Non–static hyperbolically symmetric fluids

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We present the general properties of dynamic dissipative fluid distribution endowed with hyperbolical symmetry. All the equations required for its analysis are exhibited and used to contrast the behavior of the system with the spherically symmetric case. Several exact solutions are exhibited and prospective applications to astrophysical and cosmological scenarios are discussed.

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I. INTRODUCTION

In a recent paper [1] an alternative global description of the Schwarzschild black hole has been proposed. The motivation behind such an endeavor was, on the one hand the fact that the space–time within the horizon, in the classical picture, is necessarily non–static or, in other words, that any transformation that maintains the static form of the Schwarzschild metric (in the whole space–time) is unable to remove the coordinate singularity appearing on the horizon in the line element [2]. Indeed, as is well known, no static observers can be defined inside the horizon (see [3, 4] for a discussion on this point). This conclusion becomes intelligible if we recall that the Schwarzschild horizon is also a Killing horizon, implying that the time–like Killing vector existing outside the horizon, becomes space–like inside it.

On the other hand, based on the physically reasonable point of view that any equilibrium final state of a physical process should be static, it would be desirable to have a static solution over the whole space–time.

Based on the arguments above, the following model was proposed in Ref. [1].

Outside the horizon (R > 2M) one has the usual Schwarzschild line element corresponding to the spherically symmetric vacuum solution to the Einstein equations, which in polar coordinate reads (with signature +2)

\[
\begin{align*}
    ds^2 &= -\left(1 - \frac{2M}{R}\right)dt^2 + \frac{dR^2}{\left(1 - \frac{2M}{R}\right)} + R^2 d\Omega^2, \\
    d\Omega^2 &= d\theta^2 + \sin^2 \theta d\phi^2.
\end{align*}
\]

(1)

This metric is static and spherically symmetric, meaning that it admits four Killing vectors:

\[
\begin{align*}
    \xi_{(0)} &= \partial_t, \\
    \xi_{(2)} &= -\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi, \\
    \xi_{(1)} &= \partial_\phi, \\
    \xi_{(3)} &= \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi.
\end{align*}
\]

(2)

The solution proposed for R < 2M (with signature (+ − −−)) is

\[
\begin{align*}
    ds^2 &= \left(\frac{2M}{R} - 1\right) dt^2 - \frac{dR^2}{\left(\frac{2M}{R} - 1\right)} + R^2 d\Omega^2, \\
    d\Omega^2 &= d\theta^2 + \sinh^2 \theta d\phi^2.
\end{align*}
\]

(3)

This is a static solution, meaning that it admits the time–like Killing vector ξ_{(0)}, however unlike (1) it is not spherically symmetric, but hyperbolically symmetric, meaning that it admits the three Killing vectors

\[
\begin{align*}
    K_{(2)} &= -\cos \phi \partial_\theta + \coth \theta \sin \phi \partial_\phi, \\
    K_{(1)} &= \partial_\phi, \\
    K_{(3)} &= \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi.
\end{align*}
\]

(4)

Thus the situation may be summarized as follows: if one wishes to keep sphericity within the horizon, one should abandon staticity; if one wishes to keep staticity within the horizon, one should abandon sphericity.

The classical picture of the black hole entails sphericity within the horizon, instead in [1] we have proceeded differently and have assumed staticity within the horizon.

The three Killing vectors above define the hyperbolical symmetry. Space–times endowed with hyperbolical symmetry have previously been the subject of research in different contexts (see [5]–[16] and references therein).

In [17] a general study of geodesics in the spacetime described by (4) was presented, leading to some interesting conclusions about the behavior of a test particle in this new picture of the Schwarzschild black hole, namely:

• the gravitational force inside the region R < 2M is repulsive;
• test particles cannot reach the center;
• test particles can cross the horizon outward, but only along the θ = 0 axis.

Motivated by these intriguing results we embarked into a general study of fluid distributions endowed with hyperbolical symmetry [18–20].
It is the purpose of this paper to present the main results concerning dynamic and dissipative fluids endowed with hyperbolical symmetry. This includes the general equations governing the behavior of such fluids as well as a selection of some exact analytical solutions. We shall discuss about prospective applications of these results to the study of some astrophysical and cosmological problems. We shall not consider here the static case analyzed in \cite{19}.

Finally, it is worth stressing the fact that although the main motivation to undertake the study of fluids endowed with hyperbolical symmetry was the intriguing properties of the black hole model briefly described above, the results we are going to exhibit in the sections below are completely independent on such a model.

II. BASIC EQUATIONS AND VARIABLES

We consider hyperbolically symmetric distributions of evolving fluids, which may be bounded, or not, from outside by a surface $\Sigma$. On the other hand, as we shall see below, fluids endowed with hyperbolical symmetry cannot fill the central region, therefore they should be bounded from inside by a surface $\Sigma$. The fluid is assumed to be locally anisotropic (principal stresses unequal) and undergoing dissipation in the form of heat flow (diffusion approximation).

Choosing comoving coordinates, the interior metric, admitting the three Killing vectors \cite{41}, may be written as

$$ds^2 = -A^2dt^2 + B^2dz^2 + R^2(d\theta^2 + \sinh^2 \theta d\phi^2),$$

where $A$, $B$ and $R$ are assumed positive, and due to the hyperbolical symmetry are functions of $t$ and $r$. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$.

The energy momentum tensor $T_{\alpha\beta}$ of the fluid distribution may be written in its canonical form as

$$T_{\alpha\beta} = \mu V_\alpha V_\beta + P h_{\alpha\beta} + \Pi_{\alpha\beta} + q (V_\alpha \chi_\beta + \chi_\alpha V_\beta)$$

with

$$P = \frac{P_r + 2P_\perp}{3}, \quad h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta,$$

$$\Pi_{\alpha\beta} = \Pi \left( \chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta} \right), \quad \Pi = P_r - P_\perp.$$

$$V^\alpha V_\alpha = -1, \quad V^\alpha q_\alpha = 0, \quad \chi^\alpha \chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0$$

$$V^\alpha = A^{-1} \delta^\alpha_0, \quad q^\alpha = q B^{-1} \delta^\alpha_1, \quad \chi^\alpha = B^{-1} \delta^\alpha_1,$$

where $\mu$, $P_r$, $P_\perp$, $q^\alpha$, $V^\alpha$ have the usual meaning, and $\chi^\alpha$ is unit four–vector along the radial direction. It is worth noticing that bulk or shear viscosity could be introduced by redefining the radial and tangential pressures. In addition, dissipation in the free streaming approximation can be absorbed in $\mu$, $P_r$ and $q$.

