NEW RESULTS FOR OSCILLATION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH DAMPING TERM

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Abstract. In this paper, we study the oscillatory behavior of solutions of a class of damped fractional partial differential equations subject to Robin and Dirichlet boundary value conditions. By using integral averaging technique and Riccati type transformations, we obtain some new sufficient conditions for oscillation of all solutions of this kind of fractional differential equations with damping term. Our results essentially enrich the ones in the existing literature. Finally, we also give two specific examples to illustrate our main results.

1. Introduction. The subject of the qualitative theory of differential difference equations, including stability, oscillation, periodic and homoclinic solutions, etc., has undergone a rapid development in the last three decades, we refer to [4, 5, 9, 10, 11, 12, 15, 26, 28, 29, 30]. The theory and its application of fractional differential equations, as some generalizations of classical differential equations, have been investigated extensively [1, 2, 3, 7, 8, 14, 16, 21, 25, 27, 31] in the recent years. Nowadays the interest in the study of fractional differential equations lies in the fact that fractional-order models are more accurate than integer-order ones. Fractional differential equations provide in some cases more accurate models of equations under consideration. The many important mathematical models are described by fractional differential equations. In particular, the oscillation theory of fractional partial differential equations has attracted a great portion of attention, which is evidenced by extensive study in the field, see for example [13, 17, 18, 19, 20, 22, 23, 24] and the references cited therein. However, to the best of the author’s knowledge, the study of oscillation of fractional partial differential equation which involve the Riemann-Liouville fractional partial derivative is still in its infancy. Motivated by

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all the above works, in this paper, we study the oscillation of all solutions of the following damped fractional partial differential equation

$$D_{+}^{1+\alpha}u(x, t) + p(t)D_{+}^{\alpha}u(x, t) + q(x, t) \int_{0}^{t} (t - \xi)^{-\alpha}u(x, \xi)d\xi = \alpha(t)\Delta u(x, t) + \sum_{i=1}^{m} a_i(t)\Delta u(x, t - \tau_i), \quad (x, t) \in \Omega \times R_+ = G,$$

subject to either of the following boundary value conditions

$$\frac{\partial u(x, t)}{\partial \nu} + \beta(x, t)u(x, t) = 0, \quad (x, t) \in \partial\Omega \times R_+, \quad (1.2)$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times R_+, \quad (1.3)$$

where $\Omega$ is a bounded domain in $R^n$ with a piecewise smooth boundary $\partial\Omega$, $R_+ = (0, \infty)$, $\alpha \in (0, 1)$ is a constant, $D_{+}^{\alpha}u(x, t)$ is the Riemann-Liouville fractional derivative of order $\alpha$ of $u$ with respect to $t$, $\Delta$ is the Laplacian in $R^n$, $\nu$ is the unit exterior normal vector to $\partial\Omega$ and $\beta(x, t)$ is a nonnegative continuous function on $\partial\Omega \times R_+$.

By using Riccati type transformations as well as the integral averaging technique, we establish some new sufficient conditions which can guarantee all solutions of the problems (1.1)-(1.2) and (1.1)-(1.3) to be oscillatory. These results are considered essentially new. We also provide two specific examples to illustrate the main results. The oscillatory behavior of the problems (1.1)-(1.2) and (1.1)-(1.3) for the case when $p(t) \in C([0, \infty), [0, \infty))$ was studied in Li and Sheng [20]. However, there is no results on oscillation of the problems (1.1)-(1.2) and (1.1)-(1.3) for the case when $p(t) \in C([0, \infty), (-\infty, 0))$.

For the sake of convenience, we always assume that the following conditions hold throughout this paper.

1. $a, a_i \in C([0, \infty), [0, \infty))$, and $\tau_i \geq 0 (i = 1, 2, \ldots, m)$ are some constants;
2. $q(x, t) \in C(G, R_+)$ and $q(t) = \min_{x \in \Omega} q(x, t)$;
3. $p(t) \in C([0, \infty), (-\infty, 0))$.

