NATURAL DIFFERENTIABLE STRUCTURES ON STATISTICAL MODELS AND THE FISHER METRIC

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Abstract. In this paper I discuss the relation between the concept of the Fisher metric and the concept of differentiability of a family of probability measures. I compare the concepts of smooth statistical manifolds, differentiable families of measures, $k$-integrable parameterized measure models, diffeological statistical models, differentiable measures, which arise in Information Geometry, mathematical statistics and measure theory, and discuss some related problems.

Dedicated to Professor Shun-ichi Amari on his 88th birthday.

1. Introduction

In 1945 Rao introduced the Fisher metric, also called the Fisher-Rao metric, on smooth finite dimensional parametric families of probability densities and, equipped with this tool, he derived the Cramér-Rao inequality \[32\]. A smooth structure on a set may give information about its qualitative properties as well as simplify computations involving the elements of the set. Between 1970 and 1980 Chentsov and Amari independently discovered a one-parameter family of affine connections, called $\alpha$-connections, that are crucial in study of the geometry of smooth finite dimensional parametric families of probability densities, see also Remark \[27\] below. Using the newly discovered information geometric structure consisting of the $\alpha$-connections and the Fisher metric, Amari proposed a differential geometric framework for constructing a higher order asymptotic theory of statistical inference. Inspired by these results, Lauritzen introduced the concept of (finite dimensional) statistical manifolds \[1\]. In this article we re-examine the

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\[1\] See Historical Remarks in \[2\] Subsection 1.3] for more detailed accounts.
relation between the concept of the Fisher metric and the concepts of smooth parametric and nonparametric statistical models, statistical manifolds and their generalizations, which appear in measure theory, functional analysis, mathematical statistics and Information Geometry.

The present article is organized as follows. In Section 2 we discuss the concept of a smooth statistical model, the Fisher metric and the Amari-Chentsov tensor that has been proposed by Amari in his seminal work [4], see also his book with Nagaoka [5], and independently proposed by Chentsov [13]. We also recall the related concept of a statistical manifold, proposed by Lauritzen [21]. In Section 3 we recall the concept of a differentiable family of measures, following Bogachev [9], and relate it to the concept of a smooth statistical model, the concept of a \( k \)-integrable parameterized measure model, introduced by Ay-Jost-Lê-Schwachhöfer in [1] and [3], and the concept of a differentiable measure introduced by Fomin [14] and by Skorohod [33]. In Section 4 we consider a motivating example due to Friedrich [15], and explain it by the concept of a diffeological statistical model introduced by Lê [24] that combines the concept of a smooth diffeology due to Soureau [34] and the concept of a differentiable family of measures. We compare the concept of the Fisher metric on \( C^k \)-diffeological statistical models and the related Crâmer-Rao inequality with existing concepts of the Fisher metric and Crâmer-Rao inequalities in the literature. In the final section we summarize our discussion and propose few questions for future research.

2. Smooth statistical models and statistical manifolds

Motivated by the local character of asymptotic estimation in statistical inference, in 1980 Amari introduced the information geometric structure on \( \textit{finite dimensional smooth parametric families of probability density functions} \ p(\cdot, \theta) \) on a measurable space \( \mathcal{X} \) with respect to some common dominating \( \sigma \)-finite nonnegative measure \( \mu \) on \( \mathcal{X} \) and parameterized by \( \theta \) belonging to some open subset \( \Theta \subset \mathbb{R}^n \) [4, p. 2,4,11], [5, p.26]. The smoothness of a family of probability density functions \( \{p(x, \theta), \ x \in \mathcal{X}, \theta \in \Theta\} \) is expressed via the smoothness of the function \( p(x, \theta) \) in the parameter \( \theta \in \Theta \). Let \( \mathcal{F}(\mathcal{X}) \) denote the set of all real valued functions on \( \mathcal{X} \). Amari also required that the map \( \hat{p} : \Theta \to \mathcal{F}(\mathcal{X}), \theta \mapsto p(\cdot, \theta) \), is injective. Amari called a smooth family of probability density functions satisfying the aforementioned properties an \( n \)-dimensional statistical model. In their influential book Amari-Nagaoka also assumed the following two regularity conditions (i) and (ii).
(i) The following rule

\[ \int_X \partial_V p(x, \theta) \, d\mu(x) = \partial_V \int_X p(x, \theta) \, d\mu(x) = 0 \]

holds for any tangent vector \( V \in T_\theta \Theta = \mathbb{R}^n \) \([2.4]\). (This formula is used for simplification of many computations on statistical models.)

(ii) \( p(x, \theta) > 0 \) for all \( x \in X \) and all \( \theta \in \Theta \). (This assumption makes sure that the set \( \text{sppt} p : = \{ x \in X | p(x, \theta) > 0 \} \) does not depend on \( \theta \) \([5, p. 27-28]\), and hence, all the probability measures \( p(\cdot, \theta) \mu | \theta \in \Theta \) are equivalent.

For \( V, W \in T_\theta \Theta \) we set

\[ g_\theta(V, W) = \int_X \partial_V \log p(x, \theta) \partial_W \log p(x, \theta) \, p(x, \theta) \, d\mu(x). \]

Assuming that the RHS of (2.2) is a finite number for all \( \theta \) and \( V, W \), the quadratic form \( g \) is called the Fisher metric on an \( n \)-dimensional statistical model \( \{ p(\cdot, \theta) | \theta \in \Theta \subset \mathbb{R}^n \} \). Note we can rewrite (2.2) as follows

\[ g_\theta(V, W) = 4 \int_X \partial_V \sqrt{p(x, \theta)} \partial_W \sqrt{p(x, \theta)} \, d\mu(x). \]

Formula (2.3) implies that the Fisher metric is well-defined at \( \theta \) if and only if the function \( \partial_V \sqrt{p(\cdot, \theta)} \) belongs to \( L^2(X, \mu) \) for any \( V \in \mathbb{R}^n \).

Now we assume further that \( g_\theta \) is positive definite for all \( \theta \in \Theta \). Under this assumption the Fisher metric \( g \) is a Riemannian metric on \( \Theta \).

On a smooth Riemannian manifold \( (M, g) \), one considers the Levi-Civita connection \( \nabla^{LC} = \nabla^{LC}(g) \) which is the unique torsion-free metric connection on \( M \). Any other affine connection \( \nabla \) on \( M \) is differed from the Levi-Civita connection by a tensor \( T^* \in \Gamma(T^* M \otimes T^* M \otimes TM) \), namely for any \( V \in TM \) and any vector field \( X \) on \( M \) we have

\[ \nabla_V X = \nabla^{LC}_V X + T^*(V, X). \]

Clearly, the connection \( \nabla \) is torsion-free if and only if \( T \) is symmetric.

Amari’s \( \alpha \)-connections \( \nabla^\alpha, \alpha \in \mathbb{R} \), on a statistical model of probability density functions \( \{ p(\cdot, \theta) | \theta \in \Theta \} \), where the Fisher metric \( g \) exists as a Riemannian metric, is defined as follows

\[ \nabla^\alpha = \nabla^{LC}(g) - \frac{\alpha}{2} T^*. \]

Here \( T^* \) satisfies the following equation for any \( V_1, V_2, V_3 \in T_\theta \Theta = \mathbb{R}^n \)

\[ g(T^*(V_1, V_2), V_3) = \int_X \partial_{V_1} \log p(x, \theta) \partial_{V_2} \log p(x, \theta) \partial_{V_3} \log p(x, \theta) \, p(x, \theta) \, d\mu(x). \]
(iv) Amari-Nagaoka posed the assumption that the covariant tensor $T$, defined by the equation $T(V_1, V_2, V_3) = g(T^*(V_1, V_2), V_3)$, is well-defined for any $\theta \in \Theta$. In this case, since $T$ is 3-symmetric, $\nabla^\alpha$ is a torsion-free affine connection.

