{\text{p}}^2 \frac{H_{\text{inf}}}{2\pi}}^2$ is the spectrum of the standard slow-roll inflation, $H_{\text{inf}} = \mathcal{H}_0(\eta_0)/a_0(\eta_0)$ is the Hubble parameter during inflation, and in this sense actually $\mathcal{H}_0(\eta_0) = \mathcal{H}_{\text{inf}}$. The spectral index of $\mathcal{R}$ is
\[n_{\mathcal{R}} = n_{\text{inf}} + \frac{d \ln (k |C_{0,1} - C_{0,2}|^2)}{d \ln k}, \tag{17}\]
where $n_{\text{inf}} = \frac{d \ln \mathcal{P}_{\mathcal{R}}^{\text{inf}}}{d \ln k} + 1$ is the spectral index of slow-roll inflation, which is nearly unity.

The coefficients $C_{0,1}$ and $C_{0,2}$ are determined by the recursive Eq.\,(10), thus the effects of all pre-inflationary phases are encoded in $C_{0,1}$ and $C_{0,2}$, which are non-trivial. By calculating Eq.\,(16), it can be found that the perturbations produced in phase $i$ roughly have a power spectrum
\[\mathcal{P}_{\mathcal{R}} \sim k^{3-2v_i}. \tag{18}\]

We give a proof for this result in Appendix B. It is observed in Appendix B that the power spectrum of perturbations is Eq.\,(18) but modulated with a small oscillation, which is induced by the evolution of perturbation through the matching surface between adjacent phases.

### III. THE LARGE-SCALE POWER SUPPRESSION

In this section, we will apply our universal formula (16) to the bounce inflation and the superinflation preceding slow-roll inflation, and show how the intensity of the large-scale power suppression in the CMB fluctuations is affected by the pre-inflationary evolution.
The large-scale power suppression requires that the efolding number of slow-roll inflation is just enough or less, and in the meantime the pre-inflationary era can contribute a strong blue-tilt spectrum. The spectral index of pre-inflationary perturbations is approximately

\[ n_R - 1 \simeq \frac{d \ln(|C_{0,1} - C_{0,2}|^2)}{d \ln k}. \]  

(19)

Thus the intensity of suppression can be model-dependent, since \(|C_{0,1} - C_{0,2}|^2\) is different for different parameters \(\omega_i\) and \(\eta_i\).

Here, we will focus on the cases that the pre-inflationary era consists of two phases, i.e. \(i_{\text{max}} = 2\), and set \(H_0(\eta_0) = H_{\text{inf}}\) and \(\eta_0 = 0\) is the time at the onset of inflation, thus we have \(\eta < 0\) in pre-inflationary phases and \(\eta > 0\) in the inflationary phase. According to Eq.(10), for \(i_{\text{max}} = 2\), the solutions of Eq.(5) are

\[ u_0 = \sqrt{-k \eta + \frac{k}{H_{\text{inf}}}} \left[ C_{0,1} H_{3/2}^{(1)}(-k \eta + \frac{k}{H_{\text{inf}}}) + C_{0,2} H_{3/2}^{(2)}(-k \eta + \frac{k}{H_{\text{inf}}}) \right], \]

\[ u_1 = \sqrt{-k \eta - \frac{2k}{(1 + 3 \omega_1) H_1(0)}} \left[ C_{1,1} H_{v_1}^{(1)}(-k \eta - \frac{2k}{(1 + 3 \omega_1) H_1(0)}) \right] \]

\[ u_2 = \frac{\sqrt{\pi}}{2} \sqrt{-\eta + \eta_1 - \frac{2}{(1 + 3 \omega_2) H_2(\eta_1)}} H_{v_2}^{(1)}(-k \eta + k \eta_1 - \frac{2k}{(1 + 3 \omega_2) H_2(\eta_1)}), \]

(20)

respectively, where \(\eta_1 < 0\) is the conformal time at the matching surface between phase 1 and phase 2.

