ON THE OPTIMAL CONSTANTS OF THE BOHNENBLUST–HILLE AND HARDY–LITTLEWOOD INEQUALITIES

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Abstract. We find the optimal constants of the generalized Bohnenblust–Hille inequality for $m$-linear forms over $\mathbb{R}$ and with multiple exponents $(1, 2, ..., 2)$, sometimes called mixed $(\ell_1, \ell_2)$-Littlewood inequality. We show that these optimal constants are precisely $(\sqrt{2})^{m-1}$ and this is somewhat surprising since a series of recent papers have shown that the constants of the Bohnenblust–Hille inequality have a sublinear growth, and in several cases the same growth was obtained for the constants of the generalized Bohnenblust–Hille inequality. This result answers a question raised by Albuquerque et al. (2013) in a paper published in 2014 in the Journal of Functional Analysis. Among other results, we also improve the best known constants of the generalized Hardy–Littlewood inequality in such a way that an unnatural behavior of the old estimates (that will be clear along the paper) does not happen anymore.

1. Introduction

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and $m \geq 2$ be a positive integer. In 1930 J.E. Littlewood proved his famous $4/3$ inequality ([19]) and, one year later, F. Bohnenblust and E. Hille extended Littlewood’s result to multilinear forms by proving, in the Annals of Mathematics ([8]), the following result:

Theorem (Bohnenblust and Hille, ([8], 1931). There exists a constant $B_{\mathbb{K}, m} \geq 1$ such that

\begin{equation}
(1.1) \left( \sum_{j_1, \ldots, j_m=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})| \right)^{\frac{2m}{m+1}} \leq B_{\mathbb{K}, m} \|T\|
\end{equation}

for all continuous $m$–linear forms $T : \ell_n^\infty \times \cdots \times \ell_n^\infty \to \mathbb{K}$, and all positive integers $n$.

Besides its own mathematical interest, this inequality, now called the Bohnenblust–Hille inequality, has been shown to be a key result in many areas of Mathematics and even in Physics. The exponent $\frac{2m}{m+1}$ is sharp, but the optimal values of $B_{\mathbb{R}, m}$ are still unknown. For complex scalars, the search for precise estimates for the polynomial version of $B_{\mathbb{C}, m}$ is quite important in applications in Complex Analysis and Number Theory; for real scalars, the estimates of $B_{\mathbb{R}, m}$ are important in Quantum Information Theory. For references we mention, for instance, [1, 5, 9, 10, 11, 20, 22] and the excellent survey [12].

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The best known upper and lower estimates for the constants in (1.1) appeared in [5] and [15], respectively:

\[ B_{\epsilon,m} \leq \prod_{j=2}^{m} \Gamma \left( 2 - \frac{1}{j} \right)^{\frac{1}{2j}} < m^{\frac{1}{p-m}} < m^{0.21139}, \]

\[ 2^{1-\frac{1}{m}} \leq B_{\eta,m} \leq 2^{\frac{1}{446381-14m}} \prod_{j=14}^{m} \left( \frac{\Gamma \left( \frac{3}{2} - \frac{1}{j} \right)}{\sqrt{\pi}} \right)^{\frac{1}{2j}} < 1.3 \cdot m^{\frac{2-\log 2 - \gamma}{2}}, \text{ if } m \geq 14, \]

\[ 2^{1-\frac{1}{m}} \leq B_{\eta,m} \leq \prod_{j=2}^{m} \left( 2^{\frac{1}{2j-2}} \right)^{\frac{1}{2j-2}} < 1.3 \cdot m^{\frac{2-\log 2 - \gamma}{2}}, \text{ if } m \leq 13, \]

where \( \gamma \) denotes the Euler–Mascheroni constant. The exponent \( \frac{2-\log 2 - \gamma}{2} \) above is approximately 0.36482. Nowadays the Bohnenblust–Hille inequality can be seen as a predecessor of the theory of multilinear summing operators (see [13, 23, 24] and the references therein).

From now on, to simplify the notation we will sometimes use the following definitions:

\[ \eta_{\epsilon,m} := \prod_{j=2}^{m} \Gamma \left( 2 - \frac{1}{j} \right)^{\frac{1}{2j}}, \]

\[ \eta_{\eta,m} := 2^{\frac{1}{446381-14m}} \prod_{j=14}^{m} \left( \frac{\Gamma \left( \frac{3}{2} - \frac{1}{j} \right)}{\sqrt{\pi}} \right)^{\frac{1}{2j}} \text{ for } m \geq 14, \]

\[ \eta_{\eta,m} := \prod_{j=2}^{m} 2^{\frac{1}{2j-2}} \text{ for } m \leq 13. \]

In other words, \( \eta_{\epsilon,m} \) are the best known upper estimates for the Bohnenblust–Hille constants. The replacement of \( \ell_p^m \) by \( \ell_\infty^m \) in the Bohnenblust–Hille inequality is only possible by changing the exponent, and this is not an easy task; the first step in this direction was made by Hardy and Littlewood ([17]) in 1934, followed by Praciano-Pereira ([25]) in 1981. In 2014 a generalized version was proved in [1] (see also [2, 14]). From now on, as usual, if \( f \) is a function, we define \( f(\infty) := \lim_{p \to \infty} f(p) \) whenever it makes sense.

