Conformal Blocks and Correlators in WZNW Model.
I. Genus Zero.

Kirill Saraikin

L.D.Landau Institute for Theoretical Physics, 117334, Moscow, Russia
and
Institute of Theoretical and Experimental Physics, 117259, Moscow, Russia

We consider the free field approach or bosonization technique for the Wess-Zumino-Novikov-Witten model with arbitrary Kač-Moody algebra on Riemann surface of genus zero. This subject was much studied previously, and the paper can be partially taken as a brief survey. The way to obtain well-known Schechtman-Varchenko solutions of the Knizhnik-Zamolodchikov equations as certain correlators in free chiral theory is revisited. This gives rise to simple description of space of the WZNW conformal blocks. The general $N$-point correlators of the model are constructed from the conformal blocks using non-chiral action for free fields perturbed by exactly marginal terms. The method involved generalizes the Dotsenko-Fateev prescription for minimal models. As a consequence of this construction we obtain new integral identities.

\[e-mail: saraikin@itp.ac.ru\]
1 Introduction

The Wess-Zumino-Novikov-Witten model \([1, 2, 3]\) has a long history. Its exceptional role is determined by the fact that this model can be in broad sense thought of as a generator of all the 2d conformal theories \([4]\). Since the seminal work by Knizhnik and Zamolodchikov \([3]\) much effort has been made to obtain exact solution of WZNW model \([3, 6, 7, 8, 9, 10, 11, 12]\). The most progress was achieved in solving the model by means of its representation in terms of the free fields (bosonization approach). Since the work by Wakimoto \([13]\) this approach has been developed by many groups \([3, 14, 15, 16, 17, 18]\).

In fact, it is possible to think of conformal theories as just of theories of free fields with additional constraints on the space of states. A good example of how this idea works is the well-known Dotsenko-Fateev free field representation for minimal models \([19]\). There exist a lot of serious arguments in favour of applicability of such a viewpoint to the WZNW model. For instance, in geometrical quantization approach the WZNW Lagrangian may be naturally considered as a \(d^{-1}\) of a Kirillov-Konstant form on the orbit of Kač-Moody coadjoint representation \([14]\). After choosing Gauss parameterization for the group element the action becomes diagonalized and quadratic in the corresponding fields \([9]\). Thus the free field representation of the WZNW model canonically arises. Unfortunately, since in general the transformation to the free variables is highly non-local, it is hard to describe it carefully. In fact, this way one can only conjecture expressions for the conformal blocks (see \([9]\) for more detailed discussion). It is worth mentioning that there exists an alternative approach due to Gawędzki et al. \([20]\) based on the correspondence between the Chern-Simons theory in the bulk that spans some surface and the WZNW model on this surface. This correspondence links the correlators in WZNW model and scalar product on the quantum states space of the CS theory. The last one can be calculated using the Iwasawa parameterization of the gauge field. After that the corresponding action also becomes Gaussian and explicit expressions for the scalar products can be obtained \([20]\).

However it seems that at present a brief and clear description of such a point of view on the WZNW model is absent in literature. This paper is a small step in this direction. Although the problem of primary interest is that of the WZNW model on higher genus, we start our investigation with the simplest genus zero case. This subject has been studied previously, and the paper can be partially taken as a survey. Our other purpose here is to prepare the necessary background for the second paper in the series, devoted to the higher genus case.

Our ideology of the WZNW model description is quite simple. The correlators and hence conformal blocks should satisfy a set of the operator product expansions (OPE), fixed in bootstrap approach \([3, 21]\). We realize these OPE in terms of free fields thus ensuring in correct local properties of our expressions. The correctness of the global properties follows then from the useful fact that “there are not so many good objects on Riemann surface” (a slang variant of the Riemann-Roch theorem). This rough statement turns to be remarkably confirmed: as the result, we obtain the well-known Schechtman-Varchenko solutions \([22]\) of the Knizhnik-Zamolodchikov equations.

To represent all the primary fields from the multiplet we introduce a generating function in a way which is quite similar to the exponential map from algebraic to the group elements. We also mention that one can use an alternative generating function from \([17, 18]\), which corresponds
to the algebra representation on the polynomial ring. However, to compare the answers with the solutions [22] one needs an exponential generating function.

Let us list some results:

1. For the $SU(2)_k$ WZNW model we claim that the $N$-point correlator of the spinless primary fields coincides with the $N$-point correlator of the “dressed” vertex operators in the theory with the action

$$S'_{\phi\beta\gamma} = \frac{1}{4\pi} \int d^2 z \left\{ \frac{1}{2} \partial \bar{\phi} \partial \phi - \beta \bar{\partial} \gamma - \bar{\beta} \partial \gamma + i \sqrt{\frac{2}{k+2}} R \phi + \beta \bar{\beta} \exp \left( -i \sqrt{\frac{2}{k+2}} \phi \right) \right\} \quad (1.1)$$

The last term in the integrand is exactly marginal and one should calculate correlators as a power series over this term. In fact, often after functional integration a single term survives in expansion.

The “dressed” vertex operator associated with the primary field from the highest weight $j$ representation is

$$\tilde{V}_j = e^{\gamma f_L} \cdot e^{i \sqrt{2} j \phi} \cdot e^{\Sigma f_R} \quad (1.2)$$

where $f_L$ and $f_R$ are the $su(2)$ step generators, which form the “left” and “right” representations. Thus, our claim is:

$$\langle \Phi_{\Delta_1} (z_1, \bar{z}_1) \ldots \Phi_{\Delta_N} (z_N, \bar{z}_N) \rangle_{WZNW} = \text{const} \langle \tilde{V}_{j_1} (z_1, \bar{z}_1) \ldots \tilde{V}_{j_N} (z_N, \bar{z}_N) \rangle_{S'_{\phi\beta\gamma}} \quad (1.3)$$

This construction may be easily generalized for other Lie groups. As a result one can calculate the $N$-point correlators in group $G$ WZNW model as correlators of “dressed” vertex operators in the theory of free fields perturbed by exactly marginal terms. The number of the fields necessary for bosonization is defined by the rank $G$ while the marginal terms correspond to the “squared modules” of simple screening currents.

2. As a result of the above suggestion we obtain expressions which satisfy all the requirements necessary for the WZNW correlators:

i) the holomorphic factorization property

ii) correct conformal properties; this means that the behavior of a correlator as a function of the primary fields insertion points under the change of coordinates on the surface is governed by the stress-tensor and corresponding conformal dimensions

iii) satisfy the differential (Knizhnik-Zamolodchikov) and additional algebraic equations, which reflects the null-vectors decomposition

iv) be a well-defined function of the primary fields insertion points; in other words, the monodromy of the correlator when one insertion point is moved around the others should be trivial.

3
3. We suggest a set of non-trivial integral identities, the simplest one is the following:

\[
|x|^{1/3} (1 - x)^{1/3} \int d^2t_1 |t_1|^{-2/3} |t_1 - x|^{-2/3} |t_1 - 1|^{-2/3} \times \\
\times \int d^2t_2 |t_1 - t_2|^{4/3} |t_2|^{-2/3} |t_2 - x|^{-2/3} |t_2 - 1|^{-2/3} \left\{ \frac{1}{t_1(t_1 - 1)} + \frac{1}{t_2(t_2 - 1)} \right\}^2 = \text{const} \\
\frac{\sqrt{x\bar{x}(1 - x)(1 - \bar{x})}}{\sqrt{x\bar{x}(1 - x)(1 - \bar{x})}}
\]

All such identities occur in some exceptional cases, when an alternative way to solve the WZNW model exists. For instance, (1.4) corresponds to the $SU(2)_{k=1}$ case, when due to special circumstances all correlators can be represented in terms of single scalar bosonic field with values on self-dual circle, see section 5. We have checked some of them numerically. These tests ensure us in correctness of the proposed construction.

The outline of this paper is as follows. In section 2 we fix notations and briefly review the basics of the bosonization technique.

In section 3 we describe how the Schechtman-Varchenko solutions of the Knizhnik-Zamolodchikov equations can be obtained as certain correlators in the free chiral theory. As a consequence, we obtain essential interpretation of the “resonance conditions” from [24] as a neutrality condition on vertex operators in the free field theory, which is necessary for obtaining a non-vanishing result.

After that, in section 4 we turn to the problem of “gluing” correlators from the conformal blocks. We use a simple prescription to obtain the conformal blocks: First, add to the free (non-chiral) action exactly marginal terms which are “squared modules” of screening currents integrated over whole surface in question. Second, take the power series over these terms. The method involved generalizes the Dotsenko-Fateev prescription for minimal models.

In section 5 we suggest a set of new integral identities. These identities arise when an alternative way to solve the WZNW model exists. The lhs and rhs are the correlators calculated in two different ways. They have equal conformal, algebraic and analytical properties. Moreover, they obey the same differential and algebraic equations. Thus at least at physical level we can conclude that they should be equal. This results in the non-trivial relations between (generalized) hypergeometric functions. We have checked some of them numerically performing a number of tests on proposed construction.

