Order Determination of Large Dimensional Dynamic Factor Model

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Abstract

Consider the following dynamic factor model: 
\[ R_t = \sum_{i=0}^{q} \Lambda_i f_{t-i} + e_t, \quad t = 1, \ldots, T, \]
where \( \Lambda_i \) is an \( n \times k \) loading matrix of full rank, \( \{f_t\} \) are i.i.d. \( k \times 1 \)-factors, and \( e_t \) are independent \( n \times 1 \) white noises. Now, assuming that \( n/T \to c > 0 \), we want to estimate the orders \( k \) and \( q \) respectively. Define a random matrix
\[ \Phi_n(\tau) = \frac{1}{2T} \sum_{j=1}^{T} (R_j R_{j+\tau}^* + R_{j+\tau} R_j^*), \]
where \( \tau \geq 0 \) is an integer. When there are no factors, the matrix \( \Phi_n(\tau) \) reduces to
\[ M_n(\tau) = \frac{1}{2T} \sum_{j=1}^{T} (e_j e_{j+\tau}^* + e_{j+\tau} e_j^*). \]
When \( \tau = 0 \), \( M_n(\tau) \) reduces to the usual sample covariance matrix whose ESD tends to the well known MP law and \( \Phi_n(0) \) reduces to the standard spike model. Hence the number \( k(q+1) \) can be estimated by the number of spiked eigenvalues of \( \Phi_n(0) \). To obtain separate estimates of \( k \) and \( q \), we have employed the spectral analysis of \( M_n(\tau) \) and established the spiked model analysis for \( \Phi_n(\tau) \).
1 Introduction

For a \( p \times p \) random Hermitian matrix \( A \) with eigenvalues \( \lambda_j, j = 1, 2, \cdots, p \), the empirical spectral distribution (ESD) of \( A \) is defined as

\[
F^A(x) = \frac{1}{p} \sum_{j=1}^{p} I(\lambda_j \leq x).
\]

The limiting distribution \( F \) of \( \{F^{A_n}\} \) for a given sequence of random matrices \( \{A_n\} \) is called the limiting spectral distribution (LSD). Let \( \{\varepsilon_{it}\} \) be independent identically distributed (i.i.d) random variables with common mean 0, variance 1. Consider a high dimensional dynamic \( k \)-factor model with lag \( q \), that is, \( R_t = \sum_{i=0}^{q} \Lambda_i f_{t-i} + \epsilon_t, t = 1, ..., T \), where \( \Lambda_i \) is an \( n \times k \) loading matrix of full rank, \( \{f_t\} \) are i.i.d. \( k \times 1 \)-factors with common mean 0, variance 1, whereas \( \epsilon_t \) corresponds to the noise component with \( \epsilon_t = (\varepsilon_{1t}, \cdots, \varepsilon_{nt})' \). In addition, both components of \( \epsilon_t \) and \( f_t \) are assumed to have finite 4th moment.

This model can also be thought as an information-plus-noise type model (Dozier & Silverstein, 2007a, b; Bai & Silverstein, 2012). Here both \( n \) and \( T \) tend to \( \infty \), with \( n/T \to c \) for some \( c > 0 \). Compared with \( n \) and \( T \), the number of factors \( k \) and that of lags \( q \) are fixed but unknown. An interesting and important problem to economists is how to estimate \( k \) and \( q \). To this end, define \( \Phi_n(\tau) = \frac{1}{2T} \sum_{j=1}^{T} (R_j R_j^* + R_{j+\tau} R_{j+\tau}^*), \gamma_t = \frac{1}{\sqrt{2T}} \epsilon_t \) and \( M_n(\tau) = \sum_{k=1}^{T} (\gamma_k \gamma_k^* + \gamma_{k+\tau} \gamma_{k+\tau}^*), \tau = 0, 1, \cdots \). Here * stands for the transpose and complex conjugate of a complex number and \( \tau \) is referred to be the number of lags. Denote

\[
\Lambda = (\Lambda_0, \Lambda_1, \cdots, \Lambda_q)_{n \times k(q+1)},
\]

\[
F^{\tau} = \begin{pmatrix}
\epsilon_{T+\tau} & \epsilon_{T+\tau-1} & \cdots & \epsilon_{\tau+1} \\
\epsilon_{T+\tau-1} & \epsilon_{T+\tau-2} & \cdots & \epsilon_{\tau} \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon_{T+\tau-q} & \epsilon_{T+\tau-1-q} & \cdots & \epsilon_{\tau+1-q}
\end{pmatrix}_{k(q+1) \times T},
\]

\[
e^{\tau} = (\epsilon_{T+\tau}, \epsilon_{T+\tau-1}, \cdots, \epsilon_{\tau+1})_{n \times T}.
\]

Then we have that \( \Phi_n(\tau) = \frac{1}{2T} [(\Lambda F^{\tau} + e^{\tau})(\Lambda F^0 + e^0)^* + (\Lambda F^0 + e^0)(\Lambda F^{\tau} + e^{\tau})^*] \) and \( M_n(\tau) = \)
\( \frac{1}{2\pi} (e^r e^{0*} + e^0 e^{r*}) \).

Note that essentially, \( M_n(\tau) \) and \( \Phi_n(\tau) \) are symmetrized auto-cross covariance matrices at lag \( \tau \) and generalize the standard sample covariance matrices \( M_n(0) \) and \( \Phi_n(0) \), respectively. The matrix \( M_n(0) \) has been intensively studied in the literature and it is well known that the LSD has an MP law (Marčenko and Pastur, 1967). Readers may refer to Jin et al. (2014) and Wang et al. (2015) for more details about the model.

To estimate \( k \) and \( q \), the following method can be employed. First, note that when \( \tau = 0 \) and \( \text{Cov}(f_t) = \Sigma_f \), the population covariance matrix of \( R_t \) is a spiked population model (Johnstone (2001), Baik and Silverstein (2006), Bai and Yao (2008)) with \( k(q+1) \) spikes. Therefore, \( k(q+1) \) can be estimated by counting the number of eigenvalues of \( \Phi_n(0) \) that are larger than some phase transition point. Next, the separated estimation of \( k \) and \( q \) can be achieved by investigating the spectral property of \( M_n(\tau) \) for general \( \tau \geq 1 \), using the fact that the number of eigenvalues of \( \Phi_n(\tau) \) that lie outside the support of the LSD of \( M_n(\tau) \) at lags \( 1 \leq \tau \leq q \) is different from that at lags \( \tau > q \). Thus, the estimates of \( k \) and \( q \) can be separated by counting the number of eigenvalues of \( \Phi_n(\tau) \) that lie outside the support of the LSD of \( M_n(\tau) \) from \( \tau = 0, 1, 2, \ldots, q, q+1, \ldots \).

Note that for the above method to work, the LSD of \( M_n(\tau) \) for general \( \tau \geq 1 \) must be known. This is derived in Jin et al. (2014). Moreover, it is required that no eigenvalues outside the the support of the LSD of \( M_n(\tau) \) so that if an eigenvalue of \( \Phi_n(\tau) \) goes out of the support of the LSD of \( M_n(\tau) \), it must come from the signal part. Wang et al. (2015) proved such phenomenon theoretically. Both results are included in Section 2 for readers’ reference.

The rest of the paper is structured as follows: Some known results are given in Section 2. Section 3 presents truncation of variables and Section 4 estimates \( k(q+1) \). The estimation of \( q \) is provided in Section 5, from the which the estimation of \( k \) can also be obtained. Section 6 discusses the case when the variance of the noise part is unknown. A simulation study is shown in Section 7 and some proofs are presented in Appendix.

Regarding the norm used in this paper, the norm applied to a vector is the usual Euclidean norm, with notation \( \| \ast \| \). For a matrix, two kinds of norm have been used. The operator norm,
denoted by $\| \ast \|_\alpha$, is the largest singular value. For matrices of fixed dimension, the Kolmogorov norm, defined as the largest absolute value of all the entries, has been used, with notation $\| \ast \|_K$.

2 Some known results

In this section, we present some known results.

Lemma 2.1 (Burkholder (1973)). Let \{X_k\} be a complex martingale difference sequence with respect to the increasing $\sigma$-fields \{F_n\}. Then, for $p \geq 2$, we have

$$E|\sum X_k|^p \leq K_p \left( E\left( \sum E(|X_k|^2 |F_{k-1}) \right)^{p/2} + E\sum |X_k|^p \right).$$

Lemma 2.2 (Lemma A.1 of Bai and Silverstein (1998)). For $X = (X_1, \cdots, X_n)'$ i.i.d. standardized (complex) entries, $B$ $n \times n$ Hermitian nonnegative definite matrix, we have, for any $p \geq 1$,

$$E|X^*BX|^p \leq K_p \left( (\text{tr}B)^p + rE|X_1|^{2p}\text{tr}B^p \right),$$

where $K_p$ is a constant depending on $p$ only.

Lemma 2.3 (Jin et al. (2014)). Assume:

(a) $\tau \geq 1$ is a fixed integer.

(b) $e_k = (\varepsilon_{1k}, \cdots, \varepsilon_{nk})'$, $k = 1, 2, \ldots, T+\tau$, are $n$-dimensional vectors of independent standard complex components with $\sup_{1 \leq i \leq n, 1 \leq t \leq T+\tau} E|\varepsilon_{it}|^{2+\delta} \leq M < \infty$ for some $\delta \in (0, 2)$, and for any $\eta > 0$,\n
$$\frac{1}{\eta^{2+\delta}nT} \sum_{i=1}^{n} \sum_{t=1}^{T+\tau} E(|\varepsilon_{it}|^{2+\delta} I(|\varepsilon_{it}| \geq \eta T^{1/(2+\delta)}) = o(1). \quad (2.1)$$

(c) $n/(T + \tau) \rightarrow c > 0$ as $n, T \rightarrow \infty$.

(d) $M_n(\tau) = \sum_{k=1}^{T} (\gamma_k \gamma_k^* \tau + \gamma_{k+\tau} \gamma_k^*)$, where $\gamma_k = \frac{1}{\sqrt{2T}}e_k$. 

4
Then as $n, T \to \infty$, $F_{M_n(\tau)} \overset{D}{\to} F_c$ a.s. and $F_c$ has a density function given by

$$
\phi_c(x) = \frac{1}{2c\pi} \sqrt{\frac{y_0^2}{1+y_0} - \frac{(1-c|x| + \frac{1}{\sqrt{1+y_0}})^2}{1+y_0}}, \quad |x| \leq a,
$$

where

$$
a = \begin{cases} 
\frac{(1-c)\sqrt{1+y_0}}{y_1-1}, & c \neq 1, \\
2, & c = 1,
\end{cases}
$$

$y_0$ is the largest real root of the equation: $y^3 - \frac{(1-c)^2-x^2}{x^2}y^2 - \frac{4}{x^2}y - \frac{4}{x^2} = 0$ and $y_1$ is the only real root of the equation:

$$
((1-c)^2 - 1)y^3 + y^2 + y - 1 = 0 \tag{2.2}
$$

such that $y_1 > 1$ if $c < 1$ and $y_1 \in (0, 1)$ if $c > 1$. Further, if $c > 1$, then $F_c$ has a point mass $1 - 1/c$ at the origin. Note that as long as $\tau \geq 1$, $F_c$ does not depend on $\tau$.

**Lemma 2.4** (Bai and Wang (2015)). Theorem 2.3 still holds with the $2+\delta$ moment condition weakened to 2nd moment.

**Lemma 2.5** (Wang et al. (2015)). Assume:

(a) $\tau \geq 1$ is a fixed integer.

(b) $e_k = (\varepsilon_{1k}, \ldots, \varepsilon_{nk})'$, $k = 1, 2, \ldots, T + \tau$, are $n$-vectors of independent standard complex components with $\sup_{i,t} E|\varepsilon_{it}|^4 \leq M$ for some $M > 0$.

(c) There exist $K > 0$ and a random variable $X$ with finite fourth order moment such that, for any $x > 0$, for all $n, T$

$$
\frac{1}{nT} \sum_{i=1}^n \sum_{t=1}^{T+\tau} P(|\varepsilon_{it}| > x) \leq KP(|X| > x). \tag{2.3}
$$

(d) $c_n \equiv n/T \to c > 0$ as $n \to \infty$.

