CURVATURE TRANSFORMATION

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Abstract. A transformation based on mean curvature is introduced which morphs triangulated surfaces into round spheres.

1. Introduction

Let $S \subset \mathbb{R}^3$ be a surface embedded in $\mathbb{R}^3$ which is locally defined by the smooth parametrization $\{(x, y, S(x, y)) \mid (x, y) \in U \subset \mathbb{R}^2\} \subset S$. The mean curvature $H(x, y)$ at every point $z = S(x, y)$ is given as the mean of the maximal and minimal principal curvatures:

$$H(x, y) = \frac{1}{2}(\kappa_1 + \kappa_2).$$

The sphere is known to be the unique embedded constant mean curvature (CMC) surface [Ale58], and the unique immersed CMC sphere [Hop83]. For higher genus, there are many constant mean curvature surfaces. The vector mean curvature given by $\nu H(x, y)$ where $\nu$ is the outer unit normal vector at $z = S(x, y)$ is the negative gradient of the area functional of $S$, that is

$$\nu H = -\nabla \text{area}(S).$$

The parametric family $\{S(x, y, t)\}_{t \in \mathbb{R}}$ of surfaces which evolves proportionally to the vector mean curvature, so that the area of $S(t)$ reduces with $t \to \infty$, is called mean curvature flow and is the solution of the following partial differential equation

$$\frac{\partial S(x, y, t)}{\partial t} = -\nu H(x, y, t),$$

where $H(x, y, t)$ is the mean curvature at the point $S(x, y, t)$ of the surface $S(t)$. Huisken [Hui84] proved that if $S(0)$ is a bounded convex surface, then $S(t)$ becomes more and more nearly spherical as it shrinks, and at the instant it vanishes it is asymptotic to the shrinking sphere given above.

It has always been our aim to transform and study polyhedra, triangulated surfaces and meshes in the context of mesh smoothing by elementary geometric means [VAGW08, VWP13, VB14, VH14]. This is the motivation behind studying mean curvature using elementary geometric transformations. Given a triangulated surface $S$, the discrete mean curvature is defined for each edge $e$ by

$$K(e) = l(e)\theta(e),$$

where $\theta(e) \in (-\pi, \pi]$ is the (oriented) dihedral angle between the two adjacent facets and $l(e)$ is the length of the edge. If $n_1$ and $n_2$ are the (oriented) unit surface normals, then $\cos \theta(e) = n_1 \cdot n_2$. For each vertex $p$, its mean curvature $K(p)$ is defined by averaging over the mean curvatures of the neighboring edges. We tried to find a simple transformation based on the discrete mean curvature which morphs a (reasonably shaped) triangulated surface into a round sphere.

2. The transformation

Let $S$ be an oriented, triangulated surface embedded in $\mathbb{R}^3$ given by a set of vertices $V \subset \mathbb{R}^3$ and a set of (unoriented) edges $E \subset V \times V$. Since $S$ is embedded in $\mathbb{R}^3$ there is an inside and and outside, and we choose unit face normals to point outward. The vertex normal $n_p$ at $p$ is computed by averaging the unit face normals of the adjacent faces and then normalizing.
the result. Let $K_{\min}$ and $K_{\max}$ be the minimal and maximal discrete mean curvature for the vertices of an oriented mesh. The transformation consists of applying two steps iteratively, each of which is applied to all points simultaneously.

\[
I_C(p) = p - C \frac{K(p) - K_{\min}}{K_{\max} - K_{\min}} n_p,
\]

\[
O_C(p) = p + C \left(1 - \frac{K(p) - K_{\min}}{K_{\max} - K_{\min}}\right) n_p.
\]

Transformation $I_C$ in (2) moves each point inward, and $O_C$ in (3) moves each point outward. The factors in front of $n_p$ are chosen so that the magnitude of the translating vector is a value in the interval $[0, 1]$, and it is big for small curvature values and vice versa. The dynamic process is determined by the following triple: $(k_{\text{in}}, k_{\text{out}}, C)$. The integers $k_{\text{in}}$ and $k_{\text{out}}$ determine how often the outward resp. inward transformation is applied before the other one is applied. More specifically we consider the transformation

\[
T(k_{\text{in}}, k_{\text{out}}, C) = O_C^{k_{\text{out}}} \circ I_C^{k_{\text{in}}}.
\]

This transformation $T$ will be repeated $n$ times as described in Algorithm 1.

**Algorithm 1** Curvature transformation

```python
function Morph-Step(n, k_{\text{in}}, k_{\text{out}}, C, V)
    for i ← 1, ..., n do
        for j ← 1, ..., |V| do
            p'_j ← T(k_{\text{in}}, k_{\text{out}}, C)(p_j)
        for j ← 1, ..., |V| do
            p_j ← p'_j
    return V
```

We will see in Section 3 that the constant factor $C = 0.25$ makes the transformation process described in Algorithm 2 converge towards a CMC sphere for a variety of surfaces. Even though this behavior is not surprising, it is difficult to find conditions that guarantee convergence and a proof thereof.

**Algorithm 2** Curvature transformation

1: procedure Morph(m, V)
2: for i ← 1, ..., m do
3:     V ← Morph(100, 2, 2, 0, 25, V)
4:     V ← Morph(100, 2, 1, 0, 25, V)

3. Numerical tests

This transformation has been implemented in Python using the mean curvature method provided by the VTK class `vtkCurvatures`. We have tried different parameters and found that the following process works on our examples. The number of iterations mentioned in the figures corresponds to $200 \cdot m$.

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Figure 1. Initial face mesh

Figure 2. Face mesh after 100 (left) and 200 (right) iterations

Figure 3. Face mesh after 1000 iterations
Figure 4. Face mesh after 10000 iterations

Figure 5. Initial stone mesh

Figure 6. Stone mesh after 1500 iterations
Figure 7. Initial puma mesh

Figure 8. Puma mesh after 2000 iterations

Figure 9. Sphere bar mesh initial (left) and after 7500 iterations
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Figure 10. Initial mesh (left) and result (right)

Figure 11. Cube mesh

Figure 12. Cylinder mesh initial (left) and result (right)

Figure 13. Non-convex sphere mesh (left) and result (right)
REFERENCES

[Ale58] A. D. Aleksandrov. Uniqueness theorems for surfaces in the large. V. Vestnik Leningrad. Univ., 13(19):5–8, 1958.

[Hop83] Heinz Hopf. Differential geometry in the large, volume 1000 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1983. Notes taken by Peter Lax and John Gray, With a preface by S. S. Chern.

[Hui84] Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. J. Differential Geom., 20:237–266, 1984.

[VAGW08] Dimitris Vartziotis, Theodoros Athanasiadis, Iraklis Goudas, and Joachim Wipper. Mesh smoothing using the Geometric Element Transformation Method. Comput. Methods Appl. Mech. Engrg., 197(45–48):3760–3767, 2008.

[VB14] Dimitris Vartziotis and Doris Bohnet. Convergence properties of a geometric mesh smoothing algorithm. arXiv:1411.3869 [math.NA], 2014.

[VH14] Dimitris Vartziotis and Benjamin Himpel. Efficient mesh optimization using the gradient flow of the mean volume. SIAM Journal on Numerical Analysis, 52(2):1050–1075, 2014.

[VWP13] Dimitris Vartziotis, Joachim Wipper, and Manolis Papadrakakis. Improving mesh quality and finite element solution accuracy by GETMe smoothing in solving the Poisson equation. Finite Elem. Anal. Des., 66:36–52, 2013.

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