Since the Lie derivative $\mathcal{L}$ and the partial derivative commute, then

$$\mathcal{L}_K (R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R) = 8 \pi \mathcal{L}_K T_{\alpha\beta} = 0,$$

implying that all physical variables only depend on $t$ and $r$.

Finally, it is worth mentioning that because of the hyperbolical symmetry there are only two possible unequal principal stresses ($P_r$, $P_\perp$).

A. Einstein equations and conservation laws

The Einstein equations for (5) and (6) are

$$8 \pi \mu = -\frac{1}{R^2} \frac{1}{B^2} \left[ -2B' R' \frac{R'}{R} + (\frac{R'}{R})^2 + 2R'' \right]$$

$$+ \frac{1}{A^2} \left( 2 \frac{B'}{B} \frac{R'}{R} + \frac{R'}{R} \right),$$

$$8 \pi q = -\frac{1}{AB} \left( \frac{R'}{R} \frac{B'}{B} + \frac{A'}{A} \frac{R'}{R} - \frac{R'}{R} \right),$$

$$8 \pi P_r = \frac{1}{R^2} + \frac{1}{B^2} \left[ \frac{2A' R'}{A R} + (\frac{R'}{R})^2 \right]$$

$$+ \frac{1}{A^2} \left( 2 \frac{A'}{A} \frac{R'}{R} - \frac{R'}{R} \right),$$

$$8 \pi P_\perp = \frac{1}{B^2} \left[ -\frac{A'}{A} \frac{B'}{B} + \frac{A'}{A} \frac{R'}{R} - \frac{B'}{B} \frac{R'}{R} + \frac{A''}{A} + \frac{R''}{R} \right]$$

$$+ \frac{1}{A^2} \left( \frac{A'}{A} \frac{B'}{B} + \frac{A'}{A} \frac{R'}{R} - \frac{B'}{B} \frac{R'}{R} - \frac{B}{B} - \frac{R}{R} \right),$$

where dots and primes denote derivative with respect to $t$ and $r$ respectively. It is worth noticing the difference between these equations and the corresponding to the spherically symmetric case (see for example Eqs.(7)–(10) in \cite{21}).

The conservation laws $T^\mu_{\mu \nu} = 0$, as in the spherically symmetric case, have only two independent components, which are displayed in the Appendix.

B. Kinematical variables

The four–acceleration $a_\alpha$ and the expansion $\Theta$ of the fluid are given by

$$a_\alpha = V_{\alpha;\beta} V^\beta, \quad \Theta = V^\alpha_{;\alpha}.$$
From which we obtain for the four–acceleration and its scalar $a$,

$$a_1 = \frac{A'}{A}, \quad a = \frac{A'}{AB} \Rightarrow a^\alpha = a \chi^\alpha, \quad (13)$$

and for the expansion

$$\Theta = \frac{1}{A} \left( \frac{\dot{B}}{B} + 2 \frac{\dot{R}}{R} \right). \quad (14)$$

The shear tensor $\sigma_{\alpha \beta}$ is defined by (the vorticity vanishes identically)

$$\sigma_{\alpha \beta} = V_{(\alpha: \beta)} + a_{(\alpha} V_{\beta)} - \frac{1}{3} \Theta h_{\alpha \beta}, \quad (15)$$

its non zero components are

$$\sigma_{11} = \frac{2}{3} B^2 \sigma, \quad \sigma_{22} = \frac{\sigma_{33}}{\text{sinh}^2 \theta} = -\frac{1}{3} R^2 \sigma, \quad (16)$$

and its scalar

$$\frac{3}{2} \sigma^\alpha_{\alpha} \sigma_{\alpha \beta} = \sigma^2, \quad (17)$$

reads

$$\sigma = \frac{1}{A} \left( \frac{\dot{B}}{B} - \frac{\dot{R}}{R} \right). \quad (18)$$

All the expressions above are the same, in terms of the metric functions, as in the spherically symmetric case.

### C. The Weyl tensor

The magnetic part of the Weyl tensor for our metric vanishes, whereas its electric part may be written as

$$E_{\alpha \beta} = \mathcal{E} \left( \chi_{\alpha} \chi_{\beta} - \frac{1}{3} h_{\alpha \beta} \right), \quad (19)$$

with

$$\mathcal{E} = \frac{1}{2 B^2} \left[ -\frac{A' R'}{A R} - \frac{A' B'}{A B} + \frac{R' B'}{R B} + \left( \frac{R'}{R} \right)^2 - \frac{A''}{A} - \frac{R''}{R} \right]$$

$$+ \frac{1}{2 A^2} \left[ -\frac{\dot{A} \dot{B}}{A B} + \frac{\dot{A} \dot{R}}{A R} + \frac{\dot{B} \dot{R}}{R B} - \left( \frac{\dot{R}}{R} \right)^2 - \frac{\dot{B}}{B} + \frac{\dot{R}}{R} \right] + \frac{1}{2 R^2}. \quad (20)$$

### D. The mass function

Following we may define the mass function as

$$m(r, t) = \frac{R}{2} R^3_{232} = \frac{R}{2} \left[ \left( \frac{R'}{B} \right)^2 - \left( \frac{R}{A} \right)^2 + 1 \right], \quad (21)$$

where the Riemann tensor component $R^3_{232}$ is now calculated for $\dot{E}$.

Introducing the proper time derivative $D_T$, and the proper radial derivative $D_R$ by

$$D_T = \frac{1}{A} \frac{\partial}{\partial t}, \quad (22)$$

and

$$D_R = \frac{1}{R} \frac{\partial}{\partial r}, \quad (23)$$

we may define the “areal” velocity $U$ as

$$U = D_T R, \quad (24)$$

which must be smaller than 1 (in relativistic units).

