TIME-FREQUENCY ANALYSIS AND DECAY PROPERTIES OF TSUNAMIS

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Wigner and Choi-Williams distributions produce increased time-frequency resolution and are applied to obtain the time-frequency distributions for tsunamis. In this study, a chirp signal was analyzed to examine the performance of the Wigner distribution. The Choi-Williams distribution, which suppresses the cross terms, was applied to estimate the time-frequency distributions for tsunamis. The distribution for the tsunami waveform observed at Susaki, Kochi Prefecture showed that variations in the natural time periods at Susaki Bay were predominant and the maximum peak of the time-varying spectrum appeared about 80 minutes after the first tsunami wave arrived. The decay relation of the tsunami was theoretically estimated using a wave energy equation with the rate of dissipation due to bottom friction and turbulence. A Talbot formula was obtained as the relationship of tsunami decay with time and its applicability was verified.

Key Words: time-frequency analysis, tsunami waveform, non-stationary process, Wigner distribution, Choi-Williams distribution, decay law

1. INTRODUCTION

Most natural phenomena generally arise from a non-stationary process in which the spectral content changes in time. In analyzing their time-frequency properties, a stationary process is commonly assumed for a short time interval. The spectra and autocovariance functions are generally evaluated under the assumption. This may be due to the fact that there is no existing appropriate estimation method for time-varying signals. The short-time Fourier transform (STFT) or maximum entropy method (MEM) has been used to represent time-frequency variation. After the 1970s, the wavelet transform (WT) has been used to estimate a time-varying spectrum. However, they do not provide sufficient resolution in both time and frequency distribution.

The Wigner distribution (WD) is a fundamental concept for increased resolution and has been receiving more attention among time-frequency analysis techniques in recent years. The Wigner distribution was proposed by Wigner to study quantum statistical mechanics. The distribution has the advantage of having high time and frequency resolution compared to STFT, but sometimes indicates intensity where one would expect zero values due to the cross terms for multi-component signals. Numerous studies have been made to remove spurious values caused by the cross terms. In the 1980s, Claasen and Mecklenbrauker rediscovered the Wigner distribution as a time-frequency signal analysis tool and applied it to acoustic, or speech signal processing. Choi and Williams succeeded in suppressing the cross terms by introducing an exponential kernel satisfying desirable properties of distribution, and in minimizing the spurious values.

This paper applied the Wigner distribution to a typical time-varying signal, a chirp signal, to examine its time-frequency resolution. The Choi-Williams distribution is also applied to estimate the time-frequency distribution for tsunami waveforms and to investigate the time-varying properties of tsunamis.

It is important to understand the decay process of tsunamis because there is a need to know when we may begin rescue activities. Munk was the first to deal with the decay process of large tsunamis in the Pacific Ocean and developed an acoustic analogy decay model. He proposed two decay mechanisms of diffusion scattering from coastlines and islands decaying as $t^{-1}$ and $t^{-2}$, respectively, and absorption decaying as $e^{-at}$, where $t$ is time, and $a$ is an attenuation coefficient. Hayashi et al. investigated tsunami decay along the coast of Japan and found two types of decay processes, namely, decaying as $t^m$ for near-field tsunamis, and $e^{-at}$ for trans-ocean tsunamis. The decay mechanism of tsunamis, however, are not well
understood. In the present paper, tsunami wave energy conservation law was applied to the decay of tsunami amplitude to include energy dissipation due to bottom friction and turbulence. The tsunami amplitudes estimated by the present study decay with time like a Talbot equation, i.e., $K(t+c)$. The results were compared with the observed tsunami amplitude data.

2. ANALYSIS OF TIME-FREQUENCY DISTRIBUTION

(1) Properties of the Wigner distribution

When a non-stationary sequence is denoted as $x(t)$, the Wigner distribution $W(t, f)$ is defined as

$$W(t, f) = \int_{-\infty}^{\infty} x(t + \frac{\tau}{2}) x^*(t - \frac{\tau}{2}) e^{-j2\pi ft} d\tau$$

(1)

where $t$ is the time, $f$ is the frequency, $j$ is the imaginary unit, and $*$ indicates the complex conjugate. The relationships of Eqs. (3) and (4) are important and desirable features of the Wigner distribution. However, the Wigner distribution has shortcomings such that it sometimes takes negative values and indicates spurious values caused by the cross terms. Techniques for improving them will be shown in the next section.

(2) Time-Frequency distributions of Cohen’s class to reduce the cross-term difficulty

The Wigner distribution can produce a high time and frequency resolution for non-stationary signals with a single frequency component varying slowly with time. For multicomponent signals, however, it often indicates false energy peaks in time-frequency space caused by the cross terms. To remove the cross terms, extensive studies have been made so that the distribution satisfies the desirable properties. Most of the time-frequency distributions are known to be members of Cohen’s class distributions. The distributions are expressed as

$$C(t, f) = W(t, f) * \Phi(t, f)$$

(6)

where $C(t, f)$ is the member of Cohen’s class distributions, and $\Phi(t, f)$ is an arbitrary function. The Wigner distribution is obtained by taking $\Phi(t, f) = \delta(t)\delta(f)$. Taking the two-dimensional Fourier transform of $\Phi(t, f)$ as follows:

$$\varphi(\theta, \tau) = \int_{-\infty}^{\infty} \Phi(t, f) e^{-j(\theta t - 2\pi f t)} dt df$$

