Convergence of the logarithm of the characteristic polynomial of unitary Brownian motion in Sobolev space

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Abstract
We prove that the convergence of the real and imaginary parts of the logarithm of the characteristic polynomial of unitary Brownian motion toward Gaussian free fields on the cylinder, as the matrix dimension goes to infinity, holds in certain suitable Sobolev spaces, whose regularity we prove to be optimal. Our result can be seen as the natural dynamical analogue to the stationary result for a fixed time by Hughes et al (2001 Commun. Math. Phys. 220 429–51). Further our result is related to the work of Spohn (1998 Markov Processes and Related Fields vol 4), from which the identification of the above limit as the Gaussian free field first followed, albeit in a different function space.

Keywords: unitary Brownian motion, Gaussian free field, characteristic polynomial, random matrix theory

1. Introduction

Since unitary Brownian motion preserves the Haar measure on the unitary group $U(n)$, to many results of Haar distributed unitary matrices there is a corresponding dynamical result for a unitary Brownian motion $U$ at equilibrium. This is in particular the case for some properties of the eigenvalues, whose dynamics have been studied first by Dyson [1], who computed a stochastic differential equation describing their evolution. In this paper, we intend to achieve

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such a transition from static to dynamic for the Hughes–Keating–O’Connell theorem on the large $n$ limit of the logarithm of the characteristic polynomial.

Characteristic polynomials of random matrices are fundamental objects in random matrix theory. They are closely related to the theory of log-correlated fields and to Gaussian multiplicative chaos, see [2–10] for the circular ensembles, and [11–13] for Hermitian ensembles and the Ginibre ensemble, among others. In the case of Haar-distributed matrices from the classical compact groups, there are also remarkable similarities between the statistics of the characteristic polynomial and those of the Riemann zeta function and other number-theoretic $L$-functions, see for example [14–21] for a review.

The goal of this paper is to specify Sobolev spaces of optimal regularity in which the convergence of the real and imaginary part of the logarithm of the characteristic polynomial to Gaussian free fields holds\(^3\). This is then the natural dynamical version of the corresponding stationary result for Haar-distributed unitary matrices by Hughes et al [2], who proved that for any fixed time the logarithm of the characteristic polynomial converges to a generalized Gaussian field on the unit circle.

1.1. Context

We let $U_n : \mathbb{R} \to \mathcal{U}(n)$ be a unitary Brownian motion (see section 2.1 for a definition) at its equilibrium measure, which is Haar measure, and define its characteristic polynomial as

$$ p_n(t, \theta) := \det \left( I_n - e^{-i\theta} U_n(t) \right) = \prod_{k=1}^{n} (1 - e^{i(\theta_k(t) - \theta)}), \quad (\theta, t) \in [0, 2\pi) \times \mathbb{R}, $$

where $0 \leq \theta_1(t) < \ldots < \theta_n(t) < 2\pi$ denote the eigenangles. We define its logarithm by

$$ \log p_n(t, \theta) := \sum_{k=1}^{n} \log \left( 1 - e^{i(\theta_k(t) - \theta)} \right), $$

with the branches on the RHS being the principal branches, such that $\Im \log(1 - e^{i(\theta_k(t) - \theta)}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$, with $\Im \log 0 := \pi/2$. Since

$$ \log (1 - z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}, $$

for $|z| \leq 1$, where for $z = 1$ both sides equal $-\infty$, and by the identity $\log \det = \text{Tr} \log$, we see that the Fourier expansion of $\log p_n$ w.r.t. the spatial variable $\theta$ is given as follows:

$$ \log p_n(t, \theta) = -\sum_{k=1}^{\infty} \frac{\text{Tr} \left( U^k_n(t) \right)}{k} e^{-ik\theta}. $$

By the results of Diaconis and Shahshahani [22] and Diaconis and Evans [23] it follows that for any $l \in \mathbb{N}$ and fixed $t \in \mathbb{R}$, the random variables $k^{-1} \text{Tr} \left( U^k_n(t) \right)$, $k = 1, \ldots, l$, converge in

\(^3\) Several Sobolev spaces are involved because we can improve the regularity with respect to one of the parameters at the cost of sacrificing some regularity with respect to the other parameter.
distribution to independent complex Gaussians \( A_k, k = 1, \ldots, l \), whose real and imaginary parts are independent centered real Gaussians with variance \( 1/(2k) \). By also showing tightness in the negative Sobolev spaces \( H_{0}^{-\epsilon} (S^1) \) (see section 2.2 for a definition), Hughes, Keating and O’Connell proved that for any fixed \( t \in \mathbb{R} \), \( \log p_n(t, \cdot) \) converges to a generalized Gaussian field as \( n \to \infty \). Their result, reformulated to our setting, is as follows:

**Theorem 1.1 (Hughes et al [2]).** For any \( \epsilon > 0 \) and any fixed \( t \in \mathbb{R} \), the sequence of random functions \( (\log p_n(t, f))_{n \in \mathbb{N}} \) converges in distribution in \( H_{0}^{-\epsilon} (S^1) \) to the generalized Gaussian field

\[
X(\theta) := \sum_{k=1}^{\infty} A_k e^{-i k \theta},
\]

where \( A_k \) is a complex Gaussian whose real and imaginary parts are independent centered real Gaussians with variance \( 1/(2k) \).

