Personalized Federated Learning with Multiple Known Clusters

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Abstract

We consider the problem of personalized federated learning when there are known cluster structures within users. An intuitive approach would be to regularize the parameters so that users in the same cluster share similar model weights. The distances between the clusters can then be regularized to reflect the similarity between different clusters of users. We develop an algorithm that allows each cluster to communicate independently and derive the convergence results. We study a hierarchical linear model to theoretically demonstrate that our approach outperforms agents learning independently and agents learning a single shared weight. Finally, we demonstrate the advantages of our approach using both simulated and real-world data.

Keywords: personalized federated learning, multi-task learning, distributed optimization

1 Introduction

Smart phones, voice assistants, and wearable devices are everywhere in our lives, constantly collecting data on our behavior and habits. Federated learning (FL) is a recently introduced framework developed to use this rich source of data, while minimizing the intrusion of clients’ privacy. Although traditional machine learning methods often require the aggregation of client data in a central server, federated learning avoids such a requirement, allowing models to be trained with mostly local computation and occasional server-wide communication rounds (McMahan et al., 2017; Yang et al., 2019; Bonawitz et al., 2019; Li et al., 2020; Kairouz et al., 2019).

Data sets used in FL tasks are heterogeneous in nature, as they are collected from clients who are heterogeneous in nature. Tailoring the learned model to each client through personalized FL has garnered a tremendous amount of interest in recent years (Hanzely et al., 2020a; Fallah et al., 2020; Deng et al., 2020; Dinh et al., 2020; Mansour et al., 2020). A central theme is learning a single global model and a personalized model for each client simultaneously in the training process (Hanzely and Richtárik, 2020; Dinh et al., 2020; Li et al., 2021; Fallah et al., 2020). While it is typical to assume that clients can be grouped into a single cluster in existing literature, in disciplines such as education, psychology, or economics, it is often assumed that clients can be grouped into multiple clusters using known information. Hierarchical linear models are frequently used to model

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Motivated by the prevalence of hierarchical models in social sciences, we propose a hierarchical, multi-cluster approach to personalization in federated learning. In particular, we leverage the known hierarchical structure and simultaneously learn (1) a global model for all clients, (2) a cluster-specific model for each client cluster, and (3) a personalized model for each client. More specifically, we develop a loopless algorithm for fitting the hierarchical linear model and derive its convergence rates and optimal parameters. Our algorithm allows each client cluster to communicate only within the group and allows each cluster to determine when to aggregate local updates independently of other clusters. Such an independence could improve convergence when the client clusters align with the communication graph. Existing empirical research has shown that extra communication rounds within different geographical clusters can improve convergence rates (Huang et al., 2019; Briggs et al., 2020) and the proposed algorithm can leverage this property. In the finite-sum setting, we further develop an accelerated, variance reduced, stochastic variant of the algorithm, which has been shown to be minimax optimal in terms of communication rounds and oracle calls in the single cluster setting (Hanzely et al., 2020a).

Some existing work studied federated learning under the assumption that clients can be grouped into different clusters without assuming that these clusters are known. Our work emphasizes the optimality of the proposed approach when clusters are given a priori. In particular, the prior literature mostly focuses on the convergence rates of the optimization procedure and identifies the conditions under which the underlying cluster structures can be recovered. A key problem remains unaddressed: even assuming the cluster structure is known, can we prove that these structures can improve upon simple baselines such as training using only local data or training a single model? More importantly, can we show the optimality of the clustered formulation even in the simple case where the cluster labels are given?

To answer these questions, we establish statistical properties for the estimator obtained in our framework. We show that under a simple problem setting, our approach recovers the best linear unbiased estimator of the clients’ local weights, dominating both training a single model for all clients and training a unique model for each client independently. This result complements Li et al. (2021) that studied the single-group setting. We further show that the unbiased restriction cannot be weakened. Even in the single cluster setting, we can construct a counterexample where a James-Stein estimator outperforms the proposed method. Our result also complements Chen et al. (2021) by identifying a regime in which a personalized approach is at least as good as the recommended alternatives, noting that structural information can be used to develop personalization schemes that outperform the proposed alternatives.

Finally, we demonstrate the empirical effectiveness of our model on the DMEF Customer Lifetime Value data set Blattberg et al. (2009) and compare it with a recently proposed method in targeted marketing Bumbaca et al. (2020), developing a personalized marketing model using data from a leading non-profit organization in the United States Blattberg et al. (2009).

1.1 Related work

Our work is related to the literature on personalized federated learning, distributed multi-task learning, and Bayesian hierarchical models.
There has been a lot of focus on personalization in federated learning over the past few years. Karimireddy et al. (2020) is one of the first works to discuss heterogeneity among clients. Fallah et al. (2020) adapted model-agnostic meta-learning (MAML) algorithms for personalization. Deng et al. (2020) developed a method that interpolates between client-specific parameters and global parameters. Mansour et al. (2020) suggests three different approaches for personalized federated learning and provides learning theoretic guarantees. Li et al. (2021); Hanzely and Richtárik (2020); Dinh et al. (2020) studied a personalization approach where each client has a parameter whose distance to the average of the parameters is regularized either explicitly or implicitly and developed different optimization techniques. Li et al. (2018) studied optimizing a similar regularized loss in a non-personalized setting, where a single model is trained for all clients. More specifically, Li et al. (2021) provided some theoretical justification on the robustness and fairness of the procedure in the single cluster regime, Hanzely et al. (2021) provided a unified analysis of different optimization techniques, and Dinh et al. (2020) studied the optimization problem assuming that individual clients can exactly evaluate a proximal operator. These approaches can be viewed as a special instance of distributed multi-task learning with graph regularization Wang et al. (2018).

A different line of work on personalization assumes that client-specific parameters may be drawn from an unknown mixture distribution and simultaneously group clients and learn model parameters using a single algorithm (Mansour et al., 2020; Ghosh et al., 2020; Sattler et al., 2020; Smith et al., 2017; Briggs et al., 2020; Huang et al., 2019). Similar approaches have been studied in multi-task learning (Kumar and Daume III, 2012; Jacob et al., 2008; Zhang and Yang, 2017; Zhang and Yeung, 2012; Zhou et al., 2011a,b; Bakker and Heskes, 2003). While these approaches focus on the setting where the structure and parameters of the cluster are learned simultaneously, in many domains, the clusters are known a priori and given, for example, by known covariates such as age, gender, and geographical location (Raudenbush and Bryk, 1986; Raudenbush, 1988; Bryk and Raudenbush, 1987, 1992; Hofmann, 1997; Stephen and Anthony, 2002; Lee and Nelder, 1996; Daniels and Gatsonis, 1999). Such a structure can be used to develop hierarchical models with improved personalization without having to separately cluster the clients. The data example that we investigate in detail in Section 6 comes from marketing, where hierarchical models have a long history (Naik and Peters, 2009; Bumbaca et al., 2017, 2020; Hooley et al., 1999; French and Russell-Bennett, 2015). Given the ubiquity of mobile devices, combining these marketing models with federated learning could better help the industry implement state-of-the-art marketing research.

Our optimization procedure is related to the methods used to optimize the objectives commonly found in hierarchical federated learning (Abad et al., 2020; Wang et al., 2020; Liu et al., 2020; Briggs et al., 2020; Wainakh et al., 2020). However, while the existing literature focuses on finding a single model for all clients, our optimization procedure learns personalized models for each client and each cluster and also learns a joint global model. The design of the optimization algorithm is related to loopless procedures in distributed optimization (Zhao et al., 2021; Li, 2021; Qian et al., 2021a). These procedures remove the inner loops, thereby simplifying the algorithm. In single-machine settings, such simplifications have been shown to outperform their loopy counterparts (Kovalev et al., 2020).

Concurrent to our work, Marfoq et al. (2021) studies federated multi-task learning under a mixture of distributions, focusing on nonasymptotic convergence rates. Our work further shows the optimality of our approach in terms of generalization error in addition to convergence analy-
sis. Additionally, instead of analyzing an expectation-maximization-inspired approach, our work uses a loopless gradient-based algorithm that has been shown to enjoy optimal communication complexity in the single cluster regime. Furthermore, Duan and Wang (2022) discuss a general framework for multi-task learning and shows that the approach can be adapted to various concepts of task relatedness. Although the loss function discussed here is similar, we further develop a federated optimization procedure and characterize the bounds on the communication and computation complexity of the federated learning algorithm used to minimize the loss.

1.2 Notation

For any vector \( v \in \mathbb{R}^d \), we use \( \| v \| \) to denote its \( \ell_2 \) norm. For any finite set \( A \), we use \( |A| \) to denote its cardinality and \( A[i] \) to denote the \( i \)-th element in \( A \) according to some arbitrary order. The \( d \)-dimensional identity matrix is denoted as \( I_d \in \mathbb{R}^{d \times d} \). The Kronecker product between two conforming matrices \( B, C \) is denoted as \( B \otimes C \).

2 Model Formulation

Suppose that there are \( n \) clients with their individual data sets for whom we would like to fit personalized models. Furthermore, suppose that these clients are divided into \( k \) known clusters. Generally speaking, if two clients belong to the same cluster, then we expect their models to be more similar than if the clients belong to different clusters. Let \( I_j, j = 1, \ldots, k \), be the set of clients belonging to the cluster \( j \). We use \( f_i(\cdot) \) to denote the loss function for client \( i = 1, \ldots, n \). Throughout the paper, we assume that the loss function \( f_i(\cdot) \) is strongly convex and smooth. In particular, we make the following assumption.

**Assumption 2.1.** The loss function \( f_i \) is \( \mu \)-strongly convex and \( L \)-smooth.

We use “local”, “cluster”, and “global” to denote variables, functions, and values associated with individual clients, different clusters, and the entire network, respectively.

Let \( \theta_i \in \mathbb{R}^d \) denote the model parameter for client \( i \). We focus on minimizing the following objective function:

\[
\min_{\{\theta_i\}_{i=1}^n} F(\{\theta_i\}_{i=1}^n) = \sum_{j=1}^k \sum_{i \in I_j} \left( f_i(\theta_i) + \frac{(1 - \alpha_j)\gamma_i}{2} \| \theta_i - \bar{\theta}_j \|^2 + \frac{\alpha_j \gamma_i}{2} \| \theta_i - \bar{\theta} \|^2 \right),
\]

where \( \alpha_j \in \mathbb{R}_{\geq 0}, j = 1, \ldots, k, \) are the tuning parameters that control the regularization strength in each cluster, \( \gamma_i \in \mathbb{R}_{\geq 0}, i = 1, \ldots, n, \) are the tuning parameters specific to each client, and \( \{\bar{\theta}_j\}_{j=1}^k, \bar{\theta} \) denote the weight averages of the parameters in the cluster \( j \) and the entire network, respectively. That is,

\[
\bar{\theta}_j = \frac{\sum_{i \in I_j} \gamma_i \theta_i}{\sum_{i \in I_j} \gamma_i}, \quad j = 1, \ldots, k, \quad \text{and} \quad \bar{\theta} = \frac{\sum_{j=1}^k \alpha_j \bar{\theta}_j}{\sum_{j=1}^k \alpha_j} = \frac{\sum_{j=1}^k \alpha_j \sum_{i \in I_j} \gamma_i \theta_i}{\sum_{j=1}^k \alpha_j}. \quad (2.2)
\]

The objective is comprised of three terms: the client-specific loss and two regularization terms, summed over all clients. The first regularizer penalizes the distance between the local parameter
\( \theta_i \) and the cluster averages \( \tilde{\theta}_j \), while the second regularizer penalizes the distance between the local parameter and the global average \( \bar{\theta} \). By changing the cluster-specific parameter \( \alpha_j \) from 0 to 1 we can interpolate between the two regimes: when \( \alpha_j = 0 \) for all \( j \), we train \( k \) personalized models—one for each cluster, independently of other clusters; when \( \alpha_j = 1 \) for all \( j \), we train the single-cluster model studied in Hanzely and Richtárik (2020); Dinh et al. (2020); Li et al. (2021).

The objective in (2.1) is different from commonly used objective functions in multi-task learning (Wang et al., 2018; Zhou et al., 2011a,b; Jacob et al., 2008). When the cluster structure is known, the multi-task learning objective can be written as

\[
\min_{\{ \theta_i \}_{i=1}^n, \{ w_j \}_{j=1}^k, \bar{w}} F_{MTL}(\{ \theta_i \}_{i=1}^n, \{ w_j \}_{j=1}^k, \bar{w}) := \sum_{j=1}^k \left( \sum_{i \in \mathcal{I}_j} \left( f_i(\theta_i) + \frac{\gamma_i}{2} \| \theta_i - w_j \| ^2 \right) + \frac{\lambda_j}{2} \| w_j - \bar{w} \|^2 \right),
\]

where the penalty parameter \( \gamma_i \) regularizes the distance between the local parameter and the cluster parameter \( w_j \), while the penalty parameter \( \lambda_j, j = 1, \ldots, k \), regularizes the distance between the cluster parameter and the global parameter \( \bar{w} \). Furthermore, in (2.3) we optimize both the local weights \( \{ \theta_i \}_{i=1}^n \) and the average local parameters, \( \{ w_j \}_{j=1}^k \) and \( \bar{w} \). The two forms, however, have same stationary points.

**Proposition 2.2.** Suppose that Theorem 2.1 holds for loss functions \( \{ f_i \}_{i=1}^n \). Let \( \{ \lambda_j \}_{j=1}^k, \{ \gamma_i \}_{i=1}^n \) be any set of tuning parameters for (2.3). Let \( \{ \bar{w}_j \}_{j=1}^k, \bar{w}, \) and \( \{ \hat{\theta}_i \}_{i=1}^n \) be the minimizers of (2.3). Fix

\[
\alpha_j = \frac{\lambda_j}{\lambda_j + \sum_{i \in \mathcal{I}_j} \gamma_i}, \quad j = 1, \ldots, k,
\]

and let \( \{ \hat{\theta}_i \}_{i=1}^n \) denote the unique minimizer of (2.1). We then have \( \hat{\theta}_i = \hat{\theta}_i^\ast \) for all \( i \in [n] \).

**Proof.** See Appendix A.1.

As we will demonstrate in the sequel, while (2.1) and (2.3) have the same stationary points, optimizing the former does not require us to keep track of cluster or network averages, allowing these parameters to be calculated on the fly. This feature of (2.1) better suits the federated learning setting, removing the server’s need to keep track of additional variables during the optimization process.

To further motivate the objective in (2.1), we show that the maximum likelihood estimate of all parameters in a hierarchical generalized linear model can be viewed as the minimizer of (2.1) (Lee and Nelder, 1996; Stephen and Anthony, 2002; Bryk and Raudenbush, 1992). Specifically, let

\[
\tilde{\theta}_j^\ast \sim \mathcal{N}(\theta^\ast, \sigma^2_{center} I_d), \quad j = 1, \ldots, k,
\]

where \( \mathcal{N}(\cdot, \cdot) \) denotes a Gaussian distribution and \( \theta^\ast \in \mathbb{R}^d \). Furthermore, assume that for \( j = 1, \ldots, k \) and \( i \in \mathcal{I}_j \), we have

\[
\theta_i^\ast \sim \mathcal{N}(\tilde{\theta}_j^\ast, \sigma_j^2 I_d), \quad y_i \mid \theta_i^\ast, X_i \sim p_Y(y_i; u^{-1}(X_i^T \theta_i^\ast), \tau),
\]

where \( X_i \) and \( y_i \) are the input and output data for the \( i \)-th cluster.
where $X_i \in \mathbb{R}^{n_i \times d}$ is a matrix of observations, $y_i \in \mathbb{R}^{n_i}$ is the response vector, $u$ is a known link function, $p_Y(\cdot; \cdot, \cdot)$ is the probability density function of an exponential family distribution. The negative log-likelihood for this model is

$$
\ell(\{\theta_j\}_{j=1}^k; \{\tilde{\theta}_j\}_{j=1}^k; \{X_i, y_i\}_{i=1}^n, \tau) =
\sum_{j=1}^k \left( \log p_Y(y_i; u^{-1}(X_i^T \theta_j^*), \tau) + \frac{1}{2\sigma^2_{\text{cluster}} j} \| \theta_i - \tilde{\theta}_j \|^2 \right)
+ \frac{1}{2\sigma^2_{\text{center}}} \| \tilde{\theta}_j - \bar{\theta} \|^2,
$$

which is a special instance of (2.3). Then, by Theorem 2.2, minimizing (2.1) with suitable tuning parameters corresponds to the maximum likelihood estimation.

