BARI-MARKUS PROPERTY FOR RIESZ PROJECTIONS OF HILL OPERATORS WITH SINGULAR POTENTIALS

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ABSTRACT. The Hill operators $L_y = -y'' + v(x)y$, $x \in [0, \pi]$, with $H^{-1}$ periodic potentials, considered with periodic, antiperiodic or Dirichlet boundary conditions, have discrete spectrum, and therefore, for sufficiently large $N$, the Riesz projections

$$P_n = \frac{1}{2\pi i} \int_{C_n} (z - L)^{-1} dz, \quad C_n = \{z : |z - n^2| = n\}$$

are well defined. It is proved that

$$\sum_{n>N} \|P_n - P_0^H\|_{HS} < \infty,$$

where $P_0^H$ are the Riesz projection of the free operator and $\| \cdot \|_{HS}$ is the Hilbert–Schmidt norm.

1. Introduction

We consider the Hill operator

(1.1)  \[ L_y = -y'' + v(x)y, \quad x \in I = [0, \pi], \]

with a singular complex–valued periodic potential $v$, $v(x + \pi) = v(x)$, $v \in H^{-1}_{loc}(\mathbb{R})$, i.e.,

$$v(x) = v_0 + Q'(x),$$

where

$Q \in L^2_{loc}(\mathbb{R})$, \quad $Q(x + \pi) = Q(x)$, \quad $q(0) = \int_0^\pi Q(x)dx = 0,$

so

(1.2)  \[ Q = \sum_{m \in 2Z\setminus\{0\}} q(m)e^{imx}, \quad \|v|H^{-1}\|^2 = |v_0|^2 + \sum_{m \in 2Z\setminus\{0\}} |q(m)|^2/m^2 < \infty. \]

A. Savchuk and A. Shkalikov [17] gave thorough spectral analysis of such operators. In particular, they consider a broad class of boundary conditions (bc) – see (1.6), Theorem 1.5 there – in terms of a function $y$ and its quasi–derivative

$$u = y' - Qy.$$  

The natural form of periodic or antiperiodic ($Per^\pm$) bc is the following one:

(1.3)  \[ Per^\pm : \quad y(\pi) = \pm y(0), \quad u(\pi) = \pm u(0) \]

If the potential $v$ happens to be an $L^2$-function these bc are identical to the classical ones (see discussion in [8], Section 6.2).

The Dirichlet bc is more simple:

$$Dir : \quad y(0) = 0, \quad y(\pi) = 0;$$

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it does not require quasi–derivatives, so it is defined in the same way as for \( L^2 \)-potentials \( v \).

In our analysis of instability zones of Hill and Dirac operators (see [6] and the comments there) we follow an approach ([11], [12], [2], [3], [4], [5]) based on Fourier Method. But in the case of singular potentials it may happen that the functions

\[ u_k = e^{ikx} \quad \text{or} \quad \sin kx, \quad k \in \mathbb{Z}, \]

have their \( L \)-images outside \( L^2 \). This implies, for some singular potentials \( v \), that we have \( Lf \notin L^2 \) for any smooth (say \( C^2 \)) nonzero function \( f \) (see an example in [9], between (1.3) and (1.4)).

In general, for any reasonable \( \text{bc} \), the eigenfunctions \( \{ u_k \} \) of the free operator \( L_{0 \text{bc}} \) are not necessarily in the domain of \( L_{\text{bc}} \). Yet, in [7], [8] we gave a justification of the Fourier method for operators \( L_{\text{bc}} \) with \( H^{-1} \)-potentials and \( bc = \text{Per}^\pm \) or Dir.

Our results are announced in [7], and in [8] all technical details of justification of the Fourier method are provided.

Now, in the case of singular potentials, we want to compare the Riesz projections \( P_n \) of the operator \( L_{\text{bc}} \), defined for large enough \( n \) (say \( n > N \)) by the formula

\[ P_n = \frac{1}{2\pi i} \int_{C_n} (z - L_{\text{bc}})^{-1} dz, \quad C_n = \{ |z - n^2| = n \}, \]

with the corresponding Riesz projections \( P^0_n \) of the free operator \( L_{0 \text{bc}} \) (although \( E^0_n = \text{Ran}(P^0_n) \) maybe have no common nonzero vectors with the domain of \( L_{\text{bc}} \)). In [9], Theorem 1 we showed that

\[ \| P_n - P^0_n \|_{L^1 \rightarrow L^\infty} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

In this paper, the main result is Theorem 1 which claims, for sufficiently large \( N \), that

\[ \sum_{n > N} \| P_n - P^0_n \|_{HS}^2 < \infty. \]

For a potential \( v \in L^2 \) (1.6) is "easy". Indeed, using (1.9) and (1.10) below, and estimating, as in the proof of Lemma 23 in [6], the Hilbert–Schmidt norm of \( VR_\lambda \) for \( \lambda \in C_n \) (where \( V \) is the operator of multiplication by \( v \) and \( R_\lambda \) is the resolvent of the free operator), one could get

\[ \| P_n - P^0_n \|_{HS} \leq \frac{C}{n} \| v \|_{L^2}, \quad n \geq N(\| v \|_{L^2}), \]

with \( C \) being an absolute constant, so (1.6) follows. However, for singular potentials \( v \) the proof of (1.6) and Theorem 1 now is rather complicated.

Since the Hilbert–Schmidt norm does not exceed the \( L^2 \)-norm, (1.6) implies that

\[ \sum_{n > N} \| P_n - P^0_n \|_{L^2 \rightarrow L^2}^2 \leq \infty, \]

which was proven earlier by A. Savchuk and A. Shkalikov ([17], Sect.2.4). This implies (by Bari–Markus theorem – see [10], Ch.6, Sect.5.3, Theorem 5.2) that the spectral decompositions

\[ f = f_N + \sum_{n > N} P_n f \]

converge unconditionally.
The proof of Theorem 1 is based on the perturbation theory (for example, see [13]), which gives the representation

\[ P_n - P_n^0 = \frac{1}{2\pi i} \int_{C_n} (R(\lambda) - R^0(\lambda)) \, d\lambda, \]

where \( R(\lambda) = (\lambda - L_{bc})^{-1} \) and \( R^0(\lambda) \) are the resolvents of \( L_{bc} \) and of the free operator \( L^0_{bc} \), respectively.

In many respects the constructions of this paper are parallel to constructions in [9], the proof of (1.5); see, for example, comments in the next paragraph. However, there is no direct way to use the inequalities proven in [9] and to come to the main results of the present paper.

In the classical case, where \( v \in L^2 \), one can get reasonable estimates for the norms \( \|R(\lambda) - R^0(\lambda)\| \) on the contour \( C_n \), and then by integration for \( \|P_n - P_n^0\| \).

But now, with \( v \in H^{-1} \), we use the same approach as in [9], namely, we get good estimates for the norms \( \|P_n - P_n^0\| \) after having integrating term by term the series representation

\[ R - R^0 = R^0VR^0 + R^0VR^0VR^0 + \cdots. \]

This integration kills many terms, maybe in their matrix representation. Only then we go to the norm estimates which allow us to prove our main result Theorem 1.

2. Main result

By our Theorem 21 in [8] (about spectra localization), the operator \( L_{Per^\pm} \) has, for sufficiently large \( n \), exactly two eigenvalues (counted with their algebraic multiplicity) inside the disc of radius \( n \) about \( n^2/2 \) (periodic for even \( n \) or antiperiodic for odd \( n \)). The operator \( L_{Dir} \) has one eigenvalue in every such disc for all sufficiently large \( n \).

Let \( E_n \) be the corresponding Riesz invariant subspace, and let \( P_n \) be the corresponding Riesz projection, i.e.,

\[ P_n = \frac{1}{2\pi i} \int_{C_n} (\lambda - L)^{-1} \, d\lambda, \]

where \( C_n = \{ \lambda : |\lambda - n^2| = n \} \). Further \( P_n^0 \) denotes the Riesz projections of the free operator and \( \| \cdot \|_{HS} \) denotes the Hilbert–Schmidt norm.

**Theorem 1.** In the above notations, for boundary conditions \( bc = Per^\pm \) or \( Dir \),

\[ \sum_{n>N} \|P_n - P_n^0\|_{HS}^2 < \infty, \]

**Proof.** We give a complete proof in the case \( bc = Per^\pm \). If \( bc = Dir \) the proof is the same, and only minor changes are necessary due to the fact that in this case the orthonormal system of eigenfunctions of \( L^0 \) is \( \{ \sqrt{2}\sin nx, n \in \mathbb{N} \} \) (while it is \( \{ \exp(imx), m \in 2\mathbb{Z} \} \) for \( bc = Per^+ \), and \( \{ \exp(imx), m \in 1 + 2\mathbb{Z} \} \) for \( bc = Per^- \)).