Finally, section 6 describes various open problems and directions for future research.

2 Basics of the Bosonization Technique

For most of the material presented in this section, see [9, 17] and references therein. We concentrate our attention on the $su(2)$ algebra. After careful analysis of this case the generalization for the arbitrary algebras is straightforward.
2.1 Notations for the finite dimensional algebra

Let \( g \) be a simple finite dimensional complex Lie algebra of dim \( g = d \) and rank \( g = r \). Let \( \langle , \rangle \) be an invariant scalar product (Killing form) on \( g \) normalized in such a way that \( \langle \theta, \theta \rangle = 2 \), \( \theta \) being the highest root. The set of positive roots is denoted \( \Delta_+ \). The simple roots are \( \{\alpha_i\}_{i=1, \ldots, r} \). For a vector space \( V \), \( V^\vee \) will always denote the dual space. The Cartan matrix is \( A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle \), the dual Coxeter number is \( h^\vee = \sum_{i=1}^r \alpha_i^\vee + 1 \). Commutator relations of the Chevalley generators \( e_i, h_i, f_i \) (where subscript \( i \) is for the \( \alpha_i \)) are:

\[
[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_j, \quad [h_i, e_j] = A_{ij} e_j, \quad [h_i, f_i] = -A_{ij} f_j
\] (2.1)

We will use the highest weight \( \vec{j} = \{j_1, \ldots, j_r\} \) representation of \( g \), the highest weight vector is denoted by \( |\vec{j}, \vec{0}\rangle \):

\[
|\vec{j}, \vec{m}\rangle = f_1^{m_1} \cdots f_r^{m_r} |\vec{j}, \vec{0}\rangle, \quad h_i |\vec{j}, \vec{0}\rangle = 2j_i |\vec{j}, \vec{0}\rangle, \quad e_i |\vec{j}, \vec{0}\rangle = 0. \quad (2.2)
\]

Relations (2.2) define a Verma module \( \mathcal{V}(\vec{j}) \) over \( g \).

For the algebra \( su(2) \) generators \( e, f, h \) we have explicitly:

\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \quad (2.3)
\]

The highest weight representation with spin \( j \in \mathbb{N}/2 \) is:

\[
|j, m\rangle = f^m |j, 0\rangle, \quad h |j, 0\rangle = 2j |j, 0\rangle, \quad e |j, 0\rangle = 0, \quad f^{2j+1} |j, 0\rangle = 0. \quad (2.4)
\]

Another useful algebra realization is given in terms of the differential operators acting on the polynomial ring \( \mathbb{C}[x^\alpha] \). The \( su(2) \) realization is:

\[
\begin{cases} 
  f = \partial_x \\
  h = 2x\partial_x - 2j \\
  e = -x^2\partial_x + 2jx
\end{cases} \quad (2.5)
\]

The polynomial highest weight representation follows from the identification:

\[
|j, m\rangle \leftrightarrow \frac{x^{2j-m}}{(2j-m)!}. \quad (2.6)
\]

2.2 Free fields

Let us recall some facts about free fields of use to bosonization procedure. The first is a scalar massless bosonic field \( \phi \) with values in the circle, described by the action:
\[ S_\phi = \frac{1}{4\pi} \int \frac{1}{2} \partial \phi \bar{\partial} \phi \ d^2 z. \]  
\hspace{1cm} (2.7)

With equations of motion in mind it is useful to introduce chiral parts of the \( \phi \) field:

\[ \phi(z, \bar{z}) = \phi_L(z) + \phi_R(\bar{z}) \]  
\hspace{1cm} (2.8)

Then we obtain the following OPE:

\[ \phi_L(z)\phi_L(w) = -\log(z - w) + O(1), \quad \phi_R(\bar{z})\phi_R(\bar{w}) = -\log(\bar{z} - \bar{w}) + O(1). \]  
\hspace{1cm} (2.9)

From (2.7) a useful formula for the vertex operators correlator follows\footnote{Where \( \alpha_i \) are just a complex numbers.}:

\[ \langle \prod_{i=1}^{N} : \exp(i\alpha_i \phi(z_i, \bar{z}_i)) : \rangle_{S_\phi} = \prod_{i < l} (z_i - z_l)^{\alpha_l \alpha_i} (\bar{z}_i - \bar{z}_l)^{\alpha_l \alpha_i} \delta_{\Sigma \alpha_i, 0}, \]  
\hspace{1cm} (2.10)

where the Cronecker’s symbol comes from the integration over the \( \phi \)-field zero mode. The holomorphic factorization in this expression is evident. The constraint \( \Sigma \alpha_i = 0 \) will be of great importance for us. It is conventionally referred to as a neutrality condition.

The action (2.7) can be deformed to

\[ S_R = \frac{1}{4\pi} \int \left( \frac{1}{2} \partial \phi \bar{\partial} \phi + i\alpha \phi \mathcal{R} \sqrt{g} \right) d^2 z, \]  
\hspace{1cm} (2.11)

where \( \mathcal{R} \) and \( g \) are the two-dimensional scalar curvature and metric determinant correspondingly. After choosing a special metric \( ds^2 = |\omega(z)|^2 \) on a sphere, where \( \omega(z) \) is a meromorphic 1-differential, the last term in the integrand becomes proportional to the \( \delta \)-function. It gives the nonzero contribution only in the singular point \( R \) of the \( \omega(z) \). Thus from such a point of view the term with the curvature results in the insertion of the vertex operator

\[ V_{\text{vac}}(R) = : \exp(i\alpha \phi(R)) : \]  
\hspace{1cm} (2.12)

to the point \( R \) of the surface. This operator is conventionally referred to as a "vacuum" charge \footnote{Where \( \alpha_i \) are just a complex numbers.}.

Note that the field \( \phi \) in modified action (2.11) should take values in the circle with the radius defined by identification \( \phi \sim \phi + 2\pi/\alpha \).

The second of the fields is a bosonic \( \beta\gamma \) systems with \( \beta \) of spin 1 and \( \gamma \) spin 0. There are chiral and anti-chiral versions of the corresponding action:
\[ S_{\beta\gamma} = \frac{1}{4\pi} \int \beta \bar{\partial} \gamma \ d^2z, \quad S_{\bar{\beta}\bar{\gamma}} = \frac{1}{4\pi} \int \bar{\beta} \partial \bar{\gamma} \ d^2z, \quad (2.13) \]

from which we read the following OPE

\[ \beta(z)\gamma(w) = \frac{1}{z-w} + O(1), \quad \bar{\beta}(\bar{z})\bar{\gamma}(\bar{w}) = \frac{1}{\bar{z}-\bar{w}} + O(1). \quad (2.14) \]

General N-point correlators in \( \beta\gamma \) system are calculated using the Wick’s theorem and Green’s functions of the \( \bar{\partial} \) and \( \partial \) operators corresponding to the singular parts in the rhs of (2.14)

\[ \langle \prod_{i=1}^{N} \beta(z_i) \prod_{l=1}^{M} \gamma(w_l) \rangle_{S_{\beta\gamma}} = \delta_{NM} \sum_{\text{perm}(\sigma)} \frac{1}{z_{\sigma_1} - w_{\sigma_1}} \ldots \frac{1}{z_{\sigma_N} - w_{\sigma_N}}, \quad (2.15) \]

and a similar one for the anti-chiral system. The neutrality condition now is \#\( \beta = \#\gamma \).

### 2.3 Free field realization of the Kač-Moody algebras

Let us concentrate on the holomorphic (chiral) objects. For the sake of brevity up to section 4 we will use the notation \( \phi \) for the chiral part \( \phi_L(z) \) of the \( \phi(z, \bar{z}) \). Kač-Moody algebra \( \hat{g} \) associated with the Lie algebra \( g \) can be described in terms of currents \( J^a(z) \) with OPE

\[ J^a(z)J^b(w) = \frac{k}{2(z-w)^2} \frac{q^{ab}}{z-w} J^c(w) + O(1), \quad (2.16) \]

where tensors \( f^{ab}_c \) and \( q^{ab} \) are structure constants of algebra \( g \) and invariant Killing form. For the case of \( g = su(2) \) they have components:

\[ q^{00} = \frac{1}{2} q^{+-} = 1, \quad f^{0+}_0 = -2, \quad f^{0-}_0 = -f^{+0}_0 = 1. \quad (2.17) \]

It is an easy exercise to check, that the currents

\[ J^+ = \beta \]

\[ J^0 = :\beta\gamma : - \frac{iq}{\sqrt{2}} \partial \phi \]

\[ J^- = - :\beta\gamma^2 : + i\sqrt{2} q \gamma \partial \phi + (q^2 - 2) \partial \gamma \]

form the \( su(2)_k \) algebra at the level \( k = q^2 \). Note that these currents are obtained from the differential operator representation (2.5) by the substitution

\[ \partial \rightarrow \beta, \quad x \rightarrow \gamma, \quad j \rightarrow \frac{iq}{\sqrt{2}} \partial \phi \quad (2.19) \]
and a subsequent renormalization by adding an anomalous term \((q^2 - 2)\partial \gamma\) to the last line of the (2.18).