It is natural thus to assume that in the dynamic case, i.e. when considering \( \log p_n \) also as a function of \( t \), that the limit (in an appropriate function space) would be given by the infinite-dimensional Ornstein-Uhlenbeck process

\[
X(t, \theta) := \sum_{k=1}^{\infty} A_k(t) e^{-i k \theta},
\]

where \( A_k(\cdot), k \in \mathbb{N} \), are independent complex Ornstein–Uhlenbeck processes at their stationary distribution, i.e. (up to a linear time change) solutions to the SDEs

\[
dA_k(t) = -k A_k(t) \, dt + d(W_k(t) + i \tilde{W}_k(t)),
\]

with the real and imaginary parts of \( A_k(t) \) at any fixed time \( t \) being independent real Gaussians with variance \( 1/(2k) \), and \( W_k, \tilde{W}_k, k \in \mathbb{N} \), denoting real standard Brownian motions. The identification of \( X \) as the limit of \( \log p_n \), and in particular the weak convergence of the dynamic Fourier coefficients \( t \to k^{-1} \text{Tr}(U_n^k(t)) \), \( k \in \mathbb{N} \), to the independent Ornstein–Uhlenbeck processes \( A_k, k \in \mathbb{N} \), follows from the work of Spohn [24], as we now explain.

For a function \( f : S^1 \to \mathbb{C} \), we let \( f_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i \theta}) e^{-i k \theta} d\theta \) denote its \( k \)th Fourier coefficient. For fixed \( \epsilon > 0 \), and real-valued functions \( f \in H_{0}^{3/2+\epsilon} (S^1, \mathbb{R}) \), Spohn considered the linear statistics

\[
\xi_n(t, f) := \sum_{j=1}^{n} f(e^{i \theta_j(t)}) = \sum_{j=1}^{n} \sum_{k \in \mathbb{Z} \setminus \{0\}} f_k e^{i k \theta_j(t)} = \sum_{k \in \mathbb{Z} \setminus \{0\}} f_k \text{Tr}(U_n(t)^k),
\]

of the eigenvalues \( e^{i \theta_1(t)}, \ldots, e^{i \theta_n(t)} \) of unitary Brownian motion (in fact he more generally considered interacting particles on the unit circle with different repulsion strengths). Since \( H_{0}^{-3/2-\epsilon} (S^1, \mathbb{R}) \) is the dual space of \( H_{0}^{3/2+\epsilon} (S^1, \mathbb{R}) \), one can consider \( \xi_n \) as a random variable in \( C(\mathbb{R}, H_{0}^{-3/2-\epsilon} (S^1, \mathbb{R})) \), and (with an abuse of notation) represent it as a time-dependent formal Fourier series, and see that
\[ \xi_n(t, \theta) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \text{Tr} \left( U_n(t)^{-k} \right) e^{ik\theta} = -i \partial_\theta \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\text{Tr} \left( U_n(t)^{-1} \right) e^{ik\theta}}{|k|} \right) = -2 \partial_\theta \Im \log p_n(t, \theta) \]

\[ = \sqrt{-\partial_\theta^2} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\text{Tr} \left( U_n(t)^{-1} \right) e^{ik\theta}}{|k|} \right) = 2 \sqrt{-\partial_\theta^2} \Re \log p_n(t, \theta), \]

where \( \sqrt{-\partial_\theta^2} \) denotes the operator that multiplies the \( k \)th Fourier coefficient by \( |k| \). Spohn’s result, reformulated to our setting, is then as follows:

**Theorem 1.2 (Spohn [24]).** Let \( \epsilon > 0 \) and endow \( C(\mathbb{R}, H^{-3/2-\epsilon}(S^1, \mathbb{R})) \) with the topology of locally uniform convergence. Then, as \( n \to \infty \), the random variables \( \xi_n \) converge in distribution to the infinite-dimensional Ornstein–Uhlenbeck process (where \( A_{-k} = \overline{A_k} \) for \( k \in \mathbb{N} \) and \( \text{sgn}(k) = 1_{k>0} - 1_{k<0} \))

\[ \xi(t, \theta) := \sum_{k \in \mathbb{Z} \setminus \{0\}} |k| A_{-k}(t) e^{ik\theta} = -i \partial_\theta \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} \text{sgn}(k) A_{-k}(t) e^{ik\theta} \right) = -2 \partial_\theta \Im \mathcal{X}(t, \theta) \]

\[ = \sqrt{-\partial_\theta^2} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} A_{-k}(t) e^{ik\theta} \right) = 2 \sqrt{-\partial_\theta^2} \Re \mathcal{X}(t, \theta). \]  