So, roughly speaking, the only difference is that when working with \( bc = Per^\pm \) the summation indexes in our formulas below run, respectively, in \( 2\mathbb{Z} \) and \( 1 + 2\mathbb{Z} \), while for \( bc = Dir \) the summation indexes have to run in \( \mathbb{N} \). Therefore, we consider in detail only \( bc = Per^\pm \).

Now we present the proof of the theorem up to a few technical inequalities proved in Section 3, Lemmas [6] and [7].
In [8], Section 5, we gave a detailed analysis of the representation

\[(2.2) \quad R_\lambda - R_\lambda^0 = \sum_{s=0}^{\infty} K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda,\]

where \(K_\lambda = \sqrt{R_\lambda^1}\) – see [8], (5.13-14) and what follows there.

With this definition the operator valued function \(K_\lambda\) is analytic in \(\mathbb{C} \setminus \mathbb{R}_+\). But (2.2) below and all formulas of this section – which are essentially variations of (1.11) – have always even powers of \(K_\lambda\) and \(K_\lambda^2 = R_\lambda^0\) is analytic outside on the complement of \(Sp(L^0)\). Certainly, this justifies the use of Cauchy formula or theorem when warranted.

By (1.9),

\[(2.3) \quad P_n - P_n^0 = \frac{1}{2\pi i} \int_{C_n} \sum_{s=0}^{\infty} K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda d\lambda\]

if the series on the right converges. Taking into account that the adjoint operator of \(R_\lambda(v)\) is

\[(R_\lambda(v))^* = R_{\lambda^*}(v),\]

we get

\[(P_n - P_n^0)^* = \frac{1}{2\pi i} \int_{C_n} \sum_{s=0}^{\infty} K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda d\mu,\]

where

\[(2.4) \quad \tilde{V}(m) = V(-m).\]

Since \(\|(P_n - P_n^0)e_m\|^2 = ((P_n - P_n^0)^*(P_n - P_n^0)e_m, e_m)\), it follows that

\[(2.5) \quad \|(P_n - P_n^0)e_m\|^2 = \frac{1}{4\pi^2} \int_{\Gamma_n} \sum_{s=0}^{\infty} (K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda K_\lambda V K_\lambda e_m, e_m)d\lambda d\mu,\]

where \(\Gamma_n = C_n \times C_n\). Thus,

\[(2.6) \quad \sum_{n>N} \|P_n - P_n^0\|^2_{HS} = \sum_{n>N} \sum_{m} \|(P_n - P_n^0)e_m\|^2 \leq \sum_{t,s=0}^{\infty} A(t, s),\]

where

\[(2.7) \quad A(t, s) = \sum_{n>N} \left| \frac{1}{4\pi^2} \int_{\Gamma_n} \sum_{m} (K_\lambda(K_\lambda V K_\lambda)^{t+1} K_\lambda K_\lambda V K_\lambda e_m, e_m)d\lambda d\mu \right|.\]

Notice that \(A(t, s)\) depends on \(N\) but this dependence is suppressed in the notation. Our goal is to show, for sufficiently large \(N\), that \(\sum_{t,s=0}^{\infty} A(t, s) < \infty\) which, in view of (2.6), implies (2.1).

Let us evaluate \(A(0, 0)\). From the matrix representation of the operators \(K_\lambda\) and \(V\) (see more details in [8], (5.15-22)) it follows that

\[(2.8) \quad \langle K_\lambda(K_\lambda V K_\lambda)K_\lambda V K_\lambda e_m, e_m \rangle = \sum_{p} \frac{\tilde{V}(m-p)V(p-m)}{(\mu - m^2)(\mu - p^2)(\lambda - p^2)(\lambda - m^2)}.\]
By integrating this function over \( \Gamma_n = C_n \times C_n \) we get

\[
\frac{1}{4\pi^2} \int \int \Gamma_n \cdots = \begin{cases} 
\sum_{p \neq \pm n} \frac{\|V(p \mp n)\|^2}{|n^2 - p^2|^2}, & m = \pm n, \\
\sum_{p \neq \pm n} \frac{\|V(\pm n - m)\|^2}{|n^2 - m^2|^2}, & m \neq \pm n.
\end{cases}
\]

(2.9)

Thus,

\[
A(0,0) = \sum_{n>N} \sum_{p \neq \pm n} \frac{|V(p - n)|^2}{|n^2 - p^2|^2} + \sum_{n>N} \sum_{p \neq \pm n} \frac{|V(p + n)|^2}{|n^2 - p^2|^2}
\]

\[
+ \sum_{n>N} \sum_{m \neq \pm n} \frac{|V(n - m)|^2}{|n^2 - m^2|^2} + \sum_{n>N} \sum_{m \neq \pm n} \frac{|V(-n - m)|^2}{|n^2 - m^2|^2}
\]

Let us estimate the first sum on the right. In view of (1.2),

\[
|V(m)| \leq |m|r(m), \quad r(m) = \max(|q(m)|, |q(-m)|) \quad r \in \ell^2(2\mathbb{Z}).
\]

Therefore, by Lemma 5 we have

\[
\sum_{n>N} \sum_{p \neq \pm n} \frac{|V(p - n)|^2}{|n^2 - p^2|^2} \leq \sum_{n>N} \sum_{p \neq \pm n} \frac{|p - n|^2 |r(p - n)|^2}{|n^2 - p^2|^2}
\]

\[
\leq \sum_{n>N} \sum_{p \neq \pm n} \frac{|r(n - p)|^2}{|n + p|^2} \leq C \left( \frac{\|r\|^2}{N} + (E_N(r))^2 \right),
\]

where we use the notation

\[
E_a(r) = \left( \sum_{|k| \geq a} |r(k)|^2 \right)^{1/2}, \quad a > 0.
\]

Since each of the other three sums could be estimated in the same way, we get

\[
A(0,0) \leq C \left( \frac{\|r\|^2}{N} + (E_N(r))^2 \right).
\]

(2.12)

Remark: For convenience, here and thereafter we denote by \( C \) any absolute constant.

Next we estimate \( A(t, s) \) with \( s + t > 0 \). From the matrix representation of the operators \( K_\lambda \) and \( V \) we get

\[
\langle K_\mu(K_\nu \tilde{V})^{t+1} K_\mu K_\lambda(K_\nu V K_\lambda)^{s+1} K_\lambda e_m, e_m \rangle
\]

\[
= \sum_{i_1, \ldots, i_t, j_1, \ldots, j_s} \frac{\tilde{V}(m - i_1)\tilde{V}(i_1 - i_2) \cdots \tilde{V}(i_t - p)V(p - j_1)V(j_1 - j_2) \cdots V(j_s - m)}{(\mu - m^2)(\mu - i_1^2) \cdots (\mu - i_t^2)(\mu - p^2)(\lambda - j_1^2) \cdots (\lambda - j_s^2)(\lambda - m^2)}
\]

(2.13)

Notice that if

\[
\pm n \notin \{m, p, i_1, \ldots, i_t\} \quad \text{or} \quad \pm n \notin \{m, p, j_1, \ldots, j_s\},
\]

then the integral over \( C_n \times C_n \) of the corresponding term in the above sum is zero because that term is, respectively, an analytic function of \( \mu, |\mu| \leq n \) and/or an analytic function of \( \lambda, |\lambda| \leq n \). This observation is crucial in finding good estimates for \( A(t, s) \). It means that we may ”forget” the terms satisfying (2.14).

Moreover, by the Cauchy formula, if

\[
m, p, i_1, \ldots, i_t \in \{\pm n\} \quad \text{or} \quad m, p, j_1, \ldots, j_s \in \{\pm n\},
\]

then the integral of the corresponding term vanishes.
Then the matrix representation of the operator \( A \) (2.16)

\[
A(t, s) \leq \sum_{n>N} \frac{1}{4\pi^2} \int_{I_n^*} \sum_{I^n} \frac{\tilde{V}(m - i_1) \cdots \tilde{V}(i_t - p) V(p - j_1) \cdots V(j_s - m)}{(\mu - m^2)(\mu - i_1^2) \cdots (\mu - p^2)(\lambda - j_1^2) \cdots (\lambda - m^2)} d\mu d\lambda,
\]

where \( I^* \) is the set of \( t + s + 2 \)-tuples of indices \( m, i_1, \ldots, i_t, p, j_1, \ldots, j_s \in n + 2Z \) such that (2.14) and (2.15) do not hold.

