We investigate diagonal forms of degree $d$ over the function field $F$ of a smooth projective $p$-adic curve: if a form is isotropic over the completion of $F$ with respect to each discrete valuation of $F$, then it is isotropic over certain fields $F_U$, $F_P$ and $F_p$. These fields appear naturally when applying the methodology of patching; $F$ is the inverse limit of the finite inverse system of fields $\{F_U, F_P, F_p\}$. Our observations complement some known bounds on the higher $u$-invariant of diagonal forms of degree $d$.

We only consider diagonal forms of degree $d$ over fields of characteristic not dividing $d!$.

Keywords: Forms of higher degree, diagonal forms, function fields, $p$-adic curves, $u$-invariant.

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Introduction

The fact that Springer’s Theorem holds for diagonal forms of higher degree over fields of characteristic not dividing $d!$ [9] guarantees that on occasion diagonal forms of higher degree defined over function fields behave similarly to quadratic forms. For a survey on the behaviour of (diagonal) forms of higher degree in general the reader is referred to [12].

In this note we consider diagonal forms of degree $d$ over function fields $F = K(X)$ where $X$ is a smooth, projective, geometrically integral curve over $K$ and $K$ is the fraction field of a complete discrete valuation ring with a residue field $k$ of characteristic not dividing $d!$. Let $v$ be a rank one discrete valuation of $F$, and $F_v$ the completion of $F$ with respect to $v$. It was shown by Colliot-Thélène, Parimala and Suresh [2, Theorem 3.1] that a quadratic form which is isotropic over $F_v$ for each $v$ is already isotropic over $F$, using the methodology of patching developed by Habater and Hartmann [4], i.e. viewing $F$ as the inverse limit of a finite inverse system of certain fields $\{F_U, F_P, F_p\}$.

Given a nondegenerate diagonal form $\varphi$ over $F$ of degree $d > 2$ and dimension $> 2$, it is not clear, however, whether the isotropy of $\varphi$ over $F_v$ for each $v$ implies
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that \( \varphi \) is isotropic.

Our main result proves that the isotropy of a nondegenerate diagonal form \( \varphi \)
over \( F_v \) for each \( v \) implies that at least over the field extensions \( F_{v'} \), \( F_P \) and \( F_p \) of \( F \),
\( \varphi \) is isotropic as well (Theorem 3.1). These fields depend on the choice of the form
\( \varphi = \langle a_1, \ldots, a_n \rangle \), more precisely on the choice of the regular proper model \( X \) (over
the complete discrete valuation ring \( A \)) of the curve \( X \) over \( K \), which depends on \( \varphi \):
\( X \) is selected such that there exists a reduced divisor \( D \) with strict normal crossings,
which contains both the support of the divisor of all the entries \( a_i \), \( 1 \leq i \leq n \), and
the components of the special fibre of \( X/A \). Since nondegenerate diagonal forms of
degree \( d \geq 3 \) have finite automorphism groups [6, p. 137], we are not able to apply
[5, Theorem 3.7] to conclude that the isotropy of \( \varphi \) over the \( F_{v'} \)'s and \( F_P \)'s implies
that \( \varphi \) is also isotropic over \( F \), however. This is only possible for \( d = 2 \).

After collecting the terminology and some basic results in Section 1, in particular
defining diagonal \( u \)-invariants of degree \( d \) over \( k \), we consider diagonal forms of
higher degree over valued fields in Section 2 and then study diagonal forms of
higher degree over function fields of \( p \)-adic curves using some of the ideas of [2] in
Section 3. Recall that a \( p \)-adic field is a finite field extension of \( \mathbb{Q}_p \).

As a consequence of Springer’s Theorem for diagonal forms, any diagonal form
of degree \( d \) and dimension \( \geq d^3 + 1 \) over a function field in one variable \( F = K(t) \),
where \( K \) is a \( p \)-adic field with residue field \( k \), \( \text{char}(k) \nmid d! \), is isotropic over \( F_v \) for
every discrete valuation \( v \) with residue field either a function field in one variable
over \( k \) or a finite extension of \( K \). Moreover, it is isotropic over \( F_{v'} \) for each reduced,
irreducible component \( U \subset Y \) of the complement of \( S \) in the special fibre \( Y = X \times_A k \)
of \( X/A \), and isotropic over \( F_P \) for each \( P \in S \) (Corollary 3.4), and thus isotropic
over \( F_p \) for each \( p = (U, P) \). Here, \( S \) is the inverse image under a finite \( A \)-morphism
\( f : X \to \mathbb{P}^3_A \) of the point at infinity of the special fibre \( \mathbb{P}^3_k \).

1. Preliminaries

Let \( k \) be a field such that \( \text{char}(k) \) does not divide \( d! \).

