CONVEX SEPARABLY RATIONALLY CONNECTED COMPLETE INTERSECTIONS

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Abstract. We give a generalization of a result of R. Pandharipande to
arbitrary characteristic: We prove that, if \( X \) is a convex, separably rationally
connected, smooth complete intersection in \( \mathbb{P}^N \) over an algebraically closed
field of arbitrary characteristic, then \( X \) is rational homogeneous.

1. Introduction

F. Campana and T. Peternell [3] conjectured that, if \( X \) is a smooth Fano variety
(over \( \mathbb{C} \)) with nef tangent bundle, then \( X \) is rational homogeneous. This was
answered over \( \mathbb{C} \) affirmatively by Campana and Peternell in the case of \( \dim(X) \leq 3 \)
[3] and in the case of \( \dim(X) = 4 \) with Picard number \( \rho_X > 1 \) [4], by N. Mok [10]
and J.-M. Hwang [7] for any 4-dimensional \( X \), by K. Watanabe [12] in the case
of \( \dim(X) = 5 \) with \( \rho_X > 1 \), and by A. Kanemitsu [8] for any 5-dimensional \( X \)
(moreover, in the case of \( \rho_X > \dim(X) - 5 \)).

We say that a variety \( X \) is convex if every morphism \( \mu : \mathbb{P}^1 \to X \) satisfies
\( H^1(\mathbb{P}^1, \mu^*T_X) = 0 \). R. Pandharipande [11] proved that, if \( X \) is a convex, rationally
connected, smooth complete intersection in \( \mathbb{P}^N \) over \( \mathbb{C} \), then \( X \) is rational homoge-
neous. Since a smooth Fano variety is rationally connected (over \( \mathbb{C} \)), and since
nefness of the tangent bundle implies convexity, the result solved the conjecture for
complete intersections.

Question 1.1. Does the statement of the above theorem of Pandharipande hold
in arbitrary characteristic?

In this paper, we prove:

Theorem 1.2. Let \( X \) be a convex, smooth complete intersection in \( \mathbb{P}^N \) over an
algebraically closed field of arbitrary characteristic. Assume that: (i) there exists
an immersion \( \mathbb{P}^1 \to X \). Then \( X \) is of degree \( \leq 2 \); in particular, \( X \) is rational
homogeneous.

Note that a smooth complete intersection \( X \subset \mathbb{P}^N \) satisfies the above condition
(i) if one of the following conditions (ii-iv) holds: (ii) \( X \) is separably rationally
connected; (iii) \( X \) is Fano; (iv) \( 2N - 2 - r \geq \sum_{i=1}^{r} d_i \) holds, where \( (d_1, \ldots, d_r) \) is the
type of \( X \). (See Remark 4.2.) As a corollary, a smooth Fano complete intersection
with nef tangent bundle is rational homogeneous in arbitrary characteristic.

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We can assume that the type \((d^1, \ldots, d^r)\) of \(X\) satisfies \(d^i > 1\) for all \(i\). The convexity of \(X\) is equivalent to the freeness of all the rational curves on \(X\) (see Remark 2.8), where a rational curve \(\mu : \mathbb{P}^1 \to X\) is said to be free if

\[H^1(\mathbb{P}^1, \mu^* T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0.\]

Roughly speaking, in order to prove Theorem 1.2, we show that if \(\text{deg}(X) = \prod_{1 \leq i \leq r} d^i > 2\), then \(X\) has a non-free rational curve and hence is not convex.

The paper is organized as follows. In section 2 we first study the parameter space \(\mathcal{H}\) of complete intersections of type \((d^1, \ldots, d^r)\) and the incidence variety \(I \subset \mathbb{G}(1, \mathbb{P}^N) \times \mathcal{H}\) parametrizing pairs \((L, X)\) such that \(L \subset X\); the techniques are based on [2, 7, 8, 9 V. 4]. Next we define the subset \(J \subset I\) parametrizing pairs \((L, X)\) such that \(L\) is non-free in \(X\), and we give a calculation method of the defining polynomials of \(J\). In section 3 assuming the product \(\prod d^i > 2\), we construct a pair \((L, X)\) such that the space of non-free lines in \(L, X\) is non-free in \(X\), and we give a calculation method of expected dimension at characteristic 2 case (see Remark 3.4). In consequence, we show that \(\text{pr}_2 : J \to \mathcal{H}\) is surjective. Thus, under our assumption, every complete intersection \(X\) of type \((d^1, \ldots, d^r)\) has a non-free rational curve; then we have the proof of Theorem 1.2 in section 4.

2. Parameter space of complete intersections

2.1. Incidence variety and its projection. We fix some notation. Let \(\mathcal{H}_d\) be the projectivization of the vector space \(H^0(\mathbb{P}^N, \mathcal{O}(d))\), whose general members parametrize hypersurfaces of degrees \(d\) in \(\mathbb{P}^N\). We take \(d = (d^1, \ldots, d^r)\) with \(r\) positive integers \(d^1, \ldots, d^r\), and denote by \(|d| := \sum_{1 \leq i \leq r} d^i\). Let

\[\mathcal{H} := \mathcal{H}_{d^1} \times \cdots \times \mathcal{H}_{d^r},\]

whose general member \(h = (h^1, \ldots, h^r)\) defines an \((N - r)\)-dimensional complete intersection \(X \subset \mathbb{P}^N\) of type \(d\). For a homogeneous polynomial \(\varphi \in H^0(\mathbb{P}^N, \mathcal{O}(d))\) and a line \(L \subset \mathbb{P}^N\), we denote by \(\varphi|_L\) the image of \(\varphi\) under the linear map \(H^0(\mathbb{P}^N, \mathcal{O}(d)) \to H^0(L, \mathcal{O}(d))\).

Definition 2.1. We set

\[I = \{ (L, h) \in \mathbb{G}(1, \mathbb{P}^N) \times \mathcal{H} | h^1|_L = \cdots = h^r|_L = 0 \},\]

the incidence variety whose general members parametrize pairs \((L, X)\) such that \(L \subset X\). We denote by

\[\text{pr}_1 : I \to \mathbb{G}(1, \mathbb{P}^N), \quad \text{pr}_2 : I \to \mathcal{H},\]

the first and second projections.