In view of (2.10), we may estimate \( A(t, s) \) by

\[
A(t, s) \leq \sum_{n>N} n^2 \sup_{(\mu, \lambda) \in I^*} \sum_{I^n} B(\mu, m, i_1, \ldots, i_t, p) \cdot B(\lambda, p, j_1, \ldots, j_s, m),
\]

where

\[
B(z, m, i_1, \ldots, i_t, p) = \frac{W(m - i_1)W(i_1 - i_2) \cdots W(i_t - p)}{|z - m^2||z - i_1^2| \cdots |z - i_t^2||z - p^2|},
\]

and

\[
B(z, m, p) = \frac{W(m - p)}{|z - m^2||z - p^2|}
\]

(in the degenerate case, when there are no \( i \)-indices), with

\[
W(m) = \max\{|V(m)|, |V(-m)|\}, \quad m \in 2Z.
\]

In view of (2.10) and (2.4), we have

\[
W(m) = |m|r(m), \quad \text{where} \quad r(-m) = r(m) \geq 0, \quad r = (r(m)) \in 2Z.
\]

We consider the following subsets of \( I^* \):

\[
I_0^* = \{(m, i_1, \ldots, i_t, p, j_1, \ldots, j_s) : m = \pm n, p = \pm n\},
\]

(2.22)

\[
I_1^* = \{(m, i_1, \ldots, i_t, p, j_1, \ldots, j_s) : m = \pm n, p \neq \pm n\},
\]

(2.23)

\[
I_2^* = \{(m, i_1, \ldots, i_t, p, j_1, \ldots, j_s) : m \neq \pm n, p = \pm n\},
\]

(2.24)

\[
I_3^* = \{(m, i_1, \ldots, i_t, p, j_1, \ldots, j_s) : m \neq \pm n, p \neq \pm n\}.
\]

(2.25)

Since \( I^* = \bigcup I_k^* \), \( k = 0, 1, 2, 3 \), we have

\[
A(t, s) \leq A_0(t, s) + A_1(t, s) + A_2(t, s) + A_3(t, s),
\]

where \( A_k(t, s) \) is the subsum of the sum on the right of (2.16) which corresponds to \( I_k^* \), i.e.,

\[
A_k(t, s) = \sum_{n>N} n^2 \sup_{(\mu, \lambda) \in I_k^*} \sum_{I^n} B(\mu, m, i_1, \ldots, i_t, p) \cdot B(\lambda, p, j_1, \ldots, j_s, m), \quad k = 0, 1, 2, 3.
\]

(2.26)

Let \( \mathcal{K}_z \) denote the operator with a matrix representation

\[
(\mathcal{K}_z)_{jm} = \frac{1}{|z - m^2|^{1/2} \delta_{jm}},
\]

and let \( W \) denote the operator with a matrix representation

\[
W_{jm} = W(j - m).
\]

Then the matrix representation of the operator \( \mathcal{K}_z W \mathcal{K}_z \) is

\[
(\mathcal{K}_z W \mathcal{K}_z)_{jm} = \frac{W(j - m)}{|z - j^2|^{1/2} |z - m^2|^{1/2}}.
\]

(2.27)
and we have (see the proof of Lemma 19 in [8])

\[(2.29) \quad \|K_z\| = \frac{1}{\sqrt{n}}, \quad \|K_zW\|_{HS} \leq \rho_n \quad \text{for} \quad z \in C_n, \quad n \geq 3,\]

where

\[(2.30) \quad \rho_n = C \left( E_{\sqrt{n}}(r) + \|r\|^2/n \right)^{1/2},\]

and \(\| \cdot \|_{HS}\) means the Hilbert–Schmidt norm of the corresponding operator.

Moreover, by (2.18), we have

\[(2.31) \quad \sum_{i_1, \ldots, i_t} B(z, m, i_1, \ldots, i_t, p) = \langle K_z(WK_z)^{t+1}K_z e_p, e_m \rangle\]

Estimates for \(A_0(t, s)\). Notice, that \(A_0(t, 0) = 0\) and \(A_0(0, s) = 0\) because the corresponding set of indices \(I_n^*\) is empty (see the text around (2.15), and the definition of \(I^*\)).

Assume that \(t > 0, s > 0\). In view of (2.22) and (2.27), we have

\[(2.32) \quad A_0(t, s) \leq \sum_{n > N} n^2 \sum_{m, p \in \{\pm n\}} \sup_{(\mu, \lambda) \in C_n} \sum_{i_1, \ldots, i_t} B(\mu, m, i_1, \ldots, i_t, p) \sum_{j_1, \ldots, j_s} B(\lambda, p, j_1, \ldots, j_s, m).\]

Therefore, by the Cauchy inequality,

\[(2.33) \quad A_0(t, s) \leq \left( \sum_{n > N} n^2 \sum_{m, p \in \{\pm n\}} \sup_{\mu \in C_n} \sum_{i_1, \ldots, i_t} B(\mu, m, i_1, \ldots, i_t, p) \right)^2 / 12\]

\[\times \left( \sum_{n > N} n^2 \sum_{m, p \in \{\pm n\}} \sup_{\lambda \in C_n} \sum_{j_1, \ldots, j_s} B(\lambda, p, j_1, \ldots, j_s, m) \right)^2 / 12\].

**Lemma 2.** In the above notations,

\[(2.34) \quad \sum_{n > N} n^2 \sup_{\mu \in C_n} \left| \sum_{i_1, \ldots, i_t} B(\mu, m, i_1, \ldots, i_t, p) \right|^2 \leq C\|r\|^2 \rho_n^2 \quad \text{if} \quad m, p \in \{\pm n\},\]

where \(C\) is an absolute constant and \(\rho_n\) is defined in (2.30).

**Proof.** If \(t = 1\), then, by (2.18), the sum \(\sigma\) in (2.34) has the form

\[\sigma(m, p) = \sum_{n > N} n^2 \sup_{\mu \in C_n} \left| \sum_{i} \frac{W(m-i)W(i-p)}{\mu - m^2||\mu - i^2||\mu - p^2} \right|^2, \quad m, p \in \{\pm n\}.\]

One can easily see that

\[\sigma(-n, -n) = \sigma(n, n), \quad \sigma(-n, n) = \sigma(n, -n)\]

by changing \(i\) to \(-i\) and using that \(W(-k) = W(k)\).

Taking into account that \(|\mu - n^2| = n\) for \(\mu \in C_n\), and \(W(k) = |k|r(k)\), we get, by the elementary inequality

\[(2.35) \quad \frac{1}{|\mu - i^2|} \leq \frac{2}{|n^2 - i^2|} \quad \text{for} \quad \mu \in C_n, \quad i \in n + 2\mathbb{Z}, \quad i \neq \pm n,\]
that
\[ \sigma(n, n) \leq 4 \sum_{n > N} n^2 \left( \sum_{i \neq \pm n} \frac{|n - i|}{n^2 |n + i|} r(n - i)r(i - n) + \frac{4}{n} r(2n)r(-2n) \right)^2 \]

Therefore, by the Cauchy inequality,
\[ \sigma(n, n) \leq 4 \sum_{n > N} 2n^2 \left( \sum_{i \neq \pm n} \frac{|n - i|}{n^2 |n + i|} r(n - i)r(i - n) \right)^2 + 128 \sum_{n > N} |r(2n)r(-2n)|^2 \]
\[ \leq 2 \|r\|^2 \sum_{n > N} \sum_{i \neq \pm n} \frac{|n - i|^2}{n^2 |n + i|^2} |r(n - i)|^2 + 128 \|r\|^2 \sum_{n > N} |r(2n)|^2 \leq C \rho_N^2. \]

(by (3.5) in Lemma 3). In an analogous way, we get
\[ \sigma(n, -n) = \sum_{n > N} n^2 \left( \sum_{i \neq \pm n} W(n - i)W(i + n) \right)^2 \]
\[ = \sum_{n > N} \frac{1}{n^2} \left( \sum_{i \neq \pm n} r(n - i)r(i + n) \right)^2 \leq \frac{4 \|r\|^4}{N} \leq 4 \rho_N^2. \]

This completes the proof of (2.34) for \( t = 1 \).