Let the level \(k\) be an integer. As a next step we construct a finite-dimensional \(su(2)_k\) highest weight representation with spin \(j\):

\[
\begin{align*}
V_{j,0} &= \frac{s^{2j}}{(2j)!} : \exp \left( i \frac{\sqrt{2}}{q} \phi \right) :
\end{align*}
\]

\[
\begin{align*}
V_{j,1} &= \frac{s^{2j-1}}{(2j-1)!} : \exp \left( i \frac{\sqrt{2}}{q} \phi \right) :
\end{align*}
\]

\[
\begin{align*}
\cdots
\end{align*}
\]

\[
\begin{align*}
V_{j,2j} &= : \exp \left( i \frac{\sqrt{2}}{q} \phi \right) :
\end{align*}
\]

The first operator in this series is the highest weight vector of the representation, it has no singularity in the OPE with the "rising" current \(J^-\). Acting on \(V_{j,0}\) by the "lowering" current \(J^+\) one obtains all representation step by step. The series is truncated because of the vanishing factor at the singular term in the OPE \(J^+(z)V_{j,2j}(w)\). More explicitly,

\[
\begin{align*}
J^+(z)V_{j,m}(w) &= \frac{1}{z-w} V_{j,m-1}(w) + O(1) \\
J^0(z)V_{j,m}(w) &= \frac{i-m}{z-w} V_{j,m}(w) + O(1) \\
J^-(z)V_{j,m}(w) &= \frac{m(2j-m)}{z-w} V_{j,m+1}(w) + O(1)
\end{align*}
\]

The correspondence of (2.20) with the polynomial representation (2.6) is evident.

To study conformal properties of the vertex operators (2.20), we need a stress-tensor. It is provided by the Sugawara construction:

\[
T(z) = \frac{q_{ab}}{k + h^\vee} : J^a(z)J^b(z) :
\]

(2.22)

where \(h^\vee\) is a dual Coxeter number (quadratic Casimir operator in the adjoint representation), \(h^\vee(SU(N)) = N\), and \(q_{ab}\) is dual to the \(q^{ab}\). In the case of \(su(2)\) one obtains by direct calculation:

\[
T = : \beta \partial \gamma : - \frac{1}{2} : (\partial \phi)^2 : - \frac{i}{\sqrt{2q}} \partial^2 \phi.
\]

(2.23)

Note that this stress-tensor corresponds to the action:

\[
S_{\text{chiral}} = \frac{1}{4\pi} \int \left( \frac{1}{2} \partial \phi \partial \phi - \beta \partial \gamma + i \frac{\sqrt{2}}{q} \phi R \sqrt{g} \phi \right) d^2 z,
\]

(2.24)

where the term with the curvature induces the vacuum charge
\[ V_{\text{vac}}(R) = \exp \left( i \frac{\sqrt{2}}{q} \phi(R) \right) \quad (2.25) \]

From the OPE with the stress-tensor \((2.23)\), it follows that the conformal dimensions of all the operators from the representation \((2.20)\) are equal to
\[ \Delta_j = \frac{j(j + 1)}{k + 2}. \quad (2.26) \]

Besides the primary family \((2.20)\) there is one more operator of great importance to us — the so-called screening current of the conformal dimension \((1,0)\). The integrals of this current along the closed contours are the screening charges (Feigin-Fuchs operators \([23]\)). Conformal blocks are constructed as correlators of vertex operators \((2.20)\) with the appropriate number of the screening charges insertions \([9, 10]\). The crucial property of the screening charges is that they commute with the Kač-Moody currents and have zero conformal dimension. Thus insertion of such operators does not affect the conformal and algebraic properties of the correlator, but serves to “screen” out the extra charge to satisfy the neutrality condition. For the \(su(2)\) conformal blocks we will use the following screening charge \([9]\):
\[ \oint dt S(t) = \oint \beta(z) : \exp \left( -i \frac{\sqrt{2}}{q} \phi(z) \right) : dz \quad (2.27) \]

Let us briefly discuss the free field realization for the arbitrary Kač-Moody algebra \(\mathfrak{g}\). (For more details see \([17, 18]\) and references therein.) One starts with the differential operator realization of the associated Lie algebra \(\mathfrak{g}\) on the polynomial ring \(\mathbb{C}[x^\alpha]\), given by the following expressions for the Chevalley generators:
\[
\begin{align*}
e_\alpha(x, \partial) &= W_\alpha^\beta(x) \partial_\beta \\
h_i(x, \partial, j) &= W_i^\beta(x) \partial_\beta + \lambda_i \\
f_\alpha(x, \partial, j) &= W_\alpha^\beta(x) \partial_\beta + P_\alpha^i(x) \lambda_i
\end{align*}
\quad (2.28)
\]

Then one introduces a \(r\) copies of the free \(\phi, \beta, \gamma\) fields with OPE:
\[ \phi_i(z) \phi_j(w) = -\delta_{ij} \log(z - w), \quad \beta_i(z) \gamma_j(w) = \frac{\delta_{ij}}{z - w}. \quad (2.29) \]

After that the expressions for the corresponding Kač-Moody currents are obtained by the substitution:
\[ \partial_i \rightarrow \beta_i(z), \quad x_i \rightarrow \gamma_i(z), \quad \lambda_i \rightarrow iq \partial \phi_i(z) \quad (2.30) \]
and a subsequent renormalization by adding an anomalous term $F_{i}^{\text{anom}}(\gamma(z), \partial \gamma(z))$ to the lower part of (2.28). The primary fields are given in the terms of the following vertex operators:

$$V_{j,m}(z) = \prod_{i=1}^{m} \gamma_{i}^{2j_{i}-m_{i}}(z) \exp \left( i \sqrt{2} \frac{1}{q} \bar{j}_{\bar{i}} \bar{\phi}(z) \right) : .$$

The screening currents (of the first kind, in the terminology of [17, 18]) are:

$$S_{i}(z) =: W_{,a}^{\alpha} \gamma(z) \beta_{\delta}(z) \exp \left( -i \sqrt{2} \frac{1}{q} \alpha_{i} \bar{\phi}(z) \right) : .$$

## 3 Conformal Blocks

### 3.1 Knizhnik-Zamolodchikov and additional algebraic equations

It is well known, that the correlators in WZNW theory satisfy the system of differential equations first found by Knizhnik and Zamolodchikov:

$$\left( \kappa \frac{\partial}{\partial z_{i}} - \sum_{l \neq i} t_{l}^{a} t_{l}^{a} \right) \langle \Phi_{1}(z_{1}, \bar{z}_{1}) \ldots \Phi_{N}(z_{N}, \bar{z}_{N}) \rangle_{\text{WZNW}} = 0$$

where $t_{i}^{a}$ is a generator of algebra $g$ that acts on the primary field $\Phi_{i}$ belonging to the $i$th representation. The summation over the $a$ indices in (3.1) is assumed. The (complex) parameter $\kappa$ is equal to $k + h^{\vee}$. Due to the holomorphic factorization property of the correlators in conformal field theory [21] we have:

$$\langle \Phi_{1}(z_{1}, \bar{z}_{1}) \ldots \Phi_{N}(z_{N}, \bar{z}_{N}) \rangle_{\text{WZNW}} = \sum_{a,b} C_{a}^{ab} \mathcal{F}_{a}(z_{1}, \ldots, z_{N}) \mathcal{F}_{b}(\bar{z}_{1}, \ldots, \bar{z}_{N}).$$

where the gluing constants $C_{a}^{ab}$ are related to the structure constants of operator algebra. In fact, (3.1) is a system of differential equations on the so-called conformal blocks $\mathcal{F}_{a}(\bar{z})$. From the mathematical point of view, conformal block is a multivalued function (to be more precise, section of a holomorphic bundle over the moduli space of the principal $G$-bundles over the punctured $\mathbb{C}P^{1}$) of the $N$ variables $\bar{z} \equiv (z_{1}, \ldots, z_{N})$ with values in the tensor product of $N$ Verma modules $\mathcal{V}(\jmath_{1}) \otimes \ldots \otimes \mathcal{V}(\jmath_{N})$ over $g$. In notation from the subsection 2.1 the KZ equations looks like:

$$\kappa \frac{\partial}{\partial z_{i}} \mathcal{F}(\bar{z}) = \sum_{j \neq i} \frac{\Omega_{ij}}{z_{i} - z_{j}} \mathcal{F}(\bar{z}).$$

$^{3}$There is an antiholomorphic system of equation as well.
where
\[ \Omega_{ij} \equiv \sum_{l=1}^{r} \frac{1}{2} (h_l)_i \otimes (h_l)_j + \sum_{\alpha \in \Delta_+} ((e_\alpha)_i \otimes (f_\alpha)_j + (f_\alpha)_i \otimes (e_\alpha)_j). \tag{3.4} \]

and \((a_\alpha)\) stands for the element \(1 \otimes \ldots a_\alpha \ldots \otimes 1\) with \(a_\alpha\) in the \(i\)th place.

