Spreading Speeds for a Class of Non-Local Convolution Differential Equation

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Abstract. The spatial spreading dynamics is considered for a class of convolution differential equation resulting from physical and biological problems. It is shown that this kind of equation with monostable structure admits a spreading speed, even when the nonlinear reaction terms without monotonicity. The upward convergence of spreading speed is also established under appropriate conditions.

1. Introduction. In the past years, there have been wide studies on the spatial-temporal dynamics of nonlinear integral equations and integro-differential equations modeling physical and biological phenomena. Such type of equations take into account the long-range interaction and describe the interaction via a dispersal kernel, which specifies the probability that an individual moves from one location to another in a certain time interval as function. For example, Ermentrout and Mcleod [10] considered the wave propagation dynamics for neutral network model

$$u(t, x) = \int_{-\infty}^{t} h(t-s) \int_{-\infty}^{\infty} k(x-y)g(u(s, y))dyds$$

and its differential form

$$u_t(t, x) = -u(t, x) + \int_{\mathbb{R}} k(x-y)g(u(t, y))dy,$$

which was derived by differentiating (1) with $h(t) = e^{-t}$ ($t > 0$). Similar problems were considered by Bates, Fife, Ren and Wang [3] for the phase transition model

$$u_t(t, x) = -f(u(t, x)) - u(t, x) + \int_{\mathbb{R}} k(x-y)u(t, y)dy,$$

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and by De Masi, Gobron and Presutti [20] for the Ising model
\[ u_t(t, x) = -u(t, x) + \tanh \left\{ \beta \int_{\mathbb{R}} k(x - y)u(t, y)dy + h \right\} . \]  
(4)

Chen, Ermentrout and Mcleod [5] considered the wave propagation dynamics for a more general class of integro-differential equation
\[ u_t(t, x) = r(u) + p(u)\Gamma \left( \int_{\mathbb{R}} k(x - y)g(u(t, y))dy \right) , \]  
(5)
which includes Eqs. (2)-(4) as special cases. More works on the wave propagation dynamics of integral equations and integro-differential equations can be found in [4, 6, 17, 7, 8, 24, 23, 12, 16, 22, 18, 14, 21, 25, 15, 27, 13, 31, 29] and references therein.

Another important issue in the study of spatial-temporal dynamics of integral equations and integro-differential equations is the asymptotic speed of spread (spreading speed in short). This concept was introduced by Aronson and Weinberger [1, 2] in the study of nonlinear diffusion problems in genetics, combustion and nerve propagation. Roughly speaking, a number \( c^* > 0 \) is called the spreading speed if for any \( c_1, c_2 \) with \( 0 < c_1 < c^* < c_2 \), the solution tends to zero uniformly in the region \( |x| \geq c_2 t \), while it is bounded away from zero uniformly in the region \( |x| \leq c_1 t \) for \( t \) sufficiently large. Biologically, this concept states that if one leaves the origin of the population at a speed exceeding \( c^* \), one will outrun the population, whereas, if moving at a speed less than \( c^* \), the population will overtake the observer. Diekmann [6] and Thieme [23], independently, extended the concept of asymptotic speed of spread to an integral equation of the form (1). In particular, when the recruitment function \( g(\cdot) \) is non-monotone, Thieme [24] developed an idea which is based on the representation of \( g(\cdot) \) as the diagonalization of a monotone function of two variables to show that (1) admits a spreading speed. That is quite remarkable in the study of spreading speed of non-monotone integral equations since the comparison principle is not available in this case.

Recently, Thieme and Zhao [25] extended the idea developed in [24] to a large class of non-monotone nonlinear integral equation for the existence of spreading speed. Fang and Zhao [11] further completed the results of [25] by considering more general cases. In particular, the authors in [11] showed the fact that the spreading speed coincides with the minimal wave speed of traveling waves. And under some appropriate conditions, they proved the upward convergence of the spreading speed, i.e., the solution converges to a unique positive equilibrium uniformly in the region \( |x| \leq c_1 t \) when \( c_1 < c^* \) and \( t \) tends to infinite. It should be pointed out that the recent work from Ding and Liang [9] established an abstract framework of the existence of spreading speeds for a general recursion operator with spatial periodicity, which can be applied to certain types of non-monotone evolution equations. We also point out that there have been some works on the the existence of spreading speeds for some non-monotone integro-differential equations (see, e.g. [16, 22, 30, 28, 19]).

Inspired by the above aforementioned works on the study of spatial spreading dynamics, in this paper, we explore the spatial spreading of integro-differential equation (5) in the general non-monotone case, i.e. \( \Gamma(\cdot) \) and \( g(\cdot) \) are unimodal functions (cf. [24, 33, 26] or Definition 3.1). We should mention that the theory in [9] is still applicable to establish the existence of spreading speed for (5) by verifying the abstract and universal assumptions in [9]. Here we would like to comment on some
differences. In [9], they paid attention to what kind of diffusion mechanism guaranteeing the existence of spreading speeds for a generalized convolution operator. However, our work aims to extend the results for integro-differential equation with monotone nonlinearity to non-monotone nonlinearity and compare the conclusions for these two situations. This setting enables us to reach some more concrete results. For example, we prove that the upward convergence of spreading speed by using fluctuation method. These results together with the wave propagation results established in our recent work [32] also lead to the conclusion that the spreading speed of (5) coincides with the minimal wave speed of traveling waves in the general case.

The rest of this paper is organized as follows. In Section 2, we present some existing results on the spreading speed of (5) with monotone nonlinearity. In Section 3, we apply the squeezing technique and the comparison principle to establish the existence of spreading speed for (5) without monotonicity. Then we proved the upward convergence of spreading speed by the fluctuation method. Some examples are presented in the last section to illustrate the application of the theoretical results.