Then, since $U < 1$, it follows at once from (21) that $m$ is a positive defined quantity. Also, (21) can be rewritten as

$$E = \frac{R'}{B} = \left( \frac{2m}{R} + U^2 - 1 \right)^{1/2}. \quad (25)$$

Using (21) with (22), (23), and the field equations, we obtain

$$D_T m = 4\pi (P_U + q E) R^2, \quad (26)$$

and

$$D_R m = -4\pi \left( \mu + q \frac{U}{E} \right) R^2, \quad (27)$$

producing

$$m = -4\pi \int_0^r \left( \mu + q \frac{U}{E} \right) R^2 R' dr, \quad (28)$$

satisfying the regular condition $m(t, 0) = 0$.

Integrating (28) we find

$$\frac{3 m}{R^3} = -4\pi \mu + \frac{4\pi}{R^3} \int_0^r R^3 \left( D_R \mu - 3 q \frac{U}{RE} \right) R' dr. \quad (29)$$

From (28) two important general properties of hyperbolically symmetric fluids, follow.

On the one hand, $\mu$ is necessarily negative. This follows from the (physically meaningful) condition that the fluid is not very far from thermal equilibrium, implying $q << |\mu|$, and furthermore $R' > 0$ to avoid shell crossing, $m > 0$ and $E$ is a regular function within the fluid.

On the other hand, we see from (28) that whenever the energy density is regular, then $m \sim r^3$ as $r$ tends to zero. However, in this same limit $U \sim 0$, and $R \sim r$ implying because of (25) that the central region cannot be filled with our fluid distribution. In other words, there is a central region of finite dimensions, which cannot be filled with a fluid endowed with hyperbolical symmetry. At this point many different scenarios may be envisaged.
We shall assume here that the center is surrounded by a vacuum cavity described by a Minkowski space–time. It is worth emphasizing that such a choice does not affect the general properties of the fluids endowed with hyperbolic symmetry.

The two above mentioned features of the fluid appear also in the static case [14].

Before concluding this section it is worth discussing with some detail on equation (A3), and compare it with the corresponding equation for the spherically symmetric case (see eq.(C6) in [21]).

Equation (A3) is particularly appealing because it has the “Newtonian” form \( F = \text{Mass density} \times \text{Acceleration} \). Let us now analyze its different terms. The first term on the right represents the gravitational interaction, it is the product of the passive gravitational mass density (p.g.m.d) \( \langle \mu + P_{r} \rangle \), which due to the fact that the energy density is negative, would be negative, and the active gravitational mass (a.g.m) \( (4\pi P_{r} R^{3} - m) \) which would also be negative for most equations of state. Thus the gravitational term has the same sign as in the spherically symmetric case. However its effect is the inverse of this latter case. Indeed, since the p.g.m.d is negative so is the inertial mass density, implying that the gravitational term tends to increase \( D_{T}U \), i.e. it acts as a repulsive force, instead of an attractive one as in (C6) of [21]. Also, we see that a negative pressure gradient which produces a force pointing outward, would tend to push any fluid element inwardly.

### III. THE TRANSPORT EQUATION

Since we are considering dissipative systems we have to adopt a transport equation. In order to ensure causality we shall resort to the transport equation obtained from the M"uller–Israel–Stewart theory [24–26].

Then, the corresponding transport equation for the heat flux reads

\[
\tau h^{\alpha \beta} V^\gamma q_{\beta \gamma} + q^\alpha = -\kappa h^{\alpha \beta} (T_{\beta} + T a_{\beta}) - \frac{1}{2} \kappa T^{2} \left( \frac{\tau V^{\beta}}{\kappa T^{2}} \right)_{\beta} q^{\alpha},
\]

where \( \kappa \) denotes the thermal conductivity, and \( T \) and \( \tau \) denote temperature and relaxation time, respectively.

There is only one non-vanishing independent component of Equation (30), which may be written as

\[
\tau D_{T} q = -q - \frac{\kappa}{A B} (AT)'/2 + \frac{1}{2} \tau q' q - \frac{1}{2} \kappa T^{2} D_{T} \left( \frac{\tau}{\kappa T^{2}} \right) q.
\]

In the case \( \tau = 0 \) we recover the Eckart–Landau equation [27].

Under some circumstances it is possible to adopt the so called “truncated” version where the last term in (30) is neglected [28].

\[
\tau h^{\alpha \beta} V^\gamma q_{\beta \gamma} + q^\alpha = -\kappa h^{\alpha \beta} (T_{\beta} + T a_{\beta}),
\]

and whose only non–vanishing independent component becomes

\[
\tau q + qA = -\frac{\kappa}{B} (T A)'.
\]

Two important thermodynamical properties of hyperbolically symmetric fluids can be inferred from (31).

Let us first consider the condition of thermal equilibrium.

As it was pointed out by Tolman many years ago [29], the condition of thermal equilibrium in the presence of a gravitational field must change with respect to its form in the absence of gravity since thermal energy tends to displace to regions of lower gravitational potential (independently on any temperature gradient). Thus, a temperature gradient is necessary in thermal equilibrium in order to prevent the flow of heat from regions of higher to lower gravitational potential.

Indeed, as it follows at once from (31), thermal equilibrium implies

\[
(T A)' = 0 \Rightarrow T' = -\frac{T}{A} A' = -T a B.
\]

However in our case \( a < 0 \), (the four–acceleration is now directed radially inwardly), implying the existence of a repulsive gravitational force, leading to a positive temperature gradient in order to assure thermal equilibrium. This situation is at variance with the spherically symmetric case, where a negative temperature gradient is required to assure thermal equilibrium.

A second interesting property appears from the combination of (A3) with (31), producing (A4). It brings out the effect of dissipative processes on the p.g.m.d., and by virtue of the equivalence principle, on the effective inertial mass density as well. This kind of effect was pointed out for the first time for the spherically symmetric case in [30] (see also [31] for a discussion on this effect). In our case the term \( \frac{\kappa}{A B} \) increases the absolute value of the effective p.g.m.d (which is negative), thereby increasing the absolute value of the effective inertial mass density (the term in the bracket on the left of (A3)), as a result of which any hydrodynamic force directed outward tends to push the fluid element inward but weaker than in the non–dissipative case, due to the term \( \frac{\kappa}{A B} \). Thus in the hyperbolically symmetric case the thermal effect on the inertial mass density enhances the tendency to expansion, as in the spherically symmetric case, but comes about from different way.

