Double-Diffusive Mixing-Length Theory, Semiconvection, and Massive Star Evolution

Scott A. Grossman and Ronald E. Taam
Northwestern University, Dearborn Observatory, 2131 Sheridan Rd., Evanston, IL 60208

ABSTRACT
Double-diffusive convection refers to mixing where the effects of thermal and composition gradients compete to determine the stability of a fluid. In addition to the familiar fast convective instability, such fluids exhibit the slow, direct salt finger instability and the slow, overstable semiconvective instability. Previous approaches to this subject usually have been based on linear stability analyses. We develop here the nonlinear mixing-length theory (MLT) of double-diffusive convection, in analogy to the more familiar MLT for a fluid of homogeneous composition. We present approximate solutions for the mixing rate in the various regimes, and show that the familiar Schwarzschild and Ledoux stability criteria are good approximations to the precise criteria in stellar interiors.

We have implemented the self-consistent computation of the temperature gradient and turbulent mixing rate in a stellar evolution code and solved a diffusion equation to mix composition at the appropriate rate. We have evolved 15\,\text{M}_\odot and 30\,\text{M}_\odot stars from the zero-age main sequence to the end of core He-burning. Semiconvective mixing is fast enough to alter stellar composition profiles on relevant time scales, but not so fast that instantaneous readjustment is appropriate.

Key words: convection–hydrodynamics–instabilities–stars: evolution and interiors

1 INTRODUCTION
Double-diffusive instabilities arise when the transport of two different properties compete against each other to dominate the stability of a fluid. In stars, heat and composition are the two quantities transported by mixing. Semiconvection occurs when a thermally driven fluid has a composition gradient that opposes the instability. This subject also goes by the names of thermosolutal and thermohaline convection, as motivated by its relevance to oceanography, where varying salinity causes composition gradients. In stars, nuclear burning provides the source of heat for thermal driving. It also causes composition changes as elements are transformed into species of higher molecular weight, leading to the possibility of a stabilizing composition gradient.

A fluid of homogeneous composition is unstable to convection if its temperature gradient is steeper than the adiabatic gradient,

\[ \nabla - \nabla_{\text{ad}} > 0, \]

where \( \nabla = \partial \ln T/\partial \ln P \) is the logarithmic temperature gradient and \( \nabla_{\text{ad}} \) gives the temperature change of an adiabatic displacement. This is the famous Schwarzschild criterion. In stars, convection is very rapid compared to an evolutionary time scale, and small composition gradients mix to homogeneity virtually instantaneously.

If a star is Schwarzschild unstable in a region with a significant composition gradient (such as left by the retreating convective core of a main sequence star), what happens? If the composition gradient that opposes the thermal instability satisfies the relation

\[ (\nabla - \nabla_{\text{ad}}) - \nabla_\mu < 0, \]

the rapid growth of convective motions is stabilized; this does not preclude instabilities of slower growth. \( \nabla_\mu = \partial \ln \mu/\partial \ln P \) is the nondimensional composition gradient, where the composition is measured by the molecular weight \( \mu \). This is the Ledoux stability criterion (Ledoux 1947). If the fluid is Ledoux stable, but Schwarzschild unstable, can the composition gradient be sustained indefinitely? It cannot, but the mixing is much slower than the fast convective mixing of a convective zone.

Consider a fluid blob displaced upward from its equilibrium position in a background with temperature and composition gradients such that the fluid is Ledoux stable and Schwarzschild unstable. The displaced blob will be hotter than its environment because of the Schwarzschild instability, and thermal buoyancy will drive it upward. It will also be heavier than its
environment, and it will feel a force downward. Because the fluid is Ledoux stable, the downward force wins and there is a net restoring force, so that the blob oscillates around its equilibrium position. If there is nonzero rate of thermal diffusion, then while the blob is on its upward excursion where it is hotter than its environment, it loses heat and returns to the equilibrium position with a lower temperature than it began with. When it then makes its downward excursion, negative buoyancy will make it travel somewhat deeper than it would have otherwise. While on the downward excursion, it gets hotter and returns to the equilibrium position somewhat too hot and has an excess buoyancy. Thus, the amplitude of the oscillation grows. The phase lag between the temperature and velocity oscillations is responsible for the work that increases the amplitude of oscillation. This vibrational instability or overstability grows on the time scale of thermal diffusion. It is what those who study the stability of fluids call semiconvection (Kato 1966; Baines & Gill 1969; Grossman, Narayan, & Arnett 1993, hereafter GNA), and is the description that can be found in text books (Kippenhahn & Weigert 1991) and review articles (Spiegel 1972).

Semiconvection is known to occur in massive stars and horizontal branch stars. The idea of semiconvection in stars originated in a classic paper by Schwarzschild & Härm (1958; cf. also Sakashita & Hayashi 1959). They noticed that in massive main sequence stars ($M \gtrsim 10M_\odot$) where electron scattering is the dominant source of opacity, the opacity is larger in the H rich envelope outside the He enriched convective core. Thus, outside the Schwarzschild boundary (as defined by the fully mixed core), the H rich envelope is also Schwarzschild unstable. The core, however, was growing, bringing the envelope back to stability. The issue of whether the envelope was stable or unstable was resolved by establishing a zone of partial mixing, where He rich material was mixed into the envelope until it was neutrally stable. Whether the condition for neutral stability is expressed by the Schwarzschild or Ledoux criterion was a point of contention about 30 years ago (e.g., Spiegel 1969; Gabriel 1969), but the former is now accepted. This redistribution of composition is what most authors in stellar evolution refer to as semiconvection, irrespective of the specific nature of the instability. The importance of semiconvection to massive star evolution was manifest with SN 1987A, which exploded as a blue supergiant, rather than as a red supergiant as expected. Langer, El Eid, & Baraffe (1989) and Arnett (1991) found that whether a massive star explodes while it is blue or while it is red depends sensitively on the semiconvective rate of mixing.

What is the evidence for formation of semiconvective zones (SCZs), regions of partial mixing where the envelope is Schwarzschild unstable? Comparisons of observations of the numbers of blue versus red supergiants with theoretical evolutionary calculations of massive stars favor instantaneous mixing out to the Ledoux boundary only, i.e. the boundary of the retreating H-burning core. Complete mixing out to the Schwarzschild boundary where the H enriched envelope eventually becomes stable is disfavored (Stothers & Chin 1992a,b, Stothers & Chin 1994). A model of semiconvective mixing suggests that the mixing is slow enough that instantaneous mixing does not happen in the SCZ and that the core mass is significantly smaller than in models with instantaneous mixing to the Schwarzschild boundary (Langer, El Eid, & Fricke 1985; Langer 1991).

