ON THE GROUP OF SELF-HOMOTOPY EQUIVALENCES OF AN ELLIPTIC SPACE

MAHMOUD BENKHALIFA

Abstract. Let $X$ be a simply connected rational elliptic space of formal dimension $n$ and let $\mathcal{E}(X)$ denote the group of homotopy classes of self-equivalences of $X$. If $X^{[k]}$ denotes the $k$th Postnikov section of $X$ and $X^k$ denotes its $k$th skeleton, then making use of the models of Sullivan and Quillen we prove that $\mathcal{E}(X) \cong \mathcal{E}(X^{[n]})$ and if $n > m = \max\{k \mid \pi_k(X) \neq 0\}$ and $\mathcal{E}(X)$ is finite, then $\mathcal{E}(X) \cong \mathcal{E}(X^{m+1})$. Moreover, in case when $X$ is 2-connected, we show that if $\pi_n(X) \neq 0$, then the group $\mathcal{E}(X)$ is infinite.

1. Introduction

A simply connected rational topological space $X$ is elliptic if both $H^*(X, \mathbb{Q})$ and $\pi_*(X)$ are finite dimensional. Let us call $n = \max\{i : H^i(X, \mathbb{Q}) \neq 0\}$ the formal dimension of $X$. It is well known ([3], Theorem 4.2) that such a space satisfies $\pi_i(X) = 0$, for $i \geq 2n$.

Therefore, if $X^{[k]}$ denotes the $k$th Postnikov section of $X$ and $X^k$ denotes its $k$th skeleton, on the one hand $X$ coincides with $X^{[2n-1]}$ and on the other hand, as its formal dimension is $n$, the space $X$ coincides with $X^n$.

This paper is directed towards an understanding of the group of homotopy classes of self-equivalences $\mathcal{E}(X)$, where $X$ is a simply connected rational elliptic space of formal dimension $n$. As is well known, the homotopy theory of rational spaces is equivalent, by Sullivan’s work, to the homotopy theory of minimal, differential, graded commutative $\mathbb{Q}$-algebras and by Quillen’s work to the homotopy theory of differential, graded Lie $\mathbb{Q}$-algebras. Those algebras provide an effective algebraic setting to work in, so working algebraically we establish the following theorem which will be split in Theorem 3.3 and Theorem 3.4 later on.

**Theorem 1.** Let $X$ be a simply connected rational elliptic space of formal dimension $n$. Then

$$\mathcal{E}(X) = \mathcal{E}(X^{[2n-1]}) \cong \mathcal{E}(X^{[2n-2]}) \cong \ldots \cong \mathcal{E}(X^{[n]})$$

Moreover if $n > m = \max\{k \mid \pi_k(X) \neq 0\}$ and $\mathcal{E}(X)$ is finite, then

$$\mathcal{E}(X) \cong \ldots \cong \mathcal{E}(X^{m+2}) \cong \mathcal{E}(X^{m+1}).$$

The idea of getting some information regarding the (in)finiteness of the groups $\mathcal{E}(X)$ within the framework of Sullivan model traces back to the results of Arkowitz and Lupton [2] in which they exhibited conditions under which $\mathcal{E}(X)$ is finite or
infinite, where $X$ is a rational space having a 2-stage Postnikov-like decomposition (for example, rationalizations of homogeneous spaces). In the same spirit we establish the following result which will be Theorem 3.5 later on.

**Theorem 2.** Let $X$ be a 2-connected rational elliptic space of formal dimension $n$. If $\pi_n(X) \neq 0$, then the group $E(X)$ is infinite.

In [15], Costoya and Viruel proved the remarkable result that every finite group $G$ occurs as $G = E(X)$ for some elliptic rational space $X$ having formal dimension $n = 208 + 80|V|$, where $V$ is a certain finite graph associated with $G$ and $|V|$ denotes the order of $V$. The space $X$ is constructed such that $\pi_k(X) = 0$ for all $k \geq 120$. As a consequence of the main theorem of this paper we show that every finite group $G$ can be realised by a rational space $X$ whose formal dimension does not depend on the order of $G$. Precisely, we can ameliorate Costoya and Viruel theorem by showing that $G = E(X)$ for some rational space $X$ having formal dimension $n = 120$.

For a space $X$, let $[X, X]$ denote the monoid of homotopy classes of self-maps of the space $X$ and let

$$A^k_\#(X) = \left\{ f \in [X, X] \mid \pi_i(f) : \pi_i(X) \to \pi_i(X) \text{ for any } i \leq k \right\}$$

In [14], Choi and Lee introduced the concept of the self-closeness number defined as follows

$$N \mathcal{E}(X) = \min \left\{ k \mid A^k_\#(X) = E(X) \right\}$$

Notice that the study of this numerical homotopy invariant by means of Sullivan model in algebraic setting is an interesting problem (see for instance [20]). From the main result in this paper we establish the following result which will be Theorem 3.7 later on.

