FANO SHIMURA VARIETIES WITH MOSTLY BRANCHED CUSPS

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Abstract. We show the Satake-Baily-Borel compactification of certain Shimura varieties are Fano varieties or with ample canonical divisor by means of special modular forms and give various concrete examples. Their unbranched open subsets are always quasi-affine and in Fano unitary Shimura varieties case for instance, all but one cusps are necessarily covered by the closure of branch divisors. Our examples include the moduli of (log) Enriques surfaces, those corresponding to $II_{2,26}$, and those associated to various Hermitian lattices. On the way, we also show that for some imaginary quadratic fields and any unimodular Hermitian lattices, the associated unitary Shimura varieties are unbranched in codimension 1.

1. Introduction and General claims

1.1. Introduction. We show the Satake-Baily-Borel compactification of certain Shimura varieties are Fano varieties, Calabi-Yau varieties or have ample canonical divisors with mild singularities (see Theorem 1.3), and discuss some variants statements, applications and new examples. Study of birational types of Shimura varieties is a semi-classical topic; e.g., [Tai82, Kon94, GHS07, Gri10, GH14, Gri18, Ma12, Ma18, Mac20a, Mac20b] to name a few. One of the powerful tool for it is the use of certain special modular forms, as in this paper. For this recurring theme, our standpoint here is to focus on the Satake-Baily-Borel compactification, study it through modern birational geometry adapted to singular varieties and give applications.

In particular, we are motivated by its Fanoness in certain examples and their common properties such as e.g. all but one compact cusps are shown to be contained in the closure of branch divisors and if there are no such compact cusps, two general points are connected by a rational curve i.e., rationally connected by [Zha06]. The former

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In this paper, we are not concerned with its $\mathbb{T}$-structure and Shimura varieties only refers to the (connected) complex varieties obtained as arithmetic quotients of Hermitian symmetric domains. See details below.
uses \cite{Amb03, Fjn11}, and in particular it logically relies on a vanishing theorem proven in \textit{loc.cit}. We do not know other proof which does not use vanishing theorem (Problem 1.12). See Corollaries 1.7, 1.8, 1.10 for the details and more assertions proven.

In this section we prove such general theorems and next section provides various concrete examples. First we set the scene.

1.2. \textbf{Convention and Notation.} Below, we discuss the linear equivalence class of a Cartier divisor and the corresponding holomorphic line bundle interchangably. Similarly, we do not distinguish the $\mathbb{Q}$-linear equivalence class of a $\mathbb{Q}$-Cartier divisor and the corresponding $\mathbb{Q}$-line bundle. We use the following notations throughout.

- $G$ is a simple algebraic group over $\mathbb{Q}$, not isogeneous to $\text{SL}(2)$.
- $G$ is the identity component of $G(\mathbb{R})$, which we assume to be a simple Lie group.
- $K$ is a maximal compact subgroup of $G$.
- Corresponding Hermitian symmetric domain is $G/K$.
- Take an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$ i.e., commensurable to $G(\mathbb{Z})$.
- $X := \Gamma \backslash G/K$ and its Satake-Baily-Borel compactification $\overline{X}_{\text{SBB}}$ \cite{Sat60, BB66}.
- Denote a toroidal compactification of $X$ in the sense of [AMRT], with an arbitrary fixed cone decompositions, simply as $\overline{X}$. (The choice of cone decompositions do not affect the following discussions.)
- Denote the boundary divisor $\overline{X} \setminus X$ as $\Delta$ (with coefficients 1).
- Denote the branch divisor of $G/K \to \Gamma \backslash G/K$ to be $\bigcup_i B_i \subset X$ with prime divisors $B_i$ and branch degree $d_i$. We denote the closure of $B_i$ in $\overline{X}$ (resp., $\overline{X}_{\text{SBB}}$) as $\overline{B}_i$ (resp., $\overline{B}_i^{\text{SBB}}$).
- $X^o := X \setminus \bigcup_i B_i$.
- $L := K_{\overline{X}} + \Delta + \sum_i \frac{d_i-1}{d_i} B_i \in \text{Pic}(\overline{X}) \otimes \mathbb{Q}$ and its descended (automorphic) $\mathbb{Q}$-line bundle on $\overline{X}_{\text{SBB}}$ i.e., $K_{\overline{X}_{\text{SBB}}} + \sum_i \frac{d_i-1}{d_i} B_i^{\text{SBB}}$.

We consider the following subclasses of reflective modular forms, the concept originally formulated in \cite{Gri10} for orthogonal group case. These modular forms are rare, but still various interesting examples are known (cf., \cite{Gri18}, our Section 2).

\textbf{Assumption 1.1} \textit{(Special reflective modular forms - General case).}

\emph{Consider the following subclasses of reflective modular forms:}
(i) a non-vanishing holomorphic section $s$ of
\[ O_X(N(aL - \sum_i \frac{d_i-1}{d_i}B_i)) := L^{\otimes aN} \left( - \sum_i \frac{N(d_i-1)}{d_i}B_i \right) \]
for some $N \in \mathbb{Z}_{>0}$, $a \in \mathbb{Q}_{>0}$ with $aN, \frac{N}{d_i} \in \mathbb{Z}_{>0}$. We follow the same convention below.

(ii) a non-vanishing holomorphic section $s$ of $O_X(N(aL - \sum_i c_iB_i))$
for some $N \in \mathbb{Z}_{>0}$, $a \in \mathbb{Q}_{>0}$, and $c_i \in \mathbb{Q}$ with $0 \leq c_i \leq \frac{d_i-1}{d_i}$ for all $i$, such that $Na, Nc_i \in \mathbb{Z}$.

For many orthogonal case, the former condition (i) above becomes simpler as follows (due to [GHS07, 2.12, 2.13]):

**Assumption 1.2** (Special reflective modular forms - orthogonal case). Consider the case when there is a lattice $\Lambda$ of signature $(2, n)$ such that $G = O(\Lambda \otimes \mathbb{Q})$ with $\Gamma \subset O(\Lambda)$ we concern the following subclasses of reflective modular forms:

(i) a non-vanishing holomorphic section $s$ of $O_X(N(aL - \frac{1}{2} \sum_i B_i))$
for some $N \in \mathbb{Z}_{>0}$, $a \in \mathbb{Q}_{>0}$.

Note that $N$ is unessential as it gets multiplied when replacing $s$ by its power, while $a$ is more essential and sometimes called slope in the literatures. When we work on the cases $G = O(2, n)$ or $G = U(1, n)$ and regard $s$ as a modular form, we call its arithmetic weight simply as weight from now on (cf., e.g., [GHS07]). Recall that the compact dual $D^c$ of $D$ in the orthogonal case $G = O(2, n)$ is the $n$-dimensional quadric hypersurface (resp., $D^c = \mathbb{P}^n$ in the unitary case $G = U(1, n)$), its canonical divisor is $K_{D^c} = \mathcal{O}_{\mathbb{P}^{n+1}(-n)}|_{\mathbb{P}^n}$ (resp., $K_{D^c} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$) so that the canonical weight is $n$ (resp., $n + 1$).
Therefore, the quantity $a$ in Theorem 1.3 is the arithmetic weight of the modular form $s$ divided by such canonical weight.

Below, we discuss various Shimura varieties $X$ which can be roughly divided into two types i.e., those with modular forms satisfying Assumption 1.1 (i), and those with modular forms satisfying Assumption 1.1 (ii).

The former is discussed in the next subsection §1.3 with examples given in section §2 and the latter is discussed in the subsection 1.4 while some examples are given in [GH14, Mae20b].

1.3. Fano, Calabi-Yau, K-ample modular varieties. Here is our first general theorem.

**Theorem 1.3.** We follow the notation as above. If there is a reflective modular form which satisfies Assumption 1.1 (i) with some $a \in \mathbb{Q}_{>0}$,
the Satake-Baily-Borel compactification $\overline{X}^{\text{SBB}}$ of $X = \Gamma \setminus D$ only has log canonical singularities and $X^o$ is quasi-affine.

Actually, $\overline{X}^{\text{SBB}}$ is either

(i) a Fano variety i.e., $-K_{\overline{X}^{\text{SBB}}} \text{ is ample (Q-Cartier)}$ if $a > 1$,

(ii) a Calabi-Yau variety i.e., $K_{\overline{X}^{\text{SBB}}} \sim_Q 0$ if $a = 1$,

(iii) $K_{\overline{X}^{\text{SBB}}}$ is ample if $a < 1$.

**Terminology.** In this paper, we often say a normal variety is a log canonical model (resp., canonical model) in the sense that it only has log canonical singularities (resp., canonical singularities) and the canonical class is ample. Hence, in the case (iii) above, $\overline{X}^{\text{SBB}}$ is a log canonical model. For basics of birational geometry, we refer to e.g., [KM98].

**Proof.** Note that codimension of the boundary of Satake-Baily-Borel compactification $\partial \overline{X}^{\text{SBB}} := \overline{X}^{\text{SBB}} \setminus X$ is at least 2, following from our assumption that $G$ is not isogeneous to $\text{SL}(2)$. The existence of the special reflexive modular form implies

\[ \sum_i \frac{d_i - 1}{d_i} B_i \sim_Q aL. \]

If we regard the holomorphic section satisfying Assumption 1.1 (i) as a section of ample line bundle $L \otimes aN$, it follows that the complement of the vanishing locus is affine but that is nothing but $\overline{X}^{\text{SBB}} \setminus \bigcup_i B_i^{\text{SBB}}$ which includes $X^o$. This proof reflects the idea of [Bor96].

From (1) and the definition of $L$ it follows that

\[ -K_{\overline{X}^{\text{SBB}}} \sim_Q (a - 1) L \]

in $\text{Pic}(\overline{X}^{\text{SBB}}) \otimes \mathbb{Q}$. Hence $-K_{\overline{X}^{\text{SBB}}}$ is ample and $\mathbb{Q}$-Cartier. On the other hand, from [Mum77] 3.4, 4.2 (also see 1.3)], $(\overline{X}^{\text{SBB}}, \sum_i \frac{d_i - 1}{d_i} B_i^{\text{SBB}})$ has only log canonical singularity (as a pair) and $K_{\overline{X}^{\text{SBB}}} + \sum_i \frac{d_i - 1}{d_i} B_i^{\text{SBB}}$ is ample. Thus $\sum_i \frac{d_i - 1}{d_i} B_i^{\text{SBB}}$ is also $\mathbb{Q}$-Cartier so that $X$ itself is also log canonical.

