Almost Quaternion-Hermitian Manifolds

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Abstract. Following the point of view of Gray and Hervella, we derive detailed conditions which characterize each one of the classes of almost quaternion-Hermitian 4n-manifolds, \( n > 1 \). Previously, by completing a basic result of A. Swann, we give explicit descriptions of the tensors contained in the space of covariant derivatives of the fundamental form \( \Omega \) and split the coderivative of \( \Omega \) into its \( \text{Sp}(n)\text{Sp}(1) \)-components. For \( 4n > 8 \), A. Swann also proved that all the information about the intrinsic torsion \( \nabla \Omega \) is contained in the exterior derivative \( d\Omega \). Thus, we give alternative conditions, expressed in terms of \( d\Omega \), to characterize the different classes of almost quaternion-Hermitian manifolds.

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1. Introduction

An almost quaternion-Hermitian manifold is a Riemannian 4n-manifold \( (n > 1) \) which admits a \( \text{Sp}(n)\text{Sp}(1) \)-structure, i.e., a reduction of its frame bundle to the subgroup \( \text{Sp}(n)\text{Sp}(1) \) of \( \text{SO}(4n) \). These manifolds are of special interest because \( \text{Sp}(n)\text{Sp}(1) \) is included in the list of Berger ([2]) of possible holonomy groups of locally irreducible Riemannian manifolds that are not locally symmetric. An almost quaternion-Hermitian manifold is said to be quaternion-Kähler, if its reduced holonomy group is a subgroup of \( \text{Sp}(n)\text{Sp}(1) \).

A general approach to classify \( G \)-structures compatible with the metric in a Riemannian manifold is described as follows. Given any Riemannian manifold \( M^m \) and a Lie group \( G \) that is the stabilizer of some tensor \( \varphi \) on \( \mathbb{R}^m \), that is, \( G = \{g \in \text{SO}(m) : g\varphi = \varphi\} \), a \( G \)-structure on \( M \) defines a global tensor \( \varphi \) on \( M \), and it can be shown that \( \nabla \varphi \) (the so-called intrinsic torsion of the \( G \)-structure, where \( \nabla \)
denotes the Levi-Civita connection) is a section of the vector bundle $\mathcal{W} = T^* M \otimes g^\perp$, being $\mathfrak{so}(m) = g \oplus g^\perp$. The action of $G$ splits $\mathcal{W}$ into irreducible components, say $\mathcal{W} = \mathcal{W}_1 \oplus \ldots \oplus \mathcal{W}_k$. Then, $G$-structures on $M$ can be classified in at most $2^k$ classes.

This systematic way of classifying $G$-structures was initiated by A. Gray and L. Hervella in [8], where they considered the case $G = U(n)$ (almost Hermitian structures), giving rise to sixteen classes of almost Hermitian manifolds. Later, diverse authors have studied the situation for other $G$-structures: $G_2$, $Spin(7)$, $Spin(9)$ etc. In the present paper we are concerned with the case $G = Sp(n)Sp(1)$, i.e., we consider a $4n$-manifold $M$ with an $Sp(n)Sp(1)$-structure, $n > 1$. Associated with this type of geometric structure there is a four-form $\Omega$, given by 2.1, such that $\Omega^n$ is nowhere zero. An orientation on the manifold can be defined regarding a constant multiple of $\Omega^n$ as volume form, and $Sp(n)Sp(1)$ is the stabilizer of $\Omega$ under the action of $SO(4n)$. Moreover, $\nabla \Omega$ is contained in $\mathcal{W} = T^* M \otimes (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))^\perp$. The splitting of the $Sp(n)Sp(1)$-module $\mathcal{W}$ was shown by A. Swann in [16] (see Proposition 2.1 of the present paper). Such a decomposition consists of four or six irreducible components, for $4n = 8$ or $4n > 8$, respectively. Therefore, it is already known how many classes of almost quaternion-Hermitian manifolds we have at most, 16 or 64. But actually, to our knowledge, except some partial results (see [19]), there are not explicit descriptions of each one of such classes. Thus, the main purpose of the present paper is to show conditions which define each class, and that is the content of the second column of Table 2 (page 19).

For some $G$-structures, the intrinsic torsion $\nabla \phi$ is totally determined by tools of the exterior algebra. This implies that we would have an alternative easier way to characterize each class. For an $Sp(n)Sp(1)$-structure, when $4n > 8$, this is such a case. A. Swann in [16] proved that all the information about the covariant derivative $\nabla \Omega$ is contained in the exterior derivative $d\Omega$. Thus alternative conditions expressed in terms of $d\Omega$ are displayed in the third column of Table 2 to characterize the different classes of almost quaternion-Hermitian manifolds of dimension higher than eight. Since for dimension eight, $d\Omega$ only contains partial information of $\nabla \Omega$, in this case we can only characterize, via $d\Omega$, eight of the possible sixteen classes. Such characterizations are shown in Table 3.

### 2. Almost quaternion-Hermitian structures

Almost quaternion-Hermitian manifolds have been broadly treated by diverse authors ([12],[16], etc.). In this section we recall basic definitions and known results.

A $4n$-dimensional manifold $M$ ($n > 1$) is said to be **almost quaternion-Hermitian**, ...
if $M$ is equipped with a Riemannian metric $\langle \cdot , \cdot \rangle$ and a rank-three subbundle $\mathcal{G}$ of the endomorphism bundle $\text{End } TM$, such that locally $\mathcal{G}$ has an adapted basis $I, J, K$ with $I^2 = J^2 = -1$, $K = IJ = -JI$, and $\langle AX, AY \rangle = \langle X, Y \rangle$, for $A = I, J, K$. This is equivalent to saying that $M$ has a reduction of its structure group to $Sp(n)Sp(1)$.

There are the three local Kähler-forms $\omega_A(X, Y) = \langle X, AY \rangle$, $A = I, J, K$. From these one may define a global four-form, non degenerate $\Omega$, the fundamental form, by the local formula

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$  

Now, we recall some facts about almost quaternion-Hermitian manifolds in relation with representation theory. We will follow the $E$-$H$-formalism used in [12], [16]-[19] and we refer to [6] for general information on representation theory. Thus, $E$ is the fundamental representation of $Sp(n)$ on $\mathbb{C}^{2n} \cong \mathbb{H}^n$ via left multiplication by quaternionic matrices, considered in $\text{GL}(2n, \mathbb{C})$, and $H$ is the representation of $Sp(1)$ on $\mathbb{C}^2 \cong \mathbb{H}$ given by $q, \zeta = \zeta\overline{q}$, for $q \in Sp(1)$ and $\zeta \in H$. An $Sp(n)Sp(1)$-structure on a manifold $M$ gives rise to local bundles $E$ and $H$ associated to these representation and identifies $TM \otimes_{\mathbb{R}} \mathbb{C} \cong E \otimes_{\mathbb{C}} H$.

On $E$, there is an $Sp(n)$-invariant complex symplectic form $\omega_E$ and a Hermitian inner product given by $\langle x, y \rangle_{\mathbb{C}} = \omega_E(x, \bar{y})$, for all $x, y \in E$ and being $\bar{y} = jy$ ($y \rightarrow \bar{y}$ is a quaternionic structure map on $E = \mathbb{C}^{2n}$ considered as left complex vector space). The mapping $x \rightarrow x^\omega = \omega_E(\cdot, x)$ gives us an identification of $E$ with its dual $E^*$. If $\{e_1, \ldots, e_n, \tilde{e}_1, \ldots, \tilde{e}_n\}$ is a complex orthonormal basis for $E$, then $\omega_E = e_i^\omega \wedge \overline{e}_i^\omega = e_i^\omega \overline{e}_i^\omega - \overline{e}_i^\omega e_i^\omega$, where we have used the summation convention and omitted tensor product signs. These conventions will be used throughout the paper.

The $Sp(1)$-module $H$ will be also considered as left complex vector space. Regarding $H$ as 4-dimensional real space with the Euclidean metric $\langle \cdot , \cdot \rangle$ such that $\{1, i, j, k\}$ is an orthonormal basis, the complex symplectic form $\omega_H$ is given by $\omega_H = 1^\# \wedge j^\# + k^\# \wedge i^\# + i(1^\# \wedge k^\# + i^\# \wedge j^\#)$, where $h^\#$ is the real one-form given by $q \rightarrow \langle h, q \rangle$. We also have the identification, $q \rightarrow q^\omega = \omega_H(\cdot, q)$, of $H$ with its dual $H^*$ as complex space. On $H$, we have a quaternionic structure map given by $q = z_1 + z_2 j \rightarrow \bar{q} = jq = -\overline{z}_2 + \overline{z}_1 j$, where $z_1, z_2 \in \mathbb{C}$ and $\overline{z}_1, \overline{z}_2$ are their conjugates. If $h \in H$ is such that $\langle h, h \rangle = 1$, then $\{h, \overline{h}\}$ is a basis of the complex vector space $H$ and $\omega_H = h^\omega \wedge \overline{h}^\omega$.

The irreducible representations of $Sp(1)$ are the symmetric powers $S^k H \cong \mathbb{C}^{k+1}$. An irreducible representation of $Sp(n)$ is determined by its dominant weight $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i$ are integers with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. This representation will be denoted by $V^{(\lambda_1, \ldots, \lambda_r)}$, where $r$ is the largest integer such that $\lambda_r > 0$.
notation is used for some of these modules, when possible. For instance, \( V^{(k)} = S^kE \), the \( k \)th symmetric power of \( E \), and \( V^{(1,\ldots,1)} = \Lambda_0^rE \), where there are \( r \) ones in exponent and \( \Lambda_0^rE \) is the \( Sp(n) \)-invariant complement to \( \omega_E \Lambda^{r-2}E \) in \( \Lambda^rE \). Also \( K \) will denote the module \( V^{(21)} \), which arises in the decomposition \( E \otimes \Lambda_0^3E \cong \Lambda_3^3E + K + E \), where + denotes direct sum.

As we have pointed out in the introduction an analogue of the Gray-Hervella classification may be obtained for almost quaternion hermitian manifolds by considering the covariant derivative \( \nabla \Omega \) of the fundamental 4-form \( \Omega \) (2.1). For dimension at least 12, this leads to \( 2^{6} = 64 \) classes. Such quaternionic classes come from the following \( Sp(n)Sp(1) \)-decomposition.

**PROPOSITION 2.1 (Swann [16])** The covariant derivative of the fundamental form \( \Omega \) of an almost quaternion-Hermitian manifold \( M \) of dimension at least 8, has the property

\[
\nabla \Omega \in T^*M \otimes (\text{sp}(n) \oplus \text{sp}(1))^1 \cong T^*M \otimes \Lambda_0^3E S^2H = (\Lambda_0^3E + K + E)(S^3H + H). \quad \square
\]

If the dimension of \( M \) is at least 12, all the modules of the sum are non-zero. For an eight-dimensional manifold \( M \), we have \( \Lambda_0^3ES^3H = \Lambda_0^3EH = \{0\} \).

**REMARK 2.2** Let \( V \) be a complex \( G \)-module equipped with a real structure, where \( G \) is a Lie group. In the previous proposition and most of the times throughout this paper, \( V \) also denotes the \((+1)\)-eigenspace of the structure map which is a real \( G \)-module. The context should tell us which space are referring to. However, if there is risk of confusion or when we feel that a clearer exposition is needed, we denote the second mentioned space by \([V]\).

For sake of simplicity, we are referring to a real vector space \( \mathcal{V} \) of dimension \( 4n \) with \( n > 1 \). In the final conclusions, we will apply the obtained results to an almost quaternion-Hermitian manifold \( M \) and \( \mathcal{V} \) will be \( T_pM \), the tangent space at \( p \in M \).