(7)

and inserting Eq.(7) into Eq. (6), we obtain

$$C(t, f) = \iint e^{-j(\theta t + 2\pi f t - \theta \tau)} \varphi(\theta, \tau) \cdot x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) d\mu d\nu$$

(8)

where $\varphi(\theta, \tau)$ is the kernel termed by Claasen and Mecklenbrauker. For example, the Wigner distribution is obtained by setting $\varphi(\theta, \tau) = 1$. To reduce the cross terms, the kernel $\varphi(\theta, \tau)$ is determined, minimizing spurious values so that the desirable properties of distribution are satisfied. Choi and Williams realized that by adopting the exponential kernel, one could minimize the cross terms and retain the desirable properties, such as the marginal conditions. The kernel is expressed as:

$$\varphi(\theta, \tau) = \exp\left(-\frac{\theta^2 + \tau^2}{\sigma}\right)$$

(9)

where $\sigma$ is a constant and controls the relative importance of $\tau$. If $\sigma$ tends to infinity, the Choi-Williams distribution becomes the Wigner distribution. The Choi-Williams distribution has an important property that satisfies the marginals for any value of $\sigma$. It can
also preserve a sharp resolution and significant structure in time-frequency analysis by choosing an appropriate value of $\sigma$.

3. IMPLEMENTATION OF NUMERICAL CALCULATION

(1) Wigner distribution

According to the definition of the Wigner distribution for the continuous case, it is convenient for the discrete case to rewrite the Eq. (1) as follows:

$$W(t, f/2) = \int_{-\infty}^{\infty} 2x(t + \tau)x^*(t - \tau)e^{-j2\pi f\tau}d\tau$$

(10)

The Wigner distribution can be estimated using data at time $t \pm \tau$. For the discrete case, the discrete Wigner distribution is given by

$$W(n, k) = \sum_{m=\frac{N}{2}}^{\frac{N}{2}+1} 2x(n + m)w(m)$$

(11)

where $w(m)$ represents the window function, $n$ is the discrete time, $m$ is the discrete time lag, and $k$ is the discrete frequency. There exist numerous window functions proposed by many researchers. Among them, The Hamming window is often used in signal processing to minimize leakage effects and is expressed as:

$$w(m) = 0.54 + 0.46\cos\left(\frac{2\pi m}{N}\right)$$

(12)

where $N$ is the number of discrete times. The discrete time window used for the discrete Wigner distribution adopts the analytic function, such as the Hamming window expressed as:

$$w(m) = 0.54 + 0.46e^{j\frac{2\pi m}{N}}$$

(13)

As shown in Eq. (10), it must be noted that the discrete Wigner distribution can be estimated at the frequencies up to half of the Nyquist frequency.

In calculating the discrete Wigner distribution, an adequate time lag needs to be set so that it provides the highest possible temporal and frequency resolution. The adequate time lag depends on the sampling frequency, the peak frequency, and the rate of change in the amplitude of a non-stationary sequence.

(2) Choi-Williams distribution

The Choi-Williams distribution\(^9\) was employed in this study to suppress the cross-terms and to produce a higher time-frequency resolution. The distribution was applied to the time-frequency analysis of tsunamis to investigate the properties of non-stationary variations of tsunamis.

By substituting Eq. (9) into Eq. (8) and by integrating, the Choi-Williams distribution, $W_{CW}(t, f)$, can be expressed as:

$$W_{CW}(t, f) = \int_{-\infty}^{\infty} \sqrt{\frac{\sigma}{4\pi^2}} \exp\left(\frac{(t-\mu)^2}{4\tau^2/\sigma}\right) x(\mu + \frac{\tau}{2})x^*(\mu - \frac{\tau}{2})e^{-j2\pi f\tau}d\mu d\tau$$

(14)

Discretizing the above equation and using a time window $w(m)$, we obtain

$$W_{CW}(n, k) = 2 \sum_{m=\frac{N}{2}}^{\frac{N}{2}+1} w(m)w^*(-m)$$

(15)

where $t=n\Delta t$, $\tau=m\Delta \tau$, and $\mu=\mu\Delta t$. The Hamming window is employed for the data window in this study since it produces lower side-lobes in the frequency domain.

(3) Application of the Wigner distribution to a chirp signal

To investigate the time-frequency resolution of Wigner distribution, we analyzed a chirp signal that had the instantaneous frequency changing in time. The following chirp signal $x(t)$ was employed (e.g., Inuzuka\(^{17}\)):

$$x(t) = \begin{cases} 0, & (0 \leq t \leq 200) \\ \sin\left(\frac{\pi}{3200}\right), & (200 \leq t \leq 1000) \end{cases}$$

(16)

The signal expressed by Eq.(16) and its Hilbert transform are shown in Fig. 1, indicated by the solid line and the dot-dash line, respectively. The signal by the Hilbert transform varies out of phase by $\pi/2$ compared to the real part of the signal. For the signal in the range of $(200 \leq t \leq 1000)$, the Wigner distribu-
tion can be integrated and expressed as:

\[ W(t, f) = \int_{-\infty}^{\infty} x\left(t + \frac{\tau}{2}\right) x^*\left(t - \frac{\tau}{2}\right) e^{-j2\pi f \tau} d\tau = \delta\left(f - \frac{(t - 200)}{3200}\right) \]  

(17)

where \( \delta \) is the Dirac’s delta function. This equation indicates that the instantaneous frequency of the signal changes linearly with time.

