On T violation in non-standard neutrino oscillation scenarios

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We discuss time reversal (T) violation in neutrino oscillations in generic new physics scenarios. A general parameterization is adopted to describe flavour evolution, which captures a wide range of new physics effects, including non-standard neutrino interactions, non-unitarity, and sterile neutrinos in a model-independent way. In this framework, we discuss general properties of time reversal in the context of long-baseline neutrino experiments. Special attention is given to fundamental versus environmental T violation in the presence of generic new physics. We point out that T violation in the disappearance channel requires new physics which modifies flavour mixing at neutrino production and detection. We use time-dependent perturbation theory to study the effect of non-constant matter density along the neutrino path, and quantify the effects for the well studied baselines of the DUNE, T2HK, and T2HKK projects. The material presented here provides the phenomenological background for the model-independent test of T violation proposed by us in Ref. \cite{1}.

CONTENTS

I. Introduction 1

II. Model-independent description of flavour evolution
   II.1. Transition probabilities 2

III. Properties under the T transformation
   III.1. Constant matter potential 4
   III.2. Non-constant matter potential 4

IV. Comments on T asymmetries and the disappearance channel 5

V. Estimation of non-constant density corrections
   V.1. Different average densities 6
   V.2. Non-constant density 7

VI. Conclusions 8

References 9

I. INTRODUCTION

The search for CP violation is a central goal for current and upcoming long-baseline neutrino oscillation experiments. Thanks to the CPT theorem, fundamental CP violation is closely tied to T violation. Early works on this topic are Refs. \cite{2–4}. Fundamental CP or T violation are related to complex couplings in the Lagrangian. Indeed, in the standard three-flavour scenario \cite{5}, it is described by a single complex phase in the lepton mixing matrix \cite{6–8}, the so-called Dirac phase \cite{9}. However, actual neutrino oscillation experiments involve neutrino passage through matter, and hence observable transition probabilities are subject to matter effects \cite{10}. Since a background of normal matter leads to violation of CPT in the neutrino flavour evolution \cite{11}, fundamental CP or T violation is not directly observable, see Refs. \cite{12, 13} for a recent discussion. In this respect, T violation has an advantage over CP violation for the following reason. Since the matter effect is different between neutrinos and antineutrinos, environmental CP violation is typically large and difficult to disentangle from fundamental CP violation. In contrast, it is well known that a symmetric matter density profile (symmetric between neutrino source and detector) does not introduce environmental T violation if the fundamental theory is T invariant, see for instance Refs. \cite{14, 15}. On the other hand, T violation itself is again difficult to observe experimentally, since it formally corresponds to exchanging neutrino flavours of neutrino source and detector. There is extensive literature on T violation in neutrino oscillations, see Refs. \cite{2, 13–22} for an incomplete list.

The usual search for CP violation is highly model-dependent. It relies on the standard unitary three-flavour paradigm, implying the absence of any new physics in neutrino interactions, mixing, and propagation. In this restricted framework, a parametric fit to the available data is performed in terms of the standard mass-squared differences \(\Delta m^2_{21}, \Delta m^2_{31}\), mixing angles \(\theta_{12}, \theta_{23}, \theta_{13}\), and the complex phase \(\delta\). “Discovery of CP violation” is usually identified with the situation when the fit favours values of \(\delta = 0\) and \(\pi\) at a certain confidence level. Indeed, within this approach, current results from the T2K \cite{23} and NOvA \cite{24} long-baseline experiments, combined with the global neutrino oscillation data, show already an indication for a preferred range of \(\delta\) \cite{25–27}.

In Ref. \cite{1}, we have proposed a method, with the goal to address several limitations outlined above and to search for fundamental T violation in the neutrino sector in a more model-independent way. In order to achieve this goal, we have introduced two main ingredients:

(i) We prosed a rather general parameterization of neutrino evolution, to describe the flavour system more model-independently, and

(ii) we presented a potentially realistic way to search
for fundamental T violation in long-baseline experiments.

Regarding (i), our general parameterization allows for effects of non-standard interactions in neutrino source and detection as well as arbitrary matter effect. Mixing can be non-unitary, and therefore the presence of sterile neutrinos is allowed, as long as they do not introduce additional oscillation frequencies (i.e., we restrict to two independent oscillation frequencies). We will review this parameterization in detail in sec. II below.

The main idea with respect to item (ii) is the following: we consider the oscillation probabilities within the general framework at different baselines but at the same energy. The reason for this assumption is that we want to be agnostic about the energy dependence of the new physics. Hence we need to combine data from long-baseline experiments at different baselines L at the same neutrino energy. Then we check if the data requires T-odd (or equivalently L-odd) terms in the transition probability, based on the model-independent parameterization. We have shown in Ref. [1] that this test potentially can be performed already with data from 3 different long-baseline experiments (plus data from a near detector). This opens the possibility to apply the proposed test with actually planned and proposed experiments, such as DUNE (L = 1300 km) [28, 29], T2HK (L = 295 km) [30], T2HK with a second detector in Korea, T2HKK (L = 1100 km) [31], and a long-baseline experiment at the European Spallation Source, ESS/eSB (L = 540 km) [32, 33]. The crucial requirement is the availability of measurements at the first and the second oscillation maxima (at the same energy), with sufficiently good energy reconstruction. Preliminary sensitivity estimates have been performed in Ref. [1].

