

Quantum Theory of Generation of Coherent X-Ray in a Wiggler

H.K. Avetissian, T. R. Hovhannisyan, G.F. Mkrtchian
Department of Quantum Electronics, Plasma Physics Laboratory, Yerevan State University
I. A. Manukian, Yerevan 375025, Armenia
E-mail: avetissian@ysu.am

In this paper we present the nonlinear quantum theory of X-Ray FEL in a wiggler. We present the solution of the Dirac equation in a space periodic strong magnetic field, which describes the quantum dynamics of a single electron in a wiggler. With the help of obtained wave function of an electron the self-consistent set of the Maxwell and relativistic quantum kinetic equations is obtained. Then, the process of coherent X-ray radiation generation in nonlinear quantum regime, when the photon energy is larger than resonance widths due to energetic/angular spreads of an electron beam and resonance width caused by finite interaction length, is investigated.

PACS number(s): 41.60.Cr, 42.55.Vc, 42.50.Fx

I. INTRODUCTION

Many types of stimulated radiation sources have been suggested over the years to achieve lasing in shortwave domain. A promising way for realization of X-ray laser is connected with the version of free electron laser (FEL) [1]. The main advantage of FEL lies in the fact that the emission frequency is continuously Doppler upshifted by several orders (∼ γ^2, γ being Lorenz factor) with respect to the frequency of the pump field. In particular, the X-ray FEL schemes have been proposed based on the coherent accumulation of ultrarelativistic electron beam radiation in undulator and at Compton-backscattering, channeling, transition and diffraction radiations [2]. At the present, however, among these versions the undulator scheme is actually developed and the first lasing has been carried out in this system [3]. Although the amplifying frequencies are still far from X-ray, the main hopes for an efficient X-ray FEL remain connected with the undulator scheme. The absence of the normal-incidence mirrors of high reflectivity at the short wavelengths in order of X-ray practically excepts a resonator scheme of radiation generation. In this case it is necessary to implement a single pass high gain FEL. The most attractive scheme, which is presently considered, is the Self Amplified Spontaneous Emission [4], where the spontaneous undulator radiation from the first part of an undulator is used as an input signal to the downstream part. For this purpose two international projects [5] TESLA and LCLS being currently implemented.

In the more conventional undulator devices to achieve X-ray frequencies domain one should increase the electrons energies up to several GeV, which in turn significantly reduces the small-signal gain (∼ γ^{-3}).

In contrast with conventional laser devices on atomic systems FEL is usually reckoned as a classical device. But this is not universal property of FEL as in some cases the quantum effects may play significant role, especially for X-ray frequencies. In the quantum description [6], the resonant momenta of an electron for the emission p_e and absorption p_a are different due to a quantum recoil. The probability distributions of emission and absorption are centered at p_e and p_a respectively and when these distributions much narrower than the spread of electron beam distributions f(p), the small-signal gain is proportional to the so called “population inversion” f(p_e) − f(p_a). In the quasiclassical limit when amplifying photon energy ℏω fulfills condition

\[ ℏω < < \max \{ Δεγ, Δεθ, ΔεL \} \]

where Δεγ, Δεθ, are the resonance widths due to energetic and angular spreads, and ΔεL resonance width caused by finite interaction length the quantum expression for the gain coincides with its classical counterpart being antisymmetric about the classical resonant momentum p_{ce}(p_e + p_a)/2 and proportional to the derivative of the momentum distribution function df(p)/dp at p = p_e. Resulting gain takes place only if the initial momentum distribution is centered above p_e as the electrons whose momenta are above p_e contribute on average to the small-signal gain, and those whose momenta are below p_e contribute on average to the corresponding loss. This severely limits the FEL gain performance at short wavelengths.

The efficiency of FEL at short wavelengths can be significantly increased in the quantum regime of generation:

\[ ℏω ≥ \max \{ Δεγ, Δεθ, ΔεL \} \]

In this case the absorption and emission lineshapes are separated and the simultaneous absorption of probe wave is excluded. To achieve the condition (1.2) for FEL operation is problematic as it presumes severe restrictions on the beam spreads. In any case, it may be satisfied at the emission of hard X-ray quanta.
In this work the scheme of X-ray coherent radiation generation in nonlinear quantum regime by means of relativistic high density electron beam in wiggler is investigated. The possibility to achieve quantum regime of FEL at high harmonics of Doppler-shifted “wiggler frequencies” is treated. The consideration is based on the self-consistent set of the Maxwell and relativistic quantum kinetic equations. Because of nonequidistantness of the energy-momentum levels the probe wave resonantly couples only two electrons states in wiggler and the coupled equations are solved in the slow varying envelope approximation.

This paper is organized as follows. In Section II we obtain the wave function of an electron in a wiggler. Section III describes our model with the self-consistent set of equations. The particular solutions of self-consistent set of equations for the X-ray generation are discussed in Sec. IV. Finally, conclusions are given in Sec. V.

II. WAVE FUNCTION OF AN ELECTRON IN A WIGGLER

For the quantum description of FEL dynamics we will need the wave function of an electron in a wiggler. We will consider as linear (LW) as well as helical Wiggles (HW). Here and in what follows for the four-component vectors we have chosen the following metrics

$$a = (a_0, a)$$

$ax ≡ a_\mu x_\mu = a_0 x_0 - ax.$

To describe the magnetic field of a Wiggler we will choose the following four-vector potential

$$A_H^\mu = (0, A_H), \quad (2.1)$$

where

$$A_H = (A_0 \cos(-k_0 x), g A_0 \sin(-k_0 x), 0), \quad (2.2)$$

$x = (ct, r)$ is the four-component radius vector and

$$k_0 ≡ (0, k_0) = (0, 0, 0, \frac{2\pi}{\ell}), \quad (2.3)$$

with the Wiggler step $\ell$. In (2.2) $g = \pm 1$ correspond to HW, while $g = 0$ corresponds to LW.

