EXTENDED APPLICABILITY OF THE SYMPLECTIC
PONTRYAGIN METHOD

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Abstract. The Symplectic Pontryagin method was introduced in a previous paper. This work shows that this method is applicable under less restrictive assumptions. Existence of solutions to the Symplectic Pontryagin scheme are shown to exist without the previous assumption on a bounded gradient of the discrete dual variable. The convergence proof uses the representation of solutions to a Hamilton-Jacobi-Bellman equation as the value function of an associated variation problem.

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1. INTRODUCTION

The Symplectic Pontryagin method was introduced in [7], as a numerical method designed for approximation of the value function associated with optimal control problems. Under general conditions this value function, $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}$, is a viscosity solution of an associated Hamilton-Jacobi equation,

$$u_t + H(x,u_x) = 0, \quad \text{in } \mathbb{R}^d \times (0,T),$$

$$u(x,T) = g(x), \quad (1.1)$$

where $u_t$ denotes the derivative with respect to the “time” variable $t$, and $u_x$ the gradient with respect to the state variable $x$. The function $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is denoted the Hamiltonian. This function is concave in its second argument when the Hamilton-Jacobi equation originates in optimal control. We shall therefore only deal with such concave Hamiltonians in this paper. The main ingredient in the Symplectic Pontryagin method is the
observation that, under favorable conditions, optimal paths to the control problem solve a Hamiltonian system associated with the Hamilton-Jacobi equation (1.1). More accurately, if 

\[ x(s, T) \rightarrow \mathbb{R}^d \]

is an optimal path for the optimal control problem with initial position \((y, s)\), then there exists a dual function \(\lambda(s, T) \rightarrow \mathbb{R}^d\), such that \(x\) and \(\lambda\) solve

\[ x'(t) = H_\lambda(x(t), \lambda(t)), \quad \text{for } s < t < T, \]

\[ x(s) = y, \]

\[ -\lambda'(t) = H(x(t), \lambda(t)), \quad \text{for } s < t < T, \]

\[ \lambda(T) = g'(x(T)). \]

This is the Pontryagin principle, a necessary condition for optimality, in the case where the Hamiltonian function, \(H\), is differentiable. The Hamiltonian is, however, often nondifferentiable, even in situations where the control and cost functions in the control problem are differentiable, see [7]. Even though the Pontryagin principle may be formulated for other control problems than those having differentiable Hamiltonians, the version in (1.2) is appealing as a starting point for numerical methods.

In the Symplectic Pontryagin method, the Hamiltonian system (1.2) is used with a regularized Hamiltonian, \(H_\delta\), satisfying \(|H_\delta(x, \lambda) - H(x, \lambda)| \leq \delta\), for all \((x, \lambda) \in \mathbb{R}^{2d}\). The Hamiltonian system (1.2) makes sense when \(H\) has been replaced with the differentiable \(H_\delta\). The comparison principle for Hamilton-Jacobi equations then grants that the maximum difference between the solution to the original Hamilton-Jacobi equation (1.1) and the one where \(H_\delta\) has taken the place of \(H\) is of the order \(\delta\).

The second ingredient in the Symplectic Pontryagin method is the application of the Symplectic Euler numerical scheme for the Hamiltonian system (1.2) with the regularized Hamiltonian \(H_\delta\). We shall for simplicity assume that an approximation of \(u(x_s, 0)\) is to be computed. The method to approximate \(u\) for a starting position whose time coordinate differs from zero, is similar. The time interval \([0, T]\) is split into \(N\) intervals of length \(\Delta t = T/N\). We introduce the notation \(t_n = n\Delta t\), for \(n = 0, \ldots, N\). The Symplectic Pontryagin scheme is:

\[ x_{n+1} = x_n + \Delta t H_\lambda^\delta(x_n, \lambda_{n+1}), \quad n = 0, \ldots, N - 1 \]

\[ x_0 = x_s, \]

\[ \lambda_n = \lambda_{n+1} + \Delta t H_\lambda^\delta(x_n, \lambda_{n+1}), \quad n = 0, \ldots, N - 1 \]

\[ \lambda_N = g'(x_N). \]

In [7] this method is analyzed by extending the solutions \(\{x_n\}\) to piecewise linear functions. These functions are used to define an approximate value function which is shown to solve a Hamilton-Jacobi equation equal to the initial equation, but with an additional error term. The above mentioned comparison principle gives the difference between the approximate and the exact value functions.
In this paper the Symplectic Pontryagin method is analyzed in a different way. The first step is here to consider the Hamilton-Jacobi equations with the original Hamiltonian \(H\), and the regularized variant \(H^\delta\). As mentioned above, this gives a difference of the order \(\delta\) between the corresponding solutions. Next, a representation formula of the solutions to the Hamilton-Jacobi equations as minima of a variation problem is used. It is shown that there exists one solution to the Symplectic Pontryagin method which is also a minimizer to an Euler discretized version of the variation problem. The minimizer to the discretized variation problem is then shown to give a value which is close to the value of the original variation problem.

This work extends the result in [7] in the following ways:

- A solution to the Symplectic Pontryagin method is shown to exist for the considered class of Hamiltonians.
- The result in [7] relies on the assumption that the variation \(\partial \lambda_{n+1}/\partial x_n\) is bounded everywhere but on a codimension one hypersurface. This assumption is not needed with the present approach.
- The error bound in the present paper is shown to be of the order \(\delta + \Delta t\), compared with the previous \(\delta + \Delta t^2/\delta\). The new result therefore ensures the possibility of decreasing \(\delta\) independently of \(\Delta t\), without deteriorating the error estimate.

The paper is organized as follows. In section 2 we present the main results of this paper, existence of solutions to, and convergence of, the Symplectic Pontryagin method. In sections 3 and 4 the convergence proof is given in the form of a series of lemmas.

2. Main Results

Given the Hamiltonian function \(H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}\) we define the running cost function

\[
L(x, \alpha) = \sup_{\lambda \in \mathbb{R}^d} \left\{ -\alpha \cdot \lambda + H(x, \lambda) \right\},
\]

for all \(x\) and \(\alpha\) in \(\mathbb{R}^d\). This function is convex in its second argument, and extended valued, i.e. its values belong to \(\mathbb{R} \cup \{+\infty\}\). If the Hamiltonian is real-valued and concave in its second variable it is possible to get it back when having possession of \(L\):

\[
H(x, \lambda) = \inf_{\alpha \in \mathbb{R}^d} \left\{ \lambda \cdot \alpha + L(x, \alpha) \right\}.
\]

This is a consequence of the bijectivity of the Legendre-Fenchel transform, see [6].

For Hamilton-Jacobi equations with initial data given (Cauchy problems, i.e. not as here with data given at time \(T\)) and convex Hamiltonians, the Hamiltonian and the running cost are connected via the usual Legendre-Fenchel transform. The results presented here could have been presented for Hamilton-Jacobi equations in this form, but since it is more common for
optimal control problems to include cost functions of the terminal positions
than the initial positions, the current setting is chosen.