**Remark 2.1.** The $\alpha$-connections have been studied by Chentsov in the case of discrete and finite sample space $\mathcal{X}$ [13]. Chentsov proved that the Fisher metric and the family of $\alpha$-connections on smooth families of probability measures over finite sample spaces are the unique up to multiplication Riemannian metric and family of connections that are invariant under sufficient statistics [13, Theorem 11.1, p.159], [13, Theorems 12.2, 12.3, p. 175, 178]. For a measurable space $\mathcal{X}$ we denote by $\mathcal{P}(\mathcal{X})$ the set of all probability measures on $\mathcal{X}$. Recall that a measurable mapping $f : \mathcal{X} \to \mathcal{Y}$ is called a *sufficient statistic for a family* $\{\mu_\theta \in \mathcal{P}(\mathcal{X}), \theta \in \Theta\}$, if there exists a Markov kernel $P : \mathcal{Y} \times \Sigma_\mathcal{X} \to \mathbb{R}$ such that $P(y|\cdot), y \in \mathcal{Y}$, is the conditional probability measure for $\mu_\theta$ relative to $f$ for any $\theta \in \Theta$ [13, Definition 2.5, p. 28].

Chentsov’s theorem has been generalized in various forms by Ay-Jost-Lê-Schwachhöfer [1], [2], by Lê [23] and by Bauer-Bruveris-Michor [7]. Nowadays we call the 3-symmetric tensor $T$ whose values $T(V_1, V_2, V_3)$ is defined in the RHS of (2.4) the *Amari-Chentsov tensor*.

Motivated by the importance of the Fisher metric and the $\alpha$-connections, in [21] Lauritzen proposed the concept of a *statistical manifold* which is a smooth Riemannian manifold $(M, g)$ endowed with a 3-symmetric covariant tensor $T$ [21, Section 4, p. 149]. Lauritzen asked if there is a statistical manifold which does not correspond to a smooth statistical model. In [5, Section 8.4] Amari-Nagaoka listed the Lauritzen question as one of mathematical problems posed by Information Geometry. Inspired by Amari-Nagaoka’s book and their list of problems, Lê gave an answer to Lauritzen’s question in [22], see also [2, Theorem 4.10, p. 222]. Lê’s theorem states that any smooth statistical manifold $(M, g, T)$ can be immersed into the space $\mathcal{P}(\mathcal{X})$ over a countable sample space $\mathcal{X}$, which is finite if $M$ is compact, such that $g$ is induced by the Fisher metric $g$ and $T$ is induced by the immersion from the Amari-Chentsov tensor on the image of $M$ in $\mathcal{P}(\mathcal{X})$.

In [31] Pistone-Sempi endowed each set $\mathcal{P}_\mu$ of all probability measures equivalent to a given reference probability measure $\mu \in \mathcal{P}(\mathcal{X})$ with a structure of an infinite dimensional smooth Banach manifold, which can be provided with the Fisher metric and the Amari-Chentsov tensor, see Example 3.14 in Section 3.3. Other infinite dimensional statistical models that carry a structure of a Banach or Frechét smooth manifold.
endowed with the Fisher metric have been considered recently, see e.g. Bauer-Bruveris-Michor \[7\], Newton \[28\], and reference therein.

3. Differentiable families of measures, differentiable measures and parameterized measure models

The concept of a smooth statistical model considered in the previous section is based on the concept of a smooth mapping from an open subset in a Fréchet space to a space of smooth probability density functions. In this section we shall consider a more general question, what is a differentiable mapping from an open subset in a topological vector space to the space $\mathcal{P}(\mathcal{X})$, or to the space $\mathcal{M}(\mathcal{X})$ of all finite nonnegative measures \(^2\) on $\mathcal{X}$. Families of finite measures smoothly dependent on a parameter arise in theory of random processes, functional analysis, mathematical physics and mathematical statistics see e.g. Bogachev \[9\], Borovkov \[12\], Chentsov \[13\], Pfanzagl \[29\], Pflug \[30\], Strasser \[35\]. All approaches to the concept of a differentiable mapping $f$ into $\mathcal{M}(\mathcal{X})$ exploits the possibility to approximate $f$ locally by a linear mapping into $\mathcal{S}(\mathcal{X}) \supset \mathcal{M}(\mathcal{X})$ and therefore requires to specify a convergence type on $\mathcal{S}(\mathcal{X})$.

3.1. Convergence types and natural topologies on $\mathcal{S}(\mathcal{X})$ and $\mathcal{M}(\mathcal{X})$. Recall that $\mathcal{S}(\mathcal{X})$ endowed with the total variation norm $TV$ is a Banach space and the strong topology $\tau_v$ generated by this norm is compatible with the linear structure on $\mathcal{S}(\mathcal{X})$. It is well-known that the weak topology $\tau_W$ on the Banach space $\mathcal{S}(\mathcal{X})_{TV}$ is also compatible with the linear structure on $\mathcal{S}(\mathcal{X})$. Besides these topologies, one considers also setwise convergence and the associated topology $\tau_s$ which is the weakest topology on $\mathcal{S}(\mathcal{X})$ such that for any $A \in \Sigma_{\mathcal{X}}$ the map $I_A : \mathcal{S}(\mathcal{X}) \to \mathbb{R}, \mu \mapsto \mu(A)$, is continuous. Equivalently, $\tau_s$ is generated by the duality with the space $\mathcal{F}^s(\mathcal{X})$ of all simple functions on $\mathcal{X}$. Note that a sequence of $\mu_n \in \mathcal{S}(\mathcal{X})$ converges setwise if and only if $\mu_n$ converge in the weak topology $\tau_W$, see e.g. Bogachev \[9\] Corollary 4.7.26]. Furthermore, the restriction of $\tau_s$ to $\mathcal{P}(\mathcal{X})$ is not metrizable unless $\mathcal{X}$ is countable, see e.g. Ghosal-van der Vaart \[16\] p. 513.

If $\mathcal{X}$ is a topological space then we consider the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{X})$ unless otherwise stated. Some time we also consider the Baire $\sigma$-algebra $\mathcal{Ba}(\mathcal{X})$ that is the smallest sub-$\sigma$-algebra of $\mathcal{B}(\mathcal{X})$ such that any continuous function on $\mathcal{X}$ is measurable. If $\mathcal{X}$ is a metrizable topological space, then $\mathcal{B}(\mathcal{X}) = \mathcal{Ba}(\mathcal{X})$ by Bogachev \[8\] Corollary 6.3.5, vol. 2, p. 13]. The space $C_b(\mathcal{X})$ consisting of bounded continuous

\(^2\)We shall omit the adjective “nonnegative” to a measure in this paper, and add “signed” to a measure if it is necessary.
functions on a topological space $\mathcal{X}$ is a Banach space with the sup-norm $\|f\|_\infty := \sup_{x \in \mathcal{X}} |f(x)|$. Clearly the canonical embedding $C_b(\mathcal{X})_\infty \to L^\infty(\mathcal{X}, \mu)$ is continuous for any $\mu \in \mathcal{M}(\mathcal{X})$. Hence $C_b(\mathcal{X})_\infty$ is a closed subspace of the Banach space $S(\mathcal{X})'$. We recall that the weak* topology $\tau_w$ on the space $S(\mathcal{X})$, where $\mathcal{X}$ is a topological space endowed with the Borel $\sigma$-algebra, is the weakest topology such that for each $f \in C_b(\mathcal{X})$ the map $I_f : S(\mathcal{X}) \to \mathbb{R}, \mu \mapsto \int_{\mathcal{X}} f \, d\mu$, is continuous. We also denote by $\tau_w$ the restriction of $\tau_w$ on subsets of $S(\mathcal{X})$. It is well-known that if $\mathcal{X}$ is infinite, then $(S(\mathcal{X}), \tau_w)$ is non-metrizable but the restriction of $\tau_w$ to $\mathcal{M}(\mathcal{X})$ is metrizable, if $\mathcal{X}$ is a separable metrizable space, see e.g. Bogachev [10, p. 102].

3.2. Differentiable families of measures and differentiable measures. Let us begin with the concept of a differentiable family of signed finite measures, following Bogachev [9, Section 11.2], which encompasses all notions of differentiable families of measures that have been considered in literature in mathematical statistics to the best of our knowledge, see e.g. Chentsov [13], Pfanzagl [29], Strasser [35], Pflug [30], van der Vaart [36], and references therein. Let $\tau$ be a topology on a vector space $S(\mathcal{X})$.

