Causal Quantum Gravity

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Abstract

I discuss some issues of perturbative quantum gravity, namely of a theory of self-interacting massless spin-2 quantum gauge fields, the gravitons, on flat space-time, in the framework of causal perturbation theory. The central aspects of this approach lie in the construction of the scattering matrix by means of causality and Poincaré covariance and in the analysis of the gauge structure of the theory. For this purpose, two main tools will be used: the Epstein–Glaser inductive and causal construction of the perturbation series for the scattering matrix and the concept of perturbative operator quantum gauge invariance borrowed from non-Abelian quantum gauge theories. The first method deals with the ultraviolet problem of quantum gravity and the second one ensures gauge invariance at the quantum level, formulated by means of a gauge charge, in each order of perturbation theory. The gauge charge leads to a characterization of the physical subspace of the graviton Fock space. Aspects of quantum gravity coupled to scalar matter fields are also discussed.
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1 Introduction

The central aspect of this work is the construction of the S-matrix for gravity by means of causality in the quantum field theoretical (QFT) framework. This idea goes back to Stückelberg, Bogoliubov and Shirkov and the program was carried out successfully by Epstein and Glaser [1, 2] for scalar field theories and subsequently applied to QED by Scharf [3], to non-Abelian gauge theories by Dütsch et al. [4, 5, 6] and to quantum gravity (QG), (by which we mean a QFT of self-interacting massless spin-2 quantum gauge fields on flat space-time), by Schorn [7, 8]. For this non-geometrical approach, see [9, 10, 11]. For our purpose, namely the implementation of QG as a Poincaré covariant local quantum field theory with a considerable gauge arbitrariness, two main tools will be used: the Epstein-Glaser inductive construction of the perturbation series for the S-matrix with the related causal renormalization scheme [1, 3] and the concept of perturbative operator quantum gauge invariance [6, 12]. A detailed exposition of what follows can be found in [13, 14, 15, 16].

2 Causal Perturbation Theory

In this section we give a concise review of the causal approach to QFT. We consider the S-matrix, being a formal power series in the coupling constant, as a sum of smeared operator-valued distributions of the following form [1, 2, 3]

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n T_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n).$$

(2.1)

The Schwartz test function $g \in S(\mathbb{R}^4)$ plays the rôle of adiabatically switching the interaction and provides a natural infrared cutoff in the long-range part of the interaction.

To establish the existence of the adiabatic limit $g \to 1$ in theories involving self-coupled massless particles, like QG, may be problematic. This aspect will not be considered here.

The n-point operator-valued distributions $T_n$ are well-defined renormalized time-ordered products and can be expressed in terms of Wick monomials of free fields. They are constructed inductively from the first order $T_1(x)$, which plays the rôle of the usual interaction Lagrangian in terms of free fields, by means of Poincaré covariance and causality. The latter, if correctly incorporated, leads directly to the renormalized perturbation series for the S-matrix which is UV-finite in every order.

The construction of $T_n$ requires some care: if it were simply given by the
usual time-ordering

\[ T_n(x_1, \ldots, x_n) = \mathcal{T}\{T_1(x_1) \ldots T_1(x_n)\} \]
\[ = \sum_{\pi \in \mathcal{S}_n} \Theta(x_{\pi(1)}^0 - x_{\pi(2)}^0) \ldots \Theta(x_{\pi(n-1)}^0 - x_{\pi(n)}^0) \]
\[ \times T_1(x_{\pi(1)}) \ldots T_1(x_{\pi(n)}), \] \hspace{1cm} (2.2)

then UV-divergences would appear. If the arguments \( x_1, \ldots, x_n \) are all time-ordered, \( i.e. \) if we have \( x_1^0 > x_2^0 > \ldots > x_n^0 \), then \( T_n \) is rigorously given by \( T_n(x_1, \ldots, x_n) = T_1(x_1) \ldots T_1(x_n) \). Since \( T_n \) has to be totally symmetric in \( x_1, \ldots, x_n \), we so obtain \( T_n \) everywhere except for the complete diagonal \( \Delta_n = \{ x_1 = x_2 = \ldots = x_n \} \), \( i.e. \) except for the coincident point in configuration space. The correct treatment of this point constitutes the key to control the UV-behaviour of the \( n \)-point distributions. Indeed, products of Feynman propagators with coincident arguments

\[ \Pi(x - y) = D_m^p(x - y) \cdot D_m^p(x - y) = ? \]
\[ \hat{\Pi}(p) \sim \int d^4k \frac{1}{(p - k)^2 - m^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon} \] \hspace{1cm} (2.3)

are the origin of the UV-divergences in loop graphs, because time-ordering cannot be done simply by multiplying (singular) distributions by discontinuous \( \Theta \)-distributions, since this procedure is usually ill-defined.

The distributions must be carefully split into a retarded and an advanced part for the \( T_n \) to be well-defined and finite.

Let us illustrate how the inductive construction works by means of an example in which \( T_2(x_1, x_2) \) is constructed for a massive scalar field \( \varphi \) with a \( \varphi^3 \)-coupling.

We define a QFT by giving the equation of motion of the free quantum field, the covariant commutator rule and the interaction Lagrangian \( T_1 \) with coupling strength \( g \)

\[ (\square + m^2) \varphi(x) = 0, \quad [\varphi(x_1), \varphi(x_2)] = -i D_m(x_1 - x_2), \quad T_1(x) = i g :\varphi(x)^3: ; \] \hspace{1cm} (2.4)

where \( D_m \) is the massive Jordan–Pauli distribution

\[ D_m(x) = i (2\pi)^3 \int d^4p \delta(p^2 - m^2) \text{sgn}(p^0) e^{-ip \cdot x}. \] \hspace{1cm} (2.5)

Causality of the \( S \)-matrix means

\[ S(g_1 + g_2) = S(g_2)S(g_1) \quad \text{for} \quad \text{supp}(g_1) < \text{supp}(g_2). \] \hspace{1cm} (2.6)

The notation \( < \) in the support condition means more precisely: \( \text{supp}(g_1) \cap \{ \text{supp}(g_2) + V^+ \} = \emptyset \).
Translated in terms of $T_2(x_1, x_2)$, the condition of causality becomes

\[
T_2(x_1, x_2) = \begin{cases} 
T_1(x_1)T_1(x_2) & \text{for } x_1 > x_2, \\
T_1(x_2)T_1(x_1) & \text{for } x_2 > x_1.
\end{cases}
\] (2.7)

Clearly, difficulties arise for $x_1 = x_2$.

Following the causal construction of Epstein and Glaser, we define the auxiliary distributions

\[
R'_2(x_1, x_2) := -T_1(x_2)T_1(x_1), \quad A'_2(x_1, x_2) := -T_1(x_1)T_2(x_2)
\]

\[D_2(x_1, x_2) := R'_2(x_1, x_2) - A'_2(x_1, x_2),\] (2.8)

and carry out all possible contractions in $D_2$ using Wick’s lemma, leading to

\[D_2(x_1, x_2) = \sum_{k=1}^{3} : O_k(x_1, x_2) : d_k^2(x_1 - x_2).\] (2.9)

$O_k(x_1, x_2)$ represents a normally ordered product of free field operators and $d_k^2(x_1 - x_2)$ is a numerical distribution. Expanding the result we can identify tree, loop and vacuum graph contributions (no tadpoles appear), respectively

\[D_2(x_1, x_2) = + : \varphi(x_1)\varphi(x_1)\varphi(x_2)\varphi(x_2) : d_2^{(1)}(x_1 - x_2) +
\]

\[+ : \varphi(x_1)\varphi(x_2) : d_2^{(2)}(x_1 - x_2) + d_2^3(x_1 - x_2),\] (2.10)

where the numerical distributions are given by

\[d_2^{(1)}(x_1 - x_2) = 9ig^2[D_m^{(+)p}(x_1 - x_2) + D_m^{(-p)}(x_1 - x_2)] = 9ig^2D_m(x_1 - x_2),\]

\[d_2^{(2)}(x_1 - x_2) = 18g^2[D_m^{(+)p}(x_1 - x_2)^2 - D_m^{(-p)}(x_1 - x_2)^2],\]

\[d_2^3(x_1 - x_2) = -6ig^2[D_m^{(+)p}(x_1 - x_2)^3 + D_m^{(-p)}(x_1 - x_2)^3].\] (2.11)

In addition, we define

\[R_2(x_1, x_2) := -T_1(x_2)T_1(x_1) + T_2(x_1, x_2),\]

\[A_2(x_1, x_2) := -T_1(x_1)T_1(x_2) + T_2(x_1, x_2);\] (2.12)

so that

\[D_2(x_1, x_2) = R_2(x_1, x_2) - A_2(x_1, x_2).\] (2.13)

From this last equation, it follows that

\[T_2(x_1, x_2) = R_2(x_1, x_2) - R'_2(x_1, x_2).\] (2.14)

Now, the issue is how to compute $R_2$ without using its definition (since it contains the unknown $T_2$). This can be done by analyzing the support property
of $D_2$: the most important property of $D_2$ is causality, i.e. $\text{supp}(d_2^{[k]}(x)) \subseteq V^+(x) \cup V^-(x)$, with $x := x_1 - x_2$.

In order to obtain $T_2(x_1, x_2)$ we have to split the distribution $D_2$ into a retarded part, $R_2$, and an advanced part, $A_2$, with respect to the coincident point $x = 0$, so that $\text{supp}(R_2(x)) \subseteq V^+(x)$ and $\text{supp}(A_2(x)) \subseteq V^-(x)$.

This splitting of the numerical distribution $d_2^{[k]}(x)$ must be accomplished according to the correct singular order $\omega(d_2^{[k]})$, which agrees here with the usual power-counting degree of Feynman diagrams, and describes the behaviour of $d_2^{[k]}(x)$ near $x = 0$, or that of $d_2^{[k]}(p)$ in the limit $p \to \infty$.

If $\omega < 0$, then the splitting is trivial and agrees with the standard time-ordering and we recover the Feynman rules. If $\omega \geq 0$, then the splitting is non-trivial and non-unique

$$d_2^{[k]}(x) \longrightarrow r_2^{[k]}(x) + \sum_{|a|=0}^{\omega(d_2^{[k]})} C_a D^a \delta^{(4)}(x), \quad (2.15)$$

and the retarded part $r_2^{[k]}(x)$ is obtained in momentum space by means of a subtracted dispersion integral (thus recovering the relation between causality and dispersion relation) of the form

$$r_2^{[k]}(p) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{d_2^{[k]}(tp)}{(t-i0)^{\omega+1}} \bigg|_{t=0} \bigg|_{t=\infty}, \quad p \in V^+. \quad (2.16)$$

Eq. (2.15) contains a local ambiguity in the normalization: the $C_a$’s are undetermined finite normalization constants, which multiply terms with local support $\sim \delta^{(4)}(x_1 - x_2)$. This freedom in the normalization has to be restricted by further physical conditions, e.g. unitarity, Lorentz covariance, existence of the adiabatic limit and gauge invariance in the case of gauge theories or gravity.

