Construction of indecomposable $N$-replications of Kac-modules of type I Lie superalgebra $sl(m/n)$, $osp(2/2n)$.

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ABSTRACT: Given a Kac module for a type I Lie superalgebra $sl(m/n)$ or $osp(2/2n)$ and a positive integer $N$, we construct an indecomposable representation chaining $N$ copies of the given module, modeling for example the existence of 3 generations of quarks and leptons.
1 On the simple Lie-Kac superalgebras of types A and C.

In Kac’s complete classification of the simple Lie superalgebras [1, 2], two families contain an even generator $y$ commuting with the even subalgebra, namely the $A(m−1,n−1) = sl(m/n)$ and the $C(n+1) = osp(2/2n)$ superalgebras. They admit a single Dynkin diagram with a single odd positive simple root $\beta$ [3].

The even subalgebra, in the corresponding Chevalley basis, has the structure:

$$[h_i, h_j] = 0, \quad [h_i, e_j] = C_{ij} e_j, \quad [h_i, f_j] = -C_{ij} f_j, \quad [y, h_i] = y, e_i = [y, f_i] = 0, \quad i, j = 1, 2, \ldots, r$$

(1.1)

where $r$, $h_i$, $e_i$, $f_i$ and $C_{ij}$ denote respectively the rank, the Cartan commuting generators, the raising and the lowering generators associated to the simple roots, and the Cartan matrix of the semisimple even Lie subalgebra $sl(m) \oplus sl(n)$, respectively $sp(2n)$, with rank $r = m + n - 2$, respectively $r = n$. The remaining raising (respectively lowering) generators of the even semisimple subalgebra are generated by the iterated commutators of the $e$ (respectively $f$) generators limited by the Serre rule $ad(e_i)(e_j)^{-C_{ij}+1} = 0$. Finally, the additional even generator $y$, that physicists often call the hypercharge, centralizes the even subalgebra. Even in finite dimensional representations, $y$ is not quantized, and as shown below, this is the cornerstone of our new construction of indecomposable $N$-replications of an arbitrary Kac module.

In its odd sector, the superalgebra has $P$ odd raising generators $u_i$ corresponding to the $P$ positive odd roots $\beta_i$ and $P$ odd lowering generators $v_i$ corresponding to the $-\beta_i$. $P = mn$ for $sl(m/n)$, or $P = 2n$ for $osp(2/2n)$. In both cases, the $u_i$ sit in the irreducible fundamental representation of the even subalgebra. We call $u_1$ the lowest weight vector of the $u_i$ representation; $u_1$ corresponds to the simple positive odd root $\beta = \beta_1$. Reciprocally, we call $f_1$ the highest weight vector of the $f_i$. For our following analysis, the important relations are

$$[y, u_i] = u_i, \quad [y, v_i] = -v_i,$$

$$\{u_i, u_j\} = \{v_i, v_j\} = 0, \quad \{u_i, v_j\} = \delta_{ij}.$$ 

(1.2)
where $d_{ij}^a$ and $k$ are constants ($k \neq 0$) and the $\mu_a$ span the even generators of type $(h, e, f)$. That is: the hypercharge $y$ grades the superalgebra, with eigenvalues $(0, \pm 1)$. The $u_i$ anti-commute with each other. So do the $v_i$. Finally and most important, the anticommutator of the odd raising operator $u_i$ with the odd lowering operator $v_i$ corresponding to the opposite odd root depends linearly on the hypercharge $y$. In particular, \[ \{u_1, v_1\} = h_\beta = d_1^{11}h_{a} + ky, \]

where $k$ is non zero and $h_\beta$ is the Cartan generator associated to the odd simple root $\beta$.

See for example the works of Kac [1, 2] or the dictionary on superalgebras by Frappat, Sciarrino and Sorba [4] for details.

\section{Construction of the Kac modules}

Following Kac [5], choose a highest weight vector $\Lambda$ defined as an eigenstate of the Cartan generators $(h_i, y)$, and annihilated by all the raising generators $(e_i, u_j)$. The eigenvalues $a_i$ of the Cartan operators $h_i$ are called the even Dynkin labels. The eigenvalue $b$ of the Cartan operator $h_\beta$ corresponding to the odd simple root is called the odd Dynkin label:

\[ h_i\Lambda = a_i\Lambda, \quad \{u_1, v_1\}\Lambda = h_\beta\Lambda = b\Lambda. \quad (2.1) \]

Construct the corresponding Verma module using the free action on $\Lambda$ of the lowering generators $(f, v)$ modulo the commutation relations of the superalgebra. Since the $v$ anti-commute, the polynomials in $(f, v)$ acting on $\Lambda$ are at most of degree $P$ in $v$, and hence the Verma module is graded by the hypercharge $y$ and contains exactly $P$ layers.

