The properties of the quantum supergroup $GL_{p,q}(1|1)$

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Abstract

In this paper properties of the quantum supermatrices in the quantum supergroup $GL_{p,q}(1|1)$ are discussed. It is shown that any element of $GL_{p,q}(1|1)$ can be expressed as the exponential of a matrix of non-commuting elements, like the group $GL_q(1|1)$. An explicit construction of this exponential representation is presented.
1. Introduction

In the past few years, Drinfeld [1] and Faddeev et al [2] constructed a new mathematical object, called the quantum group which later has been generalized to the quantum supergroup. This topic has been studied by many mathematicians and theoretical physicists.

The simplest supergroup is the group of 2x2 supermatrices with two even and two odd matrix elements, i.e. $GL(1|1)$. Even matrix elements commute with everything and odd matrix elements anticommute among themselves. The deformation of the supergroup of 2x2 matrices, i.e. the quantum supergroup $GL_q(1|1)$ can be found in Refs. 3-5. A two parameter deformation of $GL(1|1)$ was given in Ref. 6 and also Ref. 7.

It was shown in Ref. 5 that any element of $GL_q(1|1)$ can be written as the exponential of a matrix and this exponential form was explicitly constructed. This work will be along the lines of the work in Ref. 5. In sec. 2 we present a review of $GL_{p,q}(1|1)$ together proofs some elementary lemmas which we use in the later sections. In sec. 3 we get the matrix elements of $T^n$, the $n$-th power of $T$, for $T \in GL_{p,q}(1|1)$. Using these matrix elements we prove that $T^n \in GL_{p^n,q^n}(1|1)$ if $T \in GL_{p,q}(1|1)$. This result suggests that an element of $GL_{p,q}(1|1)$ can be expressed as an exponential of a matrix whose entries obey $(h_1, h_2)$-dependent commutation relations (here the parameters $h_1$ and $h_2$ are the logarithm of the deformation parameters $q$ and $p$, respectively). To prove this in sec. 4 we use the method of the paper of Schwenk et al [5]. We derive the explicit form of the $n$-th power of the matrix $M$ which is the natural logarithm of the matrix $T \in GL_{p,q}(1|1)$ in sec. 4. Thus, we obtain the matrix elements of $T$ in terms of $M$ and vice versa. Finally, we state that the usual relation between the superdeterminant and the supertrace, which is also satisfy in the supergroup $GL_q(1|1)$, is true again in $GL_{p,q}(1|1)$.

2. Notations and useful formulas

In Ref. 3 Manin identifies a quantum supergroup with the endomorphisms acting on quantum superplanes. In the matrix representation of these endomorphisms, the commutation relations of the space coordinates include the
We state briefly some notations and useful formulas we are going to need in this work. The quantum supergroup $GL_{p,q}(1|1)$ consists of all matrices in the form

$$ T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \quad (2.1) $$

where the elements $a, \beta, \gamma$ and $d$ obey the following commutation relations

$$ a\beta = q\beta a, \quad d\beta = q\beta d, $$

$$ a\gamma = p\gamma a, \quad d\gamma = p\gamma d, \quad (2.2) $$

$$ \beta\gamma + pq^{-1}\gamma\beta = 0, \quad \beta^2 = 0 = \gamma^2, $$

$$ ad - da = (p - q^{-1})\gamma\beta $$

and $p, q$ non-zero complex numbers, $pq \pm 1 \neq 0$.

The quantum superdeterminant is defined as

$$ sD_{p,q}(T) = ad - \beta d^{-1}\gamma d^{-1} \quad (2.3) $$

provided $d$ is invertible. We will also suppose that $a$ is invertible. Using (2.2) it is easy to show that $sD_{p,q}(T)$ is central, that is, it commutes with $a, a^{-1}, d, d^{-1}, \beta$ and $\gamma$.

If we take

$$ \Delta_1 = ad - p^{-1}\beta\gamma \quad \text{and} \quad \Delta_2 = da - q^{-1}\gamma\beta \quad (2.4) $$

the super-inverse of $T$ becomes

$$ T^{-1} = \begin{pmatrix} d\Delta_1^{-1} & -q^{-1}\beta\Delta_2^{-1} \\ -p^{-1}\gamma\Delta_1^{-1} & a\Delta_2^{-1} \end{pmatrix}. \quad (2.5) $$

Then the superdeterminant is given by

$$ sD_{p,q}(T) = a^2\Delta_2^{-1}. \quad (2.6) $$

Of course, the superdeterminant of $T^{-1}$ may also be defined and it is of the form

$$ sD_{p,q}(T^{-1}) = d^2\Delta_1^{-1}. \quad (2.7) $$
On the other hand,
\[
\left\{ a^2 \Delta_2^{-1} \right\}^{-1} = d^2 \Delta_1^{-1}. \tag{2.8}
\]
So,
\[
sD_{p,q}(T^{-1}) = \left\{ sD_{p,q}(T) \right\}^{-1}. \tag{2.9}
\]
This relation is generalized in Corollary of sec. 3.