(e) $M_n = \sum_{k=1}^T (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*)$, where $\gamma_k = \frac{1}{\sqrt{2T}}e_k$.

(f) The interval $[a, b]$ lies outside the support of $F_c$, where $F_c$ is defined as in Lemma 2.3.

Then $P$ (no eigenvalues of $M_n$ appear in $[a, b]$ for all large $n$) = 1.
3 Truncation, centralization and standardization of variables

As proved in Wang et al.(2015), we may assume that the $\varepsilon_{ij}$’s satisfy the conditions that

$$|\varepsilon_{ij}| \leq C, \ E\varepsilon_{ij} = 0, \ E|\varepsilon_{ij}|^2 = 1, \ E|\varepsilon_{ij}|^4 < M$$

for some $C, M > 0$.

For the truncation of variables in $F^\tau$, first note that for a random variable $X$ with $E|X|^4 < \infty$, we have

$$\sum_{\ell=1}^{\infty} 2^\ell P(|X| > 2^\ell/4) < \infty.$$ Given $E|F^\tau_{ij}|^4 < \infty, i = 1, \ldots, k(q + 1), j = 1, \ldots, T$, define $\hat{F}_\tau^{ij}$ as

$$\hat{F}_\tau^{ij} = \begin{cases} F^\tau_{ij}, & |F^\tau_{ij}| < T^{1/4}, \\ 0, & \text{otherwise}, \end{cases} \hat{F}^\tau = (\hat{F}_\tau^{ij})$$

$$\hat{\Phi}_n(\tau) = \frac{1}{2T}[(\Lambda \hat{F}^\tau + e^\tau)(\Lambda \hat{F}^0 + e^0)^* + (\Lambda \hat{F}^0 + e^0)(\Lambda \hat{F}^\tau + e^\tau)^*].$$

Then we have

$$P\left( \Phi_n(\tau) \neq \hat{\Phi}_n(\tau), i.o. \right)$$

$$= P\left( F^\tau \neq \hat{F}^\tau, i.o. \right)$$

$$= P\left( \bigcap_{L=1}^{\infty} \bigcup_{T=L}^{\infty} \bigcup_{i \leq k(q + 1)} \bigcup_{j \leq T} \{ |F^\tau_{ij}| \geq T^{1/4} \} \right)$$

$$\leq \lim_{L \to \infty} \sum_{\ell=L}^{\infty} P\left( \bigcup_{T=2^{\ell+1}}^{2^{\ell+1}} \bigcup_{i \leq k(q + 1)} \bigcup_{j \leq 2^{\ell+1}} \{ |F^\tau_{ij}| \geq 2^{\ell/4} \} \right)$$

$$\leq \lim_{L \to \infty} \sum_{\ell=L}^{\infty} P\left( \bigcup_{i \leq k(q + 1)} \bigcup_{j \leq 2^{\ell+1}} \{ |F^\tau_{ij}| \geq 2^{\ell/4} \} \right)$$

$$\leq k(q + 1) \lim_{L \to \infty} \sum_{\ell=L}^{\infty} 2^{\ell+1} P( |F^\tau_{11}| \geq 2^{\ell/4} )$$

$$\to 0.$$
same way as in Appendix A of Wang et al. (2015). In what follows, we may assume that

\[ |F_{ij}^r| < T^{1/4}, \ E[|F_{ij}^r|^2] = 1, \ E[F_{ij}^r|^4] < M \]

for some $M > 0$.

4 Estimation of $k(q + 1)$

In this section, we will estimate $k(q + 1)$ by an investigation of the limiting properties of eigenvalues of $\Phi_n(0)$. For simplicity, rewrite $\Phi_n(0) = \Phi(0), F^0 = F$ and $e^0 = e$. With these notations, we have $\Phi(0) = \frac{1}{T}(\Lambda F + e)(\Lambda F + e)^* + M(0) = \frac{1}{T}ee^*$. When $\Lambda = 0$, $\Phi(0)$ reduces to $M(0)$, which is a standard sample covariance matrix and thus its ESD tends to the famous MP law (Marčenko and Pastur, 1967).

Suppose $\ell$ is an eigenvalue of $\Phi(0)$, then we have

\[ 0 = \det |\ell I - \Phi(0)| = \det \begin{vmatrix} \ell I - M(0) & -\frac{1}{T}\Lambda Fe^* - \frac{1}{T}eF^*\Lambda^* - \frac{1}{T}\Lambda FF^*\Lambda^* \end{vmatrix}. \]  \hspace{1cm} (4.1)

Let $B = (B_1 : B_2)$ be an $n \times n$ orthogonal matrix such that $B_1 = \Lambda(\Lambda^*\Lambda)^{-1/2}$ and thus $\Lambda^*B_2 = 0_{k(q+1)\times(n-k(q+1))}$. Then (4.1) is equivalent to

\[ \begin{vmatrix} \ell I_{k(q+1)} & -\frac{1}{T}B_1^*(\Lambda F + e)(F^*\Lambda^* + e^*)B_1 & -\frac{1}{T}B_1^*(\Lambda F + e)e^*B_2 \\
-\frac{1}{T}B_2^*e(F^*\Lambda^* + e^*)B_1 & \ell I_{n-k(q+1)} - \frac{1}{T}B_2^*ee^*B_2 \end{vmatrix} = 0 \]  \hspace{1cm} (4.2)

If we further assume that $\ell$ is not an eigenvalue of $\frac{1}{T}B_2^*ee^*B_2$, then we have

\[ \det |I_{k(q+1)} - \frac{1}{T}B_1^*(\Lambda F + e)D^{-1}(\ell)(F^*\Lambda^* + e^*)B_1| = 0, \]  \hspace{1cm} (4.3)

where $D(\ell) = \ell I_T - \frac{1}{T}e^*B_2B_2^*e$. Denote $H(\ell) = \ell I_T - \frac{1}{T}e^*e$, then we obtain

\[ \frac{1}{T}B_1^*eD^{-1}(\ell)e^*B_1 = \left( I + \frac{1}{T}B_1^*eH^{-1}(\ell)e^*B_1 \right)^{-1} - \frac{1}{T}B_1^*eH^{-1}(\ell)e^*B_1. \]  \hspace{1cm} (4.4)

Next, we have

\[ \frac{1}{T}B_1^*eH^{-1}(\ell)e^*B_1 = \frac{1}{T}B_1^*\left( \frac{1}{T}B_1^*\left( \ell I + T(\ell I_n - M(0))^{-1} \right)B_1 \right) \]

\[ = -\ell B_1^* + \ell B_1^* (I_n - M(0))^{-1}B_1. \]  \hspace{1cm} (4.5)
Substitute (4.5) back to (4.4), and we have
\[
\frac{1}{T}B_1^*eD^{-1}(\ell)e^*B_1 = I_{k(q+1)} - \left(\ell B_1^*(I_n - M(0))^{-1}B_1\right)^{-1} = I_{k(q+1)} + \frac{1}{\ell} \left(B_1^*(M(0) - I_n)^{-1}B_1\right)^{-1}
\]
Write \(B_1 = (b_1, \ldots, b_{k(q+1)})\), then we have \(\|b_i\| = 1\). By Lemma 6 in Bai, Liu and Wong (2011), we have
\[
b_i^*(M(0) - \ell I_n)^{-1}b_i \to m, \quad a.s.
\]
and for \(i \neq j\)
\[
b_i^*(M(0) - \ell I_n)^{-1}b_j \to 0, \quad a.s.,
\]
where \(m = m(\ell) = \lim_{n \to \infty} \frac{1}{n} \text{tr}(M(0) - \ell I_n)^{-1}\) is the Stieltjes transform of the sample covariance with ratio index \(c = \lim_{n \to \infty} \frac{n}{T}\).

By Lemma 3.11 of Bai and Silverstein (2010), we have \(m\) satisfying
\[
m(\ell) = \frac{1 - \ell - \sqrt{(1 - \ell - c)^2 - 4\ell c}}{2c\ell}. \tag{4.6}
\]
Therefore, we obtain
\[
\frac{1}{T}B_1^*eD^{-1}(\ell)e^*B_1 \to \left(1 + \frac{1}{\ell m}\right)I_{k(q+1)}.
\]
Next, we want to show, with probability 1 that
\[
\frac{1}{T}B_1^*\Lambda F D^{-1}(\ell)e^*B_1 \to 0
\]
and
\[
\frac{1}{T}B_1^*eD^{-1}(\ell)F^*\Lambda^*B_1 \to 0.
\]
Note that
\[
\frac{1}{T}B_1^*\Lambda F D^{-1}(\ell)e^*B_1
\]
\[
= \frac{1}{T}(I + \frac{1}{T}B_1^*eH^{-1}(\ell)e^*B_1)^{-1}B_1^*\Lambda F H^{-1}(\ell)e^*B_1
\]
\[
= \frac{1}{\ell T} \left(B_1^*(M(0) - \ell I_n)^{-1}B_1\right)^{-1}B_1^*\Lambda F H^{-1}(\ell)e^*B_1
\]
\[
= \frac{1}{\ell T} \left(B_1^*(M(0) - \ell I_n)^{-1}B_1\right)^{-1}B_1^*\Lambda F (\ell I_T - \frac{1}{T}e^*e)^{-1}e^*B_1.
\]
Recall $M(0) = \frac{1}{T}ee^\ast$. Fix $\delta > 0$ and let event $\mathcal{A} = \{\lambda_{\text{max}}(M(0)) \leq (1 + \sqrt{c})^2 + \delta\}$ and $\mathcal{A}^c$ be the complement. By Theorem 5.9 of Bai and Silverstein (2010), we have $P(\mathcal{A}^c) = o(n^{-t})$ for any $t > 0$.

Suppose $\ell$ is an eigenvalue of $\Phi(0)$ larger than $(1 + \sqrt{c})^2 + 2\delta$. By the fact that $M(0)$ and $\frac{1}{T}ee^\ast$ have the same set of nonzero eigenvalues, we have, under $\mathcal{A}$, that $\|I_T - \frac{1}{T}ee^\ast\|_o \geq \ell - \|\frac{1}{T}ee^\ast\|_o \geq \delta > 0$, and hence $\| (I_T - \frac{1}{T}ee^\ast)^{-1} \|_o \leq \frac{1}{\delta}$.

Therefore, for any $\varepsilon > 0$, we have

$$P(\left\| I_T - \frac{1}{T}ee^\ast \right\|_K \geq \varepsilon) = E\left( P(\left\| I_T - \frac{1}{T}ee^\ast \right\|_K \geq \varepsilon) \mid e, A \right) \leq E \left( P(\left\| I_T - \frac{1}{T}ee^\ast \right\|_K \geq \varepsilon) \mid e, A \right).$$

Write $F = (\tilde{F}_1, \cdots, \tilde{F}_{k(q+1)})'$. For the first term, by Lemma 2.2 we have

$$\begin{align*}
E \left( P(\left\| I_T - \frac{1}{T}ee^\ast \right\|_K \geq \varepsilon) \mid e, A \right) & \leq \frac{1}{\varepsilon^{4r} T^{4r}} E \left( \left( \sum_{i=1}^{k(q+1)} E \left( \tilde{F}_i (I_T - \frac{1}{T}ee^\ast)^{-1} e^\ast B_i e \left( I_T - \frac{1}{T}ee^\ast \right)^{-1} \tilde{F}_i \right)^{2r} \mid e, A \right) \right) \\
& \leq \frac{1}{\varepsilon^{4r} T^{4r}} E \left( \left( \sum_{i=1}^{k(q+1)} \left( \left[ \text{tr} \left( I_T - \frac{1}{T}ee^\ast \right)^{-1} e^\ast B_i e \left( I_T - \frac{1}{T}ee^\ast \right)^{-1} \tilde{F}_i \right]^{2r} \right) \right) \mid e, A \right) \\
& \leq \frac{1}{\varepsilon^{4r} T^{4r}} E \left( \left( \sum_{i=1}^{k(q+1)} \left( \left[ \text{tr} \left( I_T - \frac{1}{T}ee^\ast \right)^{-1} e^\ast B_i e \left( I_T - \frac{1}{T}ee^\ast \right)^{-1} \tilde{F}_i \right]^{2r} \right) \right) \mid e, A \right). 
\end{align*}$$
which is summable for $r \geq 1$.

