We discuss the application of the permutation group $S_N$ to a few problems in hadron physics. In Ref. [11] a method was proposed for matching a quark model Hamiltonian onto the effective Hamiltonian of the $1=N_c$ expansion, which makes use of the transformation properties of the states and operators under $S_N$. This method is used in [13] to obtain information about the spin-flavor structure of the quark interaction Hamiltonian from the spectrum of the negative parity $L=1$ excited baryons. Assuming the most general 2-body quark Hamiltonian, we derive two correlations among the masses and mixing angles of these states which should hold in any quark model. These correlations constrain the mixing angles, and can be used to test for the presence of 3-body quark interactions. We find that the pure gluon-exchange model is disfavored by data, independently of any assumptions about the hadronic wave functions.
1. Introduction

Quark models provide a simple and intuitive picture of the physics of ground state baryons and their excitations [1, 2]. An alternative description is provided by the $1=N_c$ expansion, which is a systematic and model-independent approach to the study of baryon properties [3]. This program can be realized in terms of a quark operator expansion, which gives rise to a physical picture similar to the one of the phenomenological quark models, but is closer connected to QCD. In this context quark models gain additional significance.

The $1=N_c$ expansion has been applied both to the ground state and excited nucleons [4, 5, 6, 7]. In the system of negative parity $L=1$ excited baryons this approach has yielded a number of interesting insights. At leading order in $O(1= N_c)$, the following predictions follow from the contracted $SU(4)_c$ symmetry:

The three towers [5, 8, 9] predicted by $\mathcal{K}$-symmetry for the $L=1$ negative parity $N$ baryons, labeled by $\mathcal{K} = 0; 1; 2$ with $\mathcal{K}$ related to the isospin $I$ and spin $J$ of the $N$ ‘s by $I + J = \mathcal{K}$.

The vanishing of the strong decay width $\Gamma(N_1 2 [N \pi S])$ for $N_1$ in the $\mathcal{K} = 0$ tower, which provides a natural explanation for the relative suppression of pion decays for the $N(1535)$ [5, 8, 9].

The order $O(N_c)$ mass splitting of the $SU(3)$ singlets $\Lambda(1405) - \Lambda(1520)$ in the $[70;1]$ multiplet [7].

The $1=N_c$ expansion for the excited nucleons has been extended also to the first subleading order in $1=N_c$ [4, 5, 6, 7].

In a recent paper [11] we showed how to match an arbitrary quark model Hamiltonian onto the operators of the $1=N_c$ expansion, thus making the connection between these two physical pictures. This method makes use of the transformation of the states and operators under $S_{sp}^{fl}$, the permutation group of $N$ objects acting on the spin-flavor degrees of the quarks. This is similar to the method discussed in Ref. [12] for $N_c=3$ in terms of $S_{3}^{sp}$, the permutation group of 3 objects acting on the orbital degrees of freedom.

The main result of [11] can be summarized as follows: consider a two-body quark Hamiltonian $V_{qq} = \sum_{i<j} O_{ij} R_{ij}$, where $O_{ij}$ acts on the spin-flavor quark degrees of freedom, and $R_{ij}$ acts on the orbital degrees of freedom. Then the hadronic matrix elements of the quark Hamiltonian on a baryon state $\mathcal{B}\downarrow$ contains only the projections $O_\alpha$ of $O_{ij}$ onto irreducible representations of $S_N$, the permutation group of $N$ objects acting on the spin-flavor degrees of the quarks. Then the hadronic matrix elements of the quark Hamiltonian on a baryon state $\mathcal{B}\downarrow$ contains only the projections $O_\alpha$ of $O_{ij}$ onto irreducible representations of $S_N$, the permutation group of $N$ objects acting on the spin-flavor degrees of freedom $\mathcal{B}\downarrow V_{qq} \mathcal{B}\downarrow = \sum_\alpha C_\alpha O_\alpha \downarrow$. The coefficients $C_\alpha$ are related to reduced matrix elements of the orbital operators $R_{ij}$, and are given by overlap integrals of the quark model wave functions.