Remark 2.2. Each fiber of \(I \to \mathbb{G}(1, \mathbb{P}^N)\) at \(L\) is isomorphic to \(I_{d^1, L} \times \cdots \times I_{d^r, L}\), where we denote by \(\mathcal{I}_L \subset \mathcal{O}_{\mathbb{P}^N}\) the ideal sheaf of \(L \subset \mathbb{P}^N\) and by \(I_{d, L} \subset \mathcal{H}_d\) the projectivization of \(H^0(\mathbb{P}^N, \mathcal{I}_L(d))\). Note that \(I_{d, L} \subset \mathcal{H}_d\) parametrizes hypersurfaces of degree \(d\) containing the line \(L\). From the exact sequence

\[0 \to H^0(\mathbb{P}^N, \mathcal{I}_L(d)) \to H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(L, \mathcal{O}_L(d)) \to 0,\]

we have \(\text{codim}(I_{d, L}, \mathcal{H}_d) = d + 1\). Hence

\[\text{codim}(I, \mathcal{H} \times \mathbb{G}(1, \mathbb{P}^N)) = \text{codim}(I_{d^1, L} \times \cdots \times I_{d^r, L}, \mathcal{H}) = |d| + r.\]
In addition, 
\begin{equation}
(1) \quad \dim(I) - \dim(\mathcal{H}) = \dim \mathbb{G}(1, \mathbb{P}^N) - (|d| + r) = 2N - 2 - |d| - r.
\end{equation}

We denote by 
\[(S : T : Z_1 : Z_2 : \cdots : Z_{N-1})\]
the homogeneous coordinates on $\mathbb{P}^N$.

**Definition 2.3.** We take a line $L \subset \mathbb{P}^N$ and choose coordinates on $\mathbb{P}^N$ such that $L \cap (S = T = 0) = \emptyset$. For a homogeneous polynomial $h \in H^0(\mathbb{P}^N, \mathcal{O}(d))$ with $h|_L = 0$, we define a homomorphism $\delta_L(h) : \mathcal{O}_L(1)^{\oplus N-1} \to \mathcal{O}_L(d)$ by
\[(f_1, \cdots, f_{N-1}) \mapsto h_{Z_1}|_L \cdot f_1 + \cdots + h_{Z_{N-1}}|_L \cdot f_{N-1},\]
where $h_{Z_i} := \partial h/\partial Z_i \in H^0(\mathbb{P}^N, \mathcal{O}(d-1))$.

Now, we take $h = (h^1, \ldots, h^r) \in \mathcal{H}$ such that $(L, h) \in I$. Then, $r$ homomorphisms $\delta_L(h^1), \ldots, \delta_L(h^r)$ induce a homomorphism
\[\delta_L(h) : \mathcal{O}_L(1)^{\oplus N-1} \to \bigoplus_{1 \leq i \leq r} \mathcal{O}_L(d^i) \cdot \mathbf{e}_i.\]

**Remark 2.4.** Let $K$ be the ground field. The image of the linear map
\[H^0(\delta_L(h)(-1)) : H^0(L, \mathcal{O})^{\oplus N-1} = K^{\oplus N-1} \to \bigoplus_{1 \leq i \leq r} H^0(L, \mathcal{O}(d^i - 1)) \cdot \mathbf{e}_i\]
is equal to the vector subspace spanned by $N-1$ elements
\[\sum_{1 \leq i \leq r} h^i_{Z_1}|_L \cdot \mathbf{e}_i, \ldots, \sum_{1 \leq i \leq r} h^i_{Z_{N-1}}|_L \cdot \mathbf{e}_i.\]

Although $\delta_L(h)$ depends on the choice of coordinates on $\mathbb{P}^N$, the rank of $H^0(\delta_L(h)(-1))$ is the same because of the following lemma.

**Lemma 2.5.** Let $(L, h) \in I$ be as above. Let $X \subset \mathbb{P}^N$ be the complete intersection defined by $h$, and assume that $X$ is smooth along $L$. Then $\delta_L(h)$ can be identified with the natural map $N_{L/\mathbb{P}^N} \to N_{X/\mathbb{P}^N}|_L$.

**Proof.** We first consider the case of $r = 1$. The exact sequence $0 \to \mathcal{I}_L^T \to \mathcal{I}_L \to N_{L/\mathbb{P}^N} \to 0$ induces the following linear map:
\[
(2) \quad H^0(\mathbb{P}^N, \mathcal{I}_L(d)) \to H^0(L, N_{L/\mathbb{P}^N}^\vee \otimes \mathcal{O}(d)).
\]
Let $h \in H^0(\mathbb{P}^N, \mathcal{I}_L(d))$ and let $X \subset \mathbb{P}^N$ be the hypersurface defined by $h$. Then the image of $h$ under the above map gives the natural homomorphism $N_{L/\mathbb{P}^N} \to N_{X/\mathbb{P}^N}|_L \simeq \mathcal{O}_L(d)$ (see [4] Rem. 3.9).

Since $L \cap (S = T = 0) = \emptyset$, we can write $L = (W_1 = \cdots = W_{N-1} = 0)$, where $W_j = Z_j - a_j S - b_j T$ with $a_j, b_j \in K$. Then we can write $h = h_1 W_1 + \cdots + h_{N-1} W_{N-1}$. Here the image of $h$ in $H^0(L, N_{L/\mathbb{P}^N}^\vee \otimes \mathcal{O}(d)) \simeq H^0(L, \mathcal{O}(d-1))^{\oplus N-1}$ under (2) is given by $(h_1|_L, \ldots, h_{N-1}|_L)$. On the other hand, we have $h_{Z_i}|_L = h_j|_L$ (we also have $h_{W_j}|_L = h_j|_L$). Hence $N_{L/\mathbb{P}^N} \to N_{X/\mathbb{P}^N}$ can be identified with $\delta_L(h) : \mathcal{O}_L(1)^{\oplus N-1} \to \mathcal{O}_L(d)$.

Next, for any $r \geq 1$, considering the linear map
\[H^0(\mathbb{P}^N, \mathcal{I}_L \bigoplus \bigoplus_{1 \leq i \leq r} \mathcal{O}_N(d^i)) \to H^0(L, N_{L/\mathbb{P}^N}^\vee \bigoplus_{1 \leq i \leq r} \mathcal{O}_L(d^i)),\]
instead of (2) we have the assertion in a similar way to the case of $r = 1$. □
Remark 2.6. The linear map $H^0(\delta_L(h)(-1))$ is surjective if and only if $L$ is free in $X$, i.e., $H^1(L, N_{L/X}(-1)) = 0$. This follows from $H^1(L, N_{L/\mathbb{P}^N}(-1)) = 0$ and the following exact sequence:

$$0 \to N_{L/X} \to N_{L/\mathbb{P}^N} \xrightarrow{\delta_L(h)} N_{X/\mathbb{P}^N}|_L \to 0.$$  

Lemma 2.7. Assume $2N - 2 - r \geq |d|$. Then $pr_2 : I \to \mathcal{H}$ is surjective, which means that every complete intersection of type $d$ in $\mathbb{P}^N$ contains a line.