### 3 Algorithm and Convergence Analysis

We develop a loopless SGD-style algorithm to minimize the objective in (2.1). In each iteration, the algorithm randomly decides between descending on the loss functions $f_i$’s, descending on the regularizer controlling the distance between the local weights and the centers of the local clusters, and descending on both regularizers. The first kind of descent step can be done in parallel on all machines, the same as the local computation step in local SGD (Stich, 2018). The second kind requires clients within the same cluster to communicate with one another, but does not require between-cluster communication. Only when we simultaneously perform gradient descent on both the within-cluster and between-cluster regularizers do we communicate across clusters.

#### 3.1 Technical Preliminaries

We start by introducing an additional notation. Recall that $i = 1, \ldots, n$ is used to index clients, $j = 1, \ldots, k$ is used to index clusters, $\mathcal{I}_j$ is the set of clients belonging to the cluster $j$, with cardinality $|\mathcal{I}_j|$. Let $\Theta_j \in \mathbb{R}^{|\mathcal{I}_j| \times d}$ be the weight vector formed by stacking the weights of the clients in the cluster $j$ and $\Theta \in \mathbb{R}^{nd}$ be the weight vector formed by stacking the weights of all clients; that is,

$$
\theta_j = (\theta_{j[1]}, \ldots, \theta_{j[|\mathcal{I}_j|]}) \in \mathbb{R}^{|\mathcal{I}_j| \times d} \quad \text{and} \quad \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^{nd}.
$$

Without loss of generality, assume that clients are ordered according to the cluster to which they belong, that is, $\mathcal{I}_1 = \{1, \ldots, |\mathcal{I}_1|\}$ and $\mathcal{I}_k = \{|\mathcal{I}_{k-1}| + 1, \ldots, |\mathcal{I}_{k-1}| + |\mathcal{I}_k|\}$. We may then write the cluster-specific and global regularizers as

$$
\psi_j \left( \theta_j; \{\gamma_i\}_{i \in \mathcal{I}_j} \right) = \frac{1}{2} \sum_{i \in \mathcal{I}_j} \gamma_i \| \theta_i - \tilde{\theta}_j \|^2, \quad j = 1, \ldots, k,
$$

$$
\varphi \left( \theta; \{\gamma_i\}_{i=1}^n, \{\alpha_j\}_{j=1}^k \right) = \frac{1}{2} \sum_{j=1}^k \alpha_j \sum_{i \in \mathcal{I}_j} \gamma_i \| \theta_i - \bar{\theta} \|^2.
$$
The loss function in (2.1) can now be rewritten as
\[
\min_{\theta} F(\theta) = \sum_{j=1}^{k} \sum_{i \in I_j} f_i(\theta_i) + \sum_{j=1}^{k} (1 - \alpha_j) \psi_j\left(\{\theta_i; \{\gamma_i\}_{i \in I_j}\}\right) + \varphi\left(\{\theta; \{\gamma_i\}_{i=1}^{n}, \{\alpha\}_{j=1}^{k}\}\right).
\]

We explicitly calculate the gradients of the cluster-specific regularizers and the global regularizer in the following proposition.

Proposition 3.1. We have that
\[
\nabla_{\theta_i} \psi_j\left(\{\theta_i; \{\gamma_i\}_{i \in I_j}\}\right) = \gamma_i (\theta_i - \bar{\theta}), \quad i \in I_j, j = 1, \ldots, k;
\]
\[
\nabla_{\theta_i} \varphi\left(\{\theta; \{\gamma_i\}_{i=1}^{n}, \{\alpha\}_{j=1}^{k}\}\right) = \alpha_j \gamma_i (\theta_i - \bar{\theta}), \quad i \in I_j, j = 1, \ldots, k.
\]

Proof. See Appendix A.2.

3.2 Asynchronous Loopless Local Gradient Descent (Async-L2GD)

We formally introduce our main algorithm, Async-L2GD, in this section. Algorithm 1 provides the pseudocode. We present an accelerated, variance reduced, and stochastic variant of the algorithm for the finite-sum setting in Section 4.

At the beginning of each round in Algorithm 1, the network randomly decides whether to aggregate the averages between all clusters or not. If a global aggregation round is performed, each cluster first calculates its cluster average and then communicates with one another (or a central server) to compute the between-cluster aggregate.

In a round that does not involve communication between clusters, we allow each cluster to independently decide whether to communicate within the cluster or not by randomly sampling \(\xi_j \sim \text{Bernoulli}(p_j)\). By allowing \(p_j\) to be different across clusters, we effectively allow different clusters to have different communication schedules that are specific to them. In real-world applications, this flexibility can be appreciated. For example, when we partition clients according to their geographical locations, we effectively allow clients in different regions to communicate according to different schedules, which could reduce communication latency and energy consumption (Abad et al., 2020; Liu et al., 2020). We characterize the impact of \(p_j\) on the number of communication rounds in Theorem 3.2.

Proposition 3.2 (Expected Number of Communication Rounds). Suppose that Algorithm 1 is run for \(T\) rounds. The expected number of communication rounds between clusters is \(p_0(1 - p_0)T\). The expected number of communication rounds within the cluster \(j\) is \((1 - p_0)p_j(1 - p_j)T\).

Proof. By Lemma 4.3 in Hanzely et al. (2020a), the expected number of between-cluster communication rounds is given by \(p_0(1 - p_0)T\). A within-cluster communication round occurs only when \(\xi_0 = 0\), which occurs with probability \(1 - p_0\).

As can be seen from Algorithm 1, any communication between clusters necessarily implies that all clusters have computed within-cluster average. Intuitively speaking, it would then be more efficient to optimize \(\psi_j\left(\{\theta_i; \{\gamma_i\}_{i \in I_j}\}\right)\) whenever the cluster averages are computed in both the within-cluster and between-cluster aggregation rounds. Unfortunately, a naive implementation of this idea
Algorithm 1 Async-L2GD

1: **Input:** $\theta_0^0 = \ldots = \theta_n^0 = 0_d \in \mathbb{R}^d$, step size $\eta > 0$, probabilities $p_1, \ldots, p_k, p_0 \in [0, 1]$, fractions $\tau_1, \ldots, \tau_k \in [0, 1]$.  
2: **for** $t = 1, 2, \ldots$ **do**  
3: \hspace{1em} $\xi_0 = 1$ with probability $p_0$ and 0 with probability $1 - p_0$.  
4: \hspace{1em} **if** $\xi_0 = 1$ **then**  
5: \hspace{2em} All Clusters $j = 1, \ldots, k$ compute cluster average $\bar{\theta}_j^t = \frac{\sum_{i \in I_j} \gamma_i^t \theta_i^t}{\sum_{i \in I_j} \gamma_i}$.  
6: \hspace{2em} All Clusters communicate with one another and calculate network average 
7: \hspace{3em} $\bar{\theta}^t = \frac{\sum_{j=1}^k \sum_{i \in I_j} \alpha_j \gamma_i^t \theta_i^t}{\sum_{j=1}^k \sum_{i \in I_j} \alpha_j \gamma_i}$.  
8: \hspace{2em} All Clusters $j = 1, \ldots, k$ compute step towards both cluster average and network average for all clients $i \in I_j$ 
9: \hspace{3em} $\theta_i^{t+1} = \left(1 - \frac{\gamma_i^t (\alpha_j + \tau_j (1 - \alpha_j))}{p_0} \right) \theta_i^t + \frac{\gamma_i^t}{p_0} (\alpha_j \bar{\theta}^t + \tau_j (1 - \alpha_j) \bar{\theta}_j^t)$.  
10: **else**  
11: \hspace{2em} **for** All Clusters $j = 1, \ldots, k$ in parallel **do**  
12: \hspace{3em} $\xi_j = 1$ with probability $p_j$ and 0 with probability $1 - p_j$.  
13: \hspace{3em} **if** $\xi_j = 1$ **then**  
14: \hspace{4em} Compute cluster average $\bar{\theta}_j^t = \frac{\sum_{i \in I_j} \gamma_i^t \theta_i^t}{\sum_{i \in I_j} \gamma_i}$.  
15: \hspace{4em} Compute step towards cluster average for all clients $i \in I_j$  
16: \hspace{5em} $\theta_i^{t+1} = \left(1 - \frac{\gamma_i^t (1-\tau_j)(1-\alpha_j)}{(1-p_0)p_j} \right) \theta_i^t + \frac{\gamma_i^t (1-\tau_j)(1-\alpha_j)}{(1-p_0)p_j} \bar{\theta}_j^t$.  
17: \hspace{4em} **else**  
18: \hspace{5em} All Clients $i \in I_j$ perform a local gradient descent step  
19: \hspace{6em} $\theta_i^{t+1} = \theta_i^t - \frac{\eta}{(1-p_0)(1-p_j)} \nabla f_i(\theta_i^t)$.  
20: \hspace{4em} **end if**  
21: **end for**  
22: **end if**  
23: **end for**
would lead to a biased gradient oracle. Suppose that we make a gradient descent on \( \psi_j \) and \( \varphi \) with the same learning rate. Since we descend on \( \psi_j \) in both types of communication rounds, using the same learning rate for both \( \psi_j \) and \( \varphi \) would effectively cause us to descend twice on \( \psi_j \), as the stochastic gradient oracle is biased towards updating \( \psi_j \) more frequently. We introduce variables \( \tau_1, \ldots, \tau_j \) to scale the effective step size for \( \psi_j \) in the two different types of communication rounds, thereby ensuring that the stochastic gradient oracle is unbiased.

Algorithm 1 induces a stochastic gradient oracle that is equivalent to SGD with the following oracle. Assuming \( p_j, \tau_j \in (0, 1), j = 0, \ldots, k \), Algorithm 1 defines a stochastic gradient oracle for \( F(\theta) \), denoted

\[
G(\theta) = (G_1(\theta_1)^T, \ldots, G_n(\theta_n)^T)^T \in \mathbb{R}^{nd},
\]

where for each \( j \) and \( i \in \mathcal{I}_j \), \( G_i(\theta_i) \in \mathbb{R}^d \) and

\[
G_i(\theta_i) = \begin{cases} 
\frac{\gamma_i \alpha_j}{\hat{p}_j} (\theta_i^j - \bar{\theta}_j^0) + \frac{\gamma_i \tau_j (1-\alpha_j)}{\hat{p}_j} (\theta_i^j - \bar{\theta}_j^0), & \text{if } \xi_0 = 1 \\
\frac{\gamma_i (1-\tau_j)(1-\alpha_j)}{(1-\hat{p}_j)p_j} (\theta_i^j - \bar{\theta}_j^0), & \text{if } \xi_0 = 0 \text{ and } \xi_j = 1 \\
(1-\hat{p}_j)(1-p_j) \nabla f_i(\theta_i^j), & \text{if } \xi_0 = \xi_j = 0.
\end{cases}
\]

Note that \( G_i(\theta_i) \) is an unbiased estimator of \( \nabla_{\theta} F(\theta) \) for all \( i \), ensuring that \( G(\theta) \) is an unbiased estimator of \( \nabla_{\theta} F(\theta) \). Intuitively speaking, we can view Algorithm 1 as a stochastic gradient descent procedure on \( F(\theta) \) with step size \( \eta \), using a noisy gradient oracle \( G(\cdot) \).

The variance of \( G(\theta) \) with respect to \( \{\xi_j\}_{j=0}^k \) depends on a combination of \( \{p_j\}_{j=0}^k \) and \( \{\tau_j\}_{j=1}^k \). Fixing \( \tau_j \) to some arbitrary value is suboptimal, and we should properly tune the parameter. Intuitively speaking, when \( p_j \) is small, within-cluster communications are less frequent. A larger \( \tau_j \) allows between-cluster communication rounds to “help out” more when optimizing \( \psi_j (\theta_j; \{\gamma_i\}_{i \in \mathcal{I}_j}) \), instead of relying on within-cluster communication. We discuss how they should be adjusted according to the frequency of local and global aggregation rounds, that is, how \( \{\tau_j\}_{j=1}^k \) should be chosen given \( \{p_j\}_{j=0}^k \).

Proposition 3.3. Suppose \( \tau_j = p_0(\gamma_j + 2(1-p_0)p_j) \), \( j = 1, \ldots, k \). Then

\[
\mathbb{E}_{\{\xi_j\}_{j=0}^k} [||G(\theta) - G(\hat{\theta})||^2] \\
\leq \frac{2}{p_0} \left( \nabla_{\theta} \varphi(\theta) - \nabla_{\theta} \varphi(\hat{\theta}) \right)^2 + \sum_{j=1}^k \frac{2(1-\alpha_j)^2}{p_0 + 2(1-p_0)p_j} \left( \nabla_{\theta_j} \psi_j(\theta_j) - \nabla_{\theta_j} \psi_j(\hat{\theta}_j) \right)^2 \\
+ \frac{1}{1-p_0} \sum_{j=1}^k \frac{1}{1-p_j} \left( \nabla_{\theta_j} F_j(\theta_j) - \nabla_{\theta_j} F_j(\hat{\theta}_j) \right)^2.
\]

Proof. See Appendix A.3.

Intuitively, \( \tau_j \) balances between the probability of a communication round between clusters, \( p_0 \), and the probability of a communication round within a cluster, \((1 - p_0)p_j \). Therefore, it reduces the equivalent gradient oracle variance. Consider the extreme case where \( p_0 \neq 0 \) while \( p_j = 0 \), that is, there are no communication rounds within a cluster. Our choice for \( \tau_j \) is then exactly 1, which means that we will optimize \( \psi_j (\theta_j; \{\gamma_i\}_{i \in \mathcal{I}_j}) \) only during communication rounds between clusters, which is expected.
3.3 Convergence Analysis

We begin our convergence analysis by analyzing the convexity and smoothness of the loss function, $F(\theta)$. First, we show that the regularizers $\psi_j$ and $\varphi$ are convex and smooth.

**Proposition 3.4.** The regularizers are convex and smooth. In particular,

1. For all $j$, $\psi_j(\theta_j; \{\gamma_i\}_{i \in I_j})$ is convex and $\max_{i \in I_j} \gamma_i$-smooth in $\theta_j$.

2. $\varphi(\theta; \{\gamma_i\}_{i=1}^n, \{\alpha_j\}_{j=1}^k)$ is convex and $\max_{j=1,\ldots,k} \max_{i \in I_j} \alpha_j \gamma_i$-smooth in $\theta$.

**Proof.** See Appendix A.4. \qed

Under Theorem 2.1, from Theorem 3.4 it follows that the loss function $F(\theta)$ is $\mu$-strongly convex in $\theta$ and has a unique minimizer for any set of penalty parameters $(\{\alpha_j\}_{j=1}^k, \{\gamma_i\}_{i=1}^n)$. Let $\hat{\Theta}(\{\alpha_j\}_{j=1}^k, \{\gamma_i\}_{i=1}^n)$ be the corresponding minimizer. When the penalty parameters $\{\alpha_j\}_{j=1}^k, \{\gamma_i\}_{i=1}^n$ are clear from the context, we write

$$
\hat{\Theta} = \hat{\Theta}(\{\alpha_j\}_{j=1}^k, \{\gamma_i\}_{i=1}^n), \quad \psi_j(\cdot) = \psi_j(\cdot; \{\gamma_i\}_{i \in I_j}), \quad j = 1, \ldots, k,
$$

$$
\varphi(\cdot) = \varphi(\cdot; \{\gamma_i\}_{i=1}^n, \{\alpha_j\}_{j=1}^k), \quad j = 1, \ldots, k.
$$

Finally, we use $F_j(\theta_j) = \sum_{i \in I_j} f_i(\theta_i)$ to denote the average unregularized loss function in each cluster. The following theorem provides the convergence rate.