**Theorem 3.** If $X$ is a simply connected rational elliptic space of formal dimension $n$, then

$$N \mathcal{E}(X) \leq n$$

**Remark 1.1.** Theorem 3 is a weaker version of (14, Theorem 2) in which the same result is proved, using topological arguments, for a CW-complex $X$ of dimension $n$. Our proof is based on the analysis of the Sullivan model of $X$.

The paper is organized as follows. In section 2, we recall the basic properties of the Quillen model and Sullivan model in rational homotopy theory, the Whitehead exact sequences as well as the important properties of elliptic spaces. Then we formulate and prove the main theorems in the algebraic setting. In section 3, a mere transcription of the above results in the topological context is given and some examples are provided illustrating our results.

2. MAIN RESULTS

2.1. Quillen model and Sullivan model in rational homotopy theory. We briefly recall Quillen’s differential graded Lie algebra and Sullivan’s commutative differential algebra frameworks for rational homotopy theory. All the materials can be founded [3, 18].

If $X$ is a simply connected rational CW complex of finite type, then there exists a free commutative cochain algebra $(\Lambda V, \partial)$ called the Sullivan model of $X$, unique
up to isomorphism, which determines completely the homotopy type of the space $X$. Moreover Sulivan model recovers homotopy data via the identifications:
\[
\text{Hom}(\pi_*(X), \mathbb{Q}) \cong V^+ \quad H^*(X; \mathbb{Q}) \cong H^*(\Lambda V, \partial) \quad \text{and} \quad \mathcal{E}(X) \cong \text{aut}(\Lambda V, \partial)/\simeq,
\]
where $\text{aut}(\Lambda V, \partial)/\simeq$ is the group of homotopy cochain self-equivalences of $(\Lambda V, \partial)$ modulo the relation of homotopy between free commutative cochain algebras (see [18]). We write
\[
\mathcal{E}(\Lambda V) = \text{aut}(\Lambda V, \partial)/\simeq
\]
for this group.

Dually, if $X$ is a simply connected rational CW complex of finite type, then there exists a differential graded Lie algebra $(\mathbb{L}(W), \delta)$ called the Quillen model of $X$, unique up to isomorphism, which determines completely the homotopy type of the space $X$. The Quillen model recovers homotopy data via the identifications:
\[
\pi_*(X) \cong H_{*-1}(\mathbb{L}(W)) \quad \text{and} \quad H_*(X, \mathbb{Q}) \cong W_*.
\]
Quillen’s theory directly implies an identification
\[
\mathcal{E}(\mathbb{L}(W)) = \text{aut}(\mathbb{L}(W))/\simeq
\]
for this group.

**Definition 2.1** ([6], Definition 2.6). Given a simply connected free commutative cochain algebra $(\Lambda(\bigoplus_{q \leq n} V)^q, \partial)$, where $q > n$ and let $b^q : V^q \to H^{q+1}(\Lambda(V^\leq n))$ be the linear map defined as follows
\[
b^q(v) = [\partial(v)] \quad v \in V^q.
\]
Here $[\partial(v)]$ denotes the cohomology class of $\partial(v) \in (\Lambda V^\leq n)^{q+1}$.

We define $\mathcal{D}^q_n$ to be the subgroup of $\text{aut}(V^q) \times \mathcal{E}(\Lambda V^\leq n)$ consisting of the pairs $(\xi, [\alpha])$ making the following diagram commutes
\[
\begin{array}{ccc}
V^q & \xrightarrow{\xi} & V^q \\
\downarrow{b^q} & & \downarrow{b^q} \\
H^{q+1}(\Lambda V^\leq n) & \xrightarrow{H^{q+1}(\alpha)} & H^{q+1}(\Lambda V^\leq n)
\end{array}
\]

**Theorem 2.2** ([6], Theorem 1.1). There exists a short exact sequence of groups
\[
\text{Hom}(V^q, H^q(\Lambda(V^\leq n))) \to \mathcal{E}(\Lambda(V^q \oplus V^\leq n)) \xrightarrow{\Psi} \mathcal{D}^q_n
\]
where $\Psi([\alpha]) = (\tilde{\alpha}^q, [\alpha_n])$. Here $\tilde{\alpha}^q : V^q \to V^q$ is the isomorphism induced by $\alpha$ on the indecomposables and $\alpha_n$ is the restriction of $\alpha$ to $\Lambda V^\leq n$.

**Corollary 2.3.** Assume that the linear map $b^q$ is an isomorphism, then
\[
\mathcal{D}^q_n \cong \mathcal{E}(\Lambda V^\leq n)
\]
Proof. As \( b^q \) is an isomorphism, then from the commutative diagram \([2]\) we deduce that \( \xi = (b^q)^{-1} \circ H^{q+1}(\alpha) \circ b^q \). Therefore the map

\[
E(\Lambda^q \Omega^n) \to \mathcal{D}_n^q \quad \text{such that} \quad [\alpha] \mapsto (b^q)^{-1} \circ H^{q+1}(\alpha) \circ b^q, [\alpha]
\]

is an isomorphism. \( \square \)

Definition 2.4 (\([7]\), Definition 2.1). Given a simply connected free differential graded Lie \((L, \delta)\) where \( q > k \) and let \( b_q : W_q \to H_{q-1}(\mathbb{L}(W_{\leq k})) \) be the linear map defined as follows

\[
b_q(v) = [\delta(v)] \quad , \quad v \in W_q
\]

Here \([\delta(v)]\) denotes the homology class of \( \delta(v) \in \mathbb{L}_{q-1}(W_{\leq k}) \).

