Decomposition of spectral density in individual eigenvalue contributions

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Abstract

The eigenvalue densities of two random matrix ensembles, the Wigner Gaussian matrices and the Wishart covariant matrices, are decomposed in the contributions of each individual eigenvalue distribution. It is shown that the fluctuations of all eigenvalues, for medium matrix sizes, are described with a good precision by nearly normal distributions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Borrowing from Wishart the random matrices used by statisticians to construct their ensemble of covariant matrices [1], Wigner, in the late fifties, introduced the ensemble of Gaussian matrices of random matrix theory (RMT) [2]. Spectral properties of these two ensembles are characterized by correlations generated by the repulsion among the levels and their properties are directly connected to two classical polynomials, the Hermite ones, in the Wigner case [3], and the Laguerre ones, in the Wishart case. Both ensembles, and specially Wigner’s, have proved to be important matrix models. By the same time, Girardeau proved the mapping theorem that states that in 1D the properties of a gas of impenetrable bosons are the same of a gas of fictitious spinless fermions [4]. In the limit of negligible size, the only interaction among the atoms that remains is that it is forbidden for two atoms to occupy the same site. This generates a repulsion among them in an entirely similar way of what occurs among the eigenvalues. Placed in a harmonic trap, it follows from this equivalence that the ground-state wavefunction of the atomic system is just the joint probability distribution of RMT eigenvalues [5]. Therefore, under these conditions, the atoms of the unidimensional boson gas constitute a physical realization of eigenvalues of the matrix ensemble.

In the last decades, there has been in RMT studies a great interest in the behavior of individual eigenvalues at the edge of spectra. This interest followed the many applications
that the probability distribution derived by Tracy and Widom which describes the behavior of eigenvalues at the border of the Gaussian ensemble [6] have found. The distributions of the largest and the smallest eigenvalues of Wishart matrices have also been derived [7]. A salient feature of these edge distributions is their asymmetry that can be understood as an effect of the unbalanced repulsion that eigenvalues at the border are submitted from the other eigenvalues. Here we are interested in investigating the behavior of eigenvalues not only at the edge but also at the bulk of the spectrum. This question has been aroused some years ago in connection with the behavior of eigenvalues of the two-body random ensemble as compared with the RMT ones [8]. In the boson gas terms, this decomposition is equivalent to determine how individual atoms behave.

First we remark that, in general, the density of a set of random variables can be expressed as a sum of the individual distributions of each variable considered in an ordered sequence. Of course, this just translates the fact that a variable found at a given position has to be or the first, or the second, or the third, and so on, of the ordered sequence. Therefore, if the distribution of each variable is determined, a decomposition of the density follows. Intuitively, it is reasonable to expect that, at the bulk of the spectrum, if not exactly at least approximately in the case of large matrices, eigenvalue fluctuations will be normally distributed and in fact this has been proved in the limit when the size of the matrices goes to infinity [9, 10]. As a consequence, the exact expression of the eigenvalue density should allow a decomposition in terms of nearly Gaussian distributions. It is our purpose to show that this decomposition exists and to determine its parameters. This will complement the exact results obtained in the asymptotic limit when matrix sizes go to infinity [11]. To do this, we note that the exact density of the eigenvalues of these two ensembles show tiny fluctuations around the average density. These wiggles correspond to the peaks of the individual eigenvalue distributions, that is, to the average position of the eigenvalues. To find these positions we separate in the eigenvalue density, a smooth leading-order term of the fluctuating term which vanishes in the limit of a large matrix size. The locations of the wiggles are then determined in the fluctuating term. Comparing then this term with the one of the same order in the asymptotic of the Gaussian decomposition, the dependence of the Gaussian variances on their positions along the spectrum is determined.

The Wigner and the Wishart ensembles are characterized by Dyson index $\beta$ that takes the values 1, 2, 4 for the three symmetry classes, respectively, the orthogonal (GOE) of real symmetric matrices, the unitary (GUE) of complex Hermitian matrices and the symplectic (GSE) whose elements are real quaternion numbers. A $\beta$-ensemble has been proposed that generalizes, for arbitrary positive values of $\beta$, the Wigner and the Wishart ensembles [12]. In particular, in the limit $\beta \to \infty$, the spectrum gets frozen with eigenvalues located at the position of the zeros of the Hermite (Laguerre, in Wishart case) polynomials. As $\beta$ decreases, the eigenvalues start to vibrate around those positions. Using perturbation theory, a Gaussian decomposition of the eigenvalue density for large but finite $\beta$ was derived for both ensembles [13]. Here we are extending this decomposition for small values of $\beta$. For large $\beta$ individual distributions do not overlap while in our case there is overlapping between neighboring distributions. In the opposite limit $\beta \to 0$, the eigenvalues become a set of uncorrelated variables and, in this case, there is strong overlapping among the individual distributions.

2. Uncorrelated variables

Starting with the case of uncorrelated variables, we consider a set of $N$ independent and identically distributed (i.i.d.) variables $x_i$ with $i = 1, 2, \ldots, N$ uniformly distributed in the
interval \((-N/2, N/2)\). By a simple probabilistic argument, the density probability of finding a variable at \(t\), with \(n\) others greater and the \(N-1-n\) others smaller than it, is

\[
F(n, t) = \frac{(N-1)!}{n!(N-n-1)!} \left( \frac{1}{2} - \frac{t}{N} \right)^n \left( \frac{1}{2} + \frac{t}{N} \right)^{N-n-1}.
\]

With \(n = 0, 1, \ldots, N-1\), this family of beta distributions gives an exact description of the order statistics of the set of variables in which the \(n\)th function, \(F(n, t)\), corresponds to the density distribution of the \((n+1)\)st largest variable. Immediately, we verify that summing up all of them, the unit density is recovered as it should, namely \(\rho(t) = \sum F(n, t) = 1\). From (1), it follows that these distributions have first moment and variance given by

\[
\bar{t} = \frac{N(N-1-2n)}{2(N+1)},
\]

and

\[
\sigma^2 = \frac{(n+1)(N-n)N^2}{(N+2)(N+1)^2},
\]

respectively.

By taking the limit \(N \to \infty\) keeping \(n\) fixed, the order statistics of the largest variables is obtained. In this case it is more convenient to express the distributions using as variable the distance \(y = t - N/2\), to the right border in terms of which, the distributions \(F(n, t)\) converge, for large \(N\), to

\[
F(n, t) = \frac{(-y)^n}{n!} \exp(y).
\]

This set of functions are known to give the density distributions of the largest variables of an i.i.d. sequence with a compact support [14]. In particular, the first one, \(F(0, t) = \exp(y)\), is the Weibull distribution [14, 15]. From (4), we find that on average the \((n+1)\)st variable is located at the position \(\bar{y} = -(n+1)\).

