On the Number of Representations for Simple Lie Groups

Mohammed Barhoush

Abstract

The growth rate function \( r_N \) counts the number of irreducible representations of simple complex Lie groups of dimension \( N \). While no explicit formula is known for this function, previous works have found bounds for \( R_N = \sum_{i=1}^{N} r_i \). In this paper we improve on previous bounds and show that \( R_N = O(N) \).

1 Preliminaries

This section provides a brief overview of the required preliminaries based from [1]. A general background in Lie algebra and representation theory is assumed.

The following elementary theory in Lie algebra is crucial for this paper.

Theorem 1. For any complex simple Lie algebra \( g \),

1. Every finite dimensional representation \( V \) of \( g \) possess a highest weight vector.

2. The subspace of \( V \) generated by the images of a highest weight vector \( v \) under successive application of root spaces \( g_i \) for some negative \( i \) is an irreducible subrepresentation.

3. An irreducible representation possess a unique highest weight vector up to scalar multiplication.

What this theorem entails is that an irreducible representation of a complex simple Lie algebra is determined by the highest weight. Furthermore, we know that a weight is always an integer combination of the functions \( L_i \). This means every irreducible representation of complex simple Lie algebra is determined by
a tuple of integers. We will detonate an irreducible representation by \( V_\lambda \), where \( \lambda \) is the tuple of integers.

This paper will focus on the dimension of the irreducible representations of simple Lie groups. The simple Lie groups are \( SO_n \), \( Sl_n \) and \( Sp_{2n} \) for some \( n \in \mathbb{N} \). There are also five special cases \( E_6 \), \( E_7 \), \( E_8 \), \( F_4 \), and \( G_2 \).

**Theorem 2.** If \( \lambda = (\lambda_1, ..., \lambda_n) \), then we can find \( \dim V_\lambda \).

\( SO_{2n} \mathbb{C} : \)

\[
\dim(V_\lambda) = \prod_{1 \leq i < j \leq n} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2}
\]

Where \( l_i = \lambda_i + n - i \) and \( m_i = n - i \).

\( SO_{2n+1} \mathbb{C} : \)

\[
\dim(V_\lambda) = \prod_{1 \leq i < j \leq n} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} \prod_{1 \leq i \leq n} \frac{l_i}{m_i}
\]

Where \( l_i = \lambda_i + n - i + \frac{1}{2} \) and \( m_i = n - i + \frac{1}{2} \).

\( Sp_{2n} \mathbb{C} : \)

\[
\dim(V_\lambda) = \prod_{1 \leq i < j \leq n} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} \prod_{1 \leq i \leq n} \frac{l_i}{m_i}
\]

Where \( l_i = \lambda_i + n - i + 1 \) and \( m_i = n - i + 1 \).

\( Sl_n \mathbb{C} : \)

\[
\dim(V_\lambda) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i + ... + \lambda_{j-1}}{j - i}
\]

Now we know how to calculate the dimension of a irreducible representation of a complex simple Lie group. This begs the question of how many of these representations have a certain dimension?

We define \( r_N(G) \) to be the number of irreducible representations of a group \( G \) with dimension \( N \).

**Definition 1.**

\[
r_N = r_N(SO_{2n+1} \mathbb{C}) + r_N(Sl_n(\mathbb{C})) + r_N(SO_{2n} \mathbb{C}) + r_N(Sp_{2n} \mathbb{C}) + r_N(E_6) + r_N(E_7) + r_N(E_8) + r_N(F_4) + r_N(G_2)
\]

Unfortunately there is no known explicit formula for this function. We can try to find bounds instead. However, \( r_N \) fluctuates a lot so it is more useful to bound the sum function.
Definition 2.

\[ R_N(G) = \sum_{i=1}^{N} r_i(G) \]

\[ R_N = \sum_{i=1}^{N} r_i \]

A useful simple upper bound for this function is given by Theorem 1 in [2].

**Theorem 3.** If \( G \) is a simple compact Lie group and \( n \) is a positive integer,

\[ R_N(G) \leq N \]

The following result by [2] is the best known upper bound for the rate growth function.

**Theorem 4.** For any \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) such that if \( G \) is a simple compact Lie group of dimension \( \leq n \) then \( r_N(G) \leq N^\epsilon \) for \( N \geq 1 \).

This gives an upper bound for \( R_N \).

**Theorem 5.** For all \( \epsilon > 0 \), there exists \( M \in \mathbb{N} \) such that \( \forall N > M \)

\[ R_N < O(N^{1+\epsilon}) \]

The problem is that as \( \epsilon \) approaches 0, \( N \) approaches infinity.

In this paper, the following theorem is proved.

**Theorem 6.**

\[ R_N < O(N) \]

2. \( Sp_{2n}\mathbb{C}, SO_{2n}\mathbb{C} \) and \( SO_{2n+1}\mathbb{C} \)

These three simple lie groups have very similar Weyl character formulas. That is why I will only prove that \( R_N(SO(\mathbb{C})) \leq 13N \). The same steps can be used to prove that \( R_N(SO(\mathbb{C})) \leq 13N \) and \( R_N(Sp(\mathbb{C})) \leq 13N \).