Finally, $T_2$ is given by

$$T_2(x_1, x_2) = R_2(x_1, x_2) + T_1(x_2)T_1(x_1) = \sum_{k=0}^{3} \mathcal{O}_k(x_1, x_2) : t_2^{[k]}(x_1 - x_2)^{\text{tot}}, \quad (2.17)$$

with

$$t_2^{[k]}(p)^{\text{tot}} = t_2^{[k]}(p) + \sum_{|a|=0}^{\omega(d_2^{[k]})} \tilde{C}_a p^a. \quad (2.18)$$

Applying the described scheme to our example, we find for the operator-valued distribution $T_2(x_1, x_2)$, Eq. (2.17), the expression

$$T_2(x_1, x_2) = + : \varphi(x_1) \varphi(x_1) \varphi(x_1) \varphi(x_2) \varphi(x_2) \varphi(x_2) : (-g^2)
+ : \varphi(x_1) \varphi(x_1) \varphi(x_2) \varphi(x_2) : t_2^{[1]}(x_1 - x_2)^{\text{tot}}
+ : \varphi(x_1) \varphi(x_2) : t_2^{[2]}(x_1 - x_2)^{\text{tot}} + t_2^{[3]}(x_1 - x_2)^{\text{tot}}. \quad (2.19)$$
The first term represents the disconnected contribution coming, in Eq. (2.17), from $T_1(x_2)T_1(x_1)$. The distribution in the second term $I_{21}^{[3]}(x_1 - x_2)_{\text{tot}} = +9 i g^2 D_{\text{tot}}^F(x_1 - x_2)$ is the Feynman propagator for the tree graph contribution, whereas the loop distribution $I_{21}^{[2]}(x_1 - x_2)_{\text{tot}}$ is easily obtained in momentum space by means of Eq. (2.16) and reads

$$I_{21}^{[2]}(p)_{\text{tot}} = \frac{i}{2\pi} \frac{-9g^2\pi}{(2\pi)^4} \int_0^\infty ds \frac{\sqrt{s(s-q)}}{s^2(1-s+i0)} + c_0, \quad c_0 \in \mathbb{R}, \quad q = 4m^2/p^2. \quad (2.20)$$

The result of the evaluation of the above integral can be found in Sec. 8. Since $\omega(d_{21}^{[3]}) = 0$, the splitting is not unique and we must take the local normalization term $c_0$ into account.

We do not give here the expression for the vacuum graph contribution, the treatment of the latter can be found in Sec. 5 and in Sec. 8.

The inductive construction can be repeated for every order of perturbation theory, although the complexity increases. The most delicate step is the distribution splitting, which corresponds to a natural and mathematically well defined ultraviolet regularization in the usual terminology. The advantage of the causal scheme is that it leads directly to the renormalized perturbative expansion for the $S$-matrix without using a cutoff. It makes possible to compute finite amplitudes for various processes to a given order in the coupling constant and it does not rely on the Lagrangian approach.

3 Quantization of Gravity

3.1 From General Relativity to Quantum Gravity

Since we are interested in a quantum theory of Einstein’s general relativity, we start from the Hilbert–Einstein Lagrangian density $L_{HE}$ written in terms of the Goldberg variable $\tilde{g}^{\mu\nu} = \sqrt{-\tilde{g}} g^{\mu\nu}$ and we expand it into a power series in the coupling constant $\kappa^2 = 32 \pi G$, by introducing the graviton field $h^{\mu\nu}$ defined through $\kappa h^{\mu\nu} = \tilde{g}^{\mu\nu} - \eta^{\mu\nu}$, where $\eta^{\mu\nu} = \text{diag}(1,-1,-1,-1)$ is the flat space-time metric tensor

$$L_{HE} = -\frac{2}{\kappa^2} \sqrt{-g} g^{\mu\nu} R_{\mu\nu} = \sum_{j=0}^{\infty} \kappa^j L_{HE}^{(j)}. \quad (3.1)$$

$L_{HE}^{(j)}$ represents an interaction involving $j + 2$ gravitons. From this formulation of general relativity we extract the ingredients for the perturbative construction of causal QG.

We stress however the fact that we consider the classical Lagrangian density Eq. (3.1) only as a source of information about the fields, the couplings and the gauge which we work with. Causal perturbation theory does not rely on a quantum Lagrangian with interacting fields.
In a new approach, which has been proposed in [12], one constructs the first-order interaction essentially by the requirement of perturbative quantum gauge invariance (see Sec. 4).

By considering the Euler–Lagrange variation of $\mathcal{L}^{(0)}_{HE}$ from Eq. (3.1) in the Hilbert gauge $h^{\alpha\beta}(x),\beta = 0$ we obtain the equation of motion for the free graviton field $\Box h^{\alpha\beta}(x) = 0$.

### 3.2 Quantum Gravity as a Quantum Field Theory

In quantum gravity, we consider the free rank-2 quantum tensor field $h^{\mu\nu}(x)$, the graviton, which fulfils the free wave equation after having fixed the gauge. For the causal construction we need the commutation relation between free field operators at different space-time points and the first-order graviton self-coupling $T^h_1(x)$.

The graviton field fulfils the Lorentz covariant quantization rule

$$\left[ h^{\alpha\beta}(x), h^{\rho\sigma}(y) \right] = -\frac{i}{2} \left( \eta^{\alpha\rho} \eta^{\beta\sigma} + \eta^{\alpha\sigma} \eta^{\beta\rho} - \eta^{\alpha\beta} \eta^{\rho\sigma} \right) D_0(x-y),$$

where $D_0(x)$ is the massless Jordan–Pauli causal distribution.

The first order coupling among gravitons, being linear in the coupling constant $\kappa$, can be derived from Eq. (3.1) by taking the normally ordered product of $\mathcal{L}^{(1)}_{HE}$

$$T^h_1(x) = i\kappa : h^{\rho\sigma}(x) h^{\alpha\beta}(x) h^{\alpha\beta}(x) : +$$

$$- \frac{1}{2} : h^{\rho\sigma}(x) h(x) h(x) : + 2 : h^{\rho\sigma}(x) h^{\alpha\beta}(x) h^{\alpha\beta}(x) :$$

$$+ : h^{\rho\sigma}(x) h(x) h^{\alpha\beta}(x) : - 2 : h^{\rho\sigma}(x) h^{\alpha\beta}(x) h^{\alpha\beta}(x) :$$

The non-linearity of gravitation reflects itself in the self-coupling of gravitons. For convenience of notation, $h := h^\gamma_\gamma$ and all Lorentz indices are written as superscripts whereas the derivatives are written as subscripts. All indices occurring twice are contracted by the Minkowski metric $\eta^{\mu\nu} = \text{diag}(1,-1,-1,-1)$. Since the perturbative expansion for the $S$-matrix is in powers of the coupling constant $\kappa$, we are allowed to take for the first order cubic interaction between gravitons only the contribution coming from $\mathcal{L}^{(1)}_{HE}$.

After quantization, Eq. (3.2), the coupling (3.3), completed by a suitable ghost-graviton coupling term (see Sec. 4), can be used in perturbation theory to calculate quantum corrections to classical general relativity.

Two serious problems arise in this procedure. The first one is the non-renormalizability of quantum gravity due to presence of two derivatives on the graviton fields in (3.3) whose origin lies in the dimensionality of the coupling constant ($[\kappa] = \text{mass}^{-1}$). The second one is the non-polynomial character of...
\( \mathcal{L}_{HE} \), Eq. (3.1), which reflects itself into a proliferation of couplings, i.e. into an increasing polynomial degree in the interaction structure.

The first drawback, non-renormalizability, can be approached by means of the inductive causal construction of the \( T_n \)'s, which makes it possible to find finite and cutoff-free quantum corrections for any process describable in the \( S \)-matrix framework, although the solution is not quite clear with regard to physical predictability due to the increasing number of finite normalization terms in the distribution splitting (2.15) in each order of perturbation theory.

With regard to the second issue, we could try to generalize the result of [7] and the more recent result of [12], which suggest that the concept of perturbative quantum operator gauge invariance (see Sec. 4) may be able to explain the higher polynomial couplings: gauge invariance to second order will automatically imply the introduction of a quartic graviton interaction exactly as prescribed by the expansion of the Hilbert–Einstein Lagrangian [7], see Sec. 5.2. If we were able to repeat this step in each order of perturbation theory, we would recover the full Einstein gravity in quantum form. At this point the somewhat artificial decomposition of the metric tensor into a flat background and a dynamical variable would acquire a merely book-keeping purpose beside the fact that we consider an asymptotically flat situation.

4 Gauge Structure of Quantum Gravity

4.1 Gauge Charge

The classical gauge transformations \( h'^{\alpha\beta} \rightarrow h^{\alpha\beta} + \xi^{\alpha\beta} + \xi^{\beta\alpha} - \eta^{\alpha\beta} u^\sigma \), which corresponds to the linearized general covariance of \( g^{\alpha\beta}(x) \) [17], can be implemented on a quantum level by means of the gauge charge \( Q \)

\[
Q := \int_{x^0 = \text{const}} d^3 x \ h^{\alpha\beta}(x) \beta \overrightarrow{D_0} u_\alpha(x) . \quad (4.1)
\]

In order to get a nilpotent gauge charge \( (Q^2 = 0, \text{in order to prove unitarity of the} \ S\text{-matrix and to construct the physical subspace of the graviton Fock space}), \) we have to quantize the vector field \( u^\nu(x) \), the ghost field \( (\Box u^\nu(x) = 0) \), with its partner \( \bar{u}^\nu(x) \), the anti-ghost field \( (\Box \bar{u}^\nu(x) = 0, \text{too}) \), as free fermionic vector fields through the anti-commutator

\[
\{ u^\mu(x), \bar{u}^\nu(y) \} = i \eta^{\mu\nu} D_0(x - y) , \quad (4.2)
\]

whereas all other anti-commutators vanish.

The gauge charge \( Q \) defines an infinitesimal gauge variation by

\[
d_Q A := Q A - (-1)^{n_G(A)} A Q \quad (4.3)
\]
where \( n_G(A) \) is the number of ghost fields minus the number of anti-ghost fields in the Wick monomial \( A \). The operator \( d_Q \) obeys also the Leibniz rule

\[
d_Q(AB) = (d_Q A) B + (-1)^{n_G(A)} A d_Q B ,
\]

for arbitrary operators \( A \) and \( B \).

The infinitesimal operator gauge variations of the fundamental asymptotic free quantum fields are

\[
d_Q h^{\alpha\beta}(x) = [Q, h^{\alpha\beta}(x)] = \frac{\partial}{\partial x^\nu} h^{\nu\sigma}(x)\sigma ,
\]

\[
d_Q u^\alpha(x) = \{Q, u^\alpha(x)\} = 0 ,
\]

\[
d_Q \tilde{u}^\alpha(x) = \{Q, \tilde{u}^\alpha(x)\} = i h^{\alpha\beta}(x)\beta .
\]

### 4.2 Perturbative Operator Quantum Gauge Invariance

Formally, S-matrix gauge invariance means \( \lim_{g \to 1} d_Q S(g) = 0 \). This follows from

\[
\lim_{g \to 1} \left( S'(g) - S(g) \right) = \lim_{g \to 1} \left( -i\lambda [Q, S(g)] + \text{higher commutators} \right) = 0 , \quad (4.6)
\]

which holds true, if the condition of perturbative gauge invariance to \( n \)-th order

\[
d_Q T_n(x_1, \ldots , x_n) = [Q, T_n(x_1, \ldots , x_n)] = \text{sum of divergences} , \quad (4.7)
\]

is fulfilled for all \( n \geq 1 \).