Proof. It holds, for example, by a dimension count in the next subsection (see Remark 2.13).

Remark 2.8. A smooth projective variety $X$ is convex if and only if any rational curve on $X$ is free. The reason is as follows.

Suppose that there exists a non-free rational curve $\mu : \mathbb{P}^1 \to X$. For the splitting $\mu^*T_X \simeq \bigoplus_{i=1}^{N-r} \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \in \mathbb{Z}$, it follows $a_{i_0} \leq -1$ for some $i_0$. Taking a double cover $\iota_2 : \mathbb{P}^1 \to \mathbb{P}^1$, since the splitting of $(\iota_2 \circ \mu)^*T_X$ contains $\mathcal{O}_{\mathbb{P}^1}(2a_{i_0})$, we have $H^1(\mathbb{P}^1, (\iota_2 \circ \mu)^*T_X) \neq 0$. Then $X$ is not convex.

The converse holds in a similar way; we note that a non-constant morphism $\mu : \mathbb{P}^1 \to X$ factors as $\mathbb{P}^1 \to \mathbb{P}^1 \xrightarrow{n} \mu(\mathbb{P}^1)$, where $n$ is the normalization of the rational curve $\mu(\mathbb{P}^1) \subset X$.

Let us consider non-free lines on complete intersections. We note that $L$ is not fixed anymore.

Definition 2.9. We denote by $J$ the set of pairs $(L, h) \in I$ such that the linear map $H^0(\delta_L(h)(-1))$ is not surjective.

Assume that a complete intersection $X \subset \mathbb{P}^N$ defined by $h \in \mathcal{H}$ is smooth along a line $L$. Then $(L, h) \in J$ if and only if $L$ is non-free in $X$, because of Remark 2.6.

Theorem 2.10. Assume that $2N - 2 - r \geq |d|$, $\prod_{1 \leq i \leq r} d_i > 2$, and all $d_i > 1$. Then $pr_2|_J : J \to \mathcal{H}$ is surjective.

Remark 2.11. If $|d| \geq N$, then we have $J = I$. This immediately follows from Remark 2.4. Thus $J \to \mathcal{H}$ is surjective, because of Lemma 2.7. If $|d| = N - 1$, then the surjectivity of $pr_2|_J$ follows from a dimension count in the next subsection (see Remark 2.14).

We denote by $J_h := \text{pr}_1(\text{pr}_2^{-1}(h) \cap J) \subset \mathbb{G}(1, \mathbb{P}^N)$. In the case of $|d| \leq N - 2$, the above theorem follows from:

Proposition 2.12. Assume that $N - 2 \geq |d|$, $\prod_{1 \leq i \leq r} d_i > 2$, and all $d_i > 1$. Then there exists a pair $(L, h) \in J$ such that $J_h$ is smooth and of dimension $N - r - 2$ at the point $L \in \mathbb{G}(1, \mathbb{P}^N)$.

Our goal is to construct an expected pair $(L, h)$, satisfying the statement of Proposition 2.12. In the next subsection, as a preparation, we give defining polynomials of $J_h$ on an affine open subset of $\mathbb{G}(1, \mathbb{P}^N)$.

2.2. Defining polynomials of the space of non-free lines. Let us fix $(s : t)$ as the homogeneous coordinates on $\mathbb{P}^1$, and $(S : T : Z_1 : Z_2 : \cdots : Z_{N-1})$ on $\mathbb{P}^N$. We set $\mathbb{G}_1 \subset \mathbb{G}(1, \mathbb{P}^N)$ to be the standard open subset, which is the set of lines $L$ such that $L \cap (S = T = 0) = \emptyset$. Here $\mathbb{G}_1 \simeq \mathbb{A}^{2(N-1)}$, which maps $L \in \mathbb{G}_1$ to

$$A_L = \begin{bmatrix} a_1, \ldots, a_{N-1} \\ b_1, \ldots, b_{N-1} \end{bmatrix} \in \mathbb{A}^{2(N-1)}$$

(3)
if the line $L \subset \mathbb{P}^N$ is equal to the image of the morphism $\mathbb{P}^1 \to \mathbb{P}^N$ defined by
\begin{equation}
(s : t) \mapsto [s \ t] \cdot [E_2 \ A_L] = (s : t : sa_1 + tb_1 : \cdots : sa_{N-1} + tb_{N-1}),
\end{equation}
where $E_2$ is the unit matrix of size 2. We set
\[\xi : G_1^0 \times \mathbb{P}^1 \to \mathbb{P}^N\]
to be the morphism which sends $(L, (s : t)) \mapsto [s \ t] \cdot [E_2 \ A_L]$.

Let $X \subset \mathbb{P}^N$ be a complete intersection defined by homogeneous polynomials $h = (h_1, \ldots, h^r) \in \mathcal{I}$. The composite function $h^1 \circ \xi$ is regarded as a polynomial with variables $s, t, a_1, \ldots, a_{N-1}, b_1, \ldots, b_{N-1}$. Then $h^1 \circ \xi$ is written as
\[f^1_0 s^d + f^1_1 s^{d-1} t + \cdots + f^1_d t^d,
\]
with some $f^1_0, f^1_1, \ldots, f^1_d \in K[a_1, \ldots, a_{N-1}, b_1, \ldots, b_{N-1}]$. For a line $L \in G_1^0$, which was expressed as (3), we have $L \subset X$ if and only if
\[f^1_k|L = 0 \quad \text{for any} \quad 1 \leq i \leq r \quad \text{and} \quad 0 \leq k \leq d^i.
\]

Remark 2.13. Let $F_h := \text{pr}_1(\text{pr}_2^{-1}(h) \cap I) \subset G_1(1, \mathbb{P}^N)$, which is equal to the set of lines lying in $X$. Then $F_h \cap G_1^0$ is equal to the zero set of $(r + |d|)$-polynomials $f^i_k$'s in the affine space $G_1^0 \subset \mathbb{A}^{2(N-1)}$. In particular, by choosing coordinates on $\mathbb{P}^N$, $\deg(f^i_k) > 0$ and thus $\dim F_h \cap G_1^0 \geq 2(N-1) - (r + |d|)$.

