RECODING LIE ALGEBRAIC SUBSHIFTS

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Abstract. We study internal Lie algebras in the category of subshifts on a fixed group – or Lie algebraic subshifts for short. We show that if the acting group is virtually polycyclic and the underlying vector space has dense homoclinic points, such subshifts can be recoded to have a cellwise Lie bracket. On the other hand there exist Lie algebraic subshifts (on any finitely-generated non-torsion group) with cellwise vector space operations whose bracket cannot be recoded to be cellwise. We also show that one-dimensional full vector shifts with cellwise vector space operations can support infinitely many compatible Lie brackets even up to automorphisms of the underlying vector shift, and we state the classification problem of such brackets.

From attempts to generalize these results to other acting groups, the following questions arise: Does every f.g. group admit a linear cellular automaton of infinite order? Which groups admit abelian group shifts whose homoclinic group is not generated by finitely many orbits? For the first question, we show that the Grigorchuk group admits such a CA, and for the second we show that the lamplighter group admits such group shifts.

1. Introduction. If a subshift has a group structure defined by shift-commuting continuous operations, then we can recode it so that the group operations become cellwise [11]. In other words, up to topological conjugacy every group subshift is a subgroup \( X \) of a full shift \( A^\mathbb{Z} \) where \( A \) itself is a group and the group operation is given by \((x \cdot y)_i = x_i \cdot y_i \) for all \( x, y \in A^\mathbb{Z} \) and \( i \in \mathbb{Z} \). We gave another proof of this recodability in [19] based on a property of the variety (a class of algebraic structures defined by identities; see [5]) of groups: this variety is shallow. This means that if the variety is defined by function symbols \( f_i \) of arity \( n_i \) for \( i \in I \), then every composition of the functions \( \xi \mapsto f_i(x_1, \ldots, x_{k-1}, \xi, x_{k+1}, \ldots, x_n) \) with \( 1 \leq k \leq n_i \) and \( x_j \in X \) is equivalent to such a composition of bounded length. For groups these functions are exactly multiplication by fixed group elements from the left and right, as well as inversion, and their compositions have the form \( \xi \mapsto x \cdot \xi^\pm 1 \cdot y \) for \( x, y \in X \).

For any variety \( \mathcal{V} \) of algebras and any category \( \mathcal{C} \) with finite products and a terminal object, one can define an internal \( \mathcal{V} \)-algebra over \( \mathcal{C} \) as an object together

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with algebra operations realized as morphisms [12, Section III.6]. A natural category for one-dimensional symbolic dynamics is the one whose objects are subshifts and whose morphisms are the continuous shift-commuting functions between them. This setting was investigated in [20]. We study the aforementioned property of cellwiseability, meaning the existence of a recoding of the subshift that sends the algebra operations into ones that are defined cellwise. Cellwising a subshift makes it much easier to deal with, as we can study algebras with cellwise operations through the finite quotient algebras obtained by restricting to words of finite length, without words getting shorter as operations are applied. The actual algebra is then the inverse limit of the algebras on finite words.

We can more generally do this in the category of \(G\)-subshifts and continuous shift-commuting functions for a finitely-generated group \(G\). For many varieties \(V\) that one encounters in everyday life, the following are equivalent:

- for all f.g. groups \(G\), all internal \(V\)-algebras over \(G\)-subshifts can be cellwised,
- all internal \(V\)-algebras over \(\mathbb{Z}\)-subshifts can be cellwised,
- the variety \(V\) is \(k\)-shallow for some \(k \in \mathbb{N}\).

In [19] we proved this equivalence for groups, monoids, semigroups, rings, vector spaces over a finite field, heaps, Boolean algebras and distributive lattices (which are all shallow), and also for quasigroups, loops\(^1\) and lattices (which are not shallow). We are in fact not aware of any varieties where the equivalence fails. One interesting open case is that of modular lattices: we do not know whether modular lattice subshifts can be recoded to be cellwise. It is known that this variety is not \(k\)-shallow for any \(k\), and in [19] we only give an uncellwiseable non-modular lattice example.

In this paper, we study the internal Lie algebras, i.e. subshifts that are in the internal variety of Lie algebras over a fixed field \(K\), which we call Lie algebraic subshifts (over \(K\)). More concretely, they are subshifts with a Lie algebra structure given by shift-commuting continuous operations. Our motivation for studying them were the equivalence results mentioned in the previous paragraph. The variety of Lie algebras is not shallow, either by the characterization of free Lie algebras, or by Corollary 16 below. Initial computer experiments suggested that Lie algebraic subshifts on small alphabets tend to be cellwiseable, which made Lie algebras a potential example of a variety that does not satisfy the equivalences. Another motivation to study this class is of course the importance of Lie algebras in mathematics and the fact they are given by additional structure on top of vector spaces, and vector space structure over a finite field is already known to play well with subshift structure (indeed vector shifts are a widely studied class of subshifts).

As a simple example, let \(K\) be a finite field and consider the finite Lie algebra \(\mathfrak{gl}_n(K)\) of \(n \times n\) matrices over \(K\), with standard matrix addition and commutator bracket \([A,B] = AB - BA\). The full shift \((\mathfrak{gl}_n(K))^G\) is a cellwise Lie algebraic subshift for any group \(G\). For another example, let \(V = \mathbb{K}^3\) be a three-dimensional vector space over \(K\) and consider the \(K\)-vector space \(V^\mathbb{Z}\) defined by cellwise operations. We can define a Lie bracket on it by \([x, y]_i = (0, 0, a)\) for all \(i \in \mathbb{Z}\), where \(a = (x_i)_0(y_i)_1 + (x_{i+1})_0(y_{i+1})_1\). The resulting Lie algebraic subshift is not cellwise, but our results show that it can be recoded into one. We return to this subshift

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\(^1\)This can be obtained from Example 4 in [19] by taking the disjoint union with a singleton subshift that acts as an identity element.
in Example 1, where we prove that it is indeed a Lie algebra over $K$ and study its isomorphism class.

We prove that the variety of Lie algebras satisfies the equivalence above, i.e. there exist Lie algebraic $\mathbb{Z}$-subshifts whose Lie bracket cannot be recoded to be cellwise. The underlying vector spaces of these examples do not have dense homoclinic points, and we can produce them on any group $G$ that admits a linear cellular automaton of infinite order. We do not know whether all groups admit such cellular automata, but any group containing an element of infinite order trivially admits such a cellular automaton, and we also provide an example for the Grigorchuk group.

**Theorem.** Let $K$ be a finite field, $d \geq 1$ and $G$ a group such that $(K^d)^G$ admits a linear cellular automaton of infinite order. Then $G$ admits a non-cellwiseable Lie algebraic subshift.

As our main positive result we show that all Lie algebraic subshifts where homoclinic points are dense and well-behaved can be recoded to have a cellwise bracket.

**Theorem.** Let $G$ be a finitely-generated group and $X \subset A^G$ a Lie algebraic subshift over a finite field $K$. If the set of homoclinic points of $X$ is generated, as a vector space, by finitely many $G$-orbits, and is dense in $X$, then $X$ is cellwiseable.

This theorem shows that Lie algebras are a “near miss” for the nonequivalence of shallowness and cellwiseability: the variety is not shallow, but at least nice enough Lie algebraic subshifts are cellwiseable. In this result, a non-trivial recoding is required even in the case of a one-dimensional full shift, i.e. after the vector operations have been cellwised, there can still be infinitely many distinct brackets compatible with the vector shift structure. We briefly study this issue in Section 3.2 and state the classification problem for brackets compatible with a particular vector shift structure.

If $G$ is virtually polycyclic, then the homoclinic points of $X$ are generated by finitely many orbits (essentially due to classical results of Hall), so the condition on their density is sufficient. We show that not all groups have this property: indeed on the lamplighter group (the non-virtually polycyclic metabelian group $\mathbb{Z}_2 \wr \mathbb{Z}$) we give an example of a vector shift whose homoclinic points are not finitely orbit-generated.

In Proposition 4 we show that if $G$ is amenable and $X$ satisfies a suitable gluing property (mean TMP), then the assumption that homoclinic points are dense is equivalent to having a trivial topological Pinsker factor. In our non-cellwiseable examples, the closure of the homoclinic points is a subgroup of finite index in $X$, and the quotient is isomorphic to the additive group of the underlying field $K$.