Before passing to the next section, we give three lemmas.

**Lemma 2.1.** For any integer \( n \)
\[
(a - \beta d^{-1} \gamma)^n = a^n - \frac{q^n - p^{-n}}{q - p^{-1}} \beta a^{n-1} d^{-1} \gamma. \tag{2.10}
\]

*Proof.* The relation (2.10) can be proved by an induction procedure.

We note here that if \( f \) is any function of \( a \) and \( d \) of the form \( f(a, d) = a^n d^m \) where \( n \) and \( m \) are integer, in the products with \( \beta \) or \( \gamma \), the arguments \( a^n \) and \( d^m \) of the function \( f(a, d) \) behave as commuting quantities. So in the products by \( \beta \) or \( \gamma \), the element \( a^n \) commutes with \( d^m \), i.e.,
\[
\beta a^n d^m = \beta d^m a^n.
\]

**Lemma 2.2.** For any integers \( n \) and \( m \)
\[
a^n d^m = d^m a^n + (p^n - q^{-n}) \frac{p^m - q^{-m}}{p - q^{-1}} \gamma a^{n-1} d^{m-1} \beta. \tag{2.11}
\]

*Proof.* The proof of this lemma can also be proved by an induction procedure.

**Lemma 2.3.** For any integer \( n \)
\[
\left\{ sD_{p,q}(T) \right\}^n = a^n d^{-n} - p \frac{p^{-n} - q^n}{p - q^{-1}} a^{n-1} \gamma d^{n-1} \beta. \tag{2.12}
\]

*Proof.* With (2.6), one can write
\[
\left\{ sD_{p,q}(T) \right\}^n = a^{2n} \Delta_2^{-n}, \tag{2.13}
\]
since $a$ and $\Delta_2$ commute. On the other hand, it may be shown by using (2.11) with $n = m$ [or from (2.10)] that

$$\Delta_2^n = a^n d^n - p^\frac{n - q^n}{p - q^{-1}}a^{n-1} \gamma d^{n-1} \beta.$$ 

Hence, by replacing $n$ by $-n$ into the above equation one gets

$$a^{2n} \Delta_2^{-n} = a^n d^{-n} - p^\frac{n - q^n}{p - q^{-1}}a^{n-1} \gamma d^{-n-1} \beta = \{sD_{p,q}(T)\}^n,$$

as required.

These results will be used in the following sections.

3. The properties of $T^n$

To show that $T^n \in GL_{p^n,q^n}(1|1)$ for $T \in GL_{p,q}(1|1)$ we will explicitly obtain the matrix elements of $T^n$, the $n$-power of $T$, for a matrix $T \in GL_{p,q}(1|1)$. The matrix elements of $T^n$ in a more compact form also appear in the paper of Schwenk et al [5]. First we define the following functions. Let

$$F_n(a, q^{-1}d)\beta \gamma = \sum_{k=0}^{n-2} < n - k - 1 >_{pq} a^{n-k-2}(q^{-1}d)^k \beta \gamma, \quad (3.1)$$

$$G_n(a, q^{-1}d)\beta = \sum_{k=0}^{n-1} a^{n-k-1}(q^{-1}d)^k \beta, \quad (3.2)$$

where

$$< N >_{pq} = \frac{1 - (pq)^{-N}}{1 - (pq)^{-1}}. \quad (3.3)$$

**Lemma 3.1.** If $T \in GL_{p,q}(1|1)$ then the matrix $T^n$, the $n$-th power of $T$, has the form

$$T^n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \quad (3.4)$$

where

$$A_n = a^n + F_n(a, q^{-1}d)\beta \gamma, \quad B_n = G_n(a, q^{-1}d)\beta,$$

$$D_n = d^n + F_n(d, p^{-1}a)\gamma \beta, \quad C_n = G_n(d, p^{-1}a)\gamma. \quad (3.5)$$

**Proof.** It can be done by induction on $n$ using the fact that $T^{n+1} = T^n T$. 


Lemma 3.2. If $T \in GL_{p,q}(1|1)$ then $T^n \in GL_{p^n,q^n}(1|1)$. That is, the matrix elements of $T^n$ obey the following commutation relations

\[
A_n B_n = q^n B_n A_n, \quad D_n B_n = q^n B_n D_n, \quad D_n C_n = p^n C_n D_n, \quad B_n^2 = 0 = C_n^2, \quad q^n B_n C_n + p^n C_n B_n = 0,
\]

(3.6)

and

\[
[A_n, D_n] = (p^n - q^{-n}) C_n B_n,
\]

(3.7)

where

\[
[u, v] = uv - vu.
\]

Proof. It is not difficult to check that the relations (3.6) are satisfied. The reader can easily prove this by using the relations (2.2) and the equations (3.5). But proof of the relation (3.7) requires some operations. The proof of (3.7) can be found in the Appendix.

