Possibility of the use of Cartesian method in the proofs of fundamental theorems of school planimetry.

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Abstract

We show how Cartesian method can be used in the proof of fundamental planimetric topics of the school course, such as introduction of trigonometric functions, equation of a line and similarity of triangles.

This work also can be considered as a plan of the school course of geometry, where the Cartesian method plays the main role.

Introduction

The work [14] is published in Ukrainian and concerns the manner of use of the Thales' theorem in the courses of geometry in Ukrainian schools.

Here we present the mathematical ideas of [14] and these ideas deal with the possibility and convenience of the use of Cartesian method in the proof of the Thales’ theorem (about the ratios of various line segments that are created if two intersecting lines are intercepted by a pair of parallels), introduction of trigonometric functions and proof the criterions of similarity of triangles.

It is well known, that Thales’ theorem can be proved with the use of areas. Also is is known thousands proofs of the Pythagorean theorem. One of these proofs uses the notion of area. Another classical proof of the Pythagorean theorem uses the notion of the correctness of definition of sin and cos of obtuse angles as a relation of lengths of correspond sides of the right triangle. In the first two chapters of our work we will remind these proofs for the simplicity of reader.

As our main results we will show the following.

1. It follows from the Pythagorean theorem that the graph of the equation $y = ax$ for a parameter $a \in \mathbb{R}$ is a line.
2. Suppose that we know that every non-vertical lines on cartesian plain, passing through origin, is the graph of the equation \( y = ax \) for some \( a \). This fact directly yields the Thales’ theorem.

3. Thales’ theorem immediately follows from the Pythagorean theorem.

This work was motivated by the following facts. In Ukraine the school course of mathematics is traditionally divided to algebra and geometry.

In both modern Ukrainian books of geometry [2, 3, 5, 6, 8, 12, 11], both in more ancient Soviet [1, 8, 13] Pythagorean theorem is proved with the use of similarity of triangles, which is equivalent to correctness of trigonometric functions \( \sin \) and \( \cos \) of obtuse angle and also is equivalent to the Thales’ theorem.

The fact, that the graph of the function \( y = ax \) is a line is studied in Ukrainian schools in the course of algebra but this fact is not properly proved in the books [1, 7, 9, 10].

1 Proof of the Pythagorean theorem with the use of areas

Consider a \( ABC \) triangle with a right angle \( C \) (see pict. 1). Take points \( F \) and \( M \) on the lines \( CA \) and \( CB \) such that segments \( CF \) and \( CM \) be equal to the sum of legs of the triangle \( ABC \).

Take points \( G \) and \( L \) on the sides on the square \( CF KM \) such that \( FG = KL = AB \). Then the four new triangles will be equal, and the quadrangle \( AGLB \) will be a square.

The proof of the Pythagorean theorem can be obtained, if we subtract the area of the four right triangles from the area of the “big” square.

2 Proof of the Thales’ theorem using areas

The proof of the correctness of trigonometric functions in the most widespread school books in Ukraine is realized by via the Generalized Thales’ theorem. Thus, we will give below the proof of the correctness of trigonometric functions, which is given in the by Aleksandrov in [11, p. 95].

Take any right triangle \( ABC \) (see pict 2) and a point \( M \) on the leg \( AC \). Express the area of the triangle \( ABM \) by manners. From one hand, \( S = \frac{1}{2}ma \), where \( a = BC \) and \( m = AM \).

From another hand, \( S = \frac{1}{2}ch \), where \( h = MD \) is the height of the triangle \( ABM \) and \( c \) is the hypotenuse of the former right triangle. Since the obtain formulas represent the same area, then

\[
\frac{a}{c} = \frac{h}{m}.
\]
3 Introduction of trigonometric functions using cartesian method

Let triangles $A_1B_1C_1$ and $A_2B_2C_2$ have right angles $\angle C_1 = \angle C_2 = 90^0$ and let $\angle A_1 = \angle A_2$.

