Discrete Orbits, Recurrence and Solvable Subgroups of Diff (C², 0)

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Abstract We discuss the local dynamics of a subgroup of Diff (C², 0) possessing locally discrete orbits as well as the structure of the recurrent set for more general groups. It is proved, in particular, that a subgroup of Diff (C², 0) possessing locally discrete orbits must be virtually solvable. These results are of considerable interest in problems concerning integrable systems.

Keywords Dynamics of pseudogroups · Recurrent points · Diffeomorphisms tangent to the identity

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1 Introduction

This paper is devoted to establishing some general theorems about the dynamics of (virtually) non-solvable subgroups of Diff (C², 0). Whereas motivations for these results arise from a few different sources, certain problems concerned with integrable systems, and especially those also having a connection with Morales–Ramis–Simó differentiable Galois theory, are very directly related to our results, [19,20]. In this Introduction, we shall first state the two main theorems obtained in this paper and
then proceed to a general discussion about their motivations and applications while keeping them in perspective with some previous works.

Throughout this paper, a group will be said to be virtually solvable if it contains a normal, solvable subgroup of finite index. Now, consider finitely many local diffeomorphisms \( f_1, \ldots, f_k \) inducing elements of \( \text{Diff}(\mathbb{C}^2, 0) \). Denote by \( G_U \) the pseudogroup generated by \( f_1, \ldots, f_k \) on some chosen small neighborhood \( U \) of \((0, 0) \in \mathbb{C}^2\); see Sect. 2.1 for details. At the level of germs, the subgroup of \( \text{Diff}(\mathbb{C}^2, 0) \) generated by \( f_1, \ldots, f_k \) will be denoted by \( G \). When no misunderstanding is possible, we shall make no distinction between \( G_U \) and \( G \). With this identification, \( G \) is said to have locally discrete orbits (resp., finite orbits), if there is a sufficiently small neighborhood \( U \) of \((0, 0) \) where \( G_U \) has locally discrete orbits (resp., finite orbits). The reader is referred to Sect. 2 for accurate definitions. With this terminology, our first main result reads:

**Theorem A** Suppose that \( G \) is a finitely generated subgroup of \( \text{Diff}(\mathbb{C}^2, 0) \) with locally discrete orbits. Then \( G \) is virtually solvable.

**Remark** Although we always work with finitely generated groups, the reader will note that Theorem A also holds for groups that are infinitely generated. A simple argument to derive this slightly stronger statement from the proof of Theorem A is provided at the end of Sect. 3; see Theorem 3.7.

The notion of recurrent points allows us to accurately state Theorem A. Given \( U \) and \( G_U \) as above, a point \( p \in U \) is said to be recurrent if there exists a sequence \( \{g_n\} \) of elements in \( G_U \) such that \( g_n(p) \to p \) with \( g_n(p) \neq p \) for every \( n \). In this definition, it is implicitly assumed that \( p \) belongs to the domain of definition of \( g_n \) when \( g_n \) is viewed as an element of the pseudogroup \( G_U \). A recurrent point does not have locally finite orbit and, conversely, a point whose orbit is not locally finite must be recurrent. Thus, Theorem A can be rephrased by saying that there are always recurrent points for a non-virtually solvable group \( G \subset \text{Diff}(\mathbb{C}^2, 0) \). The size of the set formed by these recurrent points may, however, be relatively small, as it may coincide with a Cantor set (this is very similar to the case of a Kleinian group having a Cantor set as its limit set; cf. Sect. 4). To obtain a general result about the size of recurrent points, we are led to consider the normal subgroup \( \text{Diff}_1(\mathbb{C}^2, 0) \) of \( \text{Diff}(\mathbb{C}^2, 0) \) consisting of those local diffeomorphisms tangent to the identity. When \( G \) happens to be a (pseudo-) subgroup of \( \text{Diff}_1(\mathbb{C}^2, 0) \), the following stronger result holds:

**Theorem B** Consider a non-solvable group \( G \subset \text{Diff}_1(\mathbb{C}^2, 0) \) and denote by \( \Omega(G) \) the set of points that fail to be recurrent for \( G \). Then there is a neighborhood \( U \) of \((0, 0) \in \mathbb{C}^2 \) such that \( \Omega(G) \cap U \) is contained in a countable union of proper analytic subsets of \( U \) (in particular, \( \Omega(G) \cap U \) has null Lebesgue measure).

**Remark** In the above statement the reader will note that the group of germs at \((0, 0) \in \mathbb{C}^2 \) naturally associated with \( G \) is only assumed to be non-solvable as opposed to non-virtually solvable. Also it is easy to prove that for a group \( G \) generated by a random choice of \( n \geq 2 \) elements in \( \text{Diff}_1(\mathbb{C}^2, 0) \), the resulting set \( \Omega(G) \) is reduced to the origin of \( \mathbb{C}^2 \); cf. Remark 3.2.
On a different note, we know of no example of a non-solvable group $G \subset \Diff_1(C^2, 0)$ for which $\Omega(G)$ is not contained in a proper analytic set. It would be nice to know whether this stronger statement always holds.

Concerning the above theorems, it may be observed that suitable versions of them are likely to hold in arbitrary dimensions, although we have not tried to work out any of these generalizations. Indeed, we decided to restrict our attention to the 2-dimensional case partly because this setting is already full of new phenomena and partly because some proofs are already fairly involved. Yet, a careful reading of our arguments indicates that more typical arguments of complex dimension two were used only at a few points which, in turn, suggests the existence of suitable arbitrary-dimensional versions of the mentioned results.

We can now go back to the beginning of this Introduction and discuss the motivations for the above statements. The most important motivations can be ascribed to several types of Galois theories and to certain integrability problems; see below. However, we may begin by observing that very little in general is known about the dynamics of large (e.g., non-solvable) subgroups of $\Diff_n(C^n, 0)$ when $n \geq 2$. In this sense, the above results stand among the first ones in this direction. The situation contrasts with the case of the local dynamics associated with subgroups of $\Diff(C, 0)$, and a brief review of the main results in this case may be a good starting point for us. Whereas the local dynamics of subgroups of $\Diff(C, 0)$ still holds some subtle open problems, the topic can be regarded as well understood since a large body of knowledge on these dynamics can be found in the literature; see [8,14,21,29,32]. The picture changes drastically when $n \geq 2$, as many new phenomena emerge to provide a far more involved landscape. Indeed, when $n \geq 2$, there is a significant body of theory developed in the case of the dynamics associated with a parabolic germ; cf. [1,3,7,12]. For non-solvable groups, the results of [15] provide satisfactory answers in the case of non-discrete groups containing a hyperbolic contraction. The conditions of [15], however, are not always satisfied in the cases of interest.

Along the lines of the above paragraph, a first motivation for this work can broadly be described as the beginning of a systematic study of the dynamics associated with “large” subgroups of $\Diff(C^2, 0)$, where by “large” we typically mean non-solvable (and in some cases non-virtually solvable). Naturally, when considering these groups, we might be tempted to parallel the theory of Shcherbakov–Nakai vector fields applicable to non-solvable subgroups of $\Diff(C, 0)$. Although their theory remains an important guiding principle for our investigations, the very existence of free discrete subgroups of $\GL(2, C)$ is enough to ensure that Shcherbakov–Nakai vector fields cannot be associated with subgroups of $\Diff(C^2, 0)$ without additional assumptions; see Sect. 4 for details and definitions. In this direction, whereas our recurrence statements constitute a less powerful tool than vector fields approximating the dynamics of the group, they have the advantage of holding for arbitrary non-virtually solvable subgroups of $\Diff(C^2, 0)$ and, in fact, they constitute the first general result concerning the dynamics of these groups. Moreover, as far as general non-virtually solvable subgroups of $\Diff(C^2, 0)$ are concerned, Theorem A is probably not far from sharp. Also, it is worth mentioning that in a number of standard applications of Shcherbakov–Nakai theory, only the recurrent character of the dynamics is needed so that the Theorem A suffices to derive important conclusions. As an outstanding exam-
ple of these situations, we quote the work of Camacho and Scardua on the “Analytic limit set problem”, see [4,5]: the remarkable conclusion that the holonomy group of the limit set in question must be solvable requires only the fact that the dynamics of a non-solvable subgroup of $\text{Diff}(\mathbb{C}, 0)$ has recurrent points. Thus, Theorem A is strong enough to yield the analogous conclusion for suitable higher-dimensional versions of the problem in question.

The second and more important motivation for the previously stated results, however, comes from a few fundamental questions concerning the integrable character of certain systems (vector fields). Most of this goes along the connection between integrable systems and Galois differential theories in the spirit of [19,20]. It turns out, however, that our first motivation stemming from integrable systems is not related to Galois differential theories, but rather to a classical theorem due to Mattei and Moussu [17]. Mattei–Moussu’s theorem asserts that, in dimension 2, the existence of holomorphic first integrals for holomorphic foliations defined around the origin of $\mathbb{C}^2$ can be read off the topological dynamics associated with the singular point. It was recently shown in [22] that, strictly speaking, this remarkable phenomenon no longer holds in higher dimensions, and some additional curious examples were provided in [24]. These examples made it clear that a fundamental question in this problem is to decide which kind, if any, of “integrable character” can be associated with a finitely generated subgroup $G$ of $\text{Diff}(\mathbb{C}^2, 0)$ possessing finite orbits (or more generally locally finite orbits so as to allow for meromorphic as well as other types of first integrals). Indeed, the cornerstone of Mattei–Moussu’s argument [17] is the fact that a subgroup $G$ of $\text{Diff}(\mathbb{C}, 0)$ all of whose orbits are finite must be finite itself: a result no longer valid in dimension 2; see [24]. Naturally, finite groups always admit non-constant first integrals for their actions, which leads to the existence of first integrals for the initial foliation. In this sense Theorem A can be viewed as a suitable generalization of their important one-dimensional result.

We can now focus on motivation arising from differentiable Galois theories. From the point of view of Galois theories, or more precisely from the point of view of Morales–Ramis–Simó theory, solvable groups are associated with integrable systems where integrability should be understood in a type of quadrature sense slightly more general than the standard context of Arnold–Liouville’s theorem. In this sense, Theorem A provides a fully satisfactory answer to the preceding question, namely the integrable character of a subgroup of $\text{Diff}(\mathbb{C}^2, 0)$ possessing locally finite orbits lies in the fact that this group must be virtually solvable.

We are finally able to explain other aspects of Morales–Ramis–Simó theory [20] that have provided us with extra motivation for the present work. Inasmuch as Galois differentiable theories are highly developed in the linear case, and they allow us to decide whether or not a given equation is solvable by quadratures, a far more general non-integrability criterion applicable to genuinely non-linear situations is summarized by Morales–Ramis–Simó’s theorem [20]. This theorem asserts that the Galois group associated with the $k$-th variational equation arising from a periodic solution must be virtually solvable (actually virtually abelian) provided that the system is integrable in the sense of Arnold–Liouville. The passage from virtually solvable to virtually abelian being a minor issue in this context, their result is somehow very close to our Theorems A and B, and this topic deserves further elaboration.
The first main difference between the two sets of results lies in the groups considered: both Morales–Ramis and Morales–Ramis–Simó theories focus on Galois groups, which may be larger than the more commonly used holonomy groups, which are the primary concern of the results in this work. In this sense, the theories in [19,20] are more complete since they have a better chance at detecting non-integrable behavior. On the other hand, the advantage of our direct analysis of the holonomy group is the possibility of providing further information on the actual dynamics of several non-integrable systems. As a matter of fact, when the mentioned group is not (virtually) solvable, then our results allow us to derive non-trivial conclusions concerning the dynamics of the (necessarily non-integrable) system in question. Naturally, similar ideas can also be applied to Galois groups, so that our dynamical results provide a nice complement to Morales–Ramis–Simó theory.

Let us close this Introduction with an outline of the structure of the paper. The basic idea underlining most of the present work is rather simple and comes from Ghys’s recurrence theorem [11] which, however, is formulated in a slightly different context. More precisely, Ghys proves that a group of real analytic diffeomorphisms of a compact manifold generated by diffeomorphisms close to the identity has recurrent dynamics provided that the group is not pseudo-solvable. Exploiting his idea to prove Theorem A involves, however, two main issues. The first important issue is related to the assumption on closeness to the identity made in Ghys’s theorem [11]. The other fundamental difficulty is related to the notion of a pseudo-solvable group introduced in the same paper [11].

From an algebraic point of view, the main point lies in the definition of a pseudo-solvable group, which is related to the fact that certain sets of commutators degenerate into the identity. This is actually a tricky point: the geometric meaning of pseudo-solvability is not clear, especially because the notion may, in principle, depend on the generating set; see Sect. 2 for details. To overcome this difficulty, we shall prove Theorem 2.5 asserting that among subgroups of the group of (formal) diffeomorphisms of $\mathbb{C}^2$ fixing the origin, the notions of a pseudo-solvable group and of a solvable group turn out to coincide; again, see Sect. 2 for further comments.

The analytic side of the above-mentioned difficulties is related to the assumption on closeness to the identity involved in Ghys’s recurrence theorem [11]. Our approach to this issue conceptually hinges on the dichotomy involving discrete and non-discrete groups; see Sect. 4 for a detailed self-contained discussion. As already mentioned, among finitely generated subgroups of Diff ($\mathbb{C}^2$, 0) there are groups that are discrete in a natural sense as well as groups that are non-discrete in the same sense. Roughly speaking, a group is said to be non-discrete if it contains a non-trivial sequence of elements defined on some fixed neighborhood $U$ of $(0,0) \in \mathbb{C}^2$ and converging uniformly to the identity on this neighborhood. Ultimately, the importance of showing that a pseudo-solvable group is actually solvable lies in this dichotomy: the corresponding result yields a powerful criterion to detect non-discrete groups. In fact, every sequence of “iterated commutators” starting from two elements sufficiently close to the identity will converge to the identity; see Sect. 3. This phenomenon of convergence actually explains the assumption on “closeness to the identity” made in the quoted theorem due to Ghys [11]. From this point, our general argument will allow us to connect discrete subgroup of Diff ($\mathbb{C}^2$, 0) with Kleinian groups in an accurate sense. Then, by
ranging on basic facts from Kleinian group theory combined with equally basic results on stable manifold theory of a hyperbolic fixed point, we shall manage to establish Theorem A in the case of discrete groups. The complementary case of non-discrete groups can then be handled by resorting to the argument on convergence of iterated commutators close to the identity as in [11].

This paper is organized as follows. Section 2 contains some background material, including the statement of Theorem 2.5 and further explanation of the role played by the “pseudo-solvable condition” in this paper. Taking Theorem 2.5 for granted, the proofs of Theorems A and B will be supplied in Sect. 3. Additional details, examples and some definitions helping to illustrate these theorems are provided in the short Sect. 4.

The remainder of the paper, namely Sects. 5, 6, 7, and 8, is devoted to the proof of Theorem 2.5. The reader is assumed to be familiar with basic properties of formal diffeomorphisms tangent to the identity, including the fact that every such diffeomorphism can be viewed as the time-one map induced by a formal vector field. Material from [16] involving the formal classification of the corresponding solvable groups will be used as well. The arXiv version of this paper [26] contains a detailed review of the corresponding background material in a form suited to proving Theorem 2.5. A relatively raw manuscript containing the main ideas for the present paper as well as for our paper [24] is also available on arXiv [25]. Finally, Ribon recently announced a generalization of some of the main results in this paper; see [27].

2 Basic Notions

Throughout this work Diff(C^2, 0) stands for the group of germs of holomorphic diffeomorphisms fixing (0, 0) ∈ C^2, while Diff^1(C^2, 0) denotes its normal subgroup consisting of diffeomorphisms tangent to the identity. It is also convenient to consider the group of formal diffeomorphisms of (C^2, 0), which will be denoted by \( \hat{\text{Diff}}(C^2, 0) \). Similarly, \( \hat{\text{Diff}}^1(C^2, 0) \) will stand for the formal counterpart of Diff^1(C^2, 0), i.e., \( \hat{\text{Diff}}^1(C^2, 0) \) is constituted by formal diffeomorphisms tangent to the identity. In other words, an element \( F \in \hat{\text{Diff}}(C^2, 0) \) consists of a series \( (F_1(x, y), F_2(x, y)) \) of formal series \( F_1(x, y), F_2(x, y) \) satisfying the following condition: setting \( F_1(x, y) = a_1x + a_2y + \text{h.o.t.} \) and \( F_2(x, y) = b_1x + b_2y + \text{h.o.t.} \), the \( 2 \times 2 \) matrix whose entries are the coefficients \( a_1, a_2, b_1, b_2 \) is invertible. The formal diffeomorphism \( F \) is said to belong to \( \hat{\text{Diff}}^1(C^2, 0) \) when this matrix happens to coincide with the identity.

Consider a subgroup \( G \) of Diff(C^2, 0) generated by certain elements \( h_1, \ldots, h_k \). A natural way to make sense of the local dynamics of \( G \) consists in choosing representatives for \( h_1, \ldots, h_k \) as local diffeomorphisms fixing \( (0, 0) \in \mathbb{C}^2 \). These representatives are still denoted by \( h_1, \ldots, h_k \) and, having fixed them along with a sufficiently small neighborhood of \((0, 0) \in \mathbb{C}^2 \), the (local) dynamics of \( G \) is identified with the dynamics associated with the pseudogroup generated by the local diffeomorphisms \( h_1, \ldots, h_k \) on the mentioned neighborhood of the origin. It is therefore convenient to recall the definition of a pseudogroup. For this, consider a small neighborhood \( V \) of the origin where the local diffeomorphisms \( h_1, \ldots, h_k \), along with their inverses \( h_1^{-1}, \ldots, h_k^{-1} \), are all well-defined diffeomorphisms onto their images. The pseudogroup generated by \( h_1, \ldots, h_k \) (or rather by \( h_1, \ldots, h_k, h_1^{-1}, \ldots, h_k^{-1} \) if there is any risk of confu-
Suppose we are given local holomorphic diffeomorphisms \( p \) pseudogroup or as a group of germs. Whenever no misunderstanding is possible, the pseudogroup defined above will also be denoted by \( V \), open domain of definition. This domain of definition may, however, be disconnected.

Let us continue with some definitions that will be useful throughout the text. Suppose we are given local holomorphic diffeomorphisms \( h_1, \ldots, h_k, h_1^{-1} \), \( \ldots, h_k^{-1} \) fixing the origin of \( \mathbb{C}^n \). Let \( V \) be a neighborhood of the origin where all these maps yield diffeomorphisms from \( V \) onto the corresponding image. Thus, from now on, the dynamics of \( G \) is simply the dynamics of the corresponding pseudogroup on \( V \). In other words, in the sequel \( G \) is thought of as a pseudogroup operating on \( V \). Given an element \( h \in G \), the domain of definition of \( h \) (as an element of \( G \)) will be denoted by \( \text{Dom}_V(h) \).

**Definition 2.1** The \( V \)-orbit \( O_V^G(p) \) of a point \( p \in V \) is the set of points in \( V \) obtained from \( p \) by taking its image through every element of \( G \) whose domain of definition (as an element of \( G \)) contains \( p \). In other words,

\[
O_V^G(p) = \{ q \in V \; ; \; q = h(p), \; h \in G \; \text{and} \; p \in \text{Dom}_V(h) \} .
\]

For fixed \( h \in G \), the \( V \)-orbit of \( p \) can be defined as the \( V_{<h>} \)-orbit of \( p \), where \( < h > \) denotes the subgroup of \( \text{Diff}(\mathbb{C}^n, 0) \) generated by \( h \).

**Definition 2.2** Given a pseudogroup \( G \) and a point \( p \), the orbit \( O_V^G(p) \) of \( p \) under \( G \) is said to be **finite** if the set \( O_V^G(p) \) is finite. This orbit \( O_V^G(p) \) is called **locally discrete** (or **locally finite**), if there is a neighborhood \( W \subset \mathbb{C}^n \) of \( p \) such that \( W \cap O_V^G(p) = \{ q \} \). Finally, if the orbit of \( p \) is not locally discrete then it is said to be **recurrent**.

For \( G \) and \( V \) as above, we can now define the notions of **pseudogroups with finite orbits** and of **pseudogroups with locally discrete orbits** (or, equivalently, locally finite orbits).

**Definition 2.3** A pseudogroup \( G \subseteq \text{Diff}(\mathbb{C}^2, 0) \) is said to have finite orbits if there exists a sufficiently small open neighborhood \( V \) of \( 0 \in \mathbb{C}^n \) all of whose points have finite orbits. Analogously, \( h \in G \) is said to have finite orbits if the pseudogroup \( \langle h \rangle \) generated by \( h \) has finite orbits.

Similarly, a pseudogroup is said to have locally discrete orbits (or locally finite orbits) if there is \( V \) small as above such that every point in \( V \) has locally discrete orbit.

Let us now remind the reader of the definition of a solvable group. Let \( G \) be a given (abstract) group and denote by \( D^1G \) its **first derived group**, namely the subgroup generated by all elements of the form \( [g_1, g_2] = g_1 \circ g_2 \circ g_1^{-1} \circ g_2^{-1} \) where \( g_1, g_2 \in G \).
The second derived group $D^2G$ of $G$ is defined as the first derived group of $D^1G$, i.e., $D^2G = D^1(D^1G)$. More generally, we set $D^jG = D^1(D^{j-1}G)$. The group $G$ is said to be solvable if the groups $D^jG$ become reduced to $\{\text{id}\}$ for sufficiently large $j \in \mathbb{N}$. The smallest $r \in \mathbb{N}^*$ for which $D^rG = \{\text{id}\}$ is called the derived length of $G$. Equivalently, the group $G$ is also said to be step-$r$ solvable. Thus, a non-trivial abelian group is step-1 solvable. The terminology metabelian groups is sometimes also used to encompass both abelian and Step-2 solvable groups.

It is now convenient to recall the definition of pseudo-solvable groups as formulated in [11].

**Definition 2.4** Let $G$ be a group and consider a given finite generating set $S$ for $G$. With the generating set $S$, a sequence of sets $S(j) \subseteq G$ is associated as follows: $S(0) = S$ and $S(j + 1)$ is the set whose elements are the commutators written under the form $[F_1^{\pm 1}, F_2^{\pm 1}]$, where $F_1 \in S(j)$ and $F_2 \in S(j) \cup S(j - 1)$ ($F_2 \in S(0)$ if $j = 0$). The group $G$ is said to be pseudo-solvable if it admits a (finite) generating set $S$ as above for which the sequence $S(j)$ becomes reduced to the identity for $j$ large enough.

A solvable group is automatically pseudo-solvable, but the converse does not hold in full generality. This is clearly pointed out in [11], and the basic issue lies in the fact that the quotient of the free group on two generators by its second derived group is not finitely presented, though it is clearly a step-2 solvable group. In view of this, we cannot conclude that some high order derived group from a group $G$ is reduced to the identity from knowing that the sets $S(j)$ become reduced to the identity for large $j$ and some generating set of $S$. In particular, at the combinatorial level, there are pseudo-solvable groups that are not solvable. We refer the reader to [11] for further details, including an enlightening comparison with the classical Zassenhaus lemma related to nilpotent groups.

In the context of formal diffeomorphism, however, it turns out to be possible to decide whether a subgroup of $\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ (or of $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$) is solvable from the fact that the sequence of sets $S(j)$ degenerates for a given generating set. In fact, the following holds:

**Theorem 2.5** A pseudo-solvable subgroup $G$ of $\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ is necessarily solvable.

In dimension 1, Theorem 2.5 is due to Ghys [11], who proved that pseudo-solvable subgroups of $\text{Diff}(\mathbb{C}, 0)$ (or of the group of real analytic diffeomorphisms of the circle) are solvable. In dimension 2 the problem was first treated in the relatively raw first draft of this paper [25], and a more detailed treatment appears in the first version of [26]. Some remaining inaccuracies were fixed for the second version of [26]. Recently, J. Ribon also established a version of this statement valid for formal diffeomorphisms of $(\mathbb{C}^n, 0)$; see [27].

### 3 Proof of Theorems A and B

Taking for granted Theorem 2.5, we shall prove in this section Theorems A and B. As mentioned, the proof of Theorem 2.5 is deferred to Sects. 5–8. To begin with,
we will exploit Ghys’s observation [11] concerning convergence of commutators for
diffeomorphisms “close to the identity” to establish the following proposition:

**Proposition 3.1** Suppose that $G \subset \text{Diff}_1(\mathbb{C}^2, 0)$ is a finitely generated group pos-
seSSing locally discrete orbits. Then $G$ is solvable.

*Proof* Consider a finite set $S$ consisting of local diffeomorphisms of $(\mathbb{C}^2, 0)$ that
are tangent to the identity. Assume that the group $G$ generated by the set $S$ is not
solvable (at the level of groups of germs of diffeomorphisms). Then consider the
pseudogroup generated by $S$ on a sufficiently small neighborhood of the origin. For
the sake of notation, this neighborhood of $(0, 0) \in \mathbb{C}^2$ will be left implicit in the course
of the discussion. The proof of the proposition amounts to showing that the resulting
pseudogroup $G$ is non-discrete in the sense that it contains a sequence of elements $g_i$
satisfying the following conditions (cf. Sect. 4):

- $g_i \neq \text{id}$ for every $i \in \mathbb{N}$. Furthermore, $g_i$ viewed as an element of the pseudogroup
  $G$ is defined on a ball $B_\epsilon$ of uniform radius $\epsilon > 0$ around $(0, 0) \in \mathbb{C}^2$.
- The sequence of mappings $\{g_i\}$ converges uniformly to the identity on $B_\epsilon$.

Assuming the existence of a sequence $g_i$ as indicated above, it follows that each of the
sets $\text{Fix}_{g_i} = \{ p \in B_\epsilon : g_i(p) = p \}$ is a proper analytic subset of $B_\epsilon$. For every
$N \geq 1$, pose $A_N = \bigcap_{i=1}^\infty \text{Fix}_{g_i}$ so that $A_N$ is also a proper analytic set of $B_\epsilon$. Finally,
let $F = \bigcup_{N=1}^\infty A_N$. The set $F$ has null Lebesgue measure so that points in $B_\epsilon \setminus F$ can
be considered. If $p \in B_\epsilon \setminus F$ then, by construction, there is a subsequence of indices
$(i(j))_{j \in \mathbb{N}}$ such that $g_{i(j)}(p) \neq p$ for every $j$. Since $g_i$ converges to the identity on
$B_\epsilon$, the sequence $(g_{i(j)}(p))_{j \in \mathbb{N}}$ converges to $p$. This shows that the orbit of $p$ is not
locally discrete and establishes the proposition modulo verifying the existence of the
mentioned sequence $\{g_i\}$.