By a solution of the problem (1.1)-(1.2) (or (1.1)-(1.3)), we mean a function $u(x, t) \in C^{1+\alpha}(\Omega \times [0, \infty))$ such that $D_{+}^{\alpha}u(x, t)$, $\int_{0}^{t} (t - \xi)^{-\alpha}u(x, \xi)d\xi \in C^{1}(\overline{G}, R)$ and satisfies (1.1) on $\overline{G}$ and the boundary condition (1.2) (or (1.3)).

A solution of the problem (1.1)-(1.2) (or (1.1)-(1.3)) is said to be oscillatory in $G$ if it is neither eventually positive nor eventually negative, otherwise it is said to be nonoscillatory.

2. Preliminaries. In this section, we give the definitions of fractional derivatives and integrals and two lemmas which are useful throughout this paper. There are several kinds of definitions of fractional derivatives and integrals. In this paper, we use the Riemann-Liouville left-sided definition on the half-axis $R_+$.

**Definition 2.1.** ([21]) The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $y : R_+ \rightarrow R$ on the half-axis $R_+$ is given by

$$I_{+}^{\alpha}y(t) := \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha-1}y(s)ds \text{ for } t > 0$$

provided the right side exists pointwise on $R_+$, where $\Gamma$ is the Gamma function.

**Definition 2.2.** ([21]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a function $y : R_+ \rightarrow R$ on the half-axis $R_+$ is given by

$$D_{+}^{\alpha}y(t) := \frac{1}{\Gamma([\alpha] - \alpha)} \frac{d^{[\alpha]}y(t)}{dt^{[\alpha]}} \int_{0}^{t} (t - s)^{[\alpha]-\alpha-1}y(s)ds \text{ for } t > 0$$
provided the right side exists pointwise on $R_+$, where $\lceil \alpha \rceil$ is the ceiling function of $\alpha$.

**Definition 2.3.** ([16]) The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ with respect to $t$ of a function $u(x, t)$ is given by

$$D_{+}^{\alpha} u(x, t) := \frac{1}{\Gamma(1 - \alpha)} \frac{\partial}{\partial t} \int_{0}^{t} (t - s)^{-\alpha} u(x, s) ds$$

provided the right hand side is pointwise defined on $R_+$.

**Lemma 2.4.** ([16]) Let $\alpha > 0$, $m \in N$, $D = \frac{d}{dt}$. If the fractional derivative $D_{+}^{\alpha} y(t)$ and $D_{+}^{m+\alpha} y(t)$ exist, then

$$D^{m}(D_{+}^{\alpha} y(t)) = D_{+}^{m+\alpha} y(t).$$

**Lemma 2.5.** ([22]) Let

$$E(t) = \int_{0}^{t} (t - \xi)^{-\alpha} y(\xi) d\xi$$

for $\alpha \in (0, 1)$ and $t > 0$. Then $E'(t) = \Gamma(1 - \alpha) D_{+}^{\alpha} y(t)$.

In this paper, we assume that the solutions of the problems under consideration globally exist.

3. **Oscillation of the problem** (1.1)-(1.2). In this section, we give some sufficient conditions under which all solutions of the problem (1.1)-(1.2) is oscillatory.

**Theorem 3.1.** Assume that for some $t_0 > 0$,

$$\lim_{t \to \infty} \int_{t_0}^{t} q(s) \frac{\beta(s)}{\Gamma(1 - \alpha)} ds = \infty. \quad (3.1)$$

Then every solution $u(x, t)$ of the problem (1.1)-(1.2) is oscillatory in $G$.

**Proof.** The proof is by contradiction. Suppose that the problem (1.1)-(1.2) has one non-oscillatory solution $u(x, t)$ . Without loss of generality, we may assume that $u(x, t) > 0$, and $u(x, t - \tau_i) > 0$ in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$, $i = 1, 2, \ldots, m$.

