ON SYMMETRIC CR GEOMETRIES OF HYPERSURFACE TYPE

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Abstract. We study non–degenerate CR geometries of hypersurface type that are symmetric in the sense that at each point, there is a CR transformation reverting the CR distribution at the point. We show that they are either flat or homogeneous. We show that non–flat non–degenerate symmetric CR geometries of hypersurface type are covered by CR geometries with a compatible pseudo–Riemannian metric preserved by all symmetries. We construct examples of simply connected flat non–degenerate symmetric CR geometries of hypersurface type that do not carry a compatible pseudo–Riemannian metric that would be preserved by some symmetry at each point.

1. Introduction

In [9], Kaup and Zaitsev generalized Riemannian symmetric spaces to the setting of CR geometries, i.e., smooth manifolds with the so–called CR distribution carrying a complex structure. They consider a Riemannian metric compatible with the CR geometry in the sense that the complex multiplication on the CR distribution preserves the Riemannian metric. Such a manifold is symmetric in the sense of [9] if there is an isometric CR transformation at each point that fixes the point and acts as $-\text{id}$ on the CR distribution at the point [9, Definition 3.5,]. They show that such isometric CR transformations are uniquely determined by the tangent action on the CR distribution [9, Theorem 3.3]. They also show that such CR geometries are homogeneous [9, Proposition 3.6]. In fact, these CR geometries may be considered as reflexion spaces in the sense of [11].

We have studied in [8] filtered geometric structures that carry an automorphism at each point that fixes the point and acts as $-\text{id}$ on a distinguished part of the filtration at the point. Let us point out that non–degenerate CR geometries of hypersurface type, i.e., where the CR distribution has codimension 1, are among these geometries. We have studied when these filtered geometries are homogeneous and can be considered as reflexion spaces. However, for non–degenerate CR geometries of hypersurface type, the conditions [8, Theorem 5.7,] we have found are different from the conditions from [9]. In particular, it suffices that such a non–degenerate CR geometry of hypersurface type is non–flat at one point.

In this article, we study the case of non–degenerate CR geometries of hypersurface type in more detail. We consider CR transformations that preserve a point and induce $-\text{id}$ on the CR distribution at the point. We say that a non–degenerate CR geometry of hypersurface type is symmetric (in our sense) if there is such a
symmetry at each point, see the Definition 1. In particular, our definition does not require the existence of a metric compatible with the CR geometry. We adapt and significantly improve the general results of [4, 7, 8] for this class of CR geometries. Let us emphasize that every non-degenerate CR geometry of hypersurface type that is symmetric in the sense of [9] is symmetric (in our sense).

Let us say that [8, Theorem 5.7.] is written in the general setting of Cartan geometries. So we present here results of this theorem in the setting of CR geometries. We also provide new direct proofs of these results, because we will need the presented ideas to explain new results, see Lemmas 2, 3 and Propositions 1, 2.

We prove in the Theorem 2 that non-flat non-degenerate CR geometries of hypersurface type that are symmetric (in our sense) are covered by symmetric non-degenerate CR geometries of hypersurface type that carry a pseudo-Riemannian metric compatible with the CR geometry that is preserved by all our symmetries. In the Riemannian signature, these coverings are symmetric in the sense of [9], see the Theorem 3. Moreover, we show in the Theorem 4 that the CR geometry on these coverings can be always explicitly embedded into a complex manifold. In the Riemannian signature, this embedding is provided by a different construction than the one in [9, Proposition 7.3].

We construct examples of simply-connected flat non-degenerate CR geometries of hypersurface type that are symmetric (in our sense), but do not carry a pseudo-Riemannian metric that would be preserved by some symmetry at each point. Unfortunately, the construction does not work in the Riemannian signature. Therefore, there still is the open question whether there is a flat CR geometry of hypersurface type with positive definite Levi form that is symmetric (in our sense) but that is not symmetric in the sense of [9].

2. CR geometries of hypersurface type

2.1. CR geometries. Let $M$ be a smooth manifold of dimension $2n + 1$ for $n > 1$ together with a distribution $\mathcal{H} \subset TM$ of dimension $2n$ and a complex structure $J$ on $\mathcal{H}$, i.e., $J : \mathcal{H} \to \mathcal{H}$ is an endomorphism with the property that $J^2 = -\text{id}$. The triple $(M, \mathcal{H}, J)$ is called a CR geometry of hypersurface type if the $i$-eigenspace $\mathcal{H}^{1,0}$ of $J$ in the complexification of $\mathcal{H}$ is integrable, i.e., $[\mathcal{H}^{1,0}, \mathcal{H}^{1,0}] \subset \mathcal{H}^{1,0}$. The CR geometry $(M, \mathcal{H}, J)$ is called non-degenerate, if $\mathcal{H}$ is completely non-integrable.

There always is a symmetric bilinear form $h$ on $\mathcal{H}$ valued in the line bundle $TM/\mathcal{H}$ given by $h(\xi, \eta) = \frac{1}{2}\pi([\xi, J\eta])$ for all $\xi, \eta \in \Gamma(\mathcal{H})$, where $\pi : TM \to TM/\mathcal{H}$ is the natural projection. Let us recall that $h$ is the real part of the Levi form $\tilde{h}$ of $(M, \mathcal{H}, J)$, while the imaginary part of the Levi form is the map given by $\frac{1}{2}\pi([\xi, \eta])$. We assume that $M$ is orientable and denote by $(p, q)$ the signature of the Levi form, where our convention is $p \leq q$, $p + q = n$. Then the signature of $h$ is $(2p, 2q)$.

There is a standard model of such a CR geometry of arbitrary signature $(p, q)$ that is the homogeneous space $PSU(p + 1, q + 1)/P$, where the group $PSU(p + 1, q + 1)$ is the projectivization of the group of matrices preserving the pseudo-Hermitian form

$$m((u_0, \ldots, u_{n+1}), (v_0, \ldots, v_{n+1})) = u_0\overline{v}_{n+1} + u_{n+1}\overline{v}_0 + \sum_{k=1}^{p} u_k\overline{v}_k - \sum_{k=p+1}^{n} u_k\overline{v}_k$$

on $\mathbb{C}^{n+2}$ and $P$ is the stabilizer of the complex line generated by the first basis vector in the standard basis of $\mathbb{C}^{n+2}$. The standard model $PSU(p + 1, q + 1)/P$ is a
smooth real hypersurface in $\mathbb{C}P^{n+1}$ that can be also viewed as the projectivization of the null cone of $m$ in $\mathbb{C}^{n+2}$.

In the rest of the article, by a CR geometry we mean a non-degenerate CR geometry of hypersurface type of signature $(p, q)$ for $p \leq q$. Such CR geometries can be equivalently described as parabolic geometries modeled on standard models $PSU(p + 1, q + 1)/\mathbb{P}$ [1 Section 4.2.4]. We need not to present this equivalent description explicitly, because we will only use several consequences of its existence.

2.2. Distinguished connections. There exist many admissible connections, i.e., connections preserving $\mathcal{H}$ and $J$, on CR geometries. In particular, there are several distinguished classes of admissible connections given by a particular normalization condition on the torsion of admissible connections in the class. The most common class is the class of Webster–Tanaka connections [1 Section 5.2.12]. Another important class is the class of Weyl connections [1 Sections 5.1.2 and 5.2.13]. In this article, we consider the class of Weyl connections, because we use in our proofs relations between CR transformations and geodesic transformations of normal Weyl connections [1 Section 5.1.12].

