Can the fluctuations of the quantum vacuum solve the cosmological constant problem?

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The cosmological constant problem arises because the magnitude of vacuum energy density predicted by quantum mechanics is ~ 120 orders of magnitude larger than the value implied by cosmological observations of accelerating cosmic expansion. Recently some of the current authors proposed that the stochastic nature of the quantum vacuum can resolve this tension [1]. By treating the fluctuations in the vacuum seriously, and applying a high-energy cutoff at which Quantum Field Theory is believed to break down, a parametric resonance occurs that predicts a slow expansion and acceleration. In this work we more thoroughly examine the implications of this proposal by investigating the resulting dynamics. Firstly, we improve upon the numerical calculations and show that convergence issues with the original code had overshadowed some important effects. Some of the conclusions are thus reversed, however, the premise that parametric resonance can explain a very slowly accelerating expansion remains sound. After improving the resolution and efficiency of the numerical tests, we explore a wider range of cutoff energies, and examine the effects of multiple particle fields. We introduce a simple model using Mathieu’s equation, a prototypical example of parametric resonance, and find that it closely matches numerical results in regimes where its assumptions are valid. Using this model, we extrapolate to find that in a universe with 28 bosonic fields and a high-energy cutoff 40 times higher than the Planck energy, the acceleration would be comparable to what is observed.

I. INTRODUCTION

One of the greatest challenges in modern physics is to reconcile general relativity and quantum physics into a unified theory. Perhaps the most dramatic clash between the two theories lies in the cosmological constant problem [2–6]. Naive predictions of vacuum energy from quantum physics predict a magnitude so high that the expansion of the universe should have accelerated so quickly that no structure could have formed. The predicted rate of acceleration resulting from vacuum energy is famously 120 orders of magnitude larger than what is observed.

In a 2017 paper [1] some of the current authors proposed a solution to the cosmological constant problem. They proposed that rather than use the expectation value of the quantum energy density in Einstein’s equations, which would lead to the overwhelmingly large prediction for cosmic acceleration, one should instead treat the vacuum as an inhomogeneous stochastic field. Accounting for the fluctuations in the density of the vacuum energy – which are on the order of the magnitude of the vacuum energy itself – can potentially explain a slow expansion.

Here we investigate that proposal with improved computational methods. Our faster computational methods allow us to extend the model to a greater number of particle fields. With at least three fields, the exponentially small acceleration predicted by the original proposal is observed, and the magnitude of the acceleration gets smaller as more fields are added and the cutoff increased – meaning that with a sufficient number of fields at a sufficiently high cutoff, the predicted acceleration would match observation.

The paper is structured as follows. In Section II we summarise the aspects of the cosmological constant problem that are relevant to this work. In Section III we summarise the model of cosmological dynamics in the presence of a stochastic inhomogeneous vacuum that was introduced in [1], and how it attempts to resolve the problem. We also mention a caveat to the application of the adiabatic theorem in [1], which implies that the resultant analytical description is only valid with three or more scalar fields present. In Section IV, we describe our numerical methods, which are similar to those used in [1], before testing the convergence of our new results across all relevant parameters to demonstrate that they are robust to all limits. In Section V we provide corrections to numerical findings of the original paper, before using our improved methods to test greater numbers of particle fields, and a larger range of choices of cutoff frequency for vacuum oscillations. Finally, we conclude with the physical significance of the new results in Section VI. Throughout, we use $\hbar = G = c = 1$, and a metric signature of $(-, +, +, +)$.
II. THE COSMOLOGICAL CONSTANT PROBLEM

In Einstein’s equation of general relativity, a term representing the curvature of spacetime \( G_{\mu\nu} \) is related to a term describing the stress-energy of matter \( (T_{\mu\nu}) \), as well as the cosmological constant \( (\lambda) \) and metric tensor \( (g_{\mu\nu}) \) as follows:

\[
G_{\mu\nu} + \lambda g_{\mu\nu} = \frac{8\pi}{3} T_{\mu\nu}.
\] (1)

Each element of the curvature tensor and metric tensor are just classical fields, but the elements of the stress-energy tensor must be quantum operators in order to account for known quantum effects of matter. A currently undiscovered theory of quantum gravity would presumably elevate the left hand side to become quantum operators. In the meantime it is common to treat both sides as classical (known as “semiclassical” gravity). The most common way of doing this is to replace \( T_{\mu\nu} \) with \( (T_{\mu\nu}) \) (the Moller–Rosenfeld approach) \([7]\). But this approach fails in a number of ways: it allows faster-than-light communication \([8]\), it leads to a nonlinear Hamiltonian which contradicts the Born rule \([9]\), and most infamously, it predicts an overwhelming large accelerating expansion of the universe.

Here we outline the traditional approach to the cosmological constant problem, see \([2–4, 10–12]\). In a background of flat spacetime, the usual argument states that the vacuum state should be Lorentz invariant, so the expectation value of its energy-momentum tensor, \( \langle T_{\mu\nu}^{\text{vac}} \rangle \), must be a scalar multiple of the metric tensor \( g_{\mu\nu} \) (which is the only Lorentz invariant rank \((0, 2)\) tensor). Because the \( T_{00} \) component is an energy density, we can identify \( T_{00} = \rho_{\text{vac}} \), so that the vacuum contribution to the right hand side of Equation (1) is \( \langle T_{\mu\nu}^{\text{vac}} \rangle = -\langle \rho_{\text{vac}} \rangle g_{\mu\nu} \). Subtracting it from the right hand side of Equation (1) and grouping it with the cosmological constant term replaces \( \lambda \) with an “effective” cosmological constant:

\[
\lambda_{\text{eff}} = \lambda + \frac{8\pi}{3} \langle \rho_{\text{vac}} \rangle.
\] (2)

The relative acceleration of the expansion of the universe (\( \frac{\ddot{a}}{a} \), where the scalefactor \( a \) represents the overall scale of the universe, and a dot denotes a time derivative), is directly proportional to the effective cosmological constant, and is measured to be about \( 10^{-122} \) in Planck units.

Now, we determine \( \langle \rho_{\text{vac}} \rangle \) in the simplified case of a single massless spin-0 particle field. For each 4-momentum \( k = (\omega_k, \mathbf{k}) \), the field acts like a simple harmonic oscillator. The \( n \)th state, with energy \( (n + \frac{1}{2}) \omega_k \) (recalling \( h = 1 \)), contains \( n \) particles with momentum \( \mathbf{k} \) and energy \( \omega_k \), and the ground state (with no particles) has energy \( \frac{1}{2} \omega_k \). Combining the ground state energy of each mode (i.e. the harmonic oscillator for each 4-momentum) yields an infinite value for the vacuum energy density. By restricting to modes with particle energy below a certain cutoff energy \( \omega_k \leq \Lambda \) (not to be confused with \( \lambda \), the cosmological constant), a finite result for the energy density can be obtained. The result is proportional to \( \Lambda^4 \), because the number of allowed modes scales with \( \Lambda^3 \), and the average energy of the allowed modes scales linearly with \( \Lambda \). Any other fields will contribute similarly, so that if there are \( n_f \) scalar fields, the density scales with \( n_f \Lambda^4 \).

Typically, the cutoff is taken to be near \( \Lambda = 1 \) in Planck units (i.e. the Planck energy), so the vacuum energy gives a contribution to the cosmological constant on the order of at least unity according to Equation (2). Thus we see the extreme fine-tuning problem: the original cosmological constant \( \lambda \) must cancel this large vacuum energy density \( \langle \rho_{\text{vac}} \rangle \sim 1 \) to a precision of \( 1 \) in \( 10^{120} \) but not completely – to result in the observed value \( \lambda_{\text{eff}} \sim 10^{-120} \).

