(Z_2)^3 Symmetry of the Tripartite Model

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Abstract

We derive in a simple way the (Z_2)^3 symmetry which characterizes uniquely the phenomenologically successful tripartite form leading to the tribimaximal mixing in the neutrino mass matrix. We impose this symmetry in a setup including the charged leptons and find that it can accommodate all the possible patterns of lepton masses in the framework of type-I and type-II seesaw mechanisms. We also discuss the possibility of generating enough baryon asymmetry through lepton-lepton asymmetry.

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1 Introduction

The present data from neutrino oscillations determine approximately the 3 × 3 neutrino mixing matrix with the two mass-squared differences [1, 2, 3]. In the standard model (SM) of particle interactions, there are 3 lepton families. The charged-lepton mass matrix linking left-handed to their right-handed counterparts is arbitrary, but can always be diagonalized by a bi-unitary transformation:

\[ M_l = U^r_L \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U^r_R)^\dagger. \]  

(1)

Likewise, we can diagonalize the neutrino mass matrix either by a bi-unitary transformation if the neutrinos are of Dirac-type:

\[ M^D_\nu = U^r_L \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U^r_R)^\dagger, \]  

(2)

or by just one unitary transformation if it is symmetric, which is the case when the neutrinos are of Majorana-type:

\[ M^M_\nu = U^r_L \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U^r_L)^T. \]  

(3)

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The observed neutrino mixing matrix comes from the mismatch between $U_L^\dagger$ and $U_L^\prime$ in that

$$U_L = (U_L^\dagger)^\dagger U_L^\prime \simeq \begin{pmatrix} 0.83 & 0.56 & < 0.2 \\ -0.39 & 0.59 & 0.71 \\ -0.39 & 0.59 & -0.71 \end{pmatrix}. \quad (4)$$

We see that this mixing matrix is approximately equal to a specific pattern $V_0$ named tribimaximal by Harrison, Perkins, and Scott (HPS) [4]:

$$U_L \simeq V_0 = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix}. \quad (5)$$

One might ask whether or not the HPS tribimaximal form $V_0$, with its rich phenomenology [5], results from a symmetry. If we work in the flavor basis where $M_1M_1^\dagger$ is diagonal, thus $U_L^\dagger = 1$ is a unity matrix, and assume the neutrinos are of Majorana-type, then the flavor mixing matrix is simplified to $V_0 \simeq U_L = U_L^\prime$, and so, with $M_\nu^{\text{diag}} = \text{Diag}(m_1, m_2, m_3)$, we have

$$M_\nu = V_0 \cdot M_\nu^{\text{diag}} \cdot V_0^T. \quad (6)$$

A special form of $M_\nu$ in this basis was proposed by Ma in [6] which leads to the tribimaximal mixing. This “tripartite” form is

$$M_\nu = M_A + M_B + M_C, \quad (7)$$

where

$$M_A = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_B = B \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_C = C \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad (8)$$

with neutrino eigen masses:

$$m_1 = A - B, \quad m_2 = A - B + 3C, \quad m_3 = A + B. \quad (9)$$

It was shown in [6] that $M_A + M_B$ respect a $Z_3 \times Z_2$ symmetry, but the $Z_3$ is broken under $M_C$, and the whole mass matrix $M_\nu$ is $Z_2$ symmetric. One can show [7] that $M_A + M_B$ has a $U(1)$ underlying symmetry which is broken by $M_C$, however a residual $(Z_2)^3$ symmetry is invariably left unbroken and this latter symmetry characterizes uniquely the tripartite form in Eq. (7).

In fact, it was pointed out in [8] that any neutrino mass matrix possessed a $(Z_2)^3$ symmetry (reduced to $Z_2 \times Z_2$ when restricted to the “special” symmetries preserving the “orientation” of the flavor basis), which meant that this “general” kind of symmetry was devoid of any physical significance since it is just a mere consequence of the diagonalizability of $M_\nu$. However, one should notice that the specific form of this $(Z_2)^3$ depends on the form of the neutrino mass matrix. Hence, when we consider some restricted forms of neutrino mass matrix (the tripartite model in our case) then the associated symmetry will clearly have a physical significance related to the form under study. Moreover, the symmetry of the HPS tribimaximal neutrino mixing was also examined in [9] and was shown to originate from a pattern $M_\nu$ which can be cast in the tripartite form. In this paper, we redervie these results in a simple way by imposing the form invariance idea [10] and examining its implications on the diagonalized neutrino mass matrix $M_\nu^{\text{diag}}$ as the analysis in the latter case is very simple. Nonetheless, any symmetry defined in the basis $(\nu_e, \nu_\mu, \nu_\tau)$ is automatically applicable to $(e, \mu, \tau)$ in the complete Lagrangian, and thus we show, in addition, that one can implement this symmetry on the lepton sector coupled to additional scalar fields with suitably chosen Yukawa couplings and get in a natural manner the charged-lepton mass hierarchy. Furthermore, if one assumes the canonical seesaw mechanism with the Majorana neutrino mass matrix,