**Theorem (Generalized Hardy–Littlewood inequality, ([1], 2014)).** Let \( m \geq 2 \) be a positive integer, \( 2m \leq p \leq \infty \) and \( \mathbf{q} := (q_1, \ldots, q_m) \in \left[ \frac{p}{p-m}, 2 \right]^m \) be such that

\[ \frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{mp + p - 2m}{2p}. \]

Then there exists a constant \( C_{m,p,q}^\infty \geq 1 \) such that

\[ \left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} \cdots \left( \sum_{j_m=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^{q_m} \right)^{\frac{q_m-1}{q_m}} \right) \right)^{\frac{1}{q_1}} \leq C_{m,p,q}^\infty \|T\| \]

for all continuous \( m \)-linear forms \( T : \ell_p^m \times \cdots \times \ell_p^m \to \mathbb{K} \) and all positive integers \( n \).

Henceforth a sequence \( (q_1, \ldots, q_m) \) satisfying (1.2) is called multiple exponent. As it happens with the Bohnenblust–Hille inequality, the multiple exponents given by (1.2) are
sharp but the optimal constants $C_{m,p,q}^K$ are unknown. The evolution of the estimates for the constants of the Hardy–Littlewood inequality can be summarized by the following table:

\[
\begin{align*}
\text{first approaches}, \\
C_{m,p,q}^K \left( \frac{2m}{mp+p-2m} \cdots \frac{2m}{mp+p-2m} \right) &\leq (\sqrt{2})^{m-1}, \\
C_{m,p,q}^K \left( \frac{2mp}{mp+p-2m} \cdots \frac{2mp}{mp+p-2m} \right) &\leq (\sqrt{2})^{2m(m-1)} (\frac{\eta_{K,m}}{p-2m})^{p-2m}, \\
C_{m,p,q}^K &\leq \eta_{K,m}, \\
C_{m,p,q}^K &\leq 2^{(m-1)} \left( \frac{m+1}{m} \frac{\max \eta_i}{\max \eta_i} \right) (\eta_{K,m})^{m(m-1)}, \\
\text{if max } q_i &< \frac{2m^2-4m+2}{m^2-m-1} \quad (\text{[3], 2014}), \\
\text{if max } q_i &\geq \frac{2m^2-4m+2}{m^2-m-1} \quad (\text{[3], 2014}).
\end{align*}
\]

The above estimates are valid for $K = \mathbb{R}$ or $\mathbb{C}$, and for complex scalars the estimates are even smaller (see Theorem 3.1). Note that the estimates from [1] and [4] are quite better than the original ones; for instance, from these estimates we know that in several situations the estimates for the constants $C_{m,p,q}^K$ are dominated by constants with sublinear growth (of the order of $m^{0.36482}$) instead of $(\sqrt{2})^{m-1}$.

The main results of this paper are presented in the next two sections. In Section 2 we answer a question raised in [1] by showing that the optimal constants of the generalized Bohnenblust–Hille inequality (i.e., the generalized Hardy–Littlewood inequality with $p = \infty$) for real scalars and $q = (1, 2, \ldots, 2)$ are precisely $(\sqrt{2})^{m-1}$. In fact we prove a more general result which asserts that if $\alpha \in [1, 2]$ is a constant and $q = (\alpha, \beta_m, \ldots, \beta_m)$ is a multiple exponent of the generalized Bohnenblust–Hille inequality, then the associated constants $C_{m,\infty,q}^K$ have an exponential growth if and only if $\alpha \neq 2$. In Section 3 we improve the estimates of the generalized Hardy–Littlewood inequality for $q := (q_1, \ldots, q_m)$ and $\max q_i \geq \frac{2m^2-4m+2}{m^2-m-1}$. These new estimates of the Hardy–Littlewood inequality show a continuous behavior when compared with the estimates for the case $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$. The previous estimates (see the above table) present an unnatural lack of continuity when $\max q_i = \frac{2m^2-4m+2}{m^2-m-1}$ because $2^{(m-1)} \left( \frac{m+1}{m} \frac{\max \eta_i}{\max \eta_i} \right) (\eta_{K,m})^{m(m-1)} > \eta_{K,m}$ when $\max q_i = \frac{2m^2-4m+2}{m^2-m-1}$.

2. The Bohnenblust–Hille Inequality

By making $p = \infty$ in the generalized Hardy–Littlewood inequality we have the generalized Bohnenblust–Hille inequality:

**Theorem (Generalized Bohnenblust–Hille inequality, ([1], 2014)).** Let $m \geq 2$ be a positive integer, and $q := (q_1, \ldots, q_m) \in [1, 2]^m$ be such that

\[\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2}.
\]

Then there exists a constant $C_{m,\infty,q}^K \geq 1$ such that

\[
\left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} \cdots \left( \sum_{j_m=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^{q_m} \frac{2m-1}{q_m} \frac{q_2}{q_3} \frac{q_1}{q_2} \frac{1}{q_1} \right) \right) \right) \leq C_{m,\infty,q}^K \| T \|
\]

for all continuous $m$–linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \rightarrow K$ and all positive integers $n$. 
In [11, Remark 5.1] the following sentence raises a natural question: “It would be nice to know if the constants in the mixed \((\ell_1, \ell_2)\)-Littlewood inequality can also be chosen to be subpolynomial.” Our next result shows that these constants can not be subpolynomial for real scalars, since we prove that the optimal values are \((\sqrt{2})^{m-1}\). This is somewhat surprising since various recent papers have shown that the constants of the Bohnenblust–Hille inequality have a subpolynomial (sublinear) growth, and in many cases the same growth was estimated for the constants of the generalized Bohnenblust–Hille inequality. In our terminology, the mixed \((\ell_1, \ell_2)\)-Littlewood inequality is the generalized Bohnenblust–Hille inequality with \(q := (1, 2, \ldots, 2)\).

**Theorem 2.1.** The optimal constants of the generalized Bohnenblust–Hille inequality for real scalars and \(q := (1, 2, \ldots, 2)\) are \((\sqrt{2})^{m-1}\).