Superpose points $A_1$ and $A_2$ and assume that $C_1$ and $C_2$ are on the same side from $A_1 = A_2$ on the line $A_1C_1$. Take the cartesian plain with the origin in $A = A_1$ and the x-axis $A_1C_1$ such that $B_1$ and $B_2$ belong to the first quadrant (see fig. 3).

In this case semi-straight lines $AB$ and $A_1B_1$ will coincide. The equation of their line is $y = ax$, where real $a$ depends on the value of angle $\angle A = \angle A_1$.

Figure 3 is prepared for the case when $A_1C_1 < A_2C_2$. Denote $A_1C_1 = x_1$ and $A_2C_2 = x_2$. Then we can express the coordinates of the vertices of our triangles as follows: $A_1(0, 0)$, $A_2(0, 0)$, $B_1(x_1, ax_1)$, $B_2(x_2, ax_2)$, $C_1(x_1, 0)$ and $C_2(x_2, 0)$.

Obtain the lengths of the sides of triangles from the distance formula, precisely $A_1C_1 = x_1$, $A_2C_2 = x_2$, $B_1C_1 = ax_1$, $B_2C_2 = ax_2$, $A_1B_1 = x_1\sqrt{1 + a^2}$ and $A_2B_2 = x_2\sqrt{1 + a^2}$.

Due to these formulas, the equalities of ratios of the lengths of sides of the triangles, which are necessary for the introduction of trigonometric functions, are evident.

4 Similarity of right triangles

When the cartesian plain and equation of a line are introduced, the criterions of similarity of triangles become obvious facts, which follow from Cartesian method.

Remind that triangles are called similar, if their correspond sides are proportional and correspond angles are equal. Remind the criterions of similarity of right triangles.
1. By two angles (two right triangles are similar if and only if a pair of their correspond angles are equal).

2. By two pairs of proportional sides (two right triangles are similar if and only if a pair of their correspond sides have the proportional lengths).

The first of these criterions follows from reasonings, which are analogous the one from the introducing of trigonometric functions. It is necessary to consider the Cartesian plane, superpose vertices of two equal acute angles at origin, plug right angles to the x-axis and plug hypotenuses at the same line, passing through origin. Not the fact that corresponds sides of triangles are proportional follows from the distance formula and the formula of equation of the line, passing through origin.

The second criterion of the similarity of right triangles can be proved in the same manner. Suppose that sides of triangles $A_1B_1C_1$ are $A_2B_2C_2$ with right angles $\angle C_1$ and $\angle C_2$ are proportional. Like we have done above, superpose vertices $A_1$ and $A_2$ at origin, plug $C_1$ and $C_2$ at x-axis and plug $B_1$ and $B_2$ in the first quadrant. Suppose the equation of lines $A_1B_1$ and $A_2B_2$ be $y = ax$ and $y = bx$ respectively. Nevertheless, $a = \frac{B_1C_1}{A_1C_1} = \frac{B_2C_2}{A_2C_2} = b$, which implies the equality of all correspond angles of our triangles.

5 Similarity of triangles in general case

We suggest to reduce the “general case” of the question on similarity of triangles to the similarity of right triangles.

We will introduce the idea of the proof of the similarity of triangles in “general case”.

**Similarity by three angles.** Suppose that correspond angles of triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal. Then sides of triangles are proportional.

Consider the height of the biggest angle of these triangle, let it be $\angle C_1 = \angle C_2$. Since the angle if the biggest, then the base of the height will belong to the side of triangle, but not to its continuation.

It follows from the similarity of right triangles by three angles, that $\triangle B_1C_1H_1 \sim \triangle B_2C_2H_2$ and $\triangle A_1C_1H_1 \sim \triangle A_2C_2H_2$, whence

$$\frac{a_1}{a_2} = \frac{x_1}{x_2} = \frac{h_1}{h_2} = \frac{y_1}{y_2} = \frac{b_1}{b_2}.$$  

Thus, $x_1 = \frac{h_1y_2}{h_2}$ and $y_1 = \frac{h_1x_2}{h_2}$, which implies that $\frac{x_1+y_1}{x_2+y_2} = \frac{h_1}{h_2}$ and correspond sides of $A_1B_1C_1$ are $A_2B_2C_2$ proportional.
Similarity by an angle and two proportional sides. Suppose that triangles $A_1B_1C_1$ and $A_2B_2C_2$ are such that $\angle B_1 = \angle B_2$ and sides of the angle $B_1$ are proportional to side of $B_2$. Consider the heights from vertices $C_1$ and $C_2$ to the correspond side, or its continuation.