The goals of the present paper are the following. We provide a more in-depth discussion of T violation, allowing for the general non-standard physics described above to establish the theoretical basis for the test proposed in Ref. [1]. We show explicitly that several results known for the standard framework carry over to the new-physics case considered here. For instance, we prove that any non-standard matter effect does not introduce environmental T violation if the fundamental theory is T conserving, as long as the matter density profile is symmetric. Special care is given to non-standard mixing effects in source and detector. We give a careful definition of the time reversal symmetry and discuss its effect in non-standard mixing scenarios. Along this way we establish the basic assumptions of the test in [1].

In Ref. [1], we have formulated the test by assuming a constant matter density along the neutrino path, and that the matter density is the same for all experiments. These assumptions are only approximately valid for the experiments under consideration. Therefore, in the present article we provide a quantitative estimate for corrections induced by realistic matter density profiles, based on the detailed investigations for the T2HKK and DUNE baselines from Refs. [34, 35]. In general, our approach is based on a perturbation ansatz, using that both the new physics as well as the non-constant density effects are small perturbations to the standard three-flavour and constant matter case. The methods developed below allow for a straightforward correction of the T violation test with respect to non-constant density.

The outline of the remainder of the paper is as follows. In section II we review the model-independent parameterization of the neutrino flavour evolution and derive the transition amplitudes and probabilities, taking into account non-constant matter densities as a small perturbation effect. In section III we consider the time reversal transformation within this model-independent setting and discuss which properties known for the standard oscillation case still hold in our model-independent framework and which not. In section IV we provide some further discussion and comment on T violation in the disappearance channel within new physics scenarios. In section V we provide quantitative estimates of non-constant matter density profiles for T2HK, T2HKK and DUNE. We conclude in section VI.

II. MODEL-INDEPENDENT DESCRIPTION OF FLAVOUR EVOLUTION

We assume that propagation of the three SM neutrino states is described by a hermitian Hamiltonian H(E, x), which depends on neutrino energy E and in general on the matter density at the position x along the neutrino path. We follow the usual approximation in describing neutrino evolution by setting, x = t (in units where the speed of light is unity), i.e., localized neutrino wave packets propagating with the speed of light. Therefore, the space dependence of the Hamiltonian effectively becomes a time dependence. In the following we will use x and t interchangeably. The evolution of the flavour state |ψ⟩ is described by the equation

\[ i\partial_t|ψ⟩ = H(t)|ψ⟩, \]

where here and in the following we suppress the energy dependence.

We consider neutrinos propagating from a source at position x_s at time t_s to a detector at position x_d, arriving at time t_d, with x_d − x_s = t_d − t_s = L. Let us define

\[ H(t) = H_{\text{vac}} + V_{\text{tot}}(t) = H_0 + V(t), \]
\[ V_{\text{tot}}(t) = V_0 + V(t), \]
\[ H_0 = H_{\text{vac}} + V_0, \]

where H_0 contains contributions from the vacuum Hamiltonian as well as the average matter potential V_0 and is time/position independent, whereas V(t) corresponds to the matter potential due to the varying matter density, with \( \int_{x_s}^{x_d} dx V(x) = 0 \). For our purposes (long-baseline experiments) we assume that V(t) is a small
perturbation, i.e., that the matter density is roughly constant along the neutrino path. This is a good approximation for experiments with baselines less than several 1000 km [34–37]. We will quantify this in section V.

Let us diagonalize the position-independent part \( H_0 \) by \( H_0 = W A W^\dagger \), with \( W \) being a unitary matrix and \( \lambda = (\lambda_i) \) a diagonal matrix of the real eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \) of \( H_0 \). Both \( W \) and \( \lambda \) depend on the neutrino energy and they will be different for neutrinos and antineutrinos due to the matter effect as well as fundamental CP violation. The energy eigenstates \( |\nu_i\rangle \) fulfill

\[
H_0 |\nu_i\rangle = \lambda_i |\nu_i\rangle.
\]

(5)

We allow for arbitrary (non-unitary) mixing of the energy eigenstates \( |\nu_i\rangle \) with the flavour states \( |\nu_s,d\rangle \) relevant for detection and production,

\[
|\nu^{s,d}_i\rangle = \sum_{i=1}^3 (N^{s,d}_{ai})^* |\nu_i\rangle,
\]

(6)

where * denotes complex conjugation. We make no specific assumption on the coefficients \( N^{s,d}_{ai} \). They can include effects of heavy sterile neutrinos as well as non-standard interactions. Note that generically the unitary matrix \( W \) diagonalizing the Hamiltonian will contribute to these coefficients; in specific models they may be related and the new physics entering in the coefficients \( N^{s,d}_{ai} \) and will also induce non-standard contributions to the Hamiltonian \( H(t) \). Some examples for specific models are non-unitary mixing [38, 39], non-standard neutrino interactions [40–43], or the presence of sterile neutrinos [44–47]. Our oscillation formalism has some similarities with the one developed in the context of non-unitary mixing, see e.g., Refs. [48–50]. Ref. [51] discusses the parametric relation between various non-standard scenarios.

