An Efficient Global Algorithm for Single-Group Multicast Beamforming

Cheng Lu, Ya-Feng Liu

Abstract

Consider the single-group multicast beamforming problem, where multiple users receive the same data stream simultaneously from a single transmitter. The problem is NP-hard and all existing algorithms for the problem either find suboptimal approximate or local stationary solutions. In this paper, we propose an efficient branch-and-bound algorithm for the problem that is guaranteed to find its global solution. To the best of our knowledge, our proposed algorithm is the first tailored global algorithm for the single-group multicast beamforming problem. Simulation results show that our proposed algorithm is computationally efficient (albeit its theoretical worst-case iteration complexity is exponential with respect to the number of receivers) and it significantly outperforms a state-of-the-art general-purpose global optimization solver called Baron. Our proposed algorithm provides an important benchmark for performance evaluation of existing algorithms for the same problem. By using it as the benchmark, we show that two state-of-the-art algorithms, semidefinite relaxation algorithm and successive linear approximation algorithm, work well when the problem dimension (i.e., the number of antennas at the transmitter and the number of receivers) is small but their performance deteriorates quickly as the problem dimension increases.

Index Terms

Argument cuts, branch-and-bound algorithm, convex relaxation, multicasting, global optimality, transmit beamforming.

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I. INTRODUCTION

Physical layer multicasting via transmit beamforming has been recognized as a powerful technique for efficient audio and video streaming in multi-user multi-antenna wireless communication networks. For instance, multicast beamforming is a part of the Evolved Multimedia Broadcast Multicast Service (eMBMS) in the Long-Term Evolution (LTE) standard. Multicast beamforming exploits channel state information at the transmitter and utilizes multiple transmit antennas to broadcast common information to a preselected group of users.

One scenario of particular interest in this paper is single-group multicast beamforming, where all users receive the same data stream from the transmitter and the data rate is determined by the minimum received signal-to-noise-ratio (SNR). The earliest mathematical formulation of the single-group multicast beamforming problem is to maximize the average SNR subject to the total transmission power constraint [1]. This formulation is simple, and can be solved efficiently. However, the solution that maximizes the average SNR does not consider the SNR of each individual user, so that the minimum received SNR may be significantly lower than the average SNR. Hence, the solution obtained by maximizing the average SNR does not always achieve a satisfactory common data rate. To overcome this drawback, reference [2] proposed two new problem formulations, the quality of service (QoS) constrained problem formulation and the max-min fairness problem formulation, where the former one minimizes the total transmission power subject to SNR constraints of all receivers and the latter one maximizes the minimum SNR among all users subject to the total transmission power constraint. These two new formulations can guarantee QoS of each user and are shown to be equivalent from an optimization point of view [2]. Unfortunately, the two problems are NP-hard in general [3] and thus it is impossible to solve them to global optimality in polynomial time (unless P=NP) [4].

Various algorithms have been proposed to solve the single-group multicast beamforming problem; see [3], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20] and references therein. Based on our knowledge, these algorithms are either convex relaxation based algorithms or convex approximation based algorithms. Due to the NP-hardness of the problem, none of them can guarantee that they can find the global solution of the problem (except for very special problem instances [11], [21]). Moreover, since there might exist large gaps between the original problem and its convex relaxations/approximations, the quality of the returned solution...
by the aforementioned algorithms based on these convex relaxations/approximations might be poor.

The goal of this paper is to develop a computationally efficient global algorithm for solving the single-group multicast beamforming problem that is guaranteed to find the global solution of the problem.

A. Related Works

One of the state-of-the-art algorithms for the single-group multicast beamforming problem is the semidefinite relaxation (SDR) algorithm [3]. The main observation behind the SDR algorithm is that the beamforming problem can be equivalently formulated as a rank-one constrained semidefinite program (SDP). The SDR algorithm drops the rank-one constraint, solves the SDR, and then applies a Gaussian randomization strategy to generate a rank-one approximate solution of the original problem based on the obtained solution of the SDR. The SDR algorithm is capable of finding high quality approximate solutions when the number of antennas at the transmitter and the number of users are small. However, the performance of the SDR algorithm deteriorates quickly as the number of users increases. As justified in [22], the provable (worst-case) approximation accuracy of the solution returned by the SDR algorithm degrades linearly with the number of users.

One of the best algorithms for the problem of interest is the successive linear approximation (SLA) algorithm [5]. At each iteration, the SLA algorithm approximates the original nonconvex problem by using its first-order Taylor series expansion at the current iterate and then solves the resulting convex quadratic program to generate the next iterate. It has been shown in [5] that the sequence of points generated by the SLA algorithm converges to a KKT point of the original beamforming problem. Moreover, numerical simulation results in [5] have shown that the SLA algorithm performs better than the SDR algorithm.

Recently, an alternating maximization (AM) algorithm for the single-group multicast beamforming problem was proposed in [6]. Recall that the beamforming problem is equivalent to an SDP with a rank-one constraint [3]. The main contribution of [6] is that it reformulated the problem as a nonconvex problem without the rank-one constraint, which is naturally amenable to AM. Numerical simulation results in [6] show that the AM algorithm performs better than the SDR algorithm. However, further simulation results in [7] show that the AM algorithm performs
worse and has higher complexity than the SLA algorithm.

We also mention some interesting related works, which extend the previously mentioned works \cite{3, 5, 6} on the single-group multicast beamforming problem from different aspects. One extension is to develop more efficient algorithms for solving the problem when computational efficiency is a big issue, i.e., when the channel changes fast over time. Along this direction, low-complexity algorithms \cite{8, 9, 10, 11, 12} as well as adaptive (online) algorithms (which learn the channel correlation matrices) \cite{13, 14, 15, 16} have been proposed. These algorithms generally are very fast, but their performance is worse than the ones of the SDR algorithm \cite{3} and the SLA algorithm \cite{5}. Recently, a new adaptive multiplicative update (MU) algorithm and a hybrid MU-SLA algorithm based on it have been proposed in \cite{7}. The MU-SLA algorithm runs the SLA algorithm for only one iteration with the returned point by the MU algorithm as the initial point.

Another extension is to develop new physical-layer transmit strategies instead of using the beamforming transmit strategy. Along this direction, a beamformed Alamouti scheme has been independently proposed in \cite{17, 18, 19}, which can be seen as a rank-two generalization of the previous (rank-one) SDR beamforming framework. It has been shown in \cite{20} that the worst-case approximation accuracy of the beamformed Alamouti scheme degrades only at a rate of the square root of the number of the users. This improves over the beamforming strategy, where the approximation accuracy degrades at a rate of the number of users \cite{22}.

B. Our Contributions

This is the first paper that proposes a tailored efficient global algorithm for solving the single-group multicast beamforming problem, which is in sharp contrast to all existing works that focus on the design of approximation algorithms or local optimization algorithms. Our proposed algorithm is based on the branch-and-bound strategy combined with a new argument cut technique. The argument cut (see Definition \ref{def:argument_cut}) itself is interesting and is a technical contribution of this paper.

Since the single-group multicast beamforming problem is NP-hard, there does not exist a polynomial time algorithm which can solve it to global optimality (unless P=NP) \cite{4}. Therefore, our proposed algorithm has an exponential worst-case iteration complexity with respect to the number of users (see Theorem \ref{thm:complexity}). However, our simulation results show that our proposed
algorithm is highly efficient and it significantly outperforms the state-of-the-art general-purpose global optimization solver Baron [23], [24]. The high efficiency of our proposed algorithm is mainly due to the new argument cuts. More specifically, our proposed algorithm can globally solve small-scale problem instances with 2 antennas at the transmitter and 8 users within 0.3 seconds on average while Baron needs 110.4 seconds; our proposed algorithm can solve median-scale problem instances with 4 antennas at the transmitter and 8 users within 2.8 seconds on average while Baron fails to solve the same problem instances within 10 minutes.

Even though in some scenarios (i.e., both the number of antennas at the transmitter and the number of users are large) it takes our proposed algorithm a (relatively) large number of iterations (and thus long time) to find the global solution, our proposed algorithm is still attractive from the following two aspects. First, when the channel changes slowly with time and computational efficiency is not an issue, our proposed algorithm is capable of finding the global solution and therefore provides (potentially much) better performance compared to the existing algorithms. Second, when the channel changes fast with time and computational efficiency becomes an issue in this case, our proposed algorithm can serve as a benchmark to evaluate the performance of the existing heuristic/local optimization algorithms. Our simulation results show that the relative gaps between the objective values at the solutions returned by the SDR algorithm and the SLA algorithm and the optimal objective value (achieved by our proposed algorithm) exceed 50% for some problem instances with a (relatively) large number of antennas at the transmitter and a (relatively) large number of receivers.

C. Organization and Notations

The organization of this paper is as follows. In Section II, we introduce the QoS constrained formulation of the single-group multicast beamforming problem, and review two state-of-the-art algorithms for solving the problem. In Section III, we propose a global branch-and-bound algorithm for solving the problem. Simulation results are presented in Section IV to illustrate the efficiency of our proposed algorithm and the paper is concluded in Section V.