In fact, Webster–Tanaka connections and Weyl connections induce the same class of distinguished partial connections $\nabla$ on $\mathcal{H}$. Two such distinguished partial connections $\nabla$ and $\hat{\nabla}$ are related by the formula

$$\hat{\nabla}_\xi(\eta) = \nabla_\xi(\eta) + F(\xi)\eta + F(\eta)\xi - \tilde{h}(\xi, \eta)\tilde{h}^{-1}(F),$$

where $\xi, \eta$ are vector fields in $\mathcal{H}$ and $F$ is a one–form in $\mathcal{H}^*$. Here $\tilde{h}^{-1}$ is the inverse of the Levi form $\tilde{h}$. We will write shortly $\hat{\nabla} = \nabla + F$ instead of the entire formula [1].

Each Weyl connection $D$ is associated with particular decompositions $TM \simeq \mathcal{H} \oplus \ell$ and $T^*M \simeq \mathcal{H}^* \oplus \ell^*$ that are preserved by $D$, where $\ell$ is a one–dimensional distribution complementary to $\mathcal{H}$. In fact, the one–form $F$ in $\mathcal{H}^*$ from the formula [1] also describes the change of the decompositions of $TM$ and $T^*M$ associated with $D$ and $\hat{D}$. The precise formula for the change of the decompositions can be easily computed using [1 Section 5.1.5]. In general, arbitrary two Weyl connections $D$ and $\hat{D}$ are related by a suitable action of a one–form $\Upsilon = \Upsilon_1 + \Upsilon_2$ in $T^*M = \mathcal{H}^* \oplus \ell^*$, where we consider the decomposition associated with the Weyl connection $D$. We write shortly $\hat{D} = D + \Upsilon_1 + \Upsilon_2$ instead of the explicit formula for the change, which is fairly complicated and can be computed using [1 Section 5.1.6]. Let us emphasize that $\Upsilon_1$ coincides with $F$ from the formula [1] for the corresponding partial connections $\nabla, \hat{\nabla}$ determined by $D, \hat{D}$.

Let us finally point out that admissible connections provide the fundamental invariant $W$ of the CR geometries, which is known as Chern–Moser tensor or Weyl tensor and coincides with the totally trace–free part of the curvature of arbitrary Weyl or Webster–Tanaka connection. Vanishing of this invariant implies that the CR geometry is flat, meaning that the CR geometry is locally equivalent to the standard model $PSU(p + 1, q + 1)/\mathbb{P}$.

3. Symmetries of CR geometries

3.1. Definition of symmetries. A CR transformation of a CR geometry $(M, \mathcal{H}, J)$ is a diffeomorphism of $M$ such that the tangent map preserves the CR distribution
\( \mathcal{H} \) and its restriction to \( \mathcal{H} \) is complex linear. We study the following CR transformations.

**Definition 1.** A symmetry at \( x \in M \) on a CR manifold \((M, \mathcal{H}, J)\) is a CR transformation \( S_x \) of \( M \) such that:

1. \( S_x(x) = x \),
2. \( T_x S_x = -\text{id} \) on \( \mathcal{H} \).

We say that the CR geometry is symmetric, if there exists a symmetry at each point \( x \in M \). We describe each system of symmetries on \( M \) by a map \( S : M \times M \to M \) given by \( S(x, y) = S_x(y) \) for the chosen symmetries \( S_x \) at each \( x \in M \). We call the system smooth, if the map \( S \) is smooth in both variables.

Let us show that the standard model \( PSU(p+1, q+1)/P \) is symmetric. The Lie group \( PSU(p+1, q+1) \) is the group of all CR transformations of the standard model \( PSU(p+1, q+1)/P \), where we consider left action. Direct computation gives that all symmetries of the standard model \( PSU(p+1, q+1)/P \) at the origin \( eP \) are represented by \((1, n, 1)\)-block matrices of the form

\[
(2) \quad s_{Z, z} = \begin{pmatrix}
-1 & -Z & iz + \frac{1}{2}IZ^* \\
0 & E & -IZ^* \\
0 & 0 & -1
\end{pmatrix},
\]

where \( Z \in \mathbb{C}^{n*} \), \( z \in \mathbb{R}^* \) are arbitrary, \( E \) is the identity matrix of the rank \( n \) and \( I \) is the diagonal matrix with the first \( p \) entries equal to 1 and the remaining \( q \) entries equal to \(-1\).

**Lemma 1.** There exists an infinite number of symmetries at each point \( kP \) of \( PSU(p+1, q+1)/P \) given by matrices of the form \( ks_{Z, z}k^{-1} \) for all \( Z \in \mathbb{C}^{n*} \) and \( z \in \mathbb{R}^* \). In particular:

1. There exists an infinite number of involutive symmetries at each point characterized by the condition \( z = 0 \). For each such symmetry, there is a different metric preserved by this symmetry compatible with the CR geometry.
2. There exists an infinite number of non–involutive symmetries at each point characterized by the condition \( z \neq 0 \). They do not preserve any metric compatible with the CR geometry.

**Proof.** Each element \( s_{Z, z} \) satisfying \( z = 0 \) is conjugated to an element of maximal compact subgroup of \( PSU(p+1, q+1) \), and thus preserves the corresponding metric. Each element \( s_{Z, z} \) satisfying \( z \neq 0 \) is contained in a different orbit with respect to the conjugation than the maximal compact subgroup of \( PSU(p+1, q+1) \), and thus cannot preserve any compatible metric.

The standard model \( PSU(p+1, q+1)/P \) carries a pseudo–Riemannian metric compatible with the CR geometry, because the maximal compact subgroup of \( PSU(p+1, q+1) \) acts transitively on the standard model. Moreover, there is exactly one involutive symmetry at each point of this model that is contained in the maximal compact subgroup. These symmetries preserve the corresponding pseudo–Riemannian metric and form a smooth system. This means that in the Riemannian signature, the standard model \( PSU(1, n+1)/P \) is symmetric in the sense of [9].

On flat CR geometries, the set of symmetries is locally the same as the one on the standard model. However, these symmetries may not be defined globally. This means that on flat CR geometries, there locally always is a pseudo–Riemannian
metric compatible with the CR geometry preserved by some symmetry at each point. We show in the Example 1 that such a pseudo–Riemannian metric compatible with the CR geometry does not have to exist globally.

3.2. **Involutive and non–involutive symmetries.** Suppose there is a symmetry $S_x$ at $x$ on a CR geometry $(M, \mathcal{H}, J)$. If $D$ is a Weyl connection, then $S_x^* D$ is a Weyl connection, too. Therefore, there is a one–form $\Upsilon_1 + \Upsilon_2 \in \mathcal{H}^* \oplus \ell^*$ such that

$$(3) \quad S_x^* D = D + \Upsilon_1 + \Upsilon_2.$$ 

**Lemma 2.** Suppose $S_x$ is a symmetry at $x \in M$. Let $D$ be an arbitrary Weyl connection and let $\Upsilon_1 + \Upsilon_2 \in \mathcal{H}^* \oplus \ell^*$ be the one–form from the formula $(3)$. Then the following is equivalent:

1. The symmetry $S_x$ is involutive,
2. $\Upsilon_2(x) = 0$, and
3. the diffeomorphism $S_x$ is linear in the normal coordinates given by the normal Weyl connection $\tilde{D}$ at $x$ that is uniquely determined by the property that $\tilde{D}$ coincides with the Weyl connection $D + \frac{1}{2} \Upsilon_1$ at $x$.

Moreover, the partial connection $\nabla^{S_x}$ induced by the Weyl connection $D^{S_x} := D + \frac{1}{2} \Upsilon_1$ does not depend on the choice of $D$ at $x$ and satisfies

- $S_x^*(\nabla^{S_x}) = \nabla^{S_x}$ at $x$, and
- $\nabla^{S_x}W(x) = 0$.

