Ill-posedness for a two-component Novikov system in Besov space

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Abstract: In this paper, we consider the Cauchy problem for a two-component Novikov system on the line. By specially constructed initial data \((\rho_0, u_0)\) in \(B^{s-1}_{p,\infty}(\mathbb{R}) \times B^s_{p,\infty}(\mathbb{R})\) with \(s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}\) and \(1 \leq p \leq \infty\), we show that any energy bounded solution starting from \((\rho_0, u_0)\) does not converge back to \((\rho_0, u_0)\) in the metric of \(B^{s-1}_{p,\infty}(\mathbb{R}) \times B^s_{p,\infty}(\mathbb{R})\) as time goes to zero, thus results in discontinuity of the data-to-solution map and ill-posedness.

Keywords: Two-component Novikov system; ill-posedness; Besov spaces

MSC (2020): 35B30; 37K10

1 Introduction

The two-component Novikov system takes the form

\[
\begin{align*}
\rho_t &= \rho_x u^2 + \rho u u_x, \quad t > 0, \quad x \in \mathbb{R}, \\
m_t &= 3 u_x u m + u^2 m_x - \rho (u \rho)_x, \quad t > 0, \quad x \in \mathbb{R}, \\
m &= u - u_{xx}, \\
\rho(0, x) &= \rho_0, \quad u(0, x) = u_0,
\end{align*}
\]  

which was recently proposed by Popowicz [23] as the two-component generalization of the Novikov equation and can be rewritten in the Hamiltonian form (see [23] for details).

We recall that system (1.1) is well-posed in the sense of Hadamard (see [1]) in Besov spaces \(B^{s-1}_{p,r} \times B^s_{p,r}\) with \(s > \max\{1 + \frac{1}{p}, \frac{5}{2}\}\), \(1 \leq p \leq \infty\), \(1 \leq r < \infty\) as well as in the critical Besov space \(B^{\frac{1}{2}}_{2,1} \times B^{\frac{3}{2}}_{2,1}\) (see [20]). More precisely, if \((\rho_0, u_0)\) belongs to \(B^{s-1}_{p,r} \times B^s_{p,r}\) on the line, then there exists \(T = T(\|\rho_0\|_{B^{s-1}_{p,r}}, \|u_0\|_{B^s_{p,r}})\), and a unique solution \((\rho, u) \in L^\infty(0, T; B^{s-1}_{p,r} \times B^s_{p,r})\) satisfying system (1.1) with continuous dependence on initial data. Note that if we want to include the endpoint case \(r = \infty\), then we need to weaken the notion of well-posedness since the continuity of data-to-solution map in \(B^{s-1}_{p,\infty} \times B^s_{p,\infty}\) is still unknown.

If we set \(\rho = 0\), then system (1.1) becomes the famous Novikov equation

\[
m_t = 3 u_x u m + u^2 m_x, \quad m = u - u_{xx},
\]

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which was derived by Novikov [22] as a new integrable equation with cubic nonlinearities. It is shown the Novikov equation is integrable in the sense of having a Lax pair in matrix form, a bi-Hamiltonian structure as well as possessing infinitely many conserved quantities [16]. Furthermore, it admits peakon solutions given by the formula \( u(x, t) = \pm \sqrt{c} e^{\pm |x - ct|} \) (\( c=\text{const.} \) is the wave speed) and also multipeakon traveling wave solutions.

The local well-posedness for Novikov equation was initially established in the Sobolev spaces \( H^s \) with \( s > \frac{3}{2} \) on both the line and the circle by Himonas and Holliman [14], and then by Ni and Zhou [21] in the critical Besov space \( B_{2,1}^{\frac{3}{2}}(\mathbb{R}) \). Later, the well-posed space was extended to a larger class of Besov spaces \( B_{p,r}^s(\mathbb{R}) \), \( s > \max\{1 + \frac{1}{p}, \frac{3}{2}\} \), \( 1 \leq p \leq \infty \), \( 1 \leq r < \infty \) by Yan, Li and Zhang [24]. Moreover, by using the peakon traveling wave solution they showed that the local well-posedness fails in \( B_{2,\infty}^{\frac{3}{2}}(\mathbb{R}) \) due to failure of continuity. Several years later, by constructing a 2-peakon solution with an asymmetric antipeakon-peakon, Himonas, Holliman and Kenig [15] proved ill-posedness for the Novikov equation in \( H^s \) for \( s < \frac{3}{2} \) on both the line and the circle. In fact, a norm-inflation occurs (the corresponding solution has infinite \( H^s \) norm instantaneously at \( t > 0 \), though the initial \( H^s \) norm can be arbitrarily small) and gives rise to discontinuity when \( \frac{5}{4} < s < \frac{3}{2} \), while for \( s < \frac{5}{4} \), the solution is not unique, and either continuity or uniqueness fails when \( s = \frac{5}{4} \). Very recently, Li et al. [18] obtained the ill-posedness of the Novikov equation in \( B_{2,\infty}^s(\mathbb{R}) \) with \( s > \frac{7}{2} \) due to the discontinuity.