Hence, we have shown with probability 1 that

$$\frac{1}{T}B_1^*A F D^{-1}(\ell)e^*B_1 \to 0.$$ 

Similarly, we have with probability 1 that

$$\frac{1}{T}B_1^*e D^{-1}(\ell)F^*A^*B_1 \to 0.$$ 

Therefore, substituting into (4.3), we have

$$\det \left| \frac{1}{T}B_1^*AF D^{-1}(\ell)F^*A^*B_1 + \frac{1}{\ell m(\ell)} I_{k(q+1)} \right| \to 0. \quad (4.7)$$

Using Bai, Liu and Wong (2011) again, we have the diagonal elements of the matrix $T^{-1}F D^{-1}(\ell)F^*$ tend to $-m(\ell)$ and the off diagonal elements tend to 0. Here $m(\ell)$ is the Stieltjes transform of the LSD of $\frac{1}{T}e^*e$ and satisfies

$$m(\ell) = -\frac{1-c}{\ell} + cm(\ell).$$

Thus, if $A^*A \to Q$, then (4.7) can be further simplified as

$$\det \left| -Q m(\ell) + \frac{1}{\ell m(\ell)} I_{k(q+1)} \right| = 0. \quad (4.8)$$

If $\alpha$ is an eigenvalue of $Q$, and there is an $\ell$ belonging to the complement of the support of the LSD of $M(0)$ such that $\alpha = \frac{1}{\ell m(\ell)m(\ell)}$, then $\ell$ is a solution of (4.8).

From (4.6), we have

$$c\ell m^2(\ell) - (1-c-\ell)m(\ell) + 1 = 0,$$

which implies

$$\ell m(\ell)m(\ell) = \ell m(\ell)(-\frac{1-c}{\ell} + cm(\ell)) = -(1-c)m(\ell) + c\ell m^2(\ell) = -(1-c)m(\ell) + (1-c-\ell)m(\ell) - 1 = -\ell m(\ell) - 1 = \frac{1-c-\ell + \sqrt{(1-\ell-c)^2 - 4\ell c}}{2c} - 1 =: g(\ell).$$
It is easy to verify that \( g'(\ell) < 0 \), implying that \( \ell m(\ell) m'(\ell) \) is decreasing. Also note that 
\[
\ell m(\ell) m'(\ell) = \frac{1}{\sqrt{c}} \quad \text{when} \quad \ell = (1 + \sqrt{c})^2.
\]
Therefore, if \( \alpha = \frac{1}{\ell m(\ell) m'(\ell)} > \sqrt{c} \), then we have \( \ell > (1 + \sqrt{c})^2 \). This recovers the result of Baik and Silverstein (2006). Note that \([((1-\sqrt{c})^2, (1+\sqrt{c})^2]\) is the support of the MP law. Hence, if all the eigenvalues of \( Q \) are greater than \( \sqrt{c} \), we have \( k(q+1) \) sample eigenvalues of \( \Phi_n(0) \) goes outside the right boundary of the support of the MP law. Note that although the distribution of rest \( n - k(q+1) \) sample eigenvalues follows the MP law with the largest sample eigenvalue converging to the right boundary, there is still a positive probability that the largest sample eigenvalue converging to the right boundary, there is still a positive probability that the largest sample eigenvalue goes beyond the right boundary. Therefore, to completely separate the \( k(q+1) \) spiked sample eigenvalues from the rest, the threshold is set as \((1 + \sqrt{c})^2(1 + 2n^{-2/3})\). In other words, \( k(q+1) \) can be estimated by the number of sample eigenvalues of \( \Phi_n(0) \) greater than \((1 + \sqrt{c})^2(1 + 2n^{-2/3})\).

Remark 4.1 For factor models, the loading matrix is unknown. This, however, is not a concern in our estimation because compared with the noise matrix, the loading matrix is denominating, making the condition easily satisfied that all the eigenvalues of \( Q \) are greater than \( \sqrt{c} \).

Remark 4.2 The rationale of choosing \((1 + 2n^{-2/3})\) as the buffering factor of the criterion is that, according to Tracy-Widom law, the quantity of a non-spiked eigenvalue larger than \((1 + \sqrt{c})^2\) has an order of \( n^{-2/3} \). Therefore, it is good enough for us to choose \((1 + 2n^{-2/3})\) to completely separate the spikes and the bulk eigenvalues.

5 Estimation of \( q \)

Next, we want to split \( k \) and \( q \). Let \( \tau \geq 1 \) be given and assume that \( \ell \) is an eigenvalue of \( \Phi_n(\tau) \). For simplicity, write \( M_n(\tau) = M \) and for \( t = 1, 2, \cdots, T \), define \( F_t = (f_t, f_{t-1}, \cdots, f_{t-q})' \) such
that \( R_t = \Lambda F_t + e_t \). Then we have

\[
0 = \det |\ell I - \Phi_\ell(\tau)| \\
= \det |\ell I - M - \frac{1}{2T} \sum_{j=1}^{T} \left( \Lambda F_j F_{j+\tau}^* \Lambda^* + \Lambda F_{j+\tau} F_j^* \Lambda^*ight. \\
\left. + e_j F_{j+\tau} \Lambda^* + \Lambda F_{j+\tau}^* e_j^* + e_{j+\tau}^* F_j^* \Lambda^* + \Lambda F_{j+\tau}^* e_j^* \right)|.
\]

(5.1)

Define \( B, B_1 \) and \( B_2 \) the same as in the last section. Multiplying \( B^* \) from left and \( B \) from right to the above matrix and by \( \Lambda^* B_2 \), we have (5.1) equivalent to

\[
0 = \det |\ell I - S_{11} - S_{12} \ell - S_{21} - S_{22}| = \det |\ell I - S_{22}| \det |\ell I - K_n(\ell)|,
\]

where

\[
S_{11} = \frac{1}{2T} \sum_{j=1}^{T} B_j^*[\Lambda F_j^* e_j^* + (\Lambda F_j + e_j^*) \Lambda^* + \Lambda F_j + e_j^*]B_1
\]

\[
S_{12} = \frac{1}{2T} \sum_{j=1}^{T} B_j^*[\Lambda F_j e_{j+\tau}^* + \Lambda F_{j+\tau} e_j^*]B_2 + B_1^* M B_2
\]

\[
S_{21} = S_{12}^*
\]

\[
S_{22} = B_2^* M B_2
\]

\[
K_n(\ell) = S_{11} + S_{12} (\ell I_{n-k(q+1)} - S_{22})^{-1} S_{21}.
\]

Therefore, if \( \ell \) is not an eigenvalue of \( S_{22} \), by the factorization above, \( \ell \) must be an eigenvalue of \( K_n(\ell) \), i.e. \( \det |K_n(\ell) - \ell I| = 0 \).

Denote \( W = \frac{1}{2T} \sum_{j=1}^{T} (F_j e_{j+\tau}^* + F_{j+\tau} e_j^*) \). By the assumptions of \( e_j \)'s and \( F_j \)'s, the random vector \( \{F_j e_{j+\tau}^* + F_{j+\tau} e_j^*, j \geq 1\} \) is \((q + 1)\)-dependent (see Page 224, Chung 2001). It then follows with probability 1 that

\[
W B_1 = o(1)
\]

\[
B_1^* W^* = o(1)
\]

\[
\frac{1}{2T} \sum_{j=1}^{T} (F_j F_{j+\tau}^* + F_{j+\tau} F_j^*) = H(\tau) + o(1),
\]

12
where

$$H(\tau) = \begin{pmatrix}
0 & \ldots & 1 & \ldots & 0 \\
\vdots & \ddots & 0 & 1 & \vdots \\
1 & 0 & \ddots & 0 & 1 \\
\vdots & 1 & 0 & \ddots & \vdots \\
0 & \ldots & 1 & \ldots & 0
\end{pmatrix},$$

is of dimension $k(q + 1) \times k(q + 1)$ with two bands of 1’s of $k\tau$-distance from the main diagonal. Therefore, we have a.s.

$$S_{11} = B_1^* \Lambda H(\tau) \Lambda^* B_1 + B_1^* M B_1 + o(1)$$
$$S_{12} = B_1^* (M + \Lambda W) B_2$$
$$S_{21} = B_2^* (M + W^* \Lambda^*) B_1.$$

Subsequently, we have a.s.

$$K_n(\ell) = B_1^* \Lambda H(\tau) \Lambda^* B_1 + B_1^* M B_1 + B_1^* (M + \Lambda W) B_2 \left( I - B_2^* M B_2 \right)^{-1} B_2^* (M + W^* \Lambda^*) B_1 + o(1).$$

Note that

$$B_1^* M B_1 + B_1^* M B_2 \left( I - B_2^* M B_2 \right)^{-1} B_2^* M B_1$$
$$= B_1^* M B_1 + B_1^* M B_2 \frac{1}{\ell} B_2 B_2^* M \left( I - \frac{1}{\ell} B_2 B_2^* M \right)^{-1} B_1$$
$$= B_1^* M \left( I - \frac{1}{\ell} B_2 B_2^* M \right)^{-1} B_1$$
$$= \ell B_1^* M (\ell I - M + B_1 B_1^* M)^{-1} B_1$$
$$= \ell I - \ell \left( I + B_1^* M (\ell I - M)^{-1} B_1 \right)^{-1}$$
$$= \ell I - \left( B_1^* (\ell I - M)^{-1} B_1 \right)^{-1}.$$
and

\[
WB_2 \left( (I - B_2^*MB_2) \right)^{-1} B_2^*W^* \\
= \frac{1}{2T} \sum_{i,j=1}^{T} \left( F_i \gamma_i^* + F_{i+i} \gamma_i^* \right) B_2 \left( (I - B_2^*MB_2) \right)^{-1} B_2^* (\gamma_{i+i} F_j^* + \gamma_j F_{j+i}^*) \\
= \frac{1}{2T} \sum_{j=1}^{T} \left[ F_j \gamma_{j+i} B_2 \left( (I - B_2^*MB_2) \right)^{-1} B_2^* \gamma_{j+i} F_j^* + F_{j+i} \gamma_j B_2 \left( (I - B_2^*MB_2) \right)^{-1} B_2^* \gamma_j F_{j+i}^* \right] + \\
\frac{1}{2T} \sum_{j=1}^{T} \left[ F_j \gamma_{j+2i} B_2 \left( (I - B_2^*MB_2) \right)^{-1} B_2^* \gamma_{j+2i} F_j^* + F_{j+2i} \gamma_j B_2 \left( (I - B_2^*MB_2) \right)^{-1} B_2^* \gamma_j F_{j+2i}^* \right] + \\
\frac{1}{2T} \sum_{j=1}^{T} \left( F_{j-i} \gamma_{j-i}^* + F_{j-i} \gamma_{j+i}^* \right) B_2 \left( (I - B_2^*MB_2) \right)^{-1} B_2^* (\gamma_{j-i} F_j^* + \gamma_j F_{j-i}^*) + \\
\frac{1}{2T} \sum_{i,j=1, i\neq j, j \neq \pm 1}^{T} \left( F_i \gamma_i^* + F_{i+i} \gamma_i^* \right) B_2 \left( (I - B_2^*MB_2) \right)^{-1} B_2^* (\gamma_i F_j^* + \gamma_j F_i^*) \\
=: \ P_1 + P_2 + P_3 + P_4 + P_5.
\]

Next, we give a lemma on the quadratic form of \( \gamma_j \).

