Abstract

The AdS-bubble solutions interestingly mimic Schrödinger-like geometries when expressed in light-cone coordinates. These Dp bubble vacua exhibit asymmetric scaling property with a negative dynamical exponent of time $a < 0$, but are smooth geometries. Through a time-like T-duality we map these vacua to $E_p$-brane bubbles with $a > 0$ in type-II$^*$ super-strings. We obtain an expression for the entanglement entropy for ‘bubble E3-branes’. It is argued that the entropy from E3-bubbles has to be the lowest.
1 Introduction

A remarkable progress has been made towards understanding various string backgrounds which exhibit Lifshitz or Schrödinger type non-relativistic symmetries [1]-[18]. Particularly, in these solutions the time and space coordinates scale asymmetrically and therefore Lorentzian symmetry is explicitly broken in the holographic boundary theory. The Lifshitz type solutions may admit boundary field theories which can exhibit non-fermi liquid or a strange metallic behavior at ultra-low temperatures. The strange metallic effects have also been associated with the phenomenon where strongly correlated quantum systems develop hidden fermi surfaces [12, 13]. Recently estimating the entanglement entropy of quantum systems has become an effective holographic tool in order to understand strongly coupled CFT dynamics [1, 12]. On general grounds, there are several fundamental issues attached with the measurement of the entanglement within the quantum systems, including black-holes [16], and entangled pairs [17]. Even for a pure system, once it is subdivided into smaller subsystems, say $A$ and $B$, the subsystems get maximally entangled amongst themselves. The von Neumann entropy measure of the system $A$, $S_A \sim -\text{Tr} \rho_A \ln \rho_A$, is defined in terms of the reduced density matrix $\rho_A = \text{Tr}_B[\rho]$. Following AdS/CFT holographic dictionary the basic picture is well laid out. That is, if strongly correlated field theory system at critical point could be represented as a system living on the boundary of some known bulk AdS theory, the boundary system becomes phenomenologically more tractable. According to Ryu-Takayanagi proposal, in such holographic cases the entanglement entropy of the boundary (quantum) theory can also be estimated geometrically as the area of an extremal surface, $X$, embedded inside the bulk spacetime [1]

$$S_X = \frac{1}{4G_N}\text{Area}[X]$$

(1)

Following an early work on $AdS_5 \times S^5$ D3-brane vacua and $a = 3$ Lifshitz solutions [8], we generalized that very approach to include all D$p$-brane AdS vacuas and obtained Lifshitz and Schrödinger like solutions in type II A/B string theory [14]. Our main focus in this work will be on the Schrödinger-like D$p$ brane solutions of [14]. They exist in various dimensions with a line element

$$ds_{Sch}^2 = -\frac{\beta^2}{z^{2a}}(dx^+)^2 + \frac{1}{z^2}\{-dx^+ dx^- + dx_1^2 + \cdots + dx_{p-1}^2 + dz^2\},$$

(2)

which have a dynamical exponent of time as $a$. We shall mainly study vacua for which $a = \frac{2}{p-5}$. So $a$ is essentially negative for all $1 < p < 5$. Another special characteristic of these solutions is that they would involve a compact direction, namely $x^-$, which is typically null. Due to that these type of classical geometries may not be trustworthy and would require quantum corrections to be included. Although, one may choose to decompactify the null (lightcone) coordinate, but a meaningful non-relativistic Schrödinger

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1 As per our terminology, the ‘Schrödinger-like’ solutions we discuss in this paper will have appropriate conformal factors multiplying the Schrödinger metrics if an explicit compactification is performed, although it would not be required in this work. Thus we would be discussing ‘conformally Schrödinger vacua’ all along in this work.
group description [4, 5, 7], would require the lightcone coordinate to be rather compact. Otherwise also, even in the noncompact cases the horizon like unphysical surfaces may be present [18]. In our previous work [18], we studied Schrödinger-like Dp-brane solutions with $a = \frac{2}{p-5}$, but these solutions could be trusted in a limited UV region only. The reason is that the unrestricted IR geometry of such solutions [18] had horizon-like features or a conical singularity. Thus these Schrödinger solutions with $a < 0$ still remain the least understood Dp-brane vacua.

In this work we shall study AdS-bubble solutions, which are well behaved geometries and altogether avoid IR singularity. Remarkably, once these bubble solutions are expressed in the lightcone coordinates, not only do they mimic Schrödinger-like spacetimes but also exhibit the same value of $a = \frac{2}{p-5}$ as their unregulated counterpart in [14]. The dynamical exponent of time surely remains negative but the energy spectrum has no superluminosity or unrestricted blue-shifts. This becomes possible due to the presence of an in-built IR cut-off in the bubble geometries. Thus keeping a finite cut-off is thus essential for avoiding the singularity. To explore these spacetimes with $a < 0$ further, we employ a time-like T-duality [20] and obtain corresponding time-dualized ‘Ep-brane bubbles’, which are solutions in type-II* superstring theory. In these static Ep-brane bubbles the dynamical exponent of time, however, becomes positive definite. In fact, all Ep branes have a fixed dynamical exponent and that is $a_{Ep} = 1$. Further, in Ep-brane bubble geometries it is rather straightforward to pick up an static embedding extremal surface and evaluate the entanglement entropy. Particularly, E3-brane solutions are used to calculate the entropy of a strip-like subsystem in the boundary theory. To caution here, we have literally assumed that the Ryu-Takayanagi area entropy functional can be suitably applied for Ep-branes also, although these are the vacua of type II* A/B superstrings.

