Counting functions for Dirichlet series and compactness of composition operators

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Abstract
We give a sufficient condition for a composition operator with positive characteristic to be compact on the Hardy space of Dirichlet series.

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1 | INTRODUCTION
1.1 | Description of the results

Let $H^2$ be the Hilbert space of Dirichlet series $f(s) = \sum_{n \geq 1} a_n n^{-s}$ with square summable coefficients endowed with the norm $\|f\|^2 = \sum_{n \geq 1} |a_n|^2$. By the Cauchy–Schwarz inequality, Dirichlet series in $H^2$ generate holomorphic functions in $\mathbb{C}_{1/2}$, where $\mathbb{C}_\theta = \{s \in \mathbb{C} : \Re(s) > \theta\}$. Let $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ be analytic. The composition operator with symbol $\varphi$ is defined on $H^2$ by $C_\varphi(f) = f \circ \varphi$. In [10], Gordon and Hedenmalm determined which symbols $\varphi$ generate a bounded composition operator on $H^2$: this happens if and only if $\varphi$ belongs to the Gordon–Hedenmalm class $G$ of the analytic functions $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ that may be written $\varphi(s) = c_0 + \psi(s)$ where $c_0$ is a non-negative integer, $\psi$ is a Dirichlet series that converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and satisfies the following properties.

(a) If $c_0 = 0$, then $\psi_0(C_0) \subset \mathbb{C}_{1/2}$.
(b) If $c_0 \geq 1$, then either $\psi(C_0) \subset C_0$ or $\psi_0 \equiv 0$.

The nonnegative integer $c_0$ is called the characteristic of $\varphi$ and we will use the notation $G_0$ and $G_{c+1}$, respectively, for the subclasses (a) and (b).

Once you know your operator is continuous the next step is to study whether it is compact. In our context, we try to characterize compactness of $C_\varphi$ from properties of its symbol $\varphi$. This has
been investigated in many papers (like [1, 3, 4, 7–9, 13]). Following the seminal paper of Shapiro [14] for composition operators on $H^2(D)$, a natural way for doing so is to characterize compactness of $C_\varphi$ by mean of some counting function related to $\varphi$. This was recently achieved in [7] for the subclass $G_0$ of composition operators with zero-characteristic.

In this paper, we mostly concentrate on the subclass $G_{\geq 1}$. When the symbol stays in this subclass, it is not harder to work in the Hardy spaces $H^p$ for $p \in [1, +\infty)$. These spaces are defined as the closure of the set of Dirichlet polynomials, namely the finite sums $\sum_{n=1}^{N} a_n n^{-s}$, with respect to

$$
\|f\|_p^p = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p dt.
$$

It is known that if $\varphi : C_{1/2} \to C_{1/2}$ generates a bounded composition operator on $H^p$, then $\varphi$ belongs to $G$ and that the converse is true provided $\varphi \in G_{\geq 1}$ (see [2]).

Inspired by [14], the Nevanlinna counting function of $\varphi = c_0 s + \psi \in G_{\geq 1}$ was introduced in [3]. It is defined on $C_0$ by

$$
\mathcal{N}_\varphi(w) = \sum_{\varphi(s) = w} \Re(s), \ w \in C_0.
$$

In that paper, it was shown that provided $|\Im m(\psi)|$ is bounded, the condition $\mathcal{N}_\varphi(w) = o(\Re(w))$ as $\Re(w)$ tends to $0$ implies the compactness of $C_\varphi$ on $H^2$ (this was generalized to $H^p$ in [5]). Conversely, in [1], Bailleul established that if $\psi$ is supported on a finite set of prime numbers and is finitely valent, the compactness of $C_\varphi$ implies the above condition on $\mathcal{N}_\varphi$. Moreover, these two properties are equivalent if $\psi$ is supported on a single prime number (see [8]). We recall that if $Q$ is a subset of the set of prime numbers, a Dirichlet series $\sum_n a_n n^{-s}$ is supported on $Q$ provided $a_n = 0$ as soon as there exists a prime number not in $Q$ such that $p|n$.