We define

\[ P(\eta_1, \omega_1, \omega_2) = \frac{P_R}{P_{R_{\text{inf}}}} = \frac{4}{\pi} k |C_{0,1} - C_{0,2}|^2, \]

(21)

where according to Eq.(10), we have

\[ \begin{pmatrix} C_{0,1} \\ C_{0,2} \end{pmatrix} = \mathcal{M}^{(0,1)} \times \mathcal{M}^{(1,2)} \times \begin{pmatrix} \frac{\sqrt{\pi}}{2 \sqrt{k}} \\ 0 \end{pmatrix}, \]

(22)

and the components of \(\mathcal{M}^{(0,1)}\), \(\mathcal{M}^{(1,2)}\) can be obtained from Eq.(11). \(P(\eta_1, \omega_1, \omega_2)\) reflects the shape of spectrum. We will check how the shape of spectrum, as well as the intensity of suppression on large scale, changes with the parameters \(\eta_1\), \(\omega_1\) and \(\omega_2\) in the bounce inflation and the superinflation preceding slow-roll inflation.
A. The superinflationary phase before inflation

The superinflation is defined as the evolution with \( \epsilon \equiv -\dot{H}/H^2 < 0 \) or \( \omega < -1 \) e.g. [41], [42], [43], [44]. The model with one single superinflationary phase \( \omega = -5/3 \) preceding slow-roll inflation has been builded in string theory in Ref. [18]. In Refs. [29], [30], the pre-inflationary universe is in a slowly expanding genesis phase with \( \epsilon \ll -1 \), which is almost Minkowski space. This genesis phase actually also belongs to the superinflation, but since \( \epsilon \ll -1 \), the expansion is actually slow, see also [28] for the emergent universe.

We will fix the phase 2 with \( \omega_2 = -5/3 \). Thus in certain sense the phase 1 actually corresponds to an intermediate phase, which obviously must also be expanding. The duration that this intermediate phase lasts is \( \Delta \eta = |\eta_1| \). Here, the parameter \( \omega_1 \) in phase 1 is model-dependent but should satisfy \( \omega_1 < -1/3 \). By requiring the continuity of \( |H| \), we get

\[
\mathcal{H}_1(0) = \mathcal{H}_{inf}, \quad \mathcal{H}_2(\eta_1) = \frac{2\mathcal{H}_{inf}}{2 + (1 + 3\omega_1)\eta_1 \mathcal{H}_{inf}},
\]

where \( \mathcal{H}_{inf} \) is determined by the amplitude of CMB fluctuations. Thus in equation set (20) only parameters \( \eta_1 \) and \( \omega_1 \) are left to be free.

We plot \( P(\eta_1, \omega_1, -5/3) \) in Fig. 2 with different \( \eta_1 \) and \( \omega_1 \). The black solid curves, i.e. \( P(0, -1, -5/3) \), is the case with only one single superinflationary phase before slow-roll inflation, which has been studied in Ref. [18]. The perturbation mode with wavelength \( 1/k = 1/\mathcal{H}_{inf} \) is that exiting the horizon at conformal time \( \eta = 0 \), i.e. the onset of inflation. In Fig. 2 we plot \( \tilde{P}(\eta_1, \omega_1, -5/3) \) by replacing \( k \) with \( k/|\eta_1| \), which corresponds to move the suppression of spectrum to smaller scale.

In Fig. 2 the effects of an intermediate phase between superinflation and slow-roll inflation is obvious, compared with the case without the intermediate phase. The first peaks in the dashed curves will left shift with the increase of the duration \( |\eta_1| \) that the intermediate phase lasts. The height of the first peak will lower with the decrease of \( \omega_1 \), since the first peak corresponds to \( k/\mathcal{H}_{inf} < 1 \) while the spectrum is blue-tilted. The power spectrum of perturbation from the intermediate phase, i.e. phase 1, is roughly \( \mathcal{P}_R \sim \left( \frac{k}{\mathcal{H}_{inf}} \right)^{3-2\omega_1} \), which will be proved in Appendix B.
FIG. 2: The power spectrum \( P(\eta_1, \omega_1, -5/3) \) of model, in which there is an intermediate phase between superinflation and slow-roll inflation, with different \( \eta_1 \) and \( \omega_1 \). \( P(0, -1, -5/3) \) corresponds to the case in Ref. [18] without the intermediate phase.

FIG. 3: \( \tilde{P} \) is obtained by replacing \( k \) with \( k/|\eta_1| \).