In order to obtain specific solutions to the Einstein equations we shall need to impose additional restrictions. In this work we shall assume two different types of restrictions. On the one hand, we shall assume that the fluid evolves in the quasi–homologous regime and satisfies the vanishing complexity factor condition. The next section is devoted to explain these conditions in some detail. On the other hand we shall seek for spacetimes which could be considered as hyperbolically symmetric versions of Lemaitre–Tolman–Bondi spacetimes, for doing so we shall assume that the fluid is geodesic, shearing,
IV. COMPLEXITY FACTOR AND QUASI–HOMOLOGOUS EVOLUTION

The complexity factor is a scalar function intended to measure the degree of complexity of a self-gravitating system (in some cases more than one scalar function may be required). For a static, hyperbolically symmetric fluid distribution it was assumed in [32] (following the arguments developed in [32]) that the simplest system corresponds to a homogeneous (in the energy density), locally isotropic fluid distribution (principal stresses equal). Thus, a zero value of the complexity factor is assigned to such a distribution. Furthermore, it was shown that a single scalar function (hereafter referred to as $Y_{TF}$) describes the modifications introduced by the energy density inhomogeneity and pressure anisotropy to the Tolman mass, with respect to its value for the vanishing complexity factor.

This scalar belongs to a set of variables named structure scalars, defined in [33] and which appear in the orthogonal splitting of the Riemann tensor [33–35]. For our purpose here we shall need only one of the five structure scalars, defined in [33] and which appear in the or-

The scalar $Y_{TF}$ defines the trace–free part of the electric Riemann tensor $Y_{αβ}$ given by

$$Y_{αβ} = R_{αγδβ}V^γV^δ.$$  \hspace{1cm}  \text{(35)}

In our case the expression for $Y_{TF}$ reads (see [33] for details)

$$Y_{TF} = \mathcal{E} - 4\pi\Pi.$$  \hspace{1cm}  \text{(36)}

Using [10], [11] and [20], we may express $Y_{TF}$ in terms of the metric functions and their derivatives as

$$Y_{TF} = \frac{1}{B^2} \left( \frac{A''}{A} - \frac{A'}{A} \frac{R'}{R} - \frac{A'}{A} \frac{B'}{B} \right) + \frac{1}{A^2} \left( \frac{\dot{A}}{A} \frac{\dot{B}}{B} - \frac{\dot{A}}{A} \frac{\dot{R}}{R} - \frac{\dot{B}}{B} + \frac{\dot{R}}{R} \right).$$  \hspace{1cm}  \text{(37)}

In the dynamic case, we still need to provide a criterion for the definition of complexity of the pattern of evolution.

We shall consider the quasi-homologous evolution defined in [21] as the simplest mode of evolution.

The quasi–homologous condition reads

$$U = R \frac{U'_{\Sigma^r}}{R'_{\Sigma^r}},$$  \hspace{1cm}  \text{(38)}

implying

$$\frac{4\pi q}{E} + \frac{\sigma}{R} = 0.$$  \hspace{1cm}  \text{(39)}

The above condition will be used to obtain specific models, the rationale behind such a choice is based on the fact that it represents the relativistic version of the well-known homologous condition widely used in classical astrophysics, furthermore as shown in [21], it qualifies as one of the simplest patterns of evolution.

V. JUNCTION CONDITIONS

In the case that the fluid is bounded then junction conditions on the boundary have to be imposed [36], otherwise we have to cope with the presence of thin shells [37].

Thus, outside $\Sigma^r$ we assume that we have the hyperbolic version of the Vaidya spacetime, described by

$$ds^2 = -\left[ \frac{2M(v)}{r} - 1 \right] dv^2 - 2drdv + r^2(d\theta^2 + \sinh^2 \theta d\phi^2),$$  \hspace{1cm}  \text{(40)}

where $M(v)$ denotes the total mass, and $v$ is the retarded time.

The continuity of the first and the second fundamental forms imply, on $\Sigma^r$,

$$q \equiv P_r,$$  \hspace{1cm}  \text{(41)}

and

$$m(t, r) \equiv M(v),$$  \hspace{1cm}  \text{(42)}

where $\equiv$ means that both sides of the equation are evaluated on $\Sigma^r$.

In the cases where the central region is surrounded by an empty vacuole bounded by a surface $\Sigma^i$, junction conditions at the inner boundary of the fluid distribution read

$$P_r \equiv 0,$$  \hspace{1cm}  \text{(43)}

and

$$m(t, r) \equiv 0.$$  \hspace{1cm}  \text{(44)}

VI. SOME EXACT SOLUTIONS

In the following subsections we shall exhibit several families of solutions to the Einstein equations for hyperbolically symmetric fluids. More specifically, we shall consider two families of solutions. One of them will be obtained by assuming quasi–homologous evolution and the vanishing of the complexity factor. The other family of solutions corresponds to hyperbolically symmetric fl...
versions of Lemaitre–Tolman–Bondi (LTB) space–times. In each case dissipative and non–dissipative models were obtained. The purpose of the presentation of these models is not only the potential application of some of them to the study of specific astrophysical scenarios, but also to illustrate the richness of fluid distributions endowed with hyperbolical symmetry.

A. Models with vanishing complexity factor and evolving in the quasi–homologous regime

In this case we shall exhibit only non–dissipative models (for dissipative models see [18]). Then, excluding dissipative processes, and assuming the quasi–homologous condition \( \dot{q} = 0 \) we may write

\[
q = 0 \Rightarrow \sigma = 0 \Rightarrow \frac{\dot{B}}{B} = \frac{\dot{R}}{R} \Rightarrow R = rB,
\]

and

\[
U = \frac{\dot{R}}{A} = \frac{r\dot{B}}{A} = \ddot{a}(t)rB.
\]

Imposing next the condition \( Y_{TF} = 0 \) we have

\[
\frac{A''}{A} - \frac{A'B'}{A B} - \frac{A'R'}{A R} = 0.
\]

In order to exhibit specific solutions, we shall further assume some additional restrictions.

