Upper bounds for the achromatic and coloring numbers of a graph

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Abstract: Dvořák et al. introduced a variant of the Randić index of a graph $G$, denoted by $R'(G)$, where $R'(G) = \sum_{uv \in E(G)} \frac{1}{\max\{d(u),d(v)\}}$, and $d(u)$ denotes the degree of a vertex $u$ in $G$. The coloring number $col(G)$ of a graph $G$ is the smallest number $k$ for which there exists a linear ordering of the vertices of $G$ such that each vertex is preceded by fewer than $k$ of its neighbors. It is well-known that $\chi(G) \leq col(G)$ for any graph $G$, where $\chi(G)$ denotes the chromatic number of $G$. In this note, we show that for any graph $G$ without isolated vertices, $col(G) \leq 2R'(G)$, with equality if and only if $G$ is obtained from identifying the center of a star with a vertex of a complete graph. This extends some known results. In addition, we present some new spectral bounds for the coloring and achromatic numbers of a graph.

Keywords: Chromatic number; Coloring number; Achromatic number; Randić index

1 Introduction

The Randić index $R(G)$ of a (molecular) graph $G$ was introduced by Milan Randić [17] in 1975 as the sum of $1/\sqrt{d(u)d(v)}$ over all edges $uv$ of $G$, where $d(u)$ denotes the degree of a vertex $u$ in $G$. Formally,

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}$$

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This index is useful in mathematical chemistry and has been extensively studied, see [12]. For some recent results on the Randić index, we refer to [3, 13, 14, 15].

A variation of the Randić index of a graph $G$ is called the Harmonic index, denoted by $H(G)$, which was defined in [8] as follows:

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$ 

In 2011 Dvořák et al. introduced another variant of the Randić index of a graph $G$, denoted by $R'(G)$, which has been further studied by Knor et al [16]. Formally,

$$R'(G) = \sum_{uv \in E(G)} \frac{1}{\max\{d(u), d(v)\}}.$$ 

It is clear from the definitions that for a graph $G$,

$$R'(G) \leq H(G) \leq R(G).$$  \hspace{1cm} (1)$$

The chromatic number of $G$, denoted by $\chi(G)$, is the smallest number of colors needed to color all vertices of $G$ such that no pair of adjacent vertices is colored the same. As usual, $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of $G$, respectively. The coloring number $\text{col}(G)$ of a graph $G$ is the least integer $k$ such that $G$ has a vertex ordering in which each vertex is preceded by fewer than $k$ of its neighbors. The degeneracy of $G$, denoted by $\text{deg}(G)$, is defined as $\text{deg}(G) = \max\{\delta(H) : H \subseteq G\}$. It is well-known (see Page 8 in [11]) that for any graph $G$,

$$\text{col}(G) = \text{deg}(G) + 1.$$  \hspace{1cm} (2)$$

List coloring is an extension of coloring of graphs, introduced by Vizing [21] and independently, by Erdős et al. [6]. For each vertex $v$ of a graph $G$, let $L(v)$ denote a list of colors assigned to $v$. A list coloring is a coloring $l$ of vertices of $G$ such that $l(v) \in L(v)$ and $l(x) \neq l(y)$ for any $xy \in E(G)$, where $v, x, y \in V(G)$. A graph $G$ is $k$-choosable if for any list assignment $L$ to each vertex $v \in V(G)$ with $|L(v)| \geq k$, there always exists a list coloring $l$ of $G$. The list chromatic number $\chi_l(G)$ (or choice number) of $G$ is the minimum $k$ for which $G$ is $k$-choosable.

It is well-known that for any graph $G$,

$$\chi(G) \leq \chi_l(G) \leq \text{col}(G) \leq \Delta(G) + 1.$$  \hspace{1cm} (3)$$

The detail of the inequalities in (3) can be found in a survey paper by Tuza [20] on list coloring.

In 2009, Hansen and Vukicević [9] established the following relation between the Randić index and the chromatic number of a graph.
Theorem 1.1. (Hansen and Vukicević [9]) Let $G$ be a simple graph with chromatic number $\chi(G)$ and Randić index $R(G)$. Then $\chi(G) \leq 2R(G)$ and equality holds if $G$ is a complete graph, possibly with some additional isolated vertices.

Some interesting extensions of Theorem 1.1 were recently obtained.

