Let us recall some basic setting of thermodynamic formalism, referring the interested reader to [Ke98] or [PU11] for more information.

Let \((X,d)\) be a compact metric space, and \(T : X \to X\) be a continuous map with \(h_{\text{top}}(T) < \infty\). Denote by \(\mathcal{M}(X)\) the space of Borel probability measures on \(X\) endowed with the weak* topology, and let \(\mathcal{M}(T,X)\) denote the subset of \(T\)-invariant ones. For each measure \(\nu \in \mathcal{M}(T,X)\), denote by \(h_\nu(T)\) the measure-theoretic entropy for \(\nu\).

Given a upper semi-continuous\(^1\) function \(\phi : X \to \mathbb{R} \cup \{-\infty\}\), the topological pressure of \(T\) for the potential \(\phi\) is defined as

\[
P(T, \phi) := \sup \left\{ h_\nu(T) + \int_X \phi d\nu : \nu \in \mathcal{M}(T,X) \right\}.
\]

An equilibrium state of \(T\) for the potential \(\phi\) is a measure which attains the supremum.

Let \(I\) be a compact interval in \(\mathbb{R}\). For a differentiable map \(f : I \to I\), a point of \(I\) is critical if the derivative of \(f\) vanishes at it. We denote by \(\text{Crit}(f)\) the set of critical points of \(f\). We also denote by \(J(f)\) the Julia set, which is the complement of the largest open subset of \(I\) on which the family of iterates of \(f\) is normal. In particular, let \(\text{Crit}'(f) := \text{Crit}(f) \cap J(f)\).

In what follows, we denote by \(\mathcal{A}\) the collection of all non-injective differentiable maps \(f : I \to I\) such that

- The critical set is finite;
- \(Df\) is Hölder continuous;
- The Julia set \(J(f)\) is completely invariant\(^2\) (i.e., \(f(J) = f^{-1}(J) = J\)), and contains at least two points;

\(^1\)A function \(\phi : X \to \mathbb{R} \cup \{-\infty\}\) is upper semi-continuous if the sets \(\{ y \in X : \phi(y) < c \}\) are open for each \(c \in \mathbb{R}\). Since \(X\) is compact, \(\sup \phi < +\infty\).

\(^2\)In contrast with complex rational maps, the Julia set of an interval map might not completely invariant. However, it is possible to make an arbitrarily small smooth perturbation of \(f\) outside a neighborhood, so that the Julia set of the perturbed map is completely invariant, and coincides with \(J(f)\) correspondingly.
All periodic points are hyperbolic repelling (i.e., a periodic point $p$ of period $N$ with $|(f^N)'(p)| > 1$);

- $f$ is topologically exact on the Julia set $J(f)$ (i.e., for each open set $U \in J(f)$, there exists an integer $n \geq 0$ such that $J(f) \subset f^n(U)$).

Throughout the rest of this note, for each $f \in \mathcal{A}$, we restrict the action of $f$ to its Julia set $f|_{J(f)} : J(f) \rightarrow J(f)$. In particular, the topological pressure of $f$ is defined through measures supported on $J(f)$.

On the other hand, given a map $f \in \mathcal{A}$, let

$$
\mathcal{U} := \left\{ u : J(f) \rightarrow \mathbb{R} \cup \{-\infty\}, \ u(x) = g(x) + \sum_{c \in \text{Crit}'(f)} b(c) \log |x - c|, \right. 
$$

\begin{equation}
\left. \text{with } g \text{ Hölder continuous, and } b(c) \geq 0 \right\}.
\end{equation}

Obviously, Hölder and geometric potentials belong to set $\mathcal{U}$, and for each upper semi-continuous potential $G \in \mathcal{U}$, denote by

$$
\Lambda(G) := \{ x \in J(f), \ G(x) = -\infty \} = \{ c \in \text{Crit}'(f), \ b(c) > 0 \} \subseteq \text{Crit}'(f).
$$

The following is the key hypothesis.