**Proof.** Iterating the formula $(3)$ we compute

$S_x^*S_x^* D = D + \Upsilon_1 + S_x^*(\Upsilon_1) + \Upsilon_2 + S_x^*(\Upsilon_2).$

The component of the (dual) action of $T_x S_x$ on $T^*_x M$ preserving the decomposition $T^*_x M = \mathcal{H}^*(x) \oplus \ell^*(x)$ is $- id \oplus id$, and the component that maps $\mathcal{H}^*(x)$ into $\ell^*(x)$ depends linearly on $\Upsilon_1$ and is antisymmetric as a map $\mathcal{H}^*(x) \otimes \mathcal{H}^*(x) \rightarrow \ell^*(x)$.

Therefore, $S_x^*(\Upsilon_1)(x) = -\Upsilon_1(x)$ and $S_x^*(\Upsilon_2)(x) = \Upsilon_2(x)$.

If the symmetry $S_x$ is involutive, i.e., $S_x^2 = id$, then

$$0 = \Upsilon_2(x) + S_x^*(\Upsilon_2)(x) = 2\Upsilon_2(x)$$

and thus $\Upsilon_2(x) = 0$.

If $\Upsilon_2(x) = 0$, then the normal Weyl connection $\tilde{D}$ that coincides with the Weyl connection $D + \frac{1}{2} \Upsilon_1$ at $x$ satisfies

$$S_x^*(D + \frac{1}{2} \Upsilon_1) = D + \Upsilon_1 + S_x^*(\frac{1}{2} \Upsilon_1).$$

At the point $x$, we get

$$\Upsilon_1(x) + S_x^*(\frac{1}{2} \Upsilon_1)(x) = \Upsilon_1(x) - \frac{1}{2} \Upsilon_1(x) = \frac{1}{2} \Upsilon_1(x)$$

and thus $S_x^* \tilde{D} = \tilde{D}$ follows from the normality [1, Section 5.1.12]. Thus $S_x$ is an affine map, which is linear in the normal coordinates.

If the symmetry $S_x$ at $x$ is linear in the normal coordinates of a Weyl connection, then its (dual) tangent action preserves the decomposition $T^*_x M = \mathcal{H}^*(x) \oplus \ell^*(x)$ and therefore $(T_x S_x)^2 = id$. Then it follows from linearity that $S_x$ is involutive.

Finally, the last claim follows, because $(\nabla^{S_x}W)(x)$ is a tensor of type $\otimes^4 \mathcal{H}^*_2 \otimes \mathcal{H}_x$ invariant with respect to $S_x$. \qed
Lemma 3. Suppose there is a symmetry \( S_x \) at \( x \in M \). Let \( D \) be an arbitrary Weyl connection and let \( \Upsilon_1 + \Upsilon_2 \in H^r \oplus \ell^r \) be the one-form from the formula (3). If \( W(x) \neq 0 \), then \( \Upsilon_2(x) = 0 \) and the symmetry \( S_x \) is involutive.

Proof. Consider the covariant derivative of \( W \) with respect to \( D + \frac{1}{2} \Upsilon_1 \) in the direction \( \ell \) and compute \( S_x^*(D + \frac{1}{2} \Upsilon_1)_r W(x) \) for \( r \in \ell(x) \). We know that \( W(x) \) is \( S_x \)-invariant and thus

\[
S_x^*(D + \frac{1}{2} \Upsilon_1)_r W(x) = (D + \frac{1}{2} \Upsilon_1)_{S_x^*(r)} W(x) = (D + \frac{1}{2} \Upsilon_1)_r W(x).
\]

On the other hand, it generally holds \( S_x^* (D + \frac{1}{2} \Upsilon_1) = D + \frac{1}{2} \Upsilon_1 + \Upsilon_2 \) and \( \Upsilon_2(x) = a \theta(x) \) for a covector \( \theta \in \ell^*(x) \) such that \( \theta(r) = 1 \). Then

\[
S_x^*(D + \frac{1}{2} \Upsilon_1)_r W(x) = (D + \frac{1}{2} \Upsilon_1)_r W(x) + 2a W(x)
\]

and thus \( a = 0 \) which implies \( \Upsilon_2(x) = 0 \). \( \square \)

3.3. Smooth systems of involutive symmetries. Let us show here that the assumption \( W \) is nowhere vanishing not only implies that all symmetries are involutive, but it also implies that there is at most one symmetry at each point of \( M \) and that these symmetries change smoothly along \( M \).

Proposition 1. Suppose \((M, H, J)\) is a symmetric CR geometry such that \( W(x) \neq 0 \) for all \( x \in M \). Then

1. there is a unique symmetry \( S_x \) at each \( x \in M \),
2. the map \( S : x \mapsto S_x \) is smooth, and
3. \( S_x \circ S_y = S_{S_x(y)} \) holds for all \( x, y \in M \).

In particular, \((M, S)\) is a reflexion space, i.e., \( S : M \times M \rightarrow M \) is a smooth map that satisfies for all \( x, y, z \in M \) that

- \( S(x, x) = x \),
- \( S(x, S(x, y)) = y \), and
- \( S(x, S(y, z)) = S(S(x, y), S(x, z)) \).

Proof. We show that if there are two different symmetries at \( x \) on a CR geometry \((M, H, J)\), then \( W \) vanishes at \( x \). Consider two different symmetries \( S_x \) and \( S'_x \) at \( x \) (which must be both involutive). We know from the Lemma 2 that \( \nabla^{S_x} W(x) = 0 \) and \( \nabla^{S_x'} W(x) = 0 \) hold for partial connections \( \nabla^{S_x}, \nabla^{S_x'} \). These partial connections are different (at \( x \)) due to the Claim (3) of the Lemma 2 i.e., \( \nabla^{S_x'} = \nabla^{S_x} + F \) holds according to the formula (11) for \( F(x) \neq 0 \). This means that the linear map \( H_x \rightarrow H_x \) given by

\[
\eta \mapsto (F(\xi) \eta + F(\eta) \xi - \tilde{h}(\xi, \eta) \tilde{h}^{-1}(F)(x))
\]

defines a non–zero element \( \xi(F)(x) \) of a Lie algebra \( \mathfrak{su}(p, q) \) for each \( \xi \in H_x \), where we identify \( \mathfrak{su}(p, q) \) with

\[
\{X \in \mathfrak{gl}(H_x) : [X, J_x] = 0, h_x(X(\xi), \nu) + h_x(\xi, X(\nu)) = a \cdot h_x(\xi, \nu), a \in \mathbb{R}\}.
\]

Moreover, the element \( \xi(F)(x) \) of \( \mathfrak{su}(p, q) \) has to act trivially on \( W(x) \) for all vectors \( \xi \). Let us denote by \( \text{ann}(W_x) \) the set of all \( A \in \mathfrak{su}(p, q) \) such that \( A \) acts trivially on \( W(x) \). Then we get

\[
F(x) \in \text{ann}(W_x)^{(1)} := \{F : \xi(F)(x) \in \text{ann}(W_x) \text{ for all } \xi \in H_x\}.
\]
The result of [11] states that if \( W(x) \) is non–trivial, then \( \text{ann}(W_x) = 0 \), and thus \( \xi(F)(x) = 0 \) for all \( \xi \in \mathcal{H}_x \). Since \( \xi(-)(x) : \mathcal{H}_x \to \mathfrak{su}(p, q) \) is a linear map at each \( x \in M \), this implies \( F(x) = 0 \), which is a contradiction. This proves the uniqueness of symmetries at \( x \) in the case \( W(x) \neq 0 \).

Since \( S_x \circ S_y \circ S_x \) is a symmetry at \( S_x(y) \), the condition \( S_x \circ S_y \circ S_x = S_{S_x(y)} \) trivially follows from the uniqueness of symmetries. Thus it remains to prove the smoothness of \( S \).