Theorem 1.2. (Deng et al [2]) For a graph $G$, $\chi(G) \leq 2H(G)$ with equality if and only if $G$ is a complete graph possibly with some additional isolated vertices.

Theorem 1.3. (Wu, Yan and Yang [23]) If $G$ is a graph of order $n$ without isolated vertices, then $\text{col}(G) \leq 2R(G)$, with equality if and only if $G \cong K_n$.

Let $n$ and $k$ be two integers such that $n \geq k \geq 1$. We denote the graph obtained from identifying the center of the star $K_{1,n-k}$ with a vertex of the complete graph $K_k$ by $K_k \bullet K_{1,n-k}$. In particular, if $k \in \{1, 2\}$, $K_k \bullet K_{1,n-k} \cong K_{1,n-1}$; if $k = n$, $K_k \bullet K_{1,n-k} \cong K_n$. The primary aim of this note is to prove stronger versions of Theorems 1.1-1.3, noting the inequalities in (1).

Theorem 1.4. For a graph $G$ of order $n$ without isolated vertices, $\text{col}(G) \leq 2R'(G)$, with equality if and only if $G \cong K_k \bullet K_{1,n-k}$ for some $k \in \{1, \ldots, n\}$.

Corollary 1.5. For a graph $G$ of order $n$ without isolated vertices, $\chi(G) \leq 2R'(G)$, with equality if and only if $G \cong K_k \bullet K_{1,n-k}$ for some $k \in \{1, \ldots, n\}$.

Corollary 1.6. For a graph $G$ of order $n$ without isolated vertices, $\chi_l(G) \leq 2R'(G)$, with equality if and only if $G \cong K_k \bullet K_{1,n-k}$ for some $k \in \{1, \ldots, n\}$.

Corollary 1.7. For a graph $G$ of order $n$ without isolated vertices, $\text{col}(G) \leq 2H(G)$, with equality if and only if $G \cong K_n$.

The proofs of these results will be given in the next section.

A complete $k$-coloring of a graph $G$ is a $k$-coloring of the graph such that for each pair of different colors there are adjacent vertices with these colors. The achromatic number of $G$, denoted by $\psi(G)$, is the maximum number $k$ for which the graph has a complete $k$-coloring. Clearly, $\chi(G) \leq \psi(G)$ for a graph $G$. In general, $\text{col}(G)$ and $\psi(G)$ are incomparable. Tang et al. [19] proved that for a graph $G$, $\psi(G) \leq 2R(G)$.

In Section 3, we prove new bounds for the coloring and achromatic numbers of a graph in terms of its spectrum, which strengthen $\text{col}(G) \leq 2R(G)$ and $\psi(G) \leq 2R(G)$. In Section 4, we provide an example and propose two related conjectures.
2 The proofs

For convenience, an edge $e$ of a graph $G$ may be viewed as a 2-element subset of $V(G)$ and if a vertex $v$ is an end vertex of $e$, we denote the other end of $e$ by $e \setminus v$. Moreover, $\partial_G(v)$ denotes the set of edges which are incident with $v$ in $G$.

First we need the following theorem, which will play a key role in the proof of Theorem 1.4.

**Theorem 2.1.** If $v$ is a vertex of $G$ with $d(v) = \delta(G)$, then

$$R'(G) - R'(G - v) \geq 0,$$

with equality if and only if $N_G(v)$ is an independent set of $G$ and $d(w) < d(v)$ for all $w \in N(v) \setminus \{v\}$.

**Proof.** Let $k = d_G(v)$. The result is trivial for $k = 0$. So, let $k > 0$ and let $N_G(v) = \{v_1, \ldots, v_k\}$ and $d_i = d_G(v_i)$ for each $i$. Without loss of generality, we may assume that $d_1 \geq \cdots \geq d_k$. Let

$E_i = \{e : e \in \partial_G(v_i) \setminus \{vv_i\} \text{ such that } d_G(e \setminus v_i) < d_G(v_i) \text{ and } e \not\subseteq N_G(v)\},$

$E'_i = \{e : e \in \partial_G(v_i) \setminus \{vv_i\} \text{ such that } d_G(e \setminus v_i) = d_G(v_i) \text{ and } e \subseteq N_G(v)\},$

and for an integer $i \geq 2$,

$E_i = \{e : e \in \partial_G(v_i) \setminus \{vv_i\} \text{ such that } d_G(e \setminus v_i) < d_G(v_i) \text{ and } e \not\subseteq N_G(v)\},$

$E'_i = \{e : e \in \partial_G(v_i) \setminus \{vv_i\} \text{ such that } d_G(e \setminus v_i) = d_G(v_i) \text{ and } e \subseteq N_G(v)\}\setminus (\cup_{j=1}^{i-1} E'_j)$.

