DEFORMATIONS OF THE NEIGHBORHOOD COMPLEX

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Abstract. The neighborhood complexes of graphs were constructed in the context of the graph coloring problem. In this paper, we show that some deformations of graphs do not change the simple homotopy types of the neighborhood complexes. As an application, we prove that there is a graph homomorphism \( f : G \rightarrow H \) such that \( f \) induces a \( \mathbb{Z}_2 \)-homotopy equivalence between the box complexes, but the chromatic numbers of \( G \) and \( H \) are different.

1. Introduction

The neighborhood complex \( N(G) \) were considered in the context of the graph coloring problem. Lovász proved in [7] that the connectivity of the neighborhood complex gives the lower bound of the chromatic number of a graph, and determine the chromatic number of the Kneser graphs.

The \( \mathbb{Z}_2 \)-heights or \( \mathbb{Z}_2 \)-indexes of the box complexes give the lower bounds of the chromatic numbers. See [1], [3], [8], and [9] for further studies among these topics.

In this paper, we show that some deformations of graphs do not change the simple homotopy types of neighborhood complexes. As an application, we prove the followings.

Theorem 1.1. For any positive integer \( n \) with \( n \geq 3 \), there is a graph homomorphism \( K_n \rightarrow Y_n \) such that \( N(Y_n) \) collapses to \( N(K_n) \), and \( S^n \cong \text{Hom}(K_2, K_n) \hookrightarrow B(K_2, Y_n) \) induces a \( \mathbb{Z}_2 \)-homotopy equivalence, but \( \chi(Y_n) = n + 1 \).

Theorem 1.2. For any positive integer \( k \), there is a graph homomorphism \( f : G \rightarrow H \) such that \( f \) induces a simple homotopy equivalence \( N(G) \rightarrow N(H) \), and \( \mathbb{Z}_2 \)-homotopy equivalence from \( \text{Hom}(K_2, G) \) to \( \text{Hom}(K_2, H) \), but \( \chi(H) \geq \chi(G) + k \).

This paper is organized as follows. In Section 2, we review definitions and facts we need, and fix some terminologies and notations. In Section 3, we prove that some deformations do not change the simple homotopy equivalence of the neighborhood complex. In Section 4, we prove the above two theorems.

2. Preliminaries

In this section, we collect and fix some definitions and facts about topological combinatorics we need in this paper.

A graph we say in this paper is a pair \((V, E)\) such that \( V \) is a set and \( E \) is a symmetric subset of \( V \times V \), that is, \((x, y) \in E\) implies \((y, x) \in E\). For a graph \( G = (V, E) \), we write \( V(G) \) for the set \( V \), called the vertex set of \( G \), and \( E(G) \) for the set \( E \). In this paper, we assume that all graphs are finite. A graph homomorphism \( f \) from a graph \( G \) to a graph \( H \) is a set map from \( V(G) \) to \( V(H) \) such that \((f \times f)(E(G)) \subseteq E(H)\).
For a non-negative integer $n$, $K_n$ denotes the complete graph with $n$-vertices, that is, the graph defined by $V(K_n) = \{1, \ldots, n\}$ and $E(K_n) = \{(x, y) \mid x \neq y\}$. For a graph $G$, the chromatic number $\chi(G)$ of $G$ is the number $\inf\{n \mid \text{There is a graph homomorphism from } G \text{ to } K_n\}$.

Let $G$ be a graph. For a vertex $v \in V(G)$, we write $N(v)$ or $N^G(v)$ for the subset $\{w \in V(G) \mid (v, w) \in E(G)\}$. The neighborhood complex $N(G)$ of $G$ is an abstract simplicial complex whose vertex set is the set of non-isolated vertices of $G$, and the set of simplices is

$$N(G) = \{\sigma \subset V(G) \mid \#\sigma < \infty \text{ and there is } v \in V(G) \text{ such that } \sigma \subset N^G(v)\}.$$ 

Let $G$ be a graph. A vertex $v$ of $G$ is said to be dismantlable if there is $w \in V(G) \setminus \{v\}$ such that $N(v) \subset N(w)$. If $v \in V(G)$ is a dismantlable vertex, then the induced subgraph $G \setminus v$ whose vertex set is $V(G) \setminus \{v\}$ is called the folding of $G$. We need the following lemma which is easily proved.

**Proposition 2.1.** Let $G$ be a graph and $v \in V(G)$ a dismantlable vertex. Then $N(G)$ collapses to $N(G \setminus v)$.