The explicit calculation in Ref. [11] confirms the $N_c$ power counting rules of Ref. [4, 5], in particular the leading order $O(N_c^0)$ contribution to the mass coming from the spin-orbit interaction $s\mathcal{L}$, and confirms in a direct way the prediction of the breaking of the $SU(4)$ spin-flavor symmetry at leading order in $N_c$ [4]. The calculation in Ref. [11] confirms that the nonrelativistic quark model with gluon mediated quark interactions displays the same breaking phenomenon.
Table 1: The most general two-body spin-flavor quark interactions and their projections onto irreducible representations of $S_3$, the permutation group of three objects acting on the spin-flavor degrees of freedom. $C^2_F = \frac{T^2 - 1}{2}$ is the quadratic Casimir of the fundamental representation of $SU(F)$.

| Operator           | $O_{ij}$ | $O_S$ | $O_{MS}$ |
|--------------------|----------|-------|----------|
| Scalar             |          | $T^2$ | $T^2$    |
|                    | $t^a_{ij}$ | $S^2 \frac{9}{7}$ | $S^2 \frac{9}{7}$ |
|                    | $s_i t^a_j$ | $G^2 \frac{9}{7} C_2(F)$ | $3g_1 G_c \tilde{G} + \frac{3}{5} C_2(F)$ |
| Vector (symm)      |          | $L \cdot S$ | $2 \frac{1}{4} L S^1_c + (L^i S^1_c T^a + L^i t^a C^1_c)$ |
|                    | $s_i + s_j$ | $L^i j \xi G^a t^a a \xi$ | $L^i j \xi G^a t^a a \xi$ |
|                    | $s_i \xi$ | $L^i j \xi G^a t^a a \xi$ | $L^i j \xi G^a t^a a \xi$ |
|                    | $s_i \xi$ | $L^i j \xi G^a t^a a \xi$ | $L^i j \xi G^a t^a a \xi$ |
| Tensor (symm)      |          | $L^i j \xi G^a t^a a \xi$ | $L^i j \xi G^a t^a a \xi$ |
|                    | $s_i ^a t^b_j$ | $L^i j \xi G^a t^a a \xi$ | $L^i j \xi G^a t^a a \xi$ |
|                    | $s_i ^a t^b_j$ | $L^i j \xi G^a t^a a \xi$ | $L^i j \xi G^a t^a a \xi$ |
|                    | $s_i ^a t^b_j$ | $L^i j \xi G^a t^a a \xi$ | $L^i j \xi G^a t^a a \xi$ |

Another important conclusion following from the $S_N$ analysis is that operators depending on excited and core quarks are indeed required for a correct implementation of the $1=N_c$ expansion, in contrast to the approach of Ref. [17] which does not include such operators.

Any particular model of quark interactions, e.g. the one-gluon exchange model (OGE) [1], or the Goldstone boson exchange model (GBE) [14], predicts a distinct hierarchy among the coefficients $C_\alpha$ of the $1=N_c$ expansion. This prediction can be used to discriminate among models by confronting it against the observed values of the coefficients.

In a recent paper [13] we used the $S_N$ approach to study the predictions of the quark model with the most general 2-body quark interactions, and to obtain information about the spin-flavor structure of the quark interactions from the observed spectrum of the $L=1$ negative parity baryons. This talk summarizes the main results of this paper.

2. The most general two-body quark Hamiltonian

The most general 2-body quark interaction Hamiltonian in the constituent quark model can be written in generic form as $V_{qq} = \sum_{i<j} V_{qq}(i,j)$ with

$$V_{qq}(i,j) = \sum_k f_{0,k}(r_{ij}) O_S(k)(i,j) + f_{1,k}^a(r_{ij}) O^a_{V,k}(i,j) + f_{2,k}^{ab}(r_{ij}) O^{ab}_{T,k}(i,j);$$

where $O_S O^a_{V}, O^{ab}_{T}$ act on spin-flavor, and $f_k(r_{ij})$ are functions of $r_{ij} = r_i - r_j$. Their detailed form is unimportant for our considerations. $a \in \{1,2,3\}$ denote spatial indices.

We list in Table 1 a complete set of spin-flavor 2-body operators with all possible Lorentz structures allowed by the orbital angular momentum $L = 1$. Columns 3 and 4 of Table list the projections of the spin-flavor operators $O_S$, $O^a_{V}, O^{ab}_{T}$ onto the irreducible representations of the $S_3$ permutation group, computed as explained in Ref. [13]. The representation content depends on
3. Correlations

The symmetry of $O_{ij}$ under the permutation $\{ij\}$ the symmetric operators $O_{ij}$ are decomposed as $S + MS$, and antisymmetric $O_{ij}$ as $MS + A$.