**Remark 1.3.** Spohn defined \( \xi \) as a Gaussian process indexed by \( (t, f) \in \mathbb{R} \times H^{3/2+\epsilon}(S^1, \mathbb{R}) \) (as in (3)), i.e. \( \xi(t, \cdot) = (\xi(\cdot, \cdot), f) \), which satisfies a certain stochastic partial differential equation. To see that the two definitions are equivalent, one can either use Gaussianity and do a covariance calculation, see section 2.2 in (3), or see that the SDEs for \( \xi(t, e^{ik}/k) := \langle \xi(t, \cdot), e^{ik}/k \rangle = \langle \xi(t, \cdot), \cos(k \cdot)/k \rangle + i \langle \xi(t, \cdot), \sin(k \cdot)/k \rangle, k \in \mathbb{Z} \setminus \{0\} \), that follow from Spohn’s representation, are exactly the SDEs (2) that define \( A_k \). In particular, Spohn’s result implies that

\[ \xi_n(\cdot, e^{ik}/k) = |k|^{-1} \text{Tr} \left( U_n^{-k} (\cdot) \right) \to \xi(\cdot, e^{ik}/k) = A_k(\cdot), \]

weakly as random variables in \( C(\mathbb{R}, \mathbb{C}) \), i.e. it implies convergence of the dynamic Fourier coefficients of \( \log p_n \) to those of \( X \).

In this paper we prove the following theorem, which on the one hand can be seen as the natural dynamic analogue of theorem 1.1, and on the other hand as an analogue to Spohn’s result (see below remark 1.5):

**Theorem 1.4.** For any \( s \in [0, \frac{1}{4}) \), \( \epsilon > s \), and \( T > 0 \), the sequence of random fields \( (\log p_n)_n \) converges in distribution in the tensor product of Hilbert spaces \( H_s([0, T]) \otimes H_0^{-s}(S^1) \) (see section 2.2 for a definition) to the generalized Gaussian field \( X \) in (1).

Furthermore, those regularity parameters \( s \) and \( -\epsilon \) are optimal, in the sense that for \( s = 1/2 \) or \( s \geq \epsilon \geq 0 \) the sequence \( (\log p_n)_n \) almost surely does not converge and \( X \) is almost surely not an element of \( H_s([0, T]) \otimes H_0^{-s}(S^1) \).
Remark 1.5. A straightforward calculation shows that the covariance functions of $\Re X$ and $\Im X$ are given by

$$
\mathbb{E}(\Re X(t, \theta), \Re X(t', \theta')) = \mathbb{E}(\Im X(t, \theta), \Im X(t', \theta'))
$$

$$
= \sum_{k=1}^{\infty} \frac{e^{-k|t-t'|}}{2k} \cos(k(\theta - \theta')) = \frac{1}{2} \log \left| \frac{e^{i t - \theta} - e^{-i t - \theta'}}{e^{-i t - \theta} - e^{i t - \theta'}} \right|.
$$

This implies that $\Re X$ and $\Im X$ are Gaussian free fields on the infinite cylinder $\mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}$, since

$$
\frac{1}{2} \log \left| \frac{e^{i t - \theta} - e^{-i t - \theta'}}{e^{-i t - \theta} - e^{i t - \theta'}} \right| = \pi (-\Delta)^{-1}(t, \theta, t', \theta'),
$$

where $(-\Delta)^{-1}$ denotes the Green’s function associated to the Laplacian $\Delta = \partial_t^2 + \partial_\theta^2$, see [3, section 2.2].

To compare our theorem 1.4 to Spohn’s theorem 1.2, observe that since by the latter it holds that $\sqrt{-\partial_\theta^2} \Re X(t, \theta), \partial_0 \Im X \in \mathcal{C}([0, T], H_0^{3/2-\varepsilon}(S^1, \mathbb{R}))$, for $\varepsilon > 0$, it follows that $\Re X, \Im X \in \mathcal{C}([0, T], H_0^{3/2-\varepsilon}(S^1, \mathbb{R}))$ for $\varepsilon > 1/2$. Further note that the inclusions

$$
H^s([0, T]) \otimes H_0^{-s}(S^1, \mathbb{R}) \subset \mathcal{C}([0, T], H_0^{-s}(S^1, \mathbb{R})),
$$

$$
H_0^{-s}(S^1, \mathbb{R}) \subset \mathcal{C}(S^1, \mathbb{R}),
$$

are valid if and only if $s > 1/2$, and $\varepsilon > 1/2$, respectively. Thus Spohn’s theorem 1.2 implies that $X$ can be considered as a continuous function in $t$, but then it is of regularity $-\varepsilon < -1/2$ in $\theta$, and can thus only be integrated against functions of regularity $\varepsilon > 1/2$, which are in particular continuous. Our theorem 1.4 treats the case where $X$ is considered to be of regularity $s < 1/2$ in $t$, in which it is not continuous in $t$ anymore, and shows that then $X$ is of regularity $-\varepsilon < -s > -1/2$ in $\theta$, and can thus be integrated against functions of regularity $\varepsilon < s < 1/2$, which includes discontinuous functions.

