Uniqueness, existence and solution of the direct boundary heat exchange problem for a weakly non-linear temperature conductivity coefficient

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Abstract. This work is a preliminary result necessary for the formulation and solution of the inverse heat conduction problem for a weakly quasilinear equation. Nevertheless, the article has independent significance. For the direct heat conduction problem a solution is constructed in the form of a series with a small parameter, the existence of the solution, its uniqueness are proved, and a number of estimates are made - in particular, the rate of decrease of the solution at large times. The solution is built for a medium, part of which has a piecewise constant coefficient of thermal conductivity, and part of the medium has slightly non-linear thermal diffusivity. The construction of a classical solution is impossible due to the discontinuity of the thermal diffusivity, therefore, on the sections of the media the matching conditions are set - the conditions for the continuity of the solution and the condition for the continuity of the heat flux. The solution to the weakly quasilinear equation is constructed in the explicit form of a series with a small parameter, for which uniform convergence is proved. The decreasing of the solution at large times is proved as of time raised to the negative third power.

1. Introduction
In modern technology there are devices whose details are exposed to thermal effects. The properties of materials may change in parts as a result. In turn, this leads to the failure of the device [3].

You can control the surface temperature to protect components from excessive heat. And since the temperature may be too high for direct measurement [4], it is advisable to be able to solve the inverse problem of thermal diffusivity.

A protective coating can also be applied. The coating will protect the device for a while, but will make it difficult to solve the inverse problem posed already for the composite material.

Nonlinear properties of materials will add additional complexity to the solution of the inverse problem [2]. This complexity is conveniently taken into account either using the small parameter method, as in the present paper, or by the iteration method.

The above difficulties of setting and solving a mathematical problem can be overcome, for example, as follows: the thermal diffusivity for one of the regions is considered separately as a weakly non-linear function. A classic solution is available for each of the areas. It is impossible to build a general classical solution due to discontinuities in the thermal diffusivity; therefore, the solutions that are classic within each of the areas are combined into a common one by setting matching conditions on media sections.
It is easy to see that constructing a solution to even a direct problem is cumbersome and requires considerable preliminary work. We propose to divide the solution of the inverse quasilinear problem into:

(i) constructing the solution of the direct problem, proving the convergence of the method to the solution and the uniqueness of the solution thus constructed, substantiating a number of estimates that will be required to solve the inverse heat conduction problem itself;

(ii) construction of a solution of the inverse problem itself, necessary estimates and proofs [5], [6], [7], [8], [9].

We propose to consider the first part of the problem in this article.

2. Statement of the direct problem

Consider the area in which we solve the problem. Let \( x \in [0; +\infty) \) be the spatial coordinate, \( t \in [0; +\infty) \) be the temporal coordinate.

Let the spatial region be divided into four parts. We denote the temperature in each of the spatial regions as \( u_1(x, t) \), \( u_2(x, t) \), \( u_3(x, t) \), \( u_4(x, t) \). In the first part, \( x \in [0; x_0] \), the medium heats up, the heating function is \( F(x, t) \), and the thermal diffusivity \( k_1(x, t) = a^2_1 \) is constant. The protective layer is heated from the wall in the second region, \( x \in [x_0; x_1] \). Its thermal diffusivity \( k_2(x, t) = a^2_2 \) is constant. \( x \in [x_1; x_2] \) is the third area. The material itself is heated in it, and its thermal diffusivity coefficient weakly non-linearly depends on the temperature \( k_3(x, t) = a^2_3(1 + \epsilon u_3(x, t)) \). The thermal diffusivity is constant: \( k_4(x, t) = a^2_4 \) in the fourth region \( x \in [x_2; +\infty) \). Here \( a_1, a_2, a_3, a_4 \) are real numbers, \( 0 < \epsilon \ll 1 \) is a small parameter.

