IDEMPOTENCE AND DIVISORIALTY IN PRÜFER-LIKE DOMAINS

MARCO FONTANA, EVAN HOUSTON, AND MI HEE PARK

Abstract. Let \( D \) be a Prüfer \( \ast \)-multiplication domain, where \( \ast \) is a semistar operation on \( D \). We show that certain ideal-theoretic properties related to idempotence and divisoriality hold in Prüfer domains, and we use the associated semistar Nagata ring of \( D \) to show that the natural counterparts of these properties also hold in \( D \).

1. Introduction and preliminaries

Throughout this work, \( D \) will denote an integral domain, and \( K \) will denote its quotient field. Recall that Arnold [1] proved that \( D \) is a Prüfer domain if and only if its associated Nagata ring \( D[X_N] \), where \( N \) is the set of polynomials in \( D[X] \) whose coefficients generate the unit ideal, is a Prüfer domain. This was generalized to Prüfer \( v \)-multiplication domains (PvMDs) by Zafrullah [16] and Kang [14] and to Prüfer \( \ast \)-multiplication domains (P\( \ast \)MDs) by Fontana, Jara, and Santos [8].

Our goal in this paper is to show that certain ideal-theoretic properties that hold in Prüfer domains transfer in a natural way to P\( \ast \)MDs. For example, we show that an ideal \( I \) of a Prüfer domain is idempotent if and only if it is a radical ideal each of whose minimal primes is idempotent (Theorem 2.9), and we use a Nagata ring transfer “machine” to transfer a natural counterpart of this characterization to P\( \ast \)MDs. For another example, in Theorem 3.5 we show that an ideal in a Prüfer domain of finite character is idempotent if and only it is a product of idempotent prime ideals and, perhaps more interestingly, we characterize ideals that are simultaneously idempotent and divisorial as (unique) products of incomparable divisorial idempotent primes; and we then extend this to P\( \ast \)MDs.

Let us review terminology and notation. Denote by \( F(D) \) the set of all nonzero \( D \)-submodules of \( K \), and by \( F(D) \) the set of all nonzero fractional ideals of \( D \), i.e., \( E \in F(D) \) if \( E \in F(D) \) and there exists a nonzero \( d \in D \) with \( dE \subseteq D \). Let \( f(D) \) be the set of all nonzero finitely generated \( D \)-submodules of \( K \). Then, obviously, \( f(D) \subseteq F(D) \subseteq F(D) \).

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Following Okabe-Matsuda [13], a semistar operation on \( D \) is a map \( \ast : \text{F}(D) \to \text{F}(D) \), \( E \mapsto E^\ast \), such that, for all \( x \in K, x \neq 0 \), and for all \( E, F \in \text{F}(D) \), the following properties hold:

\[ (\ast_1) \ (x E)^\ast = x E^\ast; \]

\[ (\ast_2) \ E \subseteq F \text{ implies } E^\ast \subseteq F^\ast; \]

\[ (\ast_3) \ E \subseteq E^\ast \text{ and } E^{**} = (E^\ast)^\ast = E^\ast. \]

Of course, semistar operations are natural generalizations of star operations–see the discussion following Corollary 2.5 below.

The semistar operation \( \ast \) is said to have finite type if \( E^\ast = \bigcup \{ F^\ast \mid F \in f(D), F \subseteq E \} \) for each \( E \in \text{F}(D) \). To any semistar operation \( \ast \) we can associate a finite-type semistar operation \( \tilde{\ast} \), given by

\[ E^\tilde{\ast} = \bigcap \{ ED_Q \mid Q \in \text{QMax}^\ast(D) \}. \]

Then \( \tilde{\ast} \) is also a finite-type semistar operation, and we have \( I^\tilde{\ast} \subseteq I^\ast \subseteq I^\ast \) for all \( I \in \text{F}(D) \).

Let \( \ast \) be a semistar operation on \( D \). Set \( N(*) = \{ g \in D[X] \mid c(g)^\ast = D^\ast \} \), where \( c(g) \) is the content of the polynomial \( g \), i.e., the ideal of \( D \) generated by the coefficients of \( g \). Then \( N(*) \) is a saturated multiplicatively closed subset of \( D[X] \), and we call the ring \( \text{Na}(D, \ast) := D[X]\vert_{N(*)} \) the semistar Nagata ring of \( D \) with respect to \( \ast \). The domain \( D \) is called a Prüfer \( \ast \)-multiplication domain (\( \text{P} \ast \text{MD} \)) if \( (FF^{-1})^\ast = D^\ast \) (i.e., each such \( F \) is \( \ast \)-invertible).

In the following two lemmas, we assemble the facts we need about Nagata rings and \( \text{P} \ast \text{MDs} \). Most of the proofs can be found in [9, 11, or 5].

**Lemma 1.1.** Let \( \ast \) be a semistar operation on \( D \). Then:

1. \( D^\ast = D^\ast \).
2. \( \text{QMax}^\ast(D) = \text{QMax}^\tilde{\ast}(D) \).
3. The map \( \text{QMax}^\tilde{\ast}(D) \to \text{Max}(\text{Na}(D, \ast)), P \mapsto \text{PNa}(D, \ast) \), is a bijection with inverse map \( M \mapsto \text{M} \cap D \).
4. \( P \mapsto \text{PNa}(D, \ast) \) defines an injective map \( \text{QSpec}^\tilde{\ast}(D) \to \text{Spec}(\text{Na}(D, \ast)) \).
5. \( N(*) = N(*_j) = N(\tilde{\ast}) \) and (hence) \( \text{Na}(D, \ast) = \text{Na}(D, *_j) = \text{Na}(D, \tilde{\ast}) \).
6. For each \( E \in \text{F}(D) \), \( E^\tilde{\ast} = \text{ENa}(D, \ast) \cap K \), and \( E^\ast \text{Na}(D, \ast) = \text{E} \text{Na}(D, \ast) \).
7. A nonzero ideal \( I \) of \( D \) is a quasi-\( \ast \)-ideal if and only if \( I = I \text{Na}(D, \ast) \cap D \).