Next we consider the case \( t > 1 \). Since \( |\mu - n^2| = n \) for \( \mu \in C_n \), by (2.18) the sum \( \sigma \) in (2.34) can be written in the form
\[ \sigma = \sum_{n > N} \frac{1}{n^2} \sup_{\mu \in C_n} \left| \sum_{i_1, \ldots, i_t} \frac{W(m - i_1)W(i_1 - i_2) \cdots W(i_t - p)}{|\mu - i_1^2||\mu - i_2^2| \cdots |\mu - i_t^2|} \right|^2, \quad m, p \in \{ \pm n \}. \]

In view of (2.28), we have (with \( i = i_1, k = i_t \))
\[ \sigma = \sum_{n > N} \frac{1}{n^2} \sup_{\mu \in C_n} \left| \sum_{i, k} \frac{W(m - i)}{|\mu - i^2|^{1/2}} \cdot H_{ik}(\mu) \cdot \frac{W(k - p)}{|\mu - k^2|^{1/2}} \right|^2, \quad m, p \in \{ \pm n \}, \]

where \( (H_{ik}(\mu)) \) is the matrix representation of the operator \( H(\mu) = (K_\mu W K_\mu)^{t-1} \).

By (2.29),
\[ \|H(\mu)\|_{HS} = \left( \sum_{i, k} |H_{ik}(\mu)|^2 \right)^{1/2} \leq \|K_\mu W K_\mu\|_{HS}^{t-1} \leq \rho_{t-1} \leq \rho_N \] for \( \mu \in C_n, \ n > N \).

Therefore, the Cauchy inequality implies
\[ \sigma(m, p) \leq \rho_N^{2(t-1)} \cdot \sum_{n > N} \frac{1}{n^2} \sup_{\mu \in C_n} \sum_{i, k} \frac{|W(m - i)|^2}{|\mu - i^2|} \cdot \frac{|W(k - p)|^2}{|\mu - k^2|^2}. \]

By (2.35) and \( W(-k) = W(k) \), one can easily see (changing \( i \) with \(-i\), if necessary) that
\[ \max_{m = \pm n} \sup_{\mu \in C_n} \sum_{i} \frac{|W(m - i)|^2}{|\mu - i^2|} \leq \sum_{i \neq \pm n} \frac{2|W(n - i)|^2}{|n^2 - i^2|} + \frac{|W(2n)|^2}{n} \]
In an analogous way, it follows that
\[
\max_{\mu=\pm n} \sup_{\mu \in \mathcal{C}_n} \sum_k \frac{|W(k-p)|^2}{|\mu-k|^2} \leq \sum_{i \neq \pm n} \frac{2|W(n-i)|^2}{|n-i|^2} + \frac{|W(2n)|^2}{n}.
\]
Therefore, we have
\[
\sigma(m, p) \leq \rho_N^{2(t-1)} \sum_{n>N} \frac{1}{n^2} \left( \sum_{i \neq \pm n} \frac{2|W(n-i)|^2}{|n-i|^2} + \frac{|W(2n)|^2}{n} \right)^2.
\]
Since \(W(k) = |k|r(k)\), by \((a + b)^2 \leq 2a^2 + 2b^2\) and the Cauchy inequality, we get
\[
\left( \sum_{i \neq \pm n} \frac{2|W(n-i)|^2}{|n-i|^2} + \frac{|W(2n)|^2}{n} \right)^2 \leq 32 \|r\|^2 \sum_{i \neq \pm n} \frac{|n-i|^2}{|n+i|^2} |r(n-i)|^2 + 2|W(2n)|^2 \|r\|^2.
\]
Thus,
\[
\sigma(m, p) \leq 32 \|r\|^2 \rho_N^{2(t-1)} \left( \sum_{n>N} \sum_{i \neq \pm n} \frac{|n-i|^2}{|n+i|^2} |r(n-i)|^2 + \sum_{n>N} |r(2n)|^2 \right) \leq C \|r\|^2 \rho_N^{2t}
\]
(by \(3.5\) in Lemma 3).

Now, by \((2.33)\) and \((2.34)\) in Lemma 2 we get
\[
(2.36) \quad A_0(t, s) \leq C \|r\|^2 \rho_N^{t+s}, \quad t + s > 0,
\]
where \(C\) is an absolute constant.

Estimates for \(A_1(t, s)\). Assume that \(t + s > 0\). In view of \((2.23)\) and \((2.27)\), we have
\[
(2.37) \quad A_1(t, s) \leq \sum_{n>N} n^2 \sup_{m=\pm n, p \neq \pm n} \sum_{\mu \in \mathcal{C}_n} \sum_{i_1, \ldots, i_t} B(\mu, m, i_1, \ldots, i_t, p) \sup_{\lambda \in \mathcal{C}_n} \sum_{j_1, \ldots, j_s} B(\lambda, p, j_1, \ldots, j_s, m).
\]
Therefore, by the Cauchy inequality,
\[
(2.38) \quad A_1(t, s) \leq \left( \sum_{n>N} n^2 \sup_{m=\pm n, p \neq \pm n} \sum_{\mu \in \mathcal{C}_n} \left( \sum_{i_1, \ldots, i_t} B(\mu, m, i_1, \ldots, i_t, p) \right)^2 \right)^{1/2} \times \left( \sum_{n>N} n^2 \sup_{m=\pm n, p \neq \pm n} \sum_{\lambda \in \mathcal{C}_n} \left( \sum_{j_1, \ldots, j_s} B(\lambda, p, j_1, \ldots, j_s, m) \right)^2 \right)^{1/2}.
\]

Lemma 3. In the above notations,
\[
(2.39) \quad \sum_{n>N, p \neq \pm n} n^2 \sup_{\mu \in \mathcal{C}_n} \left( \sum_{i_1, \ldots, i_t} B(\mu, m, i_1, \ldots, i_t, p) \right)^2 \leq C \|r\|^2 \rho_N^{2t} \quad \text{if} \quad m \in \{\pm n\},
\]
where \(C\) is an absolute constant and \(\rho_N\) is defined in \((2.30)\).
Proof. If \( t = 0 \), then, by (2.19), the sum \( \sigma \) in (2.39) has the form
\[
\sigma(m) = \sum_{n>N, p \neq \pm n} n^2 \sup_{\mu \in C_n} \frac{|W(m-p)|^2}{n^2|\mu - p|^2}, \quad m = \pm n.
\]
By (2.35), and since \( W(-k) = W(k) = |k|r(k) \),
\[
\sigma(m) \leq \sum_{n>N, p \neq \pm n} \frac{4|W(m-p)|^2}{|n^2 - p^2|^2} = \sum_{n>N, p \neq \pm n} 4|W(n-p)|^2
\]
\[
= 4 \sum_{n>N, p \neq \pm n} \frac{|r(n-p)|^2}{|n+p|^2} \leq C \rho_N^2
\]
by (3.3) in Lemma 5 So, (2.39) holds for \( t = 0 \).
If \( t = 1 \), then, by (2.18), the sum \( \sigma \) in (2.39) has the form
\[
\sum_{n>N, p \neq \pm n} n^2 \sup_{\mu \in C_n} \left| \sum_k \frac{W(m-k)W(k-p)}{n|\mu - k^2||\mu - p^2|} \right|^2, \quad m = \pm n.
\]
By (2.35), and since \( W(-k) = W(k) = |k|r(k) \), we have
\[
\sigma(\pm n) \leq \sum_{n>N, p \neq \pm n} \left( \sum_{k \neq \pm n} \frac{4|n-k||k-p|}{|n^2-k^2||n^2-p^2|} r(n-k)r(k-p) + \frac{4r(2n)r(n+p)}{|n-p|} \right)^2
\]
(to get this estimate for \( m = -n \) one may replace \( k \) and \( p \), respectively, by \(-k\) and \(-p\)). Since \((a+b)^2 \leq 2a^2 + 2b^2\), we have
\[
\sigma(\pm n) \leq 32\sigma_1 + 32\sigma_2,
\]
where
\[
\sigma_1 = \sum_{n>N, p \neq \pm n} \left( \sum_{k \neq \pm n} \frac{|k-p|}{|n+k||n^2-p^2|} r(n-k)r(k-p) \right)^2
\]
and
\[
\sigma_2 = \sum_{n>N, p \neq \pm n} \frac{|r(2n)|^2}{|n-p|^2} \leq \|r\|^2 \cdot \sum_{n>N, p \neq \pm n} \frac{|r(n+p)|^2}{|n-p|^2} \leq C\|r\|^2 \rho_N^2
\]
by (3.3) in Lemma 5 On the other hand, the identity,
\[
\frac{k-p}{(n+k)(n+p)} = \frac{1}{n+p} - \frac{1}{n+k}
\]
implies that
\[
\sigma_1 = \sum_{n>N, p \neq \pm n} \left( \sum_{k \neq \pm n} \frac{1}{|n+p|} - \frac{1}{n+k} \frac{1}{|n-p|} r(n-k)r(k-p) \right)^2 \leq 2\sigma_1' + 2\sigma_1'',
\]
where
\[
\sigma_1' = \sum_{n>N, p \neq \pm n} \frac{1}{|n^2-p^2|^2} \left( \sum_{k \neq \pm n} r(n-k)r(k-p) \right)^2 \leq \sum_{n>N, p \neq \pm n} \frac{1}{|n^2-p^2|^2} \|r\|^2 \leq C\|r\|^2 \frac{1}{N},
\]
and
\[
\sigma_1'' = \sum_{n>N, p \neq \pm n} \frac{1}{|n^2-p^2|^2} \left( \sum_{k \neq \pm n} \frac{1}{|n+k|} \frac{1}{|n-p|} r(n-k)r(k-p) \right)^2 \leq \sum_{n>N, p \neq \pm n} \frac{1}{|n^2-p^2|^2} \|r\|^2 \leq C\|r\|^2 \frac{1}{N},
\]
and
\[
\sigma''_1 = \sum_{n > N, p \neq \pm n} \left( \sum_{k \neq \pm n} \frac{r(n-k)r(k+p)}{|n+k||n-p|} \right)^2 \\
\leq \sum_{n > N} \left( \sum_{k \neq \pm n} \frac{|r(k-p)|^2}{|n+k|^2|n-p|^2} \right) \cdot \|r\|^2 \leq C\|r\|^2 \rho_N^2
\]
(by the Cauchy inequality and (3.31) in Lemma 5). So, the above inequalities imply (2.39) for \( t = 1 \).

