A Fixed-Parameter Linear-Time Algorithm for Maximum Flow in Planar Flow Networks

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Abstract

We pull together previously established graph-theoretical results to produce the algorithm in the paper’s title. The glue are three easy elementary lemmas.
1 Introduction

We combine a previous result on what is called graph reassembling, together with a previous result on what are called network typings, in order to show the existence of an algorithm that returns the value of a maximum flow in planar flow networks in fixed-parameter linear-time. Those results are made to work together by means of three easy elementary lemmas. In this introductory section we informally explain the notions involved; formal definitions are in later sections of the report.

One way of understanding the reassembling of a simple undirected graph $G$ is this: It is the process of cutting every edge of $G$ in two halves, and then splicing the two halves of every edge, one by one in some order, in order to recover the original $G$. We thus start from one-vertex components, with one component for each vertex $v$ and each with $\deg(v)$ dangling half edges,\footnote{$\deg(v)$ is the degree of vertex $v$, the number of edges incident to $v$.} and then gradually reassemble larger and larger components of the original $G$ until $G$ is fully reassembled. One optimization associated with graph reassembling is to keep the number of dangling half edges of each reassembled component as small as possible. Graph reassembling and associated optimization problems are examined in earlier reports \cite{10,17,12,14}.

As for network typings, these are algebraic or arithmetic formulations of interface conditions that network components must satisfy for them to be safely and correctly interconnected. A particular use of network typings is to quantify desirable properties related to resource management (e.g., percentage ranges of channel utilization, mean delays between routers, etc., as well as flow conservation and capacity constraints along channels), and to enforce them as invariant properties across network interfaces. More on this use of network typings is in several reports \cite{3,9,10}. In this paper, a typing for a network component $N$ is limited to specify a range of admissible values for every combination of input ports (or “sources”) and output ports (or “sinks”) of $N$.

The parameter to be bounded in the algorithm of our main result is called the edge-outerplanarity of a planar graph. Edge-outerplanarity is distinct but closely related to the usual notion of outerplanarity, and was introduced in earlier studies for other purposes (e.g., disjoint paths in sparse graphs, as in \cite{2}). As with outerplanarity, for a fixed edge-outerplanarity $k$, the number $n$ of vertices in a graph can be arbitrarily large. Our main result can be re-phrased thus: For the class $\mathcal{C}_k$ of planar flow networks whose edge-outerplanarity is bounded by a fixed $k \geq 1$, there is an algorithm which, given an arbitrary $N \in \mathcal{C}_k$, computes the value of a maximum flow in $N$ in time $O(n)$ where $n = |N|$.

Organization of the Report. Section 2 is background material that makes precise many of the notions we use throughout the report. Section 3 includes the three elementary lemmas (Lemmas 4, 5, and 6) that we need to pull together the results on graph reassembling and network typings.

A formal definition of graph reassembling – different from, but equivalent to, the informal definition above – is in Section 4, which includes the optimization result (Theorem 7) that we need for the main result. A formal definition of network typings – also more general than the informal definition above – is in Section 4, where we present the relevant result about typings (Theorem 8) that we use in this paper.

Our main result (Theorem 9) is in Section 5. We conclude with a brief discussion of follow-up work in Section 6.

2 Preliminary Notions

In this paper we need to consider both directed and undirected graphs. We use the same letter ‘$G$’, possibly decorated, to refer to both directed and undirected graphs; the context will make clear whether $G$ is directed or undirected. We refer to the vertices and edges of a graph $G$ by writing $V(G)$ and $E(G)$.
**Directed Graphs and Undirected Graphs.** Throughout, our undirected graphs are simple graphs, i.e., they have no self-loops and no multi-edges. In particular, an edge is uniquely identified by the two-element set of its endpoints \(\{v, w\}\), which we also write as \(vw\).

In the case of directed graphs also, we disallow self-loops as well as multi-edges with the same direction. However, we allow two edges with opposite directions between the same two vertices \(v\) and \(w\), written as the ordered pairs \((v, w)\) and \((w, v)\). We also write \(\overrightarrow{vw}\) and \(\overleftarrow{vw}\) for \((v, w)\) and \((w, v)\), respectively.

The context will make clear whether \(\overrightarrow{vw}\) is an undirected edge in an undirected graph, or a directed edge in a directed graph. If \(\overrightarrow{vw}\) is undirected, then \(\overrightarrow{vw} = \overleftarrow{vw}\); if \(\overrightarrow{vw}\) is directed, then \(\overrightarrow{vw} \neq \overleftarrow{vw}\).

Let \(G\) be a directed graph. The undirected version of \(G\), denoted \(\tilde{G}\), consists in ignoring all edge directions. In the graphical representation of \(G\), all the edges are reproduced in \(\tilde{G}\), with every arrow \(\overrightarrow{vw}\) replaced by a line segment \(\overline{vw}\), with one exception: Two directed edges between the same two vertices, \(\overrightarrow{vw}\) and \(\overrightarrow{wv}\), are collapsed into a single line segment \(\overline{vw}\).²

If \(G\) is a directed graph containing two edges with opposite directions between the same two vertices \(\{v, w\}\), say \(e_1 = \overrightarrow{v_1v_2}\) and \(e_2 = \overrightarrow{v_2v_1}\), then \(\{e_1, e_2\}\) form what we call a two-edge cycle in \(G\). Two-edge cycles do not occur in undirected graphs.

For a vertex \(v\) in a directed graph, we write \(\text{deg}_\text{in}(v)\) and \(\text{deg}_\text{out}(v)\) for the in-degree and out-degree of \(v\). And we write \(\text{deg}(v)\) for \(\text{deg}_\text{in}(v) + \text{deg}_\text{out}(v)\), the total number of edges incident to \(v\), both incoming and outgoing.