1.1. Forms of higher degree

Let \( V \) be a finite-dimensional vector space over \( k \) of dimension \( n \). A \( d \)-linear form
over \( k \) is a \( k \)-multilinear map \( \theta : V \times \cdots \times V \to k \) (\( d \)-copies) which is symmetric, i.e.
\( \theta(v_1, \ldots, v_d) \) is invariant under all permutations of its variables. A form of degree \( d \)
over \( k \) (and of dimension \( n \)) is a map \( \varphi : V \to k \) such that \( \varphi(au) = a^d \varphi(v) \) for all
\( a \in k \), \( v \in V \) and such that the map \( \theta : V \times \cdots \times V \to k \) defined by
\[
\theta(v_1, \ldots, v_d) = \frac{1}{d!} \sum_{1 \leq i_1 < \cdots < i_d \leq d} (-1)^{d-i} \varphi(v_{i_1} + \cdots + v_{i_d})
\]
(\( 1 \leq l \leq d \)) is a \( d \)-linear form over \( k \). By fixing a basis \( \{e_1, \ldots, e_n\} \) of \( V \), any form \( \varphi \)
of degree \( d \) can be viewed as a homogeneous polynomial of degree \( d \) in \( n = \dim V \)
variables \( x_1, \ldots, x_n \) via \( \varphi(x_1, \ldots, x_n) = \varphi(x_1e_1 + \cdots + x_ne_n) \) and, vice versa, any
homogeneous polynomial of degree $d$ in $n$ variables over $k$ is a form of degree $d$ and dimension $n$ over $k$. Any $d$-linear form $\theta : V \times \cdots \times V \to k$ induces a form $\varphi : V \to k$ of degree $d$ via $\varphi(v) = \theta(v, \ldots, v)$. We can hence identify $d$-linear forms and forms of degree $d$.

Two $d$-linear spaces $(V_i, \theta_i)$, $i = 1, 2$, are isomorphic (written $(V_1, \theta_1) \cong (V_2, \theta_2)$) or just $\theta_1 \cong \theta_2$) if there exists a bijective linear map $f : V_1 \to V_2$ such that $\theta_2(f(v_1), \ldots, f(v_d)) = \theta_1(v_1, \ldots, v_d)$ for all $v_1, \ldots, v_d \in V_1$. A $d$-linear space $(V, \theta)$ (or the $d$-linear form $\theta$) is nondegenerate if $v = 0$ is the only vector such that $\theta(v, v_2, \ldots, v_d) = 0$ for all $v_i \in V$. A form of degree $d$ is called nondegenerate if its associated $d$-linear form is nondegenerate. A form $\varphi$ over $k$ is called anisotropic, if it does not have any non-trivial zeroes, otherwise it is called isotropic.

The orthogonal sum $(V_1, \theta_1) \perp (V_2, \theta_2)$ of two $d$-linear spaces $(V_i, \theta_i)$, $i = 1, 2$, is the $k$-vector space $V_1 \oplus V_2$ together with the $d$-linear form

$$(\theta_1 \perp \theta_2)(u_1 + v_1, \ldots, u_d + v_d) = \theta_1(u_1, \ldots, u_d) + \theta_2(v_1, \ldots, v_d)$$

$(u_i \in V_1, v_i \in V_2)$ [7].

A $d$-linear space $(V, \theta)$ is called decomposable if $(V, \theta) \cong (V_1, \theta_1) \perp (V_2, \theta_2)$ for two non-zero $d$-linear spaces $(V_i, \theta_i)$, $i = 1, 2$. If $\varphi$ is represented by the homogeneous polynomial $a_1x_1^d + \cdots + a_nx_n^d$ ($a_i \in k^\times$) we use the notation $\varphi = (a_1, \ldots, a_n)$ and call $\varphi$ diagonal. A diagonal form $\varphi = (a_1, \ldots, a_n)$ over $k$ is nondegenerate if and only if $a_i \in k^\times$ for all $1 \leq i \leq n$.

If $d \geq 3$, $a_i, b_j \in k^\times$, then $\langle a_1, \ldots, a_n \rangle \cong \langle b_1, \ldots, b_n \rangle$ if and only if there is a permutation $\pi \in S_n$ such that $\langle b_i \rangle \cong \langle a_{\pi(i)} \rangle$ for every $i$. This is a special case of [6, Theorem 2.3].

Note that for quadratic forms ($d = 2$), the automorphism group of $\varphi$ is infinite, whereas for $d \geq 3$, the automorphism group of $\varphi$ usually is finite, for instance if $\varphi$ is nonsingular in the sense of algebraic geometry [13]. In particular, nondegenerate diagonal forms of degree $d \geq 3$ have finite automorphism groups [6, p. 137], which creates a problem when trying to imitate patching arguments as it is not possible to apply [5, Theorem 3.7].

### 1.2. Higher degree $u$-invariants

The $u$-invariant (of degree $d$) of $k$ is defined as $u(d, k) = \sup \{\dim_k \varphi \}$, where $\varphi$ ranges over all the anisotropic forms of degree $d$ over $k$. The diagonal $u$-invariant (of degree $d$) of $k$ is defined as $u_{\text{diag}}(d, k) = \sup \{\dim \varphi \}$, where $\varphi$ ranges over all the anisotropic diagonal forms over $k$.

Thus the diagonal $u$-invariant $u_{\text{diag}}(d, k)$ is the smallest integer $n$ such that all diagonal forms of degree $d$ over $k$ of dimension greater than $n$ are isotropic, and the $u$-invariant $u(d, k)$ is the smallest integer $n$ such that all forms of degree $d$ over $k$ of dimension greater than $n$ are isotropic. Obviously, $u_{\text{diag}}(d, k) \leq u(d, k)$.