1. \( \mathcal{E} = 0, \Pi = 0 \)

We shall assume here that the fluid is conformally flat \( (\mathcal{E} = 0) \) and the pressure is isotropic \( (\Pi = 0) \), which combined with \( Y_{TF} = 0 \) produces \( \mu' = 0 \) (i.e. the energy density is homogeneous).

In this case the metric functions are (see [18] for details)

\[
R = \frac{\dot{R}(t)}{\cos[c_1(t) + \ln r]},
\]

\[
B = \frac{\dot{R}(t)}{r\cos[c_1(t) + \ln r]},
\]

\[
A = \gamma(t)\hat{R}^2(t)\tan[c_1(t) + \ln r] + b(t),
\]

where \( \hat{R}(t), c_1(t), \gamma(t), b(t) \) are arbitrary functions of their argument.

We shall further specify the solution by choosing the above functions as follows

\[
\dot{c}_1 = \frac{\dot{R}}{R}, \quad b(t) = \gamma(t)\hat{R}^2,
\]

producing

\[
\frac{\ddot{R}}{\dot{R}} = \frac{\dot{R}}{R}(1 + \tan u), \quad (52)
\]

\[
A = \gamma(t)\hat{R}^2(1 + \tan u), \quad \Rightarrow \quad A = \frac{\ddot{a}}{\dot{R}}, \quad (53)
\]

with \( \ddot{a} = \frac{\gamma(t)\hat{R}^2}{\dot{R}} \) and \( u = c_1(t) + \ln r \). From the above expressions we found for the physical variables and the mass function,

\[
8\pi\mu = -\frac{3}{R^2} + \frac{3}{\dot{a}^2}, \quad (54)
\]

\[
8\pi P_r = 8\pi P_\perp = -\frac{3}{\dot{a}^2} + \frac{3\tan u + 1}{\hat{R}^2(\tan u + 1)}
\]

\[
+ \frac{2\dot{R}\dot{a}}{\dot{a}^2\hat{R}(\tan u + 1)}, \quad (55)
\]

\[
m = \frac{\hat{R}}{2\cos^3 u} \left( 1 - \frac{\hat{R}^2}{\dot{a}^2} \right). \quad (56)
\]

If we choose \( \hat{R}(t), c_1(t), \gamma(t) \) such that they tend to a constant as \( t \to \infty \), then the above solution tend to the incompressible isotropic solution found in [13], which is a particular case of the hyperbolically symmetric Bowers–Liang solution found in [19].

This model might be considered as a hyperbolically symmetric version of the Friedman–Robertson–Walker space–time (FRW), since both share some similar properties e.g. \( \mathcal{E} = \Pi = \mu' = \sigma = 0 \). However our solution is not geodesic as in the spherically symmetric case. We shall next find another version of the hyperbolically symmetric FRW space–time, but satisfying the geodesic condition \( A' = 0 \).

2. \( A = 1, \mathcal{E} = 0 \)

If we further impose the geodesic condition on the fluid \( (A = 1) \), then the quasi–homologous condition implies

\[
\frac{R_I}{R_{II}} = constant, \quad (57)
\]

where \( R_I \) and \( R_{II} \) denote the areal radii of two shells \( (I, II) \) described by \( r = r_I = constant, \) and \( r = r_{II} = constant, \) respectively. In the notation of [18], conditions \( (49) \) and \( (57) \) define the homologous evolution.

From \( (57) \) it follows at once that \( R \) is a separable function.

The conditions \( A = 1 \) and \( q = 0 \) imply

\[
\frac{\dot{B}}{B} = \frac{\dot{R}'}{R'}. \quad (58)
\]
where $\mathbf{10}$ has been used. From the shear–free condition and the separability of $R$ it can be easily shown that $B$ becomes a function of $t$ alone $B = B(t)$, i.e.

$$R = rB(t).$$

Then $\mathbf{18}$ is automatically satisfied, as well as $Y_{TF} = 0$ as it follows from $\mathbf{17}$.

The physical variables and the mass function for this model read

$$8\pi \mu = -\frac{2}{r^2 \beta^2} + 3\dot{B}^2 \frac{\beta}{B^2},$$

$$8\pi P_r = \frac{2}{r^2 \beta^2} \dot{B}^2 - \frac{2\dot{B}^2}{\beta B^2},$$

$$8\pi P_\perp = -\frac{2\dot{B}^2}{\beta} - \frac{2\dot{B}}{B},$$

$$m = \frac{r\dot{B}}{2}(2 - r^2 \dot{B}^2).$$

Thus the fluid is conformally flat, shear–free, geodesic, evolves homologously and satisfies the vanishing complexity factor condition. So, it also qualifies as a hyperbolically symmetric version of FRW space–time. However, unlike the spherically symmetric case, it is anisotropic in the pressure and the energy–density is inhomogeneous.

It is worth analyzing with some detail the differences between this case and the situation in the spherically symmetric case (FRW). In the latter case we have seen $\mathbf{38}$ that for a non–dissipative fluid satisfying the homologous condition, the complexity factor vanishes and there is a single solution characterized by $\Pi = \mu' = a = \mathcal{E} = 0$ (FRW).

However in the present case, imposing homologous condition on a geodesic non–dissipative fluid we get a conformally flat, shear–free fluid with $\Pi, \mu' \neq 0$. If we want to describe an isotropic, homogeneous, shear–free non–dissipative fluid, then we have to relax the geodesic condition.

Finally, let us build a toy model with the above solution, by choosing a particular form of $B$ such that asymptotically it leads to a static regime.

Thus, let us assume

$$B = \beta \left(1 + e^{-\alpha t}\right),$$

where $\alpha, \beta$ are two positive constants.

Then it is a simple matter to check that as $t \to \infty$ we get

$$8\pi \mu = -\frac{2}{r^2 \beta^2},$$

$$8\pi P_r = \frac{2}{r^2 \beta^2} \dot{\beta}^2,$$

$$8\pi P_\perp = 0,$$

and for the mass function we get asymptotically $m = r \beta$.