Recently Bourgade and Falconet proved the first dynamical generalization of Fisher-Hartwig asymptotics [3, theorem 1.2], which allowed them to show that for $\gamma \in (0, 2\sqrt{2})$ the random measures on $\mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}$

$$
\frac{|p_{\gamma}(t, \theta)|^\gamma}{\mathbb{E}(|p_{\gamma}(t, \theta)|^\gamma)} \, dt \, d\theta,
$$

converge weakly to a Gaussian multiplicative chaos measure, which can be formally written as

$$
\frac{e^{\gamma \Re X(t, \theta)}}{\mathbb{E}(e^{\gamma \Re X(t, \theta)})} \, dt \, d\theta.
$$

Their dynamic Fisher-Hartwig asymptotics are related to this paper since they in particular imply that the exponential moments of the linear statistics

$$
\xi_n(t, f) = \langle \xi_n(t, \cdot), f \rangle,
$$

converge to the exponential moments of $\xi(t, f) := \langle \xi(t, \cdot), f \rangle$, jointly for any finite number of $f$ in a certain class of functions, and times $t \in \mathbb{R}$, thus providing another proof for the convergence
of the dynamic Fourier coefficients of $\log p_n$ to those of $X$. The specific result from Bourgade and Falconet that we use in our proof of theorem 1.4 is the following [3, corollary 3.2]:

**Corollary 1.6 (Bourgade, Falconet).** Let $(e^{i\theta_1(t)}, \ldots, e^{i\theta_n(t)})_{t \geq 0}$ denote the eigenvalue process of unitary Brownian motion, started at Haar measure. For functions $f, g \in H^1(S^1)$, $n \in \mathbb{N}$ and $t \geq 0$, it holds that

$$
\mathbb{E}[\xi_n(0,f)\xi_n(t,g)] = \sum_{|k| \leq n-1} f_k g_{-k} \text{sgn}(k) e^{-|k|t} \frac{\sinh(k^2 t)}{k} + \sum_{|k| \geq n} f_k g_{-k} \frac{\sin(k^2 t)}{k}.
$$

2. Mathematical preliminaries

2.1. Unitary Brownian motion

We define Brownian motion $(U_n(t))_{t \geq 0}$ on the unitary group $U(n)$ as the diffusion governed by the stochastic differential equation

$$
dU_n(t) = \sqrt{2} U_n(t) dB_n(t) - U_n(t) dt,
$$

with $(B_n(t))_{t \geq 0}$ denoting a Brownian motion on the space of skew-Hermitian matrices. That is

$$
B_n(t) = \sum_{k=1}^{n^2} \xi_k B^{(k)}(t),
$$

where $B^{(k)}$, $k = 1, \ldots, n^2$, are independent one-dimensional standard Brownian motions, and where the matrices $X_k$, $k = 1, \ldots, n^2$, are an orthonormal basis of the real vector space of skew-Hermitian matrices w.r.t. the scalar product $\langle A, B \rangle := n \text{Tr}(AB^*)$. One such basis is given by the matrices $\frac{1}{\sqrt{2n}} (E_{k,l} - E_{l,k})$, $\frac{1}{\sqrt{2n}} (E_{k,l} + E_{l,k})$, $1 \leq k < l \leq n$, and $\frac{1}{\sqrt{n}} E_{k,k}$, $1 \leq k \leq n$, where $(E_{k,l})_{mn} = \delta_{km} \delta_{ln}$.

**Remark 2.1.** Unitary Brownian motion is usually defined using a different normalisation, i.e. satisfying the SDE $\frac{dU_n(t)}{dt} = \frac{1}{2} U_n(t) dB_n(t)$. With this normalisation the generator is given by one half times the Laplacian on $U(n)$, which is the usual definition of Brownian motion on a Riemannian manifold. The relation between the two normalisations is $U_n(t) = U_n(t)$. We chose our normalisation to be consistent with the work of Spohn [24], and Bourgade and Falconet [3].

Unitary Brownian motion at its stationary/equilibrium distribution, which is Haar measure, is reversible and can thus be defined for all $t \in \mathbb{R}$. In this paper we always consider unitary Brownian motion at equilibrium, so $U_n(t)$ is Haar distributed for all $t \in \mathbb{R}$.