**Lemma 1.2.** Let \( \ast \) be a semistar operation on \( D \).
The following statements are equivalent.
(a) $D$ is a $P\star$MD.
(b) $Na(D,\star)$ is a Prüfer domain.
(c) The ideals of $Na(D,\star)$ are extended from ideals of $D$.
(d) $D_P$ is a valuation domain for each $P \in Q\text{Max}^{\star}(D)$.

Assume that $D$ is a $P\star$MD. Then:
(a) $\sim = \ast$, and (hence) $D^* = D^\sim$.
(b) The map $\text{QSpec}^{\ast}(D) \rightarrow \text{Spec}(Na(D,\star))$, $P \mapsto PNa(D,\star)$, is a bijection with inverse map $Q \mapsto Q \cap D$.
(c) Finitely generated ideals of $Na(D,\star)$ are extended from finitely generated ideals of $D$.

2. Idempotence

We begin with our basic definition.

**Definition 2.1.** Let $\star$ be a semistar operation on $D$. An element $E \in \mathbf{F}(D)$ is said to be $\star$-idempotent if $E^* = (E^2)^\star$.

Our primary interest will be in (nonzero) $\star$-idempotent ideals of $D$. Let $\ast$ be a semistar operation on $D$, and let $I$ be a nonzero ideal of $D$. It is well known that $I^* \cap D$ is a quasi-$\ast$-ideal of $D$. (This is easy to see: we have

$$(I^* \cap D)^* \subseteq I^{**} = I^* = (I \cap D)^* \subseteq (I^* \cap D)^*,$$

and hence $I^* = (I^* \cap D)^*$; it follows that $I^* \cap D = (I^* \cap D)^* \cap D$.) It therefore seems natural to call $I^* \cap D$ the quasi-$\ast$-closure of $I$. If we also call $I$ $\ast$-proper when $I^* \subseteq D^\ast$, then it is easy to see that $I$ is $\ast$-proper if and only if its quasi-$\ast$-closure is a proper quasi-$\ast$-ideal. Now suppose that $I$ is $\ast$-idempotent. Then

$$(I^* \cap D)^* = I^* = (I^2)^\ast = ((I^*)^2)^\ast = (((I^* \cap D)^*)^2)^\ast = (((I^* \cap D)^2)^\ast)^\ast,$$

whence $I^* \cap D$ is a $\ast$-idempotent quasi-$\ast$-ideal of $D$. A similar argument gives the converse. Thus a ($\ast$-proper) nonzero ideal is $\ast$-idempotent if and only if its quasi-$\ast$-closure is a (proper) $\ast$-idempotent quasi-$\ast$-ideal.

Our study of idempotence in Prüfer domains and $P\star$MDs involves the notions of sharpness and branchedness. We recall some notation and terminology.

Given an integral domain $D$ and a prime ideal $P \in \text{Spec}(D)$, set

$$\nabla(P) := \{M \in \text{Max}(D) \mid M \nsubseteq P\}$$
and
$$\Theta(P) := \bigcap\{D_M \mid M \in \nabla(P)\}.$$

We say that $P$ is sharp if $\Theta(P) \nsubseteq D_P$ (see [11] Lemma 1 and [3] Section 1 and Proposition 2.2). The domain $D$ itself is sharp (doublesharp) if every maximal (prime) ideal of $D$ is sharp. (Note that a Prüfer domain $D$ is doublesharp if and only if each overring of $D$ is sharp [3] Theorem 4.1.7.) Now let $\ast$ be a semistar operation on $D$. Given a prime ideal $P \in Q\text{Spec}^{\ast}(D)$, set

$$\nabla^{\ast}(P) := \{M \in Q\text{Max}^{\ast}(D) \mid M \nsubseteq P\}$$
and
$$\Theta^{\ast}(P) := \bigcap\{D_M \mid M \in \nabla^{\ast}(P)\}.$$
Call $P \star_{\sim}$-sharp if $\Theta^\sim(P) \nsubseteq D_P$. For example, if $\star = d$ is the identity, then the $\star_{\sim}$-sharp property coincides with the sharp property. We then say that $D$ is $\star_{\sim}$-(double) sharp if each quasi-$\star_{\sim}$-maximal (quasi-$\star_{\sim}$-prime) ideal of $D$ is $\star_{\sim}$-sharp. (For more on sharpness, see [10], [11], [13], [7, page 62], [3], [4, Chapter 2, Section 3], and [5].)

Recall that a prime ideal $P$ of a ring is said to be branched if there is a $P$-primary ideal distinct from $P$. Also, recall that the domain $D$ has finite character if each nonzero ideal of $D$ is contained in only finitely many maximal ideals of $D$.

We now prove a lemma that discusses the transfer of ideal-theoretic properties between (on which a semistar operation $\star$ has been defined) and its associated Nagata ring.

**Lemma 2.2.** Let $\star$ be a semistar operation on $D$.

(1) Let $E \in \mathcal{F}(D)$. Then $E$ is $\sim_{\star}$-idempotent if and only if $E\text{Na}(D,\star)$ is idempotent. In particular, if $D$ is a $P\star M D$, then $E$ is $\star_{\sim}$-idempotent if and only if $E\text{Na}(D,\star)$ is idempotent.

(2) Let $P$ be a quasi-$\sim_{\star}$-prime of $D$ and $I$ a nonzero ideal of $D$. Then:

(a) $P$ is $P$-primary in $D$ if and only if $I$ is a quasi-$\sim_{\star}$-ideal of $D$ and $INa(D,\star)$ is $P\text{Na}(D,\star)$-primary in $\text{Na}(D,\star)$.