Let $a_i = |E_i|$ and $b_i = |E'_i|$ for any $i \geq 1$. Since $a_i + b_i \leq d_i - 1$,

$$R'(G) - R'(G - v) = \sum_{i=1}^{k} \frac{1}{d_i} - \sum_{i=1}^{k} \frac{a_i + b_i}{d_i(d_i - 1)} \geq 0,$$

with equality if and only if $a_i + b_i = d_i - 1$ for all $i$, i.e., $N_G(v)$ is an independent set of $G$ and $d(w) < d(v)$ for all $w \in N(v) \setminus \{v\}$.

□

**Lemma 2.2.** If $T$ is a tree of order $n \geq 2$, then $R'(T) \geq 1$, with equality if and only if $T = K_{1,n-1}$.

**Proof.** By induction on $n$. It can be easily checked that $R'(K_{1,n-1}) = 1$. So, the assertion of the lemma is true for $n \in \{2, 3\}$. So, we assume that $n \geq 4$. Let $v$ be a leaf of $T$. Then $T - v$ is a tree of order $n - 1$. By Theorem 2.1, $R'(T) \geq R'(T - v) \geq 1$. If $R'(T) = 1$, then $R'(T - v) = 1$, and by the induction hypothesis, $T - v \cong K_{1,n-2}$. Let $u$ be the center of $T - v$. We claim that $vu \in E(T)$. If this is not, then $v$ is adjacent to a leaf of $T - v$, say $x$, in $T$. Thus $d_T(x) = 2$. However, since $R'(T) = R'(T - v)$ and $u \in N_T(x)$, by Theorem 2.1 that $d_T(x) > d_T(u) = n - 2 \geq 2$, a contradiction. This shows that $vu \in E(T)$ and $T \cong K_{1,n-1}$.

□
The proof of Theorem 1.4:

Let \( v_1, v_2, \ldots, v_n \) be an ordering of all vertices of \( G \) such that \( v_i \) is a minimum degree vertex of \( G_i = G - \{v_{i+1}, \ldots, v_n\} \) for each \( i \in \{1, \ldots, n\} \), where \( G_n = G \) and \( d_{G_i}(v_i) = \delta(G) \). It is well known that \( \text{deg}(G) = \max\{d_{G_i}(v_i) : 1 \leq i \leq n\} \) (see Theorem 12 in [1]). Let \( k \) be the maximum number such that \( \text{deg}(G) = d_{G_k}(v_k) \), and \( n_k \) the order of \( G_k \). By Theorem 2.1,

\[
2R'(G) \geq 2R'(G_{n-1}) \geq \cdots \geq 2R'(G_k).
\]

Moreover,

\[
2R'(G_k) = \sum_{uv \in E(G_k)} \left( \frac{2}{\max\{d_{G_k(u)}, d_{G_k(v)}\}} \right) \geq \sum_{uv \in E(G_k)} \frac{2}{\Delta(G_k)} \geq \frac{\Delta(G_k) + (n_k - 1)\delta(G_k)}{\Delta(G_k)} \geq \delta(G_k) + 1 = \text{col}(G).
\]

It follows that

\[
\text{if } R'(G_k) = \delta(G_k) + 1, \text{ then } G_k \cong K_k.
\]

Now assume that \( \text{col}(G) = 2R'(G) \). Observe that \( \text{col}(G) = \max\{\text{col}(G_i) : G_i \) is a component of \( G \} \) and \( R'(G) = \sum_i R'(G_i) \). Thus, by the assumption that \( \text{col}(G) = 2R'(G) \) and \( G \) has no isolated vertices, \( G \) is connected and \( \text{col}(G) \geq 2 \).