Integrating (1.1) with respect to $x$ over the domain $\Omega$, we have

$$\int_{\Omega} D_{+}^{\alpha+\alpha} u(x, t) dx + p(t) \int_{\Omega} D_{+}^{\alpha} u(x, t) dx = a(t) \int_{\Omega} \Delta u(x, t) dx + \sum_{i=1}^{m} a_i(t) \int_{\Omega} \Delta u(x, t - \tau_i) dx$$

$$- \int_{\Omega} q(x, t) \int_{0}^{t} (t - \xi)^{-\alpha} u(x, \xi) d\xi dx, \quad t \geq t_0. \quad (3.2)$$

Green’s formula and the boundary condition (1.2) yield

$$\int_{\Omega} \Delta u(x, t) dx = -\int_{\partial \Omega} \frac{\partial u(x, t)}{\partial N} dS \leq 0, \quad t \geq t_0, \quad (3.3)$$

$$-\int_{\partial \Omega} \beta(x, t) u(x, t) dS = \int_{\partial \Omega} \frac{\partial u(x, t - \tau_i)}{\partial N} dS \leq 0, \quad t \geq t_0, \quad i = 1, 2, \ldots, m, \quad (3.4)$$

where $dS$ is the surface element on $\partial \Omega$.

It follows from the assumption (C3) that

$$\int_{\Omega} q(x, t) \int_{0}^{t} (t - \xi)^{-\alpha} u(x, \xi) d\xi dx \geq q(t) \int_{\Omega} \int_{0}^{t} (t - \xi)^{-\alpha} u(x, \xi) d\xi dx$$

$$= q(t) \int_{0}^{t} (t - \xi)^{-\alpha} \int_{\Omega} u(x, \xi) dx d\xi, \quad t \geq t_0. \quad (3.5)$$
Let 
\[ U(t) = \int_{\Omega} u(x, t)dx. \]

Then \( U(t) > 0, \ t \geq t_0. \) Combining (3.2)-(3.5), we have
\[ D_t^{1+\alpha}U(t) + p(t)D_t^\alpha U(t) + q(t)E(t) \leq 0, \ t \geq t_0, \tag{3.6} \]
where
\[ E(t) = \int_0^1 (t - \xi)^{-\alpha}U(\xi)d\xi > 0. \tag{3.7} \]

Define
\[ W(t) = \frac{D_t^\alpha U(t)}{E(t)}, \ t \geq t_0. \tag{3.8} \]

Using Lemma 2.4 and Lemma 2.5, from (3.8), we have
\[ W'(t) = -\frac{(D_t^\alpha U(t))^\prime}{E(t)} + \frac{E(t)D_t^\alpha U(t)}{E^2(t)} \]
\[ = -\frac{D_t^{1+\alpha}U(t)}{E(t)} + \frac{\Gamma(1-\alpha)D_t^\alpha U(t)^2}{E(t)} \]
\[ \geq -p(t)W(t) + q(t) + \Gamma(1-\alpha)W^2(t), \ t \geq t_0. \tag{3.9} \]

Integrating (3.9) from \( t_0 \) to \( t, \) we obtain
\[ W(t) \geq W(t_0) + \int_{t_0}^t (-p(s)W(s) + q(s) + \Gamma(1-\alpha)W^2(s))ds \]
\[ = W(t_0) + \Gamma(1-\alpha) \int_{t_0}^t [W(s) - \frac{p(s)}{2\Gamma(1-\alpha)}]^2 ds \]
\[ + \int_{t_0}^t q(s) - \frac{p^2(s)}{4\Gamma(1-\alpha)}]ds. \tag{3.10} \]

In view of (3.1), there exists \( t_1 \geq t_0, \) such that
\[ W(t) > \Gamma(1-\alpha) \int_{t_0}^t [W(s) - \frac{p(s)}{2\Gamma(1-\alpha)}]^2 ds, \ t \geq t_1. \]

Let
\[ H(t) = \Gamma(1-\alpha) \int_{t_0}^t [W(s) - \frac{p(s)}{2\Gamma(1-\alpha)}]^2 ds. \tag{3.11} \]