GNA used a Boltzmann transport description of convection to derive the known results of mixing-length theory (MLT) (Böhm-Vitense 1958). Their results suffer from the same lack of rigor as all formulations of MLT, namely that damping from the turbulent cascade is approximated by damping on the largest scale only, the mixing length, which is itself unknown. Their method has the advantage that standard MLT can be extended to more complicated problems easily. They demonstrated that their method reproduces the double-diffusive linear instabilities, but the nonlinear MLT of double-diffusive convection was not developed fully. We extend the results of GNA on double-diffusive instabilities here to formulate the local MLT of double-diffusive convection, that is, the nonlinear point where the linear instabilities saturate. Our results apply in all stability regimes--convective, semiconvective, and salt finger. Some work has been done previously that gives similar results for various subsets of this problem. Linear stability analyses of semiconvection have been discussed by Kato (1966), Langer, Sugimoto, & Fricke (1983), Eggleton (1983), and Nakakita & Umezu (1994). GNA followed an approach similar to Xiong (1981) and Eggleton (1983) by writing the hydrodynamic equations for correlations of perturbations. The nonlinear MLT with composition dependence was developed by Umezu & Nakakita (1988) and Umezu (1989), but there was some confusion regarding the relevant solutions. The salt finger instability has been discussed by Ulrich (1971) and Kippenhahn, Ruschenplatt, & Thomas (1980). Relevant nonlinear studies have been made by Shibahashi & Osaki (1976) and Gabriel & Noels (1976), who computed the oscillatory modes of massive stars, and by Gough & Toomre (1982), who performed a modal analysis of semiconvection. We do not develop the nonlocal MLT theory here.

It has been most common to treat semiconvection in stellar evolution codes using an iterative scheme that maintains convective neutrality instantaneously. We have implemented the double-diffusive local MLT in a stellar evolution code to test the validity of instantaneous mixing to neutrality. To our knowledge, only Langer, El Eid, & Fricke (1985) and Langer (1991)
Figure 1. The $\nabla - \nabla_{\text{ad}}$ vs. $\nabla_{\mu}$ stability plane, with the various stability regimes labeled. The dotted curve separates solutions to eq. 3 that are purely real from ones that admit complex roots. It is the analog of the Ledoux criterion. Note, however, that no combination of parameters could reproduce the classical picture precisely, where the semiconvective regime is bounded by the Ledoux and Schwarzschild lines. In the semiconvective regime, the dominant mode of growth is overstable. In the salt finger regime, the dominant mode of growth is direct, even though oscillatory modes are present also. We use parameters taken from just outside the H-burning core of a $30M_\odot$ star.

have done something comparable by solving a diffusion equation for composition evolution using the semiconvective diffusion rate of Langer (1983). Langer’s convective time scale is derived from a linear stability analysis of the growth of semiconvective modes, whereas our time scale depends on the nonlinear velocity where the growth saturates.

We note that in the laboratory, semiconvective mixing evolves such that regions of rapid mixing are separated by boundary layers, across which the composition varies in discrete steps. The number of layers grows on a thermal diffusion time scale, until the fluid is thoroughly mixed. This is a consequence of a nonlinear instability whose onset is at smaller $\nabla - \nabla_{\text{ad}}$ than for the linear instability (Proctor 1981). This process has been modeled by Spruit (1992). Our methods cannot reproduce the composition steps and boundary layers of laboratory experiments, nor could any mixing length approximation to stellar convection. We think it likely that semiconvection in stars may be more turbulent and less ordered than the laboratory experiments due to the extremely high Reynolds numbers and low Prandtl numbers of stellar fluids (Gabriel 1970; Stevenson 1977, 1979).

In §2, we derive the MLT of fluids with composition gradients. We review results of the linear stability analysis and discuss the nonlinear point where these instabilities saturate. We show how to use flux conservation to obtain self-consistent temperature gradients and turbulent velocities simultaneously. In §3, we describe the implementation of the physics into the stellar evolution code described by Eggleton (1971, 1972). We discuss examples of massive star evolution using the extended MLT and compare results to those from the unmodified code and from other authors. In §4 we summarize and discuss the results and the possibility of using the extended MLT for horizontal branch evolution.

2 THE EXTENDED MLT

In this section we review the essential results of the linear stability analysis discussed in greater detail by GNA. We examine the nonlinear solutions for the turbulent velocity at which the linear growth saturates and show how to compute the temperature gradient $\nabla - \nabla_{\text{ad}}$ and turbulent velocity simultaneously.

2.1 The Local Stability Diagram

As shown by GNA (Appendix C), in a fluid characterized by a superadiabatic gradient $\nabla - \nabla_{\text{ad}}$ and a composition gradient $\nabla_{\mu}$, if linear perturbations vary in time like $e^{st}$, then the eigenvalues $s$ of linear growth satisfy the cubic equation

$$s^3 + as^2 + bs + c = 0.$$
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The coefficients are given by
\[ a = A + D + F, \]  \hspace{1cm} (4a)
\[ b = (AD + AF + DF) - \frac{g\alpha}{H_P}(\nabla - \nabla_{ad}) + \frac{g\phi}{H_P} \nabla_\mu, \]  \hspace{1cm} (4b)
\[ c = \frac{F g\alpha}{H_P}(\nabla - \nabla_{ad}) + \frac{D g\phi}{H_P} \nabla_\mu + ADF, \]  \hspace{1cm} (4c)
where \( g \) is gravity, \( H_P \) is a pressure scale height, and \( \alpha = -(\partial \ln \rho / \partial \ln T)_{P,\mu} \) and \( \phi = (\partial \ln \rho / \partial \ln \mu)_{P,T} \) are constants derived from the equation of state. The parameters \( A, D, \) and \( F \) define the diffusion rates of viscosity, heat, and composition according to
\[ A = 10\nu_{\text{mic}}/3\ell^2, \]  \hspace{1cm} (5a)
\[ D = 3\chi/\ell^2, \]  \hspace{1cm} (5b)
\[ F = 3\chi D/\ell^2, \]  \hspace{1cm} (5c)
where \( \nu_{\text{mic}}, \chi, \) and \( \chi D \) are measured in \( \text{cm}^2/\text{s} \). Although GNA allowed for several mixing lengths for the various turbulent diffusion rates and for potentially different horizontal and vertical dimensions of a convective eddy, here we take all mixing lengths equal to the single value \( \ell \). (Clearly it would be more appropriate for the salt finger instability to worry about the different horizontal and vertical dimensions of an eddy using the original definitions of GNA.) Results essentially equivalent to equation (3) have been derived by many authors previously, beginning with Kato (1966) and Baines & Gill (1969).

