ON SERINI’S RELATIVISTIC THEOREM

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Abstract. I expound here in a more detailed way a proof of an important Serini’s theorem, which I have already sketched in a previous Note. Two related questions are briefly discussed.

1. - Introduction
In the Note “On the beams of wavy metric tensors” [1] I sketched a simple and concise proof of an important Serini’s theorem (1918). Serini’s original demonstration was subsequently generalized by Einstein and Pauli (1943) and by Lichnerowicz (1946) (see [2]). Independently, a different proof was given by Fock [2bis].

I wish now to expound again, in a more detailed way, my proof of the above theorem, and to solve an apparent contradiction with the main result of the paper “Regular solutions of Schwarzschild problem” [3]. The Appendices contain some remarks concerning the so-called gravitational waves.

2. - The theorem
Serini’s theorem affirms the non-existence of regular (i.e. without singularities) time independent solutions of Einstein field equations for the perfectly empty space, $R_{jk} = 0$, ($j, k = 0, 1, 2, 3$), that become pseudo-Euclidean at spatial infinity. The unique time independent solution of $R_{jk} = 0$ is the trivial solution $g_{jk} = \text{const}$. Thus, in the time independent case, $R_{jk} = 0$ imply $R_{jklm} = 0$, the vanishing of Riemann curvature tensor of the spacetime manifold.

3. - Proof of the theorem
a) As it was remarked by Hilbert [4], we can always choose a Gaussian normal (or synchronous [5]) system of coordinates for the solution of any relativistic problem. In their treatise [5] Landau and Lifchitz explain in a detailed way the interesting properties of this reference system. I shall follow their treatment, but with some slight differences of notations.

A Gaussian normal (or synchronous) reference frame can be defined by the conditions:

\begin{equation}
  \begin{align*}
    g_{00} &= 1, \\
    g_{0\alpha} &= 0, \\
    (\alpha &= 1, 2, 3); \\
  \end{align*}
\end{equation}

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accordingly:

\[ (2) \quad ds^2 = (dx^0)^2 + g_{\alpha\beta}(x, x^0)dx^\alpha dx^\beta; \]
putting \( g_{\alpha\beta} \equiv -h_{\alpha\beta} \), we have:

\[ (2') \quad ds^2 = (dx^0)^2 - h_{\alpha\beta}(x, x^0)dx^\alpha dx^\beta. \]

It is easy to see that the time lines \textit{coincide} with the spacetime geodesics. Henceforth, all the operations of index displacement and covariant derivative concern \textit{only} the three-dimensional space with the metric tensor \( h_{\alpha\beta} \). If

\[ (3) \quad \kappa_{\alpha\beta} := \frac{\partial h_{\alpha\beta}}{\partial x^0}, \]
the components of the Ricci-Einstein tensor \( R_{lm} \), \((l, m = 0, 1, 2, 3)\), are:

\[ (4) \quad R_{00} = -\frac{1}{2} \frac{\partial \kappa^\beta_\alpha}{\partial x^0} - \frac{1}{4} \kappa^\beta_\alpha \kappa^\alpha_\beta, \]
\[ (4') \quad R_{0\alpha} = \frac{1}{2} \left( \kappa^\beta_\alpha - \kappa^\beta_{\beta\alpha} \right), \]
\[ (4'') \quad R_{\alpha\beta} = \frac{1}{2} \frac{\partial \kappa_{\alpha\beta}}{\partial x^0} + \frac{1}{4} \left( \kappa_{\alpha\beta} \kappa^\gamma_\gamma - 2 \kappa^\gamma_\alpha \kappa_{\beta\gamma} \right) + P_{\alpha\beta}, \]
where \( P_{\alpha\beta} \) is the three-dimensional analogue of \( R_{lm} \).

The Riemann curvature tensor \( R_{lmrs} \) is given by:

\[ (5) \quad R_{\alpha\beta\gamma\delta} = P_{\alpha\beta\gamma\delta} + \frac{1}{4} \left( \kappa_{\alpha\gamma} \kappa_{\beta\delta} - \kappa_{\alpha\beta} \kappa_{\gamma\delta} \right), \]
\[ (5') \quad R_{00\alpha\beta} = \frac{1}{2} \left( \kappa_{0\alpha\beta} - \kappa_{0\beta\alpha} \right), \]
\[ (5'') \quad R_{0\alpha0\beta} = -\frac{1}{2} \frac{\partial \kappa_{\alpha\beta}}{\partial x^0} + \frac{1}{4} \kappa_{\alpha\gamma} \kappa^\gamma_\beta, \]
where \( P_{\alpha\beta\gamma\delta} \) is the three-dimensional analogue of \( R_{lmrs} \).