Definition 3.1. ([9, Definitions 11.2.1, 11.2.11]) (i) A family of signed finite measures $\{\mu(t) \in S(\mathcal{X}) | t \in (a, b)\}$ is called $\tau$-differentiable at a point $\tau_0 \in (a, b)$ if a limit

$$
\mu'(t_0) = \lim_{s \to 0} \frac{\mu(t_0 + s) - \mu(t_0)}{s}
$$

exists in the topology $\tau$.

(ii) The definition of continuity is completely analogous. Higher order differentiability is defined inductively.\footnote{See [11] Definitions 4.1.10, 4.1.11, p. 428 and Example 4.2 below.}

(iii) Let $p \in [1, +\infty)$.

A family of finite nonnegative measures $\{\mu(t) \in \mathcal{M}(\mathcal{X}) | t \in (a, b)\}$, is called $L^p(\nu)$-differentiable, or just $L^p$-differentiable, at a point $t_0$ if there exists a nonnegative $\sigma$-finite measure $\nu$ such that $\mu(t) = f(t)\nu$ for all $t$ and the mapping $\Phi : I \to L^p(\nu), t \mapsto f(t)^{1/p}$, is differentiable at $t_0$. A family of signed finite measures is called $L^p$-differentiable, if its positive and negative parts in the Jordan-Hahn decomposition have this properties.

Remark 3.2. Similarly, we define the differentiability of a family of measures $\{\mu(m) | m \in M\}$ where $M$ is a (possibly infinite dimensional) differentiable manifold, provided that some concept of differentiability of mappings from $M$ to $S(\mathcal{X})$ is chosen (e.g. if $M$ is a normed
space, or $M$ is an open subset of a topological vector space, and the differentiability can be Gâteaux-differentiability or Hadamard differentiability, see also Bogachev-Smolyanov [11, Section 4.1] for the most general concept of differentiability with respect to a system of sets.)

**Remark 3.3.** If a family \( \{ \mu_t \in \mathcal{P}(\mathcal{X}) | t \in (a, b) \} \) is \( \tau_\sigma \)-differentiable, then for any \( t_0 \in (a, b) \) we have

\[
\int_{\mathcal{X}} d\mu'(t_0) = 0.
\]

We assume in the remainder of this Subsection that in the case \( \mathcal{X} \) is a topological space then \( \mathcal{X} \) is a completely regular topological space. Under this assumption, we have \([9, p. 379]\)

\[
\|\mu\| = \sup_{f \in C_b(\mathcal{X}), \|f\|_\infty \leq 1} \int_{\mathcal{X}} f(x) d\mu(x).
\]

Equality (3.1) allows us to formulate strong results concerning \( \tau_w \)-differentiability and compare it with \( \tau_s \)- and \( \tau_v \)-differentiability in Proposition 3.4 below, which is a combination of \([9, Lemma 11.2.3, Theorems 11.2.5, 11.2.6]\) and some assertions in their proofs. Note that (3.1) also holds if we replace \( C_b(\mathcal{X}) \) by \( F^*_\mathcal{X} \). Furthermore, the \( \tau_w \)-differentiability is weaker than the \( \tau_v \)-differentiability and stronger than the \( \tau_s \)-differentiability and shall not be considered in the present Subsection.

**Proposition 3.4.** Let \( -\infty < a < b < \infty \). (1) A family \( \{ \mu(t) \in \mathcal{M}(\mathcal{X}) | t \in (a, b) \} \) is \( \tau_w \)-differentiable at \( t_0 \in (a, b) \) if and only if it is \( \tau_s \)-differentiable at \( t_0 \) and if and only if for any \( A \in \mathcal{B}(\mathcal{X}) \) the function \( t \mapsto \mu_1(A) \) is differentiable at \( t_0 \).

(2) Assume that a family \( \{ \mu(t) \in \mathcal{M}(\mathcal{X}) | t \in (a, b) \} \) is \( \tau_w \)-differentiable. Then the family \( \{ \mu_t \} \) is \( \tau_v \)-continuous, \( \mu'(t) \ll \mu(t) \) for all \( t \in (a, b) \), moreover the family \( \{ \mu_t \} \) is \( \tau_v \)-differentiable for almost every \( t \in (a, b) \). Furthermore there exists \( \nu \in \mathcal{P}(\mathcal{X}) \) such that \( \mu(t) \ll \nu \) for all \( t \). Suppose also that the function \( t \mapsto \|\mu'(t)\| \) is Lebesgue integrable on \( (a, b) \). Then for all \( t \in (a, b) \) we have

\[
\mu(t) - \mu(a) = \int_a^t \mu'(t) \, dt
\]

where the integral on the right is an \( S(\mathcal{X}) \)-valued Bochner integral. If the \( \{ \mu'(t) \} \) is \( \tau_v \)-continuous, e.g. if \( \{ \mu(t) \} \) is twice \( \tau_s \)- or twice \( \tau_w \)-differentiable, then \( \{ \mu(t) \} \) is \( \tau_v \)-differentiable and the corresponding derivative coincides with the \( \tau_s \)-derivative or \( \tau_w \)-derivative respectively.
Remark 3.5. The proof of \( \mu'(t) \ll \mu(t) \) is given in the proof of [9, Theorem 11.2.6], see also [2, p.142], [26, Lemma 2]. The Radon-Nykodým derivative \( \frac{d\mu'(t)}{d\mu(t)} \) is defined as \( L^1(\mu(t)) \) is called the logarithmic derivative of \( \mu(t) \) at \( t \) [9, p. 383], [2, (4.3)] [36, (3.66), (3.68)].

Remark 3.6. Proposition 3.4 implies that a \( \tau_s \) or \( \tau_w \)-differentiable family \( \{\mu(t) \in \mathcal{M}(\mathcal{X}), t \in (a, b)\} \) is dominated by a probability measure \( \nu \in \mathcal{P}(\mathcal{X}) \), so we can write \( \mu(t) = f(t, \cdot) \nu \) where \( f(t, \cdot) \in L^1(\nu) \). Now assume that the function \( t \mapsto \|\mu'(t)\| \) is Lebesgue integrable. By the Lebesgue differentiation theorem, (3.2) implies that the partial derivative \( \partial_t f(t, \cdot) \) exists for a.e. \( t \in (a, b) \) and
\[
\partial_t f(t, \cdot) = \frac{d\mu'(t)}{d\mu(t)} \in L^1(\mu(t))
\]
as an equality for elements in \( L^1(\mu(t)) \).

Example 3.7. Finite dimensional statistical models \( \{p(\cdot, \theta)|, \theta \in \Theta \subset \mathbb{R}^n\} \) in the previous section are regular \( \tau_v \)-differentiable families of dominated measures \( \{\theta \mapsto p(\cdot, \theta)\mu, \theta \in \Theta\} \), where, by Proposition 3.4, we can assume that \( p \in \mathcal{P}(\mathcal{X}) \). The condition that \( \partial_t f(\theta, \cdot) \) belongs to \( L^1(\mu) \) is necessary for the existence of the LHS of (2.1). The equality (2.1) follows then from Remark 3.3. Furthermore the Fisher metric is well-defined on a statistical model \( \{p(\cdot, \theta)|, \theta \in \Theta \subset \mathbb{R}^n\} \) if and only if the family is \( L^2 \)-differentiable.

Proposition 3.8. [9, Proposition 11.2.12 and its proof] (i) If for some \( p \geq 1 \) a family \( \{\mu(t) \in \mathcal{M}(\mathcal{X})| t \in (a, b)\} \) is \( L^p(\nu) \)-differentiable at \( t_0 \), then it is \( \tau_v \)-differentiable at this point and its logarithmic derivative \( \frac{d\mu'(t_0)}{d\mu(t_0)} \) exists and belongs to \( L^p(\mu(t)) \). (ii) Conversely, if the family \( \{\mu(t) \in \mathcal{M}(\mathcal{X})| t \in (a, b)\} \) is \( \tau_s \)-differentiable or \( \tau_w \)-differentiable and the function \( \frac{d\mu'(t)}{d\mu(t)} \) is locally integrable on \( (a, b) \), then this family is \( L^p(\nu) \)-integrable for some \( \nu \in \mathcal{P}(\mathcal{X}) \) and for any \( (a_1, b_1) \subset (a, b) \) we have
\[
f(b_1)^{1/p} - f(a_1)^{1/p} = p^{-1} \int_a^{b} f(t)^{1/p} \frac{d\mu'(t)}{d\mu(t)} \mu(t) dt
\]
\( \nu \)-a.e. and as an equality for \( L^p(\nu) \)-valued Bochner integral.