The paper is arranged as follows. In section-2 we review AdS-bubble solutions in lightcone coordinates where their Schrödinger spacetime properties become quite explicit. These bubbles are smooth vacua with dynamical exponent $a < 0$, without IR (conical) singularity. The entanglement entropy is calculated next. In section-3 we employ a time-like T duality so as to convert these $a < 0$ AdS-bubble solutions into Ep-brane bubble solutions with $a > 0$. Especially we calculate the entanglement entropy for the bubble E3-branes. It is found that the entanglement entropy of ‘bubble E3-branes’ has the same functional form as it is for the ‘bubble D3-brane’ solutions. This result raises one vital question. Does our result imply that the field theory living on the boundary of E3-brane bubbles has similar entanglement information (involving physical degrees of freedom) as it does for the CFT on the boundary of D3-brane bubbles? The answer we get is positive. The summary is provided in the section-4.

\[2\] We note that a similar type of situation arises for the Schrödinger vacua with $a > 0$, namely the work [7] for $a = 2$, where the singularity (shrinking circle) appears in UV regime. However, an inclusion of black hole in $a = 2$ solutions temporarily solves the problem.

\[3\] A defining feature of the type II* superstring action [20] is that all Ramond-Ramond potentials have negative sign kinetic terms. But a complete understanding of the type II* superstrings is still lacking in string theory. This work may be taken as an attempt in that direction.
2 Bubble geometries and entropy

We start with static AdS-bubble or ‘AdS soliton’ spacetimes which are asymptotically AdS solutions in type II A/B superstring theory. The AdS bubble solutions are known to describe the low temperature phases in the holographic dual Yang-Mills theories at large ’t Hooft coupling. Typically the boundary CFT reduces to a pure confining Yang-Mills theory with an IR cut-off and it develops a mass gap \[21\]. These bubble \(D_p\)-brane solutions are written as

\[
ds^2_{\text{Bubble}} = R_p^2 r^{p-3} \left[ r^{5-p}(-dt^2 + fdy^2 + d\vec{x}^2_{(p-1)}) + \frac{dr^2}{r^2} + d\Omega^2_{(8-p)} \right],
\]

\[
e^\phi = (2\pi)^{2-p} g_{YM}^2 \left( \frac{R_p^4}{r^4} \right)^\frac{p-3}{4}
\]

with appropriate electric or magnetic flux of the Ramond-Ramond \(F_{p+2}\) form field strength, which we avoid writing explicitly as it is not required in this work. The RR-flux is measured in the units of \(N\) number of branes (\( (R_p)^4 \equiv d_p g_{YM}^2 N \)). The most notations here are the same as in the case \(N D_p\)-branes \[19\], or see \[14\]. But there is no supersymmetry in the AdS bubble solutions. We shall mainly consider the cases of \(D_p\)-branes for \(1 < p < 5\). In the above

\[
f(r) = (1 - r_0^{7-p}/r^{7-p}) \quad \text{with} \quad r_0 \leq r \leq \infty.
\]

Since the radial coordinate is restricted in IR these describe what is commonly known as ‘bubble’ geometries. The inside of the \(r = r_0\) is just empty. As \(r\) is holographic (energy) coordinate, the boundary Yang-Mills theory has an effective IR cut-off \(r = r_0\) \[21\]. At length scales larger than \(1/r_0\) there are no correlations in the field theory due to the mass gap. However the coordinate \(y\) has to be taken with right periodicity so that the metric is smooth and avoids (conical) singularity at \(r = r_0\).

Asymptotically, as \(r \to \infty\), the solution becomes conformally \(AdS_{p+2} \times S^{8-p}\), usually with a running dilaton field except for a \(D3\)-brane. The YM theory at low temperatures can be studied, in that case the Euclidean time, \(\tau (\equiv it)\), will have to be periodic. However the period of \(\tau\) can be arbitrary in these solutions.

2.1 Lightcone coordinates and Schrödinger-like bubbles

By introducing the lightcone coordinates \((x^\pm = t \pm y)\) the bubble metric (3) can be expressed as

\[
ds^2 = R_p^2 z^{p-3} \left[ \frac{1}{2} \left( 1 - \frac{f}{4} \right) [(dx^+)^2 + (dx^-)^2] - \frac{1 + f}{2} dx^- dx^+ + d\vec{x}^2_{(p-1)} \right.

\[
\left. + \frac{4}{(5-p)^2} \frac{dz^2}{f} \right] + d\Omega^2_{(8-p)}
\]

\[
e^\phi = (2\pi)^{2-p} g_{YM}^2 \left( R_p^4 z^{2-p} \right)^\frac{3-p}{4}
\]

(5)
along with the associated $F_{+-x_1\ldots x_{p-1}z}$ RR-flux component. The $z$-coordinate has been introduced through $r^2 = z_{\text{IR}}$.