**Proof.** It is well-known that the optimal constants are not bigger than \((\sqrt{2})^{m-1}\): this is a consequence of the Khinchine inequality (see, for instance, [11, Lemma 2] or the proof of [5, Proposition 3.1] and use the optimal constants of the Khinchine inequality due to Haagerup and Szarek ([16, 26]). So it suffices to show that \((\sqrt{2})^{m-1}\) is a lower bound. The proof is done by induction and follows the lines of the proof of the main result of [15].

For \(m = 2\), let \(T_2 : \ell^2_{\infty} \times \ell^2_{\infty} \to \mathbb{R}\) be defined by
\[
T_2(x, y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2.
\]
The signal minus before \(x_2y_2\) is strategic to make the norm of \(T_2\) small (for our purposes). It is not difficult to prove that \(\|T_2\| = 2\). Since
\[
\sum_{i_1=1}^{2} \left( \sum_{i_2=1}^{2} |T_2(e_{i_1}, e_{i_2})|^2 \right)^{\frac{1}{2}} = 2\sqrt{2}
\]
we conclude that
\[
C_{2,\infty, q}^R \geq \frac{2\sqrt{2}}{2} = \left(\sqrt{2}\right)^{2-1}.
\]

For \(m = 3\) let \(T_3 : \ell^4_{\infty} \times \ell^4_{\infty} \times \ell^4_{\infty} \to \mathbb{R}\) be given by
\[
T_3(x, y, z) = (z_1 + z_2) (x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) + (z_1 - z_2) (x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4)
\]
and note that \(\|T_3\| = 4\). Since
\[
\sum_{i_1=1}^{4} \left( \sum_{i_2,i_3=1}^{4} |T_3(e_{i_1}, e_{i_2}, e_{i_3})|^2 \right)^{\frac{1}{2}} = 4\sqrt{4}
\]
we have
\[
C_{3,\infty, q}^R \geq \frac{4\sqrt{4}}{4} = \left(\sqrt{2}\right)^{3-1}.
\]

In the case \(m = 4\) we consider \(T_4 : \ell^8_{\infty} \times \ell^8_{\infty} \times \ell^8_{\infty} \times \ell^8_{\infty} \to \mathbb{R}\) given by
\[
T_4(x, y, z, w) = (w_1 + w_2) \left( (z_1 + z_2) (x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) \right. \\
+ (z_1 - z_2) (x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4) \\
+ (w_1 - w_2) \left( (z_3 + z_4) (x_5y_5 + x_5y_6 + x_6y_5 - x_6y_6) \right. \\
+ (z_3 - z_4) (x_7y_7 + x_7y_8 + x_8y_7 - x_8y_8).
\]
and a similar argument shows that \( \|T_4\| = 8 \) and
\[
\sum_{i_1=1}^{8} \left( \sum_{i_2,i_3,i_4=1}^{8} |T_4(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})|^2 \right)^{\frac{1}{2}} = 8\sqrt{8}.
\]
Therefore
\[
C_{4,\infty,q}^R \geq \frac{8\sqrt{8}}{8} = (\sqrt{2})^{4-1}.
\]

The general case is proved by induction. Define the \( m \)-linear forms \( T_m : \ell_\infty^{2m-1} \times \ldots \times \ell_\infty^{2m-1} \rightarrow \mathbb{R} \) by induction as
\[
T_m(x_1, \ldots, x_m) = (x_1^m + x_m^2)T_{m-1}(x_1, \ldots, x_{m-1})
+ (x_m^1 - x_m^2)T_{m-1}(B^{2m-2}(x_1), B^{2m-2}(x_2), B^{2m-3}(x_3) \ldots, B^2(x_{m-1})),
\]
where \( x_k = (x_k^j)_n \in \ell_\infty^{2m-1} \) for \( 1 \leq k \leq m, 1 \leq n \leq 2^{m-1} \) and \( B \) is the backward shift operator in \( \ell_\infty^{2m-1} \). Then \( \|T_m\| = 2^{m-1} \) for all \( m \in \mathbb{N} \) and the proof follows straightforwardly. \( \square \)

**Remark 2.2.** A natural question is whether the above approach can give the optimal constants for other multiple exponents. We do not know the answer but, if compared to the best known upper bounds, the values are different, in general. The case of the classical exponents is more evident, since the lower bounds for the constants associated to \( (\frac{2m}{m+1}, \ldots, \frac{2m}{m+1}) \) with real scalars are \( 2^{1-\frac{1}{m}} \) (and thus not bigger than \( 2 \)) while the best known upper bounds are of the order \( m^{0.36482} \). Even for exponents close to \( (1,2,\ldots,2) \) the upper and lower estimates seem to be different. For instance, for the multiple exponents
\[
\mathbf{r} = \left( \frac{m+1}{m}, \frac{2(m+1)(m-1)}{m^2+1}, \ldots, \frac{2(m+1)(m-1)}{m^2+1} \right)
\]
the same argument used in the proof of Theorem 2.1 shows us that
\[
C_{m,\infty,\mathbf{r}}^R \geq 2^{\frac{m^2-2m+3}{2m^2+2}},
\]
while the upper bounds satisfy (see the forthcoming Theorem 3.3)
\[
C_{m,\infty,\mathbf{r}}^R \leq \left( \sqrt{2} \right)^{(m-1)} \left( \frac{1}{\left( 1 - \frac{m+1(2m-1)(m-1)^2}{(m^2-m-2)\max_q (m-1)^2} \right)} \right) \frac{\sum_{\mathbf{r}}}{\max_q \sum_{\mathbf{r}}} \frac{\left( \frac{3}{2} - \frac{1}{j} \right)}{\sqrt{\pi}}^{\frac{3}{2}-\frac{1}{j}} \frac{2^{m-2}}{m^2-2m}
\]
for \( m \geq 14 \). However for this particular case, it is interesting to note that the upper and lower estimates have the same asymptotic behavior. In fact, since
\[
2^{\frac{2^{m+1}}{m+1}} \prod_{j=14}^{m} \left( \frac{\left( \frac{3}{2} - \frac{1}{j} \right)}{\sqrt{\pi}} \right)^{\frac{3}{2}-\frac{1}{j}} < 1.3 \cdot m^{0.36482},
\]
the quotient between the upper and lower estimates tends to 1 as \( m \) grows. In fact,