Remark 3.9. Proposition 3.8(i) implies that a family \( \{\mu(t) \in \mathcal{M}(\mathcal{X}) t \in (a, b)\} \) is \( L^p(\nu) \)-differentiable, if and only if it is \( L^p(\nu') \)-differentiable where \( \nu' \in \mathcal{P}(\mathcal{X}) \) dominates \( \mu(t) \) for all \( t \in (a, b) \). The same argument
implies that, given a smooth finite dimensional manifold \( M \), a family \( \{ \mu(y) \in \mathcal{M}(\mathcal{X}), y \in M \} \) is \( L^p(\nu) \)-differentiable, if and only if it is \( L^p(\nu') \)-differentiable where \( \nu' \in \mathcal{P}(\mathcal{X}) \) dominates \( \mu(y) \) for all \( y \in M \).

**Example 3.10.** Given a measurable mapping \( \kappa: \mathcal{X} \rightarrow \mathcal{Y} \) between measurable spaces \( \mathcal{X}, \mathcal{Y} \), we denote by \( \kappa_*: \mathcal{S}(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{Y}) \) the push-forward mapping defined by \( \kappa_*\mu(B) := \mu(\kappa^{-1}(B)) \) for \( \mu \in \mathcal{S}(\mathcal{X}) \) and \( B \in \Sigma_{\mathcal{Y}} \). Now let \( \mathcal{X} \) be a linear space endowed with a \( \sigma \)-algebra \( \Sigma_{\mathcal{X}} \) that is invariant under a shift \( L_{th}: \mathcal{X} \rightarrow \mathcal{X}, x \mapsto x - th \) for some \( h \in \mathcal{X} \) and \( t \in \mathbb{R} \). In particular, the shift \( L_{th}: \mathcal{X} \rightarrow \mathcal{X} \) is a measurable mapping. Then a finite signed measure \( \mu \in \mathcal{S}(\mathcal{X}) \) is called differentiable along the vector \( h \) in Fomin’s sense, if the family \( (L_{th})_*\mu \) is \( \tau_s \)-differentiable \([14]\). Bogachev \([9, p. 69-70]\) proposed an equivalent definition of Fomin’s differentiability of \( \mu \in \mathcal{S}(\mathcal{X}) \) that requires a weaker condition, namely for every \( A \in \Sigma_{\mathcal{X}} \) there exists a finite limit

\[
d_{h,\mu}(A) := \lim_{t \to 0} \frac{\mu(A + th) - \mu(A)}{t}.
\]

A measure on \( \mathcal{B}(\mathcal{X}) \) is called a Baire measure. A Baire measure \( \mu \) on a locally convex topological vector space \( \mathcal{X} \) is called Skorohod differentiable along a vector \( h \in \mathcal{X} \), if for every \( f \in C_b(\mathcal{X}) \) the function

\[
t \mapsto \int_{\mathcal{X}} f(x - th) d\mu(x)
\]

is differentiable \([33, 9, \text{Definition 3.1.5, p.71}]\). Skorohod differentiability of a Borel measure is understood as the differentiability of its restriction to \( \mathcal{B}(\mathcal{X}) \). Note that every topological vector space is completely regular \([14, \text{Theorem 1.6.5, p. 44}]\). Using a theorem due to Alexandroff one can show that \( \mu \) is Skorohod differentiable along \( h \), if and only if the family \( (L_{th})_*\mu \) is \( \tau_w \)-differentiable \([9, p.71]\). \( \footnote{The concept of \( \tau_w \)-differentiability in Definition \ref{tau_w} \text{extends natural to families of Baire measures.}} \)

### 3.3. Parameterized measure models.

In \([2, 3] \) Ay-Jost-Lê-Schwachhöfer proposed a geometric formulation of the \( L^p \)-differentiability of a family of finite signed measures by using the concept of the \( p \)-th root of a finite nonnegative measure. Assume that \( 1 \leq p \in \mathbb{R} \). For \( \mu \in \mathcal{M}(\mathcal{X}) \) we set

\[
S^{1/p}(\mathcal{X}, \mu) := \{ \nu \in \mathcal{S}(\mathcal{X}, \mu) \left| \frac{d\nu}{d\mu} \in L^p(\mu) \} \},
\]

\[
S^{1/p}_0(\mathcal{X}, \mu) = \{ \nu \in S^{1/p}(\mathcal{X}, \mu) \left| \int_{\mathcal{X}} dv = 0 \} \}.
\]
The natural identification $S^{1/p}(\mathcal{X}, \mu) = L^p(\mu)$ defines a $p$-norm on $S^{1/p}(\mathcal{X}, \mu)$ by setting

$$\|f\mu\|_p := \|f\|_{L^p(\mu)}.$$ 

Then $S^{1/p}(\mathcal{X}, \mu)_p$ is a Banach space. For $\mu_1 \ll \mu_2$ the linear inclusion

$$S^{1/p}(\mathcal{X}, \mu_1) \to S^{1/p}(\mathcal{X}, \mu_2), \ f \mu_1 \mapsto f\left(\frac{d\mu_1}{d\mu_2}\right)^{1/p} \in S^{1/p}(\mathcal{X}, \mu_2)$$

preserves the $p$-norm. Since $(\mathcal{M}(\mathcal{X}), \ll)$ is a directed set, the directed limit

$$(3.5) \quad S^{1/p}(\mathcal{X})_p := \lim \{S^{1/p}(\mathcal{X}, \mu)_p | \mu \in \mathcal{M}(\mathcal{X})\}$$

is a Banach space. The image of $\mu \in S^{1/p}(\mathcal{X}, \mu)_p$ in $S^{1/p}(\mathcal{X})_p$ via the directed limit in $(3.5)$ is called the $p$-th root of $\mu$ and denoted by $\mu^{1/p}$. By [2, Proposition 3.2] the map $\pi^{1/p} : \mathcal{M}(\mathcal{X}) \to S^{1/p}(\mathcal{X})_p, \mu \mapsto \mu^{1/p}$, is continuous with respect to the strong topology $\tau_\nu$ on $\mathcal{M}(\mathcal{X})$. Using the Jordan-Hahn decomposition $\mu = \mu^+ - \mu^-$, we can extend the continuous map $\pi^{1/p}$ to continuous maps $\pi^{1/p}_\pm : S(\mathcal{X})_{TV} \to S^{1/p}(\mathcal{X})_p, \mu \mapsto (\mu^\pm)^{1/p} \pm (\mu^-)^{1/p}$ [2, p. 148]. Note that $\pi^{1/p}_+(S(\mathcal{X})_{TV}) = S^{1/p}(\mathcal{X})_p$. Thus the image $\pi^{1/p}_+(\mu)$ for $\mu \in S(\mathcal{X})$ is called the $p$-th root of measure $\mu$. This agrees with the concept of $L^p$-differentiability of a family of signed finite measures [9, Definition 11.2.11. p.386].

- Given topological vector spaces $V, W$ we denote by $\text{Lin}(V, W)$ the space of all continuous linear maps from $V$ to $W$. If $V$ and $W$ are Banach spaces then $\text{Lin}(V, W)$ is the Banach space of bounded linear operators from $V$ to $W$, which is endowed with the operator norm.