(b) $P$ is branched in $D$ if and only if $P\text{Na}(D,\star)$ is branched in $\text{Na}(D,\star)$.

(3) $D$ has $\star_{\sim}$-finite character (i.e., each nonzero element of $D$ belongs to only finitely many (possibly zero) $M \in \text{QMax}^\sim(D)$) if and only if $\text{Na}(D,\star)$ has finite character.

(4) Let $I$ be a quasi-$\sim_{\star}$-ideal of $D$. Then $I$ is a radical ideal if and only if $I\text{Na}(D,\star)$ is a radical ideal of $\text{Na}(D,\star)$.

(5) Assume that $D$ is a $P\star M D$. Then:

(a) If $P \in \text{QSpec}^\sim(D)$, then $P$ is $\star_{\sim}$-sharp if and only if $P\text{Na}(D,\star)$ is sharp in $\text{Na}(D,\star)$.

(b) $D$ is $\star_{\sim}$-(double) sharp if and only if $\text{Na}(D,\star)$ is (double) sharp.

**Proof.** (1) We use Lemma [1.1(6)]. If $E\text{Na}(D,\star)$ is idempotent, then $E^2 = E\text{Na}(D,\star) \cap K = E^2\text{Na}(D,\star) \cap K = (E^2)^\sim$. Conversely, if $E$ is $\sim_{\star}$-idempotent, then $(E\text{Na}(D,\star))^2 = E^2\text{Na}(D,\star) = (E^2)^\sim\text{Na}(D,\star) = E^2\text{Na}(D,\star) = E\text{Na}(D,\star)$. The “in particular” statement follows because $\star_{\sim} = \sim$ in a $P\star M D$ (Lemma [1.2(2a)]).

(2) (a) Suppose that $I$ is $P$-primary. Then $ID[X]$ is $P D[X]$-primary. Since $P$ is a quasi-$\sim_{\star}$-prime of $D$, $P\text{Na}(D,\star)$ is a prime ideal of $\text{Na}(D,\star)$ (Lemma [1.1(4)]), and then, since $\text{Na}(D,\star)$ is a quotient ring of $D[X]$, $INa(D,\star)$ is $P\text{Na}(D,\star)$-primary in $\text{Na}(D,\star)$. Also, again using the fact that $ID[X]$ is $P D[X]$-primary (along with Lemma [1.1(6)]), we have

$$I^\sim \cap D = I\text{Na}(D,\star) \cap D \subseteq ID[X]_{P D[X]} \cap D[X] \cap D = ID[X] \cap D = I,$$

whence $I$ is a quasi-$\sim_{\star}$-ideal of $D$. Conversely, assume that $I$ is a quasi-$\sim_{\star}$-ideal of $D$ and that $I\text{Na}(D,\star)$ is $P\text{Na}(D,\star)$-primary. Then for $a \in P$, there is a positive integer $n$ for which $a^n \in I\text{Na}(D,\star) \cap D = I^\sim \cap D = I$. Hence $P = \text{rad}(I)$. It now follows easily that $I$ is $P$-primary.
(b) Suppose that $P$ is branched in $D$. Then there is a $P$-primary ideal $I$ of $D$ distinct from $P$, and $I N a(D, \ast)$ is $P N a(D, \ast)$-primary by (a). Also by (a), $I$ is a quasi-$\ast$-ideal, from which it follows that $I N a(D, \ast) \neq P N a(D, \ast)$. Now suppose that $P N a(D, \ast)$ is branched and that $J$ is a $P N a(D, \ast)$-primary ideal of $N a(D, \ast)$ distinct from $P N a(D, \ast)$. Then it is straightforward to show that $J \cap D$ is distinct from $P$ and is $P$-primary.

(3) Let $\psi$ be a nonzero element of $N a(D, \ast)$, and let $N$ be a maximal ideal with $\psi \in N$. Then $\psi N a(D, \ast) = f N a(D, \ast)$ for some $f \in D[X]$, and writing $N = M N a(D, \ast)$ for some $M \in Q M a^\ast(D)$ (Lemma I.1(3)), we must have $f \in M D[X]$ and hence $c(f) \subseteq M$. Therefore, if $D$ has finite $\ast$-character, then $N a(D, \ast)$ has finite character. The converse is even more straightforward.

(4) Suppose that $I$ is a radical ideal of $D$, and let $\psi^n \in I N a(D, \ast)$ for some $\psi \in N a(D, \ast)$ and positive integer $n$. Then there is an element $g \in N(\ast)$ with $(g \psi^n) \in I D[X]$. Since $I D[X]$ is a radical ideal of $D[X]$, $g \psi \in I D[X]$ and we must have $\psi \in I N a(D, \ast)$. Therefore, $I N a(D, \ast) \ast$ is a radical ideal of $N a(D, \ast)$. The converse follows easily from the fact that $I N a(D, \ast) \cap D = I^2 \cap D = I$ (Lemma I.1(7)).

