EVERY NON-SMOOTH 2-DIMENSIONAL BANACH SPACE HAS
THE MAZUR–ULAM PROPERTY

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Abstract. A Banach space \( X \) has the Mazur–Ulam property if any isometry from the unit sphere of \( X \) onto the unit sphere of any other Banach space \( Y \) extends to a linear isometry of the Banach spaces \( X, Y \). A Banach space \( X \) is called smooth if the unit ball has a unique supporting functional at each point of the unit sphere. We prove that each non-smooth 2-dimensional Banach space has the Mazur–Ulam property.

1. Introduction

By the classical result of Mazur and Ulam \([10]\), every bijective isometry between Banach spaces is affine. This result essentially asserts that the metric structure of a Banach space determines its linear structure. In \([9]\) Mankiewicz proved that every bijective isometry \( f : B_X \to B_Y \) between the unit balls of two Banach spaces \( X, Y \) extends to a linear isometry of the Banach spaces. In \([14]\) Tingley asked if the unit balls in this result of Mankiewicz can be replaced by the unit spheres. More precisely, he posed the following problem.

Problem 1.1 (Tingley, 1987). Let \( f : S_X \to S_Y \) be a bijective isometry between the unit spheres of two Banach spaces \( X, Y \). Can \( f \) be extended to a linear isometry between the Banach spaces \( X, Y \)?

Here for a Banach space \( (X, \| \cdot \|) \) by

\[
B_X = \{ x \in X : \| x \| \leq 1 \} \quad \text{and} \quad S_X = \{ x \in X : \| x \| = 1 \}
\]

we denote the unit ball and unit sphere of \( X \), respectively.

Tingley’s Problem 1.1 remains open even for 2-dimensional Banach spaces. This problem can be equivalently reformulated in terms of the Mazur–Ulam property, introduced by Cheng and Dong \([6]\) and widely used in the literature devoted to Tingley’s problem \([6, 11, 13]\).

Definition 1.2. A Banach space \( X \) is defined to have the Mazur–Ulam property if every isometry \( f : S_X \to S_Y \) of \( S_X \) onto the unit sphere \( S_Y \) of an arbitrary Banach space \( Y \) extends to a linear isometry of the Banach spaces \( X, Y \).

In fact, Tingley’s Problem 1.1 asks whether every Banach space has the Mazur–Ulam property. There are many results on the Mazur–Ulam property in some special Banach spaces like \( C(K), c_0(\Gamma), \ell_p(\Gamma), L_p(\mu) \), see the survey \([11]\). By a result of Kadets and Martín \([8]\), every polyhedral finite-dimensional Banach space has the Mazur–Ulam property.

For 2-dimensional Banach spaces this result of Kadets and Martín was improved by Cabello Sánchez who proved the following theorem in \([3]\).

Theorem 1.3. A 2-dimensional Banach space has the Mazur–Ulam property if it is not strictly convex.

Let us recall that a Banach space \( X \) is strictly convex if each convex subset of the unit sphere \( S_X \) contains at most one point.

A Banach space \( X \) is smooth if for every point \( x \in S_X \) there exists a unique linear continuous functional \( x^* : X \to \mathbb{R} \) such that \( x^*(x) = 1 = \| x^* \| \). Geometrically this means that the unit ball \( B_X \) has a unique supporting hyperplane at \( x \).

2020 Mathematics Subject Classification. 46B04, 46B20, 52A21, 52A10, 53A04, 54E35, 54E40.

Key words and phrases. Tingley’s Problem, Mazur–Ulam property, smooth Banach space, isometry.
It is well-known [1, 7.23] that a reflexive Banach space \( X \) is strictly convex if and only if its dual Banach space \( X^* \) is smooth.

The main result of this paper is the following theorem, a kind of a dual version of Theorem 1.3.

**Theorem 1.4.** Each non-smooth 2-dimensional Banach space has the Mazur–Ulam property.

This theorem follows from Propositions 4.2 and 4.7 proved in Section 4. For piecewise \( C^1 \)-smooth Banach spaces, Theorem 1.4 was proved in [4].

**Remark 1.5.** Theorems 1.4 and Proposition 4.7 (on the Mazur–Ulam property of 2-dimensional Banach spaces whose sphere contains two linearly independent special directions) are essential ingredients in the main result of the paper [2] answering the Tingley’s Problem in the class of 2-dimensional Banach spaces.

## 2. The natural parameterization of the unit sphere of a 2-dimensional Banach space

By a 2-based Banach space we understand any 2-dimensional Banach space \( X \) endowed with a basis \( e_1, e_2 \) consisting of two linearly independent vectors in \( X \).

Let \( X \) be a 2-based Banach space and \( e_1, e_2 \) be the basis of \( X \).

The polar parameterization of the unit sphere \( S_X \) is the map

\[
p : \mathbb{R} \to S_X, \quad p : t \mapsto \frac{e^{it}}{\|e^{it}\|}, \quad \text{where } e^{it} = \cos(t)e_1 + \sin(t)e_2.
\]

The following properties of the polar parameterization were established in [2, §4].

**Lemma 2.1.** The polar parameterization \( p : \mathbb{R} \to S_X \) has the following properties:

1. \( p(t + \pi) = -p(t) \) for every \( t \in \mathbb{R} \);
2. the function \( p \) has one-sided derivatives
   \[
   p_-'(t) = \lim_{\varepsilon \to 0^-} \frac{p(t + \varepsilon) - p(t)}{\varepsilon} \quad \text{and} \quad p_+'(t) = \lim_{\varepsilon \to 0^+} \frac{p(t + \varepsilon) - p(t)}{\varepsilon}
   \]
   at each point \( t \in \mathbb{R} \);
3. the set \( \{ t \in \mathbb{R} : p_-'(t) \neq p_+'(t) \} \) is at most countable.
4. \[
   \frac{c}{C} \cdot |\sin(\varepsilon)| \leq \|p(t + \varepsilon) - p(t)\| \leq \frac{2C^2}{c^2} \cdot |\varepsilon| \quad \text{for any } t, \varepsilon \in \mathbb{R};
\]
5. \[
   \frac{c}{C} \leq \min\{\|p_-'(t)\|, \|p_+'(t)\|\} \leq \max\{\|p_-'(t)\|, \|p_+'(t)\|\} \leq \frac{2C^2}{c^2} \quad \text{for every } t \in \mathbb{R},
\]

where \( c = \min\{\|e^t\| : t \in \mathbb{R}\} \) and \( C = \max\{\|e^t\| : t \in \mathbb{R}\} \).

Lemma 2.1 implies that the function

\[
s : \mathbb{R} \to \mathbb{R}, \quad s : t \mapsto \int_0^t \|p_-'(u)\| \, du = \int_0^t \|p_+'(u)\| \, du
\]

is continuous and strictly increasing.

