AFFINE LAUMON SPACES AND A CONJECTURE OF KUZNETSOV

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Abstract. We prove a conjecture of Kuznetsov stating that the equivariant $K$–theory of affine Laumon spaces is the universal Verma module for the quantum affine algebra $U_q(\hat{\mathfrak{gl}}_n)$. We do so by reinterpreting the action of the quantum toroidal algebra $U_q(\hat{\mathfrak{sl}}_n)$ on the $K$–theory from [14] in terms of the shuffle algebra studied in [12], which constructs an embedding $U_q(\hat{\mathfrak{gl}}_n) \hookrightarrow U_q(\hat{\mathfrak{sl}}_n)$.

1. Introduction

Laumon spaces for the group $GL_n$ parametrize flags of torsion-free sheaves on $\mathbb{P}^1$:

$$ F_1 \subset \ldots \subset F_{n-1} \subset O_{\mathbb{P}^1}(1) $$

whose fibers near $\infty \subset \mathbb{P}^1$ match a fixed full flag of subspaces of $\mathbb{C}^n$. Laumon spaces are disconnected, with components indexed by vectors $d = (d_1, \ldots, d_{n-1}) \in \mathbb{N}^{n-1}$ that keep track of the first Chern classes of the sheaves in (1.1). The component indexed by $d$ coincides with the space of framed degree $d$ quasimaps into the complete flag variety, hence the interest of these objects in representation theory.

We will denote quantum affine and quantum toroidal algebras by $U_q(\hat{\mathfrak{gl}}_n)$, $U_q(\hat{\mathfrak{sl}}_n)$, respectively (we use dots instead of the more customary hats in order to avoid double hats on our symbols). The equivariant $K$–theory groups of Laumon spaces have been studied and identified with the universal Verma module of $U_q(\hat{\mathfrak{sl}}_n)$ in [1]. In the present paper, we study an “affine” version of these spaces, denoted by:

$$ \mathcal{M} = \bigsqcup_{d=(d_1,\ldots,d_n) \in \mathbb{N}^n} \mathcal{M}_d $$

whose definition we will recall in Subsection 3.1. The reference [1] constructs an action of $U_q(\hat{\mathfrak{sl}}_n) \rtimes K = K_{\text{equiv}}(\mathcal{M})$ and recalls a conjecture of Kuznetsov that this action can be extended to $U_q(\hat{\mathfrak{gl}}_n) \rtimes K$. Our main purpose is to prove this fact:

Theorem 1.1. There is a geometric action of the affine quantum algebra $U_q(\hat{\mathfrak{gl}}_n)$ on $K$, with the latter being isomorphic to the universal Verma module.

Affine Laumon spaces appear naturally in mathematical physics, geometry and representation theory as semismall resolutions of singularities of Uhlenbeck spaces for $\mathfrak{sl}_n$. In [10], we will use Theorem 1.1 to prove a conjecture of Braverman that relates the Nekrasov partition function of $\mathcal{N} = 2$ supersymmetric $U(n)$ gauge
theory with bifundamental matter in the presence of a complete surface operator to the elliptic Calogero-Moser system. The fact that the $K$–theory group of $M$ is isomorphic to the universal Verma module of $U_q(\mathfrak{gl}_n)$ will be crucial to our proof.

Reference [14] constructed an action of the bigger algebra $U_{q,\mathfrak{g}}(\mathfrak{gl}_n)$ on $K$, by generators and relations. We will recast this action in terms of the shuffle algebra realization of $U_{q,\mathfrak{g}}(\mathfrak{gl}_n)$, see [12]. Thus, any element of the shuffle algebra gives rise to an operator on $K$. In particular, we have introduced in loc. cit. the elements:

$$S^\pm_m, T^\pm_m \in U_q(\mathfrak{g}^a_n) \Rightarrow S^\pm_m, T^\pm_m \curvearrowright K$$

for any Laurent polynomial $m(z_i, \ldots, z_{j-1})$ and any pair of integers $i < j$. When $m$ is the constant Laurent polynomial 1, the operators (1.2) will give rise to the action of the root generators of $U_q(\mathfrak{g}^a_n)$ on $K$, and we will use this fact to prove Theorem 1.1.

The word “geometric” in the statement of Theorem 1.1 comes from the fact that the operators (1.2) will be given by certain explicit correspondences. Specifically, we will define three types of correspondences in Section 4:

- the fine correspondences $\mathcal{J}_{[i;j)}$ in Definition 4.3
- the eccentric correspondences $\mathcal{Y}_{[i;j)}$ in Definition 4.7
- the smooth correspondences $\mathcal{M}_k$ in (4.27)

The fine and eccentric correspondences are equipped with tautological line bundles $L_i, \ldots, L_{j-1}$. With this in mind, we may identify the operators (1.2) with those induced by the fine and eccentric correspondences:

**Theorem 1.2.** The shuffle element $S^\pm_m$ (resp. $T^\pm_m$) acts on $K$ via:

$$m(L_i, \ldots, L_{j-1}) \text{ on } \mathcal{J}_{[i;j)} \text{ (resp. } \mathcal{Y}_{[i;j)})$$

interpreted as a correspondence on $K$ in (4.17) (resp. (4.18)). Similarly, the smooth correspondence $\mathcal{M}_k$ corresponds to the shuffle element $G_{\pm(k,\ldots,k)}$ of (2.20).

The content of the present paper and of [12] is synthesized in [11], with additional details, in the related context when affine Laumon spaces are replaced by Nakajima cyclic quiver varieties. Indeed, affine Laumon spaces may be interpreted as “chainsaw quiver varieties” (see [7]), which we review in Subsection 3.7.

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2. The quantum toroidal and shuffle algebras

2.1. Let us now review the main constructions of [12], and introduce certain notation and results that will be used. In the present paper, we will encounter variables:

\[ x_{i1}, x_{i2}, \ldots \]

for arbitrary \( i \in \{1, \ldots, n\} \). The first index is called the color of the variable \( x_{ia} \), and is denoted by \( i = \text{col} x_{ia} \). Although colors come from the set \( \{1, \ldots, n\} \), in many of our formulas we will encounter arbitrary colors \( i \in \mathbb{Z} \), by the convention:

\[
\left( z \text{ of color } i \right) \quad \text{is identified with} \quad \left( z^{q^{-2}\mid \frac{i}{\bar{i}} \rangle} \text{ of color } \bar{i} \right)
\]

where \( \bar{i} \) denotes the residue class of \( i \in \mathbb{Z} \) in the set \( \{1, \ldots, n\} \). As an example, let us consider the following color-dependent rational function:

\[
\zeta \left( \frac{z}{w} \right) = \left( \frac{z^{q^{-2}\mid \frac{i}{\bar{i}} \rangle} - w^{q^{-1}}}{z^{\gamma} - w} \right)^{\delta_{i, j \mod n} - \delta_{i, j + 1 \mod n}}
\]  

for variables \( z, w \) of colors \( i, j \in \mathbb{Z} \). If we wanted to convert the right-hand sides of (2.2) into an expression that only involves variables of colors \( i, j \in \{1, \ldots, n\} \), then:

\[
\zeta \left( \frac{z}{w} \right) = \begin{cases} \frac{z^{q-wq^{-1}}}{z^{\gamma} - w} & \text{if } \bar{i} = \bar{j} \\ \frac{z^{q-wq^{-1}}}{z^{\gamma} - w} & \text{if } \bar{i} + 1 = \bar{j} \\ \frac{z^{q-wq^{-1}}}{z^{\gamma} - w} & \text{if } \bar{i} = n, \bar{j} = 1 \\ 1 & \text{otherwise} \end{cases}
\]

2.2. Let us now introduce the trigonometric shuffle algebra corresponding to the \( n \) vertex cyclic quiver (see [6] for the original inspiration for shuffle algebras in the context of elliptic quantum algebras). A rational function:

\[ R(\ldots, z_{i1}, \ldots, z_{ik}, \ldots) \]

(\( i \) goes from 1 to \( n \)) will be called color-symmetric if it is symmetric in \( z_{i1}, \ldots, z_{ik} \) for all \( i \) separately. We will refer to the vector \( k = (k_1, \ldots, k_n) \in \mathbb{N}^n \) as the degree of \( R \), and we will write \( \text{Sym}^k \rightarrow \text{Sym}^n \). Let \( \mathcal{F} = \mathbb{Q}(q, \bar{q}) \) and define the vector space:

\[ \mathcal{V} = \bigoplus_{k \in \mathbb{N}^n} \mathcal{F}(\ldots, z_{i1}, \ldots, z_{ik}, \ldots)^{\text{Sym}_{1 \leq i \leq n}} \]  

where the superscript Sym means that we only consider color-symmetric rational functions. We make (2.5) into a \( \mathcal{F} \)-algebra via the shuffle product:

\[
R(\ldots, z_{i1}, \ldots, z_{ik}, \ldots) * R'(\ldots, z_{i1}, \ldots, z_{ik'}, \ldots) = \frac{1}{k! \cdot k'!} \cdot \]  

\[
\text{Sym} \left[ R(\ldots, z_{i1}, \ldots, z_{ik}, \ldots) R'(\ldots, z_{i1}, \ldots, z_{i1+k'}, \ldots) \prod_{1 \leq j \leq k} \prod_{k_{i', j} = k_{i', j+1}} \zeta \left( \frac{z_{ij}}{z_{ij'}} \right) \right]
\]

for all rational functions \( R \) and \( R' \) in \( k \) and \( k' \) variables, respectively.

**Definition 2.3.** The shuffle algebra \( \mathcal{A}^+ \subset \mathcal{V} \) consists of rational functions:

\[
R(\ldots, z_{i1}, \ldots, z_{ik}, \ldots) = \frac{1}{\prod_{i=1}^{k} \prod_{r=1}^{k_i} (z_{ia}q - z_{i+1,a}q^{-1})}
\]
where \( r \) goes over all color-symmetric Laurent polynomials satisfying the conditions:

\[
r(\ldots, z_{i\alpha}, \ldots) \bigg|_{z_{i1} \rightarrow w, z_{i2} \rightarrow wq^{\pm 2}, z_{i1,1} \rightarrow w} = 0
\]

for all \( i \in \{1, \ldots, n\} \).

2.4. It was shown in [12] that the shuffle algebra coincides with the subalgebra of \( V \) generated by the one-variable rational functions \( z_{i\alpha}^d, \forall i \in \{1, \ldots, n\}, d \in \mathbb{Z} \). This fact allowed us to prove the isomorphism:

\[
\mathcal{A}^+ \cong U_q\mathfrak{gl}(n)
\]

between the shuffle algebra and the positive half of the quantum toroidal algebra (the precise definition of the latter will not be important to us, but can be found in [12]). The isomorphism (2.9) induces an isomorphism between Drinfeld doubles:

\[
\mathcal{A} \cong U_q\mathfrak{gl}(n)
\]

The algebra \( \mathcal{A} \) will be called the double shuffle algebra, and it is defined as:

\[
\mathcal{A} = \mathcal{A}^- \otimes \mathcal{A}^0 \otimes \mathcal{A}^+ \tag{2.11}
\]

where:

\[
\mathcal{A}^- = \left(\mathcal{A}^+\right)^{op}
\]

\[
\mathcal{A}^0 = F\left[ c, \psi_{i,d}^\pm \right]_{d \in \mathbb{N} \cup 0} \left/ \left( \psi_{i,0}^+ \psi_{i,0}^- - 1 \right) \right.
\]

where the element \( c \in \mathcal{A} \) is central. We refer the reader to [12] about details on the multiplicative relations between the three factors of (2.11). Let us write \( \psi_i = \psi_{i,0} \) and note that the definition of \( \mathcal{A}^0 \) implies that \( \psi_i \) is invertible. We will write \( R^- \) or \( R^+ \) when specifying that a particular rational function \( R \) as in (2.4) lies in either of the opposite algebras \( \mathcal{A}^- \) or \( \mathcal{A}^+ \), respectively. There exists a grading:

\[
\mathcal{A}^\pm = \bigoplus_{k \in \mathbb{N}^n} \mathcal{A}_{\pm k}
\]

which assigns to a rational function \( R^\pm \) its degree \( \pm k \), where \( k = (k_1, \ldots, k_n) \).

2.5. For any \( i < j \), let \( (i,j) \in \mathbb{N}^n \) denote the vector whose \( a \)-th entry is the number of integers in the set \( \{i, \ldots, j-1\} \) that are congruent to \( a \) mod \( n \). An important role in establishing the isomorphism (2.10) was played by the rational functions:

\[
S_m^\pm(z_i, \ldots, z_{j-1}) = \text{Sym} \left[ \frac{m(z_i, \ldots, z_{j-1})}{(1-\frac{z_i}{z_{i+1}})(1-\frac{z_{i+1}}{z_{i+2}}) \prod_{i \leq a < b < j} \zeta\left(\frac{z_b}{z_a}\right)} \right] \in \mathcal{A}_{\pm (i,j)} \tag{2.12}
\]

\[
T_m^\pm(z_i, \ldots, z_{j-1}) = \text{Sym} \left[ \frac{m(z_i, \ldots, z_{j-1})}{(1-\frac{z_i}{z_{i+1}})(1-\frac{z_{i+1}}{z_{i+2}}) \prod_{i \leq a < b < j} \zeta\left(\frac{z_a}{z_b}\right)} \right] \in \mathcal{A}_{\pm (i,j)} \tag{2.13}
\]
defined for all \( i < j \) and all \( m \in \mathbb{F}[z_i^{-1}, \ldots, z_j^{-1}] \) (see (2.1) on how to change the variables \( z_i, \ldots, z_{j-1} \) of colors \( i, \ldots, j-1 \) into variables of color \( \{1, \ldots, n\} \), so that the right-hand sides of (2.12)--(2.13) are well-defined elements of (2.5)). In particular:

\[
E_{(i;j)} := S_i^+ \\
F_{(i;j)} := T_i^-
\]  

(2.14) (2.15)

were shown in [12] to give rise to an embedding:

\[
U_q(\hat{g}_n) \hookrightarrow \mathcal{A}
\]

(2.16)

e_{(i;j)} \sim E_{(i;j)}, \quad f_{(i;j)} \sim F_{(i;j)}

2.6. Consider the following elements of the shuffle algebra, defined for all \( k \in \mathbb{N}^n \):

\[
G_k = \frac{1}{(q^{-1} - q)^{|k|}} \prod_{1 \leq i \leq n} \prod_{1 \leq a \leq k_i} \left( \frac{1}{q} - \frac{z_{ia} q}{z_{ia}} \right) \in \mathcal{A}_k
\]

(2.17)

\[
G_{-k} = \frac{1}{(1 - q^{-2})^{|k|}} \prod_{1 \leq i \leq n} \prod_{1 \leq a \leq k_i} \left( \frac{1}{q} - \frac{z_{ia} q}{z_{ia}} \right) \in \mathcal{A}_{-k}
\]

(2.18)

where \(|k| = k_1 + \ldots + k_n\). It was shown in loc. cit. that there exists an isomorphism:

\[
U_q(\hat{g}_1) \otimes U_q(\hat{g}_1) \cong U_q(\hat{g}_n)
\]

as well as an injective homomorphism:

\[
U_q(\hat{g}_1) \hookrightarrow \mathcal{A}
\]

given by sending:

\[
\text{Drinfeld-Jimbo generators of } U_q(\hat{g}_1) \sim E_{[i;i+1]}, F_{[i;i+1]}
\]

(2.19)

\[
\text{group-like generators } g_{\pm k} \text{ of } U_q(\hat{g}_1) \sim \prod q^k G_{\pm(k,\ldots,k)}
\]

(2.20)

\( \forall k \in \mathbb{N} \). We refer to loc. cit. for a discussion of the Heisenberg algebra \( U_q(\hat{g}_1) \) and its embedding into \( U_q(\hat{g}_n) \), in terms of which the defining relation takes the form:

\[
[P_k, P_l] = \delta^0_{k+l} \cdot \left( q^{nk} - q^{-nk} \right) \left( c^k - c^{-k} \right) \frac{(q^n - q^{-n})(c^k - c^{-k})}{(q^n q^{-n})(q^k q^{-k})}
\]

(2.21)

where \( \sum_{k=0}^{\infty} \prod G_{\pm(k,\ldots,k)} x^k = \exp \left( \sum_{k=1}^{\infty} \frac{P_{k+1} x^k}{k} \right) \) and \( c \) is central in \( \mathcal{A} \).
3. Laumon Spaces

3.1. To define affine Laumon spaces, consider the surface $\mathbb{P}^1 \times \mathbb{P}^1$ and the divisors:

$$D = \mathbb{P}^1 \times \{0\}, \quad \infty = \mathbb{P}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{P}^1$$

A rank $n$ parabolic sheaf $\mathcal{F}$ is a flag of rank $n$ torsion free sheaves:

$$\mathcal{F}_\bullet = \left\{ \mathcal{F}_n(-D) \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n \right\}$$

(3.1)

on $\mathbb{P}^1 \times \mathbb{P}^1$, together with a collection of isomorphisms:

$$\mathcal{F}_n(-D)|_\infty \overset{\cong}{\to} \mathcal{F}_1|_\infty \overset{\cong}{\to} \ldots \overset{\cong}{\to} \mathcal{F}_{n-1}|_\infty \overset{\cong}{\to} \mathcal{F}_n|_\infty$$

$$\mathcal{O}_\infty^{\otimes n}(-D) \to \mathcal{O}_\infty \oplus \mathcal{O}_\infty^{\otimes n-1}(-D) \to \ldots \to \mathcal{O}_\infty^{\otimes n-1} \oplus \mathcal{O}_\infty(-D) \to \mathcal{O}_\infty^{\otimes n}$$

The vertical isomorphism are called framing, and they force $c_1(\mathcal{F}_i) = -(n-i)D$. On the other hand, $-c_2(\mathcal{F}_i) =: d_i$ can vary over all non-negative integers, and we call the vector $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$ the degree of the parabolic sheaf (3.1).

**Definition 3.2.** The affine Laumon space $\mathcal{M}_d$ is the moduli space of rank $n$, degree $d$ parabolic sheaves as above (see the footnote on page 1 for the etymology of the word “affine”).

Affine Laumon spaces are smooth and quasi-projective varieties of dimension $2|d| = 2(d_1 + \ldots + d_n)$. It will be convenient to extend (3.1) to an infinite flag of sheaves, by setting $\mathcal{F}_{i+n} = \mathcal{F}_i(D)$ for all $i \in \mathbb{Z}$.

**Remark 3.3.** When $d_n = 0$, one has an isomorphism:

$$\mathcal{F}_n \cong \mathcal{O}_\mathbb{P}_n^{\otimes n}$$

for any parabolic sheaf (3.1). In this case, the parabolic sheaf (3.1) is completely determined by its restriction to the divisor $D$, so affine Laumon spaces for $d_n = 0$ parametrize framed flags of sheaves (1.1) on a projective line. This precisely matches Laumon’s original definition of the moduli spaces of flags (1.1).

3.4. An alternative presentation of affine Laumon spaces was given in [5], inspired by independent constructions of Biswas and Okounkov. Consider the $n$-to-1 map:

$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, \quad \sigma(x, y) = (x, y^n)$$

To any flag (3.1), we may associate the following rank $n$ torsion-free sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\mathcal{F} = \sigma^*(\mathcal{F}_0) + \sigma^*(\mathcal{F}_1)(-D) + \ldots + \sigma^*(\mathcal{F}_{n-1})(-(n-1)D)$$

(3.2)

Let us stress the fact that the right hand side is not a direct sum, but simply the linear span of the sheaves $\{\sigma^*(\mathcal{F}_i)(-iD)\}_{0 \leq i \leq n}$ inside the sheaf $\sigma^*(\mathcal{F}_{n-1})$. Because of the framing condition, we observe that:

$$\mathcal{F}|_\infty = \mathcal{O}_\infty(-D) \oplus \ldots \oplus \mathcal{O}_\infty(-nD)$$

(3.3)
It is elementary to show that $-c_2(F) = |d|$. The authors of [7] modify the framing of $F$ at $P^1 \times \{\infty\} \subset \subset$ and produce a one-to-one correspondence:

$$\left( F_n(-D) \subset F_1 \subset ... \subset F_{n-1} \subset F_n \right) \sim \tilde{F}$$

where $\tilde{F}$ matches $F$ off $\infty$, but the framing condition is changed to $\tilde{F}|_{\infty} = O_{\infty}^{\oplus n}$.

The advantage of this choice is that one can directly match the construction with the usual moduli space of framed sheaves, as defined below.

**Definition 3.5.** Consider the moduli space $N_d$ which parametrizes rank $n$ torsion free sheaves $\tilde{F}$ on $P^1 \times P^1$, with $-c_2(\tilde{F}) = d$ and a framing isomorphism $\tilde{F}|_{\infty} \sim = O_{\infty}^{\oplus n}$.

As explained in loc. cit., any sheaf $\tilde{F}$ on $P^1 \times P^1$ which arises from the correspondence (3.4) is invariant under the action:

$$Z/nZ \rightrightarrows P^1 \times P^1, \quad e^{2\pi i/n} \cdot (x, y) = (x, e^{-2\pi i/n} y)$$

and $Z/nZ \rightrightarrows O_{\infty}^{\oplus n}$ via:

$$e^{2\pi i/n} \mapsto \text{the matrix } \delta = \text{diag} \left( e^{2\pi i/n}, ..., e^{2\pi i(n-1)/n}, e^{2\pi i n} \right)$$

Conversely, any $Z/nZ$–invariant rank $n$ torsion free sheaf on $P^1 \times P^1$ is of the form (3.2) for some parabolic sheaf (3.1). We conclude that:

$$N_{Z/nZ}^d = \bigsqcup_{d \in \mathbb{N}^n} M_d$$

**3.6.** We note that the moduli space $N_d$ of framed sheaves on $P^1 \times P^1$ is isomorphic to the moduli space of framed sheaves on $P^2$ (see [2], Section 4).

