On small deviations of stationary Gaussian processes and related analytic inequalities

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Abstract

Let \( \{X_j, j \in \mathbb{Z}\} \) be a Gaussian stationary sequence having a spectral function \( F \) of infinite type. Then for all \( n \) and \( z \geq 0 \),

\[
P\left( \sup_{j=1}^{n}|X_j| \leq z \right) \leq \left( \int_{-z/\sqrt{G(f)}}^{z/\sqrt{G(f)}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right)^n,
\]

where \( G(f) \) is the geometric mean of the Radon-Nykodim derivative of the absolutely continuous part \( f \) of \( F \). The proof uses properties of finite Toeplitz forms. Let \( \{X(t), t \in \mathbb{R}\} \) be a sample continuous stationary Gaussian process with covariance function \( \gamma > 0 \). We also show that there exists an absolute constant \( K \) such that for all \( T > 0, a > 0 \) with \( T \geq \varepsilon(a) \),

\[
P\left( \sup_{0 \leq s, t \leq T}|X(s) - X(t)| \leq a \right) \leq \exp\left\{ -\frac{KT}{\varepsilon(a)p(\varepsilon(a))} \right\},
\]

where \( \varepsilon(a) = \min \{ b > 0 : \delta(b) \geq a \} \), \( \delta(b) = \min_{u \geq 1} \{ \sqrt{2(1 - \gamma((ub)))}, u \geq 1 \} \), and \( p(b) = 1 + \sum_{j=2}^{\infty} |2\gamma(jb) - \gamma((j-1)b) - \gamma((j+1)b)|/2(1 - \gamma(b)) \).

The proof is based on some decoupling inequalities arising from Brascamp-Lieb inequality, from which we also derive a general upper bound for the small values of stationary sample continuous Gaussian processes. A two-sided inequality for correlated suprema in the case of the Ornstein-Uhlenbeck process. We also establish an unexpected link between the Littlewood hypothesis and small values of cyclic Gaussian processes. In the discrete case, we obtain a general bound by combining Anderson’s inequality with a weighted inequality for quadratic forms. In doing so, we also clarify link between matrices with dominant principal diagonal and Geršgorin’s disks. Both approaches are developed and compared on examples.

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1 Introduction and preliminary results

The study of small deviations of continuous Gaussian processes and more general continuous processes is a very active domain of research. This is also a very specialized area, rich of many specific results, mainly concerning typical processes having strongly regular covariance structure, such as Brownian motion, Brownian sheet, fractional Brownian motions, integrated fractional Brownian motions, Hurst processes, ... This aspect of the theory has naturally many applications in statistics. It is also sometimes related to operator theory.

The small deviations problem for the class of stationary Gaussian processes is of particular interest, the way how stationary and mixing properties interact being notably not quite well understood. This is the main focus of this work. Let $X = \{X(t), t \in \mathbb{R}\}$ be throughout a sample continuous stationary Gaussian process with covariance function $\gamma(u) = \mathbb{E}X(t+u)X(t)$. The underlying problem is the study for small $z$ and $T$ large, $0 < z < z_0$, $T_0 \leq T < \infty$ say, of the probability

$$P\left\{ \sup_{0 \leq s, t \leq T} |X(s) - X(t)| \leq z \right\}.$$ 

One can also separately consider asymptotics for $T \to \infty$, $z$ being fixed, or $z \to 0$, $T$ fixed. The most celebrated example of stationary Gaussian process is naturally the Ornstein-Uhlenbeck process $U(t) = W(e^t)e^{-t/2}$, $t \in \mathbb{R}$, $W$ denoting the standard Brownian motion. And we know that for $z > 0$, there exist positive constants $K_1(z), K_2(z)$ such that for all $T \geq 1$

$$K_1(z)e^{-\lambda(z)T} \leq P\left\{ \sup_{0 \leq s \leq T} |U(s)| < z \right\} \leq K_1(z)e^{-\lambda(z)T}. \quad (1.1)$$

Further $\lambda(z) \sim \pi^2/4z^2$ as $z \to 0$. See Csáki (1994), Lemma 2.2. This precise estimate follows from earlier work of Newell in which this question is showed to be intimately linked to the Sturm-Liouville equation

$$\psi''(x) - x\psi'(x) = -\lambda\psi(x), \quad \psi(-z) = \psi(z) = 0. \quad (1.2)$$

Let $\lambda_1 \leq \lambda_2 \leq \ldots$ and $\psi_1(x), \psi_2(x), \ldots$ respectively denote the eigenvalues and normed eigenfunctions of Eq. (1.2). Here $\lambda_i, \psi_j$ depend on $z$ and it is known that $\psi_1, \psi_2, \ldots$ form an orthonormal sequence with respect to the weight function $e^{-x^2/2}$. And $\lambda(z) = \lambda_1$ in (1.1). According to Newell (1962),

$$P\left\{ \sup_{0 \leq s \leq t} |U(s)| < z \right\} = \frac{1}{(2\pi)^{1/2}} \sum_{k=1}^{\infty} e^{-\lambda_k t} \left( \int_{-z}^{z} \psi_k(x)e^{-x^2/2} dx \right)^2.$$
For many purposes, the weaker estimate below suffices, and is moreover simpler to establish: for $T \geq T_0$, $0 \leq z \leq z_0$

$$e^{-K_1 \frac{T}{z^2}} \leq \mathbb{P}\left\{ \sup_{0 \leq s, t \leq T} |U(s) - U(t)| \leq z \right\} \leq e^{-K_2 \frac{T}{z^2}}, \quad (1.3)$$

$K_1, K_2$ being absolute constants. The lower bound part follows from Talagrand's general lower bound in Talagrand (1993). See Aurzada and Lifshits (2008), Weber (2012) for recent improvements. As to the upper bound part, it can be for instance deduced from Stolz's estimate (1996) (Corollary 1.2) or (2.8). The small deviations problem of $X$ naturally relies on both the behavior of $\gamma(u)$ near 0 and near infinity. At this regard, it is worth observing that the (exponential) rate of decay of $\gamma(u)$ near infinity is hidden in (1.1) and (1.3). Let us begin with the discrete case. Let $X = \{X_j, j \in \mathbb{Z}\}$ be a stationary Gaussian sequence. If the sequence $X$ is i.i.d., then obviously for all $x$

$$\mathbb{P}\left\{ \sup_{j=1}^n |X_j| \leq x \right\} = \mathbb{P}\{|X_1| \leq x\}^n.$$

We show in Theorem 2.4 that the slightly weaker estimate

$$\mathbb{P}\{|X_1| \leq x\}^n \leq \mathbb{P}\left\{ \sup_{j=1}^n |X_j| \leq x \right\} \leq \mathbb{P}\{|X_1| \leq \gamma x\}^n, \quad (1.4)$$

in which $\gamma$ may depend on $X$, but not on $n$ nor $x$, holds for a large class of stationary Gaussian sequences, which is rather unexpected. It will suffice in effect, that the geometric mean of the Radon-Nykodim derivative of the absolutely continuous part of its spectrum is finite. This defines a very large class of stationary Gaussian sequences. Beyond this case, that question seems to lose much interest. For instance if $X$ has absolutely continuous spectrum with spectral density $f$, and $f$ has infinite geometric mean, then $X$ is deterministic. This yields extremely strong dependence between the successive variables $X_j$. The condition that $\sum_{n=1}^{\infty} |\mathbb{E} X_0 X_n| < \infty$ is also sufficient for the validity of (1.4).

We will study these questions through essentially two different ways: one is probabilistic, although based on a real analysis device, and the other of spectral nature. We shall also compare them on representative classes of examples. The first is the correlation approach, which is based on powerful correlation inequalities derived from Brascamp–Lieb’s inequality. We notably establish for the continuous parameter case a rather general upper bound integrating the rate of decay of $\gamma(u)$ near infinity.
A first relevant and little known correlation estimate is Gebelein’s inequality (Beska and Ciesielski, 2006; Veraar, 2009). Let $\nu$ be the centered normalized Gauss measure on $\mathbb{R}$. Let $(U,V)$ be a Gaussian pair with $U \overset{D}{=} V \overset{D}{=} \nu$ and let $\rho = \mathbb{E} UV$. Then for any $f,h \in L^2(\nu)$ with $\mathbb{E} f(U) = \mathbb{E} h(V) = 0,$

$$|\mathbb{E} f(U)h(V)| \leq |\rho|\|f\|_2\|h\|_2.$$  

An analog result is Nelson’s hyper-contractive estimate, which can be reformulated as follows

$$|\mathbb{E} f(U)h(V)| \leq \|f\|_p\|h\|_q,$$

where $(p-1)(q-1) \geq \rho^2$. One can take in particular $p = q = 1 + |\rho|$. We have given Guerra, Rosen and Simon formulation of Nelson’s estimate (1975), which was originally stated for the Ornstein-Uhlenbeck process. They also established for this process that

$$\left| \mathbb{E} \prod_{j=1}^n f_j(U(ja)) \right| \leq \prod_{j=1}^n \|f_j(U(0))\|_p,$$

for all integers $n$, where $a > 0$ and $p = (1 - e^{-na})^{-1}(1 + e^{-na})$. A more general form was later proved in a deep work (1982) by Klein, Landau and Shucker. See Lemma 3.1. As already mentioned, the main ingredient is a real analysis inequality due to Brascamp–Lieb (1976), which asserts that for any complex-valued functions $f_j$ and real numbers $1 \leq p_j \leq \infty$, $j = 1,\ldots,k$ with $\sum_{j=1}^k 1/p_j = n \leq k$, $n$ integer, if $f_j \in L^{p_j}(\mathbb{R})$, then for any vectors $a^j$ in $\mathbb{R}^n$, $j = 1,\ldots,k$,

$$\left| \int_{\mathbb{R}^n} \prod_{j=1}^k f_j(\langle a^j, x\rangle) \, dx \right| \leq D \prod_{j=1}^k \|f_j\|_{p_j},$$

and the constant $D$ is computable explicitly (see Brascamp and Lieb, 1976, Theorems 1,5). Inequalities of this sort were intensively investigated in the recent years, see Barthe (1998) for instance and references therein.