2.2. Sobolev spaces and their tensor product

Consider the space of square integrable $\mathbb{C}$-valued functions on the unit circle, with vanishing mean:

$$
L^2_0(S^1) = \left\{ f(\theta) = \sum_{k \in \mathbb{Z}} f_k e^{ik\theta} : \sum_{k \in \mathbb{Z}} |f_k|^2 < \infty, f_0 = 0 \right\}.
$$
For $s \geq 0$, we define $H^s_0(S^1)$ as the restriction of $L^2_0(S^1)$ w.r.t. the functions for which the inner product

$$\langle f, g \rangle_s = \sum_{k \in \mathbb{Z}} |k|^{2s} f_k \overline{g_k},$$

is finite. For $s < 0$, we define $H^s_0(S^1)$ as the completion of $L^2_0(S^1)$ w.r.t. the scalar product $\langle \cdot, \cdot \rangle_s$. Thus we see that for all $s \in \mathbb{R}$ it holds that

$$H^s_0(S^1) = \left\{ f(\theta) = \sum_{k \in \mathbb{Z}} f_k e^{ik\theta} : \sum_{k \in \mathbb{Z}} |k|^{2s} |f_k|^2 < \infty, f_0 = 0 \right\},$$

i.e. $H^s(S^1)$ can be seen as a space of (formal) Fourier series. Note that $(H^s_0(S^1), \langle \cdot, \cdot \rangle_s)$ is a Hilbert space for all $s \in \mathbb{R}$. For $s \geq 0$ it is a subspace of $H^0_0(S^1) = L^2_0(S^1)$, i.e. the space of square-integrable functions with zero mean, while for $s < 0$, $H^s_0(S^1)$ can be interpreted as the dual space of $H^{-s}_0(S^1)$, i.e. as a space of generalized functions defined up to additive constant. We denote by $H^s_0(S^1, \mathbb{R})$ the subset of $H^s_0(S^1)$ consisting of real-valued (generalized) functions, i.e. those for which $f_k \in \mathbb{R}$.

Fix $T > 0$. For $s = 0$ we set $H^s([0,T]) := L^2([0,T])$, and for $s \in (0,1)$, we define the fractional Sobolev space $H^s([0,T])$ as the subspace of $L^2([0,T])$, where the Slobodeckij inner product

$$\langle f, g \rangle_s := \int_0^T \int_0^T \frac{(f(t) - f(u))(g(t) - g(u))}{|t - u|^{1+2s}} \, dt \, du,$$

is finite. Note that $(H^s([0,T]), \langle \cdot, \cdot \rangle_s)$ is a Hilbert space for all $s \geq 0$.

**Remark 2.2.** For the fact that the fractional Sobolev spaces defined through Fourier series or through the Slobodeckij norm agree, the reader can consult e.g. [25].

For $s \geq 0$ and $\epsilon > 0$ we let $H^s([0,T]) \otimes H^{-\epsilon}_0(S^1)$ denote the tensor product of Hilbert spaces $H^s([0,T])$ and $H^{-\epsilon}_0(S^1)$. Since the inner product on that space is determined by

$$\langle f \otimes g, h \otimes k \rangle_{s,-\epsilon} = \langle f, h \rangle_s \langle g, k \rangle_{-\epsilon},$$

we obtain that

$$\langle F, G \rangle_{s,-\epsilon} = \int_0^T \int_0^T \frac{(F(t,\cdot) - F(u,\cdot))(G(t,\cdot) - G(u,\cdot))}{|t - u|^{1+2s}} \, dt \, du,$$

first when $F$ and $G$ are linear combinations of pure tensor products, and then for all $F, G \in H^s([0,T]) \otimes H^{-\epsilon}_0(S^1)$ by density and continuity.

3. **Proof of the main result theorem 1.4**

The strategy for proving convergence is as in the stationary case in [2]; we treat $(\log p_n)_{n \in \mathbb{N}}$ as a sequence of random variables in $H^s([0,T]) \otimes H^{-\epsilon}_0(S^1)$, and first establish that if any of its
subsequences has a limit then that limit has to be \( X \). This follows immediately from Spohn’s result, here theorem 1.2, as it implies in particular the convergence of the dynamic Fourier coefficients of \( \log p_n \) to those of \( X \), see remark 1.3.

We then show that the set \( \{ \log p_n \}_{n \in \mathbb{N}} \) is tight in \( H'(\{0,T\}) \otimes H_0^{-s}(S^1) \) for \( s \in [0,1/2) \) and \( \epsilon > s \). Since \( H'(\{0,T\}) \otimes H_0^{-s}(S^1) \) is complete and separable, Prokhorov’s theorem implies that the closure of \( \{ \log p_n \}_{n \in \mathbb{N}} \) is sequentially compact w.r.t. the topology of weak convergence. In particular this means that every subsequence of \( \{ \log p_n \}_{n \in \mathbb{N}} \) has a weak limit in \( H'(\{0,T\}) \otimes H_0^{-s}(S^1) \).

Since any such limit has to be \( X \) it follows that the whole sequence \( \{ \log p_n \}_{n \in \mathbb{N}} \) must converge weakly to \( X \). This implies in particular that \( X \) is almost surely an element of \( H'(\{0,T\}) \otimes H_0^{-s}(S^1) \) for \( s \in [0,1/2) \) and \( \epsilon > s \).

