On Lipschitz regularity for bounded minimizers of functionals with \((p, q)\)-growth

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Abstract

We obtain Lipschitz estimates for bounded minimizers of functionals with nonstandard \((p, q)\)-growth satisfying the dimension-independent restriction \(q < p + 2\) with \(p \geq 2\). This relation improves existing restrictions when \(p \leq N - 1\), moreover our result is sharp in the range \(N > \frac{p(2 + p)}{2} + 1\). The standard Lipschitz regularity takes the form \(W^{1,\infty}_{\text{loc}} - W^{1,p}_{\text{loc}}\), whereas we obtain \(W^{1,\infty}_{\text{loc}} - L^\infty_{\text{loc}}\) regularity estimate and then make use of existing sharp \(L^\infty_{\text{loc}}\) bounds to obtain the required conclusion.

Keywords: nonuniformly elliptic equations, local Lipschitz continuity, \((p, q)\)-growth, nonstandard growth conditions

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ be a bounded open set and we consider the problem of local regularity of minimizers of

$$\mathcal{F}[u] := \int_{\Omega} f(\nabla u) \, dx,$$

(1.1)

where $f : \mathbb{R}^N \to \mathbb{R}$ is a $C^2$ integrand satisfying $(p,q)$ growth of the form

**Assumption 1.1.** Let $2 \leq p \leq q < \infty$ and suppose there exist constants $m, M \in (0, \infty)$ such that for any $z \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^N$, the following holds

$$m |z|^p \leq f(z) \leq M (1 + |z|)^q,$$

(1.2a)

$$m |z|^p \leq \langle Df(z), z \rangle \quad \text{and} \quad |Df(z)| \leq M (1 + |z|^{q-1}),$$

(1.2b)

$$m |z|^{p-2} |\xi|^2 \leq \langle D^2f(z)\xi, \xi \rangle \leq M (1 + |z|^{q-2})|\xi|^2.$$  

(1.2c)

**Definition 1.2.** We say that $U \in W^{1,p}_{\text{loc}}(\Omega)$ is a local minimizer of (1.1) provided the following two conditions are satisfied:

(i) $f(\nabla U) \in L^1_{\text{loc}}(\Omega)$ and

(ii) $\int_{\text{spt} \varphi} f(\nabla u) \, dx \leq \int_{\text{spt} \varphi} f(\nabla u + \nabla \varphi) \, dx$ holds for all $\varphi \in W^{1,p}(\Omega)$ with $\text{spt}(\varphi) \Subset \Omega$.

Our main theorem reads as follows:

**Theorem 1.3.** Let $2 \leq p \leq q < \infty$ with $q < p + 2$ and let $U \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ be a bounded, local minimizer of $\mathcal{F}$ as in Definition 1.2, then $\nabla U \in L^\infty_{\text{loc}}(\Omega)$.

1.1. Comparison to Previous Results

Regularity theory of variational problems with nonstandard $(p,q)$ growth was pioneered by P.Marcellini in a series of seminal papers [27, 28, 29, 31, 30]. Since we are interested in Lipschitz regularity, let us recall that P.Marcellini proves Lipschitz bounds for $U \in W^{1,p}_{\text{loc}}(\Omega)$ under the restriction

$$\frac{q}{p} < 1 + \frac{2}{N}.$$  

In a recent paper, P.Bella and M.Schäffner [3] improved the restriction to

$$\frac{q}{p} < 1 + \min \left\{ 1, \frac{2}{N-1} \right\}, \quad \text{for } N \geq 2 \quad \text{and} \quad p \geq 2,$$

(1.3)

by employing a specialized test function that enables them to use Sobolev embedding on the sphere. There is a large body of work dealing with problems of $(p,q)$-growth as well as other nonstandard growth problems, for which we refer to [17, 18, 19, 20, 8, 7, 10, 1, 15, 2, 16, 12, 13, 14]. A more detailed survey on the state of the art for problems with nonstandard growth may be found in [32, 33].

It is well known that Lipschitz continuity and even boundedness for (1.1) fail when $p$ and $q$ are far apart as evidenced by the following example of Hong [25], which is a variation on the famous counterexample of Giaquinta [21]:

$$\int_{\Omega} |\nabla u|^2 + |u_{x_n}|^4 \, dx,$$

which satisfies (1.2a)–(1.2c) for $p = 2$ and $q = 4$ and admits an unbounded minimizer if $N \geq 6$ (more examples of unbounded minimizers of (1.1) may be found in [26]). It was shown in [28, Section 6] that if $q > \frac{(N-1)p}{N-1-p}$ then
one cannot expect boundedness and only recently, this restriction was found to be sharp in [24], where it is proved that the minimizer is bounded provided

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{N-1}. \tag{1.4}$$

It is easy to see that there is a gap between the restrictions in (1.4) and (1.3) and in this context, the authors in [3, 33] asked if one could obtain a Sobolev-type restriction (as in (1.4)) in order for the minimizer to be Lipschitz regular. In this regard, we improve the restriction in (1.3) in some special ranges of $p,q$ and $N$ and also partially provide an answer to the question from [3, 33] by obtaining a Sobolev type restriction when $N > \frac{p(2 + p)}{2} + 1$.

(i) For bounded minimizers, we require $q < p + 2$, see Theorem 1.3.