If $u = u(d, k)$ then each anisotropic form of degree $d$ over $k$ of dimension $u$ is universal. If $u = u_{\text{diag}}(d, k)$ then each diagonal anisotropic form of degree $d$ over $k$...
of dimension $u$ is universal. We have

$$u_{\text{diag}}(d, k) \leq \min \{ n \mid \text{all forms of degree } d \text{ over } k \text{ of dimension } \geq n \text{ are universal} \}$$

with the understanding that the “minimum” of an empty set of integers is ∞, cf. [12]. For $d = 2$, $u_{\text{diag}}(d, k) = u(d, k)$ is the $u$-invariant of quadratic forms.

For an algebraically closed field $k$, $|k^x/k^x| = 1$ and hence $u_{\text{diag}}(d, k) = u(d, k) = 1$. For a formally real field $k$, the diagonal $u$-invariant is infinite for even $d$; since $-1 \notin \sum k^2$, also $-1 \notin \sum k^d$ for any even $d$. Thus the form $m \times (1)$ of degree $d$ is anisotropic for each integer $m$, implying $u_{\text{diag}}(d, k) = u(d, k) = \infty$.

The strong diagonal $u$-invariant of degree $d$ of $k$, denoted $u_{\text{diag}, s}(d, k)$, is the smallest real number $n$ such that

1. every finite field extension $E/k$ satisfies $u_{\text{diag}}(d, E) \leq n$, and
2. every finitely generated field extension $E/k$ of transcendence degree one satisfies $u_{\text{diag}}(d, E) \leq dn$.

If these $u$-invariants are arbitrarily large, put $u_{\text{diag}, s}(d, k) = \infty$.

Analogously as observed in [5] for $d = 2$, $u_{\text{diag}, s}(d, k) \leq n$ if and only if every finitely generated field extension $E/k$ of transcendence degree $l \geq 1$ satisfies $u_{\text{diag}}(d, k) \leq d^l n$. Thus if $u_{\text{diag}, s}(d, k)$ is finite, it is at least 1 and lies in $\frac{1}{2} \mathbb{N}$.

1.3. $C_r^0$ fields

Let $r \geq 1$ be an integer. A field $F$ is a $C_r$-field if for all $d \geq 1$ and $n > d^r$, every homogeneous form of degree $d$ in $n$ variables over $F$ has a non-trivial solution in $F$. In particular, then $F$ satisfies $u(d, F) \leq d^r$. Moreover, every finite extension of $F$ is a $C_r$-field, and every one-variable function field over $F$ a $C_{r+1}$-field [14, II.4.5]. Hence $u_{\text{diag}, s}(d, F) \leq d^r$ for a $C_r$-field $F$.

A field $F$ is a $C_r^0$-field if the following holds: For any finite field extension $F'$ of $F$ and any integers $d \geq 1$ and $n > d^r$, for any homogeneous form over $F'$ of degree $d$ in $n$ variables, the greatest common divisor of the degrees of finite field extensions $F''/F'$ over which the form acquires a nontrivial zero is one. This amounts to requiring that the $F'$-hypersurface defined by the form has a zero-cycle of degree 1 over $F'$.

Assume $\text{char}(F) = 0$. For each prime $l$, let $F_l$ be the fixed field of a pro-$l$-Sylow subgroup of the absolute Galois group of $F$. Any finite subextension of $F_l/F$ is of degree coprime to $l$. The field $F$ is $C_r^0$ if and only if each of the fields $F_l$ is $C_r$. A finite field extension of a $C_r^0$-field is $C_r^0$. If $F$ is $C_r^0$ then a function field $E = F(x_1, \ldots, x_s)$ in $s$ indeterminates $x_1, \ldots, x_s$ over $F$ is $C_{r+s}^0$ [2, 2.1].

It is not known if $p$-adic fields have the $C_r^0$-property.

**Remark 1.1.** Assume that $p$-adic fields have the $C_r^0$-property. Let $K(X)$ be any function field of transcendence degree $r$ over a $p$-adic field $K$ (here we do not need to assume $p \neq 2, 3$). Suppose that there is $\ell \neq 2$ such that there exists a finite subextension of $K_\ell(X)/K(X)$ of degree 2. Then any cubic form over $K(X)$ in
strictly more than $3^{2+r}$ variables has a nontrivial zero: If the $p$-adic field $K$ is $C_3^0$, then the function field $E = K(X)$ in $r$ variables over $K$ is $C_3^{0+r}$ [2, Lemma 2.1]. Thus a cubic form over $E = K(X)$ in strictly more than $3^{2+r}$ has a nontrivial zero in each of the fields $K_{\ell}(X)$, $\ell$ a prime, hence in a finite extension of $K(X)$ of degree coprime to $\ell$, for each $\ell$ prime. Pick $\ell \neq 2$, then $[K_{\ell}(X) : K(X)]$ is even. Moreover, pick $l \neq 2$ such that there exists a finite subextension of $E_l/K(X)$ of degree 2 then the cubic form has a zero over it. By Springer’s Theorem for cubic forms and their behaviour under quadratic field extensions [8, VII], thus the cubic form has a nontrivial zero in $K(X)$. This is the analogue of [2, Proposition 2.2].

2. Diagonal forms over Henselian valued fields

2.1.

