Equivalence of Decentralized Observation, Diagnosis, and Control Problems in Discrete-event Systems

K. Ritsuka K. Rudie

Abstract

This paper demonstrates an equivalence between observation problems, control problems (with partial observation), and diagnosis problems of decentralized discrete-event systems, namely, the three classes of problems are Turing equivalent, as one class Turing reduces to another.

The equivalence allows decomposition of a control problem into a collection of simpler control sub-problems, which are each equivalent to an observation problem; and similarly allows converting a diagnosis problem to a formally simpler observation problem. Since observation problems in their most general formulation have been shown to be undecidable in previous work, the equivalence produced here demonstrates that control problems are also undecidable; whereas the undecidability of diagnosis problems is a known result.

1 Introduction

Most research in discrete-event systems (DES) falls into two categories: those concerning closed-loop systems such as control problems, and those concerning open-loop systems, such as observation problems and diagnosis problems.

For a discrete-event system plant, a closed-loop system is formed by imposing supervisory control over the plant. A control problem asks for
a supervisory control policy so that the closed-loop system meets some prescribed properties. The scheme of control problems is illustrated in Fig. 1.

Studies of control problems began with the seminal work of Ramadage and Wonham [RW87]. Partial observations [LW88] and decentralized supervision [Cie+88; RW92] were introduced in subsequent studies. Cieslak et al. [Cie+88] and Rudie and Wonham [RW92] initially introduced decentralized supervision under a constraint of available local control decisions and how overall control decisions are fused from the local ones. That constraint has been gradually relaxed [PKK97; YL02; YL04; KT05; CK11] over the past few decades.

On the other hand, open-loop systems take different forms. A concrete example is diagnosis problems [Sam+95; DLT00; ST02; QK06; WYL07] that seek distinguishing strings contain “faulty” events within a bounded delay of the occurrence of the faulty events. On the other hand, a more abstract example is observation problems that seek distinguishing strings from a prescribed collection. The earliest formalization of observation problems, as known to the author, is by Tripakis [Tri04]. The scheme of open-loop systems is illustrated in Fig. 1.

The three classes of problems, control, diagnosis, and observation, seem to be unrelated. Control problems concern closed-loop behaviour, whereas diagnosis problems allow a delay for the correct verdicts to be made, but observation problems do not. Therefore, the three classes of problems are usually studied separately.

However, results for one of the classes of problems have often been adopted to one of the other classes of problems. This suggests that there is a mutual connection between the three classes of problems. This document is intended to establish such connection as an equivalence between the
three classes of problems.

## 2 Observation Problem

An observation problem seeks to distinguish strings in a set $K$, from strings in $L - K$. Formally, an observation problem is specified as follows. Given alphabet $\Sigma$ and subalphabets $\Sigma_{i,o} \subseteq \Sigma$ called the observed alphabets, natural projections $P_i : \Sigma^* \rightarrow \Sigma_{i,o}$, for agents $i \in \mathcal{N} = \{1, \ldots, n\}$, and given languages $K \subseteq L \subseteq \Sigma^*$, the observation problem is to construct observers $f_i$ and a fusion rule $f$, such that

$$\forall s \in L. \\
\begin{align*}
  s \in K & \Rightarrow f(f_1 P_1(s), \ldots, f_n P_n(s)) = 1 \\
  \land s \in L - K & \Rightarrow f(f_1 P_1(s), \ldots, f_n P_n(s)) = 0
\end{align*}$$

(1)

An instance of the observation problem, $\text{Obs}$, is denoted by $O(L, K, \{\Sigma_{i,o}\}_{i \in \mathcal{N}})$ or more simply, $O(L, K, \Sigma_{i,o})$.

If the fusion rule $f$ is given as part of the problem, then the instance is denoted by $O(f, L, K, \Sigma_{i,o})$. Such problems are instance of the $f$-observation problem, or $f$-Obs.

Solvability of observation problems is known to be undecidable [Tri04].

We will show that diagnosis problems and control problems are both equivalent to observation problems.

## 3 Diagnosis Problem

The diagnosis problems were first studied in the centralized case by Sam-path et al. [Sam+95], and extended to the decentralized cases [DLT00; ST02; QK06; WYL07].

A diagnosis problem seeks to identify strings containing special events, known as “faulty events”, within a bounded delay of time. Formally, a diagnosis problem is specified as follows. Given alphabet $\Sigma$ and subalphabets $\Sigma_{i,o} \subseteq \Sigma$ called the observed alphabets, natural projections $P_i : \Sigma^* \rightarrow \Sigma_{i,o}$, for agents $i \in \mathcal{N} = \{1, \ldots, n\}$. For a fault alphabet $\Sigma_f \subseteq \Sigma_{uo} = \Sigma - \bigcup_i \Sigma_{io}$, a language $L$, say a string $s \in L$ is positive (faulty) if $s$ contains at least one symbol from $\Sigma_f$, and otherwise is negative. We may assume that there is a single fault event $\sigma_f$ as this assumption is inconsequential to the
3 Diagnosis Problem

hardness of the problem.