$$M_\nu = -M_\nu^D M_R^{-1} (M_\nu^D)^T, \quad (10)$$
where $M_R$ is the heavy Majorana mass matrix for the right neutrinos, then we show that one can accommodate the different possible mass patterns.

The plan of the paper is as follows. In section 2 we find a realization of the $(Z_2)^3$ symmetry underlying the tripartite, and hence the HPS tribimaximal, model. In section 3 we introduce extra scalar fields and derive the charged-lepton mass matrix. In section 4, we infer within the framework of type-I seesaw mechanism the neutrino mass matrix, whereas in section 5 we discuss another scenario for the neutrino mass matrix in the framework of type-II seesaw mechanism. In both scenarios we discuss also the possibility of explaining Baryon asymmetry using lepton asymmetry. We end up by summarizing our results in section 6.

2 The underlying symmetry of the HPS tri-bimaximal pattern

The approach of form invariance states that the neutrino mass matrix is invariant when expressed in the flavor basis and another basis related to the former by a specific unitary transformation $S$:

$$S^T M_\nu S = M_\nu.$$  \hfill (11)

In order to find the most general symmetry $S$ that imposes the form invariance property on a given $M_\nu$, we see that this invariance is equivalent, using equation (6), to

$$U^T M_\nu^{\text{diag}} U = M_\nu^{\text{diag}},$$  \hfill (12)

where $U$ is a unitary matrix related to $S$ by

$$S = V_0^* \cdot U \cdot V_0^T.$$  \hfill (13)

This means that the diagonalized mass matrix is itself form invariant under $U$, and any $U$ symmetry for the diagonalized form can appear as an $S$ symmetry in the flavor basis. However, equation (12) implies

$$U^* M_\nu^{\text{diag}} = M_\nu^{\text{diag}} U,$$  \hfill (14)

and the experimental data, implying three distinct masses $(m_1, m_2, m_3)$, would force $U$ to be of the form:

$$U = \text{Diag}(\pm 1, \pm 1, \pm 1).$$  \hfill (15)

The “geometrical” interpretation of the symmetry group is now very clear in the diagonalized basis, in that we have a group $U$ formed of eight elements $U = \{\pm I, \pm I_i\}$ where $I_i$ represents the $i$ reflection $[i = x, y, z : e.g. \ I_z = \text{Diag}(-1, +1, +1)]$. Since the inverse of any element in $U$ is itself and since $U$ is generated by the three reflections (note that $\ -I = I_x I_y I_z$ and, say, that $-I_x = I_z I_y$) we may write:

$$U = \langle I_x, I_y, I_z \rangle \cong (Z_2)^3.$$  \hfill (16)

The determinant-function from $U$ to the multiplicative group ($\{-1, +1\}$) is a group morphism whose kernel forms a subgroup $U_0$ of $U$ consisting of the unitary matrices satisfying the form invariance property and whose determinant is equal to 1, and is generated by two elements, say:

$$U_0 = \langle -I_x, -I_y \rangle \cong (Z_2)^2.$$  \hfill (17)

One can thus find a realization of $(Z_2)^3$ (or of $Z_2^2$) for any pattern in the flavor basis characterized by the mixing matrix $V$ by simply writing

$$S = V^* \cdot \text{Diag}(\pm 1, \pm 1, \pm 1) \cdot V^T.$$  \hfill (18)
Thus, the three generators of the \((Z_2)^3\) characteristic of the HPS tribimaximal mixing are

\[
S_i = V_0^* \cdot I_x \cdot V_0^T = \frac{1}{3} \begin{pmatrix}
-1 & 2 & -2 \\
2 & 2 & 1 \\
-2 & 1 & 2
\end{pmatrix},
\]

\(i = 1, 2, 3\) (21)

(20)