**Definition 3.11.** [2, Definitions 3.4, 3.7] (i) A parameterized measure model is a triple $(M, \mathcal{X}, p)$, where $M$ is a Banach manifold and $p : M \to \mathcal{M}(\mathcal{X})$ is a $C^1$-map, i.e. the composition $i \circ p : M \to S(\mathcal{X})_{TV}$ is continuously Fréchet differentiable. (ii) Let $k \geq 1$. A parameterized measure model $(M, \mathcal{X}, p)$ is called $k$-integrable if the map $p^{1/k} := \pi^{1/k} \circ p : M \to S^{1/k}(\mathcal{X})_k$ is continuously Fréchet differentiable. It is called weakly $k$-integrable, if $\pi^{1/k} \circ p$ is weakly Fréchet differentiable and its derivative is weakly continuous. In other words, on any coordinate chart $(U_j, \varphi_j : U_j \to E_j)$, where $U_j \subset M$ and $E_j$ is a Banach space modeling $M$, for any $z \in \varphi_j(U_j)$ there exists a weak differential $d_z(i \circ p^{1/k} \circ \varphi_j^{-1})_v \in \text{Lin}(E_j, S^{1/k}(\mathcal{X})_k)$ such that

$$i \circ p^{1/k} \circ \varphi_j^{-1}(z + v) - i \circ p^{1/k} \circ \varphi_j^{-1}(z) - d_z(i \circ p^{1/k} \circ \varphi_j^{-1})(v) = \|v\|_{E_j}.$$
converges to 0 ∈ \( S^{1/k}(\mathcal{X})_k \) in the weak topology as \( v \) converges to 0 ∈ \( E_j \), moreover, the map \( (\varphi_j(U_j)) \to \text{Lin}(E_j, S^{1/k}(\mathcal{X})_k), z \mapsto d_z(i \circ p^{1/k} \circ \varphi_j^{-1}) \), is weakly continuous.

(iii) A parameterized measure model \((M, \mathcal{X}, p)\) is called a parameterized statistical model, if \( p(M) \subset \mathcal{P}(\mathcal{X}) \).

(iv) A parameterized measure model \((M, \mathcal{X}, p)\) is called dominated, if there exists \( \mu \in \mathcal{P}(\mathcal{X}) \) such that \( p(m) \ll \mu \) for all \( m \in M \).

By Proposition 3.4, see also Remarks 3.6 and 3.9, we obtain the following Lemma immediately.

**Lemma 3.12.** Let \( M \) be a finite dimensional smooth manifold and \( k \geq 1 \). Then any \( k \)-integrable measure model \((M, \mathcal{X}, p)\) corresponds to a continuously \( L^k \)-differentiable family with parameters in \( M \) and any continuously \( L^k \)-differentiable family with parameters in \( M \) corresponds to a \( k \)-integrable measure model \((M, \mathcal{X}, p)\).

The value of the (weak) differential \( d_mp^{1/k} \in \text{Lin}(T_mM, S^{1/k}(\mathcal{X})_k) \) at \( V \in T_mM \) is called the (weak) derivative of \( p^{1/k} \) along \( V \). For an arbitrary (weakly) \( k \)-integrable parameterized measure model \((M, \mathcal{X}, p)\) the (weak) derivative of \( p^{1/k} \) along \( V \) is given as [2, Proposition 3.4, p. 154], cf. Proposition 3.8 (ii)

\[
\partial_V p^{1/k}(m) = \frac{1}{k} \frac{d\partial_V p(m)}{dp(m)} p^{1/k}(m) \quad \text{for } m \in M, V \in T_mM.
\]

The following Proposition is a version of Proposition 3.8 (ii).

**Proposition 3.13.** [2, Theorem 3.2, p. 155] Let \((M, \mathcal{X}, p)\) be a parameterized measure model. The model is \( k \)-integrable, if and only if the map \( V \mapsto \|\partial_V p^{1/k}\|_p \) is well-defined on \( TM \) and continuous. The model is weakly \( k \)-integrable, if for any \( V \in TM \) the weak derivative \( \partial_V p^{1/k} \) exists and weakly converges to \( \partial_V p^{1/k} \) as \( V \) converges to \( V_0 \).

By Proposition 3.13 any 2-integrable parameterized statistical model \((M, \mathcal{X}, p)\) is endowed with the continuous Fisher metric \( g \) on \( M \) defined at \( m \in M \) by

\[
g_m(V, W) = \left( \frac{d\partial_V p(m)}{dp(m)}, \frac{d\partial_W p(m)}{dp(m)} \right)_{L^2(p(m))}.
\]

**Example 3.14.** Assume that a finite dimensional statistical model of probability density functions \( \{p(\cdot, \theta) \mu| \theta \in \Theta, \mu \in \mathcal{P}(\mathcal{X})\} \) in the previous section is endowed with continuous Fisher metric in (2.3). Then it is a \( L^2 \)-differentiable family of probability measures (Example 3.7), and by Lemma 3.12 it is a 2-integrable parameterized statistical model. Furthermore, the Fisher metric \( g \) in (3.7) can be expressed
as in (2.2). The continuous Amari-Chentsov tensor on a 3-integrable parameterized measure model is defined in a similar way \cite[3.96, p. 167]{2}, cf. (2.4).

The Pistone-Sempi manifold $\mathcal{P}_\mu$ is an $\infty$-integrable parameterized statistical model $(\mathcal{P}_\mu, \mathcal{X}, i)$, where $i : \mathcal{P}_\mu \to \mathcal{P}(\mathcal{X})$ is the natural inclusion \cite[Proposition 3.14]{2}.

Newtons’ statistical manifolds of differentiable densities in \cite{28} are 3-integrable parameterized statistical models.

4. Diffeological statistical models

The concept of smooth parametric families of probability density functions considered by Amari as well as its various generalizations we considered in the previous sections is based on the notion of a smooth map from a nice smooth space $M$ to the vector space $\mathcal{S}(\mathcal{X})$. This concept leads to the notion of a smooth parameterization of a subset $S \subset \mathcal{S}(\mathcal{X})$ by $M$. Diffeology theory is created by Souriau in 1980 \cite{34} to describe consistent smooth parameterizations of a set. Diffeology language is therefore a convenient language to deal with various objects in parametric and nonparametric Information Geometry in a unified elegant framework.

4.1. Friedrich’s example. In \cite{15}, motivated by Amari’s work \cite{4} and the concept of differentiable curves in mathematical statistics \cite{35}, Friedrich supplied the space $P(\lambda) := \{\mu \in \mathcal{P}(\mathcal{X}) | \mu \ll \lambda\}$, where $\lambda$ is a $\sigma$-finite nonnegative measure on $\mathcal{X}$, with the Fisher metric as follows. He defined the tangent space $T_\mu P(\lambda)$ to be the linear space $\mathcal{S}^{1/2}(\mathcal{X}, \mu)$ and the Fisher metric on $\mathcal{S}^{1/2}(\mathcal{X}, \mu)$ is given by (cf. (3.7))

$$g_\mu(V, W) = \left\langle \frac{dV}{d\mu}, \frac{dW}{d\mu} \right\rangle_{L^2(\mu)}.$$

Note that the tangent spaces $\mathcal{S}^{1/2}_0(\mu)$ and $\mathcal{S}^{1/2}_0(\mu')$ are not isomorphic if $\mu$ and $\mu'$ are not equivalent. Hence $P(\lambda)$ does not have the structure of an infinite dimensional Fréchet manifold. There is no obvious way to parameterize $P(\lambda)$ by a single map from an infinite dimensional smooth manifold. In \cite{24} Lê proposed to endow $P(\lambda)$ with a natural diffeology, called statistical diffeology, which provides $P(\lambda)$ with a differentiable structure, whose tangent space at $\mu \in P(\lambda)$ is $\mathcal{S}^{1/2}_0(\mu)$.

4.2. Diffeological spaces. Let us recall the concept of diffeology, following Iglesias-Zemmour \cite[1.5, 1.14]{17}.
Definition 4.1. Let $X$ be a nonempty set. A parametrization of $X$ is a map from an open subset $U \subset \mathbb{R}^n$ to $X$. A diffeology of $X$ is any set $\mathcal{D}$ of parametrizations of $X$ such that the following axioms are satisfied:

D1. Covering The set $\mathcal{D}$ contains the constant parametrization $x : r \mapsto x$, defined on $\mathbb{R}^n$ for all $x \in X$ and all $n \in \mathbb{N}$.

D2. Locality Let $p : U \to X$ be a parameterization. If for every point $u \in U$ there exists an open neighborhood $V \subset U$ of $u$ such that $p|_V \in \mathcal{D}$, then the parameterization $p$ belongs to $\mathcal{D}$.

D3. Smooth compatibility. For every element $p : U \to X$ of $\mathcal{D}$, for every open subset $V \subset \mathbb{R}^m$ and for every $F \in C^\infty(U, V)$, the composition $p \circ F$ belongs to $\mathcal{D}$.