2. Preliminaries. In this section, we present the relevant definition and some preliminary results.

Definition 2.1. A number \( c^* > 0 \) is called the asymptotic speed of spread (spreading speed for short) for a solution \( u \) of (5) if there exists some \( c > 0 \) such that
\[
\lim_{t \to \infty, |x| \geq ct} u(t, x) = 0 \quad \text{for any } c > c^*, \quad \text{and} \quad \lim_{t \to \infty, |x| \leq ct} u(t, x) \geq \epsilon \quad \text{for any } \epsilon \in (0, c^*).
\]

According to the above definition, one may call \( c^* \) as the spreading speed of (5) if all the solutions of (5) with some prescribed initial functions share with the same \( c^* \) and \( \epsilon \).

Let \( X \) be the set of all bounded and uniformly continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \) with supremum norm \( | \cdot |_X \). For any \( \vartheta > 0 \), denote
\[
X_{\vartheta} := \{ \psi \in X : 0 \leq \psi(x) \leq \vartheta \quad \text{for any } x \in \mathbb{R} \}.
\]

Next, we state some known results on the spreading speeds related to (5). For simplicity of statements, we first give some basic assumptions as follows.

(H) \( k(\cdot) \) is a nonnegative even function with unit integral on \( \mathbb{R} \), \( r(\cdot), p(\cdot), g(\cdot), \Gamma(\cdot) \) are locally Lipschitz continuous on \( \mathbb{R}_+ \), and have the following properties:

(i) there exists \( \lambda_* \in (0, \infty] \) such that \( \int_{\mathbb{R}} k(y)e^{-\lambda y}dy < +\infty \) for \( \lambda \in (0, \lambda_*] \), and \( \int_{\mathbb{R}} k(y)e^{-\lambda y}dy \to +\infty \) as \( \lambda \to \lambda_*^+ \).

(ii) \( p(0) > 0, p(u) \leq p(0) \) for \( u \geq 0, p(u) = 0 \) has at most one root.

(iii) \( r(\cdot) \) is \( C^1 \) smooth in some neighborhood of 0 with \( r(0) = 0, r(u) \leq r'(0)u \) for \( u > 0 \).

(iv) \( \Gamma(\cdot) \) is \( C^1 \) smooth in some neighborhood of 0 with \( \Gamma(0) = 0, 0 < \Gamma(u) \leq \Gamma'(0)u \) for \( u > 0 \).

(v) \( g(\cdot) \) is \( C^1 \) smooth in some neighborhood of 0 with \( g(0) = 0, 0 < g(u) \leq g'(0)u \) for \( u > 0 \).

(vi) There exists exactly one positive constant \( \omega \) such that \( \Gamma(g(\omega)) = -r(\omega)/p(\omega) \), \( \Gamma(g(u)) > -r(u)/p(u) \) for \( u \in (0, \omega) \), \( -r(u)/p(u) \) is strictly increasing with \( -r(u)/p(u) \to \infty \) as \( u \to \theta \), where \( \theta \) is the root of \( p(u) = 0 \) if it exists, otherwise it is plus infinity.

From the above assumptions, it is easily seen that (5) admits two steady states \( u \equiv 0 \) and \( u \equiv w \) and \( r'(0) + p(0)\Gamma'(0)g'(0) > 0 \).

Substituting \( u(t, x) = e^{\lambda(x+ct)} \) into the linearized equation corresponding to (5) around the trivial steady state \( u = 0 \), we obtain the following characteristic equation

\[
c\lambda = r'(0) + p(0)\Gamma'(0)g'(0) \int_{\mathbb{R}} k(y)e^{-\lambda y}dy.
\]

Define

\[
c^* = \inf_{\lambda > 0} \frac{1}{\lambda} \left[ r'(0) + p(0)\Gamma'(0)g'(0) \int_{\mathbb{R}} k(y)e^{-\lambda y}dy \right].
\]

The following result is helpful for the computation of the \( c^* \) (cf. [31, Lemma 2.1]).

**Lemma 2.2.** Assume that (H) holds. Then,

(i) there exists a unique \( \lambda^* > 0 \) such that \( \Phi(\lambda^*, c^*) = 0 \), and for \( c > c^* \), \( \Phi(\lambda, c) = 0 \) has a smallest positive real root \( \lambda_1 = \lambda_1(c) \), and \( \Phi(\lambda_1 + \gamma, c) < 0 \) for some \( \gamma > 0 \);

(ii) \( c^* \) and \( \lambda^* \) are uniquely determined by \( \Phi(\lambda, c) = 0 \), \( \frac{\partial}{\partial \lambda} \Phi(\lambda, c) = 0 \), where

\[
\Phi(\lambda, c) = r'(0) - c\lambda + p(0)\Gamma'(0)g'(0) \int_{\mathbb{R}} k(y)e^{-\lambda y}dy.
\]

When \( g(\cdot) \) is monotone increasing on \([0, w]\) and in the meantime \( \Gamma(\cdot) \) is monotone increasing on \([0, g(w)]\), the following results can be deduced from [22, 30].

**Proposition 1.** Assume that (H) holds, \( g(\cdot) \) is monotone increasing on \([0, \omega]\) and \( \Gamma(\cdot) \) is monotone increasing on \([0, g(\omega)]\). Then, the following conclusions are valid:

(i) If \( u_0 \in X_\omega \) is compact supported, then the solution of (5) satisfies

\[
\lim_{t \to \infty, |x| \geq ct} u(t, x; u_0) = 0 \quad \text{for any} \quad c > c^*.
\]

(ii) If \( u_0 \in X_\omega \) with \( u_0 \not\equiv 0 \), then the solution of (5) satisfies

\[
\lim_{t \to \infty, |x| \leq ct} u(t, x; u_0) = \omega \quad \text{for any} \quad 0 < c < c^*.
\]

Next, we shall extend Proposition 1 to a more general case that both the reaction functions \( g(\cdot) \) and \( \Gamma(\cdot) \) do not have the monotonicity prescribed in Proposition 1.