Next we consider the case \( t > 1 \). Since \( |\mu - n^2| = n \) for \( \mu \in C_n \), by (2.18) the sum \( \sigma \) in (2.39) can be written in the form
\[
\sigma(m) = \sum_{n > N, p \neq \pm n} \sup_{\mu \in C_n} \left( \sum_{i_1, \ldots, i_t} W(m - i_1)W(i_1 - i_2) \cdots W(i_t - p) \right)^2, \quad m = \pm n.
\]
In view of (2.28), we have (with \( i = i_1, k = i_t \))
\[
\sigma(m) = \sum_{n > N, p \neq \pm n} \sup_{\mu \in C_n} \left( \sum_{i,k} \frac{W(m - i)W(i - k)}{|\mu - i^2|^{1/2}} \cdot H_{ik}(\mu) \cdot \frac{W(k - p)}{|\mu - k^2|^{1/2}|\mu - p^2|} \right)^2, \quad m = \pm n,
\]
where \( (H_{ik}(\mu)) \) is the matrix representation of the operator \( H(\mu) = (K_\mu W K_\mu)^t \) for \( \mu \in C_n, n > N \).

By (2.29),
\[
\|H(\mu)\|_{HS} = \left( \sum_{i,k} |H_{ik}(\mu)|^2 \right)^{1/2} \leq \|K_\mu W K_\mu\|_{HS}^{t-1} \leq \rho_N^{t-1}
\]
Therefore, the Cauchy inequality and (2.36) imply
\[
\sigma(\pm n) \leq 4\rho_N^{2(t-1)} \cdot \sum_{n > N, p \neq \pm n} \frac{1}{(n^2 - p^2)^2} \sup_{\mu \in C_n} \sum_{i,k} \frac{|W(n + i)|^2}{|\mu - i^2|} \cdot \frac{|W(k + p)|^2}{|\mu - k^2|}
\]
(one may see that the inequality holds for \( m = \pm n \) by replacing, if necessary, \( i \) by \( -i \) and \( p \) by \( -p \).

From (2.35) and \( W(k) = |k|r(k) \) it follows that
\[
\sup_{\mu \in C_n} \sum_i \frac{|W(n + i)|^2}{|\mu - i^2|} \leq 2 \sum_{i \neq \pm n} \frac{|n + i|^2}{|n - i|} |r(n + i)|^2 + 4n|r(2n)|^2
\]
and
\[
\sup_{\mu \in C_n} \sum_k \frac{|W(k + p)|^2}{|\mu - k^2|} \leq 2 \sum_{k \neq \pm n} \left[ \frac{|k + p|^2}{|n^2 - k^2|} |r(k + p)|^2 + \frac{|n + p|^2}{n} |r(n + p)|^2 + \frac{|n - p|^2}{n} |r(n - p)|^2 \right].
\]
Therefore, we have
\[
\sigma(\pm n) \leq 4\rho_N^{2(t-1)}(4\sigma_1 + 2\sigma_2 + 2\sigma_3 + 8\sigma_4 + 4\sigma_5 + 4\sigma_6),
\]
where
\[
\sigma_1 = \sum_{n > N, p \neq \pm n} \frac{1}{(n^2 - p^2)^2} \sum_{i,k \neq \pm n} \frac{|n + i|^2|p + k|^2}{|n - i|^2|n^2 - k^2|} |r(n + i)|^2|r(p + k)|^2 \leq C\|r\|^2 \rho_N^2
\]
\[ \sigma_2 = \sum_{n > N, p \neq \pm n} \frac{|n + p|^2}{|n^2 - p^2|^2} |r(n + p)|^2 \sum_{i \neq \pm n} \frac{|n + i|}{n|n - i|} |r(n + i)|^2 \leq \sum_{n > N, p \neq \pm n} \frac{|r(n + p)|^2}{|n - p|^2} \leq C\|r\|^2 \rho_N^2 \]

(since \(\frac{|n + i|}{n|n - i|} = \frac{1}{n} - \frac{1}{2n} \leq 2\), and by (3.3) in Lemma 6):

\[ \sigma_3 = \sum_{n > N, p \neq \pm n} \frac{|n - p|^2}{|n^2 - p^2|^2} |r(n - p)|^2 \sum_{i \neq \pm n} \frac{|n + i|}{n|n - i|} |r(n + i)|^2 = \sigma_2 \leq C\|r\|^2 \rho_N^2 \]

(the change \(p \rightarrow -p\) shows that \(\sigma_3 = \sigma_2\));

\[ \sigma_4 = \sum_{n > N, p \neq \pm n} \frac{n}{|n^2 - p^2|^2} |r(2n)|^2 \sum_{k \neq n} \frac{|k + p|^2}{|n^2 - k^2|^2} |r(k + p)|^2 \leq C\|r\|^2 \rho_N^2 \]

(by Lemma 7)

\[ \sigma_5 = \sum_{n > N, p \neq \pm n} \frac{|n + p|^2}{|n^2 - p^2|^2} |r(2n)|^2 |r(n + p)|^2 \leq \sum_{n > N} \frac{|r(2n)|^2}{\|r\|^2} \sum_{p \neq \pm n} |r(n + p)|^2 \leq C\|r\|^2 \rho_N^2 \]

and

\[ \sigma_6 = \sum_{n > N, p \neq \pm n} \frac{|n - p|^2}{|n^2 - p^2|^2} |r(2n)|^2 |r(n - p)|^2 = \sigma_5 \leq C\|r\|^2 \rho_N^2 \]

(the change \(p \rightarrow -p\) shows that \(\sigma_6 = \sigma_5\)). Hence

\[ \sigma(\pm n) \leq C\|r\|^2 \rho_N^{2t}. \]

which completes the proof of (2.39).

Now, by (2.38) and (2.39) in Lemma 6 we get

\[ (2.40) \quad A_1(t, s) \leq C\|r\|^2 \rho_N^{t+s}, \quad t + s > 0, \]

where \(C\) is an absolute constant.

Estimates for \(A_2(t, s)\). Since \(m\) and \(p\) play symmetric roles, the same argument that was used to estimate \(A_1(t, s)\) yields

\[ (2.41) \quad A_2(t, s) \leq C\|r\|^2 \rho_N^{t+s}, \quad t + s > 0, \]

where \(C\) is an absolute constant.

