A Note on Quantum Separability

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This short note describes a method to tackle the (bipartite) quantum separability problem. The method can be used for solving the separability problem in an experimental setting as well as in the purely mathematical setting. The idea is to invoke the following characterization of entangled states: A state is entangled if and only if there exists an entanglement witness that detects it. The method is basically a search for an entanglement witness that detects the given state.

1 Introduction

Entangled quantum states are interesting both from theoretical and practical points of view. Theoretically, entanglement is connected with the confounding issue of nonlocality. Practically, entangled states are useful in quantum cryptography and other quantum information processing tasks (for example, see [1] and references therein). The problem of determining whether a state is entangled or separable is thus important and comes in two flavors – one mathematical, and the other experimental.

The mathematical problem of separability (for bipartite systems) is defined as follows. Let $\mathcal{H}_{M,N}$ denote the set of all Hermitian operators mapping $\mathbb{C}^M \otimes \mathbb{C}^N$ to $\mathbb{C}^M \otimes \mathbb{C}^N$. The set of bipartite separable quantum states $S_{M,N}$ in $\mathcal{H}_{M,N}$ is defined as the convex hull of the separable pure states $\{ |\alpha\rangle \langle \alpha| \otimes |\beta\rangle \langle \beta| \in \mathcal{H}_{M,N} \}$, where $|\alpha\rangle$ is a norm-1 vector in $\mathbb{C}^M$ and $|\beta\rangle$ is a norm-1 vector in $\mathbb{C}^N$. The set of separable states $S_{M,N}$ may be viewed as a compact, convex subset of $\mathbb{R}^{M^2N^2}$ by expressing each density operator as a real linear combination of the canonical Hermitian generators of $SU(M)$ and $SU(N)$ [2]. The quantum separability problem is now easily defined as an instance of the Weak Membership problem [3]: Given a convex set $K \subset \mathbb{R}^n$, a point $p \in \mathbb{R}^n$, and an accuracy parameter $\delta > 0$, assert either that (i) $p \in S(K, \delta)$ (i.e. $p$ is “almost in” $K$) or that (ii) $p \notin S(K, -\delta)$ (i.e. $p$ is “not $\delta$-deep within” $K$), where $S(K, \delta)$ denotes the union of all $\delta$-balls with centers belonging to $K$, and $S(K, -\delta)$ denotes the union of all centers of all $\delta$-balls contained in $K$ (in the standard Euclidean norm). The separability problem has been shown to be NP-hard [4], thus any devised test for separability is likely

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to require a number of computing resources that scales exponentially with \( M \) and \( N \). There exist efficient “one-sided” tests for separability, where the output of some polynomial-time computable function of the matrix for \( \rho \) can indicate that \( \rho \) is certainly entangled \([5, 6, 7, 8]\) or certainly separable \([9, 10, 11]\), but not both. The algorithm in \([8]\) is in principle a one-sided test because it requires an infinite amount of computational resources to detect some entangled states.

The experimental flavor of the separability problem can be defined as follows: Given many physical copies of a completely unknown quantum state \( \rho \in \mathcal{H}_{M,N} \), determine whether \( \rho \) is separable. One way to solve this problem is to perform a full state tomography in order to construct the density matrix for \( \rho \) to some precision \( \delta > 0 \), and then solve the mathematical separability problem. If rather there is some partial knowledge of \( \rho \), then there are certainly more options, such as testing for a violation of a specific Bell-type inequality \([12, 13]\) or invoking entanglement witnesses \([14, 15]\). As well, in the case where \( MN \leq 6 \), the positive partial transpose (PPT) test \([5, 16]\) can be implemented physically \([17, 18]\), though currently this approach is not experimentally viable.

In section 2 we describe a method to tackle the mathematical separability problem in general. The idea is to invoke the following characterization \([16]\) of entangled states: A state \( \rho \) is entangled if and only if there exists an entanglement witness \([19]\) that detects it. The method is basically a search for an entanglement witness that detects the given state. In section 3 we describe how to use this method as a novel tool for solving the experimental separability problem. We conclude with a brief discussion, and point to some future directions of research in section 4.

## 2 Solving Quantum Separability

In this note we use the following definition of “entanglement witness”, which differs slightly from the definition used in the literature.

**Definition 1.** An entanglement witness is any operator \( A \in \mathcal{H}_{M,N} \) for which there exists a state \( \rho \in \mathcal{H}_{M,N} \) such that

\[
tr(A\sigma) < tr(A\rho) \quad \forall \, \sigma \in S_{M,N}.
\]

Recalling that \( \mathcal{H}_{M,N} \) is isomorphic to \( \mathbb{R}^{M^2N^2} \), the above definition implies that for entangled \( \rho \) there exists a hyperplane in \( \mathbb{R}^{M^2N^2} \) which separates \( \rho \) from the set of all separable states \( S_{M,N} \). If we define the function

\[
b_A := \max_{\sigma \in S_{M,N}} tr(A\sigma),
\]

(1)
then the set \( \{ x \in H_{M,N} : \text{tr}(Ax) = b_A \} \) is one such hyperplane. The function \( b_A \) is implicitly at the heart of the definition of “entanglement witness” as it pins down which, if any, of the hyperplanes, with normal \( A \), separate the state \( \rho \) from \( S_{M,N} \). The hyperplane defined by \( A \) and \( b_A \) is tangent to \( S_{M,N} \) and is thus the optimal hyperplane with normal vector \( A \) that separates \( \rho \) from \( S_{M,N} \).