III. COSMOLOGICAL DYNAMICS UNDER SEMICLASSICAL STOCHASTIC GRAVITY

The energy density of the vacuum state fluctuates wildly, with variations comparable to its magnitude. Thus, rather than ignoring these fluctuations by treating the vacuum energy density as constant, some of the current authors \([1]\) proposed treating it as an inhomogeneous stochastic field to better approximate a full quantum description. They show that this leads to an inhomogeneous scalefactor \( a(t, x) \) (representing the relative “size” of spacetime at each point) which oscillates rapidly from positive to negative, according to the following equation:

\[
\ddot{a}(t, x) + \Omega^2(t, x)a(t, x) = 0.
\] (3)

We can recognize Equation (3) as a harmonic oscillator equation for each \( x \), with \( \Omega \) playing the role of a frequency (not to be confused with the usual use of \( \Omega \) in cosmology to mean energy density). The square of the frequency of those oscillations is proportional to a linear combination of components of the energy tensor, which we treat as time and position dependent stochastic fields:

\[
\Omega^2(t, x) = \frac{4\pi G}{3} \left( T_{00}(t, x) + \frac{1}{a^2(t, x)} \sum_{i=1}^{3} T_{ii}(t, x) \right).
\] (4)

Several assumptions are used to produce a simple description of the stochastic fields on the right of this equation. A number \( n_f \) of massless scalar fields are used, which was two in \([1]\), but here we will extend this to a greater number of fields. Using massless scalar fields ensures that \( \Omega^2 \) is strictly positive, which is no longer true if massive particles or fermionic particles are included because they add negative terms to the expression for \( \Omega^2 \). \( \Omega^2 \) being strictly positive means that \( \Omega = \sqrt{\Omega^2} \) is well defined, and that Equation (3) will always act like a harmonic oscillator, rather than yield an explicitly exponential solution (like, for example, \( \ddot{a} = a \)).

Once \( T_{\mu\nu} \) is defined according to the choice of fields, we determine the stochastic properties of \( \Omega \) (expectation value, variance, power spectrum, etc.) by considering the
$T_{\mu\nu}$ components in Equation (4) as classical stochastic fields, whose statistical properties are described by vacuum expectations (e.g. variance $\langle 0 | \Omega^4 | 0 \rangle - \langle 0 | \Omega^2 | 0 \rangle^2$). Because we are only considering the vacuum, and no excitations, the cosmological scenario being described is a simplified model consisting only of vacuum energy. If $T_{\mu\nu}$ contained contributions from all the fields in our universe, this would be approximately equivalent to studying our own universe in the current, dark-energy-dominated epoch.

The vacuum state is not an eigenstate of the local energy density and pressure operators in Equation (4), so measurements of these variables will fluctuate with a predictable spectrum. By modelling these fluctuations stochastically, $\Omega$ becomes a quasiperiodic function in space and time – meaning that its statistical properties are constant, but there is no fixed period $T$ for which $\Omega(t, x) = \Omega(t+T, x)$ or $\mathbf{X}$ for which $\Omega(t, x) = \Omega(t, x+\mathbf{X})$, as would be the case for a strictly periodic function.

Solutions to harmonic oscillator equations with time-dependent frequency, like this one, can exhibit long-term growth or decay, a phenomenon known as parametric resonance [13, 14]. A common example of parametric resonance occurs on a swing, when one straightens and bends one’s legs to increase the amplitude. Because of the linearity and symmetry of Eq. 3, it turns out that decaying solutions will be suppressed unless the initial conditions are fine-tuned, so that the long-term solution will either grow exponentially or remain steady. This means that the general solution can be written as,

$$a(x, t) \approx e^{H t} P(x, t), \quad (5)$$

where $H \geq 0$ is a constant and $P$ is a quasiperiodic function, by which we mean that all its statistical properites are time-independent, and it has time average $\bar{P} = 0$. Note that we use, for example, $\mathcal{P}$ to denote the time average of a variable, reserving $\langle P \rangle$ to denote the expectation of $P$ as a quantum operator. Note that $\frac{\dot{P}}{P} = \frac{d \log |P|}{dt}$, so since $\log |P| = \text{const.}, (\frac{\dot{P}}{P}) = 0$. This means that taking a time average of $\frac{\dot{a}}{a} = H + \frac{\dot{P}}{P}$ gives us $H = \frac{1}{2} \frac{d}{dt} \log \frac{P}{P_0}$. This leads to a natural interpretation of $H$ as the Hubble parameter, which is defined in cosmology as $\frac{\dot{a}}{a}$. If $H$ is zero, then there is no parametric resonance, because $a = P$ and $P$ has no long-term growth or decay. Otherwise, it will result in an exponentially increasing scalefactor, resulting in observed distances scaling with $L(t) = L(0) e^{H t}$, and macroscopic acceleration obeying $\frac{\dot{L}(t)}{L(t)} = H^2$.

Thus, the key goal is to determine $H^2$; if $H^2 \sim 1$, the model has done nothing to remedy the problem of the traditional approach, as it still predicts an acceleration 120 orders of magnitude too large. If $H^2 \sim 10^{-120}$, then this would indicate that the model predicts an appropriate order of magnitude for the acceleration, and has potential to resolve the cosmological constant problem.

IIIa. Timescales of Oscillation

Parametric resonance is strongest (i.e. growth or decay is most rapid) when the timescale of frequency oscillation and amplitude oscillation are similar – e.g. when one bends one’s legs with a frequency close to the frequency of the swing itself. It is therefore important to assess the conditions under which the variations in $\Omega$ are of a similar frequency to those of $a$, since that is when accelerating expansion will be strongest. This will also provide us with expectations of the limiting behaviour when the oscillations in $\Omega$ are much slower or faster than those of $a$. Because Equation (3) contains no spatial derivatives, we will omit the label $x$ and just consider a fixed point in space from now on.

Although neither $\Omega(t)$ nor $a(t)$ are strictly periodic, their variations still occur on somewhat consistent timescales, which we can use to test the strength of parametric resonance. $a(t)$ will typically vary with a frequency comparable to $\Omega_{\text{rms}} = \sqrt{\langle \Omega^2 \rangle}$, and as shown in [1], $\langle \Omega^2 \rangle = \frac{2 \sqrt{6}}{3} \Lambda^2$. This dependence agrees with Section II, which gave justification that $T_{\Omega a} = \rho_{\text{vac}}$ (and thus $\Omega^2$) should scale with $n_{\Omega} \Lambda^4$. Thus $a$ typically varies with frequency $\sim \sqrt{n_{\Omega} \Lambda^2}$, i.e. on a timescale of about $1/(\sqrt{n_{\Omega} \Lambda^2})$.

Analysis from [1] shows that the power spectrum of $\Omega^2$, on the other hand, is given by Figure 1 (independently of the number of fields). The field amplitude oscillates at all frequencies up to the cutoff, and $\Omega^2$ is proportional to the energy of the vacuum, which scales with the square of the field, so it will oscillate at up to twice the cutoff. With the average energy of each mode scaling with frequency as $\frac{1}{2 \omega}$, we expect that the modes with larger frequencies will dominate as they fluctuate the most violently – with lower frequencies being less significant, as Figure 1 confirms. It follows that the typical timescale for oscillations of $\Omega^2$ (or $\Omega$) will be on the order of $1/\Lambda$.

As mentioned, parametric resonance is strongest when these timescales are similar; specifically, if the frequency of $a$ is $\Omega$ and that of $\Omega^2$ is $\gamma$, then it is strongest when $r \equiv $.
in the cases of \( n_f = 1 \) to 3. Nonetheless, with more fields, we still expect an exponential decrease of \( H \) with respect to \( \Lambda \), providing a mechanism for \( H^2 \sim 10^{-120} \) as desired.

