Abstract: The goal of this work, motivated by the desire to understand causality in classical and quantum gravity, is an in depth investigation of causality in classical field theories with quasilinear equations of motion, of which General Relativity is a prominent example. Several modern geometric tools (jet bundle formulation of partial differential equations (PDEs), the theory of symmetric hyperbolic PDE systems, covariant constructions of symplectic and Poisson structures) and applies them to the construction of the phase space and the algebra of observables of quasilinear classical field theories. This construction is shown to be diffeomorphism covariant (using auxiliary background fields if necessary) using categorical tools in a strong parallel with the locally covariant field theory (LCFT) formulation of quantum field theory (QFT) on curved spacetimes. In this context, generalized versions of LCFT axioms become theorems of classical field theory, which includes a generalized Causality property. Considering deformation quantization as the connection to QFT, a plausible conjecture is made about the Causal structure of quantum gravity. In the process, conal manifolds are identified as the generalization of the causal structure of Lorentzian geometry to quasilinear PDEs. Several important concepts and results are generalized from Lorentzian to conal geometry. Also, the proof of compatibility of the Peierls formula for Poisson brackets and the covariant phase space symplectic structure for hyperbolic systems is generalized to now encompass systems with constraints and gauge invariance.

1. Introduction

Great strides have been made in the past decade in terms of the definition and perturbative construction quantum field theories (QFTs) in the algebraic framework [5]. Notable successes include the extension of perturbative renormalization
of non-linear field theories to arbitrary, curved (though non-dynamical), globally hyperbolic spacetimes and the treatment of gauge theories \[64, 47, 95\]. These ingredients are, for instance, sufficient to study perturbative general relativity (GR) in this framework. Unfortunately, deeply rooted in the formalism of QFT on curved spacetimes is the reliance on a non-dynamical background metric to supply the causal structure that defines the singularity and support structures of various \(n\)-point functions, time ordered products, and retarded products.

The objects in classical field theory that are most closely related to these QFT objects are the retarded and advanced Green functions of the dynamical field equations of motion linearized at some particular solution (the dynamical linearization point). But we know that, in classical gravity, the singularity and support structures of these classical Green functions change as a function of the metric that provides the dynamical linearization point. In perturbative GR, the dynamical metric can be expressed as \(g = \bar{g} + \kappa h\), where \(\bar{g}\) is a fixed background metric and \(h\) is the dynamical perturbation, \(\kappa^2\) is proportional to Newton’s gravitational constant, which is to be treated as the formal perturbation parameter. As already mentioned, the singularity structure of the Green function of the linearized gravitational wave equation depends on the full, dynamical linearization point \(g = \bar{g} + \kappa h\). But, to order \(O(\kappa^0)\), this singularity structure is fully determined by the non-dynamical background metric \(\bar{g}\).

Clearly, the necessary dependence of this singularity structure on the dynamical perturbations \(h\) will appear at higher orders in \(\kappa\). However, at least in the case of QFT \(n\)-point functions, it is not \(a\ priori\) clear what form this modifications would take or how they are to be recognized in the result of some higher order perturbative calculation. Given the tight connection between the singularity structure of Green functions and what is commonly referred to as causal structure, it is fair to say that the causal structure of gravity is dynamical. Therefore, one might say that in the current state of affairs one does not know how to recognize a dynamical causal structure in perturbative QFT nor even how to interpret the meaning of causal structure in a non-perturbative QFT, whose classical version has dynamical causal structure.

Unfortunately, currently, QFT of GR is only accessible to us perturbatively. Eventually, though, we would like to move to a non-perturbative formulation. However, at the non-perturbative level, the axiomatic framework of locally covariant field theory (LCFT), both classical and quantum, encompasses only theories whose equations of motion are systems of PDEs with principal symbols that only depend on a non-dynamical background metric. This explicitly excludes so-called quasilinear equations like GR, hydrodynamics, elasticity and non-linear \(\sigma\)-models. Quasilinear systems are defined by the property that their principal symbols (roughly, the coefficients of the highest order derivatives, which play a major role in the properties of the previously mentioned Green functions) depend on the dynamical fields. Fortunately, at the classical level, GR and other quasilinear systems can be studied non-perturbatively using classical PDE theory. Using these tools, the causal structure can be described and understood in fairly direct terms, closely related to the geometry of domain of dependence theorems.

This work has multiple goals. In broad outline, it is concerned with obtaining a deep, non-perturbative understanding of an algebraic formulation of causality in classical field theories with dynamical causal structures, in a way that parallels
the well understood geometric formulation of well-posedness in hyperbolic PDE systems. Once such an understanding has been obtained another aim is to formulate a conjecture as to how it would translate to quantum field theory. Of course, at this point, one can only expect to make conjectures in this direction, owing to the difficulties in dealing with quantum field theories non-perturbatively. Once formulated, these notions could be adapted to a formal perturbative context, where they could be used in conjunction with practical calculations, for instance in perturbatively quantized GR.

More specifically, another goal is to present in a unified way all the necessary geometric aspects of the construction of a classical field theory. These aspects include an intrinsic geometric formulation of PDE systems using jet bundles, an application of the theory of symmetric hyperbolic PDE systems to the construction of the phase space, and the use of the covariant phase space method and Peierls formula to endow it with symplectic and Poisson structures. Each of these aspects is well known in some segments of the mathematical analysis and mathematical physics communities, but certainly is not familiar to the broad high energy theory and theoretical relativity communities. Some of these aspects are presented here also in potentially novel ways, which highlight how they are used together. In a sense, this paper can be seen as a follow that supplements with symplectic geometric and Poisson algebraic aspects the older work of Geroch [50] who essentially applied the theory of symmetric hyperbolic systems to a unified construction of the spaces of solutions of relativistic classical field theories.

Finally, this paper also aims to apply the category-theoretic methods that have proven so useful in formulating the algebraic structure of QFT on curved spacetimes (now known as LCQFT) to both the algebraic and dual geometric formulation of classical field theory. Essentially, these tools are used to draw a strong parallel or duality between the notion of well-posedness from classical PDE theory and the algebraic properties of classical field theory, so that both of which could be formulated as mutually dual, succinct theorems. These theorems could later be used as a starting point for an improved axiomatization of classical and quantum field theories.

1.1. Outline of the work. Sect. 2 recalls the axiomatic framework of LCFT and emphasizes that it currently excludes quasilinear systems like GR.

Sect. 3 introduces the geometric formulation of PDE systems in terms of jet bundles, Sect. 3.1, with reference to Apdx. A for background on jets. It then focuses on the notion of a quasilinear, hyperbolic system of PDEs, with special emphasis on first order, symmetric hyperbolic systems, Sect. 3.2. The precise formulation of the condition of symmetric hyperbolicity given there differs slightly from the standard one in order to fit better in the subsequent geometric context. It concludes with a discussion of prolongation and equivalence of PDE systems, needed for the reduction of variational equations of motion to hyperbolic form, Sect. 3.3.

Sect. 4 examines the role of characteristics of a PDE system in its causal structure. Sect. 4.1 identifies chronal and spacelike cone bundles as essential abstractions of the resulting causal order. Sect. 4.2 groups solutions together into slow patches by the property of being slower than a certain fixed chronal or spacelike cone bundle. Sect. 4.3 uses the thus far developed notions of causal structure to state some standard theorems of PDE theory, which will be used as
the basic tools in the construction of classical field theories. Sect. 4 presents some more specialized results on linear systems, including global existence and properties of Green functions. The presented material is standard, but may be formulated in a slightly novel way to fit better with the geometric context of the Peierls formula for the Poisson bracket appearing later.

Sect. 5 constitutes roughly half of the technical bulk of this work. It collects the mathematical tools presented thus far and applies them in a unified way to construct the phase space of a classical field theory and its algebra of observables. Sect. 5.1 reviews the covariant phase space method for constructing the symplectic form of a variational PDE system. Since the focus of this work is more geometrical than functional analytical, we avoid most details of the treatment of infinite dimensional manifolds, of which the space of solutions of a PDE system, to be identified with the phase space of a classical field theory, is an example. The purpose of Sect. 5.2 is to introduce enough technical machinery to formally with the tangent and cotangent spaces of the space of solutions, while keeping with the spirit of the preceding remark. Sect. 5.3 identifies the space of solutions with the phase space, by endowing it with (formal) symplectic and Poisson structures. The Poisson structure is given by a generalized Peierls formula. This Peierls formula is shown to be equivalent to the Poisson bracket obtained from the covariant phase space symplectic form, a result that, compared to previous forms of this equivalence, now also encompasses hyperbolic PDE systems with constraints. As an application of this construction, a refined version of classical microcausality is proven both for algebras of observables on a given slow patch as well as on the global phase space, Sects. 5.3.5 and 5.3.6. Also of note is a discussion in Sect. 5.3.5 of the relation between the on-shell formalism used in this work and the off-shell formalism used in other related literature.

Sect. 6 makes a connection between the constructions thus far and the covariant formalism of LCFT via the notion of natural bundles and PDE systems, relying on some background material on category theory from Apd. B. For convenience, Sect. 6.2 systematically summarizes the notation for various concepts and constructions appearing in this work, while also remarking on their categorical and functorial properties.

Sect. 7 constitutes the other half of the technical bulk of this work. It generalizes the axioms of LCFT to quasilinear field theories taking into account their dynamical causal structures, and proves them as theorems of classical field theory. Unfortunately, at this point the proofs rely on a number of sufficient technical hypotheses. The necessity of these hypotheses remain to be examined on a case by case basis or in the context of deeper investigation of the relevant functional analytical details. In particular, the generalization of the Causality axiom provides a sought deeper understanding of causality in quasilinear classical field theories and its algebraic formulation. An important check, Sect. 7.4, is that these theorems reduce to the standard axioms of LCFT in the case of a well-posed semilinear classical field theory, as is to be expected.

Sect. 8 recalls the notion of deformation quantization and remarks that it is the leading candidate for the modern formulation of what it means to quantize a classical mechanical system. This notion of quantization is then used to make a conjecture about the translation of the algebraic formulation of Causality from classical to quantum field theory. In particular, this conjecture answers an old
question about the structure of the commutator of two quantum metric field operators in the quantum field theory of GR.

Finally, Sec. 9 concludes with a discussion of the results presented in this work and lists several ideas, conjectures and hypotheses that had arisen in the process that would be fruitful to investigate further.

2. Free, Interacting and Perturbative Locally Covariant Field Theories

The currently leading axiomatic framework in the algebraic framework for quantum field theory is called locally covariant field theory (LCFT) and rests on the axioms originally proposed by Brunetti, Fredenhagen and Verch [25]. These axioms rest on simple physical principles, namely locality, causality, and the existence of a dynamical law, and have a convenient mathematical formulation. Moreover, they are flexible in that they can and have been adapted to the classical (locally covariant classical field theory abbreviated again LCFT), quantum (locally covariant quantum field theory or LCQFT), and both classical and quantum perturbative contexts (respectively abbreviated pLCFT or pLCQFT) [5].

2.1. Brunetti-Fredenhagen-Verch axioms for LCFT. To state the axioms in a succinct way, we need to appeal to some notions of category theory. Some basic information on categories and functors can be found in [109,21]. A classical mechanical system, at a bare minimum is described by a Poisson algebra over \( \mathbb{R} \), say \((F,\{\})\), called the algebra of observables. Actually, we are likely to be interested only the algebras \( F \) that correspond to smooth functions on some phase space manifold, with the bracket \( \{\} \) defined by a Poisson tensor. However, as we will not need it, we do not consider a detailed characterization of these algebras. Such Poisson algebras form the category \( \mathcal{Pois} \), with Poisson homomorphisms as morphisms. On the other hand, it will be important below to restrict possible homomorphisms to injective ones. Thus we define the following categories.

**Definition 1.** Let \( \mathcal{Pois} \) be the category of Poisson algebras over \( \mathbb{R} \) as objects and Poisson homomorphisms as morphisms. Let \( \mathcal{Pois}_i \), be the subcategory where the morphisms are restricted to injective homomorphisms.

A field theory assigns a classical mechanical system to a region of spacetime, that is, a Poisson algebra of observables whose spacetime supports overlap with that region. A field theory is locally covariant, if these assignments are coherent with respect to embedding smaller spacetimes into larger ones, in a sense to be specified below. First, however, we must define what constitutes a spacetime. The notion of a LCFT was developed in the context of quantum and classical field theory on curved spacetimes. Thus it is natural to consider Lorentzian manifolds as spacetimes. Overwhelming experience from the physics literature suggests that it is sensible to restrict our attention to globally hyperbolic spacetimes. The spacetime morphisms are restricted to ensure compatibility between the metric and causal structures on the source and target manifolds.

---

1 The relevant notion of spacetime support will be discussed in detail in Sect. 5.3.5.

2 A globally hyperbolic Lorentzian manifold can be identified in several equivalent ways. Perhaps the simplest definition is the existence of a Cauchy surface, which is a surface intersected exactly once by every inextensible timelike curve.
Definition 2. Given two oriented and time oriented Lorentzian \(n\)-manifolds \((M, g)\) and \((M', g')\), a smooth map \(\chi : M \to M'\) is called a causal isometric embedding if (i) \(\chi\) is an open embedding preserving orientation, (ii) it is an isometry, \(\chi^* g' = g\), preserving time orientation, and (iii) if any two points \(x, y \in \chi(M)\) can be joined by a causal curve in \(M'\) then they can be joined by a causal curve in \(\chi(M)\) (causal compatibility).

Let \(\text{GlobHyp}\) denote the category of oriented and time oriented globally hyperbolic Lorentzian spacetimes as objects and isometric embeddings as morphisms. Let \(\text{GlobHyp}_c\) denote the subcategory where the morphisms are restricted to causal isometric embeddings.

To express the causality property, it is convenient to introduce a tensor product on Poisson algebras and the notion of independent subsystems. First, though, a few words about tensor products in categories. The necessary formal setting is that of a symmetric monoidal category \([12]\). Such a category is equipped with a self-bifunctor \((A, B) \mapsto A \otimes B\) called tensor product (also monoidal structure), as well as an identity object. This product is required to satisfy some identities, expressed as commutative diagrams, which guarantee that associativity holds, that the identity object acts as an identity for the product, and that there existence of a canonical isomorphism \(A \otimes B \cong B \otimes A\). Categorical products and coproducts (Sect. \([B]\)) are well known examples of tensor products. In fact, we do not give the detailed list of the above axioms because the tensor products that will be used in this paper are all constructed by putting some extra structure on an underlying (co)product in a way that makes the needed axioms manifest. Like the underlying (co)products, our tensor products will be equipped with canonical inclusion, \(A, B \to A \otimes B\), or projection, \(A \otimes B \to A, B\), morphisms. An important distinction of a tensor product from a (co)product is the lack of universality. That is, the existence of a pair of morphisms \(A \to C\) and \(B \to C\) (or \(C \to A\) and \(C \to B\)) does not guarantee the existence of a canonical morphism \(A \otimes B \to C\) (\(C \to A \otimes B\)), unlike for a coproduct (or product). However, since our tensor products are based on underlying (co)products, which such a morphism exists, it is canonical. A functor between two tensor categories that the tensor product structure is called a tensor functor (also a symmetric monoidal functor).

Definition 3. Let \((F, \{\}_F)\) and \((G, \{\}_G)\) be Poisson algebras, their independent subsystems (tensor) product is defined by

\[
(F, \{\}_F) \otimes (G, \{\}_G) \cong (F \otimes G, \{\}),
\]

where \(F \otimes G\) is the tensor product (coproduct) of commutative algebras to which the Poisson bracket is extended by the rule \(\{F \otimes 1, 1 \otimes G\} = 0\). The product Poisson algebra is equipped with the canonical inclusion morphisms

\[
(F, \{\}_F) \longrightarrow (F, \{\}_F) \otimes (G, \{\}_G) \longleftarrow (G, \{\}_G),
\]

which are given respectively by \(f_1 \mapsto f_1 \otimes 1\) and \(f_2 \mapsto 1 \otimes f_2\).

Definition 4. Two subalgebras \((F_i, \{\}_{i}) \subseteq (F, \{\})\) \(i = 1, 2\), of a larger Poisson algebra are said to be independent subsystems if the inclusion morphisms factor through the independent subsystems according to the following commutative
diagram:

\[
\begin{array}{c}
(F_1, \{} \rightarrow (F_1, \{} \otimes (F_2, \{} \leftarrow (F_2, \{} \downarrow (F, \{})
\end{array}
\]

where the horizontal morphisms are the canonical inclusions, the diagonal arrows are the given subalgebra inclusion morphisms, and the vertical dotted line morphism is \( f_1 \otimes f_2 \mapsto f_1 f_2 \) and is a canonical injective Poisson homomorphism.

The axioms classical field theory can now be succinctly stated as follows.

**Definition 5 ([5]).** A locally covariant classical field theory \( F \) satisfies the following axioms:

- **Isotony** It is a covariant functor, \( F : \text{GlobHyp}_c \rightarrow \text{Poiss}_i \).
- **Time Slice** The image \( F(\chi) \) of \( \chi : M \rightarrow M' \) is a Poisson isomorphism whenever \( \chi(M) \) contains a Cauchy surface of \( M' \).
- **Causality** The images of morphisms \( F(\chi_i) : F(M_i) \rightarrow F(M) \) are independent subsystems of \( F(M) \), whenever the images of morphisms \( \chi_i : M_i \rightarrow M, i = 1, 2 \), are spacelike separated in \( M \).

The Causality axiom may also be rephrased in terms of tensor products. We can introduce the disjoint union \((M_1, g_1) \sqcup (M_2, g_2) = (M_1 \sqcup M_2, g_1 \oplus g_2)\) as a tensor product in \( \text{GlobHyp}_c \). It is based on the disjoint union (coproduct) or manifolds, such that disconnected components are considered spacelike separated from each other. As for the independent subsystems product on Poisson algebras, the canonical dotted line morphism in the following diagram does not always exist:

\[
\begin{array}{c}
(M_1, g_1) \rightarrow (M_1, g_1) \sqcup (M_2, g_2) \leftarrow (M_2, g_2) \downarrow (M, g)
\end{array}
\]

It exists only if the images of \( M_i \) are spacelike separated in \( M \). Thus, the Causality axiom simply states that the LCFT functor \( F : \text{GlobHyp}_c \rightarrow \text{Poiss}_i \) is a tensor functor.

It should also be clear by now that the above formalism explicitly ties the causal structure of a field theory to a fixed (non-dynamical) Lorentzian metric. A Lorentzian metric provides a notion of a causal relation, which appears crucially in each of the named axioms. What about field theories that have a dynamical Lorentzian metric or have no Lorentzian metric naturally associated to them at all? It is worth taking a step back and examining why a Lorentzian metric is important in the first place.

### 2.2. Linear and semilinear PDEs

We start by considering examples of field theories that can be shown to actually satisfy the LCFT axioms. There are two prominent families of examples: free waves field and globally well-posed,
interacting wave fields without derivative couplings. Prototypical representatives of each family are the free scalar field, obeying the Klein-Gordon equality

\[ \square \phi - m^2 \phi = 0, \quad (5) \]

and the \( \phi^4 \)-interacting scalar field, obeying the semilinear equation

\[ \square \phi - m^2 \phi - \lambda \phi^3 = 0. \quad (6) \]

By wave field, we mean a field theory, defined on a Lorentzian manifold, whose equations of motion have the corresponding d’Alambertian wave operator \( \square \) as the principal symbol. The principal symbol is roughly the differential operator consisting of the highest derivative order terms of the PDE (cf. Eq. 11). For the wave operator \( \square \), the principal symbol is essentially \( g^{\mu \nu} \partial_\mu \partial_\nu \), where \( g^{\mu \nu} \) is the inverse Lorentzian metric, which is contracted with two coordinate derivatives. As is usual, the term free means that the corresponding equations of motion are linear. The absence of derivative couplings is taken to mean the absence in the equations of motion of non-linear terms with derivatives. Such equations are termed semilinear. Finally, well-posedness is taken to mean, in particular, that solutions corresponding to arbitrary smooth initial data (obeying appropriate boundary conditions, of course) do not form singularities in finite time. Well-posedness of linear equations can be proven under quite general assumptions, which is not the case for semilinear ones.

These field theories depend on the background Lorentzian metric \( g_{\mu \nu} \) through the principal symbol of their equations of motion. The exclusion of interactions with derivative couplings, ensures that this principal symbol is always the same as that of the wave operator \( \square \). Since the metric is Lorentzian, these partial differential equations are special cases of hyperbolic PDE systems. Roughly, hyperbolicity is an algebraic and geometric property of the principal symbol, which is exploited by standard PDE theory to prove analytical results like (a) local-in-time well-posedness of the Cauchy initial value problem (this entails the existence of solutions for arbitrary smooth initial data, uniqueness of solutions, continuous dependence of the solution on the initial data), (b) finiteness of the speed of propagation of disturbances. In particular, as a generic result, the maximal speed of propagation in any direction happens to be given by the null directions of the background Lorentzian metric. Global well-posedness can also be proven under generic circumstances for linear PDEs and under some additional hypotheses for semilinear ones.

These results are important because, roughly, global well-posedness and finite propagation speed jointly imply the validity of the Isotony, Time Slice and Causality axioms for the corresponding classical field theory. The Causality axiom, and hence the notion of causal structure, is particularly closely related to finite propagation speed. We shall see in Sect. [7.4] in more detail how these results from PDE theory can be used to construct LCFTs and verify these axioms.

Following the logic of the last paragraph, it makes sense to look for a generalization of the notion of causal structure, and corresponding generalizations of LCFT axioms, by studying the way PDE theory establishes the properties of well-posedness and finite propagation speed in more general hyperbolic systems.

\[ ^3 \text{Actually, semilinear equations allow non-linear terms with derivatives, as long as they are not of the highest present order.} \]
Of particular interest are PDE systems where these properties are not directly tied to a background, non-dynamical Lorentzian metric. As will be shown in the following sections, such generalizations can be found in the PDE theory of quasilinear hyperbolic systems, of which GR is a special case.

3. Quasilinear Hyperbolic Systems

In this section we formulate several aspects the theory of systems of partial differential equations (PDEs) in a geometrical way, namely in terms of jets. This approach is not completely standard in the analysis or mathematical physics literatures, but it does have some advantages.

Eventually, we would like to make contact with locally covariant field theory, which assigns algebras to spacetime regions in a functorial way (a diffeomorphism-covariant way). We would like to construct these algebras as algebras of functions on spaces of solutions of some PDEs. Thus we would like to assign PDE systems to manifolds in a functorial way as well. It turns out that this is more conveniently done using natural bundles (to be introduced in Sect. 6) and jet bundles. Moreover, it is also more convenient to discuss integrability conditions and equivalence of PDE systems in the language of jets. Finally, the variational bicomplex on jet space is the natural setting for conservation laws and the covariant symplectic structure (to be discussed in Sect. 5).

Within this framework, we define PDE systems of quasilinear hyperbolic type. We rely mostly on the notion of symmetric hyperbolicity of first order systems (following [50]), but also briefly comment on the related notion of regular hyperbolicity.

From this point, we will be discussion finite dimensional spacetime manifolds, bundle manifolds and infinite dimensional manifolds of sections and solutions, as well as morphisms between them. For later convenience and to fix some notations it is helpful to define the following categories.

Definition 6. Let $\text{Man}$ denote the category of smooth manifolds (of possibly infinite dimension) and $\text{Man}_e$ the subcategory where morphisms are restricted to open embeddings. The morphism are always $C^\infty$ maps. Denote also by $\text{Man}^n$ and $\text{Man}^n_e$ the respective subcategories of manifolds of fixed dimension $n = 0, 1, 2, \ldots, \infty$. The finite dimensional manifolds are taken to be oriented.

Let $\text{Bndl}$ denote the category of smooth, finite dimensional bundles over manifolds and $\text{Bndl}_e$ be the subcategory where morphisms are restricted to open embeddings of the total space. These categories are fibered over manifolds, $\Pi: \text{Bndl} \to \text{Man}$, with $\Pi: (E \to M) \mapsto M$. Let $\text{Bndl}(M)$ denote the subcategory of $\text{Bndl}$ of bundles over the base manifold $M$ with base fixing morphisms (projection to the base is always $\text{id}: M \to M$). Thus, in an obvious way, we have defined a functor from manifolds to categories, $\text{Bndl}: \text{Man} \to \text{Cat}$.

Let $\text{VBndl}$ denote the category of vector bundles, a subcategory of $\text{Bndl}$, and $\text{VBndl}_e = \text{VBndl} \cap \text{Bndl}_e$. We similarly introduce the notations $\text{VBndl}(M)$ and $\text{VBndl}_e(M)$.

We shall adopt $\Pi$ as the generic notation for forgetful functors. More generally, from time to time, we shall reuse functor names as long as they can be distinguished by the domain category and are similar in purpose.
We shall not be specific about the precise definition of the class of infinite dimensional manifolds in question. We presume that such details can be added when applying the current formalism to specific situations. Such functional analytical questions have been recently tackled in earnest in [47,24] (cf. also [19]).

A prominent example of an infinite dimensional space that we will treat as a manifold is the space \( \Gamma(\mathcal{E}) \) of all smooth sections of the vector bundle \( \mathcal{E} \to M \), as well as some infinite dimensional subspaces thereof, like the space of all solutions of a PDE system.

### 3.1. Jet bundles and systems of PDEs

This section outlines the description of PDEs as submanifolds of the jet bundle. Jet bundles are briefly introduced in Sect. A, where also notation is fixed (not all of it being completely standard) and standard literature references are given. Such a description of PDEs is more intrinsic than than the usual one in terms of equations, but is essentially equivalent. This approach is well known in the geometric and formal theory of differential systems [102,27]. It will be later combined with the notion of natural bundles to define locally covariant field theories. The discussion of natural PDE systems in connection with natural bundles will be postponed to Sect. 6. Below we consider only fixed base manifolds.

From now on, fix \( M \) to be finite dimensional manifold and let \( n = \dim M \). Also fix a vector bundle \( \mathcal{F} \to M \). We refer to \( M \) as the spacetime manifold and to \( \mathcal{F} \) as the field bundle.

We restrict our attention to regular PDEs in the following sense.

**Definition 7.** A PDE system \( \mathcal{E} \) of order \( k \) is a smooth, closed sub-bundle (in the \( \text{Bndl} \) sense) of \( J^k \mathcal{F} \to M \), \( \mathcal{E} \subseteq J^k \mathcal{F} \).

Note that \( \mathcal{E} \) need not be a vector sub-bundle of \( J^k \mathcal{F} \). The above definition may seem unfamiliar to some, but can be cast in more recognizable form using the following

**Proposition 1.** Given a PDE system \( \mathcal{E} \) of order \( k \), there exists a vector bundle \( \mathcal{E} \to M \), a smooth sub-bundle \( \mathcal{E}' \subseteq \mathcal{E} \) containing the zero section of \( \mathcal{E} \to M \), and a smooth base fixing bundle morphism \( f: J^k \mathcal{F} \to \mathcal{E} \) (in the \( \text{Bndl} \) sense) such that the image of \( f \) is contained in \( \mathcal{E}' \), the image of \( f \) is transverse in \( \mathcal{E}' \) to the zero section of \( \mathcal{E} \) and \( \mathcal{E} \) is precisely the preimage of the zero section, that is, \( \mathcal{E} \) satisfies \( f = 0 \).

The proof follows from basic differential topology, up to a global topological obstruction [53, §7]. Clearly, the equation form is not unique. For instance, applying any invertible transformation to the equations \( f = 0 \) gives another equation form \( f' = 0 \), which describes exactly the same PDE system.

We refer to \( \mathcal{E} \to M \) as the equation bundle and to \( f \) or the pair \((f, \mathcal{E})\) as the equation form of the PDE system \( \mathcal{E} \). A section \( \phi: M \to \mathcal{F} \), also referred to as a field configuration, is said to satisfy the PDE system \( \mathcal{E} \) if the \( k \)-jet prolongation of \( \phi \) is contained in \( \mathcal{E} \), \( j^k \phi(x) \in \mathcal{E}_x \subseteq J^k_x(\mathcal{F}, M) \). Equivalently, \( j^k \phi \) is a section of \( \mathcal{E} \to M \). We denote the space of all solution sections by \( S(\mathcal{F}) \subseteq \Gamma(\mathcal{F}) \) or \( S_\mathcal{E}(\mathcal{F}) \) when the PDE system needs to be mentioned explicitly. Using the above proposition, we can equivalently say that \( \phi \) is a solution of the PDE system \( \mathcal{E} \) if

\[
f[\phi] = f(j^k \phi) = 0.
\]
Expressing the $k$-jet in local coordinates, $j^k \phi(x) = (x, \phi^a(x), \partial_i \phi^a(x), \ldots)$, it is clear that $f(x, \phi^a(x), \partial_i \phi^a(x), \ldots) = 0$ is a system of partial differential equations in the usual sense of the term. Starting with a PDE system in the usual sense, its geometric form as a sub-bundle of the jet bundle can be obtained by a converse of the above lemma. At this point, the regularity assumptions on both $\mathcal{E}$ and $f$ become important. Namely, the transversality properties of $f$ ensure that the zero set of $f = 0$ is a submanifold of $J^k F$ and vice versa.

The linear and affine structures on $J^k F$ give us the possibility of defining the notion of linear and quasilinear PDE systems.

**Definition 8.** A PDE system $\mathcal{E} \subset J^k F$ is called linear if $\mathcal{E} \rightarrow M$ is a vector sub-bundle of the vector bundle $J^k F \rightarrow M$. The PDE system is called quasilinear if $\mathcal{E} \rightarrow J^{k-1} F$ is an affine sub-bundle of the affine bundle $J^k F \rightarrow J^{k-1} F$.

The connection to the usual meanings of these terms can be seen through adapted equation forms.

**Lemma 1.** The PDE system $\mathcal{E} \subset J^k F$ is linear iff it has an equation form $(f, E)$, where $f: J^k F \rightarrow E$ is a morphism of vector bundles over $M$.

The PDE system $\mathcal{E} \subset J^k F$ is quasilinear iff it has an equation form $(f, E)$, where $f: J^k F \rightarrow E$ is a morphism of affine bundles, which fits into the commutative diagram

$$
\begin{array}{ccc}
J^k F & \xrightarrow{f} & E \\
\downarrow & & \downarrow \\
J^{k-1} F & \rightarrow & M,
\end{array}
$$

where the vertical maps define the affine bundles, with the vector bundle $E \rightarrow M$ naturally considered an affine one.

The proof is immediate. Alternatively, the quasilinear case can be cast into the form of a base fixing affine bundle morphism $f: J^k F \rightarrow (E)^{k-1}$, where both bundles are over $J^{k-1} F$. Such equation forms are called adapted.

In the more common language of adapted local coordinates, the conditions of linearity and quasilinearity are expressed as follows. Consider adapted local coordinates $(x^i, v_A)$ on the equation bundle $E$, $(x^i, u^a)$ on the field bundle $F$, and the corresponding $(x^i, u^a_I)$ on the $k$-jet bundle $J^k F$. Let $\phi: M \rightarrow F$ be a field configuration, then its $k$-jet in local coordinates is $j^k \phi(x) = (x^i, \partial_I \phi^a(x))$. The above lemma asserts the existence of an equation form that looks like

$$
f_I^A(x) \partial_I \phi^a(x) = 0. \quad (9)
$$

Note that this equation is linear in $\phi(x)$ and its derivatives and that the coefficients $f_I^A(x)$, with multi-indices $I$, depend only on the base space coordinates $x$. On the other hand, for a quasilinear equation, the lemma asserts the existence of an equation form that looks like (with $|I| = k$)

$$
f_I^A(x, j^{k-1} \phi(x)) \partial_I \phi^a(x) + f_A(x, j^{k-1} \phi(x)) = 0. \quad (10)
$$

Note that, in the linear case, the fact that the coefficients of $f: J^k(F, M) \rightarrow E$ only depend on the base space coordinates $x$ is captured by the requirement that it is a morphism of vector bundles over $M$. In the quasilinear case, the
coefficients of \( f \) can obviously depend on both \( x, \phi(x) \) as well as all derivatives \( \partial_I \phi(x) \) up to order \( |I| = k - 1 \), which is captured by allowing \( f: J^k F \to (E)^{k-1} \) to be a (base fixing) bundle morphism over \( J^{k-1} F \). It is worth remarking that any linear PDE system is also naturally quasilinear.

Recall that the affine bundle \( J^k F \to J^{k-1} F \) is modeled on the vector bundle \( (S^k T^* M \otimes_M F) \to J^{k-1} F \). Therefore, an adapted equation form \((f, E)\) of a quasilinear PDE system \( E \subset J^k F \) naturally singles out a section

\[
\bar{f}: J^{k-1} F \to (E \otimes_M F^* \otimes_M S^k TM)^{k-1}.
\] (11)

In local coordinates, \( \bar{f} \) corresponds to the coefficient \( f^I_{\lambda\alpha} \) of the highest derivative term \( \partial^I \phi(x) \) with \( |I| = k \) in Eq. (10). This section \( \bar{f} \) is called the principal symbol of the given equation form of \( E \).

Below, we will be mostly concerned with first or second order PDE systems.

### 3.2. Symmetric and regular hyperbolicity.

The notion of hyperbolicity for a PDE system is strongly linked to the ability to formulate it as an initial value problem. Locally such a formulation is constrained by the existence of so called characteristic surfaces or characteristic covectors, which are defined in detail below. Various notions of hyperbolicity are then stated either in terms of geometric conditions on the locus of characteristic covectors or in terms of equivalent algebraic conditions on the principal symbol of a special equation form of the system.

An initial value formulation consists of converting the PDE system into an infinite dimensional ODE as follows. Suppose we are given an \( n \)-dimensional manifold \( M \) that can be smoothly factored as \( M \cong R \times S \), where \( S \) is a manifold of dimension \( n-1 \), and the projection on the first factor is denoted by \( t: M \to R \), the time function. Then the space of restrictions of field configurations to a level set of \( t \) (always a codimension-1 surface diffeomorphic to \( S \)) forms an infinite dimensional space, which can be considered as a fiber of a smooth, infinite dimensional bundle over \( R \). If the PDE system can be turned into an ODE system on this bundle, one can use Picard iteration and a Gronwall-type lemma to prove local-in-\( t \) existence and uniqueness of solutions. Various notions of hyperbolicity essentially correspond to sufficient conditions under which the above construction can be carried out. Some of these conditions are local (referred to simply as hyperbolicity) and some global (referred to as global hyperbolicity). Local conditions put restrictions on the principal symbol of a PDE system and on the germs of the leaves of the \( S \)-foliations of \( M \) to which the above construction would be applicable. The global conditions require \( M \) to factor in a way similar to above, with the \( S \)-leaves satisfying the corresponding local conditions. Strictly speaking, when the \( S \)-leaves are non-compact or have a boundary, boundary conditions may have to be supplied, on top of those already mentioned, to ensure a well-posed initial value problem. Such boundary conditions will not be discussed in this work.

To obtain some necessary conditions on the geometry of initial value surfaces, we need to first look at the linear and affine geometry of \( k \)-jet space in the presence of a preferred codimension-1 subspace \( \tau_x \subset T_x M \) at a fixed \( x \in M \). This subspace can be thought of as tangent to a putative initial value surface \( S \) passing through \( x \). For convenience of notation, we promote \( \tau_x \) to be the fiber of a codim-1 vector sub-bundle \( \tau \subset TM \) (also called a tangent plane distribution).
However, all subsequent calculations will be purely local and go through equally well if $\tau$ is only defined in a neighborhood of $x$, or just at $x$ itself. In particular, the distribution $\tau$ need not be integrable.

Let $\nu \subset T^*M$ be the 1-dimensional vector sub-bundle of covectors annihilating $\tau$ (the conormals to the putative surface $S$). In turn, the 1-dimensional vector bundle $\nu$ singles out the 1-dimensional vector sub-bundle $\nu^\otimes k = \nu \otimes M \nu \otimes M \cdots \subset S^k T^* M$, and finally the vector sub-bundle $N = \nu^\otimes k \otimes M F \subset S^k T^* M \otimes M F$.

Recall that the affine bundle $J^k F \to J^{k-1} F$ is modeled on the vector bundle $(S^k T^* M \otimes M F)^{k-1}$. Note that the fibers of $J^k F \to J^{k-1} F$ are foliated by affine planes parallel to the fibers of the vector bundle $(N)^{k-1}$. We define $J^{k-1,⊥}$ to be the leaf space of this foliation. In other words, the short exact sequence of vector bundles

$$0 \to N \to S^k T^* M \otimes M F \to (S^k T^* M \otimes M F)/N \to 0$$  \hspace{1cm} (12)

induces the following affine bundle projections

$$ J^k F \to J^{k-1,⊥} F \to J^{k-1} F, $$  \hspace{1cm} (13)

where the second projection defines an affine bundle with fibers modeled on the vector bundle $(S^k T^* M \otimes M F/N)^{k-1}$ and the first projection defines an affine bundle with fiber modeled on the vector bundle $(N)^{k-1,⊥}$, with the latter notation meaning the pullback of the bundle $N \to M$ along the canonical projection $J^{k-1,⊥} \to M$. A more in depth discussion of the bundle $J^{k,⊥} F$ can be found in [103, Ch.6].

The above constructions are easily illustrated in local coordinates. Consider local coordinates $x^1 = (t, s^1)$ on a neighborhood of $x \in M$ and local adapted coordinates $(t, s^1, u^a)$ on the corresponding neighborhood of $F_x \subset F$. Suppose that the $t$ coordinate is chosen such that the level set $t = t(x)$ is tangent to the plane $\tau_x \subset T_x M$, in other words $dt(x) \in \nu_x \subset T_x M$. The $k$- and $(k-1,⊥)$-jets of a section $\phi: M \to F$ can be represented by

$$ \begin{align*}
j^k \phi(x) &= (x^1, \partial_j \phi^a(x)), \quad |J| \leq k, \\
&= (t, s^1, \partial_t \partial_j \phi^a(x)), \quad |I| + l \leq k, \\
&= (x^1, \partial_{L_k} \phi^a(x), \partial_t \partial_I \phi^a(x), \ldots, \partial_{T_k}^{-1} \partial_{I_{k-1}} \phi^a(x), \partial_{T_k}^k \phi^a(x)), \quad |I_k| \leq k - l,
\end{align*}$$  \hspace{1cm} (14)

where the multi-indices in $J$ range over all indices of $x^j$, while the multi-indices of $I$ and $I_l$ range only over the indices of $s^1$ and not $t$, and by

$$ \begin{align*}
j^{k-1,⊥} \phi^a(x) &= (t, s^1, \partial_{I_0} \phi^a(x), \partial_t \partial_I \phi^a(x), \ldots, \partial_{T_k}^{-1} \partial_{I_{k-1}} \phi^a(x)), \quad |I_1| \leq k - l.
\end{align*}$$  \hspace{1cm} (17)

In other words, the affine bundle projection $j^k \phi(x) \mapsto j^{k-1,⊥} \phi(x)$ simply discards the last component, which is the highest order derivative $\partial_{T_k}^k \phi^a(t, s)$ along the $t$-direction, which is transverse to the level set of $t$ passing through $x$.

In order to convert the PDE system into an ODE with respect to the $t$ coordinate, still working locally at $x \in M, S$, given a section $\phi: M \to F$ that satisfies the PDE system, $j^k \phi(x) \in \mathcal{E}_x$, we need to uniquely determine the highest order $t$-derivative of the dynamical part of $\phi$ at $x$ as a function of all other
derivatives of equal or lower order. So, geometrically, each fiber of the affine bundle \((J^k F)^{k-1} \rightarrow (J^{k-1,1} F)^{k-1}\) needs to intersect \(\mathcal{E}\) exactly once. On the other hand, to express this condition algebraically, we need to pick an adapted equation form \((f,E)\) for this quasilinear PDE system, with principal symbol \(\vec{f}\). Pick sections \(u: M \rightarrow J^k F\), \(u^\perp: M \rightarrow J^{k-1,1} F\) and \(\vec{u}: M \rightarrow J^{k-1} F\) such that \(u \mapsto u^\perp\) and \(u \mapsto \vec{u}\) under the affine bundle maps \(J^k F \rightarrow J^{k-1,1} F\) and \(J^k F \rightarrow J^{k-1} F\) respectively. Then by the affine structure of the \(k\)-jet space, any other \(k\)-jet section \(v\) that projects to \(u^\perp\) is given by \(v = u + p^\otimes k \otimes \psi\), with sections \(\psi: M \rightarrow D\) and \(p: M \rightarrow \nu\). If we demand that \(v\) is actually a section of \(\mathcal{E} \subset J^k F\), then a simple calculation shows that

\[
f(v) = \tilde{f}_a (p^\otimes k \otimes \psi) + f(u) = 0 \tag{18}
\]

\[
\implies (\tilde{f}_a \cdot p^\otimes k) \psi = -f(u). \tag{19}
\]

The last equality is a linear equation for \(\psi\). For it to have a unique solution, as we demanded above, the linear map \(\tilde{f}_a \cdot p^\otimes k: F \rightarrow E\) needs to be invertible. This discussion motivates the following definition.

\textbf{Definition 9.} Consider a vector bundle \(F \rightarrow M\) and a quasilinear PDE system \(\mathcal{E} \subset J^k F\), with adapted equation form \((f,E)\) and principal symbol \(f\). Given \(\vec{u} \in J^{k-1} F\), a covector \(p \in (T^* M)^{k-1}_\vec{u}\) is called \(u\)-non-characteristic (or just non-characteristic) if the contraction of the principal symbol with \(p^\otimes k\), \((\tilde{f}_a \cdot p^\otimes k): (F)^{k-1}_\vec{u} \rightarrow (E)^{k-1}_\vec{u}\), is invertible. Otherwise, \(p\) is said to be \(u\)-characteristic (or just characteristic).

Recall that in local coordinates \((x^i, u^a)\) on \(F\), a \((k-1)\)-jet \(\vec{u} \in J^{k-1} F\) is represented as \(\vec{u} = (x^i, u^a_f), \|I\| \leq k-1\). For linear and semilinear systems, the principal symbol depends only on the base space coordinates \(x^i\). This implies that for such systems a covector \(p \in T^*_x M\) can be decided to be characteristic without looking at the field value \(u^a\) or higher jet components \(u^a_f\). This is the usual situation in relativistic field theory with a fixed background metric, where characteristic covectors coincide with \textit{null} covectors of the background metric. On the other hand, for quasilinear PDE systems, the principal symbol \(\tilde{f}_a\) may depend on \(u^a\) as well as higher jet components \(u^a_f\) up to order \(k - 1\). In this sense, the notion of a characteristic covector becomes \textit{field dependent}.

As we shall see later on, the geometry of the locus of characteristic covectors of a PDE system is closely related to the causal structure of the corresponding classical field theory, in particular to the domain of dependence and finite propagation speed results. Compare now the equations of motion of relativistic field theory (say the Standard Model) with a fixed background metric and coupled with GR, which provides a dynamical metric. As remarked above, characteristic vectors and hence the causal structure of the theory on a fixed background is field independent, since the equations of motion constitute at most a semilinear PDE system. On the other hand, coupled to GR, relativistic field theory becomes quasilinear, since all principal symbols depend on the metric, which is dynamical. Thus, GR with any of the Standard Model matter coupled to it constitutes a system with \textit{field dependent causal structure}. This statement will be made more precise in the next section. At the very least, we expect the causal

\footnote{In the classical PDE literature this is known as putting the equation in \textit{Cauchy-Kovalevskaya form} \cite{32,54}.}
structure of GR to be significantly different (and more complicated) than that of other relativistic field theories.

Characteristic covectors are obstructions to converting a PDE system into ODE or Cauchy-Kovalevskaya form. An initial value formulation with data on a codim-1 surface \( \iota : S \subset M \) can be achieved if the highest \( S \)-transverse derivatives of the unknown section could be solved for in terms of data \( \varphi : S \to \iota^* J^{k-1} F \).

For this it is necessary that a non-vanishing conormal section \( p : S \to \iota^* T^* M \) is everywhere non-characteristic with respect to \( \varphi \). Such a pair \((S, \varphi)\) is referred to as non-characteristic initial data.

However, being non-characteristic is not a sufficient property to set up a well-posed initial value problem for a given initial data set \((S, \varphi)\). This is where the notion of hyperbolicity comes in. There are several different notions of hyperbolicity. We will only discuss two, which are sufficient for our purposes. First, we define symmetric hyperbolicity and then make some comments about regular hyperbolicity.

For a first order quasilinear system, the geometry of the principal symbol simplifies. It can be expressed as a morphism

\[
\tilde{f} : (F \otimes_M T^* M)^0 \to (E)^0
\]

of vector bundles over \( J^0 F \cong F \). Given a section \( p : F \to (T^* M)^0 \) and \( \psi : F \to (F)^0 \), we get a section \( \xi = f(p \otimes \psi) = (f \cdot p) \psi \) of \((E)^0 \to F \). In local coordinates \((x^i, u^a)\) on \( F \) and \((x^i, v_A)\) on \( E \), we have

\[
\xi_A(x, u) = \tilde{f}_{Aa}(x, u) p^i(x, u) \psi^a(x, u).
\]

For the next definition, we introduce some new notation. Recall that \( \Lambda^n M \to M \) is the bundle of densities on \( M \). Given any vector bundle \( V \to M \), we denote the bundle of \( V \)-valued densities by \( \tilde{V} = \Lambda^n M \otimes_M V \). The bundle of dual densities is denoted by \( \tilde{V}^* = \Lambda^n M \otimes_M V^* \). Densitized symmetric powers will be denoted by \( \tilde{S}^k V = \Lambda^n M \otimes_M S^k V \). The defining property of an orientable manifold is the existence of nowhere vanishing sections of \( \Lambda^n M \to M \). An oriented manifold chooses a privileged class of positive densities, which are nowhere vanishing and such that any two positive densities are related through multiplication by an everywhere positive scalar function.

**Definition 10.** Consider a first order, quasilinear PDE system on \( F \to M \), with \( M \) oriented and with adapted equation form \((f, F^*)\) and principal symbol \( f \). This PDE system is said to be symmetric hyperbolic if for each \( \bar{u} \in F \) there exists a covector \( \bar{p} \in (T^* M)^0_{\bar{u}} \) such that

\[
\bar{f}_\bar{u} \cdot \bar{p} \in \tilde{S}^2 F^*^0_{\bar{u}} \subset (\Lambda^n M \otimes_M F^* \otimes_M F^*)^0_{\bar{u}},
\]

that is, \( \bar{f}_\bar{u} \cdot \bar{p} \) is a symmetric bilinear form on the fiber \((F)^0_{\bar{u}}\), and moreover that \( \bar{f}_\bar{u} \cdot \bar{p} \) is positive definite with respect to the orientation on \( M \). Such a covector \( \bar{p} \) called \( \bar{u} \)-spacelike (or just spacelike) and future oriented, while \(-\bar{p}\) is \( \bar{u} \)-spacelike but past oriented.

Decoding the above definition in local coordinates \((x^i, u^a)\) on \( F \), we have that \( p_i \tilde{f}_{Aa}(x, u) \psi^a \psi^b > 0 \) (as an element of \((\Lambda^2_M)^0_{(x,u)}\) with respect to the orientation on \( M \)) with \( p \in (T^* M)^0_{(x,u)} \) and for all \( \psi \in (F)^0_{(x,u)} \).
Note that positive definiteness is stronger than invertibility. So every spacelike covector is also non-characteristic, but not every non-characteristic covector is necessarily spacelike. An \textit{initial data set} \((S, \varphi)\), with \(i: S \subset M\), is called \textit{spacelike} if a non-vanishing conormal section \(p: S \rightarrow T^*M\) is everywhere \(\varphi\)-spacelike. It is for spacelike initial data that the theory of symmetric hyperbolic PDEs establishes local well-posedness.