(19)

\[
S_1 = V_0^* \cdot I_y \cdot V_0^T = \frac{1}{3} \begin{pmatrix}
1 & -2 & 2 \\
-2 & 1 & 2 \\
2 & 2 & 1
\end{pmatrix},
\]

\[
S_2 = V_0^* \cdot I_z \cdot V_0^T = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix}.
\]

We have discovered here, in a simple perspective, the \((Z_2)^3\) symmetry characteristic of the tribimaximal mixing. Moreover, we could check that the tripartite form—which, as noted above, is equivalent to the tribimaximal pattern—can be determined uniquely by the subgroup \(S_0 = \langle -S_1, -S_2 \rangle\), or the whole group \(S \cong (Z_2)^3\) (since \(U\) appears quadratically in equation 12):

\[
(S_i^T \cdot M \cdot S_i = M) \Leftrightarrow \exists A, B, C : M = \begin{pmatrix}
A - B + C & C & -C \\
C & A + C & B - C \\
-C & B - C & A + C
\end{pmatrix},
\]

(22)

where \(i = 1, 2\) or \(i = 1, 2, 3\). Additionally, we note here that this proposed structure can be altered minimally in order to accommodate deviation from the tripartite form. In fact, had the experimental data given a maximal atmospheric mixing angle \(\theta_y = \frac{\pi}{4}\) and a solar mixing angle \(\theta_x = \alpha\) different from the tribimaximal value \(\theta_{x_0} = \arctan \left( \frac{1}{\sqrt{2}} \right)\), then the mixing matrix \(V\) would be

\[
V_\alpha = R_{23}(\theta_y = \frac{\pi}{4}) \otimes R_{12}(\theta_x = \alpha) = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} \sin(2\alpha) & \cos(2\alpha) & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(23)

(with \(s_x \equiv \sin \theta_x\), \(c_y \equiv \cos \theta_y\), and so on) where the Euler rotation matrices are given by

\[
R_{12}(\theta_x) = \begin{pmatrix}
\cos \theta_x & 0 & \sin \theta_x \\
0 & 1 & 0 \\
-\sin \theta_x & 0 & \cos \theta_x
\end{pmatrix}, \quad
R_{23}(\theta_y) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_y & \sin \theta_y \\
0 & -\sin \theta_y & \cos \theta_y
\end{pmatrix}.
\]

(24)

Substituting \(V_\alpha\) with \(V_0\) in (eq. 6), we get, in the flavor basis, a “variant” form of the tripartite model:

\[
M_\nu = \begin{pmatrix}
A_\alpha - B_\alpha + C_\alpha & \frac{1}{\sqrt{2}} \tan(2\alpha) C_\alpha & -\frac{1}{\sqrt{2}} \tan(2\alpha) C_\alpha \\
\frac{1}{2\sqrt{2}} \tan(2\alpha) C_\alpha & A_\alpha + C_\alpha & B_\alpha - C_\alpha \\
-\frac{1}{2\sqrt{2}} \tan(2\alpha) C_\alpha & B_\alpha - C_\alpha & A_\alpha + C_\alpha
\end{pmatrix},
\]

(25)

where

\[
A_\alpha = -(3/4) \cos(2\alpha)(m_2 - m_3) + (1/4)(m_2 + m_1) + (1/2)m_3,
\]

(26)

\[
B_\alpha = -(1/4) \cos(2\alpha)(m_2 + m_3) + (3/4) \cos(2\alpha)(m_2 - m_1) + (1/2)m_3,
\]

(27)

\[
C_\alpha = \cos(2\alpha)(m_2 - m_3).
\]

(28)

We can check now that this variant form is completely determined by the invariance under, say, \(S_{3\alpha}\) and \(S_{2\alpha}\), where

\[
S_{3\alpha} = V_0^* \cdot I_z \cdot V_0^T = S_3,
\]

(29)

\[
S_{2\alpha} = V_0^* \cdot I_y \cdot V_0^T = \begin{pmatrix}
\cos(2\alpha) & -1/\sqrt{2} \sin(2\alpha) & 1/\sqrt{2} \sin(2\alpha) \\
1/\sqrt{2} \sin(2\alpha) & 1/2 - 1/2 \cos(2\alpha) & -1/2 + 1/2 \cos(2\alpha) \\
1/\sqrt{2} \sin(2\alpha) & 1/2 + 1/2 \cos(2\alpha) & 1/2 - 1/2 \cos(2\alpha)
\end{pmatrix}.
\]