Therefore, we conclude that $N_d$ is a Nakajima quiver variety, which by [9] can be presented as the set of quadruples of linear maps $(X, Y, A, B)$ which satisfy certain properties. In more detail, consider the double framed Jordan quiver:

![Diagram](image)

Specifically, the picture represents the fact that we fix vector spaces $V \cong \mathbb{C}^d$ and $W \cong \mathbb{C}^n$ and consider the following vector space of linear maps between them:

$$N_d = \text{Hom}(V, V) \oplus \text{Hom}(V, V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$$

Elements of this vector space are quadruples of linear maps $(X, Y, A, B)$, whose domains and codomains are depicted in Figure 3.6. Consider the quadratic map:

$$N_d \rightarrow \text{End}(V), \quad \mu(X, Y, A, B) = [X, Y] + AB$$
and the action $GL_d := GL(V)$ on $N_d$ by conjugation. Then the well-known ADHM presentation of the moduli space of framed sheaves (see loc. cit.) asserts that:

$$N_d \cong \mu^{-1}(0)^s/GL_d$$  \hspace{1cm} (3.8)

Here and throughout this paper, the superscript $s$ means that we intersect $\mu^{-1}(0)$ with the open set of stable quadruples $(X,Y,A,B)$, i.e. those for which the vector space $V$ is generated by $X$ and $Y$ acting on $\text{Im} A$.

**3.7.** We will now recall the chainsaw quiver construction from [7]. In terms of quadruples, the action $\mathbb{Z}/n\mathbb{Z} \curvearrowright N_d$ from (3.5)–(3.6) is given by:

$$e^{\frac{2\pi}{n}} \cdot (X,Y,A,B) = (X, e^{\frac{2\pi}{n}}Y, A\delta, e^{\frac{2\pi}{n}}\delta^{-1}B)$$

For such a quadruple to be $\mathbb{Z}/n\mathbb{Z}$–fixed, i.e. for the quadruple to correspond to a point in Laumon space by (3.7), there must exist some $g \in GL_d$ such that:

$$(X, e^{\frac{2\pi}{n}}Y, A\delta, e^{\frac{2\pi}{n}}\delta^{-1}B) = (gXg^{-1}, gYg^{-1}, gA, Bg^{-1})$$  \hspace{1cm} (3.9)

If we decompose $V = \bigoplus_{\lambda \in \mathbb{C}} V(\lambda)$ in terms of the generalized eigenspaces of $g$, then (3.9) specifies that the linear maps $X,Y,A,B$ can only act non-trivially between the following pairs of eigenspaces:

$$V(\lambda) \xrightarrow{X} V(\lambda) \hspace{1cm} V(\lambda) \xrightarrow{Y} V\left(\lambda \cdot e^{\frac{2\pi}{n}}\right)$$

$$w_k \xrightarrow{A} V\left(e^{\frac{2\pi ik}{n}}\right) \hspace{1cm} V\left(e^{\frac{2\pi ik}{n}}\right) \xrightarrow{B} w_{k+1}$$  \hspace{1cm} (3.10) (3.11)

where $w_k$ denotes the basis vector of $W$ which is acted on by the character $e^{\frac{2\pi ik}{n}}$ in the action (3.6). Because we only consider stable quadruples, then (3.10)–(3.11) imply that the only non-zero eigenspaces are $V_k := V(e^{\frac{2\pi ik}{n}})$. Therefore, the maps $(X,Y,A,B)$ in a $\mathbb{Z}/n\mathbb{Z}$–fixed quadruple split up according to the following diagram:

![Diagram](image)

**Figure 3.7**

More precisely, we fix vector spaces $V_1,\ldots,V_n$ of dimensions $d_1,\ldots,d_n$ that sum up to $d$, and consider the vector space of linear maps:

$$M_d = \bigoplus_{i=1}^{n} \text{Hom}(V_i, V_i) \bigoplus_{i=1}^{n} \text{Hom}(V_{i-1}, V_i) \bigoplus_{i=1}^{n} \text{Hom}(W_i, V_i) \bigoplus_{i=1}^{n} \text{Hom}(V_{i-1}, W_i)$$  \hspace{1cm} (3.12)
where \( \mathbf{d} = (d_1, ..., d_n) \in \mathbb{N}^n \), and we identify \( V_0 \) with \( V_n \). Elements of the vector space \( M_{\mathbf{d}} \) will be quadruples \((X_i, Y_i, A_i, B_i)_{1 \leq i \leq n}\). Consider the quadratic map:

\[
M_{\mathbf{d}} \xrightarrow{\nu} \bigoplus_{i=1}^{n} \text{Hom}(V_{i-1}, V_i)
\]

\[
\nu(X_i, Y_i, A_i, B_i)_{1 \leq i \leq n} = \bigoplus_{i=1}^{n} (X_i Y_i - Y_i X_{i-1} + A_i B_i)
\]

We let \( GL_{\mathbf{d}} := \prod_{i=1}^{n} GL(V_i) \) act on the vector space (3.12) by conjugation, and with this in mind, (3.7)–(3.8) imply the following description of affine Laumon spaces:

\[
M_{\mathbf{d}} \cong \nu^{-1}(0)^s / GL_{\mathbf{d}}
\]

The superscript \( s \) refers to the open subset of stable quadruples, i.e. those where the vector spaces \( V_i \) are generated by the \( X \) and \( Y \) maps acting on the images of the \( A \) maps. Therefore, points of affine Laumon spaces are stable quadruples of linear maps as in Figure 3.7 mod conjugation, satisfying the moment map equation \( \nu = 0 \).

3.8. The maximal torus \( T_n \subset GL_n \) and the rank 2 torus \( \mathbb{C}^* \times \mathbb{C}^* \) act on \( M_{\mathbf{d}} \) by changing the trivialization at \( \infty \), respectively by multiplying the base \( \mathbb{P}^1 \times \mathbb{P}^1 \). In terms of quadruples (3.13), this action is given by:

\[
(U_1, ..., U_n, Q, \overline{Q}) \cdot (..., X_i, Y_i, A_i, B_i, ...) = \left( ..., Q^2 X_i, \overline{Q}^{2d_i} Y_i, A_i U_i^2, \frac{Q^2 \overline{Q}^{3d_i}}{U_i^2} B_i, ... \right)
\]

for all \((U_1, ..., U_n, Q, \overline{Q}) \in T_n \times \mathbb{C}^* \times \mathbb{C}^* \). We choose even powers in the expression above in order to avoid square roots later on (in other words, the torus action we consider is a \( 2^{n+1} - \)fold cover of the usual one). Write \( T = T_n \times \mathbb{C}^* \times \mathbb{C}^* \), and let

\[
K_T(pt) = \text{Rep}(T) = \mathbb{Z}[u_1^{\pm 1}, ..., u_n^{\pm 1}, q^{\pm 1}, \overline{q}^{\pm 1}]
\]

(3.15)

(where \( u_1, ..., u_n, q, \overline{q} \) denote functions on Lie \( T \) which are dual to \( U_1, ..., U_n, Q, \overline{Q} \) above). In the present paper, we will study the \( T \)-equivariant algebraic \( K \)-theory groups of affine Laumon spaces:

\[
K^\text{int}_{\mathbf{d}} := K_T(M_{\mathbf{d}})
\]

which are all modules over the ring (3.15). The superscript “int” refers to integral \( K \)-theory, as opposed from the localized \( K \)-theory that we will mostly focus on:

\[
K_{\mathbf{d}} := K_T(M_{\mathbf{d}}) \bigotimes_{K_T(pt)} \text{Frac}(K_T(pt))
\]

Our main actor is the \( \mathbb{N}^n \)-graded vector space:

\[
K = \bigoplus_{\mathbf{d} \in \mathbb{N}^n} K_{\mathbf{d}}
\]

over the field \( \mathbb{F}_u := \text{Frac}(K_T(pt)) = \mathbb{Q}(u_1, ..., u_n, q, \overline{q}) \).
3.9. For any \( d \in \mathbb{N}^n \), consider the ring of color-symmetric Laurent polynomials:

\[
\Lambda_d = \mathbb{F}[x_{i_1}^\pm 1, \ldots, x_{i_d}^\pm 1]_{1 \leq i \leq n, 1 \leq a \leq d_i}
\]

namely those polynomials \( f \) which are symmetric in \( x_{i_1}, \ldots, x_{i_d} \) for each \( i \) separately. Let \( \pi : \nu^{-1}(0)^s \to \text{pt} \) denote the standard map, and consider the composition:

\[
\Lambda_d := K_{T \times GL_d}(pt) \xrightarrow{\pi^*} K_{T \times GL_d}(\nu^{-1}(0)^s) \xrightarrow{(3.14)} K_T(M_d)
\]

\[
f \sim \mathcal{f}
\]

Elements of in the image of the map (3.17) will be called tautological classes. When \( f(..., x_{i_a}, ...) = x_{i_1} + \ldots + x_{i_d} \) is the first power sum function, then:

\[
f = [\mathcal{V}_i]
\]

is the class of the tautological vector bundle \( \mathcal{V}_i \), whose fiber over a quadruple \( \in \mathcal{M}_d \) is the vector space \( \mathcal{V}_i \) itself. To compute the tangent space to \( \mathcal{M}_d \), we recall that the tangent space to a vector space is naturally identified with itself:

\[
[TM_d] = \sum_{i=1}^{n} \left( \frac{\mathcal{V}_i}{\mathcal{V}_i q^2} + \frac{\mathcal{V}_i}{\mathcal{V}_{i-1}} + \frac{\mathcal{V}_i}{u_i^2} + \frac{u_i^2}{\mathcal{V}_{i-1} q^2} \right) \quad (3.19)
\]

The four sums above are the contributions of the \( X, Y, A, B \) linear maps, respectively. To keep formulas simple, here and throughout this paper we abuse notation and write \( \mathcal{V} \) instead of \( [\mathcal{V}] \) for the class of a vector bundle. We also write:

\[
\mathcal{V}' \quad \text{instead of} \quad [\mathcal{V}' \otimes \mathcal{V}']
\]

and set:

\[
\mathcal{V}_i = \mathcal{V}_i \cdot q^{-2[\frac{i-1}{n}]}
\]

\[
u_i = u_i \cdot q^{-1[\frac{i}{n}]}
\]

for all \( i \in \mathbb{Z} \), which absorbs the dependence of (3.19) on the equivariant parameter \( \overline{q} \). To obtain the tangent space to \( \mathcal{M}_d \), one must subtract from (3.19) the classes of the tangent direction to the “moment map” equation \( \nu = 0 \) and the gauge group \( GL_d \), which are respectively:

\[
\sum_{i=1}^{n} \frac{\mathcal{V}_i}{\mathcal{V}_{i-1} q^2} \quad \text{and} \quad \sum_{i=1}^{n} \frac{\mathcal{V}_i}{\mathcal{V}_i}
\]

We obtain the following formula for the tangent bundle to affine Laumon spaces:

\[
[TM_d] = \sum_{i=1}^{n} \left( 1 - \frac{1}{q^2} \right) \left( \frac{\mathcal{V}_i}{\mathcal{V}_{i-1}} - \frac{\mathcal{V}_i}{\mathcal{V}_i} \right) + \frac{\mathcal{V}_i}{u_i^2} + \frac{u_i^2}{\mathcal{V}_{i-1} q^2} \right)
\]

\[
(3.23)
\]

In particular, the rank of the tangent bundle, i.e. the dimension of \( \mathcal{M}_d \), is:

\[
2 \sum_{i=1}^{n} \text{rk} \mathcal{V}_i = 2(d_1 + \ldots + d_n) = 2|d|
\]
3.10. One expects that tautological classes generate the integral $K$–theory ring $K^\text{int}_d$. We do not have a proof of this claim, but it becomes quite simple once we replace integral $K$–theory with localized $K$–theory. In other words, the following Proposition establishes the fact that the map (3.17) is surjective upon localization:

**Proposition 3.11.** For any $d \in \mathbb{N}^n$, the vector space $K_d$ is spanned by the tautological classes $f$, as $f \in \Lambda_d$ goes over all color-symmetric functions.

The Proposition above will be proved in Subsection 3.18. An advantage of working with tautological classes is that we can replace the ring $\Lambda_d$ of color-symmetric functions in finitely many variables by the ring:

$$\Lambda = \mathbb{F}_u \langle x_{1a}, \ldots, x_{n,b} \rangle_{\text{Sym}}$$

of color-symmetric Laurent polynomials in infinitely many variables over the field $\mathbb{F}_u = \mathbb{Q}(u_1, \ldots, u_n, q, \bar{q})$. Then we may collect the maps (3.17) together for all $d$:

$$\Lambda \to \prod_{d \in \mathbb{N}^n} K_d, \quad f \sim \overline{f}$$

We abuse notation by writing $\overline{f}$ either for a $K$–theory class on a single moduli space $M_d$, or for the collection of these $K$–theory classes over all $d \in \mathbb{N}^n$. We will often write color-symmetric functions $f \in \Lambda$ by using the shorthand notation:

$$f(X) = f(\ldots, x_{ia}, \ldots)_{1 \leq i \leq n, a \in \mathbb{N}} \in \Lambda[[z^{\pm 1}]]$$

Recalling the color-symmetric rational function (2.2), we may write expressions such as:

$$\zeta \left( \frac{z}{X} \right) = \prod_{a=1}^{\infty} \frac{z - x_{ia} q^{-1}}{z - x_{ia}} \prod_{a=1}^{\infty} \frac{z - x_{i+1,a}}{z q - x_{i+1,a} q^{-1}} \in \Lambda[[z^{\pm 1}]]$$

(3.25)

$$\zeta \left( \frac{X}{z} \right)^{-1} = \prod_{a=1}^{\infty} \frac{x_{ia} - z}{x_{ia} q - z q^{-1}} \prod_{a=1}^{\infty} \frac{x_{i-1,a} q - z q^{-1}}{x_{i-1,a} - z} \in \Lambda[[z^{\pm 1}]]$$

(3.26)

for any variable $z$ of color $i$. The right-hand sides are interpreted as color-symmetric functions in the $x$ variables by expanding $z$ around either 0 or $\infty$, depending on the situation. Moreover, the notation (3.24) allows us to define the so-called **plethysm** homomorphism with respect to any variable $z$ of any color $i$:

$$f(X) \sim f(X + z) = f(x_{11}, \ldots, \ldots, x_{i1}, x_{i2}, \ldots, \ldots, x_{1n}, \ldots)$$

(3.27)

for any $f \in \Lambda$. The inverse operation to (3.27) is denoted by:

$$f(X) \sim f(X - z)$$

(3.28)

It does not have a closed formula akin to (3.27), but if $z$ is a variable of color $i$ and $f = x_{j1}^k + x_{j2}^k + \ldots$ is a **power sum function** of variables of color $j$, then:

$$f(X \pm z) = \pm b_i^j z^k + f(X)$$

Since arbitrary elements of $\Lambda$ are Laurent polynomials in the power sum functions, the equation above determines the correspondences (3.27) and (3.28) completely.
3.12. Plethysms can also be defined by adding or subtracting whole alphabets:

\[ Z = \sum_{1 \leq i \leq n} z_{ia} \]

by successively adding or subtracting the individual variables \( z_{ia} \). We will write:

\[ R^\pm(Z) = R^\pm(\ldots, z_{ia}, \ldots)_{1 \leq a \leq k_i} \]

for arbitrary shuffle elements \( R^\pm \in A_{\pm k} \). We define:

\[ \tau_+(Z) = \prod_{1 \leq i \leq n} \left( \frac{u_{i+1}}{q} - \frac{z_{ia}q}{u_{i+1}} \right) \]

(3.30)

\[ \tau_-(Z) = \prod_{1 \leq i \leq n} \left( u_i - \frac{z_{ia}}{u_i} \right) \]

(3.31)

and make the convention that:

\[ \zeta \left( \frac{Z}{Z} \right) = \prod_{i,j=1}^{n} \prod_{a \leq k_i, b \leq k_j} \zeta \left( \frac{z_{ia}}{z_{jb}} \right) \]

since \( \zeta(x)|_{x \to 1} \) doesn’t make sense if \( \text{col } x = 0 \). Similarly, we define:

\[ DZ = \prod_{1 \leq a \leq k_i} Dz_{ia}, \quad \text{where} \quad Dz = \frac{dz}{2\pi i z} \]

We are now ready to restate Theorem 4.13 of [14] in terms of the shuffle algebra:

**Theorem 3.13.** There is an action \( A \acts K \), where elements \( R^\pm \in A_{\pm k} \) act by:

\[ R^\pm \acts f = \frac{q^{-|k|\delta^\pm}}{k!} \int \frac{R^\pm(Z) f(X \mp Z) \zeta(Z/Z) \tau^\pm(Z)}{\zeta(X/1) \zeta(Z/1)} \tau^\pm(Z)^{\pm 1} DZ \]

(3.32)

for all \( f(X) \in \Lambda \). In the above formulas, \( \int^\pm \) denote normal-ordered integrals which will be defined in Remark 3.15. The commuting elements \( \psi^\pm_{i,d} \in A^0 \) act by:

\[ \sum_{d=0}^{\infty} \frac{\psi^\pm_{i,d}}{z^{\pm d}} = \text{multiplication by } q^{\pm i} \left( \frac{1}{u_i^{\pm 1}} - \frac{u_i^{\pm 1}}{z^{\pm 1}} \right) \cdot \zeta \left( \frac{X}{z} \right) \]

(3.33)

expanded around \( z^{\pm 1} = \infty \).

**Remark 3.14.** The central element \( c \) acts by \( q^n c \), and the leading term of (3.33) is:

\[ \psi_i = \text{multiplication by } \frac{q^{i+d_{i-1} - d_i}}{u_i} \]

(3.34)

on the graded component \( K_d \subset K \). This is quite relevant, because the various \( K_d \) are the weight subspaces of \( K \), and (3.34) tells us that the weights are prescribed by the integers \( d_1, \ldots, d_n \) and the equivariant parameters \( u_1, \ldots, u_n \).
Remark 3.15. For a rational function \( F(Z) = F(..., z_{ia}, ...) \), we define

\[
\int^+ F(Z)DZ = \sum_{\sigma: \{i,a\} \to \{1,-1\}} \int_{|z_{ia}| = \gamma^{\sigma(i,a)} |\eta|^{-1}} |q|^{\pm 1} |\eta|^{\pm 1} < 1 \ F(..., z_{ia}, ...) \prod_{(i,a)} \sigma(i,a) Dz_{ia} \tag{3.35}
\]

\[
\int^- F(Z)DZ = \sum_{\sigma: \{i,a\} \to \{1,-1\}} \int_{|z_{ia}| = \gamma^{\sigma(i,a)} |\eta|^{2} > 1} |q|^{\pm 1} |\eta|^{\pm 1} > 1 \ F(..., z_{ia}, ...) \prod_{(i,a)} \sigma(i,a) Dz_{ia} \tag{3.36}
\]

for some positive real \( \gamma \ll 1 \). In each summand, each variable \( z_{ia} \) is integrated over either a very small circle of radius \( \gamma |\eta|^{-1} \) or a very large circle of radius \( \gamma^{-1} |\eta|^{-2} \). The orientation of the contours is such that the residue theorem reads:

\[
\int_{|z| < 1} f(z)Dz - \int_{|z| > 1} f(z)Dz = - \sum_{\alpha \not= 0, \infty} \text{Res}_{z=\alpha} f(z) = \text{Res}_{z=0} f(z) + \text{Res}_{z=\infty} f(z) \tag{3.37}
\]

The meaning of the superscripts \( |q|^{\pm 1}, |\eta|^{\pm 1} < 1 \) that adorn the integral (3.35) is the following. In the summand corresponding to a particular function \( \sigma \), if:

\[
\sigma(i,a) = \sigma(j,b) = 1
\]

then the variables \( z_{ia} \) and \( z_{jb} \) are both integrated over the small circle. For any factor of the form \( z_{ia} |q| |\eta|^{2} - z_{jb} |q|^{2} |\eta|^{-2} \) appearing in the denominator of \( F(Z) \), we make the assumption that \( |q|, |\eta| < 1 \) when evaluating the integral via residues. If:

\[
\sigma(i,a) = \sigma(j,b) = -1
\]

then the variables \( z_{ia} \) and \( z_{jb} \) are both integrated over the large circle. For any factor of the form \( z_{ia} |q|^{2} |\eta| - z_{jb} |q| |\eta|^{-2} \) in the denominator of \( F(Z) \), we assume that \( |q|, |\eta| > 1 \) when evaluating the integral via residues. If \( \sigma(i,a) \not= \sigma(j,b) \), then we need not assume anything about the sizes of \( q \) and \( \eta \). One defines (3.36) similarly.