The construction of the sequence $\{g_i\}$ begins with an estimate concerning commu-
tators of diffeomorphisms that can be found in [15, p. 159], which is itself similar to
another estimate found in [11]. Let $F_1, F_2$ be local diffeomorphisms (fixing the origin
and) defined on the ball $B_r$ of radius $r > 0$ around the origin of $\mathbb{C}^2$. For small $\delta > 0$,
to be fixed later, suppose that

$$\max \left\{ \sup_{z \in B_r} \| F_1^\pm 1(z) - z \|, \sup_{z \in B_r} \| F_2^\pm 1(z) - z \| \right\} \leq \delta. \quad (1)$$

Then, given $\tau$ such that $4\delta + \tau < r$, the commutator $[F_1, F_2]$ is defined on the ball of
radius $r - 4\delta - \tau$ and, in addition, the following estimate holds:

$$\sup_{z \in B_{r-4\delta-\tau}} \| [F_1, F_2](z) - z \| \leq \frac{2}{\tau} \sup_{z \in B_r} \| F_1(z) - z \|. \sup_{z \in B_r} \| F_2(z) - z \|. \quad (2)$$

Let us apply the preceding estimate to $S(1)$. Up to conjugating the elements of $S$
by a homothety having the form $(x, y) \mapsto (\lambda x, \lambda y)$, we can assume that all of them
are defined on the unit ball. Moreover, since these diffeomorphisms are tangent to the
identity, the use of a conjugating homothety as above allows us to assume that the
diffeomorphisms in question also satisfy Estimate (1) for $r = 1$ and some arbitrarily
small $\delta > 0$ to be fixed later. Setting $\tau = 4\delta$, it then follows that every element $\overline{g}$ in $S(1)$ is defined on $B_{1-8\delta}$ and satisfies

$$\sup_{z \in B_{1-8\delta}} \| \overline{g}(z) - z \| \leq \frac{\delta}{2}.$$ 

Next, note that every element in $S(2)$ is the commutator of an element in $S(1)$ and an element in $S \cup S(1)$. Thus, again applying Estimate (2) to $r = 1 - 8\delta$, $\delta$ and $\tau = 4\delta$, we conclude that every element $\overline{g}$ in $S(2)$ is defined on $B_{1-16\delta}$. Furthermore, these elements $\overline{g}$ satisfy the estimate

$$\sup_{z \in B_{1-8\delta}} \| \overline{g}(z) - z \| \leq \frac{\delta}{2^2}.$$ 

Now, every element in $S(3)$ is the commutator of an element in $S(2)$ and an element in $S \cup S(2)$. Hence, the distance to the identity of any of these elements is bounded by $\delta/2$. Thus, choosing $\delta_1 = \delta/2$ and $\tau_1 = 4\delta_1 = 2\delta = \tau/2$, we obtain

$$\sup_{z \in B_{1-8\delta-4\delta}} \| \overline{g}(z) - z \| \leq \frac{\delta}{2^3}.$$ 

For $S(4)$ we have to consider the commutator of an element in $S(3)$ with an element in $S(2) \cup S(3)$. Now the distance to the identity of any of these diffeomorphisms is bounded by $\delta/2^2$ (on the ball of radius $1 - 8\delta - (8+4)\delta$). Hence, this time we choose $\delta_2 = \delta_1/2$ and $\tau_2 = 4\delta_2 = 2\delta_1 = \tau_1/2$ so as to conclude that the elements in $S(4)$ satisfy

$$\sup_{z \in B_{1-8\delta-(8+4+2)\delta}} \| \overline{g}(z) - z \| \leq \frac{\delta}{2^4}.$$ 

The proof continues inductively as follows: for $i \geq 3$, we divide the previous values of “$\delta$” and of “$\tau$” by 2. For every value of $i \in \mathbb{N}^*$ the radius chosen is then dictated by the choices of “$\delta$” and “$\tau$” according to Formula (1). In particular, for every $i \geq 3$ and $\overline{g}_{(i)}$ in $S(i)$, the local diffeomorphism $\overline{g}_{(i)}$ is defined on the ball of radius $1 - 8\delta - \delta \sum_{j=1}^{i-1} 2^{4-j}$. Hence, if $\delta < 1/48$, all the diffeomorphisms $\overline{g}_{(i)}, i \in \mathbb{N}^*$, are defined on the ball of radius $1/2$.

Similarly, it is also clear that elements in $S(i)$ converge uniformly to the identity on $B_{1/2}$. In fact, for $i \geq 3$ and $\overline{g}_{(i)} \in S(i)$, we have

$$\sup_{z \in B(1/2)} \| \overline{g}(z) - z \| \leq \frac{\delta}{2^i}.$$ 

Therefore, to obtain the desired sequence $g_i$, it suffices to select for every $i \in \mathbb{N}^*$ one diffeomorphism $g_i \in S(i)$ which is different from the identity. In view of Theorem 2.5, the sequence of sets $S(i)$ never degenerates into the identity alone so that the indicated choice of $g_i$ is always possible. The proof of the proposition is over.

The above argument suffices to imply Theorem B.
Proof of Theorem B Let then $G \subset \text{Diff}_1(\mathbb{C}^2, 0)$ be a given non-solvable group and again consider the sets $S(i)$ constructed above. Without loss of generality we can suppose that the sequence \( \{g_j\}_{j \in \mathbb{N}} \) actually forms an enumeration of the set \( \bigcup_{i=1}^{\infty} [S(i) \setminus \{\text{id}\}] \), where \( \text{id} \) stands for the identity map. In particular, it follows from the proof of Proposition 3.1 that all these local diffeomorphisms \( g_j \) are defined and one-to-one on the ball \( B(1/2) \) of radius 1/2 around the origin.

Now, consider the sets \( \text{Fix}_j \) given as
\[
\text{Fix}_j = \{ p \in B(1/2) ; \ g_i(p) = p \}.
\]

Let \( A_N = \bigcap_{j=N}^{\infty} \text{Fix}_j \) so that \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_N \cdots \subseteq B(1/2) \). For every fixed value of \( N \in \mathbb{N} \), note that the set \( A_N \) is a proper analytic subset of \( B(1/2) \) since it is given as a countable intersection of proper analytic subsets \( \text{Fix}_j \). Since the inclusion
\[
\Omega(G) \cap B(1/2) \subset \bigcup_{N=1}^{\infty} A_N
\]
clearly holds, the proof of Theorem B results at once. \( \square \)

Remark 3.2 In the Introduction we have claimed that a generic \( n \)-tuple, \( n \geq 2 \), of local diffeomorphisms in \( \text{Diff}_1(\mathbb{C}^2, 0) \) generates a subgroup \( G \subset \text{Diff}_1(\mathbb{C}^2, 0) \) whose set of non-recurrent points \( \Omega(G) \) is reduced to the origin. The purpose of this remark is to substantiate this claim by providing an accurate statement along with a detailed indication of proof.

For this, let \( n \geq 2 \) be fixed and consider the product \( (\text{Diff}_1(\mathbb{C}^2, 0))^n \) of \( n \) copies of \( \text{Diff}_1(\mathbb{C}^2, 0) \). Note that \( \text{Diff}_1(\mathbb{C}^2, 0) \), and hence \( (\text{Diff}_1(\mathbb{C}^2, 0))^n \), can be equipped with the Takens topology discussed in [18,23] so that these sets become Baire spaces. Now, there is a \( G_\delta \)-dense set \( \mathcal{U} \subset (\text{Diff}_1(\mathbb{C}^2, 0))^n \) whose points are \( n \)-tuples \( (F_1, \ldots, F_n) \) of diffeomorphisms in \( \text{Diff}_1(\mathbb{C}^2, 0) \) satisfying the following conditions:

- The subgroup \( \text{Diff}_1(\mathbb{C}^2, 0) \) generated by \( F_1, \ldots, F_n \) is isomorphic to the free group in \( n \) letters. In fact, the pseudogroup \( G \) generated on some fixed small neighborhood \( V \) of the origin is isomorphic to the free group in \( n \) letters in the following sense: consider an element \( F_{s_k} \circ \cdots \circ F_{s_1} \) of the pseudogroup \( G \), where each \( F_{s_i} \) lies in the set \( \{F_1^{\pm 1}, \ldots, F_n^{\pm 1}\} \) and assume that the resulting word does not represent the identity in the free group on \( n \) letters with the obvious identifications (in view of the first part of the statement, this is also equivalent to saying that the germ of \( F_{s_k} \circ \cdots \circ F_{s_1} \) at the origin does not coincide with the identity). Then the element \( F_{s_k} \circ \cdots \circ F_{s_1} \) of the pseudogroup \( G \) does not coincide with the identity on any connected component of its domain of definition.
- Every point \( P \) different from the origin is such that its stabilizer in the pseudogroup \( G \) is either trivial or infinite cyclic. In other words, consider all the elements \( F_{s_k} \circ \cdots \circ F_{s_1} \) in \( G \) such that \( F_{s_k} \circ \cdots \circ F_{s_1}(P) = P \). The germs of all these elements at \( P \) naturally form a group, and our assertion means that this group is either reduced to the identity or infinite cyclic.
Whereas [18,23] deal with local diffeomorphisms of \((\mathbb{C}, 0)\), as opposed to local diffeomorphisms of \((\mathbb{C}^2, (0, 0))\), the above claim is actually much easier to prove than the analogous statements in [18,23]. In fact, to establish the above assertions, every type of perturbation of an initial \(n\)-tuple \((F_1, \ldots, F_n)\) can be considered, while in [18,23] the construction of perturbations was constrained by the condition that they needed to preserve the analytic conjugation classes of the generators.

Finally, if \(G = \langle F_1, \ldots, F_n \rangle\) is as above, then it is clear that the set \(A_N\) is reduced to the origin for every \(N \in \mathbb{N}\). To check this assertion, consider a diffeomorphism \(g\) in some \(S(j_0)\) that is different from the identity and satisfies \(g(P) = P\). In view of the above conditions on the pseudogroup \(G\) and, in particular, the fact that its germ at the origin can be identified with the free group on \(n\) letters, there follows easily the existence of an unbounded sequence \(j_1, j_2, \ldots\) along with elements \(\overline{g}_{j_r}\) in \(S(j_r)\) which are not commensurable with \(g\) in the following sense: \(g\) and \(\overline{g}_{j_r}\) are not powers of the same element in \(G\). Here the reader is also reminded that the elements in \(S(j)\) are all defined on a uniform disc about the origin so that their domain of definition may indeed be seen as connected. Now owing to the second condition above involving stabilizers, it follows that \(\overline{g}_{j_r}(P) \neq P\). Therefore, \(A_N\) is reduced to the origin provided that \(N\) is large enough. Hence, the set \(\Omega(G)\) of non-recurrent points must be reduced to the origin as well.

In the preceding, the condition of having a group \(G\) constituted by diffeomorphisms tangent to the identity was important to fix an initial set of local diffeomorphisms sufficiently close to the identity on a fixed domain (the unit ball); cf. the proof of Proposition 3.1. Convergence of iterated commutators no longer holds when we work with diffeomorphisms that are allowed to have arbitrary linear parts. The proof of Theorem A will thus require a more elaborated discussion. We begin by pointing out another consequence of the proof of Proposition 3.1 that will be useful for the proof of Theorem A. This begins as follows.

**Lemma 3.3** A finitely generated pseudo-solvable subgroup of \(\text{GL}(2, \mathbb{C})\) is necessarily solvable.

**Proof** The analogous statement for subgroups of \(\text{GL}(2, \mathbb{R})\) was proven in [11]; the same argument applies to \(\text{GL}(2, \mathbb{C})\). \(\square\)

Now we have:

**Lemma 3.4** There is a neighborhood \(U\) of the identity matrix in \(\text{GL}(2, \mathbb{C})\) with the following property: assume that \(\Gamma \subset \text{GL}(2, \mathbb{C})\) is a non-solvable group generated by finitely many elements \(\gamma_1, \ldots, \gamma_s\) belonging to \(U\). Assume also that \(G \subset \text{Diff}(\mathbb{C}^2, 0)\) is generated by local diffeomorphisms \(f_1, \ldots, f_s\) with \(D_{(0,0)} f_i = \gamma_i\) for every \(i = 1, \ldots, s\). Then there is a neighborhood \(U\) of \((0, 0) \in \mathbb{C}^2\) and a sequence of elements \(\{g_i\}\) in the pseudogroup \(G\) generated by \(f_1, \ldots, f_s\) satisfying the following conditions:

- For every \(i \in \mathbb{N}\), \(g_i\) is defined on all of \(U\) and \(g_i \neq \text{id}\).
- The sequence \(g_i\) converges uniformly to the identity on \(U\).

**Proof** According to the proof of Proposition 3.1, there is \(\delta > 0\) such that the following holds: if \(S = S(0)\) is a finite set consisting of local diffeomorphisms that are \(\delta\)-close to
the identity on the unit ball $B_1$, then every sequence of elements $(\overline{g}_k)$, with $\overline{g}_k \in S(k)$ for every $k$, converges uniformly to the identity on the ball of radius $1/2$ (and in particular, all these diffeomorphisms are defined on the ball in question). Here the sets $S(k)$ are obtained as indicated in Definition 2.4. For a fixed value of $\delta$, the neighborhood $U$ of the identity matrix in $\text{GL}(2, \mathbb{C})$ is determined by letting

$$U = \{ \gamma \in \text{GL}(2, \mathbb{C}) : \sup_{z \in B_1} |\gamma \cdot z - z| < \delta/2 \}.$$ 

Having fixed the neighborhood $U$ of the identity matrix in $\text{GL}(2, \mathbb{C})$, suppose now that $\Gamma$ is a non-solvable group generated by elements $\gamma_1, \ldots, \gamma_s$ lying in $U$. Similarly, let $G \subset \text{Diff}(\mathbb{C}^2, 0)$ be the group generated by a set of local diffeomorphisms $f_1, \ldots, f_s$ such that $D(0,0)f_i = \gamma_i$ for every $i = 1, \ldots, s$. Up to changing coordinates by means of a suitable homothety, we can assume that $f_1, \ldots, f_s$ are defined on the unit ball of $\mathbb{C}^2$. Furthermore, in view of the definition of $U$ and since $D(0,0)f_i = \gamma_i$ with $\gamma_i \in U$, this homothety can be chosen so as to yield coordinates where we actually have

$$\sup_{z \in B_1} |f_i(z) - z| < \delta$$

for every $i = 1, \ldots, s$. Setting $S = \{ f_1, \ldots, f_s \}$ and considering the resulting sequence of sets $S(k)$, it follows that every sequence of diffeomorphisms $(\overline{g}_k)$ converges to the identity on the ball of radius $1/2$, provided that $\overline{g}_k \in S(k)$ for every $k$. Furthermore, since $\Gamma$ is non-solvable (and hence non-pseudo-solvable, cf. Lemma 3.3), every set $S(k)$ contains an element whose derivative at the origin is different from the identity so that this element itself is different from the identity. The lemma follows at once. \hfill \Box

We can now start the approach to the proof of Theorem A. We assume then that $G \subset \text{Diff}(\mathbb{C}^2, 0)$ is a subgroup with locally discrete orbits, and we need to show that $G$ is a virtually solvable group. Let $\rho$ be the homomorphism from $G$ to $\text{GL}(2, \mathbb{C})$ assigning to an element $g \in G$ its Jacobian matrix at the origin. Denoting by $\Gamma \subset \text{GL}(2, \mathbb{C})$ the image of $\rho$, consider the short exact sequence

$$0 \longrightarrow G_0 = \text{Ker}(\rho) \longrightarrow G \overset{\rho}{\longrightarrow} \Gamma \longrightarrow 0.$$ \hfill (3)

The kernel $G_0$ of $\rho$ consists of those elements in $G$ that are tangent to the identity. Since $G_0$ has locally discrete orbits, it follows from Proposition 3.1 that $G_0$ is solvable.

**Lemma 3.5** To prove Theorem A, it suffices to check that the group $\Gamma \subset \text{GL}(2, \mathbb{C})$ is virtually solvable.

**Proof** Suppose that $\Gamma$ is virtually solvable so that there is a normal, solvable subgroup $\Gamma_0 \subset \Gamma$ with finite index. Denote by $\xi$ the natural (projection) homomorphism from $\Gamma$ onto the finite group $\Gamma/\Gamma_0$ and consider the homomorphism $\xi \circ \rho : G \rightarrow \Gamma/\Gamma_0$. The
kernel $\text{Ker} (\xi \circ \rho)$ of $\xi \circ \rho$ is clearly a normal subgroup of $G$ having finite index since $\Gamma / \Gamma_0$ is finite. Moreover, this kernel is the extension of a solvable group by another solvable group (namely $\Gamma_0$ and $G_0$) and hence is itself a solvable group. Thus $G$ is virtually solvable and Theorem A is proved. \hfill \square

A last important result for the proof of Theorem A is Lemma 3.6 below. Although this lemma has an elementary nature, we will provide a detailed proof of it. In the course of this proof, we shall avoid using Tits’s alternative theorem which, in its best known form, asserts that a finitely generated subgroup of a linear group either is virtually solvable or contains a free subgroup on two generators; see [6, 30]. Whereas the use of Tits’s deep theorem would shorten the proof at certain points, this would come at the expense of making the argument less transparent, since many experts in dynamics are not familiar with the proof of Tits’s alternative. This is compounded by the fact that we are dealing only with the group of $2 \times 2$ matrices, so that a proof using only results that are well known to experts in dynamics can be obtained without making the discussion too long. Yet, the parts of the proof that can be overcome (or significantly shortened) by a direct application of Tits’s alternative will be clearly indicated in our discussion.

Before proceeding further, it is convenient to make a few general remarks aimed at showing that a given group $\Gamma \subset \text{GL} (2, \mathbb{C})$ is virtually solvable. Here Tits’s alternative would lead us directly towards the desired conclusions, though the extremely elementary nature of our remarks hardly justifies the use of Tits’s theorem. This said, consider first the surjective projection homomorphism $\sigma : \text{GL} (2, \mathbb{C}) \rightarrow \text{PSL} (2, \mathbb{C})$ which realizes $\text{GL} (2, \mathbb{C})$ as a central extension of $\text{PSL} (2, \mathbb{C})$, i.e., the kernel of $\sigma$ is contained in the center of $\text{GL} (2, \mathbb{C})$. The restriction of $\sigma$ to $\text{SL} (2, \mathbb{C}) \subset \text{GL} (2, \mathbb{C})$ will still be denoted by $\sigma$, and it also realizes $\text{SL} (2, \mathbb{C})$ as a central extension of $\text{PSL} (2, \mathbb{C})$. Now, given a subgroup $\Gamma \subset \text{GL} (2, \mathbb{C})$, in order to show that $\Gamma$ is virtually solvable, it suffices to check that $\sigma (\Gamma) \subset \text{PSL} (2, \mathbb{C})$ is virtually solvable. This is similar to Lemma 3.5: given a normal, solvable subgroup $H$ of $\sigma (\Gamma)$ with finite index, denote by $\pi$ the canonical projection $\pi : \sigma (\Gamma) \rightarrow \sigma (\Gamma)/H$ and consider the homomorphism $\pi \circ \sigma$ restricted to $\Gamma$. The kernel of $\pi \circ \sigma$ is clearly a normal subgroup of $\Gamma$ having finite index. The claim then follows from observing that $\text{Ker} (\pi \circ \sigma)$ must be a solvable group since $H$ is solvable and $\sigma$ realizes $\text{GL} (2, \mathbb{C})$ as a central extension of $\text{PSL} (2, \mathbb{C})$.

Let us now go back to $\Gamma \subset \text{GL} (2, \mathbb{C})$, which is the image by $\rho$ of the group $G \subset \text{Diff} (\mathbb{C}^2, 0)$. While $\Gamma$ is a subgroup of $\text{GL} (2, \mathbb{C})$, its standard action on $(\mathbb{C}^2, 0)$ has little to do with the action of $G$. In fact, if $\gamma$ is an element of $\Gamma$, then $\gamma$ is simply the derivative at the origin of an actual element $g \in G$ and it is $g$, rather than $\gamma$, that acts on $(\mathbb{C}^2, 0)$. Thus, the effect of the non-linear terms in $g$ must be taken into account in the following discussion. In this direction, we have the following:

**Lemma 3.6** Assume that the group $\Gamma$ is not virtually solvable. Then at least one of the following conditions holds:

1. There is a diffeomorphism $g \in G$ whose derivative at $(0, 0) \in \mathbb{C}^2$ is a hyperbolic saddle (i.e., its eigenvalues $\lambda_1$, $\lambda_2$ satisfy $0 < |\lambda_1| < 1 < |\lambda_2|$).

2. There is a sequence of elements $g_i$ in the pseudogroup $G$ satisfying the conclusions of Lemma 3.4.

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Proof Again consider the projection homomorphism $\sigma : \text{GL}(2, \mathbb{C}) \to \text{PSL}(2, \mathbb{C})$ as well as its restriction to $\text{SL}(2, \mathbb{C}) \subset \text{GL}(2, \mathbb{C})$, which realizes both $\text{GL}(2, \mathbb{C})$ and $\text{SL}(2, \mathbb{C})$ as central extensions of $\text{PSL}(2, \mathbb{C})$. As noted above, $\sigma(\Gamma) \subset \text{PSL}(2, \mathbb{C})$ is not virtually solvable since $\Gamma$ is by assumption not virtually solvable.

To describe our strategy for proving Lemma 3.6, we first consider the elements of $\text{PSL}(2, \mathbb{C})$ classified into elliptic, parabolic and loxodromic ones; see [2,10]. The reader will note that an element of $\text{GL}(2, \mathbb{C})$ having determinant equal to 1 and projecting on a loxodromic element of $\text{PSL}(2, \mathbb{C})$ must be the differential of an element in $G$ exhibiting a hyperbolic saddle at the origin of $\mathbb{C}^2$. In particular, Condition (1) in the statement holds, provided that we can find $\gamma \in D^1 \Gamma$ such that $\sigma(\gamma)$ is a loxodromic element of $\text{PSL}(2, \mathbb{C})$.

A similar observation concerning Condition (2) in Lemma 3.6 is as follows. Since $\Gamma$ is a countable subgroup of the Lie group $\text{GL}(2, \mathbb{C})$, the closure $\overline{\Gamma}$ of $\Gamma$ can be considered and $\overline{\Gamma} = \Gamma$ if and only if $\Gamma$ is discrete. Moreover, unless $\Gamma$ is discrete, $\overline{\Gamma}$ is itself a real Lie group admitting a non-trivial real Lie algebra. If this real Lie algebra is not solvable, then $\overline{\Gamma}$ contains elements $\gamma_1, \ldots, \gamma_s$ satisfying the assumptions of Lemma 3.4. The same conclusion holds for the group $\Gamma$ since $\Gamma$ is dense in $\overline{\Gamma}$ and the condition for a finite set to generate a non-solvable subgroup of $\text{GL}(2, \mathbb{C})$ is open. To check the latter claim, recall that a solvable subgroup of $\text{GL}(2, \mathbb{C})$ has solvable length at most 4; see, for example, [16]. Thus the condition for a subgroup of $\text{GL}(2, \mathbb{C})$ to be non-solvable can be expressed as the non-triviality of its fifth derived subgroup and hence it is an open condition. Therefore, Lemma 3.4 ensures that $G$ contains a sequence of elements satisfying Condition (2) in the statement, provided that the real Lie algebra associated with $\overline{\Gamma}$ is not solvable.

Summarizing the preceding, our proof of Lemma 3.6 is organized as follows. Aiming at a contradiction, we assume that no element in $D^1 \Gamma$ projects on a loxodromic element of $\text{PSL}(2, \mathbb{C})$. Furthermore, we also assume that the real Lie algebra associated with $\overline{\Gamma}$ is solvable, where it is understood that this Lie algebra is trivial (and hence solvable) if $\Gamma$ is discrete. From these two assumptions, we shall conclude that $\sigma(\Gamma)$ must be virtually solvable, hence deriving a contradiction that will complete the proof of the lemma.

To implement the above-mentioned strategy, it is natural to split the discussion into two cases according to whether or not $\sigma(\Gamma) \subset \text{PSL}(2, \mathbb{C})$ is a discrete subgroup of $\text{PSL}(2, \mathbb{C})$.

Case A Suppose that $\sigma(\Gamma)$ is not discrete.

The closure of the subgroup $\sigma(\Gamma)$ in $\text{PSL}(2, \mathbb{C})$ possesses a non-trivial real Lie algebra which will be denoted by $\mathcal{L}_{PSL}$. We then have:

Claim 1 The algebra $\mathcal{L}_{PSL}$ is solvable.

Proof of the Claim 1 Aiming at a contradiction, assume that $\mathcal{L}_{PSL}$ is not solvable. Then the Lie algebra $D^1 \mathcal{L}_{PSL}$ associated with $D^1(\sigma(\Gamma))$ is non-trivial and not solvable. This Lie algebra is, however, isomorphic to the Lie algebra $D^1 \mathcal{L}$ associated with $D^1(\overline{\Gamma}) \subset \text{SL}(2, \mathbb{C})$ since $\text{SL}(2, \mathbb{C})$ is a double covering of $\text{PSL}(2, \mathbb{C})$. Thus the Lie algebra associated with $D^1(\overline{\Gamma})$, and hence the Lie algebra associated with $\overline{\Gamma}$, is not solvable. The resulting contradiction establishes the claim. \qed
The Lie algebra $\mathcal{L}_{PSL}$ is therefore solvable. Clearly, $\mathcal{L}_{PSL}$ is also invariant by all elements in $\sigma(\Gamma)$. In the sequel PSL $(2, \mathbb{C})$ will often be identified with the corresponding automorphism group of the Riemann sphere $S^2$. With this identification, the exponential of the Lie algebra $\mathcal{L}_{PSL}$ is a solvable, connected subgroup $\text{Exp}(\mathcal{L}_{PSL})$ of the automorphism group of the Riemann sphere. This subgroup is not reduced to the identity since $\mathcal{L}_{PSL}$ is not trivial which, in fact, ensures that $\text{Exp}(\mathcal{L}_{PSL})$ must contain a real one-parameter subgroup (i.e., a “flow”). Note that, in principle, $\sigma(\Gamma)$ need not be connected so that we cannot yet derive a contradiction. It turns out, however, that the solvable, connected subgroup $\text{Exp}(\mathcal{L}_{PSL})$ must have a fixed point (as an elementary particular case of Borel’s theorem, see [28]).