**Definition 0.1.** Let $f : J(f) \rightarrow J(f)$ in $\mathcal{A}$, then

**Hyperbolicity:** a potential $G$ in $\mathcal{U}$ is hyperbolic for $f$ if for some integer $n \geq 1$, the function $S_n(G) := \sum_{j=0}^{n-1} G \circ f^j$ satisfies

\begin{equation}
\sup_{J(f)} \frac{1}{n} S_n(G) < P(f, G).
\end{equation}

**Exceptionality:** a potential $G$ in $\mathcal{U}$ is exceptional for $f$ if there is a non-empty forward invariant finite subset $\Sigma \subset J(f)$ satisfies

\begin{equation}
\frac{1}{n} \left( f^{-1}(\Sigma) \right) \subseteq \Lambda(G).
\end{equation}

Such set $\Sigma$ is a $\Lambda(G)-$exceptional set. Otherwise a potential $G$ is non-exceptional for $f$.

The aim of this note$^3$ is aiming to prove the following:

**Proposition 0.2** (Li and Rivera-Letelier). Let $f : J(f) \rightarrow J(f)$ be an interval map in $\mathcal{A}$, and $G : J(f) \rightarrow \mathbb{R} \cup \{-\infty\}$ be a upper semi-continuous potential in $\mathcal{U}$. If $G$ is hyperbolic and non-exceptional for $f$, then for every periodic point $x \in J(f)$, or every non-periodic point $x \in J(f) \setminus \bigcup_{j=-\infty}^{\infty} f^j(\Lambda(G))$, we have

\begin{equation}
P(f, G) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} \exp(S_n(G)(y))^4.
\end{equation}

This is a generalized version of Corollary 2.2 in the paper [LRL14] by Li and Rivera-Letelier, though these authors restrict $G$ to be Hölder continuous. With our convention, it is equivalent to the special case where $\Lambda(G) = \emptyset$, so that the non-exceptionality hypothesis is automatically satisfied. Since we want to state this proposition in the paper [Z15] to show the existence of a conformal measure for

$^3$Written under supervision by Juan Rivera-Letelier.

$^4$A term of tree pressure is used in [PU11] to stand for the right hand side of Equation (0.5).
non-exceptional geometric potential (i.e., $G := -t \log |Df|$) at negative spectrum (i.e., $t \leq 0$), we decide to write down the detail.

We will estimate the tree pressure in (0.5) from below and above. However, it might be worth to remark that the estimation from above is much easier than from below. In particular,

**Lemma 0.3.** [RL14] Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$, and $G : J(f) \to \mathbb{R} \cup \{-\infty\}$ be a upper semi-continuous potential in $\mathcal{U}$, then for every point $x \in J(f)$, we have

$$P(f, G) \geq \limsup_{n \to \infty} \frac{1}{n} \sum_{y \in f^{-n}(x)} \exp(S_n(G)(y)).$$

The proof was written for Hölder continuous functions, but it apply without change to the potentials in $\mathcal{U}$ by using a variational principle for upper semi-continuous functions [Ke98, Theo 4.4.11]. In the view of Lemma 0.3, to prove Proposition 0.2, it is enough to show the following.

**Proposition 0.4.** Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$, and $G$ be the upper semi-continuous potential in $\mathcal{U}$, with $\Lambda(G)$ the resulting singular set. If $G$ is hyperbolic and non-exceptional for $f$, then for every periodic point $x \in J(f)$, or non-periodic point $x \in J(f) \setminus \bigcup_{j=-\infty}^{\infty} f^j(\Lambda(G))$, we have

$$\liminf_{n \to \infty} \frac{1}{n} \log \sum_{y \in f^{-n}(x)} \exp(S_n(y)) \geq P(f, G). \tag{0.6}$$

The proof of Proposition 0.4 will occupy the rest of the note, and requires a few other lemmas.

Given an interval map $f : J(f) \to J(f)$ in $\mathcal{A}$, and a subset $\Lambda \subseteq \text{Crit}'(f)$, a point $x \in J(f)$ is said to be $\Lambda$-normal, if for any integer $n \geq 1$, there is a pre-image $y$ of $x$ by $f^n$, such that

$$\{y, f(y), \ldots, f^{n-1}(y)\} \cap \Lambda = \emptyset.$$ 

**Lemma 0.5.** [Zh15] Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$, and $G : J(f) \to \mathbb{R} \cup \{-\infty\}$ be a upper semi-continuous potential in $\mathcal{U}$ with $\Lambda(G)$ as the resulting singular set. If $G$ is non-exceptional for $f$, then for each $x \in J(f)$, there is an integer $N \geq 0$ such that $f^N(x) \notin \Lambda(G)$, and $f^N(x)$ is $\Lambda(G)$-normal. In addition, if further assume $x$ is periodic, then the integer $N = 0$.