Our aim here is to show that the result of [3] is still true without any assumption on $|\Im m(\psi)|$.

**Theorem 1.1.** Let $\varphi \in G_{\geq 1}$ and let us assume that $\mathcal{N}_\varphi(w) = o(\Re(w))$ as $\Re(w)$ tends to $0$. Then $C_\varphi$ is compact on $H^p$, $p \in [1, +\infty)$.

### 1.2 Background material

We briefly review some basic facts on Dirichlet series (see [2] or [10] and the references therein for details). Let $\mathbb{T}^\infty$ be the infinite polycircle endowed with its Haar measure $m$. It can be identified with the group of characters of $(\mathbb{Q}_+, +)$ via the prime number factorization: to any $z \in \mathbb{T}^\infty$ we associate the character $\chi$ defined by

$$
\chi(n) = z_{\alpha_1} \cdots z_{\alpha_d} \text{ for } n = \prod_{j=1}^{d} p_{\alpha_j}.
$$

For $f(s) = \sum_n a_n n^{-s}$ a Dirichlet series and $\chi$ a character, we denote by $f_\chi$ the Dirichlet series $f_\chi(s) = \sum_n a_n \chi(n) n^{-s}$. If $f$ converges uniformly in $\mathbb{C}_\theta$ for some $\theta \in \mathbb{R}$, then for any $\chi \in \mathbb{T}^\infty$, there exists a sequence of real numbers $(\tau_n)$ such that $f_\chi$ is the uniform limit in $\mathbb{C}_\theta$ of the vertical translates $(f(\cdot + i\tau_n))$. Conversely all uniform limits in $\mathbb{C}_\theta$ of vertical translates are equal to some $f_\chi$, which justifies that the functions $f_\chi$ are called vertical limit functions.
If we now assume that $f$ belongs to $H^p$, $p \geq 1$, then for almost every $\chi \in \mathbb{T}^\infty$, $f_\chi$ converges in $C_0$ and one can estimate the norm of $f \in H^p$ via the following Littlewood–Paley formula (see [5]): if $\mu$ is any finite positive measure on $\mathbb{R}$,

$$
\|f\|_p^p \simeq |f(+\infty)|^p + \int_{\mathbb{T}^\infty} \int_0^{+\infty} \int_{\mathbb{R}} |f_\chi(s)|^{p-2} |f'_\chi(s)|^2 \sigma^2 d\sigma d\mu(t) dm(\chi).
$$

As usual, the notation $u(x) \asymp v(x)$ will mean that there exists $C \geq 1$ such that $C^{-1} u(x) \leq v(x) \leq C u(x)$ for all $x$, whereas $u(x) \ll v(x)$ will mean that there exists $C > 0$ such that $u(x) \leq C v(x)$ for all $x$.

Regarding composition operators, the notion of vertical limits is extended to symbols $\varphi \in \mathcal{G}_{\geq 1}$ by defining $\varphi_\chi = c_0 s + \psi_\chi$. The composition operators $C_\varphi$ and $C_{\varphi_\chi}$ are related by the formula

$$(f \circ \varphi)_\chi = f_\chi c_0 \circ \varphi_\chi.$$

### 2 ON COUNTING FUNCTIONS FOR DIRICHLET SERIES

Let $\varphi$ belong to $\mathcal{G}_{\geq 1}$. The classical Nevanlinna counting function associated to $\varphi$ is defined on $C_0$ by

$$N_\varphi(w) = \sum_{\varphi(s) = w} \Re(s), \ w \in C_0.$$ 

It was introduced in [3] where the two following important properties were proved:

(NC1) $N_\varphi(w) \leq \frac{1}{c_0} \Re(w)$ for all $w \in C_0$.