The detailed criteria for instability depend on the diffusion rates of heat, composition, and momentum. In stars, the exchange of heat by radiative diffusion is always faster than the diffusion of composition, and the diffusion of momentum by molecular viscosity is negligible. As shown by GNA (see also Kato 1966, Baines & Gill 1969), the \( \nabla - \nabla_{ad} \) versus \( \nabla_\mu \) parameter space is divided by lines of critical stability, across which the roots of equation (3) change their character. If the roots are all real, the stability changes when the largest root equals zero. This transition is defined by
\[ F g\alpha(\nabla - \nabla_{ad}) - D g\phi \nabla_\mu = H_P ADF. \]  \hspace{1cm} (6)
Another possibility is that the dominant root is a complex conjugate pair. The stability changes when the real part of this root equals zero. This transition is defined by
\[ (A + D)g\alpha(\nabla - \nabla_{ad}) - (A + F)g\phi \nabla_\mu = H_P (A + D)(A + F)(D + F). \]  \hspace{1cm} (7)
The curve that separates solutions of these two types, i.e. pure exponential growth/decay of linear perturbations from oscillatory growth/decay is given by
\[ \left( \frac{b}{3} - \frac{a^2}{9} \right)^3 + \left( \frac{ab - 3c}{6} \right)^2 = 0. \]  \hspace{1cm} (8)
A more complete discussion of stability behavior can be found in GNA. In the limit that the diffusion rate \( D \) goes to zero and \( A \ll D, F \ll D, \) equations (6), (7), and (8) simplify to the Rayleigh-Taylor, Schwarzschild, and Ledoux criteria, respectively. The regimes of semiconvective and salt finger instability exist only as a consequence of a nonzero rate of thermal diffusion.

In Figure 1, the critical lines defined by equations (6) and (7) are drawn as solid lines, and the curve defined by equation (8) is dotted. These lines correspond very nearly to the stability criteria \( \nabla - \nabla_{ad} = 0 \) and \( \nabla_\mu = 0 \) if the thermal diffusion is much greater than the composition diffusion \( D \gg F \) and viscosity \( A \) is negligible, as is true in stellar interiors. GNA showed that the region of overstability is bounded by the dotted line and the line of equation (7), the line that approximates the Schwarzschild criterion. The dotted curve has a cusp near the origin, and does not appear to resemble the Ledoux criterion, which would trace a diagonal line from upper left to lower right in this figure. Indeed, for no combination of parameters is the fluid stable by the Ledoux line, but unstable by the Schwarzschild line. This lead GNA to suggest that the textbook description of semiconvection may not be appropriate for use in realistic calculations. We demonstrate below that this conclusion was incorrect.

2.2 The Nonlinear Turbulent Velocity

We begin by writing the equation for the turbulent velocity \( \sigma \),
\[ \sigma^2 \left[ (A + D + 2B\sigma)g\alpha(\nabla - \nabla_{ad}) - (A + F + 2B\sigma)g\phi \nabla_\mu - H_P (A + D + 2B\sigma)(A + F + 2B\sigma)(D + F + 2B\sigma) \right] \]
\[ (F \sigma) g_0(\nabla - \nabla_\text{ad}) - (D + B \sigma) g_0 \phi \nabla_\mu - H_\text{P}(A + B \sigma)(D + B \sigma)(F + B \sigma) = 0, \]  

(9)

(GNA, eq. 7.34). We define
\[ B = 2/\ell, \]  

(10)

so that \( B \sigma \) is the turbulent damping rate, in analogy to the microscopic rates given by \( A, D, \) and \( F \). Physically meaningful roots to this equation are real and non-negative. Not all these roots will be stable equilibria of the time dependent moment equations of GNA from which equation (9) was derived. In general, the fluid will seek out the most turbulent equilibrium state, and only this solution will be stable. The leading factor \( \sigma^2 \) shows that \( \sigma = 0 \) is always an equilibrium solution for a fluid; indeed, it is the only physical solution if the fluid is in the stable regime. The rest of equation (9) is the product of two terms, both cubic in \( \sigma \). Each term will have at least one real solution. If one or more are positive, the equilibrium state will evolve to the largest positive solution, and the solution \( \sigma = 0 \) will be an unstable equilibrium (cf. the stability analyses of GNA and Nakakita & Umezu 1994).

We consider approximate solutions to equation (9) in the convective, salt finger, and semiconvective regimes. In the regime of convective instability, all roots are real (see below). The largest positive root comes from the second term in brackets in equation (9), which can be expanded to read
\[ (B \sigma)^3 + a(B \sigma)^2 + b(B \sigma) + c = 0. \]  

(11)

Thus, in the convective regime defined by equation (8), all roots \( \sigma \) are real, just as for the eigenvalues \( s^\dagger \). Equation (11) also illustrates the close relationship between the linear stability analysis and the nonlinear MLT. This is not surprising since the MLT is derived by assuming eddy-damping rates balance growth rates given by \( s \) in the linear approximation.

To approximate the root \( \sigma \) in the convective regime where all roots are real, we simplify the calculation by setting the rate of molecular composition diffusion, \( F = 0 \), and consider the limit of efficient convection where the turbulent transport of heat dominates radiative transport, \( B \sigma/D \gg 1 \). In this limit, \( c \ll bB \sigma \), and equation (11) can be reduced to a quadratic relation, with positive real solution
\[ \sigma \approx \left[ \frac{g_0(\nabla - \nabla_\text{ad}) - g_0 \phi \nabla_\mu}{H_\text{P}B^2} \right]^{1/2}. \]  

(12)

If \( \nabla_\mu = 0 \), this approximation is precisely the local estimate of the turbulent velocity in the efficient convective regime as derived in GNA. This shows that the rate of turbulent diffusion \( \ell \sigma \propto \ell^2 \). In the general case that \( \nabla_\mu \) is not equal to zero, it is evident that across the Ledoux line, equation (12) becomes complex and does not give a physical solution for the semiconvective regime. Numerical solutions of the cubic equation (11) show that across the curve defined by equation (8), the two nontrivial real roots merge into a complex conjugate pair with a small real part.\(^\dagger\)

Crossing the Ledoux curve in the direction of the salt finger instability, the small real root of equation (11) is the largest real root of the higher order equation (9). Factoring out the complex conjugate pair, one can show that the value of the real root
\[ \sigma \approx -\left( \frac{D}{B} \right) \frac{\phi \nabla_\mu - \alpha(\nabla - \nabla_\text{ad})}{\phi \nabla_\mu} \]  

(13)

In the salt finger regime where the molecular weight profile is inverted, \( \nabla_\mu < 0 \) and \( \sigma \) is positive. This is the result from the linear stability analysis of Kippenhahn et al. (1980).

Crossing the Ledoux curve in the direction of the semiconvective instability where \( \nabla_\mu > 0 \), clearly equation (13) does not give a physical solution. In this case, the physical solution comes from the first cubic factor of equation (9). This real root equals the real part of the complex conjugate pair. Equations (12) approximates the imaginary part of this pair. If we take the limit of \( B \sigma/D \gg 1 \) in the first factor of equation (9), then the quadratic and zeroth order terms are small, with the cubic and linear terms dominating. In this case, the real root is small, and the dominant roots are a complex conjugate pair.

\* \*  

\* Eq. (3) is cubic because it derives from the three equations of motion for the perturbations of velocity, \( w \), temperature, \( \theta \), and composition, \( \nu \). Eq. (9), ignoring the leading \( \sigma^2 \), is sixth order because steady convection is described by the six equations for the correlations \( \sigma^2, \sigma w, \sigma \theta, \sigma \nu, \sigma^2, \) and \( \sigma^2, \). The leading term \( \sigma^2 = w^2 \) admits the trivial solution of the equations, where all correlations are zero.