Let \( u(x, t) \) be such that

\[
\begin{align*}
    u(x, t) &= \begin{cases} 
        u_1(x, t), & x \in [0; x_0], \ t \in [0; +\infty), \\
        u_2(x, t), & x \in [x_0; x_1], \ t \in [0; +\infty), \\
        u_3(x, t), & x \in [x_1; x_2], \ t \in [0; +\infty), \\
        u_4(x, t), & x \in [x_2; +\infty), \ t \in [0; +\infty). 
    \end{cases}
\end{align*}
\]

Let \( \Delta'_4 \) be such that

\[
\begin{align*}
    \Delta'_4 u(x, t) &= \begin{cases} 
        \frac{a_1^2 \partial^2 u_1(x, t)}{\partial x^2}, & x \in (0; x_0), \ t \in (0; +\infty), \\
        \frac{a_2^2 \partial^2 u_2(x, t)}{\partial x^2}, & x \in (x_0; x_1), \ t \in (0; +\infty), \\
        \frac{a_3^2 \partial}{\partial x} \left( 1 + \epsilon u_3(x, t) \right) \frac{\partial u_3(x, t)}{\partial x}, & x \in (x_1; x_2), \ t \in (0; +\infty), \\
        \frac{a_4^2 \partial^2 u_4(x, t)}{\partial x^2}, & x \in (x_2; +\infty), \ t \in (0; +\infty). 
    \end{cases}
\end{align*}
\]

Let the heat flux operator be \( D'_4 \):

\[
\begin{align*}
    D'_4 u(x, t) &= \begin{cases} 
        \frac{\partial u_1(x, t)}{\partial x}, & x \in (0; x_0), \ t \in (0; +\infty), \\
        \frac{\partial u_2(x, t)}{\partial x}, & x \in (x_0; x_1), \ t \in (0; +\infty), \\
        a_3 (1 + \epsilon u_3(x, t)) \frac{\partial u_3(x, t)}{\partial x}, & x \in (x_1; x_2), \ t \in (0; +\infty), \\
        \frac{\partial u_4(x, t)}{\partial x}, & x \in (x_2; +\infty), \ t \in (0; +\infty). 
    \end{cases}
\end{align*}
\]
Then the problem is
\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \Delta u(x,t) + F(x,t), x \in (0; +\infty), t \in (0; +\infty), \\
\left\{ \begin{array}{l}
u_1(0,t) = 0, t \in (0; +\infty), \\
u_4(+\infty,t) = 0, t \in (0; +\infty), \\
u(x,0) = 0, x \in [0; +\infty).
\end{array} \right.
\end{aligned}
\tag{2}
\]

**Definition** The function \(u(x,t)\) will be a solution to the problem (2) if:

(i) \(u(x,t) \in C(x \in [0; +\infty), t \in [0; +\infty))\),

(ii) \(D^2_t u(x,t) \in C(x \in (0; +\infty), t \in [0; +\infty))\),

(iii) \(u(x,t) \in C^{(2,1)}(x \in (0;x_0) \cup (x_0;x_1) \cup (x_1;x_2) \cup (x_2; +\infty), t > 0)\),

(iv) \(u(x,t)\) satisfies all the relations (2).

Choose a function \(F(x,t)\) in (2) as follows:
\[
F(x,t) = \begin{cases} 
F_0(t), & t \geq 0, \quad x \in (0; x_0) \\
0, & t \geq 0, \quad x \in [x_0; +\infty), \\
F_0(t) \in C(t > 0), & \exists t_0 > 0, \quad \forall t > t_0 : F_0(t) = 0, \\
\exists A_1 > 0, \quad \forall t > 0 : |F_0(t)| \leq A_1.
\end{cases}
\tag{3}
\]

3. Formal solution to the problem (2-3)

We construct a solution to problem (2-3) in the form of a formal power series:
\[
u_k(x,t) = \sum_{j=0}^{\infty} \epsilon^j u_k^{(j)}(x,t), k = 1, 2, 3, 4, x \in (s_k; s_{k+1}), t \in (0; +\infty),
\]
\[
s_1 = 0, s_2 = x_0, s_3 = x_1, s_4 = x_2, s_5 = +\infty.
\tag{4}
\]

**Lemma 3.1.** Let the conditions be satisfied:

(i) The terms of the (4) \(u_k^{(j)}(x,t) \in C^2(x \in (s_k; s_{k+1})), C^1(t \in (0; +\infty)) \forall k = 1, 2, 3, 4.

(ii) Functional series (4) converges uniformly in \(\forall x \in [0; +\infty), \quad t \geq 0\).

(iii) Functional series (4) is a solution to (2-3).

(iv) The series of by term differentiation (4) \(\sum_{j=0}^{\infty} \epsilon^j \frac{\partial u_k^{(j)}(x,t)}{\partial x}, k = 1, 2, 3, 4, \) converges uniformly in \(\forall x \in (0; +\infty), \quad t \geq 0\).