The cornerstone in the convergence analysis for the Symplectic Pontryagin
method will be the following representation theorem, taken from [4], see
also [2, 5, 6, 8]. The result is given in greater generality in that paper, but we
present a form suited for our purposes. We will assume that the Hamiltonian
satisfies the following estimates:

\[
\begin{align*}
|H(x, \lambda_1) - H(x, \lambda_2)| & \leq C_1|\lambda_1 - \lambda_2|, \quad \text{for all } x, \lambda_1, \lambda_2 \in \mathbb{R}^d, \\
|H(x_1, \lambda) - H(x_2, \lambda)| & \leq C_2|x_1 - x_2|(1 + |\lambda|), \quad \text{for all } x_1, x_2, \lambda \in \mathbb{R}^d.
\end{align*}
\]

(2.3)

Then the following representation result holds:

**Theorem 2.1.** Let the Hamiltonian \( H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be concave in its
second argument, and satisfy the bounds in (2.3). Let the running cost \( L \)
be defined by (2.1), and let \( g : \mathbb{R}^d \to \mathbb{R} \) be a continuous function such that
\( g(x) \geq -k(1 + |x|) \) for all \( x \in \mathbb{R}^d \) for some \( k > 0 \). Then the value function
\( V(y, s) = \inf \left( \int_s^T L(x(t), x'(t)) dt + g(x(T)) \right) \)

\( x : [s, T] \to \mathbb{R}^d \) absolutely continuous, \( x(s) = y \)  (2.4)

is a continuous viscosity solution to (1.1) which satisfies

\( u(x, t) \geq -k(1 + |x|) \) for all \( (x, t) \in \mathbb{R}^d \times [0, T] \).  (2.5)

Furthermore, if a function is a continuous viscosity solution to (1.1) which
satisfies (2.5), then this function must be the value function \( V \).

We also have the following result, stating that optimal solutions to the
value function \( V \) solve a Hamiltonian system. For a proof, see [3].

**Theorem 2.2.** Let the conditions in Theorem 2.1 be satisfied, and let \( (y, s) \)
be any point in \( \mathbb{R}^d \times [0, T] \). Then a minimizer \( x : [s, T] \to \mathbb{R}^d \) for
\( V(y, s) \) in (2.4) exists. If furthermore the Hamiltonian \( H \) and the terminal cost \( g \)
are continuously differentiable, there exists a dual function \( \lambda : [s, T] \to \mathbb{R}^d \), such
that \( (x, \lambda) \) solve the Hamiltonian system (1.2)

In view of Theorem 2.1 a possible approximate value of \( u(x_s, t_m) \) is the
minimizer of

\[
J_{(x_s, t_m)}(\alpha_m, \ldots, \alpha_{N-1}) = \Delta t \sum_{n=m}^{N-1} L(x_n, \alpha_n) + g(x_N)
\]

(2.6)

where \( \alpha_n \in \mathbb{R}^d \) and

\[
\begin{align*}
x_{n+1} &= x_n + \Delta t \alpha_n, \quad \text{for } m \leq n \leq N - 1, \\
x_m &= x_s.
\end{align*}
\]

(2.7)

We define a discrete value function as the optimal value of \( J \):

\[
\bar{u}(x_s, t_m) = \inf \left\{ J_{(x_s, t_m)}(\alpha_m, \ldots, \alpha_{N-1}) \left| \alpha_m, \ldots, \alpha_{N-1} \in \mathbb{R}^d \right. \right\}
\]
For the proof of Theorem 2.5, the existence theorem, we need two definitions.

**Definition 2.3.** A function \( f : \mathbb{R}^d \to \mathbb{R} \) is locally semiconcave if for every compact convex set \( V \subset \mathbb{R}^d \) there exists a constant \( K > 0 \) such that \( f(x) - K|x|^2 \) is concave on \( V \).

There exist alternative definitions of semiconcavity, see [2], but this is the one used in this paper.

**Definition 2.4.** An element \( p \in \mathbb{R}^d \) belongs to the superdifferential of the function \( f : \mathbb{R}^d \to \mathbb{R} \) at \( x \), denoted \( D^+ f(x) \), if

\[
\limsup_{y \to x} \frac{f(y) - f(x) - p \cdot (y - x)}{|y - x|} \leq 0.
\]

**Theorem 2.5.** Let \( x_0 \) be any element in \( \mathbb{R}^d \), and \( g : \mathbb{R}^d \to \mathbb{R} \) a continuously differentiable function such that \( |g'(x)| \leq L_g \) for all \( x \in \mathbb{R}^d \), and some constant \( L_g > 0 \). Let the Hamiltonian \( H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be two times continuously differentiable and concave in its second argument, and satisfy the bounds in (2.3). Let the running cost \( L \) be defined by (2.1). Then there exists a minimizer \( (\alpha_m, \ldots, \alpha_{N-1}) \) of the function \( J(x_m, t_m) \) in (2.6). Let \( (x_m, \ldots, x_N) \) be a corresponding solution to (2.7). Then there exists a discrete dual variable \( \lambda_n \), \( n = m, \ldots, N \) satisfying

\[
x_{n+1} = x_n + \Delta H(\alpha_{n+1}, x_n), \quad \text{for all } m \leq n \leq N - 1,
\]

\[
x_m = x_s
\]

\[
\lambda_n = \lambda_{n+1} + \Delta H_x(\alpha_{n+1}, x_n), \quad \text{for all } m \leq n \leq N - 1,
\]

\[
\lambda_N = g'(x_N).
\]

Hence \( \alpha_n = H(\lambda_{n+1}, x_n) \) for all \( m \leq n \leq N - 1 \).

**Proof.**

**Step 1.** By the properties of the Hamiltonian it follows that the running cost \( L \) is lower semicontinuous, see [6]. It is clear that the infimum of \( J(x_m, t_m) \) is less than infinity. Let \( \varepsilon > 0 \) and \( \alpha_n^\varepsilon, n = m, \ldots, N - 1 \) be discrete controls such that

\[
J(x_m, t_m)(\alpha_m^\varepsilon, \ldots, \alpha_{N-1}^\varepsilon) \leq \inf \{ J(x_m, t_m)(\alpha_m, \ldots, \alpha_{N-1}) \mid \alpha_n \in \mathbb{R}^d, \ n = m, \ldots, N - 1 \} + \varepsilon.
\]

Since the Hamiltonian \( H \) satisfies the bounds (2.3), it follows that all \( \alpha_n^\varepsilon \) are bounded by \( C_1 \). Hence there exist \( \{\varepsilon_i\} \) and corresponding \( \{\alpha_n^{\varepsilon_i}\} \) such that \( \varepsilon_i \to 0 \), and \( \alpha_n^{\varepsilon_i} \to \alpha_n \), where \( |\alpha_n| \leq C_1 \), for all \( n \). The lower semicontinuity of \( L \) implies that this \( \{\alpha_n\}_{n=m}^{N-1} \) is a minimizer for \( J(x_m, 0) \).