Thus, \( \mathcal{V} \) is equipped with an Euclidean metric \( \langle , \rangle \) and a three-dimensional subspace \( \mathcal{G} \) of the endomorphism space \( \text{End} \mathcal{V} \) such that \( \mathcal{G} \) admits a basis \( I, J, K \), with \( I^2 = J^2 = -1, K = IJ = -JI \) and \( \langle Ax, Ay \rangle = \langle x, y \rangle \), for \( A = I, J, K \). From the two-forms \( \omega_A(x, y) = \langle x, Ay \rangle \), a four-form \( \Omega \) is defined as in 2.1.

Moreover, we can consider \( \mathcal{V} \) as a left complex vector space by defining \((a+bi)x = ax + bIx, \ x \in \mathcal{V} \ and \ a + bi \in \mathbb{C} \). We denote \( E \) when we refer to \( \mathcal{V} \) as complex vector space and we have \( \dim E = \dim_\mathbb{C} \mathcal{V} = 2n \). The \( Sp(n) \)-representation \( E \) has a quaternionic structure map \( E \to \mathbb{H} \) given by \( x \to \bar{x} = Jx \) and is equipped with the \( Sp(n) \)-invariant complex symplectic form \( \omega_E = -\omega_J - i\omega_K \). Note that
\(x^\omega = -Jx - iKx\), where \(x\) also denote the dual form \(y \rightarrow \langle y, x \rangle\), for all \(y \in \mathcal{V}\) (the identification \(\mathcal{V} = \mathcal{V}^*\)).

The complexification of the real vector space \(\mathcal{V}\) can be identified with \(E \otimes \mathbb{C} H\) via the isomorphism \(\mathcal{V} \otimes \mathbb{R} \mathbb{C} \rightarrow E \otimes \mathbb{C} H\), defined by \(x \otimes z \rightarrow x \otimes \mathbb{C} \mathcal{V} \mathcal{H} + Jx \otimes \mathbb{C} \mathcal{H}\), where we have fixed \(h \in H\) such that \(\langle h, h \rangle = 1\). On \(E \otimes \mathbb{C} H\), we can consider the real structure \(x \otimes \mathbb{C} q \rightarrow \bar{x} \otimes \mathbb{C} \bar{q}\), which corresponds with the map \(x \otimes \mathbb{R} \mathbb{C} \rightarrow x \otimes \mathbb{R} \mathbb{C}\) on \(\mathcal{V} \otimes \mathbb{R} \mathbb{C}\). The \((+1)\)-eigenspace \([EH] \cong \mathcal{V} \otimes \mathbb{R} \mathbb{C} \cong \mathcal{V}\) is the real irreducible \(Sp(n)Sp(1)\)-representation determined by the complex irreducible \(Sp(n)Sp(1)\)-representation \(E \otimes \mathbb{C} H\) of real type. Taking \(\langle \cdot, \cdot \rangle_c = \langle \cdot, \cdot \rangle + i\omega I\) into account, for \(x \in E\), we obtain that the restrictions of \(x^\omega h^\omega, x^\omega h^\bar{\omega}\), \(\bar{x}^\omega h^\omega\) and \(\bar{x}^\omega h^\bar{\omega}\) to \(\mathcal{V} \cong [EH]\) are given by

\[
\begin{align*}
(x^\omega h^\omega)_{\mathcal{V}} &= x - iIx, \\
(\bar{x}^\omega h^\omega)_{\mathcal{V}} &= x + iIxE.
\end{align*}
\]

The following conventions will be used in the sequel. If \(b\) is a \((0, s)\)-tensor, we write

\[A_{(i)}b(X_1, \ldots, X_t, \ldots, X_s) = -b(X_1, \ldots, AX_t, \ldots, X_s),\]

\[Ab(X_1, \ldots, X_s) = (-1)^s b(AX_1, \ldots, AX_s),\]

\[i_A = (A_{(1)} + \ldots + A_{(s)})b,\]

for \(A = I, J, K\). We also consider the extension of \(\langle \cdot, \cdot \rangle\) to \((0, s)\)-tensors given by

\[\langle a, b \rangle = \frac{1}{s!} a(e_{i_1}, \ldots, e_{i_s})b(e_{i_1}, \ldots, e_{i_s}),\]

where \(\{e_1, \ldots, e_{4n}\}\) an orthonormal basis for \(\mathcal{V}\). Finally, we define the \(Sp(n)Sp(1)\)-map \(L : \Lambda^p\mathcal{V}^* \rightarrow \Lambda^p\mathcal{V}^*\) by

\[
L(b) = \sum_{A=I,J,K} \sum_{1 \leq i < j \leq p} A_{(i)}A_{(j)} b.
\]

3. The space of covariant derivatives of \(\Omega\)

In this section we will give an explicit description of the tensors which are contained in the space of covariant derivatives of the fundamental four form \(\Omega\). For such a purpose, we consider the space \(\mathcal{V}^* \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1)) \cong \mathcal{V}^* \otimes \Lambda^2 E^2 H \subseteq \mathcal{V}^* \otimes \Lambda^2 \mathcal{V}^*\) consisting of the tensors \(c\) which, for an adapted basis \(I, J, K\) of \(\mathcal{G}\), satisfy:

i) \(c + I_{(2)} I_{(3)} c + J_{(2)} J_{(3)} c + K_{(2)} K_{(3)} c = 0\);

ii) \(\langle \cdot, c, \omega_A \rangle = 0\), for \(A = I, J, K\).
Since $\Lambda^2 V^* = S^2 E + S^2 H + \Lambda^2_0 ES^2 H$, where $S^2 E \cong \mathfrak{sp}(n)$ and $S^2 H \cong \mathfrak{sp}(1)$ are the Lie algebras of $\text{Sp}(n)$ and $\text{Sp}(1)$, respectively.

Now we define the $\text{Sp}(n)\text{Sp}(1)$-map

$$
\mathcal{F} : V^* \otimes \Lambda^2_0 ES^2 H \rightarrow V^* \otimes \Lambda^4 V^*,
$$

$$
c \rightarrow \frac{1}{4} \sum_{A=I,J,K} i_A(\cdot, c) \wedge \omega_A.
$$

Since $\Lambda^2_0 ES^2 H$ is an irreducible $\text{Sp}(n)\text{Sp}(1)$-module, by Schur’s Lemma (see [6, p. 110]), $\mathcal{F} : V^* \otimes \Lambda^2_0 ES^2 H \rightarrow \mathcal{F}(V^* \otimes \Lambda^2_0 ES^2 H)$ is an $\text{Sp}(n)\text{Sp}(1)$-isomorphism. If $a = \mathcal{F}(c)$, one can check

$$
(3.1) \quad -8nc(x, y, z) = \langle x, a, y \wedge (z, \Omega) - z \wedge (y, \Omega) \rangle,
$$

for all $x, y, z \in V$. Thus the map $a \rightarrow c$, defined by 3.1, is the inverse map of $\mathcal{F}$.

**PROPOSITION 3.1** If $a \in V^* \otimes \Lambda^4 V^*$, the following conditions are equivalent:

1) $a \in V^* \otimes \Lambda^2_0 ES^2 H$.

2) There exists an adapted basis $I, J, K$ of $\mathcal{G}$ and a unique triplet $c_I, c_J, c_K \in V^* \otimes \Lambda^2 V^*$ associated with such a basis, satisfying $a = \sum_{A=I,J,K} c_A \wedge \omega_A$ and the conditions:

i) $A_2 A_3 c_A = -c_A$, for $A = I, J, K$.

ii) $I_2 J_3 c_K + J_2 K_3 c_I + K_2 I_3 c_J = 0$.

iii) $\langle \cdot, c_A, \omega_B \rangle = 0$, for $A, B = I, J, K$.

3) There exists an adapted basis $I, J, K$ of $\mathcal{G}$ and three tensors $d_I, d_J, d_K \in V^* \otimes \Lambda^2 V^*$, satisfying $a = \sum_{A=I,J,K} d_A \wedge \omega_A$ and the conditions:

i) $A_2 A_3 d_A = -d_A$, for $A = I, J, K$.

ii) $I_2 J_3 d_K + J_2 K_3 d_I + K_2 I_3 d_J = 0$.

**Proof.** If $a \in V^* \otimes \Lambda^2_0 ES^2 H$, then $a = \mathcal{F}(c)$. Therefore, $a = \sum_{A=I,J,K} c_A \wedge \omega_A$, where $c_A = \frac{1}{4} i_A(\cdot, c)$. Taking conditions i)* and ii)* into account, it follows that $c_I, c_J$ and $c_K$ satisfy conditions i), ii) and iii) of the lemma.

On the other hand, if $a = \sum_{A=I,J,K} c_A \wedge \omega_A$ with $c_I, c_J, c_K$ satisfying the conditions i), ii) and iii), we consider the tensor $c = \frac{1}{4} \sum_{A=I,J,K} i_A(\cdot, c_A)$. Taking i), ii) and iii) into account, it follows that $c$ satisfies conditions i)* and ii)* and $\mathcal{F}(c) = a$. 


Hence $a \in \mathcal{V}^{*} \otimes \Lambda_{0}^{2}ES^{2}H$. Hence, 1) is equivalent to 2). The triplet $c_{I}$, $c_{J}$ and $c_{K}$ is unique, because one can check $-4nc_{A}(x, y, z) = a(x, y, z, e_{r}, Ae_{r})$.

In order to prove 3) implies 2), let us consider the tensors $c_{I}$, $c_{J}$, $c_{K}$ given by

$$c_{I} = d_{I} + \frac{1}{2n}(d_{I}, \omega_{J})\omega_{J} + \frac{1}{2n}(d_{I}, \omega_{K})\omega_{K}.$$ 

The tensors $c_{J}$ and $c_{K}$ are also defined from this identity by cyclically permuting $I, J, K$. Throughout this paper, when we have three equations depending cyclically of $I, J, K$ as here, we will write only one equation without further comment. Note that $d_{A} \in \mathcal{V}^{*} \otimes (S^{2}H + \Lambda_{0}^{2}ES^{2}H)$ and $c_{A}$ is its projection on $\mathcal{V}^{*} \otimes \Lambda_{0}^{2}ES^{2}H$. Also note that $a = \sum_{A=I, J, K} c_{A} \wedge \omega_{A}$ and that $c_{I}$, $c_{J}$ and $c_{K}$ satisfy i), ii) and iii). 

**Remark 3.2** In the context of almost quaternion-Hermitian manifolds, we consider $a = \nabla \Omega$ and we have $d_{A} = 2\nabla \omega_{A}$. In fact, $2\nabla \omega_{A}$ satisfies condition i') ([8]) and it is an easy exercise to check that $\nabla \omega_{I}$, $\nabla \omega_{J}$, $\nabla \omega_{K}$ satisfy condition ii)'. Tensors $c_{A}$ satisfying conditions i), ii) and iii) are given by

$$c_{I} = 2\nabla \omega_{I} + \frac{1}{n}(\nabla \omega_{I}, \omega_{J})\omega_{J} + \frac{1}{n}(\nabla \omega_{I}, \omega_{K})\omega_{K}.$$

**Lemma 3.3** Let $L : \Lambda^{4}\mathcal{V}^{*} \to \Lambda^{4}\mathcal{V}^{*}$ be defined by 2.3, then for all $a \in \Lambda_{0}^{2}ES^{2}H \subseteq \Lambda^{4}\mathcal{V}^{*}$ we have $L(a) = 2a$. In particular, in a context of almost quaternion-Hermitian manifolds we have $L(\nabla_{X} \Omega) = 2\nabla_{X} \Omega$, for all tangent vector $X$.

*Proof.* By Proposition 3.1, for all four-form $a \in \Lambda_{0}^{2}ES^{2}H$, we have $a = \sum_{A=I, J, K} c_{A} \wedge \omega_{A}$, where $c_{I}$, $c_{J}$, $c_{K}$ are skew-symmetric two-forms satisfying conditions derived directly from i), ii) and iii) of Proposition 3.1. Moreover, $-4nc_{A}(x, y) = a(x, y, e_{r}, Ae_{r})$.