Then \( W(t) > H(t) > 0 \) for \( t \geq t_1. \)

Noting the assumption (C3), from (3.11), we have
\[ H'(t) = \Gamma(1-\alpha) [W(t) - \frac{p(t)}{2\Gamma(1-\alpha)}]^2 \]
\[ > \Gamma(1-\alpha)W^2(t) > \Gamma(1-\alpha)H^2(t). \]

Thus,
\[ \Gamma(1-\alpha) < \frac{H'(t)}{H^2(t)}, \ t \geq t_1. \]

Integrating both sides of this inequality from \( t_1 \) to \( t, \) we obtain
\[ \Gamma(1-\alpha) \int_{t_1}^t ds < \frac{1}{H(t_1)} - \frac{1}{H(t)} < \frac{1}{H(t_1)}. \]

Letting \( t \to \infty \) in the above inequality, we get
\[ \lim_{t \to \infty} \Gamma(1-\alpha) \int_{t_1}^t ds < \frac{1}{H(t_1)}. \]
Which is not correct. This completes the proof of Theorem 3.1. \qed
\textbf{Theorem 3.2.} Assume that there exists a function \( g(t) \in C^1([t_0, \infty), R_+) \) such that
\[
\lim_{t \to \infty} \int_{t_0}^{t} \frac{\sqrt{t}(1-\alpha)}{g(s)} ds = \infty,
\]
and
\[
\lim_{t \to \infty} [-\frac{1}{4\sqrt{t}(1-\alpha)} \int_{t_0}^{t} \Psi(s) ds + \frac{g'(t)}{2\sqrt{1-\alpha}}] = \infty,
\]
where
\[
\Psi(t) = p^2(t)g(t) + \frac{(g'(t))^2}{g(t)} - 2p(t)g'(t) - 4Gamma(1-\alpha)g(t)q(t).
\]
Then every solution \( u(x,t) \) of the problem (1.1)-(1.2) is oscillatory in \( G \).

\textit{Proof.} By way of contradiction. Suppose that \( u(x,t) \) is a nonoscillatory solution of the problem (1.1)-(1.2). Without loss of generality, we may assume that \( u(x,t) > 0 \), and \( u(x,t-\tau_i) > 0 \) in \( \Omega \times [t_0, \infty) \) for some \( t_0 > 0 \), \( i = 1, 2, \cdots, m \).

As in the proof of Theorem 3.1, we obtain that (3.6) and \( U(t), E(t) > 0 \) for \( t \geq t_0 \) hold.

Define
\[
\tilde{W}(t) = -g(t)U'(t) - g(t)q(t), \quad t \geq t_0.
\]
Then
\[
\tilde{W}'(t) = -g'(t)U'(t) - g'(t)q(t) - \frac{g(t)q(t)U'(t)}{E(t)} + g(t)q(t) + g(t)q(t)
\]

\[
\geq \frac{g'(t)}{g(t)}\tilde{W}(t) + g(t)p(t)\frac{U'(t)}{E(t)} + g(t)q(t)
\]
\[+ g(t)\Gamma(1-\alpha)\frac{U''(t)}{E(t)} - \frac{1}{4\sqrt{1-\alpha}}\Psi(t), \quad t \geq t_0.
\]

Let
\[
\check{H}(t) = \sqrt{\Gamma(1-\alpha)}\tilde{W}(t) + \frac{1}{2\sqrt{\Gamma(1-\alpha)}}g'(t).
\]

Then, we get from (3.15) that
\[
\check{H}'(t) \geq \frac{1}{g(t)}[(\check{H}(t) - \frac{g(t)p(t)}{2\sqrt{1-\alpha}})^2 - \frac{(g(t)p(t))^2}{2\sqrt{1-\alpha}}]
\]
\[+ \frac{(g'(t))^2}{4\sqrt{(1-\alpha)}} + \frac{g(t)q(t)^2}{2\sqrt{(1-\alpha)}}] + g(t)q(t)
\]
\[= \frac{1}{g(t)}[\check{H}(t) - \frac{g(t)p(t)}{2\sqrt{1-\alpha}}]^2 - \frac{1}{4\sqrt{1-\alpha}}\Psi(t), \quad t \geq t_0.
\]