Here we want to be more general and treat \( W \) and \( N^{s,d} \) as independent. In particular, \( N^{s,d} \) can be non-unitary, arbitrary functions of neutrino energy, and they can be different for processes relevant at the neutrino source and for detection (as indicated by the indices \( s \) and \( d \)). This implies that in general \( |\nu^s_i\rangle \neq |\nu^d_i\rangle \). But we do assume that \( N^{s,d}_{ai} \) are the same for different experiments (at the same energy). Note, however, that while the mixing in Eq. (6) can be non-unitary, the induced matter potential \( V_{\text{tot}} \) as well as the total Hamiltonian will be still hermitian, leading to a unitary evolution of the system via Eq. (1).

As we will see in the following, complex phases of \( N^{s,d}_{ai} \) induce fundamental CP and T violation. In the standard scenario there is only one relevant phase (the Dirac CP phase), whereas in non-standard scenarios there are several new sources for complex phases [38–47].

**II.1. Transition probabilities**

We are interested in the transition amplitude for a neutrino of flavour \( \alpha \) at the source to a neutrino of flavour \( \beta \) at the detector, \( A^{(1)}(t^d_i \rightarrow t^d_j) = A_{\alpha\beta} \), and the corresponding transition probability \( P_{\alpha\beta} = |A^{(1)}(\nu^s_{\alpha} \rightarrow \nu^d_{\beta})|^2 \). Consider first the unitary evolution operator \( S \) of the energy eigenstates \( |\nu_i(t^e)\rangle \rightarrow |\nu_j(t^d)\rangle \): \( S_{ij}(t^d, t^e) \). Then using Eq. (6) we obtain

\[
A_{\alpha\beta} = \langle \nu^d_{\beta} | S(t^d, t^e) | \nu^s_{\alpha}\rangle = \sum_{ij} S_{ij}(t^d, t^e) N^{s*}_{ai} N^{d}_{bj}.
\]

(7)

At first order in \( V(t) \) we find

\[
S^{(1)}_{ij}(t^d, t^e) = -ie^{-i\lambda_i(t^d-t^e)} \mathcal{D}_{ij},
\]

\[
A^{(1)}_{\alpha\beta} = -i \sum_{ij} N^{s*}_{ai} N^{d}_{bj} e^{-i\lambda_i(t^d-t^e)} \mathcal{D}_{ij},
\]

(8)

(9)

(10)

For the transition probability we have

\[
P_{\alpha\beta} = \left| A^{(0)}_{\alpha\beta} + A^{(1)}_{\alpha\beta}\right|^2 \approx P^{(0)}_{\alpha\beta} + P^{(1)}_{\alpha\beta},
\]

\[
P^{(0)}_{\alpha\beta} = \left| A^{(0)}_{\alpha\beta}\right|^2, \quad P^{(1)}_{\alpha\beta} = 2 \text{Re} \left[ A^{(0)}_{\alpha\beta} A^{(1)}_{\alpha\beta}\right].
\]

(11)

The parameterization discussed here allows to cover a rather broad range of new physics scenarios, including non-standard interactions in charged current and neutral current interactions, generic non-unitarity, as well as sterile neutrinos (as long as they do not introduce additional oscillation frequencies for the relevant energy and baselines). In principle we can allow for an arbitrary energy dependence of \( N^{s,d}_{ai} \) and \( \lambda_i \), although in the presence of finite energy resolution we have to demand that the energy dependence is weak at the scale of the resolution. Following Ref. [1], due to the success of the standard three-flavour paradigm, we can assume that new physics effects are a small perturbation to the standard case. To leading order, our parameterization covers also non-standard interactions within the most general effective field theory framework [53, 54].

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1 Per assumption we do not allow for the decay of energy eigenstates, which would lead to a non-unitary evolution equation.

2 We thank Martin Gonzalez-Alonso for illuminating discussions about this point.
III. PROPERTIES UNDER THE T TRANSFORMATION

With these generalized expressions for the transition amplitudes and probabilities at hand, we can now discuss their properties under the time reflection transformation.