We adopt the following notations in this paper. We use lowercase boldface and uppercase boldface letters to denote (column) vectors and matrices, respectively. We use \( \mathbb{C} \) (\( \mathbb{R} \)) to denote the complex (real) domain, \( \mathbb{C}^n \) (\( \mathbb{R}^n \)) to denote the set of the \( n \)-dimensional complex (real) column vectors, and \( \mathbb{C}^{m \times n} \) to denote the set of \( m \times n \) complex matrices. For a given complex
number $c$, the notations $\text{Re}(c)$, $\text{Im}(c)$, and $\text{arg}(c_i)$ stand for its real part, its imaginary part, and its argument, respectively. For a given (complex) vector $x$, $\|x\|$ denotes its Euclidean norm, $x^H$ denotes its Hermitian, and $x^T$ denotes its transpose. The same notations $A^H$ and $A^T$ also apply to the matrix $A$. For a given complex Hermitian matrix $A$, $A \succeq 0$ means $A$ is positive semidefinite, $\text{Trace}(A)$ denotes the trace of the matrix $A$, and $\text{Rank}(A)$ denotes the rank of the matrix $A$. For two given Hermitian matrices $A$ and $B$, $A \succeq B$ means $A - B \succeq 0$. Finally, we use $i$ to denote the imaginary unit which satisfies the equation $i^2 = -1$ and use $I_N$ to denote the $N \times N$ identity matrix.

II. PROBLEM FORMULATION AND REVIEW

In this section, we first introduce the QoS constrained formulation of the single-group multicast beamforming problem and then review two state-of-the-art algorithms for solving the problem.

A. Problem Formulation

Consider the single-group multicast beamforming problem for the multi-user multi-input single-output (MISO) downlink channel, where the base station (transmitter) is equipped with $N$ antennas, and broadcasts a common data stream to $M$ users with a single antenna. Let $h_k \in \mathbb{C}^N$ denote the channel vector between the base station and the $k$-th receiver and let $w \in \mathbb{C}^N$ denote the beamforming vector used by the base station. Assume that $s(t)$ be the broadcasting data stream. Then the transmitted signal by the base station is given by $s(t)w$ and the received signal at the $k$-th receiver is given by

$$y_k(t) = s(t)h_k^Hw + n_k(t),$$

where $n_k(t)$ is the additive white Gaussian noise (AWGN) with variance $\sigma_k^2$. Assume that the transmitted signal $s(t)$ has a unit power. Then, the SNR of the $k$-th user can be written as

$$\text{SNR}_k = \frac{|h_k^Hw|^2}{\sigma_k^2}, \quad k = 1, 2, \ldots, M.$$

This paper is interested in minimizing the total transmission power at the base station while satisfying the SNR constraints of all users. Mathematically, the problem can be formulated as follows:

$$\min_w \|w\|^2$$

s.t. $$\frac{|h_k^Hw|^2}{\sigma_k^2} \geq \gamma_k, \quad k = 1, 2, \ldots, M,$$

(1)
where $\gamma_k$ is the desired transmission SNR target of user $k$. Let $\tilde{h}_k = h_k / \sqrt{\gamma_k \sigma_k^2}$, then

$$\frac{|h_k^H w|^2}{\sigma_k^2} \geq \gamma_k \iff |\tilde{h}_k^H w| \geq 1.$$ 

For ease of notation, we drop the $\tilde{}$ and study the following single-group multicast beamforming problem in this paper:

$$\min_w \|w\|^2$$

s.t. \quad $|h_k^H w| \geq 1, \quad k = 1, 2, \ldots, M.$

A closely related problem is to maximize the minimum SNR among all users subject to the total transmission power constraint:

$$\max_w \min_{k=1,2,\ldots,M} \{|h_k^H w|^2\}$$

s.t. \quad $\|w\|^2 \leq 1.$

It has been shown in [2] that problems (P) and (2) are equivalent from an optimization perspective of view, i.e., the global solution of problem (P) can be obtained by appropriately scaling the global solution of problem (2), and vice versa. Therefore, we focus on problem (P) in this paper.

**B. Review of Two State-of-the-Art Algorithms**

In this subsection, we briefly review two state-of-the-art algorithms for solving problem (P). We shall compare our proposed algorithm with these two algorithms later in Section IV.

One of the state-of-the-art algorithm for solving problem (P) is the SDR algorithm [3]. The SDR algorithm is based on the SDR technique, which has been widely used to solve optimization problems arising from signal processing and wireless communications; see [21] and references therein. The main observation in the SDR algorithm is that the constraint $|h_k^H w| \geq 1$ can be equivalently rewritten as

$$\text{Trace}(H_k W) \geq 1, \quad W \succeq 0, \quad \text{and Rank}(W) = 1,$$

where $H_k = h_k h_k^H \in \mathbb{C}^{N \times N}$. By dropping the rank-one constraint, problem (P) is relaxed to

$$\min_w \text{Trace}(W)$$

s.t. \quad $\text{Trace}(H_k W) \geq 1, \quad k = 1, \ldots, M,$

$$W \succeq 0.$$
The above SDR problem \((3)\) can be solved in polynomial time by using the interior-point algorithm \([25]\). If the optimal solution \(W^* \succeq 0\) of the SDR problem \((3)\) is of rank one, i.e., \(W^*\) admits the decomposition \(W^* = w^*(w^*)^H\), then \(w^*\) is a global solution of problem \((P)\). However, the solution \(W^*\) of the SDR problem \((3)\) is not always of rank one and thus the global solution of problem \((P)\) might not be obtained. In this case, the SDR algorithm employs the Gaussian randomization techniques to randomly generate approximate solutions based on \(W^*\), then scales the approximate solutions to satisfy all SNR constraints, and finally picks the one that has the smallest norm as the final solution. The SDR algorithm can find high quality approximate solutions when the problem dimension, especially the number of users, is small. The worst-case approximation accuracy of the SDR algorithm was shown in \([3, 22]\).

Another state-of-the-art algorithm for solving problem \((P)\) is the SLA algorithm \([5]\). The basic idea of the SLA algorithm is to approximate the nonconvex constraint \(|h_k^H w| \geq 1\) by a linear constraint. Specifically, the SLA algorithm introduces the auxiliary variables

\[
v_k := [\text{Re}(h_k^H w), \text{Im}(h_k^H w)]^T, \quad k = 1, 2, \ldots, M.
\]

Given the current point \(\{v_k^n \in \mathbb{R}^2\}\) (at the \(n\)-th iteration), the SLA algorithm first approximates the nonconvex constraint \(|h_k^H w| \geq 1\) by the linear constraint

\[
||v_k^n||^2 + 2(v_k^n)^T(v_k - v_k^n) \geq 1,
\]

and then solves the following approximation problem to obtain the next iterate \(\{v_k^{n+1}\}\) :

\[
\min_{w, v} ||w||^2
\]

s.t. \(||v_k^n||^2 + 2(v_k^n)^T(v_k - v_k^n) \geq 1, \quad k = 1, \ldots, M,\) \(v_k = [\mathcal{R}(h_k^H w), \mathcal{I}(h_k^H w)]^T, \quad k = 1, \ldots, M,\) (4)

where \(v\) is a collection of \(\{v_k\}_{k=1}^M\). The convergence of the SLA algorithm to a KKT point has been established in \([5]\). It is worthwhile remarking that the subproblem \((4)\) in the SLA algorithm is a linearly constrained convex quadratic program and thus can be solved efficiently to global optimality \([25]\). Moreover, for any given \(v_k^n\), there holds

\[
||v_k||^2 \geq ||v_k^n||^2 + 2(v_k^n)^T(v_k - v_k^n), \quad \forall \ v_k.
\]

Therefore, the feasible region of subproblem \((4)\) is a subset of that of the original problem \((P)\) and the SLA algorithm is a convex (inner) approximation algorithm. This differs from the SDR
algorithm which is a convex relaxation algorithm. One potential drawback of the SLA algorithm is that its performance depends on the choice of the initial point \( \{v_0^k\} \). To overcome this, [5] proposed to randomly generate many points, scale them, and pick the best one as the initial point.

III. PROPOSED GLOBAL BRANCH-AND-BOUND ALGORITHM

In this section, we propose a global optimization algorithm for solving problem (P). Our proposed algorithm is based on the branch-and-bound scheme, which is a general framework for designing global optimization algorithms [26].

A typical branch-and-bound algorithm (for the minimization problem) is generally based on an enumeration procedure, which partitions the feasible region to smaller subregions and constructs sub-problems over the partitioned subregions recursively. In the enumeration procedure, a lower bound for each subproblem is estimated by solving a relaxation problem. Meanwhile, an upper bound is obtained from the best known feasible solution generated by the enumeration procedure or by some other local optimization/heuristic algorithms. A subproblem with a lower bound being larger than the obtained upper bound is called as an inactive subproblem, which does not contain the global solution of the original problem in its feasible region, and thus will not be further enumerated. The procedure terminates until all active subproblems have been enumerated, and then an optimal solution within a given error tolerance can be obtained.