On the other hand, from the construction of Baily-Borel compactification [BB66], $L$ is ample so that our latter statements of above theorem all follow from (2). We complete the proof. \[ \square \]

**Remark 1.4.** The above results are analogous to the Fanoness results in [DNS9], (resp., [Huy94, §2] also [Li94, §4]) in the context of moduli of (semi)stable bundles over curves (resp., surfaces). For the case over surfaces, in the place of automorphic line bundle $L$, the determinant
line bundle which descends to the Donaldson-Uhlenbeck compactification is used.

**Remark 1.5.** Case (iii) is a variant of so-called “low weight cusp form trick” (cf., e.g., [GHS07]). See also [Gri10], [Gri18, §5.5] and references therein.

We prepare the following notion.

**Definition 1.6.** We call a cusp $F$ of $\overline{X}^{\text{SBB}}$ is **naked** if it is not contained in $\text{Supp}(B_i^{\text{SBB}}) \cap \partial \overline{X}^{\text{SBB}}$ for any $i$. Further, we call it **minimal naked** if it is minimal with respect to the closure relation among naked cusps i.e., $F \setminus F$ is contained in $(\bigcup_i \text{Supp}(B_i^{\text{SBB}})) \cap \partial \overline{X}^{\text{SBB}}$.

Below we observe certain weakening of connected-ness of cusps closure in the case of $a > 1$ i.e., Fano case. This follows from [Amb03, 4.4, 6.6 (ii)], [Fjn11, 8.1], [Fjn10, §3], [FG12, 1.2] as the proof below, which is essentially just a review to make our logic more self-contained. Compare with our examples of the modular varieties given in the next section.

**Corollary 1.7.** Under the same assumption of Theorem 1.3 and further that $a > 1$, then

$$\partial \overline{X}^{\text{SBB}} \setminus \bigcup_i B_i^{\text{SBB}}$$

is connected, whose closure is nothing but the non-log-terminal locus of $\overline{X}^{\text{SBB}}$. More strongly, there is at most one minimal naked cusp with respect to the closure relation. Here, $\partial \overline{X}^{\text{SBB}}$ means the boundary of the Satake-Baily-Borel compactification i.e., $\overline{X}^{\text{SBB}} \setminus X$.

Furthermore, if we suppose such a minimal naked cusp $F$ exists, there is an effective $\mathbb{Q}$-divisor $D_F$ such that $(F, D_F)$ has only klt singularity and being a log Fano pair i.e., $-K_F - D_F$ is ample and $\mathbb{Q}$-Cartier. For instance, if $F$ is a modular curve, it is rational i.e., $F \simeq \mathbb{P}^1$ (with “Hauptmodul”) and more generally, with the description of $F$ as a Shimura variety $\Gamma_F \backslash G_h(F)/(K \cap G_h(F))$ (as in [AMRT, III §3]) $G_h(F)/(K \cap G_h(F)) \rightarrow F$ is at least ramified in codimension 1 i.e., has nontrivial modular branch divisors.

**Proof.** Recall that by the existence of toroidal compactification [AMRT, chapter III], especially loc.cit 6.2, the log canonical centers of $(\overline{X}^{\text{SBB}}, \sum_i \frac{d_i}{d}B_i^{\text{SBB}})$ (resp., $\overline{X}^{\text{SBB}}$) are nothing but cusps of the Satake-Baily-Borel compactification $\overline{X}^{\text{SBB}}$ (resp., cusps of the Satake-Baily-Borel compactification $\overline{X}^{\text{SBB}}$ which are not contained in
We take the union of minimal naked cusps of $X^{SBB}$ as $W$ and put reduced scheme structure on it. We denote the corresponding coherent ideal sheaf of $\mathcal{O}_{X^{SBB}}$ as $I_W$.

From a vanishing theorem of [Amb03, 4.4], [Fjn11, 8.1], whose absolute non-log version is enough for our particular purpose here, we have $H^1(X, I_W) = 0$. Hence, combined with a standard cohomology exact sequence arguments, $H^0(\mathcal{O}_W) \simeq \mathbb{C}$ follows. Hence it implies the connectivity of $W$, so that there is at most 1 minimal naked cusp $F$.

For such $F$, the existence of $D_F$ on the closure $\overline{F}$ follows from applying the log canonical subadjunction [FG12, 1.2] to $F \subset (X^{SBB}, 0)$. The last non-quasi-étaleness assertion then follows from by applying again [Mum77, 3.4, 4.2] to $F$.

We make a caution that above Corollary 1.7 does not claim the naked cusp always has log terminal singularity as possibly obstructed by existence of bigger cusps. Nevertheless, in $\mathbb{Q}$-rank 1 case, we have the following.

**Corollary 1.8** ($\mathbb{Q}$-rank 1 case). Under the same assumption of Theorem 1.3 with $a > 1$, if further $\mathbb{Q}$-rank of $G$ is 1 (e.g., when $G \simeq U(1, n)$ for some $n$ so that $G/K$ is a $n$-dimensional complex unit ball), either one of the followings hold:

(i) there is exactly one naked cusp $F$ of $X^{SBB}$ which is an isolated non-log-terminal locus but at worst log canonical. Furthermore, there is an effective $\mathbb{Q}$-divisor $D_F$ such that $(F, D_F)$ is a klt log Fano pair hence in particular, the modular branch divisor in $F$ is nonzero effective.

(ii) or no naked cusp exists and $X$ is rationally connected i.e., two general points are connected by a rational curve and has at worst log terminal singularities. Furthermore, $X \setminus \text{Supp} \cup_i B_i$ is affine (not only quasi-affine).

**Proof.** Among the above statements, the only assertion which does not follow trivially from Corollary 1.7 is the rationally connected assertion for the latter case (ii), which holds as follows: non-existence of naked cusp means $X^{SBB} \setminus X$ is included in $\cup_i \text{Supp}(B_i^{SBB})$ which implies the log terminality of $X$. Hence the rationally connectedness follows from a theorem of Zhang [Zha06]. The affine-ness assertion follows from the proof of Theorem 1.3 and that there are no naked cusps. \qed

Here is a version of converse direction of Theorem 1.3.

**Theorem 1.9.** We follow the notation of Theorem 1.3. If $X^{SBB}$ is either
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- \( K_X^{\text{SBB}} \equiv 0 \) or
- either \( K_X^{\text{SBB}} \) or \( -K_X^{\text{SBB}} \) is ample with Picard number 1,

then there are special reflective modular forms satisfying Assumption 1.1 (i) for some \( a \in \mathbb{Q}_{>0} \) and sufficiently divisible \( N \in \mathbb{Z}_{>0} \). Furthermore, if it is of certain orthogonal type i.e., \( G \) is isogeneous to \( SO(\Lambda) \) for \( \Lambda = U \oplus U(l) \oplus N \) with some negative definite lattice \( N \) and \( l \in \mathbb{Z}_{>0} \), the modular forms are necessarily Borcherds lift of some nearly holomorphic elliptic \( Mp_2(\mathbb{Z}) \)-modular forms of specific principal part of the Fourier expansion in the sense of [Bor98], [Bru02, §3.4].

Proof. Given the proof of Theorem 1.3, we can almost trace back the arguments as follows. In either cases, the automorphic line bundle \( L \) is proportional to \( K_X^{\text{SBB}} \) in \( \text{Pic}(X^{\text{SBB}}) \), hence so is it to \( \sum d_i - 1 d_i B_i^{\text{SBB}} \). Therefore, \( O(N(aL - \sum d_i - 1 d_i B_i^{\text{SBB}})) \) is trivial for some \( a, N \). The last assertion follows from [Bru02, 5.12], [Bru14, 1.2]. \( \square \)

1.4. Modular varieties with big anti-canonical classes. Recall that Gritsenko-Hulek [GH14] (resp., Maeda [Mae20b]) discuss the classes of reflective orthogonal modular forms (resp., unitary modular forms) satisfying Assumption 1.1 (ii) with \( a > 1 \) and proved uniruledness of \( X \) and constructs some examples. We note that in the common notation in loc.cit, \( a = \frac{k}{mn} \).

This subsection proves the following slight refinement of their results, which applies to the examples constructed in loc.cit.

**Theorem 1.10** (cf., [GH14] 2.1, [Mae20b] 4.1). We follow the notation of [1.3] and discuss Shimura varieties \( X = \Gamma \backslash D \) for a priori general \( G \).

If there is a reflective modular form \( \Phi \) which satisfies Assumption 1.1 (ii) with some \( a \in \mathbb{Q} \), \( a > 1 \), we define \( V_\Phi := \cup_F \overline{F} \subset \partial X^{\text{SBB}} \) where \( F \) runs through all cusps along which \( \Phi \) does not vanish (as a function, or a section of \( L^{a_0 N} \)). Then, the following holds.

(i) The Satake-Baily-Borel compactification \( \overline{X}^{\text{SBB}} \) of \( X = \Gamma \backslash D \) only has log canonical singularities, \( X^o \) is quasi-affine and \( -K_X^{\text{SBB}} \) is big.

(ii) For any two closed points \( x, y \in \overline{X}^{\text{SBB}} \), there are union of rational curves \( C \) such that \( C \cup V_\Phi \) is connected (i.e., rationally chain connected modulo \( V_\Phi \) cf., [HM07, 1.1]).

In particular, \( X \) is uniruled. If \( G = U(1,n) \) for some \( n \), then \( \overline{X}^{\text{SBB}} \) is even rationally chain connected.
(iii) If we consider the set of cusps outside $V_\Phi$, there is at most 1 minimal element (cusp) with respect to the closure relation.

Proof. The assertion (i) follows the arguments of the proof of Theorem 1.3. From the existence of $\Phi$, it follows in the same way that

$$-K_{X_{\text{SBB}}} \sim_Q (a-1)L + \sum_i \left(\frac{d_i-1}{d_i} - c_i\right)B_i^{\text{SBB}},$$

hence it is big. The proofs of other assertions in (i) are same as those of Theorem 1.3. For (ii), note that the non-klt locus of $(X_{\text{SBB}}, \sum_i \left(\frac{d_i-1}{d_i} - c_i\right)B_i^{\text{SBB}})$ is the union of log canonical centers of $(X_{\text{SBB}}, \sum_i \frac{d_i-1}{d_i}B_i^{\text{SBB}})$ which are not inside $\text{Supp}(\text{div}(\Phi))$. Hence, the assertion (ii) directly follows from [HM07, 1.2] for $(X_{\text{SBB}}, \sum_i \frac{d_i-1}{d_i}B_i^{\text{SBB}})$. The assertion for unitary case holds since the cusps are all 0-dimensional. For (iii), the same arguments as Corollary 1.7, similarly applying [Amb03, 4.4, 6.6(ii)] or [Fjn11, 8.1] to the log canonical Fano pair $(X_{\text{SBB}}, \sum_i (\frac{d_i-1}{d_i} - c_i)B_i^{\text{SBB}})$, give a proof.