**Flow Networks.** A flow network is a quadruple \((G, c, s, t)\) where \(G\) is a directed graph, \(c : E(G) \rightarrow \mathbb{R}_+\) is the capacity function on edges, and \(s\) (the source) and \(t\) (the sink) are two distinct members of \(V(G)\). Trivially, for the max flow problem from \(s\) to \(t\), there is no loss of generality in assuming that the underlying graph \(G\) is connected and contains no self-loops.³

If the underlying graph \(G\) of the network is connected, then so is its undirected version \(\tilde{G}\). Biconnectedness is a stronger requirement than connectedness ("there are at least two distinct directed paths between any two points") which we cannot impose on \(G\).

Nonetheless, we can further assume that, if \(G\) is the underlying graph of a flow network \((G, c, s, t)\), then \(\tilde{G}\) (though not \(G\) itself) is biconnected. This means there are no cut vertices in \(\tilde{G}\). Indeed, suppose \(G\) is connected but not biconnected. If the source \(s\) and the sink \(t\) are in the same component (i.e., maximal biconnected subgraph) \(G'\) of \(\tilde{G}\), we can discard all biconnected subgraphs other than \(G'\), and compute a max flow from \(s\) to \(t\) relative to \(G'\) only, where \(G'\) is the subgraph of \(G\) whose undirected version is \(\tilde{G}'\). If the source \(s\) and the sink \(t\) are in two distinct components \(\tilde{G}'\) and \(\tilde{G}''\) of \(\tilde{G}\), respectively, then there are at least \(p \geq 1\) cut vertices, say \(\{v_1, \ldots, v_p\}\), such that all directed paths from \(s\) to \(t\) in \(G\) visit the same \(p\) vertices. For simplicity, suppose \(p = 1\) and there is only one cut vertex \(v\) on the directed paths from \(s\) to \(t\); the argument extends to an arbitrary number \(p \geq 1\) in the obvious way. With one cut vertex \(v\), we compute a first max flow \(f_1\) from \(s\) to \(v\) and a second max flow \(f_2\) from \(v\) to \(t\); the max flow in the original \(G\) is \(\max\{f_1, f_2\}\).

To compute a max flow in \((G, c, s, t)\) by first identifying the biconnected components in the underlying \(\tilde{G}\) in a preprocessing phase, as suggested in the preceding paragraph, does not add more than linear sequential time \(O(m + n)\) or logarithmic parallel time \(O(\log n)\) to the overall cost, where \(m = |E(G)|\) and \(n = |V(G)|\); e.g., see [18, 7, 15].

If \(\tilde{G}\) is biconnected, there are no vertices \(v \in V(G)\) such that \(\text{deg}(v) = 1\). However, there may exist vertices \(v \in V(G)\) such that \(\text{deg}(v) = 2\). Consider a fixed \(v \in V(G) - \{s, t\}\) such that \(\text{deg}(v) = 2\), which must

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²By this reasoning and contrary to what is often done elsewhere, we do not consider here an undirected graph as a special case of a directed graph, whereby every undirected edge \(\{v, w\}\) is viewed as being two directed edges \((v, w)\) and \((w, v)\).

³We write \(\mathbb{R}_+\) for the set of non-negative real numbers and \(\mathbb{R}\) for the set of all real numbers.
therefore occur in the graphical representation of $G$ in one of three configurations \{(a), (b), (c)\} where:

(a) $v_1 \xrightarrow{e_1} v \xrightarrow{e_2} v_2$,  
(b) $v_1 \xleftarrow{e_1} v \xrightarrow{e_2} v_2$,  
(c) $v_1 \xrightarrow{e_1} v \xleftarrow{e_2} v_2,$

for some $v_1, v_2 \in V(G) - \{v\}$. We will assume configurations (b) and (c) do not occur in $G$, as they do not contribute any value to the max flow from $s$ to $t$.\footnote{We do not suggest that we can allow the presence of configurations (b) and (c) in $G$, and then eliminate them in a preprocessing phase in linear time. To do the latter in full generality, without restrictions on the topology of $G$, would take more than $O(n)$ time though not more than $O(n^2)$, but that would be enough to spoil the linear time of our final result.} As for configuration (a), we can delete the two edges $\overline{v_1v}$ and $\overline{v_2v}$, replace them by a single new edge $\overline{v_1v_2}$, and define the new capacity $c(\overline{v_1v_2}) := \min\{c(\overline{v_1v}), c(\overline{v_2v})\}$; clearly, this is can be done without affecting the final value of the max flow from $s$ to $t$, and can be done in time $O(n)$ in a preprocessing phase.

We do not exclude the possibility $\deg(s) = 2$ and/or $\deg(t) = 2$, but in constant time $O(1)$ we can slightly modify the underlying $G$ to $G'$, and update the capacity function $c$ to $c'$, so that $(G, c, s, t)$ is equivalent to $(G', c', s, t)$ and $\deg(s) = \deg(t) = 3$. For example, if $\deg(s) = 2$, we can do the following: Introduce 3 fresh vertices $\{v_1, v_2, v_3\}$ and three fresh edges $\{v_1s, v_1v_2, v_1v_3\}$, with $v_2$ and $v_3$ inserted in the two edges incident to $s$, and then set $c'(e) = c(e)$ for every edge $e \in E(G)$ and $c'(\overline{v_1s}) = c'(\overline{v_1v_2}) = c'(\overline{v_1v_3}) = 0$.

Based on the preceding comments, there is no loss of generality in making the following assumption ($\diamond$), which is to be satisfied by the underlying graph $G$ of every flow network in this paper.

**Assumption ($\diamond$).** If $G$ is a directed graph, then it satisfies three conditions:

1. $G$ has no self-loops,
2. $\deg(v) \geq 3$ for every $v \in V(G)$, and
3. the undirected version $\overline{G}$ of $G$ is biconnected.

Note that Assumption ($\diamond$) does not preclude the presence of two-edge cycles in $G$.

**Edge Outerplanarity of Plane Graphs.** A commonly used parameter of undirected plane graphs is outerplanarity. A less common parameter is edge outerplanarity, which is also only defined for undirected plane graphs. We here extend both notions to all graphs, directed and undirected.

