A Mechanism for Charge Quantization

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We analyze a potential that produces background charges which are automatically quantized. This introduces a new mechanism for charge quantization, although so far it has only been implemented for background charges. We show that this same mechanism can also lead to an alternative means of hiding extra dimensions that is analogous to the Kaluza-Klein approach.
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1. Introduction

The quantization of electric charges has been a puzzle since it was discovered by Millikan \[^1\] nearly a hundred years ago. There have been two conventional explanations of this phenomenon. The first, due to Dirac \[^2\], considered the effect of magnetic monopoles. Dirac showed that, in the presence of a magnetic monopole, only quantized electric charges are allowed.\[^1\] Thus, should a magnetic monopole be discovered, quantization of electric charge would, of necessity, follow. The alternative explanation is one based in gauge theories. If the $U(1)$ gauge group of electromagnetism is embedded in a non-abelian gauge group, then charge quantization is automatic, for group theoretic reasons \[^4\]. The commutation relations for the non-abelian group impose non-abelian charge quantization, and thus the embedding of the $U(1)$ group in the non-abelian group implies quantization for the electric charge.

We will here present a theory that exhibits a quantized electric charge, but in a very different way. The theory in question will be an example from non-relativistic quantum mechanics, and give us an entirely different way to think about the origins of charge quantization by naturally producing a situation in which there are background charges, and for which these background charges are quantized. One of the bonuses of this method is that it also gives us a novel way to think about the possibility of higher dimensions, and gives us a way in which such dimensions can appear small without being explicitly compactified.

In some sense, the structure we identify is more like that typically found when there are topological charges. We find discrete sectors of the theory, sectors which can be understood as being related by the addition of quantized potential terms. The difference is that these sectors do not arise for topological reasons in the case that we study here.

We should note that, at present, the potential function that we examine in this paper is constructed in very much an *ad hoc* way. We do not yet have a natural mechanism for causing this potential (or one like it) to appear, or incorporate it in such a way as to lead to quantization of dynamical, as opposed to background, charges. However, what we do have, even with these caveats, is a new conceptual approach to the question of what can enforce charge quantization or lead to dimensional compactification. While we would of course have preferred already to have developed the full application of this idea, we recognize that often such work must be developed in stages. It is in this light that we therefore

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\[^1\] A generalized condition follows for dyons \[^3\].
present this first stage here, presenting both the concept and its initial implementation, while we reserve for future work the further technical developments that will extend the applicability of these ideas.

2. The Digamma Potential

The function $\Gamma(x)$ is well-known, of course; it is the analytic function that generalizes the factorial, with the relation holding for all arguments

$$\Gamma(x + 1) = x \Gamma(x)$$  \hspace{1cm} (2.1)

The digamma function is then defined as

$$\psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$$  \hspace{1cm} (2.2)

This function satisfies the identity

$$\psi(x + 1) = \psi(x) + 1/x$$  \hspace{1cm} (2.3)

We warn the reader not to confuse the digamma function with the quantum mechanical wavefunction, as both are typically denoted with the same Greek letter. We will reserve for “$\psi$” for the digamma function, and will not need any particular symbol for the wavefunction.

Suppose, now, that we consider the non-relativistic quantum mechanical Hamiltonian which has the digamma function as its potential,

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \psi(x)$$  \hspace{1cm} (2.4)

where the coordinate $x$ labels the real axis. The potential has poles at every non-negative integer. Note that at large positive $x$, $\psi(x)$ behaves asymptotically as $\ln(x)$ (this can be seen from the Stirling approximation), while at each negative integer, the function diverges to $-\infty$ on the right and to $+\infty$ on the left.

Because of the infinities, the behavior of the wavefunction at these negative integer values must be controlled. One way to do this would be to specify put boundary conditions on the wavefunction at these points, so as to produce a self-adjoint extension. Our analysis
of this problem indicates that one can do so in a way that imposes the condition that at these points the wavefunction vanishes, while its derivative need not be continuous.

