Constant residual electrostatic electron plasma mode in Vlasov-Ampere system

Hua-sheng XIE

1Institute for Fusion Theory and Simulation, Department of Physics, Zhejiang University, Hangzhou, 310027, People’s Republic of China

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In a collisionless Vlasov-Poisson (V-P) electron plasma system, two types of modes for electric field perturbation exist: the exponentially Landau damped electron plasma waves and the initial-value sensitive ballistic modes. Here, the V-P system is modified slightly to a Vlasov-Ampere (V-A) system. A new constant residual mode is revealed. Mathematically, this mode comes from the Laplace transform of an initial electric field perturbation, and physically represents that an initial perturbation (e.g., external electric field perturbation) would not be damped away. Thus, this residual mode is more difficult to be damped than the ballistic mode. [Physics of Plasmas 20, 112108 (2013); doi: 10.1063/1.4831761]

I. INTRODUCTION

For the evolution of linear electron plasma wave in an unmagnetized plasma, the one-dimensional linearized Vlasov-Poisson (V-P) system is

\[
\partial_t \delta f = -ikv\delta f + \delta E \partial_v f_0, \quad (1a)
\]

\[
\frac{ik\delta E}{\delta f} = -\int \delta f dv, \quad (1b)
\]

where time \( t \) and space \( x \) have been normalized by the inverse plasma frequency \( \omega_p^{-1} = 1/\sqrt{n_0 e^2/\epsilon_0 m} \) and the Debye length \( \lambda_D = \sqrt{\epsilon_0 e B_1/n_0 e^2} \), respectively, and \( \kappa_B \) is Boltzmann’s constant. That is, \( \omega_p = 1 \) and \( \lambda_D = 1 \).

The perturbations have the harmonic dependence \( e^{ikx} \), and \( f_0 = f_M = \exp(-v^2/2)/\sqrt{2\pi} \), where \( v \) has been normalized by the electron thermal speed, is the Maxwellian equilibrium distribution.

In most cases, the plasma frequency is sufficiently high compared with the electron collision frequency \( \nu \). The collision can be neglected for time scale \( t \ll \nu^{-1} \). The typical characteristic nonlinear time scale \( t_{NL} \) is the bounce time \( t_B = \omega_p^{-1} = (k\delta E)^{-1} \) of trapped particles. Due to \( \delta E \ll 1, \omega^{-1} \approx \omega_p^{-1} = 1 \ll \omega_B^{-1} \approx t_{NL} \), we can also ignore the nonlinear effect, because we are mainly interested in the linear time scale in this paper, i.e., \( t \ll t_{NL} \). We have also assumed the immobile ion in Eq. (1) because \( m_e/m_i \leq 1/1836 \ll 1 \).

As an initial value problem, for arbitrary initial \( \delta f(v) \), the problem can be described by a superposition of a complete set of eigen solutions, as shown by van Kampen[1] and Case[2]. It is also well known that this system can support exponentially Landau damped mode[3], which is the time asymptotic behavior. This asymptotic behavior can also be described by dispersion relation[3]. For convenience, we designate the dispersion relation solution portion as the Landau part and the initial-value sensitive [referring to the moments of the perturbations, as shown below, or Eqs. (4) and (A3)] portion in Case-van Kampen (CvK) mode as the ballistic part, as the latter part is usually closely related to the \( -ikv\delta f \) term in Eq. (1). For \( f_0 = f_M \), CvK eigen solutions are undamped in the sense that no solution of the linear Vlasov-Poisson equation with \( \Im(\omega) < 0 \) exist for the harmonic perturbation \( \sim \exp(-\nu t) \), where \( \nu \) is the mode frequency. However, because of phase mixing, the (moments with respect to velocity of the) perturbations are still damped when all such modes are considered, e.g., considering \( \delta E(t) \) instead of \( \delta f(v,t) \), which can yield Landau’s solution for some initial \( \delta f(v) \) (especially, entire function[4]), which is holomorphic over the whole complex plane). That is, the ballistic modes (if not specified, here and hereafter, we are referring to the moments of these modes) usually decay faster than Landau damping. A comprehensive description of these modes and their relations in this V-P system can be found in Ref.[6].

For example, for the initial perturbation \( \delta f(v,0) = A_0 \exp(-(v-u_a)^2/u_b^2) \), where \( |u_a| > (|u_b|,|\omega_r|/k) \) with \( \omega_r \) real frequency of the least damped Landau solution, the ballistic mode decays like \( \exp(-iku_a t - k^2u_b^2 t^2) \). On the contrary, for example, a slow (relative to that of Landau) algebraic decay mode

\[
\propto \frac{1}{t} \sin(ku_a t) \exp(-iku_a t),
\]

can be excited by the initial perturbation[5]

\[
\delta f(v,0) = \begin{cases} A_0, & \text{for } |v-u_a| \leq u_b, \\ 0, & \text{otherwise}, \end{cases}
\]

which is not holomorphic, and thus not an entire function.

