Evolutionary Laws, Initial Conditions, and Gauge Fixing

in Constrained Systems

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Abstract. We describe in detail how to eliminate nonphysical degrees of freedom in the Lagrangian and Hamiltonian formulations of a constrained system. Two important and distinct steps in our method are the fixing of ambiguities in the dynamics and the determination of inequivalent initial data. The Lagrangian discussion is novel, and a proof is given that the final number of degrees of freedom in the two formulations agrees. We give applications to reparameterization invariant theories, where we prove that one of the constraints must be explicitly time dependent. We illustrate our procedure with the examples of trajectories in spacetime and with spatially homogeneous cosmological models. Finally, we comment briefly on Dirac’s extended Hamiltonian technique.

Short Title: Evolutionary Laws and Gauge Fixing

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1 Introduction

Dynamical theories exhibiting gauge freedom are described by singular Lagrangians or in the Hamiltonian formalism introduced by Dirac [1,2]. The existence of constraints reduces the true degrees of freedom in the system. This reduction has specific features depending upon whether we are in velocity space (Lagrangian formalism) or in phase space (Hamiltonian formalism). In particular, the dimensions of the constraint surfaces are different: One can prove [3] that the number of constraints in the Hamiltonian formalism equals the number of constraints in the Lagrangian formalism plus the number of independent gauge transformations. In other words, the number of degrees of freedom in phase space seems to be less than the number in velocity space, but getting rid of the gauge freedom in the phase and velocity spaces eliminates spurious degrees of freedom. We will show that therefore the number of degrees of freedom are the same in the two formalisms.

There are two different stages in the reduction of the degrees of freedom. The first is provided by the constraints that naturally arise in the formalism by pure consistency requirements. The second corresponds to the gauge fixing procedure. In our case it will consist in the introduction of new constraints in such a way that the gauge degrees of freedom are thoroughly eliminated.

In [4] an analysis was carried out on how to implement a gauge fixing procedure in the Hamiltonian formalism to obtain the true number of physical degrees of freedom. In the present paper we will proceed to general Lagrangian gauge fixing in a way completely independent from the Hamiltonian method.

We will emphasize the double role played by the gauge fixing procedure. Some constraints are needed to fix the time evolution of the gauge system, which was undetermined to a certain extent. The rest of the constraints eliminate the spurious degrees of freedom that are still present in the setting of the initial conditions of the system. This double role may be relevant for quantization (we treat only classical systems here). It has not been adequately emphasized in the literature.

The gauge fixing procedures in the Hamiltonian and Lagrangian formalisms are technically different. For instance, there is no Poisson Bracket available in velocity space for singular Lagrangians, and the relationship between velocity and position in the tangent bundle does not carry over to the cotangent bundle. This difference is why there are difficulties in implementing a Lagrangian gauge fixing procedure which is independent of a pullback of the Hamiltonian one. We here show how to fix these difficulties and how to perform in full generality gauge fixing in both formalisms. Hence our results will include a general proof of the matching of the degrees of freedom in Hamiltonian and Lagrangian formalisms for gauge theories.

Our paper is organized as follows: In section 2 we develop a detailed version of the Hamiltonian gauge fixing procedure (with some improvements to [4]). In section 3 we study the Lagrangian gauge fixing procedure and establish the theorem that the number of degrees of freedom in the Hamiltonian and Lagrangian
formalisms are equal. In section 4 we describe some details concerning reparameterization invariant theories, since in section 5 we apply our results to two such theories. These examples are the parameter-independent Lagrangian for geodesics in special relativity and the Types I and IX spatially homogeneous cosmologies, and we will emphasize the role of time-dependent gauge fixing conditions.

Throughout the paper we will assume that some regularity conditions are fulfilled: The Hessian matrix of the Lagrangian with respect to the velocities has constant rank, ineffective constraints (such that their gradient vanishes on the constraint surface) do not appear, and also the rank of the Poisson Bracket matrix of constraints remains constant in the stabilization algorithm (so that a second class constraint can never become first class by adding new constraints to the theory).

We emphasize that we always maintain the equivalence between the Lagrangian and the Hamiltonian formalisms [5]. This equivalence holds even before the implementation of the gauge fixing conditions; in particular we do not modify the Hamiltonian formalism by adding ad hoc constraint terms as Dirac has proposed [2,6,7]. This proposal has been proved to be unnecessary under our regularity conditions [4]. We will discuss this point more fully in the Conclusion.

2 Hamiltonian Gauge Fixing Procedure

We start with a canonical formalism using Dirac’s method, starting from a singular Lagrangian $L(q^i, \dot{q}^i)$ ($i = 1, \cdots, N$) ($\dot{q}^i = dq^i/dt$). The functions $\hat{p}_i(q, \dot{q}) = \partial L/\partial \dot{q}^i$ are used to define the Hessian $W_{ij} = \partial \hat{p}_i/\partial \dot{q}_j$, a matrix with rank $N - P$ (we assume this rank is constant), $P$ being the number of primary constraints. The Legendre map from velocity space (tangent bundle for configuration space) $TQ$ to phase space (cotangent bundle) $T^*Q$ defined by $p_i = \hat{p}_i(q, \dot{q})$ defines a constraint surface of dimension $2N - P$.

The function $E_L := \hat{p}_i \dot{q}^i - L$ in velocity space (the so-called energy function) is mapped to a function on the constraint surface, and in phase space a canonical Hamiltonian $H_C$ may be defined which agrees with this function on the surface. $H_C$ is not unique, and to it may be added a linear combination $\lambda^\rho \psi_\rho(q, p)$ of constraint functions $\psi_\rho(\rho = 1, \cdots, P)$, the $\lambda^\rho$ being arbitrary functions of time $t$ and the vanishing of $\psi_\rho(q, p)$ defining the primary constraint surface. These primary constraints may be chosen so that some are first class (their Poisson Brackets with all the constraints weakly vanish, that is, vanish on the constraint surface) and some are second class (the matrix of their Poisson Brackets with each other is nonsingular—there must be an even number of second class constraints or none).

Let $\psi_1, \psi_2$ denote first and second class primary constraints, $\lambda_1, \lambda_2$ being the respective $\lambda$ functions. The time derivatives of the second class constraints yield

$$\dot{\psi}_2^\rho = \{\psi_2^\rho, H_C\} + \lambda_2^\rho \{\psi_2^\rho, \psi_2^\sigma\},$$
equations which may be solved for the $\lambda^2_\sigma$ by requiring $\dot{\psi}^2_\rho = 0$. These functions $\lambda^2_\sigma$ are then inserted into the expression $H_C + \lambda^1_\rho \psi^1_\rho + \lambda^2_\rho \psi^2_\rho$ to yield a new candidate for the Hamiltonian:

$$H_C + \lambda^1_\rho \psi^1_\rho + \lambda^2_\rho \psi^2_\rho = H^1_C + \lambda^1_\rho \psi^1_\rho.$$ 

The time derivatives of the primary first class constraints yield

$$\dot{\psi}^1_\rho = \{\psi^1_\rho, H_C\},$$

which can either be zero on the constraint surface or not. In the latter case, new constraints are found. The time derivatives of these new constraints will involve their Poisson Brackets with $H^1_C$ and with the $\psi^1_\rho$ (and will also involve their partial time derivatives). The requirement that these time derivatives vanish will be equations, some of which can be solved for some of the $\lambda^1_\rho$ functions. The rest of these equations may yield more constraints, and the process of requiring the vanishing of their time derivatives is repeated at a deeper level.