(ii) Combining the restriction $q < p + 2$ with the optimal restriction for boundedness from (1.4), we see that Lipschitz regularity for minimizers holds provided

$$\frac{q}{p} < \min \left\{ 1 + \frac{p}{N - 1 - p}, 1 + \frac{2}{p} \right\}.$$  

(iii) In the case $p \leq N - 1$, we see that $2^p \geq 2^\frac{p}{N - 1}$, which suggests that Theorem 1.3 improves the restriction given in (1.3). But it must be noted that our result additionally requires that the solutions are bounded which also requires the restriction (1.4) to be satisfied. We now compare the two results in a few special cases as follows:

| Table 1: Admissible values of $q$, shaded regions denote sharp restrictions. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $N = 2$         | $N = 3$         | $N = 4$         | $N = 5$         | $N = 6$         | $N = 7$         |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $p = 2$         | $q < 4$         | $q < 4$         | $q < 4$         | $q < 4$         | $q \leq \frac{10}{3}$ | $q \leq 3$ | Theorem 1.3 + (1.4) - $C^{0,1}$ |
|                 | $q < 4$         | $q < 4$         | $q < 3$         | $q < \frac{14}{5}$ | $q < \frac{8}{3}$ | (1.3) - $C^{0,1}$ |
| $p = 3$         | $q \in (1, \infty)$ | $q \in (1, \infty)$ | $q \leq 6$ | $q \leq 4$ | $q \leq \frac{10}{3}$ | $q \leq 3$ | (1.4) - $L^\infty$ |
| $p = 4$         | $q \in (1, \infty)$ | $q \in (1, \infty)$ | $q \in (1, \infty)$ | $q \leq 12$ | $q \leq \frac{15}{7}$ | $q \leq 6$ | (1.4) - $L^\infty$ |

(iv) Since we require bounded solutions, we see that for minimizers, Lipschitz regularity would then require

$$\frac{q}{p} < 1 + \min \left\{ \frac{p}{N - 1 - p}, \frac{2}{p} \right\}.$$  

In particular, if $N > \frac{p(2 + p)}{2} + 1$, then $\frac{2}{p} > \frac{p}{N - 1 - p}$ and thus Lipschitz regularity holds for any minimizer as they are automatically bounded. In particular, due to the sharpness of the condition (1.4), we automatically obtain sharpness of the Lipschitz regularity in this range.

(v) Our theorem improves the previous restriction for bounded minimizers of (1.1) which was found to be $q < p + 1$ in [6, 9].
We now briefly describe the method of proof, first, we begin with a regularization procedure following [5] where a quadratic term is added to $f$. The regularized solution is shown to be in $C^{1,\gamma} \cap W^{2,2}$, and we exploit this $W^{2,2}$ regularity to obtain a gradient higher integrability result. After obtaining a Caccioppoli-type inequality for the gradient of $U$, we prove that $\nabla U \in L^s_{\text{loc}}$ for all $s \in (1, \infty)$. In fact, $\|\nabla U\|_s$ is estimated in terms of $\|U\|_{L^\infty}$ provided $q < p + 2$ holds. Finally, we use a Moser iteration adapted for solutions of equations with unbalanced growth to obtain the required result.

Remark 1.4. After this paper was written, we became aware that this result has been proved previously by Bildhauer and Fuchs [4] in 2002. The methods are, for the most part, similar. Instead of De Giorgi iteration, we use a Moser iteration in the last step. Also, we use a quadratic term for the regularization and we use the notion of bounded slope condition to gain Lipschitz regularity for the regularized minimizer.

2. Notations and Preliminaries

2.1. Notations

We begin by collecting the standard notation that will be used throughout the paper.

- We shall denote $N$ to be the space dimension. A point in $\mathbb{R}^N$ will be denoted by $x$.
- Let $\Omega$ be a domain in $\mathbb{R}^N$ of boundary $\partial \Omega$.
- The notation $a \lesssim b$ is shorthand for $a \leq Cb$ where $C$ is a constant independent of the regularization parameters $\sigma$ and $\varepsilon$.
- We will use the symbol $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product.

2.2. Preliminaries for Regularization

We list some of the preliminaries that are required in the subsequent sections. The regularization procedure relies on the addition of a quadratic term to the functional. Stampacchia [37] has proved some general theorems on the local Lipschitz regularity of minimizers to convex minimization problems posed on convex domains with boundary values satisfying the bounded slope condition.

**Definition 2.1 (Uniformly Convex set).** A bounded, open set $B \subset \mathbb{R}^N$ is said to be uniformly convex if there exists $\nu > 0$ such that for every boundary point $x_0 \in \partial B$ there exists a hyperplane $H_{x_0}$ passing through that point satisfying
\[
\text{dist}(y, H_{x_0}) \geq \nu|y - x_0|^2 \quad \text{for every } y \in \partial B.
\]

**Definition 2.2 (Bounded Slope Condition).** Let $K$ be a positive real number and $B$ an open bounded convex subset of $\mathbb{R}^N$. We say that a function $\phi : \partial B \to \mathbb{R}$ satisfies the bounded slope condition of rank $K$ if for any $x_0 \in \partial B$ there exists vectors $l^-_{x_0}$ and $l^+_{x_0}$ satisfying $||l^-_{x_0}|| \leq K$ and $||l^+_{x_0}|| \leq K$ such that
\[
\langle l^-_{x_0}, x - x_0 \rangle \leq \phi(x) - \phi(x_0) \leq \langle l^+_{x_0}, x - x_0 \rangle \quad \text{for every } x \in \partial B.
\]

The following proposition gives a necessary and sufficient condition for a function to satisfy the bounded slope condition, see [23, Corollary 4.3] and [34] for the details of the proof.
Proposition 2.3. Let $B \subset \mathbb{R}^N$ be a bounded, open and uniformly convex domain with the boundary being $\partial B \in C^{1,1}$ regular. Then, a necessary and sufficient condition for any function $\phi(x), \ x \in \partial B$ to satisfy bounded slope condition is that $\phi(x) \in C^{1,1}(\partial B)$.