**Proof** Let us show that formula (3.32) for shuffle elements \( R^+_1(Z_1) \) and \( R^+_2(Z_2) \) implies the same formula for the shuffle product \( R^+_1 \ast R^+_2 \), where \( Z = Z_1 + Z_2 \). Assume \( \gamma_2 \ll \gamma_1 \ll 1 \) are positive real numbers. Applying (3.32) twice gives us:

\[
R^+_1 \ast R^+_2 \ast \tilde{f} = \frac{1}{k_1! k_2!} \int_{|Z_1| = \gamma_1^{-1}, |Z_2| = \gamma_2^{-1}} \frac{R^+_1(Z_1)R^+_2(Z_2)}{\zeta(Z_1/Z_1) \zeta(Z_2/Z_2)} \cdot \frac{f(X - Z_1 - Z_2) \zeta \left( \frac{Z_1}{X} \right) \zeta \left( \frac{Z_2}{X} \right) \zeta \left( \frac{Z_2}{Z_1} \right)^{-1}} {\tau_+(Z_1) \tau_+(Z_2) DZ_1 DZ_2}
\]

where \( |Z| = \gamma \) is shorthand for \( |z_{ia}| = \gamma^{-2} \) for all \( i, a \). The poles that involve both \( Z_1 \) and \( Z_2 \) in the integral all come from the rational function \( \zeta(Z_2/Z_1)^{-1} \). The choice of contours and the assumption on the parameters \( q, \eta \) in (3.35) were made in such a way that these poles do not hinder us to move the contours together, i.e. to assume \( \gamma_1 = \gamma_2 =: \gamma \ll 1 \). With this in mind, \( R^+_1 \ast R^+_2 \ast \tilde{f} \) becomes:

\[
\frac{1}{k_1! k_2!} \int_{|Z_1| = |Z_2| = \gamma^{-1}} \frac{R^+_1(Z_1)R^+_2(Z_2) \zeta \left( \frac{Z_2}{X} \right) \zeta \left( \frac{Z_2}{Z_1} \right)^{-1}} {\tau_+(Z) \tau_+(Z) DZ} \cdot f(X - Z) \zeta \left( \frac{Z}{X} \right)
\]
Since the contours are symmetric, we can replace the integrand by its symmetrization in the $Z$ variables, which precisely amounts to \((3.32)\) for $R^+_1 \ast R^+_2$. We leave the analogous computation for negative shuffle elements $R^-_1$ to the interested reader.

A consequence of \((2.9)\) is that any shuffle element $R^\pm_1$ can be obtained as a sum of products of $z^d_{i\lambda}$, as $i \in \{1, \ldots, n\}$ and $d \in \mathbb{Z}$. Therefore, in order to prove that \((3.32)\) gives a well-defined action $A \acts K$, it is enough to show that:

\begin{align*}
(z^d_{i\lambda})^\pm \ast T &= q^{-d} \sum_{\pm'} \pm' \\
\int_{|z_{i\lambda}| = \gamma^{\pm'}_{i\lambda}} \frac{z^d_{i\lambda} \cdot f(X \mp z_{i\lambda})}{\zeta \left(\frac{z_{i\lambda} \pm 1}{X^{\pm 1}}\right)} \tau_{\pm}(z_{i\lambda})^{\pm 1} Dz_{i\lambda} \tag{3.38}
\end{align*}

give a well-defined action $U_{q,T}(\mathfrak{g}_{\lambda}) \acts K$. The best way to carry out this computation is to repackage formula \((3.38)\) in terms of the series $\delta(z) = \sum_{d \in \mathbb{Z}} z^d$:

\begin{align*}
\delta \left(\frac{z_{i\lambda}}{\pm} \right)^\pm \ast T &= q^{-\delta} f(X \mp z) \left[\zeta \left(\frac{z_{i\lambda} \pm 1}{X^{\pm 1}}\right) \tau_{\pm}(z)\right]^{\pm 1} \tag{3.39}
\end{align*}

and show that formulas \((3.33)\), \((3.39)\) satisfy the defining relations of $U_{q,T}(\mathfrak{g}_{\lambda}) \cong A$ (these relations are spelled out in \((2.72)\), \((2.73)\), \((2.74)\), \((2.81)\), \((2.82)\) of \([12]\); the proof that they are satisfied by \((3.33)\), \((3.39)\) is analogous to Theorem II.9 of \([11]\)).

**Remark 3.16.** The discussion above ensures that the operators \((3.35)\)–\((3.36)\) satisfy the relations in the algebra $A$, but there is a more basic thing that one needs to check, namely that the right-hand sides of \((3.35)\)–\((3.36)\) give rise to well-defined endomorphisms of $K$. In other words, we need to prove that if $T = 0 \in K$, then the right-hand side of \((3.32)\) is also 0 in $K$. By \((2.9)\), it is enough to check this when $R = z^d_{i\lambda}$. In this case, we show that \((3.39)\) is a well-defined operator on $K$ in the course of the proof of Theorem 1.2, by proving that the right-hand side of \((3.39)\) matches the action of the geometric operators studied in \([14]\).

**3.17.** Let us now describe the fixed points of the action $T \acts M_4$. Because \((3.7)\) realizes Laumon spaces as $\mathbb{Z}/n\mathbb{Z}$–fixed loci of $\mathcal{N}_{\lambda^d}$, and $\mathbb{Z}/n\mathbb{Z} \subset T$, such points are among the torus fixed points of the moduli space of torsion-free sheaves $\mathcal{N}_{\lambda^d}$. It is well-known that the latter are indexed by $n$–tuples of partitions:

\[\lambda = (\lambda^1, \ldots, \lambda^n) \quad \text{where} \quad \lambda^i = (\lambda^i_0 \geq \lambda^i_1 \geq \ldots) \tag{3.40}\]

are usual partitions whose sizes sum up to $|\lambda^d|$. Recall that partitions are in one-to-one correspondence with Young diagrams (see \([11]\) for an introduction to the terminology of partitions), so we will often say “the boxes of a partition $\lambda^i$" instead of “the boxes of the Young diagram corresponding to the partition $\lambda^i$". By extension, we will refer to the boxes of $\lambda$ as the disjoint union of the boxes of the constituent partitions $\lambda^i$. The quadruple $(X, Y, A, B) \in \mathcal{N}_{\lambda^d}$ that corresponds to $\lambda$ has:

\[V = \bigoplus_{\square \in \lambda} \mathbb{C} \cdot v_{\square} \]

and:

\[X \cdot v_{\square} = v_{\text{box directly to the right of } \square} \tag{3.41}\]
\[ Y \cdot v_{\Box} = v_{\text{box directly above } \Box} \quad (3.42) \]

\[ A \cdot w_i = v_{\text{corner of } \lambda^i} \quad (3.43) \]
as well as \( B = 0 \). The same description applies to \( \lambda \) as a fixed point of \( \mathcal{M}_d \), where:

\[ V_i = \bigoplus_{k=1}^n \bigoplus_{\Box = (x,y) \in \lambda^k} \mathbb{C} \cdot v_{\Box} \quad (3.44) \]
The maps \( X_i, Y_i, A_i \) are given by restricting (3.41)–(3.43) to the subspaces (3.44), according to Figure 3.7. To make the definition (3.44) simpler, for the box \( \Box = (x,y) \) in the partition \( \lambda^k \), we will refer to \( y + k \mod n \) as the color of \( \Box \). Then a basis of \( V_i \) is given by those boxes of color \( i \mod n \) in the \( n \)--tuple of partitions \( \lambda \).

3.18. Recall the following equivariant localization formula, true for any smooth variety \( X \curvearrowright T \) acted on by a torus with finitely many fixed points:

\[ c = \sum_{x \in X^T} c_x \cdot \frac{[x]}{[\wedge^*(T^*_x X)]} \quad (3.45) \]

where \( [x] \in K^{\text{int}}_T(X) \) denotes the class of the skyscraper sheaf at \( x \). For notational convenience, we renormalize \( [x] \) by the exterior class of the tangent bundle:

\[ [x] := \frac{[x]}{[\wedge^*(T^*_x X)]} \in K_T(X) := K_T(X)_{\text{loc}} \quad (3.46) \]
in terms of which the equivariant localization formula becomes:

\[ c = \sum_{x \in X^T} c_x \cdot [x] \quad \forall c \in K_T(X) \quad (3.47) \]

In order to use this formula for \( X = \mathcal{M}_d \), we will need to compute the restriction of important \( K \)--theory classes to the fixed points \( \lambda \) described in the previous Subsection. For the tautological vector bundle \( V_i \), we have:

\[ V_i|_{\lambda} = \sum_{k=1}^n \sum_{\Box = (x,y) \in \lambda^k} \chi_{\Box} \cdot \frac{q^{x+y+k-i} - i}{n} \quad (3.48) \]

where the weight of a box \( \Box = (x,y) \) situated in the partition \( \lambda^k \in \lambda \) is:

\[ \chi_{\Box} := u_k^2 q^{2x} \quad (3.49) \]

Along the same lines, the tautological class associated to any \( f \in \Lambda \) has restriction to the fixed point \( \lambda \) given by:

\[ f|_{\lambda} = f(\lambda) := f(\ldots, \chi_{\Box}, \ldots)_{\Box \in \lambda} \quad (3.50) \]

In the right-hand side, for any box \( \Box \in \lambda \) of color \( i \), we plug the weight \( \chi_{\Box} \) into an argument of the function \( f \) of color \( \tilde{i} \) (one must remember to take into account rule (2.1) to change colors within a certain residue class modulo \( n \)).

**Proof of Proposition 3.11:** By (3.47), the classes \( |\lambda| \) span the \( \mathbb{F}_u \)--vector space \( K_d \). The Proposition is a consequence of the following general statement:
Consider any vector space $V = \text{span}(v_1, \ldots, v_p)$. Take a vector $v = \sum \alpha_i v_i$ with all $\alpha_i \neq 0$, and a collection of endomorphisms $A_1, \ldots, A_n$ that are diagonal in the basis \{\$v_1, \ldots, v_p\$. The collection of vectors:

$$A_1^{\beta_1} \cdots A_n^{\beta_n} \cdot v,$$

where $\beta_1, \ldots, \beta_n \in \mathbb{N}$ spans the vector space $V$ if for any two different basis vectors $v_i$ and $v_j$, there exists an $l \in \{1, \ldots, n\}$ such that $A_l$ has distinct eigenvalues on $v_i$ and $v_j$.

The claim is an easy exercise which relies on the fact that the Vandermonde determinant is non-zero. We will apply this situation when $V = \mathbb{K}d$, $v = 1$ is the class of the structure sheaf, and $\{v_1, \ldots, v_p\}$ is the basis of torus fixed points. We will choose $A_k$ to be the operator of multiplication by the class $[V_k]$. Since for any two distinct fixed points $\lambda \neq \mu$, there will be a box $\square \in \lambda \setminus \mu$, this ensures that the hypothesis of the Claim holds.

\[\blacksquare\]

3.20. In the current Subsection, we will study the color-symmetric rational functions of (3.25) and (3.26), and particularly the $K$–theory classes they give rise to under the map (3.17). In Theorem 3.13, they are multiplied by the power series $\tau_{\pm}(z)^{\pm 1}$, and the following Proposition computes their restrictions to the fixed points. We will use multiplicative notation akin to (3.50), specifically:

$$\zeta \left(\frac{z}{\chi \lambda}\right) := \prod_{\square \in \lambda} \zeta \left(\frac{z}{\chi \square}\right)$$

(3.51)

Given an $n$–tuple of partitions $\lambda$, an inner corner will refer to any box $\square \notin \lambda$ such that $\lambda \sqcup \square$ is a valid $n$–tuple of partitions. Similarly, an outer corner of $\lambda$ refers to any box $\square \in \lambda$ such that $\lambda \setminus \square$ is a valid $n$–tuple of partitions.

**Proposition 3.21.** For any variable $z$ and for any $n$–tuple of partitions $\lambda$:

$$\zeta \left(\frac{z}{\chi \lambda}\right) \tau_+(z) = \prod_{\square \in \text{inner corner of } \lambda} \frac{\sqrt{\chi \square} q - \frac{z \sqrt{\chi \square}}{q}}{\sqrt{\chi \square} - \frac{z \sqrt{\chi \square}}{q}}$$

(3.52)

$$\zeta \left(\frac{\lambda \chi}{z}\right)^{-1} \tau_-(z)^{-1} = \prod_{\square \in \text{outer corner of } \lambda} \frac{\sqrt{\chi \square} q - \frac{z \sqrt{\chi \square}}{q}}{\sqrt{\chi \square} - \frac{z \sqrt{\chi \square}}{q}}$$

(3.53)

where the outer/inner corners of a partition $\lambda$ are those boxes which can be removed/added from/to $\lambda$ in such a way as to produce another valid partition.

**Proof** Let us first prove (3.52) and leave (3.53) as an exercise for the interested reader. Write $i = \text{col } z$, and let us recall that the rational function $\zeta(z/X)$ is explicitly given by formula (3.25). The corresponding $K$–theory class comes about by replacing the variables $x_{ia}$ by the Chern roots of the tautological vector bundle $V_i$. When one restricts this to the fixed point $\lambda$, one must replace these Chern roots
by the weights of the boxes of color $i$ in $\lambda$ (one must remember to apply rule (2.1) to change the colors of boxes within a single residue class modulo $n$). We conclude:

$$
\zeta \left( \frac{z}{\lambda} \right) \tau_+ (z) = f \left( \frac{u_{i+1}}{q} \right) \prod_{\square \in \lambda} \frac{f \left( \sqrt[\lambda] \right)}{f \left( \sqrt[i] \right)} \prod_{\square \in \lambda} \frac{f \left( \sqrt[\lambda] \right)}{f \left( \sqrt[i] \right)}
$$

where we abbreviate $f(x) = x - \frac{x}{z}$. The boxes in the $y$-th row of $\lambda$ have weights:

$$
u_k^2 q^{\frac{\chi(y)}{n}} \cdots, \nu_k^2 q^{(\lambda_k - 1) \frac{\chi(y)}{n}}$$

if we think of them as having color $i = \frac{y + k}{n}$. Therefore, we have:

$$
\zeta \left( \frac{z}{\lambda} \right) \tau_+ (z) = f \left( \frac{u_{i+1}}{q} \right) \prod_{y=i-k}^{1 \leq k \leq n} f \left( \nu_k q^{\chi(y)} \right) \prod_{y=i+1-k}^{1 \leq k \leq n} f \left( \nu_k q^{\chi(y)} \right)
$$

Changing the index $y \mapsto y + 1$ in the last product implies the equation above is:

$$
f \left( \frac{u_{i+1}}{q} \right) \prod_{y=i-k}^{1 \leq k \leq n} f \left( \nu_k q^{\chi(y)} \right) \prod_{y=i+1-k}^{1 \leq k \leq n} f \left( \nu_k q^{\chi(y)} \right) =
$$

$$
f \left( \frac{u_{i+1}}{q} \right) \prod_{y=i-k}^{1 \leq k \leq n} f (\nu_k q^{\chi(y)}) \prod_{y=i+1-k}^{1 \leq k \leq n} f (\nu_k q^{\chi(y)})
$$

The fraction on the second row equals 1 when $\lambda_k^y = \lambda_k^y+1$, so it can be $\neq 1$ only if $\lambda_k^y > \lambda_k^y+1$. This corresponds to the existence of an outer corner of color $i$ on the $y$-th row, and an inner corner of color $i + 1$ on the $(y + 1)$-th row, and thus contributes precisely the fraction claimed in (3.52).

\[ \square \]

**3.22.** For any pair of usual partitions $\lambda$ and $\mu$, we will write:

$$
\lambda \geq \mu \quad \text{if} \quad \lambda_i \geq \mu_i, \quad \forall i \geq 0
$$

In other words, $\lambda \geq \mu$ if the Young diagram of $\lambda$ contains that of $\mu$. In this case, the set of boxes $\lambda \backslash \mu$ is called a skew Young diagram. The analogous notions apply for fixed points (3.40), which are nothing but $n$-tuples of usual partitions:

$$
\lambda \geq \mu \quad \text{if} \quad \lambda^1 \geq \mu^1, \quad \ldots, \quad \lambda^n \geq \mu^n
$$

and so $\lambda \backslash \mu$ can be thought of as an $n$-tuple of skew Young diagrams. We will think of such skew Young diagrams as sets of boxes colored modulo $n$. We will write $|\lambda \backslash \mu| = (k_1, \ldots, k_n) \in \mathbb{N}^n$ if there are $k_i$ boxes of color $i$ modulo $n$ in $\lambda \backslash \mu$. By analogy with (3.50), we will write:

$$
R(\lambda \backslash \mu) := R(\ldots, \chi \square, \ldots) \square \in \lambda \backslash \mu
$$

for all color-symmetric rational functions $R$ of degree $|\lambda \backslash \mu|$. As before, one must plug the weight $\chi \square$ into an argument of $R$ of the same color as $\square$. We are now ready to write down the matrix coefficients of the operators $R^\pm$ of Theorem 3.13 in the basis of renormalized fixed points $|\lambda| \in K$. 


Proposition 3.23. For any shuffle element $R^\pm \in A_{\pm k} \cap K$, we have:

$$
\langle \lambda | R^+ | \mu \rangle = R^+ (\lambda \setminus \mu) \prod_{\square \in \lambda \setminus \mu} \left( (q^{-1} - q) \zeta \left( \frac{\chi_{\square}}{\chi_{\mu}} \right) \tau_+ (\chi_{\square}) \right) 
$$

$$
\langle \mu | R^- | \lambda \rangle = R^- (\lambda \setminus \mu) \prod_{\square \in \lambda \setminus \mu} \left( (1 - q^{-2}) \zeta \left( \frac{\chi_{\lambda}}{\chi_{\square}} \right)^{-1} \tau_- (\chi_{\square})^{-1} \right)
$$

(3.54) (3.55)

The right-hand sides only make sense if $\lambda \geq \mu$ and $|\lambda \setminus \mu| = k$. If either of these conditions fails to hold, then the corresponding matrix coefficients of $R^\pm$ are 0.

Proof Let us show that if (3.54) holds for $R_1^+$ and $R_2^+$, then it holds for $R_1^+ \ast R_2^+$. Indeed, by the definition of the shuffle product, we have:

$$
R_1^+ \ast R_2^+ (\lambda \setminus \mu) = \sum_{\lambda \setminus \mu = A \cup B} R_1^+ (A) R_2^+ (B) \prod_{\square \in A} (q^{-1} - q) \zeta \left( \frac{\chi_{\square}}{\chi_{\mu}} \right) \prod_{\square \in B} \left( (1 - q^{-2}) \zeta \left( \frac{\chi_{\lambda}}{\chi_{\square}} \right)^{-1} \tau_- (\chi_{\square})^{-1} \right)
$$

(3.56)

A priori, the sum is over all ways to partition the set of boxes $\lambda \setminus \mu$ into two disjoint parts $A$ and $B$. But recall the fact that $\zeta(\chi_{x/y})_{x|y} = 0$ when col $y = \text{col } x$ and $\zeta(\chi_{x/y})_{x|h(y)} = 0$ when col $y = \text{col } x + 1$. This implies that if there were a box $\square \in A$ to the left or below a box $\blacksquare \in B$, then the corresponding summand of (3.56) would be 0. Therefore, the only non-zero summands of (3.56) occur when $A = \lambda \setminus \nu$ and $B = \nu \setminus \mu$ for some intermediate partition $\mu \leq \nu \leq \lambda$. Hence $\langle \lambda | R_1^+ \ast R_2^+ | \mu \rangle$ equals:

$$
\sum_{\mu \leq \nu \leq \lambda} R_1^+ (\lambda \setminus \nu) R_2^+ (\nu \setminus \mu) \prod_{\square \in \lambda \setminus \nu} \left( \frac{\chi_{\square}}{\chi_{\mu}} \right) \prod_{\square \in \nu \setminus \mu} \left( (q^{-1} - q) \zeta \left( \frac{\chi_{\lambda}}{\chi_{\square}} \right) \tau_+ (\chi_{\square}) \right)
$$

The right-hand side is precisely obtained by iterating (3.54) for $R_1^+$ and $R_2^+$. Similarly, one proves that if (3.55) holds for $R_1^-$ and $R_2^-$, then it holds for $R_1^- \ast R_2^-$. With the previous paragraphs in mind, (2.10) reduces the proof of formulas (3.54)-(3.55) to the case of the shuffle elements $R^\pm = z_{i_1}^d$. Let us first prove the case $\pm = +$.

Consider a fixed point $\mu$, and pick an arbitrary element $f \in \Lambda$ such that:

$$
f(\nu) = \delta_{\nu}^\mu
$$

(3.57)

for all fixed points $\nu$. Such an $f$ exists because (3.57) imposes only finitely many linear relations on the coefficients of the symmetric polynomial $f$ in infinitely many variables. Relation (3.38) implies:

$$
\langle \lambda | z_{i_1}^d | \mu \rangle = \sum_\lambda |\lambda| \cdot \int z_{i_1}^d \cdot f(X - z_{i_1}) \zeta \left( \frac{z_{i_1}}{X} \right) \tau_+(z_{i_1}) Dz_{i_1}
$$

where the integral is taken over the difference between a very small circle and a very large circle. Using (3.52), the formula above becomes:

$$
\langle \lambda | z_{i_1}^d | \mu \rangle = \int z_{i_1}^d \cdot f(\lambda - z_{i_1}) \prod_{\square \text{inner corner of } \lambda} \left( \frac{\sqrt{\chi_{\square}}}{q} - \frac{z_{i_1}}{\sqrt{\chi_{\square}}} \right) Dz_{i_1}
$$

$$
\prod_{\square \text{outer corner of } \lambda} \left( \frac{\sqrt{\chi_{\square}}}{q} - \frac{z_{i_1}}{\sqrt{\chi_{\square}}} \right) Dz_{i_1}
$$

(3.53)
We may think of the contour as surrounding the poles of the fraction, which are all of the form \( z_i = \chi_{\[i]} \) for an outer corner \([i]\) \( \in \lambda \) of color \( i \). We obtain:

\[
\langle \lambda | z_i^d | \mu \rangle = \sum_{\lambda} \chi^d_i \cdot f(\lambda - [i]) \prod_{\text{inner corner of } \lambda} \frac{q - \chi_{[i]}}{\sqrt{q}} \prod_{\text{outer corner of } \lambda} \frac{q - \chi_{[i]}}{\sqrt{q}}
\]

Since \( \lambda - [i] = \nu \) must be a partition, condition (3.57) requires that this partition coincide with \( \mu \). We may then rewrite the above expression by changing the product over corners of \( \lambda \) to a product over corners of \( \mu \):

\[
\langle \lambda | z_i^d | \mu \rangle = \chi^d_i \cdot (q^{-1} - q) \prod_{\text{inner corner of } \mu} \frac{q - \chi_{[i]}}{\sqrt{q}} \prod_{\text{outer corner of } \mu} \frac{q - \chi_{[i]}}{\sqrt{q}} = \chi^d_i \cdot (q^{-1} - q) \zeta \left( \frac{\chi_{[i]}}{\chi_{\mu}} \right) \tau_+ (\chi_{[i]})
\]

where \([i]\) is the unique box of \( \lambda \setminus \mu \). This is precisely (3.54) for \( R^+ = z_i^d \).