### 4.3 Gauge Invariance to First Order

Already for \( n = 1 \), Eq. (4.7) is non-trivial, because \( d_Q T_1^{h,HE}(x) \neq \text{divergence} \). This requires the introduction of a anti-ghost–graviton–ghost coupling [7]

\[
T_1^{u, KO} = i\kappa \left( + : \tilde{u}^\nu(x)_{,\mu} h^{\mu\nu}(x)_{,\rho} u^\rho(x) : - : \tilde{u}^\nu(x)_{,\mu} h^{\mu\rho}(x) u^\rho(x)_{,\nu} : - : \tilde{u}^\nu(x)_{,\mu} h^{\mu\rho}(x) u^\rho(x)_{,\nu} : + : \tilde{u}^\nu(x)_{,\mu} h^{\mu\nu}(x) u^\rho(x)_{,\rho} : \right) ,
\]

which was first derived by Kugo and Ojima in [18, 19]. Therefore, we obtain

\[
d_Q \left( T_1^{h,HE}(x) + T_1^{u, KO}(x) \right) = \partial_\nu T_1^{\nu,1}(x) = \text{sum of divergences} . \quad (4.9)
\]

One explicit form of \( T_1^{\nu,1} \sim \{ : u h h : + : \tilde{u} u u : \}^{\nu} \), the so-called \( Q \)-vertex, was derived in [7]. The ghost couplings in causal quantum gravity are analyzed in great detail in [8].

The fermionic quantization of the ghost fields, usually called Faddeev-Popov ghosts, is not only necessary for having a nilpotent \( Q \), but also for perturbative gauge invariance to be fulfilled.
In the path-integral framework, the ghost fields appear as a consequence of the quantization after gauge fixing [20], but it was already noticed by Feynman [9] that without ghost fields a unitarity breakdown occurs in second order at the loop level.

Although the condition \( d_Q T_1(x) = \text{divergence} \) seems to be rather easy to fulfil, it has two important consequences. First of all, it rules out the possibility of a renormalizable theory of quantum gravity [12], because for a renormalizable interaction \( T_1(x) \), \textit{i.e.} without the two derivatives acting on the fields\(^1\), perturbative gauge invariance to first order entails only the trivial solution \( T_1(x) = 0 \).

The other interesting consequence pointed out in [12] is the following: if \( T_{h+u}^1(x) \) is the most general ansatz for the graviton coupling and the most general ansatz for the ghost coupling

\[
T_{h+u}^1(x) = \sum_j a_j : \{hh \} : + \sum_j b_j : \{uh \} : ,
\]

(with two derivatives acting on the fields), then the requirement \( d_Q T_{h+u}^1(x) = \text{divergence} \) selects a small number of possible theories and the Hilbert–Einstein graviton coupling \( T_{h,HE}^1 \), with the Kugo–Ojima ghost coupling \( T_{u,KO}^1 \), lies among them. Moreover, all the allowed couplings can be transformed in such a way that the most general coupling has now the form

\[
T_1(x) = T_{h,HE}^1(x) + T_{u,KO}^1(x) + \text{divergence couplings} + d_Q(\tilde{uh}h + \tilde{uu}u) .
\]

The last term represents the so-called \textit{coboundary terms} which, together with divergence terms, seem to play no physical rôle.

The definition of the \( Q \)-vertex from Eq. (4.9) allows us to give a precise prescription on how the right side of Eq. (4.7) has to be inductively constructed. We define the concept of \textit{perturbative quantum operator gauge invariance} by the equation

\[
d_Q T_n(x_1, \ldots , x_n) = \sum_{l=1}^{n} \frac{\partial}{\partial x'_l} T_{n/l}^{\nu}(x_1, \ldots , x_l, \ldots , x_n) .
\]

Here, \( T_{n/l}^{\nu} \) is the time-ordered renormalized product, obtained according to the inductive causal scheme, with a \( Q \)-vertex at \( x_l \), while all other \( n-1 \) vertices are ordinary \( T_1 \)-vertices.

Analysis of the condition (4.12) shows that perturbative gauge invariance can be spoiled by local terms, \textit{i.e.} terms proportional to \( : \mathcal{O}(x_1, \ldots , x_n) : \delta^{(4n-4)}(x_1 - x_n, \ldots , x_{n-1} - x_n) \), which may appear as a consequence of distribution splitting on both sides of Eq. (4.12).

\(^1\)with only one derivative it is impossible to form a Lorentz scalar interaction term
If it is possible to absorb these local terms by suitable local normalization terms $N_n$ of $T_n$ and $N^\nu_n/l$ of $T^\nu_{n/l}$ in such a way that the equation

$$dQ \left( T_n + N_n \right)(x_1, \ldots , x_n) = \sum_{l=1}^{n} \frac{\partial}{\partial x^\nu_l} \left( T^\nu_{n/l} + N^\nu_{n/l} \right)(x_1, \ldots , x_l, \ldots , x_n)$$

(4.13)

holds true, then we call the theory gauge invariant to $n$-th order.

5 Quantum Gravity in Second Order

Before undertaking the examination of the various contributions in second order perturbation theory (tree, self-energy and vacuum graphs), we give the formula for the singular order of arbitrary $n$-point distributions in perturbative quantum gravity.

We consider in the $n$-th order of perturbation theory an arbitrary $n$-point distribution $T_G^n(x_1, \ldots , x_n)$, appearing in Eq. (2.1), as a sum of normally ordered products of free field operators multiplied by numerical distributions

$$T_G^n(x_1, \ldots , x_n) = \prod_{j=1}^{n_h} h(x_{k_j}) \prod_{i=1}^{n_u} u(x_{m_i}) \prod_{l=1}^{n_{\bar{u}}} \bar{u}(x_{n_l}) : t_G^n(x_1, \ldots , x_n) .$$

(5.1)

This $T_G^n$ corresponds to a graph $G$ with $n_h$ external graviton lines, $n_u$ external ghost lines and $n_{\bar{u}}$ external anti-ghost lines.

The singular order of $G$ then reads

$$\omega(G) \leq 4 - n_h - n_u - n_{\bar{u}} - d + n .$$

(5.2)

Here $d$ is the number of derivatives on the external field operators in (5.1). The $\leq$ means that in certain cases the singular order is lowered by peculiar conditions, e.g. by the equations of motions of free fields.

In the usual QFT formulation, Eq. (5.2) implies the non-renormalizability of QG, because $\omega(G)$ increases without bound for higher orders in the perturbative expansion. This means that there is a proliferation of divergences and of counterterms (one still has to hope that the needed counterterms can be fitted into the original Lagrangian) to remove them.

The hope that QG was UV-finite to all orders failed after the two-loop calculation in [21, 22, 23], although the one-loop order is UV-finite [24, 25].

The situation is different in causal perturbation theory: we are facing in this case a non-normalizable theory. The theory has a weaker predictive power but it is still well-defined in the sense of UV finiteness.

The ambiguity in the normalization reflects itself into an increasing number of free, undetermined but finite constants in Eq. (2.15). The problem is then to find enough physical conditions or requirements to fix this increasing freedom and to investigate the effects of these local interactions for physical quantities.
5.1 Graviton Self-Energy

We investigate the graviton self-energy contribution (graviton and ghost loops) in second order. As in Sec. 2, the inductive construction of \( T^2(x_1, x_2) \) can be accomplished in two steps: in the first place we construct the causal distribution \( D^2 \) from Eq. (3.3) and (4.8) by applying Wick expansion with the contractions given by Eq. (3.2)

\[
D^2_{SE}(x_1, x_2) = \left[ T^h(x_1), T^h(x_2) \right]_{SE} = h^{\alpha\beta}(x_1)h^{\mu\nu}(x_2) \cdot d^2_{SE}(x_1 - x_2)_{\alpha\beta\mu\nu}.
\]

Because of translation invariance the C-number distribution \( d^2_{SE} \) depends only on the relative coordinate \( x = x_1 - x_2 \). In momentum space we obtain for the self-energy tensor

\[
\hat{d}^2_{SE}(p)_{\alpha\beta\mu\nu} = \hat{P}(p)_{\alpha\beta\mu\nu}^{(4)} \Theta(p^2) \text{sgn}(p^0),
\]

where \( \hat{P}(p)_{\alpha\beta\mu\nu}^{(4)} \) is a covariant polynomial of degree 4

\[
\hat{P}(p)_{\alpha\beta\mu\nu}^{(4)} = \frac{\kappa^2 \pi}{960(2\pi)^4} \left[ -656 p^\alpha p^\beta p^\mu p^\nu - 208 p^2 (p^\alpha p^\beta \eta^\mu\nu + p^\mu p^\nu \eta^\alpha\beta) 
\right.
\]
\[
+ 162 p^2 (p^\alpha p^\mu \eta^\beta\nu + p^\alpha p^\nu \eta^\beta\mu + p^\beta p^\mu \eta^\alpha\nu + p^\beta p^\nu \eta^\alpha\mu)
\]
\[
- 162 p^4 (\eta^\alpha\beta \eta^\mu\nu - \eta^\alpha\mu \eta^\beta\nu) + 118 p^4 \eta^\alpha\beta \eta^\mu\nu \right].
\]

Then, in order to obtain \( T^2_{SE}(x_1, x_2) \), we split \( d^2_{SE}(x) \) according to the singular order \( \omega(d^2_{SE}) = 4 \), obtained from Eq. (5.2) or from direct inspection of Eq. (5.5). Thus, admitting free normalization polynomial terms \( \hat{N}(p)_{\alpha\beta\mu\nu}^{(2a)} \), which correspond to local interaction terms in configuration space, we obtain

\[
\hat{T}^2_{SE}(x_1, x_2) = h^{\alpha\beta}(x_1)h^{\mu\nu}(x_2) \cdot i \hat{\Pi}(x_1 - x_2)_{\alpha\beta\mu\nu}^{tot},
\]

\[
\hat{\Pi}(p)_{\alpha\beta\mu\nu}^{tot} = \frac{\hat{P}(p)_{\alpha\beta\mu\nu}^{(4)}}{2\pi} \log \left( \frac{-(p^2 + i0)}{M_0^2} \right) + \sum_{a=0}^{2} \hat{\hat{N}}(p)_{\alpha\beta\mu\nu}^{a(2a)}.
\]

The scalar distribution

\[
\hat{\hat{t}}(p) = \frac{i}{2\pi} \log \left( \frac{-(p^2 + i0)}{M_0^2} \right),
\]

is calculated from the massless causal scalar distribution \( \hat{\hat{d}}(p) = \hat{\hat{r}}(p) - \hat{\hat{a}}(p) = \Theta(p^2) \text{sgn}(p^0) \) in Eq. (5.4) by splitting it into \( \hat{\hat{d}}(p) = \hat{\hat{r}}(p) - \hat{\hat{a}}(p) \) and subtracting \( \hat{\hat{r}}(p) \) from \( \hat{\hat{r}}(p) \), see [5, 14]. The mass scale \( M_0 \) represents a normalization constant and not a cutoff.
To get a condition for the undetermined normalization terms, we consider the sum of the proper self-energy diagrams with an increasing number of self-energy insertions. By requiring that the mass of the graviton (which is zero) and the coupling constant $\kappa$ remain unchanged under these radiative corrections, we find that all normalization terms must vanish, except for the term of degree 4 which can be absorbed in the new parameter $M_0^2$, see [14] for details.