The function

$$f(z) = 1 - \left(\frac{z}{z_0}\right)^{\frac{2p-14}{p-5}}$$

(6)

and the coordinate range is $0 \leq z \leq z_0$. From the metric (5) we see that the lightcone coordinates are quite symmetrically placed, but both have time-like signatures. Thus anyone can be tagged as the lightcone time. Picking $x^{+}$ as the time, the metric (5) can be expressed as

$$ds^2 = R_0^2 \left[ \frac{z_{\text{IR}}^{\frac{2p-14}{p-5}}}{4z^{p-5}} (dx^{+} + U dx^{-})^2 + f \frac{2}{z^{p-5}} (dx^{-})^2 + \frac{d\hat{x}^{2(p-1)}}{z^2} + \frac{4}{(5-p)^2 f z^2} + d\Omega_{(8-p)}^2 \right]$$

$$\equiv R_0^2 \left[ \frac{\chi}{4z^2} (dx^{+} + U dx^{-})^2 + \frac{f}{\chi z^2} (dx^{-})^2 + \frac{d\hat{x}^{2(p-1)}}{z^2} + \frac{4}{(5-p)^2 f z^2} + \cdots \right]$$

(7)

where we have defined

$$\chi(z) \equiv \left(\frac{z}{z_0}\right)^{\frac{2p-14}{p-5}}, \quad U(z) = \frac{(1+f)}{\chi}$$

(8)

for later convenience. Thus for the metric in (7) the effective dynamical exponent of (lightcone) time is $a = \frac{2}{p-5}$, and it is negative for $1 \leq p \leq 4$. Indeed $x^{-}$ coordinate behaves as spatial coordinate. In the neighborhood of $z = z_0$, by defining a new coordinate $u$, and since $f(z)|_{z \to z_0} \simeq u^2 \to 0$, the metric (7) becomes approximately

$$ds^2 \simeq z_0^{\frac{2p-14}{p-5}} \left[ \frac{1}{z_0^2} \left\{ -(dt)^2 + u^2(dx^{-})^2 + d\hat{x}^{2(p-1)} \right\} + \frac{4}{(7-p)^2} du^2 + d\Omega_{(8-p)}^2 \right]$$

(9)

Thus these solutions are smooth near $z = z_0$ provided we take $x^{-} \sim x^{-} + 2\pi r^{-}$, with $r^{-} = \frac{2z_0}{7-p}$. Let us below summarize below what have we achieved by introducing the lightcone coordinates in the AdS bubble metric.

- The metric (7) in lightcone coordinates represents conformally Schrödinger bubble spacetime, with dynamical exponent of time $a = \frac{2}{p-5}$. These however remain smooth geometries in the IR.

- Eventhough the dynamical exponent is negative definite for $p \leq 4$, there is no superluminosity anywhere in the valid holographic region $z_0 \geq z \geq 0$. It has got $\chi \leq 1$ in the allowed range.

- These solutions thus represent a resolved version of otherwise singular Schrödinger-like solutions with same value of $a = \frac{2}{p-5}$, reported in [14, 18]. Also see some details in the Appendix.
Figure 1: The schematic drawing of a regular Schrödinger/bubble spacetime when viewed in the light-cone coordinates. Near $z \sim 0$ the spacetime becomes conformally AdS. The (blue) central IR region ($z > z_0$) is an empty space and does not exist in the geometry.

2.2 Entanglement from bubbles

We now obtain the entanglement entropy of a subsystem on the boundary of the Schrödinger bubble vacua (5). We follow covariant Ryu-Takayanagi [3], embedding of a strip-like surface, namely $x^+ = y$, $x^- = -y$, $x_1 = x_1(z)$ inside the bulk geometry (5). The boundaries of the extremal bulk surface coincide with the two ends of the interval $-l/2 \leq x^1 \leq l/2$. The regulated size of the rest of the coordinates is taken as $0 \leq x^i \leq l_i$, with $l_i \gg l$. We shall always have $0 \leq y \leq 2\pi r^-$ as it is a bubble geometry, and our covariant embedding implies that on the embedding section $x^+$ and $x^-$ will have the same periodicity.