(NC2) If $N_\varphi(w) = o(\Re(w))$ as $\Re(w) \to 0$, then for all $\varepsilon > 0$, there exists $\theta > 0$ such that, for all $w \in C_0$ with $\Re(w) < \theta$, for all $\chi \in \mathbb{T}^\infty$, $N_{\varphi_\chi}(w) \leq \varepsilon \Re(w)$ (namely, the $o$ bound is uniform with respect to $\chi \in \mathbb{T}^\infty$).

In this paper, we will also be interested in a restricted version of the counting function, which is defined by

$$N_\varphi(w) = \sum_{\varphi(s) = w, \ |\Im(m(s)| < 1} \Re(s), \ w \in C_0.$$ 

A similar restricted counting function has been introduced in [8], when $\varphi$ is supported on a single prime number. The main interest of working with $N_\varphi$ instead of $N'_\varphi$ comes from the following enhancement of (NC1), which is inspired by [8, Lemma 6.3].

**Proposition 2.1.** Let $\varphi = c_0 s + \psi \in \mathcal{G}_{\geq 1}$. There exists $C > 0$ such that, for all $\chi \in \mathbb{T}^\infty$, for all $w \in C_0$ with $\Re(w) < c_0$,

$$N_{\varphi_\chi}(w) \leq C \frac{\Re(w)}{1 + (\Im(m(w)))^2}.$$
Proof. Let $\Theta$ be the conformal map from $\mathbb{D}$ onto the half-strip

$$S = \{s = \sigma + it : \sigma > 0, |t| < 2\}$$

normalized by $\Theta(0) = 2$ and $\Theta'(0) > 0$. By standard regularity results on conformal maps, there exists $C_1 > 0$ such that, for all $s$ with $0 < \Re(s) < 1$ and $|\Im(s)| < 1$,

$$\Re(s) \leq C_1 \log \frac{1}{|\Theta^{-1}(s)|}. \quad (2.1)$$

Fix $w \in C_0$ and $\chi \in \mathbb{T}^\infty$ with $0 < \Re(w) < c_0$. Define $G$ on $\mathbb{D}$ by

$$G(z) = G_{w,\chi}(z) : = \frac{\varphi_{\chi}(\Theta(z)) - w}{\varphi_{\chi}(\Theta(z)) + \bar{w}}, \quad (2.2)$$

which is a self-map of $\mathbb{D}$. The Littlewood inequality for the standard Nevanlinna counting function of $G$ says that

$$\sum_{z \in G^{-1}(\{0\})} \log \frac{1}{|z|} \leq \log \frac{1}{|G(0)|}. \quad (2.3)$$

Now $G(z) = 0$ if and only if $\varphi_{\chi}(\Theta(z)) = w$, so that (2.3) becomes

$$\sum_{\Theta(z) \in \varphi_{\chi}^{-1}(\{w\})} \log \frac{1}{|z|} \leq \log \left| \frac{\varphi_{\chi}(\Theta(0)) + \bar{w}}{\varphi_{\chi}(\Theta(0)) - w} \right|.$$

which itself can be rewritten as

$$\sum_{s \in \varphi_{\chi}^{-1}(\{w\})} \log \frac{1}{|\Theta^{-1}(s)|} \leq \log \left| \frac{\varphi_{\chi}(2) + \bar{w}}{\varphi_{\chi}(2) - w} \right|.$$

Observe now that, when $\varphi_{\chi}(s) = w$, then $0 < \Re(s) < 1$ so that, using (2.1),

$$N_{\varphi_{\chi}}(w) = \sum_{\varphi_{\chi}(s) = w} \Re(s)$$

$$\leq C_1 \sum_{s \in \varphi_{\chi}^{-1}(\{w\})} \log \frac{1}{|\Theta^{-1}(s)|}$$

$$\leq C_1 \log \left| \frac{\varphi_{\chi}(2) + \bar{w}}{\varphi_{\chi}(2) - w} \right|.$$