(30)
Sofar we have looked for symmetries which give results in line with the experimental data ($V_0$ in our case). However, having found a specific form of symmetry satisfying the experimental constraints we can now consider it as an ansatz for the underlying symmetry of the neutrino mass matrix and extend it, in a consistent manner, to include other parts of the lepton sector. In fact, the realization of the symmetry ($S \cong Z_2^3$) we have found on the left light neutrinos should also apply to their doublet-charged lepton partners, and it is often incompatible with a diagonal $M_l$ with three different eigenvalues. In order to avoid this difficulty, we introduce additional scalars, which are singlet under the SM symmetry while transforming non-trivially under the proposed $S \cong Z_2^3$ symmetry.

3 The charged-lepton mass matrix

We start with the normal SM mass term:

$$\mathcal{L}_1 = Y_{ij} \bar{L}_i \Phi l^c_j,$$

where the SM Higgs $\Phi$ and the charged right-handed leptons $l^c_j$ are assumed to be singlets under the $S = (Z_2)^3$ symmetry, whereas the lepton left-doublets transform component-wise faithfully:

$$L_i \rightarrow S_{ij} L_j,$$

with $i, j = 1, 2, 3$ and $S$ is the $(Z_2)^3$ symmetry given in eqs. 19–21.

The invariance under $S$ restricts the Yukawa-couplings to satisfy the matrix equation:

$$S^T \cdot Y = Y.$$

This equation can not be met for a matrix $S$ with determinant equal to $-1$, and hence we deduce that the $S \cong (Z_2)^3$-symmetry forces the term $\mathcal{L}_1$ to vanish, whereas it would have been allowed had we restricted the symmetry to $S_0 \cong (Z_2)^2$.

In order to generate lepton masses, then, we introduce three SM singlet scalar fields, $\Delta_k$, with non-trivial transformations under the symmetry $S$, one for each family (the indices $k = 1, 2, 3$ refer also to the flavors $e, \mu$ and $\tau$ respectively). The field $\Delta_k$ is coupled to the corresponding lepton left doublet $L_k = (\nu_k \ l_k)$ via the dimension 5 operator:

$$\mathcal{L}_2 = \frac{f_{ikr}}{\Lambda} \bar{L}_i \Phi \Delta_k l^c_r,$$

where $\Lambda$ is a heavy mass scale. Note here that our ad hoc assumption of the coupling of charged leptons with these additional Higgses via higher operators, and not through SM-like Yukawa terms, is apt to reduce the effects of flavor changing neutral currents, usual when many Higgs doublets exist [11]. We assume the new scalars $\Delta_k$ transforming under $S$ like the lepton left doublets:

$$\Delta_i \rightarrow S_{ij} \Delta_j.$$

Invariance of the Lagrangian under the symmetry implies

$$S_{\alpha \beta} f_{ikr} = f_{\alpha \beta r}.$$

In matrix form, we write this as

$$S^T f_r S = f_r,$$

where $f_r$, for fixed $r$, is the matrix whose $(i, j)$ entry is $f_{ijr}$. From (22), we have a solution for the new Yukawa coupling of the above equation in the form:

$$f_r = \begin{pmatrix} A_r - B_r + C_r & C_r & -C_r \\ C_r & A_r + C_r & B_r - C_r \\ -C_r & B_r - C_r & A_r + C_r \end{pmatrix}.$$
When the fields $\Delta_k$ and $\phi^0$ take the vacuum expectation values (vevs) $<\Delta_k> = \delta_k$ and $<\phi^0> = v$, the charged lepton mass matrix originating from $L_2$ becomes:

$$(M_2)_{ij} = \frac{v f_{ikr} \delta_k}{\Lambda}. \quad (39)$$

As we are concentrating on the neutrino sector without stating explicitly the $\Delta_k$ potential and since the $S$ symmetry is broken by “soft” terms in the Higgs sector, we may assume a $\Delta_3$-dominated pattern: $\delta_1, \delta_2 \ll \delta_3$, so to get the charged lepton mass matrix

$$M_l = \frac{v \delta_3}{\Lambda} \begin{pmatrix} -C_1 & -C_2 & -C_3 \\ B_1 - C_1 & B_2 - C_2 & B_3 - C_3 \\ A_1 + C_1 & A_2 + C_2 & A_3 + C_3 \end{pmatrix}. \quad (40)$$

