THE BOUNDED SPHERICAL FUNCTIONS ON THE CARTAN MOTION GROUP

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Abstract. The bounded spherical functions are determined for a complex Cartan motion group.

1. Introduction

Consider a symmetric space \( X = G/K \) of noncompact type, \( G \) being a connected noncompact semisimple Lie group with finite center and \( K \) a maximal compact subgroup. Let \( g = \mathfrak{k} + \mathfrak{p} \) be the corresponding Cartan decomposition, \( \mathfrak{p} \) being the orthocomplement of \( \mathfrak{k} \) relative to the Killing form \( B(= \langle \ , \rangle) \) of \( g \). Let \( \mathfrak{a} \subset \mathfrak{p} \) be a maximal abelian subspace, \( \Sigma \) the set of root of \( g \) relative to \( \mathfrak{a} \), \( \mathfrak{a}^+ \) a fixed Weyl chamber and \( \Sigma^+ \) the set of roots \( \alpha \) positive on \( \mathfrak{a}^+ \). Let \( \rho \) denote the half sum of the \( \alpha \in \Sigma^+ \) with multiplicity. The spherical functions on \( X \) (and \( G \)) are by definition the \( K \)-invariant joint eigenfunctions of the elements in \( \mathcal{D}(X) \), the algebra of \( G \)-invariant differential operators on \( X \). By Harish-Chandra’s result \([HC58]\) the spherical functions on \( X \) are given by

\[
(1.1) \quad \phi_{\lambda}(gK) = \int_K e^{i(\lambda - \rho)(H(gK))} \, dk, \quad \phi(eK) = 1,
\]

where \( \exp H(g) \) is the \( A \) factor in the Iwasawa decomposition \( G = KAN \) (\( N \) nilpotent) and \( \lambda \) ranges over the space \( \mathfrak{a}^*_c \) of complex-valued linear functions on \( \mathfrak{a} \). Also, \( \phi_\lambda \equiv \phi_\mu \) if and only if the elements \( \lambda, \mu \in \mathfrak{a}^*_c \) are conjugate under \( W \).

Let \( L^2(G) \) denote the (commutative) Banach algebra of \( K \)-bi-invariant integrable functions on \( G \). The maximal ideal space of \( L^2(G) \) is known to consist of the kernels of the spherical transforms

\[
f \rightarrow \int_G f(g) \phi_{-\lambda}(g) \, dg
\]

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for which $\phi_{-\lambda}$ is bounded. These bounded spherical functions were in [HJ69] found to be those $\phi_{\lambda}$ for which $\lambda$ belongs to the tube $a^* + iC(\rho)$ where $C(\rho)$ is the convex hull of the points $s\rho(s \in W)$.

This result is crucial in proving that the horocycle Radon transform is injective on $L^1(X)$ ([H70], Ch. II).

2. The boundedness criterion.

In this note we deal with the analogous question for the Cartan motion group $G_0$. This group is defined as the semidirect product of $K$ and $p$ with respect to the adjoint action of $K$ on $p$. The $X_0 = G_0/K$ is naturally identified with the Euclidean space $p$. The element $g_0 = (k,Y)$ actions on $p$ by

$$g_0(Y') = Ad(k)Y' + Y \quad k \in K, Y, Y' \in p,$$

so the algebra $\mathfrak{D}(X_0)$ of $G_0$–invariant differential operators on $X_0$ is identified with the algebra of $Ad(K)$–invariant constant coefficient differential operators on $p$. The corresponding spherical functions on $X_0$ are given by

$$\psi_{\lambda}(Y) = \int_K e^{i\lambda(Ad(k)Y)} \, dk \quad \lambda \in a_+^*,$$

and $\psi_{\lambda} \equiv \psi_{\mu}$ if and only if $\lambda$ and $\mu$ are $W$–conjugate. See e.g. [HS4], IV §4. Again, the maximal ideal space of $L^\natural(G_0)$ is up to $W$–invariance identified with the set of $\lambda$ in $a_+^*$ for which $\psi_{\lambda}$ is bounded. Since $\rho$ is related to the curvature of $G/K$ it is natural to expect the bounded $\psi_{\lambda}$ to come from replacing $C(\rho)$ by the origin, in other words $\psi_{\lambda}$ is would be expected to be bounded if and only if $\lambda$ is real, that is $\lambda \in a^*$.

The bounded criterion in [HJ69] for $X$ relies on Harish-Chandra’s expansion for $\phi_{\lambda}$, combined with the reduction to the boundary components of $X$. These are certain subsymmetric spaces of $X$. These tools are not available for $X_0$ so the “tangent space analysis” in [H80] relies on approximating $\psi_{\lambda}$ by $\phi_{\lambda}$ suitably modified. Although several papers ([BC86], [R88], [SØ05]) are directed to asymptotic properties of the function $\psi_{\lambda}$ the boundedness question does not seem to be addressed there. In this note we only give a partial solution through the following result.