We emphasize the fact that, in virtue of the causal splitting prescription, all expressions are UV-finite and Eq. (5.6) agrees exactly with the finite part obtained using standard regularization schemes [26, 27]. As a consequence it is not necessary to add counterterms [24] to the original Lagrangian in order to renormalize the theory.

The graviton self-energy satisfies the Slavnov–Ward identity for the 2-point connected Green function [26, 27]

$$\rho^\mu p^\nu \{ b_{\alpha\beta\gamma\delta} \left[ \Pi_{SE,grav.}^{SE}(p)\gamma^{\delta\rho\sigma} + \Pi_{SE,ghost}^{SE}(p)\gamma^{\delta\rho\sigma} \right] b_{\rho\sigma\mu\nu} \} = 0,$$

only if ghost and graviton loops are taken into account, as well as perturbative gauge invariance, Eq. (4.7): $d_0 T_2^{SE}(x_1, x_2) = \text{divergence}$.

The result of Eq. (5.6) can also be used to find the long range, low energy quantum corrections to the Newtonian potential between two bodies of mass $m_1$ and $m_2$ at a distance $r$ in the non-relativistic static limit (see Sec. 8 for the coupling between matter and gravity).

In the spirit of [28, 29, 30], but without resorting to any effective field theoretical calculation, we compute a matter–matter scattering diagram with exchange of one graviton. The corresponding scattering amplitude leads to the Newtonian potential

$$V(r) = -G\frac{m_1 m_2}{r}.$$

Considering also the radiative corrections coming from the graviton self-energy, we obtain quantum corrections to the Newtonian potential. More precisely, we find that the logarithm depending term in Eq. (5.6) gives the $r^{-3}$-correction

$$V(r) = -G\frac{m_1 m_2}{r} \left( 1 + \frac{206}{30} \frac{G\hbar}{c^3 r^2} \right).$$

In Sec. 8, when we consider also scalar massless matter, we will find supplementary corrections coming from these massless particle loops.

The central piece in the calculation is the distributional Fourier transform of $\log (p^2/M_0^2)$ which yields $(-2\pi r^3)^{-1}$ and the $M_0$-dependence disappears from the non-local part of the final result, being proportional to $\delta^{(3)}(x)$ with $r = |x|$.

The relevant length scale appearing in Eq. (5.9) is the Planck length $\ell_{pl} = \sqrt{G\hbar/c^5}$. Therefore, appreciable quantum corrections manifest themselves only for $r \sim \ell_{pl}$.

Our result agrees with the corresponding one in [30], although this represents only a partial correction to the Newtonian potential, because we have taken into account only the graviton self-energy contribution and not the complete
set of diagrams of order $\kappa^4$ contributing to these corrections, as, for example, the vertex correction or the double scattering. Therefore we cannot make any statement on the absolute sign of the correction in Eq. (5.9).

### 5.2 Tree Graphs and Gauge Invariance

For the tree graphs we quote briefly the result of Schorn [7, 8]: *perturbative gauge invariance to second order generates the 4-graviton coupling.*

The condition of perturbative gauge invariance to second order

$$d_Q T_2(x, y) = \partial_\sigma^x T_{2/1}^\sigma(x, y) + \partial_\sigma^y T_{2/2}^\sigma(x, y)$$

restricted to the operator structure: $\text{uhhh}$ and $\text{ăuuhh}$: can be spoiled by terms with local support, namely proportional to $\delta(4)(x - y)$: local normalization terms $N_{2/1}$, $N_{2/2}$ of $T_2$, $T_{2/1}$ and $T_{2/2}$ respectively, and local anomalies$^2$. Therefore, we have to investigate the equation

$$d_Q N_2(x, y) = \text{an} \left( \partial_\nu^x T_{2/1}^\nu(x, y) + \partial_\nu^y T_{2/2}^\nu(x, y) \right) + \partial_\nu^x N_{2/1}^\nu(x, y) + \partial_\nu^y N_{2/2}^\nu(x, y),$$

which relates the local terms appearing in Eq. (5.10).

We take advantage of the freedom in the normalization of tree diagrams with singular order $\omega \geq 0$ (they appear because of the two derivatives present in the coupling) by choosing local normalization terms $N_2(x, y)$ of the form

$$N_2(x, y) = i \kappa^2 \left\{ + :h^{\rho\sigma} h^{\alpha\beta} h^{\alpha\mu} h_{\rho\sigma}^{\rho\sigma} : - 2 :h^{\rho\sigma} h^{\alpha\beta} h^{\alpha\mu} h_{\rho\sigma}^{\rho\sigma} : + 2 :h^{\rho\sigma} h^{\alpha\beta} h^{\alpha\mu} h_{\rho\sigma}^{\rho\sigma} : + \right. \right.$$

$$\left. - 2 :h^{\rho\sigma} h^{\alpha\beta} h^{\alpha\mu} h_{\rho\sigma}^{\rho\sigma} : + \frac{1}{2} :h^{\rho\sigma} h^{\alpha\beta} h^{\alpha\mu} h_{\rho\sigma}^{\rho\sigma} : - :h^{\rho\sigma} h^{\alpha\beta} h^{\alpha\mu} h_{\rho\sigma}^{\rho\sigma} : + + :h^{\rho\sigma} h^{\alpha\beta} h^{\alpha\mu} h_{\rho\sigma}^{\rho\sigma} : \right\} \delta(4)(x - y),$$

for Eq. (5.11) to be fulfilled, see [7] for details.

Taking the factor $1/2$ for the second order of the $S$-matrix expansion into account, these quartic interactions (quadratic in $\kappa$) agree exactly with the terms of order $\kappa^2$ in the expansion of the Hilbert–Einstein Lagrangian density $\mathcal{L}_{EH}$ given by Eq. (3.1). This mechanism of generation of the higher orders works in a purely quantum framework.

$^2$Anomalies are terms arising from $\partial_\sigma^x T_{2/1}^\sigma(x, y) + \partial_\sigma^y T_{2/2}^\sigma(x, y)$ because of the following mechanism:

$$\partial_\sigma^x \left( :\mathcal{O}(x, y) : \partial_\nu^x D_\nu^\sigma(x - y) \right) = \ldots :\mathcal{O}(x, y) : \partial_\sigma^x D_\sigma^\sigma(x - y)$$

$$= \ldots :\mathcal{O}(x, y) : \delta^{(4)}(x - y)$$

local anomaly
Such a property was already observed in Yang–Mills theories [4]: starting with an interaction between three gauge fields, perturbative gauge invariance generates automatically the 4-gauge fields coupling.

Since QG is constructed starting from a non-polynomial Lagrangian, it is not clear if this scheme would also work in higher orders and the question whether perturbative gauge invariance to $n$-th order requires the introduction of local terms which turn out to agree with the $(n + 1)$-th term in the expansion of the Hilbert–Einstein Lagrangian remains unanswered.

5.3 Vacuum Graphs in Second Order

We discuss also the vacuum graphs in second order. In the causal perturbation theory they cannot be divided away as in the Gell–Mann and Low series for connected Green functions, but this is not a problem because they are finite as a consequence of the causal scheme. The corresponding distribution $\hat{T}^\nu G_2$ has been obtained by computing three contractions in Eq. (2.10). It has singular order $\omega = 6$ and reads

$$\hat{T}^\nu G_2(p) = \frac{i\kappa^2}{(2\pi)^8} \frac{\pi^2}{512} p^6 \log \left( \frac{-(p^2 + i0)}{M_0^2} \right) + \sum_{i=0}^{3} c_i (p^2)^i. \quad (5.13)$$

It is possible to prove the free vacuum stability in QG as described in [31]:

$$\lim_{g \to 1} \langle \Omega, S(g)\Omega \rangle = 1, \quad \text{where } \Omega \text{ is the Fock vacuum of free asymptotic fields.}$$

Perturbatively this means $\lim_{g \to 1} \langle \Omega, S_n(g)\Omega \rangle = 0, \forall n \geq 1$.

We perform the adiabatic limit in scaling form: $g(x) = g_0(\epsilon x)$ where $\epsilon \to 0$ and $g_0 \in S(\mathbb{R}^4)$ with $g_0(0) = 1$. For $n = 2$ we get

$$\lim_{g \to 1} \langle \Omega, S_2(g)\Omega \rangle = \frac{(2\pi)^2}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon^4} \int d^4p \left[ \hat{T}^\nu G_2(\epsilon p) + \hat{N}^\nu G_2(\epsilon p) \right] \hat{g}_0(p) \hat{g}_0(-p) = 0,$$

as a consequence of the bad UV behaviour of QG ($\hat{T}^\nu G_2(p) \sim p^6$).

At the same time, free vacuum stability forces the free normalization constants $c_i$ to vanish. This allows the graviton to show up as an asymptotic particle carrying the long range gravitational interaction.

6 The Physical Subspace $\mathcal{F}_{phys}$

An interesting feature of this approach to the gauge structure of QG is the construction of the physical Hilbert–Fock space for the asymptotic free graviton field.

In order to decouple the ghosts and the unphysical degrees of freedom of the graviton from the truly physical degrees of freedom in the theory, we could apply the Gupta-Bleuler [32] formalism with indefinite metric, but we prefer to realize
the free field representations on a Fock space with positive definite metric [33]. Lorentz covariance requires then the introduction of a Krein structure [34, 35] on the Fock space and we can characterize the physical subspace $F_{\text{phys}}$ by the following definition

$$F_{\text{phys}} := \ker \left\{ Q, Q^\dagger \right\}.$$  \hfill (6.1)

In order to verify the consistency of this formula, we need an explicit representation of the free fields appearing in the theory.

