A relation between $m_{G,N}$ and the Euler characteristic of the nerve space of some class poset of $G$

Heguo Liu$^1$, Xingzhong Xu$^{1,2,}$, Jiping Zhang$^3$

Abstract. Let $G$ be a finite group and $N \triangleleft G$ with $|G : N| = p$ for some prime $p$. In this note, to compute $m_{G,N}$ directly, we construct a class poset $\mathcal{T}_C(G)$ of $G$ for some cyclic subgroup $C$. And we find a relation between $m_{G,N}$ and the Euler characteristic of the nerve space $|N(\mathcal{T}_C(G))|$ (see the Theorem 1.3). As an application, we compute $m_{S_5,A_5} = 0$ directly, and get $S_5$ is a $B$-group.

Key Words: $B$-group; nerve space; poset of group.

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1. Introduction

In [4], Bouc proposed the following conjecture:

Conjecture 1.1. [4] Conjecture A] Let $G$ be a finite group. Then $\beta(G)$ is nilpotent if and only if $G$ is nilpotent.

Here, $\beta(G)$ a largest quotient of a finite group which is a $B$-group and the definition of $B$-group can be found in [4] or in the Section 2. Bouc has proven the Conjecture 1.1 under the additional assumption that finite group $G$ is solvable in [4]. In [10], Xu and Zhang consider some special cases when the finite group $G$ is not solvable. But this result relies on the proposition of Baumann [3], and his proposition relies on the Conlon theorem [7, (80.51)]. If we want to generalize the result of [10], we need use the new method to compute $m_{G,N}$ directly. Here, $N$ is a normal subgroup of $G$. And the definition of $m_{G,N}$ can be fined in [4] or in the Section 2.

For this aim, some attempts have been done in the paper. We begin to compute $m_{G,N}$ when $|G : N| = p$ for some prime number. But it is not easy to compute $m_{G,N}$ directly even if $|G : N| = p$. When $|G : N| = p$, we have the following

* Date: 27/01/2017.
1. Department of Mathematics, Hubei University, Wuhan, 430062, China
2. Departament de Matemàtiques, Universitat Autònom de Barcelona, E-08193 Bellaterra, Spain
3. School of Mathematical Sciences, Peking University, Beijing, 100871, China
Heguo Liu’s E-mail: ghliu@hubu.edu.cn
Xingzhong Xu’s E-mail: xuxingzhong407@mat.uab.cat, xuxingzhong407@126.com
* Corresponding author
Jiping Zhang’s E-mail: jzhang@pku.edu.cn
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observation:

\[ m_{G,N} + \frac{1}{|G|} \sum_{X \leq N} |X| \mu(X, G) \]

\[ = \frac{1}{|G|} \sum_{X = G, X \leq G} |X| \mu(X, G) + \frac{1}{|G|} \sum_{X \leq N} |X| \mu(X, G) \]

\[ = \frac{1}{|G|} \sum_{X = G, X \leq G} |X| \mu(X, G) + \frac{1}{|G|} \sum_{XN \neq G, X \leq G} |X| \mu(X, G) \]

\[ = \frac{1}{|G|} \sum_{X \leq G} |X| \mu(X, G) \]

\[ = m_{G,G} = 0, \text{ if } G \text{ is not cyclic.} \]

So to compute \( m_{G,N} \), we can compute \( \frac{1}{|G|} \sum_{X \leq N} |X| \mu(X, G) \) first. And we find there is a formula (see the proof of the Proposition 4.1) about

\[ \frac{1}{|G|} \sum_{X \leq N} |X| \mu(X, G) + \frac{1}{|G|} \sum_{X \leq N} |X| \mu(X, N) \]

\[ (= \frac{1}{|G|} \sum_{X \leq N} |X| \mu(X, G) + m_{N,N}). \]

Now, we set

\[ m'_{G,N} := \frac{1}{|G|} \sum_{XN \neq G, X \leq G} |X| \mu(X, G) = \frac{1}{|G|} \sum_{X \leq N} |X| \mu(X, G); \]

and set

\[ M'_{G,N} := \sum_{X \leq N} |X| \mu(X, G) = |G|m'_{G,N}. \]

We get the following theorem about \( M'_{G,N} \).

**Proposition 1.2.** Let \( G \) be a finite group and \( N \leq G \) such that \(|G : N| = p\) for some prime number \( p\). Then

\[ M'_{G,N} = - \sum_{C \leq N, C \text{ is cyclic}} \varphi(|C|) - \sum_{Y \leq G, Y \not\leq N} M'_{V,Y \cap N}. \]

Here, \( \varphi \) is the Euler totient function.

To compute \( M'_{G,N} \), we need compute \( M'_{V,Y \cap N} \) for every \( Y \leq G \). Since \( Y \leq G \), thus we can get \( M'_{G,N} \) by finite steps. Here, we define a new class poset of subgroups of \( G \) as following: Let \( C \) be a cyclic subgroup of \( N \), define

\[ \Sigma_{C,G} := \{ X | C \leq X \leq G, X \not\leq N \}. \]

We can see that \( \Sigma_{C,G} \) is a poset ordered by inclusion. We can consider poset \( \Sigma_{C,G} \) as a category with one morphism \( Y \rightarrow Z \) if \( Y \) is a subgroup of \( Z \). We set \( N(\Sigma_{C,G}) \) is the nerve of the category \( \Sigma_{C,G} \) and \( |N(\Sigma_{C,G})| \) is the geometric realization of \( N(\Sigma_{C,G}) \). Then we get a following computation about \( m_{G,N} \).

**Theorem 1.3.** Let \( G \) be a finite group and \( G \) not cyclic. Let \( N \leq G \) such that \(|G : N| = p\) for some prime number \( p\). Then

\[ m_{G,N} = \frac{1}{|G|} \sum_{C \leq N, C \text{ is cyclic}} (1 - \chi(|N(\Sigma_{C,G})|) \cdot \varphi(|C|)). \]

Here, \( |N(\Sigma_{C,G})| \) is a simplicial complex associated to the poset \( \Sigma_{C,G} \), and \( \chi(|N(\Sigma_{C,G})|) \) is the Euler characteristic of the space \( |N(\Sigma_{C,G})| \).
Proof. Since \( m_{G,N} + m'_{G,N} = m_{G,G} = 0 \) when \( G \) is not cyclic, thus we prove this theorem by using the Proposition 5.4.

By using the method of the Theorem 1.2-3, we compute \( m_{S_5,A_5} \) directly, and we get the following result.

**Proposition 1.4.** \( S_5 \) is a \( B \)-group.

In fact, \([6]\) had proved that \( S_n \) is a \( B \)-group. And by using \([3]\), we also get that \( S_n \) is a \( B \)-group when \( n \geq 5 \).

After recalling the basic definitions and properties of \( B \)-groups in the Section 2, we prove some lemmas about Möbius function in the Section 3. And this lemmas will be used in the Section 4 to prove the Proposition 1.2. In the Section 5, we construct a class poset \( T_C(G) \) of \( G \) for some cyclic subgroup of \( C \) and prove the Theorem 1.3. As an application, we compute \( m_{S_5,A_5} = 0 \) and get \( S_5 \) is a \( B \)-group in the Section 6.

2. **Burnside rings and \( B \)-groups**

In this section we collect some known results that will be needed later. For the background theory of Burnside rings and \( B \)-groups, we refer to \([4], [5]\).