On the other hand, recall that a real one-parameter subgroup of PSL $(2, \mathbb{C})$ must have at least one and at most two fixed points in $S^2$. In other words, the set of fixed points of $\text{Exp}(\mathcal{L}_{PSL})$ is non-empty and contains at most two points. Now note that the set formed by these fixed points is necessarily invariant under $\sigma(\Gamma)$ since $\mathcal{L}_{PSL}$ is invariant under $\sigma(\Gamma)$ (alternatively, $\text{Exp}(\mathcal{L}_{PSL})$ is normal in $\sigma(\Gamma)$ since it coincides with the connected component of $\sigma(\Gamma)$ containing the identity). Summarizing the preceding, up to passing to a (necessarily normal) subgroup of $\sigma(\Gamma)$ having index 2, we can assume the existence of $p \in S^2$ fixed by all elements in $\sigma(\Gamma)$. The subgroup of PSL $(2, \mathbb{C})$ fixing a given point in $S^2$ is, however, conjugate to the affine group of $\mathbb{C}$. In particular, all these groups are solvable. Summarizing, either $\sigma(\Gamma)$ embeds into a solvable group or it has an index 2 subgroup that does. Since index 2 subgroups are always normal, we conclude that $\sigma(\Gamma)$ is necessarily virtually solvable (in fact, it contains a normal, solvable subgroup of index 2). The resulting contradiction establishes the lemma in this first case.

Case B Suppose that $\sigma(\Gamma)$ is discrete.

A short argument relying on Tits’s alternative is as follows. Assume that it is not virtually solvable, so that it contains a free subgroup on two generators. Now note that the subgroup $\sigma(D^1 \Gamma) = D^1(\sigma(\Gamma))$ is discrete, since it is contained in the discrete group $\sigma(\Gamma)$. It is therefore a Kleinian group which, in addition, cannot be an elementary Kleinian group, since $\Gamma$ contains a free subgroup on two generators (and hence so does $D^1 \Gamma$). A non-elementary Kleinian group, however, always contains loxodromic elements, and a contradiction completing the proof of the lemma follows at once; see [2].

A more self-contained proof avoiding the use of Tits’s alternative can be obtained as follows. First, we can assume that $\sigma(\Gamma)$ is not a finite group, otherwise the kernel of $\sigma$ is an abelian (normal) subgroup of finite index and the desired contradiction arises immediately. Therefore, $\sigma(\Gamma)$ is assumed to be a finitely generated, infinite group in what follows. Being finitely generated and infinite, a result of Schur [31] asserts that $\sigma(\Gamma)$ contains an element of infinite order which will be denoted by $\sigma(\gamma)$ for some $\gamma \in \Gamma$ (again, this can be viewed as a corollary of Tits’s alternative, though Schur’s result is much more elementary). The element $\sigma(\gamma)$ is either parabolic or loxodromic, since $\sigma(\Gamma)$ is supposed to be discrete.

As already seen, the subgroup $\sigma(D^1 \Gamma) = D^1(\sigma(\Gamma))$ is discrete, and it is an elementary Kleinian group since it does not contain loxodromic elements. Moreover, we have:

Claim 2 If the group $\sigma(D^1 \Gamma)$ is not solvable, then either this group is finite or contains an element of infinite order.
Proof of Claim 2 The only difficulty in again applying Schur’s lemma [31] to conclude the statement is to ensure that $\sigma(D^1\Gamma) \subset \text{PSL}(2, \mathbb{C})$ is finitely generated (our assumption only ensures that $\Gamma$ is finitely generated). To overcome this difficulty, suppose that $\sigma(D^1\Gamma)$ is an infinite group and consider an enumeration $\gamma_1, \gamma_2, \ldots$ of its elements. Consider then the groups $(\sigma(D^1\Gamma))_n$ generated by $\gamma_1, \ldots, \gamma_n$. All the groups $(\sigma(D^1\Gamma))_n$ are finitely generated so that Schur’s lemma applies to ensure the existence of an element of finite order unless all these groups are finite. We assume then that this is the case.

Next we recall that finite subgroups of $\text{PSL}(2, \mathbb{C})$ were classified since Klein, and apart from cyclic groups and dihedral groups, there are only finitely many of them (in correspondence with the Platonic solids; see, for example, [13]). Thus, for $n$ large enough, every group $(\sigma(D^1\Gamma))_n$ must be either cyclic or dihedral. Therefore, all these groups are abelian or metabelian, i.e., their derived length is at most 2. This clearly implies that $\sigma(D^1\Gamma)$ is solvable. The resulting contradiction proves the claim. \[ \Box \]

Naturally, we can assume $\sigma(D^1\Gamma)$ to be non-solvable, otherwise $\sigma(\Gamma)$ is solvable itself. Assume also that $\sigma(\Gamma)$ is not finite and consider an element in $\sigma(D^1\Gamma)$ having infinite order. This element must be parabolic since elliptic and loxodromic elements are excluded (the existence of an elliptic element with infinite order would force the group $\sigma(D^1\Gamma)$ to be non-discrete). Elementary Kleinian groups containing parabolic elements are described in [10], and these groups possess a fixed point in $S^2$. Therefore, they are solvable as they can be realized as a subgroup of the affine group of $\mathbb{C}$. Thus we conclude $\sigma(D^1\Gamma)$ must be finite unless $\sigma(D^1\Gamma)$, and hence $\sigma(\Gamma)$, is solvable. Since $\sigma(D^1\Gamma)$ is finite, it follows that $\sigma(\Gamma)$ is amenable as a finite (and hence amenable) extension of an abelian (and hence amenable) group; cf. [9].

Summarizing the preceding, the group $\sigma(\Gamma)$ is amenable. Moreover, it was already seen that $\sigma(\Gamma)$ is a finitely generated, infinite group so that Schur’s lemma ensures it must contain an element of infinite order $\sigma(\gamma)$. In turn, $\sigma(\gamma)$ is either parabolic or loxodromic since $\sigma(\Gamma)$ is assumed to be discrete. To complete the proof of the lemma, we now proceed as follows. The action of $\sigma(\Gamma)$ on $S^2$ must preserve a probability measure $\mu$ since this group is amenable; see [9]. In particular, $\mu$ must be invariant by $\sigma(\gamma)$. Now we have:

- Suppose that $\sigma(\gamma)$ is parabolic.

The only probability measure preserved by a parabolic element (with infinite order) is the Dirac mass concentrated at the unique fixed point for the element in question. In other words, there is a point $p \in S^2$ which is fixed by the entire group $\sigma(\Gamma)$. Therefore, $\sigma(\Gamma)$ is solvable as it is conjugate to a subgroup of the affine group of $\mathbb{C}$. The desired contradiction follows at once.

- Suppose that $\sigma(\gamma)$ is loxodromic.

A loxodromic element of $\text{PSL}(2, \mathbb{C})$ has exactly two fixed points $p_1$ and $p_2$ in $S^2$. Furthermore, the only probability measures invariant under these elements are the convex combinations of Dirac masses concentrated at $p_1$ and at $p_2$. Hence, the set $\{p_1, p_2\}$ must be invariant by $\sigma(\Gamma)$. If one of these two points is fixed by all of...
\(\sigma(\Gamma)\), then we conclude as in the previous case that \(\sigma(\Gamma)\), and hence \(\Gamma\), is solvable. A contradiction then results.

Finally, in the general case, \(\sigma(\Gamma)\) contains a normal subgroup \([\sigma(\Gamma)]_{p_1}\) of index two fixing \(p_1\). Again, \([\sigma(\Gamma)]_{p_1}\) must be solvable. Thus \(\sigma(\Gamma)\) is virtually solvable. This implies that \(\Gamma\) is virtually solvable and provides the final contradiction, ending the proof of Lemma 3.6.

\(\square\)

**Proof of Theorem A** Again consider the short exact sequence (3). To prove Theorem A we will assume that \(\Gamma\) is not virtually solvable and derive from this the existence of recurrent points. Since \(\Gamma\) is not virtually solvable, the alternative provided by Lemma 3.6 holds. However, if \(G\) actually contains a sequence \(\{g_i\}\) of elements as in Condition (2), then the existence of the mentioned recurrent points follows at once from the argument employed in the proof of Theorem B. Therefore, in order to prove Theorem A, we can assume without loss of generality the existence of an element \(g \in G\) whose derivative \(D_0 g\) at the origin is a hyperbolic saddle as indicated in Condition (1) of Lemma 3.6. In fact, we can assume that \(\sigma(\Gamma)\) is a non-elementary Kleinian group. Moreover, we can also assume that the Jacobian determinant of \(D_0 g\) equals 1 since the preceding Lemma 3.6 actually ensures that the element in Condition (1) can be assumed to belong to \(D^1 \Gamma\). In this respect, however, the only role played by the fact that the Jacobian determinant of \(D_0 g\) equals 1 in the discussion below consists of helping us to abridge notation, as the reader will not fail to notice.

The eigenvalues of \(D_0 g\) at the origin are then denoted by \(\lambda\) and by \(\lambda^{-1}\), with \(|\lambda| > 1\). It follows that \(g\) has a hyperbolic fixed point at the origin with stable and unstable manifolds, \(W^s_g\), \(W^u_g\), having complex dimension 1 and intersecting transversely at \((0, 0) \in \mathbb{C}^2\). Fix then a closed annulus \(A^s \subset W^s_g\) (resp., \(A^u \subset W^u_g\)) with radii \(r_2 > r_1 > 0\) such that every point \(p \in W^s_g\) (resp., \(p \in W^u_g\)) possesses an orbit by \(g\) non-trivially intersecting \(A^s\) (resp., \(A^u\)).

Given a point \(p\) in a fixed neighborhood \(U\) of the origin, denote by \(O_G(p)\) the orbit of \(p\) by the pseudogroup \(G\). Similarly, let \(\text{Acc}_p(G)\) denote the set of *ends* of \(O_G(p)\). To define this set, we consider the closure \(\overline{O_G(p)}\) of the orbit \(O_G(p)\). We then set \(\text{Acc}_p(G) = \overline{O_G(p)} \setminus O_G(p)\), i.e., \(\text{Acc}_p(G)\) is the closure of the difference \(\overline{O_G(p)} \setminus O_G(p)\). Concerning the set \(\text{Acc}_p(G)\), note that a point \(p\) having locally finite orbit is an isolated point of both \(O_G(p)\) and \(\overline{O_G(p)}\). Hence, such a point does not belong to either \(\overline{O_G(p)} \setminus O_G(p)\) or \(\text{Acc}_p(G)\). Conversely, if \(p\) is recurrent (i.e., its orbit is not locally finite), then no point in \(O_G(p)\) is an isolated point of this set. Hence, \(\overline{O_G(p)}\) is a closed set without isolated points and, in particular, it is not countable (indeed locally not countable). Since \(O_G(p)\) is countable, \(p\) is still accumulated by points in \(\overline{O_G(p)} \setminus O_G(p)\) so that \(p\) lies in \(\text{Acc}_p(G) = \overline{O_G(p)} \setminus O_G(p)\). Summarizing, \(p\) is a recurrent point if and only if \(p \in \text{Acc}_p(G)\). Furthermore, \(\text{Acc}_p(G) = \emptyset\) provided that \(O_G(p)\) is finite. Clearly, \(\text{Acc}_p(G)\) is closed and invariant by \(G\) (viewed as a pseudogroup). The following claim is the key for the proof of Theorem A.

**Claim** For every point \(p \in A^s\), the closed set \(A^s \cap \text{Acc}_p(G)\) is not empty.

Note that the claim does not immediately imply Theorem A, for it does not assert that \(p\) itself belongs to \(A^s \cap \text{Acc}_p(G)\). However, if this were the case, then clearly the
The resulting contradiction shows that $K$ is relatively invariant by the pseudogroup $G$ if, for every point $p \in K$ and every point $q \in A^s \cap \text{Acc}_p(G)$, the point $q$ lies in $K$ as well. Next, let $\mathcal{C}$ denote the collection of non-empty closed sets in $A^s$ that are relatively invariant by the pseudogroup $G$. The above claim ensures that the collection $\mathcal{C}$ is non-empty. In fact, $A^s \cap \text{Acc}_p(G)$ in a non-empty set relatively invariant under $G$, and thus $A^s \cap \text{Acc}_p(G)$ belongs to $\mathcal{C}$ for every $p \in A^s$. Now, let the collection $\mathcal{C}$ be endowed with the partial order defined by inclusion. Finally, given a sequence $K_1 \supset K_2 \supset \cdots$ of sets in $\mathcal{C}$, the intersection $K_\infty = \bigcap_{i=1}^{\infty} K_i$ is non-empty since each $K_i$ is compact (closed and contained in the compact set $A^s$). The set $K_\infty$ is clearly closed and relatively invariant by $G$ so that it belongs to $\mathcal{C}$. Moreover, we have $K_\infty \subset K_i$ for every $i$, i.e., in terms of the fixed partial order $K_\infty$ is smaller than $K_i$ for every $i$. According to Zorn’s lemma, the collection $\mathcal{C}$ contains minimal elements, so that we can consider a minimal element $K$. Choose then $q \in K$ and consider the non-empty set $A^s \cap \text{Acc}_q(G)$. If $q \notin \text{Acc}_q(G)$, then $A^s \cap \text{Acc}_q(G)$ would be an element of $\mathcal{C}$ strictly smaller than $K$. The resulting contradiction shows that $q \in A^s \cap \text{Acc}_q(G)$ and finishes the proof of Theorem A.

It only remains to prove the Claim.

Proof of the Claim Recall that $A^s \subset W^s_g$ (resp., $A^u \subset W^u_g$) is an annulus such that every $p \in W^s_g$ (resp., $p \in W^s_g$) possesses an orbit by $g$ non-trivially intersecting $A^s$ (resp., $A^u$).

Now consider another element $\overline{g} \in G$ whose Jacobian matrix at the origin defines a hyperbolic saddle with determinant equal to 1. Again, stable and unstable manifolds for $\overline{g}$ will respectively be denoted by $W^s_{\overline{g}}$, $W^u_{\overline{g}}$. Since a (non-elementary) Kleinian group contains “many” loxodromic elements (including conjugates of $g$), the element $\overline{g}$ can be chosen so that all four invariant manifolds $W^s_{\overline{g}}$, $W^u_{\overline{g}}$, $W^s_g$, $W^u_g$ intersect pairwise transversely at the origin. The previously fixed annuli $A^s_g \subset W^s_g$ and $A^u_g \subset W^u_g$ will be denoted in the sequel by $A^s_{\overline{g}}$ and $A^u_{\overline{g}}$. An annulus $A^s_{\overline{g}} \subset W^s_{\overline{g}}$ (resp., $A^u_{\overline{g}} \subset W^u_{\overline{g}}$) with analogous properties concerning $\overline{g}$ is also fixed. To prove the claim it suffices to check that every point $p$ in $A^s_{\overline{g}}$ is such that $A^u_{\overline{g}} \cap \text{Acc}_p(G) \neq \emptyset$. Indeed, let $p^* \in A^u_{\overline{g}}$ be a point in $A^u_{\overline{g}} \cap \text{Acc}_p(G)$. The analogous argument changing the roles of $g$, $\overline{g}$ and replacing them by their inverses will ensure that $A^s_g \cap \text{Acc}_{p^*}(G) \neq \emptyset$. Since $p^*$ lies in $\text{Acc}_p(G)$ and this set is invariant under the pseudogroup $G$, it will follow that $A^s_g \cap \text{Acc}_p(G) \neq \emptyset$ as desired.

Finally, to check that $A^u_{\overline{g}} \cap \text{Acc}_p(G) \neq \emptyset$ for every point $p \in A^s_g$, we proceed as follows. Consider local coordinates $(x, y)$ about the origin of $\mathbb{C}^2$ so that $\{x = 0\} \subset W^u_g$ and $\{y = 0\} \subset W^s_g$. Recall that $W^s_g$ is smooth and intersects the coordinate axes transversely at the origin. Since this intersection is transverse, we can assume that it is the only intersection point of $W^s_g$ with the coordinate axes. In particular, a point $p \in A^s_g$ has coordinates $(u, v)$ with $u.v \neq 0$. By iterating $g$, we can find points $p_n = (u_n, v_n) = g^n(p) \in \mathbb{C}^2$ such that $|u_n| \to 0$ and
for some uniform constant $C$ related to the “angles” between $W_s^g$ and the coordinate axes at the origin. Now, for every $n$, consider the points of the form $\bar{g}(p_n), \ldots, \bar{g}^{(l(n))}(p_n)$ where $l(n)$ is the smallest positive integer for which the absolute value of the second component of $\bar{g}^{(l(n))}(p_n)$ is greater than $\sup_{z \in A_u^n} |z|$. The integer $l(n)$ exists since $\bar{g}$ has a hyperbolic fixed point at the origin and the action of $\bar{g}$ on $p_n$ is such that the first coordinate becomes smaller and smaller while the second coordinate gets larger and larger. Now it is clear that the closure of the set $\bigcup_{n=1}^{\infty} \{\bar{g}(p_n), \ldots, \bar{g}^{(l(n))}(p_n)\}$ intersects $A_u^n$ non-trivially, and this ends the proof of the Claim. The proof of Theorem A is completed as well. \hfill \Box

Let us close this section by showing how to extend Theorems A and B to encompass groups that are infinitely generated.

**Theorem 3.7** Theorems A and B remain valid for infinitely generated groups.

**Proof** We begin by justifying the case of Theorem B. We consider a finitely generated subgroup $H$ of $G$. If $H$ is not solvable, then $H$, and in particular $G$, has recurrent orbits away from a countable union of proper analytic sets. Thus we can assume that $H$ is solvable. Owing to [16], it follows that $D^3H = \{\text{id}\}$. In other words, the third derived group of every finitely generated subgroup of $G$ is trivial. We then conclude that $D^3G$ must be reduced to the identity as well, and this yields a contradiction proving our claim.

Concerning Theorem A, as has been the case before, we shall avoid using Tits’s alternative (in its version for infinitely generated groups). We then need to revisit the argument provided above. Given a finitely generated subgroup $H$ of Diff $(\mathbb{C}^2, 0)$, let $\Gamma_H \subset \text{GL}(2, \mathbb{C})$ denote the image of $H$ by $\rho$. Assuming that $H$ has locally discrete orbits, Theorem A ensures that $\Gamma_H$ possesses a normal, solvable subgroup $\Gamma_{H_0}$ having finite index. Moreover, we have seen that $H$ itself possesses a normal, solvable subgroup $H_0$ whose index equals the index of $\Gamma_{H_0}$ in $\Gamma_H$. A careful reading of the proof of Lemma 3.6 shows that the group $\Gamma_{H}$ possesses an index 2 (normal) solvable subgroup unless $\Gamma_{H}$ is a finite group. As mentioned, setting aside abelian and metabelian groups, there are only a finite number of finite subgroups of $\text{PSL}(2, \mathbb{C})$; see [13]. A similar remark applies to subgroups of $\text{GL}(2, \mathbb{C})$. Putting everything together, we conclude that all finitely generated groups $H \subset \text{Diff}(\mathbb{C}^2, 0)$ having locally discrete orbits possess a normal, solvable subgroup whose index is finite and, indeed, uniformly bounded by some constant $C$ whose exact value is not important for us. On the other hand, again owing to the main results in [16], we know that every solvable subgroup of $\text{Diff}(\mathbb{C}^2, 0)$ has derived length bounded by 5. Thus, we finally conclude that every finitely generated subgroup $H \subset G$ possesses a normal subgroup with index bounded by $C$ which has derived length no greater than 5. It follows that $G$ itself possesses a normal subgroup of index less than $C$ whose derived length is no greater than 5. In particular, this subgroup is solvable and the statement results. \hfill \Box
4 Discrete Subgroups of Diff \((\mathbb{C}^2, 0)\), Examples and Complements

Consider a finitely generated subgroup \(G \subset \text{Diff}(\mathbb{C}^n, 0)\). Up to choosing representatives for elements of \(G\) in some finite generating set, the dynamics of \(G\) is identified with the dynamics of the corresponding pseudogroup on a small neighborhood of the origin. By a standard abuse of notation, we shall also identify \(G\) and the mentioned pseudogroup. The following definition is very natural.

**Definition 4.1** The group \(G \subset \text{Diff}(\mathbb{C}^n, 0)\) is said to be non-discrete if there is an open neighborhood \(V \subseteq \mathbb{C}^n\) of the origin and a sequence of elements \(\{g_j\} \subseteq G\) satisfying the following conditions:

1. For every \(j \in \mathbb{N}\), the set \(V\) is contained in the domain of definition of \(g_j\) viewed as an element of the pseudogroup \(G\).
2. For every \(j \in \mathbb{N}\), the restriction of \(g_j\) to \(V\) does not coincide with the identity.
3. The sequence \(\{g_j\}\) converges uniformly to the identity on compact parts of \(V\).

The above definition clearly makes sense in terms of germs since it does not depend on the set of representatives chosen. The definition can be made more global at the expense of considering pseudogroups acting on open sets of \(\mathbb{C}^n\), whether or not the origin is fixed. In this sense, the pseudogroup \(G\) generated by a (finite) collection of holomorphic diffeomorphisms defined around the origin will be called **globally non-discrete** if and only if there is a non-empty open set \(V\) satisfying the conditions (1), (2) and (3) of Definition 4.1.

For pseudogroups \(G\) as above, the definition below is also standard by now.

**Definition 4.2** An analytic vector field \(X\) defined on a non-empty open set \(U\) is said to be in the closure of \(G\) if the following condition is satisfied, up to reducing \(U\): for every set \(U' \subset U\) and every \(t_0 \in \mathbb{R}_+\) so that the local flow of \(X\) is defined on \(U'\) for every \(t \in [0, t_0]\), the resulting local diffeomorphism \(\Psi^t_X : U' \to \mathbb{C}^n\) induced by this local flow is the uniform limit on \(U'\) of a sequence of elements \(\{g_j\}\) contained in \(G\).

In the case \(n = 1\), it is a simple fact that a non-solvable subgroup of \(\text{Diff}(\mathbb{C}, 0)\) is always non-discrete. Indeed, this statement can be checked by specifying to the one-dimensional case the results in the previous section valid for \(n = 2\). This phenomenon is in line with the general character of Shcherbakov–Nakai theory in \(\text{Diff}(\mathbb{C}, 0)\) asserting the existence of non-identically zero vector fields in the closure of these groups. A sort of converse also holds in the sense that a globally discrete pseudogroup cannot admit non-identically zero vector fields in its closure since the local flow \(\Psi^t_X\) converges to the identity on compact parts of \(U\) as \(t \to 0_+\). Thus the first fundamental issue opposing subgroups of \(\text{Diff}(\mathbb{C}, 0)\) to subgroups of \(\text{Diff}(\mathbb{C}^n, 0), n \geq 2\), is the fact that the latter contains globally discrete free subgroups on two generators.

**Example 1** (Schottky groups and discrete subgroups of \(\text{PSL}(2, \mathbb{C})\)) Consider a Schottky subgroup \(\Gamma\) of \(\text{PSL}(2, \mathbb{C})\). The group \(\Gamma\) is free on 2 or more generators and \(\Gamma\) is also discrete as a subgroup of \(\text{PSL}(2, \mathbb{C})\) in the classical sense (i.e., it forms a discrete set in \(\text{PSL}(2, \mathbb{C})\)). Once a lift of \(\text{PSL}(2, \mathbb{C})\) in \(\text{SL}(2, \mathbb{C})\) is chosen, \(\Gamma\) can be identified with a subgroup of \(\text{SL}(2, \mathbb{C})\). Since, in turn, \(\text{SL}(2, \mathbb{C})\) can be viewed as linear
diffeomorphisms of \( C^2 \) fixing the origin, it follows that \( \Gamma \) can also be identified with a certain subgroup of \( \text{Diff} (C^2, 0) \). The purpose of this example is to prove the following statement, which does not depend on the chosen lift of \( \text{PSL} (2, \mathbb{C}) \) in \( \text{SL} (2, \mathbb{C}) \).

**Claim** The group \( \Gamma \subseteq \text{Diff} (C^2, 0) \) is globally discrete in the sense of our previous definition.

**Proof of the Claim** This is certainly a well-known result, so that we shall content ourselves with sketching an argument. Aiming at a contradiction, assume that \( \Gamma \) is not globally discrete. Then there exists a non-empty open set \( V \subset C^2 \) and a sequence of elements \( \{ \gamma_i \} \subset \Gamma \subset \text{Diff} (C^2, 0) \) converging uniformly to the identity on \( V \) (where \( (0, 0) \notin V \)). Since the action of \( \Gamma \) on \( C^2 \) is linear, it induces a projective action of \( \Gamma \) on \( \mathbb{C} P(1) \) coinciding with the action induced by identifying \( \text{PSL} (2, \mathbb{C}) \) with the automorphism group of \( \mathbb{C} P(1) \). In particular, up to fixing three pairwise different points \( P_1, P_2, \) and \( P_3 \) in the image of the projection of \( V \) in \( \mathbb{C} P(1) \), it follows that the image of the sequence \( \{ \gamma_i \} \) in \( \text{PSL} (2, \mathbb{C}) \) (still denoted by \( \{ \gamma_i \} \)) is such that \( \gamma_i(P_1) \rightarrow P_1, \gamma_i(P_2) \rightarrow P_2, \) and \( \gamma_i(P_3) \rightarrow P_3 \). Since a projective transformation of \( \mathbb{C} P(1) \) is determined by the image of three points, it is a well-known and easy-to-check fact that the sequence \( \{ \gamma_i \} \subset \text{PSL} (2, \mathbb{C}) \) converges to the identity in the standard topology of \( \text{PSL} (2, \mathbb{C}) \) obtained through its identification with \( \text{SO} (3, \mathbb{R}) \times B^3 \), where \( B^3 \) stands for the unit ball of \( \mathbb{R}^3 \). The resulting contradiction proves the claim. \( \square \)

On the other hand, the results in Sect. 3 also show that every non-solvable subgroup of \( \text{Diff}_1 (C^2, 0) \) is non-discrete in the sense of Definition 4.1 (i.e., for a chosen neighborhood of the origin). At this level, there is no known obstruction to the existence of vector fields in the closure of these non-solvable groups, though no general affirmative result is so far available. Inasmuch as no “counterexample” is known, it seems a bit unlikely that non-trivial vector fields in the closure of the corresponding group will exist without any (at least weak) additional assumption.