Integrating both sides of the above inequality from \( t_0 \) to \( t \), we obtain
\[
\tilde{W}(t) \geq \tilde{W}(t_0) + \int_{t_0}^{t} \frac{1}{g(s)} (\check{H}(s) - \frac{g(s)p(s)}{2\sqrt{1-\alpha}})^2 ds - \frac{1}{4\sqrt{1-\alpha}} \int_{t_0}^{t} \Psi(s) ds.
\]

Using (3.16) in (3.17), we obtain
\[
\check{H}(t) \geq \sqrt{\Gamma(1-\alpha)}\tilde{W}(t_0) + \sqrt{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{1}{g(s)} (\check{H}(s) - \frac{g(s)p(s)}{2\sqrt{1-\alpha}})^2 ds
\]
\[+ \frac{1}{4\sqrt{1-\alpha}} \int_{t_0}^{t} \Psi(s) ds + \frac{1}{2\sqrt{1-\alpha}}g'(t).
\]

In view of (3.13), there exists \( t_1 \geq t_0 \), such that
\[
\check{H}(t) > \sqrt{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{1}{g(s)} (\check{H}(s) - \frac{g(s)p(s)}{2\sqrt{1-\alpha}})^2 ds, \quad t \geq t_1.
\]

Let
\[
Q(t) = \sqrt{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{1}{g(s)} (\check{H}(s) - \frac{g(s)p(s)}{2\sqrt{1-\alpha}})^2 ds,
\]
Thus, \( \sqrt{\frac{1}{2}} \). Oscillation of the problem then \( \tilde{\theta} \).

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\( \lambda \) conditions under which all solutions of the problem (1.1)-(1.3) is oscillatory. To this is positive and the corresponding eigenfunction

Theorem 4.1.

Proof. (1

the domain \( \Omega \), we have

\[
\begin{align*}
\int_\Omega (D_{x,t}^{\alpha} u(x,t))\phi(x)dx + p(t) \int_\Omega (D_x^{\alpha} u(x,t))\phi(x)dx &= a(t) \int_\Omega \Delta u(x,t)\phi(x)dx + \sum_{i=1}^m a_i(t) \int_\Omega \Delta u(x,t-\tau_i)\phi(x)dx \\
- \int_\Omega q(x,t) \int_0^t (t-\xi)^{-\alpha} u(x,\xi) d\xi \phi(x)dx, \quad t \geq t_0.
\end{align*}
\]

Green’s formula and the boundary condition (1.3) yield

\[
\int_\Omega \phi(x)\Delta u(x,t)dx = \int_\Omega u(x,t)\Delta \phi(x)dx = -\lambda_0 \int_\Omega u(x,t)\phi(x)dx \leq 0, \quad t \geq t_0,
\]

(4.2)

\[
\int_\Omega \phi(x)\Delta u(x,t-\tau_i)dx = \int_\Omega u(x,t-\tau_i)\Delta \phi(x)dx = -\lambda_0 \int_\Omega u(x,t-\tau_i)\phi(x)dx \leq 0, \quad t \geq t_0, \quad i = 1, 2, \ldots, m.
\]

(4.3)

Noting the assumption \( (C_3) \), we have

\[
\begin{align*}
\int_\Omega q(x,t) \int_0^t (t-\xi)^{-\alpha} u(x,\xi) d\xi \phi(x)dx \\
\geq q(t) \int_\Omega (\int_0^t (t-\xi)^{-\alpha} u(x,\xi) d\xi) \phi(x)dx \\
= q(t) \int_\Omega (\int_0^t (t-\xi)^{-\alpha} u(x,\xi) d\xi) \phi(x)dx, \quad t \geq t_0.
\end{align*}
\]