IV. NUMERICAL METHODS

In [1], numerical methods were employed to test Equation (6), which are also used here. We will outline the approach used, emphasising the role of resolution parameters with respect to which our results must converge, before showing detailed convergence tests.

We follow the Wigner-Weyl description of quantum mechanics as used in [1] to describe the vacuum stress-energy tensor, and by extension \( \Omega^2 \). Using this method, we define a pair of coordinates \( x_k \) and \( p_k \) for each mode of the field, indexed by momentum \( k \). These do not represent actual position and momentum coordinates (each mode has well-defined momentum and is completely localised), but instead represent the phase information of each mode.

A particular state is represented by a distribution over these variables, \( W(\{x_k\}, \{p_k\}, t) \), where \( \{x_k\} \) denotes the set \( \{x_{k_1}, x_{k_2}, \ldots\} \) with all possible momenta \( k \). Any quantum operator \( \hat{A} \) can be represented by a function over these variables, \( A(\{x_k\}, \{p_k\}, t) \), and its expectation for a state is given by integrating over the state’s corresponding distribution:

\[
\langle \hat{A} \rangle = \int \prod_k (dx_k dp_k) A(\{x_k\}, \{p_k\}, t) W(\{x_k\}, \{p_k\}, t).
\]  

(7)

In the case of the vacuum state, and using the normalised units from [1], the state distribution is a product of Gaussians:

\[
W(\{x_k\}, \{p_k\}, t) = \frac{1}{\pi} \prod_k e^{-p_k^2 - x_k^2}.
\]  

(8)
FIG. 4. The top two plots display the evolution of the scalefactor at the cutoff values $\Lambda$ used in [1]. The left uses time resolution $t_{\text{res}} = 0.15$ (i.e. the spacing between evaluations of $\Omega$), which erroneously indicates that $H$ decreases with $\Lambda$ (as shown in the bottom left). This is very similar to Fig 5 of [1]. On the right, a finer time resolution $t_{\text{res}} = 0.01$ is used, showing the correct relationship between $H$ and $\Lambda$ (which persists if time resolution is increased even further). This corrected relationship does not exponentially decay to zero as originally claimed, meaning that the cosmological constant problem cannot be resolved by simply taking the cutoff to be $\Lambda \sim 1000E_P$. 
It is quite difficult to numerically perform the integral in Equation (7), because there can be a very large number of modes (i.e., many values of \( k \)), meaning that this is an integral over many dimensions. Fortunately, there is an alternative method. If we randomly sample \( \{ x_k \} \) and \( \{ p_k \} \) from the distribution given by Equation (8), and then perform an average over the resultant solutions of \( A (\{ x_k \}, \{ p_k \}, t) \), the different regions of phase space will be appropriately weighted by their likelihood of being chosen. As the number of randomly sampled points \( N \) increases, the resultant value will converge to the true result from Equation (7).

Now, we can choose an operator \( A \) to evaluate. We wish to examine what happens on average to \( a (x, t) \) in Equation (3) at a single point in space over time. This means we must describe \( \Omega \) for \( N \) different choices for the sets of random numbers \( \{ x_k \} \) and \( \{ p_k \} \), solve for \( a (\{ x_k \}, \{ p_k \}, t) \), and then average the results to determine \( \langle a (t) \rangle \). Alternatively, one could apply Equation (7) to \( H \) instead of \( a (t) \), to compute the expectation value \( \langle H \rangle = \langle \frac{\hat{a} (t)}{\hat{a} (t)} \rangle \). We will discuss this further shortly.

The expression for \( \Omega^2 \) in terms of quantum operators contains contributions from the infinite continuum of allowed momenta values \( k \). Even if a cutoff energy (or equivalently, cutoff frequency) \( \Lambda \) is applied, there will still be continuously infinitely many modes to consider. To make it suitable for numerical calculation then, we need to discretise it, which can be done by considering a cube of width \( L \) in physical space, and restricting the allowed modes of our field to be only harmonic modes of the box. \( L \) is another parameter with respect to which our results should converge to a consistent, physical solution, specifically in the limit \( L \to \infty \). Harmonic modes in this box are proportional to \( \sin \left( \frac{n_x \pi x}{L} \right) \sin \left( \frac{n_y \pi y}{L} \right) \sin \left( \frac{n_z \pi z}{L} \right) \), for some set of integers \( (n_x, n_y, n_z) \) (each of which can be positive or negative) that we call \( n \). The corresponding frequency is \( \omega = \frac{2 \pi |n|}{L} \), so we can translate the cutoff \( \omega \leq \Lambda \) to a cutoff on \( n \) by \( n_{\text{max}} = \frac{L \Lambda}{2 \pi} \). In [1], this cutoff was applied to each component, i.e., \( n_{x,y,z} \leq n_{\text{max}} \). Whereas this would signify a cube of allowed modes in momentum space, with side length \( 2 \Lambda \) and maximum frequency \( \sqrt{3} \Lambda \), we instead apply the cutoff as a sphere in momentum space of radius \( \Lambda \) by choosing modes with \( |n| < n_{\text{max}} \). Now, our sets \( \{ x_k \} \) and \( \{ p_k \} \) are labelled as \( \{ n \} \) and \( \{ p_n \} \), and they each contain one random number for every value of \( n = (n_x, n_y, n_z) \) such that \( |n| < n_{\text{max}} \).

For a particular cutoff method, [1] shows that we can write \( \Omega^2 \) for a single massless scalar field as:

\[
\Omega^2 (\{ x_n \}, \{ p_n \}, t) = \left[ \sum_n \sqrt{n} (x_n \sin (nt) - p_n \cos (nt)) \right]^2.
\]  

(9)

The above just describes the process for a single massless scalar field. To incorporate more, it is repeated for each individual \( \Omega_j^2 \) and then the total is computed as \( \Omega^2 = \sum_{j=1}^{n_i} \Omega_j^2 \).

After randomly sampling \( \{ x_n \} \) and \( \{ p_n \} \) values for each field and computing \( \Omega^2 \) at a number of points in time (with spacing \( t_{\text{res}} \) up to a maximum \( t_f \), two more parameters to test for convergence), the differential equation in Equation (3) is solved for \( a (t) \) by interpolating \( \Omega^2 \). Doing this \( N \) times, either \( \langle \frac{\hat{a} (t)}{\hat{a} (t)} \rangle \) or \( \langle \hat{a} (t) \rangle \) can be determined from Equation (7), and then a time average taken to find \( H \).

**IVa. Correction to Previous Results**

When implementing these methods, we found substantially different results to those of [1], as shown in Figure 4. Investigation showed that due to a combination of factors, the original calculations did not properly capture the dynamics of \( \omega \) at fine enough timescales. When computing \( \Omega^2 \) from Equation (9), the spacing \( t_{\text{res}} \) must be at least as small as the timescale on which we expect oscillations in \( \Omega \) to occur—otherwise the numeric description of \( \Omega \) will not display the high-frequency behaviour of the actual function (which is particularly significant for parametric resonance, as discussed in the previous section). This was the key problem with the original calculations: \( t_{\text{res}} \) was too large to have sufficiently converged. Furthermore, it was not made finer for higher \( \Lambda \), so more and more of the significant short-timescale behaviour was lost for higher \( \Lambda \). Finally, a simple linear interpolation method was used rather than a smooth method when determining \( \Omega \), which exacerbated the resolution problem (see Fig. 10). The impact of these differences on the resultant scalefactor is shown in Figure 4, which shows that the relationship between \( H \) and \( \Lambda \) is drastically affected. We discuss the implications of these changed results in Section V.