The second approach is based on the theory of finite Toeplitz forms, especially strong Szegö limit theorem. We obtain comparable upper and lower estimates under simple conditions regarding the spectral density of the stationary Gaussian sequence. It seems by the way rather evident to assert that any reasonable attempt for developing a small deviation theory of stationary Gaussian processes cannot be undertaken without including a
large account from the asymptotic theory of eigenvalues of finite Toeplitz forms. This can be well illustrated as follows. Let \( X \) having a spectral density function \( f(t) \) and put
\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikt} f(t) dt, \quad k \in \mathbb{Z}.
\]
Let \( \Gamma_n \) denote the covariance matrix of \( (X_1, \ldots, X_n) \), obviously \( \mathbb{E} X_j X_k = c_{j-k} \). The study of the asymptotic distribution of its eigenvalues, as \( n \) tends to infinity, can be equivalently viewed as the one of the finite Toeplitz forms
\[
T_n(f) = \sum_{j,k=0}^{n} c_{j-k} a_j \overline{a}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{n} a_k e^{ikt} \right|^2 f(t) dt, \quad n = 0, 1, \ldots
\]
where \( a_k \) are arbitrary complex numbers. This is an old question. Let \( m \) and \( M \) denote the essential lower and upper bound \( f \) respectively. Assume for instance that \( 0 < m \leq M < \infty \). Denote by \( \lambda_1^n, \ldots, \lambda_{n+1}^n \), the eigenvalues of the Hermitian form \( T_n(f) \), namely the roots of the characteristic function \( T_n(f - \lambda) = 0 \). As \( \lambda_j^n \geq m > 0 \), it follows that \( \det(\Gamma_n) > 0 \). It is well-known that the sets
\[
\{ \lambda_j^n \} \quad \text{and} \quad \left\{ f\left( -\pi + \frac{2j\pi}{n+2} \right) \right\}, \quad n \to \infty,
\]
are equally distributed in the Weyl sense. According to Szegö’s limit theorem (Grenander and Szegö, 1958, Chapter 5), for any continuous function \( F \) defined on \([m, M]\),
\[
\lim_{n \to \infty} \frac{F(\lambda_1^n) + \ldots + F(\lambda_{n+1}^n)}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(f(t)) dt. \quad (1.7)
\]
A well-known fact easily derived from (1.7) is that
\[
\lim_{n \to \infty} \left[ \det(\Gamma_n) \right]^{\frac{1}{n+1}} = \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(t) dt \right\}. \quad (1.8)
\]
Indeed, as \( \det(\Gamma_n) = \lambda_1^n \ldots \lambda_{n+1}^n \), it suffices to apply (1.7) with \( F(\lambda) = \log \lambda \), \( \lambda > 0 \). This has immediate consequences concerning the small values of \( (X_1, \ldots, X_n) \), \( n \to \infty \).

Finally we also examine the non-stationary case and use the convenient notion of matrices with dominant principal diagonal. This direction was explored by Li and Shao (see Li and Shao, 2004, see also the survey Li and
Shao, 2005 and the references therein, as well the earlier work of Marcus, 1968), and some improvements of their results are established. We also clarify the relevance of this notion in the context of eigenvalues of Hermitian matrices by linking it with Geršgorin’s Theorem.

We believe that the used approaches are potentially more developable and should certainly allow to improve on the general knowledge of small deviations in the stationary case.

**Basic estimates.** Recall well-known Kathri-Sidák’s inequality implying for any Gaussian vector \((X_1, \ldots, X_J)\) that

\[
\prod_{j=1}^{J} \mathbb{P}\{|X_j| \leq z\} \leq \mathbb{P}\left\{\sup_{j=1}^{J} |X_j| \leq z\right\}.
\] (1.9)

Now recall Boyd’s precise estimate of Mills’ ratio \(R(x) = e^{x^2/2} \int_{x}^{\infty} e^{-t^2/2} dt\): for all \(x \geq 0\),

\[
\frac{\pi}{\sqrt{x^2 + 2\pi + (\pi - 1)x}} \leq R(x) \leq \frac{\pi}{\sqrt{(\pi - 2)x^2 + 2\pi + 2x}}.
\] (1.10)

Notice that both bounds tend to \((\pi/2)^{1/2}\) as \(x\) tends to 0. Let \(g\) denote throughout a standard Gaussian random variable. Mill’s ratio is clearly directly related to the Laplace transform of \(g\) since for any real \(\lambda \geq 0\),

\[
\mathbb{E} e^{-\lambda|g|} = \left(\frac{2}{\pi}\right)^{1/2} R(\lambda).
\]

It follows that \(\mathbb{E} e^{-\lambda|g|} \sim (2/\pi)^{1/2} \lambda^{-1/2}, \lambda \to \infty\). Further, for all \(\lambda > 0\)

\[
\mathbb{E} e^{-\lambda|g|} \leq \min\left(\frac{\sqrt{2}}{\lambda\sqrt{\pi}}, 1\right).
\] (1.11)

We refer for instance to Weber (2009) Section 10.1 for these facts and more details.

**Notation–Convention.** All Gaussian random variables, Gaussian sequences or processes we consider are implicitly assumed to be centered. Further, \(g_1, g_2, \ldots\) will always denote a sequence of i.i.d. Gaussian standard random variables, and the Ornstein-Uhlenbeck process is denoted by \(U(t), t \geq 0\). The notation \(f(t) \asymp h(t)\) near \(t_0 \in \mathbb{R}\) means that for \(t\) in a neighborhood of \(t_0, c|h(t)| \leq |f(t)| \leq C|h(t)|\) for some constants \(0 < c \leq C < \infty\). Finally, we convince that \(1/0 = \infty\).
2 Main results

Let \( \{X(t), t \in \mathbb{R}\} \) be a stationary Gaussian process with continuous sample paths and let \( \gamma(u) = \mathbb{E} X(0)X(u) \) denote its covariance function.

**Theorem 2.1.** Assume that \( \sum_{j=1}^{\infty} |\gamma(jb)| < \infty \), for all \( b > 0 \). Then there exists an absolute constant \( K \) such that for all \( T > 0 \), \( a > 0 \) with \( T \geq \varepsilon(a) \),

\[
\mathbb{P} \left\{ \sup_{s,t \in [0,T]^d} |X(s) - X(t)| \leq a \right\} \leq \exp \left\{ -\frac{KT}{\varepsilon(a)p(\varepsilon(a))} \right\},
\]

where \( \varepsilon(a) = \min \{ b > 0 : \delta(b) \geq a \} \), \( \delta(b) = \min \{ \sqrt{2(1 - \gamma((ub))}, u \geq 1 \} \)
and

\[
p(b) = 1 + \sum_{j=2}^{\infty} \frac{2\gamma(jb) - \gamma((j-1)b) - \gamma((j+1)b)}{2(1 - \gamma(b))}.
\]

**Remark 2.1.** It is natural to check whether Theorem 2.1 contains known upper bounds for the Ornstein-Uhlenbeck process. It can be shown in this case that \( p(b) \) tends to some positive finite limit as \( b \) tends to 0. Indeed,

\[
p(b) = 1 + \frac{2 - e^{b/2} - e^{-b/2}}{2(1 - e^{-b/2})} \sum_{j=2}^{\infty} e^{-jb/2} = 1 + e^{-b/2} \frac{(1 - e^{b/2}) + (1 - e^{-b/2})}{2(1 - e^{-b/2})^2}.
\]

By developing near \( b = 0 \), we have

\[
(1 - e^{b/2}) + (1 - e^{-b/2}) = \left( 1 - \left[ 1 + \frac{b}{2} + \frac{1}{4} \frac{b^2}{2} \right] \right) + \left( 1 - \left[ 1 - \frac{b}{2} + \frac{1}{4} \frac{b^2}{2} \right] \right) + \mathcal{O}(b^3)
\]

\[
= -\frac{b^2}{4} + \mathcal{O}(b^3),
\]

so that

\[
p(b) \sim 1 + e^{-b/2} \frac{b^2}{8(1 - e^{-b/2})^2} \sim 1 + e^{-b/2} \frac{b^2}{8(b^2/4)} \sim \frac{3}{2}, \quad b \to 0.
\]

Moreover \( \delta(b) = \sqrt{2(1 - e^{-b/2})} \sim \sqrt{b} \) as \( b \to 0 \). Theorem 2.1 thus implies the upper bound part of (1.3).

We also obtain a general upper bound for stationary sample continuous Gaussian processes.
Theorem 2.2. Let \( \{X_t, t \in \mathbb{R}^d\} \) be a stationary sample continuous Gaussian process. Assume that condition (3.2) is fulfilled. For any \( z > 0 \), any bounded interval \( B \) of \( \mathbb{R}^d \),
\[
\mathbb{P}\left\{ \sup_{t \in B} |X_t| \leq z \right\} \leq \left( e^{p} \mathbb{P}\{ |g| < z \}\right)^{|B|/p}.
\]

Consider now the similar question for correlated suprema. Let \( I_1, \ldots, I_J \) be bounded, pairwise disjoint intervals, and associate to them the sets
\[
C_j(X) = \left\{ \sup_{t \in I_j} |X(t)| \leq z_j \right\}, \quad j = 1, \ldots, J
\]
where \( z_j \) are positive reals. By Hölder’s inequality,
\[
\mathbb{P}\left\{ \bigcap_{j=1}^{J} C_j(X) \right\} \leq \prod_{j=1}^{J} \mathbb{P}\{C_j(X)\}^\sigma, \quad \sigma = \frac{1}{J}.
\]

In general that inequality cannot be improved. In particular there is no reason for \( \sigma \) to be independent of \( J \). However when \( X = U \), namely for the Ornstein-Uhlenbeck process, this can be much improved.