For \( t \in [0, \pi] \) the value \( s(t) \) can be thought as the length of the curve on the sphere \( S_X \) between the points \( p(0) \) and \( p(t) \) in the Banach space \( X \). The number

\[
L = s(\pi) = \int_0^\pi \|p_-'(t)\| \, dt = \int_0^\pi \|p_+'(t)\| \, dt
\]

is called the half-length of the sphere \( S_X \) in \( X \).

The image

\[
\tilde{S}_X = \{ p(t) : 0 \leq t \leq \pi \}
\]

is called the upper half-sphere of the 2-based Banach space \( X \).

Since the function \( s \) is continuous and increasing, there exists a unique continuous increasing function \( t : \mathbb{R} \to \mathbb{R} \) such that \( s \circ t \) is the identity map of \( \mathbb{R} \).

The function

\[
r : \mathbb{R} \to S_X, \quad r : s \mapsto p \circ t(s),
\]
Lemma 2.2. The natural parameterization \( r : \mathbb{R} \to S_X \) of the sphere \( S_X \).

The following properties of the natural parameterization were established in [1] §5.

Lemma 2.3. The natural parameterization \( r : \mathbb{R} \to S_X \) has the following properties:

1. \( r(s + L) = -r(s) \) for every \( s \in \mathbb{R} \);
2. the function \( r \) has one-sided derivatives

\[
\begin{align*}
    r'_+(s) &= \lim_{\varepsilon \to 0^-} \frac{r(s + \varepsilon) - r(s)}{\varepsilon} \\
    r'_-(s) &= \lim_{\varepsilon \to 0^+} \frac{r(s + \varepsilon) - r(s)}{\varepsilon}
\end{align*}
\]

at each point \( s \in \mathbb{R} \);
3. the set \( \{ s \in \mathbb{R} : r'_-(s) \neq r'_+(s) \} \) is at most countable;
4. \( r \) is non-expanding and has \( \| r'_+(s) \| = \| r'_-(s) \| = 1 \) for every \( s \in \mathbb{R} \).
5. If \( r \) is differentiable on some open set \( U \subseteq \mathbb{R} \), then \( r \) is continuously differentiable on \( U \).

The natural parametrization is closely related to the intrinsic metric on the half-sphere \( \tilde{S}_X \).

For two points \( x, y \in \tilde{S}_X \) the real number

\[
d(x, y) = \sup_{\varepsilon > 0} \inf \left\{ \sum_{i=1}^n \| x_{i+1} - x_i \| : x_0, \ldots, x_n \in \tilde{S}_X, x_0 = x, x_n = y, \max_{1 \leq i \leq n} \| x_{i+1} - x_i \| < \varepsilon \right\}
\]

is called the intrinsic distance between the points \( x, y \) on the half-sphere \( \tilde{S}_X \). The following lemma can be proved by analogy with Lemma 3.1 of [1].

Lemma 2.3. If \( r \) is continuously differentiable at each point \( s \in (0, L) \), then the map \( r \mid_{(0,L)} \) is an isometry of the interval \([0,L] \) onto the half-sphere \( \tilde{S}_X \) endowed with the intrinsic distance.

If \( r \) is arbitrary, then we can prove a weaker statement.

Lemma 2.4. For any \( s \in \mathbb{R} \) and small \( \varepsilon \) we have

\[
\| r(s + \varepsilon) - r(s) \| = (1 + o(1)) \cdot |\varepsilon|.
\]

Proof. If \( \varepsilon > 0 \), then \( r(s + \varepsilon) = r(s) + r'_+(s)\varepsilon + o(\varepsilon) \) and hence

\[
\| r(s + \varepsilon) - r(s) \| = \| r'_+(s) \| \cdot |\varepsilon| = (1 + o(1)) \cdot |\varepsilon|.
\]

By analogy we can show that \( \| r(s + \varepsilon) - r(s) \| \neq (1 + o(1)) \cdot |\varepsilon| \) for small \( \varepsilon < 0 \).

For every parameter \( s \in \mathbb{R} \) let

\[
r'_{\pm}(s) = \frac{1}{2} (r'_+(s) + r'_-(s)).
\]

It is easy to see that the vectors \( r(s) \) and \( r'_{\pm}(s) \) are linearly independent. Then

\[
\frac{1}{2} (r'_+(s) - r'_-(s)) = j(s) \cdot r(s) + j(s) \cdot r'_{\pm}(s)
\]

for some real numbers \( j(s) \) and \( j'(s) \), called the radial and tangential jumps of the derivative \( r' \) at \( s \), respectively.

It follows that

\[

\begin{align*}
    r'_+(s) &= j(s) \cdot r(s) + (1 + j(s)) \cdot r'_{\pm}(s) \quad \text{and} \\
    r'_-(s) &= -j(s) \cdot r(s) + (1 - j(s)) \cdot r'_{\pm}(s).
\end{align*}
\]

Lemma 2.5. \( |j(s)| < 1 \).

1. \( r'_+(s) = r'_+(s) \) iff \( j(s) = 0 \).
2. \( r'_-(s) = r'_+(s) \) iff \( j(s) = 0 \).
3. \( j(s) \leq 0 \).

Proof. 1. It is easy to see that the bases \( (r(s), r'_+(s)) \) and \( (r(s), r'_{\pm}(s)) \) have the same orientation, which implies that for the basis \( (r(s), r'_+(s)) \) the \( r'_{\pm}(s) \)-coordinates are \( 1 + j(s) \) and \( 1 - j(s) \) of the vectors \( r'_+(s) \) and \( r'_-(s) \) are positive and hence \( |j(s)| < 1 \).

2. If \( r'_-(s) = r'_+(s) \), then \( 0 = \frac{1}{2} (r'_+(s) - r'_-(s)) = j(s) \cdot r(s) + j(s) \cdot r'_{\pm}(s) \) and hence \( j(s) = 0 \). If \( j(s) = 0 \), then the vectors \( r'_+(s) = (1 + j(s)) r'_{\pm}(s) \) and \( r'_-(s) = (1 - j(s)) r'_{\pm}(s) \) are collinear and hence they are equal since they have the same norm and the bases \( (r(s), r'_+(s)) \) and \( (r(s), r'_+(s)) \) have the same orientation.
3. The inequality $j(s) \leq 0$ follows from the convexity of the ball $B_X$, see the following picture.

