Identifying a minimal flavor symmetry of the seesaw mechanism behind neutrino oscillations

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ABSTRACT: In the canonical seesaw framework flavor mixing and CP violation in weak charged-current interactions of light and heavy Majorana neutrinos are correlated with each other and described respectively by the $3 \times 3$ matrices $U$ and $R$. We show that the very possibility of $|U_{\mu i}| = |U_{\tau i}|$ (for $i = 1, 2, 3$), which is strongly indicated by current neutrino oscillation data, automatically leads to a novel prediction $|R_{\mu i}| = |R_{\tau i}|$ (for $i = 1, 2, 3$). We prove that behind these two sets of equalities and the experimental evidence for leptonic CP violation lies a minimal flavor symmetry — the overall neutrino mass term keeps invariant when the left-handed neutrino fields transform as $\nu_{eL} \rightarrow (\nu_{eL})^c$, $\nu_{\mu L} \rightarrow (\nu_{\tau L})^c$, $\nu_{\tau L} \rightarrow (\nu_{\mu L})^c$ and the right-handed neutrino fields undergo an arbitrary unitary CP transformation. Such a generalized $\mu-\tau$ reflection symmetry may help constrain the flavor textures of active and sterile neutrinos to some extent in the seesaw mechanism.

KEYWORDS: CP Violation, Discrete Symmetries, Neutrino Mixing, Sterile or Heavy Neutrinos

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1 Motivation

The discoveries of marvellous flavor oscillations in the atmospheric, solar, reactor and accelerator neutrino (or antineutrino) experiments have constituted a great success in particle physics [1]. But the Standard Model (SM) of particle physics does not really offer an opportunity for three neutrino flavors to oscillate from one type to another, simply because it has required both neutrino masses and lepton flavor mixing to vanish. It is therefore an absolute “must” to introduce tiny neutrino masses and significant lepton flavor mixing effects in a way that is beyond the SM framework, so as to fully account for current experimental data on neutrino oscillations. In this connection, however, the real theoretical challenges include how to pin down the true origin of neutrino masses and single out the most likely flavor symmetry responsible for the observed pattern of lepton flavor mixing and CP violation [2].

Regarding the issue of neutrino mass generation, it has commonly been accepted that the tiny masses of three active neutrinos should be attributed to their Majorana nature [3] and the existence of a few species of sterile neutrinos which are much heavier than the Higgs boson [4–10]. Such a popular seesaw mechanism works only in a qualitative way, since the unknown flavor structures of active and sterile neutrinos unavoidably forbid its quantitative predictability [11]. Although quite a lot of efforts have been made to constrain the neutrino textures with the help of various flavor symmetry groups [12–14], which flavor symmetry really lies behind what we have observed remains unclear. But a consensus seems to have essentially been reached: no matter how large the true flavor symmetry group is, its residual symmetry at low energies should serve as a minimal flavor symmetry in the neutrino sector. Here the meaning of “minimal” is two-fold. On the one hand, this flavor symmetry can be described by one of the simplest discrete or continuous symmetry groups. On the other hand, the pattern of lepton flavor mixing and CP violation determined or constrained by such a simple flavor symmetry should be as close as possible to the one extracted from the experimental data on neutrino oscillations.
Along this line of thought, we intend to identify a minimal flavor symmetry associated with three species of active and sterile neutrinos in the canonical seesaw mechanism by starting from the very possibility of $|U_{\mu 1}| = |U_{\tau i}|$ (for $i = 1, 2, 3$) that is strongly favored by current neutrino oscillation data for the $3 \times 3$ Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix $U$ [15–17]. Given the fact that $U$ is not exactly unitary but intrinsically correlated with another $3 \times 3$ flavor mixing matrix $R$ describing the strength of weak charged-current interactions of heavy Majorana neutrinos via the exact seesaw formula and the unitarity condition $UU^\dagger + RR^\dagger = I$, we show that $|U_{\mu 1}| = |U_{\tau i}|$ can automatically lead to a novel prediction $|R_{\mu 1}| = |R_{\tau i}|$ (for $i = 1, 2, 3$). We proceed to prove that behind these two sets of equalities and the preliminary experimental evidence for leptonic CP violation [18] lies an expected minimal flavor symmetry; namely, the overall neutrino mass term keeps invariant when the left-handed neutrino fields transform as $\nu_L \rightarrow (\nu_{cL})^c$, $\nu_{\mu L} \rightarrow (\nu_{c\mu L})^c$, $\nu_{\tau L} \rightarrow (\nu_{c\tau L})^c$ and the right-handed neutrino fields undergo an arbitrary unitary CP transformation. With the help of a full Euler-like parametrization of the PMNS matrix $U$ and its counterpart $R$, we demonstrate that such a generalized $\mu$-$\tau$ reflection symmetry may help constrain the flavor structures of active and sterile neutrinos to some extent. So it should be able to enhance both predictability and testability of the canonical seesaw mechanism.