**Step 2.** Assume that \( \tilde{u}(\cdot, t_{n+1}) \) is locally semiconcave, and that \( \lambda_{n+1} \in D^+ \tilde{u}(x_{n+1}, t_{n+1}) \). We will show that this implies

\[
\lambda_{n+1} \cdot \alpha_n + L(x_n, \alpha_n) = H(\lambda_{n+1}, x_n).
\]
By the semiconcavity we have that there exists a constant $A > 0$, such that
\[ \bar{u}(x_n + \Delta t \alpha, t_{n+1}) \leq \bar{u}(x_{n+1}, t_{n+1}) + \Delta t \lambda_{n+1} \cdot (\alpha - \alpha_n) + A|\alpha - \alpha_n|^2. \] (2.10)
for all $\alpha$ in a neighborhood around $\alpha_n$. Since we know that the function
\[ \alpha \mapsto \bar{u}(x_n + \Delta t \alpha, t_{n+1}) + \Delta t L(x_n, \alpha) \]
is minimized for $\alpha = \alpha_n$, the semiconcavity of $\bar{u}$ in (2.10) implies that the function
\[ \alpha \mapsto \lambda_{n+1} \cdot \alpha + A|\alpha - \alpha_n|^2 + L(x_n, \alpha) \] (2.11)
is also minimized for $\alpha = \alpha_n$. We will prove that the function
\[ \alpha \mapsto \lambda_{n+1} \cdot \alpha + L(x_n, \alpha) \] (2.12)
is minimized for $\alpha = \alpha_n$. Let us assume that this is false, so that there exists an $\alpha^* \in \mathbb{R}^d$, and an $\varepsilon > 0$ such that
\[ \lambda_{n+1} \cdot \alpha_n + L(x_n, \alpha_n) - \lambda_{n+1} \cdot \alpha^* - L(x_n, \alpha^*) \geq \varepsilon. \] (2.13)
Let $\xi \in [0, 1]$, and $\hat{\alpha} = \xi \alpha^* + (1 - \xi)\alpha_n$. Insert $\hat{\alpha}$ into the function in (2.11):
\[
\begin{align*}
\lambda_{n+1} \cdot \hat{\alpha} + A|\hat{\alpha} - \alpha_n|^2 + L(x_n, \hat{\alpha}) &= \xi \lambda_{n+1} \cdot \alpha^* + (1 - \xi)\lambda_{n+1} \cdot \alpha_n + A\xi^2|\alpha^* - \alpha_n|^2 + L(x_n, \xi \alpha^* + (1 - \xi)\alpha_n) \\
&\leq \xi \lambda_{n+1} \cdot \alpha^* + (1 - \xi)\lambda_{n+1} \cdot \alpha_n + A\xi^2|\alpha^* - \alpha_n|^2 + \xi L(x_n, \alpha^*) + (1 - \xi)L(x_n, \alpha_n) \\
&\leq \lambda_{n+1} \cdot \alpha_n + L(x_n, \alpha_n) + A\xi^2|\alpha^* - \alpha_n|^2 - \xi \varepsilon \\
&< \lambda_{n+1} \cdot \alpha_n + L(x_n, \alpha_n),
\end{align*}
\]
for some small positive number $\xi$. This contradicts the fact that $\alpha_n$ is a minimizer to the function in (2.11). Hence we have shown that the function in (2.12) is minimized at $\alpha_n$. By the relation between $L$ and $H$ in (2.2) our claim (2.9) follows.

**Step 3.** From the result in Step 2, equation (2.9), and the definition of the running cost $L$ in (2.1) it follows that $\alpha_n = H_\lambda(x_n, \lambda_{n+1})$, for if this equation did not hold, then $\lambda_{n+1}$ could not be the maximizer of $-\alpha_n \cdot \lambda + H(x_n, \lambda)$.

**Step 4.** In this step, we show that if $\bar{u}(\cdot, t_{n+1})$ is locally semiconcave, and $\lambda_{n+1} \in D^+\bar{u}(x_{n+1}, t_{n+1})$, then $\bar{u}(\cdot, t_n)$ is locally semiconcave, and $\lambda_n \in D^+\bar{u}(x_n, t_n)$.

By the assumed local semiconcavity of $\bar{u}(\cdot, t_{n+1})$ we have that there exists a $K > 0$ such that
\[ \bar{u}(x, t_{n+1}) \leq \lambda_{n+1} \cdot (x - x_{n+1}) + K|x - x_{n+1}|^2, \]
for all $x$ in some neighborhood around $x_{n+1}$. Let us now consider the control $H_\lambda(x, \lambda_{n+1})$ at the point $(x, t_n)$. Since this control is not necessarily optimal
except at \((x_n, t_n)\), we have
\[
\bar{u}(x, t_n) \leq \bar{u}(x + \Delta H \lambda(x, \lambda_{n+1}), t_{n+1}) + \Delta t L(x, H \lambda(x, \lambda_{n+1})) \\
\leq \bar{u}(x_{n+1}, t_{n+1}) + \lambda_{n+1} \cdot (x + \Delta H \lambda(x, \lambda_{n+1}) - x_{n+1}) + K|x + \Delta H \lambda(x, \lambda_{n+1}) - x_{n+1}|^2 + \Delta t L(x, H \lambda(x, \lambda_{n+1})),
\] (2.14)
for all \(x\) in some neighborhood around \(x_n\). By the definition of \(L\) in (2.1) it follows that
\[
L(x, H \lambda(x, \lambda_{n+1})) = -H \lambda(x, \lambda_{n+1}) \cdot \lambda_{n+1} + H(x, \lambda_{n+1}).
\]
With this fact in (2.14) we have
\[
\bar{u}(x, t_n) \leq \bar{u}(x_{n+1}, t_{n+1}) + \lambda_{n+1} \cdot (x - x_{n+1}) + K|x + \Delta H \lambda(x, \lambda_{n+1}) - x_{n+1}|^2 + \Delta t H(x, \lambda_{n+1}).
\] (2.15)
By the result in step 3 we have that
\[
x_{n+1} = x_n + \Delta t H \lambda(x, \lambda_{n+1}),
\]
so that, by the fact that \(H\) is twice continuously differentiable
\[
|x + \Delta H \lambda(x, \lambda_{n+1}) - x_{n+1}| = |x - x_n + \Delta t(H \lambda(x, \lambda_{n+1}) - H \lambda(x_n, \lambda_{n+1}))| \leq K|x - x_n|,
\] (2.16)
for some (new) \(K\) and \(x\) in a neighborhood of \(x_n\). We also need the facts that
\[
\bar{u}(x_n, t_n) = \bar{u}(x_{n+1}, t_{n+1}) + \Delta t L(x_n, \alpha_n)
\] (2.17)
and
\[
L(x_n, \alpha_n) = -\lambda_{n+1} \cdot \alpha_n + H(x_n, \lambda_{n+1}) = -\lambda_{n+1} \cdot \frac{x_{n+1} - x_n}{\Delta t} + H(x_n, \lambda_{n+1}).
\] (2.18)
We insert the results (2.16), (2.17), and (2.18) into (2.15). This implies that there are constants \(K > 0\) and \(K' > 0\), such that
\[
\bar{u}(x, t_n) \leq \bar{u}(x_n, t_n) + \lambda_{n+1} \cdot (x - x_n) + \Delta t(H(x, \lambda_{n+1}) - H(x_n, \lambda_{n+1})) + K|x - x_n|^2 \\
\leq \bar{u}(x_n, t_n) + (\lambda_{n+1} + \Delta t H \lambda(x_n, \lambda_{n+1})) \cdot (x - x_n) + K|x - x_n|^2 \\
= \bar{u}(x_n, t_n) + \lambda_n \cdot (x - x_n) + K|x - x_n|^2,
\]
for all \(x\) in a neighborhood around \(x_n\). We here used the differentiability of \(H\) in the last inequality, and the \(\lambda_n\) evolution in (2.8) in the last equality. This shows that \(u(\cdot, t_n)\) is locally semiconcave, and that \(\lambda_n \in D^+ \bar{u}(x_n, t_n)\).