Estimates for \(A_3(t, s)\). In view of (2.25) and the definition of the set \(I^*\) (see the text after (2.16)), \(I^*_t\) is the set of \(t + s + 2\)-tuples of indices \((m, i_1, \ldots, i_t, p, j_1, \ldots, j_s)\) such that \(t \geq 1\), \(s \geq 1\), and

\[ m, p \neq \pm n, \quad \{i_1, \ldots, i_t\} \cap \{\pm n\} \neq \emptyset, \quad \{j_1, \ldots, j_s\} \cap \{\pm n\} \neq \emptyset. \]

Therefore, by (2.27), we have

\[ (2.42) \quad A_3(t, s) \leq \sum_{n > N} n^2 \sup_{m, p \neq \pm n} \sup_{\mu \in C_n} \sum_{i_1, \ldots, i_t} B(\mu, m, i_1, \ldots, i_t, p) \sup_{\lambda \in C_n} \sum_{j_1, \ldots, j_s} B(\lambda, p, j_1, \ldots, j_s, m), \]
where * means that at least one of the summation indices is equal to ±n. The Cauchy inequality implies

\[
(2.43) \quad A_3(t, s) \leq \left( \sum_{n>N} n^2 \sum_{m, p \neq \pm n} \sup_{\mu \in \mathbb{C}_n} \left| \sum_{i_1, \ldots, i_t} B(\mu, m, i_1, \ldots, i_t, p) \right|^2 \right)^{1/2} \times \left( \sum_{n>N} n^2 \sum_{m, p \neq \pm n} \sup_{\mu \in \mathbb{C}_n} \left| \sum_{j_1, \ldots, j_v} B(\lambda, p, j_1, \ldots, j_v, m) \right|^2 \right)^{1/2}.
\]

Lemma 4. In the above notations,

\[
(2.44) \quad \sum_{n>N} n^2 \sup_{\mu \in \mathbb{C}_n} \left( \sum_{i_1, \ldots, i_t} B(\mu, m, i_1, \ldots, i_t, p) \right)^2 \leq C\|r\|^4 \rho_N^{2(t-1)},
\]

where C is an absolute constant and \(\rho_N\) is defined in (2.30).

Proof. Let \(\tau \leq t\) be the least integer such that \(i_\tau = \pm n\). Then, by (2.18) or (2.19), and since \(|\mu - n^2| = n\) for \(\mu \in \mathbb{C}_n\),

\[
B(\mu, m, i_1, \ldots, i_{\tau-1}, \pm n, i_{\tau+1}, \ldots, i_t, p) = nB(\mu, m, i_1, \ldots, i_{\tau-1}, \pm n) \cdot B(\pm n, i_{\tau+1}, \ldots, i_t, p).
\]

Therefore, if \(\sigma\) denotes the sum in (2.44), we have

\[
\sigma \leq \sum_{\tau=1}^t \sum_{\tilde{n}=\pm n} \sum_{n>N} n^2 \sup_{\mu \in \mathbb{C}_n} \left| \sum_{i_1, \ldots, i_{\tau-1}} B(\mu, m, i_1, \ldots, i_{\tau-1}, \tilde{n}) \right|^2 \times \sum_{p \neq \pm n} \sup_{\mu \in \mathbb{C}_n} \left| \sum_{i_{\tau+1}, \ldots, i_t} B(\mu, \tilde{n}, i_{\tau+1}, \ldots, i_t, p) \right|^2.
\]

On the other hand, by Lemma 3

\[
n^2 \sum_{p \neq \pm n} \sup_{\mu \in \mathbb{C}_n} \left| \sum_{i_{\tau+1}, \ldots, i_t} B(\mu, \tilde{n}, i_{\tau+1}, \ldots, i_t, p) \right|^2 \leq C\|r\|^2 \rho_N^{2(t-\tau)}, \quad n > N.
\]

Thus, we have

\[
\sigma \leq C\|r\|^2 \sum_{\tau=1}^t \rho_N^{2(t-\tau)} \sum_{\tilde{n}=\pm n} \sum_{n>N} n^2 \sup_{\mu \in \mathbb{C}_n} \left| \sum_{i_1, \ldots, i_{\tau-1}} B(\mu, m, i_1, \ldots, i_{\tau-1}, \tilde{n}) \right|^2.
\]

Again by Lemma 3

\[
\sum_{n>N} n^2 \sup_{\mu \in \mathbb{C}_n} \left| \sum_{i_1, \ldots, i_{\tau-1}} B(\mu, m, i_1, \ldots, i_{\tau-1}, \tilde{n}) \right|^2 \leq C\|r\|^2 \rho_N^{2(\tau-1)}
\]

(one may apply Lemma 3 because \(B(\mu, m, i_1, \ldots, i_{\tau-1}, \tilde{n}) = B(\mu, \tilde{n}, j_1, \ldots, j_{\tau-1}, m)\)) if \(j_1 = i_{\tau-1}, \ldots, j_{\tau-1} = i_1\). Hence,

\[
\sigma \leq C\|r\|^4 \sum_{\tau=1}^t \rho_N^{2(t-1)} = C\|r\|^4 \rho_N^{2(t-1)},
\]
which completes the proof. \hfill \Box

By (2.43) and (2.44) (since the roles of \(m\) and \(p\) are symmetric in (2.43)), we get

\[
A_3(t, s) \leq C \sqrt{ts} \|r\|_4 (t+s-2) \leq C(t+s) \|r\|_4 (t+s-2).
\]

Now we are ready to complete the proof of Theorem 1. Choose \(N\) so large that \(\rho N < 1\). Then, from (2.12), (2.26), (2.36), (2.40), (2.41) and (2.45) it follows that

\[
\sum_{t,s=0}^{\infty} A(t, s) < \infty,
\]

which, in view of (2.6), yields (2.1). \hfill \Box

So, Theorem 1 is proven subject to Lemmas 5, 6 and 7 in the next section.

3. TECHNICAL LEMMAS

Throughout this section we use that

\[
\sum_{n>N} 1/n^2 < \sum_{n>N} \left( \frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{N}, \quad N \geq 1.
\]

and

\[
\sum_{p \neq \pm n} \frac{1}{(n^2-p^2)^2} < \frac{4}{n^2}, \quad n \geq 1
\]

since

\[
\frac{1}{(n^2-p^2)^2} = \frac{1}{4n^2} \left( \frac{1}{n-p} + \frac{1}{n+p} \right)^2 \leq \frac{1}{2n^2} \left( \frac{1}{(n-p)^2} + \frac{1}{(n+p)^2} \right),
\]

the sum in (3.2) does not exceed

\[
\frac{1}{2n^2} \left( \sum_{p \neq \pm n} \frac{1}{(n-p)^2} + \sum_{p \neq \pm n} \frac{1}{(n+p)^2} \right) \leq \frac{1}{2n^2} \cdot \frac{2\pi^2}{3} < \frac{4}{n^2}
\]

because \(\pi^2 < 10\).

**Lemma 5.** If \(r = (r(k)) \in \ell^2(2\mathbb{Z})\) (or \(r = (r(k)) \in \ell^2(\mathbb{Z})\)), then

\[
\sum_{n>N, k \neq n} \frac{|r(n+k)|^2}{|n-k|^2} \leq C \left( \frac{\|r\|^2}{N} + (\mathcal{E}_N(r))^2 \right),
\]

\[
\sum_{n>N, k \neq n} \frac{|n+k|^2}{|n-k|^2} \frac{|r(n+k)|^2}{|n-k|^2} \leq C \left( \frac{\|r\|^2}{N} + (\mathcal{E}_N(r))^2 \right),
\]

and

\[
\sum_{n>N, p, k \neq n} \frac{|r(p+k)|^2}{|n-p|^2|n-k|^2} \leq C \left( \frac{\|r\|^2}{N} + (\mathcal{E}_N(r))^2 \right),
\]

where \(n \in \mathbb{N}, k, p \in n+2\mathbb{Z}\) (or, respectively, \(k, p \in \mathbb{Z}\)) and \(C\) is an absolute constant.
Proof. Indeed, we have (with \( \tilde{k} = n + k \), and using (3.1))
\[
\sum_{n>N, k \neq n} \frac{|r(n + k)|^2}{|n - k|^2} = \sum_{n>N, k < 0} \frac{|r(n + k)|^2}{|n - k|^2} + \sum_{n>N} \sum_{0 \leq k \neq n} \frac{|r(n + k)|^2}{|n - k|^2}
\]
\[
\leq \sum_{n>N} \frac{1}{n^2} \sum_{\tilde{k}} |r(\tilde{k})|^2 + \sum_{\tilde{k} \geq N} |r(\tilde{k})|^2 \sum_{n \neq \tilde{k}/2} \frac{1}{|2n - \tilde{k}|^2} \leq C \left( \frac{||r||^2}{N} + (E_N(r))^2 \right).
\]

Next we prove (3.4). By the identity
\[
\frac{n + k}{n(n - k)} = \frac{1}{n - k} - \frac{1}{2n},
\]
we get (using the inequality \( ab \leq (a^2 + b^2)/2 \))
\[
\sum_{n>N, k \neq n} \frac{|n + k|^2}{n^2|n - k|^2} |r(n + k)|^2 = \sum_{n>N, k \neq n} \left( \frac{1}{n - k} - \frac{1}{2n} \right)^2 |r(n + k)|^2
\]
\[
\leq \frac{1}{2} \sum_{n>N, k \neq n} \frac{|r(n + k)|^2}{|n - k|^2} + \frac{1}{2} \sum_{n>N} \frac{1}{4n^2} \sum_k |r(n + k)|^2.
\]
In view of (3.1) and (3.3), from here (3.4) follows.

