Hölder Inequalities and Bounds on the Masses of Light Quarks

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Abstract

QCD Laplace sum-rules must satisfy a fundamental (Hölder) inequality if they are to consistently represent
an integrated hadronic cross-section. After subtraction of the pion-pole, the Laplace sum-rule of pion currents
is shown to violate this fundamental inequality unless the u and d quark masses (respectively denoted by \( m_u \)
and \( m_d \)) are sufficiently large, placing a lower bound on the 1GeV \(^{\text{MS}}\) running masses.

QCD Laplace sum-rules, a technique which equates a theoretical quantity to an integrated hadronic cross-section to
determine a QCD prediction of hadronic properties \[1\], have demonstrated their utility in numerous applications to
hadronic physics. It has recently been shown that Laplace sum-rules must satisfy a fundamental inequality in order
to consistently represent an integrated hadronic cross-section \[2\]. When applied to the Laplace sum-rule related
to the pion, a strong dependence on the light quark masses \( m_u \) and \( m_d \) occurs in the analysis of this inequality
because the pseudoscalar correlation is proportional to \( (m_u + m_d)^2 \). In this paper we employ this fundamental
Hölder inequality to place lower bounds on the sum of the \( u \) and \( d \) 1.0 GeV \(^{\text{MS}}\) running masses.

Consider the correlation function of pseudoscalar currents with the quantum numbers of the pion:

\[
J_5(x) = \frac{1}{\sqrt{2}} (m_u + m_d) \left[ \bar{u}(x) i\gamma_5 u(x) - \bar{d}(x) i\gamma_5 d(x) \right]
\]

(1)

\[
\Pi_5(Q^2) = i \int d^4 x e^{i q \cdot x} \langle O | T J_{5}(x) J_{5}(0) | O \rangle
\]

(2)

To leading order in the quark masses, the perturbative contributions \( \Pi_5^{\text{pert}} \) are known to four-loop order in the \(^{\text{MS}}\) scheme \[3\]:

\[
\Pi_5^{\text{pert}}(Q^2) = \frac{3Q^2}{8\pi^2} (m_u + m_d)^2 \log \left( \frac{Q^2}{\mu^2} \right)
\left[ 1 + \frac{\alpha}{\pi} (a_0 + a_1 L) + \left( \frac{\alpha}{\pi} \right)^2 \left[ b_0 + b_1 L + b_2 L^2 \right] + \left( \frac{\alpha}{\pi} \right)^3 \left[ c_0 + c_1 L + c_2 L^2 + c_3 L^3 \right] \right]
\]

(3)

\[
L \equiv \log \left( \frac{Q^2}{\mu^2} \right) , \quad a_0 = \frac{17}{3} , \quad a_1 = -1
\]

(4)

\[
b_0 = 45.846 \quad , \quad b_1 = \frac{-95}{6} , \quad b_2 = \frac{17}{12}
\]

(5)

\[
c_0 = 465.8463 \quad , \quad c_1 = -194.2393 \quad , \quad c_2 = 38.1667 \quad , \quad c_3 = -2.3020825
\]

(6)

Divergent polynomials in \( Q^2 \) are ignored in \[3\] since they correspond to subtraction constants in dispersion relations
which do not contribute to sum-rules.

In the QCD sum-rule approach, nonperturbative effects are parametrized by the QCD condensates representing
infinite correlation-length vacuum effects \[4\]. In addition to the QCD condensate contributions, scalar and pseudo-
scalar correlation functions must also take into account the effects of instantons which represent vacuum effects

\[1\] J. Kwiecinski and C. Quigg, “QCD Laplace sum-rules,” Phys. Rev. D 30, 784 (1984).

\[2\] T.G. Steele and K. Kostuik, “Hölder Inequalities and Bounds on the Masses of Light Quarks,” hep-ph/9812497 (1998).

\[3\] J. Kwiecinski and C. Quigg, “QCD Laplace sum-rules,” Phys. Rev. D 30, 784 (1984).

\[4\] J. Kwiecinski and C. Quigg, “QCD Laplace sum-rules,” Phys. Rev. D 30, 784 (1984).
with a finite correlation length \[ L \]. Thus the nonperturbative contributions to \( \Pi^5 \) consist of the QCD condensate effects \( \Pi^\text{cond}_5 \) and instanton effects \( \Pi^\text{inst}_5 \).