In the following lemma we use the standard function $\text{sign} : \mathbb{R} \to \{-1,0,1\}$ defined by the formula

$$
\text{sign}(\varepsilon) = \begin{cases} 
1 & \text{if } \varepsilon > 0; \\
0 & \text{if } \varepsilon = 0; \\
-1 & \text{if } \varepsilon < 0.
\end{cases}
$$

**Lemma 2.6.** Let $a, b, s \in \mathbb{R}$ be such that $r$ is differentiable at $b$ and

$$
0 \neq r(b) - r(a) = \|r(b) - r(a)\| \cdot r(s).
$$

Let $r'(b) = x \cdot r(s) + y \cdot r'_{\pm}(s)$ for some $x, y \in \mathbb{R}$ with $y \neq 0$. Then $y > 0$ and for any small $\varepsilon$ we have the asymptotic formula

$$
\|r(b + \varepsilon) - r(a)\| = \|r(b) - r(a)\| + x \cdot \varepsilon - \frac{\text{sign}(\varepsilon) \cdot j(s) \cdot y}{1 + \text{sign}(\varepsilon) \cdot j(s)} \cdot \varepsilon + o(\varepsilon).
$$

**Proof.** The positivity of $y$ follows from $0 \neq r(b) - r(a) = \|r(b) - r(a)\| \cdot r(s)$ and the same orientation of the bases $(r(s), r'_{\pm}(s))$ and $(r(s), r'(b))$, see the following picture.

Since $r(b) - r(a) = \|r(b) - r(a)\| \cdot r(s)$, for a small $\varepsilon$ there exists a small $\delta$ such that

$$r(b + \varepsilon) - r(a) = \|r(b + \varepsilon) - r(a)\| \cdot r(s + \delta).$$
By the differentiability of $r$ at $b$, we obtain
\[
r(b + \varepsilon) - r(a) = r(b) + r'(b)\varepsilon + o(\varepsilon) - r(a) = \|r(b) - r(a)\| \cdot r(s) + (x \cdot r(s) + y \cdot r'_\pm(s))\varepsilon + o(\varepsilon) = \left(\|r(b) - r(a)\| + x \varepsilon + o(\varepsilon)\right) \cdot r(s) + (y \varepsilon + o(\varepsilon)) \cdot r'_\pm(s).
\]
If $\delta > 0$, then
\[
r(s + \delta) = r(s) + r'_+(s)\delta + o(\delta) = (1 + (\dot{j}(s) + o(1))\delta) \cdot r(s) + (1 + \dot{j}(s) + o(1))\delta \cdot r'_+(s).
\]
If $\delta < 0$, then
\[
r(s + \delta) = r(s) + r'_-(s)\delta + o(\delta) = (1 - (\dot{j}(s) + o(1))\delta) \cdot r(s) + (1 - \dot{j}(s) + o(1))\delta \cdot r'_-(s).
\]
In both cases we obtain
\[
r(s + \delta) = (1 + (\text{sign}(\delta) \cdot \dot{j}(s) + o(1))\delta) \cdot r(s) + (1 + \text{sign}(\delta) \cdot \dot{j}(s) + o(1))\delta \cdot r'_\pm(s).
\]
It follows that
\[
\left\|r(b) - r(a)\right\| + x \varepsilon + o(\varepsilon) \cdot r(s) + (y \varepsilon + o(\varepsilon)) \cdot r'_\pm(s) = r(b + \varepsilon) - r(a) = \left\|r(b + \varepsilon) - r(a)\right\| \cdot (1 + (\text{sign}(\delta) \cdot \dot{j}(s) + o(1))\delta) \cdot r(s) + (1 + \text{sign}(\delta) \cdot \dot{j}(s) + o(1))\delta \cdot r'_\pm(s).
\]
Writing this equation in coordinates, we obtain two equations:
\[
(1) \quad \|r(b + \varepsilon) - r(a)\| \cdot (1 + (\text{sign}(\delta) \cdot \dot{j}(s) + o(1))\delta) = \|r(b) - r(a)\| + x \cdot \varepsilon + o(\varepsilon)
\]
and
\[
(2) \quad \|r(b + \varepsilon) - r(a)\| \cdot (1 + \text{sign}(\delta) \cdot \dot{j}(s) + o(1)) \cdot \delta = y \cdot \varepsilon + o(\varepsilon).
\]
The equation (2) implies
\[
\delta = \frac{(y + o(1))\varepsilon}{\|r(b + \varepsilon) - r(a)\| (1 + \text{sign}(\delta) \cdot \dot{j}(s))} = \frac{(y + o(1))\varepsilon}{\|r(b) - r(a)\| (1 + \text{sign}(\delta) \cdot \dot{j}(s))}.
\]
Since $y > 0$ and $|\dot{j}(s)| < 1$, this implies
\[
\text{sign}(\delta) = \text{sign}(\varepsilon) + o(1).
\]
After substitution of $\delta$ into the equation (1), we obtain
\[
\|r(b + \varepsilon) - r(a)\| = \frac{\|r(b) - r(a)\| + x \cdot \varepsilon + o(\varepsilon)}{1 + (\text{sign}(\delta) \cdot \dot{j}(s) + o(1))\delta} = \left(\|r(b) - r(a)\| + x \cdot \varepsilon + o(\varepsilon)\right) \cdot \left(1 - (\text{sign}(\delta) \cdot \dot{j}(s) + o(1))\delta\right) = \left(\|r(b) - r(a)\| + x \cdot \varepsilon + o(\varepsilon)\right) \cdot \left(1 - \frac{(\text{sign}(\varepsilon) \cdot \dot{j}(s) + o(1)) \cdot \varepsilon}{\|r(b) - r(a)\| (1 + \text{sign}(\varepsilon) \cdot \dot{j}(s))}\right) = \|r(b) - r(a)\| + x \cdot \varepsilon - \frac{\text{sign}(\varepsilon) \cdot \dot{j}(s) \cdot y}{1 + \text{sign}(\varepsilon) \cdot \dot{j}(s)} \cdot \varepsilon + o(\varepsilon).
\]
\[
\square
\]
Lemmas 2.6 and 2.5 imply the following corollary.
\[
\textbf{Corollary 2.7.} \text{ Let } a, b, s \in \mathbb{R} \text{ be real numbers such that the map } r \text{ is differentiable at } b \text{ and } 0 \neq r(b) - r(a) = \|r(b) - r(a)\| \cdot r(s).
\]
The map $r$ is differentiable at the point $s$ if and only if the map
\[
\nu : \mathbb{R} \to \mathbb{R}, \quad \nu : \varepsilon \mapsto \|r(b + \varepsilon) - r(a)\|,
\]
is differentiable at zero.
3. Recognizing smooth points on the unit sphere

A point \( x \) of the unit sphere of a Banach space \( X \) is defined to be \textit{smooth} if there exists a unique linear continuous functional \( x^*: X \rightarrow \mathbb{R} \) such that \( x^*(x) = 1 = \|x^*\| \).

The following lemma proved by Cabello Sánchez [4] shows that the smooth points can be recognized by measurements of distances on the sphere.

**Lemma 3.1.** A point \( p \in S_X \) is non-smooth if and only if there exist positive real numbers \( \delta \) and \( \varepsilon_0 \) such that for every positive \( \varepsilon < \varepsilon_0 \) there are points \( x, y \in S_X \) such that

\[
\max\{\|x - p\|, \|y + p\|\} < \varepsilon \quad \text{and} \quad \|u - v\| < (2 - \delta)\varepsilon.
\]

Another smoothness criterion is given by the following lemma.