(4.4)

4. Oscillation of the problem (1.1)-(1.3). In this section, we give some sufficient conditions under which all solutions of the problem (1.1)-(1.3) is oscillatory. To this end, we need to use the fact that the first eigenvalue \( \lambda_0 \) of the Dirichlet eigenvalue problem

\[
\begin{align*}
\Delta \omega(x) + \lambda \omega(x) &= 0, \quad x \in \Omega, \\
\omega(x) &= 0, \quad x \in \partial \Omega
\end{align*}
\]

is positive and the corresponding eigenfunction \( \phi(x) \) is also positive in \( \Omega \), see [6].

Theorem 4.1. Under the conditions of Theorem 3.1, then every solution \( u(x,t) \) of the problem (1.1)-(1.3) is oscillatory in \( \Omega \).

Proof. By way of contradiction, suppose that \( u(x,t) \) is a nonoscillatory solution of the problem (1.1)-(1.2). Without loss of generality, we may assume that \( u(x,t) > 0 \) and \( u(x,t-\tau_i) > 0 \) in \( \Omega \times [t_0, \infty) \) for some \( t_0 > 0, \quad i = 1, 2, \ldots, m. \)

Multiplying both sides of (1.1) by \( \phi(x) \) and integrating with respect to \( x \) over the domain \( \Omega \), we have

\[
\begin{align*}
\int_\Omega (D_{x,t}^{\alpha} u(x,t))\phi(x)dx + p(t) \int_\Omega (D_x^{\alpha} u(x,t))\phi(x)dx &= a(t) \int_\Omega \Delta u(x,t)\phi(x)dx + \sum_{i=1}^m a_i(t) \int_\Omega \Delta u(x,t-\tau_i)\phi(x)dx \\
- \int_\Omega q(x,t) \int_0^t (t-\xi)^{-\alpha} u(x,\xi) d\xi \phi(x)dx, \quad t \geq t_0.
\end{align*}
\]

(4.1)
Consider the following fractional partial differential equation

\[ \text{Example 2.} \]

\[ \text{Theorem 4.2.} \]

Under the conditions of Theorem 3.2, then every solution of the following conclusion.

\[ \text{Examples.} \]

\[ (1) \]

\[ \text{where} \]

Combining (4.1)-(4.4), we have

\[ D_{x,t}^{1-\alpha} \tilde{U}(t) + p(t)D_{x,t}^{\alpha} \tilde{U}(t) + q(t)\tilde{E}(t) \leq 0, \quad t \geq t_0, \quad (4.5) \]

where

\[ \tilde{E}(t) = \int_0^t (t - \xi) \tilde{U}(\xi)d\xi > 0. \quad (4.6) \]

The remainder of the proof is similar to that of Theorem 3.1 and we omit it here. The proof of Theorem 4.1 is complete.

By using a similar argument to that of Theorem 4.1, it is not difficult to obtain the following conclusion.

**Theorem 4.2.** Under the conditions of Theorem 3.2, then every solution \( u(x, t) \) of the problem (1.1)-(1.3) is oscillatory in \( G \).

**5. Examples.** In this section, we give the following two specific examples to illustrate above Theorem 3.1 and Theorem 4.2.

**Example 1.** Consider the following fractional partial differential equation

\[ D_{x,t}^{2} u(x, t) - \frac{1}{2} D_{x,t}^{2} u(x, t) + e^x(1 + \frac{1}{t}) \int_0^t (t - \xi)^{-\alpha} u(x, \xi)d\xi = e^x \Delta u(x, t) + 2t \Delta u(x, t - 1), \quad (x, t) \in (0, \pi) \times R_+ \]

with the boundary value condition

\[ u_x(0, t) = u_x(\pi, t) = 0, \quad t > 0. \quad (5.2) \]