A diffeological space is a nonempty set $X$ equipped with a diffeology $\mathcal{D}$. A map $f : X \to X'$ between two diffeological spaces $(X, \mathcal{D})$ and $(X', \mathcal{D}')$ is said to be smooth, if for any $p \in \mathcal{D}$ we have $f \circ p \in \mathcal{D}'$.

We verify immediately that the composition of smooth maps between diffeological spaces is a smooth map. So the set of all diffeological spaces forms a category whose morphisms are smooth mappings.

Example 4.2. Let $\tau$ be a topology on $S(\mathcal{A})$ defined in Subsection 3.1 and $U$ an open subset in $\mathbb{R}^n$. A map $f : U \to (S(\mathcal{A}), \tau)$ will be called $\tau$-differentiable at $x \in U$, if there exists a $\tau$-differential $d_x f \in \text{Lin}(\mathbb{R}^n, (S(\mathcal{A}), \tau))$ such that

$$
\frac{f(x + v) - f(x) - d_x f(v)}{\|v\|} \xrightarrow{\tau} 0 \in S(\mathcal{A}),
$$

as $v$ goes to 0 $\in \mathbb{R}^n$. Here $\|v\|$ is the Euclidean norm of $v$, and $\xrightarrow{\tau} 0$ denotes the convergence to 0 $\in S(\mathcal{A})$ in the $\tau$-topology. If $f$ is $\tau$-differentiable at all $x \in U$, it will be called $\tau$-differentiable. A $\tau$-differentiable map $f : U \to S(\mathcal{A})$ will be called continuously $\tau$-differentiable, or $C^1_\tau$-differentiable, if its partial $\tau$-derivative map $\partial f : TU \to (S(\mathcal{A}), \tau), (x, v) \mapsto \partial_v f := d_x f(v) \in S(\mathcal{A})$ is continuous in $\tau$-topology. Inductively we define the concept of $C^k_\tau$-map from $U$ to $S(\mathcal{A})$.

Note that for any Fréchet differentiable map $f : U \to E$, where $E$ is a Banach space, the differential $df : U \to \text{Lin}(\mathbb{R}^n, E)$ is continuous if and only if its partial derivative map $\partial f : TU \to E, (m, v) \mapsto d_m f(v)$, is continuous. Denote by $\text{Lin}^2(\mathbb{R}^n, \mathbb{R}^n; E)$ the set of all continuous bilinear mappings from $\mathbb{R}^n \times \mathbb{R}^n$ to $E$. Taking into account the equality

$$
\text{Lin}(\mathbb{R}^n, \text{Lin}(\mathbb{R}^n, E)) = \text{Lin}^2(\mathbb{R}^n, \mathbb{R}^n; E)
$$

see e.g. [20], Proposition 2.4, p. 7, we conclude that $C^k_\tau$-differentiable maps from $U$ to $S(\mathcal{A})$ are exactly Fréchet $C^k$-differentiable maps.
Now we denote by $D^k_\tau$ the set consisting of all $C^k_\tau$ maps from $U \subset \mathbb{R}^n$ to $S(\mathcal{X})$, where $U$ runs over all open subsets in $\mathbb{R}^n$, $n \in \mathbb{N}$. It is straightforward to check that $D^k_\tau$ is a diffeology on $S(\mathcal{X})$.

**Example 4.3.** Let $X$ be a set and $(X', \mathcal{D}')$ be a diffeological space. Given a map $f : X \to X'$ there is a coarest diffeology of $X$ such that the map $f$ is smooth. This diffeology is called the pullback diffeology of the diffeology $\mathcal{D}'$ by $f$, and denoted by $f^*(\mathcal{D}')$ [17, Section 1.26, p. 14]. We have

$$f^*(\mathcal{D}') := \{ p : U \to X | f \circ p \in \mathcal{D}' \}.$$ 

If $X$ is a subset of $X'$ and $f$ is the natural inclusion the pullback diffeology on $X$ is also called the subset diffeology [17, Section 1.33, p. 18].

**Example 4.4.** Let $(X, \mathcal{D})$ be a diffeological space and $f : X \to X'$ a map. The finest diffeology of $X'$ such that $f$ is smooth is called the pushforward diffeology of $\mathcal{D}$ by $f$, and denoted by $f_*(\mathcal{D})$ [17, Section 1.43, p. 24]. A mapping $p : U \to X'$ belongs to $f_*(\mathcal{D})$ if and only if it satisfies the following condition. For every point $u \in U$ there exists an open neighborhood $V \subset U$ of $u$ such that either $p|_V$ is a constant map, or there exists a map $q : V \to X$ in $\mathcal{D}$ such that $p|_V = f \circ q$.

**Remark 4.5.** In [24] Lê introduced the concept of a $C^k$-diffeology, which coincides with the concept of diffeology in Definition 4.1, except that the condition $D3$ on smooth compatibility in Definition 4.1 is replaced by the $C^k$-compatibility, namely the set $C^\infty(U, V)$ is replaced by the larger set $C^k(U, V)$. Since $C^\infty(U, V)$ is a subset of $C^k(U, V)$ any $C^k$-diffeological space is a diffeological space.

### 4.3. Diffeological statistical models and diffeological Fisher metric

From now on, a statistical model is a subset $\mathcal{P}_X \subset \mathcal{P}(\mathcal{X})$ [27]. Let $\tau$ be a topology on $S(\mathcal{X})$, with specification as in Subsection 3.2.

**Definition 4.6.** Let $\mathcal{P}_X \subset \mathcal{P}(\mathcal{X})$ be a statistical model and $i : \mathcal{P}_X \to S(\mathcal{X})$ the inclusion. A diffeology $\mathcal{D}$ on $\mathcal{P}_X$ is called a $C^k_\tau$ statistical diffeology, if the embedding $i : (\mathcal{P}_X, \mathcal{D}) \to (S(\mathcal{X}), D^k_\tau)$ is a smooth embedding. In this case $(\mathcal{P}_X, \mathcal{D})$ is called a $C^k_\tau$ diffeological statistical model.

**Example 4.7.** Any statistical model $\mathcal{P}_X \to S(\mathcal{X})$ has the subset diffeology $i^*(D^k_\tau)$, which is a $C^k_\tau$ statistical diffeology.

**Example 4.8.** Let us consider all possible open subsets $U \subset \mathbb{R}^n$, $n \in \mathbb{N}$ and for $p \geq 1$ we set

$$D^{1,p}_\tau := \{ f : U \to \mathcal{P}(\mathcal{X}), |\pi^{1/p} \circ f : U \to S^{1/p}(\mathcal{X})_p \text{ is a Frechét } C^1\text{-map } \}.$$
Equivalently, by Lemma 3.12, \( f \) is a continuously \( L^p \)-differentiable map. Then we verify immediately that the set \( D^1_{\tau_v} \) is a \( C^1_\tau \) statistical diffeology of \( \mathcal{P}(\mathcal{X}) \).

**Example 4.9.** For any smooth Banach manifold \( M \) we denote by \( D^\infty_M \) the diffeology consisting of all smooth maps from all open subsets \( U \subset \mathbb{R}^n, n \in \mathbb{N} \), to \( M \). Let \( (M, \mathcal{X}, p) \) be a \( k \)-integrable parametrized measure model. Then the image \( p(M) \subset \mathcal{M}(\mathcal{X}) \) has the pushforward diffeology \( p_* (D^\infty_M) \) which is a \( C^1_\tau \) statistical diffeology.

**Definition 4.10.** [24] Definition 1] (i) Let \((\mathcal{P}_X, \mathcal{D})\) be a \( C^k_\tau \) diffeological statistical model. A vector \( v \in \mathcal{S}(\mathcal{X}) \) is called a tangent vector of \((\mathcal{P}_X, \mathcal{D})\) at \( \mu \in \mathcal{P}_X \) if there is a smooth map \( c : (-1, 1) \to (\mathcal{P}_X, \mathcal{D}) \) such that \( c(0) = \mu \) and \( c'(0) = v \).

