Supercuspidal representations in non-defining characteristics

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Abstract

We show that a mod-ℓ-representation of a p-adic group arising from the analogue of Yu’s construction is supercuspidal if and only if it arises from a supercuspidal representation of a finite reductive group. This has been previously shown by Henniart and Vigneras under the assumption that the second adjointness holds, a statement that is not yet available in the literature.

1 Introduction

The exhaustive explicit construction and parametrization of supercuspidal irreducible representations of p-adic groups with complex coefficients plays a key role in the complex representation theory of p-adic groups and beyond. For number theoretic applications it is often desirable to obtain analogous results for representations whose coefficients are valued in an algebraically closed field R of characteristic ℓ different from p. In that setting one needs to distinguish between cuspidal and supercuspidal representations. An exhaustive construction of the former, more general notion, is known if the p-adic group is tame and p does not divide the order of the Weyl group ([Fin]). This paper concludes the exhaustive construction of supercuspidal irreducible representations for p-adic groups in the same setting by determining which of the cuspidal representations are supercuspidal.

More precisely, we show that if an R-representation arising from the analogue of Yu’s construction is supercuspidal, then the representation of a finite reductive group that forms part of the input for the construction has to be supercuspidal as well (Theorem 1). Combined with the reverse implication proved by Henniart and Vigneras [HV, Theorem 6.10 and §6.4.2] and the result that all cuspidal R-representations arise from the analogue of Yu’s construction ([Fin, Theorem 4.1]), we obtain an exhaustive explicit construction of all supercuspidal irreducible R-representations.

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Theorem 1 has previously been proven by Henniart and Vigneras (Theorem 6.10 and §6.4.2 in [HV]) using different techniques, but only under the assumption that the second adjointness holds in this setting. The second adjointness is so far only proven in the literature for depth-zero representations or if \( G \) is a general linear group, a classical group (with \( p \neq 2 \)) or a group of relative rank 1 ([Dat09]). Our approach does not rely on the second adjointness to hold.

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## 2 The main theorem and corollaries

Let \( F \) be a non-archimedean local field of residual characteristic \( p \) with ring of integers \( \mathcal{O} \) and residue field \( \mathbb{F}_q \). Let \( G \) be a (connected) reductive group over \( F \). Let \( R \) be an algebraically closed field of characteristic \( \ell \) different from \( p \). All representations in this paper are representations with coefficient field \( R \).

For a point \( x \) in the enlarged Bruhat–Tits building \( \mathcal{B}(G, F) \) of \( G \) over \( F \), we write \([x]\) for the image of the point \( x \) in the reduced Bruhat–Tits building, \( G[x] \) for the stabilizer of \([x]\), \( G_{x,0} \) for the parahoric subgroup, and \( G_{x,r} \) for the Moy–Prasad filtration subgroup of depth \( r \) for a positive real number \( r \).

Let \( ((G_i)_{1 \leq i \leq n+1}, x \in \mathcal{B}(G_{n+1}, F) \subset \mathcal{B}(G,F), (r_i)_{1 \leq i \leq n}, \rho, (\phi_i)_{1 \leq i \leq n}) \) be an input for the construction of a cuspidal \( R \)-representation as in [Fin Section 2.2], i.e. following Yu’s construction [Yu01] adapted to the mod-\( \ell \) setting. If \( n > 0 \), then we assume that \( G \) and \( G_{n+1} \) split over a tamely ramified extension of \( F \) and \( p \neq 2 \), as in Yu’s construction. In the case of \( n = 0 \), the construction recovers the depth-zero representations and we allow \( G \) to be wildly ramified and/or \( p = 2 \). The irreducible \( R \)-representation \( \rho \) of \( (G_{n+1})[x] \) is trivial on \( (G_{n+1})_{x,0}^+ \) and its restriction to \( (G_{n+1})_{x,0} \) is a cuspidal representation of \( (G_{n+1})_{x,0}/(G_{n+1})_{x,0}^+ \). Note that the restriction of \( \rho \) to \( (G_{n+1})_{x,0} \) is semisimple because \( (G_{n+1})[x] \) normalizes \( (G_{n+1})_{x,0} \). Hence it makes sense to talk about the restriction being supercuspidal or not.

From the tuple \( ((G_i)_{1 \leq i \leq n+1}, x \in \mathcal{B}(G_{n+1}, F) \subset \mathcal{B}(G,F), (r_i)_{1 \leq i \leq n}, \rho, (\phi_i)_{1 \leq i \leq n}) \) we obtain a representation \( (\tilde{\rho} = \rho \otimes \kappa, V_\rho \otimes V_\kappa) \) of

\[
\tilde{K} = (G_1)_x \otimes_{\mathcal{F}} (G_2)_x \otimes_{\mathcal{F}} \cdots (G_n)_x \otimes_{\mathcal{F}} (G_{n+1})_x
\]

([Fin Section 2.3]), where \( \rho \) also denotes the extension of the depth-zero representation \( \rho \) from \( (G_{n+1})[x] \) to \( \tilde{K} \) that is trivial on \( (G_1)_x \otimes_{\mathcal{F}} (G_2)_x \otimes_{\mathcal{F}} \cdots (G_n)_x \otimes_{\mathcal{F}} \), such that \( c\text{-ind}^G_{\tilde{K}}(\rho \otimes \kappa) \) is irreducible and cuspidal.

