Wick rotation in the tangent space

Joseph Samuel

Raman Research Institute, Bangalore 560 080, India

E-mail: sam@rri.res.in

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Abstract

Wick rotation is usually performed by rotating the time coordinate to imaginary values. In a general curved spacetime, the notion of a time coordinate is ambiguous. We note here, that within the tetrad formalism of general relativity, it is possible to perform a Wick rotation directly in the tangent space using considerably less structure: a timelike, future pointing vector field, which need not be killing or hypersurface orthogonal. This method has the advantage of yielding real Euclidean metrics, even in spacetimes which are not static. When applied to a black hole exterior, the null generators of the event horizon reduce to points in the Euclidean spacetime. Requiring that the Wick rotated holonomy of the null generators be trivial ensures the absence of a ‘conical singularity’ in the Euclidean space. To illustrate the basic idea, we use the tangent space Wick rotation to compute the Hawking temperature by Euclidean methods in a few spacetimes including the Kerr black hole.

Keywords: wick rotation, Euclidean methods, black holes

(Some figures may appear in colour only in the online journal)

1. Introduction

Euclidean spacetimes are analytic continuations of Lorentzian ones. This complex rotation of the time coordinate to imaginary values yields many insights into the quantum nature of space and time. Euclidean methods are motivated by the formal similarity between the propagator for quantum mechanics and the partition function in statistical mechanics. Results obtained by Euclidean methods agree with those obtained by completely different techniques, examples being the Hawking temperature and entropy of black holes [1, 2].

In the usual approach, we identify a time coordinate \( t \) and perform the replacement \( t \rightarrow it \). In quantum field theory in flat spacetime, the time coordinate of an inertial observer is usually selected for Wick rotation. In a general curved spacetime, there need be no preferred time coordinate. In stationary spacetimes, the Killing vector \( \xi \) does give us a preferred notion
of a time \textit{direction}, but does not in general give us a time coordinate. One can use a time function \(t\) partly defined by the condition \(\mathcal{L}_\xi t = 1\). But, if \(t\) is a time function, so is \(t' = t + f(\xi)\), where \(f(\xi)\) is any function constant along the integral curves of \(\xi\). Coordinate Wick rotation (CWR) in \(t\) and \(t'\) gives us different metrics which in general are complex.

In the special case of static spacetimes, the killing vector \(\xi\) is hypersurface orthogonal and determines a time function via \(\xi_a = \lambda \nabla_a t\), where \(t\) is determined up to affine transformations \(t \rightarrow at + b\), with \(a, b\) constants. These transformations commute with Wick rotation and result in the same Euclidean metric in transformed coordinates. This permits us to define CWR for static spacetimes. A good example is the Schwarzschild exterior, where the timelike Killing vector is hypersurface orthogonal. The method results in a real Euclidean metric and correctly gives the thermodynamic properties of the Schwarzschild black hole. Except in special cases \([2]\) like static spacetimes, the procedure does not result in a real Euclidean metric.

The purpose of the present paper is to explore Wick rotation in the tangent space, using a tetrad frame. Wick rotation in flat space involves not only rotating the time coordinate, but also correspondingly rotating tensor field components to imaginary values. In theories with spinor fields, Wick rotation also entails rotating the spinor fields \([3, 4]\). As is well known, in order to handle spinors in curved spacetime, one has to introduce tetrads. One might as well formulate Wick rotation in the tangent space from the beginning using tetrads. As we will see, our method leads to real Euclidean metrics. We do not need a globally defined time coordinate, but can make do with slightly less structure. What we \textit{do} need is a local notion of ‘time’ in each tangent space: a timelike, future pointing vector field \(u^i\). In the special case that \(u^i\) is killing \(D_i u_j = 0\) and hypersurface orthogonal \((u_i = \lambda D_i t)\), these local notions of time mesh together to give a global time coordinate and our method reduces to the usual Wick rotation. We shall also be interested in black hole exterior solutions, where the timelike vector field becomes null on the boundary. Indeed this is the most interesting application of Euclidean methods. In this case, we shall also require that the integral curves of the timelike vector field lie in the exterior region.

In section 2, we describe the formalism we use and in section 3, apply this to some simple two dimensional examples to illustrate the difference between CWR and tangent space Wick rotation (TSWR). We show that unlike CWR, TSWR yields real Euclidean metrics. In section 4, we apply TSWR to the Kerr black hole, arrive at a real Euclidean metric and compute the Hawking temperature. Section 5 is a summary.