(ii) The tangent cone \( C^\mu_\tau (\mathcal{P}(\mathcal{X}), \mathcal{D}) \) consists of all tangent vectors \( v \) of \((\mathcal{P}(\mathcal{X}), \mathcal{D})\) at \( \mu \).

(iii) The tangent space \( T^\mu_\tau (\mathcal{P}(\mathcal{X}), \mathcal{D}) \) is the linear hull of \( C^\mu_\tau (\mathcal{P}(\mathcal{X}), \mathcal{D}) \).

**Example 4.11.** Let us revisit Friedrich’s example of the space \( P(\lambda) \) where \( \lambda \) is a \( \sigma \)-finite nonnegative measure on \( \mathcal{X} \). Denote by \( \iota : P(\lambda) \to \mathcal{S}(\mathcal{X}) \) the natural inclusion. By Remark 3.3 and Proposition 3.4 the tangent space \( T^\iota_\tau (P(\lambda), i^*(D^1_{\tau_v})) \) is a subset of \( \mathcal{S}^1_0(\mu) \). We shall show that \( T^\iota_\tau (P(\lambda), D^1_{can}) = \mathcal{S}^1_0(\mu) \). Let \( \phi \mu \in \mathcal{S}^1_0(\mu) \).

Set \( \tilde{\mu}(t) = p(x, t) \mu \), where

\[
(4.2) \quad p(x, t) = \begin{cases} 
1 + t\phi(x) & \text{if } t\phi(x) \geq 0 \\
\exp(t\phi(x)) & \text{if } t\phi(x) < 0.
\end{cases}
\]

In [2] p. 142] Ay-Jost-Lé-Schwachhöfer showed that the curve

\[
(4.3) \quad \mu(t) = \frac{\tilde{\mu}(t)}{\|\tilde{\mu}(t)\|}
\]

belongs to \( \mathcal{S}^1_0(\mu) \) is continuously \( \tau_v \)-differentiable, and \( \mu(0) = \mu, \mu'(0) = \phi \mu \). Thus \( T^\iota_\tau (P(\lambda), i^*(D^1_{\tau_v})) = \mathcal{S}^1_0(\mu) \).

Now let us consider the diffeology \( i^*(D^1_{\tau_v}) \) of \( \mathcal{P}(\lambda) \). By Lemma 3.12, Remark 3.3, and Proposition 3.8 any tangent vector in \( T^\iota_\tau (P(\lambda), i^*(D^1_{\tau_v})) \) belongs to \( \mathcal{S}^{1/2}_0(\mathcal{X}, \mu) \). We claim that \( T^\iota_\tau (P(\lambda), i^*(D^1_{\tau_v})) = \mathcal{S}^{1/2}_0(\mathcal{X}, \mu) \).

Let \( \phi \mu \in \mathcal{S}^{1/2}_0(\mathcal{X}, \mu) \). We consider the same curve \( \tilde{\mu}(t) = p(x, t) \mu \), where \( p(x, t) \) is defined by (4.2) and set \( \mu(t) \) by (4.3), cf. [2] p. 147. By Proposition 3.13, \( \mu(t) \) is continuously \( L^2 \)-differentiable, and \( \partial_t \mu^{1/2}(0) = \frac{1}{2} \phi \mu^{1/2} \) by (3.6). This proves our claim.

**Definition 4.12.** [24] Definition 4. A \( C^k_\tau \) diffeological statistical model \((\mathcal{P}_X, \mathcal{D})\) is called almost \( 2 \)-integrable, if for all \( \mu \in \mathcal{P}_X \) we have \( T^\mu_\tau (\mathcal{P}_X, \mathcal{D}) \subset \mathcal{S}^{1/2}_0(\mathcal{X}, \mu) \).
$S_0^{1/2}(\mu)$. A $C^k_r$-diffeological statistical model $(\mathcal{P}_X, \mathcal{D})$ is called 2-integrable, if the inclusion $i : (\mathcal{P}_X, \mathcal{D}) \rightarrow (\mathcal{P}(\mathcal{X}), \mathcal{D}_{1,2}^{\tau_v})$ is smooth.

Clearly any 2-integrable $C^k_r$-statistical model is almost 2-integrable but there is an almost 2-integrable statistical model which is not 2-integrable, see [26, Example 4].

Now we define the diffeological Fisher metric on an almost 2-integrable $C^k_r$-diffeological statistical model $(\mathcal{P}_X, \mathcal{D})$ by the same formula in (4.1). Namely for $V, W \in T^\mu(\mathcal{P}_X, \mathcal{D}) \subset S_{0}^{1/2}(\mathcal{X}, \mu)$ (the inclusion is a consequence of Proposition 3.8 and Remark 3.3) we set

$g^\mu(V, W) = \langle \frac{dV}{d\mu}, \frac{dW}{d\mu} \rangle_{L^2(\mu)}$.

**Example 4.13.** Let $(M, \mathcal{X}, p)$ be a 2-integrable parameterized statistical model. Then $(p(M), p_*(\mathcal{D}_M^\infty))$ is a 2-integrable $C^1_r$-statistical model. The tangent space $T_{p(m)}(p(M), p_*(\mathcal{D}_M^\infty))$ consists of tangent vectors $\partial_V p(m), V \in T_m M$. Then we have

$g_{p(m)}(\partial_V p(m), \partial_W p(m))) = \langle \frac{d\partial_V p(m)}{dp(m)}, \frac{d\partial_W p(m)}{dp(m)} \rangle_{L^2(p(m))}$.

Thus the Fisher metric $g_m(V, W)$ defined on the parameter space $M$ by (3.7) is the pull back of the diffeological Fisher metric in (4.1) on the image $(p(M), p_*(\mathcal{D}_M^\infty))$ and its degeneracy is caused by the kernel of the differential $d_m p : TM \rightarrow S_0^{1/2}(\mathcal{X}, p(m))$.

**Remark 4.14.** Since the diffeological Fisher metric $g$ is nondegenerate, Lê-Tuzhilin used $g$ to define the Fisher distance on 2-integrable $C^k_r$-diffeological statistical models $(\mathcal{P}_X, \mathcal{D})$. The length of a smooth curve $c : [0, 1] \rightarrow (\mathcal{P}_X, \mathcal{D})$ is defined by [26, Definition 5]

$L(c) = \int_0^1 |c'(t)|_g \, dt$.

The length of a piece-wise smooth curve is set to be the sum of the lengths of its smooth sub-intervals. The diffeological Fisher distance $d_g$ between two-points is defined as the infimum of the lengths over the space of piece-wise $[26]$. Lê-Tuzhilin showed that a 2-integrable $C^k_r$ diffeological statistical model $(\mathcal{P}_X, \mathcal{D})$ endowed with the Fisher distance $d_g$ is a length space [26, Theorem 1], moreover $d_g(x, y) \geq \|x - y\|_{TV}$ [26, Lemma 4]. Thus the topology on $\mathcal{P}_X$ generated by $d_g$ is not weaker than the strong topology $\tau_v$. If the Hausdorff dimension of $(\mathcal{P}_X, \mathcal{D}, d_g)$ is finite, then $(\mathcal{P}_X, \mathcal{D}, d_g)$ is endowed with the Hausdorff-Jeffrey measure, which coincides with the unnormalized Jeffrey prior measure defined on smooth $n$-dimensional statistical models [26, Theorem 3].
4.4. Diffeological and parametric Cramér-Rao inequalities. In density estimation problems, given a statistical model $\mathcal{P}_X \subset \mathcal{P}(\mathcal{X})$, we wish to measure the accuracy of a nonparametric estimator $\hat{\sigma} : \mathcal{X} \to \mathcal{P}_X$, or its $\varphi$-coordinate, formalized as a map $\varphi : \mathcal{P}_X \to V$, where $V$ is a topological vector space. If $\mathcal{P}_X = p(M)$, where $(M, \mathcal{X}, p)$ is a parameterized statistical model, then it is convenient to have a parametric estimator $\hat{\sigma} : \mathcal{X} \to M$, or its $\varphi$-coordinate, given a map $\varphi : M \to V$. We measure the deviation of a parametric and nonparametric $\varphi$-estimator $\varphi \circ \hat{\sigma} : \mathcal{X} \to V$ from its mean value using its covariance

$$V^\varphi_\mu[\hat{\sigma}](l, l) := \int_\mathcal{X} (l \circ \varphi \circ \hat{\sigma}(x) - \mathbb{E}_\mu(l \circ \varphi \circ \hat{\sigma}))^2 d\mu(x),$$

$$V^\varphi_m[\hat{\sigma}](l, l) := \int_\mathcal{X} (l \circ \varphi \circ \hat{\sigma}(x) - \mathbb{E}_p(m)(l \circ \varphi \circ \hat{\sigma}))^2 dp(m)(x),$$

assuming that they are well-defined.