III.1. Constant matter potential

Let us first assume that the matter potential is constant, i.e., we work at zeroth order in the time-dependent perturbation theory. In this case we can define the T transformation by

\[ T : t \rightarrow -t. \]  

Then we obtain from Eq. (8):

\[ T A_{\alpha\beta}^{(0)} = T A^{(0)}(\nu^s_{\alpha} \rightarrow \nu^d_{\beta}) = \left[ A^{(0)}(\nu^d_{\beta} \rightarrow \nu^s_{\alpha}) \right]^*. \]  

Note that if \( N^s_{ai} \neq N^d_{ai} \) then \( A^{(0)}(\nu^d_{\beta} \rightarrow \nu^s_{\alpha}) \neq A^{(0)}(\nu^s_{\alpha} \rightarrow \nu^d_{\beta}) \). Therefore, the usual result \( TP_{\alpha\beta} = P_{\beta\alpha} \) holds only if the mixing is the same for the processes relevant for neutrino production and detection. If there is new physics distinguishing between neutrino production and detection, the transformation (12) is not equivalent to exchanging only the neutrino flavours of source and detector, but also the type of interaction needs to be exchanged (formally \( d \leftrightarrow s \)).

For real \( N^s_{ai} \), it follows from Eq. (8) that \( T A_{\alpha\beta}^{(0)} = A_{\alpha\beta}^{(0)*} \) and therefore \( TP_{\alpha\beta} = P_{\alpha\beta} \). As expected, the T transformation tests complex phases in the theory.

For the transition probabilities at zeroth order in \( V(t) \) we obtain

\[ P_{\alpha\beta}^{(0)} = \left| \sum_i c_i e^{-i\omega_i(t_d-t_s)} \right|^2 \]

\[ = \sum_i |c_i|^2 + 2 \sum_{j<i} \text{Re}(c_i c_j^*) \cos(\omega_{ij} L) \]

\[ - 2 \sum_{j<i} \text{Im}(c_i c_j^*) \sin(\omega_{ij} L), \]

with the abbreviation \( c_i \equiv N^s_{ai} N^d_{bi} \) and the frequencies \( \omega_{ij} = \lambda_j - \lambda_i \). As usual we identify the baseline by \( L = t_d - t_s \). Therefore, T is formally equivalent to \( L \rightarrow -L \).

We see that the first line of Eq. (15) is invariant under T, whereas the second line is T-odd. It is also apparent that T violation will be present only for non-zero \( \text{Im}(c_i c_j^*) \), i.e., non-trivial complex phases of \( N^s_{ai} \).

Hence, fundamental T violation can be established by proving the presence of the \( L \)-odd term in the probability. Or, put in other words, if data cannot be described by an \( L \)-even transition probability

\[ P_{\alpha\beta}(L) = \sum_i c_i^2 + 2 \sum_{j<i} c_i c_j \cos(\omega_{ij} L) \]

with \( c_i \) real, fundamental T violation needs to be present in the theory. This is the test proposed in Ref. [1].

III.2. Non-constant matter potential

Let us now consider the first-order correction in the case of a non-constant matter potential \( V(t) \). We recall that \( V(t) \) is defined between the locations of the source \( x_s \) and the detector \( x_d \). Therefore, we have to make sure that \( V(t) \) is evaluated only for times in the interval \( [t_s, t_d] \). This requirement has to be respected also when applying the T transformation in Eq. (12). One possible choice is to replace Eq. (12) by

\[ T : t \rightarrow t_s + t_d - t. \]

This implies that T leads to \( t_s \leftrightarrow t_d \) and that T is still equivalent to \( L \rightarrow -L \). Applying this transformation to Eq. (10) we find

\[ T A^{(1)} = i \sum_{ij} N_{ai}^s N_{aj}^d e^{i\lambda_j t_d - i\lambda_i t_s} \int_{t_s}^{t_d} dt V^s_j(t) e^{-i(\lambda_j - \lambda_i)t}. \]

Here we have re-named the indices \( i \leftrightarrow j \) and used the hermiticity of the Hamiltonian, \( V_{ij} = V_{ji}^* \). We observe that Eq. (13), which we have obtained for \( A^{(0)} \), as well as the comments thereafter hold also for the first-order correction \( A^{(1)} \).

As we have seen in sec. III.1 for the case of constant matter density, the mixing coefficients \( N^s_{ai} \) are the only sources of complex phases relevant for the transition probabilities and therefore fundamental T violation/conservation can be characterized by the presence/absence of (non-trivial) complex phases in \( N^s_{ai} \). Note, however, that \( N^s_{ai} \) are defined with respect to the eigenbasis of the Hamiltonian \( H_0 \) for constant density, see Eqs. (5) and (6). Therefore, in general the non-constant perturbation \( V(t) \) may contain non-trivial complex phases even if \( N^s_{ai} \) are real. Hence, we will define “fundamental T conservation” in the following as real \( N^s_{ai} \) and real \( V_{ij}(t) \).

Comparing Eqs. (10) and (18), we see that even for \( N^s_{ai} \) and \( V_{ij}(t) \) real, we still have \( T A^{(1)} \neq A^{(1)*} \) in general. Hence, we recover the result that a non-constant matter density induces environmental T violation in neutrino oscillations, even if the fundamental theory is T conserving. This is a well-known result in the standard oscillation scenario, e.g., [15].