We define \( \mathcal{R}_k^q \) to be subgroup of \( \text{aut}(W_q) \times E(\mathbb{L}(W_{\leq k})) \) consisting of the pairs \((\xi, [\alpha])\) making the following diagram commutes

\[
\begin{array}{ccc}
W_q & \xrightarrow{\xi} & W_q \\
\downarrow{b_q} & & \downarrow{b_q} \\
H_{q-1}(\mathbb{L}(W_{\leq k})) & \xrightarrow{H_{q-1}(\alpha)} & H_{q-1}(\mathbb{L}(W_{\leq k}))
\end{array}
\]

(5)

Theorem 2.5 (\([7]\), Theorem 2.6). There exists a short exact sequence of groups

\[
\text{Hom}(W_q, H_q(\mathbb{L}(W_{\leq k}))) \to E(\mathbb{L}(W_q \oplus W_{\leq k})) \xrightarrow{\lambda} \mathcal{R}_k^q \tag{6}
\]

where \( \lambda([\alpha]) = (\tilde{\alpha}_q, [\alpha_k]) \). Here \( \tilde{\alpha}_q : W_q \to W_q \) is the isomorphism induced by \( \alpha \) on the indecomposables and \( \alpha_k \) is the restriction of \( \alpha \) to \( \mathbb{L}(W_{\leq k}) \)

Corollary 2.6. Assume that the linear map \( b_q \) is an isomorphism, then

\[
\mathcal{R}_k^q \cong E(\mathbb{L}(W_{\leq k}))
\]

Proof. As \( b_q \) is an isomorphism, then from the commutative diagram \([5]\) we deduce that \( \xi = (b_q)^{-1} \circ H_{q-1}(\alpha) \circ b_q \). Therefore the map

\[
E(\mathbb{L}(W_{\leq k})) \to \mathcal{R}_k^q \quad \text{such that} \quad [\alpha] \mapsto (b_q)^{-1} \circ H_{q-1}(\alpha) \circ b_q, [\alpha]
\]

is an isomorphism. \( \square \)

2.2. Whitehead exact sequences in rational homotopy theory. To every free differential graded Lie algebra \((\mathbb{L}(W), \delta)\) such that \( W_1 = 0 \), we can assign (see \([4, 5, 7, 11]\)) the following long exact sequence

\[
\cdots \to W_{n+1} \xrightarrow{b_{n+1}} H_n(\mathbb{L}(W_{\leq n-1})) \to H_n(\mathbb{L}(W)) \to W_n \xrightarrow{b_n} \cdots \tag{7}
\]

called the Whitehead exact sequence of \((\mathbb{L}(W), \delta)\), where \( b_n \) is the graded linear map defined in \([1]\). Hence if \( X \) is a 2-connected rational space of finite type and if \((\mathbb{L}(W), \delta)\) is its Quillen’s model, then the properties of this model imply

\[
\pi_n(X) \cong H_{n-1}(\mathbb{L}(W)) \quad , \quad H_n(X, \mathbb{Q}) \cong W_{n-1} \quad , \quad \pi_n(X^{n-1}) \cong H_{n-1}(\mathbb{L}(W_{\leq n-2}))
\]
Here $X^n$ for the $n^{th}$ skeleton of $X$. Therefore the Whitehead exact sequence of this model can be written as

$$\cdots \rightarrow H_{n+1}(X) \rightarrow \pi_n(X^{n-1}) \rightarrow \pi_n(X) \rightarrow H_n(X) \rightarrow \cdots$$  \hfill (8)

Likewise, let $(\Lambda V, \partial)$ be a simply connected free commutative cochain algebra. In $[\ref{14}]$, it is shown that with $(\Lambda V, \partial)$ we can associate the following long exact sequence

$$\cdots \rightarrow V^k \xrightarrow{\delta^k} H^{k+1}(\Lambda V) \rightarrow H^{k+1}(\Lambda V^\leq k-1) \rightarrow V^{k+1} \xrightarrow{\delta^{k+1}} \cdots$$  \hfill (9)

called the Whitehead exact sequence of $(\Lambda V, \partial)$, where $\delta^k$ is the graded linear map defined in $[\ref{14}]$. Thus, if $X$ is a simply connected rational space of finite type and $(\Lambda(V), \partial)$ is its Sullivan minimal model, then by virtue of the properties of this model we obtain the following identifications

$$H^k(X, \mathbb{Q}) \cong H^k(\Lambda V) \ , \ H^{k+1}(X[k], \mathbb{Q}) \cong H^{k+1}(\Lambda V^\leq k) \ , \ V^k \cong \text{Hom}(\pi_k(X), \mathbb{Q})$$

