TO THE PROBLEM OF $1/N_c$ APPROXIMATION IN THE NAMBU-JONA-LASINIO MODEL

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Abstract

In this article, the gap equation for the constituent quark mass in the $U(2) \times U(2)$ Nambu-Jona-Lasinio model for the $1/N_c$ approximation is investigated. It is shown that taking into account scalar isovector mesons plays an important role for the correct description of quark masses in this approximation. The role of the Ward identity in calculations of $1/N_c$ corrections to the meson vertex functions is shortly discussed.
The NJL model in the leading $1/N_c$ approximation, Hartree approximation, allows us to obtain a relatively complete picture of low-energy meson physics [1-5] ($N_c$ is the number of quark colors). However, in the last time, there have been undertaken some attempts to describe next to the leading $1/N_c$ approximation in the NJL model and to consider mesons not in the tree diagrams only but also in the loop diagrams [6-12]. Interesting results have been obtained in this direction for the description of the behaviour of the thermodynamical potential and the bulk of thermodynamical quantities in the vicinity of the critical temperature. It has been shown that mesonic degrees of freedom play the dominant role at low $T$, and the quark degrees of freedom are most relevant at high $T$.

Thus, it seems to be very useful to continue these investigations, to study more carefully the $1/N_c$ approximation in the NJL model by using different methods. Here, we consider the perturbation theory and calculate $1/N_c$ corrections to the gap equation. We will show how to correctly use the perturbation theory for the description of constituent quark mass in the $1/N_c$ approximation. Our results are remarkably different from those obtained in the series of previous papers (see e.g. [7]). It will be shown that the inclusion of scalar isovector mesons $a_0(980)$ plays an important role in the description of the $1/N_c$ approximation.

We consider the NJL model for the $U(2) \times U(2)$ chirally symmetric case [1-2]

$$L(\bar{q}, q) = \bar{q}(i\hat{\partial} - m^0)q + \frac{G}{2} [\bar{q}(\lambda^a q)^2 + (\bar{q}i\gamma^5 \lambda^a q)^2],$$

(1)

where $q$ are the fields of u and d quarks, $m^0$ is the current quark mass, $\lambda^0 = 1$ is the unique matrix and $\lambda^a = \tau^a$ ($a = 1, 2, 3$) are the Pauli matrices.

After the introduction of meson fields by using the technique of generating functional [1-3] and performing the integration over quark fields in the functional integral, we come to the Lagrangian

$$L'(\tilde{\sigma}, \tilde{\phi}) = -\frac{\tilde{\sigma}^2_a + \phi^2_a}{2G} - i\text{Tr} \ln S^{-1}(x - y),$$

(2)

where $\tilde{\sigma}_a$ and $\phi_a$ are the scalar and pseudoscalar meson fields, respectively, $\tilde{\sigma}_0 = \sigma_0 - m + m_0$, $\tilde{\sigma}_a = \sigma_a$ ($a = 1, 2, 3$)

$$S^{-1}(x, y) = [i\partial_x - m + \sigma_a \lambda^a + i\gamma_5 \lambda^a \phi_a] \partial^4(x - y).$$

(3)

To get the $\sigma$-model, it is enough to consider the divergent quark loops, depicted in Fig.1, and to perform the renormalization of meson fields [1-3]

As a result, we obtain the meson Lagrangian of the following type:

$$L''(\sigma, \phi) = \frac{1}{4} \text{Tr} \left\{ (\partial_{\mu} \bar{\sigma})^2 + (\partial_{\mu} \tilde{\phi})^2 + 2g \left( \frac{m - m_0}{G} - 8m I_1(m, \Lambda) \right) \tilde{\sigma} -
- g^2 \left( \frac{1}{G} - 8I_1(m, \Lambda) \right) (\bar{\sigma}^2 + \bar{\phi}^2) - g^2 \left[ \bar{\sigma}^2 - \frac{2m^2}{g} \bar{\sigma} + \bar{\phi}^2 \right]^2 \right\} -
- i\text{Tr} \ln \left( 1 + \frac{g}{i\hat{\partial} - m} \right) \left[ \bar{\sigma} + i\gamma_5 \tilde{\phi} \right],$$

(4)
where \( g = [4I_2(m, \Lambda)]^{-1/2}, \) \( \bar{\sigma} = \sigma^\alpha \lambda_\alpha, \) \( \bar{\phi} = \phi^\alpha \lambda_\alpha, \) and \( I_1(m, \Lambda) \) and \( I_2(m, \Lambda) \) are divergent integrals (\( \Lambda \) is the cut-off parameter)

\[
I_n(m, \Lambda) = -i \frac{N_c}{(2\pi)^4} \int_\Lambda^\Lambda \frac{d^4k}{(m^2 - k^2)^n}.
\] (5)

From (5) we can see that the coupling constant \( g^2 \) has the order \( 1/N_c \). Remind that the coupling constant \( G \) also has the order \( 1/N_c \).