**Lemma 5.1** Let \( i, j \in \mathbb{N} \) be given, we have almost surely and uniformly in \( i \) and \( j \) that

\[
\gamma_i^* B_2 \left( (I - B_2^*MB_2) \right)^{-1} B_2^* \gamma_j \to \begin{cases} 
\frac{cm}{1 - \frac{c}{4m}} \left( \frac{cm}{1 - \frac{c}{4m}} \right)^p & i = j \pm p \tau \\
0, & \text{otherwise.}
\end{cases}
\]

The proof of the lemma is postponed in the Appendix.
First, we have

\[
E(P_1) = E[E(P_1 | \gamma_1, \cdots, \gamma_{T+\tau})]
\]
\[
= \frac{1}{2T} \sum_{j=1}^{T} \mathbb{E}\left\{ \left[ F_j \gamma_j^* B_2 \left( \ell I - B_2^2 MB_2 \right)^{-1} B_2^* \gamma_{j+\tau} F_j^* + F_{j+\tau} \right] \mathbb{E} \left[ \gamma_1, \cdots, \gamma_{T+\tau} \right] \right\}
\]
\[
= \frac{1}{2T} \text{Etr} \left[ \left( \ell I - B_2^2 MB_2 \right)^{-1} B_2^* \sum_{j=1}^{T} (\gamma_j^* \gamma_j + \gamma_j^* \gamma_{j+\tau} + \gamma_{j+\tau}^* \gamma_j) B_2 \right] I_{k(q+1)}
\]
\[
= \frac{1}{2T} \sum_{j=1}^{T} \mathbb{E}\gamma_j^* B_2 \left( \ell I - B_2^2 MB_2 \right)^{-1} B_2^* \gamma_{j+\tau} I_{k(q+1)}
\]
\[
= C_0 I_{k(q+1)} + o(1).
\]

Similarly, we have

\[
E(P_2) = \frac{1}{2} C_1 H(\tau) + o(1),
\]
\[
E(P_3) = C_1 H_L(\tau) + \frac{1}{2} (C_0 H_L(2\tau) + C_2 I_{k(q+1)}) + o(1),
\]
\[
E(P_4) = C_1 H_U(\tau) + \frac{1}{2} (C_0 H_U(2\tau) + C_2 I_{k(q+1)}) + o(1).
\]

Here \( H_L(\tau) \) and \( H_U(\tau) \) denote the lower and upper part of \( H(\tau) \) with the rest entries being 0.

Furthermore, denote \( H_L(0) \equiv H_U(0) \equiv I_{k(q+1)} \) and hence \( H(0) = H_L(0) + H_U(0) = 2I_{k(q+1)} \).

Consider \( i = j \pm p\tau \) for \( p = 0, 1, \cdots, \left[ \frac{T}{\tau} \right] \).

When \( p = 0 \), we have \( E(P_1) + E(P_2) = \frac{1}{2} C_0 H(0) + \frac{1}{2} C_1 H(\tau) + o(1) \).

When \( p = 1 \), we have \( E(P_3) + E(P_4) = \frac{1}{2} C_2 H(0) + C_1 H(\tau) + \frac{1}{2} C_0 H(2\tau) + o(1) \).

When \( p = 2 \), we have part of \( E(P_5) \) is \( \frac{1}{2} C_3 H(\tau) + C_2 H(2\tau) + \frac{1}{2} C_1 H(3\tau) + o(1) \).

\( \vdots \)
When \( p = \left[ \frac{q}{\tau} \right] \), we have part of \( E(P5) \) is
\[
\frac{1}{2} C_{\left[ \frac{q}{\tau} \right]} + 1 H\left( \left[ \frac{q}{\tau} \right] \tau - \tau \right) + C_{\left[ \frac{q}{\tau} \right]} + \frac{1}{2} C_{\left[ \frac{q}{\tau} \right] - 1} H\left( \left[ \frac{q}{\tau} \right] \tau + \tau \right) + o(1)
\]
\[
= \frac{1}{2} C_{\left[ \frac{q}{\tau} \right]} + 1 H\left( \left[ \frac{q}{\tau} \right] \tau - \tau \right) + C_{\left[ \frac{q}{\tau} \right]} + o(1).
\]

When \( p = \left[ \frac{q}{\tau} \right] + 1 \), we have part of \( E(P5) \) is
\[
\frac{1}{2} C_{\left[ \frac{q}{\tau} \right]} + 1 H\left( \left[ \frac{q}{\tau} \right] \tau + 2 \tau \right) + C_{\left[ \frac{q}{\tau} \right]} + 1 H\left( \left[ \frac{q}{\tau} \right] \tau + \tau \right) + \frac{1}{2} C_{\left[ \frac{q}{\tau} \right] + 2} H\left( \left[ \frac{q}{\tau} \right] \tau \right) + o(1)
\]
\[
= \frac{1}{2} C_{\left[ \frac{q}{\tau} \right]} + 2 H\left( \left[ \frac{q}{\tau} \right] \tau \right) + o(1).
\]

Next, we want to show that \( P_i \to E(P_i) \) a.s. Since all the \( P_i \)'s are of finite dimension, it suffices to show the a.s convergence entry-wise. Denote the \((u, v)\)-entry of a matrix \( A \) by \( A_{(u, v)} \). For \( i = 1 \), define \( \alpha_j = \gamma_j^* B_2 \left( (I - B_2^* MB_2)^{-1} \right) B_2^* \gamma_j \). Then for any positive integer \( s \), applying Lemma [2.1], we have
\[
E(\|P_{1(i1,i2)}\| - \frac{\alpha_j}{2} \delta_{(i1,i2)} - \frac{\alpha_{j+\tau}}{2} \delta_{(i1,i2)})^{2s}
\]
\[
= E\left\{ \frac{1}{2T} \sum_{j=1}^{T} \left[ \alpha_{j+\tau} (F_j F_j^*)_{(i1,i2)} + \alpha_j (F_{j+\tau} F_{j+\tau}^*)_{(i1,i2)} \right] - \frac{\alpha_j}{2} \delta_{(i1,i2)} - \frac{\alpha_{j+\tau}}{2} \delta_{(i1,i2)} \right\}^{2s}
\]
\[
\leq 2^{2s-1} E\left\{ \frac{1}{2T} \sum_{j=1}^{T} \alpha_{j+\tau} (F_j F_j^*)_{(i1,i2)} - \frac{\alpha_{j+\tau}}{2} \delta_{(i1,i2)} \right\}^{2s} + 2^{2s-1} E\left\{ \frac{1}{2T} \sum_{j=1}^{T} \alpha_j (F_{j+\tau} F_{j+\tau}^*)_{(i1,i2)} - \frac{\alpha_j}{2} \delta_{(i1,i2)} \right\}^{2s}
\]
\[
= \frac{1}{2} E\left\{ \frac{1}{T} \sum_{j=1}^{T} \alpha_{j+\tau} (F_j F_j^*)_{(i1,i2)} - \alpha_j \delta_{(i1,i2)} \right\}^{2s} + \frac{1}{2} E\left\{ \frac{1}{T} \sum_{j=1}^{T} \alpha_j (F_j F_j)_{(i1,i2)} - \alpha_j \delta_{(i1,i2)} \right\}^{2s}
\]
\[
= \frac{1}{2} E\left\{ \frac{1}{T} \sum_{j=1}^{T} F_j^* \left[ \alpha_{j+\tau}^{(i1,i2)} \right] F_j - \text{tr} \left[ \alpha_{j+\tau}^{(i1,i2)} \right] \right\}^{2s} + \frac{1}{2} E\left\{ \frac{1}{T} \sum_{j=1}^{T} F_{j+\tau}^* \left[ \alpha_j^{(i1,i2)} \right] F_{j+\tau} - \text{tr} \left[ \alpha_j^{(i1,i2)} \right] \right\}^{2s}.
\]
Here \([a^{(u,v)}]\) denotes the matrix with the \((u,v)\)-entry being \(a\) and 0 elsewhere. By the truncation of \(\varepsilon_{ij}\) and the fact that \(||(\ell I - B^*MB_2)^{-1}||_o \leq \eta^{-1}\) with \(\eta = \ell - \delta > 0\), both \(|\alpha_j|\) and \(|\alpha_{j+\tau}|\) are bounded from above, say, by \(C\). Also notice that \(|F^\tau_{ij}| < T^{1/4}\) and \(E|F^\tau_{ij}|^4 < M\). Similar to the proof of Lemma 9.1 in Bai and Silverstein (2010), we have

\[
\mathbb{E}\left\{ \frac{1}{T} \sum_{j=1}^{T} F^s_j \left[ \alpha\left(\alpha_j \right) F_j - \text{tr}\left[ \alpha\left(\alpha_j \right) F_j \right] \right] \right\}^{2s} \leq \frac{C^{2s}}{T^s} \sum_{l=1}^{s} (M^l l^{2s} + l^{4s})
\]

\[
\mathbb{E}\left\{ \frac{1}{T} \sum_{j=1}^{T} F^s_{j+\tau} \left[ \alpha\left(\alpha_j \right) F_{j+\tau} - \text{tr}\left[ \alpha\left(\alpha_j \right) F_{j+\tau} \right] \right] \right\}^{2s} \leq \frac{C^{2s}}{T^s} \sum_{l=1}^{s} (M^l l^{2s} + l^{4s}).
\]

Substituting the above back to (5.2) and choosing \(s \geq 2\), we have

\[
P_{1(i_1,i_2)} - \frac{\alpha_j}{2} \delta_{(i_1,i_2)} - \frac{\alpha_{j+\tau}}{2} \delta_{(i_1,i_2)} = o_{a.s.}(1).
\]

Again, by the almost sure and uniform convergence of \(\alpha_j\) and \(\alpha_{j+\tau}\) to \(C_0\), we have

\[
\frac{\alpha_j}{2} \delta_{(i_1,i_2)} + \frac{\alpha_{j+\tau}}{2} \delta_{(i_1,i_2)} - C_0 \delta_{(i_1,i_2)} = o_{a.s.}(1).
\]

Therefore, we have shown that \(P_1 - \mathbb{E}(P_1) = o_{a.s.}(1)\). Results for \(i = 2, 3, 4, 5\) can be shown in a similar way.

Denote \(\alpha = \frac{-c_{\beta}}{1 - c^2_{\beta} x^2_{\beta}}\) and \(\beta = \frac{-c_{\beta}}{1 - c^2_{\beta} x^2_{\beta}}\), then we have \(C_\beta = \alpha \beta^p\). Note that \(H(p\tau) = 0\) for
$p > [q/\tau]$, and we have, with probability 1 that

$$
WB_2\left(\beta I - B_2^*MB_2\right)^{-1}B_2^*W^*
\rightarrow \left(\frac{1}{2}C_0 + \frac{1}{2}C_2\right)H(0) + \left(\frac{3}{2}C_1 + \frac{1}{2}C_3\right)H(\tau) + \sum_{p=2}^{\infty} \left(\frac{1}{2}C_{p-2} + C_p + \frac{1}{2}C_{p+2}\right)H(p\tau)
= \frac{\alpha}{2}\left[(1 + \beta^2)H(0) + (3\beta + \beta^3)H(\tau) + (1 + \beta^2)^2 \sum_{p=2}^{\infty} \beta^{p-2}H(p\tau)\right]
= \frac{\alpha}{2}\left[(1 + \beta^2)H(0) + (3\beta + \beta^3)H(\tau) + (1 + \beta^2)^2 \sum_{p=2}^{\infty} \beta^{p-2}(H_L(p\tau) + H_U(p\tau))\right]
= \frac{\alpha}{2}\left[(1 + \beta^2)H(0) + (3\beta + \beta^3)H(\tau) + (1 + \beta^2)^2 \sum_{p=2}^{\infty} \beta^{p-2}(J_L^p(p\tau) + J_U^p(p\tau)) \otimes I_k\right]
= \frac{\alpha}{2}\left[(1 + \beta^2)H(0) + (3\beta + \beta^3)H(\tau) + (1 + \beta^2)^2 \left(J_L^2(\tau)(I - \beta J_L(\tau))^{-1} + J_U^2(\tau)(I - \beta J_U(\tau))^{-1}\right) \otimes I_k\right].
$$

Note that

$$
J_L(\tau)(I - \beta J_L(\tau))^{-1} = \frac{1}{\beta}[I - (I - \beta J_L(\tau))(I - \beta J_L(\tau))^{-1}]
= \frac{1}{\beta}[(I - \beta J_L(\tau))^{-1} - I]
$$

$$
J_L^2(\tau)(I - \beta J_L(\tau))^{-1} = \frac{1}{\beta^2}J_L(\tau)((I - \beta J_L(\tau))^{-1} - I]
= \frac{1}{\beta^2}[(I - \beta J_L(\tau))^{-1} - I] - \frac{1}{\beta}J_L(\tau).
$$