Proposition 2.1. For any pairwise disjoint bounded intervals \( I_1, \ldots, I_J \), any positive reals \( z_j \),
\[
\prod_{j=1}^{J} \mathbb{P}\left\{ \sup_{t \in I_j} |U(t)| \leq z_j \right\}^{1/p} \leq \mathbb{P}\left\{ \bigcap_{j=1}^{J} \left\{ \sup_{t \in I_j} |U(t)| \leq z_j \right\} \right\} \leq \prod_{j=1}^{J} \mathbb{P}\{\sup_{t \in I_j} |U(t)| \leq z_j \}^{p},
\]
where
\[
p = \frac{1 + e^{-|I_1| - \cdots - |I_J|}}{1 - e^{-|I_1| - \cdots - |I_J|}}.
\]

Now let \( I_j = n_j + I \) where \( I \) is some fixed bounded interval and \( n_j \uparrow \infty \) with \( j \) and such that \( n_{j+1} - n_j \geq |I|, j \geq 1 \). Proposition 2.1 can be strengthened in that case. Put
\[
M(I, n_1, \ldots, n_J) = \sup_{t \in I} \sup_{1 \leq j \leq J} |U(t + n_j)|
\]
Theorem 2.3 (Existence of the Limit). For \( z > 0 \),
\[
\lim_{J \to \infty} \frac{\log \mathbb{P}\{M(I, n_1, \ldots, n_J) \leq z\}}{J} = \log \mathbb{P}\left\{ \sup_{t \in I} |U(t)| \leq z \right\}.
\]

We will also establish that

Corollary 2.1. For \( z > 0 \),
\[
\lim_{J \to \infty} \frac{\log \mathbb{P}\{M(I, n_1, \ldots, n_J) \leq z\}}{J} = \inf_{J \geq 1} \frac{\log \mathbb{P}\{M(I, n_1, \ldots, n_J) \leq z\}}{J},
\]
where \( M(I, n_1, \ldots, n_J) \) is defined in (2.1).

Now we pass to results concerning stationary sequences satisfying Szegő spectral type conditions. Recall some basic facts. Let \((X_1, \ldots, X_n)\) be a Gaussian vector with associated covariance matrix (or Gram matrix) \( \Gamma = \{\gamma_{i,j}\}_{1 \leq i,j \leq n} \). Assume that \( \Gamma \) is invertible and let \( \Gamma_j = \Gamma(X_1, \ldots, X_j) \) be the \( j \)-th principal minor of \( \Gamma \). Define \( \rho_j = \det(\Gamma_{j-1})/\det(\Gamma_j), j = 1, \ldots, n, \Gamma_0 = 1 \). By Gram-Schmidt orthogonalization process we obtain from \( X_1, \ldots, X_n \) an orthogonal sequence \( Y_1, \ldots, Y_n \), which may be expressed as follows

\[
Y_j = \frac{1}{\sqrt{\Gamma_{j-1}\Gamma_j}} \begin{bmatrix} \gamma_{1,1} & \gamma_{2,1} & \cdots & \gamma_{j,1} \\ \gamma_{1,2} & \gamma_{2,2} & \cdots & \gamma_{j,2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1,j-1} & \gamma_{2,j-1} & \cdots & \gamma_{j,j-1} \\ X_1 & X_2 & \cdots & X_j \end{bmatrix}, \quad j = 1, \ldots, n.
\]

Developing along the last line gives

\[
Y_j = \mathcal{L}(X_1, \ldots, X_{j-1}) + \rho_j X_j \quad j = 1, \ldots, n.
\]

Here we have denoted by \( \mathcal{L} \) some linear form of the random variables \( X_1, \ldots, X_{j-1} \). From this and (1.9), we easily deduce the following basic estimate: for \( z_j > 0 \) arbitrary,

\[
\prod_{j=1}^n \left( \int_{-z_j}^{z_j} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right) \leq \mathbb{P}\left\{ \sup_{j=1}^n \frac{|X_j|}{z_j} \leq 1 \right\} \leq \prod_{j=1}^n \left( \int_{-z_j^{\sqrt{\rho_j}}}^{z_j^{\sqrt{\rho_j}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right).
\]

(2.2)

The search of suitable bounds of \( \rho_j \) is consequently a fundamental question. There are some special inequalities involving the Gram determinants \( \det(\Gamma_j) \). For instance (Kurepa, 1967, p.382–383),

\[
\det \Gamma(X_1, \ldots, X_j) \leq \prod_{i=1}^j \|X_i\|_2^2.
\]
\[ \det \Gamma(X_1, \ldots, X_j) \leq \det \Gamma(X_1, \ldots, X_k) \det \Gamma(X_{k+1}, \ldots, X_j). \]  

(2.3)

Hence,

\[ \rho_j \geq \frac{1}{\|X_j\|^2}. \]  

(2.4)

See the upper bound (2.10), see also Davies (1965), Everitt (1957). Let \( \{X_j, j \in \mathbb{Z}\} \) be a Gaussian stationary sequence with spectral function \( F \). It is natural to wonder which spectral conditions may be imposed on \( F \) to get upper and lower bounds to the probability \( \mathbb{P}\{\sup_{j=1}^n |X_j| \leq z\} \) (or to its logarithm), which are comparable and remain valid for some range of values of type \( 0 < z \leq z_0, n \geq n_0 \). Let

\[ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\lambda} F(d\lambda), \]

so that \( \mathbb{E} X_j X_k = c_{j-k} \). The corresponding Hermitian forms are also called the Toeplitz forms associated with \( F \), and we have the representation

\[ T_n = \sum_{\mu, \nu=0}^{n} c_{\nu-\mu} u_\mu u_\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u_0 + u_1 e^{i\lambda} + u_2 e^{2i\lambda} + \ldots + u_n e^{in\lambda} \right|^2 F(d\lambda). \]

Recall that \( F \) is said of finite type if its range consists of a finite number of values. In the opposite case, it is called of infinite type. The forms \( T_n \) are positive definite unless \( F \) is of finite type (Grenander and Szegö, 1958, §1.11). If \( F \) is of infinite type, all determinants of the forms \( T_n \) are positive, namely \( \det \Gamma_n > 0 \) for all \( n \).

**Theorem 2.4.** Assume \( F \) is of infinite type. Let \( f \) be the Radon-Nykodim derivative of the absolutely continuous part of \( F \), and put

\[ G(f) = \begin{cases} \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(t) dt \right\} & \text{if } \log f(t) \text{ is integrable} \\ 0 & \text{otherwise.} \end{cases} \]

Then for all \( n \) and \( z > 0 \),

\[ \left( \int_{-z}^{z} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right)^n \leq \mathbb{P}\{\sup_{j=1}^n |X_j| \leq z\} \leq \left( \int_{-z/\sqrt{G(f)}}^{z/\sqrt{G(f)}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right)^n. \]

**Remark 2.2.** More explicit formulations can be deduced from estimate (1.10). The quantity \( \exp \left\{ 1/2\pi \int_{-\pi}^{\pi} \log f(t) dt \right\} \) is by definition the geometric mean of \( f \). The condition that \( \log f \) be integrable is satisfied by a remarkable
class of functions. Let \( u(z) = \sum_{n=0}^{\infty} c_n z^n \) be an analytic function, regular in the open unit disk \( |z| < 1 \) and belonging to \( H_2 \), namely the integral
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{it})|^2 \, dt
\]
is bounded for every \( r < 1 \). This is equivalent to the fact that \( \sum_{n=0}^{\infty} |c_n|^2 < \infty \). Then the limit
\[
\lim_{r \to 1^{-}} u(re^{it}) = h(t)
\]
exists for almost every \( t \). Let \( f(t) = |h(t)| \). We furthermore have that \( \log f \) is (Lebesgue) integrable and (see Grenander and Szegö, 1958, §1.13),
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |\log |f(e^{it})|| \, dt \leq \sum_{n=0}^{\infty} |c_n|^2.
\]

**Remark 2.3.** A direct use of (1.8) would have provided a less precise result. Much later, Szegö also showed that a rate of convergence can be associated to (1.8) in presence of reasonable smoothness assumptions. Suppose that \( f \) has a derivative which satisfies a Lipschitz condition of order \( \alpha \), \( 0 < \alpha < 1 \). Then,
\[
\lim_{n \to \infty} \left[ \log \det \Gamma_n - \frac{n+1}{2\pi} \int_{-\pi}^{\pi} \log f(t) \, dt \right] = \frac{1}{\pi} \iint |h(z)|^2 \, d\sigma,
\]
where the function \( h(z) \) is analytic in \( z \) and is defined by the equality
\[
h(z) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \log f(\lambda) \frac{1 + ze^{-i\lambda}}{1 - ze^{-i\lambda}} \, d\lambda,
\]
and the integration in the right-hand-side in (2.5) is along the unit circle. See Libkind (1972) for some generalization.

Consider the following examples.

**Example 2.1.** Assume that the spectral density exists, \( f(t) = a_0 + \sum_{n \in \mathbb{Z}^*} a_n e^{int}, a_{-n} = a_{|n|}, \) and
\[
\sum_{n \in \mathbb{Z}^*} |a_n| < |a_0|.
\]
Let \( m \) and \( M \) denote the essential lower and upper bound \( f \) respectively. Then \( m > 0 \). The conclusion of Theorem 2.4 holds. The link between a
(square integrable) spectral function and its corresponding correlation function being given by
\[ f(t) = \mathbb{E} X_0^2 + \sum_{n \in \mathbb{Z}^*} (\mathbb{E} X_0 X_{|n|}) e^{int}, \]
this holds in particular for the Ornstein-Uhlenbeck sequence \( \{U(n), n \geq 0\} \).
Indeed, in this case \( f(t) = 1 + \sum_{n \in \mathbb{Z}^*} e^{-|n|/2} e^{int} \).

**Remark 2.4.** In the lacunary case \( f(t) = a_0 + \sum_{k \in \mathbb{Z}^*} a_k e^{i k t} \), \( \lambda > 1, a_{-k} = a_{|k|} \), condition \( m > 0 \) is equivalent to (2.6), since the maxima of the polynomials \( \sum_{|k| \leq N} a_k e^{i k t} \) verify
\[
\sup_{-\pi \leq t \leq \pi} \left| \sum_{|k| \leq N} a_k e^{i k t} \right| = \sum_{|k| \leq N} |a_k|.
\]
This follows from a well-known theorem of Sidon.