Remark 1.11. We can also show a variant of Corollary 1.7, Theorem 1.10 (iii) under general meromorphic modular forms if we replace the use of [Amb03, 6.6(ii)] by [Amb03, 4.4] or [Fjn11, 6.1.2]. However, because the obtained statement is rather complicated and no interesting applications have been found (yet at least), we omit it in this paper.

We conclude this section by posing a natural problem.

Problem 1.12. In specific situations e.g., when $G = SO(\Lambda \otimes \mathbb{Q})$ for a quadratic lattice $\Lambda$, or in the unitary modular case corresponding to a Hermitian lattice as later subsection 2.4, the assertions of Corollaries 1.7, 1.8, Theorem 1.10 (iii) can be phrased in a purely lattice theoretic manner. Is there a more lattice theoretic or number theoretic proof without use of vanishing theorem in algebraic geometry?

2. Examples of Fano and K-ample cases

We provide examples to which Theorems 1.3, Corollary 1.7, Corollary 1.8, Theorem 1.9 in §1.3 apply. In the examples, the compactified modular varieties are either Fano varieties or with ample canonical classes. There are also some examples with $a = 1$, for instance [FSM10] (cf., also earlier [BN94] with a weaker statement) but we do not focus such cases in this paper.
2.1. Siegel modular cases. We start with confirming some known examples, which we review, fit our picture. Actual use of Theorem 1.3 is from \S 2.3. Here is a couple of examples of Siegel modular varieties whose Satake-Baily-Borel compactifications are Fano varieties.

Example 2.1 ([Igu64]). The Satake-Baily-Borel compactification of the moduli of principally polarized abelian surfaces $\mathcal{A}_{2}^{\text{SBB}}$ is known to be weighted projective hypersurface in non-well-formed $\mathbb{P}(4, 6, 10, 12, 35)$ of degree 70 with the coarse moduli isomorphic to $\mathbb{P}(2, 3, 5, 6)$ by relating to the invariants of genus 2 curves, hence binary sextics. Note that the adjunction does not work due to non-well-formedness, as indeed one has non-trivial isotropy ($\mu_2$) along a divisor in the moduli stack. The reduction of the natural Faltings-Chai model over $\mathbb{F}_p$ are also determined (cf., [Ich09, vdG21]).

Example 2.2 (cf., [vdG82, 5.2] (also [Igu64])). The Satake-Baily-Borel compactification of the moduli of principally polarized abelian surfaces with level 2 structure $\Gamma(2)\backslash\mathcal{H}^{\text{SBB}}$ is known to be a quartic 3-fold

\[ \sum_{i=0}^{5} x_i = (\sum_{i=0}^{5} x_i^2)^2 - 4(\sum_{i=0}^{5} x_i^4) = 0, \]

with non-isolated singularities along 15 lines.

2.2. Orthogonal modular cases, Part I. Below, we consider the cases when $G = SO(\Lambda\otimes\mathbb{Q})$ for a quadratic lattice $(\Lambda, (\ , \ ))$ of signature $(2, n)$ with $n \in \mathbb{Z}_{>0}$. We realize the Hermitian symmetric domain $X = G/K$ as $G/K \simeq D_\Lambda$ which is defined as one of the isomorphic two connected component of

\[ \{ v \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (v, v) = 0, \ (v, \overline{v}) > 0 \}. \]

We keep this notation throughout in the discussion of orthogonal modular varieties. Our first two examples in this Part I are understood via moduli-theoritic methods and GIT as follows.

Example 2.3 (Hilbert). The GIT compactification of the moduli of cubic surfaces ([OSS16, §4.2]) is known to be isomorphic to the Satake-Baily-Borel compactification of the stable locus which admits uniformization of complex ball. (cf., [ACT02]). Hilbert’s invariant calculation in his thesis tells this is $\mathbb{P}(1, 2, 3, 4, 5)$ hence the only cusp is not naked because of the log terminality. Obviously it is also a (Q-)Fano variety. This is also one of simplest examples of the K-moduli variety of Fano varieties ([OSS16, §4.2]).

Given [Ma18], it is reasonable to ask the following problem in general.
Problem 2.4. Classify the lattices $\Lambda$ of signature $(2, n)$ such that the Satake-Baily-Borel compactification $\Gamma \backslash \mathcal{D}_\Lambda$ are Fano varieties, especially when $\Gamma = O^+(\Lambda)$ or $\tilde{O}^+(\Lambda)$.

From what follows, our arithmetic subgroup satisfies $\Gamma$ is either $O^+(\Lambda)$ or the stable orthogonal group $\tilde{O}^+(\Lambda)$.

Example 2.5 (Moduli of elliptic K3 surfaces). We consider the moduli $M_W$ of Weierstrass elliptic K3 surfaces, which is open subset of $O^+(\Lambda) \backslash \mathcal{D}_\Lambda$ for $\Lambda := U^2 \oplus E_8(-1)^{\oplus 2}$. We consider its Satake-Baily-Borel compactification ([O018, Theorem 7.9]), which we denote $\overline{M_W}^{\text{SBB}}$ here. Recall from loc.cit §7.1 that there are exactly two 1-cusps intersecting at the only 0-cusp. Two 1-cusps are $M_W^{\text{mn}}$ with canonical Gorenstein singularity and $M_W^{\text{seg}}$ with toroidal singularity (including the intersection) hence $\overline{M_W}^{\text{SBB}}$ also only has log terminal singularity ([Od20, Part I, §2]). From the singularity type along 1-cusp $M_W^{\text{mn,o}}$ loc.cit in its Theorem 2.2, it easily follows that the local fundamental group along the transversal slice is $(\mathbb{Z}/2\mathbb{Z})^4$ hence not cyclic. In particular, $\overline{M_W}^{\text{SBB}}$ can not be weighted projective space.

We can take a complete curve $C$ in $\overline{M_W}$ which does not meet the closure of the 1-cusp $M_W^{\text{seg}}$. If we take a resolution $\overline{M_W}$ of $\overline{M_W}^{\text{SBB}}$ and consider the strict transform of $C$ as $C'$, then $(K_{\overline{M_W}}.C') < 0$ because of the unirationality of $\overline{M_W}$. Hence it would contradict if $K_{\overline{M_W}}$ is either trivial or ample. Therefore, since $\overline{M_W}$ has Picard rank 1 because of the GIT quotient description, we conclude that $M_W$ is also a (log terminal) Fano variety and all cusps are non-naked as confirmed in loc.cit Part I, §2. [Lej93] proved rationality of $M_W$.

2.3. Orthogonal modular cases, Part II. From here, we use the Borchers products to show various Satake-Baily-Borel compactifications of orthogonal modular varieties are Fano varieties or log canonical models.

Notation. Let

$$\mathcal{H}(\ell) := \{v \in \mathcal{D}_\Lambda \mid (v, \ell) = 0\}$$
be a special divisor with respect to \( \ell \in \Lambda \) with \( (\ell, \ell) < 0 \). We also define
\[
\mathcal{H}_{-2} := \bigcup_{\ell \in \Lambda, \ell^2 = -2} \mathcal{H}(\ell)
\]
\[
\mathcal{H}_{-4} := \bigcup_{\ell \in \Lambda, \ell^2 = -4} \mathcal{H}(\ell)
\]
\[
\mathcal{H}_{-4, \text{special-even}} := \bigcup_{\ell \in \Lambda : \text{special-even}, \ell^2 = -4} \mathcal{H}(\ell).
\]
Here we say a vector \( r \in \Lambda \) is special-even (also called even type e.g., in [Kon02]) if \( (\ell, r) \) is even for any \( \ell \in \Lambda \), i.e., \( \text{div}(r) \) is even integer, so that the corresponding reflection lies in \( \Gamma \). For any element \( r \in \Lambda \) satisfying \( (r, r) < 0 \), we define the reflection \( \sigma_r \in O^+(\Lambda)(\mathbb{Q}) \) with respect to \( r \) as follows:
\[
\sigma_r(\ell) := \ell - \frac{2(\ell, r)}{(r, r)} r.
\]
The union of ramification divisors of \( \pi_\Gamma : \mathcal{D}_\Lambda \to \Gamma \mathcal{D}_\Lambda \) is
\[
\bigcup_{r \in \Lambda / \pm: \text{primitive}} \mathcal{H}(r)
\]
by [GHS07] for \( \Gamma \subset O^+(\Lambda) \) and \( n > 3 \). We sometimes denote \( \pi_\Gamma \) as \( \pi \).

**Example 2.6.** Let \( \Pi_{2,26} = U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus E_8(-1) \) be an even unimodular lattice of signature \((2, 26)\). We consider the case \( \Gamma = O^+(\Lambda) \). There is the modular form \( \Phi_{12} \) of weight 12 on \( \mathcal{D}_{\Pi_{2,26}} \) by Borcherds [Bor95] with \( \text{div} \Phi_{12} = \mathcal{H}_{-2} \).