This brief description of the stabilization algorithm is not meant to be rigorous, but the process does eventually finish. Once the stabilization algorithm has been performed, we end up with [2,5,8]:

1: A certain number, $M$, of constraints. These constraints may be more numerous than the ones introduced above ($M \geq P$), but they are arranged into first and second classes. The first class constraints have weakly vanishing Poisson Brackets with all the constraints, and the matrix of the second class constraint Poisson Brackets is nonsingular. These constraints restrict the dynamics to a constraint surface within $T^*Q$ of dimension $2N - M$.

2: A dynamics (with some gauge arbitrariness) on the constraint surface which is generated, through Poisson Brackets, by the so called Dirac Hamiltonian:

$$H_D := H_{FC} + \lambda^\mu \phi^1_\mu.$$ 

$H_{FC}$ is the first class Hamiltonian, obtained by adding to the canonical Hamiltonian $H_C$ pieces linear in the primary second class constraints. $\phi^1_\mu (\mu = 1, \cdots, P_1)$ are the primary (hence the superscript 1) first class constraints. The secondary and higher first class constraints, obtained from the time derivatives of the $\phi^1_\mu$, are not used here. The $\lambda^\mu$ are arbitrary functions of time (or spacetime in field theories).

3: A certain number ($P_1$) of independent gauge transformations generated, through Poisson Brackets,

$$\delta^i_\mu q^i = \{q^i, G_\mu\}, \quad \delta^i_\mu p_i = \{p_i, G_\mu\},$$

by functions $G_\mu (\mu = 1, \cdots, P_1)$ which have the following form [4,9,10,11]:

$$G_\mu = \epsilon^i_\mu \phi^{K^i} + \epsilon^{(1)}_\mu \phi^{(K^i)^{-1}} + \epsilon^{(2)}_\mu \phi^{(K^i)^{-2}} + \cdots + \epsilon^{(K^i^{-1})}_\mu \phi^{1},$$
where \( \epsilon_\mu \) is an arbitrary infinitesimal function of time; \( \epsilon^{(r)}_\mu \) is its \( r \)-th time derivative; \( K_\mu \) is the length of the stabilization algorithm for the primary first class constraint \( \phi^{1}_\mu \); and \( \phi^{2}_\mu, \ldots, \phi^{K}_\mu \) are secondary through \( K_\mu \)-ary, first class constraints. It turns out [12] that one can take these gauge generators in such a way that all the first class constraints are involved once and only once in the \( G_\mu \), and so their total number equals

\[
F := \sum_{\mu=1}^{P_1} K_\mu .
\]

Now we are ready for the gauge fixing procedure. Even though the order of introducing the gauge fixing constraints is irrelevant, we will proceed in the way that makes the whole procedure more illuminating from the theoretical point of view. As we said in the Introduction, we will distinguish two different steps in the gauge fixing procedure, corresponding to evolutionary laws and initial conditions [4]. In the first step we fix the laws of evolution, which otherwise have a certain amount of mathematical arbitrariness. In the second step we eliminate the redundancy of initial conditions that are physically equivalent.

The arbitrariness in the dynamics is represented by the \( P_1 \) functions \( \lambda_\mu \). To get rid of this arbitrariness, we introduce a set of \( P_1 \) constraints \( \chi^{1}_\mu \simeq 0 \) (\( \mu = 1, \ldots, P_1 \)), defined so that their own stability equations, under dynamical evolution, will determine the functions \( \lambda_\mu \). To this end we must require that the matrix

\[
C_{\mu\nu} := \{ \chi^{1}_\mu, \phi^{1}_\nu \}
\]

be non-singular. The conservation in time of this new set of constraints leads to

\[
\dot{\chi}^{1}_\mu = 0 = \frac{\partial \chi^{1}_\mu}{\partial t} + \{ \chi^{1}_\mu, H_{FC} \} + \lambda_\nu \{ \chi^{1}_\mu, \phi^{1}_\nu \},
\]

which determines \( \lambda_\nu \) as

\[
\lambda_\nu = -(C^{-1})^{\nu\mu}(\{ \chi^{1}_\mu, H_{FC} \} + \frac{\partial \chi^{1}_\mu}{\partial t}).
\]

The dynamical evolution thus becomes completely determined. The imposition of these constraints causes the dynamics to be further restricted to the \( (2N-M-P_1) \)-dimensional constraint surface defined by \( \chi^{1}_\mu = 0.\)

The gauge fixing procedure is not yet finished. It is necessary to address the issue of initial conditions, which we call “point gauge equivalence,” our second step. Let us clarify this crucial point: Even though the dynamics has now been fixed, there is still the possibility of gauge transformations which take one trajectory into another. To check whether these gauge transformations do exist, we need only check their action at a specified time. That is, the points on the set of trajectories at a specified time are unique initial data for the trajectories. If a gauge transformation exists which relates two initial data points, then these two points are physically equivalent. We will obtain the generators of the transformations which take initial points into equivalent ones (“point gauge transformations”) and use them to fix the gauge finally.
Consider the gauge generators given above at, say for simplicity, \( t = 0: G_{\mu}(0) \). The most arbitrary point gauge transformation at \( t = 0 \) will be generated by \( G(0) = \sum_{\mu=1}^{P_1} G_{\mu}(0) \). The arbitrary functions \( \epsilon_\mu \) and their derivatives become, at the given time \( t = 0 \), (infinitesimal) independent arbitrary parameters (there are \( F \) in number). We redefine them as 

\[
\alpha_{\mu,i_\mu} := \epsilon_{\mu}^{(K_\mu - i_\mu)}(0); \ (\mu = 1, \cdots, P_1) \ (i_\mu = 1, \cdots, K_\mu).
\]

These point gauge transformation generators must be consistent with the new constraints \( \chi^{1}_{\mu} \). This requirement introduces relations among the \( \alpha_{\mu,i_\mu} \):

\[
0 = \{ \chi^{1}_{\nu}, G(0) \} = \{ \chi^{1}_{\nu}, \sum_{\mu=1}^{P_1} \sum_{i_\mu=1}^{K_\mu} \alpha_{\mu,i_\mu} \phi^{i_\mu}_{\mu} \} = \sum_{\mu=1}^{P_1} \left( \sum_{i_\mu=2}^{K_\mu} \alpha_{\mu,i_\mu} \{ \chi^{1}_{\nu}, \phi^{i_\mu}_{\mu} \} + \alpha_{\mu,1} C_{\nu\mu} \right).
\]

Remember that the matrix \( C_{\nu\mu} = \{ \chi^{1}_{\nu}, \phi^{1}_{\mu} \} \) is nonsingular. These relations imply

\[
\alpha_{\nu,1} = -(C^{-1})_{\nu\rho} \sum_{\mu=1}^{P_1} \sum_{i_\mu=2}^{K_\mu} \alpha_{\mu,i_\mu} \{ \chi^{1}_{\nu}, \phi^{i_\mu}_{\mu} \}.
\]

As a consequence, the independent point gauge generators are

\[
\tilde{\phi}^{i_\mu}_{\mu} := \phi^{i_\mu}_{\mu} - \phi^{1}_{\mu} (C^{-1})_{\nu\rho} \{ \chi^{1}_{\nu}, \phi^{i_\mu}_{\mu} \} \ (\mu = 1, \cdots, P_1) \ (i_\mu = 2, \cdots, K_\mu).
\]

Notice that new point gauge generators only exist when there are secondary first-class constraints, that is, when the length \( K_\mu \) of at least one of the \( G_{\mu} \) is greater than one.