**IVb. Convergence Tests**

Having found that the discrepancy between our results and those of [1] was due to different time resolution parameter values, we sought to validate that no other resolution parameters were being overlooked. Let us recap the roles of the relevant parameters: we generate \( N \) instances of the random sets \( \{ x_n \} \) and \( \{ p_n \} \), which each contain a random number for every integer vector \( n \) with magnitude \( |n| < \frac{L \Lambda}{2 \pi} \), where \( L \) is the size of the box and \( \Lambda \) is the maximum frequency permitted. These produce a frequency function \( \Omega (t) \) using Equation (9), which we evaluate at evenly spaced points between \( t = 0 \) and \( t = t_f \), with spacing \( t_{\text{res}} \). Then, we interpolate between those points to solve the differential equation \( \ddot{a} (t) = -2 \Omega^2 (t) a (t) \) with initial conditions
(a(0), ˙a(0)) = (1, 0). Averaging over the N different samples, we then determine our average expansion rate H.

There are five variables here with respect to which our results should converge: the box width L, the final time t_f, the time resolution t_{res}, the number of samples being averaged N, and the relative tolerance of the ODE solver, which we will denote ε. There are also several qualitative choices which may affect the results: whether the cutoff should be implemented as a cube or sphere in momentum space, how to interpolate Ω when solving the differential equation Equation (3), and how to determine H given the solution a(t). We present discussions for each of these in Appendix B, except for that of H which we present now.

IVc. New Method of Determining H

In our new tests, we made a number of changes to the implementation to improve the efficiency. Most of these did not represent physical differences in what was being computed, but one exception is the method of determining H. Physically, the Hubble Constant H is defined as \( H = \frac{v}{d} \), in which v is the radial outwards velocity of a remote astronomical object and d is its distance to the earth. To determine v and d one needs to measure at least two properties, the redshift of a galaxy as well as an independent measure of its distance, such as the luminosity of a type Ia supernova, or the length of a standard ruler. So in principle, we need to study the behavior of a long wave photon field propagating on our wildly fluctuating metric to determine H. Technically, we need to solve the wave equation in our inhomogeneous “FLRW” metric (23) in [1]:

\[
\nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0 \tag{10}
\]

Unfortunately, this is a non-trivial calculation which is beyond the scope of this article. The usual definition of H in cosmology, i.e. \( H = \frac{\dot{a}}{a} \), is not in the homogenous FLRW metric. For the generalized inhomogeneous FLRW metric (23) in [1], we can have a similar definition as:

\[
H(t) = \frac{\dot{L}}{L} = \frac{\int_{x_1}^{x_2} \frac{\dot{a}}{a}(t, x) \sqrt{a^2(t, x)} dx}{\int_{x_1}^{x_2} \sqrt{a^2(t, x)} dx} \tag{11}
\]

The macroscopic Hubble constant is acquired by taking both the spatial and temporal average on both sides of (11), as well as the average in the phase space using the Wigner-Weyl representation to get its quantum expectation value.

In [1], the expectation value of the scale factor \( \langle a(t) \rangle \) is determined first by Wigner-Weyl formulation, and then H is calculated as the time average of \( H(t) = \frac{\dot{a}(t)}{a(t)} \). However, we can also change the sequence of averaging and directly compute the expectation value of H, by using Equation (7) to calculate \( H(t) = \langle \frac{\dot{a}(t)}{a(t)} \rangle \). In this way, we actually define H as the time average

\[
H = \langle \frac{\dot{a}(t)}{a(t)} \rangle \tag{12}
\]

It is more physical compared to the original case in [1] since the scale factor \( a(t) \) (being an arbitrary distance scale) is less fundamental than the actual distance between objects. Given that \( \dot{a}/a \) is also equivalent to \( \frac{d \log |a|}{dt} \), this choice means that an average is computed in logarithmic space with respect to a, rather than linear space. Not only does this method lead to a different value for H which is physically better justified, but computation of this value is also much easier and more stable, as shown in Figure 5. This is because the linear method is heavily biased towards the samples with the largest H, resulting in high sensitivity to the occasional outlier, so it has much slower convergence. Instead, the logarithmic method (averaging H) quickly converges to a consistent result about which the distribution of individual samples appears to be roughly symmetric.

As we mentioned before, our new definition of H, which is based on the distance definition (11), is not necessarily equivalent to the observed Hubble constant in astronomy. The observed Hubble constant should be acquired by solving the Equation (10) for the actual redshift and intensity damping of a macroscopic light signal. However, we believe that the calculation of H based on Equation (12) can still provide useful insight about how the actual Hubble Constant behaves in this metric.

V. NEW RESULTS

In Figure 6, we see that the relationship between \( \Lambda \) and H is quite complex, with the behaviour of the curve...
FIG. 6. Here we see the relationship between $H$ and $\Lambda$ with $n_f$ fields. On the left it is shown as a log-log plot, so that all the regimes can be seen at once. On the right, the top shows more clearly the linear increase for one field and the logarithmic increase for two (in linear space). The bottom right shows the beginning of the turning point for three fields, and exponential decay for several higher $n_f$ cases (which Equation (6) predicted for all numbers of fields). Note that we have used $\sqrt{n_f}\Lambda$ on the $x$-axis for each of these rather than just $\Lambda$, because this is the term on which the adiabatic limit depends, as well as the resonances described in Section III.
depending significantly on the number of fields. Figure 7 then shows the relationship between $H$ and $n_t$ for several choices of fixed $\Lambda$, to examine what happens to $H$ if we enforce an approximate Planck cutoff ($\Lambda \sim 1$) and then vary the number of fields. We will first compare the findings of Figure 6 to the proposed relationship Equation (6), and check that the limiting behaviours predicted in Section III are satisfied. After this, we will introduce a model which captures important features of the behaviour of $H$ vs $n_t$, shown in Figures 7 and 8, and use it to estimate the number of scalar fields required to achieve $H \sim 10^{-60}$, such that $\lambda_{\text{eff}} \sim 10^{-120}$.

As explained in Section III, we expect that $\lim_{\Lambda \rightarrow 0} H = 2\Omega_0$, which seems to hold in all cases. Equation (6) also predicts that, for a given number of fields and at large enough $\Lambda$, there will be an exponentially decreasing relationship $H \sim e^{-\beta H}$. On the left, in log-log space, such a relationship appears as $\log(H) \sim -\beta \log(\Lambda)$, which upon inspection, seems to match the large-$\Lambda$ behaviour for $n_t \geq 4$. As discussed in Section III, this relied upon the adiabatic theorem which is only valid when $n_t \gtrsim 4$. Indeed, this trend does not seem to hold for $n_t < 4$, (the behaviour for $n_t = 1$ at large $\Lambda$ appears to be linear, and for $n_t = 2$ and 3 it appears to be logarithmic). While there may be some turnaround at higher $\Lambda$ (and the linear behaviour of $n_t = 1$ may become logarithmic at some higher $\Lambda$), this does not occur in the regime checked, which is up to $\Lambda \approx 1500$.