2 Proofs

2.1 The seesaw mechanism

The trivial reason for vanishing neutrino masses in the SM is an absence of the right-handed neutrino fields and thus an absence of the Yukawa interactions between the Higgs and neutrino fields. It is therefore natural to minimally extend the SM by adding three right-handed neutrino fields $N_{\alpha R}$ (for $\alpha = e, \mu, \tau$) and allowing for lepton number violation [4–10]. In this case the gauge- and Lorentz-invariant neutrino mass terms can be written as

$$-\mathcal{L}_\nu = \overline{\ell}_L Y_\nu \tilde{H} N_R + \frac{1}{2} (N_R)^c \nu_M N_R + \text{h.c.},$$

where $\ell_L$ denotes the $SU(2)_L$ doublet of the left-handed lepton fields, $\tilde{H} \equiv i\sigma_2 H^\ast$ with $H$ being the Higgs doublet and $\sigma_2$ being the second Pauli matrix, $N_R = (N_{eR}, N_{\mu R}, N_{\tau R})^T$ is the column vector of three right-handed neutrino fields which are the $SU(2)_L$ singlets, $(N_R)^c \equiv C N_R^T$ with $C$ being the charge-conjugation operator and satisfying $C \gamma_\mu C^{-1} = -\gamma_\mu$ and $C^{-1} = C^T = -C$, $Y_\nu$ represents an arbitrary $3 \times 3$ Yukawa coupling matrix, and $M_R$ stands for a symmetric $3 \times 3$ Majorana mass matrix. After spontaneous electroweak gauge symmetry breaking, eq. (2.1) becomes

$$-\mathcal{L}_\nu = \frac{1}{2} \nu_L (N_R)^c \left( \begin{array}{cc} 0 & M_D \\ M_D^T & M_R \end{array} \right) \left[ \begin{array}{c} (\nu_L)^c \\ N_R \end{array} \right] + \text{h.c.},$$

where $\nu_L = (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})^T$ denotes the column vector of three left-handed neutrino fields, $M_D \equiv Y_\nu \langle H \rangle$ with $\langle H \rangle$ being the vacuum expectation value of the Higgs field, and $0$ denotes the $3 \times 3$ zero matrix. Note that $M_D$ is in general neither Hermitian nor symmetric. The
overall symmetric $6 \times 6$ neutrino mass matrix in eq. (2.2) can be diagonalized by the unitary transformation

$$
\begin{pmatrix}
U & R \\
S & Q
\end{pmatrix}^\dagger
\begin{pmatrix}
0 & M_D \\
M_D^T & M_R
\end{pmatrix}
\begin{pmatrix}
U & R \\
S & Q
\end{pmatrix}^* =
\begin{pmatrix}
D_\nu & 0 \\
0 & D_N
\end{pmatrix},
$$

(2.3)

where $D_\nu \equiv \text{Diag} \{m_1, m_2, m_3\}$ and $D_N \equiv \text{Diag} \{M_1, M_2, M_3\}$ with $m_i$ and $M_i$ (for $i = 1, 2, 3$) being the masses of active and sterile Majorana neutrinos, respectively. The $3 \times 3$ submatrices $U, R, S$ and $Q$ in eq. (2.3) satisfy the unitarity conditions:

$$
UU^\dagger + RR^\dagger = SS^\dagger + QQ^\dagger = I,
$$

$$
U^\dagger U + S^\dagger S = R^\dagger R + Q^\dagger Q = I,
$$

$$
US^\dagger + RQ^\dagger = U^\dagger R + S^\dagger Q = 0;
$$

(2.4)

and the exact 

seesaw formula that characterizes a kind of balance between the light and heavy neutrino sectors can also be obtained from eq. (2.3) [19]:

$$
UD_\nu U^T + RD_N R^T = 0.
$$

(2.5)

So the smallness of $m_i$ is essentially ascribed to the highly suppressed magnitude of $R$ which signifies the largeness of $M_i$ with respect to the electroweak scale (i.e., $R \sim O(M_D/M_R) \ll 1$ in the leading-order approximation [4–10]). In this canonical seesaw framework $\nu_\alpha$ (for $\alpha = e, \mu, \tau$) can be expressed as a linear combination of the mass eigenstates of three active (light) neutrinos and three sterile (heavy) neutrinos (i.e., $\nu_i = (\nu_j)^c$ and $N_i = (N_j)^c$ for $i = 1, 2, 3$). The standard weak charged-current interactions of these six Majorana neutrinos is therefore given by

$$
-L_{cc} = \frac{g}{\sqrt{2}} (e \mu \tau)_L \gamma^\mu \left[ U \begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}_L + R \begin{pmatrix}
N_1 \\
N_2 \\
N_3
\end{pmatrix}_L \right] W^-_\mu + \text{h.c.}
$$

(2.6)

It is obvious that the PMNS matrix $U$ describes flavor mixing and CP violation of three active neutrinos (e.g., in neutrino oscillations), and its counterpart $R$ measures the strength of weak charged-current interactions of three sterile neutrinos (e.g., in precision collider physics). The validity of such a seesaw mechanism means that $R \neq 0$ must hold no matter how small its nine elements may be, otherwise the active neutrinos would have no chance to acquire their masses from eqs. (2.1) and (2.5). As a consequence of $UU^\dagger = I - RR^\dagger \neq I$, the PMNS matrix $U$ must be non-unitary in the seesaw framework although its deviation from unitarity is expected to be very small. A careful analysis of current electroweak precision measurements and neutrino oscillation data has constrained $U$ to be unitary at the $O(10^{-2})$ sensitivity level [20–24]. In other words, possible non-unitarity effects hiding in $U$ are expected to be at most of $O(10^{-2})$ and hence cannot be identified up to the accuracy level of today’s neutrino oscillation experiments.
Figure 1. An illustration of the relative moduli of nine PMNS matrix elements, where the colored area of each circle is proportional to $|U_{\alpha i}|$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) on the same scaling.