**Theorem 1.** With the above notation, if \( c\text{-ind}^G_{\tilde{K}}(\rho \otimes \kappa) \) is supercuspidal, then the restriction of \( \rho \) to \( (G_{n+1})_{x,0} \) is supercuspidal as a representation of \( (G_{n+1})_{x,0}/(G_{n+1})_{x,0}^+ \).

This theorem has been proven by Henniart and Vigneras ([HV Theorem 6.10 and §6.4.2]) under the assumption that the second adjointness holds in this setting, which is so far only
proven in the literature for depth-zero representations or if $G$ is a general linear group, a classical group (with $p \neq 2$) or a reductive group of relative rank 1 ([Dat09]).

Combining Theorem 1 with the unconditional result of Henniart and Vigneras that $\text{c-ind}^G_K(\rho \otimes \kappa)$ is supercuspidal when the restriction of $\rho$ to $(G_{n+1})_{x,0}$ is supercuspidal as a representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ ([HV, Theorem 6.10 and §6.4.2]), we obtain the following unconditional Corollary 2.

**Corollary 2.** With the above notation, $\text{c-ind}^{G(F)}_K(\rho \otimes \kappa)$ is supercuspidal if and only if the restriction of $\rho$ to $(G_{n+1})_{x,0}$ is supercuspidal as a representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$.

**Proof.**
Combine Theorem 1 with Theorem 6.10 and §6.4.2 of [HV].

Combined with [Fin, Theorem 4.1] we obtain the following result.

**Corollary 3.** Suppose that $G$ splits over a tamely ramified field extension of $F$ and $p$ does not divide the order of the (absolute) Weyl group of $G$. Then all supercuspidal representations of $G(F)$ are of the form $\text{c-ind}^{G(F)}_K(\rho \otimes \kappa)$ as above where the restriction of $\rho$ to $(G_{n+1})_{x,0}$ is supercuspidal as a representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$.

**Proof.**
This follows from Corollary 2 and [Fin, Theorem 4.1].

## 3 Proof of Theorem 1

Let $\pi := \text{c-ind}^{G(F)}_K(\rho \otimes \kappa)$ be a cuspidal irreducible representation as in the previous section. In this section we will prove that $\pi$ being supercuspidal implies that $\rho$ is supercuspidal, i.e. Theorem 1. This statement is trivially true if $G_{n+1}$ is anisotropic as in this case $(G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ is also anisotropic and hence all its semisimple representations are supercuspidal. Thus we assume throughout that $G_{n+1}$ is not anisotropic.

We will eventually prove the desired result by assuming that $\rho$ is not supercuspidal and proving that then $\pi$ is a subquotient of a parabolically induced representation, i.e. is not supercuspidal either. However, we will not make this assumption until the end of this section to first prove a series of results that hold without this assumption and might be useful on its own for other applications.

Let $T$ be a maximally split, maximal torus of $G_{n+1}$ such that $x$ is contained in the apartment $\mathcal{A}(T, F)$ of $T$. Let $\lambda$ be a cocharacter of $T$. We write $P_{G_i}(\lambda)$ for the parabolic subgroup of $G_i$ ($1 \leq i \leq n + 1$) attached to $\lambda$ as in Section 2.1 and 2.2, in particular Proposition 2.2.9, of [CGP15]. This means $P_{G_i}(\lambda)(F)$ consists of the elements $g \in G_i(F)$ for which the limit of $\lambda(t)g\lambda(t)^{-1}$ as $t$ goes to zero exists (i.e. extends to a map from the affine line to $G_i$). Then the centralizer $Z_{G_i}(\lambda)$ of $\lambda$ is a Levi subgroup of $P_{G_i}(\lambda)$, which we also denote by
Let \( U_{G_i}(\lambda) \) be the unipotent radical of \( P_{G_i}(\lambda) \) and \( \bar{U}_{G_i}(\lambda) \) the unipotent radical of the opposite parabolic \( \bar{P}_{G_i}(\lambda) \) of \( G_{G_i}(\lambda) \) with respect to \( M_{G_i}(\lambda) \).