By straightforward computation, we obtain

$$L(a)(x, y, e_{r}, Ie_{r}) = -4n \left(-2c_{I}(x, y) + c_{I}(Jx, y) + c_{I}(Kx, y) + 2c_{J}(Kx, y) + 2c_{J}(Kx, y) - 2c_{K}(Jx, z) - 2c_{K}(Jx, y)\right).$$

Now, making use of the conditions satisfied by $c_{I}$, $c_{J}$ and $c_{K}$, we will have $L(a)(x, y, e_{r}, Ie_{r}) = -8nc_{I}(x, y)$. In a similar way, we can obtain $L(a)(x, y, e_{r}, J e_{r}) = -8nc_{J}(x, y)$ and $L(a)(x, y, e_{r}, K e_{r}) = -8nc_{K}(x, y)$.

If $a \neq 0$, then $c_{I} \neq 0$ or $c_{J} \neq 0$ or $c_{K} \neq 0$. Therefore, $L(a) \neq 0$. Since there is only one copy of $\Lambda_{0}^{2}ES^{2}H$ in the $Sp(n)Sp(1)$-decomposition of $\Lambda^{4}\mathcal{V}^{*}$ (see [14, p. 127]), by Schur’s Lemma, $L$ is an $Sp(n)Sp(1)$-isomorphism on $\Lambda_{0}^{2}ES^{2}H$. Thus $L(a) = 2\sum_{A=I, J, K} c_{A} \wedge \omega_{A} = 2a$. 

\[\square\]
There is an alternative description of $\mathcal{V}^* \otimes \Lambda^2_0 ES^2 H$ as a subspace of $\mathcal{V}^* \otimes \Lambda^4 (EH)$. In fact, we have $\Lambda^2_0 ES^2 H \cong (\Lambda^2_0 E \wedge \omega_E) \otimes \Lambda^2 S^2 H \subseteq (EH)^4$. Since the restriction to $(\Lambda^2_0 E \wedge \omega_E) \otimes \Lambda^2 S^2 H$ of the alternation map $a : (EH)^4 \to \Lambda^4 (EH)$ is non-zero, by Schur’s Lemma, we have

\begin{equation}
\mathcal{V}^* \otimes \Lambda^2_0 ES^2 H \cong \mathcal{V}^* \otimes a \left( (\Lambda^2_0 E \wedge \{\omega_E\}) \otimes \Lambda^2 S^2 H \right) \subseteq \mathcal{V}^* \otimes \Lambda^4 (EH).
\end{equation}

Next we obtain an $Sp(n)Sp(1)$-automorphism on $\mathcal{V}^* \otimes \Lambda^2_0 ES^2 H$, which will play significant rôle in explicit descriptions of classes of almost quaternion Hermitian manifolds. Thus, we consider $\mathcal{L} : \mathcal{V}^* \otimes \Lambda^2_0 ES^2 H \to \mathcal{V}^* \otimes \Lambda^2_0 ES^2 H$, defined by

\begin{equation}
\mathcal{L}(a) = \sum_{A=1,12} A(1)(A(2) + \ldots + A(5))a.
\end{equation}

Relative to the map $\mathcal{L}$, we have the following results.

**PROPOSITION 3.4**

i) $\mathcal{L}$ is an $Sp(n)Sp(1)$-isomorphism.

ii) $(\Lambda^2_0 E + K + E)H$ consists of $a \in \mathcal{V}^* \otimes \Lambda^2_0 ES^2 H$ such that $\mathcal{L}(a) = 4a$.

iii) $(\Lambda^2_0 E + K + E)S^3 H$ consists of $a \in \mathcal{V}^* \otimes \Lambda^2_0 ES^2 H$ such that $\mathcal{L}(a) = -2a$.

**Proof.** Let us consider the $Sp(1)$-maps $Id \otimes \omega_{23} : H \otimes \Lambda^2 S^2 H \to H \otimes S^2 H \cong S^3 H + H$, $s : H \otimes S^2 H \to S^3 H$ and $\omega_{12} : H \otimes S^2 H \to H$, where $Id$ is the identity on $H$, $s$ is the symmetrization and $\omega_{a,b}$ is the $\alpha\beta$- contraction in $\Lambda^2(S^2 H)$ by $\omega_H$. Note that $Id \otimes \omega_{23}$ is an isomorphism, $H \cong \ker s \circ (Id \otimes \omega_{23})$, $S^3 H \cong \ker \omega_{12} \circ (Id \otimes \omega_{23})$ and both are $Sp(1)$-subspaces of $H \otimes \Lambda^2 S^2 H$. Taking this into account, it follows that

\[
\omega_H \otimes \omega_H \omega_H - \overline{\omega}_H \otimes \omega_H \omega_H \in \ker s \circ (Id \otimes \omega_{23}) \cong H,
\]

\[
\omega_H \otimes \omega_H \omega_H \in \ker \omega_{12} \circ (Id \otimes \omega_{23}) \cong S^3 H.
\]

For $x, y, z \in E$, we regard the tensor

\[
\eta = \{ x^\omega \otimes (y^\omega \wedge z^\omega) \wedge \omega_E \} \otimes \left\{ \omega_H \otimes \omega_H \omega_H \wedge \overline{\omega}_H \overline{\omega}_H - \overline{\omega}_H \otimes \omega_H \omega_H \wedge \overline{\omega}_H \overline{\omega}_H \right\}.
\]

It is obvious that $\eta \in E \otimes \Lambda^2 E \wedge \{\omega_E\} \otimes \ker s \circ (I \otimes \omega_{23}) \subseteq EH \otimes (EH)^4$.

Now, if we consider the $Sp(n)Sp(1)$-map $Id \otimes a : EH \otimes (EH)^4 \to EH \otimes \Lambda^4 (EH)$, where $Id$ is the identity on $EH$ and $a$ is the alternation map, we will obtain

\[
(Id \otimes a)(\eta) = x^\omega \overline{\omega}_H \otimes y^\omega \omega_H \wedge z^\omega \omega_H + e_i^* \overline{\omega}_H \wedge c_{ij}^* \overline{\omega}_H - x^\omega \omega_H \otimes y^\omega \overline{\omega}_H \wedge z^\omega \overline{\omega}_H + e_i^* \omega_H \wedge c_{ij}^* \omega_H - x^\omega \overline{\omega}_H \otimes y^\omega \omega_H \wedge z^\omega \overline{\omega}_H + e_i^* \omega_H \wedge c_{ij}^* \omega_H \]

\[
- x^\omega \overline{\omega}_H \otimes y^\omega \omega_H \wedge z^\omega \omega_H + e_i^* \overline{\omega}_H \wedge c_{ij}^* \overline{\omega}_H - x^\omega \overline{\omega}_H \otimes y^\omega \omega_H \wedge z^\omega \omega_H + e_i^* \overline{\omega}_H \wedge c_{ij}^* \omega_H \]

\[
+ x^\omega \overline{\omega}_H \otimes y^\omega \omega_H \wedge z^\omega \omega_H + e_i^* \overline{\omega}_H \wedge c_{ij}^* \overline{\omega}_H + x^\omega \overline{\omega}_H \otimes y^\omega \overline{\omega}_H \wedge z^\omega \overline{\omega}_H + e_i^* \overline{\omega}_H \wedge c_{ij}^* \overline{\omega}_H.
\]
Using identities 2.2, we can obtain the restriction \((Id \otimes a)(\eta)|_V\) of \((Id \otimes a)(\eta)\) to \(V^5 \cong ([EH])^5\). Furthermore, the real part \(Re(Id \otimes a)(\eta)|_V\) of this complex valued map belongs to \(V^* \otimes \Lambda^3V^*\) and, by a straightforward computation, one can check
\[
\mathcal{L} \left( Re(Id \otimes a)(\eta)|_V \right) = 4Re(Id \otimes a)(\eta)|_V.
\]

If we choose \(x, y, z \in E\) such that \(x \otimes y \wedge z\) belongs successively to \(\Lambda^3E\), \(K\) and \(E\), then \(Re(Id \otimes a)(\eta)|_V\) is an element of the irreducible real \(Sp(n)Sp(1)\)-representation determined by \(\Lambda_0^3EH, KH\) and \(EH\) respectively. Therefore, by Schur’s Lemma, any element \(a\) of these \(Sp(n)Sp(1)\)-modules satisfies \(\mathcal{L}(a) = 4a\).

On the other hand, if we consider, for \(x, y, z \in E\), the tensor
\[
\zeta = \left\{ x^\omega \otimes (y^\omega \wedge z^\omega) \wedge \omega_E \right\} \otimes \left\{ h^\omega \otimes h^\omega h^\omega \wedge (\overline{h}^\omega h^\omega + h^\omega \overline{h}^\omega) \right\},
\]
we have \(\zeta \in (E \otimes \Lambda^2E \wedge \{\omega_E\}) \otimes \ker \omega_{12} \circ (I \otimes \omega_{23}) \subseteq EH \otimes (EH)^4\). Therefore,
\[
(Id \otimes a)(\zeta) = 2x^\omega h^\omega \otimes y^\omega h^\omega \wedge z^\omega h^\omega \wedge \varepsilon\overline{h}^\omega \wedge \overline{\varepsilon\overline{h}^\omega} + 2x^\omega h^\omega \otimes \overline{y}^\omega h^\omega \wedge z^\omega h^\omega \wedge \overline{\varepsilon\overline{h}^\omega} \wedge \overline{\varepsilon\overline{h}^\omega} - 2x^\omega h^\omega \otimes y^\omega h^\omega \wedge z^\omega h^\omega \wedge \varepsilon\overline{h}^\omega \wedge \varepsilon\overline{h}^\omega - 2x^\omega h^\omega \otimes \overline{y}^\omega h^\omega \wedge z^\omega h^\omega \wedge \varepsilon\overline{h}^\omega \wedge \varepsilon\overline{h}^\omega.
\]

Taking equations 2.2 into account, we obtain the restriction \((Id \otimes a)(\zeta)|_V\) of \((Id \otimes a)(\zeta)\) to \(V^5 \cong ([EH])^5\). The real part \(Re(Id \otimes a)(\zeta)|_V\) of this complex valued map is in \(V^* \otimes \Lambda^3V^*\). As before, one can check \(\mathcal{L} \left( Re(Id \otimes a)(\zeta)|_V \right) = -2Re(Id \otimes a)(\zeta)|_V\).

If we choose \(x, y, z \in E\) such that \(x \otimes y \wedge z\) belongs successively to \(\Lambda^3E\), \(K\) and \(E\), then \(Re(Id \otimes a)(\zeta)|_V\) is an element of the irreducible real \(Sp(n)Sp(1)\)-representation determined by \(\Lambda_0^3ES^3H, KS^3H\) and \(ES^3H\), respectively. Therefore, by Schur’s Lemma, any element \(a\) of these \(Sp(n)Sp(1)\)-modules satisfies \(\mathcal{L}(a) = -2a\).

Since \(\mathcal{L}\) is bijective on each irreducible \(Sp(n)Sp(1)\)-module of \(EH \otimes \Lambda_0^3ES^3H\), then \(\mathcal{L}\) is an \(Sp(n)Sp(1)\)-isomorphism.

\[\square\]

4. \(Sp(n)Sp(1)\)-spaces of skew-symmetric three-forms

In this section we study the irreducible modules which take part in the \(Sp(n)Sp(1)\)-decomposition of \(\Lambda^3V^* \cong \Lambda^3(EH)\), given by \(\Lambda^3V^* = (K+E)H + (\Lambda_0^3E+E)S^3H([17])\).
For such a purpose, we consider the \(Sp(n)Sp(1)\)-map \(L\) on \(\Lambda^3V^*\) defined by 2.3.
Relative to this \(Sp(n)Sp(1)\)-map, we have the following results.