\^ One way to estimate \( \sigma \) from the linear analysis is to replace \( s \) in eq. (3) by \( \sigma/\ell \sim B \sigma \) (Eggleton 1983). In the regime of overstability where \( s \) is complex, \( \sigma \) will be too. Eggleton (1983) suggested one should use the real part of this complex \( \sigma \) to describe the mixing rate. As we show, this supposition is correct, although in the sixth order eq. (9), the appropriate solution \( \sigma \) is strictly real, as must be any value of the rms turbulent velocity.

\* Across the Ledoux curve, the second cubic factor has roots in the pattern \( 2R_1, 2R_2 \pm i2I_2 \), and the first cubic factor has roots in the pattern \( 2R_2, (R_1 + R_2) \pm i2I_2 \). Thus, the real part of the complex conjugate pair approximated by eq. (12) as \( \pm i2I_2 \) is the same as the real root of the first cubic factor. In the salt finger regime, \( 2R_1 > 0 \) and \( 2R_2 < 0 \). In the semiconvective regime, \( 2R_1 < 0 \), and the physical root is \( 2R_2 > 0 \).
Clearly the solution we require is the small real root. Factoring out the approximate complex roots, we can show that the real root must be

\[ \sigma \approx \left( \frac{D}{2B} \right) \frac{\alpha(\nabla - \nabla_{ad})}{\phi\nabla_{\mu} - \alpha(\nabla - \nabla_{ad})}. \] (14)

Numerical solutions show this approximation is good, even if \( B\sigma/D \ll 1 \), so long as \( \nabla - \nabla_{ad} \) is not close to the critical line of equation (6), i.e., \( (\nabla - \nabla_{ad})/(\nabla - \nabla_{ad})_{\text{crit}} \gg 1 \), where \( (\nabla - \nabla_{ad})_{\text{crit}} = F\phi\nabla_{\mu}/D\alpha \). Taking the semiconvective diffusion rate as \( \ell \sigma \), equation (14) essentially gives Langer et al.’s (1983) result, and illustrates the essential feature that the rate of semiconvective mixing depends on the thermal diffusion rate \( D \). Langer et al. (1983), however, multiplied their diffusion rate by an arbitrary constant, \( aL \), to account for uncertainties in mixing length theory and the translation of a linear growth rate to a rate of nonlinear mixing. The effect of semiconvection is to mix regions toward neutral stability, so that \( \nabla - \nabla_{ad} \) approaches zero.

Equation (14) shows \( \sigma \propto \ell^{-1} \), so that the diffusion rate, \( \ell \sigma \), does not depend on the value of the mixing length. Thus, for decreasing \( \ell \), \( \sigma \) increases, approaching the solution approximated by equation (12). It reaches a maximum, and for \( \ell \) yet smaller, \( \sigma \) decreases. As long as a fluid is in the semiconvective regime, the rate of turbulent mixing is independent of the particular choice of mixing length. For sufficiently small \( \ell \), the fluid is actually in the convective regime, but the diffusion rate is even smaller.

The complex condition, equation (8), will approximate the Ledoux criterion in the limit that the term \( b/3 \) dominates the others, that is, in the limit that

\[ \frac{g\alpha(\nabla - \nabla_{ad})}{H_F} - \frac{g\phi\nabla_{\mu}}{H_F} \gg \mathcal{O}(A, D, F)\mathcal{O}(A, D, F), \] (15)

where the right-hand-side represents all permutations of the microscopic diffusion rates for momentum, heat, and composition. If a convective zone, bordering on a semiconvective region, is in the regime of efficient convection, we can use equation (12) to show that the left-hand-side of equation (15) \( \sim B^2\sigma^2 \) according to standard MLT. But \( B^2\sigma^2 \sim \sigma^2/\ell^2 \sim 1/\tau_{\text{conv}}^2 \), where \( \tau_{\text{conv}} \) is the characteristic time for the turnover of a convective eddy. On the other hand, the right-hand-side of equation (15) can be represented as \( 1/\tau_{\text{mic}}^2 \), where \( \tau_{\text{mic}} \) is the characteristic damping time by microscopic diffusion processes. Thus, equation (15) translates into the condition

\[ \tau_{\text{conv}} \ll \tau_{\text{mic}}. \] (16)

Thus, when the convection zone is in the efficient regime where fluid blobs move nearly adiabatically and losses by microscopic diffusion are negligible, the transition to semiconvection occurs across a boundary that closely approximates the Ledoux criterion. Only when a SCZ borders on a region of inefficient convection is it important to worry about the detailed cubic relation of equation (8). This never happens in stellar interiors, and thus the Ledoux criterion is an excellent approximation to the exact result. It is not necessary to compute the detailed stability criteria from equations (6)–(8), as GNA erroneously claimed.

The stability diagram is shown again in Figure 2, but extended to values of \( \nabla - \nabla_{ad} \) and \( \nabla_{\mu} \) relevant for the convective and semiconvective zones of stars. Contours of constant turbulent velocity \( \sigma \) are drawn. In the convective core of a massive star (see §3), typical convective velocities are of order \( 10^3 \text{ cm/s} \). For a pressure scale height or mixing length of order several\( \times 10^{19} \text{ cm} \), a characteristic mixing time is \( \tau_{\text{conv}} \sim 0.01 \text{ yr} \). Since the size of the convective core is only a few times larger than the pressure scale height, instantaneous mixing is an excellent approximation on evolutionary time scales.

Crossing into the semiconvective zone, the mixing rate falls dramatically. In the SCZ of a massive star, typical velocities are of order \( 10^{-3} \text{ cm/s} \), giving a mixing time \( \tau_{\text{conv}} \sim 10^6 \text{ yr} \). The size of the SCZ is of order a pressure scale height, so the time scale to mix the SCZ is only a few times shorter than the evolutionary time scale. In fact, since only a partial redistribution of composition is required to suppress the mixing, the time scale to mix to neutrality is at least a factor of several shorter. Nevertheless, it is clear that instantaneous mixing is not appropriate in the semiconvective region.