Here $X[k]$ for the $k^{th}$ Postnikov section of $X$. Therefore the Whitehead exact sequence of this model can be written as

$$\cdots \rightarrow \text{Hom}(\pi_k(X), \mathbb{Q}) \rightarrow H^{k+1}(X[k]) \rightarrow H^{k+1}(X) \rightarrow \text{Hom}(\pi_{k+1}(X), \mathbb{Q}) \rightarrow \cdots$$  \hfill (10)

**Theorem 2.7.** If $X$ is a 2-connected rational space of finite type, then

$$H^{k+1}(X[k], \mathbb{Q}) \cong \text{Hom}(\pi_k(X^{k-1}), \mathbb{Q}) \quad , \quad k \geq 2$$  \hfill (11)

**Proof.** Applying the exact functor $\text{Hom}(\cdot, \mathbb{Q})$ to the exact sequence (8) we obtain

$$\cdots \leftarrow H^{n+1}(X, \mathbb{Q}) \leftarrow \text{Hom}(\pi_n(X^{n-1}), \mathbb{Q}) \leftarrow \text{Hom}(\pi_n(X), \mathbb{Q}) \leftarrow H^n(X, \mathbb{Q}) \leftarrow \cdots$$  \hfill (12)

Taking into account that

- All groups involved are vector spaces of finite dimensions
- The two maps $H^n(X, \mathbb{Q}) \rightarrow \text{Hom}(\pi_n(X), \mathbb{Q})$ appearing in (10) and (12) are the same morphism
- $\text{Hom}(H_n(X, \mathbb{Q}), \mathbb{Q}) = H^*(X, \mathbb{Q})$

and by comparing the sequences (10), (12) we get (11). \hfill $\square$

### 2.3. Elliptic algebras.

Recall that (see $[\ref{10}]$) a simply connected free differential graded commutative algebra $(\Lambda V, \partial)$ is called elliptic if both $H^*(\Lambda V)$ and $V^*$ are finite dimensional. Let us call $n = \max\{i : H^i(\Lambda V) \neq 0\}$ the formal dimension of $(\Lambda V, \partial)$. The following theorem mentions some important properties of elliptic algebras.

**Theorem 2.8.** ($[\ref{10}]$ Theorem 7.4.2). Suppose $(\Lambda V, \partial)$ is simply connected and elliptic of formal dimension $n$. Then

1. $\dim V^{\text{odd}} \geq \dim V^{\text{even}}$
2. If $\{x_j\}$ is a basis of $V^{\text{odd}}$ and $\{y_j\}$ is a basis of $V^{\text{even}}$, then
   $$n = \sum |x_j| - \sum (|y_j| - 1).$$
3. $\sum |y_j| \leq n$ and $\sum |x_j| \leq 2n - 1$.
4. $V^i = 0$, for $i \geq 2n$.

From the property (3) we can derive
Corollary 2.9. Suppose \((\Lambda V, \partial)\) is simply connected and elliptic of formal dimension \(n\). Then \(V^i = 0\), for \(i > n\) and \(i\) even.

Lemma 2.10. Suppose \((\Lambda V, \partial)\) is simply connected and elliptic of formal dimension \(n\). For every \(k\) such that \(2k > n\) we have

\[
H^{2k+1}(\Lambda V \leq 2k-1) = 0
\]

Proof. From the long exact sequence of cohomology associated to the inclusion \((\Lambda V \leq 2k-1, \partial) \subseteq (\Lambda V, \partial)\) we get

\[
\ldots \rightarrow H^{2k}(\Lambda V/\Lambda V \leq 2k-1) \rightarrow H^{2k+1}(\Lambda V \leq 2k-1) \rightarrow H^{2k+1}(\Lambda V) \rightarrow \ldots
\]

But \(H^{2k}(\Lambda V/\Lambda V \leq 2k-1) \cong V^{2k}\) and by Corollary 2.9 we have \(V^{2k} = 0\) for \(2k > n\). Therefore the map \(H^{2k+1}(\Lambda V \leq 2k-1) \rightarrow H^{2k+1}(\Lambda V)\) is injective. As \((\Lambda V, \partial)\) has formal dimension \(n\), it follows that \(H^{2k+1}(\Lambda V) = 0\) for \(2k \geq n\), implying that \(H^{2k+1}(\Lambda V \leq 2k-1) = 0\). \(\square\)

Using the exact sequence \((\ref{eq:13})\), Corollary 2.9 and Lemma 2.10 we derive the following result

Corollary 2.11. Suppose \((\Lambda V, \partial)\) is simply connected and elliptic of formal dimension \(n\). Then the linear map \(b^i : V^i \rightarrow H^{i+1}(\Lambda V \leq i-1)\) is

- An isomorphism for \(i > n\) and \(i\) odd.
- Nil for \(i > n\) and \(i\) even.