However, we can also adopt a simpler approach that achieves the same results. All the properties of the potential that will be necessary in our analysis can be preserved if we add a periodic term to the digamma function potential. We therefore can consider the generalized potential

\[ \tilde{U}(x) = \psi(x) + \frac{1}{\sin^2 \pi x} . \] (2.5)

This potential still satisfies (2.3), but at the same time causes the potential to diverge to a positive infinite value at every negative integer, whether one approaches from the left or the right, as there is now a double pole at each of these points. We thus imagine that we have added such a term to the potential. Note that the potential we have added is, when considered by itself, exactly soluble, and related by shape invariance \[5\] to the infinite square well. This guarantees that the boundary conditions in the presence of this new term are such that the wavefunction vanishes whenever \( x \) is a negative integer, with the wavefunction generally being discontinuous at these points, as the new term dominates the digamma function at these locations, and hence determines the behavior of the wavefunction.

Whichever approach one takes – either constructing self-adjoint extensions or adding the extra term to the potential – will have the same effects, and will lead to the conclusions presented in this paper. For the sake of simplicity of presentation, we will suppress the addition of the \( 1/\sin^2(\pi x) \) term, as it does not affect the mechanism behind the key results we obtain, but the reader should be aware that such a term can, if one chooses the latter scheme, be added throughout the analysis of this paper, although we will our frame our results without reference to a specific choice between these two options.

Each interval \(-(m + 1) < x < -m\), where \( m \) is a non-negative integer, we will term a sector. Since the wavefunction vanishes at the endpoints of each sector, and the derivatives need not be continuous at these points, we see that wavefunctions which are non-zero in only one sector can be perfectly good eigenstates of the Hamiltonian, and so we can choose to focus on such functions as the eigenfunctions of interest. One then solves the Schrödinger equation in each sector, finding the corresponding wavefunctions. Due to the boundary conditions imposed by either of our techniques, it is acceptable to consider this theory exclusively on the negative real axis, which for simplicity we will do here (although we will see later that we can remove this limitation). Thus the eigenfunctions for the different sectors, when put together, form a complete basis on the space in question.
What would a theory on this half-line look like? As we discussed above, one way to think of it would be as a set of separate sectors. However, this theory, as we will show shortly, is equivalent to another theory: one with only a single spatial sector, but with different background charges. The important feature here is that the background charges must be quantized.

Let us refer to the region in the interval \(- (m + 1) < x < - m\) as the \(m^{th}\) sector of the theory. Then when we are studying the wavefunctions confined to this sector, we need simply solve the Schrödinger equation with the potential \(V_m(x) = \psi(x)\) in the \(m^{th}\) sector, and with barriers preventing penetration into the adjacent sectors.

We can, however, label the sectors using shifted coordinates, so that in each sector, we restrict the spatial variable \(x\) to the interval \(-1 < x < 0\). Then in each sector, we must solve the Schrödinger equation with a suitable potential. We denote the potential in the \(m^{th}\) sector as \(U_m(x)\). Then we have

\[
U_0(x) = \psi(x) ,
\]

\[
U_1(x) = \psi(x - 1) = \psi(x) - \frac{1}{x - 1} = U_0(x) - \frac{1}{x - 1} ,
\]

\[
U_2(x) = \psi(x - 2) = \psi(x - 1) - \frac{1}{x - 2} = U_0(x) - \frac{1}{x - 2} - \frac{1}{x - 1} ,
\]

and, in general,

\[
U_k(x) = U_0(x) - \frac{1}{x - 1} - \frac{1}{x - 2} - \cdots - \frac{1}{x - k} .
\]

Consequently, we can think of the theory in sector \(m - 1\) as equivalent to the theory in sector \(m\) provided that an additional charge, with its associated “Coulombic” potential\(^2\) having been added to the theory. The coefficient of the attendant \(1/(x - m)\) term is a charge of the theory, as it determines the strength of the force on the dynamical particle.

Iterating this process as we have in (2.9), we see that every sector is equivalent to the \(0^{th}\) sector with a series of additional charges added to the theory through the background potential, with each of these additional charges having strength \(-1\). Thus, the \(m^{th}\) sector can be understood as equivalent to the \(0^{th}\) sector in the presence of \(m\) background charges of charge \(-1\). (These charges also appear at discrete locations.) This fixed charge strength tells us, then, that these background charges are quantized.

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\(^2\) We put the term *Coulombic* in quotation marks, as it is only in three spatial dimensions that the Coulomb potential goes as \(1/r\).
Thus we have a theory which, although initially formulated on the semi-infinite negative real-axis, should actually be understood as being confined to a single finite interval, but in the presence of various background charges, with these charges quantized. This is the crux of our mechanism: that the potential itself automatically generates a quantization of charge. As a consequence, we have a distinct and novel mechanism for explaining the appearance of charge quantization. Even though the present model cannot by itself be a realistic model of nature, the mechanism itself may well prove fruitful when developed in more elaborated settings.