Recall that \( F \) is the number of first class constraints in the original theory, including primary, secondary, and higher constraints; we conclude that there are \( F - P_1 \) generators that relate physically equivalent initial conditions. To eliminate the extraneous variables, we will select a unique representative of each equivalence class by introducing a new set of \( F - P_1 \) gauge fixing constraints, \( \chi^{i_\mu}_{\nu} \simeq 0 \ (\mu = 1, \cdots, P_1) \ (i_\mu = 2, \cdots, K_\mu) \), such that

\[
\det |\{ \chi^{i_\mu}_{\nu}, \tilde{\phi}^{j_\nu}_{\nu} \}| \neq 0, \ i_\mu \neq 1, \ j_\nu \neq 1,
\]

in order to prevent any motion generated by \( \tilde{\phi}^{j_\nu}_{\nu} \). The stability requirement is

\[
\frac{\partial}{\partial t} \chi^{i_\mu}_{\nu} + \{ \chi^{i_\mu}_{\nu}, H_D \} \simeq 0,
\]

which \( \simeq 0 \) means vanishing on the constraint surface; this requirement simply dictates how the \( \chi^{i_\mu}_{\nu} \) evolve off the initial data surface.

Notice that we have explicitly allowed time dependence in the \( \chi^{i_\mu}_{\nu} \) constraints. In fact, time dependence is necessary in the special case when \( H_{FC} \) is a constraint (first class, of course). This point will be clarified in Section 4.
This ends the gauge fixing procedure. Now we can count the physical number of degrees of freedom: The \(M\) constraints left after the stabilization algorithm restricted motion to a \(2N - M\)-dimensional surface in \(T^*Q\). The gauge fixing constraints needed to fix the evolutionary equations number \(P_1\). Finally there are \(F - P_1\) point gauge fixing constraints needed to select physically inequivalent initial points. (The total number of gauge fixing constraints equals the number \(F\) of first class constraints in the original theory.) The final number of degrees of freedom is \(2N - M - F\). Notice that \(M - F\) is the original number of second class constraints and is therefore even. Consequently, \(2N - M - F\) is even; this result agrees with the fact that the above procedure makes all constraints into second class ones, and in this case the constraint surface is symplectic [13].

### 3 Lagrangian Gauge Fixing Procedure

We first use a stabilization algorithm similar to the one used in the Hamiltonian formalism. (In the equations below, we use the summation convention for configuration space indices \(i = 1, \ldots, N\).) The equations of motion obtained from the Lagrangian \(L\) are (assuming for simplicity no explicit time dependence):

\[
W_{is} \dddot{q}^s = \alpha_i ,
\]

where

\[
W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} ; \quad \alpha_i = - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^s} \dot{q}^s + \frac{\partial L}{\partial q^i} .
\]

If \(W_{ij}\) is singular, it possesses \(P\) null vectors \(\gamma^i_\rho\), giving up to \(P\) (these relations may not be independent) constraints

\[
\alpha_i \gamma^i_\rho \simeq 0 .
\]

It is easily shown that there exists at least one \(M^\gamma\) and \(\tilde{\gamma}^\rho_i\) such that

\[
W_{is} M^{\gamma^i} = \delta^i_j + \tilde{\gamma}^\rho_i \gamma^i_\rho ,
\]

and therefore [14]

\[
\dddot{q}^i = M^{\gamma^i} \alpha_s + \tilde{\eta}^\rho_i \gamma^i_\rho \quad \text{(with } \tilde{\eta}^\rho = \tilde{\gamma}^\rho_i \dot{q}^i) ,
\]

where \(\tilde{\eta}^\rho\) are arbitrary functions of \(t\).

The stabilization algorithm starts by demanding that time evolution preserve the \(\alpha_i \gamma^i_\rho\) constraints. Sometimes new constraints are found; sometimes some of the \(\tilde{\eta}^\rho\) are determined; eventually the dynamics is described by a vector field in velocity space

\[
X := \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial \dot{q}^i} + a^i(q, \dot{q}) \frac{\partial}{\partial q^i} + \eta^\mu \Gamma_\mu =: X_0 + \eta^\mu \Gamma_\mu ;
\]

the \(a^i\) are determined from the equations of motion and the stabilization algorithm; \(\eta^\mu\) \((\mu = 1, \ldots, P_1)\) are arbitrary functions of time; and

\[
\Gamma_\mu = (1) \gamma^i_\mu \frac{\partial}{\partial \dot{q}^i} ,
\]
where \( (1) \gamma^i_\mu \) are a subset of the null vectors of \( W_{ij} \), corresponding to the primary first class constraints found in the Hamiltonian formalism. It is not necessary to use the Hamiltonian technique to find the \( \Gamma_\mu \), but it does facilitate the calculation:

\[
(1) \gamma^i_\mu = \frac{\partial \phi^1_\mu}{\partial p_i}(q, \hat{p}) ,
\]

where the \( \phi^1_\mu \) are the primary first class constraints, and \( \hat{p}_i \) stands for the Lagrangian definition of the momenta \( \hat{p}_i = \frac{\partial L}{\partial \dot{q}^i} \). The \( P_1 \) number of \( \eta^\mu \) is the same number as in the Hamiltonian formalism. There are left \( M - P_1 \) constraints [3].

At this point it is useful to appeal to the Hamiltonian formalism for the computation of the \( P_1 \) independent gauge transformations. The result includes the definitions of the functions \( \phi^0_\mu \), and then in the Lagrangian formalism we define

\[
f^i_{\mu,j_\nu}(q, \dot{q}) := \frac{\partial \phi_\mu^0}{\partial p_i}(q, \hat{p}) .
\]

These functions give the infinitesimal Lagrangian gauge transformations as

\[
\delta_{\mu}q^i = \sum_{j_\nu=1}^{K_\nu} \epsilon^{(K_\nu - j_\nu)}(q, \dot{q}) f^i_{\mu,j_\nu} ,
\]

the \( \epsilon_\mu \) being arbitrary functions of time.

As we did in the Hamiltonian formalism, the first step in the gauge fixing procedure will be to fix the dynamics to determine the arbitrary functions \( \eta^\mu \). To this end we introduce \( P_1 \) constraints, \( \omega^0_\nu \simeq 0 \), such that \( D_{\mu\nu} := \Gamma_\mu \omega^0_\nu \) has non-zero determinant: \( \det |\Gamma_\mu \omega^0_\nu| = \det |D_{\mu\nu}| \neq 0 \). Then the functions \( \eta_\mu \) become determined by requiring the stability of these new constraints:

\[
0 = X\omega^0_\nu = X_0\omega^0_\nu + \eta^\mu \Gamma_\mu \omega^0_\nu .
\]

This relation gives

\[
\eta^\mu = -(D^{-1})^\nu_\mu (X_0\omega^0_\nu) ,
\]

which determines the dynamics as

\[
X_F = X_0 - (X_0\omega^0_\nu)(D^{-1})^\nu_\mu \Gamma_\mu .
\]

Although the time evolution is fixed, as in the previous section there still remain some point gauge transformations in the constraint surface that we should get rid of. Again, these transformations may be thought as affecting the space of initial conditions. In fact, we can extract those transformations at \( t = 0 \) that preserve the gauge fixing constraints \( \omega^0_\nu \simeq 0 \) from the general gauge transformations. This general transformation is

\[
\delta \omega^0_\nu = \sum_{\mu=0}^{P_1} \delta_\mu \omega^0_\nu = \sum_{\mu=0}^{P_1} \left( \frac{\partial \omega^0_\nu}{\partial q^i} \delta_\mu q^i + \frac{\partial \omega^0_\nu}{\partial \dot{q}^i} \delta_\mu \dot{q}^i \right) ,
\]
where now
\[ \delta q^i = \frac{d}{dt} \delta q^i = F_\mu \delta q^i + \frac{\partial \delta q^i}{\partial t}. \]

In this expression we use the values at \( t = 0 \):
\[
\begin{align*}
\delta_\mu q^i(0) & = \sum_{\mu=1}^{K_\mu} \epsilon_{\mu}^{(K_\mu-i_\mu)}(0) f^i_{\mu,i_\mu}, \\
\frac{\partial \delta_\mu q^i}{\partial t}(0) & = \sum_{\mu=1}^{K_\mu} \epsilon_{\mu}^{(K_\mu-i_\mu+1)}(0) f^i_{\mu,i_\mu+1}.
\end{align*}
\]

after redefining \( \alpha_{\mu,i_\mu} := \epsilon_{\mu}^{(K_\mu-i_\mu)}(0) \) (we have defined \( f^i_{\mu,K_\mu+1} = 0 \)).