**PROPOSITION 4.1**

\[\text{i)} \ L \text{ is an } Sp(n)Sp(1)\text{-isomorphism such that } L^2 = 9Id.\]

\[\text{ii)} \ (K+E)H \text{ consists of those three-forms } b \in \Lambda^3V^* \text{ such that } L(b) = 3b.\]
iii) \((\Lambda^2_0 E + E)S^3 H\) consists of those three-forms \(b \in \Lambda^3 \mathcal{V}^*\) such that \(L(b) = -3b\).

**Proof.** Part i) follows by a straightforward computation. Now, we consider the complex tensor \(\eta = a \left( (x^\omega \otimes y^\omega \wedge z^\omega) \otimes \left( 2h^\omega \otimes h^\omega h^\omega - h^\omega \otimes (\tilde{h}^\omega h^\omega + h^\omega \tilde{h}^\omega) \right) \right),\) where \(a : (EH)^3 \to \Lambda^3(EH)\) is the alternation map.

Since \(2h^\omega \otimes h^\omega h^\omega - h^\omega \otimes (\tilde{h}^\omega h^\omega + h^\omega \tilde{h}^\omega) \in \ker s \cong H,\) where \(s : H \otimes S^2 H \to S^3 H\) is the symmetrization map, then \(\eta \in (K + E)H \subseteq \Lambda^3(EH)\).

The real part \(Re(\eta)\) of the restriction of \(\eta\) to \([EH]\) is an element of the real \(Sp(n)Sp(1)\)-module determined by \((K + EH) ((+1)\text{-eigenspace of the real structure map}, \text{i.e.,})\)

\[
Re(\eta) = -2Jx \wedge y \wedge z - 2Kx \wedge Iy \wedge z - 2Kx \wedge y \wedge Iz + 2Jx \wedge Iy \wedge Iz \\
+ x \wedge Jy \wedge z + Ix \wedge Ky \wedge z - Ix \wedge Jy \wedge Iz + x \wedge Ky \wedge Iz \\
+ x \wedge y \wedge Jz - Ix \wedge Iy \wedge Jz + Ix \wedge y \wedge Kz + x \wedge Iy \wedge Kz
\]

is contained in \([KH] + [EH] \subseteq \Lambda^3[EH]\). Now, by straightforward computation we get \(L(Re(\eta)) = 3Re(\eta)\). Hence we have part ii).

Finally, we consider \(\zeta = a \left( (x^\omega \otimes y^\omega \wedge z^\omega) \otimes (h \otimes hh) \right)\). Taking \(h \otimes hh \in \ker \omega_{12} \cong S^3 H\) into account, where \(\omega_{12} : H \otimes S^2 H \to H,\) we have \(\zeta \in (\Lambda^3_0 E + E)S^3 H\). The real part \(Re(\zeta)\) of the restriction of \(\zeta\) to \([EH]\) is an element of the real \(Sp(n)Sp(1)\)-module determined by \((\Lambda^3_0 E + E)S^3 H\), i.e.,

\[
Re(\zeta) = x \wedge y \wedge z - Ix \wedge Iy \wedge z - Ix \wedge y \wedge Iz - x \wedge Iy \wedge Iz
\]

belongs to \([\Lambda^3_0 ES^3 H] + [ES^3 H] \subseteq \Lambda^3[EH]\). By straightforward computation we get \(L(Re(\zeta)) = -3Re(\zeta)\). Thus part iii) follows. \(\square\)

**REMARK 4.2** Note that the condition \(L(b) = 3b\) is equivalent to \((A_{(1)}A_{(2)} + A_{(2)}A_{(3)} + A_{(3)}A_{(1)})b = b,\) for \(A = I, J, K\) and, on the other hand, the condition \(L(b) = -3b\) is equivalent to \(\sum_{A=I,J,K}A_{(2)}A_{(3)}b = -b.\) Therefore, we also have these alternative ways to describe the spaces \((K + E)H\) and \((\Lambda^3_0 E + E)S^3 H\).

The \(Sp(n)Sp(1)\)-subspace \(EH\) of \(\Lambda^3 \mathcal{V}^*\) consists of the three-forms \(x \cdot \Omega,\) where \(x \in \mathcal{V}.\) For \(b \in \Lambda^3 \mathcal{V}^*\), its projection on \(EH\) is given by \(\pi_{EH}(b) = \xi_b \cdot \Omega,\) where \(\xi_b\) is defined by

\[
\xi_b(x) = \frac{1}{12(2n+1)} \sum_{A=I,J,K} b(e_r, A e_r, Ax) = -\frac{1}{6(2n+1)} \sum_{A=I,J,K} \langle Ax, b, \omega_A \rangle,
\]

for all \(x \in \mathcal{V},\) where \(\{e_1, \ldots, e_{4n}\}\) is an orthonormal basis for \(\mathcal{V}.\) Note that the map \(\Lambda^3 \mathcal{V}^* \to \mathcal{V}^*,\) given by \(b \to \xi_b,\) is an \(Sp(n)Sp(1)\)-map.
The subspace of $\Lambda^3 V^*$ consisting of the three-forms $b = -2 \sum_{A=I, J, K} A\xi_A \wedge \omega_A$, where $\xi_I, \xi_J, \xi_K \in V^*$, is an $\text{Sp}(n)\text{Sp}(1)$-module of dimension $12n$. By reasons of dimension, such an $\text{Sp}(n)\text{Sp}(1)$-module must coincide with $E(H + S^3H)$. Moreover, if $\xi_b, \Omega$ is the $EH$-projection of $b = -2 \sum_{A=I, J, K} A\xi_A \wedge \omega_A$, then $\xi_b = \frac{1}{3}(\xi_I + \xi_J + \xi_K)$.

Therefore, the subspace $ES^3H$ of $\Lambda^3 V^*$ consists of the three-forms $b = -2 \sum_{A=I, J, K} A\xi_A \wedge \omega_A$ such that $\xi_I + \xi_J + \xi_K = 0$. That is, the kernel of the map $E(H + S^3H) \to EH$ given by $b \to \xi_b$. Hence the subspace $EH$ can be also described as consisting of those three-forms $b = -2 \sum_{A=I, J, K} A\xi_A \wedge \omega_A$ such that $\xi_I = \xi_J = \xi_K$.

Note that, in such a situation, $\xi_b = \xi_I = \xi_J = \xi_K$ and $b = \xi_b, \Omega$.

For $b \in \Lambda^3 V^*$, $\pi_{E(H + S^3H)}(b) = -2 \sum_{A=I, J, K} A\xi_{bA} \wedge \omega_A$ is its projection on $E(H + S^3H)$, where

$$
\xi_{bA}(x) = -\frac{3}{2(n-1)}\xi_b(x) - \frac{1}{4(n-1)}(A x, b, \omega_A),
$$

for all $x \in V$. Note that each one-form $\xi_{bA}$ is not global, because its definition depends on the chosen basis $\{I, J, K\}$. However, the three-form $-2 \sum_{A=I, J, K} A\xi_{bA} \wedge \omega_A$ is global and $b \to -2 \sum_{A=I, J, K} A\xi_{bA} \wedge \omega_A$ constitutes an $\text{Sp}(n)\text{Sp}(1)$-map.

After these considerations and taking Proposition 4.1 into account, in Table 1 we display the corresponding conditions which describe the $\text{Sp}(n)\text{Sp}(1)$-subspaces of $\Lambda^3 V^* \cong \Lambda^3(EH)$. In case of $V$ eight-dimensional, we would have $\Lambda^3_0ES^3H = \{0\}$ and a smaller corresponding table would be obtained.

| $b = 0$ | $b = \xi_b, \Omega$ |
|--------|------------------|
| $L(b) = 3b$ or $\sum_{1 \leq i < j \leq 3} A_{(i)A_{(j)}}b = b$, for $A = I, J, K$, and $\xi_b = 0$ | $L(b) = -3b$ or $\sum_{A=I, J, K} A^{(2)}A^{(3)}b = -b$, and $\xi_{bI} = \xi_{bJ} = \xi_{bK} = 0$ |
| $b = -2 \sum_{A=I, J, K} A\xi_{bA} \wedge \omega_A$ and $\xi_b = 0$ | $b = 3b$ or $\sum_{1 \leq i < j \leq 3} A_{(i)A_{(j)}}b = b$, for $A = I, J, K$ |
| $L(b) = 3b + 12 \sum_{A=I, J, K} \xi_{bA} \wedge \omega_A$ | $L(b) = -3b + 6\xi_{b, \Omega}$ and $\xi_{bI} = \xi_{bJ} = \xi_{bK}$ |
| $b = -2 \sum_{A=I, J, K} A\xi_{bA} \wedge \omega_A$ | $b = 3b + 12 \sum_{A=I, J, K} \xi_{bA} \wedge \omega_A$ |

Table 1: $\text{Sp}(n)\text{Sp}(1)$-subspaces of $\Lambda^3 V^* \cong \Lambda^3(EH)$
5. Classes of almost quaternion-Hermitian manifolds

In this section, our aim is to deduce conditions which characterize the different classes of almost quaternion-Hermitian manifolds. In other words, we are going to show the explicit conditions which describe the \(Sp(n)Sp(1)\)-subspaces of the space \(\mathcal{V}^* \otimes \Lambda^3_0 ES^2 H\) of covariant derivatives of the fundamental four-form \(\Omega\).

For such a purpose, we consider the \(Sp(n)Sp(1)\)-map \(d^* : \mathcal{V}^* \otimes \Lambda^3_0 ES^2 H \to \Lambda^3 \mathcal{V}^*\), defined by \(d^* a = -C_{12}(a)\), where \(C\) is the metric contraction. In [19] it is proved

\[
\ker d^* = \Lambda^3_0 EH + KS^3 H. \tag{5.1}
\]

If \(b\) is a three-form, one can check that the tensors \(i_A(\cdot, b)\) satisfy i') and ii") of Proposition 3.1. Then \(\sum_{A=I,J,K} i_A(\cdot, b) \wedge \omega_A \in \mathcal{V}^* \otimes \Lambda^3_0 ES^2 H\). Therefore, we can consider the \(Sp(n)Sp(1)\)-map \(\hat{d}^* : \Lambda^3 \mathcal{V}^* \to (K + E)H + (\Lambda^3_0 E + E)S^3 H\), defined by

\[
\hat{d}^*(b) = \frac{1}{18} \sum_{A=I,J,K} i_A(\cdot, -L(b)) \wedge \omega_A - \frac{2k_1}{3k_2} \sum_{A,B=I,J,K} i_A(\cdot, B\xi_{b,B} \wedge \omega_B) \wedge \omega_A
\]

\[
- \frac{4k_2^2 + k_2^2}{12k_1k_2} \{ \cdot \wedge (\xi_b, \Omega) - \xi_b \wedge (\cdot, \Omega) \},
\]

where \(k_1 = n - 1, k_2 = 2n + 1\) and \(L, \xi_b, \xi_{b,A}\) are defined by 2.3, 4.1, 4.2, respectively.

By straightforward computation one can check \(d^* \circ \hat{d}^*(b) = b\). Therefore, \(d^*\) is surjective and \(\hat{d}^*\) is injective.

Next we will show a relation involving the maps \(L, d^* \) and \(L\).