**Step 5.** Since \(u(x, T) = g(x)\) and \(g\) is differentiable, it follows that \(\bar{u}(\cdot, T)\) is semiconcave. Induction backwards in \(n\) shows that \(\bar{u}(\cdot, t_n)\) is locally semiconcave for all \(n\). Thereby the conclusions in step 2 and 3 hold for all \(n\). □
By the results from sections 3 and 4, Lemmas 3.7 and 4.2 we have the following convergence result.

**Theorem 2.6.** Let the Hamiltonian $H$ be concave in its second argument, be continuously differentiable and satisfy (2.3). Let $g : \mathbb{R}^d \to \mathbb{R}$ be differentiable and satisfy $|g'(x)| \leq C_3$, for all $x \in \mathbb{R}^d$. Let $x_s$ be any element in $\mathbb{R}^d$. Then the viscosity solution $u$ satisfying the conditions in Theorem 2.1, and the approximate value function $\bar{u}$ satisfy

\[
-\frac{C_1C_2T}{2}((e^{C_2T} - 1)\Delta t + \Delta t^2) - \frac{C_1C_3}{2}(e^{C_2T} - 1)\Delta t \\
\leq u(x_s, 0) - \bar{u}(x_s, 0) \\
\leq \frac{1}{2}C_1C_2(C_3 + 1)e^{C_2T}T\Delta t.
\]

If the Hamiltonian is not differentiable, the following result may be used.

**Theorem 2.7.** Let $H$ be a Hamiltonian which is concave in its second argument, and satisfies (2.3). Let $H^\delta$ be a Hamiltonian that satisfies the same conditions, and which is continuously differentiable, and satisfies

\[
\|H - H^\delta\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \leq \delta.
\]

Then

\[
-\frac{C_1C_2T}{2}((e^{C_2T} - 1)\Delta t + \Delta t^2) - \frac{C_1C_3}{2}(e^{C_2T} - 1)\Delta t - T\delta \\
\leq u(x_s, 0) - \bar{u}(x_s, 0) \\
\leq \frac{1}{2}C_1C_2(C_3 + 1)e^{C_2T}T\Delta t + T\delta,
\]

where $\bar{u}$ is computed using $H^\delta$.

**Proof.** The comparison principle for solutions to Hamilton-Jacobi equations gives that the two solutions to (1.1) with $H$ and $H^\delta$ differ by $T\delta$, see [7]. The result follows from Theorem 2.6. \(\square\)

3. **Lower bound of** $u(x_s, 0) - \bar{u}(x_s, 0)$

In order to be able to prove the inequality, we need the result in Theorem 3.2 for convex functions. The theorem uses the following definition.

**Definition 3.1.** Let $f$ be a convex real-valued function defined on $\mathbb{R}^d$. A vector $y \neq 0$ is called a direction of recession of $f$ if $f(x + ty)$ is a nondecreasing function of $t$, for every $x \in \mathbb{R}^d$.

It is shown in [6] that if $f(x + ty)$ is a nondecreasing function of $t$ for one $x \in \mathbb{R}^d$, then this property holds for all $x \in \mathbb{R}^d$. The following thorem is taken from [6], but in a slightly simplified form.
Theorem 3.2. Let \( \{ f_i \mid i \in I \} \), where \( I \) is an arbitrary index set, be a collection of real-valued convex functions on \( \mathbb{R}^d \) which have no common direction of recession. Then one and only one of the following alternatives holds:

a) There exists a vector \( x \in \mathbb{R}^d \) such that
\[
 f_i(x) \leq 0, \quad \text{for all } i \in I.
\]

b) There exist non-negative real numbers \( l_i \), only finitely many non-zero, such that, for some \( \varepsilon > 0 \), one has
\[
 \sum_{i \in I} l_i f_i(x) \geq \varepsilon, \quad \text{for all } x \in \mathbb{R}^d.
\]

Lemma 3.3. Assume that the Hamiltonian \( H \) satisfies equation (2.3) and is concave in its second argument. Let \( x_1, x_2, \) and \( \alpha_1 \) be any elements in \( \mathbb{R}^d \), and \( \beta \) any positive number. Then there exists an element \( \alpha_2 \) in \( \mathbb{R}^d \) such that
\[
 |\alpha_2 - \alpha_1| \leq C_2 |x_2 - x_1| + \beta,
\]
and
\[
 L(x_2, \alpha_2) \leq L(x_1, \alpha_1) + C_2 |x_2 - x_1|.
\]

Proof. If \( L(x_1, \alpha_1) = \infty \), we can let \( \alpha_2 = \alpha_1 \). We henceforth assume that \( L(x_1, \alpha_1) < \infty \). From the definition of \( L \) in (2.1) it follows that \( H(x_1, 0) \leq L(x_1, \alpha_1) \). By (2.3) we have
\[
 H(x_2, 0) \leq L(x_1, \alpha_1) + C_2 |x_2 - x_1|.
\]

We start by assuming that a strict inequality holds in (3.4), and then use the result for this case to prove the result in the general situation where we also allow equality in the equation.

When we assume that \( H(x_2, 0) > L(x_1, \alpha_1) + C_2 |x_2 - x_1| \), it is possible to use Theorem 3.2 with the functions
\[
 \lambda \mapsto -H(x_2, \lambda) + \alpha \cdot \lambda + L(x_1, \alpha_1) + C_2 |x_2 - x_1|.
\]

We let \( \alpha \) be any element in the ball of radius \( C_2 |x_2 - x_1| + \beta \) centered at \( \alpha_1 \), the set we will use as the index set \( I \) in Theorem 3.2.

We now show that the functions in (3.5) have no common direction of recession. Let \( \lambda = av \), where \( v \) is a unit vector in \( \mathbb{R}^d \), and let \( \alpha = \alpha_1 + (C_2 |x_2 - x_1| + \beta) v \). The corresponding function from (3.5) then satisfies
\[
 -H(x_2, av) + a \alpha_1 \cdot v + (C_2 |x_2 - x_1| + \beta) a + L(x_1, \alpha_1) + C_2 |x_2 - x_1|
\]
\[
 = -H(x_1, av) + a \alpha_1 \cdot v + (C_2 |x_2 - x_1| + \beta) a + H(x_1, av) - H(x_2, av)
\]
\[
 + L(x_1, \alpha_1) + C_2 |x_2 - x_1| \geq \beta a \to \infty \quad \text{when } a \to \infty,
\]

since \( L(x_1, \alpha_1) \geq -a \alpha_1 \cdot v + H(x_1, av) \) by the definition of \( L \) in (2.3), and
\[
 H(x_1, a \cdot v) - H(x_2, a \cdot v) \geq -C_2 |x_2 - x_1|(1 + a) \text{ by equation } (2.3).
We can therefore apply Theorem 3.2 on the set of functions in (3.5) with the \( \alpha \)'s taken from the ball of radius \( C_2|x_2-x_1|+\beta \), centered at \( \alpha_1 \). Since all functions in (3.5) are positive at \( \lambda = 0 \), and at least one function is positive at every other point, by (3.6), alternative a) in Theorem 3.2 can not hold. Therefore the set of functions satisfy alternative b). We may assume that the numbers \( l_i \) in (3.1) satisfy
\[
\sum_{i \in I} l_i = 1.
\]
This corresponds to a multiplication of the inequality (3.1) by a positive number. This changes the number \( \varepsilon \), but that is not important here. Hence we have that there exist a finite number of vectors \( \alpha^i \), with
\[
|\alpha^i - \alpha_1| \leq C_2|x_2-x_1| + \beta,
\]
such that
\[
\sum_{i} l_i \left( -H(x_2, \lambda) + \alpha^i \cdot \lambda + L(x_1, \alpha_1) + C_2|x_2-x_1| \right) = -H(x_2, \lambda) + \left( \sum_{i} l_i \alpha^i \right) \cdot \lambda + L(x_1, \alpha_1) + C_2|x_2-x_1| \geq \varepsilon.
\]
Since every vector \( \alpha^i \) satisfy (3.7), so does the convex combination
\[
\sum_{i} l_i \alpha^i.
\]
This convex combination can be taken as the \( \alpha_2 \) in the lemma.