The motivation to investigate the Novikov equation is that it can be viewed as a cubic generalization of the classical Camassa-Holm (CH) equation

\[
m_t + 2u_x m = um_x, \quad m = u - u_{xx}. \tag{1.3}
\]

The CH equation with quadratic nonlinearity was initially derived by Foka and Fuchssteiner [12] as a bi-Hamiltonian system in the context of the KdV model and gained prominence after Camassa-Holm [4] independently re-derived it as an approximation to the Euler equations of hydrodynamics. Constantin and Lannes [8] later educated the CH equation from the water waves equations. The CH equation is completely integrable in the sense of having a Lax pair, a bi-Hamiltonian structure as well as possessing an infinity of conservation laws, and it also admits exact peakon solutions of the form \( ce^{-|x-ct|} \) [4, 5, 6, 7, 9, 12].

The local existence, uniqueness and continuity for the CH equation was initially established in the Sobolev spaces \( H^s(\mathbb{R}) \) with \( s > \frac{3}{2} \) by Li and Olver [17] as also Blanco [2], and then was extended by Danchin to a larger class of Sobov spaces \( B_{p,r}^s(\mathbb{R}) \) with \( s > \max\{1 + \frac{1}{p}, \frac{3}{2}\} \), \( 1 \leq p \leq \infty \) and \( 1 \leq r < \infty \) [10] and in the critical Besov space \( B_{2,1}^{\frac{3}{2}}(\mathbb{R}) \) [11]. It was also showed in [11] that the CH equation is ill-posed in \( B_{2,\infty}^{\frac{3}{2}}(\mathbb{R}) \) due to discontinuity by using peakon solution. As the CH equation is ill-posed in the Sobolev spaces \( H^s \) for \( s < \frac{3}{2} \) in the sense of norm-inflation [3], it means that the critical Sobolev exponent for well-posedness is \( \frac{3}{2} \) and it was recently solved by Guo et al. in [13], where they proved the ill-posedness of the CH equation in \( B_{p,r}^{1+\frac{1}{p}} \) with \( 1 \leq p \leq \infty \) and \( 1 < r \leq \infty \) in the sense of norm-inflation, especially in \( H^s \). Very recently, Li et al. [18] showed that the CH equation is ill-posed in \( B_{p,\infty}^s \) for \( 1 \leq p \leq \infty \), \( s > 2 + \max\{1 + \frac{1}{p}, \frac{3}{2}\} \) due to the failure of continuity.

The above results do not extend clearly to the two-component Novikov system (1.1), however, using some ideas from [19, 20], we are able to deal with the coupled system with these two components of the solution in different Besov spaces and establish the ill-posedness for the two-component Novikov system (1.1) in a broader range of Besov spaces.
For studying the ill-posedness of the two-component Novikov system, it is more convenient to rewrite (1.1) in the following equivalent nonlocal form

\[
\begin{aligned}
\rho_t &= u^2 \rho_x + \rho uu_x, \\
{u}_t &= u^2 u_x + \mathcal{P}(u) + \mathcal{Q}(u, \rho), \\
\rho(0, x) &= \rho_0, \ u(0, x) = u_0,
\end{aligned}
\]  

(1.4)

where \( \mathcal{P}(u) = \mathcal{P}_1(u) + \mathcal{P}_2(u) + \mathcal{P}_3(u) \), \( \mathcal{Q}(u, \rho) = \mathcal{Q}_1(u, \rho) + \mathcal{Q}_2(u, \rho) \) and

\[
\begin{aligned}
\mathcal{P}_1(u) &= \partial_x (1 - \partial_x^2)^{-1}(u^3), \\
\mathcal{P}_2(u) &= \frac{3}{2} \partial_x (1 - \partial_x^2)^{-1}(uw_x^2), \\
\mathcal{P}_3(u) &= \frac{1}{2} (1 - \partial_x^2)^{-1}(u_x^3), \\
\mathcal{Q}_1(u, \rho) &= -\frac{1}{2} \partial_x (1 - \partial_x^2)^{-1}(u \rho^2), \\
\mathcal{Q}_2(u, \rho) &= -\frac{1}{2} (1 - \partial_x^2)^{-1}(u_x \rho^2).
\end{aligned}
\]

Now our main result is stated as follows.