Now we apply [7, Lemma 2.3] to get

$$N_{\varphi_{\chi}}(w) \leq 2C_1 \frac{\Re(\varphi_{\chi}(2))\Re(w)}{|w - \varphi_{\chi}(2)|^2}$$

$$\leq 2C_1 \frac{\Re(\varphi_{\chi}(2))\Re(w)}{(\Re(\varphi_{\chi}(2)) - \Re(w))^2 + (\Im(\varphi_{\chi}(2)) - \Im(w))^2}.$$
Our restriction on $\Re e(w)$ shows that $\Re e(\varphi_\chi(2)) - \Re e(w) \geq c_0$. On the other hand, as $\psi$ is a Dirichlet series uniformly convergent in each $C_\varepsilon, \varepsilon > 0, \psi(2 + i\varepsilon)$ is bounded. $\varphi_\chi$, being a vertical limit function of $\psi$, there exists $C_2 > 0$ such that $|\varphi_\chi(2)| \leq C_2$ for all $\chi \in \mathbb{T}^\infty$. If $|\Im m(w)| \leq 2C_2$, then we write

$$N_{\varphi_\chi}(w) \leq \frac{2C_1C_2\Re e(w)}{c_0^2} \leq \frac{2C_1C_2}{c_0^2} \frac{\Re e(w)}{(1 + (\Im m(w))^2)},$$

whereas, if $|\Im m(w)| > 2C_2$, then $\Im m(w) - \Im m(\varphi_\chi(2)) \geq \frac{1}{2}\Im m(w)$ that yields

$$N_{\varphi_\chi}(w) \leq 2C_1C_2 \frac{\Re e(w)}{c_0^2 + \frac{1}{4}(\Im m(w))^2}.$$

Therefore, in all cases, Proposition 2.1 is proved.

Proposition 2.1 will be used to give a uniform bound on $N_{\varphi_\chi}$, in the spirit of (NC2), upon the assumption $\mathcal{N}_\varphi(w) = o(\Re e(w))$.

**Corollary 2.2.** Let $\varphi \in \mathcal{G}_{\geq 1}$ and $\delta \in (0, 1)$. Let us assume that $\mathcal{N}_\varphi(w) = o(\Re e(w))$ as $\Re e(w)$ goes to 0. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\chi \in \mathbb{T}^\infty$, for all $w \in C_0$ with $\Re e(w) < \delta$,

$$N_{\varphi_\chi}(w) \leq \varepsilon \frac{\Re e(w)}{(1 + (\Im m(w))^2)^{(1+\delta)/2}}. \quad (2.4)$$

**Proof.** Assume that the result is false. Then we can find $\varepsilon > 0$, a sequence $(w_k)$ in $C_0$ with $\Re e(w_k) \to 0$ and a sequence $(\chi_k)$ in $\mathbb{T}^\infty$ such that, for all $k$,

$$N_{\varphi_\chi_k}(w_k) \geq N_{\varphi_\chi_k}(w_k) > \varepsilon \frac{\Re e(w_k)}{(1 + (\Im m(w_k))^2)^{(1+\delta)/2}}.$$

If $\Im m(w_k)$ is unbounded, this contradicts Proposition 2.1, whereas if $\Im m(w_k)$ is bounded, this contradicts that $\mathcal{N}_\varphi(w_k) = o(\Re e(w_k))$ and in particular (NC2).