2. Preliminaries.

2.1. **Symbolic dynamics on groups.** Let $G$ be a group, usually a finitely generated infinite one, and $A$ a finite alphabet. The **full $G$-shift over** $A$ is the set $A^G$, whose elements are **configurations**, equipped with the product topology. It is a compact metric space where $G$ acts from the left by shifting: $(gx)_h = x_{g^{-1}h}$ for $x \in A^G$, $g, h \in G$. A $G$-invariant closed subset $X \subset A^G$ is a **$G$-subshift**. If $Y \subset B^G$ is another $G$-subshift, a **block code** is a $G$-equivariant continuous map $f : X \rightarrow Y$. Each block code is defined by a finite neighborhood $N \subset G$ and a local function $F : A^N \rightarrow B$ by $f(x)_g = F((g^{-1}x)|_N)$, and the **radius** of $f$ is the smallest $r \geq 0$ such that the $r$-ball $B_r(1_G) \subset G$ can be a neighborhood of $f$. If $X = Y$, then $f$ is a **cellular automaton** (CA) on $X$. A bijective block code is a **topological conjugacy**,
and if one exists, we say $X$ and $Y$ are conjugate. A finite product $\prod_{i=1}^{n} A_i^G$ of full shifts is naturally isomorphic to the full shift $\prod_{i=1}^{n} A_i$ over the product alphabet, and we frequently identify them in order to talk about $n$-ary block codes of the form $f : X^n \to X$ for $X \subset A^G$. If $n = 0$, we see $f$ as a $G$-invariant element of $X$.

A continuous function $f : X \to X$ on a subshift $X$ has bounded radius if there exists a finite set $N \subset G$ such that for all $g \in G$ and $x \in X$, $f(x)_g$ only depends on $g$ and $(g^{-1}x)|_N$. Note that such an $f$ will generally not commute with the action of $G$. The minimal $r \geq 0$ such that we can choose $N = B_r(1_G)$ (ball of radius $r$) is called the radius of $f$.

2.2. Universal algebra. A type of algebras is a family $\mathcal{T}$ of function symbols $(f_i)_{i \in I}$ of respective arities $(n_i)_{i \in I}$. A term of type $\mathcal{T}$ over a variable set $X$ is either an element $x \in X$ or an expression $f_i(t_1, \ldots, t_m)$ where $f_i$ is a function symbol of arity $n_i$ in $\mathcal{T}$ and each $t_j$ is a term. An algebra of type $\mathcal{T}$ (or a $\mathcal{T}$-algebra) is a set $A$ together with functions $F_i : A^{n_i} \to A$ for each function symbol $f_i$. We usually identify $F_i$ with $f_i$, and use the same symbol for both. A function between ($\mathcal{T}$-)algebras that intertwines the respective functions is a ($\mathcal{T}$-)homomorphism, and a bijective homomorphism is an isomorphism.

Remark 1. Often we say that a set (or a subshift) $X$ is a $\mathcal{T}$-algebra, without listing the operations. This typically does not lead to confusion, but when $X$ is implicitly carrying some structure, this convention makes it somewhat awkward to drop the structure. This only comes up in Example 1 where we consider a Lie algebraic subshift without its underlying vector space structure.

A variety $\mathcal{V}$ of algebras of type $\mathcal{T}$ is defined by a set of identities $(L_j \approx R_j)_{j \in J}$, where $L_j, R_j$ are terms over some abstract set of variables. An algebra belongs to $\mathcal{V}$ if it satisfies each of its identities for all choices of values for each variable. For example, the variety of abelian groups is defined by the function symbols $(+, -, 0)$ of arities 2, 1 and 0, as well as the identities $(x + y) + z \approx x + (y + z)$, $x + 0 \approx x$, $x + (0) \approx 0$ and $x + y \approx y + x$. A standard reference on this topic is [5].

We recall some definitions from [19]. If $\mathcal{V}$ is a variety as above, a $\mathcal{V}$-subshift over a group $G$ is a subshift $X \subset A^G$ equipped with block codes $f_i : X^{n_i} \to X$ that give $X$ a $\mathcal{V}$-algebra structure. This is equivalent to $(X, (f_i)_{i \in I})$ being an internal $\mathcal{V}$-algebra in the category of $G$-subshifts and block codes, as stated in the introduction. For example, group subshifts are subshifts with a group structure given by block codes, and they have been studied extensively in the literature [11, 21, 4]. We say $X$ is cellwise if each $f_i$ admits $\{1_G\}$ as a neighborhood. In this case $A$ is a finite $\mathcal{V}$-algebra and $X$ is a sub-algebra of the direct product $A^G$. If there exists a cellwise $\mathcal{V}$-algebra $Y \subset B^G$ and a topological conjugacy $\phi : X \to Y$ that is also a $\mathcal{V}$-isomorphism, we say $X$ is cellwiseable.

The affine maps on a $\mathcal{V}$-algebra $X$ are terms over the variable set $X \cup \{\xi\}$ defined inductively as follows:

- The term $\xi$ is affine. Its depth is 0.
- If $t$ is an affine map of depth $d$, $f_i : X^{n_i} \to X$ is an $n_i$-ary algebra operation, $0 \leq j < n_i$, and $y_k$ for $0 \leq k < n_i$, $k \neq j$, are elements of $X$, then $f_i(y_0, \ldots, y_{k-1}, t, y_{k+1}, \ldots, y_{n-1})$ is an affine map of depth $d + 1$.

Each affine map defines a function $X \to X$ by substituting $\xi$ with the function argument and evaluating the operations in $X$. For example, if $R$ is a ring and $r, s \in R$, then the term $t = r \cdot \xi + s$ is an affine map, and defines a function
a \mapsto ra + s$ on $R$. We often identify affine maps with the functions they define, when there is no danger of confusion. We say $X$ is $k$-shallow if each affine map on $X$ defines a function that is equivalent to the function of some affine map of depth at most $k$, and shallow if it is $k$-shallow for some $k \in \mathbb{N}$.

If $X$ is a $\mathcal{V}$-subshift, then each affine map on $X$ has bounded radius. The following result combines Theorem 2 and Theorem 3 in [19].

Lemma 2. Let $\mathcal{V}$ be a variety of algebras, $G$ be a finitely-generated group and $X \subset A^G$ a $\mathcal{V}$-subshift. Then the following are equivalent:

- $X$ is cellwiseable.
- There is a uniform bound on the radius of affine maps on $X$.

If $X$ is shallow, then it is cellwiseable.

2.3. Homoclinic points and entropy. Let $G$ be a group that acts on a compact metrizable group $X$ by continuous automorphisms. A point $x \in X$ is homoclinic if $gx \to 1_X$ as $g \in G$ escapes finite subsets of $G$, meaning that for each neighborhood $U$ of $1_X$ there is a co-finite $F \subset G$ with $Fx \subset U$. If $X \subset A^G$ is a group shift, this is equivalent to the support $\text{supp}(x) = \{g \in G \mid x_g \neq 1_A\}$ being finite. We denote by $\Delta_X$ the set of homoclinic points of $X$, and call $X/\Delta_X$ the co-homoclinic factor of $X$. Homoclinic points play an important role in our results, as they do in the general theory of group dynamical systems: our main results depend on the nontriviality of the co-homoclinic factor.

In the rest of this section, we explain the connection between homoclinic points and the topological Pinsker factor, for amenable groups. This discussion is included as background information, and is not used in the proofs of the main results. Recall that a countable group $G$ is amenable if it admits a left Følner sequence, which is a sequence $F_1, F_2, \ldots$ of finite subsets of $G$ with $\bigcup_n F_n = G$ and $|gF_n \triangle F_n|/|F_n| \to 0$ for each $g \in G$, where $\triangle$ denotes symmetric difference. We symmetrically define right Følner sequences. The entropy $h(X)$ of a topological $G$-dynamical system $X$ (a continuous action of $X$ on a compact metric space $(X, d)$) has several equivalent definitions (see [10, Section 9.9]), but in the case where $X \subset A^G$ is a $G$-subshift, it equals $\lim_{n \to \infty} |F_n|^{-1} \log \{|x|_{F_n} \mid x \in X\}$ for any left Følner sequence $F_n$.