**Theorem 3.5** (Convergence Rate). Suppose Theorem 2.1 holds and $\{\tau_j\}_{j=1}^k$ are set according to Theorem 3.3. Let

$$
\mathcal{L} = \max \left\{ \frac{2}{p_0} \max_{j=1,\ldots,k} \max_{i \in I_j} \alpha_j \gamma_i, \max_{j=1,\ldots,k} \frac{2(1 - \alpha_j) \max_{i \in I_j} \gamma_i}{p_0 + 2(1 - p_0)p_j}, \frac{L}{1 - p_0} \max_{j=1,\ldots,k} \frac{1}{1 - p_j} \right\}, \quad (3.3)
$$

and

$$
\sigma^2 = \frac{2}{p_0} \|\nabla_{\theta} \varphi(\hat{\Theta})\|^2 + \sum_{j=1}^k \frac{2(1 - \alpha_j)^2}{p_0 + 2(1 - p_0)p_j} \|\nabla_{\theta_j} \psi_j(\hat{\Theta}_j)\|^2 \quad (3.4)
$$

$$
+ \frac{1}{1 - p_0} \sum_{j=1}^k \frac{1}{1 - p_j} \|\nabla_{\theta_j} F_j(\hat{\Theta}_j)\|^2.
$$

If the step size satisfies $\eta \leq \frac{1}{2\mathcal{L}}$, then

$$
\mathbb{E}[\|\theta^t - \hat{\Theta}\|^2] \leq (1 - \eta\mu)^t \|\theta^0 - \hat{\Theta}\|^2 + \frac{2\eta\sigma^2}{\mu}.
$$

**Proof.** See Appendix A.5. \qed

Compared with the single cluster result in Hanzely and Richtárik (2020), our convergence rate involves a few additional terms. By allowing each cluster to determine when to communicate individually, the expected smoothness coefficient and the variance of the gradient oracle at the
optimum are more complex, as they incorporate both local and global communication frequencies, \( \{p_j\}_{j=1}^k \) and \( p_0 \). When we set \( p_j = 0, \alpha_j = 1 \) for all \( j \), we recover the single cluster convergence rates given in Theorem 4.5 of Hanzely and Richtárik (2020) up to constant factors.

The parameters \( p_j, j = 0, \ldots, k \), that control the frequency of communication can be tuned by minimizing \( \mathcal{L} \) in principle. Unfortunately, as \( \mathcal{L} \) is effectively a maximum taken over \( 2k + 1 \) different terms, directly minimizing the expression is infeasible. We instead consider minimizing the following upper bound on \( \mathcal{L} \):

\[
\tilde{\mathcal{L}} = \max \left\{ \frac{2}{p_0} C_1, \max_{j=1,\ldots,k} \frac{2C_2}{p_0 + 2(1 - p_0)p_j}, \frac{L}{1 - p_0} \max_{j=1,\ldots,k} \frac{1}{1 - p_j} \right\},
\]

where \( C_1 = \max_{j=1,\ldots,k} \max_{i \in I_j} \alpha_j \gamma_i \) and \( C_2 = \max_{j=1,\ldots,k} \max_{i \in I_j} (1 - \alpha_j) \gamma_i \). The choice of parameters depends on the relationship between \( C_1 \) and \( C_2 \).

**Corollary 3.6.** Suppose that \( \{\tau_j\}_{j=1}^k \) are set according to Theorem 3.3.

When \( C_2 > C_1 \), setting \( \eta = \frac{1}{2C_2}, p_0 = \frac{2C_1}{C_1 + C_2 + L} \), and \( p_j = \frac{C_2 - C_1}{C_2 - C_1 + L} \) ensures that the optimal number of iterations is in \( \mathcal{O}\left( \frac{C_1(C_2+C_1+L)}{C_1(C_2+C_1+L)+\mu} \log \frac{1}{\varepsilon} \right) \), the number of communication rounds between clusters is in \( \mathcal{O}\left( \frac{C_1(C_2+C_1+L)}{C_1(C_2+C_1+L)+\mu} \log \frac{1}{\varepsilon} \right) \), and the number of communication rounds within a cluster is in \( \mathcal{O}\left( \frac{L}{(C_2-C_1)^2} \log \frac{1}{\varepsilon} \right) \) for all clusters.

When \( C_2 \leq C_1 \), setting \( \eta = \frac{1}{2C_2}, p_0 = \frac{2C_1}{C_1 + L} \), and \( p_j = 0 \) ensures that the optimal number of iterations is in \( \mathcal{O}\left( \frac{C_1+L}{\mu} \log \frac{1}{\varepsilon} \right) \), the number of communication rounds between clusters is in \( \mathcal{O}\left( \frac{C_1L}{(C_1+L)^2} \log \frac{1}{\varepsilon} \right) \), and the number of communication rounds within a cluster is 0.

**Proof.** See Appendix A.6.

We conclude the section by emphasizing that Algorithm 1 does not require a central server to aggregate information across all clusters. The central server in Algorithm 1 only serves two purposes: flipping a coin (\( \zeta_0 \)) to determine whether a communication round between clusters is necessary and calculating the global average should there be a communication round between clusters. The former can be easily decentralized across clusters by asking all cluster servers to flip a coin and take a majority vote, while the latter can be implemented by asking all cluster servers to communicate with one another.

### 4 Asynchronous Accelerated Loopless Local SGD with Variance Reduction

The convergence rate of Algorithm 1 when minimizing the loss in (2.1) is suboptimal due to the lack of acceleration and variance reduction. In this section, we propose an accelerated, variance reduced, stochastic variant of Algorithm 1 tailored to the finite-sum setting. The variant enjoys optimal communication complexity in a single cluster setting Hanzely et al. (2020a), regardless of the relationship between regularization strength and smoothness of loss functions \( f_i \), outperforming popular alternatives discussed in Dinh et al. (2020); Hanzely et al. (2021); Li et al. (2021); Mansour...
Algorithm 2 Async-AL2SGD+

**Input:** Step size $\eta$, probabilities $p_1, \ldots, p_k, p_0, \rho \in [0,1]$, fractions $\tau_1, \ldots, \tau_k \in [0,1].$

**Initialize:** $1 < a_0, a_1 < 1$, $b_1, b_2 > 0$, $x_i^0 = y_i^0 = z_i^0 = \theta_i^0 = 0_d \in \mathbb{R}^d$

for $t = 1, 2, \ldots$ do

All Clients $i = 1, \ldots, n$ perform local update

$\theta_i^t = a_1 z_i^t + a_2 x_i^t + (1 - a_1 - a_2) y_i^t.$

$\xi_0 = 1$ with probability $p_0$ and $0$ with probability $1 - p_0$

if $\xi_0 = 1$ then

All Clusters $j = 1, \ldots, k$ compute cluster average $\bar{\theta}_j^t = \frac{\sum_{i \in I_j} \gamma_i \theta_i^t}{\sum_{i \in I_j} \gamma_i}$

All Clusters aggregate network average $\bar{\theta}^t = \frac{\sum_{j=1}^k \sum_{i \in I_j} \alpha_j \gamma_i \theta_i^t}{\sum_{j=1}^k \sum_{i \in I_j} \alpha_j \gamma_i}$

All Clients calculate gradient estimate according to (4.1)
Set $y_i^{t+1} = \theta_i^t - \eta g_i^t$

else

for All Clusters $j = 1, \ldots, k$ in parallel do

$\xi_j = 1$ with probability $p_j$ and $0$ with probability $1 - p_j$

if $\xi_j = 1$ then

Compute cluster average $\bar{\theta}_j^t = \frac{\sum_{i \in I_j} \gamma_i \theta_i^t}{\sum_{i \in I_j} \gamma_i}$ and send it back to each client

All Clients calculate gradient estimate according to (4.1)
Set $y_i^{t+1} = \theta_i^t - \eta g_i^t$

else

All Clients calculate gradient estimate according to (4.1)
Set $y_i^{t+1} = \theta_i^t - \eta g_i^t$

end if

end for

end if

All Clients $i = 1, \ldots, n$ update: $z_i^{t+1} = b_1 z_i^t + (1 - b_1) y_i^t + \frac{b_2}{\eta} (y_i^{t+1} - \theta_i^t)$

$\xi' = 1$ with probability $\rho$ and $0$ with probability $1 - \rho$

if $\xi' = 0$ then

For all Clients $i = 1, \ldots, n$: $x_i^{t+1} = x_i^t$

else

For all Clients $i = 1, \ldots, n$ update $x_i^{t+1} = y_i^{t+1}$, and evaluate and store $\nabla f_i(x_i^{t+1})$

All Clusters communicate, compute averages $\bar{x}^t = \frac{\sum_{j=1}^k \sum_{i \in I_j} \alpha_j \gamma_i x_i^t}{\sum_{j=1}^k \sum_{i \in I_j} \alpha_j \gamma_i}$, $\bar{x}_j^t = \frac{\sum_{i \in I_j} \gamma_i x_i^t}{\sum_{i \in I_j} \gamma_i}$ for all $j$,
and send them back to the clients.

end if

end for
et al. (2020). We further hypothesize that the optimality holds when extended to multi-cluster setup studied here.

We assume that the local loss has a finite sum structure over smooth and strongly convex functions, a common assumption in the literature on accelerated variance reduced algorithms (Hanzely et al., 2020a; Hanzely and Richtárik, 2020; Kovalev et al., 2020). We formally characterize our assumption below.

**Assumption 4.1.** The loss function $f_i$, $i = 1, \ldots, n$, has the following finite structure:

$$f_i(\theta) = \frac{1}{n_i} \sum_{l=1}^{n_i} \tilde{f}_{i,l}(\theta),$$

where $\tilde{f}_{i,l}$ is $\tilde{L}$-smooth and $\mu$-strongly convex, $l = 1, \ldots, n_i$.

We define a stochastic gradient estimate for all clients, similar to the construction for Algorithm 1. For a client $i$ that belongs to the cluster $j$, the variance reduced stochastic gradient is

$$g_i^t = \nabla f_i(x_i^t) + \alpha_j \gamma_i(x_i^t - \bar{x})^t + (1 - \alpha_j) \gamma_i(x_i^t - \bar{x}^t)$$

$$+ \mathbb{I}\{\xi_0 = 1\} \frac{\gamma_i \alpha_j}{p_0} (\theta_i^t - \bar{\theta} - (x_i^t - \bar{x}^t))$$

$$+ \mathbb{I}\{\xi_0 = 0\} \mathbb{I}\{\xi_j = 1\} \frac{\gamma_i (1 - \tau_j)(1 - \alpha_j)}{(1 - p_0)p_j} ((\theta_i^t - \bar{\theta}_j) - (x_i^t - \bar{x}_j^t))$$

$$+ \mathbb{I}\{\xi_0 = 0\} \mathbb{I}\{\xi_j = 0\} \frac{1}{(1 - p_0)(1 - p_j)} \left( \nabla \bar{f}_{i,l}(\theta_i^t) - \nabla \tilde{f}_{i,l}(x_i^t) \right),$$

where $l$ is selected uniformly at random at each iteration for every client. At a high level, (4.1) defines a stochastic gradient oracle for a finite-sum composite optimization problem. The different realizations of $\{\xi_0, \ldots, \xi_j\}$ determine the type of communication round to execute, if any, at any given step. For example, when $\xi_0 = 1$, a communication round between the client clusters is executed, while when $x_0 = 0$ and $\xi_1 = 1$, the first client cluster executes a communication round within the cluster. The following lemma provides a bound on the variance of the stochastic gradient oracle in (4.1).

**Lemma 4.2.** Suppose that Theorem 4.1 holds and that $\{\tau_j\}_{j=1}^k$ are selected as in Theorem 3.3. Let

$$\mathcal{L} = \max \left\{ \frac{2}{p_j} \max_{j=1, \ldots, k} \alpha_j \gamma_i, \max_{j=1, \ldots, k} \frac{2(1 - \alpha_j) \max_{i \in I_j} \gamma_i}{p_0 + 2(1 - p_0)p_j}, \max_{j=1, \ldots, k} \frac{\tilde{D}}{1 - p_0} \right\},$$

Then

$$\mathbb{E} \left[ \|g^t - \nabla F(x^t)\|^2 \right] \leq 2\mathcal{L}D_F(\theta^t, x^t),$$

(4.2)

where $g^t = (g_1^t, \ldots, g_n^t)^T \in \mathbb{R}^{nd}$ is the variance reduced stochastic gradient oracle, where $g_i^t$ is defined in (4.1), and $D_F(x_1, x_2) := F(x_1) - F(x_2) - \langle \nabla F(x_2), x_1 - x_2 \rangle$ is the Bregman divergence induced by the loss function $F$ in (2.1).

**Proof.** See Appendix A.7.
We use the oracle to construct an instance of L-Katyusha, a loopless, variance-reduced, accelerated algorithm Qian et al. (2021b). Algorithm 2 provides the pseudocode, while Theorem 4.3 provides the convergence rate for Algorithm 2.

**Theorem 4.3.** Suppose that the conditions of Theorem 4.2 are satisfied. Let

\[
L_F = \tilde{L} + \max_{i=1,...,n} \gamma_i, \quad \eta = \frac{1}{4} \max \{L_F, \mathcal{L} \}^{-1},
\]

\[
a_1 = \min \left\{ \frac{1}{2}, \sqrt{\eta \mu \max \left\{ \frac{1}{2}, \frac{a_2}{\rho} \right\}} \right\}, \quad a_2 = \frac{\mathcal{L}}{2 \max \{L_F, \mathcal{L} \}},
\]

\[
b_1 = 1 - b_2 \mu, \quad b_2 = \frac{1}{\max\{2\mu, 4a_1/\eta\}}.
\]

Then the iteration complexity of Algorithm 2 is

\[
\mathcal{O} \left( \left( \frac{1}{\rho} + \sqrt{\rho \mu} \right) \log \frac{1}{\epsilon} \right).
\]

**Proof.** See Appendix A.8. \qed

In addition to the iteration complexity, it is easy to obtain communication bounds, gradient complexity, as well as optimal parameters. Similar to Theorem 3.6, directly minimizing \( \mathcal{L} \) over all the parameters is infeasible and we consider the following upper bound on \( \mathcal{L} \) instead:

\[
\tilde{\mathcal{L}} = \max \left\{ \frac{2C_1}{p_0} \frac{2C_2}{p_0 + 2(1 - p_0)p_j}, \frac{\tilde{L}}{1 - p_0} \frac{1}{\max_{j=1,...,k} 1 - p_j} \right\},
\]

where \( C_1 = \max_{j=1,...,k} \max_{i \in \mathcal{I}_j} \alpha_j \gamma_i \) and \( C_2 = \max_{j=1,...,k} \max_{i \in \mathcal{I}_j} (1 - \alpha_j) \gamma_i \). The upper bound is virtually the same as the one in (3.5), and we have the following.

**Corollary 4.4.** Consider Algorithm 2 with a fixed \( \rho \) and the tuning parameters set as:

\[
L_F = \tilde{L} + \max_{i=1,...,n} \gamma_i, \quad \eta = \frac{1}{4} \max \{L_F, \mathcal{L} \}^{-1},
\]

\[
a_1 = \min \left\{ \frac{1}{2}, \sqrt{\eta \mu \max \left\{ \frac{1}{2}, \frac{a_2}{\rho} \right\}} \right\}, \quad a_2 = \frac{\mathcal{L}}{2 \max \{L_F, \mathcal{L} \}},
\]

\[
b_1 = 1 - b_2 \mu, \quad b_2 = \frac{1}{\max\{2\mu, 4a_1/\eta\}}.
\]

Furthermore, \( \{\tau_j\}_{j=1}^k \) is set according to Theorem 3.3.