The symmetric $S$ projection depends only on quantities acting on the entire hadron $S^i; T^a & G^ia$, while the mixed-symmetric $MS$ operators depend on operators acting on the core and excited quarks. We express them in a form commonly used in the application of the $1=N_\ell$ expansion \footnote{2}, according to which their matrix elements are understood to be evaluated on the spin-flavor state $\Phi(SI)j$ constructed as a tensor product of an excited quark with a symmetric core with spin-flavor $S_c = L_c$. The antisymmetric operators contain also an $A$ projection; its orbital matrix element vanishes for $N_c = 3$ because of $T$-invariance \footnote{1,\footnote{2}}, such that these operators do not contribute, and are not shown in Table \footnote{1}.

The orbital matrix elements yield factors of $L_i^j; L_2^{ij} = \frac{1}{3} \epsilon L_i^j; L_j^i \gamma _5 \frac{1}{3} \delta ^{ij}L(L + 1)$, which are the only possible structures which can carry the spatial index.

From Table \footnote{1} one finds that the most general form of the mass operator in the presence of 2-body quark interactions is a linear combination of 10 operators

$$
O_1 = T^2_1; O_2 = S^2_1; O_3 = s_1 S_c; O_4 = L_1 S_c; O_5 = L_1; I_5 O_6 = L^1 g^ia \ ;
$$

$$
O_7 = L^1 g^{ia} T^a_c; O_8 = L^1 g^{ia} S^i_c; O_9 = L^{ij}_2 S^i_c; O_{10} = L^{ij}_2 g^{ia} G^ia.
$$

This gives the most general form of the hadronic mass operator of the negative parity $L = 1$ states allowing only 2-body quark operators.

3. Correlations

The $L = 1$ quark model states include the following SU(3) multiplets: two spin-1/2 octets \footnote{1}, \footnote{2}, two spin-3/2 octets \footnote{1}, \footnote{2}, one spin-5/2 octet \footnote{1}, \footnote{2}, two decuplets \footnote{1}, \footnote{2} and two singlets \footnote{1}, \footnote{2}. States with the same quantum numbers mix, and we define the relevant mixing angles in the nonstrange sector as

$$
N(1535) = \cos \theta_{N1} N_{1=2} + \sin \theta_{N1} N_{1=2}^0 \ ;
$$

$$
N(1520) = \cos \theta_{N3} N_{3=2} + \sin \theta_{N3} N_{3=2}^0 \ ;
$$

$$
N(1650) = \sin \theta_{N1} N_{1=2} + \cos \theta_{N1} N_{1=2}^0 \ ;
$$

$$
N(1700) = \sin \theta_{N3} N_{3=2} + \cos \theta_{N3} N_{3=2}^0.
$$

It turns out that the 11 coefficients $C_0$ to $C_{10}$ contribute to the mass operator of the negative parity $N$ states only in 9 independent combinations: $C_0; C_1; C_2 = 2; C_3 = C_4; C_5 = C_6 = C_7 = C_8$ + $C_9 = 2; C_{10} = 3$. This implies the existence of two universal relations among the masses of the 9 multiplets plus the two mixing angles, which must hold in any quark model containing only 2-body quark interactions.

The first universal relation involves only the nonstrange hadrons, and requires only isospin symmetry. It can be expressed as a correlation among the two mixing angles $\theta_{N1}$ and $\theta_{N3}$ (see Fig. \footnote{1} left)

$$
\frac{1}{2} (N(1535) + N(1650)) + \frac{1}{2} (N(1535) - N(1650)) (3 \cos \theta_{N1} + \sin 2 \theta_{N1})
$$

$$
= \frac{7}{9} (N(1520) + N(1700)) + (N(1520) - N(1700)) \frac{3}{5} \cos 2 \theta_{N3} + \frac{e}{5} \sin 2 \theta_{N3}
$$

$$
= 2 \Delta_{s=2} + 2 \Delta_{3=2} + \frac{9}{5} N_{5=2}.
$$
The yellow square corresponds to the values used in Ref. \[6, 7\]. The second point is favored by a 1\(\exp\) compared against the experimental value \(\Lambda = 1481 \pm 15\) MeV, with \(\Lambda = 1481 \pm 15\) MeV.