Let us now prove (3.55) when \( R^- = z_i^d \). Consider a fixed point \( \lambda \), and pick an arbitrary element \( f \in \Lambda \) such that:

\[
f(\nu) = \delta^\lambda_\nu \quad \text{(3.58)}
\]

for all fixed points \( \nu \). Relation (3.38) implies:

\[
z_i^d | \lambda \rangle = \frac{1}{q} \sum_{|\mu|} \langle |\mu| | z_i^d | \mu \rangle \prod_{\text{inner corner of } \mu} \frac{q - \chi_{[i]}}{\sqrt{q}} \prod_{\text{outer corner of } \mu} \frac{q - \chi_{[i]}}{\sqrt{q}} \tau_-(z_i) D z_i
\]

where the integral is taken over the difference between a very small circle and a very large circle. Using (3.53), the formula above becomes:

\[
\langle |\mu| | z_i^d | \lambda \rangle = \frac{1}{q} \int \langle |\mu| | z_i^d | \mu + z_i \rangle \prod_{\text{inner corner of } \mu} \frac{q - \chi_{[i]}}{\sqrt{q}} \prod_{\text{outer corner of } \mu} \frac{q - \chi_{[i]}}{\sqrt{q}} \tau_-(z_i) D z_i
\]

We may think of the contour as surrounding the poles of the fraction, which are all of the form \( z_i = \chi_{\[i]} \) for an inner corner \([i]\) \( \in \lambda \) of color \( i \). We obtain:

\[
\langle |\mu| | z_i^d | \lambda \rangle = \frac{1}{q} \sum_{\text{inner corner of } \mu} \chi^d_i \cdot f(\mu + [i]) \prod_{\text{inner corner of } \mu} \frac{q - \chi_{[i]}}{\sqrt{q}} \prod_{\text{outer corner of } \mu} \frac{q - \chi_{[i]}}{\sqrt{q}}
\]

Since \( \mu + [i] = \nu \) must be a partition, condition (3.58) requires that this partition coincide with \( \lambda \). We may then rewrite the above expression by changing the product over corners of \( \mu \) to a product over corners of \( \lambda \):

\[
\langle |\mu| | z_i^d | \lambda \rangle = \chi^d_i \cdot (q - q^{-1}) \prod_{\text{inner corner of } \lambda} \frac{q - \chi_{[i]}}{\sqrt{q}} \prod_{\text{outer corner of } \lambda} \frac{q - \chi_{[i]}}{\sqrt{q}} = \chi^d_i \cdot (1 - q^{-2}) \zeta \left( \frac{\chi_{[i]}}{\chi_{\lambda}} \right) \tau_-(\chi_{[i]})^{-1}
\]

\( \square \)
4. Geometric Correspondences

4.1. In the present Section, we will prove Theorem 1.2. Specifically, we construct correspondences which act on $K$ in the same way as the shuffle elements $S_m^+, T_m^+$, $G_{1 \times k}$ of (2.12), (2.13), (2.17), (2.18) act via Theorem 3.13. Since the root generators (2.14)-(2.15) of $U_q(gl_n)$ are particular cases of the shuffle elements of $S_m^+$ and $T_m^+$, this will also give a geometric interpretation of:

$$U_q(gl_n) \subset U_q(gl_n) \cong A$$

acting on $K$, thus yielding Theorem 1.1. Historically, the conjecture of Kuznetsov that motivated Theorem 1.1 (see [5] for some background) suggests that the action of $gl_n$ on the cohomology of affine Laumon spaces is given by the correspondences:

$$C_{[i;j]} = \left\{ (F^+_o \subset \bigcup^{[i;j]} F^-_o) \right\} \quad (4.1)$$

where $o = (0,0)$ denotes the origin of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \infty$, and the notation $F^+_o \subset \bigcup^{[i;j]} F^-_o$ means that the individual component sheaves of $F^+_o$ and $F^-_o$ are contained inside each other, with the quotient giving a type $[i;j]$ indecomposable representation of the cyclic quiver supported at the origin $o$. The main problem with the correspondences $C_{[i;j]}$ is that they are not local complete intersections, and thus their structure sheaves do not give the “right” operators in $K$–theory. Even worse, it is not clear how to define an appropriate virtual structure sheaf on $C_{[i;j]}$.

4.2. Instead, in Definition 4.3 we will introduce an analogue of the fine correspondence studied in [13]. Afterwards, in Definition 4.7 we will introduce an analogue of the eccentric correspondence studied in [11]. These two types of constructions will be the correct replacements for (4.1) in $K$–theory:

**Definition 4.3.** The **fine correspondence** is the locus of collections of $j - i + 1$ parabolic sheaves, sitting inside each other:

$$\mathfrak{F}_{[i;j]} = \left\{ (F^+_o \subset^{i-1} F^+_o \subset^{j-2} \ldots \subset^{i+1} F^+_o \subset_j F^-_o) \right\} \quad (4.2)$$

where the notation $F^+_o \subset^{k} F^-_o$ is shorthand for $F^+_o \subset F^-_o$ for all integers $l$, and the quotient is non-trivial only if $l \equiv k$ mod $n$, in which case it is a length 1 skyscraper sheaf supported at the origin $o$.

Because the successive quotients in a flag (4.2) are length one, the parabolic sheaves $F^+_o, \ldots, F^-_o$ that make up such a flag must have degrees $d^i, \ldots, d^i$, respectively, where:

$$d^{k+1} = d^k + \zeta^k \quad \forall k \in \{ i, \ldots, j - 1 \}$$

and $\zeta^k \in \mathbb{N}^n$ is the vector with entry 1 on position $k$, and 0 everywhere else. We will often write $d^i = d^+\cdot i$ and $d^- = d^-\cdot i$, so we must have $d^+ = d^- + [i;j]$. Although $\mathfrak{F}_{[i;j]}$ is also quite far from being smooth, one expects that the forgetful map:

$$\mathfrak{F}_{[i;j]} \to C_{[i;j]}$$

$$\left( F^+_o \subset^{i-1} \ldots \subset^{j} F^-_o \right) \to \left( F^+_o \subset^{[i;j]} F^-_o \right)$$

behaves like a resolution of singularities, and thus the correspondences $\mathfrak{F}_{[i;j]}$ and $C_{[i;j]}$ give rise to the same operator in cohomology. However, this argument does not work in $K$–theory, and to fix this issue, one needs to define an appropriate
Recall from Subsection 3.7 that Laumon spaces $\mathcal{M}_d$ parametrize quadruples:
\[(X, Y, A, B) = (... , X_k, Y_k, A_k, B_k, ... )_{1 \leq k \leq n} \tag{4.3}\]
of linear maps between $n$–tuples of vector spaces $V = (V_1 , ..., V_n)$ and $\bigoplus_{i=1}^n \mathbb{C} w_i$. It is easy to see that $\mathfrak{Z}_k := \mathfrak{Z}_{i ; i + 1}$ parameterizes quadruples of maps (4.3) which preserve a fixed quotient $V^+ \to V^-$, by which we mean a fixed collection of quotients of vector spaces $V^+_k \to V^+_k$, of codimension $\delta_k$. Then consider the diagram:

![Diagram](image)

Figure 4.4

We claim that a collection of maps as in the picture above amounts to a point of $\mathfrak{Z}_k$. In more detail, we use the codimension 1 surjection (the vertical dotted arrow, which we henceforth call $\iota$) to produce the maps $Y_i, A_i$ to $V^+_i$ and the maps $Y_{i+1}, B_{i+1}$ from $V^{-}_{i}$. The endomorphisms $X^+_i$ and $X^-_i$ are required to satisfy $X^-_i \circ \iota = \iota \circ X^+_i$, and induce the zero map on the one-dimensional kernel of $\iota$.

4.5. Iterating the above argument, points of $\mathfrak{Z}_{i ; j}$ are quadruples (4.3) which preserve a fixed flag of subspaces:

\[V^+_i = V^j \to V^j \to ... \to V_i, V^i \to V^i \tag{4.4}\]

If we fix a surjection of $n$–tuples of vector spaces $V^+ \to V^-$ of codimension $[i ; j)$, then the datum (4.4) is equivalent to a full flag of subspaces:

\[0 = U^{[j : j)]} \subset U^{[j-1 : j)} \subset U^{[j-2 : j)} \subset ... \subset U^{[i ; j)]} = \text{Ker}(V^+ \to V^-) \tag{4.5}\]

Here, $U^{[a : j]}$ is itself an $n$–tuple of vector spaces of dimension $[a ; j)$, and the cokernel $U^{[a : j)}/U^{[a+1 : j)}$ is an $n$–tuple of vector spaces of dimension $\zeta^a$. This means that the cokernel is non-zero only on the $a$–th place, where it is one-dimensional:

\[L_a := U^{[a : j)]}/U^{[a+1 : j)} \tag{4.6}\]

Then the $n$–tuple of vector spaces $V^a$ which appears in (4.4) is precisely $V^+/U^{[a : j)}$. Recall the vector space $M_{d+}$ of quadruples of linear maps $(X, Y, A, B)$ between the
of quadruples that preserve the quotient $V^+ \twoheadrightarrow V^-$, and such that:

- the $X$ maps preserve the flag (4.5) and act nilpotently on it:
  \[ X \left( U^{[i,a]} \right) \subset U^{[i,a+1]} \quad \forall a \in \{i, \ldots, j-2\} \]
- the $Y$ maps preserve the flag (4.5). Since $V_{a-1} \xrightarrow{Y_a} V_a$ for all $a$, this forces:
  \[ Y \left( U^{[i,a]} \right) \subset U^{[i,a+1]} \quad \forall a \in \{i, \ldots, j-2\} \]

As a consequence of the above bullets, we note that the commutator $[X, Y]$ maps the vector space $U^{[a]}$ to $U^{[a+2]}$ for all $a$. We then define the linear map $\eta$:

\[
\begin{array}{c}
 Z_{[i,j]} \\
 M_{d^+}
\end{array} \xrightarrow{\eta} \begin{array}{c}
 a_{[i,j]} \\
 \oplus_{k=1}^n \text{Hom}(V^+_k, V^+_k)
\end{array}
\]

in order to make the diagram commute. Above, we define $a_{[i,j]}$ to consist of those $n$-tuples of homomorphisms $V_{k-1}^+ \to V_{k}^+$ which not only preserve the flag (4.5), but map $U^{[i,a]}$ to $U^{[a+2]}$ for all $a$. Finally, consider the subgroup $P_{[i,j]} \subset GL_{d^+}$ of automorphisms of $V^+$ which preserves (4.5). The above discussion establishes:

\[ Z_{[i,j]} = \eta^{-1}(0)^s/P_{[i,j]} \] (4.7)

where the superscript “$s$” denotes, as always, the open subset of stable quadruples.

### 4.6. We will now give a related, but different construction, which admits a description as in Subsection 4.5 but not as in Subsection 4.2. Consider a quotient of $n$-tuples of vector spaces $V^+ \twoheadrightarrow V^-$, and fix a full flag of subspaces:

\[ 0 = U^{[i;i]} \subset U^{[i;i+1]} \subset U^{[i;i+2]} \subset \ldots \subset U^{[i;j]} = \text{Ker}(V^+ \twoheadrightarrow V^-) \] (4.8)

Note the difference between the flag above and (4.5). In (4.8), $U^{[i;a]}$ is itself an $n$-tuple of vector spaces of dimension vector $[i; a]$, and the cokernel $U^{[i;a+1]}/U^{[i;a]}$ is an $n$-tuple of vector spaces of dimension vector $\varsigma^a$. This means that the cokernel is non-zero only on the $a$-th place, where it is one-dimensional:

\[ L_a := U^{[i;a+1]}/U^{[i;a]} \] (4.9)

Recall the vector space $M_{d^+}$ of quadruples of linear maps $(X, Y, A, B)$ between the various components of the $n$-tuple of vector spaces $V^+$. Consider the affine space:

\[ \mathcal{Z}_{[i,j]} \subset M_{d^+} \]

of quadruples that preserve the quotient $V^+ \twoheadrightarrow V^-$, such that:

- the $X$ maps preserve the flag (4.8) and act nilpotently on it:
  \[ X \left( U^{[i;a]} \right) \subset U^{[i;a-1]} \quad \forall a \in \{i+1, \ldots, j-1\} \]
• the $Y$ maps “almost” preserve the flag (4.8), in the sense that:
$$Y \left( U^{[i:a]} \right) \subset U^{[i:a+1]} \quad \forall a \in \{i + 1, ..., j - 1\}$$  (4.10)

As a consequence of the above bullets, note that the commutator $[X, Y]$ takes $U^{[i:a]}$ to $U^{[i:a]}$ for all $a$, and so preserves the flag (4.8). We define the linear map $\eta$:

$\eta^{[i;j]} : Z^{[i;j]} \rightarrow \mathfrak{a}^{[i;j]}$

$\mathfrak{a} \rightarrow \bigoplus_{k=1}^{n} \text{Hom}(V_{k-1}^+, V_k^+)$

in order to make the diagram commute. Above, we define $\mathfrak{a}^{[i;j]}$ to consist of those $n$–tuples of homomorphisms $V_{k-1}^+ \rightarrow V_k^+$ which preserve the flag (4.8). Consider the subgroup $\mathfrak{P}^{[i;j]} \subset GL_{d+}$ of automorphisms of the $n$–tuple of vector spaces $V^+$, which preserve the flag of subspaces (4.8). Then we define:

$$\mathfrak{Z}^{[i;j]} = \eta^{-1}(0) / \mathfrak{P}^{[i;j]}$$  (4.11)

where the superscript “s” denotes the open subset of stable quadruples.

**Definition 4.7.** Call the variety $\mathfrak{Z}^{[i;j]}$ of (4.11) an eccentric correspondence.

4.8. In Subsections 4.5 and 4.6, we started from affine spaces $Z^{[i;j]}$ and $\mathfrak{Z}^{[i;j]}$, imposed on them the equations $\eta = 0$ and $\eta = 0$, respectively, and then took the quotient under a parabolic group action. The latter operation does not affect the smoothness of the spaces in question, because the action is free on the open locus of stable points, and stability in our case is precisely the notion derived from GIT. But a priori, imposing a number of equations on an affine space may yield a “bad” space.

**Definition 4.9.** Consider a section $s$ of a vector bundle $E$ on a smooth variety $X$. The structure sheaf of the scheme-theoretic zero locus $\iota : Z = \{s = 0\} \hookrightarrow X$ is isomorphic to the right-most cohomology group $\mathcal{O}_X / \text{Im } s^\vee$ of the Koszul complex:

$$\wedge^\bullet(E, s) = \left[ ... \xrightarrow{s^\vee} \wedge^2 E^\vee \xrightarrow{s^\vee} E^\vee \xrightarrow{s^\vee} \mathcal{O}_X \right]$$

(if the section $s$ is regular, then all other cohomology groups vanish). In general, we define the virtual structure sheaf as the $K$–theory class:

$$[Z] = \sum_{k=0}^{\text{rank } E} [\wedge^k E^\vee] \in \text{Im} \left( K(Z) \xrightarrow{\iota_*} K(X) \right)$$

In the following, we will abuse notation and refer to $[Z]$ as an element in $K(Z)$.

Applying Definition 4.9 gives us virtual structure sheaves $\mathcal{Z}^{[i;j]}$ and $\mathfrak{Z}^{[i;j]}$ on the fine and eccentric correspondences of (4.7) and (4.11), respectively.
**Definition 4.10.** The tautological line bundles $L_i$, ..., $L_{j-1}$ on $3_{[i;j]}$, $\overline{3}_{[i;j]}$ have fibers given by the one-dimensional quotients (4.6), (4.9), respectively.

We will adjust the virtual structure sheaves $[\overline{3}_{[i;j]}]$ and $[\overline{3}_{[i;j]}]$ by multiplying them with certain combinations of the line bundles of 4.10 and equivariant constants. This is simply a cosmetic change, meant to match our shuffle algebra conventions:

\[
[3^+_{[i;j]}] = [3_{[i;j]}] \cdot (-1)^{j-i-1}u_{i+1} \cdots u_j \frac{L_i \cdot q^{\frac{i-j}{2}}}{L_{j-1} \cdot q^{d_i - d_j}} \tag{4.12}
\]

\[
[3^-_{[i;j]}] = [3_{[i;j]}] \cdot (-1)^{j-i-1} \frac{u_i \cdots u_{j-1} \cdot q^{\frac{i-j}{2}}}{L_i \cdots L_{j-1} \cdot q^{d_i - d_{j-1}}} \tag{4.13}
\]

\[
[\overline{3}^+_{[i;j]}] = [3_{[i;j]}] \cdot u_{i+1} \cdots u_j \frac{q^{-j+i+1} \cdot d_j!}{q^{d_i - d_j}} \tag{4.14}
\]

\[
[\overline{3}^-_{[i;j]}] = [3_{[i;j]}] \cdot \frac{u_i \cdots u_{j-1} \cdot q^{-j+i+1} \cdot d_j!}{L_i \cdots L_{j-1} \cdot q^{d_i - d_{j-1}}} \tag{4.15}
\]

where $d_k^\pm$ are the dimensions of the vector spaces $V_k^\pm$ in the definition of $3_{[i;j]}$, $\overline{3}_{[i;j]}$.

**4.11.** Consider the following projection maps $p^\pm$ or $\overline{p}^\pm$:

\[ p^\pm \text{ or } \overline{p}^\pm \quad \begin{array}{c} \overset{3_{[i;j]}}{\longrightarrow} \text{ or } \overline{3}_{[i;j]} \\ \text{ or } \overline{3}_{[i;j]} \overset{\text{or } \overline{p}^\pm}{\longrightarrow} \text{ or } \overline{3}_{[i;j]} \end{array} \quad \begin{array}{c} \overset{\text{or } p^\pm}{\longrightarrow} \text{ or } \overline{3}_{[i;j]} \\ \text{ or } \overline{3}_{[i;j]} \overset{\text{or } p^\pm}{\longrightarrow} \text{ or } \overline{3}_{[i;j]} \end{array} \]

which remember only the quadruple of linear maps on the $n$–tuple of vector spaces $V^\pm$. For any Laurent polynomial $m(z_i, ..., z_{j-1})$, we define the operators:

\[
s_m^\pm : K \rightarrow K, \quad \alpha \mapsto p_m^\pm \left( m(L_i, ..., L_{j-1}) \cdot [3^\pm_{[i;j]}] \cdot p^\pm(\alpha) \right) \tag{4.17}
\]

\[
t_m^\pm : K \rightarrow K, \quad \alpha \mapsto \overline{p}_m^\pm \left( m(L_i, ..., L_{j-1}) \cdot \overline{3}^\pm_{[i;j]} \cdot \overline{p}^\pm(\alpha) \right) \tag{4.18}
\]

The way the virtual structure sheaves in (4.17)–(4.18) give rise to endomorphisms of $K$ is well-known. In more detail, recall from (4.7) that $3_{[i;j]}$ is cut out from the affine space $Z_{[i;j]}$ (modulo the group $P_{[i;j]}$) by the section $\eta$ of the vector bundle with fibers $a_{[i;j]}$. Therefore, Definition 4.9 gives rise to a virtual fundamental class inside the $P_{[i;j]}$ equivariant $K$–theory of $Z_{[i;j]}$, which then maps to the equivariant $K$–theory group of $\mathcal{M}_{d^+} \times \mathcal{M}_{d^-}$, via the projection maps (4.16). This $K$–theory class produces the operator (4.17).

**Remark 4.12.** The correspondence $3_i := 3_{[i;i+1]} = \overline{3}_{[i;i+1]}$ is smooth of middle dimension in $\mathcal{M}_{d^+} \times \mathcal{M}_{d^-}$. We call $3_i$ a simple correspondence, and note that it was used in [5] and [14] to construct operators in the cohomology and $K$–theory of affine Laumon spaces, respectively. In more detail, *loc. cit.* used the operators (4.17) and (4.18) for $j = i + 1$ to construct an action $U_{q,p}(u_n) \rtimes K$. Because $U_{q,p}(u_n)$ is isomorphic to $\mathcal{A}$ via (2.10), this action coincides with that of Theorem
3.13, as one can see from the $j = i + 1$ case of Proposition 4.22.