The efficiency of a branch-and-bound algorithm considerably relies on the quality of the lower bound as well as the upper bound. The quality of the lower bound depends on the tightness of the convex relaxation and the one of the upper bound depends on the local optimization or heuristic algorithms which are employed to generate the feasible solutions (to the original problem). With better lower and upper bounds, more inactive subproblems can be detected and more unnecessary enumerations can be avoided.

In the remaining part of this section, we first propose a convex quadratic programming relaxation in Section III-A, which provides efficient lower bounds in our branch-and-bound algorithm. Then, we present our proposed branch-and-bound algorithm for solving problem (P) in Section III-B. Finally, we show that our proposed branch-and-bound algorithm indeed can find the global solution of problem (P) within any given positive error tolerance and analyze its worst-case iteration complexity in Section III-C.
A. New Argument Cut based Relaxation

As is well known, the SDR (3) is the tightest convex relaxation of problem (P) and its optimal value provides high quality lower bounds on that of problem (P). However, the decision variable in the SDR is lifted to an $N \times N$ matrix, whose dimension is much larger than the dimension $N$ of the original variable $w$. In comparison, the subproblem (4) is a convex linearly constrained quadratic program with $N$ variables, which can be solved much more efficiently than the SDR (3). However, the subproblem (4) is a convex inner approximation of problem (P) but not a convex relaxation, which makes its objective value not suitable for serving as a lower bound. In the next, we propose a new convex relaxation, which achieves much higher computational efficiency than the SDR and provides valid lower bounds with satisfactory tightness.

Without loss of generality, we first change the constraint $|h_M^H w| \geq 1$ to $h_M^H w \geq 1$ in problem (P). This is because that, for any $w$ satisfying $|h_M^H w| \geq 1$, we can always find an appropriate $\theta \in \mathbb{R}$ such that $\exp(i\theta)h_M^H w \geq 1$. Second, we introduce a new variable

$$c = [c_1, c_2, \ldots, c_{M-1}]^T$$

with $c_k = h_k^H w$. Then, problem (P) is transformed to the following problem (P’):

$$\begin{align*}
\min_{w,c} & \quad \|w\|^2 \\
\text{s.t.} & \quad c_k = h_k^H w, \quad k = 1, \ldots, M - 1, \\
& \quad |c_k| \geq 1, \quad k = 1, \ldots, M - 1, \\
& \quad h_M^H w \geq 1.
\end{align*}$$

(P’)

In problem (P’), $|c_k| \geq 1$ ($k = 1, \ldots, M - 1$) are the only nonconvex constraints. To develop the branch-and-bound algorithm, we need to relax these nonconvex constraints to convex ones.

Let us consider the set

$$\{c_k \in \mathbb{C} \mid |c_k| \geq 1\}. \quad (5)$$

It is simple to see that the convex envelope of set (5) is the whole complex set $\mathbb{C}$. Obviously, it is loose, if we we directly relax set (5) to its convex envelope, i.e., relax the constraint $|c_k| \geq 1$ to $c_k \in \mathbb{C}$.

Next, we develop a tighter convex relaxation for (5). To do so, we introduce $x_k = \text{Re}(c_k)$ and $y_k = \text{Im}(c_k)$, and assume that the argument of $c_k$ satisfies $\arg(c_k) \in [l_k, u_k]$. Let

$$D_{[l_k,u_k]} = \{(x_k, y_k) \mid c_k = x_k + y_ki, |c_k| \geq 1, \arg(c_k) \in [l_k, u_k]\} \quad (6)$$
and let $\text{Conv}(\mathcal{D}_{[l_k,u_k]})$ be the convex envelope of the set $\mathcal{D}_{[l_k,u_k]}$. The following proposition characterizes how $\text{Conv}(\mathcal{D}_{[l_k,u_k]})$ looks like if $u_k - l_k \leq \pi$.

**Proposition 1.** If $l_k$ and $u_k$ in (6) satisfying $u_k - l_k \leq \pi$, then

\[
\text{Conv}(\mathcal{D}_{[l_k,u_k]}) = \{(x, y) \mid \sin(l_k)x - \cos(l_k)y \leq 0, \sin(u_k)x - \cos(u_k)y \geq 0, a_kx + b_ky \geq a_k^2 + b_k^2, (7)\}
\]

where

\[
a_k = \frac{\cos l_k + \cos u_k}{2} \quad \text{and} \quad b_k = \frac{\sin l_k + \sin u_k}{2}.
\]

Instead of giving a rigorous (but lengthy) proof, we give an illustration of Proposition 1 by using Fig. 1, which shows that the convex envelope of set $\mathcal{D}_{[l_k,u_k]}$ with $[l_k, u_k] = [0, \pi/2]$ is

\[
\text{Conv}(\mathcal{D}_{[0,\pi/2]}) = \{(x, y) \mid x \geq 0, y \geq 0, x + y \geq 1\}.
\]

The linear inequalities in (7) are introduced due to the argument constraints. Hence, we name these linear inequalities the argument cuts in this paper.

**Definition 1 (Argument Cuts).** The linear inequalities in (7) are called the argument cuts.

To the best of our knowledge, the argument cuts have not been used in the literatures. With the help of the argument cuts, we are able to develop efficient convex relaxations for problem (P’). More specifically, it follows from Proposition 1 that

\[
\mathcal{F}_{[l,u]} := \{x + yi \mid (x, y) \in \text{Conv}(\mathcal{D}_{[l,u]})\}
\]

is the convex envelope of the nonconvex set

\[
\{x + yi \mid (x, y) \in \mathcal{D}_{[l,u]}\}.
\]

Assume $\text{arg}(c_k) \in [l_k, u_k]$ for all $k = 1, 2, \ldots, M - 1$ in problem (P’). Then we obtain the following convex argument cut based relaxation (ACR) of problem (P’):

\[
\min_{\bf w, c} \|\bf w\|^2 \\
\text{s.t.} \quad c_k = h_k^H \bf w, \quad k = 1, \ldots, M - 1, \\
\quad c_k \in \mathcal{F}_{[l_k,u_k]}, \quad k = 1, \ldots, M - 1, \\
\quad h_M^H \bf w \geq 1.
\]
Fig. 1. An illustration of the convex envelope $\text{Conv}(D_{[0, \pi/2]})$ and the proof of Propositions 1 and 2.

We consider the following two cases: whether or not the width of the interval $[l_k, u_k]$, i.e., $d_k := u_k - l_k$, is greater than $\pi$.

- Case I: $d_k > \pi$. In this case, $c_k \in F_{[l_k, u_k]}$ is equivalent to $c_k \in C$ and thus the corresponding argument cuts do not take effect. Therefore, we can drop the constraint $c_k \in F_{[l_k, u_k]}$ in problem (8).

- Case II: $d_k \leq \pi$. It follows from Proposition 1 that $c_k \in F_{[l_k, u_k]}$ can be represented by at most three (real) linear inequality constraints. It is simple to see that the argument cuts are effective in this case.

To sum up, we know that ACR problem (8) is a linearly constrained convex quadratic program with $2N + 2M - 2$ (real) variables, at most $2M - 1$ (real) linear equality constraints, and at most $3M - 2$ (real) linear inequality constraints. Therefore, ACR problem (8) can be solved efficiently and globally by using the interior-point algorithm within $O(N^3M^{3.5})$ operations [25, Page 423].

We conclude this subsection with a tightness measure of the ACR. Let us focus on the case

1. Notice that the three linear constraints in (7) are the same to each other when $d_k = \pi$.

2. Recall that the constraint $h_k^H w \geq 1$ corresponds a (real) linear inequality constraint and a (real) linear equality constraint.
\(d_k \leq \pi\). Note that the inclusion
\[
\{x + yi \mid (x, y) \in D[l_k, u_k]\} \supseteq F[l_k, u_k]
\]
generally is not true, which implies that there exist some point \(c_k \in F[l_k, u_k]\) such that \(|c_k| < 1\). Therefore, the smallest norm of the points in \(F[l_k, u_k]\) can be used to measure the tightness of the ACR. The following theorem shows that the smallest norm of the points in \(F[l_k, u_k]\) can be computed in a closed form.

**Proposition 2.** Given any interval \([l_k, u_k]\) with \(u_k - l_k \leq \pi\). We have

\[
\min_{c_k \in F[l_k, u_k]} |c_k| = \cos \left(\frac{u_k - l_k}{2}\right).
\]

**Proof:** The point that has the smallest norm in a convex set is the projection of the origin onto the corresponding set. We first consider the special case where \([l_k, u_k] = [0, \pi/2]\), as shown in Fig. 1. In this case, the projection of the origin \(O\) onto the set \(F[0, \pi/2]\) is the point \(C\), which satisfies \(OC \perp AB\). Notice that the coordinates of \(A\) and \(B\) in Fig. 1 are \((0, 1)\) and \((1, 0)\), respectively. It is simple to see that the coordinate of the point \(C\) is \((0.5, 0.5)\). In the general case of \(F[l_k, u_k]\), the coordinate of the point \(c_k\) is

\[
\left(\frac{\cos l_k + \cos u_k}{2}, \frac{\sin l_k + \sin u_k}{2}\right)
\]

and its norm is \(\cos \left(\frac{u_k - l_k}{2}\right)\). The proof is completed.