Lemma 0.5 permits us to deduce other lemmas.

**Lemma 0.6.** Let $f : J(f) \to J(f)$ be an interval map in $\mathcal{A}$. Let $G : J(f) \to \mathbb{R} \cup \{-\infty\}$ be the upper semi-continuous potential in $\mathcal{U}$ with $\Lambda(G)$ as the resulting singular set.

Then for every integer $N \geq 1$, and any compact set $K \subset J(f) \setminus \bigcup_{j=0}^{N-1} f^{-j}(\Lambda(G))$, there is a constant $C_K > 1$, such that the potential $\bar{G} := 1_N S_N(G)$ satisfies the following:

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5This partially answers a question about the existence of a conformal measure for geometric potential at negative spectrum imposed in [GPR14, §2.3].
For each integer $n \geq 1$, we have

\begin{equation}
\sup_{K} |S_n(G) - S_n(\tilde{G})| < C_K.
\end{equation}

**Proof.** For each integer $N \geq 1$, put

$$h := -\frac{1}{N} \sum_{j=0}^{N-1} (N - 1 - j)G \circ f^j,$$

then $\tilde{G} = G + h - h \circ f$.

1. Since $G \in \mathcal{U}$, and $f$ is Lipschitz, we have $\tilde{G}$ is upper semi-continuous and has log poles solely inside the set $\bigcup_{j=0}^{N-1} f^{-j}(\Lambda(G))$. For each measure $\nu \in \mathcal{M}(f, J(f))$, choose $\nu'$ by one of its ergodic component if necessary. There are two cases.

   - Suppose $\nu'$ is atomic, then the topological exactness on $J(f)$ yields that $\nu'$ supports on a periodic orbit $O_x$ in $J(f)$. Since no critical point of $f$ in $J(f)$ is periodic, the set $O_x \cap \bigcup_{j=0}^{\infty} f^{-j}(\Lambda(G)) = \emptyset$. This implies functions $G$, $G'$, and $h$ have no log poles on $O_x$. Thus

   \begin{align*}
   \int_{J(f)} \tilde{G} d\nu' &= \int_{J(f)} G + h - h \circ f d\nu' = \int_{J(f)} G d\nu'.
   \end{align*}

   - Suppose $\nu'$ is non-atomic, then the topological exactness on $J(f)$ yields that $\nu'$ supports on entire $J(f)$, and $\nu'\left(\bigcup_{j=0}^{\infty} f^{-j}(\Lambda(G))\right) = 0$. This also implies that

   \begin{align*}
   \int_{J(f)} \tilde{G} d\nu' &= \int_{J(f)} G + h - h \circ f d\nu' = \int_{J(f)} G d\nu'.
   \end{align*}

   So in both cases, we have $P(f, G) = P(f, \tilde{G})$, and thus $G, \tilde{G}$ share the same equilibrium states.

2. Since for each $n \geq N$,

   $$S_n(\tilde{G}) = S_n(G + h - h) = S_n(G) + h \circ f^n - h,$$

   and $h \circ f^n, h$ are Hölder continuous on the compact set $K$, so we have the desired inequality (0.7) with $C_K := (N - 1)(\sup K - \inf K)$.

From now on, given an interval map $f : J(f) \to J(f)$ in $\mathcal{A}$ and a upper semi-continuous potential $G : J(f) \to \mathbb{R} \cup \{-\infty\}$, denote by $L_G$ the transfer operator acting on the space of bounded functions on $J(f)$ and taking values in $\mathbb{C}$, defined as

$$L_G(\psi)(x) := \sum_{y \in f^{-1}(x)} \exp(G(y))\psi(y).$$

We are ready to prove Proposition 0.4. In informal term, the proof is split into 3 parts. In Part 1, we construct a new potential $\tilde{G}$ by the Birkhoff average of $G$, and show that $\tilde{G}$ is also hyperbolic and non-exceptional for $f$. In part 2, we rely on the hyperbolicity to ensure the existence of an ergodic measure with positive Lyapunov exponent. Applying the Pesin Theory and Katok Theory on this measure, we will obtain a low bound estimation on the tree pressure by the
differeomorphic pull-backs on a neighborhood of a \( \text{Crit}'(f) \)–normal point. In Part 3, we will use the non-exceptionality and topological exactness to move the desired points inside a neighbor of a \( \text{Crit}'(f) \)–normal point, and away from the singular set.