Theorem 2.12. Suppose \((\Lambda V, \partial)\) is simply connected and elliptic of formal dimension \(n\). Then

\[
\mathcal{E}(\Lambda V) \cong \mathcal{E}(\Lambda V \leq n)
\]

Proof. First note that since \((\Lambda V, \partial)\) is elliptic of formal dimension \(n\), by the property (3) of Theorem 2.8 we derive that \(\mathcal{E}(\Lambda V) = \mathcal{E}(\Lambda V \leq 2n-1)\). Next using Theorem 2.2 we obtain the following short exact sequence

\[
\text{Hom}(V^{2n-1}, H^{2n-1}(\Lambda V \leq 2n-2)) \rightarrow \mathcal{E}(\Lambda V \leq 2n-1) \rightarrow D_{2n-2}^{2n-1} \tag{13}
\]

But \(V^{2n-2} = 0\) and \(b^{2n-1}\) is an isomorphism, so using Corollary 2.3 and Lemma 2.10 the sequence \((\ref{eq:13})\) implies

\[
\mathcal{E}(\Lambda(V \leq 2n-1)) \cong \mathcal{E}(\Lambda(V \leq 2n-3))
\]

Again using theorem 2.2 we obtain the following short exact sequence

\[
\text{Hom}(V^{2n-3}, H^{2n-3}(\Lambda V \leq 2n-4)) \rightarrow \mathcal{E}(\Lambda V \leq 2n-3) \rightarrow D_{2n-2}^{2n-3}
\]

but \(V^{2n-4} = 0\) and \(b^{2n-3}\) is an isomorphism, so using Corollary 2.3 and Lemma 2.10 the sequence \((\ref{eq:13})\) implies

\[
\mathcal{E}(\Lambda(V \leq 2n-3)) \cong \mathcal{E}(\Lambda(V \leq 2n-5))
\]

Continuing this process by using the same arguments, we end up with the following formula

\[
\mathcal{E}(\Lambda(V \leq 2n-3)) \cong \mathcal{E}(\Lambda(V \leq 2n-5)) \cong \ldots \cong \mathcal{E}(\Lambda(V \leq n)), \quad \text{if } n \text{ is odd}
\]

and

\[
\mathcal{E}(\Lambda(V \leq 2n-4)) \cong \mathcal{E}(\Lambda(V \leq 2n-6)) \cong \ldots \cong \mathcal{E}(\Lambda(V \leq n+1)), \quad \text{if } n \text{ is even.}
\]
In the case when $n$ is even, according to Corollary 2.3 and Lemma 2.10, the sequence (13) implies
\[ \text{Hom}(V_{n+1}, H_{n+1}(\Lambda V \leq n)) \to E(\Lambda V \leq n+1) \to E(\Lambda V \leq n) \]
Finally, Lemma 2.10 assures that $H_{n+1}(\Lambda V \leq n) = 0$. Hence $E(\Lambda V \leq n+1) \cong E(\Lambda V \leq n)$

**Proposition 2.13.** Let $(\Lambda V, \partial)$ be a simply connected free differential graded algebra. If the group $E(\Lambda V \leq n)$ is finite, then the linear map $b^n$ is injective.

**Proof.** First Theorem 2.2 implies that
\[ \text{Hom}(V_n, H_n(\Lambda V \leq n-1)) \to E(\Lambda V \leq n) \to D_{n-1} \quad (14) \]
Assume that $b^n$ is not injective and let $v \neq 0 \in V_n$ such that $b^n(v) = 0$. Choose \{\(v, v_1, \ldots, v_k\}\} as a basis of $V^n$ and define
\[ \xi^a(v) = av, \quad a \neq 0 \in \mathbb{Q}, \quad \xi^a(v_i) = v_i \]
Clearly the pair $(\xi^a, [id]) \in \text{aut}(V^n) \times E(\Lambda V \leq n-1)$ for every $a \neq 0 \in \mathbb{Q}$ and makes following diagram commute
\[
\begin{array}{ccc}
V^n & \xrightarrow{\xi^a} & V^n \\
\downarrow{b^n} & & \downarrow{b^n} \\
H^{n+1}(\Lambda V \leq n-1) & \xrightarrow{id} & H^{n+1}(\Lambda V \leq n-1)
\end{array}
\]
Therefore $(\xi^a, [id]) \in D_{n-1}^n$ for every $a \neq 0 \in \mathbb{Q}$ implying that the group $D_{n-1}^n$ is infinite. Consequently, the group $E(\Lambda V \leq n)$ is also infinite according to the exact sequence (14)

### 3. Topological applications

**Definition 3.1.** A simply connected rational space $X$ is called elliptic if its Sullivan model is elliptic.

In this case the formal dimension of $X$ is defined as the formal dimension of its Sullivan model.

**Remark 3.2.** By virtue of Theorem 2.8 we conclude that if $X$ is a simply connected rational elliptic space of formal dimension $n$, then its Quillen model can be written as $(L(W \leq n-1), \delta)$ and its Sullivan model as $(\Lambda V \leq 2n-1, \partial)$.