**B. The bounce inflation**

The pre-inflationary universe may be contracting. After the bounce, the slow-roll inflation begins, which is the so-called bounce inflation scenario [11], [14], [15], [16], [17], and [21] with G-bounce, and [22] with the quintom. It has been found that in this scenario the power spectrum of primordial perturbations will get a large-scale cutoff, which may lead to the power deficit of CMB TT-mode on large angular scale.

However, around the bounce the evolution of universe is complicated, it might be possible that after bounce the universe enters into an intermediate phase prior to the slow-roll inflation. We will check how the shape of spectrum changes with this intermediate phase. We fixed \( \omega_2 = 1 \) and assume that the intermediate phase is characterized by the constant
FIG. 4: The power spectrum $P(\eta_1, \omega_1, 1)$ of model, in which there is an intermediate phase in bounce inflation scenario, with different $\eta_1$ and $\omega_1$. $P(0, -1, 1)$ corresponds to the case in Ref. [18] where there is only one single contracting phase before inflation.

state parameter $\omega_1$, which is model-dependent but satisfies $\omega_1 < -1/3$. By requiring the continuity of $|H|$, we get

$$H_1(0) = H_{inf}, \quad H_2(\eta_1) = \frac{2H_{inf}}{2 + (1 + 3\omega_1)\eta_1H_{inf}}, \quad (24)$$

where $H_{inf}$ is determined by the amplitude of CMB fluctuations. Thus in equation set (20) only parameters $\eta_1$ and $\omega_1$ are left to be free.

We plot $P(\eta_1, \omega_1, 1)$ with different $\eta_1$ and $\omega_1$ in Fig. 4 in which the black solid curves, i.e. $P(0, -1, 1)$, corresponds to the bounce inflation studied explicitly in [11] [16]. In Fig. 4 the effects of an intermediate phase between contraction and slow-roll inflation is obvious. The case is similar to III A. In Fig. 5 we plot $\bar{P}(\eta_1, \omega_1, 1)$ by replacing $k$ with $k/|\eta_1|$, which corresponds to move the suppression of spectrum to smaller scale. Actually, similar shapes of power spectrum have been also obtained in Ref. [20].
IV. DISCUSSION

The large-scale power deficit in CMB TT-mode spectrum may imply certain pre-inflationary physics, e.g. a contracting phase followed by the bounce or a superinflationary phase before slow-roll inflation, which can provide a singular-free realization of inflation. However, the physics of the pre-inflationary era might be complex, sometimes a single phase can hardly reflect the drastic change of the background dynamics. Thus it is interesting to have a quantitative estimate for the power spectrum of primordial perturbations from arbitrary pre-inflationary era, involving multiple phases with different background dynamics.

We perform a model-independent calculation for the power spectrum of primordial perturbations produced during the pre-inflation with different background evolutions. We require the relevant physical quantities continuously pass through the matching surface between adjacent phases, and obtain a universal formula (16) for the primordial spectrum in terms of the recursive Bogoliubov coefficients.

We apply our formula to the bounce inflation and the superinflation preceding slow-roll inflation, and show how the intensity of the CMB power suppression on large scale is affected by the pre-inflationary physics. It is found that due to the existence of the intermediate phase, the intensity of the power suppression becomes model-dependent.

We, with the power spectrum in Fig.4, plot CMB TT-mode spectrum in Fig.6, in which Planck 2013 data is used and the model with an intermediate phase is called as the extended bounce inflation. It has been found in Ref.[16] that the bounce inflation model can improve the fit to the data with $\Delta \chi^2_{\text{eff}} \approx -4.6$ with respect to the standard inflation model with power-law spectrum. Thus it is interesting to have a global fitting analysis with the Planck new data, which is left in the upcoming work.

In addition, both the bounce inflation and the superinflation preceding slow-roll inflation may also explain a large dipole power asymmetry at low-$l$ in CMB TT-mode spectrum [16,18], see also other attempts [57,58,59,60,61,62], and also [63,64]. Actually, there may be also dipole power asymmetry in CMB polarization [65,66], which might be larger than those in TT-mode power spectrum, thus it is interesting to has an estimate for dipole power asymmetry of CMB polarization in the scenarios discussed here.