We can see from Proposition 2 that: the smaller the width of the interval, the tighter the ACR; as the width of the interval goes to zero, the set \(F[l_k, u_k]\) becomes \(\{x + yi \mid (x, y) \in D[l_k, u_k]\}\) and the ACR becomes tight. Hence, an effective approach to tightening the ACR is to reduce the width of the corresponding interval.

**B. Proposed Branch-and-Bound Algorithm**

In this subsection, we propose a branch-and-bound algorithm for globally solving problem \((P')\). The basic idea of the proposed algorithm is to relax the original problem (with appropriate argument constraints) to ACR \((8)\) and gradually tighten the relaxation by reducing the width of the associated intervals.

For ease of presentation, we introduce the following notations. Let \(A = \prod_{k=1}^{M-1} [l_k, u_k]\) and let ACR(\(A\)) denote the ACR problem defined over the set \(A\); let \(P\) denote the constructed problem
list and let \( \{A, c, L\} \) denote a problem instance from the list \( \mathcal{P} \), where \( L \) is the optimal value of \( \text{ACR}(A) \) and \( c \) is its optimal solution; let superscript \( t \) denote the iteration number; and let \( w^* \) denote the best known feasible solution and let \( U^* \) denote the objective value of problem \( (P') \) at \( w^* \).

We are now ready to present the main steps of the proposed branch-and-bound algorithm.

**Initialization.** We initialize all intervals \([l_k^0, u_k^0]\) for all \( k = 1, 2, \ldots, M - 1 \) to be \([0, 2\pi]\), i.e., set \( A^0 = [0, 2\pi]^{M-1} \). In this case, problem (8) reduces to

\[
\min_w \|w\|^2 \\
\text{s.t. } h_h^H w \geq 1.
\]

Its optimal solution \((w^0, c^0)\) and its optimal value \( L^0 \) are

\[
w^0 = \frac{h_M}{\|h_M\|^2}, \quad L^0 = \frac{1}{\|h_M\|^2},
\]

\[
c^0_k = \frac{h_k^H h_M}{\|h_M\|^2}, \quad k = 1, 2, \ldots, M - 1.
\]

**Termination.** Let \( \{A^t, c^t, L^t\} \) be the problem instance that has the least lower bound in the problem list \( \mathcal{P} \). If

\[
(U^* - L^t)/L^t \leq \epsilon,
\]

where \( \epsilon \) is the preselected error tolerance, we terminate the algorithm; otherwise we branch some interval of the above problem instance according to some rule. We can see from (10) that, both lower and upper bounds are important to avoid unnecessary branches and enumerations and good lower and upper bounds can significantly improve the efficiency of our proposed algorithm.

Below, we shall introduce our branch rule as well as lower and upper bounds one by one.

**Branch.** Again, let \( \{A^t, c^t, L^t\} \) be the problem instance that has the least lower bound in the problem list \( \mathcal{P} \) and suppose that the stopping criterion (10) is not satisfied. In this case, we first select the interval that leads to the largest gap to be branched to smaller sub-intervals, i.e., the interval with index

\[
k^* = \arg\min_{k \in \{1, \ldots, M-1\}} \{|c^t_k|\};
\]

then we partition \( A^t \) into two sets (denoted as \( A^t_l \) and \( A^t_r \)), with its \( k^* \)-th interval partitioned into two equal intervals and all the others unchanged. It follows from Proposition 2 that the ACR
problems defined over the newly obtained two sets, i.e., the two children problems, are tighter than the one defined over the original set \( A' \).

**Lower Bound.** Obviously, for any problem instance \( \{A, c, L\} \), \( L \) is a lower bound of the optimal value of the original problem \( (P') \) defined over \( A \). Therefore, the smallest lower bound among all bounds is a lower bound of the optimal value of the original problem. This statement will be formally summarized as Lemma 1 and proved in Section III-C.

**Upper Bound.** An upper bound of the original problem \( (P') \) can be obtained by appropriately scaling the solution of any ACR problem instance. More specifically, let \( \{w^t, c^t\} \) be the solution of problem ACR\((A^t)\). Then,

\[
\hat{w}^t = \text{Scale}(w^t, c^t) := \frac{w^t}{\min \{|c^t_1|, |c^t_2|, \ldots, |c^t_{M-1}|, 1\}}
\]

(11)
is feasible to the original problem and \( \|\hat{w}^t\|^2 \) is an upper bound of the original problem. In our proposed algorithm, the upper bound \( U^* \) is chosen as the best objective values at all of the known feasible solutions.

By judiciously combining the above main steps, we can obtain our proposed branch-and-bound algorithm for solving problem \( (P) \) (equivalent to problem \( (P') \)). The pseudo-code of our proposed algorithm can be found below. We will call the algorithm ACR-BB for short from now on. To make the ACR-BB algorithm more clear, an illustration on how it works is given in Appendix A.

We emphasize again that both lower and upper bounds play important roles in the efficiency of the proposed ACR-BB algorithm, because good lower and upper bounds can effectively detect inactive subproblems and avoid unnecessary branches and enumerations. Lines 9 and 11 of the proposed ACR-BB algorithm ensure that it will never branch and enumerate subproblems with the lower bound \( L \) being greater than or equal to the upper bound \( U^* \). The efficiency of the proposed ACR-BB algorithm will be shown in Section IV.
ACR-BB Algorithm for Single-Group Multicast Beamforming Problem (P)

1: **input:** An instance of problem (P), and an error tolerance $\epsilon > 0$.

2: Initialize $P = \emptyset$, $A^0 = \prod_{k=1}^{M-1} [t_k^0, u_k^0] = [0, 2\pi]^{M-1}$, and set $t = 0$. // Initialization.

3: Solve ACR($A^0$) for its optimal solution $(w^0, c^0)$ and its optimal value $L^0$.

4: Compute $\hat{w}^0 = \text{Scale}(w^0, c^0)$, where the operator Scale($\cdot, \cdot$) is defined in [II].

5: Set $U^* = \|\hat{w}^0\|^2$ and $w^* = \hat{w}^0$. // Initial Upper Bound and Optimal Solution.

6: Add \{$A^0, c^0, L^0$\} into the problem list $P$.

7: **loop**

8: Set $t \leftarrow t + 1$.

9: Choose a problem from $P$, denoted as \{$A^t, c^t, L^t$\}, such that the bound $L^t$ is the smallest one in $P$. // Lower Bound.

10: Delete the chosen subproblem from $P$.

11: if $(U^* - L^t)/L^t < \epsilon$ then

12: return $w^*$ and terminate the algorithm. // Termination.

13: end if

14: Set $k^* = \arg \min_{k \in \{1, \ldots, M-1\}} \{c_k\}$ and $z_{k^*} = 1/2(t_k^t + u_k^t)$.

15: Branch $A^t$ into two sets $A^t_1 = \{\theta \in A^t | \theta_{k^*} \leq z_{k^*}\}$ and $A^t_2 = \{\theta \in A^t | \theta_{k^*} \geq z_{k^*}\}$, where $\theta_{k^*}$ is the $k^*$-th component of $\theta \in \mathbb{R}^{M-1}$. // Branch.

16: Solve ACR($A^t_1$) for its optimal solution $(w^t_1, c^t_1)$ and its optimal value $L^t_1$.

17: Compute $\hat{w}^t_1 = \text{Scale}(w^t_1, c^t_1)$, where Scale($\cdot, \cdot$) is defined in [II].

18: if $L^t_1 \leq U^*$ then

19: add \{$A^t_1, c^t_1, L^t_1$\} into $P$.

20: end if

21: if $U^* > \|\hat{w}^t_1\|^2$ then

22: set $U^* = \|\hat{w}^t_1\|^2$ and $w^* = \hat{w}^t_1$. // Update Upper Bound and Optimal Solution.

23: end if

24: Solve ACR($A^t_2$) for its optimal solution $(w^t_2, c^t_2)$ and its optimal value $L^t_2$.

25: Compute $\hat{w}^t_2 = \text{Scale}(w^t_2, c^t_2)$, where Scale($\cdot, \cdot$) is defined in [II].

26: if $L^t_2 \leq U^*$ then

27: add \{$A^t_2, c^t_2, L^t_2$\} into $P$.

28: end if

29: if $U^* > \|\hat{w}^t_2\|^2$ then

30: set $U^* = \|\hat{w}^t_2\|^2$ and $w^* = \hat{w}^t_2$. // Update Upper Bound and Optimal Solution.

31: end if

32: end loop
C. Global Convergence and Worst-Case Iteration Complexity

In this subsection, we present some theoretical results of our proposed ACR-BB algorithm. We first define the $\epsilon$-optimal solution of problem (P).

**Definition 2 ($\epsilon$-Optimal Solution).** Given any $\epsilon > 0$, a feasible point $w$ is called an $\epsilon$-optimal solution of problem (P) if it satisfies

$$\frac{\|w\|^2 - \nu^*}{\nu^*} \leq \epsilon,$$

where $\nu^*$ is the optimal value of problem (P).

The following Lemma 1 shows that the sequence $\{L^t\}$ generated by the ACR-BB algorithm is a lower bound of the optimal value of the original problem.