(5) (a) This is part of [5 Proposition 3.5], but we give here a proof more in the spirit of this paper. Let $P \in Q S e c^\ast(D)$. If $P$ is $\ast$-sharp, then by [5 Proposition 3.1] $P$ contains a finitely generated ideal $I$ with $I \nsubseteq M$ for all $M \in \nabla^\ast(P)$, and, using the description of $M a x(N a(D, \ast))$ given in Lemma I.1(3), $I N a(D, \ast)$ is a finitely generated ideal of $N a(D, \ast)$ contained in $P N a(D, \ast)$ but in no element of $\nabla(P N a(D, \ast))$. Therefore, $P N a(D, \ast)$ is sharp in the Prüfer domain $N a(D, \ast)$. For the converse, assume that $P N a(D, \ast)$ is sharp in $N a(D, \ast)$. Then $P N a(D, \ast)$ contains a finitely generated ideal $J$ with $J \subseteq P N a(D, \ast)$ but $J \nsubseteq N$ for $N \in \nabla(P N a(D, \ast))$ (Corollary 2). Then $J = I N a(D, \ast)$ for some finitely generated ideal $I$ of $D$ (necessarily) contained in $P$ by Lemma I.2(2c), and it is easy to see that $I \nsubseteq M$ for $M \in \nabla^\ast(D)$. Then by [5 Proposition 3.1], $P$ is $\ast$-sharp. Statement (b) follows easily from (a) (using Lemma I.2). □

Let $D$ be an almost Dedekind domain with a non-finitely generated maximal ideal $M$. Then $M^{-1} = D$, but $M$ is not idempotent (since $M D_M$ is not idempotent in the Noetherian valuation domain $D_M$). Our next result shows that this cannot happen in a sharp Prüfer domain.

**Theorem 2.3.** Let $D$ be a Prüfer domain. If $D$ is $(d\text{-})$-sharp and $I$ is a nonzero ideal of $D$ with $I^{-1} = D$, then $I$ is idempotent.

**Proof.** Assume that $D$ is sharp. Proceeding contrapositively, suppose that $I$ is a nonzero, non-idempotent ideal of $D$. Then, for some maximal ideal $M$ of $D$, $I D_M$ is not idempotent in $D_M$. Since $D$ is a sharp domain, we may choose a finitely generated ideal $J$ of $D$ with $J \subseteq M$ but $J \nsubseteq N$ for all maximal ideals $N \neq M$. Since $I D_M$ is a non-idempotent ideal in the valuation domain $D_M$, there is an element $a \in I$ for which $I^2 D_M \subseteq a D_M$. Let $B := J + Da$. Then $I^2 D_M \subseteq B D_M$ and, for $N \in M a x(D) \setminus \{M\}$, $I^2 D_N \subseteq D_N = B D_N$. Hence $I^2 \subseteq B$. Since $B$ is a proper finitely generated ideal, we then have $(I^2)^{-1} \supseteq B^{-1} \supseteq D$. Hence $(I^2)^{-1} \neq D$, from which it follows that $I^{-1} \neq D$, as desired. □
Lemma 2.4. Kang [14, Proposition 2.2] proves that, for a nonzero ideal $I$ of $D$, we always have $I^{-1} \text{Na}(D, v) = (\text{Na}(D, v)) : I$. This cannot be extended to general semistar Nagata rings; for example, if $D$ is an almost Dedekind domain with non-invertible maximal ideal $M$ and we define a semistar operation $\ast$ by $E^\ast = ED_M$ for $E \in \overline{F}(D)$, then $(D : M) = D$ and hence $(D : M)\text{Na}(D, \ast) = \text{Na}(D, \ast) = D[X]_{M[X]} = D_M(X) \subseteq (D_M : M)D_M(X) = (\text{Na}(D, \ast) : M\text{Na}(D, \ast))$ (where the proper inclusion holds because $MD_M$ is principal in $D_M$). At any rate, what we really require is the equality $(D^\ast : E)\text{Na}(D, \ast) = (\text{Na}(D, \ast) : E)$ for $E \in \overline{F}(D)$. In the next lemma, we show that this holds in a $P\ast MD$ but not in general. The proof of part (1) of the next lemma is a relatively straightforward translation of the proof of [14, Proposition 2.2] to the semistar setting. In carrying this out, however, we discovered a minor flaw in the proof of [14, Proposition 2.2]. The flaw involves a reference to [12, Proposition 34.8], but this result requires that the domain $D$ be integrally closed. To ensure complete transparency, we give the proof in full detail.

Lemma 2.4. Let $\ast$ be a semistar operation on $D$. Then:

1. $(D^\ast : E)\text{Na}(D, \ast) \supseteq (\text{Na}(D, \ast) : E)$ for each $E \in \overline{F}(D)$.
2. The following statements are equivalent:
   a. $(D^\ast : E)\text{Na}(D, \ast) = (\text{Na}(D, \ast) : E)$ for each $E \in \overline{F}(D)$.
   b. $D^\ast = D^\ast$.
   c. $D^\ast \subseteq \text{Na}(D, \ast)$.
3. $(D^\ast : E)\text{Na}(D, \ast) = (\text{Na}(D, \ast) : E)$ for each $E \in \overline{F}(D)$.
4. If $D$ is a $P\ast MD$, then the equivalent conditions in (2) hold.

Proof. (1) Let $E \in \overline{F}(D)$, and let $\psi \in (\text{Na}(D, \ast) : E)$. For $a \in E$, $a \neq 0$, we may find $g \in N(\ast)$ such that $\psi a g \in D[X]$. This yields $\psi a g \in a^{-1}D[X] \subseteq K[X]$, and hence $\psi = f/g$ for some $f \in K[X]$. We claim that $c(f) \subseteq (D^\ast : E)$. Granting this, we have $f \in (D^\ast : E)D[X]$, from which it follows that $\psi = f/g \in (D^\ast : E)\text{Na}(D, \ast)$, as desired. To prove the claim, take $b \in E$, and note that $fb \in \text{Na}(D, \ast)$. Hence $fbb \in D[X]$ for some $h \in N(\ast)$, and so $c(fh)b \subseteq D$. By the content formula [12, Theorem 28.1], there is an integer $m$ for which $c(f)c(h)^{m+1} = c(fh)c(h)^m$. Using the fact that $c(h)^* = D^\ast$, we obtain $c(f)^* = c(fh)^*$ and hence that $c(b) \subseteq c(fh)^*b \subseteq D^\ast$. Therefore, $c(f) \subseteq (D^\ast : E)$, as claimed.