For the \(su(u)\) case the additional algebraic equation is:
\[
\left( \sum_{\subfrac{i=1}{k}}^{N} \frac{f_i}{z - z_i} \right)^{k-2j+1} \langle \Phi_j(z) \Phi_{j_1}(z_1) \ldots \Phi_{j_N}(z_N) \rangle_{WZW} = 0 \tag{3.5} \]

### 3.2 \(su(2)\) case

To represent all the primary fields from the multiplet, we will introduce a generating function that contains all vertex operators from the \((2.20)\). We are interested in generating function of a special form, having the following OPE with the Kač-Moody currents:
\[
J^a(z) \Phi(w) = \frac{t^a}{z - w} \Phi(w) + O(1). \tag{3.6} \]

This form of OPE is fixed in the bootstrap approach \([3]\). It is easy to check that the generating function\(^4\)
\[
\tilde{V}_j = \sum_{m=-\infty}^{2j} V_{j,m} f^{2j-m} \tag{3.7} \]

indeed has OPE \((3.6)\) with the \(su(2)_k\) currents \((2.18)\):
\[
\begin{cases} 
J^+(z) \tilde{V}_j(w) = \frac{t^a}{z - w} \tilde{V}_j(w) + O(1) \\
H(z) \tilde{V}_j(w) = \frac{b}{z - w} \tilde{V}_j(w) + O(1) \\
J^-(z) \tilde{V}_j(w) = \frac{e}{z - w} \tilde{V}_j(w) + O(1)
\end{cases} \tag{3.8} \]

The sum \((3.7)\) can be rewritten in a more suitable form
\[
\tilde{V}_j(z) \equiv \exp \left( i \frac{\sqrt{2}}{q} j \phi(z) \right) : \exp (\gamma(z)f), \tag{3.9} \]

\(^4\)The terms which contain \(f^n, n > 2j\) vanish when \(\tilde{V}_j\) acts on the vacuum vector \(|j, 0\rangle\), so they are not important.
to which we will refer as to a “dressed” vertex operator. Note that this form of the generating function is quite similar to the exponential mapping from algebra to a group.

Now we are ready to calculate the conformal blocks. The prescription is quite simple — we should take the holomorphic part of the “dressed” vertex operator $N$-point correlator with a proper number $n$ of the screening charges:

$$F_a(\vec{z}) = \oint_a dt_1 \ldots dt_n \left\langle S(t_1) \ldots S(t_n) \tilde{V}_{j_1}(z_1) \ldots \tilde{V}_{j_N}(z_N) \right\rangle_{\text{chiral}} \bar{v}. \quad (3.10)$$

where the average with the chiral action $(2.24)$ is assumed. The label $a$ marks different integration contours. The number $n$ in this formula is dictated by the neutrality condition:

$$n = \sum_{i=1}^{N} j_i + 1, \quad (3.11)$$

where the unity is due to the vacuum charge contribution. Functional integral over the $\phi$ field in $(3.10)$ gives the factor

$$\prod_{p<q} (t_p - t_q)^{\frac{2\mu}{\kappa}} \prod_{i<l} (z_i - z_l)^{\frac{\mu}{\kappa}} \frac{n}{N} \prod_{p=1}^{N} \prod_{l=1}^{N} (t_p - z_l)^{-\frac{\mu}{\kappa}}, \quad (3.12)$$

while the integral over the $\beta\gamma$ fields yields

$$\sum_{\text{perm}(\sigma)} \prod_{m_i=n} \frac{1}{m_1! \ldots m_N!} \langle \beta(t_1) \ldots \beta(t_n) \gamma^{m_1}(z_1) \ldots \gamma^{m_N}(z_N) \rangle f_1^{m_1} \ldots f_N^{m_N} = \quad (3.13)$$

where $\sum_{\text{perm}(\sigma)}$ denotes the sum over the permutation group of the numbers $\{\sigma(1), \ldots, \sigma(N)\}$ such that $\#i = m_i$ among them $(i = 1, \ldots, N)$. It is useful to introduce a notation $[25]$:

$$Y(\vec{z}, t) = \prod_{l=1}^{N} (t - z_l)^{-\frac{\mu}{\kappa}} \sum_{i=1}^{N} \frac{f_i}{t - z_i}. \quad (3.14)$$

Then, putting $(3.12)$ and $(3.13)$ together we obtain:

$$F_a(\vec{z}) = \prod_{i<l} \frac{(z_i - z_l)^{\frac{\mu}{\kappa}}}{z_i} \oint_a dt_1 \ldots dt_n \prod_{p<q} (t_p - t_q)^{\frac{2\mu}{\kappa}} Y(\vec{z}, t_1) \ldots Y(\vec{z}, t_n) \bar{v}. \quad (3.15)$$

One can check using the “brute force” method of $[22, 24]$ that $(3.13)$ indeed satisfies the KZ equation which for the $su(2)$ case takes the form:
κ \frac{\partial}{\partial z_i} F_a(\vec{z}) = \sum_{j \neq i}^{N} \frac{h_i h_j + e_i f_j + f_i e_j}{z_i - z_j} F_a(\vec{z}), \quad (3.16)

and the algebraic equation (3.5):

\left( \sum_{i=1}^{N} \frac{f_i}{z - z_i} \right)^{k-2j+1} F_a^{(j)}(z_0, \vec{z}) = 0, \quad (3.17)

where $F_a^{(j)}(z_0, \vec{z})$ denotes the conformal block with insertion of the vertex operator from spin $j$ representation in the point $z_0$.

**Polynomial representation**

Note that one can use an alternative generating function [17, 18] which corresponds to the algebra representation on the polynomial ring:

$$\tilde{V}_j(z) = (1 + x \gamma(z))^j : \exp \left( i \sqrt{\frac{2}{q}} j \phi(z) \right) :. \quad (3.18)$$

This generating function satisfies the OPE (3.8) where $f, e, h$ generators are given by (2.5). One can reduce the problem of the product $\prod (1 + x \gamma(z_i))^j$ correlator calculation to the problem of exponents $\exp(x \gamma(z_i))$ correlator calculation using the formula [17]:

$$(1 + x \gamma)^j = \frac{j!}{2\pi i} \oint_{u=0} du \frac{u^{-j-1} \exp[(1 + x \gamma)u]}{u} \quad (3.19)$$

which follows from the Cauchy theorem. After that one obtains the following expression for the conformal block (3.10) in polynomial representation:

$$F_a(\vec{x}; \vec{z}) = \left( \prod_{i \neq l}^{N} (z_i - z_l)^{-\frac{j_i}{2\pi i}} \right) \phi_a^{\left( \prod_{i=1}^{n} dt_i \right)} \left( \prod_{p < q}^{n} (t_p - t_q)^{\frac{2}{\pi}} \right) \left( \prod_{p=1}^{n} \prod_{l=1}^{N} (t_p - z_l)^{-\frac{j_l}{2\pi i}} \right) \times \quad (3.20)$$

$$\times \left( \prod_{r=1}^{N} \frac{j_r}{2\pi i} \oint_{u_r=0} du_r \frac{u_r^{-j_r-1} e^{u_r}}{u_r} \right) \prod_{p=1}^{n} \sum_{l=1}^{N} \frac{u_r x_i}{t_p - z_i}$$

However, to compare resulting conformal blocks with the solutions of the KZ equations, obtained in [22], one needs to use the first construction for the generation function.

### 3.3 Simple complex Lie algebras

As we already mentioned, generalization of this construction to an arbitrary simple complex Lie algebra $\mathfrak{g}$ is straightforward. One needs just to pass through three stages:
i) introduce the free chiral action:

\[ S_r = \frac{1}{4\pi} \int d^2 z \left( \frac{1}{2} \partial \vec{\phi} \cdot \bar{\partial} \vec{\phi} - \bar{\partial} \partial \vec{\gamma} + i \frac{R \sqrt{g}}{q} \vec{\rho} \vec{\phi} \right), \quad (3.21) \]

where \( \vec{\phi} = \{ \phi_1, \ldots, \phi_r \} \), \( r = \text{rank } g \), \( \bar{a} \bar{b} \equiv \sum_{i=1}^{r} a_i b_i \), \( q = \sqrt{k + h^\vee} \) and \( \vec{\rho} \) stands for the half-sum of all positive roots.

ii) introduce the generating function:

\[ \tilde{V}_{\vec{j}}(z) = \exp \left( \vec{\gamma}(z) \vec{f}_L \right) : \exp \left( i \frac{\sqrt{2}}{q} \vec{j} \vec{\phi}(z) \right) : \quad (3.22) \]

iii) introduce the screening currents

\[ S_i(z) =: W^{\delta}_{\alpha_i} (-\gamma(z)) \beta_{\delta}(z) \exp \left( -i \frac{\sqrt{2}}{q} \vec{\alpha}_i \vec{\phi}(z) \right) : \quad (3.23) \]

As a result for the algebra \( g \) conformal blocks one obtains:

\[ \mathcal{F}_a^{g}(\vec{z}) = \oint_a dt \left( \prod_{i=1}^{r} (S_i(t_i[1]) \ldots S_i(t_i[k_i])) \right) \tilde{V}_{\vec{j}_1}(z_1) \ldots \tilde{V}_{\vec{j}_N}(z_N) \right)_{S_r} \vec{v}, \quad (3.24) \]

for some \( \{ k_1, \ldots, k_r \} \) defined by a neutrality condition. After functional integration the expression which coincides with the Schechtman-Varchenko solution arises (see also [11, 26]).