We make a distinction between planar graphs and plane graphs. $G$ is a plane graph if it is drawn on the plane without any edge crossings. $G$ is a planar graph if it is isomorphic to a plane graph; i.e., it is embeddable in the plane in such a way that its edges intersect only at their endpoints. To keep the distinction between the two notions, we define the outerplanarity index of a planar graph and the outerplanarity of a plane graph.

If $G$ is a plane graph, directed or undirected, then the outerplanarity of $G$ is the number $k$ of times that all the vertices on the outer face (together with all their incident edges) have to be removed in order to obtain the empty graph. In such a case, we say that the plane graph $G$ is $k$-outerplanar.

If $G$ is a planar graph, directed or undirected, then the outerplanarity index of $G$ is the minimum of the outerplanarities of all the plane embeddings $G'$ of $G$.

Deciding whether an arbitrary graph is planar can be carried out in linear time $O(n)$ and, if it is planar, a plane embedding of it can also be carried out in linear time [16]. Given a planar graph $G$, the outerplanarity index $k$ of $G$ and a $k$-outerplanar embedding of $G$ in the plane can be computed in time $O(n^2)$, and a 4-approximation of its outerplanarity index can be computed in linear time [8].

We give a formal definition of edge outerplanarity, less common than standard outerplanarity, now also extended to directed graphs.

\[\text{decide whether } \overline{G} \text{ is embeddable in the plane in such a way that its edges intersect only at their endpoints.} \]
**Definition 1 (Edge-Outerplanarity).** Let $G$ be a plane graph, directed or undirected. If $E(G) = \emptyset$ and $G$ is a graph of isolated vertices, the edge outerplanarity of $G$ is $0$. If $E(G) \neq \emptyset$, we pose $G_0 := G$ and define $K_0$ as the set of edges lying on $\text{OuterFace}(G_0)$.

For every $i > 0$, we define $G_i$ as the plane graph obtained after deleting all the edges in $K_0 \cup \cdots \cup K_{i-1}$ from the initial $G$ and $K_i$ the set of edges lying on $\text{OuterFace}(G_i)$.

The edge outerplanarity of $G$, denoted $E$-outerplanarity($G$), is the least integer $k$ such that $G_k$ is a graph without edges, i.e., the edge outerplanarity of $G_k$ is $0$. This process of peeling off the edges lying on the outer face $k$ times produces a $k$-block partition of $E(G)$, namely, $\{K_0, \ldots, K_{k-1}\}$.

To keep outerplanarity and edge outerplanarity clearly apart, we call the first vertex outerplanarity, or more simply V-outerplanarity, and the second edge outerplanarity, or more simply E-outerplanarity.

There is a close relationship between V-outerplanarity and E-outerplanarity (Theorem 4 in Section 5.1 in [2]). In the case of three-regular plane graphs, the relationship is much easier to state. This is Proposition 2 next.

**Proposition 2.** If $G$ is a 3-regular plane graph, directed or undirected, then:

$$V\text{-outerplanarity}(G) \leq E\text{-outerplanarity}(G) \leq 1 + V\text{-outerplanarity}(G).$$

Thus, for 3-regular plane graphs, V-outerplanarity and E-outerplanarity are “almost the same”.

**Proof Sketch.** For a 3-regular plane graph, the difference between V-outerplanarity($G$) and E-outerplanarity($G$) occurs in the last stage in the process of repeatedly removing (in the case of standard V-outerplanarity) all vertices on the outer face and all their incident edges. The corresponding last stage in the case of E-outerplanarity may or may not delete all edges; if it does not, then one extra stage is needed to delete all remaining edges.

The preceding result is not true for arbitrary plane graphs, even if they are regular. Consider, for example, the four-regular plane graph $G$ in Figure 1, where $V\text{-outerplanarity}(G) = 2$ while $E\text{-outerplanarity}(G) = 4$.

![Figure 1: A four-regular plane graph $G$, with $V$-outerplanarity($G$) = 2 and $E$-outerplanarity($G$) = 4.](image)

**3 A Flow-Preserving and Planarity-Preserving Transformation**

We define a transformation which, given an arbitrary directed graph $G$ satisfying Assumption (♦) on page 3, returns a directed graph $G^*$ where:

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5 There is an unessential difference between our definition here and the definition in [2]. In Section 2.2 of that reference, “a $k$-edge-outerplanar graph is a planar graph having an embedding with at most $k$ layers of edges.” In our presentation, we limit the definition to plane graphs and say “a $k$-edge-outerplanar plane graph has exactly $k$ layers of edges.” Our version simplifies a few things later.
• $\deg(v) = 3$ for every vertex $v \in V(G^*)$, and
• there are no two-edge cycles,

where $\deg(v) = \deg_{in}(v) + \deg_{out}(v)$, the total number of edges incident to vertex $v$, both incoming and outgoing. The transformation $G \rightarrow G^*$ is defined in terms of an operation which we call **expand**.

**Definition 3 (Expand).** The operation **expand** is applied to vertices of degrees $\geq 3$. Given a vertex $v$ such that $\deg(v) = p \geq 3$, there are $p$ edges incident to $v$, say $\{e_1, \ldots, e_p\}$. The expansion of $v$ consists in constructing a simple cycle with $p$ fresh vertices $\{v_1, \ldots, v_p\}$ and $p$ fresh edges $\{e'_1, \ldots, e'_p\}$, and then attaching the original edges $e_1, \ldots, e_p$ to the cycle thus constructed at the new vertices $v_1, \ldots, v_p$, respectively. An example when $p = 4$ is shown in Figure 2.

![Figure 2](image)

Figure 2: Applying the **expand** operation to a degree-4 vertex $v$ (on the left) produces a cycle with four new vertices $\{v_1, v_2, v_3, v_4\}$ and four new edges $\{e'_1, e'_2, e'_3, e'_4\}$ (on the right), while preserving planarity.

The transformation $G \rightarrow G^*$ has two stages in sequence. Stage 1 eliminates all vertices $v$ such that $\deg(v) \geq 4$, and Stage 2 eliminates all two-edge cycles.