**Proof of Proposition 0.4.** 1. Since \( G \) is hyperbolic for \( f \), there exists an integer \( N \geq 1 \), so that the function \( \tilde{G} := \frac{1}{N}S_N(G) \) satisfies that \( \sup_{j(f)} \tilde{G} < P(f,G) \).

Applying Part (1) of Lemma 0.6, the function \( \tilde{G} \) is upper semi-continuous and has log poles solely inside the set \( \bigcup_{j=0}^{N-1} f^{-j}(\Lambda(G)) \), and

\[
(0.8) \quad P(f,\tilde{G}) = P(f,G) > \sup_{j(f)} \tilde{G}.
\]

Next, we show that the potential \( \tilde{G} \) is also non-exceptional for \( f \). This can be proved by contradiction. Suppose on the contrast, then there exists a non-empty finite forward invariant subset \( \Sigma \subset J(f) \), such that

\[
(0.9) \quad f^{-1}(\Sigma) \setminus \Sigma \subset \Lambda(\tilde{G}) = \bigcup_{j=0}^{N-1} f^{-j}(\Lambda(G)).
\]

Without loss of generality, we can assume that \( \Sigma \) contains only one periodic point \( p \), and every point in \( \Sigma \) is pre-periodic and will map to \( p \). Using (0.9), it follows that

\[
\forall x \in f^{-1}(\Sigma) \setminus \Sigma, \ \exists j := j_x \geq 0, \ s.t. \ f^j(x) \in \Lambda(G) \subset \text{Crit}'(f).
\]

Note also that no critical point in \( J(f) \) is periodic, so for each \( x \in f^{-1}(\Sigma) \setminus \Sigma \), we can define by \( j^* \) the unique index \( j \) such that \( f^j(x) \in \Lambda(G) \), but \( f^i(x) \notin \Lambda(G), \ \forall i > j \).

Put also

\[
A := \{ f^{j^*}(x) : x \in f^{-1}(\Sigma) \setminus \Sigma \}, \ \text{and} \ \Sigma' := \bigcup_{i=1}^{\infty} f^i(A).
\]

With this convention, to complete the proof, it is sufficient to show that \( \Sigma' \) is a \( \Lambda(G) \)–exceptional set, so that it is a contradiction to the hypothesis that the potential \( G \) is non-exceptional for \( f \). This contradiction yields that the potential \( \tilde{G} \) is non-exceptional for \( f \).

It is straightforward to see that the set \( \Sigma' \) is non-empty, finite and forward invariant, so it only remains to verify that \( f^{-1}(\Sigma') \setminus \Sigma' \subseteq \Lambda(G) \). Note that \( \Sigma' \subseteq \Sigma \), so it follows from (0.9), that

\[
(0.10) \quad \forall y \in \Sigma', \ \forall z \in f^{-1}(y) \Rightarrow z \in \Sigma \setminus \Lambda(\tilde{G}), \ \text{or} \ z \in \Lambda(\tilde{G}).
\]

- Suppose \( z \in \Sigma \setminus \Lambda(\tilde{G}) \). Note that by definition \( \Sigma' \cap \Lambda(G) = \emptyset \), so the forward orbit of \( y \), and \( z \) are outside the set \( \Lambda(G) \). This means there must exists \( z' \in f^{-1}(\Sigma) \setminus \Sigma \) and a integer \( d \geq 1 \) such that \( z = f_1z^{1+d}(z') \). In other words \( z \in \Sigma' \).
- Suppose \( z \in \Lambda(\tilde{G}) \). This implies that there is an integer \( j \geq 0 \), with

\[
f^j(z) = f^j(\Lambda(G)) = f^j-1(\Lambda(G)) \subseteq \Lambda(G);
\]

On the other hand, since \( y \in \Sigma' \), there is \( x \in f^{-1}(\Sigma) \setminus \Sigma \) and integer \( d \geq 1 \) such that \( y = f^{j+d}(x) \). Therefore

\[
f^{j-1+j+d}(x) \in \Lambda(G).
\]
By the maximality of \( j^* \), we have \( j - 1 + d \leq 0 \), so \( d = 1 \) and \( j = 0 \). In other words, \( z \in \Lambda(G) \).