We can also prove the result known for the standard case, namely that a symmetric matter profile does

\[ N^s_{ai} \text{ includes possible phases from the constant matter potential. Therefore, having complex phases in } V(t) \text{ but not in } N^s_{ai} \text{ would require rather special CP violating new physics, coupling only to density variations.} \]

\[ \]
not induce $T$ violation in the absence of fundamental $T$ violation. Following Ref. [15], this can most easily be seen by setting the time $t = 0$ at the baseline midpoint, such that the time interval $[t_s, t_d]$ becomes symmetric: $[-L/2, L/2]$. The transformation (17) still remains $L \to -L$. Rewriting Eq. (10) we obtain

\[ A^{(1)}_{\alpha \beta} = -i \sum_{ij} N^*_{\alpha i} N^d_{\beta j} e^{-i(\lambda_j + \lambda_i) L/2} \int_{-L/2}^{L/2} dt V_{ij}(t) e^{i(\lambda_j - \lambda_i) t}, \]  \hspace{1cm} (19)

\[ T A^{(1)}_{\alpha \beta} = i \sum_{ij} N^*_{\alpha i} N^d_{\beta j} e^{i(\lambda_j + \lambda_i) L/2} \int_{-L/2}^{L/2} dt V_{ij}(-t) e^{-i(\lambda_j - \lambda_i) t}, \]  \hspace{1cm} (20)

where in the second line we have performed the variable transformation $t \to -t$ in the integral. Comparing the two expressions, we see that if $N^*_{\alpha i}^d$ and $V_{ij}$ are real (no fundamental $T$ violation) and the matter potential is symmetric, $V_{ij}(t) = V_{ij}(-t)$, then $T A^{(1)}_{\alpha \beta} = A^{(1)*}_{\alpha \beta}$. Since the same holds for $A_{\alpha \beta}^{(0)}$, we have $T P_{\alpha \beta} = P_{\alpha \beta}$ in this case. Hence, the above statement is proven. As in the standard scenario, in order to introduce environmental $T$ violation in the absence of fundamental $T$ violation an asymmetric matter potential is needed.

Here we derived this result at first order in perturbation theory. In Ref. [15] this statement was proven for the standard oscillation scenario for arbitrary matter profile. It is straightforward to generalize the prove given in [15] also to the non-standard scenario considered here, which we briefly outline in the following. We depart from Eq. (7) and use general properties of the evolution operator:

\[ S(t_d, t_s) S(t_s, t_d) = 1, \quad S(t_d, t_s) S^\dagger(t_d, t_s) = 1. \]  \hspace{1cm} (21)

The second relation follows from the unitarity of the evolution due to the hermiticity of $H(t)$. Using now the $T$ transformation $T$: $t_s \leftrightarrow t_d$ and combining the two properties above we find

\[ TS(t_d, t_s) = S(t_s, t_d) = S^\dagger(t_d, t_s). \]  \hspace{1cm} (22)

Using again a symmetric time coordinate, we consider the evolution operator $S(t, -t)$. Its time evolution is given by [15]:

\[ i \frac{d}{dt} S(t, -t) = H(t) S(t, -t) + S(t, -t) H(-t). \]  \hspace{1cm} (23)

Take now the transpose of this equation. If the Hamilton operator is real (no fundamental $T$ violation), then it is symmetric. If in addition the density profile is symmetric, $H(t) = H(-t)$, we see that $S(t, -t)$ and $S^\dagger(t, -t)$ follow the same evolution equation and are therefore equal, i.e., $S$ is symmetric. Using this in Eq. (22), we obtain $TS(t_d, t_s) = S^\dagger(t_d, t_s)$. With real $N^*_{\alpha i}^d$ we obtain then from Eq. (7) that $T A_{\alpha \beta} = A^{\dagger*}_{\alpha \beta}$ and therefore, $T P_{\alpha \beta} = P_{\alpha \beta}$.

\[ IV. COMMENTS ON T ASYMMETRIES AND THE DISAPPEARANCE CHANNEL \]

Let us collect a few relations regarding time reversal asymmetries. We define the $T$ asymmetry as

\[ A_{\alpha \beta} = P_{\alpha \beta} - T P_{\alpha \beta}. \]  \hspace{1cm} (24)

From Eq. (15) we find for the zeroth-order asymmetry

\[ A_{\alpha \beta}^{(0)} = 4 \sum_{i < j} \text{Im} \left( N^*_{\alpha i} N^d_{\beta j} N^*_{\beta j} N^d_{\alpha i} \right) \sin(\omega_{ij} L). \]  \hspace{1cm} (25)