Considering the metric (7), we get the area functional of extremal surface as

$$A \equiv \frac{2\pi r^- V_{p-2} \Theta_8 (L_p)^8}{2G_{10}} \int_{z_*}^{z_c} dz \ z^{\frac{p-2}{2-p}} \sqrt{f} \left( \frac{1}{4} \right) + (\partial_z x^1)^2$$

where $G_D$ is $D$-dimensional Newton’s constant, $\Theta_n$ is the complete solid angle of $n$-sphere, and $V_{p-2} \equiv l_2 l_3 \cdots l_{p-1}$ is the spatial volume of the subsystem. The new constant $L_p$ is defined as

$$L_p \equiv \frac{(2\pi)^{\frac{p-2}{4}}}{\sqrt{g_{YM}}(R_p)^{\frac{p+1}{4}}}$$

Note $(L_p)^2$ is an overall constant factor multiplying the metric (7) when written in Einstein frame. Note that the integrand in the action is well defined in the region, $z \leq z_0$. In our notation $z_\epsilon \approx 0$ denotes the UV cut-off to regulate the divergences near the boundary, and $z_*$ is the point of return of the extremal surface inside the bulk. From (10) it follows that a minimal surfaces will satisfy

$$\frac{dx^1}{dz} = \frac{2}{5 - p} \left( z_c \right) \frac{1}{\sqrt{f - \left( \frac{z}{z_c} \right)^{18 - 2p}}}$$

The constant $z_c$ is fixed only in terms of the turning point relation

$$f^* - \left( \frac{z_c}{z_\epsilon} \right)^{18 - 2p} = 0$$
where \( f^* = f(z)|_{z_*} \). The identification of the boundary leads to

\[
\frac{l}{2} = \frac{2}{5 - p} \int_{z_*}^{0} \left( \frac{z}{z_*} \right)^{\frac{9-p}{5-p}} \frac{\sqrt{f}}{\sqrt{f^2 - ff^*(\frac{z}{z_*})^{\frac{18-2p}{5-p}}}}
\]

which related \( l \) with \( z_* \). While the turning-point has the value \( x^1(z_*) = 0 \). Evaluating it, the expression of the entanglement entropy for these bubble solutions is

\[
S_{EE}(Dp) = \frac{2\pi r V_{p-2}(L_p)^p}{G_{p+2}} \frac{1}{5 - p} \int_{z_*}^{z_c} dz \frac{1}{z^{\frac{9-p}{5-p}}} \frac{\sqrt{f}}{\sqrt{f^2 - ff^*(\frac{z}{z_*})^{\frac{18-2p}{5-p}}}}
\]

For \( p = 3 \) the above result gives

\[
S_{EE}(D3) = \frac{\pi r}{G_5} l_2(L_3)^3 \int_{z_*}^{z_c} dz \frac{1}{z^3} \frac{\sqrt{f}}{\sqrt{f^2 - ff^*(\frac{z}{z_*})^6}}
\]

which was initially derived in [2] involving \( N \) D3-brane bubbles. The expression (16) gives the entanglement entropy for a strip-like subsystem in a confining Yang-Mills theory.

3 \ Time T-duality and Ep branes

We would like to time-dualize the AdS-bubble solutions to obtain the dual solutions (with \( a > 0 \)) which can only be described as vacua of type II* A(B) string theory. However, an useful property of the time-dual solutions would be that they admit unambiguous constant time hypersurfaces inside the bulk geometry, and that choice allows us to estimate the entanglement entropy of the boundary theory of Ep-branes. We are expecting that the entropy will have similar result in the time-dual cases, eventhough it requires for us to work in type II* string framework.

For our purpose we wish to explore the Schrödinger-like vacua \( \{3\} \) with \( a < 0 \). We shall employ the time-like T-duality (simply TT-duality) proposed by C. Hull [20] for these vacua. Note that unlike standard T-duality, \( R \rightarrow \frac{a'}{R} \), which is performed along an spatial circle, the time-like T-duality is performed along a (periodic) time direction. In static solutions the time is an isometry direction, so it will not be difficult to make it periodic.\( \) The rules of implementing TT-duality are discussed in the works [20]. Using these duality maps one can generate brane-like Ep solutions of the type II* A(B) theory

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4 One can understand this duality on the basis of Euclidean time as well. Let us consider that ‘Euclidean time’ has got a period, \( \tau \sim \tau + \beta \) then under TT-duality the ‘dual Euclidean time’ (\( \tilde{\tau} \)) will have the period \( \tilde{\tau} \sim \tilde{\tau} + \frac{a'}{\beta} \) as viewed in the string coordinates.
starting from the known Dp-brane solutions of (ordinary) type II B(A) string theory, and vice-versa.

Let us consider the AdS-bubble solutions in lightcone coordinates \([5]\). Note that we have singled out \(x^+\) as the (lightcone) time coordinate in these solutions. We seek to make TT-duality along \(x^+\) and obtain the corresponding time-dual solutions in type II* string theory. Adopting the TT duality rules given in \([20]\), the time-dual solutions are obtained as, following from \([7]\) and \([5]\),

\[
ds^2(E_p) = R_p^2 z^{\frac{p-5}{2}} \left[ \frac{1}{z^2} \left\{ \frac{f}{\chi} (dx^-)^2 + d\mathbb{x}^2_{(p-1)} + \frac{4}{(5-p)^2} \frac{dz^2}{f} \right\} - \frac{4z_0^{2p-14}}{R_p^4 z^2} (d\bar{x}^+)^2 + d\Omega^2_{(8-p)} \right]
\]