\textbf{Remark 1.} Our definition of a symmetric hyperbolic system is slightly different from the standard one, where the principal symbol is valued in \(S^2F^*\) rather than the densitized \(\tilde{S}^2F^*\). For a manifold with a fixed orientation, as we are considering, the difference consists of tensoring with a positive density. Densitizing the equation form turns out to be more convenient in the current setting, in particular in Sect. 5. However, densitization is also natural in the original context where symmetric hyperbolicity is used, the construction of an energy norm by integrating over a future oriented, spacelike surface \(\iota: \Sigma \subset M\),
\[
\|\psi\|^2 = \int_{\Sigma} \iota^* \text{tr} \bar{f}(\psi, \psi),
\]
where the trace naturally converts the principal symbol into a current density (the single contravariant index of \(\bar{f}\) is contracted with one of its \(n\) antisymmetric covariant ones), which can be naturally integrated when pulled back to the codim-1 surface \(\Sigma\).

\textbf{Remark 2.} Note also that our usage of the word \textit{spacelike} differs from the standard one in Lorentzian geometry, where our spacelike covectors would be referred to as “timelike”. However, this terminology is consistent with the literature on hyperbolic PDEs. Moreover, outside pseudo-Riemannian geometry, there is no natural identification between vectors and covectors. On the other hand, covectors are still naturally identified with codimension-1 tangent planes. As expected a spacelike plane consists of spacelike vectors we merely extend this terminology to a corresponding covector. The terminology introduced for vectors below is standard in either literature.

It is convenient to introduce the following geometric notions as well.

\textbf{Definition 11.} Consider \(\bar{u} \in F\). A tangent vector \(v \in (TM)_{\bar{u}}^0\) is called \textit{causal} and \textit{future directed} if \(p \cdot v \geq 0\) for every spacelike and future oriented \(p \in (T^*M)_{\bar{u}}^0\). If the inequality is reversed, \(p \cdot v \leq 0\), then \(v\) is called \textit{causal} and \textit{past directed}.

As noted above, the notion of symmetric hyperbolicity applies only to first order, quasilinear systems. This is not too strong of a restriction in practice, as often higher order PDE systems can be reduced to first order ones by introducing extra fields (increasing the dimension of the field bundle fibers). In particular, this exercise was carried out for GR, all Standard Model fields, as well as relativistic hydrodynamics in [50]. On the other hand, many equations obtained as the Euler-Lagrange equations of an action are not directly in symmetric hyperbolic form. Such equations can actually be conveniently treated directly, without reduction to first order form. The relevant notion which comes with a well-posedness theory is \textit{regular hyperbolicity}, which was developed much more recently by Christodoulou in [31]. It would be ideal if the class of regularly hyperbolic systems, upon reduction to first order form, were included in the class of symmetric hyperbolic ones. Unfortunately, this does not appear to be the case due to subtle differences between the two definitions [15]. On the other hand, the
similarities between symmetric and regular hyperbolicity include similar definitions of characteristic and spacelike covectors and similar conal properties of the geometry of their loci, which are explored in the next section. These properties are all we need for the purposes of this paper. Therefore, in the sequel, we refer only to results for symmetric hyperbolic systems, even though similar results can be obtained for regular hyperbolic ones.

At this point it is worth making a few comments on the so-called energy methods used to establish well-posedness for symmetric and regular hyperbolic systems of order \( k = 1, 2 \). A central role is played by a family of local, horizontally conserved, positive definite, coercive energy currents densities \( \varepsilon^\tau \), parametrized by \( \tau > 0 \). Locality means that it is a section of the horizontal \((n - 1)\)-form bundle over \( J^k F \), \( \varepsilon^\tau : J^{k-1} F \to (\mathbb{A}^{n-1} M)^{k-1} \). Equivalently, \( \varepsilon^\tau \in \Omega^{n-1,0}(F) \) and is projectable to \( J^{k-1} F \). Horizontally conserved means that \( d(j^k \phi)^* \varepsilon^\tau = (j^k \phi)^* d_h \varepsilon^\tau = 0 \), when \( \phi : M \to F \) is a solution of the given PDE system. This conservation criterion can actually be relaxed up to lower order terms, that is, we allow \((j^k \phi)^* d_h \varepsilon^\tau = \gamma (j^k \phi) + \beta (j^k \phi)\) to be a function of \( \phi \) and its derivatives. Here, both terms on the right hand side are of lower order: \( \gamma \) because it depends only on derivatives up to order \( k-1 \) and \( \beta \) because it is proportional to \( \tau \), which can be made arbitrarily small. Positive definite means that the pullback \( \iota^*[(j^k \phi)^* \varepsilon^\tau] \) onto a future oriented, spacelike codim-1 surface \( \Sigma^\tau \subset \Sigma \) gives a positive density, so that the energy integral \( E^\tau_\Sigma[\phi] = \int_{\Sigma^\tau} \iota^*[(j^k \phi)^* \varepsilon^\tau] \) is also positive. Coercivity means that the algebraic structure of \( \varepsilon^\tau \) is such that it allows \( E^\tau_\Sigma[\phi] \) to dominate \( \|j^{k-1} \phi\|_{\Sigma^\tau} \), some \( L^2 \)-norm on the restrictions of \((k - 1)\)-jets to \( \Sigma^\tau \), in order to prove an inequality of Gronwall type,

\[
\|j^{k-1} \phi\|_{\Sigma^\tau} \leq C e^{C \tau} \|j^{k-1} \phi\|_{\Sigma_0} , \quad C > 0 , \tag{23}
\]

where the surfaces \( \Sigma_t \) are spacelike surfaces with common boundary, whose interiors foliate a domain in \( M \). Such domains are called lens-shaped and, as we shall see later on, they essentially determine domains of dependence. This inequality is crucial in the proofs of well-posedness.

The main difference between the symmetric and regular hyperbolic equations is in how the energy current density \( \varepsilon^\tau \) is obtained. In the symmetric hyperbolic case, \( \varepsilon^\tau \) is constructed directly from the principal symbol, namely \( \varepsilon^\tau[\phi] = e^{-\tau t} \tau \int_0^\tau \phi(\phi, \phi) \), where \( t \) is a time function whose spacelike level sets foliate a lens-shaped domain. In the regular hyperbolic case, Christodoulou extends Nöther’s theorem to convert a timelike vector field into a corresponding conserved (up to lower order terms) “symmetry current density” \( \varepsilon \), which satisfies all the desired properties of an energy current density. See Sect. 5.0 of [31], as well as [15], for a more detailed comparison of the two cases.

From the above discussion, it follows that it is not so much the structure of a special equation form that defines the PDE system that matters directly. Rather what is important to establish well-posedness using energy methods is the presence of (almost) conserved, local energy current densities. The algebraic structure of the PDE system is then important in so far as it gives rise to such energy current densities. In the geometric theory of PDE systems (or equivalently of so-called differential systems) the study of such (almost) conserved currents (or conservation laws) has been named characteristic cohomology [28, 30] (this meaning of the overloaded term characteristic is distinct from its use in characteristic covector). For some PDE systems, these conservation laws can be
classified exhaustively, in an intrinsic manner (that is, independent of the equation form used to define the PDE system) [20]. This observation leads one to speculate that a more general notion of hyperbolicity can be defined in terms of intrinsic invariants of a PDE system given by its characteristic cohomology, such that both symmetric and regular hyperbolicity become special cases thereof.

3.3. Prolongation, integrability, hyperbolization. One reason to discuss PDE systems as submanifolds of a jet bundle is independence of a particular equation form. Any two equation forms are equivalent if they define the same PDE system manifold. We should specify our notion of equivalence.

Definition 12. Consider two field bundles \( F_i \to M, i = 1, 2 \), and two PDE systems \( \mathcal{E}_i \subseteq J^k F_i \). Denote the corresponding spaces of smooth solution sections by \( \mathcal{S}_i(F_i) \). The PDE systems \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are said to be equivalent if there exist bundle morphisms \( e_{ij}: J^l F_i \to F_j, i \neq j \), such that

\[
\phi_i \in \mathcal{S}_i(F_i) \quad \text{and} \quad \phi_j = e_{ij} \circ j^l \phi_i \implies \phi_j \in \mathcal{S}_j(F_j),
\]

as well as that \( e_{12} \circ j^1 \) and \( e_{21} \circ j^2 \) are mutual inverses when restricted to the solution spaces \( \mathcal{S}_1(F_1) \) and \( \mathcal{S}_2(F_2) \).

We can easily extend the notion of equivalence to equation forms of PDE systems. In that case two different equations forms that define the same PDE system manifold are trivially equivalent. Note that neither the field bundles nor the orders of the PDE systems need to be same for equivalence to hold.

Let us restrict to the case that will be of importance in a later section, namely of \( F_1 = F_2 = F \) and \( e_{12} \) and \( e_{21} \) respectively equal to the canonical projections \( J^l F \to F \) and \( J^m F \to F \), which are in a sense trivial. In this case, it is certainly sufficient that \( \mathcal{E}_1 = \mathcal{E}_2 \) for equivalence to hold, but it is not necessary. In fact \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) could be of different orders. To obtain necessary conditions for equivalence, we need to consider prolongation of PDE systems and the possible resulting integrability conditions.

A discussion of these notions in the setting of the jet bundle description of PDE systems can be rather technical. On the other hand, the theory of equivalence of PDE systems formulated in these terms has become quite mature and has yielded some important results. The technical details of this theory can be found elsewhere [27,102]. Below we give a brief non-technical introduction to this theory and state some simplified results relevant for hyperbolic systems.

The step by step derivation and inclusion of integrability conditions into a PDE is called prolongation. It is easiest to define prolongation in equation form and in local coordinates. Consider an equation form \( (f, E) \) of a PDE system \( \mathcal{E} \subseteq J^k F \), as well as local coordinates \((x^i, u^a)\) on \( F \) and \((x^i, v_A)\) on \( E \). If the section \( \phi: M \to F \) satisfies the PDE system, we have the following system of equations holding in local coordinates

\[
f_A(x^i, \partial_j \phi^a) = 0.
\]

These equations hold for each point \( x \in M \), therefore when both sides are differentiated with respect to the coordinates on \( M \), the resulting equations are still satisfied,

\[
\partial_i f_A(x^i, \partial_j \phi^a) = (\partial_i f_A)(x^i, \partial_j \phi^a) = 0,
\]

(26)
where \( \partial_t f_A \) are functions on \( J^{k+1}F \) obtained by pulling back the functions \( f_A \) from \( J^kF \) to \( J^{k+1}F \) and applying the horizontal vector field \( \partial_t \). These new functions \( f_A = \partial_t f_A \), together with the old \( f_A \) ones, constitute the local coordinate expression for the equation form \((p^1 f, J^1 E)\), where \( p^1 \) is the 1-prolongation defined in Sect. [A]. We call the corresponding PDE system \( E^1 = E^1 \subset J^{k+1}F \) the first prolongation of \( E \) or also its prolongation to order \( k + 1 \). Prolongations to any higher order, \((p^l f, J^l E)\) and \( E^l \subset J^{k+l}F \), are defined iteratively.

Let \( p_l : J^{k+l}F \to J^kF \) be the canonical jet projection, which restricts to \( p_l : E^l \to E \). Notice that we necessarily have \( p_l(E^l) \subseteq E \), since the prolonged system contains the original one as a subsystem. We have just shown that sections satisfying \( E \) automatically satisfy \( E^1 \), and vice versa. In other words, \( S_{E^l}(F) = S_{E^l}(\tilde{F}) \) and the two PDE systems are equivalent. However, the inclusion \( p_l E^l \subseteq E \) may be strict, which would mean that there exist non-trivial integrability conditions. There exists an equation form \((f \oplus g, E \oplus G)\) for \( p_l(E^l) \subset J^kF \), where \((g, G)\) is an equation form for the integrability conditions. These observations provide another sufficient condition for the equivalence of two PDE systems, namely that there exists an order \( l \geq k_1, k_2 \) such that \( E^1_{\leq k_1} = E^1_{\leq k_2} \) as subsets of \( J^lF \).

Prolongation can be iterated indefinitely. Taking this process to its limit, we obtain the infinite order prolongation \( E^\infty \subset J^\infty F \) from the equation form \((p^\infty f, J^\infty E)\), which takes all possible integrability conditions into account. One can then show that the equality \( E^\infty_1 = E^\infty_2 \), as subsets of \( J^\infty F \), is both a necessary and a sufficient condition for the equivalence of two PDE systems. It is a deep theorem of the geometric theory of PDE systems [102,27,53] that, for any given PDE system, there exists a finite order \( l \) such that prolongations above that order introduce no new integrability conditions. Therefore, this restricted version of the equivalence problem can be decided in finitely many steps. Finally, it can also be shown that any PDE system is equivalent to one of first order, though usually defined on a different field bundle.

It is a well known fact that many PDE systems of mathematical physics are not given directly in symmetric hyperbolic equation form, though for somewhat different reasons. The Klein-Gordon equation, though regularly hyperbolic, in its variational form is not first order. The Dirac equation, though first order in its variational form, does not have a symmetric principal symbol. Maxwell equations, in terms of the vector potential, cannot be hyperbolic because of local gauge invariance (which spoils uniqueness in the Cauchy problem). The Proca equation, again in its variational form, Maxwell equations, in terms of the field strength or in terms of a gauge fixed vector potential, are only equivalent to a hyperbolic system with additional constraints.

Consider a quasilinear PDE system \( E \subset J^kF \) with an equation form \((f \oplus c, \tilde{F}^* \oplus E)\), where \((f, \tilde{F}^*)\) is an adapted, first order, quasilinear, symmetric hyperbolic equation form. We refer to \((f, \tilde{F}^*)\) as the hyperbolic subsystem and to \((c, E)\) as the constraints subsystem. The constraints are said to be (symmetric) hyperbolically integrable if there exists a first order, linear PDE system \( E^l \subset J^1E \) with adapted (symmetric) hyperbolic equation form \((h, E^*)\), the consistency subsystem, such that the following identity is satisfied: \( h \circ c = q \circ h \), with some differential operator \( q: J^1F^* \to E^* \) satisfying \( q(0) = 0 \). In other words, for any
section $\phi: M \to F$ we have

$$h[c[\phi]] = q[f[\phi]],$$  \hspace{1cm} (27)$$

and $h[c[\phi]] = 0$ if $f[\phi] = 0$. The consistency subsystem $(h, \tilde{E}^*)$ is linear in the sense that $h[c[\phi]]$ depends linearly on $c[\phi]$, but it may depend non-linearly on $\phi$, such that the compound system with equation form $(f \oplus h, \tilde{F}^* \oplus \tilde{E}^*)$ is first order, quasilinear, symmetric hyperbolic. A PDE system with such an adapted equation form $(f \oplus c, \tilde{F}^* \oplus E)$ is called (symmetric) hyperbolic with constraints. When the constraints are hyperbolically integrable, by studying the properties of the compound hyperbolic PDE system on $F \oplus_M E$, we shall see in Sect. 4.3 that all the well-posedness results for symmetric hyperbolic systems also apply in the presence of constraints.

We call a PDE system (symmetric) hyperbolizable if it is equivalent to one that is (symmetric) hyperbolic with constraints. We call the equivalence map that brings a PDE system into such a form a (symmetric) hyperbolization.

3.4. Examples. Many examples of reductions of relativistic field theories are given in Appendix A of [50]. Another source of examples of field theories described by hyperbolic systems with constraints is [58]. In the latter reference a slightly different notion of hyperbolicity is used, but the given examples still fit into our framework provide they are first reduced to symmetric hyperbolic form.

4. Characteristic Geometry, Causality, Domain of Dependence

The term Lorentzian geometry refers to the study of structures induced on spacetime manifolds by the presence of a Lorentzian metric. One example of this kind of structure are the cones of null vectors. In particular, it is these cones that determine causal relationships between points in Lorentzian spacetimes. Below we will be similarly interested in the geometry of characteristic covectors of a hyperbolic PDE system, which also form cones and also induce a causal order on the points of a manifold. We refer to the investigation of the geometry of cones of characteristic covectors and related structures as characteristic geometry. Even more generally, we are interested in conal geometry, which is concerned with the differential topology of conal manifolds [78, 87]. Each point of a conal manifold is smoothly assigned an open cone of tangent or cotangent vectors. The study of conal manifolds, referred to below as cone bundles, though still an immature field, has the potential to capitalize on and then subsume much of the earlier work on causal order on spacetime manifolds [81, 82, 85, 86]. The generalization from Lorentzian cones to more general ones, for the purposes of describing causality in quantum field theory has been considered before [81, 82, 85], but not in a concrete way.

Fix a vector bundle $F \to M$ and a first order, quasilinear, symmetric hyperbolic PDE system $\mathcal{E} \subset J^1F$ on it with adapted equation form $(f, \tilde{F}^*)$ and a symmetric adapted principal symbol $\tilde{f}$. We mostly follow [14] with the regards characteristic geometry terminology.
4.1. Geometry of cone bundles.

Definition 13. For \( \bar{u} \in F \), denote by \( \Gamma^\bar{u} \subset (T^*M)_\bar{u}^0 \) the set of \( \bar{u} \)-spacelike, future oriented covectors. Similarly, denote by \( \bar{C}_\bar{u} \subset (T^*M)_\bar{u}^0 \) the set of \( \bar{u} \)-characteristic covectors. Let a covector be called cocausal, future oriented if it belongs to \( \Gamma^\bar{u} = \bar{C}_\bar{u} \), where the bar denotes closure. Finally, let a covector be called inner \( \bar{u} \)-characteristic and future oriented if it belongs to \( \bar{C}_\bar{u} = \bar{C}_\bar{u} \cap \Gamma^\bar{u} \).

Also, denote
\[
\Gamma^{\bar{u}} = \bigcup_{\bar{u} \in F} \Gamma^\bar{u}, \quad \Gamma^{\bar{u}}_* = \bigcup_{\bar{u} \in F} \Gamma^\bar{u}_*, \quad \hat{C}^* = \bigcup_{\bar{u} \in F} \hat{C}_\bar{u}^*, \quad \text{and} \quad C^* = \bigcup_{\bar{u} \in F} C_\bar{u}^*.
\] (28)

It is easy to show that \( \Gamma^{\bar{u}} \subset (T^*M)^0 \) is an open subset and hence a submanifold, as well as that \( \partial \Gamma^{\bar{u}} = \partial \Gamma^{\bar{u}} \subset \hat{C}^* \). Moreover, each \( \Gamma^{\bar{u}}_* \) is a convex cone (invariant under multiplication by positive scalars). Note that, since the interior of an open convex cone is diffeomorphic to a ball, \( \Gamma^{\bar{u}} \to M \) is actually a smooth bundle, in the sense of \( \textbf{CBndl} \). On the other hand, \( \Gamma^{\bar{u}} \) has much more structure than a generic smooth bundle. To take this structure into account we introduce another category.

Definition 14. Let the category of cone bundles \( \textbf{CBndl} \) be the subcategory of \( \textbf{Bndl} \) described as follows. An object \( C \to M \) of \( \textbf{CBndl} \), termed a cone bundle, is a smooth bundle for which there exists a vector bundle \( E \to M \) and an inclusion bundle morphism \( \iota: C \subset E \), such that each fiber \( C_x, x \in M, \) is an open convex cone in the corresponding fiber \( E_x \) (where we have implicitly identified \( C \) with its image \( \iota(C) \subset E \)). Given two cone bundles \( C \to M \) and \( C' \to M' \), with corresponding enveloping vector bundles \( E \to M \) and \( E' \to M' \), for every \( \textbf{CBndl} \) morphism \( \chi: C \to C' \) there exists a vector bundle morphism \( \psi: E \to E' \) such that \( \chi = \psi \iota |_C \), namely the following diagram commutes
\[
\begin{array}{ccc}
C & \xrightarrow{\iota} & E \\
\downarrow{\chi} & & \downarrow{\psi} \\
C' & \xrightarrow{\iota'} & E'.
\end{array}
\] (29)

Thus, \( \Gamma^{\bar{u}} \to F \) is obviously a cone bundle enveloped by \( (T^*M)^0 \to F \). We refer to \( \Gamma^{\bar{u}} \to F \) as the cone bundle of future oriented, spacelike covectors. Unfortunately, the closure \( \Gamma^* = \Gamma^{\bar{u}} \) is in general not expected to be a manifold, since its boundary, consisting of the inner characteristic covectors \( \hat{C}^* \), is fiberwise a piecewise algebraic variety and hence can have corners and other singularities. However, even though \( \hat{C}^*, \Gamma^* \subset (T^*M)^0 \) are not objects of \( \textbf{CBndl} \) (by lack of convexity or by presence of a boundary) we refer to them as cone bundles anyway, namely the cone bundle of future oriented, inner characteristic covectors and the cone bundle of future oriented, cocausal covectors, respectively.

Next we turn to causal and related vectors. These are most easily described using the following geometric notion of duality that is often used in convex geometry.

Definition 15. Given a finite dimensional vector space \( V \) and a convex cone \( C \subset V \), denote its closure by \( \bar{C} \) and its open interior by \( \bar{C} \). We define the convex dual \( C^* \subset V^* \) as the set
\[
C^* = \{ u \in V^* \mid u \cdot v \geq 0 \ \text{for all} \ v \in C \}.
\] (30)
We define the strict convex dual $C^\circ \subset V^*$ as the set
\[
C^\circ = \{ u \in V^* \mid u \cdot v > 0 \text{ for all } v \in \bar{C} \setminus \{0\}\}.
\] (31)

To attribute strict may be dropped from the description of $C^\circ$ when it is clear from context. It is easy to check the following

**Proposition 2.** Consider a convex cone $C$.

(i) The convex dual $C^*$ is always closed and also convex. In addition, $C^{**} = \bar{C}$.

The strict convex dual $C^\circ$ is always open and convex.

(ii) $C^* \setminus \{0\}$ is non-empty iff $C$ is contained in a closed half space. $C^\circ$ is non-empty iff $C$ contains no affine line (it is salient).

(iii) If $C$ is open and salient, then $C^{**} = \hat{C}$.

(iv) The inclusion of cones $C_1 \subseteq C_2$ implies the reverse inclusion of their duals, $C_1^\circ \supseteq C_2^\circ$.

(v) The convex dual of the intersection of closures of cones $C_1$ and $C_2$ is the convex union (convex hull of the union) of their duals, $(\bar{C}_1 \cap \bar{C}_2)^* = \bar{C}_1^\circ + \bar{C}_2^\circ$, where the right hand side is written as a Minkowski sum, which for cones coincides with the convex hull of the union. The converse identity holds as well, $(C_1 + C_2)^* = \bar{C}_1^\circ \cap \bar{C}_2^\circ$. Similarly, if $C_1$ and $C_2$ are open and salient, then $(C_1 \cap C_2)^\circ = C_1^\circ + C_2^\circ$ and $(C_1 \cup C_2)^\circ = C_1^\circ \cap C_2^\circ$.

For convenience, we extend this duality to cone bundles. That is, if $C \rightarrow M$ is a cone sub-bundle of a vector bundle $E \rightarrow M$, then the convex dual cone bundle $C^* \rightarrow M$ is the cone sub-bundle of $E^* \rightarrow M$ such that each fiber $C^*_x$ is the convex dual of the corresponding fiber $C_x$, for $x \in M$. Similarly, we can define the strict convex dual cone bundle $C^\circ \rightarrow M$. The operations of intersection (\cap) and convex union (+) are extended to cone bundles in the same way. Clearly both $C \rightarrow M$ and $C^\circ \rightarrow M$ are cone bundles, that is objects of $\mathcal{CBnd}$. On the other hand, $C^* \rightarrow M$ may not be, though again we shall refer to it as a cone bundle anyway.

**Definition 16.** The cone bundle of future directed, timelike vectors $\Gamma \subset (TM)^0$ is defined to be strict convex dual of the cone bundle of future directed, spacelike covectors, $\Gamma = (\Gamma^\circ)^\circ$. Let $\hat{\Gamma} \subset (TM)^0$ be called the cone bundle of future directed, causal vectors. Incidentally, $\hat{\Gamma} = (\Gamma^\circ)^\circ$. Let $\mathcal{C} = \partial \Gamma = \partial \hat{\Gamma} \subset (TM)^0$ be called the cone bundle of future directed, outer ray vectors. The past directed versions are obtained by reflecting the cones through the origin.

While the definition of $\Gamma^\circ$ is primitive, it is also straightforward to show that in fact it is the strict convex dual of $\Gamma$, that is, $\Gamma^\circ = (\Gamma^*)^\circ$.

These definitions extend straightforwardly to hyperbolic systems with constraints. Suppose the adapted equation form $(f \oplus h, \bar{F}^* \oplus_M \bar{E}^*)$ defines the corresponding compound system, with $(f, \bar{F}^*)$ the hyperbolic and $(h, \bar{E}^*)$ the consistency subsystems. The compound system is quasilinear, but the consistency subsystem is linear in the constraints (sections of $E \rightarrow M$). Therefore the cones of inner characteristic and spacelike covectors, as well as outer ray and timelike vectors depend only on the field bundle $F$ and not on the constraint bundle $E$. Therefore, for a hyperbolic system on $F \rightarrow M$ with hyperbolically integrable constraints, we define the cone bundles $C^*$, $\Gamma^\circ$, and $\bar{\Gamma}^\circ$ over $F$ as those associated to the corresponding compound system.
Note that these cones are defined intrinsically (independently of any background metric) by the geometry of a first order, quasilinear, symmetric hyperbolic PDE system, or equivalently algebraic and geometric properties of its principal symbol. Of course, the dependence on a background metric (if one is present) may appear implicitly through the principal symbol, as it does for linear and semi-linear wave equations. This remark and the significance of the geometry of $Γ$ and $Γ^\circ$ in the domain of dependence theorem to be described later is sufficient to motivate the fact that such cones are the central objects of the causal structure of classical field theory. The following definition leads to the study of conal manifolds and conal geometry alluded to in the introduction to this section. Many of the results in conal geometry do not depend on the provenance of these cones from the characteristic or spacelike covectors of a hyperbolic PDE system. However, the later results concerning causal relations in PDE and classical field theories do use that connection.

**Definition 17.** Given a manifold $M$, a chronal cone bundle on $M$ is a cone bundle $C \to M$ (in the sense of $\mathcal{CBndl}$) enveloped by the tangent bundle $TM \to M$ such that each fiber $C_x$, $x \in M$, is a proper cone (non-empty, open, convex, salient). The elements of $C$ are future directed, timelike vectors. The strict convex dual $C^\circ \to M$, enveloped by the cotangent bundle $T^*M \to M$, is the corresponding spacelike cone bundle. The elements of $C^\circ$ are future oriented, spacelike covectors. The corresponding cone bundles $\bar{C} \to M$ and $\bar{C}^\circ \to M$ are referred to, respectively, as the causal and cocausal cone bundles.

Note that the open convex dual of a proper cone is again a proper cone. Prototypical examples of chronal and spacelike cone bundles are the pullbacks of the cone bundles of timelike vectors and spacelike covectors along a section $\phi: M \to F$, $C = \phi^* \Gamma$ and $C^\circ = \phi^* \Gamma^\circ$.

**Definition 18.** Let the category of chronal cone bundles $\mathcal{ChrBndl}$ be the subcategory of $\mathcal{CBndl}$ where each object $C \to M$ must be enveloped by the tangent bundle $TM \to M$ and the morphisms must be restrictions of the natural maps between tangent bundles induced by base space maps. Similarly, let the category of spacelike cone bundles $\mathcal{SpBndl}$ be the subcategory of $\mathcal{CBndl}$ where each object $C^\circ \to M$ must be enveloped by the cotangent bundle $T^*M \to M$ and the morphisms must be restrictions of the natural maps between cotangent bundles induced by base space maps.

Note that the morphisms are restricted to be open embeddings between base spaces to make sure that their natural extensions to the tangent and cotangent bundles are defined unproblematically.

At this point, one may recall some standard notions of Lorentzian geometry, as long as they are defined only in terms of timelike cones, and apply them to the geometry cone bundles. Many of the standard theorems translate as well, some directly and others with some extra effort. We restrict ourselves to those that are relevant to the issues at hand.

Fix a chronal cone bundle $C \to M$ on a spacetime manifold $M$. Note that any chronal cone bundle is time oriented, since by definition the fibers consist of single cones rather than double cones like in the Lorentzian case. By assumptions the cones are directed into the future.
Definition 19. A smooth curve $\gamma$ in $M$ is called future directed, timelike if the tangent to $\gamma$ is everywhere contained in $C$. The chronal precedence relation $I^+ \subseteq M \times M$ (also $I^+_C$) is defined as

$$I^+ = \{(x, y) \in M \times M \mid \exists \gamma, \text{ future directed, timelike curve from } x \text{ to } y\}. \quad (32)$$

When $(x, y) \in I^+$, we say that $x$ chronologically precedes $y$ and also write $x \ll y$.

If we replace $C$ by $\bar{C}$ in the above definitions, we obtain causal curves and the causal precedence relation $J^+ \subseteq M \times M$, denoted $x < y$. The inverse relations are written $I^-$ and $J^-$. It is immediate that $I^+$ is an open, transitive relation ($x \ll y$ and $y \ll z$ implies $x \ll z$). Using this relation, we can define the usual causal hierarchy.

Definition 20. (i) If $I^+$ is irreflexive ($x \not\ll x$ or, equivalently, no closed timelike curves exist), then $C$ is chronological.

(ii) Given an open $N \subseteq M$, $I^+_C \mid N$ and $I^+_C \cap (N \times N)$ are both relations on $N \times N$. We say that $N$ is chronologically compatible if both of these relations coincide. More conventionally, this means that any two points of $N$ that can be joined by a timelike curve in $M$ can also be joined by a timelike curve in $N$.

(iii) An open $N \subseteq M$ is called chronologically convex if any timelike curve that joins any two points of $N$ must also lie in $N$, which is a stronger condition than chronological compatibility.

(iv) The chronal cone bundle $C$ is said to be strongly chronological if it is chronological and for every $x \in M$ and every neighborhood $N \subseteq M$ of $x$, there exists a smaller open neighborhood $L \subseteq N$ that is chronologically convex.

(v) The chronal cone bundle $C$ is said to be stably chronological if there exists another chronal cone bundle $C'$ that is itself chronological and an open neighborhood of the closure of $C$, that is, $C \setminus \{0\} \subset C'$. For spacelike cone bundles, stable chronology is equivalent to the reverse inclusion $C^{\ominus} \setminus \{0\} \subset C^{\ominus}$. The full subcategories $\text{ChrBndl}_{sc}$ and $\text{SpBndl}_{sc}$ of $\text{ChrBndl}$ and $\text{SpBndl}$ contain only stably chronological cone bundles as objects.

Each of these definitions has an obvious analog when the adjectives chronological or timelike are replaced by causal.

Note that stably chronological is equivalent to stably causal, so these terms will be used interchangeably. Moreover, the chronological chronal cone bundle $C'$ such that contains a stably chronological chronal cone bundle $C$ can itself be chosen to be stably chronological. Next we turn from curves to surfaces.

Definition 21. Each of the following concepts may be prefaced with $C$- or $C^{\ominus}$- to be more specific.

(i) An oriented codim-1 surface $S \subseteq M$ is called future oriented, spacelike if its oriented conormals are everywhere contained in $C^\ominus$.

(ii) A codim-1 surface $S \subseteq M$ is called achronal if it $(S \times S) \cap I^+ = \emptyset$, that is, no two points of $S$ are connected by a timelike curve. Similarly, $S$ is acausal when $(S \times S) \cap J^+ = \emptyset$.

(iii) A codim-1 surface $S \subseteq M$ is called Cauchy if it is acausal and every inextendible causal curve intersects $S$ exactly once.
(iv) A chronal cone bundle $C \to M$ is called globally hyperbolic if there exists a Cauchy surface $S \subseteq M$. A spacelike cone bundle $C^\circ \to M$ is called globally hyperbolic if $C \to M$ is. The full subcategories $\text{ChrBndl}_H$ and $\text{SpBndl}_H$ of $\text{ChrBndl}$ and $\text{SpBndl}$ contain only globally hyperbolic cone bundles as objects.

Next we define some commonly used domains. Fix $S \subset M$ to be a $C$-acausal codim-1 submanifold, such that either $S$ is closed or $\bar{S} \subset M$ is a submanifold with boundary.

Definition 22. (i) The future/past domain of influence $I^\pm(N)$ of a subset $N \subseteq M$ is the set of points $y \in M$ such that there exists $x \in N$ with either $x \ll y$ (+) or $y \ll x$ (−). Let $I(N) = I^+(N) \cup I^-(N)$.

(ii) The domain of dependence $D(S)$ is the largest open subset of $M$ for which $S$ is a Cauchy surface. More commonly, $D(S)$ is the set of points $y \in M$ such that every inextensible timelike curve through $y$ intersects $S$. Let $D^\pm = D(S) \cap I^\pm(S)$.

(iii) An open subset $L \subseteq M$ is lens-shaped with respect to $S$ if it can be smoothly factored as $L \cong (-1,1) \times S$, with $t: L \to (-1,1)$ denoting the projection onto the first factor (the time function), such that the level set $t = 0$ is $S$ and all other level sets are spacelike as well as share the same boundary as $S$ in $M$ (which may be empty).

The literature in relativity and Lorentzian geometry mostly makes use of the notion of global hyperbolicity as given above, but specialized to Lorentzian cone bundles. On the other hand, the literature on symmetric (and regular) hyperbolic PDE systems mostly makes use of lens-shaped domains. It is a non-trivial fact that these two notions coincide. The argument is essentially that the time function of a lens-shaped domain foliates it with Cauchy surfaces. Conversely, a globally hyperbolic cone bundle admits a time function and a smooth factorization that turns it into a lens-shaped domain (glossing over some details related to spatial compactness). The original argument establishing the converse link in Lorentzian geometry is due to Geroch [49]. However, his argument only established the existence of a continuous time function. The details necessary to establish the smooth version of the result are due to more recent work of Bernal and Sanchez [18,17]. The very recent result by Fathi and Siconolfi [43], using completely different methods, established the existence of a smooth time function and factorization for a class of cone bundles sufficient for the current discussion.

Proposition 3. An open subset $D \subseteq M$ is $C$-lens-shaped with respect to $S \subseteq D$ iff it is globally $C|_D$-hyperbolic, with $S$ being $C|_D$-Cauchy.

The last point we address in this section is the kind of maps we allow between chronal cone bundles. Let $\text{ChrBndl}(M)$ and $\text{SpBndl}(M)$ denote the respective subcategories of $\text{ChrBndl}$ and $\text{SpBndl}$ where the objects are restricted to have the same base manifold $M$ and all morphisms must restrict to the identity on the base $M$. Essentially, the only allowed morphisms between the cone bundles in these categories are inclusions. These inclusions, and hence the morphisms, of the corresponding categories, implement a partial order on chronal and spacelike cone bundles, ordering the bundles by speed. Consider chronal cone bundles $C_1 \to M$ and $C_2 \to M$, as well as the corresponding spacelike cone bundles
The morphism
\[
\begin{array}{c}
C_1 \rightarrow C_2 \\
\downarrow \text{id} \quad \downarrow \text{id} \\
M \quad M
\end{array}
\] should read as
\[
\begin{array}{c}
\text{slow} \subseteq \text{fast} \\
\downarrow \text{id} \quad \downarrow \text{id} \\
M \quad M
\end{array}
\] (33)

Fast and slow are used in the conventional sense: \(C_2\)-timelike curves can propagate signals faster than \(C_1\)-timelike ones. This ordering is reversed for spacelike cone bundles. The morphism
\[
\begin{array}{c}
C_2^\circ \rightarrow C_2^\circ \\
\downarrow \text{id} \quad \downarrow \text{id} \\
M \quad M
\end{array}
\] should read as
\[
\begin{array}{c}
\text{fast} \subseteq \text{slow} \\
\downarrow \text{id} \quad \downarrow \text{id} \\
M \quad M
\end{array}
\] (34)

**Definition 23.** Given a chronal cone bundle \(C \rightarrow M\), a section \(\phi: M \rightarrow F\) is called \(C\)-slow (also slower than \(C\)) if \(\phi^* \Gamma \subseteq C\), that is, signals propagated by \(C\)-timelike curves can travel at least as fast as signals propagated by \(\phi\)-timelike curves. Also, \(\phi\) is said to be strictly slower than \(C\) if \(\phi^* \Gamma \subset C\). Given a convex dual spacelike cone bundle \(C^\circ \rightarrow M\), the section \(\phi\) is (strictly) \(C^\circ\)-slow (also (strictly) slower than \(C^\circ\)) if it is (strictly) \(C\)-slow.

In terms of spacelike cone bundles, \(C^\circ\)-slowness is equivalent to \(C^\circ \subseteq \phi^* \Gamma^\circ\), while strict \(C^\circ\)-slowness is equivalent to \(C^\circ \subset \phi^* \Gamma^\circ\).

Consider two spacelike cone bundles \(C_1^\circ \subseteq C_2^\circ\) and two sections \(\phi_i: M \rightarrow F\), \(i = 1, 2\), such that \(\phi_i\) is \(C_i^\circ\)-slow. Then clearly \(\phi_2\) is also \(C_1^\circ\)-slow, while \(\phi_1\) need not be \(C_2^\circ\)-slow. This observation shows that slow sections can be pulled back to slow sections along morphisms in the \(\mathcal{SpBndl}(M)\) category. It is elaborated on in the next section.

Let us expand our attention to the larger categories \(\mathcal{ChrBndl}\) and \(\mathcal{SpBndl}\), where the base manifold is no longer fixed, however only open embeddings are allowed as morphisms between base spaces (which naturally extend to morphisms of the total spaces). Consider a pair of objects and a morphism between them in the \(\mathcal{ChrBndl}\) category.

\[
\begin{array}{c}
C_1 \xrightarrow{T \chi} C_2 \\
\downarrow \chi \quad \downarrow \chi \\
M_1 \quad M_2
\end{array}
\] (35)

It is interesting to note that the same base space morphism \(\chi: M_1 \rightarrow M_2\) extends to a morphism between the convex dual cone bundles in the \(\mathcal{SpBndl}\) category,

\[
\begin{array}{c}
C_1^\circ \xrightarrow{(T^* \chi)^{-1}} C_2^\circ \\
\downarrow \chi \quad \downarrow \chi \\
M_1 \quad M_2
\end{array}
\] (36)
only if the two cone bundles agree exactly, that is $\chi_1 C_1 = C_2$, or equivalently $C_1^\# = \chi^\ast C_2^\#$. In this case, it is clear that the relations $I_1^+ C_1^\ast$ and $I_2^+ C_2^\ast$ on the source manifold $M_1$ are the same. However, it may not be true that the chronological precedence relation $I_{C_1}^+ (M_1) = \chi \times \chi (I_{C_1}^+)$ is necessarily the same as the restriction $I_2^+ \cap (\chi (M_1) \times \chi (M_1))$ on the target manifold $M_2$. The two are only equal when $\chi (M_1)$ is a $C_2$-chronologically compatible subset of $M_2$.

**Definition 24.** A $\mathsf{ChrBndl}$ morphism $T \colon C_1 \to C_2$ in is called chronologically compatible if $\chi (M_1)$ is a $C_2$-chronologically compatible subset of $M_2$. Similarly, the morphism is called chronologically convex if $\chi (M_1)$ is a $C_2$-chronologically convex subset of $M_2$.

A $\mathsf{SpBndl}$ morphism $T^\ast \chi \colon C_1^\# \to C_2^\#$ is called chronologically compatible or convex if the corresponding $\mathsf{ChrBndl}$ morphism $T \chi \colon C_1 \to C_2$ exists and is respectively chronologically compatible or convex.

When $C_1 \to M$ is a globally hyperbolic chronal cone bundle, there is no distinction between and chronally convex $\mathsf{ChrBndl}$ morphisms. Thus, in this context, these terms will sometimes be used interchangeably.

**Lemma 2.** Consider chronal cone bundles $C_i \to M_i$, $i = 1, 2$, and an open embedding $\chi_1 \colon M_1 \to M_2$ such that the induced morphism $T \chi \colon C_1 \to C_2$ in $\mathsf{ChrBndl}$ is chronally related. Then $T \chi$ is also chronally convex.

**Proof.** Consider any two points $p, q \in M_1$ that are $C_1$-chronally related. Then, by chronal compatibility, $\chi (p), \chi (q) \in M_2$ are also $C_2$-chronally related. By a simple application of Zorn’s lemma, among all timelike curves passing through $\chi (p)$ and $\chi (q)$ there must exist some maximal (inextensible) curves $\gamma_2$. For each such curve, the preimage $\gamma_1 = \chi^{-1} (\gamma_2 \cap \chi (M_1))$ is also inextensible. Assume for the moment that chronal convexity fails. Then we can chose $\gamma_2$ such that it has at least one point not contained in $\chi (M)$, which implies that $\gamma_1$ consists of more than one connected component, each of which is inextensible. On the other hand, global hyperbolicity implies that each component of $\gamma_1$ intersects a $C_1$-Cauchy surface $\Sigma_1 \subseteq M$ and so the same can be said about $\gamma_2$ and $\Sigma_2 = \chi (\Sigma_1)$. In other words, $\gamma_2$ intersects $\Sigma_2$ more than once, which is impossible, since from chronal compatibility $M_2$ must be achronal. Therefore, $T \chi$ must also be chronally convex. $\Box$

These conditions on cone bundle morphisms will become important in the formulation of the generalized classical or quantum LCFT functors. In particular, it would be extremely convenient for the construction of an LCFT, namely verifying the Isotony property (cf. Sect. 7.1), if it were possible, in a spacetime with a globally hyperbolic chronal cone bundle, to be able to construct a Cauchy surface by extending an existing acausal surface, which may not be intersected by all inextensible causal curves. However, this question does not seem to have received much attention, even in the more restricted setting of Lorentzian geometry. Therefore, we simply formulate this property as a plausible conjecture, whose validity would have to be explored in future work.

**Conjecture 1.** Consider globally hyperbolic spacelike cone bundles $C^\# \to M$ and $C'^\# \to M'$, together with an embedding $\chi_1 \colon M \to M'$ that induces a chronologically compatible morphism $T^\ast \chi \colon C^\# \to C'^\#$. If $\Sigma \subseteq M$ is a $C^\#$-Cauchy surface,
then for any compact subset \( K \subseteq \Sigma \), there exists a \( C^{0,\infty} \)-Cauchy surface \( \Sigma' \subset M' \) such that \( \Sigma' \) agrees with \( \chi(\Sigma) \) on \( \chi(U) \), where \( U \) is a neighborhood of \( K \) in \( \Sigma \).

We say that the \( C^{0,\infty} \)-Cauchy surface \( \Sigma' \) extends \( K \). If there exists a single \( C^{0,\infty} \)-Cauchy surface \( \Sigma' \) that extends every compact \( K \subseteq \Sigma \), we say that \( \Sigma' \) extends \( \Sigma \). It is possible that there exists no \( \Sigma' \) that extends a given \( \Sigma \). Think of a lens-shaped domain in \( M' \) whose spacelike boundary is not smooth.

4.2. Slow sections. As we have noticed above, a distinctive feature of quasilinear PDE systems, compared to linear ones, is the field dependence of their causal structure, as embodied in the geometry of the cone bundles of timelike vectors and spacelike covectors \( \Gamma \) and \( \Gamma^\circ \), whose base space is the field bundle \( F \) instead of just the manifold \( M \). This means that any two sections \( \phi, \psi : M \to F \) of the field bundle (solution or not) define by pullback the cone bundles \( \phi^* \Gamma \to M \) and \( \psi^* \Gamma \to M \). A priori, these cone bundles need not be related to each other in any way. This complicates the analysis of the causal structure in contexts where multiple solutions or sections have to be considered simultaneously. In particular, such contexts arise when considering nontrivial subsets of the space of solution sections in classical field theory, probability distributions on the space of solution sections in classical statistical mechanics, and deformation quantization of the space of solution sections in quantum field theory.

It is fortunate that multiple cone bundles can be easily combined via intersection or convex union. Thus, if we have two sections \( \phi, \psi : M \to F \), we can form a chronal cone bundle from their convex union, \( C_{\phi,\psi} = (\phi^* \Gamma + \psi^* \Gamma) \). Then clearly both \( \phi \) and \( \psi \) are \( C_{\phi,\psi} \)-slow, or equivalently \( C_{\phi,\psi} \) is faster than both \( \phi \) and \( \psi \). Similarly, in terms of spacelike cone bundles, both \( \phi \) and \( \psi \) are \( C_{\phi,\psi}^\circ \)-slow, where \( C_{\phi,\psi}^\circ = \phi^* \Gamma^\circ \cap \psi^* \Gamma^\circ \). The only caveat is that the combined cones may fail to be proper, \( C_{\phi,\psi} \) could fail to be salient or \( C_{\phi,\psi}^\circ \) could be empty. In this case, we call the sections \( \phi \) and \( \psi \) chronologically incomparable. We can generalize this construction to any collection of sections \( \{\phi_i\}_{i \in I} \). Define \( C_I = \sum_{i \in I} \phi_i^* \Gamma \) and \( C_I^\circ = (\bigcap_{i \in I} \phi_i^* \Gamma^\circ)^\circ \). If either of these cones fails to be proper, we again call the collection of sections \( \{\phi_i\}_{i \in I} \) chronologically incomparable.

This correspondence between sections and collectively faster cones can be easily inverted.

**Definition 25.** Given a chronal cone bundle \( C \to M \) or the corresponding spacelike cone bundle \( C^\circ \to M \), we denote the set of all sections strictly slower than \( C \to M \) by \( \Gamma(F,C) \subseteq \Gamma(F) \), or also \( \Gamma(F,C^\circ) \). We can similarly restrict the space of solutions, \( S(F,C) = S(F,C^\circ) = S(F) \cap \Gamma(F,C) \). It follows immediately from the definition that given two chronal cone bundles \( C_1 \subseteq C_2 \) (equivalently \( C_2^\circ \subseteq C_1^\circ \) or \( C_1 \) slower than \( C_2 \)), we have the inclusions

\[
\Gamma(F,C_1) \subseteq \Gamma(F,C_2) \quad \text{and} \quad S(F,C_1) \subseteq S(F,C_2).
\]

(37) Incidentally, we denote by \( \Gamma_{sc}(F) \subseteq \Gamma(F) \) the space of stably causal sections and by \( \Gamma_{sh}(F) \subseteq \Gamma_{sc}(F) \) the space of all globally hyperbolic sections. The same notation also extends to solution spaces: \( S_{sc}(F) = S(F) \cap \Gamma_{sc}(F) \), and \( S_{sh}(F) = S(F) \cap \Gamma_{sh}(F) \).
If $C \to M$ is stably causal, then all sections in $\Gamma(F,C)$ are a fortiori also stably causal. It then follows immediately from basic properties of stable causality that

$$\Gamma_{sc}(F) = \bigcup_{\text{stably causal } C} \Gamma(F,C),$$

(38)

where the union is over all stably causal cone bundles $C \to M$. It would be desirable to make a similar statement about $\Gamma_H(F)$. However, that would entail showing that global hyperbolicity is stable in a certain technical sense. This is a known result for Lorentzian cone bundles \[16\], but remains to be proven in the general case. However, since $\Gamma_H(F) \subseteq \Gamma_{sc}(F)$ (and both are topologized by their inclusion in $\Gamma(F)$), the sets $\Gamma_H(F,C) = \Gamma_H(F) \cap \Gamma(F,C)$, where $C$ is stably causal, do furnish a cover of $\Gamma_H(F)$. We show below that each $\Gamma(F,C)$ is open in $\Gamma(F)$ in the Whitney fine topology. Thus, in this topology, $\Gamma(F,C)$ and $\Gamma_H(F,C)$, with $C$ stably causal, furnish open covers of $\Gamma_{sc}(F)$ and $\Gamma_H(F)$, respectively. An observation that is important for later developments in Sect. \[.\]

While we could continue to work with an open cover by the sets $\Gamma_H(F,C)$, it is simpler and likely not to make a large difference to instead assume the following

**Conjecture 2 (Stability of global hyperbolicity).** The space of globally hyperbolic sections $\Gamma_H(F)$ is open in the space of all sections $\Gamma(F)$, provided with the Whitney fine $C^0$ topology. Moreover, the following identity holds

$$\Gamma_H(F) = \bigcup_{\text{globally hyperbolic } C} \Gamma(F,C).$$

(39)

This conjecture also implies the stability of global hyperbolicity in a common Fréchet topology on $\Gamma(F)$, which coincides with the Whitney fine $C^\infty$ topology, as well, as it is finer (has more open sets) than the Whitney fine $C^0$ topology.