Instead of relating a highest weight vector with a tuple \( \lambda = (\lambda_1, ..., \lambda_n) \) where \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \), we will relate it to the tuple \( l = (l_1, ..., l_n) \) where \( l_i = \lambda_i + n - i \). So every highest weight vector has a unique tuple \( l \) such that \( l_1 > l_2 > ... l_n \geq 0 \).

**Theorem 7.** \( \forall N \in \mathbb{N}, R_N(SO(\mathbb{C})) \leq 13N. \)
Proof.

I will prove this by strong induction on $N$. The base cases are satisfied since $R_1(SO(3)) = 0 \leq 13$ and $R_2(SO(3)) = 1 \leq 26$.

Assume that $\forall M < N, M \in \mathbb{N}, R_M(SO_{2n}(\mathbb{C})) \leq 13M$.

Now we need to prove that $R_N(SO(3)) \leq 13N$. I will do this by creating a map from tuples with dimensions less than or equal to $N$ to tuples with dimensions less than or equal to $\frac{9N}{16}$ and then use the inductive hypothesis.

For any tuple $l = (l_1, ..., l_n)$ let

$$\prod(l_1, ..., l_n) = \prod_{1 \leq i < j \leq n} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} = dim(V_l)$$

Consider a tuple $l$ that satisfies $\prod(l_1, ..., l_n) \leq N$. There are three cases.

1. $l_1 = l_2 + 1$

   In this case we will remove the $l_1$ term from the tuple.

   **Lemma 1.** $\prod(l_2, ..., l_n) \leq \frac{1}{4} \prod(l_1, ..., l_n) = \frac{1}{4}N$

   **Proof.**

   $$\frac{\prod(l_1, ..., l_n)}{\prod(l_2, ..., l_n)} = \prod_{1 < j \leq n} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2}$$

   Notice that the above equation is minimized when $l_1$ is minimized and all the other $l_i$’s are maximized. Although there is a bigger decrease when minimizing $l_1$ than when maximizing the other $l_i$’s. But since we are not considering the case where $\lambda_i = 0, \forall \ 1 \leq i \leq n$ and since $l_1 > l_2 > ...l_n$ we get $l_1 \geq n$. So all in all to minimize the product above we choose $l_i = i$.

   $$\prod_{1 < j \leq n} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} > \prod_{1 < i \leq n} \frac{n^2 - i^2}{(n-1)^2 - (i-1)^2} = \prod_{1 < i \leq n} \frac{n + i}{n + i - 2} = \frac{(2n)(2n-1)}{(n)(n-1)} > 4$$

   $$\therefore \prod(l_2, ..., l_n) < \frac{\prod(l_1, ..., l_n)}{4}$$

2. $l_1 > l_2 + 1$ and $n > 6$

   We now consider $(\lfloor l_1 + l_2 \rfloor, l_2, ..., l_n)$. Notice that since $l_1 > l_2 + 1$ we get $\lfloor l_1 + l_2 \rfloor \geq \frac{l_2 + l_2 + l_2}{2} \geq l_2 + 1$. 

   ...
Lemma 2. \[ \prod(\lfloor \frac{l_1 + l_2}{2} \rfloor, l_2, ..., l_n) \leq (3/4)^6 \times N \]

**Proof.**

\[
\prod(\lfloor \frac{l_1 + l_2}{2} \rfloor, l_2, ..., l_n) = \prod_{1 < j \leq n} \frac{(\lfloor \frac{l_1 + l_2}{2} \rfloor)^2 - l_j^2}{m_i^2 - m_j^2} \prod_{1 < i < j \leq n} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2}
\]

\[
\leq \prod_{1 < j \leq n} \frac{(\frac{l_1 + l_2}{2})^2 - l_j^2}{m_i^2 - m_j^2} \prod_{1 < i < j \leq n} \frac{l_i^2 - l_j^2}{m_i^2 - m_j^2} = \prod_{1 < j \leq n} \frac{(\frac{l_1 + l_2}{2})^2 - l_j^2}{m_i^2 - m_j^2} \prod_{1 < i < j \leq n} \frac{3(l_i^2 - l_j^2)}{m_i^2 - m_j^2}
\]

\[
\leq \prod_{1 < j \leq n} \frac{(3/4)^6 \prod(d_1, l_2, ..., l_n)}{m_i^2 - m_j^2}
\]

The last inequality is because we consider only the case \( n > 6 \).

3. \( n = 2, 3, 4, 5, 6 \)

We will use the following result for this case.

**Theorem 8.** If \( G \) is a simple compact Lie group and \( N \) is a positive integer,

\[ R_N(G) \leq N \]

Applying this theorem to \( SO_{2n}\mathbb{C} \) we deduce that \( R_N(SO_{2n}\mathbb{C}) \leq N \) for \( n = 2, 3, 4, 5, 6 \). These cases sum up to \( 5N \).