3. **Main results.** For a given \( b \geq \omega \), let \( L_r, L_p \) be the Lipschitz constants of \( r(\cdot) \) and \( p(\cdot) \) on \([0, b]\), respectively, and let

\[
m = \max_{u \in [0, b]} g(u), \quad \alpha = \max_{v \in [0, m]} \Gamma(v), \quad \beta = L_r + \alpha L_p.
\]

For convenience, denote

\[
k_g(u)(t, x) = \int_{\mathbb{R}} k(x-y)g(u(t, y))dy.
\]

It then follows that Eq. (5) can be rewritten as

\[
u_t(t, x) + \beta u(t, x) - H(u)(t, x) = 0, \tag{8}
\]

where

\[
H(u)(t, x) = \beta u + r(u) + p(u)\Gamma(k_g(u)(t, x)).
\]

It is convenient to write (8) into the following equivalent integral form

\[
u(t, x) = u(0, x)e^{-\beta t} + \int_0^t e^{-\beta(t-z)}H(u)(z, x)dz, \quad t \geq 0, x \in \mathbb{R}. \tag{9}
\]

By the integral form (9), the existence and uniqueness of solutions of (5) can be obtained by standard arguments (cf.[30]).
Definition 3.1. A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is called unimodal on $[0, b]$ if there is some $\tilde{a} \in (0, b)$ such that $g$ is increasing on $[0, \tilde{a}]$ and decreasing on $[\tilde{a}, \infty)$. The number $\tilde{a}$ is called a mode of $g$.

Now, we state and prove the following main result, which shows that the $c^*$ defined in (6) is also the spreading speed of (5) even when $g(\cdot)$ and $\Gamma(\cdot)$ are unimodal functions. Moreover, the upward convergence as stated in Proposition 1 (ii) is also holds under the additional assumption

\begin{align*}
\text{(K)} \quad & \text{For } v, u \text{ satisfying } 0 < v \leq \omega \leq u \leq \theta, \Gamma(g(u)) = -r(u)/p(u) \text{ and } \Gamma(g(\omega)) \geq -r(u)/p(u), \text{ it follows that } v = u, \text{ where } \theta \text{ is the point such that } -r(\theta)/p(\theta) = m, \quad m = \max_{v \in [0, \max_{u > 0} g(u)]} \Gamma(v).
\end{align*}

Theorem 3.2. Assume that (H) holds, $g(\cdot)$ is unimodal on $[0, \omega]$, and either $\Gamma(\cdot)$ is a monotone or unimodal function. Then, the following conclusions are valid:

(i) If $u_0 \in X_\omega$ is compact supported, then the solution of (5) satisfies

\begin{equation}
\lim_{t \to \infty, \ |x| \geq ct} u(t, x; u_0) = 0 \quad \text{for } c > c^*.
\end{equation}

(ii) If $u_0 \in X_\omega$ with $u_0 \neq 0$, then there exists some $\epsilon > 0$ such that the solution of (5) satisfies

\begin{equation}
\lim \inf_{t \to \infty, \ |x| \leq ct} u(t, x; u_0) \geq \epsilon \quad \text{for } 0 < c < c^*.
\end{equation}

If, in addition, (K) holds, then

\begin{equation}
\lim_{t \to \infty, \ |x| \leq ct} u(t, x; u_0) = \omega.
\end{equation}

Proof. Since $g$ is unimodal on $[0, \omega]$, we may assume that $a \in (0, \omega)$ is the mode of $g$. We consider the following two cases:

Case I. $\Gamma$ is monotone on $[0, g(a)]$. Since $\Gamma(g(a)) \geq \Gamma(g(\omega)) = -r(\omega)/p(\omega)$, $-r(0)/p(0) = 0$ and $-r(u)/p(u) \to \infty$ as $u \to \theta$, there exists a $\omega^0 \in (\omega, \theta)$ such that $-r(\omega^0)/p(\omega^0) = \Gamma(g(a))$. Let $\tilde{u} \in (0, a)$ be the smallest number such that $g(\tilde{u}) = \min_{u \in [a, \omega^0]} g(u) = g(\omega^0)$. Similarly, there exists $\omega_0 \leq \omega$ such that $-r(\omega_0)/p(\omega_0) = \Gamma(g(\tilde{u}))$. Define

$$g^+(u) = \begin{cases} g(u), & u \in [0, a], \\ g(a), & u \in [a, \omega^0], \end{cases} \quad g^-(u) = \begin{cases} g(u), & u \in [0, \tilde{u}], \\ g(\tilde{u}), & u \in [\tilde{u}, \omega^0]. \end{cases}$$

Then, both $g^+(u)$ and $g^-(u)$ are nondecreasing on $[0, \omega^0]$, and

$$g^-(u) \leq g(u) \leq g^+(u) \quad \text{for } u \in [0, \omega^0].$$

Furthermore,

$$\Gamma(g^+(\omega^0)) = \Gamma(g(a)) = -r(\omega^0)/p(\omega^0), \quad \Gamma(g^+(0)) = \Gamma(g(0)) = 0.$$ \text{Note that}

$$\Gamma(g^+(u)) \geq \Gamma(g(u)) > -r(u)/p(u) \quad \text{for } u \in (0, \omega),$$

and

$$\Gamma(g^+(u)) = \Gamma(g(a)) = -r(\omega^0)/p(\omega^0) > -r(u)/p(u) \quad \text{for } u \in [\omega, \omega^0].$$
Thus, \( \Gamma \left( g^+(u) \right) > -r(u)/p(u) \) for \( u \in (0, \omega^0) \).
Moreover, \( \Gamma \left( g^-(\omega^0) \right) = \Gamma \left( g(\bar{u}) \right) = -r(\omega^0)/p(\omega^0) \) and \( \Gamma \left( g^-(0) \right) = \Gamma \left( g(0) \right) = 0 \).