Notice that now, due to the presence of the time derivative of \( \delta_\mu q^i \), \( i_\mu \) runs from 0 to \( K_\mu \). This is a key difference with respect to the Hamiltonian case, where the \( \alpha_{\mu,i_\mu} \) parameters had indices \( i_\mu \) running from 1 to \( K_\mu \). We call the result for the independent point gauge transformations (at \( t = 0 \)) \( \delta(0)\omega_\nu^0 \):
\[
\delta(0)\omega_\nu^0 = \sum_{\mu=1}^{P_1} \left( \sum_{i_\mu=1}^{K_\mu} \left( \frac{\partial \omega_\mu^0}{\partial q^i} f^i_{\mu,i_\mu} + \frac{\partial \omega_\mu^0}{\partial q^i} (X_F f^i_{\mu,i_\mu}) + \frac{\partial \omega_\mu^0}{\partial q^i} f^i_{\mu,i_\mu+1} \right) \alpha_{\mu,i_\mu} + \alpha_{\mu,0} \Gamma^0_\mu \right),
\]

where \( f^i_{\mu,1} = \gamma^i_\mu \). At this point, recalling that \( \det [\Gamma_\mu \omega_\nu^0] \neq 0 \), we see that the stability conditions \( \delta(0)\omega_\nu^0 = 0 \) allow the determination of \( \alpha_{\mu,0} \) in terms of \( \alpha_{\mu,i_\mu} \) (\( \mu = 1, \ldots, P_1 \) \( i_\mu = 1, \ldots, K_\mu \)).

We conclude that the independent point gauge transformations \( \delta(0) \) that still remain, relating physically equivalent initial conditions, are parameterized by \( \alpha_{\mu,i_\mu} \) (\( \mu = 1, \ldots, P_1 \) \( i_\mu = 1, \ldots, K_\mu \)). Their number equals \( F \), the total number of first class constraints in the Hamiltonian theory. To eliminate these transformations we introduce \( F \) new gauge fixing constraints \( \omega_\mu^{i_\mu} \simeq 0 \) (\( \mu = 1, \ldots, P_1 \) \( i_\mu = 1, \ldots, K_\mu \)), with the conditions:

1: The system \( \delta(0)\omega_\mu^{i_\mu} = 0 \), which is linear in the \( \alpha_{\mu,i_\mu} \) (\( \mu = 1, \ldots, P_1 \) \( i_\mu = 1, \ldots, K_\mu \)) has only the solution \( \alpha_{\mu,i_\mu} = 0 \) (so that no point gauge transformations are left).

2: \( X_F(\omega_\mu^{i_\mu}) \simeq 0 \) (the requirement of stability under evolution).

Now we have completed the gauge fixing procedure. For reasons similar to the ones raised in the Hamiltonian formalism, there are cases where a time dependent constraint shows up necessarily. Our examples in section 5 will be two of these cases, and we discuss these cases in the next section.

Summing up, the gauge fixing constraints introduced in velocity space (that is, in the Lagrangian formalism) are \( \omega_\mu^{i_\mu} \) (\( \mu = 1, \ldots, P_1 \) \( i_\mu = 0, \ldots, K_\mu \)). Their number is \( F + P_1 \), and therefore the total number of constraints becomes \( (M - P_1) + (F + P_1) = M + F \). The number of degrees of freedom is then \( 2N - M - F \). Comparison with the results of the previous section shows that we have proved
Theorem. The number of physical degrees of freedom in constrained Hamiltonian and Lagrangian formalisms is the same.

Observe that this result, which was obviously expected on physical grounds, is nontrivial. In fact, before introducing the gauge fixing constraints, the dimensions of the constraint surface were different in the two formalisms. This means that the gauge fixing procedure has to make up for this difference—and we see that it does.

4 Reparameterization Invariant Theories

Reparameterization invariant theories provide interesting cases for the application of the preceding sections. Examples of this kind, including spatially homogeneous cosmologies of Types I and IX, will be treated in the next section.

If we consider the infinitesimal reparameterization \( t \to t' = t - \epsilon(t) \), with \( \epsilon \) an arbitrary (infinitesimal) function, the trajectories \( q^i(t) \) (any trajectory, not necessarily solutions of the equations of motion) change accordingly, \( q^i(t) \to q'^i(t') \). If we define the functional infinitesimal transformation \( \delta q^i = q'^i(t) - q^i(t) \), the transformations we will consider are of the type \( \delta q^i = \epsilon \dot{q}^i + h^i \), where the \( h^i \) terms involve time derivatives of \( \epsilon \). The theory is reparameterization-invariant when the Lagrangian remains form-invariant under these changes:

\[
L(q(t), \dot{q}(t)) dt = L(q'(t'), \dot{q}'(t')) dt'.
\]

In such a case, we define

\[
\delta L := \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i.
\]

It is easy to check that \( \delta L \) is a total derivative:

\[
\delta L = \frac{d}{dt}(\epsilon L).
\]

This equality can be transformed into

\[
[L]_i \delta q^i + \frac{d}{dt} G = 0,
\]

where \([L]_i\) are the Euler-Lagrange derivatives and where \( G \) is the conserved quantity of Noether’s theorem:

\[
[L]_i = \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i, \quad G = \frac{\partial L}{\partial \dot{q}^i} \delta q^i - \epsilon L.
\]

\( G \) can be expanded as a sum of an \( \epsilon \) term and terms involving derivatives of \( \epsilon \). Each of these terms must be a constraint of the theory because \( G \) is a constant of motion for whatever arbitrary values we give to \( \epsilon \) (this is a general argument for gauge symmetries). Using the form of \( \delta q^i \) introduced above, we obtain, for the \( \epsilon \) coefficient of \( G \),

\[
\frac{\partial L}{\partial \dot{q}^i} \delta q^i - L = E_L,
\]
namely the Lagrangian energy function $E_L$. Its corresponding canonical quantity is the canonical Hamiltonian $H_C$; therefore we have stated the following:

**Theorem.** The canonical Hamiltonian (if it is non-zero) in a reparameterization-invariant theory is a constraint.

In general $H_C$ will be a secondary, first class constraint, but a particular case is worth mentioning: When all the configuration space variables transform as $\delta q^i = \epsilon \dot{q}^i$ ("scalars"), then $G$ becomes $G = \epsilon E_L$. The relation $[L]_i \delta q^i + \frac{d}{dt} G = 0$ gives

$$[L]_i \epsilon \dot{q}^i + \epsilon \frac{d}{dt} E_L + \epsilon E_L = 0.$$  

Since this relation is identically zero for any function $\epsilon$, we conclude that the coefficient of $\dot{\epsilon}$, namely $E_L$, is identically zero. Therefore the Lagrangian is homogeneous of first degree in the velocities: The canonical Hamiltonian vanishes in this case. The remaining pieces tell us that $\dot{q}^i$ is a null vector of the Hessian matrix of the Lagrangian, and in case this is the only null vector, that there are no Lagrangian constraints. This situation occurs exactly in the case of the relativistic free particle, which is described by the Lagrangian $L = \sqrt{\dot{x}^\mu \dot{x}_\mu}$ and which will be treated in the next section.