Since a symmetric tensor field with arbitrary trace transforms under the proper Lorentz group $L^\uparrow_+$ according to the tensor product of two spinor representations $D^{(1/2,1/2)}$, we decompose $h^{\alpha\beta}(x)$ according to the irreducible reduction of the representations

$$h^{\alpha\beta}(x) = H^{\alpha\beta}(x) + \frac{1}{4} \eta^{\alpha\beta} \Phi(x),$$  \hfill (6.3)

where $H^{\alpha\beta}(x)$ represents a traceless symmetric tensor field defined as

$$H^{\alpha\beta}(x) := h^{\alpha\beta}(x) - \frac{1}{4} \eta^{\alpha\beta} h(x)/4$$  \hfill (9 degrees of freedom) and $\Phi(x)$ a scalar field with $h^{\gamma\gamma} = \Phi$. From Eq. (3.2) we obtain the following commutation relations

$$[\Phi(x), \Phi(y)] = 4 i D_0(x - y), \quad [H^{\alpha\beta}(x), \Phi(y)] = 0,$$

$$[H^{\alpha\beta}(x), H^{\mu\nu}(y)] = \frac{i}{2} \eta^{\alpha\mu} \eta^{\beta\nu} + \eta^{\alpha\nu} \eta^{\beta\mu} - \frac{1}{2} \eta^{\alpha\beta} \eta^{\mu\nu} D_0(x - y)$$

$$= -i t^{\alpha\beta\mu\nu} D_0(x - y).$$  \hfill (6.4)

For the quantization of $H^{\alpha\beta}(x)$ and $\Phi(x)$ we choose the following free field representations

$$H^{\alpha\beta}(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\omega}} \left( A^{\alpha\beta}(k)e^{-ikx} + A^{\alpha\beta}(k)K e^{+ikx} \right),$$

$$\Phi(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\omega}} \left( a(k)e^{-ikx} + a(k)K e^{+ikx} \right),$$  \hfill (6.5)

where the Krein conjugation $K$ is defined by the Krein operators $\eta_\mu$ and $\eta_\phi$

$$(A^{\alpha\beta})^K = \eta_\mu A^{\alpha\beta}\eta_\mu, \quad \eta_\mu = \prod_{i=1}^{3} (-1)^{N_{0i}}, \quad N_{0i} = 2 \int d^3k A^{0i}(k)^\dagger A^{0i}(k);$$

$$(a)^K = \eta_\phi a^\dagger \eta_\phi, \quad \eta_\phi = (-1)^{N_\phi}, \quad N_\phi = \frac{1}{4} \int d^3k a(k)^\dagger a(k).$$  \hfill (6.6)

The fields are then K-selfconjugate and not $^\dagger$-selfadjoint, but the field components which spoil the selfadjointness turn out to be the unphysical ones and
therefore these will be absent in the physical subspace $\mathcal{F}_{\text{phys}}$, so that on $\mathcal{F}_{\text{phys}}$ one has $H^{\alpha\beta\gamma} = H^{\alpha\beta\gamma\dagger}$. The absorption and creation operators $A^{\alpha\beta} = A^{\beta\alpha}, A^{\alpha\beta\dagger}$ and $a, a^{\dagger}$ satisfy the canonical commutation relations

$$\begin{align*}
[A^{\alpha\beta}(k), A^{\mu\nu}(p)^\dagger] &= \frac{1}{2}(\delta^{\alpha\mu}\delta^{\beta\nu} + \delta^{\alpha\nu}\delta^{\beta\mu} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}) \delta^{(3)}(k - p) \\
&= \tilde{t}^{\alpha\beta\mu\nu} \delta^{(3)}(k - p), \\
[a(k), a^{\dagger}(p)] &= 4 \delta^{(3)}(k - p).
\end{align*}$$

(6.7)

The $\tilde{t}^{\alpha\beta\mu\nu}$-tensor has the following values

| $\tilde{t}^{\alpha\beta\mu\nu}$ | 00;00 | 00;ii | 0i;0i | ii;ii | ii;jj | ij;ij | otherwise |
|---------------------------|--------|-------|------|------|------|------|------------|
| value                    | 3/4    | 1/4   | 1/2  | 3/4  | -1/4 | 1/2  | 0          |

with $i, j = 1, 2, 3; i \neq j$. From this table we see that the $\tilde{t}^{\alpha\beta\mu\nu}$-tensor is neither diagonal nor positive definite, although it is positive for the diagonal terms. In order to remedy these defects, we define new absorption operators

$$\begin{align*}
A^{00} &= \frac{1}{2}(+a^{11} + a^{22} + a^{33}), & A^{11} &= \frac{1}{2}(-a^{11} + a^{22} + a^{33}), \\
A^{22} &= \frac{1}{2}(+a^{11} - a^{22} + a^{33}), & A^{33} &= \frac{1}{2}(+a^{11} + a^{22} - a^{33});
\end{align*}$$

(6.8)

and analogously for the creation operators. Then we obtain the commutation relations

$$\begin{align*}
[a^{ii}(k), a^{jj}(p)^\dagger] &= \delta^{ij} \delta^{(3)}(k - p).
\end{align*}$$

(6.9)

Note that the operators $a^{00}$ and $a^{00\dagger}$ do not appear here because this operator pair is superfluous due to the trace condition $H^{\alpha\gamma} = 0$.

Now we want to specify the physical subspace with the help of the gauge charge $Q$, which now reads

$$Q = \int d^3x \left( H^{\alpha\beta}(x,\beta) + \frac{1}{4}\Phi(x,\alpha) \right) \overleftarrow{\partial_0} u^\dagger(x) \eta_{\alpha\gamma}.$$  

(6.10)

For this purpose, we need the free field representations of the ghost fields. We follow in our discussion the analysis of the scalar ghost fields for Yang–Mills theories carried out in [34]. Here we are dealing with vector ghost fields and we choose the following free field representations

$$\begin{align*}
u^{-}(x) &= (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} \left( + b^\nu(p)e^{-ipx} - \eta^{\nu\rho} c^\rho(p)^\dagger e^{ipx} \right), \\
\bar{u}^{-}(x) &= (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} \left( - c^\nu(p)e^{-ipx} - \eta^{\nu\rho} b^\rho(p)^\dagger e^{ipx} \right);
\end{align*}$$

(6.11)
which satisfy the covariant commutation rule Eq. (4.2), whereas the absorption and creation operators satisfy the commutation relations
\[
\{ c^\mu(p), c^\nu(k) \} = \delta^{\mu\nu}\delta^{(3)}(p-k),
\{ b^\mu(p), b^\nu(k) \} = \delta^{\mu\nu}\delta^{(3)}(p-k).
\]
(6.12)

Extending the \( \dagger \)-conjugation to the K-conjugation we obtain the most symmetric form
\[
u(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} \left( b^\nu(p)e^{-ipx} + b^\nu(p)K e^{ipx} \right),
\]
(6.13)
\[
\bar{u}^\nu(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} \left( -c^\nu(p)e^{-ipx} + c^\nu(p)K e^{ipx} \right).
\]
This implies \( (u^\nu)^K = u^\nu \) and \( (\bar{u}^\nu)^K = -\bar{u}^\nu \), or equivalently: \( (b^\mu)^K = c^{\mu\dagger} \), \( (c^\mu)^K = b^{\mu\dagger} \), \( (b^0)^K = -c^0\dagger \) and \( (c^0)^K = -b^0\dagger \).

The construction of the Krein operator \( \eta_G \) on the ghost Fock space which generates the transformations \( (O)^K = \eta_G O\dagger \eta_G \) from the \( \dagger \)-conjugation to the K-conjugation requires more work, see [14], and reads
\[
\eta_G = \exp \left( \frac{\pi}{2} (N_g - \Gamma_g) \right)
\]
(6.14)
where
\[
N_g = N^{(0)}_g - \sum_{i=1}^{3} N^{(i)}_g \quad \text{with} \quad N^{(\mu)}_g = \int d^3 p \left( b^{\mu\dagger}b^\mu + c^{\mu\dagger}c^\mu \right), \quad \text{and}
\]
\[
\Gamma_g = -3 \sum_{\mu=0}^{3} \Gamma^{(\mu)}_g \quad \text{with} \quad \Gamma^{(\mu)}_g = \int d^3 p \left( b^{\mu\dagger}c^\mu + c^{\mu\dagger}b^\mu \right).
\]
(6.15)

\( N^{(\mu)}_g \) represents the the \( \mu \)-ghost number operator whereas \( \Gamma^{(\mu)}_g \) represents the \( \mu \)-ghost transfer operator.

Calculating \( \{ Q, Q^\dagger \} \) in momentum space, we obtain
\[
\{ Q, Q^\dagger \} = \int d^3k \omega^2(k) \left[ + 2 \sum_{\mu=0}^{3} \left( A^{0\mu}(k) A^{0\mu}(k) + A^{\mu}(k) A^{\mu}(k) \right) + \right.
\]
\[+ \frac{1}{4} a^{\dagger}(k)a(k) + \sum_{\alpha=0}^{3} \left( c^{\alpha}(k) c^{\alpha}(k) + b^{\alpha}(k) b^{\alpha}(k) \right) + \]
\[+ \frac{1}{2} \left( \sum_{\alpha=0}^{3} \frac{k_\alpha}{k_0} c^{\alpha}(k) \right) \left( \sum_{\beta=0}^{3} \frac{k_\beta}{k_0} c^{\beta}(k) \right) + \]
\[+ \frac{1}{2} \left( \sum_{\alpha=0}^{3} \frac{k_\alpha}{k_0} b^{\alpha}(k) \right) \left( \sum_{\beta=0}^{3} \frac{k_\beta}{k_0} b^{\beta}(k) \right) \]'',
\]
(6.16)
where $A^\mu_{\parallel}$ represent the absorption operator for the $\mu$-longitudinal mode

$$A^\mu_{\parallel}(k) := \frac{k_i}{\omega(k)} A^{\mu i}(k). \quad (6.17)$$

Apparently there is an over-counting in the graviton sector: we have four $0\mu$- and four $\mu$-longitudinal modes number operators, as well as the scalar component $a$ number operator, but $A^0_{\parallel}$ is not independent, being a linear combination of the $A^{0i}$-operators and we have not taken into account that $\eta_{\alpha\beta} A^{\alpha\beta} = 0$.

For this purpose let us choose a reference frame in which $k^\mu = (\omega, 0, 0, \omega)$ is parallel to the third axis, because obviously the unphysical graviton modes depend on $k$, and substitute the $A^\mu_{\mu}'s$ by the $a^{ii}$'s, Eq. (6.8), so that the integrand of $\{Q, Q^\dagger\}$ restricted to the graviton sector becomes

$$2\omega^2 \left[ + 2 A^{03\dagger} A^{03} + A^{01\dagger} A^{01} + A^{02\dagger} A^{02} + A^{13\dagger} A^{13} + A^{23\dagger} A^{23} \right] +$$
$$+ \omega^2 \left[ + a^{11\dagger} a^{11} + a^{22\dagger} a^{22} + a^{33\dagger} a^{33} + a^{11\dagger} a^{22} + a^{22\dagger} a^{11} + \frac{1}{4} a^{\dagger} a \right]. \quad (6.18)$$

With the definitions

$$J^{\pm}(k) := \frac{a^{11}(k) \pm a^{22}(k)}{\sqrt{2}},$$

$$[J^{\pm}(k), J^{\pm}(p)^\dagger] = \delta^{(3)}(k - p), \quad [J^{\pm}(k), J^{\mp}(p)^\dagger] = 0, \quad (6.19)$$

we find that the integrand of $\{Q, Q^\dagger\}$ restricted to the graviton sector now reads

$$2\omega^2 \left[ + A^{01\dagger} A^{01} + A^{02\dagger} A^{02} + 2 A^{03\dagger} A^{03} + A^{13\dagger} A^{13} + \right.$$
$$+ A^{23\dagger} A^{23} + \frac{1}{8} a^{\dagger} a + \frac{1}{2} a^{33\dagger} a^{33} + J_+^\dagger J_+ \right], \quad (6.20)$$

which is manifestly the sum of particle number operators for unphysical modes of the graviton field in the chosen reference frame: the two remaining physical modes for fixed $k$ are created from the Fock vacuum $|\Omega\rangle$ by $J_-(k)^\dagger$ and $A^{12}(k)^\dagger$ in close analogy to the classical reduction of the degrees of freedom in a plane gravitational tensor wave $h_{cl.}^{\alpha\beta}(x) = \varepsilon_{cl.}^{\alpha\beta}(k) e^{-i k \cdot x}$ with polarization tensor $\varepsilon_{cl.}^{\alpha\beta}(k)$.

