Abstract. In this paper, we define the recurrence and “non-wandering” for decompositions. The following inclusion relations hold for codimension one foliations on closed 3-manifolds: p.a.p. $\subset$ recurrent $\subset$ non-wandering. Though each closed 3-manifold has codimension one foliations, no codimension one foliations exist on some closed 3-manifolds.

1. Introduction and preliminaries

In 1927, Birkhoff introduced the concepts of non-wandering points and recurrent points [Bi]. Using these concepts, we can describe and capture sustained or stationary dynamical behaviors and conservative dynamics. In this paper, we define the recurrence and “non-wandering” for decompositions. As usual dynamical systems, the following relations hold:

pointwise almost periodic $\subset$ recurrent $\subset$ non-wandering.

In particular, the above inclusions hold for codimension one foliations on closed 3-manifolds. Though each closed 3-manifold has codimension one foliations, no codimension one foliations exist on some closed 3-manifolds. Let $\mathcal{F}$ be a codimension one non-wandering foliation on a closed 3-manifold. When the ends of each leaf of $\mathcal{F}$ is countable, we show that $\mathcal{F}$ is $R$-closed if and only if $\mathcal{F}$ has either no compact leaves or no locally dense leaves. On the other hand, there are codimension one smooth foliations without compact leaves which are not $R$-closed.

By a decomposition, we mean a family $\mathcal{F}$ of pairwise disjoint nonempty subsets of a topological space $X$ such that $X = \sqcup \mathcal{F}$. For $L \in \mathcal{F}$, we call that $L$ is proper if $\overline{L} - L$ is closed, and it is recurrent if it is either closed or non-proper. Denote by $Cl$ (resp. $P$) the set of closed elements (resp. the set of elements which are not closed but proper). Then the complement of $P$ is the set of recurrent. A decomposition $\mathcal{F}$ is recurrent (resp. non-wandering) if $P = \emptyset$ (resp. $\text{int} P = \emptyset$). Then a decomposition is recurrent if and only if each element of it is recurrent. For any $x \in X$, denote by $L_x$ the element of $\mathcal{F}$ containing $x$. Recall that $\mathcal{F}$ is pointwise almost periodic (p.a.p.) if the set of all closures of elements of $\mathcal{F}$ also is a decomposition and it is $R$-closed if $R := \{(x, y) \mid y \in \overline{L_x}\}$ is closed.

Lemma 1.1. Let $\mathcal{F}$ be a decomposition on $X$. If $\mathcal{F}$ is pointwise almost periodic, then $\mathcal{F}$ is recurrent. If $\mathcal{F}$ is recurrent, then $\mathcal{F}$ is non-wandering.

Proof. By definition, recurrence implies non-wandering property. Suppose that $\mathcal{F}$ is pointwise almost periodic. Fix any non-closed element $L \in \mathcal{F}$. Since $L$ is not
closed, there is an element \( x \in \overline{L} - L \). Since \( \mathcal{F} \) is pointwise almost periodic, we obtain \( \overline{\mathcal{F}} = \overline{L} \). Then \( \overline{\mathcal{F}} = \mathcal{F} \supseteq \overline{L} - L = \overline{L} - L \supseteq \overline{L} - L \). Thus \( \overline{L} - L \) is not closed. This shows that \( L \) is not proper. Therefore \( P = \emptyset \). \( \square \)

2. Codimension one foliations on 3-manifolds

Note that each codimension one non-wandering foliation on a closed surface is either minimal or compact because of Theorem 2.3 [Y2]. However the 3-dimensional case is different from the 2-dimensional case. For instance, in [H], the author has constructed a codimension one continuous non-wandering foliation on a closed 3-manifold \( M \) such that \( \overline{LD} = E = M = LD \sqcup E \) (resp. \( E = M \)), where \( LD \) is the union of locally dense leaves and \( E \) is the union of exceptional leaves. This shows that \( LD \) (resp. \( E \sqcup Cl \)) is neither open nor closed in general. Recall that \( \mathcal{F} \) is \( \pi_1 \)-injective if each inclusion of a leaf \( L \) of \( \mathcal{F} \) induces an injective map \( \pi_1(L) \to \pi_1(M) \) for some base point in \( L \). Now we state some properties of non-wandering codimension one continuous foliations.

**Lemma 2.1.** Let \( \mathcal{F} \) be a codimension one non-wandering continuous foliation on a closed 3-manifold \( M \). Then \( \mathcal{F} \) is \( \pi_1 \)-injective. Moreover if \( \pi_2(M) \) is trivial, then the universal cover of \( M \) is homeomorphic to \( \mathbb{R}^3 \).