Now we will prove another result for reparameterization-invariant theories, the need for time dependence in some gauge fixing constraint. Suppose that $H_C$ vanishes. Then to fix the dynamics (that is, to determine functions $\lambda^\mu$ in $H_D$ which do not all vanish) by using the conditions $\dot{\chi}_\mu^1 = 0$, it is necessary that at least one of the constraints $\chi_\mu^1$ have explicit time dependence. If $H_C$ doesn’t vanish, then it is a constraint; the first class Hamiltonian, $H_{FC}$, will necessarily be a first class constraint of the original theory. After the first step of the gauge fixing procedure (in which the dynamics is determined), the final Hamiltonian $H_D$ will be a first class constraint that generates motions tangent to the first-step gauge fixing surface.

This latter result means that $H_D$ will become a part of $G(0)$, the generator of point gauge transformations which relate physically equivalent initial conditions. Then, in order to fulfill the two requirements introduced in the second step of the gauge fixing procedure, it is mandatory that at least one of the gauge fixing constraints be time-dependent: Otherwise there is no way to satisfy the gauge fixing conditions. By choosing variables appropriately, we can always end up with only one time-dependent gauge fixing constraint. Therefore we have proved:

**Theorem.** Reparameterization-invariant theories necessarily require that one of the gauge fixing constraints be time-dependent.

This result is clearly expected from the physical interpretation of this kind of theory: The existence of
reparameterization invariance as a gauge symmetry implies that the evolution parameter —the “time”—is an unphysical variable.

5 Examples

We will first discuss the case of a relativistic free particle and then spatially homogeneous cosmological models of Bianchi Types I and IX.

One Lagrangian for a free particle in special relativity is (τ is the path parameter)

\[ L = \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \]

where \( \dot{\cdot} \) means \( d/d\tau \). (For convenience we take \( \eta_{\mu\nu} = \text{diag}(1,-1,-1,-1) \).) The action integral \( \int Ld\tau \) is invariant under arbitrary reparameterizations, so \( \tau \) is not necessarily proper time. The conjugate momenta functions in velocity space are

\[ \hat{p}_\mu = \frac{\eta_{\mu\nu} \dot{x}^\nu}{\sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}, \]

and therefore the velocity-space energy function is

\[ E_L = \hat{p}_\mu \dot{x}^\mu - L = 0. \]

First, we examine this system from the Lagrangian point of view. The definition of \( \hat{p}_\mu \) implies

\[ \eta^{\mu\nu} \hat{p}_\mu \hat{p}_\nu = 1. \]

The equations of motion imply that \( \hat{p}_\mu = \text{const} \). The dynamics vector is

\[ X = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \lambda (\dot{x}^\mu \frac{\partial}{\partial \dot{x}^\mu}) + \frac{\partial}{\partial t}. \]

To fix the dynamics (to determine the arbitrary function \( \lambda \)), we use the constraint that the path parameter is proper time:

\[ \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 1; \]

this constraint implies \( \lambda = 0 \).

We must now set initial data, which will be seen to be equivalent to fixing the zero point of proper time. Because the Lagrangian is homogeneous of first degree in the velocities, the gauge transformation is of the form \( \delta x^\mu = \epsilon x^\mu \). At \( \tau = 0 \), this becomes

\[ \delta_0 x^\mu = \alpha \dot{x}^\mu, \quad \delta_0 \dot{x}^\mu = \beta \dot{x}^\mu, \]
where $\alpha, \beta$ are infinitesimal constants. To be compatible with the proper time constraint, we must have $\beta = 0$. We will choose a constraint $\chi$ such that $\delta_0 \chi = 0$ implies $\alpha = 0$ (to prevent any point gauge transformation); therefore

$$\dot{x}^\mu \frac{\partial \chi}{\partial x^\mu} \neq 0.$$  

The evolution of $\chi$ obeys

$$\dot{\chi} + \frac{\partial \chi}{\partial \tau} = 0,$$

and so $\chi$ must be explicitly time-dependent. One convenient choice is

$$\chi = x^0 - \dot{x}^0 \tau.$$  

The zero point of the path parameter is set by this requirement: $x^0(0) = 0$. There are then a six-parameter set of paths which are solutions, the six parameters being the three positions and three spatial components of the velocity at $\tau = 0$.

We now turn to the Hamiltonian discussion. The Legendre map to phase space is a map onto the surface defined by $\eta^{\mu\nu} p_\mu p_\nu = 1$. The canonical Hamiltonian $H_C$ is a function on the surface, but a trivial one: $H_C = 0$. The actual Hamiltonian in the Dirac procedure is the addition of an appropriate function of the constraint to $H_C$, namely

$$H_D = \frac{1}{2} \lambda(\tau)(\eta^{\mu\nu} p_\mu p_\nu - 1).$$

The equations of motion in phase space are $\dot{p}_\mu = 0$ and $\dot{x}^\mu = \lambda \eta^{\mu\nu} p_\nu$. The one primary constraint ($M = 1$)

$$\phi := \frac{1}{2}(\eta^{\mu\nu} p_\mu p_\nu - 1)$$

is first class, and there are no secondary constraints ($P_1 = 1$). The gauge transformations are generated by the one function

$$G = \epsilon(t) \phi.$$  

The gauge fixing procedure has two steps. The arbitrariness in the equations of motion represented by $\lambda$ requires a constraint function $\chi$ defined so that

$$\{ \chi, \phi \} \neq 0.$$  

Clearly this function must be time dependent, or else the result of requiring $\dot{\chi} = 0$ will be $\lambda = 0$. One choice is

$$\chi = \tau - x^0 \implies \lambda = \frac{1}{p_0}.$$  

The Hamiltonian is now definite:

$$H = \frac{1}{2p_0}(\eta^{\mu\nu} p_\mu p_\nu - 1).$$
The second step is to consider $G(0) = \alpha \phi$, where $\alpha = \epsilon(0)$, as a generator of point gauge transformations. In order for the Poisson Bracket of $\chi$ with $G(0)$ to vanish, clearly $\alpha$ must be zero, so that there are no further gauge fixing steps to perform.

The result of gauge fixing in the Hamiltonian treatment of this free particle is therefore that motion in phase space is within the surface defined by $\eta^{\mu \nu} p_\mu p_\nu = 1$ (a 7-dimensional space), and the motion is given by

$$p_\mu = \text{const} \ , \ x^\mu = \frac{\eta^{\mu \nu} p_\nu}{p_0} \tau + \text{const} .$$

Note that there are six free parameters for these paths, since the $\chi$ constraint fixes the $x^0$ motion to be $x^0 = \tau$. Of course, this is the same number as found in the Lagrangian treatment, though here it was convenient to normalize $\tau$ by the requirement $\dot{x}^0 = 1$, rather than by the requirement that $\tau$ be proper time (to which it is proportional, anyway).

Our next set of examples are spacetime metrics which are invariant under a three-dimensional isometry group which is transitive on spacelike three surfaces [15]. The metric is best expressed in a basis of differential forms which is invariant under the group. One such basis consists of the four one-forms $\{dt, \omega^i\}$, where

$$d\omega^i = \frac{1}{2} C^j_{ts} \omega^s \land \omega^t ,$$

where the $C^j_{st}$ are the structure constants of the Lie algebra of the group and obey

$$C^i_{st} = -C^i_{ts} \ , \ C^j_{s[t} C^s_{j]} = 0 .$$

The second relation is the Jacobi identity, and in the case of a three-dimensional Lie algebra is exactly equivalent to

$$C^d_{st} C^r_{ij} = 0 .$$

In what follows, Greek indices range over 0,1,2,3, while Latin indices range over 1,2,3. The summation convention will be followed even if both indices are superscripts or subscripts.