The property in the Lemma 3.2 gives an opportunity for the matrix $T \in GL_{p,q}(1|1)$ to be represented as an exponential of a matrix. This will be done in the last section.

Now we want to calculate the superdeterminant of $T^n$. For this, we write the matrix $T^n$ in the form

\[
T^n = \begin{pmatrix} A_n & 0 \\ C_n & D_n - C_n A_n^{-1} B_n \end{pmatrix} \begin{pmatrix} 1 & A_n^{-1} B_n \\ 0 & 1 \end{pmatrix}
\]

(3.8)

using the Crout decomposition. Then the superdeterminant of $T^n$ becomes

\[
sD_{p,q}(T^n) = A_n (D_n - C_n A_n^{-1} B_n)^{-1} = (A_n - B_n D_n^{-1} C_n) D_n^{-1}.
\]

(3.9)

After some calculations one gets

\[
sD_{p,q}(T^n) = a^n d^{-n} - p \frac{p^{-n} - q^n}{p - q} a^{n-1} \gamma d^{-n-1} \beta
\]

(3.10)

which is the same with (2.12). Thus we have:

Corollary 3.3. For any integer $n$

\[
\{sD_{p,q}(T)\}^n = sD_{p,q}(T^n).
\]

(3.11)
4. The exponential parametrization of $GL_{p,q}(1|1)$

The implication $T^n \in GL_{q^n}(1|1)$ for $T \in GL_q(1|1)$ suggests that $T$ can be represented by exponentiating a matrix. This was shown by Schwenk et al [5]. However, in sec. 3 it is show that $T^n \in GL_{p^n,q^n}(1|1)$ if $T \in GL_{p,q}(1|1)$ [see, Lemma 3.2]. Thus the matrix $T \in GL_{p,q}(1|1)$ can be written as the exponential of a matrix with non-commuting entries. Let

$$q = e^{h_1} \quad \text{and} \quad p = e^{h_2}. \quad (4.1)$$

Suppose that

$$T = e^{hM}, \quad h = \frac{h_1 + h_2}{2} \quad (4.2)$$

where

$$M = \begin{pmatrix} x & \mu \\ \nu & y \end{pmatrix}. \quad (4.3)$$

To find the commutation relations of the matrix elements of $M$ we write the exponent as

$$M = \frac{1}{h} \ln T. \quad (4.4)$$

The logarithm of the matrix $T$ is defined by

$$\ln T = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (T - I)^n \quad (4.5)$$

as a series expansion. Now let

$$(T - I)^n = \begin{pmatrix} \tilde{A}_n & \tilde{B}_n \\ \tilde{C}_n & \tilde{D}_n \end{pmatrix}. \quad (4.6)$$

Then some calculations show that

$$\tilde{A}_n = (a - 1)^n + \sum_{j=0}^{n-2} \sum_{k=0}^{n-j-2} (a - 1)^k (p^{-1} q^{-1} a - 1)^{n-k-j-2} (q^{-1} d - 1)^j \beta \gamma, \quad (4.7)$$

$$\tilde{D}_n = (d - 1)^n + \sum_{j=0}^{n-2} \sum_{k=0}^{n-j-2} (d - 1)^k (p^{-1} q^{-1} d - 1)^{n-k-j-2} (p^{-1} a - 1)^j \gamma \beta, \quad (4.7)$$

$$\tilde{B}_n = \sum_{j=0}^{n-1} (a - 1)^{n-j-1} (q^{-1} d - 1)^j \beta, \quad \tilde{C}_n = \sum_{j=0}^{n-1} (d - 1)^{n-j-1} (p^{-1} a - 1)^j \gamma.$$
We want to obtain the matrix elements of $M$ in an explicit form. For the sake of simplicity we define

\[
f_q(a,d,p)_{\beta\gamma} = \frac{q^2}{q-p^{-1}} \left( \frac{\ln a}{a(qa-d)} - \frac{\ln(p^{-1}q^{-1}a)}{a(p^{-1}a - d)} \right)_{\beta\gamma} + \frac{q^2}{(p^{-1}a - d)(qa - d)}_{\beta\gamma}
\]