A mere transcription of the Theorem 2.12 in the topological context, using the properties of the Sullivan model, implies the following theorem.

**Theorem 3.3.** Let $X$ be a rational elliptic space of formal dimension $n$. Then
\[ E(X) = E(X^{[2n-1]}) \cong E(X^{[2n-2]}) \cong \ldots \cong E(X^{[n]}) \]
Here $X^{[k]}$ denotes the $k^{th}$ Postnikov section of $X$.

Combining the model of Quillen and the model of Sullivan we obtain
Theorem 3.4. Let $X$ be a simply connected rational elliptic space of formal dimension $n$ such that $n > m = \max \{k \mid \pi_k(X) \neq 0\}$ and $\mathcal{E}(X)$ is finite. Then
\[ \mathcal{E}(X) \cong \ldots \cong \mathcal{E}(X^{m+2}) \cong \mathcal{E}(X^{m+1}) \] (15)

**Proof.** By hypothesis the Quillen model of $X$ has the form $(\mathbb{L}(W_{\leq n-1}), \delta)$, where $W_{n-1} \neq 0$ and its Sullivan model has the form $(\Lambda V^{\leq m}, \partial)$, where $V^m \neq 0$. Recall that
\[ V^* \cong \text{Hom} \left( H_{n-1}(\mathbb{L}(W_{\leq n-1})), \mathbb{Q} \right) \] (16)
Since $n > m$ and by (17), we deduce that
\[ H_k(\mathbb{L}(W_{\leq n-1})) = 0 \quad , \quad k \geq m. \] (17)

It follows
\[ H_{n-1}(\mathbb{L}(W_{\leq n-1})) = 0. \] (18)
Let us consider the Whitehead exact sequence of $(\mathbb{L}(W_{\leq n-1}), \delta)$, namely
\[ \cdots \to H_k(\mathbb{L}(W_{\leq n-1})) \to W_k \xrightarrow{b_k} H_{k-1}(\mathbb{L}(W_{\leq k-2})) \to H_{k-1}(\mathbb{L}(W_{\leq n-1})) \to \cdots \] (19)
which implies
\[ \cdots \to W_n = 0 \xrightarrow{b_n} H_{n-1}(\mathbb{L}(W_{\leq n-2})) \to H_{n-1}(\mathbb{L}(W_{\leq n-1})) = 0 \to \cdots \]
As a result, we obtain
\[ H_{n-1}(\mathbb{L}(W_{\leq n-2})) = 0 \] (20)
Now, according to (17), the map $b_{n-1}$ is an isomorphism and according to Corollary 2.3, it follows that
\[ \mathcal{R}_{n-2} \cong \mathcal{E}(\mathbb{L}(W_{\leq n-2})). \]
Using Theorem 2.2 (for $q = n - 1$ and $k = n - 2$) and (20), we conclude that $\mathcal{E}(\mathbb{L}(W_{\leq n-1})) \cong \mathcal{R}_{n-2}$. Consequently $\mathcal{E}(\mathbb{L}(W_{\leq n-1})) \cong \mathcal{E}(\mathbb{L}(W_{\leq n-2}))$.

Using the same arguments and taking into account the relation (17), which implies that $b_k$ is an isomorphism for $k \geq m + 1$, by iterating the above process it follows that
\[ \mathcal{E}(\mathbb{L}(W_{\leq n-1})) \cong \mathcal{E}(\mathbb{L}(W_{\leq n-2})) \cong \ldots \cong \mathcal{E}(\mathbb{L}(W_{\leq m})) \]
Finally, by the properties of the Sullivan and Quillen models and taking into consideration Theorem 3.3, we obtain \[ \Box \]

**Theorem 3.5.** Let $X$ be a 2-connected rational elliptic space of formal dimension $n$. If $\pi_n(X) \neq 0$, then the group $\mathcal{E}(X)$ is infinite.

**Proof.** Since the formal dimension of $X$ is $n$, we can choose $(\mathbb{L}(W_{\leq n-1}), \delta)$ as the Quillen model of $X$ with $W_{n-1} \neq 0$ and $(\Lambda V^{\leq 2n-1}, \partial)$ as its Sullivan model. Next, taking $q = n - 1$ and $k = n - 2$ in (14), we derive the following short exact sequence
\[ \text{Hom}(W_{n-1}, H_{n-1}(\mathbb{L}(W_{\leq n-2}))) \to \mathcal{E}(\mathbb{L}(W_{\leq n-1})) \to \mathcal{R}_{n-2} \] (21)
Assume by contradiction that $\mathcal{E}(X)$ is finite. By Theorem 2.12 we have $\mathcal{E}(\Lambda V^{\leq 2n-1}) \cong \mathcal{E}(\Lambda V^{\leq n})$, so $\mathcal{E}(\Lambda V^{\leq n})$ is also finite and the short exact sequence (14) implies that $H^n(\Lambda V^{\leq n-1}) = 0$ because $V^n \cong \pi_n(X) \neq 0$. Taking in account that $X$ is 2-connected, the formula (14) implies that $H^n(\Lambda V^{\leq n-1}) \cong H_{n-2}(\mathbb{L}(W_{\leq n-3}))$. So $H_{n-2}(\mathbb{L}(W_{\leq n-3})) = 0$. Now recall that $\mathcal{R}_{n-2}$ is the subgroup of $\text{aut}(W_{n-1}) \times$
$\mathcal{E}(\mathbb{L}(W_{\leq n-3}))$ consisting of the pairs $(\xi, [\alpha])$ making the following diagram commutes