If \( \text{col}(G) = 2 \), then \( G \) is a tree. By Corollary 2.1, \( G \cong K_{1,n-1} \). Next we assume that \( \text{col}(G) \geq 3 \). By (4) and (5) we have \( R'(G) = \cdots = R'(G_k), G_k \cong K_k \) and thus \( \text{col}(G) = k \). We show \( G \cong K_k \cdot K_{1,n-k} \) by induction on \( n - k \). It is easy to check that \( \text{col}(K_n) = n + 1 = 2R'(K_n) \) and thus the result is for \( n - k = 0 \). If \( n - k = 1 \), then \( G_k = G - v_n \) and \( R'(G_k) = R'(G) \), by Theorem 2.1, \( N_G(v_n) \) is an independent set of \( G \). Combining with \( G_k \cong K_k \) (\( k \geq 3 \), \( d_{G_n}(v_n) = 1 \). Thus \( G = K_k \cdot K_{1,1} \).

Next assume that \( n - k \geq 2 \). We consider \( G_{n-1} \). Since \( \text{col}(G_{n-1}) = 2R'(G_{n-1}) \), by the induction hypothesis \( G_{n-1} \cong K_k \cdot K_{1,n-1-k} \). Without loss of generality, let \( N_{G_{n-1}}(v_{k+1}) = \cdots = N_{G_{n-1}}(v_{n-1}) = \{v_k\} \). So, it remains to show that \( N_G(v_n) = \{v_k\} \).

Claim 1. \( N_G(v_n) \cap \{v_{k+1}, \ldots, v_{n-1}\} = \emptyset \).

If it is not, then \( \{v_{k+1}, \ldots, v_{n-1}\} \subseteq N_G(v_n) \), because \( d_G(v_n) = \delta(G) \). By Theorem 2.1, \( d_G(v_{n-1}) > d_G(v_k) \). But, in this case, \( d_G(v_{n-1}) = 2 < d_G(v_k) \), a contradiction.
By Claim 1, $N_G(v_n) \subseteq \{v_1, \ldots, v_k\}$. Since $N_G(v_n)$ is an independent set and $\{v_1, \ldots, v_k\}$ is a clique of $G$, $|N_G(v_n) \cap \{v_1, \ldots, v_k\}| = 1$. If $N_G(v_n) = \{v_i\}$, where $i \neq k$, then $d_G(v_i) = k$. By Theorem 2.1, $d_G(v_i) > d_G(v_k)$. However, $d_G(v_k) \geq n - 2 \geq k$, a contradiction. This shows $N_G(v_n) = \{v_k\}$ and hence $G \cong K_k \cdot K_{1,n-k}$.

It is straightforward to check that

$$2R'(K_k \cdot K_{1,n-k}) = 2 \times \left(k - \frac{2}{2} + 1\right) = k = \text{col}(G).$$

**The proofs of Corollaries 1.5 and 1.6:**

By (3) and Theorem 1.4, we have $\chi(G) \leq \chi_l(G) \leq 2R'(G)$. If $\chi(G) = 2R'(G)$ (or $\chi_l(G) = 2R'(G)$), then $\text{col}(G) = 2R'(G)$. By Theorem 1.4, $G \cong K_k \cdot K_{1,n-k}$. On the other hand, it is easy to check that

$$\chi(K_k \cdot K_{1,n-k}) = \chi_l(K_k \cdot K_{1,n-k}) = k = 2R'(K_k \cdot K_{1,n-k}).$$

**The proof of Corollary 1.7:**

By (1) and Theorem 1.4, we have $\text{col}(G) \leq 2H(G)$. If $\text{col}(G) = 2H(G)$, then $\text{col}(G) = 2R'(G)$. By Theorem 1.4, $G \cong K_k \cdot K_{1,n-k}$. It can be checked that $k = 2H(K_k \cdot K_{1,n-k})$ if and only if $k = n$, i.e., $G \cong K_n$.

### 3 Spectral bounds

#### 3.1 Definitions

Let $\mu = \mu_1 \geq \ldots \geq \mu_n$ denote the eigenvalues of the adjacency matrix of $G$ and let $\pi, \nu$ and $\gamma$ denote the numbers (counting multiplicities) of positive, negative and zero eigenvalues respectively. Then let

$$s^+ = \sum_{i=1}^{\pi} \mu_i^2 \text{ and } s^- = \sum_{i=n-\nu+1}^{n} \mu_i^2.$$ 

Note that $\sum_{i=1}^{n} \mu_i^2 = s^+ + s^- = tr(A^2) = 2m$.