Recall that **Whitney fine $C^0$ topology** (also known as the **Whitney strong $C^0$ topology** or the **wholly open $C^0$ topology**) on the space of sections of the bundle $F \to M$ is defined as follows \[62, 75\]. Consider sets of the form $U(\phi, V)$, with $\phi : M \to F$ a section and $V \subseteq F$ an open neighborhood of the image $\phi(M)$, consisting of all sections $\psi : M \to F$ such that $\psi(M) \subseteq V$. These sets form a sub-base for the open sets of the Whitney fine topology on $\Gamma(F)$. This topology should be contrasted with the **compact open $C^0$ topology** (also known as the Whitney weak $C^0$ topology), where a sub-base for the open sets consists of sets of the form $U(\phi, K, V)$, with $\phi : M \to F$ a section, $K \subseteq M$ compact and $V \subseteq F|_K$ an open neighborhood of $\phi(K)$, consisting of all sections $\psi : M \to F$ such that $\psi(K) \subseteq V$. These two topologies coincide only if $M$ is compact.

The $C^\infty$ versions of these topologies are easily defined in terms of jet prolongations. As before, consider the sets $U_k(\phi, V)$, but now with $V \subseteq J^kF$ an open neighborhood of $J^k\phi(M)$, and $U_k(\phi, K, V)$, but now with $V \subseteq J^kF|_K$ an open neighborhood of $J^k\phi(K)$. They form sub-bases for the open sets of the Whitney fine $C^k$ and compact open $C^k$ topologies, respectively. The index $k$ may also take on the value $\infty$. With these topologies, of which the compact open one is somewhat more commonly used, $\Gamma(F)$ is a Fréchet space.

**Theorem 1.** Given a chronological cone bundle $C \to M$, the subset of smooth sections strictly slower than $C$, $\Gamma(F,C) \subseteq \Gamma(F)$, is open in the Whitney fine $C^0$ topology.
Proof. Pick a section $\phi \in \Gamma(F, C)$. We will construct a Whitney fine $C^0$ neighborhood of it that is contained entirely in $\Gamma(F, C)$. Every element $\psi$ of such a neighborhood would satisfy $\tilde{C}^0 \subseteq \psi^*\Gamma^0$ (recall that convex duality reverses inclusion).

The fibers of the cone bundle $\tilde{C}^0$ are cones of compact section. We can collect these sections into a continuous fibration $K \to M$ with compact fibers as follows. Pick an affine sub-bundle $\tau^* \subset T^*M$ whose fibers are of fiber codim-1 and intersect the fibers of $\tilde{C}^0$ on compact sets with non-empty interior. Such an affine sub-bundle is easily constructed by modeling it on the vector sub-bundle of $T^*M$ that is annihilated by a section of $C \to M$, which always exists, since each fiber is contractible. Let $K$ be the union of the boundaries of these fiberwise cross sections, $K = \partial(\tau^* \cap \tilde{C}^0)$. Essentially, $K \to M$ is a topological sphere bundle due to convexity of the fibers of $\tilde{C}^0$. From the strict $C$-slowness hypotheses, we can conclude that $\phi^*\Gamma^0$ is an open neighborhood of $K$.

Next we pick special open neighborhoods for each point $(x, u, p) \in F \times_M T^*M$ with $u = \phi(x)$ and $(x, p) \in K$. Let $j: M \to \tilde{S}^2F^* \times_M TM$ be the adapted principal symbol of the symmetric hyperbolic PDE system under consideration. Consider the following composition of continuous bundle maps

$$
F \times_M T^*M \to (\tilde{S}^2F^* \times_M TM) \times_M T^*M \to \tilde{S}^2D^* \quad \eta \mapsto (\tilde{f}(\eta), q) \mapsto (\eta, \tilde{f}(\eta) \cdot q).
$$

We note the following facts: (i) the subset of $\tilde{S}^2F^*$ consisting of orientation positive definite bilinear forms in the fibers is open, (ii) the above composition of continuous maps sends $(x, u, p) \in F \times_M T^*M$ to $(x, u) \times_M \tilde{C}^0$ in this open set, (iii) $F \times_M (\phi^*\Gamma^0)$ is an open neighborhood of $(x, u, p)$. These observation allow us to conclude that there exists a neighborhood $\tilde{U}$ of $(x, u, p)$ that maps via the composition $[\tilde{f}]$ to orientation positive definite bilinear forms in $\tilde{S}^2F^*$, such that $\tilde{U} \subseteq F \times_M (\phi^*\Gamma^0)$. Choosing a local bundle trivialization $\nu$ contained within the base projection of $\tilde{U}$ shows the existence of an open neighborhood of $(x, u, p)$ of the form $\nu(U \times V \times W) \subseteq \tilde{U}$, where $U$ is open in $M$, $V$ is adapted to the fibers of $F$ and $W$ is adapted to the fibers of $T^*M$. This means that for any $(y, v, q) \in \nu(U \times V \times W)$ we have that $q$ is $(y, v)$-spacelike and future oriented.

Next, for each $x \in M$, we let $u = \phi(x)$ and we build special neighborhoods of $(x, u) \in F$ and $(x, u) \times_M K_x \in F \times_M T^*M$. For each $(x, u, p) \in (x, u) \times_M K_x$, let $\iota_{x,p}(U_{x,p} \times V_{x,p} \times W_{x,p})$ be the special neighborhood described in the previous paragraph. By compactness of the fiber $K_x$, we can choose finitely many $p_j$ such that the corresponding neighborhoods cover $(x, u) \times_M K_x$. Denote by $\pi: F \times_M T^*M \to F$ the canonical projection. Then, let

$$
\tilde{V} = \bigcap_j \pi \circ \iota_{x,p_j}(U_{x,p_j} \times V_{x,p_j} \times W_{x,p_j}),
$$

$$
\tilde{W} = \pi^{-1}(\tilde{V}) \cap \bigcup_j \iota_{x,p_j}(U_{x,p_j} \times V_{x,p_j} \times W_{x,p_j}).
$$

That is, $\tilde{V}$ is an open neighborhood of $(x, u) \in F$ and $\tilde{W}$ is an open neighborhood of $(x, u) \times_M K_x \subseteq F \times_M T^*M$. Moreover, by construction, for each $(x, v) \in \tilde{V}$ and each $(x, p) \in K_x$, the covector $p$ is $(x, v)$-spacelike, since $(x, v, p) \in \tilde{W}$. 


Next, by continuity of the section $\phi: M \to F$ and the bundle $K \to M$, for each $x \in M$, we can find an open neighborhood $U_x$ of $x$, trivializations $t_x$ of $F$ and $t'_x$ of $T^*M$ on $U_x$ (with $t_x = \pi \circ t'_x$), an open neighborhood $t_x(U_x \times V_x)$ of $(x, u)$ in $F$, and an open neighborhood $t'_x(U_x \times V_x \times W_x)$ of $(x, u, q)$ in $T^*M$, such that $t'(U_x) \subseteq t'_x(U_x \times V_x)$ and $K|U_x \subseteq t'_x(U_x \times V_x \times W_x)$. By construction, for every $(y, v, q) \in t'_x(U_x \times V_x \times W_x)$, and a fortiori for every $(y, q) \in K|U_x$, we have that $q$ is $(y, v)$-spacelike.

Finally, let

$$\mathcal{N} = \bigcup_{x \in M} t_x(U_x \times V_x),$$

which is an open neighborhood of $\phi(M)$ in $F$ such that, for any other section $\psi: M \to F$ with $\psi(M) \subseteq \mathcal{N}$, each $(x, p) \in K$ is $\psi$-spacelike. By properties of convex cones, this implies that $\mathcal{C}^\circ \subset \psi^* \mathcal{C}^\circ$. The set $U(\phi, \mathcal{N})$ of sections whose images are contained in $\mathcal{N}$ is then a Whitney fine $C^0$ neighborhood of $\phi$ that is contained in $\Gamma(F, \mathcal{C})$. □

This theorem will be used in the following section when discussing the geometry and topology of the phase space of classical field theory.

An analogous result for Lorentzian cone bundles was first given in [82]. The above proof is significantly more complicated because the fibers of both $\mathcal{C}^\circ$ and $\phi^* \Gamma^\circ$ cone bundles need not have elliptical cross sections, like they do in the Lorentzian case.

### 4.3. PDE theory

As we shall see in Sect 5, PDE theory can be seen as a tool for a non-perturbative construction of classical theory. The following theorems are the main workhorses in this construction: local existence, possible global existence, as well as uniqueness, domain of dependence and finite propagation speed. The details of these results can be found in the standard literature on hyperbolic PDEs [69,79,67]. See also more in depth discussion and bibliography in [24].

**Theorem 2 (Local Existence).** Consider a spacelike initial data set $(S, \varphi)$. There exists an open neighborhood $U$ of $S$ in $M$, $S \subset U \subseteq M$, and a solution section $\phi: U \to F|_U$ that agrees with $(S, \varphi)$, that is $\phi|_S = \varphi$ on $U$.

Recall that, for a linear PDE system, cone bundle $\Gamma^\circ$ does not depend on the dynamical fields. Therefore, it can be seen as a cone bundle over the space-time manifold $M$, $\Gamma^\circ \to M$. For linear equations, the existence result can be strengthened to a global one [65, Ch.XIII], [79], [104, Ch.IV], [99, Ch.7], [7], [108].

**Theorem 3 (Global Existence).** Consider a spacelike initial data set $(S, \varphi)$ and suppose that the PDE system is linear. If the spacelike cone bundle $\Gamma^\circ$ is globally hyperbolic and the surface $S$ is $\Gamma^\circ$-Cauchy, then, under generic conditions, there exists a global solution section $\phi: M \to F$ that agrees with $(S, \varphi)$, that is $\phi|_S = \varphi$.

**Theorem 4 (Uniqueness).** Suppose that $\phi: M \to F$ is a solution section, that the spacelike cone bundle $C^\circ = \phi^* \Gamma^\circ$ is globally hyperbolic and that $S \subset M$ is a $C^\circ$-Cauchy surface. Then $\phi$ is the unique solution section on $M$ that agrees with the initial data $(S, \phi|_S)$. 
There is a more local, though equivalent, version of the uniqueness theorem, that usually goes under a different name.

**Corollary 1 (Domain of Dependence).** Let \((S, \phi)\) be an initial data set and \(\phi: U \to F|_U\) be solution section defined on an open neighborhood \(U \supset S\) such that \(\phi|_S = \phi\). If \(V \subseteq U\) is a lens-shaped domain with respect to \(S\) and the spacelike cone bundle \(\phi^* \Gamma^\oplus\), then \(\phi\) is the unique solution section on \(V\) that agrees with \((S, \phi)\).

*Proof.* By Prop. 3, the interior of \(V\) is globally hyperbolic and \(S\) is Cauchy with respect to \(\phi^* \Gamma^\oplus\). The result then follows directly from Thm. 4. \(\Box\)

The same result can also be interpreted as showing finite speed of propagation of disturbances.

**Corollary 2 (Finite Propagation Speed).** Suppose \(\phi: M \to F\) is a solution section such that the spacelike cone bundle \(C^\oplus = \phi^* \Gamma^\oplus\) is globally hyperbolic and the surface \(S \subseteq M\) is \(C^\oplus\)-Cauchy. If \(\phi'\) is another solution section whose restriction to \(S\) differs from \(\phi\) only on an open submanifold \(S' \subseteq S\), then \(\phi\) and \(\phi'\) agree in the interior of the complement of the domain of influence \(I_C(S')\).

In other words, if a disturbance is confined to \(S'\), it does not propagate faster than allowed by the chronological cones \(C = \phi^* \Gamma\) of the undisturbed solution.

*Proof.* It is a consequence of the definitions that the interior of the complement \(M \setminus I(S')\) is the domain of dependence \(D(S'')\), where \(S'' = (S \setminus S')^\circ\) and the interior is taken with respect to the submanifold topology on \(S\). Since \(D(S'')\) is globally hyperbolic, the result follows directly from Thm. 4. \(\Box\)

What the preceding theorems allow us to do is parametrize globally hyperbolic solution sections in terms of initial data sets. The parametrization is unfortunately not bijective. Any solution section \(\phi\) on \(M\) corresponds to many different initial data, one for each \(\phi\)-Cauchy surface \(S\), namely \((S, \phi|_S)\). On the other hand, not all initial data sets correspond to global solutions. There may exist data sets \((S, \phi)\) for which the corresponding solution, which is guaranteed to exist in a neighborhood of \(S\) develops singularities within \(\phi\) and so does not extend to a global solution. It is also possible that there exists a solution \(\phi\) that agrees with the given initial data, but is not globally hyperbolic on \(M\). Let us refer as globally hyperbolic initial data sets to those that come from the restriction of a globally hyperbolic solution section \(\phi\) to a \(\phi\)-Cauchy surface \(S \subseteq M\). Two globally hyperbolic initial data sets that come from the same solution section are said to be equivalent, which defines an equivalence relation. Thus, by the above uniqueness theorem, globally hyperbolic solution sections are in bijective correspondence with equivalence classes of globally hyperbolic initial data.

The last paragraph refers to solutions of hyperbolic PDE systems. What about those with hyperbolically integrable constraints? Consider the adapted equation form \((f \oplus c, F^* \oplus_M E)\), with \((f, F^*)\) and \((c, E)\) respectively the hyperbolic and constraints subsystems, and the corresponding hyperbolic compound system \((f \oplus h, F^* \oplus_M E^*)\) with identity \(h \circ c = q \circ f\). Certainly, if a section \(\phi: M \to F\) satisfies both \(f[\phi] = 0\) and \(c[\phi] = 0\), so does its restriction (or the restriction of \(f^k \phi\) for sufficiently high \(k\)) to any \(\phi\)-Cauchy surface. On the other hand, consider initial data \((S, \varphi)\) such that any extension of \(\varphi\) to a solution \(\phi\)
of \( f[\phi] = 0 \) on an open neighborhood of \( S \) satisfies the constraints \( c[\phi]|_S = 0 \). Is it necessarily true that an extension of \( \varphi \) to a solution \( \phi \) of \( f[\phi] = 0 \) on all of \( M \) also satisfies \( c[\phi] = 0 \) on all of \( M \)? That is indeed the case if the section \( \phi \) is globally hyperbolic on \( M \) with respect to the compound system, which includes the consistency subsystem \((h, \tilde{E}^*)\). To see this, notice that \( \phi \oplus c[\phi] \) gives a solution of the compound system on \( M \), since the consistency identity and \( q[0] = 0 \) yield
\[
h[c[\phi]] = q[f[\phi]] = q[0] = 0.
\] (44)

On the other hand, we know that the section \( \phi \oplus c[\phi] \) restricts to the initial data set \((S, \phi \oplus 0)\) for the compound system. But the consistency subsystem is linear and zero is always a solution, implying that \( \phi \oplus 0 \) is also a solution with the same initial data. Finally, if the section \( \phi \) is in fact globally hyperbolic with respect to the compound system, the uniqueness Thm. 4 shows that the two solutions are identical, that is that \( c[\phi] = 0 \) on \( M \). So, for symmetric hyperbolic systems with symmetric hyperbolically integrable constraints, globally hyperbolic solution sections \( \phi: M \to F \) are in bijective correspondence with equivalence classes of globally hyperbolic initial data sets \((S, \varphi)\) satisfying the constraints. Note that the notion of global hyperbolicity is in both cases with respect to the hyperbolic structure of the compound system \((f \oplus h, \tilde{F}^* \oplus_M \tilde{E}^*)\).

### 4.4. Linear inhomogeneous problems

In the preceding section we have discussed the Cauchy problem for quasilinear systems. If the system under consideration is linear, we can also discuss the linear algebra of the corresponding inhomogeneous problem
\[
f[\phi] = \tilde{\alpha}^*,
\] (45)
where \( \tilde{\alpha}^* \) is a compactly supported dual density, that is, a section of the bundle \( \tilde{F}^* \to M \).

The field independent spacelike cone bundle \( I^\otimes \) defined by the principal symbol \( f_{ab}^\mu \) endows \( M \) with causal structure. Let us assume that \( M \) is globally hyperbolic with respect to \( F^\otimes \). It is convenient to introduce spaces of sections with restricted supports, in particular in ways related to the causal structure.

**Definition 26.** Consider a vector bundle \( V \to M \). We define the following subspaces of the space of sections \( \Gamma(V) \):
\[
\Gamma_0(V) = \{ \phi \in \Gamma(V) \mid \text{supp} \phi \text{ is compact} \},
\] (46)
\[
\Gamma_+(V) = \{ \phi \in \Gamma(V) \mid \text{supp} \phi \text{ is retarded} \},
\] (47)
\[
\Gamma_-(V) = \{ \phi \in \Gamma(V) \mid \text{supp} \phi \text{ is advanced} \},
\] (48)
\[
\Gamma_{SC}(V) = \{ \phi \in \Gamma(V) \mid \text{supp} \phi \text{ is spacelike compact} \},
\] (49)
where retarded support, advanced support, or spacelike compact support means, respectively, that \( \text{supp} \phi \subset I^+(K) \), \( \text{supp} \phi \subset I^-(K) \), or \( \text{supp} \phi \subset I(K) \) for some compact \( K \subset M \).

The corresponding subspaces of the solution space \( \mathcal{S}(F) \) are defined as
\[
\mathcal{S}_{0,\pm,SC}(F) = \mathcal{S}(F) \cap \Gamma_{0,\pm,SC}(F).
\] (50)
4.4.1. Duhamel’s principle. It is a well known fact of classical PDE theory that for some equations (like the heat and wave equations) the solutions of Cauchy and inhomogeneous problems are equivalent. That is, a solution of one problem can be obtained from the other. This is usually known as Duhamel’s Principle. For completeness, we present it here in a form appropriate for our geometric formulation of symmetric hyperbolic systems. In addition, once we have the retarded and advanced Green functions for the inhomogeneous problem, we can construct an exact sequence that conveniently parametrizes the solution space of the linear PDE. This parametrization will be useful in Sects. 5.3.3 and 5.3.4 for the construction and analysis of the symplectic and Poisson structures of classical field theory.

Consider a Cauchy surface $\Sigma \subset M$ (where the Cauchy property holds with respect to $\Gamma^\otimes \to M$). Let $t: M \to \mathbb{R}$ be a time function, such that $t|_\Sigma = 0$ and $dt$ is future oriented. The ability to solve the Cauchy problem on $\Sigma$ means that we have access to

**Definition 27.** The Cauchy Green function $\tilde{G}_\Sigma: \Gamma_0(F|_\Sigma) \to \Gamma_{SC}(F)$ is a linear map uniquely defined by the requirement

$$\phi = \tilde{G}_\Sigma[\varphi] \implies f[\phi] = 0 \quad \text{and} \quad \phi|_\Sigma = \varphi.$$  \hspace{1cm} (51)

More explicitly, its action is given by the contraction of a bitensor distribution on $M \times \Sigma$ with the initial data on $\Sigma$:

$$\tilde{G}_\Sigma[\varphi]^a(x) = \int_{\Sigma} d\tilde{s} G_{ab}^a(x,s) \varphi^b(s),$$  \hspace{1cm} (52)

where we have used local coordinates $(x^i, u^a)$ on $F$ and $(s^j, v^b)$ on $F|_\Sigma$, and $d\tilde{s}$ is the coordinate volume $(n-1)$-form on $\Sigma$.

Consider a compactly supported dual density $\tilde{\alpha}^* \in \Gamma_0(\tilde{F}^*)$. The ability to solve the inhomogeneous problem $f[\phi] = \tilde{\alpha}^*$ means that we have access to

**Definition 28.** The retarded/advanced Green function $G_{\pm}: \Gamma_0(\tilde{F}^*) \to \Gamma_{SC}(F)$ is a linear map defined uniquely by the requirement

$$\phi_{\pm} = G_{\pm}[\tilde{\alpha}^*] \implies f[\phi_{\pm}] = \tilde{\alpha}^* \quad \text{and} \quad \text{supp} \phi_{\pm} \subseteq I^\pm(\text{supp} \tilde{\alpha}^*),$$  \hspace{1cm} (53)

where $+$ denotes the retarded and $-$ the advanced the boundary condition.

More explicitly, its action is given by the contraction of a bitensor distribution on $M \times M$ with the source term,

$$G_{\pm}[\tilde{\alpha}^*]^a(x) = \int_M G_{\pm}^{ab}(x,y) \alpha^*_b(y) dy,$$  \hspace{1cm} (54)

where we have used the local coordinates $(x^i, u^a)$ and $(y^j, v^b)$ on the two copies of $F$, and $\alpha^*_b(y) dy$ are the components of $\tilde{\alpha}^*$ in local coordinates, with $dy$ the coordinate volume form on $M$. We also use the notation $G(x|y) = G(x,y)$ for the Cauchy or retarded/advanced Green functions, to highlight the different roles of the arguments.

Consider any section of the field bundle $\phi: M \to F$ and define $\theta_{\pm} = \Theta(\pm t)$, with $\Theta(t)$ the Heaviside step function, which are the characteristic functions of
wherefrom the desired result follows immediately.

Global hyperbolicity of Cauchy Green function by $\tilde{\theta}$ identities

\[ f[\theta \pm \phi] = (f \cdot d\theta) \phi + \theta \pm f[\phi]. \]  

(55)

A calculation in local coordinates $(x^i, u^a)$ reveals $d\theta \pm = \pm \delta(t) \, dt$ and

\[ [(f \cdot d\theta) \phi]_a = \pm \delta(t) \, dt \int \tilde{f}_{ab} \delta^b = \pm \delta(t) \, dt \int \tilde{f}_{ab} \phi^b, \]  

(56)

where for convenience, we have defined $\tilde{f}_{ab}$ by the equation $dt \wedge \tilde{f}_{ab} = dt_i \tilde{f}_{ab}^i$.

We are now ready to prove what is known in classical PDE theory as

**Lemma 3 (Converse Duhamel’s Principle).** In local coordinates $(x^i, u^a)$ on $(t, s^l, v^b)$ on the two copies of $F$, with the second restricting to $(s^l, v^b)$ on $F|\Sigma$, the Cauchy Green function can be expressed in terms of the retarded/advanced Green functions as follows

\[ G^a_b(x, s) \, d\delta = \pm \sum_{\pm} \pm G^a_b(x|0,s) \tilde{f}_{ab}(0,s), \]  

(57)

where the pullback is along the inclusion $\iota : \Sigma \subset M$, corresponding to $t = 0$.

**Proof.** Consider $\phi \in \Gamma_0(F|\Sigma)$. We will construct a solution $\phi$ agreeing with this initial data, $\phi|\Sigma = \varphi$, using retarded/advanced Green functions. If such a solution $\phi$ existed, then from the identity $\text{id}$, it would satisfy

\[ f[\theta \pm \phi]_a = \pm \delta(t) \, dt \wedge \tilde{f}_{ab} \phi^b = \pm \delta(t) \, dt \wedge \tilde{f}_{ab} \phi^b, \]  

(58)

where the $\delta(t)$ prefactor allows us to consider only the values of $\phi|\Sigma = \varphi^b$. Conversely, knowing only the initial data on $\Sigma$, we can define the following solutions to distributional inhomogeneous problems:

\[ f[\phi \pm] = \delta(t) \, dt \wedge \tilde{f} \cdot \phi \quad \implies \quad \phi \pm = G_{\pm}[\delta(t) \, dt \wedge \tilde{f} \cdot \varphi]. \]  

(59)

Clearly, the section $\phi = \phi_+ - \phi_-$ is then a solution of $f[\phi] = 0$ that satisfies the identities $\theta \pm \phi = \pm \phi$ and hence, by Eq. $\text{id}$, agrees with initial data $\phi|\Sigma = \varphi$. Global hyperbolicity of $M$ implies that $\phi$ is unique. Hence we can define the Cauchy Green function by $G_{\Sigma}[\varphi] = \phi$. More explicitly, in local coordinates $(x^i, u^a)$ and $(t, s^l, v^b)$ on the two copies of $F$, we find

\[ \phi^a(x) = \pm \int_M G^a_{\pm}(x|t,s) \delta(t) \, dt \wedge \tilde{f}_{ab}(t,s) \phi^b(t,s) \]  

(60)

\[ = \int_\Sigma \varphi^b(s) \pm \int_M G^a_{\pm}(x|0,s) \tilde{f}_{ab}(0,s) \]  

(61)

wherefrom the desired result follows immediately. \(\square\)
To obtain the retarded/advanced Green functions from the Cauchy ones, we need only set up an initial value problem whose solution determines the retarded and advanced responses to the point source: $f[\psi_{\pm}] = \text{id}_F \delta(x, y)$. In local coordinates $(x^i, u^a)$ and $(y^i, v^b)$ on the two copies of $F$, $(\text{id}_F)^b_a = \delta^b_a$ (Kronecker delta) and $\delta(x, y) = \prod_i \delta(x^i - y^i) \, \text{d} \overline{x}$ (Dirac delta densitized on the first argument). Without loss of generality, we introduce local coordinates $(t, r^i, v^b)$ on $F$ such that the point $y$ lies on the surface $t: \Sigma \subset M$, $y = (0, r)$ with $\Sigma$ a Cauchy surface defined by $t = 0$. We define also $\bigwedge^{\pm}_{ab} = (\iota^* \bigwedge^{\pm}_{ab})^{-1}$, an $(n-1)$-multivector field on $\Sigma$, which means that
\[ (\iota^* \bigwedge^{\pm}_{ab}) \cdot \bigwedge^{bc} = \bigwedge^{ab} \cdot (\iota_* \bigwedge^{bc}) = \delta^c_a, \] where $\iota^*$ denotes the pullback of $(n-1)$-forms from $M$ to $\Sigma$, $\iota_*$ denotes the pushforward of $(n-1)$-multivectors from $\Sigma$ to its image in $M$, and the $\cdot$ denotes the contraction of the $(n-1)$-form indices with the corresponding $(n-1)$-multivector indices. We then have the direct form of

**Lemma 4 (Duhamel’s Principle).** In local coordinates $(x^i, u^a) = (t, s^i, u^a)$ and $(y^i, v^b) = (t, r^i, v^b)$ on the two copies of $F$, restricting to $(s^i, u^a)$ and $(r^i, v^b)$ on the corresponding copies of $F|\Sigma$, the retarded/advanced Green function can be expressed in terms of the Cauchy Green function as follows:

\[ G^{ab}_{\pm}(x, y) = \pm \theta^a_{\pm}(x) G^{ab}_{\Sigma^c}(x, r) \, \text{d} \overline{r} \cdot \bigwedge^{bc}(r). \] (63)

**Proof.** The result can be verified by direct calculation. First, note that when $x = (0, s)$ the Cauchy Green function satisfies $G_{\Sigma^c}(0, s| r) \, \text{d} \overline{r} = \delta^c_a \prod_j \delta(s^j - r^j) \, \text{d} \overline{r}$. Let

\[ \phi^{ab}(x, r) = G_{\Sigma^c}(x, r) \, \text{d} \overline{r} \cdot \bigwedge^{bc}(r). \] (64)

It is clear that $\phi$ is a solution of $f[\phi] = 0$ with initial data
\[ (\phi|_{\Sigma})^{ab}(s, r) = \prod_j \delta(s^j - r^j) \, \text{d} \overline{r} \cdot \bigwedge^{ab}(r). \] (65)

If $\phi^{ab}_{\pm}(x, r) = \pm \theta^a_{\pm}(x) \phi^{ab}(x, r)$, then Eq. (65) implies
\[ f[\phi^{ab}]_{\pm}(x, r) = \delta(t) \, \text{d}t \wedge \bigwedge^{ab}(x) \phi^{bc}(t, s| r) \]
\[ = \delta(t) \, \text{d}t \wedge \bigwedge^{ab}(x) \prod_j \delta(s^j - r^j) \, \text{d} \overline{r} \cdot \bigwedge^{bc}(r) \]
\[ = \delta(t) \prod_j \delta(s^j - r^j) \, \text{d}t \wedge d\overline{s} (\iota^* \bigwedge^{ab})(r) \cdot \bigwedge^{bc}(r) \]
\[ = \delta^c_a \delta(x|0, r). \] (69)

Since $y = (0, r)$, we find the desired identity from $G^{ab}_{\pm}(x, y) = G^{ab}_{\pm}(x|0, r) = \phi^{ab}_{\pm}(x, r)$. For other values of $t$ we can find $G^{ab}_{\pm}(x|t, r)$ in a similar way. \qed
4.4.2. Causal Green function (without constraints). Now that we are sure to have access to the retarded/advanced green functions $G_{\pm}$ for the linear, symmetric hyperbolic system $f[\phi] = 0$, we can define the so-called causal Green function

$$G = G_+ - G_-.$$  

(70)

This new Green function helps to conveniently parametrize the space of solutions $S_{SC}(F) \cong \ker f \subset \Gamma_{SC}(F)$ by featuring in the following

**Proposition 4.** The sequence

$$0 \longrightarrow \Gamma_0(F) \xrightarrow{f} \Gamma_0(\hat{F}^*) \xrightarrow{G} \Gamma_{SC}(F) \xrightarrow{f} \Gamma_{SC}(\hat{F}^*) \longrightarrow 0,$$  

(71)

is exact (in the sense of linear algebra).

The proof given in [4] Thm.3.4.7 and [107] Lem.3.2.1 (which excludes the final surjection) for wave equations directly carries through to the symmetric hyperbolic case. The final surjection is covered by the proof in [5] Ch.3,Cor.5.

We can interpret the above proposition in the following way. Since $S_{SC}(F) \cong \ker f$, we can express any solution to the homogeneous problem as $\phi = G[\tilde{\alpha}^*]$, where $\alpha \in \Gamma_0(\hat{F}^*)$ is some smooth dual density of compact support. Also, since $\Gamma_{SC}(F) \cong \ker f$, for any dual density $\tilde{\alpha}^*$ with spatially compact support, there exists a solution $\phi$ with spatially compact support of the inhomogeneous problem $f[\phi] = \tilde{\alpha}^*$.

**Definition 29.** Consider one Cauchy surface $\Sigma \subset M$ and two more Cauchy surfaces $\Sigma^\pm \subset M$ to the past and future of $\Sigma$, where $\Sigma^\pm \subset I^\pm(\Sigma)$, and let $S^\pm = I^\pm(\Sigma^\mp)$. Let $\{\chi_+, \chi_-\}$ be a partition of unity adapted to the open cover $\{S^+, S^-\}$ of $M$, that is, $\chi_+ + \chi_- = 1$ and $\text{supp } \chi_\pm \subset S^\pm$. We call $\{\chi_+, \chi_-\}$ a partition of unity adapted to the Cauchy surface $\Sigma$.

**Lemma 5.** The exact sequence of Prop. 4 splits at

$$\Gamma_0(\hat{F}^*) \cong \Gamma_0(F) \oplus S_{SC}(F) \quad \text{and} \quad \Gamma_{SC}(F) \cong S_{SC}(F) \oplus \Gamma_{SC}(\hat{F}^*).$$  

(72)

Given a partition of unity $\{\chi_+, \chi_-\}$ adapted to a Cauchy surface $\Sigma$, there exist (noncanonical) splitting maps

$$f_\chi: \text{im } G \rightarrow \Gamma_0(\hat{F}^*), \quad f_\chi[\phi] = \pm f_{\chi}^\pm[\phi] = \pm f[\chi_\pm \phi],$$

(73)

$$G_\chi: \Gamma_{SC}(\hat{F}^*) \rightarrow \Gamma_{SC}(F), \quad G_\chi[\tilde{\alpha}^*] = G_+[\chi_+ \tilde{\alpha}^*] + G_-[\chi_- \tilde{\alpha}^*].$$

(74)

**Proof.** We demonstrate that the maps defined in Def. 29 are precisely the splitting maps needed for this Lemma. Note that the splitting maps are not canonical, as they depend on the choice of a Cauchy surface and a partition of unity adapted to it.

When $\phi \in S_{SC}(F) \cong \ker f$, the identity 55 shows that

$$f_{\chi}^\pm[\phi] = f[\chi_\pm \phi] = (\hat{f} \cdot d\chi_\pm)\phi$$

(75)

does in fact have compact support, as $\text{supp } \phi$ is spacelike compact while $\text{supp } d\chi_\pm \subset S^+ \cap S^-$ is timelike compact. Also, since $d(\chi_+ + \chi_-) = 0$, we have $\int_{\chi_+} [\phi] + \int_{\chi_-} [\phi] = 0$, which means that the map $f_\chi = \pm f_{\chi}^\pm$ is well defined. On the other
hand, we have $G_±[f|\chi±\phi] = \chi±\phi$ from the uniqueness of solutions to the inhomogeneous problem with retarded/advanced support. The definition of the causal Green function then immediately implies that $G \circ f_\chi = \pm i d$ on $S_{SC}(F)$. Also, a direct calculation shows that $f \circ G_\chi = id$ on $\Gamma_{SC}(\tilde{F}^*)$:

$$f \circ G_\chi[\tilde{\alpha}^*] = \chi_+\tilde{\alpha}^* + \chi_-\tilde{\alpha}^* = \tilde{\alpha}^*. \quad (76)$$

This concludes the proof. \(\Box\)

We conclude this section by noting a simple but important fact.

**Lemma 6 (Covariance).** Consider two manifolds $M$ and $M'$ with globally hyperbolic, linear, symmetric hyperbolic PDE systems on them, with respective equation forms $(f, \tilde{F}^*)$ and $(f', \tilde{F}'^*)$. Suppose that the open embedding $\chi: M \to M'$ is such that $(f, \tilde{F}^*) = (\chi^* f', \chi^* \tilde{F}'^*)$ and, moreover, the induced morphism of chronal cone bundles is chronally compatible. Then the Causal Green function of the two systems agree: $G = (\chi \times \chi)^*G'$. 

**Proof.** Since $G = G_+ - G_-$, the result follows trivially if we know that $G_± = (\chi \times \chi)^*G_±'$. The latter identity follows from the uniqueness of the retarded/advanced Green functions on $M$ and the fact that the pullbacks $(\chi \times \chi)^*G_±'$ do in fact satisfy the desired Green function identity with appropriate boundary conditions, which is implied by chronal compatibility. \(\Box\)

#### 4.4.3. Causal Green function (with constraints)

The presence of constraints complicates the parametrization of solution spaces to the homogeneous and inhomogeneous problems. Recall that a symmetric hyperbolic system with hyperbolically integrable constraints consists of the hyperbolic subsystem $(f, \tilde{F}^*)$, a constraints subsystem $(c, E)$, and a consistency subsystem $(h, \tilde{E}^*)$ satisfying the identity $h \circ c = q \circ h$ for some differential operator $q$. Since in this section we are concerned with linear systems, we take all differential operators to be linear. Moreover, the causal structure is presumed to be determined by the symbol of the linear symmetric hyperbolic compound system $(f \oplus h, \tilde{F}^* \oplus \tilde{E}^*)$.

We will not discuss the most general kind of constraints and restrict our attention only to parametrizable ones. By the term parametrizable, we mean that there exist an additional vector bundle $E' \to M$ and additional differential operators $h', c'$ and $q'$, which fit into the following commutative diagram

$$\begin{align*}
\Gamma(E') &\xrightarrow{c'} \Gamma(F) &\xrightarrow{c} \Gamma(E) \\
\mid h' &\downarrow &\downarrow h \\
\Gamma(E') &\xrightarrow{q'} \Gamma(\tilde{F}^*) &\xrightarrow{q} \Gamma(\tilde{E}^*)
\end{align*} \quad (77)
$$

such that $(h', \tilde{E}'^*)$ is symmetric hyperbolic, and that the horizontal complexes of differential operators are formally exact (meaning that $c \circ c' = 0$, $q \circ q' = 0$ and that the corresponding principal symbols form an exact sequence of vector bundle maps), that is, they form an elliptic complex [66, §XIX.4]. Since both $(h, \tilde{E}^*)$ and $(h', \tilde{E}'^*)$ are symmetric hyperbolic, we can define their retarded/advanced Green functions, $H_±$ and $H_±'$, as well as their causal Green
functions, \( H = H_+ - H_- \) and \( H' = H'_+ - H'_- \). All these operators then fit into the following commutative diagram:

\[
\begin{align*}
0 & \longrightarrow \Gamma_0(E') \xrightarrow{H'} \Gamma_0(E') \xrightarrow{H'_-} \Gamma_0(E') \xrightarrow{q'} \Gamma_0(E') \xrightarrow{q'} 0 \\
0 & \longrightarrow \Gamma_0(F') \xrightarrow{f} \Gamma_0(F') \xrightarrow{G} \Gamma_0(F') \xrightarrow{f} \Gamma_0(F') \xrightarrow{G} \Gamma_0(F') \xrightarrow{G} 0 \\
0 & \longrightarrow \Gamma_0(E) \xrightarrow{h} \Gamma_0(E) \xrightarrow{H} \Gamma_0(E) \xrightarrow{h} \Gamma_0(E) \xrightarrow{H} \Gamma_0(E) \xrightarrow{H} 0
\end{align*}
\]

We call the constraints subsystem globally parametrizable if the elliptic complexes constituting the columns of the above diagram are all exact (their cohomologies are trivial).

**Lemma 7.** The retarded/advanced inhomogeneous problem

\[
\begin{align*}
f[\phi] &= \tilde{\beta}^*, \\
c[\phi] &= \gamma,
\end{align*}
\]

with \( \tilde{\beta}^* \in \Gamma_0(\tilde{E}^*) \) and \( \gamma \in \Gamma_{\pm}(E) \), is solvable for \( \phi \in \Gamma_{\pm}(F) \) iff \( h[\gamma] = q[\tilde{\beta}^*] \).

**Proof.** Let \( \phi = G_{\pm}[\tilde{\beta}^*] \). We obviously have \( f[\phi] = \tilde{\beta}^* \). It remains to check

\[
c[\phi] = c[G_{\pm}[\tilde{\beta}]] = H_{\pm}[q[\tilde{\beta}^*]] = H_{\pm}[h[\gamma]] = \gamma.
\]

This concludes the proof. \( \square \)

### 4.4.4. Green functions and adjoints.

We conclude this section by remarking the identities

\[
(G_{\pm})^* = G_{\mp}^*,
\]

where on the left hand side \((G_{\pm})^*\) denotes the adjoint of the retarded/advanced Green function \( G_{\pm} \) of the equation \( f[\phi] = 0 \), and on the right hand side \( G_{\mp}^* \) denotes the advanced/retarded Green function of the adjoint equation \( f^*[\phi] = 0 \).

**Definition 30.** Given two differential operators \( f, f^*: \Gamma(F) \to \Gamma(\tilde{E}^*) \) are said to be mutually adjoint if there exists a bilinear differential operator \( G: \Gamma(F) \times \Gamma(F) \to \Omega^{n-1}(M) \) such that

\[
f[\phi] \cdot \psi - \phi \cdot f^*[\psi] = dG(\phi, \psi).
\]

for any sections \( \phi, \psi: M \to F \). The \((n - 1)\)-form valued bilinear differential operator \( G(\phi, \psi) \) is called a Green form associated to \( f \) and \( f^* \) [22] §IV.5.5, [22] §V.1.3. Note that Eq. [22] defines \( G \) up to the addition of an exact form \( f [\tilde{H}] G \sim G + \Delta H \). Denote by \([G]\) the uniquely defined equivalence class modulo exact local bilinear forms \( \Delta H (-, -) \).
Notice that the principal symbols of mutually adjoint, first order, linear, symmetric hyperbolic differential operators $f$ and $f^*$ are negatives of each other

$$\bar{f}^* = -\bar{f}. \quad (84)$$

Also, the Green form of $f$ and $f^*$, given local coordinates $(x^i, u^a)$ on $F$, has the following representative

$$G(\phi, \psi) = \phi \cdot (\text{tr} f) \cdot \psi = (\text{tr} f)_{ab} \phi^a \psi^b, \quad (85)$$

where $(\text{tr} f)_{ab}$ are $(n-1)$-forms obtained by contracting the single contravariant index of the symbol $\bar{f}$ with one of its covariant $n$-form indices. When pulled back to a codim-1 surface $\iota: \Sigma \subset M$, the Green form forms a density that can be integrated over $\Sigma$. Let $\Sigma$ be defined as the zero set $t = 0$ of a smooth function $t$. Then a straightforward coordinate calculation shows that

$$\iota^* G(\phi, \psi) = \iota^* (\text{tr} f)_{ab} \phi^a \psi^b = \iota^* \bar{f}_{ab} \phi^a \psi^b, \quad (86)$$

where $\bar{f}_{ab}$ was defined in Eq. $(56)$. In particular, when $\Sigma$ is spacelike and future oriented, this shows that for a symmetric hyperbolic differential operator $f$ the pulled back Green form $\iota^* G(-, -)$ defines an orientation positive definite, symmetric bilinear form on the fibers of the restricted field bundle $F|\Sigma \to \Sigma$. This fact will be used in Sect. 5.2.5.

If we keep the orientation on $M$ fixed, from the relation between their principal symbols, the symmetric hyperbolic equation forms $(f, \tilde{F}^*)$ and $(f^*, \tilde{F})$ define the same spacelike covectors, except for the future/past orientation, which gets flipped. This means that the causal relations defined by $f^*$ are simply the reverse of those defined by $f$. In this section, when using $\pm$ indices to denote retarded or advanced support, we always refer to the notions of past and future defined by $f$.

Recall that we may introduce a natural pairing between elements $\phi \in \Gamma(F)$ and $\tilde{\alpha} \in \Gamma(\tilde{F}^*)$ given by

$$\langle \phi, \tilde{\alpha} \rangle = \langle \tilde{\alpha}, \phi \rangle = \int \phi \cdot \tilde{\alpha}^*. \quad (87)$$

The pairing is only partially defined, that is, for simplicity, only for those pairs of sections for which the integrand $\phi \cdot \tilde{\alpha}^*$ has compact support. Its properties, including non-degeneracy, will discussed in more detail in Sects. 5.2.4, 5.2.5.

It is easy to show that the adjoint $f^*$ coincides with the adjoint of $f$ with respect to this natural pairing: $\langle f[\phi], \psi \rangle = \langle \phi, f^*[\psi] \rangle$. This natural pairing allows us to define adjoints for integral operators like Green functions, namely $\langle G_\pm[\tilde{\alpha}^*], \beta^* \rangle = \langle \tilde{\alpha}^*, (G_\pm)^*[\beta^*] \rangle$.

It is straightforward to check that the identities $f \circ G_\pm = G_\pm \circ f = \text{id}$ hold on $\Gamma_\pm(F)$ as well as characterize the retarded/advanced Green functions, and also

$^5$ It is at this point that it becomes convenient to have chosen the adapted equation form of symmetric hyperbolic systems to be densitized. With the standard, non-densitized definition, the adjoint operator $f^*$ is in any case densitized. So it is only in symmetric hyperbolic form after contraction with a nowhere vanishing degree $n$ multivector field.
that the natural pairing \((-,-)\) is non-degenerate on the spaces \(\Gamma_{\pm}(F) \times \Gamma_{\mp}(\tilde{F}^*)\).

It is now easy to verify the adjoint identities (82) since

\[
\int_M \phi \cdot f \circ G^* \circ \tilde{\alpha}^*_{\mp} = \langle \phi_{\mp}, \tilde{\alpha}^*_{\mp} \rangle = \int_M (G_{\pm})^* \circ f^*[\phi_{\mp}] \cdot \tilde{\alpha}^*_{\mp}
\]

(88)

\[
\int_M \tilde{G}_{\mp} \circ f[\phi_{\pm}] \cdot \tilde{\alpha}^*_{\mp} = \langle \phi_{\pm}, \tilde{\alpha}^*_{\mp} \rangle = \int_M \phi_{\mp} \cdot f^* \circ (G_{\mp})^*[\tilde{\alpha}^*_{\mp}],
\]

(89)

for any \(\phi_{\pm} \in \Gamma_{\pm}(F)\) and \(\tilde{\alpha}^*_{\mp} \in \Gamma_{\pm}(F)\). The causal Green functions then satisfy \((G)^* = -G^*\), where \(G^*\) is the causal Green function for \(f^*\).

5. Construction of the Classical Field Theory

Classical mechanical systems, and field theories in particular, have three standard levels of description: spacetime, phase space, and observables. At the spacetime level, the mechanical system is specified by the underlying spacetime as a manifold \(M\), by the dynamical degrees of freedom as a field bundle \(F \rightarrow M\), and by the dynamics as a PDE system \(\mathcal{E} \subseteq J^k(F,M)\). At the phase space level, the set of all possible solutions forms a (possibly infinite dimensional) symplectic manifold, referred to as the phase space. At the level of observables, the mechanical system is associated with the Poisson algebra of smooth functions on the phase space, known as the algebra of observables. For locally covariant field theories, all of these descriptions should be associated functorially to the given spacetime manifold \(M\). A discussion of these functorial aspects is delayed until Sect. 6. For now, we discuss individual PDE systems.

So far, we have only discussed field theories at the spacetime level, as PDE systems. To move on to the phase space level, we must consider the space of solutions of the PDE system. To become a viable phase space, it must be equipped with symplectic structure, preferably in a spacetime-local way (which is precised later on). It is well known that this is possible when the PDE system has an equation form of Euler-Lagrange equations of a local variational principle (or local action principle). The corresponding spacetime-local symplectic structure is given by a horizontally conserved symplectic current density on the jet space \(J^\infty F\). If this symplectic current density is provided along with the PDE system, its space of solutions can be directly treated as the phase space of a classical field theory, without the need to introduce a local action. What is less well known \([71]\), and perhaps a little surprising, is that a local action can be recovered from a symplectic current density. In other words, the two ways of specifying a classical field theory are essentially equivalent.

The construction of the phase space of the classical field theory is broken up below into three sections. Sect. 5.1 starts with a local Lagrangian and extracts from it a local presymplectic form, which is used the construct the symplectic and Poisson tensors on the space of solutions in Sect. 5.3. Because the space of a PDE system is in general infinite dimensional, we first establish some formal properties of its tangent and cotangent spaces necessary for the discussion of these phase space structures in Sect. 5.2, which constitutes the bulk of this Section. Making the discussion the relevant infinite dimensional geometry non-formal would require substantially more functional analytical detail, which would detract from the geometric focus of this paper. More details in this direction can be found in \([19,47,95,24]\).
Before proceeding, it is worth remarking that the construction of the symplectic and Poisson structures on the classical phase space more commonly carried out in a Hamiltonian framework. However, such an approach usually requires a non-canonical 3+1 decomposition of an underlying 4 dimensional spacetime (and similarly in higher dimensions), which destroys manifest 4-dimensional spacetime covariance. On the other hand, we hold spacetime covariance as an important guiding principle underlying the construction of locally covariant field theories, cf. Sects. 2 and 6. Fortunately, it has been known for a long time, that the symplectic and Poisson structures on the phase space can be built directly from the Lagrangian without giving up spacetime covariance [80,38,90]. In fact, it is known from general principles that the usual Hamiltonian formalism is subsumed as a special case of this Lagrangian formalism [10,61,46].

5.1. Variational systems. Consider a field vector bundle $F \to M$ over an $n$-dimensional manifold $M$. A local action functional of order $k$ on $F \to M$ is a function $S[\phi]$ of sections $\phi: M \to F$,

$$S[\phi] = \int_M (j^k \phi)^* L, \quad (90)$$

where $L$, the Lagrangian density, is a section of the bundle $(\Lambda^a M)^k \to J^k F$ densities, which could depend on jet coordinates of order up to $k$. The Lagrangian density is called local because, given a section $\phi$ and local coordinates $(x^i, u^a_I)$ on $J^k F$, the pullback at $x \in M$ can be written as

$$(j^k \phi)^* L(x) = L(x^i, \partial_I \phi^a(x)), \quad (91)$$

which depends only on $x$ and on the derivatives of $\phi$ at $x$ up to order $k$. For the most part, the integral over $M$ can be considered formal, since all the necessary properties will be derived from $L$. On the other hand, the finiteness of $S[\phi]$ or related quantities may be important while discussing boundary conditions in spacetimes with non-compact spatial extent. However, we will no discuss these issues below.