The dynamics of an electron in Wiggler can be described by the Dirac equation, which in the quadratic form [7], taking into account (2.2), is the following

$$\left\{ \left( i \hbar \partial_\mu + \frac{e}{c} A_H^\mu \right)^2 - m^2 c^2 + \frac{e}{c} \hat{\Sigma} H \right\} \psi = 0, \quad (2.4)$$

where $\hbar$ is the Plank constant $m$ and $e$ are the particle mass and charge respectively, $c$ is the light speed in vacuum and $\partial_\mu \equiv \partial/\partial x^\mu$ denotes the first derivative of a function with respect to four-component radius vector $x$. Here

$$\hat{\Sigma} = \left( \begin{array}{cc} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{array} \right) \quad (2.5)$$

is the spin operator with the $\hat{\sigma}$ Pauli matrices and

$$H = \text{rot} A_H \quad (2.6)$$

is the magnetic field of a Wiggler.

As the magnetic field depends only on the $\tau = -k_0 x = k_0 r$ then raising from the symmetry, we seek a solution of Eq. (2.4) in the form

$$\psi(x) = f(\tau) \exp \left[ -i \frac{p}{\hbar} x \right], \quad (2.7)$$

where $p = (\varepsilon/c, p)$ is the four-momentum of a free Dirac particle.

To solve Eq. (2.4) we will consider $f(\tau)$ as a slowly varying bispinor function of $\tau$ (in the scale of $pk_0/(\hbar k_0^2)$ and neglect the second derivative compared with the first order (see condition (2.10)). So from (2.4) and (2.7) for $f(\tau)$ we will have the following equation:
\[-2i\hbar(pk_0) \frac{df(\tau)}{d\tau} + \left\{ -\frac{2e(pA_H)}{c} + \frac{e^2A_H^2}{c^2} + \frac{e}{c} \tilde{\Sigma}H \right\} f(\tau) = 0. \quad (2.8)\]

The solution of Eq. (2.8) we can write in the operator form

\[f(\tau) = \exp \left\{ \frac{i}{\hbar c(pk_0)} \int_0^\tau \left( e(pA_H) - \frac{e}{2c}A_H^2 \right) d\tau' - \frac{ie}{2c(pk_0)} \tilde{\Sigma} [k_0A_H], \right\} w(p) \quad (2.9)\]

where \(w(p)\) is the arbitrary bispinor amplitude. Then taking into account the property of spin operator

\[\exp \left[ \tilde{\Sigma}a \right] = \frac{1}{2} (\exp(a) + \exp(-a)) + \tilde{\Sigma}a \frac{1}{2a} (\exp(a) - \exp(-a))\]

and putting condition \(a << 1\), which in our case restricts the magnetic field strength by the condition

\[K = \frac{eA_0}{mc^2} = \frac{eH_\ell}{2\pi mc^2} << \gamma L. \quad (2.10)\]

Here \(K\) is the so called Wiggler parameter.

Hence, for the wave function we will have the following expression

\[\psi(x) = \left[ 1 + \frac{\hat{e}k_0A_H}{2c(k_0p)} \right] w(p) \exp \left\{ -\frac{i}{\hbar} \left\{ px - \frac{i}{\hbar c(pk_0)} \int_0^\tau \left( e(pA_H) - \frac{e}{2c}A_H^2 \right) d\tau' \right\} \right\}. \quad (2.11)\]

Here we have introduced the following notation \(\hat{a} = a^\dagger \gamma_\mu\), where \(\gamma^\mu = (\gamma_0, \gamma)\) are Dirac matrices. Note that (2.10) is also the condition of slowly varying function \(f(\tau)\) over \(\tau\). The wave function (2.8) is an analogy of Volkov wave function [7]. The main difference in this case is that \(k_0^2 \neq 0\) but taking into account (2.10) we can neglect the terms which come from \(k_0^2 \neq 0\) (this will be more evident in the Waizsäcker-Williams approach, when in the frame connected with electron the wiggler field is well enough described by a plane wave one).

Making integration in the (2.11), taking into account (2.2), for the wave function we will have

\[|q, \sigma\rangle = \left[ 1 + \frac{\hat{e}k_0A_H}{2c(k_0p)} \right] \frac{u_\sigma(p)}{\sqrt{2\pi}} \times \exp \left\{ -\frac{i}{\hbar} \left\{ qx - \frac{eA_0}{c(pk_0)} (p_x \sin(-k_0x) - gp_y \cos(-k_0x)) - \frac{e^2A_H^2}{8c^2(pk_0)} (1 - g^2) \sin(-2k_0x) \right\} \right\} \quad (2.12)\]

where by further analogy with Volkov state we have introduced the quasimomentum

\[q = p + k_0 \frac{m^2c^2}{4k_0 \cdot p} (1 + g^2)K^2; \quad (2.13)\]

and for arbitrary bispinor we have put

\[w(p) = \frac{u_\sigma(p)}{\sqrt{2\pi}} \]

where \(u_\sigma(p)\) is the bispinor amplitude of a free Dirac particle with polarization \(\sigma\). It is assumed that

\[\pi u = 2mc^2, \]

where \(\pi = u^\dagger \gamma_0; u^\dagger\) denotes the transposition and complex conjugation of \(u\) (in what follows we will put the volume of the periodicity \(V = 1\) ).

So the state of the particle in Wiggler (2.12) is characterized by the quasimomentum and polarization \(\sigma\). The wave function (2.12) is normalized by the condition

\[\langle q', \sigma' \mid q, \sigma \rangle = \delta_{q,q'}\delta_{\sigma,\sigma'}, \]

where \(\delta_{\mu\nu'}\) is the Kronecker symbol.
III. SELF-CONSISTENT SET OF THE MAXWELL AND RELATIVISTIC QUANTUM KINETIC EQUATIONS.