(4.8)

\[
g(a, q^{-1}d)_{\beta} = \frac{\ln a - \ln(q^{-1}d)}{a - q^{-1}d} \beta,
\]

(4.9)

where the logarithms of $a$ and $d$ exist and the denominators are non-zero. Then we have

**Lemma 4.1.** If $T \in GL_{p,q}(1|1)$ then the expressions for the matrix elements of $M$ in terms of $T$ are as follows:

\[
x = \frac{1}{\hbar} \{ \ln a + f_q(a,d,p)_{\beta\gamma} \}, \quad \mu = \frac{1}{\hbar} g(a, q^{-1}d)_{\beta},
\]

\[
y = \frac{1}{\hbar} \{ \ln d + f_p(d,a,q)_{\gamma\beta} \}, \quad \nu = \frac{1}{\hbar} g(d, p^{-1}a)_{\gamma}.
\]

(4.10)

**Proof.** Use the equation (4.4) with (4.5).

Note that the matrix elements $a$ and $d$ are behave as commuting quantities when they are in a product case by $\beta$ or $\gamma$. Thus it is not necessary to order the arguments in the equations (4.8) and (4.9).

**Proposition 4.2.** If $T \in GL_{p,q}(1|1)$ then the matrix elements of $M$ obey the following commutation relations

\[
[x, \mu] = \frac{2h_1}{h_1 + h_2} \mu, \quad [y, \mu] = \frac{2h_1}{h_1 + h_2} \mu, \quad \mu^2 = 0,
\]

\[
x, \nu] = \frac{2h_2}{h_1 + h_2} \nu, \quad [y, \nu] = \frac{2h_2}{h_1 + h_2} \nu, \quad \nu^2 = 0,
\]

\[
xy - yx = 0, \quad \mu \nu + \nu \mu = 0.
\]

(4.11)

**Proof.** It is easy to see that the relations (4.11) [if we except that the relation $xy - yx = 0$] is satisfied. Let us prove the last relation in (4.11), here.
Let
\[ X = [\ln a, \ln d], \]
\[ Y = [\ln a, f_p(d, a, q)\gamma\beta], \]
\[ Z = [\ln d, f_q(a, d, p)\beta\gamma]. \]

After some calculations one gets
\[ Y - Z = 4h^2 \frac{1}{1 - pq} \gamma a^{-1} \beta d^{-1}, \]  
(4.12a)
and
\[ X = \frac{\ln^2(pq)}{pq - 1} \gamma a^{-1} \beta d^{-1}. \]  
(4.12b)

[The proof of (4.12b) is rather lengthy but straightforward]. Thus we have
\[ X + Y - Z = 0. \]

Here we used the relations
\[ [\ln a, \beta] = h_1 \beta = [\ln d, \beta], \]
\[ [\ln a, \gamma] = h_2 \gamma = [\ln d, \gamma]. \]  
(4.13)

If \( T \in GL_q(1|1) \) and \( T = e^{\theta M} \) where the matrix \( M \) is given by (4.3) and \( q = e^\theta \), we know that \( x - y \) is the central element, which is known as the supertrace of the matrix \( M \). This case is true in the supergroup \( GL_{p,q}(1|1) \) too, i.e. the supertrace of \( M \), \( strM = x - y \), is the central element in the algebra (4.11).

Now we will obtain the matrix elements of \( T \) in terms of \( M \). First we derive the explicit form of \( M^n \), the \( n \)-th power of \( M \). It will simplify the elements of \( M^n \) to define a transformation \( \tau \) by
\[ \tau : \tau(x, y, \mu, \nu, h_1, h_2) \mapsto (y, x, \mu, \nu, h_2, h_1). \]  
(4.14)

Then the relation (4.11) are preserved by \( \tau \). For example,
\[ [x, \mu]^\tau = (\frac{h_1}{h} \mu)^\tau \implies [y, \nu] = \frac{h_2}{h} \nu. \]
Let
\[
\phi = \frac{h_1}{h} \quad \text{and} \quad \varphi = \frac{h_2}{h}.
\] (4.15)

The following lemma can be proved by mathematical induction. We denote the algebra (4.11) by \( \mathcal{M}_{h_1, h_2} \).