Recall that Sect. A introduces the variational bicomplex $\Omega^{h,v}(F)$ of vertically and horizontally graded differential forms on $J^\infty F$. Below, we use the notation introduced in that section. A Lagrangian density is then an element $L \in \Omega^{n,0}(F)$ that can be projected to $J^k F$. Incidentally the usual variational derivative of variational calculus can be put into direct correspondence with the vertical differential $d_v$ on this complex, which is how the name variational bicomplex was established [12].

Let $(x^i, u^a_I)$ be a set of adapted coordinates on the $\infty$-jet bundle $J^\infty F$, where all the following calculations can be lifted. Any result that depends only on jets of finite order can then be projected on to the appropriate finite dimensional jet bundle. Using the integration by parts identity if necessary, we can always write the first vertical variation of the Lagrangian density as

$$d_v L = EL_a \land d_v u^a - d_h \theta. \quad (92)$$

All terms proportional to $d_v u^a_I, |I| > 0$, have been absorbed into $d_h \theta$. In the course of the performing the integrations by parts, $EL_a$ can acquire dependence
on jets up to order $2k$, and $\theta$ on jets up to order $2k - 1$. Note that $\text{EL}_a = 0$ are the Euler-Lagrange equations associated with the action functional $S[\phi]$ or the Lagrangian density $L$. We can identify the form $\text{EL}_a \wedge d_\nu u^a$ with a bundle morphism $\text{EL} : J^{2k} F \to \tilde{F}^*$. Therefore, $(\text{EL}, \tilde{F}^*)$ is an equation form of a PDE system $\mathcal{E}_{\text{EL}} \subset J^{2k} F$ on $F$ of order $2k$. A PDE system with an equation form given by Euler-Lagrange equations of a Lagrangian density is said to be variational.

Also, the form $\theta$ is an element of $\Omega^{n-1,1}(F)$, projectable to $J^{2k-1} F$. It is referred to as the presymplectic potential current density. Applying the vertical exterior differential to $\theta$ we obtain the presymplectic current density (or the presymplectic current density defined by $L$ if the extra precision is necessary).

$$\omega = d_\nu \theta,$$  \hspace{1cm} (93)

with $\omega \in \Omega^{n-1,2}(F)$. This terminology implies that $\omega$ can be integrated over a codim-1 spacetime surface to construct a presymplectic form, Sect. 5.3.3, which is then necessarily local. This method of construction a symplectic form on the phase space of classical field theory is sometimes referred to as the covariant phase space method [80,35,4].

The following lemma is an easy consequence of the definition of $\omega$.

**Lemma 8.** The form $\omega \in \Omega^{n-1,2}(F)$ defined in Eq. (93) is both horizontally and vertically closed when pulled back to $\iota_\infty : \mathcal{E}_{\text{EL}} \subseteq J^{2k} F$:

$$d_h \iota_\infty^* \omega = 0,$$  \hspace{1cm} (94)

$$d_\nu \iota_\infty^* \omega = 0.$$  \hspace{1cm} (95)

**Proof.** The horizontal and vertical differentials on $\mathcal{E}_{\text{EL}}$ are defined by pullback along $\iota_\infty$, that is, $d_h \iota_\infty^* = \iota_\infty^* d_h$ and $d_\nu \iota_\infty^* = \iota_\infty^* d_\nu$. Since $\omega = d_\nu \theta$ is already vertically closed on $J^{2k-1} F$, it is a fortiori vertically closed on $\mathcal{E}$. The rest is a consequence of the nilpotence and anti-commutativity of $d_h$ and $d_\nu$:

$$0 = d_h^2 \mathcal{L} = d_\nu \text{EL}_a \wedge d_\nu u^a - d_\nu d_h \theta,$$  \hspace{1cm} (96)

$$d_h \omega = d_h d_\nu \theta = -d_\nu \text{EL}_a \wedge d_\nu u^a,$$  \hspace{1cm} (97)

$$d_h \iota_\infty^* \omega = \iota_\infty^* d_h \omega = -\iota_\infty^* d_\nu \text{EL}_a \wedge d_\nu u^a = 0,$$  \hspace{1cm} (98)

since $\text{EL}_a$ and $d_\nu \text{EL}_a$ generate the differential ideal in $\Omega^*(J^\infty F)$ annihilated by the pullback $\iota_\infty^*$. $\square$

In fact, we will promote the name presymplectic current density to any form satisfying these properties.

**Definition 31.** Given a PDE system $\mathcal{E} \subset J^k F$ we call a form $\omega$ a presymplectic current density compatible with $\mathcal{E}$ if $\omega \in \Omega^{n-1,2}(F)$ and it is both horizontally and vertically closed on solutions:

$$d_h \iota_\infty^* \omega = 0,$$  \hspace{1cm} (99)

$$d_\nu \iota_\infty^* \omega = 0.$$  \hspace{1cm} (100)

The particular form $\omega$ defined by Eq. (93) will be referred to as the presymplectic current density associated to or obtained from the Lagrangian density $\mathcal{L}$, if there is any potential confusion.
5.2. Formal differential geometry of solution spaces. Before describing the symplectic and Poisson structures on the space of solutions, we should say something about the differential geometry of the manifold of solutions of a PDE system as well as its tangent and cotangent spaces. As usual for infinite dimensional manifolds, there are some subtleties.

The main goal of this section is to describe the formal tangent and formal cotangent spaces of the manifold of arbitrary field sections and the manifold of solution sections. The adjective formal, in the last sentence, alludes to the fact that we avoid most technical issues of infinite dimensional analysis and concentrate on what would be dense subspaces of the true tangent and cotangent spaces with a reasonable for their topologies. Results are algebraic and (finite dimensional) geometric identities that would form the core of an earnest functional analytical formulation of their non-formal versions. The formal tangent and cotangent spaces have a natural dual pairing, which we prove to be non-degenerate, as a substitute for the absence of true topological duality between them. In the presence of constraints, the proof is carried out under some additional sufficient conditions.

We start with Sect. 5.2.1, which discusses the choices of topology on the infinite dimensional spaces of field configurations and solutions. Sects. 5.2.2 and Sect. 5.2.3 discuss sufficient conditions on the constraints and gauge transformations needed for later results. Sects. 5.2.4, 5.2.5 and 5.2.6 define the formal tangent and cotangent spaces in the progressively more complicated cases of the space of field configurations, the space of solutions (without constraints), and the space of solutions (with constraints). Finally, Sect. 5.2.7 uses the preceding discussion to define formal local differential forms, of which the symplectic form will be an example.

5.2.1. Choice of topology. What we really want to do is describe the space of solutions \( S_H(F) \). However, we first start with the space \( \Gamma_H(F) \) or arbitrary field sections, because it has a simpler structure. In fact, since \( F \to M \) is a vector bundle, \( \Gamma(F) \) is a vector space. It can be turned into a topological vector space for several reasonable choices of topology. One such choice is the compact open topology (or more precisely the \( C^\infty \) compact open topology). When the base manifold \( M \) is not compact, the Whitney topology (also known as the Whitney \( C^\infty \) topology, or the wholly open topology) is another natural choice. Unfortunately, \( \Gamma(F) \) ceases to be a topological vector space\(^6\), because multiplication by scalars fails to be continuous. These two topologies coincide iff the base manifold \( M \) is compact. The Whitney topology is naturally singled out by Thm. \(1\) which shows that the sets of slow sections \( \Gamma(F,C) \) are Whitney-open in \( \Gamma_H(F) \). For the purposes of the discussion in Sect. \(13\) it is advantageous to consider a cover of the space of solutions by open sets of slow sections. On the other hand, the differential geometry of infinite dimensional manifolds not modeled on a topological vector space becomes significantly more complicated. Also, it becomes more difficult to make contact with the approach of \[41,95,24\], who use the compact open topology.

\(^6\) With this choice of topology, \( C^\infty(M) \) is still a topological ring, but not a topological algebra over \( \mathbb{R} \). The space of sections \( \Gamma(F) \) is then a topological module over \( C^\infty(M) \), where both have the Whitney topology.

\(^7\) They are proved to be open in the Whitney \( C^0 \) topology. They are also open in the Whitney \( C^\infty \) topology, since the latter is finer (it has more open sets).
One can think of different ways out of this impasse. One could bite the bullet and consider infinite dimensional manifolds modeled on topological $C^\infty(M)$-modules. One could also restrict the spacetime manifold $M$ to be the interior of a compact manifold with boundary and restrict $\Gamma(F)$ to the set of sections that extend continuously in some way to the boundary. At the moment, we remain agnostic about these or other possibilities, as they do not affect the results presented below, and simply assume that some choice has been made so that the sets $\Gamma(F, C)$ are open in $\Gamma_H(F)$, or could be effectively treated as such. For example, the choice of the functor $C^\infty(-)$ that assigns the algebra of smooth functions to our infinite dimensional manifolds is insensitive to the difference between the compact open and Whitney topologies. Therefore, we presume (assuming also that Conj. 2 holds) the following hypothesis

**Hypothesis 1.** The space $\Gamma_H(F)$ of globally hyperbolic sections is open in $\Gamma(F)$, and hence a manifold modeled on $\Gamma(F)$, with an open cover by $\Gamma(F, C)$, with globally hyperbolic $C \to M$, as in Eq. (59).

Now, the space of solutions can be topologized as a subset $S(M) \subset \Gamma(F)$. We are not interested in all solutions, only in the open subset of globally hyperbolic ones, $S_H(M) = S(M) \cap \Gamma_H(M)$. This will be our classical phase space.

### 5.2.2. Constraints.

Recall that we are considering a symmetric hyperbolic PDE system with constraints defined on the field bundle $F \to M$, whose equation form is given by the hyperbolic subsystem $(f, \bar{F}^*)$, the constraint subsystem $(c, E)$ and the consistency subsystem $(\bar{h}, \bar{E}^*)$, satisfying the identity $\bar{h} \circ c = q \circ f$ for some differential operator $q$. The formal tangent vectors on the space of solutions will essentially consist of solutions of the linearized version of this PDE system. Supposing that some dynamical linearization point $\phi \in S_H(M)$ is held fixed, the PDE subsystems linearized about $\phi$ will consist of $(\bar{f}, \bar{F}^*)$, $(\bar{c}, E)$ and $(\bar{h}, \bar{E}^*)$, satisfying $\bar{h} \circ \bar{c} = \bar{q} \circ \bar{f}$, respectively. All of $\bar{f}$, $\bar{c}$, $\bar{h}$, $\bar{q}$ are now linear differential operators. The essential properties of linear, symmetric hyperbolic systems with constraints that we will refer to are laid out in Sect. 4.4.

Dually, the formal cotangent space will be defined using the adjoint operators $f^*: \Gamma(F) \to \Gamma(\bar{F}^*)$, $\bar{c}^*: \Gamma(\bar{E}^*) \to \Gamma(\bar{F}^*)$, $\bar{h}^*: \Gamma(E) \to \Gamma(\bar{E}^*)$, $\bar{q}^*: \Gamma(E) \to \Gamma(F)$, which satisfy $\bar{c}^* \circ h^* = \bar{f}^* \circ q^*$. Note that $(f^*, \bar{F}^*)$ and $(\bar{h}^*, \bar{E}^*)$ are also symmetric hyperbolic.

When dealing with constrained systems, some results covered in this section will require the further sufficient condition that the constraints be parametrizable so that we can extend both the linearized system and its adjoint to the following commutative diagrams:

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{0} & \Gamma_0(E') & \xrightarrow{h'} & \Gamma_0(\bar{E}^*) & \xrightarrow{H'} & \Gamma_{SC}(E') & \xrightarrow{\bar{h}'} & \Gamma_{SC}(\bar{E}^*) & \xrightarrow{0} \\
\downarrow{0} & & \downarrow{\bar{c}^*} & & \downarrow{\bar{q}^*} & & \downarrow{\bar{c}^*} & & \downarrow{\bar{q}^*} & \\
0 & \xrightarrow{0} & \Gamma_0(F) & \xrightarrow{f} & \Gamma_0(\bar{F}^*) & \xrightarrow{G} & \Gamma_{SC}(F) & \xrightarrow{f} & \Gamma_{SC}(\bar{F}^*) & \xrightarrow{0} \\
\downarrow{0} & & \downarrow{\bar{h}} & & \downarrow{\bar{q}} & & \downarrow{\bar{h}} & & \downarrow{\bar{q}} & \\
0 & \xrightarrow{0} & \Gamma_0(E) & \xrightarrow{h} & \Gamma_0(\bar{E}^*) & \xrightarrow{H} & \Gamma_{SC}(E) & \xrightarrow{\bar{h}} & \Gamma_{SC}(\bar{E}^*) & \xrightarrow{0}
\end{array}
\]
The rows form exact sequences, while the columns form elliptic complexes, as described in Sect. 4.4.3. Note that the adjoint diagram also describes a symmetric hyperbolic system with hyperbolically integrable constraints, except that the role of the constraint subsystem is now played by \((\hat{q}^*, E')\) and the consistency subsystem is \((\hat{h}^*, \hat{E}^*)\), which satisfies the consistency identity \(\hat{h}^* \circ \hat{q}^* = \hat{q}^* \circ \hat{f}^*\).

Since these systems are linear, their causal structures are field independent. Since we will be discussing both the linearized system and its adjoint system, we will refer to the causal structure defined by the extended compound systems \((h' \oplus \hat{f} \oplus \hat{h}, \hat{E}^* \oplus M \hat{E}^*)\) and \((\hat{h}^* \oplus \hat{f}^* \oplus \hat{h}^*, \hat{E}^* \oplus M \hat{E}^*)\). In fact, it is easy to check that their causal structures coincide and in turn coincide with that of the dynamical linearization point \(\phi \in \mathcal{S}_H(M)\) and hence are globally hyperbolic.

Recall that adjoint symmetric hyperbolic systems have opposite notions of future and past. We shall always take future and past to be defined by \(\hat{h}^* \oplus \hat{f} \oplus \hat{h}\), rather than its adjoint.

5.2.3. Gauge transformations. Many important classical field theories exhibit gauge invariance, like Maxwell theory, Yang-Mills theory, and GR. A gauge transformation is a family of maps \(g_\varepsilon: \Gamma(F) \to \Gamma(F)\), parametrized by sections \(\varepsilon \in \Gamma(P)\) of the gauge parameter bundle \(P \to M\), that take solutions to solutions, while not changing modifying a field section outside the support of \(\delta\), \(g_\varepsilon[\phi](x) = \phi(x)\) if \(x \notin \text{supp} \varepsilon\), which may be compact. If we linearize about some pair of background section \(\delta \to \delta + \varepsilon\), we obtain a linearized gauge transformation \(g_\varepsilon[\phi] \to g_\varepsilon[\phi] + [\dot{\phi}][\varepsilon]\). It is another requirement on gauge transformations that the generator of linearized gauge transformations \(\dot{g}: \Gamma(P) \to \Gamma(F)\) is a differential operator, which may depend on the background sections \(\delta\) and \(\phi\).

Equivalence classes of sections under gauge transformations are considered physically equivalent. Therefore, physical observables will consist only of those functions on phase space that are gauge invariant (constant on orbits of gauge transformations). Equivalently, observables are annihilated by the action of linearized gauge transformations. Another way to look at it, is to consider observables as functions on the space of gauge orbits, denoted \(\Gamma_H(F)\) for the space of field configurations and \(\mathcal{S}_H(F)\) for the space of solutions. We call the space of globally hyperbolic solution sections modulo gauge transformations, \(\mathcal{S}_H(F) / \sim\), the physical phase space.

Often it is convenient to impose subsidiary conditions on field sections, called gauge fixing, that restrict the choice of representatives of gauge equivalence classes. The gauge fixing is called full if they only allow a unique representative.
from each equivalence class, and otherwise called partial. The gauge transformations that are compatible with a partial gauge fixing are called residual.

Unfortunately, PDE systems with gauge invariance cannot have a well-posed initial value problem, and hence cannot be hyperbolized. However, the addition of subsidiary conditions on field sections can make the new PDE system equivalent to a hyperbolic one, usually with constraints. In practice, many hyperbolic systems with constraints arise after adding such gauge fixing conditions to a non-hyperbolic system with gauge invariance. However, there may remain non-trivial residual gauge freedom. For later convenience, as we did with constraints, we restrict our attention to what we call recognizable gauge transformations. That is, given linearized gauge transformations of the form \( \dot{g} \), we can fit them into the following commutative diagram, whose columns form elliptic complexes:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \Gamma_0(P) & \xrightarrow{k} & \Gamma_0(\tilde{P}^*) & \xrightarrow{K} & \Gamma_{SC}(P) & \xrightarrow{k} & \Gamma_{SC}(\tilde{P}^*) & \rightarrow & 0 \\
\downarrow{\dot{g}} & & \downarrow{\dot{s}} & & \downarrow{\dot{g}} & & \downarrow{\dot{s}} & & \downarrow{\dot{g}} & & \downarrow{\dot{s}} \\
0 & \rightarrow & \Gamma_0(F) & \xrightarrow{f} & \Gamma_0(\tilde{F}^*) & \xrightarrow{G} & \Gamma_{SC}(F) & \xrightarrow{f} & \Gamma_{SC}(\tilde{F}^*) & \rightarrow & 0 \\
\downarrow{\dot{g}'} & & \downarrow{\dot{s}'} & & \downarrow{\dot{g}'} & & \downarrow{\dot{s}'} & & \downarrow{\dot{g}'} & & \downarrow{\dot{s}'} \\
0 & \rightarrow & \Gamma_0(P') & \xrightarrow{k'} & \Gamma_0(\tilde{P}^*) & \xrightarrow{K'} & \Gamma_{SC}(P') & \xrightarrow{k'} & \Gamma_{SC}(\tilde{P}^*) & \rightarrow & 0 \\
\end{array}
\]

(103)

Their adjoints fit into the adjoint diagram whose columns are also elliptic complexes:

\[
\begin{array}{ccccccccc}
0 & \leftarrow & \Gamma_{SC}(\tilde{P}^*) & \xleftarrow{k^*} & \Gamma_{SC}(P) & \xleftarrow{K^*} & \Gamma_0(\tilde{P}^*) & \xleftarrow{k^*} & \Gamma_0(P) & \leftarrow & 0 \\
\uparrow{\dot{g}'} & & \uparrow{\dot{s}'} & & \uparrow{\dot{g}'} & & \uparrow{\dot{s}'} & & \uparrow{\dot{g}'} & & \uparrow{\dot{s}'} \\
0 & \leftarrow & \Gamma_{SC}(\tilde{F}^*) & \xleftarrow{f^*} & \Gamma_{SC}(F) & \xleftarrow{G^*} & \Gamma_0(\tilde{F}^*) & \xleftarrow{f^*} & \Gamma_0(F) & \leftarrow & 0 \\
\uparrow{\dot{g}''} & & \uparrow{\dot{s}''} & & \uparrow{\dot{g}''} & & \uparrow{\dot{s}''} & & \uparrow{\dot{g}''} & & \uparrow{\dot{s}''} \\
0 & \leftarrow & \Gamma_{SC}(\tilde{P}^*) & \xleftarrow{k'^*} & \Gamma_{SC}(P') & \xleftarrow{K'^*} & \Gamma_0(\tilde{P}^*) & \xleftarrow{k'^*} & \Gamma_0(P') & \leftarrow & 0 \\
\end{array}
\]

(104)

The systems \((k, \tilde{P}^*)\) and \((k'^*, \tilde{P}^*)\) are required to be symmetric hyperbolic and \(P' \rightarrow M\) is called the gauge invariant field bundle, while \(\dot{g}'\) is called the operator of gauge invariant field combinations.

The above commutative diagrams are formally similar to those of symmetric hyperbolic systems with parametrizable constrains, as described in Sect. \[1.3.3\] By analogy, we say that the gauge transformations are globally recognizable if all the vertical elliptic complexes in the two diagrams above are exact.
Definition 32. We define the formal full tangent space at $\phi$ as the set of spacelike compact sections. We define and the formal full cotangent space at $\phi$ as the set

$$T_{\phi}\Gamma \cong \Gamma_{SC}(F) \quad \text{and} \quad T^*_{\phi}\Gamma \cong \Gamma_0(\tilde{F}^*). \quad (105)$$

The natural pairing $\langle -, - \rangle : T_{\phi}\Gamma \times T^*_{\phi}\Gamma \to \mathbb{R}$ is

$$\langle \psi, \tilde{\alpha}^* \rangle = \int_M \psi \cdot \tilde{\alpha}^*. \quad (106)$$

Lemma 9. The natural pairing between $T_{\phi}\Gamma$ and $T^*_{\phi}\Gamma$ is non-degenerate.

This is essentially the fundamental lemma of the calculus of variations and the proof is standard [33, §IV.3.1].

Since the physical phase space will be identified with the space of gauge orbits $\bar{S}_H(F)$ in the solution space $S_H(F)$, given a solution section $\phi \in S_H(F)$, the formal tangent space $T_{\phi}\bar{S} = T_{\phi}\bar{S}_H(F)$ at the corresponding equivalence class $[\phi] \in \bar{S}_H(F)$ in the space of gauge orbits consists of equivalence classes of linearized solutions up to linearized gauge transformations. Dually, the formal cotangent space $T^*_{\phi}\bar{S} = T^*_{\phi}\bar{S}_H(F)$ will consist of dual densities annihilated by the adjoint of infinitesimal gauge transformation generator.

Since gauge transformations act on field configurations and not just solutions, it makes sense to consider all field configurations related by gauge transformations as physically equivalent. Thus, we also introduce $\Gamma_H(F)$ as the space of gauge orbits of field configurations as well as its formal tangent and cotangent spaces. The natural pairing between them is shown to be non-degenerate under the condition of global recognizability, that is, the vertical elliptic complexes in diagrams (103) and (104) are exact. We deal with field configurations first and delay the discussion of solutions to the next section.

The exactness of the composition $\dot{g}' \circ \dot{g} = 0$ ensures that we can recognize pure gauge field configurations, which are of the form $\psi = \dot{g}[\epsilon]$ for some $\dot{g}$-spacelike compact section $\epsilon : M \to P$, precisely as those $\dot{g}$-spacelike compact field sections $\psi : M \to F$ that give vanishing gauge invariant field combinations $\dot{g}'[\psi] = 0$. On the other hand, the exactness of the dual composition $\dot{g}^* \circ \dot{g}'^* = 0$ ensures that we can parametrize gauge invariant, compactly supported dual densities $\tilde{\alpha}^* : M \to \tilde{F}^*$, those satisfying $\dot{g}^*[\alpha] = 0$, precisely as the image of the differential operator $\dot{g}^*$ acting on compactly supported sections of $\tilde{P}^* \to M$.

Since gauge transformations act on arbitrary sections, not just solutions, we introduce the formal tangent and cotangent spaces for arbitrary and solution sections at the same time. As before, fix a section $\phi$ in $\Gamma_H(F)$ or $S_H(F)$, as needed.

Definition 33. The formal gauge invariant full tangent space at $\phi$ is the set of gauge equivalence classes of $\dot{g}$-spacelike compact sections,

$$T_{\phi}\bar{\Gamma} = T_{\phi}\bar{\Gamma}_H(F) = \{[\psi] \mid \psi \in \Gamma_{SC}(F)\}, \quad (107)$$

$$[\psi] \sim \psi + \dot{g}[\epsilon], \quad \text{with } \epsilon \in \Gamma_{SC}(P). \quad (108)$$

The formal gauge invariant full cotangent space at $\phi$ is the set of compactly supported gauge invariant dual densities,

$$T_{\phi}\bar{\Gamma}^* = T_{\phi}\bar{\Gamma}_H(F) = \{\tilde{\alpha}^* \in \Gamma_0(\tilde{F}^*) \mid \dot{g}^*[\alpha] = 0\}. \quad (109)$$
The natural pairing \( \langle -, - \rangle : T_\phi \Gamma \times T_\phi^* \Gamma \to \mathbb{R} \) is
\[
\langle [\psi], \alpha^* \rangle = \int_M \psi \cdot \alpha^*.
\]

Lemma 10. If the gauge transformations are globally recognizable, the natural pairing between the gauge invariant spaces \( T_\phi \Gamma \) and \( T_\phi^* \Gamma \) is non-degenerate.

Proof. Non-degeneracy in the second argument follows once again from the fundamental lemma of the calculus of variations: \( \langle [\psi], \alpha \rangle = \langle \psi, \alpha \rangle = 0 \) for all \( \psi \in T_\phi \Gamma \) implies that \( \alpha = 0 \).

Non-degeneracy in the first argument requires an appeal to the global recognizability of the gauge transformations. Suppose that \( \langle [\psi], \alpha \rangle = 0 \) for all \( \alpha \in T_\phi^* \Gamma \). We need to show that this implies \( \psi = \hat{g}[\varepsilon] \) is pure gauge, for some \( \phi \)-spacelike compactly supported \( \varepsilon : M \to P \). From the elliptic complex property of the first column of (104), we are free to use gauge invariant dual densities of the form \( \alpha = \hat{g}'[\varepsilon'] \) for arbitrary \( \varepsilon' \in \Gamma_0(\hat{P}^*) \) and find
\[
\langle [\psi], \alpha^* \rangle = \langle \psi, \alpha^* \rangle = \langle \psi, \hat{g}'[\varepsilon'] \rangle = \langle \hat{g}'[\psi], \varepsilon' \rangle.
\]
Since \( \varepsilon' \) could be arbitrary, the vanishing of \( \langle \hat{g}'[\psi], \varepsilon' \rangle \) implies that \( \hat{g}'[\psi] = 0 \). On the other hand, from the exactness of the third column (103), we can then conclude that \( \psi = \hat{g}[\varepsilon] \) for some \( \varepsilon \in \Gamma_{SC}(P) \). This shows that \( \psi \) is pure gauge and completes the proof. \( \square \)

5.2.5. Formal \( T \) and \( T^* \) for solutions (without constraints). The formal tangent space \( T_\phi \mathcal{S} \) will consist of linearized solutions, that is solutions of the linearized constrained hyperbolic system \( \hat{f}[\psi] = 0 \) and \( \hat{c}[\psi] = 0 \). The formal cotangent space will naturally consist of equivalence classes of dual densities up to the images of the adjoints of \( \hat{f} \) and \( \hat{c} \). After giving the precise definitions below, we prove that that the natural pairing between these formal tangent and cotangent spaces is non-degenerate, provided the constraints \( \hat{c}[\psi] = 0 \) are trivial. The discussion of the case with non-trivial constraints is deferred to the next section.

Definition 34. We define the formal solutions tangent space at \( \phi \) as the set of spacelike compact linearized solution sections,
\[
T_\phi \mathcal{S} = T_\phi \mathcal{S}_H(F) = \{ \psi \in \Gamma_{SC}(F) \mid \hat{f}[\psi] = \hat{c}[\psi] = 0 \}.
\]

We define the formal solutions cotangent space at \( \phi \) as the set of equivalence classes of compactly supported dual densities,
\[
T_\phi^* \mathcal{S} = T_\phi^* \mathcal{S}_H(F) = \{ [\alpha^*] \mid \alpha^* \in \Gamma_0(\hat{F}^*) \},
\]
\[
[\alpha^*] \sim \tilde{\alpha}^* + \hat{f}^*[\xi] + \hat{c}^*[\tilde{\varepsilon}^*], \quad \text{with} \quad \xi \in \Gamma_0(F), \ \tilde{\varepsilon}^* \in \Gamma_0(\hat{E}^*).
\]

The natural pairing \( \langle -, - \rangle : T_\phi \mathcal{S} \times T_\phi^* \mathcal{S} \to \mathbb{R} \) is
\[
\langle \psi, [\alpha^*] \rangle = \int_M \psi \cdot \alpha^*.
\]

Characteristics, Conal Geometry and Causality in Locally Covariant Field Theory 49
Before proving the non-degeneracy of the above pairing, we introduce an auxiliary pairing and prove its non-degeneracy first. Recall that the Green form (Def. 30) associated to $\dot{f}$ and $\dot{f}^*$ is orientation positive definite when pulled back to a future oriented, spacelike surface, or at least the representative in Eq. (86) is.

**Definition 35.** Let $\iota: \Sigma \subset M$ be a future oriented, $\phi$-Cauchy surface and $G$ the Green form associated to $\dot{f}$ and $\dot{f}^*$ is orientation positive definite when pulled back to a future oriented, spacelike surface, or at least the representative in Eq. (86) is.

The Green pairing on $\Sigma$ is a bilinear form $\langle \cdot, \cdot \rangle_{G, \Sigma}: \Gamma_0 (F|_{\Sigma})^2 \to \mathbb{R}$ given by

$$
\langle \varphi_1, \varphi_2 \rangle_{G, \Sigma} = \int_{\Sigma} \iota^* G(\varphi_1, \varphi_2) = \int_{\Sigma} \iota^* \dot{f}_ab \varphi_1^a \varphi_2^b.
$$

(116)

**Lemma 11.** The Green pairing $\langle \cdot, \cdot \rangle_{G, \Sigma}$ depends only on the equivalence class $[G]$ and is positive definite (hence non-degenerate).

**Proof.** Any two representatives $G_1$ and $G_2$ of $[G]$ will differ by an exact term $dH$, with $H(\cdot, \cdot)$ a bilinear bidifferential operator. Therefore, the integrands $\iota^* G_i(\varphi_1, \varphi_2)$ will differ by the exact term $d \iota^* H(\varphi_1, \varphi_2)$, with necessarily compact support. Therefore, since $\Sigma$ has no boundary, we can use any representative of $[G]$ to evaluate the pairing.

As already mentioned above, the representative given by Eq. (86) is orientation positive, $\iota^* G(\varphi, \varphi) > 0$ at $s \in \Sigma$, whenever $\varphi(s) \neq 0$. It then an elementary conclusion that

$$
\langle \varphi, \varphi \rangle_{G, \Sigma} = \int_{\Sigma} \iota^* G(\varphi, \varphi) > 0
$$

whenever $0 \neq \varphi \in \Gamma_0 (F|_{\Sigma})$. Therefore, the Green pairing on $\Sigma$ is positive definite and non-degenerate, which concludes the proof. \(\square\)

**Lemma 12.** If the constraints $\dot{c}[\phi] = 0$ are trivial, then the natural pairing between $T_{\phi} S$ and $T^*_\phi S$ is non-degenerate.

**Proof.** Non-degeneracy in the first argument follows again from the fundamental lemma of the calculus of variations: $\langle \psi, [\tilde{\alpha}^*] \rangle = 0$ for all $\tilde{\alpha}^* \in \Gamma_0 (F^*)$ implies that $\psi = 0$.

Non-degeneracy in the second argument is more tricky. Suppose we have $\langle \psi, [\tilde{\alpha}^*] \rangle = 0$ for all $\phi$-spatially compact linearized solutions $\psi \in T_{\phi} S$. From this we need to deduce that $[\tilde{\alpha}^*] = [0]$, which means $\tilde{\alpha}^* = \dot{f}^* [\xi]$ for some compactly supported $\xi \in \Gamma_0 (F)$. Let $K = \text{supp} \tilde{\alpha}^*$ and $t_+ : \Sigma^+ \subset M$ a future oriented, $\phi$-Cauchy surface in the future of $K$ and not intersecting it. Let $\theta_-$ be the characteristic function of $I^- (\Sigma^+)$, the past of $\Sigma^+$. If $\Sigma^+$ is defined as a regular zero set $t = 0$ of smooth function $t$ on $M$, then $d \theta_- = -t dt$.

Also, let $\xi_+ = G^*_+ [\tilde{\alpha}^*]$ be the retarded solution of $\dot{f}^* [\xi_+] = \tilde{\alpha}^*$, so then supp $\xi_+ \subseteq I^+ (K)$. Note that the intersection $I^- (\Sigma^+) \cap I^+ (K)$ is compact and
contains $K$. In particular, we have $\theta_\ast \tilde{\alpha}^* = \tilde{\alpha}^*$, from which follows

$$\langle \psi, [\tilde{\alpha}^*] \rangle = \langle \psi, \theta_\ast \tilde{\alpha}^* \rangle = \int_M \theta_\ast \psi \cdot \hat{f}^\ast [\xi_+] \quad (118)$$

$$= - \int_M \theta_\ast (\hat{f}^\ast [\xi_+] - \psi \cdot \hat{f}^\ast [\xi_+]) = - \int_M \theta_\ast dG(\psi, \xi_+) \quad (119)$$

$$= \int_M d\theta_\ast \wedge G(\psi, \xi_+) = - \int_M \delta(t) dt \wedge G(\psi, \xi_+) \quad (120)$$

$$= - \int_{\Sigma_+} \iota_\ast G(\psi, \xi_+) = - \langle \psi|_{\Sigma_+}, \xi_+|_{\Sigma_+}\rangle_{G, \Sigma_+}. \quad (121)$$

Since $\psi$ is allowed to be any $\phi$-spatially compactly supported linearized solution, its restriction $\psi|_{\Sigma_+}$ to the Cauchy surface $\Sigma_+$ could be any element of $\Gamma_0(F|_{\Sigma_+})$. The non-degeneracy of the Green pairing, Lem. 11, then implies that the restriction $\xi_+|_{\Sigma_+}$ is identically zero.

On the other hand, by global hyperbolicity, we can foliate the future of $\Sigma_+$ with other $\phi$-Cauchy surfaces and apply the same argument to each of them. Then $\xi_+$ must vanish identically in the future of $\Sigma_+$, so its support is contained in the intersection $I^- (\Sigma_+ \cap I^+ (K))$, which we have already noted is compact. Finally, we have the desired conclusion that $\tilde{\alpha}^* = \hat{f}^\ast [\xi]$ with $\xi = \xi_+ \in \Gamma_0(F)$.

In the presence of gauge symmetries, the formal tangent space consists of equivalence classes of linearized solutions up to gauge transformations. On the other hand, the formal cotangent space is restricted to equivalence represented by gauge invariant dual densities. After giving the precise definitions below, we prove that the natural pairing between these formal tangent and cotangent spaces is non-degenerate, provided the constraints $\hat{c}[\phi] = 0$ are trivial and the gauge transformation are globally recognizable. The discussion of the case with non-trivial constraints is deferred to the next section.

**Definition 36.** We define the formal gauge invariant solutions tangent space at $\phi$ as the set of gauge equivalence classes of $\phi$-spacelike compact linearized solution sections,

$$T_\phi \hat{S} = T_\phi \hat{S}_H (F) = \{ [\psi] \mid \psi \in \Gamma_{SC} (F), \hat{f}[\psi] = 0, \hat{c}[\phi] = 0 \}, \quad (122)$$

with $\psi \sim \psi + \hat{g} [\varepsilon]$, with $\varepsilon \in \Gamma_{SC} (P)$.

The formal gauge invariant solutions cotangent space at $\phi$ is the set of equivalence classes of compactly supported gauge invariant dual densities,

$$T_\phi^* \hat{S} = T_\phi^* \hat{S}_H (F) = \{ [\tilde{\alpha}^*] \mid \tilde{\alpha}^* \in \Gamma_0 (\tilde{F}^*), \hat{g}^* [\tilde{\alpha}^*] = 0 \}, \quad (123)$$

$$[\tilde{\alpha}^*] \sim \tilde{\alpha}^* + \hat{f}^* [\xi] + \hat{c}^*[\varepsilon]^*, \quad (124)$$

with $\xi \in \Gamma_0 (F), \tilde{\epsilon}^* \in \Gamma_0 (\tilde{E}^*)$ and $\hat{g}^*[\hat{f}^* [\xi] + \hat{c}^*[\varepsilon]^*] = 0. \quad (125)$

The natural pairing $\langle -, - \rangle : T_\phi \hat{S} \times T_\phi^* \hat{S} \to \mathbb{R}$ is

$$\langle [\psi], [\tilde{\alpha}^*] \rangle = \int_M \psi \cdot \tilde{\alpha}^*. \quad (126)$$
Lemma 13. If the constraints $\dot{\iota}(\phi) = 0$ are trivial and the gauge transformations are globally recognizable (cf. diagrams (103) and (104)), then the natural pairing between $T_\phi S$ and $T^*_{\phi}S$ is non-degenerate.

Proof. Unfortunately, we cannot directly rely on the fundamental lemma of the calculus of variations to prove non-degeneracy in the second argument of $\langle [\psi], [\tilde{\alpha}^\ast] \rangle = \langle \psi, \tilde{\alpha}^\ast \rangle$, since we are only allowed to use equivalence classes represented by gauge invariant dual densities, $\dot{g}^*[\tilde{\alpha}^\ast] = 0$. Fortunately, any dual density of the form $\tilde{\alpha}^\ast = \dot{g}^*[\beta^\ast]$, with unrestricted choice of $\beta^\ast \in \Gamma_0(\tilde{P}^\ast)$, is an allowed representative. Then the fundamental lemma of the calculus of variations and

$$
\langle [\psi], [\tilde{\alpha}^\ast] \rangle = \langle \psi, \dot{g}^*[\beta^\ast] \rangle = \langle \dot{g}^*[\psi], \beta^\ast \rangle = 0,
$$

(127)

for arbitrary $\dot{\beta}^\ast$, imply that $\dot{g}^*[\psi] = 0$. By global recognizability of gauge transformations, or more precisely the exactness of the third column of diagram (103), we have $\psi = \dot{g}^*[\epsilon]$ for some $\epsilon \in \Gamma_{SC}(P)$. That is, $\psi$ must be pure gauge, $[\psi] = [0]$, which proves non-degeneracy of the natural pairing in the second argument.

To prove non-degeneracy in the first argument, we proceed as in the proof of Lem.12. Namely, assume that we have a gauge invariant dual density $\dot{g}^*[\tilde{\alpha}^\ast] = 0$, such that $\langle [\psi], [\tilde{\alpha}^\ast] \rangle$ for all $[\psi] \in T_\phi S$, and let $\xi_+ = G_+[\tilde{\alpha}^\ast]$, so that $f^*[\xi+] = \tilde{\alpha}^\ast$. Also, let $K = \text{supp} \tilde{\beta}^\ast$, $\Sigma^+ \subset M$ a future oriented, $\phi$-Cauchy surface that does not intersect $K$ itself but does intersect $I^+(K)$, and $\theta_-$ the characteristic function of $I^-(\Sigma^+)$. Then

$$
\langle [\psi], [\tilde{\alpha}^\ast] \rangle = \langle \psi, \theta_- \dot{f}^*[\xi_+] \rangle = -\langle \psi|_{\Sigma^+}, \xi_+|_{\Sigma^+} \rangle_{G, \Sigma^+},
$$

(128)

which implies that $\text{supp} \xi_+ \subset I^-(\Sigma^+)$ and hence is compact. In other words, with $\xi = \xi_+ \in \Gamma_0(F)$, we have $[\tilde{\alpha}^\ast] = [\dot{f}^*[\xi]] = [0]$ and the that natural pairing is non-degenerate in the first argument as well. □

5.2.6. Formal $T$ and $T^*$ for solutions (with constraints). When the constraints are non-trivial, the proof that the natural pairing between formal tangent and cotangent spaces is non-degenerate complicates a bit. In fact, it requires the sufficient condition that the constraints be globally parametrizable. We first handle the case without gauge invariance.

Lemma 14. If the constraints $\dot{\iota}(\phi) = 0$ are globally parametrizable (cf. diagrams (101) and (102)), then the natural pairing between $T_\phi S$ and $T^*_{\phi}S$ is non-degenerate.

Proof. Non-degeneracy in the first argument follows again from the fundamental lemma of the calculus of variations: $\langle \psi, [\tilde{\alpha}^\ast] \rangle = \langle \psi, \tilde{\alpha}^\ast \rangle = 0$ for all $\alpha \in \Gamma_0(\tilde{F}^\ast)$ implies that $\psi = 0$.

Non-degeneracy in the second argument means that $\langle \psi, [\tilde{\alpha}^\ast] \rangle = 0$ for all $\psi \in T_\phi S$ implies $[\tilde{\alpha}^\ast] = [0]$, which means that $\dot{g}^*[\tilde{\alpha}^\ast] = 0$ and $\tilde{\alpha}^\ast = \dot{f}^*[\xi] + \tilde{\epsilon}^*[\tilde{\epsilon}]$ for some $\xi \in \Gamma_0(F)$ and $\tilde{\epsilon}^* \in \Gamma_0(\tilde{E}^\ast)$. Using the same approach as in the unconstrained case, define $\xi_+ = G_+^{\ast}[\tilde{\alpha}^\ast]$, so that $\dot{f}^*[\xi_+] = \tilde{\alpha}^\ast$. Also, let $K = \text{supp} \alpha$ and $\Sigma^+ \subset M$ a future oriented, $\phi$-Cauchy surface that does not intersect $K$ itself but does intersect $I^+(K)$. Unfortunately, we will not be able
to show that $\xi_+$ vanishes outside a compact set, but instead we will show that $\hat{q}^*\lbrack \xi_+ \rbrack$ vanishes outside a compact set.

To do that, we first need the identity

$$\langle \hat{c}'[\psi'], \xi_+ \rangle_{G,S^+} = \langle \psi', \hat{q}^*\lbrack \xi_+ \rbrack \rangle_{H',S^+},$$

where $H'$ is the Green form associated to the symmetric hyperbolic operator $\hat{h}'$ and $\psi' \in \Gamma_{SC}(E')$ satisfies $\hat{h}'[\psi'] = 0$, hence also $f \circ \hat{c}'[\psi] = 0$. Let $C'$ denote the Green forms associated to the operator $\hat{c}'$. This identity then follows from

$$dG(\hat{c}'[\psi'], \xi_+) = -\hat{c}'[\psi'] \cdot \hat{f}^*\lbrack \xi_+ \rbrack$$

$$= -\psi' \cdot \hat{c}'[\hat{f}^*\lbrack \xi_+ \rbrack] - dC'(\psi', \hat{f}^*\lbrack \xi_+ \rbrack)$$

$$= -\psi' \cdot \hat{h}^*[\hat{q}^*\lbrack \xi_+ \rbrack] - dC'(\psi', \hat{\alpha}^*)$$

and the fact that $\text{supp} C'(\psi', \hat{\alpha}^*) \subseteq \text{supp} \hat{\alpha}^*$ does not intersect $\Sigma^+$. Since we are certainly allowed to use $\psi = \hat{c}'[\psi']$, using the same argument as in Lem. 12 we have

$$\langle \hat{c}'[\psi], \lbrack \hat{\alpha}^* \rbrack \rangle = -\langle \hat{c}'[\psi'], \xi_+, \xi_+ \rangle_{G,S^+} = -\langle \psi', \hat{q}^*\lbrack \xi_+ \rbrack \rangle_{H',S^+},$$

which forces $\hat{q}^*\lbrack \xi_+ \rbrack = 0$ on $\Sigma^+$. In fact, we can also conclude that the same equality holds on a neighborhood $S^+$ of $\Sigma^+$ that contains $I^+(\Sigma^+)$, on which we have $\hat{f}^*\lbrack \xi_+ \rbrack = 0$ as well.

Now, let $\eta$ denote the unique solution of $\hat{f}^*\lbrack \eta \rbrack = 0$ on $M$ such that $\eta = \xi_+$ on $S^+$. Since, by the global parametrizability assumption (cf. diagram 102) the constraint $\hat{q}^*\lbrack \eta \rbrack = 0$ is hyperbolically integrable, it is satisfied everywhere in $M$ as well. Therefore, from the results of Sect. 4.3.3 it follows that there exists $\hat{\varepsilon}^* \in \Gamma_0(E^*)$ such that $\eta = \hat{q}^*[H^*\lbrack \hat{\varepsilon}^* \rbrack]$.

Let $\eta_+ = \hat{q}^*[H^+_+\lbrack \hat{\varepsilon}^* \rbrack]$ and $\xi_+ = \eta_+ - \eta_+$. Then, by construction, $\xi$ has compact support and

$$\hat{\alpha}^* = \hat{f}^*\lbrack \xi_+ \rbrack = \hat{f}^*\lbrack \xi \rbrack + \hat{f}^*\lbrack \eta_+ \rbrack$$

$$= \hat{f}^*\lbrack \xi \rbrack + \hat{f}^*[\hat{q}^*[H^+_+\lbrack \hat{\varepsilon}^* \rbrack]] = \hat{f}^*[\xi] + \hat{\varepsilon}^*[H^+_+\lbrack \hat{\varepsilon}^* \rbrack]]$$

$$= \hat{f}^*[\xi] + \hat{\varepsilon}^*[\hat{\varepsilon}^*],$$

as desired, implying $\lbrack \hat{\alpha}^* \rbrack = 0$, which completes the argument that the natural pairing is non-degenerate in the second argument. $\square$

The final non-degeneracy result is for the case where both non-trivial constraints and gauge transformations are present.

**Lemma 15.** If the constraints are globally parametrizable (cf. diagrams 101 and 102) and the gauge transformations are globally recognizable (cf. diagrams 103 and 104), then the natural pairing between $T_\phi S$ and $T_\psi S$ is non-degenerate.

The proof follows straight forwardly by combining the methods of the preceding non-degeneracy lemmas, namely Lem. 13 and Lem. 14.
5.2.7. Formal local differential forms. In this section, we use differential forms on $J^\infty F$ to construct formal differential forms on $\Gamma_H(F)$ or $S_H(F)$.

Consider any form $\alpha \in \Omega^{h,v}(F)$ and suppose that $\phi: M \rightarrow F$ is any section. We already know that the pullback along the jet prolongation $j^\infty \phi$ intertwines the horizontal differential $d^h$ with the de Rham differential on $M$, $d(j^\infty \phi)^* \alpha = (j^\infty \phi)^* d^h \alpha$. However, if $\alpha$ has non-zero vertical degree, $v \neq 0$, then automatically $(j^\infty \phi)^* \alpha = 0$ from the defining properties of vertical forms. Though, if $\alpha$ is first contracted with $v$ evolutionary vector fields $\hat{\psi}_j$ before effecting the pullback, the result

$$ (j^\infty \phi)^* \iota_{\hat{\psi}_1} \cdots \iota_{\hat{\psi}_v} \alpha, $$

being the pullback of vertical degree 0 form, is not-necessarily zero. Recall that, a section $\psi: M \rightarrow F$ defines a corresponding evolutionary vector field $\hat{\psi}$ that has the action $\hat{\psi}(x^i) = 0$, $\hat{\psi}(u^a_I) = \partial_I \psi^a$ in local coordinates $(x^i, u^a_I)$.

In fact, if we were concerned with calculus on $\Gamma_H(F)$ or $S_H(F)$, the infinite dimensional manifolds of sections of $F \rightarrow M$ and of solution sections of a PDE system, we could use vertical differential forms on $J^\infty F$ to define local differential forms on the manifold of sections. A point local form $A$ depends only on a specific point $x \in M$. That is, evaluated at $\phi \in \Gamma_H(F)$, $A$ depends only on $\phi(x)$ and finitely many of its derivatives at the same point. A local form is an integral over an $M$-family of point local forms, with respect to some smooth density on $M$. A degree $v$ form on $\Gamma_H(F)$ or $S_H(F)$ would be a functional of $v$ tangent vectors $\psi_j$, which would correspond to sections of $F \rightarrow M$ or linearized solution sections. Since we are not delving into the details of how these infinite dimensional objects are defined, we are content to formally associate a local differential form $A_N[\phi]$ of degree $v$ on $\Gamma_H(F)$ to each differential form $\alpha$ of degree $(h,v)$ on $J^\infty F$ as follows:

$$ A_N(\psi_1, \ldots, \psi_v)[\phi] = \int_N (j^\infty \phi)^* \iota_{\hat{\psi}_1} \cdots \iota_{\hat{\psi}_v} \alpha, $$

where $N \subseteq M$ is a submanifold of dimension $h$.