**4.13.** The equivariant localization formula (3.45) may be adapted to the case when the smooth variety $X$ is replaced by the zero locus $Z \hookrightarrow X$ of a regular section of a vector bundle $E$ (as in Definition 4.9), and it reads:

$$
c = \sum_{z \in \mathbb{Z}^X} c_z \cdot [z] \cdot \frac{[\wedge^* (E^\vee)]}{[\wedge^* (T^\vee_Z X)]} = \frac{c_z \cdot [z]}{[\wedge^* (T^\vee_Z X - E^\vee)]} \quad (4.19)
$$

The difference $TZ := TX - E$ is considered to be the virtual tangent space to the zero locus $Z \hookrightarrow X$. Therefore, formula (3.47) still holds for $X$ replaced with $Z$, if the $K$–theory classes of the skyscraper sheaves $[z]$ are replaced by:

$$
|z| = \frac{[z]}{[\wedge^* (T^\vee_Z Z)]} = \frac{[z]}{[\wedge^* (T^\vee_Z X - E^\vee)]} \quad (4.20)
$$

Even when the section is non-regular, (4.19) and (4.20) continue to hold, at the price of replacing the scheme-theoretic zero locus $Z$ by the virtual structure sheaf of Definition 4.9. In the particular cases of the fine and eccentric correspondences, it follows from (4.7) and (4.11) that their virtual tangent spaces are given by:

$$
[T \mathcal{Z}_{[i,j]}] = [TZ_{[i,j]}] - [\mathcal{O}_{[i,j]}] - [\mathcal{P}_{[i,j]}] \quad (4.21)
$$

$$
[T \mathcal{F}_{[i,j]}] = [T \mathcal{F}_{[i,j]}] - [\mathcal{P}_{[i,j]}] - [\mathcal{P}_{[i,j]}] \quad (4.22)
$$

Since the Lie algebra of an algebraic group is the space of invariant vector fields, the reason why we subtract the Lie algebras $\mathcal{P}_{[i,j]}$ and $\mathcal{P}_{[i,j]}$ from (4.21)–(4.22) is to account for killing those tangent vectors that come from the $P_{[i,j]}$ and $P_{[i,j]}$ actions in the quotients (4.7) and (4.11), respectively. All the summands in (4.21) and (4.22) may be expressed in terms of the tautological vector bundles:

$$
\{V^+ \}_1 \leq k \leq n \quad \text{and} \quad \{V^- \}_1 \leq k \leq n
$$

pulled back from Laumon spaces via the maps $p^+$ or $p^-$ of (4.16), and in terms of the tautological line bundles $L_i, \ldots, L_{j-1}$ from Definition 4.10. The precise formula is given in the following Proposition, which will be proved in the Appendix.

**Proposition 4.14.** The virtual tangent space to the fine correspondence $\mathcal{Z}_{[i,j]}$ is:

$$
[T \mathcal{Z}_{[i,j]}] - [T \mathcal{M}_{d+}] = i - j + \left(1 - \frac{1}{q^2}\right) \left(\sum_{i \leq a < j} \frac{V^+_{a}}{L_a} - \sum_{i \leq a < j} \frac{V^+_{a+1}}{L_a}\right)
$$

$$
- \sum_{i \leq a < b < j} \frac{L_b}{L_a} + \sum_{i \leq a < b < j} \frac{L_b}{L_a} - \sum_{i \leq a < b < j} \frac{L_b}{L_a} - \sum_{i \leq a < j} \frac{V^+_{a+1}}{L_a} q^2 \quad (4.23)
$$

or equivalently:

$$
[T \mathcal{Z}_{[i,j]}] - [T \mathcal{M}_{d-}] = i - j + \left(1 - \frac{1}{q^2}\right) \left(- \sum_{i \leq a < j} \frac{L_a}{V^+_{a}} + \sum_{i \leq a < j} \frac{L_a}{V^+_{a-1}}\right)
$$
\[
- \sum_{i \leq a < b < j} \frac{\mathcal{L}_b}{\mathcal{L}_a} + \sum_{i \leq a < b < j} \frac{\mathcal{L}_b}{\mathcal{L}_a} \left( \frac{1}{q^2} \right) \left( \sum_{i \leq a < j} \frac{\mathcal{V}_a^+}{\mathcal{L}_a} - \sum_{i \leq a < j} \frac{\mathcal{V}_{a+1}^+}{\mathcal{L}_a} \right) + \sum_{a = i+1}^{j-1} \frac{\mathcal{L}_a}{\mathcal{L}_{a-1} q^2} + \sum_{a = i}^{j-1} \frac{\mathcal{L}_a}{u_a^+} \quad (4.24)
\]

Meanwhile, the virtual tangent space to the eccentric correspondences \( \mathfrak{F}_{i;j} \) is:

\[
[T \mathfrak{F}_{i;j}] - [T \mathcal{M}_{d^+}] = i - j + \left( 1 - \frac{1}{q^2} \right) \left( \sum_{i \leq a < j} \frac{\mathcal{V}_a^+}{\mathcal{L}_a} - \sum_{i \leq a < j} \frac{\mathcal{V}_{a+1}^+}{\mathcal{L}_a} \right)
\]

\[
- \sum_{i \leq a < b < j} \frac{\mathcal{L}_b}{\mathcal{L}_a} + \sum_{i \leq a < b < j} \frac{\mathcal{L}_b}{\mathcal{L}_a} \left( 1 - \frac{1}{q^2} \right) \left( \sum_{i \leq a < j} \frac{\mathcal{L}_a}{\mathcal{V}_a} + \sum_{i \leq a < j} \frac{\mathcal{L}_a}{\mathcal{V}_{a-1}^-} \right)
\]

\[
- \sum_{i \leq a < b < j} \frac{\mathcal{L}_b}{\mathcal{L}_a} + \sum_{i \leq a < b < j} \frac{\mathcal{L}_b}{\mathcal{L}_a} \left( 1 - \frac{1}{q^2} \right) \left( \sum_{a = i+1}^{j-1} \frac{\mathcal{L}_a^+}{\mathcal{L}_{a-1} q^2} + \sum_{a = i}^{j-1} \frac{\mathcal{L}_a^+}{u_a^+} \right) \quad (4.25)
\]

or equivalently:

\[
[T \mathfrak{F}_{i;j}] - [T \mathcal{M}_{d^-}] = i - j + \left( 1 - \frac{1}{q^2} \right) \left( \sum_{i \leq a < j} \frac{\mathcal{V}_a^+}{\mathcal{L}_a} - \sum_{i \leq a < j} \frac{\mathcal{V}_{a+1}^+}{\mathcal{L}_a} \right)
\]

\[
- \sum_{i \leq a < b < j} \frac{\mathcal{L}_b}{\mathcal{L}_a} + \sum_{i \leq a < b < j} \frac{\mathcal{L}_b}{\mathcal{L}_a} \left( 1 - \frac{1}{q^2} \right) \left( \sum_{a = i+1}^{j-1} \frac{\mathcal{L}_a^-}{\mathcal{L}_{a-1} q^2} + \sum_{a = i}^{j-1} \frac{\mathcal{L}_a^-}{u_a^+} \right) \quad (4.26)
\]

The tangent bundles to \( \mathcal{M}_{d^+}, \mathcal{M}_{d^-} \) may be computed with (3.23), but for the sake of localization computations, it makes sense to present our formulas as above.

4.15. We will introduce another geometric operator, which will be shown to correspond to the shuffle elements (2.17)–(2.18). Consider the locus:

\[ \mathfrak{M}_{d^+, d^-} \subset \mathcal{M}_{d^+} \times \mathcal{M}_{d^-} \quad (4.27) \]

consisting of pairs of parabolic sheaves \( \mathcal{F}_{i}^+ \subset \mathcal{F}_{i}^- \) such that the quotient \( \mathcal{F}_{i}^- / \mathcal{F}_{i}^+ \) is scheme-theoretically supported on the divisor \( D^i = \{ 0 \} \times \mathbb{P}^1 \), i.e.:

\[ \mathcal{F}_{i}^+ \subset \mathcal{F}_{i}^- \subset \mathcal{F}_{i}^+(D^i), \quad \forall i \in \mathbb{Z} \]

Note that \( \mathfrak{M}_{d^+, d^-} \) is the \( \mathbb{Z}/n\mathbb{Z} \)-fixed locus of the space \( \mathfrak{V}_{d^+, d^-} \) defined in Section 6.2 of [13]. The space \( \mathfrak{V}_k \) was itself realized as a \( \mathbb{Z}/2\mathbb{Z} \)-fixed locus of a certain moduli space of sheaves, and this argument was used in Section 6.10 of loc. cit. to show that \( \mathfrak{M}_k \) is smooth. Refining this argument allows us to show that \( \mathfrak{M}_{d^+, d^-} \) is smooth, and we leave the details to the interested reader.

Since we do not wish to prove that \( \mathfrak{M}_{d^+, d^-} \) is smooth, we will work with its virtual structure sheaf instead, as in Subsections 4.8 and 4.13. To do so, we introduce the quiver presentation of \( \mathfrak{M}_{d^+, d^-} \). Take a quotient of \( n \)-tuples of vector spaces:

\[ \mathbf{V}^+ \to \mathbf{V}^- \quad (4.28) \]

of dimension vectors \( d^+ \) and \( d^- \), and consider the vector space of quadruples:

\[ W_{d^+, d^-} = \left\{ (X, Y, A, B) \right\} \subset \mathcal{M}_{d^+} \]
which preserve the quotient \((4.28)\), and such that \(X_i\) vanishes on the kernel of \((4.28)\). Consider the restriction of the moment map \((3.13)\) to the vector space \(W_{d^+,d^-}\):

\[
\begin{array}{ccc}
W_{d^+,d^-} & \overset{\nu}{\rightarrow} & a := \oplus_{i=1}^{n} \text{Hom}(V_i^{-1}, V_i^+) \\
M_{d^+} & \overset{\mu}{\rightarrow} & \oplus_{i=1}^{n} \text{Hom}(V_i^{+1}, V_i^+)
\end{array}
\]

The reason why the moment map \(\mu|_W\) factors through the space on the top right is the fact that the maps \(X_i\) vanish on \(L_i := \text{Ker}(V_i^+ \rightarrow V_i^-)\). Therefore:

\[
W_{d^+,d^-} = \nu^{-1}(0)/P
\]

where \(P \subset GL_{d^+}\) is the subgroup of automorphisms which preserve the quotient \((4.28)\). As in Definition 4.9, the presentation \((4.29)\) gives us a virtual structure sheaf \([W_{d^+,d^-}]\) as an element in the \(K\)-theory group of \(W_{d^+,d^-}\). Moreover, the discussion in Subsection 4.13 implies that the virtual tangent bundle is given by:

\[
[TW_{d^+,d^-}] = [TW_{d^+,d^-}] - [a] - [p]
\]

(4.30)

where \(d^\pm\) are the degrees of the moduli spaces in \((4.27)\), and we write:

\[
u_k = \prod_{i=1}^{n} u_{k_i}^{i} \quad \text{ and } \quad u_{k+1} = \prod_{i=1}^{n} u_{k_i+1}^{i}
\]

(4.33)

\[
\langle k, k' \rangle = \sum_{i=1}^{n} k_i k'_i - k_i k'_{i+1}
\]

(4.35)

for all \(k, k' \in \mathbb{N}^n\), and \(\mathcal{L}_i = \text{Ker}(V_i^+ \rightarrow V_i^-)\) denotes the rank \(d_i^+ - d_i^-\) tautological quotient bundle on \(\mathfrak{M}\), which parametrizes the vector spaces \(L_i\). Let us abbreviate:

\[
\mathfrak{M}_k = \bigsqcup_{d^+ - d^- = k} \mathfrak{M}_{d^+,d^-} \quad \text{ and } \quad \mathfrak{M} = \bigsqcup_{k \in \mathbb{N}^n} \mathfrak{M}_k
\]

Take the projections \(\mathfrak{M}_k \rightarrow \mathfrak{M}_{d^+,d^-}\) induced by \((4.27)\), and construct the operators:

\[
g_{\pm k} : K \rightarrow K
\]

(4.36)

\[
\alpha \mapsto \pi^\pm_+ \left( \mathfrak{M}_k \cdot \pi^\pm_+ (\alpha) \right)
\]

To work with the operators \((4.36)\), we need to compute the tangent bundle to \(\mathfrak{M}\).
Proposition 4.16. The virtual tangent bundle to $\mathcal{M}$ is given by:

$$[T\mathcal{M}] - [T\mathcal{M}_d^+] = \sum_{i=1}^{n} \left[ \left( 1 - \frac{1}{q^2} \right) \left( \frac{V_i^+}{L_i} - \frac{V_i^+}{L_i-1} \right) + \frac{L_i}{L_i-1} - \frac{L_i}{L_i} - \frac{u_i^2}{u_i^2} \right]$$  \hspace{1cm} (4.37)

or equivalently:

$$[T\mathcal{M}] - [T\mathcal{M}_d^-] = \sum_{i=1}^{n} \left[ \left( 1 - \frac{1}{q^2} \right) \left( \frac{L_i+1}{V_i} - \frac{L_i}{V_i} \right) + \frac{L_i}{L_i-1} - \frac{L_i}{L_i} + \frac{L_i}{u_i^2} \right]$$  \hspace{1cm} (4.38)

4.17. In the remainder of the present Section, we will compute the operators (4.17), (4.18) and (4.36) in the basis of fixed points, as well as in the integral notation of Theorem 3.13. Recall from Subsection 3.17 that fixed points of $\mathcal{M}_d$ are indexed by $n-\text{tuples of partitions} \lambda$ as in (3.40). As explained in [5], fixed points of the simple correspondence $Z_{[i;i+1]} = Z_{[i;i+1]}$ are pairs of such tuples:

$$\lambda \geq_1 \mu$$

by which we mean that $|\lambda \setminus \mu| = \varsigma_i$

In other words, $\lambda$ is obtained from $\mu$ by adding a box of color $i$. Iterating this logic shows us that fixed points of the fine correspondence $\mathfrak{F}_{[i;j]} = \mathfrak{F}_{[i;j]}$ are given by collections:

$$\lambda = \nu^i \geq_{j-1} \nu^{i-1} \geq_{j-2} \ldots \geq_{i+1} \nu^{i+1} \geq_i \nu^i = \mu$$  \hspace{1cm} (4.39)

where $\nu^{i-1}, \ldots, \nu^{i+1}$ are all $n-\text{tuples of Young diagrams}$.

Definition 4.18. A standard Young tableau (abbreviated SYT) of shape $\lambda \setminus \mu$ is a way to label the boxes of $\lambda \setminus \mu$ with the integers $i, \ldots, j-1$:

$$\lambda \setminus \mu = \{ \Box_i, \ldots, \Box_{j-1} \}$$  \hspace{1cm} (4.40)

such that the labels match the colors of the boxes modulo $n$, and the labels increase along the rows and columns of the skew diagram $\lambda \setminus \mu$.

It is easy to see that a SYT is the same datum as the flag (4.39), so they also parametrize fixed points of the fine correspondences $\mathfrak{F}_{[i;j]}$. Then the restriction of the tautological line bundles of Definition 4.10 to such a fixed point are given by:

$$L_k|_{\text{SYT of shape } \lambda \setminus \mu} = \chi_k := \chi_{\Box_k}$$  \hspace{1cm} (4.41)

where the box $\Box_k$ is assumed to have color $k$ and weight given by formula (3.49).

We will refer to $\lambda$ (resp. $\mu$) as the upper (resp. lower) shape of the given SYT.

4.19. As for the fixed points of eccentric correspondences $\mathfrak{F}_{[i;j]}$ of (4.11), there is a description similar to the previous Subsection. One must replace the flag (4.5) by the flag (4.8) and the two bullets in Subsection 4.5 by the two bullets of Subsection 4.6. This leads to the following definition:

Definition 4.20. An almost standard Young tableau (abbreviated ASYT) of shape $\lambda \setminus \mu$ is a way to label the boxes of $\lambda \setminus \mu$ with the integers $i, \ldots, j-1$:

$$\lambda \setminus \mu = \{ \Box_i, \ldots, \Box_{j-1} \}$$  \hspace{1cm} (4.42)
such that the labels match the colors of the boxes modulo \( n \), and the labels decrease along the rows and columns of \( \lambda \setminus \mu \), with the only possible exception that:

\[
\Box_a \text{ is allowed to be directly above } \Box_{a-1}
\]

\[\text{(4.43)}\]

for all \( a \in \{ i + 1, \ldots, j - 1 \} \).

The exception (4.43) arises from the fact that the \( Y \) maps “almost” preserve the flag (4.8), as in (4.10). This is the reason why one cannot build a full flag of intermediate partitions analogous to (4.39) from the datum of an ASYT. However, it does make sense to construct a partial flag of intermediate partitions:

\[
\lambda = \nu^0 \geq \nu^1 \geq \ldots \geq \nu^{t-1} \geq \nu^t = \mu
\]

\[\text{(4.44)}\]

where \( \nu^{s-1} \setminus \nu^s \) is a vertical strip of boxes of colors \( k_{s-1}, k_{s-1} + 1, \ldots, k_s - 1 \) for some:

\[i = k_0 < k_1 < \ldots < k_{t-1} < k_t = j\]

\[\text{(4.45)}\]

The data (4.44) and (4.45) determine an ASYT completely.

Finally, we must describe fixed points of the smooth correspondences \( \mathcal{W} \) of Subsection 4.15. As in (4.27), such a fixed point consists of two \( n \)–tuples of partitions \( \lambda \) and \( \mu \). Since the corresponding parabolic sheaves must be contained inside each other, we must have \( \lambda \geq \mu \). The condition that the \( X \) maps vanish on the kernel of (4.28) is equivalent to no two boxes of \( \lambda \setminus \mu \) being right next to each other, hence:

\[
\lambda \setminus \mu = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_t
\]

\[\text{(4.46)}\]

where every \( S_i \) is a vertical strip of boxes, and no two strips have a common edge. The strips in (4.46) are unordered, as opposed from the setup of (almost) standard Young tableaux. To summarize,

\[
\mathcal{W}^{\text{fixed}} = \left\{ (\lambda \geq \mu) \text{ s.t. } \lambda \setminus \mu \text{ is a union of strips (4.46)} \right\}
\]

\[\text{(4.47)}\]

4.21. We will use the description of the fixed points of \( \mathcal{Z}_{[i,j]} \), \( \mathcal{Z}^\pm_{(i,j)} \) and \( \mathcal{W} \) to prove formulas for the push-forward of classes under the projection maps \( p^\pm, p^\pm = \pi^\pm \) from each of these varieties to \( \mathcal{M}_{d^\pm} \). The subsequent Propositions will be key to computing the matrix coefficients of the operators (4.17), (4.18) and (4.36).

**Proposition 4.22.** For any Laurent polynomial \( M(z_i, \ldots, z_{j-1}) \) with coefficients pulled-back from \( K_T(\mathcal{M}_{d^\pm}) \), we have the following push-forward formulas:

\[
p^+_\ast \left( M(\mathcal{L}_i, \ldots, \mathcal{L}_{j-1}) \mathcal{Z}^+_i \right) =
\]

\[
= \int_{z_{i-1} < \ldots < z_i < \{0, \infty\}} \frac{M(z_i, \ldots, z_{j-1}) \prod_{a=i}^{j-1-1} \left[ \zeta \left( \frac{z_{a+1}}{z_a} \right) \tau_{\pm}(z_a) \right]^{\pm 1}}{q^{(j-i)+1} \prod_{a=i+1}^{j-1} \left( 1 - \frac{z_a}{z_{a-1}} \right) \prod_{i \leq b < j} \zeta \left( \frac{z_b}{z_a} \right)} Dz_a
\]

\[\text{(4.48)}\]

Recall that the modified virtual classes \( \mathcal{Z}^+_i \) were defined in (4.12)–(4.13). The normal ordered integral (4.48) is defined as in Remark 3.15, but the contours may be moved without changing the value of the integral to the following:

\[
p^+_\ast \left( \ldots \mathcal{Z}^+_i \right) = \int_{z_{j-1} < \ldots < z_i < \{0, \infty\}} \ldots
\]

\[\text{(4.49)}\]
4.24. As for the correspondence $\mathcal{M}_k$ of Subsection 4.15, one has the tautological rank $k_i$ vector bundle $L_i$, $i \in \{1, ..., n\}$. Let us formally write $L_i = x_{i1} + ... + x_{ik_i}$. For any color-symmetric Laurent polynomial $M$ in $k = (k_1, ..., k_n)$ variables, define:

\[ M(\ldots, L_i, \ldots) := M(\ldots, x_{i1}, ..., x_{ik_i}, \ldots) \in K_T(\mathcal{M}_k) \quad \text{(4.54)} \]
Then the analogue of Proposition 4.22 is the following result.