Note that there are peaks corresponding to those predicted in Section III, near $\sqrt{m_0\Lambda} \approx 4$, and a weaker one near 8. These resonances draw a direct parallel with the behaviour of Mathieu’s equation, a simple prototypical example of parametric resonance [13, 16]. Mathieu’s equation takes the following simplified form, in which the fluctuations to $\Omega^2$ are strictly periodic with constant amplitude:

$$a(t) = -\Omega_0^2 (1 + \epsilon \cos(\gamma t)) a(t).$$

Solutions of Mathieu’s equation take the following form, similar to the right hand side of Equation (5):

$$a(t) = e^{Ht} P(t),$$

for $P(t) = P \left( t + \frac{2\pi}{\gamma} \right)$. (13)

$H$ can either be real (an unstable solution with exponentially growing solutions) or imaginary (representing stable quasiperiodic solutions with no long-term growth or decay). The stable and unstable regions depend on $\epsilon$ and $r = \frac{2\Omega_0}{\gamma}$, as shown in Figure 9. Although no closed-form expressions exist, there are efficient methods of computing both the region boundaries and the magnitude of the exponents [16]. For our purposes, it will be beneficial to approximate Equation (3) using the above form to obtain an approximate model in terms of the simpler, better-understood Mathieu equation. Full details are shown in Appendix C, showing that we obtain $\Omega_0^2 = \langle \Omega^2 \rangle = \frac{n_t \Lambda^4}{6\pi}$, $\epsilon = \sqrt{\text{Var}(\Omega^2)} / \langle \Omega^2 \rangle = \sqrt{2/n_t}$, and $\gamma$ taking on a range from 0 to $2\Lambda$ according to Figure 1. With fixed $\epsilon$ and $r$ varying as $r = \frac{2\Omega_0}{\gamma} = \frac{2\Lambda^2}{\gamma} \sqrt{\frac{\pi}{n_t}}$, each choice of $n_t$ and $\Lambda$ excites a range of resonances from $\gamma = 0$ to $\gamma = 2\Lambda$ as indicated by the dashed line in Figure 9. Our approximation is to select out the $\gamma$ with the most significant parametric resonance effect, weighted by the strength it oscillates at according to Figure 1.

Using these methods, we obtain the dotted lines shown in Figure 7, which capture many of the key properties (e.g. existence and size of the “steps” that arise as a result of resonance). This method explains the step-like behaviour of Figure 7, because these “steps” occur when a resonance band leaves the region of allowed $\gamma$ (e.g. when the dashed black line in Figure 9 moves high up enough that it does not cover the second band). These methods can also explain why we see divergence as $\Lambda$ increases for $n_t = 1$ through 3. Looking at the top of Figure 9, one and two fields correspond to $\epsilon > 1$, and in this region the higher-order bands (further from the origin) have a larger amplitude. In these cases, as $\Lambda$ increases and the ratio between frequencies of oscillation for $a$ and $\Omega$ increases, the parametric resonance effect gets stronger and $H$ diverges. At lower numbers of fields, $\epsilon$ decreases and the trend reverses: on the left of Figure 9, increasing $r$ leads to exponentially weaker resonance, and $H \rightarrow 0$. At high $\Lambda$, the model predicts logarithmic divergence for $1 \leq n_t \leq 3$, and an asymptotically uniform $H$ for $n_t = 4$.

In some regions this model clearly does not fit as well as others. It approximates $\Omega^2$ as only oscillating at one frequency, and simplifies $a$ by ignoring any squared frequencies outside of the range $\Omega^2 \pm \epsilon$. The latter approximation explains why the model fails at low $\Lambda$ and $n_t$ in Figure 8, and for low $n_t$ when $\Lambda = 0.1$ in Figure 7: in these regimes, none of the frequencies that $a$ oscillates at excite resonances, and it is in fact oscillations outside this range which drive the dominant resonances. On the other hand, when the number of fields increases, the power spectrum for $a$ (see Figure 2) becomes much narrower, and the Mathieu model is a better description. This explains the very tight fit for $n_t = 100$ in Figure 8. The approximation that $\Omega^2$ oscillates only at one frequency fails at low fields for the same reason that the adiabatic limit does: in these cases, the distribution of $\Omega^2$ values is too broad (see Figure 2). This is why the model does not fit as well for 1 and 4 fields in Figure 8 at high $\Lambda$.

The advantage of this model is that compared to the full simulations, it is much easier to calculate for small $H$. Even though these methods are still restricted by machine precision to $H \sim 10^{-16}$, the trends are consistent and can be extended all the way down to $H = 10^{-60}$ so that we can test what cutoff and number of fields would be required to match observation. If the trend for $\Lambda = 1$ continues as shown, then $H = 10^{-60}$ will be achieved with $n_t \approx 6000$. Similarly, extending the $n_t = 28$ line (because 28 is the number of bosonic field components in the Standard Model), we get $H = 10^{-60}$ (i.e. we match observation) when $\Lambda \approx 40$ (i.e. cutoff at 40 times the Planck energy).
FIG. 7. Here we see $H$ against $n_f$ for fixed cutoff, which exhibits a decreasing step-like relationship. The dotted lines represent the results of an approximation described in the text which allows an analytical prediction of the behaviour of $H$. We see that although they do not precisely match the results, these approximations do predict the existence and approximate size of the periodic steps downward, and the relationship between the two curves. For low numbers of fields, the error is dramatic, but the fit improves as $n_f$ increases, because the approximation that $\Omega^2$ is roughly constant vastly improves.

FIG. 8. A similar model to that of Figure 7 is shown in dashed lines, and some of the simulation results from Figure 6 are shown in solid lines. Again, the results are not matched precisely, but the fit is quite good as $n_f$ increases, because the approximation that only one resonance contributes significantly becomes vastly more accurate. At low $\Lambda$, resonances no longer occur near the mean of $\Omega^2$, and the approximation also worsens.
VI. DISCUSSION AND CONCLUSION

We will now review the typical assumptions that are made in the usual formulation of the cosmological constant problem (which we refer to as the “traditional approach”), in order to provide a framework with which we can discuss the significance of our new findings. In Section II, we described the problem using a simple case with a single scalar field, but its conclusions hold in a much broader range of contexts. We summarise the key assumptions (as relating to calculations of vacuum energy) before discussing them in further detail:

1 The total effective cosmological constant $\lambda_{\text{eff}}$ is on at least the order of magnitude of the vacuum energy density generated by zero-point fluctuations of particle fields.

2 QFT is an effective field theory description of a more fundamental, discrete theory, which becomes significant at some high energy scale $\Lambda$.

3 The vacuum stress-energy tensor is Lorentz invariant.

4 The Møller–Rosenfeld approach to semiclassical gravity (using an expectation value for the energy-momentum tensor) is sound.

With these assumptions, one arrives at the usual value of $\lambda_{\text{eff}} \sim 1 \sim 10^{120}\lambda_{\text{obs}}$. However, it has been noted [4, 17–19] that there is an inconsistency between Assumptions 2 and 3: the vacuum state cannot be Lorentz invariant if modes are ignored above some high-energy cutoff $\Lambda$, because a mode that is high-energy in one reference frame will be low-energy in another appropriately boosted frame.

In the new approach proposed by [1], Assumption 3 is not used and this contradiction is avoided, and Assumption 4 is modified, which we denote as Assumption 4'. Their simple toy model also introduces Assumption 5:

4' The **semiclassical stochastic approach** to gravity (using a stochastic field for the energy-momentum tensor) is sound.

5 The universe can be effectively modelled by a single massless scalar field.

VIa. Different contributions to $\lambda_{\text{eff}}$

Assumption 1 is well justified in the case of the traditional problem, because the contribution from zero-point fluctuations is on the order of 1 in Planck units and no other known contributions are as large [4] – thus, assuming no significant cancellation of terms (e.g. fine tuning of the bare cosmological constant $\lambda$), the total $\lambda_{\text{eff}}$ should be at least on the order of the largest contribution. In the case of the new approach introduced in [1] and used here, this assumption is also reasonable: any other contributions would also presumably fluctuate and result in similar effects to what we have found here.

VIb. Effective Field Theory and Lorentz Invariance

To prevent the vacuum energy density from diverging, the traditional approach also assumes that performing a high-energy cutoff is acceptable. This type of regularisation is a common step in renormalisation procedures, which aim to eventually arrive at a physical, cutoff-independent result. However, in the case of the vacuum energy density, the result is inherently cutoff-dependent, scaling quartically with the cutoff.