(v) A series of second derivatives (4) \(\sum_{j=0}^{\infty} \epsilon^j \frac{\partial^2 u_k^{(j)}(x,t)}{\partial x^2}, k = 1, 2, 3, 4, \) converges uniformly in \(\forall x \in (s_0; s_1) \cup (s_1; s_2) \cup (s_2; s_3) \cup (s_3; s_4) \cup (s_4; s_5), \quad t > 0\).

(vi) The series
\[
\sum_{i,j=0}^{\infty} \epsilon^i \epsilon^j u_3^{(i)}(x,t) \frac{\partial u_3^{(i)}}{\partial x}
\tag{5}
\]

converges uniformly in \(\forall x \in (s_0; s_1) \cup (s_1; s_2) \cup (s_2; s_3) \cup (s_3; s_4) \cup (s_4; s_5), \quad t > 0\) as well as its derivatives series.
The series (4) is a solution to (2-3).

Lemma 3.4. Let the following conditions be satisfied:

3.1. The uniform continuity of the series

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Lemma 3.2. Let the series (4)

Then the heat equation (2) can be written as

\[ u(x,t) = e^{a_3 x} + a_4 e^{-a_3 x} \]

Lemma 3.3. Let the following conditions be satisfied:

(i) \( u_k^{(j)}(x,t) \in C^1(x \in (0; +\infty), t \in (0; +\infty)), k = 1, 2, 3, 4, j = 0, 1, 2, \ldots\).

(ii) Series (4), as well as a series of its term derivatives in coordinate, converges uniformly in

\[ \forall x \in (0; x_0) \cup (x_0; x_1) \cup (x_1; x_2) \cup (x_2; +\infty), \ t \in (0; +\infty)), k = 1, 2, 3, 4. \]

(iii) The series (4) is a solution to (2-3).

Then the continuity condition for the heat flux \( D_4 u(x,t) \in C(x \in (0; +\infty), t \in (0; +\infty)) \) is:

\[
\begin{align*}
    a_1 \frac{\partial u_1^{(j)}(x_0,t)}{\partial x} &= a_2 \frac{\partial u_2^{(j)}(x_0,t)}{\partial x}, \ t > 0, j = 0, 1, 2, \ldots, \\
    a_2 \frac{\partial u_2^{(j)}(x_1,t)}{\partial x} &= a_3 \frac{\partial u_3^{(j)}(x_1,t)}{\partial x} + a_4 G_j(x_1,t), \ t > 0, j = 0, 1, 2, \ldots, \\
    a_3 \frac{\partial u_3^{(j)}(x_2,t)}{\partial x} + a_4 G_j(x_2,t) &= a_4 \frac{\partial u_4^{(j)}(x_2,t)}{\partial x}, \ t > 0, j = 0, 1, 2, \ldots,
\end{align*}
\]

where \( G_j(x,t) \) is defined by (7).

Therefore, we have to prove:

(i) The uniform converges of (4) in \( x > 0, t > 0 \),

(ii) The uniform converges of term derivatives (first and second order) series (4) in \( x > 0, t > 0 \),

(iii) The uniform converges of (7) and the uniform converges of series term derivatives (7).

3.1. The uniform continuity of the series

Lemma 3.4. Let the following conditions be satisfied:

(i) Function \( F_2(x,t) \) in \( x \in [a;b], t \in [0;+\infty) \) is determined by

\[
F_2(x,t) = \int_0^t \frac{d\tau}{2a_3 \sqrt{\pi(t-\tau)}} \int_a^b \exp \left( -\frac{(x-s)^2}{4a_3^2(t-\tau)} \right) F_1(s,\tau) ds,
\]
We get zero, going to the limit at $t \to 0$.

**Proof.** We calculate the first internal integral in (10) to prove the first part.

Differentiate $J_3(x, t)$ with respect to $t$: 

$$\frac{\partial J_3(x, t)}{\partial t} = \int_0^t F_0(\tau)(x - s_1) d\tau \left\{ e^{-\left(\frac{(x-s_1)^2}{4a_3^2(\tau-t)}\right)} - e^{-\left(\frac{(x-s_0)^2}{4a_3^2(\tau-t)}\right)} \right\}, x \in (s_2; s_3), t \geq 0,$$

The integrand is continuous for every $\tau \in [0; t)$. We get zero passing to the limit as $t \to 0+0$. Note that due to the finite integration interval over $t$ and the continuity of the integrand, $\forall \tau \in (0; t)$ this derivative is uniformly bounded, that is, a continuous function in $x \in (s_2; s_3), t \geq 0$.