Therefore Eq. (6.1) defines the physical subspace in a correct manner. A different formulation of the graviton quantization can be found in [36].

In addition, we can compute the generators of the time evolutions of the fields $H^{\alpha\beta}(x)$ and $\Phi(x)$, respectively. These free quantum fields satisfy the Heisenberg equations of motion

$$-i H^{\alpha\beta}(x) = [H_{\alpha\beta}, H^{\alpha\beta}(x)], \quad -i \dot{\Phi}(x) = [H_{\Phi}, \Phi(x)]. \quad (6.21)$$
The Hamilton operators are easily found by (6.7) and read

\[
H_H = \int d^3 p \omega(p) \left[ + \sum_{\nu=0}^3 A^{\nu\nu}(p)^\dagger A^{\nu\nu}(p) + 2 \sum_{i=1}^3 A^{0i}(p)^\dagger A^{0i}(p) \\
+ 2 \sum_{i,j=1}^3 A^{ij}(p)^\dagger A^{ij}(p) \right],
\]

\[
H_\Phi = \frac{1}{4} \int d^3 p \omega(p) a(p)^\dagger a(p).
\]

(6.22)

If we restrict these operators to the physical subspace \( \mathcal{F}_{phys} \) and, in addition, choose the special reference frame as before, we find that the integrand of \((H_H + H_\Phi)\) reads

\[
\omega \left[ J^\dagger J_\mp + 2A^{12\dagger}A^{12} \right].
\]

(6.23)

This expression can be recast with

\[
a_{\pm}(p)^\dagger := \frac{1}{\sqrt{2}} J^\dagger(p)^\mp \mp i A^{12}(p)^\dagger,
\]

\[
[a_\pm(k), a_\pm(p)^\dagger] = \delta^{(3)}(k - p), \quad [a_\pm(k), a_\mp(p)^\dagger] = 0,
\]

(6.24)

into the form

\[
\omega \left[ a_\dagger_+ a_+ + a_\dagger_- a_- \right],
\]

(6.25)

which confirms that graviton states in \( \mathcal{F}_{phys} \) have only two independent components, the other eight being unphysical. The four operators \( a_{\pm}, a_{\pm}^\dagger \) absorb or create physical states with helicities \( \pm 2 \).

### 7 Unitarity

The property of unitarity in QG, as in non-Abelian gauge theories, is very important (and usually very difficult to prove), because the Fock space contains a lot of unphysical states (see Sec. 6).

Nevertheless, there exists a physical subspace \( \mathcal{F}_{phys} \), such that the \( S \)-matrix restricted to \( \mathcal{F}_{phys} \) is unitary. Because of the unphysical degrees of freedom involved in the theory, unitarity does not hold on the whole Fock space. There, the theory is pseudo-unitary, namely unitary with respect to the K-conjugation.

Since \( (u^\nu)^K = u^\nu, \ (\tilde{u}^\nu)^K = -\tilde{u}^\nu, \ H^{\alpha\beta K} = H^{\alpha\beta} \) and \( \Phi^K = \Phi \), it follows that \( T_1^{h+u} \) is skew-K-conjugate

\[
\left( T_1^{h+u}(x) \right)^K = -T_1^{h+u}(x) = \tilde{T}_1^{h+u}(x),
\]

(7.1)
and this holds for all the $n$-point distributions $T_n$ by induction \[3, 35\] if the normalization constants in the distribution splitting Eq. (2.15) are chosen appropriately:

$$T_n(X)^K = \tilde{T}_n(X),$$  \hspace{1cm} (7.2)

where $X := \{x_1, \ldots, x_n\}$ and $\tilde{T}_n(X)$ is the $n$-point distribution belonging to the perturbative expansion of the inverse $S$-matrix. According to Eq. (7.2), we get pseudo-unitarity

$$S(g)^K = S(g)^{-1}.$$  \hspace{1cm} (7.3)

We cannot expect unitarity on the whole Fock space because the scalar graviton $\Phi$, the 0i-components of $H^{\alpha\beta}$ and the ghosts are not hermitian (with respect to $\dagger$), but only skew-hermitian.

With unitarity on the physical subspace we mean the heuristic equation

$$\lim_{g \uparrow 1} \left[ P_{\text{phys}} S(g) P_{\text{phys}} \right]^{-1} = P_{\text{phys}},$$

where $P_{\text{phys}}$ stands for the projection operator onto $F_{\text{phys}}$. In [14], we were able to prove the perturbative version of Eq. (7.4), namely

$$\tilde{T}_n^P(X) = P_{\text{phys}} T_n(X)^\dagger P_{\text{phys}} + \text{divergences},$$

where $\tilde{T}_n^P(X)$ is the $n$-point distribution of the $S$-matrix inverted on $F_{\text{phys}}$

$$(P_{\text{phys}} S(g) P_{\text{phys}})^{-1} = P_{\text{phys}} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4 x_1 \ldots d^4 x_n \tilde{T}_n^P(x_1, \ldots, x_n)$$

$$\times g(x_1) \ldots g(x_n).$$

The sum of divergences appearing on the right side of Eq. (7.5) does not harm, because the divergences can be integrated out in the adiabatic limit $g \rightarrow 1$.

## 8 Scalar Matter Coupled to Quantum Gravity

In this section we investigate scalar massive matter fields coupled to QG; also the massless limit will be discussed.

Expanding $\mathcal{L}_M = \frac{1}{2} \sqrt{-g} \left( g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 \right)$ as in Sec. 3, we find

$$\mathcal{L}_M = \frac{1}{2} \left( \eta^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - m^2 \phi^2 \right) + \frac{\kappa}{2} h^{\mu\nu} \left( \phi_{,\mu} \phi_{,\nu} - \frac{m^2}{2} \eta_{\mu\nu} \phi^2 \right)$$

$$= \mathcal{L}^{(0)}_M + \frac{m^2 \kappa^2}{8} \left( h^{\alpha\beta} h^{\alpha\beta} \phi - \frac{1}{2} h h \phi \phi \right) + O(\kappa^3).$$

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From the first term we obtain the Klein-Gordon equation of motion
\[(\Box + m^2)\phi(x) = 0.\]  
(8.2)

We quantize the scalar field by imposing the commutation rule
\[[\phi(x), \phi(y)] = -i D_m(x - y).\]  
(8.3)

The first order matter-graviton coupling reads
\[T^M_{1}(x) = i \kappa :L_M^{(1)}(x): = i\kappa \{ :h^{\alpha\beta}\phi_{,\alpha\beta}\phi: - \frac{m^2}{2} :h\phi\phi: \} = \frac{i\kappa}{2} :h^{\alpha\beta}h_{\alpha\beta\mu\nu}T^\mu\nu_M:,\]
(8.4)

where \(T^\mu\nu_M\) is the conserved energy-momentum tensor of the matter field \(T^\mu\nu_M = \phi^{,\mu}\phi^{,\nu} - \eta^{\mu\nu}L_M^{(0)}\). Gauge invariance to first order is readily established
\[d_Q T^M_{1}(x) = \partial^\mu \kappa \{ :u^{,\mu}\phi_{,\mu}\phi: - \frac{1}{2} :u^{,\mu}\phi_{,\mu}\phi^{,\mu}: + \frac{m^2}{2} :u^{,\mu}\phi\phi: \} = \partial^\mu T^M_{1/1}(x),\]
(8.5)

### 8.1 Tree Graph Sector

Gauge invariance to second order in the tree graph sector
\[d_Q T^\text{tree}_2(x, y) = \text{divergence} - \frac{\kappa^2 m^2}{2} :u^{\alpha}(x),_{\beta}h^{\alpha\beta}(x)\phi(x)\phi(x): \delta^{(4)}(x - y),\]
(8.6)

is spoiled by the local term on the right side that cannot be written as a divergence. By exploiting the ambiguity in the normalization, in order to restore gauge invariance we can add on both sides of Eq. (8.6)
\[d_Q N^\text{tree}_2(x, y) = \frac{\kappa^2 m^2}{2} :u^{\alpha}_{,\beta}h^{\alpha\beta}\phi\phi: \delta^{(4)}(x - y),\]
(8.7)

which is the gauge variation of the normalization term
\[N^\text{tree}_2(x, y) = \frac{i\kappa^2 m^2}{4} \left(:h^{\alpha\beta}h^{\alpha\beta}\phi\phi: - \frac{1}{2} :hh\phi\phi: \right) \delta^{(4)}(x - y),\]
(8.8)

so that we arrive at \(d_Q T^\text{tree}_2(x, y) + d_Q N^\text{tree}_2(x, y) = \text{divergence},\) see [16].

Also in this case, gauge invariance to second order requires the introduction of a new matter–graviton local interaction which turns out to agree with the second order in the classical expansion of \(L_M\) given in Eq. (8.1), up to the factor 1/2 coming from the scattering matrix.

In the massless case we obtain directly \(d_Q T^\text{tree}_2(x, y) = \text{divergence},\) because the local anomaly of Eq. (8.6) does not appear. This agrees with the fact that for massless matter \(L_M^{(j)} = 0, \forall j \geq 2.\)
8.2 Graviton Self-Energy

As in the calculation of Sec. 5, we find a contribution to the graviton self-energy tensor if we perform two matter field contractions in

\[ D^2(x, y) = \left[ T^I_1(x), T^I_1(y) \right] \]

\[ d^2SE_{\mu\nu}(x, y) = h^{\alpha\beta}(x) h^{\mu\nu}(y) : d^2SE(x - y) \alpha\beta\mu\nu : \]

\[ \hat{\Pi}(p)_{\alpha\beta\mu\nu} = \frac{\kappa^2 \pi}{960(2\pi)^5} \left[ \hat{P}(p)_{\alpha\beta\mu\nu} + \frac{m^2}{p^2} \hat{Q}(p)_{\alpha\beta\mu\nu} + \frac{m^4}{p^4} \hat{R}(p)_{\alpha\beta\mu\nu} \right] \]

\[ \times \sqrt{1 - \frac{4m^2}{p^2}} \Theta(p^2 - 4m^2) \operatorname{sgn}(p^0), \quad (8.9) \]

where the three polynomials of degree 4 have the same structure as in Eq. (5.5) with the coefficients: \( \hat{P}_{\alpha\beta\mu\nu} = [-8, -4, 1, -1, -1] \), \( \hat{Q}_{\alpha\beta\mu\nu} = [-16, -8, -8, 8, -12] \) and \( \hat{R}_{\alpha\beta\mu\nu} = [-48, -24, +16, -16, +4] \).