$$
\begin{array}{ccc}
W_{n-1} & \xrightarrow{\xi} & W_{n-1} \\
\downarrow b_{n-1} & & \downarrow b_{n-1} \\
H_{n-2}(\mathbb{L}(W_{\leq n-3})) & \xrightarrow{H_{n-2}(\alpha)} & H_{n-2}(\mathbb{L}(W_{\leq n-3}))
\end{array}
$$

As $H_{n-2}(\mathbb{L}(W_{\leq n-3})) = 0$, we deduce that $\mathcal{R}_{n-2}^{n-1} = \text{aut}(W_{n-1}) \times \mathcal{E}(\mathbb{L}(W_{\leq n-3}))$ implying that $\mathcal{R}_{n-2}^{n-1}$ is infinite and by (21) the group $\mathcal{E}(\mathbb{L}(W_{\leq n-3}))$ is also infinite contradicting the fact that $\mathcal{E}(X) \cong \mathcal{E}(\mathbb{L}(W_{\leq n-1}))$ is finite. $\square$

3.1. **Self-closeness number $N\mathcal{E}(X)$**. For a space $X$, let $[X, X]$ denote the monoid of homotopy classes of self-maps of the space $X$ and let

$$A^k_\#(X) = \{ f \in [X, X] \mid \pi_i(f) : \pi_i(X) \to \pi_i(X) \text{ for any } i \leq k \}$$

In [14], Choi and Lee introduced the following concept:

**Definition 3.6.** The self-closeness number, denoted by $N\mathcal{E}(X)$, is defined as follows

$$N\mathcal{E}(X) = \min \{ k \mid A^k_\#(X) = \mathcal{E}(X) \}$$

**Theorem 3.7.** If $X$ is a simply connected rational elliptic space of formal dimension $n$, then

$$N\mathcal{E}(X) \leq n$$

**Proof.** As $X$ is of formal dimension $n$, by remark 3.2 its Sullivan model has the form $(\Lambda V, \partial) = (\Lambda V^{\leq 2n-1}, \partial)$. Define

$$A^k_\#(\Lambda V) = \{ [\alpha] \in [\Lambda V, \Lambda V] \mid \tilde{\alpha}^i : V^i \to V^i \text{ for any } i \leq k \},$$

where $[\Lambda V, \Lambda V]$ denotes the monoid of homotopy cochain algebras of self-equivalences classes of $\Lambda V$ and where $\tilde{\alpha}$ is the graded linear isomorphism induced by $\alpha$ on the graded vector space of indecomposables $V$ (see [13, 12, 15. (d)]).

Clearly by virtue on the properties of the Sullivan model we can identify the two sets $A^k_\#(\Lambda V)$ and $A^k_\#(X)$.

First we have $A^{2n-2}_\#(\Lambda V) = \mathcal{E}(\Lambda V)$ and $A^n_\#(\Lambda V^{\leq n}) = \mathcal{E}(\Lambda V^{\leq n})$. Next it is easy to see that the set $A^n_\#(\Lambda V^{\leq n})$ can be identified as a subset of $A^{2n-1}_\#(\Lambda V)$ by considering the following injective map

$$\theta : A^n_\#(\Lambda V^{\leq n}) \to A^{2n-2}_\#(\Lambda V), \quad [\alpha] \mapsto \theta([\alpha]) = [\beta] \quad (22)$$

where $\beta = \alpha$ on $V^{\leq n}$ and $\beta = id$ on $V^{> n}$. Finally using Theorem 3.3 and (22) we get

$$A^n_\#(\Lambda V^{\leq n}) \subseteq A^{2n-2}_\#(\Lambda V) = \mathcal{E}(\Lambda V) = \mathcal{E}(\Lambda V^{\leq n}) = A^n_\#(\Lambda V^{\leq n})$$

Therefore $A^n_\#(\Lambda V^{\leq n}) = \mathcal{E}(\Lambda V)$ implying that $N\mathcal{E}(\Lambda V) \leq n$. $\square$