**Stage 1:** All vertices $v$ such that $\deg(v) \geq 4$ are eliminated by applying the **expand** operation repeatedly, until it cannot be applied.

After Stage 1 there are only degree-3 vertices in the transformed directed graph. But we still want to eliminate every two-edge cycle, *i.e.*, two edges of the form $v \overset{w}{\rightarrow} v$ and $w \overset{v}{\rightarrow} v$ where $v \neq w$; we want to eliminate such a two-edge cycle because $v \overset{w}{\rightarrow} v$ and $w \overset{v}{\rightarrow} v$ collapse into a single edge $\{v, w\}$ in the undirected version of the graph. This is the purpose of Stage 2, to prevent such a collapse.

**Stage 2:** Every two-edge cycle $\{v \overset{w}{\rightarrow} v, w \overset{v}{\rightarrow} v\}$ where $v \neq w$ is eliminated by applying the **expand** operation twice, once to each of its endpoints $v$ and $w$, where necessarily $\deg(v) = \deg(w) = 3$ after Stage 1.

Stage 1 and Stage 2 complete the transformation $G \rightarrow G^*$. In words, we have transformed the original $G$ into a 3-regular $G^*$ by adding “a few” directed edges to the former.

**Lemma 4.** Let $G$ be a directed graph satisfying **Assumption (◊)**, where $|E(G)| = m$ and $|V(G)| = n$. We have the following facts:

1. The transformation $G \rightarrow G^*$ is carried out in linear time $O(n)$.
2. $|E(G^*)| \leq 3m$
3. $|V(G^*)| \leq n + 2m$.

**Proof.** The proof of part 1 of the lemma is straightforward, with Stage 1 and Stage 2 each requiring $O(n)$ time.
to do its work. Each of the two stages needs to visit each vertex \( v \) only once, to test whether \( v \) satisfies the condition calling for a local transformation at \( v \) and costing \( O(1) \) time.

For the proof of part 2 of the lemma, note that Stage 1 works on vertices \( v \) such that \( \deg(v) \geq 4 \) such that none of the new edges it introduces are involved in two-edge cycles; that is, every two-edge cycle that is present after Stage 1 is a two-edge cycle that is already present before Stage 1. Stage 2 works on degree-3 vertices that are endpoints of two-edge cycles, none of them introduced in the Stage 1.

Let \( q \) be the number of edges \( e = wv \) or \( e = wv \) with one or two endpoints satisfying one of the two conditions:

- \( \deg(v) \geq 4 \), or
- \( \deg(v) = 3 \) and \( v \) is one of two vertices on a two-edge cycle;

these are the endpoints/vertices worked on during Stage 1 and Stage 2. Each edge \( e \) of these \( q \) edges is associated with one or two new edges, depending on whether one or two of \( e \)'s endpoints are expanded. We conclude:

\[
| E(G^*) | \leq m + 2q \leq m + 2m = 3m.
\]

For the proof of part 3 of the lemma, we use the same reasoning as for part 2, to show that:

\[
| V(G^*) | \leq n + 2q \leq n + 2m.
\]

We omit the straightforward details.\(^6\)

The next lemma specializes Lemma 4 to the case of plane directed graphs. It makes clear that for plane directed graphs, the transformation \( G \mapsto G^* \) produces a (small) linear growth in the size.

**Lemma 5.** If \( G \) is a plane directed graph satisfying Assumption \((\Diamond)\), with \( | E(G) | = m \) and \( | V(G) | = n \) where \( n \geq 3 \), then:\(^7\)

1. \( | E(G^*) | \leq 18n - 36 \),
2. \( | V(G^*) | \leq 13n - 24 \), and
3. \( G^* \) is a plane directed graph satisfying Assumption \((\Diamond)\) such that
   - (3.a) there are no two-edge cycles in \( G^* \),
   - (3.b) \( \deg(v) = 3 \) for every \( v \in V(G^*) \), and
   - (3.c) \( E\text{-outerplanarity}(G) = E\text{-outerplanarity}(G^*) \).

**Proof.** Euler’s formula (Theorem 4.2.7 and its corollaries in [6]) is usually proved for undirected plane graphs (no self-loops, no multi edges) and written as \( m \leq 3n - 6 \) when \( n \geq 3 \). But our \( G \) is a directed plane graph, which may contain two-edge cycles (but no self-loops). If every double-edge cycle in \( G \) is collapsed into a single edge, we can write \( m/2 \leq 3n - 6 \), because there are at least \( m/2 \) edges in \( \overline{G} \). Hence, \( m \leq 6n - 12 \). Hence also, by parts 2 and 3 in Lemma 4, we have:

\[
| E(G^*) | \leq 3m \leq 3(6n - 12) = 18n - 36,
\]
\[
| V(G^*) | \leq n + 2m \leq n + 2(6n - 12) = 13n - 24.
\]

\(^6\) The upper bound \( 3m \) on \( | E(G^*) | \) is tight, in that there are directed graphs \( G \) satisfying Assumption \((\Diamond)\) on page 3 such that \( | E(G^*) | = 3m \); this happens when the two endpoints of every edge in \( G \) are expanded in Stage 1 or Stage 2. However, the upper bound \( n + 2m \) on \( | V(G^*) | \) is not tight; this is so because, if vertex \( v \) of degree \( p \) is expanded, then each of the \( p \) incident edges \( \{e_1, \ldots, e_p\} \) contributes one new vertex on the cycle replacing \( v \), but \( v \) itself has to be removed from the total count of vertices.

\(^7\) Again here, the upper bounds are not tight. See footnote 6. But they are easy to compute and good enough for our main result.
as claimed for parts 1 and 2 of the lemma.

For part 3, first note that the transformation \( G \rightarrow G^* \) is defined to guarantee (3.a) and (3.b). Moreover, it is readily checked that planarity is an invariant of every step of the transformation: If \( G \) is a plane graph (not just planar), then so is \( G^* \). Finally, it is readily checked that the equality:

\[
E\text{-outerplanarity}(G) = E\text{-outerplanarity}(G^*)
\]

is also an invariant of every step of the transformation \( G \rightarrow G^* \). The desired conclusion follows.