2.2 Implications of $|U_{\mu i}| = |U_{\tau i}|$

A global analysis of the latest atmospheric, solar, reactor and accelerator neutrino oscillation data yields the $3\sigma$ intervals of nine elements of the PMNS matrix $U$ as follows [25, 26]:

$$
|U_{e1}| = 0.801 \rightarrow 0.845, \quad |U_{e2}| = 0.513 \rightarrow 0.579, \quad |U_{e3}| = 0.143 \rightarrow 0.155, \\
|U_{\mu 1}| = 0.234 \rightarrow 0.500, \quad |U_{\mu 2}| = 0.471 \rightarrow 0.689, \quad |U_{\mu 3}| = 0.637 \rightarrow 0.776, \\
|U_{\tau 1}| = 0.271 \rightarrow 0.525, \quad |U_{\tau 2}| = 0.477 \rightarrow 0.694, \quad |U_{\tau 3}| = 0.613 \rightarrow 0.756.
$$

(2.7)

As argued above, these numerical results are actually insensitive to small unitarity violation of $U$ although they were originally obtained by assuming $U$ to be unitary. The relative magnitudes of $|U_{\alpha i}|$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) are more intuitively illustrated in figure 1, where the colored area of each circle is proportional to the size of $|U_{\alpha i}|$ on the same scaling. We see that the very possibility of $|U_{\mu i}| = |U_{\tau i}|$, which is strongly favored by current experimental data, constitutes the most salient feature of $U$ known to us.\textsuperscript{1} This observation motivates us to consider how $U_{\mu i}$ is directly related to $U_{\tau i}$ or $U_{\tau i}^*$ in a given flavor basis and whether their rephasing-dependent relationship is naturally attributed to an underlying flavor symmetry. In the same flavor basis $U_{ei}$ is expected to either stay unchanged or become its complex conjugate, such that $|U_{ei}|$ keeps invariant. We find that there are two typical possibilities of this kind that allow the PMNS matrix elements $U_{ei}$, $U_{\mu i}$ and $U_{\tau i}$ to transform together.

\textsuperscript{1}Note that a somewhat weaker possibility, $|U_{e2}| = |U_{\mu 2}| = |U_{\tau 2}| = 1/\sqrt{3}$, has also attracted some interest of model building [27] and is apparently consistent with the case of $|U_{\mu i}| = |U_{\tau i}|$ (for $i = 1, 2, 3$) under discussion.
Case A: \( U = \mathcal{P} U^* \), where the real orthogonal \((\mu, \tau)\)-permutation matrix \( \mathcal{P} \) takes the form

\[
\mathcal{P} = \mathcal{P}^T = \mathcal{P}^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

is certainly a simple but nontrivial solution to \( |U_{\mu i}| = |U_{\tau i}| \). Substituting \( U = \mathcal{P} U^* \) into the exact seesaw formula in eq. (2.5) and taking the complex conjugate for the whole equation, we obtain

\[
UD_N U^T + \mathcal{P} R^* D_N (\mathcal{P} R^*)^T = 0.
\]

Then a comparison between eqs. (2.5) and (2.9) leads us to \( R = \mathcal{P} R^* \), This observation implies that the novel and rephasing-invariant prediction \( |R_{\mu i}| = |R_{\tau i}| \) is actually a natural consequence of \( |U_{\mu i}| = |U_{\tau i}| \) in the canonical seesaw mechanism. Now let us proceed to substitute \( U = \mathcal{P} U^* \) and \( R = \mathcal{P} R^* \) into the unitarity conditions listed in eq. (2.4). After a careful but straightforward check, we find that \( S = T S^* \) and \( Q = T Q^* \) with \( T \) being an arbitrary unitary matrix satisfy all the normalization and orthogonality relations in eq. (2.4). In short, we have

\[
\begin{aligned}
U = \mathcal{P} U^* \quad \rightarrow R = \mathcal{P} R^* \\
S = T S^* \quad \downarrow \\
Q = T Q^*
\end{aligned}
\]

as constrained by eqs. (2.4) and (2.5). One will see that these relations can help fix the flavor textures of active and sterile Majorana neutrinos to a large extent.

Case B: \( U = \mathcal{P} U^\zeta \), where \( \zeta = \text{Diag}\{\eta_1, \eta_2, \eta_3\} \) with \( \eta_i = \pm 1 \), is the other simple but typical solution to \( |U_{\mu i}| = |U_{\tau i}| \) (for \( i = 1, 2, 3 \)). Note that replacing \( \zeta \) with a diagonal phase matrix is also allowed in this regard, but such a possibility will be disfavored when the corresponding expression of \( U \) is substituted into the exact seesaw relation in eq. (2.5). In this case one may start from \( U = \mathcal{P} U^\zeta \) to similarly prove that \( R = \mathcal{P} R^\zeta \), \( S = T^S S^\zeta \) and \( Q = T^Q Q^\zeta \) hold with the help of eqs. (2.4) and (2.5), where \( \zeta' = \text{Diag}\{\eta'_1, \eta'_2, \eta'_3\} \) (for \( \eta'_i = \pm 1 \)) and \( T' \) is another arbitrary unitary matrix. Concentrating on the \((\mu, \tau)\)-associated orthogonality condition, we obtain