If we neglect the term proportional to the electric field in Eq. (1), the solution Eq. (2) will be straightforward and precise. However, the difference is that, for Eq. (2), the electric field is kept, which holds for very large \( t \), i.e., \( t \to \infty \).

For the linear collisionless initial value problem of the V-P system, the above pictures are complete.
In this paper, we find that if we change the equations slightly to Vlasov-Ampere (V-A) equations, the dominant mode will be a new constant residual mode, and not the Landau and ballistic modes.

II. VLASOV-AMPERE SYSTEM

From the charge continuity equation \( \partial_t \rho + \partial_x J = 0 \), where \( \rho = \int \delta f dv \) and \( J = \int v \delta f dv \), the V-A and V-P systems are equivalent (the proof is straightforward) if the Poisson’s equation is used to supplement the initial condition in the V-A system.

The reasons below fueled our interest to obtain a more complete description of the linearized V-A system, where the Poisson’s equation Eq. (11) in the V-P system has been replaced by Ampere’s law Eq. (12) and the whole system of Eq. (1) is changed to

\[
\partial_t \delta f = -ikv\delta f + \delta E \partial_x f_0, \tag{4a}
\]

\[
\partial_t \delta E = \int v \delta f dv, \tag{4b}
\]

where the normalization and all other definitions are unchanged.

First, in the work of Horne and Freeman \[10\], they needed to use the V-P equations for the first few steps of the numerical integration before continuing with the V-A equations. Without the Poisson start, recovering the Landau damping solutions is difficult. How do we explain this finding?

Second, in a toroidal system, a well-known residual mode called the Rosenbluth-Hinton residual zonal flow \[11, 12\], was found. The authors showed that poloidal flows driven by ion-temperature gradient (ITG) turbulence will not damp to zero by linear collisionless processes, and the final poloidal velocity \( u_p \neq 0 \). Usually, physics in complicated systems can be understood using simple system. For example, the phase mixing in velocity space can also be found in a real space non-uniform system (see e.g., \[13, 14\]), and rich physics are associated. Thus, phase mixing is a useful concept for understanding continuum damping, from Alfvén wave \[12\] to geodesic acoustic mode \[14\]. Can a constant residual mode be found in a simple system?

A. Initial value solutions

Before to discuss the new constant residual mode, one can refer to the review of the derivations and the verification of the Landau and ballistic solutions in V-P system in the Appendix A which will be used for comparison.

Similar to the V-P system, the dispersion relation of plasma waves for the V-A system can be derived as

\[
D_{VA}(\omega, k) \equiv \omega + \int_C \frac{v \partial f_0}{\omega - kv} dv = 0. \tag{5}
\]

Eqs. (5) and (11) are equivalent, so they should both yield the normal modes given by the dispersion relation, which (the asymptotic solution) is independent of the initial condition. That is, we should also be able to obtain the Landau damped solutions from the V-A equations without using the Poisson start. However, as mentioned, existing simulations indicate that this process is difficult.

Applying Laplace transform \((L_p)\) in time and Fourier transform \((F_v)\) in space to the V-A system, we can obtain

\[
\delta E(t, k) = L_p^{-1} \delta \hat{E}(\omega, k) = \int_{C_{\omega}} \frac{e^{-i\omega t} d\omega}{2\pi} \left[ \int dv \frac{v \delta f_k(0)}{(\omega - kv)D(\omega, k) - \delta E_k(0)} \right], \tag{6}
\]

We are interested in asymptotic behavior, so only the \( \omega = kv \) pole and the maximum-3\( \omega_m \) normal mode are kept. From Eq. (6) we have,

\[
\lim_{t \to \infty} \delta E = \frac{1}{2\pi} \int_{C_{\omega}} dv \left[ \frac{ve^{-ikvt} \delta f_k(0)}{D_{VA}(kv, k)} + \frac{ve^{-i\omega_m t}}{(\omega_m - kv) \partial_{\omega_m D}} \right] \text{ballistic part} \tag{7a}
\]

\[
- \frac{1}{2\pi} \int_{C_{\omega}} \delta E_k(0) D_{VA}(\omega, k) \text{Landau part} \tag{7b}
\]

\[
- \frac{1}{2\pi} \int_{C_{\omega}} \frac{e^{-i\omega t} d\omega}{2\pi} \delta E_k(0) D_{VA}(\omega, k) \text{initial } \delta E \text{ part}
\]

Noting the relation \( k D_{VA} = \omega D_{VP} \), the main difference between Eqs. (7) and (13) is in the initial \( \delta E \) or \( \delta E(0) \), which is a constant. To determine the typical asymptotic behavior, we let \( D_{VA} \approx \omega \) in the \( \delta E \) part of the integral and obtain the residual mode

\[
\lim_{t \to \infty} \left[ \delta E_{VA}(t) - \delta E_{VP}(t) + \delta E_{Poisson}(0) \right] \propto \delta E(0). \tag{8}
\]