We will conclude by proving optimality of the regularity parameters \( s \) and \( -\epsilon \), by showing in lemma 3.3 that when \( s = 1/2 \) or \( s > \epsilon \geq 0 \) the field \( X \) is almost surely not an element of \( H'(\{0,T\}) \otimes H_0^{-s}(S^1) \). Thus for \( s = 1/2 \) or \( s = \epsilon > 0 \) the sequence \( \{ \log p_n \}_{n \in \mathbb{N}} \) almost surely cannot converge (and cannot be tight) in \( H'(\{0,T\}) \otimes H_0^{-s}(S^1) \) as its limit would have to be \( X \).

We now show tightness of \( \{ \log p_n \}_{n \in \mathbb{N}} \) in \( H'(\{0,T\}) \otimes H_0^{-s}(S^1) \), i.e. that for every \( \delta > 0 \) we construct a compact \( K_\delta \subset H'(\{0,T\}) \otimes H_0^{-s}(S^1) \) for which

\[
\sup_{n \in \mathbb{N}} P(\{ \log p_n \notin K_\delta \}) < \delta.
\]

We let \( 0 < s' < s' < \epsilon' < \epsilon \), and choose

\[
K_\delta = \left\{ F \in H'(\{0,T\}) \otimes H_0^{-s'}(S^1) : ||F||_{\ell', \ell''} \leq C \right\},
\]

for a \( C_\delta \) depending on \( \delta \). While \( K_\delta \) is not compact in \( H'(\{0,T\}) \otimes H_0^{-s'}(S^1) \), we see by lemma 3.1 below that \( K_\delta \) is relatively compact in \( H'(\{0,T\}) \otimes H_0^{-s}(S^1) \), as the image of a bounded set under a compact operator. Since \( ||.||_{\ell', \ell''} \leq C ||.||_{\ell', \ell'''} \) for some constant \( C \), it is easy to see that \( K_\delta \) is also closed and thus compact. By lemma 3.2 below we see that \( \sup_{n \in \mathbb{N}} E(\|\log p_n\|_{\ell', \ell''}) < \infty \).

Thus, when choosing \( C_\delta \) large enough, we see that

\[
\sup_{n \in \mathbb{N}} P(\{ \log p_n \notin K_\delta \}) = \sup_{n \in \mathbb{N}} E(\|\log p_n\|_{\ell', \ell''}^2) < \delta,
\]

which shows tightness of \( \log p_n \).

**Lemma 3.1.** Let \( 0 \leq s < s' < \epsilon' < \epsilon \). Then, the inclusion of \( H'(\{0,T\}) \otimes H_0^{-s'}(S^1) \) into \( H'(\{0,T\}) \otimes H_0^{-s}(S^1) \) is a compact operator.

**Proof.** From the Kondrachov embedding theorem, the inclusion \( \iota_1 \) of \( H'(\{0,T\}) \) into \( H'(\{0,T\}) \) is compact, as well as the inclusion \( \iota_2 \) from \( H_0^0(S^1) \) into \( H_0^0(S^1) \). Then the dual operator \( \iota_2^* : H_0^{-s'}(S^1) \rightarrow H_0^{-s}(S^1) \) is also compact. On Hilbert spaces, the tensor product of two compact operators is also compact (see e.g. [24]\(^4\)), so that \( \iota_1 \otimes \iota_2^* \) is compact indeed. \( \square \)

**Lemma 3.2.** For all \( s \in [0,1/2) \) and all \( \epsilon > s \), it holds that \( \sup_{n \in \mathbb{N}} E(\|\log p_n\|_{\ell', \ell''}^2) < \infty \).

**Proof.** We see that

\[
E(\|\log p_n\|_{\ell', \ell''}^2) = E\left( \int_0^T \int_0^T \|\log p_n(\cdot,t)\|^2 \cdot dt \right) + 1_{s \neq 0} E\left( \int_0^T \int_0^T \frac{\|\log p_n(\cdot,t) - \log p_n(\cdot,u)\|^2 \cdot dt}{|t-u|^{\delta+1}} \cdot du \right).
\]