As we see, our toy model converges to the static solution corresponding to the stiff equation of state ($P_r = |\mu|$) found in $\mathbf{19}$ (Eqs.(138-139) in that reference).

B. Hyperbolically symmetric versions of Lemaitre–Tolman–Bondi spacetimes

We shall now consider a family of solutions sharing some basic properties with LTB space–times, namely.

- $\mu', \sigma, \mathcal{E} \neq 0, A = 1$.
- We shall NOT require homologous condition.

Let us first consider the simplest case (non–dissipative dust) and afterward we shall consider extensions to the dissipative anisotropic case.

1. The non–dissipative dust case

In this case it follows from the field equations

$$B(t, r) = \frac{R'}{(k(r) - 1)^{1/2}},$$

where $k$ is an arbitrary function of $r$.

Next, from the definition of the mass function we may write

$$\dot{R}^2 = -\frac{2m}{R} + k(r),$$

implying $k(r) > \frac{2m}{R}$. Thus unlike the spherically symmetric LTB space–time we now have only one case $k(r) > 0$.

The solution to $\mathbf{69}$ may be written as:

$$R = \frac{m}{k}(\cosh \eta + 1), \quad \frac{m}{k^{3/2}}(\sinh \eta + \eta) = t - t_0(r),$$

and the line element reads

$$ds^2 = -dt^2 + \frac{\left(R'\right)^2}{k(r) - 1}dr^2 + R^2(d\theta^2 + \sinh^2 \theta d\phi^2).$$

In order to prescribe an explicit model we have to provide the three functions $k(r), m(r)$ and $t_0(r)$. However, since $\mathbf{21}$ is invariant under transformations of the form $r = r(\tilde{r})$, we only need two functions of $r$.

Assuming as example $m_0 = \frac{m}{k} = \text{constant}$, and $t_0(r) = \text{constant}$ the expressions for $\Theta$ and $\sigma$ read

$$\Theta = \sqrt{k} \frac{\sinh \eta}{m_0} \left(\frac{\sinh \eta}{\sinh \eta + \eta} + \cosh \eta + 2 \sinh \eta + 1 \right),$$

$$\sigma = \sqrt{k} \frac{\sinh \eta}{m_0} \left(\frac{\sinh \eta}{\sinh \eta + \eta} + \cosh \eta - \sinh \eta + 1 \right),$$

from where it is clear that the expansion is always positive.

The only non–trivial conservation law in this case reads

$$\dot{\mu} + \mu \Theta = 0,$$
In order to obtain specific models we shall assume that the complexity factor has the same form as in the non-dissipative case, implying

\[
\frac{\dot{K}}{2(K-1)} + \frac{K}{K-1} \left( \frac{\dot{R}'}{R'} - \frac{3}{4} \frac{K}{K-1} \right) = 0.
\]  

(82)

The integration of (82) produces

\[
\frac{R'}{\sqrt{K}} = C_1(r),
\]

(83)

where \(C_1\) is an integration function.

From the above, using the field equations we may write for \(q\)

\[
2\pi q = \frac{1}{R \left( R' C_1(r) \int \frac{dt}{(R')^2} \right)^2}.
\]

(84)

Now, it is worth noticing that this class of solutions does not admit pure dust condition (pressure cannot vanish).

Indeed, from the dust condition and (82) we obtain

\[
\frac{\dot{R} C_1^2 \sqrt{K-1}}{R' \cdot 2(R')^2} = 0,
\]

(85)

which cannot be satisfied, implying that there are no radiating dust solutions in this case. Therefore we have to relax the dust condition, and we shall consider models of radiating anisotropic fluids. A simple model of this kind, may be obtained by assuming \(P_\perp = 0\), \(P_r \neq 0\).

Then, from the condition \(P_\perp = 0\), we obtain

\[
\dot{B} = 0, \quad \Rightarrow B = b_1(r)t + b_2(r),
\]

(86)

where \(b_1\) and \(b_2\) are two arbitrary functions.

Using (75) we find for \(R\)

\[
R' - B\dot{R} = 0, \quad \Rightarrow \quad R = \Phi [a_1(r)t + a_2(r)],
\]

(87)

where \(\Phi\) is an arbitrary function of its argument and

\[
a_1(r) = e^{\int b_1(r)dr}, \quad a_2(r) = \int b_2(r)e^{\int b_1(r)dr} dr.
\]

(88)

For specifying further the model let us choose \(b_1(r)\) and \(b_2(r)\) as

\[
b_1(r) = \frac{\beta_1}{r + \beta_2}, \quad b_2(r) = (r + \beta_2)^\alpha,
\]

(89)

\[
B = \frac{\beta_1 t}{r + \beta_2} + (r + \beta_2)^\alpha,
\]

(90)

\[
R = (a_1 t + a_2)^\rho.
\]

(91)
where \( \beta_1, \beta_2, \alpha \) and \( n \) are arbitrary constants, chosen positive to ensure the positivity of \( B \) and \( R \).

Then the physical variables and the complexity factor for this model read

\[
8\pi\mu = -\frac{1}{(a_1t + a_2)^2n} + \frac{2(n-1)na_1^2}{(a_1t + a_2)^2}, \quad (92)
\]

\[
4\pi q = \frac{(n-1)na_1^2}{(a_1t + a_2)^2}, \quad (93)
\]

\[
8\pi P_r = -\frac{1}{(a_1t + a_2)^2n} - \frac{2(n-1)na_2^2}{(a_1t + a_2)^2}, \quad (94)
\]

\[
m = \frac{(a_1t + a_2)^n}{2}, \quad (95)
\]

\[
\Theta = \frac{b_1}{b_1t + b_2} + \frac{na_1}{a_1t + a_2}, \quad (96)
\]

\[
\sigma = \frac{b_1}{b_1t + b_2} - \frac{na_1}{a_1t + a_2}, \quad (97)
\]

\[
Y_{TF} = \frac{(n-1)na_1^2}{(a_1t + a_2)^2}. \quad (98)
\]

It is worth noticing that \( Y_{TF} \) has exactly the same expression as \( q \), as given by (93). Therefore any solution of this family satisfying the vanishing complexity factor is necessarily non–dissipative. On the other hand \( Y_{TF} \) is zero if \( n = 1 \). Thus the solution of this family with vanishing complexity factor is characterized by \( n = 1 \), which using (92) and (93) produces

\[
P_r = -\mu. \quad (99)
\]

Finally, let us mention that the the expansion scalar is always positive.