Now we assume $2(N-1) - (r + |d|) \geq 0$. For a homogeneous polynomial $h \in K[S, T, Z_1, \ldots, Z_{N-1}]$ of degree $d$ with $h \circ \xi = f_0 s^d + f_1 s^{d-1} t + \cdots + f_d t^d$, we have
\begin{equation}
(h \circ \xi)_{a_j} = f_{0,a_j} s^d + f_{1,a_j} s^{d-1} t + \cdots + f_{d,a_j} t^d,
\end{equation}
\begin{equation}
(h \circ \xi)_{b_j} = f_{0,b_j} s^d + f_{1,b_j} s^{d-1} t + \cdots + f_{d,b_j} t^d,
\end{equation}
where $(h \circ \xi)_{a_j} := \partial(h \circ \xi)/\partial a_j$ and so on. In addition, it follows that
\begin{equation}
(h \circ \xi)_{a_j} = s \cdot (h_{Z_j} \circ \xi) \quad \text{and} \quad (h \circ \xi)_{b_j} = t \cdot (h_{Z_j} \circ \xi).
\end{equation}
This is shown as follows: We write the morphism $\xi$ by the coordinates $(\eta_0 : \eta_1 : \xi_1 : \cdots : \xi_{N-1})$, which are given as in the right hand side of (4). Then $(h \circ \xi)_{a_j}$ is equal to $\partial \eta_0/\partial a_j \cdot (\partial h/\partial S) \circ \xi + \partial \eta_1/\partial a_j \cdot (\partial h/\partial T) \circ \xi + \sum_{1 \leq i \leq N-1} \partial \xi_i/\partial a_j \cdot (\partial h/\partial Z_i) \circ \xi = s \cdot (h_{Z_j} \circ \xi)$. Similarly, $(h \circ \xi)_{b_j}$ is also obtained.

We denote by $M(h)$ the $(N-1) \times d$ matrix such that
\[
\begin{bmatrix}
h_{Z_1} \circ \xi \\
h_{Z_2} \circ \xi \\
\vdots \\
h_{Z_{N-1}} \circ \xi
\end{bmatrix} = M(h) \cdot
\begin{bmatrix}
s^{d-1} \\
s^{d-2} t \\
\vdots \\
t^{d-1}
\end{bmatrix},
\]
where each entry of $M(h)$ is a polynomial with variables $a_1, \ldots, a_{N-1}, b_1, \ldots, b_{N-1}$.

Now, for $h = (h^1, \ldots, h^r)$, let us consider the $(N-1) \times |d|$ matrix
\begin{equation}
M(h) := [M(h^1) \ \cdots \ M(h^r)],
\end{equation}
where $M(h^i)$ is an $(N-1) \times d^i$ submatrix of $M(h)$ and recall $|d| = \sum d^i$.

For a line $L \in G_1^0$ expressed as (3), it follows from Remark 2.4 that $M(h)|_L$ is of rank $|d|$ if and only if $H^0(\delta_L(h)(-1))$ is surjective. Then $J_h \cap G_1^0$ is the zero set in $F_h \cap G_1^0$ of the polynomials given as the $|d| \times |d|$ minors of $M(h)$. 

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Remark 2.14. If \(|d| = N - 1\), then \(M(h)\) is a square matrix, and hence \(J_h \cap G^0\) is the zero set in \(F_h \cap G^0\) of the determinant of \(M(h)\). In this case, if \(|d| > 2\) and \(h\) is general, then by choosing coordinates on \(\mathbb{P}^N\), the determinant is a polynomial of positive degree in \(K[a_1, \ldots, a_{N-1}, b_1, \ldots, b_{N-1}]\), and then \(\dim J_h \geq \dim F_h - 1 \geq 2(N - 1) - (r + |d|) - 1 = |d| - r - 1\), where the right hand side is \(\geq r - 1\) because of all \(d^i > 1\); hence \(\dim J_h \geq 0\). If \(|d| = N - 1 = 2\), then the determinant can be constant, and then \(J_h = \emptyset\); for example, if \(h = SZ_1 + TZ_2\), the defining equation of a quadric surface in \(\mathbb{P}^3\), then \(M(h) = E_2\) and the determinant is 1.

For a polynomial \(f\) with varieties \(a_1, \ldots, a_{N-1}, b_1, \ldots, b_{N-1}\), we set
\[
D_a(f)_L := \begin{bmatrix} a_1 & f_a_2 & \cdots & f_{a_{N-1}} \\ a_1 & f_a_2 & \cdots & f_{a_{N-1}} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & f_a_2 & \cdots & f_{a_{N-1}} \end{bmatrix}_L,
\]
\[
D_b(f)_L := \begin{bmatrix} b_1 & f_b_2 & \cdots & f_{b_{N-1}} \\ b_1 & f_b_2 & \cdots & f_{b_{N-1}} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & f_b_2 & \cdots & f_{b_{N-1}} \end{bmatrix}_L.
\]

We denote by \(e_i = [e_{i,1} \ e_{i,2} \ \ldots \ e_{i,N-1}]\) the row vector of size \(N - 1\) such that \(e_{i,i} = 1\) and \(e_{i,j} = 0\) for \(i \neq j\), and by \(0\) the zero vector of size \(N - 1\).

Remark 2.15. Let \(g_1, \ldots, g_m\) be a minimal subset of the \(|d| \times |d|\) minors of \(M(h)\) such that \(J_h \cap G^0\) is locally defined in \(G^0 \simeq \mathbb{A}^{2(N-1)}\) around \(L\) by the \(|d| + r + m\) polynomials \(g_1, \ldots, g_m\) and \(f^j_k (1 \leq i \leq r, 0 \leq k \leq d^i)\). We consider the \((|d| + r + m) \times 2(N - 1)\) matrix
\[
\begin{bmatrix}
D_a(f^j_1) & D_b(f^j_1) \\
\vdots & \vdots \\
D_a(g_1) & D_b(g_1) \\
\vdots & \vdots \\
D_a(g_m) & D_b(g_m) \\
\end{bmatrix},
\]
(8)
where we recall that \(|d| + r \leq 2(N - 1)\). Then \(J_h\) is smooth at \(L\) if and only if the rank of the above matrix \([8]\) is equal to \(|d| + r + m\). We note that the rank can be calculated by using formulae \([5], [6], [7]\).

Lemma 2.16. Every irreducible component of \(J\) is of codimension \(\leq N - |d|\) in \(I\).