Let $K$ be a valued field with valuation $v$, valuation ring $R$ and maximal ideal $m$. Let $\Gamma$ be the value group. Assume that $d!$ is not divisible by the characteristic of the residue field $k = R/m$. For $u \in R$, denote by $\bar{u}$ the image of $u$ in $k$. For a polynomial $f \in R[X]$, $f = a_nx^n + \cdots + a_1x + a_0$ define the polynomial $\bar{f} = a_nx^n + \cdots + a_1x + a_0$ over $k$. If $\varphi = \langle a_1, \ldots, a_n \rangle$ is a nondegenerate diagonal form with entries $a_i \in R$, define the diagonal form $\varphi' = \langle a_1, \ldots, a_n \rangle$ over $k$. $\varphi$ is called a unit form, if $\varphi'$ is nondegenerate. Choose a set $\{ \pi_\gamma \in R \mid \gamma \in I \}$ such that the values of the $\pi_\gamma$’s represent the distinct cosets in $\Gamma/d\Gamma$. We may decompose a diagonal form $\varphi$ as $\varphi = \varphi' \varphi''$ by taking $\varphi'_\gamma$ to be the diagonal form whose entries comprise all $a_i$ with $v(a_i) = v(\pi_\gamma) \mod d\Gamma$. By altering the slots by $d$-powers if necessary, we may then write $\varphi'' = \pi_\gamma \varphi''$ with each $\varphi', \varphi''$ a diagonal unit form. There are only finitely many non-trivial $\varphi_\gamma$ [9]. If $\Gamma = \mathbb{Z}$, the set $\{ \pi_\gamma \mid \gamma \in I \}$ can be chosen to be $\{ \pi^i \mid i = 0, \ldots, d - 1 \}$ and $|\Gamma/d\Gamma| = d$ is finite.

If $R$ satisfies Hensel’s Lemma then $(K,v)$ is called a Henselian valued field and $R$ a Henselian valuation ring. Every complete discretely valued field is Henselian.

Let $\varphi$ be a diagonal form over a Henselian valued field $(K,v)$. Write $\varphi = \pi_1\varphi_1 \perp \cdots \perp \pi_r\varphi_r$ with each $\varphi_i$ a diagonal unit form and the $\pi_i$ having distinct values in $\Gamma/d\Gamma$. Then $\varphi$ is isotropic if and only if some $\varphi_i$ is isotropic [9, Proposition 3.1]. This is because for a diagonal unit form $\varphi$ over a Henselian valued field $(K,v)$, $\varphi$ is isotropic if and only if $\varphi$ is isotropic [9, Lemma 2.3].

**Theorem 2.1.** ([9] or [12, Theorem 4, Corollary 2]) Suppose that $\text{char}(k) \nmid d!$.

(i) Let $(K,v)$ be a Henselian valued field. Then $u_{\text{diag}}(d,K) = |\Gamma/d\Gamma|^{-d}u_{\text{diag}}(d,k)$.

(ii) Let $(K,v)$ be a Henselian valued field. If every diagonal form of degree $d$ of dimension $n + 1$ over $k$ is isotropic, then every diagonal form of degree $d$ and dimension $dn + 1$ over $K$ is isotropic. If $K$ has an anisotropic form of degree $d$ and dimension $m$, then $K$ has an anisotropic form of degree $d$ and dimension $dm$.

(iii) Let $K$ be a discretely valued field. Then $u_{\text{diag}}(d,K) \geq d^{n}u_{\text{diag}}(d,k)$.

(iv) Let $F$ be a field extension of finite type over $k$ of transcendence degree $n$. Then $u_{\text{diag}}(d,F) \geq d^n u_{\text{diag}}(d,k')$ for a suitable finite field extension $k'/k$. 


The (in)equalities in (i), (iii), (iv) also hold when the values are infinite.

For $d = 2$, (ii) is Springer’s Theorem for quadratic forms over Henselian valued fields \cite{15}. Springer’s Theorem does not hold for non-diagonal forms of higher degree than $2$ \cite{9, 2, 7}. Theorem 2.1 is a major ingredient in our proofs, for instance we can show:

**Proposition 2.2.** Let $A$ be a discrete valuation ring with fraction field $K$ and residue field $k$ such that $\text{char}(k) \not\mid d!$.

(i) $u_{\text{diag}}(d, K) \geq d u_{\text{diag}}(d, k)$ and $u_{\text{diag}, s}(d, K) \geq d u_{\text{diag}, s}(d, k)$.

(ii) If $A$ is Henselian then $u_{\text{diag}}(d, K) = d u_{\text{diag}}(d, k)$.

(iii) If $A$ is a complete discrete valuation ring then every finite extension of $K$ has diagonal $u$-invariant at most $d u_{\text{diag}}(d, k)$.

The first assertions of (i) as well as (ii) and (iii) follow from Theorem 2.1. The proof of the second claim in (i) is analogous to the one of \cite[4.9]{5}, employing Theorem 2.1 instead of Springer’s Theorem.

2.2.

A field $K$ is called an $m$-local field with residue field $k$ if there is a sequence of fields $k_0, \ldots, k_m$ with $k_0 = k$ and $k_m = K$, and such that $k_i$ is the fraction field of an excellent Henselian discrete valuation ring with residue field $k_{i-1}$ for $i = 1, \ldots, m$. Recall that a discrete valuation ring $R$ is called excellent, if the field extension $\hat{K}/K$ is separable, where $\hat{K}$ denotes the quotient field of $R$ and $K$ is its completion. (This condition is trivially satisfied if $K$ has characteristic 0 or $R$ is complete.)