Figure 2 shows the estimated Wigner distribution functions with the horizontal time axis and the inclined instantaneous frequency axis. The results were calculated for each 25 s interval with the Hamming window of N=128 and \( \Delta t=1s \). As shown in Fig.2, the predominant instantaneous frequency changes linearly with time. The energy distributions for the chirp signal at particular times, such as 400 s, 600 s, and 800 s are shown in Fig. 3. The vertical dot-dash line indicates theoretical values as shown in Eq. (17). The Wigner distribution estimated is not the Dirac’s delta function, but has a sharp energy peak. The peak instantaneous frequencies agree well with the theoretical values, while the distributions take negative values at the frequencies on both sides of the peaks.

4. TIME-FREQUENCY ANALYSIS OF TSUNAMI WAVEFORMS

Tsunamis generated by earthquakes in the ocean and/or submarine slides appear to be in the non-stationary process. Their instantaneous power spectra vary with time. It is difficult to know when the variation having a particular instantaneous frequency is predominant. In this study, Wigner and Choi-Williams distributions were applied to tsunami waveforms to verify the applicability of their distributions.

(1) Tsunami observed at Susaki due to the 2011 Great off the Pacific Coast of Tohoku Earthquake

On March 11, 2011, a huge major earthquake occurred in the Pacific Ocean northeast of Japan. The moment magnitude of the earthquake was 9.0 \(^{18}\). A massive tsunami struck the Pacific Coast of Japan and caused devastating damage and took more than 18,000 lives. Tsunami waveforms at Susaki in Kochi Prefecture far from the epicenter of the earthquake more than 1000km varied slowly in amplitude and instantaneous frequency. We tried to analyze the time-frequency distribution for the tsunami waveform using Wigner and Choi-Williams distributions.

Figure 4 shows the observed tsunami waveform and its Hilbert transform indicated by solid line and dot-dash line, respectively. The tsunami waveform varies effectively and slowly in instantaneous frequency and period.
The Wigner distribution for the tsunami waveform was calculated with a Hamming window of $N=128$, $\Delta t=1\text{min}$ and is shown in Fig. 5. The distributions are indicated by solid, dotted, short dot, broken, and dot-dash lines at times $t=100\text{ min}, 150\text{ min}, 200\text{ min}, 250\text{ min},$ and $278\text{ min}$, respectively. After the Wigner distribution in the figure, tsunami waves having a period of 50 mins ($f=0.02\text{ cycle/min}$) were predominant around the time that they arrived at Susaki. The tsunami wave period reduced by about 33 min to 35 min, while the energy peaks became larger during the interval of $t=150\text{ min}$ to $t=300\text{ min}$.

The Choi-Williams distribution ($\sigma=3$) for the tsunami waveform is shown in Fig. 6. Most of the energy density spectra diminished their magnitudes by about 20% to 30% compared to those of the Wigner distribution. Among them, the distribution at time, $t=100\text{ min}$ changes its form significantly. This appears to result from the spurious values by the cross terms becoming small by an appropriate choice of $\sigma$. The energy peaks in the instantaneous frequency of $f=0.03\text{ cycle/min}$ (a period of about 33 min) are predominant during the interval of $t=150\text{ min}$ to $t=278\text{ min}$. We investigated why the tsunami having a particular instantaneous frequency of 0.03 cycle/min was predominant.

Umeda et al. 19) carried out field observation of sea level changes in the Susaki Bay for 15 months from November 2010 to obtain the amplification and predominant period of the tsunami. They found that energy peak periods of 85, 73, 51, 47, 37, and 32 min were predominantly observed three days after the occurrence of the 2011 off the Pacific Coast of Tohoku earthquake. They also found that variations with the periods of 85, 57, 47, 37, 32, and 25 min prevailed in the stormy weather, and those with the periods of 85, 49, 34, 31, 26 min in fine weather. Among these periods, the variation with the period of 85 min always appeared. The period of 85 min can be related to the natural period of continental seiche in the Tosa Bay.

Imai et al. 20) found that the continental seiche was generated with a period of 85 min in the Tosa Bay through their tsunami numerical simulation. The angular frequencies or the periods of continental edge waves can be estimated by the dispersion relationship proposed by Ursell 21) and Eckart 22). In the case of bottom slopes $s<<1.0$, they are expressed as 23):

$$\sigma_{n}^2 = g k (2n + 1) s, \quad (n = 0, 1, 2, \cdots) \quad (18)$$

$$\frac{T_n}{T_0} = \frac{1}{\sqrt{2n + 1}} \quad \frac{f_n}{f_0} = \sqrt{2n + 1} \quad (19)$$

where $\sigma_n = 2\pi/T_n$ is the angular frequency of mode $n$, $T_n$, and $f_n$ is the period and frequency of mode $n$, respectively; $k$ is the longshore wavenumber; and $g$ is the gravitational acceleration. If the period of the edge wave with the mode, $n=0$, is 85 min, the periods of modes 1 to 3 of the edge wave are obtained as 49 min, 38 min, and 32 min, respectively. Their periods coincide with the periods observed by Umeda et al. in the stormy weather. Therefore, edge waves of the modes 0 to 3 may likely appear in stormy weather.