Eq. (8) indicates that for almost all kinds of initial perturbations \( \delta f(v, 0) \), the electric field perturbation \( \delta E \) will not be damped away, except that when the initial \( \delta E \) satisfies the Poisson’s equation Eq. (11). Therefore, in the V-A system, the dominant mode would be this constant residual mode, and Landau damping will not be evident. Similar to the ballistic mode, this mode is also from the initial perturbation. We shall designate it as the E mode, to distinguish it from ballistic mode.

This result indicates that the linear residual mode can also be found in simple systems. The issue on why we can hardly find Landau damping in V-A system is also answered.

However, in contrast to the zero poloidal and toroidal mode numbers for residual zonal flow \( m = n = 0 \) with perturbation quantities factor \( e^{i(m\theta-n\phi)} \), the wave vector \( k \) for E mode is not necessarily zero.

Now, we verify the above calculation using simulation. Eq. (4) is solved numerically as an initial value problem. The simulation scheme and the initial \( \delta f \) perturbation are similar to that in the Appendix A. Fig. 1I shows the residual E mode, where \( \delta E(0) = 0.02 \).

The results for a V-A simulation of Landau damping using a Poisson start is given in Fig. 2. The parameters
are the same as those in Fig. 1 except that a V-P run is added in the initial \( t_i \) time step(s). In this linear simulation, \( t_i = 1 \) is sufficient. For nonlinear simulations, one may need more time steps [17]. In Fig. 2, we can see that both the real and imaginary parts of \( \delta E \) match the Landau result.

In the V-A simulation, the residual E mode is dominant and the initial perturbation Eq. (3) also cannot prevent it (not shown here).

Compared with the V-P system, the free parameter \( \delta E(0) \) in the V-A system change the picture completely. Mathematically, the residual mode comes from the Laplace transform of initial electric field perturbation and physically represents that an initial mode (e.g., external electric field perturbation) would not be damped away. Here, we solved the initial value problem. Experimentally, we usually address the boundary value problem, which means the electric field will not be zero but remains a constant at large distance. The solution for this boundary value problem can be found in Landau[2].

**B. Eigenmodes in the V-A system**

Other phenomena are related to Landau damping. For example, the Landau damped normal mode is not an eigenmode in the V-P system, although a growing normal mode can be an eigenmode (see, e.g., Ref. [15]. The eigenmode problem with collision term and the connections between collisionless case have been studied by several authors [15–19]. Bratanov et al. [15, 16] reported new numerical investigations regarding the connection between the CvK eigenmode and the Landau normal mode. Spectral density accumulation occurs around the real frequency of the Landau-damped mode.

We rewrite the governing equations into the matrix form \( M \cdot \vec{F} = \omega \vec{F} \), with \( \partial_t = -i\omega \). The eigenvector \( \vec{F} \) for the V-P system is \( \vec{F} = \{ F_j \} = \{ \delta f(v_j) \} \), where \( v_j = v_{\min} + (j - 1)\Delta v \) and \( j = 1, 2, 3, \ldots, N_v + 1 \). Bratanov[15] verified numerically that although all the (CvK) eigenvalue solutions of \( \vec{F} \) are undamped, their integral, which is related to \( \delta E \), can yield Landau damping because of phase mixing.

A question then arises: what if \( \delta E \) (the moment) is also contained in the eigenvector \( \vec{F} \)? Will it give the Landau solution directly, so that the Landau damped mode can also be an eigenmode? The V-A system can combine \( \delta E \) into \( \vec{F} \) directly. The eigenvector of the V-A system then becomes

\[
\vec{F} = \{ F_j \} = \begin{cases} \delta f(v_j), & j = 1, 2, 3, \ldots, N_v + 1, \\ \delta E, & j = N_v + 2. \end{cases}
\]  

Using Eqs. (1) and (4), we can easily find the elements of the eigenvalue matrices \( M \) for the V-P and V-A systems, respectively.

The solutions are shown in Fig. 3. No eigenmodes and the normal modes are identical in panel (a). That is, our result also indicates that the Landau damped mode in the V-A system is not an eigenmode.

The only difference between the V-A and V-P systems is in the spectral density around \( \omega_r = 0 \): an extra accumulating point exists at \( \omega_r = 0 \) of the V-A system [see panel (b)]. Noting Eq. (7), we can attribute this new accumulating point to the E mode, particularly to the constant residual E mode. That is, two \( \omega = 0 \) solutions exist: one from the continuum CvK mode and another from the new residual E mode, which causes the singularity at \( \omega_r = 0 \) in the spectral density figure [panel (b)]. Panels (c) and (d) show the corresponding eigen functions \( \delta f(v_j) \). As expected, the eigen function of the continuum CvK mode is singular [1, 2], whereas the eigen function of the E mode is smoother.
FIG. 3: Eigenmodes in the V-A system. (a) compared with the Landau damping normal modes (dispersion relation solutions), (b) the spectral density, (c) and (d) the corresponding eigen functions for $\omega = 0$ E mode and continuum CvK mode.