Consider now the situation in which, in the same limit of a large number of variables, the ratio \(n/N\) is kept fixed. In this case, the beta distributions converge to the Gaussian

\[
F(n, t) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(t-\bar{t})^2}{2\sigma^2} \right]
\]

give the order statistics of the variables at the bulk of the sequence. Taking the limit of large \(N\) in expressions (2) and (3) the first moment and the variance become

\[
\bar{t} = \frac{(N-1)}{2} - n
\]

and

\[
\sigma^2 = \frac{n(N-n)}{N}.
\]

The above equation shows that the variance of the individual distributions at the bulk scales as \(\sigma \sim \sqrt{N}\). The case of i.i.d. variables with an arbitrary symmetric density distribution \(\rho(x)\) can be mapped into the above case by the transformation \(t = \int_0^x dx' \rho(x')\). The density distribution of the \((n+1)\)st variable of the ordered sequence is then

\[
F(n, x) = \rho(x) \frac{(N-1)!}{n!(N-n-1)!} \left[ \frac{1}{2} - \frac{t(x)}{N} \right]^n \left[ \frac{1}{2} + \frac{t(x)}{N} \right]^{N-n-1}.
\]

In figure 1, it is shown that the individual contribution decomposition for a set of \(N = 20\) i.i.d. variables sorted from the common Gaussian distribution

\[
\rho(x) = \frac{N}{\sqrt{\pi}} \exp(-x^2).
\]
In this case

\[ t(x) = \frac{N}{2} \text{erf}(x), \]  

(10)

where \( \text{erf}(x) \) is the error function.

It is seen that there is a strong overlapping among the individual distributions that reflects the dependence of the variances with the square root of the number \( N \) of variables. The largest variable is expected to be distributed according to the Gumbel density distribution [16]

\[ F_{\text{Gumbel}}(z) = \exp\left[ -\exp(-z) \right] \exp(-z) \]  

(11)
in the new variable

\[ z = -\ln \left[ \frac{N}{2} \text{erfc}(x) \right], \]  

(12)

where \( \text{erfc}(x) \) is the complementary error function. Indeed, one can see that this is true from the very good agreement exhibited in figure 2, in which the largest variable in a sequence of \( N = 100 \) Gaussian variables is compared against the above Gumbel density distribution, equation (11).

3. Correlated eigenvalues

The Wigner and the Wishart ensembles belong to a class of ensembles whose joint probability density distribution of the eigenvalues has the form

\[ P(x_1, x_2, \ldots, x_N) = K_N \exp \left[ -\frac{\beta}{2} \sum_{k=1}^{N} V(x_k) \right] \prod_{j > i} |x_j - x_i|^{\beta}, \]  

(13)

where \( N \) is the number of eigenvalues, \( \beta \) is Dyson index and \( K_N \) is a normalization constant. In (13), \( V(x) \) is a confining potential which makes the above distribution normalizable by keeping the eigenvalues inside a potential. The exact expression for the eigenvalue density obtained integrating all variables but one is given by [3]

\[ \rho(x) = \exp[-V(x)] \sum_{0}^{N-1} P^2 \rho(x), \]  

(14)
where the $P_n(x)$ are normalized polynomials orthogonal with respect to the weight $\exp[-V(x)]$. In fact, this is the density of the unitary ensemble, $\beta = 2$, but extra terms have to be added in the case of the orthogonal and the symplectic ensembles. The above sum can be reduced to the contribution of the last term using the Christoffel–Darboux relation

$$\sum_{0}^{N-1} P_n(x) P_n(y) = \frac{k_{N-1}}{k_N} \frac{P_N(x) P_{N-1}(y) - P_N(y) P_{N-1}(x)}{x - y},$$  \hspace{1cm} (15)$$

where $k_n$ is the highest coefficient of $P_n(x)$, and in the limit $y \to x$, (15) becomes

$$\sum_{0}^{N-1} P_n^2(x) = \frac{k_{N-1}}{k_N} [P_N'(x) P_{N-1}(x) - P_N(x) P_{N-1}'(x)].$$  \hspace{1cm} (16)$$

The density described by equation (14), in the case we are interested in of Wigner’s and Wishart’s ensembles, has a central part with wiggles separated by inflection points, namely where the curvature changes sign, from the decaying tails at the edges. In the following, we use equation (16) to show that, in the central bulk of the spectra, the density for these two ensembles can be written as a sum

$$\rho(x) = \rho_s(x) + \rho_f(x)$$  \hspace{1cm} (17)$$

of a smooth leading term, $\rho_s(x)$, and a fluctuating term, $\rho_f(x)$, which vanishes in the limit of large matrices. From this decomposition of the density in smooth and fluctuating terms, the parameters which define individual eigenvalue distributions are extracted. However, this procedure cannot be used beyond the inflection point that is for the first and the last eigenvalues. But, we remark that for these extreme eigenvalues, the tail of their densities coincides with the density itself and does not need to be calculated. With this procedure, we are able to obtain the individual distribution of all eigenvalues.
3.1. Wigner ensembles

The joint distribution of the eigenvalues of the Gaussian random matrices for the orthogonal, unitary and symplectic ensembles is

$$P(x_1, x_2, \ldots, x_N) = Z_N^{-1} \exp \left( -\frac{\beta}{2} \sum_{k=1}^{N} x_k^2 \right) \prod_{j>i} |x_j - x_i|^{\beta},$$

where

$$Z_N(\beta) = (2\pi)^{N/2} \beta^{-N/2 - \beta N(N-1)/4} \prod_{j=1}^{N} \frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)}.$$  

Therefore, from equation (13), the confining potential for this ensemble is the parabola $V(x) = x^2/2$ and the orthogonal polynomials are the Hermite polynomials $H_n(x)$.

3.1.1. Unitary ensemble ($\beta = 2$). The eigenvalue density is [3]

$$\rho(x) = \sum_{n=0}^{N-1} \phi_n^2(x),$$

where

$$\phi_n(x) = \frac{\exp(-x^2/2)H_n(x)}{\sqrt{n!}2^n \sqrt{n!}}$$

satisfies the equation

$$\frac{d^2\phi_n(x)}{dx^2} + (2n + 1 - x^2)\phi_n(x) = 0,$$

which, in appropriate units, is the Schrödinger equation for the quantum harmonic oscillator.

To be able to later calculate individual distributions of i.i.d. random variables with the above density, equation (20), we define the counting function

$$N(x) = \int_0^x dx' \rho(x').$$

To calculate this function the recurrence relation

$$\phi_n(x) = \sqrt{\frac{2}{n}} x \phi_{n-1}(x) - \sqrt{\frac{n-1}{n}} \phi_{n-2}(x),$$

which follows from the known recurrence relation

$$H_n(x) = 2xH_{n-1}(x) - (2n - 1)H_{n-2}(x)$$

of the Hermite polynomials can be used. Integrating the square of (24) and using the relation

$$\phi_n'(x) = -x\phi_n(x) + \sqrt{2n}\phi_{n-1}(x),$$

derived taking the derivative of (21) together with

$$H_n' = nH_{n-1},$$

we deduce the recursion relation

$$\int_0^1 dt \phi_n^2(t) = \frac{x}{n} \phi_{n-1}^2(x) + \frac{1}{n} \int_0^x dt \phi_{n-1}^2(t) + \frac{n-1}{n} \int_0^x dt \phi_{n-2}^2(t)$$

which provides an efficient way to calculate the counting function starting from the ground-state function $\phi_0(x)$.
Before analyzing the case of large matrices, it is instructive to consider the simple case of matrices of size $N = 2$ for which equation (20) gives

$$
\rho(x) = \frac{\exp(-x^2)}{\sqrt{\pi}} (1 + 2x^2).
$$

(29)