The topological Pinsker factor of a dynamical system, as defined in [3] for $\mathbb{Z}$-actions, is the largest factor that has zero topological entropy. In the case of an action of an amenable group $G$ by automorphisms of a compact group $X$ it is exactly the factor group $X/\text{IE}(X)$, where $\text{IE}(X)$ is the closed and $G$-invariant $\text{IE}$-group of $X$. To define it, we recall some auxiliary notions from [10] (except they use IE-pairs instead of orbit IE-pairs; the notions are equivalent for actions of amenable groups [9, Theorem 4.8]). A subset $I \subset G$ is an independence set for a pair of sets $U, V \subset X$ if for all $\pi : I \to \{U, V\}$ the intersection $\bigcap_{g \in I} g^{-1} \pi(g)$ is nonempty. The independence density of $U$ and $V$ is the maximal $q \geq 0$ such that every finite $F \subset G$ contains an independence set of size at least $q|F|$ for $U$ and $V$. A pair $(x, y) \in X^2$ is an orbit IE-pair if $U$ and $V$ have positive independence density whenever $U$ is a neighborhood of $x$ and $V$ one of $y$. Finally, the subgroup $\text{IE}(X)$ consists of those $x \in X$ for which $(x, 1_X)$ is an orbit IE-pair.

The group ring $R[G]$ for a ring $R$ is the ring of finite-support configurations $x \in R^G$, where addition is defined cellwise and multiplication as convolution: $(xy)_k = \sum_{k=g_n} x_g y_h$. If the group ring $\mathbb{Z}[G]$ is left Noetherian, as it is for virtually polycyclic

\[2\text{In the statement of Theorem 2, affine maps are mistakenly referred to “block maps” although they are not } G\text{-equivariant. See Theorem 4.1.2 in [16] for a more careful statement.}\]
groups $G$, and $X$ is abelian (i.e. $G \curvearrowright X$ is an algebraic action), then the group $\Delta_X$ is dense in $\text{IE}(X)$, and hence $X/\Delta_X$ is the topological Pinsker factor. See [10, Section 13] and [6] for more information on the entropy theory of group actions by group automorphisms, and [4] for the special case of $\mathbb{Z}^d$-group shifts.

The characterization of the Pinsker factor as the co-homoclinic factor extends to all $G$-group shifts with an additional technical property, as we show below.

**Definition 3.** Let $G$ be a group, $X \subset A^G$ a subshift and $K, F, C \subset G$ finite with $1_G \in K$. Suppose that whenever $x, y \in X$ satisfy $x|_{C \setminus F} = y|_{C \setminus F}$, there exists $z \in X$ with $z|_C = x|_C$ and $z|_{(G \setminus F)K} = y|_{(G \setminus F)K}$. Then $C$ is a $K$-memory set for $F$.

We say $X$ has the weak topological Markov property (TMP) if every finite $F \subset G$ admits a \{1_G\}-memory set. We say $X$ has the mean TMP if for each finite $K$ there exists an increasing sequence of finite sets $F_n$ with $G = \bigcup_{n \geq 0} F_n$ that admit $K$-memory sets $C_n$ with $|C_n \setminus F_n|/|F_n| \to 0$ as $n \to \infty$.

Weak TMP was defined in [1], where it was also proved that all group shifts on countable groups satisfy it. Mean TMP was defined in the preprint [2] in a more general setting; the above definition is its special case. Shifts of finite type on amenable groups have the mean TMP. The following was proved in [2], but we present a proof for completeness.

**Proposition 4.** Let $G$ be an amenable group and $X \subset A^G$ a group shift. Then every zero-entropy factor of $X$ factors through $X/\Delta_X$. If $X$ has the mean TMP, then $X/\Delta_X$ has zero entropy.

In particular, in the latter case the co-homoclinic factor and the topological Pinsker factor coincide.

**Proof.** By Lemma 2 may assume $X$ is cellwise, i.e. $A$ is a group and operations of $X$ are cellwise operations of $A$. We first show $\Delta_X \subset \text{IE}(X)$, implying the first claim. Let $x \in \Delta_X$, and let $K, F \subset G$ be finite with $\text{supp}(x) \subset K$. There exists a subset $I \subset F$ with $|I| \geq |F|/|K|^2$ such that the sets $gK$ for $g \in I$ are pairwise disjoint. For any subset $J \subset I$, we have

$$\prod_{g \in J} g^{-1} x \in \bigcap_{g \in J} g^{-1}[x|_K] \cap \bigcap_{g \in I \setminus J} g^{-1}[1^K],$$

so the intersection is nonempty. Note that the multiplicands on the left hand side commute, so the order of the product is irrelevant. Thus $I$ is an independence set for $([x|_K], [1^K])$ and the independence density of the pair is at least $|K|^{-2} > 0$. Such pairs of sets form a neighborhood basis for $(x, 1_X)$, so $x \in \text{IE}(X)$.

Suppose then that $X$ has the mean TMP. We prove that $\Delta_X$ is dense in $\text{IE}(X)$, implying the second claim. Let $x \in \text{IE}(X)$ be arbitrary, and let $K \subset G$ be finite. Let $F_n$ and $C_n$ be given for $K$ by the mean TMP. Since $x \in \text{IE}(X)$, there exists $q > 0$ such that for all $n \geq 0$, we have an independence set $I_n \subset F_n$ for $([x|_K], [1^K])$ with $|I_n| \geq q|F_n|$. For each $\pi : I_n \to \{x|_K, 1^K\}$ there exists a configuration $y^\pi \in X$ with $(g^{-1}y^\pi)|_K = \pi(g)$ for each $g \in I_n$.

If $|x|_K \neq 1^K_A$, there are $2|I_n|$ choices for the map $\pi$. For large enough $n$ we have $|A|^{C_n \setminus F_n} < 2^{|I_n|} \leq 2^{|I_n|}$, so $y^\pi|_{C_n \setminus F_n} = y^\rho|_{C_n \setminus F_n}$ holds for some $\pi \neq \rho$. We may assume $\pi(g) = x|_K$ and $\rho(g) = 1^K_A$ for some $g \in I_n$. Using the mean TMP we find a configuration $z \in X$ with $z|_{(G \setminus F_n)K} = y^\pi|_{(G \setminus F_n)K}$ and $z|_{C_n K} = y^\rho|_{C_n K}$. Denote $y = g^{-1}(z^{-1}y^\pi)$. Then $y \in \Delta_X$, and since $gK \subset C_n K$ and multiplication in $X$ is defined cellwise, we have $y|_K = x|_K$. Since $K$ can be arbitrarily large, $x \in \Delta_X$. \qed
An example of Meyerovitch [15] shows that for subshifts without the mean TMP, over the infinite direct product $\oplus\mathbb{Z}_2$ one can have an abelian group shift $X$ with positive entropy and trivial $\Delta X$. For a finitely-generated example, note that as in Proposition 19, by acting independently on cosets, one obtains an abelian group shift with the same property on the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$.

2.4. Lie algebras. Fix a field $K$, which in this article will always be finite. The variety of Lie algebras over $K$ is the variety with a scalar multiplication operation $x \mapsto a \cdot x$ (which we usually write as just $ax$) for each $a \in K$, a nullary operation 0, a binary addition $(x, y) \mapsto x + y$, and a binary bracket operation $(x, y) \mapsto [x, y]$, such that scalar multiplication, 0 and addition satisfy the vector space axioms (when $K$ is finite, this is a finite list of axioms), and the bracket satisfies bilinearity

$$[ax + by, z] = a[x, z] + b[y, z], \quad [z, ax + by] = a[z, x] + b[z, y]$$

for all scalars $a, b \in K$, reflexivity $[x, x] = 0$ and the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$  

The first two properties imply anticommutativity $[x, y] = -[y, x]$.

Two vectors $u, v$ in a Lie algebra commute if $[u, v] = 0$. An ideal in a Lie algebra $A$ is a subalgebra $B \subset A$ satisfying $[B, A] \subset B$. A subspace of a Lie algebra $A$ is finite-dimensional if it is finite-dimensional as a vector space.

We write the ideal generated by any set $B \subset A$ as $[B, A]$, and the vector space generated by $B \subset A$ as $\langle B \rangle$. Inductively define $[a_0, ..., a_k] = [[a_0, ..., a_{k-1}], a_k]$ for $k \geq 3$. We refer to such expressions as simply brackets, and $k$ is the depth of the bracket. We call $a_0$ the base of the bracket.