Proof. We consider the morphism \(\Phi : I \cap (G^0 \times \mathcal{H}) \to \mathbb{A}^{(N-1)d}\) which sends \((L, h) \to M(h)|_L\). Let \(M|_{d-1} \subset \mathbb{A}^{(N-1)d}\) be the locus of matrix of rank \(\leq |d| - 1\). Then \(J \cap (G^0 \times \mathcal{H})\) is equal to \(\Phi^{-1}(M|_{d-1})\). Since \(M|_{d-1}\) is of codimension \(N - |d|\) in \(\mathbb{A}^{(N-1)d}\) (see [I, p. 67, II, §2, Prop.]), \(J \cap (G^0 \times \mathcal{H})\) is of codimension \(\leq N - |d|\) in \(I\), which implies the assertion. □

3. Construction of Expected Pairs

3.1. Hypersurfaces. In this subsection, we assume \(r = 1\) and give construction of pairs satisfying the statement of Proposition 2.12

Remark 3.1. Let \((L, h) \in G(1, \mathbb{P}^N) \times \mathcal{H}_d\) satisfy \(h|_L = 0\). Then the linear map \(H^0(\mathcal{H}_d(-1)) : K^{\oplus N-1} \to H^0(L, \mathcal{O}(d-1))\) given in Definition 2.3 is not surjective if and only if
\[
\psi(h_{Z_1}|_L) = \cdots = \psi(h_{Z_{N-1}}|_L) = 0
\]
holds for some linear functional \( \psi : H^0(\mathbb{P}^1, \mathcal{O}(d-1)) \to K \). A linear functional \( \psi \) is represented by the row vector

\[
\begin{bmatrix}
c_0 & c_1 & \cdots & c_{d-1}
\end{bmatrix} \quad (c_0, \ldots, c_{d-1} \in K);
\]

this means that \( \psi(s^{d-1-k}t^k) = c_k \) for each \( k \).

First we consider the case \((d, N) = (4, 6)\). Let \((S : T : Z_1 : \cdots : Z_5)\) be the homogeneous coordinates on \( \mathbb{P}^6 \), and \((s : t)\) be on \( \mathbb{P}^1 \).

**Example 3.2.** Let \( \psi : H^0(\mathbb{P}^1, \mathcal{O}(3)) \to K \) be a general linear functional, where we can assume that \( c_0 = 1 \) and other \( c_i \)'s are general elements of \( K \) in the expression \( \psi \). Let \( X \subset \mathbb{P}^6 \) be a hypersurface defined by the following homogeneous polynomial of degree \( 4 \):

\[
h := h_1Z_1 + h_2Z_2 + h_3Z_3 + T^2Z_4Z_5,
\]

where we set

\[
h_j := c_j s^3 - s^{3-j}T^j \quad (j = 1, 2, 3).
\]

Let \( L = (Z_1 = \ldots Z_5 = 0) \subset \mathbb{P}^6 \). Then \((L, h) \in J \), and \( J_h \) is smooth and of dimension \( N - 3 = 3 \) at the point \( L \in \mathbb{G}(1, \mathbb{P}^6) \). The reason is as follows.

It follows that \( h_j \circ \xi = h_j \circ \xi = c_j s^3 - s^{3-j}T^j \quad (j = 1, 2, 3) \). In particular, we find \((L, h) \in J \) as in [Remark 3.1](#) From [5] [6] since \( s \cdot h_j \circ \xi = c_j s^4 - s^4-jT^j \), it follows that \( f_{0, a_j} = c_j, f_{j, a_j} = -1, \) and other \( f_{j', a_j} = 0 \). In addition, since \( t \cdot h_j \circ \xi = c_j s^3t - s^{3-j}T^{j+1} \), it follows that \( f_{1, b_j} = c_j, f_{j+1, b_j} = -1, \) and other \( f_{j', b_j} = 0 \). Hence, we have

\[
\begin{bmatrix}
D_a(f_0) \\
D_a(f_1) \\
D_a(f_2) \\
D_a(f_3) \\
D_a(f_4)
\end{bmatrix}_L =
\begin{bmatrix}
c_1 e_1 + c_2 e_2 + c_3 e_3 & -e_1 \\
-e_2 \\
-e_3 \\
0
\end{bmatrix},
\begin{bmatrix}
D_b(f_0) \\
D_b(f_1) \\
D_b(f_2) \\
D_b(f_3) \\
D_b(f_4)
\end{bmatrix}_L =
\begin{bmatrix}
0 & c_1 e_1 + c_2 e_2 + c_3 e_3 \\
e_1 \\
e_2 \\
e_3
\end{bmatrix}.
\]

Here, \( \begin{bmatrix} D_a(f_0) & D_b(f_0) \end{bmatrix}_L \) is transformed to

\[
\begin{bmatrix}
0 & (c_1^2 - c_2) e_1 + (c_1 c_2 - c_3) e_2
\end{bmatrix}
\]

under elementary row operations of \( \begin{bmatrix} D_a(f_k) & D_b(f_k) \end{bmatrix}_L \) with \( 1 \leq k \leq 4 \).

On the other hand, from [7] we have

\[
M(h) =
\begin{bmatrix}
c_1 & -1 \\
c_2 & -1 \\
c_3 & -1 \\
a_5 & b_5 \\
a_4 & b_4
\end{bmatrix}
\sim
\begin{bmatrix}
0 & -1 \\
0 & -1 \\
0 & -1 \\
c_2 a_5 + c_3 b_5 & a_5 & b_5 \\
c_2 a_4 + c_3 b_4 & a_4 & b_4
\end{bmatrix}
\]

where the right hand side is a transformation under elementary column operations, and where we consider blank entries in the matrix to be filled by 0. Here \( \text{rk } M(h) < 4 \) if and only if \( 4 \times 4 \) minors \( g_1 = c_2 a_4 + c_3 b_4 \) and \( g_2 = c_2 a_5 + c_3 b_5 \) are equal to zero. Now we have

\[
\begin{bmatrix}
D_a(g_1) \\
D_a(g_2)
\end{bmatrix}_L =
\begin{bmatrix}
c_2 e_4 \\
c_2 e_5
\end{bmatrix},
\begin{bmatrix}
D_b(g_1) \\
D_b(g_2)
\end{bmatrix}_L =
\begin{bmatrix}
c_3 e_4 \\
c_3 e_5
\end{bmatrix}.
\]

Hence the matrix [8] in [Remark 2.15](#) is of rank 7 (we take \( c_i \)'s are general), which implies that \( J_h \cap \mathbb{G}_1 \), the zero set of 7 polynomials \( f_0, \ldots, f_4, g_1, g_2 \), is smooth and of codimension 7 in \( \mathbb{G}_1 \simeq \mathbb{A}^{10} \) at \( L \).
Recall that \((S : T : Z_1 : \cdots : Z_{N-1})\) are the homogeneous coordinates on \(\mathbb{P}^N\).