Let us fix a partial Weyl connection \( \nabla \). For each \( y \in M \), there is \( F(y) \) such that \( (\nabla^{S_y} - \nabla)(y) = F(y) \) by the formula [11], which is well–defined due to the uniqueness of \( \nabla^{S_y} \) at \( y \). Thus \( \nabla W(y) \) is given by the algebraic action (3) of \( \xi(F(y)) \) on \( W(y) \) for each \( \xi \in \mathcal{H}_y \). Since \( \nabla W(y) \) is smooth, the image of \( \xi(F(y)) \) in \( \mathfrak{su}(p, q) \) depends smoothly on \( y \) for each \( \xi \in \mathcal{H}_y \). Since the kernel of the action coincides with \( \text{ann}(W_y) = 0 \), we conclude that \( F(y) \) depends smoothly on \( y \).

Let \( D \) be an arbitrary Weyl connection inducing the partial Weyl connection \( \nabla \). Then \( S_y \) is linear in the normal coordinates of the normal Weyl connection \( D \) constructed for \( D + \frac{1}{2} F(y) \) due to the Claim (3) of the Lemma [2]. Since \( D \) depends smoothly on \( y \), we get that \( S \) is smooth.

It clearly holds \( S_x(x) = x \) and \( S_x^2 = \text{id} \) for all \( x \in M \). We have proved that \( S \) is smooth and satisfies \( S_x \circ S_y = S_{S_x(y)} \circ S_x^{-1} = S_{S_x(y)} \circ S_x \) for all \( x, y \in M \). Thus it follows that \( (M, S) \) satisfies the conditions of the reflexion space. \( \square \)

The Proposition [1] has the following consequence.

**Proposition 2.** Suppose \( (M, \mathcal{H}, J) \) is a symmetric CR geometry. Then either

1. \( W = 0 \) and the CR geometry is locally equivalent to the standard model, or
2. \( W \neq 0 \) and the group generated by symmetries is a Lie group that acts transitively on \( M \), i.e., the CR geometry is homogeneous. In particular, the reflexion space \( (M, S) \) from the Proposition [1] is a homogeneous reflexion space.

**Proof.** Suppose \( U \subset M \) consists of all points with non–trivial \( W \). It suffices to prove that the group generated by symmetries at points in \( U \) acts transitively on \( U \) to obtain the claim of the Theorem, because then \( W \) is constant on \( U \) due to the homogeneity. The fact that the group generated by symmetries on a reflexion space is a Lie group is a result of [11].

Let \( c(t) \) be a curve in \( U \) such that \( c(0) = x \) and \( \frac{d}{dt}|_{t=0} c(t) = X \in \mathcal{H}_x \). Then \( \frac{d}{dt}|_{t=0} S_{c(t)}(x) \) is tangent to the orbit of the action of the group generated by symmetries at points in \( U \). Differentiating of the equality \( c(t) = S_{c(t)} c(t) \) gives

\[
X = \frac{d}{dt}|_{t=0} S_{c(t)}(c(t)) = \frac{d}{dt}|_{t=0} S_{c(t)}(x) + T_x S_x . X,
\]

and we get

\[
\frac{d}{dt}|_{t=0} S_{c(t)}(x) = X - T_x S_x . X = 2X.
\]

Thus at all \( x \in U \), the CR distribution \( \mathcal{H} \) is tangent to the orbit of the group generated by symmetries at points in \( U \). Therefore the group generated by symmetries at points in \( U \) acts transitively on \( U \). \( \square \)

Flat symmetric CR geometries do not have to be homogeneous. We construct an explicit example in the Section [3].
4. Non–flat symmetric CR geometries

4.1. Homogeneous CR geometries and their symmetries. There are several possible ways, how to describe a homogeneous CR geometry. We will use the description from [11 Section 1.5.15] that is closely tight with the setting of Cartan geometries, but as we show in this section, it can be treated independently of the general theory. We need only to recall that the Lie algebra $\mathfrak{su}(p+1, q+1)$ of $PSU(p+1, q+1)$ consists of the $(1, n, 1)$–block matrices

$$
\begin{pmatrix}
\alpha & Z & -iz^* \\
ix & -Z^* & -i\bar{z}
\end{pmatrix},
$$

where $\mathfrak{csu}(p, q) = \{(a, A) : a \in \mathbb{C}, A \in \mathfrak{u}(n), a + tr(A) - \bar{a} = 0\}, X \in \mathbb{C}^n, Z \in \mathbb{C}^{n*}, x \in \mathbb{R}$ and $z \in \mathbb{R}^*$. This means that we have the following decomposition

$$
\mathfrak{su}(p+1, q+1) = \mathbb{R} \oplus \mathbb{C}^n \oplus \mathfrak{csu}(p, q) \oplus \mathbb{C}^{n*} \oplus \mathbb{R}^*.
$$

The Lie algebra $\mathfrak{p}$ of $P$ corresponds to $(1, n, 1)$–block upper triangular part and decomposes as $\mathfrak{p} = \mathfrak{csu}(p, q) \oplus \mathbb{C}^{n*} \oplus \mathbb{R}^*$. In fact, $P \cong CSU(p, q) \exp(\mathbb{C}^{n*} \oplus \mathbb{R}^*)$, where $CSU(p, q)$ consists of all elements of $P$ preserving the above decomposition.

**Lemma 4.** Let $K$ be an arbitrary transitive Lie group of CR transformations of a homogeneous CR geometry $(M, \mathcal{H}, J)$ and let $L \subset K$ be the stabilizer of a point. Then there is a pair of maps $(\alpha, i)$ such that $i$ is an injective Lie group homomorphism $i : L \to P$ and $\alpha$ is a linear map $\alpha : \mathfrak{l} \to \mathfrak{su}(p+1, q+1)$ satisfying the following conditions:

1. $\alpha : \mathfrak{l} \to \mathfrak{su}(p+1, q+1)$ is a linear map extending $T_e i : l \to \mathfrak{p}$.
2. $\alpha$ induces an isomorphism $\mathbb{t}/l = \mathfrak{su}(p+1, q+1)/\mathfrak{p}$ of vector spaces.
3. $\text{Ad}(i(l)) \circ \alpha = \alpha \circ \text{Ad}(l)$ holds for all $l \in L$.
4. The linear map $\wedge^2 \mathfrak{l} \to \mathfrak{su}(p+1, q+1)$ given by the formula $[\alpha(X), \alpha(Y)] - \alpha([X, Y])$ for all $x, y \in \mathfrak{l}$ has values in $\mathfrak{p}$ and defines a $K$–invariant two–form $\kappa$ with values in $K \times_{\text{Ad}i} \mathfrak{p}$.
5. The component of $\kappa$ in $K \times_{\text{Ad}i} \mathfrak{su}(p, q)$ is a tensor that coincides with $W$, where $\text{Ad}$ is the induced action of $P$ on $\mathfrak{csu}(p, q) \cong \mathfrak{p}/(\mathbb{C}^{n*} \oplus \mathbb{R}^*)$.

Conversely, suppose $(\alpha, i)$ is such a pair of maps from $(K, L)$ to $(PSU(p+1, q+1), P)$. Then there is a $K$–homogeneous CR geometry $(K/L, \mathcal{H}, J)$ satisfying $\mathcal{H}_{eL} = \alpha^{-1}((\mathbb{C}^n \oplus \mathfrak{p})/l$ and $J_{eL} = \alpha^*(J)$, where $J$ is the complex structure on $\mathbb{C}^n$.