Takayama and Hiraishi 24) carried out a hydraulic model experiment on tsunamis in the Susaki Bay to investigate the effect of tsunami breakwaters. According to their experiment, the predominant period of the Susaki Bay might be 34 min. The energy peaks of the Choi-Williams distribution function with periods of 33 min to 35 min may not be attributed to edge waves, but to natural oscillation because a natural period of the Susaki Bay is about 34 min and slightly different from the representative periods of edge wave.
(2) Tsunami observed at Onahama due to the 2016 Fukushima Earthquake

On November 22, 2016, an earthquake occurred 37 km east of Fukushima Prefecture, and triggered a tsunami of up to 1.4 m height to the northeast of Japan\(^{25}\). The Japan Meteorological Agency (JMA) placed the magnitude of the earthquake at 7.4\(^{25}\). Its mechanism was estimated to be a reverse fault-type. The Tsunami Warnings/Advisories, including the Tsunami Warning for Fukushima Prefecture, were issued by JMA three minutes after the earthquake. Two hours after the earthquake, Miyagi Prefecture was upgraded from Tsunami Advisories to Tsunami Warning because a maximum wave amplitude of 1.4 m was recorded in Sendai Port\(^{26}\).

The fault due to the earthquake struck at 238 degrees\(^{25}\) and was significantly different from those of the earthquakes that occurred in the offshore area of Fukushima Prefecture. Along the coasts in Sendai Bay, tsunami waves were unexpectedly amplified and brought damages to fishery harbors.

The tsunami waveform observed at Onahama in Fukushima Prefecture is shown in Fig. 7. A tsunami amplitude of up to 60 cm was observed at Onahama. It became smaller with an amplitude of about 20 cm after the arrival of the second tsunami wave.

The tsunami waveform was recorded with a sampling interval of 5 min. The maximum instantaneous frequency of the Wigner distribution estimated is 0.05 min\(^{-1}\) since it is equal to half of the Nyquist frequency, \(f_N=1/(2At)=0.1\) min\(^{-1}\). Therefore, the minimum period of the Wigner distribution is 20 min and a little bit longer. Under these circumstances, we show the Choi-Williams distribution for the near-field tsunami waves as an example.

**Figure 8** shows the Choi-Williams distribution (\(\sigma=3.0\)) for the tsunami waveform at times \(t=445\) min, 500 min, 600 min, 700 min, and 800 min. The periods of edge wave for 0th and 1st modes are also shown as 42 min and 24 min marked by the vertical arrows. There are two energy peaks in the distributions at the instantaneous frequencies of 0.025 and 0.04 cycle/min\(^{-1}\). They almost coincide with the low-mode frequencies of the edge wave. This means that edge waves of 0th and 1st modes were developed by \(t=600\) min, i.e., 200 min after the arrival of the first tsunami wave because remarkable energy peaks evolved at time \(t=600\) min, as shown in Fig. 8.

(3) Tsunami observed at off Fukushima due to the 2011 Great off the Pacific Coast of Tohoku Earthquake

On March 11, 2011, great tsunami waves were recorded by the offshore Global Positioning System...
The Choi-Williams distribution was calculated for $\sigma=5.0$ and is shown in Fig. 10. At time $t=160$ min when the largest tsunami waves came, the tsunami developed with a frequency of 0.015 min$^{-1}$, i.e., with a period of about 60 min. They gradually decreased their energy.

There exists an oddly shaped energy distribution at 160 min in the range of the instantaneous frequencies 0.02 to 0.04 cycle/min. That may be caused by the cross terms because the Wigner distribution function changes its sign in the time domain and the effect of the cross terms may remain.

(4) Applicability of the marginal conditions

The Wigner distribution satisfies the two marginal conditions as shown in Eqs. (3) and (4). The Choi-Williams distribution also satisfies them for any values of $\sigma^8$. Confirmation was needed whether they were valid even for discrete time series with a window function of a finite length. Integrals of their distributions with respect to time and/or frequency were evaluated and compared with the spectral density function and/or the instantaneous power of $x(t)$.

Figure 11 compares the time averages of Wigner and Choi-Williams distribution ($\sigma=1.0$) and the spectral density function estimated by the FFT, indicated by the symbols $\bullet$ and $\circ$, and solid line, respectively. The Hamming window is a window function with the number of discrete points, $N=128$. Only the result of the Choi-Williams distribution for $\sigma=1.0$ is shown because the distributions for $\sigma>3.0$ are almost the same as those of the Wigner distribution.

The integrated Wigner distribution with respect to time is similar to the spectral density function but shows a slight difference indicating negative values even in the integrated distribution. On the other hand, the Choi-Williams distribution for $\sigma=1.0$ showed smaller values by about 20% compared to the spectral density function estimated by the FFT. That may be attributed to the usage of the window function of a finite lag length.