**Lemma 5.** Suppose $A$ is a Lie algebra over a field $K$ generated by vectors $(e_i)_{i \in I}$. Let $B \subset A$ be arbitrary. Then

$$[B, A] = \{\langle [b, e_{i_1}, ..., e_{i_n}] \mid b \in B, \forall j : [b, e_{i_j}] \neq 0 \rangle\}.$$  

Note that directly by bilinearity of the bracket, $[B, A]$ is generated by such expressions where $e_{i_j}$ does not commute with $[b, e_{i_1}, ..., e_{i_{j-1}}]$. What requires the Jacobi identity is that none of the $e_{i_j}$ commute with the base $b$.

**Proof.** Let $a \in [B, A]$. By bilinearity and anticommutativity of the bracket and the definition of the ideal generated by $B$, we can write $a$ as a linear combination of brackets of the form $[b, e_{i_1}, ..., e_{i_k}]$ where $b \in B$. Thus, it is enough to make the brackets satisfy $[b, e_{i_j}] \neq 0$ for all $j$. Say a bracket is bad if $[b, e_{i_j}] = 0$ for some $j$ and call the minimal such $j$ the bad index.

Suppose that whenever $a$ is written as a linear combination of brackets of the form above, at least one bracket is bad. Consider then all possible ways to write $a$ as a linear combination of such brackets, and to each such expression associate the tuple $(d, \ell, p)$ where $d$ is the maximal depth among the bad brackets, $\ell$ is the number of brackets of maximal depth, and $p$ is the minimal bad index among bad brackets $[b, e_{i_1}, ..., e_{i_k}]$ achieving $k = d$.

Write $a$ as a linear combination of brackets so that $(d, \ell, p)$ is lexicographically minimal (in particular $d$ is globally minimal), and consider a bad bracket $[b, e_{i_1}, ..., e_{i_k}]$ with $[b, e_{i_p}] \neq 0$.

If $p = 1$, then $[b, e_{i_1}] = 0$ and by bilinearity we may remove the bracket entirely, making $\ell$ smaller but without increasing $d$. Suppose then $p \geq 2$. Then writing Jacobi’s identity as $[[x, y], z] = [x, [y, z]] + [[x, z], y]$,
setting \( x = [b, e_{i_1}, \ldots, e_{i_p-2}], \ y = e_{i_p-1} \), \( z = e_{i_p} \) and applying to the corresponding subbracket of \([b, e_{i_1}, \ldots, e_{i_p-2}, e_{i_p-1}, \ldots, e_{i_k}]\) gives
\[
[b, e_{i_1}, \ldots, e_{i_p-2}, e_{i_p-1}, e_{i_p}, \ldots, e_{i_k}] = [b, e_{i_1}, \ldots, e_{i_p-2}, [e_{i_p-1}, e_{i_p}], e_{i_p+1} \ldots e_{i_k}]
+ [b, e_{i_1}, \ldots, e_{i_p-2}, e_{i_p}, e_{i_p-1}, e_{i_p+1}, \ldots, e_{i_k}].
\]

The second term is clearly still bad, but its bad index is smaller. Thus if we replace the LHS by the RHS in the minimal expression for \( a \), we obtain a contradiction since \( d \) and \( \ell \) are not modified, but \( p \) is decreased.

While the technical statement of Lemma 5 is what we use in our proofs, we note the following slightly nicer proposition that contains the essence of it. Our first main result, Theorem 7 in the next section, could be proved using the proposition instead of the lemma in the special case \( G = \mathbb{Z} \).

**Proposition 6.** Suppose \( A \) is a Lie algebra over a field \( K \) generated by vectors \((e_i)_{i \in I}\) such that every \( e_i \) commutes with all but finitely many \( e_j \). Then every finitely-generated ideal of \( A \) is finite-dimensional.

**Proof.** Under the assumptions of the proposition, for any fixed \( a \in A \), the brackets \([a',a]\) for \( a' \in A \) generate a finite-dimensional space. Namely, for \( a = e_i \), writing \( a' \) in terms of generators and applying bilinearity, the fact that all but finitely many generators commute with \( e_i \) implies that we see only finitely many generators in the simplified expression. For general \( a \), write \( a \) in terms of generators and again apply bilinearity.

To prove the proposition, it is enough to show that \( B = \{e_i \mid i \in F\} \) generates a finite-dimensional ideal for any finite \( F \subset I \). The previous lemma gives
\[
[B, A] = \langle \{[b, e_{i_1}, \ldots, e_{i_k}] \mid b \in B, \forall j : [b, e_j] \neq 0\} \rangle,
\]
so by the assumption \( [B, A] \) is generated as a vector space by brackets involving only finitely many generators \( e_i \). In particular the top element \( e_{i_k} \) can take at most finitely many values, and we conclude by the previous paragraph. \( \square \)

3. Lie algebraic subshifts.

3.1. **Cellwiseability.** Let \( X \subset A^G \) be a shift-invariant subset of a group shift. We say \( X \) is orbit-generated by a set \( Y \) if it is generated (in the algebraic sense, as a discrete group) by the \( G \)-orbits of elements of \( Y \). We denote this by \( X = \langle GY \rangle \). We say \( X \) is finitely orbit-generated if it is orbit-generated by a finite set.

**Theorem 7.** Let \( G \) be a finitely-generated group and \( X \subset A^G \) a Lie algebraic \( G \)-subshift over a finite field \( K \). If \( X = \Delta_X \) and \( \Delta_X \) is finitely orbit-generated as a subgroup of \( (X,+) \), then the operations of \( X \) can be recoded to be cellwise.

**Proof.** We may assume the vector space operations (addition and scalar multiplication) of \( X \) are cellwise: vector spaces over \( K \) are shallow so Lemma 2 applies. So assume \( A \) is a finite-dimensional vector space over \( K \) and \( X \subset A^G \) has the vector space operations induced by those of \( A \). The Lie bracket of \( X \) may still have a larger neighborhood. An isomorphism of vector space \( G \)-subshifts (which is a topological conjugacy that commutes with vector space operations) preserves homoclinic points, \( G \)-orbits and subgroups of the additive group, so \( \Delta_X \) stays finitely orbit-generated. Hence there exist \( x_1, \ldots, x_k \in \Delta_X \) such that \( \Delta_X = \langle gx_i \mid g \in G, i \in [1,k] \rangle \).

We now claim that affine maps of \( X \) have bounded radius, which suffices by Lemma 2. First for affine maps in a Lie algebra we show the normal form \( \xi \mapsto \)
formulation states that the group ring $a$ of a finite group $G$ is a unary operation of the algebra). By induction, if $T$ is a term in this normal form, we need to show the same for an affine map with depth one higher. The most difficult case of the induction step is the bracket operation, and indeed

$$[T,y_{k+1}] = [a \cdot [\xi,y_1,\ldots,y_{k-1}] + y_k, y_{k+1}]$$

$$= a \cdot [[\xi,y_1,\ldots,y_{k-1}] + a^{-1} \cdot y_k, y_{k+1}]$$

$$= a \cdot [[\xi,y_1,\ldots,y_{k-1}], y_{k+1}] + [y_k, y_{k+1}].$$

is of the claimed form.

Suppose $x,y \in X$ agree in a large ball $B_r(1_G) \subset G$ around the origin with respect to the word metric $d$ of $G$. It is enough to show that if $r$ is large enough, then $t(x)_{1_G} = t(y)_{1_G}$ for any affine map $t(\xi) = a \cdot [\xi,y_1,\ldots,y_{k-1}] + y_k$. Since the bracket is bilinear, we may assume $y = y_k = 0$.