Let \( \epsilon > 0 \) be sufficiently small so that \( G_{x,+} \subset G_{y,s} \subset G_{x,s} \) for \( s \in \{ \lambda, \frac{\lambda}{2}, \ldots, \frac{\lambda}{2^n}, 0 \} \) and \( y = x + \epsilon \lambda \in \mathcal{A}(T, F) \). While there is in general no canonical embedding of the Bruhat–Tits building of \( M_{n+1} := M_{G_{n+1}}(\lambda) \) into the Bruhat–Tits building of \( G_{n+1} \), the embedding is unique up to translation by \( X_s(Z(M_{n+1})) \otimes \mathbb{Z} \mathbb{R} \), where \( X_s(Z(M_{n+1})) \) denotes the cocharacters of the center \( Z(M_{n+1}) \) of \( M_{n+1} \), and we fix an embedding of Bruhat–Tits buildings throughout the paper to view \( \mathcal{B}(M_{n+1}, F) \) as a subset of \( \mathcal{B}(G_{n+1}, F) \). We will do the same for all twisted Levi subgroups of \( G \) to view all Bruhat–Tits buildings over \( F \) as subsets of \( \mathcal{B}(G, F) \). Then we have \( y = x + \epsilon \lambda \in \mathcal{B}(M_{n+1}, F) \subset \mathcal{B}(G_{n+1}, F) \subset \mathcal{B}(G, F) \), and

\[
(G_{n+1})_{y,0}/(G_{n+1})_{y,0+} \simeq (M_{n+1})_{y,0}/(M_{n+1})_{y,0+}.
\]

For \( z \in \{ x, y \} \) and \( s \in \{ 0, 0+ \} \), we set

\[
K_{z,s} = (G_1)_{z, \frac{s}{2}} (G_2)_{z, \frac{s}{2}} \ldots (G_n)_{z, \frac{s}{2}} (G_{n+1})_{z,s},
\]

\[
K_+ = (G_1)_{x, \frac{s}{2}+(G_2)_{x, \frac{s}{2}+} \ldots (G_n)_{x, \frac{s}{2}+} (G_{n+1})_{x,0+}.
\]

We might abbreviate the groups \( K_{x,0} \) and \( K_{x,0+} \) by \( K_0 \) and \( K_{0+} \), respectively, and write \( P = P_G(\lambda) \), \( M = M_G(\lambda) \), \( U = U_G(\lambda) \), \( \bar{P} = \bar{P}_G(\lambda) \) and \( \bar{U} = \bar{U}_G(\lambda) \). Then \( K_{y,0} \subset K_0 \) and

\[
K_{y,0} \cap U(F) = K_{y,0+} \cap U(F) = K_0 \cap U(F),
\]

\[
K_{y,0} \cap \bar{U}(F) = K_{y,0+} \cap \bar{U}(F) = K_+ \cap \bar{U}(F),
\]

\[
K_{y,0} = (K_{y,0} \cap \bar{U}(F))(K_{y,0} \cap M(F))(K_{y,0} \cap U(F)).
\]

**Lemma 4.**

(a) The space of \((K_0 \cap U(F))\)-fixed vectors \( V^K_{\kappa,0}(U(F)) \) of the representation \((\kappa, V_\kappa)\) is non-trivial.

(b) The representation \((\kappa, V_\kappa)\) is trivial when restricted to the subgroup \( K_{y,0} \cap \bar{U}(F) \).

(c) The subspace \( V^K_{\kappa,0}(U(F)) \) is preserved under the action of \( K_{y,0} \) via \( \kappa \).

**Proof.**

If \( \pi \) has depth-zero, \( \kappa \) is the trivial one dimensional representation, and hence all statements are trivially true. Thus we may assume \( n > 0 \) and hence that we are in the setting where \( G_{n+1} \) splits over a tamely ramified field extension. Let \( E \) be the splitting field of \( T \). Since \( \lambda \) factors through a maximal split torus, which is contained in a maximal torus the splits over a tamely ramified extension, we may assume without loss of generality that \( E \) is tamely ramified over \( F \). For \( 1 \leq i \leq n \), we define

\[
U_i = G(F) \cap \left< U_{\alpha}(E)_{x, \frac{s}{2}} \mid \alpha \in \Phi(G_i, T) \setminus \Phi(G_{i+1}, T), \lambda(\alpha) > 0 \right>,
\]

\[
U_{n+1} = G(F) \cap \left< U_{\alpha}(E)_{x,0} \mid \alpha \in \Phi(G_{n+1}, T), \lambda(\alpha) > 0 \right>.
\]
where \( \Phi(G_i, T) \) denotes the root system of \( G_i \) with respect to \( T \) over the field \( E \) (for \( 1 \leq i \leq n + 1 \)) and \( U_\alpha(E) \) denotes the depth-\( \frac{\alpha}{2} \) filtration subgroup of the root group \( U_\alpha(E) \) of \( G(E) \) corresponding to \( \alpha \) and normalized with respect to the valuation on \( E \) that extends the valuation on \( F \) used to define the Moy–Prasad filtration. Then

\[
K_0 \cap U(F) = U_1 U_2 \ldots U_{n+1}.
\]