**Example 2.2.** Let \( b > 0 \), and consider again \( Y_j = U(jb) - U((j - 1)b), \)
\( j = 1, 2, \ldots \). We compute the corresponding geometric mean. Recall that
\[
\mathbb{E} Y_t Y_{t+u} = \begin{cases} 
2(1 - e^{-b/2}) & \text{if } u = 0, \\
(2 - e^{b/2} - e^{-b/2}) e^{-ub/2} & \text{if } u \geq 1.
\end{cases}
\]
Let \( r = e^{-b/2} \). Then \( \delta := 2 - e^{b/2} - e^{-b/2} = -(1 - r)^2 / r - b^2 / 4 + O(b^3), \)
as \( b \to 0 \). Now introduce the Poisson kernel
\[
gr(t) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{int} = \frac{1 - r^2}{1 - 2r \cos t + r^2}, \quad 0 < r < 1.
\]
It is well-known that log \( g_r(t) \) is integrable. Further,
\[
\int_0^\pi \log(1 - 2r \cos x + r^2) \, dx = 0, \quad (= \pi \log r^2 \text{ if } r > 1). \quad (2.7)
\]
The spectral function, call it \( h(t) \), verifies
\[
h(t) = 2(1 - r) + \delta \sum_{u \in \mathbb{Z}^*} r^{|u|} e^{i ut} = 2(1 - r) - \delta + \delta \sum_{u \in \mathbb{Z}} r^{|u|} e^{i ut}
\]
\[
= 2(1 - r) + \frac{(1 - r)^2}{r} - \frac{(1 - r)^2}{r} \frac{1 - r^2}{1 - 2r \cos x + r^2}
\]
\[
= \frac{1 - r^2}{r} \left( 1 - \frac{(1 - r)^2}{1 - 2r \cos x + r^2} \right) \]
\[
\frac{2(1-r^2)(1-\cos x)}{1-2r\cos x + r^2}.
\]

We have from (2.7)
\[
\int_{-\pi}^{\pi} \log h(t)dt = \int_{-\pi}^{\pi} \log \left(2(1-r^2)(1-\cos x)\right)dt
= 2\pi \log[2(1-r^2)] + \int_{-\pi}^{\pi} \log \left(2\sin^2 \frac{t}{2}\right)dt
= 2\pi \log[4(1-r^2)] + 4 \int_{0}^{\pi} \log \sin \frac{t}{2}dt.
\]

But \[\int_{0}^{\pi} \log \sin \frac{t}{2}dt = -\pi \log 2.\] Therefore
\[\frac{1}{2\pi} \int_{-\pi}^{\pi} \log h(t)dt = \log[2(1-r^2)] + \log 2 - 2 \log 2 = \log(1-r^2).
\]

Thus \(G(h) = 1-r^2 = 1-e^{-b}\) and by Theorem 2.4,
\[
P\left\{ \sup_{j=1}^{n} |U(jb) - U((j-1)b)| \leq z \right\} \leq \left( \int_{\frac{-z}{\sqrt{1-e^{-b}}}}^{\frac{z}{\sqrt{1-e^{-b}}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right)^n. \tag{2.8}
\]

More generally, let \(\xi(t), t \geq 0\) be a Gaussian stationary process and \(b\) being a positive real, let \(\xi_b(j) = \xi((j+1)b) - \xi(jb), \ j = 0, 1, \ldots\) Let also \(\gamma(h) = \mathbb{E}\xi(0)\xi(h), \gamma(0) = 1, \sigma(h) = \sqrt{2(1-\gamma(h))}.\)

**Proposition 2.2.** Assume that \(\gamma(h)\) is convex decreasing and let \(f = -\gamma'.\) Then \(f(t) = 2\sin(t/2)g(t)\) where
\[
g(t) < \frac{\sigma^2(b) + \sigma^2(2b) + \ldots + \sigma^2((m-1)b)}{m} + \sigma^2(mb)
\]
as \(t \to +0,\) and we write \(m = |\pi/|t||\) for brevity. Further \(\log f\) is integrable.

**Remark 2.5.** Assume \(f\) be integrable, \(f \not\equiv 0.\) The condition that \(\log f\) be integrable characterizes the fact that there exists an analytic function \(h(z)\) of the class \(H_2\) such that \(f(t) = |h(z)|^2, \ z = e^{it}.\) This is well-known extension of Fejér-Riesz’s representation theorem for non negative trigonometric polynomials. It also characterizes the property that \(\{X_j, j \in \mathbb{Z}\}\) be non-deterministic.

Recall that \(\Gamma_j = \Gamma(X_1, \ldots, X_j).\) Let \(E_k\) be the subspace of \(L^2\) linearly generated by \(X_1, \ldots, X_{j-1}\) and put
\[
\vartheta_j = \|X_j - E_{j-1}\|,
\]
namely the distance from \(X_j\) to \(E_{j-1}.\) Theorem 2.4 can reformulated in a more abstract and less handy way, using the characteristics \(\vartheta_j.\)
Proposition 2.3. i) Let \((X_1, \ldots, X_n)\) be Gaussian with invertible covariance matrix. Then
\[
\mathbb{P}\left\{ \sup_{j=1}^{n} \frac{|X_j|}{z_j} \leq 1 \right\} \leq \prod_{j=1}^{n} \left( \int_{-\frac{z_j}{\sqrt{2}}}^{\frac{z_j}{\sqrt{2}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right).
\]

ii) Let \(\{X_j, j \in \mathbb{Z}\}\) be a Gaussian stationary sequence having an absolutely continuous spectrum, with spectral density function \(f\). Then \(\vartheta_j^{-2} = \sum_{k=0}^{j} |\varphi_k(0)|^2\) where \(\{\varphi_k, k \in \mathbb{Z}\}\) is the orthonormal sequence of polynomials associated to the weight function \(f(x)\).

See also Lifshits (1995), Proposition 3, Section 3 where a more complicated proof is given. For applications of strong Szegő limit theorems to linear prediction of stationary processes, we refer to Chapter 10 of Grenander and Szegő (1958), which is entirely devoted to this question.

Finally, we also consider the non-stationary case. For an important class of matrices the parameters \(\rho_j\) in (2.2) turn up to be easily controlable. An \(n \times n\) matrix \(A = \{a_{i,j}, 1 \leq i, j \leq n\}\) has dominant principal diagonal if
\[
|a_{i,i}| > \sum_{j=1}^{n} |a_{i,j}|, \quad i = 1, 2, \ldots, n. \tag{2.9}
\]

This notion already appeared in Minkowski and Hadamard works (see the overview in Taussky, 1988). Matrices with dominant principal diagonal define a quite remarkable class: they are invertible and their determinants are easy to estimate. Put for \(i = 1, 2, \ldots, n,\)
\[
A_i = \sum_{j=1}^{n} |a_{i,j}|, \quad m_i = |a_{i,i}| - A_i, \quad M_i = |a_{i,i}| + A_i.
\]

The following basic estimate is due to Price (1951, Theorem 1), the lower bound being previously proved by Ostrowski in 1951, (see also Brenner, 1954; Feingold and Varga, 1962; Haynsworth, 1953; Ky Fan, 1971 for various refinements).
\[
0 < m_1 \ldots m_n \leq |\det(A)| \leq M_1 \ldots M_n.
\]

If \(A\) is a Gram matrix, it follows from this and inequality (2.4) that
\[
\rho_j \leq \frac{(1 + \tau)^{j-1}}{a_{j,j}}, \quad \text{where} \quad \tau = \max_i \frac{A_i}{a_{i,i}} < 1. \tag{2.10}
\]
Then by (2.10) and (2.2),
\[
\mathbb{P}\left\{ \sup_{j=1}^{n} \frac{|X_j|}{z_j} \leq 1 \right\} \leq \prod_{j=1}^{n} \left( \int_{-\frac{z_j}{\sqrt{a_{j,j}(1+\tau)}}}^{\frac{z_j}{\sqrt{a_{j,j}(1+\tau)}}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \right)^{j-1}.
\] (2.11)

This can be however improved.

\textbf{Proposition 2.4.} Let \((X_1, \ldots, X_n)\) be a Gaussian vector and assume that for some \(r < 1\),
\[
\sum_{j=1}^{n} |E X_i X_j| \leq r E X_i^2, \quad i = 1, \ldots, n.
\] (2.12)

Then,
\[
\mathbb{P}\left\{ \sup_{j=1}^{n} |X_j| \leq z \right\} \leq \prod_{j=1}^{n} \mathbb{P}\left\{ |X_j| \leq \frac{z}{\sqrt{1-r}} \right\}.
\]

Condition (2.12) can be compared with condition (2.3) in Li and Shao (2004) where a bound of \(\sup_{j=1}^{n} X_j\) is given. This condition however has to be slightly modified.

\textbf{Remark 2.6.} Condition (2.9) has to be related with famous Geršgorin’s theorem, which states that the eigenvalues of an \(n \times n\) matrix with complex entries lie in the union of the closed disks (Geršgorin disks)
\[
|z - a_{i,i}| \leq A_i \quad (i = 1, 2, \ldots, n)
\]

in the complex plane. This result has naturally many concrete applications. An example to the analysis of flutter phenomenon in aircraft design is described in Taussky (1988). There is an analog result due to Brauer on ovals of Cassini stating that
\[
|z - a_{i,i}| |z - a_{j,j}| \leq A_i A_j \quad (i, j = 1, 2, \ldots, n, \ i \neq j).
\]

See Brauer (1947). In relation with this, we have that if
\[
|a_{i,i}| |a_{k,k}| > A_i A_k \quad (i, k = 1, 2, \ldots, n, \ i \neq k)
\] (2.13)

then \(\text{det}(A) > 0\). Note that the relations (2.13) imply \(|a_{i,i}| > A_i\) for all \(i\) but one. See also Varga (1965).
Matrices with dominant principal diagonal are used in a crucial way in Marcus (1968) starting from (2.2), see proof of Lemma 2. Assume $\gamma(u)$ is convex on $[0, \delta]$ for some $\delta > 0$, and let $\sigma^2(x) = 2(1 - \gamma(x))$. Let also $t_0 < t_1 < \ldots < t_n$ with $t_n - t_0 \leq \delta$. By using convexity of $\gamma$, $\mathbb{E}(X(t_i) - X(t_{i-1})(X(t_j) - X(t_{j-1})) \leq 0$, if $j \neq i$, so that

$$A_i := \sum_{j=1}^{n} \left| \mathbb{E}(X(t_i) - X(t_{i-1})(X(t_j) - X(t_{j-1})) \right|$$

$$= -\mathbb{E}(X(t_i) - X(t_{i-1})[X(t_n) - X(t_i) + X(t_{i-1}) - X(t_0)]$$

$$= \sigma^2(t_i - t_{i-1}) + \frac{1}{2}[\sigma^2(t_{i-1} - t_0) - \sigma^2(t_i - t_0) + \sigma^2(t_n - t_i) - \sigma^2(t_{n-1} - t_i)]$$

$$< \sigma^2(t_i - t_{i-1}).$$

However the ratio $A_i/\sigma^2(t_i - t_{i-1})$ has to be estimated in order to adjust with assumption (2.12), and we don’t see how this can be done. It seems therefore that inequality (7) (and thereby (8)) in Marcus (1968) needs a correction. A strictly weaker estimate can be deduced from (2.11). A comparable estimate (without absolute values) however trivially follows from Slepian’s lemma since the process has negatively correlated increments, see Li and Shao (2005) Theorem 4.5.