From the condition \( \frac{\delta L''(\sigma, \phi)}{\delta \sigma} |_{\sigma, \phi=0} = 0 \) (absence of linear in \( \sigma \) terms in \( L''(\sigma, \phi) \)) we obtain the gap equation

\[
m = m^0 + 8mGI_1(m, \Lambda). \tag{6}
\]

How does the gap equation change, if we permit the existence of meson propagators inside the quark loops? \((1/N_c \) approximation). From Fig.1 one can easily see that in this case, in addition to the tadpole 1a there appear complementary terms (linear in \( \sigma \)) from the diagram 1c which lead to the appearance of additional terms in the gap equation (6) (see Fig.2)

\[
m = m^0 + 8mGI_1(m, \Lambda) + \Delta =
\]

\[
m^0 + 2G \frac{iN_c}{(2\pi)^4} \text{Tr} \int_\Lambda^\Lambda \frac{d^4k}{k - m} + 2G \frac{iN_c}{(2\pi)^4} \text{Tr} \int_\Lambda^\Lambda \frac{1}{k - m} \Sigma(k) \frac{1}{k - m} + ... \tag{7}
\]

The last two terms in (7) can be written in the form of one tadpole with the modified quark mass:

\[
m = m^0 + 2G \frac{iN_c}{(2\pi)^4} \text{Tr} \int_\Lambda^\Lambda \frac{d^4k}{k - m - \Sigma(k)}, \tag{7a}
\]

where \( \Sigma(k) \) is the operator of quark self-energy

\[
\Sigma(k) = 3\Sigma_\pi(k) + \Sigma_{\sigma_0}(k) + 3\Sigma_{a_0}(k), \tag{8}
\]

\[
\Sigma_\pi(k) = i \frac{g_\pi^2}{(2\pi)^4} \int_\Lambda^\Lambda d^4q \frac{\hat{q} - m}{(m^2 - q^2)(M_\pi^2 - (k - q)^2)}, \tag{9}
\]

\[
\Sigma_{\sigma_0}(k) = i \frac{g_\sigma^2}{(2\pi)^4} \int_\Lambda^\Lambda d^4q \frac{\hat{q} + m}{(m^2 - q^2)(M_{\sigma_0}^2 - (k - q)^2)}. \tag{10}
\]

Here \( M_\pi \) and \( M_{\sigma_0} \) are the masses of pions and \( \sigma \)-particles \((\sigma_i = \sigma_0, a_0^+, a_0^-, a_0^-), \) respectively, \( g_\pi = \frac{F_\pi}{M_\pi}, \) where \( F_\pi = 93 \) MeV is the pion decay constant.

The gap equation (7a) can be written in the form of the Schwinger - Dyson equation for the new quark mass \( \tilde{m} = m + \Sigma(m). \) For this purpose, we add the term \( \Sigma(\tilde{m}) \) to both the sides of equation (7a) and write it in the form

\[
\text{\footnotesize 1After accounting } \pi - a_1 \text{ transitions the constants } g_\sigma \text{ and } g_\pi \text{ will be different from each other } [1b]. \text{ Here } a_1 \text{ is the axial vector meson.}
\]

In the general case the cut-off parameters \( \Lambda \) and \( \bar{\Lambda} \) are not equal to each other. Here, we assume that \( \Lambda = \bar{\Lambda} = 1.2 \)GeV.
\[ \bar{m} = m^0 + 2G \frac{iN_c}{(2\pi)^4} \text{Tr} \int_0^\Lambda \frac{d^4k}{k - \bar{m}} + \Sigma(\bar{m}) = \]
\[ = m^0 + 8G\bar{m}I_1(\bar{m}, \Lambda) + \Sigma(\bar{m}). \]  
\text{(11)}