The 1st order correction to the probabilities from Eq. (11) can be written in the following way:

\[ P_{\alpha \beta} = \sum_{ijk} \sin \left( \frac{\lambda_{ij}}{2} \right) \text{Re} G_{ij}^k \]
\[ \quad + \sum_{ijk} \cos \left( \frac{\lambda_{ij}}{2} \right) \text{Im} G_{ij}^k, \]  \hspace{1cm} (26)

where we have defined

\[ \lambda_{ij}^k = 2 \lambda_{ij} - (\lambda_i + \lambda_j), \]  \hspace{1cm} (27)

\[ G_{ij}^k = 2 N^*_{\alpha k} N^d_{\beta k} N^*_{\beta k} N^d_{\alpha k} I_{ij}, \]  \hspace{1cm} (28)

\[ I_{ij} = \int_{-L/2}^{L/2} dt V_{ij}(t) e^{i(\lambda_j - \lambda_i) t}, \]  \hspace{1cm} (29)

with $\lambda_{ij}^k = \lambda_{ij}^k$ and $I_{ij} = I_{ji}^k$. The $T$ transformation corresponds to $L \to -L$ and it follows that $T I = -I$, $TG = -G$, and therefore the first (second) line in Eq. (26) is $T$ even (odd). Hence, we obtain for the 1st order asymmetry

\[ A_{\alpha \beta}^{(1)} = 2 \sum_{ijk} \cos \left( \frac{\lambda_{ij}}{2} \right) \text{Im} G_{ij}^k. \]  \hspace{1cm} (30)

For a symmetric profile $V_{ij}(t) = V_{ij}(-t)$, we have

\[ I_{ij}^{\text{sym}} = \int_{-L/2}^{L/2} dt V_{ij}(t) \cos(\omega_{ij} t). \]  \hspace{1cm} (31)
In the absence of fundamental T violation, with \( N^s_{\alpha i} \) and \( V_{ij} \) real, \( R^\text{sym} \) and \( G^k_{ij} \) are real as well, and \( A^{(1)}_{\alpha\beta} = 0 \), in agreement with our results in sec. III.2.

Consider now the disappearance probabilities \( \beta = \alpha \). In the standard oscillation scenario, the T transformation becomes trivial, since exchanging initial and final flavour has no effect. Let us re-consider this case in the extended new physics scenario. First we assume that mixing is identical at source and detector \( N^s_{\alpha i} = N^d_{\alpha i} \). Then we find that both, \( A^{(0)}_{\alpha\alpha} = 0 \) and \( A^{(1)}_{\alpha\alpha} = 0 \). The first follows directly from Eq. (25). The second follows from Eq. (30) by noting that for \( N^s_{\alpha i} = N^d_{\alpha i} \) and \( \alpha = \beta \) we have \( G^k_{ij} = G^k_{ji} \). Hence, we conclude that for \( N^s_{\alpha i} = N^d_{\alpha i} \) (which includes also standard mixing) no T asymmetry can be observed in the disappearance channel. This holds even in presence of complex phases as well as asymmetric matter density profiles.

On the contrary, if \( N^s_{\alpha i} \neq N^d_{\alpha i} \), in the presence of non-trivial complex phases we obtain \( A^{(0)}_{\alpha\alpha} \neq 0 \). If \( N^s_{\alpha i} \neq N^d_{\alpha i} \) but both real, then \( A^{(1)}_{\alpha\alpha} \neq 0 \) for an asymmetric density profile.\(^4\) We conclude that

- the observation of T violation in a disappearance channel would be a signal of new physics inducing different flavour mixing at source and detector;
- if effects of asymmetric matter densities can be neglected, it requires fundamental T violation (in addition to \( N^s_{\alpha i} \neq N^d_{\alpha i} \)).

Let us briefly comment on the possible observability of such an effect, at least in principle. One can follow the approach of Ref. [1] and imagine measurements of \( P_{\alpha\alpha}(L_b) \) at a number of baselines \( L_b \) at a fixed energy, and in this way study the \( L \) dependence of the probability. Then one can check if this shape is consistent with an even function of \( L \), or if data require the presence of \( L \)-odd terms. However, to map out the \( L \) dependence for the disappearance channel, one would need several data points, covering at least 1st and 2nd oscillation maxima. Therefore, currently such an analysis seems not feasible with the proposed long-baseline experiments. The test studied in Ref. [1] is based on the interplay of disappearance and appearance channel, and therefore works already with 4 baselines (including the near detector). We leave for future studies whether the disappearance test could potentially be performed with atmospheric neutrinos.

\(^4\) Note, however, that this would be a second order effect, being suppressed by the small density variations and the new physics responsible for \( N^s_{\alpha i} \neq N^d_{\alpha i} \).

\[\text{FIG. 1. Matter density along the baseline of the T2HK (top) T2HKK (middle) and DUNE (bottom) experiments. Data taken from Refs. [34, 35].}\]

V. ESTIMATION OF NON-CONSTANT DENSITY CORRECTIONS

In this section we are going to use this formalism to estimate the impact of a non-constant density for our T violation test. We will address the following two points:

1. when the average matter densities for different baselines are not exactly the same, and
2. an asymmetric density profile at a given baseline.