This correlation holds also model independently in the 1=\(N_c\) expansion, up to corrections of order 1=\(N_c^2\), since for non-strange states the mass operator to order \(O(1=\Lambda)\) \[8\] is generated by the operators in Eq. \(3.2\). An example of an operator which violates this correlation is \(L^j g^{ja} \xi S^j_c \xi G^{ia} c\), which can be introduced by 3-body quark forces.

On the same plot we show also the values of the mixing angles obtained in several analyses of the \(N^+\) \(N\pi\) strong decays and \(N\) hadron masses. The two black dots correspond to the mixing angles \((\theta_{N1} ; \theta_{N3}) = (22 \beta ; 136 \delta )\) and \((22 \beta ; 161 \delta )\) obtained from a study of the strong decays in Ref. \[15\]. The second point is favored by a 1=\(N_c\) analysis of photoproduction amplitudes Ref. \[14\]. The yellow square corresponds to the values used in Ref. \[8\] \((\theta_{N1} ; \theta_{N3}) = (35 \delta ; 174 \omega )\), and the triangle gives the angles corresponding to the solution 1\(^0\) in the large \(N_c\) analysis of Ref. \[8\], \((\theta_{N1} ; \theta_{N3}) = (114 \delta ; 80 \omega )\). All these determinations (except the triangle) are compatible with the ranges \(\theta_{N1} = 0 \quad 35 \quad 0 \quad 135 \quad 180\). They are also in good agreement with the correlation Eq. \(3.2\), and provide no evidence for the presence of 3-body quark interactions.

The second universal expression relays the spin-weighted SU(3) singlet mass \(\Lambda = 1/6(2\Lambda_{1=2} + 4\Lambda_{3=2})\) in terms of the nonstrange hadronic parameters

\[
\Lambda = \frac{1}{6} (N(1535) + N(1650)) + \frac{17}{15} (N(1520) + N(1700)) + \frac{3}{5} N_{5=2}(1675) + \Delta_{1=2}(1620) \]

\[
\frac{1}{6} (N(1535) + N(1650)) \cos 2\theta_{N1} + \sin 2\theta_{N1}) + (N(1520) + N(1700)) \cos 2\theta_{N3} \]

The rhs of Eq. \(3.3\) is plotted as a function of \(\theta_{N1}\) in the right panel of Fig. \[1\], where it can be compared against the experimental value \(\Lambda = 1481 \pm 15\) MeV. Allowing for SU(3) breaking effects 100 MeV, this constraint is also compatible with the range for \(\theta_{N1}\) obtained above from direct determinations of the mixing angles.

Combining the Eqs. \(3.2\) and \(3.3\) gives a determination of the mixing angles from hadron masses alone, in contrast to their usual determination from \(N^+\) \(N\pi\) decays. The green area in Fig. \[1\] shows the allowed region for \((\theta_{N1} ; \theta_{N3})\) compatible with a positive SU(3) breaking correction in \(\Lambda\) of 100 30 MeV. One notes a good agreement between this determination of the mixing angles and that from \(N^+\) \(N\pi\) strong decays.
4. Spin-flavor structure of the quark interactions

We derive next constraints on the spin-flavor structure of the quark interaction, which can discriminate between models of effective quark interactions. There are two popular models used in the literature. The first model is the one-gluon exchange model (OGE) \([1]\) which includes operators in Table 1 without isospin dependence. Expressed in terms of the operator basis \(O_{1\ 10}\) this gives the constraints

\[
OGE: \quad C_4 = C_6 = C_7 = C_{10} = 0 : \quad (4.1)
\]

An alternative to the OGE model is the Goldstone boson exchange model (GBE) \([14]\). In this model quark forces are mediated by Goldstone boson exchange, and the quark Hamiltonian contains all the operators in Table 1 which contain the flavor-dependent factor \(t^a_t^j\). The coefficients of the hadronic Hamiltonian \(C_i\) satisfy the constraints \((F = 3\) is the number of light quark flavors\)

\[
GBE: \quad C_1 = \frac{F}{4} C_3 ; \quad C_5 = C_9 = 0 : \quad (4.2)
\]