\[
1 \leq \frac{m^2}{2^{m+1}} \frac{(\sqrt{2})^{(m-1)\left(1 + \frac{2m}{m^2-m-2}\right)}}{2^{m^2-2m+1}} \left( \frac{\Gamma\left(\frac{3}{2} \cdot \frac{1}{j} \right)}{\sqrt{\pi}} \right)^j \to 1.
\]

The same argument of the proof of Theorem 2.1 furnishes the following theorem:

**Theorem 2.3.** Let \( \alpha \in [1, 2] \) be a constant and \( q = (\alpha, \beta_m, ..., \beta_m) \) be a multiple exponent of the generalized Bohnenblust–Hille inequality. Then

\[
C_{m,\infty}^R \geq 2^{\frac{2m-\alpha m-4+3\alpha}{2\alpha}}.
\]

**Proof.** Note that

\[
\beta_m = \frac{2\alpha m - 2\alpha}{\alpha m - 2 + \alpha}.
\]

Plugging the \( m \)-linear forms defined in the proof of Theorem 2.1 into the Bohnenblust–Hille inequality we obtain

\[
C_{m,\infty}^R \geq \frac{2^{m-1} \left(2^{m-1} \right) \left(2m-2\alpha\right) \left(\frac{2m-2}{m} \right) \left(\frac{1}{\sqrt{\pi}}\right) \Gamma\left(\frac{3}{2} \cdot \frac{1}{j} \right)}{2^{m-1}} = 2^{\frac{2m-\alpha m-4+3\alpha}{2\alpha}}.
\]

\( \square \)

Note that when \( \alpha < 2 \) the above theorem shows that the respective optimal constants have an exponential growth. The following corollary shows that the result is sharp in the sense that when \( \alpha = 2 \) the growth is sublinear.

**Corollary 2.4.** Let \( \alpha \in [1, 2] \) be a constant and \( q = (\alpha, \beta_m, ..., \beta_m) \) be a multiple exponent of the generalized Bohnenblust–Hille inequality. Then the optimal constants associated to \( q \) have an exponential growth if and only if \( \alpha < 2 \).

**Proof.** The case \( \alpha < 2 \) is done in the above theorem (note that the growth of the optimal constants can not be bigger than exponential because from \[1\] we know that \( C_{m,\infty}^R \leq \left(\sqrt{2}\right)^{m-1} \) regardless of the \( q \).

If \( \alpha = 2 \) we obtain \( q = (2, \frac{2m-2}{m}, ..., \frac{2m-2}{m}) \), and in this case the optimal constants have a sublinear growth. In fact, from the proof of \[5\] Proposition 3.1, using the optimal constants of the Khinchine inequality we conclude that the optimal constants associated to \( (\frac{2m-2}{m}, ..., \frac{2m-2}{m}, 2) \) have a sublinear growth; this is a by now classic consequence of the Khinchine inequality. By using the Minkowski inequality we can move the number 2 to the first position and conclude that the constants associated to \( q = (2, \frac{2m-2}{m}, ..., \frac{2m-2}{m}) \) are dominated by the constants associated to \( (\frac{2m-2}{m}, ..., \frac{2m-2}{m}, 2) \), and thus have a sublinear growth. \( \square \)

### 3. The Hardy–Littlewood Inequality

Up to now the best known estimates for the Hardy–Littlewood inequality are given by the following theorem:
Theorem 3.1. ([3, Theorem 4.1]) Let \( m \geq 2 \) be a positive integer and \( 2m < p \leq \infty \). Let also \( q := (q_1, \ldots, q_m) \in \left[ \frac{p}{p-m}, 2 \right]^m \) be such that

\[
\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{mp + p - 2m}{2p}.
\]

(i) If \( \max q_i < \frac{2m^2 - 4m + 2}{m^2 - m - 1} \), then the constants in (1.3) satisfy

\[
C_{m,p,q}^R \leq \eta_{\mathbb{R},m}.
\]

(ii) If \( \max q_i \geq \frac{2m^2 - 4m + 2}{m^2 - m - 1} \), then the constants in (1.3) satisfy

\[
C_{m,p,q}^C \leq \left( \frac{2}{\sqrt{\pi}} \right)^{2(m-1)} \left( \frac{\eta_{\mathbb{C},m}}{\max q_i} \right)^m \left( \eta_{\mathbb{R},m} \right)^m \leq 2^{(m-1)} \left( \frac{\eta_{\mathbb{C},m}}{\max q_i} \right)^m \left( \frac{\eta_{\mathbb{R},m}}{\max q_i} \right)^m.
\]

Remark 3.2. The case \( p = 2m \) is not included in the statement of Theorem 3.1 in [3] but it could be included, although in this case there is no improvement in the original estimates. In fact, if \( p = 2m \), then \( q_1 = \cdots = q_m = 2 \) and we would be in the case (ii) of Theorem 3.1 and from the estimates of (ii) with \( \max q_i = 2 \) we obtain \( (\sqrt{2})^{m-1} \) and \( (2/\sqrt{\pi})^{m-1} \), which are implicit in in [1].