Theorem 2.4 ([37], Theorem 9.2). Let $H \in C^2(\mathbb{R}^N)$ and assume it satisfies $\langle D^2H(p)\zeta,\zeta \rangle \geq \nu|\zeta|^2$ for all $\zeta \in \mathbb{R}^N$. If $\Omega \subset \mathbb{R}^N$ is uniformly convex and $C^{1,1}$, then the integral given by $I(u) = \int_{\Omega} H(\nabla u) \, dx$ attains its minimum in the class of all Lipschitz functions in $\Omega$ assuming that the boundary values satisfy the bounded slope condition and are the trace of a $W^{2,p}$ function for some $p > n$.

Let us now recall some basic facts from calculus of variations that will be needed later on. The first concerns the existence of a minimizer, the proof of which can be found in [35, Theorem 2.7].

Theorem 2.5. Let $f : \mathbb{R}^N \to [0,\infty)$ be a $C^2$ function such that

(i) $f$ satisfies the $r$-coercivity bound, i.e., $f(z) \geq m|z|^r$ holds for all $z \in \mathbb{R}^N$ and some $r \in (1,\infty)$.

(ii) $f$ is a convex function.

Then, the associated functional $F(u) = \int_{\Omega} f(\nabla u) \, dx$ has a minimizer over $W^{1,r}_g(\Omega) = \{u \in W^{1,r}(\Omega) : u|_{\partial \Omega} = g\}$, where $g \in W^{1-1/r,r}(\partial \Omega)$.

The second result discusses when does there exist a unique solution, the proof of which can be found in [35, Proposition 2.10].

Theorem 2.6. Let $F : W^{1,r}(\Omega) \to \mathbb{R}$, $r \in [1,\infty)$, be an integral functional with a $C^2$ integrand $f : \mathbb{R}^N \to \mathbb{R}$. If $f$ is strictly convex, i.e.,

$$f(\theta z_1 + (1-\theta)z_2) < \theta f(z_1) + (1-\theta)f(z_2),$$

holds for all $z_1, z_2 \in \mathbb{R}^N$ with $z_1 \neq z_2$ and any $\theta \in (0,1)$, then the minimizer $u_* \in W^{1,r}_g(\Omega) = \{u \in W^{1,r}(\Omega) : u|_{\partial \Omega} = g\}$ of $F$, where $g \in W^{1-1/r,r}(\partial \Omega)$, if it exists, is unique.

The next theorem gives a criterion for the integrand to be strictly convex, the proof of which can be found in [36, Theorem 1.5].

Theorem 2.7. Let $\Omega$ be an open convex subset of $\mathbb{R}^N$ and let $f : \Omega \to \mathbb{R}$ be $C^2$. Suppose that for all $x \in \Omega$ the Hessian matrix $D^2f(x)$ is strictly positive-definite, then $f$ is strictly convex.

We end this subsection by recalling a maximum principle, whose proof may be found in [37, Theorem 2.1].

Theorem 2.8 (Maximum Principle). Let $a_{ij}(x), i, j = 1, 2, \ldots, N$ be measurable and bounded functions in $B$ such that

$$a_{ij}\xi_i\xi_j \geq \mu|\xi|^2,$$  

a.e. $x \in B$, for all $\xi \in \mathbb{R}^N$.

If $u \in H^1(B)$ satisfies

$$\int_B a_{ij}(x) u_{x_i}(x)v_{x_j}(x) \, dx = 0, \text{ for all } v \in H^1_0(B),$$

then we have

$$u(x) \leq \max_{x \in \partial B} u(x) \text{ a.e. } x \in B.$$
We shall make use of the following well-known iteration lemma whose proof may be found in [22, Lemma 6.1].

**Lemma 2.9.** Let $Z(t)$ be a bounded non-negative function in the interval $\rho, R$. Assume that for $\rho \leq t < s \leq R$ we have

$$ Z(t) \leq [A(s-t)^{-\alpha} + B(s-t)^{-\beta} + C] + \partial Z(s) $$

with $A, B, C \geq 0$, $\alpha, \beta > 0$ and $0 \leq \partial < 1$. Then,

$$ Z(\rho) \leq c(\alpha, \partial)[A(R-\rho)^{-\alpha} + B(R-\rho)^{-\beta} + C]. $$