A pair $(\alpha, i)$ satisfying the conditions (1)–(3) of the Lemma 4 is usually called an extension of $(K, L)$ to $(PSU(p+1, q+1), P)$. The two–form $\kappa$ from the condition (4) is the curvature of the Cartan connection given by the extension $(\alpha, i)$. Finally, the condition (5) is the normalization condition on the curvature $\kappa$ that can be also expressed as $\partial^* \kappa = 0$, where $\partial^*$ is the Kostant’s co–differential [11 Section 3.1.11].

**Proof.** It is shown in [11 Section 1.5.15] that each homogeneous Cartan (and thus parabolic) geometry can be described by a particular extension. It follows from [11 Section 1.5.16] that the conditions (4) and (5) on the curvature $\kappa$ are necessary and sufficient to obtain the claim from results in [11 Section 4.2.4].

**Definition 2.** The pair $(\alpha, i)$ from the Lemma 4 is called the normal extension of $(K, L)$ to $(PSU(p+1, q+1), P)$ describing the homogeneous CR geometry $(M, \mathcal{H}, J)$.

Examples of normal extensions describing certain homogeneous CR geometries and the explicit formula from the condition (5) of the Lemma 4 can be found in [3].
It is clear from the second part of the Lemma [3] that only the maps $i$ and $\alpha$ are sufficient to determine the CR geometry. This means that there are many normal extensions $(\alpha, i)$ of $(K, L)$ to $(PSU(p + 1, q + 1), P)$ describing the same CR geometry. The other parts of $\alpha$ are completely determined by the condition (5) from the Lemma [4] and carry the information about Weyl connections. The remaining freedom (for fixed $i$) is in the choice of a complex basis of $\alpha^{-1}(\mathbb{C}^n)$. In general, if $h \in P$, then the pair $(Ad(h) \circ \alpha, conj(h) \circ i)$ is also a normal extension of $(K, L)$ to $(PSU(p + 1, q + 1), P)$ describing the same CR geometry as the normal extension $(\alpha, i)$.

Let us summarize results characterizing symmetric non–flat homogeneous CR geometries following from [6, 7].

**Proposition 3.** Let $K$ be the Lie group of all CR transformations of a non–flat homogeneous CR geometry $(M, H, J)$. Then the following is equivalent:

1. There is a (unique) symmetry at each point.
2. There is $s \in L$ such that the triple $(K, L, s)$ is a (non–prime) homogeneous reflection space, i.e.,
   - $s$ commutes with all elements of $L$,
   - $s^2 = e$, where $e$ is the identity element of $L$, and
   - all symmetries are of the form $S_{KL} = ksk^{-1}$ for $k \in K$.
3. There is a normal extension $(\alpha, i)$ of $(K, L)$ to $(PSU(p + 1, q + 1), P)$ describing $(M, H, J)$ such that $i(L) \subset CSU(p, q)$ and $s_{0,0} \in i(L)$ (see the formula [3]).
4. For each normal extension $(\alpha, i)$ of $(K, L)$ to $(PSU(p + 1, q + 1), P)$ describing $(M, H, J)$, there is a (unique) $Z \in \mathbb{C}^{n+}$ such that $\text{Ad}(\exp(Z)) \alpha(x)$ is preserved by $\text{Ad}(s_{0,0})$, and the Lie algebra automorphism of $\mathfrak{k}$ given by $\text{Ad}(s_{0,0})$ defines an automorphism of the Lie group $K$.

The condition (3) of the Proposition [3] immediately implies that there are $K$–invariant Weyl connections on a symmetric non–flat CR geometry $(M, H, J)$. According to [1] Proposition 1.4.8, a $K$–invariant connection on $T(K/L)$ can be described by a map $\gamma : \mathfrak{k} \rightarrow \mathfrak{gl}(\mathfrak{t}/\mathfrak{l})$ such that

- $\gamma|_l = \text{ad}$, and
- $\gamma(Ad(h)(X)) = Ad(h) \circ \gamma(X) \circ Ad(h)^{-1}$

hold for all $X \in \mathfrak{k}$ and $h \in L$, where $\text{Ad} : L \rightarrow GL(\mathfrak{k}/\mathfrak{l})$ is induced by the adjoint representation.

**Proposition 4.** Let $K$ be the Lie group of all CR transformations of a non–flat symmetric CR geometry $(M, H, J)$. Let $(\alpha, i)$ be a normal extension of $(K, L)$ to $(PSU(p + 1, q + 1), P)$ describing $(M, H, J)$ such that $i(L) \subset CSU(p, q)$ and $s_{0,0} \in i(L)$. Then $\gamma := \alpha^* (\text{ad} \circ r_0)$ describes a $K$–invariant Weyl connection, where $r_0 : \mathfrak{su}(p + 1, q + 1) \rightarrow \mathfrak{su}(p, q)$ is the projection along $\mathbb{R} \oplus \mathbb{C}^n \oplus \mathbb{C}^{n+} \oplus \mathbb{R}^*$.

In particular, there is a bijection between the set of $K$–invariant Weyl connections on $M$ and the set of $z \in \mathbb{R}^+$ such that $\text{conj}(\exp(z)) \circ i(L) \subset CSU(p, q)$ holds for the extension $(\alpha, i)$.

**Proof.** We proved the existence of $K$–invariant Weyl connections on non–flat symmetric CR geometries in [7]. Here we only check that they can be described by the functions $\gamma$. Since $i(L) \subset CSU(p, q)$, the projection $r_0$ is $i(L)$–equivariant and $\gamma|_l = \text{ad}$ holds. Therefore each such $\gamma$ describes a $K$–invariant connection. The
Lemma 4. improved the characterization of non–flat symmetric homogeneous CR geometries given by Propositions 2 and 3.

4.2. Groups generated by symmetries. The following Theorem significantly improves the characterization of non–flat symmetric homogeneous CR geometries given by Propositions [2] and [3].

Theorem 1. Let $K$ be the Lie group generated by all symmetries of a non–flat symmetric CR geometry $(M, H, J)$. Let $(\alpha, i)$ be a normal extension of $(K, L)$ to $(PSU(p + 1, q + 1), P)$ describing the CR geometry that satisfies $i(s) = s_{0,0}$ and $i(L) \subset CSU(p, q)$. Denote by $\mathfrak{h}$ the 1–eigenspace of $s$ in $\mathfrak{k}$ and by $\mathfrak{m}$ the −1–eigenspace of $s$ in $\mathfrak{k}$. Then:

1. It holds
   - $\alpha(l) \subset u(p, q)$,
   - $\alpha(m) \subset C^n \oplus C^{n*}$, and
   - $\alpha(h) \subset R \oplus \mathfrak{su}(p, q) \oplus R^*$ is a Lie subalgebra.

2. There is a basis of $\mathfrak{h}/l \oplus \mathfrak{m}$ such that for a vector in $\mathfrak{h}/l \oplus \mathfrak{m}$ with coordinates $(x, X)$ holds

   $$\alpha((x, X) + l) = \text{Ad}(\exp(z)) \circ \left( \begin{array}{ccc} ax & \mathfrak{P}_1(X) & \mathfrak{P}_2(x) \\ X & Ax & -IX^* \\ x & -IX & ax \end{array} \right) + \alpha(l),$$

   where $z \in R^*$, $\mathfrak{P}_1 : C^n \to C^{n*}$, $\mathfrak{P}_2 : R \to R^*$ and $(a, A) \in u(p, q)$ normalizes $\alpha(l)$.

3. The maps $\mathfrak{P}_1, \mathfrak{P}_2$ and the matrix $(a, A)$ are completely determined by the condition (5) from the Lemma [4].

Proof. We know from the Proposition [3] that there is always a normal extension $(\alpha, i)$ of $(K, L)$ to $(PSU(p + 1, q + 1), P)$ satisfying our assumptions.