Similarly,

$$
J_U^2(\tau)(I - \beta J_U(\tau))^{-1} = \frac{1}{\beta}J_U(\tau)((I - \beta J_U(\tau))^{-1} - I]
= \frac{1}{\beta^2}[(I - \beta J_U(\tau))^{-1} - I] - \frac{1}{\beta}J_U(\tau).
$$
Therefore, we have

\[
(1 + \beta^2)^2 \left( J_L^2(\tau)(I - \beta J_L(\tau))^{-1} + J_U^2(\tau)(I - \beta J_U(\tau))^{-1} \right) \otimes I_k
\]

\[
= (1 + \beta^2)^2 \left( \frac{1}{\beta^2}(I - \beta J_L(\tau))^{-1} + \frac{1}{\beta^2}(I - \beta J_U(\tau))^{-1} \right.
\]

\[
- \frac{2}{\beta^2} I - \frac{1}{\beta}(J_L(\tau) + J_U(\tau)) \left) \otimes I_k
\]

\[
= \frac{(1 + \beta^2)^2}{\beta^2} \left[ (I - \beta J_L(\tau))^{-1} \left( 2I - \beta J_L(\tau) - \beta J_U(\tau) \right) \right]
\]

\[
- \frac{(1 + \beta^2)^2}{\beta^2} H(0) - \frac{(1 + \beta^2)^2}{\beta} H(\tau)
\]

\[
=: \frac{(1 + \beta^2)^2}{\beta^2} G(\tau) - \frac{(1 + \beta^2)^2}{\beta^2} H(0) - \frac{(1 + \beta^2)^2}{\beta} H(\tau).
\]

Hence, we have a.s.

\[
B^\ast_1 AWB_2 \left( I - B^\ast_2 MB_2 \right)^{-1} B^\ast_2 W^* A^\ast B_1
\]

\[
\rightarrow \frac{\alpha}{2} Q^{1/2} \left[ \left( 1 + \beta^2 - \frac{(1 + \beta^2)^2}{\beta^2} \right) H(0) + \left( 3\beta + \beta^3 - \frac{(1 + \beta^2)^2}{\beta} \right) H(\tau) \right.
\]

\[
+ \frac{(1 + \beta^2)^2}{\beta^2} G(\tau) \right] Q^{1/2}.
\]

Last, we want to show that with probability 1,

\[
B^\ast_1 AWB_2 \left( I - B^\ast_2 MB_2 \right)^{-1} B^\ast_2 MB_1 \rightarrow 0
\]

and

\[
B^\ast_1 MB_2 \left( I - B^\ast_2 MB_2 \right)^{-1} B^\ast_2 W^* A^\ast B_1 \rightarrow 0.
\]

Note that

\[
B^\ast_1 AWB_2 \left( I - B^\ast_2 MB_2 \right)^{-1} B^\ast_2 MB_1 = B^\ast_1 AW \left( I - \frac{1}{\ell} B_2 B_2^\ast M \right)^{-1} B_1 - B^\ast_1 AW B_1
\]

and that

\[
B^\ast_1 MB_2 \left( I - B^\ast_2 MB_2 \right)^{-1} B^\ast_2 W B_1 = B^\ast_1 \left( I - \frac{1}{\ell} MB_2 B_2^\ast \right)^{-1} W^* A^\ast B_1 - B^\ast_1 W^* A^\ast B_1.
\]
Hence, by $WB_1 = o_{a.s.}(1)$ and $B_1^*W^* = o_{a.s.}(1)$, it suffices to show with probability 1 that,

$$B_1^*W\left(I - \frac{1}{\ell}B_2B_2^*M\right)^{-1}B_1 \to 0$$

and

$$B_1^*\left(I - \frac{1}{\ell}MB_2B_2^*\right)^{-1}W^*\Lambda^*B_1 \to 0.$$  

By $B_1 = \Lambda(\Lambda^*\Lambda)^{-1/2}$, we have

$$B_1^*\Lambda W\left(I - \frac{1}{\ell}B_2B_2^*M\right)^{-1}B_1 \to Q^1/2W\left(I - \frac{1}{\ell}B_2B_2^*M\right)^{-1}\Lambda Q^{-1/2}.$$  

By law of large numbers, we have with probability 1 that,

$$W\left(I - \frac{1}{\ell}B_2B_2^*M\right)^{-1}\Lambda$$

$$= \frac{1}{2T} \sum_{j=1}^T (F_j^*\varepsilon_{1+r}^* + F_j^*\varepsilon_1^*)\left(I - \frac{1}{\ell}B_2B_2^*M\right)^{-1}\Lambda$$

$$\to E(F_1\varepsilon_{1+r}^* + F_1\varepsilon_1^*)\left(I - \frac{1}{\ell}B_2B_2^*M\right)^{-1}\Lambda$$

$$= EE\left((F_1\varepsilon_{1+r}^* + F_1\varepsilon_1^*)\left(I - \frac{1}{\ell}B_2B_2^*M\right)^{-1}\Lambda\right|\varepsilon_1, \cdots, \varepsilon_{T+r})$$

$$= o(1).$$

Hence, we have with probability 1

$$B_1^*AW\left(I - \frac{1}{\ell}B_2B_2^*M\right)^{-1}B_1 = Q^{1/2}W\left(I - \frac{1}{\ell}B_2B_2^*M\right)^{-1}\Lambda Q^{-1/2} = o(1).$$

Similarly,

$$B_1^*\left(I - \frac{1}{\ell}MB_2B_2^*\right)^{-1}W^*\Lambda^*B_1 = o_{a.s.}(1).$$
Therefore, $\ell$ should satisfy

$$\det \left| Q^{1/2}H(\tau)Q^{1/2} + \left( B^*_1 (M - \ell I)^{-1} B_1 \right)^{-1} \right| + \frac{\alpha}{2} Q^{1/2} \left[ \left( 1 + \beta^2 - \frac{(1 + \beta^2)^2}{\beta^2} \right) H(0) + \left( 3\beta + \beta^3 - \frac{(1 + \beta^2)^2}{\beta} \right) H(\tau) \right. $$

$$\left. + \frac{(1 + \beta^2)^2}{\beta^2} G(\tau) \right] Q^{1/2} \rightarrow 0.$$ 

Recall $B_1 = \Lambda(\Lambda^* \Lambda)^{-1/2}$. Our next goal is to find the limit of $B^*_1 (M - \ell I)^{-1} B_1$. 

Define $A = M - \ell I$ and $A_k = A - (\gamma_{k+\tau} + \gamma_{k-\tau})\gamma_k - \gamma_k(\gamma_{k+\tau} + \gamma_{k-\tau})^*$, then we have the following lemmas, with proofs given in the Appendix.

**Lemma 5.2** Let $x \in C^n_1 := \{ x \in C^n : \|x\| = 1 \}$ be given. For $r \geq 1$, we have

$$E|\hat{\gamma}_k^* A_k^{-1} x|^{2r} \leq KT^{-r}$$

for some $K > 0$.

**Lemma 5.3** For any $x, y \in C^n_1$, we have $x^* A^{-\ell} y \rightarrow \frac{x^* y}{1 - c^2 m^2 (\ell) + \sqrt{1 - c^2 m^2 (\ell)}}$ a.s.

Finally, we have

$$\det \left| Q^{1/2}H(\tau)Q^{1/2} - \left( \frac{cm(\ell)}{1 - c^2 m^2 (\ell) + \sqrt{1 - c^2 m^2 (\ell)}} + \ell \right) I_{k(q+1)} \right| + \frac{\alpha}{2} Q^{1/2} \left[ \left( 1 + \beta^2 - \frac{(1 + \beta^2)^2}{\beta^2} \right) H(0) + \left( 3\beta + \beta^3 - \frac{(1 + \beta^2)^2}{\beta} \right) H(\tau) \right. $$

$$\left. + \frac{(1 + \beta^2)^2}{\beta^2} G(\tau) \right] Q^{1/2} = 0,$$

or equivalently

$$\det \left| \frac{\alpha}{2} \left[ \left( 1 + \beta^2 - \frac{(1 + \beta^2)^2}{\beta^2} \right) H(0) + \left( 3\beta + \beta^3 - \frac{(1 + \beta^2)^2}{\beta} + \frac{2}{\alpha} \right) H(\tau) \right. $$

$$\left. + \frac{(1 + \beta^2)^2}{\beta^2} G(\tau) \right] - \left( \frac{cm(\ell)}{1 - c^2 m^2 (\ell) + \sqrt{1 - c^2 m^2 (\ell)}} + \ell \right) Q^{-1} \right| = 0. \quad (5.3)$$
When $\tau > q$, one has $H(\tau) = 0$, $G(\tau) = 2I$ and (5.3) reduces to
\[
\det \left| \alpha(1 + \beta^2)I - \left( \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} + \ell \right)Q^{-1} \right| = 0,
\]
or
\[
\det \left| \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}}(Q + I) + \ell I \right| = 0.
\]
Let $\lambda$ be an eigenvalue of $Q$, then we have
\[
\frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}}(1 + \frac{1}{\lambda}) + \frac{\ell}{\lambda} = 0. \quad (5.4)
\]
When $\tau = q$, (5.3) reduces to
\[
\det \left| \alpha(1 + \beta^2)I + \left(1 + \frac{\alpha}{2}(3\beta + \beta^3)\right)H(q) - \left( \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} + \ell \right)Q^{-1} \right| = 0.
\]
Writing
\[
H(q) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \otimes I_k,
\]
one can easily verify that the eigenvalues of $H(q)$ are 1, $-1$ and 0, with multiplicity $k$, $k$ and $k(q - 1)$, respectively.

Suppose that $H(q)$ and $Q$ are commutative, that is, there is a common orthogonal matrix $O$ simultaneously diagonalizing the two matrices, i.e., we have $H(q) = ODH^O'$ and $Q = ODQ^O'$, where $D^H = \text{diag}[a_1, \cdots, a_{k(q+1)}]$ and $D^Q = \text{diag}[\lambda_1, \cdots, \lambda_{k(q+1)}]$. Then, (5.3) further reduces to
\[
(1 + \frac{\alpha}{2}(3\beta + \beta^3))a_j = \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} \left(1 + \frac{1}{\lambda_j} \right) + \frac{\ell}{\lambda_j}, \quad j = 1, \cdots, k(q + 1).
\]
Substituting \( \alpha = \frac{-cm}{1 - c^2m^2} \) and \( \beta = \frac{-cm}{1 - c^2m^2} \), for \( j = 1, \ldots, k(q + 1) \), we have

\[
a_j = \left( \frac{1}{2} + \frac{1}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} \right)^{-1} \times \\
\quad \left[ \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} \left( 1 + \frac{1}{\lambda_j} \right) + \frac{\ell}{\lambda_j} \right] \\
=: g_j(\ell). \tag{5.5}
\]

Notice that (5.4) is a special case of (5.5) for \( a_j = 0 \).

Note that when \( x \) is outside the support \([-d(c), d(c)]\) of the LSD of \( M_n(\tau), m_2(x) \neq 0 \). Hence, we have

\[
cm_1(x)((1 - c - ckm_1(x))^2 - c^2x^2m_2(x)) \\
+ x(1 - c^2m_1^2(x) + c^2m_2^2(x))(1 - c - ckm_1(x)) = 0.
\]

Let \( x \downarrow d(c) := d \) and we have \( m_2(x) \to 0 \) and \( m_1(x) \to m_1(d) \) satisfying

\[
cm_1(d)(1 - c - cdm_1(d))^2 + d(1 - c^2m_1^2(d))(1 - c - cdm_1(d)) \\
= [1 - c - cdm_1(d)][cm_1(d)(1 - c - cdm_1(d)) + d(1 - c^2m_1^2(d))] = 0,
\]

from which we have \( m_1(d) = \frac{1-c-\sqrt{(1-c)^2+8d^2}}{4cd} \).

Rewrite

\[
g_j(\ell) = \frac{cm(\ell)}{\frac{3}{2} - \frac{1}{2}c^2m^2(\ell) + \frac{1}{2} \sqrt{1 - c^2m^2(\ell)}} \left( 1 + \frac{1}{\lambda_j} \right) \\
\quad + \left( \frac{1}{2} + \frac{1}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)}} \right)^{-1} \frac{\ell}{\lambda_j} \\
:= g_{j1}(\ell) + g_{j2}(\ell).
\]

We will show that \( g_j(\ell) \) is increasing over \((d(c), \infty)\) by showing so are \( g_{j1} \) and \( g_{j2} \). By definition, over \((d(c), \infty)\), \( m \) is an increasing function taking negative values and \( m^2 \) is a decreasing function taking positive values. Hence, it is easy to see that \( g_{j1}(\ell) \) is increasing over \((d(c), \infty)\).