For a faulty string \( s \), if \( s = \pi \sigma_f \tau \) for some strings \( \pi \) and \( \tau \), where \( |\tau| \geq m \), we say that \( s \) is faulty for at least \( m \) steps. In other words, a string \( st \) is faulty for at least \( m \) steps if \( s \) is faulty and \( |t| \geq m \).

Then the diagnosis problem is, given an upper bound of delay as an integer \( m \), construct observers \( f_i \) and a fusion rule \( f \), such that

\[
\forall s \in L.
\]

\[
\begin{align*}
s \text{ is positive for at least } m \text{ steps} & \Rightarrow f(f_1P_1(s), \ldots, f_nP_n(s)) = 1 \\
\text{and } s \text{ is negative} & \Rightarrow f(f_1P_1(s), \ldots, f_nP_n(s)) = 0
\end{align*}
\]

That is, faulty strings are diagnosed after at most \( m \) steps of the fault.

The problem statement above is a simplification: A subtlety in the problem statement is that there may exist a faulty string \( s \in L \) that is positive for less than \( m \) steps but has no extension in \( L \) which is positive for at least \( m \) steps. Since we nonetheless want to diagnose such faulty strings, the phrase “\( s \) is positive for at least \( m \) steps” should be augmented to include such strings.

An instance of diagnosis problem, \( D_x \), is denoted by \( D(L, \{ \Sigma_{i,o} \}_{i \in \mathbb{N}}, \sigma_f, m) \) or more simply, \( D(L, \Sigma_{i,o}, \sigma_f, m) \).

If the fusion rule \( f \) is given as part of the problem, then the instance is denoted by \( D(f, L, \Sigma_{i,o}, \sigma_f, m) \). Such problems are instance of the \( f \)-diagnosis problem, or \( f\text{-Dx} \).

3.1 Equivalence of Diagnosis Problems and Observation Problems

Theorem 3.1

Given a fusion rule \( f \), the class of \( f \)-diagnosis problems — \( f\text{-Dx} \) — reduces to the class of \( f \)-observation problems — \( f\text{-Obs} \).

Proof. For a given \( f \)-diagnosis problem \( D(f, L, \Sigma_{i,o}, \sigma_f, m) \), construct the following \( f \)-observation problem \( O(f, L, K, \Sigma_{i,o}) \), where \( K = \{ s \in L \mid s \text{ is positive for at least } m \text{ steps} \} \).

By construction, (1) and (2) coincide. \( \square \)

Theorem 3.2

Given a fusion rule \( f \), \( f\text{-Obs} \) reduces to \( f\text{-Dx} \). \( \square \)
**Proof.** For a given $f$-observation problem $O(f, L, K, \Sigma_{i,o})$, construct the following $f$-diagnosis problem $D(f, L', \Sigma_{i,o}, \sigma_f, 0)$, where $L' = (L - K) \cup \{ s \sigma_f \mid s \in K \}$ and where we have chosen $m = 0$.

Notice that negative strings in $L'$ are exactly strings in $L - K$, and a string in $L'$ that is positive (for at least 0 steps) — i.e., one in $\{ s \sigma_f \mid s \in K \}$ — uniquely corresponds to a string $s \in K$ and satisfies $s' = s \sigma_f$, hence $P_i(s) = P_i(s' \sigma_f') = P_i(s')$. Thus, by construction, (1) and (2) coincide. □

**Theorem 3.3**

The classes of problems $f$-Obs and $f$-Dx are equivalent. Moreover, Obs and Dx are equivalent.

**Proof.** By Thms. 3.1 and 3.2. □

It is known that solvability of diagnosis problems is undecidable [ST02]. The reduction Thm. 3.1 offers an alternative route to proving that undecidability. Namely, we showed that observation problems reduce to diagnosis problems, and from Tripakis [Tri04] we know that observation problems are undecidable.

## 4 Control Problem

Recall that the control problem is to construct controllers $f_{i}^{\sigma}$ and fusion rules $f^{\sigma}$, for each event $\sigma \in \Sigma_{c}$, such that

$$\forall s \in K .$$

$$s \sigma \in K \Rightarrow f^{\sigma}(f_{1}^{\sigma}P_{1}(s), \ldots, f_{n}^{\sigma}P_{n}(s)) = 1$$

$$\wedge s \sigma \in L - K \Rightarrow f^{\sigma}(f_{1}^{\sigma}P_{1}(s), \ldots, f_{n}^{\sigma}P_{n}(s)) = 0$$  \hspace{1cm} (3)

To avoid trivial unsolvable instances, we assume that an instance is always controllable.

An instance of control problem, $C_{\text{Con}}$, is denoted by $C(L, K, \{ \Sigma_{i,o} \}_{i \in N}, \{ \Sigma_{i,c} \}_{i \in N})$, or more simply, $C(L, K, \Sigma_{i,o}, \Sigma_{i,c})$.