\[
e^{2\phi} = e^{2\phi} z^{\frac{p-7}{2}} = (2\pi R_p)^{4-2p} g_{YM}^2 \frac{z^{\frac{(p-7)(4-p)}{2p}}}{\chi^{\frac{p-5}{8}}} \chi^{-1}
\]

\[
\bar{F}_{-x_1\cdots x_{p-1}z} = -\tilde{F}_{+x_1\cdots x_{p-1}z}, \quad \bar{B}_{++} = U(z), \tag{17}
\]

where we have used \(\tilde{\cdot}\) sign to denote the background fields in type II* theory. The functions \(f, \chi, U\) are the same as given in \([8]\). The \(\bar{x}^+\) is new (dual) time coordinate. The solutions \(\text{(17)}\) are generally recognized as Ep-branes\footnote{An Ep-brane has \(p\)-dimensional Euclidean world-volume and appears as a solution in type II* A/B string theories \([20]\). Note, for all Ep-branes, especially the time coordinate counts as one of the \((10-p)\) transverse (Dirichlet) directions, whose all world-volume (Neumann) coordinates are otherwise Euclidean. An E0-brane is localized in 10-dimensional spacetime.}. The coordinates patch \((x_1, \cdots, x_{p-1}, x^-)\) in eq. \((17)\) defines a \(p\)-dimensional Euclidean world-volume of an Ep-brane. In the present case the Ep-branes also have nontrivial \(B_{\mu\nu}\) field, which implies the presence of a fundamental string, stretched along \(x^-\) direction along the world-volume. Note that the time coordinate \(\bar{x}^+\) should be treated as the Dirichlet coordinate and is obviously transverse to the Ep-brane world-volume. The Ep-branes are fundamentally charged under \(\tilde{A}_{(p)}\), the \(p\)-form RR potential. It is remarkable that the dynamical exponent of time is simply \(a = 1\) for all the above Ep-branes in \((17)\), while one spatial coordinate, namely \(x^-\), has an effective dynamical exponent \(\frac{2p-12}{p-5}\) (note \(\chi \equiv \left(\frac{z_0}{z}\right)^{\frac{2p-14}{p-5}}\)). That generates asymmetric scaling which may be alluded to the presence of fundamental string in the Ep solutions.

### 3.1 ‘D3 with \(a = -1\)’ to ‘E3 with \(a = 1\)’

For the simplification purpose, we shall only discuss the D3-brane backgrounds here, but the procedure can be repeated for all other Dp brane cases also. The Schrödinger-type \((a = -1)\) D3-brane bubble vacua in type IIB string theory, read from eq. \((5)\), is

\[
ds^2 = R_3^2 \left[ -\frac{1}{4} \frac{z^2}{z_0^2} [dx^+]^2 + (1 + f) \frac{z_0^4}{z^4} [dx^-]^2 + \frac{z_0^4}{z^6} f (dx^-)^2 + \frac{d\bar{x}^2_{(2)}}{z^2} + \frac{dz^2}{f z^2} + d\Omega^2_{(5)} \right]
\]

\[
e^{2\phi} = \frac{g_{YM}^2}{2\pi}, \quad F_{(5)}^b = 4R_3^4(1 + \star)\omega_5. \tag{18}
\]

where \(F_{(5)}\) is self-dual, \(\omega_5\) is the unit volume element over unit \(S^5\), and \(f(z) = 1 - \frac{z^4}{z_0^4}\). Note that \(x^+\) is the time coordinate, say with its (Euclidean) period \(\beta\). Then the period
of Euclidean time, fixed by a measurement say at \( z = z_0 \) inside the bulk, will be \( \frac{\beta R_3}{z_0} \). By making TT-duality along \( x^+ \) direction and we shall obtain the corresponding time-dual solution of type II*A theory. From (17) we obtain the time dual solution as

\[
d\tilde{s}^2_{E3} = R_3^2 \left[ \frac{z_0^4}{z^6} f(dx^-)^2 + \frac{d\tilde{x}^2(2)}{z^2} + \frac{dz^2}{z^2} - \frac{4z_0^4}{R_3^4 z^2} (d\tilde{x}^+)^2 + d\Omega_5^2 \right]
\]

\[
e^{2\phi_a} = e^{2\phi_b} \frac{4z_0^4}{R_3^2 z^2}, \quad \tilde{B}_{+-} = (1 + f) \frac{z_0^4}{z^4},
\]

\[
\tilde{F}^a_{-x_1 x_2 z} = -F^b_{(+) - x_1 x_2 z}
\]  

(19)

We have used the suffix \( a(b) \) in order to distinguish type II*A and type IIB solutions. The above solution (19) is an E3-brane bubble. Three coordinates \((x^-, x_1, x_2)\) in (19) define 3-dimensional Euclidean world-volume of E3-brane. Note that the dual time coordinate \( \tilde{x}^+ \) will be treated as one of the Dirichlet coordinates and is obviously transverse to the E3-brane world-volume. Also note that the period of dual (Euclidean) time, would be taken \( \tilde{\beta} = \frac{1}{\beta} \). Correspondingly the period of dual time inside the bulk, measured at \( z = z_0 \), becomes \( \frac{2z_0}{R_3} \tilde{\beta} \). The E3-branes are fundamentally charged with \( F_{(4)} \) RR field strength.