Let $(\mathcal{P}_X, \mathcal{D})$ be a 2-integrable $C^k$-diffeological statistical model. We call $\varphi \circ \hat{\sigma} : \mathcal{X} \to V$ a regular $\varphi$-estimator, if for all $l \in V'$ and for all $\mu_0 \in \mathcal{P}_X$

$$\lim_{\mu \to \mu_0} \sup \|l \circ \varphi \circ \hat{\sigma}\|_{L^2(\mathcal{X}, \mu)} < \infty. \quad (4.4)$$

If $\varphi \circ \hat{\sigma}$ is regular, then the function $\varphi^l_{\hat{\sigma}} : (\mathcal{P}_X, \mathcal{D}) \to \mathbb{R}, \mu \mapsto \mathbb{E}_\mu(l \circ \varphi \circ \hat{\sigma})$, is differentiable [21, Proposition 2]. Similarly, for a 2-integrable parameterized statistical model $(M, \mathcal{X}, p)$ we call $\varphi \circ \hat{\sigma} : \mathcal{X} \to V$ a regular $\varphi$-estimator if for all $l \in V'$ and all $m_0 \in M$

$$\lim_{m \to m_0} \sup \|l \circ \varphi \circ \hat{\sigma}\|_{L^2(\mathcal{X}, p(m))} < \infty. \quad (4.5)$$

If $\varphi \circ \hat{\sigma}$ is regular, then the function $\varphi^l_{\hat{\sigma}} : M \to \mathbb{R}, m \mapsto \mathbb{E}_p(m)(l \circ \varphi \circ \hat{\sigma})$, is Gâteaux-differentiable [2 Lemma 5.2, p. 282].

Let $T^g_\mu(\mathcal{P}_X, \mathcal{D})$ be the completion of $T_\mu(\mathcal{P}_X, \mathcal{D})$ by $g$. Since $T^g_\mu \mathcal{P}_X$ is a Hilbert space, the map

$$L_g : T^g_\mu \mathcal{P}_X \to (T^g_\mu \mathcal{P}_X)', L_g(v)(w) := \langle v, w \rangle_g,$$

is an isomorphism. We define the inverse $g^{-1}$ of $g$ on $(T^g_\mu \mathcal{P}_X)'$ as follows:

$$\langle L_g v, L_g w \rangle_{g^{-1}} := \langle v, w \rangle_g.$$
The diffeological Cramér-Rao inequality asserts that under the above conditions we have [24, Theorem 3]
\[ V_{\mu}^\varphi[\hat{\sigma}](l, l) - \|d_{\mu}\varphi^l\|_{g^{-1}}^2 \geq 0. \]
(4.6)

The parametric Cramér-Rao inequality asserts that under the above conditions we have [2, Theorem 5.7], [25], [19]
\[ V_{m}^\varphi[\hat{\sigma}](l, l) - \|\hat{d}_{m}\varphi^l\|_{\hat{g}^{-1}}^2 \geq 0. \]
(4.7)

If \( M = \mathcal{P}_X \) and \( \varphi : M \to \mathbb{R}^n \) is a coordinate mapping in a neighborhood \( U(x) \) of \( x \in M \) i.e. \( \{x^l := l \circ \varphi(x)|, l = [1, n]\} \) are local coordinates of \( x \), then the diffeological and parametric Cramér-Rao inequalities (4.6), (4.7) become the classical Cramér-Rao inequality [5, Theorem 2.2, p. 32]:
\[ V_m[\hat{\sigma}] \geq g_m^{-1}. \]

The proof of the diffeological and parametric Cramér-Rao inequalities is based on the explicit expression of the dual of \( d_{\mu}\varphi^l \) in \( T^0_\mu(\mathcal{P}_X, \mathcal{D}) \), and the dual of \( \hat{d}_{m}\varphi^l \) in \( T^0_mM \), respectively, as the orthogonal projection of the measure \( (l \circ \varphi \circ \hat{\sigma} - \mathbb{E}_{\mu}(l \circ \varphi \circ \hat{\sigma})) \mu \in \mathcal{S}^{1/2}(\mathcal{X}, \mu) \), and the measure \( (l \circ \varphi \circ \hat{\sigma} - \mathbb{E}_{p(m)}(l \circ \varphi \circ \hat{\sigma})) p(m) \in \mathcal{S}^{1/2}(\mathcal{X}, p(m))_2 \), respectively, to the closed subspace \( T^0_\mu(\mathcal{P}_X, \mathcal{D}) \subset \mathcal{S}^{1/2}(\mathcal{X}, \mu) \) and the close subspace \( T^0_mM \), identified with the closure of its image via \( d_m p \) in \( \mathcal{S}^{1/2}(\mathcal{X}, p(m))_2 \), respectively.

**Remark 4.15.** In [18] Janssen showed that \( \varphi^l : \mathcal{P}_X \to \mathbb{R} \) is differentiable under the weaker assumption that smooth curves in \( (\mathcal{P}_X, \mathcal{D}) \) are \( L^2 \)-differentiable and the regularity of \( \varphi \circ \hat{\sigma} \) in (4.4) also holds. Thus his nonparametric Cramér-Rao inequality assumes the weakest condition, according to our best knowledge.

5. Conclusions and final remarks

(1) The fruitful concept of a smooth statistical model endowed with the Fisher metric and the Chentsov tensor has been investigated and generalized using different, but closely related, formalisms of statistical manifolds and natural differentiable structures on statistical models. In particular, the concept of the Fisher metric is naturally related to the concept of \( L^2 \)-differentiability, the concept of a 2-integrable parametrized measure model, and the concept of statistical diffeology \( \mathcal{D}^{1,2} \). Since the Amari-Chentsov tensor on statistical models is a covariant tensor of finite degree, it can be defined and described similarly using statistical diffeologies. We refer the reader to [6] and [2] for many applications of information geometric structures in different fields of sciences.
In this article we omitted a discussion on the monotonicity of the Fisher metric under Markov kernels. The set of all Markov kernels from a measurable space $\mathcal{X}$ to a measurable space $\mathcal{Y}$ encompasses the set of all measurable mappings from $\mathcal{X}$ to $\mathcal{Y}$, and as measurable mappings from $\mathcal{X}$ to $\mathcal{Y}$, Markov kernels induce (smooth) transformations between (smooth) statistical models over $\mathcal{X}$ to (smooth) statistical models over $\mathcal{Y}$. It is well-known that the pull back of the Fisher metric under the Markov kernel is weaker than the Fisher metric on the domain, and this property is called the monotonicity, which can be used to characterize Fisher metrics, see Chentsov [13], Lê [23]. The monotonicity also justifies the alternative name of Fisher-Rao metric as the Fisher information metric [5]. It is a natural problem to investigate smooth families of Markov kernels and their geometry. Smooth families of measurable mappings, e.g. neural networks, and smooth families of Markov kernels play important role in statistical learning theory, in particular in supervised learning theory.

By Remark 4.14 the Fisher metric generates the topology on statistical models $\mathcal{P}_X \subset S(\mathcal{X})$ that is not weaker than the topology $\tau_v$. For many problems in statistics the weak topology $\tau_w$ on $\mathcal{P}(\mathcal{X})$ is more relevant. It is an interesting problem to describe statistical diffeologies that support metrics generating $\tau_w$.

In [5, Chapter 7] Amari and Nagaoka discussed Information Geometry for quantum systems, in particular they defined a quantum version of the Fisher metric and Amari’s $\alpha$-connections and applied them to quantum estimation theory. It is an interesting problem to develop diffeologies that carry the quantum Fisher metric and the quantum Amari’s $\alpha$-connections.

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The author states that there is no conflict of interest.

Data availability statement

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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