Consider the antisymmetrized product $w^-$ of all the odd lowering generators $(v_i, i = 1, 2, ..., P)$. The state $\bar{\Lambda} = w^-\Lambda$ is a highest weight with respect to the even subalgebra $e_i\Lambda = 0$. Indeed $e_i$ annihilates $\Lambda$ and each term in the Leibniz development of $[e_i, w^-]$ contains a repetition of one of the $v$ generators and hence vanishes.

Let $\rho$ be the half supersum of the even and odd positive roots

\[ \rho = \rho_0 - \rho_1 = \frac{1}{2}(\sum \alpha - \sum \beta). \quad (2.2) \]

Let $w^+$ be the antisymmetrized product of all the $u$ generators. As shown by Kac [5], we have

\[ w^+\bar{\Lambda} = w^+w^-\Lambda = \pm \prod_i <\Lambda + \rho|\beta_i> \Lambda \quad (2.3) \]

where the product iterates over the $P$ positive odd roots $\beta_i$, the sign depends on the relative ordering of $w^+$ and $w^-$ and the bilinear form $<|>$ is a symmetrized version of the Cartan metric. If this product is non-zero, the Verma module is called typical. $\Lambda$ belongs to the orbit of $\bar{\Lambda}$ and vice-versa, hence they both belong to the same irreducible submodule. If the scalar product $<\Lambda + \rho|\beta_i>$ vanishes for one or more odd positive root $\beta_i$, the Verma module is indecomposable since $\Lambda$ is not in the orbit of $\bar{\Lambda}$. It is then called atypical of type $i$ and there exists a state $\omega_i$ with Cartan eigenvalues $\Lambda_i - \beta_i$ which is a sub highest weight annihilated by all the even and odd raising operators $(e, u)$. In the present study, we do not quotient out by this submodule but preserve the indecomposable Verma module construction because we want to preserve the continuity in $b$. Notice that in the $A$ and $C$ superalgebras that we are studying the odd roots are on the light-cone of the Cartan root.
space: $\langle \beta_i | \beta_i \rangle = 0$. Therefore, if $\Lambda$ is atypical $i$, the secondary highest weight $\Lambda - \beta_i$ is also atypical $i$.

As in the Lie algebra case, this Verma module is infinite dimensional, because of the acceptable iterated action of the even lowering generators $f$. But as we just discussed, the iterated action of the anticommuting odd lowering operators $v$ saturates at layer $P$.

Let us now recall for completeness the usual procedure to extract a finite dimensional irreducible module from a Lie algebra Verma module. All the states with negative even Dynkin labels which are annihilated by the even raising generators can be quotiented out. For example, given a Chevalley basis $(h, e, f)$ for the Lie algebra $sl(2)$ and a Verma module with highest weight $\Lambda$, we have

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = 2h,$$

$$h\Lambda = a\Lambda, \quad e\Lambda = 0; \quad (2.4)$$

hence

$$hf^n\Lambda = (a - 2n)f^n\Lambda, \quad ef^n\Lambda = n(a - n + 1)f^{n-1}\Lambda. \quad (2.5)$$

If $a$ is a positive integer, the Verma module can be quotiented by the orbit of the state $f^{a+1}\Lambda$, and the equivalence classes form an irreducible module of finite dimension $a + 1$.

Generalizing to a superalgebra, all the even Dynkin labels $a_i$ associated to the Cartan operators $h_i, i = 1, 2, ..., r$ are restricted to non negative integers. We pass to the quotient in each even submodule and define the Kac module as the resulting finite dimensional quotient space. The crucial observation is that the identification of the even sub highest weights $\omega$ requests to solve a set of equations involving the even Dynkin labels $a_i$, but independent of the odd Dynkin weight $b$, which remains non-quantized. For example, in $sl(2/1)$, the state $\omega = (afv - (a + 1)v\Lambda)$ is an even highest weight [6]. But please remember that we do not quotient out the atypical submodules.

This procedure does not extend to the type II Lie-Kac superalgebras $B(m, n), D(m, n), F(4)$ and $G(3)$, because these algebras contain even generators with hypercharge $y = \pm 2$. For example the generator associated to the lowest weight of the adjoint representation, i.e. to the supplementary root of the (affine) extended Dynkin diagram, is even. Thus, the Kac module is finite dimensional if and only if its hidden extended Dynkin label is also a non negative integer. This integrality constraint involves $b$. So the representations of the type II superalgebras are finite dimensional only for quantized values of $b$, see Kac [5] for the original proofs and [7, 8] for examples.