Note that
\[
\Gamma \left( g^{-}(u) \right) = \Gamma \left( g(u) \right) > -r(u)/p(u) \quad \text{for} \quad u \in (0, \bar{u}].
\]
Thus,
\[
\Gamma \left( g^{-}(u) \right) > -r(u)/p(u) \quad \text{for} \quad u \in (0, \omega^0).
\]

By the above definitions of \( g^- \) and \( g^+ \), it is easily seen that
\[
\Gamma \left( g^{-}(u) \right) \leq \Gamma \left( g(u) \right) \leq \Gamma \left( g^+(u) \right) \quad \text{for} \quad u \in [0, \omega^0].
\]

Consider the following two convolution equations
\[
u_t(t, x) = r(u) + p(u)\Gamma \left(k \ast g^+(u)(t, x)\right), \quad (13)\]
and
\[
u_t(t, x) = r(u) + p(u)\Gamma \left(k \ast g^-(u)(t, x)\right). \quad (14)\]

For a given \( u_0 \in \mathbb{R}_\omega \) with compact support, let \( u(t, x) = u(t, x; u_0) \) and \( u^+(t, x) = u(t, x; u_0) \) be the solutions of (5) and (13), respectively. Observe that the solution of (5) is the subsolution of (13). By the comparison principle (cf. [30]), it follows that
\[
0 \leq u(t, x) \leq u^+(t, x) \quad \text{for} \quad t \geq 0, x \in \mathbb{R}.
\]

Since \( \lim_{t \to \infty, |x| \leq ct} u^+(t, x) = 0 \) for \( c > c^* \) by Proposition 1, we then have
\[
\lim_{t \to \infty, |x| \leq ct} u(t, x) = 0 \quad \text{for} \quad c > c^*.
\]

For a given \( u_0 \in \mathbb{R}_\omega \setminus \{0\} \), define \( u^-_0(x) = \min\{u_0(x), \omega^0\}, \) \( x \in \mathbb{R} \). Clearly, \( u^-_0 \in \mathbb{R}_\omega \) and \( u^-_0 \neq 0 \).

Let \( u(t, x; u_0) \), \( u^+(t, x; u_0) \) and \( u^-(t, x) = u(t, x; u^-_0) \) be the solutions of (5), (13) and (14), respectively. By the comparison principle, we obtain
\[
0 \leq u^-(t, x) \leq u(t, x) \leq u^+(t, x) \quad \text{for} \quad t \geq 0, x \in \mathbb{R}.
\]

Note by Proposition 1, we have
\[
\lim_{t \to \infty, |x| \leq ct} u^-(t, x) = \omega^0 \quad \text{and} \quad \lim_{t \to \infty, |x| \leq ct} u^+(t, x) = \omega_0 \quad \text{for} \quad 0 < c < c^*.
\]

It then follows that
\[
\omega_0 \leq \liminf_{t \to \infty, |x| \leq ct} u(t, x) \leq \limsup_{t \to \infty, |x| \leq ct} u(t, x) \leq \omega^0 \quad \text{for} \quad 0 < c < c^*.
\]

**Case II.** \( \Gamma \) is unimodal on \([0, g(a)]\). Since \( \Gamma \) is unimodal on \([0, g(a)]\), we may assume that \( u_0 \in (0, a) \) is the point such that \( g(u_0) \) is the mode of \( \Gamma \). Then, \( \Gamma(v) \) is increasing on \([0, g(u_0)]\). Note that \( \min_{v \in [g(u_0), g(a)]} \Gamma(v) = \Gamma(g(a)) \). There exists \( \hat{u} \in (0, u_0) \) such that
\[
g(\hat{u}) \leq g(u_0) \quad \text{and} \quad \Gamma(g(\hat{u})) = \min_{v \in [g(u_0), g(a)]} \Gamma(v).
\]

Define
\[
\Gamma^+(v) = \begin{cases} \Gamma(v), & v \in [0, g(u_0)], \\
\Gamma(g(u_0)), & v \in [g(u_0), g(a)] \end{cases}, \quad \Gamma^-(v) = \begin{cases} \Gamma(v), & v \in [0, g(\hat{u})], \\
\min_{s \in [v, g(a)]} \Gamma(s), & v \in [g(\hat{u}), g(a)]. \end{cases}
\]
Then, both \( \Gamma^+(v) \) and \( \Gamma^-(v) \) are nondecreasing on \([0, g(a)]\), and
\[
\Gamma^-(v) \leq \Gamma(v) \leq \Gamma^+(v) \quad \text{for} \quad u \in [0, g(a)].
\]
Since \(-r(0)/p(0) = 0\) and \(-r(u)/p(u) \to \infty\) as \(u \to \theta\), there exists \(\omega_2 \in (0, \theta)\) such that \(-r(\omega_2)/p(\omega_2) = \Gamma(g(u_0))\). Since \(-r(u)/p(u)\) is increasing on \((0, \theta)\) and
\[
-r(\omega_2)/p(\omega_2) = \Gamma(g(u_0)) \geq \Gamma(g(\omega)) = -r(\omega)/p(\omega),
\]
it follows that \(\omega_2 \geq \omega\).
Choose \(u_1 \in (0, a)\) such that \(g(u_1) \leq \min\{g(\hat{u}), g(\omega_2)\}\). Since \(-r(0)/p(0) = 0\) and \(-r(u)/p(u) \to \infty\) as \(u \to \theta\), there exists \(\omega_1 \in (0, \theta)\) such that \(-r(\omega_1)/p(\omega_1) = \Gamma(g(u_1))\). Since \(\Gamma(g(u_1)) > -r(u_1)/p(u_1)\), \(-r(u)/p(u)\) is increasing on \((0, \theta)\) and
\[
-r(\omega_1)/p(\omega_1) = \Gamma(g(u_1)) \leq \Gamma(g(\omega)) = -r(\omega)/p(\omega),
\]
it follows that \(u_1 < \omega_1 \leq \omega\).
Define
\[
g^+(u) = \begin{cases} g(u), & u \in [0, u_1], \\ \max_{s \in [u_1, u]} g(s), & u \in [u_1, \omega_2], \end{cases} \quad g^-(u) = \begin{cases} g(u), & u \in [0, u_1], \\ g(u_1), & u \in [u_1, \omega_2]. \end{cases}
\]
Then, both \(g^+(u)\) and \(g^-(u)\) are nondecreasing on \([0, \omega_2]\), and
\[
g^-(u) \leq g(u) \leq g^+(u) \quad \text{for} \quad u \in [0, \omega_2].
\]
Furthermore, \(\Gamma^+(g^+(0)) = \Gamma(g(0)) = 0\), and
\[
\Gamma^+(g^+(\omega_2)) = \Gamma^+(g(a)) = \Gamma(g(u_0)) = -r(\omega_2)/p(\omega_2).
\]
Note that
\[
\Gamma^+(g^+(u)) \geq \Gamma(g(u)) > -r(u)/p(u) \quad \text{for} \quad u \in (0, w),
\]
and
\[
\Gamma^+(g^+(u)) = \Gamma^+(g(a)) = \Gamma(g(u_0)) > -r(u)/p(u) \quad \text{for} \quad u \in [w, \omega_2).
\]
Thus,
\[
\Gamma^+(g^+(u)) > -r(u)/p(u) \quad \text{for} \quad u \in (0, \omega_2).
\]
It is easily seen that \(\Gamma^-(g^-(0)) = \Gamma(g(0)) = 0\), and
\[
\Gamma^-(g^-(\omega_1)) = \Gamma(g(u_1)) = -r(\omega_1)/p(\omega_1).
\]
Observe that
\[
\Gamma^-(g^-(u)) = \Gamma^-(g(u)) = \Gamma(g(u)) > -r(u)/p(u) \quad \text{for} \quad u \in (0, u_1].
\]
Moreover, since \(g^-(u) = g(u_1)\) for \(u \in (u_1, \omega_1)\), it follows that
\[
\Gamma^- (g^-(u)) = \Gamma^- (g(u_1)) = -r(\omega_1)/p(\omega_1) > -r(u)/p(u), \quad u \in (u_1, \omega_1).
\]
Thus,
\[
\Gamma^- (g^-(u)) > -r(u)/p(u) \quad \text{for} \quad u \in (0, \omega_1).
\]
In view of the definitions of \(g^-, g^+, \Gamma^-\) and \(\Gamma^+\), it is easy to see that
\[
\Gamma^-(g^-(u)) \leq \Gamma(g^-(u)) \leq \Gamma^+(g(u)) \leq \Gamma^+(g^+(u)) \quad \text{for} \quad u \in [0, \omega_2).
\]
Consider the following two convolution equations
\[
u_t(t,x) = r(u) + p(u)\Gamma^+(k \ast g^+(u)(t,x))
\]
and
\[
u_t(t,x) = r(u) + p(u)\Gamma^-(k \ast g^-(u)(t,x)).
\]
By similar arguments to those in the case I, we would obtain that
\[
\lim_{t \to \infty, |x| \geq ct} u(t, x) = 0 \quad \text{for} \quad c > c^*,
\]
and
\[
\omega_1 \leq \liminf_{t \to \infty, |x| \leq ct} u(t, x) \leq \limsup_{t \to \infty, |x| \leq ct} u(t, x) \leq \omega_2 \quad \text{for} \quad 0 < c < c^*.
\]