**Lemma 1.** For any given instance of problem (P), let $\nu^*$ be its optimal value. Then, we have

$$0 < L^t \leq \nu^*, \quad \forall \ t \geq 1.$$

**Proof:** The optimal value of problem (8) is always positive because zero is not a feasible solution to it. Hence, $L^t > 0$ for all $t \geq 1$. Next, we show $L^t \leq \nu^*$ for all $t \geq 1$. Let $w^*$ be the global solution of the given problem instance. At the beginning of the $t$-th iteration of the ACR-BB algorithm, the set $A^0$ has been partitioned into $t$ small subsets, and $w^*$ must lie in one of them (we denote the corresponding subset as $A^*$). Then, the optimal value of problem $ACR(A^*)$ must be less than or equal to $\nu^*$. Moreover, since $L^t$ is the smallest lower bound of all subproblems in $P$ at the $t$-th iteration, it follows that $L^t$ is less than or equal to the optimal value of problem $ACR(A^*)$. Therefore, we get $L^t \leq \nu^*$ for all $t \geq 1$.

From lemma [1] we immediately get

$$\frac{U^* - L^t}{L^t} \geq \frac{U^* - \nu^*}{\nu^*}, \quad \forall \ t \geq 1.$$

This further shows that, if the ACR-BB algorithm terminates, i.e., condition (10) is satisfied, the returned solution $w^*$ by the algorithm is an $\epsilon$-optimal solution of problem (P).

The following Lemma 2 shows that the ACR-BB algorithm will terminate.

**Lemma 2.** For any given instance of problem (P), let $\{A^t, c^t, L^t\}$ be the subproblem chosen in Line 9 and let $k^*$ be the index chosen in Line 14. If

$$u^t_{k^*} - l^t_{k^*} \leq 2\delta,$$
where
\[ \delta = \arccos \left( \frac{1}{\sqrt{1 + \epsilon}} \right), \] (14)
then condition (10) holds true and the ACR-BB algorithm will terminate in Line 12.

**Proof:** It follows from (13) and Proposition 2 that
\[ |c_{k^*}^t| \geq \cos \left( \frac{u_{k^*}^t - l_{k^*}^t}{2} \right) \geq \frac{1}{\sqrt{1 + \epsilon}}. \]
By the above inequality and the definition of \( k^* \) (see Line 14 of the ACR-BB algorithm), we obtain
\[ |c_i| \geq |c_{k^*}^t| \geq \frac{1}{\sqrt{1 + \epsilon}}, \quad \forall \ i = 1, 2, \ldots, M - 1. \] (15)
Let \( (w^t, c^t) \) be the solution of problem ACR\( (\mathcal{A}^t) \). Then, it follows from (11) and (15) that the scaled feasible solution \( \hat{w}^t = \text{Scale}(w^t, c^t) \) satisfies
\[ \|\hat{w}^t\|^2 = \frac{\|w^t\|^2}{\min \{ |c_1|^2, |c_2|^2, \ldots, |c_{M-1}|^2, 1 \}} \leq \|w^t\|^2(1 + \epsilon). \]
Moreover, since \( U^* \) is the objective value at the best known feasible solution, we get
\[ U^* \leq \|\hat{w}^t\|^2 \leq \|w^t\|^2(1 + \epsilon). \] (16)
Now, we can use the fact \( L^t = \|w^t\|^2 \) and (16) to obtain
\[ \frac{U^* - L^t}{L^t} = \frac{U^* - \|w^t\|^2}{\|w^t\|^2} \leq \frac{\|w^t\|^2(1 + \epsilon) - \|w^t\|^2}{\|w^t\|^2} = \epsilon. \]
The proof is completed.

Based on Lemmas 1 and 2, we obtain the main result of this subsection.

**Theorem 1** (Iteration Complexity). For any given \( \epsilon > 0 \) and any given instance of problem (P) with \( M \) users, the ACR-BB algorithm will return an \( \epsilon \)-optimal solution of the given instance within at most
\[ T := \left\lceil \left( \frac{2\pi}{\delta} \right)^{M-1} \right\rceil + 1 \] (17)
iterations, where \( \delta \) is defined in (14).

**Proof:** It follows from Lemmas 1 and 2 that, if (13) is satisfied, the ACR algorithm will terminate and return an \( \epsilon \)-optimal solution of the given problem instance. To show the theorem,
it remains to show that the algorithm will terminate within $T$ iterations, where $T$ is defined in (17). Next, we show this based on the contradiction principle.

Suppose that the algorithm does not terminate within $T$ iterations. This fact, together with Lemma 2 implies that the interval that is chosen to be partitioned at the $t$-th iteration must satisfy $u_k^t - l_k^t > 2\delta$ for all $t = 1, 2, \ldots, T$. Then, after the partition, the width of the two sub-intervals $[l_k^t, z_k^t]$ and $[z_k^t, u_k^t]$ is greater than $\delta$. Based on this, we can conclude that, for each subset $\mathcal{A} = \prod_{k=1}^{M-1} [l_k, u_k]$ partitioned from the original set $\mathcal{A}^0$, there holds $u_k - l_k > \delta$ for all $k = 1, \ldots, M - 1$. Hence, the volume of each subset $\mathcal{A}$ is not less than $\delta^{M-1}$ and the total volume of all $T$ subsets is not less than $T\delta^{M-1}$. Obviously, the volume of $\mathcal{A}^0$ is $(2\pi)^{M-1}$. By the choice of $T$, we get $T\delta^{M-1} > (2\pi)^{M-1}$, which further implies that the total volume of all $T$ subsets is greater than the one of the original set $\mathcal{A}^0$. This is a contradiction. Hence, the algorithm will terminate within at most $T$ iterations.

Theorem 1 shows that the total number of iterations for our proposed ACR-BB algorithm to return an $\epsilon$-optimal solution of any instance of problem (P) is exponential with respect to the number of receivers $M$. The iteration complexity of our proposed algorithm seems high at first sight. However, as will be shown in Section IV, its practical iteration complexity is actually significantly less than the worst-case bound in (17). It is also worthwhile remarking that there is no polynomial time algorithm which can globally solve the problem (unless P=NP) [4], because the problem is NP-hard.

IV. SIMULATION RESULTS

In this section, we present some numerical simulation results of our proposed ACR-BB algorithm for solving problem (P). More specifically, we first present some simulation results to show the convergence behaviors of the proposed algorithm in Section IV-A. Then, we show the efficiency of the proposed algorithm by comparing it with a state-of-the-art general-purpose global optimization solver called Baron [23], [24]. Last but not least, we use the proposed algorithm as the benchmark to evaluate the performance of two state-of-the-art algorithms [3], [5] for solving the same problem.

In all of our simulations, the channel vectors $\mathbf{h}_k$ are randomly generated according to the distribution $\mathcal{CN}(0, \mathbf{I}_N)$ as done in [3], [5], [20]. We performed all numerical experiments on a PC with a 3.40-GHz Intel Core i7-2600 processor with access to 4 GB of RAM. We implemented
our proposed ACR-BB algorithm in Matlab 7.10. We use the built-in function “quadprog” in Matlab to solve all linearly constrained convex quadratic programming problems, and use Sedumi \[27\] to solve all SDPs.

A. Convergence Behaviors of the ACR-BB Algorithm

In this subsection, we generate a problem instance with \((N, M) = (4, 40)\), apply our proposed ACR-BB algorithm to solve it, and study the convergence behaviors of our proposed algorithm, i.e., the convergence behaviors of the lower bounds \(\{L^t\}\) and the upper bounds \(\{U^t\}\), where \(U^t\) denotes the best known upper bound at the \(t\)-th iteration. In this simulation, we set \(\epsilon = 1e-7\) in our proposed algorithm such that the algorithm can find the “real” optimal solution when it terminates. We denote the objective value at the returned solution by \(\bar{\nu}\) and define

\[
E_1^t = \frac{|L^t - \bar{\nu}|}{\bar{\nu}}, \quad E_2^t = \frac{|U^t - \bar{\nu}|}{\bar{\nu}}, \quad \text{and} \quad E_3^t = \frac{|U^t - L^t|}{L^t}.
\]

The above three relative errors (gaps) versus the number of iterations are illustrated as Fig. 2.

It can be seen from Fig. 2 that \(E_2\) converges much faster than \(E_1\) and \(E_3\) at the beginning stage of our proposed algorithm. For instance, \(E_2\) becomes smaller than \(1e-1\) at the 6702-th iteration, while \(E_1\) and \(E_3\) become smaller than \(1e-1\) at the 9653-th and 12221-th iteration, respectively. The above results imply that an 0.1-optimal solution has been found at the 6702-th
iteration but it is verified (to be an 0.1-optimal solution) at the 12221-th iteration. These results show that:

- The upper bounds \( \{U^i\} \) converge faster than the lower bounds \( \{L^i\} \) at the beginning stage of our proposed algorithm; and more importantly,
- A low-accuracy solution can be more easily found than verified by our proposed algorithm.