The remaining even highest weights $\Lambda_{\ell_0}^\ell$ are spread over the $P$ layers with hypercharge decreasing from $y$ down to $y - P$. On the zeroth layer, we have $\Lambda^0 = \Lambda$, on the first layer we have the $P$ weights $\Lambda^1_i = \Lambda - \beta_i$, on the second layer we have the $P(P - 1)/2$ weights $\Lambda^2_{ij} = \Lambda - \beta_i - \beta_j, i \neq j$, down to the $P$th layer $\Lambda^{P}_{12...P} = \overline{\Lambda}$, each time excluding the even highest weight vectors with negative even Dynkin labels, since they have been quotiented out. For an explicit construction of the matrices of the indecomposable representations of $sl(2/1)$, we refer the reader to our study [6] and references therein.

To conclude, if the Kac module with highest weight $\Lambda$ is typical, it is irreducible. If it is atypical, it is indecomposable. In both cases, its even highest weights are the $\Lambda_{\ell_0}^\ell$ and
the whole module is given by the even orbits of the \( \Lambda^p \) with non negative even Dynkin labels and hypercharge \( y - p \).

### 3 On the derivative of the odd raising generators

Consider a finite \( D \) dimensional Kac module with highest weight \( \Lambda \), typical or atypical, as described in the previous section. Call \( a_i \) the even Dynkin labels and \( b \) the odd Dynkin label. Notice that, using (1.2) and (2.1), the hypercharge generator \( y \) is linear in \( h_\beta \), hence the matrix representing \( y \) is linear in \( b \): 

\[
y = \alpha b + \alpha a_i \quad \text{where the} \quad (\alpha, \alpha a_i) \quad \text{are constants independent of} \quad b \quad \text{and} \quad \alpha \neq 0.
\]

Notice that by construction the matrices representing the \((e,f,v)\) generators in the Verma module are also independent of \( b \), and that remains true in the Kac module because the quotient operation does not involve \( b \). Finally, the matrices representing the odd raising generators \( u \) are linear in \( b \) because, when we push an odd raising generator \( u \) acting from the left through an element of the Kac module, i.e. through a polynomial in \((f,v)\) acting on \( \Lambda \), we must contract \( u \) with one of the \( v \) generators before \( u \) touches \( \Lambda \).

Now consider the scaled derivatives \( u'_i \) of the \( u_i \) matrices

\[
u'_i(a) = \alpha^{-1} \partial_b u_i(a,b) .
\]

Using (2), we derive the anticommutation relations

\[
\{u'_i, v_j\} = \alpha^{-1} \partial_b \{u_i, v_j\} = \alpha^{-1} \partial_b (d^a_{ij} u_a + k y \delta_{ij}) = k \delta_{ij} ,
\]

where the \( \mu \) matrices span the even generators \((h,e,f)\), and where \( \mu(a) \) and \( v(a) \) are independent of \( b \). Another way of seeing the same results is to compute the \( \{u_i(a,b), v_j(a)\} \) anticommutator, divide by \( ab \) and take the limit when \( b \) goes to infinity. Since the matrix elements of the even generators are all bounded when \( b \) diverges, except the hypercharge \( y \) with spectrum \( ab, ab - 1, ... , ab - N \), we arrive at the same conclusion: the \( \{u', v\} \) anticommutator is proportional to the identity on the whole Kac module. Many explicit examples of the matrices \( u(a,b), u'(a), v(a) \) can be found in our extensive study of \( sl(2/1) \) [6].

This result holds for typical and atypical-indecomposable Kac modules of type I superalgebras, but does not hold for the type II superalgebras or for the irreducible atypical modules of the type I superalgebras because we need continuity in \( b \). Indeed, we have proved elsewhere by a cohomology argument [9] that the fundamental atypical triplet of \( sl(2/1) \) cannot be doubled.

### 4 Construction of an indecomposable N-replication of a Kac module

Given a finite \( D \) dimensional Kac module, typical or atypical-indecomposable, represented by \( D \times D \) matrices \((\mu, y, u, v)\), constructed as above and where \( \mu \) collectively denotes the even matrices of type \((h,e,f)\), consider the doubled matrices of dimension \( 2D \times 2D \):

\[
M = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} , \quad Y = \begin{pmatrix} y & I \\ 0 & y \end{pmatrix} , \quad U = \begin{pmatrix} u & u' \\ 0 & u \end{pmatrix} , \quad V = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} .
\]
where we used the $D \times D$ matrices $u'$ constructed in the previous section. By inspection, the matrices $(M, Y, U, V)$ have the same super-commutation relations as the matrices $(\mu, y, u, v)$ and therefore form an indecomposable representation of the same superalgebra of doubled dimension $2D$. This representation cannot be diagonalized since the matrix $Y$ representing the hypercharge cannot be diagonalized because of its block Jordan structure.