**Definition 2.1.** \([5, \text{Notation 5.2.2}]\) Let \( G \) be a finite group and \( N \triangleleft G \). Denote by \( m_{G,N} \) the rational number defined by:

\[
m_{G,N} = \frac{1}{|G|} \sum_{XN=G} |X|\mu(X, G),
\]

where \( \mu \) is the Möbius function of the poset of subgroups of \( G \).

**Remark 2.2.** If \( N = 1 \), we have

\[
m_{G,1} = \frac{1}{|G|} \sum_{X1=G} |X|\mu(X, G) = \frac{1}{|G|}|G|\mu(G, G) = 1 \neq 0.
\]

**Definition 2.3.** \([4, \text{Definition 2.2}]\) The finite group \( G \) is called a \( B \)-group if \( m_{G,N} = 0 \) for any non-trivial normal subgroup \( N \) of \( G \).

**Proposition 2.4.** \([5, \text{Proposition 5.4.10}]\) Let \( G \) be a finite group. If \( N_1, N_2 \triangleleft G \) are maximal such that \( m_{G,N} \neq 0 \), then \( G/N_1 \cong G/N_2 \).

**Definition 2.5.** \([4, \text{Notation 2.3}]\) When \( G \) is a finite group, and \( N \triangleleft G \) is maximal such that \( m_{G,N} \neq 0 \), set \( \beta(G) = G/N \).

**Theorem 2.6.** \([5, \text{Theorem 5.4.11}]\) Let \( G \) be a finite group.

1. \( \beta(G) \) is a \( B \)-group.

2. If a \( B \)-group \( H \) is isomorphic to a quotient of \( G \), then \( H \) is isomorphic to a quotient of \( \beta(G) \).

3. Let \( M \triangleleft G \). The following conditions are equivalent:

   (a) \( m_{G,N} \neq 0 \).

   (b) The group \( \beta(G) \) is isomorphic to a quotient of \( G/M \).

   (c) \( \beta(G) \cong \beta(G/N) \).

We collect some properties of \( m_{G,N} \) that will be needed later.
Proposition 2.7. [4 Proposition 2.5] Let $G$ be a finite group. Then $G$ is a $B$-group if and only if $m_{G,N} = 0$ for any minimal (non-trivial) normal subgroup of $G$.

Proposition 2.8. [5 Proposition 5.6.1] Let $G$ be a finite group. Then $m_{G,G} = 0$ if and only if $G$ is not cyclic. If $P$ be cyclic of order $p$ and $p$ be a prime number, then $m_{P,P} = \frac{p^2 - 1}{p}$.

Remark 2.9. If $G$ is a finite simple group, then $G$ is a $B$-group if and only if $G$ is not abelian.

Proposition 2.10. [5 Proposition 5.3.1] Let $G$ be a finite group. If $M$ and $N$ are normal subgroup of $G$ with $N \leq M$, then

$$m_{G,M} = m_{G,N}m_{G/N,M/N}.$$  

We collect two results that will be needed later.

When $p$ is a prime number, recall that a finite group $G$ is called cyclic modulo $p$ (or $p$-hypo-elementary) if $G/O_p(G)$ is cyclic. And M. Baumann has proven the Conjecture under the additional assumption that finite group $G$ is cyclic modulo $p$ in [3].

Theorem 2.11. [3 Theorem 3] Let $p$ be a prime number and $G$ be a finite group. Then $\beta(G)$ is cyclic modulo $p$ if and only if $G$ is cyclic modulo $p$.

In [4], S. Bouc has proven the Conjecture under the additional assumption that finite group $G$ is solvable.

Theorem 2.12. [4 Theorem 3.1] Let $G$ be a solvable finite group. Then $\beta(G)$ is nilpotent if and only if $G$ is nilpotent.

3. Some lemmas about the M"obius function

In this section, we prove some lemmas about the M"obius function. The main lemma is the Lemma 3.4, and the Lemma 3.2-3 are prepared for it. In fact, the Lemma 3.4 will be used in computing $m_{G,N}$ where $G$ is a finite group and $N \leq G$.

Let $G$ be a finite group and let $\mu$ denote the M"obius function of subgroup lattice of $G$. We refer to [1], [7, p.94]:

Let $K, D \leq G$, recall the Zeta function of $G$ as following:

$$\zeta(K, D) = \begin{cases} 1, & \text{if } K \leq D; \\ 0, & \text{if } K \not\leq D. \end{cases}$$

Set $n := |\{K|K \leq G\}|$, we have a $n \times n$ matrix $A$ as following:

$$A := (\zeta(K, D))_{K,D \leq G}.$$ 

It is easy to find that $A$ is an invertible matrix, so there exists $A^{-1}$ such that $AA^{-1} = E$.

Here, $E$ is an identity element. Recall the M"obius function as following:

$$(\mu(K, D))_{K,D \leq G} = A^{-1}.$$ 

Now, we set the subgroup lattice of $G$ as following:

$$\{K|K \leq G\} := \{1 = K_1, K_2, \ldots, K_n = G\}$$

where $n = |\{K|K \leq G\}|$. 

Definition 3.1. Let $K \leq G$, there exists a proper subgroup series of $K$ as following:

$$\sigma : 1 = K_1 \leq K_2 \leq K_3 \leq \cdots \leq K_t = K$$

where $K_i$ are subgroups of $G$ and $K_i$ is a proper subgroup of $K_{i+1}$ for all $i$. Set $\mathcal{X}_K$ is the set of elements like above $\sigma$. And we call $t$ is the length of $\sigma$ and set $t := l(\sigma)$.

We define the height of $K$ as following:

$$ht(K) = \max \{l(\sigma) | \sigma \in \mathcal{X}_K\}.$$  

It is easy to see that $ht(1) = 1$.

Lemma 3.2. Let $G$ be a finite group and $K, L$ be subgroups of $G$. If $ht(K) \geq ht(L)$ and $K \neq L$, then $\zeta(K, L) = 0$.

Proof. Suppose that $\zeta(K, L) \neq 0$, by the definition of Zeta function, we have that $K \leq L$. Set $ht(K) = t$, there exists a proper subgroup series of $K$ as following:

$$1 = K_1 \leq K_2 \leq K_3 \leq \cdots \leq K_t = K$$

Also $K \leq L$ and $K \neq L$, thus we have the following series of $L$:

$$1 = K_1 \leq K_2 \leq K_3 \leq \cdots \leq K_t = K \leq L.$$  

Hence $ht(L) \geq t + 1 \geq t = ht(K)$. That is a contradiction to $ht(K) \geq ht(L)$.

Lemma 3.3. Let $G$ be a finite group. Let $\{K_i\}_{i = 1, 2, \ldots, n}$ be the set of all subgroups of $G$. And Set $K_1 = 1, K_n = G$. Then we can reorder the sequence $1 = K_1, K_2, \ldots, K_n = G$ as the sequence $1 = K_{j(1)}, K_{j(2)}, \ldots, K_{j(n)} = G$ such that $(\zeta(K_{j(b)}, K_{j(b)}))_{n \times n}$ is an invertible upper triangular matrix. Here, $\{l^{(1)}, l^{(2)}, \ldots, l^{(n)}\} = \{1, 2, \ldots, n\}$.