Let us now consider the remaining case where \( H(x_2, 0) = L(x_1, \alpha_1) + C_2|x_2-x_1| \). We now modify the functions in (3.5) to be
\[
\lambda \mapsto -H(x_2, \lambda) + \alpha \cdot \lambda + L(x_1, \alpha_1) + C_2|x_2-x_1| + \gamma,
\]
where \( \gamma \) is a positive number. The same analysis as for (3.5) shows that there is a vector \( \alpha^\gamma \) whose distance from \( \alpha_1 \) is less than or equal to \( C|x_2-x_1| + \beta \), such that
\[
-H(x_2, \lambda) + \alpha^\gamma \cdot \lambda + L(x_1, \alpha_1) + C_2|x_2-x_1| + \gamma \geq 0.
\]
Since all \( \alpha^\gamma \) are contained in a compact set it is possible to find a sequence \( \{\gamma_n\}_{n=1}^\infty \) converging to zero such that \( \alpha^{\gamma_n} \) converges to an element we may call \( \alpha_2 \). It is straightforward to check that this \( \alpha_2 \) satisfies the conditions in the lemma. \( \square \)

The idea is now to use Lemma 3.3 in order to show that there on each interval \((t_n, t_{n+1})\) exists a function \( \hat{\alpha}(t) \), which satisfies
\[
|\hat{\alpha}(t) - \alpha(t)| \leq C_2|x_n-x(t)| + \beta,
\]
\[
L(x_n, \hat{\alpha}(t)) \leq L(x(t), \alpha(t)) + C_2|x_n-x(t)|,
\]
where the function \( x(t) \) is a minimizer to the value function \( V \) in (2.4). While Lemma 3.3 provides the existence of such an \( \hat{\alpha}(t) \) for each particular time \( t \), it does not provide us with a measurable function \( \hat{\alpha} \). In order to show
that such a measurable function exists, the result in Theorem 3.5 Michael’s Selection Theorem, is needed. It can be found in a more general form in e.g. [1]. Before stating it we need the following definition, which can also be found in a more general form in [1].

**Definition 3.4.** A function $F$ from the interval $(t_1, t_2)$ into the nonempty subsets of $\mathbb{R}^d$ is lower semicontinuous at $t^* \in (t_1, t_2)$ if for any $z^* \in F(t^*)$ and any neighborhood $N(z^*)$ of $z^*$, there exists a neighborhood $N(t^*)$ of $t^*$ such that for all $t \in N(t^*)$,

$$F(t) \cap N(z^*) \neq \emptyset.$$  

We say that $F$ is lower semicontinuous if it is lower semicontinuous at every $t \in (t_1, t_2)$.

**Theorem 3.5.** Let the set-valued function $F$ from the interval $(t_1, t_2)$ into the closed convex subsets of $\mathbb{R}^d$ be lower semicontinuous. Then there exists $f : (t_1, t_2) \rightarrow \mathbb{R}^d$ which is a continuous selection of $F$, i.e. which satisfies $f(t) \in F(t)$ for all $t_1 < t < t_2$.

**Lemma 3.6.** Let $\beta$ be any positive number, $x_n$ any element in $\mathbb{R}^d$, the terminal cost $g$ and the Hamiltonian $H$ be continuously differentiable, and $H$ be concave in its second argument and satisfy (2.3). Let $x : [0, T] \rightarrow \mathbb{R}^d$ be a minimizer for the value $V(x, 0)$, defined in (2.1). Let $A(t)$ for $t_n < t < t_{n+1}$ be the set of all $\alpha \in \mathbb{R}^d$, such that the following conditions are satisfied:

$$|\alpha - x'(t)| \leq C_2|x_n - x(t)| + \beta, \quad (3.9)$$

and

$$H(x_n, \lambda) \leq \alpha \cdot \lambda + L(x(t), x'(t)) + C_2|x_n - x(t)|, \quad \text{for all } \lambda \in \mathbb{R}^d. \quad (3.10)$$

The set-valued function $A(t)$ is lower semicontinuous.

**Proof.** We use the notation $\alpha(t) = x'(t)$. By Theorem 2.2 we know that $x$ solves the Hamiltonian system (1.2). Hence $\alpha$ is continuous. Let $t_n < t^* < t_{n+1}$ and $\alpha^*$ be an element in $A(t^*)$. Similarly as in the proof of Lemma 3.3 we have that

$$H(x_n, 0) \leq L(x(t^*), \alpha(t^*)) + C_2|x_n - x(t^*)|. \quad (3.11)$$

We first consider the case where there is equality in (3.11), and later the case of strict inequality.

**Case 1.** We assume here that

$$H(x_n, 0) = L(x(t^*), \alpha(t^*)) + C_2|x_n - x(t^*)|, \quad (3.12)$$

and let $t_n < t < t_{n+1}$. We know by Lemma 3.3 that $A(t)$ is nonempty. In order for equation (3.10) to be satisfied for an element in $A(t)$, we must have

$$L(x(t), \alpha(t)) + C_2|x_n - x(t)| \geq H(x_n, 0) = L(x(t^*), \alpha(t^*)) + C_2|x_n - x(t^*)|. \quad (3.13)$$

We now show that there exists an element in $A(t)$ which is close to $\alpha^*$ when $|t - t^*|$ is small. First note that since $\alpha^*$ satisfies (3.10) at $t^*$, and we
Figure 3.1. The circle has radius $C_2|x_n - x(t)| + \beta$. When the dimension $d$ is two, the set $A(t)$ therefore consists of the points within the circle that satisfies (3.10). Since the set of points that satisfy this equation is convex, and $\alpha^*$ is one of them, it follows that if the point $a$ in the figure belongs to $A(t)$, then so does $b$. We therefore see that there exists a point in $A(t)$ that is no farther away from $\alpha^*$ than $c$, i.e. a distance $\sqrt{|\alpha^* - \alpha(t)|^2 - C_2^2(|x_n - x(t)| + \beta)^2}$. The situation in higher dimension is analogous.

have assumption (3.12), it follows that $\alpha^* = H_\lambda(x_n, 0)$. By equation (3.13) it follows that $\alpha^*$ satisfies (3.10) also at $t$. The set of $\alpha$:s that solve (3.10) with $t$ fixed is convex. This convexity together with the fact that $\alpha^*$ satisfies (3.10), and the existence of an element in $A(t)$, by Lemma 3.3, implies that there exists an element in $A(t)$ which is not farther away from $\alpha^*$ than

$$\sqrt{|\alpha^* - \alpha(t)|^2 - C_2^2(|x_n - x(t)| + \beta)^2},$$

see Figure 3.1. We now use that

$$|\alpha(t) - \alpha^*| \leq |\alpha(t) - \alpha(t^*)| + |\alpha(t^*) - \alpha^*| \leq C_2|x_n - x(t^*)| + \beta + o(|t - t^*|)$$

and

$$|x_n - x(t)| \geq |x_n - x(t^*)| - |x(t^*) - x(t)| \geq |x_n - x(t^*)| - O(|t - t^*|)$$

in (3.14) to see that we can make the distance from $\alpha^*$ to an element in $A(t)$ arbitrarily small if $|t - t^*|$ is small. The case where the dimension $d = 1$ can
not be treated as in Figure 3.1 but then the same line of reasoning shows that there is an element in $A(t)$ which is a distance

$$\max\{|\alpha(t) - \alpha^*| - C|x_n - x(t)|, 0\}$$

from $\alpha^*$. It follows that $A$ is lower semicontinuous at $t^*$.