Integrals of Wigner and Choi-Williams distribution with respect to frequency were compared with the instantaneous power $|x(t)|^2$ as shown in Fig. 12. The agreement between them was very high, even if...
the window functions were employed. This meant that the values obtained by integrating their distributions with respect to frequency were equal to the instantaneous power $|x(t)|^2$ and almost satisfied the marginal conditions, even if a window function was utilized.

5. DECAY PROPERTIES OF EDGE WAVES AND TSUNAMIS

(1) Tsunami energy decay due to bottom friction and turbulence

Long waves, such as edge waves on a continental shelf generated by tsunamis, decrease their amplitudes by energy dissipation due to bottom friction and turbulence. To predict a tsunami energy decay rate, we used a wave energy equation, including the energy dissipation rate due to bottom friction and turbulence proposed by Izumiya and Horikawa (27) and expressed as:

$$\frac{\partial E}{\partial t} + dJ(EC_g) = -\sqrt{2}C_f \frac{\partial}{\partial z} \left( \frac{E^2}{\rho^2 h^2} - \gamma \beta \frac{E^2}{\rho^2 h^2} \right) \tag{20}$$

where $E$ is the wave energy per unit surface area, $C_g$ is the group velocity vector of long waves, $H$ is the water depth, and $\rho$ is the water density. Coefficient $C_f$ is a friction factor and $\gamma, \beta$ is a proportional coefficient related to a turbulent intensity. They are assumed to be constant in this study. The values of $\gamma, \beta$ are in the range of $0 \leq \gamma/\beta < 5\sqrt{2} s^*/\gamma$, from Eq.(47) in Izumiya and Horikawa (27), where $y$ is the water height to the mean water depth, and $s^*$ is the beach slope modified to include wave set-up. The wave energy $E$ is given by potential energy $E_p$ plus kinematic energy $E_k$ and expressed as:

$$E = E_p + E_k = \frac{1}{2} \rho g \zeta^2 + \frac{1}{2} \rho (u^2 + v^2) h \tag{21}$$

where $\zeta$ is the water surface elevation, and $u$ and $v$ denote the horizontal velocity components in the $x$- and $y$- directions, respectively. The overbar denotes taking time average over a wave period.

Let us consider an area enclosed by a dot-dashed line as shown in Fig.13. A part of the line includes shoreline and across an area with a sufficient depth compared to 200 m. Taking the average of Eq. (20) over the area $A$ and using the Gauss divergence theorem, we obtain:

$$\frac{\partial}{\partial t} \left( \frac{1}{A} \int_0^A E dA \right) + \frac{1}{A} \int_0^A E C_g \cdot nds = - \frac{\sqrt{2} C_f + \gamma \beta}{\rho \frac{1}{h^2}} \int_0^A \frac{E^2}{\rho^2 h^2} dA \tag{22}$$

Along the shoreline, an integral of the second term on the left side of Eq. (22) vanishes because $EC_g = 0$.

For trapped mode long waves, the energy flux at a deep offshore site will vanish because the energy $E$ tends to zero. For a continental shelf with a parallel contour depth, the integrals along two normal lines on the left and right sides of Fig. 13 may vanish because the absolute value integrated along the line for each side is equal.

Fig.13 An integration area enclosed by the dotted-dashed line. The normal unit vectors are indicated by $n$. The dashed line shows the equi-depth contour line of $h=200$m.

(2) Energy decay of near-field tsunamis

Since as seen in section 4.2, the edge waves of modes 0 and 1 are dominant, the trapped waves would be predominant on a continental shelf. Then, the energy flux expressed in Eq. (22) will vanish or negligible at a deep water and a shoreline. For a continental shelf with straight and parallel bottom contours, the amount of wave energy flux will vanish, i.e., $A^{-1} \int_0^A E C_g \cdot nds = 0$.

If the energy flux at each point on a continental shelf was approximately isotropic, and the energy decay rate at all points was uniform (11), the tsunami wave energy density could be written as:
The solid line indicates a linear proportional relation to time.

![Fig. 14 Relationship between inverse values of maxima of tidal elevation at Onahama and time. The solid line indicates a linear proportional relation to time.](image1)

\[ F(x) \cdot n dA = \frac{1}{A_d} \int_{s_i} E C_{gi} \cdot n d s_i \]

where \( C_{gi} \) is the mean energy density in deep water ocean, \( s_i \) is the width of the window of the \( i \)th ocean.