Proposition 2.2 implies (compare the next two results with \cite[Corollary 4.13, 4.14]{5} for quadratic forms):

**Corollary 2.3.** Suppose that $K$ is an $m$-local field whose residue field $k$ is a $C_r$-field with $\text{char}(k) \not\mid d!$. Let $F$ be a function field over $K$ in one variable.

(i) $u_{\text{diag}}(d, k) = u_{\text{diag}, s}(d, k) = d^r$ and $u_{\text{diag}}(d, K) = d^{r+m}$.

Moreover, if some normal $K$-curve with function field $F$ has a $K$-point, then $u_{\text{diag}}(d, F) \geq d^{r+m+1}$.

(ii) If $u_{\text{diag}}(d, k') = d^r$ for every finite extension $k'/k$, then $u_{\text{diag}}(d, F) \geq d^{r+m+1}$.

**Proof.** (i) Since $k$ is a $C_r$-field, $u_{\text{diag}}(d, k) = d^r$, thus $u_{\text{diag}}(d, k) \leq u_{\text{diag}, s}(d, k) \leq d^r$ and the first two equations follow. Applying Proposition 2.2 and induction yields that $u_{\text{diag}}(d, K) \geq d^m u_{\text{diag}}(d, k) = d^{r+m}$. Let $X$ be a normal $K$-curve with function field $F$ and let $\xi$ be a $K$-point on $X$. The local ring at $\xi$ has fraction field $F$ and residue field $K$. So Proposition 2.2 implies that $u_{\text{diag}}(d, K) = d^m u_{\text{diag}}(d, k)$ and $u_{\text{diag}}(d, F) \geq d u_{\text{diag}}(d, K) = d^{r+m+1}$.

(ii) Choose a normal or equivalently a regular $K$-curve $X$ with function field $F$, and a closed point $\xi$ on $X$. Let $R$ be the local ring of $X$ at $\xi$ with residue field $k(\xi)$. Then the fraction field of $R$ is $F$, and $k(\xi)$ is a finite extension of $K$. Hence $k(\xi)$
is an $m$-local field whose residue field $k'$ is a finite extension of $k$. By assumption, $u_{\text{diag}}(d, k') = d^r$ and $k'$ is a $C_r$-field since $k$ is. So applying part (ii) to $k'$ and $\kappa(\xi)$, it follows that $u_{\text{diag}}(d, \kappa(\xi)) \geq d^{r+m}$. Proposition 2.2 yields $u_{\text{diag}}(d, F) \geq d^{r+m+1}$. □

**Corollary 2.4.** (i) Let $F$ be a one-variable function field over an $m$-local field $K$ with residue field $k$ such that $\text{char}(k) \nmid d!$ and $k$ is algebraically closed. Then $u_{\text{diag}}(d, F) \geq d^{m+1}$.

(ii) If $k$ is a finite field and $u_{\text{diag}}(d, k) = d^r$ with $r \in \{0, 1\}$, then $u_{\text{diag}}(d, k) = u_{\text{diag}, s}(d, k) = d^r$ and $u_{\text{diag}}(d, K) \geq d^{r+m}$. Moreover, if some normal $K$-curve with function field $F$ has a $K$-point, then $u_{\text{diag}}(d, F) \geq d^{r+m+1}$.

**Proof.** (i) This is a special case of Corollary 2.3 using that an algebraically closed field $k$ is $C_0$, satisfies $u_{\text{diag}}(d, k) = 1$, and has no non-trivial finite extensions.

(ii) A finite field is $C_1$ [14, II.3.3(a)], hence $u_{\text{diag}}(d, k) \leq d$. Here $u_{\text{diag}}(d, k) = u_{\text{diag}, s}(d, k) = d^r$ and Corollary 2.3 yields the assumption. □

In general, for any finite field $k = \mathbb{F}_q$ we obviously do not have $u_{\text{diag}}(d, k) = d^r$, $r \in \{0, 1\}$: for instance, if $-1 \in \mathbb{F}_q^*d$ and $d \geq 4$ then $u_{\text{diag}}(d, \mathbb{F}_q) \leq d - 1$ by [10]. Or, if $d^* = \gcd(d, q - 1)$, then $u_{\text{diag}}(d, \mathbb{F}_q) \leq d^*$. This implies that $u_{\text{diag}}(d, \mathbb{F}_q) = 1$, if $d$ is relatively prime to $q - 1$ and that for $q > (d^* - 1)^4$, $u_{\text{diag}}(d, \mathbb{F}_q) = 2$ [12, 5.1].

3. The behaviour of diagonal forms of higher degree over function fields of $p$-adic curves

Whenever we write ‘discrete valuation ring’ and ‘discrete valuation’ we mean a discrete valuation ring of rank one and a valuation with value group $\mathbb{Z}$.

3.1. Let $A$ be a complete discrete valuation ring with fraction field $K$ and residue field $k$ with $\text{char}(k) \nmid d!$. Let $X$ be a smooth, projective, geometrically integral curve over $K$ and $F = K(X)$ be the function field of $X$. Let $t$ denote a uniformizing parameter for $A$. For each (rank one) discrete valuation $v$ of $F$, let $F_v$ denote the completion of $F$ with respect to $v$.