When deriving (16), we assume that around and at matching surface the perturbation mode has no ghost and gradient instabilities. How to implement such a requirement is an
FIG. 6: The CMB TT-mode power spectra for inflation with power-law power spectrum (red solid), bounce inflation model (green dashed) and extended bounce inflation (blue dashed), in which $\Omega_b h^2 = 0.02203$, $\Omega_c h^2 = 0.1204$, $\tau = 0.09$, $100 \Theta_s = 1.04$, $A_s = 2.1 \times 10^{-9}$, $n_s = 0.961$. The orange points are the Planck 2013 data with 1σ errors.

interesting issue, e.g. [30], [56]. However, if this requirement is not satisfied, which actually is not allowed physically, the result of the perturbation spectrum will be strongly affected by the physical details around the matching surface. In addition, for simplicity, we set $c_s^2 = 1$ and $w$ constant for each phase, it is also interesting to investigate the case with varying $c_s^2$, like e.g. [67], [68] for slow-roll inflation, and $w$, which might be closer to the pre-inflationary physics. We will studied these issues later.

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Appendix A: The Wronskian constraint for $C_{i,1}$ and $C_{i,2}$

Initially $k \ll \mathcal{H}$, the perturbation should be in its minimal energy state, i.e. BD vacuum. Thus the BD vacuum is usually regarded as the initial state of the primordial perturbations. The effects of non-BD initial states on the primordial perturbation has been discussed in
e.g. Refs. [69], [70], [71], [72], [73]. Here, the Bogoliubov coefficients are

\[ \alpha_i = 2\sqrt{\frac{k}{\pi}} C_{i,1}, \]
\[ \beta_i = 2\sqrt{\frac{k}{\pi}} C_{i,2}. \]  

The Wronskian (or canonical normalization) constraint requires \(|\alpha_i|^2 - |\beta_i|^2 = 1\). The initial state, i.e. BD vacuum, corresponds to \(\alpha_{i_{\text{max}}} = 1\) and \(\beta_{i_{\text{max}}} = 0\), i.e.

\[ C_{i_{\text{max}},1} = \frac{\sqrt{\pi}}{2\sqrt{k}}, \quad C_{i_{\text{max}},2} = 0. \]  

We will prove that if \(C_{i_{\text{max}},1}\) and \(C_{i_{\text{max}},2}\) satisfy the Wronskian constraint, for any phase \(i\), \(C_{i,1}\) and \(C_{i,2}\) also satisfy this constraint. This is equivalent to the statement that for any phase \(i\), if

\[ \frac{4k}{\pi} |C_{i+1,1}|^2 - |C_{i+1,2}|^2 = 1 \]  

is satisfied, we always have

\[ \frac{4k}{\pi} |C_{i,1}|^2 - |C_{i,2}|^2 = 1. \]  

It equals to proof that

\[ \left( \overline{C}_{i,1} \overline{C}_{i,2} \right) \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} C_{i,1} \\ C_{i,2} \end{pmatrix} = \left( \overline{C}_{i+1,1} \overline{C}_{i+1,2} \right) \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} C_{i+1,1} \\ C_{i+1,2} \end{pmatrix} \]  

where \(\overline{C}\) is the complex conjugation of \(C\). Due to

\[ \begin{pmatrix} C_{i,1} \\ C_{i,2} \end{pmatrix} = \mathcal{M}^{(i,i+1)} \times \begin{pmatrix} C_{i+1,1} \\ C_{i+1,2} \end{pmatrix}, \]  

we have

\[ \left( \overline{C}_{i,1} \overline{C}_{i,2} \right) = \left( \overline{C}_{i,1} \overline{C}_{i,2} \right)^\dagger = \left( \overline{C}_{i+1,1} \overline{C}_{i+1,2} \right)^\dagger \times (\mathcal{M}^{(i,i+1)})^\dagger = \left( \overline{C}_{i+1,1} \overline{C}_{i+1,2} \right) \times (\mathcal{M}^{(i,i+1)})^\dagger. \]  

Thus after substituting Eqs. (30) and (31) into Eq. (29), we have

\[ (\mathcal{M}^{(i,i+1)})^\dagger \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times (\mathcal{M}^{(i,i+1)}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  

where \( M^{(i,i+1)} \) is defined in Eq. (11).