![Fig. 15 Relationship between maxima of tidal elevation and \( t+c \). The upper solid line shows the decay law that decreases like \((t+c)^{-1}\).](image2)
On the other hand, the tsunami energy is partly absorbed on continental shelves where the energy dissipation occurs due to turbulent bottom friction and turbulence. We evaluated the tsunami wave energy flux to continental shelf areas, using energy absorption coefficients, $\delta_j$ $(j=1, M)$, as follows:

$$\frac{1}{A_d}\int_E EC_g \cdot n ds = \frac{1}{A_d}\sum_{j=1}^M \delta_j \bar{E} C_{gj/s_j}$$

where $M$ is the number of continental shelves, $C_{gj}$ is the $j$th mean group velocity at a depth of $h = 200$ m, and $s_j$ is the length of the $j$th continental shelf. The energy absorption coefficient, $\delta_j$, is expressed as:

$$\delta_j = \int_{s_j} EC_g \cdot n ds / C_{gj/s_j} \bar{E}$$

Applying the tsunami wave energy equation to an ocean deeper than 200 m, and neglecting the energy dissipation in the deep ocean that is small compared to the energy flux to other oceans and seas, the Eq. (22) can be rewritten as:

$$\frac{\partial \bar{E}}{\partial t} = -\frac{1}{A_d}\left\{\sum_{i=1}^N \beta_i C_{gi}\bar{s}_i + \sum_{j=1}^M \delta_j \bar{E} C_{gj/s_j}\right\}$$

When the energy decay rates at all points in the ocean were assumed to be uniform $^{11,30}$, and the energy density $\bar{E}$ was expressed by Eq. (25), the tsunami amplitude, $\zeta_0$, could be expressed as:

$$\zeta_0 = \zeta\alpha e^{-\alpha t}$$

where $\zeta_0$ is the amplitude at $t=0$, and $\alpha$ is the amplitude decay coefficient, expressed as:

$$\alpha = \frac{1}{2A_d}\left\{\sum_{i=1}^N \beta_i C_{gi}\bar{s}_i + \sum_{j=1}^M \delta_j \bar{E} C_{gj/s_j}\right\}$$

Equation (36) is derived from the wave energy equation, and shows that the amplitude decays exponentially with time, as Munk $^{10}$ and Van Dorn $^{11,30}$ indicated. The latter obtained the tsunami energy decay time for the Pacific Ocean to be 22 h. Since $\alpha$ is the amplitude decay coefficient, the coefficient is determined based on this result as $\alpha = 0.02$ h$^{-1}$ and shows a relatively small value.

The tsunami energy on a continental shelf is dissipated due to bottom friction and turbulence. If there was no energy supply from the deeper ocean, the energy should decrease rapidly. This means that the tsunami wave energy in the deep water area was partly provided on a continental shelf. Figure 17 illustrates the tsunami energy balance on a continental shelf. When we consider a continental shelf with a unit width, the energy flux from the deep ocean is assumed to be $\delta C_0 \bar{E}/A_d$, where $\bar{E}$ is the mean energy density in the deep ocean.

The energy equation for a tsunami on a continental shelf can be expressed as:

$$\frac{\partial}{\partial t} \left\{ \frac{1}{A_c} \int_A E dA \right\} = \frac{1}{A_c} \int_A \frac{E^3}{\rho^2} dA + \delta C_0 \bar{E}/A_d$$

(38)

where $A_c$ is the area of a continental shelf with a unit width. Inserting Eq. (25) and Eq. (36) into Eq. (38), we obtain the following equation.

$$2\zeta P \frac{d\zeta P}{dt} = -\frac{1}{\rho^{1/2} F^2} \bar{E}^3 + \frac{1}{A_c} \int_A \frac{F^2}{\rho^{3/2}} dA$$

(39)

where

$$F_1 = \frac{1}{A_c} \int_A f(x,y)dA, \quad F_2 = \frac{1}{A_c} \int_A \frac{f^2(x,y)}{h^{3/2}} dA$$

(40)

By introducing $K^*$ and $G$ thus:

$$K^* = \frac{\sqrt{2} C_f + \gamma \bar{E} F^2}{\rho^{3/2} F^2}, \quad G = \frac{1}{A_c} \int_A \frac{F^2}{\rho^{3/2}} dA$$

(41)

Eq. (39) can be rewritten as:

$$2\zeta P \frac{d\zeta P}{dt} = -K^* \xi^3 + G e^{-2\alpha t}$$

(42)

Although the above equation is a nonlinear ordinary differential equation, the solution of the equation can be easily obtained by introducing a small parameter $\epsilon$ equivalent to $\alpha$; i.e., $\alpha = O(\epsilon)$, and by using the perturbation method. The approximate solution of Eq. (42) can be expressed as follows:

$$\zeta P = \left( \frac{G}{K^*} \right)^{1/3} e^{-\frac{2\alpha t}{9K^*}} + \frac{4\alpha}{9K^*}$$
\[
\begin{equation}
\left( \frac{G}{K} \right)^{1/3} e^{-2at/3} + O(\varepsilon)
\end{equation}
\]
An constant is added in Eq. (43), but it can be negligible compared to the first term because of the very small quantity. The energy dissipation on a continental shelf can be found to be a little larger than the incident energy flux from Eqs. (42) and (43), i.e.,
\[
K^* \zeta_p^3 = Ge^{-2at} + O(\varepsilon) > Ge^{-2at}
\]
Since the tsunami amplitude can be approximately expressed by \(\left( \frac{G}{K} \right)^{1/3} e^{-2at/3}\), the amplitude decay time becomes 1.5 times longer than that of the deep ocean. Saito et al.31) and Rabinovich et al.32) investigated the energy decay of trans-Pacific tsunamis using data recorded by a lot of DART stations, and found that shorter period waves attenuate much faster than longer period waves. Imai et al.33) also showed that the decay time of tsunami waves observed at nearshore sites was longer than that of waves at offshore sites. The decay time appears to depend on the energy dissipation rate of tsunami waves.