Since we have \( x^- \sim x^- + 2\pi r^- \), the solutions (19) are smooth near \( z = z_0 \) with the radius \( r^- = z_0/2 \). While near the boundary, as \( z \rightarrow 0 \), the E3 solution (19) simply becomes

\[
ds^2_{E3} \simeq R_3^2 \left[ \frac{z_0^4}{z^6} f(dx^-)^2 + \frac{d\tilde{x}^2(2)}{z^2} + \frac{dz^2}{z^2} - \frac{4z_0^4}{R_3^4 z^2} (d\tilde{x}^+)^2 + d\Omega_5^2 \right],
\]

\[
e^{2\phi_a} = \frac{g_5^2 M}{2\pi} \frac{4z_0^4}{R_3^2 z^2}, \quad B_{+-} \simeq 2\frac{z_0^4}{z^4},
\]

\[
F^a_{-x_1 x_2 z} \simeq F^b_{(+) - x_1 x_2 z}
\]  

(20)

which is an anisotropic spacetime. This asymptotic form of metric (20) has an explicit asymmetric scaling property:

\[
z \rightarrow \zeta z, \quad \tilde{x}^+ \rightarrow \zeta \tilde{x}^+, \quad x^- \rightarrow \zeta^3 x^-, \quad x^1 \rightarrow \zeta x^1, \quad x^2 \rightarrow \zeta x^2
\]  

(21)

Note the scaling however requires \( r^- \rightarrow \zeta^3 r^- \), so the compactification radius ought to change alongwith the scaling. It thus is a source of scaling violation. It essentially results in hyperscaling violation in lower dimensional action. Overall the world-volume coordinate \( x^- \) scales differently as compared to the rest, namely \( x^1 \) and \( x^2 \). Also the presence of \( B_{+-} \) breaks the \( SO(3) \) invariance down to rotation in \( x_1 - x_2 \) plane on E3-brane world-volume. However, the dynamical exponent of time is \( a = 1 \), so it behaves like a relativistic field theory. Actually above E3 branes are delocalized along the time \( \tilde{x}^+ \) direction, so the time is not essentially a decoupled direction unlike the 5-sphere in the geometry (19). The field theory thus lives on the boundary of 5-dimensional spacetime

\[
\left[ \frac{z_0^4}{z^6} f(dx^-)^2 + \frac{d\tilde{x}^2(2)}{z^2} + \frac{dz^2}{z^2} - \frac{4z_0^4}{R_3^4 z^2} (d\tilde{x}^+)^2 \right]
\]

In the next section our aim is to determine the entanglement entropy of the boundary theory. Also note that all constant time surfaces in (19) are smooth spatial geometries.
3.2 The entanglement entropy of E3-brane bubbles

Here we shall first assume that the Ryu-Takayanagi proposal works also for the case of Ep-brane solutions of type II* string theories. The reason for this belief is that the entanglement entropy is calculated geometrically as an area of an extremal entanglement surface. There exist unambiguous spatial slices described by \( x^+ = \text{constant surfaces} \) in \((19)\). So we look for a static (constant \( x^+ \)) embedding of a strip-like 3D spatial surface, namely \((x^-, x^1(z), x^2)\) inside the bulk geometry described by \((19)\). The end points of the 3D surface coincide with the boundaries of the strip \(-l/2 \leq x^1 \leq l/2\). The size of the rest of spatial coordinates is taken as \(0 \leq x^- \leq 2\pi r, 0 \leq x^2 \leq l_2\), with the length \(l_2 \gg l\). Considering the metric in \((19)\), we determine the area functional of entanglement surface

\[
\hat{A} \equiv \frac{\pi r^{-l_2} \Theta_5 R_3^8}{G_{10}} \int_{z_*}^{z_e} dz e^{-2\phi_0} \frac{z_0^2}{z^5} \sqrt{f} \sqrt{\frac{1}{f} + \left(\frac{dx^1}{dz}\right)^2} \nonumber \\
= \frac{\pi r^{-l_2} \Theta_5 R_3^8 e^{-2\phi_0} R_2^2}{4z_0^2} \int_{z_*}^{z_e} dz \sqrt{\frac{z_0^2}{z^5}} \sqrt{f} \frac{1}{\sqrt{f} + \left(\frac{dx^1}{dz}\right)^2} \nonumber \\
= \frac{\pi r^{-l_2} L_3^3 R_3^2}{G_5} \sqrt{\frac{f}{z^3}} \int_{z_*}^{z_e} dz \sqrt{f - f^*(\frac{z}{z_0})^6} 
\]

where \(\frac{1}{G_5} = \Theta_5 L_3^3 G_{10}\). Note that \(z_e \approx 0\) denotes the UV cut-off and \(z_*\) is the turning point of the extremal surface inside the bulk. From \((22)\) we can determine that the minimal surface must satisfy the equation

\[
\frac{dx^1}{dz} = \left(\frac{z}{z_*}\right)^3 \sqrt{\frac{f}{f^*(\frac{z}{z_*})^6}} 
\]

where we have denoted \(f^* = f(z_*).\) The identification of the boundary is \(x^1(z_0) = l/2\), while the turning-point itself has the mid-point value \(x^1(z_*) = 0\). Note that we shall have to take \(z_* \leq z_0\) always. Thus we get the expression of the entanglement entropy for the YM theory living on the boundary of the E3 solutions

\[
\hat{S}_{EE}(E3) = \frac{\pi r^{-l_2} (L_3)^3 R_3^2}{G_5} \sqrt{\frac{f}{f^*(\frac{z}{z_*})^6}} 
\]