The block $u'$ can be rescaled via a change of variables

$$Q = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1/\lambda & 0 \\ 0 & 1 \end{pmatrix},$$

(4.2)

Furthermore, we can construct a module of dimension $ND$, for any positive integer $N$ by iterating the previous construction. By changing variables we can then introduce a complex parameter $\lambda$ at each level. For example, for $N = 3$, we can construct

$$M = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad Y = \begin{pmatrix} y & I & 0 \\ 0 & y & I \\ 0 & 0 & y \end{pmatrix}, \quad \tilde{Q}Y\tilde{Q}^{-1} = \begin{pmatrix} y & \lambda_1 I & 0 \\ 0 & y & \lambda_2 I \\ 0 & 0 & y \end{pmatrix},$$

(4.3)

Theorem: Given any finite dimensional, typical or atypical, Kac module of a type I superalgebra, $A(m/n)$ or $C(n)$, using the derivative $u'$ of the odd raising generators with respect to the hypercharge $y$ which centralizes the even subalgebra, we can construct an indecomposable extension joining $N$ replications of the original module.

5 Discussion

Representation theory of Lie algebras and superalgebras involves three increasingly difficult steps: classification, characters and construction. In Lie algebra theory, we can rely on three major results: all finite dimensional representations of the semisimple Lie algebras are completely reducible, their irreducible components are classified by the Dynkin labels of their highest weight state, their characters are given by the Weyl formula. Nevertheless, the actual construction of the matrices, although known in principle, remains challenging. We only know the matrices in closed analytic form in the case of $sl(2)$.

Finite dimensional simple Lie superalgebras have been classified by Kac [1, 2]. In the present study, we only consider the basic classical superalgebras of type 1, called $A(m-1, n-1) = sl(m/n)$ and $C(n) = osp(2, 2n)$ which are characterized by the existence of an even generator, the hypercharge $y$, commuting with the even subalgebra. As for Lie algebras, their irreducible modules can be classified by the Dynkin labels of their highest weight and Kac [5] has discovered in 1977 an elegant generalization of the Weyl formula.
But there are two additional difficulties. First, as found by Kac, the hypercharge $y$ of the finite dimensional modules is not quantized, but for certain discrete values, the Kac module ceases to be irreducible but becomes indecomposable. One can quotient out one or several invariant submodules and the Weyl-Kac formula of the irreducible quotient module is not known in general [10, 11]. Furthermore, there is a rich zoology of finite dimensional indecomposable modules which were progressively discovered by Kac [5], Scheunert [12], Marcu [13, 14], Su[15], and others, culminating in the classification of Germoni [16–18]. See [6] for an explicit description of the indecomposable $sl(2/1)$ modules.

A particular class, first described by Marcu [14], is of great interest in physics because it has implications for the standard model of leptons and quarks. These particles are well described by $sl(2/1)$ irreducible modules graded by chirality [19–24]. However, experimentally, they appear as a hierarchy of three quasi identical families, for example the muon and the tau behave as heavy electrons. This hierarchical structure has no clear explanation in Lie algebra theory. Furthermore, the three families leak into each other in a subtle way first described by Cabibbo (C) for the strange quarks and generalized to all three families by Kobayashi and Maskawa (KM). In a certain technical sense, the axis of the electroweak interactions is not orthogonal to the axes of the strong interactions, but tilted by small angles, called the CKM angles. As a result the weak interactions are not truly universal because the heavier quarks leak into the lighter quarks. Again, this experimental phenomenon has no explanation in Lie algebra theory precisely because all representations are completely reducible.

Marcu found in 1980 [14] that the fundamental $sl(2/1)$ quartet can be duplicated and triplicated in an indecomposable way. Coquereaux, Haussling, Scheck and coworkers [25–27] have proposed in the 90’s to interpret these representations as a description of the CKM mechanism. This raises several questions: is the construction of Marcu limited to three generations, as observed experimentally in the case of the quarks and leptons, or does there exist indecomposable modules involving more layers? Is this property specific of $sl(2/1)$, or is it applicable to other simple Lie-Kac superalgebras?

We have previously partially answered these questions. In [16, 17] the existence of multi-generations indecomposable modules is indicated. In [9], we proved, using cohomology, that any Kac module of a type I superalgebra can be duplicated. But these were just proofs of existence.

In the present study, using the derivative of the odd generators relative to the hypercharge, we show that any Kac module of any type I Lie-Kac superalgebra can be replicated any desired number of times in an indecomposable way, and for the first time we actually construct the matrices of the replicated representation in terms of the matrices of the original Kac module. This result is interesting for physics, surprising relative to Lie algebra theory, and very specific as we actually construct the matrices of these indecomposable modules rather than limit our analysis to their existence, classification, or the calculation of their characters.
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