**Final remark.** Although not presented here, the results admit some extensions to Gaussian random fields defined on $\mathbb{R}^n$ with values in $\mathbb{R}^d$.

## 3 Intermediate results and proofs

We begin with recalling some decoupling inequalities (Klein et al., 1982, Theorems 1 and 2) due to Klein, Landau and Shucker, and which turn up to be not so known. Introduce a definition. Write for $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, $dt = dt_1 \ldots dt_d$, $\|t\| = (t_1^2 + \ldots + t_d^2)^{1/2}$ and let $0 = (0, \ldots, 0)$. A continuous function $r : \mathbb{R}^d \to \mathbb{R}$ is said to be Riemann approximable if,

$$\lim_{a \downarrow 0} \sum_{n \in \mathbb{Z}^d} |r(an)| = \int_{\mathbb{R}^d} |r(t)| dt.$$

It suffices for instance that $|r(t)| \leq C(1 + \|t\|)^{-\delta}$, for some $C > 0$ and $\delta > d$.

**Lemma 3.1.** a) Let $X = \{X_t, t \in \mathbb{Z}^d\}$ be a stationary Gaussian process with finite decoupling coefficient $p$, that is:

$$p = \sum_{k \in \mathbb{Z}^d} \frac{\mathbb{E}|X_0X_k|}{\mathbb{E}X_0^2} < \infty. \quad (3.1)$$
Let \( \{f_k, k \in \mathbb{Z}^d\} \) be a sequence of complex-valued measurable functions. Then for each finite subset \( J \) of \( \mathbb{Z}^d \),

\[
\left| \mathbb{E} \prod_{j \in J} f_j(X_j) \right| \leq \prod_{j \in J} \| f_j(X_0) \|_p.
\]

b) Let \( \{X_t, t \in \mathbb{R}^d\} \) be a stationary Gaussian process, continuous in mean, with Riemann approximable covariance function. Let \( V \) be a \( \mathbb{C} \)-valued measurable function of a real variable. Assume that \( V(X_0) \) is integrable. Then, for all bounded measurable subsets \( B \) of \( \mathbb{R}^d \),

\[
\left| \mathbb{E} \exp \left\{ \int_B V(X_t)dt \right\} \right| \leq \left\| \exp \{V(X_0)\} \right\|_p |B|,
\]

where

\[
p = \int_{\mathbb{R}^d} \frac{\mathbb{E} (X_0 X_t)}{\mathbb{E} X_0^2} dt < \infty,
\]

and \( |B| \) denotes the Lebesgue measure of \( B \).

In either case, the proof relies on inequality (1.6). It is of matter to briefly explain its principle. At first, a similar result (see Lemma 3.3) is established for cyclic stationary Gaussian processes. The proof is next achieved by approximating \( X \) with cyclic stationary Gaussian processes. A key observation is then that

\[
r_N(n) = \sum_{k \in \mathbb{Z}^d} r(n + kN), \quad \text{where} \quad r(u) = \mathbb{E} X_0 X_u,
\]

is, under condition (3.1), an \( N \)-periodic covariance function, and \( \lim_{N \to \infty} r_N(n) = r(n) \) for all \( n \), which is a remarkable fact. The proof for the continuous parameter case is similar.

Proof of Theorem 2.1. Notice that for each fixed real \( b > 0 \), the Gaussian sequence

\[
\xi_b(j) = X(jb) - X((j - 1)b), \quad j = 1, 2, \ldots
\]

is stationary. Let indeed \( \ell, u \geq 1 \), then

\[
\mathbb{E} \xi_b(\ell) \xi_b(\ell + u) = \mathbb{E} \left( X((\ell + u)b) - X((\ell + u - 1)b) \right) \left( X(\ell b) - X((\ell - 1)b) \right) = 2\gamma(ub) - \gamma((u - 1)b) - \gamma((u + 1)b),
\]

which only depends on \( u \). It has finite decoupling coefficient, and more precisely

\[
\sum_{j=1}^{\infty} \frac{\left| \mathbb{E} \xi_b(1) \xi_b(j) \right|}{\mathbb{E} \xi_b(1)^2} = 1 + \sum_{j=2}^{\infty} \frac{2\gamma(jb) - \gamma((j - 1)b) - \gamma((j + 1)b)}{2(1 - \gamma(b))} = p(b) < \infty.
\]
Further if $F$ denotes the spectral function of $X$, $\gamma(u) = \int_{\mathbb{R}} e^{iu\lambda} F(d\lambda)$, then

$$\mathbb{E} \xi_b(\ell) \xi_b(\ell + u) = \int_{\mathbb{R}} e^{-i\lambda ub} e^{ib\lambda} - 1 |^2 F(d\lambda).$$

And

$$\sum_{j=1}^{\infty} \frac{|\mathbb{E} \xi_b(1) \xi_b(j)|}{\mathbb{E} \xi_b(1)^2} = 1 + \frac{\int_{\mathbb{R}} \sum_{j=2}^{\infty} e^{-i\lambda ub} e^{ib\lambda} - 1 |^2 F(d\lambda)}{\int_{\mathbb{R}} e^{ib\lambda} - 1 |^2 F(d\lambda)}.$$

Let $T \geq b$. Consider on $[0, T]$ the subdivision $t_j = jb$, $0 \leq j \leq n := \lfloor T/b \rfloor$. We have

$$\|X((j + u)b) - X(jb)\|_2^2 = 2(1 - \gamma((ub)) \geq \min_{u \geq 1} (1 - \gamma((ub))) = \delta^2(b).$$

Let $c = \sqrt{2/\pi}$. Let $a > 0$ and choose $b$ so that $\delta(b) \geq a$. By Lemma 3.1, recalling that $g$ denote a Gaussian standard random variable,

$$\mathbb{P}\left\{ \max_{1 \leq j \leq n} |\xi_b(j)| \leq a \right\} = \mathbb{E} \prod_{i=1}^{n} \chi_{[-a, a]}(\xi_b(j)) \leq \left( \prod_{i=1}^{n} \mathbb{P}\{ |\xi_b(j)| \leq a \} \right)^{1/p(b)}$$

$$\leq \mathbb{P}\left\{ |g| \leq \frac{a}{\delta(b)} \right\} = \left( \sqrt{\frac{2}{\pi}} \int_{0}^{\frac{a}{\delta(b)}} e^{-x^2/2} dx \right)^{n/p(b)}$$

$$\leq e^{n/p(b)} = e^{-\frac{T/b}{p(b)} \log \frac{1}{c}} \leq e^{-\frac{T}{2p(b)b} \log \frac{1}{c}}.$$

As

$$\mathbb{P}\left\{ \sup_{0 \leq s, t \leq T} |X(s) - X(t)| \leq a \right\} \leq \mathbb{P}\left\{ \max_{1 \leq j \leq n} |\xi_b(j)| \leq a \right\}$$

by taking $b = \varepsilon(a)$, we obtain

$$\mathbb{P}\left\{ \sup_{0 \leq s, t \leq T} |X(s) - X(t)| \leq a \right\} \leq e^{-K \frac{T}{\varepsilon(a)p(\varepsilon(a))}},$$

with $K = 1/2 \log 1/c = 1/4 \log \pi/2$.

**Remark 3.1.** A direct application of the decoupling inequality to the sequence $X(jb)$ instead of $X(jb) - X((j - 1)b)$ only provides a bound with a decoupling coefficient which may tend to infinity when $b \to 0$. So is in particular the case when $X$ is the Ornstein-Uhlenbeck process.
Proof of Theorem 2.2. Let $f : \mathbb{R} \to \mathbb{C}$ be measurable, such that $\mathbb{E}|f(X_0)| < \infty$, and let $\lambda, \theta$ be positive reals. Applying part b) of Lemma 3.1 with $V(x) = -\lambda f(x)$ gives

$$\mathbb{E} \exp \left\{ -\lambda \int_B f(X_t) dt \right\} \leq \left\| e^{-\lambda f(X_0)} \right\|_p^{|B|} = \left( \mathbb{E} e^{-p\lambda f(g)} \right)^{|B|/p}.$$ 

Thereby,

$$P \left\{ \int_B f(X_t) dt \leq \theta \right\} = P \left\{ -\lambda \int_B f(X_t) dt \geq -\lambda \theta \right\} \leq \min \left( e^{\lambda \theta} \mathbb{E} \exp \left\{ -\lambda \int_B f(X_t) dt \right\}, 1 \right) \leq \min \left( e^{\lambda \theta} \left( \mathbb{E} e^{-p\lambda f(g)} \right)^{|B|/p}, 1 \right).$$

Apply this to $f(x) = |x|^r$, $0 < r < \infty$. Put

$$\|X\|_{r,B} = \left( \frac{1}{|B|} \int_B |X_t|^r dt \right)^{1/r}, \quad \|X\|_{\infty,B} = \sup_{t \in B} |X_t|. \tag{3.3}$$

Notice first that $\|X\|_{\infty,B} = \lim_{r \to \infty} \|X\|_{r,B}$, almost surely, since $X$ is sample continuous. Take $\theta = z^r |B|$. This gives

$$P \{ \|X\|_{r,B} \leq z \} \leq \inf_{\lambda > 0} e^{\lambda z^r |B|} \left( \mathbb{E} e^{-p\lambda |g|^r} \right)^{|B|/p}.$$