3. Regularization

3.1. Approximation Scheme

Let us fix a ball $B \Subset \Omega$ such that $4B \Subset \Omega$. Let $\varepsilon_0 = \min \left\{ 1, \frac{\text{diam}(B)}{2} \right\} > 0$. For any $\varepsilon \in (0, \varepsilon_0)$, using a standard mollifier $\rho_\varepsilon$ supported in a ball of radius $\varepsilon$ centered at the origin, we define $U_\varepsilon := U \ast \rho_\varepsilon$. For $0 < \sigma < 1$, we define the regularized functional

$$ \mathfrak{F}_\sigma(w) := \int_\Omega f_\sigma(\nabla w) \, dx := \int_\Omega f(\nabla w) + \frac{\sigma}{2} |\nabla w|^2 \, dx, \quad (3.1) $$

where $f_\sigma \in C^2(\mathbb{R}^N)$ satisfies the following growth and ellipticity conditions: From Assumption 1.1, we see that for $2 \leq p \leq q < \infty$, $z \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^N$, the following is satisfied:

$$ \frac{\sigma}{2} |z|^2 + m |z|^p \leq f_\sigma(z) \leq M(1 + |z|)^q + \frac{\sigma}{2} |z|^2 \quad (3.2a) $$

$$ \frac{\sigma}{2} |z|^2 + m |z|^p \leq \langle D f_\sigma(z), z \rangle \quad \text{and} \quad |D f_\sigma(z)| \leq M(1 + |z|^{q-1}) \quad (3.2b) $$

$$ m |z|^{p-2} |\xi|^2 + \sigma |\xi|^2 \leq \langle D^2 f_\sigma(z) \xi, \xi \rangle \leq M(1 + |z|^{q-2}) |\xi|^2 + \sigma |\xi|^2. \quad (3.2c) $$

An application of Theorem 2.5 shows that there exists a minimizer $u_{\sigma, \varepsilon}$ of $\mathfrak{F}_\sigma$ in $B$, i.e., the following holds:

$$ \mathfrak{F}_\sigma(u_{\sigma, \varepsilon}) = \min_{v \in U_\varepsilon + W_0^1(B)} \int_\Omega f(\nabla v) + \frac{\sigma}{2} |\nabla v|^2 \, dx. \quad (3.3) $$

From (3.2c) and Theorem 2.7, we see that $f_\sigma$ is strictly convex and thus, Theorem 2.6 shows that $u_{\sigma, \varepsilon}$ is unique.

3.2. Regularity of minimizers

**Lemma 3.1.** The unique minimizer $u_{\sigma, \varepsilon}$ of (3.1) belongs to $u_{\sigma, \varepsilon} \in L_0^\infty(B) \cap C_0^{0,1}(B) \cap W_0^{2,2}(B)$. Moreover, for all $0 < \varepsilon < \varepsilon_0$ and $0 < \sigma < 1$, the following holds:

$$ ||u_{\sigma, \varepsilon}||_{L^\infty(B)} \leq ||U||_{L^\infty(2B)}. $$

**Proof.** Since the functional $\mathfrak{F}_\sigma$ and boundary data $U_\varepsilon$ given in (3.3) satisfies the hypothesis from Theorem 2.4, we have the existence of a minimizer $u_{\sigma, \varepsilon}$ which is Lipschitz regular. Moreover, the minimizer is unique which follows from Theorem 2.6 and (3.2c) applied with Theorem 2.7.

Clearly the function $u_{\sigma, \varepsilon}$ satisfies the following Euler-Lagrange equation

$$ \nabla \cdot (D f_\sigma(u_{\sigma, \varepsilon})) = 0 \quad \text{in} \ B. $$

To prove the bound, we invoke the Maximum principle from Theorem 2.8 along with (3.2b) to conclude that

$$ \max_B |u_{\sigma, \varepsilon}| \leq \max_{\partial B} |u_{\sigma, \varepsilon}| = \max_{\partial B} |U_\varepsilon| \leq \max_{2B} |U|. $$
Thus, we get $u_{\sigma,\varepsilon} \in L^\infty(B) \cap W^{1,\infty}(B)$. Note that the $L^\infty$ estimate is uniform and independent of $\sigma$ and $\varepsilon$, whereas the Lipschitz bound could possibly depend on $\sigma$ and $\varepsilon$.

Noting that the functional $f_\sigma$ satisfies (3.2c), by a standard argument involving difference quotients, we can prove that $u_{\sigma,\varepsilon} \in W^{2,2}(B)$. Note that the $W^{2,2}(B)$ estimate comes from the regularizing term $\frac{\sigma}{2} |z|^2$ in (3.2a) and depends on the parameter $\sigma$. In particular, the $W^{2,2}(B)$ estimate could possibly blow up as $\sigma \to 0$.

**Remark 3.2.** In subsequent sections excepting Section 7, we shall suppress the subscript of $u_{\sigma,\varepsilon}$ for ease of notation.

4. Caccioppoli Inequality

We shall prove the following Caccioppoli inequality for the gradient of $u$. Note that the proof is only formal and everything can be made rigorous using difference quotients and the a priori regularity from **Lemma 3.1**.