This is a complete expression for the entanglement entropy of the boundary theory of E3-brane bubbles, where the boundary subsystem is a flat strip of width \(l\) along \(x^1\) and covers the whole of \(x^-\) and \(x^2\). It is interesting that, apart from numerical factors outside the integral in \((24)\), the integral expression of the entanglement entropy is the same as it is found in the case of AdS5-bubble vacua involving D3-branes in \((16)\).

Clearly the two entropy expressions \((24)\) and \((16)\) look similar, but differ in respective numerical factors multiplying them, which we do not expect to be same given that we have evaluated entropy using constant time slices. But the two integrals in them remain essentially the same. It is a well known fact that static AdS-bubble vacua have lowest entropy amongst the asymptotically AdS vacua with same symmetry \([21,2]\). Taking this
fact as our guiding principle, we expect that the measurements of entropy for E3-bubbles should also not deviate from this. The integral form of expression in (24) essentially favours the proposal that the entanglement entropy of E3-brane bubbles is also the lowest amongst all E3-brane vacua having same asymptotic symmetry.

4 Summary

We have studied Dp-brane bubble solutions having conformally AdS asymptotic geometries. Once these are expressed in lightcone coordinates the solutions mimic Schrödinger-like geometries with dynamical exponent of time given by \( a = \frac{2}{p-5} \). Thus \( a \) is essentially negative for \( p < 5 \). These solutions are smooth geometries in the IR and have a cut-off. Thus the entanglement entropy is well defined. To convert these solutions into the solutions with exponent \( a > 0 \), we employed time-like T-duality. The time-like duality of course gives rise to Ep-brane solutions with \( a > 0 \). In these Ep-brane bubble geometries finding static embedding surfaces is rather straightforward. Particularly, we have estimated the entanglement entropy of a strip-like subsystem on the boundary of E3-brane bubble solution. It is found that, barring an overall numerical factor, the entanglement entropy has the same functional form as it were in the case of D3-bubble solutions [2]. This new result involving Ep-branes is interesting but it leaves some unanswered questions. First, why the entanglement entropy has to have the same functional form for E3-bubbles and the AdS5-bubble cases. Does it imply that the entropy of E3-branes counts the right physical degrees of freedom, eventhough the Lagrangian formulation of type II\(^*\) superstrings are not so well understood, as these theories come with negative sign kinetic terms in the RR-sector. We may note that E3 branes preserve supersymmetry but E3-bubbles do not. Secondly, does the entropy (except an overall constant) mean that we could trust type II\(^*\) string theories in asymptotically AdS Ep-brane background, like the ones presented here. Our minimal entropy result for E3-brane bubbles is encouraging and will have some important implications on these issues. We hope to report on some of them in the next work [22].

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A Conformally Schrödinger spacetimes with \( a < 0 \) and Entanglement

A class of conformally Schrödinger spacetimes with negative dynamical exponents of time were reported in [14]. These could be generated by taking double limits of the boosted bubble solutions or by Wick rotations of the corresponding Lifshitz type solutions. These
solutions are (for $p \neq 5$)
\begin{align*}
    ds_{Sch}^2 &= R_{p}^2 z^{p-5} \left[ -\frac{1}{4z^2} \left( \frac{z_s}{z} \right)^{2p-14} (dx^+)^2 + \frac{-dx^+ dx^- + dx^2}{z^2} + \frac{4}{(5-p)^2} \frac{dz^2}{z^2} \right] + d\Omega_{(8-p)}^2 \\
    e^\phi &= (2\pi)^{2-p} R^{p+2} (R_{p}^4 z^{p-5})^{\frac{3-p}{4}}
\end{align*}
with $(p+2)$ form RR-flux. The $z_s$ is an intermediate scale in the IR regime. One can see that there is an asymmetric scaling involving the coordinates as [14]
\begin{align*}
    z \rightarrow \xi z, \quad x^- \rightarrow \xi^{2-a} x^-, \quad x^+ \rightarrow \xi^a x^+, \quad \vec{x} \rightarrow \xi \vec{x}
\end{align*}
with dynamical exponent
\begin{align*}
    a = \frac{2}{p-5},
\end{align*}
under which 10-dimensional dilaton and the string metric conformally scale as
\begin{align*}
    g_{MN} \rightarrow \xi^{\frac{p-3}{p-5}} g_{MN}, \quad e^\phi \rightarrow \xi^{\frac{(7-p)(p-3)}{2(p-5)}} e^\phi
\end{align*}
The latter equation is the standard Weyl (conformal) scaling behavior of the near-horizon D$p$-brane vacua [19] and it remains unchanged. Notice that, since $x^-$ is to be taken compact for a nonrelativistic (Schrödinger group) realisation of the CFT, the scaling, namely $x^- \rightarrow \xi^{2-a} x^-$ above, involves jumps in the compactification radius. Thus the rescaling will take us from one compactification radius to another, but preserving the Weyl scaling (28) of the metric. The solutions (25) are invariant under space and time translations and rotations in the Euclidean patch $\mathcal{E}_{(p-1)}$.