Proof. As the above definition of height of subgroups, we can set

$$\mathcal{T}_1 = \{K \leq G|ht(K) = 1\} = \{1 = K_1\};$$
$$\mathcal{T}_2 = \{K \leq G|ht(K) = 2\} := \{K_2, K_2, \ldots, K_{22}\};$$
$$\ldots$$
$$\mathcal{T}_{m-1} = \{K \leq G|ht(K) = m - 1\} := \{K_{(m-1)_1}, K_{(m-1)_2}, \ldots, K_{(m-1)_{m-1}}\};$$
$$\mathcal{T}_m = \{K \leq G|ht(K) = m\} = \{K_n = G\};$$
$$\mathcal{T}_{m+1} = \{K \leq G|ht(K) = m + 1\} = \emptyset.$$  

Now, we can reorder $1 = K_1, K_2, \ldots, K_n = G$ as

$$1 = K_1;$$
$$K_{21}, K_{22}, \ldots, K_{22};$$
$$\ldots$$
$$K_{(m-1)_1}, K_{(m-1)_2}, \ldots, K_{(m-1)_{m-1}};$$
$$K_n = G.$$  

We can set

$$l^{(1)} := 1, l^{(2)} := 2, \ldots, (m - 1)_{t_{m-1}} := l^{(n-1)}, l^{(n)} := n.$$  

Let $s \geq r$, we want to prove that

$$\zeta(K_{j(a)}, K_{j(b)}) = 0.$$  

Here, we can set $K_{j(a)} = K_{k_a}$ and $K_{j(b)} = K_{j_b}$ for $1 \leq a \leq t_k, 1 \leq b \leq t_j$. Since $s \geq r$, thus $k \geq j$. Then by the Lemma 3.2, we have

$$\zeta(K_{k_a}, K_{j_b}) = 0$$
Lemma 3.4.

Proposition 4.1 in the Section 4. □

Let us list the main lemma as following, and this lemma is used to prove the Proposition 4.1 in the Section 4.

Lemma 3.4. Let $G$ be a finite group. Let $\{K_i|i=1,2,\ldots,n\}$ be the set of all subgroups of $G$. And Set $K_1=1, K_n=G$. Then we have $\mu(K_i, K_{i'})=0$ if $K_i \not\leq K_{i'}$.

Proof. By the proof of the above lemma, we can suppose that $\{K_i|i=1,2,\ldots,n\}$ be the set of all subgroups of $G$ and set $K_1=1, K_n=G$ such that $(\zeta(K_j, K_k))_{n \times n}$ is an invertible upper triangular matrix. Here, $1 \leq j,k \leq n$.

To prove the lemma, we need to adjust the positions of some $K_j$ such that the sequence $1=K_1, K_2, \ldots, K_n = G$ are reordered as $1=K_{j^{(1)}}, K_{j^{(2)}}, \ldots, K_{j^{(n)}}=G$, and we can get that $(\zeta(K_{j^{(1)}}, K_{j^{(2)}}))_{n \times n}$ is an invertible upper triangular matrix. Here, $j^{(1)},k^{(1)} \in \{1^{(1)}, 2^{(1)}, \ldots, n^{(1)}\} = \{1,2,\ldots,n\}$.

First, we can set $ht(K_i)=k$ and $ht(K_{i'})=j$. Moreover, we set $K_i=K_k$ and $K_{i'}=K_{k'}$ for $1 \leq a \leq t_k$, $1 \leq b \leq t_j$. If $k \geq j$, it is easy to see that $\mu(K_k, K_{k'})=\mu(K_k, K_{k'})=0$. Now, we will consider the cases when $k=j$ and $k \leq j$ as following.

Case 1. $k=j$. We will prove this case when $k \geq k_b$ and $k \leq k_b$ as following.

Case 1.1. If $k_a \geq k_b = j_b$, we can set

\begin{align*}
\mathfrak{T}_1 &= \{K \leq G||ht(K)|=1\} = \{1=K_1\}; \\
\mathfrak{T}_2 &= \{K \leq G||ht(K)|=2\} := \{K_{2_1}, K_{2_2}, \ldots, K_{2_{t_2}}\}; \\
\mathfrak{T}_k &= \{K \leq G||ht(K)|=k\} := \{K_{k_1}, K_{k_2}, \ldots, K_{k_{t_k}}\}; \\
\mathfrak{T}_{m-1} &= \{K \leq G||ht(K)|=m-1\} := \{K_{(m-1)_1}, K_{(m-1)_2}, \ldots, K_{(m-1)_{t_{m-1}}}\}; \\
\mathfrak{T}_m &= \{K \leq G||ht(K)|=m\} = \{K_n=G\}; \\
\mathfrak{T}_{m+1} &= \{K \leq G||ht(K)|=m+1\} = \emptyset.
\end{align*}

It is easy to see that we can reorder of $1=K_1, K_2, \ldots, K_n = G$ as

\begin{align*}
K_1 &= 1, \\
K_{2_1}, K_{2_2}, \ldots, K_{2_{t_2}}, \\
\mathfrak{T}_k &= K_{k_1}, K_{k_2}, \ldots, K_{k_{t_k}}, \\
\mathfrak{T}_{m-1} &= K_{(m-1)_1}, K_{(m-1)_2}, \ldots, K_{(m-1)_{t_{m-1}}}, \\
K_n &= G.
\end{align*}

And we set this order of subgroup series of $G$ as

\begin{align*}
1 &= K_{r^{(1)}}, K_{r^{(2)}}, \ldots, K_{r^{(n)}} = G.
\end{align*}
We have

\[ B := (\zeta(K_{r(1)}, K_{r(n)}))_{n \times n} \]

is an invertible upper triangular matrix by the proof of the Lemma 3.2. Hence \( B^{-1} \) is also invertible upper triangular matrix, thus \( \mu(K_{j_1}, K_{k_n}) = 0 \). That is \( \mu(K_i, K_{i'}) = 0 \).

**Case 1.2.** If \( k_a \leq k_b = j_b \), we can set

\[ \mathfrak{T}_1 = \{ K \leq G | \text{ht}(K) = 1 \} = \{ 1 = K_1 \}; \]
\[ \mathfrak{T}_2 = \{ K \leq G | \text{ht}(K) = 2 \} := \{ K_{2_1}, K_{2_2}, \ldots, K_{2_2} \}; \]
\[ \cdots \cdots \]
\[ \mathfrak{T}_k = \{ K \leq G | \text{ht}(K) = k \} := \{ K_{k_1}, K_{k_2}, \ldots, K_{k_1} \}; \]
\[ \cdots \cdots \]
\[ \mathfrak{T}_{m-1} = \{ K \leq G | \text{ht}(K) = m - 1 \} := \{ K_{(m-1)_1}, K_{(m-1)_2}, \ldots, K_{(m-1)_{m-1}} \}; \]
\[ \mathfrak{T}_m = \{ K \leq G | \text{ht}(K) = m \} = \{ K_n = G \}; \]
\[ \mathfrak{T}_{m+1} = \{ K \leq G | \text{ht}(K) = m + 1 \} = \emptyset. \]

It is easy to see that we can reorder \( 1 = K_1, K_2, \ldots, K_n = G \) as

\[ K_1(= 1), \]
\[ K_2, K_2, \ldots, K_2_2, \]
\[ \cdots \cdots \]
\[ K_{1_1}, K_{1_2}, \ldots, K_{k_1}, K_{k_2}, \ldots, K_{k_1_1}, \ldots, K_{k_n_1}, K_{k_{n-1_1}}, \ldots, K_{k_{n_1}}; \]
\[ \cdots \cdots \]
\[ K_{(m-1)_1}, K_{(m-1)_2}, \ldots, K_{(m-1)_{m-1}}, \]
\[ K_n(= G). \]

That is, \( K_{k_a} \) and \( K_{k_b} \) switch places. And we set this order of subgroups of \( G \) as

\[ 1 = K_{r(1)}, K_{r(2)}, \ldots, K_{r(n)} = G. \]

We have

\[ B := (\zeta(K_{r(1)}, K_{r(n)}))_{n \times n} \]

is an invertible upper triangular matrix by the Lemma 3.2. Hence \( B^{-1} \) is also invertible upper triangular matrix, thus \( \mu(K_{k_a}, K_{k_b}) = 0 \). That is \( \mu(K_i, K_{i'}) = 0 \).