Recall that, given local coordinates $(x^i, u^a)$ on $F$, an evaluation functional is a function of $\Gamma_H(F)$ of the form $\Phi^a_{\psi}(\phi) = \phi^a(x)$. A vector field $\Psi$ on $\Gamma_H(F)$ with $(L_{\Psi} \Phi^a_{\psi})[\phi]$ depending only on $\phi(x)$ and finitely many derivatives at the same point is called a local vector field. We can formally associate a local vector field to any evolutionary vector field $\hat{\psi}$ on $J^\infty F$, by defining its action on formal local 0-forms on $\Gamma_H(F)$ by

$$ (L_{\phi} A_N)[\phi] = \int_N (j^\infty \phi)^* L_{\phi} \alpha. $$

This is enough to define a formal exterior derivative $\delta$ on $\Gamma_H(F)$, whose action on formal local forms can be formally given by the standard expressions with
contractions and Lie derivatives. It boils down to

\[
(\delta A_N)(\Psi_0, \ldots, \Psi_v)[\phi] = \sum_{j=0}^{v} (-)^j [\mathcal{L}_{\Psi_j}(i_{\Psi_v} \cdots i_{\Psi_{j+1}} \cdots i_{\Psi_0} A_N)[\phi]
\]

\[
- \mathcal{L}_{\Psi_j}(i_{\Psi_v} \cdots i_{\Psi_{j+1}} \cdots i_{\Psi_0} A_N[\phi])
\]

\[
= \sum_{j=0}^{v} (-)^j \int_{\mathcal{N}} (j^\infty \phi)^* [\mathcal{L}_{\tilde{\psi}_j}(i_{\tilde{\psi}_v} \cdots i_{\tilde{\psi}_{j+1}} \cdots i_{\tilde{\psi}_0} \alpha)]
\]

\[
- \mathcal{L}_{\tilde{\psi}_j}(i_{\tilde{\psi}_v} \cdots i_{\tilde{\psi}_{j+1}} \cdots i_{\tilde{\psi}_0} \alpha]
\]

\[
= (j^\infty \phi)^* i_{\tilde{\psi}_v} \cdots i_{\tilde{\psi}_0} d\alpha,
\]

where ` means that the marked symbol is omitted. The last formula gives the real significance of the vertical differential. To derive the first equality, one must make use of the formal identification (139) and the identity (299). The second equality results from simplifications in the usual differential calculus on \(J^\infty F\).

5.3. Symplectic and Poisson structure. In this section, we endow the space of solutions \(\mathcal{S}_H(F)\) of a variational PDE system with the structures of both a symplectic and a Poisson manifold (or rather formal versions of these structures), really turning it into the phase space of classical field theory. First, a variational system has to be put into symmetric hyperbolic forms, possibly with constraints. If the constraints are globally parametrizable, the gauge transformations globally recognizable and the hyperbolization of the variational Euler-Lagrange system satisfies an extra sufficient condition (which identifies part of the constraints with gauge fixing conditions), then we can use a generalized Peierls formula to construct the (formal) Poisson bivector and hence show that the covariant presymplectic structure is invertible and hence actually symplectic.

5.3.1. Constraints and gauge fixing. Consider a Lagrangian density \(L \in \Omega^{n,0}(F)\) of order \(k\) and the corresponding Euler-Lagrange PDE system \(\mathcal{E}_{EL} \subset J^{2k} F\), with equation form \((\mathcal{E}_L, \tilde{F}^*)\). More often then not, the EL equation form is not directly an adapted equation form for a symmetric hyperbolic system. That could be true for several reasons. It is possible that \(\mathcal{E}_{EL}\) is in fact symmetric hyperbolic and that the EL equations need only be hyperbolized. However, if the EL equations are not of first order, then would need to be first reduced to first order form (this may be done already at the level of the Lagrangian, by introducing auxiliary fields). On the other hand, the EL equations may only admit a hyperbolization after several prolongations that identify all the relevant integrability conditions. Finally, if the Lagrangian is singular (it has non-trivial gauge invariance), the EL system cannot be hyperbolized at all, since systems with gauge invariance do not have a well-posed initial value problem. In that case, one would first have to introduce gauge fixing conditions, whose conjunction with the EL system is hyperbolizable (Sect. 3.3).

Below, we postulate a condition on the hyperbolization of a gauge fixed variational EL system that in the next section will be shown to be sufficient to construct certain Green functions needed for the Peierls formula for the Poisson bracket. By a hyperbolization we mean an equivalence between the gauge fixed
EL equations for \((EL \oplus c_g, \tilde{F}^* \oplus E_g)\) and a constrained symmetric hyperbolic system \((f \oplus c, \tilde{F}^* \oplus E)\). Here, \(c_g: \Gamma(F) \to \Gamma(E_g)\) is a differential operator constituting the gauge fixing condition. The equivalence is presumed to have the following form

\[
\begin{align*}
\{ \text{EL} = R \circ (f \oplus c) \} & \iff \{ f = \tilde{R} \circ (EL \oplus c_g) \} , \\
\{ c_g = R_g \circ c \} & \iff \{ c = R_g \circ (EL \oplus c_g) \},
\end{align*}
\]

(144)

where the \(R, \tilde{R}, R_g\) and \(\tilde{R}_g\) are (possibly non-linear) differential operators.

Since we will be concerned with symplectic and Poisson tensors, we can work with individual (formal) tangent and cotangent spaces at a fixed dynamical linearization point \(\phi \in \mathcal{S}_H(F)\), which is a globally hyperbolic solution of the PDE system defined by the equation form \((f \oplus c, \tilde{F}^* \oplus E)\). In other words, we need to work with the linearized versions of each of the hyperbolic system, the constraints, the Euler-Lagrange system, the gauge fixing conditions, the gauge transformations, as well as the hyperbolization. As before, we denote the adapted equation form of the linearized hyperbolic system \((\tilde{f}, \tilde{F}^*)\). The linearized constraints are presumed to be globally parametrizable and fit into the commutative diagrams \((101)\) and \((102)\). The linearized gauge transformations are presumed to be globally recognizable and fit into the commutative diagrams \((103)\) and \((104)\). The linearized EL equations are denoted \((J, \tilde{F}^*)\) and are also called the Jacobi system, with \(J\) the Jacobi operator \([38]\), while the linearized gauge fixing conditions are denoted by the equation form \((\tilde{c}_g, E_g)\). In local coordinates \((x^a, u^a)\) on \(F\), the components of the Jacobi operator satisfy the identity

\[
J^I_{ab} \wedge d_a u^b_j = d_j \text{EL}_{\alpha}. 
\]

(145)

The equivalence of the linearized systems takes the following form:

\[
\begin{align*}
\{ J = r \circ \tilde{f} + r_c \circ \tilde{c} \} & \iff \{ \tilde{f} = \tilde{r} \circ J + \tilde{r}_c \circ \tilde{c}_g \} , \\
\{ \tilde{c}_g = r_g \circ \tilde{c} \} & \iff \{ \tilde{c} = \tilde{r}_\alpha \circ J + \tilde{r}_g \circ \tilde{c}_g \}
\end{align*}
\]

(146)

If the operator \(\tilde{r}_\alpha\) is non-vanishing, it means that part of the constraints consist of integrability conditions of the Jacobi system.

Note that, strictly speaking, the \(r\)- and \(\tilde{r}\)- differential operators effecting the equivalence are not inverses of each other. Their compositions may differ from the identity by some differential operator that factors through a differential identity, that is, \(\tilde{q} \circ \tilde{f} - h \circ \tilde{c} = 0\) or \(\tilde{q}^* \circ J = 0\). In other words, we must have

\[
\begin{align*}
r \circ \tilde{r} + r_c \circ \tilde{r}_g & = id + p_{f1} \circ \tilde{q}^*, \\
r \circ \tilde{r}_c + r_c \circ \tilde{r}_g & = 0, \\
r_g \circ \tilde{r}_\alpha & = p_g \circ \tilde{q}^*, \\
r_g \circ \tilde{r}_g & = id.
\end{align*}
\]

(147-150)

for some differential operators \(p_{f1}, p_f, p_g\) and \(p_c\). Also, the identity \(\tilde{q} \circ \tilde{f} - h \circ \tilde{c} = 0\), when expressed in terms of the \(J\) and \(\tilde{c}_g\) operators, is identically satisfied when

\[
\begin{align*}
\tilde{q} \circ \tilde{r} - h \circ \tilde{r}_\alpha & = q_1 \circ \tilde{q}^*, \\
\tilde{q} \circ \tilde{r}_c - h \circ \tilde{r}_g & = 0.
\end{align*}
\]

(151-152)
It is worth noting that the above relations involving the \( r \)- and \( \bar{r} \)-operators follow from the equivalence (146) only when \( J \) and \( \dot{c}_g \) satisfy no additional differential identities. However, we will simply presume that they hold as needed sufficient conditions for the derivation of the Peierls formulas later in Sect. 5.3.4.

Finally, to make sure that the condition \( \dot{c}_g[\psi] = 0 \) in fact constitutes a gauge fixing condition, we require the following compatibility between the gauge transformation operator and the constraints that we shall refer to as the gauge fixing compatibility condition:

\[
\dot{s}' \circ \bar{r}_c \circ \dot{c}_g = 0
\]

(153)

This condition connects constraints (represented by \( \dot{c}_g \)) and gauge transformations (represented by \( \dot{s}' \)). Essentially, this condition says that the part of the constraints \( \dot{c}[\psi] = 0 \) that comes from \( \dot{c}_g[\psi] = 0 \) is sufficient, when adjoined to \( J[\psi] = 0 \) to make the gauge fixed system hyperbolizable.

To summarize, we require that the PDE system of interest satisfies the following conditions: (a) variationality, (b) hyperbolizability, (c) global parametrizability of constraints, (d) global recognizability of gauge transformations, and (e) gauge fixing compatibility. The consequences of imposing these conditions are explored in the following section. We stress that these conditions are sufficient for our purposes and can in fact satisfied by most fundamental physical field theories, but some of the same results could also hold under weaker conditions.

5.3.2. Causal Green functions. The goal of this section is to use the gauge fixing compatibility condition (153) to construct a causal Green function for the gauge fixed Jacobi system.

First, recall that the Jacobi system, due to its variational character is easily shown to be self-adjoint:

\[
J^* = J.
\]

(154)

Also, gauge invariance and Nöther’s second theorem imply the identities

\[
J \circ \dot{g} = 0 \quad \text{and} \quad \dot{g}^* \circ J = 0.
\]

(155)

The equivalence of the gauge fixed Jacobi system and a constrained hyperbolic system postulated in (146) then gives

\[
\dot{s} \circ \dot{k} = \dot{f} \circ \dot{g} = \bar{r}_c \circ \dot{c}_g \circ \dot{g}.
\]

(156)

Suppose that \( \psi \in \Gamma_{\text{SC}}(F) \) such that \( \dot{c}_g[\psi] = 0 \) and \( \psi = \dot{g}[\varepsilon'] \) for some \( \varepsilon' \in \Gamma_{\text{SC}}(P) \). Then \( \dot{s} \circ \dot{k}[\varepsilon'] = \bar{r}_c \circ \dot{c}_g \circ \dot{g}[\varepsilon'] = 0 \), or \( \dot{k}[\varepsilon'] = \tilde{\beta}^* \), where \( \tilde{\beta}^* \in \ker \dot{s} \subset \Gamma_{\text{SC}}(\tilde{P}^*) \). Let \( \{\chi_{\pm}\} \) be a partition of unity adapted to a Cauchy surface (Def. 29) and recall the associated splitting maps (Lem. 5), which lead to the identity

\[
\dot{g}[K\chi[\tilde{\beta}^*]] = \sum \pm \frac{1}{4} G_{\pm} [s[\chi_{\pm}\tilde{\beta}^*]] = G[s[\tilde{\beta}^*]],
\]

(157)

where \( s[\tilde{\beta}^*] = \pm s[\chi_{\pm}\tilde{\beta}^*] \in \Gamma_0(\tilde{F}^*) \). Note that \( \dot{s}'[s[\tilde{\beta}^*]] = 0 \), which by global recognizability implies that there exists a \( \tilde{\gamma}^* \in \Gamma_0(\tilde{P}^*) \) such that \( \dot{s}[\tilde{\gamma}^*] = s[\tilde{\beta}^*] \).
Note that $\tilde{\gamma}^*$ cannot be just $\chi_+\tilde{\beta}^*$ or $-\chi_-\tilde{\beta}^*$, because their supports need not be compact. Now, let $\varepsilon = \varepsilon' - K_\chi[\tilde{\beta}^*] + K[\tilde{\gamma}^*]$ and notice that $\hat{k}[\varepsilon] = 0$ as well as

$$\hat{g}[\varepsilon] = \hat{\gamma}[\varepsilon'] - \hat{g}[K_\chi[\tilde{\beta}^*]] + \hat{g}[K[\tilde{\gamma}^*]]$$  \hspace{1cm} (158)$$

$$= \psi - G[\hat{s}_\chi[\tilde{\beta}^*]] + G[\hat{s}[\tilde{\gamma}^*]]$$  \hspace{1cm} (159)$$

$$= \psi.$$  \hspace{1cm} (160)$$

We have just proven

**Lemma 16.** If there exists $\psi \in \Gamma_{SC}(F)$ such that $\hat{c}_g[\psi] = 0$ and $\psi \in \text{im} \hat{g}$, there exists $\varepsilon \in \Gamma_{SC}(P)$ such that $\hat{k}[\varepsilon] = 0$ and $\psi = \hat{g}[\varepsilon]$.

Equivalence with a constrained hyperbolic system now allows us to solve the inhomogeneous problem

$$J[\psi] = \tilde{\alpha}^*, \quad \hat{c}_g[\psi] = 0,$$  \hspace{1cm} (161)$$

where the source must necessarily satisfy the gauge invariance condition $\hat{g}^*[\tilde{\alpha}^*] = 0$. The equivalent inhomogeneous problem in symmetric hyperbolic form is

$$\hat{f}[\psi] = \hat{r}[\tilde{\alpha}^*], \quad \hat{c}[\psi] = \hat{r}_J[\tilde{\alpha}^*].$$  \hspace{1cm} (162)$$

Recall from Lem. 7 that this system is solvable iff the sources satisfy the consistency identity:

$$\hat{q}[\hat{r}[\tilde{\alpha}^*]] - \hat{h}[\hat{r}_J[\tilde{\alpha}^*]] = p_J \circ \hat{g}^*[\tilde{\alpha}^*] = 0,$$  \hspace{1cm} (163)$$

which is obviously satisfied, after using identity (151), for any gauge invariant source. The retarded and advanced solutions to this inhomogeneous problem are then $\psi_+ = G_\pm[\hat{r}[\tilde{\alpha}^*]]$. This means that $\hat{r}_J[\tilde{\alpha}^*] = \hat{c}[G_\pm[\hat{r}[\tilde{\alpha}^*]]]$ and, in particular, $\hat{c}_g[\psi] = 0$. Motivated by this formula, we introduce the following retarded, advanced and causal Green functions for the gauge fixed Jacobi system:

$$E_\pm = G_\pm \circ \hat{r} \quad \text{and} \quad E = E_+ - E_- = G \circ \hat{r}.$$  \hspace{1cm} (164)$$

One can immediately check that $\psi = E[\tilde{\alpha}^*]$ satisfies both $\hat{f}[\psi] = 0$ and $\hat{c}[\psi] = 0$, whenever $\tilde{\alpha}^*$ is a gauge invariant dual density. By the equivalence (158), the same solution also satisfies $J[\psi] = 0$ and $\hat{c}_g[\psi] = 0$.

**Theorem 5.** Provided the gauge fixed Jacobi system $J[\psi] = 0$, $\hat{c}_g[\psi] = 0$ is equivalent to a constrained hyperbolic system obeying the gauge fixing compatibility condition (153), as described in Sect. 5.3.1, we can define the Jacobi causal Green function $E$, Eq. (164), such that the following diagram

\[
\begin{array}{ccccccccc}
\Gamma_0(P) & 0 & \Gamma_{SC}(P) & 0 \\
\downarrow g & & \downarrow g & & \downarrow g & & \downarrow g \\
\Gamma_0(F) & \Gamma_0(F^*) & \Gamma_{SC}(F) & \Gamma_{SC}(F^*) & 0 \\
\downarrow \hat{g}^* & \downarrow \hat{g}^* & \downarrow \hat{g}^* & \downarrow \hat{g}^* & \downarrow \hat{g}^* \\
\Gamma_0(\tilde{F}^*) & \Gamma_0(\tilde{F}^*) & \Gamma_{SC}(\tilde{F}^*) & \Gamma_{SC}(\tilde{F}^*) & 0
\end{array}
\]  \hspace{1cm} (165)$$
becomes an exact sequence after taking the vertical cohomologies. Moreover, we have the following splittings at $\Gamma_0(\tilde{F}^*)$ and $\Gamma_{SC}(F)$:

$$\ker \tilde{g}^* \cong \text{Im} J \oplus S_{SC}(F) \quad \text{and} \quad \text{coker} \tilde{g} \cong S_{SC}(F) \oplus \ker \tilde{g}^*, \quad (166)$$

where a $\phi$-Cauchy surface $\Sigma \subset M$ and a partition of unity $\{\chi_\pm\}$ adapted to it define the splitting maps (cf. Def. 29 and Lem. 4).

\begin{align*}
J_\chi : S_{SC}(F) &\to \Gamma_0(\tilde{F}^*), \\
J_\chi[\psi] &= \pm J[\chi_\pm \psi] = r \circ \tilde{f}_\chi[\psi] + r_c \circ \tilde{c}_\chi[\psi], \quad (167) \\
E_\chi : \Gamma_{SC}(\tilde{F}^*) &\to \Gamma_{SC}(F), \\
E_\chi[\tilde{\alpha}^*] &= \sum \pm G_{\pm}[\tilde{g}^*[\chi_\pm \tilde{\alpha}^*]], \quad (168)
\end{align*}

where $S_{SC}(F)$ denotes the spacelike compactly supported solutions of $\tilde{f}[^0_\phi] = 0$, $d\text{ote}[^0_\phi] = 0$, and $\tilde{c}_\chi[\psi] = \tilde{c}[^0_\pm \chi_\pm \psi]$.

**Proof.** The proof consists of checking the desired conclusions at each object of horizontal sequence in the above diagram.

1. If $\psi \in \Gamma_0(F)$ and $J[\psi] = 0$, then $\psi = \tilde{g}[\varepsilon]$ for some $\varepsilon \in \Gamma_0(P)$.

   Let $\tilde{\alpha}^* = \tilde{f}[^0_\psi]$. From the equivalence (146), we have

$$\tilde{\alpha}^* = \tilde{r} \circ J[\psi] + \tilde{r}_c \circ \tilde{c}_g[\psi] = \tilde{r}_c \circ \tilde{c}_g[\psi]. \quad (169)$$

Then the gauge fixing compatibility condition (153) implies $\tilde{s}'[\tilde{\alpha}^*] = 0$. Now, note the identity $\tilde{\psi} = G_+[\tilde{\alpha}^*]$, which holds from the uniqueness of solutions with retarded support and from the fact that $\text{supp} \, \tilde{\psi}$ is automatically retarded by virtue of being compact. Then

$$\tilde{g}'[\psi] = \tilde{g}'[G_+[\tilde{\alpha}^*]] = K'_+ [\tilde{s}'[\tilde{\alpha}^*]] = 0, \quad (170)$$

which, together with global recognizability of the gauge transformations allows us to conclude that $\psi = \tilde{g}[\varepsilon]$, for some $\varepsilon \in \Gamma_0(P)$.

2. Every $\tilde{\alpha}^* \in \Gamma_0(\tilde{F}^*)$, such that $E[\tilde{\alpha}^*] = \tilde{g}[\varepsilon]$, with $\varepsilon \in \Gamma_0(P)$, and $\tilde{g}^*[\tilde{\alpha}^*] = 0$, can be written as $\tilde{\alpha}^* = J[\psi]$ for some $\psi \in \Gamma_0(F)$. We also have $\text{Im} E \subset \text{Im} \tilde{g}$.

   The condition

$$E[\tilde{\alpha}^*] = G[\tilde{r}[\tilde{\alpha}^*]] = \tilde{g}[\varepsilon] \quad (171)$$

implies that $\tilde{s} \circ \tilde{k}[\varepsilon] = \tilde{f} \circ \tilde{g}[\varepsilon] = 0$, which in turn implies that $\tilde{g}[\varepsilon] = \tilde{g}[K[\tilde{r}^*]] = G[\tilde{s}[\tilde{r}^*]]$, for some $\tilde{r}^* \in \Gamma_0(\tilde{F}^*)$. We then have

$$G[\tilde{r}[\tilde{\alpha}^*] - \tilde{s}[\tilde{r}^*]] = 0, \quad (172)$$

which implies that $\tilde{r}[\tilde{\alpha}^*] - \tilde{s}[\tilde{r}^*] = \tilde{f}[\psi]$, where $\psi \in \Gamma_0(F)$ is unique and given by

$$\psi = G_+ [\tilde{r}[\tilde{\alpha}^*] - \tilde{s}[\tilde{r}^*]] = G_- [\tilde{r}[\tilde{\alpha}^*] - \tilde{s}[\tilde{r}^*]], \quad (173)$$
since compact support is both retarded and advanced. Then the following calculation gives the desired conclusion:

\[ J[\psi] = r \circ \hat{f}[\psi] + r_c \circ \tilde{c}[\psi] \tag{174} \]

\[ = r \circ \hat{r}[\tilde{\alpha}^*] + r_c \circ \tilde{c} \circ G_+ \circ \hat{r}[\tilde{\alpha}^*] \tag{175} \]

\[- r \circ \hat{s}[\tilde{\alpha}^*] - r_c \circ \tilde{c} \circ G_+ \circ \hat{s}[\tilde{\alpha}^*] \]

\[ = (\text{id} + p_1 \circ \hat{g}^* - r_c \circ \hat{r}_1)[\tilde{\alpha}^*] + r_c \circ H_+ \circ \hat{g} \circ \hat{r}[\tilde{\alpha}^*] \tag{176} \]

\[- r \circ \hat{s}[\tilde{\alpha}^*] - r_c \circ (\hat{r}_1 \circ J + \hat{r}_g \circ \hat{c}_g) \circ \hat{g} \circ K_+[\tilde{\alpha}^*] \]

\[ = \tilde{\alpha}^* + r_c \circ (- \hat{r}_1 + H_+ \circ \hat{h} \circ \hat{r}_1 + H_+ \circ q_3 \circ \hat{g}^*)[\tilde{\alpha}^*] \tag{177} \]

\[- r \circ \hat{s}[\tilde{\alpha}^*] + r \circ (r_c \circ \hat{c}_g \circ \hat{g}) \circ K_+[\tilde{\alpha}^*] \]

\[ = \tilde{\alpha}^*, \tag{178} \]

where we have used the definition \((164)\) and identities \((146), (147), (151)\) and the commutativity of diagrams \((101)\) and \((103)\). Now, consider the following composition of operators applied to any \(\psi \in \Gamma_0(F)\):

\[ \hat{g}'[E \circ J[\psi]] = \hat{g}' \circ G \circ \hat{r} \circ J[\psi] \tag{179} \]

\[ = \hat{g}' \circ (G \circ \hat{f})[\psi] - K' \circ (s' \circ \hat{r}_c \circ \hat{c}_g)[\psi] \tag{180} \]

\[ = 0, \tag{181} \]

where the last two terms vanish by the exactness of the causal Green function sequence \((71)\) and the gauge fixing compatibility condition \((154)\). Global recognizability then dictates that there exists a section \(\epsilon \in \Gamma_0(P)\) such that \(E \circ J[\psi] = \hat{g}[\epsilon]\), which is the desired result.

3. Every solution \(\psi \in \Gamma_{SC}(F)\) of \(J[\psi] = 0\) is of the form \(\psi = E[\tilde{\alpha}^*] + \hat{g}[\epsilon]\) for some \(\epsilon \in \Gamma_{SC}(P)\) and \(\tilde{\alpha}^* \in \Gamma_0(F^*)\), with \(\hat{g}^*[\tilde{\alpha}^*] = 0\).

Let \(\Sigma \subset M\) be a \(\phi\)-Cauchy surface and \(\{\chi_\pm\}\) a partition of unity adapted to it. Let \(\tilde{\alpha}^* = J[\chi]\) and \(\psi' = E[\tilde{\alpha}^*]\), where now \(J[\psi'] = 0\) and \(\hat{c}[\psi'] = 0\). Also, denote \(\hat{c}_\chi[\psi] = \hat{c}_\chi[\pm \chi_\pm \psi]\). It remains to show that \(\psi - \psi'\) is pure gauge:

\[ \hat{g}'[\psi'] = \hat{g}' \circ E[\tilde{\alpha}^*] = \hat{g}' \circ G \circ \hat{r}[\tilde{\alpha}^*] \tag{182} \]

\[ = \hat{g}' \circ G \circ \hat{r} \circ \hat{f}_\chi[\psi] + r_c \circ \hat{c}_\chi[\psi] \tag{183} \]

\[ = \hat{g}' \circ G[\hat{f}_\chi[\psi] + r_c \circ r_g \circ \hat{c}_\chi[\psi]] \tag{184} \]

\[ = \hat{g}'[\psi] + K' \circ (s' \circ \hat{r}_c \circ \hat{c}_\chi[\psi]) \tag{185} \]

\[ = \hat{g}'[\psi], \tag{186} \]

where we have appealed to identities \((147), (148)\) and the gauge fixing compatibility condition \((153)\). Now, we have \(\hat{g}'[\psi - \psi'] = 0\), which implies, together with global recognizability, that \(\psi - \psi' = \hat{g}^*[\epsilon]\) for some \(\epsilon \in \Gamma_{SC}(P)\).

4. Every \(\tilde{\alpha}^* \in \Gamma_{SC}(F^*)\), such that \(\hat{g}^*[\tilde{\alpha}^*] = 0\), can be written as \(\tilde{\alpha}^* = J[\psi]\) for some \(\psi \in \Gamma_{SC}(F)\).

Global recognizability implies that there exists a \(\tilde{\alpha}^* \in \Gamma_{SC}(F^*)\) such that \(\tilde{\alpha}^* = \hat{g}^*[\tilde{\alpha}^*]\). Let \(\psi = E[\chi]\). Then the following calculation gives the desired
conclusion:

\[
\begin{align*}
J[\psi] &= \sum_k (r \circ \tilde{f} + r_c \circ \tilde{c}) \circ G_k \circ \tilde{r}[\tilde{\gamma}^*]|_{\chi_k \tilde{\epsilon}^*}] \\
&= r \circ \tilde{r}[\tilde{\alpha}^*] + r_c \circ \tilde{r}_f[\tilde{\alpha}^*] = (id + p_3 \circ \tilde{g}^*)[\tilde{\alpha}^*] \\
&= \tilde{\alpha}^*. 
\end{align*}
\]

(187)

(188)

(189)

5. We have \( E \circ J = \text{id} \mod \text{im} \tilde{g} \) on \( S_{SC}(\tilde{P}) \).

Let \( \psi \in S_{SC}(\tilde{P}) \). Direct calculation shows

\[
\begin{align*}
\dot{g}'[E \circ J[\psi]] &= \dot{g}'[G \circ \tilde{r} \circ J[\psi]] \\
&= \dot{g}'[G \circ \tilde{f}[\psi]] + H \circ (\dot{s}' \circ \tilde{r}_c \circ \dot{i}_g)[\psi] \\
&= \dot{g}'[\psi]
\end{align*}
\]

(190)

(191)

(192)

where we have used a splitting identity from Lem. 5 and the gauge fixing compatibility condition (153). It follows that \( \psi - E \circ J[\psi] \in \text{im} \dot{g} \) is pure gauge.

6. We have \( J \circ E[\dot{\psi}] = \dot{g}^* \) on \( \Gamma_{SC}(\tilde{P}) \).

This was already checked in item 4 above. \( \Box \)

It is easy to see from its variational nature that the Jacobi operator is self-adjoint \((J)^* = J\). If it were directly invertible, the Green functions \( E_{\pm} \) would satisfy the same relation with their adjoints as shown in Sect. (4.4.4), making the causal Green function anti-self-adjoint, \( E^* = -E \). However, due to gauge invariance the relation of the gauge fixed Green functions to their adjoints is more complicated.

**Lemma 17.** When restricted to act on gauge invariant dual densities, the causal Green function of the gauge fixed Jacobi system is anti-self-adjoint up to gauge:

\[
(E)^* = E \mod \text{im} \tilde{g}.
\]

(193)

**Proof.** First, note that from identities (156) and (157) we have

\[
\begin{align*}
J \circ E_{\pm}[\tilde{\alpha}^*] &= \sum_k (r \circ \tilde{f} + r_c \circ \tilde{c}) \circ G_k \circ \tilde{r}[\tilde{\alpha}^*] \\
&= r \circ \tilde{r}[\tilde{\alpha}^*] + r_c \circ \tilde{r}_f[\tilde{\alpha}^*] \\
&= (id + p_3 \circ \tilde{g}^*)[\tilde{\alpha}^*].
\end{align*}
\]

(194)

(195)

(196)

It follows that

\[
\begin{align*}
(E_{-})^* \circ J \circ E_{\pm} &= (E_{-})^* \circ (J \circ E_{\pm}) = (E_{-})^* \circ p_3 \circ \tilde{g}^*, \\
(E_{+})^* \circ J \circ E_{\pm} &= ((E_{+})^* \circ J^*) \circ E_{\pm} = E_{\pm} + \tilde{g} \circ p_3 \circ E_{\pm},
\end{align*}
\]

(197)

(198)

and hence \( E_{\pm} = (E_{+})^* + (E_{-})^* \circ p_3 \circ \tilde{g}^* - \tilde{g} \circ p_3 \circ E_{\pm} \).

(199)

Given that \( E = E_{+} - E_{-} \), we then have

\[
E = -(E)^* - \tilde{g} \circ p_3 \circ E - (E)^* \circ p_3 \circ \tilde{g}^*,
\]

(200)

which gives the desired conclusion. \( \Box \)
As noted earlier in this section, there is considerable gauge freedom left that is compatible with the partial gauge fixing considered above. We call this kind of gauge fixing purely hyperbolic. Any further gauge fixing conditions are then called residual. We leave the consideration of residual gauge fixing to future work. A principal difficulty in dealing with residual gauge fixing conditions is that the resulting constraints are no longer parametrizable (such as operators that are elliptic on a family of spatial slices). Thus, the kernel of the gauge fixing conditions may contain very few, if any solutions with spacelike compact support, which would be difficult to fit into the current formal framework for tangent and cotangent spaces to the space of solutions.

5.3.3. Formal symplectic structure. In this section we define a formal presymplectic structure on a slow patch \( S(F, C) \) of the solution space \( S_H(F) \); using the language of Sect. 5.2.7, it consists of a formal local differential 2-form \( \Omega_\Sigma \) that is formally closed, \( \delta \Omega_\Sigma = 0 \). In a later section, Sect. 5.3.4, we shall see that some sufficient conditions ensure that \( \Omega_\Sigma \) is invertible and hence symplectic. Thus, the solution space becomes the phase space of classical field theory, as it can be endowed with both formal symplectic and Poisson structures. In the presence of gauge symmetries, this presymplectic form projects to an actual symplectic form on the space of gauge equivalence classes of solutions, that is, the physical phase space, or rather a slow patch \( \bar{S}(F, C) \) thereof. The symplectic forms will also be seen in a later section, Sect. 5.3.6, to agree on overlapping slow patches and hence constitute symplectic structure on the global phase space \( S_H(F) \).

Fix a globally hyperbolic chronal cone bundle \( C \to M \), as well as a \( C \)-Cauchy surface \( \Sigma \subset M \).

Definition 37. Given a presymplectic potential current density \( \omega \in \Omega^{n-1,2}(F) \) compatible with a PDE system \( \iota: E \subseteq J^kF \) and a closed codim-1 surface \( \Sigma \subset M \), we can define the following local differential 2-form on the solution space \( S(F, C) \):

\[
\Omega_\Sigma(\psi, \xi)[\phi] = \int_\Sigma \omega(\psi, \xi)[\phi] = \int_\Sigma (j^\infty \phi)^*[\iota^\Sigma\hat{\psi} \iota^\Sigma\hat{\xi} \omega]. \tag{201}
\]

Recall that two formal tangent vectors \( \psi, \xi \in T_\phi S \) correspond to linearized solutions with spatially compact support, while \( \hat{\psi} \) and \( \hat{\xi} \) are evolutionary vector fields on \( J^\infty F \) defined by them.

Ideally, we would then prove that \( \Omega_\Sigma[\phi] \) is a smooth differential 2-form on the infinite dimensional manifold \( S(F, C) \). However, that would necessitate a level of functional analytic detail going beyond the formal approach we have adopted. Instead, we note that the dependence of \( \Omega_\Sigma[\phi] \) on the solution section \( \phi \) is entirely through the ordinary differential form \( \omega \) on the jet bundle \( J^\infty F \) (or even one of its finite dimensional projections \( J^kF \)). Since \( \omega \) is smooth (by hypothesis), we also declare \( \Omega_\Sigma \) formally smooth. In addition to \( \phi \), the form \( \Omega_\Sigma[\phi] \) depends on the surface \( \Sigma \). For every \( \phi \in S(F, C) \), the supports of the linearized solutions \( \psi \) and \( \xi \) (formal tangent vectors) will have compact intersection with any surface that is \( \phi \)-Cauchy. A fortiori, since \( \Sigma \) is \( \phi \)-Cauchy for every \( \phi \in S(F, C) \), the integral defining \( \Omega_\Sigma(\psi, \xi)[\phi] \) converges. Moreover, because the presymplectic current density \( \omega \) is horizontally closed on solutions, the value of the integral remains well defined and does not change as \( \Sigma \) is varied within the same homology class.

It is now straightforward to prove the following
Lemma 18. The formal local 2-form $\Omega^*_\Sigma$ is formally closed, $\delta \Omega^*_\Sigma = 0$, and it depends only on the homology class of $\Sigma$.

Proof. Fix a background solution $\phi \in S_H(F)$ and consider linearized solutions $\chi, \xi, \psi \in T_\phi \mathcal{S}$. It is straightforward to check that $\Omega^*_\Sigma$ is vertically closed:

$$
(\delta \Omega^*_\Sigma)(\chi, \xi, \psi)[\phi] = \int_\Sigma (j^\infty \phi)^*[\iota_\psi \iota_\xi \iota_\chi \delta \omega] = \int_\Sigma (j^\infty \phi)^*[\iota_\psi \iota_\xi \iota_\chi \delta \omega]
$$

(202)

$$
= \int_\Sigma (j^\infty \phi)^*[\iota_\psi \iota_\xi \iota_\chi (\delta \omega)] = 0,
$$

(203)

where we have used the vertical closure $\iota^*_\infty \omega$, from the presymplectic hypothesis.

Independence of the representative of the homology class of $\Sigma$ follows if we can show that the integrand of $\Omega^*_\Sigma(\chi, \xi)[\phi] = \int_\Sigma (j^\infty \phi)^*[\iota_\psi \iota_\xi \iota_\chi \delta \omega]$ is de Rham closed on $M$. This follows directly from the horizontal closure of $\iota^*_\infty \omega$, again from the presymplectic hypothesis:

$$
d[(j^\infty \phi)^*[\iota_\psi \iota_\xi \iota_\chi \omega]] = (j^k \phi)^*[\iota_\psi \iota_\xi \iota_\chi \omega] = (j^k \phi)^*[\iota_\psi \iota_\xi \iota_\chi (\delta \omega)] = 0.
$$

(204)

Note that, strictly speaking, on needs to apply Stokes’ theorem to establish independence of the representative $\Sigma$, which requires the convergence of all intermediate integrals. This property will actually hold in the main application of this result, which is Def. 38 below, so we do not discuss it in more detail here.

In both cases, we have introduced the pullback $\iota^*_\infty$ because $\phi$ is a solution and $\chi, \xi, \psi$ are linearized solutions, so that the pullback $(j^\infty \phi)^*$ factors through $\iota^*_\infty$. □

Definition 38. Given a (gauge fixed, if necessary) variational PDE system on the field bundle $F \to M$, with Lagrangian density $\mathcal{L}$, that is equivalent to a quasilinear, symmetric hyperbolic system on $F \to M$ with constraints and a globally hyperbolic chronological cone bundle $C \to M$, we define the formal variational presymplectic form on the space $\mathcal{S}(F, C)$ of $C$-slow solutions as $\Omega = \Omega_\Sigma$, where we use the presymplectic current density $\omega$ associated to $\mathcal{L}$ and $\Sigma$ is $C$-Cauchy.

Using the smooth splitting of $M$ induced by a globally hyperbolic cone bundle, it is straightforward to show that all $C$-Cauchy surfaces belong to the same homology class. That, together with the fact that a $C$-Cauchy surface will compactly intersect the support of any $\phi$-spacelike compact linearized solution, with $\phi \in \mathcal{S}(F, C)$, shows that $\Omega$ is well defined, independently of the choice of $\Sigma$.

A bilinear form defines a linear map from a vector space to its algebraic dual. A similar statement holds for a continuous bilinear form and to topological dual space. However, our formal cotangent space $T^*_\psi \mathcal{S}$ is neither the algebraic nor the topological dual of the formal tangent space $T_\phi \mathcal{S}$. Thus we have to check this property for $\Omega$ by hand. This will be accomplished using an analog of a splitting map for the Jacobi system from Thm. 3, which is analogous Lem. 38 for purely hyperbolic systems. The argument in the proof was inspired by [26].

---

\[8\] The appropriate homology theory here should correspond to a variant of locally finite Borel-Moore homology, where one considers only chains whose intersection with every $C$-spatially compact set is compact. This variant does not appear to have gotten any attention in the literature and thus deserves further study.
Lemma 19. The presymplectic form $\Omega$ defines the following map from the formal tangent space to the formal cotangent space:

$$\Omega : T_\phi S \to T_\phi^* S$$

$$\psi \mapsto [\tilde{\alpha}^*],$$

$$\text{with } \tilde{\alpha}^* = J_\chi[\psi] = \pm J[\chi^\pm \psi],$$

where $J : \Gamma(F) \to \Gamma(\tilde{F}^*)$ the Jacobi differential operator at the dynamical linearization point $\phi \in S_H(F)$ and $\{\chi^\pm\}$ is a partition of unity adapted to a $C^\infty$ Cauchy surface $\Sigma$.

Proof. Using the adapted partition of unity, we can write any solution of $\dot{f}[\psi] = 0$ as $\psi = \psi_+ + \psi_-$, with $\psi_\pm = \chi^\pm \psi$. If the solution also satisfies the purely hyperbolic gauge fixing condition $\dot{s} \circ r[\psi] = 0$, by the equivalence discussed earlier in this section it also satisfies the Jacobi equation $J[\psi] = 0$. Hence $J[\psi_+] + J[\psi_-] = 0$ or $J[\psi_+] = -J[\psi_-] = J_\chi[\psi]$. Note that the support of $J[\psi_\pm]$ is compact, since $\psi_\pm$ satisfy the Jacobi equation away from the intersection $S^+ \cap S^- \cap \text{supp } \psi$.

Next, we want to find a dual density $\tilde{\alpha}^*$ that satisfies $\Omega[\psi][\phi] = [\xi, \tilde{\alpha}^*]$ for any $\xi \in T_\phi S$, which in particular satisfies $J[\xi] = 0$. Recall that an adapted partition of unity also depends on two additional $C^\infty$-Cauchy surfaces $\Sigma^\pm \subset I^\pm(\Sigma)$ and the supports of the partition are contained in $\text{supp } \chi^\pm \subset S^\pm = I^\pm(\Sigma^\mp)$.

$$\Omega[\xi, \psi][\phi] = \int_{\Sigma^\pm} \omega[\xi, \psi][\phi] = \sum_{\pm} \int_{\Sigma^\pm} \omega[\xi, \psi_\pm][\phi]$$

$$= \sum_{\pm} \int_{\Sigma^\pm} \omega[\phi][\xi, \psi_\pm] + \sum_{\pm} \int_{S^\pm \cap I^\mp(\Sigma)} d\omega[\xi, \psi_\pm][\phi]$$

$$= \sum_{\pm} \int_{I^\mp(\Sigma)} (d_\kappa \omega)[\xi, \psi_\pm][\phi]$$

$$= \sum_{\pm} \int_{I^\mp(\Sigma)} (-d_\xi EL_m \land d_\phi u^a)(\xi, \psi_\pm)[\phi]$$

$$= \sum_{\pm} \int_{I^\mp(\Sigma)} [(J^a_{\phi} \partial_\xi \psi^a_\pm) - (J^a_{\phi} \partial_\xi \psi^b a))]$$

$$= \int_{I^-(\Sigma)} \xi \cdot J[\psi_+] + \int_{I^+(\Sigma)} \xi \cdot J[\psi_-]$$

$$= \int_{I^-(\Sigma)} \xi \cdot J_\chi[\psi] + \int_{I^+(\Sigma)} \xi \cdot (-J_\chi[\psi])$$

$$= \int_M \xi \cdot J_\chi[\psi]$$

After the integration by parts, the boundary integrals over $\Sigma^\pm$ were dropped since they did not intersect the support of their integrands. Since the $\text{supp } \psi_\pm \subset S^\pm$, the integration over $S^+ \cap I^\mp(\Sigma)$ could be extended to all of $I^\mp(\Sigma)$. The term $\psi_\pm \cdot J[\xi]$ was dropped since $\xi$ is a linearized solution.
Recall that we are not interested in the dual density $\tilde{\alpha}^* = J_\xi[\psi]$ specifically, which explicitly depends on the adapted partition of unity, but rather the equivalence class $[\tilde{\alpha}^*] \in T^*_\phi S$, defined modulo $\text{im} J$. Consider another adapted partition of unity $\{\chi'\}$.

Because they both provide splitting maps, we have $\psi = E \circ J_\chi[\psi] = E \circ J_{\chi'}[\psi]$. Then $E[J_{\chi'}[\psi] - J_\chi[\psi]] = 0$. By exactness, $J_\chi[\psi]$ and $J_{\chi'}[\psi]$ must differ by an element of $\text{im} J$; in other words, they represent the same equivalence class in $T^*_\phi S$.

To complete the proof, we use the non-degeneracy of the natural pairing between $T^*_\phi S$ and $T^*_\phi S$ to define the operator $\Omega$ by the formula

$$\langle \xi, \Omega \psi \rangle = \Omega[\phi](\xi, \psi) = \langle \xi, \tilde{\alpha}^* \rangle,$$

so that $\Omega \psi = [\tilde{\alpha}^*] \in T^*_\phi S$, with $\tilde{\alpha}^* = J_\chi[\psi]$. \hfill \Box

Notice that in the presence of gauge symmetries (the gauge fixing is only partial), the form $\Omega$ is degenerate, since every pure gauge solution lies in its kernel:

$$\Omega(\dot{g}[\varepsilon], \psi)[\phi] = \langle \dot{g}[\varepsilon], \pm J[\chi \pm \psi] \rangle = \langle \varepsilon, \pm \dot{g}^* \circ J[\chi \pm \psi] \rangle = 0$$

for any $\psi$, since Noether’s second theorem implies that $\dot{g}^* \circ J = 0$ [80].

**Corollary 3.** The 2-form $\Omega$ on $T_\phi S$ projects to a 2-form $\bar{\Omega}$ on $T_\phi \bar{S}$ and hence defines a map

$$\bar{\Omega}: T_\phi S \to T^*_\phi S \tag{218}$$

$[\psi] \mapsto [\tilde{\alpha}^*], \tag{219}$

with $\tilde{\alpha}^* = J_\chi[\psi]. \tag{220}$

In other words, formally, the quotient projection to the physical phase space effects a presymplectic reduction $(S_H(F), \Omega) \to (\bar{S}_H(F), \bar{\Omega})$. We shall see later on that $\bar{\Omega}$ is non-degenerate and hence symplectic.

### 5.3.4. Formal Poisson bivector, Peierls formula

In this section we show that the formal symplectic form $\bar{\Omega}$ defined above is invertible and that its inverse, the formal Poisson bivector $\Pi$, is given by the Peierls formula

$$\Pi = E, \tag{221}$$

where $E$ is the causal Green function of the Jacobi operator $J$ as defined in Sect. 5.3.1. To show that $\Pi$ is indeed a Poisson bivector, it suffices to show that (a) it is an antisymmetric bilinear form on the formal cotangent space, (b) it defines a map from the formal cotangent space to the formal tangent space and (c) it is a two-sided inverse of $\bar{\Omega}$ defined in Cor. 3. The fact that $\Pi$ defines a Poisson bracket, with its Leibniz and Jacobi identities, then formally follows from standard arguments.

**Lemma 20.** The Peierls formula specifies a map from the formal cotangent space to the formal tangent space:

$$\Pi: T^*_\phi S \to T_\phi S \tag{222}$$

$[\tilde{\alpha}^*] \mapsto [\psi], \quad \text{with} \quad \dot{g}^*[\tilde{\alpha}^*] = 0 \tag{223}$

and $\psi = E[\tilde{\alpha}^*]. \tag{224}$
Proof. The challenge is to show that $II$ maps equivalence classes to equivalence classes (Def. 36). That is, that any representative $\tilde{\alpha} + f[\xi] + \hat{c}^*[\gamma^*]$ of an equivalence class $[\tilde{\alpha}^*] \in T^*_0S$, with $\hat{g}^*[\tilde{\alpha}^*] = 0$, gets mapped to the same equivalence class in $T^*_0S$. By linearity, it suffices to check that $[0] \in T^*_0S$ is mapped to $[0] \in T^*_0S$. Recall that any solution representing $[0] \in T^*_0S$ is pure gauge $\hat{g}[\epsilon]$. Note that the equivalence (146) of the (140) equation forms, together with the self-adjointness of the Jacobi operator $(J)$, allows us to rewrite any representative of $[0] \in T^*_0S$ as $J[\xi] + \hat{c}^*[\gamma^*]$, for some $\xi \in \Gamma_0(F)$ and $\gamma^* \in \Gamma_0(F^g)$. This representative will also satisfy the identity

$$\hat{g}^* \circ \hat{c}^*[\gamma^*] = \hat{g}^*[J[\xi] + \hat{c}^*[\gamma^*]] = 0.$$  \hspace{1cm} (225)

Direct calculation then shows that

$$II(J[\xi] + \hat{c}^*[\gamma^*]) = E \circ J[\xi] + E \circ \hat{c}^*[\gamma^*]$$

$$= \hat{g}[\epsilon] - (\hat{c}_g \circ E)^*[\gamma^*] - \hat{g} \circ p_1^* \circ E \circ \hat{c}_g^*[\gamma^*]$$

$$- (E)^* \circ p_1(\hat{g}^* \circ \hat{c}_g^*[\gamma^*])$$

$$= \hat{g}[\epsilon - q_g \circ (H)^* \circ r_g^*[\gamma^*] - p_1^* \circ E \circ \hat{c}_g^*[\gamma^*]]$$  \hspace{1cm} (226)

is pure gauge. We have used the identity that $E \circ J[\xi] = \hat{g}[\epsilon]$ for some $\epsilon \in \Gamma_0(P)$ (Thm. 5), the anti-self-adjointness identity (200), that (Eqs. (140) and (151))

$$\hat{c}_g \circ E = r_g \circ (\hat{c} \circ G) \circ \tilde{r} = r_g \circ (H \circ \hat{q} \circ \tilde{r})$$

$$= (r_g \circ H \circ q_1) \circ \hat{g}^*$$  \hspace{1cm} (227)

and the identity (225).