When \( C_2 > C_1 \), setting \( p_0 = \frac{2C_1}{C_1 + C_2 + \tilde{L}} \) and \( p_j = \frac{C_2 - C_1}{C_2 - C_1 + \tilde{L}} \) ensures that the optimal number of iterations is in \( \mathcal{O} \left( \sqrt{\frac{C_1 + C_2 + \tilde{L}}{\mu}} \log \frac{1}{\epsilon} \right) \), the number of communication rounds between clusters and within a cluster is in \( \mathcal{O} \left( \frac{C_1(C_2 - C_1 + \tilde{L})}{(C_1 + C_2 + \tilde{L})\sqrt{(C_1 + C_2 + \tilde{L})\mu}} \log \frac{1}{\epsilon} \right) \) and \( \mathcal{O} \left( \frac{(C_2 - C_1)\tilde{L}}{(C_2 - C_1 + \tilde{L})\sqrt{(C_1 + C_2 + \tilde{L})\mu}} \log \frac{1}{\epsilon} \right) \), respectively.
When $C_2 \leq C_1$, setting $p_0 = \frac{2C_1}{2C_1 + L}$ and $p_j = 0$ ensures the optimal number of iterations is in $O\left(\sqrt{\frac{C_1 + L}{\mu}} \log \frac{1}{\epsilon}\right)$, the number of communication rounds between clusters is in $O\left(\frac{C_1 L}{(C_1 + L)\sqrt{(C_1 + L)\mu}} \log \frac{1}{\epsilon}\right)$, and the number of communication rounds within a cluster is 0.

Proof. See Appendix A.9.

5 Case Study: Hierarchical Linear Model

In the previous section, we have answered how to minimize the objective in (2.1). Next, we provide a statistical model of personalization under which the minimizer of (2.1) corresponds to an estimator that outperforms the common alternatives. More precisely, we show that the minimizer of (2.1) strictly outperforms both (a) training a single global model for all clients and (b) training a separate model for each client independent of the data of other clients. Unlike the analysis in Li et al. (2021), we consider the hierarchical, multi-cluster regime. Existing approaches to personalized federated learning often use loss functions similar to those discussed in (2.1) and (2.3). Therefore, it is important to understand the statistical properties of the corresponding minimizers.

Although the minimizer (2.1) outperforms commonly used alternatives, we also provide two alternative estimators that are hard to efficiently compute in a federated learning setting, yet dominate our proposed estimator. Specifically, they achieve a lower mean squared error. The efficient implementation of the two alternatives remains an open question for future research.

The statistical model we consider in this section is based on a hierarchical linear model with Gaussian priors (Stephen and Anthony, 2002). Nature first draws the cluster centers from a Gaussian distribution with unknown mean and then draws each client’s parameter from a Gaussian distribution centered at the cluster center the client belongs to. More precisely, for an unknown parameter $\bar{\theta}^* \in \mathbb{R}^d$, our model is:

\[\bar{\theta}^*_j = \bar{\theta}^* + \bar{\xi}_j, \quad \bar{\xi}_j \sim \mathcal{N}(0, \sigma_j^2 I_d), \quad j = 1, \ldots, k,\]

\[\theta^*_i = \bar{\theta}^*_j + \xi_i, \quad \xi_i \sim \mathcal{N}(0, \bar{\sigma}_j^2 I_d), \quad i \in I_j,\]

\[y_i = X_i \theta^*_i + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, \sigma_i^2 I_{n_i}), \quad i \in I_j,\]

where $\bar{\theta}^*_j \in \mathbb{R}^d$ represents the center of the cluster $j$, $\theta^*_i \in \mathbb{R}^d$ represents the client-specific parameter, and $(X_i, y_i) \in \mathbb{R}^{n_i \times d} \times \mathbb{R}^{n_i}$ is the data set on the $i$-th client.

When estimating all client parameters simultaneously in (5.1), we obtain the following maximum likelihood estimation problem:

\[
\min_{(\theta_i)} \sum_{j=1}^{k} \sum_{i \in I_j} \left( \frac{1}{\sigma_j^2} \|y_i - X_i \theta_i\|^2 + \frac{\gamma_j \alpha_j}{2} \|\theta_i - \bar{\theta}\|^2 + \frac{\gamma_j (1 - \alpha_j)}{2} \|\theta_i - \bar{\theta}_j\|^2 \right),
\]
where
\[
\bar{\theta} = \left( \sum_{j' = 1}^{k} \sum_{i' \in I_{j'}} \gamma_{i'} \alpha_{j'} \right)^{-1} \sum_{j = 1}^{k} \sum_{i \in I_{j}} \gamma_{i} \alpha_{j} \theta_{i};
\]
\[
\hat{\theta}_{j} = \left( \sum_{i' \in I_{j}} \gamma_{i'} \right)^{-1} \sum_{i \in I_{j}} \gamma_{i} \theta_{i}, \quad j = 1, \ldots, k.
\]

The objective is an instance of (2.1), and by Theorem 2.2, is equivalent to
\[
\min_{\{\theta_{i}\}_{i=1}^{n}, (w_{j})_{j=1}^{k}, \bar{w}} \sum_{j=1}^{k} \left( \frac{\lambda_{j}}{2} \|w_{j} - \bar{w}\|^{2} + \sum_{i \in I_{j}} \left( \frac{1}{2 \sigma_{i}^{2}} \|y_{i} - X_{i} \theta_{i}\|^{2} + \frac{\gamma_{i}}{2} \|\theta_{i} - w_{j}\|^{2} \right) \right). \tag{5.2}
\]

We focus on (5.2) for convenience and show that, when \(\{\lambda_{j}\}_{j=1}^{k}, \{\gamma_{i}\}_{i=1}^{n}\) are properly tuned and \(\bar{\sigma}^{2}, \sigma^{2}\) are known, the resulting minimizers \(\{\hat{\theta}_{i}\}_{i=1}^{n}\) attain the smallest mean squared error among a class of linear unbiased estimators (Kariya and Kurata, 2004).

**Theorem 5.1.** Suppose that \(\lambda_{j} = (\bar{\sigma}^{2})^{-1}, \gamma_{i} = (\sigma^{2})^{-1}\), and \(X_{i}^{T} X_{i} = \beta_{i} I_{d}\) for some \(\beta_{i} \in \mathbb{R}, i \in I_{j}\), \(j = 1, \ldots, k\). Then \(\hat{\theta}_{i}\), obtained as the minimizer of (5.2), is the best linear unbiased estimator of \(\theta_{i}^{\ast}\) given \(\{(X_{i}, y_{i})\} \cup \{(\hat{\theta}_{i}^{d})_{i' \neq i}\}, \) where \(\hat{\theta}_{i}^{d} = (X_{i}^{T} X_{i})^{-1} (X_{i} y_{i})\).

**Proof.** See Appendix A.10.

To provide an intuition behind Theorem 5.1, we first consider the single cluster setting in Li et al. (2021). Focusing on an arbitrary cluster \(j\), we know that for all \(i \in I_{j}\), \(\theta_{i}\) consists of two parts: the first part estimates \(\theta_{i}^{p}\) using only data on the \(i\)-th client and the second part estimates \(\theta_{i}^{p}\) using other clients’ parameters. Since (5.2) aggregates information across clients only by regularizing the distance between weight estimates, the estimate for \(\hat{\theta}_{i}^{p}\) cannot depend directly on \(\{(X_{i'}, y_{i'})\}_{i' \neq i}\) and is constructed instead through \(\{(\hat{\theta}_{i}^{d})_{i' \neq i}\}_{i' \neq i}\). In our multi-cluster setting, the estimator \(\hat{\theta}_{i}\) operates in a similar fashion, as can be seen from the proof of Theorem 5.1. Since we do not have direct access to the data of other clients, the estimates for \(\hat{\theta}_{i}^{p}, \theta_{i}^{p}\) are constructed indirectly using \(\{(\hat{\theta}_{i}^{d})_{i' \neq i}\}_{i' \neq i}\).

A direct consequence of Theorem 5.1 is the following.

**Corollary 5.2.** Suppose that the conditions of Theorem 5.1 hold. Let \(\hat{\theta}_{i}^{all} = (\sum_{i=1}^{n} X_{i}^{T} X_{i})^{-1} (\sum_{i=1}^{n} X_{i} y_{i})\). Then
\[
\mathbb{E}[\|\hat{\theta}_{i} - \theta_{i}^{\ast}\|^{2}] \leq \min \left\{ \mathbb{E}[\|\hat{\theta}_{i}^{d} - \theta_{i}^{\ast}\|^{2}], \mathbb{E}[\|\hat{\theta}_{i}^{all} - \theta_{i}^{\ast}\|^{2}] \right\},
\]
where \(\hat{\theta}_{i}\) and \(\hat{\theta}_{i}^{d}\) are defined in Theorem 5.1, and the expectation is taken over the randomness in (5.1).

Theorem 5.2 illustrates existence of a regime under which a personalized estimator consistently outperforms the alternatives, learning a single model for all clients without any personalization \(\hat{\theta}_{i}^{all}\) and learning a model independently for each client \(\hat{\theta}_{i}^{d}\) proposed in Chen et al. (2021), highlighting the effectiveness and necessity of personalization within highly structured problems. Note that \(\hat{\theta}_{i}^{d}\) and \(\hat{\theta}_{i}^{all}\) can both be written as unbiased linear estimators of \(\theta_{i}^{\ast}\) given \(\{(X_{i}, y_{i})\} \cup \{(\hat{\theta}_{i}^{d})_{i' \neq i}\}\). The optimality of \(\hat{\theta}_{i}\) among this class of estimators ensures that its mean squared error is no greater than these alternatives.
5.1 Limitations of Unbiased Estimators

We have shown that solving (5.2) consistently outperforms common alternatives \( \hat{\theta}^{all} \) and \( \hat{\theta}_i^d \). In particular, regularizing the distance between client model parameters and average model parameters provides a viable approach for personalization. See also Hanzely and Richtárik (2020); Li et al. (2021); Hanzely et al. (2021); Dinh et al. (2020). Furthermore, our result complements Chen et al. (2021), identifying a regime in which personalization consistently outperforms learning a single model and learning models independently. Unfortunately, we cannot guarantee that our approach is optimal among all possible estimators. We discuss two alternative estimators that result in a lower mean squared error, but are hard to implement in federated learning setting.

First, we note that (5.2) is the maximum likelihood estimator when we simultaneously estimate all clients’ parameters. However, for any particular client, we can derive an unbiased linear estimator with smaller mean squared error, by marginalizing other clients’ and clusters’ parameters.

**Proposition 5.3.** Suppose \( \{(X_i, y_i)\}_{i=1}^n \) are generated according to the model in (5.1). For any \( i \), there exists a linear unbiased estimator \( \hat{\theta}_i \) of \( \theta^*_i \) that satisfies \( \mathbb{E}|\hat{\theta}_i - \theta^*_i|^2 \leq \mathbb{E}|\hat{\theta}_i - \theta^*_i|^2 \).

**Proof.** See Appendix A.11.

By marginalizing other parameters, we derive \( \hat{\theta}_i \), the best linear unbiased estimator of \( \theta^*_i \) given \( \{(X_{i'}, y_{i'})\}_{i'=1}^n \), which includes data from other clients. An explicit form for the equation that \( \hat{\theta}_i \) solves can be found in Appendix A.11. By contrast, \( \hat{\theta}_i \) only has access to other clients’ weight estimates, \( \{\hat{\theta}_i^{d'}\}_{i' \neq i} \), but does not have direct access to their data. We emphasize, however, that doing so for all the clients is costly, as we need to solve a separate generalized least squares problem for all clients. On the other hand, learning \( \hat{\theta}_i \) for all clients can be done simultaneously by optimizing (5.2).

An alternative estimator of \( \theta^*_i \) can be constructed based on James-Stein estimator James and Stein (1992). Suppose that \( X \sim N(\xi, I) \) is a \( p \)-dimensional Gaussian random vector with mean \( \xi \) and covariance \( I_p \). The James-Stein Estimator of \( \xi \) is defined as

\[
\hat{\xi}^{JS} = \left( 1 - \frac{p-2}{\|X\|^2} \right) X.
\]

Let \( \hat{\xi}^{MLE} \) be the maximum likelihood estimator of \( \xi \), that is, \( \hat{\xi}^{MLE} = X \). Then, for all \( p \geq 3 \),

\[
\mathbb{E}\left(\|\hat{\xi}^{JS} - \xi\|^2\right) \leq \mathbb{E}\left(\|\hat{\xi}^{MLE} - \xi\|^2\right).
\]

See James and Stein (1992) for a proof.

We construct a biased estimator that dominates \( \hat{\theta}_i \) even in a simplified, single cluster regime, under which we are effectively solving a \( d \)-dimensional point estimation problem.

**Proposition 5.4.** Consider a single-cluster model under which \( y_i \sim N(\theta^*_i, I_d) \), where \( \theta^*_i \sim N(\theta^*, I_d) \), \( i = 1, \ldots, n \), and \( \theta^* \in \mathbb{R} \) is an unknown parameter. If \( d > 3 \), then there exists a biased estimator \( \hat{\theta}_i^{JS} \) such that \( \mathbb{E}|\hat{\theta}_i^{JS} - \theta^*_i|^2 \leq \mathbb{E}|\hat{\theta}_i - \theta^*_i|^2 \), where \( \hat{\theta}_i \) is the best linear unbiased estimator for \( \theta^*_i \) given \( \{(y_i)\}_{i=1}^n \).

**Proof.** See Appendix A.12.
Efficiently implementing the biased estimator is non-trivial. The estimator requires careful adjustment of a shrinkage coefficient to achieve a smaller mean squared error. How to tune this shrinkage coefficient efficiently in a federated learning setting is unclear. As a result, the minimizer of (5.2) is a great practical alternative.

The two estimators provided in this section also point out that alternative approaches to personalized federated learning (Deng et al., 2020; Li et al., 2021; Hanzely and Richtárik, 2020) are not optimal from a statistical point of view under a hierarchical linear model. While these two estimators are impractical in federated learning setting, it remains an open question how to approximate them with efficient computation and communication, while also respecting the privacy concerns in federated learning.

6 Numerical Results

We illustrate the performance of our algorithm on both simulated data and a real-world marketing data set. We focus on the generalization error of the minimizer of (2.1), rather than on the optimization performance of Algorithms 1 and 2, since related approaches have been studied in the single-cluster regime Li et al. (2021); Hanzely and Richtárik (2020); Hanzely et al. (2020a, 2021); Dinh et al. (2020). Our aim is to complement those studies.

We consider a modified version of Algorithm 1, where one coin toss is used to determine whether the clients should perform a local step or a communication round. During a communication round, each machine minimizes its distance to the cluster and global average simultaneously. Our experiments focus on the generalization behavior of the estimator rather than on the optimization error. Therefore, such a simplification does not affect the validity of our results and simplifies the implementation. Scripts for replicating the experiments can be found this GitHub repository.

6.1 Simulation Studies

We compare our algorithm with the tuning parameters set as in Theorem 5.1 against three baselines: i) learning a single model for all clients, ii) learning each client’s model independently, and iii) learning personalized models centered around a single point Hanzely and Richtárik (2020); Li et al. (2021); Dinh et al. (2020).