**Case 2.** \( k \leq j \). We will prove this case when \( j - k = 1 \) and \( j - k \geq 2 \) as following.

**Case 2.1.** If \( j - k = 1 \), we can set

\[ \mathfrak{T}_1 = \{ K \leq G | \text{ht}(K) = 1 \} = \{ 1 = K_1 \}; \]
\[ \mathfrak{T}_2 = \{ K \leq G | \text{ht}(K) = 2 \} := \{ K_{2_1}, K_{2_2}, \ldots, K_{2_2} \}; \]
\[ \cdots \cdots \]
\[ \mathfrak{T}_k = \{ K \leq G | \text{ht}(K) = k \} := \{ K_{k_1}, K_{k_2}, \ldots, K_{k_1} \}; \]
\[ \mathfrak{T}_{k+1} = \{ K \leq G | \text{ht}(K) = k + 1 \} := \{ K_{(k+1)_1}, K_{(k+1)_2}, \ldots, K_{(k+1)_{k+1}} \}; \]
\[ \cdots \cdots \]
\[ \mathfrak{T}_{m-1} = \{ K \leq G | \text{ht}(K) = m - 1 \} := \{ K_{(m-1)_1}, K_{(m-1)_2}, \ldots, K_{(m-1)_{m-1}} \}; \]
\[ \mathfrak{T}_m = \{ K \leq G | \text{ht}(K) = m \} = \{ K_n = G \}; \]
\[ \mathfrak{T}_{m+1} = \{ K \leq G | \text{ht}(K) = m + 1 \} = \emptyset. \]

We can set \( a = t_{k}, \) that is \( K_{k_a} = K_{k_{t_{k}}} \). And we can set \( b = 1, \) that is \( K_{j_b} = K_{(k+1)_1}. \).
Now, we reorder the sequence $1 = K_1, K_2, \ldots, K_n = G$ as

\[
K_1(= 1), \\
K_2, K_2', \ldots K_{2e}, \\
\ldots \ldots \\
K_{k_1}, K_{k_2}, \ldots, K_{k_{t_k-2}}, K_{k_{t_k-1}}, K_{(k+1)1}(= K_{j_1}), \\
K_{k_{t_k}}(= K_{j_n}), K_{(k+1)2}, K_{(k+1)3}, \ldots, K_{(k+1)t_{k+1}}, \\
\ldots \ldots \\
K_{(m-1)1}, K_{(m-1)2}, \ldots K_{(m-1)t_{m-1}}, \\
K_n(= G).
\]

That is, $K_{j_n}$ and $K_{j_1}$ switch places. And we set the above sequence as

\[
1 = K_{r(1)}, K_{r(2)}, \ldots, K_{r(n)} = G.
\]

That is

\[
r^{(1)} = 1, r^{(2)} = 2, \ldots, r^{(n-1)} = (m-1)t_{m-1}, r^{(n)} = n.
\]

Since $K_{j_n} \not\leq K_{j_1}$, we can set $K_{j_n} = K_{r(1)}, K_{j_n} = K_{r(n)}$, we have $B := (\zeta(K_{r(i)}, K_{r(i+1)}))_{n \times n}$ is an invertible upper triangular matrix by the Lemma 3.2. Hence $B^{-1}$ is also invertible upper triangular matrix, thus $\mu(K_{j_n}, K_{j_1}) = 0$. That is $\mu(K_1, K_{r}) = 0$.

**Case 2.2.** If $j - k \geq 2$, set $c := j - k$ and we have $j = k + c$. Here, we can set

\[
\begin{align*}
\mathfrak{T}_1 &= \{K \leq G|\mathrm{ht}(K) = 1\} = \{1 = K_1\}; \\
\mathfrak{T}_2 &= \{K \leq G|\mathrm{ht}(K) = 2\} := \{K_2, K_2', \ldots K_{2e}\}; \\
\ldots \ldots \\
\mathfrak{T}_k &= \{K \leq G|\mathrm{ht}(K) = k\} := \{K_{k_1}, K_{k_2}, \ldots K_{k_k}\}; \\
\mathfrak{T}_{k+1} &= \{K \leq G|\mathrm{ht}(K) = k + 1\} := \{K_{(k+1)1}, K_{(k+1)2}, \ldots K_{(k+1)t_{k+1}}\}; \\
\ldots \ldots \\
\mathfrak{T}_{k+c-1} &= \{K \leq G|\mathrm{ht}(K) = k + c - 1\} := \{K_{(k+c-1)1}, K_{(k+c-1)2}, \ldots K_{(k+c-1)t_{k+c-1}}\}; \\
\mathfrak{T}_{k+c} &= \{K \leq G|\mathrm{ht}(K) = k + c\} := \{K_{(k+c)1}, K_{(k+c)2}, \ldots K_{(k+c)t_{k+c}}\}; \\
\ldots \ldots \\
\mathfrak{T}_{m-1} &= \{K \leq G|\mathrm{ht}(K) = m - 1\} := \{K_{(m-1)1}, K_{(m-1)2}, \ldots K_{(m-1)t_{m-1}}\}; \\
\mathfrak{T}_m &= \{K \leq G|\mathrm{ht}(K) = m\} = \{K_n = G\}; \\
\mathfrak{T}_{m+1} &= \{K \leq G|\mathrm{ht}(K) = m + 1\} = \emptyset.
\end{align*}
\]

We can set $a = t_k$, that is $K_{j_n} = K_{k_{t_k}}$. And we can set $b = 1$, that is $K_{j_1} = K_{(k+c)1}$.

For each $k + 1 \leq l \leq k + c - 1$, we consider $\mathfrak{T}_l$ and we can suppose that there exists $1 \leq s_l \leq t_l$ such that

\[
K_{s_l} \begin{cases} 
\leq K_{(k+c)1}, & \text{if } 1 \leq d \leq s_l; \\
\not\leq K_{(k+c)1}, & \text{if } s_l + 1 \leq d \leq t_l.
\end{cases}
\]
First, we reorder $1 = K_1, K_2, \ldots, K_n = G$ as

$K_1 = (1)$,
$K_2, K_3, \ldots, K_{2g}$,
$
\ldots \ldots \ldots$
$K_{k_1}, K_{k_2}, \ldots, K_{k_{k_2}}, K_{k_{k_2}}$,
$K_{(k+1)}_1, K_{(k+1)}_2, \ldots, K_{(k+1)}_{r_{k+1}}$,
$\ldots \ldots \ldots$
$K_{(k+c-1)}_1, K_{(k+c-1)}_2, \ldots, K_{(k+c-1)}_{r_{k+c-1}}$,
$K_{(k+c)}_1 (= K_{j_1}), K_{k_{k_2}} (= K_{k_0})$,
$K_{(k+1)}_{sk_{k+1}+1}, K_{(k+1)}_{sk_{k+1}+2}, \ldots, K_{(k+1)}_{sk_{k+1}}$,
$\ldots \ldots \ldots$
$K_{(m-1)}_1, K_{(m-1)}_2, \ldots, K_{(m-1)}_{r_{m-1}},$
$K_n (= G)$.