The probability of the largest eigenvalue be less than a value $x$ is obtained by integrating the joint distribution $P(x_1, x_2)$ in the two variables $x_1$ and $x_2$ from $-\infty$ to $x$. Taking then the derivative of this probability, we find that the density distribution of the largest eigenvalue is

$$
F(0, x) = \frac{\exp(-x^2)}{2\sqrt{\pi}} \left[ (1 + 2x^2)(1 - \text{erf}(x)) - \frac{2x \exp(-x^2)}{\sqrt{\pi}} \right].
$$

(30)

For the smallest eigenvalue distribution it is simpler to determine it by taking the difference $\rho(x) - F(0, x)$. For the uncorrelated case (see the previous section), with $n = 0, 1$ and $\rho(x)$ given by (29) the density distributions of the largest and the smallest are given by

$$
F_U(n, x) = \frac{\rho(x)}{2} \left[ 1 + (-1)^n \left( \text{erf}(x) - \frac{x \exp(-x^2)}{\sqrt{\pi}} \right) \right].
$$

(31)

In figure 3, these distributions are shown together with the density. One can clearly see the reduction produced by the correlations in the range of fluctuations of the eigenvalues.

Turning now to the case of large matrices, by using the (16), (27) and (21) we rewrite (20) as

$$
\rho(x) = \sqrt{\frac{N}{2}} \left[ \phi'_N(x)\phi_{N-1}(x) - \phi_N(x)\phi'_{N-1}(x) \right].
$$

(32)

If the harmonic oscillator functions are expressed in terms of amplitude and phase as

$$
\phi_n(x) = A_n(x) \cos \theta_n(x),
$$

(33)

then, when substituted in (32), the density becomes a sum of cosines and sines whose arguments are either addition or subtraction of the function phases. We observe that as the function $\phi_n(x)$ has $n$ zeros its phase $\theta_n(x)$ must have a range of variation of order $n\pi$. Therefore, by subtracting two adjacent phases we get a function whose variation is smaller than $\pi$ producing therefore...
no wiggles. We can then argue that the smooth part of the density, $\rho_s(x)$, is obtained by collecting the terms in which the phases subtract while, the fluctuating part, $\rho_f(x)$, comes from those in which they add. Explicitly, this leads to the expressions

$$\rho_s = \sqrt{\frac{N}{8}} A_N A_{N-1} \left( \frac{A_N}{A_N} - \frac{A_{N-1}}{A_{N-1}} \right) \cos(\theta_N - \theta_{N-1}) - (\theta_N' + \theta_{N-1}') \sin(\theta_N - \theta_{N-1})$$  \hspace{1cm} (34)

and

$$\rho_f = \sqrt{\frac{N}{8}} A_N A_{N-1} \left( \frac{A_N}{A_N} - \frac{A_{N-1}}{A_{N-1}} \right) \cos(\theta_N + \theta_{N-1}) - (\theta_N' - \theta_{N-1}') \sin(\theta_N + \theta_{N-1})$$  \hspace{1cm} (35)

(all quantities $\theta_n$'s and $A_n$'s above and others below are $x$-dependent, but to keep the notation less heavy, this dependence is often dropped). An asymptotic expression for these two density terms can be deduced from the semi-classical formalism described in the appendix. We find that, asymptotically, the harmonic oscillator function is given by

$$\phi_n(x) = \sqrt{\frac{2}{\pi}} \cos \left[ \xi_n(x) - \frac{n}{2\pi} \right]$$  \hspace{1cm} (36)

where

$$\xi_n(x) = \frac{2n + 1}{2\pi} \left[ \arcsin \left( \frac{x}{\sqrt{2n + 1}} \right) + \frac{x}{\sqrt{2n + 1}} \sqrt{1 - \frac{x^2}{2n + 1}} \right]$$  \hspace{1cm} (37)

is the classical mechanical action obtained integrating the momentum $p = \sqrt{2n + 1 - x^2}$. In (36), the phase $n\pi/2$ has been introduced to fix the parity of the function. Equation (36) together with (37) determine the phases and amplitudes in (34) and (35). Substituting them and neglecting the derivatives of the amplitude, we find, after neglecting higher order terms, that the density takes the simple analytic form

$$\rho(x) = \rho_W(x) - \frac{\sqrt{2N} \cos[N - 2\xi(x)\pi]}{2\pi^3 \rho_W^2(x)}$$  \hspace{1cm} (38)

where

$$\rho_W(x) = \begin{cases} 
\frac{1}{\pi} \sqrt{2N - x^2}, & |x| < \sqrt{2N} \\
0, & |x| > \sqrt{2N}
\end{cases}$$  \hspace{1cm} (39)

is Wigner’s semi-circle law and

$$\xi(x) = \begin{cases} 
\frac{-N}{2}, & x < -\sqrt{2N} \\
\frac{N}{2} \left[ \arcsin \left( \frac{x}{\sqrt{2N}} \right) + \frac{x}{\sqrt{2N}} \sqrt{1 - \frac{x^2}{2N}} \right], & |x| < \sqrt{2N} \\
\frac{N}{2}, & x > \sqrt{2N}
\end{cases}$$  \hspace{1cm} (40)

In deriving these equations, $N$ was assumed to be large enough to treat indexed quantities as continuous functions of the indices in such a way that the approximations $f(N) + f(N-1) = 2f(N - 1/2)$ and $f(N) - f(N-1) = f'(N - 1/2)$ can be made.

Equation (38) shows that averaging out the wiggles, the fluctuating term vanishes and the density becomes the semi-circle. The quantity $\xi(x)$ is the average staircase function from
Figure 4. The relative difference \((\rho_{\text{asym}} - \rho)/\rho\) between the exact density \(\rho\) and its asymptotic is plotted versus the unfolded variable \(\xi\).

which the so-called unfolded spectrum is calculated. This transformation leads to a new spectrum with density

\[
\rho(\xi) = \frac{\rho(x)}{\rho_W(x)} = 1 - \frac{\sqrt{2N \cos(N - 2\xi)\pi}}{2\pi^3 \rho_W^3(x)}
\]  

(41)

whose average is equal to 1. Since \(\xi(x)\) is a counting function, in this variable the wiggles are equally spaced with a unit distance between them. In figure 4, the density calculated using the approximated expression (38) is compared with the density calculated with equation (20). One can see that with the exception of the wiggles at the very edges, those at the bulk are well described by the asymptotic density.