Going back to subgroups of \( \text{Diff} (C^n, 0) \), the notion of *global non-discrete* is less suited than the notion of a non-discrete set forth by Definition 4.1, since the former depends on the representatives chosen. Actually, even for a given finite set of local diffeomorphisms (fixing the origin), it may happen that the pseudogroup they generate on an open set \( U \) is non-discrete, while it becomes discrete on a smaller open set. Furthermore, from a technical point of view, the effects of non-linear terms away from the origin can easily become out of control. Let us close this discussion with a remark showing that “many” discrete subgroups of \( \text{Diff} (C^2, 0) \) can be produced by “higher order perturbations” of discrete subgroups of \( \text{GL} (2, \mathbb{C}) \).

**Example 2** (Non-linear perturbations of discrete subgroups of \( \text{GL} (2, \mathbb{C}) \)) Given a subgroup \( G \subset \text{Diff} (C^2, 0) \), again consider the natural homomorphism \( \rho : G \rightarrow \text{GL} (2, \mathbb{C}) \) and the associated exact sequence

\[
0 \longrightarrow \text{Ker} (\rho) \longrightarrow G \longrightarrow \rho(G) \subset \text{GL} (2, \mathbb{C}) \longrightarrow 0 ,
\]

where \( \rho(g) \) is the derivative \( D_0 g \) at the origin. Then we have:
Claim Suppose that $\rho(G) \subset \text{GL}(2, \mathbb{C})$ is a discrete subgroup and that $\text{Ker}(\rho) \subset \text{Diff}_1(\mathbb{C}^2, 0)$ is discrete as well (this happens, for example, when the homomorphism $\rho$ is one-to-one). Then $G$ is discrete.

Proof Suppose that \{${g_j}$\}, $g_j \neq \text{id}$ for every $j \in \mathbb{N}$, is a sequence of elements in $G$ converging uniformly to the identity on some neighborhood $V$ of $(0, 0) \in \mathbb{C}^2$. Then the sequence of derivatives $\{D_0g_j\} \subset \text{GL}(2, \mathbb{C})$ must converge to the identity matrix $I$ by virtue of the Cauchy formula. Since $\rho(G) \subset \text{GL}(2, \mathbb{C})$ is discrete, it follows that $D_0g_j$ equals $I$ for large $j \in \mathbb{N}$. Hence, modulo dropping finitely many terms in the mentioned sequence, we have $g_j \in \text{Ker}(\rho)$ for every $j \in \mathbb{N}$. A contradiction then arises from the fact that $\text{Ker}(\rho) \subset \text{Diff}_1(\mathbb{C}^2, 0)$ is discrete. The claim is proved. $\square$

5 Lie Algebras for Subgroups of $\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ and Additional Formal Computations

As stated in the Introduction, the remainder of the paper is totally devoted to the proof of Theorem 2.5. In order to prove that a pseudo-solvable group is, indeed, solvable there is a standard strategy which was put forward in [11]. This is as follows. Consider a pseudo-solvable group $G$ along with a finite generating set $S = S(0)$ leading to a sequence of sets $S(j)$ that degenerates into $\{\text{id}\}$ for large enough $j \in \mathbb{N}$. Denote by $G(j)$ (resp., $G(j, j - 1)$) the subgroup generated by $S(j)$ (resp., $S(j) \cup S(j - 1)$). Let $k$ be the largest integer for which $S(k)$ is not reduced to the identity. It then follows that $G(k)$ is abelian. Similarly, the group $G(k, k - 1)$ is solvable. In particular, the smallest integer $m$ for which $G(m, m - 1)$ is solvable can be considered. Furthermore, $m = 1$ means that the initial group $G$ is solvable and hence there is nothing else to be proved. Suppose then that $m \geq 2$ and note that every element $F$ in $S(m - 2)$ satisfies the condition

$$F^{\pm 1} \circ G(m - 1) \circ F^{\mp 1} \subset G(m, m - 1).$$

(4)

To derive a contradiction with the fact that $m \geq 2$ (so that $G$ is not solvable), we only need to show that $G(m - 1, m - 2)$ must be solvable as well. In other words, we need to show that the group generated by

$$G(m, m - 1) \cup S(m - 2)$$

is still solvable. To establish this statement, we are, however, allowed to exploit the assumption that the elements $F$ of $S(m - 2)$ satisfy the condition expressed in Eq. (4), where $G(m - 1)$ and $G(m, m - 1)$ are both solvable groups with $G(m) \subset G(m, m - 1)$. Furthermore, neither $G(m - 1)$ nor $G(m)$ is reduced to $\{\text{id}\}$. Indeed, we can be slightly more precise by saying that for every $F \in S(m - 2)$ and $g \in S(m - 1)$, we have

$$F \circ g \circ F^{-1} = g \circ \overline{g},$$

(5)

for some $\overline{g} \in S(m)$. In any event, we are then led to investigate the structure of the solutions “$F$” of the functional relation expressed by (4). Besides, and inasmuch
as we shall apply Theorem 2.5 only to subgroups of $\text{Diff}_1(\mathbb{C}^2, 0)$, the issue about convergence of power series will play no role in the course of the discussion. This explains why Theorem 2.5 is stated for formal subgroups of $\text{Diff}_1(\mathbb{C}^2, 0)$. In the present case, both $G(m-1)$ and $G(m, m-1)$ are subgroups of $\text{Diff}_1(\mathbb{C}^2, 0)$, and this explains why the implementation of the above-mentioned strategy requires detailed information on solvable subgroups of $\text{Diff}_1(\mathbb{C}^2, 0)$. At this point, we shall have occasion to take advantage of the results established in [16].

To effectively begin our discussion, let $\mathbb{C}[[x, y]]$ denote the ring of formal series in two variables and without component of degree zero. Consider the group $\widehat{\text{Diff}_1}(\mathbb{C}^2, 0)$ of formal diffeomorphisms of $(\mathbb{C}^2, 0)$ that are tangent to the identity. Consider also the set $\widehat{\mathcal{X}}$ of formal vector fields at $(\mathbb{C}^2, 0)$ so that every element (formal vector field) in $\widehat{\mathcal{X}}$ has the form $a(x, y)\partial/\partial x + b(x, y)\partial/\partial y$, where $a, b \in \mathbb{C}[[x, y]]$. The space of formal vector fields whose first jet at the origin vanishes is going to be denoted by $\widehat{\mathcal{X}}_2$.

It is well known that the exponential $\exp(tX)$ of a vector field $X$ in $\widehat{\mathcal{X}}_2$ can be defined so that for each $t \in \mathbb{C}$, $\exp(tX)$ is an element of $\widehat{\text{Diff}_1}(\mathbb{C}^2, 0)$. Moreover, the assignment $t \mapsto \exp(tX)$ is a homomorphism, i.e., $\exp((t_1 + t_2)X) = \exp(t_1X) \circ \exp(t_2X)$. Conversely, every element of $F \in \widehat{\text{Diff}_1}(\mathbb{C}^2, 0)$ coincides with $\exp(X)$ (i.e., $\exp(tX)$ for $t = 1$) for some (unique) formal vector field $X \in \widehat{\mathcal{X}}_2$. The vector field $X$ is then called the infinitesimal generator of $F$, and the notation $X = \log(F)$ is also used in the literature. Indeed, if we fix $t = 1$ and consider the resulting map $\text{Exp} : \widehat{\mathcal{X}}_2 \to \widehat{\text{Diff}_1}(\mathbb{C}^2, 0)$ associating with a vector field $X \in \widehat{\mathcal{X}}_2$ the induced time-one formal diffeomorphism $\exp(X)$, then $\text{Exp}$ settles a bijection between $\widehat{\mathcal{X}}_2$ and $\widehat{\text{Diff}_1}(\mathbb{C}^2, 0)$.

Recall that the order $\text{ord}(f)$ at $(0, 0)$ of an element $f \in \mathbb{C}[[x, y]]$ is nothing but the degree of the first non-zero homogeneous component of the formal series of $f$. The order of a formal vector field is analogously defined as the minimum between the orders of its components. Next, if $F \neq \text{id}$ is a formal diffeomorphism tangent to the identity, the order of the (formal) function $F - \text{id}$ is called the contact order with the identity of $F$. Here the order of $F - \text{id}$ is again defined as the minimum of the order of its components.

An elementary and yet fundamental observation concerning vector fields in $\widehat{\mathcal{X}}_2$ is that the order of a (non-trivial) commutator $[X, Y]$ is strictly greater than the orders of both $X$ and $Y$ provided that $X, Y \in \widehat{\mathcal{X}}_2$. In particular, this lends sense to the Campbell–Hausdorff formula (see, for example, [28]), stating that the infinitesimal generator of $F_1 \circ F_2$ satisfies

$$\log(F_1 \circ F_2) = \log(\text{Exp}(X_1) \circ \text{Exp}(X_2)) = X_1 + X_2 + \frac{1}{2}[X_1, X_2] + \cdots$$

where $X_1, X_2$ stand for the infinitesimal generators of $F_1, F_2$, respectively. In fact, whereas the right-hand side of the Campbell–Hausdorff formula involves adding up an infinite number of formal series, for every $k \in \mathbb{N}^*$ fixed, the mentioned sum contains only finitely many formal series having orders smaller than $k$. In other words, for a fixed monomial $x^{k_1}y^{k_2}$ the corresponding coefficients appearing in the above series are equal to zero except for finitely many series. This immediately gives a well-defined...
meaning to the right side of the Campbell–Hausdorff formula. The lemma below is a simple application of these ideas.

**Lemma 5.1** Consider two elements \( F_1, F_2 \) in \( \text{Diff}_1(\mathbb{C}^2, 0) \) together with their respective infinitesimal generators \( X_1, X_2 \). Then the following holds:

1. \( F_1, F_2 \) commute if and only if so do \( X_1, X_2 \).
2. If \( F_1, F_2 \) do not commute, then the contact order with the identity of \( [F_1, F_2] = F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1} \) is strictly greater than the corresponding orders of \( F_1 \) and \( F_2 \).

**Proof** Consider the first claim in the above statement. It suffices to show that \([X_1, X_2] = 0\) provided that \( F_1 \) and \( F_2 \) commute since the converse is clear. For this, denote by \( Z_+ \) (resp., \( Z_- \)) the infinitesimal generator of \( F_1 \circ F_2 \) (resp., \( F_1^{-1} \circ F_2^{-1} \)). The diffeomorphisms \( F_1, F_2 \) commute if and only if \( F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1} = \text{Exp} (Z_+) \circ \text{Exp} (Z_-) = \text{id} \). Denoting by \( Z \) the infinitesimal generator of \( F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1} \) and twice applying the Campbell–Hausdorff formula, we obtain

\[
Z = [X_1, X_2] + \frac{1}{2} [X_1, [X_1, X_2]] + \frac{1}{2} [X_2, [X_1, X_2]] + \cdots .
\]

(6)

Assuming that \([X_1, X_2] \) does not vanish identically, we can write \([X_1, X_2] = \sum_{j \geq k} Y_j \) where \( Y_j \) is a degree \( j \) homogeneous vector field and where \( Y_k \) is not identically zero. The orders of the higher iterated commutators appearing in Equation (6) are strictly greater than \( k \), since the orders of \( X_1, X_2 \) at the origin are at least 2. In other words, we have \( Z = Y_k + \text{h.o.t.} \). Since \( F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1} = \text{Exp} (Z) \), it follows that \( F_1, F_2 \) do not commute either and this establishes the first assertion.

As to item (2), assuming that \( F_1 \) and \( F_2 \) do not commute, it follows from the preceding that the order of contact with the identity of \( [F_1, F_2] \) coincides with the order of \([X_1, X_2] \) at the origin. However, if \( r \geq 2 \) (resp., \( s \geq 2 \)) stands for the order of \( X_1 \) (resp., \( X_2 \)) at the origin, then the order of \([X_1, X_2] \) is greater than or equal to \( r + s - 1 \). The statement follows at once. \( \square \)

Next let \( \mathbb{C}((x, y)) \) denote the quotient field of \( \mathbb{C}[[x, y]] \) and consider an element \( h \in \mathbb{C}((x, y)), h = f/g \) with \( f, g \in \mathbb{C}[[x, y]] \). The order of \( h \) at \((0, 0)\) can be defined as the unique integer \( n \in \mathbb{Z} \) for which the limit \( \lim_{\lambda \to 0} h(\lambda x, \lambda y)/\lambda^n \) is a non-identically zero quotient of two homogeneous polynomials. Clearly this value of \( n \) is simply the difference \( \text{ord} (f) - \text{ord} (g) \). The extension of this definition to formal vector fields with coefficients in \( \mathbb{C}((x, y)) \) is immediate: the order at \((0, 0)\) of the vector field in question is the minimum between the orders of its components. Clearly the order is well defined since it does not depend on the choice of the formal coordinates.

In what follows, a formal vector field \( X \) with coefficients in \( \mathbb{C}[[x, y]] \) will often be referred to as a (formal) vector field belonging to \( \hat{\mathbb{K}} \) (or to \( \hat{\mathbb{K}}_2 \)). Unless otherwise mentioned, whenever we talk about formal vector fields without specifying that they belong to either \( \hat{\mathbb{K}} \) or \( \hat{\mathbb{K}}_2 \), they are allowed to have coefficients in \( \mathbb{C}((x, y)) \).

Two formal vector fields \( X, Y \in \hat{\mathbb{K}}_2 \) are said to be everywhere parallel if \( X \) is a multiple of \( Y \) by an element in \( \mathbb{C}((x, y)) \) (or if the converse holds). When \( X, Y \in \hat{\mathbb{K}}_2 \)
are not everywhere parallel, then every formal vector field $Z \in \hat{\mathfrak{X}}$ can be expressed as a linear combination of $X, Y$ with coefficients in $\mathbb{C}((x, y))$. More precisely, for $X, Y$ and $Z$ as above, there are $f, g \in \mathbb{C}((x, y))$ such that $Z = fX + gY$. In fact, by setting $X = A \partial/\partial x + B \partial/\partial y, Y = C \partial/\partial x + D \partial/\partial y$ and $Z = P \partial/\partial x + Q \partial/\partial y$, we obtain:

$$f = \frac{PD - QC}{AD - BC} \quad \text{and} \quad g = \frac{QA - PB}{AD - BC}. \quad (7)$$

Another well-known result that will be useful in our discussion is Hadamard’s lemma, expressing the pull-back of a vector field $X$ by a formal diffeomorphism $F$ in terms of the infinitesimal generator $Z$ of $F$; see [28]. Hadamard’s lemma can be summarized by the following formula:

$$F^* X = X + [Z, X] + \frac{1}{2} [Z, [Z, X]] + \frac{1}{3!} [Z, [Z, [Z, X]]] + \cdots. \quad (8)$$

Given a subgroup $G \subset \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$, we can now discuss the notion of the Lie algebra associated with $G$. This discussion will also allow us to quickly review some aspects of Martelo and Ribon’s construction in [16].

Consider a collection of vector fields $\{X_i\}_{i \in I}$ contained in $\hat{\mathfrak{X}}_2$. The algebraic Lie algebra $\mathcal{A}$ generated by $\{X_i\}_{i \in I}$ is the smallest vector space stable under the Lie bracket and satisfying the following condition on infinite sums of elements: if $\{Y_j\}_{j \in J}$ is a family of elements in $\mathcal{A}$ then the (possibly infinite) sum $\sum_{j \in J} Y_j$ belongs to $\mathcal{A}$ provided that for every fixed monomial in the variables $x, y$ the corresponding coefficients in the series of the vector fields $Y_j$ equal zero for all but finitely many values of $j$. The reader will immediately note that under this condition the sum $\sum_{j \in J} Y_j$ naturally yields an element of $\hat{\mathfrak{X}}_2$. The difference between the algebraic Lie algebra $\mathcal{A}$ and the Lie algebra generated by $\{X_i\}_{i \in I}$—as we want to define—lies in the fact that we shall require the latter to be closed in a suitable sense (as is often the case in infinite-dimensional situations). The topology to be used here is sometimes referred to as the Krull topology and an accurate statement is provided by the definition below. We begin by recalling that for every $N \in \mathbb{N}^*$ the space $J^N(\hat{\mathfrak{X}}_2)$ formed by the jets of order $N$ (or $N$-jets) of elements in $\hat{\mathfrak{X}}_2$ is immediately identified with the space of polynomials of degree bounded by $N$ (and without degree 0 and degree 1 components). In particular, this space has finite dimension and is endowed with a natural topology of (finite-dimensional) vector space arising from comparing coefficients.

**Definition 5.2** The Lie algebra $\mathfrak{g}$ generated by $\{X_i\}_{i \in I}$ is the closure of $\mathcal{A}$ with respect to the Krull topology. In other words, a formal vector field $X \in \hat{\mathfrak{X}}_2$ belongs to $\mathfrak{g}$ if and only if the following condition holds: for every $N \in \mathbb{N}^*$, the $N$-jet $J^N(X)$ of $X$ at the origin is a limit in the topology of $J^N(\hat{\mathfrak{X}}_2)$ of $N$-jets $J^N(Z_i)$ for some sequence of elements $\{Z_i\}_{i \in \mathbb{N}}$ in $\mathcal{A}$. Since $J^N(\mathcal{A})$ is naturally closed, the preceding condition is indeed equivalent to saying that $J^N(X)$ belongs to $J^N(\mathcal{A})$.

Now assume we are given a group $G \subset \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$. For every element $f \in G$, let $X_f \in \hat{\mathfrak{X}}_2$ denote the infinitesimal generator of $f$. The Lie algebra $\mathfrak{g} \subset \hat{\mathfrak{X}}_2$ associated...
with \( G \) is then defined as the Lie algebra generated by the collection of formal vector fields \( \{ X_f \} _{ f \in G} \) in the sense of Definition 5.2.

The construction above provides an alternative perspective in Martelo and Ribon’s point of view \([16]\) which can be summarized as follows. Let \( m \) denote the maximal ideal of \( \mathbb{C}[[x, y]] \) and note that every formal diffeomorphism \( f \in \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \) acts on the vector space \( m/m^k \) of \( k \)-jets of elements in \( \mathbb{C}[[x, y]] \). More precisely \( f \) defines an element \( f_k \in \text{GL}(m/m^k) \) whose action on the vector space \( m/m^k \) is given by \( g + m^k \mapsto g \circ f + m^k \). Next, let \( D_k \subset \text{GL}(m/m^k) \) be the subgroup consisting of those automorphisms having the form \( \{ f_k \} \in \text{GL}(m/m^k) \) for some \( f \in \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \). It is easy to check that \( D_k \) is an algebraic group. Furthermore, there are natural (restriction) morphisms \( \pi_k : D_{k+1} \rightarrow D_k \) of algebraic groups for every \( k \in \mathbb{N}^* \).

Suppose now that we are given a group \( G \subset \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \). For a fixed \( k \in \mathbb{N}^* \), we can consider all automorphisms in \( \text{GL}(m/m^k) \) having the form \( \{ f_k \} \) for some \( f \in G \). The Zariski-closure \( G_k \) of this group is the smallest algebraic subgroup of \( D_k \) containing all the mentioned automorphisms. Clearly \( G_k \) is itself an algebraic group and the natural character of the preceding constructions ensures that \( \pi_k \) sends \( G_{k+1} \) to \( G_k \). It is a standard fact \([16]\) that the groups \( G_k \) are connected for every \( k \in \mathbb{N}^* \).

Next set

\[
\overline{G} = \{ f \in \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) : f_k \in G_k \text{ for every } k \in \mathbb{N}^* \}.
\]

The group \( \overline{G} \) is closed for the Krull topology and it clearly contains \( G \). Furthermore, \( \overline{G} \) is connected since so is \( G_k \) for every \( k \in \mathbb{N}^* \). By slightly abusing notation, the group \( \overline{G} \) defined above will often be referred to as the Zariski-closure of \( G \).

For every \( k \in \mathbb{N}^* \), let \( g_k \) denote the Lie algebra associated with the algebraic group \( G_k \). Consider the Lie algebra \( \mathfrak{g} \subset \widehat{X}_2 \) defined as follows:

\[
\mathfrak{g} = \{ X \in \widehat{X}_2 : X_k \in g_k \text{ for every } k \in \mathbb{N}^* \}.
\]

The Lie algebra \( \mathfrak{g} \) is, by definition, the Lie algebra associated with the initial group \( G \subset \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \) according to \([16]\).

Thus whereas a priori there are two different constructions for the Lie algebra associated with a given group \( G \subset \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \), a straightforward comparison of these constructions complemented by the well-known fact that a closed Lie subgroup of an algebraic group is itself algebraic shows that the resulting Lie algebras coincide. Hence, we obtain a uniquely defined Lie algebra associated with \( G \). In particular, no misunderstanding involving the use of Lie algebras will arise in our discussion.

It is convenient to summarize the main properties of the Lie algebra \( \mathfrak{g} \subset \widehat{X}_2 \) associated with a subgroup \( G \subset \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \) as they will be useful for our purposes. Given a Lie algebra \( \mathfrak{g} \subset \widehat{X}_2 \), the exponential of \( \mathfrak{g} \) is the image of \( \mathfrak{g} \) by the exponential map \( \text{Exp} \). In other words, it is the subgroup of \( \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \) consisting of all formal diffeomorphisms that are time-\( t \) maps induced by the exponential of some vector field in \( \mathfrak{g} \). In particular, if \( \mathfrak{g} \) is associated with a group \( G \), then it is clear that \( G \) is contained in the exponential of \( \mathfrak{g} \). In fact, assuming that \( \mathfrak{g} \) is a Lie algebra associated with a group \( G \subset \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \), the following holds (see \([16]\)).
Lemma 5.1 admits a generalization stating that the Lie algebra associated with a solvable Lie group is itself solvable and has the same derived length as the Lie group in question. An analogous statement holds for nilpotent Lie groups though this will not be necessary for the discussion in this paper.

The group \( \overline{G} \) is generated by \( \exp(g) \). In fact \( \exp : g \to \overline{G} \) is a bijection.

Owing to the second statement above, the formal classification of solvable subgroups of \( \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \) becomes reduced to the classification of solvable Lie algebras of formal vector fields with zero linear parts. A description of the structure of these algebras can be found in [16] (Theorem 6); see also Sect. 6.3.

6 Abelian Groups, Normalizers, and General Solvable Subgroups of \( \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \)

This section is divided into three subsections and, in the first one, some additional elementary results concerning abelian subgroups of \( \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \) are provided. The second subsection concerns more elaborate results on normalizers of abelian subgroups of \( \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \); see also Lemma 6.8. Finally, the third section is essentially devoted to stating a detailed version of Theorem 6 in [16] in the specific case of subgroups of \( \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \).

6.1 Elementary Facts on Abelian Groups

To begin with, consider an abelian subgroup \( G \subset \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \) which, in principle, need not be finitely generated. The group \( G \) is necessarily torsion-free since all of its elements are tangent to the identity. Hence, a basis \( \{ g_i \}_{i=1}^N \subset G \) for this group can be considered where \( N \in \mathbb{N} \cup \{ \infty \} \) (at this point we only assume the group is countably generated).

With the group \( G \) is associated an abelian Lie algebra \( g \subset \mathfrak{X}_2 \) which is generated (both as Lie algebra and as vector space) by the infinitesimal generators \( X_i \) of the elements \( g_i \) in the above-mentioned basis. Let us first consider the case in which \( g \) contains two formal vector fields \( X \) and \( Y \) which are not everywhere parallel. In particular, every vector field in \( \mathfrak{X} \) can be written as a linear combination of \( X \) and \( Y \) with coefficients in the field \( \mathbb{C}((x, y)) \); see Sect. 5. Then we have:

**Lemma 6.1** Under the above assumption, the abelian Lie algebra \( g \) is generated over \( \mathbb{C} \) by \( X \) and \( Y \). In particular, \( G \) is contained in the exponential of \( g \) though \( G \) is not contained in the exponential of a single vector field in \( g \).

**Proof** Consider \( Z \in g \) and let \( Z = aX + bY \) with \( a, b \in \mathbb{C}((x, y)) \). Since \( g \) is abelian, it follows that \( [Z, X] = [Z, Y] = 0 \) which in turn leads to

\[
\frac{\partial a}{\partial X} = \frac{\partial a}{\partial Y} = \frac{\partial b}{\partial X} = \frac{\partial b}{\partial Y} = 0.
\]

Since \( X \) and \( Y \) are not everywhere parallel, it follows that \( a, b \) are both constants, i.e., \( a, b \in \mathbb{C} \) proving the first part of the statement.
To conclude that $G$ cannot be contained in the exponential of a single vector field just note that, if this were the case, the Lie algebra $g$ would coincide with the one-dimensional vector space spanned by the vector field in question. This clearly contradicts the existence of two non-everywhere parallel vector fields $X$ and $Y$ in $g$. $\Box$

The argument above also yields the following corollary:

**Corollary 6.2** Assume that $g \subset \hat{\mathfrak{X}}_2$ is a (non-trivial) abelian Lie algebra. Then one of the following holds:

(1) Suppose that $g$ contains two vector fields $X$ and $Y$ that are not everywhere parallel. Then $g$ can be identified with the two-dimensional vector space spanned by $X$ and $Y$ over $\mathbb{C}$.