\[
\sum_{i=1}^{3} \left( U_{\mu i} U_{\tau i}^* + R_{\mu i} R_{\tau i}^* \right) = \sum_{i=1}^{3} \left( \eta_i |U_{\mu i}|^2 + \eta'_i |R_{\mu i}|^2 \right) = 0,
\]

where the magnitude of \( |R_{\mu i}|^2 \) is expected to be of or below \( \mathcal{O}(10^{-2}) \). Figure 2 is an illustration of the unitarity polygon defined by the above orthogonality relation in the complex plane. It becomes clear that the possibility of \( \eta_1 = \eta_2 = \eta_3 \) has to be abandoned, otherwise eq. (2.11) would be in conflict with the corresponding normalization condition of \( U_{\mu i} \) and \( R_{\mu i} \) (for \( i = 1, 2, 3 \)). Given the fact that \( |U_{\mu 3}| \) is definitely greater than \( |U_{\mu 1}| \)
Figure 2. An illustration of the unitarity polygon defined by the $(\mu, \tau)$-associated orthogonality condition in the complex plane, where the effective vertex in sky blue denotes small corrections of the active-sterile flavor mixing effects to the three-flavor unitarity triangle in the seesaw framework.

and most likely greater than $|U_{\mu 2}|$, as one can see from eq. (2.7), we find that the choice of $\eta_1 = \eta_2 = -\eta_3$ is interesting because it results in

$$|U_{\mu 1}|^2 + |U_{\mu 2}|^2 = \frac{1}{2} \left[ 1 - \sum_{i=1}^{3} (1 + \eta_i \eta'_i) |R_{\mu i}|^2 \right],$$

$$|U_{\tau 3}|^2 = |U_{\mu 3}|^2 = \frac{1}{2} \left[ 1 - \sum_{i=1}^{3} (1 - \eta_i \eta'_i) |R_{\mu i}|^2 \right].$$

(2.12)

However, this possibility indicates that $|U_{e 3}|$ must be remarkably smaller than its true value (i.e., $|U_{e 3}| \simeq 0.15$) that has been measured by the Daya Bay Collaboration [28]. On the other hand, a highly suppressed value of $|U_{e 3}|$ would give rise to a strong suppression of leptonic CP violation in neutrino oscillations and thus contradict the $3\sigma$ evidence for CP violation established recently by the T2K Collaboration [18]. So our subsequent discussions will focus only on Case A, which is more likely to lead us to a minimal flavor symmetry of the seesaw mechanism.

2.3 A minimal flavor symmetry

Now that the rephasing-dependent relation $U = \mathcal{P}U^*$ does assure that the rephasing-independent equalities $|U_{\mu i}| = |U_{\tau i}|$ and $|R_{\mu i}| = |R_{\tau i}|$ hold, it is very likely to hint at a kind of flavor symmetry of the seesaw mechanism which just lies behind the observed pattern of $U$. To pin down such a possible flavor symmetry, let us first substitute $U = \mathcal{P}U^*$ and its counterparts $R = \mathcal{P}R^*$, $S = \mathcal{T}S^*$ and $Q = \mathcal{T}Q^*$ into eq. (2.3) and then take the complex conjugate for the whole equation. In this way we are immediately left with

$$
\begin{pmatrix}
U & R \\
S & Q
\end{pmatrix}^\dagger
\begin{pmatrix}
0 & \mathcal{P}M_D^T \mathcal{T} \\
\mathcal{T}^T M_D^T \mathcal{P} & \mathcal{T}^T M_R^T \mathcal{T}
\end{pmatrix}
\begin{pmatrix}
U & R \\
S & Q
\end{pmatrix}^* =
\begin{pmatrix}
D_{\nu} & 0 \\
0 & D_N
\end{pmatrix},
\tag{2.13}
$$

A comparison between eqs. (2.3) and (2.13) leads us to the following constraint equations:

$$M_D = \mathcal{P}M_D^T \mathcal{T}, \quad M_R = \mathcal{T}^T M_R^T \mathcal{T}.$$  

(2.14)

This new result is in principle powerful to constrain the flavor textures of $M_D$ and $M_R$, but its application is in practice limited by the fact that $\mathcal{T}$ is an arbitrary unitary matrix. For this reason the assumption of a specific form of $\mathcal{T}$ actually corresponds to a specific seesaw model with the $\mu$-$\tau$ reflection symmetry. Here let us consider two simple but instructive scenarios of $\mathcal{T}$ for example.
Scenario A: \( T = I \). In this simplest case \( M_R = M^*_R \) is a real symmetric matrix and can always be arranged to be diagonal in the beginning (i.e., \( M_R = D_N \)) by taking a proper basis for the right-handed neutrino fields in eq. (2.1), and \( M_D = \mathcal{P} M^*_D \) is constrained as

\[
M_D = \begin{pmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X^*_1 & X^*_2 & X^*_3
\end{pmatrix},
\]

(2.15)

where \( X_{11}, X_{12} \) and \( X_{13} \) are all real. It is obvious that the texture of \( M_D \) is exactly analogous to the pattern of \( U \) in such a specific \( \mu - \tau \) reflection symmetry scenario.