In this section we will consider the quantum kinetic equation for a spinor particles interacting with the classical probe electromagnetic (EM) wave in a Wiggler.

We assume the probe EM wave to be linearly polarized with the carrier frequency $\omega$ and four-vector potential

$$A_w = \varepsilon_1 \left\{ A_c(t, r) e^{i k x} + k.c. \right\}, \quad (3.1)$$

where $A_c(t, r)$ is a slowly varying envelope, $k = (\frac{\omega}{c}, k)$ is the four-wave vector and $\varepsilon_1$ is the unit polarization four vector $\varepsilon_1 k = 0$.

Rising from the second quantization formalism, the Hamiltonian is

$$\tilde{H} = \int \tilde{\Psi}^+ \tilde{H}_0 \tilde{\Psi} + \tilde{H}_{int} \quad (3.2)$$

where $\tilde{\Psi}$ is the fermionic field operator, $\tilde{H}_0$ is the one-particle Hamiltonian in Wiggler and interaction Hamiltonian is

$$\tilde{H}_{int} = \frac{e}{c} \int \tilde{j} A_w \, d\mathbf{r} \quad (3.3)$$

with the current density operator

$$\tilde{j} = \tilde{\Psi}^+ e \gamma_0 \gamma_\mathbf{\tau} \tilde{\Psi} \quad (3.4)$$

We pass to the Furry representation and write the Heisenberg field operator of the electron in the form of an expansion over the stationary states of type (2.12)

$$\tilde{\Psi}(\mathbf{x}, t) = \sum_{\mathbf{q}, \sigma} \tilde{a}_{\mathbf{q}, \sigma}(t) |\mathbf{q}, \sigma\rangle \quad (3.5)$$

where we have excluded the antiparticle operators, since the contribution of particle-antiparticle intermediate states will lead only to small corrections to the processes considered. The creation and annihilation operators $\tilde{a}_{\mathbf{q}, \sigma}(t)$ and $\tilde{a}_{\mathbf{q}, \sigma}^+(t)$ associated with the positive energy solutions satisfy the anticommutation rules at equal times

$$\{\tilde{a}_{\mathbf{q}, \sigma}^+(t), \tilde{a}_{\mathbf{q}', \sigma'}(t')\}_{t=t'} = \delta_{\mathbf{q}, \mathbf{q}'} \delta_{\sigma, \sigma'} \quad (3.6)$$

$$\{\tilde{a}_{\mathbf{q}, \sigma}^+(t), \tilde{a}_{\mathbf{q}', \sigma'}(t')\}_{t=t'} = \{\tilde{a}_{\mathbf{q}, \sigma}(t), \tilde{a}_{\mathbf{q}', \sigma'}(t')\}_{t=t'} = 0 .$$

Taking into account (3.5), (3.4), (3.3) and (2.12) the second quantized Hamiltonian can be expressed in the form

$$\tilde{H} = \sum_{\mathbf{q}, \sigma} \varepsilon(\mathbf{q}) \tilde{a}_{\mathbf{q}, \sigma}^+ \tilde{a}_{\mathbf{q}, \sigma} + \frac{e}{2c} \varepsilon \sum_{s} \sum_{\mathbf{q}_1, \sigma_1, \sigma_4} \tilde{a}_{\mathbf{q}_1 - \mathbf{h} k + s \mathbf{h k}_0, \sigma_4}^+ \tilde{a}_{\mathbf{q}_1, \sigma_1} (\mathbf{q}_1 - \mathbf{h} k + s \mathbf{h k}_0, \sigma_4) |\mathbf{q}_1, \sigma_4\rangle e^{i \Delta(\mathbf{q}_1 - \mathbf{h} k + s \mathbf{h k}_0, \mathbf{q}_1)t}$$

$$+ \frac{e}{2c} \varepsilon \sum_{s} \sum_{\mathbf{q}_1, \sigma_1, \sigma_4} \tilde{a}_{\mathbf{q}_1 + \mathbf{h} k + s \mathbf{h k}_0, \sigma_4}^+ \tilde{a}_{\mathbf{q}_1, \sigma_1} (\mathbf{q}_1 + \mathbf{h} k + s \mathbf{h k}_0, \sigma_4) |\mathbf{q}_1, \sigma_4\rangle e^{-i \Delta(\mathbf{q}_1 + \mathbf{h} k + s \mathbf{h k}_0, \mathbf{q}_1)t}. \quad (3.7)$$

Here

$$\langle \mathbf{q}_2, \sigma_2 | s | \mathbf{q}_1, \sigma_1 \rangle = \frac{\pi_s \varepsilon_2(p_2)}{2 \sqrt{\varepsilon_1 \varepsilon}} \left\{ \left( \varepsilon_1 - \frac{y^2 e^2 A_0^2(\mathbf{k}_0, e_1)\mathbf{k}_0}{2c^2(p_1 k_0)(p_2 k_0)} \right) A_0 \right\} u_{\sigma_1}(p_1)$$

$$- e A_0 \left( \frac{\gamma_2 \mathbf{k}_0 \gamma_1}{2c(p_1 k_0)} + \frac{\gamma_1 \mathbf{k}_0 \gamma_2}{2c(p_2 k_0)} \right) A_1 - e A_0 \left( \frac{\gamma_y \mathbf{k}_0 \gamma_1}{2c(p_1 k_0)} + \frac{\gamma_1 \mathbf{k}_0 \gamma_y}{2c(p_2 k_0)} \right) A_1' + \frac{e^2 A_0^2(\mathbf{k}_0, e_1)\mathbf{k}_0}{2c^2(p_1 k_0)(p_2 k_0)} A_2 \right\} u_{\sigma_1}(p_1) \quad (3.8)$$