Similarity $\triangle B_1C_1H_1 \sim \triangle B_2C_2H_2$ follows from the criterion “by three angles”. This, together with proportionality of sides of angles $B_1$ and $B_2$ imply that

\[
\begin{align*}
\frac{h_1}{h_2} &= \frac{x_1}{x_2} = \frac{a_1}{a_2}, \\
\frac{a_1}{a_2} &= \frac{x_1 + y_1}{x_2 + y_2}.
\end{align*}
\]

It follows from these equalities that legs of right triangles $C_1H_1A_1$ and $C_2H_2A_2$ are proportional, whence triangles are similar.

Fig. 4: On the similarity of triangles
Similarity of right triangles imply that correspond angles of $A_1B_1C_1$ and $A_2B_2C_2$ are equal, whence triangles are similar.

Similarity by proportional sides. Suppose that sides of triangles $A_1B_1C_1$ and $A_2B_2C_2$ are proportional and prove the equality of angles.

Consider the triangle $\tilde{A}_2\tilde{B}_2\tilde{C}_2$, which is similar to $A_1B_1C_1$ and equal to $A_2B_2C_2$. For this deal take an arbitrary points $\tilde{A}_2$ and $\tilde{B}_2$ such that $\tilde{A}_2\tilde{B}_2 = A_2B_2$ and take rays to the same semi plain of $\tilde{A}_2\tilde{B}_2$, with angles $\angle A_1$ and $\angle B_1$ between rays and the line. Denote by $\tilde{C}_2$ the point of intersection of these rays.

Triangle $\tilde{A}_2\tilde{B}_2\tilde{C}_2$ is similar to $A_1B_1C_1$ by three angles, whence sides of triangles are proportional. From another hand, triangles $\tilde{A}_2\tilde{B}_2\tilde{C}_2$ and $A_2B_2C_2$ are equal by three sides, since $\tilde{A}_2\tilde{B}_2 = A_2B_2$ and correspond sides of these triangles are proportional to sides of $A_1B_1C_1$.

6 Proof of the Thales’ theorem as a corollary of Pythagorean theorem

We will show in this section how Thales’ theorem implies from the Pythagorean theorem.

Use the figure 5 where small letters $a, b, c, d$ denote segments $AD = HF$, $DC$, $HB$ and $AH = DF$. Start from the case when lines $BA$ and $FD$, which intersect sides of the angle $\angle BCA$, are perpendicular to $AC$.

Use Pythagorean theorem to express $FC$ from the right triangle $DFC$, $FB$ from $HFB$ and $BC$ from $ABC$. Then equality

\[
FC + FB = BC.
\]
can be rewritten as
\[ \sqrt{d^2 + b^2} + \sqrt{a^2 + c^2} = \sqrt{(a + b)^2 + (d + c)^2}. \]

After evident transformations obtain \((ad - bc)^2 = 0\). Use Pythagorean theorem ones more, rewrite this equality as
\[ \frac{\sqrt{a^2 + c^2}}{a} = \frac{\sqrt{b^2 + d^2}}{b}, \]
which is one of the forms of the Thales’ theorem.

The proof of Thales’ theorem without the assumption that lines are perpendicular to one of the side of an angle is the following. Use the notation of the figure [6], which is prepared for the case when perpendicular from the vertex of the angle to parallel lines is inside, but not outside of the angle. For another case the proof is almost the same.

Apply Thales’ theorem for the angles \(BCH\) and \(HCA\) obtain equalities
\[ \frac{x_2}{x_1} = \frac{h_2}{h_1} = \frac{y_2}{y_1}, \]
which imply Tales theorem for general case.

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