In conclusion, the right hand side of (0.10) yields that \( z \in \Sigma' \) or \( z \in \Lambda(G) \), namely \( \Sigma' \) is a \( \Lambda(G) \)-exceptional set, as we wanted.

2. We are aiming to prove following Claim in this section.

Claim: For every \( \varepsilon > 0 \), and every Crit\((f)\)–normal point \( x \) of \( J(f) \), there is \( \delta > 0 \) such that

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{W \in \mathcal{D}_n} \inf_{W \cap J(f)} \exp(S_n(\tilde{G})) \geq P(f, \tilde{G}) - \varepsilon,
\]

where \( \mathcal{D}_n \) is the collection of diffeomorphic pull-backs of \( B(x, \delta) \) by \( f^n \).

On one hand, Inequality (0.8) yields that there is \( \varepsilon > 0 \) so that \( \varepsilon < P(f, \tilde{G}) - \sup_{J(f)} \tilde{G} \). Let \( \nu \) be a measure in \( \mathcal{M}(J(f), f) \) such that

\[
h_\nu(f) + \int_{J(f)} \tilde{G}d\nu \geq P(f, \tilde{G}) - \varepsilon > \sup_{J(f)} \tilde{G}.
\]

Replacing \( \nu \) by one of its ergodic components if necessarily, assuming that \( \nu \) is ergodic. We thus have

\[
h_\nu(f) > \sup_{J(f)} \tilde{G} - \int_{J(f)} \tilde{G}d\nu \geq 0,
\]

and then the Ruelle’s inequality yields that the Lyapunov exponent of \( \nu \) is strictly positive. Applying [PU11, Theo4.4.3]\(^6\), there is a compact and forward invariant subset \( Y \) of \( J(f) \) on which \( f \) is topological transitive, so \( f \) is open and uniformly expanding, and so that

\[
P(f|_Y, \tilde{G}|_Y) \geq P(f, \tilde{G}) - \varepsilon.
\]

Therefore, [PU11, Theo4.4.3] implies that there is \( \delta_0 > 0 \) such that the desired property (0.11) holds for every \( x \in Y \) with \( \delta = \delta_0 \).

One the other hand, the hypothesis that \( x \) is Crit\((f)\)-normal and \( f \) is topological exact on \( J(f) \) imply that there is a non-critical pre-image \( x' \in B(Y, \delta_0) \) of \( x \) such that all the pre-images are non-critical and \( \{x', f(x'), \cdots, x\} \cap \text{Crit}\((f)\) = \emptyset \). Therefore, \( \tilde{G} \) is finite along the orbit \( \{x', f(x'), \cdots, x\} \) and there exists \( \delta > 0 \) such that the pull-back of \( B(x_0, \delta) \) by \( f^n \) that contains \( x' \) is contained in \( B(x, \delta_0) \), and the desired assertion (0.11) directly follows from the previous discussion. So, the proof of the claim is completed.

3. Let \( x \) be a periodic point or a non-periodic point \( J(f) \setminus \bigcup_{j=-\infty}^{\infty} f^j(\Lambda(G)) \), and recall \( N \) to be the integer given in Part 1. Since no critical point of \( f \) in \( J(f) \) is periodic, following the discussions in Part 1, there are a compact subset \( \tilde{K} \subset J(f) \setminus \bigcup_{j=0}^{N-1} f^{-j}(\Lambda(G)) \) and a constant \( C_{\tilde{K}} \), such that \( x \in \tilde{K} \), and

\[
|S_n(G)(x) - S_n(\tilde{G})(x)| \leq \sup_{\tilde{K}} |S_n(G) - S_n(\tilde{G})| = C_{\tilde{K}}, \ \forall n \geq 1.
\]