4 Correlators

4.1 SU(2) case

Now we turn to the problem of constructing correlators \( \langle \Phi_1(z_1, \bar{z}_1) \ldots \Phi_N(z_N, \bar{z}_N) \rangle_{WZW} \).

In more detail the spinless primary field multiplet \( \Phi_{\Delta_j}(z, \bar{z}) \) corresponding to the conformal dimension \( \Delta_j = \frac{j(j+1)}{k+2} \) looks like

\[ \Phi_{\Delta_j}(z, \bar{z}) = \sum_{m, \bar{m}} c_m c_{\bar{m}} \Phi_{\Delta_j}^{m\bar{m}}(z, \bar{z}) f^m_L \otimes f^{\bar{m}}_R \quad (4.1) \]

with some constants \( c_m, c_{\bar{m}} \) defined in such a way that OPE (3.8) for \( \Phi_{\Delta_j}(z, \bar{z}) \) holds. Upper indices \( m \) and \( \bar{m} \) are for different primary fields from the “left” and “right” multiplet respectively.

To “glue” correlators from the conformal blocks according to the (3.2) we need to know the gluing constants \( C^{ab} \). They are determined by the natural physical condition on correlators to be single-valued functions of the primary fields complex coordinates \((z, \bar{z})\). In mathematical language, the monodromy with respect to a moving point in the correlator around the others
should be trivial. The most naive way to obtain an expression with such a property is simply
to get the “squared modules” of the conformal blocks and replace the integration over contours
with the integration over the whole surface. Provided that there is a single choice of \( C^{ab} \) (this
is natural from the physicist’s point of view) these prescription will give a correct answer. Thus
formally we can write:

\[
\left\langle \Phi_{\Delta_1} (z_1, \bar{z}_1) \ldots \Phi_{\Delta_N} (z_N, \bar{z}_N) \right\rangle_{WZW} \tilde{v} \otimes \bar{v} \sim \\
\sim \int \prod_{i=1}^n d^2 t_i \left| \left\langle S(t_1) \ldots S(t_n) \tilde{V}_{j_1}(z_1) \ldots \tilde{V}_{j_N}(z_N) \right\rangle \right|^2 \tilde{v} \otimes \bar{v}.
\] (4.2)

One can restore original conformal blocks and gluing constants using the following formula
which expresses the integral over the Riemann surface \( \Sigma \) in terms of the integrals over canonical
A, B circles on this surface:

\[
\int_\Sigma \omega \wedge \bar{\omega}' = \sum_{A,B} \left( \oint_A \omega \oint_B \bar{\omega}' - \oint_B \omega \oint_A \bar{\omega}' \right).
\] (4.3)

where \( \omega \) and \( \omega' \) are arbitrary holomorphic 1-differentials. The Riemann surface \( \Sigma \) in our case is
a branch covering of \( \mathbb{CP}^1 \) defined by the multivalued form (3.12) connected with the proper con-
formal block. Corresponding methods of calculation are in fact well known from the Dotsenko-
Fateev [19, 27] representation for minimal models where the same construction is used. As an
example of operator algebra structure constants calculation see [19, 28].

To be more precise we should substitute the following “dressed” vertex operator

\[
\tilde{V}_j(z, \bar{z}) = \exp (\gamma(z) f_L) : \exp \left( i \sqrt{2} q j \phi_L(z) \right) : : \exp \left( i \sqrt{2} q j \phi_R(\bar{z}) \right) : \exp (\bar{\gamma}(\bar{z}) f_R)
\] (4.4)

for the “squared modulus” of the \( \tilde{V}_j(z) \) from (4.2). Here \( f_L \) and \( f_R \) are the \( su(2) \) step generators
which form the “left” and “right” representations. We should also substitute expression

\[
\text{Scr}(t, \bar{t}) = \beta(t) \bar{\beta}(\bar{t}) \exp \left( -i \frac{\sqrt{2}}{q} \phi(t, \bar{t}) \right)
\] (4.5)

for the “squared modulus” of the screening currents. Therefore,

\[
\left\langle \Phi_{\Delta_1} (z_1, \bar{z}_1) \ldots \Phi_{\Delta_N} (z_N, \bar{z}_N) \right\rangle_{WZW} \tilde{v} \otimes \bar{v} = \\
= \text{const} \int \prod_{i=1}^n d^2 t_i \left\langle \text{Scr}(t_1, \bar{t}_1) \ldots \text{Scr}(t_n, \bar{t}_n) \tilde{V}_{j_1}(z_1) \ldots \tilde{V}_{j_N}(z_N) \right\rangle_{S_0} \tilde{v} \otimes \bar{v},
\] (4.6)

where averaging with the free action

\[
S_0 = \frac{1}{4\pi} \int d^2 z \left\{ \frac{1}{2} \phi \partial \bar{\phi} - \beta \partial \bar{\gamma} - \bar{\beta} \partial \gamma + i \frac{\sqrt{2}}{q} \mathcal{R} \sqrt{g} \phi \right\}
\] (4.7)

15
Remarkable, expression (4.6) can be rewritten in more simple and profound form using an old idea suggested by A. M. Polyakov and developed by Dotsenko and Fateev in [19]. Namely, consider the model with the action

\[ S'_\phi = S_0 + S_{int}, \]

where interacting term is equal to

\[ S_{int} = \int d^2z \beta \bar{\beta} \exp \left( -\frac{i \sqrt{2}}{q} \phi \right) \]

(4.8)

and is exactly marginal because of the zero conformal dimension of the exponent. Explicitly,

\[ S'_\phi = \frac{1}{4\pi} \int d^2z \left\{ \frac{1}{2} \partial \phi \bar{\partial} \bar{\phi} - \beta \bar{\partial} \gamma - \bar{\beta} \partial \bar{\gamma} + i \frac{\sqrt{2}}{q} R \sqrt{g} \phi + \beta \bar{\beta} \exp \left( -\frac{i \sqrt{2}}{q} \phi \right) \right\}. \]

(4.9)

All correlators in this model are calculated by expanding in \( S_{int} \). As we know, the correlator

\[ \langle \prod_{i=1}^n \exp (i\alpha_i \phi) \rangle \]

(4.10)

will not vanish only if the neutrality condition \( \sum l \alpha_l = 0 \) holds. Therefore, only one term in the series survive, and this is exactly the term in rhs of the (4.6). Thus, we identify the WZNW correlators of primary fields \( \Phi_{\Delta(j_i)} \) with the “dressed” vertex operator correlators in theory with the action \( S'_\phi \): \[
\langle \Phi_{\Delta(j_1)}(z_1, \bar{z}_1) \cdots \Phi_{\Delta(j_N)}(z_N, \bar{z}_N) \rangle_{WZNW} \bar{\nu} \otimes \bar{\nu} = \]
\[
= \int D[\phi, \beta, \gamma, \bar{\beta}, \bar{\gamma}] \exp (-S'_\phi) \tilde{V}_{j_1}(z_1, \bar{z}_1) \cdots \tilde{V}_{j_N}(z_N, \bar{z}_N) \bar{\nu} \otimes \bar{\nu}. \]

(4.11)

Straightforward calculation gives:

\[
\langle \Phi_{\Delta(j_1)}(z_1, \bar{z}_1) \cdots \Phi_{\Delta(j_N)}(z_N, \bar{z}_N) \rangle_{WZNW} \bar{\nu} \otimes \bar{\nu} = \text{const} \prod_{i<l} |z_i - z_l|^\frac{\hbar}{2} \times \]
\[
\times \left( \prod_{p<q} |t_p - t_q|^\frac{2}{\hbar} \right) \left( \prod_{p=1}^n \prod_{i=1}^N |t_p - z_i|^\frac{2}{\hbar} \right) \left( \prod_{p=1}^n \sum_{i=1}^N (f^L)_i \right) \left( \prod_{q=1}^n \sum_{i=1}^N (f^R)_l \right) \bar{\nu} \otimes \bar{\nu}. \]

(4.12)

**Polynomial representation**

One can do all the above steps using the polynomial representation (3.18) for the generating function. As a result in the \( SU(2) \) case the following formula arises:

16
\begin{align}
\langle \Phi^{(j_1)}(x_1, \bar{x}_1; z_1, \bar{z}_1) \ldots \Phi^{(j_N)}(x_N, \bar{x}_N; z_N, \bar{z}_N) \rangle_{W_{ZNW}} = & \quad \text{const} \left( \prod_{i<l} |z_i - z_l|^{-\frac{2j_{ij}}{\kappa}} \right) \int \left( \prod_{i=1}^n d^2t_i \right) \left( \prod_{p<q} |t_p - t_q|^{\frac{2}{\kappa}} \right) \left( \prod_{p=1}^n \prod_{l=1}^N |t_p - z_l|^{-\frac{2j_{pl}}{\kappa}} \right) \\
& \times \left( \prod_{r=1}^N \frac{1}{2\pi i} \int_{u_r=0} du_r u_r^{-j_r-1} e^{u_r} \right) \left( \prod_{s=1}^n \frac{1}{2\pi i} \int_{\bar{u}_s=0} \bar{u}_s^{-j_s-1} e^{\bar{u}_s} \right) \left| \prod_{p=1}^n \sum_{l=1}^N \frac{u_i x_i}{t_p - z_l} \right|^2.
\end{align}

\[ (4.13) \]

4.2 Simple complex Lie groups

Generalization of the construction from previous subsection for other groups is obvious. Let us introduce the action

\[ S_{\phi\bar{\phi}\gamma\bar{\gamma}} = \frac{1}{4\pi} \int d^2z \left\{ \frac{1}{2} \partial \bar{\phi} \partial \bar{\phi} - \beta \partial \bar{\gamma} - \bar{\beta} \partial \gamma + i \frac{R}{q} \sqrt{g} \bar{\rho} \bar{\phi} + \right. \]

\[ + \sum_i W_{\alpha_i}^\delta (-\gamma) W_{\alpha_i}^\delta (-\bar{\gamma}) |\beta| \bar{\beta} |\bar{\bar{\gamma}}| \exp \left( -i \frac{\sqrt{2}}{q} \bar{\alpha}_i \bar{\phi}(z) \right) \left\} \right. \]

and the generating function:

\[ \bar{V}_j(z, \bar{z}) = \exp \left( \bar{\gamma}(z) \bar{f}_L \right) \exp \left( i \frac{\sqrt{2}}{q} \bar{\phi}(z, \bar{z}) \right) \exp \left( \bar{\bar{\gamma}}(z) \bar{f}_R \right) \]

\[ (4.15) \]

Then for N-point correlator we obtain the following representation:

\begin{align}
\langle \Phi_{\Delta(j_1)}(z_1, \bar{z}_1) \ldots \Phi_{\Delta(j_N)}(z_N, \bar{z}_N) \rangle_{W_{ZNW}} \bar{v} \otimes \bar{v} = & \quad \int \mathcal{D}[\phi, \bar{\phi}, \gamma, \bar{\gamma}, \beta, \bar{\beta}, \bar{\bar{\gamma}}] \exp(-S_{\phi\bar{\phi}\gamma\bar{\gamma}}) \bar{V}_{j_1}(z_1, \bar{z}_1) \ldots \bar{V}_{j_N}(z_N, \bar{z}_N) \bar{v} \otimes \bar{v}.
\end{align}

\[ (4.16) \]

To obtain expression for the correlator in polynomial representation on needs just to substitute exponential generating function by the corresponding polynomial one.
5 New integral identities

5.1 A simple test

Now we do some tests on the proposed construction. It is well known that the $SU(2)_{k=1}$ WZNW model can be represented in terms of one $\phi$ field with values in the self-dual circle. The $su(2)_{k=1}$ currents are:

\[ J^+ = e^{i\sqrt{2}\phi}, \quad J_0 = i\sqrt{2}\partial\phi, \quad J^- = e^{-i\sqrt{2}\phi} \quad (5.1) \]

The are only two non-trivial primary fields correspond to the spin 1/2 representation:

\[ \Phi_\uparrow = e^{i\sqrt{2}\phi}, \quad \Phi_\downarrow = e^{-i\sqrt{2}\phi} \quad (5.2) \]

The 4-point correlator obviously is

\[ \langle \Phi_\uparrow(0)\Phi_\downarrow(x, \bar{x})\Phi_\uparrow(1)\Phi_\downarrow(\infty) \rangle_{WZNW} = \frac{\text{const}}{\sqrt{x\bar{x}(1-x)(1-\bar{x})}} \quad (5.3) \]

From the other hand, the general expression (4.11) in this case gives:

\[
\langle \Phi_\uparrow(0)\Phi_\downarrow(x)\Phi_\uparrow(1)\Phi_\downarrow(\infty) \rangle_{WZNW} \sim |x|^{1/3} |1-x|^{1/3} \int d^2t_1 |t_1|^{-2/3} |t_1-x|^{-2/3} |t_1-1|^{-2/3} \times
\]
\[
\times \int d^2t_2 |t_1-t_2|^{4/3} |t_2|^{-2/3} |t_2-x|^{-2/3} |t_2-1|^{-2/3} \left| \frac{1}{t_1(t_1-1)} + \frac{1}{t_2(t_2-1)} \right|^2.
\quad (5.4)
\]

The equality of this two different expressions for the correlator seems to be a non-trivial fact. It can be easily checked numerically. Let us describe some steps which reduce the problem to the statement that some 2-fold integrals over the surface must vanish. This statement was checked with the help of the Mathematica 3.0 program. As a first step we change the integration variables:

\[ t_1 = t'_1 x, \quad t_2 = t'_2 - (t'_2 - 1)x, \quad (5.5) \]

so that the rhs of (5.4) takes the form
$$\frac{1}{\sqrt{x(1-x)(1-\bar{x})}} \int d^2t' |t'_1|^{-2/3} |1-t'_1|^{-2/3} |1-xt'_1|^{-2/3} \times$$

$$\times \int d^2t'_2 |1-t'_2|^{-2/3} \left| 1-x \frac{t'_2 + t'_1 - 1}{t'_2} \right|^{4/3} \left| 1-x \frac{t'_2 - 1}{t'_2} \right|^{-2/3} \times$$

$$\times \left| \frac{1}{t'_1(t'_1-1)} + \frac{x(1-x)}{(t'_2-x(t'_2-1))(xt'_1-1)} \right|^2.$$

If the rhs of (5.3) and (5.4) are equal, integral in (5.6) should not depend on $x$. In order to check it, we expand this integral to the series in $x$. As a result of the numerical calculation it turns out that at least 8 first coefficients $c_n$ ($n \geq 1$) in front of the $x^n$ terms in this series vanish.

### 5.2 Applications

New integral identities arise when an alternative way to solve the WZNW model exists. In these identities lhs and rhs are the correlators calculated in two different ways. They have equal conformal, algebraic and analytical properties. Moreover, they obey the same differential and algebraic equations. Thus at least at physical level we can conclude that they should be equal.

The $SU(2)_{k=1}$ WZNW 2N-point correlator, calculated with help of bosonization (5.2) is

$$\left< \prod_{i=1}^{N} \Phi_+(z_i, \bar{z}_i) \prod_{i=1}^{N} \Phi_1(w_i, \bar{w}_i) \right>_{WZNW} = \text{const} \frac{\prod_{i,j} |z_i - z_j|^{1/2} \prod_{i,j} |w_i - w_j|^{1/2}}{\prod_{i,j} |z_i - w_j|^{1/2}} \ (5.7)$$

Comparing with the general expression (1.11) for this correlator gives an identity:

$$\left( \prod_{i<j} |z_i - z_j|^{1/3} |w_i - w_j|^{1/3} \right) \left( \prod_{i,j} |z_i - w_j|^{1/3} \right) \int \left( \prod_{p=1}^{N+1} d^2t_p \right) \left( \prod_{p<q} |t_p - t_q|^{4/3} \right) \times$$

$$\times \left( \prod_{p=1}^{N+1} \prod_{i=1}^{N} |t_p - z_i|^{-2/3} |t_p - w_i|^{-2/3} \right) \left| \sum_{\text{perm}\{\sigma\}} \frac{1}{(t_{\sigma(1)} - z_1) \cdots (t_{\sigma(N)} - z_N)} \right|^2 = (5.8)$$

$$= \text{const} \frac{\prod_{i<j} |z_i - z_j|^{1/2} \prod_{i,j} |w_i - w_j|^{1/2}}{\prod_{i,j} |z_i - w_j|^{1/2}}$$
Another integral identity arises when one compares the prediction (4.13) for the 4-point $SU(2)_k$ WZNW correlator in polynomial representation:

\[
\left\langle \Phi^{(j_1)}(x_1, \bar{x}_1; z_1, \bar{z}_1) \cdots \Phi^{(j_4)}(x_4, \bar{x}_4; z_4, \bar{z}_4) \right\rangle_{WZNW} \sim \left( \prod_{i<j} |z_i - z_j|^{\frac{2j_i + 2j_j}{k+2}} \right) \int \left( \prod_{i=1}^{\Sigma_{j_i+1}} d^2 t_i \right) \left( \prod_{p<q} |t_p - t_q|^{\frac{4}{k+2}} \right) \left( \prod_{p=1}^{\Sigma_{j_i+1}} \prod_{l=1}^{4} |t_p - z_l|^{-\frac{2j_l}{k+2}} \right) \times (5.9)
\]