### 2.3 Flux Conservation

If convection carries some fraction of the energy flux, the true temperature gradient \( \nabla \) is modified from the value \( \nabla_{\text{Rad}} \) it would have in the absence of convection. The true temperature gradient can be found by writing the equation for flux conservation,

\[ \nabla_{\text{Rad}} - \nabla_{ad} = (\nabla - \nabla_{ad}) + H_F\overline{\mathbf{u}\nabla T}/T\chi. \] (17)

The term \( \overline{\mathbf{u}\nabla T} \) is the correlation between turbulent velocity and turbulent temperature excess, and measures the convective flux. It can be written in terms of the turbulent velocity \( \overline{\mathbf{u}} \). By solving the local moment equations (GNA, eqs. 7.28, 7.31–33), we obtain
Figure 2. The stability diagram, just as in Fig. 1, except extended out by many orders of magnitude to values of $\nabla - \nabla_{\text{ad}}$ and $\nabla_\mu$ relevant for stars. On this scale, the dotted line resembles the classical Ledoux criterion to excellent approximation. This line is not at a $45^\circ$ angle because the coefficient of thermal expansion $\alpha \approx 1$ due to radiation pressure. The light, solid curves are contours of constant turbulent velocity $\sigma$. Contour levels are $10^{-2}$, $10^{-1}$, $10^0$, $2 \times 10^7$ cm/s. The rate of turbulent mixing changes very rapidly across the Ledoux line and is much slower in the semiconvective and salt finger regimes. The open symbols correspond to examples of the hypothetical radiative temperature gradient of $\nabla_{\text{Rad}} - \nabla_{\text{ad}} = 0.05$ and $\nabla_\mu = 0, 0.025, 0.15$. The corresponding solid symbols show the solutions to eq. 17 for the true temperature gradient $\nabla - \nabla_{\text{ad}}$. In the SCZ, negligible flux is carried by turbulence, and the temperature gradient is very nearly equal to the radiative gradient. This extremely small difference has been exaggerated here.

\[
\frac{\alpha (\nabla - \nabla_{\text{ad}}) + \frac{\mu_\mu}{\rho} (D + F + 2B\sigma)}{\alpha (\nabla - \nabla_{\text{ad}}) + \frac{\mu_\mu}{\rho} (D + F + 2B\sigma)} \left( A + F + 2B\sigma + \frac{\rho g}{\rho T} \frac{\nabla_\mu}{(F + 2B\sigma)} \right) - \phi \nabla_\mu \frac{T}{\alpha g} (D + F + 2B\sigma) \sigma^2.
\]

In the limit that $F = \nabla_\mu = 0$, one can show that equations (17) and (18) simplify to the standard mixing length result for a homogeneous fluid (cf. GNA eqs. 6.16 and 6.19). For a given value of $\nabla_\mu$, only one value of $\nabla - \nabla_{\text{ad}}$ gives $\sigma$ and $\overline{w\theta}$ (by eqs. 9 and 18) that satisfies flux conservation (eq. 17).

In Figure 2 we show a few solutions to equation (17). We use parameters appropriate to the SCZ of a 30$M_\odot$ main sequence star, but choose the total flux $\nabla_{\text{Rad}}$ and composition gradient $\nabla_\mu$ ourselves for the sake of example. We choose a total flux $\nabla_{\text{Rad}} - \nabla_{\text{ad}} = 0.05$ and composition gradients $\nabla_\mu = 0, 0.025, 0.15$. For $\nabla_\mu = 0$, the fluid is homogeneous and convectively unstable. Convection is efficient and carries nearly the total energy flux. The temperature gradient is very nearly adiabatic, $\nabla - \nabla_{\text{ad}} = 3.3 \times 10^{-7}$, and $\sigma = 3.5 \times 10^4$ cm/s. For $\nabla_\mu = 0.025$, the fluid is again convectively unstable and $\nabla - \nabla_{\text{ad}} = 0.014 \approx \phi \nabla_\mu/\alpha$, almost on the Ledoux line. The temperature gradient is as nearly adiabatic as it can be without the convective efficiency dropping precipitously. In stars, however, this situation does not occur since mixing is fast and is thought to evolve to the $\nabla_\mu = 0$ configuration nearly instantaneously. Finally, for $\nabla_\mu = 0.15$, the fluid is in the semiconvective regime, where $\nabla - \nabla_{\text{ad}} \approx \nabla_{\text{Rad}} - \nabla_{\text{ad}}$ to high precision (the difference is exaggerated in Fig. 2) and $\sigma = 0.13$ cm/s. In this case, turbulent mixing is much slower, and convection carries a negligible fraction of the flux. We note that in the salt finger regime, the turbulent heat flux is actually reversed from the convective and semiconvective regimes, but is negligibly small.
like in the semiconvective region. In this case, the temperature gradient is also very nearly radiative.

3 TESTS OF THE EXTENDED MLT

To demonstrate the feasibility of using the extended MLT in stellar evolution calculations and to understand its consequences, we have implemented it in the stellar evolution code of Eggleton (1971, 1972, 1973). In addition to replacing the standard MLT (i.e., for homogeneous composition) with a new routine to self-consistently compute the temperature gradient and mixing rate, we have added a routine to mix the composition at the appropriate rate. The modifications are described below. We evolve $15 M_\odot$ and $30 M_\odot$ stars, characterized by an initial composition $X = 0.7$, $Z = 0.02$, from the zero-age main sequence to the end of core He-burning. The constant $\alpha$ is computed from the equation of state, and can be somewhat larger than unity, mainly because of radiation pressure. The constant $\phi = 1$. The rate of thermal diffusion $D$ is computed from the expression for radiative diffusion, and viscosity and composition diffusion are zero, $A = F = 0$.

Our study is primarily a differential investigation of the consequences of using the new mixing scheme. Detailed comparisons with other authors must be made with caution, since various other physics may not be the same. Nevertheless, we compare our results with those in Langer, El Eid, & Fricke (1985), who also evolved $15 M_\odot$ and $30 M_\odot$ stars for the similar composition $X = 0.701$, $Z = 0.019$. They used a semiconvective diffusion rate that essentially is given by the approximation of equation (14), but multiplied by a coefficient, $\alpha_L$, that allowed them to vary the rate of semiconvective mixing by orders of magnitude.

3.1 Code Modification

In its original form, the Eggleton code identifies a region outside the H-burning convective core of sufficiently massive stars as unstable according to the Schwarzschild criterion, but stabilized by the composition gradient created by the retreating convective core. Using a somewhat contrived mixing rate, this semiconvective region is slowly mixed. Indeed, as we illustrate below, the mixing in this zone is so slow that composition readjustment does not occur on evolutionary time scales, and at no time during main sequence evolution does this region reach neutral stability. Furthermore, the mixing rate does not even depend on the composition gradient and does not contain the physics of semiconvection (Eggleton 1972).

We modified Eggleton’s code to use the extended local MLT to compute the temperature gradient and convective mixing rate in semiconvective and convective regions. To mix the composition, the original code solves a diffusion equation for composition evolution. The diffusion time, however, is not derived from the convective time scale. In fact, the code is known not to converge for mixing times that are too short (Eggleton 1972), and it is not possible to use our newly computed mixing times here. The minimum mixing time that can be used is large enough (around $10^5$ yr) that convective regions clearly are not mixed instantaneously. The consequences of this can be seen in some short-lived phases of evolution that turn out to be of crucial importance for evolution. In particular, post-main sequence evolution depends sensitively on the treatment of mixing.

To remedy this inability to modify the mixing time in the code as appropriate, we have added a diffusion routine that mixes composition after each evolutionary step. The routine is implicit, and accuracy is best if the diffusion equation is solved on a grid of constant $\Delta r$. This requires that the diffusion grid have more points than the evolution code, and we interpolate back and forth between the two as required. During main sequence evolution, the diffusion grid needs only about 15 times more points than the stellar evolution code, for which we used 200 mesh points. Later in the giant phases, the diffusion grid may require 400 times more points.