Scenario B: \( T = \mathcal{P} \). This interesting possibility has been discussed for a concrete seesaw model based on the \( \mu - \tau \) reflection symmetry (see, e.g., refs. [29, 30]). Needless to say, the flavor textures of \( M_D \) and \( M_R \) can be well constrained in this case:

\[
M_D = \begin{pmatrix}
X_{11} & X_{12} & X^*_{12} \\
X^*_{21} & X_{22} & X_{23} \\
X^*_{21} & X^*_2 & X^*_2
\end{pmatrix},
\]

(2.16)

\[
M_R = \begin{pmatrix}
Z_{11} & Z_{12} & Z_{12} \\
Z_{12} & Z_{22} & Z_{23} \\
Z^*_{12} & Z^*_2 & Z^*_{22}
\end{pmatrix},
\]

where \( X_{11}, Z_{11} \) and \( Z_{23} \) are real. We find that the number of real free parameters of \( M_D \) is reduced by half, from eighteen to nine; and that of \( M_R \) is also reduced by half, from twelve to six.

It has been shown in ref. [29] that the observed baryon number asymmetry of the Universe can be interpreted by combining Scenario B with the leptogenesis mechanism [31]. Some other forms of \( T \), which respect the \( S_3 \) symmetry, have also been discussed in ref. [30] to constrain the seesaw flavor textures and in turn the validity of leptogenesis. Instead of assuming a special form of \( T \) as done in the literature, here we have derived the constraint conditions for \( M_D \) and \( M_R \) in eq. (2.14) with the help of only the data-inspired conjecture \( U = \mathcal{P} U^* \) in the canonical seesaw framework. So our result is certainly more generic and thus could open up some more possibilities of model building along this line of thought.

After eq. (2.14) is substituted into eq. (2.2), the overall neutrino mass term \( \mathcal{L}'_\nu \) becomes

\[
\mathcal{L}'_\nu = \frac{1}{2} \left[ \mathcal{P} \nu_L T^* (N_R)^c \right] \begin{pmatrix}
0 & M_D^* \\
M_D & M_R
\end{pmatrix} \begin{pmatrix}
\mathcal{P} (\nu_L)^c \\
T N_R
\end{pmatrix} + \text{h.c.}
\]

\[
= \frac{1}{2} \left[ \mathcal{P} (\nu_L)^c \right] T (N_R)^c \begin{pmatrix}
0 & M_D \\
M_D^T & M_R
\end{pmatrix} \begin{pmatrix}
\mathcal{P} \nu_L \\
T^* (N_R)^c
\end{pmatrix} + \text{h.c.},
\]

(2.17)

in which the apparent mass term of the second equation comes from the Hermitian conjugation term of the first equation. Comparing between eq. (2.2) and eq. (2.17), we easily find that \( \mathcal{L}'_\nu \) keeps unchanged under the transformations

\[
\nu_L \to \mathcal{P} (\nu_L)^c, \quad N_R \to T^* (N_R)^c.
\]

(2.18)
That is, the neutrino mass term $\mathcal{L}_\nu'$ is invariant when the left-handed neutrino fields transform as

$$
\begin{pmatrix}
\nu_{eL} \\
\nu_{\mu L} \\
\nu_{\tau L}
\end{pmatrix}
\rightarrow \mathcal{P}
\begin{pmatrix}
(\nu_{eL})^c \\
(\nu_{\mu L})^c \\
(\nu_{\tau L})^c
\end{pmatrix},
$$

while the right-handed neutrino fields transform as in eq. (2.18). It is well known that CP would be a good symmetry in the canonical seesaw mechanism if $\mathcal{L}_\nu'$ were invariant under the charge conjugation and parity transformations $\nu_{aL}(t, x) \rightarrow [\nu_{aL}(t, -x)]^c$ and $N_{aR}(t, x) \rightarrow [N_{aR}(t, -x)]^c$ (for $\alpha = e, \mu, \tau$) [32]. But the charge-conjugated interchange between the $(\mu, \tau)$-associated left-handed neutrino fields in eq. (2.19), together with an arbitrary unitary CP transformation of the right-handed neutrino fields, makes CP violation possible even though $\mathcal{L}_\nu'$ itself keeps invariant in this case. The issue will be more specific and transparent if a specific form of $\mathcal{T}$ is taken. Here let us consider two simple examples as above.

**Scenario A:** $\mathcal{T} = I$ means that the right-handed neutrinos have a normal CP transformation,

$$
\begin{pmatrix}
N_{eR} \\
N_{\mu R} \\
N_{\tau R}
\end{pmatrix}
\rightarrow I
\begin{pmatrix}
(N_{eR})^c \\
(N_{\mu R})^c \\
(N_{\tau R})^c
\end{pmatrix} =
\begin{pmatrix}
(N_{eR})^c \\
(N_{\mu R})^c \\
(N_{\tau R})^c
\end{pmatrix};
$$

where the right-handed neutrino fields transform in the same way as the left-handed ones in eq. (2.19). This parallelism between active and sterile sectors might be helpful for model building.