The function \( b_A \) leads naturally to an algorithm for quantum separability as follows. For \( A \) such that \( \text{tr}(A^2) = 1 \), define the function \( d_{A,\rho} \) as
\[
d_{A,\rho} := b_A - \text{tr}(A\rho).
\]
Geometrically, \( d_{A,\rho} \) is the signed distance from the state \( \rho \) to the hyperplane defined by \( b_A \). By using equations 1 and 2 and the definition of “entanglement witness”, it follows that \( \rho \) is entangled if and only if
\[
\exists A : d_{A,\rho} < 0.
\]

Thus we have reduced quantum separability to the task of finding an \( A \) such that \( d_{A,\rho} \) is negative. By observing that the ability to calculate \( b_A \) gives an oracle for the Weak Optimization problem, Theorem 4.4.7 from [3] shows that it is possible to solve quantum separability with only polynomially many evaluations of the function \( b_A \). Thus, the “hardness” of quantum separability is contained in the “hardness” of evaluating \( b_A \). In practice, and for low-dimensional applications, well-known sophisticated techniques (such as simulated annealing or interval analysis) for finding global extrema would likely be sufficient to calculate \( b_A \). Unfortunately the algorithm derived in [3] is not implementable on fixed precision computers, and it uses many more evaluations of \( b_A \) than is strictly necessary. In a later paper, we hope to demonstrate a better algorithm which uses far fewer evaluations of \( b_A \) and thus, hopefully, is of practical use for low-dimensional problems.

3 Solving Separability with Partial Information

In this section, we briefly show how the above approach may be used when only partial information about the state \( \rho \in H_{M,N} \) is available. This is of particular use in an experimental setting.

The state \( \rho \) can be written
\[
\rho = \sum_{i=0}^{M^2-1} \sum_{j=0}^{N^2-1} \rho_{ij} \lambda_i^M \otimes \lambda_j^N,
\]
where \( \rho_{ij} \in \mathbb{R} \) and the \( \lambda^M_i \) and \( \lambda^N_j \) (as defined in [2]) respectively generate the special unitary groups \( SU(M) \) and \( SU(N) \). In this case, the coefficients \( \rho_{ij} \) are simply related to the expected values of \( \lambda^M_i \otimes \lambda^N_j \):

\[
\rho_{ij} = \langle \lambda^M_i \otimes \lambda^N_j \rangle / 4 := \text{tr}(\lambda^M_i \otimes \lambda^N_j \rho) / 4.
\]

(5)

Let \( \Lambda := \{ \lambda^M_i \otimes \lambda^N_j \}_{i=0,1,...,M^2-1; j=0,1,...,N^2-1} \). The expected values of all elements of \( \Lambda \) constitute complete information about \( \rho \). Suppose only partial information about \( \rho \) has been obtained by an experimental procedure, that is, only the expected values of the elements of a proper subset \( T \) of \( \Lambda \) are known.

It helps to think of each density operator as a real vector of its expected values. With only \(|T|\) expected values known, we now effectively project all the density operators onto \( \text{span}(T) \) by ignoring the components of the real vectors that correspond to the unknown expected values of \( \rho \). Now each density operator, including our unknown \( \rho \), is represented by a point in a \(|T|\)-dimensional “expectation space”. Note that the set of points in this projective space representing all separable density operators is still a convex set. Call this convex set \( \bar{S}_T \) and denote its elements by \( \bar{\sigma} \in \mathbb{R}^{|T|} \). Similarly, let \( \bar{\rho} \) be the \(|T|\)-dimensional real vector of known expected values of \( \rho \). To represent the Hermitian operator \( A = \sum_{X \in T} a_XX \) in this space, we can use the real vector \( \bar{A} \) of the coefficients \( a_X \) in order to have the correspondence \( \text{tr}(A\rho) = \bar{A} \cdot \bar{\rho} \), where “\( \cdot \)” denotes the standard dot-product of two real vectors.

It is clear that the functions \( b_A \) and \( d_{A,\rho} \) in the previous section can be redefined for the \(|T|\)-dimensional space:

\[
\bar{b}_{\bar{A}} := \max_{\bar{\sigma} \in \bar{S}_T} \bar{A} \cdot \bar{\sigma} \quad (6)
\]

and

\[
\bar{d}_{\bar{A},\bar{\rho}} := \bar{b}_{\bar{A}} - \bar{A} \cdot \bar{\rho}. \quad (7)
\]

If there exists \( \bar{A} \) such that \( \bar{d}_{\bar{A},\bar{\rho}} < 0 \), then \( \rho \) is entangled; otherwise, more information is needed to determine the separability of \( \rho \). As expected values are being gathered through experimental observation, they may be input to a computer program that searches for \( \bar{d}_{\bar{A},\bar{\rho}} < 0 \). Finally, we point out that the idea of searching for an entanglement witness in the span of operators whose expected values are known was discovered independently and applied, in a special case, to quantum cryptographic protocols in [20].

4 Discussion

By looking at quantum separability as a mathematical problem in the real Euclidean space \( \mathbb{R}^{M^2N^2} \) and slightly altering the definition of entanglement witness, we have show that quan-
tum separability can be solved in oracle-polynomial time, for a rather natural looking oracle. This allows us to highlight the fact that the “hard” part of quantum separability is contained in the function $b_A$. The method we describe for solving quantum separability also gives experimentalists a tool for potentially determining if an unknown state is entangled by measuring only a subset of the expected values which completely describe the state. This method effectively trades quantum resources (additional copies of $\rho$) for classical resources (a computer able to calculate $\bar{b}_A$). As well as providing a practical, implementable algorithm for low-dimensional quantum separability, some open questions which we hope to address in the future include getting a tight (exponential) bound on the complexity of calculating $b_A$, and determining the average number of expected values required to detect a random unknown quantum state.

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