This is acceptable under the philosophy of Assumption 2, which treats QFT as a low-energy effective field theory and not a fundamental theory. This approach
draws parallels with the case of the ultraviolet catastrophe: the equipartition theorem (a key feature of classical physics) made a rapidly divergent prediction when high-energy modes were considered, but a new high-energy theory (quantum mechanics) resolved this problem, and showed classical mechanics to be only an effective low-energy theory. Similarly, it is presumed here that a high-energy discrete theory would not display the zero-point fluctuations that are characteristic of QFT, and hence that the divergence caused by oscillations above the corresponding cutoff frequency is unphysical. In this case, the cutoff is no longer an intermediate mathematical construct, but instead a physical scale at which the smooth, continuous behaviour of QFT breaks down.

Although it is naturally difficult to speculate about a non-existent theory, it is generally believed that such a theory would emerge at a scale comparable to that of the Planck energy [20]. Several theories describe a spacetime made of “quantum foam” which violates Lorentz invariance at very high energy scales [21–23], which would imply that the vacuum (which is dominated by these high energy modes) need not be Lorentz invariant, justifying the relaxation of Assumption 3.

This relaxation of Lorentz invariance is crucial to the new approach: as discussed in Section II, Lorentz-invariance would require $T_{00} = -T_{ii}$ for $i = 1, 2, 3$ (i.e. if energy density is positive, pressure is negative), which from Equation (4), would prevent $\Omega^2$ from being positive definite and exhibiting the harmonic oscillator behaviour that we describe. One caveat is that fermionic and massive fields contribute negative terms to $\Omega^2$, meaning that perhaps a Lorentz-invariant cutoff in combination with such fields would result in an $\Omega^2$ which is positive once more, and can exhibit parametric resonance. This is an avenue for possible future exploration.

VIId. Number and Type of Fields

The traditional approach as presented in Section II used only a single scalar field with no interactions, but this is not crucial to its conclusions because adding more fields simply combines the uniform energy density linearly, which has a linear effect on the cosmic acceleration. A full description of the real vacuum requires bosonic and fermionic fields (the latter of which contribute negatively to the vacuum energy), with all the interactions of the Standard Model. An important goal of this work has been to begin to relax Assumption 5 by incorporating a greater number of fields.

A single field predicts a similar outcome in the new approach as it did in the traditional approach: $H \sim 1$ with a Planck scale cutoff, and it diverges as the cutoff is increased (see Figure 6). However, with the new approach, adding more fields no longer worsens the problem, but instead dramatically ameliorates it! As the number of fields increases, the relative magnitude of the fluctuations to the energy density tends to cancel out, decreasing as the inverse root of the number of fields (as per the central limit theorem). This causes parametric resonance to weaken, and the resultant acceleration to become smaller and smaller.

In our tests, numerical instability became more significant than the growth from $H$ below about $H \approx 10^{-6}$, i.e. when the acceleration $H^2$ is about 12 orders of magnitude smaller than the traditional approach. Because of the exponential relationship between $H$ and $\Lambda$ (which only begins past about $\sqrt{\eta_1} \Lambda \approx 6$), increasing the cutoff or number of fields marginally beyond this point would result in dramatically smaller acceleration, approaching the observed value $H^2 = \frac{1}{2} \sim 10^{-120}$. We found using a simple model based on the Mathieu equation, that if these trends continue then we can expect $H$ to match observation when (for example) $n_1 = 28$ and $\Lambda = 40$, or when $n_1 = 6000$ and $\Lambda = 1$.

Of course, our description has still been restricted to massless scalar fields, and is not a complete description of the real universe. Our description is actually sufficient for bosonic fields, even if they are not scalar and massless. Introducing a mass adds a term of the form $-m^2c^2$ to Equation (4), which can result in $\Omega^2$ becoming negative. But the masses of all observed particles are vastly smaller than the Planck scale, meaning this correction will have an insignificant effect on the dynamics. Furthermore, even if a boson is not a scalar, but rather, has polarisation modes like the photon, then each component still contributes to the vacuum in a manner like that of an individual scalar field. Given the large number of bosonic field components in the Standard Model [24], this amounts to a significant number of fields that our model is able to describe.

Nonetheless, this description is not sufficient for describing fermionic fields, or interactions between fields. Fermionic fields contribute to the vacuum energy negatively, with the same magnitude as bosonic fields. With
a number of fermionic fields $n_F$ and bosonic fields $n_B$, the mean $(\Omega_0^2)$ would become $\frac{(n_0-n_F)\Lambda^4}{b^2}$, while the variance remains related to the total number of fields (as adding more fields cannot reduce variance): $\text{Var}(\Omega^2) = \frac{n_B + n_F}{\Lambda^2} \Lambda^4$. Thus we expect that the relative magnitude of the fluctuations will become larger with a mix of bosonic and fermionic fields, increasing the strength of parametric resonance and making it harder to reach the observed $H$.

With developments to our analytical description of parametric resonance, one could relate $n_F$ and $n_B$ to corresponding values of $H$, allowing a relationship between the observed $H$ and the number of fields. Because these numbers must obviously be integers, there would be a kind of “quantisation” of allowed $H$ values, providing both a test for this theory and a method of relating $H$ to the number of particle fields in the universe – potentially probing dark matter fields, supersymmetric fields, etc.

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**Appendix A: Probability Distribution of $\Omega^2$**

As explained in Section III, the probability distribution of $\Omega^2$ is very important in determining the validity of the adiabatic limit. To evaluate the probability distribution, we appeal to the Wigner formulation as described in Section IV. To start with we will follow [1], for which the calculations are just for one field. The Weyl transform of the non-dimensionalised $\tilde{\Omega}^2 = \frac{X^2}{s^2} \Omega^2$ operator is given by Eq. (B31) of [1]. In the chosen non-dimensionalised units used there, $\tilde{x}$ and $\tilde{p}$ are standardised normal random variables, $X(0,1)$ (we will use the notation that $X(\mu, s^2)$ is a random variable sampled from the normal distribution with mean $\mu$ and variance $s^2$, and make use of the properties $cX(0, s^2) = X(0, c^2 s^2)$ and $X(0, s^2) + X(0, s'^2) = X(0, s^2 + s'^2)$ where each variable is independent). Then:

$$\tilde{\Omega}^2 = \left( \sum_{\vec{n}} \sqrt{n} (\tilde{x}_\vec{n} \sin \theta \tilde{t} - \tilde{p}_\vec{n} \cos \theta \tilde{t}) \right)^2$$

$$= \left( \sum_{\vec{n}} \sqrt{n} (X(0,1) \sin \theta \tilde{t} - X(0,1) \cos \theta \tilde{t}) \right)^2$$

$$= \left( \sum_{\vec{n}} \sqrt{n} (X(0, s^2) \sin \theta \tilde{t} - X(0, s^2) \cos \theta \tilde{t}) \right)^2$$

$$= \left( \sum_{\vec{n}} \sqrt{n} (X(0, s^2) \cos \theta \tilde{t} + \cos \theta \tilde{t}) \right)^2$$

$$= \left( \sum_{\vec{n}} \sqrt{n} (X(0, s^2) \cos \theta \tilde{t} + \cos \theta \tilde{t}) \right)^2$$

$$= \left( \sum_{\vec{n}} \sqrt{n} (X(0, s^2) \cos \theta \tilde{t} + \cos \theta \tilde{t}) \right)^2$$

$$= \left( X(0, s^2) \right)^2$$

$$= \left( \sum_{\vec{n}} n \right) X(0, s^2)$$

$$= \left( \sum_{\vec{n}} n \right) X(0, s^2)$$
Thus, generalising to \( n_t \) fields, we have:

\[
\hat{\Omega}^2 = \left( \sum_{n}^{n_{\text{max}}} n \right) \sum_{i=1}^{n_t} X(0,1)^2
\]