Now, we set the above sequence as

$
1 = K_{\gamma(1)}, K_{\gamma(2)}, \ldots, K_{\gamma(n)} = G.$

To prove

$B := (\zeta(K_{\gamma(j)}, K_{\gamma(j')}))_{n \times n}$

is an invertible upper triangular matrix, we will prove the following (1)-(3) first:

(1) For each

$K_v \in \{ K_{(k+1)}_{sk_{k+1}+1}, K_{(k+1)}_{sk_{k+1}+2}, \ldots, K_{(k+1)}_{sk_{k+1}} \}$

and

$K_u \in \{ K_{(k+1)}_1, K_{(k+1)}_2, \ldots, K_{(k+1)}_{sk_{k+1}} \}$

we can see that

$\zeta(K_v, K_u) = 0.$

Suppose $\zeta(K_v, K_u) \neq 0$, we have $K_v \leq K_u$. But $K_u \leq K_{(k+c)}_1$ and $K_v \not\leq K_{(k+c)}_1$, that is a contradiction. So $\zeta(K_v, K_u) = 0.$

(2) For each

$K_v \in \{ K_{(k+1)}_{sk_{k+1}+1}, K_{(k+1)}_{sk_{k+1}+2}, \ldots, K_{(k+1)}_{sk_{k+1}} \}$

we have $K_v \not\leq K_{(k+c)}_1$, thus

$\zeta(K_v, K_{(k+c)}_1) = 0.$
(3) For each
\[ K_u \in \{ K_{(k+1)_1}, K_{(k+1)_2}, \ldots, K_{(k+1)c_1}, \ldots, K_{(k+c-1)_1}, K_{(k+c-1)_2}, \ldots, K_{(k+c-1)c_{c-1}} \}, \]
we can see that
\[ \zeta(K_{k_{i_k}}, K_u) = 0. \]
Suppose \( \zeta(K_{k_{i_k}}, K_u) \neq 0 \), that is \( K_{k_{i_k}} \leq K_u \). But \( K_{k_{i_k}} \leq K_{(k+c)_1} \), thus \( K_{k_{i_k}} \leq K_{(k+c)_1} \). And we know \( K_{(k+c)_1} = K_{j_k}, K_{k_{i_k}} = K_{a_{k_{i_k}}} \) and \( K_{a_{k_{i_k}}} \neq K_{j_k} = K_{j_h} \).
That is a contradiction. So \( \zeta(K_{k_{i_k}}, K_u) = 0. \)

Since (1) – (3) hold, we have
\[ B := (\zeta(K_{r_1(l)}, K_{r_1(l)}'))_{n \times n} \]
is an invertible upper triangular matrix. It implies \( B^{-1} \) is also an invertible upper triangular matrix.
We can set \( K_{j_k} = K_{r_1(l)}, K_{a_{k_{i_k}}} = K_{r_1(l+1)}. \)
That is \( \mu(K_{r_1(l+1)}, K_{r_1(l)}) = 0 \) because \( B^{-1} \) is an invertible upper triangular matrix.
So \( \mu(K_{a_{k_{i_k}}}, K_{j_k}) = 0, \) it implies \( \mu(K_{i_k}, K_{r_1(l)}) = 0. \)

4. Computing the \( m_{G,N} \) when \( |G : N| = p \) for some prime number \( p \)

Let \( G \) be a finite group and \( N \trianglelefteq G \) with \( |G : N| = p \). In this section we will compute \( m_{G,N} \). First, we set
\[ m'_{G,N} := \frac{1}{|G|} \sum_{XN \neq G, X \leq G} |X| \mu(X, G) = \frac{1}{|G|} \sum_{X \leq N} |X| \mu(X, G); \]
and set
\[ M'_{G,N} := \sum_{X \leq N} |X| \mu(X, G) = |G|m'_{G,N}. \]
We can see that
\[ m_{G,N} + m'_{G,N} = \frac{1}{|G|} \sum_{XN \leq G, X \leq G} |X| \mu(X, G) + \frac{1}{|G|} \sum_{XN \neq G, X \leq G} |X| \mu(X, G) \]
\[ = \frac{1}{|G|} \sum_{X \leq G} |X| \mu(X, G) = m_{G,G}. \]

So to compute \( m_{G,N} \), we only need to compute \( M'_{G,N} \). Thus we have the following propositions.

**Proposition 4.1**. Let \( G \) be a finite group and \( N \trianglelefteq G \) such that \( |G : N| = p \) for some prime number \( p \). Then
\[ M'_{G,N} = - \sum_{Y \leq G, X \leq N \cap Y} |X| \mu(X, Y) \]
Proof. Let \{1 = X_1, X_2, \ldots, X_n\} be the poset of subgroups of \(G\) and we set \(X_{n-1} = N, X_n = G\). We have the following matrix:

\[
A := (\zeta(X_i, X_j)) := \begin{pmatrix}
A_1 & \alpha & \beta \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

where \(A_1 := (\zeta(X_i, X_j))_{i,j \leq n-2}\) and

\[
\alpha := \begin{pmatrix}
\zeta(X_1, N) \\
\zeta(X_2, N) \\
\vdots \\
\zeta(X_{n-2}, N)
\end{pmatrix}, \quad \beta := \begin{pmatrix}
\zeta(X_1, G) \\
\zeta(X_2, G) \\
\vdots \\
\zeta(X_{n-2}, G)
\end{pmatrix} = \begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}_{(n-2) \times 1}
\]

Here, we set \(B := (\mu(X_i, X_j))\). We know that \(B = A^{-1}\). We can set

\[
B := (\mu(X_i, X_j)) := \begin{pmatrix}
B_1 & \gamma & \delta \\
0 & 1 & -1 \\
0 & 0 & 1
\end{pmatrix}
\]

where \(B_1 := (\mu(X_i, X_j))_{i,j \leq n-2}\) and

\[
\gamma := \begin{pmatrix}
\mu(X_1, N) \\
\mu(X_2, N) \\
\vdots \\
\mu(X_{n-2}, N)
\end{pmatrix}, \quad \delta := \begin{pmatrix}
\mu(X_1, G) \\
\mu(X_2, G) \\
\vdots \\
\mu(X_{n-2}, G)
\end{pmatrix}.
\]

Since \(AB = 1\), we have \(A_1 B_1 = 1\),

\[A_1 \gamma + \alpha = 0,\]

and,

\[A_1 \delta - \alpha + \beta = 0.\]

So we have \(A_1 (\gamma + \delta) = -\beta\), that is \(\gamma + \delta = -A_1^{-1} \beta = -B_1 \beta\).