Following [Fin21a, Section 2.5] we write \( V_\kappa = \bigotimes_{i=1}^{n} V_{\omega_i} \) so that the action of \( \kappa \) restricted to \( U_j \) (\( 1 \leq j \leq n \)) is given by \( U_j \) acting on \( V_{\omega_k} \) for \( k \neq j \) via the character \( \hat{\phi}_k \) defined in loc. cit. and on \( V_{\omega_j} \) via a Heisenberg representation. The action of \( \kappa \) restricted to \( U_{n+1} \) arises from \( U_{n+1} \) acting on \( V_{\omega_k} \) via \( \phi_k \) tensored with a composition with a Weil representation, see loc. cit. for a precise definition. For \( 1 \leq j < k \leq n \), the restriction of \( \hat{\phi}_k \) to \( U_j \) is trivial by the construction of \( \hat{\phi}_k \). For \( 1 \leq k < j \leq n + 1 \), the restriction of \( \hat{\phi}_k \) to \( U_j \) equals the restriction of the character \( \phi_k \) from \( G_{k+1}(F) \) to \( U_j \). Since \( U_j \) is contained in the unipotent radical of a parabolic subgroup of \( G_{k+1}(F) \), we conclude that the restriction of \( \phi_k \) to \( U_j \) is trivial ([Tit64, Tit78]). Thus

\[
V_\kappa U_1 U_2 \ldots U_n = \bigotimes_{i=1}^{n} (V_{\omega_i})^{U_i}.
\]

Using the same arguments as in the proof of [Fin21a, Theorem 3.1], we obtain that the space \( (V_{\omega_i})^{U_i} \) is nontrivial and that \( U_{n+1} \) acts on \( (V_{\omega_i})^{U_i} \) via the restriction of the character \( \phi_i \) to \( U_{n+1} \) for \( 1 \leq i \leq n \), which we observed above is trivial. Hence

\[
V_\kappa^{K_0 \cap U(F)} = V_\kappa U_1 U_2 \ldots U_n U_{n+1} = \bigotimes_{i=1}^{n} (V_{\omega_i})^{U_i} \neq \{0\}.
\]

For (b) recall that \( \kappa \) restricted to \( K_+ \) acts via the character \( \prod_{1 \leq i \leq n} \hat{\phi}_i \) (times identity). For \( 1 \leq i \leq n \), we define

\[
\bar{U}_i^+ = G(F) \cap \left\{ U_\alpha(E) \mid \alpha \in \Phi(G_i, T) \setminus \Phi(G_{i+1}, T), \lambda(\alpha) < 0 \right\},
\]

\[
\bar{U}_{n+1}^+ = G(F) \cap \left\{ U_\alpha(E) \mid \alpha \in \Phi(G_{n+1}, T), \lambda(\alpha) < 0 \right\}.
\]

Then

\[
K_{y,0} \cap \bar{U}(F) = K_+ \cap \bar{U}(F) = \bar{U}_1^+ \bar{U}_2^+ \ldots \bar{U}_{n+1}^+.
\]

For \( 1 \leq j \leq i \leq n \), the restriction of \( \hat{\phi}_i \) to \( \bar{U}_j^+ \) is trivial by the construction of \( \hat{\phi}_i \) and the definition of \( \bar{U}_j^+ \). For \( 1 \leq i < j \leq n + 1 \), the restriction of \( \hat{\phi}_i \) to \( \bar{U}_j^+ \) equals the restriction of the character \( \phi_i \) from \( G_{i+1}(F) \) to \( \bar{U}_j^+ \). Since \( \bar{U}_j^+ \) is contained in the unipotent radical of a parabolic subgroup of \( G_{k+1}(F) \), the restriction of \( \phi_i \) to \( \bar{U}_j^+ \) is trivial ([Tit64, Tit78]). Hence the restriction of \( (\kappa, V_\kappa) \) to \( K_{y,0} \cap \bar{U}(F) = \bar{U}_1^+ \bar{U}_2^+ \ldots \bar{U}_{n+1}^+ \) is trivial.

Claim (c) follows now from Equation (4) and the observation that \( K_{y,0} \cap M(F) \) normalizes \( K_{y,0} \cap U(F) = K_0 \cap U(F) \).

\( \square \)
Lemma 5. Let \((\rho', V_{\rho'})\) be a representation of \(K_{y,0}K_{0+}\) that is trivial on \(K_{0+}\). Then there exists a surjection of \(\tilde{K}\)-representations

\[
\text{pr} : \text{c-ind}_{K_{y,0}}^{\tilde{K}} (V_{\rho'} \otimes V_{\kappa}^{K_{0}\cap U(F)}) \rightarrow (\text{c-ind}_{K_{y,0}K_{0+}}^{\tilde{K}} V_{\rho'}) \otimes V_{\kappa}.
\]

Proof.
To ease notation, we abbreviate \((\text{c-ind}_{K_{y,0}}^{\tilde{K}} (\rho' \otimes \kappa), \text{c-ind}_{K_{y,0}}^{\tilde{K}} (V_{\rho'} \otimes V_{\kappa}^{K_{0}\cap U(F)})\)) by \((\sigma_1, V_{\sigma_1})\) and denote \((\text{c-ind}_{K_{y,0}K_{0+}}^{\tilde{K}} \rho', \text{c-ind}_{K_{y,0}K_{0+}}^{\tilde{K}} V_{\rho'}\) by \((\sigma_2, V_{\sigma_2})\). Using this notation we need to construct a surjection \(\text{pr}\) from \((\sigma_1, V_{\sigma_1})\) to \((\sigma_2 \otimes \kappa, V_{\sigma_2} \otimes V_{\kappa})\).