**LEMMA 5.1** Let \(L\) and \(L\) be the maps defined by 3.3 and 2.3, respectively. Then

\[
d^* \circ L = d^* + L \circ d^*. \tag{5.2}
\]
Proof.- In fact, let \( a = a_1 + a_2 \in V^* \otimes \Lambda_0^2 ES^2(H) \), where \( a_1 \in (\Lambda_0^3 E + K + E)H \) and \( a_2 \in (\Lambda_0^3 E + K + E)S^3H \). Taking Proposition 3.4 into account, we have \( d^*L(a) = 4d^*a_1 - 2d^*a_2 \). Since \( d^*a_1 \) and \( d^*a_2 \) are the projections of \( d^*a \) on \((K + E)H \) and \((\Lambda_0^3 E + E)S^3H \), respectively, we have \( d^*a_1 = \frac{1}{6}(3d^*a + L(d^*a)) \) and \( d^*a_2 = \frac{1}{6}(3d^*a - L(d^*a)) \) and the required identity holds. \( \square \)

By reiterated use of the maps \( L, \hat{d}^* \) and Table 1, taking Schur’s Lemma into account, we can get the conditions which define each \( \text{Sp}(n)Sp(1) \)-subspace of \( V^* \otimes \Lambda_0^2 ES^2H \). Identity 5.2 is used to eliminate some redundant conditions. Thus, the different classes are characterized by conditions displayed in the second column of Table 2, where we denote \( k_1 = n - 1, k_2 = 2n + 1, \xi = \xi_d^3 \Omega \) and \( \xi_A = \xi_{d^3\Omega, A} \).

We will show some examples to illustrate the method we follow.

EXAMPLE 1: Let \( \nabla \Omega \in KH \), then \( \nabla \Omega = \hat{d}^*(d^*\Omega) \) (we are writing \( d^*\Omega = d^*(\nabla \Omega) \), in the context of manifolds \( d^*\Omega \) means the coderivative of \( \Omega \)). From Table 1, we have \( L(d^*\Omega) = 3d^*\Omega \) and \( \xi = 0 \), then

\[
(5.3) \quad \nabla_x \Omega = \frac{1}{6} \sum_{A=I,J,K} i_A(x, d^*\Omega) \wedge \omega_A.
\]

for all \( x \in V \). Thus, we could say:

\( \nabla \Omega \in KH \) if and only if \( \nabla \Omega \) satisfies 5.3, \( L(d^*\Omega) = 3d^*\Omega \) and \( \xi = 0 \).

But some of these conditions are redundant. In fact, from the equation 5.3, taking 4.1 and 4.2 into account, we have

\[
(5.4) \quad d^*\Omega = \frac{1}{3} L(d^*\Omega) - \xi_3 \Omega + \frac{4(n-1)}{3} \sum_{A=I,J,K} A\xi_A \wedge \omega_A.
\]

Since \( L^2(d^*\Omega) = 9d^*\Omega \), \( L(\xi_3 \Omega) = 3\xi_3 \Omega \) and \( L \left( \sum_{A=I,J,K} A(\xi_A - \xi) \wedge \omega_A \right) = -3 \sum_{A=I,J,K} A(\xi_A - \xi) \wedge \omega_A \), we have

\[
(5.5) \quad L(d^*\Omega) = 3d^*\Omega - (4n - 1)\xi_3 \Omega - 4(n-1) \sum_{A=I,J,K} A\xi_A \wedge \omega_A.
\]

Now, equations 5.4 and 5.5 imply \( \xi_3 \Omega = 0 \). Therefore, \( \xi = 0 \).

Taking \( \xi = \frac{1}{3}(\xi_I + \xi_J + \xi_K) \) and equation 5.4 into account, it follows next description of the \( KH \)-subspace:

\( \nabla \Omega \in KH \) if and only if \( \nabla \Omega \) satisfies 5.3 and \( \xi_I = \xi_J = \xi_K \).

EXAMPLE 2: Let \( \nabla \Omega \in \Lambda_0^3 E(H + S^3H) \), then \( \nabla \Omega = (\nabla \Omega)_1 + (\nabla \Omega)_2 \), where \( (\nabla \Omega)_1 \in \Lambda_0^3 EH \) and \( (\nabla \Omega)_2 \in \Lambda_0^3 ES^3H \). By 5.1, \( d^*\Omega = d^*(\nabla \Omega)_2 \). Since \( d^* \Omega \in \Lambda_0^3 ES^3H \)
$\Lambda^3_0 ES^3H$, then $L(d^*\Omega) = -3d^*\Omega$ and $\xi_I = \xi_J = \xi_K$ (see Table 1). Therefore, we have

$$(\nabla\Omega)_2 = d^*(d\Omega) = -\frac{1}{6} \sum_{A=I,J,K} i_A(\cdot \lrcorner d^*\Omega) \wedge \omega_A.$$  

From Proposition 3.4 we obtain

$$(5.6) \quad \mathcal{L}(\nabla\Omega) = 4(\nabla\Omega)_1 - 2(\nabla\Omega)_2 = 4\nabla\Omega + \sum_{A=I,J,K} i_A(\cdot \lrcorner d^*\Omega) \wedge \omega_A.$$  

Thus, taking 5.2 into account, we could say:

$$\nabla\Omega \in \Lambda^3_0 E(H + S^3H) \text{ if and only if } \nabla\Omega \text{ satisfies } 5.6, \quad L(d^*\Omega) = -3d^*\Omega \quad \text{and} \quad \xi_I = \xi_J = \xi_K.$$  

But again, some of these conditions are redundant. In fact, from the equality 5.6, taking 4.1 and 4.2 into account, we have

$$(5.7) \quad d^*\mathcal{L}(\nabla\Omega) = d^*\Omega + L(d^*\Omega) = 4d^*\Omega + 2L(d^*\Omega) - 6\xi_J \Omega + 8(n-1) \sum_{A=I,J,K} A\xi_A \wedge \omega_A.$$  

Applying the map $L$ to both sides of this equation, we get

$$(5.8) \quad 0 = 3d^*\Omega + L(d^*\Omega) - 2(4n - 1)\xi_J \Omega - 8(n-1) \sum_{A=I,J,K} A\xi_A \wedge \omega_A.$$  

From equations 5.7 and 5.8, it follows $0 = 2 \sum_{A=I,J,K} A\xi_A \wedge \omega_A$. Therefore $\xi_I = \xi_J = \xi_K = \xi$.

Now, taking equation 5.7 into account, we obtain the following description of the subspace $\Lambda^3_0 E(H + S^3H)$:

$$\nabla\Omega \in \Lambda^3_0 E(H + S^3H) \text{ if and only if } \nabla\Omega \text{ satisfies } 5.6 \text{ and } \xi = 0.$$  

As we have already pointed out, by similar considerations to those contained in the exposed examples, we will get the conditions given in Table 2 which describe how must be $\nabla\Omega$ to belong to the different $Sp(n)Sp(1)$-subspaces of $V^* \otimes \Lambda^3_0 ES^2H$.

In the mentioned Table 2, the conditions are displayed for $V = T_pM$, the tangent space at a point $p$ of an almost quaternion-Hermitian manifold $M$ of dimension higher than eight. In case $M$ were eight-dimensional, we would have $\Lambda^3_0 EH = \Lambda^3_0 ES^3H = \{0\}$ and an smaller corresponding table would be resultant.

A. Swann in [16] showed that, for dimension higher than eight, $\nabla\Omega$ is totally determined by the exterior derivative $d\Omega$. Therefore, our purpose now is to describe the different classes of almost quaternion-Hermitian manifolds by conditions on $d\Omega$. For this aim, we consider the alternation map $a : T^*M \otimes \Lambda^3_0 ES^2H \rightarrow \Lambda^5 T^*M$, defined by $a(a)(X, Y, Z, U, V) = \mathcal{S}_{X^Y Z^U V} a(X, Y, Z, U, V)$, where $\mathcal{S}$ denotes cyclic sum. For dimension $4n > 8$, it was shown in [16] that $a$ is non-zero on each irreducible summand. Thus, by Schur’s Lemma, applying $a$ to the conditions already
obtained in terms of $\nabla \Omega$, we will get conditions in terms of $d\Omega$ which characterize, for dimension $4n > 8$, the different classes of almost quaternion-Hermitian manifolds. Proceeding in this way, taking Lemma below into account, we obtain the conditions displayed in the third column of Table 2.

If the almost quaternion-Hermitian manifold is eight-dimensional, we have $\Lambda^3_0 E = \{0\}$ and $\Lambda^5 T^* M \cong \Lambda^3 T^* M = (K + E)H + ES^3 H$. Therefore, a partial set of classes can be characterized via $d\Omega$. Table 3 contains such characterizations.

**Lemma 5.2**

Given any one-form $\zeta$ and any three-form $b$ on an almost quaternion-Hermitian manifold $M$, and the alternation map $a : T^* M \otimes \Lambda^2_0 ES^2 H \to \Lambda^5 T^* M$, then

\begin{align*}
(5.9) & \quad a(\mathcal{L}(\nabla \Omega)) + 2d\Omega = L(d\Omega), \\
(5.10) & \quad a(\cdot \wedge (\zeta \wedge \Omega) - \zeta \wedge (\cdot \wedge \Omega)) = 4\zeta \wedge \Omega, \\
(5.11) & \quad a \left( \sum_{A=I,J,K} i_A(\cdot \wedge b) \wedge \omega_A \right) = 2 \sum_{A=I,J,K} i_A b \wedge \omega_A. 
\end{align*}

**Proof.** Equations 5.11 and 5.10 follow by straightforward computation. Taking Lemma 3.3 into account, equation 5.9 also follows by direct computation.

The coderivative $d^* \Omega$ and one-forms $\xi$, $\xi_I$, $\xi_J$, $\xi_K$ are involved in some of the conditions contained in the third column of Table 2. Therefore, it is a natural question whether such objects can be computed by using the exterior derivative $d\Omega$. We will answer positively to such a question. In fact, one can check

$$\frac{(-1)^{n+1}}{(2n+1)!} \Omega^n = e_1 \wedge \ldots \wedge e_n \wedge I e_1 \wedge \ldots \wedge I e_n \wedge J e_1 \wedge \ldots \wedge J e_n \wedge K e_1 \wedge \ldots \wedge K e_n.$$ 

Thus, taking as volume form $Vol = \frac{(-1)^{n+1}}{(2n+1)!} \Omega^n$, we obtain

$$d^* \Omega = - * d * \Omega = \frac{(-1)^n(n-1)}{(2n+1)!} * (\Omega^{n-2} \wedge d\Omega).$$

For a one-form $a$ on $M$, it can be checked that

$$\ast (\ast (a \wedge \Omega) \wedge \Omega) = 12(n-1)(2n+1)a.$$

If we consider the $Sp(n)Sp(1)$-map $T^*_p M \otimes \Lambda^2_0 ES^2 H \to T^*_p M$ given by $\nabla \Omega \to \ast (\ast (a(\nabla \Omega)) \wedge \Omega)$, where $a$ is the alternation map above defined, then we get

$$\ast (\ast (a(\nabla \Omega)) \wedge \Omega) = \ast (\ast (a(\pi_{\text{EH}}(\nabla \Omega)) \wedge \Omega) = - \frac{1}{n-1} \ast (\ast (\xi \wedge \Omega) \wedge \Omega),$$

Now, from Table 2 we have

$$\pi_{\text{EH}}(\nabla \Omega) = - \frac{1}{4(n-1)} \{ \cdot \wedge (\xi \wedge \Omega) - \xi \wedge (\cdot \wedge \Omega) \}.$$
Taking \(d\Omega = a(\nabla \Omega)\, 5.13\) and 5.14 into account, we obtain the identity
\[
\xi = \frac{-1}{12(2n+1)} * (d\Omega \wedge \Omega).
\]

Now, if \(\xi_I, \xi_J\) and \(\xi_K\) are three one-forms, it can be checked
\[
\sum_{B=I,J,K} * (B\xi_B \wedge \omega_B) \wedge \omega_A = 4n A\xi_A.
\]

If we consider the \(Sp(n)Sp(1)\)-map \(\Lambda^5 T^*_p M \rightarrow EH + ES^3 H\) given by \(d^* \Omega \rightarrow \frac{1}{4n} \sum_{A=I,J,K} * (d^* \Omega \wedge \omega_A) \wedge \omega_A\), then, taking identity 5.16 into account, we have
\[
\sum_{A=I,J,K} * (d^* \Omega) \wedge \omega_A = \sum_{A=I,J,K} * (\pi_{E(H+S^3 H)}(d^* \Omega) \wedge \omega_A) \wedge \omega_A
\]
\[
= -8n \sum_{A=I,J,K} A\xi_A \wedge \omega_A.
\]
Hence \(A\xi_A = \frac{-1}{8n} * (d^* \Omega) \wedge \omega_A\). Therefore, by 5.12, \(\xi_A\) can be computed via \(d\Omega\).