**Lemma 3.2.** Let \( X \) be a strictly convex 2-dimensional Banach space and \( a, b, c \in S_X \) be points such that \( r \) is differentiable at \( b \) and \( 0 \neq b - a = \|b - a\| \cdot c \). The point \( c \) is not smooth if and only if there exist positive real numbers \( \delta \) and \( \varepsilon_0 \) such that for every positive \( \varepsilon < \varepsilon_0 \) there are two distinct points \( x, y \in S_X \) such that \( \|x - b\| = \varepsilon = \|y - b\| \) and \( \|x - a\| + \|y - a\| > 2\|b - a\| + \delta \cdot \varepsilon \).

**Proof.** Assume that the point \( b \) is smooth and fix any basis \( e_1, e_2 \) for the space \( X \) such that \( e_1 = c \). Let \( r : \mathbb{R} \rightarrow S_X \) be the natural parameterization of the 2-based Banach space \((X, e_1, e_2)\). Then \( r(0) = e_1 = c \). Find real numbers \( \alpha, \beta \) such that \( a = r(\alpha) \) and \( b = r(\beta) \). Since the point \( b \) is smooth, the function \( r \) is differentiable at \( \beta \). Write the derivative \( r'(\beta) \) as \( x \cdot r(0) + y \cdot r'(0) \) for some real numbers \( x, y \). Since \( X \) is strictly convex and \( 0 \neq b - a = \|b - a\| \cdot c \), the vector \( r'(\beta) \) is not parallel to the vector \( c \) and hence \( y \neq 0 \). By Lemma 2.6, \( y > 0 \).

For every \( \varepsilon \in [0, 2) \) let

\[\varepsilon_+ = \min\{s \in [0, +\infty) : \|r(\beta + s) - r(\beta)\| = \varepsilon\}\]

and

\[\varepsilon_- = \max\{s \in (-\infty, 0] : \|r(\beta + s) - r(\beta)\| = \varepsilon\}\].

By Lemma 2.4

\[\varepsilon_+ = \varepsilon + o(\varepsilon) = \varepsilon + o(\varepsilon_+) \quad \text{and} \quad \varepsilon_- = -\varepsilon + o(\varepsilon) = -\varepsilon + o(\varepsilon_-).
\]

If the point \( c \) is not smooth, then \( r'(0) \neq r'_+(0) \) and \( j(0) < 0 \) by Lemma 2.6. Then the number

\[\delta = \frac{-j(s) \cdot y}{1 - j(s)^2}\]

is positive.

By Lemma 2.6, for a small \( \varepsilon > 0 \) we have

\[
\|r(\beta + \varepsilon_+) - r(\alpha)\| + \|r(\beta + \varepsilon_-) - r(\alpha)\| =
\]

\[
= \|b - a\| + x \cdot \varepsilon_+ - \frac{j(s) \cdot y \cdot |\varepsilon_+|}{1 + j(s)} + o(\varepsilon) + \|b - a\| + x \cdot \varepsilon_- - \frac{j(s) \cdot y \cdot |\varepsilon_-|}{1 - j(s)} + o(\varepsilon) =
\]

\[
= 2 \cdot \|b - a\| + x (\varepsilon_+ + \varepsilon_-) \frac{j(s) \cdot y \cdot |\varepsilon_+| (1 + o(1))}{1 + j(s)} + o(\varepsilon) =
\]

\[
= 2 \cdot \|b - a\| + o(\varepsilon) + 2\delta |\varepsilon| + o(\varepsilon) = 2 \cdot \|b - a\| + (2\delta + o(1)) |\varepsilon|.
\]

and hence there exists \( \varepsilon_0 > 0 \) such that for any positive \( \varepsilon < \varepsilon_0 \) and the points \( x = r(\beta + \varepsilon_+) \) and \( y = r(\beta + \varepsilon_-) \) we have

\[
\|x - a\| + |y - a| = \|r(\beta + \varepsilon_+) - r(\alpha)\| + \|r(\beta + \varepsilon_-) - r(\alpha)\| = 2 \cdot \|b - a\| + (2\delta + o(1)) |\varepsilon| > 2 \cdot \|b - a\| + \delta |\varepsilon|.
\]

The choice of \( \varepsilon_+ \) and \( \varepsilon_- \) guarantees that \( \|x - a\| = \varepsilon = \|y - a\| \). This completes the proof of the “only if” part of the lemma.

To prove the “only if” part, assume that there exist positive \( \delta \) and \( \varepsilon_0 \) such that for any positive \( \varepsilon < \varepsilon_0 \) there exist distinct points \( x, y \in S_X \) such that \( \|x - b\| = \varepsilon = \|y - b\| \) and

\[
\|x - a\| + \|y - a\| > 2 \|b - a\| + \delta \cdot \varepsilon.
\]
We can assume that $\varepsilon_0$ is so small that for any positive $\varepsilon < \varepsilon_0$ the set $\{x \in S_X : \|x - b\| = \varepsilon\}$ coincides with the doubleton $\{r(\beta + \varepsilon_+), r(\beta + \varepsilon_-)\}$. Assuming that the point $c$ is smooth, we conclude that $r'_- (0) = r'_+(0)$ and hence $j(0) = 0 = j(0)$. By Lemma 2.6
\[
\|r(\beta + \varepsilon_+) - a\| + \|r(\beta + \varepsilon_-) - a\| = 2 \cdot \|b - a\| + x \cdot (\varepsilon_+ + \varepsilon_-) + o(\varepsilon) = 2 \cdot \|b - a\| + o(\varepsilon).
\]
Replacing $\varepsilon_0$ by a smaller positive number, we can assume that
\[
\|r(\beta + \varepsilon_+) - a\| + \|r(\beta + \varepsilon_-) - a\| = 2 \cdot \|b - a\| + o(\varepsilon) < 2 \cdot \|b - a\| + \delta \cdot |\varepsilon|
\]
for all positive $\varepsilon < \varepsilon_0$. But this contradicts our assumption. This contradiction shows that the point $c$ is not smooth.

\[\square\]

4. Special directions on the unit sphere

**Definition 4.1.** Let $X$ be a Banach space. A point $x \in S_X$ is called **special** if for any bijective isometry $f : S_X \to S_Y$ to the unit sphere of a Banach space $Y$ and any points $y, z \in S_X$ with $y - z = \|y - z\| \cdot x$ we have
\[
f(y) - f(z) = \|f(y) - f(z)\| \cdot f(x) = \|y - z\| \cdot f(x).
\]

**Proposition 4.2.** Every non-smooth point of the unit sphere of a strictly convex 2-dimensional Banach space is special.

**Proof.** Let $c \in S_X$ be a non-smooth point of the sphere $S_X$. Since the Banach space $X$ is strictly convex, the set $c^\perp = \{x \in S_X : \{x\} = S_X \cap (x + \varepsilon Y)\}$ contains exactly two points. The complement $S_X \setminus c^\perp$ has exactly two connected components. Let $A$ be the connected component of $S_X \setminus c^\perp$ containing the point $c$. It follows that $S_X = (-A) \cup p^+ \cup A$.