To leading order in the quark mass, \( \Pi^\text{cond}_5 \) is given by
\[
\Pi^\text{cond}_5(Q^2) = (m_u + m_d)^2 \left[ \frac{\langle \alpha G^2 \rangle}{8\pi Q^2} - \frac{\langle m\bar{q}q \rangle}{Q^2} + \frac{\pi \langle O_6 \rangle}{4Q^4} \right]
\]  
(7)

where we have used \( SU(2) \) symmetry for the dimension-four quark condensates (i.e. \( (m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle = 4m\langle \bar{q}q \rangle \)), and \( \langle O_6 \rangle \) denotes the dimension six quark condensates
\[
\langle O_6 \rangle = \alpha_s \left[ 2\langle \bar{u}\sigma_{\mu\nu}\gamma_5 T^a u \bar{u}\sigma^{\mu\nu}\gamma_5 T^a u \rangle + u \rightarrow d \right] - 4\langle \bar{u}\sigma_{\mu\nu}\gamma_5 T^a u \bar{d}\sigma^{\mu\nu}\gamma_5 T^a d \rangle \\
+ \frac{2}{3} \langle \bar{q}\gamma_{\mu}T^a u + \bar{d}\gamma_{\mu}T^a d \rangle \sum_{u,d,s} \bar{q}\gamma^\mu T^a q \right]
\]  
(8)

The vacuum saturation hypothesis \[ 1 \] will be used as a reference value for \( \langle O_6 \rangle \)
\[
\langle O_6 \rangle = f_{vs} \frac{448}{27} \alpha \langle \bar{q}q \rangle = f_{vs} 3 \times 10^{-3}\text{GeV}^6
\]  
(9)

where \( f_{vs} = 1 \) for exact vacuum saturation. Larger values of effective dimension-six operators found in \[ 7, 8 \] imply that \( f_{vs} \) could be as large as 2. The quark condensate is determined by the GMOR relation \[ 6 \]
\[
(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle = 4m\langle \bar{q}q \rangle = -2f_2^2m_\pi^2
\]  
(10)

where \( f_\pi = 93 \text{MeV} \). We will use recent determinations of the gluon condensate \( \langle \alpha G^2 \rangle \) \[ 11 \]
\[
\langle \alpha G^2 \rangle = (0.07 \pm 0.01) \text{GeV}^4
\]  
(11)

However, it should be noted that there is some discrepancy between \[ 10 \] and the smaller value \( \langle \alpha G^2 \rangle = (0.047 \pm 0.014) \text{GeV}^4 \) found in \[ 8 \].

The direct single-instanton contribution \( \Pi^\text{inst}_5 \) in the instanton liquid model is
\[
\Pi^\text{inst}_5(Q^2) = \frac{3}{2\pi^2} (m_u + m_d)^2 Q^2 \left[ K_{-1} \left( \rho_c \sqrt{Q^2} \right) \right]^2
\]  
(12)

where \( K_{\nu}(x) \) denotes the modified Bessel function and \( \rho_c \approx 1/(600\text{MeV}) \) is a fundamental scale in the instanton liquid model \[ 4 \].

The correlation function \( \Pi_5(Q^2) \) satisfies a twice-subtracted dispersion relation
\[
\Pi_5(Q^2) = \Pi_5(0) - Q^2 \Pi_5'(0) + \frac{Q^4}{\pi} \int_0^\infty \frac{I_m \Pi_5(t)}{t^2(t + Q^2)} dt
\]  
(13)

\[
\Pi_5(Q^2) = \Pi^\text{pert}_5(Q^2) + \Pi^\text{cond}_5(Q^2) + \Pi^\text{inst}_5(Q^2)
\]  
(14)

Laplace sum-rules are formed from the dispersion relation by applying the Borel transform operator \( \hat{B} \) to \[ 13 \]
\[
\hat{B} = \lim_{N \to \infty} \frac{1}{Q^2/N!M^2} \frac{1}{\Gamma(N)} (-Q^2)^N \left( \frac{d}{dQ^2} \right)^N
\]  
(15)