For \(v \in V_{\rho'}, w \in V_{\kappa}^{K_{0}\cap U(F)}\), we write \(f_{v \otimes w}\) for the element of \(V_{\sigma_1} = \text{c-ind}_{K_{y,0}}^{\tilde{K}} (V_{\rho'} \otimes V_{\kappa}^{K_{0}\cap U(F)})\) that is supported on \(K_{y,0}\) and satisfies \(f_{v \otimes w}(1) = v \otimes w\). Then an arbitrary element of \(V_{\sigma_1}\) can be written as

\[
\sum_{1 \leq i \leq j} c_i \sigma_1(g_i) f_{v_i \otimes w_i}
\]

with \(j \in \mathbb{N}\) and \(c_i \in R, g_i \in \tilde{K}, v_i \in V_{\rho'}, w_i \in V_{\kappa}^{K_{0}\cap U(F)}\) for \(1 \leq i \leq j\). For \(v \in V_{\rho'}\), we write \(f_v\) for the element of \(V_{\sigma_2} = \text{c-ind}_{K_{y,0}K_{0+}}^{\tilde{K}} V_{\rho'}\) that is supported on \(K_{y,0}K_{0+}\) and satisfies \(f_v(1) = v\). Then we define the morphism \(\text{pr} : V_{\sigma_1} \rightarrow V_{\sigma_2} \otimes V_{\kappa}\) by

\[
\text{pr} \left( \sum_{1 \leq i \leq j} c_i \sigma_1(g_i) f_{v_i \otimes w_i} \right) = \sum_{1 \leq i \leq j} c_i (\sigma_2(g_i) f_{v_i} \otimes \kappa(g_i) w_i).
\]

In order to see that this linear morphism is well defined, we assume that

\[
\sum_{1 \leq i \leq j} c_i \sigma_1(g_i) f_{v_i \otimes w_i} = 0
\]

and need to show that \(\text{pr}(\sum_{1 \leq i \leq j} c_i \sigma_1(g_i) f_{v_i \otimes w_i}) = \sum_{1 \leq i \leq j} c_i (\sigma_2(g_i) f_{v_i} \otimes \kappa(g_i) w_i) = 0\). By considering the support we reduce to the case that \(g_i \in K_{y,0}\) for \(1 \leq i \leq j\). Let \(u_1, u_2, \ldots, u_j\) be a basis for the \(R\)-linear span of \(\{\kappa(g_i) w_i\}_{1 \leq i \leq j}\) and write \(\kappa(g_i) w_i = \sum_{1 \leq i' \leq j'} d_{i,i'} u_{i'}\) for some \(d_{i,i'} \in R\) for \(1 \leq i \leq j\). Then

\[
0 = \sum_{1 \leq i \leq j} c_i \sigma_1(g_i) f_{v_i \otimes w_i}(1) = \sum_{1 \leq i \leq j} c_i (\rho' \otimes \kappa)(g_i)(v_i \otimes w_i) = \sum_{1 \leq i \leq j} c_i (\rho'(g_i) v_i \otimes \kappa(g_i) w_i)
\]

\[
= \sum_{1 \leq i \leq j} \sum_{1 \leq i' \leq j'} c_i d_{i,i'} (\rho'(g_i) v_i \otimes u_{i'}),
\]

which implies

\[
\sum_{1 \leq i \leq j} c_i d_{i,i'} (g_i) v_i = 0 \quad \text{for} \quad 1 \leq i' \leq j.
\]
Hence
\[
\sum_{1 \leq i \leq j} c_i (\sigma_2(g_i) f_{v_i} \otimes \kappa(g_i) w_i) = \sum_{1 \leq i \leq j} \sum_{1 \leq i' \leq j'} c_i (f_{\rho'(g_i)} v_i \otimes d_{i,i'} w_{i'})
\]
\[
= \sum_{1 \leq i' \leq j'} \left( \sum_{1 \leq i \leq j} c_i d_{i,i'} f_{\rho'(g_i)} v_i \otimes w_{i'} \right) = 0.
\]