We split the scalar distribution in Eq. (8.9) according to their singular order \( \omega = 0, \omega = -2 \) and \( \omega = -4 \), respectively, because the polynomials can be neglected in the splitting ([16]), and find

\[ T^2_{SE}(x, y) = h^{\alpha\beta}(x) h^{\mu\nu}(y) : i \Pi(x - y)_{\alpha\beta\mu\nu} : \quad (8.10) \]

The above calculation shows that, in our approach, scalar matter coupled to QG does not require the introduction of a non-renormalizable counterterm \([24, 25]\), i.e. of a counterterm that cannot be absorbed in the redefinitions of bare parameters appearing in the original Lagrangian of the theory.
Further, gauge invariance

\[ d_Q T^S_E(x, y) = \partial_\sigma^x (\cdot u^\rho(x)h^{\mu\nu}(y) : \frac{\partial^{\alpha\beta\rho\sigma}}{\alpha\beta\mu\nu} \Pi(x - y)_{\alpha\beta\mu\nu} \cdot (x \leftrightarrow y), \quad (8.12) \]

implies the identity

\[ \frac{\partial^{\alpha\beta\rho\sigma}}{\alpha\beta\mu\nu} \Pi(x - y)_{\alpha\beta\mu\nu} = 0, \quad (8.13) \]

which corresponds to the transversality of the 2-point connected Green function (or Slavnov–Ward identity)

\[ p^\alpha \hat{G}(p)_{\alpha\beta\mu\nu} = p^\alpha \left[ b_{\alpha\beta\gamma\delta} \hat{D}^E_0(p) \tilde{\Pi}(p)_{\gamma\delta\rho\sigma} b_{\rho\sigma\mu\nu} \hat{D}^E_0(p) \right] = 0. \quad (8.14) \]

The attached line represents a free graviton Feynman propagator

\[ \langle \Omega | T \{ h^{\alpha\beta}(x)h^{\mu\nu}(y) \} | \Omega \rangle = -i \frac{\kappa^2}{960(2\pi)^5} \left( \frac{p^4}{6p^2} \hat{R}(p)_{\alpha\beta\mu\nu} + z_1 \hat{Z}_1(p)_{\alpha\beta\mu\nu} + z_2 \hat{Z}_2(p)_{\alpha\beta\mu\nu} \right), \quad (8.15) \]

This latter is affected by radiative corrections due to self-energy insertions. If we require that the mass of the graviton and the coupling constant remain unchanged, we find for the graviton self-energy tensor

\[ \tilde{\Pi}(p)_{\alpha\beta\mu\nu} = \frac{\kappa^2}{960(2\pi)^5} \left( \frac{p^4}{6p^2} \hat{R}(p)_{\alpha\beta\mu\nu} + \frac{m^2}{p^2} \hat{Q}(p)_{\alpha\beta\mu\nu} + \frac{m^4}{p^4} \hat{R}(p)_{\alpha\beta\mu\nu} \right) \tilde{\Pi}(p) \]

\[ + \frac{m^2}{6p^2} \hat{R}(p)_{\alpha\beta\mu\nu} + z_1 \hat{Z}_1(p)_{\alpha\beta\mu\nu} + z_2 \hat{Z}_2(p)_{\alpha\beta\mu\nu} \right), \quad (8.16) \]

where \( \hat{Z}_i(p)_{\alpha\beta\mu\nu}, i = 1, 2 \) are 2 fixed gauge invariant polynomials and \( z_i \in \mathbb{R}, i = 1, 2 \).

In the massless case we obtain

\[ \tilde{\Pi}(p)_{\alpha\beta\mu\nu} = \frac{\kappa^2}{960(2\pi)^5} \hat{P}(p)_{\alpha\beta\mu\nu} \log \left( \frac{-(p^2 + i0)}{M_0^2} \right). \quad (8.17) \]

This massless particle loop gives also a correction to the Newtonian potential

\[ V(r) = \frac{-G m_1 m_2}{r} \left( 1 + \frac{1}{10} \frac{Gh}{r^2} \right). \quad (8.18) \]

### 8.3 Matter Self-Energy

In \( D_2 \) we can isolate also the matter self-energy contribution by performing one graviton and one matter field contraction, in this case we obtain

\[ d_2^{\text{MSE}}(x, y) = \phi(x)\phi(y) : d_2^{\text{MSE}}(x - y), \]

\[ d_2^{\text{MSE}}(p) = \frac{-\kappa^2 m^2}{2(2\pi)^4} \left( p^2 - m^2 \right) \left( 1 - \frac{m^2}{p^2} \right) \Theta(p^2 - m^2) \text{sgn}(p^0). \quad (8.19) \]
We split the numerical distribution $\hat{d}_2^{MSE}$ with $\omega(\hat{d}_2^{MSE}) = 2$ and obtain

$$T_2^{MSE}(x, y) =: \phi(x)\phi(y): i\Sigma(x - y),$$

$$\hat{\Sigma}(p) = -\frac{\kappa^2 m^2\pi}{2(2\pi)^5} \left\{ \left( \frac{p^2 - 3m^2}{2} + \frac{m^4}{2p^2} \right) \left\{ \log \left( \frac{p^2 - m^2}{m^2} \right) - i\pi \Theta(p^2 - m^2) \right\} \right. + \left. \frac{m^2}{2} - \frac{5p^2}{4} + c_0 + c_2p^2 \right\}. \quad (8.20)$$

If we formally sum the series of graphs with 0, 1, 2, ... self-energy insertions, we obtain the matter propagator

$$\hat{\Sigma}(p)^{tot} = +\hat{D}^\phi_m(p) + \hat{D}^\phi_m(p) (2\pi)^4 \hat{\Sigma}(p) \hat{D}^\phi_m(p) + \quad \text{or} \quad +\hat{D}^\phi_m(p) (2\pi)^4 \hat{\Sigma}(p) \hat{D}^\phi_m(p) (2\pi)^4 \hat{\Sigma}(p) \hat{D}^\phi_m(p) + \ldots \quad (8.21)$$

$$= -(2\pi)^{-2} \left( p^2 - m^2 + i0 + (2\pi)^2 \hat{\Sigma}(p) \right)^{-1}.$$ 

The mass-normalization condition reads $\hat{\Sigma}(p^2 = m^2) = 0$ and fixes $c_0 = m^2(\frac{3}{2} - c_2)$. To find a condition for $c_2$ one should consider the vertex function $\Lambda(p, q)_{\alpha\beta}$ to the third order in the 3-point distribution $T_3(x, y, z) =: \phi(x)\phi(y)\hat{h}^{\alpha\beta}(z) : \Lambda(x, y, z)_{\alpha\beta}$ as in QED [3].

In the massless case we obtain $D_2^{MSE}(x, y) = T_2^{MSE}(x, y) = 0$.

### 8.4 Vacuum Graphs

If we perform three contractions in $D_2$, we get the vacuum graph contribution

$$\hat{D}_2^{VG}(p) = \frac{\kappa^2 m^2}{(2\pi)^5} \Theta(p^2 - 4m^2) \text{sgn}(p^0) \hat{f}(p),$$

$$\hat{f}(p) = \frac{1}{384} \sqrt{1 - \frac{4m^2}{p^2}} \left( p^4 - 7m^2p^2 + 6m^4 \right) + \quad \text{or} \quad + \frac{m^6}{16p^2} \log \left( \sqrt{\frac{p^2}{4m^2}} + \sqrt{\frac{p^2}{4m^2} - 1} \right). \quad (8.22)$$

After distribution splitting with $\omega(\hat{D}_2^{VG}) = 4$ we obtain

$$\hat{T}_2^{VG}(p) = \hat{X}(p)^{an} + \frac{\kappa^2 m^2}{2(2\pi)^5} \hat{f}(p) \Theta(p^2 - 4m^2), \quad (8.23)$$

where $\hat{X}(p)^{an}$ is the analytic continuation of $\hat{T}_2^{V}\hat{G}(p)$

$$\hat{X}(p)^{an} = \frac{i\kappa^2 m^2}{384(2\pi)^6} \left[ -3p^4 + \frac{31}{2} m^3p^2 - 6m^4 + \left( -p^4 + 7m^2p^2 - 6m^4 \right) \sqrt{1 - \frac{4m^2}{p^2}} \times \log \left( \frac{\sqrt{1 - 4m^2/p^2} - 1}{\sqrt{1 - 4m^2/p^2} + 1} \right) + \frac{24m^6}{p^2} \log^2 \left( \sqrt{-\frac{p^2}{4m^2} + \sqrt{1 - \frac{p^2}{4m^2}}} \right) \right].$$

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Since we are interested in the adiabatic limit of the vacuum graphs as in Sec. 5.3, we isolate in Eq. (8.23) the leading behaviour in the limit $p^2 \to 0$ (IR-regime)

$$T_2^{V\,G}(p) \sim -\frac{i\kappa^2}{5120(2\pi)^6} p^6 + O(p^8),$$

so that the adiabatic limit becomes in scaling form

$$\lim_{g \to 1} (\Omega, S_2(g)\Omega) = \frac{(2\pi)^2}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon^4} \int d^4p \left[ T_2^{V\,G}(\epsilon p) + N_2^{V\,G}(\epsilon p) \right] g_0(p) g_0(-p)$$

$$= \frac{(2\pi)^2}{2} \lim_{\epsilon \to 0} \frac{1}{\epsilon^4} \int d^4p \left[ B\epsilon^6 p^6 + O(p^8) + c_0 + c_2 \epsilon^2 p^2 + c_4 \epsilon^4 p^4 \right] g_0(p) g_0(-p)$$

$$= 0.$$

The existence of the above limit requires $c_0 = c_2 = 0$ and is assured by the IR-behaviour of the massive theory $T_2^{V\,G}(p) \sim p^6$ for $p^2 \to 0$. Independence from the test functions $g_0$ is reached by choosing $c_4 = 0$.

## 9 Abelian Gauge Fields Coupled to Quantum Gravity

We discuss very briefly the coupling between gravitons and $U(1)$-Abelian gauge fields (photons), see [15] for the details.