Equation (41) gives analytical expressions for the smooth and the fluctuating parts of the density which figure 4 shows are very precise at the bulk of the spectrum. To extend this precision up to the edge, removing the singularities at the classical turning points, an improvement is needed to go beyond the asymptotic expression (36). To this end, one must also consider the exact second independent solution of the equation expressed as

\[
\tilde{\phi}_n(x) = A_n(x) \sin \theta_n(x).
\]  

(42)

From the pair of independent solutions, modulus and phase of the functions can be extracted. This second independent solution can be determined by integrating from the origin, \(x = 0\), the differential equation with initial conditions provided by the Wronskian relation

\[
W(\phi_n, \tilde{\phi}_n) = \phi_n(x)\tilde{\phi}_n'(x) - \phi_n'(x)\tilde{\phi}_n(x) = 2/\pi.
\]  

(43)

In fact, since solutions of the equation can be constructed with a definite parity, for even \(n\), we take \(\tilde{\phi}_n(x)\) to be odd such that \(\tilde{\phi}_n(0) = 0\) and, from (43), \(\tilde{\phi}_n'(0) = 2/\pi \phi_n(0)\). Inversely, for \(n\) odd, \(\tilde{\phi}_n(x)\) is taken to be even, \(\tilde{\phi}_n'(0) = 0\) and, from (43), \(\phi_n(0) = -2/\pi \phi_n'(0)\).

Once the pair of solutions is determined, modulus and phase are obtained as

\[
A_n = \sqrt{\phi_n^2 + \tilde{\phi}_n^2}
\]  

(44)

and

\[
\theta_n = \arctan(\tilde{\phi}_n/\phi_n)
\]  

(45)
and their derivatives as
\[ A_nA'_n = \phi_n\phi'_n + \tilde{\phi}_n\tilde{\phi}'_n \quad (46) \]
and
\[ \theta'_n = \frac{2}{\pi}A^{-2} \quad (47) \]
The exact expression of the scaled density becomes
\[ \rho = \frac{\rho(x)}{\rho_s(x)} = 1 + \frac{B\cos(\theta_N + \theta_{N-1} + \theta)}{\rho_s}, \quad (48) \]
where
\[ B = \sqrt{\frac{N}{8}}A_NA_{N-1}\sqrt{\left(\frac{A'_N}{A_N} - \frac{A'_{N-1}}{A_{N-1}}\right)^2 + \left(\theta'_N - \theta'_{N-1}\right)^2} \quad (49) \]
and
\[ \theta = \arctan \left[ \frac{A_NA_{N-1}(\theta'_N - \theta'_{N-1})}{A_{N-1}A'_N - A_NA'_{N-1}} \right]. \quad (50) \]
Guided by intuition and numerics, let us make the ansatz that the eigenvalue density in the scaled variable can be decomposed as
\[ \rho(\nu) = \sum_{k=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\nu - \nu_k)^2}{2\sigma^2}\right], \quad (51) \]
where, according to equation (38),
\[ \nu_k = \frac{N + 1}{2} - k, \quad (52) \]
with \( k = 1, 2, \ldots, N \). Since the quantities \( \nu \) and \( \sigma \) may depend on position, each term in the above sum is not a true Gaussian; however, they can be considered as nearly Gaussian distributions as that dependence is expected to be weak. In order to determine this dependence, we turn the summation in (51) into an infinite sum and rewrite it as
\[ \rho(\nu) = \sum_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\nu - \nu_k)^2}{2\sigma^2}\right] - R, \quad (53) \]
where
\[ R = \sum_{k=-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\nu - \nu_k)^2}{2\sigma^2}\right] + \sum_{k=N+1}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\nu - \nu_k)^2}{2\sigma^2}\right]. \quad (54) \]
This \( R \) quantity is expected to affect only the tail of the distribution of the extreme eigenvalues which, as explained in the beginning of this section, practically coincide with the total density. Therefore, this reminder can be neglected even for relatively small values of \( N \). The first term on the right-hand side of (53) can be transformed into an infinite sum of integrals using the Poisson sum formula
\[ \sum_{n=-\infty}^{\infty} f(t+n) = \sum_{m=-\infty}^{\infty} F(2\pi m)\exp(2\pi imt), \quad (55) \]
where \( F(s) \) is the Fourier transform
\[ F(s) = \int_{-\infty}^{\infty} f(t)\exp(-ist). \quad (56) \]
Then, after performing all integrals we obtain
\[
\rho(\nu) = 1 + 2 \sum_{m=1}^{\infty} (-1)^m \exp[-2(\pi m \sigma)^2] \cos \left(2\pi m \left(\nu + \frac{N}{2}\right)\right),
\]
where the term \( R \) in (53) was neglected. Due to the presence of the exponential factor, the sum is dominated by its first term \( m = 1 \).
Comparing this term with the oscillating term in equation (41), we find that the phase \( \nu \) and the variance \( \sigma \) depend on their positions as
\[
\nu(x) = \frac{\theta_N + \theta_{N-1} + \theta - \pi}{2\pi} - \frac{N}{2},
\]
and
\[
\sigma^2(x) = -\frac{1}{2\pi^2} \ln \left(\frac{B(x)}{2\rho_s(x)}\right).
\]
At the bulk of the spectrum, the phases take their asymptotic values \( \theta_n = (\xi_n - n/2)\pi \) and \( \theta = \pi/2 \) such that \( \nu(x) = \xi(x) \) and
\[
\sigma^2(x) = \frac{3}{2\pi^2} \ln \left(\frac{\pi \sqrt{2\rho_W(x)}}{N^{1/6}}\right).
\]
Once the variances have been determined, the decomposed density in the actual spectrum variable is
\[
\rho(x) = \rho_s(x) \sum_{k=1}^{N} \frac{1}{\sqrt{2\pi \sigma^2(x)}} \exp \left[ -\frac{(x - \nu_k)^2}{2\sigma^2(x)} \right],
\]
In figure 5, these \( N \) individual distributions are compared with the distributions obtained by performing numerical simulations for matrices of size \( N = 20 \) with a very good agreement. In figure 6, these density distributions are compared with those of uncorrelated variables with the same density exhibiting the great effect of the correlations.
Motivated by the good agreement between simulations and nearly Gaussian distributions, we compare the nearly Gaussian distribution at the edge with Tracy and Widom’s prediction...
for the largest eigenvalue. They proved that, when \( N \to \infty \), in a new variable \( s \) defined by
the linear relation
\[
x = \sqrt{2N} + \frac{s}{2^{1/2}N^{1/6}},
\] (62)
the distribution probability, \( E_2(s) \), of the largest eigenvalue of the unitary ensemble, \( \beta = 2 \),
is given by [6]
\[
E_2(s) = \exp \left[ - \int_{s}^{\infty} (x - s)q^2(x) \, dx \right],
\] (63)
where \( q(s) \) satisfies the Painlevé II equation
\[
q'' = sq + 2q^3
\] (64)
with the boundary condition
\[
qu(s) \sim \text{Ai}(s) \quad \text{when} \quad s \to \infty,
\] (65)
where \( \text{Ai}(s) \) is the Airy function.