2. Wick rotation in frames

Let \((g, \mathcal{M})\) be a Lorentzian spacetime. The real metric \(g_{ij}, i, j = 0, 1, 2, 3\) can be locally expressed in real orthonormal tetrads \(e^a, a = 0, 1, 2, 3\)

\[g_{ij} = e^a_i e^b_j \eta_{ab},\]

where \(\eta_{ab}\) is the Minkowski metric. The tetrad fields uniquely define the \(SO(3, 1)\) connection 1-form \(A^a_b\) via the Cartan structure equation:

\[de^a + A^a_b \wedge e^b = 0.\]

Suppose now that we are also given a timelike future pointing vector field \(u^i\). Let us choose an orthonormal tetrad frame so that \(e^0 = \hat{u}\), which is normalised to \(\hat{u}. \hat{u} = -1\). (Our metric signature is mostly plus.) A Wick rotation in the tangent space consists of the replacement \(e^0 \rightarrow ie^0\), which of course results in a real Euclidean metric.
and the Cartan structure equation yields a Euclidean $SO(4)$ connection $\mathbb{A}^a_b$, which differs from the Lorentzian connection only in that the spacetime components in the internal indices are multiplied by $-i$: $\mathbb{A}^a_b = A^a_b$ if $a, b = 1, 2, 3$ and $\mathbb{A}^a_b = -iA^a_b$ if $a$ or $b$ is 0. ($A^0_0$ vanishes since the connection preserves the Minkowski tensor $\eta_{ab}$.) Thus the holonomy of the Wick rotated frame is just the Wick rotation of the holonomy of the Lorentzian connection. The relation between the Euclidean and Lorentzian metrics can also be expressed without reference to frames:

$$\mathcal{G}_{ij} = g_{ij} + 2\hat{a}_i\hat{a}_j,$$

where the reality of $\mathcal{G}$ is manifest.

An interesting application of Wick rotation is a black hole exterior, the region from which escape to infinity is possible. This region has a null boundary $N = \partial \mathcal{E}$, the event horizon, which has signature $0, +, +$. The null boundary $N$ is ruled by null generators. We suppose the black hole exterior is axisymmetric as obtains for the Kerr metric and indeed all stationary black holes. Since we require that $u'$ is a timelike vector field whose integral curves remain in $\mathcal{E}$, it follows from $u, u < 0$ that $u$ must approach a null generator of $N$ on the boundary.

Let us choose a $u'$ timelike everywhere, but degenerating to null at $N$. We can also choose it to agree with the coordinate time translation at infinity. Performing a Wick rotation leads to a Euclidean metric in the bulk. But at the horizon, we find that each null generator $N$ of $N$ projects down to a single point $p(N)$ in the Euclidean space. A timelike curve $C$ that just grazes the horizon close to $N$ must therefore correspond to a closed curve encircling this point. This demands a periodic identification of the ends of $C$ in the Euclidean space and thus an identification of the whole space by an isometry. In order to avoid a conical singularity at the image of the $N$, we must demand that the Wick rotated holonomy of the null generator be trivial. Indeed, that the holonomy is $H_N = \exp 2\pi J = 1$, where $J$ is a generator in the lie algebra of $SO(4)$. As we will see this condition will give us the Hawking temperature of the black hole.

### 3. Some two-dimensional examples

In this section we describe two simple toy examples to bring out the difference between CWR and TSWR. These are all essentially two-dimensional examples, in which the calculations and visualisation are easy. Of course we must replace $SO(3, 1)$ and $SO(4)$ above by $SO(1, 1)$ and $SO(2)$ to make the correspondence with the two-dimensional case. These are all cases in which both methods CWR and TSWR are possible, (because in two-dimensions, stationary spacetimes are also static). But to make our point we use non-static Painleve–Gullstrand coordinates.

#### 3.1. Rindler and Painleve–Gullstrand

Consider Rindler spacetime (with $g$ an arbitrary positive constant, $R > 0$)

$$ds^2 = -g^2R^2dT^2 + dR^2.$$  

The CWR replacement give us

$$ds^2 = g^2R^2dT^2 + dR^2.$$
The horizon at $R = 0$ is a null curve that goes over to a point in the Euclidean spacetime. Requiring the absence of a conical deficit at $R = 0$ fixes $T$ to be a periodic coordinate with period $\beta = 2\pi/g$. This gives the correct Unruh temperature $T_U = g/(2\pi)$.