From equation (11) we can find the correction \( \delta m \) to the quark mass \( m_H \), obtained in the Hartree approximation, after taking account of the first order in \( 1/N_c \) expansion. That is why we write the mass \( \bar{m} \) in the form

\[ \bar{m} = m_H + \delta m \]  
\text{(12)}

and expand the second term in the r.h.s. of (11) over \( \delta m \), conserving the terms of first order over \( 1/N_c \)

\[ m_H + \delta m = m_0 + (m_H + \delta m)8G \left[ I_1(m_H, \Lambda) + \delta m \frac{\delta I_1}{\delta m} \bigg|_{m=m_H} \right] + \Sigma(m_H). \]  
\text{(13)}

By using the formulae

\[ \frac{\delta I_1(m, \Lambda)}{\delta m} = -2mI_2(m, \Lambda) = -\frac{m}{2g^2} \]  
\text{(14)}

and the gap equation in the Hartree approximation (see formula (6))

\[ m_H = m_0 + 8Gm_HI_1(m_H, \Lambda), \]

we find for \( \delta m \) the following expression:

\[ \delta m = Z^{-1}\Sigma(m_H), \]
\text{(15)}

where

\[ Z = 16Gm_H^2I_2(m_H, \Lambda) + \frac{m_0}{m_H} = \left( \frac{2m_H}{g} \right)^2 G + \frac{m_0}{m_H}. \]  
\text{(16)}

For the parameters we use here [1b]: \( m_H = 280 \text{ MeV}, \ m_0 = 3.3 \text{ MeV}, \ \Lambda = 1.2 \text{ GeV}, \ G = 5.4 \text{ GeV}^{-2}, \) and \( g^2 \approx 2\pi, \) we get \( Z^{-1} = 3.6, \ \delta m = 3.6 \Sigma(m_H, \Lambda). \)

Now we have to determine the operators \( \Sigma_\pi(p, \Lambda) \) and \( \Sigma_\pi(p, \Lambda) \) at the point \( \hat{p} = m_H \).

One can easily evaluate the integrals in formulae (9) and (10) and get the following expressions:

\[ \Sigma_\pi(p, \Lambda) = \frac{g^2}{(4\pi)^2} \int_0^1 dx \ (m - x\hat{p}) \left[ \ln \left( \frac{\Lambda^2}{m^2} + 1 \right) + \right. \]

\text{2The results, obtained in the papers [7,8], correspond to the value } Z = 1. \]
+ \ln \frac{1 + \bar{b}_\pi x + \bar{c}x^2}{1 + b_\pi x + cx^2} - \left(1 + \frac{m^2}{\Lambda^2}\right)^{-1} \frac{1}{1 + b_\pi x + cx^2} \right] = \\
= \frac{g_\pi^2}{(4\pi)^2} \left[ mC_1^\pi(p, \Lambda) - \hat{p}C_2^\pi(p, \Lambda) \right], \quad (17)

\Sigma_{\sigma_i}(p, \Lambda) = - \frac{g^2}{(2\pi)^4} \int_0^1 dx \left( m + x\hat{p} \right) \left[ \ln \left( \frac{\Lambda^2}{m^2} + 1 \right) + \\
+ \ln \frac{1 + \bar{b}_{\sigma_i} x + \bar{c}x^2}{1 + b_{\sigma_i} x + cx^2} - \left(1 + \frac{m^2}{\Lambda^2}\right)^{-1} \frac{1}{1 + b_{\sigma_i} x + cx^2} \right] = \\
= - \frac{g^2}{(4\pi)^2} \left[ mC_{1\sigma_i}(p, \Lambda) + \hat{p}C_{2\sigma_i}(p, \Lambda) \right], \quad (18)

where

\begin{align*}
 b_i &= \frac{M_i^2 - m^2 - p^2}{m^2}, \quad c = \frac{p^2}{m^2}, \quad \bar{b}_i = \frac{M_i^2 - m^2 - p^2}{a}, \quad \bar{c} = \frac{p^2}{a}, \quad a = m^2 + \Lambda^2, \\
 C_1^i &= \ln \left( \frac{\Lambda^2}{m^2} + 1 \right) + \left(1 + \frac{\bar{b}_i}{2c}\right) \ln(1 + \bar{b}_i + \bar{c}) - \left(2 + \frac{b_i}{c}\right) \ln \frac{M_i}{m} + \\
&\quad + \left(1 - \frac{\bar{b}_i^2}{2c}\right) I_0 + \left(\frac{b_i^2}{2c} - 2\right) I_0, \quad (19)
\end{align*}