#### 3.2 Bounds for $\psi(G)$ and $\text{col}(G)$

**Theorem 3.1.** For a graph $G$, $\psi(G) \leq 2m/\sqrt{s^+} \leq 2m/\mu \leq 2R(G)$.

**Proof.** Ando and Lin [1] proved a conjecture due to Wocjan and Elphick [22] that $1 + s^+/s^- \leq \chi$ and consequently $s^+ \leq 2m(\chi - 1)/\chi$. It is clear that $\psi(\psi - 1) \leq 2m$. Therefore:
Taking square roots and re-arranging completes the first half of the proof.
Favaron et al. [7] proved that \( R(G) \geq m/\mu \). Therefore \( \psi(G) \leq 2m/\mu \leq 2R(G) \).

Note that for regular graphs, \( 2m/\mu = 2R' = 2H = 2R = n \) whereas for almost all regular graphs \( 2m/\sqrt{s^+} < n \).

**Lemma 3.2.** For all graphs, \( \text{col}(\text{col} - 1) \leq 2m \).

**Proof.** As noted above, \( s^+ \leq 2m(\chi - 1)/\chi \) and \( \chi(\chi - 1) \leq 2m \). Therefore:

\[
\sqrt{s^+}(\sqrt{s^+} + 1) = s^+ + \sqrt{s^+} \leq \frac{2m(\chi - 1)}{\chi} + \frac{2m}{\chi} = 2m.
\]

We can show that \( \text{deg}(G) \leq \mu(G) \) as follows.

\[
\text{deg}(G) = \max(\delta(H) : H \subseteq G) \leq \max(\mu(H) : H \subseteq G) \leq \mu(G).
\]

Therefore \( \text{col}(G) \leq \mu + 1 \), so:

\[
\text{col}(\text{col} - 1) \leq \mu(\mu + 1) \leq \sqrt{s^+}(\sqrt{s^+} + 1) \leq 2m.
\]

We can now prove the following theorem, using the same proof as for Theorem 3.1.

**Theorem 3.3.** For a graph \( G \), \( \text{col}(G) \leq 2m/\sqrt{s^+} \leq 2m/\mu \leq 2R(G) \).

### 3.3 Bounds for \( s^+ \)

The proof of Lemma 3.2 uses that \( s^+ + \sqrt{s^+} \leq 2m \), from which it follows that:

\[
\sqrt{s^+} \leq \frac{1}{2}(\sqrt{8m+1} - 1).
\]

This strengthens Stanley’s inequality [18] that:

\[
\mu \leq \frac{1}{2}(\sqrt{8m+1} - 1).
\]

However, \( \sqrt{2m-n+1} \leq (\sqrt{8m+1} - 1)/2 \), and Hong [10] proved for graphs with no isolated vertices that \( \mu \leq \sqrt{2m-n+1} \). Elphick et al. [5] recently conjectured that for connected graphs \( s^+ \leq 2m - n + 1 \), or equivalently that \( s^- \geq n - 1 \).
4 Example and Conjectures

A Grundy $k$-coloring of $G$ is a $k$-coloring of $G$ such that each vertex is colored by the smallest integer which has not appeared as a color of any of its neighbors. The Grundy number $\Gamma(G)$ is the largest integer $k$, for which there exists a Grundy $k$-coloring for $G$. It is clear that for any graph $G$,

$$\Gamma(G) \leq \psi(G) \leq \chi(G) \leq \Delta(G) + 1. \quad (6)$$

Note that each pair of $\text{col}(G)$ and $\psi(G)$, or $\text{col}(G)$ and $\Gamma(G)$, or $\psi(G)$ and $\Delta(G)$ is incomparable in general.

As an example of the bounds discussed in this paper, if $G = P_4$, then

$$\chi(G) = \text{col}(G) = 2$$
$$\Gamma(G) = \psi(G) = \Delta(G) + 1 = 2R'(G) = 3$$
$$2H(G) = 3.67 \text{ and } 2R(G) = 3.83$$
$$\mu_1(G) = 1.618 \text{ and } \mu_2 = 0.618$$
$$2m/\mu_1(G) = 3.71 \text{ and } 2m/\sqrt{s^+} = 3.46.$$ 

We believe the following conjectures to be true.

**Conjecture 4.1.** For any graph $G$, $\psi(G) \leq 2R'(G)$.

In view of (6), a more tractable conjecture than the above is as follows.

**Conjecture 4.2.** For any graph $G$, $\Gamma(G) \leq 2R'(G)$.

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