Therefore \(pr\) is well defined and is by construction a \(\tilde{K}\)-homomorphism. It remains to show that the map \(pr : V_{\sigma_1} \to V_{\sigma_2} \otimes V_{\kappa}\) is surjective. Let \(v\) be a non-zero element of \(V_{\sigma_2} \otimes V_{\kappa} = (\text{c-ind}^R_{K_{y,0}K_{0+}} V_{\rho'}) \otimes V_{\kappa}\). Then there exists a positive integer \(j\), and elements \(v_i \in V_{\rho'}, w_i \in V_{\kappa}\) and \(g_i \in \tilde{K}\) for \(1 \leq i \leq j\) such that
\[
v = \sum_{1 \leq i \leq j} \sigma_2(g_i) f_{v_i} \otimes w_i.
\]
Since the restriction of \(\kappa\) to \(K_{0+}\) is irreducible, there exist an element \(w \in V_{\kappa}^{K_0 \cap U(F)}\), an integer \(j'\), elements \(h_{i'} \in K_{0+}\) and \(c_{i,i'} \in R\) for \(1 \leq i \leq j\) and \(1 \leq i' \leq j'\) such that
\[
\kappa(g_i^{-1})(w_i) = \sum_{1 \leq i' \leq j'} c_{i,i'} \kappa(h_{i'})(w).
\]
Thus, using that \((\rho', V_{\rho'})\) is trivial on \(K_{0+}\), we obtain
\[
v = \sum_{1 \leq i \leq j} \sigma_2(g_i) f_{v_i} \otimes w_i
\]
\[
= \sum_{1 \leq i \leq j} (\sigma_2 \otimes \kappa)(g_i)(f_{v_i} \otimes \kappa(g_i^{-1})(w_i))
\]
\[
= \sum_{1 \leq i \leq j} \sum_{1 \leq i' \leq j'} (\sigma_2 \otimes \kappa)(g_i)(f_{v_i} \otimes c_{i,i'} \kappa(h_{i'})(w))
\]
\[
= \sum_{1 \leq i \leq j} \sum_{1 \leq i' \leq j'} c_{i,i'} (\sigma_2 \otimes \kappa)(g_i h_{i'})(f_{v_i} \otimes w)
\]
\[
= \text{pr} \left( \sum_{1 \leq i \leq j} \sum_{1 \leq i' \leq j'} c_{i,i'} \sigma_1(g_i h_{i'})(f_{v_i} \otimes w) \right).
\]

Hence \(pr\) is surjective. \(\Box\)

This allows us to prove the following key lemma for the proof of Theorem 1.

**Lemma 6.** If \(\rho\) is not supercuspidal, then there exists a maximally split, maximal torus \(T\) of \(G_{n+1}\) whose apartment contains \(x\), a cocharacter \(\lambda\) of \(T\) and a representation \((\rho', V_{\rho'})\) of \(K_{y,0}\) (with \(y = x + \epsilon\lambda\) as above) that is trivial on \(K_{y,0+}\) such that the representation \((\rho \otimes \kappa, V_{\rho} \otimes V_{\kappa})\) is a subquotient of \(\text{c-ind}^R_{K_{y,0}} (V_{\rho'} \otimes V_{\kappa}^{K_0 \cap U(F)})\).
The cocharacter $\lambda$ can be chosen so that $M := M_G(\lambda)$ is the centralizer of the maximal split torus in the center of $M_{n+1} := M_{G_{n+1}}(\lambda)$ and $\epsilon > 0$ can be chosen so that the point $y = x + \epsilon \lambda \in \mathcal{B}(M_{n+1}, F) \subset \mathcal{B}(G, F)$ is contained in a facet of minimal dimension of $\mathcal{B}(M_{n+1}, F)$ and

$$\sum_{i=1}^{n} \left( \dim \left( \frac{(G_i)_y^{G_i}}{(G_i)_{y, x_i^{G_i}}} \right) - \dim \left( \frac{(M_G(\lambda))_y^{M_G(\lambda)}}{(M_G(\lambda))_{y, x_i^{M_G(\lambda)}}} \right) \right) = 0. \quad (5)$$