Therefore, the thing left is to prove

\[
\begin{align*}
|M_{11}|^2 - |M_{21}|^2 &= 1 \\
M_{11} \times M_{12} - M_{21} \times M_{22} &= 0 \\
M_{12} \times M_{11} - M_{22} \times M_{21} &= 0 \\
|M_{12}|^2 - |M_{22}|^2 &= -1
\end{align*}
\]

(33)

This actually can be obtained by using

\[
H^{(1)}_v(\xi) = H^{(2)}_v(\xi),
\]

(34)

and

\[
H^{(1)}_v(\xi) H^{(2)}_{-1+v}(\xi) - H^{(1)}_{-1+v}(\xi) H^{(2)}_v(\xi) = -\frac{4i}{\pi \xi}.
\]

(35)

Hence, for any phase \( i \), if (27) is satisfied, we always have (28).

The Wronskian constrain for \( C_{0,1} \) and \( C_{0,2} \) is

\[
\frac{4k}{\pi} (|C_{0,1}|^2 - |C_{0,2}|^2) = 1.
\]

(36)

Thus \( P_R \) in (16) becomes

\[
P_R = P_R^{inf} \frac{4}{\pi} k |C_{0,1} - C_{0,2}|^2 = P_R^{inf} |\alpha_0 - \beta_0|^2,
\]

(37)

where \( \beta_0 \neq 0 \). Thus in certain sense the scenario with pre-inflationary era is equivalent to the inflation scenario with the non-BD initial state.

**Appendix B: The perturbation spectrum from pre-inflationary phase \( i \)**

In this appendix, we will prove that the power spectrum of perturbations produced in phase \( i \) is approximately

\[
P_R \sim \left( \frac{k}{\mathcal{H}_{inf}} \right)^{n_i - 1},
\]

(38)

where

\[
n_i - 1 = 3 - 2v_i,
\]

(39)

and \( v_i \) is defined as \( v_i = \frac{3}{2} \left| \frac{1 - \omega_i}{1 + 3\omega_i} \right| \).
First, we give a proof for the case with $i_{\text{max}} = 1$. The wave number of the perturbations produced in phase $i$ satisfies $k \ll \mathcal{H}_{\text{inf}}$. We assume that phase 1 is an expanding phase, thus $\omega_1 < -1/3$. According to Sec. II we have

$$u_0 = \sqrt{-k\eta + \frac{k}{\mathcal{H}_{\text{inf}}}} \left( C_{0,1} H_{3/2}^{(1)}(-k\eta + \frac{k}{\mathcal{H}_{\text{inf}}}) + C_{0,2} H_{3/2}^{(2)}(-k\eta + \frac{k}{\mathcal{H}_{\text{inf}}}) \right),$$

$$u_1 = \sqrt{-k\eta - \frac{2k}{(1 + 3\omega_1)\mathcal{H}_{\text{inf}}}} \left( C_{1,1} H_{v_1}^{(1)}(-k\eta - \frac{2k}{(1 + 3\omega_1)\mathcal{H}_{\text{inf}}}) + C_{1,2} H_{v_1}^{(2)}(-k\eta - \frac{2k}{(1 + 3\omega_1)\mathcal{H}_{\text{inf}}}) \right), \tag{40}$$

where

$$C_{1,1} = \frac{\sqrt{\pi}}{2\sqrt{k}}, \quad C_{1,2} = 0. \tag{41}$$

Because $\mathcal{H}_{\text{inf}}$ always appears with $k$ as $k/\mathcal{H}_{\text{inf}}$ bellow, we will set $\mathcal{H}_{\text{inf}} = 1$ for convenience, and actually can easily get it back. We have

$$\mathcal{P}_R = \mathcal{P}_R^{\text{inf}} \frac{4}{\pi} k |C_{0,1} - C_{0,2}|^2, \tag{42}$$

where

$$C_{0,1} = -\frac{\pi k}{8i} \left\{ \sqrt{-\frac{\pi}{(1 + 3\omega_1)(2k)^3}} \left[ 2k \left( -H_{-1+\nu_1}^{(1)}\left( -\frac{2k}{1 + 3\omega_1} \right) + H_{1+\nu_1}^{(1)}\left( -\frac{2k}{1 + 3\omega_1} \right) \right) H_{3/2}^{(2)}(k) 