We consider these two cases using existing density profile studies for the T2HK(K) [34] and DUNE [35] experiments.\(^5\) The corresponding density profiles are shown in

\(^5\) As shown in Ref. [1], the ESS\(\nu\)/SB experiment contributes only very little to the sensitivity of the T violation test. Therefore, we focus here on the DUNE and T2HK(K) experiments.
V.1. Different average densities

Let us first assume that the matter density can be considered constant for each experiment, even if the average values are not the same. From the profiles shown in Fig. 1 we obtain

\[ \bar{\rho}_{\text{HK}} = 2.6 \text{ g/cm}^3, \]
\[ \bar{\rho}_{\text{DUNE}} = 2.85 \text{ g/cm}^3, \]
\[ \bar{\rho}_{\text{HKK}} = 3.0 \text{ g/cm}^3. \]  

The Hamiltonian of the system is thus reduced to the time-independent \( H_0 \) in Eq. (4). Assuming the standard neutrino model, it reads

\[ H_0 = \frac{1}{2E} \begin{bmatrix} 0 & \Delta m_{21}^2 \\ \Delta m_{31}^2 & 0 \end{bmatrix} U^\dagger + \begin{bmatrix} \bar{v}_b & 0 \\ 0 & 0 \end{bmatrix} \]  

in the flavour basis, where \( E \) is the neutrino energy, \( U \) is the standard PMNS mixing matrix, \( \Delta m_{ij}^2 \equiv m_i^2 - m_j^2 \) are the neutrino mass-squared differences, and \( v(\rho) = \sqrt{2} G_F n_e(\rho) \approx 3.78 \times 10^{-14} \text{ eV} \left( \frac{\rho}{\text{g/cm}^3} \right) \).

\[ (33) \]

\( \bar{v}_b \) denotes the potential corresponding to the average density \( \bar{\rho}_b \) and the index \( b \) labels the different baselines, which emphasizes that each baseline may have a different mean density, and thus a different Hamiltonian. We study the time evolution of the system by diagonalizing this Hamiltonian numerically, which leads to a set of effective masses \( m_b^2 = 2E\lambda_b \) and mixings \( N_b^e = N_b^d = W_b \) for each experiment.

In Tab. I we give the values of disappearance and appearance oscillation probabilities for the three baselines, assuming different values for the mean densities for a few choices of the CP phase \( \delta \). The relative size of the effect for the appearance probabilities is shown in the upper panel of Fig. 2. For the table and the figure we have chosen the neutrino energy \( E = 0.75 \text{ GeV} \), which has been found to provide the best sensitivity in Ref. [1]. For the figure we assume the mean density \( \bar{\rho} = 2.85 \text{ g/cm}^3 \), corresponding to the DUNE experiment, and show the relative error induced for the T2HK and T2HKK baselines. As mentioned above, we can use the standard oscillation scenario to estimate this effect, considering only leading order terms in density variations and new physics. We show the size of the correction as a function of the CP phase \( \delta \); all other oscillation parameters are set to their best fit values [25]. We see the effect is below 1% for all values of the CP phase \( \delta \). For the disappearance probabilities the effect is even smaller, compare Tab. I.

Regarding the T violation test [1] based on Eq. (16), the main effect of the different mean densities is that the fit parameters, the amplitudes \( c_i \) and frequencies \( \omega_{ij} \), are no longer the same for all baselines. Since the density dependence is small, we can include it in the fit assuming

\[ c_i^b = \bar{c}_i + \delta c_i^b, \quad \omega_{ij}^b = \bar{\omega}_{ij} + \delta \omega_{ij}^b. \]  

(35)

with the reference parameters \( \bar{c}_i \) and \( \bar{\omega}_{ij} \) corresponding to standard oscillations and a common density \( \bar{\rho} \) taken the same for all baselines. Expanding up to first order in these \( L \)-dependent perturbations, the \( L \)-even probability in Eq. (16) becomes

\[ P_{\alpha \beta}^{\text{even}, b} = \sum_i \bar{c}_i^2 + 2 \sum_{j < i} \bar{c}_i \bar{c}_j \cos(\bar{\omega}_{ij} L) \]
\[ + 2 \sum_i \bar{c}_i \delta c_i^b + 2 \sum_{j < i} (\bar{c}_i \delta c_j^b + \bar{c}_j \delta c_i^b) \cos(\bar{\omega}_{ij} L) \]
\[ - 2 \sum_{j < i} \bar{c}_i \bar{c}_j \delta \omega_{ij}^b L \sin(\bar{\omega}_{ij} L) \].  

(36)

Thus the crucial effect of different (mean) densities is the appearance of new terms in the (previously) \( L \)-even oscillation probability. Note the corrections shown in the 2nd and 3rd line of Eq. (36) are known and fixed and can be just included in the test described in Ref. [1] as constant correction terms for each baseline. The fit itself can be performed with an expression equivalent to the 1st line in Eq. (36), consistent with the leading order perturbation approach mentioned above. From Fig. 2 we see
that at $\delta = 0$ and $\pi$ relevant for the test, the corrections are $\lesssim 0.5\%$ on the appearance probabilities, which themselves are only few $\%$. Therefore, with realistic statistical uncertainties these corrections are negligible.

V.2. Non-constant density

Let us now discuss the effect of a non-constant and non-symmetric matter profile at a given baseline. We use the formalism developed section III.2 to calculate how much this affects the probabilities to be probed in T2HK(K) and DUNE, assuming the standard neutrino model (using the same perturbative argument as above).