**Proposition 4.1.** Let $\alpha \geq 0$. Let $u$ be the solution to (3.3). Then it holds that

$$
\int_B |\nabla u|^{p-2+\alpha} |\nabla^2 u|^{\gamma^2} \, dx \leq C(M,m) \left\{ \int_B \left( |\nabla u|^{\eta^2} + |\nabla u|^2 \right) |\nabla \eta|^2 \, dx \right\}.
$$

**Proof.** The minimizer $u$ of (3.3) satisfies the following Euler-Lagrange equation:

$$
\int_B \langle Df_\sigma(\nabla u), \nabla \phi \rangle \, dx = 0.
$$

By choosing $\phi = \psi_{x_j} \in H^1_0(B)$ and integrating by parts, we get

$$
\int_B \langle D^2 f_\sigma(\nabla u) \nabla u_{x_j}, \nabla \psi \rangle \, dx = 0.
$$

Now, for $\kappa > 0$, we choose $\psi = u_{x_j}(\kappa + |u|^2)^{\frac{\eta}{2}}$, where $\eta \in C^0_0(B)$ such that $0 \leq \eta \leq 1$ in $B$, to get

$$
\int_B \langle D^2 f_\sigma(\nabla u) \nabla u_{x_j}, \nabla \psi \rangle \, dx + \int_B \langle D^2 f_\sigma(\nabla u) \nabla u_{x_j}, \nabla \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \rangle u_{x_j} \eta^2 \, dx
$$

$$
= -2 \int_B \langle D^2 f_\sigma(\nabla u) \nabla u_{x_j}, \nabla \eta \rangle u_{x_j} \eta \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \, dx.
$$

Now observe that due to the coercivity of $D^2 f_\sigma(z)$, we can apply the Cauchy-Schwarz inequality for positive definite Hermitian matrix $O$ given by $\langle Ox, y \rangle \leq \langle Ox, x \rangle^{1/2} \langle Oy, y \rangle^{1/2}$ along with Young’s inequality to get

$$
\left\{ \int_B \langle D^2 f_\sigma(\nabla u) \nabla \eta, \nabla \eta \rangle u_{x_j}^2 \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \, dx \right\} + \int_B \langle D^2 f_\sigma(\nabla u) \nabla u_{x_j}, \nabla \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \rangle u_{x_j} \eta^2 \, dx
$$

$$
\leq \frac{1}{2} \int_B \langle D^2 f_\sigma(\nabla u) \nabla \eta, \nabla \eta \rangle u_{x_j}^2 \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \, dx.
$$

Substituting (4.3) in (4.2) and summing over $j \in \{1, 2, \ldots, N\}$, we get

$$
\frac{1}{2} \sum_{j=1}^N \int_B \langle D^2 f_\sigma(\nabla u) \nabla u_{x_j}, \nabla u_{x_j} \rangle \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \eta^2 \, dx + \frac{1}{2} \sum_{j=1}^N \int_B \langle D^2 f_\sigma(\nabla u) \nabla u_{x_j}, \nabla \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \rangle u_{x_j} \eta^2 \, dx
$$

$$
\leq 2 \sum_{j=1}^N \int_B \langle D^2 f_\sigma(\nabla u) \nabla \eta, \nabla \eta \rangle u_{x_j}^2 \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \, dx.
$$

Observing that $\sum_{j=1}^N u_{x_j} \nabla u_{x_j} = (\kappa + |u|^2)^{\frac{\eta}{2}} \nabla \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right)$, we can rewrite the previous estimate as

$$
\frac{1}{2} \sum_{j=1}^N \int_B \langle D^2 f_\sigma(\nabla u) \nabla u_{x_j}, \nabla u_{x_j} \rangle \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \eta^2 \, dx
$$

$$
+ \alpha \int_B \langle D^2 f_\sigma(\nabla u) \nabla \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right), \nabla \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \rangle \left( (\kappa + |u|^2)^{\frac{\eta}{2}} \right) \eta^2 \, dx
$$

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Now, we shall apply (3.2c) to get
\[
\frac{m}{2} \int_B |\nabla u|^{p-2} |\nabla^2 u|^2 (\kappa + |\nabla u|^2)^{\frac{p}{2}} \, dx + \alpha m \int_B |\nabla u|^{p-2} \left| \nabla \left( (\kappa + |\nabla u|^2)^{\frac{p}{2}} \right) \right|^2 (\kappa + |\nabla u|^2)^{\frac{p}{2}} \eta^2 \, dx
\leq 2(M+1) \int_B \left( |\nabla u|^q + |\nabla u|^2 \right) (\kappa + |\nabla u|^2)^{\frac{p}{2}} |\nabla \eta|^2 \, dx.
\]
We can derive the required inequality by dropping the second term on the left hand side and passing to the limit \( \kappa \to 0 \) by an application of Dominated Convergence Theorem on the right hand side and Fatou’s lemma on the left hand side.

5. Higher Integrability of gradient

In this section, we will prove that \( \nabla u \in L^s_{\text{loc}}(B) \) for all \( s \in (1, \infty) \) provided \( q < p + 2 \).

**Proposition 5.1.** Assume that \( q < p + 2 \) and let \( u \) be the unique solution of (3.3) (recall Remark 3.2). For every \( \beta \geq 2 \), there exists a constant \( C = C(N, p, q, \beta, M, m) > 0 \) such that for every \( 0 < \sigma < 1, 0 < \varepsilon < \varepsilon_0 \) and every pair of concentric balls \( B_{r_0} \subset B_{R_0} \subset B \), we have

\[
\int_{B_{r_0}} |\nabla u|^{p+\beta} \leq C R_0^N \left( \frac{||u||_{L^\infty(B_{R_0})}}{R_0 - r_0} \right)^{\frac{2(\beta+p)}{p+2}} + \left( \frac{||u||_{L^\infty(B_{R_0})}}{R_0 - r_0} \right)^{\frac{2(\beta+p)}{p+2}} + \left( \frac{||u||_{L^\infty(B_{R_0})}}{R_0 - r_0} \right)^{p+\beta}. \tag{5.1}
\]