**Case 2.** We assume here that $H(x_n, 0) < L(x(t^*), \alpha(t^*)) + C_2|x_n - x(t^*)|$. As in the proof of Lemma 3.6 there exists an $\alpha^\#$ which satisfies $|\alpha^\# - \alpha(t^*)| \leq C_2|x_n - x(t^*)| + \beta$, and an $\epsilon > 0$ such that

$$\alpha^\# \cdot \lambda + L(x(t^*), \alpha(t^*)) + C_2|x_n - x(t^*)| \leq H(x_n, \lambda) + \epsilon, \quad \text{for all } \lambda \in \mathbb{R}^d.$$  

The linear combination $\xi\alpha^\# + (1 - \xi)\alpha^*$, where $0 \leq \xi \leq 1$, satisfies

$$(\xi\alpha^\# + (1 - \xi)\alpha^*) \cdot \lambda + L(x(t^*), \alpha(t^*)) + C_2|x_n - x(t^*)| \leq H(x_n, \lambda) + \xi\epsilon,$$

for all $\lambda \in \mathbb{R}^d$. Along the optimal path the running cost satisfies

$$L(x(t), \alpha(t)) = H(x(t), \lambda(t)) - \lambda(t) \cdot H_\lambda(x(t), \lambda(t)).$$

This follows by the fact that $\alpha(t) = H_\lambda(x(t), \lambda(t))$. Since the Hamiltonian is assumed to be continuously differentiable, we therefore have

$$\left| L(x(t), \alpha(t)) + C|x_n - x(t)| - \left( L(x(t^*), \alpha(t^*)) + C|x_n - x(t^*)| \right) \right| \leq o(|t - t^*)|.$$ 

If $\xi$ is small enough, $\hat{\alpha} = \xi\alpha^\# + (1 - \xi)\alpha^*$ therefore satisfies

$$\hat{\alpha} \cdot \lambda + L(x(t), \alpha(t)) + C_2|x_n - x(t)| \leq H(x_n, \lambda), \quad \text{for all } \lambda \in \mathbb{R}^d$$

Since $\hat{\alpha}$ satisfies equation (3.10), similar reasoning as in case 1 shows that the distance from $\alpha^*$ to an element in $A(t)$ can be made arbitrarily small when $|t - t^*|$ is made small. \qed

We are now ready to state the error estimate.

**Lemma 3.7.** Let the Hamiltonian $H$ be continuously differentiable, concave in its second argument, and satisfy (2.3), and the terminal cost $g$ be continuously differentiable with $|g'(x)| \leq C_3$, for all $x \in \mathbb{R}^d$. Then

$$u(x_s, 0) - \bar{u}(x_s, 0) \geq -\frac{C_1C_2T}{2}((e^{C_2T} - 1)\Delta t + \Delta t^2) - \frac{C_1C_3}{2}(e^{C_2T} - 1)\Delta t.$$  

(3.15)

**Proof.** As in Lemma 3.6 we let $x : [0, T] \to \mathbb{R}^d$ be a minimizer for the value $V(x_s, 0)$, defined in (2.4), and denote $x'(t)$ by $\alpha(t)$. It is easy to check that the set-valued function $A(t)$ in Lemma 3.6 has closed and convex values. Hence Theorem 3.3 and Lemma 3.7 show that for every $\beta > 0$ and any $x_n \in \mathbb{R}^d$, there exists a measurable function $\hat{\alpha}(t)$, such that for $t_n < t < t_{n+1}$

$$|\hat{\alpha}(t) - \alpha(t)| \leq C_2|x_n - x(t)| + \beta,$$

$$L(x_n, \hat{\alpha}(t)) \leq L(x(t), \alpha(t)) + C_2|x_n - x(t)|.$$
We now define the vectors \( \{ \tilde{x}_n \}_{n=0}^{N-1} \) and \( \{ \tilde{\alpha}_n \}_{n=0}^{N-1} \) as follows. Let \( \tilde{x}_0 = x_s \). Then there exists a \( \tilde{\alpha} : (t_0, t_1) \to \mathbb{R}^d \) as above. Let

\[
\tilde{\alpha}_0 = \frac{1}{\Delta t} \int_{t_0}^{t_1} \tilde{\alpha}(t) dt,
\]

and \( \tilde{x}_1 = \tilde{x}_0 + \Delta t \tilde{\alpha}_0 \). This \( \tilde{x}_1 \) now takes the role of \( x_1 \) in Lemma 3.6. We can therefore find \( \tilde{\alpha} : (t_1, t_2) \to \mathbb{R}^d \), and define \( \tilde{\alpha}_1 \) and \( \tilde{x}_2 \) as before, and then iterate this process further.

**Step 1.** This step consists of a proof that

\[
\max_{0 \leq n \leq N} |x(t_n) - \tilde{x}_n| \leq (e^{CT} - 1) \left( \frac{A}{2} \Delta t + \delta/C \right). \tag{3.16}
\]

To see this, we write

\[
x(t_n) - \tilde{x}_n = x(t_{n-1}) - \tilde{x}_{n-1} + \int_{t_{n-1}}^{t_n} (\alpha(t) - \tilde{\alpha}(t)) dt. \tag{3.17}
\]

We now use that

\[
|\alpha(t) - \tilde{\alpha}(t)| \leq C_2 |\tilde{x}_{n-1} - x(t)| + \beta \leq C_2 |x(t_{n-1}) - \tilde{x}_{n-1}| + C_2 |x(t) - x(t_{n-1})| + \beta, \tag{3.18}
\]

when \( t_{n-1} \leq t < t_n \). We now use that \( |x'(t)| \leq C_1 \) by \( (3.3) \), and \( (3.18) \) in \( (3.17) \) to obtain

\[
|x(t_n) - \tilde{x}_n| \leq (1 + C_2 \Delta t)|x(t_{n-1}) - \tilde{x}_{n-1}| + \frac{C_1 C_2}{2} \Delta t^2 + \beta \Delta t
\]

\[
\leq (1 + C_2 \Delta t)^2 |x(t_{n-2}) - \tilde{x}_{n-2}| + (1 + C_2 \Delta t) \left( \frac{C_1 C_2}{2} \Delta t^2 + \beta \Delta t \right) + \ldots \leq (1 + C_2 \Delta t)^n |x(0) - \tilde{x}_0|
\]

\[
+ (1 + C_2 \Delta t)^{n-1} + (1 + C_2 \Delta t)^{n-2} + \ldots + 1) \left( \frac{C_1 C_2}{2} \Delta t^2 + \beta \Delta t \right)
\]

\[
= \frac{(1 + C_2 \Delta t)^n - 1}{C_2 \Delta t} \left( \frac{C_1 C_2}{2} \Delta t^2 + \beta \Delta t \right) = ((1 + C_2 \Delta t)^n - 1) \left( \frac{C_1}{2} \Delta t + \beta/C_2 \right)
\]

\[
\leq e^{C_2 T n/N} \left( \frac{C_1}{2} \Delta t + \beta/C_2 \right) \leq (e^{C_2 T} - 1) \left( \frac{C_1}{2} \Delta t + \beta/C_2 \right),
\]

where the second last inequality follows from \( \Delta t = T/N \), and the second last equality follows from \( x(0) = \tilde{x}_0 \), and the formula for a geometrical sum.