where we have introduced the following functions [8]
\[
\{\sin \varphi, \cos^n \varphi\} \exp \left[ i (\alpha \sin(\varphi - \varphi_0) - \beta \sin 2\varphi) \right] = \sum_s \{ A'_s(\alpha, \beta, s), A_n(\alpha, \beta, s) \} \exp(is\varphi)
\] (3.9)

and parameters are defined as following

\[
\alpha = \frac{eA_0}{\hbar c} \left\{ \left( \frac{p_{1x}}{(p_1 k_0)} - \frac{p_{2x}}{(p_2 k_0)} \right)^2 + g^2 \left( \frac{p_{1y}}{(p_1 k_0)} - \frac{p_{2y}}{(p_2 k_0)} \right)^2 \right\}^{1/2}
\] (3.10)

\[
\beta = (g^2 - 1) \frac{e^2 A_0^2}{8e^2\hbar} \left( \frac{1}{(p_1 k_0)} - \frac{1}{(p_2 k_0)} \right),
\] (3.11)

\[
\sin \varphi_0 = \frac{eA_0}{\alpha \hbar c} g \left( \frac{p_{1y}}{(p_1 k_0)} - \frac{p_{2y}}{(p_2 k_0)} \right),
\] (3.12)

and

\[
\Delta(q_1 - \hbar k + \hbar k_0, q_1) = \frac{\epsilon(q_1 - \hbar k + \hbar k_0) - \epsilon(q_1) + \hbar \omega}{\hbar}
\] (3.13)

is the resonance detuning.

We will use Heisenberg representation where operators evolution are given by the following equation

\[
i\hbar \frac{\partial \hat{L}}{\partial t} = [\hat{L}, \hat{H}],
\] (3.14)

and expectation values are determined by the initial density matrix \( \hat{D} \)

\[
< \hat{L} > = Sp \left( \hat{D}\hat{L} \right).
\] (3.15)

The equations (3.14) should be supplemented by the Maxwell equation for \( \overrightarrow{A}_c \) which is reduced to

\[
\frac{\partial A_c}{\partial t} + \frac{c^2 k}{\omega} \frac{\partial A_c}{\partial r} = -4\pi e \frac{\rho}{\omega} < j e_1 > \exp(ikx)
\] (3.16)

where bar denotes averaging over time and space much larger than \((1/\omega, 1/k)\) and

\[
< j e_1 > = Sp \left( e_{1j}\hat{D} \right)
\] (3.17)

\[
\epsilon_{1j} = e \sum_s \sum_{q_1, \sigma_1, q_2, \sigma_2} \hat{a}_{q_1, \sigma_1}^\dagger \hat{a}_{q_2, \sigma_2} (q_2, \sigma_2 \parallel s \parallel q_1, \sigma_1) e^{i\epsilon(q_1 - q_2 + \hbar k_0)t} e^{i\Delta(q_1 - q_2)}
\] (3.18)

As we are interested in amplification of the wave with given \( \omega, k \), then we will keep only the resonant terms in (3.18) with \( q_2 = q_1 - \hbar k + \hbar k_0 \). In principle, due to electron beam energy and angular spreads different harmonics may contribute to the process considered, but in the quantum regime (see below (3.40), (3.41)) we can keep only one harmonic \((s_0)\). And for the resonant current amplitude we will have the following expression

\[
-i(\epsilon_{1j}) \exp(ikx) = \sum_q \hat{\Pi}(q)
\] (3.19)

where

\[
\hat{\Pi}(q) = -ie \sum_{\sigma_1, \sigma_2} \hat{a}_{q_-, \sigma_1}^\dagger \hat{a}_{q, \sigma_2} (q_-, \sigma_2 \parallel s_0 \parallel q, \sigma_1) \frac{e^{i\Delta(q_1 - q_2)}}{q}.
\] (3.20)

Here we have introduced the following notation

\[
q_- = q - \hbar k + s_0 \hbar k_0
\] (3.21)
The physical meaning of Eq. (3.20) is obvious: it describes the process where a particle with quazimomentum \( q \) is annihilated and is created in the state with the quazimomentum \( q - \hbar k + s_0 \hbar k_0 \). Taking into account Eqs. 3.7, 3.14, 3.6 for the operator \( \hat{\Pi}(q) \) we obtain the following equation

\[
\frac{\partial \hat{\Pi}(q)}{\partial t} - i \Delta(q_{-}, q_{1}) \hat{\Pi}(q) = \frac{\epsilon^2}{2\hbar c} A_e \sum_{\sigma_1, \sigma_2, \sigma_3} \left\{ (q, \sigma_1 || -s_0 || q_{-}, \sigma_3) (q_{-}, \sigma_2 || s_0 || q, \sigma_1) \hat{a}^+_{q_{-}, \sigma_2} \hat{a}_{q_{-}, \sigma_3} \right\} \tag{3.22}
\]

where we have kept only the resonant terms as far as these terms are predominant in near-resonant emission/absorption, since their respective detuning are much smaller than those for non-resonant terms, which are detuned from the resonance due to \( \omega >> |\Delta(q_{-}, q)| \).