\(^6\)Actually, the proof is written for complex rational maps with geometric potential, but they apply without changes to interval function \( \tilde{G} \) by applying [Do08, Theo 6] instead of [PU11, Coro1.2.4].
With this convention, to complete the proof of the lemma, it suffices to prove that for every \( \varepsilon > 0 \), there is \( N_0 > 0 \) such that for every \( n \geq N_0 \), we have

\[
L^n_G(1)(x) \geq \exp(C_R) \exp(n(P(f, \tilde{G}) - \varepsilon)).
\]

On one hand, let \( x_0 \) be a \( \text{Crit}'(f) \)-normal point. Let \( \delta > 0 \) and for each \( n \geq 1 \), let \( \mathcal{D}_n \) be as in (0.11) with \( \varepsilon \) replacing by \( \varepsilon/2 \). Then there is \( n_0 \geq 1 \) such that for every integer \( n \geq n_0 \), we have

\[
\frac{1}{n} \log \sum_{W \in \mathcal{D}_n} \inf_{W \cap J(f)} \exp(S_n)(\tilde{G}) \geq P(f, \tilde{G}) - \varepsilon/2.
\]

This implies that for each \( n \geq n_0 \), and every \( x^* \in B(x_0, \delta) \cap J(f) \), we have

\[
L^n_G(1)(x^*) \geq \exp(n(P(f, \tilde{G}) - \varepsilon/2)).
\]

On the other hand, note that it follows from Part 1 that the potential \( \tilde{G} \) is non-exceptional for \( f \). Applying Lemma 0.5, point \( x \) must be \( \Lambda(\tilde{G}) \)-normal. Together with the topological exactness on \( J(f) \), there exists \( n_1 \geq 1 \), and \( x' \in B(x_0, \delta) \cap J(f) \) with

\[
f^{n_1}(x') = x \text{ and } f^i(x') \notin \Lambda(\tilde{G}), \forall i = 0, 1, 2, \ldots, n_1.
\]

Therefore, for every \( n \geq n_1 + n_0 \),

\[
L^n_G(1)(x) = \sum_{y \in f^{-n}(x)} \exp(S_n(\tilde{G})(y))
\]

\[
= \sum_{y' \in f^{-n_1}(x)} \sum_{y \in f^{-n-n_1}(y')} \exp(S_{n-n_1}(\tilde{G})(y) + S_{n_1}(\tilde{G})(y'))
\]

\[
\geq \exp(S_{n_1}(\tilde{G})(x')) \prod_{i=0}^{n-n_1} L^n_{\tilde{G}}(1)(x').
\]

Using (0.13), we can choose another compact set \( K \subset J(f) \) contains \( \{f^i(x')\}_{i=0}^{n_1} \) but away from \( \Lambda(\tilde{G}) \), so that \( \inf_K(\tilde{G}) > -\infty \). Thus

\[
\prod_{i=0}^{n-n_1} \geq \exp(n_1 \inf_K \tilde{G}) \prod_{i=0}^{n-n_1} L^n_{\tilde{G}}(1)(x')
\]

\[
\geq \exp(n_1 \inf_K \tilde{G}) \exp((n - n_1)(P(f, \tilde{G}) - \varepsilon/2)).
\]

Let \( N_2 > 0 \) be such that

\[
\exp(n_1 \inf_K \tilde{G}) \geq \exp(C_R) \exp(n_1(P(f, \tilde{G}) - (\varepsilon N_2)/2).
\]

Therefore for every integer \( n \geq \max\{n_1 + n_0, N_2\} \), we have

\[
L^n_G(1)(x) \geq \exp(C_R) \exp(n_1(P(f, \tilde{G}) - (\varepsilon N_2)/2 + (n - n_1)(P(f, \tilde{G}) - \varepsilon/2))
\]

\[
\geq \exp(C_R) \exp(n(P(f, \tilde{G}) - \varepsilon) + (\varepsilon/2)(n + n_1 - N_2))
\]

\[
\geq \exp(C_R) \exp(n(P(f, \tilde{G}) - \varepsilon)).
\]

This provides the desired inequality (0.12) with \( N_0 = \max\{n_1 + n_0, N_2\} \), and the proof of this lemma is completed. \( \square \)
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