Let $\theta : A \to -A$ be the function assigning to each point $x \in A$ the unique point $y \in A$ such that $x - y = \|x - y\| \cdot c$. It is clear that the function $\theta$ is injective.

Now take any bijective isometry $f : S_X \to S_Y$ of $S_X$ onto the unit sphere of a Banach space $Y$ and consider the continuous function
\[
g : A \to S_Y, \quad g : x \mapsto \frac{f(x) - f(\theta(x))}{\|f(x) - f(\theta(x))\|} \quad \text{and} \quad f(x) - f(\theta(x)) = f(x) - f(x) = 0.
\]

To prove that the point $c$ is special, it suffices to check that the function $g$ is constant. To derive a contradiction, assume that the function $g$ is not constant. Then for some compact connected subset $K \subseteq A$ the image $g(K)$ is not a singleton. It is clear that $K$ is homeomorphic to the closed interval $[0, 1]$. By a classical result of Sierpiński [12], the unit interval $[0, 1]$ cannot be covered by countably many pairwise disjoint closed proper subsets.

Let $\Lambda_X, \Lambda_Y$ be the sets of non-smooth points on the spheres $S_X$ and $S_Y$, respectively. By Lemma 2.2 the sets $\Lambda_X, \Lambda_Y$ are almost countable. For every $y \in \Lambda_Y$ consider the closed set
\[
K_y = K \cap g^{-1}(y),
\]
which is not equal to $K$ as $g(K) \neq \{y\}$. Then the countable family $K = \{K_y : y \in \Lambda_Y\} \cup \{\{x\} : x \in \Lambda_X \setminus \bigcup_{y \in \Lambda_Y} K_y\}$ consists of pairwise disjoint closed subsets of $K$. By the mentioned result of Sierpiński, $K \neq \bigcup K$. Consequently, there exists a point $b \in K \setminus \bigcup K$. It follows that the image $g(b)$ is a smooth point of $S_Y$ and $b \notin \Lambda_X$, which means that the point $b$ is smooth. Let $a = \theta(b) \in -A$. The definition of the map $\theta$ ensures that $b - a = \|b - a\| \cdot c$.

Since the point $c$ is not smooth, we can apply Lemma 3.1 and find positive $\delta$ and $\varepsilon_0$ such that for every positive $\varepsilon < \varepsilon_0$ there exist two distinct points $x, y \in S_X$ such that $\|x - b\| = \varepsilon = \|y - b\|$ and $\|x - a\| + \|y - a\| > 2 \|b - a\| + \delta \varepsilon$.

Since $f : S_X \to S_Y$ is an isometry and $b$ is a smooth point of the sphere $S_X$, its image $f(b)$ is a smooth point of the sphere $S_Y$, according to Lemma 3.1. Observe that
\[
0 \neq f(b) - f(a) = f(b) - f(\theta(b)) = \|b - \theta(b)\| \cdot g(b) = \|b - a\| \cdot g(a) = f(b) - f(a) \cdot g(a).
\]

Since the point $g(a) \in S_X$ is smooth we can apply Lemma 3.1 and find a positive $\varepsilon > 0$ such that for any distinct points $u, v \in S_Y$ with $\|u - f(b)\| = \varepsilon = \|v - f(b)\|$ we have $\|u - f(a)\| + \|v - f(a)\| \neq 2 \cdot \|f(b) - f(a)\| + \delta \varepsilon$. 

Since $f(b) - f(a) = f(b) - f(\theta(b)) = \|b - \theta(b)\| \cdot g(b) = \|b - a\| \cdot g(a) = f(b) - f(a) \cdot g(a)$.
By the choice of $\delta$ and $\varepsilon_0$, there exist points $x, y \in S_X$ such that $\|x - b\| = \varepsilon = \|y - b\|$ and $\|x - b\| + \|y - b\| > 2 \cdot \|b - a\| + \delta \varepsilon$. Since $f : S_X \to S_Y$ is an isometry, for the points $u = f(x)$ and $v = f(y)$, we obtain

$$\|f(x) - f(b)\| = \|x - b\| = \varepsilon = \|y - b\| = \|f(y) - f(b)\|$$

and

$$\|f(x) - f(a)\| + \|f(y) - f(a)\| = \|x - a\| + \|y - a\| > 2 \cdot \|b - a\| + \delta \varepsilon = 2 \cdot \|f(b) - f(a)\| + \delta \varepsilon,$$

which contradicts the choice of $\varepsilon$. \hfill \Box

**Definition 4.3.** Let $x, y \in S_X$ be two points on the unit sphere of a 2-dimensional Banach space $X$. The pair $(x, y)$ is called

- *singular* if $\{x, -x\} \cap \{y, -y\} \neq \emptyset$ or there exists a point $z \in S_X$ such that $S_X \cap (z + \mathbb{R}x) = \{z\} = S_X \cap (z + \mathbb{R}y)$;
- *regular* if it is not singular.

**Lemma 4.4.** Let $u, v \in S_X$ be two points on the unit sphere of a strictly convex 2-dimensional Banach space $X$. If the pair $(u, v)$ is regular, then for any point $w \in S_X$ there exist points $x, y, z \in S_X$ such that

$$x \in z + \mathbb{R}u \quad y \in z + \mathbb{R}v \quad \text{and} \quad x - y \in \mathbb{R}w.$$

**Proof.** By the strict convexity of $X$, for every $z \in S_X$ there exist unique points $\bar{u}(z), \bar{v}(z) \in S_X$ such that $\{z, \bar{u}(z)\} = S_X \cap (z + \mathbb{R}u)$ and $\{z, \bar{v}(z)\} = S_X \cap (z + \mathbb{R}v)$. It is easy to see that the function

$$\varphi : S_X \to X, \quad \varphi : z \mapsto \bar{u}(z) - \bar{v}(z),$$

is continuous. Since the pair $(u, v)$ is regular, $\varphi(z) \neq 0$ for any $z \in S_X$. Then the function

$$\psi : S_X \to S_X, \quad \psi : z \mapsto \frac{\varphi(z)}{\|\varphi(z)\|} = \frac{\bar{u}(z) - \bar{v}(z)}{\|\bar{u}(z) - \bar{v}(z)\|}$$

is well-defined and continuous.