**Theorem 1.1** Let

\[ s > \max\{2 + \frac{1}{p}, \frac{5}{2}\}, \ 1 \leq p \leq \infty. \]

Then the two-component Novikov system (1.1) is ill-posed in the Besov spaces \( B^{s-1}_{p,\infty}(\mathbb{R}) \times B^s_{p,\infty}(\mathbb{R}) \). More precisely, there exist \((\rho_0, u_0) \in B^{s-1}_{p,\infty} \times B^s_{p,\infty}\) and a positive constant \( \delta \) for which the Cauchy problem of system (1.1) has a unique solution \((\rho, u) \in L^{\infty}(0, T; B^{s-1}_{p,\infty} \times B^s_{p,\infty})\) for some \( T = T(\|\rho_0\|_{B^{s-1}_{p,r}}, \|u_0\|_{B^s_{p,r}}) \), while

\[
\liminf_{t \to 0} (\|\rho - \rho_0\|_{B^{s-1}_{p,\infty}} + \|u - u_0\|_{B^s_{p,\infty}}) \geq C\delta.
\]

That is to say, any energy bounded solution starting from \((\rho_0, u_0)\) does not converge back to \((\rho_0, u_0)\) in the metric of \( B^{s-1}_{p,\infty}(\mathbb{R}) \times B^s_{p,\infty}(\mathbb{R}) \) as time goes to zero, thus results in discontinuity of the data-to-solution map and ill-posedness.

**Notations:** For \( f = (f_1, f_2, \ldots, f_n) \in X \),

\[ \|f\|_X^2 = \|f_1\|_X^2 + \|f_2\|_X^2 + \ldots + \|f_n\|_X^2. \]

Throughout this paper, \( C, C_i (i = 1, 2, 3, \ldots) \) stand for universal constant which may vary from line to line.

## 2 Littlewood-Paley analysis

In this section, we give a summary of the definition of Littlewood-Paley decomposition and non-homogeneous Besov space, and then list some useful properties. For more details, the readers can refer to [1].

There exists a couple of smooth functions \((\chi, \varphi)\) valued in \([0, 1]\), such that \( \chi \) is supported in the ball \( B \triangleq \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3} \} \), \( \varphi \) is supported in the ring \( C \triangleq \{ \xi \in \mathbb{R}^d : \frac{2}{3} \leq |\xi| \leq \frac{5}{3} \} \). Moreover,

\[
\forall \ \xi \in \mathbb{R}^d, \ \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1,
\]

\[
\forall \ \xi \in \mathbb{R}^d \setminus \{0\}, \ \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1,
\]

\[ \chi, \varphi \leq 1. \]
Then, we can define the nonhomogeneous dyadic blocks $\Delta_j$ as follows:

$$
\Delta_j u = 0, \text{ if } j \leq -2, \quad \Delta_{-1} u = \chi(D) u = \mathcal{F}^{-1}(\chi \mathcal{F} u),
$$

$$
\Delta_j u = \varphi(2^{-j} D) u = \mathcal{F}^{-1}(\varphi(2^{-j}) \mathcal{F} u), \text{ if } j \geq 0.
$$

**Remark 2.1 ([1])** Following the construction of $\chi$ and $\varphi$, one has $\varphi(\xi) \equiv 1$ for $\frac{4}{3} \leq |\xi| \leq \frac{2}{3}$.

**Definition 2.1 ([1])** Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ consists of all tempered distribution $u$ such that

$$
||u||_{B^s_{p,r}(\mathbb{R}^d)} \triangleq \left\| (2^{js} ||\Delta_j u||_{L^p(\mathbb{R}^d)})_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.
$$

Next, we list some basic lemmas and properties about Besov space which will be frequently used in proving our main result.

**Lemma 2.1 ([1])**

1. **Algebraic properties:** $\forall s > 0$, $B^s_{p,r}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a Banach algebra. $B^s_{p,r}(\mathbb{R}^d)$ is a Banach algebra $\iff B^s_{p,r}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \iff s > \frac{d}{p}$ or $s = \frac{d}{p}$, $r = 1$.

2. **Embedding:** $B^s_{p_1,r_1} \hookrightarrow B^s_{p_2,r_2}$ for $s > t$ or $s = t$, $r_1 < r_2$.