**Example 3.3.** Assume that \(d \geq 3\) and \(d \leq N - 2\). Let \(\psi : H^0(\mathbb{P}^1, O(d-1)) \to K\) be a general linear functional with \(c_0 = 1\) in \([9]\). Let \(X \subset \mathbb{P}^N\) be a hypersurface of degree \(d\) defined by the homogeneous polynomial \(h := h_1Z_1 + h_2Z_2 + \cdots + h_{d-1}Z_{d-1} + \tilde{h}\), where \(h_j := c_jS^{d-j} - S^{d-j-1}T^j\) \((1 \leq j \leq d - 1)\), and where

\[
\tilde{h} := \begin{cases} 
T^{d-2}(Z_dZ_{d+1} + Z_{d+2}Z_{d+3} \cdots + Z_{N-2}Z_{N-1}), & \text{if } N - d \text{ is even}, \\
ST^{d-3}Z_dZ_{d+1} + T^{d-2}(Z_{d+1}Z_{d+2} + Z_{d+3}Z_{d+4} \cdots + Z_{N-2}Z_{N-1}), & \text{if } N - d \text{ is odd}.
\end{cases}
\]

Let \(L = (Z_1 = \ldots Z_{N-1} = 0)\). Then \((h, L) \in J\), and \(J_h\) is smooth and of dimension \(N - 3\) at the point \(L \in G(1, \mathbb{P}^N)\). It can be shown in a similar way to the previous example.

We check the calculation of the dimension of \(J_h\) and smoothness at \(L\) in the case where \(N - d\) is odd. From \([5]\) \([6]\) we have

\[
\begin{bmatrix}
D_a(f_0) \\
D_a(f_1) \\
\vdots \\
D_a(f_{d-1}) \\
D_a(f_d)
\end{bmatrix}_L = \begin{bmatrix}
c_1e_1 + \cdots + c_{d-1}e_{d-1} \\
- e_1 \\
\vdots \\
- e_{d-1} \\
0
\end{bmatrix}, \quad \begin{bmatrix}
D_b(f_0) \\
D_b(f_1) \\
\vdots \\
D_b(f_{d-1}) \\
D_b(f_d)
\end{bmatrix}_L = \begin{bmatrix}
0 \\
- e_1 \\
\vdots \\
- e_{d-1}
\end{bmatrix}.
\]

From \((7)\) we have that the \((N - 1) \times d\) matrix \(M(h)\) is

\[
\begin{bmatrix}
c_1 & -1 \\
\vdots & \ddots \\
c_{d-3} & -1 & -1 \\
c_{d-2} & a_{d+1} & b_{d+1} \\
c_{d-1} & \frac{a_d}{a_d + a_{d+2}} & \frac{b_d}{b_d + a_{d+2}}
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0 \\
0 & c_{d-3}a_{d+1} + c_{d-2}b_{d+1} \\
0 & c_{d-2}a_{d+1} + c_{d-1}b_{d+2} \\
\vdots & \vdots \\
0 & c_{d-2}a_{N-2} + c_{d-1}b_{N-2}
\end{bmatrix}.
\]

Hence \(M(h)\) is of rank \(d\) if and only if the \(d \times d\) minors \(g_1, \ldots, g_{N-d}\) are zero, where \(g_1 = c_{d-3}a_{d+1} + c_{d-2}b_{d+1}\), \(g_2 = c_{d-3}a_d + c_{d-2}(b_d + a_{d+2}) + c_{d-1}b_{d+2}\), \(g_3 = c_{d-2}a_{d+1} + c_{d-1}b_{d+4}\), and

\[
g_4 = c_{d-2}a_{d+3} + c_{d-1}b_{d+3}, \quad g_5 = c_{d-2}a_{d+4} + c_{d-1}b_{d+4}, \ldots, g_{N-d} = c_{d-2}a_{N-1} + c_{d-1}b_{N-1}.
\]

Here we have:

\[
\begin{bmatrix}
D_a(g_1) \\
D_a(g_2) \\
D_a(g_3) \\
D_a(g_4) \\
\vdots \\
D_a(g_{N-d})
\end{bmatrix}_L = \begin{bmatrix}
c_{d-3}e_{d+1} \\
c_{d-3}e_d + c_{d-2}e_{d+2} \\
c_{d-2}e_{d+1} \\
c_{d-2}e_{d+3} \\
\vdots \\
c_{d-2}e_{N-1}
\end{bmatrix}, \quad \begin{bmatrix}
D_b(g_1) \\
D_b(g_2) \\
D_b(g_3) \\
D_b(g_4) \\
\vdots \\
D_b(g_{N-d})
\end{bmatrix}_L = \begin{bmatrix}
c_{d-2}e_{d+1} \\
c_{d-2}e_d + c_{d-1}e_{d+2} \\
c_{d-1}e_{d+1} \\
c_{d-1}e_{d+3} \\
\vdots \\
c_{d-1}e_{N-1}
\end{bmatrix}.
\]

Since \(c_1, \ldots, c_{d-1} \in K\) are general, the matrix \([8]\) in Remark 2.15 is of rank \(N + 1\), which implies that \(J_h \cap G_1^0\) is smooth and of codimension \(N + 1\) in \(G_1^0 \simeq \mathbb{A}^{2(N-1)}\) at the point \(L\).
Remark 3.4. Let the characteristic be not equal to 2. Then, by taking $\tilde{h}$ to be

$$T^{d-2}(Z_2^2 + \cdots + Z_{N-1}^2),$$

we can show the same statement of Example 3.3 with easier calculations. However, this does not work in characteristic 2.

3.2. Complete intersections. First we assume that $d^i \geq 3$ for some $i$. Then Example 3.3 is generalized to the case of $(r, N, (d^1, d^2)) = (2, 9, (4, 3))$, as follows.

Example 3.5. Let $\psi : H^0(\mathbb{P}^1, \mathcal{O}(3)) \to K$ be a general linear functional with $c_0 = 1$ in (9) Let $X \subset \mathbb{P}^9$ be a complete intersection defined by the following homogeneous polynomials $h^1, h^2$ of degrees 4, 3:

$$h^1 := h_1^1Z_1 + h_2^1Z_2 + h_3^1Z_3 + T^2Z_4Z_5 \quad (\tilde{h}_j := c_jS^3 - S^{3-j}T^j),$$

$$h^2 := S^2Z_6 + T^2Z_7 + T^2Z_8.$$

Let $L = (Z_1 = \ldots = Z_8 = 0) \subset \mathbb{P}^9$. Then $(L, (h^1, h^2)) \in J$, and $J(h^1, h^2)$ is smooth and of dimension $N - r - 2 = 5$ at the point $L \in G(1, \mathbb{P}^9)$. The reason is as follows.