We will adapt some ideas from [2] to diagonal forms of higher degree: take a nondegenerate form $\varphi = \langle a_1, \ldots, a_n \rangle$ of degree $d$ over $F$. Then choose a regular proper model $\mathcal{X}/A$ of $X/K$, such that there exists a reduced divisor $D$ with strict normal crossings which contains both the support of the divisor of all the entries $a_i$, $1 \leq i \leq n$, and the components of the special fibre of $X/A$. (Note that this implies that the regular proper model $\mathcal{X}/A$ depends on the form $\varphi$, and thus so do $Y, Y_i, S_0, S, F_P, F_U, \ldots$ as defined in the following.)

Let $Y = \mathcal{X} \times_A k$ be the special fibre of $X/A$. Let $x_i$ be the generic point of an irreducible component $Y_i$ of $Y$. Then there is an affine Zariski neighbourhood $W_i \subset \mathcal{X}$ of $x_i$, such that the restriction of $Y_i$ to $W_i$ is a principal divisor. Let $S_0$
be a finite set of closed points of $Y$ containing all singular points of $D$, and all the points that lie on some $Y_i$, but not in the corresponding $W_i$.

Choose a finite $k$-morphism $f : X \to \mathbb{P}^1_k$ as in [4, Proposition 6.6]. Let $S$ be the inverse image under $f$ of the point at infinity of the special fibre $\mathbb{P}^1_k$. Then the set $S_p$ is contained in $S$. All the intersection points of two components $Y_i$ are in $S$. Each component $Y_i$ contains at least one point of $S$. Let $U \subset Y$ run through the reduced irreducible components of the complement of $S$ in $Y$. Then each $U$ is a regular affine irreducible curve over $k$ and we define $k[U]$ to be its ring of regular functions and $k(U)$ to be its fraction field. $k[U]$ is a Dedekind domain and $U = \text{Spec } k[U]$. Each $U$ is contained in an open affine subscheme $\text{Spec } R_U$ of $X$ and is a principal effective divisor in $\text{Spec } R_U$. Moreover, $R_U$ is the ring of elements in $F$ which are regular on $U$ and also a regular ring, since it is the direct limit of regular rings. The ring $R_U$ is a localisation of $\mathcal{O}_U$ and so $U$ is a principal effective divisor on $\text{Spec } R_U$ given by the vanishing of an element $s \in R_U$. The $t$-adic completion $\hat{R}_U$ of $R_U$ is a domain and coincides with the $s$-adic completion of $R_U$, since $t = us^r$ for some integer $r \geq 1$ and a unit $u \in R_U^\times$. By definition, $F_U$ is the field of fractions of $\hat{R}_U$. We have $k[U] = R_U/s = \hat{R}_U/s$. For $P \in S$, the completion $\hat{R}_P$ of the local ring $R_P$ of $X$ at $P$ is a domain and $F_P$ is the field of fractions of $\hat{R}_P$. Let $P = (U, P)$ be a pair with $P \in S$ in the closure of an irreducible component $U$ of the complement of $S$ in $Y$. Then let $R_p$ be the complete discrete valuation ring which is the completion of the localisation of $\hat{R}_P$ at the height one prime ideal corresponding to $U$. Then $F_p$ is the field of fractions of $R_p$ and $F$ is the inverse limit of the finite inverse system of fields $\{F_U, F_P, F_p\}$ by [4, Proposition 6.3].

The following can be seen as a weak generalization of [2, Theorem 3.1] to diagonal forms of higher degree. Here we are not able to conclude that under the given assumptions, $\varphi$ is isotropic over $F$, only over the $F_U$’s and $F_P$’s:

**Theorem 3.1.** Let $\varphi$ be a nondegenerate diagonal form of degree $d$ over $F$. If $\varphi$ is isotropic over the completion $F_v$ of $F$ with respect to each discrete valuation $v$ of $F$ with residue field either a function field in one variable over $k$ or a finite extension of $K$, then:

(i) $\varphi$ is isotropic over $F_U$ for each reduced irreducible component $U \subset Y$ of the complement of $S$ in $Y$,

(ii) $\varphi$ is isotropic over $F_P$ for each $P \in S$.

**Proof.** Suppose $\varphi = (a_1, \ldots, a_n)$.

(i) Each entry $a_i$ of $\varphi$ is supported only along $U$ in $\text{Spec } R_U$, thus has the form $us^j$ where $u \in R_U^\times$. We sort the entries $a_i = u_is^j$ by the power $j$ of $s$ and use them to define new diagonal forms $\rho_j$ which have all the $u_i$’s belonging to those $a_i$ where $s$ occurred in the $j$th power as their diagonal entries. Hence $\varphi$ is isomorphic to the diagonal form

$$\rho_0 \perp s\rho_1 \perp \cdots \perp s^{d-1}\rho_{d-1}$$

over $F$, where the $\rho_i$ are nondegenerate diagonal forms of degree $d$ over $R_U$. Note
that if for some \( j \in \{0, 1, \ldots, d - 1\} \) there is no \( a_j \) with \( a_i = u_i s^j \), then there is no corresponding form \( \rho_j \) and a \( \rho_j \) does not appear as a component in the sum.