On the other hand, the ramification divisors of the map \( \pi : \Pi_{2,26} \to X := O^+(\Pi_{2,26}) \backslash \mathcal{D}_{\Pi_{2,26}} \) are \( \mathcal{H}_{-2} \) by the even unimodularity of \( \Lambda \) and [GHS07]. Now \( \Phi_{12}^{2 \times 26} \) satisfies Assumption 1.2 (i) with \( N = 104 \) and \( a = \frac{3}{13} \) and by Theorem 1.3 (iii) so that the Satake-Baily-Borel compactification \( X^{\text{SBB}} \) of the 26-dimensional orthogonal modular variety \( X = O^+(\Pi_{2,26}) \backslash \mathcal{D}_{\Pi_{2,26}} \) is a log canonical model i.e., with ample canonical divisor \( K_X \) and at worst log canonical singularities. Furthermore, we can specify what is the non-log-terminal locus or the log canonical center. There are exactly 24 1-dimensional log canonical centers, which corresponds to Niemeier lattices and all intersect at a common closed point (cf., [Gri12, 1.1]). By [Bor95, \S 10], [Gri12, 1.2], and our arguments in the proof of Theorem 1.3 the only non-log-terminal locus is exactly one 1-cusp which is the compactification of the modular curve.
Corollary 2.7 (II_{26} case). The Satake-Baily-Borel compactification $X_{\text{SBB}}$ of the 26-dimensional orthogonal modular variety $X = O^+(II_{2,26}) \backslash \mathcal{D}_{II_{2,26}}$ is a log canonical model i.e., with ample canonical divisor $K_{X_{\text{SBB}}}$ and at worst log canonical singularities. Further, the non-log-terminal locus is the single $C \simeq \mathbb{P}^1$ in the boundary $\partial X_{\text{SBB}}$ which compactifies 1-cusp $SL(2,\mathbb{Z}) \backslash \mathbb{H}$ and is characterized by that the corresponding isotropic plane $p \subset II_{2,26} \otimes \mathbb{R}$ satisfies that $(p^\perp \cap II_{2,26})/(p \cap II_{2,26})$ is the Leech lattice i.e., contains no roots. Its degree is $(K_{X_{\text{SBB}}}, C) = 20$. (resp., $(H_{-2}, C) = 12$).

Later in Example 2.27, we also construct a 13-dimensional unitary modular subvariety which also compactifies with ample canonical class as the Satake-Baily-Borel compactification.

Example 2.8. Let $\Lambda := U \oplus U \oplus E_8(-1)$ be an even unimodular lattice of signature $(2,10)$. We again consider the case $\Gamma = O^+(\Lambda)$. Borcherds constructed a reflective modular form on $\mathcal{D}_{\Lambda}$.

Theorem 2.9 (Bor95, 10.1, 16.1). There is a reflective modular form $\Phi_{252}$ of weight 252 on $\mathcal{D}_{\Lambda}$ such that $\text{div} \Phi_{252} = H_{-2}$.

Here, by the map $\pi : \mathcal{D}_\Lambda \to X := O^+(\Lambda) \backslash \mathcal{D}_\Lambda$, the divisors $\mathcal{H}_{-2}$ maps to the unique branch divisors (cf., GHS07, §2). Hence $\Phi_{252}$ satisfies Assumption [1.2] (i) with $N = 20c$ and $a = \frac{6a}{5}$ for some $c \in \mathbb{Z}$, and by Theorem [1.3] (i), the compactified Shimura variety $\overline{X}_{\text{SBB}}$ is a Fano variety. Actually, [HUL14, 1.1], [DKW19, 4.1] (also attributed to H.Shiga and Loo84) shows it is a weighted projective space $\mathbb{P}(2,5,6,8,9,11,12,14,15,18,21)$.

Example 2.10. The well-studied moduli space $M_{\text{Enr}}$ of (unpolarized) Enriques surfaces (cf., e.g., Nam85, Ste91, Bor96, Kon94) also fit into our setting. Let $\Lambda_{\text{Enr}} := U \oplus U(2) \oplus E_8(-2)$ be an even lattice of signature $(2,10)$. Then the Shimura variety $M_{\text{Enr}} := O^+(\Lambda_{\text{Enr}}) \backslash \mathcal{D}_{\Lambda_{\text{Enr}}}$.
is a 10-dimensional quasi-projective variety. Now we review the ramification divisors of the natural map $\pi : D_{\Lambda_{\text{Enr}}} \to M_{\text{Enr}}$ and moduli description. By the unimodularity of $\Lambda_{\text{Enr}}$ and \[ \text{[GHS07]}, \text{or [GH16]}, \] the ramification divisors are

$$\mathcal{H}_{-2} \cup \mathcal{H}_{-4, \text{special--even}}.$$ 

On the other hand, let $\tilde{M}_{\text{Enr}} := \tilde{O}^+(\Lambda_{\text{Enr}}) \setminus D_{L_{\text{Enr}}}$. Then the following are known.

**Proposition 2.11.**

(i) $M_{\text{Enr}} \setminus \pi(\mathcal{H}_{-2})$ is the so-called moduli space of Enriques surfaces (cf., e.g., \[ \text{[Nam85]}. \] Moreover this is rational (Kondo \[ \text{[Kon94]}. \])

(ii) $\tilde{M}_{\text{Enr}} \setminus \pi(\mathcal{H}_{-2})$, which is a finite cover of $M_{\text{Enr}}$, is the moduli space of Enriques surfaces with a certain level-2 structure. Moreover $\tilde{M}_{\text{Enr}}$ and $\tilde{M}_{\text{Enr}} \setminus \pi(\mathcal{H}_{-2})$ are of general type (Gritsenko cf., \[ \text{[GH16]}. \])

(iii) $M_{\text{Enr}} \setminus (\pi(\mathcal{H}_{-2}) \cup (\pi(\mathcal{H}_{-4, \text{special--even}})))$ is the moduli space of non-nodal Enriques surfaces.

Going back to our situation, we need special reflective modular forms satisfying Assumption 1.2 (i). Our input here is following.

**Lemma 2.12 (\[ \text{[Bor96, Kon02]}. \])**. There exist two reflective modular forms $\Phi_4$ and $\Phi_{124}$ on $D_{L_{\text{Enr}}}$ of weights 4, 124 respectively such that;

$$\text{div}\Phi_4 = \mathcal{H}_{-2}$$

$$\text{div}\Phi_{124} = \mathcal{H}_{-4, \text{special--even}}.$$ 

We put $F_{128} := \Phi_4 \Phi_{124}$. Then this is a weight 128 modular form on $D_{L_{\text{Enr}}}$ and $\text{div}(F_{128})$ is exactly the ramification divisors of the map $\pi : D_{L_{\text{Enr}}} \to M_{\text{Enr}}$ with coefficients 1. Now $F_{128}^2$ has a trivial character. Hence $F_{128}^{20}$ satisfies Assumption 1.2 (i) with $N = 40$ and $a = \frac{32}{5}$ and by Theorem 1.3 (i), $\tilde{M}_{\text{Enr}}^{\text{SBB}}$ is a log canonical Fano variety.

Actually, it is even log terminal without naked cusps as we confirm as follows. By \[ \text{[Ste91, 3.3, 4.5]}, \] there are only two 0-cusps which correspond to an isotropic vector $e$ in the first summand $U$ and an isotropic vector $e'$ the second summand $U(2)$ of $\Lambda_{\text{Enr}}$. They belong to the same 1-cusp which corresponds to isotropic plane $\mathbb{Q}e \oplus \mathbb{Q}e'$. That 1-cusp is contained in the closure of $\mathcal{H}_{-4, \text{special--even}}$ since $e$ and $e'$ are orthogonal to the (norm-doubled) root of $E_8(-2)$, the third summand of $L_{\text{Enr}}$. By \textit{loc.cit}, the only other 1-cusp corresponds to another isotropic plane $p = \mathbb{Q}e' \oplus \mathbb{Q}(2e + 2f + \alpha)$.
where $e, f$ is the standard basis of the first summand $U$ and $\alpha$ is norm $-8$ integral vector in the third summand $E_8(-2)$. Since $p$ is obviously orthogonal to the $-2$ vector $e - f \in U$, the corresponding 1-cusp is also contained in the closure of the Coble locus $H_{-2}$. Hence there are no naked cusps so that we conclude the following.

**Corollary 2.13.** The Satake-Baily-Borel compactification $\overline{M_{\text{Enr}}}^{\text{SBB}}$ of the moduli of Enriques surfaces $M_{\text{Enr}}$ is a log terminal Fano variety.

**Example 2.14.** For each $1 \leq k \leq 10$ $(k \neq 2)$, let $\Lambda_{\text{logEnr}, k} := U(2) \oplus A_1 \oplus A_1(-1)^{\otimes 9-k}$ be an even lattice of signature $(2, 10 - k)$. Then the associated Shimura variety $O^+(\Lambda_{\text{logEnr}, k})/D_{\text{logEnr}}$ is a (partial compactification of) the moduli space of log Enriques surface with $k \frac{1}{4}(1, 1)$ singularities. Yoshikawa [Yos09] and [Ma12] constructed reflective modular forms on $D_{\text{logEnr}}$ for $k \leq 7$ which we use.

**Theorem 2.15 ([Yos09, Theorem 4.2(i)])**. There is a reflective modular form $\Psi_4$ of weight $4 + k$ on $D_{\Lambda_{\text{logEnr}, k}}$ with
\[
\text{div}\Psi_{4+k} = H_{-2}.
\]

**Theorem 2.16 ([Ma12, Appendix by Yoshikawa; A.4, proof of A.5])**. There is a reflective modular form $\Psi_{124}$ of weight $(124 - k^2 - 9k)$ on $D_{\Lambda_{\text{logEnr}, k}}$ with
\[
\text{div}\Psi_{124} = H_{-4}.
\]

On the other hand, the ramification divisors of the map $\pi : D_{\text{logEnr}, k} \rightarrow X := O^+(L_{\text{logEnr}, k})/D_{\text{logEnr}}$ is the union of special divisors with respect to $(-2)$-vectors and $(-4)$-vectors by the same discussion. As $(\Psi_{4+k}\Psi_{124})^{c(10-k)}$ with $c \in \mathbb{Z}_{>0}$ satisfies Assumption 1.2 (i) with $N = 2c(10 - k)$ and $a = \frac{128 - k^2 - 8k}{2(10 - k)}$ for $k \leq 7$, by Theorem 1.3 (i), we conclude the following.

**Corollary 2.17.** For the above (partially compactified) moduli spaces of log Enriques surface with $k \frac{1}{4}(1, 1)$ singularities with $1 \leq k \leq 7$ $(k \neq 2)$ $X = O^+(\Lambda_{\text{logEnr}, k})/D_{\text{logEnr}}$, the Satake-Baily-Borel compactifications $\overline{X}^{\text{SBB}}$ are Fano varieties.

Actually they are also unirational, by [Ma12].

**Example 2.18 (Simple lattices case).** Let $L$ be a quadratic lattice over $\mathbb{Z}$ of signature $(2, n)$. We recall from [BEF14] that $L$ is called simple if the space of cusp forms of weight $1 + \frac{n}{2}$ associated with a finite quadratic form $\Lambda^\vee / L$ is zero. Then the special divisors on the Hermitian symmetric domain associated with $L$ are all given by the divisors of Borcherds lift, so that we can apply Theorem 1.3.
In fact, Wang-Williams [WW20] showed that for every simple lattice $L$ of signature $(2, n)$ with $3 \leq n \leq 10$, the graded algebra of modular forms for certain subgroups of the orthogonal group is freely generated. From this, we have the associated Shimura varieties are the weighted projective spaces, in particular, log terminal $\mathbb{Q}$-Fano.