6. CONCLUSIONS

The properties of Wigner and Choi-Williams distributions which have been receiving increasing attention as time-frequency analysis tools were described. By applying Wigner distribution to a chirp signal, the resolution of both time and frequency were examined. The time-varying properties of tsunami waves were investigated through the application of Winger and Choi-Williams distributions. The decays of tsunamis on a continental shelf and major trans-ocean tsunamis were treated theoretically based on the wave energy equation and were compared with the observed data. These are summarized as follows:

(1) The discrete Wigner distribution was evaluated for a chirp signal and the instantaneous frequencies of the energy peaks were compared with the theoretical values. They were in good agreement with the theoretical estimate.

(2) The Choi-Williams distribution was evaluated for the tsunami waveform observed at Susaki due to the 2011 off the Pacific coast of Tohoku Earthquake. The distribution showed a predominant energy peak with instantaneous frequency similar to the natural frequency of the Susaki Bay, 100 minutes after the arrival of the first tsunami wave.

(3) Through the application of the Choi-Williams distribution for the tsunami waveform at Onahama due to the 2016 Fukushima earthquake, it was found that edge waves of mode 0 to 1 on a continental shelf were developed.

(4) Despite the use of the Hamming window is employed, the time-averaged values of Wigner and Choi-Williams distribution for \(\sigma \geq 3.0\) showed to be very close to the spectral density calculated by the FFT.

(5) The integrals of Wigner and Choi-Williams distributions for \(\sigma \geq 1.0\) with respect to frequency coincident with the instantaneous power \(|x(t)|^2\) as expected.

(6) The decay of near-field tsunamis was estimated using the wave energy equation with the rate of dissipation due to bottom friction and turbulence. Talbot equation decaying with time was obtained and compared was in fairly good agreement with the observation of tsunamis.

(7) For major trans-ocean tsunamis, by evaluating the transmitted tsunami energy into other oceans and seas, and by considering the energy flux absorbed on continental shelves, the tsunami energy was found to decay exponentially with time.

ACKNOWLEDGMENT: The author would like to express his sincere thanks to the Ministry of Land, Infrastructure, Transport and Tourism, and the Port and Airport Research Institute for providing the tsunami data of the 2011 off the Pacific Coast of Tohoku Earthquake. He also expresses his appreciation to the Japan Meteorological Agency for providing tide level data for the tsunami due to the 2016 Fukushima earthquake.

REFERENCES

1) Cohen, L.: *Time-frequency Analysis*, New York, Prentice-Hall, ISBN 978-0135945322, p. 299, 1995.

2) Burg, J. P.: Maximum Entropy Spectral Analysis, the 37th Annual Meeting Society Explore. Geophysics., Oklahoma City, Oct. 1967.

3) Mallat, S. G.: A theory for multiresolution signal decomposition: the wavelet representation, *IEEE Trans. Pattern Anal. Machine Intell.*., Vol. 11, pp. 674-693, 1989.

4) Chui, C. K.: *An Introduction to Wavelets*, Academic Press, San Diego, 1992.

5) Wigner, E.: On the Quantum Correction for Thermodynamic Equilibrium, *Phys. Rev.*, Vol. 40, pp. 749-759, 1932.

6) Ville, J.: Theorie et Applications de la Notion de Signal Analytique, *Cables et Transmissions*, Vol. 2a, pp. 61-74, 1948.

7) Claassen, T. A. C. M. and Mechenbrucker, W. F. G.: Wigner Distribution-Tool for Time-frequency Signal Analysis – PART 1, 2, 3, *Philips J. Rec.*, Vol. 35, pp. 217-250, pp. 276-300, pp. 372-389, 1980.

8) Cohen, L.: Time-frequency distributions- A review, *Proc. IEEE*, Vol. 77, pp. 941-981, 1989.

9) Choi, H. and Williams, W.: Improved Time-frequency Representation of Multicomponent Signals Using Exponential Kernels, *IEEE Transactions on Acoustics, Speech and Signal Processing*, Vol. 37, pp. 862-871, 1989.

10) Munk, W. H.: Some comments regarding diffusion and absorption of tsunamis, *Proc. of Tsunami Meetings, Tenth Pacific Science Congress*, Honolulu, IUGG Monogr., No. 24, Paris, pp. 53-72, 1963.

11) Van Dorn, W. G.: Some tsunami characteristics deducible from tide records, *J. of Physical Oceanography*, Vol. 13, pp. 353-363, 1984.
12) Hayashi, Y., Koshimura, S. and Imamura, F.: Time -dependent decay model of MRMS tsunami amplitudes for forecasting decay of far-field tsunamis, *J. of Japan Society of Civil Engineers*, Ser. B2 (Coastal Engineering), Vol. 67, No. 2, pp. I_216-I_220, 2011 (in Japanese with English abstract).

13) Hino, M.: *Spectral Analysis*, Asakura Publishing, p. 320, 1977.