(A8)

\[
\check{\Omega}^2 = \left( \sum_{n}^{n_{\text{max}}} n \right) \chi_{n_t}^2
\]

(A9)

Where we used the definition of \( \chi^2 \) as the sum of \( k \) standard normal random variables. To compute the sum over \( \tilde{n} \), we have (for a spherical cutoff, see next section),

\[
\sum_{\tilde{n}}^{n_{\text{max}}} n \approx \int_0^{n_{\text{max}}} n d\tilde{n} = \pi n_{\text{max}}^4 = \pi \left( \frac{L}{2\pi} \right)^4.
\]

Computing \( \hat{\Omega}^2 \) now (noting that \( L \) drops out, as it should):

\[
\check{\Omega}^2 = \frac{8\pi^2}{3L^4} \Omega^2
\]

(A10)

\[
\check{\Omega}^2 = \frac{8\pi^2}{3L^4} \left( \frac{L}{2\pi} \right)^4 \chi_{n_t}^2
\]

(A11)

\[
\check{\Omega}^2 = \frac{\Lambda^4}{6\pi} \chi_{n_t}^2
\]

(A12)

Appendix B: Convergence Tests

Before beginning discussion of convergence, we must discuss our desired precision for determining \( H(\Lambda) \). Some parameters resulted in relative uncertainty, while others give absolute uncertainty values. In order to see overall trends in \( H \) with confidence, we aimed for 1% uncertainty in \( H \) from each parameter, or an absolute precision of \( 10^{-6} t_f^{-1} \), whichever was higher (here we reintroduce the unit of the Planck time \( t_f \)).

1. Cutoff Method

Whereas a cubic cutoff was used in [1], i.e. each component \( i \) satisfies \( |n_i| < n_{\text{max}} \), we used a spherical cutoff \( n = |n| < n_{\text{max}} \). This difference does not affect the results greatly, except that it slightly modifies the effective \( \Lambda \) being tested – with a cubic cutoff, the highest actual frequency is \( \sqrt{3} \Lambda \) instead of \( \Lambda \) itself.

2. Interpolation method and \( t_{\text{res}} \)

The method of interpolation turns out to be crucially important for convergence, in particular when a larger \( t_{\text{res}} \) is being used. We found, as shown in Figure 10, that of three inbuilt MATLAB interpolation methods (linear method, pchip method, and spline method), a spline interpolation converged most quickly. It appears that the salient feature of the spline method which gives this advantage is that it extends past the upper and lower extremes of the sample points, increasing the magnitude of fluctuations of \( \hat{\Omega} \), as seen in the upper left panel of Figure 10. The other methods underestimate the deviations to \( \check{\Omega} \), which typically results in a weaker parametric resonance effect, as seen in the lower panel of Figure 10. Because oscillations of \( \check{\Omega}^2 \) occur on a timescale of \( 1/\Lambda \), as discussed in Section III, \( t_{\text{res}} \) should be fixed in proportion to this time period. From Figure 10, we see that setting \( t_{\text{res}} = 1/3\Lambda \) is sufficient for uncertainty to remain within 1%.

3. Dependence on ODE solver tolerance

The ODE solver being used, MATLAB’s \texttt{ode45}, accepts a choice of relative tolerance, which we denote \( \varepsilon \). This represents the acceptable relative error in the solution per unit time, relative to its own magnitude, so it is another parameter we can tune to maximise accuracy and computational efficiency. Within the accepted tolerance range, the amplitude of \( a(t) \) may deviate from its true value (typically, it will decrease) by a fairly consistent factor each cycle, which we call \( r \) (defined as a ratio, i.e. a perfect solution would have \( r = 1 \)). Thus \( \log |a| \) is misestimated by an increment of \( \log(r) \) per cycle, which means that as time goes on, our estimation of \( \log |a| \) will linearly deviate from its true value with time. Because \( H \) is calculated as the slope of \( \log |a| \), the effect of this numerical artifact will be to modify the observed \( H \) by a constant \( \Delta H \) compared to the correct result. As the number of cycles increases, i.e. when \( \sqrt{\mu} \Lambda^2 \) increases, this will occur more quickly, so we need a smaller tolerance. For this reason, we choose the parametrisation:

\[
\varepsilon(\Lambda) = \frac{10^{-\varepsilon_0}}{\sqrt{\mu} \Lambda^2}
\]

(B1)

and investigate the dependence of \( H \) on \( \varepsilon_0 \). In Figure 11, in the lower right, this dependence is displayed for a number of cutoffs, and we see that \( \varepsilon_0 = 4 \) (i.e. \( \varepsilon = 0.0001 \)) is enough to constrain \( |\Delta H| < 10^{-6} t_f^{-1} \).

4. Dependence on duration of simulation and number of samples

The duration of simulation and the number of samples are closely linked – both result in an approximate linear increase in computational difficulty (in both parts of the calculation: determining \( \Omega \), in which there will be linearly more timesteps or modes needed for calculation; and for determining \( a(t) \) from \( \Omega \), because of the number and length of differential equations needing to be solved increasing linearly). Furthermore, both result in an inverse-square-root relationship between uncertainty in \( H \) and size of \( N \) or \( t_f \) respectively. This is because the total number of timesteps being computed, \( "n_t" \), is proportional to \( N t_f \), and the uncertainty in estimating the average \( H \) across all timesteps can be computed using the usual formula, \( \Delta H = \sigma / \sqrt{n_t} \), where \( \sigma \) is the standard deviation. As shown in Figure 11, 100 samples with \( t_f = 50000 \Lambda^{-1} \) is sufficient to constrain \( \Delta H / H < 1\% \).
FIG. 10. Here we see the effects of three different interpolation methods for $\Omega$. In the top left panel, the true variation in $\Omega^2$ is contrasted with three interpolation methods: linear, pchip, and spline. Here $\Lambda = 5$, and the resolution is very coarse ($t_{\text{res}} = 0.2$) to exaggerate the effect. The resultant solutions of $a(t)$ are shown in the bottom left. On the right, the resulting error in $H$ is shown for each interpolation method, for a range of time resolutions and two cutoff values ($\Lambda = 5$ above and $\Lambda = 10$ below). For both cutoffs, the results converge much more quickly for the spline method, indicating that $t_{\text{res}} = 1/3\Lambda$ is sufficient to constrain uncertainty within 1%.
5. Dependence on width of box

Finally, let us consider the dependence on the width of the box $L$. The error for low $L$ stems from the way in which the modes are discretised in $n$ space. The sphere of allowed modes for a given field has volume given by

$$\frac{4\pi n^3_{\text{max}}}{3} = \frac{\Lambda^3}{3\pi^2},$$

and because the modes are spaced as an integer lattice, the number of modes should approximate this volume. At low $n_{\text{max}}$, the difference between the actual number of modes and the volume of the sphere in $n$-space becomes significant, but the approximation improves for larger $n_{\text{max}}$. This means that the accuracy improves for both higher $L$ and higher $\Lambda$, as is shown in Figure 12, and $L\Lambda = 50$ is sufficient for convergence within a few percent (note that this graph also includes the error from $t_f$ and $N$, so it will not completely converge as $L \to \infty$).