We expand the Lagrangian $L_A = -\sqrt{-g} F_{\mu\nu} F^{\alpha\beta} g^{\alpha\mu} g^{\beta\nu}/4$ in powers of the coupling constant $\kappa$ and isolate the first order coupling

$$T_1^A(x) = i \cdot \mathcal{L}_A^{(1)}(x) := \frac{i\kappa}{2} : h^{\alpha\beta} (x) T_A(x)_{\alpha\beta} :$$

$$= \frac{i\kappa}{2} : h^{\alpha\beta} \left( - F_{\beta\nu} F^{\alpha\nu} + \frac{\eta_{\alpha\beta}}{4} F_{\mu\nu} F^{\mu\nu} \right),$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The photon field is quantized according to

$$[A^\mu(x), A^\nu(y)] = i \eta^{\mu\nu} D_0(x-y).$$

First order gauge invariance, $d_Q T_1^E M(x) = \text{divergence}$, holds true because of $T_1^{\mu\nu, A, A} = 0$ and $\eta_{\alpha\beta} T_{A, A}^{\alpha\beta} = 0$.

We evaluate some loop contributions [37, 38, 39] in second order of perturbation theory. The photon loop graviton self-energy contribution [37] reads

$$T_2^{S\,G}(x, y) = : h^{\alpha\beta}(x) h^{\mu\nu}(y) : i \Pi(x-y)_{\alpha\beta\mu\nu},$$

$$27$$
where the self-energy tensor reads

\[ \Pi(p)^{\alpha \beta \mu \nu} = \frac{\kappa^2 \pi}{960(2\pi)^5} \left[ -16 p^\alpha p^\beta p^\mu p^\nu - 8 p^2 (p^\alpha p^\beta \eta^{\mu \nu} + p^\mu p^\nu \eta^{\alpha \beta}) 
+ 12 p^2 (p^\alpha p^\mu \eta^{\beta \nu} + p^\alpha p^\nu \eta^{\beta \mu} + p^\beta p^\mu \eta^{\alpha \nu} + p^\beta p^\nu \eta^{\alpha \mu}) 
- 12 p^4 (\eta^{\alpha \mu} \eta^{\beta \nu} + \eta^{\alpha \nu} \eta^{\beta \mu}) + 8 p^4 \eta^{\alpha \beta} \eta^{\mu \nu} \right] \log \left( \frac{-(p^2 + i0)}{M^2} \right) \]

(9.29)

and satisfies the perturbative gauge invariance condition (and as a consequence the Slavnov–Ward identity) as in Sec. (8.2), and, in addition, is transversal \( p_\alpha \Pi(p)^{\alpha \beta \mu \nu} = 0 \) and traceless \( \eta_{\alpha \beta} \Pi(p)^{\alpha \beta \mu \nu} = 0 \).

The photon loop graviton self-energy contributes also to the corrections of the Newtonian potential as the graviton and ghost loop:

\[ V(r) = \frac{-G m_1 m_2}{r} \left( 1 + \frac{4}{15} \frac{G \hbar}{c^3 \pi r^2} \right). \]

(9.30)

The photon self-energy contribution through a graviton-photon loop reads

\[ T^{SE}_2(x, y) =: A^\gamma(x) A^\rho(y) : \left( -i \Pi(x - y)_{\gamma \rho} \right), \]

(9.31)

where the photon self-energy tensor is

\[ \Pi^{\gamma \rho}(p) = \frac{\kappa^2 \pi}{12(2\pi)^5} \left( p^2 p^\gamma p^\rho - \eta^{\gamma \rho} p^4 \right) \log \left( \frac{-(p^2 + i0)}{M^2} \right). \]

(9.32)

In both cases, we find UV-finite and cutoff-free results for our one-loop calculations. Therefore the introduction of counterterms (that cannot be renormalized away, see [25, 38, 39]) is not necessary.

10 General Ansatz for Matter Coupling and Perturbative Gauge Invariance

In this section we adopt a new strategy [6, 12] in order to construct a gauge invariant theory of quantum gravity coupled to matter fields.

This purely quantum approach relies merely on the inductive causal construction of \( T_n \) (see Sec. 2) and on the perturbative quantum gauge invariance condition (see Sec. 4.2). It does not appeal to any classical Lagrangian density and uses only free quantum fields. In [12] this idea was implemented for pure QG, as already explained above.
10.1 Massive Case

We adopt the same strategy by choosing the following ansatz for the most general massive matter coupling (disregarding non-relevant divergence couplings) between one graviton and two matter fields

\[ T^M \equiv T_{1}(x) = i \kappa \left( + \tilde{x} : h^{\alpha \beta} \phi_{\alpha} \phi_{\beta} : + y : h \phi_{\gamma} \phi_{\gamma} : + z : h^{\alpha \beta} \phi_{\alpha} \phi_{\beta} : + w m^2 : h \phi \phi : \right), \]

(10.33)

where \( \tilde{x}, y, z, w \in \mathbb{R} \) are undetermined coefficients. The quantized graviton and matter fields satisfy the commutation rules Eq. (3.2) and Eq. (8.3), respectively.

The condition of perturbative gauge invariance to first order, \( d_Q T^M_1 = \text{divergence} \), implies

\[ y = z/2 - w - \tilde{x}/2. \]

In second order, for the graviton self-energy \( T^{SE}_2 \), gauge invariance \( d_Q T^{SE}_2 = \text{divergence} \) (which is equivalent to the Slavnov–Ward identity) implies \( w = -\tilde{x}/2 \) so that the general matter coupling becomes

\[ T^M_1 (x) = i \frac{\kappa}{2} \left( + \tilde{x} : h^{\alpha \beta} \phi_{\alpha} \phi_{\beta} : + \frac{z}{2} : h \phi_{\gamma} \phi_{\gamma} : + z : h^{\alpha \beta} \phi_{\alpha} \phi_{\beta} : - \frac{\tilde{x}}{2} m^2 : h \phi \phi : \right), \]

(10.34)

with only two undetermined coefficients instead of four.

The analysis of perturbative gauge invariance to second order in the tree sector gives two possible solutions:

\[ d_Q T^{tree}_2 (x_1, x_2) + d_Q N^{tree}_2 (x_1, x_2) = \text{divergence} \iff \tilde{x} = z \text{ or } \tilde{x} = z + 1. \]

(10.35)

Both of the conditions on the right side are in agreement with the natural assumption that the first order coupling can be written as

\[ T^M_1 (x) = i \frac{\kappa}{2} : h^{\alpha \beta} (x) b_{\alpha \beta \mu \nu} \Theta^\mu \nu_M (x) : \]

(10.36)

for an improved energy-momentum tensor with \( \Theta^\mu \nu_M (x)_{\mu \nu} = 0 \). The \( b_{\alpha \beta \mu \nu} \)-tensor appears here because we are using the expansion of the Goldberberg variable.

Thus, we have seen that if we start with the most general ansatz for \( T^M_1 \), Eq. (10.33), with four undetermined parameters, then perturbative gauge invariance up to the second order is able to reduce this number to one. Analysis of the third order should then fix unambiguously this last parametric freedom.

10.2 Massless Case

As in the previous section, we investigate if the condition of perturbative quantum gauge invariance is strong enough to select by itself, among all the possible couplings between massless matter fields and gravitons, the right coupling, namely to select only one coupling which, in addition, should agree with the expansion of the classical Lagrangian.
Let us write the most general ansatz for the massless matter coupling (disregarding unimportant divergence couplings) as

\[ T^M_1(x) = \frac{i}{2} \left( + \tilde{x} : h^{\alpha\beta} \phi,\alpha \phi,\beta : + y : h \phi,\gamma \phi,\gamma : + z : h^{\alpha\beta} \phi,\alpha \phi,\beta : \right) \]

\[ = \frac{i}{2} h^{\alpha\beta}(x) b_{\alpha\beta\mu\nu} \Theta_M^{\mu\nu}(x) , \]

(10.37)

where \( \tilde{x}, y, z \in \mathbb{R} \) are undetermined coefficients and \( \Theta_M^{\mu\nu} \) an improved energy-momentum tensor with \( \Theta_M^{\mu\nu}(x)_{,\nu} = 0 \).

To establish a connection between our undetermined coefficients \( \tilde{x}, y, z \) and the classical theory, we expand the non-minimally matter coupled Lagrangian

\[ \tilde{L}_M = \frac{1}{2} \sqrt{-g} \left( g^{\mu\nu} \phi,\mu \phi,\nu + \xi R \phi^2 \right) \]

(10.38)
in terms of the graviton field and compare the coefficients. We obtain the relations: \( \tilde{x} = 1 - 2\xi, y = \xi \) and \( z = -2\xi \).

If \( \xi = 1/6 \) we obtain the Callan-Jackiw improved energy-momentum tensor [40].

On the other side we can also consider in Eq. (10.37) the most general conserved and traceless energy-momentum tensor

\[ \Theta_M^{\mu\nu} = \alpha \phi^{\mu} \phi^{\nu} - \frac{\alpha}{4} \eta^{\mu\nu} \phi,\gamma \phi,\gamma - \frac{\alpha}{2} \phi^{\mu\nu} , \quad \alpha \in \mathbb{R} , \]

(10.39)

which gives the relations \( \tilde{x} = \alpha, y = -\alpha/4 \) and \( z = -\alpha/2 \).

Gauge invariance to first order, \( d_Q T^M_1 = divergence \) is then always satisfied.

Gauge invariance to second order for the matter loop graviton self-energy, \( d_Q T^{SE}_2 = divergence \), requires that \( y = z/2 \).

Since the particle circulating in the loop is massless, we expect the self-energy tensor to be traceless, too. This implies \( y = -\tilde{x}/4 \) and \( z = -\tilde{x}/2 \).

With these relations among the parameters we can undertake the investigation of perturbative gauge invariance to second order in the tree graph sector. We find again

\[ d_Q T^{tree}_2(x_1, x_2) + d_Q N^{tree}_2(x_1, x_2) = divergence \iff \tilde{x} = z \text{ or } \tilde{x} = z + 1 . \]

(10.40)

Obviously the first relation \( \tilde{x} = z \) cannot be satisfied by our coefficients in both cases, Eq. (10.38) and Eq. (10.39), therefore should be rejected.

The second relation \( \tilde{x} = z + 1 \) is satisfied \( \forall \xi \in \mathbb{R} \) in the case of non-minimal matter coupling \( \xi R \phi^2 \). The reason is that this term has zero gauge variation so that its addition to the term \( : h^{\alpha\beta} \phi,\alpha \phi,\beta : \), which is already gauge invariant to first and second order alone, does not change the theory from the point of view of the gauge structure.

On the other side, if we examine the relation \( \tilde{x} = z + 1 \) in view of Eq. (10.39), we find that it has only one solution, namely \( \alpha = 2/3 \). Therefore, according to
our strategy, perturbative quantum gauge invariance to first and second order, together with some assumptions about the structure of the massless matter energy-momentum tensor, Eq. (10.39), leads to the coupling

\[ T^M_1(x) = i \frac{\kappa}{2} \left( \frac{2}{3} :h^{\alpha\beta}\phi_{\alpha}\phi_{\beta} : - \frac{1}{6} :h\phi_{\gamma}\phi_{\gamma} : - \frac{1}{3} :h^{\alpha\beta}\phi_{\alpha}\phi_{\beta} : \right) \].

(10.41)

This result is equivalent to the choice of \( \xi = 1/6 \) in Eq. (10.38) and, equivalently, to the use of the Callan-Jackiw improved energy-momentum tensor.

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