From Theorem 1.3, all Borcherds product satisfying Assumption 1.2 (i) should have $a > 1$. Also from Corollary 1.7 the boundary of the Satake-Baily-Borel compactification is in the closure of the ramification divisors. See the tables of examples in [WW20].

We remark that before [WW20], [BEF14] showed there are only finitely many isometry classes of even simple lattices $L$ of signature $(2, n)$.

2.4. Preparation for unitary case - Hermitian lattice. Here we review some material on Hermitian lattices from [Hof14] to prepare for constructing some examples of unitary modular varieties from next subsection. There, we similarly apply Theorem 1.3 to certain restriction of Borcherds products to explore their birational properties.

Here is the setup. For a square-free negative integer $d$, let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field which we assume to be of the class number 1 i.e., the ring of integers $\mathcal{O}_F$ is PID. Let $\delta$ be the inverse different of $F$, i.e.,

$$\delta := \begin{cases} \frac{1}{\sqrt{d}} & (d \equiv 2, 3 \mod 4) \\ \frac{1}{2\sqrt{d}} & (d \equiv 1 \mod 4) \end{cases}.$$ 

Let $(\Lambda, \langle , \rangle)$ be a Hermitian lattice of signature $(1, n)$ over $\mathcal{O}_F$ in the sense of [Hof14] i.e., a finitely generated free $\mathcal{O}_F$-module with an Hermitian form which is $\delta \mathcal{O}_F$-valued. We define the dual lattice $\Lambda^\vee$ as

$$\Lambda^\vee := \{ v \in \Lambda \otimes_{\mathcal{O}_F} F \mid \langle v, w \rangle \in \delta \mathcal{O}_F \ (\forall w \in \Lambda) \}.$$ 

We say $\Lambda$ is unimodular if $\Lambda = \Lambda^\vee$ and $\Lambda$ is even if $\langle v, v \rangle \in \mathbb{Z}$ for all $v \in \Lambda$. It is also known that the quotient $A_\Lambda := \Lambda^\vee/\Lambda$ is a finite $\mathcal{O}_F$-module, called the discriminant group. Then, $\widetilde{U}(\Lambda) := \{ g \in U(\Lambda) \mid g|_{A_\Lambda} = 1_{A_\Lambda} \}$ is the so-called discriminant kernel or the stable unitary group. For a Hermitian lattice $\Lambda$, we define $\Lambda(a) := (\Lambda, a \langle , \rangle)$ for $a \in \delta \mathcal{O}_F$. Analogously to quadratic forms, we also have the following proposition.

**Proposition 2.19.** There exists a unimodular Hermitian lattice $M$ and an element $a \in \mathcal{O}_F$ such that $\Lambda = M(a)$ if and only if the ideal $\{ \langle v, w \rangle \in \delta \mathcal{O}_F \mid w \in \Lambda \}$ with respect to $v \in \Lambda$ is equal $a\delta \mathcal{O}_F$ for every primitive element $v \in \Lambda$. 

Let $D_{\Lambda}$ be the Hermitian symmetric domain (complex ball) with respect to $U(\Lambda)(\mathbb{R})$, equivalently,

$$D_{\Lambda} := \{ v \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid \langle v, v \rangle > 0 \}$$

and $H(v)$ be a special divisor with respect to $v \in \Lambda$. For any element $r \in \Lambda$ satisfying $\langle r, r \rangle < 0$ and $\xi \in \mathcal{O}_F^\times \setminus \{1\}$, we define the quasi-reflection $\tau_{r, \xi} \in U(\Lambda)(\mathbb{Q})$ with respect to $r$, $\xi$ as follows:

$$\tau_{r, \xi}(\ell) := \ell - (1 - \xi) \frac{\langle \ell, r \rangle}{\langle r, r \rangle} r.$$

We remark that for $\xi = -1$, we have the usual reflection. See also [AF02]. We also remark that, for example, for $F = \mathbb{Q}(\sqrt{-1})$, we get $\tau_{2r, \sqrt{-1}} = \tau_{r, -1}$ and for $F = \mathbb{Q}(\sqrt{-3})$, we get $\tau_{2r, \omega} = \tau_{r, \omega}$ for any $r \in \Lambda$ where $\omega$ is a primitive third root of unity.

The union of ramification divisors of $\pi_\Gamma: D_{\Lambda} \to \Gamma \setminus D_{\Lambda}$ is

$$\bigcup_r H(r)$$

by [Beh12, Corollary 3] for $\Gamma \subseteq U(\Lambda)$ and $n > 1$. Here the union runs thorough primitive elements $r \in \Lambda/\mathcal{O}_F^\times$ with $\langle r, r \rangle < 0$ such that $\tau_{r, \xi} \in \Gamma$ or $-\tau_{r, \xi} \in \Gamma$ for some $\xi \in \mathcal{O}_F^\times \setminus \{1\}$. We consider the natural embedding of the unitary Hermitian symmetric domain to the quadratic Hermitian symmetric domains

$$\iota: D_{\Lambda} \hookrightarrow D_{\Lambda_Q}$$

where $(\Lambda_Q, (\ , \ ))$ is a quadratic lattice associated with $(\Lambda, (\ , \ ))$, i.e., $\Lambda_Q := \Lambda$ as a $\mathbb{Z}$-module and $(\ , \ ) := \text{Tr}_{F/\mathbb{Q}}(\ , \ )$. By [Mae20a, Lemma 2.4], when $d \equiv 2, 3 \mod 4$ or $d = -3$, we have

$$\iota(\text{Ramification divisors on } D_{\Lambda} \text{ with respect to } \Gamma_U)$$

\subset \text{Ramification divisors on } D_{\Lambda_Q} \text{ with respect to } \Gamma_{O^+}

where the pair $(\Gamma_U, \Gamma_{O^+}) = (U(\Lambda), O^+(\Lambda_Q)), (\tilde{U}(\Lambda), \tilde{O}^+(\Lambda_Q))$.

For computation of multiplicities of unitary modular forms later, we need the following converse to [Hof14, Remark after 6.1].

**Lemma 2.20.** For any $r \in \Lambda$ as above i.e., which is primitive with $\langle r, r \rangle < 0$ such that $\tau_{r, \xi} \in \Gamma$ for some $\xi \in \mathcal{O}_F^\times \setminus \{1\}$.

(i) Then, $H(r)$ is contained in exactly $\#\mathcal{O}_F^\times$ special divisors of the form $\mathcal{H}(r') \subset D_{\Lambda_Q}$ for some $r' \in \Lambda_Q$.

(ii) For $r$ as above, $\mathcal{H}_{r'}|_{D_{\Lambda}}$ is $H(r)$ with multiplicity 1 i.e., reduced.

**Proof.** We fix $\sqrt{-d} \in \mathbb{C}$ and corresponding embedding $F \hookrightarrow \mathbb{C}$. First we prove (i). Note $\mathcal{H}(r)|_{D_{\Lambda}} = \mathcal{H}(r')|_{D_{\Lambda}}$ if and only if $Cr' = Cr$ for $r, r' \in \Lambda$. 

\( \mathbb{C}r \cap \Lambda \) is a torsion free \( \mathcal{O}_F \)-module of rank 1, hence isomorphic to a (fractional) ideal of \( \mathcal{O}_F \). Therefore, \( x(\mathbb{C}r \cap \Lambda) = \mathbb{C}r \cap \Lambda \) for \( x \in \mathbb{C} \) if and only if \( x \in \mathcal{O}_F^\times \). As \( \mathcal{H}(r') \) only depends on \( \mathbb{R}r' \) so that \( \mathcal{H}(r') = \mathcal{H}(-r') \), the number we concern is \( \# \mathcal{O}_F^\times \).

The proof of (ii) is as follows. Since \( \langle r, r \rangle < 0 \), \( \mathcal{H}(r) \) is again an orthogonal symmetric domain which is an (analytic) open subset of a quadric hypersurface, say \( Q^{n-1} \subset Q^n \subset \mathbb{P}^{n+1} \). Thus the restriction of Cartier divisor \( r = 0 \) to \( Q^n \) is reduced and \( \mathcal{H}(r) \) is its open subset. \( H(r) \) is also an open subset of the restriction of \( r = 0 \) to the linear subspace, which is also clearly reduced. Hence the assertion follows. \( \square \)

2.5. Unramifiedness of unitary modular varieties.

**Theorem 2.21.** Let \( F = \mathbb{Q}(\sqrt{d}) \) (\( d \neq -1 \)) be an imaginary quadratic field and \( \Lambda \) be a Hermitian lattice over \( \mathcal{O}_F \) of signature \( (1, n) \) for \( n > 1 \). We assume \( d \equiv 2, 3 \mod 4 \) or \( d = -3 \). Then for any arithmetic subgroup \( \Gamma \subset U(\Lambda) \), the canonical map \( \pi_\Gamma : D_\Lambda \to \Gamma \backslash D_\Lambda \) does not ramify in codimension 1, so that \( X_{\text{sBB}}^{\mathbb{Q}} \) is a log canonical model.