2. Under the assumption in (c), $D^\ast \subseteq \text{Na}(D, \ast) \cap K = D^\ast$ (Lemma [14, 6]). Hence (c) $\Rightarrow$ (b). Now assume that $D^\ast = D^\ast$. Then for $E \in \overline{F}(D)$, we have $(D^\ast : E)\text{Na}(D, \ast) \subseteq D^\ast \subseteq \text{Na}(D, \ast)$; using (1), the implication (b) $\Rightarrow$ (a) follows. That (a) $\Rightarrow$ (c) follows upon taking $E = D$ in (a).

3. This follows easily from (2), because $\text{Na}(D, \ast) = \text{Na}(D, \ast)$ by Lemma [14, 5].

4. This follows from (2), since if $D$ is a $P\ast MD$, then $D^\ast = D^\ast$ by Lemma [12, 2a).

The conditions in Lemma [24, 2] need not hold: Let $F \subseteq k$ be fields, $V = k[[x]]$ the power series ring over $V$ in one variable, and $D = F + M$, where $M = xk[[x]]$. Define a (finite-type) semistar operation $\ast$ on $D$ by $A^\ast = AV$ for $A \in \overline{F}(D)$. Then $D^\ast = V \supseteq D = D_M = D^\ast$.

We can now extend Theorem 2.3 to $P\ast MD$s.
Corollary 2.5. Let $\ast$ be a semistar operation on $D$ such that $D$ is a $\ast_f$-sharp $P\ast MD$, and let $I$ be a nonzero ideal of $D$ with $(D^\ast : I) = D^\ast$. Then $I$ is $\ast_f$-idempotent.

Proof. By Lemma 2.4(3), we have

$$(\text{Na}(D, \ast) : I\text{Na}(D, \ast)) = (D^\ast : I)\text{Na}(D, \ast) = D^\ast\text{Na}(D, \ast) = \text{Na}(D, \ast).$$

Hence $I\text{Na}(D, \ast)$ is idempotent in the Prüfer domain $\text{Na}(D, \ast)$ by Theorem 2.3 Lemma 2.2(1) then yields that $I$ is $\ast_f$-idempotent.

Many semistar counterparts of ideal-theoretic properties in domains result in equations that are “external” to $D$, since for a semistar operation $\ast$ on $D$ and a nonzero ideal $I$ of $D$, it is possible that $I^\ast \not\subseteq D$. Of course, $\ast$-idempotence is one such property. Often, one can obtain a “cleaner” counterpart by specializing from $P\ast MDs$ to “ordinary” $PMDs$. We recall some terminology. Semistar operations are generalizations of star operations, first considered by Krull and repopularized by Gilmer [12, Sections 32, 34]. Roughly, a star operation is a semistar operation restricted to the set $F(D)$ of nonzero fractional ideals of $D$ with the added requirement that one has $D^\ast = D$. The most important star operation (aside from the $d$-, or trivial, star operation) is the $v$-operation: For $E \in F(D)$, put $E^{-1} = \{x \in K \mid xE \subseteq D\}$ and $E^v = (E^{-1})^{-1}$. Then $v_f$ (restricted to $F(D)$) is the $t$-operation and $\tilde{v}$ is the $w$-operation. Thus a $PMD$ is a domain in which each nonzero finitely generated ideal is $t$-invertible. Corollary 2.5 then has the following restricted interpretation (which has the advantage of being internal to $D$).

Corollary 2.6. If $D$ is a $t$-sharp $PMD$ and $I$ is a nonzero ideal of $D$ for which $I^{-1} = D$, then $I$ is $t$-idempotent.

Our next result is a partial converse to Theorem 2.3.

Proposition 2.7. Let $D$ be a Prüfer domain such that $I$ is idempotent whenever $I$ is a nonzero ideal of $D$ with $I^{-1} = D$. Then, every branched maximal ideal of $D$ is sharp.

Proof. Let $M$ be a branched maximal ideal of $D$. Then $MD_M = \text{rad}(aD_M)$ for some nonzero element $a \in M$ [12, Theorem 17.3]. Let $I := aD_M \cap D$. Then $I$ is $M$-primary, and since $ID_M = aD_M$, $(ID_M$ and hence $I$ is not idempotent. By hypothesis, we may choose $u \in I^{-1} \setminus D$. Since $Iu \subseteq D$ and $ID_N = D_N$ for $N \in \text{Max}(D) \setminus \{M\}$, then $u \in \bigcap\{D_N \mid N \in \text{Max}(D), N \neq M\}$. On the other hand, since $u \notin D$, $u \notin D_M$. It follows that $M$ is sharp.

Now we extend Proposition 2.7 to $PMDs$.

Corollary 2.8. Let $\ast$ be a semistar operation on $D$, and assume that $D$ is a $P\ast MD$ such that $I$ is $\ast_f$-idempotent whenever $I$ is a nonzero ideal of $D$ with $(D^\ast : I) = D^\ast$. Then, each branched quasi-$\ast_f$-maximal ideal of $D$ is $\ast_f$-sharp. (In particular if $D$ is a $PMD$ in which $I$ is $t$-idempotent whenever $I$ is a nonzero ideal of $D$ with $I^{-1} = D$, then each branched maximal $t$-ideal of $D$ is $t$-sharp.)
Proof. Let \( J \) be a nonzero ideal of the Prüfer domain \( \text{Na}(D, \ast) \) with \( \text{Na}(D, \ast) : J = \text{Na}(D, \ast) \). By Lemma 1.2(1c), \( J = I \text{Na}(D, \ast) \) for some ideal \( I \) of \( D \). Applying Lemma 2.4(3) and Lemma 1.1(6), we obtain \( (D^\ast : I) = D^\ast \). Hence, by hypothesis, \( I \) is \( \ast \)-idempotent, and this yields that \( J = I \text{Na}(D, \ast) \) is idempotent in the Prüfer domain \( \text{Na}(D, \ast) \) (Lemma 2.2(1)). Now, let \( M \) be a branched quasi-\( \ast \)-maximal ideal of \( D \). Then, by Lemma 2.2(2), \( M \text{Na}(D, \ast) \) is a branched maximal ideal of \( \text{Na}(D, \ast) \). We may now apply Proposition 2.7 to conclude that \( M \text{Na}(D, \ast) \) is sharp. Therefore, \( M \) is \( \ast \)-sharp in \( D \) by Lemma 2.2(5). \(\square\)