+ H_{v_1}^{(1)}\left( -\frac{2k}{1 + 3\omega_1} \right) \left( 2k H_{1/2}^{(2)}(k) + 3(1 + \omega_1) H_{3/2}^{(2)}(k) - 2k H_{5/2}^{(2)}(k) \right) \right] \right\}, \tag{43}$$

$$C_{0,2} = \frac{\pi k}{8i} \left\{ \sqrt{-\frac{\pi}{(1 + 3\omega_1)(2k)^3}} \left[ 2k \left( H_{1/2}^{(1)}(k) - H_{3/2}^{(1)}(k) \right) H_{v_1}^{(1)}\left( -\frac{2k}{1 + 3\omega_1} \right) 

+ H_{3/2}^{(1)}(k) \left( -2k H_{-1+\nu_1}^{(1)}\left( -\frac{2k}{1 + 3\omega_1} \right) + 3(1 + \omega_1) H_{v_1}^{(1)}\left( -\frac{2k}{1 + 3\omega_1} \right) + 2k H_{1+\nu_1}^{(1)}\left( -\frac{2k}{1 + 3\omega_1} \right) \right) \right] \right\}. \tag{44}$$

The special case is $\omega_1 = -5/3$, which gives $\nu_1 = 1, -1 + \nu_1 = 0$. When $k \ll 1$, i.e. $\frac{k}{\mathcal{H}_{\text{inf}}} \ll 1$, we get

$$\mathcal{P}_R \approx \mathcal{P}_R^{\text{inf}} \frac{4k}{\pi \mathcal{H}_{\text{inf}}^2} \left( 1 + \frac{k^2}{12 \mathcal{H}_{\text{inf}}^2} \ln \frac{k}{4 \mathcal{H}_{\text{inf}}} - \frac{k^2}{6 \mathcal{H}_{\text{inf}}^2} \right)^2 \sim \frac{k}{\mathcal{H}_{\text{inf}}}. \tag{45}$$
If \(-5/3 < \omega_1 < -1/3\), then \(v_1 > 1, -1 + v_1 > 0\). When \(\frac{k}{H_{in}} \ll 1\), we get

\[
P_R \approx P_{inf}^{in} \frac{1}{\pi} \left( \frac{-1}{1 + 3 \omega_1} \right)^{3 - 2v_1} \cdot \left( \frac{k}{H_{in}} \right)^{3 - 2v_1} \times \left\{ - (1 + 3 \omega_1) \Gamma(v_1) + \frac{k^2}{6H_{in}} \left[ (1 + 3 \omega_1) \Gamma(v_1) - \Gamma(-1 + v_1) \right] \right\}^2 \approx \left( \frac{k}{H_{in}} \right)^{3 - 2v_1}.
\]

If \(\omega_1 < -5/3\), then \(v_1 < 1, -1 + v_1 < 0\), \(H_{1+v_1}^{(1)}\) should replaced by \(e^{i(-1+v_1)\pi} H^{(1)}_{1-v_1}\), when \(\frac{k}{H_{in}} \ll 1\), we get

\[
P_R \approx P_{inf}^{in} \frac{1}{\pi} \left( \frac{k}{H_{in}} \right)^{3 - 2v_1} \times \left[ \Gamma^2(v_1)(-1 - 3 \omega_1)^{-1 + 2v_1} + \frac{1}{3} \left( \frac{k}{H_{in}} \right)^{2v_1} \cos(v_1 \pi) \Gamma(1 - v_1) \Gamma(v_1) \right] \approx \left( \frac{k}{H_{in}} \right)^{3 - 2v_1}.
\]

Thus we always have \(P_R \sim \left( \frac{k}{H_{in}} \right)^{n_i - 1}\). This result can also apply to the case with \(\omega_1 > -1/3\).

Then, we focus on the case with \(i_{\text{max}} > 1\). Though the spectra index seems nontrivial, based on some assumptions, we could get the similar result.