Therefore, we can conclude that if $[\tilde{\alpha}^*] = [0]$, then $[E[\tilde{\alpha}^*]] = [0]$. \hspace{1cm} \Box

**Lemma 21.** The Peierls formula defines an antisymmetric bilinear form on the formal cotangent space:

$$II([\tilde{\alpha}^*], [\tilde{\beta}^*]) = \langle II[\tilde{\alpha}^*], [\tilde{\beta}^*] \rangle = -\langle II[\tilde{\beta}^*], [\tilde{\alpha}^*] \rangle,$$  \hspace{1cm} (231)

for any $[\tilde{\alpha}^*], [\tilde{\beta}^*] \in T^*_0S$.

**Proof.** Recall that the representatives always satisfy $\hat{g}^*[\tilde{\alpha}^*] = \hat{g}^*[\tilde{\beta}^*] = 0$. Appealing directly to the anti-self-adjointness identity (200) we have

$$II([\tilde{\alpha}^*], [\tilde{\beta}^*]) = \langle II[\tilde{\alpha}^*], [\tilde{\beta}^*] \rangle = \langle E[\tilde{\alpha}^*], [\tilde{\beta}^*] \rangle = \langle E^*[\tilde{\beta}^*], [\tilde{\alpha}^*] \rangle$$  \hspace{1cm} (232)

$$= -\langle E[\tilde{\beta}^*] + \hat{g} \circ p_1^* \circ E \circ \hat{c}_g^*[\gamma^*], [\tilde{\alpha}^*] \rangle$$

$$= -\langle E[\tilde{\beta}^*], [\tilde{\alpha}^*] \rangle = -\langle II[\tilde{\beta}^*], [\tilde{\alpha}^*] \rangle$$

$$= -II([\tilde{\beta}^*], [\tilde{\alpha}^*]).$$ \hspace{1cm} \Box

**Theorem 6.** The Peierls formula gives a two-sided inverse to the formal symplectic form, $\tilde{\Omega} II = \text{id}$ on $T^*_0S$ and $II \tilde{\Omega} = \text{id}$ on $T^*_0S$.

The argument in the proof below was inspired by [46] and [107, Lem.3.2.1]. However, the argument has been generalized to handle hyperbolic systems with constraints and gauge invariance (presuming purely hyperbolic gauge fixing).
Proof. The proof uses in an essential way the splitting identities of Thm. 5. Consider any \( [\psi] \in T_{\phi} \mathcal{S} \) and \( [\tilde{\alpha}^*] \in T_{\phi}^* \mathcal{S} \). To use these splitting identities, we introduce a \( \phi \)-Cauchy surface \( \Sigma \subset M \) and a partition of unity \( \{ \chi_\pm \} \) adapted to it. Then

\[
(\Pi \tilde{\Omega}[\psi], [\tilde{\alpha}^*]) = (E \circ J_\chi[\psi], \tilde{\alpha}^*)
\]

(236)

\[
= (\psi + \dot{g}[\epsilon], \tilde{\alpha}^*) \quad \text{(for some } \epsilon \in \Gamma_{SC}(P)\text{)}
\]

(237)

\[
= (\{\psi\}, [\tilde{\alpha}^*]).
\]

(238)

Therefore, from the non-degeneracy of the natural pairing between \( T_{\phi} \mathcal{S} \) and \( T_{\phi}^* \mathcal{S} \), we concluded that \( \Pi \tilde{\Omega} = \text{id} \). Similarly,

\[
([\psi], \Omega \Pi [\tilde{\alpha}^*]) = (\psi, J_\chi \circ E[\tilde{\alpha}^*]).
\]

(239)

But then

\[
E[J_\chi \circ E[\tilde{\alpha}^*]] = (E \circ J_\chi) \circ E[\tilde{\alpha}^*] = E[\tilde{\alpha}^*] + \dot{g}[\epsilon],
\]

(240)

for some \( \epsilon \in \Gamma_0(P) \). But, by the cohomological exactness of Thm. 5, this means that \( J_\chi \circ E[\tilde{\alpha}^*] = J[\xi] \) for some \( \xi \in \Gamma_0(F) \). In other words,

\[
([\psi], \Omega \Pi [\tilde{\alpha}^*]) = (\psi, \tilde{\alpha}^* + J[\xi]) = ([\psi], [\tilde{\alpha}^*]).
\]

(241)

Therefore, from the non-degeneracy of the natural pairing between \( T_{\phi} \mathcal{S} \) and \( T_{\phi}^* \mathcal{S} \), we concluded that \( \Omega \Pi = \text{id} \). \( \square \)

5.3.5. Algebra of observables, locality. This section and the next complete the construction of a classical field theory by constructing, at least formally, the Poisson algebra of observables associated with it. Along the way, several important points are discussed. (i) Since the focus of this work is more geometric than analytical, some parts of the construction are kept formal. (ii) In the physics literature, the given construction is often called on-shell. We discuss its relation to and trade offs with respect to the off-shell formalism, which has received considerable attention in the recent literature on perturbative algebraic quantum field theory (pAQFT). (iii) We define the notion of spacetime support, corresponding to the spacetime localization of observables, and use it to prove a classical version of microcausality that is generalized to field theories with dynamical causal structure.

We will use the Peierls formula for the Poisson bivector \( \Pi \) to construct a formal Poisson bracket on the formal algebra of observables on the phase space \( (\mathcal{S}_H(F), \Omega) \) of a classical field theory. This section will actually concentrate on a single slow patch \( \mathcal{S}(F, C) \subseteq \mathcal{S}_H(F) \) of full phase space. Globalization to the full phase space \( \mathcal{S}_H(F) \) by covering it with slow patches will be considered in the following section, Sect. 5.3.6.

In the last paragraph, the word formal was used in two different ways. The algebra that we will consider is the polynomial algebra generated by local functionals with compact spacetime support. This algebra is much smaller than a reasonable space of smooth functions on the infinite dimensional manifold of solutions \( \mathcal{S}_H(F) \), and hence is only a formal substitute. However, this polynomial algebra is much less complicated from the analytical point of view and is expected to be dense in the larger, ultimately desired algebra \( C^\infty(\mathcal{S}_H(F)) \), however it is rigorously defined. Also, the Poisson bracket of two elements of
Igor Khavkine

this polynomial algebra, if defined using the bivector $\Pi$, is not an element of the same algebra, though still a well defined function on $S_H(F)$. However, just as the polynomial algebra is expected to be dense in $C^\infty(S_H(F))$, the Poisson bracket is expected to be continuous, so nested applications of the Poisson bracket can be considered by replacing each evaluation thereof by an approximating sequence within the polynomial algebra. In this sense, the definition of the Poisson bracket is given only formally. The hypotheses needed to justify its non-formal use are summarized in Hyp. [2].

Before moving on to the technical part of this section, briefly discuss the relation between our approach to the algebra of observables and the one taken in the recent literature on pAQFT [64,23,47,95,24]. That work is in the so-called off-shell formalism, while we work in the on-shell one. The space of all globally hyperbolic solution sections $S_H(F) \subset \Gamma_H(F)$ is a submanifold of the space of all globally hyperbolic sections and is sometimes referred to as the shell. Thus, on-shell field configurations correspond to solutions, while the off-shell ones are not so restricted. Dually, on the algebraic side, the algebra of smooth functions on $S_H(F)$ is a quotient $C^\infty(S_H(F)) \cong C^\infty(\Gamma_H(F))/I_E$ by the ideal $I_E$ generated by the equations of motion (an equation form of the PDE system $J^k F \supset \mathcal{E} \to M$). Correspondingly, $C^\infty(S_H(F))$ is called the on-shell algebra while $C^\infty(\Gamma_H(F))$ is called the off-shell algebra. The on-shell formalism attempts to construct the on-shell algebra directly as the algebra of functions on the space of solutions (or gauge equivalence classes thereof), while the off-shell formalism rather constructs it as the above quotient of the off-shell algebra (with a further quotient to factor out gauge).

The off-shell formalism is advantageous in perturbative algebraic quantum field theory (pAQFT) as it dramatically simplifies the perturbative renormalization of interacting theories [64,23,47,95]. On the other hand, the on-shell formalism connects more directly with the mathematical PDE literature. It is the theorems on the well-posedness of hyperbolic PDE systems, as quoted in Sec. 4.3, that will allow us to establish the validity of the (generalized) Causality and Time Slice axioms in Sec. 7. I am not aware of any methods that can be used to establish these results directly in the off-shell formalism (particularly for quasilinear systems) without first establishing its equivalence with the on-shell construction. Since understanding causality in field theories with quasilinear equations of motion is the main motivation for this work, we feel our choice to work in the on-shell formalism is justified. Of course, it is fully expected that future work will realize an exact duality between the two approaches.

First suppose that the gauge transformations are trivial. Consider a globally hyperbolic chronal cone bundle $C \to M$ and a local functional $A[\phi]$ (Sect. 5.2.7) on the slow patch $\Gamma(F,C)$ of the space of field configurations. Fix some $\phi \in \Gamma(F,C)$ and consider the formal exterior differential $\delta A[\phi]$ and denote its value in $T^*_\phi \Gamma$ by $\tilde{\alpha}^*$. Then $\delta A[\phi](\psi) = \langle \psi, \tilde{\alpha}^* \rangle$ for any tangent vector $\psi \in T^*_\phi \Gamma$.

**Definition 39.** We refer to the support of the dual density $\tilde{\alpha}^*$ as the local spacetime support of $A[\phi]$ at $\phi$, $\text{supp} M,\phi A = \text{supp} \tilde{\alpha}^*$. The global spacetime

---

[2] This discussion should be considered formal and analogous to the situation where all manifolds are finite dimensional. The justification of its conclusions is a matter of ongoing work in this field.
support of $A[\phi]$ is the closure of the union of all local spacetime supports, 
\[ \text{supp}_M A = \bigcup_{\phi \in \Gamma(F,C)} \text{supp}_{M,\phi} A. \]

We may sometimes speak simply of spacetime support of $A$ when the precise version of the notion is clear from context.

More than functionals on field configurations, we are interested in functionals on the slow patch $S(F,C)$ of the space of solutions $\mathcal{S}_H(F)$. So a functional $A[\phi]$ on $S(F,C)$ is called local if it is the pullback of a local functional on field configurations along the inclusion $S(F,C) \subset \Gamma(F,C)$. The definition of spacetime support for a functional on solutions is complicated by the fact that many local functionals on field configurations may pull back to the same functional $A[\phi]$ on solutions. This difficulty is resolved by deciding that $A[\phi]$ admits multiple spacetime supports.

**Definition 40.** Consider a local functional $A[\phi]$ on $S(F,C)$ and any local functional $A'[\phi]$ on $\Gamma(F,C)$, which pulls back to $A[\phi]$ on solutions. Then we say that $\text{supp}_{M,\phi} A'$ is a local spacetime support of $A$ at $\phi$ and that $\text{supp}_M A'$ is a global spacetime support of $A$. The collections of all local and global spacetime supports of $A$ are denoted $\text{supp}_{M,\phi} A = [\text{supp}_{M,\phi} A']$ and $\text{supp}_M A = [\text{supp}_M A']$, where the square brackets enclose a representative.

Essentially, this definition states that $\text{supp}_{M,\phi}$ is the collection of supports of the representatives $\tilde{\alpha}^*$ of the value of the formal exterior differential $\delta A[\phi] = \tilde{\alpha}^*$ in $T^*_\phi \mathcal{S}$. When referring to the properties of the spacetime support of $A[\phi]$, we are free to refer to any representative. For instance we say that a local functional $A[\phi]$ on $S(F,C)$ has a compact local spacetime support at $\phi$ if $\text{supp}_{M,\phi} A$ contains a compact representative. The same goes for global spacetime support. A subtly, but importantly different notion is of globally compact local spacetime support, which means that the local spacetime support $\text{supp}_{M,\phi} A$ has a compact representative for every $\phi \in S(F,C)$. Also two such local functionals $A[\phi]$ and $B[\phi]$ have spacelike separated local spacetime supports at $\phi$ if $\text{supp}_{M,\phi} A$ and $\text{supp}_{M,\phi} B$ have representatives that are $\phi$-spacelike separated. If $A[\phi]$ and $B[\phi]$ have spacelike separated local spacetime supports at every $\phi \in S(F,C)$, we say that $A[\phi]$ and $B[\phi]$ have globally spacelike separated local spacetime supports. The same terminology also works for functionals on field configurations. On the other hand, if we say that $A[\phi]$ and $B[\phi]$ have C-spacelike separated global spacetime supports if $\text{supp}_M A$ and $\text{supp}_M B$ have representatives that are C-spacelike separated. Of course, C-spacelike separation of global spacetime supports implies global spacelike separation of local spacetime supports, but the converse is does not hold. In fact, one can imagine local functionals $A[\phi]$ and $B[\phi]$ with globally spacelike separated local spacetime supports but with $\text{supp}_M A = \text{supp}_M B = [M]$, because of the way $\text{supp}_{M,\phi} A$ and $\text{supp}_{M,\phi} B$ vary as a function of $\phi$.

In the presence of non-trivial gauge transformations, the gauge invariant cotangent spaces, $T^*_\phi \mathcal{S}$, are distinct from the ordinary ones, $T^*_\phi \mathcal{S}$.

**Definition 41.** A local functional $A'[\phi]$ on $\Gamma(F,C)$ is called gauge invariant if its formal exterior derivative $\delta A'[\phi]$ takes values in the gauge invariant cotangent space $T^*_\phi \mathcal{S} \subset T^*_\phi \mathcal{S}$. Similarly, a local functional $A[\phi]$ on $S(F,C)$ is gauge invariant if its formal exterior derivative $\delta A[\phi]$ takes values in $T^*_\phi \mathcal{S} \subset T^*_\phi \mathcal{S}$. 
In particular, a local functional $A[\phi]$ (whether on solutions or field configurations) satisfies $\delta A[\phi](\dot{g}[\varepsilon])$ for any linearized gauge transformation at $\phi$ with gauge parameter section $\varepsilon \in \Gamma_0(P)$. The notions of compact spacetime support and $C$-spacelike separated supports specialize straightforwardly to gauge invariant local functionals.

We are now ready to define the Poisson bracket of two gauge invariant local functionals $A[\phi]$ and $B[\phi]$, which we write simply as

$$\{A, B\} = \Pi(\delta A, \delta B). \quad (242)$$

In more detail, let $\phi \in S(F, C)$ and denote the values of the formal exterior differentials at $\phi$ by $\delta A[\phi] = [\tilde{\alpha}^*]$ and $\delta B[\phi] = [\tilde{\beta}^*]$, both in $T^*_\phi \bar{\mathcal{S}}$. We then have

$$\{A, B\}[\phi] = \Pi([\tilde{\alpha}^*], [\tilde{\beta}^*]) = \langle E[\tilde{\alpha}^*], \tilde{\beta}^* \rangle, \quad (243)$$

where we have invoked Lem. \[21\] and $E$ is the causal Green function of the Jacobi system at the dynamical linearization point $\phi$. Incidentally, the last formula in Eq. \[243\] allows the extension of the Poisson bracket off-shell to gauge invariant functionals on field configurations \[84,83,40,24\], though we will not discuss this possibility at the moment.

A problem with this definition, and the reason we announced at the beginning of this section that the Poisson bracket would be defined only formally, is that $\{A, B\}[\phi]$ is in general not a local functional. That is because the Green function $E$ is a non-local integral operator. Consider two local coordinate charts $(x^i, u^a)$ and $(y^j, u^b)$ on $F$ so that the dual densities $\tilde{\alpha}^*$ and $\tilde{\beta}^*$ have the coordinate expressions $\alpha^*_a(x) \, d\tilde{x}$ and $\beta^*_b(y) \, d\tilde{y}$, while $E$ has the integral kernel $E^{ab}(x, y)$. Then

$$\{A, B\}[\phi] = \int_{M \times M} E^{ab}(x, y) \, \alpha^*_a(x) \, \beta^*_b(y) \, d\tilde{x} \wedge d\tilde{y}. \quad (244)$$

As $E^{ab}(x, y)$ can be seen as a vector valued distribution on $M \times M$, the above integral is well defined, but cannot be re-expressed as a local functional unless $E^{ab}(x, y)$ is supported only on the total diagonal $x = y$, which it obviously is not. It should be also clear that the product of two local functionals $A[\phi]B[\phi]$ is also not a local functional.

To address both of the above problems, we expand the set of functionals under consideration to polynomials in local functionals.

**Definition 42.** Let $\text{Loc}(F, C)$ denote the space of local functionals on $\Gamma(F, C)$ with compact local spacetime support and $\text{PolyLoc}(F, C)$ the polynomial algebra generated by $\text{Loc}(F, C)$. Similarly, $\text{Loc}(F)$ and $\text{Loc}(F)$ refer to functionals on $\Gamma_H(F)$. The elements of $\text{PolyLoc}(F, C)$ and $\text{PolyLoc}(F)$ are referred to as polylocal functionals.

As before for local functionals, polylocal functionals may be defined on field configurations as well as solutions. Recall that we did not give a precise definition of infinite dimensional manifolds in our category $\text{Man}$ nor of the algebra of smooth functions on them. However, whatever precise definitions that are ultimately chosen, we require the following
Hypothesis 2. The functor of smooth functions $\mathcal{C}^\infty: \text{Man} \to \text{CAlg}$ is such that $\mathcal{C}^\infty(\Gamma(F,C))$ is a topological algebra containing $\text{PolyLoc}(F,C)$ as a dense sub-algebra, which also contains the Poisson brackets $\{A, B\}$ of local functionals and such that $\{-, -\}$ extends continuously to all of $\mathcal{C}^\infty(\Gamma(F,C))$. Analogous statements hold for $\mathcal{C}^\infty(\Gamma_H(F))$ and $\text{PolyLoc}(F)$ and upon replacement of field configurations with solutions.

Remark 3. It is straightforward to generalize the formal exterior derivative from local functionals to polylocal ones, as well as to more general $\mathcal{C}^\infty$ functionals. The same goes for the notions of local spacetime support and global spacetime support. We silently make use of this generalization from now on.

So, we will content ourselves with approximating the Poisson bracket $\{A, B\}$ by a sequence (or net or filter) of polylocal functionals $C_i[\phi]$, with $i$ from some index set $I$. This concludes the formal definition of the algebra of observables and the Poisson bracket on it.

Definition 43. We call the algebra of gauge invariant smooth functions on the slow patch $\mathcal{S}(F,C)$ together with the Poisson bracket (243) the Poisson algebra of observables of the classical field theory and denote it $\mathcal{F}(F,C) \cong (\mathcal{C}_c^\infty(\bar{\mathcal{S}}(F,C)), \{\})$, where the subscript $\text{cst}$ means that we only take functionals with compact local spacetime support. (Note that we have identified the gauge invariant functions on $\mathcal{S}(F,C)$ with functions on $\bar{\mathcal{S}}(F,C)$, the space of gauge equivalence classes of solutions.) Denote the underlying commutative algebra of $\mathcal{F}(F,C)$ by $\mathcal{A}(F,C) \cong \mathcal{C}_c^\infty(\bar{\mathcal{S}}(F,C))$, which will also be referred to as the algebra of observables.

Of course, any results concerning $\mathcal{F}(F,C)$ and $\mathcal{A}(F,C)$ that we can establish in this paper will only be at the level of $\text{PolyLoc}(F,C)$, which will be strengthened by Hyp. 2.

Finally, we are ready to state and prove the main result of this section. In quantum field theory, it usually goes under the name of microcausality: observables that depend on fields in spacelike separated regions have vanishing commutators. The classical version replaces the commutator by the Poisson bracket. Note, however, that the result below is an important generalization of the known microcausality property of field theories with semilinear equations of motion, i.e., those with a non-dynamical causal structure. Usually, only the global spacetime support is considered and, even if the local spacetime support were to be considered, the notion of space-like separation is independent of the dynamical fields and so does not change when the observables are linearized about different solutions (for instance, in perturbation theory). For field theories with quasi-linear equations of motion, and hence dynamical causal structures, the notion of spacelike separation is intrinsically changes from solution to solution on the phase space. Thus, to even have a suitable generalization of the spacelike separation hypothesis for microcausality, we are forced to consider local spacetime support.

Theorem 7 (Classical microcausality). Consider two observables $A$ and $B$ belonging to the Poisson algebra of observables $\mathcal{F}(F,C)$. If their local spacetime supports are spacelike separated, then their Poisson bracket vanishes identically,

$$\{A, B\} = 0.$$
Proof. Consider any solution \( \phi \in \mathcal{S}(F, C) \). Let the dual densities \( \tilde{\alpha}^*, \tilde{\beta}^* \in \Gamma_0(\hat{F}^*) \) represent the formal exterior derivatives of the observables, \( \delta A[\phi] = [\tilde{\alpha}^*] \) and \( \delta B[\phi] = [\tilde{\beta}^*] \). Pick a pair of local coordinates \((x^a, u^a)\) and \((y^b, u^b)\) on \( F \) such that the dual densities have the components \( \alpha^*_a(x) \, d\bar{x} \) and \( \beta^*_b(y) \, d\bar{y} \). By hypothesis, \( \text{supp} \alpha^*_a \) and \( \text{supp} \beta^*_b \) are \( \phi \)-spacelike separated.

From the Peierls formula for the Poisson bracket (243)

\[
\{A, B\}[\phi] = \int_{M \times M} E^{ab}(x, y) \alpha^*_a(x) \beta^*_b(y) \, d\bar{x} \wedge d\bar{y},
\]

where \( E^{ab}(x, y) \) is the integral kernel of the causal Green function \( E \) of the Jacobi system linearized at \( \phi \). However, by Eq. (164), \( E = G \circ \tilde{r} \), where \( G \) is the causal Green function of a linear symmetric hyperbolic system with adapted equation form \((f, F^*)\), whose principal symbol precisely determines the causal structure of the solution \( \phi \) (Sect. 4.3). Recall that the causal Green function is the difference of the retarded and advanced Green functions, \( G = G_+ - G_- \). Thus, the support of \( G \), and hence \( E \) and \( E^{ab}(x, y) \), since \( \tilde{r} \) is a differential operator and does not increase supports, excludes any pair of points \((x, y)\) that are \( \phi \)-spacelike separated.

It is now an easy conclusion that, according to the formula (247), the Poisson bracket \( \{A, B\}[\phi] \) vanishes at \( \phi \in \mathcal{S}(F, C) \). Since the solution \( \phi \) was arbitrary, it follows that the Poisson bracket vanishes identically, \( \{A, B\} = 0 \). □

A coarser version of the above result, though stated in the more familiar terms of global spacetime support follows immediately once the definitions are unwound.

Corollary 4. Consider two observables \( A \) and \( B \) belonging to the Poisson algebra of observables \( \mathcal{F}(F, C) \). If their global spacetime supports are \( C \)-spacelike separated, then their Poisson bracket vanishes identically, \( \{A, B\} = 0 \).

Remark 4. The notion of spacetime support in previous work on the functional approach to classical and quantum field theory [23,47,95] corresponds to what we call global spacetime support. The notion of local spacetime support and of spacelike separation of local spacetime supports seems to have been overlooked until now, with the notable exception of [24,97]. The distinction between local and global spacetime support is particularly important for gauge theories and theories with dynamical causal structures, both exemplified by GR. It is well known that there are no gauge invariant observables in GR with compact global spacetime support, a significant technical complication. However, the door is still open to discover a large, technically convenient class of observables with compact local spacetime support.

Another important but simple result is

Lemma 22 (Covariance). Consider two globally hyperbolic chronal cone bundles \( C' \to M' \) and \( C \to M \), where \( M' \subseteq M \) and \( C' \) is faster than \( C \). If we restrict our Lagrangian and PDE system to \( M' \), there are induced maps of solution spaces \( S(F, C) \to S(F', C') \) and algebras \( \mathcal{A}(F', C') \to \mathcal{A}(F, C) \), where \( F' = F|_{M'} \) is the field bundle over \( M' \). These maps induce morphisms of Poisson manifolds \( (S(F, C), \Pi) \to (S(F', C'), \Pi') \) and a Poisson homomorphism \( (\mathcal{A}(F', C'), \{\cdot, \cdot\}') \to \mathcal{A}(F, C), \{\cdot, \cdot\}) \), where \( \Pi' \) is also defined by the Peierls formula and in turn defines \( \{-, -\}' \).
Proof. The agreement of the Poisson bivectors $\Pi'$ and $\Pi$ follows directly from the Peierls formula and the covariance lemma for the causal Green function, Lem 6. This agreement immediately implies that the induced algebra homomorphism $\mathcal{A}(F, C') \to \mathcal{A}(F, C)$ is automatically a Poisson homomorphism with respect to the brackets $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}$. □

5.3.6. The global phase space. Here we take the final step in the construction of the phase space of the classical field theory and the Poisson algebra of observables on it. We appeal to the results obtained for each slow patch $\mathcal{S}(F, C)$ and recall that these patches form an open cover of the total space of solutions $\mathcal{S}_H(F)$, according to Hyp. 1. Moreover, each of the open patches carries a formally smooth presymplectic form $\Omega_C[\phi]$ (which becomes a symplectic one $\Omega_C[\phi]$) when projected to $\mathcal{S}(F, C)$, Sects. 5.3.3 and 5.3.4. The symplectic structure is translated to a Poisson structure $\Pi_C[\phi]$ by pointwise-inversion (smoothness and Jacobi identity are presumed to follow from the corresponding properties of the symplectic form), Sect. 5.3.3. It remains only to check that the symplectic and Poisson structures agree as these patches are glued together. In a later section, Sect. 6.3, this construction via slow patches will be interpreted in categorical terms as a colimit, which will play an important role in the generalized Causality property of classical field theory with quasilinear equations of motion.

Consider two chronally comparable globally hyperbolic chronal cone bundles $C_1 \to M$ and $C_2 \to M$, as well as their intersection $C_1 \cap C_2 = C_3 \to M$, also a globally hyperbolic cone bundle. The each we can associate the corresponding patch of $C_3$-slow solutions $\mathcal{S}(F, C_3)$ modulo gauge transformations. On each such patch, we have the symplectic and Poisson tensors $\Omega_{C_3}[\phi]$ and $\Pi_{C_3}[\phi]$. The slow patches intersect as $\mathcal{S}(F, C_1) \cap \mathcal{S}(F, C_2) = \mathcal{S}(F, C_3)$. Therefore, we must check whether $\Omega_{C_1}[\phi]$ all agree when restricted to $\mathcal{S}(F, C_3)$. Recall that each of the symplectic forms was defined with the help of a $C_3$-Cauchy surface $\Sigma_1 \subset M$. Each of the three symplectic forms is defined by exactly the same formula, Cor. 3, with the exception that the integration surface $\Sigma$ is replaced by the corresponding $\Sigma_1$. By construction, $\Sigma_1$ and $\Sigma_2$ are both $C_3$-Cauchy and so lie in the same $C_3$-spacelike homology class. Since $\Omega_{C_3}[\phi]$ depends only on that homology class, we can directly establish that

$$\Omega_{C_1}[\phi] = \Omega_{C_3}[\phi] = \Omega_{C_2}[\phi]$$

(248)

on $\mathcal{S}(F, C_3)$. The Poisson tensors $\Pi_{C_1}[\phi]$ necessarily satisfy the same property.

From the above discussion, we can conclude that the global space of globally hyperbolic solutions $\mathcal{S}_H(F)$ is endowed with both symplectic and Poisson structures, $\Omega[\phi]$ and $\Pi[\phi]$, defined by their agreement with the corresponding tensors on the slow patches $\mathcal{S}(F, C) \subseteq \mathcal{S}_H(F)$. We call the symplectic manifold $\mathcal{S}_H(F) = (\mathcal{S}_H(F), \Omega)$ the global phase space of the classical field theory under consideration. The Poisson algebra of smooth functions on it, with Poisson structure defined by $\Pi[\phi]$, is called the global algebra of observables $\mathcal{F}_H(F) = (\mathcal{A}_H(F), \{\cdot, \cdot\})$, where $\mathcal{A}_H(F) = C_c^{\infty}(\mathcal{S}_H(F))$ is the underlying commutative algebra.

The notions of local spacetime supports of observables and their global spacelike separation, as in Defs. 29 and 30, translate directly from the slow patches to the global phase space. Therefore, as a direct consequence of Thm. 10, we have...
Corollary 5 (Global classical microcausality). Consider two observables $A$ and $B$ belonging to the global Poisson algebra of observables $\mathcal{F}_H(F)$. If their local spacetime supports are globally spacelike separated, then their Poisson bracket vanishes identically,

$$\{A, B\} = 0.$$  

(249)

6. Natural Variational PDE Systems and Functoriality

In this section we take account of the various categories and functors introduced so far and introduce a few more. The main goal is to make explicit the functorial nature of all the steps in the construction of a classical field theory. For reference, some basic information about categories and functors can be found in [109,21].

6.1. Natural bundles. So far, we have looked at a fixed spacetime manifold $M$, a fixed field vector bundle over $F \to M$, and a fixed PDE system $E \to M$ on it, without concerning ourselves how the field bundle of the PDE system should change when the spacetime manifold is changed. To make a connection with covariant field theory, which functorially assigns algebras to spacetime manifolds, it is helpful to look at systems of partial differential equations that can also be assigned to spacetime manifolds functorially. As before, we presume that all fields are sections of vector bundles. The discussion below can be easily adapted to arbitrary smooth bundles. There are two issues that need to be addressed here: vector bundles and differential operators on them. Given an open embedding of manifolds $M \to M'$, there is a priori no natural map from vector bundles $F \to M$ and $F' \to M'$, even if the fibers of $F$ and $F'$ are modeled on the same vector space. The absence of such maps is an obstruction to relating sections of $F \to M$ to sections of $F' \to M$. The needed bundle maps must to be specified along with the typical fiber and the total space topology, which leads to the notion of a natural bundle. It is common practice to use a connection to define differential operators acting on sections of a vector bundle (that practice was adopted in [50], instance). But, once again, for an arbitrary vector bundle, there is no natural choice of connection. It then becomes convenient to work with jet bundles instead, as we have elected to do here. The switch between these points of view is straightforward [73, §17].

A field configuration corresponds to a section of a vector bundle $F \to M$ over a spacetime manifold $M$. To implement covariance with respect to arbitrary base space morphisms $\chi: M \to M'$ we need to be able to turn sections of $F \to M$ into sections of the corresponding bundle $F' \to M'$, restricted to the image $\chi(M)$. This requires a uniquely determined corresponding bundle morphism, $F \to F'$ that fits in the commutative diagram

$$\begin{array}{ccc}
F & \xrightarrow{\cong} & F' \\
\downarrow & & \downarrow \\
M & \xrightarrow{\chi} & M'.
\end{array}$$

(250)

In essence, we cannot allow $F$ and $F'$ to be arbitrary vector bundles; we need them to be associated functorially to $M$ and $M'$ respectively, which motivates the following definition.
Definition 44. A natural vector bundle $F$ over $n$-dimensional manifolds is a functor from the category of manifolds to the category of vector bundles, $F: \text{Man}^n \to \text{VBndl}$, that is right-inverse to the forgetful base space functor, $\Pi: (A \to M) \mapsto M$, that is, $\Pi \circ F = \text{id}$.

Examples of natural vector bundles are the tangent bundle $T$, the cotangent $T^\ast$. Operations on natural vector bundles like direct sums, direct products, linear duals, tensor products and jet extensions also define natural bundles. These and other examples are discussed extensively in [73]. Given a natural vector bundle $F: \text{Man} \to \text{VBndl}$, its $k$-jet bundle $M \mapsto J^k F(M)$ is also natural.

As we have seen in Sect. 3.1, to specify a $k$-th order PDE system on a bundle $F$, we need only specify a bundle map from $J^k F$ to an equation bundle $E$. If $F$ and $E$ are natural bundles, then a natural transformation between $J^k F$ and $E$ defines precisely these kinds of morphisms, $J^k F(M) \to E(M)$ in a way covariant with diffeomorphisms of $M$. Recall that it is sometimes convenient to see $M \xrightarrow{id} M$, a trivial bundle with zero dimensional fiber, as an object in $\text{Bndl}$ and $\text{id}: M \mapsto (M \xrightarrow{id} M)$ as the natural identity bundle.

Definition 45. A natural bundle map $f$ between natural ($\text{Bndl}$-valued) bundles $F$ and $E$ is a natural transformation $f: F \to E$.

A natural section $f$ of a natural bundle $F$, is a natural bundle map $f: M \to F$, where the domain is treated as the identity bundle.

Two examples of natural sections, given say a natural vector bundle $F$, are the zero section and the identity section of $F^\ast \otimes F$. An example of a natural bundle map, is the dual pairing $\langle -, - \rangle: F^\ast \otimes F \to \mathbb{R}$, where $\mathbb{R}$ is the trivial real line bundle. Other examples are provided by smooth real functions $f: \mathbb{R} \to \mathbb{R}$, which can be promoted to a natural bundle map by applying it fiberwise to the natural trivial bundle $M \times \mathbb{R}$.

We now have the necessary concepts to introduce natural variational hyperbolic PDE systems.

Definition 46. Let $F, B, V = F \times B$ and $E$ be respectively natural dynamical field, background field, total field and constraint vector bundles.

A natural first order, quasilinear, symmetric hyperbolic PDE system with constraints $f \oplus c$ is a natural bundle map $f \oplus c: J^1 F \times J^\infty B \to F^\ast \oplus E$, such that the bundle maps $f_M$, for any $M$ in $\text{Man}^n$, satisfy Def. 10, for some natural, non-empty subset of each fiber of $J^\infty B(M)$.

A natural Lagrangian density is a natural bundle map $\mathcal{L}: J^1 F \times J^\infty B \to \Lambda^n M$.

A natural symmetric hyperbolic variational PDE system is a pair $(f \oplus c, \mathcal{L})$ where $\mathcal{L}$ is a natural Lagrangian density and $f$ is a natural first order, quasilinear, symmetric hyperbolic PDE system that is naturally equivalent (Def. 13) to gauge fixed Euler-Lagrange equations of $\mathcal{L}$ (Sect. 5.5.2).

The introduction of background fields is sometimes necessary to make the Lagrangian natural. One may also need extra background fields to put the corresponding Euler-Lagrange equations into symmetric hyperbolic form in a natural way, as discussed for example in [59]. The example of a scalar wave equation on a curved background is discussed in some detail at the end of the next section.

A field section $\psi$ on $M$ will from now on always be a section of the total field bundle $\psi = (\phi, \beta): M \to V = V \times B$. As a consequence of the dependence
of the principal symbol $f$ of $f$ on the value of background fields, implies that
the naturally defined chronal and spacelike cone bundles $\Gamma(M)$ and $\Gamma^\oplus(M)$ are
cone bundles over $F \times J^\infty B(M)$.

An unfortunate complication in the presence of background fields is that not
all background field configurations are compatible either with the intrinsic inte-
grability conditions of the PDE system (like the restriction to Einstein vacuum
backgrounds in the Rarita-Schwinger system [57]) or with symmetric hyper-
bolicity (like the restriction to Lorentzian signature of the background metric
for wave equations). With this in mind, we say that a background field sec-
tion $\beta: M \to B(M)$ is admissible when (a) there exists a total field section
$\psi = (\phi, \beta): M \to V$ that is globally hyperbolic and solves the PDE system
$f = 0$, (b) for any $\psi$-Cauchy surface $\Sigma \subset M$, there exists an open neighborhood
of $N \supset \Sigma$ such that the initial data $|\xi|$ of the space of globally hyperbolic solu-
tions of the form $(\xi, \beta)$ on $N$ forms an open neighborhood of $|\phi|$ in the space of
initial data allowed by the constraints $c = 0$ (without taking further integrability
conditions into account).

6.2. Functoriality. In this section we (a) systematically summarize some of the
concepts and notations used throughout this paper, (b) system atically summa-
rized various constructions that appear as steps in the construc tion of a classical
field theory, and (c) remark on the categorical and functorial properties of these
objects and constructions. This section will serve as a reference for the later
discussion of how LCFT axioms, including causality, translate to field theories
with dynamical causal structure.

1. $\mathsf{Man}$, $\mathsf{Man}^n$, $\mathsf{Man}^\infty$: Category of smooth manifolds, subcategories of $n$-di-
dimensional and infinite dimensional manifolds. Subscript $e$ stands for restric-
tion of morphisms to open embeddings.

2. $\mathsf{Bndl}$: Category of smooth bundles, a subcategory of $\mathsf{Man}$. Fibered over $\mathsf{Man}$
with the base space functor $\Pi: (E \to M) \mapsto M$.

3. $\mathsf{VBndl}$: Category of vector bundles, a subcategory of $\mathsf{Bndl}$.

4. $\mathsf{CBndl}$, $\mathsf{ChrBndl}$, $\mathsf{SpBndl}$: Subcategories of $\mathsf{Bndl}$ of cone bundles, chronal
cone bundles, spacelike cone bundles. Subscripts $sc$ and $H$ indicate stable
chronality and global hyperbolicity and superscript $n$ denotes the restriction
to $n$-dimensional base manifolds. Morphisms in $\mathsf{ChrBndl}$ and $\mathsf{SpBndl}$ must
be chronally convex. As functors from $\mathsf{Man}$, the images of $M \mapsto \mathsf{ChrBndl}(M)$
and $M \mapsto \mathsf{SpBndl}(M)$ indicate the respective subcategories of cone bundles
over $M$.

5. $J^k: \mathsf{VBndl} \to \mathsf{VBndl}$: Functor of jet prolongation of a vector bundle.

6. $\Gamma, \Gamma_0: \mathsf{VBndl} \to \mathsf{Man}^\infty$: Contravariant functors of sections, sections with
compact support.

7. $\Gamma_{SC,+,-}: \mathsf{VBndl} \times_\Pi \mathsf{SpBndl}_H \to \mathsf{Man}^\infty$: Contravariant functors of sections
with spacelike compact, retarded (+), or advanced (−) supports. The objects of
the domain category can be identified with pairs $(F, C^\oplus)$, with $F \to M$
a vector bundle and $C^\oplus \to M$ is a spacelike cone bundle over the same $M$.
Throughout the paper we have also used the notation $\Gamma_{SC,+,-}(F, C)$, where
$C \to M$ is a chronal cone bundle. These notations can be used interchangeably

\[10\] We will re-use the symbol $\Pi$ to generically denote a forgetful functor.
because of the duality between chronal and spacelike cone bundles. However, these sections functors are covariant only if applied to spacelike cone bundles.

8. $B,F,V,E: \text{Man}_n^\sigma \to \text{VBndl}$: Natural bundle functors of background, dynamical, and total ($V = B \oplus F$) field bundles as well as the equation bundle.

9. $\mathcal{L}: J^\infty F \times J^\infty B \to \Lambda^n M$: Natural bundle section, defining a covariant Lagrangian density. It can also be interpreted as a differential form $\mathcal{L} \in \Omega^{n,0}(F \oplus B)$ in the variational bicomplex.

10. $f \oplus c: J^1 F \times J^\infty B \to F^* \oplus E$: Natural bundle section defining a covariant first order, quasilinear, symmetric hyperbolic PDE system, which is equivalent to gauge fixed Euler-Lagrange equations of $\mathcal{L}$.

11. $\mathfrak{Bkgr}$: Category of manifolds augmented by collections of admissible background field configurations. Objects are pairs $(M, B)$ with $M$ an $n$-manifold and $B \subseteq \Gamma(B(M))$. Each element $\beta \in B$ must satisfy all the necessary integrability conditions and be compatible with the existence of globally hyperbolic dynamical field solutions. A morphism $\chi: (M, B) \to (M', B')$ determines an open embedding $\chi: M \to M'$ compatible with background fields, $\chi^*(B') \subseteq B$ being a closed embedding (to play nice with $C^\infty$, as discussed in item 15 below). $\mathfrak{Bkgr}$ is fibered over $\text{Man}_n^\sigma$ with respect to the forgetful functor to $\Pi: (M, B) \mapsto M$.

12. $\mathfrak{SpBkgr} = (\mathfrak{SpBndl}_n^\sigma \times n \mathfrak{Bkgr})_c$: Category of admissible background field configurations, equipped with a globally hyperbolic spacelike cone bundle. As functor from $\mathfrak{Bkgr}$, the image of $M \mapsto \mathfrak{SpBkgr}(M, B)$ indicates the subcategory of different ways of equipping $(M, B)$ with a spacelike cone bundle. The objects of $\mathfrak{SpBkgr}$ will be denoted as $\mathcal{M} = (M, C^\sigma, B)$. The forgetful functor is again denoted $\Pi: \mathfrak{SpBkgr} \to \mathfrak{Bkgr}$, with $\Pi(M) = \Pi(M, C^\sigma, B) = (M, B)$. The subscript $c$ indicates that morphisms are restricted to those that are chronally compatible (Def. 24).

13. $\mathfrak{Symp}$: The generalized category of symplectic manifolds, a subcategory of $\text{Man}$. It includes symplectic manifolds as well as manifolds foliated by symplectic leaves (which is actually the same as the category of regular Poisson manifolds). The morphisms are leaf preserving symplectomorphisms. The forgetful functor to manifolds is $\Pi: \mathfrak{Symp} \to \text{Man}$.

14. $\mathfrak{CAlg}, \mathfrak{Poiss}$: Categories of commutative algebras and Poisson algebras, with the forgetful functor $\Pi: \mathfrak{Poiss} \to \mathfrak{CAlg}$. Though, keeping away from functional analytic details, we implicitly treat these categories as though the objects are equipped with topology and the homomorphisms are continuous. Such continuity requirements also identify a tensor product $\otimes$ (identified with the categorical coproduct), which is compatible with the categorical product in $\text{Man}$, as specified in Hyp. 2 below. The extension of the tensor product to $\mathfrak{Poiss}$ is the independent subsystems tensor product.

15. $C^\infty, C^\infty_{\text{ext}}, \mathfrak{POISS}$: Functors of smooth functions of ordinary manifolds, that restricted to compact spacetime support, and that equipped with Poisson bracket on symplectically foliated manifolds, which fit in the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{Symp} & \mathfrak{POISS} & \mathfrak{Poiss} \\
\downarrow_{\Pi} & \downarrow_{\Pi} & \\
\text{Man} & C^\infty_{\text{ext}} & \mathfrak{CAlg}.
\end{array}
\]
The definition of $C^\infty_{\text{cst}}(M)$ is unambiguous when $M$ is a finite dimensional manifold. On the other hand, its definition requires some functional analytical detail not tackled here. However, we have already postulated that, whatever ultimate definition is adopted for it, it will have to satisfy Hyp. 2. In addition to that, let us make explicit additional hypotheses, of a more category theoretic nature, that we presume it would satisfy.

**Hypothesis 3.** The functor $C^\infty_{\text{cst}}$ gives a sheaf of commutative algebras on a manifold $M$ when applied to the open subsets of $M$. Moreover, $C^\infty_{\text{cst}}$ maps surjections of manifolds to injective homomorphisms ("left exactness") and maps closed embeddings of manifolds to surjective homomorphisms ("right exactness" or smooth Urysohn lemma). Products of manifolds are taken to tensor products of algebras, $C^\infty_{\text{cst}}(M \times N) \cong C^\infty_{\text{cst}}(M) \otimes C^\infty_{\text{cst}}(N)$. "Transverse" pullbacks of manifolds are taken to pushouts of algebras.

**Remark 5.** The statement about transverse pullbacks is standard in differential geometry of finite dimensional manifolds. However, the notion of transversality is much more subtle for infinite dimensional manifolds. In fact, the notions of transversality discussed in these references may not be general enough to encompass the situation where it is needed in the formulation of the generalized Time Slice property in Sect. 7.2. At this point, we have no choice but to leave the existence of the appropriate notion of transversality as a conjecture.

16. $S: \text{SpBkgr} \to \text{Man}, \mathcal{P}: \text{SpBkgr}_c \to \text{Symp}$: The contravariant functor assigning $S(M)$, the set of all $C^\ast$-slow solutions of the gauge fixed Euler-Lagrange equations (equivalently of the corresponding constrained symmetric hyperbolic system) compatible with the background field configurations in $B$, to the object $M = (M, C^\ast, B)$ of $\text{SpBkgr}$. The contravariant functor $\mathcal{P}$ augments the images of $S$ with symplectic structure defined by the Lagrangian $L$. A fixed element $\beta \in B$ singles out a symplectic leaf in $S(M)$. It is the dependence of solutions on background fields that forces us to consider generalized symplectic manifolds. Note, however, that a symplectically foliated manifold must have at least dimension two. On the other hand, many of the $S(M)$ spaces will be empty sets, hence cannot be the underlying manifolds of images of $\mathcal{P}$. For that reason, the domain category of $\mathcal{P}$ is restricted to $\text{SpBkgr}_c$, where the subscript $c$ indicates the largest subcategory such that no object in the image $S(\text{SpBkgr}_c)$ is empty.

17. $\mathcal{P}(M, B): \text{SpBkgr}(M, B) \to \text{Man}, \mathcal{P}(M, B): \text{SpBkgr}_c(M, B) \to \text{Symp}$: The contravariant functor whose image in $\text{Man}$, resp. $\text{Symp}$, consists of the diagram $S(M) \to S(M')$, resp. $\mathcal{P}(M) \to \mathcal{P}(M')$, of all slow patches on the total solution space $S_H(M, B)$, resp. phase space $P_H(M, B)$, with the natural inclusions between them.

18. $S_H: \text{Bkgr} \to \text{Man}, P_H: \text{Bkgr}_c \to \text{Symp}$: The contravariant functor of all globally hyperbolic solutions that are compatible with specified background field configurations. The contravariant functor $P_H$ augments the images of $S_H$ with symplectic structure, as above, where we have defined $\text{Bkgr}_c = \Pi(\text{SpBkgr}_c)$. Each $S(M)$ is an open slow patch of the total solution space $S_H(M, B)$. The natural isomorphisms $S_H \cong \lim \circ \mathcal{P}$ and $P_H \cong \lim \circ \mathcal{P}$ are verified in Sect. 6.3. These functors fit into the following commutative dia-
gram:

\begin{align*}
\text{SpBkgr} & \hookrightarrow \text{SpBkgr} \\
\Pi & \quad \Pi \\
\text{Symp} & \quad \text{Man}.
\end{align*}

(252)

19. \(F = \text{POISS} \circ \mathcal{P}, A = C^\infty_{\text{st}} \circ S = \Pi \circ F\): Covariant functors of the Poisson algebra of observables on \(S(M)\) and its underlying commutative algebra, specified by a slow patch \(M = (M, C^\infty, B)\).

20. \(\mathcal{A}(M, B): \text{SpBkgr}(M, B) \rightarrow \mathcal{CAlg}, \mathcal{F}(M, B): \text{SpBkgr}_c(M, B) \rightarrow \text{Poiss}\): The covariant functor whose image in \(\mathcal{CAlg}\), resp. \(\text{Poiss}\), consists of the diagram \(\mathcal{A}(M) \rightarrow \mathcal{A}(M')\), resp. \(\mathcal{F}(M) \rightarrow \mathcal{F}(M')\), of all algebras of observables associated to slow patches in the total algebra \(\mathcal{A}_H(M, B)\), resp. Poisson algebra \(\mathcal{F}_H(M, B)\), with the natural projections between them.

21. \(\mathcal{F}_H = \text{POISS} \circ \mathcal{P}_H, \mathcal{A}_H = C^\infty_{\text{st}} \circ S_H = \Pi \circ \mathcal{F}_H\): The covariant functors of the Poisson algebra of observables and its underlying commutative algebra of the total phase space \(S_H(M, B)\). The natural isomorphisms \(\mathcal{F}_H = \lim \circ \mathcal{F}\) and \(\mathcal{A}_H = \lim \circ \mathcal{A}\) are verified in Sect. 6.3. These functors fit in the following commutative diagram:

\begin{align*}
\text{SpBkgr} & \hookrightarrow \text{SpBkgr} \\
\Pi & \quad \Pi \\
\text{Poiss} & \quad \mathcal{CAlg}.
\end{align*}

(253)

22. \(\text{Bkgr}, \text{Bkgr}^*, \text{Bkgr}'\): Subcategories of \(\text{Bkgr}\). As defined in a preceding item, \(\text{Bkgr} = \Pi(\text{SpBkgr})\). We define \(\text{Bkgr}^*\) as the subcategory where the collections of admissible background configurations are singletons, \(\mathcal{B} = \{\beta\}\), and \(\text{Bkgr}'\) is the image of the category \(\text{Bkgr}^*\), where the auxiliary components of the background fields have been thrown away.