**Proposition 4.25.** For any color-symmetric Laurent polynomial $M$ as in (4.54) with coefficients pulled back from $K_T(M_{d+})$, we have:

$$
\pi^\pm_* \left( M(\ldots, \mathcal{L}_i, \ldots) \right) = \frac{1}{k!} \int_{\pi^\pm} M(\ldots, z_{ia}, \ldots) \prod_{1 \leq i \leq n, 1 \leq a < b \leq k} \left( 1 - \frac{z_{ia}}{z_{ib}} \right) \prod_{1 \leq i \leq n, 1 \leq a \leq k} \left[ \frac{z_{ia}^\pm}{X^\pm} \right] \tau_\pm(z_{ia}) \, Dz_{ia} \tag{4.55}
$$

The adjusted virtual classes $[M_{k}^\pm]_k$ were defined in (4.31) and (4.32). In the normal-ordered integral (4.55), one may move the contours without changing the value of the integral as follows:

$$
\pi^\pm_* \left( M(\ldots, \mathcal{L}_i, \ldots) \right) = \sum_{\text{decompositions}} \frac{1}{\#B_1, \ldots, B_t} \int_{y_1, \ldots, y_t < (0, \infty)} M(\ldots, z, \ldots)(\pm 1)^{|\mathbb{k}|-t} \prod_{s, s' = 1}^t \prod_{z \in B_s} \left( 1 - \frac{z}{z'} \right)^{\delta^\pm(z, z')} \prod_{1 \leq i \leq t} \left( \frac{z^\pm}{X^\pm} \right)^{\tau_\pm(z)} \left| z_{a} \right|_{[\mathbb{k}]} Dz_{a} \tag{4.56}
$$

where the number $\#B_1, \ldots, B_t$ is defined in (5.28). Each summand in (4.56) corresponds to a way to partition the set of variables $\{z_{ia}\}_{1 \leq a \leq k}$ into sets $B_s$ of variables of consecutive colors, then specialize all the $z_{ia}$ from a given set $B_s$ to the same value $y_s$, and finally evaluate the corresponding integral when $y_1, \ldots, y_t$ are far apart.

**Proof of Theorem 1.2:** We will prove the required statement for the shuffle element $S_m^\pm$, since the other cases are analogous. By (4.17), we have:

$$
s^\pm_m \cdot \mathcal{J} = p^\pm_* \left( m(\mathcal{L}_i, ..., \mathcal{L}_{j-1}) \cdot [3_{(i,j)}^\pm] \cdot f(X^\pm) \right) \tag{4.57}
$$

where we use the symbols $X^+$ and $X^-$ as placeholders for Chern classes of tautological bundles, pulled-back to $3_{(i,j)}^\pm$ from $M_{d+}$ and $M_{d-}$, respectively. We have:

$$
X^+|_{3_{(i,j)}} = X^-|_{3_{(i,j)}} + \mathcal{L}_i + \ldots + \mathcal{L}_{j-1}
$$

in the sense of the plethysm (3.27). Therefore, (4.57) becomes:

$$
s^\pm_m \cdot \mathcal{J} = p^\pm_* \left( m(\mathcal{L}_i, ..., \mathcal{L}_{j-1}) \cdot [3_{(i,j)}^\pm] \cdot f(X^\pm \mp (\mathcal{L}_i + \ldots + \mathcal{L}_{j-1})) \right)
$$

Because the variables $X^\pm$ are pulled-back under $p^\pm$, they pass unhindered through the push-forward $p^\pm$. Then we may use Proposition 4.22 to obtain:

$$
s^\pm_m \cdot \mathcal{J} = \int_{m(z_i, ..., z_{j-1})} f(X^\mp + \sum \frac{\prod_{a=1}^{j-1} \left( 1 - \frac{z_a}{z_{a-1}} q^a \right)}{q^{(j-1)\delta} \prod_{a=i+1}^{j-1} \prod_{1 \leq a < b < j} \left( 1 - \frac{z_a}{z_b} \right)} \prod_{1 \leq a \leq k} \left[ \frac{z_{ia}^\pm}{X^\pm} \right] \tau_\pm(z_{ia}) \, Dz_{ia}
$$

for all $f \in \mathcal{A}$. Comparing the above formula with the shuffle elements (2.12) and Theorem 3.13, we obtain the following equality of operators on $K$:

$$
s^\pm_m = S^\pm_m \tag{4.58}
$$
The analogous computations for $\mathfrak{g}_{\mu_k}$ and $\mathfrak{g}_{\mu_{\pm k}}$ gives us $t_m^\pm = T_m^\pm$ and $g_{\pm k} = G_{\pm k}$. 

**Proof of Theorem 1.1:** According to (2.14), (2.15), the assignment:

\[
e_{i:j} \sim E_{i:j} = S^+_1 \curvearrowright K
\]

\[
f_{i:j} \sim F_{i:j} = T^-_1 \curvearrowright K
\]

yields an action of $U_q(\hat{\mathfrak{g}}_{\mathfrak{n}}) \to A$ on $K$. To be completely precise, we must also specify that the Cartan elements $\psi_i$ of the quantum affine algebra act as:

- Multiplication by \( q^{i+d_{i-1}-d_i} \) on the subspace $K_d \subset K$

Clearly, the unit class $|\emptyset\rangle \in K(a,...,0)$ is a lowest weight vector, in the sense that:

\[
f_{i:j} \cdot |\emptyset\rangle = 0 \quad (4.59)
\]

\[
\psi_i \cdot |\emptyset\rangle = \frac{q^i}{u_i} |\emptyset\rangle \quad (4.60)
\]

Letting $M$ denote the universal Verma module freely generated over $F_u$ by a lowest weight vector $|\emptyset\rangle$ under properties (4.59)–(4.60), we obtain a morphism:

\[M \to K\]

of $U_q(\hat{\mathfrak{g}}_{\mathfrak{n}})$–modules. Because the universal Verma module is irreducible for generic lowest weight, proving that $M \cong K$ reduces to showing that the modules $M$ and $K$ have the same dimension in every graded component. As a graded vector space:

\[M \cong U_q^+ (\hat{\mathfrak{g}}_{\mathfrak{n}})\]

and so that the dimension of $M_d$ equals the number of partitions of the degree vector $d$ into arcs $[i; j)$ ([12]). The dimension of $K_d$ is the number of fixed points of Laumon spaces, which were interpreted in [5] as collections $(d_{j,i})_{i \leq j \in \mathbb{Z}}$ such that:

\[d_{j,i} \geq d_{j',i} \text{ if } j \leq j', \quad d_{j+n,i+n} = d_{j,i}, \quad d_{j,i} = 0 \text{ for } j-i \gg 0, \quad d_j = \sum_{i \leq j} d_{j,i}\]

To such a collection, we associate the following partition of $d$:

\[d = \sum_{i < j} [i; j) \cdot (d_{j-1,i} - d_{j,i})\]

Conversely, to a partition of $d$ into arcs, we associate the collection:

\[d_{j,i} = \# \text{ of arcs } [i; a) \text{ with } a > j\]

It is easy to see that these two assignments are inverses of each other, and therefore produce the required bijection between partitions of $d$ into arcs and collections $(d_{j,i})$. Therefore $\dim M_d = \dim K_d$, $\forall d \in \mathbb{N}^n$, and so we conclude that $M \cong K$. 

\[\square\]
4.26. In the remainder of this Section, we will study the analogue of the Carlsson-Okounkov vector bundle of [3]. In the setup of affine Laumon spaces, it was defined in [5] as the following rank $|d| + |d'|$ vector bundle $E$ on $M_d \times M_{d'}$, for any pair of degree vectors $d, d' \in \mathbb{N}^n$:

$$E|_{\mathcal{F}_{\bullet} \times \mathcal{F}'_{\bullet}} = \text{Ext}^1(F'_{\bullet}, F_{\bullet}(-\infty))$$ (4.61)

The notion of Ext space of flags of sheaves is defined in [5], where it is explained why the twist by $\infty$ forces the corresponding $\text{Ext}^0$ and $\text{Ext}^2$ spaces vanish. The following result is proved in loc. cit.:

**Proposition 4.27.** The vector bundle $E$ has a section $s$ which vanishes on:

$$C = \{ (F_{\bullet} \supset F'_{\bullet}) \} \subset M_d \times M_{d'}$$ (4.62)

set-theoretically, but not necessarily scheme-theoretically.

**Remark 4.28.** When $d' = d$, the locus $C$ is the diagonal.

**Remark 4.29.** When $d' = d + \varsigma^i$ for some $1 \leq i \leq n$, the locus $C$ coincides with $\mathfrak{Z}_i$. We conclude that $C$ is smooth and middle-dimensional inside $M_d \times M_{d'}$.

**Remark 4.30.** If $d' - d \notin \mathbb{N}^n$, the locus $C$ is empty, and hence the vector bundle $E$ has a nowhere vanishing section. This implies the exterior power of $E$ has class 0:

$$[\wedge^*(E^\vee)] := \sum_{k=0}^{\text{rank } E} (-1)^k [\wedge^k E^\vee] = 0 \in K_d \otimes K_{d'}$$ (4.63)

In general, we need to compute the $K$–theory class of the vector bundle $E$ inside the ring $K_d \otimes K_{d'}$. We will do so in terms of the tautological vector bundles $\mathcal{V}_k$ and $\mathcal{V}'_{\cdot}$ pulled-back from the two factors, and the formula we obtain is:

$$[E] = \left(1 - \frac{1}{q^2}\right) \sum_{k=1}^{n} \left(\frac{\mathcal{V}_k}{\mathcal{V}_{k-1}'} - \frac{\mathcal{V}_k'}{\mathcal{V}'_{k-1}}\right) + \sum_{k=1}^{n} \left(\frac{\mathcal{V}_k}{u_k^2} + \frac{u_k^2}{\mathcal{V}_k} + 1\right)$$ (4.64)

Together with (3.23), the above formula may be rewritten as:

$$[T.M_d] - [E] = \left(1 - \frac{1}{q^2}\right) \sum_{k=1}^{n} (\mathcal{V}_k - \mathcal{V}_{k+1}) \left(\frac{1}{\mathcal{V}_k'} - \frac{1}{\mathcal{V}_k}\right) - \sum_{k=1}^{n} \frac{u_k^2 + 1}{q^2} \left(\frac{1}{\mathcal{V}_k'} - \frac{1}{\mathcal{V}_k}\right)$$ (4.65)

If we restrict the formula above to the diagonal (i.e. $\mathcal{V}_k = \mathcal{V}_{k}'$), then we obtain 0, as one would expect from Remark 4.28. If we restrict it to the simple correspondence $\mathfrak{Z}_i \subset M_d \times M_{d'}$ (where we have $\mathcal{V}_k' = \mathcal{V}_k + \delta^i_{\cdot} \mathcal{L}_i$), then then the right-hand side of (4.65) matches (4.23), as one would expect from Remark 4.29.
4.31. As a consequence of (4.64), we infer that the exterior class of $\mathcal{E}^\vee$ is given by:

$$\wedge^\bullet(\mathcal{E}^\vee) = \frac{n}{k=1} \left(1 - \frac{V_k}{V_k} \frac{V_k}{V_k} \right) \left(1 - \frac{V_{k+1}}{V_{k+1}} \frac{V_{k+1}}{V_{k+1}} \right) \left(1 - \frac{V_{k+1}^2}{V_{k+1}^2} \frac{V_{k+1}^2}{V_{k+1}^2} \right) \left(1 - \frac{V_{k+1}^2}{V_{k+1}^2} \frac{V_{k+1}^2}{V_{k+1}^2} \right) \left(1 - \frac{V_{k+1}^2}{V_{k+1}^2} \frac{V_{k+1}^2}{V_{k+1}^2} \right)$$

We will work with the following adjustment of the above exterior class:

$$\wedge^\bullet(\mathcal{E}^\vee) := \wedge^\bullet(\mathcal{E}^\vee) \cdot (-1)^{|d'|-|d|} u_{d'-d} q^{(d',d'-d)} \prod_{i=1}^n \frac{\det V_i}{\det V_i} \quad (4.66)$$

on $\mathcal{M}_d \times \mathcal{M}_{d'}$. Let $p_1$ and $p_2$ be the projections from $\mathcal{M}_d \times \mathcal{M}_{d'}$ to the two factors, and define the operators:

$$a_{d,d'} : K_{d'} \rightarrow K_d$$

$$\alpha \sim p_{d'} \left(\wedge^\bullet(\mathcal{E}^\vee) \cdot \rho_1(\alpha)\right)$$

According to (4.63), $a_{d,d'} = 0$ unless $d' - d \in \mathbb{N}^n$. Therefore, we will write:

$$a_k = \bigoplus_{d \in \mathbb{N}^n} a_{d,d+k}, \quad a = \bigoplus_{k \in \mathbb{N}^n} a_k$$

as an endomorphism of $K$, and prove the following:

**Proposition 4.32.** The operator $K \overset{a_k}{\rightarrow} K$ coincides with the action of the element:

$$(1 - q^{-2})^{-|k|} \in A_{-k} \bowtie K \quad (4.68)$$

Above, $(1 - q^{-2})^{-|k|}$ is a constant rational function in $(k_1, ..., k_n)$ variables, which is an element of $A^*$ as in Definition 2.3, and the action $\bowtie$ is the one of Theorem 3.13.

**Proof** If $\lambda \not\geq \mu$, then Remark (4.63) implies that $\langle \mu | a | \lambda \rangle = 0$. Otherwise:

$$\langle \mu | a | \lambda \rangle = \frac{\wedge^\bullet(\mathcal{E}^\vee_{\lambda \geq \mu})}{\wedge^\bullet(\mathcal{M}_d \times \mathcal{M}_{d'})} = \frac{(-1)^{|d'|-|d|} u_{d'-d} q^{(d',d'-d)} \prod_{i=1}^n \frac{\det V_i}{\det V_i}}{\wedge^\bullet(\mathcal{T}_{\lambda \geq \mu} \mathcal{M}_d \times \mathcal{M}_{d'})}$$

Formula (4.65) is equivalent to the following:

$$[T,\mathcal{M}_d] - [\mathcal{E}] = \left(1 - \frac{1}{q^2} \right) \sum_{k=1}^n \left(\frac{1}{V_k} - \frac{1}{V_k} \right) (V_k' - V_k) - \sum_{k=1}^n \frac{V_k' + V_k}{u_k^2} \quad (4.69)$$

on $\mathcal{M}_d \times \mathcal{M}_{d'}$, where $d' = d + k$. Therefore, we conclude that:

$$\langle \mu | a | \lambda \rangle = \prod_{\mu \in \lambda \mu} \left[\zeta \left(\frac{\lambda \mu}{\mu \mu} \right) \tau_-(\chi \mu)\right]^{-1}$$

Comparing the above with (3.55) gives us the required equality of operators.
5. Appendix

Proof of Proposition 4.14: The proof is mostly based on the following claim:

Claim 5.1. Consider vector spaces $V_i^+ \to V_i^-$ equipped with a flag of subspaces:

$$0 = U_i^0 \subset U_i^1 \subset \cdots \subset U_i^k = \text{Ker} (V_i^+ \to V_i^-)$$

(5.1)

for $i \in \{1, 2\}$. The vector space $W_k$ of homomorphisms $V_1^+ \to V_2^+$ which preserve the flag (5.1) is given by:

$$[W_k] = \frac{V_2^+}{V_1^+} - \sum_{a=1}^{k} \frac{V_2^+}{L_1^a} + \sum_{1 \leq a \leq b \leq k} \frac{L_2^a}{L_1^b}$$

(5.2)

where $L_1^a = U_i^a/U_i^{a-1}$. Formula (5.2) holds in the Grothendieck group, i.e. we identify $V$ with $V_1 + V_2$ for any short exact sequence $0 \to V_1 \to V \to V_2 \to 0$.

Proof The flag (5.1) is equivalent to a flag of quotients:

$$V_i^+ = V_i^0 \to V_i^1 \to \cdots \to V_i^{k-1} \to V_i^k = V_i^-$$

where $L_1^a = \text{Ker}(V_i^{a-1} \to V_i^a)$. Consider the vector space $W_a$ of homomorphisms $V_1^+ \to V_2^+$ which preserve the truncated flags $V_i^+ = V_i^0 \to \cdots \to V_i^{a-1}$. We have:

$$W_0 := \text{Hom}(V_1^+, V_2^+) = (V_1^+)^\vee \otimes V_2^+ = \frac{V_2^+}{V_1^+}$$

(5.3)

where the last equality is simply a matter of notation for us. In general, we have:

$$W_a = \text{Ker}(W_{a-1} \to \text{Hom}(L_1^a, V_2^a))$$

and so we have the following equality in the Grothendieck group:

$$W_a = W_{a-1} - \frac{V_2^a}{L_1^a} = W_{a-1} - \frac{V_2^+}{L_1^a} + \sum_{a' \leq a} \frac{L_2^{a'}}{L_1^a}$$

Iterating the preceding relation for $a \in \{1, \ldots, k\}$ gives us formula (5.2).

Recall that the affine space $Z_{[i;j]}$ parametrizes quadruples of linear maps $X, Y, A, B$ to/from an $n-$tuple of vector spaces $V^+ = (V_1^+, \ldots, V_n^+)$ which interact with the flag (4.5) as in the two bullets of Subsection 4.5. Applying Claim 5.1, we see that:

$$[Z_{[i;j]}] = \left( \sum_{k=1}^{n} \frac{Y_k^+}{V_k^+ q^2} - \sum_{a=1}^{i-1} \frac{Y_a^+}{L_a q^2} + \sum_{a \equiv b \leq i \leq b < j} \frac{L_b}{L_a q^2} \right) +$$

$$+ \left( \sum_{k=1}^{n} \frac{u_k^+}{V_k^+ - \sum_{a=1}^{j-1} \frac{V_{a+1}}{L_a} + \sum_{i \leq a < b < j} \frac{L_b}{L_a q^2}} + \sum_{k=1}^{n} \frac{V_k^+}{u_k^+} + \left( \sum_{k=1}^{n} \frac{u_k^2}{V_k^+ q^2} - \sum_{a=1}^{j-1} \frac{u_{a+1}^2}{L_a q^2} \right) \right)$$

(5.4)
where the 4 parentheses keep track of the contributions of the maps $X, Y, A, B$, respectively. The codomain of the map $\eta$ and the Lie algebra of $P_{[i,j]}$ contribute:

$$[\mathfrak{a}_{[i,j]}] = \sum_{k=1}^{n} \frac{V_k^+}{V_k^{-1} q^2} - \sum_{a=1}^{j-1} \frac{V_a^+}{L_a q^2} + \sum_{b=a+1}^{b \equiv a+1} \frac{L_b}{L_a q^2} \quad (5.5)$$

$$[\mathfrak{p}_{[i,j]}] = \sum_{k=1}^{n} \frac{V_k^+}{V_k^{-1} q^2} - \sum_{a=1}^{j-1} \frac{V_a^+}{L_a q^2} + \sum_{i \leq a \leq b < j} \frac{L_b}{L_a q^2} \quad (5.6)$$

By formula (4.21), the class of the (virtual) tangent space to $\mathfrak{z}_{[i,j]}$ is:

$$\left( \text{LHS of (5.4)} \right) - \left( \text{LHS of (5.5)} \right) - \left( \text{LHS of (5.6)} \right)$$

In the right hand sides, the summands which only involve $V_k^+$ contribute precisely the class of the tangent space to $\mathcal{M}_{A^+}$, by (3.23). The remaining summands precisely constitute the RHS of (4.23), as we needed to prove. Formula (4.24) is proved similarly, so we leave it as an exercise to the interested reader.