\[
\times \left( \prod_{r=1}^{4} \frac{j_r!}{2\pi i} \oint_{u_r=0} du_r \frac{u_r^{-j_r-1} e^{u_r}}{\prod_{s=1}^{\Sigma_{j_s+1}} \prod_{l=1}^{4} \left| \bar{u}_s - j_s u_s \right|^2} \right) \left( \prod_{p=1}^{\Sigma_{j_i+1}} \sum_{i=1}^{4} \frac{u_i x_i}{t_p - z_i} \right)^2
\]

with the following expression suggested by Zamolodchikov and Fateev [3]:

\[
\left\langle \Phi^{(j_1)}(x_1, \bar{x}_1; z_1, \bar{z}_1) \cdots \Phi^{(j_4)}(x_4, \bar{x}_4; z_4, \bar{z}_4) \right\rangle_{WZNW} = |x_{14}|^{4j_1} |x_{24}|^{2(j_1+j_2+j_4-j_3)} \times (5.10)
\]

\[
\times |x_{34}|^{2(j_1+j_2+j_3-j_4)} |x_{32}|^{2(j_1+j_2+j_3-j_4)} |z_{14}|^{2\nu_1} |z_{24}|^{2\nu_2} |z_{34}|^{2\nu_3} |z_{32}|^{2\mu} U_{j_1j_2j_3j_4}(x, \bar{x}; z, \bar{z}),
\]

where

\[
\nu_1 = -2\Delta^{(j_1)}, \quad \nu_2 = \Delta^{(j_2)} - \Delta^{(j_1)} - \Delta^{(j_3)} - \Delta^{(j_4)},
\]

\[
\nu_3 = \Delta^{(j_3)} - \Delta^{(j_2)} - \Delta^{(j_1)} - \Delta^{(j_4)},
\]

\[
\mu = \Delta^{(j_1)} + \Delta^{(j_4)} - \Delta^{(j_3)} - \Delta^{(j_2)},
\]

(5.11)

and $U_{j_1j_2j_3j_4}(x, \bar{x}; z, \bar{z})$ is a function of the projective invariants:

\[
x = \frac{x_{12} x_{34}}{x_{14} x_{32}}, \quad \bar{x} = \frac{\bar{x}_{12} \bar{x}_{34}}{\bar{x}_{14} \bar{x}_{32}}, \quad z = \frac{z_{12} z_{34}}{z_{14} z_{32}}, \quad \bar{z} = \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{14} \bar{z}_{32}}.
\]

(5.12)

given by the following multiple integral

\[
U_{j_1j_2j_3j_4}(x, \bar{x}; z, \bar{z}) = |z|^{\frac{4j_1j_2}{k+2}} \left| 1 - z \right|^{\frac{4j_3j_4}{k+2}} N(j_1j_2j_3j_4) \times (5.13)
\]

\[
\times \int \prod_{i=1}^{2j_1} d^2 t_i \left| t_i - z \right|^{\frac{2j_1}{k+2}} \left| t_i - z \right|^{\frac{2j_2}{k+2}} \left| t_i - 1 \right|^{\frac{2j_3}{k+2}} \left| t_i - 1 \right|^{\frac{2j_4}{k+2}} |x - t_i|^2 |D(t)|^{\frac{1}{k+2}}
\]

where $N(j_1j_2j_3j_4)$ is certain normalization factor (see [3]), and
\[ D(t) = \prod_{i<j}(t_i - t_j), \]
\[ \beta_1 = j_1 + j_2 + j_3 + j_4 + 1, \]
\[ \beta_2 = k + j_1 + j_2 - j_3 - j_4 + 1, \]
\[ \beta_3 = k + j_1 + j_3 - j_2 - j_4 + 1. \]

(5.14)

To obtain new integral identities one can also use expressions for the \(SU(N)_k\) 4-point correlators of the primary fields belonging to the fundamental representation \([5]\). As a result proper multiple integrals \((1.11)\) become expressed in terms of the basic hypergeometric functions \(2F_1\).

6 Conclusions, Speculations and Outlook

In this paper we considered the free field approach or bosonization technique for the Wess-Zumino-Novikov-Witten model with arbitrary Kač-Moody algebra on genus zero Riemann surface. We show how to obtain Schechtman-Varchenko solutions of the Knizhnik-Zamolodchikov equations as certain correlators in free chiral theory thus providing a simple description of space of the WZNW conformal blocks. We also propose a simple prescription for “gluing” correlators from the conformal blocks, quite similar to the Dotsenko-Fateev prescription for the minimal models \([19]\).

This construction has a simple interpretation in the functional integral language. Namely, one can attribute the additional marginal terms in action (screening generators) to the functional measure by means of the corresponding \(\delta\)-function insertions. Thus WZNW model becomes essentially embedded in a certain subsector of the free fields theory. It is worth mentioning the analogy of such description with the powerful “projection method” from the integrable systems theory \([29]\).

Interesting insight on bosonized action arises from the non-critical string theory. Let us discuss the simplest case. Consider the bosonic Polyakov string propagating in two dimensions. For many reasons it is useful to compactify it in a certain way. For instance, one can consider as a target space \(\mathbb{CP}^1\) with the complex coordinates \((\gamma, \bar{\gamma})\). The conformal anomaly results in metric dependence governed by the Liouville action — this is the way how the \(\phi\) field arises. All fields become “dressed” by the Liouville field — they acquire factors like \(e^{\alpha\phi}\) where \(\alpha\) defines the anomalous dimension. At the moment the possible dynamics of such a non-perturbative process is absolutely unknown. But if we are interested in the case when the resulting string theory has a current algebra on the world-sheet, the form of the possible terms in effective action is strongly constrained. For instance, the Liouville interaction term is forbidden (!) since it results in screening charge which does not commute with the currents. In fact, the only possibility is the action \(S'_{\phi\beta\gamma}\) corresponding to the \(SU(2)\) theory or “conjugated” action \(S'_{\phi\beta\gamma}\) corresponding to the \(SL(2,\mathbb{C})\) theory. (Note that the \(\beta\) field can be easily integrated out.) Of course, this is nothing but speculation. However, it gives us an alternative way to think about our construction and brings an interesting link to such a long-standing problem of mathematical physics as Liouville theory.

It is worth mentioning renewed interest to the bosonization of the WZNW models related to
the strings propagating on $AdS_3 \times S^3, AdS_{2d+1} \times S^{2d'+1}$ [30, 31, 32]. This interest is motivated by dualities between certain CFT’s and string theories on anti-de-Sitter spaces with RR fluxes [33].

An intriguing direction for future research comes from the observation [22] that KZ equations and their SV solutions can be generalized to the case when $g$ itself is an arbitrary Kač-Moody algebra associated with the symmetriable Cartan matrix. These generalized KZ equations should correspond to the loop group WZNW model while generalized conformal blocks should correspond to the some two-loop algebra representations. Such objects are very important for the unification of conformal and two-dimensional integrable models as points on the String Theory configuration space [34].

We also believe that the tools described in the paper will help us to understand deeper and generalize the intriguing interplay between Langlands duality and Sklyanin’s separation of variables [26, 35].

To summarize, we suggest the following prescription, which generalizes that one from [8, 19]:

**N-point correlators of the spinless primary fields in the genus zero WZNW model coincide with the N-point correlators of the “dressed” vertex operators in theory of free ($\phi, \beta, \gamma$)-fields perturbed by the exactly marginal terms corresponding to the “squared modules” of simple screening currents.**

As one of the possible applications of this prescription, we have obtained a set of new integral identities between (naively) different hypergeometric functions.

Further development of these methods and generalization of the proposed construction to the higher genus case will appear elsewhere.

**Acknowledgements**

I am grateful to A. Marshakov and A. Mironov for critical comments and deeply indebted to A. Morozov and A. Losev for illuminating discussions and friendly guidance throughout the work. I wish to thank Dima Lyubshin and Ira Vashkevich for technical support. The work was partly supported by the Russian President’s grant 96-15-9639 and RFBR grant 98-02-16575.
References

[1] S. Novikov, Usp. Mat. Nauk 37 (1982) 3.

[2] E. Witten, “Non-Abelian Bosonization in Two Dimensions,” Comm. Math. Phys. 92 (1984) 455.

[3] A. Polyakov and P. Wiegmann “Theory of Non-Abelian Goldstone Bosons,” Phys. Lett. 131B (1983) 121; “Goldstone Fields in Two-Dimensions with Multivalued Actions,” Phys. Lett. 141B (1984) 223.

[4] G. Moore and N. Seiberg, “Taming the Conformal Zoo,” Phys. Lett. B220 (1989) 422.

[5] V. Knizhnik and A. Zamolodchikov, “Current Algebra and Wess-Zumino Model in Two Dimensions,” Nucl. Phys. B247 (1984) 83.

[6] A. Zamolodchikov and V. Fateev, “Operator Algebra and Correlation Functions in the Two-Dimensional Wess-Zumino SU(2) × SU(2) Chiral Model,” Sov. J. Nucl. Phys. 43 (1986) 657, Yad. Fiz. 43 (1986) 1031.