In figure 7, the distribution of the largest eigenvalue of matrices of size \( N = 20 \) obtained
performing numerical simulation is compared with both: Tracy–Widom density distribution,
\( E_2(s) \) and our nearly Gaussian distribution. Both give a reasonable fit although \( N = 20 \)
can be considered a relatively small size. In table 1, the cumulants of the two distributions are
shown [17] and, as one would expect, the figures point that ours is indeed more normal.
3.1.2. Orthogonal ensemble ($\beta = 1$). For the orthogonal case, we have to add to (14) the term

$$\gamma(x) = \sqrt{\frac{N}{2}} \phi_{N-1}(x) \int_{-\infty}^{\infty} dt \frac{1}{2} \text{sgn}(x-t) \phi_N(t) = \sqrt{\frac{N}{2}} \phi_{N-1}(x) \int_{0}^{x} dt \phi_N(t),$$

where $\text{sgn}(x)$ is the sign function, if $N$ is even, and a further term

$$\phi_N(x) \int_{-\infty}^{\infty} \phi_N(t) dt,$$

if $N$ is odd [3].

As done in the unitary case, it is instructive to start discussing the decomposition of the spectral density of matrices of size $N = 2$. This will illustrate the differences between the unitary and the orthogonal cases. Evaluating the integral in (66) we find that the extra term is

$$\gamma(x) = \frac{-2x^2 \exp(-x^2)}{\sqrt{\pi}} + \frac{\exp(-x^2/2)x}{\sqrt{\pi}} \int_{0}^{x} dt \exp \left( \frac{-t^2}{2} \right).$$

Adding this term to (29), its first term cancels the second term in (29) and the density becomes

$$\rho(x) = \frac{\exp(-x^2)}{\sqrt{\pi}} + \frac{\exp(-x^2/2)}{\sqrt{x}} \text{erf} \left( \frac{x}{\sqrt{2}} \right).$$

The important point is that this cancellation of terms removes the wiggles in the unitary density in such a way that the orthogonal becomes a flat function. The individual distributions of the two eigenvalues are easily calculated to be given by

$$F(0, x) = \frac{\sqrt{3}}{4} \exp(-x^2) \text{erfc} \left( \frac{-x}{\sqrt{2}} \right) + \frac{\exp(-x^2/2)}{2\sqrt{\pi}}$$

for the greater while distribution of the smaller is obtained by subtracting the above $F(0, x)$ from equation (69). For the uncorrelated case, with $n = 0, 1$ and $\rho(x)$ given by (69), the density distributions are

$$F_U(n, x) = \frac{\rho(x)}{2} \left[ 1 + (-1)^n \left( \text{erf}(x) - \frac{\exp(-x^2/2)}{\sqrt{\pi}} \text{erf} \left( \frac{x}{\sqrt{2}} \right) \right) \right].$$
Figure 8 shows the spectral density of matrices of size \( N = 2 \) of the orthogonal ensemble decomposed in eigenvalue individual contributions. In contrast to the unitary case, figure 3 in the orthogonal case, the two individual contributions add in such a way that density becomes flat at the top.

Turning now to the case of matrices of large sizes, we have to calculate the integral

\[ I_N(x) = \int_0^x dt \phi_N(t). \]  

(72)

Starting with the derivation of an asymptotic expression for it, we use the differential equation satisfied by the \( \phi_N(x) \) functions to rewrite it as

\[ I_N(x) = - \int_0^x dt \frac{\phi_N''(t)}{2N + 1 - t^2}. \]  

(73)

We recall that the denominator in the above integrand is the square of the classical momentum supposed to be large. Therefore, integration by parts can be used to obtain a series in inverse powers of the momentum or, equivalently, of the density, whose first two terms are

\[ I_N(x) = - \frac{\phi_N'(x)}{2N + 1 - x^2} + \frac{2x\phi_N(x)}{(2N + 1 - x^2)^2}. \]  

(74)

Substituting this term in equation (66) and replacing the functions by their semi-classical approximation, we find that the first additional extra term cancels the oscillating term of the unitary case. This canceling makes it necessary to take into account higher order terms, which can be done using the expansion

\[ (2N \pm 1 - x^2)^\mu = (2N - x^2)^\mu \pm \mu(2N - x^2)^{\mu-1}. \]  

(75)

By doing this, we end up with the following asymptotic expression for the density

\[ \rho(x) = \rho_W - \frac{1}{2\pi^2 \rho_W} + \frac{\sqrt{2N}}{2\pi^2 \rho_W^2} \cos \left( 2\xi - \frac{N}{2} \right) \pi + \frac{3x\sqrt{2N}}{8\pi^5 \rho_W^3} \sin \left( 2\xi - \frac{N}{2} \right) \pi \]  

(76)

in the orthogonal case. The first two terms in (76) correspond to the smooth part of the density while the two last ones to its fluctuating part.
In order to go beyond this asymptotic expression, removing, as done in the unitary case, its singularities, we assume that the integral $I_N(x)$ can be written as

$$I_N(x) = Q_1(x) \cos \theta_N(x) + Q_2(x) \sin \theta_N(x),$$

where $\theta_N(x)$ is the phase of the function $\phi_N(x)$ and $Q_1(x)$ and $Q_2(x)$ are smooth functions of the position. These functions can be determined considering the function $\tilde{I}_N(x)$ related to the second independent solution as $\tilde{I}_N(x) = \tilde{\phi}_N(x)$. From the asymptotic analysis, we deduce that it satisfies, at the origin, the condition

$$\tilde{I}_N(0) = -\frac{\phi_N'(0)}{2N + 1},$$

and can be written as

$$I_N(x) = Q_1(x) \sin \theta_N(x) - Q_2(x) \cos \theta_N(x).$$

Equations (77) and (79) can be inverted to give

$$Q_1(x) = \frac{I_N(x) \cos \theta_N(x) + \tilde{I}_N(x) \sin \theta_N(x)}{\cos \theta_N(x)},$$

and

$$Q_2(x) = \frac{I_N(x) \sin \theta_N(x) - \tilde{I}_N(x) \cos \theta_N(x)}{\sin \theta_N(x)}.$$

Once these two functions are determined the integral (77) is expressed in terms of amplitudes and phase and can be replaced in the additional term $\gamma(x)$ which can also be decomposed as a sum of smooth and fluctuating terms as

$$\gamma_s = \sqrt{\frac{N}{8}} A_{N-1} \left[ Q_1 \cos (\theta_N - \theta_{N-1}) + Q_2 \sin (\theta_N - \theta_{N-1}) \right],$$

and

$$\gamma_f = \sqrt{\frac{N}{8}} A_{N-1} \left[ Q_1 \cos (\theta_N + \theta_{N-1}) + Q_2 \sin (\theta_N + \theta_{N-1}) \right].$$

By adding $\gamma_s$ and $\gamma_f$ to equations (34) and (35), respectively, the scaled density is

$$\rho = \frac{\rho_1(x)}{\rho_{1s}(x)} = 1 + \frac{B \cos (\theta_N + \theta_{N-1} + \theta)}{\rho_{1s}},$$

where

$$\rho_{1s}(x) = \rho_s(x) + \gamma_s(x),$$

$$B = \sqrt{\frac{N}{8}} A_N A_{N-1} \sqrt{\left( \frac{A_N'}{A_N} - A_{N-1}' + \frac{N}{A_N} \right)^2 + \left( \theta_N' - \theta_{N-1}' - \frac{Q_2}{A_N} \right)^2},$$

and

$$\theta = \arctan \left[ \frac{\theta_N' - \theta_{N-1}' - \frac{Q_2}{A_N}}{\theta_N'/A_N - \theta_{N-1}'/A_{N-1} + \frac{Q_1}{A_N}} \right].$$