Now we deduce alternative ways to characterize some classes of almost quaternion-Hermitian manifolds. For such a purpose, we consider the \(Sp(n)Sp(1)\)-module of \(\Lambda^5 T^*_p M\) given by \(E(H+S^3 H+S^5 H)\) (see [17]), which consists of those five-forms \(\varphi\) such that \(\varphi = \sum_{A=I,J,K} a_{AB} \wedge \omega_A \wedge \omega_B\), where \(a_{AB}\) are one-forms. Therefore, the orthogonal complement \((E(H+S^3 H+S^5 H))^\perp\) consists of those five-forms \(\varphi\) such that \(\langle a \wedge \omega_A \wedge \omega_B, \varphi \rangle = 0\), for \(A, B = I, J, K\) and for all one-form \(a\). Since, for all \(p\)-forms \(\psi, \phi\), we have \(\psi \wedge * \phi = \langle \psi, \phi \rangle \text{Vol}\), it follows part i) of next lemma.

**LEMMA 5.3** An almost quaternion-Hermitian manifold satisfies:

i) For all five-form \(\varphi, \varphi \in (E(H+S^3 H+S^5 H))^\perp\) if and only if, for \(A, B = I, J, K\), \(* \varphi \wedge \omega_A \wedge \omega_B = 0\).

ii) For \(A = I, J, K\), we have \(2A(A \wedge \omega_A, \omega_A) = * (d\Omega \wedge \omega_A \wedge \omega_A)\).

**Proof.** It remains to prove ii). For all three one-forms \(\xi_I, \xi_J, \xi_K\) and \(A = I, J, K\), by direct computation we obtain
\[
\sum_{B,C=I,J,K} * (i_B(C\xi_B \wedge \omega_C) \wedge \omega_A) \wedge \omega_A = -4(n-1)(2n+1)\xi_A.
\]

Furthermore, we have \(d\Omega = (d\Omega)_1 + (d\Omega)_2\), where \((d\Omega)_1 \in (\Lambda^3 E + K)(H+S^3 H)\) and \((d\Omega)_2 \in E(H+S^3 H)\). By part i), we have \(* (d\Omega)_1 \wedge \omega_A \wedge \omega_A = 0\). Now, from Table 2, taking 4.2, \(k_1 = n - 1\) and \(k_2 = 2n + 1\) into account, we deduce that
\[
(d\Omega)_2 = \sum_{A,B=I,J,K} i_A (B\eta_B \wedge \omega_B) \wedge \omega_A,
\]
where the one-forms $\eta_A$ are given by

$$\eta_A = \frac{1}{2k_1k_2} \langle A \cdot d\omega_A, \omega_A \rangle.$$

Hence, taking 5.17 and 5.18 into account, it follows

$$\ast (\ast d\Omega \wedge \omega_A \wedge \omega_A) = \ast (\ast (d\Omega)_2 \wedge \omega_A \wedge \omega_A) = 2 \langle A \cdot d\omega_A, \omega_A \rangle.$$

Next corollary is an immediate consequence of 4.1, 4.2 and Lemma 5.3.

**Proposition 5.4**

i) $\pi_{EH}(\nabla \Omega) = 0$ if and only if $\ast d\Omega \wedge \Omega = 0$.

ii) $\pi_{E_3H}(\nabla \Omega) = 0$ if and only if $\ast d\Omega \wedge \omega_I \wedge \omega_J \wedge \omega_J = \ast d\Omega \wedge \omega_K \wedge \omega_K$. 

iii) $\pi_{E(H+S^3H)}(\nabla \Omega) = 0$ if and only if $\ast d\Omega \wedge \omega_A \wedge \omega_A = 0$, for $A = I, J, K$.  

In the following lines we derive an expression for $d^* \Omega$ which is useful to handle examples. We recall the following identity, given in [7],

$$2\nabla \omega_I = d\omega_I - I_2(I_3)d\omega_I - I_2(N_I),$$

where $N_I(X,Y,Z) = \langle X, N_I(Y,Z) \rangle$ and the $(1,2)$-tensor $N_I$ is the Nijenhuis tensor for $I$ defined by $N_I(X,Y) = [X,Y] + I[IX,Y] + I[X,IY] - [IX,IY]$, for all vector fields $X, Y, Z$. Moreover, we have the following fact, noted in [1],

$$N_I(e_i, e_i, \cdot) = 0.$$

Now, identities 5.19 and 5.20 imply

$$I d^* \omega_I = -\langle \cdot \cdot d\omega_I, \omega_I \rangle.$$

On the other hand, from $d^* \Omega = -\nabla_{\omega_I} \Omega(e_i, \cdot, \cdot, \cdot)$ directly follows the expression $d^* \Omega = 2 \sum_{A=I,J,K} (d^* \omega_A \wedge \omega_A - Ad\omega_A)$. Thus, taking 5.21 into account, we obtain

$$d^* \Omega = -2 \sum_{A=I,J,K} \langle A \cdot d\omega_A, \omega_A \rangle \wedge \omega_A - 2 \sum_{A=I,J,K} Ad\omega_A.$$

Finally, we briefly point out a relation of examples already indicated in some references and others studied by ourselves. Because it would take up a lot of space in the present exposition, we reserve a more detailed presentation of these latter examples to future paper. If $\nabla \Omega = 0$, the manifold is said to be quaternionic Kähler (q.K.) and its metric $\langle \cdot, \cdot \rangle$ is Einstein. The model example of such manifolds is the quaternionic projective space $\mathcal{HP}(n)$. For more q.K. examples, there is a relative extensive bibliography of them (for instance, [12]). If $\nabla \Omega \in EH$, then the manifold is
said to be locally conformal quaternionic Kähler (l.c.q.K.). The manifolds $S^{4n+3} \times S^1$ are locally conformal hyperKähler which is an special case of l.c.q.K. More examples of these manifolds are given by L. Ornea and P. Piccinni in [11]. The theory of quaternionic manifolds have been independently developed by S. Salamon in [13] and by L. Berard Bergery and T. Ochoiai in [3]. An almost quaternion-Hermitian (a.q.H.) manifold is quaternionic if and only if $\nabla \Omega \in (\Lambda_0^3 E + K + E)H$ (see [13]). Compact examples of quaternionic manifolds are given by D. Joyce in [10].

An example of eight-dimensional a.q.H. manifold such that $d\Omega = 0$ and $\nabla \Omega \neq 0$ ($KS^3H$ type) is given by S. Salamon in [15]. Likewise one can check that the manifolds $N_1$, $N_2$ and $N_3$ given by I. Dotti and A. Fino in [5] are a.q.H. manifolds of type $KH$. The quaternionic Heisenberg group studied by L. Cordero, M. Fernández and M. de León in [4], is an a.q.H. manifold of type $(\Lambda_0^3 E + K)H$. Quaternion Kähler manifolds with torsion, introduced by S. Ivanov in [9], can be identified with the class $(K + E)H$. Special cases of these manifolds are hyperKähler manifolds with torsion (HKT). The manifold $(S^1 \times S^3)^3$ is HKT and hence $(K + E)H$. Because the torsion one-form is closed, doing local conformal changes of metric one can obtain open submanifolds of type $KH$. On $S^3 \times T^9$ and $(S^3)^4$, one can find a.q.H. structures of type $\Lambda_0^3 E(S^3 H + H)$. On $T^3 \times M^3$, with $M$ a three-dimensional Lie group, either nilpotent or solvable, one can define a.q.H. structures of type $(K + E)(S^3 H + H)$, $(\Lambda_0^3 E + K)(S^3 H + H)$, $(K + E)S^3 H$ and $(\Lambda_0^3 E + K)H + KS^3 H$. In the case of $T^3 \times M^3$ with a.q.H. of type $(K + E)(S^3 H + H)$, the one-form $\xi$ is closed. Therefore, doing local conformal changes of metric, one can obtain open submanifolds with a.q.H. structure of type $K(S^3 H + H) + ES^3 H$.

**REMARK 5.5** The results here contained are also valid for almost quaternion pseudo-Hermitian manifolds as it happens with Swann’s results in [16]. In such a case, we have an $\text{Sp}(p, q)\text{Sp}(1)$-structure and $(4p, 4q)$ is the signature of the metric $\langle \cdot, \cdot \rangle$. Since $4q$ is even the expressions involving the Hodge $*$-operator still are valid. In the present text, we would only need to do slight modifications in those expressions involving an orthonormal basis $\{e_1, \ldots, e_{4n}\}$ for vectors, where we would have to write $\epsilon_r = \langle e_r, e_r \rangle$. For instance, the complex symplectic form $\omega_E$ and the extension of $\langle \cdot, \cdot \rangle$ to $(0, s)$-tensors would be respectively given by

$$\omega_E = \epsilon_i e^\omega_i \wedge \bar{e}_i^\omega, \quad \langle a, b \rangle = \frac{1}{s!} a(e_{i_1}, \ldots, e_{i_s}) b(e_{i_1}, \ldots, e_{i_s}) \epsilon_{i_1} \ldots \epsilon_{i_s}.$$
Table 2: Classes of almost quaternion-Hermitian manifolds of dimension $\geq 12$