Theorem 2.1. Assume the group $G$ complex. The spherical function $\psi_{\lambda}$ on $G_0$ is bounded if and only if $\lambda$ is real, i.e. $\lambda \in a^*$. 
For $\lambda \in \mathfrak{a}^*$ let $\lambda = \xi + i\eta$ with $\xi, \eta \in \mathfrak{a}^*$. It remains to prove that if $\lambda_0 = \xi_0 + i\eta_0$ with $\eta_0 \neq 0$ then $\psi_{\lambda_0}$ is unbounded. For $\lambda \in \mathfrak{a}^*$ let $A_\lambda \in \mathfrak{a}_c$ be determined by $\langle A_\lambda, H \rangle = \lambda(H) (H \in \mathfrak{a})$. With $i\lambda_0 = i\xi_0 - \eta_0$ we may by the $W$-invariance of $\psi_\lambda$ in $\lambda$ assume that $-A_{\eta_0} \in \mathfrak{a}^+$ (the closure of $\mathfrak{a}^+$).

Let $U \subset W$ be the subgroup fixing $\lambda_0$ and $V \subset W$ the subgroup fixing $\eta_0$. Then $U \subset V$ and

$$\psi_{s\xi_0 + i\eta_0} = \psi_{\xi_0 + i\eta_0} \quad \text{for } s \in V.$$  

In addition we assume that for the lexicographic ordering of $\mathfrak{a}^*$ defined by the simple roots $\alpha_1, \ldots, \alpha_\ell$ we have $\xi_0 \geq s\xi_0$ for $s \in V$.

In particular,

$$\alpha(A_{\xi_0}) \geq 0 \quad \text{for } \alpha \in \Sigma^+ \text{ satisfying } \alpha(A_{\eta_0}) = 0.$$  

**Lemma 2.2.** The subgroup $U$ of $W$ fixing $\lambda_0$ is generated by the reflections $s_\alpha$, where $\alpha_i$ is a simple root vanishing at $A_{\lambda_0}$.

**Proof.** We first prove that some of the $\alpha_i$ vanishes at $A_{\lambda_0}$. The group $U$ is generated by the $s_\alpha$ for which $\alpha > 0$ vanishes on $\lambda_0$ ([H78], VII, Theorem 2.15). If $\alpha$ is such then $\alpha(-A_{\eta_0}) = 0$ and since $\alpha = \sum n_j \alpha_j$ ($n_j \neq 0$ in $\mathbb{Z}^+$) and $\alpha_j(-A_{\eta_0}) \geq 0$ we see that each of these $\alpha_j$ vanishes on $A_{-\eta_0}$. Since $\alpha(A_{\xi_0}) = 0$ and $\alpha_j(A_{\xi_0}) \geq 0$ by (2.3) for each $j$ we deduce $\alpha_j(A_{\xi_0}) = 0$.

Let $U'$ denote the subgroup $U$ generated by those $s_{\alpha_i}$ with $\alpha_i$ vanishing at $\lambda_0$. For each $\alpha > 0$ mentioned above we shall prove $\alpha = s_{\alpha_p}$ where $s \in U'$ and $\alpha_p$ is simple and vanishes at $A_{\lambda_0}$. We shall prove this by induction on $\sum m_i$ if $\alpha = \sum m_i \alpha_i$ ($m_i \neq 0$ in $\mathbb{Z}^+$). The statement is clear if $\sum m_i = 1$ so assume $\sum m_i > 1$. Since $\langle \alpha, \alpha \rangle > 0$ we have $\langle \alpha, \alpha_k \rangle > 0$ for some $k$ among the indices $i$ above. Then $\alpha \neq \alpha_k$ (by $\sum m_i > 1$). Since $s_{\alpha_k}$ permutes the positive roots $\neq \alpha_k$ we have $s_{\alpha_k} \alpha \in \sum^+$ and $s_{\alpha_k} \alpha = \sum j m_j' \alpha_j (m_j' \in \mathbb{Z}^+)$ and by the choice of $k, \sum m_j' < \sum m_i$. Now $\alpha(A_{\lambda_0}) = 0$ and $\alpha_i(A_{-\eta_0}) \geq 0$ so for each $i$ in the sum for $\alpha$ above, $\alpha_i(A_{\eta_0}) = 0$. Hence by (2.3) $\alpha_i(A_{\xi_0}) = 0$. In particular $s_{\alpha_k} \in U$. Thus the induction assumption applies to $s_{\alpha_k} \alpha$ giving a $s' \in U'$ for which $s_{\alpha_k} \alpha = s' \alpha_p$. Hence $\alpha = s_{\alpha_p}$ with $s \in U'$. But then $s_{\alpha} = ss_{\alpha_k}s^{-1}$ proving the lemma.  \( \square \)