We assume that the pre-inflationary era is phase \(\tilde{i}\) dominated, i.e. the phase \(\tilde{i}\) lasts long enough, nearly, \((\eta_{i-1} - \eta_i) \to \infty\). The modes of the perturbations produced in phase \(\tilde{i}\) satisfy \(H_{\tilde{i}}(\eta_i) \ll k \ll H_{\tilde{i}}(\eta_{i-1})\). Thus, for \(i \geq \tilde{i}\), we always have \(x \gg 1\) and \(y \gg 1\), \(x\) and \(y\) are those in Eq. (10), and obtain

\[
\mathcal{M}_{11} = \left( 1 - \frac{x^{-1} - y^{-1}}{4i} \right) e^{-i(x-y-v_{i-v_{i+1}} \pi)} \approx e^{-i(x-y-v_{i-v_{i+1}} \pi)};
\]

\[
\mathcal{M}_{12} = -\frac{x^{-1} - y^{-1}}{4} e^{-i(x+y-v_{i-v_{i+1}} \pi)} \approx 0;
\]

\[
\mathcal{M}_{21} = -\frac{x^{-1} - y^{-1}}{4} e^{i(x+y-v_{i-v_{i+1}} \pi)} \approx 0;
\]

\[
\mathcal{M}_{22} = \left( 1 + \frac{x^{-1} - y^{-1}}{4i} \right) e^{i(x-y-v_{i-v_{i+1}} \pi)} \approx e^{i(x-y-v_{i-v_{i+1}} \pi)}.
\]
Thus, when \( i \geq \tilde{i} \), the recursive matrixes \( \mathcal{M}^{(i,i+1)} \) are diagonal. We get

\[
\begin{pmatrix}
C_{i,1} \\
C_{i,2}
\end{pmatrix} = \mathcal{M}^{(\tilde{i},i+1)} \times \mathcal{M}^{(\tilde{i}+1,i+2)} \times \cdots \times \mathcal{M}^{(i_{\text{max}}-1,i_{\text{max}})} \times 
\begin{pmatrix}
C_{i_{\text{max}},1} \\
C_{i_{\text{max}},2}
\end{pmatrix}
\]

\[
= \begin{pmatrix} e^{-iE} & 0 \\ 0 & e^{iE} \end{pmatrix} \times \begin{pmatrix} \sqrt{\pi} \over 2 \sqrt{k} \\ 0 \end{pmatrix} = e^{-iE} \begin{pmatrix} C_{i_{\text{max}},1} \\ 0 \end{pmatrix},
\] (49)

where \( E \) is real, \( C_{i_{\text{max}},1} = \frac{\sqrt{\pi}}{2 \sqrt{k}} \). Because what we concern is \( |C_{0.1} - C_{0.2}| \) in the final result, \( e^{-iE} \) makes no difference, we can neglect it and write that

\[
\begin{pmatrix}
C_{i,1} \\
C_{i,2}
\end{pmatrix} = \begin{pmatrix} \sqrt{\pi} \over 2 \sqrt{k} \\ 0 \end{pmatrix} = \begin{pmatrix} C_{i_{\text{max}},1} \\ 0 \end{pmatrix}.
\] (50)

For \( i < \tilde{i} \), we always have \( x \ll 1, y \ll 1 \). By expanding the Hankel functions to second order, we have

\[ C_{i_1} + C_{i_2} = (\mathcal{M}_{11} + \mathcal{M}_{12})C_{i+1,1} + (\mathcal{M}_{12} + \mathcal{M}_{22})C_{i+1,2} \]

\[ \approx A_{i,1} \cdot (C_{i+1,1} + C_{i+1,2}) \left( \frac{k}{\mathcal{H}_i(\eta_i)} \right)^{v_{i+1} - v_i} \]

\[ + A_{i,2} \cdot (C_{i+1,1} - C_{i+1,2}) \left( \frac{k}{\mathcal{H}_i(\eta_i)} \right)^{-v_{i+1} - v_i}, \]

\[ C_{i_1} - C_{i_2} = (\mathcal{M}_{11} - \mathcal{M}_{21})C_{i+1,1} + (\mathcal{M}_{12} - \mathcal{M}_{22})C_{i+1,2} \]

\[ \approx A_{i,3} \cdot (C_{i+1,1} + C_{i+1,2}) \left( \frac{k}{\mathcal{H}_i(\eta_i)} \right)^{v_{i+1} + v_i} \]

\[ + A_{i,4} \cdot (C_{i+1,1} - C_{i+1,2}) \left( \frac{k}{\mathcal{H}_i(\eta_i)} \right)^{-v_{i+1} + v_i}, \] (51)