| Class | $\nabla \Omega = 0$ | $d\Omega = 0$ |
|-------|------------------|----------------|
| $\Lambda_3^0 \text{EH}$ | $L(\nabla \Omega) = 4\nabla \Omega$ and $d^\ast \Omega = 0$ | $L(d\Omega) = 6d\Omega$ and $d^\ast \Omega = 0$ |
| $KH$ | $\nabla \Omega = \frac{1}{6} \sum_{A=1,J,K} i_A (\cdot d^\ast \Omega) \wedge \omega_A$ and $\xi_I = \xi_J = \xi_K$ | $d\Omega = \frac{1}{3} \sum_{A=1,J,K} i_A (d^\ast \Omega) \wedge \omega_A$ and $\xi_I = \xi_J = \xi_K$ |
| $EH$ | $\nabla \Omega = -\frac{1}{k_2} \{ \wedge (\xi_J \Omega) - \xi (\cdot \Omega) \}$ | $d\Omega = -\frac{1}{k_3} \xi \wedge \Omega$ |
| $\Lambda_3^0 \text{ES}^3 \text{H}$ | $\nabla \Omega = \frac{1}{6} \sum_{A=1,J,K} i_A (\cdot d^\ast \Omega) \wedge \omega_A$ and $\xi = 0$ | $d\Omega = -\frac{1}{3} \sum_{A=1,J,K} i_A (d^\ast \Omega) \wedge \omega_A$ and $\xi = 0$ |
| $ES^3 \text{H}$ | $\nabla \Omega = \frac{1}{k_2} \sum_{A,B=1,J,K} i_A (\cdot (B \xi_B \wedge \omega_B)) \wedge \omega_A$ and $\xi = 0$ | $d\Omega = \frac{2}{k_2} \sum_{A,B=1,J,K} i_A (B \xi_B \wedge \omega_B) \wedge \omega_A$ and $\xi = 0$ |
| $\Lambda_3^0 (E + K)H$ | $L(\nabla \Omega) = 4\nabla \Omega$ and $\xi = 0$ | $L(d\Omega) = 6d\Omega$ and $\xi = 0$ |
| $\Lambda_3^0 (E + E)H$ | $L(\nabla \Omega) = 4\nabla \Omega$ and $d^\ast \Omega = \Omega \xi \Omega$ | $L(d\Omega) = 6d\Omega$ and $d^\ast \Omega = \xi \cdot \Omega$ |
| $\Lambda_3^0 (H + S^3 \text{H})$ | $L(\nabla \Omega) = 4\nabla \Omega + \sum_{A=1,J,K} i_A (\cdot d^\ast \Omega) \wedge \omega_A$ and $\xi = 0$ | $L(d\Omega) = 6d\Omega + 2 \sum_{A=1,J,K} i_A (d^\ast \Omega) \wedge \omega_A$ and $\xi = 0$ |
| $\Lambda_3^0 \text{EH} + KS^3 \text{H}$ | $d^\ast \Omega = 0$ or $\Omega^p = 2 \wedge d\Omega = 0$ | Idem |
| $\Lambda_3^0 \text{EH} + ES^3 \text{H}$ | $L(\nabla \Omega) = 4\nabla \Omega$ and $d^\ast \Omega = -2 \sum_{A=1,J,K} A \xi_A \wedge \omega_A$ | $L(d\Omega) = 6d\Omega - \frac{12}{k_2} \sum_{A,B=1,J,K} i_A (B \xi_B \wedge \omega_B) \wedge \omega_A$ and $d^\ast \Omega = -2 \sum_{A=1,J,K} A \xi_A \wedge \omega_A$ |
| $(K + E)H$ | $\nabla \Omega = \frac{1}{6} \sum_{A=1,J,K} i_A (\cdot d^\ast \Omega) \wedge \omega_A$ and $\xi = 0$ | $d\Omega = \frac{1}{3} \sum_{A=1,J,K} i_A (d^\ast \Omega) \wedge \omega_A - \frac{k_2}{k_3} \xi \wedge \Omega$ and $\xi_I = \xi_J = \xi_K$ |
| $KH + \Lambda_3^0 \text{ES}^3 \text{H}$ | $L(\nabla \Omega) = -2\nabla \Omega + \sum_{A=1,J,K} i_A (\cdot d^\ast \Omega) \wedge \omega_A$ and $\xi_I = \xi_J = \xi_K$ | $L(d\Omega) = 2 \sum_{A=1,J,K} i_A (d^\ast \Omega) \wedge \omega_A$ and $\xi_I = \xi_J = \xi_K$ |
| $K(H + ES^3 \text{H})$ | $\nabla \Omega = \frac{1}{6} \sum_{A=1,J,K} i_A (\cdot d^\ast \Omega) \wedge \omega_A$ and $\xi_I = \xi_J = \xi_K$ | $d\Omega = \frac{1}{3} \sum_{A=1,J,K} i_A (d^\ast \Omega) \wedge \omega_A + \frac{2}{3k_2} \sum_{A,B=1,J,K} i_A (B \xi_B \wedge \omega_B) \wedge \omega_A$ |
| $KH + ES^3 \text{H}$ | $\nabla \Omega = \frac{1}{6} \sum_{A=1,J,K} i_A (\cdot d^\ast \Omega) \wedge \omega_A + \frac{k_2 + 3}{3k_2} \sum_{A,B=1,J,K} i_A (\cdot (B \xi_B \wedge \omega_B)) \wedge \omega_A$ | $d\Omega = \frac{1}{3} \sum_{A=1,J,K} i_A (d^\ast \Omega) \wedge \omega_A + \frac{2}{3k_2} \sum_{A,B=1,J,K} i_A (B \xi_B \wedge \omega_B) \wedge \omega_A$ |
| Equation | Description |
|----------|-------------|
| \( EH + \Lambda_3^e ES^3 H \) | \( \nabla \Omega = - \frac{1}{6} \sum_{A=I,J,K} i_A(\cdot d^* d^* \Omega) \land \omega_A + \frac{k_2}{k_2 - 6} \left\{ \cdot \land (\xi d^* \Omega) - \xi \land (\cdot d^* \Omega) \right\} \) | \( d\Omega = - \frac{1}{3} \sum_{A=I,J,K} i_A(d^* \Omega) \land \omega_A + \frac{k_2 - 6}{3k_1} \xi \land \Omega \) |
| \( EH + KS^3 H \) | \( \mathcal{L}(\nabla \Omega) = -2 \nabla \Omega + \frac{3}{2k_1} \left\{ \cdot \land (\xi d^* \Omega) - \xi \land (\cdot d^* \Omega) \right\} \) and \( d^* \Omega = \xi d^* \Omega \) | \( L(d\Omega) = \frac{6}{k_1} \xi \land \Omega \) and \( d^* \Omega = \xi d^* \Omega \) |
| \( EH + ES^3 H \) | \( \nabla \Omega = \frac{1}{k_2} \sum_{A,B=I,J,K} i_A(\cdot (B \xi_B \land \omega_B)) \land \omega_A - \frac{3}{4k_1 k_2} \left\{ \cdot \land (\xi d^* \Omega) - \xi \land (\cdot d^* \Omega) \right\} \) and \( \xi = 0 \) | \( d\Omega = \frac{2}{k_2} \sum_{A,B=I,J,K} i_A(B \xi_B \land \omega_B) \land \omega_A - \frac{3}{k_1 k_2} \xi \land \Omega \) |
| \( (\Lambda_3^e E + K)S^3 H \) | \( \mathcal{L}(\nabla \Omega) = -2 \nabla \Omega \) and \( \xi = \xi_J = \xi_K \) | \( L(d\Omega) = 0 \) and \( \xi = \xi_J = \xi_K \) |
| \( (\Lambda_3^e E + E)S^3 H \) | \( \nabla \Omega = - \frac{1}{6} \sum_{A=I,J,K} i_A(\cdot d^* d^* \Omega) \land \omega_A - \frac{k_2}{3k_2} \sum_{A=I,J,K} i_A(\cdot (B \xi_B \land \omega_B)) \land \omega_A \) and \( \xi = 0 \) | \( d\Omega = \frac{1}{3} \sum_{A=I,J,K} i_A(d^* \Omega) \land \omega_A - \frac{4k_1}{3k_2} \sum_{A,B=I,J,K} i_A(B \xi_B \land \omega_B) \land \omega_A \) and \( \xi = 0 \) |
| \( (K + E)S^3 H \) | \( \mathcal{L}(\nabla \Omega) = -2 \nabla \Omega \) and \( d^* \Omega = -2 \sum_{A=I,J,K} A \xi_A \land \omega_A \) | \( L(d\Omega) = 0 \) and \( d^* \Omega = -2 \sum_{A=I,J,K} A \xi_A \land \omega_A \) |
| \( (\Lambda_3^e E + K + E)H \) | \( \mathcal{L}(\nabla \Omega) = 4 \nabla \Omega \) | \( L(d\Omega) = 6d\Omega \) |
| \( (\Lambda_3^e K + \Lambda_3^e E)H + ES^3 H \) | \( \mathcal{L}(\nabla \Omega) = 4 \nabla \Omega - \frac{1}{2} \sum_{A=I,J,K} i_A(\cdot d^* \Omega) \land \omega_A - \frac{1}{6} \sum_{A=I,J,K} i_A(\cdot (B \xi_B \land \omega_B)) \land \omega_A \) and \( \xi = 0 \) | \( L(d\Omega) = 6d\Omega + \frac{12}{k_2} \sum_{A,B=I,J,K} i_A(B \xi_B \land \omega_B) \land \omega_A \) and \( \xi = 0 \) |
| \( (\Lambda_3^e E + K)H + ES^3 H \) | \( \mathcal{L}(\nabla \Omega) = 4 \nabla \Omega - \frac{6}{k_2} \sum_{A,B=I,J,K} i_A(\cdot (B \xi_B \land \omega_B)) \land \omega_A \) and \( d^* \Omega = -2 \sum_{A=I,J,K} A \xi_A \land \omega_A \) | \( L(d\Omega) = 6d\Omega + 2 \sum_{A=I,J,K} i_A(d^* \Omega) \land \omega_A - 4\xi \land \Omega \) |
| \( (\Lambda_3^e E + K)H + KS^3 H \) | \( \mathcal{L}(\nabla \Omega) = 4 \nabla \Omega - \frac{6}{k_2} \sum_{A,B=I,J,K} i_A(\cdot (B \xi_B \land \omega_B)) \land \omega_A \) and \( d^* \Omega = -2 \sum_{A=I,J,K} A \xi_A \land \omega_A \) | \( L(d\Omega) = 6d\Omega - \frac{12}{k_2} \sum_{A,B=I,J,K} i_A(B \xi_B \land \omega_B) \land \omega_A - 4\xi \land \Omega \) and \( d^* \Omega = -2 \sum_{A=I,J,K} A \xi_A \land \omega_A \) |
| \( (\Lambda_3^e E + E)H + KS^3 H \) | \( \mathcal{L}(\nabla \Omega) = 4 \nabla \Omega - \frac{6}{k_2} \sum_{A,B=I,J,K} i_A(\cdot (B \xi_B \land \omega_B)) \land \omega_A - \frac{3}{k_2} \left\{ \cdot \land (\xi d^* \Omega) - \xi \land (\cdot d^* \Omega) \right\} \) and \( d^* \Omega = -2 \sum_{A=I,J,K} A \xi_A \land \omega_A \) | \( L(d\Omega) = 6d\Omega - \frac{12}{k_2} \sum_{A,B=I,J,K} i_A(B \xi_B \land \omega_B) \land \omega_A - \frac{12}{k_2} \xi \land \Omega \) and \( d^* \Omega = -2 \sum_{A=I,J,K} A \xi_A \land \omega_A \) |
| \( (\Lambda_3^e E + E)H + KS^3 H \) | \( \mathcal{L}(d^* \Omega) = -3d^* \Omega \) and \( \xi = \xi_J = \xi_K \) | \( L(d^* \Omega) = 3d^* \Omega \) and \( \xi = 0 \) |
\[
\begin{array}{|c|c|c|}
\hline
\text{Alm Quat-Hermitian Manifolds} & \text{Idem} \\
\hline
\text{L}(\nabla \Omega) = 4\nabla \Omega + \frac{4k_1}{k_2} \sum_{A,B=I,J,K} i_A(\cdot d^* \Omega) \wedge \omega_A + \frac{8k_1}{k_2} \sum_{A,B=I,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
\text{and } \xi = 0 \\
\hline
\left( K + E \right) S^3 H + \Lambda_0^3 \text{ES}^3 H & d^* \Omega = -2 \sum_{A=I,J,K} \Lambda_0^3 A_A \wedge \omega_A \\
\hline
\left( K + E \right) H + \Lambda_0^3 \text{ES}^3 H & \nabla \Omega = \frac{1}{18} \sum_{A=I,J,K} i_A(\cdot d \Omega) \wedge \omega_A \\
& - \frac{k_2}{12k_1} \cdot \{ \cdot \Omega \} - \xi \wedge (\cdot \Omega) \\
& + \frac{k_2}{2k_1} \cdot \{ \cdot \Omega \} - \xi \wedge (\cdot \Omega) \\
& + \frac{2k_2}{3k_1} \xi \wedge \Omega \\
& - \frac{3}{4k_1k_2} \cdot \{ \cdot \Omega \} - \xi \wedge (\cdot \Omega) \\
& + \frac{2k_2}{3k_1} \xi \wedge \Omega \\
\hline
\text{L}(\nabla \Omega) = -2\nabla \Omega - \frac{2}{3} \sum_{A=I,J,K} i_A(\cdot d^* \Omega) \wedge \omega_A \\
\hline
\text{L}(d \Omega) = 2 \sum_{A=I,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
\text{and } \xi = \xi \wedge \Omega \\
\hline
\text{E}(H + S^3 H) + \text{KH} & \nabla \Omega = \frac{1}{6} \sum_{A=I,J,K} i_A(\cdot d^* \Omega) \wedge \omega_A \\
& + \frac{k_2 + 3}{3k_2} \sum_{A,B=I,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
& - \frac{3}{4k_1k_2} \cdot \{ \cdot \Omega \} - \xi \wedge (\cdot \Omega) \\
\hline
\text{L}(\nabla \Omega) = -2\nabla \Omega + \frac{1}{2} \sum_{A=I,J,K} i_A(\cdot d^* \Omega) \wedge \omega_A \\
\text{and } \xi = \xi \wedge \Omega \\
\hline
\text{L}(d \Omega) = \sum_{A=I,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
\text{and } \xi = \xi \wedge \Omega \\
\hline
\text{K}(H + S^3 H) + \text{KH} & \nabla \Omega = \frac{1}{18} \sum_{A=I,J,K} i_A(\cdot d \Omega) \wedge \omega_A \\
& - \frac{2k_1}{3k_2} \sum_{A,B=I,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
& + \frac{4k_1}{3k_2} \sum_{A,B=I,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
\hline
\text{L}(\nabla \Omega) = -2\nabla \Omega - \frac{2}{3} \sum_{A=I,J,K} i_A(\cdot d^* \Omega) \wedge \omega_A \\
\hline
\text{L}(d \Omega) = 2 \sum_{A=I,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
\text{and } \xi = \xi \wedge \Omega \\
\hline
\text{E}(H + S^3 H) + \text{KH} & \nabla \Omega = \frac{1}{6} \sum_{A=I,J,K} i_A(\cdot d^* \Omega) \wedge \omega_A \\
& + \frac{2k_2}{3k_1} \sum_{A,B=I,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
& - \frac{3}{4k_1k_2} \cdot \{ \cdot \Omega \} - \xi \wedge (\cdot \Omega) \\
\hline
\text{L}(\nabla \Omega) = -2\nabla \Omega - \frac{2}{3} \sum_{A=I,J,K} i_A(\cdot d^* \Omega) \wedge \omega_A \\
\text{and } \xi = \xi \wedge \Omega \\
\hline
\text{L}(d \Omega) = \sum_{A=I,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
\text{and } \xi = \xi \wedge \Omega \\
\hline
\end{array}
\]
\[
\begin{array}{|c|c|c|}
\hline
E(H + S^3H) + KS^3H & L(\nabla \Omega) = -2\nabla \Omega - \frac{3}{2k_1} \{ \cdot (\xi \cdot \Omega) - \xi \cdot (\cdot \Omega) \} & L(d\Omega) = -\frac{6}{2k_1} \xi \cdot \Omega \\
& \text{and } d^* \Omega = -2 \sum_{A=L,J,K} A \xi_A \wedge \omega_A & \text{and } d^* \Omega = -2 \sum_{A=L,J,K} A \xi_A \wedge \omega_A \\
\hline
(A_2^3E + K + E)S^3H & L(\nabla \Omega) = -2\nabla \Omega & L(d\Omega) = 0 \\
\hline
(L_2^3E + H + S^3H) + (K + E)H & L(\nabla \Omega) = 4\nabla \Omega + \frac{1}{2} \sum_{A=L,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
& - \frac{1}{6} \sum_{A=L,J,K} i_A(\cdot \Omega(d\Omega)) \wedge \omega_A & L(d\Omega) = 6d\Omega + \frac{1}{3} \sum_{A=L,J,K} i_A(d^* \Omega) \wedge \omega_A \\
\hline
K(H + S^3H) + (A_2^3E + E)H & L(d^* \Omega) = 3d^* \Omega & \text{Idem} \\
\hline
(A_2^3E + K)(H + S^3H) & L(\nabla \Omega) = 4\nabla \Omega + \frac{1}{2} \sum_{A=L,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
& - \frac{1}{6} \sum_{A=L,J,K} i_A(\cdot \Omega(d\Omega)) \wedge \omega_A & L(d\Omega) = 6d\Omega + \frac{1}{3} \sum_{A=L,J,K} i_A(d^* \Omega) \wedge \omega_A \\
& + \frac{4k_1}{k_2} \sum_{A=L,J,K} i_A(\cdot (\Omega(B\xi_B \wedge \omega_B))) \wedge \omega_A & - \frac{1}{3} \sum_{A=L,J,K} i_A(B\xi_B \wedge \omega_B) \wedge \omega_A \\
\hline
(A_2^3E + E)(H + S^3H) & L(d^* \Omega) = 3d^* \Omega + 12 \sum_{A=L,J,K} A \xi_A \wedge \omega_A & \text{Idem} \\
\hline
(A_2^3E + E)H + (A_2^3E + K)S^3H & \text{Idem} \\
\hline
(A_2^3E + E)(H + S^3H) & L(\nabla \Omega) = 4\nabla \Omega + \sum_{A=L,J,K} i_A(\cdot \Omega) \wedge \omega_A \\
& - \frac{3}{2k_1} \{ \cdot (\xi \cdot \Omega) - \xi \cdot (\cdot \Omega) \} & L(d\Omega) = 6d\Omega + 2 \sum_{A=L,J,K} i_A(d^* \Omega) \wedge \omega_A \\
& + \frac{4k_1}{k_2} \sum_{A=L,J,K} i_A(\cdot (\Omega(B\xi_B \wedge \omega_B))) \wedge \omega_A & - \frac{12}{3k_1} \xi \cdot \Omega \\
\hline
(L_2^3E + E)(H + S^3H) + (K + E)S^3H & \text{Idem} \\
\hline
(K + E)H + (A_2^3E + K)S^3H & L(d^* \Omega) = -3d^* \Omega & \text{Idem} \\
\hline
\end{array}
\]
\[(K + E)H + \{A_3^0 E + E\}S^3H\]
\[
\nabla \Omega = \frac{1}{18} \sum_{A=1, J, K} i_A (\dot{,} L(d^* \Omega)) \wedge \omega_A - \frac{4k_1^2 + k_2^2}{12k_1k_2} \left\{ \cdot \wedge (\xi, \Omega) - \xi \wedge (\cdot , \Omega) \right\} - \frac{2k_1}{3k_2} \sum_{A, B=1, J, K} i_A (\dot{,} (B \xi_B \wedge \omega_B)) \wedge \omega_A
\]
\[
d\Omega = \frac{1}{9} \sum_{A=1, J, K} i_A (L(d^* \Omega)) \wedge \omega_A - \frac{4k_1^2 + k_2^2}{3k_1k_2} \xi \wedge \Omega - \frac{4k_1}{3k_2} \sum_{A, B=1, J, K} i_A (B \xi_B \wedge \omega_B) \wedge \omega_A
\]
\[(K + E)(H + S^3H)\]
\[
\mathcal{L}(\nabla \Omega) = -2 \nabla \Omega + \frac{1}{2} \sum_{A=1, J, K} i_A (\cdot , d^* \Omega) \wedge \omega_A + \frac{2}{3} \sum_{A=1, J, K} i_A (\cdot , (B \xi_B \wedge \omega_B)) \wedge \omega_A
\]
\[
L(d\Omega) = \sum_{A=1, J, K} i_A (d^* \Omega) \wedge \omega_A + \frac{4}{3} \sum_{A, B=1, J, K} i_A (B \xi_B \wedge \omega_B) \wedge \omega_A
\]
\[K(H + S^3H) + \{A_3^0 E + E\}S^3H\]
\[
\mathcal{L}(\nabla \Omega) = -2 \nabla \Omega - \frac{3}{2k_1} (\cdot \wedge (\xi, \Omega) - \xi \wedge (\cdot \Omega)) + \frac{1}{6} \sum_{A=1, J, K} i_A (\cdot , L(d^* \Omega)) \wedge \omega_A
\]
\[
L(d\Omega) = \sum_{A=1, J, K} i_A (d^* \Omega) \wedge \omega_A + \frac{1}{3} \sum_{A, B=1, J, K} i_A (L(d^* \Omega)) \wedge \omega_A
\]
\[E(H + S^3H) + \{A_3^0 E + K\}H + \{A_3^0 E + E\}S^3H + KH\]
\[
\mathcal{L}(\nabla \Omega) = 4 \nabla \Omega + \frac{1}{2} \sum_{A=1, J, K} i_A (\cdot , d^* \Omega) \wedge \omega_A - \frac{1}{6} \sum_{A=1, J, K} i_A (\cdot , L(d^* \Omega)) \wedge \omega_A - \frac{2k_1}{k_2} (\cdot \wedge (\xi, \Omega) - \xi \wedge (\cdot \Omega))
\]
\[
L(d\Omega) = 6d\Omega + \sum_{A=1, J, K} i_A (d^* \Omega) \wedge \omega_A - \frac{1}{3} \sum_{A=1, J, K} i_A (L(d^* \Omega)) \wedge \omega_A - \frac{8k_1}{k_2} \xi \wedge \Omega + \frac{8k_1}{k_2} \sum_{A, B=1, J, K} i_A (B \xi_B \wedge \omega_B) \wedge \omega_A
\]
\[(K + E)(H + S^3H) + A_3^0 EH\]
\[
L(d^\ast \Omega) = 3d^\ast \Omega + 6 \xi, \Omega + 12 \sum_{A=1, J, K} A \xi_A \wedge \omega_A
\]
\[\xi = 0 \text{ or } s \xi \wedge \omega_{J,} = 0\]
\[(K + E)(H + S^3H) + ES^3H\]
\[
L(d^\ast \Omega) = -3d^\ast \Omega + 6 \xi, \Omega
\]
\[(K + E)(H + S^3H) + A_3^0 ES^3H\]
\[
\mathcal{L}(\nabla \Omega) = -2 \nabla \Omega + \frac{1}{2} \sum_{A=1, J, K} i_A (\cdot , d^* \Omega) \wedge \omega_A + \frac{1}{6} \sum_{A=1, J, K} i_A (\cdot , L(d^* \Omega)) \wedge \omega_A
\]
\[
L(d\Omega) = \sum_{A=1, J, K} i_A (d^* \Omega) \wedge \omega_A + \frac{1}{3} \sum_{A=1, J, K} i_A (L(d^* \Omega)) \wedge \omega_A - \frac{2k_2}{k_1} \xi \wedge \Omega
\]
\[\{A_3^0 E + K\}H + \{A_3^0 E + E\}S^3H + KH\]