Hom$(K_2, G)$ is a poset defined by

$$\{(\sigma, \tau) \mid \sigma, \tau \subset V(G), \sigma \times \tau \subset E(G)\}.$$ 

with the ordering $(\sigma, \tau) \leq (\sigma', \tau') \iff \sigma \subset \sigma'$ and $\tau \subset \tau'$. As the notation shows, the Hom$(K_2, G)$ is one of the complex Hom$(T, G)$ called the Hom complex, see [4, 6] for example. But we only need Hom$(K_2, G)$ in this paper, so the precise definitions of Hom complexes are omitted. The fact we need about Hom$(K_2, G)$ is that there is a natural homotopy equivalence $|\text{Hom}(K_2, G)| \to |N(G)|$, see [3] for the proof.

The terminology and facts about simple homotopy theory is found in [2]. We need the following elementary fact of topology, called the gluing lemma.

**Proposition 2.2.** Let $X$ and $Y$ be CW-complexes, $A_1, A_2$ subcomplexes of $X$, $B_1, B_2$ subcomplexes of $Y$, and $f : X \to Y$ a continuous map. Suppose the following conditions hold.

1. $X = A_1 \cup A_2$, $Y = B_1 \cup B_2$
2. $f(A_i) \subset B_i$ ($i = 1, 2$), and $f|_{A_i} : A_i \to B_i$ ($i = 1, 2$) and $f|_{A_1 \cap A_2} : A_1 \cap A_2 \to B_1 \cap B_2$ are homotopy equivalence.

Then $f : X \to Y$ is a homotopy equivalence.

3. Deformations

In this section, we prove that some types of deformations of graphs do not change the simple homotopy type of the neighborhood complexes.

Let $L_3$ denote the graph defined by

$$V(L_3) = \{0, 1, 2, 3\}, \quad E(L_3) = \{(x, y) \in V(L_3)^2 \mid |x - y| = 1\}.$$ 

**Proposition 3.1.** Let $G$ be a graph, $x, w \in V(G)$. Suppose the followings hold.

1. $x \neq w$.
2. One of $x$ or $w$ is non-looped, namely, $(x, x) \notin E(G)$ or $(w, w) \notin E(G)$.
3. There is only one graph homomorphism $f : L_3 \to G$ such that $f(0) = x$ and $f(3) = w$. 


Let $H$ denote the graph defined by $V(H) = V(G)$ and $E(H) = E(G) \cup \{(x, w), (w, x)\}$. Then $N(G)$ is a deformation retract of $N(H)$. Moreover, if $G$ and $H$ are finite, then $N(H)$ collapses to $N(G)$.

Proof. Put $f(1) = y$ and $f(2) = z$. Let $A$ denote the subcomplex of $N(H)$ defined by $\{\sigma \in N(H) \mid \sigma \subset N^H(x)\}$ and $B$ denote the subcomplex of $N(H)$ defined by $\{\sigma \in N(H) \mid \sigma \subset N^H(w)\}$. Obviously, $A$ and $B$ are contractible.

Remark that for $v \in V(G)$ such that $v \neq x$ and $v \neq w$, we have $N^H(v) = N^G(v)$, and $N^H(x) = N^G(x) \cup \{w\}$, and $N^H(w) = N^G(w) \cup \{x\}$.

Claim 1. $N(G) \cap A = \{\sigma \in N(G) \mid \sigma \subset N^G(x)\}$ or $\sigma \subset \{y, w\}$ and $N(G) \cap B = \{\sigma \in N(G) \mid \sigma \subset N^G(w)\}$ or $\sigma \subset \{x, z\}$.

Let us prove Claim 1. We only give the proof of the case of $N(G) \cap A$. The other is similar. The proof of the part “$\subset$” is obvious. Suppose $\sigma \in N(G) \cap A$ such that $\sigma \subset N^G(x)$ and $\sigma \not\subset \{y, w\}$. Since $\sigma \subset N^H(x) = N^G(x) \cup \{w\}$, this implies that $w \in \sigma$ and there is $y' \in N^G(x)$ such that $y' \in \sigma$ and $y' \neq y$. Since $\sigma \in N(G)$, there is $z' \in V(G)$ such that $\sigma \subset N^G(z')$. Then the map $f' : V(L_3) \to V(G)$ defined by $f(0) = x$, $f(1) = y'$, $f(2) = z'$, and $f(3) = w$ is a graph homomorphism. This contradicts the assumption (3). This completes the proof of Claim 1.

