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Nucleation of Market Shocks in Sornette-Ide model

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Abstract: The Sornette-Ide differential equation of herding and rational trader behaviour together with very small random noise is shown to lead to crashes or bubbles where the price change goes to infinity after an unpredictable time. About 100 time steps before this singularity, a few predictable roughly log-periodic oscillations are seen.

Bubbles and crashes are a property of market prices since centuries. Some may arise from external perturbations like changes of the interest rate by the central bank, government decisions on war and peace, or for stocks of single companies the introduction of new (un)successful products. Some, like the October 1987 crash on Wall Street, may arise from intrinsic market mechanisms [1]. While we cannot make a clear distinction between external and intrinsic reasons for a crash in reality, we can do so at least in computer simulations. There a single external event can be put in explicitly into the program, the multitude of individual decisions can be approximated by a small random noise while the general rational as well as psychological (“herding”) behaviour of investors and traders can be approximated by deterministic algorithms. We show here that in the Sornette-Ide [2,3] approach this small noise can escalate to a singularity (bubble or crash) where the price change goes to $\pm \infty$.

The Sornette-Ide approach [2] used two different contributions. One is the nonlinear fundamentalist assumption of [3] that traders buy (sell) if the prices are low (high). Thus a positive difference $x$ between the actual and the perceived fundamental price encourages selling and thus causes a downwards tendency in $v = dx/dt$ such that $dv/dt$ is proportional to $-x^n$. This effect stabilizes prices. But a rising price, $v > 0$, creates the hope or illusion that the price will increase further and thus will encourage further buying, with $dv/dt$ varying as $v^m$; this second term alone leads to a divergence $v \propto (t_c-t)^{-1/(m-1)}$.
at some critical time $t = t_c$ (bubble or crash). The combination of both terms leads to roughly $4$ log-periodic oscillations $\propto \sin[\text{const } \log(t_c - t)]$ preceding the singularity, which may be used to make profit or to prevent this singularity.

![Graph showing long-time behaviour of one example. Slow weak oscillations are followed by strong rapid oscillations.](image)

Figure 1: Long-time behaviour of one example. Slow weak oscillations are followed by strong rapid oscillations.

The deterministic nonlinear differential equation for the “return” $x(t)$ (the logarithm of the current price $P$ to the initial “fundamental” or “equilibrium” price) and its velocity (short-time return) $v = dx/dt$ is

$$\frac{dv}{dt} = c_m v^m - c_n x^n$$

if $m$ and $n$ are odd integers. The damped harmonic oscillator has $m = n = 1$, $c_m < 0$, $c_n > 0$. Suitable absolute values are needed if $m$ and $n$ are not odd integers [2]. Noise can be introduced here in various ways: Chang et al [5] added a small random value to $x$; then after a long time regular oscillations appeared, of increasing strength and decreasing period, until $x$
Three examples, $m=3$, $n=5$, return multiplied with $1+0.0001 \cdot \text{noise}$, $-1 < \text{noise} < +1$

Figure 2: Behaviour shortly before the crash of the same sample as in Fig.1 plus two other examples differing only by the random number seed. Only for a few oscillations the three curves are reasonable synchronized; for longer times before the crash some show a maximum where the others show a minimum.

diverged. While this behaviour is what we want, the assumption about the noise has little justification except that it is close to Langevin equations.

We instead first made $c_m$ vary randomly between 0 and +1, but then the curves were too regular, and different initial seeds for the random number generator gave nearly the same curves. This multiplicative assumption was supposed to simulate the volatile psychology of the traders who follow more or less the current trend. When we applied the random noise to $c_n$ instead of $c_m$, this noise influenced the prices more strongly but we got oscillations right from the beginning while we want them to emerge only later. Also applying noise to $v$ did not give satisfactory results.

To get better results, in our second method we made a multiplication for $x$ itself, not for a contribution to its time derivatives. Thus the $x$ resulting
from the discretized differential equation was multiplied by a factor which differed from unity by a small amount between –0.0001 and +0.0001. Since we use multiplication instead of addition, we cannot start with \( x \) and \( v \) both zero, and thus took \( x = 0, v = 0.0001 \) initially, to create some fluctuations. We thus assume that traders decide in a deterministic way on fundamental value through \( c_n x^n \) and on herding through \( c_m v^m \), but finally some small randomness also affects the market.

Figure 1 shows a run over all nearly 5000 time steps, Fig.2 is the same run over the last 200 time steps before the crash, together with two other runs using the same parameters but different random numbers. We see that first the prices fluctuate in a rather random way about the fundamental value. In other runs, the noise can make the deviations larger leading to a few nearly regular oscillations which die down again. Finally, again such oscillations occur, but now they become stronger, faster and finally lead to a crash at time \( = 0 \). Averaging over thousands of samples gives smooth oscillations before the crash, Fig.3. If we increase the noise, the times needed for a crash
Figure 4: Distribution of the times after initialization needed for a crash to appear. A parabola in this log-log plot means a log-normal distribution. The bin sizes increase by a factor 2. Besides our normal noise 0.0001 on the right, also ten and hundred times stronger noise is shown (center and left curves.)

to build up get shorter and there are less oscillations; these times are roughly log-normally distributed, Fig.4.

Mathematically it is easy to understand why this second method works better than our first method. In the first method, the noise affects the time derivative only, meaning it changes the curvature or slope of $x(t)$ only. Our second method affects $x$ directly.

Without any noise, the origin $x = v = 0$ of the $(v, x)$ phase space plays a special role as the unstable fixed point around which spiral structures of trajectories are organized. It corresponds to the case of fundamental prices; there is no trend and the market does not know which direction to take.

Our parameters for this simulation were $m = 3$, $n = 5$, $c_m = c_n = 1$. We used the leap frog method of molecular dynamics with 1000 iterations per
unit time. Thus first we calculate from the above equation the acceleration 
\[ a = \frac{d^2 x}{dt^2}, \]
from \( a \) the change 0.001\( a \) in the velocity \( v = \frac{dx}{dt} \), and finally
the new \( x \) is obtained by first adding 0.001\( v \) to it using the new velocity, and
then multiplying \( x \) by \( 1 + \epsilon \), with \( \epsilon \) selected randomly between \(-0.0001 \) and
\[ + 0.0001. \]

The “noise” \( \epsilon \) symbolizes all the new information coming in every day (or
every simulation interval). We also tried to simulate insider trading: Half of
the market movement uses not today’s noise but the one of tomorrow. Not
much is changed since the equations describe the overall market behaviour,
not the profits of some at the expense of others.

Of course, this method is similar to that of Chang et al; the main dif-
ference is that the present noise in \( x \) is proportional to \( x \) while for Chang
et al it is independent of \( x \). An additional linear friction term,
\[ \frac{dv}{dt} = c_n x^n - c_f v, \]
restricts the fluctuations \( x \) mostly to one sign, except for
very small friction coefficients \( c_f \).

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