(2) There is a basis $\{X_i\}_{i=1}^N$, where $N \in \mathbb{N} \cup \{\infty\}$ and where $X_1 = X$, for $g$ such that $X_i = h_i X$ for every $2 \leq i \leq N$ where $h_i \in \mathbb{C}((x, y))$ is a first integral of $X$ (i.e., $\partial h_i / \partial X = 0$).

The Lie algebra $g$ spanned by vector fields $X$ and $Y$ as in item (1) above will be referred to as the **linear span of $X$ and $Y$** so that the phrase “the linear span of $X$ and $Y$” implies that $X$ and $Y$ commute and that they are not everywhere parallel. Concerning item (2), we note that $N \in \mathbb{N}$ if and only if $g$ is finitely generated as vector space. More importantly, although $X$ and $X_i$, $i \geq 2$, belong to $\hat{\mathfrak{X}}_2$, the equation $X_i = h_i X$ does not imply that $h_i$ lies in $\mathbb{C}[[x, y]]$—as opposed to $\mathbb{C}((x, y))$—since $X$ is not supposed to have isolated singularities.

Recall that the **centralizer** of an element $F \in \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ is the group formed by those elements in $\hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ commuting with $F$ (and hence commuting with every element in the cyclic group generated by $F$). To characterize the centralizer of an element $F \in \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$, we denote by $X$ its infinitesimal generator. Note that there may or may not exist another vector field $Y$ commuting with $X$ while not everywhere parallel to $X$. When this vector field $Y$ exists, it is never unique since every linear combination of $X$ and $Y$ will have similar properties. Furthermore, if $X$ happens to admit some non-constant first integral $h$, then $h Y$ will also commute with $X$. When both $h$ and $Y$ exist, then every element of $\hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ whose infinitesimal generator $Z$ has the form $Z = aX + bY$, where $a$ and $b$ are first integrals of $X$, automatically belongs to the centralizer of $F$; cf. Lemma 5.1. With this notation, the centralizer of $F$ admits the following characterization.

**Lemma 6.3** Let $F \in \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ be given and denote by $X$ its infinitesimal generator. Then the centralizer of $F$ in $\hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ coincides with one of the following groups.

**Case 1** Suppose that every vector field $Y \in \hat{\mathfrak{X}}_2$ commuting with $X$ is everywhere parallel to $X$. Then the centralizer of $F$ consists of the subgroup of $\hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ whose elements have infinitesimal generators of the form $hX$, where $h \in \mathbb{C}((x, y))$ is a formal first integral of $X$. In particular, if $X$ admits only constants as first integrals, then the centralizer of $F$ is reduced to the exponential of $X$.

**Case 2** Suppose there is $Y \in \hat{\mathfrak{X}}_2$ which is not everywhere parallel to $X$ and still commutes with $X$. Then the centralizer of $F$ coincides with the subgroup of...
\(\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\) consisting of those elements \(F \in \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\) whose infinitesimal generators have the form \(aX + bY\), where \(a, b \in \mathbb{C}((x, y))\) are first integrals of \(X\).

**Proof** Suppose that \(H\) is an element of \(\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\) commuting with \(F\). Denoting by \(Z\) the infinitesimal generator of \(H\), it follows from Lemma 5.1 that \([X, Z] = 0\). Conversely the 1-parameter group obtained as the exponential of \(Z\) is automatically contained in the centralizer of \(F\).

Next, suppose that the assumption in Case 1 is verified. Then the quotient \(h\) between \(Z\) and \(X\) can be defined as an element of \(\mathbb{C}((x, y))\) satisfying \(Z = hX\). Therefore, the condition \([X, Z] = 0\) becomes \(dh.X = 0\), i.e., \(h\) is a first integral for \(X\).

Now consider the existence of \(Y\), not everywhere parallel to \(X\), verifying \([X, Y] = 0\). It is clear that the elements of \(\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\) described in Case 1 belong to the centralizer of \(F\). Thus only the converse needs to be proved. Since \(H\) commutes with \(F\), Lemma 5.1 again yields \([X, Z] = 0\). Since \(Y\) is not a multiple of \(X\), there are functions \(a(x, y), b(x, y) \in \mathbb{C}((x, y))\) such that \(Z = aX + bY\). Now the equation \([X, Z] = 0\) yields

\[
(\partial a/\partial X).X + (\partial b/\partial X).Y = 0. 
\]

Thus the fact that \(Y\) is not a multiple of \(X\) ensures that \((\partial a/\partial X) = (\partial b/\partial X) = 0\). In other words, both \(a, b\) are first integrals of \(X\). The lemma follows. \(\square\)

Concerning the situation described in Case 2 of Lemma 6.3, it is already known that \(Y\) is not uniquely defined. However, the reader will note that the resulting group of formal diffeomorphisms commuting with \(F\) does not depend on the choice of \(Y\).

Here is an easy consequence of Lemma 6.3.

**Lemma 6.4** Suppose that \(h\) is a non-constant first integral of \(X\) and let \(F_1 = \text{Exp} (X)\) and \(F_2 = \text{Exp} (hX)\) be elements in \(\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\). The intersection of the centralizers of \(F_1\) and of \(F_2\) coincides with the subgroup of \(\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\) formed by those elements whose infinitesimal generators have the form \(aX\), where \(a\) is a first integral of \(X\). In particular, the intersection of these centralizers is an abelian group.

**Proof** Let \(F \in \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\) be an element commuting with both \(F_1\) and \(F_2\) and denote by \(Z\) the infinitesimal generator of \(F\). If \(Z\) is everywhere parallel to \(X\), then the statement follows from Lemma 6.3, Case 1. Assume now that \(Z\) is not everywhere parallel to \(X\) and note that we must have \([Z, X] = [Z, hX] = 0\) (Lemma 5.1). From this, it follows that \(\partial h/\partial Z = 0\). Since \(\partial h/\partial X = 0\) and \(X, Z\) are not everywhere parallel, we conclude that \(h\) must be constant which gives the desired contradiction. \(\square\)

### 6.2 On Normalizers of Certain Abelian Groups

Recall that the normalizer of a group \(G \subset \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\) is the maximal subgroup \(N_G\) of \(\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\) containing \(G\) as a normal subgroup. Similarly, the centralizer of an abelian group \(G\) is the maximal subgroup of \(\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)\) containing \(G\) in its
center. Given $F \in \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$, we shall refer to the normalizer (resp., centralizer) of $F$ meaning the normalizer (resp., centralizer) of the cyclic group generated by $F$. This section is intended to establish certain results concerning normalizers of abelian groups. In the sequel we shall freely use the following consequence of the Lie algebra constructions detailed in Sect. 5: given a subgroup $G \subset \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ whose Lie algebra is denoted by $\mathfrak{g}$, the normalizer of $G$ in $\hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ naturally acts by pull-backs on $\mathfrak{g}$.

We begin with an easy observation:

**Lemma 6.5** The normalizer and the centralizer of a formal diffeomorphism $F \in \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ coincide and hence are described by Lemma 6.3.

**Proof** It suffices to show that the normalizer of $F$ is contained in the centralizer of $F$. For this, denote by $Z$ the infinitesimal generator of $F$ and consider an element $g$ in the normalizer of $F$. The Lie algebra associated with the cyclic group generated by $F$ has dimension 1 and consists of constant multiples of $Z$. Since $g$ acts on this Lie algebra by pull-backs, it follows that $g^*Z = cZ$ for some $c \in \mathbb{C}$. However, Hadamard’s lemma [Formula (8)] shows that $g^*Z = Z$ has order at the origin strictly greater than the order of $Z$ unless $g^*Z = Z$. Therefore, the latter possibility must hold.

Now consider an abelian group $G \subset \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ whose Lie algebra coincides with the linear span of two vector fields $X, Y$ as in item (1) of Corollary 6.2. Concerning the Lie algebra $\mathfrak{g}$ of $G$, two different situations may occur, namely: all linear combinations of $X, Y$ may or may not have the same order at the origin. Clearly, when not all these vector fields have the same order at the origin, there is a unique (up to a multiplicative constant) vector field $Z$ in $\mathfrak{g}$ whose order at the origin is strictly greater than the orders of all remaining vector fields in $\mathfrak{g}$.

**Lemma 6.6** Let $G \subset \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ be an abelian group whose Lie algebra $\mathfrak{g}$ is isomorphic to the linear span of vector fields $X, Y \in \tilde{\mathfrak{X}}_2$. Then one of the following holds:

1. **Assume that all vector fields in the linear span of $X, Y$ have the same order at the origin.** Then the normalizer $N_G$ of $G$ is contained in the exponential of $\mathfrak{g}$.
2. **Assume that there is a vector field $Z$ in the linear span of $X, Y$ whose order is greater than the orders of the remaining vector fields.** Then the normalizer $N_G$ of $G$ is either abelian or metabelian. Also $N_G$ is necessarily a nilpotent group.

**Proof** Consider the subgroup $\Gamma_{\text{abelian}}^{-1}$ of $\hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ consisting of all elements in $\hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ that act on $\mathfrak{g}$. In other words, a formal diffeomorphism $F \in \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ lies in $\Gamma_{\text{abelian}}^{-1}$ if and only if $F^*\mathfrak{g} \subset \mathfrak{g}$. Recalling that the normalizer $N_G \subset \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ of $G$ naturally acts on $\mathfrak{g}$, it follows that $N_G \subset \Gamma_{\text{abelian}}^{-1}$. To prove the lemma it is therefore sufficient to study the group $\Gamma_{\text{abelian}}^{-1}$.

Then let $F \in \Gamma_{\text{abelian}}^{-1} \subset \hat{\text{Diff}}_1(\mathbb{C}^2, 0)$ and consider the action of $F$ on $\mathfrak{g}$. Since $F$ is tangent to the identity, it follows from Hadamard’s lemma that the eigenvalues of the automorphism induced by $F$ on $\mathfrak{g}$ are equal to 1 (i.e., $F$ is unipotent). In fact, for every $Z \in \mathfrak{g}$, Hadamard’s lemma ensures that $F^*Z = Z$ has order strictly greater than
the order of \( Z \) unless it vanishes identically. Thus whenever \( F^*Z = cZ \) it follows that \( c = 1 \). Summarizing, the eigenvalues of the automorphism induced by \( F \) on \( g \) are equal to 1 so that this automorphism either coincides with the identity or it is non-diagonalizable. In the former case \( F \) belongs to the exponential of \( g \) since it commutes with both \( X \) and \( Y \).

Now suppose that the above-mentioned action of \( F \) is not diagonalizable. Up to a change of basis, we can assume that \( F^*X = X \) and \( F^*Y = Y + X \). In particular, Hadamard’s lemma applied to \( F^*Y - Y \) shows that the order of \( X \) is strictly larger than the order of \( Y \) so that we are in the situation described in item (2). In other words, if \( X \) and \( Y \) are as in item (1), then the group \( \Gamma_{\text{abelian}} - 1 \) coincides with the exponential \( \text{Exp} (g) \) of \( g \) and the lemma follows at once.

It remains to study the case where \( X \) and \( Y \) are as in item (2). Without loss of generality, we can assume that the order of \( X \) is strictly larger than the order of \( Y \) so that \( X \) is distinguished in \( g \) as the unique vector field (up to a multiplicative constant) having maximal order at the origin. In particular, for every \( F \in \Gamma_{\text{abelian}} - 1 \), we have \( F^*X = X \).

Next fix a vector field \( Z \in g \) which is not a constant multiple of \( X \). The vector field \( F^*Z - Z \) lies in \( g \) and has order strictly larger than the order of \( Z \). Thus \( F^*Z - Z \) must be a constant multiple of \( X \), i.e., we have

\[
F^*Z = Z + cX
\]  

for some constant \( c \in \mathbb{C} \) depending only on \( F \) (note that \( c \) may equal zero, and this is certainly the case when \( F \) lies in the exponential of \( g \)). In particular, we have obtained a map \( \sigma \) from \( \Gamma_{\text{abelian}} - 1 \) to \( \mathbb{C} \) that assigns to \( F \in \Gamma_{\text{abelian}} - 1 \) the constant \( c \in \mathbb{C} \) appearing in Eq. (10). However, since every element \( F \in \Gamma_{\text{abelian}} - 1 \) fulfills the condition \( F^*X = X \), it also follows that \( \sigma : \Gamma_{\text{abelian}} - 1 \rightarrow \mathbb{C} \) is a group homomorphism. Furthermore, the kernel of \( \sigma \) consists of those elements fixing both \( Z \) and \( X \) so that this kernel can be identified with the exponential \( \text{Exp} (g) \) of \( g \). Summarizing, the group \( \Gamma_{\text{abelian}} - 1 \) can alternatively be defined by the short exact sequence

\[
0 \rightarrow \text{Exp} (g) \cong \mathbb{C}^2 \rightarrow \Gamma_{\text{abelian}} - 1 \xrightarrow{\sigma} \mathbb{C} \rightarrow 0. \tag{11}
\]

This sequence realizes \( \Gamma_{\text{abelian}} - 1 \) as an abelian extension of an abelian group so that \( \Gamma_{\text{abelian}} - 1 \) must be step 2 solvable. Since \( N_G \) naturally sits inside \( \Gamma_{\text{abelian}} - 1 \), we conclude that \( N_G \) is either abelian or metabelian.

It only remains to check that the group \( \Gamma_{\text{abelian}} - 1 \) is, in fact, nilpotent. For this, note that \( \Gamma_{\text{abelian}} - 1 \) is a complex Lie group of dimension 3, as follows from sequence (11). Denoting by \( g_{\text{abelian}} - 1 \) its Lie algebra, we see that \( g_{\text{abelian}} - 1 \) is neither abelian (otherwise there is nothing to be proved) nor isomorphic to the Lie algebra of \( \text{PSL}(2, \mathbb{C}) \) since \( X \) commutes with \( Y \). Furthermore, the image of \( g_{\text{abelian}} - 1 \) by the adjoint representation must be contained in \( g \), i.e., the commutator of a vector field in \( g \) with a vector field in \( g_{\text{abelian}} - 1 \) must lie in \( g \). Since \( X \) is distinguished in \( g \) for its maximal order at the origin, it follows that \( X \) lies in the center of \( g_{\text{abelian}} - 1 \). Finally, by considering a third element \( \tilde{Z} \) in \( g_{\text{abelian}} - 1 \) so that \( X, Y, \tilde{Z} \) form a basis for \( g_{\text{abelian}} - 1 \), we also conclude that \([\tilde{Z}, Y] = cX\) since \([\tilde{Z}, Y] \) lies in \( g \) and has order strictly larger than
the order of $Y$. From this, it follows that $\mathfrak{g}_{\text{abelian}^{-1}}$ is isomorphic to the Lie algebra of strictly upper-triangular $3 \times 3$ matrices or, equivalently, that $N_G$ is isomorphic to a subgroup of the group of unipotent upper-triangular $3 \times 3$ matrices. The lemma is proved. □

The next lemma completes the description of the normalizers of (non-trivial) finitely generated abelian groups; cf. Corollary 6.2.

**Lemma 6.7** Let $G \subset \widetilde{\text{Diff}}_1(\mathbb{C}^2, 0)$ be a finitely generated abelian group all of whose elements have infinitesimal generators parallel to a certain formal vector field $X$. Assume that the rank of $G$ is at least 2. Then the normalizer $N_G$ of $G$ in $\widetilde{\text{Diff}}_1(\mathbb{C}^2, 0)$ is either abelian or metabelian.

**Proof** Recall that the Lie algebra $\mathfrak{g}$ of $G$ is generated both as Lie algebra and as vector space by the infinitesimal generators of a set of elements forming a basis for $G$. Denote by $n \geq 2$ the dimension of this Lie algebra and let $X$ be a vector field in $\mathfrak{g}$ whose order at $(0, 0) \in \mathbb{C}^2$ is maximal among all vector fields in $\mathfrak{g}$. The existence of $X$ is guaranteed by the fact that $\mathfrak{g}$ has finite dimension.

Next let $\Gamma_{\text{abelian}^{-2}} \subset \widetilde{\text{Diff}}_1(\mathbb{C}^2, 0)$ be the group formed by all those diffeomorphisms $F$ in $\text{Diff}_1(\mathbb{C}^2, 0)$ for which $F^* \mathfrak{g} \subset \mathfrak{g}$. In particular, the normalizer $N_G$ of $G$ is contained in $\Gamma_{\text{abelian}^{-2}}$. Furthermore, for every $F \in \Gamma_{\text{abelian}^{-2}}$, we have $F^*X = X$ since $X$ has maximal order in $\mathfrak{g}$ and $F$ is unipotent. In particular, if there is more than one vector field (up to multiplicative constants) in $\mathfrak{g}$ having maximal order at $(0, 0) \in \mathbb{C}^2$, the group $\Gamma_{\text{abelian}^{-2}}$ must be abelian since it will lie in the intersection of the centralizers of $X$ and another vector field $hX$, where $h$ is a non-constant first integral of $X$; see Lemma 6.4. In fact, $\Gamma_{\text{abelian}^{-2}}$ will coincide with the exponential of the infinite-dimensional Lie algebra formed by vector fields of the form $aX$ where $a$ is a first integral of $X$.

Suppose now that up to multiplicative constants $X$ is the unique vector field in $\mathfrak{g}$ whose order at the origin is maximal. Consider a vector field $Y \in \mathfrak{g}$ whose order at the origin is the “second largest possible” in the sense that the condition of having a vector field $Z$ in $\mathfrak{g}$ whose order at $(0, 0) \in \mathbb{C}^2$ is strictly greater than the order of $Y$ implies that $Z$ must be a constant multiple of $X$. Note that the vector field $Y$ clearly exists since $n \geq 2$, though it is not necessarily unique (always up to multiplicative constants). For $F \in \Gamma_{\text{abelian}^{-2}}$, we consider $F^*Y - Y \in \mathfrak{g}$. As previously seen, Hadamard’s lemma implies that $F^*Y = Y + cX$ for some constant $c \in \mathbb{C}$ and for every $F \in \Gamma_{\text{abelian}^{-2}}$. Thus, arguing as in the proof of Lemma 6.6, we conclude that the assignment $F \in \Gamma_{\text{abelian}^{-2}} \mapsto c \in \mathbb{C}$ such that $F^*Y = Y + cZ$ defines a homomorphism from $\Gamma_{\text{abelian}^{-2}}$ to $\mathbb{C}$ whose kernel is an abelian group. It follows that $\Gamma_{\text{abelian}^{-2}}$ is either abelian or metabelian. In fact, the group $\Gamma_{\text{abelian}^{-2}}$ can alternatively be defined by the exact sequence

$$0 \longrightarrow G_X \longrightarrow \Gamma_{\text{abelian}^{-2}} \longrightarrow \mathbb{C} \longrightarrow 0 ,$$

where $G_X \subset \widetilde{\text{Diff}}_1(\mathbb{C}^2, 0)$ is the abelian group all of whose elements have infinitesimal generator of the form $aX$ where $a$ is a first integral for $X$. The lemma follows. □
6.3 Classification of Solvable Groups in \( \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \)

In this section the classification of finitely generated solvable subgroups of \( \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \) obtained in the work of Martelo and Ribon [16] will be detailed; cf. Theorem 6 in [16]. However, we begin with a more general lemma.

**Lemma 6.8** Suppose that \( G_0 \subset \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \) is an abelian group whose Lie algebra coincides with the linear span of two vector fields \( X \) and \( Y \). Suppose also that \( G_1 \) is a non-abelian group containing \( G_0 \) as a normal subgroup. Then the normalizer of \( G_1 \) is metabelian.

*Proof* Since \( G_1 \) is non-abelian so is the normalizer of \( G_0 \). Therefore, it follows from Lemma 6.6 that \( G_1 \) is isomorphic to a (non-abelian) subgroup of \( \Gamma_{\text{abelian}}^{-1} \) which, in turn, is a step 2 solvable (and nilpotent) group. Also, still owing to Lemma 6.6, we can assume without loss of generality that the order of \( X \) at \((0, 0) \in \mathbb{C}^2\) is strictly larger than the corresponding order of \( Y \) and that the exponential of \( X \) contains the center of \( G_1 \).

Consider the Lie algebra associated with \( G_1 \) and note that this Lie algebra cannot be abelian. Also, this Lie algebra contains (strictly) the linear span of \( X \) and \( Y \) so that its dimension is at least 3. On the other hand, the Lie algebra of \( G_1 \) is isomorphic to a sub-algebra of the Lie algebra of \( \Gamma_{\text{abelian}}^{-1} \). Since the latter algebra has dimension 3 (Lemma 6.6), it follows that the two Lie algebras coincide. In particular, \( G_1 \) is Zariski-dense in \( \Gamma_{\text{abelian}}^{-1} \). In turn, the Zariski-denseness of \( G_1 \) in \( \Gamma_{\text{abelian}}^{-1} \) implies that these two groups share the same normalizer in \( \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \). Therefore, to prove Lemma 6.8, it suffices to establish the lemma below concerning the normalizer of \( \Gamma_{\text{abelian}}^{-1} \). \( \square \)

**Lemma 6.9** The normalizer of \( \Gamma_{\text{abelian}}^{-1} \) is a metabelian group.

*Proof* We know that every element \( F \in \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \) lying in the normalizer of \( \Gamma_{\text{abelian}}^{-1} \) acts by pull-backs on the Lie algebra \( \mathfrak{g}_{\text{abelian}}^{-1} \) which, in turn, is spanned as a vector space by three vector fields \( X, Y, \) and \( Z \) where \( X \) lies in the center and where \([Y, Z] = X\).

We claim that \( X \) is distinguished in \( \mathfrak{g}_{\text{abelian}}^{-1} \) as the vector field of maximal order at the origin. To check this assertion, note that every vector field in \( \mathfrak{g}_{\text{abelian}}^{-1} \) having maximal order at the origin must lie in the center of \( \mathfrak{g}_{\text{abelian}}^{-1} \). If this center were not spanned by constant multiples of \( X \), then the Lie algebra \( \mathfrak{g}_{\text{abelian}}^{-1} \) would be contained in the centralizer of two vector fields, which would force \( \mathfrak{g}_{\text{abelian}}^{-1} \) to be abelian; cf. Sect. 6.1. The resulting contradiction proves the claim. From this, it also follows that every \( F \in \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \) lying in the normalizer of \( \Gamma_{\text{abelian}}^{-1} \) must satisfy \( F^*X = X \).

To finish the proof of Lemma 6.8, we proceed as follows. Consider the family \( \mathfrak{F} \) of vector fields contained in the Lie algebra of \( \Gamma_{\text{abelian}}^{-1} \) and having the form \( c_1 Y + c_2 Z \) with \( c_1, c_2 \in \mathbb{C} \). Let \( W \) denote a vector field in \( \mathfrak{F} \) having maximal order at the origin among vector fields in this family. The existence of \( W \) is clear, although it need not be unique. Note, however, that \( W \) does not coincide with \( X \) since \( X, Y \) and \( Z \) are linearly independent over \( \mathbb{C} \). Now, for \( F \) in the normalizer of \( \Gamma_{\text{abelian}}^{-1} \), consider \( F^*W - W \in \mathfrak{g}_{\text{abelian}}^{-1} \). By construction, we also have \( F^*W - W = a_1 X + a_2 Y + a_3 Z \),
for certain $a_1, a_2, a_3 \in \mathbb{C}$. However, the order of both $X$ and $F^*W - W$ are strictly greater than the order of $a_2Y + a_3Z$. Thus we must have $F^*W - W = a_1X$. The statement now follows from repeating the arguments of Lemma 6.6 (if $X$ and $W$ are not everywhere parallel) or of Lemma 6.7 (if $X$ and $W$ are everywhere parallel). Lemma 6.8 is proved.

We can now provide the formal classification of non-abelian solvable subgroups of $\hat{\operatorname{Diff}}_1(\mathbb{C}^2, 0)$. As mentioned, the list below is a consequence of Theorem 6 in [16] specified for the case of subgroups of $\hat{\operatorname{Diff}}_1(\mathbb{C}^2, 0)$. Consider then a finitely generated solvable non-abelian group $G \subset \hat{\operatorname{Diff}}_1(\mathbb{C}^2, 0)$ and denote by $D^kG$ the non-trivial derived subgroup of $G$ having highest order $k$. In other words, $k \geq 1$ is such that $D^kG$ is abelian and not reduced to the identity. The reader is reminded that, although abelian, the group $D^kG$ need not be finitely generated even if $G$ is so. In view of Lemma 6.8, if $D^kG$ contains vector fields that are not everywhere parallel then $D^kG$ is a span of two vector fields and $G$ must coincide with $D^{k-1}G$ which, in turn, is a subgroup of the group $\Gamma_{\text{abelian-1}}$; cf. also Remark 6.11. Thus we can assume that every element in $D^kG$ has an infinitesimal generator of the form $hX$ where $X$ is some fixed vector field and where $h$ is some first integral of $X$. In particular, these infinitesimal generators form the abelian Lie algebra $D^k\mathfrak{g}$ associated with $D^kG$. Note also that $X$ can be supposed to belong to $D^k\mathfrak{g}$ since the quotient between two first integrals still is a first integral. These assumptions will be made without further comments in what follows.

A last remark is needed before the classification of solvable subgroups of $\hat{\operatorname{Diff}}_1(\mathbb{C}^2, 0)$ can be stated. Consider two vector fields $X$ and $Y$ such that the commutator $[X, Y]$ has the form $aX$, i.e., it is everywhere parallel to $X$. Then the very definition of commutator for two vector fields yield the following “generalized Schwarz theorem”

$$\frac{\partial}{\partial Y} \left( \frac{\partial f}{\partial X} \right) - \frac{\partial}{\partial X} \left( \frac{\partial f}{\partial Y} \right) = \frac{\partial f}{\partial [X, Y]} = a \frac{\partial f}{\partial X}$$

for every $f \in \mathbb{C}((x, y))$. In particular, $Y$ derives first integrals of $X$ into first integrals of $X$, i.e., if $\varphi$ is a first integral of $X$ then so is $\partial \varphi / \partial Y$.