First, consider $x \in \Delta X$. We have $\{(x)\}, \Delta_X = \{(x), X\} \subset \Delta_X$ by continuity and shift-commutation of the Lie bracket. It is also easy to see that $t(x)_{1_G} = 0$ holds for every choice of $k$ and $y_1,\ldots,y_k-1$ if and only if the ideal generated by $x$ contains an element that is nonzero at the origin of $G$. By Lemma 5, we have

$$\{(x), \Delta_X\} = \{\{x, g_m x_i, \ldots, g_m x_i\} \mid \forall j : [x, g_m x_i] \neq 0\}. \quad (1)$$

Let $F$ be the union of supports (group elements containing nonzero values) of the homoclinic generators $x_i$. If

$$d_{\min}(\text{supp}(x), \text{supp}(g_m x_i)) \geq d_{\min}(\text{supp}(x), g_m F) \geq 2R$$

then $[x, g_m x_i] = 0$, where $R$ is the radius of the bracket as a block map, and $d_{\min}(K, K') = \min_{k \in K, k' \in K'} d(k,k')$. Namely, the local rule of the bracket satisfies $[P, Q] = 0$ if at least one of $P, Q \in A^{B_R(1_G)}$ is the all-0 pattern. Then for each $j$ in (1), we have $d_{\min}(\text{supp}(x), \text{supp}(g_m x_i)) < 2R$, and since $\text{supp}(x) \cap B_r(G) = \emptyset$, this implies $\text{supp}(g_m x_i) \cap B_R(G) = \emptyset$ if $r$ is large enough. Then the local rule of the outermost bracket in $[x, g_m x_i, \ldots, g_m x_i]_{1_G}$ has the form $[P, 0]^{B_R(G)} = 0$ for some pattern $P$. Thus $z_{1_G} = 0$ for all $z \in \{(x), \Delta_X\}$.

For general $x \in X$ with $x|_{B_r(G)} = 0|_{B_r(G)}$, if some $z \in \{(x), X\}$ satisfies $z_{1_G} \neq 0$, then by continuity of the operations and $X = \Delta_X$ we also have $z_{1_G} \neq 0$ for some $z \in \{(x), \Delta_X\}$ and $x \in \Delta_X$.

The following is essentially a classical result, see e.g. [8, Theorem 1]. Its usual formulation states that the group ring $\mathbb{Z}[G]$ of a virtually polycyclic group $G$ is Noetherian, thus has only finitely-generated ideals. We sketch a version of the proof for subgroups of $A^G$ for a finite group $A$.

**Lemma 8.** Let $G$ be a virtually polycyclic group, $A$ a finite group and $Y \subset \Delta_A$ a shift-invariant subgroup of $A^G$. Then $Y$ is finitely orbit-generated. In particular, $\Delta_X$ is finitely orbit-generated for any group shift $X$ over $G$.

**Proof.** First, we may assume $G$ is strongly polycyclic, i.e. admits a subnormal series $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{1_G\}$ with infinite cyclic factors. This is because $G$ has such a finite-index subgroup and the properties of being a shift-invariant subgroup and being finitely orbit-generated are not affected by moving between $G$ and its finite-index subgroups. We proceed by induction on $n$, assuming that $H =: G_1$ has this property. Let $g \in G$ be such that $G = \bigcup_{n \in \mathbb{Z}} g^n H$. 

}\end{document}
Define $\Delta^+_A$ as the set of those $y \in \Delta_A$ with $\text{supp}(y) \in \bigcup_{n \geq 0} g^n H$. This is an $H$-invariant set since each $g^n H$ is. Let $Z = \{ y \in Y \cap \Delta^+_A \}$, which is an $H$-invariant subgroup of $\Delta_A$. By the inductive assumption there is a finite set $F \subset Y \cap \Delta^+_A$ such that $\{ y \in F \}$ orbit-generates $Z$. Let $k \geq 0$ be minimal such that $\text{supp}(y) \subset \bigcup_{i=0}^{k-1} g^i H$ for all $y \in F$ and denote $K = \{ 1_G, g, g^2, \ldots, g^{k-1} \}$.

Let $y \in Y$ be arbitrary and consider $\langle GF \rangle \subset X$, the subgroup generated by $G$-orbits of $F$. We claim that there exists $z \in \langle GF \rangle$ with $\text{supp}(zy) \subset g^n KH$ for some $n \in Z$. We may assume $y \in \Delta^+_A$, shifting by some $g^n$ for $n > 0$ if this is not the case, and proceed by induction on the minimal $p \geq 0$ with $\text{supp}(y) \subset \bigcup_{i=0}^{p-1} g^i H$. If $p \leq k$, then $\text{supp}(y) \subset KH$ and we are done. Otherwise, observe that $y|_H \in Z$, so that some $z \in \langle F \rangle$ satisfies $(zy)|_H = 1$. Then $g^{-1} \cdot zy \in \Delta^+_A$ has a smaller value of $p$, so there exists $x \in \langle GF \rangle$ with $\text{supp}(x(g^{-1} \cdot zy)) \subset g^n KH$ for some $n$, and hence $\text{supp}((g \cdot x)zy) \subset g^{n+1} KH$.

Let $Z' = \{ y \in Y \mid \text{supp}(y) \subset KH \}$. This is an $H$-invariant subgroup isomorphic to a subgroup of $\Delta_B H$ for some finite group $B$ with $|B| = |A|^k$, so it is orbit-generated by a finite set $F' \subset Z'$ in the sense that $Z' = \langle HF' \rangle$. For the $n$, $y$ and $z$ above, this implies $g^{-n}(zy) \in \langle HF' \rangle$, so $Y$ is orbit-generated by $F \cup F'$.

**Theorem 9.** Let $G$ be a virtually polycyclic group and suppose $X \subset A^G$ is a Lie algebraic subshift over a finite field $K$ in the category of $G$-subshifts. If the homoclinic points are dense, then the operations of $X$ can be recoded to be cellwiseable.

**Proof.** Direct from Lemma 8 and Theorem 7.

We do not know whether virtual polycyclic is needed in the theorem.

**Question 10.** Are Lie algebraic subshifts with dense homoclinic points cellwiseable, on every group?

### 3.2. Brackets compatible with a particular vector shift.

After cellwiseability of the bracket, an interesting question is to try to understand what the bracket can actually do. We show some preliminary results. We concentrate on full vector shifts $V^G$. When $V$ has dimension at least 2, there are always infinitely many compatible brackets:

**Proposition 11.** Let $V$ be a $d$-dimensional vector space over a finite field $K$, with $d \geq 2$. If $G$ is an infinite f.g. group then the full vector shift $V^G$ admits infinitely many compatible Lie brackets.

**Proof.** If $\langle \cdot, \cdot \rangle : V^2 \to V$ is any compatible Lie bracket, then $\langle x, y \rangle = f^{-1}(\langle f(x), f(y) \rangle)$ is a compatible Lie bracket for any linear automorphism $f$ of $V^G$ (i.e. automorphism of the subshift and the vector space structures), by a direct calculation. Identify $V$ with $K^d$ and then $V^G$ with $(K^G)^d$, and denote the standard basis of $V$ by $\vec{e}_1, \ldots, \vec{e}_d$. On $V$, define a bracket by $\langle \vec{e}_1, \vec{e}_2 \rangle = \vec{e}_1$ and extend by bilinearity and reflexivity and mapping everything else to zero. This defines a Lie bracket on $V$. Namely, by bilinearity and reflexivity it is enough to check the Jacobi identity on triples of distinct basis vectors. One of them must be $\vec{e}_i$ for some $i \geq 3$, which commutes with the others, so each term of the Jacobi sum equals 0. We extend the bracket to $V^G$ by cellwise operations.

Now, pick $f_g$ to be the partial shift

$$f_g(x_1, x_2, \ldots, x_d)_h = ((x_1)_{hg}, (x_2)_h, \ldots, (x_d)_h),$$
which is obviously an automorphism. Then it is easy to show that the brackets 
\[ [x, y] = f_g^{-1}([f_g(x), f_g(y)]) \] are distinct for each \( g \in G \).

The linear automorphism groups of \((K^d)^G\) can be quite complex once \( d \geq 2\). Already if \( K = \mathbb{Z}_2\) (the two-element field), \( G = \mathbb{Z}\) and \( d = 2\), the automorphism group of the vector shift \((K^d)^G\) is a finitely-generated group containing a two-generator free group and a copy of the lamplighter group \( \mathbb{Z}_2 \wr \mathbb{Z} \) [17, Sections 4.3, 4.4]. (If \( d = 1\), the linear automorphisms are easily seen to be shift maps.)