Proof.
Suppose $\rho$ is not supercuspidal. Recall that $\rho|_{(G_{n+1})_{x,0}}$ is semisimple as $(G_{n+1})_{x,0}$ is normal inside $(G_{n+1})_{[x]}$. Let $\rho_1$ be an irreducible quotient of $\rho|_{(G_{n+1})_{x,0}}$, viewed as a representation of $(G_{n+1})_{x,0}/(G_{n+1})_{x,0^+}$. We denote by $G$ the connected reductive group over $\mathbb{F}_q$ that satisfies for any unramified field extension $E$ of $F$ with residue field $\mathfrak{f}_E$ that $G(\mathfrak{f}_E) = G_{n+1}(E)_{x,0}/G_{n+1}(E)_{x,0^+}$. Let $P$ be a proper parabolic subgroup of $G$ with Levi subgroup $M_i$, and $\rho'$ a representation of $M(\mathbb{F}_q)$ such that $\rho_1$ is a subquotient of the parabolic induction $\text{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \rho'$. Let $S$ be a maximal split torus of $M$ and $\mathcal{S}$ the split torus defined over $\mathcal{O}$ contained in the parahoric group scheme attached to $G_{n+1}$ and $x$ such that the special fiber of $\mathcal{S}$ is $S$. We denote the generic fiber $\mathcal{S}_G$ of $\mathcal{S}$ by $S$. Note that $S$ is a maximal split torus of $G_{n+1}$. Let $C$ be the split subtorus of $S$ whose special fiber $C := C_{\mathbb{F}_q}$ is the maximal split torus in the center of $M$. Let $M$ be the centralizer of $C := C_{\mathbb{F}_q}$ in $G$. Then $M$ is a Levi subgroup of $G$ and there exists a cocharacter $\lambda \in X_*(S)$ such that $M = M_G(\lambda)$ (e.g. by [CGP15 Proposition 2.2.9] combined with the fact that Levi subgroups of a fixed parabolic are rationally conjugate). Choosing a maximally split, maximal torus $T$ of $G_{n+1}$ containing $S$, we can perform the above constructions to obtain a parabolic subgroup $P_G(\lambda)$ of $G_i$ ($1 \leq i \leq n+1$) with Levi subgroup $M_i := M_{G_i}(\lambda) = Z_{G_i}(\lambda)$ and a point $y = x + \epsilon \lambda$ in the apartment $\mathcal{A}(T, F)$. Note that $M_{n+1}$ is the centralizer of $C$ in $G_{n+1}$, because $M$ is the centralizer of $C$ in $G$. Hence by Equation (1) and [MP96 Proposition 6.4(1)], the point $y$ is a minimal facet of the building $\mathcal{B}(M_{n+1}, F)$. Moreover, since $C$ is the maximal split torus in the center of $M$ and $M(\mathbb{F}_q) = (M_{n+1})_{y,0}/(M_{n+1})_{y,0+}$, the torus $C$ is the maximal split torus in the center of $M_{n+1}$. Hence $M$ is the centralizer of the maximal split torus in the center of $M_{n+1}$, as desired. Moreover, by the definition of $M_{G_i}(\lambda)$, Equation (5) is satisfied by all but finitely many $\epsilon$ in the open interval $(0, 1)$. Hence we may choose $\epsilon > 0$ such that Equation (5) holds true.

Since we have

$$M(\mathbb{F}_q) = K_{y,0}/K_{y,0^+} = K_{y,0}K_{0^+}/K_{y,0^+}K_{0^+},$$

we may view $\rho'$ as a representation of $K_{y,0}K_{0^+}$ via inflation. Note that the image of $K_{y,0}K_{0^+}$ in $K_{x,0}/K_{0^+} \simeq (G_{n+1})_{x,0}/(G_{n+1})_{x,0+}$ is $P(\mathbb{F}_q)$. Viewing $\rho$ and $\rho_1$ as representations of $\tilde{K}$ and $K_{x,0}$, respectively, by asking them to be trivial on $(G_1)_{x,0}(G_2)_{x,0} \cdots (G_n)_{x,0}$, we have by Frobenius reciprocity that the irreducible representation $\rho$ is a quotient of $\text{c-ind}_{K_{x,0}}^{\tilde{K}} \rho_1$ and therefore a subquotient of

$$\text{c-ind}_{K_{x,0}}^{\tilde{K}} \text{c-ind}_{K_{y,0}K_{0^+}}^{K_{y,0}K_{0^+}} \rho' = \text{c-ind}_{K_{y,0}K_{0^+}}^{\tilde{K}} \rho'.$$
Therefore \((\rho \otimes \kappa, V_\rho \otimes V_\kappa)\) is a subquotient of \(((\text{c-ind}^{K}_{K_{y,0}})^{K_{y,0}}_{K_{y,0}^+} \rho') \otimes \kappa, \text{c-ind}^{K}_{K_{y,0}}\ (V_\rho' \otimes V_\kappa) \otimes V_\kappa).\) From Lemma 3 we deduce that \((\rho \otimes \kappa, V_\rho \otimes V_\kappa)\) is a subquotient of \(\text{c-ind}^{K}_{K_{y,0}}(V_\rho \otimes V_\kappa)^{K_0 \cap U(F)}\).

**Proof of Theorem 1.** Suppose \(\rho\) is not supercuspidal. We need to prove that \(\pi := \text{c-ind}^{G(F)}(\rho \otimes \kappa)\) is not supercuspidal. We let \(\lambda\) be as given by Lemma 3 which provides us with a point \(y = x + \epsilon \lambda\) and a parabolic subgroup \(P = P_G(\lambda)\) of \(G\) with Levi \(M = M_G(\lambda)\) and unipotent radical \(U\) as above. Then the representation \((\rho \otimes \kappa, V_\rho \otimes V_\kappa)\) is a subquotient of \(\text{c-ind}^{K}_{K_{y,0}}(V_\rho \otimes V_\kappa)^{K_0 \cap U(F)}\).

Hence \(\pi\) is a subquotient of \(\text{c-ind}^{G(F)}(V_\rho' \otimes V_\kappa^{K_0 \cap U(F)})\). We will show that the latter is isomorphic to a parabolic induction of a smooth representation from \(P(F)\), which will imply that \(\pi\) is not supercuspidal and hence finish the proof.