Within our perturbation theory in $V(t)$, the zeroth-order result corresponds to the diagonalization of the Hamiltonian with the constant mean density of the previous subsection. Therefore, the procedure described above yields the eigenvalues $\lambda_k$ and mixing matrices $N^b_k = N^d_b = W_k$ for each experiment. As above, we obtain the mean probabilities from Eq. (8) as $P_{\alpha\beta}^{(0)} = |A^{(0)}_{\alpha\beta}|^2$. The first-order correction to the oscillation amplitudes is then given by Eq. (19) in terms of these parameters and the matrix elements of the perturbation in the $H_0$ eigenbasis are $V_{ij}(t) = W_{ei}W^*_{ej} [v(t) - \bar{v}]$. With this we can calculate the 1st order correction to the probabilities given in Eq. (11). The relative size of this correction is shown in the lower panel of Fig. 2 as a function of $\delta$. We observe that for T2HK these corrections are negligible and not visible on the scale of the plot. For DUNE the effect is sub-percent for all values of $\delta$. For T2HKK it can become as large as $3.5\%$ for $\delta \simeq 110^\circ$; for $\delta = 0$ and $\pi$ it is around $1.5\%$. Similar as in sec. V.1, these are calculable and fixed corrections to the probabilities which can be taken into account in the test proposed in Ref. [1]. However, considering that these are percent-level correction on probabilities which themselves are only a few $\%$, this effect is again negligibly small, given realistic statistical errors.

Notice, however, that the size of such corrections does not directly determine the amount of environmental T violation induced by the matter profile. Even for the case of a symmetric profile with real mixings, where no extra T violation is introduced, the oscillation probabilities themselves get a non-vanishing correction. In order to get a feeling for the size of environmental T violation at the DUNE and T2HKK baselines we calculate the asymmetries (24) for the standard oscillation case considered above:

\[
\begin{align*}
T2HKK: \quad & A^{(1)}_{\mu e} \approx 3.5 \times 10^{-4}, \\
DUNE: \quad & A^{(1)}_{\mu e} \approx -2.9 \times 10^{-4},
\end{align*}
\]

for $\delta = 0 (180^\circ)$. These can be compared to the case of $\delta = 90^\circ$, where we find

\[
\begin{align*}
T2HKK: \quad & A^{(0)}_{\mu e} \approx 7.1 \times 10^{-2}, \quad A^{(1)}_{\mu e} \approx -9.1 \times 10^{-4}, \\
DUNE: \quad & A^{(0)}_{\mu e} \approx 6.5 \times 10^{-2}, \quad A^{(1)}_{\mu e} \approx 3.8 \times 10^{-4}.
\end{align*}
\]

We conclude that for these realistic density profiles, environmental T violation is typically a $\%$ level effect compared to generic fundamental T violation [34, 35].

VI. CONCLUSIONS

In this paper we have studied some aspects of the time reversal transformation in a generic non-standard neutrino oscillation framework. The motivation for our study is the model-independent T violation test proposed recently in Ref. [1]. This test can potentially be performed with three long-baseline experiments, such as T2HK, DUNE and the proposed T2HKK. Here we provide a theoretical discussion of the formalism for the model-independent new physics parameterization proposed in Ref. [1]. We derive the relevant flavour transition amplitudes and probabilities and study their behaviour under the T transformation.

The proposed parameterization covers a wide range of new-physics scenarios in a model-independent way,
including non-standard neutrino interactions with arbitrary Lorentz structures in the charged and neutral-current interaction, generic non-unitarity as well as sterile neutrinos. We provide a discussion of fundamental versus environmentally induced T violation, where the former is related to complex phases in the theory while the latter is due to (standard or non-standard) matter effects along the neutrino path. We show that a result well known for standard oscillations holds also in our extended scenario: in the absence of fundamental T violation, environmental T violation can only be induced by an asymmetric matter density profile.

We show that in general new-physics scenarios, also the disappearance channel can be sensitive to T violating effects. This requires new physics generating different flavour mixing at neutrino source and detector. Although difficult to realise in practice, such an observation offers in principle a clear signal of new physics, since in the standard oscillation scenario no T violation is expected in the disappearance channel.

Focusing on long-baseline accelerator neutrino experiments, we have treated density variations along the neutrino path as a small perturbation. Using detailed matter density profile studies for the DUNE and T2HK(K) baselines from the literature, we have provided some quantitative estimates on the corrections induced by a non-constant matter density. Typically they are of order few percent or smaller. Considering that appearance probabilities are themselves typically only few percent, these corrections are much smaller than realistic experimental uncertainties and hence, do not affect the test proposed in Ref. [1].

To conclude, the material presented here provides background information to the model-independent T violation test from Ref. [1]. This lies out the basis for the possibility to test one of the fundamental symmetries of nature, the time reversal symmetry, in a model-independent way using actually planned neutrino oscillation experiments.

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