We would like to determine the coefficients \(C_i\), and compare their values with the predictions of the two models Eqs. \((4.1), (4.2)\). As mentioned, only 9 combinations of the 11 coefficients can be determined from the available data: \(C_0 \neq C_1 = C_3 = 0\), \(C_{2} = 2 C_2 + C_3 C_4 C_5 C_6 C_7 C_8 + C_{10} = 4 C_9\). In particular, as the coefficients of the spin-orbit interaction terms \(C_{4,7}\) can be determined, we propose to use their values to discriminate between different models of quark interaction.

The values of \(C_{4,7}\) can be compared with the hierarchy expected in each model. In the OGE model the flavor-dependent operators have zero coefficients \(C_{6,7} = 0\) while in the GBE model the spin-orbit interaction of the excited quark vanishes \(C_5 = 0\).

The coefficient \(C_5 = 75 \text{ MeV}\) is fixed by the \(\Lambda_{3-2}\) splitting. This indicates the presence of the operators \(s_i\) in the quark Hamiltonian, which is compatible with the OGE model. A suppression of the coefficients \(C_{6,7}\) would be further evidence for the OGE model. We show in Fig. 2 the coefficients of the spin-orbit operators \(C_{6,7}\) as functions of \(\theta_{N1}\). Within errors small values for \(C_7\) are still allowed, however no suppression is observed for \(C_6\). This indicates the presence of the operators \(s_i\) in the quark Hamiltonian. These results show that the quark Hamiltonian is a mix of the OGE and GBE interactions.

In the pure OGE model Eq. \((4.1)\) the 7 nonvanishing coefficients \(C_i\) can be determined from the 7 nonstrange \(N, \Delta\) masses (assuming only isospin symmetry but no specific form of the wave functions). This fixes the mixing angles, and the \(\Lambda_{3-2}\) splitting, up to a 2-fold ambiguity. The allowed region for mixing angles is shown as the violet region in Fig. 1 left, and the central values as diamonds \((\theta_{N1}, \theta_{N3}) = (64.2, 98.2); (114.5, 88.2)\). Note that they are different from the angles obtained in the Isgur-Karl model \((31.7, 73.6)\) in Refs. \([2, 18, 19]\).

The violet region near \(\theta_{N1} = 0 \text{ is consistent with the determinations from strong decays and from the SU(3) universal relation Eq. (3.3), but is ruled out by the prediction for the } \Lambda \text{ splitting, in agreement with the non-zero value of } C_6 \text{ that can be read off from Fig. 3. This implies that the pure OGE model is disfavored}^1.

\(^1\) Note that this argument neglects possible long-distance contributions to the \(\Lambda\) splitting, due to the proximity of the \(\Lambda(1405)\) to the \(KN\) threshold. Such threshold effects are not described by the quark Hamiltonian Eq. (2.1), and their presence could invalidate the prediction of the \(\Lambda\) splitting in the OGE model.
Figure 2: The coefficients of the spin-orbit operators $C_{6,7}$ as functions of the mixing angle $\theta_{N1}$, in the quark model with the most general 2-body interactions. The green area is obtained by imposing the $\bar{\Lambda}$ constraint.

5. Conclusions

We discussed a few applications of the permutation group $S_N$ to the study of baryonic properties in the quark model. The applications are based on a simple result: the spin-flavor contents of the mass operator is directly related to the projections of the spin-flavor part of the quark interaction onto irreducible representations of $S_N$. Using this result, any quark Hamiltonian can be matched onto the effective Hamiltonian of the $1=N_c$ expansion.

Following Ref. [13], we discussed the predictions of the most general 2-body quark Hamiltonian for the spin-flavor structure of the negative parity $L=1$ excited baryons, without making any assumptions about the orbital hadronic wave functions. We derive two universal correlations among masses and mixing angles, which constrain the mixing angles, and can test for the presence of 3-body quark interactions. In addition, we derive constraints on the spin-flavor structure of the quark forces from the observed spectrum, and conclude that the gluon-exchange model is disfavored by data, independently on any assumptions about the hadronic wave functions.

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