\( b) \) For a \textit{time-independent} metric tensor \( h_{\alpha\beta}(x) \), we have:

\[ (6) \quad R_{00} = R_{0\alpha} = 0, \]
\[ (6') \quad R_{\alpha\beta} = P_{\alpha\beta}; \]
\[ (7) \quad R_{\alpha\beta\gamma\delta} = P_{\alpha\beta\gamma\delta}, \]
\[ (7') \quad R_{00\alpha\beta} = R_{0\alpha0\beta} = 0; \]
now, it is (see e.g. Fock [2bis, App. G]):

\[ P_{\alpha\beta\gamma\delta} = \left( \mathbf{P}_{\rho\sigma} - \frac{1}{2} h_{\rho\sigma} \mathbf{P} \right) E_{\rho\alpha} E_{\sigma\gamma}, \]

where

\[ (8') \quad E_{\alpha\beta\gamma} := h^{1/2} e_{\alpha\beta\gamma}, \]

if \( h \equiv \det \| h_{\alpha\beta} \| \), and \( e_{\alpha\beta\gamma} \) is a system of antisymmetric quantities with \( e_{123} = 1. \)

c) For a perfectly empty space, we have:

\[ R_{lm} = 0, \]

and therefore, as an immediate consequence of eqs. (6), (6'), (7), (7'), (8), (8'):

\[ R_{lmrs} = 0, \quad \text{q.e.d.;} \]

the unique time-independent solution of \( R_{lm} = 0 \) is \( g_{lm} = \text{const} \). This result is obviously quite intuitive, because the curvature of spacetime is created by matter, and if the matter is absent . . . –

(Remark that the above proof does not require the hypothesis that \( g_{jk} \) is pseudo-Euclidean at spatial infinity.)

4. - An apparent contradiction

At the first sight, it seems that Serini’s theorem denies, in particular, the existence of those regular solutions of Schwarzschild problem – i.e., of the problem to determine the gravitational field of a point mass at rest – which have been exhibited in paper [3]. However, the contradiction is only apparent: indeed, all forms of solution of Schwarzschild problem are in reality relative to a matter tensor \( T_{jk} \) different from zero, and precisely: to a matter tensor involving a Dirac delta-distribution [6], or to the matter tensor of the limiting case of a concentrated mass, according to Fock’s procedure [7], which was also followed in [3].

**APPENDIX A**

In sect.3. of paper [1] I have given an intuitive demonstration of the physical unreality of the gravitational waves (GW’s). I have considered there a spatially limited train \( L \) of running (hypothetical) GW’s – the source of which is at spatial infinity –, satisfying exactly the equations \( R_{jk} = 0 \). (It was implicitly assumed that the \( g_{jk} \)’s of \( L \) do not possess any singularity of any kind whatever.) Then, the proof rested on a characteristic property of general relativity (GR), that distinguishes it from Maxwell theory: the absence of any limitation to the velocities of the reference frames. Thus, we can ideally consider an observer \( \Omega \), who moves together with our train \( L \). For \( \Omega \) the metric tensor of \( L \) is time independent; consequently, Serini’s
theorem tells us that its Riemann curvature tensor is zero: the GW’s of \( L \) are mere coordinate undulations.

Of course, this demonstration of the \textit{physical} non-existence of GW’s is a little bold. But there exist absolutely trenchant proofs, as e.g. the proofs of the non-existence, in the \textit{exact} formulation of GR, of “mechanisms” capable of generating GW’s, \textit{in primis} the \textbf{fact} that the purely gravitational motions of bodies are \textit{geodesic} \cite{3}. Quite generally, even the \textit{non}-purely gravitational motions cannot generate GW’s, see \cite{9}.