## 3 | COMPACT COMPOSITION OPERATORS

### 3.1 | Compactness on Hardy spaces

**Proof of Theorem 1.1.** Let $(f_n)$ be a sequence in $H^p$ that converges weakly to zero. Let $f_{n,\chi}$ denote the vertical limit function of $f_n$ with respect to the character $\chi$. By the Littlewood–Paley formula applied with $d\mu(t) = \frac{1}{2}1_{[-1,1]}dt$, setting $s = \sigma + it$,

$$\|C_\varphi(f_n)\|_p^p \leq |f_n(+\infty)|^p + \int_{-\infty}^{\infty} \int_{\mathbb{R}^+} \int_{-1}^{1} |f_{n,\chi}^{-1}(\varphi_\chi(s))|^2 |f_{n,\chi}^{-1}(\varphi_\chi(s))|^{p-2} |\varphi_\chi'(s)|^2 |\sigma| \, ds \, dt \, dm(\chi).$$
Our assumption on \((f_n)\) implies that \((f_n(+\infty))\) tends to zero. In the inner-most integrals, we do the nonunivalent change of variables \(w = u + iv = \varphi(\sigma + it)\). Observe that this change of variables involves the restricted Nevanlinna counting function \(N_{\varphi}\), whereas in [3] we used \(\tilde{N}_{\varphi}\) (but we only obtained an inequality). Hence,

\[
\|C_{\varphi}(f_n)\|_p^p \lesssim |f_n(+\infty)|^p + \int_{T^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}} |f_{n,\mathcal{C}_0}'(w)|^2 |f_{n,\mathcal{C}_0}(w)|^{p-2} N_{\varphi'}(w) \text{d}u \text{d}v \text{d}m(\mathcal{C}).
\]

Now let \(\varepsilon > 0\) and let \(\theta > 0\) be given by Corollary 2.2 for \(\delta = 1/2\). We split the integral over \(\mathbb{R}_+\) into \(\int_0^\theta + \int_\theta^{+\infty}\). For the first integral, say \(I_0\), we use (2.4) to get

\[
I_0 \lesssim \varepsilon \int_{T^\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}} |f_{n,\mathcal{C}_0}'(w)|^2 |f_{n,\mathcal{C}_0}(w)|^{p-2} u \frac{dv}{(1+v^2)^{3/4}} \text{d}m(\mathcal{C})
\lesssim \varepsilon \|f_n\|^2,
\]

where we have used the Littlewood–Paley equality with the finite measure \(d\mu(v) = \frac{dv}{(1+v^2)^{3/4}}\). To handle the integral over \([\theta, +\infty]\), say \(I_1\), we now use Proposition 2.1 to write

\[
I_1 \lesssim \int_{T^\infty} \int_{\theta/2}^{+\infty} \int_{\mathbb{R}} |f_{n,\mathcal{C}_0}'(w)|^2 |f_{n,\mathcal{C}_0}(w)|^{p-2} u \frac{dv}{1+v^2} \text{d}m(\mathcal{C})
\lesssim \int_{T^\infty} \int_{\theta/2}^{+\infty} \int_{\mathbb{R}} |f_{n,\mathcal{C}}'(w + \theta/2)|^2 |f_{n,\mathcal{C}}(w + \theta/2)|^{p-2} u \frac{dv + \theta/2}{1+v^2} \text{d}m(\mathcal{C})
\lesssim \int_{T^\infty} \int_{\theta/2}^{+\infty} \int_{\mathbb{R}} |f_{n,\mathcal{C}}'(w + \theta/2)|^2 |f_{n,\mathcal{C}}(w + \theta/2)|^{p-2} u \frac{dv}{1+v^2} \text{d}m(\mathcal{C})
\lesssim \|f_n(\cdot + \theta/2)\|^p.
\]

This last quantity goes to zero because the horizontal translation operator \(f(\cdot) \mapsto f(\cdot + \theta/2)\) acts compactly on \(H^p\).

\[\square\]

Remark 3.1. Proposition 2.1 provides also an alternative approach to the continuity of \(C_{\varphi}\) when \(\varphi \in \mathcal{G}_{\varphi > 1}\). Nevertheless, as it is written, we lose the fact that \(C_{\varphi}\) is a contraction.