If \( P \) is a prime ideal of a Prüfer domain \( D \), then powers of \( P \) are \( P \)-primary by [12, Theorem 23.3(b)]; it follows that \( P \) is idempotent if and only if \( PD_P \) is idempotent. We use this fact in the next result.

It is well known that a proper idempotent ideal of a valuation domain must be prime [12, Theorem 17.1(3)]. In fact, according to [12, Exercise 3, p. 284], a proper idempotent ideal in a Prüfer domain must be a radical ideal. We (re-)prove and extend this fact and add a converse:

**Theorem 2.9.** Let \( D \) be a Prüfer domain, and let \( I \) be an ideal of \( D \). Then \( I \) is idempotent if and only if \( I \) is a radical ideal each of whose minimal primes is idempotent.

**Proof.** The result is trivial for \( I = (0) \) and vacuously true for \( I = D \). Suppose that \( I \) is a proper nonzero idempotent ideal of \( D \), and let \( P \) be a prime minimal over \( I \). Then \( IP_P \) is idempotent, and we must have \( IP_P = PD_P \) [12, Theorem 17.1(3)]. Hence \( PD_P \) is idempotent, and therefore, by the comment above, so is \( P \). Now let \( M \) be a maximal ideal containing \( I \). Then \( IM_M \) is idempotent, hence prime (hence radical). It follows (checking locally) that \( I \) is a radical ideal.

Conversely, let \( I \) be a radical ideal each of whose minimal primes is idempotent. If \( M \) is a maximal ideal containing \( I \) and \( P \) is a minimal prime of \( I \) contained in \( M \), then \( IP_M = PD_M \). Since \( P \) is idempotent, this yields \( IM_M = P^2M_M \). It follows that \( I \) is idempotent. \(\square\)

We next extend Theorem 2.9 to \( P \)-MDs.

**Corollary 2.10.** Let \( D \) be a \( P \)-MD, where \( \ast \) is a semistar operation on \( D \), and let \( I \) be a quasi-\( \ast \)-ideal of \( D \). Then \( I \) is \( \ast \)-idempotent if and only if \( I \) is a radical ideal each of whose minimal primes is \( \ast \)-idempotent. (In particular, if \( D \) is a \( P \)-MD and \( I \) is a \( t \)-ideal of \( D \), then \( I \) is \( t \)-idempotent if and only if \( I \) is a radical ideal each of whose minimal primes is \( t \)-idempotent.)

**Proof.** Suppose that \( I \) is \( \ast \)-idempotent. Then \( I \text{Na}(D, \ast) \) is an idempotent ideal in \( \text{Na}(D, \ast) \) by Lemma 2.2(1). By Theorem 2.9, \( I \text{Na}(D, \ast) \) is a radical ideal of \( \text{Na}(D, \ast) \), and hence, by Lemma 2.2(4), \( I \) is a radical ideal of \( D \). Now let \( P \) be a minimal prime of \( I \) in \( D \). Then \( P \) is a quasi-\( \ast \)-prime of \( D \). By Lemma 1.2(2b) \( P \text{Na}(D, \ast) \) is minimal over \( I \text{Na}(D, \ast) \), whence \( P \text{Na}(D, \ast) \) is idempotent, again by Theorem 2.9. The \( \ast \)-idempotence of \( P \) now follows from Lemma 2.2(1).

The converse follows by similar applications of Theorem 2.9 and Lemma 2.2. \(\square\)
Recall that a Prüfer domain is said to be strongly discrete (discrete) if it has no nonzero (branched) idempotent prime ideals. Since unbranched primes in a Prüfer domain must be idempotent [12, Theorem 23.3(b)], a Prüfer domain is strongly discrete if and only if it is discrete and has no unbranched prime ideals. We have the following straightforward application of Theorem 2.9.

**Corollary 2.11.** Let $D$ be a Prüfer domain.

1. If $D$ is discrete, then an ideal $I$ of $D$ is idempotent if and only if $I$ is a radical ideal each of whose minimal primes is unbranched.
2. If $D$ is strongly discrete, then $D$ has no proper nonzero idempotent ideals.

Let us call a P⋆MD *f*-strongly discrete (*f*-discrete) if it has no (branched) *f*-idempotent quasi-*f*-prime ideals. From Lemma 2.2(1,2), we have the usual connection between a property of a P⋆MD and the corresponding property of its *-Nagata ring:

**Proposition 2.12.** Let * be a semistar operation on $D$. Then $D$ is *f*-discrete if and only if $Na(D,*)$ is a (strongly) discrete Prüfer domain.

Applying Corollary 2.10 and Lemma 2.2(1,2), we have the following extension of Corollary 2.11.

**Corollary 2.13.** Let $D$ be a domain.

1. Assume that $D$ is a P⋆MD for some semistar operation * on $D$.
   a. If $D$ is *f*-discrete, then a nonzero quasi-*f*-ideal $I$ of $D$ is *f*-idempotent if and only if $I$ is a radical ideal each of whose minimal primes is unbranched.
   b. If $D$ is *f*-strongly discrete, then $D$ has no *f*-proper *f*-idempotent ideals.
2. Assume that $D$ is a P⋆MD.
   a. If $D$ is t-discrete, then a $t$-ideal $I$ of $D$ is $t$-idempotent if and only if $I$ is a radical ideal each of whose minimal primes is unbranched.
   b. If $D$ is $t$-strongly discrete, then $D$ has no $t$-proper $t$-idempotent ideals.