We come to the conclusion, repeating the differentiation, that there are every order derivatives in $t$ for the function $J_3(x, t)$. It is easy to see that similar considerations are true for $J_4(x, t)$.

Thus, the first two statements of the lemma are true.
Let’s prove the third statement of the lemma. To do this, we estimate the derivatives found for sufficiently large $t$. Select the first term in the expression for $J_3(x, t)$ and denote its kernel for the integral over $t$ as $J_{10}(x, t)$. Then for its derivative of order $k$ we get:

$$\frac{\partial^k J_{10}(x, t)}{\partial t^k} = \exp \left( -\frac{(x-s_1)^2}{2a_3\sqrt{t-\tau}} \right) \left( -1 \right)^k F_0(\tau) \left( \frac{(-1)^{2k+1}}{(2a_3(t-\tau))^k} \right) + \ldots + \left( \frac{(2k+1)!!}{2^k} \right).$$

If $t \to +\infty$, then the largest term is $(2k+1)!!/2^k$, because $0 \leq \tau \leq t_0$.

Therefore,

$$\left| \frac{\partial^k J_1}{\partial t^k} \right| \leq \exp \left( -\frac{(x-s_1)^2}{2a_3\sqrt{t-\tau}} \right) F_0(\tau) \left( \frac{2(2k+1)!!}{(2k)^k} \right).$$

Let $\tau \in [0; t_0]$. If $x \in [s_2; s_3]$ and $|F_0(\tau)| \leq A_1$,

$$\int_0^{t_0} \frac{(x-s_1)}{a_3^2\sqrt{t-\tau}} \exp \left( -\frac{(x-s_1)^2}{2a_3\sqrt{t-\tau}} \right) F_0(\tau) d\tau \leq A_1 \frac{2(2k+1)!}{2^k (2a_3(t-t_0))^{k-0.5}}.$$  

We obtain an upper bound, repeating the argument for the second term in $J_3$, and then for $J_4$.

Thus $\frac{\partial^k u_3^{(0)}(x, t)}{\partial t^k}$ in $x \in (s_2; s_3)$, $t > 0$ is uniformly bounded, and the third statement is true. \hfill $\Box$

**Lemma 3.6.** $u_3^{(0)}(x, t) \in C^\infty(t \in (0; +\infty), x \in (0; +\infty)$.

Now let’s build a solution to the problem (2-3). Let a new function $w^{(k)}(x, t)$ be

$$\forall k = 1, 2, 3, \ldots :$$

$$w^{(k)}_1(x, t) = u^{(k)}_1(x, t), \quad x \in [s_1; s_2], \ t > 0,$$

$$w^{(k)}_2(x, t) = u^{(k)}_2(x, t), \quad x \in [s_2; s_3], \ t > 0,$$

$$w^{(k)}_3(x, t) = u^{(k)}_3(x, t) + \frac{(x-s_1)^2(x-s_4)G_{k_1}(x, t)}{(s_4-x)^2} \frac{x-s_2}{(s_3-x)^2}, \ x \in [s_3; s_4], \ t > 0,$$

$$w^{(k)}_4(x, t) = u^{(k)}_4(x, t), \quad x \in [s_2; s_3], \ t > 0.$$

**Lemma 3.7.** Let the function $F(x, t)$ in (2) satisfy the condition (3). Then the solution to (2-3) is:

$$w^{(j)}_1(x, t) = \int_0^t \frac{d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_{s_3}^{s_4} G_{j}(s, \tau) \left\{ -\frac{(k_2x-x_2)^2}{4a_3^2(t-\tau)} - e^{-\frac{(k_3x-x_3)^2}{4a_3^2(t-\tau)}} \right\} ds, x \in [s_1; s_2], t > 0,$$

$$w^{(j)}_2(x, t) = \int_0^t \frac{d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_{s_3}^{s_4} G_{j}(s, \tau) \left\{ -\frac{(k_2x-x_2)^2}{4a_3^2(t-\tau)} - e^{-\frac{(k_3x-x_3)^2}{4a_3^2(t-\tau)}} \right\} ds, x \in [s_2; s_3], t > 0,$$