Here, \( \Omega = (0, \pi), \) \( n = m = 1, \) \( a(t) = e^t, \) \( a_3(t) = \sqrt{t}, \) \( \tau_1 = 1, \) \( p(t) = -\frac{1}{t^2}, \) and \( q(x, t) = e^x(1 + \frac{1}{t}). \) Hence \( q(t) = 1 + \frac{1}{t^2} \) and \( \Gamma(1 - \alpha) = \Gamma(\frac{1}{2}) = \sqrt{\pi}. \) We easily see that

\[ \lim_{t \to \infty} \int_0^t [q(s) - \frac{p^2(s)}{4\Gamma(1-\alpha)}]ds = \lim_{t \to \infty} \int_0^t \left[ (1 + \frac{1}{s}) - \frac{1}{4\sqrt{\pi s}} \right] ds = \infty. \]

Therefore, the conditions of Theorem 3.1 are satisfied. Then every solution of problem (5.1)-(5.2) oscillates in \((0, \pi) \times R_+.

**Example 2.** Consider the following fractional partial differential equation

\[ D_{x,t}^{3} u(x, t) - \frac{1}{2} D_{x,t}^{3} u(x, t) + e^x \left[ 1 + \frac{1}{t^2} \right] \int_0^t (t - \xi)^{-\alpha} u(x, \xi)d\xi = e^{-t} \Delta u(x, t) + \frac{1}{2} \Delta u(x, t - \frac{1}{2}), \quad (x, t) \in (0, \pi) \times R_+ \]

with the boundary value condition

\[ u(0, t) = u(\pi, t) = 0, \quad t > 0. \quad (5.4) \]

Here, \( \Omega = (0, \pi), \) \( n = m = 1, \) \( a(t) = e^{-t}, \) \( a_3(t) = \frac{1}{t}, \) \( \tau_1 = \frac{1}{2}, \) \( p(t) = -\frac{1}{t}, \) and \( q(x, t) = e^x \left[ 1 + \frac{1}{t^2} \right]. \) Hence \( q(t) = \frac{1}{\sqrt{\pi}} \left( 1 + \frac{1}{t^2} \right) \) and \( \Gamma(1 - \alpha) = \Gamma(\frac{1}{2}) = \sqrt{\pi}. \) Take \( g(t) = t, \) it is easy to see that

\[ \lim_{t \to \infty} \int_0^t \frac{\sqrt{\Gamma(1-\alpha)}}{g(s)} ds = \lim_{t \to \infty} \int_0^t \frac{\sqrt{\pi}}{s} ds = \infty, \]

and

\[ \Psi(t) = p^2(t) g(t) + \left(\frac{g'(t)}{g(t)}\right)^2 - 2p(t) g'(t) - 4\sqrt{\Gamma(1-\alpha)} g(t) = \frac{1}{t} + \frac{1}{t} + \frac{2}{t} - 4 \sqrt{\pi t} \frac{1}{\sqrt{\pi}} \left( 1 + \frac{1}{t^2} \right) = -4t, \]
\[
\lim_{t \to \infty} \left\{ \frac{1}{4\sqrt{T(1-r)}} \int_{t_0}^{t} \Psi(s)ds + \frac{g'(t)}{2\sqrt{T(1-r)}} \right\} = \lim_{t \to \infty} \left\{ -\frac{1}{4\sqrt{T}} \int_{t_0}^{t} (-4s)ds + \frac{1}{2\sqrt{T}} \right\} = \infty.
\]

Therefore, the conditions of Theorem 4.2 are satisfied. Then every solution of problem (5.3)-(5.4) oscillates in \((0, \pi) \times R_+.

**Remark.** The oscillatory behavior of the problems (1.1)-(1.2) and (1.1)-(1.3) was studied in Li and Sheng [20] for the case when \(p(t) \in C([0, \infty), [0, \infty))\). The results in this paper need to assume that \(p(t) \in C([0, \infty), (-\infty, 0))\) is satisfied. It remains one question how to find sufficient conditions on the oscillation of solutions of the problems (1.1)-(1.2) and (1.1)-(1.3) for the case when \(p(t)\) is neither positive nor negative, or it is sign-changing.

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