Simulation data are generated from the following hierarchical linear model. The center of all clusters is $\hat{\theta}^* = \mathbf{0}_{20} \in \mathbb{R}^{20}$, where $\mathbf{0}_{20}$ is an all-zero vector. There are 20 clusters, each with 20 clients. The cluster centers and client parameters are generated as:

$$
\hat{\theta}^*_j \sim \mathcal{N}(\hat{\theta}^*, I_{20}), \quad j = 1, \ldots, k,
$$

$$
\theta^*_i \sim \mathcal{N}(\hat{\theta}^*_j, I_{20}), \quad i \in \mathcal{I}_j.
$$

On each client, we generate a data matrix $X_i \in \mathbb{R}^{m \times 20}$, where $m$ is the number of samples on the $i$-th client and is selected from $\{1, 5, 25, 50, 100, 200\}$. Each entry in $X_i$ is drawn i.i.d. from the standard Gaussian distribution, $\mathcal{N}(0, 1)$. Subsequently, the response $y_i$ is drawn from the linear model:

$$
y_i \mid X_i \sim \mathcal{N}(X_i \theta^*_i, I_{n_i}).
$$
Under the data generating procedure described above, the three baselines take the following form:

1. Training a single-model for all clients: \( \hat{\theta}_{\text{sm}} = (\sum_{i=1}^{n} X_i^T X_i)^{t} (\sum_{i=1}^{n} X_i^T y_i) \).

2. Entirely locally-trained estimator: for each client \( i \), \( \hat{\theta}_i^{\text{lt}} = (X_i^T X_i)^{t} X_i^T y_i \), where \( (X_i^T X_i)^{t} \) is the Moore-Penrose pseudo-inverse of the empirical covariance matrix.

3. Training a single-cluster personalized model (Hanzely and Richtárik, 2020; Li et al., 2021; Dinh et al., 2020): the objective function is given by

\[
\min_{\theta_i} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{2} \| X_i \theta_i - y_i \|^2 + \frac{\lambda^{\text{sc}}}{2} \| \theta_i - \bar{\theta} \|^2 \right),
\]

where \( \bar{\theta} = n^{-1} \sum_{i=1}^{n} \theta_i \) and \( \lambda^{\text{sc}} > 0 \) is user-chosen parameter that controls the strength of personalization. The minimizer of the objective can be obtained as \( \hat{\theta}_i^{\text{sc}} = (X_i^T X_i + \lambda^{\text{sc}} I_{20})^{-1} \left( (X_i^T X_i) \hat{\theta}_i^{\text{lt}} + \lambda^{\text{sc}} \hat{\theta}_i^{\text{sc}} \right) \), where

\[
\hat{\theta}_{\text{sm}} = \left( I_{20} - \frac{\lambda^{\text{sc}}}{n} \sum_{i=1}^{n} (X_i^T X_i + \lambda^{\text{sc}} I_{20})^{-1} \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} (X_i^T X_i + \lambda^{\text{sc}} I_{20})^{-1} (X_i^T X_i) \hat{\theta}_i^{\text{lt}} \right).
\]

We tune \( \lambda^{\text{sc}} \) over 20 evenly spaced points in \([0.01, 2]\) using cross validation.

Following Theorem 5.1, we set \( \lambda_j = 1 \) for all \( j \) and \( \gamma_i = 1 \) for all \( i \). We use Theorem 2.2 to convert the maximum likelihood estimation problem into the form in (2.1), which we minimize using the simplified algorithm with communication probability \( p = 0.1 \), stepsize \( \eta = 10^{-4} \), and maximum number of iterations 50000. We use \( \{ \hat{\theta}_{\text{our}} \} \) to denote the estimators produced by our model.

We measure the performance of different estimators using the \( \ell_2 \) distance between the estimates of the parameters of the clients and their actual parameters, that is, \( \| \hat{\theta}_i - \theta_i^* \|_2^2 \) for \( \hat{\theta}_i \in \{ \hat{\theta}_{\text{sm}}, \hat{\theta}_i^{\text{lt}}, \hat{\theta}_i^{\text{sc}}, \hat{\theta}_i^{\text{our}} \} \). The results are averaged over five independent runs.

To demonstrate that our model consistently outperforms baselines, we pick two specific choices of \( m \), \( m = 10 \), and \( m = 100 \). Intuitively, as \( m \) increases, local training becomes more viable, whereas a smaller \( m \) means that training a single model could be more beneficial. Here, we show that our suggested approach outperforms both these alternatives regardless of \( m \). Figure 1 and Table 1 confirm that our proposed method consistently outperforms alternatives for \( m \in \{ 1, 5, 10, 25, 100, 200 \} \).

### 6.2 Application: Donor Response

We illustrate our algorithm on a real data set that contains donations and solicitation histories from a leading nonprofit organization in the US Blattberg et al. (2009). We follow the experimental setup described in Bumbaca et al. (2017). For each solicitation record, we use its recency and frequency as covariates, where recency is defined as the number of days since the donor’s last donation and frequency is the number of past donations. We define a solicitation as successful when the donor has made a donation after the current solicitation attempt and before the next solicitation attempt and
We view each ZIP code as an individual client and group the clients using the median household income of the clients in the ZIP code, based on the data obtained from www.unitedstateszipcodes.org. Using the income brackets defined in Snider (2019), we group the ZIP code into 4 different clusters: poor-or-near-poor, lower-middle-class, middle-class, and upper-middle-class. ZIP codes with no recorded median household income are grouped into a fifth category, and ZIP codes with less than 5 solicitations are removed. We retain 29490 clients and 5 clusters after processing.

We tune the parameters $\{\lambda_j\}_{j=1}^k$, $\{\gamma_i\}_{i=1}^n$, defined in (2.3), by performing a grid search over $\left\{10^{-2}, 10^{-1.875}, 10^{-1.750}, \ldots, 10^{-1.875}, 10^2\right\}$. The chosen values are then used to calculate the corresponding values for $\{\alpha_j\}_{j=1}^k$. We set the test-train ratio to 0.2, communication probability to 0.1, stepsize to $10^{-3}$, and the maximum number of iterations to 5000.

We record the accuracy and cross-entropy for each individual client. We set the cross-entropy to 100 for clients whose cross-entropy overflows. We then tune $\lambda, \gamma$ with cross-validation, based on the

\[\text{Avg. } (\pm \text{SD.}) \quad \text{Max} \quad \text{Avg. } (\pm \text{SD.}) \quad \text{Max}\]

|       | $m = 10$ |       | $m = 100$ |
|-------|---------|-------|-----------|
|       | Avg. (±SD.) | Max | Avg. (±SD.) | Max |
| $\hat{\theta}_{lt}$ | 4.50 (±0.981) | 8.224 | 0.494 (±0.093) | 0.768 |
| $\hat{\theta}_{sm}$ | 6.11 (±0.968) | 9.025 | 6.243 (±1.003) | 9.479 |
| $\hat{\theta}_{sc}$ | 4.46 (±0.958) | 8.112 | 0.494 (±0.093) | 0.763 |
| $\hat{\theta}_{our}$ | 3.46 (±0.689) | 8.676 | 0.489 (±0.093) | 0.748 |

Table 1: The average $\ell_2$ distance versus the number of local samples.
Figure 2: Distribution of cross-entropy losses at each ZIP code. From left to right: personalized model using selected parameters, models trained using entirely local data, and a single model trained on all data.

|                | Avg. (± SD.) | 25%  | 50%  | 75%  | Max  |
|----------------|-------------|------|------|------|------|
| Locally-trained | 0.413(±0.932) | 0.173 | 0.247 | 0.313 | 6    |
| Single-model   | 0.211(±0.176) | 0.058 | 0.201 | 0.273 | 2.76 |
| **Our Model**  | 0.215(±0.179) | 0.069 | 0.209 | 0.283 | 2.668|

Table 2: Summary statistics of cross-entropy losses at each ZIP code. First column is the client-specific cross entropy, second column the first quartile, third column second quartile (median), and fourth column third quartile. In the fifth column we report the maximum client-specific cross-entropy.

average cross-entropy taken across all the clients. Using cross-validation, we decide on $\lambda_j = 0.01$ for all $j$ and $\gamma_i = 1.360$ for all $j$.

Figure 2 visualizes the performance of our model compared to naive baselines. Specifically, it characterizes the distribution of the clients’ cross-entropy. To complement Figure 2, we include more quantitative results in Table 2, which contains summary statistics on the clients’ cross-entropy. From the table, we observe that our model is comparable to training a single model on all data in terms of averages and quartiles, while it has a lower maximum client-specific cross-entropy. In particular, we can see from Figure 2 that our model has a smaller percentage of clients with average cross-entropy above 1.0: 183 ZIP codes have cross-entropy loss greater than 1 in the personalized model, whereas 202 have cross-entropy greater than 1 in the single-model alternative. Our model also significantly outperforms training a local model for each ZIP code separately.

Although our model attains slightly higher cross-entropy than training a single model for all clients, the accuracy of both models are almost the same, as we can see from Section 6.2. We conjecture that the higher cross-entropy is due to a lack of emphasis on tuning the learning rates and optimizing until convergence for our proposed personalized model.

7 Conclusions and Future Directions

We propose a new approach to personalization in federated learning when there are multiple known clusters among the clients. Our algorithm is based on a variant of loopless local gradient descent that allows each cluster to have its own communication schedule. The estimator is shown to be optimal among a class of unbiased linear estimators and performs better than commonly used
| Alternative         | Avg. (± SD.)     | 25%  | 50%  | 75%  | Perf. Ratio |
|--------------------|------------------|------|------|------|-------------|
| Locally-trained    | 0.940 (± 0.070)  | 0.919| 0.947| 1    | 0.972       |
| Single-model       | 0.942 (± 0.067)  | 0.921| 0.948| 1    | 0.986       |
| **Our Model**      | 0.942 (± 0.066)  | 0.921| 0.948| 1    | 1           |

Table 3: Summary statistics of accuracy on each ZIP code. First column is the average client-specific accuracy, second column the first quartile, third column second quartile (median), and fourth column third quartile. We define performance ratio as the proportion of clients with accuracy no less than the model being compared to, and record it in the fifth column.

alternatives. We empirically demonstrated our estimator on both simulated and real-world data.

In the future, we will investigate how to obtain an efficient implementation of alternative approaches described in Section 5.1. We have identified two estimators that are costly to obtain, yet outperform our proposed method. Studying their efficient implementation could yield even better methods for personalized federated learning.
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References

M. S. H. Abad, E. Ozfatura, D. Gunduz, and O. Ercetin. Hierarchical federated learning across heterogeneous cellular networks. In ICASSP 2020-2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pages 8866–8870. IEEE, 2020.

B. Bakker and T. Heskes. Task clustering and gating for bayesian multitask learning. Journal of Machine Learning Research, 4:83–99, 2003.

R. C. Blattberg, E. C. Malthouse, and S. A. Neslin. Customer lifetime value: Empirical generalizations and some conceptual questions. Journal of Interactive Marketing, 23(2):157–168, 2009.

M. E. Bock. Minimax estimators of the mean of a multivariate normal distribution. The Annals of Statistics, pages 209–218, 1975.

K. Bonawitz, H. Eichner, W. Grieskamp, D. Huba, A. Ingerman, V. Ivanov, C. Kiddon, J. Konečný, S. Mazzocchi, H. B. McMahan, et al. Towards federated learning at scale: System design. arXiv preprint arXiv:1902.01046, 2019.

C. Briggs, Z. Fan, and P. Andras. Federated learning with hierarchical clustering of local updates to improve training on non-iid data. In 2020 International Joint Conference on Neural Networks (IJCNN), pages 1–9. IEEE, 2020.

A. S. Bryk and S. W. Raudenbush. Application of hierarchical linear models to assessing change. Psychological bulletin, 101(1):147, 1987.

A. S. Bryk and S. W. Raudenbush. Hierarchical linear models: Applications and data analysis methods. Sage Publications, Inc, 1992.

F. Bumbaca, S. Misra, and P. E. Rossi. Distributed markov chain monte carlo for bayesian hierarchical models. Available at SSRN 2964646, 2017.

F. Bumbaca, S. Misra, and P. E. Rossi. Scalable target marketing: Distributed markov chain monte carlo for bayesian hierarchical models. Journal of Marketing Research, 57(6):999–1018, 2020.

S. Chen, Q. Zheng, Q. Long, and W. J. Su. A theorem of the alternative for personalized federated learning. arXiv preprint arXiv:2103.01901, 2021.

M. J. Daniels and C. Gatsonis. Hierarchical generalized linear models in the analysis of variations in health care utilization. Journal of the American Statistical Association, 94(445):29–42, 1999.
Y. Deng, M. M. Kamani, and M. Mahdavi. Adaptive personalized federated learning. arXiv preprint arXiv:2003.13461, 2020.

C. T. Dinh, N. H. Tran, and T. D. Nguyen. Personalized federated learning with moreau envelopes. arXiv preprint arXiv:2006.08848, 2020.

Y. Duan and K. Wang. Adaptive and robust multi-task learning. arXiv preprint arXiv:2202.05250, 2022.

A. Fallah, A. Mokhtari, and A. Ozdaglar. Personalized federated learning: A meta-learning approach. arXiv preprint arXiv:2002.07948, 2020.

J. French and R. Russell-Bennett. A hierarchical model of social marketing. Journal of Social Marketing, 2015.

A. Ghosh, J. Chung, D. Yin, and K. Ramchandran. An efficient framework for clustered federated learning. arXiv preprint arXiv:2006.04088, 2020.

R. M. Gower, N. Loizou, X. Qian, A. Sailanbayev, E. Shulgin, and P. Richtárik. Sgd: General analysis and improved rates. In International Conference on Machine Learning, pages 5200–5209. PMLR, 2019.

F. Hanzely and P. Richtárik. Federated learning of a mixture of global and local models. arXiv preprint arXiv:2002.05516, 2020.

F. Hanzely, S. Hanzely, S. Horváth, and P. Richtarik. Lower bounds and optimal algorithms for personalized federated learning. Advances in Neural Information Processing Systems, 33, 2020a.

F. Hanzely, D. Kovalev, and P. Richtarik. Variance reduced coordinate descent with acceleration: New method with a surprising application to finite-sum problems. In International Conference on Machine Learning, pages 4039–4048. PMLR, 2020b.

F. Hanzely, B. Zhao, and M. Kolar. Personalized federated learning: A unified framework and universal optimization techniques. arXiv preprint arXiv:2102.09743, 2021.

D. A. Hofmann. An overview of the logic and rationale of hierarchical linear models. Journal of management, 23(6):723–744, 1997.

G. Hooley, J. Fahy, T. Cox, J. Beracs, K. Fonfara, and B. Snoj. Marketing capabilities and firm performance: a hierarchical model. Journal of market-focused management, 4(3):259–278, 1999.

L. Huang, A. L. Shea, H. Qian, A. Masurkar, H. Deng, and D. Liu. Patient clustering improves efficiency of federated machine learning to predict mortality and hospital stay time using distributed electronic medical records. Journal of biomedical informatics, 99:103291, 2019.

L. Jacob, F. Bach, and J.-P. Vert. Clustered multi-task learning: A convex formulation. arXiv preprint arXiv:0809.2085, 2008.

W. James and C. Stein. Estimation with quadratic loss. In Breakthroughs in statistics, pages 443–460. Springer, 1992.
P. Kairouz, H. B. McMahan, B. Avent, A. Bellet, M. Bennis, A. N. Bhagoji, K. Bonawitz, Z. Charles, G. Cormode, R. Cummings, et al. Advances and open problems in federated learning. *arXiv preprint arXiv:1912.04977*, 2019.

S. P. Karimireddy, S. Kale, M. Mohri, S. Reddi, S. Stich, and A. T. Suresh. Scaffold: Stochastic controlled averaging for federated learning. In *International Conference on Machine Learning*, pages 5132–5143. PMLR, 2020.