$$w^{(j)}_3(x, t) = \int_0^t \frac{d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_{s_3}^{s_4} G_{j}(s, \tau) \left\{ -\frac{(k_2x-x_2)^2}{4a_3^2(t-\tau)} - e^{-\frac{(k_3x-x_3)^2}{4a_3^2(t-\tau)}} \right\} ds, x \in [s_3; s_4], t > 0,$$

$$w^{(j)}_4(x, t) = \int_0^t \frac{d\tau}{2a_3\sqrt{\pi(t-\tau)}} \int_{s_3}^{s_4} G_{j}(s, \tau) \left\{ -\frac{(k_2x-x_2)^2}{4a_3^2(t-\tau)} - e^{-\frac{(k_3x-x_3)^2}{4a_3^2(t-\tau)}} \right\} ds, x \in [s_4; s_5], t > 0,$$

where

$$k_1 = \frac{a_3}{a_1}, \quad k_2 = \frac{a_3}{a_2}, \quad k_3 = \frac{a_3}{a_4}, \quad k_4 = \frac{a_3}{a_2},$$

$$x_2 = (k_1 - k_2)x_0, \quad x_3 = x_2 + (k_2 - 1)x_1, \quad x_4 = x_3 + (1 - k_4)x_2,$$
\[
\tilde{G}_j(x, t) = 2a_3^3 \partial_x \left\{ G_j(x, t) + \frac{(s_4 - x)^2(x - s_3)G_j(s_3, t)}{(s_4 - s_3)^3} + \frac{(x - s_3)^2(x - s_4)G_j(s_4, t)}{(s_4 - s_3)^3} \right\}.
\]

**Theorem 3.8.** Let the function \( F(x, t) \in (s_2; s_3) \), \( t > 0 \) of the problem (2). Then the solution to the problem (2-3) \( \forall j = 0, 1, \ldots \) is given by (12). The solution is twice continuously differentiable \( \forall x \in (s_1; s_2), (s_2; s_3), (s_3; s_4), (s_4; s_5), \forall t > 0 \), is continuously adjacent to the boundary condition and satisfies all the matching conditions of Lemma 2 and Lemma 7.

### 3.2. Estimates and Convergence of the Series (4)

We now consider the questions of convergence of the series (4). To do this, we need to evaluate its terms, each of which, in turn, is a sum (7).

**Lemma 3.9.** Let \( b_0 = 1, b_{j+1} = \sum_{k=0}^{j} b_k b_{j-k} \). Then \( \forall j \) \( b_j \leq 4b_{j-1} \).

Thus, if \( \exists A_5 > 0, \forall j = 1, 2, 3, \ldots \max_{t > 0, x \in (0; +\infty)} |\partial G_j(x, t)/\partial x| < A_5 j \) is true, then series (3) will converge uniformly \( \forall t, 0 \leq t < 1/(4A_5) \).

**Lemma 3.10.** Suppose that the conditions (3) are satisfied in the problem (2), and the function \( W(x, t) \) is defined as follows:

\[
W(x, t) = \int_{0}^{t} \frac{F_0(\tau)d\tau}{2a_3^3 \pi(t-\tau)} \int_{s_3}^{s_4} \left\{ e^{-\frac{(x-s)^2}{4A_5^2(t-\tau)}} - e^{-\frac{(x-s)^2}{4A_5^2(t-\tau)}} \right\} ds.
\]

Then

(i) \( W(x, t) \in C^{(2,1)}(x \in (s_3; s_4), t \in (0; +\infty)) \),

(ii) \( W(x, 0) = 0 \) \( \forall x \in (s_3; s_4) \),

(iii) \( \exists t_1 > 0 \) \( \forall t > t_1, \exists A_8 > 0, |W(x, t)| \leq \frac{A_8}{t^{3/2}} \) \( \forall x \in (s_3; s_4) \),

(iv) \( \exists t_1 > 0 \) \( \forall t > t_1, \exists A_9 > 0, \)

\[
\frac{\partial}{\partial x} W(x, t) \frac{\partial W(x, t)}{\partial x} \leq \frac{A_9}{t^{3/2}}.
\]

**Lemma 3.11.** Let function \( W(x, t) \) satisfy the conditions:

(i) \( W(x, t) \in C(x \in (s_3; s_4), t > 0) \),

(ii) \( W(x, 0) = 0 \) \( \forall x \in (s_3; s_4) \),

(iii) \( \exists t_2 > 0, \forall t > t_2, \exists A_{10} > 0 |W(x, t)| \leq \frac{A_{10}}{t^{3/2}} \) \( \forall x \in (s_3; s_4) \).