**Step 2.** We now provide a lower bound for the term \( \int_0^T L(x(t), \alpha(t)) dt \), present in \( u(x_s, 0) \). For this purpose we use that for \( t_n \leq t < t_{n+1} \),

\[
L(\tilde{x}_n, \tilde{\alpha}(t)) \leq L(x(t), \alpha(t)) + C_2 |\tilde{x}_n - x(t)|
\]

\[
\leq L(x(t), \alpha(t)) + C_2 |\tilde{x}_n - x(t_n)| + C_2 |x(t) - x(t_n)|. \tag{3.19}
\]

From the boundedness of \( |x'| \leq C_1 \), we have

\[
\int_{t_n}^{t_{n+1}} |x(t) - x(t_n)| dt \leq \frac{C_1}{2} \Delta t^2 \tag{3.20}
\]
We now use the result from step 1, equation (3.16), together with (3.19) and (3.20):

\[
\int_0^T L(x(t), \alpha(t)) dt \geq \sum_{n=0}^{N-1} \left( \int_{t_n}^{t_{n+1}} L(\tilde{x}_n, \tilde{\alpha}(t)) dt \right) - C_2 T (e^{C_2 T} - 1) \left( \frac{C_1}{2} \Delta t + \beta/C_2 \right) - \frac{C_1 C_2 T}{2} \Delta t^2
\]

\[
\geq \Delta t \sum_{n=0}^{N-1} L(\tilde{x}_n, \tilde{\alpha}_n) - C_2 T (e^{C_2 T} - 1) \left( \frac{C_1}{2} \Delta t + \beta/C_2 \right) - \frac{C_1 C_2 T}{2} \Delta t^2 \quad (3.21)
\]

**Step 3.** Since

\[
\bar{u}(x,s,0) \leq \Delta t \sum_{n=0}^{N-1} L(\tilde{x}_n, \tilde{\alpha}_n) + g(\tilde{x}_N)
\]

we can now achieve the desired lower bound,

\[
u(x_s,0) - \bar{u}(x_s,0) \geq \Delta t \sum_{n=0}^{N-1} L(\tilde{x}_n, \tilde{\alpha}_n) - C_2 T (e^{C_2 T} - 1) \left( \frac{C_1}{2} \Delta t + \beta/C_2 \right) - \frac{C_1 C_2 T}{2} \Delta t^2 + g(x(T))
\]

\[\quad - \left( \Delta t \sum_{n=0}^{N-1} L(\tilde{x}_n, \tilde{\alpha}_n) + g(\tilde{x}_N) \right) \]

\[\geq -C_2 T (e^{C_2 T} - 1) \left( \frac{C_1}{2} \Delta t + \beta/C_2 \right) - \frac{C_1 C_2 T}{2} \Delta t^2 - C_3 |x(T) - \tilde{x}_N| \]

\[\geq -C_2 T (e^{C_2 T} - 1) \left( \frac{C_1}{2} \Delta t + \beta/C_2 \right) - \frac{C_1 C_2 T}{2} \Delta t^2 - C_3 (e^{C_2 T} - 1) \left( \frac{C_1}{2} \Delta t + \beta/C_2 \right). \]

Since the number \( \beta \) can be made arbitrarily small, the lower bound (3.15) follows.

\[\square\]

4. **Lower Bound of \( \bar{u}(x_s,0) - u(x_s,0) \)**

In order to be able to prove this lower bound, we first need to establish a bound on the discrete dual variable.

**Lemma 4.1.** Suppose \( g : \mathbb{R}^d \to \mathbb{R} \) is differentiable and satisfies \(|g'(x)| \leq C_3\) for all \( x \in \mathbb{R}^d \). Suppose \( H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) is continuously differentiable and satisfies (2.3). Let \( \{x_n, \lambda_n\}_{n=0}^N \) be an optimal discrete solution and dual which satisfy (2.8). Then

\[
|\lambda_n| \leq (C_3 + 1) e^{C_2 T} - 1. \quad (4.1)
\]
\[ |\lambda_n| = |\lambda_{n+1} - \Delta t H(x_n, \lambda_{n+1})| \leq (1 + C_2 \Delta t)|\lambda_{n+1}| + C_2 \Delta t \]
\[ \leq \ldots \leq (1 + C_2 \Delta t)^{N-n}|\lambda_N| + \left( (1 + C_2 \Delta t)^{N-n-1} + \ldots + 1 \right) C_2 \Delta t \]
\[ \leq C_3(1 + C_2 \Delta t)^N + \frac{(1 + C_2 \Delta t)^N - 1}{C_2 \Delta t} \]
\[ \leq C_3 e^{C_2 T} + e^{C_2 T} - 1. \]

\[ \square \]

**Lemma 4.2.** Let the conditions in Theorem 2.1 be satisfied. Let \( g : \mathbb{R}^d \to \mathbb{R} \) be differentiable and satisfy \( |g'(x)| \leq C_3 \), for every \( x \in \mathbb{R}^d \). Let \( H \) be continuously differentiable. Then, for any \( x_0 \in \mathbb{R}^d \),
\[ \bar{u}(x_0, 0) - u(x_0, 0) \geq - \frac{1}{2} C_1 C_2 (C_3 + 1) e^{C_2 T} \Delta t. \]

**Proof.** \( \bar{u}(x_0, 0) = \Delta t \sum_{n=0}^{N-1} L(x_n, \alpha_n) + g(x_N), \)
for some \( \{x_n, \alpha_n\} \). The solution \( \{x_n\}_{n=0}^{N} \) is extended to a piecewise linear function \( \bar{x} : [0, T] \to \mathbb{R}^d \), defined by
\[ \bar{x}(t) = \frac{t - t_n}{\Delta t} x_{n+1} + \frac{t_{n+1} - t}{\Delta t} x_n, \quad \text{when } t_n \leq t \leq t_{n+1}. \]
Using this extended approximate solution, and the fact that \( g(x_N) = u(x_N, T) \),
we can represent the difference between the original and the approximate value functions as
\[ \bar{u}(x_0, 0) - u(x_0, 0) = \Delta t \sum_{n=0}^{N-1} L(x_n, \alpha_n) + u(\bar{x}(T), T) - u(\bar{x}(0), 0) \]
\[ = \Delta t \sum_{n=0}^{N-1} L(x_n, \alpha_n) + \int_0^T \frac{d}{dt} u(\bar{x}(t), t) dt. \quad (4.2) \]

The last equality in the above formula follows by the fact that the function \( t \mapsto u(\bar{x}(t), t) \) is absolutely continuous since \( x(t) \) is Lipschitz continuous, and \( u \) is locally semiconcave. That \( u \) is locally semiconcave is shown in \([2]\) for a slightly less general case than is considered here, but the proof generalizes easily to the present conditions.