T. Kariya and H. Kurata. *Generalized least squares*. John Wiley & Sons, 2004.

D. Kovalev, S. Horváth, and P. Richtárik. Don’t jump through hoops and remove those loops: Svrg and katyusha are better without the outer loop. In *Algorithmic Learning Theory*, pages 451–467. PMLR, 2020.

T. Kubokawa. An approach to improving the james-stein estimator. *Journal of Multivariate Analysis*, 36(1):121–126, 1991.

A. Kumar and H. Daume III. Learning task grouping and overlap in multi-task learning. *arXiv preprint arXiv:1206.6417*, 2012.

Y. Lee and J. A. Nelder. Hierarchical generalized linear models. *Journal of the Royal Statistical Society: Series B (Methodological)*, 58(4):619–656, 1996.

T. Li, A. K. Sahu, M. Zaheer, M. Sanjabi, A. Talwalkar, and V. Smith. Federated optimization in heterogeneous networks. *arXiv preprint arXiv:1812.06127*, 2018.

T. Li, A. K. Sahu, A. Talwalkar, and V. Smith. Federated learning: Challenges, methods, and future directions. *IEEE Signal Processing Magazine*, 37(3):50–60, 2020.

T. Li, S. Hu, A. Beirami, and V. Smith. Ditto: Fair and robust federated learning through personalization. In *International Conference on Machine Learning*, pages 6357–6368. PMLR, 2021.

Z. Li. Anita: An optimal loopless accelerated variance-reduced gradient method. *arXiv preprint arXiv:2103.11333*, 2021.

L. Liu, J. Zhang, S. Song, and K. B. Letaief. Client-edge-cloud hierarchical federated learning. In *ICC 2020-2020 IEEE International Conference on Communications (ICC)*, pages 1–6. IEEE, 2020.

Y. Mansour, M. Mohri, J. Ro, and A. T. Suresh. Three approaches for personalization with applications to federated learning. *arXiv preprint arXiv:2002.10619*, 2020.

O. Marfoq, G. Neglia, A. Bellet, L. Kameni, and R. Vidal. Federated multi-task learning under a mixture of distributions. *arXiv preprint arXiv:2108.10252*, 2021.

B. McMahan, E. Moore, D. Ramage, S. Hampson, and B. A. y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial intelligence and statistics*, pages 1273–1282. PMLR, 2017.
P. A. Naik and K. Peters. A hierarchical marketing communications model of online and offline media synergies. *Journal of Interactive Marketing*, 23(4):288–299, 2009.

Y. Nesterov et al. *Lectures on convex optimization*, volume 137. Springer, 2018.

K. B. Petersen, M. S. Pedersen, et al. The matrix cookbook. *Technical University of Denmark*, 7(15):510, 2008.

X. Qian, H. Dong, P. Richtárik, and T. Zhang. Error compensated loopless svrg, quartz, and sdca for distributed optimization. *arXiv preprint arXiv:2109.10049*, 2021a.

X. Qian, Z. Qu, and P. Richtárik. L-svrg and l-katyusha with arbitrary sampling. *Journal of Machine Learning Research*, 22:1–49, 2021b.

S. Raudenbush and A. S. Bryk. A hierarchical model for studying school effects. *Sociology of education*, pages 1–17, 1986.

S. W. Raudenbush. Educational applications of hierarchical linear models: A review. *journal of Educational Statistics*, 13(2):85–116, 1988.

F. Sattler, K.-R. Müller, and W. Samek. Clustered federated learning: Model-agnostic distributed multitask optimization under privacy constraints. *IEEE transactions on neural networks and learning systems*, 2020.

V. Smith, C.-K. Chiang, M. Sanjabi, and A. Talwalkar. Federated multi-task learning. *arXiv preprint arXiv:1705.10467*, 2017.

S. Snider. Where do I fall in the American economic class system. *US News & World Report*, 2019.

R. Stephen and B. Anthony. Hierarchical linear models, 2002.

S. U. Stich. Local sgd converges fast and communicates little. *arXiv preprint arXiv:1805.09767*, 2018.

A. Wainakh, A. S. Guinea, T. Grube, and M. Mühlhäuser. Enhancing privacy via hierarchical federated learning. In *2020 IEEE European Symposium on Security and Privacy Workshops (EuroS&PW)*, pages 344–347. IEEE, 2020.

J. Wang, S. Wang, R.-R. Chen, and M. Ji. Local averaging helps: Hierarchical federated learning and convergence analysis. *arXiv preprint arXiv:2010.12998*, 2020.

W. Wang, J. Wang, M. Kolar, and N. Srebro. Distributed stochastic multi-task learning with graph regularization. *arXiv preprint arXiv:1802.03830*, 2018.

Q. Yang, Y. Liu, Y. Cheng, Y. Kang, T. Chen, and H. Yu. Federated learning. *Synthesis Lectures on Artificial Intelligence and Machine Learning*, 13(3):1–207, 2019.

Y. Zhang and Q. Yang. A survey on multi-task learning. *arXiv preprint arXiv:1707.08114*, 2017.
Y. Zhang and D.-Y. Yeung. A convex formulation for learning task relationships in multi-task learning. *arXiv preprint arXiv:1203.3536*, 2012.

H. Zhao, Z. Li, and P. Richtárik. Fedpage: A fast local stochastic gradient method for communication-efficient federated learning. *arXiv preprint arXiv:2108.04755*, 2021.

J. Zhou, J. Chen, and J. Ye. Clustered multi-task learning via alternating structure optimization. *Advances in neural information processing systems*, 2011:702, 2011a.

J. Zhou, J. Chen, and J. Ye. Malsar: Multi-task learning via structural regularization. *Arizona State University*, 21, 2011b.
A Missing Lemmas and Proofs

We provide detailed proofs for the results in the main text.

A.1 Proof of Theorem 2.2

The gradient of $F_{\theta_i} \| \theta_i \|$ w.r.t. $\theta_i$ for any $i_0 \in \{1, \ldots, n\}$ is given by

$$\nabla \theta_i F_{\theta_i} \theta_i = \nabla f_i(\theta_i) + \alpha_{j_0} \gamma_i(\theta_i - \bar{\theta}) + (1 - \alpha_{j_0}) \gamma_i(\theta_i - \bar{\theta}_{j_0}),$$

where we let $j_0$ denote the cluster client $i_0$ belongs to. The gradients of (2.3) w.r.t. $\theta_j$ are given by

$$\nabla \theta_j F_{MTL_{\theta_j}} \theta_j = \sum_{i \in I_j} (\theta_i - w_j) + \lambda_j (w_j - \bar{\theta}), \quad j = 1, \ldots, k,$$

$$\nabla \theta_j F_{MTL_{\theta_j}} \theta_j = \sum_{j=1}^k \lambda_j (w_j - \bar{\theta}).$$

Setting the gradients to zero and solving the resulting linear system, give us

$$w_j = \frac{\lambda_j}{\lambda_j + \sum_{i \in I_j} \gamma_i} w + \frac{\sum_{i \in I_j} \gamma_i \theta_i}{\lambda_j + \sum_{i \in I_j} \gamma_i}, \quad j = 1, \ldots, k,$$

$$\bar{w} = \left( \sum_{j=1}^k \lambda_j \sum_{i \in I_j} \gamma_i \right)^{-1} \left( \sum_{j=1}^k \lambda_j \sum_{i \in I_j} \gamma_i \theta_i \right).$$

Plugging $\alpha_j = \frac{\lambda_j}{\lambda_j + \sum_{i \in I_j} \gamma_i}$ and recalling (2.2), we note that

$$\bar{w} = \frac{\sum_{j=1}^k \sum_{i \in I_j} \alpha_j \gamma_i \theta_i}{\sum_{j=1}^k \sum_{i \in I_j} \alpha_j \gamma_i} = \bar{\theta},$$

and

$$w_j = \alpha_j \bar{\theta} + (1 - \alpha_j) \bar{\theta}_j, \quad j = 1, \ldots, k.$$

Therefore, the first order condition for any $\theta_{i_0}$ is

$$\nabla f_{i_0}(\theta_{i_0}) + \alpha_{j_0} \gamma_{i_0}(\theta_{i_0} - \bar{\theta}) + (1 - \alpha_{j_0}) \gamma_{i_0}(\theta_{i_0} - \bar{\theta}_{j_0}) = 0,$$

which is exactly the first order condition $\nabla \theta_i F_{\theta_i} \theta_i = 0$. The proof is now complete.

A.2 Proof of Theorem 3.1

We prove the first statement for some $j$ and $i \in I_j$. Since

$$\nabla \theta_i \bar{\theta}_j = \frac{\gamma_i}{\sum_{i' \in I_j} \gamma_{i'}},$$
we have

\[
\nabla_{\theta_j} \psi_j \left( \theta_j ; \{ \gamma_i \}_{i \in I_j} \right) = \gamma_i (\theta_i - \tilde{\theta}_j) \left( 1 - \frac{\gamma_i}{\sum_{i' \in I_j} \gamma_{i'}} \right) + \sum_{i' \in I_j \setminus \{i\}} \gamma_{i'} (\theta_{i'} - \tilde{\theta}_j) \left( - \frac{\gamma_i}{\sum_{i' \in I_j} \gamma_{i'}} \right)
\]

\[
= \gamma_i (\theta_i - \tilde{\theta}_j) - \sum_{i' \in I_j} \frac{\gamma_{i'}}{\sum_{i' \in I_j} \gamma_{i'}} (\theta_{i'} - \tilde{\theta}_j)
\]

\[
= \gamma_i (\theta_i - \tilde{\theta}_j).
\]

The second statement is proven similarly.

A.3 Proof of Theorem 3.3

Note that the gradient oracle, \( G(\theta) \), can be written as

\[
G(\theta) = \frac{\mathbb{I}_{\{\xi_0 = 1\}}}{p_0} \left( \nabla_{\theta_1} \varphi(\theta) + \nabla_{\theta_2} \varphi(\theta) + \cdots + \nabla_{\theta_k} \varphi(\theta) \right) + \frac{\mathbb{I}_{\{\xi_0 = 0\}}}{1 - p_0} \left[ \frac{\mathbb{I}_{\{\xi_1 = 1\}}}{p_1} \left( \nabla_{\theta_1} \psi_1(\theta_1) - \nabla_{\theta_1} \psi_1(\theta_1^*) \right) + \frac{\mathbb{I}_{\{\xi_2 = 1\}}}{p_2} \left( \nabla_{\theta_2} \psi_2(\theta_2) - \nabla_{\theta_2} \psi_2(\theta_2^*) \right) + \cdots + \frac{\mathbb{I}_{\{\xi_k = 1\}}}{p_k} \left( \nabla_{\theta_k} \psi_k(\theta_k) - \nabla_{\theta_k} \psi_k(\theta_k^*) \right) \right].
\]

We bound the two terms separately using the law of total expectation. Note that

\[
\mathbb{E} \left[ \| G(\theta) - G(\theta^*) \|^2 | \xi_0 = 1 \right] \mathbb{P}(\xi_0 = 1)
\]

\[
= \frac{1}{p_0} \left\| \nabla_{\theta_j} \varphi(\theta) - \nabla_{\theta_j} \varphi(\theta^*) \right\|^2 + \frac{2}{p_0} \left\| \frac{\mathbb{I}_{\{\xi_1 = 1\}}}{p_1} \left( \nabla_{\theta_1} \psi_1(\theta_1) - \nabla_{\theta_1} \psi_1(\theta_1^*) \right) \right\|^2 + \cdots + \frac{2}{p_0} \left\| \frac{\mathbb{I}_{\{\xi_k = 1\}}}{p_k} \left( \nabla_{\theta_k} \psi_k(\theta_k) - \nabla_{\theta_k} \psi_k(\theta_k^*) \right) \right\|^2
\]

\[
\leq \frac{2}{p_0} \left\| \nabla_{\theta_j} \varphi(\theta) - \nabla_{\theta_j} \varphi(\theta^*) \right\|^2 + \frac{2}{p_0} \sum_{j=1}^k (1 - \alpha_j)^2 \left\| \nabla_{\theta_j} \psi_j(\theta_j) - \nabla_{\theta_j} \psi_j(\theta_j^*) \right\|^2
\]

where the second line uses the fact that \( \mathbb{E} \) is taken over \( \{\xi_j\}_{j=0}^k \), the third line uses the triangle inequality, and the fourth line follows directly using the fact that \( \| \cdot \|^2 \) is the squared \( \ell_2 \)-norm.
Similarly, we have
\[
E \left[ \|G(\theta) - G(\theta^*)\|^2 | \xi_0 = 0 \right] P(\xi_0 = 0) = \frac{1}{1 - p_\theta} \sum_{j=1}^k \mathbb{E}_{\xi_j} M_j,
\]
where
\[
M_j = \begin{pmatrix} \frac{1}{p_j} & (1 - \alpha_j) (1 - \tau_j) (\nabla_{\theta_j} \psi_j(\theta_j) - \nabla_{\theta_j} \psi_j(\theta_j^*)) \\ 
\frac{1}{1 - p_j} & (\nabla_{\theta_j} F_j(\theta_j) - \nabla_{\theta_j} F_j(\theta_j^*)) \end{pmatrix}^2.
\]
For a fixed \( j \), we have
\[
\mathbb{E}_{\xi_j} [M_j] = \mathbb{E}_{\xi_j} [M_j | \xi_j = 1] P(\xi_j = 1) + \mathbb{E}_{\xi_j} [M_j | \xi_j = 0] P(\xi_j = 0)
\]
\[
= \frac{1}{p_j} (1 - \alpha_j)^2 (1 - \tau_j)^2 \| \nabla_{\theta_j} \psi_j(\theta_j) - \nabla_{\theta_j} \psi_j(\theta_j^*) \|^2
\]
\[
+ \frac{1}{1 - p_j} \| \nabla_{\theta_j} F_j(\theta_j) - \nabla_{\theta_j} F_j(\theta_j^*) \|^2.
\]
Combining the two bounds, we further have
\[
E[\|G(\theta) - G(\theta^*)\|^2]
\]
\[
= \mathbb{E} \left[ \|G(\theta) - G(\theta^*)\|^2 | \xi_0 = 1 \right] P(\xi_0 = 1) + \mathbb{E} \left[ \|G(\theta) - G(\theta^*)\|^2 | \xi_0 = 0 \right] P(\xi_0 = 0)
\]
\[
\leq \frac{2}{p_0} \| \nabla_{\theta_j} \varphi(\theta) - \nabla_{\theta_j} \varphi(\theta^*) \|^2
\]
\[
+ \sum_{j=1}^k (1 - \alpha_j)^2 \left( \frac{2}{p_0} \tau_j^2 + \frac{(1 - \tau_j)^2}{p_j (1 - p_0)} \right) \| \nabla_{\theta_j} \psi_j(\theta_j) - \nabla_{\theta_j} \psi_j(\theta_j^*) \|^2
\]
\[
+ \sum_{j=1}^k \frac{1}{1 - p_j} \| \nabla_{\theta_j} F_j(\theta_j) - \nabla_{\theta_j} F_j(\theta_j^*) \|^2.
\]
In the above display, only the second term depends on \( \tau_j \). This term is minimized when
\[
\tau_j = \frac{p_0}{p_0 + 2(1 - p_0)p_j}.
\]
The result immediately follows now.