We can also see from Fig. 2 that it takes our proposed ACR-BB algorithm 3339 iterations to reduce \( E_3 \) from \( 1e^{-1} \) to \( 1e^{-3} \) and 96 iterations to reduce it from \( 1e^{-3} \) to \( 1e^{-6} \), respectively. This shows that the performance of our proposed algorithm in terms of the number of iterations is not sensitive to the choice of the relative error tolerance \( \epsilon \) (when it is small).

In the remaining part of this section, we set \( \epsilon = 5e^{-3} \).

\[ \text{B. Efficiency of the ACR-BB Algorithm} \]

In this subsection, we study the efficiency of our proposed ACR-BB algorithm for solving problem (P).

We first compare our proposed algorithm with Baron\(^3\) \cite{23, 24}, which is a state-of-the-art

\[ \text{TABLE I} \]

\text{COMPARISON OF OUR PROPOSED ACR-BB ALGORITHM AND BARON} \\

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Instance ID & \multicolumn{3}{|c|}{ACR-BB} & \multicolumn{3}{|c|}{Baron} \\
 & Value & \# of Iter. & Time & Value & \# of Iter. & Time \\
\hline
01 & 1.2344 & 90 & 0.7 & 1.2344 & 6297 & 327.9 \\
02 & 1.1412 & 40 & 0.2 & 1.1412 & 2297 & 147.4 \\
03 & 1.3541 & 68 & 0.4 & 1.3541 & 2917 & 94.5 \\
04 & 1.4251 & 49 & 0.3 & 1.4251 & 2847 & 261.5 \\
05 & 1.4136 & 61 & 0.4 & 1.4136 & 1825 & 49.8 \\
06 & 0.8540 & 40 & 0.3 & 0.8540 & 1303 & 31.9 \\
07 & 0.9777 & 48 & 0.3 & 0.9777 & 457 & 19.0 \\
08 & 5.2469 & 1 & 0.0 & 5.2469 & 4305 & 58.4 \\
09 & 11.9555 & 16 & 0.1 & 11.9555 & 5358 & 87.1 \\
10 & 1.3258 & 62 & 0.4 & 1.3258 & 653 & 26.5 \\
\hline
Average & 2.6928 & 47.5 & 0.3 & 2.6928 & 2821.1 & 110.4 \\
\hline
\end{tabular}
\end{table}

\(^3\)To the best of our knowledge, our proposed ACR-BB algorithm is the first tailored global algorithm for problem (P) and there is no existing global algorithms specially designed for the problem that we can compare our proposed algorithm with.

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general-purpose global optimization solver and has been widely applied to solve problems arising from various applications [28], [29]. Baron is also a branch-and-bound algorithm but based on linear programming relaxation [23], [24].

We apply our proposed ACR-BB algorithm and Baron to solve 10 randomly generated problem instances with \((N, M) = (2, 8)\). The comparison results are summarized in Table I where the first column shows the ID of the corresponding problem instance, the second column shows the objective values and the number of iterations and CPU time results obtained by our proposed algorithm, the third column shows the objective values and the number of iterations and CPU time results obtained by Baron, and the last row shows the average results over the 10 instances. We can observe from Table I that our proposed ACR-BB algorithm performs 366 times faster than Baron (on average) to find the same solutions. In fact, we have tried to use Baron to solve problem instances with larger \(N\) and/or \(M\). Unfortunately, we found that Baron fails to solve all problem instances with \(N \geq 4\) and \(M \geq 8\) within 10 minutes. These observations demonstrate that our specially designed ACR-BB algorithm achieves significantly higher efficiency on globally solving problem (P) than Baron.

Next, we present more numerical results on applying our proposed ACR-BB algorithm to solve problem (P) with larger \(N\) and/or \(M\) without comparing it with other general-purpose global optimization solvers. All the results to be shown from now on are obtained by averaging over or choosing from 50 randomly generated problem instances for each pair \((N, M)\). Table II reports the number of iterations and CPU time results, where the first column shows the setup of the problem, the second column shows the average results (over the 50 problem instances), and the third column shows the worst-case result (among the 50 problem instances). Figs. 3 and 4 plot the average CPU time and the average number of iterations versus the number of users with \(N = 4\).

We can observe from the above simulation results (Table II and Figs. 3 and 4) that our proposed ACR-BB algorithm is capable of finding the global solution (with the relative error being not greater than \(5e^{-3}\)) within several seconds when \(N\) is small and \(M\) is not too large. In particular, the proposed algorithm can solve all generated problem instances with \(N = 2\) and \(M\) ranging from 8 to 64 within 2.4 seconds. When \(N = 4\), the proposed algorithm can solve problem instances with \(M \leq 16\) within 20 seconds on average and problem instances with \(M\) ranging from 24 to 40 within 185 seconds on average. The proposed algorithm can solve all 50
**TABLE II**

*Average and worst-case results of number of iterations and CPU time (in seconds) under different setups*

| Setting $(N, M)$ | Average Performance | Worst-Case Performance |
|------------------|---------------------|------------------------|
|                  | # of Iter. | Time | # of Iter. | Time |
| (2,8)            | 47.9      | 0.3  | 103        | 0.7  |
| (2,16)           | 72.7      | 0.5  | 147        | 1.0  |
| (2,24)           | 100.6     | 0.6  | 201        | 1.2  |
| (2,32)           | 136.0     | 0.9  | 225        | 1.4  |
| (2,40)           | 151.3     | 1.0  | 298        | 2.0  |
| (2,48)           | 171.7     | 1.2  | 278        | 1.8  |
| (2,56)           | 199.9     | 1.4  | 343        | 2.3  |
| (2,64)           | 215.4     | 1.5  | 341        | 2.4  |
| (4,8)            | 330.1     | 2.8  | 886        | 8.1  |
| (4,16)           | 1721.2    | 16.6 | 3763       | 37.8 |
| (4,24)           | 4492.1    | 46.4 | 11627      | 125.8|
| (4,32)           | 8579.6    | 98.2 | 17442      | 216.0|
| (4,40)           | 16108.7   | 184.9| 34675      | 419.9|
| (6,8)            | 1728.6    | 21.5 | 6388       | 83.4 |
| (6,16)           | 13233.1   | 199.9| 36804      | 593.3|
| (6,24)           | 63691.6   | 1159.7| 175041     | 3679.3|
| (8,8)            | 2090.0    | 30.1 | 6301       | 95.1 |
| (8,16)           | 64168.9   | 1396.9| 235309     | 5786.3|

**Fig. 3.** Average CPU time of our proposed ACR-BB algorithm versus the number of users with $N = 4$. 

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problem instances with $N = 4$ and $M$ ranging from 8 to 40 within 7 minutes. Moreover, we can observe from Fig. 4 that the number of iterations of our proposed algorithm is extremely smaller than the worst-case bound in (17). These results demonstrate that our proposed ACR-BB algorithm is highly efficient for problem (P) when $N$ is small and $M$ is not too large. The high efficiency of our proposed ACR-BB algorithm is due to the newly developed ACR (8), which effectively avoids unnecessary branches that do not contain global optimal solutions.

When $N$ and $M$ (especially $N$) are large, it takes our proposed algorithm relatively long time (and a relatively large number of iterations) to find the global solution. For instance, our proposed algorithm needs 1396.9 seconds on average to solve problem instances with $(N, M) = (8, 16)$. Even though the efficiency of our proposed algorithm is not very high when $N$ and $M$ (especially $N$) are large, our proposed algorithm is still useful in some scenarios. For instance, in the scenario where the channel is stationary/constant and computational efficiency is not a big issue, our proposed algorithm can find the global solution and thus can provide the best performance in terms of transmitting the minimum total power. Another key application of our proposed ACR-BB algorithm is that it provides an important benchmark for performance evaluation of other heuristic/local optimization algorithms for the same problem, as will be done in the next subsection.
C. Performance Evaluation of Existing Heuristic/Local Optimization Algorithms

In this subsection, we evaluate the performance of two state-of-the-art algorithms for problem (P), the SDR algorithm [3] and the SLA algorithm [5], by using our proposed ACR-BB algorithm as the benchmark. In our simulations, the SDR algorithm and the SLA algorithm are implemented by following the descriptions in [3] and [5], respectively. In particular, 1000 points are randomly generated based on the solution of the SDR [3] to obtain the final approximate solution in [3] and also 1000 points are randomly generated to obtain a good initial point for the SLA algorithm in [5]. Recall that we have generated 900 instances of problem (P) in Table II and obtained their optimal solution (with the relative errors being not greater than $5e-3$) by using our proposed ACR-BB algorithm. We apply the SDR algorithm and the SLA algorithm to solve these instances and evaluate their performance.