Consider the canonical decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is 1–eigenspace of $s$ and $\mathfrak{m}$ is −1–eigenspace of $s$. Then $\alpha(m) \subset C^n \oplus C^{n*}$ and $\alpha(h) \subset R \oplus \mathfrak{su}(p, q) \oplus R^*$ follow from the assumption $i(s) = s_{0,0}$ and $\alpha(h)$ is a Lie subalgebra, because $dim(\mathfrak{h}/l) = 1$. We can identify $\mathfrak{m}$ with $C^n$ via $\alpha$, because the restriction of $\alpha$ to the map $\mathfrak{m} \to C^n$ is injective. Indeed, if the restriction would not be injective, then the elements in its kernel would be further symmetries at $eL$, but we know that there is only one symmetry. This identification uniquely determines the map $i : L \to CSU(p, q)$.

Further, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ holds and we have the corresponding symmetric space $K/H^0$, where $H^0$ is the connected component of identity of the fixed point set of the conjugation by $s$. Therefore $\exp([X, Y]) \in H^0$ for each $X, Y \in \mathfrak{m}$. The map $\text{Ad} : H^0 \to GL(\mathfrak{m})$ can be restricted to the connected component of identity $L^0$ of $L$ and the restriction coincides with $i$. Therefore, it suffices to show that the element $\text{ad}([X, Y]) \in \mathfrak{gl}(\mathfrak{m})$ belongs to $\mathfrak{sl}(\mathfrak{m})$ for all $X, Y \in \mathfrak{m}$. But we have $\text{ad}([X, Y]) = \text{ad}(X) \circ \text{ad}(Y) - \text{ad}(Y) \circ \text{ad}(X)$ and the trace equals to

$$\text{tr}(\text{ad}([X, Y])) = \text{tr}(\text{ad}(X) \circ \text{ad}(Y) - \text{ad}(Y) \circ \text{ad}(X)) = B(X, Y) - B(Y, X),$$
where $B$ denotes the Killing form, which is symmetric. Therefore $i(L^0) \subset U(p, q)$ and $T_i(l) \subset u(p, q)$. In particular, the Claim (1) holds. The map $\alpha$ can be expressed as in the Claim (2), because there is always $z \in \mathbb{R}^*$ such that the extension $(\text{Ad}(\exp(-z)) \circ \alpha, \text{conj}(\exp(-z)) \circ i)$ satisfies

$$\text{Ad}(\exp(-z)) \circ \alpha((x, 0) + l) = \begin{pmatrix} aix & 0 & P_2(x) \\ 0 & Ax & 0 \\ x & 0 & aix \end{pmatrix} + \text{Ad}(\exp(-z)) \circ \alpha(l).$$

Since the CR geometry $(M, H, J)$ does not depend on parts $P_1, P_2$ and $(a, A)$ of $\alpha$, these parts are completely determined by the condition (5) from the Lemma 4. □

Let us remark that although the Lie algebra homomorphism $i$ is uniquely determined by the isomorphism $m \cong \mathbb{C}^n$ given by $\alpha$, the converse is not true. See [3] for examples of non–equivalent CR geometries described by extensions with the same Lie group homomorphism $i$.

Let us further remark that we do not know any example of an extension $(\alpha, i)$ of $(K, L)$ to $(P SU(p + 1, q + 1), F)$ where $z \in \mathbb{R}^*$ from the Claim (2) of the Theorem 4 would not correspond to an invariant Weyl connection. The main reason for this is the following result.

**Proposition 5.** Suppose $\text{Ad}(L^0)|_{h/l} = \text{Ad}(L)|_{h/l}$. Then $i(L) \subset U(p, q)$ and there is a bijection between $\mathbb{R}^*$ and the set of $K$–invariant Weyl connections. In particular, there is a unique $K$–invariant Weyl connection corresponding to the normal extension $(\alpha, i)$ satisfying $i(L) \subset U(p, q)$, $i(s) = s_{0,0}$ and

$$\alpha((x, X) + l) = \begin{pmatrix} aix & P_1(X) & P_2(x) \\ X & Ax & -IP_1(X)^* \\ x & -IX^* & aix \end{pmatrix} + \alpha(l).$$

This particularly holds when the transitive group $K$ is semisimple.

**Proof.** If $\text{Ad}(L^0)|_{h/l} = \text{Ad}(L)|_{h/l}$, then $i(L) \subset U(p, q)$ holds and the claim follows. It follows from the classification of semisimple symmetric spaces that $H$ is reductive and there is a complement to $l$ in the center of $h$. Consequently $\text{Ad}(L^0)|_{h/l} = \text{Ad}(L)|_{h/l}$. □

5. **Metrizability and CR Embeddings**

In this section, we always consider the $K$–invariant Weyl connection $D$ corresponding to a normal extension $(\alpha, i)$ describing a homogeneous CR geometry $(M, H, J)$ that satisfies $i(L) \subset U(p, q)$, $i(s) = s_{0,0}$ and

$$\alpha((x, X) + l) = \begin{pmatrix} aix & P_1(X) & P_2(x) \\ X & Ax & -IP_1(X)^* \\ x & -IX^* & aix \end{pmatrix} + \alpha(l).$$

Moreover, we always assume $\text{Ad}(L^0)|_{h/l} = \text{Ad}(L)|_{h/l}$, where $L^0$ is the component of identity of $L$. This gives almost no restriction, because this condition is always satisfied on the symmetric CR geometry on the covering $K^0/L^0 \to K/L$. 


5.1. **Distinguished metrics compatible with the CR geometry.** The symmetric bilinear form \( h \) generally does not define a pseudo–Riemannian metric on \( \mathcal{H} \), because there is no natural way, how to measure the length of elements of \( T\mathcal{M}/\mathcal{H} \).

The situation is different, if there is a Weyl connection preserving not only the decomposition \( \mathcal{H} \oplus \ell \), but also a non–zero vector field \( r \) in \( \ell \). Such a Weyl connection is called **exact** and the vector field \( r \) is called the **Reeb field**. Equivalently, each exact Weyl connection corresponds to the **contact form** \( \theta \) that annihilates \( \mathcal{H} \) and satisfies \( \theta(r) = 1 \) for the Reeb field \( r \). If there is an exact Weyl connection, then \( \theta \circ h \) is a pseudo–Riemannian metric on \( \mathcal{H} \). This metric is compatible with the CR–structure, because the form \( h \) satisfies \( h(J\xi, J\nu) = h(\xi, \nu) \) for all sections \( \xi, \nu \) of \( \mathcal{H} \). The exact Weyl connection preserves this metric and the Reeb field can be used to construct a pseudo–Riemannian metric on \( T\mathcal{M} \), for which the connection is a metric connection. This metric is usually called a **Webster metric**. However, the Webster metric does not have to exist or needs not to be compatible with the symmetries. Therefore, if we want to find a metric compatible with the CR geometry that is preserved by all symmetries, we need to show that the distinguished Weyl connection \( D \) is exact.

**Theorem 2.** Let \( K \) be the Lie group generated by all symmetries of a non–flat symmetric CR geometry \((\mathcal{M}, \mathcal{H}, J)\). Suppose \( \text{Ad}(L^0)|_{h/l} = \text{Ad}(L)|_{h/l} \). The distinguished Weyl connection \( D \) is exact and there are

- \( K \)-invariant contact form \( \theta \),
- \( K \)-invariant pseudo–Riemannian metric \( \bar{g} := \theta \circ h \) on \( \mathcal{H} \), and
- \( K \)-invariant Webster metric \( g := \theta \circ h + \theta \otimes \theta \) on \( T\mathcal{M} \)

such that

1. \( D\bar{g} = 0, Dg = 0 \),
2. \( g|_{\mathcal{H}} = \bar{g} \) and the Reeb field of \( D \) is orthogonal to \( \mathcal{H} \) and has size 1,
3. choosing the Reeb field of \( D \) as a trivialization of \( T\mathcal{M}/\mathcal{H} \otimes \mathbb{C} \), the pseudo–Riemannian metric \( \bar{g} \) on \( \mathcal{H} \) coincides with the real part of the Levi form up to a constant multiple,
4. the symmetry at \( x \) is linear in geodesic coordinates of \( D \) at \( x \), reverts the directions of \( \mathcal{H}_x \) and preserves the direction of the Reeb field of \( D \) at \( x \).