We will assume that electron beam is non-polarized. This means that initial one particle density matrix in the momentum space is

\[
\rho_{\sigma_1, \sigma_2}(q_1, q_2, 0) = \langle \hat{a}^+_{q_2, \sigma_2}(0) \hat{a}_{q_1, \sigma_1}(0) \rangle = \rho_0(q_1, q_2) \delta_{\sigma_1, \sigma_2}. \tag{3.23}
\]

Here \( \rho_0(q, q) \) is connected with the classical momentum distribution function \( n(q) \) by the following formula

\[
\rho_0(q, q) = \frac{(2\pi \hbar)^3}{2} n(q). \tag{3.24}
\]

For the expectation value of \( \hat{\Pi}(q) \) from (3.22) we will have

\[
\frac{\partial \Pi(q)}{\partial t} - i \Delta(q_{-}, q_{1}) \Pi(q) = \frac{\epsilon^2 M^2}{2\hbar c} (\rho(q_{-}, q_{-}, t) - \rho(q, q, t)), \tag{3.25}
\]

where

\[
\rho(q_{1}, q_{1}, t) = \langle \hat{a}^+_{q_{1}, \sigma_2}(t) \hat{a}_{q_{1}, \sigma_1}(t) \rangle >
\]

\[
M^2 = \sum_{\sigma_1, \sigma_2} (q, \sigma_1 || -s_0 || q_{-}, \sigma_2) (q_{-}, \sigma_2 || s_0 || q, \sigma_1).
\]

The quantity \( M^2 \) is reduced to the usual calculation of trace [7], [8] and in our notations we have

\[
M^2 = \frac{2\epsilon^4}{\epsilon - \epsilon} \left| (pe')_1 \Lambda_0 + \frac{e}{c} \epsilon_0' \left( e'_{1x} \Lambda_1 + g e'_{1y} \Lambda_1' \right) \right|^2, \tag{3.26}
\]

where

\[
e'_{1} = \epsilon_{1} - \frac{k x_{1}}{\hbar k_{0} k} \tag{3.27}
\]

Here we have neglected terms in order of \((\hbar \omega / \epsilon)^2 << 1\) as far as for the FEL this condition is always satisfied. Taking into account Eqs. (3.7, 3.14, 3.6) for the \( \rho(q, q, t) \) and \( \rho(q_{-}, q_{-}, t) \) we will obtain

\[
\frac{\partial \rho(q, q, t)}{\partial t} = \frac{1}{4\hbar c} (A^*_{e} \Pi + A_{e} \Pi') \tag{3.28}
\]

\[
\frac{\partial \rho(q_{-}, q_{-}, t)}{\partial t} = -\frac{1}{4\hbar c} (A^*_{e} \Pi + A_{e} \Pi') \tag{3.29}
\]

To take into account the pulse propagation effects we can replace the time derivatives by the following expression

\[
\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \nabla \frac{\partial}{\partial r},
\]

where \( \nabla \) is the mean velocity of electron beam and convectional part of derivative expresses the pulse propagation effects. Introducing new variables
\[ \delta n = \frac{2}{(2\pi\hbar)^3} [\rho(\mathbf{q}_-, \mathbf{q}_-, t) - \rho(\mathbf{q}, \mathbf{q}, t)] \] (3.30)

\[ \frac{1}{(2\pi\hbar)^3} \Pi(\mathbf{q}) = J(\mathbf{q}) \] (3.31)

and replacing summation in (3.16) by integration, the self-consistent set of equations reads

\[ \frac{\partial J(\mathbf{q})}{\partial t} + \nabla \frac{\partial J(\mathbf{q})}{\partial \mathbf{r}} - i \Delta J(\mathbf{q}) = \frac{e^2 M^2}{4\hbar c} A_e(x, z, t) \delta n(\mathbf{q}) \]

\[ \frac{\partial \delta n(\mathbf{q})}{\partial t} + \nabla \frac{\partial \delta n(\mathbf{q})}{\partial \mathbf{r}} = -\frac{1}{\hbar c} \left( A_e^* J(\mathbf{q}) + A_e J^*(\mathbf{q}) \right) \] (3.32)

These equations yield to the conservation laws for the energy of the system and particle number:

\[ \frac{\partial |A_e|^2}{\partial t} + \frac{e^2 k}{\omega} \frac{\partial |A_e|^2}{\partial \mathbf{r}} = -\frac{4\pi c^2 e^2}{\omega} \int d\mathbf{q} \left( \frac{\partial}{\partial t} + \nabla \frac{\partial}{\partial \mathbf{r}} \right) \delta n(\mathbf{q}) \]

\[ \left( \frac{\partial}{\partial t} + \nabla \frac{\partial}{\partial \mathbf{r}} \right) \left( \delta n(\mathbf{q})^2 + \frac{8}{e^2 M^2} |J(\mathbf{q})|^2 \right) = 0 \] (3.33)

From the set of equations (3.32) by the perturbation theory one can obtain the small signal gain, which in the quasiclassical limit will coincide with the classical one.

The emission and absorption are characterized by the following widths

\[ \Delta_e = s \omega_0 \frac{v}{c} \cos \theta_1 - \omega \left( 1 - \frac{v}{c} \cos \theta \right) - \frac{ee^2 A_0^2 \omega'}{2e^2 v \cos \theta_1} \cos \theta_0 \]

\[ - \frac{s_0 \hbar \omega_0 \omega}{\epsilon} \cos \theta_0 \]

\[ \Delta_a = \Delta_e + \frac{2s_0 c \omega_0 \omega}{\epsilon} \cos \theta_0 \] (3.35)

where \( \omega_0 = 2\pi c/\ell \), \( \theta \) and \( \theta_0 \) are the scattering angles of probe photons with respect to the electron beam direction of motion and undulator axis, respectively, \( \theta_1 \) is the angle of the electron beam direction of motion with respect to undulator axis.

The quantum regime assumes

\[ \Delta_a - \Delta_e = \frac{2s_0 c \omega_0 \omega}{\epsilon} \cos \theta_0 > \max \left\{ \left| \frac{\partial \Delta_e}{\partial \eta_i} \delta \eta_i + \frac{\partial^2 \Delta_e}{\partial \eta_i^2} (\delta \eta_i)^2 \right|, \frac{\omega_0}{N} \right\} \] (3.37)

where by \( \eta_i \) we denote the set of quantities characterizing electron beam and pump field and by \( \delta \eta_i \) their spreads. The second term in the figured brackets of Eq. (3.37) expresses resonance width caused by finite interaction length, \( N \) is the number of periods of pump field.