Choose now $\lambda = z^{-r}$. Then

$$P \{ \|X\|_{r,B} \leq z \} \leq \left( e^p \mathbb{E} e^{-p |g|^r} \right)^{|B|/p}.$$

But

$$\lim_{r \to \infty} e^{-p |g|^r} \text{ a.s.} = \begin{cases} 1, & \text{if } |g| < z \\ 0, & \text{if } |g| > z. \end{cases} \tag{3.4}$$

Thus $p$ disappears from the limit. By using the dominated convergence theorem, we get

$$\lim_{r \to \infty} \mathbb{E} e^{-p |g|^r} = P\{ |g| < z \}.$$

Hence,

$$P \{ \|X\|_{\infty,B} \leq z \} \leq \liminf_{r \to \infty} P \{ \|X\|_{r,B} \leq z \} \leq \left( e^p P\{ |g| < z \} \right)^{|B|/p}.$$
Remark 3.2. (Ergodic maximal equality) Introduce the ergodic maximal function

\[ M^*(X) = \sup_{T>0} M_T(X) \quad \text{where} \quad M_T(X) = \frac{1}{T} \int_0^T |X_t|dt. \]

As a special case of a fine result from ergodic theory, namely Marcus-Petersen’s maximal equality for ergodic flows (Weber, 2009, p.133), we have

\[ P\{M^*(X) \leq \alpha\} = 0, \quad (3.5) \]

if \( \alpha < \sqrt{2/\pi} \). A slightly less precise result can be directly derived from the first part of the above proof, in which only assumptions of Lemma 3.1, part b) are used. A simple modification of this one, also yields for all \( \theta > 0, B \) with \( |B| > 0 \),

\[ P\left\{ \frac{1}{|B|} \int_B |X_t|dt \leq \theta \right\} \leq \min \left( e^{(2/\pi)^{1/2} \theta}, 1 \right) \frac{|B|}{\theta^p}. \]

Indeed, using (1.11) we have with \( c = (2/\pi)^{1/2} \),

\[ P\left\{ \int_B |X_t|dt \leq z \right\} = P\left\{ -\lambda \int_B |X_t|dt \geq -\lambda z \right\} \]
\[ \leq \min \left( e^{\lambda z} \mathbb{E} \exp \left\{ -\lambda \int_B |X_t|dt \right\}, 1 \right) \]
\[ \leq \min \left( e^{\lambda z} \left( \frac{c}{\lambda p} \right)^{|B|/p}, 1 \right). \]

Letting \( z = \theta |B|, \lambda = 1/p \theta \), we deduce

\[ P\left\{ \frac{1}{|B|} \int_B |X_t|dt \leq \theta \right\} \leq \min \left( e c \theta, 1 \right) \frac{|B|}{\theta^p}. \]

By taking \( B = [0, T] \), it follows that for all \( \theta < \sqrt{\pi/2}/e, (e \approx 2.71828 \text{ being the Neper number}) \)

\[ P\{M^*(X) \leq \theta\} \leq \limsup_{T \to \infty} P\left\{ \frac{1}{T} \int_0^T |X_t|dt \leq \theta \right\} \leq \limsup_{T \to \infty} (e \sqrt{2/\pi \theta})^{T/p} = 0. \]

As \( 2e > \pi \), this is slightly less precise than (3.5).
Proof of Proposition 2.1. The first inequality follows by proceeding by approximation. Indeed, by the general theory of Gaussian processes, there exists a non negative integrable random variable $Y$ and a continuous increasing function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Omega(0) = 0$ such that

$$
P \left\{ \forall s, t \in \bigcup_{j=1}^J I_j, \quad |U(s) - U(t)| \leq Y \Omega(|s - t|) \right\} = 1$$

Hence $\sup_{t \in I_j} |U(t)| \leq \sup_{\ell/N \in I_j} |U(\ell/N)| + Y \Omega(1/N) \ a.s.$. This suffices to imply

$$
P \{ C_j(U) \} = \lim_{N \to \infty} P \{ C_{j,N}(U) \} \quad \text{where} \quad C_{j,N}(U) = \left\{ \sup_{\ell/N \in I_j} \left| U \left( \frac{\ell}{N} \right) \right| \leq z_j \right\}.$$ 

Let $N > 0$ be some large integer. Let $\nu_N = \# \{ \ell : \ell/N \in \bigcup_{j=1}^J I_j \}$ and set

$$p_N = \frac{1 + e^{-\nu_N/N}}{1 - e^{-\nu_N/N}}.$$ 

By using inequality (1.9) and next inequality (1.5). Indeed

$$
P \left\{ \bigcap_{j=1}^J C_{j,N}(U) \right\} \geq \prod_{j=1}^J \prod_{\ell/N \in I_j} P \left\{ \left| U \left( \frac{\ell}{N} \right) \right| \leq z_j \right\}$$

$$= \prod_{j=1}^J \left( \prod_{\ell/N \in I_j} P \left\{ \left| U \left( \frac{\ell}{N} \right) \right| \leq z_j \right\} \right)^{1/p_N}$$

$$\geq \prod_{j=1}^J \left( P \left\{ \sup_{\ell/N \in I_j} \left| U \left( \frac{\ell}{N} \right) \right| \leq z_j \right\} \right)^{1/p_N}.$$ 

But

$$\lim_{N \to \infty} \frac{\nu_N}{N} = |I_1| + \ldots + |I_J|.$$ 

Therefore $p_N \to p$ with $N$. Letting $N$ tend to infinity in the above inequality allows to conclude.

Now, by using (1.5) again, we have

$$
P \left\{ \bigcap_{j=1}^J C_{j,N}(U) \right\} = \mathbb{E} \prod_{j=1}^J \prod_{\ell/N \in I_j} \chi \left\{ \left| U \left( \frac{\ell}{N} \right) \right| \leq z_j \right\}$$
\[
\leq \prod_{j=1}^{J} \prod_{N \in I_j} \mathbb{P} \left\{ \left| U \left( \frac{\ell}{N} \right) \right| \leq z_j \right\}^{p_N}
\]
(by (1.9))
\[
\leq \prod_{j=1}^{J} \mathbb{P} \left\{ \sup_{N \in I_j} \left| U \left( \frac{\ell}{N} \right) \right| \leq z_j \right\}^{p_N} = \prod_{j=1}^{J} \mathbb{P} \{ C_{j,N}(U) \}^{p_N}.
\]

Letting \( N \) tend to infinity in the above inequality achieves the proof.

Now let \( I_j = n_j + I \) where \( I \) is some fixed bounded interval and \( n_j \uparrow \infty \) with \( j \) and such that \( n_{j+1} - n_j \geq |I|, j \geq 1 \) and recall that
\[
M(I, n_1, \ldots, n_J) = \sup_{t \in I} \left| U(t + n_j) \right|
\]

PROOF OF THEOREM 2.3. We have from Proposition 2.1, for \( J \geq 1 \),
\[
\mathbb{P} \left\{ \sup_{t \in I} |U(t)| \leq \right\}^{J/p_J} \leq \mathbb{P} \left\{ M(I, n_1, \ldots, n_J) \leq z \right\} \leq \mathbb{P} \left\{ \sup_{t \in I} |U(t)| \leq z \right\}^{J/p_J},
\]
where \( p_J = 1 + e^{-J|I|}/(1 - e^{-J|I|}) \). Taking logarithms and using the fact that \( p_J \to 1 \) with \( J \) gives the result.

Introduce a notion. Let \( \varphi = \{ c_n, n \geq 1 \} \) be positive reals tending to \( c \geq 1 \). We say that a sequence \( \{ \varphi_n, n \geq 1 \} \) of real numbers is \( c \)-subadditive, if
\[
\varphi_{n_1 + \ldots + n_k} \leq c_{n_1 + \ldots + n_k} (\varphi_{n_1} + \ldots + \varphi_{n_k})
\]
for all integers \( n_1, \ldots, n_k, k \geq 1 \).

**Lemma 3.2. (Extended Subadditive Lemma)** If \( \{ \varphi_n, n \geq 1 \} \) is a \( c \)-subadditive sequence of real numbers, then
\[
\inf_{n \geq 1} \frac{\varphi_n}{n} \leq \liminf_{n \to \infty} \frac{\varphi_n}{n} \leq \limsup_{n \to \infty} \frac{\varphi_n}{n} \leq c^2 \inf_{n \geq 1} \frac{\varphi_n}{n}.
\]

When \( c_n \equiv 1 \), this is a well-known device having many applications, in ergodic theory notably.

**Proof.** It is a simple modification of the classical proof of the case \( c_n \equiv 1 \). Fix an arbitrary positive integer \( N \) and write \( n = j_n N + r_n \) with \( 1 \leq r_n \leq N \). Then,
\[
\inf_{n \geq 1} \frac{\varphi_n}{n} \leq \frac{\varphi_n}{n} \leq c_{j_n N + r_n} \frac{\varphi_n}{n} \leq c_{j_n N + r_n} \frac{\varphi_n}{j_n N} + c_{j_n N + r_n} \frac{\varphi_r}{n}
\]
\[
\leq c_{j_n N + r_n} c_{j_n N} \frac{\varphi_n}{j_n N} + c_{j_n N + r_n} \frac{\varphi_r}{n}.
\]
\[
\leq c_{jn}N + r_n c_{jn} N \frac{\varphi N}{N} + c_{jn}N + r_n \left( \max_{r \leq N} |\varphi_r| \right)/n.
\]

When \(n\) tends to infinity, we have that \(j_n/n \to 1/N\). As \(c_{jn}N + r_n \to c^2\), we get
\[
\inf_{n \geq 1} \frac{\varphi n}{n} \leq \liminf_{n \to \infty} \frac{\varphi n}{n} \leq \limsup_{n \to \infty} \frac{\varphi n}{n} \leq c^2 \frac{\varphi N}{N}.
\]

Since \(N\) was arbitrary, the lemma is proved.