It follows that $h_1^jZ_1 \circ \xi = h_2^j \circ \xi = c_jS^3 - S^{3-j}T^j$ ($j = 1, 2, 3$). Thus we find that $H^0(\delta_L(h^1)(-1))$ is not surjective as in Remark 3.1. This implies that $H^0(\delta_L(h^1, h^2)(-1))$ is also not surjective (see Remark 2.4); hence $(L, (h^1, h^2)) \in J$. The calculation of $D_a(f^1_k)_{L}$ and $D_b(f^2_k)_{L}$ with $0 \leq k \leq 4$ is the same as Example 3.2.

On the other hand, from (5) [6] we have

$$\begin{bmatrix}
D_a(f^2_0) \\
D_a(f^2_1) \\
D_a(f^2_2) \\
D_a(f^2_3)
\end{bmatrix}_{L} =
\begin{bmatrix}
e_6 \\
e_7 \\
e_8 \\
0
\end{bmatrix},
= 
\begin{bmatrix}
D_b(f^2_0) \\
D_b(f^2_1) \\
D_b(f^2_2) \\
D_b(f^2_3)
\end{bmatrix}_{L} =
\begin{bmatrix}
e_6 \\
e_7 \\
e_8
\end{bmatrix}.$$

Next, from (7) we have

$$M(h^1, h^2) = \begin{bmatrix}M' \\
E_3\end{bmatrix},$$

where $M'$ is a $5 \times 5$ matrix which is the same as the left hand side of (10). Thus, in a similar way to Example 3.2 we find that $M(h^1, h^2)$ is of rank less than 7 if and only if the $7 \times 7$ minors $g_1, g_2$ are zero. As a result, the matrix (8) in Remark 2.12 is of rank 11, which implies that $J(h^1, h^2) \cap \mathbb{G}_1^0$ is smooth and of codimension 11 in $\mathbb{G}_1^0 \simeq \mathbb{A}^{16}$ at $L$.

We denote by $|d|_{i_0} := \sum_{i=10}^{r} d^i$. Then $|d| = |d|_1$ and $|d|_r = d^r$.

Example 3.6. Assume that $d^1 \geq 3$ and $\sum d^i \leq N - 2$. Let $\psi : H^0(\mathbb{P}^1, \mathcal{O}(d^1 - 1)) \to K$ be a general linear functional with $c_0 = 1$ in (9) Let $X \subset \mathbb{P}^N$ be a hypersurface defined by the following homogeneous polynomials $h^1, \ldots, h^r$ of degrees $d^1, \ldots, d^r$. We set $h^1 = h^1_1Z_1 + h^1_2Z_2 + \cdots + h^1_{d^1-1}Z_{d^1-1} + \tilde{h}_1$, where $h^1_1 := c_jS^{d^1-1} - S^{d^1-j-1}T^j$, and where

$$\tilde{h}_1 := T^{d^1-2}(Z_{d^1}Z_{d^1+1} + Z_{d^1+2}Z_{d^1+3} + \cdots + Z_{N-2-|d|_2}Z_{N-1-|d|_2})$$

if $N - |d|$ is even, or

$$\tilde{h}_1 := ST^{d^1-3}Z_{d^1}Z_{d^1+1} + T^{d^1-2}(Z_{d^1+1}Z_{d^1+2}Z_{d^1+3}Z_{d^1+4} + \cdots + Z_{N-2-|d|_2}Z_{N-1-|d|_2})$$
On the other hand, we have

\[ L = (Z_1 = \ldots Z_{N-1} = 0). \text{ Then, in a similar way to Example 3.3 and Example 3.5 one can show that } (L, (h^1, \ldots, h^r)) \in J \text{ and that } J_{(h^1, \ldots, h^r)} \text{ is smooth and of dimension } N - r - 2 \text{ at the point } L \in G(1, \mathbb{P}^N). \]

### 3.3. Complete intersections of hyperquadrics

We complete the case where \( r \geq 2 \) and \( d^1 = \cdots = d^r = 2 \). For example, for \((N, r, (d^1, d^2)) = (7, 2, (2, 2))\), we have:

**Example 3.7.** Let \( X \subset \mathbb{P}^7 \) be the complete intersection defined by the following homogeneous quadric polynomials:

\[
h^1 = SZ_1 + TZ_2 + Z_6, \quad h^2 = SZ_2 + TZ_3 + Z_4 Z_5.
\]

Let \( L = (Z_1 = \ldots Z_6 = 0) \). Then \((L, (h^1, h^2)) \in J\), and \(J_{(h^1, h^2)}\) is smooth and of dimension \( N - r - 2 = 3 \) at the point \( L \in G(1, \mathbb{P}^7)\). The reason is as follows.

It follows that

\[
[h^1 \mid L, h^2 \mid L] = [s, 0], \quad [h^1 \mid L, h^2 \mid L] = [t, s], \quad [h^1 \mid L, h^2 \mid L] = [0, t]
\]

and that other \([h^1 \mid L, h^2 \mid L]'s\) are zero. Thus the above elements do not span the 4-dimensional vector space \( H^0(L, \mathcal{O}(1)) + H^0(L, \mathcal{O}(1))\). Then, from Remark 2.4 we have \((L, (h^1, h^2)) \in J\).

Next, we have

\[
\begin{bmatrix}
D_a(f_0^1) \\
D_a(f_1^1) \\
D_a(f_2^1) \\
D_a(f_3^1)
\end{bmatrix}_L = \begin{bmatrix}
e_1 \\
e_2 \\
0 \\
e_3
\end{bmatrix}, \quad \begin{bmatrix}
D_b(f_0^1) \\
D_b(f_1^1) \\
D_b(f_2^1) \\
D_b(f_3^1)
\end{bmatrix}_L = \begin{bmatrix}
e_1 \\
e_2 \\
0 \\
e_3
\end{bmatrix}.
\]