Let $\mathcal{I}_X$ denote the field formed by all first integrals of $X$ (we may assume this field contains non-constant elements). The first case in the classification is the following:

**Case 1** Suppose that $G$ is metabelian and that all of its elements have infinitesimal generator parallel to a same vector field (necessarily $X$). Then the Lie algebra $\mathfrak{g}$ of $G$ is constituted by vector fields of the form $uX$. Moreover, $X$ can be chosen so as to ensure the existence of $f \in \mathbb{C}((x, y))$ such that $\partial f / \partial X = \overline{h}$ is a non-zero element in $\mathcal{I}_X$. Furthermore, the assignment $uX \in \mathfrak{g} \mapsto u$ identifies $\mathfrak{g}$ with a differential algebra $\mathcal{A}$ which, in turn, is constituted by functions having the form $\varphi_1 f + \varphi_2$ where $f$ is as above and $\varphi_1, \varphi_2$ belong to $\mathcal{I}_X$.

**Case 2** Suppose that $G$ is metabelian but contains an element whose infinitesimal generator is not everywhere parallel to $X$. Then there is a vector field $Y$ not everywhere parallel to $X$ and possessing the following property:
(•) \[ [\vec{Y}, X] = \tilde{h}X \] where \( \tilde{h} \) is a first integral for \( X \) (in particular, if \( \tilde{h} \equiv 0 \) then \( X, \vec{Y} \) commute).

Furthermore, there is a function \( f \in \mathbb{C}((x, y)) \) such that \( \partial f / \partial X \) is a non-identically zero first integral of \( X \) and there is a certain first integral \( h \) of \( X \) such that every vector field in \( g \) has the form \( (\varphi_1 f + \varphi_2)X + \alpha h \vec{Y} \) where \( \varphi_1, \varphi_2 \) are first integrals of \( X \). Moreover, for every pair \( (\varphi_1 f + \varphi_2)X + \alpha_1 h \vec{Y} \) and \( (\varphi_3 f + \varphi_4)X + \alpha_2 h \vec{Y} \) of elements in \( g \), the function

\[
\alpha_2 \frac{\partial(\varphi_1 f + \varphi_2)}{\partial \vec{Y}} - \alpha_1 \frac{\partial(\varphi_3 f + \varphi_4)}{\partial \vec{Y}}
\]

lies in \( \mathcal{I}_X \) (this condition is necessary to have a metabelian group as stated in the beginning).

**Case 3** Suppose that \( G \) is not metabelian so that \( k = 2 \). Then the Lie algebra of \( G \) contains a vector field \( \vec{Y} \) not everywhere parallel to \( X \) and satisfying the same condition as in Case 2 (namely \( [\vec{Y}, X] = \tilde{h}X \) where \( \tilde{h} \) is a first integral for \( X \)). Moreover, the solvable Lie algebra of \( G \) can be identified with (a sub-algebra of) the algebra \( g_{\text{step}-3} \) consisting of all formal vector fields having the form

\[
(\varphi_1 f + \varphi_2)X + \alpha h \vec{Y}
\]

where \( \alpha \in \mathbb{C}, \varphi_1, \varphi_2 \) are first integrals of \( X \), and where \( h \) is a fixed first integral of \( X \).

**Remark 6.10** Consider Case 2 and Case 3 above along with the corresponding vector field \( \vec{Y} \). For every first integral \( h \) of \( X \), note that the vector field \( h \vec{Y} \) satisfies the same conditions as \( \vec{Y} \) (as indicated in Case 2). Hence, up to changing the vector field \( \vec{Y} \), we can say that the Lie algebra \( g_{\text{step}-3} \) consists of the vector fields having the form

\[
(\varphi_1 f + \varphi_2)X + \alpha \vec{Y}
\]

where \( \varphi, \varphi_2, \) and \( \alpha \) are as in Case 3. A similar simplification is possible in Case 2.

In view of the above normal form for Lie algebras as in Cases 2 and 3, the following observation will also be useful: suppose that we are given vector fields \( Z_1 \) and \( Z_2 \) in this Lie algebra such that \( Z_1 = fZ_2 \) for some non-constant formal function \( f \). Then \( Z_1 \), and hence \( Z_2 \), must be everywhere parallel to \( X \).

**Remark 6.11** Let us consider more closely the Lie algebra \( g_{\text{abelian}-1} \) of dimension 3. We already known that this Lie algebra is generated by vector fields \( X, Y, \) and \( Z \) such that \( X \) is central and \( [Y, Z] = X \). Moreover, we can assume that the vector fields \( X \) and \( Y \) are not everywhere parallel. Since \( [Z, X] = 0 \), it follows that \( Z \) can be written as \( Z = aX + bY \) for some first integrals \( a \) and \( b \) of \( X \). However the equation \( [Y, Z] = X \) implies that \( b \) is a first integral for \( Y \) as well so that \( b \) is actually a constant. Since in addition \( a \) is a first integral for \( X \), this Lie algebra can be seen as a particular situation of Case 2.
Towards Theorem 2.5: Induced Lie Algebra Maps

In the remainder two sections of this paper, the proof of Theorem 2.5 will finally be completed. In the present section, we shall obtain a number of general auxiliary results allowing us to derive properties about infinitesimal generators from properties involving formal diffeomorphisms in \( \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \). Some of these results hold interest in their own and, in any event, they will come in handy for the proof of Theorem 2.5 provided in the next section.

We begin the discussion with a rather general lemma.

**Lemma 7.1** Suppose we are given a Lie subalgebra \( g_1 \) of \( \hat{X}_2 \) along with a formal diffeomorphism \( F \in \hat{\text{Diff}}_1(\mathbb{C}^2, 0) \) whose infinitesimal generator is denoted by \( Z \). Assume that \( F^*g_1 \subseteq g_1 \). Then there is a well-defined homomorphism \( [Z, .] : g_1 \rightarrow g_1 \) assigning to \( X \in g_1 \) the commutator \( [Z, X] \in g_1 \).

**Proof** The proof amounts to checking that the commutator \( [Z, X] \) lies in \( g_1 \) provided that so does \( X \). To do this, the complex one-parameter group given by the exponential of \( Z \) will be denoted by \( F_t \), \( t \in \mathbb{C} \), so that \( F_1 = F \). The proof of the lemma depends on the following claim:

**Claim** We have \( F_t^*g_{k,1} \subseteq g_{k,1} \) for every \( t \in \mathbb{C} \) and every \( k \in \mathbb{N} \). Up to passing to some conveniently chosen Grassmann space where \( g_{k,1} \) becomes identified with a point, the condition \( F_t^*g_{k,1} \subseteq g_{k,1} \) becomes an algebraic equation on the variable \( t \). Here the fundamental observation leading to the algebraic nature of this equation is the fact that \( F_t \) is unipotent: its infinitesimal generator in the Lie algebra of \( D_k \) is a nilpotent vector field. In turn, the exponential of a nilpotent vector field has polynomial entries on \( t \) since a sufficiently large powers of the corresponding matrix will vanish identically. From this it follows that the subset of \( \mathbb{C} \) consisting of those \( t \in \mathbb{C} \) for which \( F_t^*g_{k,1} \subseteq g_{k,1} \) is a Zariski-closed set. However, this set contains the positive integers \( \mathbb{Z}_+ \) and hence must coincide with all of \( \mathbb{C} \). The claim is proved. \( \square \)

The rest of the proof of Lemma 7.1 relies on Hadamard’s lemma. Note that for every \( t \in \mathbb{C} \) and every vector field \( X \in g_1 \), the vector field \( F_t^*X \) lies in \( g_1 \) as a consequence of the Claim. Therefore, Hadamard’s lemma yields

\[
\frac{1}{t}(F_t^*X - X) = [Z, X] + \frac{t}{2}[Z, [Z, X]] + \cdots.
\]

For every value of \( t \in \mathbb{C}^* \), the left-hand side of the preceding equation lies in \( g_1 \) since both \( F_t^*X \) and \( X \) belong to the Lie algebra \( g_1 \). However, since \( g_1 \) is closed, the limit
of the left-hand side when $t \to 0$ also belongs to $\mathfrak{g}_1$. This limit, however, is clearly equal to $[Z, X]$. The proof of the lemma is completed. \hfill ☐

The following consequence of Lemma 7.1 is worth stating:

**Corollary 7.2** Suppose we are given $F \in \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ and $X \in \mathfrak{X}_2$ such that $F^*X$ is everywhere parallel to $X$. Then the infinitesimal generator $Z$ of $F$ is such that the commutator $[Z, X]$ is everywhere parallel to $X$.

**Proof** Consider the smallest Lie algebra $\mathfrak{g}$ stable under pull-backs by $F$ and containing the vector field $X$. Since $F^*X$ is everywhere parallel to $X$, this Lie algebra is fully constituted by vector fields everywhere parallel to $X$. Now apply Lemma 7.1 to $\mathfrak{g}_1 = \mathfrak{g}$ to conclude that $[Z, X]$ must belong to $\mathfrak{g}$. The corollary follows. \hfill ☐

Another very useful by-product of Lemma 7.1 is as follows:

**Corollary 7.3** Let $\mathfrak{g}_1$ and $F \in \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ be as in Lemma 7.1. Assume that $Z$ is the infinitesimal generator of $F$ and consider a vector field $X$ in $\mathfrak{g}_1$. Then all the iterated commutators $[Z, \ldots [Z, [Z, X]] \ldots]$ lie in $\mathfrak{g}_1$.

**Proof** We already know that $[Z, X]$ belongs to $\mathfrak{g}_1$. Let us check that $[Z, [Z, X]]$ belong to $\mathfrak{g}_1$ as well. Keeping the notation used in the proof of Lemma 7.1, we have that $F_i^*\mathfrak{g}_1 \subseteq \mathfrak{g}_1$ for every $t \in \mathbb{C}$. Now note that

$$\frac{2}{t} \left( \frac{1}{t} (F_i^* X - X) - [Z, X] \right) = [Z, [Z, X]] + \mathcal{O}(t).$$

Again, for every $t \in \mathbb{C}$ the left side of the above equation lies in $\mathfrak{g}_1$ since both $F_i^* X - X$ and $[Z, X]$ do so. By taking the limit as $t \to 0$ we then conclude that $[Z, [Z, X]] \in \mathfrak{g}_1$ as desired. The rest of the proof is a simple induction argument. \hfill ☐

Our next lemma is also rather general and, although slightly technical, it will be very useful in our discussion.

**Lemma 7.4** Assume we are given a set $S \subseteq \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ consisting of $s \geq 2$ formal diffeomorphisms $F_1, \ldots, F_s$. Denote by $Z_i$ the infinitesimal generator of $F_i$, $i = 1, \ldots, s$. Assume that every element in the set $S(1) = \{ [F_i^{\pm 1}, F_j^{\pm 1}] ; \ F_i, F_j \in S \}$ has infinitesimal generator coinciding with a constant $c_{i, j}^{\pm 1}$ multiple of a certain vector field $Y \in \mathfrak{X}_2$. Then for every pair $i, j \in \{1, \ldots, s\}$, the commutator $[Z_i, Z_j]$ coincides with $Y$ times a certain constant in $\mathbb{C}$ (depending on $i$ and $j$). Moreover, in this case, we must have $[Z_i, Y] = [Z_j, Y] = 0$ unless $[Z_i, Z_j] = 0$.

**Proof** According to Campbell–Hausdorff formula in (6), the infinitesimal generator $c_{i, j} Y$ of $F_i \circ F_j \circ F_i^{-1} \circ F_j^{-1}$ is given by

$$c_{i, j} Y = [Z_i, Z_j] + \frac{1}{2} ([Z_i, [Z_i, Z_j]] + [Z_j, [Z_i, Z_j]]) + \cdots.$$  \hfill (14)

Naturally we can assume that $c_{i, j} \neq 0$, otherwise the statement follows from Lemma 5.1. On the other hand, as observed in the proof of Lemma 5.1, the first
non-zero homogeneous component of $[Z_i, Z_j]$ coincides with the first non-zero homogeneous component of the entire right-hand side of (14). In particular, the value of $c_{i,j}$ is determined by comparing the first non-zero homogeneous component of $[Z_i, Z_j]$ with the first non-zero homogeneous component of $Y$.

Now consider the commutator $F_i \circ F_j^{-1} \circ F_i^{-1} \circ F_j$ whose infinitesimal generator is $c_{i,j}^{-1}Y$ where

$$c_{i,j}^{-1}Y = -[Z_i, Z_j] + \frac{1}{2} \left( [Z_i, [Z_i, -Z_j]] + [-Z_j, [Z_i, -Z_j]] \right) + \cdots. \quad (15)$$

Again, $c_{i,j}^{-1}$ is determined by comparing the first non-zero homogeneous components of $Y$ and of $[Z_i, Z_j]$ so that we must have $c_{i,j}^{-1} = -c_{i,j}$. Adding up equations (14) and (15), we obtain

$$0 = [Z_j, [Z_i, Z_j]] + \cdots$$

where the ellipsis stand for terms whose orders are greater than the order of $[Z_j, [Z_i, Z_j]]$. From this, we conclude that $[Z_j, [Z_i, Z_j]]$ must vanish identically. Analogously $[Z_i, [Z_i, Z_j]]$ vanishes identically as well. In turn, the right-hand side of (14) [resp., (15)] becomes reduced to $[Z_i, Z_j]$. The lemma follows at once. \(\square\)

We can now begin a direct approach to the proof of Theorem 2.5 by recalling the general strategy to prove this type of statement. Consider a pseudo-solvable group $G$ along with a finite generating set $S = S(0)$ leading to a sequence of sets $S(j)$ that degenerates into $\{id\}$ for large enough $j \in \mathbb{N}$. Denote by $G(j)$ [resp., $G(j, j-1)$] the subgroup generated by $S(j)$ [resp., $S(j) \cup S(j-1)$]. Let $k$ be the largest integer for which $S(k)$ is not reduced to the identity. It then follows that $G(k)$ is abelian. Similarly, the group $G(k, k-1)$ is solvable. Next denote by $m$ the smallest integer for which $G(m, m-1)$ is solvable. Unless otherwise mentioned, we shall always assume aiming at a contradiction that $m \geq 2$. Recall also that every element $F$ in $S(m-2)$ satisfies the condition

$$F^{\pm 1} \circ G(m-1) \circ F^{\mp 1} \subset G(m, m-1). \quad (16)$$

Actually a slightly more precise formulation of this property is provided by condition (5). Our aim will be to prove that the group generated by $G(m, m-1) \cup S(m-2) = G(m-1, m-2)$ is still solvable which, in turn, will contradict the fact that $m \geq 2$.

At this juncture, it is convenient to single out a couple of simple consequences stemming from condition (16). These are as follows.

- Assume that the group $G(m-1)$ is Zariski-dense in $G(m, m-1)$. Then the two groups share the same Lie algebra and, in fact, they are both Zariski-dense in the exponential of this common Lie algebra. In this case condition (16) implies that $F \in S(m-2)$ must belong to the normalizer of $G(m, m-1)$. This remark will simplify the discussion in Sect. 8 at a couple of points.
- Let $g(m-1)$ (resp., $g(m)$) denote the Lie algebra associated with the group $G(m-1)$ (resp., $G(m)$) while $g(m, m-1)$ will denote the Lie algebra associated with
Clearly \( g(m-1) \subseteq g(m, m-1) \). Condition (16) then implies that \( F^*(g(m-1)) \subseteq g(m, m-1) \).

Keeping the above notation, let us consider in closer detail the fact that \( F^*(g(m-1)) \subseteq g(m, m-1) \). Note that this situation is close to the content of Lemma 7.1 except that we are not certain to also have \( F^*(g(m, m-1)) \subseteq g(m, m-1) \). To overcome this difficulty and be able to exploit Lemma 7.1, a further elaboration on these conditions is needed. To begin the discussion, recall that neither \( G(m) \) nor \( G(m-1) \) is reduced to the identity so that the corresponding Lie algebras \( g(m) \) and \( g(m-1) \) are non-trivial. First, we have:

**Lemma 7.5** Without loss of generality, we can always assume that the dimension of the Lie algebra \( g(m-1) \) is at least 2.

**Proof** The proof amounts to checking that Theorem 2.5 holds whenever the dimension of \( g(m-1) \) is exactly 1. For this we assume once and for all that the dimension of \( g(m-1) \) equals 1 so that every element in \( S(m-1) \) has the same infinitesimal generator \( Y \) up to a multiplicative constant. Consider the formal diffeomorphisms \( F_1, \ldots, F_s \) in the set \( S(m-2) \). The infinitesimal generator of \( F_i \) is denoted by \( Z_i, i = 1, \ldots, s \).

Assume first that to every \( i = 1, \ldots, s \) there corresponds \( j(i) \in \{1, \ldots, s\} \) such that the commutator \([Z_i, Z_{j(i)}] \) does not vanish identically. Under this assumption, Lemma 7.4 immediately implies that the Lie algebra generated by \( Y, Z_1, \ldots, Z_s \) is solvable (actually nilpotent). The proof of Theorem 2.5 follows at once.

Now consider the more general case where there is \( r \leq s - 2 \) such that \( Z_1, \ldots, Z_r \) commute with every \( Z_i, i = 1, \ldots, s \). Moreover, to every \( i \in \{r + 1, \ldots, s\} \) there corresponds \( j(i) \in \{1, \ldots, r\} \) so that \([Z_i, Z_{j(i)}] \) does not vanish identically. The difficulty to apply Lemma 7.4 in this situation lies in the fact that this lemma provides no information on the commutators \([Z_i, Y] \) for \( i = 1, \ldots, r \). The desired information, however, can be derived from Jacobi identity as follows. Given \( Z_i \) with \( i = 1, \ldots, r \), choose two non-commuting vector fields \( Z_{j_1} \) and \( Z_{j_2} \) (in particular, \( j_1, j_2 \in \{r + 1, \ldots, s\} \)). Jacobi identity then yields

\[
0 = \left[ Z_i, \left[ Z_{j_1}, Z_{j_2} \right] \right] + \left[ Z_{j_1}, \left[ Z_{j_2}, Z_i \right] \right] + \left[ Z_{j_2}, \left[ Z_i, Z_{j_1} \right] \right].
\]

Since \([Z_{j_2}, Z_{j_1}] \) is a constant multiple of \( Y \) and \([Z_{j_2}, Z_i] = [Z_i, Z_{j_1}] = 0 \), it follows that \([Z_i, Y] = 0 \). Therefore, the Lie algebra generated by \( Y, Z_1, \ldots, Z_s \) must still be solvable and this yields Theorem 2.5 in the situation in question.

Finally, suppose that \([Z_i, Z_j] = 0 \) for every pair \( i, j \in \{1, \ldots, s\} \). Since \( G(m-1) \) is not reduced to the identity, there must exist an element \( \overline{F} \in S(m-3) \) which does not commute with, say, \( F_1 \). Denoting by \( \overline{Z} \) the infinitesimal generator of \( \overline{F} \), the argument used in the proof of Lemma 7.4 can still be applied to ensure that \([Z_1, \overline{Z}] \) coincides with a constant multiple of \( Y \) whereas \([Z_1, Y] = [\overline{Z}, Y] = 0 \). In particular, \([Z_i, Y] = 0 \) for all those vector fields \( Z_i \) for which \([Z_i, \overline{Z}] \) does not vanish identically \((i \in \{1, \ldots, s\})\). On the other hand, if \([Z_{i_0}, \overline{Z}] = 0 \) for some \( i_0 \in \{1, \ldots, s\} \), then Jacobi identity gives us again

\[
0 = \left[ Z_{i_0}, \left[ Z_1, \overline{Z} \right] \right] + \left[ Z_1, \left[ \overline{Z}, Z_{i_0} \right] \right] + \left[ \overline{Z}, \left[ Z_{i_0}, Z_1 \right] \right].
\]
Since \([Z_{i0}, Z] = [Z_{i0}, Z_1] = 0\) (by assumption) and \([Z_1, Z]\) coincides with a constant multiple of \(Y\), we conclude that \([Z_{i0}, Y]\) = 0 so that the Lie algebra generated by \(Y, Z_1, \ldots, Z_s\) is again solvable. The proof of the lemma is completed. \(\square\)

The technical character of Lemma 7.6 below is due to the fact that it deals with two solvable Lie algebras, namely \(g(m - 1)\) and \(g(m, m - 1)\).

**Lemma 7.6** There is a maximal solvable Lie algebra \(g^\infty(m, m - 1)\) containing \(g(m, m - 1)\) along with another subalgebra \(g^\infty,*(m, m - 1)\) which satisfies the following conditions:

- \(g^\infty,*(m, m - 1)\) contains \(g(m - 1)\)
- \(g^\infty,*(m, m - 1)\) is invariant under the action of \(F\) by pull-backs. Moreover, \(g^\infty,*(m, m - 1)\) is also uniform in the sense that it can be chosen so as to be simultaneously invariant by every formal diffeomorphism \(F \in Diff_1(\mathbb{C}^2, 0)\) fulfilling condition (16).

**Proof** Consider the non-trivial solvable (isomorphic) Lie algebras \(g(m - 1)\) and \(F^*(g(m - 1))\) which are both contained in the solvable Lie algebra \(g(m, m - 1)\). The proof of the lemma relies on the classification of solvable Lie algebras as described in Sect. 6.3. To begin with consider a non-zero vector field \(X \in g(m - 1)\).

**Case 1.** Assume that all vector fields in \(g(m - 1)\) are everywhere parallel to \(X\).

In this case \(F^*(g(m - 1))\) is a Lie algebra formed by mutually everywhere parallel vector fields. Owing to Lemma 7.5 it also follows that the dimension of both Lie algebras \(g(m - 1)\) and \(F^*(g(m - 1))\) is at least 2. Finally, both \(g(m - 1)\) and \(F^*(g(m - 1))\) are subalgebras of the solvable Lie algebra \(g(m, m - 1)\). Direct inspection in the classification of solvable Lie algebras provided in Sect. 6.3 (cf. also Remark 6.10) then shows that both \(g(m - 1)\) and \(F^*(g(m - 1))\) are contained in a Lie algebra of the form \((\varphi_1 f + \varphi_2)X\) (with the notation of Sect. 6.3). In particular \(F^*X\) is everywhere parallel to \(X\). It also follows that first integrals of \(X\) are preserved by \(F\) since \(F^*X\) is everywhere parallel to \(X\). Consider then the sub-algebra of \((\varphi_1 f + \varphi_2)X\) generated by the union of \(g(m - 1)\) and \(F^*(g(m - 1))\). If this Lie algebra has the form \{\(\varphi X\)\}, with \(\varphi\) first integral of \(X\), then it is clear that the (maximal) Lie algebra of this form \{\(\varphi X\)\} is invariant by \(F\) and hence can be taken as \(g^\infty,*(m, m - 1)\).

Suppose now that algebra generated by the union of \(g(m - 1)\) and \(F^*(g(m - 1))\) contains elements of the form \((\varphi_1 f + \varphi_2)X\), with \(\varphi_1\) not identically zero. The claim below shows that the (maximal) Lie algebra \{\((\varphi_1 f + \varphi_2)X\)\} can be chosen as \(g^\infty,*(m, m - 1)\).

**Claim** Under the above conditions the algebra \{\((\varphi_1 f + \varphi_2)X\)\} is invariant by \(F\).

**Proof of the Claim** Recall that \(X\) can be relabeled as any vector field of the form \(\varphi X\) where \(\varphi\) is a first integral for \(X\). To prove the claim it suffices to check that \(F\) takes \(X\) to a vector field of the form \(\varphi X\) as opposed to \((\varphi_1 f + \varphi_2)X\) (where \(\varphi\) stands for some first integral of \(X\)). In particular, if \(g(m - 1)\) is not abelian, then its derived algebra consists of vector fields having the form \(\varphi X\) and \(F\) takes this derived algebra to the derived algebra of \(F^*(g(m - 1))\) which also consists of vector fields having the same form. The claim then follows.
Suppose now that $g(m - 1)$ is abelian with dimension at least 2. Note that these two conditions force this algebra to have the form $\varphi X$ ($\varphi$ first integral of $X$). Thus we can assume that $X$ belongs to $g(m - 1)$. Hence, $F^*X = (\varphi_1 f + \varphi_2)X$ and the claim is reduced to showing that $\varphi_1$ must vanish identically.

Assume aiming at a contradiction that $\varphi$ does not vanish identically and denote by $Z$ the infinitesimal generator of $F$. Note also that $[Z, X] = \alpha X$; cf. Corollary 7.2. Note that the order $\text{ord} \ (\alpha)$ of $\alpha$ at the origin is at least 1 since $X$ and $Z$ lies in $\hat{x}_2$. Next, Hadamard’s lemma gives us

$$F^*X = hX = \left[1 + \alpha + \frac{1}{2} \left(\frac{\partial \alpha}{\partial Z} + \alpha^2\right) + \cdots\right]X.$$ \hspace{1cm} (17)

Thus, twice applying the operator $\partial/\partial X$ to the function $h$ must yield zero. However, a straightforward use of Formula (13) combined with the facts that $\varphi$ and $\phi$ are first integrals of $X$ and that $\text{ord} \ (\alpha) \geq 1$ shows that the order of $\partial \alpha/\partial X^2$ is equal to the order of $\partial^2 h/\partial X^2$. Thus $\partial^2 \alpha/\partial X^2$ must vanish identically so that $\alpha$ itself has the form $\varphi_3 f + \varphi_4$ (with $\varphi_3$ and $\varphi_4$ first integrals of $X$). However, at this point the presence of terms in $\alpha^2$ and so on for the expression (in between brackets) of $h$ shows that $\partial^2 h/\partial X^2$ cannot vanish identically unless $\varphi_1$ does so. The resulting contradiction proves the claim. \hfill \Box

Summarizing we have obtained a suitable algebra $g^{\infty,*}(m, m - 1)$. Note however that our construction does not ensure that $g^{\infty,*}(m, m - 1)$ also contains $g(m, m - 1)$. However, it is again clear from the classification in Sect. 6.3 that the smallest Lie algebra containing both $g^{\infty,*}(m, m - 1)$ and $g^{\infty}(m, m - 1)$ is still a solvable Lie algebra. Thus we can choose $g^{\infty}(m, m - 1)$ to coincide with the Lie algebra generated by the union of $g^{\infty,*}(m, m - 1)$ and $g^{\infty}(m, m - 1)$. This proves the lemma in Case 1.