### 3. Divisoriality

According to [7, Corollary 4.1.14], if $D$ is a doublesharp Prüfer domain and $P$ is a nonzero, nonmaximal ideal of $D$, then $P$ is divisorial. The natural question arises: If $D$ is a *f*-doublesharp P⋆MD and $P \in QSpec^\gamma(D) \setminus QMax^\gamma(D)$, is $P$ necessarily divisorial? Since * is an arbitrary semistar operation and divisoriality specifically involves the $\gamma$-operation, one might expect the answer to be negative. Indeed, we give a counterexample in Example 3.4 below. However, in Theorem 3.2 we prove a general result, a corollary of which does yield divisoriality in the “ordinary” P⋆MD case. First, we need a lemma, the first part of which may be regarded as an extension of [14, Proposition 2.2(2)].

**Lemma 3.1.** Let * be a semistar operation on $D$. Then

1. $(D^*: (D^*: E))Na(D,*) = (Na(D,*) : (Na(D,*) : E))$ for each $E \in \mathcal{F}(D)$, and
If $I$ is a nonzero ideal of $D$, then $\tilde{I}^\ast$ is a divisorial ideal of $D^\ast$ if and only if $\text{INa}(D, \ast)$ is a divisorial ideal of $\text{Na}(D, \ast)$.

In particular, if $D$ is a $P\ast MD$, then $(D^\ast : (D^\ast : E))\text{Na}(D, \ast) = (\text{Na}(D, \ast) : (\text{Na}(D, \ast) : E))$ for each $E \in \mathbb{F}(D)$; and, for a nonzero ideal $I$ of $D$, $I^\ast$ is divisorial in $D^\ast$ if and only if $\text{INa}(D, \ast)$ is divisorial in $\text{Na}(D, \ast)$.

**Proof.** Set $N = \text{Na}(D, \ast)$. For (1), applying Lemma 2.4 we have

$$(D^\ast : (D^\ast : E))N = (N : (D^\ast : E)) = (N : (N : E)).$$

(2) Assume that $I$ is a nonzero ideal of $D$. If $\tilde{I}^\ast$ is divisorial in $D^\ast$, then (using (1))

$$N : (N : I)\cap K = \tilde{I}^\ast.$$

Now suppose that $IN$ is divisorial. Then

$$(D^\ast : (D^\ast : I))N = (N : (N : I)) = IN,$$

whence

$$(D^\ast : (D^\ast : I)) \subseteq IN \cap K = \tilde{I}^\ast.$$

The “in particular” statement follows from standard considerations. \hfill \Box

**Theorem 3.2.** Let $\ast$ be a semistar operation on $D$ such that $D$ is a $\ast_f$-doublesharp $P\ast MD$, and let $P \in \text{QSpec}^\ast(D) \setminus \text{QMax}^\ast(D)$. Then $P^\ast$ is a divisorial ideal of $D^\ast$.

**Proof.** Since $\text{Na}(D, \ast)$ is a doublesharp Prüfer domain (Lemma 2.2(5)), $P\text{Na}(D, \ast)$ is divisorial by [2, Corollary 4.1.14]. Hence $P^\ast$ is divisorial in $D^\ast$ by Lemma 3.1. \hfill \Box

**Corollary 3.3.** If $D$ is a $t$-doublesharp $PvMD$, and $P$ is a non-$t$-maximal $t$-prime of $D$, then $P$ is divisorial.

**Proof.** Take $\ast = v$ in Theorem 3.2. (More precisely, take $\ast$ to be any extension of the star operation $v$ on $D$ to a semistar operation on $D$, so that $\ast_f$ (restricted to $D$) is the $t$-operation on $D$.) Then $P = P^\ast = P^t$ is divisorial by Theorem 3.2. \hfill \Box

**Example 3.4.** Let $p$ be a prime integer and let $D := \text{Int}(\mathbb{Z}(p))$. Then $D$ is a 2-dimensional Prüfer domain by [2, Lemma VI.1.4 and Proposition V.1.8]. Choose a height 2 maximal ideal $M$ of $D$, and let $P$ be a height 1 prime ideal of $D$ contained in $M$. Then $P = qQ[X] \cap D$ for some irreducible polynomial $q \in Q[X]$ [2, Proposition V.2.3]. By [2, Theorems VIII.5.3 and VIII.5.15], $P$ is not a divisorial ideal of $D$. Set $E^\ast = ED_M$ for $E \in \mathbb{F}(D)$. Then, $\ast$ is a finite-type semistar operation on $D$. Clearly, $M$ is the only quasi-$\ast$-maximal ideal of $D$, and, since $D_M$ is a valuation domain, $D$ is a $P\ast MD$ by Lemma 1.3. Moreover, $\text{Na}(D, \ast) = D_M(X)$ is also a valuation domain and hence a doublesharp Prüfer domain, which yields that $D$ is a $\ast_f$-doublesharp $P\ast MD$ (Lemma 2.2). Finally, since $P = PD_M \cap D = P^\ast \cap D$, $P$ is a non-$\ast_f$-maximal quasi-$\ast_f$-prime of $D$. \hfill \Box

In the remainder of the paper, we impose on Prüfer domains ($P\ast MD$s) the finite character (finite $\ast_f$-character) condition. As we shall see, this allows improved versions of Theorem 2.9 and Corollary 2.10. It also allows a type of unique factorization for (quasi-$\ast_f$-)ideals that are simultaneously ($\ast_f$-)idempotent and ($\ast_f$-)divisorial.
Theorem 3.5. Let $D$ be a Prüfer domain with finite character, and let $I$ be a nonzero ideal of $D$. Then:

1. $I$ is idempotent if and only if $I$ is a product of idempotent prime ideals.
2. The following statements are equivalent.
   (a) $I$ is idempotent and divisorial.
   (b) $I$ is a product of non-maximal idempotent prime ideals.
   (c) $I$ is a product of divisorial idempotent prime ideals.
   (d) $I$ has a unique representation as the product of incomparable divisorial idempotent primes.