\(^4\) In [26], the result is stated for endomorphisms, but this extra assumption is not used in the proof.
For the first summand it holds that (with $k \wedge n$ denoting $\min\{k, n\}$)

\[
\mathbb{E} \left( \int_0^T \|\log p_u (\cdot, t)\|_{\mathcal{F}_u}^2 \right) = \int_0^T \mathbb{E} \left( \sum_{k=1}^\infty k^{-2s} \left| \mathbf{Tr} \left( U_n (t)^k \right) \right| \right) dt \\
= T \sum_{k=1}^\infty k^{-2-2s} \mathbb{E} \left( \left| \mathbf{Tr} \left( U_n (0)^k \right) \right| \right) \\
= T \sum_{k=1}^\infty k^{-2-2s} (k \wedge n) \\
< T \sum_{k=1}^\infty k^{-1-2s} < \infty,
\]

where in the third equality we used that $\mathbb{E} \left( \left| \mathbf{Tr} (U_n (0)^k) \right| \right) = k \wedge n$, by the results by Diaconis and Shahshahani and Diaconis and Evans \[22, 23\]. For the second summand it holds that:

\[
\mathbb{E} \left( \int_0^T \int_0^T \frac{\|\log p_u (\cdot, t) - \log p_u (\cdot, u)\|_{\mathcal{F}_u}^2}{|t-u|^{2s+1}} du dt \right) \\
= \sum_{k=1}^\infty k^{-2-2s} \int_0^T \mathbb{E} \left( \mathbf{Tr} \left( U_n^k (t) - U_n^k (u) \right) \right) dt \\
\leq CT \sum_{k=1}^\infty k^{-2-2s} \int_0^T \mathbb{E} \left( \mathbf{Tr} \left( U_n^k (t) - U_n^k (0) \right) \right) dt \\
\leq CT \sum_{k=1}^\infty k^{-2-2s} \int_0^T \mathbb{E} \left( \mathbf{Tr} \left( U_n^k (t) - U_n^k (0) \right) \right) dt \\
+ CT \sum_{k=1}^\infty k^{-2-2s} \int_{k-1}^\infty \frac{4 \mathbb{E} \left( \mathbf{Tr} \left( U_n^k (0) \right) \right)}{p^{2s+1}} dt.
\]

For the second summand in (5) we get

\[
\int_{k-1}^\infty \frac{4 \mathbb{E} \left( \mathbf{Tr} \left( U_n^k (0) \right) \right)}{p^{2s+1}} dt = 8s (n \wedge k) k^{2s},
\]

which is sufficient since $\sum_{k=1}^\infty k^{-2-2s+1+2s}$ is finite as soon as $s < c$.

For the first sum in (5) we use corollary 1.6 with $f(z) = z^\delta$ and $g(z) = z^{-k}$, which implies that for all $k \geq 1$

\[
\mathbb{E} \left( \mathbf{Tr} \left( U_n^k (t) \right) \mathbf{Tr} \left( U_n^k (0) \right) \right) = 1_{k < \alpha} e^{-k^\delta} \frac{\sinh \left( \frac{k^\delta}{\alpha} \right)}{\sinh \left( \frac{\delta}{\alpha} \right)} + 1_{k \geq \alpha} e^{-k \delta} \frac{\sinh (k \delta)}{\sinh \left( \frac{\delta}{2} \right)} \\
= e^{-\frac{k(k+\alpha)}{2}} \frac{\sinh \left( \frac{k(k+\alpha)}{2} \right)}{\sinh \left( \frac{\delta}{2} \right)},
\]
with \( k \lor n := \max\{k, n\} \). Using this, and the fact that \( \sinh x \geq x \) and \( 1/\sinh x \geq 1/x - x/6 \) for all \( x > 0 \), we see that for \( t < k^{-1} \):

\[
\mathbb{E} \left( |\text{Tr} \left( T^k_n (t) - T^k_n (0) \right) |^2 \right)
\]
\[
= \mathbb{E} \left( |\text{Tr} \left( T^k_n (0) \right) |^2 \right) + \mathbb{E} \left( |\text{Tr} \left( T^k_n (t) \right) |^2 \right) - 2 \mathbb{E} \left( \text{Tr} \left( T^k_n (t) \right) \text{Tr} \left( T^k_n (0) \right) \right)
\]
\[
= 2(k \land n) - 2e^{-\frac{k(k \land n)t}{n}} \sinh \left( \frac{k(k \land n)t}{n} \right)
\]
\[
\leq 2(k \land n) - 2e^{-\frac{k(k \land n)t}{n}} \left( \frac{k}{n} \right)^{-1} - \frac{kt}{6n}
\]
\[
= 2(k \land n) - 2e^{-\frac{k^2t^2(k \land n)}{6n^2}} - 2(k \land n) \left( 1 - e^{-\frac{k^2t^2(k \land n)}{6n^2}} \right)
\]
\[
\leq 2k^2t^2 - 2(k \land n) \frac{k(k \land n)t}{n}
\]
\[
\leq 4k^2t^2.
\]

Thus we see that when \( s < 1/2 \), the first sum in (5) is bounded by \( T \sum_{k=1}^\infty k^{-1-2s+2s} \), which is finite for \( s < \varepsilon \). This finishes the proof. \( \square \)

**Lemma 3.3.** For \( s = 1/2 \) or \( s \geq \varepsilon \geq 0 \) the field \( X \) is almost surely not an element of \( H^s([0, T]) \otimes H^{s-1} \). **Proof.** We see that

\[
||X||^2_{L^2} = \sum_{k=1}^\infty k^{-2s} \int_0^T |A_k(t)|^2 dt + 1_{\rho \neq 0} \sum_{k=1}^\infty k^{-2s} \int_0^T \int_0^T \frac{|A_k(t) - A_k(u)|^2}{|t-u|^{1+2s}} \, dt \, du. \quad (6)
\]