By the central symmetry of $S_X$, for every point $z \in S_X$ we have $\bar{u}(-z) = -\bar{u}(z)$ and $\bar{v}(-z) = -\bar{v}(z)$, which implies $\psi(-z) = -\psi(z)$. Therefore, the image $\psi(S_X)$ of $S_X$ contains two opposite points. By the connectedness of $\psi(S_X)$, for any $w \in S_X$ there exists $z \in S_X$ such that

$$w = \psi(z) = \frac{\bar{u}(z) - \bar{v}(z)}{\|\bar{u}(z) - \bar{v}(z)\|}.$$

Then the points $x = \bar{u}(z), y = \bar{v}(z)$ and $z$ have the required properties. \hfill \Box

**Lemma 4.5.** If the unit sphere of a strictly convex 2-dimensional Banach space $X$ contains two linearly independent special points, then $S_X$ contains two special points $x, y$ such that the pair $(x, y)$ is regular.

**Proof.** Let $x, y \in S_X$ be two linearly independent special points on the unit sphere of $X$. If the pair $(x, y)$ is regular, then we are done. So, assume that $(x, y)$ is singular. Then there exists a point $z \in S_X$ such that $S_X \cap (z + \mathbb{R}x) = \{z\} = S_X \cap (z + \mathbb{R}y)$. It follows that $z$ is a non-smooth point of the sphere $S_X$. By Proposition 1.2, the point $z$ is special. Taking into account that the Banach space $X$ is strictly convex, it can be shown that the pairs $(x, z)$ and $(y, z)$ are regular.
The following helpful fact was proved by Tingley in [14].

**Lemma 4.6.** If \( f : S_X \to S_Y \) is a bijective isometry between unit spheres of finite-dimensional Banach spaces, then \( f(-x) = -f(x) \) for all \( x \in S_X \).

**Proposition 4.7.** A 2-dimensional Banach space \( X \) has the Mazur–Ulam property if its sphere contains two linearly independent special points.

**Proof.** If \( X \) is not strictly convex, then \( X \) has the Mazur–Ulam property by Theorem 1.3. So, we assume that \( X \) is strictly convex. Let \( u, v \in S_X \) be two linearly independent special points on \( S_X \). By Lemma 4.4, we can additionally assume that the pair \((u, v)\) is regular.

Let \( f : S_X \to S_Y \) be a bijective isometry of \( S_Y \) onto the unit sphere \( S_Y \) of an arbitrary Banach space \( Y \). Let \( L : X \to Y \) be a unique linear operator such that \( L(u) = f(u) \) and \( L(v) = f(v) \). We claim that \( L|S_X = f \). Fix any point \( w \in S_X \) and using Lemma 4.4 find points \( x, y, z \in S_X \) such that \( x = z + au, y = z + bv \) and \( x - y \in cw \) for some real numbers \( a, b, c \). Replacing \( x, y, z \) by \(-x, -y, -z\), we can assume that \( c > 0 \). Observe that \( |a| = \|x - z\| \). If \( a > 0 \), then by the special property of \( u \), the equality \( x = z + au = z + \|x - z\|u \) implies \( f(x) = f(z) + \|x - z\| \cdot f(u) = a \cdot f(u) \).

If \( a < 0 \), then the equality \( x = z + au = z - \|x - z\|u \) implies \( z = x + \|x - z\|u \). By the special property of \( u \), we have \( f(z) = f(x) + \|x - z\| \cdot f(u) = f(x) - a \cdot f(u) \) and hence \( f(x) = f(z) + a \cdot f(u) \). In both cases we obtain \( f(x) = f(z) + a \cdot f(u) \). By analogy we can show that \( f(y) = f(z) + b \cdot f(v) \).

Then

\[
f(x) - f(y) = (f(z) + a \cdot f(u)) - (f(z) + b \cdot f(v)) = a \cdot f(u) - b \cdot f(v) = a \cdot L(u) - b \cdot L(v).
\]

It follows from \( x - y = c \cdot w = \|x - y\| \cdot w \) that

\[
L(u) = \frac{L(x) - L(y)}{\|x - y\|} = \frac{L(z + au) - L(z + bv)}{\|x - y\|} = \frac{aL(u) - bL(v)}{\|x - y\|} = \frac{f(x) - f(y)}{\|x - y\|}
\]

and finally

\[
\|L(u)\| = \frac{\|f(x) - f(y)\|}{\|x - y\|} = 1.
\]

Therefore, the linear operator \( L : X \to Y \) is an isometry.

Consider the isometry \( g = L^{-1} \circ f : S_X \to S_X \) and observe that \( g(u) = u \) and \( g(v) = v \). By Lemma 4.4, \( g(-u) = -g(u) = -u \) and \( g(-v) = -g(v) = -v \). Since the space \( X \) is strictly convex, for any distinct points \( x, y \in X \) and positive real numbers \( a, b \) the intersection \( (x + aS_X) \cap (y + bS_X) \) contains at most two distinct points. This fact can be used to show that each point \( x \in S_X \) is the unique point of the intersection

\[
(u + \|x - u\|S_X) \cap (-u + \|x + u\|S_X) \cap (v + \|x - v\|S_X) \cap (-v + \|x + v\|S_X)
\]
which implies that \( g(x) = x \) and hence \( f = L|S_X \).

5. Banach spaces with exactly two non-smooth points on the unit sphere

Let \( X \) be a strictly convex 2-dimensional Banach space whose unit sphere contains exactly two non-smooth points. Let \( e_1 \in S_X \) be one of these non-smooth points. Take any vector \( e_2 \in X \)
which is linearly independent with \( e_1 \) and consider the natural parametrization \( r : \mathbb{R} \to S_X \) of the 2-based Banach space \( (X, e_1, e_2) \). For this parametrization we have \( r(0) = e_1 \) and \( r'(0) \neq r'_\perp(0) \) as \( e_1 = r(0) \) is a non-smooth point of the unit sphere. Replacing the vector \( e_2 \) by \( r'_\perp(0) = \frac{1}{2}(r'_+(0) + r'_-(0)) \), we can assume that \( e_2 = r'_-(0) \).

We recall that

\[
\frac{1}{2}(r'_+(0) - r'_-(0)) = j(0) \cdot r(0) + j(0) \cdot r'_\perp(0) = j(0) \cdot e_1 + j(0) \cdot e_2
\]

and the numbers \( j(0), j'(0) \) are called radial and tangential jumps of the derivative \( r' \) at zero. By Lemma 2.5 \( |j(0)| < 1 \). We claim that those jumps are determined by the metric of the unit sphere.

For every point \( x \in S_X \) let \( \overline{r} \) be the unique point of the sphere such that \( \{x, \overline{r}\} = S_X \cap (x + \mathbb{R}e_1) \). The uniqueness of \( \overline{r} \) follows from the strict convexity of \( X \).