3. Let $m \in \mathbb{R}$ and $f$ be an $S^m$-multiplier (i.e., $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth and satisfies that $\forall \alpha \in \mathbb{N}^d$, there exists a constant $C_\alpha$ such that $|\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}$ for all $\xi \in \mathbb{R}^d$). Then the operator $f(D)$ is continuous from $B^s_{p,r}(\mathbb{R}^d)$ to $B^{-m}_{p,r}(\mathbb{R}^d)$.

4. Let $1 \leq p, r \leq \infty$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$. Then we have

$$
||uv||_{B^{s-2}_{p,r}(\mathbb{R})} \leq C||u||_{B^{s-2}_{p,r}(\mathbb{R})} ||v||_{B^{s-1}_{p,r}(\mathbb{R})}.
$$

**Lemma 2.2 ([1])** Let $s > 0$, $1 \leq p \leq \infty$, then we have

$$
\left\| 2^{js} ||\Delta_j u||_{L^p} \right\|_{L^\infty} \leq C(||\partial_x u||_{L^\infty} ||v||_{B^s_{p,\infty}} + ||\partial_x u||_{L^\infty} ||u||_{B^s_{p,\infty}}).
$$

Here, $[\Delta_j u]|_{\partial_x v} = \Delta_j (u |_{\partial_x v}) - u |_{\Delta_j (\partial_x v)}$.

### 3 Ill-posedness

In this section, we will give the proof of Theorem 1.1.

Firstly, choose the initial data

$$
\rho_0(x) = \sum_{n=0}^{\infty} 2^{-n(s-1)} \phi(x) \cos(\lambda 2^n x),
$$

$$
u_0(x) = \sum_{n=0}^{\infty} 2^{-ns} \phi(x) \cos(\lambda 2^n x),
$$

here $\lambda \in [\frac{67}{10}, \frac{69}{10}]$ and $\hat{\phi} \in C^\infty_0(\mathbb{R})$ is an even, real-valued and non-negative function satisfying

$$
\hat{\phi}(x) = \begin{cases} 
1, & \text{if } |x| \leq \frac{1}{4}, \\
0, & \text{if } |x| \geq \frac{1}{2}.
\end{cases}
$$
It is easy to show that
\[ \text{supp } \phi(\cdot) \cos(\lambda 2^n \cdot) \subset \{ \xi : -\frac{1}{2} + \lambda 2^n \leq |\xi| \leq \frac{1}{2} + \lambda 2^n \}, \]
then it can be verified in terms of Remark 2.1 that for \( j \geq 3 \)
\[ \Delta_j(\phi(x) \cos(\lambda 2^n x)) = \begin{cases} 
\phi(x) \cos(\lambda 2^n x), & j = n, \\
0, & j \neq n,
\end{cases} \quad (3.1) \]
hence we have
\[ \|\rho_0\|_{B^{s-1}_{p,r}} \leq C, \quad \|u_0\|_{B^{s}_{p,r}} \leq C. \quad (3.2) \]

Next, we present several estimates which play an important role in the proof of Theorem 1.1.

**Lemma 3.1** Let \( s > 0 \). Then for the above constructed initial data \((\rho_0, u_0)\), we have
\[ \|u_0^2 \partial_x \Delta_n u_0\|_{L^p} \geq C 2^{-n(s-1)}, \quad (3.3) \]
\[ \|u_0^2 \partial_x \Delta_n \rho_0\|_{L^p} \geq C 2^{-n(s-2)}, \quad (3.4) \]
for large enough \( n \).

**Proof** We just show (3.4) here, since (3.3) can be performed in a similar way.
Firstly, according to (3.1), we have
\[ \Delta_n \rho_0 = 2^{-n(s-1)} \phi(x) \cos(\lambda 2^n x), \]
hence, one has
\[ u_0^2 \partial_x \Delta_n \rho_0 = 2^{-n(s-1)} u_0^2 \partial_x \phi \cos(\lambda 2^n x) - \lambda 2^{-n(s-2)} u_0^2 \phi(x) \sin(\lambda 2^n x). \]
Since \( u_0^2(x) \) is a real valued continuous function on \( \mathbb{R} \), then there exists \( \sigma > 0 \),
\[ |u_0^2(x)| \geq \frac{1}{2} |u_0^2(0)| = \frac{1}{2} \left( \sum_{n=0}^{\infty} 2^{-n} \phi(0) \right)^2 = \frac{2^{2s-1} \phi^2(0)}{(2^s - 1)^2}, \text{ for any } x \in B_{\sigma}(0), \quad (3.5) \]
thus we obtain from (3.5) that
\[ \|u_0^2 \partial_x \Delta_n \rho_0\|_{L^p} \geq C 2^{-n(s-2)} \|\phi(\cdot) \sin(\lambda 2^n \cdot)\|_{L^p(B_{\sigma}(0))} - 2^{-n(s-1)} \|u_0^2 \partial_x \phi \cos(\lambda 2^n \cdot)\|_{L^p} \]
\[ \geq (C 2^n - C_1) 2^{-n(s-1)}, \]
which can yield (3.4) by choosing \( n \) large enough such that \( C_1 < C 2^{n-1} \). Thus we finish the proof of Lemma 3.1.