where the constants \( A_{i_1,1}, A_{i_1,2}, A_{i_1,3}, A_{i_1,4} \) are independent with \( k \), and \( v_i \) is defined as

\[ v_i = \frac{3}{2} \left| \frac{1 - \omega_i}{1 + 3 \omega_i} \right|. \] Because \( k \ll \mathcal{H}(\eta_{i-1}) < \mathcal{H}(\eta_{i-2}) < \cdots < \mathcal{H}(\eta_0) \), we will not care \( \mathcal{H}(\eta_{i-1}), \cdots, \mathcal{H}(\eta_0) \) bellow, and just regard \( \frac{k}{\mathcal{H}(\eta_i)} \ll 1 \) as \( k \ll 1 \) instead.

According to Eq. (50) and Eq. (51), we get

\[ C_{i-1,1} + C_{i-1,2} = C_{i_{\text{max}},1} \cdot \left( A_{i-1,1} \cdot k^{v_{i-1} - v_{i-1}} + A_{i-1,2} \cdot k^{-v_{i-1} - v_{i-1}} \right), \]

\[ C_{i-1,1} - C_{i-1,2} = C_{i_{\text{max}},1} \cdot \left( A_{i-1,3} \cdot k^{v_{i+1} + v_{i-1}} + A_{i-1,4} \cdot k^{-v_{i+1} + v_{i-1}} \right), \] (52)

and then

\[ C_{i-2,1} + C_{i-2,2} = C_{i_{\text{max}},1} \cdot \left( A_{i-2,1} \cdot k^{v_{i-2} - v_{i-2}} + A_{i-2,2} \cdot k^{-v_{i-2} - v_{i-2}} \right), \]

\[ C_{i-2,1} - C_{i-2,2} = C_{i_{\text{max}},1} \cdot \left( A_{i-2,3} \cdot k^{v_{i+2} + v_{i-2}} + A_{i-2,4} \cdot k^{-v_{i+2} + v_{i-2}} \right), \]
\[ C_{0,1} + C_{0,2} = C_{i_{\text{max},1}} \cdot (A_{0,1} \cdot k^{v_i - v_0} + A_{0,2} \cdot k^{-v_i + v_0}), \]
\[ C_{0,1} - C_{0,2} = C_{i_{\text{max},1}} \cdot (A_{0,3} \cdot k^{v_i + v_0} + A_{0,4} \cdot k^{-v_i + v_0}). \]

After neglecting the high order terms, we have
\[ C_{0,1} - C_{0,2} \approx C_{i_{\text{max},1}} \cdot A_{0,4} \cdot k^{-v_i + v_0} \sim \frac{\sqrt{\pi}}{2k} k^{-v_i + v_0}, \tag{54} \]
where \( v_0 = 3/2 \) for the inflationary phase. Therefore
\[ P_R = P_R^{\text{inf}} \frac{4}{\pi} k \left| C_{0,1} - C_{0,2} \right|^2 \sim P_R^{\text{inf}} \cdot k^{3 - 2v_i}. \tag{55} \]

It should be noted that (55) is a good approximation only if the phase \( \tilde{i} \) lasts long enough, i.e. \( \mathcal{H}_i(\eta_i) \ll k \ll \mathcal{H}_i(\eta_{i-1}) \). The reason is apparent, since if the phase \( \tilde{i} \) lasts only a very short time, the oscillations of spectrum around the matching surface would destroy the relation we proofed above.

We plot \( n_i - 1 \) with respect to \( \omega_i \) in Fig.7 see also [74], which provides a guidance for building a pre-inflationary model leading to a cut-off spectrum on large scale. We see that the power spectrum is strong blue-tilt only for the expansion with \( \omega_i < -1 \) and the contraction with \( \omega_i > 0 \).

![FIG. 7: The figure of \( n_i - 1 \) with respect to \( \omega_i \), which is plotted based on \( n_i - 1 = 3 - 2v_i \) and \( v_i = \frac{3}{2} \left| \frac{1 - \omega_i}{1 + 3\omega_i} \right| \). The magenta curves is \( n_i - 1 \). And the blue dashed line is \( \omega_i = -1/3 \).](image-url)
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