Integrating out those heavy degrees of freedom in the canonical seesaw mechanism, one may arrive at the following effective mass term for three active neutrinos:

$$
-\mathcal{L}_{\text{mass}} = \frac{1}{2} \overline{\nu}_L M_\nu (\nu_L)^c + \text{h.c.,}
$$

where

$$
M_\nu \simeq -M_D M_R^{-1} M_D^T
$$

is known as an approximate seesaw formula for the effective Majorana neutrino mass matrix $M_\nu$. In this regard a unitary matrix $U_0$ used to diagonalize $M_\nu$ just serves as the PMNS matrix (i.e., $U_0^T M_\nu U_0^* = \mathcal{D}_\nu$). Substituting eq. (2.14) into eq. (2.22) and taking the complex conjugate, we obtain $M_\nu = \mathcal{P} M_\nu^* \mathcal{P}$ and hence $U_0 = \mathcal{P} U_0^*$ as a self-consistent result. It is then easy to check that $\mathcal{L}_{\text{mass}}$ will be unchanged under the transformations
$\nu_{eL} \rightarrow (\nu_{eL})^c$, $\nu_{\mu L} \rightarrow (\nu_{\tau L})^c$ and $\nu_{\tau L} \rightarrow (\nu_{\mu L})^c$. Namely, the flavor texture of $M_\nu$ can be constrained as [33–36]:

$$
M_\nu = \begin{pmatrix}
\langle m \rangle_{ee} & \langle m \rangle_{em} & \langle m \rangle_{e\mu}^* \\
\langle m \rangle_{em} & \langle m \rangle_{\mu\mu} & \langle m \rangle_{\mu\tau}^* \\
\langle m \rangle_{e\mu} & \langle m \rangle_{\mu\tau} & \langle m \rangle_{\mu\mu}^*
\end{pmatrix}
$$

(2.24)

with $\langle m \rangle_{ee} = \langle m \rangle_{ee}^*$ and $\langle m \rangle_{\mu\tau} = \langle m \rangle_{\mu\tau}^*$ being real. Such a minimal flavor symmetry has commonly been referred to as the $\mu$-$\tau$ reflection symmetry [27]. Here we have proved possible existence and correctness of its generalized version by taking account of the non-unitarity of the PMNS matrix $U$ and combining the equalities $|U_{\mu i}| = |U_{\tau i}|$ with the canonical seesaw mechanism.

Of course, the equalities $|U_{\mu i}| = |U_{\tau i}|$ are most likely to hold as a good approximation at low energies. This issue will become clear after the parameters of flavor mixing and CP violation are measured to a much better degree of precision in the upcoming experiments of neutrino oscillations. So it is more reasonable to conjecture that the equalities $|U_{\mu i}| = |U_{\tau i}|$ are exact and originate from the $\mu$-$\tau$ reflection symmetry at the seesaw scale, and they become approximate at the electroweak scale as a natural consequence of the renormalization-group-equation (RGE) running effects (see, e.g., refs. [36–42] for some detailed analyses and discussions).

### 2.4 An Euler-like parametrization

In the canonical seesaw framework a full Euler-like parametrization of $U$, $R$, $S$ and $Q$ in terms of fifteen rotation angles and fifteen phase angles has been proposed in refs. [43, 44] with a kind of decomposition like

$$
\begin{pmatrix}
U \\
R \\
S \\
Q
\end{pmatrix} = \begin{pmatrix}
P_l \\
A \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
U_0 \\
U_0' \\
U_0 \\
U_0
\end{pmatrix} = \begin{pmatrix}
P_l U_0 & P_l U_0' \\
U_0 R U_0 & U_0' B
\end{pmatrix},
$$

(2.25)

where $P_l = \text{Diag}\{e^{i\varphi_e}, e^{i\varphi_\mu}, e^{i\varphi_\tau}\}$ is an arbitrary phase matrix associated with three charged-lepton fields, $U_0$ and $U_0'$ are the unitary matrices which describe the respective primary flavor mixing effects of active and sterile neutrinos, and $A$ (or $B$) characterizes a slight deviation of $U = P_l A U_0$ (or $Q = U_0' B$) from $U_0$ (or $U_0'$). But here we are only interested in $U$ and $R$ that appear in the weak charged-current interactions of massive neutrinos as shown in eq. (2.6). To be explicit [44],

$$
U_0 = \begin{pmatrix}
c_{12} c_{13} & \tilde{s}_{12} c_{13} & \tilde{s}_{13} \\
-s_{12} c_{23} - c_{12} \tilde{s}_{13} \tilde{s}_{23} & c_{12} c_{23} - \tilde{s}_{12} \tilde{s}_{13} \tilde{s}_{23} & c_{13} \tilde{s}_{23} \\
\tilde{s}_{12} \tilde{s}_{23} - c_{12} \tilde{s}_{13} c_{23} & -c_{12} \tilde{s}_{23} - \tilde{s}_{12} \tilde{s}_{13} c_{23} & c_{13} c_{23}
\end{pmatrix},
$$

(2.26)
and

\[
A = \begin{pmatrix}
    c_{14}c_{15}c_{16} & 0 & 0 \\
    -c_{14}c_{15}s_{16}s_{26} - c_{14}s_{15}s_{25}c_{26} & c_{24}c_{25}c_{26} & 0 \\
    -s_{14}s_{24}c_{25}c_{26} & -c_{24}s_{25}s_{26} + c_{24}s_{25}s_{35}c_{36} & c_{34}s_{35}c_{36} \\
    -c_{14}s_{15}s_{16}c_{26}s_{36} + c_{14}s_{15}s_{25}s_{26}s_{36} & -c_{14}s_{15}s_{24}c_{25}s_{26}s_{36} & c_{34}s_{35}c_{36} \\
    +s_{14}s_{24}s_{25}s_{35}c_{36} - c_{14}s_{24}s_{34}s_{35}c_{36} & -s_{24}s_{34}c_{35}c_{36} & c_{34}s_{35}c_{36}
\end{pmatrix},
\]