We shall next obtain some models by imposing a specific condition on \( B \) (\( B = 1 \)). The reason to assume such a condition comes from the fact that, as shown in [40], it is particularly suitable for describing situations where a vacuum cavity surrounds the central region. In this case the field equations read

\[
8\pi\mu = -\frac{1}{R^2} - \frac{2R''}{R} - \left( \frac{R'}{R} \right)^2 + \frac{\dot{R}^2}{R^2}, \quad (100)
\]

\[
4\pi q = \frac{\dot{R}}{R}, \quad (101)
\]

\[
8\pi P_r = \frac{1}{R^2} + \left( \frac{R'}{R} \right)^2 - \left( \frac{\dot{R}}{R} \right)^2 + \frac{2\dot{R}}{R^2}, \quad (102)
\]

\[
8\pi P_\perp = \frac{R''}{R} - \frac{\dot{R}}{R}, \quad (103)
\]

Let us first consider the non–dissipative case \( (q = 0) \). Then it follows at once from (101) that

\[
R = R_1(t) + R_2(r), \quad (105)
\]

where \( R_1 \) and \( R_2 \) are arbitrary functions of their arguments.

In order to exhibit an exact solution let us further assume \( P_\perp = 0 \). Using this condition in (101) produces

\[
R_1(t) = a_1t^2 + b_1t + c_1, \quad R_2(r) = a_1r^2 + b_2r + c_2, \quad (106)
\]

where \( a_1, b_1, c_1, b_2, c_2 \) are positive arbitrary constants.

The physical and kinematical variables for this model are

\[
8\pi\mu = \frac{1}{\alpha^2} \left( -1 - 4a_1\alpha - \beta^2 + \gamma^2 \right), \quad (107)
\]

\[
8\pi P_r = \frac{1}{\alpha^2} \left( 1 - 4a_1\alpha + \beta^2 - \gamma^2 \right), \quad (108)
\]

\[
P_r + \mu = -\frac{\alpha_1}{\pi\alpha}, \quad (109)
\]

\[
\Theta = \frac{2\gamma}{\alpha}, \quad (110)
\]

\[
\sigma = -\frac{\gamma}{\alpha}, \quad (111)
\]

\[
m = \frac{\alpha}{2} \left( \beta^2 - \gamma^2 + 1 \right), \quad (112)
\]

where

\[
\alpha \equiv a_1(t^2 + r^2) + b_1t + b_2r + c_1 + c_2; \quad \beta \equiv 2a_1r + b_2; \quad \gamma \equiv 2a_1t + b_1. \quad (113)
\]

For this model the expression for \( Y_{TF} \) reads

\[
Y_{TF} = \frac{2\alpha_1}{\alpha}. \quad (114)
\]

Therefore the vanishing complexity factor condition implies \( \alpha_1 = 0 \), producing because of (107) and (108)

\[
P_r = -\mu. \quad (115)
\]

Thus the solution of this family with the vanishing complexity factor condition is also characterized by the stiff equation of state. Besides, the expansion scalar is always positive.

Finally, let us now consider the dissipative case \( (q \neq 0) \). If we impose the condition \( P_\perp = 0 \), then we get the equation \( R = R'' \), whose general solution is of the form

\[
R(t, r) = c_1\Psi(t + r) + c_2\Phi(t - r), \quad (116)
\]
where $c_1, c_2$ are arbitrary constants, and $\Psi, \Phi$ arbitrary functions.

As an example let us choose

$$R = a_1(t - r)^n,$$

(117)

where $a_1, n$ are positive constants, and the solution applies in the region $t > r$.

The ensuing physical and kinematical variables are in this case:

$$8\pi\nu = -\frac{1}{a_1^2(t - r)^{2n}} \frac{2n(n - 1)}{(t - r)^2},$$

(118)

$$4\pi q = -\frac{n(n - 1)}{(t - r)^2};$$

(119)

$$8\pi P_r = \frac{1}{a_1^2(t - r)^{2n}} - \frac{2n(n - 1)}{(t - r)^2},$$

(120)

$$\Theta = \frac{2n}{t - r}, \quad \sigma = -\frac{n}{t - r},$$

(121)

$$m = \frac{a_1(t - r)^n}{2},$$

(122)

$$T(t, r) = \frac{n(n - 1)}{4\pi\kappa(t - r)} - \frac{n(n - 1)}{4\pi\kappa(t - r)^2} + T_0(t),$$

(123)

where the temperature has been calculated using the truncated transport equation (33).

The corresponding expression of the complexity factor for this case reads

$$Y_{TF} = \frac{n(n - 1)}{(t - r)^2}.$$

(124)

Thus, the vanishing complexity factor conditions requires $n = 1$, implying because of (119) that the fluid is non–dissipative, and because of (118) and (120) that the fluid satisfies the stiff equation of state $P_r = -\mu$.

As in the previous model, the expansion scalar is always positive.

**VII. CONCLUSIONS**

The general approach hereby described to analyze the dynamics of hyperbolically symmetric fluids, including dissipative processes, brings out some remarkable features of hyperbolically symmetric fluids, namely:

1. The energy density is necessarily negative.

2. The Tolman condition for thermodynamic equilibrium implies in this case the presence of a positive temperature gradient, unlike the negative temperature gradient required in the spherically symmetric case.

3. The thermal modification of the inertial mass density reported for the spherically symmetric case in [31], produces an effect that is similar to the one obtained in the spherically symmetric case (to enhance the tendency to expansion) but comes about through the increasing of the absolute value of the effective inertial mass density.

4. The fluid cannot fill the central region.

The violation of the weak energy condition ($\mu < 0$) should not scare us. Indeed, while it is true that at classical level we do not expect negative energy density in a realistic fluid, the situation is quite different in presence of quantum effects. In fact there is abundant theoretical evidence illustrating the appearance of negative energy density in some astrophysical and cosmological scenarios (see [41–45] and references therein). Thus the type of fluids considered in this manuscript might be useful for studying systems under extreme conditions where quantum effects are expected to play a relevant role.