Appendix C: Mathieu’s Equation

We wish to use $\Omega^2 = \Omega^2_0 (1 + \varepsilon \cos \gamma t)$, from Equation (13), as an approximation to Equation (4). There is obviously some choice about how to implement this, but we will start by ensuring that the variance and mean of the two $\Omega^2$ functions agree. First, let us evaluate these

$$f(\gamma) = -\frac{16\pi^2}{\Lambda^4} \int_0^\Lambda \frac{\text{d}^3k_1 \text{d}^3k_2}{(2\pi)^3} \sqrt{\omega_1 \omega_2} \frac{\cos \gamma t \sqrt{\omega_1 \omega_2}}{2} \delta(\gamma - \omega_1 - \omega_2)$$

On the vacuum, $\langle f(\gamma) \rangle = 0$ so $\langle \Omega^2_t \rangle = \Omega^2_{0,1}$, and:

$$\text{Var}(\Omega^2_t) = \Omega^4_{0,1} \left( \langle 1 \rangle + \langle \left( \int_0^{2\Lambda} f(\gamma)^2 \right) \right) - \Omega^4_{0,1}$$

$$= \Omega^4_{0,1} \left( \langle \int_0^{2\Lambda} f(\gamma)^2 \right)$$

This expectation value simplifies to exactly 2, i.e.:

$$\text{Var}(\Omega^2_t) = 2\Omega^4_{0,1}$$

Now, this was for one field, but because multiple fields act as multiple identical and independent variables identical to $\Omega^2_t$, we get more generally:

$$\Omega^2 = \langle \Omega^2 \rangle = n_t \Omega^2_{0,1} = \frac{n_t \Lambda^4}{6\pi}$$

for Mathieu’s equation:

$$\langle \Omega^2 \rangle = \Omega^2_0$$

$$\text{Var}(\Omega^2) = \Omega^4_0 \left( 1 + 2\varepsilon \langle \cos \gamma t \rangle + \varepsilon^2 \langle \cos^2 \gamma t \rangle \right) - \Omega^4_0$$

$$\text{Var}(\Omega^2) = \frac{\Omega^4_0 \varepsilon^2}{2}$$

Thus we can determine $\varepsilon$ for our approximation by setting $\varepsilon = \sqrt{2\text{Var}(\Omega^2)}$. As shown in [1], with just one field, Equation (4) can be written in the form:

$$\Omega^2_t = \Omega^2_{0,1} \left( 1 + \int_0^{2\Lambda} f(\gamma) \cos \gamma t + g(\gamma) \sin \gamma t \text{ d}\gamma \right),$$

where $\Omega^2_{0,1} = \frac{\Lambda^4}{6\pi^2}$, and $f$ and $g$ are operator-valued functions.

We can exploit the fact that the expectation values and statistical properties of $\Omega^2$ are invariant under time translations to select $t = 0$ for the sake of determining variance, etc. Then we only need $f(\gamma)$:

$$f(\gamma) = \frac{16\pi^2}{\Lambda^4} \int_0^\Lambda \frac{\text{d}^3k_1 \text{d}^3k_2}{(2\pi)^3} \sqrt{\omega_1 \omega_2} \frac{\cos \gamma \sqrt{\omega_1 \omega_2}}{2} \delta(\gamma - \omega_1 - \omega_2)$$

$$= \Omega^4_{0,1} \left( \langle \int_0^{2\Lambda} f(\gamma)^2 \rangle \right)$$

Var($\Omega_{n_0}^2$) = $n_t$ Var($\Omega_1^2$)

Var($\Omega_{n_1}^2$) = $2n_t$ Var($\Omega_{0,1}^2$)

$\varepsilon = \frac{\sqrt{2 (2n_t \Omega_{0,1}^2)}}{n_t \Omega_{0,1}^2} = \frac{2}{\sqrt{n_t}}$

With these values set, then, we have $r = \frac{2\Omega_{0,1}^2}{\gamma \sqrt{6\pi^2}}$, for a variety of $\gamma$ values between 0 and $2\Lambda$ as per Figure 1. At the highest $\gamma$, this corresponds to $r = \Lambda \sqrt{\frac{\pi}{6\pi}}$. To make it into the form of Equation (13), we should choose the “most important” $\gamma$ and then replace $f(\gamma)$ with a Dirac delta function $\delta(\gamma - \gamma^*)$. It is important to account for two factors: the strength of the resonance (as we do not want to select a $\gamma$ with no resonance at all, i.e. a white region of Figure 9), and also the amplitude of $\Omega^2$’s oscillations at that frequency, as given by (see Figure 1):

$$P(\gamma) = \frac{\langle f(\gamma)^2 \rangle}{2} = \left\{ \begin{array}{ll}
\frac{2}{35\pi} \left( \frac{\gamma}{\Lambda} \right)^7, & 0 \leq \gamma \leq \Lambda \\
\frac{2}{35\pi} \left( -\frac{\gamma^2}{\Lambda^2} + 70\frac{\gamma^2}{\Lambda^2} - 168\frac{\gamma^2}{\Lambda^2} + 140\frac{\gamma}{\Lambda} - 40 \right), & \Lambda \leq \gamma \leq 2\Lambda
\end{array} \right.$$
FIG. 11. On the left, contour maps are shown for the percentage uncertainty in $H$ dependent on the number of samples ($N$) and duration of simulation ($t_f$). The top shows $\Lambda = 5$, with $L = 10$, and the bottom $\Lambda = 10$ and $L = 5$; this means that they will have the same maximum $|n| = \frac{L\Lambda}{2\pi}$, and thus the same number of modes for consistency. Note that they have very similar contours, indicating that the precision depends mainly on $t_f\Lambda$ (the $y$-axis), $N$ (the $x$-axis), and $L\Lambda = 2\pi n_{\text{max}}$ (which is the same between the two). In the bottom right, we show the relationship between the absolute uncertainty in $H$ and the ODE tolerance $\varepsilon$, parametrised by $\varepsilon_0$ as described in Equation B1.

FIG. 12. Here we examine the dependence of the simulations on the box length $L$, for $\Lambda = 5$ on the left and $\Lambda = 10$ on the right. Notice that once $L$ is multiplied by $\Lambda$, the convergence appears to occurs at a consistent rate between the two graphs, with $L\Lambda = 50$ being sufficient for convergence within 1%.
We can quantify the resonance using the Mathieu exponent $H$, which is computed according to [16] using:

$$ H(\gamma) = H \left( r = \frac{2\Lambda^2}{\gamma} \sqrt{\frac{n_f}{6\pi}}, \varepsilon = \frac{2}{\sqrt{n_f}} \right) \quad \text{(C15)} $$

We then choose $\gamma^*$ such that it maximises the product $F(\gamma) = P(\gamma)H(\gamma)$.

Now, we should consider the dimensions of these quantities in order to normalise $F(\gamma^*)$ and quantify the actual growth of the $H$ in the simulations. Because of the way that the Mathieu functions are computed, $H(\gamma)$ quantifies the growth in non-dimensionalised units of time, specifically $e^H$ is the growth factor per time unit $2/\gamma$.

Given that our actual $H$ is a frequency, to rescale it appropriately we need to multiply by $\gamma^*/2$ to reinstate units of frequency.

The units of $P(\gamma^*)$ are inverse frequency, because it is integrated to give a normalised total power. Thus we should multiply by the width of frequencies which all contribute to excite the resonance – i.e. multiply by the width of the relevant resonance band from Figure 9. For example, if $\gamma_{\text{min}}$ and $\gamma_{\text{max}}$ denote the lowest and highest $\gamma$ which lie in the resonance band, then we multiply by $\Delta \gamma = \gamma_{\text{max}} - \gamma_{\text{min}}$. All in all, we have:

$$ H_{\text{estimate}} = P(\gamma^*)H(\gamma^*) \frac{\gamma^*}{2} \Delta \gamma \quad \text{(C16)} $$

This is the estimate used in Figure 7.