This results in the Laplace sum-rule which exponentially suppresses the high-energy region.
\[
\mathcal{R}_0(M^2) = M^2 \hat{B} \left[ \Pi_5(Q^2) \right]
\]  
(16)

\[
\mathcal{R}_0(M^2) = \frac{1}{\pi} \int_0^\infty Im \Pi_5(t)e^{-t/M^2} dt
\]  
(17)
Using the expressions (13, 14, 15) for the correlation function, the definition of $\hat{B}$ and the results of [11], the sum-rule $\mathcal{R}_0(M^2)$ is obtained

$$
\mathcal{R}_0(M^2) = \frac{3(m_u + m_d)^2 M^4}{8\pi^2} \left( 1 + 4.821098 \frac{\alpha}{\pi} + 21.97646 \left( \frac{\alpha}{\pi} \right)^2 + 53.14179 \left( \frac{\alpha}{\pi} \right)^3 \right) + (m_u + m_d)^2 \left( -\langle m\bar{q} \rangle + \frac{1}{8\pi} (\alpha G^2) + \frac{\pi(\mathcal{O}_0)}{4 M^2} \right) + (m_u + m_d)^2 \frac{3\rho^2 M^6}{8\pi^2} e^{-\rho^2 M^2/2} \left[ K_0 (\rho^2 M^2/2) + K_1 (\rho^2 M^2/2) \right]
$$

(18)

Renormalization group improvement of (18) implies that $\alpha$, $m_u$ and $m_d$ are running quantities evaluated at the mass scale $M$ in the $\overline{\text{MS}}$ scheme [12]. The currently accepted range for the bottom threshold [13].

For energies above the charm threshold, $\alpha$ has a negligible effect on the evolution in $M$ [14].

Using the expressions (3, 7, 12, 14) for the correlation function, the definition of $\hat{B}$ and the results of [11], the sum-rule $\mathcal{R}_0(M^2)$ is obtained.

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The running coupling can be directly related to the experimental value [13]

$$
\alpha(M_Z) = 0.119 \pm 0.002
$$

(19)

by using (19) as an initial condition for the evolution of $\alpha$ to the scale $M$ via the 4-loop $\text{MS}$ beta function [14]

$$
\mu^2 \frac{d\alpha}{d\mu^2} = \beta \left( \frac{\alpha}{\pi} \right) = -\frac{\alpha}{2} \sum_{i=0}^{\infty} \beta_i \left( \frac{\alpha}{\pi} \right)^i, \quad \beta \equiv \frac{\alpha}{\pi}
$$

(20)

$$
\beta_0 = \frac{11 - \frac{2}{3} n_f}{4}, \quad \beta_1 = \frac{102 - \frac{38}{3} n_f}{16}, \quad \beta_2 = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{24} n_f^2, \quad \beta_3 = 114.23033 - 27.133944 n_f + 1.5823791 n_f^2 - 5.8566958 \times 10^{-3} n_f^3
$$

(21)

The only subtlety in this approach is the location of flavour thresholds where the number of effective flavour degrees of freedom $n_f$ change. In general, matching conditions must be imposed at these thresholds to relate QCD with $n_f$ quarks to an effective theory with $n_f - 1$ light quarks and a decoupled heavy quark [15]. Using the matching threshold $\mu_{th}$ defined by $m_q(\mu_{th}) = \mu_{th}$, where $m_q$ is the running quark mass, the matching condition to three-loop order is [16]

$$
\alpha^{(n_f-1)}(\mu_{th}) = \frac{1}{2} \left[ \alpha^{(n_f)}(\mu_{th}) + \langle q\bar{q} \rangle + 0.9721 - 0.0847 (n_f - 1) \right] \left[ \alpha^{(n_f)}(\mu_{th}) \right]^3
$$

(23)

For energies above the charm threshold, $\alpha(M)$ depends only on the initial condition (19) and the bottom threshold. The currently accepted range for the bottom threshold [13]

$$
4.1 \text{ GeV} \leq m_b(m_b) \leq 4.4 \text{ GeV}
$$

(24)

has a negligible effect on the evolution in $\alpha(M)$ in comparison with the uncertainty (19) in $\alpha(M_Z)$.

Similarly, the running quark masses can be parametrized by their (unknown) value at 1 GeV. Defining

$$
m = \frac{1}{2} (m_u + m_d)
$$

(25)

$$
w(M) = \frac{m(M)}{m(1 \text{ GeV})}, \quad w(1 \text{ GeV}) = 1
$$

(26)

allows the quark mass at the scale $M$ to be determined via the four-loop $\overline{\text{MS}}$ scheme anomalous mass dimension [17]

$$
\frac{dw}{d\mu} = w(\mu) \gamma \left( \frac{\alpha(\mu)}{\pi} \right) \beta \left( \frac{\alpha(\mu)}{\pi} \right)
$$

(27)
\[
\gamma(x) = -x[1 + \sum_{i=1}^{n} \gamma_i x^i]
\]

\[
\gamma_1 = 4.20833 - 0.138889 n_f, \quad \gamma_2 = 19.5156 - 2.28412 n_f - 0.0270062 n_f^2
\]

\[
\gamma_3 = 98.9434 - 19.1075 n_f + 0.276163 n_f^2 - 0.0057932 n_f^3
\]