In order to prove (3.5), we set \( \tilde{p} = n - p \) and \( \tilde{k} = n - k \). Then
\[
\sum_{n>N, p, k \neq n} \frac{|r(p + k)|^2}{|n - p|^2|n - k|^2} = \sum_{\tilde{p}, \tilde{k} \neq 0} \frac{1}{\tilde{p}^2 \tilde{k}^2} \sum_{n>N} |r(2n - \tilde{p} - \tilde{k})|^2
\]
\[
\leq \sum_{0 < |\tilde{p}, |\tilde{k}| \leq N/2} \frac{1}{\tilde{p}^2 \tilde{k}^2} \sum_{n>N} |r(2n - \tilde{p} - \tilde{k})|^2 + \sum_{|\tilde{p}| > N/2} \sum_{|\tilde{p}| \neq 0} \sum_{|\tilde{k}| > N/2} \cdots \sum \cdots
\]
\[
\leq C(E_N(r))^2 + \frac{C}{N} ||r||^2 + \frac{C}{N} ||r||^2,
\]
which completes the proof. \( \square \)

Lemma 6. Suppose that \( r = (r(k)) \in \ell^2(2\mathbb{Z}) \) (or \( r = (r(k)) \in \ell^2(\mathbb{Z}). \) Then
(3.6)
\[
\sum_{n>N, p \neq \pm n} \frac{1}{|n^2 - p|^2} \sum_{i, k \neq \pm n} \left| n + i \right| \left| k + p \right|^2 \frac{|r(n + i)|^2 |r(n + k)|^2 \leq C ||r||^2 \left( \frac{||r||^2}{N} + (E_N(r))^2 \right),
\]
where \( C \) is an absolute constant.

Proof. Let \( \Sigma \) be the sum in (3.5). Taking into account that
\[
\frac{k + p}{(n - p)(n + k)} = \frac{1}{n - p} - \frac{1}{n + k}, \quad \frac{k + p}{(n + p)(n - k)} = \frac{1}{n - k} - \frac{1}{n + p}
\]
and \( (n + i)/(n - i) = 2n/(n - i) - 1 \), we get
\[
\Sigma \leq \sum \left| n^2 - p^2 \right| \left| \frac{1}{n - p} - \frac{1}{n + k} \right| \left| \frac{1}{n - k} - \frac{1}{n + p} \right| \left| \frac{2n}{n - i} - 1 \right| \left| r(n + i) \right|^2 |r(n + k)|^2
\]
Therefore,
(3.7)
\[
\Sigma \leq \sum_{\nu=1}^{8} \Sigma_{\nu},
\]
From here it follows, in view of (3.1) and (3.3), that we have

\[ \Sigma_1 = \sum \frac{1}{|n^2 - p^2|^2} \left| \frac{2n}{n-i} \right| r(n+i)^2 |r(k+p)|^2, \]

(3.8)

\[ \Sigma_2 = \sum \frac{1}{|n^2 - p^2|^2} \left| \frac{2n}{n-k^2} \right| r(n+i)^2 |r(k+p)|^2, \]

(3.9)

\[ \Sigma_3 = \sum \frac{1}{|n^2 - p^2|^2} \left| \frac{1}{|n-p|^2} \right| r(n+i)^2 |r(k+p)|^2, \]

(3.10)

\[ \Sigma_4 = \sum \frac{1}{|n^2 - p^2|^2} \left| \frac{1}{|n-k|^2} \right| r(n+i)^2 |r(k+p)|^2, \]

(3.11)

\[ \Sigma_5 = \sum \frac{1}{|n^2 - p^2|^2} |r(n+i)|^2 |r(k+p)|^2, \]

(3.12)

\[ \Sigma_6 = \sum \frac{1}{|n^2 - p^2|^2} \left| \frac{1}{|n^2-k^2|} \right| r(n+i)^2 |r(k+p)|^2, \]

(3.13)

\[ \Sigma_7 = \sum \frac{1}{|n^2 - p^2|^2} \left| \frac{1}{|n-p|^2} \right| r(n+i)^2 |r(k+p)|^2, \]

(3.14)

\[ \Sigma_8 = \sum \frac{1}{|n^2 - p^2|^2} \left| \frac{1}{|n+p|^2} \right| r(n+i)^2 |r(k+p)|^2, \]

(3.15)

where the summation is over \( n > N \) and \( i, k, p \neq \pm n \).

After summation over \( k \) in (3.8) we get, in view of (3.2),

\[
\Sigma_1 \leq \|r\|^2 \sum_{n>N, i \neq \pm n} \frac{2n}{n-i} |r(n+i)|^2 \sum_{p \neq \pm n} \frac{1}{|n^2 - p^2|^2} \leq C \|r\|^2 \sum_{n>N, i \neq \pm n} \frac{1}{n-i} \frac{1}{n} |r(n+i)|^2
\]

\[
\leq C \|r\|^2 \left( \sum_{n>N, i \neq \pm n} \frac{|r(n+i)|^2}{|n-i|^2} + \sum_{n>N, i \neq \pm n} \frac{|r(n+i)|^2}{n^2} \right).
\]

From here it follows, in view of (3.1) and (3.3), that

\[ \Sigma_1 \leq C_1 \|r\|^2 \left( \frac{\|r\|^2}{N} + (\mathcal{E}_N(r))^2 \right). \]

(3.16)

By the inequality \( 2ab \leq a^2 + b^2 \), considered with \( a = 1/|n^2 - p^2| \) and \( b = 1/|n^2 - k^2| \),

one can easily see that

\[ \Sigma_2 \leq \Sigma_1, \]

(3.17)

Since

\[
\frac{2n}{n^2 - p^2} = \frac{1}{n-p} + \frac{1}{n+p},
\]

we have

\[ \Sigma_3 \leq \Sigma'_3 + \Sigma''_3, \]

where

\[ \Sigma'_3 = \sum \frac{1}{|n-p|^2} \left| \frac{1}{|n-k|^2} \right| r(n+i)^2 |r(k+p)|^2. \]
and
\[
\sum'' = \sum \frac{1}{n^2 - p^2} \frac{1}{|n - k|} \frac{1}{|n - i|} |r(n + i)|^2 |r(k + p)|^2.
\]

The inequality \(2ab \leq a^2 + b^2\), considered with \(a = 1/|n - k|\) and \(b = 1/|n - i|\), yields
\[
\sum'' \leq \frac{1}{2} \sum_{n>N; p, k \neq n} \frac{|r(k + p)|^2}{|n - p|^2 |n - k|^2} \sum_i |r(n + i)|^2 + \frac{1}{2} \sum_{n>N; i \neq n} \frac{|r(n + i)|^2}{|n - i|^2} \sum_{p \neq n} \frac{1}{|n - p|^2} \sum_k |r(k + p)|^2 \leq C \left( \frac{\|r\|^2}{N} + (E_N(r))^2 \right) \|r\|^2.
\]
(by (3.18) and (3.5) in Lemma 5). In an analogous way, by the Cauchy inequality and (3.3) and (3.5) in Lemma 5 we get
\[
\sum'' \leq \left( \sum_{n>N; p, k \neq n} \frac{|r(k + p)|^2}{|n - p|^2 |n - k|^2} \sum_i |r(n + i)|^2 \right)^{1/2} \times \left( \sum_{n>N; i \neq n} \frac{|r(n + i)|^2}{|n - i|^2} \sum_{p \neq n} \frac{1}{|n - p|^2} \sum_k |r(k + p)|^2 \right)^{1/2} \leq C \left( \frac{\|r\|^2}{N} + (E_N(r))^2 \right) \|r\|^2.
\]