| Setting $(N, M)$ | Average Objective Value | Average Relative Gap |
|------------------|-------------------------|----------------------|
|                  | ACR-BB | SLA | SDR | ACR-BB | SLA | SDR |
| (2,8)            | 1.629  | 1.629 | 1.642 | 0.02% | 0.79% |
| (2,16)           | 2.800  | 2.803 | 2.827 | 0.10% | 0.96% |
| (2,24)           | 3.389  | 3.398 | 3.463 | 0.26% | 2.20% |
| (2,32)           | 4.457  | 4.489 | 4.642 | 0.73% | 4.16% |
| (2,40)           | 5.720  | 5.731 | 5.871 | 0.20% | 2.65% |
| (2,48)           | 5.679  | 5.739 | 5.892 | 1.06% | 3.77% |
| (2,56)           | 5.461  | 5.491 | 5.758 | 0.55% | 5.44% |
| (2,64)           | 5.914  | 5.967 | 6.291 | 0.89% | 6.37% |
| (4,8)            | 0.514  | 0.525 | 0.577 | 2.01% | 12.17% |
| (4,16)           | 0.837  | 0.902 | 1.121 | 7.83% | 33.97% |
| (4,24)           | 1.132  | 1.256 | 1.710 | 10.95% | 51.03% |
| (4,32)           | 1.328  | 1.587 | 2.045 | 19.48% | 54.04% |
| (4,40)           | 1.525  | 1.770 | 2.587 | 16.08% | 69.64% |
| (6,8)            | 0.334  | 0.341 | 0.393 | 2.30% | 17.75% |
| (6,16)           | 0.531  | 0.578 | 0.799 | 8.90% | 50.39% |
| (6,24)           | 0.667  | 0.779 | 1.113 | 16.77% | 66.99% |
| (8,8)            | 0.246  | 0.253 | 0.278 | 2.95% | 12.93% |
| (8,16)           | 0.376  | 0.420 | 0.618 | 11.68% | 64.11% |
Table III reports the average objective values and average relative gaps of the SDR algorithm [3] and the SLA algorithm [5]. We can see from Table III that the average relative gaps of both algorithms are small when $N$ and $M$ (especially $N$) are small and the average relative gaps of the SLA algorithm is smaller than the ones of the SDR algorithm. This shows that, the quality of the returned solutions by both algorithms is high when $N$ and $M$ (especially $N$) are small and the quality of the returned solution by the SLA algorithm is generally better than the one of the returned solution by the SDR algorithm. However, as $N$ and/or $M$ increase, the average relative gaps of both algorithms quickly become large. For instance, when $(N, M) = (4, 40)$, the relative gap of the SDR algorithm is 69.64%; when $(N, M) = (4, 32)$, the relative gap of the SLA algorithm is 19.48%. These results show that the quality of the solutions obtained by the two algorithms are not good when $N$ and/or $M$ are large and the performance of both algorithms degrades quickly as $N$ and/or $M$ increase. This makes sense, because both of the algorithms are not global algorithms and they are more likely to get stuck at a local solution when $N$ and/or $M$ are large.

To study the robustness of the two algorithms, we list the worst-case relative gaps of the two algorithms in Table IV. We can see from Table IV that the worst-case relative gap of the SLA algorithm exceeds 70% in two setups $(4, 40)$ and $(6, 24)$; and the worst-case relative gap of the SDR algorithm exceeds 100% in five setups $(4, 32)$, $(4, 40)$, $(6, 16)$, $(6, 24)$, and $(8, 16)$. These results clearly demonstrate that both of the algorithms are not robust (especially when $N$ and/or $M$ are relatively large) because their performance for some problem instance is bad and the SLA algorithm is more robust than the SDR algorithm.

We also test the probability that the two algorithms can find the global solution. We call that an algorithm finds the global solution of a problem instance if the relative gap (between the objective value at its returned solution and the objective value at the solution returned by our proposed algorithm) is less than or equal to $1e-3$. The probability that an algorithm finds the global solution is defined as the ratio of the number of instances of which it finds the global solution and the total number of instances of which it is applied to solve. The latter is 50 in our case. The probability that the two algorithms find the global solution is reported in Table V. As can be seen from the table, the probability that the SLA algorithm and the SDR algorithm can find the global solution is (relatively) large when both $N$ and $M$ are small and the probability that the SLA algorithm finds the global solution is generally much larger than the one that the
TABLE IV
WORST-CASE RELATIVE GAPS OBTAINED BY SLA AND SDR.

| $(N, M)$ | SLA | SDR |
|---------|-----|-----|
| $(2,8)$ | 1%  | 6%  |
| $(2,16)$ | 6%  | 10% |
| $(2,24)$ | 4%  | 18% |
| $(2,32)$ | 13% | 30% |
| $(2,40)$ | 8%  | 22% |
| $(2,48)$ | 10% | 28% |
| $(2,56)$ | 12% | 25% |
| $(2,64)$ | 14% | 25% |
| $(4,8)$  | 34% | 42% |
| $(4,16)$ | 39% | 82% |
| $(4,24)$ | 65% | 100%|
| $(4,32)$ | 70% | 103%|
| $(4,40)$ | 74% | 104%|
| $(6,8)$  | 22% | 92% |
| $(6,16)$ | 62% | 126%|
| $(6,24)$ | 73% | 128%|
| $(8,8)$  | 23% | 46% |
| $(8,16)$ | 51% | 124%|

SDR algorithm finds the global solution. However, the probability that the SLA algorithm and the SDR algorithm find the global solution decreases quickly as $N$ and/or $M$ increase. These results are intuitive and agree well with our previous simulation results.

In summary, we can make the following conclusions on the performance of the two state-of-the-art algorithms for problem (P), the SLA algorithm and the SDR algorithm, by using our proposed ACR-BB algorithm as benchmark:

- both of the algorithms perform well when $N$ and $M$ are small;
- the performance of the SLA algorithm is generally better than the one of the SDR algorithm in terms of (average and worst-case) relative gaps, robustness, and probability of finding the global solutions; and
- the performance of both of the algorithms degrades quickly as $N$ and/or $M$ increase.
TABLE V

Probability of obtaining the global solution by SLA and SDR

| (N, M) | SLA  | SDR  |
|-------|------|------|
| (2,8) | 98%  | 74%  |
| (2,16)| 96%  | 68%  |
| (2,24)| 92%  | 48%  |
| (2,32)| 88%  | 28%  |
| (2,40)| 92%  | 30%  |
| (2,48)| 82%  | 22%  |
| (2,56)| 86%  | 24%  |
| (2,64)| 88%  | 14%  |
| (4,8) | 80%  | 38%  |
| (4,16)| 50%  | 6%   |
| (4,24)| 42%  | 0%   |
| (4,32)| 28%  | 0%   |
| (4,40)| 14%  | 0%   |
| (6,8) | 74%  | 40%  |
| (6,16)| 44%  | 0%   |
| (6,24)| 18%  | 0%   |
| (8,8) | 72%  | 28%  |
| (8,16)| 42%  | 2%   |

V. CONCLUSION

In this paper, we proposed the ACR-BB algorithm for solving the single-group multicast beamforming problem. The proposed algorithm is guaranteed to find the global solution of the problem within any given relative error tolerance. To the best of our knowledge, the proposed ACR-BB algorithm is the first specially designed global algorithm for solving the single-group multicast beamforming problem. The proposed algorithm is based on the branch-and-bound strategy as well as a newly developed argument cut technique. We show that the argument cut based convex relaxation problem can be solved efficiently and its optimal value can provide a lower bound that is needed in the design of the branch-and-bound type of algorithms. Furthermore, we show that the tightness of the argument cut based relaxation mainly depends on the width of the argument set, and this motives us to recursively partition the width of the argument set which results in the largest relaxation gap (in order to tighten the relaxation), leading to the ACR-BB
Although the worst-case iteration complexity of the ACR-BB algorithm is exponential in the number of receivers (as shown in Theorem 1), our simulation results show that the proposed algorithm is highly efficient and it significantly outperforms a state-of-the-art general-purpose solver Baron [23], [24] when the problem dimension (i.e., the number of antennas at the transmitter and the number of receivers) is small. The high efficiency of the proposed algorithm is mainly due to the use of the argument cut based relaxation, which itself is interesting and can potentially be used in other related algorithms/problems.

An important role that the proposed algorithm can play is that it can be used as a benchmark for performance evaluation of other heuristic/local optimization algorithms for the same problem. With the help of the proposed algorithm (as the benchmark), we have done extensive simulations in this paper to evaluate the performance of two state-of-the-art algorithms, the SDR algorithm [3] and the SLA algorithm [5], and have gained deeper understanding of the two algorithms. Our simulation results show that the SLA algorithm generally performs better than the SDR algorithm in terms of relative gaps, robustness, and probability of finding the global solutions, and both of the algorithms perform well when the problem dimension is small. However, the performance of both of the algorithms degrades quickly as the problem dimension increases.

One of our future works is to further improve the efficiency of the proposed algorithm, as it takes the proposed algorithm relatively long time to solve problem instances with large dimension. One possible way to do so is to solve the relaxation problem heuristically instead of globally as done in this paper, because our goal to solve the relaxation problem is to obtain a lower bound. Another of our future works is to extend the current results to other scenarios.

**APPENDIX A**

**AN ILLUSTRATION OF THE ACR-BB ALGORITHM**

To make the proposed ACR-BB algorithm clear, an illustration of applying it to solve the following problem instance with \((N, M) = (2, 3)\) is given:

\[
\mathbf{h}_1 = [1.3514 + 2.5260\mathbf{i}, -0.2938 - 1.2571\mathbf{i}]^T,
\]

\[
\mathbf{h}_2 = [-0.2248 + 1.6555\mathbf{i}, -0.8479 - 0.8655\mathbf{i}]^T,
\]

\[
\mathbf{h}_3 = [-0.7145 - 1.1201\mathbf{i}, -0.5890 + 0.3075\mathbf{i}]^T.
\]
We set $\epsilon = 0.1$.