The new diffusion routine is decoupled from the rest of the evolution equations, and this can lead to convergence difficulties. Therefore, to maintain numerical stability, we continue to use also the coupled diffusion as in the original code. In both convective and semiconvective regions, the mixing times in the decoupled diffusion routine are generally shorter than in the coupled equation, so the mixing time is determined by the decoupled routine, with the original diffusion only providing numerical stability.

3.2 $30 M_\odot$ Evolution

We evolve a $30 M_\odot$ star using three variations of the stellar evolution code. In Case A, we use only the composition mixing of the original code. We do, however, use the new MLT routine to determine the temperature gradient. This evolution is essentially identical to the evolution of the unaltered code, and serves as our reference model. Case B uses the new mixing routine, where convective and semiconvective zones are mixed at the rates computed in the new MLT routine. Mixing in both convective and semiconvective regions is significantly faster than the mixing in the original code. In Case C, we ignore the composition gradient in the MLT routine. This is equivalent to using the Schwarzschild criterion for convection. There are no semiconvective bins, and those bins that would have been semiconvective instead are mixed at the much faster convective rate. The H- and He-burning times, $\tau_H$ and $\tau_{He}$ and core masses, $M_H$ and $M_{He}$, at the times of core H exhaustion and core
Figure 3. The convective structure of the $30M_\odot$ model for three schemes of convective mixing. Case A uses only the standard mixing of Eggleton’s code, but the temperature gradient derived from the extended MLT. This result is virtually identical to that from the unmodified Eggleton code. Case B includes mixing at the rate derived from the extended MLT. There is nearly instantaneous mixing of convective regions, and mixing to convective neutrality in semiconvective regions. Case C uses the standard MLT temperature gradient and mixing rate derived by setting the composition gradient to zero in the MLT routine. This is equivalent to using only the Schwarzschild criterion for convection. Convective regions are contoured by a dotted line (which, unfortunately, looks solid in some places, particularly the convective envelope). Semiconvective regions are contoured by a solid line. In Case A, this is a region of growing size outside the H-burning core, and a small region near the start of He core burning (at about $20 - 25M_\odot$ and age of $5.6 \times 10^6$ yr). In Case B, semiconvection occurs in the broken-up cells outside the H-burning core. In both these cases, there is semiconvection in a thin shell surrounding the convective core. In this and all subsequent figures, we use a mixing length $\ell = 1.5 H_p$. 
Figure 4. The composition profile, superadiabatic temperature gradient (radiative and actual), and mixing rate for the $30M_\odot$ model for the three mixing schemes at the time when hydrogen has burned to $X = 0.2$. Squares indicate mass bins identified as semiconvective, for which mixing times are of order $10^6$ yr. The stars indicate convective regions, where mixing is virtually instantaneous.

He exhaustion are presented in Table 1 for mixing lengths $\ell = 0.5H_P, 1.5H_P, 5.0H_P$.

Figure 3 shows the evolution of the convective structure of the $30M_\odot$ star. Figure 4 shows the composition profile, temperature gradient, and mixing times at the instant when the central composition $X = 0.2$. In Case A, during core H-burning, there is a thin semiconvective shell bordering on the convective core. Slightly separated from the convective core is a growing SCZ. (This gap is probably a numerical artifact, since a higher resolution calculation does not show one.) When only
the original diffusion scheme operates, mixing is slow enough in the SCZ that convective neutrality is not reached, although Figure 4 shows that the departure from neutrality is small. As a consequence, the SCZ grows as a connected region throughout the phase of core H-burning. Occasionally the semiconvective region, which usually is slightly detached from the convective core, makes contact with the core, but the slow mixing does not mix significant unburnt material into the interior. This case resembles most closely the evolution in Langer et al. (1985) where semiconvection is suppressed (\( \alpha_L = 0.01 \)).

In Case B where the composition is mixed at the rate determined by the extended MLT, the composition of the SCZ is able to make the readjustments necessary to become neutrally stable and turn off the semiconvection. In Figure 3 the effect is to break up the SCZ into smaller zones of semiconvection that form and disappear. Not surprisingly, this evolution is most similar to Langer et al.’s (1985) evolution with \( \alpha_L = 1 \), since we do not scale the semiconvective mixing rate by any coefficient. As seen in Table 1, semiconvection is effective enough to mix some material into the core and increase the mass of the core at H exhaustion. A consequence of the larger core mass is a shorter H-burning time, \( \tau_H \), and a slightly more luminous main sequence phase. The He-burning core mass, \( M_{\text{He}} \), is smaller and \( \tau_{\text{He}} \) longer for reasons discussed below. Interestingly, Langer et al. (1985) found the opposite result, that an enlarged H core leads to a longer \( \tau_H \) and a shorter \( \tau_{\text{He}} \). Although we are uncertain why this is, one possibility is that in Langer et al.’s case, mass is accreted onto the core mainly near the end of the main sequence, so that extra fuel can only extend the main sequence lifetime, whereas in our case, mass is accreted sooner, so that the evolution more closely resembles the evolution of a more massive star.

Case C has the fastest mixing in the SCZ, with mixing at the convective rate. In the SCZ, convective bins tend to propagate inward because mixing increases the H abundance at smaller radii, thereby inducing instability. The H abundance decreases at large radii, causing stability. The effect is akin to a rain of H enriched material onto the convective core, and is like the episodic accretion onto the He core of a horizontal branch star from the surrounding SCZ (Sweigart & Renzini 1979). As seen in Figure 4, the composition profile in the SCZ is not so smooth as in Case A or B. Lamb, Iben, & Howard (1976), however, were able to maintain convective neutrality and a smooth profile in the SCZ by using sufficiently fine gridding. Case C models have the largest core mass \( M_H \) at the end of H-burning. The core H-burning time, \( \tau_H \), is again shorter than in Case A.

Only Case A evolution exhibits a blue loop in the H-R diagram, and this is revealed in Figure 3 by the formation, retreat, and reestablishment of a fully convective envelope during core He-burning. In Cases B and C, core He-burning occurs mainly while the star is a blue supergiant. The envelope does not become fully convective until near the end of core He-burning. The way this evolution proceeds has mainly to do with the structure of the intermediate convection zone (ICZ) that forms immediately following core H exhaustion (Lauterborn, Refsdal, & Weigert 1971). Following the end of core H-burning, an ICZ is established outside the formerly convective core. In Case A, this zone is relatively small in extent (cf. Fig. 3), and there is also an outer composition plateau (resulting from the convective “finger” at time 5.6 - 5.8 \times 10^6 yr and mass 24 - 14M_\odot). In Cases B and C, however, because of the more effective mixing by both semiconvection and convection, the ICZ and outer convective plateau that were disconnected in Case A become connected, making these ICZs as large as the formerly semiconvective zone.