Then \( \forall t \geq 0 \) \( \exists f_0(t) \geq 0 \) an integrable function that \( \forall t, \forall x \in (s_3; s_4) |W(x, t)| \leq f_0(t) \), and \( f_0(t) \) satisfies the conditions:

(i) \( f_0(0) = 0 \),

(ii) \( \exists A_{11} > 0, \forall t \geq 0 |f_0(t)| \leq A_{11} \),

(iii) \( \exists t_2 > 0, \forall t > t_2 |f_0(t)| \leq \frac{A_{12}}{t^{3/2}} \).

**Proof.** Let

\[
f_0(0) = \begin{cases} 0, & t = 0, \\ A_{11}, & t \in (0; t_2), \\ A_{10}/t^{3/2}, & t \geq t_2. \end{cases}
\]

The constructed function is integrable and satisfies all conditions.
Lemma 3.12. Let the function $f_0(t)$ be determined by (14). Then

$$U(x, t) = \int_0^t \frac{f_0(\tau) d\tau}{2a_3\sqrt{\pi(t-\tau)}} \left\{ \frac{-(x-s)^2}{4a_3^2(t-\tau)} - \frac{-(x-s)^2}{4a_3^2(t-\tau)} \right\} ds \leq A_{11} \frac{d^2}{dx^2} \int_0^t \frac{d\tau}{2a_3\sqrt{\pi(t-\tau)}} \left\{ \frac{-(x-s)^2}{4a_3^2(t-\tau)} - \frac{-(x-s)^2}{4a_3^2(t-\tau)} \right\} ds.$$  \hspace{1cm} (15)

Lemma 3.13. Let

$$U(x, t) = \int_0^t \frac{d\tau}{2a_3\sqrt{\pi(t-\tau)}} \left\{ \frac{-(x-s)^2}{4a_3^2(t-\tau)} - \frac{-(x-s)^2}{4a_3^2(t-\tau)} \right\} ds,$$

then $\exists t_3 > 0 \forall t > 0 \exists A_{13} > 0 \ |U(x, t)| \leq \frac{s_3^2-s_4^2}{2a_3^2} \times \frac{t^2}{4a_3^2\sqrt{\pi t}} \forall x \in (s_3; s_4)$.

Proof. The proof is carried out by direct calculation. \hfill \Box

Lemma 3.14. Let the function $U(x, t)$ satisfy the condition (15). Then $\exists t_2 > 0 \forall t > t_2 \exists A_{14} > 0$, that $|U(x, t)| < \frac{A_{14}}{t^3} \forall x \in (s_3; s_4)$.

Proof. The proof follows from the lemmas (3.12) and (3.13). \hfill \Box

The following theorem is true:

Theorem 3.15. Suppose that the conditions (3) are true in the problem (2). Then $\forall j = 1, 2, 3, \ldots$ the functions of $G_j(x, t)$ have the following properties:

(i) $\exists A_{14} > 0 \ |G_j(x, t)| < A_{14} \forall t > 0, x \in (s_3; s_4)$,

(ii) $\exists t_2 > 0 \forall t > t_2 \exists A_{15} > 0 \ |G_j(x, t)| \leq \frac{A_{15}}{t^3} \forall x \in (s_3; s_4)$.

Proof. The boundedness of each of $G_j(x, t)$ follows from their continuity and that each of its terms decreases at infinity as $t^{-3}$.

We can achieve the conditions of Lemma 8 by choosing $\epsilon$ such that each of the sums $G_j$ components would be less than 1. And this means the uniform convergence of $G_j$ series. \hfill \Box

Thus, the constructed series converges uniformly and decreases at $t \to +\infty$ as $t^{-3}$.

4. Conclusion: the existence and uniqueness of a solution

Theorem 4.1. Suppose that the conditions (3) are true in the problem (2). Then solution to (2) exists and is unique.