We note, too, that there are natural ways to generalize this phenomenon, ways that are already straightforward to characterize. For example, as we have already discussed, one can add any periodic potential to this system in the original definition of the Hamiltonian (2.4) without altering our analysis; thus we are not restricted specifically to the unmodified digamma function potential. Indeed, this idea underlay our addition in (2.3) of a $1/\sin^2(\pi x)$ term in lieu of constructing a self-adjoint extension, but one can of course also add other periodic potentials that are everywhere finite, and still maintain the mechanism for the appearance of quantized background charges. One can also easily imagine extending this to higher dimensions, using, for example, a potential in three-dimensions such as $U(x, y, z) = \psi(x)\psi(y)\psi(z)$. This gives rises to a theory which can be viewed as consisting of various cubical sectors, although the potentials will not be properly Coulombic, but rather involve terms like $1/(x - a)$, $1/(y - b)$, and $1/(z - c)$. One can imagine more complicated variations, such as a spherically symmetric three-dimensional potential, say $U(r) = \psi(-r)$, in which case there would be relationships among different spherical shells (which would also involve the centrifugal term).

In addition, we can use integrals or derivatives of the digamma function to get sectors related by the addition of quantized non-Coulombic terms. One interesting example is $U(r) = \psi'(-r)$, as here the extra term from shifting $r$ by 1 can be absorbed as a modification to the centrifugal term. As a consequence, shifting from one spherical shell to another can be reinterpreted as a modification of the angular momentum term in the radial Schrödinger equation to a non-standard value. Exploration of this theory is left as an exercise for the interested reader.

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3 In fact, now that we have presented the main analysis, one sees that the division into distinct sectors with quantized background charges will hold for all $x$, not just the negative half of the $x$-axis, when we add the $1/\sin^2(\pi x)$ term to the potential.
3. Extra Dimensions

The notion of extra dimensions — extra, that is, beyond the usual four of spacetime — goes back to the 1920s and the work of Kaluza [6] and Klein [7]. The Kaluza-Klein approach has become one of the standard tools in modern attempts at explaining the origins of the forces, and is especially important in the context of string theory.

The model we have presented gives a way besides compactification for large dimensions to look small. Suppose we have a theory in which the fifth dimension has the topology of a ray, and is endowed with a potential $\psi(y)$, where $y$ is the coordinate along this ray, with the ray stretching along $y < 0$. In this extra dimension, which is infinite, the particle will be confined to some sector of, in suitable units, length 1. Alternatively, one can view this as a fifth dimension which is of length 1, but for which one has the possibility of different background charges being inserted. By exchanging the extent of space for background charges, we get charge quantization and dimensional compactification automatically, without invoking the familiar Kaluza-Klein mechanism, and thus without having to seek a dynamical explanation for the compactification of the extra dimensions. Thus what started out as a new mechanism for generating charge quantization has turned into a new mechanism for compactifying extra dimensions.

4. Conclusions

We have shown that a perfectly reasonable quantum mechanical theory in one infinite dimension is equivalent to a quantum theory on a sector of length 1, with the addition of background charges, the values of which must be quantized. The addition or subtraction of these charges arises based on which sector one is considering. The essential observation is that the amount of charge that can be added or subtracted is quantized.

We note, too, that this same mechanism helps us see a new way to have extra dimensions, with the price for the apparent compactification of the extra dimensions being the appearance of quantized background charges. For both the charge quantization and dimensional reduction applications, it is clear that one can add any periodic potential to the theory without modifying the results.

It is interesting to see sectors of different charge arising in this case without topology. The similarity to topological charges rests in the appearance of sectors and the quantization

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4 Or imagine that a $1/\sin^2(\pi y)$ term has been added, so that we may include the whole $y$-axis.
of charge. However, the mechanism is entirely different, and arises here independent of any particular topological considerations.

Transforming this mechanism into an effective and realistic approach to the question of charge quantization will clearly take some work. Given the importance of the problem, however, having an alternative way to think about charge quantization (and dimensional reduction/compactification) is clearly of great interest, and we are currently considering ways to enlarge the applicability of this idea to higher dimensional quantum mechanics and to quantum field theory. We will leave a more careful consideration of these possibilities to future papers.

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