One may ask whether the proposition essentially produces all Lie algebra structures. We show that the construction above does not give all brackets, indeed there can exist infinitely many non-isomorphic Lie algebraic structures with the same underlying vector shifts:

**Example 1.** Consider the two-element finite field \( K = \{0_K, 1_K\}\), let \( V = K^3\) and identify \( X = (K^3)^\mathbb{Z}\) with \((K^2)^3\) so configurations of \( X\) are triples \((x, y, z)\) over \( K^2\). We refer to the elements \( x, y, \) and \( z \) as tracks of the triple. Write \( e_1 \) for \((\chi_{\{0\}}, 0_K^2, 0_K^2)\) where \( \chi_B \in K^2\) is the characteristic function of \( B\) and write \( e_2\) for the corresponding topological generator for the second track. For \( B \subset \mathbb{Z}\) finite, define a bracket by

\[ [e_1, e_2]_B = (0_K^2, 0_K^2, \chi_B). \]

Extend this using shift-commutation, continuity, reflexivity and bilinearity, mapping everything else to 0. Bilinearity and reflexivity are then obvious. The Jacobi identity trivially holds because the bracket satisfies \([X, X, X] \subset \{(0_K^2) \times \{0_K^2\} \times K^2, X \} = \{0_K\}\), hence \([a, b, c] = 0_K\) for all \( a, b, c \in X\).

Consider the brackets \([\cdot, \cdot]_i := [\cdot, \cdot]_{(0,i)}\) for \( i \in \mathbb{Z}\). It is easy to see that \([\cdot, \cdot]_i\) and \([\cdot, \cdot]_{-i}\) are conjugate by an automorphism of the vector shift structure of \( X\) (the partial shift by \( i \) in the third track). We claim the Lie algebraic subshifts \((X, [\cdot, \cdot]_i)\) and \((X, [\cdot, \cdot]_j)\) are not isomorphic when \( 0 \leq i < j\). In fact, even if we ignore the vector space structure, the algebras \((X, [\cdot, \cdot]_i, 0)\) where we consider only the algebra operation \([\cdot, \cdot]_i\) and the zero element \( 0 \in X\) as structure are not isomorphic as algebraic subshifts.

To see this, let \( p_{i,n}\) be the number of pairs \((x, y) \in X^2\) of \( n\)-periodic points (points with orbit size divisible by \( n\)) with \([x, y]_i = 0\), where \( i \geq 0\) and \( n \geq 1\). The function \( n \mapsto p_{i,n}\) is clearly an isomorphism invariant for \((X, [\cdot, \cdot]_i, 0)\). The number of words \( abcd \in \{0, 1\}^4\) such that \([ae_1 + be_2, ce_1 + de_2]\) contributes \( \chi_{(0,i)}\) on the third track is 6; the other choices contribute 0. The third tracks of the arguments are irrelevant, which produces a factor of \(4^n\). It is then easy to calculate

\[
\begin{align*}
p_{0,n} &= 24^n \\
p_{1,n} &= 24^n + 40^n \\
p_{2,2n} &= (24^n + 40^n)^2, \quad p_{2,2n+1} = 24^{2n+1} + 40^{2n+1} \\
p_{3,3n} &= (24^n + 40^n)^3, \quad p_{3,3n+1} = 24^{3n+1} + 40^{3n+1}, \quad p_{3,3n+2} = 24^{3n+2} + 40^{3n+2}
\end{align*}
\]

and in general

\[
p_{k,n} = (24^n + 40^n)^{n/m}\text{ where } m = n/\gcd(k, n).
\]

The functions \( n \mapsto p_{k,n}\) are pairwise distinct for distinct values of \( k\). Indeed, if \( p^h \mid k_1\) and \( p^h \not\mid k_2\) for a prime \( p\), let \( p^h \mid k_2\) with \( k\) maximal. Picking \( n = p^h\) we have \( m_1 = n/\gcd(k_1, n) = 1\), \( m_2 = n/\gcd(k_2, n) = p^{h-k}\) and then

\[
p_{k_1,n} = (24 + 40)^p^h = ((24 + 40)^{p^{h-k}})^p > (24^{p^{h-k}} + 40^{p^{h-k}})^p = p_{k_2,n}.
\]
Since \( n \mapsto p_{k,n} \) is an isomorphism invariant, the Lie algebraic subshifts \((X, [,]_i)\) and \((X, [,]_j)\) are not isomorphic if \( i \neq j \).

We leave open three classification problems, in increasing order of generality (and presumably difficulty).

**Conjecture 12.** Let \( X = K^Z \) be the full vector shift, where \( K \) is a finite field as a vector space over itself. Then \( X \) is not compatible with a nontrivial (i.e. noncommutative) Lie algebraic subshift structure.

**Problem 13.** Classify Lie algebraic subshift structures consistent with the cellwise vector shift structure of \( V^Z \), for \( V \) a finite-dimensional vector space over a finite field \( K \), up to automorphisms of \( V^Z \).

**Problem 14.** Let \( G \) be a f.g. group. Classify Lie algebraic subshift structures consistent with the cellwise vector shift structure of \( V^G \), for \( V \) a finite-dimensional vector space over a finite field \( K \), up to automorphisms of \( V^G \).

Note that the previous problem includes all vector shifts over all f.g. groups also in the general sense of internal vector spaces, since vector spaces are a shallow variety.

4. On the failure of cellwiseability.

4.1. **Nontrivial co-homoclinic factor.** If the co-homoclinic factor is nontrivial, Theorem 9 does not necessarily hold, even if \( G = \mathbb{Z} \). A linear cellular automaton on a vector shift is a cellular automaton (shift-invariant continuous self-map) which is linear.

**Theorem 15.** Let \( G \) be a group and suppose that for some \( d \geq 1 \) the full vector shift \((K^d)^G\) over a finite field \( K \) admits a linear cellular automaton of infinite order. Then \( G \) admits a Lie algebraic subshift with co-homoclinic factor isomorphic to the additive group of \( K \) that cannot be recoded to be cellwise.

**Proof.** Let \( X = (K^d)^G \times \{a^G \mid a \in K\} \), and observe that \( X \) is a vector shift with the cellwise vector space operations of \( K^{d+1} \). Observe that the co-homoclinic factor is indeed isomorphic to the additive group of \( K \), as \( \Delta X = (K^d)^G \times \{0^G\} \). Write \( e_g = \chi_{\{g\}} \). Let \( f : (K^d)^G \to (K^d)^G \) be a linear cellular automaton of infinite order. To simplify notation, identify 0 with \((0^d)^G\) on left sides of tuples, and \( a \in K \) with \( a^G \) on right sides.

Let \([((x,0),(0,a)] = (af(x),0)\) and extend this using shift-commutation, reflexivity and bilinearity, mapping everything else to 0. In other words, set

\[
[(x,a),(y,b)] = [(x,0),(y,b)] + [(0,a),(y,b)] = [(x,0),(y,0)] + [(x,0),(0,b)] + [(0,a),(y,0)] + [(0,a),(0,b)] = (bf(x),0) - (af(y),0).
\]

Not surprisingly, bilinearity and reflexivity hold. For three-element brackets we have

\[
[[x,a),(y,b),(z,c)] = [(bf(x) - af(y),0),(z,c)] = (bcf(x) - acf(y),0),
\]
so the Jacobi identity reduces to
\[[x, a], (y, b), (z, c)] + [(y, b), (z, c), (x, a)] + [(z, c), (x, a), (y, b)]
= (bcf(x) - acf(y), 0) + (acf(y) - abf, 0) + (abf(z) - bcf(x), 0) = (0, 0).