### 3.2 Compactness on Bergman spaces

In this section, we turn to the Bergman spaces of Dirichlet series introduced by McCarthy in [12]. For \(\alpha > -1\) define

\[
A_\alpha = \left\{ f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} : \|f\|_\alpha = \sum_{n=1}^{+\infty} \frac{|a_n|^2}{(1 + \log n)^{1+\alpha}} < +\infty \right\}
\]
(as in the case of the disc, the Hardy space $\mathcal{H}^2$ corresponds to the limiting case $\alpha = -1$). The norm of an element $f$ of $\mathcal{A}_\alpha$ can be evaluated thanks to the following Littlewood–Paley formula (see [3]):

$$
\|f\|_{\mathcal{A}_\alpha}^2 \leq |f(+\infty)|^2 + \int_{\mathbb{T}^\infty} \int_0^{+\infty} \int_{\mathbb{R}} |f'(\sigma + it)|^2 \sigma^{2+\alpha} d\mu(t) d\sigma dm(\chi),
$$

where $\mu$ is any finite positive measure on $\mathbb{R}$. Moreover, if $f$ is a Dirichlet series that converges uniformly in each half-plane $\mathbb{C}_\varepsilon$, $\varepsilon > 0$, an easy adaptation of [7, Lemma 2.2] shows that

$$
\|f\|_{\mathcal{A}_\alpha}^2 \leq |f(+\infty)|^2 + \lim_{\sigma_0 \to 0^+} \lim_{T \to +\infty} \frac{1}{T} \int_{\sigma_0}^{\sigma_0} \int_{0}^{-T} |f'(\sigma + it)|^2 \sigma^{2+\alpha} dt d\sigma.
$$

(3.1)

Compactness of composition operators on $\mathcal{A}_\alpha$ has been studied in [3] (see also [1]). The interest of working in Bergman spaces is that one often can replace a sufficient condition on some counting function by a sufficient condition on the symbol itself. For instance, it is known that if $\varphi \in \mathcal{G}_{\varphi_1}$ such that $\mathfrak{R}m(\varphi)$ is bounded and $\mathfrak{R}e(\varphi(s))/\mathfrak{R}e(s)$ tends to $+\infty$ as $\mathfrak{R}e(s)$ goes to zero, then $\mathcal{C}_\varphi$ is compact on $\mathcal{A}_\alpha$. The converse is true if $\varphi$ is supported on a finite set of prime numbers. The case $\varphi \in \mathcal{G}_0$ is thoroughly studied in [11], using a counting function in the spirit of [7].

We now show the following sufficient condition that do not require any assumption on $\mathfrak{R}m(\varphi)$.

**Theorem 3.2.** Let $\varphi \in \mathcal{G}$ and $\alpha > -1$.

a) If $\varphi \in \mathcal{G}_{\varphi_1}$ and $\mathfrak{R}e(\varphi(s))/\mathfrak{R}e(s) \to +\infty$ as $\mathfrak{R}e(s) \to 0$, then $\mathcal{C}_\varphi$ is compact on $\mathcal{A}_\alpha$.

b) If $\varphi \in \mathcal{G}_0$ and $\mathfrak{R}e(\varphi(s))^{-1/2}/\mathfrak{R}e(s) \to +\infty$ as $\mathfrak{R}e(s) \to 0$, then $\mathcal{C}_\varphi$ is compact on $\mathcal{A}_\alpha$.

**Proof.** Let us start with (a). Let $N_{\alpha,\varphi} = \sum_{\varphi(s)=w} (\mathfrak{R}e(s))^{2+\alpha}$ be the appropriate Nevanlinna counting function for $\mathcal{A}_\alpha$. We first observe that, for all $\chi \in \mathbb{T}^\infty$ and for all $w \in \mathbb{C}_0$,

$$
N_{\alpha,\varphi_\chi}(w) \leq \left( \sum_{\varphi_\chi(s)=w} \mathfrak{R}e(s) \right)^{2+\alpha} \leq C \frac{(\mathfrak{R}e(w))^{2+\alpha}}{1 + (\mathfrak{R}m(w))^{2+\alpha}}
$$