Consider now the metric

$$ds^2 = -dt^2 + (dr - v(r)dt)^2,$$

where $v(r) = \sqrt{1 - 2gr}$ and the range of coordinates is $-\infty < t < \infty$, $0 < r < 1/(2g)$. (One can cover Rindler spacetime with two coordinate patches, the Painlevé form for $r < 1/(2g)$ and the Rindler form overlapping with it and continuing to the region beyond.) In these coordinates, the metric appears stationary and not static, because of the presence of the cross terms $drdt$. In fact, the metric (7) is simply a part of Rindler spacetime (5) expressed in Painlevé–Gullstrand coordinates, which are non-singular on the horizon, $r \to 0$. The coordinate transformation $t = T + f(R)$, $r = gR^2/2$, where

$$f(R) = -1/g[\sqrt{1 - g^2R^2} + \log gR - \log(1 + \sqrt{1 - g^2R^2})]$$

relates the two metrics (7) and (5). If we were to use CWR and replace $t$ with $it$ in (7), we would get a complex metric from the terms linear in $dt$.

However using TSWR, we set $e^{i\theta} = u = dr$, $e^{i\phi} = (dr - v(r)dt)$ and the Euclidean metric has the real form

$$ds^2 = dr^2 + (dr - v(r)dt)^2.$$  (9)

The line at $r = 0$ is a null generator of the horizon and goes over to a point in the Euclidean spacetime. This also removes the ‘kink’ in the horizon at the origin of Minkowski space. From (2) we work out the Lorentzian $SO(1, 1)$ connection as

$$A = -v'(r)(dr - v(r)dt)K,$$

where $K$ is the boost, the $2 \times 2$ Pauli matrix $\sigma_i$, which generates the group. Computing the holonomy of the Lorentzian connection along the null curve $r = 0$, we find

$$H(A) = \exp \int_0^\beta K(v'(r))(-v(r)dt) = \exp[-(v^2(r))'\beta K/2]$$

(11)

where $\beta$ is the range of the integration. We require that the holonomy of this null curve when continued to Euclidean values be trivial, i.e., we impose the condition

$$-(v^2(r))'\beta/2 = 2\pi$$

or $\beta^{-1} = T_U = g/(2\pi)$ as before.

**3.2. Schwarzschild and Painleve–Gullstrand**

Very similar considerations apply to the Schwarzschild metric. By spherical symmetry, this is essentially a two-dimensional situation

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2d\Omega^2.$$  (13)

CWR results in a real Euclidean metric \[2\] and the correct Hawking temperature $T_H = 1/(8\pi M)$.

However, if we present the Schwarzschild metric in Painleve–Gullstrand form

$$ds^2 = -dt^2 + (dr - v(r)dt)^2 + r^2d\Omega^2$$

with $v(r) = \sqrt{2M/r}$ the CWR would result in a complex metric. But if we set $e^{i\theta} = u = dt$, $e^{i\phi} = (dr - v(r)dt)$ and then compute the Euclidean metric, it takes the form
\[ ds^2 = dr^2 + (dr - \nu(r)dt)^2 + r^2d\Omega^2. \] (15)

The computation of the Holonomy is then identical to the earlier case. We find that (10) is still true, but now with \((\nu(r))^2 = 2Mr/r\). The condition that the Euclideanized holonomy of the null generator is trivial gives us again (12). Using the new form of \(\nu^2\) gives us the correct Hawking temperature. In both the above examples, it was possible to transform to static coordinates and use CWR rather than TSWR. The spacetimes were static, but viewed in non-static coordinates.

In both these examples, we had a Lorentzian manifold which was geodesically incomplete. Geodesic completion of the Lorentzian manifold would result in recovering the full spacetime—all of Minkowski space in the Rindler example (3.1) and the maximally extended Schwarzschild spacetime in (3.2). However if we perform a Wick rotation, we get a Euclidean metric which is also geodesically incomplete. Completion then leads to a smooth Riemannian manifold, provided certain global identifications are made. These identifications involve modding out by an element of the isometry group and leads to a periodic time coordinate and a well defined temperature. The Euclidean completion of the Schwarzschild exterior has no counterpart of the Schwarzschild interior or the curvature singularity.