\[
\tilde{R} = \begin{pmatrix}
    s_{14}c_{15}c_{16} & s_{15}c_{16} & s_{16} \\
    -s_{14}c_{15}s_{16}s_{26} - s_{14}s_{15}s_{25}c_{26} & -s_{15}s_{16}s_{26} + c_{15}s_{25}c_{26} & c_{16}s_{26} \\
    +c_{14}s_{24}c_{25}c_{26} & -s_{14}s_{15}s_{16}c_{26}s_{36} + c_{14}s_{15}s_{25}s_{26}s_{36} & c_{14}s_{15}s_{24}s_{25}s_{35}c_{36} + c_{15}s_{25}s_{35}c_{36} \\
    -c_{14}s_{15}s_{16}c_{26}s_{36} + c_{14}s_{15}s_{25}s_{26}s_{36} & -s_{15}s_{16}s_{26}s_{36} + c_{15}s_{25}s_{26}s_{36} & c_{16}s_{26}s_{36}
\end{pmatrix},
\]

(2.27)

where $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$ and $\hat{s}_{ij} \equiv s_{ij}e^{i\delta_{ij}}$ with $\delta_{ij}$ lying in the first quadrant (for $ij = 12, 13, \ldots$). It is clear that the triangular matrix $A$ characterizes to what extent the PMNS matrix $U$ may deviate from $U_0$. Since the interplay between active and sterile neutrino sectors is expected to be weak enough, it is rather reasonable to make the following approximations to $A$ and $\tilde{R}$ [44]:

\[
A \approx I - \frac{1}{2} \begin{pmatrix}
    s_{14}^2 + s_{15}^2 + s_{16}^2 & 0 & 0 \\
    2s_{14}s_{24} + 2s_{15}s_{34} + 2s_{16}s_{26} & s_{24}^2 + s_{25}^2 + s_{26}^2 & 0 \\
    2s_{14}s_{34} + 2s_{15}s_{35} + 2s_{16}s_{36} & 2s_{24}s_{34} + 2s_{25}s_{35} + 2s_{26}s_{36} & s_{34}^2 + s_{35}^2 + s_{36}^2
\end{pmatrix} + O\left(s_{ij}^4\right),
\]

\[
\tilde{R} \approx \begin{pmatrix}
    \hat{s}_{14}^4 & \hat{s}_{15}^4 & \hat{s}_{16}^4 \\
    s_{24}^2 + s_{25}^2 + s_{26}^2 & s_{34}^2 + s_{35}^2 + s_{36}^2
\end{pmatrix} + O\left(s_{ij}^4\right),
\]

(2.28)

where all the $s_{ij}$ terms (for $i = 1, 2, 3$ and $j = 4, 5, 6$) are expected to be below or even far below $O(10^{-1})$. So $U$ can be treated as a unitary matrix to a good degree of accuracy (i.e., $A \approx I$ and $U \approx P_U U_0$ up to the $O(s_{ij}^4)$ level). In this case combining eqs. (2.26) and (2.28) with $U = P U^*$ and $R = P R^*$ allows us to obtain some approximate constraint conditions:

\[
e^{i2\varphi_\tau - i2(\varphi_\tau - \delta_{12})} \simeq e^{i2(\varphi_\tau - \delta_{13})} \simeq 1
\]

(2.29)

from $U_{ei} \simeq U_{ei}^*$ (for $i = 1, 2, 3$), and

\[
\begin{align*}
(s_{12}c_{23} + c_{12}s_{13}s_{23}e^{i\delta}) e^{i(\varphi_\tau + \varphi_\tau + 2\delta_{12} + \delta_{23})} & \simeq - (s_{12}s_{23} - c_{12}s_{13}c_{23}e^{-i\delta}), \\
(c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta}) e^{i(\varphi_\tau + \varphi_\tau + \delta_{23})} & \simeq - (c_{12}s_{23} + s_{12}s_{13}c_{23}e^{-i\delta}), \\
s_{23}e^{i(\varphi_\tau - \varphi_\tau - \delta_{23})} & \simeq c_{23}
\end{align*}
\]

(2.30)
from $U_{\mu i} \simeq U_{\tau i}^*$ (for $i = 1, 2, 3$), where $\delta = \delta_{13} - \delta_{12} - \delta_{23}$ has been defined; as well as

$$e^{i\delta(\varphi_{\mu} - \delta_{14})} \simeq e^{i\delta(\varphi_{e} - \delta_{15})} \simeq e^{i\delta(\varphi_{\tau} - \delta_{16})} \simeq 1$$  \hspace{1cm} (2.31)

from $R_{ei} \simeq R_{ei}^*$ (for $i = 1, 2, 3$), and

$$s_{24}e^{i(\varphi_{\mu} + \varphi_{e} - \delta_{24} - \delta_{34})} \simeq s_{34},$$
$$s_{25}e^{i(\varphi_{\mu} + \varphi_{e} - \delta_{25} - \delta_{35})} \simeq s_{35},$$
$$s_{26}e^{i(\varphi_{\mu} + \varphi_{e} - \delta_{26} - \delta_{36})} \simeq s_{36}$$ \hspace{1cm} (2.32)

from $R_{\mu i} \simeq R_{\tau i}^*$ (for $i = 1, 2, 3$). As a consequence, we arrive at\(^2\)

$$\theta_{23} \simeq \frac{\pi}{4}, \quad \delta \simeq \pm \frac{\pi}{2}, \quad \delta_{12} \simeq 0 \text{ or } \pi, \quad \delta_{13} \simeq 0 \text{ or } \pi, \quad \delta_{23} \simeq \pm \frac{\pi}{2},$$ \hspace{1cm} (2.33)