A last remark. One could object that – as a matter of fact – there are wavy solutions of Einstein equations \( R_{jk} = 0 \), the curvature tensor of which is different from zero. \textbf{Answer}: i) all solutions of \( R_{jk} = 0 \) do not possess an energy-momentum endowed with a \textit{true} tensor character: accordingly, they are unphysical objects; ii) any undulatory character can be obliterated by a sequence of suitable coordinate transformations; iii) the \textit{mathematical} existence of wavy solutions of \( R_{jk} = 0 \), having a curvature tensor \( R_{jklm} \neq 0 \), can be easily understood: let \( W \) be a solution of this kind; it owes its computative existence to a given gravity source \( S \) (explicitly or implicitly postulated) at a very large distance from an ideal observer \cite{10}. Of course, \( W \) retains “memory” of the spacetime curvature produced by \( S \) – for a detailed and analytical corroboration of this statement, see e.g. the treatment given by Fock in Ch. VII of his book \cite{2bis}; on the other hand, no motion of a gravity source, no cataclysmic disruption of it can give origin to GW’s, as it has been proved.

\textbf{APPENDIX B}

The analogy between Maxwell e.m. theory and Einstein general relativity is a misleading analogy. This is, in particular, clarified also by the intuitive proof of the non-existence of physical GW’s, which I have recalled in App.A. In his splendid \textit{Autobiographisches} \cite{11}, at page 53, Einstein emphasized the following paradox of classical time conception, which was discovered by him when he was only 16 years old: “Wenn ich einem Lichtstrahl nacheile mit der Geschwindigkeit \( c \) (Lichtgeschwindigkeit im Vacuum), so sollte ich einen solchen Lichtstrahl als ruhendes, räumlich oszillatorisches elektromagnetisches Feld wahrnehmen. So etwas scheint es aber nicht zu geben, weder auf Grund der Erfahrung noch gemäß den Maxwell’schen Gleichungen.” In the English translation by P.A. Schilpp: “If I pursue a beam of light with the velocity \( c \) (velocity of light in a vacuum), I should observe such a beam of light as a spatially oscillatory electromagnetic field at rest. However, there seems to be no such thing, whether on the basis of experience or according to Maxwell’s equations.” Now, in \textit{general} relativity the paradoxical character of the above consideration \textit{disappears} if in particular the beam of light is substituted by a beam of (hypothetical) GW’s: indeed, in GR there is no limitation to the velocity of the reference frames.

\textbf{References}

[1] A. Loinger, \textit{On Black Holes and Gravitational Waves} (La Goliardica Pavese, Pavia) 2002, p.82; also on \textit{arXiv:physics/0102011} (February 6th, 2001).
[2] W. Pauli, *Teoria della Relatività* (Boringhieri, Torino) 1958, p.274 of sect. 62; C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford) 1972, p.441; and the literature quoted there.

[2bis] V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon Press, Oxford, etc.) 1964, p.209 of sect.56.

[3] A. Loinger, *On Black Holes and Gravitational Waves* (La Goliardica Pavese, Pavia) 2002, p.26; also on arXiv:physics/0104064 (April 20th, 2001).

[4] D. Hilbert, *Mathem. Annalen*, 92 (1924) 1.

[5] L. Landau et E. Lifchitz, *Théorie du Champ* (Édition Mir, Moscou) 1966, sects. 99 and 110.

[6] Cf. L. Infeld, *Acta Phys. Polonica*, 13 (1954) 187. – The Newtonian analogue: we can affirm either that $M/r$ is a solution, *singular at* $r = 0$, of $\nabla^2 \varphi(r) = 0$, or that $M/r$ is solution of $\nabla^2 \varphi(r) = -4\pi M \delta(r)$, with Dirac’s $\delta(r) \equiv \delta(x)\delta(y)\delta(z)$. In GR the delta-distributions must be handled with some caution, see L. Infeld and J. Plebanski, *Motion and Relativity* (Pergamon Press, Oxford, etc.), 1960.

[7] V. Fock, 2bis, sect.57.

[8] A. Loinger, *Nuovo Cimento B*, 115 (2000) 679; also in *On Black Holes and Gravitational Waves* (La Goliardica Pavese, Pavia) 2002, p.76; also in arXiv:astro-ph/0003230 (March 16th, 2000).

[9] See: A. Loinger, arXiv:physics/0502089 v1 (February 16th, 2005) – to be published in *Spacetime & Substance*; and the literature quoted there sub [3], [5], [6], [10], [11], [14].

[10] The controversial limiting case of the *plane* GW’s, which has given origin to several learned papers – see e.g. H. Bondi, F.A.E. Pirani, and I. Robinson, *Proc. Roy. Soc.*, A, 251 (1959) 519 –, does not represent an exception, from the conceptual standpoint.

[11] See *Albert Einstein: Philosopher-Scientist*, ed. by P.A. Schilpp (Tudor Publ. Company, New York) 1949.