Now we compute the following:

\[
M'_G, N + |N| m_{G,N} = \sum_{X \leq N} |X| \mu(X, G) + \sum_{X \leq N} |X| \mu(X, N)
= \sum_{X \leq N} |X| (\mu(X, N) + \mu(X, G))
= \sum_{X \leq N} |X| (\mu(X, N) + \mu(X, G)) + (\mu(N, N) + \mu(N, G))
= \sum_{X \leq N} |X| (\mu(X, N) + \mu(X, G)).
\]
Here, $\mu(N, N) = 1$, $\mu(N, G) = -1$ because $N$ is a maximal subgroup of $G$ by $|G/N| = p$ for some prime number $p$. Since

$$
\gamma + \delta = \left(\begin{array}{cccc}
\mu(X_1, N) + \mu(X_1, G) \\
\mu(X_2, N) + \mu(X_2, G) \\
\vdots \\
\mu(X_{n-2}, N) + \mu(X_{n-2}, G)
\end{array}\right),
$$

and, $\gamma + \delta = -B_1\beta$. It implies

$$
\gamma + \delta = -\left(\begin{array}{cccc}
\mu(X_1, X_1) & \mu(X_1, X_2) & \cdots & \mu(X_1, X_{n-2}) \\
\mu(X_2, X_1) & \mu(X_2, X_2) & \cdots & \mu(X_2, X_{n-2}) \\
\vdots & \vdots & \ddots & \vdots \\
\mu(X_{n-2}, X_1) & \mu(X_{n-2}, X_2) & \cdots & \mu(X_{n-2}, X_{n-2})
\end{array}\right) \left(\begin{array}{c}
1 \\
1 \\
\vdots \\
1
\end{array}\right)_{(n-2)\times 1}.
$$

Thus, we have

$$
\mu(X_i, N) + \mu(X_i, G) = - \sum_{Y \leq G, Y \neq N} |X|\mu(X_i, Y)
$$

for $1 \leq i \leq n-2$. Hence, we have

$$
M'_{G, N} + |N|m_{N, N} = - \sum_{X \leq N} |X|\mu(X, N) + \mu(X, G)
$$

$$
= - \sum_{Y \leq G, Y \neq N} \sum_{X \leq N} |X|\mu(X, Y) + |N|\mu(N, Y) + \sum_{1 \leq X \leq N} |X|\mu(X, N).
$$

By the Lemma 3.4, we have $\mu(N, Y) = 0$ when $N \not\sqsubseteq Y$. And since

$$
\sum_{1 \leq X \leq N} |X|\mu(X, N) = |N|m_{N, N},
$$

we have

$$
\sum_{Y \leq G} \sum_{X \leq N} |X|\mu(X, Y) = \sum_{Y \leq G, Y \neq N} \sum_{X \leq N} |X|\mu(X, Y) + |N|m_{N, N}.
$$

Hence,

$$
M'_{G, N} + |N|m_{N, N} = - \sum_{X \leq N} |X|\mu(X, N) + \mu(X, G)
$$

$$
= - \sum_{Y \leq G, Y \neq N} \sum_{X \leq N} |X|\mu(X, Y)
$$

$$
= - (\sum_{Y \leq G} \sum_{X \leq N} |X|\mu(X, Y) - |N|m_{N, N}).
$$

So we have

$$
M'_{G, N} = - \sum_{Y \leq G} \sum_{X \leq N} |X|\mu(X, Y).
$$
Also \( \mu(X, Y) = 0 \) if \( X \not\subseteq Y \), thus we have

\[
M'_{G,N} = - \sum_{Y \leq G \atop X \leq N} |X| \mu(X, Y) \\
= - \sum_{Y \leq G \atop X \leq N \cap Y} |X| \mu(X, Y).
\]

Proposition 4.2. Let \( G \) be a finite group and \( N \trianglelefteq G \) such that \( |G : N| = p \) for some prime number \( p \). Then

\[
M'_{G,N} = - \sum_{C \leq N, C \text{ is cyclic}} \varphi(|C|) - \sum_{Y \leq G \atop Y \not\subseteq N} M'_{Y,Y \cap N}.
\]

Proof. By the Proposition 4.1, we have

\[
M'_{G,N} = - \sum_{Y \leq G \atop X \leq N \cap Y} |X| \mu(X, Y)
\]

We will compute \( \sum_{X \leq N \cap Y} |X| \mu(X, Y) \) by considering the cases when \( Y \leq N \) and \( Y \not\subseteq N \) in the following.

Case 1. \( Y \leq N \). We have

\[
\sum_{X \leq N \cap Y} |X| \mu(X, Y) = \sum_{X \leq Y} |X| \mu(X, Y) = |Y|m_{Y,Y}.
\]

If \( Y \) is not cyclic, we have \( m_{Y,Y} = 0 \). If \( Y \) is cyclic, we have \( m_{Y,Y} = \varphi(|Y|) / |Y| \). Hence, we have

\[
\sum_{X \leq N \cap Y} |X| \mu(X, Y) = \begin{cases} 
\varphi(|Y|), & \text{if } Y \text{ is cyclic;} \\
0, & \text{if } Y \text{ is not cyclic.}
\end{cases}
\]

Case 2. \( Y \not\subseteq N \). That is \( YN = G \), it implies \( |Y : Y \cap N| = |G : N| = p \). So we have

\[
\sum_{X \leq N \cap Y} |X| \mu(X, Y) = M'_{Y,Y \cap N}
\]

by the Definition of \( M'_{Y,Y \cap N} \).
Hence,
\[ M'_{G,N} = - \sum_{Y \leq G, Y \leq N, Y \text{ cyclic}} \left( \sum_{X \leq N \cap Y} |X| \mu(X,Y) \right) \]
\[ - \sum_{Y \leq G, Y \leq N, Y \text{ not cyclic}} \left( \sum_{X \leq N \cap Y} |X| \mu(X,Y) \right) \]
\[ - \sum_{Y \leq G, Y \notin N} \sum_{X \leq N \cap Y} |X| \mu(X,Y) \]
\[ = - \sum_{Y \leq G, Y \leq N, Y \text{ cyclic}} \varphi(|Y|) \]
\[ - \sum_{Y \leq G, Y \leq N, Y \text{ not cyclic}} 0 \]
\[ - \sum_{Y \leq G, Y \notin N} M'_{Y,Y \cap N} \]
\[ = - \sum_{C \leq N, C \text{ is cyclic}} \varphi(|C|) - \sum_{Y \leq G, Y \notin N} M'_{Y,Y \cap N}. \]

\[ \square \]

**Remark 4.3.** (1) When \( Y \notin N \), we have \( Y \cap N = G \) because \( |G : N| = p \) for some prime number \( p \). It implies that \( |Y : Y \cap N| = |Y/(Y \cap N)| = |Y \cap N| = |G/N| = p \), we can repeat the operations on \( M'_{Y,Y \cap N} \) as the Proposition 4.1.

(2) To compute \( M'_{G,N} \), we need compute \( M'_{Y,Y \cap N} \) for every \( Y \subseteq G \), thus we can get \( M'_{G,N} \) by finite steps.

**Remark 4.4.** It is easy to see that
\[ \sum_{C \leq N, C \text{ is cyclic}} \varphi(|C|) = |N|. \]

5. A class poset of subgroups of \( G \)

To compute \( M'_{Y,Y \cap N} \) for every \( Y \leq G \), we define a new class poset of subgroups of \( G \) in this section. And we find the relation between \( M'_{G,N} \) and this class poset.

**Definition 5.1.** Let \( G \) be a finite group and \( N \leq G \). Let \( C \) be a cyclic subgroup of \( N \), define
\[ \Sigma_{C}(G) := \{ X | C \leq X \leq G, X \notin N \}. \]
We can see that \( \Sigma_{C}(G) \) is a poset ordered by inclusion. We can consider poset \( \Sigma_{C}(G) \) as a category with one morphism \( Y \to Z \) if \( Y \) is a subgroup of \( Z \). We set \( N(\Sigma_{C}(G)) \) the nerve of the category \( \Sigma_{C}(G) \) and \( |N(\Sigma_{C}(G))| \) is the geometric realization of \( N(\Sigma_{C}(G)) \). More detail of topology can be seen in [10].