The formalism developed in the unitary case still applies and, with the above expressions, the scaled variable and the variance are deduced to be given by

$$\nu(x) = \frac{\theta_N + \theta_{N-1} + \theta - \pi}{2\pi} - \frac{N}{2},$$

and

$$\sigma^2(x) = -\frac{1}{2\pi^2} \ln \left[ \frac{B(x)}{2\rho_{1s}(x)} \right].$$
At the bulk, from equation (76), the scaled variable becomes

$$\nu(x) = \xi(x) + \arctan \left( \frac{3x}{4\pi \rho_w(x)} \right)$$

(90)

and the density smooth term and the amplitude are given by

$$\rho_1(x) = \rho_w(x) - \frac{1}{2\pi^2 \rho_w(x)}$$

(91)

and

$$B(x) = \frac{\sqrt{2N}}{2\pi^3 \rho_w^5} \sqrt{1 + \frac{9x^2}{16\pi^2 \rho_w^2}},$$

(92)

respectively, which replaced in equation (89) gives the asymptotic expression of the variance.

In figure 9, the density distributions of individual eigenvalues of matrices of size $N = 20$ are shown together with distributions obtained performing numerical simulations. The result is good specially at the bulk of the spectrum.

For the largest eigenvalue, Tracy and Widom predict that for the orthogonal ensemble ($\beta = 1$), the probability distribution function in the same scaled variable $s$ of the unitary case is given by

$$[E_1(s)]^2 = E_2(s) \exp[-\mu(s)],$$

(93)

where

$$\mu(s) = \int_s^\infty q(x) \, dx.$$  

(94)

In figure 10, this prediction is compared with our nearly Gaussian density distribution and with results of simulations. It is clear that the nearly Gaussian distribution gives a better description. However, comparing the cumulants [17] depicted in table 2, we see that actually, apart from a shift to the right of the Tracy–Widom distribution, the two distributions are quite alike.
3.2. Wishart matrices

Consider a rectangular matrix \( X \) of size \((M \times N)\) whose elements are sorted independently from a Gaussian distribution; a Wishart square matrix \( W \) of size \( N \) is then defined by taking the product \( W = X'X \). It can be shown that for \( M \geq N \), the joint probability distribution of the positive eigenvalues of the random matrices \( W \), for the three symmetry classes, is given by

\[
P(x_1, x_2, \ldots, x_N) = K_N \exp\left(-\frac{\beta}{2} \sum_{k=1}^{N} x_k \right) \prod_{i=1}^{N} x_i^{\frac{1}{2}(1+M-N)-1} \prod_{j>i} |x_j - x_i|^{\beta}.
\]  

(95)

From (13), the positive eigenvalues of this ensemble are confined by the potential \( V(x) = x - (1 + M - N - 2/\beta) \log(x) \) and the polynomials are the generalized Laguerre polynomials. For the unitary case, \( \beta = 2 \), the eigenvalue density is

\[
\rho(x) = \sum_{n=0}^{N-1} \psi_n^2(x),
\]  

(96)

where, with \( \alpha = M - N \),

\[
\psi_n^u(x) = \sqrt{\frac{n!}{(n + \alpha)!}} \exp\left(-x/2\right)x^{\alpha/2}L_n^\alpha(x)
\]  

(97)

in which the \( L_n^\alpha(x) \) are the Laguerre polynomials:

\[
L_n^\alpha(x) = \sum_{j=0}^{n} (-1)^j \frac{(n + \alpha)!}{(n - j)! \alpha! j!} x^j.
\]  

(98)
From (98), we derive
\[
\frac{dL_n^{\alpha}}{dx} = -L_{n-1}^{\alpha+1} \quad (99)
\]
which used in (16) yields
\[
\rho(x) = \exp(-x) x^\alpha \frac{\Gamma(N)}{\Gamma(N+\alpha)} \left[ L_{N-1}^{\alpha}(x)L_{N-1}^{\alpha+1}(x) - L_{N}^{\alpha}(x)L_{N-2}^{\alpha+1}(x) \right]. \quad (100)
\]
Inverting (97) to express the polynomials in terms of the $\psi$-functions, the density becomes
\[
\rho(x) = \sqrt{\frac{N(N+\alpha)}{x}} \left[ \psi_{N-1}^{\alpha}(x)\psi_{N-1}^{\alpha+1}(x) - \sqrt{\frac{N-1}{N}} \psi_{N}^{\alpha}(x)\psi_{N-2}^{\alpha+1}(x) \right]. \quad (101)
\]
Finally the derivative of equation (97) gives the relation
\[
\frac{d\psi_n^{\alpha}}{dx} = \left( \frac{\alpha - x}{2x} \right) \psi_n^{\alpha} - \sqrt{\frac{n}{x}} \psi_{n-1}^{\alpha+1} \quad (102)
\]
which used in (101) allows us to write the density as
\[
\rho(x) = \sqrt{\frac{N(N+\alpha)}{4}} \left[ \frac{A_{N-1}}{A_N} \frac{d\psi_{N-1}^{\alpha}}{dx} - \frac{\psi_{N-1}^{\alpha}}{\psi_{N}^{\alpha}} \frac{d\psi_{N}^{\alpha}}{dx} \right]. \quad (103)
\]
and substitute, in the expression of the density, the two functions to extract its smooth part
\[
\rho_s = \sqrt{\frac{N(N+\alpha)}{4}} \left[ \frac{A_{N-1}'}{A_{N-1}} \frac{A_N'}{A_N} \cos(\theta_{N} - \theta_{N-1}) + (\theta_{N}' + \theta_{N-1}') \sin(\theta_{N} - \theta_{N-1}) \right] \quad (106)
\]
and its fluctuating part
\[
\rho_f = \sqrt{\frac{N(N+\alpha)}{4}} \left[ \frac{A_{N-1}'}{A_{N-1}} \frac{A_N'}{A_N} \cos(\theta_{N} + \theta_{N-1}) + (\theta_{N}' - \theta_{N-1}') \sin(\theta_{N} + \theta_{N-1}) \right]. \quad (107)
\]
In order to apply the semi-classical formalism, we first transform equation (104) in the differential equation
\[
\frac{d^2(\sqrt{x}\psi)}{dx^2} + \frac{-(\alpha - x)(\alpha - x + 2\alpha + 1) - \alpha^2 + 1}{4x^2} (\sqrt{x}\psi) = 0 \quad (108)
\]
satisfied by the function $\sqrt{x}\psi(x)$. In this equation, the associated classical moment is
\[
\rho(x) = \frac{1}{2\alpha} \sqrt{(x - x_1)(x_2 - x)}. \quad (109)
\]
where
\[ x_{1,2} = 2n + \alpha + 1 \mp \sqrt{(2n + \alpha + 1)^2 + 1 - \alpha^2} \] (110)
and the asymptotic wavefunction is
\[ \psi_n(x) = \sqrt{\frac{2}{\pi}} \frac{\cos(\xi_n - \pi/4)}{[(x_2 - x)(x - x_1)]^{1/4}}, \] (111)
where
\[ \xi_n = \frac{1}{4\pi} \left[ -4\sqrt{x_1 x_2} \arctan \frac{x_2(x - x_1)}{x_1(x_2 - x)} + (x_2 + x_1) \arccos \left( \frac{x_2 + x_1 - 2x}{x_2 - x_1} \right) \right. \]
(112)
Substituting this approximate wavefunction in the smooth and the fluctuating parts of the density and neglecting derivatives of the amplitudes, we arrive at the asymptotic expression
\[ \rho(x) = \rho_{MP}(x) - \frac{x_+ - x_-}{16\pi^3 x^2 \rho_{MP}(x)} \cos[2\xi(x)]\pi, \] (113)
where the first term is the Marchenko–Pastur density [18]
\[ \rho_{MP}(x) = \frac{1}{2\pi x} \sqrt{(x_+ - x)(x - x_-)}, \] (114)
in which, with \( c = \sqrt{\frac{M}{N}}, x_\pm = N(c \pm 1)^2 \). Similarly, the function \( \xi(x) \) appearing in the cosine argument is the counting number function
\[ \xi(x) = \frac{1}{4\pi} \left[ -4\sqrt{x_1 x_2} \arctan \frac{x_2(x - x_1)}{x_1(x_2 - x)} + (x_2 + x_1) \arccos \left( \frac{x_2 + x_1 - 2x}{x_2 - x_1} \right) \right. \]
(115)
associated with the Marchenko–Pastur density. As in the previous case, approximations were made by treating indexed quantities as continuous functions of their indices. Equation (113) shows that the Marchenko–Pastur density plays, for the Wishart matrices, the same role as Wigner’s semi-circle does for the Gaussian ensembles and, by averaging out the wiggles produced by the oscillating term, it is obtained. We remark that the above expression for the next to leading order term of the asymptotic, has an analog structure to that of the Gaussian expression (41), namely, both are, basically, the ratios between the superior limit and the cubic of the asymptotic density. As before, we use the counting function \( \xi \) as an independent variable with density
\[ \rho(\xi) = \frac{\rho(x)}{\rho_{MP}(x)} = 1 - \frac{x_+}{16\pi^3 x^2 \rho_{MP}(x)} \cos[2\xi(x)]\pi. \] (116)
In figure 11, we compare the above density approximation with its exact expression. It is seen that it is indeed a very good approximation with the exception of the regions near the two borders.