4. The Kerr black hole

We now deal with the Kerr metric which is genuinely non-static. In fact, the situation is slightly worse: the metric is not even globally stationary, since the killing vector which is timelike at infinity turns spacelike in the ergoregion. We apply the same procedure, by computing the Holonomy along the null generators of the Kerr horizon and requiring that its Euclidean continuation be trivial. The computation of the holonomy is entirely straightforward if a little tedious. For clarity, we suppress the details and present only the starting point and the final expressions.

The calculation is aided by explicit formulae given in Chandrasekhar [5], whose notation we use below. The useful short forms \(\Delta(r) = r^2 - a^2 - 2Mr = 0\), \(\bar{\rho} = r + ia\cos\theta\) and \(\rho^2 = r^2 + a^2\cos\theta^2\) are standard in the Kerr metric. Note however that our metric signature is opposite to [5]. The Kerr metric in standard Boyer–Lindquist coordinates is described by the null tetrads

\[
\begin{align*}
  l &= 1/\Delta(r^2 + a^2, \Delta, 0, a), \\
  n &= 1/\rho^2(r^2 + a^2, -\Delta, 0, a), \\
  m &= 1/(\bar{\rho}\sqrt{2})(ia\sin\theta, 0, 1, i\cosec\theta),
\end{align*}
\]

from which we find the inverse metric

\[ g^{ij} = (-l'l^j - n'l^j + m'm^j + \bar{m}\bar{m}^j)/2. \] (19)

The null generators of the event horizon at \(\Delta(r) = 0\) are tangential to \(n\). We transform from null tetrads to Minkowskian ones by the transformation \(e^0 = (l + n)/\sqrt{2}\), \(e^1 = (m + \bar{m})/\sqrt{2}\), \(e^2 = (m - \bar{m})/(\sqrt{2}i)\). Writing \(n' = \frac{dx'}{d\tau}\), where \(\tau\) is a parameter along the null generator, we need the integral

1 The curvature singularity obstructs a smooth completion.
along a stretch of the null generators of the Kerr horizon, where $A_{a b}$ is defined by \( (2) \). This matrix $F_{a b} = n^a A_{a b}$ is a $4 \times 4$ matrix

$$
\begin{pmatrix}
0 & \mu & \nu_1 - \nu_2 \\
\mu & 0 & \nu_1 - \nu_2 \\
\nu_1 - \nu_1 & 0 & 0 \\
-\nu_2 & \nu_2 & 0 & 0
\end{pmatrix},
$$

where $\nu = \nu_1 + i\nu_2 = \frac{ia \sin \theta}{\sqrt{2} \rho}$ and $\mu = (r - M)/\rho^2$. The eigenvalues of this matrix are \( \{\mu, -\mu, 0, 0\} \) corresponding to eigenvectors $\nu_1, \nu_2, \nu_3, \nu_4$. $F_{a b}$ generates the transformation

$$
F = \mu K,
$$

where $K$ is a boost $s_1$ in the $\nu_1 - \nu_2$ plane. The $\nu_3 - \nu_4$ plane is left invariant by this transformation.

From the condition of trivial Euclidean holonomy we find

$$
\mu \tau = 2\pi.
$$
We can express the parameter $\tau$ in terms of $\beta$ the $t$ difference between the endpoints of $C$

$$\int_{0}^{\tau} n^i \, dt = \beta = \frac{r^2 + a^2}{\rho^2} \tau.$$

Eliminating $\tau$ in favour of $\beta$ gives

$$\beta^{-1} = T_{\text{H}} = \frac{r - M}{2\pi\left(r^2 + a^2\right)},$$

where all quantities are evaluated at the image of the horizon. This expression gives the correct Hawking temperature for the Kerr black hole.

Geometrically, the Kerr exterior $\mathcal{E}$ is a geodesically incomplete spacetime, since there are infalling geodesics that leave $\mathcal{E}$ within a finite affine parameter (or proper time). Completing $\mathcal{E}$ would lead to the maximally extended Kerr spacetime [5]. If we Euclideanise the Kerr
exterior, along the lines above, we find again that the Euclidean metric is also geodesically incomplete. The completion of the Euclidean metric gives us a smooth Euclidean manifold provided certain global identifications are made. All the points along a null generator of the horizon have to be identified as a single point in the Euclidean manifold. Thus the Euclidean image of the $S^2 \times R$ Kerr horizon is just an $S^2$, which has codimension 2. Geodesics which meet this $S^2$ normally define a planar geometry very similar to the $t - r$ plane of Schwarzschild. The requirement that the holonomy of a small circle encircling this point is trivial gives us the Hawking temperature.