The determinant of $M_l$ is equal to: $-\left(\frac{v \delta_3}{\Lambda}\right)^3 A \cdot (B \times C)$, where $A$ is the vector of components $A_i$ (similarly for $B, C$), which means that these three vectors should not be coplanar in order to have a nonsingular lepton mass matrix. We get then

$$M_1, M_l^\dagger = \frac{v^2 \delta_3^2}{\Lambda^2} \begin{pmatrix} C', C' & C', B' & C', A' \\ B', C' & B', B' & B', A' \\ A', C' & A', B' & A', A' \end{pmatrix}, \quad (41)$$

where $A' = A + C, B' = B - C$ and $C' = -C$. If we just assume the magnitudes of the three vectors coming in ratios comparable to the lepton mass ratios:

$$C'^2 : B'^2 : A'^2 \sim m_e^2 : m_\mu^2 : m_\tau^2, \quad (42)$$

then one can show that the mixing $U^L$, such that $U^L_i M_l U^L_i^\dagger$ is diagonal, will be naturally very close to the identity matrix with off-diagonal terms of order $(m_e/m_\mu \sim 5 \times 10^{-3}, m_e/m_\tau \sim 3 \times 10^{-4}, m_\mu/m_\tau \sim 6 \times 10^{-2})$, which would mean that our basis is the flavor basis to a very good approximation and that the hierarchical charged lepton masses can be obtained from a hierarchy on the a priori arbitrary Yukawa couplings $(C'^2 \ll B'^2 \ll A'^2)$.

In order to clarify the last point, let us assume

$$\frac{|C'|}{|A'|} = \lambda_e, \quad \frac{|B'|}{|A'|} = \lambda_\mu, \quad (43)$$

where $\lambda_e, \mu$ are small parameters of order $m_e/m_\tau$. This yields the squared mass matrix to be written as:

$$Q_\lambda \equiv M_l M_l^\dagger = \frac{v^2 |A'|^2 \delta_3^3}{\Lambda^2} \begin{pmatrix} \lambda_e^2 & \lambda_e \lambda_\mu \cos \psi & \lambda_e \cos \phi \\ \lambda_e \lambda_\mu \cos \psi & \lambda_\mu^2 & \lambda_\mu \cos \theta \\ \lambda_e \cos \phi & \lambda_\mu \cos \theta & 1 \end{pmatrix}, \quad (44)$$

where $\theta, \phi$ are the angles between the vector $A'$ and the vectors $B', C'$ respectively, while $\psi$ is the angle between $B'$ and $C'$. The diagonalization of $M_l M_l^\dagger$ by means of an infinitesimal rotation amounts to seeking an antisymmetric matrix

$$L_e = \begin{pmatrix} 0 & \epsilon_1 & \epsilon_2 \\ -\epsilon_1 & 0 & \epsilon_3 \\ -\epsilon_2 & -\epsilon_3 & 0 \end{pmatrix}, \quad (45)$$

with small parameters $\epsilon$'s, satisfying:

$$(Q_\lambda + [Q_\lambda, L_e])_{ij} = 0, \quad i \neq j. \quad (46)$$

Solving this equation analytically, one can express the $\epsilon$'s in terms of $(\lambda_e, \mu, \cos(\psi, \phi, \theta))$. One finds, apart from “fine tuned” situations corresponding to coplanar vectors $A', B', C'$, that we get: $\epsilon_3 \sim \lambda_\mu, \epsilon_2 \sim \lambda_e$.
and \( e_1 \sim \lambda_e / \lambda_\mu \), which indicates that a consistent solution for \( U_L^i \) close to the identity matrix is given by \( U_L^i = e^{i \epsilon_i} \approx I + I_e \). For example, taking the numerical values \( \lambda_e = 3 \times 10^{-4}, \lambda_\mu = 6 \times 10^{-2} \) and a common value \( \pi/3 \) for the angles formed by the vectors, we get: \( m_\nu^2 : m_\mu^2 : m_\tau^2 = 6 \times 10^{-8} : 3 \times 10^{-3} : 1 \), with the ‘exact’ unitary diagonalizing matrix given by:

\[
U_L^i \sim \begin{pmatrix}
1 & 10^{-3} & 10^{-4} \\
-1.6 \times 10^{-3} & 1 & 3 \times 10^{-2} \\
-10^{-4} & -3 \times 10^{-2} & 1
\end{pmatrix}.
\]

(47)

Thus, the deviations due to the rotations are, in general, small, but could justify measuring a nonzero small value of \( U_{e3} \) which is restricted by the reactor data [12] to be less than 0.16 in magnitude.