A.4 Proof of Theorem 3.4

We prove the first statement for a fixed \( j \). From the proof of Theorem 3.1, we can write the gradient of \( \psi_j \left( \theta_j; \{ \gamma_i \}_{i \in I_j} \right) \) w.r.t. \( \theta_j \) as
\[
\nabla_{\theta_j} \psi_j \left( \theta_j; \{ \gamma_i \}_{i \in I_j} \right) = \theta_j - \begin{bmatrix} \hat{\theta}_j \\ \hat{\theta}_j \\ \vdots \end{bmatrix}
\]
and the Hessian as \( \nabla_{\theta, \theta_j} \psi_j \left( \theta_j; \gamma_i \right) \) = \( I_d \otimes H_j \), where \( H_j \in \mathbb{R}^{|I_j| \times |I_j|} \) is a symmetric matrix with element in row \( i_1 \) and column \( i_2 \) \((i_1, i_2 \in I_j)\) denoted as
\[
H_j[i_1, i_2] = \begin{cases} 
\gamma_{i_1} \left(1 - \frac{\gamma_{i_1}}{\sum_{i \in I_j} \gamma_i}\right), & \text{if } i_1 = i_2, \\
-\frac{\gamma_{i_1} \gamma_{i_2}}{\sum_{i \in I_j} \gamma_i}, & \text{otherwise}.
\end{cases}
\]

For an arbitrary fixed vector \( v \in \mathbb{R}^{|I_j|} \), we have
\[
v^T H_j v = \sum_{l=1}^{|I_j|} \gamma_{I_j[l]}(v[l])^2 - \frac{1}{\sum_{l=1}^{|I_j|} \gamma_{I_j[l]}} \left(\sum_{l=1}^{|I_j|} \gamma_{I_j[l]}(v[l]) \right)^2 \\
= \left( \frac{1}{\sum_{l=1}^{|I_j|} \gamma_{I_j[l]}} \right) \left(\sum_{l=1}^{|I_j|} \gamma_{I_j[l]}(v[l]) \right)^2 - \left( \sum_{l=1}^{|I_j|} \gamma_{I_j[l]} \right) (v[l])^2 \geq 0,
\]
which shows that \( H_j \succeq 0 \). Let \( \Gamma_j \in \mathbb{R}^{|I_j| \times |I_j|} \) be a diagonal matrix with diagonal elements \( \{ \gamma_{I_j[l]} \}_{l=1}^{|I_j|} \). Then
\[
v^T \Gamma_j v - v^T H_j v = \frac{1}{\sum_{l=1}^{|I_j|} \gamma_{I_j[l]}} \left(\sum_{l=1}^{|I_j|} \gamma_{I_j[l]}(v[l]) \right)^2 \geq 0,
\]
which shows that \( H_j \preceq \Gamma_j \preceq \max_{i \in I_j} \gamma_i I_{|I_j|} \). The first statement now follows, since the eigenvalues of the Kronecker product of two matrices are the products of the pairs of eigenvalues of two matrices.

The second statement is established in the same way.

### A.5 Proof of Theorem 3.5

First, we show that the gradient estimator defined in (3.2) satisfies the expected smoothness condition. We then bound the gradient estimator’s second moment, and apply Theorem 3.1 from Gower et al. (2019) to complete the proof.

**Lemma A.1** (Expected Smoothness). Suppose conditions of Theorem 3.5 hold and \( \mathcal{L} \) is defined in Theorem 3.5. Then
\[
\mathbb{E}[\|G(\theta) - G(\theta^*)\|^2] \leq 2\mathcal{L}\left(F(\theta) - F(\theta^*)\right),
\]
where the expectation is taken over the randomness in \( \{\xi_j\}_{j=0}^k \).

**Proof.** For a convex function \( g : \mathbb{R}^d \rightarrow \mathbb{R} \), let \( D_g(x, y) = g(x) - g(y) - (\nabla g(y))^T(x - y) \), \( x, y \in \mathbb{R}^d \).

By Theorem 3.4 and Theorem B.1, we have
\[
\| \nabla_{\theta} \phi(\theta) - \nabla_{\theta} \phi(\theta^*) \|^2 \leq 2 \max_{j=1, \ldots, k} \max_{i \in I_j} \alpha_j \gamma_i D_{\phi}(\theta, \theta^*),
\]
\[
\| \nabla_{\theta_j} \psi_j(\theta_j) - \nabla_{\theta_j} \psi_j(\theta_j^*) \|^2 \leq 2 \max_{i \in I_j} \gamma_i D_{\psi_j}(\theta_j, \theta_j^*),
\]

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while Theorem 2.1 states that
\[
\left\| \nabla_{\theta_j} F_j(\theta_j) - \nabla_{\theta_j} F_j(\theta^*_j) \right\|^2 \leq 2LD_F\left(\theta_j, \theta^*_j\right).
\]

Plugging into the result of Theorem 3.3, we have
\[
\mathbb{E}\left[\|G(\theta) - G(\theta^*)\|^2\right] \leq \frac{4}{p_0} \max_{j=1, \ldots, k} \max_{i \in \mathcal{I}_j} \alpha_j \gamma_i \nabla \psi_j(\theta^*_j) + \frac{1}{p_0 + 2(1 - p_0)p_j} \sum_{j=1}^k \nabla \psi_j(\theta^*_j)
\]
\[
+ \frac{2L}{1 - p_0} \sum_{j=1}^k \frac{1}{1 - p_j} \nabla \psi_j(\theta^*_j).
\]

Since
\[
F(\theta) - F(\theta^*) = D_F(\theta, \theta^*)
\]
\[
= D_F(\theta, \theta^*) + \sum_{j=1}^k (1 - \alpha_j) \nabla \psi_j(\theta^*_j) + \sum_{j=1}^k D_F(\theta_j, \theta^*_j),
\]
we have
\[
\mathbb{E}\left[\|G(\theta) - G(\theta^*)\|^2\right] \leq 2L \left( D_F(\theta, \theta^*) + \sum_{j=1}^k (1 - \alpha_j) \nabla \psi_j(\theta^*_j) + \sum_{j=1}^k D_F(\theta_j, \theta^*_j)\right)
\]
\[
= 2L(F(\theta) - F(\theta^*)),
\]
which completes the proof.

**Corollary A.2** (Bounded Second Moment). Suppose conditions of Theorem 3.5 hold and \( L \) and \( \sigma_{\theta^*}^2 \) are defined in Theorem 3.5. Then
\[
\mathbb{E}\|G(\theta)\|^2 \leq 4L\mathbb{E}[(F(\theta) - F(\theta^*))^2] + 2\sigma_{\theta^*}^2,
\]
where the expectation is taken over the randomness in \( \{\xi_j\}_{j=0}^k \).

**Proof.** From the proof of Theorem 3.3, we have \( \mathbb{E}\|G(\theta^*)\|^2 \leq \sigma_{\theta^*}^2 \). The result follows from Lemma 2.4 of Gower et al. (2019). \( \square \)

**A.6 Proof of Theorem 3.6**

When \( C_2 > C_1 \), \( \mathcal{L} \) is minimized when \( p_0 = \frac{2C_1}{C_1 + C_2 + L} \) and \( p_j = \frac{C_2 - C_1}{C_2 - C_1 + L} \). When \( C_2 \leq C_1 \), \( \mathcal{L} \) is minimized when \( p_0 = \frac{2C_1}{2C_1 + L} \) and \( p_j = 0 \) for all \( j \). Then
\[
\mathcal{L} \leq \mathcal{L} \leq \begin{cases} 
C_1 + C_2 + L & \text{if } C_2 > C_1, \\
2C_1 + L & \text{if } C_2 \leq C_1.
\end{cases}
\]
With the upperbounds on $\mathcal{L}$ determined for when $C_2 > C_1$ and when $C_2 \leq C_1$, we are left with deriving the optimal $\tau_j$ under the two settings. For the choices of $(p_j)_{j=0}^k$ we have derived for the two settings, by Theorem 3.3, we know that when $C_2 > C_1$ we should set $\tau_j = C_2$ for all $j$ and when $C_2 \leq C_1$ we set $\tau_j = 1$ for all $j$.

For any $\epsilon > 0$, set $\eta = (2\mathcal{L})^{-1}$ and $t = 2\mathcal{L}/\mu \log(1/\epsilon)$. Then

$$(1 - \eta \mu)^t \leq \exp\{-t\eta \mu\} = \epsilon.$$ 

By Theorem 3.2, we have the following.

1. When $C_2 > C_1$, $\mathcal{L} \leq C_1 + C_2 + L$ and $t \leq \frac{2(C_1+C_2+L)}{\mu} \log \frac{1}{\epsilon}$. Since $p_0 = \frac{2C_2}{C_2-C_1+L}$, the expected number of between-cluster communication is at most $\frac{4C_1(C_2-C_1+L)}{(C_2-C_1+L)\mu} \log \frac{1}{\epsilon}$, and the expected number of within-cluster communication is at most $\frac{2L(C_2-C_1)}{(C_2-C_1+L)\mu} \log \frac{1}{\epsilon}$.

2. When $C_2 \leq C_1$, we have $\mathcal{L} \leq 2C_1 + L$, therefore $t \leq \frac{2(2C_1+L)}{\mu} \log \frac{1}{\epsilon}$. Recalling the choices for $p_0$, the number of between-cluster communication is at most $\frac{4C_1L}{(2C_1+L)\mu} \log \frac{1}{\epsilon}$.

**A.7 Proof of Theorem 4.2**

Similarly to the proof of Theorem A.1, we start by conditioning $\mathbb{E}[\|g^t - \nabla F(x^t)\|^2]$ on $\xi_0 = 1$ and $\xi_0 = 0$, respectively. For $\mathbb{E}[\|g^t - \nabla F(x^t)\|^2 \mid \xi_0 = 0]$, we further expand $\mathbb{E}[\|g^t_i - \nabla_i F(x^t)\|^2]$ and condition on $\xi_j = 1$ and $\xi_j = 0$, $j = 1, \ldots, k$, $i \in I_j$. These conditional expectations can then be bounded by Bregman divergences, which completes the proof.

**A.8 Proof of Theorem 4.3**

Similarly to the proof of Theorem 3.4, we can show that $\sum_{j=1}^k \psi_j \left( \theta_j; \{\gamma_i\}_{i \in I_j} \right) + \varphi \left( \theta; \{\gamma_i\}_{i=1}^n; \{\alpha\}_{j=1}^k \right)$ is $\max_{i=1,\ldots,n} \gamma_i$-smooth and convex in $\theta$. The function $F(\theta)$ is $\tilde{L} + \max_{i=1,\ldots,n} \gamma_i$-smooth and $\mu$-strongly convex in $\theta$ under Theorem 4.1.

Algorithm 2 is a special instance of L-Katyusha Hanzely et al. (2020b). For each $t = 1, 2, \ldots$, we obtain an unbiased stochastic gradient estimate $g_i^t$ for each client. The local updates of the clients follow the form of the updates in L-Katyusha. The random variable $\xi^t$ then controls how often the algorithm updates the full gradient. Plugging the expected smoothness of the stochastic gradient oracle, given in Theorem 4.2, into Theorem 4.1 from Hanzely et al. (2020b) completes the proof.

**A.9 Proof of Theorem 4.4**

The proof is similar to the proof of Theorem 3.6 and is omitted.

**A.10 Proof of Theorem 5.1**

Under our model, we have

$$\tilde{\theta}_i^t = \theta_i^* + \frac{1}{\beta_i} X_i^T \epsilon_i = \tilde{\theta}_i^* + \xi_i + \frac{1}{\beta_i} X_i^T \epsilon_i = \tilde{\theta}_i^* + \tilde{\xi}_j + \xi_i + \frac{1}{\beta_i} X_i^T \epsilon_i.$$
By a direct calculation, solution to (5.2) can be written as

\[
\hat{\theta}_i = \frac{\beta_i/\sigma_i^2}{\beta_i/\sigma_i^2 + \gamma_i} \hat{\theta}_i^d + \frac{\gamma_i}{\beta_i/\sigma_i^2 + \gamma_i} \hat{w}_j,
\]

\[
\hat{w}_j = \frac{\sum_{i \in \mathcal{I}_j} \gamma_i \hat{\theta}_i}{\sum_{i \in \mathcal{I}_i} \gamma_i + \lambda_j} + \frac{\lambda_j}{\sum_{i \in \mathcal{I}_i} \gamma_i + \lambda_j} \hat{w},
\]

\[
\hat{w} = \frac{1}{\sum_{j=1}^k \lambda_j} \sum_{j=1}^k \lambda_j \hat{w}_j.
\]

Expanding the right hand side of \( \hat{w}_j \), we have

\[
\hat{w}_j = \frac{1}{\sum_{i \in \mathcal{I}_j} \gamma_i + \lambda_j} \left( \sum_{i \in \mathcal{I}_j} \frac{\gamma_i \beta_i/\sigma_i^2}{\beta_i/\sigma_i^2 + \gamma_i} \hat{\theta}_i^d + \sum_{i \in \mathcal{I}_j} \frac{\gamma_i^2}{\beta_i/\sigma_i^2 + \gamma_i} \hat{w}_j \right) + \frac{\lambda_j}{\sum_{i \in \mathcal{I}_i} \gamma_i + \lambda_j} \hat{w}.
\]

Let

\[
C_i = \frac{\gamma_i \beta_i/\sigma_i^2}{\beta_i/\sigma_i^2 + \gamma_i} \quad \text{and} \quad \hat{\theta}_j = \left( \sum_{i \in \mathcal{I}_j} C_i \right)^{-1} \left( \sum_{i \in \mathcal{I}_j} C_i \hat{\theta}_i^d \right).
\]

With this notation, we have

\[
\hat{w}_j = \frac{\sum_{i \in \mathcal{I}_j} C_i}{\lambda_j + \sum_{i \in \mathcal{I}_j} C_i} \hat{\theta}_j + \frac{\lambda_j}{\lambda_j + \sum_{i \in \mathcal{I}_j} C_i} \hat{w}
\]

and

\[
\hat{w} = \frac{1}{\sum_{j=1}^k \lambda_j} \sum_{j=1}^k \lambda_j \left( \frac{\sum_{i \in \mathcal{I}_j} C_i}{\lambda_j + \sum_{i \in \mathcal{I}_j} C_i} \hat{\theta}_j + \frac{\lambda_j}{\lambda_j + \sum_{i \in \mathcal{I}_i} C_i} \hat{w} \right).
\]

Let

\[
D_j = \frac{\lambda_j \sum_{i \in \mathcal{I}_j} C_i}{\lambda_j + \sum_{i \in \mathcal{I}_j} C_i}.
\]

Then

\[
\hat{w} = \left( \sum_{j=1}^k D_j \right)^{-1} \sum_{j=1}^k D_j \hat{\theta}_j.
\]

Without loss of generality, \( \mathcal{I}_1 = \{1, \ldots, |\mathcal{I}_1|\} \), \( \mathcal{I}_2 = \{|\mathcal{I}_1| + 1, |\mathcal{I}_1| + 2, \ldots, |\mathcal{I}_1| + |\mathcal{I}_2|\} \), and so on. For all \( i' \in \mathcal{I}_1, i' \neq 1 \), we have

\[
\hat{\theta}_i^{d*} = \theta_1^* - \xi_1 + \xi_{i'} + \frac{1}{\beta_{i'}} X_{i'}^{T} \epsilon_{i'}.
\]