In this basis, the line element is

$$ds^2 = -N^2 dt^2 + g_{st} (N^s dt + \omega^s)(N^t dt + \omega^t) ,$$

where $N$ is the lapse function and $N^s$ is the shift vector; $N$, $N^s$, and $g_{st}$ are functions only of the time $t$. The models are classified according to a standard listing of the possible structure coefficients into nine basic classes, called Bianchi Types.

Our examples will be the models of Type I ($C^i_{jk} = 0$) and Type IX ($C^i_{jk} = k \epsilon_{ijk}$, $\epsilon_{ijk}$ is the completely antisymmetric Levi-Civita symbol defined by $\epsilon_{123} = 1$). The constant $k$ in the latter models is redundant but
is included to allow the limit $k = 0$. In both cases the shift vector may be taken to be zero and the spatial metric $g_{st}$ may be taken to be diagonal (we treat only vacuum models). In both cases (as well as in any model which has $C_{si} = 0$), the Lagrangian of the system may be calculated from the spatially homogeneous form of the scalar curvature $R$.

Instead of the above basis, it is more convenient to use an orthonormal basis $\{\sigma^\mu\}$. Since we take $N^i = 0$ and $g_{ij}$ diagonal, this basis is defined by

$$\sigma^0 = Ndt, \sigma^i = e^{-\Omega} e^\beta_i \omega^i \quad \text{(no sum on } i),$$

where $\Omega$ and $\beta_i$ are functions only of $t$, with $\sum \beta_i = 0$. The metric components $g_{ij}$ are given by

$$(g_{ij}) = \text{diag}(e^{-2\Omega+2\beta_i}).$$

In this basis the line element is

$$ds^2 = \eta_{\mu\nu}\sigma^\mu\sigma^\nu, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1).$$

Since $\sum \beta_i = 0$, we define $\beta_{\pm}$ by

$$\beta_1 = \beta_+ + \sqrt{3}\beta_-, \quad \beta_2 = \beta_+ - \sqrt{3}\beta_- \quad \text{and} \quad \beta_3 = -2\beta_+.$$

It is also possible to reparameterize time to make $N = 1$ (or some other function), but it will be seen that $N$ must remain a dynamical variable, at least at first, in order not to spoil the Lagrangian procedure.

The field equations for the functions $N, \Omega, \beta_i$ are the Einstein equations (for a vacuum). The Ricci tensor coefficients $R_{\mu\nu}$ are functions only of $t$ in this case, so the equations are ordinary differential equations. The most convenient form of these equations for our purposes will involve the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \eta_{\mu\nu}$. (The equations may be written either as $G_{\mu\nu} = 0$ or $R_{\mu\nu} = 0$.)

Our first example is Bianchi Type I, in which $C_{jk} = 0$. In this case the one-forms $\omega^i$ are expressible in terms of coordinates as $\omega^i = dx^i$. The line element is

$$ds^2 = \eta_{\mu\nu}\sigma^\mu\sigma^\nu, \quad \sigma^0 = Ndt, \quad \sigma^i = e^{-\Omega} e^\beta_i dx^i \quad \text{(no sum on } i).$$

The appropriate Einstein tensor components (or rather, independent linear combinations of them), which are to be set equal to zero, are

$$\frac{N^2}{3} G_{00} = -\ddot{\Omega} + \dot{\beta}_+^2 + \dot{\beta}_-^2 = 0,$$

$$\frac{N^2}{6}(G_{11} + G_{22} - 2G_{33}) = \dot{\beta}_+ - \frac{\dot{N}}{N} \dot{\beta}_+ - 3\dot{\beta}_+ \dot{\Omega} = 0,$$

$$\frac{N^2}{2\sqrt{3}}(G_{11} - G_{22}) = \dot{\beta}_- - \frac{\dot{N}}{N} \dot{\beta}_- - 3\dot{\beta}_- \dot{\Omega} = 0,$$

$$\frac{N^2}{6}(G_{11} + G_{22} + G_{33} + 3G_{00}) = \ddot{\Omega} - \frac{\dot{N}}{N} \dot{\Omega} - 3\dot{\Omega}^2 = 0.$$
The action integral for general relativity is 
\[ I = \int R \sqrt{|g|} d^4x \] (up to an irrelevant proportionality constant), where \( R \) is the Ricci scalar and \( g \) is the determinant of the metric in coordinates \( \{x^\mu\} \). In our case we have
\[ I = \int R \sigma^0 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 = \int RN e^{-3\Omega} d^4x . \]
The result here is
\[ I = \int \frac{6e^{-3\Omega}}{N}(-\dot{\Omega} + \frac{\dot{N}}{N}\Omega + 2\dot{\Omega}^2 + \dot{\beta}_+^2 + \dot{\beta}_-^2) d^4x . \]
We take the volume of space \( \int d^3x = 1/12 \) and integrate by parts with respect to time, dropping the endpoint contributions:
\[ I = \int \frac{e^{-3\Omega}}{2N}(-\ddot{\Omega}^2 + \dot{\beta}_+^2 + \dot{\beta}_-^2) dt . \]
Variations of \( I \) with respect to \( N, \beta_+, \beta_-, \Omega \) give the Einstein equations listed above.

Note that reparameterization of time can be used to set \( N = 1 \) (or some other function), but if \( N \) is eliminated from the Lagrangian, the \( G_{00} = 0 \) field equation (which is a constraint equation) will not be derivable.

But of course the Lagrangian \( L \),
\[ L = \frac{e^{-3\Omega}}{2N}(-\dot{\Omega}^2 + \dot{\beta}_+^2 + \dot{\beta}_-^2) , \]
is singular, since \( \dot{N} \) doesn’t appear:
\[ \hat{p}_N = \frac{\partial L}{\partial \dot{N}} = 0 . \]
An attempt to form the Hamiltonian thus yields
\[ H_C = \frac{1}{2} Ne^{3\Omega}(-p_\Omega^2 + p_+^2 + p_-^2) . \]
\( H_C \) is a function in \( T^*Q \), namely in principle a function of \( N, \Omega, \beta_+, \beta_- \), and \( p_N, p_\Omega, p_+, p_- \), but it happens to be independent of \( p_N \). The Legendre map from \( TQ \) to \( T^*Q \) is
\[ \{N, \Omega, \beta_+, \beta_-\} = \{N, \Omega, \beta_+, \beta_-\} , \]
\[ p_N = 0 , p_\Omega = -\frac{e^{-3\Omega}}{N} \dot{\Omega} , p_+ = \frac{e^{-3\Omega}}{N} \dot{\beta}_+ , p_- = \frac{e^{-3\Omega}}{N} \dot{\beta}_- , \]
and thus maps the 8-dimensional \( TQ \) into the 7-dimensional subspace of \( T^*Q \) defined by \( p_N = 0 \). We thus identify the primary constraint in \( T^*Q \):
\[ p_N \simeq 0 , \]
where \( \simeq 0 \) means that every solution of the equations of motion has to satisfy the constraint. Since there is only the one primary constraint, it is first class.
Therefore, to \( H_C \) may be added an arbitrary function of \( p_N \) which vanishes when \( p_N = 0 \), the simplest being \( p_N \) itself:

\[
H = H_C + \lambda p_N .
\]

This arbitrariness means that we have a gauge-type freedom. The time derivative of \( p_N \) is given by

\[
\dot{p}_N = \{p_N, H\} = -\frac{\partial H}{\partial N} = -\frac{1}{2} e^{3\Omega}(-p^2 + p^2_+ + p^2_-) .
\]

The requirement that \( \dot{p}_N \simeq 0 \) thus implies that \( H_C/N \simeq 0 \) or \( -p^2 + p^2_+ + p^2_+ \simeq 0 \).