To evaluate the quark mass above the charm threshold the currently accepted values in the \( \overline{\text{MS}} \) scheme \[13\] will be used

\[1.1 \text{ GeV} \leq m_c(m_c) \leq 1.4 \text{ GeV}\]

In sum-rule applications, the \( M \) dependence of the theoretical result \( \mathcal{R}_0(M^2) \) \[18\] over a range of \( M \) values is used to predict the resonance parameters contained in \( \text{Im} \Pi(t) \) through the relation \[17\]. Since \( \text{Im} \Pi(t) \) is in general related to a physical cross-section, fundamental inequalities for Laplace sum-rules can be constructed using the property that \( \text{Im} \Pi(t) > 0 \). Thus the simplest inequality that can be obtained is

\[
\mathcal{R}_0(M^2) = \int \text{Im} \Pi(t) e^{-t/M^2} dt \geq 0 \rightarrow \mathcal{R}_0(M^2) \geq 0
\]

More stringent sum-rule inequalities can be developed using integral inequalities. In particular, Hölder inequalities for Laplace sum-rules have been shown to be valuable in studying self-consistency of QCD sum-rules and in obtaining bounds on the electromagnetic polarizability of charged pions \[14\]. Hölder’s inequality for integrals defined over a measure \( d\mu \) is \[18\]

\[
\left| \int_{t_1}^{t_2} f(t)g(t)d\mu \right| \leq \left( \int_{t_1}^{t_2} |f(t)|^p d\mu \right)^{1/p} \left( \int_{t_1}^{t_2} |g(t)|^q d\mu \right)^{1/q},
\]

\[
\frac{1}{p} + \frac{1}{q} = 1; \quad p, q \geq 1
\]

When \( p = q = 2 \) the Hölder inequality reduces to the well known Schwarz inequality. The key idea in applying Hölder’s inequality to sum-rules is recognizing that since \( \text{Im} \Pi(t) \) is positive it can serve as the measure \( d\mu = \text{Im} \Pi(t) dt \) in \[18\]. With the definitions

\[
\tau = \frac{1}{M^2}
\]

\[
S_k(\tau) = \int_{\mu \nu} \text{Im} \Pi(t) t^k e^{-t\tau} dt
\]

\[18\] with \( d\mu = \text{Im} \Pi(t) dt, f(t) = t^a e^{-at\tau}, g(t) = t^b e^{-bt\tau} \), leads to the following inequalities for \( S_k(\tau) \)

\[
S_{\alpha + \beta}(\tau) \leq S_{\alpha p}^{1/p}(a\tau) S_{\beta q}^{1/q}(b\tau); \quad a + b = 1
\]

Imposing restrictions that we have the integer value \( k = 0 \) in our sum-rule \[17\] leads to the following inequality.

\[
S_0[\omega \tau_{\min} + (1 - \omega) \tau_{\max}, s_0] \leq S_0^{\omega}[\tau_{\min}, s_0] S_0^{1-\omega}[\tau_{\max}, s_0], \quad 0 \leq \omega \leq 1; \quad \delta \tau = \tau_{\max} - \tau_{\min} > 0
\]

If the pion pole contribution to \( \text{Im} \Pi_5(t) \) is explicitly included, then the sum-rule \[17\] becomes

\[
\mathcal{R}_0(\tau) = 2 f_\pi^4 m_\pi^4 e^{-m_\pi^2 \tau} + \int_{m_\pi^2}^{\infty} \text{Im} \Pi_5(t) e^{-t\tau} dt
\]

\[39\]
where $m_{\pi}$ represents the pion mass and $f_\pi = 93$ MeV is the pion decay constant. Since for any reasonable range of $\tau$, $\exp(-m_{\pi}^2 \tau) \approx 1$, we find

$$S_0(\tau) \equiv \mathcal{R}_0(\tau) - 2f_\pi^2 m_{\pi}^4 = \int_{m_{\pi}^2}^{\infty} \text{Im} \Pi_5(t)e^{-t\tau} \, dt \quad (40)$$

Thus if the sum-rule is a valid and consistent representation of the integration of $\text{Im} \Pi_5(t)$ in (17) then after subtraction of the pion pole, the sum-rule $S_0(\tau)$ must satisfy the fundamental inequality (37).

$$\rho_0 = \frac{S_0[\tau + (1 - \omega)\delta \tau]}{S_0[\tau]S_0[\tau + \omega]} \leq 1 \, , \quad \forall \, 0 \leq \omega \leq 1 \quad (41)$$

Provided that $\delta \tau$ is reasonably small (in QCD $\delta \tau \approx 0.1 \text{GeV}^{-2}$ appears sufficient) these inequalities are insensitive to the value of $\delta \tau$, permitting a simple analysis of the inequality as a function of the energy scale $M$.

