Viewing buoyant force as an application of principle of minimum potential energy

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Abstract. Principle of minimum potential energy (PMPE) as one of the fundamental concept in physics and engineering will be used in this work to explain the phenomenon of buoyancy, why object with lower mass density can float on fluid with higher mass density, while object with higher mass density can sink in fluid with lower mass density. Before that some examples in mechanics, e.g. object with different vertical position, object on incline, object with spring moving in vertical direction, and final condition of seesaw-like system, will be discussed to show the consequence of PMPE. Then, in the system of solid object and fluid, including the surrounding fluid with the object and treat them as a whole object or system we can easily show that the floating or sinking phenomena are simply a state where system potential energy is minimum (both object and fluid). For simplicity, the theoretical demonstration will only use the form of block, where the other form, e.g. cylinder, sphere, cone, irregular form, are the matter of geometry difficulty only. Students engagement to the topic has not been performed but the instrument is proposed for future work.

1. Introduction

One of the fundamental concept used in physics and engineering is the principle of minimum potential energy (PMPE), where a system tends to have minimum potential energy. PMPE has been widely used in solid with elastic property, as in elastodynamics for a cantilevered beam (Tabarrok & Assamoi, 1987), in a modified couple stress theory for Bernoulli-Euler beam model (Park & Gao, 2006), in derivation of beam equation for bending stiffness rig (Komperød, 2018), and in a non-classical Kirchhoff rod model for equilibrium of a helical rod and buckling of a straight rod (Zhang & Gao, 2019). It also applicable in solid-liquid interaction as in derivation of equation of static mantle density distribution on Earth structure (Yun, 2020) an even in discrete solids as in rising of intruder in Brazil-nut effect (Viridi et al., 2015), and in a granular-based system of solid and liquid (Muliyati et al., 2019). Unfortunately, in elementary physics this concept has not been stressed clearly, even though it has many application, e.g. in mechanics, which we will discuss them in brief before go to the fluid part. In this work we try to explain the buoyant force from the point of view of PMPE, where we should include the surrounding fluid with the object in applying the concept.
2. Methode

Theoretical background, examples of system and the formulation of buoyancy from PMPE point of view are presented in this part.

2.1. Concepts

Potential energy is the main concept for PMPE, where we will deal only with gravitational potential energy

\[ U_G = m g (y - y_0) \]

and elastic potential energy of a spring

\[ U_S = \frac{1}{2} k (y - y_0)^2, \]

where both concepts require a reference point \( y_0 \). In Eqn. (1) \( m \) is mass and \( g \) is gravitational acceleration, while in Eqn. (2) \( k \) is spring constant. There is also the third potential energy related to contact between two solid objects, e.g. soft-sphere method for granular materials (Cundall & Strack, 1979)

\[ U_M = \frac{1}{2} \kappa \zeta^2, \]

where \( \zeta \) is overlap between objects with value of elastic constant \( \kappa \) is very large compared to previous spring elastic constant \( k \).

2.2. Potential energy of some systems

In this part we will discuss about some systems and their potential energy function in showing that the stable position is when the potential energy is minimum.

\[ 0 \quad 20 \quad 40 \quad 60 \quad 80 \quad 100 \]
\[ 0 \quad 20 \quad 40 \quad 60 \quad 80 \quad 100 \]

**Figure 1.** System of a mass in different vertical position with \( m = 0.1 \) kg and \( y_0 = 0 \).

The first system is a mass \( m \) with several different vertical position and at the bottom it encounters a solid horizontal surface as shown in Fig. 1 (left). Total potential energy \( U \) as function of vertical position \( y \) is given in Fig. 1 (right), which consists of two part. The first part is linear curve is for \( U_G \) from Eqn. (1) and the second part is vertical curve along vertical axis for \( U_M \) from Eqn. (3). The last part actually also a linear curve but it seems vertical since \( \kappa \gg k \). The mass \( m \) will be stable at the minimum position \( y = 0 \).

An object on a frictionless incline is the second system. Instead of vertical position \( y \) we use coordinate on the incline \( s \), where

\[ y = s \sin \theta, \]

where the system and the function of \( U = U(s) \) is given in Fig. 2, which is similar to Fig. 1.
As the third system, we choose system of two masses connected with an ideal string through an ideal pulley hung on ceiling as shown in Fig. 3 (left). The string will set a constraint to vertical position of both masses

\[(y_p - y_1) + (y_p - y_2) = 2l, \quad (5)\]

where \(y_1\) and \(y_2\) are vertical positions for \(m_1\) and \(m_2\), respectively. Both \(y_1\) and \(y_2\) are better measured relatively from reference point \(y_0\) as shown in Fig. 3 (left), which gives the \(U = U(y)\) in Fig. 3 (right). If we choose \(y_p = y_0 + l\), then we can have from Eqn. (5) that

\[y_0 - y_1 = -y_0 + y_2, \quad (6)\]

or we can change the coordinate \(+y\) to the top for left side and \(+y\) to the bottom for the right side, which give us the equation in Fig. 3 (left).