together with $\varphi_{e} \simeq 0$ or $\pi$ and $\varphi_{\mu} + \varphi_{\tau} \simeq \delta_{23} \simeq \pm \pi/2$; and in the same time,

$$\theta_{2 j} \simeq \theta_{3 j}, \quad \delta_{1 j} \simeq 0 \text{ or } \pi, \quad \delta_{2 j} + \delta_{3 j} \simeq \varphi_{\mu} + \varphi_{\tau} \simeq \pm \frac{\pi}{2},$$ \hspace{1cm} (2.34)

where $j = 4, 5, 6$. Note that $\varphi_{e}$, $\varphi_{\mu}$ and $\varphi_{\tau}$ are not observable in any realistic experiments, but they should not be ignored when using the $\mu$-$\tau$ reflection symmetry to constrain those observable flavor mixing angles and CP-violating phases as discussed above.

So far a lot of attention has been paid to $\theta_{23} \simeq \pi/4$ and $\delta \simeq \pm \pi/2$ obtained in eq. (2.33) with the help of the $\mu$-$\tau$ reflection symmetry of $\mathcal{L}_{\text{mass}}$; either from the viewpoint of model building or in the in-depth studies of neutrino phenomenology [36]. When applying such a minimal flavor symmetry to the seesaw mechanism, one should carefully examine whether its generalized version advocated in the present work should be taken into account or not. In particular, the correlation between $U$ and $R$ via the exact seesaw formula and unitarity conditions are sensitive to some lepton-number-violating processes (e.g., the neutrinoless double-beta decays [19] and leptogenesis [31]) and lepton-flavor-violating processes (e.g., $\mu^- \rightarrow e^- + \gamma$) in which the interplay between active and sterile Majorana neutrinos cannot be neglected [30, 45–48].

At this point it is worth mentioning that a generalized $\mu$-$\tau$ reflection symmetry is also applicable to the minimal seesaw model which contains only two species of right-handed neutrinos [49]. In this simpler case the overall neutrino mass term keeps invariant when the three left-handed neutrino fields transform as $\nu_{eL} \rightarrow (\nu_{eL})^c$, $\nu_{\mu L} \rightarrow (\nu_{\mu L})^c$, $\nu_{\tau L} \rightarrow (\nu_{\tau L})^c$ and the two right-handed neutrino fields undergo an arbitrary unitary CP transformation (see, e.g., refs. [50–52] for a few concrete models of this kind), and the approximate relations obtained in eqs. (2.31), (2.32) and (2.34) keep valid only if the active-sterile flavor mixing parameters $\theta_{26}$ and $\delta_{16}$ (for $i = 1, 2, 3$) are switched off. Needless to say, such a simplified version of the canonical seesaw mechanism is expected to be much more predictive and thus can be more easily tested after it is combined with the $\mu$-$\tau$ reflection symmetry in a consistent manner.

\(^2\)Note that the phase convention of $U_{\nu 0}$ used in eq. (2.26) is somewhat different from those used by some other authors (see, e.g., refs. [1, 2, 38]). Of course, it is trivial to establish a direct relationship between any two kinds of phase conventions for the PMNS matrix when it is assumed to be exactly unitary.
3 Summary

In the canonical seesaw framework we have identified a minimal flavor symmetry lying behind the equalities $|U_{\mu i}| = |U_{\tau i}|$ (for $i = 1, 2, 3$) that are strongly favored by current experimental data on neutrino oscillations. In view of the fact that the PMNS matrix $U$ is correlated with the $3 \times 3$ flavor mixing matrix $R$ characterizing the strength of weak charged-current interactions of heavy Majorana neutrinos via the exact seesaw formula and the unitarity condition $UU^\dagger + RR^\dagger = I$, we have shown that $|U_{\mu i}| = |U_{\tau i}|$ can automatically lead to $|R_{\mu i}| = |R_{\tau i}|$ (for $i = 1, 2, 3$). We have further proved that behind these two sets of equalities and the T2K evidence for leptonic CP violation is a very simple flavor symmetry — the overall neutrino mass term keeps invariant when the left-handed neutrino fields transform as $\nu_{eL} \rightarrow (\nu_{eL})^c$, $\nu_{\mu L} \rightarrow (\nu_{\tau L})^c$, $\nu_{\tau L} \rightarrow (\nu_{\mu L})^c$ and the right-handed neutrino fields undergo an arbitrary unitary CP transformation. Such a generalized $\mu$-$\tau$ reflection symmetry is expected to help a lot to constrain the flavor textures of active and sterile neutrinos and hence enhance predictability and testability of the seesaw mechanism.

We have illustrated how the flavor mixing angles and CP-violating phases are explicitly constrained by making use of a full Euler-like parametrization of the PMNS matrix $U$ and its counterpart $R$.

We must stress that the canonical seesaw mechanism may serve as the most natural and popular theoretical framework for the origin of tiny neutrino masses at a very low cost for going beyond the SM. So a possible minimal flavor symmetry underlying this mechanism, like the generalized $\mu$-$\tau$ reflection symmetry that we have explored in this work, certainly deserves our serious attention. A further and comprehensive study of such a flavor symmetry, including how to naturally break it, how to naturally embed it into a viable gauge model and how to link it with various measurements of lepton-number-violating and lepton-flavor-violating processes, is well worth the wait.

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