In Cases A, B, and C, H shell burning occurs initially at the base of the H discontinuity created by the ICZ, and since the shell is fed efficiently by the overlying convection, the star stabilizes as a blue supergiant, and a fully convective envelope does
not form immediately. * In Case A, the small ICZ soon disappears and the H shell burning becomes less efficient since the shell is convectively stable. The H shell luminosity decreases, a fully convective envelope forms above the formerly convective ICZ, and the star becomes a red supergiant (cf. Stothers & Chin 1975). When the H shell burns through the former ICZ to the base of the convective envelope, H shell burning is again fed by convection and the star returns to the blue. A secondary ICZ forms, but eventually the shell luminosity drops slightly as the shell gets to sufficiently large radius and a stable region forms above the shell. This triggers a return to the red, and soon after the H shell luminosity falls to zero.

In Cases B and C, the ICZ is large enough that the H-burning shell cannot burn through the composition plateau in the lifetime of the star. The shell is fed by convection and the star remains a blue supergiant, gradually getting redder, for most of core He-burning, in agreement with the 30M☉ evolution of Simpson (1971), Stothers & Chin (1976), and Langer et al. (1985). As in Case A, when the H shell is at sufficiently large radius, a drop in the H shell luminosity causes the shell to be fed by a stable zone, triggering the formation of a fully convective envelope. Because H shell burning initially occurs at the base of an ICZ in Cases B and C, the H-burning shell is thinner and the growing H-depleted core is smaller, making the He-burning core mass, MHe, smaller and τHe longer.

We consider how these results depend on the mixing length ℓ. Case A is effectively independent of the mixing length. The mixing length, ℓ, only bears on the temperature gradient of the convective core, which is so close to adiabatic for any reasonable value that the interior evolution is the same. According to equation (14), the semiconvective diffusion rate does not depend on ℓ, so that Case B main sequence evolution is independent of ℓ within numerical uncertainty. Following core H depletion, the ICZ forms more quickly for large ℓ, however, making the H-depleted core and MHe somewhat smaller for larger values of ℓ. This effect is more clearly illustrated in Case C. The main sequence evolution is hardly affected by the choice of ℓ, but the faster formation of the ICZ for larger ℓ reduces MHe and increases τHe.

3.3 15M☉ Evolution

We evolve a 15M☉ star for the same three convection schemes described above. The evolution of the convective structure is shown in Figure 5, and the composition and temperature profiles at the onset of core He-burning (Y = 0.97) are shown in Figure 6. The times τH and τHe and corresponding core masses, MHe and MHe, are listed in Table 2.

As seen in Figure 5, there is no semiconvective region during core H-burning, except for the thin shell surrounding the core. The only semiconvection that is important for the evolution occurs during the brief interval between core H-burning and core He-burning, outside the formerly convective core. Although evolution through this phase occurs on the fast thermal time scale, typical semiconvective time scales of order 10^4 yr are fast enough to effect the internal structure of the star. In Case A, where only the original mixing is used, semiconvection does not cause any significant mixing outside the formerly convective core. Because convection effectively does not feed the H-burning shell, the shell must be relatively thick to generate the required luminosity. In Cases B and C, an ICZ creates a composition plateau. The ICZ feeds the H-burning shell, which is thinner than in Case A. Thus, in Figure 6 the H-depleted core has a smaller mass in Case B, so that the He-burning core mass is smaller. The H-depleted core and mass MHe are even smaller in Case C. The He-burning time τHe is shortest in Case A.

The 15M☉ model becomes a red giant when the ICZ disappears. For all mixing schemes, the star remains a red giant throughout core He-burning. This is in contrast to most other results, where the star either starts evolution as a blue supergiant (Stothers & Chin 1976; Simpson 1971; Lamb et al. 1976) or makes a blue loop (Stothers & Chin 1975; Langer et al. 1985), returning to the red only after central convection has ceased.

As with the 30M☉ model, Case A is independent of the mixing length, and Case B nearly so. The value of the mixing length is most important for Case C evolution. In this case a larger ℓ makes the ICZ more efficient at delivering fuel to the H-burning shell, making the shell thinner. The subsequent He core mass, MHe, is slightly smaller, and τHe is slightly longer.

4 SUMMARY AND DISCUSSION

We have derived the extended local MLT for fluids with composition gradients. Our method has been to solve the local moment equations of GNA. As is well-known from linear stability analyses, fluids with both temperature and composition gradients can experience double-diffusive instabilities, where competing diffusion rates determine the nature of the instability. The salt finger and semiconvective instabilities grow much more slowly than the dynamical convective instability, at a rate

* As a general rule, an H-burning shell fed by convection is stable as a blue supergiant, and one fed by a stable zone is a red supergiant (Simpson 1971; Lauterborn, Refsdal, & Weigert 1971). A shell fed by convection is thinner and generates more luminosity than one at the base of a stable zone because of the substantial ΔX at the core/shell boundary. The H-depleted cores of these blue supergiants are smaller, therefore. Shells fed by a stable zone with a composition gradient must be thicker. For these shells to generate a comparable luminosity, they must be hotter, and, therefore, the core has a smaller radius. Then, by the “mirror principle,” the envelope expands to red giant dimensions.
Figure 5. The convective structure of the $15M_\odot$ model for the three schemes of convective mixing. Except for the thin semiconvective shell at the edge of the H- and He-burning cores, the only semiconvective region occurs in Cases A and B during the brief period between core H-burning and core He-burning outside the formerly convective core. In Cases B and C, mixing during this time causes the formation of an ICZ, the extent of which determines when the star becomes a red supergiant. Dotted contours outline convective regions, and solid contours indicate semiconvection.

proportional to the radiative diffusion. Our extended MLT provides the characteristic turbulent velocities at which the linear instabilities saturate. This nonlinear theory applies to all stability regimes: convective, semiconvective, salt finger, and stable.
Figure 6. The composition profile, superadiabatic temperature gradient (radiative and actual), and mixing rate for the $15M_\odot$ model for the three mixing schemes at the time where core He-burning is just beginning ($Y = 0.97$). Squares indicate mass bins identified as semiconvective, for which mixing times are of order $10^4$ yr. The stars indicate convective regions, where mixing is virtually instantaneous. A composition plateau where an ICZ fed the H-burning shell is apparent for Cases B and C. Note that the H-depleted core is largest in Case A and smaller in Cases B and C. Consequently, Cases B and C have smaller He-burning cores, $M_{\text{He}}$, and longer $\tau_{\text{He}}$.

Standard MLT for a fluid of homogeneous composition is a limiting case.