**Proof.** We begin with the integral
\[
\int_B |\nabla u|^{p+\beta} \eta^2 \, dx = \int_B \nabla u \cdot (|\nabla u|^{p+\beta-2} \nabla u \eta^2) \, dx,
\]
where \( \eta \in C_c^\infty(B) \) to be eventually chosen appropriately. After integrating by parts, we get
\[
\int_B |\nabla u|^{p+\beta} \eta^2 \, dx = - \int_B u \nabla \cdot (|\nabla u|^{p+\beta-2} \nabla u \eta^2) \, dx
\leq ||u||_{L^\infty(B)} (p + \beta - 1) \underbrace{\int_B |\nabla u|^{p+\beta-2} |\nabla^2 u|^2 \eta^2 \, dx}_{A_1} + 2 ||u||_{L^\infty(B)} \underbrace{\int_B |\nabla u|^{p+\beta-1} |\nabla \eta|^2 \, dx}_{A_2}.
\]
Applying Young’s inequality, we get
\[
A_1 \leq \tau \int_B |\nabla u|^{p+\beta} \eta^2 \, dx + \frac{1}{4\tau} \int_B |\nabla u|^{p+\beta-4} |\nabla^2 u|^2 \eta^2 \, dx,
\]
\[
A_2 \leq \tau \int_B |\nabla u|^{p+\beta} \eta^2 \, dx + \frac{1}{4\tau} \int_B |\nabla u|^{p+\beta-2} |\nabla \eta|^2 \, dx.
\]
Choosing \( \tau = \frac{1}{4||u||_{L^\infty(B)}(p + \beta - 1)} \), we get
\[
\int_B |\nabla u|^{p+\beta} \eta^2 \, dx \leq C(p, \beta) ||u||_{L^\infty(B)}^2 \left\{ \int_B |\nabla u|^{p+\beta-4} |\nabla^2 u|^2 \eta^2 \, dx + \int_B |\nabla u|^{p+\beta-2} |\nabla \eta|^2 \, dx \right\}.
\]
For the first integral on the RHS, we will apply Proposition 4.1 to get
\[
\int_B |\nabla u|^{p+\beta} \eta^2 \, dx \leq C(p, \beta, M, m) ||u||_{L^\infty(B)}^2 \left\{ \int_B (|\nabla u|^{q+\beta-2} + |\nabla u|^{\beta} + |\nabla u|^{p+\beta-2}) |\nabla \eta|^2 \, dx \right\}.
\]
Let us now fix a pair of concentric balls \( B_r \subset B_{R_0} \subset B \) and choose the cut-off function \( \eta \in C_c^\infty(B_{R_0}) \) such that \( \eta \equiv 1 \) on \( B_r \) and \( |\nabla \eta| \leq \frac{C}{R - r} \) to get
\[
\int_{B_r} |\nabla u|^{p+\beta} \, dx \leq C(p, \beta, M, m) \frac{||u||_{L^\infty(B)}}{(R - r)^2} \left\{ \int_{B_{R_0}} (|\nabla u|^{q+\beta-2} + |\nabla u|^{\beta} + |\nabla u|^{p+\beta-2}) \, dx \right\}. \tag{5.2}
\]
Now, for $\tau > 0$ to be chosen, we obtain each of the terms on the right hand side of (5.2) as follows:

\[ |\nabla u|^{p+\beta-2} \leq \tau |\nabla u|^{p+\beta} + \frac{C(p, \beta)}{\tau^{\frac{p}{2}+\alpha}}, \]  
(5.3a)

\[ |\nabla u|^{\beta} \leq \tau |\nabla u|^{p+\beta} + \frac{C(\beta)}{\tau^p}, \]  
(5.3b)

\[ |\nabla u|^{q+\beta-2} = |\nabla u|^{q+\beta-2} \left( \frac{\tau}{\tau+\beta} \right)^{\frac{q}{p}+\beta-2} \right) \leq \tau |\nabla u|^{p+\beta} + \frac{C(p, q, \beta)}{\tau^{\frac{q}{2}+\beta}}. \]  
(5.3c)

For the last inequality (5.3c), we apply Young’s inequality with exponents $\frac{p+\beta}{q+\beta-2}$ and $\frac{p+\beta}{2+p-q}$, which requires the condition $q < p + 2$. Substituting (5.3a)–(5.3c) in (5.2), we get

\[ \int_{B_r} |\nabla u|^{p+\beta} \, dx \leq \frac{3C||u||_{L^\infty(B)}^2}{(R-r)^2} \tau \int_{B_R} |\nabla u|^{p+\beta} \, dx + \frac{C||u||_{L^\infty(B)}^2}{(R-r)^2} \left\{ \left[ \frac{1}{\tau^{\frac{p+\beta}{2}}} + \frac{1}{\tau^{\frac{p+\beta}{2}}} \right] R^N \right\}, \]

where $C = C(p, q, m, \beta)$. Now, we choose $\tau = \frac{(R-r)^2}{6C||u||_{L^\infty(B)}^2}$ to get

\[ \int_{B_r} |\nabla u|^{p+\beta} \, dx \leq \frac{1}{2} \int_{B_R} |\nabla u|^{p+\beta} \, dx + CR^N \left( \frac{||u||_{L^\infty(B)}}{R-r} \right)^2 + \left( \frac{||u||_{L^\infty(B)}}{R-r} \right)^{\frac{2(\beta+p)}{\beta+2}} + \left( \frac{||u||_{L^\infty(B)}}{R-r} \right)^{\frac{2(\beta+p)}{\beta}}. \]

Now, we fix $r_0 < R_0$, then by the iteration Lemma 2.9 for $r_0 \leq r < R \leq R_0$ and the maximum principle in Lemma 3.1, we obtain (5.1).