Particularly for energetic (\( \Delta \varepsilon \)) and angular (\( \Delta \vartheta \)) spreads from Eq. (3.37) (for \( \theta_0, \theta << 1 \)) we will have

\[ \Delta \varepsilon < \hbar \omega \] (3.38)

\[ \left| \theta \Delta \vartheta + \frac{\Delta \vartheta^2}{2} \right| < \frac{s_0 \hbar \omega_0 \omega}{\epsilon} \] (3.39)

The conditions of keeping only one harmonic in resonant current, are:

\[ \frac{\Delta \varepsilon}{\epsilon} << 1/s_0 \] (3.40)

\[ \left| \theta \Delta \vartheta + \frac{\Delta \vartheta^2}{2} \right| < \frac{\omega_0}{\omega} \] (3.41)

As we see, these conditions are weaker than the conditions of quantum regimes (3.38) and are well enough satisfied for actual beams.
IV. STEADY-STATE REGIMES OF AMPLIFICATION

Our goal is to determine the conditions under which we will have non-linear amplification. We will assume steady-state operation, according to which will drop with partial time derivatives in (3.32). The considered setup is either a single-pass amplifier, for which it is necessary injected input signal or self-amplified coherent spontaneous emission, for which it is necessary initially modulated beam.

Besides we will consider the case of exact resonance neglecting detuning in Eqs. (3.32) assuming that electron beam momentum distribution is centered at \( \Delta_e = 0 \). To achieve maximal Doppler-shift and optimal conditions of amplification we will assume that electron beam propagates along undulator axis (\( Z \) axis). In this case optimal condition for LW is \( \theta = 0 \), while for HW \( \theta \sim K/\gamma L \) (\( \theta << 1 \)). For both cases we will assume that the envelope of probe wave depends only on \( z \). Then the set of equations (3.32) and conservation laws (3.33, 3.34) are reduced to

\[
\frac{\partial J}{\partial z} = \frac{e^2 M^2}{4 \hbar c \nu_z} A_0 \Delta n
\]

\[
\frac{\partial \delta n}{\partial z} = -\frac{2}{\hbar c \nu_z} A_0 J
\]

\[
\frac{\partial A_0}{\partial z} = \frac{4\pi}{\omega} J
\]

\[
\delta n^2 + \frac{8}{e^2 M^2} | \Pi |^2 = N_0^2
\]

\[
I = I_0 + \frac{\hbar \omega \nu_z}{2} (\delta n_0 - \delta n),
\]

where \( N_0 \) is the beam density, \( I \) is the probe wave intensity and \( I_0 \) is the initial one. From Eq.(4.1) we have the following expressions for \( J \) and \( \delta n \):

\[
\delta n = N_0 \cos \left( \frac{e |M|}{2 \sqrt{2} \hbar c \nu_z} \int_0^z A_0 dz + \varphi_0 \right)
\]

\[
J = \frac{e |M|}{2 \sqrt{2} \hbar c \nu_z} N_0 \sin \left( \frac{e |M|}{2 \sqrt{2} \hbar c \nu_z} \int_0^z A_0 dz + \varphi_0 \right),
\]

where \( \varphi_0 \) is determined by boundary conditions. Denoting

\[
\varphi = \frac{e |M|}{2 \sqrt{2} \hbar c \nu_z} \int_0^z A_0 dz + \varphi_0
\]

we arrive to the nonlinear pendulum equation

\[
\frac{\partial^2 \varphi}{\partial z^2} = \sigma^2 \sin \varphi,
\]

where

\[
\sigma^2 = \frac{\pi e^2 M^2 N_0}{\hbar \omega c \nu_z}
\]

is the main characteristic parameter of amplification: \( L_c = 1/\sigma \) is the characteristic length of amplification. For the LW from Eqs.(3.9), (3.10), (3.11), (3.12) and (3.26) we have

\[
\sigma_L = \frac{K \Lambda_1(0, \beta, s_0)}{\gamma_L^2} \sqrt{\frac{\alpha_0 e^4}{2 \beta_0 \nu_z} N_0(1 + K^2/2)}
\]
where $\alpha_0$ is the fine structure constant and the function $\Lambda_1(0, \beta, s)$ is expressed by the ordinary Bessel functions:

$$
\Lambda_1(0, \beta, s_0) \simeq \frac{1}{2} \left| J_{s_0-1} \left( \frac{s_0K^2}{4+2K^2} \right) - J_{s_0+1} \left( \frac{s_0K^2}{4+2K^2} \right) \right|.
$$

(4.7)

In this case only odd harmonics are possible. For the HW we have

$$
\sigma_L = \frac{K}{\gamma_L} \left( \frac{\theta \gamma_L \alpha}{K} + \frac{s_0}{\alpha} \right) |J_{s_0}(\alpha)| \sqrt{\frac{\alpha_0 \sqrt{2s_0v_z}}{2s_0v_z}} N_0(1 + K^2 + \theta^2 \gamma^2_L)
$$

(4.8)

and the argument of Bessel function is

$$
\alpha \simeq \frac{2s_0K\gamma_L\theta}{1 + K^2 + \theta^2 \gamma^2_L}.
$$

(4.9)

We will consider two regimes of amplification, which are determined by initial conditions. For the first regime the initial macroscopic transition current of the electron beam is zero and it is necessary to have an seeding electromagnetic wave. In this case the following boundary conditions are imposed

$$
\delta n \mid_{z=0} = N_0; \quad J \mid_{z=0} = 0; \quad I \mid_{z=0} = I_0
$$

(4.10)

The solution of Eq.(3.31) in this case reads

$$
I(z) = \frac{I_0}{\frac{d\alpha}{dz}} \left( \frac{z}{\kappa \alpha} \right)
$$

(4.11)

with

$$
\kappa = \left( 1 + \frac{I_0}{N_0 \hbar \omega v_z} \right)^{-\frac{1}{2}}
$$

(4.12)

where $dn(z, \kappa)$ is the Elliptic function of Jacobi and $\kappa$ its module.