By hypothesis, \( \varphi \) is isotropic over the field of fractions of the completed local ring of \( X \) at the generic point of \( U \). By Theorem 2.1, this implies that the image of at least one of the forms \( \rho_0, \rho_1 \) or \( \rho_{d-1} \) under the composite homomorphism \( R_U \to k[U] \to k(U) \) is isotropic over \( k(U) \). Since the residue characteristic \( p \) does not divide \( d! \), the forms \( \rho_0, \rho_1, \ldots, \rho_{d-1} \) define a smooth projective variety over \( R_U \). In particular, all of them define a smooth variety over \( k(U) \). Since \( k[U] \) is a Dedekind domain, if such a projective variety has a point over \( k(U) \), it has a point over \( k(U) \). Since the variety is smooth over \( R_U \), a \( k[U] \)-point lifts to an \( \hat{R}_U \)-point (cf. the discussion after [5, Lemma 4.5]). Thus \( \varphi \) has a nontrivial zero over \( F_U \).

(ii) Let \( P \in S \). The local ring \( R_P \) of \( X \) at \( P \) is regular. Its maximal ideal is generated by two elements \((x,y)\) with the property that any \( a_i \) is the product of a unit, a power of \( x \) and a power of \( y \). Thus over \( F \), the fraction field of \( R_P \), \( \varphi \) is isomorphic to

\[
\varphi_1 \perp x\varphi_2 \perp y\varphi_3 \perp xy\varphi_4 \perp x^2\varphi_5 \perp y^2\varphi_6 \perp x^2y^2\varphi_7 \perp x^2y^2\varphi_8 \perp xy^2\varphi_9 \perp \cdots \perp x^{d-1}y^{d-1}\varphi_{d^2},
\]

where each \( \varphi_i \) is a nondegenerate diagonal form over \( R_P \). Let \( R_y \) be the localization of \( R_P \) at the prime ideal \((y)\). \( R_y \) is a discrete valuation ring with fraction field \( F \). The residue field \( E \) of \( R_y \) is the field of fractions of the discrete valuation ring \( R_P/(y) \). By hypothesis, the form

\[
(\varphi_1 \perp x\varphi_2 \perp x^2\varphi_5 \perp \ldots) \perp y(\varphi_3 \perp x\varphi_4 \perp \ldots)
\]

\[
\perp y^2(\varphi_6 \perp \ldots) \perp \cdots \perp y^{d-1}(\cdots \perp x^{d-1}\varphi_{d^2})
\]

is isotropic over the field of fractions of the completion of \( R_y \). By Theorem 2.1, the reduction of one of the forms

\[
(\varphi_1 \perp x\varphi_2 \perp x^2\varphi_5 \perp \ldots), (\varphi_3 \perp x\varphi_4 \perp \ldots), \ldots,
\]

is isotropic over \( E \). Since \( x \) is a uniformizing parameter for \( R_P/(y) \), by Theorem 2.1 this implies that over the residue field of \( R_P/(y) \), the reduction of one of the forms \( \varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_{d^2} \) is isotropic. But then one of these forms is isotropic over \( \hat{R}_P \), hence over the field \( F_P \) which is the fraction field of \( \hat{R}_P \).

\[ \square \]

**Remark 3.2.** (i) In the proof of Theorem 3.1, one of the forms is isotropic over \( \hat{R}_P \) and since \( R_y \) is the complete discrete valuation ring which is the completion of the localisation of \( \hat{R}_P \) at the height one prime ideal corresponding to \( U \) when \( p = (U, P) \), this form is also isotropic over \( R_y \) and therefore over the field of fractions \( F_p \) of \( \hat{R}_P \). This implies that if \( \varphi \) is isotropic over the completion of \( F \) with respect to each discrete valuation of \( F \), then \( \varphi \) is isotropic over \( F_U \) for each reduced irreducible component \( U \subset Y \) of the complement of \( S \) in \( Y \), over \( F_P \) for each \( P \in S \) and over \( R_y \) for each \( p = (U, P) \). Since \( F \) is the inverse limit of the finite inverse system of
fields \{F_U, F_P, F_p\}, \varphi is isotropic over all overfields used in the inverse limit.

(ii) The discrete valuation rings used in the above proof are the local rings at a point of codimension 1 on a suitable regular proper model \(X\) of \(X\) determined by the choice of \(\varphi\) (analogously as noted in [2, Remark 3.2]).

Given a nondegenerate diagonal form \(\varphi\) of degree \(d\) and dimension greater than two over \(F\), it is not clear whether the isotropy of \(\varphi\) over \(F_v\) for each \(v\) (respectively, of \(\varphi\) over all \(F_U, F_P\) and \(F_p\)) implies that \(\varphi\) is isotropic (the fact that \(\dim \varphi > 2\) is necessary: it is easy to adjust the example in [2, Appendix] to two-dimensional diagonal forms of even degree).

**Corollary 3.3.** Let \(r \geq 1\) be an integer and \(d \geq 3\). Assume that any diagonal form in strictly more than \(dr\) variables over any function field in one variable over \(k\) is isotropic. Then:

(i) Any diagonal form of degree \(d\) and dimension \(> d^2r\) over the function field \(F = K(X)\) of a curve \(X/K\) is isotropic over \(F_v\), for every discrete valuation \(v\) with residue field either a function field in one variable over \(k\) or a finite extension of \(K\).