Proof. (1) Suppose that $I$ is idempotent. By Theorem 2.9, $I$ is the intersection of its minimal primes, each of which is idempotent. Since $D$ has finite character, $I$ is contained in only finitely many maximal ideals, and, since no two distinct minimal primes of $I$ can be contained in a single maximal ideal, $I$ has only finitely many minimal primes and they are comaximal. Hence $I$ is the product of its minimal primes (and each is idempotent). The converse is trivial.

(2) (a) $\Rightarrow$ (b): Assume that $I$ is idempotent and divisorial. By (1) and its proof, $I = P_1 \cdots P_n = P_1 \cap \cdots \cap P_n$, where the $P_i$ are the minimal primes of $I$. We claim that each $P_i$ is divisorial. To see this, observe that $(P_1) \cap \cdots \cap P_n = I = I \subseteq P_1$. Since the $P_i$ are incomparable, this gives $(P_i) \subseteq P_1$, that is, $P_1$ is divisorial. By symmetry each $P_i$ is divisorial. It is well known that in a Prüfer domain, a maximal ideal cannot be both idempotent and divisorial. Hence the $P_i$ are non-maximal.

(b) $\Rightarrow$ (c): Since $D$ has finite character, it is a ($d$)-doublesharp Prüfer domain [13, Theorem 5], whence nonmaximal primes are automatically divisorial by [7, Corollary 4.1.14].

(c) $\Rightarrow$ (a): Write $I = Q_1 \cdots Q_m$, where each $Q_j$ is a divisorial idempotent prime. Since $I$ is idempotent (by (1)), we may also write $I = P_1 \cdots P_n$, where the $P_i$ are the minimal primes of $I$. For each $i$, we have $Q_1 \cdots Q_m = I \subseteq P_i$, from which it follows that $Q_j \subseteq P_i$ for some $j$. By minimality, we must then have $Q_j = P_i$. Thus each $P_i$ is divisorial, whence $I = P_1 \cap \cdots \cap P_n$ is divisorial.

Finally, we show that (d) follows from the other statements. We use the notation in the proof of (c) $\Rightarrow$ (a). In the expression $I = P_1 \cdots P_n$, the $P_i$ are (divisorial, idempotent, and) incomparable, and it is clear that no $P_i$ can be omitted. To see that this is the only such expression, consider a representation $I = Q_1 \cdots Q_m$, where the $Q_i$ are divisorial, idempotent, and incomparable. Fix a $Q_k$. Then $P_1 \cdots P_n = I \subseteq Q_k$, and we have $P_i \subseteq Q_k$ for some $i$. However, as above, $Q_j \subseteq P_i$ for some $j$, whence, by incomparability, $Q_k = P_i$. The conclusion now follows easily. □

We note that incomparability is necessary for uniqueness above—for example, if $D$ is a valuation domain and $P \subseteq Q$ are non-maximal (necessarily divisorial) primes, then $P = PQ$.

We close by extending Theorem 3.5 to P*MDs and then to “ordinary” PrMDs. We omit the (by now) straightforward proofs.
Corollary 3.6. Let $\star$ be a semistar operation on $D$ such that $D$ is a P-$\star$MD with finite $\star_f$-character, and let $I$ be a quasi-$\star_f$-ideal of $D$. Then:

1. $I$ is quasi-$\star_f$-idempotent if and only if $I^{\star_f}$ is a quasi-$\star_f$-product of quasi-$\star_f$-idempotent quasi-$\star_f$-prime ideals in $D$, that is, $I^{\star_f} = (P_1 \cdot \ldots \cdot P_n)^{\star_f}$, where the $P_i$ are quasi-$\star_f$-idempotent quasi-$\star_f$-primes of $D$.

2. The following statements are equivalent.
   (a) $I$ is quasi-$\star_f$-idempotent and divisorial ($I^{\star_f}$ is divisorial in $D^{\star_f}$).
   (b) $I$ is a quasi-$\star_f$-product of non-quasi-$\star_f$-maximal idempotent quasi-$\star_f$-prime ideals.
   (c) $I$ is a quasi-$\star_f$-product of quasi-$\star_f$-divisorial quasi-$\star_f$-idempotent prime ideals.
   (d) $I$ has a unique representation as a quasi-$\star_f$-product of incomparable quasi-$\star_f$-idempotent primes.

Corollary 3.7. Let $D$ be a P-vMD with finite $t$-character, and let $I$ be a nonzero $t$-ideal of $D$. Then:

1. $I$ is $t$-idempotent if and only if $I$ is a $t$-product of $t$-idempotent $t$-prime ideals in $D$.

2. The following statements are equivalent.
   (a) $I$ is $t$-idempotent and divisorial.
   (b) $I$ is a $t$-product of non-$t$-maximal $t$-idempotent $t$-primes.
   (c) $I$ is a $t$-product of divisorial $t$-idempotent $t$-primes.
   (d) $I$ has a unique representation as a $t$-product of incomparable divisorial $t$-idempotent $t$-primes.

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Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo, 1, 00146 Roma, Italy
E-mail address: fontana@mat.uniroma3.it

Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223 U.S.A.
E-mail address: eghousto@uncc.edu

Department of Mathematics, Chung-Ang University, Seoul 06974, Korea
E-mail address: mhpark@cau.ac.kr