As in the Wigner case, a complete decomposition of the density which includes eigenvalues at the border can be achieved using the second independent solution of the wave equation which can be obtained from the integral representation
\[ \psi_n(x) = \frac{2^n (-1)^n}{\pi x^{n/2}} \sqrt{(n + \alpha)! / n!} \int_0^{\pi} d\theta \cos^{n-1}\theta \cos \left[ \frac{x \tan \theta}{2} - (2n + \alpha + 1)\theta \right] \] (117)
The relative difference \((\rho_{\text{asym}} - \rho)/\rho\) between the exact density \(\rho\) and its asymptotic is plotted versus the unfolded variable \(\xi\).

of the first solution. That this function, with \(\alpha > 1\), is equivalent to (97) can be seen by changing the integration variable to \(u = \tan \theta\) that transforms the integral in (117) into the complex integral

\[
\psi_n(x) = \frac{2^{\alpha}(-1)^n}{2\pi x^{\alpha/2}} \sqrt{(n + \alpha)!/n!} (-1)^{2n+1} \int_{-\infty}^{\infty} du \frac{\exp \left( \frac{u^2}{2} \right)(u + i)^n}{(u - i)^{n+\alpha+1}}
\]

which performed by residues reproduces (97). This integral representation suggests that the other independent solution is provided by replacing in the integrand the cosine of the oscillating factor by the sine. However, this is not enough, and yet another term has to be subtracted to construct the solution

\[
\bar{\psi}_n(x) = \frac{2^{\alpha}(-1)^n}{\pi x^{\alpha/2}} \sqrt{(n + \alpha)!/n!} \left\{ \int_0^{\pi/2} \sin^{\alpha-1} \theta \sin \left( \frac{\tan \theta}{2} - (2n + \alpha + 1) \theta \right) \right. \\
\left. + \int_0^{\infty} \cos^{\alpha-1} \theta \exp \left[ \frac{\tanh \theta}{2} - (2n + \alpha + 1) \theta \right] \right\}.
\]

Now, we have a pair of independent solutions and the modulus and phase are determined through the relations

\[
A_n = \sqrt{\psi_n^2 + \bar{\psi}_n^2}
\]

and

\[
\theta_n = \arctan(\bar{\psi}_n/\psi_n)
\]

with derivatives given by

\[
A_n A_n' = \psi_n \psi_n' + \bar{\psi}_n \bar{\psi}_n'
\]

and, using the Wronskian \(W(\psi \bar{\psi})(x) = 2/\sqrt{\pi}\),

\[
\theta_n' = \frac{1}{\pi} A_n^{-2}.
\]

In calculating these quantities, the two recurrence relations are

\[
Z_{n-2} = \frac{1}{\sqrt{(n-1)(n+\alpha-1)}} \left[(2n + \alpha - 1 - x)Z_{n-1} - \sqrt{n(n+\alpha)}Z_n \right]
\]
and
\[ Z_n \frac{dZ_n}{dx} = \left( \frac{2n + \alpha - x}{2} \right) Z_n^\alpha - \sqrt{n(n + \alpha)} Z_{n-1}, \]  
(125)

where \( Z_n \) denotes that any one of the two solutions is useful.

The precise expression for the decomposition of the density in smooth and fluctuating parts is
\[
\rho = \frac{\rho(x)}{\rho_s(x) = 1 - B \cos(\theta_N + \theta_{N-1} - \theta)},
\]  
(126)

where
\[
B = \sqrt{\frac{N(N + \alpha)}{4} A_N A_{N-1} \left( \frac{A'_N}{A_N} - \frac{A'_{N-1}}{A_{N-1}} \right)^2 + (\theta'_N - \theta'_{N-1})^2},
\]  
(127)

and
\[
\theta = \arctan \left[ \frac{A_N A_{N-1}(\theta'_N - \theta'_{N-1})}{A_{N-1} A'_N - A_N A'_{N-1}} \right].
\]  
(128)

Turning now to the decomposition of the Wishart eigenvalue density in a sum of individual nearly Gaussian distributions, we write
\[
\rho(v) = \sum_{k=0}^{N-1} \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left[ -\frac{(v - \nu_k)^2}{2\sigma^2} \right],
\]  
(129)

where
\[
\nu_k = \frac{1}{2} + k
\]  
(130)
with \( k = 0, 1, 2, \ldots, N-1 \). As was done in the Wigner case, we turn the above sum into an infinite sum which can be transformed, after neglecting the rest, into the infinite sum
\[
\rho(v) = 1 + 2 \sum_{m=1}^{\infty} (-1)^m \exp[-2(\pi m \sigma)^2] \cos[2\pi m(v)],
\]  
(131)

using the Poisson sum formula. The dependence of the phase and the variance on their positions along the spectrum is then determined to be given by
\[
\nu(x) = \frac{\theta_N + \theta_{N-1} - \theta}{2\pi}
\]  
(132)
and
\[
\sigma^2(x) = \frac{1}{2\pi^2} \ln \left[ \frac{B(x)}{2\rho_s(x)} \right].
\]  
(133)

At the bulk, these quantities approach their asymptotic expressions \( \nu(x) = \xi(x) \) and
\[
\sigma^2(x) = \frac{3}{2\pi^2} \ln \left[ \frac{2\pi(2x)^{2/3} \rho_{MP}(x)}{x^{1/3}} \right].
\]  
(134)

In figure 12, the individual eigenvalue distributions of \( N = 20 \) Wishart matrices are shown. It is seen that also for this ensemble our formalism based on nearly Gaussian distributions gives an account of the results obtained performing numerical simulations.