All in all, we have a map that takes a tuple \( l \) of type 1 satisfying \( \prod(l_1, l_2, ..., l_n) \leq N \) and sends it to a tuple \( m \) satisfying \( \prod(m_1, m_2, ..., m_n) \leq \frac{3}{4} \). This map also takes tuples of type 2 and sends them to tuples \( b \) satisfying \( \prod(b_1, b_2, ..., b_n) \leq (\frac{3}{4})^6 N \), where no more than two tuples are sent to the same tuple. Finally we add the tuples \( l \) of type 3. Together this covers all tuples satisfying \( \prod(l_1, l_2, ..., l_n) \leq N \). We can use this map along with the inductive hypothesis to deduce that:

\[ R_N(SO(\mathbb{C})) \leq 2 \times R_{(\frac{3}{4})^6 N} + R_{N/4} + 5N \]

\[ \leq 2 \times 13 \times (\frac{3}{4})^6 N + 13 \times \frac{13}{4} N + 5N \leq 13N \]

\[ \square \]
3  $SL_n\mathbb{C}$

**Theorem 9.** $R_N(Sl(\mathbb{C})) < 43N$

**Proof.** First the function is split into 3 parts.

$$R_N(Sl(\mathbb{C})) = \sum_{n=1}^{n=33} R_N(Sl_n(\mathbb{C})) + \sum_{n=34}^{n=\sqrt{N}} R_N(Sl_n(\mathbb{C})) + \sum_{n=\sqrt{N}+1}^{n=N-1} R_N(Sl_n(\mathbb{C}))$$

Each part is bounded separately.

1. Using Theorem 1 it can deduce that:

$$\sum_{n=1}^{n=33} R_N(Sl_n(\mathbb{C})) < 33N$$

2. Applying Theorem 4 with $\epsilon = \frac{1}{2}$ then,

$$R_N(Sl_n(\mathbb{C})) < \sqrt{N}$$

for $n > 33$.

Now using this bound to all the terms in the sum.

$$\sum_{n=34}^{n=\sqrt{N}} R_N(Sl_n(\mathbb{C})) < (\sqrt{N} - 34)\sqrt{N} < N - 34\sqrt{N}$$

3. I will bound this sum by bounding each term in the sum.

First consider the tuple $(4,1,1,...,1)$ of length $\sqrt{N}$.

$$\prod (4,1,1,...,1) = \frac{(3 + \sqrt{N})(2 + \sqrt{N})(1 + \sqrt{N})}{6} > N$$

The last inequality is because $(3 + \sqrt{N}) > 6$ since we are only considering $N > 9$. Therefore the first term can only be 3 or 2.

Now consider the tuple $(1,3,1,1,...1)$ of length $\sqrt{N}$.

$$\prod (1,3,1,...,1) = \frac{(1 + \sqrt{N})^2(\sqrt{N})(2 + \sqrt{N})}{12} > N$$

Therefore the second term can only be 2.

Now consider the tuple $(1,1,2,...,1)$.

$$\prod (1,1,2,...,1) > N$$
Therefore the third term can only take the value 1.

Notice that the size of $\prod (a_1, ..., a_{\sqrt{N}})$ is increased more when you increase an $a_i$ closer to rather than further from the middle. This means that all terms other than the first two and last two can only take the value 1 in order to satisfy $\prod (a_1, ..., a_n) \leq N$. This along with the constriction of what values the first two and last two terms can take, leaves very few options for tuples satisfying $\prod (a_1, ..., a_{\sqrt{N}}) \leq N$. With a little algebra you can cancel out some of these possibilities and you can show there is at most 7 different tuples satisfying $\prod (a_1, ..., a_{\sqrt{N}}) \leq N$.

The same reasoning works on tuples of length $> \sqrt{N}$, therefore

$$R_N(SL_n(\mathbb{C})) < 7$$

for $n > \sqrt{N}$.

We can now bound the sum.

$$\sum_{n=\sqrt{N}+1}^{n=N-1} R_N(SL_n(\mathbb{C})) < 7\sqrt{N}$$

$$R_N(SL(\mathbb{C})) < (33N) + (N - 34\sqrt{N}) + (7\sqrt{N}) = 34N - 27\sqrt{N}$$

4 Bound on $R_N$

The bounds for $R_N(G)$ have been established for the four main simple Lie groups but there are still the five exceptions. These can be bounded using Theorem 1.

$$R_N(E_6) + R_n(E_7) + R_N(E_8) + R_N(F_4) + R_N(G_2) < 5N$$

Combining all the linear bounds obtained in the paper,

$$R_N \leq 3 * (13N) + 34N - 27\sqrt{N} + 5N = O(N)$$

Notice that this bound is tight since $R_N(SL)$ alone is bigger than $3N$.

References

[1] Fulton, W. and Harris, J., 2013. Representation theory: a first course (Vol. 129). Springer Science and Business Media.
[2] Guralnick, R., Larsen, M. and Manack, C., 2012. Low degree representations of simple Lie groups. Proceedings of the American Mathematical Society, 140(5), pp.1823-1834.