**Lemma 4.3.** If \( M \in \mathcal{M}_{h_1, h_2} \) then the matrix \( M^n \), has the form
\[
M^n = \begin{pmatrix}
x^n - \mu \nu F_n & \mu G_n \\
\nu G_n^T & y^n - \nu \mu F_n^T
\end{pmatrix}
\] (4.16)
where
\[
F_n = \frac{F_n(x, y, \phi, \varphi)}{2(x - y - \varphi)} - \frac{(x + \phi + \varphi)^n}{2(x - y + \phi)} - \frac{(y + \varphi)^n}{(x - y + \phi)(x - y - \varphi)} (4.17)
\]
\[
G_n = \frac{G_n(x, y, \phi)}{x - y + \phi}.
\] (4.18)

Now we easily obtain the expressions for the matrix elements of \( T \) in terms of \( M \) using the equation (4.2):

**Lemma 4.4.** If \( M = e^M \) then one has
\[
a = e^{hx} - \frac{\mu \nu}{(x - y + \phi)(x - y - \varphi)} \left\{ \frac{\phi + pq \varphi}{2} - \frac{pq - 1}{2} (x - y) \right\} e^{hx} - pe^{hy},
\]
\[
d = e^{hy} - \frac{\nu \mu}{(x - y + \phi)(x - y - \varphi)} \left\{ \frac{\varphi + pq \phi}{2} + \frac{pq - 1}{2} (x - y) \right\} e^{hy} - qe^{hx},
\]
\[
\beta = \frac{\mu}{x - y + \phi} (qe^{hx} - e^{hy}), \quad \gamma = \frac{\nu}{\varphi - (x - y)} (pe^{hy} - e^{hx}).
\] (4.19)

**Proposition 4.5.** If \( M \in \mathcal{M}_{h_1, h_2} \) and \( T = e^M \) then \( T \in GL_{p,q}(1|1) \).

**Proof.** To prove that the matrix \( T \) is in \( GL_{p,q}(1|1) \) the reader can be verified the relations (2.2).
In Ref. 5 it has been shown that the usual relation between the superdeterminant and supertrace is valid, i.e.

\[ sD_q(T) = e^{\theta \text{str} M}, \quad q = e^\theta. \]

Finally with direct calculation we can show that

\[ sD_{p,q}(T) = e^{h(x-y)} = e^{h \text{str} M}, \] (4.20)

where \( h = \frac{1}{2} \ln(pq) \).

**Remarks.** In the equations (4.11) if we take \( h_1 = h_2 \) we obtain the algebra in Ref. 5 (the equ.s (5.7)) where \( q \) replaces \( q^{-1} \) for \( p = q \). In this case the relations (4.19) identify the equations (5.9) in Ref. 5. Thus our work can be considered as a generalization of Ref. 5.

**Appendix: the proof of eq. (3.7)**

Now we will show that

\[ [A_n, D_n] = (p^n - q^{-n})C_nB_n. \] (A1)

For this, we will use the fact that

\[ T^{k+1} = T^kT = TT^k. \] (A2)

It is proved by induction on \( n \).

1. For \( n = 1 \), the equality (A1) identifies with the last relation in (2.2).

2. Assume that the equation is true for \( n = k \).

3. With (A2) we write that

\[ A_{k+1} = A_1A_k + B_1C_k, \quad C_{k+1} = C_1A_k + D_1C_k, \]
\[ B_{k+1} = A_1B_k + B_1D_k, \quad D_{k+1} = D_1D_k + C_1B_k. \]

Now some calculations show that

\[
[A_{k+1}, D_{k+1}] = \left( (pq)^{k+1} - 1 \right) \left( C_1B_kA_kA_k - (pq)^{-k-1}D_1D_kB_1C_k \right) + K
= \left( p^{k+1} - q^{-k-1} \right) C_{k+1}B_{k+1} + K - L,
\]

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where

\[K = \left( p^{2k+1}q^k - q^{-k-1} \right) C_1 A_k B_1 D_k + \left( p^{k+1} - pq^{-k} \right) C_k A_1 B_k D_1 + p^{k+1} \left( C_k A_1 A_k B_1 - C_1 A_k A_1 B_k \right) + p^{k+1} \left( D_k C_1 B_k D_1 - D_1 C_k B_1 D_k \right)\]

and

\[L = \left( p^{k+2}q - pq^{-k} \right) C_k A_1 B_k D_1 + \left( p^{k+1} - q^{-k-1} \right) C_1 A_k B_1 D_k\]

Thus it must be

\[K - L = 0.\]

In fact,

\[K - L = p^{k+1} \left( p^k q^k - 1 \right) C_1 A_k B_1 D_k + p^{k+1} \left( 1 - pq \right) C_k A_1 B_k D_1 + p^{k+1} \left( C_k A_1 A_k B_1 - C_1 A_k A_1 B_k \right) + p^{k+1} \left( D_k C_1 B_k D_1 - D_1 C_k B_1 D_k \right) = 0.\]

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