For all \( i' \in \mathcal{I}_j, j' \neq 1 \), we have

\[
\hat{\theta}_i^{d*} = \theta_1^* - \xi_1 - \bar{\xi}_1 + \bar{\xi}_{i'} + \xi_{i'} + \frac{1}{\beta_{i'}} X_{i'}^{T} \epsilon_{i'}.
\]
Then
\[
\begin{bmatrix}
y_1 \\
\beta_i \\
\vdots \\
\beta_m
\end{bmatrix} = \begin{bmatrix}
X_1 \\
I \\
\vdots \\
I
\end{bmatrix} \theta_i^T + \begin{bmatrix}
\epsilon_1 \\
-\xi_1 + \xi_2 + \frac{1}{\beta_2} X_2^T \epsilon_2 \\
\vdots \\
-\xi_1 - \xi_d + \xi_n + \frac{1}{\beta_n} X_n^T \epsilon_n
\end{bmatrix},
\]
where
\[
\zeta_1 \sim N\left(0, \begin{bmatrix}
\sigma_1^2 & 0^T_{n-1} \\
0 & \Omega_1
\end{bmatrix} \otimes I_d\right)
\]
and for any \(i, i' \in \{1, \ldots, n\},\)
\[
\Omega_1[i, i'] = \begin{cases}
2\sigma_i^2 + \sigma_i^2/\beta_i & \text{if } i = i' \in \mathcal{I}_1 \\
\bar{\sigma}_i^2 + \bar{\sigma}_j^2 + 2\bar{\sigma}_i^2 + \sigma_i^2/\beta_i & \text{if } i = i' \in \mathcal{I}_j, j \neq 1 \\
\bar{\sigma}_i^2 & \text{if } i \in \mathcal{I}_1 \text{ or } i' \in \mathcal{I}_1, i \neq i' \\
\bar{\sigma}_i^2 + \bar{\sigma}_j^2 & \text{if } i \in \mathcal{I}_j, i' \notin \mathcal{I}_j, i, i' \notin \mathcal{I}_1, i \neq i' \\
\bar{\sigma}_i^2 + 2\bar{\sigma}_j^2 & \text{if } i, i' \in \mathcal{I}_j, j \neq 1, i \neq i'
\end{cases}
\]
The matrix \(\Omega_1\) can be expressed as
\[
\Omega_1 = \begin{bmatrix}
\text{diag}((\sigma_i^2/\beta_i + \sigma_j^2)_{i = 2}^{|\mathcal{I}_1|}) & 0_{|\mathcal{I}_1|-1}^T 0_{n-|\mathcal{I}_1|}^n \Omega_{-1} \\
0_{n-|\mathcal{I}_1|}^n & \Omega_{-1}
\end{bmatrix} + \bar{\sigma}_1^2 1_{n-1}^T 1_{n-1}^T,
\]
where
\[
\Omega_{-1} = \begin{bmatrix}
\Omega_{-1}^{(2)} & 0_{|\mathcal{I}_2|}^T 0_{|\mathcal{I}_2|}^n & \ldots & 0_{|\mathcal{I}_2|}^T 0_{|\mathcal{I}_k|}^n \\
0_{|\mathcal{I}_2|}^T 0_{|\mathcal{I}_2|}^T & \Omega_{-1}^{(3)} & \ldots & 0_{|\mathcal{I}_2|}^T 0_{|\mathcal{I}_k|}^n \\
\vdots & \vdots & \ddots & \vdots \\
0_{|\mathcal{I}_k|}^T 0_{|\mathcal{I}_k|}^T & 0_{|\mathcal{I}_k|}^T 0_{|\mathcal{I}_k|}^n & \ldots & \Omega_{-1}^{(k)}
\end{bmatrix} + \bar{\sigma}_1^2 1_{n-|\mathcal{I}_1|}^T 1_{n-|\mathcal{I}_1|}^T,
\]
and
\[
\Omega_{-1}^{(j)} = \text{diag}((\sigma_i^2/\beta_i + \sigma_j^2)_{i \in \mathcal{I}_j}) + \bar{\sigma}_1^2 1_{|\mathcal{I}_j|}^T 1_{|\mathcal{I}_j|}^T, \quad j \neq 1.
\]
By Woodbury matrix identity (Petersen et al., 2008),
\[
\left(\Omega_{-1}^{(j)}\right)^{-1} = \text{diag}((\sigma_i^2/\beta_i + \sigma_j^2)^{-1})_{i \in \mathcal{I}_j} - \left((\sigma_2^2)^{-1} + \sum_{i \in \mathcal{I}_j} (\sigma_i^2/\beta_i + \sigma_j^2)^{-1}\right)^{-1}
\times \begin{bmatrix}
(\sigma_i^2_{|\mathcal{I}_j|}/\beta_{|\mathcal{I}_j|} + \sigma_j^2)^{-1} \\
(\sigma_i^2_{|\mathcal{I}_j|}/\beta_{|\mathcal{I}_j|} + \sigma_j^2)^{-1} \\
\vdots \\
(\sigma_i^2_{|\mathcal{I}_j|}/\beta_{|\mathcal{I}_j|} + \sigma_j^2)^{-1}
\end{bmatrix}^T
\times \begin{bmatrix}
(\sigma_i^2_{|\mathcal{I}_j|}/\beta_{|\mathcal{I}_j|} + \sigma_j^2)^{-1} \\
(\sigma_i^2_{|\mathcal{I}_j|}/\beta_{|\mathcal{I}_j|} + \sigma_j^2)^{-1} \\
\vdots \\
(\sigma_i^2_{|\mathcal{I}_j|}/\beta_{|\mathcal{I}_j|} + \sigma_j^2)^{-1}
\end{bmatrix}.
\]
With this, we have

\[
1_{|\mathcal{I}_j|}^T (\Omega_{-1}^{(j)})^{-1} 1_{|\mathcal{I}_j|} = \left( \left( \sum_{i' \in \mathcal{I}_j} \frac{1}{\sigma_i^2 / \beta_i + \bar{\sigma}^2} \right)^{-1} + \bar{\sigma}^2 \right)^{-1},
\]

\[
(\Omega_{-1}^{(j)})^{-1} 1_{|\mathcal{I}_j|} = \frac{1}{\left( \sum_{i' \in \mathcal{I}_j} \frac{1}{\sigma_i^2 / \beta_i + \bar{\sigma}^2} \right)^{-1} + \bar{\sigma}^2} \left[ \frac{(\sigma_{I_2}^2/\beta_{I_2}[2] + \bar{\sigma}^2)^{-1}}{(\sigma_{I_2}^2/\beta_{I_2}[1] + \bar{\sigma}^2)^{-1}} \right],
\]

and

\[
\Omega_{-1}^{-1} = \begin{bmatrix}
  (\Omega_{-1}^{(2)})^{-1} & 0_{|\mathcal{I}_2|} 0_{|\mathcal{I}_3|}^T & \ldots & 0_{|\mathcal{I}_2|} 0_{|\mathcal{I}_3|}^T \\
  0_{|\mathcal{I}_3|} 0_{|\mathcal{I}_2|}^T & (\Omega_{-1}^{(3)})^{-1} & \ldots & 0_{|\mathcal{I}_3|} 0_{|\mathcal{I}_2|}^T \\
  \vdots & \vdots & \ddots & \vdots \\
  0_{|\mathcal{I}_k|} 0_{|\mathcal{I}_2|}^T & 0_{|\mathcal{I}_3|} 0_{|\mathcal{I}_2|}^T & \ldots & (\Omega_{-1}^{(k)})^{-1}
\end{bmatrix} - \left( (\bar{\sigma}^2)^{-1} + \sum_{j=2}^k \left( \sum_{i' \in \mathcal{I}_j} \frac{1}{\sigma_i^2 / \beta_i + \bar{\sigma}^2} \right)^{-1} \right) \left( \Omega_{-1}^{-1} \right)^T
\]

Similarly, we have

\[
1_{m-|\mathcal{I}_1|}^T (\Omega_{-1}^{-1} 1_{m-|\mathcal{I}_1|}) = \left( \bar{\sigma}^2 + \left[ \sum_{j=2}^k \left( \sum_{i' \in \mathcal{I}_j} \frac{1}{\sigma_i^2 / \beta_i + \bar{\sigma}^2} \right)^{-1} \right] \right)^{-1},
\]

\[
(\Omega_{-1}^{-1} 1_{m-|\mathcal{I}_1|}) = \frac{1}{\bar{\sigma}^2 + \left[ \sum_{j=2}^k \left( \sum_{i' \in \mathcal{I}_j} \frac{1}{\sigma_i^2 / \beta_i + \bar{\sigma}^2} \right)^{-1} \right]^{-1}} \left[ \begin{array}{c}
(\Omega_{-1}^{(2)})^{-1} 1_{|\mathcal{I}_2|} \\
(\Omega_{-1}^{(3)})^{-1} 1_{|\mathcal{I}_3|} \\
\vdots \\
(\Omega_{-1}^{(k)})^{-1} 1_{|\mathcal{I}_k|}
\end{array} \right]
\]

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and

\[
\Omega^{-1}_1 = \left[ \begin{array}{cccc}
\text{diag}\{(\sigma_i^2/\beta_i + \hat{\sigma}^2)^{-1}\}_{i=2}^{m} & 0_{|I_i| - 1} & 0^T_{m-|I_i|} & \Omega^{-1}_{m-|I_i|} \\
0_{m-|I_i|} & 0^T_{|I_i| - 1} & \Omega^{-1}_{m-|I_i|} & \Omega^{-1}_{m-|I_i|} \\
(\hat{\sigma}^2_i) & \sum_{i=2}^{m} (\sigma_i^2/\beta_i + \hat{\sigma}^2)^{-1} + 1^T_{m-|I_i|} \Omega^{-1}_{m-|I_i|} 1_{m-|I_i|} \\
(\hat{\sigma}^2_i) & \sum_{i=2}^{m} (\sigma_i^2/\beta_i + \hat{\sigma}^2)^{-1} + 1^T_{m-|I_i|} \Omega^{-1}_{m-|I_i|} 1_{m-|I_i|} \\
\end{array} \right]^{-1}
\]

Combining the above expressions, we obtain the generalized least squares estimate for \( \theta_1^* \), which satisfies

\[
\begin{aligned}
\left( \beta_1/\sigma_1^2 + \left\{ \left[ 1^T_{m-|I_i|} \Omega^{-1}_{m-|I_i|} 1_{m-|I_i|} + \sum_{i=2}^{m} \frac{1}{\sigma_i^2/\beta_i + \hat{\sigma}^2} \right]^{-1} + \hat{\sigma}^2_i \right\}^{-1} \right) \hat{\theta}^{GLS}_1 = \\
\frac{1}{\sigma_1^2} X_1^T y_1 + \left( \beta_1/\sigma_1^2 + \left\{ \left[ 1^T_{m-|I_i|} \Omega^{-1}_{m-|I_i|} 1_{m-|I_i|} + \sum_{i=2}^{m} \frac{1}{\sigma_i^2/\beta_i + \hat{\sigma}^2} \right]^{-1} + \hat{\sigma}^2_i \right\}^{-1} \right) \sum_{i=2}^{m} \frac{1}{\sigma_i^2/\beta_i + \hat{\sigma}^2} \hat{\theta}^d_i \\
+ \left( \beta_1/\sigma_1^2 + \left\{ \left[ 1^T_{m-|I_i|} \Omega^{-1}_{m-|I_i|} 1_{m-|I_i|} + \sum_{i=2}^{m} \frac{1}{\sigma_i^2/\beta_i + \hat{\sigma}^2} \right]^{-1} + \hat{\sigma}^2_i \right\}^{-1} \right) \sum_{i=2}^{m} \frac{1}{\sigma_i^2/\beta_i + \hat{\sigma}^2} \hat{\theta}^d_i \\
\times \left( \sum_{j=2}^{k} \left( \left( \sum_{i \in I_j} \frac{1}{\sigma_i^2/\beta_i + \hat{\sigma}^2} \right)^{-1} + \hat{\sigma}^2 \right)^{-1} \right)^{-1} \\
\times \frac{1^T_{m-|I_i|} \Omega^{-1}_{m-|I_i|} 1_{m-|I_i|} + \sum_{i=2}^{m} \frac{1}{\sigma_i^2/\beta_i + \hat{\sigma}^2} \hat{\theta}^d_{i,j}}{\left( \sum_{i \in I_j} \frac{1}{\sigma_i^2/\beta_i + \hat{\sigma}^2} \right)^{-1} + \hat{\sigma}^2}.
\end{aligned}
\]

Observe that \( \hat{\theta}^{GLS}_1 \) is exactly \( \hat{\theta}_1 \) when \( \gamma_i = \frac{1}{\sigma_i^2} \) for all \( i \in I_j, j = 1, \ldots, k \) and \( \lambda_j = \frac{1}{\sigma^2} \). Our claim holds by the Gauss-Markov theorem (Kariya and Kurata, 2004).
A.11 Proof of Theorem 5.3

Under our model, for \( i \neq 1, i \in I_j \),
\[
y_i = X_i \theta_1^* - X_i (\xi_1 + \tilde{\xi}_1 - \tilde{\xi}_j - \xi_i) + \epsilon_i.
\]
Therefore, similar to (A.1), we have
\[
\begin{bmatrix}
y_1 \\
\theta_2^* \\
\vdots \\
\theta_m^*
\end{bmatrix} =
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix} \theta_1^* +
\begin{bmatrix}
\epsilon_1 \\
-X_2 (\xi_1 - \xi_2) + \epsilon_2 \\
\vdots \\
-X_n (\xi_1 + \tilde{\xi}_1 - \tilde{\xi}_k - \xi_n) + \epsilon_n.
\end{bmatrix}
\]

Let \( \tilde{\theta}_1 \) be the solution to the generalized least squares problem defined by the equation above. In general, \( \tilde{\theta}_1 \neq \theta_1^* \). For example, when \( X_i \neq I \) for all \( i \). Since \( \theta_1^* \) are linear in \( X_i, y_i \), \( \tilde{\theta}_1 \) is linear in \( \{X_i, y_i\}_{i=1}^n \). Our proposition then holds by the Gauss-Markov theorem (Kariya and Kurata, 2004).

A.12 Proof of Theorem 5.4

Similarly to (A.1), we have
\[
\tilde{\theta}_1 = \frac{n-1}{2n} y_1 + \frac{n+1}{2n} \bar{y}_{-1},
\]
where
\[
\bar{y}_{-1} = \frac{1}{n-1} \sum_{i=2}^n y_i.
\]
Then
\[
E[\|\tilde{\theta}_1 - \theta_1^*\|^2] = \frac{(n-1)^2}{4n^2} d + \frac{(n+1)^2}{4n^2} E[\|\bar{y}_{-1} - \theta_1^*\|^2].
\]
The estimator \( \tilde{\theta}_1^{JS} \) is obtained as
\[
\tilde{\theta}_1^{JS} = \frac{n-1}{2n} y_1 + \frac{n+1}{2n} C \bar{y}_{-1},
\]
where \( C \in [0, 1] \) is a shrinkage parameter. Then
\[
E[\|\tilde{\theta}_1^{JS} - \theta_1^*\|^2] = \frac{(n-1)^2}{4n^2} d + \frac{(n+1)^2}{4n^2} E[\|C \bar{y}_{-1} - \theta_1^*\|^2].
\]
When estimating \( \theta_1^* \) and \( d > 3 \), there is a \( C < 1 \) such that
\[
E[\|C \bar{y}_{-1} - \theta_1^*\|^2] \leq E[\|\bar{y}_{-1} - \theta_1^*\|^2],
\]
where the exact form of \( C \) is discussed in James and Stein (1992), Bock (1975), Kubokawa (1991). Consequently,
\[
E[\|\tilde{\theta}_1^{JS} - \theta_1^*\|^2] \leq E[\|\tilde{\theta}_1 - \theta_1^*\|^2].
\]
Since
\[
E[\tilde{\theta}_1^{JS} | \theta_1^*] = \left( \frac{n-1}{2n} + \frac{C(n+1)}{2n} \right) \theta_1^* = \left( 1 - \frac{n+1}{2n} (1 - C) \right) \theta_1^*
\]
and \( C < 1 \), \( \tilde{\theta}_1^{JS} \) is biased.
B Useful Results

Theorem B.1. Let \( g(x) : \mathbb{R}^d \rightarrow \mathbb{R} \) be an \( L_g \)-smooth and convex function. Let

\[
D_g(x, y) = g(x) - g(y) - (\nabla g(y))^T (x - y), \quad x, y \in \mathbb{R}^d.
\]

Then, for all \( x, y \in \mathbb{R}^d \), we have

\[
\|\nabla g(x) - \nabla g(y)\|_2 \leq 2L_g D_g(x, y).
\]

Proof. Directly follows from (2.1.10) in Theorem 2.1.5 of Nesterov et al. (2018). \( \square \)