**Example 3.8.** Given a finite group $G$. According to a Theorem of Frucht [19], there exists a connected finite graph $G = (V, E)$, where $V$ denotes the set of the
vertices of $\mathcal{G}$ and $E$ the set of its edges, such that $\text{aut}(\mathcal{G}) \cong G$. Recall that in Costoya and Viruel constructed a free commutative differential graded algebra

$$\left( \Lambda(x_1, x_2, y_1, y_2, y_3, z, \{z_v, x_v\}_{v \in V}), \partial \right)$$

where the degrees of the elements are

$$|x_1| = 8, \quad |x_2| = 10, \quad |x_v| = 40, \quad |z| = |z_v| = 119$$

$$|y_1| = 33, \quad |y_2| = 35, \quad |y_3| = 37$$

and where the differential is given by

$$\partial(x_1) = \partial(x_2) = \partial(x_v) = 0, \quad \partial(y_1) = x_1^3x_2, \quad \partial(y_2) = x_1^2x_2^2, \quad \partial(y_3) = x_1x_2^3$$

$$\partial(z_v) = x_1 + \sum_{(v, w) \in E} x_vx_wx_w^2,$$

$$\partial(z) = y_1y_2x_1^4x_2^2 - y_1y_3x_1^5x_2 + y_2y_3x_1^6 + x_1^{15} + x_2^{12}$$

and proved that $\mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, z, \{z_v, x_v\}_{v \in V}), \partial)$ is an elliptic of formal dimension $n = 208 + 80|V|$ with $|\mathcal{E}|$ the order of the graph $\mathcal{G}$.

Let $X$ be a rational space whose admits $(\Lambda(x_1, x_2, y_1, y_2, y_3, z, \{z_v, x_v\}_{v \in V}), \partial)$ as Sullivan model. On one hand and as $X$ has formal dimension $n = 207+80|V|$ the Quillen model of $X$ can be written as $(\mathbb{L}(W_{\leq 208+80|V|}), \partial)$ and because $X = X^{120}$ we deduce that $\mathcal{N}(X) \leq 120$.

On the other hand we have $m = \max\left\{ k \mid \pi_k(X) \neq 0 \right\}$, $\max\left\{ k \mid V^k \neq 0 \right\} = 119$. Therefore applying Theorem 3.4 leads to

$$\mathcal{E}(X) \cong \mathcal{E}(X^{207+80|V|}) \cong \ldots \cong \mathcal{E}(X^{120}) \cong G$$

Therefore we can ameliorate Costoya and Viruel Theorem by reducing the formal dimension of $X$ showing that every finite group $G$ occurs as $G = \mathcal{E}(X)$ for some rational space $X^{120}$ having formal dimension $n = 120$.

**Remark 3.9.** It is important to notice that the space $Z = X^{120}$ is not elliptic. Indeed, if $Z$ were elliptic, $H^*(Z, \mathbb{Q})$ would be Poincaré duality (134, Theorem A), and thus $\dim(H^{120}(Z, \mathbb{Q})) = 1$. But an easy computation shows that $\dim(H^{120}(Z, \mathbb{Q})) \geq |V| > 1$.

**Example 3.10.** In (I, Example 5.2), Arkowitz and Lupton constructed a free commutative differential graded algebra $(\Lambda(x_1, x_2, y_1, y_2, y_3), \partial)$ with $|x_1| = 10, |x_2| = 12, |y_1| = 41, |y_2| = 43, |y_3| = 45$ and $|z| = 119$. The differential is given by

$$\partial(x_1) = \partial(x_2) = 0, \quad \partial(y_1) = x_1^3x_2, \quad \partial(y_2) = x_1^2x_2^2, \quad \partial(y_3) = x_1x_2^3$$

$$\partial(z) = y_1y_2x_2^2 - y_1y_3x_1x_2^2 + y_2y_3x_1^2x_2 + x_1^{12} + x_2^{10}$$

and proved that $\mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3), \partial)$ is an elliptic of formal dimension 188.

If $X$ is a rational space whose admits $(\Lambda(x_1, x_2, y_1, y_2, y_3), \partial)$ as Sullivan model, then its Quillen model can be written as $(\mathbb{L}(W_{\leq 187}), \delta)$. Since $m = \max\left\{ k \mid \pi_k(X) \neq 0 \right\}$, $\max\left\{ k \mid V^k \neq 0 \right\} = 119$. Therefore applying Theorem 3.4 leads to

$$\mathcal{E}(X) \cong \mathcal{E}(X^{187}) \cong \ldots \cong \mathcal{E}(X^{120}) \cong \mathbb{Z}_2$$
Example 3.11. Define \((\Lambda(x, y), \partial)\) with \(|x| = 2p\). The differential is given by
\[
\partial(x) = 0, \quad \partial(y) = x^a, \quad a \geq 2
\] (23)
Obviously \((\Lambda(x, y), \partial)\) is elliptic and it easy to see that its formal dimension is \(n = 2(a - 1)p\) and \(m = \max\{k \mid V^k \neq 0\} = 2ap - 1\), so \(m > n\). Now if \(X\) is a rational space having \((\Lambda(x, y), \partial)\) as the Sullivan model, then by applying Theorem \(X\) we get \(E(X) \cong E(\Lambda(x)) \cong \mathbb{Q} - \{0\} \).

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Department of Mathematics. Faculty of Sciences, University of Sharjah. Sharjah, United Arab Emirates

E-mail address: mbenkhelifa@sharjah.ac.ae