We next further to prove the upward convergence (7) holds if the condition (K) is satisfied. Since the proof is similar for the above two cases, we only prove the conclusion for the Case II. Our proof is mainly based on the fluctuation method (cf. [24, 25]). For any \((v, u) \in \mathbb{R}_+^2\), define two continuous functions as follows
\[
f(v, u) = \begin{cases} \min_{s \in [v, u]} \Gamma(s), & \text{if } v \leq u, \\ \max_{s \in [v, u]} \Gamma(s), & \text{if } u \leq v, \end{cases}
\]
\[
h(v, u) = \begin{cases} \min_{s \in [v, u]} g(s), & \text{if } v \leq u, \\ \max_{s \in [v, u]} g(s), & \text{if } u \leq v. \end{cases}
\]
It is easy to see that \(f\) and \(h\) are nondecreasing with respect to the first variable \(v\), and nonincreasing with respect to the second variable \(u\). Furthermore, \(f(u, v) = \Gamma(u)\) and \(h(u, u) = g(u)\). Then by (9), we have
\[
u(t, x) = u(0, x)e^{-\beta t} + \int_0^t e^{\beta (t-z)} \left[ \beta u(t-z, x) + r(u(t-z, x)) + p(u(t-z, x)) + \Gamma(k*g(u)(t-z, x)) \right] dz
\]
\[
= u(0, x)e^{-\beta t} + \int_0^t e^{\beta (t-z)} \left[ \beta u(t-z, x) + r(u(t-z, x)) + p(u(t-z, x)) \times f(k*h(u(t-z, x), u(t-z, x)), k*h(u(t-z, x), u(t-z, x))) \right] dz.
\]