A close look at the above estimates shows us a surprising lack of continuity between (i) and (ii). In fact, when we make \( \max q_i \) tend to \( \frac{2m^2 - 4m + 2}{m^2 - m - 1} \) from below using the estimates of (i) we do not recover the estimates of (ii) when \( \max q_i = \frac{2m^2 - 4m + 2}{m^2 - m - 1} \), and this is somewhat unnatural. For instance, for complex scalars, the limit of the estimates in (i) as \( \max q_i \) goes to \( \frac{2m^2 - 4m + 2}{m^2 - m - 1} \) is \( \eta_{\mathbb{C},m} \), whereas from (ii), when \( \max q_i = \frac{2m^2 - 4m + 2}{m^2 - m - 1} \), the estimates for \( C_{m,p,q}^C \) are

\[
\beta_m := \left( \frac{2}{\sqrt{\pi}} \right)^{\frac{1}{m-1}} \left( \frac{\eta_{\mathbb{C},m}}{\max q_i} \right)^m \left( \frac{\beta_m}{\max q_i} \right)^m
\]

and a simple inspection shows us that \( \eta_{\mathbb{C},m} < \beta_m \) for all \( m > 2 \). For real scalars the lack of continuity still happens.

In this section we obtain better estimates for (ii) of Theorem 3.1 which now will behave continuously when compared with (i). The results presented here are a kind of complement of the papers [1, 3]. The enhancements presented here are due to a technical (but absolutely not evident) variation in the proof of the estimates from [3]. In view of the technicality of the results we decided to write this proof the more self-contained as possible. Our result is the following:

Theorem 3.3. Let \( m \geq 2 \) be a positive integer and \( 2m \leq p \leq \infty \). Let also \( q := (q_1, \ldots, q_m) \in \left[ \frac{p}{p-m}, 2 \right]^m \) be such that

\[
\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{mp + p - 2m}{2p}
\]

and

\[
\max q_i \geq \frac{2m^2 - 4m + 2}{m^2 - m - 1}.
\]

For all continuous \( m \)-linear forms \( T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K} \) and all positive integers \( n \), we
have

\[
\left( \sum_{j=1}^{n} \sum_{j=1}^{n} \cdots \sum_{j=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{q_m} \right)^{\frac{1}{q_m}} \leq C_{m,p,q}^{K} \| T \|,
\]

with

\[
C_{m,p,q}^{C} \leq \left( \frac{2}{\sqrt{\pi}} \right)^{(m-1)\theta_1} (\eta_{C,m})^{\theta_2},
\]

\[
C_{m,p,q}^{R} \leq \left( \sqrt{2} \right)^{(m-1)\theta_1} (\eta_{R,m})^{\theta_2},
\]

where \((\theta_1, \theta_2) = (1,0)\) if \(m = 2\) and

\[(3.3) \quad \theta_1 = 1 - \frac{(m+1)(2 - \max q_i)(m-1)^2}{(m^2 - m - 2) \max q_i} \quad \text{and} \quad \theta_2 = \frac{(m+1)(2 - \max q_i)(m-1)^2}{(m^2 - m - 2) \max q_i}\]

for \(m \geq 3\).

**Remark 3.4.** Note that now we have continuity between the estimates of Theorem 3.3 and (i) of Theorem 3.1 since

\[
\frac{(m+1)(2 - \max q_i)(m-1)^2}{(m^2 - m - 2) \max q_i} = 1
\]

when \(\max q_i = \frac{2m^2 - 4m + 2}{m^2 - m - 1}\). Also, whenever \(\max q_i < 2\) we have \(\theta_2 > m \left( \frac{2}{\max q_i} - 1 \right)\) and thus our estimates in fact improve the estimates of Theorem 3.1.

### 3.1. The proof of Theorem 3.3

The proof follows the lines of the argument used in [3, Theorem 4.1], but with a technical change in the interpolation argument. We need the following two lemmata proved in [3]:

**Lemma 3.5.** ([3, Lemma 2.1]) Let \((q_1, \ldots, q_m) \in [1,2]^m\) be such that \(\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2}\). If \(q_i \geq \frac{2m-2}{m}\) for a certain index \(i\) and \(q_k = q_i\) for all \(k \neq i\) and \(l \neq i\), then

\[
C_{m,\infty, (q_1, \ldots, q_m)}^{K} \leq \eta_{K,m}.
\]

**Lemma 3.6.** ([3, Lemma 2.2]) Let \(m \geq 2\) be a positive integer, \(2m \leq p \leq \infty\), and \(q_1, \ldots, q_m \in \left[ \frac{p}{p-m}, 2 \right]\). If

\[
\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{mp + p - 2m}{2p},
\]

then, for all \(s \in (\max q_i, 2]\), the vector \((q_1^{-1}, \ldots, q_m^{-1})\) belongs to the convex hull in \(\mathbb{R}^m\) of

\[
\left\{ \sum_{k=1}^{m} a_{1k} e_k, \ldots, \sum_{k=1}^{m} a_{mk} e_k \right\},
\]

where

\[
a_{jk} = \begin{cases} 
  s^{-1}, & \text{if } k \neq j \\
  \lambda_{m,s}, & \text{if } k = j
\end{cases}
\]
and

\( \lambda_{m,s} = \frac{2ps}{mps + ps + 2p - 2mp - 2ms} \).

We just consider the proof for the case of real scalars; as in [3, Theorem 4.1] the complex case is similar, using the Khinchine inequality with Steinhaus variables (the optimal constants of the Khinchine inequality for complex Steinhaus variables are due to H. König [13]).