6. Moser’s Iteration

Now, we are in a position to use Moser’s iteration for an unbalanced Caccioppoli inequality as in (4.1). The difference from a standard Moser’s iteration is that the starting point of our iteration must require an exponent of $|\nabla u|$ higher than $p$. In order to do this, we follow the same scheme as laid out in [6].

**Theorem 6.1.** Assume that $q < p + 2$ and let $u$ be the solution to (3.3). Let $R_0 > 0$ be such that $B_{2R_0} \subset B$ and we fix the following two exponents:

\[ 2^* := \begin{cases} \frac{2N}{N-2}, & N > 2 \\ (2, \infty), & N = 2. \end{cases} \]

and $\alpha_0 := \max \left\{ \frac{2q-2^*p}{2^*+p-2}, 2 \right\}$. Then, there exists $C = C(p, q, N, m, R_0, ||U||_{L^\infty(2B)})$ such that

\[ ||\nabla u||_{L^\infty(B_{\frac{R_0}{2}})} \leq C. \]  
(6.1)

**Proof.** We restate the Caccioppoli inequality from Proposition 4.1 as:

\[ \int_B \left| \nabla \left( |\nabla u|^{\frac{p+\alpha}{2}} \right) \right|^2 \eta^2 \, dx \leq C(M, m)(p + \alpha)^2 \left\{ \int_B (|\nabla u|^{q+\alpha} + |\nabla u|^{2+\alpha}) |\nabla \eta|^2 \, dx \right\}, \]

which may be further revised to

\[ \int_B \left| \nabla \left( |\nabla u|^{\frac{p+\alpha}{2}} \right) \right|^2 \eta^2 \, dx \leq C(M, m)(p + \alpha)^2 \left\{ \int_B (|\nabla u|^{q+\alpha} + 1) |\nabla \eta|^2 \, dx \right\}. \]

By Sobolev’s embedding, we have

\[ \left( \int_B \left| \nabla u \left( |\nabla u|^{\frac{p+\alpha}{2}} \right)^2 \eta^2 \, dx \right|^{\frac{2}{2^*}} \right) \leq C(M, m)(p + \alpha)^2 \left\{ \int_B (|\nabla u|^{q+\alpha} + 1) |\nabla \eta|^2 \, dx \right\}. \]
Finally, for two concentric balls $B_{\rho} \subset B_{\rho}$ and $\eta \in C^\infty_0(B_{\rho})$ satisfying $\eta \equiv 1$ in $B_{\rho}$ and $|\nabla \eta| \leq \frac{C}{R - r}$, we have

$$
\left( \int_{B_{\rho}} |\nabla u|^{\frac{p+\alpha}{2}} dx \right)^{2/2^*} \leq \frac{C(M, m)(p+\alpha)^2}{(R - \rho)^2} \left\{ \int_{B_{\rho}} |\nabla u|^{q+\alpha} + 1 \ dx \right\}.
$$

(6.2)

Now, for $n = 1, 2, 3, \ldots$, we define $\rho_n := \frac{R_0}{2} \left( 1 + \frac{1}{2^n} \right)$ and choose $\alpha_n$ to satisfy $(p + \alpha_n)\frac{2^*}{2} = \alpha_{n+1} + q$. Therefore, we have

$$\alpha_n = \frac{(2^*)^n}{2} \alpha_0 + \left( \frac{2^*}{2} - q \right) \left( \frac{2^*}{2} - 1 \right).
$$

(6.3)

From (6.3), it is easy to see that if $\alpha_0 + \frac{2^*}{2} p - 2q > 0$, then $\lim_{n \to \infty} \alpha_n = \infty$. Thus, we can rewrite (6.2) as

$$
\int_{B_{\rho_{n+1}}} |\nabla u|^{\alpha_n+q} dx \leq C \left\{ \int_{B_{\rho_n}} |\nabla u|^{q+\alpha_n} dx + 1 \right\}^{2^{*}/2},
$$

(6.4)

where $C = C(R_0, \alpha_0, N, M, m, p, q)$ is independent of $n$. Defining $Y_n := \int_{B_{\rho_n}} |\nabla u|^{\alpha_n+q} dx$, estimate (6.4) becomes

$$
Y_{n+1} \leq C^n (Y_n + 1)^{\frac{2^*}{2}}.
$$

(6.5)

By iterating (6.5), we get

$$
Y_{n+1} \leq C^n [Y_n + 1]^{2^{*}/2} \leq C^n \left\{ C^{n-1} [Y_{n-1} + 1]^{2^{*}/2} + 1 \right\}^{2^{*}/2} \leq C^{n+1} \left[ Y_{n-1}^{2^{*}/2} \right]^{2^{*}/2}.
$$