We need one more easy lemma. Let \((G, c, s, t)\) be a flow network. We define a new flow network \((G^*, c^*, s^*, t^*)\). The transformation \( G \rightarrow G^* \) is already defined. We still have to define \( c^* \), \( s^* \), and \( t^* \). In the transformation \( G \rightarrow G^* \), every edge \( G \) is preserved in \( G^* \), which allows us to view \( E(G) \subseteq E(G^*) \). So we define:

\[
c^*(e) := \begin{cases} 
  c(e) & \text{if } e \in E(G), \\
  \text{`a very large capacity'} & \text{if } e \in E(G^*) - E(G).
\end{cases}
\]

The idea of assigning `a very large capacity’ to every new edge introduced in the transformation \( G \rightarrow G^* \) is to make these new edges have no effect in restricting the flow through the network.

If the source \( s \) was not expanded into a cycle in the transformation \( G \rightarrow G^* \), then \( s^* := s \), else \( s^* := \) any of the new vertices on the cycle that replaces \( s \). And similarly for the definition of \( t^* \) from the original sink \( t \).

Two flow networks \((G_1, c_1, s_1, t_1)\) and \((G_2, c_2, s_2, t_2)\) are equivalent iff for every flow \( f_i : E(G_i) \rightarrow \mathbb{R}_+ \) there is a flow \( f_j : E(G_j) \rightarrow \mathbb{R}_+ \) such that \( |f_i| = |f_j| \) for all \( i, j \in \{1, 2\} \).

**Lemma 6.** Let \((G, c, s, t)\) be a flow network, where \( G \) is a plane directed graph \( G \) satisfying Assumption (\( \bigdiamond \)) and \( |V(G)| = n \), and consider the derived flow network \((G^*, c^*, s^*, t^*)\) as defined above. It then holds that:

1. The transformation \((G, c, s, t) \rightarrow (G^*, c^*, s^*, t^*)\) is carried out in linear time \( \mathcal{O}(n) \).
2. \((G, c, s, t)\) and \((G^*, c^*, s^*, t^*)\) are equivalent flow networks.

**Proof.** The transformation \( G \rightarrow G^* \) takes time \( \mathcal{O}(n) \), by part 1 of Lemma 4. The updating from \( c \) to \( c^* \) takes time \( \mathcal{O}(m) \), where \( |E(G)| = m \), and therefore time \( \mathcal{O}(n) \) by Euler’s formula (as in the proof of Lemma 5). And setting \( s^* \) and \( t^* \) takes time \( \mathcal{O}(1) \). The conclusion of part 1 follows.

The proof of part 2 is straightforward, since \( E(G) \subseteq E(G^*) \), with the edges in \( G \) preserving their capacities in \( G^* \) and the edges not in \( G \) assigned each ‘a very large capacity’. All formal details omitted.

Note that part 2 in Lemma 6 holds even if \( G \) is not a plane graph, but we do not need this fact for our main result. That \( G \) is a plane graph is only used in the proof of part 1 in Lemma 6 to change the complexity bound from \( \mathcal{O}(m + n) \) to \( \mathcal{O}(n) \).

## 4 Two Previous Results

The first result below (Theorem 7) is about the **reassembling problem**, which was studied in earlier reports and is here stated in terms of simple undirected graphs (no multi-edges, no self loops), but which applies equally well to directed graphs satisfying Assumption (\( \bigdiamond \)) on page 3.
Graph Reassembling. The reassembling of a simple undirected graph $G$ is an abstraction of a problem arising in studies of network analysis [3, 9, 10, 17]. There are several equivalent definitions of graph reassembling. An informal intuitive definition was already given in Section 1. A formal definition consists in constructing a rooted binary tree $B$ whose nodes are subsets of $V(G)$ and whose leaf nodes are singleton sets, with each of the latter containing a distinct vertex of $G$. The parent of two nodes in $B$ is the union of the two children’s vertex sets. The root node of $B$ is the full set $V(G)$. If $n = |V(G)|$, there are thus $n$ leaf nodes in $B$ and a total of $(2n - 1)$ nodes in $B$. We denote the reassembling of $G$ according to $B$ by writing $(G, B)$.

The edge-boundary degree of a node in $B$ is the number of edges that connect vertices in the node’s set to vertices not in the node’s set. Following a terminology used in earlier reports, the $\alpha$-measure of the reassembling $(G, B)$, denoted $\alpha(G, B)$, is the largest edge-boundary degree of any node in the tree $B$. We say $\alpha(G, B)$ is optimal if it is minimum among all $\alpha$-measures of $G$’s reassemblings, in which case we also say $B$ is $\alpha$-optimal.

The problem of constructing an $\alpha$-optimal reassembling $(G, B)$ of a simple undirected graph $G$ in general was already shown NP-hard [12, 14, among others]. However, restricting attention to plane graphs, we have the following positive result.

**Theorem 7.** There is an algorithm which, given a plane 3-regular simple undirected graph $G$ as input, returns a reassembling $(G, B)$ in time $O(n)$ such that $\alpha(G, B) \leq 2k$, where $k = E$-outerplanarity$(G)$ and $n = |V(G)|$.

The value of $\alpha(G, B)$ returned by the algorithm in Theorem 7 is independent of $n$; more precisely, for a fixed $k = E$-outerplanarity$(G)$, the value of $n$ can be arbitrarily large. Note that the algorithm in the theorem only returns an upper bound $2k$ on $\alpha(G, B)$ and does not claim that $\alpha(G, B)$ is optimal.

Theorem 7 and its proof are in the report [13], which also discusses conditions under which the bound $2k$ is optimal; specifically, it defines families of plane 3-regular simple graphs such that, for any graph $G$ in these families, $2k$ is an optimal $\alpha(G, B)$. We do not use the latter fact in this paper.