Recall that the affine space $\overline{Z}_{[i,j]}$ parametrizes quadruples of linear maps $X, Y, A, B$ to/from an $n$-tuple of vector spaces $\mathbb{V}^+ = (V_1^+, ..., V_n^+)$ which preserve the flag (4.8) as in the two bullets of Subsection 4.6. Applying Claim 5.1, we see that:

$$[\overline{Z}_{[i,j]}] = \left( \sum_{k=1}^{n} \frac{V_k^+}{V_k^{-1} q^2} - \sum_{a=1}^{j-1} \frac{V_a^+}{L_a q^2} + \sum_{i \leq a < b < j} \frac{L_b}{L_a q^2} \right) + \left( \sum_{k=1}^{n} \frac{V_k^+}{V_k^{-1} q^2} - \sum_{a=1}^{j-1} \frac{V_a^+}{L_a q^2} + \sum_{i \leq a < b < j} \frac{L_b}{L_a q^2} \right)$$

$$+ \left( \sum_{k=1}^{n} \frac{V_k^+}{V_k^{-1} q^2} - \sum_{a=1}^{j-1} \frac{V_a^+}{L_a q^2} + \sum_{i \leq a < b < j} \frac{L_b}{L_a q^2} \right) + \left( \sum_{k=1}^{n} \frac{V_k^+}{V_k^{-1} q^2} - \sum_{a=1}^{j-1} \frac{V_a^+}{L_a q^2} + \sum_{i \leq a < b < j} \frac{L_b}{L_a q^2} \right)$$

where the 4 parentheses keep track of the contributions of the maps $X, Y, A, B$, respectively. The codomain of the map $\overline{\eta}$ and the Lie algebra of $\overline{\mathcal{P}}_{[i,j]}$ contribute:

$$[\overline{\mathfrak{a}}_{[i,j]}] = \sum_{k=1}^{n} \frac{V_k^+}{V_k^{-1} q^2} - \sum_{a=1}^{j-1} \frac{V_a^+}{L_a q^2} + \sum_{i \leq a < b < j} \frac{L_b}{L_a q^2} \quad (5.8)$$

$$[\overline{\mathfrak{p}}_{[i,j]}] = \sum_{k=1}^{n} \frac{V_k^+}{V_k^{-1} q^2} - \sum_{a=1}^{j-1} \frac{V_a^+}{L_a q^2} + \sum_{i \leq a < b < j} \frac{L_b}{L_a q^2} \quad (5.9)$$

By formula (4.22), the class of the (virtual) tangent space to $\overline{\mathfrak{z}}_{[i,j]}$ is:

$$\left( \text{LHS of (5.7)} \right) - \left( \text{LHS of (5.8)} \right) - \left( \text{LHS of (5.9)} \right)$$

In the right hand sides, the summands which only involve $V_k^+$ contribute precisely the class of the tangent space to $\mathcal{M}_{A^+}$, by (3.23). The remaining summands precisely constitute the RHS of (4.25), as we needed to prove. Formula (4.26) is proved similarly, so we leave it as an exercise to the interested reader.

\[ \square \]

**Proof of Proposition 4.16:** We will use formula (4.30), and to do so, we will need to compute the class of the vector space $W$ in the Grothendieck group. Claim
5.1 gives us:

\[
[W] = \sum_{i=1}^{n} \left( \frac{V_i^+}{V_i^+ q^2} - \frac{V_i^+}{L_i q^2} \right) + \sum_{i=1}^{n} \left( \frac{V_i^+}{V_{i-1}^+} - \frac{V_i^+}{L_{i-1}} + \frac{L_i}{L_{i-1}} \right) +
\]
\[+ \sum_{i=1}^{n} \frac{V_i^+}{u_i^2} + \sum_{i=1}^{n} \left( \frac{u_{i+1}^2}{V_i^+ q^2} - \frac{u_{i+1}^2}{L_{i+1} q^2} \right) \quad (5.10)
\]

The four sums in \((5.10)\) correspond to the \(X, Y, A, B\) maps, respectively. Moreover:

\[
[a] = \sum_{i=1}^{n} \left( \frac{V_i^+}{V_{i-1}^+ q^2} - \frac{V_i^+}{L_{i-1} q^2} \right) \quad (5.11)
\]
\[
[p] = \sum_{i=1}^{n} \left( \frac{V_i^+}{V_i^+} - \frac{V_i^+}{L_i} + \frac{L_i}{L_i} \right) \quad (5.12)
\]

According to \((4.30)\), the difference:

\[
\left( \text{LHS of } (5.10) \right) - \left( \text{LHS of } (5.11) \right) - \left( \text{LHS of } (5.12) \right)
\]
gives us a formula for \([T \mathcal{M}]\). In the corresponding difference of right-hand sides, the terms that do not involve \(L_i\) produce \([T \mathcal{M}_{d+}]\), according to formula \((3.23)\). The remaining terms are precisely the right-hand side of \((4.37)\). Formula \((4.38)\) is proved similarly, so we leave it as an exercise to the interested reader.

\[\square\]

**Proof of Proposition 4.22:** Let us first prove \((4.48)\) when the sign is \(\pm = +\). The virtual equivariant localization formula \((4.19)\) reads:

\[
\text{LHS of } (4.48) = \sum_{\lambda \vdash d} \left. \sum_{\text{shape } \lambda} \text{SYT of upper } M(L_{i_1}, \ldots, L_{j-1}) : \left[3_{[i,j]}^+ \right] \right|_{\text{SYT}}
\]

Recall from \((4.41)\) that the restriction of \(L_a\) to the fixed point corresponding to a SYT is just the weight \(\chi_a\) of the box labeled \(a\) in the SYT. We will use \((4.12)\) and \((4.23)\) to compute the adjusted virtual class in the numerator, respectively the exterior class of the normal bundle in the denominator. Therefore, LHS of \((4.48)\)|\(\lambda\) =

\[
\text{SYT of upper } \sum_{\text{shape } \lambda} \text{M}(\chi_{i_1}, \ldots, \chi_{j-1}) u_{i+1} \ldots u_j \chi_{j-1} (-1)^{j-i-1} q^{[i-j]_{\lambda}} (1)_{j-i}^{d_i-d_j} \prod_{a=i+1}^{j} (1 - \frac{\chi_a q^2}{\chi_a})
\]
\[
\begin{align*}
\prod_{a<b}^{a=b} 1 - \frac{\chi_a \chi_b}{\chi_a} & \prod_{a<b}^{a=b} 1 - \frac{\chi_a \chi_b}{\chi_a} \prod_{a<b}^{a=b} 1 - \frac{\chi_a \chi_b}{\chi_a} = 1 - \frac{\chi_a q^2}{u_{a+1}^2} \prod_{a \in \lambda} (1 - \frac{\chi_a q^2}{\chi_a})
\end{align*}
\]

where \(d \) is the degree of \(\lambda\). The factor \((1 - 1)^{j-i}\) in the numerator of the first row is meant to cancel \(j - i\) zeroes in the denominator of the second row. Explicitly, if we write a SYT in the form \((4.39)\), then we have \(\chi_a = \chi_{i_1} + \chi_{i_2} + \ldots + \chi_{j-1}\), and so the formula above implies that LHS of \((4.48)\)|\(\lambda\) =

\[
\sum_{(4.39)} \text{M}(\chi_{i_1}, \ldots, \chi_{j-1}) u_{i+1} \ldots u_j \chi_{j-1} (-1)^{j-i-1} q^{[i-j]_{\lambda}} (1)_{j-i}^{d_i-d_j} \prod_{a=i+1}^{j} (1 - \frac{\chi_a q^2}{\chi_a})
\]
Because of the assumptions on the sizes of \(q\), the finite poles, i.e. those other than 0 and \(\infty\) as prescribed by (4.49). We will compute the right hand side by integrating over moving the contours of \(z\) one very large, which separate the set \(S\) integral, by which we mean all the poles that arise from the functions \(\zeta\) and \(\tau_+\). This completes the proof of (4.48) in the case \(z\) arises when computing the integral of \(\zeta\) and \(\tau_+\). The only poles that involve two variables \(j\) is integrated over the difference of two circles, one very small and one very large, which separate the set \(S = \{0, \infty\}\) from the finite poles of the integral, by which we mean all the poles that arise from the functions \(\zeta\) and \(\tau_+\). The only poles that involve two variables \(z\) and \(b\) with \(a < b\) are of the form:

\[
z_a - z_b q^{-\frac{2(b-a-1)}{n}} \quad \text{for } a \equiv b - 1
\]

\[
z_a - z_b q^{-\frac{2(b-a)}{n}} \quad \text{for } a \equiv b
\]

Because of the assumptions on the sizes of \(q, q\), none of these poles hinder us from moving the contours of \(z_i, ..., z_j\) very far away from each other. Therefore:

\[
\text{RHS of (4.48)}|_\lambda = \int_{z_j \prec ... \prec z_i} M(z_i, ..., z_j) \prod_{a=i}^{j-1} \zeta \left( \frac{z_a}{\chi_a} \right) \tau_+(z_a) Dz_a
\]

as prescribed by (4.49). We will compute the right hand side by integrating over the finite poles, i.e. those other than 0 and \(\infty\). The first variable to be integrated is \(z_{j-1}\), and as shown in (3.52), the finite poles are of the form \(z_{j-1} = \chi_{j-1} := \text{weight of an outer corner} \square_{j-1} \in \lambda\) of color \(j - 1\). If we write \(\nu^{j-1} = \lambda \setminus \square_{j-1}\), then:

\[
\text{RHS of (4.48)}|_\lambda = \sum_{\lambda \geq \lambda_j^{j-1} \nu^{j-1}} \int_{z_j \prec ... \prec z_i} (q^{-1} - q) M(z_i, ..., z_j) \prod_{a=i}^{j-1} \zeta \left( \frac{z_a}{\chi_a} \right) \tau_+(z_a) Dz_a
\]

(see the proof of Proposition 3.23 for the way the factor \(q^{-1} - q\) on the first row arises when computing the integral of \(z_{j-1}\) by residues). One needs now to repeat the argument by integrating \(z_{j-2}\) over the finite poles, which are now of the form \(z_{j-2} = \chi_{j-2} := \text{weight of an outer corner} \square_{j-2} \in \nu^{j-1}\) of color \(j - 2\). Iterating this argument gives rise to a flag of partitions (4.39), and the RHS of (4.48)| \(\lambda\) matches (5.13). This completes the proof of (4.48) in the case \(\pm = +\).
Let us now prove the case $\pm = -$ of (4.48). Formula (4.19) reads:

$$\text{LHS of (4.48)} = \sum_{\mu} |\mu| \text{ SYT of lower } M(L_{i_1}, \ldots, L_{j-1}) \cdot |\mathcal{Z}_{i,j}| \bigg|_{\text{SYT}}$$

Recall from (4.41) that the restriction of $L_a$ to the fixed point corresponding to a SYT is just the weight $\chi_a$ of the box labeled $a$ in the SYT. We will use (4.13) and (4.24) to compute the adjusted virtual class in the numerator, respectively the exterior class of the normal bundle in the denominator. Therefore, LHS of (4.48)|$_{\mu} =$

$$\sum_{\text{SYT of lower}} M(\chi_{i_1}, \ldots, \chi_{j-1}) \prod_a 1 - \frac{\chi_a q}{\chi_a} \prod_a \left[ 1 - \frac{\chi_a q}{\chi_a} \right]$$

where $d$ is the degree of $\mu$. The factor $(1 - 1)^{j-i}$ in the numerator of the first row is meant to cancel $j - i$ zeroes in the denominator of the second row. Explicitly, if we write a SYT in the form (4.39), then we have $\chi_{\mu} = \chi_{\nu+1} - \chi_a - \ldots - \chi_i$, and so the formula above implies that LHS of (4.48)|$_{\mu} =$

$$\sum_{\text{SYT as in (4.39)}} M(\chi_{i_1}, \ldots, \chi_{j-1}) \prod_a \left[ 1 - \frac{\chi_a q}{\chi_a} \right]$$

Absorbing all the powers of $q$ into the second row, we obtain the formula:

$$\text{LHS of (4.48)} \bigg|_{\mu} = \left( 1 - \frac{\chi_a}{\chi_{a-1} q} \right)^{-1}$$

Recalling the definition of $f^-$ in Remark 3.15, we note that the RHS of (4.48)|$_{\mu} =$

$$\sum_{\sigma : \{i_1, \ldots, j-1\} \rightarrow \{1, \ldots, \nu\}} \int_{|\sigma| = \gamma(\sigma)} M(z_{i_1}, \ldots, z_{j-1}) \prod_{a=1}^{j-1} \left[ \zeta \left( \frac{z_a}{z_{a+1}} \right) \tau_-(z_a) \right]^{-1} \sigma(\alpha) Dz_a$$

Each variable $z_a$ is integrated over the difference of two circles, one very small and one very large, which separate the set $S = \{0, \infty\}$ from the finite poles of the integral, by which we mean all the poles that arise from the functions $\zeta$ and $\tau_-$. The only poles that involve two variables $z_a$ and $z_b$ with $a < b$ are of the form:

$$z_a - z_b q^{\frac{2(b-a-1)}{n}} \quad \text{for } a \equiv b - 1$$

$$z_a - z_b q^{2\frac{(b-a)}{n}} \quad \text{for } a \equiv b$$
Because of the assumptions on the sizes of $q, \overline{q}$, none of these poles hinder us from moving the contours of $z_i, \ldots, z_{j-1}$ very far away from each other. Therefore:

$$\text{RHS of } (4.48) \bigg|_{\mu} = \int_{z_i < \ldots < z_{j-1} < \{0, \infty\}} \frac{M(z_i, \ldots, z_{j-1}) \prod_{a=1}^{j-1} \left[ \zeta \left( \frac{\omega_{a+b}}{z_a} \right) \tau_\nu(z_a) \right]^{-1} Dz_a}{q^{j-i-1} \prod_{a=i+1}^{j-1} \left( 1 - \frac{\omega_{a+b}}{z_a \bar{q}^a} \right) \prod_{1 \leq a < b < j} \zeta \left( \frac{\omega_a}{\bar{q}^a} \right)}$$

as prescribed by (4.50). We will compute the right-hand side by integrating over the finite poles, i.e. those other than 0 and $\infty$. The first variable to be integrated is $z_i$, and as shown in (3.53), the finite poles are of the form $z_i = \chi_i := \text{weight of an inner corner } \square_i \in \mu$ of color $i$. If we write $\nu^{i+1} = \mu \cup \square_i$, then:

$$\text{RHS of } (4.48) \bigg|_{\mu} = \sum_{\nu^{i+1} \geq \mu} \int_{z_{i+1} < \ldots < z_{j-1} < \{0, \infty\}} (1 - q^{-2}).$$

(see the proof of Proposition 3.23 for the way the factor $1 - q^{-2}$ on the first row arises when computing the integral of $z_i$ by residues). One needs now to repeat the argument by integrating $z_{i+1}$ over the finite poles, which are of the form $z_{i+1} = \chi_{i+1} := \text{weight of an outer corner } \square_{i+1}$ of $\nu^{i+1}$ of color $i+1$. Iterating this argument gives rise to a flag of partitions (4.39), and we conclude that the RHS of $(4.48)|_{\mu}$ equals (5.14). This completes the proof of (4.48) when $\pm = -$.

\[ \square \]

**Proof of Proposition 4.23:** Let us first prove (4.51) when the sign is $\pm = +$. The virtual equivariant localization formula (4.19) reads:

$$\text{LHS of } (4.51) = \sum_{\lambda \vdash d^+} |\lambda| \sum_{\text{shape } \lambda} \text{ASYT of upper } M(L_1, \ldots, L_{j-1}) \cdot \chi^+_{\lambda(j)} \left| \wedge^*(T^\vee \chi^+_{ij}) - T^\vee M_{d^+} \right|_{\text{ASYT}}$$

Use (4.14) to compute the adjusted virtual class in the numerator, and (4.25) to compute the exterior class in the denominator. Therefore, LHS of (4.51)$|_{\lambda} =$

$$
\sum_{\text{shape } \lambda} \text{ASYT of upper } M(\chi_1, \ldots, \chi_{j-1}) \cdot u_{i+1} \ldots u_j \frac{q^{-j+i-i} \left[ \omega_a \right]}{q^d d_j} \frac{(1 - 1)^{j-i}}{\prod_{a=i+1}^{j-1} \left( 1 - \frac{\omega_{a-1}}{\chi_a} \right)}
$$

$$
\prod_{a < b} \frac{1 - \frac{\omega_a}{\chi_a}}{1 - \frac{\omega_a}{\chi_a}} \prod_{a < b} \frac{1 - \frac{\omega_a^2}{\chi_a}}{1 - \frac{\omega_a}{\chi_a}} \prod_{a = i}^{j-1} \left[ 1 - \frac{\omega_a q^2}{\bar{q}^a} \right] \prod_{a \in \lambda} \left[ 1 - \frac{\omega_a}{\chi_a} \right] \prod_{\square \in \lambda} \left[ 1 - \frac{\omega_a}{\chi_a} \right] \prod_{\square \in \lambda} \left[ 1 - \frac{\omega_a}{\chi_a} \right]
$$

As before, the factor $(1 - 1)^{j-i}$ in the first row is meant to cancel $j - i$ zeroes in the denominator of the rational function on the second row. Explicitly, we may write an ASYT in the form (4.44), where $\lambda = \nu^a \cup B_1 \cup \ldots \cup B_s$. Here, each $B_s$ is a vertical strip of boxes of colors $k_{s-1}, \ldots, k_s - 1$, all of which have the same weight.
χₙ if we apply rule\(^3\) (2.1). Therefore, the formula above implies LHS of (4.51)\(_\lambda\) =

\[
\sum_{(4.44)} M(\chi_1, ..., \chi_j-1) \cdot u_{i+1} \cdots u_j \cdot q^{-j+i-\left\lfloor \frac{i}{\lambda} \right\rfloor} \cdot \prod_{t=1}^{j-1} \left(1 - \frac{q^2 t^2}{\lambda_k} \right) \prod_{a=1}^{j-1} \left(1 - \frac{\chi_a q^2}{\mu_a + 1} \right)
\]

Absorbing all the powers of \(q\) into the second row, we obtain the formula:

\[
\text{LHS of (4.51)} \bigg|_\lambda = \sum_{(4.44)} M(\chi_1, ..., \chi_j-1)(q^{-1} - q) \cdot \prod_{s=1}^{t} \left(1 - \frac{\chi_k - 1}{\chi_{k_s}} \right) \prod_{i=1}^{j-1} \left(1 - \frac{\chi_i}{\lambda_i} \right) \prod_{a=k_{s-1}}^{k_{s+1}} \left(1 - \frac{\chi_a q^2}{\mu_a + 1} \right)
\]

where:

\[
\beta_k = \prod_{0 \leq a < b < \lambda < \kappa} \zeta(1-a) = \prod_{i=1}^{k-1} \left(1 - \frac{q^2 - q \cdot \sigma^{-2i}}{1 - q \cdot \varphi^{-2i}} \right)
\]

where \(1_{a-b}\) refers the value 1 as a variable of color \(a - b\).

Recalling the definition of \(\int^+\) in Remark 3.15, we note that the RHS of (4.51)\(_\lambda\) =

\[
\sum_{\sigma: (i, ..., \lambda - 1) \rightarrow \{1, -1\}} \int_{|\tilde{z}_{i-1}| = \gamma^\sigma(a) \varphi} M(z_i, ..., z_{\lambda - 1}) \prod_{a=1}^{\lambda - 1} \zeta \left(\frac{z_a}{\lambda_i} \right) \tau_\sigma(z_a) \sigma(a) Dz_a
\]

Each variable \(z_a\) is integrated over the difference of two circles, one very small and one very large, which separate the set \(S = \{0, \infty\}\) from the finite poles of the integral, by which we mean all the poles that arise from the functions \(\zeta\) and \(\tau_\sigma\). The only poles that involve two variables \(z_a\) and \(z_b\) with \(a < b\) are of the form:

\[
z_a - z_b q^{2(b-a)+1} \quad \text{for } a \equiv b + 1
\]

\[
z_a - z_b q^{2(b-a)+1} \quad \text{for } a \equiv b
\]

as well as \(z_a = z_a - 1\). Because of the assumptions on the sizes of \(\tilde{q}, \varphi\), the poles (5.17)–(5.18) do not hinder us from moving the contours of \(z_i, ..., z_{\lambda - 1}\) very far away from each other (with \(z_b\) closer to 0 and \(\infty\) than \(z_a\), if \(a < b\)). However, the poles \(z_a = z_a - 1\) do hinder us, and we have to consider the corresponding residues:

\[
\text{RHS of (4.51)} \bigg|_\lambda = \sum_{i=k_0 < k_1 < \cdots < k_r} \int_{z_{i-1} = z_{i-1} - 1}^{z_{i+1} = z_{i+1} - 1} M(z_i, ..., z_{\lambda - 1}) \prod_{a=1}^{\lambda - 1} \zeta \left(\frac{z_a}{\lambda_i} \right) \tau_\sigma(z_a) \sigma(a) Dz_a
\]
Each summand in (5.19) corresponds to a chain $i = k_0 < ... < k_t = j$. Such a chain can be thought of as consisting of links, namely collections of variables:

$$z_{k_0-1} = ... = z_{k_{t-1}}$$

(5.20)

which are set equal to each other in (5.19). We will think of such a link as a vertical strip consisting of boxes of colors $k_0 - 1, ..., k_t - 1$ (indeed, the weights of the boxes in such a vertical strip are all equal, given the convention (4.41)). We will compute the integral (5.19) as a sum of residues over the finite poles, namely those poles other than 0 and $\infty$. The first variable which we have to integrate is $y_1 := z_i = ... = z_{k_1-1}$, and by (3.52) we have a pole in this variable whenever:

$$y_1 = \chi^\square$$

for $\square$ an outer corner of color $c$ of $\lambda$

for some $c \in \{i, ..., k_1 - 1\}$. As explained above, this corresponds to taking a vertical strip consisting of boxes of colors $k_1 - 1, ..., i$ and placing the box of color $c$ in this strip on top of the outer corner $\square \in \lambda$. Because of the linear factors $1 - \frac{z_{k_1-1}}{z_a}$ in (5.19), the same residue can be obtained in other ways, by considering the “finer” chain where the link $z_i = ... = z_{k_1-1}$ is subdivided into any $r + 1$ smaller links:

$$z_i = ... = z_c \quad z_{c+1} = ... = z_{u_1-1} \quad ... \quad z_{u_r-1} = ... = z_{k_1-1}$$

The sign with which the residue appears from the finer chain is $(-1)^r$. Since there are $2^{k_1-1-c}$ finer chains, the total contribution of the given residue is:

$$\sum_{r=0}^{k_1-1-c} (-1)^r \binom{k_1-1-c}{r} = \delta_c^c = 1$$

We conclude that the only non-zero residue one obtains is when the box of color $k_1 - 1$ of the vertical strip is placed on top of the outer corner $\square \in \lambda$. In other words, this corresponds to removing a vertical strip from the partition $\lambda$:

$$\chi^\square \sqcup \left( \text{vertical strip } \square_{k_1-1}, ..., \square_i \right)$$

for some partition $\nu^1$. Letting $\chi_a$ denote the weight of the box $\square_a$, we see that:

$$\text{RHS of (4.51)} \bigg|_{\lambda} = \sum_{i=0}^{t \geq 1} \sum_{i=k_0 < k_1 < ... < k_t = j} \int_{z_{k_t} = ... = z_{k_{t-1}} < ... < z_{k_1} = ... = z_{j-1} < 0, \infty} M(z_i, ..., z_{j-1})$$

$$\frac{(q^{-1} - q)\beta_{k_1-1} \prod_{a=i}^{k_1-1} \zeta \left( \frac{z_{a}}{\chi^{\mu}} \right)}{\prod_{s=1}^{t} \left( 1 - \frac{z_{k_1-1}}{z_{k_s}} \right)} \prod_{k_1 \leq a < b < j} \zeta \left( \frac{z_{a}}{\chi^{\mu}} \right) \prod_{s=2}^{t} D_{z_{k_s-1}} \bigg|_{z_{k_1-1} \mapsto \chi_{k_1-1}, ..., z_k \mapsto \chi_i}$$

One needs now to integrate $z_{k_2} = ... = z_{k_1}$ over the finite poles, which by the same reasoning corresponds to removing a whole vertical strip from $\nu^1$. Iterating this argument gives rise to a flag of partitions (4.44), in other words an almost standard Young tableau of upper shape $\lambda$, and the RHS of (4.51) $|_{\lambda}$ equals (5.15). This completes the proof of (4.51) when $\pm = +$.