The situation \( p = 2m \) has no novelty; it recovers the old estimates (the same comments from Remark [3,2] are valid here). The case \( \max q_i = 2 \) is also immediate, and just recovers the old estimates (as in Theorem [3,1]). So, from now on we consider \( p > 2m \) and \( \max q_i < 2 \) and, since \( \max q_i < 2 \) implies \( m > 2 \), we also also considering \( m > 2 \).

First of all, it is important to note that from (3.4) we straightforwardly have

\( s_q := \max q_i \geq \frac{2mp}{mp + p - 2m} \).

For \( s \in (s_q, 2] \), let

\( \lambda_{0,s} := \frac{2s}{ms + s + 2 - 2m} \)

and

\( \lambda_{m,s} := \frac{2ps}{mps + ps + 2p - 2mp - 2ms} \).

Note that the definition of \( \lambda_{m,s} \) is the same of (3.5) in Lemma [3,6]. From (3.6) and since \( \frac{2m^2 - 4m + 2}{m^2 - m - 1} \leq s_q < s \leq 2 \) we can also easily verify that \( \frac{p}{p-m} \leq \lambda_{m,s} < s \leq 2 \) and \( 1 \leq \lambda_{0,s} < s \leq 2 \).

Since

\( \frac{m - 1}{s} + \frac{1}{\lambda_{0,s}} = \frac{m + 1}{2} \),

the generalized Bohnenblust–Hille inequality tells us that there is a constant \( C_{m,\infty,(\lambda_{0,s},s,s,...,s)}^{\mathbb{R}} \geq 1 \) such that

\( \left( \sum_{j=1}^{n} \left( \sum_{j_i=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s \right)^{\frac{1}{s}} \right)^{\frac{1}{\lambda_{0,s}}} \leq C_{m,\infty,(\lambda_{0,s},s,s,...,s)}^{\mathbb{R}} \|T\| \)

for all \( m \)-linear forms \( T : \ell_\infty^n \times \cdots \times \ell_\infty^n \rightarrow \mathbb{K} \) and all \( i = 1, ..., m \) (here \( \sum_{j_i} \) denotes \( \sum_{j_1, ..., j_i-1, j_{i+1}, ..., j_m} \)). Since \( s > s_q \), and using our hypothesis (3.2), we know that

\( s > \frac{2m^2 - 4m + 2}{m^2 - m - 1} \).

Let

\( \theta_1 = 1 - \frac{(m + 1) (2 - s) (m - 1)^2}{s (m^2 - m - 2)} \),

\( \theta_2 = \frac{(m + 1) (2 - s) (m - 1)^2}{s (m^2 - m - 2)} \).
Recalling that we are now dealing with the case $m \geq 3$, and using (3.10), by means of elementary calculations we can show that $\theta_1, \theta_2 \in [0, 1]$ and it is obvious that $\theta_1 + \theta_2 = 1$. Since
\[
\frac{\theta_1}{1} + \frac{\theta_2}{\frac{2m-2}{m}} = \frac{1}{\lambda_{0,s}}
\]
and
\[
\frac{\theta_1}{2} + \frac{\theta_2}{\frac{2m^2+4m+2}{m^2-m-1}} = \frac{1}{s}
\]
we conclude that the multiple exponent
\[
(\lambda_{0,s}, s, s, \ldots, s)
\]
can be obtained by interpolating the multiple exponents
\[
(1, 2, \ldots, 2)
\] and \[
\left(\frac{2m-2}{m}, \frac{2m^2-4m+2}{m^2-m-1}, \ldots, \frac{2m^2-4m+2}{m^2-m-1}\right)
\]
with, respectively, the weights $\theta_1$ and $\theta_2$ in the sense of [1, Section 2]. Note also that both
\[
(t_1, \ldots, t_m) = (1, 2, \ldots, 2) \text{ or } \left(\frac{2m-2}{m}, \frac{2m^2-4m+2}{m^2-m-1}, \ldots, \frac{2m^2-4m+2}{m^2-m-1}\right)
\]
are exponents of the generalized Bohnenblust–Hille inequality, because
\[
\frac{1}{t_1} + \cdots + \frac{1}{t_m} = \frac{m+1}{2}.
\]
From Lemma 3.5, the multiple exponent \[
\left(\frac{2m-2}{m}, \frac{2m^2-4m+2}{m^2-m-1}, \ldots, \frac{2m^2-4m+2}{m^2-m-1}\right)
\]
is associated to the constant $\eta_{R,m}$ and the optimal constants associated to $(1, 2, \ldots, 2)$ are exactly $(\sqrt{2})^{m-1}$ (see Theorem 2.1). Therefore, the optimal constant associated to the multiple exponent $(\lambda_{0,s}, s, s, \ldots, s)$ is less than or equal to \[
(\sqrt{2})^{m-1} \theta_1 (\eta_{R,m})^{\theta_2}, \text{ i.e.,}
\]
\[
C_{R,\infty, (\lambda_{0,s}, s, s, \ldots, s)} \leq \left(\sqrt{2}\right)^{m-1} \theta_1 (\eta_{R,m})^{\theta_2}.
\]
(3.11)

Now the proof is finished by the following steps, similar to [3, Theorem 4.1]:
- Using (3.11) we can change $\infty$ by $p$ and and $\lambda_{0,s}$ by $\lambda_m$, and prove that
\[
C_{R,p, (\lambda_{0,s}, s, s, \ldots, s)} \leq \left(\sqrt{2}\right)^{(m-1)\theta_1} (\eta_{R,m})^{\theta_2}
\]
for all $s \in (s_q, 2]$ (it is not simple, but the tricky argument is the same as the one used in [3, Theorem 4.1] and [4, Theorem 1.1]). The interested reader can consult the Appendix of this paper.
- Since $\lambda_{m,s} < s$, from (3.12) and from the Minkowski inequality as in [1, Proposition 3.1] the position of $\lambda_{m,s}$ can be moved to other places in $(\lambda_{m,s}, s, \ldots, s)$ and thus
\[
C_{R,\infty, (s, s, \lambda_{m,s}, s, \ldots, s)} \leq \left(\sqrt{2}\right)^{(m-1)\theta_1} (\eta_{R,m})^{\theta_2}
\]
for all $s \in (s_q, 2]$ with $\lambda_{m,s}$ in the $j$–th position, for all $j = 1, \ldots, m$. 