For the largest eigenvalue, the prediction is that its distribution follows, for large \( N \), the Tracy–Widom distribution in the scaled variable \[ s = \frac{x - x_r}{x_r^{2/3}(MN)^{-1/6}}. \]  
(135)

In figure 13, the two theoretical predictions are compared with the results from numerical simulations. Finally, in figure 14, the density distributions of the smallest eigenvalue is magnified to show that the nearly Gaussian distribution gives a good fit to the numerical simulation.
4. Concluding remarks

We have performed a decomposition in individual contributions of the exact density of eigenvalues of matrices of the unitary and the orthogonal Gaussian ensembles and of the unitary matrices of the Wishart ensemble. This decomposition works for relatively small values of matrix sizes and shows that eigenvalues are well described by nearly Gaussian distributions. As the matrix sizes increase, these distributions tend, at the bulk of the spectra, to the true Gaussian such that the exact results of [9, 10] are recovered. These results should apply to spectra of systems whose classical analog are chaotic and be experimentally tested. The present analysis should also work for ensembles connected to others orthogonal polynomials.
Recently, invariant non-ergodic ensembles have been introduced whose ensemble densities are averages over the Wigner’s semi-circle [19], in the Gaussian case, and the Marchenko–Pastur density [20], in the Wishart case. In the Gaussian case, the effect of non-ergodicity on the Tracy–Widom distribution has been investigated [21]. It seems possible to extend this study to the individual distributions of the eigenvalues at the spectral bulk. This is interesting in connection with the behavior of the eigenvalues of the embedded ensembles [8, 22]. On the other hand, the high development in the experimental physics achieved in the field of cold atoms should perhaps reach the point in which individual atoms can be observed. In this case, the behavior of each atom of a Girardeau gas can be measured and the decomposition in individual contributions be checked.

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Appendix. Semi-classical wavefunction approximation

Consider a wavefunction $\Psi(x)$ that satisfies the wave equation

$$\frac{d^2 \Psi}{dx^2} + p^2(x)\Psi = 0,$$

(A.1)

where the momentum $p(x)$ is such that $p^2(x) > 0$ in the interval $(x_1, x_2)$. Let us search a solution of (A.1) in this interval of the form

$$\Psi(x) = M(x) \cos \beta(x).$$

(A.2)

Substituting (A.2) in (A.1) and neglecting the second derivative of the modulus, we find that modulus and phase satisfy the equations

$$\beta' = p(x)$$

(A.3)
and

\[ M\beta'' + 2M'\beta' = 0 \] (A.4)

with solutions \( \beta = \int_{x_1}^{x_2} dx \rho(x) + \beta_0 \) and \( M = C^{-1/2} \). Finally the constant \( C \) is fixed by the normalization condition \( \int \Psi^2(x) dx = 1 \) that gives

\[ C = \left[ \frac{1}{2} \int_{x_1}^{x_2} \frac{dx}{p(x)} \right]^{-1/2} = \left[ \frac{1}{2} \int_{x_1}^{x_2} \frac{dx}{p(x)} \right]^{-1/2}, \] (A.5)

where, consistently with the semi-classical method, the oscillating term in the integrand is neglected.

As a second-order differential equation, equation (A.1) has a second independent solution which is the same approximation that we assumed to be given by

\[ \tilde{\Psi}(x) = M(x) \sin \beta(x). \] (A.6)

These two independent solutions satisfy the Wronskian relation

\[ W(\Psi, \tilde{\Psi})(x) = \Psi(x)\tilde{\Psi}'(x) - \Psi'(x)\tilde{\Psi}(x) = C^2 \] (A.7)

in the same order of approximation.

References

[1] Wishart J 1928 Biometrika 20 32
[2] Porter C S 1965 Statistical Theories of Spectra (New York: Academic)
[3] Mehta M L 2004 Random Matrices 3rd edn (Amsterdam: Elsevier)
[4] Girardeau M 1960 J. Math. Phys. 1 516
[5] Kolomeisky E B, Newman T J, Straley J P and Qi X 2000 Phys. Rev. Lett. 85 1146
[6] Tracy C A and Widom H 1994 Commun. Math. Phys. 159 151
Tracy C A and Widom H 1996 Commun. Math. Phys. 177 727
[7] Johansson K 2000 Commun. Math. Phys. 209 437
Johnstone I M 2001 Ann. Stat. 29 295
Forrester P 2006 Nonlinearity 19 2989
[8] Bohigas O and Flores J 1971 Phys. Lett. B 35 383
[9] Gustavsson J 2005 Ann. Inst. H. Poincaré-Prob. Stat. 41 151
[10] O’Rourke S 2009 arXiv:0909.2677v2 [math.PR]
[11] Tao T and Vu V 2009 arXiv:0906.0510 [math.PR]
[12] Dumitriu I and Edelman A 2002 J. Math. Phys. 43 5830
[13] Dumitriu I and Edelman A 2005 Ann. Inst. Henri Poincaré-Prob. Stat. 41 1083
[14] Coles S 2001 An Introduction to Statistical Modeling of Extreme Values (Springer Series in Statistics) (Berlin: Springer)
[15] Weibull W 1951 J. Appl. Mech.-Trans. ASME 18 293
[16] Gumbel E J 1958 Statistics of Extremes (New York: Columbia University Press)
[17] Tracy C A and Widom H 2002 Proc. ICM (Beijing) vol 1 pp 587–96
[18] Marchenko V A and Pastur L A 1967 Math. USSR-Sb. 1 457
[19] Bertuola A C, Bohigas O and Pato M P 2004 Phys. Rev. E 70 065102
Toscano F, Vallejos R O and Tsallis C 2004 Phys. Rev. E 69 066131
Bohigas O, de Carvalho J X and Pato M P 2008 Phys. Rev. E 77 011122
[20] Akemann G and Vivo P 2008 J. Stat. Mech. P09002
Abul-Magd A Y, Akemann G and Vivo P 2009 J. Phys. A: Math. Theor. 42 175207
Akemann G, Fischemann J and Vivo P 2010 Physica A 389 2566
[21] Bohigas O, de Carvalho J X and Pato M P 2009 Phys. Rev. E 79 031117
[22] Asaga T, Benet L, Rupp T and Weidenmueller H A 2001 Europhys. Lett. 56 340