4 The neutrino mass matrix and type-I seesaw scenario

In this scenario the effective light left neutrino mass matrix is generated through seesaw mechanism as described in eq. 10. The Dirac neutrino mass matrix comes from the Yukawa term:

\[
g_{ij} L_i \hat{\Phi} \nu_{Rj},
\]

(48)

where \( \hat{\Phi} = i \tau_2 \Phi^* \). As to the right neutrino, we will assume that it transforms faithfully as

\[
\nu_{Rj} \rightarrow S_j \gamma \nu_{Rj},
\]

(49)

since, as we shall see, this assumption will put constraints on the right Majorana mass matrix. The invariance of the Lagrangian under \( S \) implies in matrix form:

\[
S^T g S = g.
\]

(50)

Thus, the relation (22), with \( S \) standing for the \( S_i \)'s, enforces the Yukawa couplings \( g \) to be symmetric and the neutrino Dirac mass matrix to have the form:

\[
M^D_\nu = v \begin{pmatrix}
A_D - B_D + C_D & C_D & -C_D \\
C_D & A_D + C_D & B_D - C_D \\
-C_D & B_D - C_D & A_D + C_D
\end{pmatrix}.
\]

(51)

As to the right-handed Majorana mass matrix, it comes from the term:

\[
\frac{1}{2} \nu_{iR} C (M_R)_{ij} \nu_{jR},
\]

(52)

where \( C \) is the charge conjugation matrix. Again, the invariance under the transformation (49) implies

\[
S^T M_R S = M_R,
\]

(53)

and thus \( M_R \), which has to be symmetric, has the form:

\[
M_R = \Lambda_R \begin{pmatrix}
A_R - B_R + C_R & C_R & -C_R \\
C_R & A_R + C_R & B_R - C_R \\
-C_R & B_R - C_R & A_R + C_R
\end{pmatrix}.
\]

(54)

Using equations (10,51,54), we have the effective neutrino mass matrix:

\[
M_\nu = -\frac{v^2}{\Lambda_R} \begin{pmatrix}
A_\nu - B_\nu + C_\nu & C_\nu & -C_\nu \\
C_\nu & A_\nu + C_\nu & B_\nu - C_\nu \\
-C_\nu & B_\nu - C_\nu & A_\nu + C_\nu
\end{pmatrix},
\]

(55)
where

\[ A_\nu = \frac{-2A_D B_R B_D + A_R A_D^2 + A_R B_D^2}{(A_R - B_R)(A_R + B_R)}, \]
\[ B_\nu = \frac{-B_D^2 B_R - A_D^2 B_R + 2A_R A_D B_D}{(A_R - B_R)(A_R + B_R)}, \]  \hspace{1cm} (56)
\[ C_\nu = \frac{-C_R (A_D - B_D)^2 + C_D [3C_D + 2 (A_D - B_D)] (A_R - B_R)}{(A_R - B_R)(A_R + 3C_R)}. \]

The neutrino mass eigenvalues are given by equation (9):

\[ \frac{v^2}{A_R} (A_\nu - B_\nu, A_\nu - B_\nu + 3C_\nu, A_\nu + B_\nu). \]  \hspace{1cm} (57)

All different patterns of the neutrino masses can accommodated, as follows:

- **Normal hierarchy.** It suffices to have

\[ A_i \simeq B_i, \quad C_i \ll B_i, \quad i = R, D, \]  \hspace{1cm} (58)

for getting a normal hierarchy with

\[ A_\nu \simeq \frac{A_D^2}{A_R}, \quad B_\nu \simeq \frac{B_D^2}{B_R}, \quad C_\nu \simeq \frac{C_D^2}{C_R}. \]  \hspace{1cm} (59)

We see that one can arrange the Yukawa couplings to enforce \( A_\nu \simeq B_\nu, \quad C_\nu \ll B_\nu. \)

- **Inverted hierarchy.** It is sufficient to have

\[ A_i \simeq -B_i, \quad C_i \ll B_i, \quad i = R, D, \]  \hspace{1cm} (60)

so that one gets an inverted hierarchy with

\[ A_\nu \simeq \frac{A_D^2}{A_R}, \quad B_\nu \simeq \frac{B_D^2}{B_R}, \quad C_\nu \simeq \frac{-2C_D A_D A_R - C_R A_D^2}{A_R^2}. \]  \hspace{1cm} (61)