**Remark 5.2.** Let \( A, B, D \in \Sigma_{C}(G) \), if \( A \leq B, A \leq D \), then \( B \cap D \in \Sigma_{C}(G) \).

Since we use the Euler characteristic of \( |N(\Sigma_{C}(G))| \) in Proposition 5.4, thus we recall the definition of the Euler characteristic as following:

**Definition 5.3.** [15 §22] The Euler characteristic (or Euler number) of a finite complex \( K \) is defined, classically, by the equation
\[ \chi(K) = \sum_{i} (-1)^{i} \text{rank}(C_{i}(K)). \]
Here, we also have some prime number $p$.

**Proof.** Let $Y \leq G$ and $Y \not\leq N$, since $YN = G$, we have $Y/(Y \cap N) \cong G/N$. So by the Proposition 4.2 and the Remark 4.3, we also have

$$M^I_{G,N} = - \sum_{C \leq N, C \text{ is cyclic}} \varphi(|C|) - \sum_{Y \leq G, Y \not\leq N} M^I_{Y/Y \cap N}$$

$$= - \sum_{C \leq N, C \text{ is cyclic}} \left(1 - \chi(|N(\mathcal{C}_C(G))|) \cdot \varphi(|C|) \right).$$

Here, $|N(\mathcal{C}_C(G))|$ is a simplicial complex associated to the poset $\mathcal{C}_C(G)$, and $\chi(|N(\mathcal{C}_C(G))|)$ is the Euler characteristic of the space $|N(\mathcal{C}_C(G))|$. 

**Proposition 5.4.** Let $G$ be a finite group and $N \not\leq G$ such that $|G:N| = p$ for some prime number $p$. Then

$$M^I_{Y,Y \cap N} = \sum_{Y \leq G, Y \not\leq N} \left(\sum_{C \leq Y \cap N, C \text{ is cyclic}} \varphi(|C|) - \sum_{Y \leq Y \cap N} M^I_{Y,Y \cap N}\right)$$

$$= - \sum_{Y \leq G, Y \not\leq N} \left(\sum_{C \leq Y \cap N, C \text{ is cyclic}} \varphi(|C|) \right)$$

$$- \sum_{Y \leq G, Y \not\leq N} \sum_{Y \leq Y \cap N} M^I_{Y,Y \cap N}$$

$$= - \sum_{Y \leq G, Y \not\leq N} \left(\sum_{C \leq Y \cap N, C \text{ is cyclic}} \varphi(|C|) \right)$$

$$- \sum_{Y \leq G, Y \not\leq N} \sum_{Y \leq Y \cap N} (-1)^1 \cdot \varphi(|C|)$$

$$= \sum_{C \leq N, C \text{ is cyclic}} \chi(|N(\mathcal{C}_C(G))|) \cdot \varphi(|C|)$$

Said differently, $\chi(K)$ is the alternating sum of the number of simplices of $K$ in each dimension.
Here, $\sigma$ is a 1-simplex of nerve $N(\Sigma_C(G))$ and $\sigma$ is not degenerate.

**Theorem 5.5.** The Euler characteristic of a contractible space is 1.

**Proof.** Since the Euler characteristic is a topological invariant and the contractible space is homotopy-equivalent to a point, thus we have the Euler characteristic of a contractible space is 1.

**Proposition 5.6.** Let $G$ be a finite group and $G$ be not cyclic. Let $N \leq G$ such that $|G : N| = p$ for some prime number $p$. If the space $|N(\Sigma_C(G))|$ is contractible for each cyclic subgroup $C$ of $N$, then $m_{G,N} = 0$.

**Proof.** By the Proposition 4.4, we have

$$M_{G,N} = - \sum_{C \leq N, C \text{ is cyclic}} \varphi(|C|) - \sum_{Y \leq G, Y \nsubseteq N} M_{Y \cap Y,N}.$$  

Since for each cyclic subgroup $C$ of $N$, we have $|N(\Sigma_C(G))|$ is contractible. It implies $\chi(|N(\Sigma_C(G))|) = 1$ by the Theorem 5.5, thus $M_{G,N} = 0$.

By the definition of $M_{G,N}^r$, we know that

$$m_{G,N} + \frac{1}{|G|} M_{G,N}^r = m_{G,G} = 0.$$  

So $m_{G,N} = 0$.

Now, we post the following problems about $S_n$.

**Conjecture 5.7.** Let $C$ be a cyclic subgroup of $A_n$, then the space $|N(\Sigma_C(S_n))|$ is contractible for $n \geq 5$.

**Remark 5.8.** If the Conjecture 5.7 holds, then we have $m_{S_n, A_n} = 0$. That is $\beta(S_n) = S_n$ is not solvable for $n \geq 5$. It implies that the Conjectures 1.1-2 hold when $G$ is $S_n$ for $n \geq 5$.

6. Connected simplicial complex of a poset

To compute $\chi(|N(\Sigma_C(G))|)$, we prove the following theorem. Here, the background of the topology, we refer to [13, 15].

**Proposition 6.1.** Set $C = \Sigma_C(G)$. Let $N(C)$ be the nerve of category $C$. If the space $|N(C)|$ is connected, then $H_0(|N(C)|) \cong Z$ and $H_i(|N(C)|) = 0$ for $i \geq 1$.

**Proof.** Let $X \in \text{Ob}(C)$, set $C_X$ be the full subcategory of $C$ with object set $\{Y \in \text{Ob}(C) | Y \leq X\}$.

Let $M_1, M_2, \ldots, M_n$ be all maximal elements of the poset $C$.

**Case 1.** $n = 2$. We can see that

$$|N(C)| \cong |N(C_{M_1})| \cup |N(C_{M_1 \cap M_2})| \cup |N(C_{M_2})|.$$  

Here, $M_1 \cap M_2 \in \text{Ob}(C)$ because $|N(C)|$ is connected.

We can see $|N(C_{M_1})|, |N(C_{M_1 \cap M_2})|, |N(C_{M_2})|$ are all contractible because each category has a terminal object. Hence by [15] Theorem 25.1, we have $H_0(|N(C)|) \cong Z$ and $H_i(|N(C)|) = 0$ for $i \geq 1$. 

Remark 5.2). Connected, we can delete one vertex of $D$ if and only if there exists $X \in \text{Ob}(C)$ such that $X \leq M_i$ and $X \leq M_j$ (It implies $M_i \cap M_j \in \text{Ob}(C)$ by the Remark 5.2).

Since $|N(C)|$ is connected, thus the graph $D$ is connected. Since the graph $D$ is connected, we can delete one vertex of $D$ and remaining part of the graph $D$ is also connected. Here, we can set that this deleted vertex is $M_i$.

First, set $C_{M_i}'$ be the full subcategory of $C$ with object’s set $\{X \in \text{Ob}(C)|X \leq M_i$ for some $2 \leq i \leq n\}$. And set $\hat{C}_{M_i}$ be the full subcategory of $C$ with object’s set $\{X \in \text{Ob}(C)|X \leq M_i \} \cap \{X \in \text{Ob}(C)|X \leq M_i\}$. Then we have

$$|N(C)| = |N(C_{M_i})| \cap |N(\hat{C}_{M_i})|.$$  

We can see $|N(C_{M_i}')|$ is connected, it implies $H_0(|N(C_{M_i}')|) \cong \mathbb{Z}$ and $H_i(|N(C_{M_i}')|) = 0$ for $i \geq 1$ by induction on $n$.

Now, we want to prove that $|N(\hat{C}_{M_i})|$ is also connected. By the definition of $\hat{C}_{M_i}$, each maximal object of $\hat{C}_{M_i}$ is in the set $\{M_1 \cap M_2, M_1 \cap M_3, \ldots, M_1 \cap M_n\}$.