For any \(a \in (0, c^*)\), define
\[
w_*(a) = \liminf_{t \to \infty, |x| \leq at} u(t, x), \quad w^*(a) = \limsup_{t \to \infty, |x| \leq at} u(t, x).
\]
Let \(c \in (0, c^*)\) be given, and fix a number \(\gamma \in (c, c^*)\). Define
\[
v_*(c, \gamma) = \inf_{c < a < \gamma} w_*(a), \quad v^*(c, \gamma) = \sup_{c < a < \gamma} w^*(a).
\]
It then follows that
\[
v_*(c, \gamma) \leq w_*(a) \leq w^*(a) \leq v^*(c, \gamma), \quad \forall a \in [c, \gamma].
\]
Since \(a \in (0, c^*)\), it follows from (15) that
\[
0 < \omega_1 \leq w_*(a) \leq w^*(a) \leq \omega_2 \quad \forall a \in (0, c^*).
\]
Hence,
\[
0 < \omega_1 \leq v_*(c, \gamma) \leq v^*(c, \gamma) \leq \omega_2.
\]
For any \(a \in (c, \gamma)\), choose two sequences \(\{t_j\} \subset [0, \infty)\) and \(\{x_j\} \subset \mathbb{R}\) such that \(|x_j| \leq at_j, t_j \to \infty\) as \(j \to \infty\), and \(\lim_{j \to \infty} w(t_j, x_j) = w_*(a)\). For any given \(\sigma \in \mathbb{R}_+\) and \(\eta \in \mathbb{R}\), we have
\[
w_*(\gamma) \leq \liminf_{j \to \infty} u(t_j - \sigma, x_j - \eta) \leq \limsup_{j \to \infty} u(t_j - \sigma, x_j - \eta) \leq w^*(\gamma).
\]
By Fatou’s lemma, it follows from (16) that
\[ w_*(a) \geq \int_0^\infty e^{-\beta z} \liminf_{t \to \infty} \left[ \beta u(t-z,x) + r(u(t-z,x)) + p(u(t-z,x)) \right] \times f(k \ast h(u(t-z,x),u(t-z,x)),k \ast h(u(t-z,x),u(t-z,x))) \] \[ dz. \]
Note that \( l(u) = \beta u + r(u) + \exp(u) \) is nondecreasing in \( u \in [0,\omega_2] \). Then by the monotonicity of \( f, \ h \), it follows that
\[ w_*(a) \geq \int_0^\infty e^{-\beta z} \left[ \beta v_*(gamma) + r(v_*(gamma)) + p(v_*(gamma)) f(h(v_*(gamma),v_*(gamma)),h(v_*(gamma),v_*(gamma))) \right] \] \[ \geq \int_0^\infty e^{-\beta z} \left[ \beta v_*(c,\gamma) + r(v_*(c,\gamma)) + p(v_*(c,\gamma)) \times \right. \] \[ \left. f(h(v_*(c,\gamma),v_*(c,\gamma)),h(v_*(c,\gamma),v_*(c,\gamma))) \right] \] \[ = v_*(c,\gamma) + \frac{1}{\beta} [r(v_*(c,\gamma)) + p(v_*(c,\gamma)) f(h(v_*(c,\gamma),v_*(c,\gamma)),h(v_*(c,\gamma),v_*(c,\gamma))]. \]
Thus, we have
\[ v_*(c,\gamma) \geq v_*(c,\gamma) \] \[ + \frac{1}{\beta} [r(v_*(c,\gamma)) + p(v_*(c,\gamma)) f(h(v_*(c,\gamma),v_*(c,\gamma)),h(v_*(c,\gamma),v_*(c,\gamma))], \]
and then
\[ r(v_*(c,\gamma)) + p(v_*(c,\gamma)) f(h(v_*(c,\gamma),v_*(c,\gamma)),h(v_*(c,\gamma),v_*(c,\gamma))) \leq 0. \] (17)
Similarly, we have
\[ r(v^*(c,\gamma)) + p(v^*(c,\gamma)) f(h(v^*(c,\gamma),v_*(c,\gamma)),h(v^*(c,\gamma),v_*(c,\gamma))) \geq 0. \] (18)
According to the definition of \( h \), there exists \( v, u \in [v_*(c,\gamma), v^*(c,\gamma)] \subset [\omega_1, \omega_2] \) such that
\[ h(v_*(c,\gamma),v^*(c,\gamma)) = g(u), \quad h(v^*(c,\gamma),v_*(c,\gamma)) = g(v). \]
It then follows from (17) and (18) that
\[ r(v_*(c,\gamma)) + p(v_*(c,\gamma)) f(g(u),g(v)) \leq 0, \]
\[ r(v^*(c,\gamma)) + p(v^*(c,\gamma)) f(g(v),g(u)) \geq 0. \]
By the definition of \( f \), there exists \( v_0, u_0 \) lying between \( v \) and \( u \) such that
\[ f(g(u),g(v)) = \Gamma(g(u_0)), \quad f(g(v),g(u)) = \Gamma(g(v_0)). \]
Thus,
\[ r(v_*(c,\gamma)) + p(v_*(c,\gamma)) \Gamma(g(u_0)) \leq 0, \quad r(v^*(c,\gamma)) + p(v^*(c,\gamma)) \Gamma(g(v_0)) \geq 0. \] (19)
Since \(-r(u)/p(u)\) is strictly increasing on \([0,\omega_2]\), it follows that
\[ \Gamma(g(u_0)) \leq -r(v_*(c,\gamma))/p(v_*(c,\gamma)) \leq -r(u_0)/p(u_0) \]
and
\[ \Gamma(g(v_0)) \geq -r(v^*(c,\gamma)) + p(v^*(c,\gamma)) \geq -r(v_0)/p(v_0), \]
which imply that \( v_0 \leq \omega \leq u_0 \) according to assumption (vi) in (H). Note also from (19) that
\[ \Gamma(g(u_0)) \leq -r(v_0)/p(v_0), \quad \Gamma(g(v_0)) \geq -r(u_0)/p(u_0). \]
By property (K), it follows that \( v_0 = u_0 \). Hence, \( v_0 = u_0 = \omega \). It then follows from (19) that
\[
\Gamma (g(\omega)) = -r(v_*(c, \gamma)) / p(v_*(c, \gamma)) \leq -r(v^*(c, \gamma)) / p(v^*(c, \gamma)) \leq \Gamma (g(\omega)).
\]
Thus, \( v_*(c, \gamma) = v^*(c, \gamma) = \omega \). It then follows that
\[
\omega = v_*(c, \gamma) \leq w_*(c) \leq v^*(c, \gamma) = \omega,
\]
which implies that
\[
\lim_{t \to \infty, |x| \leq ct} u(t, x) = \omega \quad \text{for} \quad c \in (0, c^*).
\]
This completes the proof. \( \square \)

**Theorem 3.3.** Assume that (H) holds, \( g(\cdot) \) is a monotone function, and either \( \Gamma(\cdot) \) is a monotone or unimodal function. Then, the following conclusions are valid:

(i) If \( u_0 \in \mathcal{X}_\omega \) is compact supported, then the solution of (5) satisfies
\[
\lim_{t \to \infty, |x| \geq ct} u(t, x; u_0) = 0 \quad \text{for} \quad c > c^*.
\]
(ii) If \( u_0 \in \mathcal{X}_\omega \) with \( u_0 \not\equiv 0 \), then there exists some \( \epsilon > 0 \) such that the solution of (5) satisfies
\[
\liminf_{t \to \infty, |x| \leq ct} u(t, x; u_0) \geq \epsilon \quad \text{for} \quad 0 < c < c^*.
\]

If, in addition, (K) holds, then
\[
\lim_{t \to \infty, |x| \leq ct} u(t, x; u_0) = \omega.
\]

**Proof.** We divide the proof into two cases.

**Case I.** \( \Gamma \) is monotone on \([0, g(\omega)]\). In this case, the existence of spreading speed has been showed by Proposition 1.