The first applications of inequalities to quark-mass bounds used the simple positivity constraint $S_0(\tau) \geq 0$ [24, 19]. We extend this analysis by using the more stringent H"older inequality, higher-loop perturbative corrections, and inclusion of instanton effects which are important for the pseudoscalar channel.

For a fixed value of $M = 1/\sqrt{\tau}$ the H"older inequality can be used to find the minimum value of $m(1 \text{ GeV}) \equiv [m_u(1 \text{ GeV}) + m_d(1 \text{ GeV})/2$ for which the inequality (11) is satisfied. This lower bound on $m(1 \text{ GeV})$ depends on the energy scale $M$ as indicated in Figure 4. However, the increasingly large coefficients in the perturbative portion of (18) suggests that these mass bounds could depend strongly upon higher-order perturbative effects. Asymptotic Padé approximation methods [21] applied to the perturbative part of (18) result in the following estimate of $\mathcal{R}_0(M^2)$ with perturbative effects to five-loop order.

$$\mathcal{R}_0(M^2) = \frac{3}{8\pi^2} \left(1 + 4.821098 \frac{\alpha}{\pi} + 21.97646 \left(\frac{\alpha}{\pi}\right)^2 + 53.14179 \left(\frac{\alpha}{\pi}\right)^3 + 137.6 \left(\frac{\alpha}{\pi}\right)^4 \right)$$

$$+ (m_u + m_d)^2 \left(-\langle m \bar{q} \rangle + \frac{1}{8\pi} (\alpha G^2) + \frac{\pi \langle O_6 \rangle}{4M^2} \right)$$

$$+ (m_u + m_d)^2 \frac{3\rho_0^2 M^6}{8\pi^2} e^{-\rho_0^2 M^2/2} \left[K_0 (\rho_0^2 M^2/2) + K_1 (\rho_0^2 M^2/2) \right] \quad (42)$$

By comparing mass bounds with and without inclusion of this estimated five-loop perturbative contribution, the uncertainty in the mass bounds of Figure 4 devolving from truncation of the perturbative expansion can be estimated. In addition to this perturbative uncertainty, a 50% uncertainty in the vacuum saturation hypothesis (parameterized by $f_{\pi}$), a 15% uncertainty in the instanton size $\rho_0$, and the effects of varying the input parameters within the ranges [1, 19, 24, 31] must also be considered. Figure 5 illustrates the effect of all sources of uncertainty upon the mass bounds obtained from the H"older inequality. As indicated by the figure, the analysis is extremely stable.

The increase of the quark mass bound $m_{\pi\pi}$ with decreasing $M$ implies that the minimum energy scale $M$ at which the pseudoscalar sum-rule is considered valid will provide the most stringent bound on the quark mass $M$. The analysis of the $\tau$ hadronic width [21] and hadronic contributions to $\alpha EM(M_Z)$ and the anomalous magnetic moment of the muon [22] provide excellent evidence for the the validity of QCD sum-rule methods at the energy scale $M \tau \approx 1.8 \text{ GeV}$. Furthermore, the largest value of $M$ at which the uncertainties in the analysis are visible in Figure 4 corresponds to $M \approx M_\tau$, providing support for the validity of the mass bounds at $M = M_\tau$. Thus a valid and conservative bound on the 1.0 GeV $\overline{M/S}$ quark masses is obtained from Figure 5 at $M = M_\tau$

$$m(1 \text{ GeV}) = \frac{1}{2} [m_u(1 \text{ GeV}) + m_d(1 \text{ GeV})] \geq 3 \text{ MeV} \quad (43)$$

This conservative bound is phenomenologically significant since the Particle Data Group quotes a lower bound of $m(1 \text{ GeV}) \geq 2.25 \text{ MeV}$ [13].

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Figure 1: The quantity $m_{\text{min}}$, representing the minimum value of $m$ for which the H"older inequality is satisfied, is plotted as a function of the mass scale $M$ using central values of all input parameters.
Figure 2: The effect of uncertainties in the input parameters on $m_{\text{min}}$ is plotted as a function of the mass scale $M$. Higher-loop effects are estimated by the inclusion of the five-loop perturbative correction in (42).