**Proof of Theorem 2.3.** We have from Proposition 2.1, for \(J \geq 1\),
\[
P \left\{ \sup_{t \in I} |U(t)| \leq z \right\} \leq P \left\{ M(I, n_1, \ldots, n_J) \leq z \right\} \leq P \left\{ \sup_{t \in I} |U(t)| \leq z \right\}^{J/p_J},
\]
where \(p_J = 1 + e^{-J|I|}/1 - e^{-J|I|}\). Taking logarithms and using the fact that \(p_J \to 1\) with \(J\) gives the result.

**Proof of Corollary 2.1.** Apply this to \(\varphi_J = \log P\{M(I, n_1, \ldots, n_J) \leq z\}\). By Corollary 2.3 and stationarity,
\[
\varphi_{J+K} = \log P \left\{ \sup_{j \leq J + K} \sup_{t \in I} |U(n_j + t)| \leq z \right\} \leq \frac{1}{p_{J+K}} \log P \left\{ \sup_{t \in I} |U(t)| \leq z \right\}^{J+K}
\]
\[
= \frac{1}{p_{J+K}} \log \left( \prod_{j \leq J} P \left\{ \sup_{t \in I} |U(n_j + t)| \leq z \right\} \cdot \prod_{j \leq K} P \left\{ \sup_{t \in I} |U(n_j + t)| \leq z \right\} \right)
\]
\[
\leq \frac{1}{p_{J+K}} \log P \left\{ \sup_{j \leq J} \sup_{t \in I} |U(n_j + t)| \leq z \right\} \cdot P \left\{ \sup_{j \leq K} \sup_{t \in I} |U(n_j + t)| \leq z \right\}
\]
\[
= \frac{1}{p_{J+K}} (\varphi_J + \varphi_K).
\]

But \(p_J = 1 + e^{-J|I|}/1 - e^{-J|I|}\). Similarly, \(\varphi_{J_1 + \ldots + J_s} \leq 1/p_{J_1 + \ldots + J_s}(\varphi_{J_1} + \ldots + \varphi_{J_s})\). Thus \(\{g_n, n \geq 1\}\) is \(\mathfrak{c}\)-subadditive with \(\mathfrak{c} = \{p_J, J \geq 1\}\). Now \(p_J = 1 + e^{-J|I|}/1 - e^{-J|I|} \to 1\) as \(J\) tends to infinity. By Lemma 3.2, we deduce that
\[
\inf_{J \geq 1} \frac{\log P\{M(I, n_1, \ldots, n_J) \leq z\}}{J} \leq \liminf_{J \to \infty} \frac{\log P\{M(I, n_1, \ldots, n_J) \leq z\}}{J}
\]
\[
\leq \limsup_{J \to \infty} \frac{\log P\{M(I, n_1, \ldots, n_J) \leq z\}}{J}
\]
\[
\leq \inf_{J \geq 1} \frac{\log P\{M(I, n_1, \ldots, n_J) \leq z\}}{J}.
\]
Cyclic Gaussian Processes. As already mentioned, these processes played a key role in Klein et al. (1982). The following lemma, which we state for our need is the crux of the proof of Lemma 3.1. Although it is valid for cyclic stationary Gaussian processes \( \{X_t, t \in \mathbb{R}^d\} \) with an arbitrary period \((b_1, \ldots, b_d)\), we state it in the standard case of period \((1, \ldots, 1)\), namely with fundamental index \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d = [0, 1]^d \).

**Lemma 3.3.** (Klein et al., 1982, Theorem 3) Let \( \{X_t, t \in \mathbb{R}^d\} \) be a 1-periodic stationary Gaussian process, continuous in quadratic mean. Let \( V : \mathbb{R} \to \mathbb{C} \) be measurable and such that \( V(X_0) \) is integrable. Then, for all measurable subsets \( B \) of \( \mathbb{T}^d \),

\[
\left| \mathbb{E} \left( \exp \left\{ \int_B V(X_t) \right\} \right) \right| \leq \| \exp(V(X_0)) \|_p |B|
\]

where

\[
p = \int_0^1 \frac{\| \mathbb{E} X_0 X_t \|}{\mathbb{E} X_0^2} dt.
\]

**Proposition 3.1.** Let \( Y_t = \sum_{n=1}^N a_n (g_1^n \cos 2\pi nt + g_2^n \sin 2\pi nt) \), \( t \in \mathbb{T} = \mathbb{R} / \mathbb{Z} = [0, 1[ \), where \( a_n \) are reals and \( g_1^n, g_2^n \) are mutually independent Gaussian standard random variables. Let \( s^2 = \sum_{n=1}^N a_n^2 \). For \( \theta > 0 \) and \( B \subset \mathbb{T} \) interval of length \( |B| \),

\[
\mathbb{P} \left\{ \sup_{t \in B} |Y_t| \leq \theta \right\} \leq \left( e^{p} \mathbb{P} \left\{ |g| < \frac{\theta}{s} \right\} \right)^{|B|},
\]

where

\[
p = \int_0^1 \frac{\| \sum_{n=1}^N a_n^2 \cos 2\pi nt \|}{\sum_{n=1}^N a_n^2} dt.
\]

**Proof.** Notice that \( \mathbb{E} Y_0 Y_t = \sum_{n=1}^N a_n^2 \cos 2\pi n(s-t) \). The proof is very similar to that of Theorem 2.2, except that we have a different decoupling coefficient:

\[
p = \int_0^1 \frac{\| \mathbb{E} Y_0 Y_t \|}{\mathbb{E} Y_0^2} dt = \int_0^1 \frac{\| \sum_{n=1}^N a_n^2 \cos 2\pi nt \|}{\sum_{n=1}^N a_n^2} dt.
\]

We only indicate the necessary changes. The proof is identical with \( Y_t \) in place of \( X_t \) until (3.3), where there is a slight modification due to the fact that \( Y_0 = \sum_{n=1}^N a_n g_1^n \overset{\mathbb{D}}{=} s g \), \( (s^2 = \sum_{n=1}^N a_n^2) \). Using Tchebycheff’s inequality and Lemma 3.3, gives

\[
\mathbb{P} \left\{ \int_B f(Y_t) dt \leq \theta \right\} \leq \min \left( e^{\lambda \theta} \mathbb{E} \exp \left\{ - \lambda \int_B f(Y_t) dt \right\}, 1 \right)
\]
\[
\leq \min \left( e^{-\lambda \theta} \left( E e^{-p \lambda f(sg)} \right)^{\frac{|B|}{p}}, 1 \right).
\]

Applying this with \( f(x) = |x|^r \), \( \theta = z^r |B| \), \( \lambda = z^{-r} \) gives in exactly the same manner, with the notation (3.3),
\[
P\{ \| Y \|_{r,B} \leq z \} \leq \left( e^p E e^{-p(|g|)^r} \right)^{\frac{|B|}{p}}.
\]

Hence, by using (3.4),
\[
P\left\{ \sup_{t \in B} |Y_t| \leq z \right\} \leq \liminf_{r \to \infty} P\{ \| Y \|_{r,B} \leq z \} \leq \left( e^p \mathbb{P}\{|sg| < z\}\right)^{\frac{|B|}{p}}.
\]

An immediate consequence of Proposition 3.1 is that

**Corollary 3.1.** With the notation from Proposition 3.1, for \( z > 0 \),
\[
\int_0^1 \left| \sum_{n=1}^N a_n^2 \cos 2\pi n t \right| dt \geq \left( \sum_{n=1}^N a_n^2 \right)^{\frac{\log \frac{1}{e}}{\log \frac{1}{\mathbb{P}\{|g| < z\}}} \frac{|B|}{p}}.
\]

**Remark 3.3.** (Littlewood hypothesis) Let \( n_1 < n_2 < \ldots \) be integers. Consider the (generalized) Lebesgue constants
\[
\vartheta_N = \int_0^1 \left| \sum_{k=1}^N e^{2i\pi nk t} \right| dt, \quad N = 1, 2, \ldots
\]

Littlewood hypothesis (Olevskii, 1975, p.12 for instance) essentially concerns the behavior of Lebesgue constants of arbitrary ordered trigonometric systems, and can be formulated as follows: for any increasing sequence of integers,
\[
\vartheta_N \geq C \log N,
\]
where \( C > 0 \) is an absolute constant. This was proved independently by Konyagin (1981) and McGehee, Pigno and Smith (1981) in 1981. Consideration of the Dirichlet kernel shows that the above lower bound is best possible. See Zygmund (2002) p. 67.

We shall deduce from Corollary 3.1

**Corollary 3.2.** For all positive integers \( N \), all \( z > 0 \) and \( B \subset \mathbb{T} \) interval,
\[
\vartheta_N \geq N |B| \left( \frac{\log \frac{1}{\mathbb{P}\{|g| < z\}}}{\log \frac{1}{\mathbb{P}\{|g| < z\}} \frac{|B|}{p}} \right).
\]
Proof. Apply Corollary 3.1 with the choice $a_n = 1/\sqrt{N}$, if $n = n_k$ for some $k \leq N$, and equal to 0 otherwise. We deduce

$$
\vartheta_N \geq N \log \log \frac{1}{P\{|g| < z\}}
$$

as claimed.

The above link between $L^1$-norms of trigonometric sums and Gaussian random trigonometric sums, seems unexpected. This suggests to examine it more closely using results in Weber (2006, 2012). This question will be investigated elsewhere. We conclude with a remarkable example in which Anderson’s inequality is used and Talagrand’s well-known lower bound since the corresponding entropy numbers are very simple.