For \( g_{j2}(\ell) \), define \( h(\ell) = 1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell)} \) and rewrite \( g_{j2}(\ell) = \frac{\ell h(\ell)}{\lambda_j(1+\frac{h(\ell)}{2})} \). It is easy
to see that $h(\ell) > 0$ and $h'(\ell) > 0$ over $(d(c), \infty)$. Hence we have

$$g_{j2}'(\ell) = \frac{[h(\ell) + \ell h'(\ell)](1 + \frac{h'(\ell)}{2}) - \ell h(\ell) h''(\ell)}{\lambda_j(1 + \frac{h'(\ell)}{2})^2} = \frac{h(\ell) + \ell h'(\ell) + \frac{h^2(\ell)}{2}}{\lambda_j(1 + \frac{h'(\ell)}{2})^2} > 0.$$  

By symmetry, $g_j(\ell)$ is increasing over $(-\infty, -d(c))$ as well. Therefore, based on the sign of $g_j(d(c))$, we have the following cases to consider.

Case I. $g_j(d(c)) \geq 0$:

i. If $a_j > g_j(d(c))$, then (5.5) has one solution in $(d(c), \infty)$ and no solution in $(-\infty, -d(c))$.

ii. If $g(d(c)) \geq a_j \geq -g_j(d(c)) = g_j(-d(c))$, then (5.5) has no solution in $(d(c), \infty)$ and $(-\infty, -d(c))$.

iii. If $a_j < -g_j(d(c))$, then (5.5) has one solution in $(-\infty, -d(c))$ and no solution in $(d(c), \infty)$.

Case II. $g_j(d(c)) < 0$:

i. If $a_j \geq -g_j(d(c))$, then (5.5) has one solution in $(d(c), \infty)$ and no solution in $(-\infty, -d(c))$.

ii. If $g(d(c)) < a_j < -g_j(d(c)) = g_j(-d(c))$, then (5.5) has one solution in $(d(c), \infty)$ and one solution in $(-\infty, -d(c))$.

iii. If $a_j \leq g_j(d(c))$, then (5.5) has one solution in $(-\infty, -d(c))$ and no solution in $(d(c), \infty)$.

**Remark 5.1** In real application, compared with the noise component, the loading matrix $\Lambda$ dominates. As a result, all the eigenvalues of $Q = \Lambda^*\Lambda$ are large (more precisely, they are of the same order as $n$). Hence, we can assume that $Q^{-1} = 0$. Thus the commutative assumption of $H(q)$ and $Q$ can be relaxed. Moreover, under this case, we always have $g_j(d(c)) < 0$.

**Remark 5.2** For the same reason as stated before Remark 4.1, $d_c$ is replaced by $(1 + an_b)d_c$ in practice. Simulation indicates a fit of $a = 0.1, b = -1/3$.

Notice that all the eigenvalues of $H(q + 1)$ are 0, while for $H(q)$, $k$ eigenvalues are 1 and $k$ eigenvalues are $-1$, with the rest being 0. Making use of such difference and applying the above analysis to the cases that $\tau = q$ and $\tau = q + 1$ gives an estimate of $q$. Together with the estimation of $k(q + 1)$, we easily obtain the estimate of $k$. A numerical demonstration is given in the simulation.

24
6 Estimate of $\sigma^2$

The above estimation is based on the assumption that $\sigma^2$, the variance of the noise part is given. In practice, it is often the case that $\sigma^2$ is unknown. To this end, we can estimate $\sigma^2$ by employing the properties of the MP law. More precisely, we first estimate the left boundary of the support of the MP law by the smallest sample eigenvalue of $\Phi_n(0)$, say $\hat{\lambda}_1$ (the eigenvalues are arranged in ascending order), and estimate the right boundary by $(1+\sqrt{c})^2 \lambda_1$. An iteration is then applied.

The initial estimator of $\sigma^2$, say $\hat{\sigma}^2(0)$, is obtained as the sample mean of the sample eigenvalues of $\Phi_n(0)$ that lie within the interval $[\hat{\lambda}_{m+1}, \hat{\lambda}_{n-m}]$, where $m \geq 0$ is such that $\hat{\lambda}_{n-m}$ is the largest eigenvalue of $\Phi_n(0)$ less than $(1+\sqrt{c})^2 (1+2n^{-2/3}) \hat{\sigma}^2(1)$. For $i \geq 1$, we obtain the updated estimator $\hat{\sigma}^2(i)$ by taking the sample mean of the sample eigenvalues of $\Phi_n(0)$ that lie within the interval $[(1-\sqrt{c})^2 \hat{\sigma}^2(i-1), (1+\sqrt{c})^2 (1+2n^{-2/3}) \hat{\sigma}^2(i-1)]$. The iteration stops once we have $\hat{\sigma}^2(\ell-1) = \hat{\sigma}^2(\ell)$ for some $\ell$ and our estimator $\hat{\sigma}^2 := \hat{\sigma}^2(\ell)$. As shown in the simulation, our estimation of $k$ and $q$ still works well with such estimator.

7 Simulation

Table 1 presents a simulation about the result discussed above, displaying the largest 13 absolute values of the eigenvalues for lags $\tau$ from 0 to 5. Here

$$R_t = \sum_{i=0}^{q} \Lambda_i f_{t-i} + e_t, \quad t = 1, ..., T \tag{7.1}$$

where $f_t$’s are factors of length $k$; $\Lambda_i$, $i = 0, ..., q$ is a constant time-invariant matrix of size $n \times k$, $e_t$ is the error term and $q$ is the lag of the model. In addition, assume that: $e_t$ are i.i.d. random variables with $e_t \sim N(0, \sigma^2 I_n)$ and $f_t$ are i.i.d. random variables with $f_t \sim N(0, \sigma_f^2 I_k)$, independent of $e_t$. $\Lambda_i = \begin{bmatrix} \Lambda^i_1 & \Lambda^i_2 & \ldots & \Lambda^i_k \end{bmatrix}$ where $\Lambda^i_j, i = 0, ..., q; j = 1, ..., k$ is a vector of length $n$ and is given by $\Lambda^i_j = \beta I_n + \varepsilon_{ij}$ where $1_n$ is a vector of 1’s and $\varepsilon_{ij}$ are i.i.d. random variables with $\varepsilon_{ij} \sim N(0, \sigma^2 \varepsilon I_n)$. For $n = 450$, $T = 500$, $k = 2$, $q = 2$, $\beta = 1.0$, $\sigma_f = 0.25$.
\[ \sigma_f^2 = 4, \sigma^2 = 1 \text{ and } \sigma_e^2 = 0.25, \text{ we have } c = 0.9, b_c = (1 + \sqrt{c})^2 = 3.7974 \text{ and } d_c = 1.8573. \]

Eigenvalues of \( Q \) are from 95 to 285, making \( Q^{-1} \sim \mathbf{0} \).

When \( \tau = 0 \), and \( \sigma^2 = 1 \) is known, using the phase transition point \( b_c = (1 + \sqrt{c})^2 \sigma^2 = 3.7974 \), we see that the number of spotted spikes is 6, which estimates \( k(q + 1) \). When for \( \tau = q + 1 \), we have \( H(\tau) = \mathbf{0} \). Moreover, as \( Q^{-1} \sim \mathbf{0} \), we have \( g_j(d(c)) \sim -0.4284 < 0 \). That is, our Case II (ii) applies for all the \( k(q + 1) \) eigenvalues of \( H(q + 1) \), making the number of spikes \( 2k(q + 1) \) as verified by applying the phase transition point \( d_c = 1.8573 \). For \( \tau = q \), \( H(\tau) \) has \( k \) eigenvalues of 1, \( k \) eigenvalues of \(-1 \) and \( k(q - 1) \) eigenvalues of 0 with Case II (i),(iii) and (ii) applicable, respectively. Thus, we have \( k + k + 2k(q - 1) = 2kq < 2k(q + 1) \) eigenvalues in this case. Again, this agrees with the use of the phase transition point \( d_c = 1.8573 \).

The estimation of \( k \) is obvious.

When \( \sigma^2 = 1 \) is unknown, using technique as in Section 6, one has \( \hat{\sigma}^2 = 0.9894 \). It then follows that \( \hat{b}_c = (1 + \sqrt{c})^2(1 + 2n^{-2/3})\hat{\sigma}^2 = 3.8851, \hat{d}_c = 1.8616 \) (with rescale factor \( 1 + 0.1n^{-1/3} \)), which gives the same estimates as above.

### A Some proofs

#### A.1 Proof of Lemma 5.1

Define \( M_k = M - \gamma_k(\gamma_{k+\tau} + \gamma_{k-\tau})^* - (\gamma_{k+\tau} + \gamma_{k-\tau})\gamma_k^* \), and

\[
M_{k,k+\tau,\ldots,k+l\tau} = M_{k,k+\tau,\ldots,k+(l-1)\tau} - \gamma_{k+(l+1)\tau}\gamma_{k+l\tau}^* - \gamma_{k+l\tau}\gamma_{k+(l+1)\tau}^*, l \geq 1.
\]
Table 1: Absolute values of the largest eigenvalues of the empirical covariance matrix at various lags with parameters: $n = 450$, $T = 500$, $k = 2$, $q = 2$, $\beta = 1.0$, $\sigma_f^2 = 4$, $\sigma^2 = 1$ and $\sigma_e^2 = 0.25$. Note that $c = 0.9$, $b_c = (1 + \sqrt{c})^2 = 3.7974$ and $d_c = 1.8573$. When $\sigma^2 = 1$ is unknown, one has $\hat{\sigma}^2 = 0.9894$, $\hat{b}_c = 3.8851$ and $\hat{d}_c = 1.8616$. 

| $\tau$ | 0       | 1       | 2       | 3       | 4       | 5       |
|--------|---------|---------|---------|---------|---------|---------|
| $10031.2366$ | $6227.5906$ | $2865.1554$ | $640.5761$ | $155.9377$ | $128.6870$ |
| $534.5839$ | $363.7782$ | $258.4859$ | $48.9667$ | $46.5224$ | $92.3756$ |
| $473.1639$ | $325.8391$ | $224.9755$ | $22.7478$ | $46.0225$ | $53.7072$ |
| $458.2226$ | $305.6334$ | $214.9755$ | $21.6373$ | $45.6650$ | $26.3564$ |
| $435.2661$ | $13.1683$ | $45.7319$ | $21.3150$ | $25.4884$ | $25.6006$ |
| $392.6272$ | $11.1482$ | $17.8374$ | $19.2596$ | $18.0820$ | $19.3930$ |
| $3.6928$ | $9.0674$ | $11.7423$ | $10.3580$ | $15.5876$ | $14.6107$ |
| $3.5809$ | $7.9537$ | $8.0837$ | $9.9668$ | $12.9088$ | $12.0980$ |
| $3.5449$ | $1.7375$ | $1.7988$ | $9.5028$ | $10.5568$ | $8.5791$ |
| $3.4579$ | $1.7326$ | $1.7895$ | $8.5483$ | $4.7840$ | $7.0596$ |
| $3.4312$ | $1.7015$ | $1.7388$ | $5.5931$ | $4.3896$ | $4.5411$ |
| $3.3829$ | $1.6957$ | $1.7242$ | $3.5968$ | $4.3843$ | $3.6744$ |
| $3.3701$ | $1.6751$ | $1.6724$ | $1.8215$ | $1.7944$ | $1.7468$ |
Suppose that $i \geq j$, then we have

$$
\gamma_i^* B_2 \left( (I - B_2^* M B_2)^{-1} B_2^* \gamma_j \right) = \gamma_i^* B_2 \left( (I - B_2^* M B_2 - B_2^* \gamma_j + \gamma_j^* B_2 - B_2^* \gamma_j^* (\gamma_j + \gamma_j^*) B_2)^{-1} B_2^* \gamma_j \right)
$$

$$
= \frac{\gamma_i^* B_2 \left( (I - B_2^* M B_2 - B_2^* \gamma_j + \gamma_j^* B_2)^{-1} B_2^* \gamma_j \right)}{1 - (\gamma_j + \gamma_j^*) B_2 \left( (I - B_2^* M B_2 - B_2^* \gamma_j + \gamma_j^* B_2)^{-1} B_2^* \gamma_j \right)} + o_a.s. (1)
$$

$$
= \left\{ \begin{array}{ll}
\frac{-c_{m/2}}{1 - \frac{c_{m/2}}{2^j}} + o_a.s. (1), & i = j \\
\frac{-c_{m/2}}{1 - \frac{c_{m/2}}{2^j}}, & i \neq j
\end{array} \right.
$$