**Lemma 5.1.**

\[
\lim_{\varepsilon \to 0^+} \frac{\|r(\varepsilon) - r(0)\|}{\|r(\varepsilon) + r(0)\|} = \frac{1 - j(0)}{1 + j(0)} \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{\|r(\varepsilon) - \overline{r}(\varepsilon)\| - 2 \varepsilon}{\|r(\varepsilon) + r(0)\|} = \frac{j(0)}{1 - j(0)}.
\]

**Proof.** For a small positive \( \varepsilon \), find a positive \( \delta \) such that \( \overline{r}(\varepsilon) = -r(-\delta) \). Observe that

\[
r(\varepsilon) = r(0) + r'_+(0)\varepsilon + o(\varepsilon) = e_1 + (j(0)e_1 + (1 + j(0))e_2)\varepsilon + o(\varepsilon) = (1 + j(0)\varepsilon + o(\varepsilon))e_1 + (1 + j(0) + o(1))\varepsilon e_2
\]

and

\[
r(-\delta) = r(0) - r'_-(0)\delta + o(\delta) = e_1 - (j(0)e_1 + (1 - j(0))e_2)\delta + o(\delta) = (1 + j(0)\delta + o(\delta))e_1 - (1 - j(0) + o(1))\delta e_2.
\]

The equality \(-r(-\delta) = \overline{r}(\varepsilon)\) implies \((1 + j(0) + o(1))\varepsilon = (1 - j(0) + o(1))\delta\) and

\[
\varepsilon \delta = \frac{1 - j(0) + o(1)}{1 + j(0) + o(1)}.
\]

Then

\[
\lim_{\varepsilon \to 0^+} \frac{\|r(\varepsilon) - r(0)\|}{\|r(\varepsilon) + r(0)\|} = \lim_{\varepsilon \to 0^+} \frac{\|r'_+(0) + o(1)\varepsilon\|}{\|-r'(-\delta) + r(0)\|} = \lim_{\varepsilon \to 0^+} \frac{\|r'_+(0) + o(1)\varepsilon\| \cdot \varepsilon}{\|r'(-\delta) + o(1)\delta\|} = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\delta} = \frac{1 - j(0)}{1 + j(0)}.
\]

On the other hand,

\[
\|r(\varepsilon) - \overline{r}(\varepsilon)\| = \|r(\varepsilon) + r(-\delta)\| = \|(1 + j(0)\varepsilon + o(\varepsilon))e_1 + (1 + j(0) + o(1))\varepsilon e_2 + (1 + j(0)\delta + o(\delta))e_1 - (1 - j(0) + o(1))\delta e_2\| = \|(2 + j(0)(\varepsilon + \delta) + o(\varepsilon + \delta))e_1\| = 2 + (j(0) + o(1))\varepsilon \left(1 + \frac{j(0) + o(1)}{1 - j(0) + o(1)}\right) = 2 + 2\varepsilon j(0) + o(1)
\]

and hence

\[
\lim_{\varepsilon \to 0^+} \frac{\|r(\varepsilon) - \overline{r}(\varepsilon)\| - 2 \varepsilon}{\|r(\varepsilon) + r(0)\|} = \frac{j(0)}{1 - j(0)}.
\]

□

**Lemma 5.2.** Let \( s, \bar{s} \in \mathbb{R} \) be two distinct real numbers such that \( 0 \neq r(s) - r(\bar{s}) = \|r(s) - r(\bar{s})\| \cdot e_1 \) and \( r(s) \notin \{e_1, -e_1\} \). Let \( r'(s) = xe_1 + ye_2 \) and \( r'(\bar{s}) = \overline{x}e_1 + \overline{y}e_2 \) for some real numbers \( x, y, \overline{x}, \overline{y} \).

For a small real number \( \varepsilon \) let \( \tilde{\varepsilon} \) be the unique small real number such that \( r(\bar{s} + \tilde{\varepsilon}) = \overline{r}(s + \varepsilon) \). Then

1. \( y > 0 \) and \( \overline{y} < 0 \);

□
The equation (2) allows us to find $\bar{y}$ and then $\bar{\tau}$ can be found from the equation (3).

**Proof.** It follows from $0 \neq \bar{r}(s) - \bar{r}(\bar{s}) = \|\bar{r}(s) - \bar{r}(\bar{s})\| \cdot e_1$ that $y > 0$ and $\bar{y} < 0$. For a small number $\varepsilon$ we have

$$\bar{r}(s + \varepsilon) - \bar{r}(s) = (\bar{r}'(s) + o(1))\varepsilon = (x + o(1))\varepsilon e_1 + (y + o(1))\varepsilon e_2$$

and

$$\bar{r}(\bar{s} + \varepsilon) - \bar{r}(\bar{s}) = (\bar{r}'(\bar{s}) + o(\varepsilon))\varepsilon = (\bar{x} + o(1))\varepsilon \bar{e}_1 + (\bar{y} + o(1))\varepsilon \bar{e}_2.$$ 

The equality $\bar{r}(s + \varepsilon) = \bar{r}(\bar{s} + \varepsilon)$ implies

$$(y + o(1))\varepsilon = (\bar{y} + o(1))\varepsilon$$

and then

$$\lim_{\varepsilon \to 0} \frac{\|\bar{r}(s + \varepsilon) - \bar{r}(\bar{s} + \varepsilon)\|}{\|\bar{r}(s + \varepsilon) - \bar{r}(s)\|} = \lim_{\varepsilon \to 0} \frac{\|\bar{r}'(s) + o(1)\| \cdot \varepsilon}{\|\bar{r}'(s) + o(1)\| \cdot \varepsilon} = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{|y + o(1)|}{|\bar{y}|} = \frac{|y|}{|\bar{y}|} = \frac{y}{\bar{y}}.$$

Also

$$\bar{r}(s + \varepsilon) - \bar{r}(\bar{s} + \varepsilon) = \bar{r}(s) + \bar{r}'(s)\varepsilon + o(\varepsilon) - (\bar{r}(\bar{s}) + \bar{r}'(\bar{s}) + o(\varepsilon)) =$$

$$= (\bar{r}(s) - \bar{r}(\bar{s})) + (x + o(1))\varepsilon e_1 + (y + o(1))\varepsilon \bar{e}_1 - (\bar{x} + o(1))\varepsilon \bar{e}_1 - (\bar{y} + o(1))\varepsilon \bar{e}_2 =$$

$$= (\|\bar{r}(s) - \bar{r}(\bar{s})\| e_1 + (x\varepsilon - x\bar{e}_1 + o(\varepsilon + \bar{\varepsilon})) \cdot e_1 =$$

and hence

$$\lim_{\varepsilon \to 0} \frac{\|\bar{r}(s + \varepsilon) - \bar{r}(\bar{s} + \varepsilon)\|}{\|\bar{r}(s) - \bar{r}(\bar{s})\|} = x - \bar{x} \cdot \frac{y}{\bar{y}}.$$ 

By Lemma 2.6,

$$\|\bar{r}(s + \varepsilon) - \bar{r}(s)\| = \|\bar{r}(s) - \bar{r}(\bar{s})\| + x\varepsilon - \frac{\text{sign}(\varepsilon) \cdot j(0) \cdot y}{1 + \text{sign}(\varepsilon) j(0)} \varepsilon + o(\varepsilon)$$

and hence

$$\lim_{\varepsilon \to 0} \frac{\|\bar{r}(s + \varepsilon) - \bar{r}(s)\| - \|\bar{r}(s) - \bar{r}(\bar{s})\|}{\varepsilon} = \frac{j(0) \cdot y}{1 + j(0)}$$

and

$$\lim_{\varepsilon \to 0} \frac{\|\bar{r}(s + \varepsilon) - \bar{r}(s)\| - \|\bar{r}(s) - \bar{r}(\bar{s})\|}{\varepsilon} = x + \frac{j(0) \cdot y}{1 - j(0)}.$$ 

Therefore, the items (1)–(5) of Lemma 5.2 are proved.