One can arrange the Yukawa couplings to enforce \( A_\nu \simeq -B_\nu, \quad C_\nu \ll B_\nu. \)

- **Degenerate case.** If we have

\[ A_i \gg B_i \gg C_i, \quad i = R, D, \]  \hspace{1cm} (62)

then we get

\[ A_\nu \simeq \frac{A_D^2}{A_R}, \quad B_\nu \simeq \frac{2A_D B_D}{A_R} - \frac{A_D^2 B_R}{A_R^2}, \quad C_\nu \simeq \frac{A_D A_R C_D - C_R A_D^2}{A_R^2}. \]  \hspace{1cm} (63)

One can arrange the Yukawa couplings to enforce \( A_\nu \gg B_\nu \gg C_\nu, \) so that we have a degenerate spectrum.

Thus, we see that a certain pattern occurring in both the Dirac and the right-handed Majorana mass matrices can resurface in the effective neutrino mass matrix.

The right handed (RH) neutrino mass term violates lepton number by two units. The out of equilibrium decay of the lightest RH neutrino to standard model particles can be a natural source of lepton asymmetry [13] and it is given by

\[ \epsilon \simeq \frac{3}{16\pi v^2} \frac{1}{(M_{D}^{\nu}M_{D}^{\nu})_{11}} \sum_{j=2,3} \text{Im}[(M_{D}^{\nu}M_{D}^{\nu})_{j1}]^2 \frac{M_{Rj}}{M_{R1}}, \]  \hspace{1cm} (64)

where \( M_{Ri}, \quad i = 1 \cdots 3 \) are the masses for right handed neutrinos. Explicitly we have

\[ (M_{D}^{\nu}M_{D}^{\nu})_{12} = 3|C_D|^2 + (C_D A_D^2 + C_D A_D) - (C_D B_D^2 + C_D B_D), \]
\[ (M_{D}^{\nu}M_{D}^{\nu})_{13} = -3|C_D|^2 + (C_D A_D^2 + C_D A_D) + (C_D B_D^2 + C_D B_D), \]  \hspace{1cm} (65)

which gives a vanishing lepton asymmetry. Thus, in this seesaw type mechanism the baryon asymmetry is zero if \((Z_2)^3\) is an exact symmetry.
5 The neutrino mass matrix and type-II seesaw scenario

In this scenario we introduce two SM triplet fields $\Sigma_A$, $A = 1, 2$ which are also assumed to be singlet under the flavor symmetry ($Z_2)^3$. The Lagrangian part relevant for the neutrino mass matrix is

$$\mathcal{L} = \lambda_A^A L^T C \Sigma_A i \tau_2 L + \mathcal{L}(H, \Sigma_A) + h.c.,$$

(66)

where $A = 1, 2$ and

$$\mathcal{L}(H, \Sigma_A) = \mu_H^2 H^\dagger H + \frac{\lambda_H}{2} (H^\dagger H)^2 + M_A \text{Tr} \left( \Sigma_A^\dagger \Sigma_A \right) + \frac{\lambda_{\Sigma A}}{2} \left[ \text{Tr} \left( \Sigma_A^\dagger \Sigma_A \right) \right]^2 + \lambda_H \Sigma_A (H^\dagger H) \text{Tr} \left( \Sigma_A^\dagger \Sigma_A \right) + \mu_A H^T \Sigma_A i \tau_2 H + h.c.,$$

(67)

where $H = \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right)$, and

$$\Sigma_A = \begin{pmatrix} \Sigma_1^+ \\ \Sigma_2^+ \\ -\Sigma_1^+ \end{pmatrix}_A.$$

(68)

The neutrino mass matrix due to the exchange of the two triplets, $\Sigma_1$ and $\Sigma_2$, is

$$(M_\nu)^A_{\alpha \beta} \simeq v^2 \left[ \lambda_{\alpha \beta}^1 \frac{\mu_1}{M_{\Sigma_1}^2} + \lambda_{\alpha \beta}^2 \frac{\mu_2}{M_{\Sigma_2}^2} \right],$$

(69)

where $M_{\Sigma_i}$ is the mass of the neutral component $\Sigma_0^A_i$ of the triplet $\Sigma_i$, $i = 1, 2$.