On the other hand, we have

\[
M(h^1, h^2) = \begin{bmatrix}
1 & 1 & 1 \\
a_5 & b_5 & 1 \\
a_6 & b_6 & a_4 & b_4 \\
a_5 & b_5 & a_4 & b_4
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 1 \\
-a_5 & a_5 & b_5 \\
a_6 & b_6 & a_4 & a_4 & b_4 \\
a_5 & b_5 & a_4 & b_4
\end{bmatrix},
\]

where the right matrix is obtained by subtracting the third column from the second column. Hence \( \text{rk}(M(h^1, h^2)) < 4 \) if and only if \( 4 \times 4 \) minors \( g_1 = -a_5, g_2 = b_6 - a_4, g_3 = b_5 \) are zero. Here, it follows that

\[
\begin{bmatrix}
D_a(g_1) \\
D_a(g_2) \\
D_a(g_3)
\end{bmatrix}_L = \begin{bmatrix}
-e_5 \\
-e_4 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
D_b(g_1) \\
D_b(g_2) \\
D_b(g_3)
\end{bmatrix}_L = \begin{bmatrix}
e_6 \\
e_5
\end{bmatrix}.
\]

Therefore, the matrix \( (8) \) in Remark 2.15 is of rank 9; then \( J_{(h^1, h^2)} \cap G_7^2 \) is smooth and of codimension 9 in \( G_7^2 \cong \mathbb{A}^{12} \) at \( L \).
Example 3.8. Assume that $r \geq 2$ and $2r \leq N - 2$. We set $X \subset \mathbb{P}^N$ to be the complete intersection defined by the following homogeneous quadric polynomials $h^1, \ldots, h^r$; we set $h^1, h^2$ by

$$h^1 = SZ_1 + TZ_2 + \tilde{h}^1, \quad h^2 = SZ_2 + TZ_3 + \tilde{h}^2,$$

where

$$\tilde{h}^1 := Z_{2r+1}Z_{2r+2} + \cdots + Z_{N-2}Z_{N-1}, \quad \tilde{h}^2 := Z_{2r}Z_{2r+1},$$

if $N - 2r$ is odd,

and set $h^3, \ldots, h^r$ by

$$h^3 = SZ_4 + TZ_5, \ldots, h^i = SZ_{2i-2} + TZ_{2i-1}, \ldots, h^r = SZ_{2r-2} + TZ_{2r-1}.$$

Let $L = (Z_1 = \ldots Z_N = 0)$. Then $(L, (h^1, \ldots, h^r)) \in J$, and $J(h^1, \ldots, h^r)$ is smooth and of dimension $N - r - 2$ at the point $L \in \mathbb{G}(1, \mathbb{P}^N)$. This assertion follows in a similar way to Example 3.7.

3.4. Surjectivity of the projection from $J$ to $\mathcal{H}$.

Proof of Proposition 2.12. It follows from Examples 3.6 and 3.8.

Now, we have the surjectivity of $pr_2|_J : J \to \mathcal{H}$.

Proof of Theorem 2.10. The case of $|d| \geq N - 1$ follows as in Remark 2.11.

Assume $|d| \leq N - 2$. From Proposition 2.12, there exists a pair $(L, h) \in J$ such that $pr_2^{-1}(h) \cap J \cong J_h$ is smooth and of dimension $N - r - 2$ at $(L, h)$. We consider the exact sequence

$$0 \to T_{(L,h)}(pr_2^{-1}(h) \cap J) \to T_{(L,h)}J \xrightarrow{d_{(L,h)}(pr_2|_J)} T_h \mathcal{H}.$$

From Lemma 2.16 and the equality (1) in Remark 2.2 we have

$$\dim(T_{(L,h)}J) - \dim(T_{(L,h)}(pr_2^{-1}(h) \cap J))$$

$$\geq (\dim(I) - (N - |d|)) - (N - r - 2) = \dim(\mathcal{H}),$$

which implies that the linear map $d_{(L,h)}(pr_2|_J)$ is surjective. Hence the morphism $pr_2|_J$ is smooth on an open neighborhood of $(L, h)$ in $J$, and hence, is surjective.

4. Non-free rational curves

From Theorem 2.10, there exists a non-free line lying in a complete intersection $X \subset \mathbb{P}^N$ in the case of $2N - 2 - r \geq |d|$ and $\deg(X) > 2$. On the other hand, for $|d| > N$, the following lemma is immediately obtained:

Lemma 4.1. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of type $d = (d_1, \ldots, d_r)$. If $|d| > N$, then every immersion $\mu : \mathbb{P}^1 \to X$ is non-free.

Proof. Let $\mu^*T_X \cong \bigoplus_{i=1}^{N-r} \mathcal{O}_{\mathbb{P}^1}(a_i)$ be the splitting on $\mathbb{P}^1$ with integers $a_1 \geq a_2 \geq \cdots \geq a_{N-r}$. Taking the positive integer $b$ with $\mu^*(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathcal{O}_{\mathbb{P}^1}(b)$, since $\bigwedge_{i=1}^{N-r} T_X = \mathcal{O}_X(N + 1 - |d|)$, we have $\sum_{1 \leq i \leq N-r} a_i = b(N + 1 - |d|) \leq 0$. Since $T_{\mathbb{P}^1} \to \mu^*T_X$ is injective, we find that $a_1 \geq 2$. Therefore $a_i < 0$ for some $i_0$. Then $H^1(\mathbb{P}^1, \mu^*T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \neq 0$. 


Remark 4.2. We check that each of the conditions (ii-iv) in Section 1 implies the condition (i) in Theorem 1.2. First, (iv) implies (i); this is because $X$ contains a line if (iv) holds, as in Lemma 2.7. Next, (ii) or (iii) implies (iv), as follows. From [9, IV, Thm. 3.7], $X$ is separably rationally connected if and only if there exists $\mu : \mathbb{P}^1 \to X$ such that $\mu^* T_X$ is ample, where the latter condition implies $|d| \leq N$. On the other hand, $X$ is Fano if and only if $|d| \leq N$.

Proof of Theorem 1.2. Let $X \subset \mathbb{P}^N$ be a complete intersection of type $d = (d^1, \ldots, d^r)$ satisfying the assumption of Theorem 1.2. Without loss of generality, we may assume that all $d^i > 1$.

Suppose that $\prod d^i = \deg(X) > 2$. Then we can find a non-free immersion $\mu : \mathbb{P}^1 \to X$, as follows. Assume $2N - 2 - r \geq |d|$. Taking defining polynomials $h = (h^1, \ldots, h^r)$ of $X$, from Theorem 2.10 we have that $J \cap \text{pr}_2^{-1}(h) \neq \emptyset$; this means that a non-free line in $X$ exists. Assume $2N - 2 - r < |d|$. We may also assume $N - r \geq 2$. Then $N < |d|$ holds. By the condition (i) in Theorem 1.2, an immersion exists and is non-free because of Lemma 4.1.

Then $X$ is not convex as in Remark 2.8 a contradiction. Hence we have $\deg(X) \leq 2$. \hfill $\square$

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