**Case II.** \( \Gamma \) is unimodal on \([0, g(\omega)]\). Since \( \Gamma \) is unimodal on \([0, g(\omega)]\), there exists \( \bar{u} \in (0, \omega) \) such that \( g(\bar{u}) \) is the mode of \( \Gamma \), i.e., \( \Gamma(g(\bar{u})) = \max_{u \geq 0} \Gamma(u) \). By \( \Gamma(g(\bar{u})) = \Gamma(g(\omega)) = -r(\omega)/p(\omega) \) and the monotonicity of \( -r(u)/p(u) \), it follows that there exists \( \omega^* \geq \omega \) such that
\[
\Gamma(g(\bar{u})) = -r(\omega^*)/p(\omega^*).
\]
Define
\[
\Gamma^+(v) = \begin{cases} 
\Gamma(v), & v \in [0, g(\bar{u})], \\
\Gamma(g(\bar{u})), & v \in [g(\bar{u}), g(\omega^*)].
\end{cases}
\]
Then, \( 0 < \Gamma^+(v) \leq \Gamma(g(\bar{u})) \), \( \Gamma^+(v) \leq \Gamma'(0)v \) for \( v > 0 \). Furthermore,
\[
\Gamma^+(g(\omega^*)) = \Gamma(g(\bar{u})) = -r(\omega^*)/p(\omega^*).
\]
Note that
\[
\Gamma^+(g(u)) = \Gamma(g(u)) > -r(u)/p(u) \quad \text{for} \quad u \in (0, \bar{u}],
\]
and
\[
\Gamma^+(g(u)) = \Gamma(g(\bar{u})) = -r(\omega^*)/p(\omega^*) > -r(u)/p(u) \quad \text{for} \quad u \in (\bar{u}, \omega^*).
\]
It follows that
\[
\Gamma^+(g(u)) > -r(u)/p(u) \quad \text{for} \quad u \in (0, \omega^*).
\]
Let \( \hat{u} \in (0, \bar{u}) \) be the smallest number such that \( \Gamma(g(\hat{u})) = \min_{v \in [g(\bar{u}), g(\omega^*)]} \Gamma(v) \). By \( \Gamma(g(\hat{u})) \leq \Gamma(g(\omega)) = -r(\omega)/p(\omega) \) and the monotonicity of \(-r(u)/p(u)\), it follows that there exists \( \omega_* \leq \omega \) such that 
\[
\Gamma(g(\hat{u})) = -r(\omega_*)/p(\omega_*).
\]
Since \( \Gamma(g(\omega_*)) \geq -r(\omega_*)/p(\omega_*), \) it follows that \( \hat{u} \leq \omega_* \leq \omega \). Define
\[
\Gamma^-(v) = \begin{cases} 
\Gamma(v), & v \in [0, g(\hat{u})], \\
\Gamma(g(\hat{u})), & v \in [g(\bar{u}), g(\omega^*)].
\end{cases}
\]
Clearly, \( 0 < \Gamma^-(v) \leq \Gamma(g(\hat{u})), \Gamma^-(v) \leq \Gamma^+(0)v \) for \( v > 0 \), and
\[
\Gamma^-(g(\omega_*)) = \Gamma(g(\hat{u})) = -r(\omega_*)/p(\omega_*).
\]
Note that
\[
\Gamma^-(g(u)) = \Gamma(g(u)) > -r(u)/p(u) \quad \text{for} \quad u \in (0, \hat{u}],
\]
and
\[
\Gamma^-(g(u)) = \Gamma(g(\hat{u})) = -r(\omega_*)/p(\omega_*) > -r(u)/p(u) \quad \text{for} \quad u \in (\hat{u}, \omega_*).
\]
It follows that 
\[
\Gamma^-(g(u)) > -r(u)/p(u) \quad \text{for} \quad u \in (0, \omega_*).
\]
By the above definitions of \( \Gamma^- \) and \( \Gamma^+ \), it is easily seen that 
\[
\Gamma^- (g(u)) \leq \Gamma^+(g(u)) \leq \Gamma^+(0) \quad \text{for} \quad u \in [0, \omega^*].
\]
Consider the following two convolution equations 
\[
u_t(t, x) = r(u) + p(u)\Gamma^+(k * g(u))(t, x)
\]
and 
\[
u_t(t, x) = r(u) + p(u)\Gamma^- (k * g(u))(t, x).
\]
By arguments similar to those in the Theorem 3.2, we would obtain the conclusions. This completes the proof.

4. Applications. According to the general setting for the nonlinearity of (5). The results established in Section 3 can be applied to a variety of integro-differential equations. Here, we only present three examples to illustrate the application of the theoretical results.

Example 4.1. Consider the convolution equation
\[
u_t(t, x) = -du(t, x) + \frac{\alpha k * u}{1 + \beta (k * u)^m}, \quad (23)
\]
where \( \beta, m > 0, \alpha > d > 0, \) and 
\[
k * u = \int_{\mathbb{R}} k(y)u(t - r, x - y)dy.
\]
It is easy to see that (23) admits two equilibria \( 0 \) and \( \omega := (\frac{\alpha - d}{\alpha \beta})^{\frac{1}{m}} > 0 \). By direct computation, it follows that 
\[
\Gamma(v) = \frac{\alpha v}{1 + \beta v^m}
\]
is strictly monotone on \( \mathbb{R}_+ \) when \( 0 < m \leq 1 \), and is non-monotone with unimodality at \( v_0 = (\frac{1}{\beta(m-1)})^{\frac{1}{m}} \) when \( m > 1 \). Moreover, \( \Gamma(v) \) is strictly monotone on \([0, \omega]\) when 
\( m \in (1, \frac{\alpha}{\alpha - d}] \).
Next, we verify the property (K). It is easy to see that \( \max_{v \in \mathbb{R}^+} \frac{\alpha v}{1 + \beta v} = \frac{\alpha(m-1)}{m} \left( \frac{1}{\beta(m-1)} \right)^{\frac{1}{m}} \) when \( m > 1 \). Let \( \vartheta = \frac{\alpha(m-1)(1 - \beta)}{\beta(m-1)} \). For any \( v, u \) satisfying \( 0 < v \leq \omega \leq u \leq \vartheta \), \( \Gamma(u) \leq dv \) and \( \Gamma(v) \geq du \), we have
\[
\vartheta \Gamma(v) \geq u \Gamma(u).
\]