Since $|N(C_{M_i}')|$ is connected, thus we have for each $M_1 \cap M_i, M_1 \cap M_j, 2 \leq i \neq j \leq n$, there exist

$$K_1, K_2, \ldots, K_s \in \{M_1 \cap M_2, M_1 \cap M_3, \ldots, M_1 \cap M_n\},$$

$$T_1, T_2, \ldots, T_{s+1} \in \text{Ob}(C_{M_i}')$$

such that

$$K_1 = M \cap M_1, K_s = M \cap M_1,$$

either $T_r \leq K_r, T_r \leq K_{r+1}$,

or $K_r \leq T_r, K_{r+1} \leq T_r$

for each $1 \leq r \leq s$. If $T_r \leq K_r, K_{r+1}$, that is $T_r \in \text{Ob}(\hat{C}_{M_i})$. If $K_r, K_{r+1} \leq T_r$, thus $K_r \leq M_1 \cap T_r$, it implies $M_1 \cap T_r \in \text{Ob}(C)$. Thus $M_1 \cap T_r \in \text{Ob}(\hat{C}_{M_i})$ and $K_r, K_{r+1} \leq M_1 \cap T_r$. That is $|N(\hat{C}_{M_i})|$ is connected. It implies $H_0(|N(\hat{C}_{M_i})|) \cong \mathbb{Z}$ and $H_i(|N(\hat{C}_{M_i})|) = 0$ for $i \geq 1$ by induction on $n$. Since $|N(C)| \cong |N(C_{M_i})| \cap |N(\hat{C}_{M_i})|\cong |N(C_{M_i}')|$, thus we have $H_0(|N(C)|) \cong \mathbb{Z}$ and $H_i(|N(C)|) = 0$ for $i \geq 1$. □

Remark 6.2. By the above theorem, if the space $|N(C)|$ is connected, then $\chi(|N(C)|) = 1$.

7. To compute $m_{S_n, A_n}$

First, we list some results about symmetric group. Let $S_n$ be a symmetric group of degree $n$ and $A_n$ be an alternating group of degree $n$.

Theorem 7.1. [14][2] Appendix|Let $S_n$ act on a set $\Omega$ of size $n$. Then every maximal subgroup $G(\neq A_n)$ of $S_n$, is of one of the types (a)-(f) below:

(a) $G = S_m \times S_k$, with $n = m + k$ and $m \neq k$ (intransitive case);

(b) $G = S_m \wr S_k$, with $n = m^k$, $m \geq 1$ and $k \geq 1$ (imprimitive case);

(c) $G = \text{AGL}_k(p)$, with $n = pk$ and $p$ prime (affine case);

(d) $G = T^k \cdot (\text{Out} T \times S_k)$, with $T$ a nonabelian simple group, $k \geq 2$ and $n = |T|^{k-1}$ (diagonal case);
\[(e) \ G = S_m \wr S_k, \text{ with } n = m^k, \ m \geq 5 \text{ and } k \geq 1, \text{ excluding the case where } X = A_n, \text{ and } G \text{ is imprimitive on } \Omega \text{ (wreath case-see Remark 2 below);}\]

\[(f) \ T \trianglelefteq G \leq \text{Aut}T, \text{ with } T \text{ a nonabelian simple group}, \ T \neq A_n, \text{ and } G \text{ acting primitively on } \Omega \text{ (almost simple case).}\]

Now, we consider the symmetric group $S_5$ and prove the following

**Theorem 7.2.** Let $C$ be a cyclic subgroup of $A_5$, then the space $|\Sigma_C(S_5)|$ is connected.

**Proof.** Since $C$ is a cyclic subgroup of $A_5$, thus $C$ likes one of the following types:

- (1) $C = 1$;
- (2) $C = \langle (123) \rangle$;
- (3) $C = \langle (12345) \rangle$;
- (4) $C = \langle (12)(34) \rangle$.

**Case 1.** $C = 1$. By the Theorem 7.1, we can see that the maximal subgroup($\neq A_5$) of $S_5$ is the following types:

\[S_m \times S_k, \ m + k = 5;\]
\[AGL_1(5) \cap S_5.\]

If $M_1, M_2(\neq A_5)$ are maximal subgroups of $S_5$ and $M_1, M_2$ are type $S_m \times S_k, \ m + k = 5$, then $M_1 \cap M_2$ contains a subgroup which is isomorphic to $S_2$. And we can see one of type $AGL_1(5) \cap S_5$ as

\[\langle (12345), (2354) \rangle.\]

Let $M_3 \cong S_4$ and act on the set $\{2, 3, 4, 5\}$, then $M_3$ is a maximal subgroup of $S_5$ and $M_3 \cap \langle (12345), (2354) \rangle \geq \langle (2354) \rangle \in \Sigma_C(S_5)$. So $|\Sigma_C(S_5)|$ is connected.

**Case 2.** $C = \langle (123) \rangle$. By the Theorem 7.1, we can see that the maximal subgroup($\neq A_5$) of $S_5$ is the following type:

\[S_m \times S_k, \ m + k = 5.\]

If $M_1, M_2(\neq A_5)$ are maximal subgroups of $S_5$ and $M_1, M_2$ contain $C = \langle (123) \rangle$, then $M_1 \cap M_2$ contains a subgroup which is isomorphic to $S_3$ which acts on the set $\{1, 2, 3\}$. So $|\Sigma_C(S_5)|$ is connected.

**Case 3.** $C = \langle (12345) \rangle$. By the Theorem 7.1, we can see that the maximal subgroup($\neq A_5$) of $S_5$ is the following type:

\[AGL_1(5) \cap S_5.\]

We can see this type subgroup which contains $C$ is only

\[\langle (12345), (2354) \rangle.\]

So $|\Sigma_C(S_5)|$ is connected.

**Case 4.** $C = \langle (12)(34) \rangle$. By the Theorem 7.1, we can see that the maximal subgroup($\neq A_5$) of $S_5$ is the following types:

\[S_m \times S_k, \ m + k = 5;\]
\[AGL_1(5) \cap S_5.\]
If $M_1, M_2(\neq A_5)$ are maximal subgroups of $S_5$ and $M_1, M_2$ are type $S_m \times S_k$, $m + k = 5$, then $M_1 \cap M_2$ contains a subgroup which is isomorphic to $S_2$. And we can see one of type $AGL_1(5) \cap S_5$ as
\[
\langle (25341), (1324) \rangle.
\]
Let $M_3 \cong S_4$ and act on the set $\{1, 2, 3, 4\}$, then $M_3$ is a maximal subgroup of $S_5$ and $M_3 \cap \langle (25341), (1324) \rangle \cong \langle (1324) \rangle \in \mathcal{T}_{C}(S_5)$. So $|\mathcal{T}_{C}(S_5)|$ is connected. □

**Proposition 7.3.** Let $G$ be the symmetric group $S_5$, and $N$ be the alternating group $A_5$ with $N \triangleleft G$. Then $m_{G,N} = 0$.

**Proof.** By the Theorem 7.2, the Theorem 6.1 and the Lemma 6.2, we know that $|\mathcal{T}_{C}(G)| = 1$ for each cyclic subgroup of $N \cong A_5$. Hence by the Proposition 5.4, we have $M'_{G,N} = 0$. That is $m_{G,N} = 0$. □

**Remark 7.4.** In [6], Bouc had proved that $S_n$ is a $B$-group. And by using [3], we also get that $S_n$ is a $B$-group.

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