As is known $dn(z, \kappa)$ is the periodic function with the period $2\Xi(\kappa)$, where $\Xi(\kappa)$ is the Complete Elliptic Integral of first order. At the distances $L = (2r + 1)\kappa \cdot \Xi(\kappa)/\sigma$ ($r = 0, 1, 2, ...$) the wave intensity reaches its maximal value which equals to

$$
I_{\text{max}} = I_0 + N_0 \hbar \omega v_z.
$$

(4.13)

For the short interaction lengths $z \ll L_c$ from Eq.(3.31) we have

$$
I(z) = I_0 \left( 1 + \sigma^2 z^2 \right)
$$

and the wave gain is rather small. To extract maximal energy from electron beam the interaction length should be at least order of half of spatial period of the wave envelope variation - $\kappa \cdot \Xi(\kappa)/\sigma$. At this condition the intensity value $I_{\text{max}} = I_0 + N_0 \hbar \omega v_z$ is achieved, because all electrons make contribution in radiation field. Taking into account that seed power is much more smaller than $I_{\text{max}}$ and when $1 - \kappa << 1$

$$
\Xi(\kappa) \to \frac{1}{2} \ln \left[ \frac{16}{1 - \kappa^2} \right]
$$

for amplification length we will have

$$
L \simeq L_c \left( 1 + \ln \left( \frac{I_{\text{max}}}{I_0} \right) \right).
$$

(4.14)

Let us now consider the other regime of wave amplification when electron beam is modulated - "macroscopic transition current" $J$ differs from zero. This regime can operate without any initial seeding power ($I_0 = 0$). So we will consider the optimal case with the following initial conditions

$$
J \mid_{z=0} = J_0 \quad \delta n \mid_{z=0} = \delta n_0; \quad I \mid_{z=0} = 0
$$

(4.15)

Then the wave intensity is expressed by the formula
\[ I(z) = \frac{N_0 \hbar \omega v_z}{2} \left( 1 - \frac{\Delta n_0}{N_0} \right) \left[ \frac{1}{dn^2(\alpha z; k)} - 1 \right] \] (4.16)

and module is

\[ \kappa^2 = \frac{1}{2} \left( 1 + \frac{\Delta n_0}{N_0} \right) \] (4.17)

As is seen from (4.16) in this case the intensity varies periodically with the distances as well, with the maximal value of intensity

\[ I_{\text{max}} = \frac{N_0 \hbar \omega v_z}{2} \left( 1 + \frac{\Delta n_0}{N_0} \right). \] (4.18)

The second regime is more interesting. It is the regime of amplification without initial seeding power and has superradiant nature. For the short interaction length \( z \ll L_c \) according to (4.16)

\[ I(z) = \frac{N_0 \hbar \omega v_z \alpha^2 z^2}{4} \left( 1 - \frac{\Delta n_0}{N_0} \right) \] (4.19)

The intensity is scaled as \( N_0^2 \), which means that we have superradiation. The radiation intensity in this regime reaches significant value even at \( z \ll L_c \).

**V. DISCUSSION**

The coherent interaction time of electrons with probe radiation is confined by the several relaxation processes. To be more precise in the self consistent set of equations (3.32) we should add the terms describing spontaneous transitions and other relaxation processes. Since we have not taken into account the relaxation processes, this consideration is correct only for the distances \( L \lesssim c \tau_{\text{min}} \), where \( \tau_{\text{min}} \) is the minimum of all relaxation times. Due to spontaneous radiation electron will lose the energy \( \sim \hbar \omega \) at the distances

\[ L_s \simeq c \omega \frac{\hbar}{I_s} = \frac{3}{2 \pi v_0} \frac{s_0 \ell}{(1 + K^2/2)K^2}, \] (5.1)

where \( I_s \) is the intensity of spontaneous radiation (this is for LW; for HW one should replace \( K^2 \rightarrow 2 K^2 \)). Although the cutoff harmonic is increased \( s_c \sim K^3 \) with increasing of \( K \) for the large wiggler parameter \( K \gtrsim 1 \) the rule of spontaneous radiation is increased \( L_s \sim K^{-4} \) and above mentioned regimes will be interrupted. The obtained solutions are correct for the distances \( \sim L_s \).

In the Tables 1, 2 we give the parameters for the different setups of a beam and undulator as for LW as well as for HW. The beam current have been chosen 5 kA and the beam radius \( 10^{-3} \text{cm} \). Maximum intensity \( I_{\text{max}} \) scales as \( \sim (\hbar \omega/mc^2) \cdot 7 \cdot 10^{14} \text{ W/cm}^2 \). As we see from these tables for the high harmonics \( L_c \) is decreased and simultaneously \( \hbar \omega/\varepsilon \) is increased, but \( L_s < L_c \). In this case wiglter tapering is necessary to keep the resonance condition. But how it will look like for quantum regime will be the subject of the future work.

The second regime may be more promising as it allows considerable output intensities even for small interaction lengths. For the \( s = 51 \) (Table 1. \( \hbar \omega \sim 1.1 \text{MeV} \)) at the \( L = L_s (\alpha^2 z^2 < < 1, 1 - \delta n_0/N_0 \approx 10^{-2} (1\% \text{ modulation}) \) from (4.19) we have \( I \sim 10^{10} \text{ W/cm}^2 \). It is expected that the effects of energy and angular spreads will not have significant influence upon this regime as it is governed by the initial current and only Doppler dephasing and spontaneous lifetime may interrupt the superradiation process. Note, that necessary for this regime quantum moduluation of the particle beams at the above optical frequencies can be obtained through multiphoton transitions in the laser field at the presence of a "third body". The possibilities of quantum modulation at hard X-ray frequencies in the induced Compton, undulator and Cherenkov processes have been studied in the works [9].