**Proof.** It suffices to show the claim for \( \Gamma = U(\Lambda) \). The ramification divisors are defined by \( \tau_{r, \xi} \) for some \( r \in L \) and \( \xi \in \mathcal{O}_F^\times \\backslash \{1\} \) and by [Mae20a Lemma 2.4], they are included in the set

\[
\bigcup_{r \in \Lambda, a \in \mathbb{Z}, \xi \in \mathcal{O}_F^\times \\backslash \{1\}} \bigcup_{(r, r) = -\frac{a}{2}, \tau_{r, \xi} \in U(\Lambda)} H(r).
\]

Now

\[
\tau_{r, \xi}(\ell) = \ell - (1 - \xi) \frac{\langle \ell, r \rangle}{\langle r, r \rangle} r.
\]

We assume that \( r \in \Lambda \) is a reflective element, that is, \( \tau_{r, \xi} \in U(\Lambda) \) for some \( \xi \in \mathcal{O}_F^\times \\backslash \{1\} \). Then

\[
(1 - \xi) \frac{\langle \ell, r \rangle}{\langle r, r \rangle} = -\frac{2(1 - \xi) \langle \ell, r \rangle}{a}.
\]

Since \( r \) is primitive and \( \Lambda \) is unimodular, by Proposition 2.19 there exists an \( \ell \in \Lambda \) such that \( \langle \ell, r \rangle = \frac{1}{2\sqrt{d}} \), so we have

\[
(1 - \xi) \frac{\langle \ell, r \rangle}{\langle r, r \rangle} = -\frac{1 - \xi}{a\sqrt{d}} \notin \mathcal{O}_F
\]

for \( F \neq \mathbb{Q}(\sqrt{-1}) \). This implies \( \tau_{r, \xi} \notin U(\Lambda) \) and this is contradiction. The last assertion then follows from [Mum77] (or as a special case of our Theorem 1.3 (iii)). \( \square \)
Corollary 2.22. Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field and $(\Lambda, \langle , \rangle) = M(a)$ be a Hermitian lattice over $\mathcal{O}_F$ of signature $(1, n)$ for $n > 1$ where $M$ is a unimodular Hermitian lattice and $a \in \mathcal{O}_F$. We assume $d \equiv 2, 3 \mod 4$ or $d = -3$, and $\frac{a}{\sqrt{d}} \not\in \mathcal{O}_F$. Then for any arithmetic subgroup $\Gamma \subset U(\Lambda)$, the canonical map $\pi_\Gamma: D_\Lambda \to \Gamma \backslash D_\Lambda$ does not ramify in codimension 1.

2.6. Unitary examples, Part I - Fano cases.

Example 2.23. For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda := \Lambda_{U \oplus U} \oplus \Lambda_{E_8(-1)}$ be an even unimodular Hermitian lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ of signature $(1, 5)$ whose associated quadratic lattice is $\Lambda_Q = U \oplus U \oplus E_8(-1)$. Here the Hermitian lattices $\Lambda_{U \oplus U}$ and $\Lambda_{E_8(-1)}$ are defined in Appendix A.1.

The only ramification divisors of the map $D_\Lambda \to X := U(\Lambda) \backslash D_\Lambda$ are

$$\bigcup_{r \in \Lambda/\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}^\times: \text{primitive}} H(r)$$

with branch degree 2. For more details, see Example 2.27.

By Example 2.28, $f := \Phi_{252}|_{D_\Lambda}$ is a weight 252 modular form with

$$\text{div}(f) = 2 \sum_{r \in \Lambda/\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}^\times: \text{primitive}} H(r)$$

whose coefficient comes from Lemma 2.20. Therefore applying Theorem 1.3 (i) for $f^{12}$ with $N = 48$ and $a = \frac{21}{2}$, we have the following.

Corollary 2.24. The Satake-Baily-Borel compactification $X_{SBB}$ of the Shimura variety $X := U(\Lambda) \backslash D_\Lambda$ is a Fano variety, where $\Lambda := \Lambda_{U \oplus U} \oplus \Lambda_{E_8(-1)}$ for $F = \mathbb{Q}(\sqrt{-1})$.

Example 2.25. For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda := \Lambda_{U \oplus U (2)} \oplus \Lambda_{E_8(-1)(2)}$ be an even Hermitian lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ of signature $(1, 5)$ whose associated quadratic lattice is $\Lambda_Q = U \oplus U (2) \oplus E_8(-2)$. Here the Hermitian lattices are defined in Appendix A.1. The ramification divisors on $D_\Lambda$ with respect to $O^+(\Lambda_Q)$ is the union of special divisors with respect to $(-2)$-vectors and special-even $(-4)$-vectors, so the ramification divisors on $D_\Lambda$ with respect to $U(\Lambda)$ is included in the union of special divisors with respect to $(-1)$-vectors and special-even $(-2)$-vectors since $\langle v, v \rangle$ is real for all $v \in \Lambda$. Here we say a vector $r \in \Lambda$ is special-even if $\Re \langle r, v \rangle \in \mathbb{Z}$ for any $v \in \Lambda$. The only ramification divisors of $\pi$ are

$$\bigcup_{r \in \Lambda/\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}^\times: \text{primitive}} H(r) \bigcup \bigcup_{\langle r, r \rangle = -1, \tau_r \in U(\Lambda)} H(r) \bigcup \bigcup_{\langle r, r \rangle = -2, \tau_r \in U(\Lambda)} H(r)$$
with branch degree $d_i = 2$ and

$$ \bigcup_{r \in \Lambda / \mathcal{O}_F^\times : \text{primitive}} H(r) \cup \bigcup_{r \in \Lambda / \mathcal{O}_F^\times : \text{special-even, primitive}} H(r).$$

with branch degree $d_i = 4$. For any primitive element $r \in \Lambda$ with $\langle r, r \rangle = -1$, we have

$$ \tau_{r,-1}(\ell) = \ell + 2\langle \ell, r \rangle r.$$ 

By the description of Hermitian lattices $\Lambda_{U \oplus U(2)}$ and $\Lambda_{E_8(-1)(2)}$,

$$ 2\langle \ell, r \rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}. $$

Hence $\tau_{r,-1} \in U(\Lambda)$ for any $(-1)$-primitive element $r \in \Lambda$. For any special-even primitive element $r \in \Lambda$ with $\langle r, r \rangle = -2$, we have

$$ \tau_{r,-1}(\ell) = \ell + \langle \ell, r \rangle r.$$ 

By calculation and definition of $\Lambda_{U \oplus U(2)}$, if $\Re(\ell, r) \in \mathbb{Z}$ then $\Im(\ell, r) \in \mathbb{Z}$ for any $\ell \in \Lambda$. Also by definition of $\Lambda_{E_8}(-2)$, we have $\langle \ell, r \rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ for any $\ell \in \Lambda$. Hence $\tau_{r,-1} \in U(\Lambda)$ for any special-even ($-2$)-primitive vector $r \in \Lambda$. Therefore the map $D_\Lambda \to X := U(\Lambda) \setminus D_\Lambda$ branches along

$$ \bigcup_{r \in L / \mathcal{O}_F^\times : \text{primitive}} H(r) \cup \bigcup_{r \in L / \mathcal{O}_F^\times : \text{special-even, primitive}} H(r).$$

For $(-1)$-primitive vector $r \in \Lambda$,

$$ \tau_{r,\sqrt{-1}}(\ell) = \ell + (1 - \sqrt{-1})\langle \ell, r \rangle r.$$ 

If $r \in \Lambda_{E_8(-1)(2)}$, then by the description of the Hermitian matrix defining $\Lambda_{E_8(-2)}$, we have $\langle \ell, r \rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, so $(1 - \sqrt{-1})\langle \ell, r \rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. If $r \in \Lambda_{U \oplus U(2)}$, then the ideal $\{ \langle \ell, r \rangle : \ell \in \Lambda_{U \oplus U(2)} \}$ is generated by $\frac{1 + \sqrt{-1}}{2}$ since $\det(L_{U \oplus U(2)}) = \frac{1}{2}$, so $(1 - \sqrt{-1})\langle \ell, r \rangle \in \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. By the above discussion, we have $\tau_{r,\sqrt{-1}} \in U(\Lambda)$ for any $(-1)$-primitive vector $r \in \Lambda$. For special-even ($-2$)-primitive vector $r \in \Lambda$,

$$ \tau_{r,\sqrt{-1}}(\ell) = \ell + \frac{(1 - \sqrt{-1})}{2} \langle \ell, r \rangle r.$$ 

If $r \in \Lambda_{E_8(-1)(2)}$, then there exists an $\ell \in \Lambda_{E_8(-1)(2)}$ such that $\langle \ell, r \rangle = 1$, so we have $\frac{(1 - \sqrt{-1})}{2} \langle \ell, r \rangle = \frac{1 - \sqrt{-1}}{2} \notin \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. If $r \in \Lambda_{U \oplus U(2)}$, then there exists an $\ell \in \Lambda_{U \oplus U(2)}$ such that $\langle \ell, r \rangle = \frac{1 + \sqrt{-1}}{2}$, so we have $\frac{(1 - \sqrt{-1})}{2} \langle \ell, r \rangle = \frac{1}{2} \notin \mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. By the above discussion, we have $\tau_{r,\sqrt{-1}} \notin U(\Lambda)$ for any special-even ($-2$)-primitive vector $r \in \Lambda$. 


Therefore the ramification in codimension 1 only occurs along
\[ \bigcup_{r \in \Lambda / \mathcal{O}_F^*; \text{primitive}} H(r) \]
with branch degree 2, and along
\[ \bigcup_{r \in \Lambda / \mathcal{O}_F^*; \text{special--even primitive}} H(r) \]
with branch degree 4.

This example implies Theorem 2.21 does not hold for non-unimodular lattices and \( F = \mathbb{Q}(\sqrt{-1}) \). By Example 2.10, we have modular forms \( \Phi_4|_{D_\Lambda} \) and \( \Phi_{124}|_{D_\Lambda} \) such that
\[ \text{div}(\Phi_4|_{D_\Lambda}) = 2 \sum_{r \in \Lambda / \mathcal{O}_F^*; \text{primitive}} H(r) \]
\[ \text{div}(\Phi_{124}|_{D_\Lambda}) = 2 \sum_{r \in \Lambda / \mathcal{O}_F^*; \text{special--even primitive}} H(r) \]
whose coefficient again comes from Lemma 2.20.

Hence applying Theorem 1.3 (i) to \( (\Phi_4|_{D_\Lambda} \Phi_{124}|_{D_\Lambda})^{12} \) with \( N = 96 \) and \( a = \frac{97}{12} \), we have the following.

**Corollary 2.26.** The Satake-Baily-Borel compactification \( \overline{X}^{\text{SBB}} \) of the Shimura variety \( X := U(\Lambda) \backslash D_\Lambda \) is a Fano variety, where \( \Lambda := \Lambda_{U\oplus U(2)} \oplus \Lambda_{E_8(-1)} \oplus E_8(-1) \oplus E_8(-1) \) for \( F = \mathbb{Q}(\sqrt{-1}) \).