(3.13)
• For any sufficiently small \( \varepsilon > 0 \) (chosen so that \( s_q + \varepsilon < 2 \), and this is possible because we are dealing with the case \( s_q < 2 \)), consider

\[
s_{q+\varepsilon} := s_q + \varepsilon = \max q_i + \varepsilon.
\]

Of course, (3.13) is valid for \( s = s_{q+\varepsilon} \). Since \( s_{q+\varepsilon} > s_q \), from Lemma 3.6 it follows that \((q_i^{-1}, \ldots, q_m^{-1})\) belongs to the convex hull of

\[
\left\{ \left( \lambda_{m,s_{q+\varepsilon}^{-1}}, s_{q+\varepsilon}^{-1}, \ldots, s_{q+\varepsilon}^{-1} \right), \ldots, \left( s_{q+\varepsilon}^{-1}, \ldots, s_{q+\varepsilon}^{-1} \right) \right\}
\]

and from the interpolative technique from [1, Section 2] (or using a variant of Hölder’s inequality for mixed sums that can be traced back to [6]), there are \( \theta_{1,s_{q+\varepsilon}} \), \( \theta_{2,m,s_{q+\varepsilon}} \in [0,1] \) such that

\[
\theta_{1,s_{q+\varepsilon}} + \cdots + \theta_{m,s_{q+\varepsilon}} = 1
\]

and

\[
C_{R,m,p,q} \leq \left( C_{R,m,p,(s_{q+\varepsilon}^{-1}, s_{q+\varepsilon}^{-1}, \ldots, s_{q+\varepsilon}^{-1})} \right)^{\theta_{1,s_{q+\varepsilon}}} \cdots \left( C_{R,m,p,(s_{q+\varepsilon}^{-1}, \ldots, s_{q+\varepsilon}^{-1})} \right)^{\theta_{m,s_{q+\varepsilon}}} \leq \left( \sqrt{2} \right)^{\left( m-1 \right) \theta_{1} \left( \eta_{R,m} \right)^{\theta_{2}}}
\]

Now we just make \( \varepsilon \to 0 \) and the proof is done.

**APPENDIX: THE PROOF OF THE INEQUALITY (3.12)**

We shall prove (3.12) with \( \theta_1 \) and \( \theta_2 \) defined as in the statement of Theorem 3.3, for all \( s \in (s_q, 2] \). Observe that if \( p = \infty \) then \( \lambda_{m,s} = \lambda_{0,s} \) (recall the definitions in (3.7) and (3.8)) and the inequality (3.12) is precisely (3.11), so it remains to prove (3.12) for \( p < \infty \). As we already mentioned, the proof is essentially taken from [3, Theorem 4.1] and [4, Theorem 1.1].

From (3.9) and (3.11) we know that for all \( m \)-linear forms \( T : \ell_{\infty}^n \times \cdots \times \ell_{\infty}^n \to K \) we have, for all \( i = 1, \ldots, m \),

\[
\left( \sum_{j=1}^{n} \left( \sum_{j=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^{s} \right)^{\frac{1}{s}} \right)^{\frac{1}{s}} \leq \left( \left( \sqrt{2} \right)^{m-1} \right)^{\theta_{1} \left( \eta_{R,m} \right)^{\theta_{2}}} \|T\|.
\]

For \( 2m < p < \infty \), let

\[
\lambda_{j,s} = \frac{\lambda_{0,s} p}{p - \lambda_{0,s} j}
\]

for all \( j = 1, \ldots, m \) (and observe that the case \( j = m \) is compatible with our definition of \( \lambda_{m,s} \)). The sequence \((\lambda_{j,s})_{j=0}^{m}\) is increasing and thus

\[
\lambda_{j,s} < \lambda_{m,s} < s
\]

for all \( j = 0, \ldots, m - 1 \). Moreover, observe that

\[
\left( \frac{p}{\lambda_{j,s}} \right)^{*} = \frac{\lambda_{j+1,s}}{\lambda_{j,s}}
\]