Recall from Equations (2), (3) and (4) that

\[
K_{y,0} = (K_{y,0} \cap \hat{U}(F))(K_{y,0} \cap M(F))(K_{y,0} \cap U(F))
\]

and that \(K_{y,0} \cap \hat{U}(F) = K_{y,0}^+ \cap \hat{U}(F)\) and \(K_{y,0} \cap U(F) = K_{y,0}^+ \cap U(F)\). Moreover, by Lemma 3 and since \(K_0 \supset K_{y,0}\) and \((\rho', V_{\rho'})\) is trivial on \(K_{y,0}^+\), the restriction of \(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}\) to \(K_{y,0} \cap \hat{U}(F)\) and to \(K_{y,0} \cap U(F)\) is trivial. Hence the pair

\[
(K_{y,0}, (\rho' \otimes \kappa, V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}))
\]

is decomposed over the pair

\[
(K_{y,0} \cap M(F), ((\rho' \otimes \kappa)|_{K_{y,0} \cap M(F)}, V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}))
\]

with respect to \(\hat{P}\) as in the notation of [Blo05 p. 245].

We write

\[
K_{y,+} = (G_1)_{y, \frac{y+1}{2}}(G_2)_{y, \frac{y+1}{2}} \cdots (G_n)_{y, \frac{y+1}{2}} + (G_n+1)_{y, 0}
\]

and note that the action of \(K_{y,+}\) on \(V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}\) via \(\rho' \otimes \kappa\) is given by \(\prod_{1 \leq i \leq n} \hat{\phi}_i\) (times identity). Let \((\pi', V')\) be an irreducible smooth representation of \(G(F)\). Then we write \(V^{(K_{y,+}, \Pi \hat{\phi}_i)}\) for the subspace of \(V'\) on which \(K_{y,+}\) acts via \(\prod_{1 \leq i \leq n} \hat{\phi}_i\). Since \(y\) is contained in a facet of minimal dimension of \(\mathcal{B}(M_{n+1}, F)\) and Equation (5) holds by Lemma 3 (which ensures that the embedding of the Bruhat–Tits buildings is \((0, \frac{y}{2}, \ldots, \frac{y}{2})\)-generic relative to \(y\) as defined by [KY17 3.5 Definition], see also [Fin21], p. 341) we can apply the proof of [KY17 6.3 Theorem] to obtain that the restriction of the Jacquet functor with respect to \(\hat{U}\) to the subspace \(V^{(K_{y,+}, \Pi \hat{\phi}_i)}\) is injective. Note that while Kim and Yu work with complex coefficients in [KY17, their proof and [MP96, Proposition 6.7], on which the proof relies, also work with coefficients in the field \(R\). Hence the Jacquet functor with respect to \(\hat{U}\) is also injective when restricted to the maximal subspace of \(V'\) that is isomorphic to a direct sum of copies of \((\rho' \otimes \kappa, V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}))\) as a \(K_{y,0}\)-representation. Therefore, the pair

\[
(K_{y,0}, (\rho' \otimes \kappa, V_{\rho'} \otimes V_\kappa^{K_0 \cap U(F)}))
\]
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is a cover of

$$(K_{y,0} \cap M(F), ((\rho' \otimes \kappa)|_{K_{y,0} \cap M(F)}, V_{\rho'} \otimes V_{\kappa}^{K_0 \cap U(F)}))$$

with respect to $\bar{P}$ as in [Blo05, p. 246 and Corollaire de Proposition 2]. Thus by [Blo05, Théorème 2] we have an isomorphism of $G(F)$-representations

$$c\text{-ind}^{G(F)}_{K_{y,0}}(V_{\rho'} \otimes V_{\kappa}^{K_0 \cap U(F)}) \simeq c\text{-ind}^{G(F)}_{K_{y,0} \cap M(F)}(V_{\rho'} \otimes V_{\kappa}^{K_0 \cap U(F)}),$$

where $U(F)$ acts trivially on $V_{\rho'} \otimes V_{\kappa}^{K_0 \cap U(F)}$. Therefore $\pi$ is a subquotient of

$$c\text{-ind}^{G(F)}_{K_{y,0} \cap M(F)}U(F)(V_{\rho'} \otimes V_{\kappa}^{K_0 \cap U(F)}) \simeq c\text{-ind}^{G(F)}_{P(F)}(c\text{-ind}^{M(F)}_{K_{y,0} \cap M(F)}(V_{\rho'} \otimes V_{\kappa}^{K_0 \cap U(F)})) \simeq \text{Ind}^{G(F)}_{P(F)}\left( c\text{-ind}^{M(F)}_{K_{y,0} \cap M(F)}(V_{\rho'} \otimes V_{\kappa}^{K_0 \cap U(F)}) \right),$$

where $\text{Ind}^{G(F)}_{P(F)}$ denotes the (unnormalized) parabolic induction. This is a contradiction to $\pi$ being supercuspidal.  

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