We make the following three observations:

$$
\frac{(2^*)^n}{\alpha_n + q} = \frac{(2^*)^n}{\alpha_0 + \frac{2^*}{2} p - 2q} \left( \frac{(2^*)^n}{2} - 1 \right) \leq \frac{(2^*)^n}{\alpha_0 + \frac{2^*}{2} p - 2q} \left( \frac{(2^*)^n}{2} - 1 \right) \leq C(N, \alpha_0, p, q) < \infty,
$$

(6.6)

$$
\frac{\sum_{j=0}^{n-1} \frac{(2^*)^j}{\alpha_n + q} (n-j)}{\sum_{j=0}^{n-1} \frac{2^*}{2} (n-j)} \leq \frac{\sum_{j=0}^{n-1} \frac{2^*}{2} (n-j)}{\sum_{j=0}^{n-1} \frac{2^*}{2} (n-j)} \leq C(\alpha_0, N, p, q) < \infty.
$$

Hence, following the observations from (6.7) and passing to the limit as $n \to \infty$ in (6.6), we get

$$
\|\nabla u\|_{L^\infty(B_{R_0}/2)} \leq C \left( \int_{B_{R_0}} |\nabla u|^{\alpha_0+q} dx + 1 \right)^{\frac{1}{\alpha_0+q+\frac{1}{2}(1-\frac{2p}{2})}},
$$

(6.8)

where $C = C(R_0, \alpha_0, M, m, p, q, N)$.

Now, we make use of (5.1) with the choice of $\beta = \alpha_0 + q - p \geq 2$, estimate (6.8) becomes

$$
\|\nabla u\|_{L^\infty(B_{R_0}/2)} \leq C (\Gamma_1 + 1)^{\frac{1}{\alpha_0+q+\frac{1}{2}(1-\frac{2p}{2})}},
$$

where $C = C(R_0, \alpha_0, M, m, p, q, N)$. 

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where
\[
\Gamma_1 = R_0^{N} \left[ \left( \frac{|U|_{L^\infty(2B)}}{R_0} \right)^{2\frac{(\alpha_0+q)}{p}} + \left( \frac{|U|_{L^\infty(2B)}}{R_0} \right)^{2\frac{(\alpha_0+q)}{2-\alpha+q}} + \left( \frac{|U|_{L^\infty(2B)}}{R_0} \right)^{\alpha_0+q} \right].
\]

This completes the proof of the theorem.

7. Proof of Theorem 1.3

It remains to obtain the Lipschitz bound for \( U \) from the Lipschitz bound (6.1) for the regularized minimizer (recall Remark 3.2). We follow the scheme of the proof in [3], which is similar to the double approximation procedure in [17]. Observe that

\[
\|\nabla u_{\sigma,\varepsilon}\|_{L^\infty(B_{R_0/2})} \leq C,
\]

where \( C = C(p, q, N, M, m, |U|_{L^\infty(2B)}) \). One can also obtain

\[
m \int_B |\nabla u_{\sigma,\varepsilon}|^p dx \leq \int_B f(\nabla u_{\sigma,\varepsilon}) dx \leq \int_B f_\sigma(\nabla u_{\sigma,\varepsilon}) dx \leq \int_B f_\sigma(\nabla U_\varepsilon) dx
\]

(3.1)

where the last inequality follows from the convexity of \( f \) and Jensen’s inequality. Now, for fixed \( \varepsilon > 0 \), we can find \( w_\varepsilon \in U_\varepsilon + W^{1,p}_0(B) \) such that, for a subsequence, as \( \sigma \to 0 \), we have

\[
\nabla u_{\sigma,\varepsilon} \rightharpoonup \nabla w_\varepsilon \text{ weak-}\ast \text{ in } L^\infty(B_{R_0/2}),
\]

\[
u_{\sigma,\varepsilon} \rightharpoonup w_\varepsilon \text{ in } W^{1,p}(B)\text{-weak}.
\]

Passing to the limit, as \( \sigma \to 0 \), we obtain, on account of weak and weak-\ast lower semicontinuity of norms,

\[
\|\nabla w_\varepsilon\|_{L^\infty(B_{R_0/2})} \leq C, \text{ and}
\]

\[
m \int_B |\nabla w_\varepsilon|^p dx \leq \int_B f(\nabla w_\varepsilon) dx \leq \int_{(1+\varepsilon)B} f(\nabla U) dx.
\]

Once again, using the fact that, for a subsequence, \( w_\varepsilon \rightharpoonup w \) in \( U + W^{1,p}_0(B) \)-weak, we obtain by lower semicontinuity,

\[
\|\nabla w\|_{L^\infty(B_{R_0/2})} \leq C, \text{ and}
\]

\[
m \int_B |\nabla w|^p dx \leq \int_B f(\nabla w) dx \leq \int_B f(\nabla U) dx.
\]

(7.1)

Finally, by the strict convexity of \( f \) (see Remark 7.1), (7.2) and the fact that \( w \in U + W^{1,p}_0(B) \), we have \( w = U \).

Hence, the Lipschitz continuity of \( U \) follows from (7.1).

Remark 7.1. In order for uniqueness to hold, we want to make use of Theorem 2.6 which requires strict convexity. But in our situation, we have the additional condition (1.2c) and this implies strict convexity of the functional, as can be seen in the calculation from [2, Proof of Theorem 4.10].

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