for some $C > 0$. Moreover, let $\varepsilon > 0$. By assumption there exists $\vartheta > 0$ such that $\mathfrak{R}e(s) < \vartheta \Rightarrow \mathfrak{R}e(s) < \varepsilon \mathfrak{R}e(\varphi(s))$. For all $\chi \in \mathbb{T}^\infty$, one gets by vertical limit $\mathfrak{R}e(s) \leq \varepsilon \mathfrak{R}e(\varphi(s))$. Let $w \in \mathbb{C}_0$ with $\mathfrak{R}e(w) < \vartheta$ and let $\chi \in \mathbb{T}^\infty$. If $\varphi^{-1}(\{w\}) = \emptyset$, then $N_{\alpha,\varphi_\chi}(w) = 0$. Otherwise, any $s \in \varphi^{-1}(\{w\})$ satisfy $\mathfrak{R}e(s) < \vartheta$ so that

$$
N_{\alpha,\varphi_\chi}(w) = \sum_{\varphi_\chi(s)=1} (\mathfrak{R}e(s))^{2+\alpha}
$$

\[\sum_{\varphi_\chi(s)=1} (\mathfrak{R}e(s))^{2+\alpha} \leq C \frac{(\mathfrak{R}e(w))^{2+\alpha}}{1 + (\mathfrak{R}m(w))^{2+\alpha}}\]
\[ \leq \varepsilon^{\alpha+1} (\text{Re}(w))^{\alpha+1} \sum_{\varphi(s)=1 \atop |\text{Im}(s)|<1} \text{Re}(s) \]
\[ \leq C \varepsilon^{\alpha+1} \frac{(\text{Re}(w))^{2+\alpha}}{1 + (\text{Im}(w))^2}. \]

where we have used Proposition 2.1. We then conclude exactly as in the proof of Theorem 1.1. Details are left to the reader.

Let us turn to (b). We are inspired by [7] but working in a Bergman space and using this very strong assumption will avoid most of the technical difficulties that appear here. First we observe that for any \( f \in A_\alpha \), \( f \circ \varphi \) is bounded on each half-plane \( C_\varepsilon, \varepsilon > 0 \). Therefore by Bohr’s theorem, \( f \circ \varphi \) converges uniformly on these half-planes, which implies that we can estimate the norm of \( f \circ \varphi \) using (3.1). We then consider, for \( \sigma_0, T > 0 \) and \( w \in \mathbb{C}_0 \), the counting function

\[ M_{\alpha, \varphi}(\sigma_0, T; w) = \frac{1}{T} \sum_{\varphi(s)=w \atop |\text{Im}(s)|<T, \text{Re}(s)\sigma_0} (\text{Re}(s))^{2+\alpha}. \]

The nonunivalent change of variables \( w = \varphi(s) \) leads to

\[ \|C_{f}f\|_\alpha^2 \simeq |f(\varphi(+\infty))|^2 + \lim_{\sigma_0 \to 0^+} \lim_{T \to +\infty} \int_{C_{1/2}} |f'(w)|^2 M_{\alpha, \varphi}(\sigma_0, T; w) d\sigma dt. \]

Let \( \varepsilon > 0 \). By assumption, we may find \( \theta \in (0, \text{Re}(\varphi(+\infty))/2) \) such that \( \text{Re}(\varphi(s)) < \frac{1}{2} + \theta \) implies \( \text{Re}(s) < \varepsilon(\text{Re}(\varphi(s)) - \frac{1}{2}) \). Then

\[ M_{\alpha, \varphi}(\sigma_0, T; w) \leq \frac{1}{T} \sum_{\varphi(s)=w \atop |\text{Im}(s)|<T} (\text{Re}(s))^{2+\alpha} \]
\[ \leq \varepsilon^{1+\alpha} \left( \text{Re}(w) - \frac{1}{2} \right)^{1+\alpha} \frac{1}{T} \sum_{\varphi(s)=w \atop |\text{Im}(s)|<T} \text{Re}(s) \]
\[ \leq C \frac{\varepsilon^{1+\alpha} \left( \text{Re}(w) - \frac{1}{2} \right)^{2+\alpha}}{|w - \varphi(+\infty)|^2} \]