The equations (4), (5) determine the numbers $x, y$ uniquely because

$$\begin{bmatrix} 1 & -\frac{j(0) \cdot y}{1 + j(0)} \\ 1 & \frac{j(0) \cdot y}{1 - j(0)} \end{bmatrix} = \begin{bmatrix} j(0) \cdot y \\ j(0) \cdot y \end{bmatrix} = \frac{2 \cdot j(0) \cdot y}{1 - j(0)^2} \neq 0.$$ 

The equation (2) allows us to find $\bar{y}$ and then $\bar{\tau}$ can be found from the equation (3). □
6. Proof of Theorem 1.4

Given any non-smooth 2-dimensional Banach space, we should prove that $X$ has the Mazur–Ulam property. If $X$ is not strictly convex, then $X$ has the Mazur–Ulam property by Theorem 1.3. If the sphere $S_X$ contains more than two non-smooth points, then $X$ has the Mazur–Ulam property by Propositions 4.2 and 4.7. So, we assume that $X$ is strictly convex and $S_X$ contains exactly two non-smooth points. Let $e_1$ be one of them. Then $-e_1$ is the other non-smooth point of $X$.

Take any vector $e_2 \in X \setminus (\mathbb{R} \cdot e_1)$ and consider the natural parameterization $\tilde{r} : \mathbb{R} \to X$ of the 2-based Banach space $(X, e_1, e_2)$. Since $e_1$ is a non-smooth point of $S_X$, the one-sided derivatives $r'_-(0)$ and $r'_+(0)$ are distinct. Replacing the vector $e_2$ by $r'_-(0)$, we can assume that $e_2 = r'_+(0)$. Let $L = \min \{ s \in [0, \infty) : r(s) = -e_1 \}$ be the half-length of the sphere $S_X$, and $\tilde{S}_X = r([0, L])$ be the upper half-sphere of $X$. By Lemmas 2.2 and 2.3, the restriction $r|_{[0, L]} : [0, L] \to \tilde{S}_X$ in an isometry of $[0, L]$ onto the half-sphere $\tilde{S}_X$ endowed with the intrinsic metric.

To show that the space $X$ has the Mazur–Ulam property, fix any bijective isometry $f : S_X \to S_Y$ of $S_X$ onto the unit sphere of an arbitrary Banach space $Y$. It is clear that the space $Y$ is 2-dimensional. Lemma 3.2 implies that $e_1 = f(e_1)$ and $-e_1$ are unique non-smooth points of the sphere $S_Y$. Repeating the above argument, we can find a vector $\tilde{e}_2 \in Y \setminus (\mathbb{R} \cdot \tilde{e}_1)$ such that for the natural parameterization $\tilde{r} : \mathbb{R} \to Y$ of the 2-based Banach space $(Y, \tilde{e}_1, \tilde{e}_2)$ we have $\tilde{r}'_+(0) = \tilde{e}_2$. Replacing $\tilde{e}_2$ by $-\tilde{e}_2$ we can additionally assume that $\tilde{r}(\varepsilon) \in f(\tilde{S}_X)$ for small positive numbers $\varepsilon$.

Then $f(\tilde{S}_X)$ coincides with the half-sphere $\tilde{S}_Y$ of the 2-based Banach space $(Y, \tilde{e}_1, \tilde{e}_2)$. Since $\tilde{S}_X$ is isometric to $\tilde{S}_Y$, the spheres $S_X$ and $S_Y$ have the same half-length. By Lemma 2.3, the restriction $\tilde{r}_|[0,L] : [0, L] \to \tilde{S}_Y$ is an isometry of $[0, L]$ onto the half-sphere $\tilde{S}_Y$ endowed with the intrinsic metric. The isometry $f|_{\tilde{S}_X} : \tilde{S}_X \to \tilde{S}_Y$ remains an isometry with respect to the intrinsic metrics on the half-spheres. Then $(\tilde{r}_|[0,L])^{-1} \circ f \circ r|_{[0,L]}$ is an isometry of $[0, L]$ which does not move zero and hence is an identity map of $[0, L]$. Consequently, $\tilde{r}(s) = f \circ r(s)$ for all $s \in [0, L]$. Using this fact and Lemma 2.2(1), we can show that $\tilde{r}(s) = f \circ r(s)$ for all $s \in \mathbb{R}$.

Let $I : X \to Y$ be the linear operator such that $I(e_1) = \tilde{e}_1$ and $I(e_2) = \tilde{e}_2$. Applying Proposition 4.2 and Lemmas 4.6 and 5.1 we can show that the spheres $S_X$ and $S_Y$ have the same radial and tangential jumps $j(0)$ and $\dot{j}(0)$. Lemma 4.2 implies that $I(r'(s)) = \tilde{r}'(s)$ for every $s \in (0, L)$ with $r(s) \neq \tilde{r}(s)$. Since $X$ is strictly convex, the set $\{ s \in [0, L] : r(s) = \tilde{r}(s) \}$ is a singleton. Now the continuity of the functions $r'$ and $\tilde{r}'$ on $(0, L)$ implies that $I(r'(s)) = \tilde{r}'(s)$ for all $s \in (0, L)$. Since $r$ is continuously differentiable on $[0, L]$, for every $s \in [0, L]$ we have $r(s) = r(0) + \int_0^s r'(t) \, dt$ and hence

$$I(r(s)) = I(r(0)) + \int_0^s I(r'(t)) \, dt = \tilde{r}(0) + \int_0^s \tilde{r}'(t) \, dt = \tilde{r}(s) = f \circ r(s).$$

By Lemmas 4.6 and 2.2(1),

$$I(r(s + L)) = I(-r(s)) = -I(r(s)) = -f(r(s)) = f(-r(s)) = f(r(s + L))$$

for every $s \in [0, L]$ and hence $I \circ r|_{[L, 2L]} = \tilde{r}|_{[L, 2L]}$. Therefore, $I$ is a linear operator extending the isometry $f$. The equality $I(S_X) = S_Y$ implies $I(B_X) = B_Y$, which means that $I$ is a linear isometry of the Banach spaces $X, Y$.

Acknowledgements

The second author has been partially supported by Junta de Extremadura programs GR-15152 and IB-16056 and DGICYT projects MTM2016-76958-C2-1-P and PID2019-103961GB-C21 (Spain).
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