The affine maps \( \phi_i \) defined inductively by \( \phi_0(\xi) = \xi, \phi_{i+1}(\xi) = [\phi_i(\xi), (0, 1)] \) do not have bounded radius, in fact \( \phi_i((x, 0)) = (f^i(x), 0) \) and since \( f \) has infinite order as a cellular automaton, the radii of the maps \( f^i \) are not uniformly bounded. Lemma 2 implies that \( X \) is not cellwiseable.

\[\textbf{Corollary 16.} \text{ Suppose } G \text{ is a finitely-generated group that is not torsion, i.e. some } g \in G \text{ has infinite order. Then for any finite field } K \text{ there is a Lie algebraic G-subshift over } K \text{ which is not cellwiseable and has co-homoclinic factor } K.\]

\[\textbf{Proof.}\] The right shift \( f(x)_h = x_{hg} \) is a linear CA on \( K^G \) and has infinite order. \( \square \)

Note that if \( G \) is nonabelian, there are \( G \)-subshifts on which the right shift map is not a cellular automaton, even though it is one on full shifts.

\[\textbf{Example 2.} \text{ On } \mathbb{Z} \text{ consider } X = \{0, 1\}^2 \times \{0^2, 1^2\} \text{ with componentwise and cellwise vector operations induced from those of } \mathbb{Z}_2. \text{ Write } e_i = \chi_{\{i\}} \times \{0^2\} \text{ where } \chi_B \in \{0, 1\}^B \text{ is the characteristic function of } B, \text{ and write } a = a^2 \text{ for } a \in \{0, 1\}. \text{ Pick any finite-support configuration } y \in \{0, 1\}^2. \text{ Define } [e_0, (0, 1)] = y, \text{ extend by shift-commutation, reflexivity and bilinearity and set all other images to } (0, 0). \text{ Now } x_0 = e_0, x_{i+1} = [x_i, (0, 1)] \text{ has 0 on its second track for all } i, \text{ and on the first track it has the } i \text{th image of } e_0 \text{ in the linear cellular automaton defined by } f(e_0) = y. \text{ For all but two choices of } y, \text{ this orbit is infinite, and thus the Lie bracket cannot be recoded to be cellwise.} \]

4.2. Groups admitting infinite-order linear CA. We do not know which groups admit linear cellular automata of infinite order:

\[\textbf{Question 17.} \text{ Let } G \text{ be a } f.g. \text{ infinite group and } V \text{ a finite-dimensional vector space over a finite field } K. \text{ Does } V^G \text{ necessarily admit a linear cellular automaton of infinite order?} \]

If the answer to Question 17 is always positive, then every f.g. infinite group admits a non-cellwiseable Lie algebraic subshift. If \( G \) has a non-torsion element \( g \), then the right shift \( f_g \) defined by \( f_g(x)_h = x_{hg} \) is linear and of infinite order, so the question is only interesting for torsion groups. Of course, every f.g. group \( G \) admits an infinite f.g. group of linear cellular automata on \( V^G \) (these right shifts), but the bracket construction in the proof of Theorem 15 only extends directly to commutative linear cellular automata actions.

Linear cellular automata can be seen as elements of the group ring \( K[G] \). Recall Kaplansky’s conjectures on group rings of torsion-free groups \( G \) over a field \( K \): the unit conjecture states that \( K[G] \) has no units other than \( kg \) for \( k \in K^\times \) and \( g \in G \); the zero-divisor conjecture states that \( K[G] \) has no zero divisors other than 0; the idempotent conjecture states that \( K[G] \) has no idempotents other than 0 and 1. See [13] for an overview of these. As a variant of Kaplansky’s conjectures, one can conjecture that \( K[G] \) contains no element \( p \) with \( p^n = p^m \) for any \( 0 \leq n < m \). This implies the idempotent conjecture and is implied by the unit conjecture. Question 17 is, in some sense, the dual of this conjecture on torsion groups, since in the case \( V = K \) it asks whether the condition holds for all \( p \in K[G] \). We do not know the answer to Question 17 in general, but for the Grigorchuk group it is positive.
Example 3. Recall the first Grigorchuk group $G = \langle a, b, c, d \rangle$, whose elements are permutations of $\{0,1\}^\mathbb{N}$ defined by $a(xv) = (1-x)v$ and

$$
\begin{align*}
  b(0v) &= 0a(v), & c(0v) &= 0a(v), & d(0v) &= 0v, \\
  b(1v) &= 1c(v), & c(1v) &= 1d(v), & d(1v) &= 1b(v)
\end{align*}
$$

for all $x \in \{0,1\}$ and $v \in \{0,1\}^\mathbb{N}$. It is the Grigorchuk group $G_\omega$ defined by the periodic sequence $\omega = (012)^\infty$. This family of groups was introduced in [7]. Let $K$ be a finite field. We show that $K[G]$ has an element of infinite order.

Take $p = ada + dad + c \in K[G]$ and denote $T = \{ada, dad, c\} \subset G$. We claim $p$ has infinite multiplicative order. For this, recall the form of the Schreier graphs of the Grigorchuk group for its natural action on $\{0,1\}^\mathbb{N}$, with the commonly used generators $a, b, c, d$. The Schreier graph $S$ defined by the action of $G$ on the orbit of $111\ldots$ is the following infinite 4-regular multigraph $S$ (see e.g. [14]):

We have a group action $V(S) \curvearrowright G$ of $G$ on the vertices of $S$ which just follows the edge with the respective label. Apart from the two leftmost ones, the vertices of $S$ can be partitioned into induced subgraphs that are isomorphic to one of the following, connected at their endpoints by double edges with labels $b$ and $c$:

$$
S_1 = \begin{array}{c}
  \text{d} \\
  a \\
  \text{d}
\end{array} \quad \quad S_2 = \begin{array}{c}
  \text{d} \\
  a \quad \text{b} \\
  \text{c} \quad \text{b} \\
  \text{d} \\
  \text{d}
\end{array} \quad \quad S_3 = \begin{array}{c}
  \text{d} \\
  a \quad \text{b} \\
  \text{c} \\
  \text{c} \quad \text{d} \\
  \text{d}
\end{array}
$$

Take a geodesic path $\gamma_v$ of some length $n_v$ along the generator set $T$ to some vertex $v$ that is the rightmost vertex of a copy of some $S_i$. We claim that $\gamma_v$ is the unique geodesic path to $v$ in $S$ along $T$. The path must begin with $dad, c$ in order to reach the leftmost copy of $S_2$. After this, the only way to cross from one $S_i$ to the next $S_j$ is to use $c$, the only way to cross an $S_1$ is $dad$, and the unique shortest way to cross an $S_2$ or $S_3$ is $ada$. Since $G$ acts on $S$, every geodesic path to the element $g_v \in G$ represented by $\gamma_v$ must form a geodesic path to $v$ in $S$ as well, hence $\gamma_v$ is also the unique geodesic path in $G$ to $g_v$ along $T$. The coefficient of $p^n$ at $g \in G$ is precisely the number of geodesics to $g$ in $G$ along $T$ modulo $\text{char}(K)$, hence for $g_v$ it equals $1_K$. Since we also have $n_v \neq n_w$ and thus $g_v \neq g_w$ for $v \neq w$, it follows that $p$ has infinite multiplicative order in $K[G]$.

A similar proof shows that $p = a + b + c + d$ has infinite multiplicative order whenever $q = \text{char}(K) \neq 2$: the number of geodesics to the vertex at distance $2n$ from the left end of $S$ is $2^n$, which is not divisible by $q$, so at least one of the respective elements of $G$ is reached by a number of geodesics that is also not divisible by $q$. The same strategy cannot be used when $q = 2$, but we conjecture that $p$ has infinite order in this case as well.

A similar analysis goes through for all the Grigorchuk groups $G_\omega$, in each case using one of the polynomials $p \in \{ada + dad + c, aca + cac + b, aba + bab + d\}$.

4.3. **Infinitely orbit-generated homoclinics.** If $G$ is f.g. and such that every group shift (or even just vector shift) on $X$ has finitely orbit-generated $\Delta_X$, then
all Lie algebraic subshifts on $G$ with trivial co-homoclinic factor are cellwiseable by Theorem 7. We do not know which groups $G$ have this property, but we can show that not all groups do.

**Lemma 18.** Let $H \leq G$ be groups, $K$ a finite field and $X \subset V^H$ a $K$-vector shift over $H$ such that $\Delta_X$ is not finitely orbit-generated. Then there exists a $K$-vector shift over $G$ with the same property.

**Proof.** Let $Y = \{y \in V^G \mid \forall g \in G : (gy)|_H \in X\}$, which is a $K$-vector shift. Suppose $\Delta_Y = \langle GZ \rangle$ for a finite $Z \subset \Delta_Y$, and let $Z' = \{(gz)|_H \mid g \in G, 1_G \in \text{supp}(gz)\}$. Then $Z' \subset \Delta_X$ is finite, and we claim $\Delta_X = \langle HZ' \rangle$, contradicting the assumptions. For any $x \in \Delta_X$ the configuration $y \in V^G$ defined by $y|_H = x$ and $y_g = 0$ for all $g \notin H$ is in $\Delta_Y$, so there are $g_1, \ldots, g_n \in G$ and $z_1, \ldots, z_n \in Z$ with $y = \sum_i g_i z_i$. Then $x = y|_H = \sum_i (g_i z_i)|_H$ and for each $i$, either $(g_i z_i)|_H = 0_Y$ or $(g_i z_i)|_H \in \langle HZ' \rangle$, hence $x \in \langle HZ' \rangle$. \hfill $\square$

**Proposition 19.** On $\oplus_{\mathbb{N}}\mathbb{Z}_2$ and $\mathbb{Z}_2 \wr \mathbb{Z}$ there exist vector shifts $X$ over the two-element field, such that $\Delta_X$ is not finitely orbit-generated.