**Proof.** By non-wandering property, there are no Reeb components. By the \( C^0 \) Novikov Compact Leaf Theorem [Sa], there are no vanishing cycles. By Th3.4.VIII [HH], we have that \( \mathcal{F} \) is \( \pi_1 \)-injective. Suppose that \( \pi_2(M) \) is trivial. By Corollary 2.4 [P], the universal cover of \( M \) is homeomorphic to \( \mathbb{R}^3 \). \( \square \)

This implies the non-existence of codimension one foliations on homological spheres.

**Corollary 2.2.** There are no codimension one non-wandering continuous foliations on homology 3-spheres.

From now on, we consider \( C^2 \) foliations on closed 3-manifolds.

**Proposition 2.3.** Let \( \mathcal{F} \) be a codimension one non-wandering \( C^2 \) foliation on a closed connected 3-manifold \( M \). Suppose there are no leaves of \( \mathcal{F} \) whose ends are uncountable. Then the following holds:
1) \( \mathcal{F} \) is \( R \)-closed.
2) \( \mathcal{F} \) is either minimal or compact.
3) \( \mathcal{F} \) either has no compact leaves or has no locally dense leaves.

**Proof.** By Theorem 5.2 [Y], we have that 1) and 2) are equivalent. Suppose that \( \mathcal{F} \) is minimal or compact and so \( \mathcal{F} \) either has no compact leaves or has no locally dense leaves. Conversely, we show that 3) implies 2). Suppose that \( \mathcal{F} \) has either no compact leaves or no locally dense leaves. By the Duminy theorem for ends [CtC2], there are no exceptional leaves. If there are no compact leaves, then the minimal set is the whole manifold \( M \) and so \( \mathcal{F} \) is minimal. Thus we may assume that there are no locally dense leaves. Assume \( \mathcal{F} \) is not compact. Since the union of compact leaves are closed, the union of non-compact leaves are nonempty open and consists of non-compact proper leaves. This contradicts to non-wandering property. Thus \( \mathcal{F} \) is compact. \( \square \)

The following statement shows that the above countable condition is necessary and that pointwise almost periodicity does not correspond to recurrence.
Proposition 2.4. There is a smooth codimension one foliation $F$ on a closed 3-manifold $\Sigma_4 \times S^1$ which is not pointwise almost periodic but recurrent such that $F$ consists of exceptional leaves and locally dense leaves, where $\Sigma_k$ is the genus $k$ closed orientable surface.

Proof. Let $G$ be the group generated by a circle diffeomorphisms $f, g$ in [Sac] with a unique Cantor minimal set $\mathcal{M}$ and $f_1, f_2 : (1/3, 2/3) \rightarrow (1/3, 2/3)$ smooth diffeomorphisms such that each orbit of the group generated by $f_1, f_2$ is dense. Note that $(1/3, 2/3)$ is a connected component of $S^1 - \mathcal{M}$. We can choose $f_1, f_2$ such that the extensions of $f_i$ are circle smooth diffeomorphism $F_i : S^1 \rightarrow S^1$ whose supports are $(1/3, 2/3)$. Consider the product foliation $\{\Sigma_4 \times \{x\} \mid x \in S^1\}$ and four disjoint loops $\gamma_f, \gamma_g, \gamma_1, \gamma_2$ in $\Sigma_4$ such that $\Sigma_4 - \sqcup_i=1,2 \gamma_i$ is a punctured disk. Taking holonomy maps $\text{id} \times F_i : \gamma_i \times S^1 \rightarrow \gamma_i \times S^1$ for a circle bundle over $\Sigma$, we obtain a codimension one foliation $F$ such that each leaf is exceptional or locally dense. Therefore $F$ is not pointwise almost periodic but recurrent. \qed

The following statement shows that recurrence does not correspond to non-wandering property.

Proposition 2.5. There is a smooth codimension one foliation on $\Sigma_3 \times S^1$ which is not recurrent but non-wandering.

Proof. Let $f, g_1, g_2$ be circle homeomorphisms such that each orbit of $f$ is non-compact proper except one fixed point and that the set of fixed points of $g_i$ is the union of the fixed point of $f$ and some non-compact proper orbit of $f$. Moreover we can require each regular orbits of the group generated by $g_1$ and $g_2$ is locally dense. As above, we can construct a codimension one foliation on $\Sigma_3 \times S^1$ which consists of one compact leaf, one non-compact proper leaf, and locally dense leaves. Hence $F$ is not recurrent but non-wandering. \qed

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Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan,

E-mail address: yokoyama@math.sci.hokudai.ac.jp