Next, we have

$$
\gamma_i^* B_2 \left( (I - B_2^* M B_2)^{-1} B_2^* \gamma_{j+\tau} \right) = \gamma_i^* B_2 \left( (I - B_2^* M_{j+\tau} B_2 - B_2^* \gamma_{j+2\tau} \gamma_{j+\tau}^* B_2 - B_2^* \gamma_{j+\tau}^* \gamma_{j+2\tau} B_2)^{-1} B_2^* \gamma_{j+\tau} \right)
$$

$$
= \frac{\gamma_i^* B_2 \left( (I - B_2^* M_{j+\tau} B_2 - B_2^* \gamma_{j+2\tau} \gamma_{j+\tau}^* B_2)^{-1} B_2^* \gamma_{j+\tau} \right)}{1 - \gamma_{j+2\tau} B_2 \left( (I - B_2^* M_{j+\tau} B_2 - B_2^* \gamma_{j+2\tau} \gamma_{j+\tau}^* B_2)^{-1} B_2^* \gamma_{j+\tau} \right)} + o_a.s. (1)
$$

$$
= \left\{ \begin{array}{ll}
\frac{-c_{m/2}}{1 - \frac{c_{m/2}}{2^{j+\tau}}} + o_a.s. (1), & i = j + \tau \\
\frac{-c_{m/2}}{1 - \frac{c_{m/2}}{2^{j+\tau}}}, & i \neq j + \tau
\end{array} \right.
$$
\[ \gamma_i^r B_2((I - B_2^*M_jB_2)^{-1}B_2^*\gamma_{j-\tau}) \]

\[ = \gamma_i^r B_2((I - B_2^*M_{j,j-\tau}B_2 - B_2^*\gamma_{j-2\tau}B_2 - B_2^*\gamma_{j-2\tau}B_2 - B_2^*\gamma_{j-2\tau}B_2)^{-1}B_2^*\gamma_{j-\tau}) \]

\[ = \gamma_i^r B_2((I - B_2^*M_{j,j-\tau}B_2 - B_2^*\gamma_{j-2\tau}B_2 - B_2^*\gamma_{j-2\tau}B_2)^{-1}B_2^*\gamma_{j-\tau}) \]

\[ = \gamma_i^r B_2\left(\frac{(I - B_2^*M_{j,j-\tau}B_2)^{-1}B_2^*\gamma_{j-\tau}}{1 - \gamma_i^r B_2((I - B_2^*M_{j,j-\tau}B_2 - B_2^*\gamma_{j-2\tau}B_2)^{-1})B_2^*\gamma_{j-\tau}}\right) + o_{a.s.}(1) \]

\[ = \frac{c_{m2}}{1 - c_{m2}} \gamma_i^r B_2((I - B_2^*M_{j,j-\tau}B_2)^{-1}B_2^*\gamma_{j-2\tau}) + o_{a.s.}(1). \]

Note that \( \left| \frac{c_{m2}}{1 - c_{m2}} \right| = \left| \frac{d}{x_1} \right| < 1 \), by induction, we have

\[ \gamma_i^r B_2((I - B_2^*M_jB_2)^{-1}B_2^*\gamma_{j-\tau}) = o_{a.s.}(1). \]

Then result then follows by induction. By symmetry, it holds when \( i < j \). The proof of the lemma is complete.

### A.2 Proof of Lemma 5.2

Let \( A_k^{-1}x = b = (b_1, \ldots, b_n)' \). Noting \( |\varepsilon_{ij}| < C \) and \( E|\varepsilon_{ij}|^2 = 1 \), we have

\[ E(\gamma_k^r A_k^{-1}x)^{2r} = \frac{1}{2^r T^r} E\left(\sum_{i=1}^{n} \varepsilon_{ki} b_i\right)^{2r} \]

\[ \leq \frac{1}{2^r T^r} E \left( \sum_{l=1}^{r} \sum_{1 \leq i_1 < \cdots < i_l \leq n, i_1 + \cdots + i_l = 2r} \frac{(2r)!}{i_1! \cdots i_l!} \varepsilon_k^{i_1} b_1^{i_1} \cdots \varepsilon_k^{i_l} b_1^{i_l} \right) \]

\[ = \frac{1}{2^r T^r} E \left( \sum_{l=1}^{r} \sum_{1 \leq i_1 < \cdots < i_l \leq n, i_1 + \cdots + i_l = 2r, i_1 \geq 2, \ldots, i_l \geq 2} \frac{(2r)!}{i_1! \cdots i_l!} \varepsilon_k^{i_1} b_1^{i_1} \cdots \varepsilon_k^{i_l} b_1^{i_l} \right) \]

\[ \leq \frac{K}{2^r T^r} E \left( \sum_{l=1}^{r} \sum_{1 \leq i_1 < \cdots < i_l \leq n, i_1 + \cdots + i_l = 2r, i_1 \geq 2, \ldots, i_l \geq 2} \frac{(2r)!}{i_1! \cdots i_l!} |b_1|^{i_1} \cdots |b_l|^{i_l}. \right) \]
By \( \sum_{j=1}^{n} |b_j|^2 = \|A_k^{-1}x\|^2 \) and Cauchy-Schwartz inequality, we have

\[
\sum_{1 \leq j_1 < \cdots < j_l \leq n, \ i_1 \geq 2, \ldots, i_l \geq 2} \frac{(2r)!}{i_1! \cdots i_l!} |b_{j_1}|^{i_1} \cdots |b_{j_l}|^{i_l} \\
\leq \sum_{i_1 + \cdots + i_l = 2r, \ i_1 \geq 2, \ldots, i_l \geq 2} \frac{(2r)!}{i_1! \cdots i_l!} (\sum_{j=1}^{n} |b_j|^2)^r \\
\leq l^{2r} \|A_k^{-1}\|_o^2 \|x\|^{2r} \\
\leq \frac{l^{2r}}{\eta^{2r}}.
\]

Here \( \eta := \ell - d_c > 0 \). Therefore, we have

\[ E|\gamma_k^*A_k^{-1}x|^{2r} \leq KT^{-r} \]

for some \( K > 0 \). The proof of the lemma is complete.

### A.3 Proof of Lemma 5.3

**Proof.** Let \( x, y \in \mathbb{C}^n \) be given. Define 
\[
\tilde{A}_k = A_k + \gamma_k(\gamma_{k+\tau} + \gamma_{k-\tau})^*, \quad \tilde{A}_{k,k+\tau} = A_k - \gamma_{k+2\tau}^* \gamma_{k+\tau}^* \gamma_{k-\tau}.
\]

First we have

\[
x^*A^{-1}y - Ex^*A^{-1}y \\
= x^* \sum_{k=1}^{T} (E_k - E_{k-1}) \left( A^{-1} - A_k^{-1}(\ell) \right) y \\
= \sum_{k=1}^{T} (E_k - E_{k-1}) \left( - \frac{x^* \tilde{A}_k^{-1}(\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k^* \tilde{A}_k^{-1} y}{1 + \gamma_k^* \tilde{A}_k^{-1}(\gamma_{k+\tau} + \gamma_{k-\tau})} - \frac{x^* \gamma_k^* A_k^{-1}(\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k^* A_k^{-1} y}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau})^* A_k^{-1} \gamma_k} \right) \\
= \sum_{k=1}^{T} (E_k - E_{k-1}) \left( - \alpha_k^1 - \alpha_k^2 \right).
\]
Using Lemma 2.1, we have, for $i = 1, 2$

$$
\mathbb{E}\left[ \sum_{k=1}^{T} (E_k - E_{k-1}) \alpha_{ki}^2 \right] \\
\leq K_l \left[ \mathbb{E} \left( \sum_{k=1}^{T} E_{k-1} (E_k - E_{k-1}) \alpha_{ki}^2 \right) \right]^{l} + \sum_{k=1}^{T} \mathbb{E}[E_k - E_{k-1}] \alpha_{ki}^2 \\
\leq K'_l \left[ \mathbb{E} \left( \sum_{k=1}^{T} E_{k-1} \alpha_{ki}^2 + \sum_{k=1}^{T} E_{k-1} |\alpha_{ki}|^2 \right) \right]^{l} + \sum_{k=1}^{T} \mathbb{E}[E_k \alpha_{ki}^2] + \sum_{k=1}^{T} \mathbb{E}[E_{k-1} \alpha_{ki}^2] \\
\leq 2^l K'_l \left[ \mathbb{E} \left( \sum_{k=1}^{T} E_{k-1} |\alpha_{ki}|^2 \right) \right]^{l} + \sum_{k=1}^{T} \mathbb{E}[|\alpha_{ki}|^2] \right]. \quad (A.1)
$$

Note that

$$
A_k^{-1} = (\tilde{A}_{k,k+\tau} + \gamma_{k+2\tau} \gamma_{k+\tau}^*)^{-1} = \tilde{A}_{k,k+\tau}^{-1} - \frac{\tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1}}{1 + \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}}.
$$

Hence, we have

$$
\gamma_{k+\tau}^* A_k^{-1} = \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} - \frac{\gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1}}{1 + \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}} = \frac{\gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1}}{1 + \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}}.
$$

Next, we have

$$
\gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} = \frac{\gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} - \frac{\gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1}}{1 + \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}}}{1 + \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}} = \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} - \frac{cm}{2} \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} + R_{k1},
$$

where

$$
R_{k1} = \frac{cm}{2} \gamma_{k+2\tau}^* \tilde{A}_{k,k+\tau}^{-1} - \frac{\gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+\tau} \gamma_{k+2\tau} \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1}}{1 + \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}} = \left( \frac{\gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} - \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+\tau} \gamma_{k+2\tau} \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1}}{1 + \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}} \right) \gamma_{k+2\tau}^* \tilde{A}_{k,k+\tau}^{-1}.
$$

Substitute back, we obtain

$$
\gamma_{k+\tau}^* A_k^{-1} y = \frac{\gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} y - \frac{\gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+\tau} \gamma_{k+2\tau} \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} y}{1 + \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau}}}{1 + \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} - \frac{cm}{2} \gamma_{k+\tau}^* \tilde{A}_{k,k+\tau}^{-1} \gamma_{k+2\tau} + R_{k1} \gamma_{k+2\tau}}, \quad (A.2)
$$

31
with
\[
R_{k_1} \gamma_{k+2r} = \left( \frac{cm}{2} \gamma_{k+2r} - \gamma_{k+\tau} A_{k,k+\tau}^{-1} \gamma_{k+\tau} + \gamma_{k+2r} A_{k,k+\tau}^{-1} \gamma_{k+\tau} \right) \gamma_{k+2r} A_{k,k+\tau}^{-1} \gamma_{k+2r} \gamma_{k+2r}.
\]

Note that
\[
\lim_{n \to \infty} \left| \frac{cm}{2} A_{k,k+\tau}^{-1} \gamma_{k+\tau} \right| = \left| \frac{cm}{2} \right| < 1
\]

and
\[
\lim_{n \to \infty} 1 + \gamma_{k+\tau} A_{k,k+\tau}^{-1} \gamma_{k+2r} - \frac{cm}{2} \gamma_{k+2r} A_{k,k+\tau}^{-1} \gamma_{k+2r} + R_{k_1} \gamma_{k+2r} = 1 - \frac{cm}{2} \left| \frac{cm}{2} \right|,
\]

which is bounded. Using induction, we have \(|\gamma_{k+\tau} A_{k}^{-1} y| \leq K |\gamma_{k+\tau} A_{k+\tau}^{-1} y|\).

Similarly, we have \(|\gamma_{k-\tau} A_{k}^{-1} y| \leq K |\gamma_{k-\tau} A_{k-\tau}^{-1} y|\), \(|x^* A_{k}^{-1} \gamma_{k+\tau}| \leq K |x^* A_{k+\tau}^{-1} \gamma_{k+\tau}|\) and
\(|x^* A_{k}^{-1} \gamma_{k-\tau}| \leq K |x^* A_{k-\tau}^{-1} \gamma_{k-\tau}|\).