### 4.1 Equivalence of Control Problems and Observation Problems

We first revise the problem specification of the control problems.
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**Theorem 4.1**
Define the following two languages

\[ L_\sigma = \{ s \in K \mid s\sigma \in L \} \]
\[ K_\sigma = \{ s \in K \mid s\sigma \in K \}. \] (4)

Then (3) is equivalent to

\[ \forall \sigma \in \Sigma_c, s \in L_\sigma. \]
\[ s \in K_\sigma \quad \Rightarrow \quad f(f^\sigma_1 P_1(s), \ldots, f^\sigma_n P_n(s)) = 1 \] (5)
\[ \wedge s \in L_\sigma - K_\sigma \Rightarrow f(f^\sigma_1 P_1(s), \ldots, f^\sigma_n P_n(s)) = 0. \]

**Proof.** By definition, for all \( s \in L_\sigma, \)

\[ s\sigma \in L - K \Leftrightarrow s \in L_\sigma - K_\sigma. \]

This concludes the proof. □

**Theorem 4.2**
The classes of problems Con reduces to Obs

**Proof.** For a given control problem \( C(L, K, \Sigma_{i,o}, \Sigma_{i,c}) \), construct the following observation problems

\[ \{ O(L_\sigma, K_\sigma, \{ \Sigma_{i,o} \}_{i \in \N_\sigma}) \}_{\sigma \in \Sigma_c}. \]

By construction, (1) and (5) coincide. □

From the proof we can see that it is appropriate to decompose a control problem into a collection of individual (control) sub-problems, each one dealing with a specific event.

**Theorem 4.3**
The class of problems Obs reduces to Con'.

**Proof.** For a given observation problem \( O(L, K, \Sigma_{i,o}) \), construct a control problem as follows. First add to the alphabet a distinguished letter \( \gamma \), and let \( \Sigma_{i,c} = \{ \gamma \} \) for all \( i \in \N \). Henceforth, let \( pr(M) \) stands for the prefix-
4.1 Equivalence of Control Problems and Observation Problems

closure of language \( M \). Now let

\[
L' := \text{pr}(L\gamma) \\
= \text{pr}(L) \cup L\gamma \\
K' := \text{pr}(K\gamma \cup L) \\
= \text{pr}(K) \cup K\gamma \cup \text{pr}(L).
\]

The control problem is then

\[
C(L', K', \Sigma_{i,o}, \Sigma_{i,c}).
\]

It should be verified that the control problem is well-posed. First, it is clear that \( L' \) and \( K' \) are indeed prefix-closed. To verify controllability, let \( \Sigma \) be the alphabet of \( L \), and hence \( \Sigma_{uc} = \Sigma \). Then, for any \( \sigma \in \Sigma_{uc} \) and string \( s \in K' \), suppose that \( s\sigma \in L' \). If \( s\sigma \in \text{pr}(L) \), \( s\sigma \in K' \) as desired. If \( s\sigma \in L\gamma \), then \( \sigma = \gamma \), which contradicts the fact that \( \gamma \) is a controllable event.

Now compute the languages in (4). First, we have

\[
L'_\gamma = \{ s \in K' \mid s\gamma \in L' \} \\
= \{ s \in K' \mid s\gamma \in \text{pr}(L) \cup L\gamma \} \\
= \{ s \in K' \mid s \in L \} \\
= L
\]

where the third line is due to \( \gamma \) being a distinguished letter that is not in \( L \), and consequently not in \( \text{pr}(L) \); the fourth line is due to the facts that \( L \subseteq K\gamma \cup L \subseteq \text{pr}(K\gamma \cup L) = K' \). Similarly, we have

\[
K'_{\gamma} = \{ s \in K' \mid s\gamma \in K' \} \\
= \{ s \in K' \mid s\gamma \in \text{pr}(K) \cup K\gamma \cup \text{pr}(L) \} \\
= \{ s \in K' \mid s \in K \} \\
= K
\]

where the third line is due to \( \gamma \) being a distinguished letter that is not in \( L \), and also \( K \) being a subset of \( L \). The last line is due to \( K \subseteq \text{pr}(K) \subseteq K' \). Then (5) coincides with (1). □
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Theorem 4.4
The classes of problems Obs and Con are equivalent.

Proof. By Thms. 4.2 and 4.3. □

The approach of Lin and Wonham [LW88] in dealing with centralized control problems under partial observation can be interpreted as a special case of the reduction of control problems to observation problems (i.e., $\text{CON} \leq_T \text{OBS}$).

Corollary 4.5
Solvability of control problems are undecidable in general.

Proof. We have just shown that the observation problems reduces to control problems, whereas Tripakis demonstrated that solvability of observation problems is undecidable [Tri04]. □

Corollary 4.5 only states the undecidability of control problems when no restriction is placed on the fusion rule. However, in special cases when the fusion rule is restricted, such as for the architecture given by Cieslak et al. [Cie+88] and Rudie and Wonham [RW92], solvability can still be decided [RW95].

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