**Proof.** By Lemma 18 it is enough to show that this is true for $H = \oplus_{\mathbb{N}}\mathbb{Z}_2$ (the direct sum of infinitely many copies of $\mathbb{Z}_2$), since it is a subgroup of $\mathbb{Z}_2 \wr \mathbb{Z}$. The left shift action of $H$ on the infinite direct product $Z_2^\mathbb{N}$ can be interpreted as follows: Identify $H$ with $\mathbb{N}$ by identifying elements of $H$ with binary expansions of numbers, so we can write elements of the direct product $Z_2^\mathbb{N}$ as infinite binary words. Then $H = \langle \{2^i \mid i \in \mathbb{N}\} \rangle$ and the group operation is bitwise XOR. The action of a generator $2^k$ on $Z_2^\mathbb{N}$ is to swap the contents of intervals

$$[2^{k+1} n, 2^{k+1} n + 2^k) \leftrightarrow [2^{k+1} n + 2^k, 2^{k+1} (n + 1))$$

where $k, n \geq 0$. For convenience, we define also a right shift map on $Z_2^\mathbb{N} = Z_2^\mathbb{Z}$ by $s(x)_i = x_{i-1}$ (with $x_{-i} = 0$ for all $i > 0$). Define the subgroups $H_n = \{0, 2^n\}$ for $n \geq 0$.

Define $m_i = 4^i$ and define a sequence of binary words inductively by $u_0 = 1$ and $u_n = 0^{m_{n-1}}0^{m_{n-1}}u_{n-1}u_{n-1}$ for $n \geq 1$. The word $u_i$ is of length $m_i$ and the word $v_i = u_i^4$ is of length $m_{i+1}$. We identify a finite word $v$ with the infinite word $v0^\mathbb{N} \in Z_2^\mathbb{N}$. Define now $X \subset Z_2^\mathbb{N}$ as the topological closure of the subgroup generated by the configurations $s^{km_{i+1}}(v_i)$ for $k \geq 0, i \geq 0$.

We now show the following claims:

- $X$ is a group shift over $H$, i.e. the translation of $H$ is well-defined on it,
- $X$ has no “new” homoclinics, i.e. $\Delta_X = \langle \{s^{km_{i+1}}(v_i) \mid i, n \in \mathbb{N}\} \rangle$, and
- $X$ is not finitely orbit-generated.

For the first claim, we show the stronger fact that for each $n \in \mathbb{N}$, the subgroup $X_n = \langle \{s^{km_{i+1}}(v_i) \mid i \leq n, k \in \mathbb{N}\} \rangle$ is closed under the action of $H$. For this we show by induction that every $H$-translate of $v_n$ with support contained in $[0, m_{n+1})$, in other words its every $H_{2n+2}$-translate, is generated by $v_n$ together with $H$-translates of $v_i$ for $i < n$. This is clearly true for $v_0 = 1111$ which has no such nontrivial translates.

Consider then

$$v_n = u_n^4 = (0^{m_{n-1}}0^{m_{n-1}}u_{n-1}u_{n-1})^4.$$  

Every $H_{2n+2}$-translate of $v_n$ is obtained by first permuting the four $u_n$-blocks by the Klein four-group $(2^{2n}, 2^{2n+1})$ (which fixes $v_n$, thus is useless), and then applying $H_{2n}$-translations which permute the four $u_n$-blocks separately. On the other hand
every $H_{2n}$-translate of the blocks $0^{m_n-1}0^{m_n-1}u_{n-1}u_{n-1}$ can be realized by adding vectors from $X_{n-1}$: adding $v_{n-1} = u_{n-1}u_{n-1}u_{n-1}u_{n-1}$ realizes the translation that swaps the largest intervals, and any $H_{2n-2}$-translation inside an $u_{n-1}$-block is obtained by adding vectors from $X_{n-2}$, by induction. This concludes the claim that $X$ is a group shift.

Observe that $\dim(X_{n-1}|[0,m_{n+1}]) = 4 \cdot \dim(X_{n-1}|[0,m_n])$ since $X_{n-1}$ is generated by translates of vectors with support contained in $[0,m_n)$. Next, we show an auxiliary claim that $\dim(X_n|[0,m_{n+1}]) = \dim(X_{n-1}|[0,m_{n+1}]) + 1$. The upper bound follows from the fact that $X_n|[0,m_{n+1}) = X_{n-1}|[0,m_{n+1}) + \langle v_n \rangle$. We prove the lower bound by induction on $n$. For $n=1$ this is clear since no $H$-translate of $v_0 = 1111$ contains a subword $0011$ aligned at a multiple of 4. Consider now $n > 1$, and suppose for a contradiction that $v_n \in X_{n-1}$. Since $v_n$ is composed of four $0^{m_n-1}0^{m_n-1}u_{n-1}u_{n-1}$-blocks, all elements of $\langle s^{km_n}(v_n-1) \mid k \geq 0 \rangle + v_n$ consist of blocks of the form $aabb$ with $a,b \in \{0^{m_n-1}, u_{n-1}\}$. Thus $u_{n-1} \in X_{n-2}$, and since $X_{n-2}$ is $H$-invariant, this implies $v_{n-2} \in X_{n-2}$, a contradiction. This concludes the claim.

In particular, $X_\infty = \bigcup_n X_n$ is not finitely-generated. If the closure $X = \overline{X_\infty}$ does not contain any new homoclinic points, i.e. $\Delta_X = X_\infty$, then the last two items follow. For this, we show that if $x \in X_\infty$ and $x|[0,m_{i+1}) \notin X_i$ then $x|[m_{i+1},m_{i+2})$ is nonzero. The result $\Delta_X = X_\infty$ follows from this, since if $x \in X \setminus X_i$ and $\mathrm{supp}(x) \subseteq [0,m_{i+1})$, then we can find an approximation $y \in X_\infty$ such that $y|[0,m_{i+2}) = x|[0,m_{i+2})$, and since $x|[m_{i+1},m_{i+2}) = 0^{m_{i+2}-m_{i+1}}$ we necessarily have $y \in X_i$, so also $x \in X_i \subseteq X_\infty$.

To show the claim, suppose that $x \in X_\infty$ and $x|[0,m_{i+1}) \notin X_i$. Observe that $X_{i+n|[0,m_{i+1}) = X_{i+1|[0,m_{i+1})}$ for all $n \geq 1$, and if $x|[0,m_{i+1}) \in X_{i+1} \setminus X_i$ then we have used a nontrivial shift of $v_{i+1}$ to produce $x$, and as seen above we cannot cancel any of the four $m_{i+1}$-blocks $u_{i+1}$ with vectors from $X_i$, so $x|[m_{i+1},m_{i+2})$ is nonzero as claimed.

Note that the example above only shows that the proof of Theorem 7 does not extend to $\mathbb{Z}_2 \wr \mathbb{Z}$. We do not know whether all Lie algebraic subshifts on $\mathbb{Z}_2 \wr \mathbb{Z}$ which have trivial co-homoclinic factor are cellwiseable.

One may wonder if for Proposition 19 it is sufficient that $\oplus_n \mathbb{Z}_2$ is infinitely-generated, i.e. $\mathbb{Z}_2 \wr \mathbb{Z}$ does not have the maximal condition on subgroups. In [18] it was shown that all such groups admit non-SFT group shifts. The above construction does not seem to directly adapt to all such groups. As a concrete example, we do not know whether $\mathbb{Z}[1/2] \wr \mathbb{Z}$ admits group shifts $X$ such that $\Delta_X$ is not finitely orbit-generated. Here, $\mathbb{Z}[1/2]$ is the dyadic rationals under addition and $\mathbb{Z}$ acts by multiplication by 2.

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