\begin{align*}
 C_2^i &= - \frac{1}{2} \left( \frac{\bar{b}_i}{c} - \frac{b_i}{c}\right) + \frac{1}{2} \ln \left( \frac{\Lambda^2}{m^2} + 1 \right) + \frac{1}{2} \left(1 - \frac{\bar{b}_i^2}{2c}\right) \ln(1 + \bar{b}_i + \bar{c}) - \\
&\quad - \left[1 + \frac{1}{c} \left(1 - \frac{\bar{b}_i^2}{2c}\right) \right] \ln \frac{M_i}{m} - \frac{\bar{b}_i}{2c} \left(1 - \frac{\bar{b}_i^2}{2c}\right) I_0 + \frac{b_i}{2c} \left(2 - \frac{b_i^2}{2c}\right) I_0, \quad (20)
\end{align*}

\begin{align*}
 I_0 &= \int_0^1 \frac{dx}{1 + b_i x + cx^2}, \quad \bar{I}_0 = \int_0^1 \frac{dx}{1 + \bar{b}_i x + \bar{c}x^2}.
\end{align*}

Scalars and pions give contributions to the quark mass with the opposite signs and strongly compensate each other. Therefore, it is important to take into account contributions of all mesons corresponding to the considered group of symmetry. In our case of the group U(2)×U(2), to the three pions there correspond four scalar mesons in the scalar sector (scalar isoscalar $\sigma_0(700)$ and three scalar isovectors $a_0(980)$).

[^3]: The isoscalar partner of pions appears only in the U(3)×U(3) group in the form of a $\eta$ meson. Therefore, we will not consider it here. Scalar mesons have the masses: $m_{\sigma_0} = 700\text{MeV}$ and $m_{a_0} = 980\text{MeV}$.
Table 1 gives the coefficients $C_i^1$ and $C_i^2$ evaluated for all these mesons at $p^2 = m^2$.

| $C_{\sigma_0}^1$ | $C_{\sigma_0}^2$ | $C_{\pi}^1$ | $C_{\pi}^2$ |
|-----------------|-----------------|------------|------------|
| 1.06            | 0.42            | 0.63       | 0.5        |
| 2.8             | 1.5             |            |            |

Then, for $\Sigma(m, \Lambda)$ we obtain

$$
\Sigma(m, \Lambda) = \frac{m}{(4\pi)^2}[-g^2(1.48 + 3 \times 1.13) + g_\pi^2(3 \times 1.3) = -30.6 + 35.4 = 5]
$$

$$
\Sigma(m, \Lambda) = 0.03 , \quad \delta m = 0.11 \, m .
$$

As a result, the mass of a constituent quark increases by 11% and is equal to 310 MeV, which completely corresponds to the standard value. If we consider only one scalar meson $\sigma_0(700)$, the corrections rapidly increase, amounting to 60% ($\delta m = 0.60 \, m$), which does not correspond to the $1/N_c$ approximation.

These calculations have shown, that for the correct estimates performed in the $1/N_c$ approximation, it is very important to take into account all real contributions of mesons from the scalar and pseudoscalar sectors.

As an example of another approach to estimation of the quark mass in the $1/N_c$ approximation within the NJL model we can illustrate the paper [7]. In this article two incorrect actions have been done, in our opinion: the first when the additional contributions (in the $1/N_c$ approximation) from the leading tadpole term in the Schwinger-Dyson equation were not taken into account. This led to the lowered result which did not correspond to the real $1/N_c$ approximation. At the second step, the contribution of only one scalar isoscalar meson was considered instead of four scalar mesons. This step substantially increased their estimate. As a result of these two operations, the final $1/N_c$ corrections to the quark mass did not go beyond the limit of 20% of the Hatree approximation.

One of the interesting tasks is the construction of chirally symmetric perturbation theory for the $1/N_c$ expansion. The positive results in this direction have been obtained by G.S.Guralnik with coauthors already in 1976 [12]. They showed that in the $1/N_c$ approximation for the NJL model with one scalar and one pseudoscalar mesons the pion mass was equal to zero when the current quark mass was vanishing. Therefore, the pion remains the Goldstone particle in this approximation as well.