III. SUMMARY

In this paper, we have presented a relatively complete picture for the modes in the linearized electrostatic 1D V-A system. Besides the usual Landau mode and the ballistic mode in the V-P system, a new constant residual mode is found, and this mode is usually dominant. Analytical asymptotic solutions, eigenmode solutions, and linear simulations are consistent. In contrast to ballistic mode, this residual mode is more robust, i.e., more difficult to be damped.

The finding indicates that the residual mode would be common, and not merely exists in complicated system such as the Rosenbluth-Hinton residual zonal flow found in toroidal system.

One may also be interested in other applications of V-A system. A successful example of the application of extended V-A equation with collision, source, and sink is the Berk-Breizman model[20, 21] for discussing the non-linear single Alfvén mode driven by an energetic injected beam, especially for Alfvén eigenmodes in tokamak.

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Appendix A: Vlasov-Poisson system

For Landau mode, the dispersion relation of the V-P system Eq.(1) is

$$D_{VP}(\omega, k) \equiv k + \int_{C} \frac{\partial f_{0}/\partial v}{\omega - kv} dv = 0,$$

where $C$ is the proper integral contour[3, 5–8].

Similar to Chen[5] or Jackson[6], by applying Laplace transform ($L_{\omega}$) in time and Fourier transform ($F_{v}$) in space to the V-P system, we can obtain

$$\delta E(t, k) = \int_{C_{\omega}} \frac{e^{-i\omega t} \omega}{2\pi} \left[ \int_{k} dv \delta f_{0}(0) \right].$$  \hspace{1cm} (A2)

We are interested in asymptotic behavior, so that only the $\omega = kv$ pole and the maximum-3$\omega_m$ normal mode are kept. From Eq.(A2) we have

$$\lim_{t \to \infty} \delta E = \frac{1}{2\pi} \int_{v_{m}} \delta f_{0}(0) \frac{(\omega_{m} - kv)}{2\pi} \frac{e^{-i\omega_{m} t}}{\omega_{m} v_{m}} D_{VP}(kv, k).$$  \hspace{1cm} \text{Landau part} \hspace{1cm} \text{ballistic part} \hspace{1cm} (A3)

Focusing on the ballistic part, one can easily solve for the Gaussian perturbation or Eq.(3), and obtain the $\propto \exp(-t^{2})$ or $\propto 1/t$ solutions.

The analytical solutions from Eqs.(A2) or (A3) are usually approximations, so we would like to verify them numerically.

We solve Eq.(1) as an initial value problem from $t = 0$ to $t_{end} = N_{t}\Delta t$ using a 4th-order Runge-Kutta scheme. The discrete velocity space is from $v_{\min}$ to $v_{\max}$. There are $N_{v}$ uniform grids of size $\Delta v = (v_{\max} - v_{\min})/N_{v}$.

As mentioned, for most initial perturbations $\delta f(t = 0)$, the asymptotic behavior of $\delta E(t)$ is determined by the normal mode that is Landau damped. Using a Gaussian initial perturbation $\delta f_{w}$ at $A_{0} = 0.05$, $u_{w} = 1.0$, and $w_{w} = 1.0$, we have successfully reproduced the Landau damped solution (not shown here, or see Fig.2). The electric field $\delta E(t = 0)$ and $\delta E(t)$ are obtained from the Poisson equation using the initially given and the calculated $\delta f$, respectively. We should use a small $\Delta v$ to avoid non-physical recurrence effect (the Poincaré recurrence) at $T_{R} = 2\pi/(k\Delta v)$, which is due to the discreteness of
The $1/t$ damped ballistic mode from the initial perturbation Eq. (3) can be seen in Fig. 4 for $A_0 = 0.05$, $u_a = 0.0$, and $u_b = 2.0$. As reference, the red dashed line in panel (d) is for $y = 3.3x$. $|\delta E|$ decays as $c/t$, as analytically predicted in Eq. (2). Note that the constant $c$ depends on $A_0$.

Notably, our linear analysis and simulation are carried out in the $k$-space, i.e., all perturbation quantities have an $e^{ikx}$ factor. However, one can obtain the corresponding $(x,v)$ phase space figure by direct mapping using the relation $f(x,v) = f_0(v) + \Re[\delta f(k,v)e^{ikx}]$.

The above analytical calculations and simulations show a complete picture of the typical Landau and ballistic modes in the V-P system.

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