**2.7. Unitary examples, Part II - with ample canonical class.**

**Example 2.27.** For \( F = \mathbb{Q}(\sqrt{-1}) \), let \( \Lambda := \Lambda_{U\oplus U} \oplus \Lambda_{E_8(-1)} \oplus \Lambda_{E_8(-1)} \oplus E_8(-1) \oplus E_8(-1) \) be an even unimodular Hermitian lattice of signature \((1,13)\) whose associated quadratic lattice is \( \Lambda_Q = II_{2,26} = U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus E_8(-1) \). Here the Hermitian lattices are defined in Appendix A.1. The ramification divisors on \( D_{\Lambda_Q} \) with respect to \( O^+(\Lambda_Q) \) is the union of special divisors with respect to \((-2)\)-vectors, so the ramification divisors on \( D_\Lambda \) with respect to \( U(\Lambda) \) is included in the union of special divisors with respect to \((-1)\)-vectors since \( \langle v, v \rangle \) is real for all \( v \in \Lambda \). There exist possibly double ramification divisors i.e., those with \( d_i = 2 \), and quadruple ramification divisors i.e., those with \( d_i = 4 \),
of the natural morphism \( \pi : D_\Lambda \to X := U(\Lambda) \backslash D_\Lambda \). It ramifies in codimension 1 along

\[
\bigcup_{r \in \Lambda / O_F^\times; \text{primitive} \atop \langle r, r \rangle = -1, \tau_{r,-1} \in U(\Lambda)} H(r)
\]

with branch degree 2, and

\[
\bigcup_{r \in \Lambda / O_F^\times; \text{primitive} \atop \langle r, r \rangle = -1, \tau_{r,\sqrt{-1}} \in U(\Lambda)} H(r)
\]

with branch degree 4.

For any primitive element \( r \in \Lambda \) with \( \langle r, r \rangle = -1 \), we have

\[
\tau_{r,\sqrt{-1}}(\ell) = \ell + (1 - \sqrt{-1})\langle \ell, r \rangle r,
\]

but by Proposition \ref{prop:branch-degree} and unimodularity of \( \Lambda \), \( \langle \ell, r \rangle = \frac{1}{2\sqrt{-1}} \) for some \( \ell \in \Lambda \). Hence \( \tau_{r,-1} \not\in U(\Lambda) \) for any \((-1)\)-primitive element \( r \in \Lambda \), that is, there is no quadruple ramification divisors.

For any primitive element \( r \in \Lambda \) with \( \langle r, r \rangle = -1 \), we have

\[
\tau_{r,-1}(\ell) = \ell + 2\langle \ell, r \rangle r.
\]

Here

\[
\langle \ell, r \rangle \in \delta O_F = \frac{1}{2\sqrt{-1}} O_{\mathbb{Q}(\sqrt{-1})},
\]

so \( 2\langle \ell, r \rangle \in O_{\mathbb{Q}(\sqrt{-1})} \). Hence \( \tau_{r,-1} \in U(\Lambda) \) for any \((-1)\)-primitive element \( r \in \Lambda \), that is, there are only double ramification divisors along

\[
\bigcup_{r \in \Lambda / O_F^\times; \text{primitive} \atop \langle r, r \rangle = -1} H(r)
\]

with branch degree 2. By Example \ref{ex:weight-12} \( f := \Phi_{12}|_{D_\Lambda} \) is a weight 12 modular form whose divisors are equal to double ramification divisors;

\[
\text{div}(f) = 2 \sum_{r \in \Lambda / O_F^\times; \text{primitive} \atop \langle r, r \rangle = -1} H(r)
\]

whose coefficient again comes from Lemma \ref{lem:coefficient}. Therefore applying Theorem \ref{thm:main}(iii) to \( f^{26} \) with \( N = 104 \) and \( a = \frac{3}{13} \), we have the following.

\textit{Corollary 2.28.} \textit{The Satake-Baily-Borel compactification} \( X_{\text{SBB}} \) \textit{of the Shimura variety} \( X := U(\Lambda) \backslash D_\Lambda \) \textit{is a log canonical model, where} \( \Lambda := \Lambda_{U} \oplus \Lambda_{E_8(-1)} \oplus \Lambda_{E_8(-1)} \oplus \Lambda_{E_8(-1)} \) \textit{for} \( F = \mathbb{Q}(\sqrt{-1}) \). Recall from
Terminology after Theorem 1.3 that a log canonical model in this paper means it has only log canonical singularities and ample canonical class.

Example 2.29. For $F = \mathbb{Q}(\sqrt{-2})$, let $\Lambda := \Lambda'_U \oplus \Lambda'_{E_8(-1)}(2)$ be an even Hermitian lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$ of signature $(1,5)$. Here the Hermitian lattices are defined in Appendix A.2. By the same calculation as above, the union of branch divisors of the map $\pi : D_\Lambda \to X := U(\Lambda) \setminus D_\Lambda$ are the union of special divisors with respect to $(-1)$-vectors only, unlike $F = \mathbb{Q}(\sqrt{-1})$ case. Of course, these divisors ramify with branch degree 2, so we can also show $X^{SBB}$ is a log canical model. (Applying Theorem 1.3 (iii) to $f_{12}$ with $N = 48$ and $a = \frac{16}{3}$.) This example implies Theorem 2.21 does not hold for non-unimodular lattices.

Corollary 2.30. The Satake-Baily-Borel compactification $X^{SBB}$ of the Shimura variety $X := U(\Lambda) \setminus D_\Lambda$ is a log canonical model, where $\Lambda := \Lambda'_U \oplus \Lambda'_{E_8(-1)}(2)$ for $F = \mathbb{Q}(\sqrt{-2})$.

Remark 2.31. For $F = \mathbb{Q}(\sqrt{-2})$, let $\Lambda := \Lambda'_U \oplus \Lambda'_{E_8(-1)} \oplus \Lambda'_{E_8(-1)}$ be an even unimodular Hermitian lattice over $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}(= \mathbb{Z}[\sqrt{-2}])$ of signature $(1,13)$, whose associated quadratic lattice $\Lambda_Q$ is $U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus E_8(-1)$. Here the Hermitian lattices are defined in Appendix A.2.

By the same calculation as above, we know that for any arithmetic subgroup $\Gamma \subset U(\Lambda)$, the map $\pi : D_\Lambda \to \Gamma \setminus D_\Lambda$ does not ramify in codimension 1. This is exactly an example of Theorem 2.21. Thus the Satake-Baily-Borel compactification $\Gamma \setminus D_\Lambda^{SBB}$ is a log canonical model.

Remark 2.32. For any imaginary quadratic field with class number 1, we can construct $\Lambda_U \oplus \Lambda$; see [Mae20a, Section 7] and [Mae20b, Appendix A]. As in Theorem 2.21 we can show that the corresponding map does not ramify in codimension 1 for any arithmetic subgroup so that the Satake-Baily-Borel compactification is log canonical model again.

Remark 2.33. By the same reason as Remark 2.31 for $F \neq \mathbb{Q}(\sqrt{-1})$, the map $\pi : D_\Lambda \to \Gamma \setminus D_\Lambda$ does not ramify in codimension 1, where $\Lambda := \Lambda_U \oplus \Lambda_{E_8(-1)}$ and $\Gamma \subset U(\Lambda)$ is any arithmetic subgroup. This is also an example of Theorem 2.21 and $\Gamma \setminus D_\Lambda^{SBB}$ is a log canonical model.

2.8. More examples. For $F = \mathbb{Q}(\sqrt{-1})$, let $\Lambda_{-1} := \Lambda_U \oplus \Lambda_{E_8(-1)}(2)$ and by the same calculation, the map $\pi : D_{\Lambda_{-1}} \to U(\Lambda) \setminus D_{\Lambda_{-1}}$ branches at the union of special divisors with respect to
\(-1\)-vectors and \((-2)\)-special-even vectors. By [Yos13, Theorem 8.1], there exists a reflective modular form \(\Psi_{12}\) of weight 12 on \(D_{(\Lambda_{-1})}\), such that
\[
\text{div}(\Psi_{12}|_{D_{\Lambda}}) = 2 \sum_{r \in (\Lambda_{-1})/\pm: \text{primitive}} \mathcal{H}(r)
\]
whose coefficient again comes from Lemma 2.20. Then the restriction \(\iota^*\Psi_{12} = \Psi_{12}|_{D_{\Lambda_{-1}}}\) is a reflective modular form on \(D_{\Lambda_{-1}}\), but this does not satisfy (ii) because the ramification divisors properly include the divisors of \(\Psi_{12}|_{D_{\Lambda_{-1}}}\), so we can not show the Fano-ness of \((U(\Lambda_{-1}) \backslash D_{\Lambda_{-1}})_{\text{SBB}}\) in this way (but we can show the uniruledness or more strongly, rationally-chain-connectedness of \(U(\Lambda_{-1}) \backslash D_{\Lambda_{-1}}\) by [Mae20b, Theorem 5.1]).

On the other hand, for \(F = \mathbb{Q}(\sqrt{-2})\), let \(\Lambda_{-2}\) be the Hermitian lattice over \(\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}\) of signature \((1, 5)\) whose associated quadratic lattice is \(U \oplus U \oplus E_8(-2)\). Then the map \(\pi : D_{\Lambda_{-2}} \to U(\Lambda) \backslash D_{\Lambda_{-2}}\) has no ramification divisors, so we can not even show the uniruledness.

**Appendix A. Matrix definitions**

The following matrices are taken from [Mae20a, Section 7] and [Mae20b, Appendix A].

A.1. \(\mathbb{Q}(\sqrt{-1})\) cases. Let \(\Lambda_{U \oplus U}\) be an even unimodular Hermitian lattice of signature \((1, 1)\) over \(\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}\) defined by the matrix
\[
\frac{1}{2\sqrt{-1}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
whose associated quadratic lattice \((\Lambda_{U \oplus U})_Q\) is \(U \oplus U\).

Let \(\Lambda_{U \oplus U(2)}\) be an even Hermitian lattice of signature \((1, 1)\) over \(\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}\) defined by the matrix
\[
\frac{1}{2} \begin{pmatrix} 0 & 1 + \sqrt{-1} \\ 1 - \sqrt{-1} & 0 \end{pmatrix}
\]
whose associated quadratic lattice \((\Lambda_{U \oplus U(2)})_Q\) is \(U \oplus U(2)\).

Let \(\Lambda_{E_8(-1)}\) be an even unimodular Hermitian lattice of signature \((0, 4)\) over \(\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}\) defined by the matrix
\[
-\frac{1}{2} \begin{pmatrix} 2 & -\sqrt{-1} & -\sqrt{-1} & 1 \\ \sqrt{-1} & 2 & 1 & \sqrt{-1} \\ \sqrt{-1} & 1 & 2 & 1 \\ 1 & -\sqrt{-1} & 1 & 2 \end{pmatrix}
\]
whose associated quadratic lattice \((\Lambda_{E_8(-1)})_Q\) is \(E_8(-1)\). This matrix is called Iyanaga’s matrix.