For \( s = \varepsilon = 0 \) we see that

\[
||X||^2_{L^2} = \int_0^T ||X(t, \cdot)||^2_2 \, dt.
\]

If this were almost surely finite, then \( \mathbb{P}(\text{for almost all } t : ||X(t, \cdot)||_0 < \infty) = 1 \), and

\[
T = \mathbb{E} \left( \int_0^T 1_{||X(t, \cdot)||_0 < \infty} \, dt \right) = \int_0^T \mathbb{P}(||X(t, \cdot)||_0 < \infty) \, dt,
\]

which is a contradiction since for all \( t \) we know that \( \mathbb{P}(||X(t, \cdot)||_0 < \infty) = 0 \). Thus we see that almost surely \( ||X||_{L^2} = \infty \).

For \( s = \varepsilon > 0 \) we use the fact that stationary Ornstein–Uhlenbeck processes \( A_k(t), k \in \mathbb{N} \), can be represented as scaled, time-changed Brownian motions:

\[
A_k(t) = \frac{e^{-kt}}{\sqrt{2k}} \left( B_k \left( e^{2i \gamma t} \right) + iB_k \left( e^{2i \gamma t} \right) \right),
\]
where $B_k, \hat{B}_k, k \in \mathbb{N}$, are real standard Brownian motions. Thus it follows that the second summand in $\|X\|^2_{\ell^2}$ is bounded below by

$$\frac{1}{2} \sum_{k=1}^{\infty} k^{-1-2\epsilon} \int_0^T \int_0^T \frac{|e^{-t} B_k (e^{2iu}) - e^{-u} B_k (e^{2iu})|^2}{|t-u|^{1+2\epsilon}} \, dt \, du$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} k^{-1-2\epsilon+2\epsilon} \int_0^T \int_0^T \frac{|e^{-t} B_k (e^{2iu}) - e^{-u} B_k (e^{2iu})|^2}{|t-u|^{1+2\epsilon}} \, dt \, du$$

$$\geq \frac{1}{2} \sum_{k=1}^{\infty} k^{-1} \int_0^T \int_0^T \frac{|e^{-t} B_k (e^{2iu}) - e^{-u} B_k (e^{2iu})|^2}{|t-u|^{1+2\epsilon}} \, dt \, du.$$

We see that $Y_k := \int_0^T \int_0^T \frac{|e^{-t} B_k (e^{2iu}) - e^{-u} B_k (e^{2iu})|^2}{|t-u|^{1+2\epsilon}} \, dt \, du, k \in \mathbb{N}$, are an i.i.d. family of almost surely positive random variables. Let $\delta > 0$ be such that $\mathbb{P}(Y_k \geq \delta) \geq \frac{1}{2}$, and set $Z_k = \delta Y_k$, so that $Y_k \geq Z_k$ for all $k$. Notice that $(\delta - Z_k)$ is stochastically dominated by $Z_k$, so $\mathbb{P}(\sum_{k=1}^{\infty} k^{-1} Z_k < \infty) \leq \mathbb{P}(\sum_{k=1}^{\infty} k^{-1} (\delta - Z_k) < \infty)$. Since $\sum_{k=1}^{\infty} k^{-1} = +\infty$, these two events cannot happen at the same time, so $\mathbb{P}(\sum_{k=1}^{\infty} k^{-1} Z_k < \infty) \leq \frac{1}{2}$. By Kolmogorov’s zero-one law, $\mathbb{P}(\sum_{k=1}^{\infty} k^{-1} Z_k < \infty) = 0$. Since $Y_k \geq Z_k$, we deduce that, for $c \leq s$, $\|X\|^2_{\ell^2} \geq \sum_{k=1}^{\infty} k^{-1} Y_k = +\infty$ almost surely.

For $s \geq \frac{1}{2}$, we consider the event $E$ in which $X$ is an element of $H^s([0,T]) \times H^{-s}([S])$. Then, since $\theta \mapsto e^{-i\theta t}$ is in $H^s([S])$, by duality in $L^2([S])$, we deduce that the function $t \mapsto \int_0^T X(s, \theta)e^{-i\theta t} \, dt$ belongs to $H^s([0,T])$. This function is nothing but $A_1$, which is an Ornstein–Uhlenbeck process. By Girsanov’s theorem, its distribution on a bounded set $[0,T]$ is absolutely continuous with respect to the one of the Brownian motion restricted to $[0,T]$. Thus, the probability of $E$ is equal to one if and only if the Brownian motion almost surely belong to $H^s([0,T])$, which is known not to be the case [27].

\[ \square \]

Data availability statement

No new data were created or analysed in this study.

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