This negative energy density implies the appearance of a repulsive gravitational force which has two important thermodynamic consequences mentioned in points 2 and 3 above.

Finally, the fact that the fluid distribution cannot fill the central region is consistent with the result obtained in [17], indicating that test particles are not allowed to reach the center for the line element (8). Here we have assumed that the central region is surrounded by an empty vacuole, however it could be assumed as well that the central region is filled with a fluid endowed with a different type of symmetry.

Once the set of equations describing the dynamics of hyperbolically symmetric fluids have been obtained, we have presented a selection of exact solutions. These were found under two types of conditions. A first class of solutions emerge from the condition of the vanishing complexity factor defined in (32) ($Y_{TF} = 0$) and the quasi–homologous evolution (39) as defined in (33). A second class of solutions was found by looking for models characterized by non–vanishing shear, inhomogeneous energy density and vanishing four–acceleration (geodesics). This class of solutions are entitled to be considered as hyperbolically symmetric versions of LTB space–times.

For the first class of models two exact solutions were found in the non–dissipative case. One of them, (158–160), describes a fluid distribution satisfying conditions $\mathcal{E} = \sigma = 0 = \Pi = \mu' = 0$, which is a reminiscence of the usual FRW space–time. However, unlike the latter it is not geodesic. The second one appears from the geodesic condition, then the quasi–homologous evolution becomes homologous, and the solution is described by (59–63). 

This is a geodesic fluid, satisfying also the conditions $\mathcal{E} = \sigma = 0$, and therefore is also a good candidate to be regarded as a hyperbolical version of the FRW spacetime, however unlike the latter, it is anisotropic in the pressure and inhomogeneous in the energy–density.

Dissipative models belonging to this first class of solutions may be found in [18].

Among the second class of solutions we have presented first, solutions corresponding to non–dissipative dust configurations. Comparing with the spherically symmetric case we observe that only one family of solutions $(k(r) > 0)$ exists, instead of the three families existing in this latter case $(k(r) \lesssim 0)$. 

These solutions cannot satisfy the vanishing complexity factor, neither can they evolve in the quasi–homologous regime. Also, the scalar expansion is positive as expected for pure dust submitted to a repulsive gravity.

Next we have analyzed the case of dissipative anisotropic fluids. For this doing we generalized the expression (65) by assuming (68). Different specific models were found from two different conditions. One family of solutions was obtained from a condition imposed on the complexity factor (82). In this case the pressure must be anisotropic. A specific solution was found assuming $P_\perp = 0$. The subclass of this type of solution satisfying the vanishing complexity factor is necessarily non–dissipative, and satisfies the stiff equation of state $P_\tau = -\mu$.

The other family of solutions was found under the condition $B = 1$. For the non–dissipative case a family of solutions was found under the additional condition $P_\perp = 0$. In this case too, the vanishing complexity factor condition implies the stiff equation of state $P_\tau = -\mu$. In the dissipative case, assuming (117) we may identify two types of contributions in the expression of the temperature. On the one hand contributions in the stationary dissipative regime (non containing $\tau$) and on the other hand, contributions from the transient regime (terms proportional to $\tau$). If we assume further the vanishing complexity factor condition then the fluid becomes non–dissipative and satisfies the stiff equation of state $P_\tau = -\mu$.

The expansion scalar for all the above models are always positive.

Finally we would like to conclude with some general remarks:

- All the models exhibited above, although not directly related to specific scenarios, suggest their potential applications in cosmology and astrophysics. We have in mind the construction of more sophisticated models of the Universe beyond the standard FRW (see for example [40] [47]). More specifically, the repulsive character of gravitation exhibited in hyperbolically symmetric fluids, suggests its possible use in the modeling of very early stages of the Universe.
- The particular behavior of test particles within the horizon, assuming the line element (3), suggests a possible model of relativistic jets (see [17] for a discussion on this issue).
- It is worth mentioning that, unlike the spherically symmetric case, there is not a Birkhoff theorem for hyperbolical symmetry, and therefore neither is the metric uniquely unique, nor is the line element the most general to describe hyperbolically symmetric fluids. It would be very interesting to delve deeper into this issue, considering other hyperbolically symmetric space–times.

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Appendix A: Conservation laws

The two independent components of the conservation laws $T^\mu_\nu = 0$, read

$$\dot{\mu} + (\mu + P_r) \frac{\dot{B}}{B} + 2(\mu + P_\perp) \frac{\dot{R}}{R} + \dot{q} \frac{\dot{A}}{A} + 2q \frac{A'}{A} \left( \frac{A'}{A} + \frac{\dot{R}}{R} \right) = 0,$$

and

$$P_r' + (\mu + P_r) \frac{A'}{A} + 2(P_r - P_\perp) \frac{\dot{R}}{R} + \dot{q} \frac{B}{A} + 2q \frac{B}{A} \left( \frac{\dot{B}}{B} + \frac{\dot{R}}{R} \right) = 0.$$

This last equation may be rewritten as

$$(\mu + P_r) D_T U = - (\mu + P_r) (4\pi P_r R^3 - m) \frac{1}{R^2}$$

$$- E^2 \left[ D_R P_r + \frac{2}{R} (P_r - P_\perp) \right]$$

$$- E \left[ D_T q + \frac{2}{3} q (2\Theta + \sigma) \right].$$

Combining the above equation with the transport equation (31) one obtains

$$\left( \mu + P_r - \frac{\kappa T}{\tau} \right) D_T U =$$

$$- (\mu + P_r - \frac{\kappa T}{\tau}) (4\pi R^3 P_r - m) \frac{1}{R^2}$$

$$- E^2 \left[ D_R P_r + \frac{2}{R} (P_r - P_\perp) - \frac{\kappa}{\tau} D_R T \right]$$

$$+ E q \left[ \frac{1}{\tau} \left( 1 + \frac{2}{3} D_T \ln \left( \frac{\tau}{\kappa T^2} \right) - \frac{5}{6} \Theta - \frac{2}{3} \sigma \right) \right].$$

(A4)