The second result below (Theorem 8) is about flow networks and what are called network typings. It is better stated in terms of what we here call extended flow networks, which have an upper bound function on edges $\underline{\tau}$, a lower bound function on edges $\underline{\ell}$, a set of source vertices $S$, and a set of sink vertices $T$.

Extended Flow Networks and their Typings. An extend flow network is denoted by a quintuple of the form $(G, \underline{\tau}, \underline{\ell}, S, T)$ where $G$ is a directed graph satisfying Assumption (◊) on page 3 and:

- $\underline{\tau} : E(G) \to \mathbb{R}_+$ and $\underline{\ell} : E(G) \to \mathbb{R}_+$, with $0 \leq \underline{\ell}(e) \leq \underline{\tau}(e)$ for every $e \in E(G)$, and

- $\emptyset \neq S \subseteq V(G)$ and $\emptyset \neq T \subseteq V(G)$, with $S \cap T = \emptyset$.

As usual, a flow in the network is a function $f : E(G) \to \mathbb{R}_+$. A flow $f$ is feasible iff $\underline{\ell}(e) \leq f(e) \leq \underline{\tau}(e)$ for every $e \in E(G)$ and $f$ satisfies flow conservation at every vertex $v \in V(G) - (S \cup T)$.

An input-output assignment (or an IO assignment) for such a network is a function $g : S \cup T \to \mathbb{R}_+$, which expresses the excess flow entering $S$ and exiting $T$. A typing for such a network is a map $\tau$ such that:

$$\tau : \mathcal{P}(S \cup T) \to \mathcal{I}(\mathbb{R})$$

where

$$\mathcal{P}(S \cup T) := \{ A \mid A \subseteq S \cup T \} \text{ and } \mathcal{I}(\mathbb{R}) := \{ [r_1, r_2] \mid r_1, r_2 \in \mathbb{R} \text{ and } r_1 \leq r_2 \},$$

i.e., $\mathcal{I}(\mathbb{R})$ is the set of bounded closed intervals of reals; such a typing must satisfy certain soundness conditions (not spelled out here). An IO assignment $g$ satisfies the typing $\tau$ iff for every $A \in \mathcal{P}(S \cup T)$:

$$\left( \sum g(A \cap S) - \sum g(A \cap T) \right) \in \tau(A)$$

To keep apart $B$ and $G$, we reserve the words ‘node’ and ‘branch’ for the tree $B$, and the words ‘vertex’ and ‘edge’ for the graph $G$. 

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where $\sum g(X)$ means $\sum \{g(x) \mid x \in X\}$. In particular, if $A = S \cup T$ and $\tau(A) = [r_1, r_2]$, then:

$$r_1 \leq \sum g(S) - \sum g(T) \leq r_2.$$  

Hence, one condition for the soundness of the typing $\tau$ is that we must have $r_1 = r_2 = 0$ when $A = S \cup T$, i.e., $\tau(S \cup T) = [0, 0] = \{0\}$, expressing the fact that the flow entering the network must equal the flow exiting it.

Given a flow $f : E(G) \rightarrow \mathbb{R}_+$, it induces an IO assignment $f^\# : S \cup T \rightarrow \mathbb{R}_+$ as follows:

- for every $s \in S$,
  $$f^\#(s) := \sum \{ f(e) \mid e = \overrightarrow{sv} \text{ for some } v \in V(G) \} - \sum \{ f(e) \mid e = \overrightarrow{vu} \text{ for some } v \in V(G) \},$$
- for every $t \in T$,
  $$f^\#(t) := \sum \{ f(e) \mid e = \overrightarrow{tv} \text{ for some } v \in V(G) \} - \sum \{ f(e) \mid e = \overrightarrow{vu} \text{ for some } v \in V(G) \}.$$  

i.e., $f^\#(s)$ is the total excess flow entering the source $s$ and $f^\#(t)$ is the total excess flow exiting the sink $t$. Thus, $\sum f^\#(S)$ and $\sum f^\#(T)$ are the total flows entering and exiting the network.

As noted in the opening paragraph of this section, a reassembling $\mathcal{B}$ can be defined equally well for a directed graph $G$ satisfying Assumption (\ integ) and containing no two-edge cycles. This allows us to use $(G, \mathcal{B})$ and its measure $\alpha(G, \mathcal{B})$ in the statement of the next theorem.

**Theorem 8.** If $(G, \tau, \xi, S, T)$ is an extended flow network as defined above and $(G, \mathcal{B})$ is a reassembling of the underlying $G$, then we can compute in time $m \cdot 2^{O(\delta)}$ a typing $\tau : \mathcal{P}(S \cup T) \rightarrow \mathcal{I}(\mathbb{R})$, where $m = |E(G)|$ and $\delta = \max\{\alpha(G, \mathcal{B}), |S \cup T|\}$, such that:

1. If $f : E(G) \rightarrow \mathbb{R}_+$ is a feasible flow, then $f^\# : S \cup T \rightarrow \mathbb{R}_+$ satisfies $\tau$.
2. If $g : S \cup T \rightarrow \mathbb{R}_+$ satisfies $\tau$, then there is a feasible flow $f : E(G) \rightarrow \mathbb{R}_+$ such that $f^\# = g$.

In particular, the typing $\tau$ is such that $\tau(S) = [r_1, r_2]$ and $\tau(T) = [-r_2, -r_1]$ for some $r_1, r_2 \in \mathbb{R}_+$, with $r_1$ and $r_2$ being, respectively, the minimum value and the maximum value of feasible flows in the network.

Theorem 8 and its proof are in the report [11, Theorem 4 on pp. 7-8], which examines other aspects of network typings and their applications.

For a simpler presentation of our main result (Theorem 9 below), we use Theorem 8 with the following restrictions: $S = \{s\}$ and $T = \{t\}$ are singleton sets, and the lower bound $\xi(e) = 0$ for every $e \in E(G)$. With these restrictions, the definition of a network as a quintuple $(G, \tau, \xi, S, T)$ in Theorem 8 matches the definition of a network as a quadruple in Section 2. But these restrictions can be lifted and our result re-stated in a more general setting, as in Theorem 10 below.