(ii) Any diagonal form of degree \(d\) and dimension \(> d^2r\) over the function field \(F = K(X)\) of a curve \(X/K\) is isotropic over \(F_U\) for each reduced, irreducible component \(U \subset Y\) of the complement of \(S\) in \(Y\) and is isotropic over \(F_P\) for each \(P \in S\).

Note that \(Y\) and \(S\) depend on \(\varphi\).

**Proof.** (i) Let \(L\) be a finite field extension of \(K\). This is a complete discretely valued field with residue field a finite extension \(\ell\) of \(k\). The assumption made on diagonal forms of degree \(d\) over functions fields in one variable over \(k\), in particular diagonal forms of degree \(d\) over the field \(\ell(t)\), and Theorem 2.1 applied to \(\ell((t))\) show that any diagonal form of dimension \(> r\) over \(\ell\) has a zero. A second application of Theorem 2.1 yields that any diagonal form of degree \(d\) of dimension \(> dr\) over \(L\) is isotropic. Let \(\varphi\) be a diagonal form of dimension \(n\) over \(F\) with \(n > d^2r\). By the assumption and Theorem 2.1, \(\varphi\) is isotropic over \(F_v\) for every discrete valuation \(v\) with residue field either a function field in one variable over \(k\) or a finite extension of \(K\).

(ii) follows from Theorem 3.1. \(\square\)

This shows that trying to extend [2, Corollary 3.4] from quadratic to diagonal forms of higher degree results in a much weaker version.

### 3.2.

Let \(K\) be a \(p\)-adic field with residue field \(k\) such that \(\text{char}(k) \nmid d!\).

**Corollary 3.4.** Any diagonal form of degree \(d\) and dimension \(> d^3 + 1\) over a function field in one variable \(F = K(t)\) is
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(i) isotropic over $F_v$, for every discrete valuation $v$ with residue field either a function field in one variable over $k$ or a finite extension of $K$;
(ii) isotropic over $F_U$ for each reduced, irreducible component $U \subset Y$ of the complement of $S$ in $Y$ and isotropic over $F_P$ for each $P \in S$.

Proof. Every finite field $k$ is $C_1$ and so every diagonal form of degree $d$ and dimension $> d^2$ over any function field in one variable over $k$ (which is $C_2$) is isotropic. Assertion (i) is a direct consequence of Theorem 2.1 and (ii) follows from Corollary 3.3 (ii).

So if $\varphi$ is a diagonal form of degree $d$ in at least $d^3 + 1$ variables over $\mathbb{Q}(t)$ then $\varphi$ is isotropic over $(\mathbb{Q}_p(t))_v$ for any $p \nmid d!$, for each reduced, irreducible component $U \subset Y$ of the complement of $S$ in $Y$, and isotropic over $(\mathbb{Q}_p(t))_P$ for each $P \in S$.

Remark 3.5. Let us compare Corollary 3.4 with the Ax-Kochen-Ersov Transfer Theorem [1]: given a degree $d$, for almost all primes $p$, a form of degree $d$ over $\mathbb{Q}_p$ of dimension greater than or equal to $d^2 + 1$ is isotropic [G, (7.4)]. Moreover, for any form $\varphi$ of degree $d \geq 2$ and dimension greater than $d^3$ over $\mathbb{Q}(t)$, for almost all primes $p$ the form $\varphi$ is isotropic over $\mathbb{Q}_p(t)$ ([16] for $d = 2$, [12] for $d \geq 3$). The model-theoretic proofs of both results do not allow for a more concrete observation on which primes exactly are included here, nor can they be extended to other base fields.

Stronger upper bounds on $u_{\text{diag}}(d, K)$ will yield stronger results on its dimension, since we only used the upper bound in the well known inequality $d \cdot u_{\text{diag}}(d, K) \leq u_{\text{diag}}(d, \mathbb{F}_q(t)) \leq d^2$ to prove Corollary 3.4, for instance we obtain:

Corollary 3.6. Assume that $u_{\text{diag}}(d, k(t)) = dr < d^2$ for some $r \in \{1, \ldots, d - 1\}$. Let $\varphi$ be a diagonal form of degree $d$ and dimension $> d^2 r + 1$ over a function field in one variable $F = K(t)$. Then:
(i) $\varphi$ is isotropic over $F_v$, for every discrete valuation $v$ with residue field either a function field in one variable over $k$ or a finite extension of $K$;
(ii) $\varphi$ is isotropic over $F_U$ for each reduced, irreducible component $U \subset Y$ of the complement of $S$ in $Y$ and over $F_P$ for each $P \in S$.

It is well known that $u_{\text{diag}}(d, K) \leq d^2$ for a $p$-adic field $K$ with residue field $k = \mathbb{F}_q$ [3]. Indeed, $u_{\text{diag}}(d, K) = d u_{\text{diag}}(d, \mathbb{F}_q)$ by Theorem 2.1, assuming that $\text{char } \mathbb{F}_q = p \nmid d!$ as before, which shows that clearly $u_{\text{diag}}(d, K)$ can be smaller than $d^2$. On the other hand, Artin’s conjecture that $\mathbb{Q}_p$ is a $C_2$-field is false for instance for forms of degree 4.

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