As the mass \(m_1\) goes up (raising gravitational potential energy), the mass \(m_2\) goes down (reducing gravitational potential energy), where total of the potential energy will be reduced and at the end the system stops. There can be two possibility, the first is when \(m_1\) reaches the pulley and the second is when \(m_2\) hits the floor. These final condition is represented by vertical curve on the right side of Fig. 3 (right) according to Eqn. (3) for large value of \(\kappa\).
Figure 4. System of two masses connected through a seesaw with \( m_1 = 0.1 \text{ kg}, \ m_2 = 0.2 \text{ kg}, \ l_1 = l_2 = 1 \text{ m}, \ y_0 = 5 \text{ m}, \) where red dotted-dashed line for \( m_1, \) green dashed line for \( m_2, \) and solid blue line for total of \( m_1 \) and \( m_2. \)

The fourth system is two masses on the both sides of a seesaw as shown in Fig. 4 (left). We take that height of the pivot or origin of the coordinate system is \( y_0 = 5 \text{ m}, \) so that both masses can rotate freely with \( \theta \) from \( 0^\circ \) to \( 360^\circ. \) Fig. 4 (right) shows that minimum and maximum points of \( m_1 \) and \( m_2 \) is always in different angle, but the minimum of total potential energy is determined by \( m_2 \) since \( m_2 > m_1. \)

Figure 5. System of a mass attached vertically to a spring with \( m = 0.1 \text{ kg}, \ k = 100 \text{ N/m}, \ y_0 = 0.1 \text{ m}, \) where red dotted-dashed line for gravitation potential energy \( U_G, \) green dashed line for elastic potential energy \( U_S, \) due to spring, and solid blue line for total potential energy \( U = U_G + U_S. \)

Mass \( m \) attached vertically on a spring with constant \( k \) hung on the ceiling is the fifth system, which is given in Fig. 5 (left). In this system elastic potential \( U_S \) is dominating compared to gravitational potential \( U_G. \)
2.3. Immersed object and its surrounding fluid

As focus in this work we choose a system of immersed object and fluid to shown that PMPE still holds. Illustration of the system is shown in Fig. 6. Immersed object uses subscript $io$, while the surrounding fluid in container uses subscript $sf$.

![Diagram of immersed object and its surrounding fluid](image)

**Figure 6.** System of fluid and an immersed object.

Volume of immersed object is

$$V_{io} = L_{io} \cdot W_{io} \cdot H_{io}$$  \hspace{1cm} (7)

and the surrounding fluid

$$V_{sf} = L_{sf} \cdot W_{sf} \cdot H_{sf} \cdot$$  \hspace{1cm} (8)

A 3-d view of the system in Fig. 6 is shown in Fig. 7, where block of surrounding fluid and immersed object are drawn in different color, blue and red.

![Diagram of system in Fig. 6](image)

**Figure 7.** Block of surrounding fluid $sf$ and immersed object $im$ (top) and the system before and after the object is immersed in the surrounding fluid (bottom).

Base area of immersed object is

$$A_{io} = L_{io} \cdot W_{io},$$  \hspace{1cm} (9)

Where base area of initial surrounding fluid is

$$A_{sf} = L_{sf} \cdot W_{sf}. \hspace{1cm} (10)$$

Using Eqns. (9) and (10) we can have the area of surrounding fluid when the object starts to immerse in it
\[ A'_q = A_q - A_{wo}. \]  
(11)

By setting \( y \) coordinate at the bottom surface of immersed object, the object will have immersed volume

\[ V'_m = A_{wo} (H_q - y), \]  
(12)

which adds

\[ \Delta H_q = \frac{V'_m}{A_q}, \]  
(13)

to the surrounding fluid. Then height of surrounding fluid after it starts to embrace the object can be calculated

\[ H_q + \Delta H_q, \]  
(14)

which holds only for \( H_{wo} - H_{io} \leq y \leq H_q \). And the height of surrounding fluid will be

\[
H = \begin{cases} 
H_q, & H_q \leq y, \\
H_q + \frac{H_q}{A_q} \left( H_q - y \right), & H_{wo} - H_{io} \leq y \leq H_q, \\
H_q + \frac{A_{wo}}{A_q} H_{wo}, & 0 \leq y \leq H_q - H_{wo}, 
\end{cases} 
\]  
(15)

that holds for all value of \( y \) or vertical coordinate of the bottom surface of the immersed object.

3. Results and discussion

The system of fluid and immersed object is represented using an aquarium with square base of 13 cm \( \times \) 13 cm and height of 17.5 cm. Initial and final heights of water in the aquarium is 10.2 cm and 11.8 cm, respectively. All quantities with length dimension are measured with measuring tape with smallest division of 1 mm.

A red brick with mass of 400 g and dimension of 9.6 cm \( \times \) 7.2 cm \( \times \) 4 cm is used as the immersed object. From calculation we have the volume of 276.48 ml, while the measurement using a 500 ml laboratory measuring cup with smallest division of 50 ml gives (275 \( \pm \) 12.5) ml.
Table 1. Parameters and their value from experiment.

| Parameters | Value | Unit |
|------------|-------|------|
| $L_{io}$   | 9.6   | cm   |
| $W_{io}$   | 7.2   | cm   |
| $H_{io}$   | 4     | cm   |
| $L_{sf}$   | 13    | cm   |
| $W_{sf}$   | 13    | cm   |
| $H_{sf}$   | 17.5  | cm   |

Using Eqns. (9), (10), (15), (1), (2), and values in Table 1, we can have there results that the red brick at the bottom of the aquarium is simply the state where total potential energy of the system (surrounding fluid and immersed object) is minimum. It has been successfully shown that the PMPE do confirm the final state of the immersed object, that it should be stay at the bottom of the container. For future work, we will try it for different immersed objects and fluid with different mass density to shown that PMPE will always work.

4. Conclusion

It has been shown from this work that buoyant force is simply a consequence of principle of minimum potential energy with an aquarium filled with water and a red brick immersed in it.

Acknowledgments

Authors wishing to acknowledge assistance or encouragement from colleagues, special work by technical staff and financial support from organizations.

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