Now is the time to generalize this example, so as not to repeat a lot of material. The generalization will still be quite concrete: The Type IX cosmology has

\[
C^{i}_{j k} = k \epsilon_{i j k} .
\]

In this case, the action integral is

\[
\mathcal{I} = \int R \sigma^0 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3 ,
\]

where the orthonormal frame of one-forms is

\[
\sigma^0 = N dt ; \sigma^i = e^{-\omega} e^{\beta_i} \Omega^i \text{ (no sum on } i) ; d\omega^i = \frac{1}{2} k \epsilon_{i j k} \omega^j \wedge \omega^k .
\]

Thus \( \mathcal{I} \) is

\[
\mathcal{I} = \int R N e^{-3\Omega} dt \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 ,
\]

and the spatial integral may be set equal to 1/12 as before.

The result for the Lagrangian is (\( \beta_+, \beta_- \) are defined as before):

\[
L = \frac{e^{-3\Omega}}{2N} (-\ddot{\Omega}^2 + \dot{\beta}_+^2 + \dot{\beta}_-^2) - k^2 N e^{-9\Omega} V(\beta_+, \beta_-) ,
\]

where the function \( V \) is

\[
V(\beta_+, \beta_-) = \frac{1}{2} e^{-8\beta_+} - 2 e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + e^{4\beta_+} (\cosh(4\sqrt{3}\beta_-) - 1) .
\]

The particular form of \( V \) is not important, nor are the specific forms of the field equations.

Needless to say \( L \) is still singular:

\[
\dot{p}_N = 0 .
\]

There is still just one primary, first class constraint, \( p_N \simeq 0 \), and the Hamiltonian is given by

\[
H = H_C + \lambda p_N ,
\]
where
\[ H_C = \frac{1}{2} N e^{3\Omega} (-p^2 + p^2_+ + p^2_-) + k^2 N e^{-\Omega} V(\beta_+, \beta_-). \]

Stability of the primary constraint gives
\[ \dot{p}_N = \{p_N, H_C\} = -\psi \simeq 0, \]
where \( \psi \) is the secondary constraint
\[ \psi = \frac{1}{2} e^{3\Omega} (p^2_+ + p^2_- - p^2_\Omega) + k^2 e^{-\Omega} V(\beta_+, \beta_-). \]

At this point we can see that the canonical Hamiltonian is just a constraint: \( H_C = N \psi \). No further constraints appear.

Primary and secondary constraints are both first class, the gauge generator is made out of them:
\[ G = \dot{\epsilon} p_N + \epsilon \psi. \]

After the redefinition of the arbitrary function, \( \epsilon = N \eta \), the transformations are
\[ \delta \beta_+ = \dot{\beta}_+ \eta, \delta \beta_- = \dot{\beta}_- \eta, \delta \Omega = \dot{\Omega} \eta, \delta N = \dot{N} \eta + \dot{N} \eta. \]
(Thus, \( \beta_+, \beta_-, \Omega \) behave as scalars under reparameterizations, and \( N \) behaves as a vector.)

According to the theory developed in the previous sections, there will be two gauge fixing constraints, one of them time-dependent. These two constraints have to make the two original constraints second class in the theory in order to determine the arbitrary function in the Dirac Hamiltonian. The simplest way to proceed is to reverse our formal methodology and first to write down the initial data gauge fixing constraint. If this constraint does not depend on \( N \), its stability will give us a new constraint which is \( N \)-dependent. Then the stabilization of this new constraint will fix the dynamics.

We start with the time-dependent constraint (a simple and common choice):
\[ \chi^{(1)} := \Omega - t \simeq 0. \]

Its stability leads to the requirement
\[ \dot{\chi}^{(1)} = \{\Omega, H_C\} - 1 = -N e^{-3\Omega} p_\Omega - 1 \simeq 0. \]

From this expression we get the new gauge fixing constraint
\[ \chi^{(2)} := N + \frac{e^{3\Omega}}{p_\Omega} \simeq 0. \]
Notice that since $N$ is required to be positive (on physical grounds), this constraint implies $p_\Omega < 0$.

In its turn, stability of $\chi^{(2)}$, because of the presence of $N$, will determine the arbitrary function $\lambda$ in the Dirac Hamiltonian $H_D = H_C + \lambda p_N$. Actually, in order to fix the gauge in the equations of motion, we do not need to know the value of $\lambda$: it only matters for the equations of motion for the variable $N$, and the constraint $\chi^{(2)} \simeq 0$ already relates $N$ to other variables. Therefore, since we are concerned with the evolution of $\beta_+, \beta_-, \Omega$ and their associated momenta, the Hamiltonian is simply:

$$H = -\frac{1}{2p_\Omega}(p_+^2 + p_-^2 - p_\Omega^2 + 2k^2 e^{-4\Omega}V(\beta_+, \beta_-)),$$

where we have used the constraint $\chi^{(2)} \simeq 0$ to substitute for $N$. This procedure is correct because $N$ was an overall factor of a constraint in $H_C$: $H_C = N\psi$. This result can be rephrased as follows: We have gotten rid of a couple of canonical variables, $N, p_N$, by using the Dirac Bracket. In this case, the Dirac Bracket for the rest of the variables is the usual Poisson Bracket.

The constraint $\psi \simeq 0$ can be used to define the evolution of $p_\Omega$ in terms of the other variables (since the constraint $\Omega = t$ fixes the time dependence of $\Omega$). It is conveniently factorized as

$$\psi = -\frac{1}{2}e^{4\Omega}(p_\Omega + H_R)(p_\Omega - H_R),$$

where

$$H_R := \sqrt{p_+^2 + p_-^2 + 2k^2 e^{-4\Omega}V(\beta_+, \beta_-)}.$$

This factorization implies

$$H = \frac{1}{2p_\Omega}(p_\Omega - H_R)(p_\Omega + H_R).$$

Satisfaction of the constraint $\psi \simeq 0$ implies $p_\Omega + H_R \simeq 0$ (the physical interpretation of the model requires $p_\Omega \leq 0$). Then $p_\Omega - H_R \simeq 2p_\Omega$, and the Hamiltonian can be equivalently written as

$$H = p_\Omega + H_R.$$

Observe that this constraint defines the evolution of $p_\Omega$ in terms of the other variables. Thus if we restrict ourselves to the evolution of the variables $\beta_+, \beta_-$ and their canonical conjugates $p_+, p_-$, then the dynamics in this reduced space is described by the Hamiltonian $H_R$. Once the evolution of these variables is determined, the rest of the variables (four in number) have a time-evolution dictated by the constraints (also four in number) of the theory.

With regard to the Lagrangian formulation, there is only one constraint in velocity space: It is the pullback of the secondary constraint $\psi$, since the pullback of the primary one is identically zero. The dynamics (given by a vector field in velocity space) has only one arbitrary function, which multiplies the vector field $\Gamma = \partial/\partial \dot{N}$.
Let us briefly sketch the Lagrangian gauge fixing procedure in this case. If we start again with the time dependent gauge fixing constraint $\Omega - t \simeq 0$, its stabilization will require $\dot{\Omega} - 1 \simeq 0$. The stabilization of this new gauge fixing constraint will give a new constraint which is dependent on $\dot{N}$ (plus other velocity space variables). Finally, the stabilization of this last gauge fixing constraint determines the arbitrary function of the dynamics. We end up with three gauge fixing constraints that add to the original single constraint of the theory to give the elimination of four degrees of freedom. This numbering is the same as in the Hamiltonian case.