**Case 2** Assume that $g(m - 1)$ contains a vector field that is not everywhere parallel to $X$.

According to the discussion in Sect. 6.3, the solvable Lie algebra $g(m, m - 1)$ either is isomorphic to the 3-dimensional Lie algebra $g_{\text{abelian} - 1}$ or is as in Case 2 or Case 3 of Sect. 6.3. However owing to Remark 6.11, the case in which $g(m, m - 1)$ coincides with $g_{\text{abelian} - 1}$ can be included in the discussion of Case 2. Hence to complete the proof of the lemma, it only remains to consider the possibility of having $g(m, m - 1)$ as in the mentioned Cases 2 and 3. If the dimension of $g(m - 1)$ equals 3 or greater, then this Lie algebra must contain two linearly independent vector fields everywhere parallel to $X$. Hence, as in Case 1, $F$ preserves the Lie algebra formed by vector fields of the form $(\varphi_1 f + \varphi_2)X$ unless the sub-algebra containing all vector fields everywhere parallel to $X$ has the form $\{\varphi X\}$. In any event we obtain a suitable invariant algebra. On the other hand, since $g(m - 1)$ also contains a vector field of the form $(\varphi_1 f + \varphi_2)X + eY$, we also conclude that $F^*(eY)$ must coincide with a constant multiple of $Y$ up to adding another vector field of the form $(\varphi_1 f + \varphi_2)X$. The lemma results as once in this case.

Suppose now that the dimension of $g(m - 1)$ equals 2 and consider non-everywhere parallel vector fields $X$ and $\bar{Y}$ in $g(m - 1)$. As already seen, the Lie algebra $g(m - 1)$ must be abelian. In particular, $\bar{Y}$ yields a representation of the Lie algebra $g(m, m - 1)$ by vector fields of the form $\partial Z/\partial Z$.\hfill \Box
that this is not the case. Thus $F$ a priori need not take $X$ to a vector field everywhere parallel to $X$. However if it can be shown that $F^*X$ and $X$ are everywhere parallel then the argument used in the previous cases still apply.

To show that $F^*X$ and $X$ are indeed everywhere parallel, suppose for a contradiction that this is not the case. Thus $F^*X = aX + cY$ for some $c \in \mathbb{C}$. Up to replacing $Y$ by a suitable linear combination of the initial vector fields $X$ and $\overline{Y}$, we can assume that $F^*Y = (\varphi_1 f + \varphi_2)X$, i.e., $F^*Y$ is everywhere parallel to $X$. This immediately implies that $F$ takes first integrals of $X$ to first integrals of $Y$. In fact, $D(\varphi \circ F).Y = D\varphi.DF.Y = D\varphi.hX = 0$ provided that $\varphi$ is a first integral of $X$. Furthermore, the (non-trivial) vector field induced by $X$ in the (formal) leaf space of $Y$ coincides (up to diffeomorphism of leaf spaces) with the vector field induced by $cY$ on the (formal) leaf space of $X$. From here it is straightforward to reduce singularities as in [17] and to conclude that $F$ must preserve a suitable sub-algebra of $g(m, m - 1)$. The proof of the lemma is completed.

The combination of Lemma 7.1, Corollary 7.3, and Lemma 7.6 immediately yields the following lemma:

**Lemma 7.7** With the preceding notation, consider an element $F \in S(m - 2)$ and denote by $Z$ its infinitesimal generator. Then for every vector field $X \in g(m - 1)$, the commutator $[Z, X]$ lies in $g^{\infty,*}(m, m - 1) \subseteq g^{\infty}(m, m - 1)$. In fact, all the iterated commutators $[Z, \ldots [Z, [Z, X]] \ldots]$ lie in $g^{\infty,*}(m, m - 1) \subseteq g^{\infty}(m, m - 1)$. \hfill \square

We close this section with Lemma 7.8 below. Whereas this lemma is slightly unrelated to the preceding material, it will be rather useful in the next section.

**Lemma 7.8** Assume that $Z_1$, $Z_2$, and $X$ are vector fields in $\mathcal{X}_2$ satisfying the following conditions:

- $Z_2 = aX + bZ_1$ and $[Z_1, X]$ is everywhere parallel to $X$.
- $b$ is a first integral of $X$.
- The time-one maps $F_1$ and $F_2$ induced respectively by $Z_1$ and $Z_2$ are such that the infinitesimal generator of $F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1}$ has the form $hX$ for some first integral $h$ of $X$.

Then the commutator $[Z_1, Z_2]$ is everywhere parallel to $X$ or, equivalently, $b$ is a first integral of $Z_1$.

**Proof** We can assume that $b$ is not identically zero, otherwise the statement is clear. Similarly we can assume that $Z_1$ is not everywhere parallel to $X$. Note that $Z_1$ and $Z_2$ have similar properties. More precisely both $[Z_1, X]$ and $[Z_2, X]$ are everywhere parallel to $X$ since $b$ is a first integral of $X$. In particular, they both derive first integrals of $X$ into first integrals of $X$. Also none of these vector fields is everywhere parallel to $X$. Denoting by ord $(b)$ the order of the formal function $b$ at $(0, 0) \in \mathbb{C}^2$, first note the following:

**Claim** Without loss of generality we can assume that ord $(b) \geq 0$.  

\[ (\varphi_1 f + \varphi_2)X + ch\overline{Y} \]

where $c \in \mathbb{C}$ and $h$ is some fixed first integral of $X$. The difference in this case lies in the fact that $F$ a priori need not take $X$ to a vector field everywhere parallel to $X$. However if it can be shown that $F^*X$ and $X$ are everywhere parallel then the argument used in the previous cases still apply.
Proof of the Claim

As observed above, the roles of $Z_1$ and $Z_2$ are interchangeable. Thus we can work either with $Z_2 = aX + bZ_1$ or with $Z_1 = \tilde{a}X + \tilde{b}Z_2$. A direct inspection in the formulas for the coefficients $a$, $b$ and $\tilde{a}, \tilde{b}$ then shows that $\tilde{b} = 1/b$. In fact, modulo considering the obvious extensions of these vector fields to $\mathbb{C}^3$ the vector product (denoted by $\wedge$) of the various vector fields in question becomes well defined. All these vector products are pairwise parallel since their only non-zero component necessarily corresponds to the “third” (added) component. Now just note that $b$ equals the ratio of $X \wedge Z_2$ and $X \wedge Z_1$ whereas $\tilde{b}$ is the ratio of $X \wedge Z_1$ and $X \wedge Z_2$. The claim results at once. □

Assuming then $\text{ord}(b) \geq 0$, we shall use the Campbell–Hausdorff formula in (6). More precisely, note that

$$[Z_1, Z_2] = a_1 X + \frac{\partial b}{\partial Z_1} Z_1.$$ 

Assume aiming at a contradiction that $\partial b/\partial Z_1$ does not vanish identically and denote by $\text{ord}(\partial b/\partial Z_1)$ the order of $\partial b/\partial Z_1$ at the origin. Note that $\text{ord}(\partial b/\partial Z_1)$ is strictly greater than the order of $b$ since the linear part of $Z_1$ at the origin vanishes. Hence, we have $\text{ord}(\partial b/\partial Z_1) \geq 1$.

The proof is reduced to check that the components in the direction of $Z_1$ of all the remaining terms in Campbell–Hausdorff formula (6) have order strictly larger than the order of $(\partial b/\partial Z_1)Z_1$. In the sequel the reader is reminded that $b$ and all its derivatives with respect to $Z_1$ are first integrals for $X$. We begin with the term

$$\frac{1}{2} ([Z_1, [Z_1, Z_2]] + [Z_2, [Z_1, Z_2]]).$$

Recalling that $[Z_1, Z_2] = a_1 X + (\partial b/\partial Z_1)Z_1$, we first obtain

$$[Z_1, [Z_1, Z_2]] = [Z_1, a_1 X + (\partial b/\partial Z_1)Z_1] = a_2 X + \frac{\partial^2 b}{\partial Z_1^2} Z_1.$$

Since the linear part of $Z_1$ at the origin equals zero, it follows that the order of $(\partial^2 b/\partial Z_1^2)Z_1$ is strictly greater than the order of $(\partial b/\partial Z_1)Z_1$ as desired. Concerning the term $[Z_2, [Z_1, Z_2]]$, we have

$$[Z_2, [Z_1, Z_2]] = [aX + bZ_1, a_1 X + (\partial b/\partial Z_1)Z_1]$$

$$= a_2 X + \left( b \left( \frac{\partial^2 b}{\partial Z_1^2} - \left( \frac{\partial b}{\partial Z_1} \right)^2 \right) \right) Z_1.$$ 

Since $\text{ord}(b) \geq 0$, it follows again that the order of $b(\partial^2 b/\partial Z_1^2)Z_1$ is strictly greater than the order of $(\partial b/\partial Z_1)Z_1$. Similarly, since $\text{ord}(\partial b/\partial Z_1) \geq 1$, the order of $(\partial b/\partial Z_1)^2 Z_1$ is strictly greater than the order of $(\partial b/\partial Z_1)Z_1$. The proof of the lemma now results from a straightforward induction argument. □
8 Proof of Theorem 2.5

To better organize the discussion, Theorem 2.5 will be proved by gradually increasing the complexity of the solvable group \( G(m, m - 1) \). The simplest possible structure for \( G(m, m - 1) \) corresponds to an abelian group and this case is handled by Proposition 8.1 below. In the sequel we always keep the notation used in Sect. 6.

**Proposition 8.1** Assume that the group \( G(m, m - 1) \subset \text{Diff}_1(\mathbb{C}^2, 0) \) is abelian. Then the initial group \( G \) is solvable.

**Proof** Note that, by definition, none of the sets \( S(m) \) and \( S(m - 1) \) is reduced to the identity. The dimension of the Lie algebra \( g(m, m - 1) \) is finite since \( G(m, m - 1) \) is abelian and finitely generated. On the other hand, Lemma 7.5 ensures that the dimension of \( g(m - 1) \), and hence the dimension of \( g(m, m - 1) \), is at least 2.

According to Corollary 6.2, either \( g(m, m - 1) \) coincides with the linear span of two vector fields \( X \) and \( Y \) or it is generated by vector fields having the form \( hX \) where \( h \) is a first integral of \( X \). Since the dimension of \( g(m, m - 1) \) is at least 2, in the latter case it also follows that \( g(m, m - 1) \) contains \( X \) and some other vector field \( Y = hX \), where \( h \) is a non-constant first integral of \( X \).

Assume first that \( g(m, m - 1) \) coincides with the linear span of vector fields \( X \) and \( Y \). Since the dimension of \( g(m - 1) \) is at least 2, it follows that these two Lie algebras should coincide. In other words, relation (16) implies that every diffeomorphism \( F \in S(m - 2) \) should leave \( g(m, m - 1) \) invariant. By virtue of the material in Sect. 6.2, we conclude that \( G(m - 1, m - 2) \) is a subgroup of \( \Gamma_{\text{abelian-1}} \) and hence it is solvable.

The proof of our proposition is now reduced to the case in which \( g(m, m - 1) \) consists of vector fields having the form \( \{hX\}, h \) first integral of \( X \). Also, we can assume that both \( X \) and some vector field \( Y = hX \) lie in \( g(m - 1) \), where \( h \) is a non-constant first integral of \( X \).

Assume first that \( g(m - 1) \) coincides with \( g(m, m - 1) \). In this case relation (16) again implies that every diffeomorphism \( F \in S(m - 2) \) should leave \( g(m, m - 1) \) invariant. Thus \( G(m - 1, m - 2) \) is a subgroup of \( \Gamma_{\text{abelian-2}} \) and hence solvable. In other words, we can assume that \( g(m - 1) \) is strictly contained in \( g(m, m - 1) \).

Let \( S(m - 2) = \{F_1, \ldots, F_s\} \) and denote by \( Z_i \) the infinitesimal generator of \( F_i \), \( i = 1, \ldots, s \). For a fixed \( i \), we know that \( F_i^+X \) and \( F_i^+Y = F_i^+(hX) \) are both multiples of \( X \) by first integrals. It follows that the Lie algebra \( \{\varphi X\} \) consisting of all vector fields of the form \( \varphi X \), where \( \varphi \) is a first integral of \( X \), is left invariant by all the formal diffeomorphisms \( F_i \), \( i = 1, \ldots, s \). In other words, in the statement of Lemma 7.7, we can choose \( g^\infty,*(m, m - 1) = g^\infty(m, m - 1) = \{\varphi X\} \). In particular, Lemma 7.7 yields

\[
[Z_i, X] = h_iX
\]

for some first integral \( h_i \) of \( X \), \( i = 1, \ldots, s \).

Assume now that all the vector fields \( Z_i \) are everywhere parallel to \( X \). Let \( Z_i = a_iX \).

Since \( [Z_i, X] = h_iX \), it follows that \( \partial a_i/\partial X \) is a first integral of \( X \). Therefore, the Lie algebra \( g(m - 1, m - 2) \) is as in Case 1 of Sect. 6.3. In particular, \( g(m - 1, m - 2) \) is solvable and the proposition follows.
To complete the proof of the proposition there only remains to consider the case where not all the vector fields \(Z_1, \ldots, Z_s\) are everywhere parallel to \(X\). We can then assume that \(Z_1\) is not everywhere parallel to \(X\). Note, however, that all the vector fields \(Z_i\) still derive first integrals of \(X\) into first integrals of \(X\) since \([Z_i, X] = h_i X\) [see Formula (13)]. Now for \(i \in \{2, \ldots, s\}\), we set \(Z_i = a_i X + b_i Z_1\). Since \([Z_i, X] = h_i X\), we conclude that both \(\partial a_i / \partial X\) and \(b_i\) are first integral of \(X\). Owing to Lemma 7.8, we therefore conclude that \([Z_1, Z_i]\) is everywhere parallel to \(X\), for every \(i = 1, \ldots, s\). However, the condition of having \([Z_1, Z_i]\) everywhere parallel to \(X\) implies that \(b_i\) must be a first integral for \(Z_1\). Therefore, \(b_i\) is actually constant since it is also a first integral for \(X\) (and \(X\) and \(Z_1\) are not everywhere parallel). The solvable nature of the Lie algebra in question is now clear and this completes the proof of the proposition. □

From now on we always assume that the finitely generated solvable group \(G(m, m - 1)\) is not abelian. Denote by \(D^s G(m, m - 1)\) the non-trivial derived subgroup of \(G(m, m - 1)\) having highest order \(s\). \(D^s G(m, m - 1)\) is also the only non-trivial abelian derived subgroup of \(G(m, m - 1)\). Furthermore, we already know that \(s \in \{1, 2\}\). Note however that the non-trivial abelian group \(D^s (m, m - 1)\) may fail to be finitely generated. The abelian Lie algebra associated with \(D^s G(m, m - 1)\) will be denoted by \(D^s g(m, m - 1)\). Then, we have:

**Lemma 8.2** Suppose that \(D^s g(m, m - 1)\) coincides with the linear span of two vector fields \(X\) and \(Y\). Then the initial group \(G\) is solvable.

**Proof** To begin with let \(S(m - 2) = \{F_1, \ldots, F_s\}\). The infinitesimal generator of \(F_i\) will be denoted by \(Z_i, i = 1, \ldots, s\). Recall that \(D^s G(m, m - 1)\) is a normal subgroup of \(D^{s-1} G(m, m - 1)\) which, in turn, is not an abelian group. Thus Lemma 6.6 ensures that \(D^{s-1} G(m, m - 1) \subset G(m, m - 1)\) is isomorphic to a non-abelian group of \(\Gamma_\text{abelian}^{-1}\). In turn, Lemma 6.8 shows that the normalizer of \(D^{s-1} G(m, m - 1)\) is metabelian which implies that \(D^{s-1} G(m, m - 1) = G(m, m - 1), i.e., we necessarily have \(s = 1\). In other words, \(G(m, m - 1)\) is isomorphic to a non-abelian subgroup of \(\Gamma_\text{abelian}^{-1}\) and hence \(g(m, m - 1)\) coincides with the Lie algebra \(g_\text{abelian}^{-1}\) of \(\Gamma_\text{abelian}^{-1}\).

Consider the Lie algebra \(g(m - 1)\). We can assume that \(g(m - 1)\) is strictly contained in \(g(m, m - 1)\) otherwise \(S(m - 2)\) is contained in the normalizer of \(\Gamma_\text{abelian}^{-1}\) which is again a solvable group (see Lemma 6.9). Similarly, owing to Lemma 7.5, we can assume that the dimension of \(g(m - 1)\) is strictly larger than \(1\). Hence, the Lie algebra \(g(m - 1)\) must have dimension equal to \(2\) and, since it is a sub-algebra of \(g(m, m - 1) \simeq g_\text{abelian}^{-1}\), it must be abelian.

Being abelian, \(g(m - 1)\) must contain \(X\) since \(X\) lies in the center of \(g_\text{abelian}^{-1}\). Recalling that \(X\) is distinguished in \(g_\text{abelian}^{-1}\) as the vector field of maximal order (up to constant multiples), it follows that \(X\) also belongs to \(F_i^* g(m - 1) \subset g(m, m - 1) \simeq g_\text{abelian}^{-1}\), for every \(i = 1, \ldots, s\). Indeed, for every \(F_i \in S(m - 2)\), we must have \(F_i^* X = X\) since \(F_i\) is unipotent.

Now consider another vector field \(\bar{Y}\) in \(g(m - 1)\) which is linearly independent with \(X\). If \(\bar{Y}\) is everywhere parallel to \(X\), then the abelian sub-algebra generated by \(X\) and by \(\bar{Y}\) is the (unique) maximal abelian sub-algebra of \(g_\text{abelian}^{-1}\) consisting of vector fields everywhere parallel to \(X\) (by assumption \(Y\) is not everywhere parallel to \(X\); see also Remark 6.11). A similar conclusion holds for the Lie algebra \(F_i^* g(m - 1) \subset...
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$\mathfrak{g}(m, m - 1) \simeq \mathfrak{g}_{\text{abelian}} - 1$, since $F_i^* X = X$. From the maximal character of the Lie algebras in question, we therefore conclude that $F_i^* \mathfrak{g}(m - 1) = \mathfrak{g}(m - 1)$; i.e., $\mathfrak{g}(m - 1)$ is invariant by $F_i$, for every $i = 1, \ldots, s$. In other words, $S(m - 2)$ is contained in the normalizer of a two-dimensional abelian Lie algebra and hence the group generated by $S(m - 1) \cup S(m - 2)$ is solvable; see Lemma 6.7.

We can now assume that $X$ and $\bar{Y}$ are not everywhere parallel. We still have $F_i^* X = X$ which implies that $[Z_i, X] = 0$. On the other hand, $\bar{Y} = \bar{a}X + \text{const}Y$, cf. Remark 6.11. To complete the proof we now proceed as follows. For $i$ fixed, $F_i^* (\bar{Y}) = (\bar{a} \circ F_i)X + \bar{b} F_i^* Y$. Since this vector field still belongs to $\mathfrak{g}(m, m - 1) \simeq \mathfrak{g}_{\text{abelian}} - 1$, we conclude that $F_i^* Y$ is still a constant. Since $F_i$ is tangent to the identity this constant must be 1. Hence, $F_i$ preserves both $X$ and $Y$. As already seen, this implies that $F_i$ is contained in the abelian group generated by the exponentials of $X$ and $Y$. It is now clear that $G(m - 1, m - 2)$ is still solvable and this completes the proof of the lemma. □

Owing to Lemma 8.2 we assume in what follows that $D^s \mathfrak{g}(m, m - 1)$ is constituted by vector fields having the form $\{ \varphi X \}$, where $\varphi$ is a first integral for $X$ (by way of notation, $X$ is assumed to belong to $D^s \mathfrak{g}(m, m - 1)$). Note that the dimension of $D^s \mathfrak{g}(m, m - 1)$ is finite if and only if $D^s G(m, m - 1)$ is finitely generated. In this case, the group $D^{s-1} G(m, m - 1)$ has non-trivial center: since inner automorphisms of $D^{s-1} G(m, m - 1)$ leave $D^s G(m, m - 1)$ invariant, they must also leave invariant those vector fields in $D^s \mathfrak{g}(m, m - 1)$ having maximal order at the origin of $\mathbb{C}^2$ (as follows from Hadamard’s lemma; cf. Sect. 5). In turn, the center of $D^{s-1} G(m, m - 1)$ must be contained in the exponential of a single vector field $X$ otherwise a contradiction would arise from Lemma 6.4. Thus those elements in the intersection of the exponential of $X$ with the group $D^s G(m, m - 1)$ lie in the center of $D^{s-1} G(m, m - 1)$ proving our assertion. Next note that the center of $G(m, m - 1)$ is non-trivial if and only if the center of $D^{s-1} G(m, m - 1)$ is non-trivial. In fact, if the center of $D^{s-1} G$ is non-trivial, then the chain of normal subgroups

$$D^s G(m, m - 1) \triangleleft D^{s-1} G(m, m - 1) \triangleleft \cdots \triangleleft G(m, m - 1)$$

implies that $D^{s-1} G$ normalizes $D^s G$ so that $D^{s-1} G$ should also normalize the center of $D^s G$. Hence, the center of $D^{s-1} G$ lies also in the center of $G$. The converse is clear.

In the general case, however, the center of $G(m, m - 1)$ may be trivial. Furthermore, (non-abelian) solvable subgroups of $\text{Diff}_1(\mathbb{C}^2, 0)$ having non-trivial center are easy to characterize. In fact, let $G$ be a (non-abelian) solvable subgroup of $\text{Diff}_1(\mathbb{C}^2, 0)$ and denote by $D^s G$ the (non-trivial) abelian derived subgroup of $G$ (it is the non-trivial derived subgroup of maximal order $s$).

**Lemma 8.3** Let $G$ and $D^s G$ be as above. Assume that the Lie algebra $D^s \mathfrak{g}$ associated with $D^s G$ is constituted by vector fields everywhere parallel to a certain vector field $X$. Assume also that $G$ has non-trivial center. Then $s = 1$. Moreover, the Lie algebra $\mathfrak{g}$ associated with $G$ is constituted by vector fields of the form $\alpha X + a Y$ where $a$ is a first integral of $X$ and where $\alpha \in \mathbb{C}$. Moreover, $X, Y$ are non-everywhere parallel commuting vector fields. In particular, the center of $G$ is (non-trivial and) contained in $\text{Exp}(t X)$.
Proof Whereas this result can be derived from the discussion in Sect. 6.3, it is convenient to provide a self-contained argument. Owing to the previous discussion, we assume that $X \in D^1\mathfrak{g}$ is such that its exponential contains the center of $D^{s-1}G$ and of $G$. Therefore, every vector field in $\mathfrak{g}$ has the form $aX + bY$ where $a, b$ are first integrals of $X$ and where $Y$ is a vector field commuting with $X$ and not everywhere parallel to $X$. The reader will also note that a vector field $Y$ as indicated must exist since $G$ would be abelian otherwise.

Now consider the Lie algebra $D^{s-1}\mathfrak{g}$ associated with $D^{s-1}G$. The commutator of two vector fields $Z_1, Z_2 \in D^{s-1}\mathfrak{g}$ must be contained in $D^s\mathfrak{g}$ and hence it must have the form $hX$ where $h$ is some first integral of $X$. Setting $Z_1 = a_1X + b_1Y$ and $Z_2 = a_2X + b_2Y$, the preceding implies that $b_1/b_2$ must be a constant unless one between $b_1, b_2$ vanishes identically. In other words, there must exist a function $f \in \mathbb{C}((x, y))$ such that the following holds:

**Claim** Every vector field $Z \in D^{s-1}\mathfrak{g}$ has the from $Z = aX + \alpha fY$ where $a$, $f$ are first integrals of $X$ and $\alpha$ is a constant in $\mathbb{C}$ depending on $Z$. \hfill $\square$

We also note that the general form of the quotient $b_1/b_2$ satisfies the co-cycle relation $(b_1/b_2)(b_2/b_3) = b_1/b_3$ which is necessary to have a well-defined Lie algebra. Furthermore, the vector fields in $D^s\mathfrak{g}$ sits inside the above-mentioned form (just take $\alpha = 0$).

Suppose now that $s \geq 2$ so that the Lie algebra $D^{s-2}\mathfrak{g}$ can be considered. The preceding argument can thus be repeated: let $Z_1, Z_2$ be vector fields in $D^{s-2}\mathfrak{g}$ leading to a commutator $[Z_1, Z_2]$ in $D^{s-1}\mathfrak{g} \setminus D^s\mathfrak{g}$. Letting $Z_1 = a_1X + b_1Y$ and $Z_2 = a_2X + b_2Y$, we obtain $\partial/(b_1/b_2)/\partial Y = \alpha f$ for some $\alpha \in \mathbb{C}$ and for $f$ as in the above claim. Naturally it can be supposed that $f$ is not a constant. If $H$ is a specific function satisfying $\partial H/\partial Y = f$, then the quotient $b_1/b_2$ has the general form $\alpha H + \varphi$ where $\varphi$ is a first integral of $Y$. Nonetheless, to have a well-defined Lie algebra, we still need to check the co-cycle relation $(b_1/b_2)(b_2/b_3) = b_1/b_3$. In particular, $b_2/b_1$ must admit the same pattern, i.e., we must have $b_2/b_1 = \alpha \varphi + \varphi$, for a suitable constant $\alpha \in \mathbb{C}$ and first integral $\varphi$ of $Y$. Furthermore, the fact that $(b_1/b_2)(b_2/b_1) = 1$ immediately leads to $\alpha \varphi H^2 + H(\alpha \varphi + \alpha \varphi) + \varphi^2 = 1$. Therefore, by taking the derivative with respect to $Y$, we obtain

$$
(2\alpha \varphi H + \alpha \varphi + \alpha \varphi) \cdot \frac{\partial H}{\partial Y} = (2\alpha \varphi H + \alpha \varphi + \alpha \varphi) \cdot f = 0.
$$

Since $f$ is not identically zero, it follows that $H$ must be a first integral for $Y$ since $\varphi, \varphi$ are so. In any event, a contradiction arises at once. From this contradiction, we conclude that $s$ equals 1. The lemma then follows by replacing $Y$ by $fY$; cf. Remark 6.10. \hfill $\square$

In what follows we always set $S(m-2) = \{F_1, \ldots, F_s\}$ while the infinitesimal generator of $F_i$ will be denoted by $Z_i$. Before discussing the case in which the group $G(m, m - 1) \subset \text{Diff}_1(\mathbb{C}^2, 0)$ has non-trivial center, it is convenient to settle the following special case:

\[ \text{Springer} \]
Lemma 8.4 Assume that $G(m, m - 1)$ is as in Case 1 of Sect. 6.3; i.e., the infinitesimal generator of every element in $G(m, m - 1)$ is parallel to a certain vector field $X$ (and $G(m, m - 1)$ is not abelian). Then the initial group $G$ is solvable.