### 5 The Main Result

We first state and prove the result which is this paper’s title, and then explain how it generalizes to extended flow networks as defined in Section 5. The time complexity in Theorem 9 can be written as $O(n \cdot f(k))$ where $k$ is an edge-outerplanarity, $n$ a number of vertices, and $f(k)$ a function of $k$ independent of $n$ – which thus makes the algorithm in Theorem 9 ‘fixed-parameter linear-time’ where $k$ is the parameter to keep fixed.

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9By convention, $\sum \emptyset = 0$.

10There are minor differences between the terminology in this paper and the terminology in the report [11]. What is called a binding schedule $\sigma$ of a graph $G$ and its index($\sigma$) in that report are here a reassembling $(G, \mathcal{B})$ and its measure $\alpha(G, \mathcal{B})$. 

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Theorem 9. There is a fixed-parameter linear-time algorithm to compute the value of a max flow in plane flow networks \((G, c, s, t)\) where the parameter bound not to be exceeded is \(k = E\text{-outerplanarity}(G)\).

Proof. We can assume the underlying graph \(G\) satisfies Assumption (\(
\diamondsuit\)) on page 3. First, we carry out the transformation \((G, c, s, t) \mapsto (G^*, c^*, s^*, t^*)\) in time \(O(n)\) where \(n = |V(G)|\), as described in Lemma 6, also according to which \((G, c, s, t)\) and \((G^*, c^*, s^*, t^*)\) are equivalent networks. According to Lemma 5, we have \(E\text{-outerplanarity}(G) = E\text{-outerplanarity}(G^*) = k\) as well as \(|V(G^*)| = O(n)\) and \(|E(G^*)| = O(n)|\). To obtain the stated result, it now suffices to apply Theorems 7 and 8 to the transformed network \((G^*, c^*, s^*, t^*)\).

Remark. It is important to note that what is returned by the algorithm in Theorem 9 is the value \(r\) of a max flow, not a particular max flow \(f : E(G) \to \mathbb{R}_+\) such that \(|f| = r\). It is an additional problem, not considered in this paper but worthy of study, to compute a particular max flow \(f : E(G) \to \mathbb{R}_+\) given that its value \(|f|\) must be \(r\). While the value \(r\) is unique, there are generally many max flows \(f\) such that \(|f| = r\).

The next result implies the preceding Theorem 9 and illustrates the flexibility of our method. Theorem 10 is about extended flow networks, each of the form \((G, \tau, \underline{c}, S, T)\) where the graph \(G\) is a plane directed graph satisfying Assumption (\(
\diamondsuit\)) and the extra assumption that \(|S \cup T| = O(k)\) where \(k = E\text{-outerplanarity}(G)\). A typing \(\tau : \mathcal{P}(S \cup T) \to I(\mathbb{R})\) for such a network includes an interval for each \(A \in \mathcal{P}(S \cup T)\); with the extra assumption, the typing has size \(2^{O(k)}\). We impose the extra assumption in order to keep the complexity linear in \(n = O(|V(G)|)\), though exponential in the parameter \(k = E\text{-outerplanarity}(G)\).

Theorem 10. There is a fixed-parameter linear-time algorithm which, given a plane extended flow network \((G, \tau, \underline{c}, S, T)\) as described in the preceding paragraph, computes for every \(A \in \mathcal{P}(S \cup T)\) a bounded closed interval \([r_1, r_2]\) of reals such that for every feasible flow \(f : E(G) \to \mathbb{R}_+\) it holds that:

\[
r_1 \leq \sum f^\#(A \cap S) - \sum f^\#(A \cap T) \leq r_2.
\]

In particular, if \(A = S\), then \(r_2\) is the value of a max flow in the extended network, which is simultaneously returned with the value \(r_1\) of a min flow at no extra cost. The fixed parameter not to be exceeded for the algorithm to work as claimed is \(k = E\text{-outerplanarity}(G)\).

Proof Sketch. This is a minor variation on the proof of Theorem 9. The algorithm starts with the transformation \((G, \tau, \underline{c}, S, T) \mapsto (G^*, \overline{\tau}, \underline{c}^*, S^*, T^*)\) in time \(O(n)\), which is carried out just like the transformation \((G, c, s, t) \mapsto (G^*, c^*, s^*, t^*)\). One subtle point here: For every new edge \(e\) introduced in the transformation \(G \mapsto G^*\), we make \(\underline{c}^*(e) := 0\) just as we make \(\overline{\tau}^*(e) := \text{‘a very large capacity’, in this way the capacities on the new edges have no effect in restricting the flow in the transformed network. The rest of the proof proceeds like the proof of Theorem 9. Details omitted.}

The same Remark after Theorem 9 applies to Theorem 10: What is returned by the algorithm are the values \(r_1\) of a min flow and \(r_2\) of a max flow, not a particular min flow \(g : E(G) \to \mathbb{R}_+\) and not a particular max flow \(f : E(G) \to \mathbb{R}_+\) such that \(|g| = r_1\) and \(|f| = r_2\).

Compare our result in Theorem 10 with the main result in [4], where it is shown that there exists an algorithm that solves the max-flow problem with multiple sources and multiple sinks in an \(n\)-vertex directed plane graph in \(O(n \log^3 n)\) time (with only upper bounds, no lower bounds, on edge capacities).
6 Future Work

The method proposed in this paper for computing the value of a maximum flow in planar networks, in fixed-parameter linear time, can be extended to other more general forms of flows in planar networks without much trouble, where the parameter bound not to be exceeded is again edge-outerplanarity. Under preparation are the four following extensions:

- **multicommodity flows** (formal definitions in [1, Chapt. 17]),
- **minimum-cost flows, minimum-cost max flows**, and variations (definitions in [1, Chapt. 9-11]),
- **flows with multiplicative gains and losses**, also called **generalized flows** (definitions in [1, Chapt. 15]),
- **flows with additive gains and losses** (definitions in [5]).