Here some remarks are in order. First, the symmetry $Z_2^3$ implies that $\lambda_1$ and $\lambda_2$ have the same tripartite structure. Second, due to the ‘tadpole’ term (the $\mu_A$-term) in $\mathcal{L}(H, \Sigma_A)$, which would forbid the “unwanted” spontaneous breaking of the lepton number, one can arrange the parameters so that minimizing the potential gives a non-zero vev for the neutral component $\Sigma_0^A$ of the triplet. This would generate a mass term for the neutrinos, however, the procedure here is equivalent to integrating out the the heavy triplets and both ways lead to the mass formula above. Third, the flavor changing neutral current due to the triplet is highly suppressed due to the heaviness of the triplet mass scale, or equivalently the smallness of the neutrino masses.

Now let us discuss the baryon asymmetry. We will show that even though the neutrino Yukawa couplings are real it is possible to generate a baryon to photon density consistent with the observations. Since the triplet $\Sigma_A$ can decay into lepton pairs $L_\alpha L_\beta$ and $HH$, it implies that these processes violate total lepton numbers (by two units) and may establish a lepton asymmetry. As the universe cools further, the sphaleron interaction [14] converts this asymmetry into baryon asymmetry. At temperature of the order $\max\{M_1, M_2\}$, the heaviest triplet would decay via lepton number violating interactions. However, no asymmetry will be generated from this decay since the rapid lepton number violating interactions due to the lightest Higgs triplet will erase any previously generated lepton asymmetry. Thus, only when the temperature becomes just below the mass of the lightest triplet Higgs the asymmetry would be generated.

With just one triplet, the lepton asymmetry will be generated at the two loop level and it is highly suppressed. The reason is that one can always redefine the phase of the Higgs field to render the $\mu$ real which will result in the vanishing of the absorptive part of the self energy diagram. The choice of having more than one Higgs triplet is necessary to generate the asymmetry [15]. In this case, the CP asymmetry in the decay of the lightest Higgs triplet (which we choose to be $\Sigma_1$) is generated at one loop level due to the interference between the tree and the one loop self energy diagram \footnote{There is no one loop vertex correction because the triplet Higgs is not self conjugate} and it is given by

$$\epsilon_{CP} \approx -\frac{V_1}{8\pi^2} \text{Im} \left[ \frac{\mu_1 \mu_2^* \text{Tr} \left( \lambda^1 \lambda^{*2} \right)}{M_1^2} \right] \frac{M_1}{\Gamma_1},$$

(70)
where $\Gamma_1$ is the decay rate of the lightest Higgs triplet and it is given by
\[
\Gamma_1 = \frac{M_1}{8\pi} \left[ \text{Tr} (\lambda^1\lambda^1) + \frac{\mu_1^2}{M_1^2} \right].
\] (71)

The baryon to photon density is approximately given by
\[
\eta_B \equiv \frac{n_B}{s} = \frac{1}{3} \eta_L \simeq \frac{1}{3} g_* K \epsilon_{CP},
\] (72)

where $g_* \sim 100$ is the number of relativistic degrees of freedom at the time when the Higgs triplet decouples from the thermal bath and $K$ is the efficiency factor [16] defined as
\[
K = \frac{\Gamma_1}{H}(T = M_1),
\] (73)

($H$ is the Hubble parameter) which takes into account the fraction of out-of-equilibrium decays and the washout effect. For $\mu_1 \approx M_{Z_1} \sim 10^{12}$ GeV the efficiency factor is of order $10^{-3}$ and the baryon asymmetry is
\[
\eta_B \approx 10^{-7} \frac{\text{Tr} (\lambda^1\lambda^{21})}{\text{Tr} (\lambda^1\lambda^1)} + 1 \sin(\phi_2 - \phi_1).
\] (74)

Thus one can produce the correct baryon-to-photon ratio of $\eta_B \simeq 10^{-10}$ by choosing $\lambda$'s of order 0.1 and not too small relative phase between the $\mu$'s.

6 Summary

We showed in a simple way that the underlying symmetry of the tripartite model is $Z_3^2$ and we implemented this symmetry in a complete setup including the charged leptons, the neutrinos, and extra scalar fields. We showed that this setup can accommodate the different patterns of charged leptons and neutrino mass matrices. We showed that one can produce the correct baryon-to-photon ratio in type-II seesaw mechanism by choosing appropriate couplings and not too small relative phase between the vacuum expectation values of the extra scalar fields.

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