It can be noted that the dynamical exponent $a$ is negative for most of the interesting cases of $p=2, 3, 4$ branes in (25). But there are some problems in interpreting these naive Schrödinger vacua.

- The $a < 0$ vacua would give rise to the spectrum which has unrestricted blue-shift in the IR ($g_{++}$ blows up in IR).
- Once $x^-$ is compactified, it leads to an existence of a conical singularity [18].
- Even if $x^-$ is noncompact, there would exist ‘horizon’ like surfaces in the IR region ($z \geq z_s$) [18]. Thus for solutions (25) things start getting worse near $z \sim z_s$.

All above issues are very much interrelated! We try to resolve them by implementing minimal changes in the solutions in this work and have employed solutions with cut-off.

### B Entanglement Entropy

Let us also discuss here the entanglement entropy of a subsystem on the boundary of the Schrödinger vacua (25), at least in the UV region where we can make some definite conclusions. We look for a covariant embedding of a strip-like surface, namely
\begin{align*}
    x^+ = 2y, \quad x^- = -y/2, \quad x^1 = x^1(z)
\end{align*}
inside the bulk geometry (25). The boundaries of the extremal bulk surface do coincide with the boundaries of the strip having the width \(-l/2 \leq x^1 \leq l/2\). The volume of the rest of the coordinates is taken as \(0 \leq x^i \leq l_i\), with \(l_i \gg l\), and we shall take \(0 \leq y \leq 2\pi R\). Considering the metric (25), we determine the area functional of entanglement surface as

\[
A \sim \frac{2\pi R^{-V_{p-2}}\Theta_{8-p}L_p^8}{2G_{10}} \int_{z_*}^{z_c} dz \, z^{9-p} \sqrt{K} \sqrt{\frac{4}{(5-p)^2} + (\partial_z x^1)^2} \tag{29}
\]

where \(G_{10}\) is 10-dimensional Newton’s constant, \(V_{p-2} = l_2l_3\cdots l_{p-1}\) is the total volume, and the function

\[
K(z) = 1 - \left(\frac{z}{z_s}\right)^{\frac{14-2p}{5-p}}.
\]

Note that the integrand of the action becomes undefined in the region \(z \geq z_s\), where \(K\) becomes negative. This is also an indication of the fact that the bulk geometry has unphysical region [18], where the superluminal (blue-shift) effects become prominent in IR. In our notation \(z_* \approx 0\) denotes the UV cut-off to regulate the divergences near the boundary, and \(z_c\) is the turning point of the extremal surface. From (29) we can determine that the minimal surfaces must satisfy the equation

\[
\frac{dx^1}{dz} \equiv \frac{2}{5-p} \left(\frac{z}{z_c}\right)^{\frac{9-p}{5-p}} \frac{1}{\sqrt{K - \left(\frac{z}{z_c}\right)^{\frac{18-2p}{5-p}}}} \tag{30}
\]

where \(z_c\) is a constant and is fixed by the turning point \((z = z_*)\) relation

\[
K^* - \left(\frac{z_*}{z_c}\right)^{\frac{18-2p}{5-p}} = 0 \tag{31}
\]

We have denoted \(K^* = K(z)|_{z = z_*}\). The identification of the boundaries gives \(x^1(z_*) = l/2\), while the turning-point has the mid-point value \(x^1(z_*) = 0\).

Evaluating it, we get the expression of the entanglement entropy for the solutions (25)

\[
S_{EE} \sim \frac{\pi R^{-V_{p-2}}(L_p)^p}{G_{p+2}} \int_{z_*}^{z_c} dz \, z^{9-p} \frac{K}{\sqrt{K - K^*(z)}} \frac{1}{\left(\frac{z}{z_c}\right)^{\frac{18-2p}{5-p}}} \tag{32}
\]

Given that \(K = 1 - \left(\frac{z}{z_c}\right)^{\frac{14-2p}{5-p}}\), the above result is the same as the expression obtained in [18] by doing an explicit compactification along the lightcone of Schrödinger vacua. But note that this formula can only be trusted for the small size subsystems for which \(z_s \ll z_0\), due to the reasons of unregulated IR bulk region. But the generic nature of the expression of \(K\) is such that, it results in reduced entanglement, when compared to the case with \((z_s)^{-1} = 0\) (pure conformally AdS cases).

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