by [7, Lemmas 2.3 and 2.4] where the constant \( C \) is uniform in \( T \) for all \( T \geq \sigma_1 \), for some \( \sigma_1 > 0 \). The compactness of \( C_{\varphi} \) now follows from an argument similar to that of [7, Theorem 1.4], using that the estimate on \( M_{\alpha, \varphi}(\sigma_0, T; w) \) for \( \text{Re}(w) < \theta \) is uniform with respect to \( \sigma_0 \) and \( T \).

Taking into account the necessary conditions obtained in [3], we finally get:

**Corollary 3.3.** Let \( \varphi \in \mathcal{G} \) and \( \alpha > -1 \).
(1) If $\varphi \in C_{\geq 1}$ and $\varphi$ is supported on a finite set of prime numbers, then

$$C_{\varphi} \text{ is compact on } A_{\alpha} \iff \frac{\Re(\varphi(s))}{\Re(s)} \xrightarrow{\Re(s) \to 0} +\infty.$$ 

(2) If $\varphi \in C_0$ and $\varphi$ is supported on a single prime number, then

$$C_{\varphi} \text{ is compact on } A_{\alpha} \iff \frac{\Re(\varphi(s)) - 1/2}{\Re(s)} \xrightarrow{\Re(s) \to 0} +\infty.$$ 

Remark 3.4. For the case of positive characteristic, we do not know whether the condition $\frac{\Re(\varphi(s))}{\Re(s)} \to +\infty$ is always necessary for compactness.

3.3 | Concluding remarks

**Question 3.5.** On $H^2$, can we get a necessary condition using some counting function without any extra assumption on $\varphi \in C_{\geq 1}$? Or at least, if $\varphi \in C_{\geq 1}$ is supported on a finite set of prime numbers, do we have

$$C_{\varphi} \text{ is compact on } H^2 \iff \mathcal{N}_{\varphi}(w) = o(\Re(w)) \text{ as } \Re(w) \to 0?$$

Remark 3.6. The condition $\frac{\Re(\varphi(s)) - 1/2}{\Re(s)} \xrightarrow{\Re(s) \to 0} +\infty$ is not necessary for compactness on $A_{\alpha}$ when the symbol is not supported on a single prime number. For instance, $\varphi(s) = \frac{5}{2} - 2^{-s} - 3^{-s}$ generates a compact operator on $A_{\alpha}$ (the proof of [4, Theorem 2], done in $H^2$, can be adapted to $A_{\alpha}$) that does not satisfy the above condition.

Remark 3.7. We can also use the restricted Nevanlinna counting function introduced here to simplify the results of [6], deleting an unnecessary assumption of boundedness. For instance, we can replace Theorem 5.5 of [6] by: let $\varphi_0$ and $\varphi_1 \in C_{\geq 1}$ and write them $\varphi_0 = c_0 s + \psi_0$, $\varphi_1 = c_0 s + \psi_1$. Assume moreover that there exists $C > 0$ such that

- $|\varphi_0 - \varphi_1| \leq C \min(\Re\varphi_0, \Re\varphi_1)$;
- $|\varphi'_0 - \varphi'_1| \leq C$.

Then $C_{\varphi_0}$ and $C_{\varphi_1}$ belong to the same component of $C(H)$, the set of composition operators on $H$. If moreover we assume that

$$|\varphi_0 - \varphi_1| = o(\min(\Re\varphi_0, \Re\varphi_1)) \text{ and } \varphi'_0 - \varphi'_1 \to 0 \text{ as } \min(\Re\varphi_0, \Re\varphi_1) \to 0,$$

then $C_{\varphi_0} - C_{\varphi_1}$ is compact.

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