Using Harish-Chandra’s integral formula [HC57] Theorem 2 we have
\[ \psi_\lambda(\exp H) = c_0 \sum_{s \in W} \epsilon(s)e^{i(sA_\lambda, H)} \langle \pi(H), \pi(A_\lambda) \rangle, \quad (H \in \mathfrak{a}), \]

where \( c_0 \) is a constant, \( \langle , \rangle \) the Killing form, \( \epsilon(s) = \det s \) and \( \pi \) the product of the positive roots. If \( \eta_0 \) is regular so \(-A_\eta_0 \in \mathfrak{a}^+\) then \( V = U = \{e\} \) and \( \pi(A_\lambda_0) \neq 0 \). Fix \( H_0 \in \mathfrak{a}^+ \) and \( \lambda = \lambda_0 \) in the sum (2.4). With \( H = tH_0(t > 0) \) the term in (2.4) with \( s = e \) will outweigh all the others as \( t \to +\infty \) so \( \psi_\lambda \) is unbounded.

We now consider the case \( \pi(A_\lambda_0) = 0 \).

Let \( \pi' \) denote the product of the positive roots \( \beta_1, \ldots, \beta_r \) vanishing at \( \lambda_0 \) and \( \pi'' \) the product of the remaining positive roots. For \( \lambda = \lambda_0 \) we want to divide the factor \( \pi'(\lambda_0) \) into the numerator of (2.4). We do this by multiplying (2.4) by \( \pi'(\lambda) \), then applying the differential operator \( \partial(\pi') \) in the variable \( \lambda \) and finally setting \( \lambda = \lambda_0 \). The theorem then follows from the following lemma.

**Lemma 2.3.** Let \( \eta_0 \neq 0 \). Then the function

\[ \zeta_\lambda(H) = \sum_{s \in W} \epsilon(s)e^{i(sA_\lambda, H)} \pi(A_\lambda) \]

is for the case \( \lambda = \lambda_0 \) unbounded on \( \mathfrak{a}^+ \).

**Proof.** We have

\[ \pi'(\lambda)\zeta_\lambda(H) = \frac{1}{\pi''(\lambda)} \sum_{s \in W} \epsilon(s)e^{i(sA_\lambda, H)}. \]

Applying \( \partial(\pi') = \partial(\beta_1) \ldots \partial(\beta_r) \) in \( \lambda \) and putting \( \lambda = \lambda_0 \) we see that

(2.5)
\[ c \zeta_{\lambda_0}(H) = \sum_{s \in W} P_s(H)e^{i(sA_{\lambda_0}, H)}. \]

Here \( c \) is a constant and \( P_s \) the polynomial

\[ P_s(H) = \left[ \partial(\pi')_\lambda \left( \epsilon(s)\frac{1}{\pi''(\lambda)}e^{is\lambda(H)} \right) \right]_{\lambda=\lambda_0} e^{-is\lambda_0(H)} \]

whose highest degree term is a constant times

(2.6)
\[ \epsilon(s)\frac{1}{\pi''(\lambda_0)}(s\pi')(H). \]
We do not need the exact value of $c$ but for $r = 2, 3$, respectively, it equals (with $x_{ij} = \langle \alpha_i, \alpha_j \rangle$)

\[
x_{12}^2 + x_{11}x_{22} + x_{11}x_{23}^2 + x_{22}x_{13}^2 + x_{33}x_{12}^2 + x_{11}x_{22}x_{33} + 2x_{12}x_{13}x_{23}.
\]

We break the sum (2.5) into two parts, sum over $V$ and sum over $W \setminus V$. For the first we consider $\Sigma_{V}$ as $\Sigma_{V/U} \Sigma_{U}$. Then (2.5) can be written

\[
(2.7) \quad c \zeta_\lambda_0(H) = e^{-\eta_0(H)} \left[ \sum_{V/U} e^{i\xi_0(H)} \sum_{\sigma \in U} P_{\sigma}(H) \right] + \sum_{W \setminus V} P_{s}(H) e^{i\lambda_0(H)}.
\]

We put here $H' = -A_{\eta_0}$, let $H_0 \in \mathfrak{a}^+$ be arbitrary and set $H = tH_0 (t > 0)$. Then the second term in (2.7) equals

\[
(2.8) \quad \sum_{s \notin V} P_{s}(tH_0) e^{i\xi_0(tH_0)} e^{(sH', tH_0)}.
\]