With all the categorical notions summarized above, it should now be clear how to recover the standard formulation of the LCFT axioms given in Sect. 2.

Consider a semilinear wave equation on a curved background with Lagrangian density

\[ L = \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \sqrt{-g} \, \text{d} \tilde{x}. \]  

(254)

The trivial dynamical field bundle \(F: M \mapsto \mathbb{R} \times M\) is clearly natural. However, the Lagrangian density \(L\) only becomes natural if we include the metric tensor among the background fields, that is we set \(B: M \mapsto S^2 T^* M\), which is also
clearly natural. The Euler-Lagrange equations automatically give a natural hyperbolic PDE system. (This system is \textit{normally hyperbolic }\cite{7,24}. For simplicity we do not put this system into symmetric hyperbolic form, which would simply require extending the dynamical and background field bundles \cite{50}.)

It is simple to check that restricting to the subcategory $\mathcal{Bkgr}^*_c$ of $\mathcal{Bkgr}_c$, where objects $(M, B)$ are equipped with a single background field configuration $B = \{g\}$, forces the allowed morphisms to be only causal isometries (cf. Def. 2, Def. 24, Lem. 2 and items 11, 12 above). This means that we have an equivalence of categories $\mathcal{Bkgr}^*_c \cong \text{GlobHyp}_c$. By construction then the covariant functor

$$F_H: \mathcal{Bkgr}^*_c \cong \text{GlobHyp}_c \to \text{Pois}$$

has the potential to be a classical LCFT. It remains only to check the axioms of Def. \cite{6}. These checks are carried out at the end of Sect. \cite{7,3}.

6.3. \textit{Limits, colimits and the global phase space.} We have seen earlier, more specifically in Sect. \cite{5.3.6}, that the slow patches $S(M)$ constitute an open cover of the global solution space $S_H(M, B)$. How can this relation be restated for the algebras $A_H(M, B)$ and $A(M, B)$, or the Poisson algebras $F_H(M, B)$ and $F(M)$? The answer is not immediately obvious. However, the situation becomes more clear when one realizes that the desired constructions can be formulated in terms of the categorical notions of \textit{limit} and \textit{colimit}. Categorical limits (also projective or inverse limits) and colimits (also inductive or direct limits) are briefly introduced in Sect. \cite{B}.

A great advantage of considering the Poisson algebras $F(M)$ is the simplified microcausality property, Cor. \cite{5}, while only the more refined microcausality property of Cor. \cite{4} survives for $F_H(M, B)$. On the other hand, the $F(M)$ for different cone bundles $C \to M$ are included as subalgebras of $F_H(M, B)$, though not as independent ones. The situation is strongly parallel to the fact that the slow patches $S(M)$ are open subsets of $S_H(M, B)$, though with non-trivial overlaps. The parallel is made precise by recognizing that the globally hyperbolic slow patches $S(M)$, as well as their symplectic analogs $\mathcal{P}(M)$, and the inclusions between them constitute the diagram $\mathcal{S}(M, B)$ in $\text{Man}$, and respectively $\mathcal{P}(M, B)$. Similarly, the algebras $A(M)$ and $F(M)$ and the projections between them constitute diagrams $\mathcal{A}(M, B)$ and $\mathcal{F}(M, B)$ in $\mathcal{CAlg}$ and $\text{Pois}$, respectively. The open cover of $S_H(M, B)$ by slow patches and the agreement of symplectic structure on their overlaps, Sect. \cite{5.3.6} ensures the following limit and colimit identities hold

$$S_H(M, B) \cong \lim_{\to} \mathcal{S}(M, B) \quad \text{and} \quad \mathcal{P}_H(M, B) \cong \lim_{\leftarrow} \mathcal{P}(M, B).$$

(256)

Similarly, in the algebraic categories, the sheaf property of $C^\infty_{\text{cst}}$ ensures that the following colimit identities hold

$$A_H(M, B) \cong \lim_{\to} \mathcal{A}(M, B) \quad \text{and} \quad F_H(M, B) \cong \lim_{\leftarrow} \mathcal{F}(M, B).$$

(257)

The same category $\text{SpBkgr}(M, B)$ contravariantly indexes the geometric colimit and covariantly indexes the algebraic limit in the left column, while the same category $\text{SpBkgr}_c(M, B)$ contravariantly indexes the geometric colimit and covariantly indexes the algebraic limit in the right column. Recall that objects
\( \mathcal{M} = (M, C^\circ, B) \) of these indexing categories specify a spacelike cone bundle \( C^\circ \to M \). If a chronal cone bundle were to be specified instead, the above limits and colimits would have been swapped, since \( \text{SpBndl}(M) \) is the opposite category of \( \text{ChrBndl}(M) \). However, it is more convenient to use spacelike cone bundles, in light of the considerations below.

**Remark 6.** Colimits appear in the framework of LCFT already at the level of spacetimes. Namely, consider a globally hyperbolic Lorentzian manifold \((M, \{g\})\) in the category \( \text{GlobHyp}_c \). Let \( \mathcal{M} \) be the diagram whose image in \( \text{GlobHyp}_c \) consists of all globally hyperbolic Lorentzian submanifolds \((M_i, \{g_i\})\) of \((M, \{g\})\), with inclusions as morphisms. The colimit \( \lim_{\rightarrow} \mathcal{M} \) is isomorphic to \((M, \{g\})\) itself. In fact one recovers \((M, \{g\})\) as the colimit even if one restricts the objects of \( \mathcal{M} \) to be simple in particular ways, like being topologically trivial, having bounded radius, being the interior of a causal diamond, etc. Each kind of simplicity translates to a correspondingly simple property of the algebra of observables \( \mathcal{F}(M_i, \{g_i\}) \) associated by a LCFT. Thus, just as the non-trivial topological or geometric structure of \((M, \{g\})\) is encoded in the diagram \( \mathcal{M} \) in \( \text{GlobHyp}_c \), the non-trivial algebraic structure of \( \mathcal{F}(M, \{g\}) \cong \lim_{\rightarrow} \mathcal{F} \circ \mathcal{M} \) is encoded in the diagram \( \mathcal{F} \circ \mathcal{M} \) in \( \text{Poiss}_c \). (We use the colimit in both categories because \( \mathcal{F} \) is a covariant functor.) This approach underlies the investigation of superselection sectors of LCQFT [56, Ch.IV], [26] and the quantization of vector field theories on manifolds of non-trivial topology [64, Apx.A], [36].

It is sometimes convenient to simultaneously display both the limit and colimit identities:

\[
\begin{align*}
S_H(M, B) &= \lim \left( \lim_{\rightarrow} \mathcal{F} \circ \mathcal{M} \right), & P_H(M, B) &= \lim \left( \lim_{\rightarrow} \mathcal{P} \circ \mathcal{M} \right), \\
A_H(M, B) &= \lim \left( \lim_{\rightarrow} \mathcal{A} \circ \mathcal{M} \right), & F_H(M, B) &= \lim \left( \lim_{\rightarrow} \mathcal{F} \circ \mathcal{M} \right),
\end{align*}
\]

where we define \( \mathcal{M} \) (\( \mathcal{M}_c \)) is the diagram in \( \text{Bkgr} \) (\( \text{Bkgr}_c \)) that consists of all subobjects \((M', B') \to (M, B)\). Similar ideas about constructing the global phase space and algebra of observables, using an open cover of the spacetime and a cover of the space of solutions also recently appeared in [24]. Though, in distinction, the covers on \( S_H(M, B) \) are considered there more general than ours. The cover elements are not required to be open (only \( G_\delta \)) and could be much more refined than our slow patches.

### 7. Causality Axiom in Locally Covariant Classical Field Theory

The axioms in the standard definition of LCFT, Def. 5, make reference to a fixed causal structure induced by the future timelike cones of a background Lorentzian metric. It is clear that they cannot be translated to direct conditions on the functor \( \mathcal{F}_H \) (or \( \mathcal{P}_H \)), which can be constructed for theories without a fixed background causal structure. On the other hand, these axioms can be straightforwardly translated to conditions on the functor \( \mathcal{F} \) (or \( \mathcal{P} \)), since objects in its domain category \( \text{SpBkgr} \) all have an externally fixed causal structure given by a spacelike cone bundle. Below, we prove theorems about the properties of the functor \( \mathcal{F} \), which lead naturally to generalizations of the LCFT axioms. The functor \( \mathcal{F}_H = \lim_{\rightarrow} \mathcal{F} \) inherits these properties through the categorical limit construction.
7.1. Isotony. Given a morphism $\chi : \mathcal{M} \to \mathcal{M}'$ in $\text{SpBkgr}$, is the corresponding morphism $\chi^* : S(\mathcal{M}) \to S(\mathcal{M}')$ surjective? The answer is not always positive; it depends on the properties of the field theory in question. In terms of solutions, $\chi^*$ is surjective precisely when every $C^k$-slow solution on $\mathcal{M}$ is the pullback along $\chi$ of a $C^{k'}$-slow solution on $\mathcal{M}'$ (equivalently, can be extended with respect to $\chi$ to a $C^{k'}$-slow solution on $M'$, where there is no presumption of uniqueness of the extension). Broadly speaking, the extension of a particular solution on $\mathcal{M}$ fails if the would be extension develops singularities in $\mathcal{M}'$.

There are different kinds of singularities, including topological, smooth, geometric and analytical. (T) For instance, depending on the structure of the equations of motion, there may be a topological obstruction in extending a solution from $\mathcal{M}$ to $\mathcal{M}'$, if $\mathcal{M}$ is not contractible [30]. (S) Also, it is possible that a solution becomes unbounded or non-smooth near the boundary of $\chi(\mathcal{M})$ in $\mathcal{M}'$ and hence cannot be extended continuously to $\mathcal{M}'$. (A) The solution may even be completely regular on $\chi(\mathcal{M})$ and its boundary, yet be forced by the dynamics of the equations of motion to blow up when extended to $\mathcal{M}'$. The formation of shocks in fluid dynamics and of black hole singularities in GR are prime examples of this phenomenon. (C) On the other hand, it is possible for a solution to extend from $\chi(\mathcal{M})$ to $\mathcal{M}'$, yet the fixed cone bundle $C^k_\ast$ is not fast enough to be compatible with the extension or simply that the image $\chi(\mathcal{M})$ is not chronally convex in $\mathcal{M}'$ (Def. [21]).

It is of course desirable to formulate a set of necessary and sufficient conditions for extensibility to hold. Unfortunately, that is in general a very difficult problem. In particular, the study of blow ups of type (A) constitute a major active branch of modern PDE theory. Moreover, the existence singularities of type (T) and (A) is determined by the structure of the PDEs, which is fixed by the relevant physics. As such, we may just have to learn to live with them. Inextensibility of type (S) depends on whether we allow asymptotically irregular solutions in $S(\mathcal{M})$. This choice is related to the issues of the choices of the structure of the domains $\mathcal{M}$ and topologies on the space of solutions, which were discussed in Sect. [4] in relation to the openness of slow patches. It is also possible that the failure of surjectivity of $\chi^*$ due to blow up of type (S) is precisely masked by the application of the $C^\infty_{\text{nst}}$ functor that constructs the algebra of observables on the given slow patch of $S_H(\mathcal{M}, B)$ [17, 45]. As in this preceding discussion, we do not address it here directly. Finally, blow up of type (C) is entirely within our control. Since the problematic causal cone bundle $C^\infty_\ast$ plays only an auxiliary role in the ultimate construction of $S_H(\mathcal{M}', B')$, it is no loss to introduce a faster cone bundle, say $C^k_{\ast 1}$ that might be fast enough to be compatible with an extension of every solution in $S(\mathcal{M})$.

Unfortunately, as can be seen from the above discussion we cannot prove a general theorem about the surjectivity of the morphism $\chi^*$. Therefore, we introduce this property as an additional hypothesis.

Definition 47. A morphism $\chi$ in $\text{SpBkgr}$, is said to be extensible if this morphism factors through

\[
\begin{align*}
&\chi^* : S(\mathcal{M}) \to S(\mathcal{M}') \\
&\chi : \mathcal{M} \to \mathcal{M}'
\end{align*}
\]
such that $M' = (M', C', B')$, $M'_1 = (M', C'_1, B')$, and the morphism $\chi^*_1 = S(\chi_1)$ is surjective.

Provided this hypothesis holds, there are no other obstacles in proving the following generalization of the Isotony property for the functor $F$.

**Theorem 8 (Generalized Isotony).** Consider a morphism $\chi$ in $\SpBkgr$. If $\chi$ is extensible, the morphism $A(\chi_1)$ is an injective homomorphism. Moreover, if $S(\chi)$ is not empty, $\chi$ and $\chi_1$ are also morphisms in $\SpBkgr$, the morphisms $S(\chi)$ and $S(\chi_1)$ lift to $\mathcal{P}(\chi)$ and $\mathcal{P}(\chi_1)$ in $\Symp$, and $A(\chi)$ and $A(\chi_1)$ lift to morphisms in $\Poiss$.

**Proof.** Since $\chi$ is extensible, the induced morphism $S(M'_1) \rightarrow S(M)$ is surjective. By Hyp. 3, $C'_{\infty}$ takes this surjection to the injective homomorphism $A(M) \rightarrow A(M'_1)$. If the corresponding solution spaces are not empty, the covariance lemma, Lem. 22, implies that the appropriate morphisms lift to $\Symp$ and $\Poiss$. $\square$.

### 7.2. Time Slice.

Given a morphism $\chi: M \rightarrow M'$ in $\SpBkgr$, is the corresponding morphism $S(M') \rightarrow S(M)$ injective? If the answer were positive, it would mean that every $C'^\ Rightsarrow$-slow solution on $M'$ is uniquely determined by its restriction to $\chi(M) \subseteq M'$. Clearly, there is no hope that this is the case unless $\chi(M)$ contains a $C'^\ Rightsarrow$-Cauchy surface in $M'$.

**Definition 48.** A morphism $\chi$ in $\SpBkgr$ is called Cauchy surjective or a Cauchy surjection if for the corresponding morphism in $\SpBndl_H$,

\[
\begin{array}{c}
C'^\ Rightsarrow \\
\downarrow \\
M \\
\downarrow \\
\chi \\
\downarrow \\
M' \\
\end{array}
\]

there exists a $C'^\ Rightsarrow$-Cauchy surface $\Sigma \subset M$ whose image $\Sigma' = \chi(\Sigma) \subset M'$ is a $C'^\ Rightsarrow$-Cauchy surface. (Recall that all cone bundles in $\SpBndl^*_H$ are globally hyperbolic.)

Unfortunately, injectivity of $S(\chi)$ does not guarantee surjectivity of $A(\chi)$, which would imply that every smooth function on the image of $S(M')$ extends to a smooth function on $S(M)$. However, this property is known to fail unless $S(\chi)$ is a closed embedding. If $S(\chi)$ is also surjective (\chi is extensible), this is automatic. However, if $S(\chi)$ fails to be injective, then the complement of its image (the solutions that blow up when extended from $\chi(M)$ to $M'$) is rarely an open set.

Fortunately, a slightly more complicated diagram in $\SpBkgr$ does guarantee a closed embedding.

---

11 A simple illustration is the inclusion $(0, 1) \subset \mathbb{R}$ and any function that is unbounded near an end of the open interval.
Definition 49. Consider morphisms $\chi_1$ and $\chi_2$ in $\text{SpBkgr}$ that fit into a pullback-pushout diagram in $\text{SpBkgr}$,

\[
\begin{array}{c}
\chi_1 & \xrightarrow{\sigma} & \chi_2 \\
\downarrow & & \downarrow \\
\chi_3 & \xrightarrow{\tau} & M' \\
\end{array}
\]

That is, the images $\chi_i(M_i)$ form an open cover of $M'$ and $M_3$ is isomorphic to their intersection. In terms of the spacelike cones, we have $C'^{\circ} = C_1^{\circ} + C_2^{\circ}$ and $C_3^{\circ} = C_1^{\circ} \cap C_2^{\circ}$ (cf. Sect. [7]), where for brevity we have omitted the pullbacks or pushforwards with respect to the appropriate morphisms, while $B_3 = B_1 \cup B_2$ and $B' = B_1 \cap B_2$ on $M_3$, with the same shorthand. We call the diagram (262) a Cauchy pushout for $M'$ if all morphisms are Cauchy surjections.

Given a Cauchy pushout for $M'$, the maps $S(M') \to S(M_i)$ are injective by Cauchy surjectivity. However, they are not guaranteed to be closed embeddings. On the other hand, consider the following diagram of spaces of field configurations in $\text{MFn}$, where $V_i = V(M_i)$ and $V' = V(M')$ are total field bundles naturally associated to their base manifolds,

\[
\begin{array}{c}
\Gamma(V_3) & \xrightarrow{\chi_2} & \Gamma(V_2) \\
\downarrow & & \downarrow \\
\Gamma(V_1) & \xrightarrow{\chi_3} & \Gamma(V') \\
\end{array}
\]

which is most definitely a pullback diagram, with the canonical dotted line morphism a closed embedding. The set theoretic pullback property is assured because any pair of smooth sections on $M_1$ and $M_2$ that agree on $M_3$ must glue together to a smooth section on $M_3$. Now, consider a point $(\psi_1, \psi_2)$ in $\Gamma(V_1) \times \Gamma(V_2)$ that is not in the image of $\Gamma(V')$, that is, it determines two sections, $\psi_i$ on $M_i$ and $\psi_2$ on $M_2$, that do not agree on $M_3$. It is then simple to construct neighborhoods (in either the compact open or Whitney fine topologies) of the graphs of these sections in $V_3$ that do not intersect over at least one point in $M_3$, which define neighborhoods of $\psi_1$ and $\psi_2$ and hence a neighborhood of $(\psi_1, \psi_2)$ that does not intersect the image of $\Gamma(V')$. Therefore, the complement of the image of $\Gamma(V')$ is open. In other words, the image of $\Gamma(V')$ in the product space $\Gamma(V_1) \times \Gamma(V_2)$ is closed.

The above discussion will be sufficient to establish a closed embedding of the solution space $S(M')$ in natural ambient space determined by the Cauchy pushout. From that, we can construct a surjective homomorphism onto the algebra $A(M')$ from a natural algebra also determined by the Cauchy pushout. Unfortunately, that is not quite satisfactory, as it is irresistible to try to formulate the generalized Time Slice property as a stronger result, namely that

\[\text{This is sometimes known as the sheaf descent property of smooth sections.}\]
the Cauchy pushout (in the spacetime category) is taken to a pullback (in the phase space category) and again a pushout (in the algebraic category) by the appropriate successive contravariant functors P and F (or S and A). However, to achieve that using an appeal to Hyp. 3 requires an additional transversality hypothesis on the interaction of the solution spaces S(M_i) and S(M'). It is not clear whether this notion has already appeared in the standard PDE literature.

**Definition 50.** A Cauchy pushout as in Eq. (262) is said to satisfy transverse descent if the smooth maps α_i: S(M_1) × S(M_2) → S(M_i) → S(M_3), i = 1, 2, are transverse to each other.

**Remark 7.** Note that the images of the maps χ^*_i: S(M_i) → S(M_3) intersect precisely on a subspace of S(M_3) that can be identified with S(M'). This property follows from the uniqueness of the Cauchy problem and the sheaf descent property inherited from the sheaf of smooth functions. As in the remark following Hyp. 3 we do not precisely specify the desired notion of transversality. However, we conjecture that such a notion can be formulated so that the maps α_i agree precisely on the subset of S(M_1) × S(M_2) identified with S(M') and, moreover, that a corresponding implicit function theorem establishes that S(M') is in fact a submanifold. The reason the standard notions of infinite dimensional transversality [30,77,59] may not be applicable here is that (precisely due to the possible blow up of solutions) the images of the tangent maps Tα_i may not be closed subspaces of the tangent space TS(M_3). This phenomenon appears already for linear PDE systems, such as, for example, the Cauchy-Riemann equations. Though, admittedly, that is not an example of a hyperbolic PDE system.

**Lemma 23.** Consider a Cauchy pushout diagram satisfying transverse descent, Defs. 49 and 50. The following diagram contains a pullback square for S(M') and the canonical dotted line morphism is a closed embedding of manifolds:

\[
\begin{array}{ccc}
S(M_3) & \xleftarrow{S(\chi_2)} & S(M_2) \\
\downarrow & & \downarrow \\
S(M_1) & \xleftarrow{S(\chi_1)} & S(M') \\
\end{array}
\]

If none of S(M_i) or S(M') are empty, then the above pullback square lifts to the following pullback square in \(\text{Symp}:\)

\[
\begin{array}{ccc}
P(M_3) & \xleftarrow{P(\chi_2)} & P(M_2) \\
\downarrow & & \downarrow \\
P(M_1) & \xleftarrow{P(\chi_1)} & P(M') \\
\end{array}
\]

**Proof.** The set theoretic pullback property holds again from the gluing property of solution sections, with which the background fields do not interfere. Now, recall that all section spaces are topologized as subspaces of \(\Gamma(-)\) with
appropriate vector bundle argument. In particular this means that the subset $S(M_1) \times S(M_2) \cap \Gamma(V') \subseteq \Gamma(V_1) \times \Gamma(V_2)$ is closed in $S(M_1) \times S(M_2)$. On the other hand, the gluing property ensures that $S(M') \cong S(M_1) \times S(M_2) \cap \Gamma(V')$. (Again, the obvious pushforwards and pullbacks have been omitted for brevity.)

This shows that the canonical dotted line morphism is a closed embedding of topological spaces. This much we can establish without appeal to the transverse descent property, on the other hand we must appeal to it to establish that it is a manifold embedding, and hence a closed embedding. The covariance lemma, Lem. 22, guarantees that the morphisms lift to $\text{Symp}$ whenever the corresponding solution spaces are not empty. \(\square\)

With the above discussion in mind, we can formulate the following generalization of the Time Slice property for the functor $F$.

**Theorem 9 (Generalized Time Slice).** Consider a Cauchy pushout diagram that satisfies transverse descent, Defs. 49 and 50. Then the following is a pushout diagram in $\text{CAlg}$:

$$
\begin{array}{ccc}
A(M_3) & \rightarrow & A(M_2) \\
\downarrow & & \downarrow A(\chi_2) \\
A(M_1) & \rightarrow & A(M')
\end{array}
$$

Moreover, if none of the $S(M_i)$ or $S(M')$ are empty, the above diagram also lifts to a pushout diagram in $\text{Poiss}$:

$$
\begin{array}{ccc}
F(M_3) & \rightarrow & F(M_2) \\
\downarrow & & \downarrow F(\chi_2) \\
F(M_1) & \rightarrow & F(M')
\end{array}
$$

**Proof.** According to Lem. 23, we can realize $S(M')$ as a closed submanifold of $S(M_1) \times S(M_2)$, with the aid of the transverse pullback diagram (264). On the other hand, Hyp. 6 immediately implies that this transverse manifold pullback is taken by $\mathbb{C}^\infty_{\text{cst}}$ to the algebraic pushout diagram (266). Finally, whenever the diagram (264) in $\text{Man}$ lifts to diagram (265) in $\text{Symp}$, then the diagram (266) in $\text{CAlg}$ automatically lifts to diagram (267) in $\text{Poiss}$. \(\square\)

### 7.3. Causality.

Given two morphisms $\chi_i : M_i \rightarrow M$, $i = 1, 2$, in $\text{SpBkgr}$, do the corresponding morphisms surjectively factor through the product $S(M_1) \times S(M_2)$? Do the corresponding morphisms factor injectively through the tensor product $A(M_1) \otimes A(M_2)$? Finally, when do the corresponding morphisms factor through the *independent subsystems* tensor product $F(M_1) \otimes F(M_2)$?

There are several obstacles to positive answers to the above questions. First, let us consider surjectivity for the spaces of solutions and, correspondingly, injectivity for the algebras. If we allow disconnected manifolds as base manifolds in $\text{Bkgr}$ and $\text{SpBkgr}$, while making sure that morphisms in $\text{SpBkgr}$ treat disconnected components as spacelike separated, we can introduce a tensor product,
referred to as the *disjoint union*:

\[
\mathcal{M}_1 \sqcup \mathcal{M}_2 = (\mathcal{M}_1 \sqcup \mathcal{M}_2, C_1^\oplus \sqcup C_2^\oplus, B_1 \times B_2).
\]  

(268)

Note that this is not a coproduct in the categorical sense, since dotted line morphisms in the diagram

\[
\begin{array}{ccc}
\mathcal{M}_1 & \xrightarrow{\chi_1} & \mathcal{M} & \xleftarrow{\chi_2} & \mathcal{M}_2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_1 \sqcup \mathcal{M}_2
\end{array}
\]  

(269)

exists only if the images \(\chi_1(M_1)\) and \(\chi_2(M_2)\) are \(C^\ast\)-spacelike separated in \(M\). However, when this morphism exists, it is canonical and is denoted by \(\chi_1 \sqcup \chi_2\).

On the other hand, \(\text{Man}\) does have a categorical product, which means that the following diagram always exists:

\[
\begin{array}{ccc}
\mathcal{M}_1 & \xrightarrow{\chi_1} & \mathcal{M}_1 \sqcup \mathcal{M}_2 & \xleftarrow{\chi_2} & \mathcal{M}_2 \\
\downarrow & & \downarrow & & \downarrow \\
S(\mathcal{M}_1) & \times & S(\mathcal{M}_2)
\end{array}
\]  

(270)

Since there is no obstacle to specifying a solution independently in each connected component, the canonical dotted line morphism is in fact an isomorphism, \(S(\mathcal{M}_1 \sqcup \mathcal{M}_2) \cong S(\mathcal{M}_1) \times S(\mathcal{M}_2)\). In the \(\text{Symp}\) category, we can also define the *independent subsystems product* given by

\[
(N_1, \Omega_1) \times (N_2, \Omega_2) = (N_1 \times N_2, \Omega_1 \oplus \Omega_2).
\]  

(271)

When \(N_i = S(\mathcal{M}_i)\), the corresponding Poisson bivector \(\Pi\) on the product is characterized by \(\Pi(x, y) = \Pi_i(x, y)\) when \(x, y \in \mathcal{M}_i\) and \(\Pi(x, y) = 0\) when \(x\) and \(y\) belong to different connected components. Note that this is not a categorical product in \(\text{Symp}\), for reasons very similar to why \(\sqcup\) is not a coproduct in \(\text{SpBkgr}\). However, given the above information, we do have the identity \(\mathcal{P}(\mathcal{M}_1 \sqcup \mathcal{M}_2) = \mathcal{P}(\mathcal{M}_1) \times \mathcal{P}(\mathcal{M}_2)\).

Now, given the \(\text{SpBkgr}\) diagram

\[
\begin{array}{ccc}
\mathcal{M}_1 & \xrightarrow{\chi_1} & \mathcal{M} & \xleftarrow{\chi_2} & \mathcal{M}_2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_1 \sqcup \mathcal{M}_2
\end{array}
\]  

(272)

we always obtain the diagram in \(\text{Man}\) below

\[
\begin{array}{ccc}
S(\mathcal{M}_1) & \xrightarrow{S(\chi_1)} & S(\mathcal{M}) & \xleftarrow{S(\chi_2)} & S(\mathcal{M}_2) \\
\downarrow & & \downarrow & & \downarrow \\
S(\mathcal{M}_1) \times S(\mathcal{M}_2)
\end{array}
\]  

(273)
where dotted lines denote the canonical morphisms. Note that the canonical morphism in $\text{Man}$ always exists, even if $\chi_1 \sqcup \chi_2$ does not exist in $\text{SpBkgr}$. The surjectivity of the canonical dotted line morphism in the last diagram is equivalent the following: given a pair of solution sections, $\psi_1$ on $M_1$ and $\psi_2$ on $M_2$, there exists a solution section $\psi$ on $M$ that restricts to $\psi_1$ on $M_1$ and $\psi_2$ on $M_2$. Already in the earlier discussion of the Isotony property, we have noticed this surjectivity property fails when at least one of the $\chi_i$ morphisms fails to be extensible. However, canonical dotted line morphism may fail to be surjective even if both $\chi_1$ and $\chi_2$ are extensible. Namely, there may exist a pair $(\psi_1, \psi_2) \in S(M_1) \times S(M_2)$ such that both $\psi_1$ and $\psi_2$ may be extended to $M$ individually, but not jointly.

The most common reason for that to happen is that the images $\chi_1(M_1)$ and $\chi_2(M_2)$ are $C^s$-causally related in $M$, so that, for example, the value of $\psi_1$ in $M_1$ is not consistent with the causal influence of $\psi_2$ on $M_2$. Thus, to even have a hope of canonical factorization, we should require the images $\chi_1 \sqcup \chi_2$ does exist in $\text{SpBkgr}$, that is the images $\chi_1(M_1)$ and $\chi_2(M_2)$ are $C^s$-spacelike separated in $M$. However, even with spacelike separation, surjectivity can still fail. Intuitively, this happens when the solution data specified by $\psi_1$ and $\psi_2$ always produces a singularity while scattering in a joint extension to $M$, so that the canonical morphism $\chi_1 \sqcup \chi_2$ is itself not extensible.

**Definition 51.** We say that two extensible morphisms $\chi_1$ and $\chi_2$ in $\text{SpBkgr}$ have regular scattering if they fit in the commutative diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\chi_1} & M_2 \\
\downarrow & & \downarrow \\
M_1 \sqcup M_2 & \xrightarrow{\chi_1 \sqcup \chi_2} & M_2
\end{array}
\]

such that the canonical dotted line morphism $\chi_1 \sqcup \chi_2$ exists and is extensible.

Once the surjectivity in $\text{Man}$ and $\text{Symp}$ is taken care of, the injectivity in the algebraic categories follow in the manner indicated in the earlier discussion of the Isotony property. With the above discussion in mind we can formulate the following generalization of the Causality property for the functor $\mathcal{F}$.

**Theorem 10 (Generalized Causality).** Consider two morphisms $\chi_i : M_i \to M$, $i = 1, 2$, in $\text{SpBkgr}$ such that $\chi_1 \sqcup \chi_2$ exists with regular scattering. Then there exists an object $M' = (M, C^s, \mathcal{B})$ in $\text{SpBkgr}$ such that $\chi_i$ and $\chi_1 \sqcup \chi_2$ factor according to the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\chi_1} & M' \\
\downarrow & & \downarrow \\
M_1 \sqcup M_2 & \xrightarrow{\chi_1 \sqcup \chi_2} & M_2
\end{array}
\]

\[\text{(275)}\]
the canonical dotted line morphism in the corresponding diagram in $\text{Man}$ is surjective,

\[
\begin{array}{ccc}
S(M) & \xrightarrow{\cup} & S(M') \\
\downarrow^S(\chi_1) & & \downarrow^S(\chi_2) \\
S(M_1) & \xleftarrow{\cup} & S(M_2) \\
\end{array}
\quad (276)
\]

and the canonical dotted line morphism in the corresponding diagram in $\text{CAlg}$ is injective,

\[
\begin{array}{ccc}
A(M) & \xrightarrow{\cup} & A(M') \\
\downarrow^A(\chi_1) & & \downarrow^A(\chi_2) \\
A(M_1) & \xleftarrow{\cup} & A(M_2) \\
\end{array}
\quad (277)
\]

Moreover, if none of the objects in diagram \(276\) are empty, the diagram \(276\) lifts to $\text{SpBkgr}$, the diagram \(276\) lifts to $\text{Symp}$ via $\mathcal{P}$, and the diagram \(277\) lifts to $\text{Poiss}$ via $\mathcal{F}$.

**Proof.** Regular scattering hypothesis implies that both $\chi_i$, $i = 1, 2$, morphisms are extensible, using the same auxiliary object $M'$, where the images of $\chi_i'$ are also spacelike separated. The existence of the canonical dotted line morphism then follows from the definition of $\cup$. Applying the $\mathcal{S}$ functor to diagram \(275\) produces diagram \(276\), where the surjectivity of the dotted line morphism, again, follows from the regular scattering hypothesis. Finally, applying the $C^\infty$ functor to diagram \(276\) gives diagram \(277\), where the canonical dotted line morphism is injective because, by Hyp. 3, $C^\infty$ takes surjections to injective homomorphisms.

If the diagram \(275\) can be lifted to $\text{SpBkgr}$, then it follows from definitions and the covariance of the Poisson structures, Lem. 22, that the diagrams \(276\) and \(277\) lift to $\text{Symp}$ and $\text{Poiss}$, respectively. \(\Box\)

### 7.4. Connection with standard LCFT

The above results can be applied to a semilinear, well-posed PDE system to construct a LCFT functor satisfying the standard axioms given in Def. 5. Consider a natural, variational, semilinear PDE system, whose background fields include a globally hyperbolic, time oriented Lorentzian metric $g$ and such that its cone bundle of future directed timelike vectors $\Gamma_M \rightarrow V(M)$ coincides with the bundle of future directed timelike cones of $g$. Let $S: \text{SpBkgr} \rightarrow \text{Man}$ and $\mathcal{P}: \text{SpBkgr} \rightarrow \text{Symp}$ be its solutions and phase space functors. We will allow other background fields but only such that they influence neither the solution space nor the symplectic structure on it. The reason is that to achieve a natural symmetric hyperbolic form extra background
fields may need to be introduced, as has already been discussed and can be seen explicitly from the examples in [50]. However, since that is their only purpose, they do not influence the dynamics. We call such background fields auxiliary. For the purposes of comparing with a standard LCFT, we can safely ignore them. Recall that we introduced the category $\mathcal{SpBkgr}^*$, with objects that are endowed with background field collections consisting of a single section. Using this category to construct a standard LCFT is slightly problematic, because of the presence of auxiliary background fields. An easy solution is to introduce the forgetful functors $\mathcal{SpBkgr}^* \to \mathcal{SpBkgr}'$ and $\mathcal{Bkgr}^* \to \mathcal{Bkgr}'$ that throw away the auxiliary background fields, where $\mathcal{SpBkgr}'$ and $\mathcal{Bkgr}'$ now has only the metric as background field. The subscript $c$ retains the same meaning as before. By stipulation, the functors $\mathcal{S}$ and $\mathcal{P}$ project to functors $\mathcal{S}: \mathcal{SpBkgr}' \to \text{Man}$ and $\mathcal{P}: \mathcal{SpBkgr}' \to \text{Symp}$, and similarly with $\mathcal{S}_H$ and $\mathcal{P}_H$, up to isomorphisms in the target categories. Let a natural PDE system that satisfies all the conditions in the preceding discussion be called a semilinear geometric wave equation.

We conclude this section by showing that the tools assembled so far in this paper allow us to construct a standard classical LCFT from a well-posed semilinear geometric wave equation. Unfortunately, a sufficiently detailed notion of well-posedness that would allow us to directly construct a standard LCFT functor based on the results of the preceding section is somewhat difficult to formulate precisely. One obstacle is the fact the properties of the $\mathcal{C}^\infty_{\text{cst}}$ functor that constructs the algebra of observables as the algebra of smooth functions on an infinite dimensional phase space have so far been mostly hypothesized (cf. Hyp. 2 and Hyp. 3) rather than constructively proven. Another is related to the boundary conditions that one is expected to impose on solutions in spacetimes with non-compact Cauchy surfaces (cf. Sect. 5.2.1) and their interplay with the standard notions of well-posedness in the PDE literature. As such, we make no special attempt to be precise and merely formulate the following hypothesis, whose application is made clear in the proof of the theorem below.

**Hypothesis 4 (Well-posedness).** We consider our classical field theory defined by a natural, variational, semilinear PDE system to be globally well-posed in the following sense. All regular initial Cauchy data (where regularity includes any necessary boundary conditions) extend to the entire spacetime manifold (no finite time blow up). Moreover, in the presence of constraints and gauge symmetry, the PDE system can be augmented with purely hyperbolic gauge fixing (Sects. 5.3.1 and 5.3.2) such that global parametrizability (Sect. 5.2.2) and global recognizability (Sect. 5.2.3) are satisfied. Finally, the $\mathcal{C}^\infty_{\text{cst}}$ functor constructs the algebras of observables in a way that is not sensitive to the behavior of solutions near the open ends of the spacetime manifold.

To substantiate the above hypothesis, let us note that there exist global well-posedness results for what we consider fundamental bosonic fields (namely Yang-Mills [41] and non-linear scalar [24] fields) other than gravity (GR definitely exhibits finite time blow up, as exemplified by the singularities of black hole and cosmological solutions). On the other hand, the algebras of observables consisting of microlocal functionals of compact spacetime support [6,64,47,95] provide a candidate for the functor $\mathcal{C}^\infty_{\text{cst}}$, which seems to have the desired properties, at least when applied to the solution spaces of linear theories. Also, the sufficient conditions imposed on the constraints and gauge transformations are actually
very similar to those considered in previous treatments, as exemplified in \[50\] and \[58\].

**Theorem 11.** Consider a semilinear geometric wave equation that is well-posed (Hyp. \[4\]). Then we have an equivalence of categories \( \mathfrak{B} \mathfrak{C} \cong \mathfrak{G} \mathfrak{O} \mathfrak{L} \mathfrak{H} \mathfrak{Y} \mathfrak{P} \mathfrak{N} \mathfrak{P} \), and the corresponding algebra of observables functor \( \mathcal{F} : \mathfrak{B} \mathfrak{C} \rightarrow \mathfrak{P} \mathfrak{o} \mathfrak{i} \mathfrak{s} \) satisfies the standard axioms of classical LCFT given in Def. \[5\].

**Proof.** The equivalence of categories \( \mathfrak{B} \mathfrak{C} \cong \mathfrak{G} \mathfrak{O} \mathfrak{L} \mathfrak{H} \mathfrak{Y} \mathfrak{P} \mathfrak{N} \mathfrak{P} \) is straightforward to establish. The objects are identical and consist of pairs \((M, g)\) (equivalently \((M, \{g\})\)), where \(g\) is a globally hyperbolic Lorentzian metric on the manifold \(M\). There exists a morphism \((M, g) \rightarrow (M', g')\) in \( \mathfrak{B} \mathfrak{C} \) if there is a morphism \(M \rightarrow M'\) in \( \mathfrak{S} \mathfrak{p} \mathfrak{B} \mathfrak{C} \), between objects that project to \(\Pi(M) = (M, g)\) and \(\Pi(M') = (M', g')\) and neither solution patch \(S(M)\) or \(S(M')\) is empty. For semilinear equations, a solution \(\phi \in S(M)\) iff the corresponding spacelike cone bundle \(C^\circ \rightarrow M\) is strictly faster than the bundle of future oriented Lorentzian cones. Now, there exists a morphism \(M \rightarrow M'\) only if the underlying open embedding \(\chi : M \rightarrow M'\) satisfies \(\chi^* g' = g\) (is an isometry) and is chronally compatible with respect to \(C^\circ\) and \(C'^\circ\), which a fortiori implies that \(\chi\) is also causally compatible with respect to \(g\) and \(g'\). This concludes the proof of the equivalence of the two categories.

Next, we establish a crucial property that helps us prove the remaining conclusions. For any object \(\mathcal{M}\) with \(\Pi(\mathcal{M}) = (M, g)\) and spacelike cone bundle \(C^\circ \rightarrow M\) that is faster than the bundle of future oriented Lorentzian cones, we have the identity \(S_{\mathcal{M}}(M, g) = S(M)\). That is, all solutions are already \(C^\circ\)-slow. This tells us that the image of the morphism \(F(\mathcal{M}, g)\) in \( \mathfrak{M} \mathfrak{a} \mathfrak{n} \) consists of all objects and morphism \(S(M) \rightarrow S(M')\) with \(\Pi(M) = \Pi(M') = (M, g)\), where each \(S(M)\) is either empty (if the cone bundle \(C^\circ\) is slower than the Lorentzian cones) or \(S(M) \cong S_{\mathcal{M}}(M, g)\) (if the cone bundle \(C^\circ\) is faster than the Lorentzian one). The same goes for the respective objects in other categories, \(\mathfrak{P}(\mathcal{M}), \mathfrak{A}(\mathcal{M})\), and \(\mathcal{F}(\mathcal{M})\), except that the \(\mathcal{P}(\mathcal{M})\) are never empty. In other words, the limit construction \(\mathcal{F}_H(M, g) = \lim F(\mathcal{M}, g)\) becomes somewhat superfluous, since the image of the diagram \(\mathcal{F}(M, g)\) already contains the object that is equal to the limit.

With the above simplification, which means that we can simply replace the Poisson algebra \(\mathcal{F}_H(M, g)\) with the algebra \(F(\mathcal{M})\) for some appropriate \(\mathcal{M}\), the diagrams defining the generalized Isotony (Thm. \[8\]), Time Slice (Thm. \[9\]) and Causality (Thm. \[10\]) properties collapse to the usual notions thereof as stated in the standard definition of LCFT, Def. \[5\]. It remains only to show that the key hypotheses needed for the above theorems, namely extensibility of morphisms in \( \mathfrak{S} \mathfrak{p} \mathfrak{B} \mathfrak{C} \), transverse descent for Cauchy pushouts in \( \mathfrak{S} \mathfrak{p} \mathfrak{B} \mathfrak{C} \), and regular scattering for morphisms in \( \mathfrak{S} \mathfrak{p} \mathfrak{B} \mathfrak{C} \), all follow from global well-posedness, Hyp. \[1\].

Given a morphism \(\chi : \mathcal{M} \rightarrow \mathcal{M}'\), extensibility (Def. \[13\]) implies Isotony. Extensibility almost follows from well-posedness. Namely, if there exist Cauchy surfaces \(\Sigma \subset M\) and \(\Sigma' \subset M'\) such that \(\Sigma'\) extends \(\Sigma\) and we have initial data \(\varphi\) on \(\Sigma\) such that its pushforward \(\chi_* \varphi\) smoothly extends to some initial data \(\varphi'\) on \(\Sigma'\), then the corresponding solution extends from \(\chi(M)\) to \(M'\). However, even if Conj. \[1\] holds, the best we can expect is the existence of a Cauchy surface \(\Sigma' \subset M'\) that extends a compact subset \(K \subset \Sigma\). Also, even if we can extend all
of $\Sigma$ to $\Sigma'$, the initial data could behave near an open end of $\Sigma$ in a way that is not smoothly extendible to $\Sigma'$. So to recover Isotony at the algebraic level, we must appeal to the part of Hyp. 4 according to which the algebra of observables constructed by applying $C^\infty_{\text{cst}}$ is not sensitive to the behavior of solutions in the neighborhood of the open ends of $\mathcal{M}$ and, in particular, near the open ends of $\Sigma$. In that case, using Conj. 1 and global well-posedness as above, every smooth solution $\phi$ can be extended from an arbitrarily large compact subset $K \subset \Sigma$ to all of $\mathcal{M}$. The details of this argument would have to await a more precise formulation of 4, and 1 and of the choices discussed in Sect. 5.2.1.

Given a Cauchy surjective morphism $\chi: \mathcal{M} \rightarrow \mathcal{M}'$ (Def. 48), it could always be completed to a Cauchy pushout for $\mathcal{M}'$ (Def. 49). Given global well-posedness, no solution blows up in finite time. This means that all the Cauchy surjections, including induce isomorphisms in the diagrams (265) and (267). In other words, transverse descent holds trivially. The generalized Time Slice property then reduces to its standard version from Def. 5.

Finally, since global well-posedness (in particular the absence of any finite time blow up) also implies regular scattering, Def. 51, the generalized Causality property holds, Thm. 10, and reduces to the corresponding standard notion from Def. 5. □

The theorems presented in this section may be seen as the translation of the well-posedness properties of a natural variational PDE system into the algebraic setting given by the corresponding Poisson algebras of observables. One may even promote some of these properties to an axiom system generalizing the existing LCFT axioms, which are only applicable to semilinear wave systems, to the more general class of quasilinear systems, which includes GR. We do not do so immediately because several aspects of the current formalism, to be discussed in Sect. 9, leave room for substantial improvement, which could facilitate an optimal axiomatization.

8. Causality Axiom in Locally Covariant Quantum Field Theory

This section contains some remarks on extending the results presented so far to quantum field theory. The remarks essential consist of the conjecture that the categorical colimit construction of the algebra of observables will persist through perturbative deformation quantization of the classical field theory. Since rigorous, non-perturbative construction of quantum field theory is in general not yet possible, that is the best we can hope for at the moment.

Recall that a classical mechanical system essentially consists of a real Poisson algebra of observables $\mathcal{F} = (\mathcal{A}, \{\})$, where $\mathcal{A} = C^\infty_{\text{cst}}(P)$ on a symplectic manifold $P$ (the phase space), whose states are normed positive linear functionals on $\mathcal{A}$, which can be expressed as probability measures on $P$. On the other hand, a quantum mechanical system essentially consists of a complex, associate, non-commutative $*$-algebra $\hat{\mathcal{F}} = (\hat{\mathcal{A}}, *^\ast)$, whose states are normed positive linear functionals on $\hat{\mathcal{A}}$. Strong physical and mathematical arguments, which were already eloquently expressed in the original papers [11, 12, 13], indicate that deformation quantization is the right way to define an $\hbar$-parametrized family of quantum systems that quantize a given classical one. Briefly, a deformation quantization of a classical mechanical system $(\mathcal{A}, \{\})$ is a family of
quantum mechanical systems \( (\mathcal{A}_h, \star_h, \star_h) \), where the underlying vector spaces are isomorphic to \( \mathbb{C} \otimes A \cong \mathcal{A}_h \), the \( \star \)-involution reduces to standard complex conjugation, \( A^\star \rightarrow \overline{A} \), in the limit \( \hbar \rightarrow 0 \), and the non-commutative product commutator satisfies the classical limit, \( \frac{1}{\hbar} [A, B]_\hbar \rightarrow \{A, B\} \) as \( \hbar \rightarrow 0 \). The non-commutative product \( \star_h \) is referred to as the \( \star \)-product. The details of the kind of dependence on \( \hbar \) is allowed and the sense in which the limits are taken can be found in the standard literature [76].

From the constructive point of view, deformation quantization has a number of successes to its name. The Poisson algebra of any symplectic \([37,45,44]\) and even any Poisson manifold \([74]\) possesses a formal deformation quantization. The deformations are rigid and physically inequivalent classes of deformations are controlled by the second de Rham cohomology group of the underlying manifold \([55]\). Constructions of formal deformation quantization commute with reduction by quotienting out gauge symmetries, via the BV-BRST method, obstructed only by well defined anomalies \([22,47,95]\). Formal deformations are possible also for symplectic supermanifolds and infinite dimensional symplectic manifolds \([105]\) (and references therein). The formal deformations of standard \( \mathbb{R}^{2n} \) symplectic spaces can be made strict via the Wigner-Weyl-Moyal \( \star \)-product formula \([76]\). Similar results are available for some other special classes of finite dimensional symplectic manifolds \([98]\).

As discussed in Sect. 2, a classical field theory assigns classical mechanical systems to spacetimes in a way coherent over spacetime embeddings, tentatively summarized in the axioms for a locally covariant field theory (LCFT) functor, Def. 5, and similarly for a quantum field theory (QFT or LCQFT). By a quantization of a classical field theory, we mean an LCQFT functor that reduces to a classical LCFT functor in the classical limit \( \hbar \rightarrow 0 \), in the sense of deformation quantization, also in a way coherent with spacetime embeddings. The recent literature on pAQFT deals specifically with the perturbative quantization of classical field theory in the above sense. From the relevant literature, it is clear that the formal deformation quantization point of view is compatible with perturbative renormalization of quantum field theories \([39,64,47,95]\).

Note, however, that the field theories, whose quantization has been studied in the above way, have only been of semilinear type or whose non-linearities have been treated perturbatively as well. Thus, their Green functions (of appropriately linearized equations) and their Poisson brackets (via the Peierls formula) have causal supports fixed by external background fields (essentially a Lorentzian metric), as reflected in the Causality axiom of Def. 5. The quantum version of the Causality axiom uses \( \star \)-product commutators, which obey a similar causal support condition. The precise form of this condition has been known for a long time (in fact going back to the original axioms of Haag and Kastler) and has been checked to hold at each perturbative \( \hbar \)-order of the deformation quantization \([39,95]\).