Claim 2. $A \cap B \subset N(G)$

Since $x$ and $w$ are not equal, we have $\{x\} \cap \{w\} = \emptyset$. Suppose $x$ and $w$ are non-looped. Then we have $N^G(x) \cap \{x\} = \emptyset$ and $\{w\} \cap N^G(w) = \emptyset$. Hence we have

$$N^H(x) \cap N^H(w) = (N^G(x) \cup \{w\}) \cap (N^G(w) \cup \{x\}) = N^G(x) \cap N^G(w)$$

Hence we have $A \cap B \subset N(G)$.

Next suppose $x$ is non-looped and $w$ is looped. By the same way of the previous paragraph, we have $N^H(x) \cap N^H(w) = (N^G(x) \cap N^G(w)) \cup \{w\} \subset N^G(w)$. Hence we have $A \cap B \subset N(G)$. This completes the proof of Claim 2.

By Claim 2, we have $(FA \setminus FN(G)) \cap (FB \setminus FN(G)) = \emptyset$, and any element of $FA \setminus FN(G)$ and any element of $FB \setminus FN(G)$ are not comparable. Hence the pairs

$$\sigma \leftrightarrow \sigma \cup \{y\}, (\sigma \in FA \setminus FN(G), y \notin \sigma),$$
$$\sigma \leftrightarrow \sigma \cup \{z\}, (\sigma \in FB \setminus FN(G), z \notin \sigma)$$

give the acyclic partial matching of $(FA \cup FB) \setminus FN(G) = FN(H) \setminus FN(G)$ which has no critical point. Hence $N(H)$ collapses to $N(G)$. □

Let $G$ be a graph and $e = \{(x, w), (w, x)\}$ an edge of $G$. Define the graph $G_e$ by $V(G_e) = V(G) \cup \{1, 2\}$,

$$E(G_e) = (E(G) \cup \{(x, 1), (1, x), (1, 2), (2, 1), (2, w), (w, 2)\}) \setminus e.$$ 

There is a graph homomorphism $r_e : G_e \to G$, which maps 1 to $w$, and 2 to $x$.

**Proposition 3.2.** Let $G$ be a graph and $e = \{(x, w), (w, x)\}$ an edge of $G$ satisfying that there is no graph homomorphism $L_3 \to G \setminus e$ from $x$ to $w$. Then the graph homomorphism $r_e$ induces a simple homotopy equivalence $N(G_e) \to N(G)$.

**Proof.** Let $H$ denote the graph obtained from $G_e$ by adding an edge connecting $x$ with $w$. By Theorem 3.2, we have that $N(H)$ collapses to $N(G_e)$. By Proposition 2.1, $N(H)$ collapses to $N(G)$. This implies that the retraction $s_e : H \to G$, defined
by \( s_e(v) = r_e(v), \) \( v \in V(G) \), induces a simple homotopy equivalence \( \text{N}(H) \rightarrow \text{N}(G) \).

Hence the composition \( \text{N}(G_e) \rightarrow \text{N}(H) \rightarrow \text{N}(G) \) which is equal to the simplicial map induced by \( r_e \) is a simple homotopy equivalence. \( \square \)

4. Construction

In this section, we prove theorems announced in Section 1.
Before giving the proofs, we recall the following fact of equivariant homotopy theory.

**Theorem 4.1.** Let \( \Gamma \) be a discrete group, \( X \) and \( Y \) free \( \Gamma \)-CW-complexes, \( f \) a \( \Gamma \)-equivariant continuous map from \( X \) to \( Y \). Then \( f \) is a \( \Gamma \)-homotopy equivalence if and only if \( f \) is a homotopy equivalence.

Recall that if \( G \) is a graph having no looped vertex, then \( \text{Hom}(K_2, G) \) is a free \( \mathbb{Z}_2 \)-complex. Hence for a graph homomorphism \( f : G \rightarrow H \) induced a \( \mathbb{Z}_2 \)-homotopy equivalence \( f_* : \text{Hom}(K_2, G) \rightarrow \text{Hom}(K_2, H) \) if \( f \) induces a homotopy equivalence \( \text{N}(G) \rightarrow \text{N}(H) \).

Let us start to prove Theorem 1.1. Define the graph \( Z \) by

\[
V(Z) = \{(x, y) \in \mathbb{Z} \mid 0 \leq x \leq 9, 0 \leq y \leq 1\} \cup \{(k, 2) \mid k = 1, 2, 7, 8\},
\]

\[
E(Z) = \{(x, y), (x', y') \mid |x - x'| + |y - y'| = 1\}
\]

The graph \( Y \) is obtained from \( Z \) by identifying the following vertices.