A.2. \(\mathbb{Q}(\sqrt{-2})\) cases. Let \(\Lambda'_{U\oplus U}\) be an even unimodular Hermitian lattice of signature \((1, 1)\) over \(\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}\) defined by the matrix

\[
\frac{1}{2\sqrt{-2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

whose associated quadratic lattice \((\Lambda'_{U\oplus U})_Q\) is \(U \oplus U\).

Let \(\Lambda'_{U\oplus U(2)}\) be a Hermitian lattice of signature \((1, 1)\) over \(\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}\) defined by the matrix

\[
\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}
\]

whose associated quadratic lattice \((\Lambda'_{U\oplus U(2)})_Q\) is \(U \oplus U(2)\).

Let \(\Lambda'_{E_8(-1)}\) be an even unimodular Hermitian lattice of signature \((0, 4)\) over \(\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}\) defined by the matrix

\[
-\frac{1}{2} \begin{pmatrix} 2 & 0 & \sqrt{-2} + 1 & \frac{1}{2}\sqrt{-2} \\ 0 & 2 & \frac{1}{2}\sqrt{-2} & 1 - \sqrt{-2} \\ 1 - \sqrt{-2} & \frac{1}{2}\sqrt{-2} & 2 & 0 \\ -\frac{1}{2}\sqrt{-2} & \sqrt{-2} + 1 & 0 & 2 \end{pmatrix}
\]

whose associated quadratic lattice \((\Lambda'_{E_8(-1)})_Q\) is \(E_8(-1)\).

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**References**

[Amb03] F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova 240 (2003), Biration. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220-239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214-233.

[AF02] D.Allcock, E.Freitag, Cubic surfaces and Borcherds products, Comment. Math. Helv. 77 (2002), no. 2, 270-296.

[ACT02] D. Allcock, J. A. Carlson, D. Toledo, The complex hyperbolic geometry of the moduli space of cubic surfaces, J. Algebraic Geom. 11 (2002), 659-724.

[AMRT] A. Ash, D. Mumford, M. Rapoport, Y.-S. Tai, Smooth Compactification of Locally Symmetric Varieties, Cambridge Mathematical Library.

[BB66] W.Baily, A.Borel, Compactification of arithmetic quotients of bounded symmetric domains, Annals of Mathematics, 2, Annals of Mathematics, 84 (3): 442-528.
[BN94] W. Barth, I. Nieto, Abelian surfaces of type (1, 3) and quartic surfaces with 16 skew lines, J. Alg. Geom. 3 (1994), 173-222.
[Beh12] N. Behrens, Singularities of ball quotients, Geom. Dedicata 159 (2012), 389-407.
[BEF14] J. H. Bruinier, S. Ehlen, E. Freitag, Lattices with many Borcherds products, preprint (2014), arXiv:1408.4148
[Bor95] R. Borcherds, Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products, Invent. Math. 120 (1995), no. 1, 161-213.
[Bor96] R. Borcherds, The moduli space of Enriques surfaces and the fake Monster Lie superalgebra, Topology 35. No.3 pp. 699-710.
[Bor98] R. Borcherds, Automorphic forms with singularities on Grassmannians, Invent. Math. 132 (1998).
[BEF14] J. H. Bruinier, S. Ehlen, E. Freitag, Lattices with Borcherds products, arXiv:1408.4148
[Bru14] J. H. Bruinier, On the converse theorem for Borcherds products, Jour. of Algebra 397 (2014), 315-342.
[Bru02] J. H. Bruinier, Borcherds products on $O(2, l)$ and Chern classes of Heegner divisors, Lecture Notes in Mathematics 1780, Springer-Verlag (2002).
[DKW19] C. Dieckmann, A. Krieg, and M. Woitalla, The graded ring of modular forms on the Cayley half-space of degree two, Ramanujan J. 48 (2019), no. 2, 385-398.
[DN89] J.-M. Drezet, M. S. Narasimhan, Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, Invent. Math. 97 (1989), no. 1, 53-94.
[Fjn10] O. Fujino, Some problems on Fano varieties, Proceeding to the conference “Fukuso-kikagaku no shomondai (2010)” in Japanese.
[Fjn11] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. 47 (2011), no. 3, 727-789.
[Fjn17] O. Fujino, Foundations of the Minimal Model Program, Mathematical Society of Japan Memoirs Vol. 35, 1-15 (2017).
[FG12] O. Fujino, Y. Gongyo, On canonical bundle formulas and subadjunctions, Michigan Math. J. 61 (2012), no. 2, 255-264.
[FSM10] E. Freitag, R. Salvati Manni, Some Siegel threefolds with a Calabi-Yau model, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) vol. (2010), 833-850.
[GHS07] V. Gritsenko, K. Hulek, G. Sankaran, The Kodaira dimension of the moduli spaces of K3 surfaces, Invent. math. 169 (2007), 519-567.
[Gri10] V. Gritsenko, Reflexive modular forms in algebraic geometry, arXiv:1012.4155
[Gri12] V. Gritsenko, 24 faces of the Borcherds modular forms $\Phi_{12}$, arXiv:1203.6503
[GH14] V. Gritsenko, K. Hulek, Uniruledness of orthogonal modular varieties, J. Alg. Geom. 23 (2014), 711-725.
[GH16] V. Gritsenko, K. Hulek, Moduli of polarized Enriques surfaces, in K3 surfaces and their moduli, 55-72, Progr. Math., 315, 2016.
[Gri18] V. Gritsenko, Reflexive modular forms and applications, Russian Math. Surv. 73 (2018) 797-864.
[GN18] V. Gritsenko, V. Nikulin, Lorentzian Kac-Moody algebras with Weyl groups of 2-reflections. Proc. Lond. Math. Soc. (3) 116 (2018), no. 3, 485-533
[HM07] C. Hacon, J. McKernan, On Shokurov’s rational connectedness conjecture, Duke Math. J. 138 (1) 119 - 136, 15 (2007).

[HU14] K. Hashimoto, K. Ueda, The ring of modular forms for the even unimodular lattice of signature (2, 10), arXiv:1406.0332.

[Hof14] E. Hofmann, Borcherds products on unitary groups. Math. Ann. 358 (2014), no. 3-4, 799-832.

[Huy94] D. Huybrechts, Complete curves in moduli spaces of stable bundles on surfaces, Math. Ann. 298 (1994), no. 1, 67-78.

[Ich09] T. Ichikawa, Siegel modular forms of degree 2 over rings, Journal of Number Theory 129 (2009) 818-823.

[Igu64] J. Igusa, On Siegel modular forms of genus two (II) Am. J. Math. 86, 392-412 (1964).

[KM98] J. Kollár, S. Mori, Birational Geometry of Algebraic Varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998.

[Kon94] S. Kondo, The rationality of the moduli space of Enriques surfaces, Compositio Math. 91 (1994), 159-173.

[Kon02] S. Kondo, The moduli space of Enriques surfaces and Borcherds products, J. Alg. Geom. 11 (2002), 601-627.

[Lej93] P. Lejjrraga, The moduli of Weierstrass fibrations over $\mathbb{P}^1$: Rationality, Rocky Mt. J. Math. 23:2, 649-650 (1993).

[Li94] J. Li, Kodaira dimension of moduli space of vector bundles on surfaces, Invent. Math. 115 (1994), no. 1, 1-40.

[Loo84] E. Looijenga, The Smoothing Components of a Triangle Singularity. II. Math. Ann., 269 : 357-387, 1984.

[Ma12] S. Ma, The unirationality of the moduli spaces of 2-elementary K3 surfaces. With an appendix by Ken-Ichi Yoshikawa. Proc. Lond. Math. Soc. (3) 105 (2012), no. 4, 757-786.

[Ma18] S. Ma, On the Kodaira dimension of orthogonal modular varieties, Invent. Math. (2018), pp. 859-911.

[Mae20a] Y. Maeda, The singularities and Kodaira dimension of unitary Shimura varieties, arXiv:2008.08095.

[Mae20b] Y. Maeda, Uniruledness of unitary Shimura varieties associated with Hermitian forms of signatures (1, 3), (1, 4), (1, 5), arXiv:2008.13106, Master thesis (Department of Mathematics, Kyoto university).

[Mum77] D. Mumford, Hirzebruch’s proportionality principle in the non-compact case, Invent Math. 42 (1977), 239-277.

[Nam85] Y. Namikawa, Periods of Enriques surfaces, Math. Ann., 270 (1985), 201-222.

[NU21] A. Nagano, K. Ueda, The ring of modular forms for the even unimodular lattice of signature (2, 18), arXiv:2102.09224.

[OO18] Y. Odaka, Y. Oshima, Collapsing K3 surface, Tropical geometry and Moduli compactifications of Satake, Morgan-Shalen type, arXiv:1810.07685.

[Od20] Y. Odaka, PL density invariant for type II degenerating K3 surfaces, Moduli compactification and hyperKahler metrics, arXiv:2010.00416.

[OSS16] Y. Odaka, C. Spotti, S. Sun, Compact moduli spaces of Del Pezzo surfaces and Kähler-Einstein metrics, J. Differential Geom. 102 (2016), no. 1, 127-172.
I. Satake, On compactifications of the quotient spaces for arithmetically defined discontinuous groups, Ann. of Math. (2) 72 (1960), 555-580.

H. Sterk, Compactifications of the period space of Enriques surfaces I, Math. Z., 207 (1991), 1-36.

Y. S. Tai, On the Kodaira dimension of the moduli space of abelian varieties, Invent. Math. 68 (1982), 425-439.

G. van der Geer, On the geometry of a Siegel modular threefold. Math. Ann. 260 (1982), no. 3, 317-350.

G. van der Geer, Siegel modular forms of degree two and three and invariant theory, arXiv:2102.02245.

K. Yoshikawa, Calabi-Yau threefolds of Borcea-Voisin, analytic torsion, and Borchers products, Asterisque 328 (2009).

K. Yoshikawa, $K3$ surfaces with involution, equivalent analytic torsion, and automorphic forms on the moduli spaces, II: A structure theorem for $r(M) > 10$, J. Reine Angew. Math. 677 (2013), 15-70.

H. Wang, B. Williams, Simple lattices and free algebras of modular forms, arXiv:2009.13343.

D-Q. Zhang, Logarithmic Enriques surfaces, J. Math. Kyoto Univ. 31 (1991), 419-466.

Q. Zhang, Rational connectedness of log $Q$-Fano varieties, J. Reine Angew. Math. 590 (2006) 131-142.