At the 0-th iteration, we initialize $A^0 = [0, 2\pi]^2$. The ACR-BB algorithm solves problem $\text{ACR}(A^0)$ and obtains its optimal solution

$$w^0 = [-0.3238 + 0.5076i, -0.2669 - 0.1394i],$$
$$c^0 = [-1.8166 + 0.2446i, -0.6618 - 0.3010i],$$
$$L^0 = 0.4532.$$

Then, $w^0$ is scaled to obtain the feasible point

$$\hat{w}^0 = [-0.4454 + 0.6982i, -0.3671 - 0.1917i].$$

Now,

$$U^* = ||\hat{w}^0||^2 = 0.8573, \ w^* = \hat{w}^0,$$

and

$$\mathcal{P} = \{\{A^0, c^0, L^0\}\}.$$

At the 1-th iteration, we have

$$\{A^1, c^1, L^1\} = \{A^0, c^0, L^0\}.$$

Since

$$\frac{U^* - L^1}{L^1} = \frac{0.8573 - 0.4532}{0.4532} = 0.8917 > \epsilon,$$

the ACR-BB algorithm starts executing Line 14. Recall $|c^1| = [1.8330, 0.7270]$, where $|\cdot|$ denotes the component-wise absolute value operator. We have

$$k^* = 2, \ z^1_2 = \pi,$$

$$A^1_l = [0, 2\pi] \times [0, \pi] \text{ and } A^1_r = [0, 2\pi] \times [\pi, 2\pi].$$

The ACR-BB algorithm solves problems $\text{ACR}(A^1_l)$ and $\text{ACR}(A^1_r)$ and obtains their optimal solutions

$$w^1_l = [-0.2558 + 0.4725i, -0.3486 - 0.2688i],$$
$$c^1_l = [-1.7747 + 0.5097i, -0.6618 + 0.0000i],$$
$$L^1_l = 0.4825,$$
and
\[
\begin{align*}
\mathbf{w}_r^1 &= \left[ -0.3238 + 0.5076i, -0.2669 - 0.1394i \right], \\
\mathbf{c}_r^1 &= \left[ -1.8166 + 0.2446i, -0.6618 - 0.3010i \right], \\
L_r^1 &= 0.4532.
\end{align*}
\]
Then, \( \mathbf{w}_l^1 \) and \( \mathbf{w}_r^1 \) are scaled to obtain two feasible points
\[
\begin{align*}
\hat{\mathbf{w}}_l^1 &= \left[ -0.3864 + 0.7139i, -0.5267 - 0.4062i \right], \\
\hat{\mathbf{w}}_r^1 &= \left[ -0.4454 + 0.6982i, -0.3671 - 0.1917i \right],
\end{align*}
\]
respectively. Now, since \( \|\hat{\mathbf{w}}_l^1\|^2 = 1.1015 > U^* \) and \( \|\hat{\mathbf{w}}_r^1\|^2 = 0.8573 = U^* \) and \( w^* \) are not updated; since \( L_l^1 < U^* \) and \( L_r^1 < U^* \), we have
\[
\mathcal{P} = \{ \{A_1^1, c_1^1, L_1^1\}, \{A_1^1, c_1^1, L_1^1\}\}.
\]
At the 2-th iteration, since \( L_l^1 < L_l^1 \), we have
\[
\{A_2^1, c_2^1, L_2^1\} = \{A_1^1, c_1^1, L_1^1\}.
\]
Since
\[
\frac{U^* - L^2}{L^2} = \frac{0.8573 - 0.4532}{0.4532} = 0.8917 > \epsilon,
\]
the ACR-BB algorithm starts executing Line 14. Recall \( |c^2| = [1.8330, 0.7270] \). We have
\[
k^* = 2, \ z_2^1 = \frac{3}{2}\pi,
\]
\[
\mathcal{A}_l^2 = [0, 2\pi] \times \left[ \pi, \frac{3}{2}\pi \right] \text{ and } \mathcal{A}_r^2 = [0, 2\pi] \times \left[ \frac{3}{2}\pi, 2\pi \right].
\]
The ACR-BB algorithm solves problems ACR(\( A_l^2 \)) and ACR(\( A_r^2 \)) and obtains their optimal solutions
\[
\begin{align*}
\mathbf{w}_l^2 &= \left[ -0.3258 + 0.5140i, -0.2539 - 0.1364i \right], \\
\mathbf{c}_l^2 &= \left[ -1.8355 + 0.2308i, -0.6804 - 0.3196i \right], \\
L_l^2 &= 0.4534,
\end{align*}
\]
and
\[
\begin{align*}
\mathbf{w}_r^2 &= \left[ -0.5569 + 0.4331i, -0.3750 + 0.3380i \right],
\end{align*}
\]
\[ c_t^2 = [-1.3117 - 0.4494i, 0.0186 - 0.9814i], \]
\[ L_t^2 = 0.7526. \]

Then \( w_t^2 \) and \( w_r^2 \) are scaled to obtain two feasible points
\[
\hat{w}_t^2 = [-0.4334 + 0.6837i, -0.3377 - 0.1815i],
\]
\[
\hat{w}_r^2 = [-0.5674 + 0.4413i, -0.3820 + 0.3443i],
\]
respectively. Now, since \( \|\hat{w}_t^2\|^2 = 0.8023 < U^* \) and \( \|\hat{w}_r^2\|^2 = 0.7811 < U^* \), \( U^* \) is updated to 0.7811 and \( w^* \) is updated to \( \hat{w}_r^2 \); since \( L_t^2 < U^* \) and \( L_r^2 < U^* \), we have
\[
\mathcal{P} = \{ \{A_1, c_1, L_1^1\}, \{A_2, c_2, L_2^1\}, \{A_r, c_r, L_r^1\} \}.
\]
At the 3-th iteration, since \( L_t^3 < L_1^1 < L_r^2 \), we have
\[
\mathcal{A} = \{A_3, c_3, L_3\} = \{A_t^2, c_t^2, L_t^1\}.
\]
Since
\[
\frac{U^* - L^3}{L^3} = \frac{0.7811 - 0.4534}{0.4534} = 0.7228 > \epsilon,
\]
the ACR-BB algorithm starts executing Line 14. Because \( |c^3| = [1.8500, 0.7517] \), we have
\[
k^* = 2, \quad z_2^1 = \frac{5}{4}\pi,
\]
\[
\mathcal{A}_t^3 = [0, 2\pi] \times \left[ \pi, \frac{5}{4}\pi \right] \quad \text{and} \quad \mathcal{A}_r^3 = [0, 2\pi] \times \left[ \frac{5}{4}\pi, \frac{3}{2}\pi \right].
\]
The ACR-BB algorithm solves problems ACR(\( \mathcal{A}_t^3 \)) and ACR(\( \mathcal{A}_r^3 \)) and obtains their optimal solutions
\[
w_t^3 = [-0.3196 + 0.5576i, -0.1681 - 0.1563i],
\]
\[
c_t^3 = [-1.9876 + 0.2034i, -0.8441 - 0.3765i],
\]
\[ L_t^3 = 0.4658, \]
and
\[
w_r^3 = [-0.4103 + 0.5652i, -0.1372 + 0.0230i],
\]
\[
c_r^3 = [-1.9129 - 0.1069i, -0.7071 - 0.7071i],
\]
\[ L_r^3 = 0.5072. \]
Then $w_l^3$ and $w_r^3$ are scaled to obtain two feasible points

\[
\begin{align*}
\hat{w}_l^3 &= [-0.3458 + 0.6033i, -0.1818 - 0.1692i], \\
\hat{w}_r^3 &= [-0.4103 + 0.5652i, -0.1372 + 0.0230i],
\end{align*}
\]

respectively. Now, since $\|\hat{w}_l^3\|^2 = 0.5453 < U^*$ and $\|\hat{w}_r^3\|^2 = 0.5072 < U^*$, $U^*$ is updated to 0.5072 and $w^*$ is updated to $\hat{w}_r^3$; since $L_l^3 < U^*$ and $L_r^3 = U^*$, we have

\[ P = \{ \{A_1^l, c_1^l, L_1^l\}, \{A_2^l, c_2^l, L_2^l\}, \{A_3^l, c_3^l, L_3^l\}, \{A_3^r, c_3^r, L_3^r\} \}. \]

At the 4-th iteration, since $L_l^3 < L_1^3 < L_3^3 < L_r^3$, we have

\[ \{A^4, c^4, L^4\} = \{A_3^l, c_3^l, L_3^l\}. \]

Since

\[ \frac{U^* - L^4}{L^4} = \frac{0.5072 - 0.4658}{0.4658} = 0.0889 < \epsilon, \]

the ACR-BB algorithm terminates and return $U^* = 0.5072$ and $w^* = \hat{w}_r^3$.

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