Thus,
\[
(3.18) \quad \sum_3 \leq C \left( \frac{\|r\|^2}{N} + (E_N(r))^2 \right) \|r\|^2.
\]

Next we estimate \(\sum_7\). After summation over \(i\) we get
\[
\sum_7 = \|r\|^2 \cdot \sum_{n>N; p \neq n} \frac{1}{n^2 - p^2} \frac{1}{|n - p|} \frac{1}{|n - k|} |r(p + k)|^2.
\]
Now the Cauchy inequality implies
\[
\sum_7 \leq \|r\|^2 \left( \sum_{n>N; p \neq n} \frac{1}{n^2 - p^2} \sum_k |r(p + k)|^2 \right)^{1/2} \left( \sum_{n>N; p, k \neq n} \frac{|r(p + k)|^2}{|n - k|^2 |n - p|^2} \right)^{1/2}.
\]
Therefore, by (3.18), (3.3), and (3.5) in Lemma 5
\[
(3.19) \quad \sum_7 \leq C \left( \frac{\|r\|^2}{N} + (E_N(r))^2 \right) \|r\|^2.
\]

To estimate \(\sum_4\) and \(\sum_8\), notice that if \(|r(-k)| = |r(k)| \forall k\) (which we can always assume because otherwise one may replace \((r(k))\) by \((|r(k)| + |r(-k)|)\)), then the change of indices \(p \rightarrow -p\) and \(k \rightarrow -k\) leads to \(\sum_4 = \sum_3\) and \(\sum_8 = \sum_7\). Thus
\[
(3.20) \quad \sum_4 \leq C \left( \frac{\|r\|^2}{N} + (E_N(r))^2 \right) \|r\|^2, \quad \sum_8 \leq C \left( \frac{\|r\|^2}{N} + (E_N(r))^2 \right) \|r\|^2.
\]

By the inequality \(2ab \leq a^2 + b^2\), considered with \(a = 1/|n^2 - p^2|\) and \(b = 1/|n^2 - k^2|\), one can easily see that
\[
(3.21) \quad \sum_6 \leq \sum_5.
\]
Finally, by (3.18) and (3.20), we get
\[
(3.22) \quad \sum_5 = \sum_{n>N; p \neq n} \frac{1}{n^2 - p^2} \sum_k |r(k + p)|^2 \sum_i |r(n + i)|^2 \leq C \frac{\|r\|^4}{N}.
\]
Now, (3.7)–(3.22) imply (3.6), which completes the proof. □

Lemma 7. In the above notations, we have

\[
\sum_{n > N, p \neq \pm n} \frac{n}{|n^2 - p^2|} |r(2n)|^2 \sum_{k \neq \pm n} \frac{|k + p|^2}{|n^2 - k^2|} |r(k + p)|^2 \leq C\|r\|^2 (E_N(r))^2.
\]

Proof. Let \( \Sigma \) be the sum in (3.23). The identities

\[
\frac{k + p}{(n - p)(n + k)} = \frac{1}{n - p} - \frac{1}{n + k}, \quad \frac{k + p}{(n + p)(n - k)} = \frac{1}{n - k} - \frac{1}{n + p},
\]

and the inequality \( n \leq |n^2 - p^2|, \ p \neq \pm n, \) imply that

\[
\Sigma \leq \sum_{n > N} \sum_{k, p \neq \pm n} \left| \frac{1}{n - p} - \frac{1}{n + k} \right| \left| \frac{1}{n - k} - \frac{1}{n + p} \right| |r(2n)|^2 |r(k + p)|^2 \leq \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4,
\]

where

\[
\Sigma_1 = \sum_{n > N} |r(2n)|^2 \sum_{p \neq \pm n} \frac{1}{|n^2 - p^2|} \sum_{k \neq \pm n} |r(k + p)|^2 \leq C(E_N(r))^2 \|r\|^2;
\]

\[
\Sigma_2 = \sum_{n > N} |r(2n)|^2 \sum_{k \neq \pm n} \frac{1}{|n^2 - k^2|} \sum_{p \neq \pm n} |r(k + p)|^2 \leq C(E_N(r))^2 \|r\|^2;
\]

\[
\Sigma_3 = \sum_{n > N} \sum_{k, p \neq \pm n} \frac{1}{|n - p|} \frac{1}{|n - k|} |r(2n)|^2 |r(k + p)|^2
\]

and

\[
\Sigma_4 = \sum_{n > N} \sum_{k, p \neq \pm n} \frac{1}{|n + p|} \frac{1}{|n + k|} |r(2n)|^2 |r(k + p)|^2.
\]

The inequality \( 2ab \leq a^2 + b^2 \) yields \( \Sigma_3 \leq \Sigma_3'' + \Sigma_3''' \) with

\[
\Sigma_3'' = \sum_{n > N} |r(2n)|^2 \sum_{p \neq \pm n} \frac{1}{|n - p|^2} \sum_{k \neq \pm n} |r(k + p)|^2 \leq C(E_N(r))^2 \|r\|^2
\]

and

\[
\Sigma_3''' = \sum_{n > N} |r(2n)|^2 \sum_{k \neq \pm n} \frac{1}{|n - k|^2} \sum_{p \neq \pm n} |r(k + p)|^2 \leq C(E_N(r))^2 \|r\|^2.
\]

Therefore,

\[
\Sigma_3 \leq C \|r\|^2 (E_N(r))^2.
\]

The same argument shows that

\[
\Sigma_4 \leq C \|r\|^2 (E_N(r))^2,
\]

which completes the proof. □
4. UNCONDITIONAL CONVERGENCE OF SPECTRAL DECOMPOSITIONS

1. To be accurate we should mention that in Formula (1.8) the first vector-term $f^N$ is defined as $P^N f$, where (see [9], (5.40))

$$P^N = \frac{1}{2\pi i} \int_{\partial R_N} (z - L_{bc})^{-1} dz,$$

and $R_N$ is the rectangle

$$R_N = \{ z \in \mathbb{C} : -N < Rez < N^2 + N, |Imz| < N \}.$$

The Bari–Markus Theorem ([11], [19], Section 5.2) gives us the claim (1.8) if the following hypotheses hold:

- (a) $\sum_{n>N} \| P_n - P_0^0 \|_{L^2 \to L^2} < \infty$ for some $N$,
- (b) $\text{Codim} H_m = \text{Codim} H_0^m$ for sufficiently large $m$,

where

$$H_m = \text{Lin Span}\{\text{Ran}P_k, k \geq m\}, \quad H_0^m = \text{Lin Span}\{\text{Ran}P_0^0 k \geq m\}.$$

Theorem 1 implies (a). On the other hand (b) is proven in details in [9], see Theorem 21, in particular, (5.54) and (5.56). Therefore we come to the following.

**Proposition 8.** Under the conditions of Theorem 1, if $N$ is sufficiently large, then for any $f \in L^2(I)$

$$f = P^N f + \sum_{n>N} P_n f;$$

these series converge unconditionally in $L^2(I)$.

This statement has been given in [17], Section 2.4. Our alternative proof is based on Fourier method which has been justified in the analysis of Hill operators with $H^{-1}$ potentials in our paper [8] (see [7] as well).

2. In this context it is worth to mention a version of the Bari–Markus theorem in the case of 1D periodic Dirac operators

$$L y = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + V(x) y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where

$$V(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}, \quad V(x + \pi) = V(x), \quad P, Q \in L^2([0, \pi]).$$

For Riesz projections (in the case of $bc = Per^{\pm}$ and $Dir$ - see definitions and details in [16] or [6], Sect. 1.1) Theorem 8.8 in [16] or Theorem 4 in [15] claims the following:

**Proposition 9.** Let $\Omega = (\Omega(k)), k \in \mathbb{Z}$, be a weight such that

$$\sum \frac{1}{(\Omega(k))^2} < \infty.$$
If $V \in H(\Omega)$, then

$$F = P^N F + \sum_{|n| > N} P_n F \quad \forall F \in L^2;$$

these spectral decompositions converge unconditionally.

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