This work is supported by International Science and Technology Center (ISTC) Project No. A-353.

**Table 1 (LW)**

| s  | \( \gamma_L \) | \( K \) (cm) | \( \hbar \omega/\varepsilon \) | \( L_s \) (m) | \( L_s/L_c \) |
|----|----------------|--------------|-----------------|--------------|-------------|
| 1  | 1.5 \times 10^4 | 0.1 1.5      | 1.6 \times 10^{-7} | 12           | 8.5         |
| 1  | 1 \times 10^9   | 0.1 1.5      | 3.2 \times 10^{-10} | 47           | 2.14        |
| 3  | 4 \times 10^4   | 1 3          | 1.3 \times 10^{-5} | 2.8          | 1.1         |
| 9  | 2 \times 10^4   | 1.5 4        | 10^{-10}        | 2            | 1.5         |
| 21 | 2.5 \times 10^4 | 2.7 5        | 10^{-5}        | 1.2          | 0.7         |
| 51 | 5 \times 10^4   | 3 5          | 4.5 \times 10^{-8} | 24           | 0.05        |
Table 2. (HW)

| s  | $\gamma_L$ | $K$ | $L_{\omega}/\varepsilon$ | $L_c/m$ | $L_s/L_c$ |
|----|------------|-----|-------------------|--------|--------|
| 1  | $5 \cdot 10^4$ | 0.04 | 1.5 | 1.6 \cdot 10^{-6} | 16.8 | 12.3 |
| 3  | $6 \cdot 10^4$ | 0.2 | 1.5 | $3 \cdot 10^{-5}$ | 21.5 | 1.1 |
| 5  | $6 \cdot 10^4$ | 1 | 3 | $10^{-5}$ | 1.9 | 1 |
| 10 | $2 \cdot 10^4$ | 1.05 | 3 | $10^{-5}$ | 2.2 | 2.58 |
| 20 | $4 \cdot 10^4$ | 1.8 | 5 | $10^{-5}$ | 2 | 1.4 |
| 50 | $6 \cdot 10^4$ | 2.3 | 5 | $2 \cdot 10^{-5}$ | 5.8 | 0.5 |

[1] Free-Electron Generators of Coherent Radiation, eds. S. F. Jacobs, H. S. Pilloff, M. Sargent III, M.O. Scully, R. Spetzer, Physics of Quantum Electronics, v. 5,7-9 (Addison-Wesley, Reading, 1982); T.C. Marshall, "Free Electron Lasers" (MacMillan, 1985); C. A. Brau, Free-Electron Lasers (Academic, Boston, 1990); W. B. Colson, C. Pellegrini and A. Renieri, Laser Handbook Vol. 6, Elsevier (1990); G. Dattoli, A. Renieri, and A. Torre, Lectures on the Free Electron Laser Theory and Related Topics (World Scientific, London, 1993); Proceedings of the annual International Free Electron Laser Conferences published in Nucl. Inst. Meth. Vols. A445, A407, A358, A341, A331, A318, A304.

[2] J. M. J. Madey, J. Appl. Phys. 42, 1906 (1971); R.M. Terhune, R.H. Pantell, Appl. Phys.Lett. 30, 265 (1977); R.H. Pantell, M.J. Alguard, J. Appl. Phys. 50, 798 (1979); H.K.Avetissian, A.K.Avetissian, K.Z.Hatsagortsian, Phys.Lett., A 137, 463 (1989); K. Nakajima et al., Nucl. Instr. and Meth. A 375, 593 (1996); F. Glotin et al., Phys. Rev. Lett. 77, 3130 (1996); H.K.Avetissian et al., Phys. Rev. A 56, 4121 (1997).

[3] L. R. Elias et al., Phys. Rev. Lett. 36, 771 (1976); D.A.Deacon et al., Phys.Rev.Lett. 38, 892 (1977).

[4] A.M. Kondratenko, E.L. Saldin, Part. Accelerators 10, 207 (1980); R. Bonifacio, C. Pellegrini, L.M. Narducci, Opt. Commun. 50, 373 (1984); K.J. Kim, Phys. Rev. Lett. 57, 1871 (1986); S. Krinsky, L.H. Yu, Phys. Rev. A 35, 3406 (1987); E.L. Saldin, E.A. Schneidmiller, M.V. Yurkov, The Physics of Free Electron Lasers, Springer (1999).

[5] Brinkmann, G. Materlik, J. Rossbach, A. Wagner, ed., Conceptual Design of a 500 GeV e+e- Linear Colider with Integrated X-ray Laser Facility, 1997. DESY, 1997-048 and ECF A 1997-182; The LCLS Design Study Group, LCLS-Design Study Report, April 1998. SLAC-R-521.

[6] A. Friedman et al., Rev. Mod. Phys. 60, 471 (1988).

[7] L.D. Landau and E.M. Lifshitz, Quantum Electrodynamics (Nauka, Moscow, 1989).

[8] V.I. Ritus, Trudi Fiz. Inst. Akad. Nauk 111, 141 (1979).

[9] V. M. Haroutunian, H. K. Avetissian, Phys. Lett.A 44, 281, (1973); H.K.Avetissian, Phys. Lett.A 63, 9, (1977); H.K. Avetissian, Phys. Lett. A 67, 101 (1978); H.K. Avetissian, et al., Phys. Lett. A 85, 263 (1981); H.K. Avetissian, G.F. Mkrtchian, Phys. Rev. E. 65, 016506 (2002).