We now introduce the function \( \bar{H} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \), defined by
\[ \bar{H}(x, \lambda) = H(x, \lambda) - K \left( \max \left( |\lambda| - ((C_3 + 1)e^{C_2 T} - 1), 0 \right) \right)^2, \]
where $K$ is a positive constant. Define the corresponding running cost

$$
\tilde{L}(x, \alpha) = \sup_{\lambda \in \mathbb{R}^d} \left( -\alpha \cdot \lambda + \tilde{H}(x, \lambda) \right). \tag{4.3}
$$

Because $\tilde{H}(x, \lambda) \leq H(x, \lambda)$ for all $x$ and $\lambda$, it follows that $\tilde{L}(x, \alpha) \leq L(x, \alpha)$ for all $x$ and $\alpha$. But using Lemma 4.1 we also establish the opposite inequality for the optimal state and control variables:

$$
\tilde{L}(x_n, \alpha_n) \geq -\alpha_n \cdot \lambda_{n+1} + \tilde{H}(x_n, \lambda_{n+1}) = -\alpha_n \cdot \lambda_{n+1} + H(x_n, \lambda_{n+1}) = L(x_n, \alpha_n).
$$

Hence we may exchange $L$ with $\tilde{L}$ in (4.2).

In order to bound the integral in (4.2) from above, we now use the local semiconcavity of the value function $u$. This property implies that the superdifferential $D^+u(\bar{x}(t), t)$ is nonempty for all $0 < t < T$. Let $p = (p_x, p_t)$ be an element in $D^+u(\bar{x}(t), t)$. Let $t \in (0, T)$ satisfy $t \neq t_n$ for $n = 0, \ldots, N$, so that $\bar{x}$ is differentiable at $t$. We split the difference quotient approximating the backward derivative at $t$:

$$
\frac{u(\bar{x}(t), t) - u(\bar{x}(t-h), t-h)}{h} = \frac{-[u(\bar{x}(t-h), t-h) - u(\bar{x}(t), t) - p_t(-h) - p_x \cdot (\bar{x}(t-h) - \bar{x}(t))] + p_t + p_x \cdot \frac{\bar{x}(t) - \bar{x}(t-h)}{h}}{h}.
$$

The semiconcavity of $u$ implies that there is a constant $K$, such that the quotient involving the square bracket above is greater than or equal to

$$
-K \left( h^2 + (\bar{x}(t) - \bar{x}(t-h))^2 \right)/h.
$$

When $h$ is small enough, the difference $\bar{x}(t) - \bar{x}(t-h) = \alpha_n$, for some $n$. If we temporarily let $d/dt$ denote the backward derivative, we therefore obtain, when letting $h \to 0$

$$
\frac{d}{dt}u(\bar{x}(t), t) \geq p_t + p_x \cdot \alpha_n. \tag{4.4}
$$

The double sided and the backward derivative of $u(\bar{x}(t), t)$ differ on a set of measure zero, so there is no problem in using the backward derivative in (4.2). By the relation (4.3) between the running cost and the Hamiltonian, the following inequality holds:

$$
p_x \cdot \alpha_n + \tilde{L}(\bar{x}(t), \alpha_n) \geq \tilde{H}(\bar{x}(t), p_x). \tag{4.5}
$$

The fact that $u$ is a viscosity solution of (1.1) implies that

$$
p_t + H(\bar{x}(t), p_x) \geq 0. \tag{4.6}
$$
When combining equations (4.4), (4.5), and (4.6) with the error representation (4.2), we find that

\[ \bar{u}(x_0, 0) - u(x_0, 0) \geq \sum_{n=0}^{N-1} \left( \Delta t \bar{L}(x_n, \alpha_n) - \int_{t_n}^{t_{n+1}} \bar{L}(\bar{x}(t), \alpha_n) \, dt \right) \]

\[ = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} (\bar{L}(x_n, \alpha_n) - \bar{L}(\bar{x}(t), \alpha_n)) \, dt. \quad (4.7) \]

With the quadratic term added to the Hamiltonian, the supremum in the definition of \( \tilde{L} \) in (4.3) is always attained. Denote by \( \lambda^* \) a maximum point in \( \lambda \) for the coordinates \( (\bar{x}(t), \alpha_n) \), i.e.

\[ \tilde{L}(\bar{x}(t), \alpha_n) = -\alpha_n \cdot \lambda^* + \tilde{H}(\bar{x}(t), \lambda^*). \]

We thereby have the lower bound

\[ \tilde{L}(x_n, \alpha_n) - \tilde{L}(\bar{x}(t), \alpha_n) \geq -\alpha_n \cdot \lambda^* + \tilde{H}(x_n, \lambda^*) - \left( -\alpha_n \cdot \lambda^* + \tilde{H}(\bar{x}(t), \lambda^*) \right) \]

\[ = \tilde{H}(x_n, \lambda^*) - \tilde{H}(x(t), \lambda^*) = H(x_n, \lambda^*) - H(x(t), \lambda^*). \quad (4.8) \]

It is straightforward to show that for each \( \varepsilon > 0 \), there exists a \( K \), such that every maximizer \( \lambda^* \) in the definition of \( \tilde{L} \), (4.3), satisfies

\[ |\lambda^*| \leq (C_3 + 1)e^{C_2 T} - 1 + \varepsilon. \quad (4.9) \]

When we combine the bound on \( H \) in (2.3) with (4.7), (4.8), and (4.9), we obtain

\[ \bar{u}(x_0, 0) - u(x_0, 0) \geq -\frac{1}{2} C_1 C_2 (C_3 + 1)e^{C_2 T} T \Delta t. \]

\[ \square \]

References

[1] Jean-Pierre Aubin and Arrigo Cellina. *Differential inclusions*, volume 264 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Set-valued maps and viability theory.

[2] Piermarco Cannarsa and Carlo Sinestrari. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*. Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston Inc., Boston, MA, 2004.

[3] Frank H. Clarke. *Optimization and nonsmooth analysis*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1983. , A Wiley-Interscience Publication.

[4] Grant N. Galbraith. Extended Hamilton-Jacobi characterization of value functions in optimal control. *SIAM J. Control Optim.*, 39(1):281–305 (electronic), 2000.

[5] Hitoshi Ishii. Representation of solutions of Hamilton-Jacobi equations. *Nonlinear Anal.*, 12(2):121–146, 1988.

[6] R. Tyrrell Rockafellar. *Monotone processes of convex and concave type*. Memoirs of the American Mathematical Society, No. 77. American Mathematical Society, Providence, R.I., 1967.

[7] Mattias Sandberg and Anders Szepessy. Convergence rates of symplectic Pontryagin approximations in optimal control theory. *M2AN Math. Model. Numer. Anal.*, 40(1):149–173, 2006.
[8] Thomas Strömberg. Hamilton-Jacobi equations having only action functions as solutions. *Arch. Math. (Basel)*, 83(5):437–449, 2004.

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