6 Conclusions

A singular Lagrangian, that is with singular Hessian $W_{ij} = \partial^2 L / \partial \dot{q}^i \partial \dot{q}^j$, results in constraints, a dynamics with some arbitrariness, and gauge transformations which reflect this arbitrariness. The Legendre map to phase space defined by $p_i = \partial L / \partial \dot{q}^i$ therefore maps the $2N$ dimensional velocity space $TQ$ to a lower dimensional surface in phase space $T^*Q$. The requirement that the dynamics on this surface be consistent can be used to reduce its dimensionality somewhat, but there still remain the same three ingredients: constraints, arbitrariness in the dynamics, and gauge transformations. Consistency requirements in the Lagrangian formalism can also be used to reduce somewhat the dimensionality of the constraint surface in velocity space, but in general the constraint surface in phase space will have smaller dimensionality than the constraint surface in velocity space. The number of gauge transformations in $TQ$ and in $T^*Q$ is the same.

In this paper we show how to determine the dynamics and fix the gauge in both the Hamiltonian and Lagrangian formalisms. We believe the Lagrangian discussion is new and useful. The result includes a proof that the final number of degrees of freedom in the two formalisms is the same. Important parts of our methods are the two steps of determining the dynamics and determining the independent initial data.

The first step in both the Lagrangian and Hamiltonian formalisms is the stabilization algorithm (which ensures consistent dynamics). The result is $M$ constraints in the Hamiltonian case (a constraint surface of dimension $2N - M$ in $T^*Q$) and $M - P_1$ constraints in the Lagrangian case (a constraint surface of dimension $2N - M + P_1$ in $TQ$); $P_1$ is the number of primary, first class constraints.

There are $P_1$ gauge transformations (in both the Lagrangian and Hamiltonian cases), and they are formed using a total of $F$ functions (first class constraints in the Hamiltonian formalism). In the second step, in both the Lagrangian and Hamiltonian cases, $P_1$ gauge-fixing functions are used to determine the dynamics (to obtain the dynamics vector $X_F$ or the Hamiltonian $H_D$ respectively). The dimensionality of the constraint surfaces in each case is reduced by $P_1$.

The third and final step is to use gauge fixing functions to determine physically inequivalent initial data. In the Hamiltonian case $F - P_1$ functions are used, so that the number of degrees of freedom is
\[2N - M - P_1 - (F - P_1) = 2N - M - F.\] In the Lagrangian case \(F\) functions are used, and again the number of degrees of freedom is \(2N - M + P_1 - P_1 - F = 2N - M - F.\) The Hamiltonian method directly shows that \(2N - M - F\) is an even number.

We have emphasized the double role played by the gauge fixing procedure. Some constraints are needed to fix the time evolution of the gauge system, which was undetermined to a certain extent. The rest of the constraints eliminate the spurious degrees of freedom that are still present in the setting of the initial conditions of the system. This double role has not been adequately emphasized in the literature. Although we only treat classical systems in this paper, our approach should also be relevant to quantum ones.

Let us briefly comment on Dirac’s extended Hamiltonian formulation [2]. Dirac suggests that the canonical dynamics be modified by adding all secondary first class constraints to the Hamiltonian in an \textit{ad hoc} manner. The result is as many arbitrary functions in the dynamics as first class constraints, and every first class constraint generates an independent gauge transformation. In our dynamics fixing step, we would introduce \(F\) constraints, and the process would then be finished (the initial data fixing step would be empty). The counting of degrees of freedom still agrees with ours. Furthermore, if we take one of our sets of gauge fixing constraints \(\chi^{\mu}_i \simeq 0\), then this set works for the extended formalism also, and we end up with the same dynamics. However, one can also use a more general gauge fixing procedure for the extended Hamiltonian theory (without the stabilization condition we required for the \(\chi^{\mu}_i \simeq 0\)); in this case the extended Hamiltonian theory will not necessarily be equivalent to the Lagrangian formalism.

We have discussed, also, the special case of reparameterization-independent theories. In those cases, as we showed, the canonical Hamiltonian \(H_C\) is a constraint of the motion. In general \(H_C\) will be a secondary, first class constraint, but in some cases it will vanish. In reparameterization-independent theories, we showed that one of the gauge fixing constraints must be time-dependent. In these theories, the “time” is defined through some dynamical variables of the system. Since there is a good deal of choice for this definition of time, one has to be very careful in verifying that the time-dependent constraint is consistent with the physical interpretation of the model.

We have illustrated these ideas by considering two interesting cases. The first is the motion of a free particle in special relativity using the Lagrangian \(L = \sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}.\) This Lagrangian is reparameterization-invariant, and the canonical Hamiltonian vanishes. The result of applying our procedures is that the paths are parameterized by six parameters (three positions in space and three velocity or three momentum initial values), and that time may be parameterized by proper time, as one does expect.

The second case included two spatially homogeneous cosmological models in general relativity, the vacuum Bianchi Type I and Type IX models. In these models, the number of degrees of freedom is found to
be reducible to four, and the role of the time-dependent gauge fixing procedure is clarified.

We have previously [16] described some of the above results and intend to extend many of these ideas, for instance by looking into aspects of field theory. For example, the various gauges used in electromagnetic theory, including the Coulomb ($\nabla \cdot \mathbf{A} = 0$), Lorentz ($A^\sigma \sigma = 0$), and radiation ($A_0 = 0$) gauges, apply in different formulations, either Lagrangian or Hamiltonian. How they affect the true degrees of freedom of the electromagnetic field may be clarified by our methods.

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References

[1] Dirac P A M 1950 *Can. J. Math.* 2, 129

[2] Dirac P A M 1964 *Lectures on Quantum Mecanics* (New York: Yeshiva Univ. Press)

[3] Pons J M 1988 *J. Phys. A: Math. Gen.* 21, 2705

[4] Gracia X and Pons J M 1988 *Annals of Physics (N.Y.)* 187, 2705

[5] Batlle C, Gomis J, Pons J M, and Roman N 1986 *J. Math. Phys.* 27, 2953

[6] Sundermeyer K 1982 “Constrained Dynamics” in *Lecture Notes in Physics* 169 (Berlin: Springer Verlag)

[7] Henneaux M, Teitelboim C, and Zanelli J 1990 *Nucl. Phys.* B332, 169

[8] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley)

[9] Anderson J L and Bergmann P G 1951 *Phys. Rev.* 83, 1018

[10] Castellani L 1982 *Annals of Physics (N.Y.)* 143, 357

[11] Sugano R, Saito Y, and Kimura T 1986 *Prog. Theor. Phys.* 76, 283

[12] Gomis J, Henneaux M, and Pons J M 1990 *Class. Quantum Grav.* 7, 1089

[13] Gotay M, Nester J M, and Hinds G 1978 *J. Math. Phys.* 19, 2388

[14] Kamimura K 1982 *Nuovo Cimento* B69, 33

[15] Ryan M P and Shepley L C 1975 *Homogeneous Relativistic Cosmologies* (Princeton: Princeton Univ. Press)

[16] Pons J M and Shepley L C 1994 “Gauge Fixing in Constrained Systems” to appear in Charap J M, Ed, *Geometry of Constrained Dynamical Systems* (Cambridge: Cambridge Univ. Press)