On the other hand, the situation for quasilinear field theories, which have a dynamical, field-dependent causal structure (GR being a prominent example), has been much less clear. The classical notion of causality, relying on the causal support of the Green functions of the linearized equations about a fixed background solution (hence, via the Peierls formula, also of the Poisson brackets), is unproblematic and has been understood for a long time. However, it relies the notion of a fixed background solution or, in other words, of a fixed point in
the classical phase space. Since the standard formalism of quantum mechanics does not make use of a phase space, the question of translation of the notion of causality to the quantization of a quasilinear field theory has been generally considered open, especially outside the perturbative context, where it is sometimes referred to as quantum fluctuation/smearing of light cones \[72,81\].

In this paper, we have attempted to precisely formulate the analog of the standard Causality axiom for quasilinear classical field theories. This formulation has been given in terms of localizing the algebra of observables to open subsets of the phase space (the space of all solutions modulo gauge transformations) whose elements (individual solutions) all define causal structures that are compatible with some externally specified chronal cone bundle (in a way that is standard for symmetric hyperbolic, quasilinear PDE systems, Sects. 3 and 4). A cover of the total phase space by such slow patches, can be interpreted in categorical terms as the identification of the total phase space with the pushout of a diagram in the category of symplectic manifolds corresponding to the slow patches and the intersections between them. This categorical formulation makes it obvious that the corresponding algebraic formulation of the analog of the Causality axiom is the identification of the total algebra of observables with a limit of the diagram in the category of Poisson algebras corresponding to the algebras of observables of the slow phase space patches, with the Poisson bracket respecting the causal structure of the algebra of observables of each individual patch. The details of this formulation occupy Sect. 7.3 and culminate in Thm. 10. Similarly, an algebraic formulation is also given to the analog of the Time Slice axiom, Thm. 9 in Sect. 7.2. An important check on these generalized Causality and Time Slice properties is that they reduce to the corresponding standard notions when specialized to semilinear field theories, Thm. 11.

Finally, the above purely algebraic formulation of a generalized Causality property of classical LCFT motivates the following conjecture for an LCQFT deformation quantization of a classical LCFT:

**Conjecture 3.** Let \(\ast\text{-Alg}\) be the category of non-commutative, associative, \(\ast\)-algebras, while \(\text{Bkgr}_c\) and \(\text{SpBkgr}_c\) are the categories of spacetime manifolds endowed admissible background fields and, respectively, also with a globally hyperbolic spacelike cone bundles (as in Sect. 6.2). An generalized LCQFT is a covariant functor \(\hat{F}_H:\text{Bkgr}_c \to \ast\text{-Alg}\), such that there exists a functor \(\hat{F}:\text{SpBkgr}_c \to \ast\text{-Alg}\), fitting into the following commutative diagram

\[
\begin{array}{ccc}
\text{SpBkgr}_c & \xrightarrow{\hat{F}_H} & \ast\text{-Alg} \\
\downarrow{\hat{F}} & & \\
\text{Bkgr}_c & \xrightarrow{\hat{F}} & \ast\text{-Alg}
\end{array}
\]  

(278)

and satisfying the identity \(\hat{F}_H(M,B) = \lim \hat{F}(M,B)\), where the limit is indexed by the subcategory \(\text{SpBkgr}_c(M,B) \subseteq \text{SpBkgr}_c\), whose objects are of the form \(\mathcal{M} = (M,C^\infty,B)\). This LCQFT also satisfies the following Causality property.
The existence of the diagram

\[
\begin{array}{c}
\chi_1 \downarrow \chi_2 \\
\cup \\
\chi_1' \downarrow \chi_2' \\
\cup \\
M_1 \sqcup M_2,
\end{array}
\]

implies the existence of the diagram

\[
\begin{array}{c}
\hat{F}(M) \downarrow \hat{F}(M') \\
\hat{F}(M_1) \sqcup \hat{F}(M_2) \qquad \hat{F}(M_1') \sqcup \hat{F}(M_2')
\end{array}
\]

Similarly, the Time Slice property can be straightforwardly translated to LCQFT as well. Though the above formulation is rather abstract, we can see how it looks in rather concrete terms in the example of GR. Consider a gauge fixed version of GR. It may include fermionic ghost fields, if necessary, though whose properties we do not discuss at the moment (that is left to future work \[70\]).

At the moment, we also ignore the issue of diffeomorphism invariant observables and simply consider components of the quantized, gauge fixed metric field \(\hat{g}_{ab}(x)\) as observable. That is, once smeared, they constitute elements of the algebra \(\hat{F}_H(M, \{\beta\})\), for a spacetime manifold \(M\) endowed with admissible background field \(\beta\). What can we say about the \(*\)-product commutator \([\hat{g}_{ab}(x), \hat{g}_{cd}(y)]\)? As is, we cannot say very much, since we do not have complete solution for the theory and due to dynamical causal structure, we cannot appeal to the standard Causality axiom. However, since \(\hat{F}_H(M, \{\beta\}) = \lim_{\leftarrow} \hat{F}(M)\), with \(M = (M, C^\otimes, \{\beta\})\), there exist canonical projections \(\hat{F}_H(M, \{\beta\}) \to \hat{F}(M)\) for each such \(M\). Denote the images of the metric fields under these projections by \(\hat{g}_{ab}(x) \mapsto \hat{g}^M_{ab}(x)\). Then, if we select a spacelike cone bundle \(C^\otimes \to M\) such that two given points \(x, y \in M\) are \(C^\otimes\)-spacelike separated, our conjecture about the Causality property in LCQFT implies that

\[\hat{g}^M_{ab}(x), \hat{g}^M_{cd}(y)] = 0,\]

in the usual distributional sense.

As we have seen, the Causality property is a theorem of classical field theory. On the other hand, its version in quantum field theory is only a conjecture. Unfortunately, we cannot do better at the moment, due to the general difficulty
9. Discussion

In this work, we have in a sense taken up a thread left loose in the previous work of Geroch [50] on the application of the PDE theory of symmetric hyperbolic systems to classical field theory. This theory provides an unambiguous, intrinsic notion of causality not tied a priori to a Lorentzian metric. This notion of causality is especially useful for theories with quasilinear (rather than linear or semilinear) equations of motions (of which General Relativity is a prominent example), whose causal structures are field dependent. This is discussed in detail in Sects. 3 and 4. See also [8,94,93,51] for related earlier ideas.

Moreover, it provides the theorems cited in Sect. 4.3 as constructive tools to build the solution space of the theory as an infinite dimensional manifold (the infinite dimensional differential geometry aspect, as discussed in Sect. 5.2, was treated only formally here, though is tackled in earnest in the recent works [47,95,24]). Geroch’s casting of most classical relativistic field theories in symmetric hyperbolic form was then supplemented by constructing (formal) symplectic and Poisson structures (Sect. 5.3), turning the space of solutions into the genuine phase space of classical field theory. This was done by making use of the well known covariant phase space method and the Peierls formula, that was generalized to treat field theories that possess gauge invariance and constraints when cast into symmetric hyperbolic form.

Together with some reasonable technical hypotheses on the structure of these infinite dimensional phase spaces and algebras of smooth functions on them (Hyps. 1, 2 and 3), the generalized Isotony, Time Slice and Causality properties were established in Sect. 4. As an important check, these notions specialize (Thm. 11) to the synonymous axioms of classical relativistic field theory (Def. 5). In formulating the generalized properties of these axioms, which are essentially algebraic in nature, we identified strong parallels between them and aspects of the notion of well-posedness in classical PDE theory, which are more geometric in nature. In particular, we have highlighted the importance of the on-shell or phase space point of view that is dual to the off-shell point of view adopted in the recent works [47,95,24]. While the off-shell approach has proven invaluable in the modern formulation of perturbative renormalization of quantum field theories...
theories \cite{64,23,47,95}, the on-shell approach together with the above mentioned results from PDE theory is currently the only constructive method of building non-perturbatively interacting classical field theories. The parallels between the axioms of classical relativistic field theory and well-posedness also show that it is sometimes too much to hope for the axioms to hold in every physically reasonable example, especially in the presence of finite time blow up singularities (which occur in GR, for example). In particular the standard Isotony property fails in the presence of finite time blow (failure of global well-posedness) up (Sect. \ref{sec:isotony}). On the other hand, local well-posedness is sufficient for the Time Slice property to hold, unless a more technical condition of \textit{transverse descent} is also violated (Sect. \ref{sec:transverse}). On the other hand, the Causality property holds as long as a weaker version of global well-posedness holds that we called \textit{regular scattering} (Sect. \ref{sec:regular}).

The formulation of the generalized Causality property relies on the key result Thm. \ref{thm:main}, which could be called a generalized Microcausality property and follows from the Peierls formula for the Poisson bracket (Sect. \ref{sec:peierls}) and the domain of dependence theorem for hyperbolic PDE systems (Cor. \ref{cor:domain}). This connection can be seen as one of the main results of this work when coupled with the following speculation. If a quantum field theory is constructed from the deformation quantization of a classical field theory, then the Poisson algebra formulation of classical causality captured by the generalized Causality property should be replaced by a non-commutative $\ast$-algebra version of the same (Conj. \ref{conj:quantum}). If deformation quantization is carried out using Fedosov’s method, it is expected that the conjectured formulation of quantum causality at the very least holds formally at each order in $\hbar$ \cite{63,97}. Finally, if this conjecture holds, it provides a concrete answer (Eq. \ref{eq:commutator}) to this old question in quantum gravity: What can we say about the commutator of two metric field operators in quantum GR?

One of the motivations for this work had been to translate the notion of well-posedness from PDE theory to the algebraic setting for classical field theory, so that this translation could serve as a basis of a more realistic axiomatization thereof (see preceding paragraph for a discussion of why the axioms in Def. \ref{def:well-posed} are not completely adequate). However, such a translation attempted in this paper has lead to a number of necessary but somewhat tangential ideas, conjectures and hypotheses. We believe that it is essential to address and clarify some of them before attempting to formulate a more realistic axiomatization.

For example, we have chosen the notion of symmetric hyperbolicity because of the large existing literature on this subject, making it easy to use to leverage the existing results of PDE theory in the non-perturbative construction of classical field theories (Sec. \ref{sec:symmetric}). However, reducing the Euler-Lagrange equations of a field theory to (constrained) symmetric hyperbolic form is not always an obvious task, and, even if so, can be rather laborious and require the introduction of auxiliary background fields to keep everything natural. On the other hand, the notion of regular hyperbolicity \cite{31} seems to be much better adapted to Lagrangian field theories, but is not yet currently general enough to handle all cases of interest (such as for example higher order theories) and has a substantially smaller literature devoted to it. However, since both notions rely (underneath the hood of their respective well-posedness theorems) on so-called energy methods, it is likely that it would be possible to subsume both notions under a more general one that uses intrinsic information about the characteristic cohomology
of a PDE system \cite{28,3,9} to identify approximate conservation laws and automate their use in an energy method that would establish local well-posedness. Pursuing such a comprehensive notion of hyperbolicity is a promising avenue of investigation.

Another set of ideas naturally follows the notion of a conal manifold, some of which have already been explored in Sect. 4. Conal manifolds abstract the notion of causal order from Lorentzian geometry, but remain within the realm of differential geometry. It is interesting to see which results generalize from Lorentzian to conal manifolds. One important result that already exists is the splitting theorem for globally hyperbolic cone bundles (Prop. 3). This result is essentially differential topological in nature, very reminiscent of results that have been obtained as instances of the so-called \textit{h-principle} \cite{54,103}. On the other hand, the known proofs of the classic \cite{49} and the (very recent) generalized \cite{43} result use methods rather removed from differential topology (in the former case appealing to non-continuous measures and in the latter to Weak KAM theory).

A notable variation on the classic result uses purely order-theoretic and topological methods \cite{85}. It would be very interesting to formulate a proof using the methods of differential topology. In particular, such methods could then be adapted to the PL (piecewise-linear) and topological manifolds. The latter are of interest, for example, in the study of the causal structure of rough Lorentzian metrics \cite{82}. Another important set of results that should be generalized concern the addition of boundaries consistent with causal structure (in particular, causal compactifications) \cite{48}. There is ample evidence from conformal compactifications in relativity that understanding the structure of a causal boundary is an important ingredient in understanding the boundary conditions and asymptotic behavior of solutions to hyperbolic equations. On a more elementary side, it seems fruitful to investigate a version of de Rham cohomology with spacelike compact supports, as well as the dual homology, whose classes seem to be naturally represented by Cauchy surfaces. Such theories may give us a better understanding of the topological and order-theoretic ambiguities in the covariant phase space method (Sect. 5.3.3) and even give us a better understanding of the structure of the space of non-globally hyperbolic solutions of a hyperbolic PDE system. Finally, it is important to verify the validity of Conjs. 2 and 1, given their importance for the topological structure of the phase space (Sect. 5.2.1) and for the generalized Isotony property (Sect. 7.1).

The refinement of the notion of \textit{spacetime support} to that of \textit{local spacetime support} (Defs. 39 and 40) invites us to reconsider the problem of local observables in GR. Without this refinement, it is a well known fact that there are no gauge (diffeomorphism) invariant observables in GR with compact spacetime support \cite{47,105}. On the other hand, the possibility of gauge invariant observables with globally compact local spacetime support is yet to be considered in detail.

There is still a substantial number of not-completely-resolved questions on the topology and differential geometry of the infinite dimensional space of solutions of a hyperbolic PDE system and the algebra of smooth functions on it. Given the geometric, rather than the functional analytical focus of this work, we have avoided most of such details and instead have simply made several reasonable hypotheses (Hyp. 1, 2, and 3). To complete the program of constructive classical field theory, it is crucial to identify the precise theorems that would replace these hypotheses.
Finally, note that our discussion of classical field theory has been restricted to bosonic fields. On the other hand, to take fermionic matter into account as well as ghost fields that feature in the BV-BRST formulation of gauge theories, we also need to consider fermi fields. We believe that the best way to do that is shift from the setting of manifolds to supermanifolds \[101\], so that fermi fields are sections of odd vector bundles over spacetime (which is equivalent to more common but less precise phrasing that classical fermi fields are Grassmann valued). This approach is more geometric than and complementary to the algebraic approach adopted in \[64,96,47,95\]. The fermionic field theories considered so far in this setting have only been of semilinear type. An extension of this formulation to also encompass quasilinear field theories (along the lines of the older work \[29\]) and their dynamical causal structure will be reported elsewhere \[70\].

Acknowledgements. The author would like to thank Urs Schreiber for many interesting discussions on the nature of classical and quantum field theory. The author also thanks the following people for their interest in and helpful comments on this work: Romeo Brunetti, Claudio Dappiaggi, Thomas Hack, Klaus Fredenhagen, and Katarzyna Rejzner. The author is also grateful to Pedro L. Ribeiro and Stefan Hollands for communicating some of their unpublished results in the summer of 2012.

A. Jet bundles and the variational bicomplex

In this section we briefly introduce jet bundles and fix the relevant notation. For simplicity, we restrict ourselves to fields taking values in vector bundles. However, the discussion could be straightforwardly generalized to general smooth bundles.

We briefly introduce \(k\)-jets, mostly to recall some basic facts and fix notation. More details, as well as a coordinate independent definition, can be found in the standard literature \[88,73,103\]. Fix a vector bundle \(F \rightarrow M\), with \(\dim M = n\), with fibers modeled on a vector space \(U\), and consider an adapted coordinate patch \(\mathbb{R}^n \times U\), with coordinates \((x^i, u^a)\). Extend this patch to a \(k\)-jet patch \(\mathbb{R}^n \times U \times U^{n_k}\) by adding extra copies of \(U\), with new coordinates \((x^i, u^a, u^b_i, u^c_{ij}, \ldots, u^a_{i_1\cdots i_k})\), which formally denote the derivatives of \(\partial_{i_1} \cdots \partial_{i_k} \phi^a(x)\) of a section \(\phi\) at \(x\). To keep track of all the derivatives, we introduce multi-index notation. A multi-index \(I = i_1 i_2 \cdots i_k\) replaces the corresponding set of symmetric covariant coordinate indices (the multi-index does not change when the defining \(i\)'s are permuted). The order of this multi-index is given by \(|I| = k\), with \(|\emptyset| = 0\). To augment a multi-index by adding another index, we use the notation \(Ij = jI = i_1 \cdots i_k j\). Thus we can write higher order derivatives as \(\partial_{i_1 \cdots i_k} \phi(x) = \partial_{i_1} \cdots \partial_{i_k} \phi(x)\), the higher order jet coordinates as \(u^a_{i_1 \cdots i_k} = u^a_{i_k} \cdots u^a_{i_1}\) and the total set of coordinates on a \(k\)-jet patch as \((x^i, u^a_i)\), \(|I| \leq k\). In particular the empty multi-index \(I = \emptyset\) corresponds to \(u^a_0 = u^a\).

Since the higher derivatives are symmetric in all indices, the number of extra coordinates is given by \(n_k = \sum_{i=1}^k \dim S^k \mathbb{R}^n\), with \(S^k\) denoting the symmetric tensor product. Given two different coordinate patches on \(F\), we define the transition maps between the corresponding \(k\)-jet patches according to the usual calculus chain rule applied to higher order derivatives. These \(k\)-jet patches can be glued together into the total space of the \(k\)-jet bundle \(J^k F \rightarrow M\), which includes \(J^0 F \cong F\).
Since $F \to M$ is a vector bundle, so is $J^k F \to M$. It is isomorphic to $F \otimes_M (F \otimes_M S^1 T^* M) \otimes_M \cdots \otimes_M (F \otimes_M S^k T^* M)$, but not naturally. Jet bundles come with natural projections $J^k F \to J^{k-1} F$, which simply discard all derivatives of order $k$. This projection gives $J^k F$ the structure of an affine bundle over the base $J^{k-1} F$, with fibers modeled on the vector bundle $(F \otimes_M S^k T^* M)^k-1 \to J^{k-1} F$.

The bundle $J^k F \to J^{k-1} F$ is affine because, in general, bundle morphisms of $J^k F \to J^k F$ induced by vector bundle automorphisms of $F$ are not linear, but are affine.

Given a vector bundle $E \to M$ it can be pulled back to the $k$-jet bundle along the projection $J^k F \to M$. We introduce a convenient notation for this pullback.

**Definition 52.** We denote by $(E)^k \to J^k F$ the pullback of $E \to M$ to $J^k F$, which then fits into the pullback commutative square

$$
\begin{array}{ccc}
(E)^k & \to & E \\
\downarrow & & \downarrow \\
J^k F & \to & M
\end{array}
$$

Any smooth section $\phi : M \to F$ automatically gives rise to its $k$-jet prolongation or $k$-prolongation $j^k \phi : M \to J^k F$. Namely $j^k \phi$ is a section of the bundle $J^k F \to M$ that is defined in a local adapted coordinate patch as

$$
\phi^a(x) = (x^i, \phi^a(x), \partial_i \phi^a(x), \ldots, \partial_{i_1 \cdots i_k} \phi^a(x)) = (x^i, \partial_i \phi^a(x)), \quad |I| \leq k.
$$

One can think of the $k$-prolongation symbol as a differential operator

$$
j^k : \Gamma(F) \to \Gamma(J^k F)
$$

of order $k$. In fact, any (not necessarily linear) differential operator of order $k$,

$$
f : \Gamma(F) \to \Gamma(E), \quad f : \phi \mapsto f[\phi],
$$

can be written as a composition of $j^k$ with an order 0 (not necessarily linear) operator $f : J^k F \to E$, such that $f[\phi] = f(j^k \phi)$. Note that we are slightly abusing notation by denoting both the differential operator and the bundle morphism by the same symbol $f$.

Further, we can define an $l$-prolongation of a differential operator $f$ of order $k$,

$$
p^\infty f : J^{k+l} F \to J^l E,
$$

which is then a differential operator of order $k + l$, by composing with $j^l$:

$$
p^\infty f[\phi] = j^l f[\phi].$$

Prolongation is discussed briefly using coordinate-wise operations in Sect. 3.3. The $k$-jet prolongation $j^k \phi$ can now be thought of as a special case of bundle morphisms, that is, $j^k \phi = p^k \phi$, where on the right hand

---

13 Though both the $k$-jet and the "direct sum" bundles can be constructed by applying a functor $\mathcal{B} \to \mathcal{B}$ to a bundle $F \to M$, a vector bundle automorphism $\chi : F \to F$ induces a vector bundle automorphism $J^k(\chi) : J^k F \to J^k F$ that need not be block diagonal in a basis adapted to the "direct sum" bundle, while the bundle automorphism induced by the "direct sum" bundle is. Therefore, the isomorphism of the "direct sum" and $k$-jet bundles cannot depend on the object $F \to M$ alone, and hence cannot be chosen naturally.
side we interpret $\phi$ as the base fixing bundle morphism to $F \to M$ from the trivial 0-dimensional bundle $id: M \to M$.

\[
\begin{array}{c}
M \\
id
\end{array} \xrightarrow{\phi} \begin{array}{c}
F \\
M \\
\end{array}
\] (287)

Given the sequence of projections $k$-jet bundles over $M$,

\[
\cdots \to J^2 F \to J^1 F \to J^0 F \cong F,
\] (288)

it is convenient to introduce the infinite jet order (or $\infty$-jet) bundle $J^\infty F$ defined as the projective limit over the jet order $k$

\[
J^\infty F = \lim_{\leftarrow} J^k F.
\] (289)

This limit implicitly defines $J^\infty F$ as an infinite dimensional smooth manifold. The main advantage of working with $\infty$-jets is that any function or tensor on $J^k F$ for finite $k$ can be pulled back to $J^\infty F$. Conversely, any smooth function or tensor on $J^\infty F$ depends only on jets up to some finite order, say $k$, and can be faithfully projected to $J^k F$. Another major convenience of working on $J^\infty F$ is the ability to decompose the usual de Rham differential into its horizontal and vertical parts

\[
d = d_h + d_v.
\] (290)

The defining property of $d_h$ is the following. Given a section $\phi: M \to F$, we must have the identity

\[
(j^\infty \phi)^* d_h \alpha = d(j^\infty \phi)^* \alpha,
\] (291)

where $\alpha$ is any differential form on $J^\infty F$ and $d$ is the usual de Rham differential on $M$. On the other hand, $d_v$ is characterized by the fact that its image is annihilated by the pullback to $M$ along any section $\phi$,

\[
(j^\infty \phi)^* d_v \alpha = 0.
\] (292)

It can be checked that the horizontal and vertical differentials anti-commute and are separately nilpotent,

\[
d_h^2 = 0 = d_v^2, \quad d_h d_v + d_v d_h = 0.
\] (293)

Note that, to apply $d_v$ or $d_h$ to forms defined on a finite order jet bundle $J^k F$, the pullback and projection operations mentioned above will often be applied implicitly. Thus the application of say $d_h$ to a differential form on $J^k F$ may yield that a differential form that projects to $J^{k+1} F$ but not to $J^k F$. In local coordinates $(x^i, u^a)$ on $F$, and the induced coordinates $(x^i, u^a_I)$ on $J^\infty F$, a convenient basis for differential forms is

\[
d_h x^i = dx^i, \quad d_v u^a_I = du^a_I - d_h u^a_I = du^a_I - u^a_I dx^i.
\] (294)

We can also define two special kinds of vector fields. A vector field $\hat{\xi}$ is horizontal if its action in local coordinates is

\[
\hat{\xi}(x^i) = \xi^i, \quad \hat{\xi}(u^a_I) = \xi^i u^a_I.
\] (295)
for some $\xi^i = \xi^i(x,u^j)$. In particular, the vector field $\hat{\xi}_j$, with $\xi^i = \delta^i_j$, is horizontal. Note that $[\hat{\xi}_i, \hat{\xi}_j] = 0$. A vector field $\hat{\psi}$ is \textit{evolutionary} if its action in local coordinates is

$$\hat{\psi}(x') = 0, \quad \hat{\psi}(u_I^a) = \hat{\xi}_I(\psi^a), \quad (296)$$

for some $\psi^a = \psi^a(x, u_I^a)$, where $\hat{\xi}_I(f) = \hat{\xi}_{i_1} \cdot \cdots \cdot \hat{\xi}_{i_k} (f)$ for multi-index $I = i_1 i_2 \cdots i_k$ (the order of application of these vector fields does not matter since they commute). Note that the $\psi^a$ can be seen as the fiber coordinate components of a section of the bundle $(F)^\infty \to J^\infty F$. These definitions can be checked to be coordinate independent.

One can show that for a horizontal vector field $\hat{\xi}$ on $J^\infty F$ there exists a vector field $\xi_\phi$ on $M$ such that their actions on scalar functions are intertwined by the pullback along the jet prolongation $j^\infty \phi$ of a section $\phi: M \to F$,

$$\hat{\xi}(f)(j^\infty \phi) = \xi_\phi(f(j^\infty \phi)), \quad (297)$$

for any scalar function $f$ on $J^\infty F$. Namely, in local coordinates, $\xi_\phi = \xi_i^j \partial_j$ with $\xi^i = (\epsilon^i_i dx^i)(j^\infty \phi) = \xi^i(x)(j^\infty \phi) = \xi^i(j^\infty \phi)$. On the other hand, evolutionary vector fields $\hat{\psi}$ satisfy the identities

$$\iota_\psi (d_{h\alpha}) + d_h (\iota_\psi \alpha) = 0, \quad (298)$$

$$L_\psi (j^\infty \phi)^* \alpha = \left. \frac{d}{dx} \right|_{x=0} [j^\infty(\phi + \varepsilon \psi)]^* \alpha = \iota_\psi d_{\alpha} \alpha = L_\phi \alpha, \quad (299)$$

for any form $\alpha \in \Omega^*(J^\infty F)$ and section $\psi: M \to F$. Actually, $\psi$ could be a section of $(F)^k \to J^k F$, that is, it could depend on $\phi^a(x)$ and its derivatives and not only on $x \in M$. The only corresponding change in the above formula would be to replace $\varepsilon \psi$ by $\varepsilon(j^k \phi)^* \psi$. Ostensibly, $L_\psi$ should stand for the Lie derivative on the infinite dimensional manifold of sections of $F \to M$, where the section $\psi$ is identified with the vector field whose action on local coordinates is $L_\psi \phi^a(x) = \psi^a(x)$. However, since we do not delve into the differential geometry of infinite dimensional manifolds here, we keep the symbol $L_\psi(j^\infty \phi)^* \alpha$ primitive and defined as above.

Integrations or differentiations by parts are carried out using the following basic identity

$$d_{\psi} u_I^a \wedge dx^i \wedge \alpha = d_{\psi}(u_I^a dx^i) \wedge \alpha \quad (300)$$

$$= (d_{h\alpha} u_I^a) \wedge \alpha \quad (301)$$

$$= -(d_h d_{\psi} u_I^a) \wedge \alpha \quad (302)$$

$$= -d_{\psi} u_I^a \wedge d_{h\alpha} - d_h (d_{\psi} u_I^a \wedge \alpha). \quad (303)$$

This split of the de Rham differential into horizontal and vertical differentials also splits the de Rham complex $\Omega^*(J^\infty F)$ of differential forms on $J^\infty F$ into a \textit{bicomplex} 211. Since the horizontal and vertical 1-forms generate the graded commutative algebra of differential forms, any form $\lambda \in \Omega^*(J^\infty F)$ can be uniquely written as

$$\lambda = \sum_{h,v} \lambda_{h,v}, \quad (304)$$
where $0 \leq h \leq n$ and $0 \leq v$ are respectively the horizontal and vertical degrees form degrees. We have thus turned the differential forms into a bigraded complex $\Omega^*(J^\infty F) = \bigoplus_{h,v} \Omega^{h,v}(F)$, with the $d_h$ differential increasing $h$ by 1 and the $d_v$ differential increasing $v$ by 1. This complex is called the variational bicomplex \cite{21}. As with any bicomplex, we can consider its cohomology with respect to either or any combination of the two differentials. The horizontal cohomology is $H^{h,v}(d_h) = H(\Omega^*(J^\infty F), d_h)$ in degrees $(h,v)$. The vertical cohomology is $H^{h,v}(d_v) = H(\Omega^*(J^\infty F), d_v)$ in degrees $(h,v)$. Both $(H^{h,*}(d_h), d_v)$ and $(H^{*,v}(d_v), d_h)$ still form complexes, therefor we can also consider their cohomologies. The relative cohomologies are $H^{h,*}(d_v|d_h) = H(H^{h,*}(d_h), d_v)$ and $H^{*,v}(d_h|d_v) = H(H^{*,v}(d_v), d_h)$.

B. Limits and colimits in category theory

Some basic information on category theory can be found in \cite{21} and more specifically about categorical limits and colimits in \cite{21,10}.

Consider categories $\mathcal{J}$ and $\mathcal{C}$ and a functor $\mathcal{D}: \mathcal{J} \to \mathcal{C}$. The image $\mathcal{D}(\mathcal{J})$ (also denoted by just $\mathcal{D}$) forms a diagram in $\mathcal{C}$ with index category $\mathcal{J}$. A cone to the diagram $\mathcal{D}$ is an object $C$ of $\mathcal{C}$, the vertex, together with a morphism $c_i: C \to D_i = \mathcal{D}(i)$ for each object $i$ of $\mathcal{J}$, such that diagram

\begin{equation}
\begin{array}{ccc}
D_i & \xrightarrow{c_i} & D_j \\
\downarrow & & \downarrow \\
C & \xrightarrow{c_j} & D_j
\end{array}
\end{equation}

is commutative for each morphism $i \to j$ in $\mathcal{J}$. If it exists, the limit (also inverse or projective limit) is an object $\lim \leftarrow \mathcal{D}$ of $\mathcal{C}$ that is the vertex of a cone of canonical morphisms $u_i: \lim \leftarrow \mathcal{D} \to D_i$ such that the following universal property holds: any cone $(C, c)$ to $\mathcal{D}$ factors through it with a unique mediating morphism (also canonical morphism) $u_C$, which makes the following diagram commute:

\begin{equation}
\begin{array}{ccc}
C & \xrightarrow{c_i} & \lim \leftarrow \mathcal{D} \\
\downarrow & \xrightarrow{u_C} & \downarrow \\
& u_i & \quad
\end{array}
\end{equation}

For an illustrative example, consider the index category $\mathcal{J}$ consisting of only the objects and morphisms $1 \to 3 \leftarrow 2$, while $\mathcal{D}$ is a functor from $\mathcal{J}$ to the category of sets. The diagram $\mathcal{D}$, the limit and the cone of canonical morphisms fit into the following commutative diagram

\begin{equation}
\begin{array}{ccc}
\lim \leftarrow \mathcal{D} & \xrightarrow{u_1} & D_2 \\
\downarrow & \xrightarrow{u_2} & \downarrow \\
D_1 & \xrightarrow{u_3} & D_3
\end{array}
\end{equation}

The universality of the limit identifies the set $\lim \leftarrow \mathcal{D}$ with the subset of the Cartesian product $D_1 \times D_2$ consisting of pairs $(d_1, d_2)$ that get mapped to the same element in $D_3$, $d_1 \mapsto d_3 \leftarrow d_2$. If the object $3$ and the morphisms to it
were absent from the index category $\mathcal{I}$, we would simply have $\varinjlim \mathcal{G} \cong D_1 \times D_2$.

In other categories, we may keep the same intuition, but replace the Cartesian product $\times$ by the categorical product.

The notion of a colimit is dual. As before, consider a diagram $\mathcal{G}: \mathcal{I} \to \mathcal{C}$. A co-cone of the diagram $\mathcal{G}$ is an object $C$ of $\mathcal{C}$, the vertex, together with a morphism $c_i: \mathcal{G}(i) = D_i \to C$ for each object $i$ of $\mathcal{I}$. If it exists, the colimit (also direct or inductive limit) is an object $\varinjlim \mathcal{G}$ of $\mathcal{C}$ that is the vertex of a cone of canonical morphisms $u_i: D_i \to \varinjlim \mathcal{G}$ such that the following universal property holds: any co-cone $(C, c_i)$ of $\mathcal{G}$ factors through it with a unique mediating morphism $u_{\mathcal{G}}$, which makes the following diagram commute:

$$D_i \xrightarrow{u_i} \varinjlim \mathcal{G} \xrightarrow{u_{\mathcal{G}}} C.$$  \hfill (308)

For an illustrative example, consider the index category $\mathcal{I}$ consisting of only the objects and morphisms $1 \leftarrow 3 \rightarrow 2$, while $\mathcal{G}$ is a functor from $\mathcal{I}$ to the category of sets. The diagram $\mathcal{G}$, the colimit and the co-cone of canonical morphisms fit into the following commutative diagram

$$\begin{array}{ccc}
D_3 & \xrightarrow{u_3} & D_2 \\
\downarrow & & \downarrow \quad u_2 \\
D_1 & \xrightarrow{u_1} & \varinjlim \mathcal{G}.
\end{array}$$ \hfill (309)

The universality of the limit identifies the set $\varinjlim \mathcal{G}$ with the quotient of the disjoint union $D_1 \sqcup D_2$ by the equivalence relation that identifies elements $d_1 \sim d_2$, respectively of $D_1$ and $D_2$, provided they are images of the same element $d_3 \in D_3$, $d_1 \leftarrow d_3 \rightarrow d_2$. If the object 3 and the morphisms from it were absent from the index category $\mathcal{I}$, we would simply have $\varinjlim \mathcal{G} \cong D_1 \sqcup D_2$. In other categories, we may keep the same intuition, but replace the disjoint union $\sqcup$ by the categorical coproduct.

References

1. Ian M. Anderson. The variational bicomplex. 1989.
2. Ian M. Anderson. Introduction to the variational bicomplex. In Mark J. Gotay, Jerrod E. Marsden, and Vincent Moncrief, editors, *Mathematical aspects of classical field theory*, volume 132 of *Contemporary Mathematics*, pages 51–73. American Mathematical Society, Providence, Rhode Island, 1992.
3. Ian M. Anderson and Charles G. Torre. Asymptotic conservation laws in classical field theory. *Physical Review Letters*, 77:4109–4113, November 1996.
4. Abhay Ashtekar, Luca Bombelli, and Oscar Reula. The covariant phase space of asymptotically flat gravitational fields. In M. Francaviglia and D. Holm, editors, *Mechanics, analysis and geometry: 200 years after Lagrange*, North-Holland Delta Series, pages 417–450, Amsterdam, 1991. North-Holland.
5. Christian Baer, Christian Becker, Romeo Brunetti, Klaus Fredenhagen, Nicolas Ginoux, Frank Pfaffle, and Alexander Strohmaier. Quantum Field Theory on Curved Spacetimes: Concepts and Methods, volume 786 of *Lecture Notes in Physics*. Springer, 2009.
6. Christian Baer and Nicolas Ginoux. Classical and quantum fields on Lorentzian manifolds. April 2011, 1104.1158.
7. Christian Baer, Nicolas Ginoux, and Frank Pfaffle. *Wave Equations on Lorentzian Manifolds and Quantization*, volume 2 of *ESI lectures in mathematics and physics*. European Mathematical Society, June 2007, 0806.1036.
8. Ulrich Bannier. On generally covariant quantum field theory and generalized causal and dynamical structures. *Communications in Mathematical Physics*, 118(1):163–170, March 1988.

9. Glenn Barnich, Friedemann Brandt, and Marc Henneaux. Local BRST cohomology in gauge theories. *Physics Reports*, 338(5):439–589, November 2000, hep-th/0002245.

10. Glenn Barnich, Marc Henneaux, and Christiane Schomblond. Covariant description of the canonical formalism. *Physical Review D*, 44(4):R939–R941, August 1991.

11. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Quantum mechanics as a deformation of classical mechanics. *Letters in Mathematical Physics*, 1(6):321–330, June 1977.

12. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. I. deformations of symplectic structures. *Annals of Physics*, 111(1):61–110, March 1978.

13. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. II. physical applications. *Annals of Physics*, 111(1):111–151, March 1978.

14. Robert Beig. Concepts of hyperbolicity and relativistic continuum mechanics. In Jörg Frauendiener, Domenico Giulini, and Volker Perlick, editors, *Analytical and Numerical Approaches to Mathematical Relativity*, volume 692 of *Lecture Notes in Physics*, chapter 5, pages 101–116. Springer Berlin / Heidelberg, Berlin/Heidelberg, 2006, gr-qc/0411092.

15. Robert Beig and Bernd G. Schmidt. Relativistic elasticity. *Classical and Quantum Gravity*, 20(5):889–9+, February 2003.

16. J. J. Benavides Navarro and E. Minguzzi. Global hyperbolicity is stable in the interval topology. *Journal of Mathematical Physics*, 52(11):112504+, August 2011, 1108.5120.

17. Antonio Bernal and Miguel Sánchez. Further results on the smoothability of Cauchy hypersurfaces and Cauchy time functions. *Letters in Mathematical Physics*, 77(2):183–197, August 2006, gr-qc/0512095.

18. Antonio N. Bernal and Miguel Sánchez. Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. *Communications in Mathematical Physics*, 257(1):43–50, May 2005, gr-qc/0411092.

19. E. Binz, J. Sniatycki, and H. Fischer. *Geometry of Classical Fields*, volume 154 of *Notas de Matemática*. North-Holland, 1988.

20. A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khorskprimekova, I. S. Krasilshchik, A. V. Samokhin, Yu Torkhov, A. M. Verbovetsky, and A. M. Vinogradov. *Symmetries and conservation laws for differential equations of mathematical physics*, volume 182 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1999.

21. F. Borceux. *Handbook of Categorical Algebra: Volume 1, Basic Category Theory*, Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 2008.

22. Martin Bordemann, Hans-Christian Herbig, and Stefan Waldmann. BRST cohomology and phase space reduction in deformation quantisation. *Communications in Mathematical Physics*, 210(1):107–144, September 1999, math/9901015.

23. Romeo Brunetti, Michael Dütsch, and Klaus Fredenhagen. Perturbative algebraic quantum field theory and the renormalization groups. *Advances in Theoretical and Mathematical Physics*, 13(5):1541–1599, July 2009, 0901.2038.

24. Romeo Brunetti, Klaus Fredenhagen, and Pedro L. Ribeiro. Algebraic structure of classical field theory I: Kinematics and linearized dynamics for real scalar fields, September 2012, 1209.2148.

25. Romeo Brunetti, Giuseppe Ruzzi. Quantum charges and spacetime topology: The emergence of new superselection sectors. *Communications in Mathematical Physics*, 287(2):523–563, March 2008, 0801.3365.

26. R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths. *Exterior Differential Systems*, volume 18 of *Mathematical Sciences Research Institute Publications*. Springer, 2011.

27. Robert L. Bryant and Phillip A. Griffiths. Characteristic cohomology of differential systems. I. General theory. *J. Amer. Math. Soc.*, 8(3), 1995.

28. Yvonne Choquet-Bruhat. The cauchy problem in classical supergravity. *Letters in Mathematical Physics*, 7(6):459–467, November 1983.
30. Yvonne Choquet-Bruhat and Cécile Dewitt-Morette. *Analysis, Manifolds and Physics, Part I*. North Holland, 2 edition, January 2004.

31. D. Christodoulou. *The Action Principle and Partial Differential Equations*, volume 146 of *Annals of Mathematics Studies*. Princeton University Press, 1999.

32. Piotr T. Chruściel and James D. E. Grant. On Lorentzian causality with continuous metrics. *Classical and Quantum Gravity*, 29(14):145001+, June 2012.

33. R. Courant and D. Hilbert. *Methods of Mathematical Physics, Volume 1*. Wiley Classics Library. John Wiley & Sons, 2008.

34. R. Courant and D. Hilbert. *Methods of Mathematical Physics, Volume 2*. Wiley Classics Library. John Wiley & Sons, 2008.

35. Čedomir Crnković and Edward Witten. Covariant description of canonical formalism in geometrical theories. In S. W. Hawking and W. Israel, editors, *Three hundred years of gravitation*, pages 676–684. Cambridge University Press, Cambridge, 1987.

36. Claudio Dappiaggi and Benjamin Lang. Quantization of Maxwell’s equations on curved backgrounds and general local covariance, May 2012, 1104.1374.

37. Marc De Wilde and Pierre B. A. Lecomte. Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds. *Letters in Mathematical Physics*, 7(6):487–496, November 1983.

38. Bryce DeWitt. *The Global Approach to Quantum Field Theory: I & II*. Oxford University Press, USA, February 2003.

39. Michael Duetsch and Klaus Fredenhagen. Perturbative algebraic field theory, and deformation quantization. In Roberto Longo, editor, *Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects*, volume 38 of *Fields Institute Communications*, pages 151–160. American Mathematical Society, January 2001, hep-th/0101079.

40. Michael Duetsch and Klaus Fredenhagen. The master Ward identity and generalized Schwinger-Dyson equation in classical field theory. *Communications in Mathematical Physics*, 243(2):275–314, June 2003, hep-th/0211242.

41. Douglas M. Eardley and Vincent Moncrief. The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space. *Communications in Mathematical Physics*, 83(2):171–191, February 1982.

42. L. C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, June 1998.

43. Albert Fathi and Antonio Siconolfi. On smooth time functions. *Mathematical Proceedings of the Cambridge Philosophical Society*, 152(02):303–339, November 2011.

44. B. Fedosov. Deformation quantization and index theory, volume 9 of *Mathematical Topics*. Akademie Verlag, 1996.

45. Boris V. Fedosov. A simple geometrical construction of deformation quantization. *Journal of Differential Geometry*, 40(2):213–238, 1994.

46. Michael Forger and Sandro V. Romero. Covariant Poisson brackets in geometric field theory. *Communications in Mathematical Physics*, 256(2):375–410.

47. Klaus Fredenhagen and Katarzyna Rejzner. Batalin-Vilkovisky formalism in the functional approach to classical field theory. January 2011, 1101.5112v1.

48. Alfonso García-Parrado and José M. M. Senovilla. Causal structures and causal boundaries. March 2005, gr-qc/0501069.

49. Robert Geroch. Domain of dependence. *Journal of Mathematical Physics*, 11(2):437–449, 1970.

50. Robert Geroch. Partial differential equations of physics. In G. S. Hall and J. R. Pulham, editors, *Proceedings of the Forty-Sixth Scottish Summer School in Physics*, Edinburgh, February 1996, SISSP Publ, gr-qc/9602055.

51. Robert Geroch. Faster than light? May 2010, 1005.1614.

52. G. Giachetta, L. Mangiarotti, and G. Sardanashvily. Cohomology of the infinite-order jet space and the inverse problem. *Journal of Mathematical Physics*, 42(9):4272–4282, 2001, math/0006074.

53. Hubert Goldschmidt. Integrability criteria for systems of nonlinear partial differential equations. *Journal of Differential Geometry*, 1(3-4):269–307, 1967.

54. M. Gromov. *Partial Differential Relations*, volume 9 of *Ergebnisse Der Mathematik Und Ihrer Grenzgebiete*. Springer, 2010.

55. Simone Gutt and John Rawnsley. Equivalence of star products on a symplectic manifold; an introduction to Deligne’s ćech cohomology classes. *Journal of Geometry and Physics*, 29(4):347–392, March 1999.

56. R. Haag. *Local Quantum Physics: Fields, Particles, Algebras*. Texts and Monographs in Physics. Springer, 1996.
57. Thomas-Paul Hack and Mathias Makedonski. A No-Go theorem for the consistent quantization of spin 3/2 fields on general curved spacetimes, September 2012, 1106.6327.
58. Thomas-Paul Hack and Alexander Schenkel. Linear bosonic and fermionic quantum gauge theories on curved spacetimes, May 2012, 1205.3484.
59. Richard S. Hamilton. The inverse function theorem of Nash and Moser. Bulletin of the American Mathematical Society, 7(1):65–223, July 1982.
60. Stephen W. Hawking and G. F. R. Ellis. The Large Scale Structure of Space-Time. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, London, March 1973.
61. M. Henneaux. Elimination of the auxiliary fields in the antifield formalism. Physics Letters B, 238(2-4):299–304, April 1990.
62. A. Jaffe and E. Witten. Quantum Yang-Mills Theory. American Mathematical Society, 2006.
63. F. John. Partial Differential Equations, volume 1 of Applied Mathematical Sciences. Springer, 1981.
64. Igor Khavkine. Supergeometry and classical field theory with fermions. in preparation.
65. Igor Khavkine. Presymplectic current and the inverse problem of the calculus of variations, October 2012, 1210.0802.
66. Igor Khavkine. Quantum astrometric observables i: time delay in classical and quantum gravity. Physical Review D, 85(12), June 2012, 1111.7127.
67. Peter J. Olver. Applications of Lie groups to differential equations, volume 107 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1993.
89. R. S. Palais. *Seminar on the Atiyah-Singer Index Theorem*, volume 57 of *Annals of Mathematics Studies*. Princeton University Press, 1965.
90. R. E. Peierls. The commutation laws of relativistic field theory. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 214:143–157, 1952.
91. Roger Penrose. *Techniques in Differential Topology in Relativity*, volume 7 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, PA, January 1972.
92. V. Perlick. *Ray Optics, Fermat's Principle, and Applications to General Relativity*, volume 61 of *Lecture notes in physics: Monographs*. Springer, 2000.
93. Martin Rainer. Algebraic quantum theory on manifolds: a Haag-Kastler setting for quantum geometry. *Classical and Quantum Gravity*, 17(9):1935+, May 2000, gr-qc/9911076.
94. Martin Rainer. Cones and causal structures on topological and differentiable manifolds, January 2000, gr-qc/9905106.
95. Katarzyna Rejzner. *Batalin-Vilkovisky formalism in locally covariant field theory*. PhD thesis, Hamburg, November 2011, 1111.5130.
96. Katarzyna Rejzner. Fermionic fields in the functional approach to classical field theory. January 2011, 1101.5126.
97. Pedro L. Ribeiro. Private communication.
98. M. A. Rieffel. *Deformation quantization for actions of R^d*, volume 506 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 1993.
99. H. Ringström. *The Cauchy Problem in General Relativity*, volume 6 of *ESI Lectures in Mathematics and Physics*. European Mathematical Society, 2009.
100. R. T. Rockafellar. *Convex Analysis*, volume 28 of *Princeton Mathematical Series*. Princeton University Press, 1996.
101. T. Schmitt. Supergroups and quantum field theory, or: What is a classical configuration? *Reviews in Mathematical Physics*, 9(8):993–1052, 1997, hep-th/9607132.
102. Werner M. Seiler. *Involution: The Formal Theory of Differential Equations and its Applications in Computer Algebra*, volume 24 of *Algorithms and Computation in Mathematics*. Springer Berlin Heidelberg, 2010.
103. D. Spring. *Convex Integration Theory: Solutions to the H-Principle in Geometry and Topology*, volume 92 of *Monographs in Mathematics*. Birkhäuser, 1998.
104. M. E. Taylor. *Pseudo differential operators*, volume 416 of *Lecture notes in mathematics*. Springer-Verlag, 1974.
105. José A. Vallejo. Symplectic connections and Fedosov’s quantization on supermanifolds. *Journal of Physics: Conference Series*, 343:012124+, February 2012, 1110.5700.
106. Robert M. Wald. *General Relativity*. University Of Chicago Press, Chicago, 1st edition, June 1984.
107. Robert M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics (Chicago Lectures in Physics)*. University Of Chicago Press, 1 edition, November 1994.
108. Stefan Waldmann. Geometric wave equations, August 2012, 1208.4706.
109. Wikipedia. Category theory, 2012. [Online; accessed 08-Nov-2012].
110. Wikipedia. Limit (category theory), 2012. [Online; accessed 08-Nov-2012].
111. Wikipedia. Sheaf (mathematics), 2012. [Online; accessed 08-Nov-2012].
112. Wikipedia. Symmetric monoidal category, 2012. [Online; accessed 08-Nov-2012].