14) Kawakami, T., Ohama, Y. and Tadauchi, Y.: Reconstruction of non-stationary signals by inverse transform of Wigner distribution, Transactions of Japan Society of Mechanical Engineering, Ser. C, Vol. 63, No. 607, pp. 341-347, 1997 (in Japanese with English abstract).

15) Honda, R. and Ohama, Y.: Wave composition from Wigner distribution using wavelets, *J. of Japan Society of Civil Engineers*, No. 696/I-58, pp. 273-283, 2002 (in Japanese with English abstract).

16) Cohen, L.: Generalized phase-space distribution functions, *J. Math. Phys.*, Vol. 7, pp. 781-786, 1966.

17) Inuzuka, H., Nagai, T. and Tsukishima, T.: Analysis of non-stationary plasma experimental data by the Wigner distribution, *J. of the Japan Society of Plasma and Nuclear Fusion Research*, Vo. 60, No. 8, pp. 217-228, 1988 (in Japanese).

18) Fujii, Y., Satake, K., Sakai, S., Shimohara, M. and Kanazawa, T.: Tsunami source of the 2011 off the Pacific coast of Tohoku Earthquake, *Earth Planets Space*, Vol. 63, pp. 815-820, 2011.

19) Umeda, Y., Itaba, T. and Hosoi, Y.: The sea level observation at Susaki Bay in Kochi Prefecture – To validate the sea level change before the 1946 Nankai earthquake -, *Bull. Geol. Surv. Japan*, Vol. 67, No. 1, pp. 11-25, 2016 (in Japanese with English abstract).

20) Imai, K., Satake, K. and Furumura, T.: Tsunami duration on the south coast of Shikoku from large earthquakes along the Nankai Trough, *J. of Japan Society of Civil Engineers*, Ser. B2 (Coastal Engineering), Vol. 65, No. 1, pp. 281-285, 2009 (in Japanese with English abstract).

21) Ursell, F.: Edge waves on a sloping beach, *Proc. Roy. Soc.*, A, Vol. 214, pp. 79-95, 1952.

22) Eckart, C.: Surface waves on water of variable depth, *Wave Rept. No. 100, Scripps. Inst. Oceanogr.*, Univ. Calif.,, 99p, 1951.

23) Izumiya, T., Haku, K. and Ishibashi, K.: Analysis of high wave disaster mechanism caused in Niigata and Toyama Prefectures on February 24, 2008, *Proc. of Coastal Engineering*, JSCE, Vol. 55, pp. 181-185, 2008 (in Japanese with English abstract).

24) Takayama, T. and Hiraishi, T.: Hydraulic Model Tests on Tsunami at Susaki-port, *Technical Note of the Port and Harbour Research Institute*, Ministry of Transport, Japan, No. 549, pp. 1-131, 1986 (in Japanese with English abstract).

25) Anawat Suppasri, Yamashita, K., Latcharote, P., Roeber, V., Hayashi, A., Ohira, H., Fukui, K., Hisamatsu, A. and Imamura, F.: Numerical analysis and field survey of the 2016 Fukushima earthquake and tsunami, *J. of Japan Society of Civil Engineers*, Ser. B2 (Coastal Engineering), Vol. 73, No. 2, pp. I_1597-I_1602, 2017 (in Japanese with English abstract).

26) Japan Meteorological Agency: The Tsunami of 22 November 2016 in Fukushima, http://www.jma.go.jp/jma/press/1611/22b/kaisetu201611221100.pdf (accessed 2016-11-28).

27) Izumiya, T. and Horikawa, K.: Wave Energy Equation Applicable in and outside the Surf Zone, *Coastal Eng. in Japan*, JSCE, Vol. 27, pp. 119-137, 1984.

28) Chow, V. T., Maidment, D. R. and Mays, L. W.: *Applied Hydrology*, McGraw-Hill, p. 340, 1988.

29) Utsu, T., Ogata, Y. and Matsu’ura, R.: The Centenary of the Omori Formula for a Decay Law of Aftershock Activity, *J. Phys. Earth*, Vol. 43, pp. 1-33, 1995.

30) Van Dorn, W. G.: Tide gage response to tsunamis, Part II: Other oceans and smaller seas, *J. Phys. Oceanogr.*, Vol. 17, pp. 1507-1516, 1987.

31) Saito, T., Inazu, D., Tanaka, S. and Miyoshi, T.: Tsunami coda across the Pacific Ocean following the 2011 Tohoku-Oki Earthquake, *Bull. Seism. Soc. Am.*, Vol. 103, No. 2B, doi:10.1785/0120120183, 2013.

32) Rabinovich, A. B., Candella, R. N. and Thomson, R. E.: The open ocean energy decay of three recent trans-Pacific tsunamis, *Geophys. Res. Lett.*, Vol. 40, pp. 3157-3162, doi: 10.1002/grl.50625, 2013.

33) Imai, K., Tanobe, A., Hayashi, Y. and Imamura, F.: Decay process of the tsunami due to the 2011 Great off the Pacific Coast of Tohoku Earthquake along the Pacific coast of Japan, *J. of Japan Society of Civil Engineers*, Ser. B2 (Coastal Engineering), Vol. 70, No. 2, pp. I_276-I_280, 2014 (in Japanese with English abstract).

(Received August 5, 2020)
(accepted January 4, 2021)