Note that function \( s \Gamma(s) = \frac{\alpha s^2}{1 + \beta s^m} \) is strictly increasing for \( s \in [0, +\infty) \) when \( m \in (0, 2] \), and is strictly increasing for \( s \in [0, \left( \frac{2}{\beta(m-2)} \right)^{\frac{1}{m}}] \) when \( m > 2 \). Thus, it follows that \( u = v \) when \( m \in (0, 2] \), or when \( m > 2 \) and \( \vartheta \leq \left( \frac{2}{\beta(m-2)} \right)^{\frac{1}{m}} \), namely, \( \vartheta \leq \frac{m-2}{m-1} \left( \frac{2(m-1)}{m-2} \right)^{\frac{1}{m}} \).

Let \( c^* \) be defined as in (6) with \( r(u) = -du, \ g(u) = u, \ p(u) = 1 \) and \( \Gamma(v) = \frac{\alpha v}{1 + \beta v} \). Applying Theorem 3.3, we have the following conclusions.

**Theorem 4.1.** Assume that \( k \) satisfies the assumptions in (H). Then the \( c^* \) is the spreading speed of (23) in the sense that (20) and (21) in Theorem 3.3 hold for (23). Moreover, the equality (22) in Theorem 3.3 is valid when \( m \in (0, \max\{2, \frac{\alpha}{\alpha - \beta}\}) \), or \( m > \max\{2, \frac{\alpha}{\alpha - \beta}\} \) and \( \frac{2}{\alpha} \leq \frac{m-2}{m-1} \left( \frac{2(m-1)}{m-2} \right)^{\frac{1}{m}} \).

**Example 4.2.** Consider the following specific case of (2)
\[
u(t, x) = -u(t, x) + \int_{\mathbb{R}} k(x - y)g(u(t, y))dy, \tag{24}
\]
where \( g(u) = \alpha e^{-\beta u}, \ \alpha > 1, \ \beta > 0 \).

It is easy to see that (24) admits exactly two constant steady states 0 and \( \omega = \frac{\ln \alpha}{\beta} > 0 \) provided that \( \alpha > 1 \). Note that
\[
g'(u) = \alpha e^{-\beta u}(1 - \beta u),
\]
It follows that \( g(u) \) is monotone on \([0, \frac{\ln \alpha}{\beta}]\) if \( \alpha \leq e \) and is unimodal on \([0, \frac{\ln \alpha}{\beta}]\) if \( \alpha > e \).

We further to verify the property (K). It holds for \( \alpha \in (e, e^2] \). For any \( u, v \) satisfying \( 0 < v \leq \omega \leq u \leq \frac{\alpha}{\beta e}, g(u) \leq v \) and \( g(v) \geq u \), since \( g(v) \) is decreasing on \([\omega, +\infty)\), we have \( v \geq g(u) \geq g(g(v)) \), and then
\[
g(g(v))/v \leq 1 = g(g(\omega))/\omega. \tag{25}
\]
Let \( f(u) = g(g(u))/u = e^{-\beta(u + \alpha u e^{-\beta c})} \). Direct calculation shows that for \( \alpha \in (e, e^2] \), the function \( f(u) \) is strictly decreasing. Thus it follows from (25) that \( v = \omega \). On the other hand, since \( \omega \leq u \leq g(v) = g(\omega) = \omega \), it follows that \( u = \omega \). Thus, we have \( u = v \).

Let \( c^* \) be defined as in (6) with \( r(u) = -u, \ g(u) = \alpha e^{-\beta u}, \ p(u) = 1 \) and \( \Gamma(v) = v \). Applying Theorem 3.2, we have the following conclusions.

**Theorem 4.2.** Assume that \( k \) satisfies the assumptions in (H). Then the \( c^* \) is the spreading speed of (24) in the sense that (10) and (11) in Theorem 3.2 hold for (24). Moreover, the equality (12) in Theorem 3.2 holds for \( \alpha \in (1, e^2] \).

**Example 4.3.** Consider the integro-differential equation
\[
u_u(t, x) = -du(t, x) + F \left( \int_{\mathbb{R}} k(x - y)u(t, y)dy \right), \tag{26}
\]
where \( d > 0 \), and \( F(v) = av(1 - v) \). Clearly, the saptially homogeneous equation of (26) is the logistic type equation.
Assume that $d < a < 4d$ so that (26) admits two equilibria $0$ and $ω := 1 - \frac{d}{a} > 0$, $F'(0) > d$ and $F([0, 1]) \subset [0, d]$ (restricted in the interval $[0, 1]$ such that $F(v)$ is nonnegative). Since $F'(v) = a(1 - 2v)$, which implies that $F(v)$ is increasing for $0 < v < \frac{1}{2}$ and is decreasing for $v < \frac{1}{2}$. Consequently, $F(v)$ is monotone on $[0, ω]$ if $d < a \leq 2d$, while it is unimodal on $[0, ω]$ if $2d < a < 4d$. Moreover, if $a \in (2d, 3d]$, then it can be easily verified the property (K) holds by arguments similar to those in Example 4.2.

Let $c^*$ be defined as in (6) with $r(u) = -du$, $g(u) = u$, $p(u) = 1$ and $Γ(v) = F(v)$. Applying Theorem 3.3, we have the following conclusions.

**Theorem 4.3.** Assume that $k$ satisfies the assumptions in (H) and $d < a < 4d$. Then the $c^*$ is the spreading speed of (26), i.e., (20) and (21) in Theorem 3.3 hold for (26). Moreover, the equality (22) in Theorem 3.3 holds when $d < a \leq 3d$.

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