5. Conclusion

Whether or not one wants to consider complex metrics is largely a matter of taste. Especially in dealing with the Kerr metric and the Newman–Penrose formalism, considerable use is made of complex analytic techniques. While these techniques have their value, the physical interpretation often demands that we deal with real metrics. It is entirely possible to use CWR to analyse the Kerr metric, by going into the complex domain. Our main point here is that a real domain analysis is also possible.

In the Lorentzian Kerr geometry, the horizon has topology $S^2 \times \mathbb{R}$, where $\mathbb{R}$ represents the null generators. In the Euclidean version the image of the horizon is an $S^2$ embedded in four-dimensional Euclidean space. Since $S^2$ has codimension two, the picture is locally as shown in figure 1b, where the $S^2$ is represented by a point. Timelike curves in the Lorentzian geometry that graze the horizon map to curves encircling the horizon. A timelike three surface with topology $S^2 \times \mathbb{R}$ just outside the horizon ($\Delta$ slightly positive) maps to a three manifold of topology $S^2 \times S^1$, which encircles the image of the Lorentzian horizon. As would be expected, the region within the Lorentzian horizon disappears entirely in the Euclidean description.

One sometimes writes CWR as $t \rightarrow (\exp i\theta)t$, where $\theta$ varies from 0 to $\pi/2$ and interpolates between Lorentzian and Euclidean manifolds. In the case of TSWR we would have $e_0 \rightarrow (\exp i\theta)e_0$. The main difference is that CWR complexifies the manifold, while TSWR complexifies the (tetrad) fields living on the manifold. If $e_0$ is hypersurface orthogonal ($e_0\kappa = \lambda \nu \kappa t$), one may try to view this as complexifying $t$, the time coordinate. However, this interpretation works only if $\frac{\partial}{\partial t}$ is killing. Otherwise the metric would also have $t$ dependence and we would need to complexify $t$ in the metric functions as well. TSWR works even when $u$ is neither killing nor hypersurface orthogonal over the manifold.

One may wonder about the role of $u$ and the arbitrariness involved in the choice of $u$. $u$ provides us with a local notion of ‘time’, which in semiclassical gravity determines a division of modes into positive and negative frequencies. (See for instance [6] which deals with field theory near a boundary.) Note however, that it is only the behaviour of $u$ in the neighbourhood of the black hole horizon that enters into our discussion. In this neighbourhood, $u$ is essentially equal to the vector field whose integral curves are those of a locally non-rotating observer. The arbitrariness in the choice of $u$ away from the horizon does not affect the calculation of the Hawking temperature.

Another situation in which the CWR fails is the black hole interior. This region is not static even for the Schwarzschild black hole. However, the geometrical ideas of this paper still apply. Let us suppress the angular coordinates for simplicity. The region has a singularity at the origin and a null generator of the horizon at $r = 2M$. The null generator becomes a point in the Euclidean spacetime. A spacelike curve just inside the horizon must therefore encircle this
point. The requirement of trivial holonomy yields the same value for $\beta$ as in the exterior. In this case the entire exterior region disappears from the Euclidean description. We have a Euclidean disc, whose centre is the image of the horizon and the singularity runs around the rim of the disc. The corresponding picture for the Kerr case is far from clear. Even for the Schwarzschild interior, the physical interpretation of this Euclidean picture is unclear.

We have shown that Wick rotation does not need a global notion of time. A timelike vector field is enough. Such a vector field induces a local notion of time in each tangent space. In static situations these local notions of time mesh together to create a global time coordinate, but there are interesting situations where this does not happen. In black hole exteriors, where there is a null boundary, the idea of Wick rotation in the tangent space can be used to derive real Euclidean metrics. The condition that the Euclideanised holonomy of the null generators of the boundary be trivial ensures that there is no conical singularity in the Euclidean space and correctly gives the Hawking (or Unruh) temperature in the examples we have considered.

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**Note added in proof.** After this paper was submitted I learned of [7] and [8], which also attempt to generalise the idea of Wick rotation. I thank Leo Stein for drawing my attention to these very relevant references.

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