It is interesting to consider the changes of the meson coupling constants $g$ and $g_\pi$ in the $1/N_c$ approximation. As we have shown in the Appendix, the scalar meson coupling constant $g$ does not change in the $1/N_c$ approximation. A more complicated situation took place for the coupling constants $g_\pi$ and the Goldberger-Treiman identity.

\(^4\)If we use the model values for the masses of the scalar mesons:$m_{\sigma_i}^2 = m_\pi^2 + 4m^2$, $m = 580$ MeV, we get the negative value for $\delta m$. ($C_1^{\sigma_i} = 1.3$, $C_2^{\sigma_i} = 0.55$)
When this work has been fulfilled, we found out that a very interesting paper appeared just now [13]. In this work, a chirally symmetric self-consistent \(1/N_c\) approximation scheme to the NJL model was developed. The authors used the correct \(1/N_c\) approximation for the gap equation and demonstrated explicitly that their scheme fulfills all the chiral symmetry theorems - the Goldstone theorem, Goldberger-Treiman relation and the conservation of the quark axial current.

This paper is very close to ref. [12]. In contrast with our work they considered the \(SU(2) \times SU(2)\) chiral symmetry Lagrangian with only one scalar isoscalar meson and the case when the current quark mass was equal to zero.

In conclusion, we would like to say that the papers [12-13] and this one give the full picture of the chirally symmetric \(1/N_c\) approximation in the NJL model.

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**APPENDIX**

The scalar vertex function and Ward identity

Let us show that \(1/N_c\) corrections to the scalar coupling constant \(g\) are equal to zero. For this aim we consider diagrams depicted in Fig.3. The scalar vertex function for \(\sigma_0\) meson in \(1/N_c\) approximation takes the form

\[
\Gamma^{(1/N_c)}(p, p'|q) = g_\sigma + \Gamma^{b}_{\sigma_0}(p, p'|q) + \Gamma^{(c+d)}_{\sigma_0}(p, p'|q) + 3\Gamma^{b}_{\sigma_0}(p, p'|q) + 3\Gamma^{(c+d)}_{\sigma_0}(p, p'|q) + \]

\[
+3\Gamma^{b}_{\pi}(p, p'|q) + 3\Gamma^{(c+d)}_{\pi}(p, p'|q). \tag{A.1}
\]

Now consider the case when \(q = 0, p = p'\). Then

\[
\Gamma^{(b)}_{\sigma_0}(p, p|0) = -i \frac{g^2}{(2\pi)^4} \int \frac{d^4k}{(k + \hat{p} - m)^2(M^2_{\sigma_0} - k^2)}, \tag{A.2}
\]

\[
\Sigma_{\sigma_0}(p + k) = -i \frac{g^2}{(2\pi)^4} \int \frac{d^4k}{(k + \hat{p} - m)(M^2_{\sigma_0} - k^2)}, \tag{A.3}
\]

\[
\Gamma^{(c+d)}_{\sigma_0}(p, p|0) = \frac{\Sigma_{\sigma_0}(p) - \Sigma_{\sigma_0}(m)}{\hat{p} - m} |_{\hat{p} = m} = \frac{\delta \Sigma_{\sigma_0}(p)}{\delta \hat{p}} |_{\hat{p} = m} =
\]

\[
= i \frac{g^2}{(2\pi)^4} \int \frac{d^4k}{(k + \hat{p} - m)^2(M^2_{\sigma_0} - k^2)} = -\Gamma^{(b)}_{\sigma_0}(p, p|0). \tag{A.4}
\]

The similar situation takes place for \(\Gamma_{\sigma_0}^{(c)}\) and \(\Gamma_{\pi}^{(c+d)}\). As a result all contributions of the diagrams depicted in Fig.3b-d cancel each other and finally we got

\[
\Gamma^{(1/N_c)}(p, p|0) = g_\sigma. \tag{A.5}
\]
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Figure captions

Fig.1 The quadratically (1a, 1b) and logarithmically (1c, 1d) divergent quark loop diagrams in the NJL model.

Fig.2 The additional tadpole diagram in the 1/N_c approximation. The Σ is the self-energy part of the quark propagator with pion and scalar meson internal lines.

Fig.3 The scalar vertex diagrams in the 1/N_c approximation.
This figure "fig1-1.png" is available in "png" format from:

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