**Proof.** The image of \( \alpha \) is contained in \( \mathbb{R} \oplus \mathbb{C}^n \oplus u(p, q) \oplus \mathbb{C}^n^* \oplus \mathbb{R}^* \) and thus \( \gamma \) describing the corresponding \( K \)-invariant Weyl connection has values in \( \text{ad}(u(p, q)) \). Further, the assumption \( \text{Ad}(L^0)|_{h/l} = \text{Ad}(L)|_{h/l} \) implies that \( i(L) \subset U(p, q) \) and therefore the maps \( \text{ad}^{-1} \circ \gamma \) and \( i \) satisfy all the conditions of \([1, \text{Theorem 1.4.5}]\). This means that the Weyl connection \( D \) is an associated connection to a \( K \)-invariant principal connection on the bundle \( K \times_{i(L)} U(p, q) \to K/L \). Therefore it is an exact Weyl connection, because its holonomy is contained in \( U(p, q) \). The remaining claims then follow from the general theory.

In the Riemannian signature, the Theorem 2 particularly allows to compare symmetric CR geometries (in our sense) with the symmetric CR geometries in the sense of \([9]\), because we have found a metric compatible with the CR geometry that is preserved by all symmetries.

**Theorem 3.** Suppose \( p = 0 \). Then each non–flat symmetric CR geometry is covered by a symmetric CR geometry in the sense of \([9]\), where the covering is a CR map that intertwines the symmetries.
5.2. CR embeddings. Consider the fiber bundle $K \times_i CSU(p, q)/U(p, q) \to K/L$. If \( \text{Ad}(L^0)|_{h/1} = \text{Ad}(L)|_{h/1} \) holds, then this bundle is trivial, i.e.,
\[
K \times_i CSU(p, q)/U(p, q) = K/L \times \mathbb{R}.
\]

Let us prove the following statement:

**Theorem 4.** Let \( K \) be the Lie group generated by all symmetries of a non-flat symmetric CR geometry \((M, \mathcal{H}, J)\). Suppose \( \text{Ad}(L^0)|_{h/1} = \text{Ad}(L)|_{h/1} \). Then:

1. The manifold \( K/L \times \mathbb{R} \) is a complex manifold, and
2. The inclusion \( K/L \to K/L \times \mathbb{R} \) given as a zero section is a CR embedding.

*Proof.* We need some more details from the theory of Cartan geometries from [1] Sections 1.5.13 and 3.1.2 to give the proof. Firstly, there is a natural complement of \( u(p, q) \) in \( \mathfrak{su}(p, q) \) given by the so-called grading element, which is the unique element \( Z \in \mathfrak{su}(p, q) \) acting by \(-2\) on \( \mathbb{R} \), \(-1\) on \( \mathbb{C}^n \), \( 0 \) on \( \mathfrak{su}(p, q) \), \( 1 \) on \( \mathbb{C}^{n*} \) and \( 2 \) on \( \mathbb{R}^* \). Further, there is a Cartan connection on \( K/L \times \mathbb{R} \) induced by the CR geometry, where we identify \( \mathbb{R} \) (via exp) with the multiples of the grading element \( Z \). Then the Weyl connection \( D \) provides a reduction of this Cartan connection to \( U(p, q) \), which allows us to identify the tangent space of \( K/L \times \mathbb{R} \) with the fiber bundle \((K \times \mathbb{R}) \times_i (\mathbb{R} \oplus \mathbb{C}^n \oplus \mathfrak{su}(p, q)/u(p, q))\). We can extend the complex structure on \( \mathbb{C}^n \) to \( \mathbb{R} \oplus \mathbb{C}^n \oplus \mathfrak{su}(p, q)/u(p, q) \) by declaring \( \mathbb{R} \) to be the imaginary part of \( \mathbb{C} \) and the multiples of the grading element in \( \mathfrak{su}(p, q)/u(p, q) \) to be the real part of \( \mathbb{C} \). This definition is clearly \( U(p, q) \)-invariant (and thus \( K \)-invariant) and defines an almost complex structure \( J \) on \( K \times_i CSU(p, q)/U(p, q) \).

Let us compute the Nijenhuis tensor \( [\xi, \eta] - [J\xi, J\eta] + J([J\xi, \eta] + [\xi, J\eta]) \) of \( J \) for \( \xi, \eta \in T(K/L \times \mathbb{R}) \). For each \( x \in K/L \times \mathbb{R} \), there are vector fields \( \tilde{\xi}, \tilde{\eta} \) such that \( \xi(x) = \tilde{\xi}(x), \eta(x) = \tilde{\eta}(x) \) and that the element \( [\tilde{\xi}, \tilde{\eta}](x) \) is identified with the element \n
\[
[X, Y] - [\alpha(X + h), \alpha(Y + h)] + \alpha([X + h, Y + h]) \mod u(p, q) + \mathbb{C}^{n*} + \mathbb{R}^*,
\]

where \( \xi(x), \eta(x) \) are identified with \( X, Y \in \mathbb{R} \oplus \mathbb{C}^n \oplus \mathfrak{su}(p, q)/u(p, q) \). This identification can be obtained using the proof analogous to [1] Proposition 3.1.8 for \( T(K/L \times \mathbb{R}) \) instead of \( T(K/L) \). Indeed, the Cartan connection in the background remains the same and we only need to restrict ourselves to normal Weyl connections that coincide with \( D \) at \( x \) and project the results given by the Cartan connection to \( T(K/L \times \mathbb{R}) \) instead of \( T(K/L) \). However,

\[
[X, Y] - [\alpha(X + h), \alpha(Y + h)] + \alpha([X + h, Y + h]) = [X, Y] \mod u(p, q) + \mathbb{C}^{n*} + \mathbb{R}^*
\]

due to the condition (5) from the Lemma. Therefore we have

\[
([\xi, \eta] - [J\xi, J\eta] + J([J\xi, \eta] + [\xi, J\eta])(x) = [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]).
\]

Let us now discuss possible values of this expression for all possible incomes:

- For \( X, Y \in \mathbb{C}^n \) we have \( [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]) = 0. \)
- For \( X \in \mathbb{C}^n \) and \( Y = JZ \in \mathbb{R} \) we have \( [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]) = [JX, Z] - J([X, Z]) = 0. \)
- For \( X \in \mathbb{C}^n \) and \( Y = Z \) we have \( [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]) = [X, Z] + J([JX, Z]) = 0. \)
- For \( X = JZ \in \mathbb{R} \) and \( Y = Z \) we have \( [X, Y] - [JX, JY] + J([JX, Y] + [X, JY]) = [JZ, Z] + [Z, JZ] = 0. \)
The remaining possibilities vanish trivially. Thus the complex structure is integrable. Then the zero section is a CR embedding, because it is a closed orbit. □

In holomorphic coordinates on $U \subset K/L \times \mathbb{R}$, the hypersurface $K/L \cap U \subset \mathbb{C}^{n+1}$ can be described as a zero set of a function $F : U \to \mathbb{R}$. The Theorem 4 and the Lemma 2 provide a distinguished holomorphic coordinates in which the function $F$ has a specific form.