Proof. The existence of a solution is proved by the construction of the solution. Let us prove the uniqueness of the solution to the problem.

Suppose that the solution (2-3) is not unique: let there exist two solutions (2-2), $v(x, t)$ and $u(x, t)$. Let their difference $z(x, t) = u(x, t) - v(x, t)$. Then for $z(x, t)$ the problem (2-3) takes the form:

$$\begin{align*}
\frac{\partial z(x, t)}{\partial t} &= \Delta_0 z(x, t) + \frac{\epsilon}{2} D_1^0 \left\{ z(x, t)(u(x, t) + v(x, t)) \right\}, x \in (0; +\infty), t \in (0; +\infty), \\
z_1(0, t) &= 0, t \in (0; +\infty), z_4(+\infty, t) = 0, t \in (0; +\infty), \\
z(x, 0) &= 0, x \in [0; +\infty). \hspace{1cm} (16)
\end{align*}$$
We construct a solution to the problem (16) in the form of a series similar to (4):

$$z(x, t) = \sum_{j=0}^{\infty} \epsilon^j z^{(j)}(x, t).$$  \hfill (17)

Substituting (17) into (16) and combining the same degrees of $\epsilon$ in one equation, we obtain for the zero degree $\epsilon^0$:

$$\begin{aligned}
\frac{\partial z^{(0)}(x, t)}{\partial t} &= \Delta_4 z^{(0)}(x, t), x \in (0; +\infty), t \in (0; +\infty), \\
z^{(0)}(0, t) &= 0, t \in (0; +\infty), z^{(0)}_4(+\infty, t) = 0, t \in (0; +\infty), \\
z^{(0)}(x, 0) &= 0, x \in [0; +\infty).
\end{aligned}$$  \hfill (18)

It was previously proved that the solution (18) $z^{(0)}(x, t) \in C^{(2,1)}(x \in (s_0; s_1) \cup (s_1; s_2) \cup (s_2; s_3) \cup (s_3; s_4) \cup (s_4; s_5), t > 0)$ and that its second derivative decreases at $t \to \infty$ as $1/t^3$, and the solution itself decreases as $1/t^2$. Therefore, $z^{(0)}(x, t)$ allows the Fourier transform:

$$\hat{z}^{(0)}(x, \tau) = \int_{-\infty}^{+\infty} z^{(0)}(x, t)e^{-i\tau t}dt,$$

where $z^{(0)}(x, t)$ is extended by zero to the region of $t < 0$, $\tau \in (-\infty; +\infty)$.

We can transform the whole equation (18) in time using the Fourier transform. The result is the ratio:

$$\begin{aligned}
-ix \hat{z}^{(0)}(x, \tau) &= \Delta_4 \hat{z}^{(0)}(x, \tau), x \in (0; +\infty), \tau \in (-\infty; +\infty), \\
\hat{z}^{(0)}_4(0, \tau) &= 0, \tau \in (-\infty; +\infty), \hat{z}^{(0)}_4(+\infty, \tau) = 0, \tau \in (-\infty; +\infty).
\end{aligned}$$

This problem is an ordinary differential linear homogeneous equation, the solution of which is trivial. By the one-to-one correspondence between the function and its Fourier transform, we obtain that $z^{(0)}(x, t) = 0$ $\forall x \in (0; +\infty), t > 0$.

Now consider the relations for $\epsilon^1$:

$$\begin{aligned}
\frac{\partial z^{(1)}(x, t)}{\partial t} &= \Delta_4 z^{(1)}(x, t), x \in (0; +\infty), t \in (0; +\infty), \\
z^{(1)}(0, t) &= 0, t \in (0; +\infty), z^{(1)}_4(+\infty, t) = 0, t \in (0; +\infty), \\
z^{(1)}(x, 0) &= 0, x \in [0; +\infty).
\end{aligned}$$

We conclude that $z^{(1)}(x, t) = 0$ $\forall x \in (0; +\infty), t > 0$, by repeating the above reasoning.

Note that the same reasoning will be valid for all the subsequent degrees of $\epsilon$.

Thus, we get that the difference of two arbitrary solutions can only be zero. Therefore, the solution is unique, and the theorem is true. \hfill \Box

**Acknowledgments**

The work was supported by the Ministry of Science and Higher Education of the Russian Federation (government order FENU-2020-0022).

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