no relation

no relation
Table 3: Partial classification via $d\Omega$, for $4n = 8$.

| $K\Sigma^3H$                      | $d\Omega = 0$                                      |
|-----------------------------------|---------------------------------------------------|
| $K(H + S^4H)$                     | $\xi_I = \xi_J = \xi_K = 0$ or $*d\Omega \wedge \omega_A \wedge \omega_A = 0$, for $A = I, J, K$ |
| $EH + KS^4H$                      | $d\Omega = -\xi \wedge \Omega$                   |
| $(K + E)S^4H$                     | $d\Omega = \frac{2}{5} \sum_{A,B=I,J,K} i_A(B\xi_B \wedge \omega_B) \wedge \omega_A$ and $\xi = 0$, or $L(d\Omega) = 0$ |
| $K(H + S^4H) + EH$                | $\xi_I = \xi_J = \xi_K$ or $L(d\Omega) = 6d\Omega$ |
| $K(H + S^4H) + ES^4H$             | $\xi = 0$ or $*d\Omega \wedge \Omega = 0$        |
| $(K + E)(H + S^4H)$               | $d\Omega = \frac{1}{3} \sum_{A=I,J,K} i_A(d^*\Omega) \wedge \omega_A + \frac{16}{15} \sum_{A,B=I,J,K} i_A(B\xi_B \wedge \omega_B) \wedge \omega_A - \frac{3}{5} \xi \wedge \Omega$ |

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