**Corollary 1.** Let $K$ be the Lie group generated by all symmetries of a non-flat symmetric CR geometry $(M, \mathcal{H}, J)$. Suppose $\text{Ad}(L^0)|_{h/1} = \text{Ad}(L)|_{h/1}$. Then for each point $x \in M$, there is a holomorphic coordinate system on $U \subset K/L \times \mathbb{R}$ centred at $x$ such that the function $F(z, w)$ defining $M$ satisfies $F(z, w) = F(-z, w)$.

6. **Locally flat conformal symmetric spaces**

Let us apply the construction from [13, 5] to CR geometries. We start with the standard model $PSU(p + 1, q + 1)/P$. Consider the CR manifold $M := PSU(p + 1, q + 1)/P - \{\langle u \rangle, \langle v \rangle\}$, where $u, v \in \mathbb{C}^{n+2}$ are arbitrary non-zero null vectors of $m$. The group $K(u, v)$ of CR transformations of the flat CR geometry on $M$ has two connected components. The component of identity of $K(u, v)$ is the intersection of the stabilizer of $\langle u \rangle$ and the stabilizer of $\langle v \rangle$. The other component contains the elements that swap $\langle u \rangle$ and $\langle v \rangle$. We check, whether there is a symmetry at each $K(u, v)$–orbit on $M$. Let us emphasize that if all symmetries at one point of a $K(u, v)$–orbit preserve or swap the points $\langle u \rangle$ and $\langle v \rangle$ then all symmetries at all points of the whole orbit have the same property. The orbits of the action of $K(u, v)$ on $M$ are characterized by the fact that the action preserves

- the subspace $\langle u, v \rangle$, and
- the (non)–isotropy with respect to the Hermitian form $m$.

Moreover, the action of $K(u, v)$ on $\langle u, v \rangle$ depends on whether $\langle u, v \rangle$ is isotropic subspace or not, which can provide further orbits.

**Example 1.** Assume $p, q > 1$, i.e., not the Riemannian signature. Consider the CR manifold $M = PSU(p + 1, q + 1)/P - \{\langle u \rangle, \langle v \rangle\}$ for arbitrary non-zero null vectors $u, v \in \mathbb{C}^{n+2}$ isotropic with respect to $m$, i.e., $m(u, v) = 0$. Then $\langle u, v \rangle - \{\langle u \rangle, \langle v \rangle\}$ consists of single orbit of $K(u, v)$. Further $K(u, v)$–orbits of points $\langle x \rangle$ such that $x \notin \langle u, v \rangle - \{\langle u \rangle, \langle v \rangle\}$ depend only on the (non)–isotropy of $x$ with respect to $u, v$.

We show that there exist symmetries at all points of each orbit of $K(u, v)$. Instead of fixing $\langle u \rangle$, $\langle v \rangle$ and discussing symmetries at various points $\langle x \rangle$, we fix the point $\langle x \rangle$ as the point $\langle e_0 \rangle$ given by the first vector of the standard basis $e_0, \ldots, e_{n+1}$ of $\mathbb{C}^{n+2}$ and we choose admissible $\langle u \rangle$ and $\langle v \rangle$ such that $\langle e_0 \rangle$ lies in the right orbit. Then we find all symmetries at $\langle e_0 \rangle$. Let us remind that all symmetries of the standard model at the origin $\langle e_0 \rangle$ are of the form

$$s_{Z, z} = \begin{pmatrix} -1 & -Z & iz + \frac{1}{2}ZIZ^* \\ 0 & E & -IZ^* \\ 0 & 0 & -1 \end{pmatrix},$$

where $Z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z \in \mathbb{R}$ are arbitrary, and involutive are those satisfying $z = 0$.

1. Let us start with the orbit corresponding to the situation $m(e_0, u) \neq 0$ and $m(e_0, v) \neq 0$. We can choose $u = ie_0 + \sqrt{2}e_1 - i e_{n+1}$ and $v = ie_0 - \sqrt{2}e_n + ie_{n+1}$.
Direct computation gives that there is exactly one symmetry \( s_{Z,z} \), where \( Z = (-i\sqrt{2},0,\ldots,0,-i\sqrt{2}) \) and \( z = 0 \). This symmetry is involutive and swaps \( \langle u \rangle \) and \( \langle v \rangle \). There is no symmetry preserving them.

(2) Let us now consider the orbit for the situation \( m(e_0,u) = 0 \) and \( m(e_0,v) \neq 0 \) (which is the same orbit as the orbit for the situation \( m(e_0,u) \neq 0 \) and \( m(e_0,v) = 0 \)). We can choose \( u = e_1 + e_n \) and \( v = ie_{n+1} \). Direct computation gives that there is exactly one symmetry \( s_{Z,z} \), where \( Z = (0,\ldots,0) \) and \( z = 0 \). This symmetry is involutive and preserves \( \langle u \rangle \) and \( \langle v \rangle \). There is no symmetry swapping them.

(3) The next possibility is the orbit for the situation \( m(e_0,u) = m(e_0,v) = 0 \) and \( e_0 \in \langle u,v \rangle \). We can choose \( u = e_1 + e_n \) and \( v = e_0 + e_1 + e_n \). Computation gives that there are (many) symmetries \( s_{Z,z} \), where \( Z = (z_1,\ldots,z_n) \) with components \( z_k = a_k + ib_k \) for \( k = 1,\ldots,n \) satisfies \( a_1 + a_n + 1 = 0 \) and \( b_1 + b_n = 0 \), and \( a_k, b_k \) for \( k = 2,\ldots,n-1 \) and \( z \) are arbitrary. All these symmetries swap \( \langle u \rangle \) and \( \langle v \rangle \), and there are no symmetries preserving them. In particular, there are also non-involutive symmetries for \( z \neq 0 \).

In fact, this covers all possible orbits for the case \( p = 1 \) or \( q = 1 \), i.e., the Lorentzian signature. In the other cases, there is one more orbit.

(4) Consider the orbit for the situation \( m(u,e_0) = m(v,e_0) = 0 \) and \( e_0 \notin \langle u,v \rangle \). We can choose \( u = e_1 + e_n \) and \( v = e_2 + e_{n-1} \). Computation gives that there are (many) symmetries \( s_{Z,z} \), where \( Z = (z_1,\ldots,z_n) \) satisfies \( a_1 + a_n = 0 \), \( b_1 + b_n = 0 \), \( a_2 + a_{n-1} = 0 \) and \( b_2 + b_{n-1} = 0 \) and \( a_k, b_k \) for \( k = 3,\ldots,n-2 \) and \( z \) are arbitrary. All these symmetries preserve \( \langle u \rangle \) and \( \langle v \rangle \) and there are no symmetries swapping them. In particular, there are also non-involutive symmetries for \( z \neq 0 \).

Altogether, symmetries at different orbits behave differently. Therefore, there is no smooth system of symmetries. In particular, there is no pseudo-Riemannian metric compatible with the CR geometry that would be preserved by some symmetry at every point.

Let us show that this principle does not work, if we remove two points corresponding to non-isotropic vectors.

**Example 2.** Consider the manifold \( M = PSU(p+1,q+1)/P - \{\langle u \rangle, \langle v \rangle \} \) for arbitrary non-zero null vectors \( u,v \in \mathbb{C}^{n+2} \) non-isotropic for \( m \), i.e. \( m(u,v) \neq 0 \). We can choose \( u = e_{n+1} \) and \( v = e_0 + \sqrt{2}e_1 + (1+i)e_n \). Computation gives that there is no symmetry at \( \langle e_0 \rangle \) preserving or swapping \( \langle u \rangle \) and \( \langle v \rangle \). Let us remark that the component of identity of \( K(u,v) \) is isomorphic to the group \( CSU(p,q) \) and \( K(u,v) \) does not act transitively on \( \langle u,v \rangle - \{\langle u \rangle, \langle v \rangle \} \).

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