To put the relevance of this work in sharper focus, there is no known algorithm to compute a max flow in any of these four extensions in linear time in general; in the case of the fourth extension (flows with additive gains and losses), the problem is known to be NP-hard [5].

We conclude with an open problem. In the **Remark** in Section 5, we pointed out that our method produces the **value** of a maximum flow, rather than a particular flow with that value, in contrast to the many other approaches to the maximum-flow problem in the extant literature.

**Open Problem.** Let \((G, c, s, t)\) be an arbitrary plane flow network. We can tackle the problem according to one of two approaches:

1. Let the value \(r\) of a max flow in \((G, c, s, t)\) be given already. Can we determine in linear time a particular max flow \(f : E(G) \rightarrow \mathbb{R}_+\) such that \(|f| = r\)?

   Alternatively:

2. How can we extend our proposed method so that it simultaneously produces the value \(r\) of a max flow in \((G, c, s, t)\) and a particular max flow \(f : E(G) \rightarrow \mathbb{R}_+\) such that \(|f| = r\) in linear time?

A further qualification on the first approach above is whether the determination of \(f\) in linear time can be carried out without reference to a fixed bound \(k = E\)-outerplanarity\((G)\); if this is possible, it will be a stronger result.

In the second approach, since \(r\) and \(f\) are to be simultaneously determined, it will be a direct extension of our proposed method which will therefore make explicit reference to a fixed bound \(k = E\)-outerplanarity\((G)\) for both \(r\) and \(f\).

References

[1] R.K. Ahuja, T. L. Magnanti, and J.B. Orlin. *Network Flows: Theory, Algorithms, and Applications*. Prentice Hall, Englewood Cliffs, N.J., 1993.

[2] Cedric Bentz. Disjoint paths in sparse graphs. *Discrete Applied Mathematics*, 157(17):3558–3568, 2009.

[3] Azer Bestavros and Assaf Kfoury. A Domain-Specific Language for Incremental and Modular Design of Large-Scale Verifiably-Safe Flow Networks. In *Proc. of IFIP Working Conference on Domain-Specific Languages (DSL 2011), EPTCS Volume 66*, pages 24–47, Sept 2011.

[4] Glencora Borradaile, Philip N. Klein, Shay Mozes, Yahav Nussbaum, and Christian Wulff-Nilsen. Multiple-Source Multiple-Sink Maximum Flow in Directed Planar Graphs in Near-Linear Time. In *Proceedings of the 2011 IEEE 52Nd Annual Symposium on Foundations of Computer Science, FOCS ’11*, pages 170–179, Washington, DC, USA, 2011. IEEE Computer Society.

[5] Franz J Brandenburg and Mao-cheng Cai. Shortest Path and Maximum Flow Problems in Networks with Additive Losses and Gains. *Theoretical Computer Science*, 412(4):391–401, 2011.

[6] Reinhard Diestel. *Graph Theory*. Springer Verlag, 2012.
[7] Dorit S. Hochbaum. Why Should Biconnected Components Be Identified First. *Discrete Applied Mathematics*, 42(2):203 – 210, 1993.

[8] Frank Kammer. Determining the Smallest $k$ Such That $G$ Is $k$-Outerplanar. In Lars Arge, Michael Hoffmann, and Emo Welzl, editors, *Proc. of 15th Annual European Symposium on Algorithms*, ESA 2007, pages 359–370. LNCS 4698, Springer Verlag, September 2007.

[9] Assaf Kfoury. The Denotational, Operational, and Static Semantics of a Domain-Specific Language for the Design of Flow Networks. In *Proc. of SBLP 2011: Brazilian Symposium on Programming Languages*, Sept 2011.

[10] Assaf Kfoury. The Syntax and Semantics of a Domain-Specific Language for Flow-Network Design. *Science of Computer Programming*, 93(Part A):19–38, November 2014.

[11] Assaf Kfoury. A Compositional Approach to Network Algorithms. *CoRR*, abs/1805.07491, 2018. preprint, http://arxiv.org/abs/1805.07491v1.

[12] Assaf Kfoury and Saber Mirzaei. Efficient Reassembling of Graphs, Part 1: The Linear Case. *J. of Combinatorial Optimization*, 33(3):1057–1089, April 2017.

[13] Assaf Kfoury and Benjamin Sisson. Efficient Reassembling of Three-Regular Planar Graphs. *CoRR*, abs/1807.03479, July 2018. preprint, https://arxiv.org/abs/1807.03479v1.

[14] Saber Mirzaei and Assaf Kfoury. Efficient Reassembling of Graphs, Part 2: The Balanced Case. *CoRR*, abs/1602.02863, 2016. preprint, http://arxiv.org/abs/1602.02863v1.

[15] Stavros D. Nikolopoulos and Leonidas Palios. On the Parallel Computation of the Biconnected and Strongly Connected Components of Graphs. *Discrete Applied Mathematics*, 155(14):1858 – 1877, 2007. 3rd Cologne/ Twente Workshop on Graphs and Combinatorial Optimization.

[16] Maurizio Patrignani. Planarity Testing and Embedding. In Roberto Tamassia, editor, *Hanbook of Graph Drawing and Visualization*, pages 1–42. CRC Press, Baton Rouge, FL, 2013.

[17] Nate Soule, Azer Bestavros, Assaf Kfoury, and Andrei Lapets. Safe Compositional Equation-based Modeling of Constrained Flow Networks. In *Proc. of 4th Int’l Workshop on Equation-Based Object-Oriented Modeling Languages and Tools*, Zürich, September 2011.

[18] Robert Endre Tarjan and Uzi Vishkin. Finding Biconnected Components and Computing Tree Functions in Logarithmic Parallel Time (Extended Summary). In *25th Annual Symposium on Foundations of Computer Science*, West Palm Beach, Florida, USA, 24-26 October 1984, pages 12–20, 1984.