for all \( j = 0, \ldots, m - 1 \) (the notation \( \left( \frac{p}{\lambda_{j,s}} \right)^{*} \) represents the conjugate number of \( \left( \frac{p}{\lambda_{j,s}} \right) \)).
We proceed by induction: first we suppose that \(1 \leq k \leq m\) and that
\[
\left( \sum_{j_1=1}^{n} \left( \sum_{j_i=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \right)^{\frac{1}{\lambda_k - 1, s}} \leq \left( \left( \sqrt{2} \right)^{m-1} \right)^{\theta_1} (\eta_{\mathcal{R}, m})^{\theta_2} \|T\|^{s \lambda_k - 1, s}
\]
is true for all continuous \(m\)-linear forms \(T : \ell_p^n \times \cdots \times \ell_p^n \times \ell_\infty^m \times \cdots \times \ell_\infty^m \to \mathbb{K}\) and for all \(i = 1, \ldots, m\). Let us prove that
\[
\left( \sum_{j_1=1}^{n} \left( \sum_{j_i=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \right)^{\frac{1}{\lambda_k, s}} \leq \left( \left( \sqrt{2} \right)^{m-1} \right)^{\theta_1} (\eta_{\mathcal{R}, m})^{\theta_2} \|T\|
\]
for all continuous \(m\)-linear forms \(T : \ell_p^n \times \cdots \times \ell_p^n \times \ell_\infty^m \times \cdots \times \ell_\infty^m \to \mathbb{K}\) and for all \(i = 1, \ldots, m\). To simplify the notation we define
\[
C_m := \left( \left( \sqrt{2} \right)^{m-1} \right)^{\theta_1} (\eta_{\mathcal{R}, m})^{\theta_2}.
\]
The initial case (the case in which all \(p = \infty\)) is precisely (3.14).
Consider \(T \in \mathcal{L}(\ell_p^n, \ldots, \ell_p^n, \ell_\infty^m, \ldots, \ell_\infty^m; \mathbb{R})\) and for each \(x \in B_{\ell_p^n}\) define
\[
T^{(x)} : \ell_p^n \times \cdots \times \ell_p^n \times \ell_\infty^m \times \cdots \times \ell_\infty^m \to \mathbb{R}
\]
with \(x z^{(k)} = (x_j z_j^{(k)})_{j=1}^n\). Observe that \(\|T\| \geq \sup\{\|T^{(x)}\| : x \in B_{\ell_p^n}\}\). By applying the induction hypothesis to \(T^{(x)}\), we obtain
\[
(3.16) \quad \left( \sum_{j_1=1}^{n} \left( \sum_{j_i=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \right)^{\frac{1}{\lambda_k - 1, s}} \leq C_m \|T^{(x)}\| \leq C_m \|T\|
\]
for all \(i = 1, \ldots, m\). For more details on (3.16) we refer to ([1, Theorem 4.1] or [4]). Now, the proof is divided in two cases:

**First case:** \(i = k\).
Since \( (\frac{p}{\lambda_{j-1,s}})^* = \frac{\lambda_{j,s}}{\lambda_{j-1,s}} \) for all \( j = 1, \ldots, m \), and using (3.16), we conclude that

\[
\left( \sum_{j_k=1}^{n} \left( \sum_{j_{k} = 1}^{n} |T(e_{j_{1}}, \ldots, e_{j_{m}})|^{s} \right)^{\frac{1}{s}} \lambda_{k,s} \right)^{\frac{1}{s}} \lambda_{k-1,s} \left( \frac{n}{\gamma_{k-1,s}} \right)^{\frac{1}{s}} \lambda_{k-1,s} \gamma_{k-1,s} \leq C_{m} \|T\|.
\]

\[\sum_{j_{i}=1}^{n} \left( \sum_{j_{i} = 1}^{n} |T(e_{j_{1}}, \ldots, e_{j_{m}})|^{s} \right)^{\frac{1}{s}} \lambda_{k,s} = \sum_{j_{i} = 1}^{n} S_{i}^{\lambda_{k,s}} = \sum_{j_{i} = 1}^{n} S_{i}^{\lambda_{k,s}-S} S_{i}^{s}
\]

\[= \sum_{j_{i} = 1}^{n} \sum_{j_{i} = 1}^{n} \frac{|T(e_{j_{1}}, \ldots, e_{j_{m}})|^{s}}{S_{i}^{\gamma_{k,s}}} = \sum_{j_{i} = 1}^{n} \sum_{j_{i} = 1}^{n} \frac{|T(e_{j_{1}}, \ldots, e_{j_{m}})|^{s}}{S_{i}^{\gamma_{k,s}}}
\]

\[= \sum_{j_{i} = 1}^{n} \sum_{j_{i} = 1}^{n} \frac{|T(e_{j_{1}}, \ldots, e_{j_{m}})|^{s}}{S_{i}^{\gamma_{k,s}}}|T(e_{j_{1}}, \ldots, e_{j_{m}})|^{s} = \sum_{j_{i} = 1}^{n} \sum_{j_{i} = 1}^{n} \frac{|T(e_{j_{1}}, \ldots, e_{j_{m}})|^{s}}{S_{i}^{\gamma_{k,s}}}
\]

Second case: \( i \neq k \).

Since \( \lambda_{k-1,s} < \lambda_{k,s} < s \) for all \( 1 \leq k \leq m \), denoting, for \( i = 1, \ldots, m \),

\[S_{i} = \left( \sum_{j_{i} = 1}^{n} |T(e_{j_{1}}, \ldots, e_{j_{m}})|^{s} \right)^{\frac{1}{s}},
\]

we have
From Hölder’s inequality used twice in the equalities above we obtain

\[
\sum_{j_1=1}^{n} \left( \sum_{j_k=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \lambda_{k,s}^{-1} \leq \sum_{j_1=1}^{n} \left( \sum_{j_k=1}^{n} \frac{|T(e_{j_1}, \ldots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}} \right)^{\frac{1}{s}} \lambda_{k,s}^{-1} \sum_{j_1=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^s \left( \sum_{j_k=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \lambda_{k,s}^{-1} \lambda_{k-1,s}^{-1}.
\]

(3.17)

From the case \(i = k\) we have

\[
\left( \sum_{j_1=1}^{n} \left( \sum_{j_k=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \lambda_{k,s}^{-1} \right) \leq \left( C_m \|T\| \right) \lambda_{k-1,s}^{-1}.
\]

(3.18)

So, now we investigate the first factor in (3.17). From Hölder’s inequality and (3.16) it follows that (for details we refer to (4))

\[
\left( \sum_{j_1=1}^{n} \left( \sum_{j_k=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \lambda_{k,s}^{-1} \right) \leq \left( C_m \|T\| \right) \lambda_{k-1,s}^{-1}.
\]

(3.19)

Replacing (3.18) and (3.19) in (3.17), the proof of (3.12) is done for all \(s \in (s_q, 2]\).

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