Therefore, by noting \((\gamma_{k+\tau} + \gamma_{k-\tau}) A_{k}^{-1} \gamma_{k} = o_a.s.(1), |\varepsilon_{it}| < C, E|\varepsilon_{it}|^2 = 1\) and \(x^* A_{k}^{-1} \bar{A}_{k} x\) being bounded, we have
\[
E\left( \sum_{k=1}^{T} E_{k+1} |\alpha_{k2}|^2 \right)^l 
\leq KE\left( \sum_{k=1}^{T} E_{k+1} |x^* A_{k}^{-1} \gamma_{k+\tau} A_{k}^{-1} y|^2 \right)^l 
= KE\left( \sum_{k=1}^{T} \frac{1}{2T} E_{k+1} x^* A_{k}^{-1} \bar{A}_{k} x^* x|\gamma_{k+\tau} A_{k}^{-1} y|^2 \right)^l 
\leq K \max_k E|\gamma_{k+\tau} A_{k}^{-1} y|^{2l} 
\leq \frac{K}{T^l}
\]
and

\[
\sum_{k=1}^{T} E|\alpha_{k2}|^{2l} = \sum_{k=1}^{T} E \left( |x^* A_k^{-1} \gamma_k y A_k^{-1} y|^2 \right)^{l/2} \leq \frac{K}{T^{l-1}} \max_k E|\gamma_{k+\tau} A_k^{-1} y|^{2l} \leq \frac{K}{T^{2l-1}}
\]

For \( i = 1 \), by \( \tilde{A}_k^{-1} = A_k^{-1} - \frac{A_k^{-1} \gamma_k (\gamma_k + \gamma_{k-\tau}) A_k^{-1}}{1 + (\gamma_k + \gamma_{k-\tau}) A_k^{-1} \gamma_k} \), we have

\[
\begin{align*}
&x^* \tilde{A}_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k \tilde{A}_k^{-1} y \\
= &x^* \left( A_k^{-1} - \frac{A_k^{-1} \gamma_k (\gamma_k + \gamma_{k-\tau}) A_k^{-1}}{1 + (\gamma_k + \gamma_{k-\tau}) A_k^{-1} \gamma_k} \right) (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k \left( A_k^{-1} - \frac{A_k^{-1} \gamma_k (\gamma_k + \gamma_{k-\tau}) A_k^{-1}}{1 + (\gamma_k + \gamma_{k-\tau}) A_k^{-1} \gamma_k} \right) y \\
= &x^* A_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau}) \gamma_k A_k^{-1} y - \frac{(\gamma_{k+\tau} + \gamma_{k-\tau}) A_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau})}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau}) A_k^{-1} \gamma_k} x^* A_k^{-1} \gamma_k A_k^{-1} y \\
&+ \frac{\gamma_k A_k^{-1} \gamma_k}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau}) A_k^{-1} \gamma_k} x^* A_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau}) A_k^{-1} y - \frac{\gamma_k A_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau})}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau}) A_k^{-1} \gamma_k} x^* A_k^{-1} \gamma_k A_k^{-1} y \\
=: &\alpha_{k11} - \alpha_{k12} - \alpha_{k13} + \alpha_{k14}
\end{align*}
\]

It is easy to see that work on \( \alpha_{k11} \) and \( \alpha_{k14} \) is the same as that on \( \alpha_{k2} \).

For \( \alpha_{k12} \), by Cauchy-Schwartz’s inequality, we have

\[
E \left( \sum_{k=1}^{T} E_{k-1} |\alpha_{k12}|^2 \right)^l \leq K E \left( \sum_{k=1}^{T} E_{k-1} |x^* A_k^{-1} \gamma_k| E_{k-1} |\gamma_k A_k^{-1} y|^2 \right)^{l/2} \leq K E \left( \sum_{k=1}^{T} \frac{1}{4T^2} E_{k-1} x^* A_k^{-1} \tilde{A}_k^{-1} y E_{k-1} y^* A_k^{-1} \tilde{A}_k^{-1} y \right)^{l/2} \leq \frac{K}{T^l}
\]

33
and by \(|\varepsilon| < C\), \(E[\varepsilon|t|^2] = 1\) and \(x^*A_k^{-1}\tilde{A}_k^{-1}x\) being bounded, we have

\[
\sum_{k=1}^{T} E|\alpha_{k12}|^{2l} \leq K \sum_{k=1}^{T} E \left(|x^*A_k^{-1}\gamma_k|^2|\gamma_k^*A_k^{-1}y|^2\right)^l \leq \frac{K}{T^{n-1}} \max_k E|\gamma_k A_k^{-1}y|^{2l} \leq \frac{K}{T^{2l-1}}.
\]

By the fact that \(|\gamma_k^*A_k^{-1}y| \leq K|\gamma_k^*A_k^{-1}\gamma_k^{-1}y|\) and \(|x^*A_k^{-1}\gamma_k^{-1}\gamma_k| \leq K|x^*A_k^{-1}\gamma_k|\), the similar result for \(\alpha_{k13}\) follows by the same reason.

Substituting all the above results into \((A.1)\) and choosing \(l\) large enough, we have

\[
x^*A^{-1}y - E x^*A^{-1}y \to 0 \text{ a.s.}
\]

Next, we want to show the convergence of \(E x^*A^{-1}y\).

By

\[
A = \sum_{k=1}^{T} (\gamma_k^*\gamma_k + \gamma_k^*\gamma_k^*) - I_n
\]

we have

\[
I_n = \sum_{k=1}^{T} (\gamma_k^*\gamma_k^*A^{-1} + \gamma_k^*\gamma_k^*A) - \ell A^{-1}.
\]

Multiplying \(x^*\) from left and \(y\) from right and taking expectation, we obtain

\[
x^*y = \sum_{k=1}^{T} (E x^*\gamma_k^*\gamma_k A^{-1}y + E x^*\gamma_k^*\gamma_k^*A^{-1}y) - \ell E x^*A^{-1}y
\]

\[
= \sum_{k=1}^{T} E x^* (\gamma_k^* + \gamma_k) \gamma_k A^{-1}y - \ell E x^*A^{-1}y.
\]

By \(A = \tilde{A}_k + (\gamma_k^* + \gamma_k)\gamma_k^*, \tilde{A}_k = A_k + \gamma_k (\gamma_k^* + \gamma_k)^* + \gamma_k^* (\gamma_k^* + \gamma_k)\gamma_k = o_{a.s.}(1),\)
\[ \gamma_k^* A_k^{-1} y = \frac{\gamma_k^* \tilde{A}_k^{-1} y}{1 + \gamma_k^* A_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau})} \]

\[ = \frac{\gamma_k^* A_k^{-1} y}{1 + \gamma_k^* A_k^{-1} (\gamma_{k+\tau} + \gamma_{k-\tau}) (\gamma_{k+\tau} + \gamma_{k-\tau})} \]

\[ = \frac{\gamma_k^* A_k^{-1} y + \gamma_k^* A_k^{-1} y (\gamma_{k+\tau} + \gamma_{k-\tau}) A_k^{-1} y - \gamma_k^* A_k^{-1} y (\gamma_{k+\tau} + \gamma_{k-\tau})^2 A_k^{-1} y}{1 + (\gamma_{k+\tau} + \gamma_{k-\tau}) A_k^{-1} y} \]

\[ = \frac{-\frac{c_2 m^2}{2(1 - \gamma_k^* A_k^{-1} y)^2}}{1 - \frac{c_2 m^2}{2(1 - \gamma_k^* A_k^{-1} y)^2}}. \tag{A.4} \]

Next, we have

\[ \gamma_{k+\tau}^* A_k^{-1} y x^* \gamma_{k+\tau} = \left( 1 + \gamma_{k+\tau}^* A_k^{-1} y x^* \gamma_{k+\tau} \right) \gamma_{k+\tau}^* A_k^{-1} y - \gamma_{k+\tau}^* A_k^{-1} y x^* \gamma_{k+\tau} \gamma_{k+\tau}^* A_k^{-1} y \]

\[ = \frac{\gamma_{k+\tau}^* A_k^{-1} y x^* \gamma_{k+\tau} + o_a.s.(1)}{1 - \gamma_{k+\tau}^* A_k^{-1} y x^* \gamma_{k+\tau} + o_a.s.(1)} \]

Similarly, we can show that \( \gamma_{k-\tau}^* A_k^{-1} y x^* \gamma_{k-\tau} = \frac{x^* A_k^{-1} y + o_a.s.(1)}{2(1 - \gamma_k^* A_k^{-1} y)^2}, \) \( \gamma_{k+\tau}^* A_k^{-1} y x^* \gamma_{k-\tau} = o_a.s.(1) \)

and \( \gamma_{k-\tau}^* A_k^{-1} y x^* \gamma_{k+\tau} = o_a.s.(1) \). Next, we will show that \( \text{E} \gamma_{k+\tau}^* A_k^{-1} y - \text{E} A^{-1} y = o(1) \). By writing

\[ \text{E} \gamma_{k+\tau}^* A_k^{-1} y - \text{E} A^{-1} y = \text{E} \gamma_{k+\tau}^* A_k^{-1} y - \text{E} \gamma_{k+\tau}^* A_k^{-1} y + \text{E} \gamma_{k+\tau}^* A_k^{-1} y - \text{E} A^{-1} y, \]

it is sufficient to show \( \text{E} \gamma_{k+\tau}^* A_k^{-1} y - \text{E} A^{-1} y = o(1) \). Note that

\[ \text{E} \gamma_{k+\tau}^* A_k^{-1} y - \text{E} A^{-1} y \]

\[ = \text{E} \left( \frac{x^* A_k^{-1} y + o_a.s.(1)}{2(1 - \gamma_k^* A_k^{-1} y)^2} \right) \]

\[ = \text{E} \alpha_{k1} + \text{E} \alpha_{k2} \]

Previous calculation shows that \( \text{E} |\alpha_{k1}| = o(1) \) and \( \text{E} |\alpha_{k2}| = o(1) \). Substituting these back to \( (A.3) \) and \( (A.4) \), we finish proving the lemma.
References

[1] Bai, Z.D., Liu H.X. and Wong, W.K. (2011) Asymptotic properties of eigenmatrices of a large sample covariance matrix. Ann. Appl. Probab. 21, 1994–2015.

[2] Bai, Z.D. and Silverstein, J.W. (1998) No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. Ann. Probab. 26, 316–345.

[3] Bai, Z.D. and Silverstein, J.W. (2010) Spectral Analysis of Large Dimensional Random Matrices, 2nd ed. Springer Verlag, New York.

[4] Bai, Z.D. and Yao, J.F. (2008) Central limit theorems for eigenvalues in a spiked population model. Ann. Inst. H. Poincaré Probab. Statist. Volume 44, Number 3, 447–474.

[5] Bai, Z.D. and Wang, C. (2015) A note on the limiting spectral distribution of a symmetrized auto-cross covariance matrix. Statistics and Probability Letters, 96, 333 – 340.

[6] Baik, Jinho and Silverstein, J. W. (2006) Eigenvalues of large sample covariance matrices of spiked population models. J. Multivariate Anal. 97(6), 1382-1408.

[7] Burkholder, D.L. (1973) Distribution function inequalities for martingales. Ann. Probab. 1, 19–42.

[8] Chung, K.L. (2001) A Course in Probability Theory, 3rd ed. Academic Press, New York.

[9] Johnstone, I. (2001) On the distribution of the largest eigenvalue in principal components analysis. Ann. Statist. 29, 295 – 327.

[10] Jin, B. S., Wang, C., Bai, Z.D., Nair, K.K. and Harding, M.C. (2014) Limiting spectral distribution of a symmetrized auto-cross covariance matrix. Ann. Appl. Probab. 24, 1199–1225.
[11] Marčenko, V.A. and Pastur, L.A. (1967) Distribution of eigenvalues for some sets of matrices. *Mat. Sb. 72*, 507 – 536.

[12] Wang, C., Jin, B. S., Bai, Z.D., Nair, K.K. and Harding, M.C. (2015) Strong limit of the extreme eigenvalues of a symmetrized auto-cross covariance matrix. *Ann. Appl. Probab. 25*, 3624 – 3683.