By a standard property of $\mathfrak{a}^+$ we have

\[
\langle H_1, H_2 \rangle \geq \langle sH_1, H_2 \rangle \quad \text{if } H_1, H_2 \in \mathfrak{a}^+
\]

so taking limit,

\[
\langle sH' - H', H \rangle \leq 0, \quad H \in \mathfrak{a}^+.
\]

If $s \notin V$ then $sH' - H' \neq 0$. Thus the map $H \to \langle sH' - H', H \rangle$ is open from $\mathfrak{a}$ to $\mathbb{R}$ mapping $\mathfrak{a}^+$ into $\{ t \leq 0 \}$, not taking there the boundary value 0. Hence we get

\[
(2.9) \quad \langle H', H_0 \rangle > \langle H', sH_0 \rangle \quad \text{for } s \notin V.
\]

Equivalently, dist $(H_0, H') <$ dist $(H_0, sH')$ for $s \notin V$.

Consider (2.7) with $H = tH_0$. Assume the expression in the bracket has absolute value with $\limsup_{t \to +\infty} \neq 0$. Considering (2.9) the first term in (2.7) would have exponential growth larger than that of each term in (2.8).

Thus $c \neq 0$ and

\[
\lim_{t \to +\infty} \zeta_\lambda_0(tH_0) = \infty
\]

implying Lemma 2.3 in this case.
We shall now exclude the possibility that the quantity in the bracket in (2.7) (with \( H = tH_0 \)) has absolute value with \( \limsup_{t \to \infty} = 0 \). For this we use the following elementary result of Harish–Chandra [HC58], Corollary of Lemma 56: Let \( a_1, \ldots, a_n \) be nonzero complex numbers and \( p_0, \ldots, p_n \) polynomials with complex coefficients.

Suppose

\[
(2.10) \quad \limsup_{t \to \infty} \left| p_0(t) + \sum_{j=1}^{n} p_j(t) e^{a_j t} \right| \leq a
\]

for some \( a \in \mathbb{R} \). Then \( p_0 \) is a constant and \( |p_0| \leq a \). This implies the following result.

Let \( k_1 \ldots k_n \in \mathbb{R} \) be different and \( p_1, \ldots, p_n \) polynomials. If

\[
(2.11) \quad \limsup_{t \to +\infty} \left| \sum_{1}^{n} e^{i k_r t} p_r(t) \right| = a < \infty
\]

then each \( p_r \) is constant. If \( a = 0 \) then each \( p_r = 0 \). This follows from (2.10) by writing the above sum as

\[
e^{i k_r t} \left( p_r(t) + \sum_{j \neq r} e^{i(k_j - k_r)t} p_j(t) \right).
\]

Note that in the sum

\[
(2.12) \quad \sum_{V/U} e^{is \xi_0 (tH_0)} \sum_{\sigma \in U} P_{s\sigma}(tH_0)
\]

all the terms \( s \xi_0 \) are different (\( s_1, s_2 \in V \) with \( s_1 \xi_0 = s_2 \xi_0 \) implies \( s_2^{-1} s_1 \in U \)). Thus we can choose \( H_0 \in a^+ \) such that all \( s \xi_0(H_0) \) are different.

We shall now show that one of the polynomial in (2.12), namely the one for \( s = e \),

\[
(2.13) \quad \sum_{\sigma \in U} P_{\sigma}(tH_0)
\]

is not identically 0. For this note that the highest degree term in \( P_{\sigma} \) is a constant (independent of \( \sigma \)) times
\[ (2.14) \quad \epsilon(\sigma) \frac{1}{\pi''(\lambda_0)}(\sigma \pi'')(tH_0). \]

Now each \( \sigma \) permutes the roots vanishing at \( A\lambda_0 \). Hence \( \sigma \pi' = \epsilon'(\sigma) \pi' \) where \( \sigma \rightarrow \epsilon'(\sigma) \) is a homomorphism of \( U \) into \( \mathbb{R} \). We now use Lemma 2.2. Since each \( s_{\alpha_i} \in U \) maps \( \alpha_i \) into \( -\alpha_i \) and permutes the other positive roots vanishing at \( \lambda_0 \) we see that \( \epsilon'(s_{\alpha_i}) = -1 = \epsilon(s_{\alpha_i}) \). Thus by Lemma 2.2 \( \epsilon'(\sigma) = \epsilon(\sigma) \) for each \( \sigma \in U \). Thus (2.14) reduces to

\[ \frac{1}{\pi''}(tH_0). \]

This shows that the polynomial in (2.13) is not identically 0. In view of (2.11) this shows that the lim sup discussed is \( \neq 0 \) and Lemma 2.3 established.

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