We have demonstrated that although the precise stability criteria require solving equations (6)--(8), the familiar Schwarzschild and Ledoux criteria are adequate approximations for stellar interiors. Likewise, although calculation of the precise mixing rate requires solving equation (9), the approximate solutions given by equations (12) for convection, (13) for
Table 2. 15M⊙ Evolution.

\[
\begin{array}{ccc}
\ell = 0.5H_\odot & A & B & C \\
\hline
\tau_H/10^7\text{yr} & 1.112 & 1.087 & 1.072 \\
\tau_{He}/10^7\text{yr} & 0.224 & 0.259 & 0.221 \\
M_H/M_\odot & 2.80 & 2.80 & 2.80 \\
M_{He}/M_\odot & 2.35 & 2.20 & 1.95 \\
\end{array}
\]

\[
\begin{array}{ccc}
\ell = 1.5H_\odot & A & B & C \\
\hline
\tau_H/10^7\text{yr} & 1.112 & 1.087 & 1.072 \\
\tau_{He}/10^7\text{yr} & 0.224 & 0.262 & 0.239 \\
M_H/M_\odot & 2.80 & 2.80 & 2.80 \\
M_{He}/M_\odot & 2.35 & 2.25 & 1.90 \\
\end{array}
\]

\[
\begin{array}{ccc}
\ell = 5.0H_\odot & A & B & C \\
\hline
\tau_H/10^7\text{yr} & 1.112 & 1.085 & 1.070 \\
\tau_{He}/10^7\text{yr} & 0.224 & 0.264 & 0.249 \\
M_H/M_\odot & 2.80 & 2.80 & 2.80 \\
M_{He}/M_\odot & 2.35 & 2.25 & 1.90 \\
\end{array}
\]

salt finger mixing, and (14) for semiconvective mixing are good in most circumstances. The rate of semiconvective diffusion does not depend on the value of the mixing length.

We have written a subroutine that solves the extended MLT for the self-consistent temperature gradient and mixing rate, and used it in Eggleton’s stellar evolution code. It has also been necessary to supply a subroutine that permits faster mixing than the unmodified code allows. We have evolved 30M⊙ stars from the start of core H-burning to the end of core He-burning. The results presented here mainly confirm results found previously by others (cf. the review of massive star evolution by Chiosi & Maeder 1986). The 30M⊙ star has a semiconvective region outside the H-burning convective core. The 10^6 yr mixing time scale is fast enough that semiconvection maintains the composition profile at convective neutrality, so that semiconvection is continually initiated and quenched during main sequence evolution. This broken-up structure of the SCZ most closely resembles the evolution of Langer et al. (1985) with \( \alpha_L = 1 \). Of greater importance for subsequent evolution is the development of an ICZ during the brief period after core H-burning stops and before core He-burning begins.

Semiconvection allows the formation of a substantial ICZ, causing He-burning to begin while the star is a blue supergiant. The star does not become a red supergiant until near the end of core He-burning.

Although the 15M⊙ star does not have an extended semiconvective zone during main sequence evolution, it does have one during the brief time between core H- and core He-burning. The treatment of mixing at this time determines the evolution of the ICZ, which in turn determines evolution during later stages. Core He-burning occurs mainly in the red and makes no blue loops, in contrast to most other authors, who find that core He-burning either occurs mostly in the blue, or makes a blue loop if it starts in the red. It has not been our intention to perform state-of-the-art evolutionary calculations for stars of these masses, and we are not making claims about the formation of blue loops. Our aim has been to demonstrate the feasibility of using the extended MLT in stellar evolution calculations and to show that semiconvective mixing time scales are in the interesting range where mixing is important on evolutionary time scales, but not so fast that instantaneous readjustment of composition is appropriate.

Horizontal branch (HB) stars also develop semiconvective zones (Paczyński 1970; Schwarzschild 1970). Increasing C/O abundance in the He-burning core increases the opacity. Thus, during the early stages of evolution, mixing at the core boundary causes instability just outside the core, and the core grows in an overshooting phase (Castellani, Giannone, & Renzini 1971a). Eventually a time comes when an unstable, high opacity region forms around the convective core (in this case due to C/O enrichment), that is, beyond the radius of Schwarzschild stability, much as the 30M⊙ star does during main sequence evolution. This region is thought to experience a readjustment of composition, where mixing with the He-rich material of the envelope restores convective neutrality. This region of partial mixing is the semiconvective zone of HB stars (Castellani, Giannone, & Renzini 1971b). The prescription for maintaining instantaneous neutrality in this region is the “canonical semiconvective scheme” commonly used in horizontal branch evolution. An algorithm has been described by Robertson & Faulkner (1972), and a recent discussion can be found in Dorman & Rood (1993). The ratio of the number of AGB to HB stars depends strongly on the rate of mixing in the SCZ (Renzini & Fusi-Pecci 1988), and evolution with the canonical scheme is consistent with observed ratio (Buzzoni et al. 1983). It would be interesting to verify whether or not mixing using the extended MLT is consistent with the observations, whether the canonical scheme is appropriate, and whether there are any consequences to the finite rate of mixing in the SCZ.

An issue we have not addressed so far is the possibility of nonlocal semiconvection. That is, do fast convective velocities penetrate far into adjacent semiconvective regions (convective overshooting)? Do slow semiconvective velocities slowly mix
into adjacent stable regions (semiconvective overshooting, cf. Auré 1971)? Answers to these questions would require that we develop the nonlocal moment equations for fluids with composition gradients. GNA developed the nonlocal equations for fluids of homogeneous composition, and it would be straight-forward to extend those results to this more complicated problem. The nonlocal moment equations are likely to be quite difficult to solve, however. This problem requires solving a large number of coupled differential equations, rather than the algebraic equations of the local theory. Experience has shown that solutions to the simpler problem of nonlocal convection in a homogeneous fluid are difficult to obtain (Grossman 1996). In principle, one could also simulate nonlocal mixing in a fluid with a composition gradient using the GSPH technique of Grossman & Narayan (1993). The current consensus is that composition gradients are effective barriers to rapid mixing (cf. Shibahashi & Osaki 1976), and we think this is likely to be proved correct when more detailed results are available. We think it possible, however, that slow semiconvective mixing may extend significantly into stable zones.

Finally, we emphasize again a central shortcoming of all the preceding discussion, that the theory of semiconvective mixing proposed here has all the same shortcomings as standard mixing length theory. In particular, the effective viscosity of the turbulent cascade of a fluid at high Reynolds number is approximated by an eddy-damping rate, a rate that accounts for turbulent dissipation at only the largest scale. This rate depends on a single, unknown parameter—the mixing length. Stevenson (1979) has proposed a mechanism for semiconvection where linear growth develops on the scale of maximum instability, which is much smaller than a pressure scale height, and energy cascades into even smaller scales. This raises the question of whether an appropriate mixing length for semiconvection is of order a pressure scale height or is much smaller. We have shown that if motions are dominated by the scale of maximum instability or larger, the rate of semiconvective diffusion is independent of $\ell$ and bigger than if motions are dominated by the smaller scales. Thus, in semiconvective regions, one can continue to use the same, large mixing length as in convective regions; it is not necessary to adopt a smaller value.

A more accurate treatment of semiconvection requires solving the multidimensional hydrodynamic equations. The first 2-dimensional simulation of semiconvection has been presented by Merryfield (1995). Unfortunately, the vertical depth of the simulation is comparable to the size of maximum growth, and is much smaller than the size of semiconvective regions in stars, thus giving little information about composition transport over distances comparable to a pressure scale height. Even if one could simulate semiconvection in a fluid of appropriate depth, the simulation would represent only one instant in the evolution of a star; one cannot evolve a star using multidimensional hydrodynamics. Consequently, highly-simplified prescriptions for mixing, such as MLT, will continue to be useful for a long time yet.

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