- The vertex \((k, 0) (k = 0, 1, 2)\) is identified with \((k + 3, 0)\) and \((k + 6, 0)\).
- The vertex \((0, k) (k = 0, 1)\) is identified with \((9, k)\).
- The vertex \((k, 2) (k = 1, 2)\) is identified with \((9 - k, 2)\).

The graph \( Z \)

It is easy to show that \( \chi(Y) = 4 \).

Let \( X \) denote the induced subgraph of \( Y \) with the vertices represented by \((0, 0)\), \((1, 0)\), or \((2, 0)\). Remark that \( X \cong K_3 \). By Proposition 2.1 and Proposition 3.1, we can prove that \( N(Y) \) collapses to \( N(X) \) (\( \cong S^1 \)).

For a graph \( G \), we write the graph \( G^+ \) for the graph defined by \( V(G^+) = V(G) \cup \{\ast\} \) and \( E(G^+) = E(G) \cup (V(G) \times \{\ast\}) \cup (\{\ast\} \times V(G)) \). It is obvious that \( \chi(G^+) = \chi(G) + 1 \).

**Lemma 4.2.** The followings hold.

1. For any non-empty graph \( G \), \( N(G^+) \) is homotopy equivalent to the suspension of \( N(G) \).

2. Let \( f : G \rightarrow H \) be a graph homomorphism between non-empty graph such that \( f \) induces a homotopy equivalence \( N(G) \rightarrow N(H) \). Then the map \( N(G^+) \rightarrow N(H^+) \) induced by \( f^+ \) is a homotopy equivalence.

**Proof.** The proof of (1) is found in [3], and we give only the proof of (2).

Let \( A_G \) and \( B_G \) for the subcomplex of \( N(G) \) defined by
\[ A_G = \{ \sigma \subset V(G^+) \mid \sharp \sigma < \infty \text{ and there is } v \in V(G) \text{ such that } \sigma \subset N^{G^+}(v) \}. \]

\[ B_G = \{ \sigma \subset V(G) \mid \sharp \sigma < \infty \}. \]

It is easy to see that \( B_G \) and \( A_G \) are contractible, and \( A_G \cup B_G = N(G^+) \) and \( A_G \cap B_G = N(G) \). Remark that \( f^+ \) induces simplicial maps \( A_G \to A_H \) and \( B_G \to B_H \). Since the simplicial map \( N(G) \to N(H) \) induced by \( f \) is a homotopy equivalence, we have that the simplicial maps \( A_G \to A_H, B_G \to B_H, \) and \( A_G \cap B_G \to A_H \cap B_H \) induced by \( f^+ \) are homotopy equivalences. Hence we have that the map \( N(G^+) \to N(H^+) \) induced by \( f^+ \) is a homotopy equivalence. \( \square \)

We define \( Y_k (k \geq 3) \) and \( X_k (k \geq 3) \) as follows. First put \( Y_3 = Y \) and \( X_3 = X \). For \( k \geq 4 \), we put \( Y_k = Y_{k-1}^+ \) and \( X_k = X_{k-1}^+ \). If \( k = 3 \), the inclusion \( N(X_3) \hookrightarrow N(Y_3) \) is a simple homotopy equivalence since \( N(Y_3) \) collapses to \( N(X_3) \).

Suppose \( k \geq 4 \). By Lemma 4.2, we have that the inclusion \( X_k \hookrightarrow Y_k \) induces a homotopy equivalence \( N(X_k) \to N(Y_k) \), and is a simple homotopy equivalence since they are 1-connected. Since \( X_k = K_k \), this completes the proof of Theorem 1.1.

Next let us prove Theorem 1.2. Let \( n \) be any positive integer. By the fact proved by Erdös, there is a finite graph \( G \) whose girth is greater than 4 and \( \chi(G) \geq n \). Let \( e_1, \ldots, e_n \) be the all edges of \( G \). Since the girth of \( G \) is greater than 4, we have that each \( e_i \) satisfies the condition of Proposition 3.2. Put \( H = G_{e_1 \cdots e_n} \) and \( f : H \to G \) be the composition of

\[ G_{e_1 \cdots e_n} \to G_{e_1 \cdots e_{n-1}} \to \cdots \to G_{e_1} \to G. \]

By Proposition 3.2, \( f \) induces a simple homotopy equivalence \( N(H) \to N(G) \). But it is obvious that \( \chi(H) = 3 \). This completes the proof of Theorem 1.2.

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