Corollary 3.3. There exists an absolute constant $C$ such that for any set of integers $J$,

$$
\int_0^1 \left| \sum_{n \in J} \frac{1}{n^2} \cos nt \right| \, dt \geq C \left( \sum_{n \in J} \frac{1}{n^2} \right)^2.
$$

Proof. Let

$$
X_t = \sum_{n=1}^\infty \frac{1}{n} \left( g_n^1 \cos nt + g_n^2 \sin nt \right), \quad Y_t = \sum_{n \in J} \frac{1}{n} \left( g_n^1 \cos nt + g_n^2 \sin nt \right)
$$

Then $E X_s^2 = \sum_{n=1}^\infty 1/n^2 = \pi^2/6$ and

$$
E X_s X_t = \sum_{n=1}^\infty \frac{\cos n(s-t)}{n^2} = \frac{3|s-t|^2 - 6\pi|s-t| + 2\pi^2}{12}.
$$

Thus $d^2(s, t) = E (X_s - X_t)^2 = \pi|s-t| - 1/2|s-t|^2 \sim \pi|s-t|$ as $|s-t| \to 0$. It follows that $N([0, 1], d, \varepsilon) \asymp \varepsilon^{-2}$. By using Talagrand’s lower bound (see Talagrand, 1993),

$$
P \left\{ \sup_{0 \leq t \leq 1} |X_t| \leq \varepsilon \right\} \geq e^{-K \varepsilon^{-2}}.
$$

Now since $Y$ and $X - Y$ are independent, by using Anderson’s inequality, we get

$$
P \left\{ \sup_{0 \leq t \leq 1} |X_t| \leq \varepsilon \right\} \leq P \left\{ \sup_{0 \leq t \leq 1} |Y_t| \leq \varepsilon \right\}.
$$
Therefore
\[ P \left\{ \sup_{0 \leq t \leq 1} |Y_t| \leq \varepsilon \right\} \geq e^{-K \varepsilon^{-2}}. \]

We have \( Y_0 = \sum_{n \in J} 1/n g_n^1 \) and \( s(J) = (\sum_{n \in J} 1/n^2)^{1/2} \),
\[ p = s(J)^{-2} \int_0^1 \left| \sum_{n \in J} \frac{1}{n^2} \cos nt \right| dt. \]

Applying Proposition 3.1 with \( B = [0, 1] \) gives,
\[ e^{-K \theta^{-2}} \leq P \left\{ \sup_{0 \leq t \leq 1} |Y_t| \leq \theta \right\} \leq e P \left\{ |g| < \frac{\theta}{s(J)} \right\}^{1/p}, \]

By taking logarithms in both sides, we get
\[ p \geq \frac{1}{\log \frac{1}{P\{ |g| < \frac{\theta}{s(J)} \}}} \frac{1}{1 + K \theta^{-2}}. \]

Consequently,
\[ \int_0^1 \left| \sum_{n \in J} \frac{1}{n^2} \cos nt \right| dt \geq \frac{s(J)^2 \theta^2}{\theta^2 + K} \log \frac{1}{P\{ |g| < \theta / s(J) \}}, \]

In particular, if \( \theta = s(J) \),
\[ \int_0^1 \left| \sum_{n \in J} \frac{1}{n^2} \cos nt \right| dt \geq C \frac{s(J)^4}{s(J)^2 + K} \geq C \left( \sum_{n \in J} \frac{1}{n^2} \right)^2, \]

since \( s(J)^2 \leq \pi^2 / 6 \).

Now we give proofs concerning results for stationary sequences verifying Szegő spectral type conditions.

Proof of Theorem 2.4. According to (1.9), only the second inequality has to be proven. We have the explicit formula
\[ \frac{1}{\rho_j} = \frac{\det(\Gamma_j)}{\det(\Gamma_{j-1})} = \min_p \int |p(e^{i\lambda})|^2 \mu(d\lambda), \]

where the minimum is taken over all polynomials \( p \) of degree \( j - 1 \), of type \( a_0 + a_1 z + \ldots + a_{j-1} z^{j-1} \) with \( |a_{j-1}| = 1 \). See (Grenander and Szegő,
Further, when \( j \) tends to infinity, these minima are decreasing and in fact

\[
\frac{\det(\Gamma_j)}{\det(\Gamma_{j-1})} \downarrow \exp \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(t) dt \right\}.
\]

Consequently, by (2.2) and monotonicity,

\[
\mathbb{P} \left\{ \sup_{j=1}^{n} |X_j| \leq z \right\} \leq \left( \int_{-z/\sqrt{G(f)}}^{z/\sqrt{G(f)}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right)^n.
\]

**Proof of Proposition 2.2.** Set \( \Delta_n = \sigma^2(nb) - \sigma^2((n-1)b) \), \( n = 1, 2, \ldots, \Delta_0 = 0 \). Then

\[
\mathbb{E} \xi_b(0) \xi_b(n) = \frac{1}{2} \left\{ -2\sigma^2(nb) + \sigma^2((n-1)b) + \sigma^2((n+1)b) \right\} = \frac{1}{2} \{ \Delta_{n+1} - \Delta_n \},
\]

and \( f(t) = \mathbb{E} \xi_b(0) + \sum_{n \in \mathbb{Z}} (\mathbb{E} \xi_b(0) \xi_b(|n|)) e^{int} = \sum_{n=0}^{\infty} (\Delta_{n+1} - \Delta_n) \cos nt \). By using Abel summation and the relation \( \cos jt - \cos(j + 1)t = 2 \sin t/2 \sin (2j + 1)t/2 \), \( f(t) \) can be rewritten as \( f(t) = 2 \sin t/2 g(t) \) where

\[
g(t) = \Delta_1 \sin \frac{3t}{2} + \Delta_2 \sin \frac{5t}{2} + \ldots
\]

As \( \sigma^2 \) is concave increasing, it follows that \( \Delta_1 \geq \Delta_2 \geq \ldots \) The behavior of sine series with non increasing coefficients were studied by Salem. We refer to Popov’s article (2003) for instance, for the result below (Telyakovskii’s estimate) and recent sharpenings,

\[
g(t) \asymp t \sum_{k=1}^{\lfloor \pi/|t| \rfloor} k \Delta_k \quad t \to +0.
\]

The constants involved in the symbol \( \asymp \) are absolute. By using again Abel summation,

\[
\sum_{k=1}^{m} k \Delta_k = \sigma^2(b) + \sigma^2(2b) + \ldots + \sigma^2((m-1)b) + m\sigma^2(mb).
\]

Letting \( m = \lfloor \pi/|t| \rfloor \), we deduce

\[
g(t) \asymp \frac{\sigma^2(b) + \sigma^2(2b) + \ldots + \sigma^2((m-1)b)}{m} + \sigma^2(mb),
\]
as $t \to +0$. Therefore

$$f(t) \asymp \sin \frac{t}{2} \quad t \to +0.$$  

It follows that $\log f$ is integrable, as claimed.

**Proof of Proposition 2.3.** First notice that

$$\Gamma_j = \Gamma_{j-1} \theta_j^2.$$  

For a reference, see Achieser and Glasman (1954) p.13. According to (2.2),

$$\mathbb{P} \left\{ \sup_{j=1}^{n} \frac{|X_j|}{z_j} \leq 1 \right\} \leq \prod_{j=1}^{n} \left( \int_{-\frac{z_j}{\theta_j}}^{\frac{z_j}{\theta_j}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right).$$

As to b), this follows from Grenander and Szegö (1958), (10) p.40.

We conclude by giving the proof of Proposition 2.4. We use the following general estimate for quadratic forms

**Lemma 3.4.** For any systems of reals $\{x_i\}, \{a_{i,j}\}$ with $a_{i,j} = a_{j,i}$,

$$\sum_{i=1}^{n} x_i^2 \left( a_{i,i} + \sum_{j \neq i}^{n} |a_{i,j}| \right) \geq \sum_{i,j=1}^{n} x_i x_j a_{i,j} \geq \sum_{i=1}^{n} x_i^2 \left( a_{i,i} - \sum_{j=1}^{n} |a_{i,j}| \right).$$

**Proof.** At first we have

$$\left| \sum_{1 \leq i < j \leq n} x_i x_j a_{i,j} \right| \leq \sum_{1 \leq i < j \leq n} \left( \frac{x_i^2 + x_j^2}{2} \right) |a_{i,j}|$$

$$\leq \frac{1}{2} \sum_{j=1}^{n} x_j^2 \left( \sum_{1 \leq \ell < j} |a_{\ell,j}| + \sum_{j < \ell \leq n} |a_{j,\ell}| \right).$$

$$= \sum_{j=1}^{n} x_j^2 \left( \sum_{1 \leq \ell \leq n} \sum_{\ell \neq j} |a_{j,\ell}| \right).$$

And next

$$\sum_{i,j=1}^{n} x_i x_j a_{i,j} = \sum_{i=1}^{n} x_i^2 a_{i,i} + 2 \sum_{1 \leq i < j \leq n} x_i x_j a_{i,j} \geq \sum_{i=1}^{n} x_i^2 \left( a_{i,i} - \sum_{\ell=1}^{n} |a_{i,\ell}| \right).$$
This yields the first inequality. The second follows similarly.

**Proof of Proposition 2.4.** Let \( 0 < \alpha < 1 - r, \ Y = \{ \sqrt{\alpha} a_{i,i} g_i, 1 \leq i \leq n \} \) and \( B = \{ \mathbb{E} X_i X_j - \mathbb{E} Y_i Y_j, 1 \leq i, j \leq n \} \). By applying Lemma 3.4 to \( B \), we get

\[
\sum_{i,j=1}^{n} x_i x_j (\mathbb{E} X_i X_j - \mathbb{E} Y_i Y_j) = \sum_{i,j=1}^{n} x_i x_j b_{i,j} = \sum_{i=1}^{n} x_i^2 \left( a_{i,i} (1 - \alpha) - \sum_{j=1 \atop j \neq i}^{n} |a_{i,j}| \right)
\]

\[
\geq \sum_{i=1}^{n} x_i^2 a_{i,i} (1 - \alpha - r) \geq 0.
\]

Thus by Anderson’s inequality Tong (1980) p.55, for any convex set \( C \) symmetric around 0,

\[
P\{ X \in C \} \leq P\{ Y \in C \}.
\]

By choosing \( C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq z\} \), we deduce

\[
P\left\{ \sup_{j=1}^{n} |X_j| \leq z \right\} \leq P\left\{ \sup_{j=1}^{n} |Y_j| \leq z \right\} = \prod_{j=1}^{n} P\left\{ |a_{j,j} g_j| \leq \frac{z}{\sqrt{\alpha}} \right\}.
\]

Letting next \( \alpha \) tend to \( 1 - r \), finally leads to

\[
P\left\{ \sup_{j=1}^{n} |X_j| \leq z \right\} \leq \prod_{j=1}^{n} P\left\{ |a_{j,j} g_j| \leq \frac{z}{\sqrt{1 - r}} \right\} = \prod_{j=1}^{n} P\left\{ |X_j| \leq \frac{z}{\sqrt{1 - r}} \right\}.
\]

as claimed.

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