Proof The Lie algebra $\mathfrak{g}(m, m - 1)$ associated with $G(m, m - 1)$ is formed by vector fields having the form $(\varphi_1 f + \varphi_2)X$ where $\varphi_1, \varphi_2$ are first integrals of $X$ and where $f$ satisfies $\partial f/\partial X = \tilde{h}$ for some non-identically zero first integral $\tilde{h}$ of $X$. Furthermore, by virtue of Lemma 7.5, we can assume that the dimension of $\mathfrak{g}(m - 1)$ is at least 2.

Given $Y \in \mathfrak{g}(m - 1)$ and $i \in \{1, \ldots, s\}$, the vector field $F_i^* Y$ lies in $\mathfrak{g}(m, m - 1)$ and hence is everywhere parallel to $X$. In particular, the solvable Lie algebras $\mathfrak{g}^{\infty,*}(m, m - 1)$ and $\mathfrak{g}^{\infty}(m, m - 1)$ (in Lemma 7.6) are constituted by vector fields everywhere parallel to $X$. Hence, they are still contained in the above-indicated Lie algebra whose elements have the form $\{(\varphi_1 f + \varphi_2)X\}$. Now, for $Y \in \mathfrak{g}(m - 1)$, Lemma 7.7 ensures that the commutator $[Z_i, Y]$ lies in $\{(\varphi_1 f + \varphi_2)X\}$. Since all the vector fields are everywhere parallel to $X$, we conclude that $[Z_i, X]$ is everywhere parallel to $X$ as well. In particular, $Z_i$ derives first integrals of $X$ into first integrals of $X$.

Suppose now that all the vector fields $Z_i$ are everywhere parallel to $X$. Set $Z_i = a_i X$ and let $[Z_i, X] = (-2\varphi_{1,i} \tilde{h} f - \varphi_{2,i} \tilde{h}) X$. We then have $\partial a_i / \partial X = 2 \varphi_{1,i} \tilde{h} f + \varphi_{2,i} \tilde{h}$ so that $a_i = \varphi_{1,i} f^2 + \varphi_{2,i} f + \varphi_{3,i}$, where $\varphi_{3,i}$ is another first integral for $X$. Another application of Lemma 7.7 ensures that $[Z_i, [Z_i, X]]$ belongs to $\{(\varphi_1 f + \varphi_2)X\}$ as well. A direct computation of $[Z_i, [Z_i, X]]$, however, yields

$$[Z_i, [Z_i, X]] = 2\varphi_{1,i}^2 \tilde{h}^2 f^2 + \varphi_{2,i} f + \varphi_{3,i}$$

for suitable first integrals $\varphi_{2,i}, \varphi_{3,i}$ of $X$. Since $[Z_i, [Z_i, X]]$ lies in $\mathfrak{g}(m, m - 1)$, we conclude that $\varphi_{1,i}$ vanishes identically. Therefore, the Lie algebra generated by $\mathfrak{g}(m - 1)$ and the vector fields $Z_1, \ldots, Z_s$ is still solvable.

It remains to consider the case in which not all the vector fields $Z_i$ are everywhere parallel to $X$. We can then assume that $Z_1$ is not everywhere parallel to $X$. We then set $Z_i = a_i X + b_i Z_1$, for $i = 2, \ldots, s$. Since $[Z_i, X]$ is everywhere parallel to $X$, we still conclude that $b_i$ is a first integral of $X$. On the other hand, the Lie algebra $\mathfrak{g}(m - 1)$ contains the infinitesimal generators of the commutators $F_i \circ F_i \circ F_i^* \circ F_i^{-1}$ so that Lemma 7.8 ensures that $[Z_1, Z_i]$ is everywhere parallel to $X$. In other words, all the coefficients $b_i$ are constants in $\mathbb{C}$. Hence, to complete the proof of the lemma it suffices to check that $a_i$ has the form $\varphi_{1,i} f + \varphi_{2,i}$ for suitable first integrals $\varphi_{1,i}$ and $\varphi_{2,i}$ of $X$. This straightforward verification is left to the reader since it amounts to keeping track of the components parallel to $X$ of the preceding vector fields. The proof of the lemma is completed.

Now we state:

Proposition 8.5 Keeping the preceding notation, assume that the non-abelian solvable group $G(m, m - 1) \subset \text{Diff}_1(\mathbb{C}^2, 0)$ has non-trivial center. Then the initial group $G$ is solvable.
Proof We keep the preceding notation so that \( S(m - 2) = \{ F_1, \ldots, F_s \} \) and the infinitesimal generator of \( F_i \) is denoted by \( Z_i \). By assumption, the solvable Lie algebra \( \mathfrak{g}(m, m - 1) \) is an in Lemma 8.3. Recall also Lemma 7.5 allows us to assume that the dimension of \( \mathfrak{g}(m - 1) \subset \mathfrak{g}(m, m - 1) \) is at least 2.

The proof of the proposition will be split into two cases according to whether or not \( \mathfrak{g}(m - 1) \) is abelian.

Case A. Assume that \( \mathfrak{g}(m - 1) \) is abelian.

We begin by considering the abelian sub-algebras of \( \mathfrak{g}(m, m - 1) \) having dimension at least 2. Owing to the description of \( \mathfrak{g}(m, m - 1) \) provided by Lemma 8.3, these algebras fall into two classes, namely:

1. Lie algebras of dimension 2 containing non-everywhere parallel vector fields. This type of Lie algebra has one of the following forms:
   - It may be generated by \( X \) and by another vector field \( Y \) having the form \( aX + \alpha Y \), with \( \alpha \in \mathbb{C}^* \).
   - It may be generated by vector fields of the form \( aX + \alpha Y \) and \( cX + \beta Y \) where \( c, \alpha \) and \( \beta \) are all constants. Moreover, \( c \neq \beta/\alpha \).

2. Lie algebras constituted by vector fields that are everywhere parallel to \( X \) (and hence of the form \( \varphi X \) for some first integral \( \varphi \) of \( X \)).

Consider first the case where \( \mathfrak{g}(m - 1) \) is as in item (1) above. Consider also the Lie algebras \( \mathfrak{g}^{\infty,*}(m, m - 1) \) and \( \mathfrak{g}^{\infty}(m, m - 1) \) provided by Lemma 7.6. The solvable Lie algebra \( \mathfrak{g}^{\infty,*}(m, m - 1) \) contains \( \mathfrak{g}(m - 1) \) and hence it is not fully constituted by vector fields everywhere parallel to \( X \). Therefore, it must be as in Cases 2 or 3 of Sect. 6.3.

Fix \( \overline{Y} \in \mathfrak{g}(m - 1) \) which is not everywhere parallel to \( X \). Hence, we have \( \overline{Y} = aX + \alpha Y \) for some \( \alpha \in \mathbb{C}^* \). According to Lemma 7.7, all the iterated commutators \( [Z_i, \ldots [Z_i, \overline{Y}] \ldots] \) lie in \( \mathfrak{g}^{\infty,*}(m, m - 1) \). However, as we iterate these commutators, the orders of the resulting vector fields keep increasing strictly since the linear parts of all the involved vector fields are zero. Since the components in the direction \( Y \) have all fixed order (they only differ by a multiplicative constant), it follows that some sufficiently high commutator will be everywhere parallel to \( X \). A further iteration of this commutator will still be everywhere parallel to \( X \) for the same reason. From this it follows that \( [Z_i, X] \) must be everywhere parallel to \( X \). Moreover, we also have:

Claim \( [Z_i, Y] \) is everywhere parallel to \( X \).

Proof of the Claim Consider the first iterated commutator \( [Z_i, [Z_i, \ldots [Z_i, \overline{Y}] \ldots]] \) which is everywhere parallel to \( X \). The preceding iterated commutator \( [Z_i, \ldots [Z_i, \overline{Y}] \ldots] \) then still has the form \( aX + \beta Y \) for some \( \beta \in \mathbb{C}^* \). Therefore, the commutator \( [Z_i, aX + \beta Y] \) is everywhere parallel to \( X \). However the commutator \( [Z_i, aX] \) is everywhere parallel to \( X \) as well since so is \( [Z_i, X] \). Therefore, the commutator \( [Z_i, \beta Y] \) must be everywhere parallel to \( X \) as well and this completes the proof of the claim.

Next set \( Z_i = a_iX + b_iY \). Since \( [Z_i, X] \) is everywhere parallel to \( X \), it follows that \( b_i \) is a first integral of \( X \). Similarly \( b_i \) is also a first integral of \( Y \) since \( [Z_i, Y] \) is everywhere parallel to \( X \). In other words, \( b_i \) is constant. Finally, \( a_i \) must have the
form $\varphi_1 f + \varphi_2$ as now follows from considering commutators $[Z_i, \overline{Y}_1]$ and $[Z_i, \overline{Y}_2]$ in $g^{\infty,*}(m, m - 1)$ for two linearly independent vector fields $\overline{Y}_1$ and $\overline{Y}_2$ in $\mathfrak{g}(m - 1)$. Therefore, the group $G(m - 1, m - 2)$ is again solvable and this prove the proposition in the present case.

To finish the discussion of Case A, suppose now that $\mathfrak{g}(m - 1)$ is constituted by vector fields having the form $\{\varphi X\}$. In this case $F^*_i(\mathfrak{g}(m - 1)) \subset \mathfrak{g}(m, m - 1)$ is an abelian sub-algebra of $\mathfrak{g}(m, m - 1)$ fully constituted by pairwise everywhere parallel vector fields. Since the dimension of $\mathfrak{g}(m - 1)$ is at least 2, the description above of the abelian sub-algebras of $\mathfrak{g}(m, m - 1)$ ensures that $F^*_i(\mathfrak{g}(m - 1))$ is again formed by vector fields everywhere parallel to $X$. In other words, the commutator $[Z_i, X]$ is everywhere parallel to $X$. In particular, $\mathfrak{g}^{\infty,*}(m, m - 1)$ has the form $(\varphi_1 f + \varphi_2)X$ as in Case 1 of Sect. 6.3.

Again, let $Z_i = a_i X + b_i Y$ so as to conclude that $b_1$ is a first integral of $X$ from the fact that $[Z_i, X]$ is everywhere parallel to $X$. The crucial point here compared to the previous case lies in the fact that only commutators of $Z_i$ with vector fields everywhere parallel to $X$ are controlled which, in turn, prevents us from repeating the above argument to conclude that $b_1$ is a constant. To overcome this difficulty we proceed as follows.

Assume first that $s = 1$ so that $Z_1 = a_1 X + b_1 Y$ with $b_1$ being a first integral of $X$. To conclude that $G(m - 1, m - 2)$ is solvable, it is therefore sufficient to check that $a_1$ has the above-indicated form $\varphi_1 f + \varphi_2$. This, however, follows from the same computations carried out in the proof of Lemma 8.4. More precisely, consider two linearly independent vector fields $(\varphi_3 f + \varphi_4)X$ and $(\varphi_5 f + \varphi_6)X$ in $\mathfrak{g}(m - 1)$. Owing to Lemma 7.7, the commutators $[Z_1, (\varphi_3 f + \varphi_4)X]$ and $[Z_1, (\varphi_5 f + \varphi_6)X]$ still possesses the general form $(\varphi_1 f + \varphi_2)X$. From this it follows that $a_1$ has the general form $\varphi_1 f + \varphi_2$ and completes the proof of the proposition in the present case.

Assume now that $s \geq 2$. Without loss of generality, we can assume that $b_1$ is not identically zero. In other words, $Z_1$ is not everywhere parallel to $X$. Now, following the argument given at the end of the proof of Lemma 8.4, we set $Z_i = \overline{a}_i X + \overline{b}_i Z_i$, for $i = 2, \ldots, s$. Since $[Z_i, X]$ is everywhere parallel to $X$, we still conclude that $\overline{b}_i$ is a first integral of $X$. On the other hand, the Lie algebra $\mathfrak{g}(m - 1)$ contains the infinitesimal generators of the commutators $F_1 \circ F_i \circ F_i^{-1} \circ F_1^{-1}$ so that Lemma 7.8 ensures that $[Z_1, Z_i]$ is everywhere parallel to $X$. In other words, all the coefficients $\overline{b}_i$ are constants in $\mathbb{C}$. The proof of Proposition 8.5 in Case A is completed.

Case B. Assume that $\mathfrak{g}(m - 1)$ is not abelian (and thus it is metabelian).

Since $\mathfrak{g}(m - 1)$ is not abelian, it necessarily contains a vector field $\overline{Y}$ of the form $a X + \alpha Y$ with $\alpha \neq 0$. Moreover, $D^1(\mathfrak{g}(m - 1))$ is non-trivial and automatically constituted by vector fields of the form $h X$, $h$ first integral of $X$. Clearly $F^*_i(D^1(\mathfrak{g}(m - 1)))$ is contained in $D^1(F^*_i(\mathfrak{g}(m - 1))) \subset D^1(\mathfrak{g}(m, m - 1))$. It follows again that $F^*_i(X)$ is everywhere parallel to $X$ (or equivalently that $[Z_i, X]$ is everywhere parallel to $X$). Note however that this conclusion can also be obtained by repeating the argument in the beginning of the proof of Case A. Similarly, the argument employed in the proof of the Claim also applies to the present situation and ensures that $[Z_i, Y]$ is everywhere parallel to $X$. Hence, we can again set $Z_i = a_i X + b_i Y$ where $b_i$ is a constant (for all $i = 1, \ldots, s$). Finally, $a_i$ must have the form $\varphi_1 f + \varphi_2$ as now follows from considering commutators...
Proof of Claim 1

\[ [Z_i, aX + \alpha Y] \text{ and } [Z_i, hX] \text{ in } \mathfrak{g}^{\infty,*}(m, m - 1) \text{ for } \bar{Y} = aX + \alpha Y \text{ as above and some other vector field } hX \in \mathfrak{g}(m - 1), \text{ where } h \text{ is some first integral of } X. \] The proof of Proposition 8.5 is now completed. \[ \square \]

To prove Theorem 2.5 it only remains to discuss the general case of a solvable group with trivial center. Since Case 1 of Sect. 6.3 was settled by Lemma 8.4, we can assume that \( G(m, m - 1) \) is as in Case 2 or in Case 3. In particular, there exists a vector field \( \bar{Y} \) which is not everywhere parallel to \( X \) and satisfies the condition indicated in the above-mentioned Case 2. The reader is also reminded that the highest order non-trivial derived Lie algebra \( D^\infty \mathfrak{g}(m, m - 1) \) of \( \mathfrak{g}(m, m - 1) \) consists of vector fields of the form \( \{ \varphi X \} \). Without loss of generality we also assume that \( X \in D^\infty \mathfrak{g}(m, m - 1) \). We are finally able to prove Theorem 2.5.

**Proof of Theorem 2.5** As mentioned, it suffices to consider the situations where \( G(m, m - 1) \) is as in Case 2 and in Case 3. However, to abridge notation, we shall only deal with Case 3 since Case 2 can be regarded as a particular one.

Therefore, the Lie algebra \( \mathfrak{g}(m, m - 1) \subset \mathfrak{g}_{\text{step}-3} \) consists of vector fields having the form

\[(\varphi_1 f + \varphi_2)X + \alpha \bar{Y}\]

where \( \alpha \in \mathbb{C} \) and \( \varphi_1, \varphi_2, f \) are first integrals of \( X \). Moreover, \([\bar{Y}, X] = hX\) where \( h \) is a first integral of \( X \), possibly vanishing identically. In particular, \( \bar{Y} \) derives first integrals of \( X \) into first integrals of \( X \).

Next note that \( \mathfrak{g}_{\text{step}-3} \) is the largest solvable Lie algebra of \( \hat{X}_2 \) so that both solvable Lie algebras \( \mathfrak{g}^{\infty,*}(m, m - 1) \) and \( \mathfrak{g}^{\infty}(m, m - 1) \) are naturally contained in \( \mathfrak{g}_{\text{step}-3} \). Lemma 7.7 then ensures that all iterated commutators \([Z_i, \ldots [Z_i, [Z_i, \bar{Z}]]\ldots]\) belong to \( \mathfrak{g}_{\text{step}-3} \) provided that \( \bar{Z} \in \mathfrak{g}(m - 1) \).

Consider the Lie algebra \( \mathfrak{g}(m - 1) \) and recall that its dimension can be assumed greater than or equal to 2. Again, the discussion will be split into two cases according to whether or not \( \mathfrak{g}(m - 1) \) is abelian.

**The abelian case**: assume that \( \mathfrak{g}(m - 1) \) is abelian.

By assumption \( \mathfrak{g}(m - 1) \) is an abelian sub-algebra of \( \mathfrak{g}_{\text{step}-3} \) whose dimension is at least 2. Thus either \( \mathfrak{g}(m - 1) \) is a linear span of two vector fields or has the form \( \{ \varphi X \} \), with \( \varphi \) first integral of \( X \). Being contained in \( \mathfrak{g}_{\text{step}-3} \) yields:

**Claim 1** Without loss of generality we can assume that \( \mathfrak{g}(m - 1) \) either is spanned by \( X \) and \( \bar{Y} \) or it consists of vector fields having the form \( \{ \varphi X \} \).

**Proof of Claim 1** Suppose first that \( \mathfrak{g}(m - 1) \) is a linear span of two vector fields \((\varphi_1 f + \varphi_2)X + \alpha \bar{Y}\) and \((\varphi_3 f + \varphi_4)X + \beta \bar{Y}\). By taking a suitable linear combination of them, \( \mathfrak{g}(m - 1) \) is also spanned by \((\varphi_1 f + \varphi_2)X + \alpha \bar{Y}\) and \((\varphi_5 f + \varphi_6)X\). Now set \((\varphi_1 f + \varphi_2)X + \alpha \bar{Y}\) as your “new vector field \( \bar{Y} \)” and \((\varphi_5 f + \varphi_6)X\) as the “new vector field \( X \)”.

Suppose now that all vector fields in \( \mathfrak{g}(m - 1) \) are pairwise everywhere parallel. Since the dimension of \( \mathfrak{g}(m - 1) \) is at least 2, it follows that these vector fields have to be everywhere parallel to \( X \). Now it is clear that the quotient between two of these
vector fields must be a first integral of $X$ so that the claim follows from choosing one of these vector fields as the “new vector field $X$”. □

We begin with the case in which $g(m-1)$ can be identified with the linear span of the vector fields $X$ and $\overline{Y}$. The extra difficulty arising in the present case when compared to the proof of Proposition 8.5 (Case A) lies in the fact that the vector field $X$ is no longer “canonical”. Indeed, in the context of Proposition 8.5, the corresponding vector field $X$ was naturally associated with the center of the Lie algebra $g(m, m-1)$ and hence was uniquely determined (up to multiplicative constants). This no longer holds here since the center of $g(m-1)$ is supposed to be trivial. Yet Claim 2 below makes up for this additional difficulty.

**Claim 2** Without loss of generality we can assume that $[Z_i, X]$ is everywhere parallel to $X$.

**Proof of Claim 2** It is known that $[Z_i, X] = a_{X,1,i} X + \alpha_{1,i} \overline{Y}$ with $\alpha_{1,i} \in \mathbb{C}$ (Lemma 7.7). We assume that $\alpha_{1,i} \neq 0$ otherwise there is nothing to be proved. Lemma 7.7 also ensures that $[Z_i, \overline{Y}] = a_{\overline{Y},1,i} X + \beta_{1,i} \overline{Y}$ with $\beta_{1,i} \in \mathbb{C}$.

Now consider the commutator $[Z_i, [Z_i, X]]$. This vector field belongs to $g_{\text{step-3}}$ (again owing to Lemma 7.7) and thus has the form $a_{X,2,i} X + \alpha_{2,i} \overline{Y}$ with $\alpha_{2,i} \in \mathbb{C}$. On the other hand, a direct computation yields

$$[Z_i, [Z_i, X]] = \left( \frac{\partial a_{X,1,i}}{\partial Z_i} \right) X + a_{X,1,i}[Z_i, X] + \alpha_{1,i}[Z_i, \overline{Y}] = \tilde{a}_X X + (a_{X,1,i} \alpha_{1,i} + \beta_{1,i} \alpha_{1,i}) \overline{Y}.$$ 

Thus $\alpha_{2,i} = a_{X,1,i} \alpha_{1,i} + \beta_{1,i} \alpha_{1,i}$ so that $a_{X,1,i}$ is a constant since $\alpha_{1,i} \neq 0$.

Similarly $[Z_i, [Z_i, \overline{Y}]] = a_{\overline{Y},2,i} X + \beta_{2,i} \overline{Y}$ has the form

$$[Z_i, [Z_i, \overline{Y}]] = \tilde{a}_{\overline{Y}} X + \left( a_{\overline{Y},1,i} \alpha_{1,i} + \beta_{1,i}^2 \right) \overline{Y}.$$ 

Thus $a_{\overline{Y},1,i}$ is constant as well. Summarizing the preceding, every $Z_i$ induces an endomorphism of the linear span of $X$ and $\overline{Y}$. Therefore, the Lie algebra generated by $X, \overline{Y}, Z_1, \ldots, Z_s$ is solvable by virtue of Lemma 6.6. □

We assume in the sequel that $[Z_i, X]$ is everywhere parallel to $X$ so that $Z_i = a_i X + b_i \overline{Y}$ where $b_i$ is a first integral of $X$. However, by resorting to the argument given in the proof of Proposition 8.5, we see that $[Z_i, \overline{Y}]$ is everywhere parallel to $X$ as well. In fact, all the iterated commutators $[Z_i, \ldots [Z_i, [Z_i, \overline{Y}]] \ldots]$ belong to $g_{\text{step-3}}$ while their orders at the origin becomes arbitrarily large. It follows that some high enough iterate must be everywhere parallel to $X$. This fact combined with Claim 2 ensures that the commutator $[Z_i, \overline{Y}]$ must be everywhere parallel to $X$. In turn, this implies that $b_i$ is a first integral of $Y$ as well so that we actually have $Z_i = a_i X + a_i \overline{Y}$ with $a_i \in \mathbb{C}$. To prove that the Lie algebra generated by $X, \overline{Y}, Z_1, \ldots, Z_s$ still is solvable, there only remains to check that $a_i$ has the form $a_i = \phi_{1,i} f + \phi_{2,i}$ ($\phi_{1,i}, \phi_{2,i}$ being first integrals of $X$). This, however, follows from the same argument employed...
in the proof of Lemma 8.4. The theorem then follows, provided that \( g(m-1) \) coincides with the linear span of two vector fields.

To complete the discussion of the case in which \( g(m-1) \) is abelian, it remains to consider the situation in which \( g(m-1) \) consists of (two or more) vector fields having the form \( hX \) where \( h \) is a first integral of \( X \). The argument is essentially the same used in the analogous situation occurring in the proof of Proposition 8.5. We summarize the discussion in the sequel. For every \( i = 1, \ldots, s \), \( F_i^* g(m-1) \) is again an abelian Lie algebra of whose vector fields are pairwise everywhere parallel. Since \( F_i^* g(m-1) \subseteq g_{\text{step}-3} \), we conclude that \( F_i^* g(m-1) \) again has the form \( h(\varphi_1 f + \varphi_2)X \) with \( \varphi_1, \varphi_2 \) first integrals of \( X \) (in particular, \([Z_i, X]\) is everywhere parallel to \( X \)). As already seen, this also implies that \([Z_i, \bar{Y}]\) is everywhere parallel to \( X \). Assembling this information, it follows that \( Z_i \) has the form \( Z_i = a_i X + \alpha_i \bar{Y} \) with \( \alpha_i \in \mathbb{C} \) and where \( a_i \) is a first integral of \( X \). Therefore, the Lie algebra generated by \( Z_1, \ldots, Z_s \) along with vector fields of the form \( hX \) (\( h \) first integral of \( X \)) is solvable and this ends the proof of Theorem 2.5 in the case where \( g(m-1) \) is an abelian algebra.

The non-abelian case. Suppose that \( g(m-1) \) is not abelian.

The fundamental observation explaining why the non-abelian case discussed below is somewhat simpler than the abelian one lies in the fact that the derived Lie algebra \( D^1 g_{\text{step}-3} \) consists of vector fields everywhere parallel to \( X \). To exploit this remark, we proceed as follows.

Since \( g(m-1) \) is not abelian, its derived Lie algebra \( D^1 g(m-1) \) is not trivial. Furthermore, \( D^1 g(m-1) \) clearly consists of vector fields everywhere parallel to \( X \). For a fixed \( i \in \{1, \ldots, s\} \), consider then the map from \( g(m-1) \) to \( g(m, m-1) \) consisting of taking the commutator with \( Z_i \). This map clearly sends \( D^1 g(m-1) \) in \( D^1 g(m, m-1) \) so that there is \( aX \in D^1 g(m-1) \) such that \([Z_i, aX]\) is again everywhere parallel to \( X \). Thus we obtain once and for all that \([Z_i, X]\) is everywhere parallel to \( X \). In turn, this implies that \([Z_i, \bar{Y}]\) is everywhere parallel to \( X \) as well (cf. Claim 2 in the proof of Proposition 8.5). Thus \( Z_i a_i X + \alpha_i \bar{Y} \) with \( \alpha_i \in \mathbb{C} \). Once again to conclude that \( a_i \) has the form \( a_i = \varphi_{1,i} f + \varphi_{2,i} \) (\( \varphi_{1,i}, \varphi_{2,i} \) first integrals of \( X \)) it suffices to repeat the argument employed in the proof of Lemma 8.4. The proof of Theorem 2.5 is completed.

\( \square \)

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