[7] P. Christe and R. Flume, “The Four Point Correlations Of All Primary Operators Of The D = 2 Conformally Invariant SU(2) Sigma Model With Wess-Zumino Term,” Nucl. Phys. B282 (1987) 466.

[8] D. Bernard and G. Felder, “Fock Representations And BRST Cohomology In Sl(2) Current Algebra,” Commun. Math. Phys. 127 (1990) 145.

[9] A. Gerasimov, A. Marshakov, A. Morozov, M. Olshanetsky and S. Shatashvili, “Wess-Zumino-Witten Model as a Theory of Free Fields”, Int. J. Mod. Phys. A5 (1990) 2495.

[10] V. Dotsenko, “The free field representation of the SU(2) conformal field theory,” Nucl. Phys. B338 747 (1990); “Solving the SU(2) conformal field theory with the Wakimoto free field representation,” Nucl. Phys. B358 (1991) 547.

[11] H. Awata, A. Tsuchiya and Y. Yamada, “Integral Formulas for the WZNW Correlation Functions,” Nucl. Phys. B365 (1991) 680; H. Awata, “Screening Currents Ward Identity and Integral Formulas for the WZNW Correlation Functions,” Prog. Theor. Phys. Suppl. 110 (1992) 303 [hep-th/9202032].

[12] P. Furlan, A. Ganchev, R. Paunov and V. Petkova, “Reduction of the rational spin sl(2, C) WZNW conformal theory,” Phys. Lett. B267 (1991) 63; “Solutions of the Knizhnik-Zamolodchikov equation with rational isospins and the reduction to the minimal models,” Nucl. Phys. B394 (1993) 665 [hep-th/9201080]; A. Ganchev and V. Petkova, “Reduction of the Knizhnik-Zamolodchikov equation: A Way of producing Virasoro singular vectors,” Phys. Lett. B293 (1992) 56 [hep-th/9207032].

[13] M. Wakimoto, “Fock Representations of the Affine Lie Algebra A(1) 3,” Comm. Math Phys. 104 (1986) 605.
[14] A. Alekseev and S. Shatashvili, “Path Integral Quantization of the Coadjoint Orbits of the Virasoro Group and 2-D Gravity,” Nucl. Phys. B323 (1989) 719; “From Geometric Quantization To Conformal Field Theory,” Commun. Math. Phys. 128 (1990) 197; “Quantum Groups And WZW Models,” Commun. Math. Phys. 133 (1990) 353.

[15] B. Feigin, E. Frenkel, “The family of representations of affine Lie algebras,” Uspekhi Mat. Nauk. 43 (1988) 227-228, Russ. Math. Surv. 43 (1989) 221–222; “Affine Kač-Moody algebras and semi-infinite flag manifolds,” Commun. Math. Phys. 128 (1990) 161–189; “Representations of affine Kač-Moody algebras, bosonization and resolutions,” Lett. Math. Phys. 19 (1990) 307-317; “Representations of affine Kač-Moody algebras and bosonization,” pp. 271–316 in: Physics and Mathematics of Strings, eds. L. Brink et al, World Scientific, Singapore, 1990; E. Frenkel, “Free field realizations in representation theory and conformal field theory,” in: Proceedings of the ICM, Zürich 1994, hep-th/9408109.

[16] P. Bouwknegt, J. McCarthy and K. Pilch, “Free Field Realizations Of WZNW Models: BRST Complex And Its Quantum Group Structure,” Phys. Lett. B234 (1990) 297; “Quantum Group Structure In The Fock Space Resolutions Of SU(N) Representations,” Commun. Math. Phys. 131 (1990) 125; “Free Field Approach To Two-Dimensional Conformal Field Theories,” Prog. Theor. Phys. Suppl. 102 (1990) 67; “Some aspects of free field resolutions in 2-D CFT with application to the quantum Drinfeld-Sokolov reduction,” hep-th/9110007.

[17] J. Petersen, J. Rasmussen and M. Yu, “Free field realization of SL(2) correlators for admissible representations, and hamiltonian reduction for correlators,” Nucl. Phys. Proc. Suppl. 49 (1996) 27 hep-th/9512175, “Conformal blocks for admissible representations in SL(2) current algebra,” Nucl. Phys. B457 (1995) 309 hep-th/9504127, hep-th/9510059, J. Rasmussen, “Applications of free fields in 2D current algebra,” PhD Thesis, hep-th/9610167.

[18] J. Petersen, J. Rasmussen and M. Yu, “Free field realizations of 2D current algebras, screening currents and primary fields,” Nucl. Phys. B502 (1997) 649 hep-th/9704052.

[19] V. Dotsenko and V. Fateev, “Conformal Algebra and Multipoint Correlation Functions in 2D Statistical Models,” Nucl. Phys. B240 (1984) 312; “Four Point Correlation Functions and the Operator Algebra in the Two-Dimensional Conformal Invariant Theories with the Central Charge c < 1,” Nucl. Phys. B251 (1985) 691.

[20] K. Gawedzki, “Quadrature of Conformal Field Theories,” Nucl. Phys. B328 (1989) 733; “Constructive Conformal Field Theory,” In *Karpacz 1989, Proceedings, Functional integration, geometry and strings* 277-302., “Geometry of Wess-Zumino-Witten models of conformal field theory,” Nucl. Phys. Proc. Suppl. 18B (1991) 78; F. Falceto, K. Gawedzki and A. Kupiainen, “Scalar product of current blocks in WZW theory,” Phys. Lett. B260 (1991) 101.

[21] A. Belavin, A. Polyakov and A. Zamolodchikov, “Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory,” Nucl. Phys. B241 (1984) 333.
[22] V. Schechtman and A. Varchenko, “Hypergeometric Solutions of Knizhnik-Zamolodchikov Equations,” Lett. Math. Phys. 20 (1990) 279; “Arrangements of Hyperplanes and Lie Algebra Homology,” Invent. Math., 106 (1991) 139.

[23] B. Feigin and D. Fuks, “Verma Modules Over the Virasoro Algebra,” Funct. Anal. Appl. 17 241 (1983).

[24] B. Feigin, V. Schechtman and A. Varchenko, “On Algebraic Equations Satisfied by Hypergeometric Correlators in WZW Models. I.,” Comm. Math. Phys. 163 (1994) 173; “On algebraic equations satisfied by hypergeometric correlations in WZW models. 2,” Commun. Math. Phys. 170 (1995) 219 hep-th/9407010.

[25] P. Etingof, I. Frenkel and A. Kirillov Jr., Lectures on representation theory and Knizhnik-Zamolodchikov equations, AMS, 1998.

[26] B. Feigin, E. Frenkel and N. Reshetikhin, “Gaudin model, Bethe ansatz and correlation functions at the critical level,” Commun. Math. Phys. 166 (1994) 27 hep-th/9402022.

[27] V. Dotsenko, “Lectures on Conformal Field Theory,” Adv. Stud. in Pure Math. 16 (1988) 123.

[28] O. Andreev, “Operator Algebra of the SL(2) Conformal Field Theories,” Phys. Lett. B363 (1995) 166, hep-th/9504082.

[29] M. Olshanetsky and A. Perelomov, Inv. Math. 31 (1976) 93.

[30] J. de Boer and S. Shatashvili, “Two-Dimensional Conformal Field Theories on AdS_{2d+1} Backgrounds,” hep-th/9905032.

[31] N. Berkovits, C. Vafa and E. Witten, hep-th/9902098, D. Kutasov and N. Seiberg, hep-th/9903219, A. Giveon, D. Kutasov and N. Seiberg, hep-th/9806194.

[32] O. Andreev, “Unitary Representations of Some Infinite Dimensional Lie Algebras Motivated by String Theory on AdS_3,” hep-th/9905002; “On Affine Lie Superalgebras, AdS_3/CFT Correspondence And World-Sheets For World-Sheets,” Nucl.Phys. B552 (1999) 169-193 hep-th/9901118; “Probing AdS_3/CFT Correspondence via World-Sheet Methods and 2d Gravity Like Scaling Arguments, hep-th/9909222.

[33] J. Maldacena, “The Large N Limit of Superconformal Field Theories and Supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231 hep-th/9711200, S. Gubser, I. Klebanov and A. Polyakov, “Gauge Theory Correlators from Non-Critical String Theory,” hep-th/9802109, E. Witten, “Anti De Sitter Space and Holography,” hep-th/9802150.

[34] A. Gerasimov, D. Lebedev and A. Morozov, “On Possible Implications of Integrable Systems for String Theory,” Int. J. Mod. Phys. A6 (1991) 977.

[35] E. Frenkel, “Affine Algebras, Langlands Duality and Bethe Ansatz,” q-alg/9506003, B. Enriquez, B. Feigin and V. Rubtsov, “Separation of variables for Gaudin-Calogero systems,” q-alg/9605030, A. Gorsky, N. Nekrasov and V. Rubtsov, “Hilbert schemes, separated variables, and D-branes,” hep-th/9901089.