Let us now prove the case $\pm = -$ of (4.51). Formula (4.19) reads:

$$\text{LHS of (4.51)} = \sum_{\mu \in \mathcal{D}^*} \sum_{\text{shape } \mu} M(\mathcal{L}_1, ..., \mathcal{L}_{j-1}) \frac{\left( \mathcal{F}(\mathcal{L}_1) \right)}{\text{ASYT}}$$

where $\mathcal{F}(\mathcal{L}_1)$ is the standard Young tableau of lower shape $\mathcal{L}_1$. This completes the proof of (4.51) when $\pm = -$. 
Use (4.15) to compute the adjusted virtual class in the numerator, and (4.26) to compute the exterior class in the denominator. Therefore, LHS of (4.51)\(\big|_\lambda\) =

\[
\sum_{\text{shape } \mu} M(\chi_1, \ldots, \chi_{j-1}) \cdot \frac{u_i \ldots u_{j-1}}{\chi_1 \ldots \chi_{j-1}} q^{-j+i} \cdot \frac{1}{q^{d_{i+1}-d_{j-1}}} \cdot \frac{(1-1)^{j-i}}{\prod_{a=i+1}^{j-1} \left(1 - \frac{\chi_{a-1}}{\chi_a}\right)} \cdot \prod_{a \leq b} \frac{1 - \frac{\chi_a^2}{\chi_b}}{1 - \frac{\chi_a^2}{\chi_b}}
\]

As before, the factor \((1-1)^{j-i}\) in the first row is meant to cancel \(j-i\) zeroes in the denominator of the rational function on the second row. Explicitly, we may write an ASYT in the form (4.44), where \(\mu = \nu^{\lambda-1}(B_s \sqcup \ldots \sqcup B_t)\), where each \(B_s\) is a vertical strip of boxes of colors \(k_s-1, \ldots, k_s-1\). Therefore, the formula above implies LHS of (4.51)\(\big|_\lambda\) =

\[
\sum_{\text{ASYT as in } (4.44)} M(\chi_1, \ldots, \chi_{j-1}) \cdot \frac{u_i \ldots u_{j-1}}{\chi_1 \ldots \chi_{j-1}} q^{-j+i} \cdot \frac{1}{q^{d_{i+1}-d_{j-1}}} \cdot \frac{(1-1)^{j-i}}{\prod_{a=i+1}^{j-1} \left(1 - \frac{\chi_{a-1}}{\chi_a}\right)} \cdot \prod_{a \leq b} \frac{1 - \frac{\chi_a^2}{\chi_b}}{1 - \frac{\chi_a^2}{\chi_b}}
\]

Absorbing all the powers of \(q\) into the second row, we obtain the formula:

\[
\text{LHS of (4.51)\(\big|_\lambda\) = } \sum_{\text{ASYT as in } (4.44)} M(\chi_1, \ldots, \chi_{j-1})(-1)^{j-i}(q^2 - q)^t \prod_{s=1}^{t} \left[ \beta_{k_s} \prod_{a=k_{s-1}+1}^{k_s-1} \left(1 - \frac{\chi_a}{\chi_s}\right) \prod_{a \in \mu_s} \frac{1 - \frac{\chi_a^2}{\chi_s}}{1 - \frac{\chi_a^2}{\chi_s}} \right]
\]

where \(\beta_k\) is defined in (5.16). Recalling the definition of \(\bar{f}^-\) in Remark 3.15, we note that the RHS of (4.51)\(\big|_\mu\) =

\[
\sum_{\text{functions } \sigma : (\ldots \rightarrow (1, -1)} \int |q|^{\pm 1} \tau_q^{\pm 1} > \int_{|z_a| = \gamma(a)} \frac{\zeta}{\chi_a} \cdot \frac{2\pi i}{q^j} \prod_{a=i+1}^{j-1} \left(1 - \frac{\chi_{a-1}}{\chi_a}\right) \prod_{1 \leq a < b} \zeta \left(\frac{2\pi i z_a}{z_a}\right)
\]

Each variable \(z_a\) is integrated over the difference of two circles, one very small and one very large, which separate the set \(S = \{0, \infty\}\) from the finite poles of the integral, by which we mean all the poles that arise from the functions \(\zeta\) and \(\tau_q\). The only poles that involve two variables \(z_a\) and \(z_b\) with \(a < b\) are of the form (5.17)–(5.18), as well as \(z_a = z_{a-1}\). Because of the assumptions on the sizes of \(q, \bar{q}\), the poles (5.17)–(5.18) do not hinder us from moving contours of \(z_1, \ldots, z_{j-1}\) very far away from each other (with \(z_a\) closer to 0 and \(\infty\) than \(z_b\), if \(a < b\)). However, the poles \(z_a = z_{a-1}\) do hinder us, and we have to consider the corresponding residues:

\[
\text{RHS of (4.51)\(\big|_\mu\) = } \sum_{i = k_0 < k_1 < \ldots < k_t} \int |z_{k_t-1} = \ldots = z_{j-1} = z_i = \ldots = z_{k_1-1} = \{0, \infty\}|(1-1)^{j-i-t}
\]
This completes the proof of (4.51) when formula (4.19) allows to compute the LHS of (4.55): \[ - \]

**Proof of Proposition 4.25:**

We will only prove (4.55) when the sign is +, and leave the case of − to the interested reader. The virtual equivariant localization formula (4.19) allows to compute the LHS of (4.55):

\[
\pi^+_a \left( M(\ldots, \mathcal{L}_i, \ldots) \cdot [\mathfrak{M}^+_k] \right) = \sum_{\lambda} |a| \sum_{\mu \text{ such that } \lambda_{\chi_a} \mu \text{ as in (4.47)}} \frac{M(\ldots, \mathcal{L}_i, \ldots) \cdot [\mathfrak{M}^+_k]}{\lambda^* \left( [\mathcal{T}^\vee \mathfrak{M}^+_k] - [\mathcal{T}^\vee \mathcal{M}^+_k] \right)} \big| \lambda_{\chi_a} (\lambda, \mu)
\]

We will use \( \mathcal{L}_i(\lambda, \mu) = \sum_{\sigma} \mathfrak{M}^+_k \chi^{\sigma} \) and the definition of the adjusted virtual class (4.31) to compute the numerator, and (4.37) to compute the denominator:

\[
\text{LHS of (4.55)} \bigg|_{\lambda_{\chi_a} (\lambda, \mu)} = \sum_{\mu \text{ such that } \lambda_{\chi_a} \mu \text{ as in (4.47)}} M(\lambda, \mu) \cdot \frac{u_{k+1}}{q^{k+1}(k, d)} \cdot \prod_{\sigma \in \lambda_{\chi_a} (\lambda, \mu)} \left( 1 - \frac{\lambda_{\chi_a} q^2}{u_{\text{col}} (k, d+1)} \right)
\]
\[
\prod_{\square \in \Lambda \setminus \mu} \left( 1 - \frac{\chi_{\square}}{q} \right)^{g_{\square}^\text{col}} \prod_{\square \in \Lambda \setminus \mu} \left( 1 - \frac{\chi_{\square}}{1 - \chi_{\square}} \right)^{g_{\square}^\text{col}} \prod_{\square \in \mu} \left( 1 - \frac{\chi_{\square}}{1 - \chi_{\square}} \right)^{g_{\square}^\text{col}}
\]

If we replace the sum over \( \square \in \Lambda \) to a sum over \( \square \in \mu \), the expression above equals:

\[
\text{LHS of (4.55)} \bigg|_{\lambda} = \sum_{\lambda \setminus \mu \text{ as in (4.47)}} M(\lambda \setminus \mu) \cdot \frac{u_{k+1}}{q^{k+1}} \cdot \prod_{\square \in \Lambda \setminus \mu} \left( 1 - \frac{\chi_{\square}}{1 - \chi_{\square}} \right)^{g_{\square}^\text{col}}
\]

and if we absorb all powers of \( q \) from the first row into the second row, we obtain:

\[
\text{LHS of (4.55)} \bigg|_{\lambda} = \sum_{\lambda \setminus \mu \text{ as in (4.47)}} M(\lambda \setminus \mu) \cdot \left( \frac{1}{q} \right)^{\gamma(i,a)} \prod_{\square \in \Lambda \setminus \mu} \left( 1 - \frac{\chi_{\square}}{1 - \chi_{\square}} \right)^{g_{\square}^\text{col}} \prod_{\square \in \lambda \setminus \mu} \left( \frac{\chi_{\square}}{\lambda_{\square}} \right)^{\tau(\chi_{\square})}
\]

We will compute the right hand side of (4.55) by changing the contours in the integral, and showing that the resulting residue computation equals (5.23). To this end, recall that the normal ordered integral was defined in Remark 3.15 as:

\[
\text{RHS of (4.55)} \bigg|_{\lambda} = \frac{1}{k!} \sum_{\sigma \in \{i,a\}^{1 \leq i \leq n}} \int_{|z_{ia}| = \gamma(i,a)} \prod_{1 \leq i \leq n} \left( 1 - \frac{\chi_{\square}}{z_{ia}} \right) \frac{\chi_{\square}}{\lambda_{\square}}^{\tau(\chi_{\square})} Dz_{ia}
\]

The only poles that involve two of the \( z \) variables are of the form \( z_{ia} = z_{i-1,b} \). We will seek to move the contours of \( \{z_{ia}\} \) far apart from each other by the following procedure: pick an arbitrary \( z_{ia} \), \( 1 \leq a \leq k_i \), which we will henceforth call the \textbf{starting variable}, and seek to move its contour toward \( \{0, \infty\} \). The only poles we can pick up along the way are of the form \( z_{ia} = z_{i-1,b} \) for some \( 1 \leq b \leq k_{i-1} \). For the corresponding residue, try to move the contour of \( z_{i-1,b} \) toward \( \{0, \infty\} \). As there are zeroes at \( z_{i-1,b} = z_{i-1,b'} \) in the integrand, the only poles we can pick up are of the form \( z_{ia} = z_{i-1,b} = z_{i-2,c} \) for some \( 1 \leq c \leq i-2 \). Iterating this argument shows that the only residues one obtains are given by partitions of the variable set:

\[
\{z_{ia}\}_{1 \leq a \leq k_i} = B_1 \sqcup B_2 \sqcup \ldots \sqcup B_t
\]

where:

\[
B_s = \{z_{i_s,a_s}, \ldots, z_{j_s-1,b_s}\}
\]

for various \( i_s < j_s \), and we underline the starting variable. The variables in each group are set equal to each other \( z_{i_s,a_s} = \cdots = z_{j_s-1,b_s} : y_s \), so we interpret each
$B_s$ as the set of weights of a vertical strip of boxes of colors $i_s, \ldots, j_s - 1$. With the above discussion in mind, (5.24) yields:

$$\text{RHS of (4.55)} = \sum_{\text{decompositions}} \int_{y_1 < \cdots < y_t < (0, \infty)} \frac{M(\ldots, z_{i_1a}, \ldots)}{\# B_1 \ldots B_t}$$

$$\prod_{s, s' = 1} z \in B_s \prod_{1 \leq s \leq t} \zeta \left( \frac{z}{\chi_{\lambda}} \right) \tau_+(z) |_{y \to y_s} \prod_{s, s' = 2} z \in B_s \prod_{s' = 1} \delta_{\text{col } z'} \frac{1}{z^s} |_{y \to y_s}$$

(5.27)

While we will not mention this explicitly in order to keep the notation legible, one must take care to remove all 0 factors from the numerators or denominators of our products, such as the situation when $z = z'$ in the first product on the second row of (5.27). The denominator that appears on the first row of (5.27) is:

$$\# B_1 \ldots B_t = \prod_{a = 1}^{t} (j_a - 1)\text{-th entry of the vector } k - [i_{a+1}; j_{a+1}) - \ldots - [i_t; j_t) \in \mathbb{N}^t$$

(5.28)

The appearance of $\# B_1 \ldots B_t$ in (5.27) stems from the fact that the second indices of the variables $\{z_{i,a}\}$ can be permuted at random, except for the second indices of the starting variables, which are fixed by our algorithm.

To compute the integral (5.27), we will first integrate $y_1$ over the finite poles of the integral (namely those poles $\neq 0, \infty$). According to (3.52), these are of the form $y_1 = \chi_1 = \text{weight of an outer corner } \Box \text{ of the partition } \lambda$. This corresponds to a vertical strip of boxes of colors $i_1, \ldots, j_1 - 1$, such that the outer corner $\Box$ has color $c \in \{i_1, \ldots, j_1 - 1\}$. As in the computation following (5.19) in the proof of Proposition 4.23, the corresponding residue is canceled by an inclusion-exclusion argument, unless:

$$c = j_1 - 1 \Rightarrow \lambda = \nu^1 \sqcup \left( \text{vertical strip } B_1 \right)$$

With this in mind, we may rewrite (5.27) by breaking up the products in terms of whether each variable $z$ lies in $B_2 \sqcup \ldots \sqcup B_t$ or in $B_1$. Therefore, RHS of (4.55)$|_\lambda = $

$$= \sum_{\text{decompositions}} \int_{y_2 < \cdots < y_t < (0, \infty)} \frac{M(\ldots, z_{i_1a}, \ldots)}{\# B_1 \ldots B_t} \prod_{2 \leq s \leq t} \left( 1 - \frac{w}{w'} \right) \delta_{\text{col } z} \frac{1}{z^s} |_{y \to y_s} \prod_{w,w' \in B_1 \oplus B_1} \left( 1 - \frac{w}{w'} \right) \delta_{\text{col } z} \frac{1}{z^s} |_{y \to y_s}$$

$$\prod_{w \in B_1} \left( 1 - \frac{w}{w'} \right) \delta_{\text{col } z} \frac{1}{z^s} |_{y \to y_s} \prod_{s, s' = 2} z' \in B_s \prod_{s' = 1} \delta_{\text{col } z'} \frac{1}{z^s} |_{y \to y_s}$$

(5.29)

We may now analyze the finite poles of the integrand in $y_2$, which corresponds to those variables in $B_2$. Note that the finite poles come from $\chi_{\nu^1}$ as well as the $w$ variables. Relation (3.52) tells us that the third row of (5.29) has a pole when $y_2 = \chi_2$ = weight of an outer corner of the partition $\nu^1 = \lambda \sqcup B_1$, but also a zero
when \( y_2q^2 \) = weight of an inner corner of the partition \( \nu^1 \). The latter zero precisely cancels out the pole produced by the first row of (5.29), so we can repeat our argument to conclude that the variables in \( B_{t-1} \) correspond to removing a strip:

\[
\nu^1 = \nu^2 \cup \text{(vertical strip } B_2) \]

which does not share a common vertical wall with \( B_1 \). Iterating this argument shows that integrating over the various remaining variables \( y_3, \ldots, y_t \) amounts to removing vertical strips from \( \lambda \), no two next to each other. We obtain:

\[
\text{RHS of (4.55)} \bigg|_{\lambda} = \sum_{\text{decompositions as in (5.25)}} \frac{M(..., z_{1a}, \ldots)}{\# B_1, \ldots, B_t} \prod_{1 \leq s < s' \leq t} \frac{(1 - \frac{w}{z}) \delta_{\text{col}} z \delta_{\text{col}} z'}{(1 - \frac{w}{z}) \delta_{\text{col}} z \delta_{\text{col}} z'} \prod_{z \in B_s} \left( \frac{z}{\chi_{\mu^{s-1}}} \tau_+ (z) \right) \text{z-weight of \( s \)-th strip if } z \in B_s
\]

In the formula above, the variables \( z \in B_s \), \( \forall s \in \{1, \ldots, t\} \). If we let \( \mu = \nu_t \), hence:

\[
\nu_{s-1} = \mu \sqcup B_s \sqcup \ldots \sqcup B_t
\]

for all \( s \), then the expression above becomes:

\[
\text{RHS of (4.55)} \bigg|_{\lambda} = \sum_{\mu \text{ such that } \lambda \setminus \mu = B_1 \sqcup \ldots \sqcup B_t} \frac{M(\lambda \setminus \mu)}{\# B_1, \ldots, B_t} \prod_{\square \in \lambda \setminus \mu} \left( \frac{1}{q - \frac{\chi_{\square}}{\chi_{\lambda}}} \right) \prod_{\square \in \lambda \setminus \mu} \left( \frac{\chi_{\square}}{\chi_{\mu}} \right) \tau_+ (\chi_{\square}) \tag{5.30}
\]

Besides \( \# B_1, \ldots, B_t \), the only difference between formulas (5.23) and (5.30) is that:

- In (5.23), \( \lambda \setminus \mu \) is decomposed into strips \( S_1, \ldots, S_t \) with no common edge
- In (5.30), each summand corresponds to a decomposition of \( \lambda \setminus \mu \) into vertical strips \( B_1, \ldots, B_t \) which may have a horizontal common edge, but no vertical common edges (the factors in the numerator of the first line of (5.29) kill poles corresponding to vertical strips \( B_t \) which may share vertical common edges).

It is clear that any skew diagram \( \lambda \setminus \mu \) can be decomposed at most once as in the first bullet. Meanwhile, any decomposition as in the second bullet can only be a refinement of a decomposition in the first bullet (meaning that every \( S_\alpha \) splits up into several of the \( B_i \)'s). Therefore, the fact that (5.23) equals (5.30) (which would complete the proof of the Proposition) follows from the fact that for any collection of vertical strips \( S_1, \ldots, S_t \) with no common edge, we have the equality:

\[
\frac{B_{1 \sqcup \ldots \sqcup B_t}}{\# B_{1 \sqcup \ldots \sqcup B_t}} = 1 \quad \tag{5.31}
\]

with the understanding that the decomposition \( \sqcup B_t \) is a refinement of the decomposition \( \sqcup S_\alpha \). Our algorithm prescribes that for any first chosen starting variable
z_{i a}, a box of color \( i \) must be at the bottom of the vertical strip \( B_1 \), which in turn must be at the top of the vertical strip \( S_a \) for some \( a \in \{1, \ldots, k\} \). Therefore:

\[
S_a = S'_a \sqcup_i B_1
\]

where the symbol \( \sqcup_i \) means that the vertical strip \( S'_a \) is directly below \( B_1 \), and the bottom-most box of \( B_1 \) has color \( \equiv i \mod n \). Therefore, the LHS of (5.31) equals:

\[
\frac{1}{g} \sum_{1 \leq \sigma \leq t} \sum_{S_a = S'_a \sqcup_i B_1} = \frac{1}{\# B_2 \sqcup \ldots \sqcup B_t} \tag{5.32}
\]

where \( g \) is the number of boxes of color \( \equiv i \mod n \) in the collection \( S_1 \sqcup \ldots \sqcup S_t \). By the induction hypothesis, all inner sums are equal to 1. Since the number of outer sums is precisely equal to \( g \), we conclude that the entire (5.32) equals 1, as desired.

\[\Box\]

References

[1] Braverman A., Finkelberg M. Finite difference quantum Toda lattice via equivariant K−theory, Transform. Groups 10 (2005), no. 3-4, 363-386
[2] Braverman A., Finkelberg M., Gaitsgory D. Uhlenbeck spaces via affine Lie algebras, The unity of mathematics, Progr. Math., 244, Birkhauser Boston, Boston, MA, 2006, 17–135
[3] Carlsson E., Okounkov A. Ets and Vertex Operators, Duke Math. J. 161 (2012), no. 9, 1797-1815
[4] Ding J., Frenkel I. Isomorphism of two realizations of quantum affine algebra \( U_q(\widehat{\mathfrak{g}}_n) \), Comm. Math. Phys. 156 (1993), no. 2, 277-300
[5] Feigin B., Finkelberg M., Neguț A., Rybnikov L. Yangians and cohomology rings of Laumon spaces Selecta Math. (N.S.) 17 (2011), no. 3, 573-607
[6] Feigin B., Odesskii A. Quantized moduli spaces of the bundles on the elliptic curve and their applications, Integrable structures of exactly solvable two-dimensional models of quantum field theory (Kiev, 2000), 123-137, NATO Sci. Ser. II Math. Phys. Chem., 35, Kluwer Acad. Publ., Dordrecht, 2001
[7] Finkelberg M., Rybnikov L. Quantization of Drinfeld Zastava in type A, Alg. Geom. Vol 9 (2010), doi: 10.4171/JEMS/432
[8] Gou L., Molev A. Representations of twisted \( q \)−Yangians, Selecta Math. 16 (2010), 439-499
[9] Nakajima H. Lectures on Hilbert schemes of points on surfaces, University Lecture Series, 1999, Vol 18, 132 pp
[10] Neguț A. Affine Laumon spaces and the Calogero-Moser integrable system, arxiv:1112.1756
[11] Neguț A. Quantum Algebras and Cyclic Quiver Varieties, PhD thesis, Columbia University, 2015, arxiv:math/1504.06525
[12] Neguț A. Quantum toroidal and shuffle algebras, arxiv:1302.0202
[13] Neguț A. Moduli of flags of sheaves and their K−theory, Algebraic Geometry, 2015, Vol 2, pp 19-43
[14] Tsybulevski A. Quantum affine Gelfand-Tsetlin bases and quantum toroidal algebra via K−theory of affine Laumon spaces, Selecta Math. (N.S.) 16 (2010), no. 2, 173-200

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