**Lemma 3.2** Let \( s > \max\{1 + \frac{1}{p}, \frac{3}{2}\} \). For the above constructed initial data \((\rho_0, u_0)\), then there exists some \( T = T(\|\rho_0\|_{B^{s-1}_{p,r}}, \|u_0\|_{B^{s}_{p,r}}) \), for \( 0 \leq t \leq T \), we have
\[ \|\rho(t) - \rho_0\|_{B^{s-2}_{p,r}} \leq C t, \quad \|u(t) - u_0\|_{B^{s-1}_{p,r}} \leq C t. \quad (3.6) \]
Proof Since \((\rho_0, u_0) \in B_{p,\infty}^{s-1} \times B_{p,\infty}^s\), according to the local existence result [20], the Cauchy problem of system (1.1) has a unique solution \((\rho, u) \in L^\infty(0, T; B_{p,\infty}^{s-1} \times B_{p,\infty}^s)\) for some \(T = T(\|\rho_0\|_{B_{p,r}^{s-1}}, \|u_0\|_{B_{p,r}^s})\), and

\[
\sup_{0 \leq t \leq T} (\|\rho(t)\|_{B_{p,\infty}^{s-1}} + \|u(t)\|_{B_{p,\infty}^s}) \leq C (\|\rho_0\|_{B_{p,\infty}^{s-1}} + \|u_0\|_{B_{p,\infty}^s}). \tag{3.7}
\]

For \(t \in [0, T]\), using the differential mean value theorem, the Minkowski inequality, Lemma 2.1 together with (3.7), we obtain that

\[
\|\rho - \rho_0\|_{B_{p,\infty}^{s-2}} \leq \int_0^t \|\partial_\tau \rho\|_{B_{p,\infty}^{s-2}} d\tau \leq \int_0^t \|\rho_u \partial_x u\|_{B_{p,\infty}^{s-2}} d\tau \leq C t,
\]

and

\[
\|u - u_0\|_{B_{p,\infty}^{s-1}} \leq \int_0^t \|\partial_\tau u\|_{B_{p,\infty}^{s-1}} d\tau \leq \int_0^t \|\rho_u \partial_x u\|_{B_{p,\infty}^{s-1}} d\tau \leq C t.
\]

Thus, we complete the proof of Lemma 3.2.

Lemma 3.3 Under the assumption of Theorem 1.1, for all \(0 \leq t \leq T\), we have

\[
\|\rho(t) - \rho_0 - tv_0\|_{B_{p,\infty}^{s-3}} \leq Ct^2, \quad \|u(t) - u_0 - tw_0\|_{B_{p,\infty}^{s-2}} \leq Ct^2. \tag{3.8}
\]

Here, \(v_0 = u_0^2 \partial_x \rho_0 + \rho_0 u_0 \partial_x u_0\), \(w_0 = \mathcal{P}(u_0) + \mathcal{Q}(u_0) + u_0^2 \partial_x u_0\).

Proof For simplicity, denote

\[
\begin{cases}
\tilde{\rho} = \rho(t) - \rho_0 - tv_0, \\
\tilde{u} = u(t) - u_0 - tw_0.
\end{cases}
\]

For \(t \in [0, T]\), firstly, using the differential mean value theorem and the Minkowski inequality, we arrive at

\[
\|\tilde{\rho}\|_{B_{p,\infty}^{s-3}} \leq \int_0^t \|\partial_\tau \rho - v_0\|_{B_{p,\infty}^{s-3}} d\tau \leq \int_0^t \|\rho_u \partial_x u - \rho_0 u_0 \partial_x u_0\|_{B_{p,\infty}^{s-3}} d\tau, \tag{3.9}
\]
By the definition of the Besov norm, we have

\[
\| \tilde{u} \|_{B^{s-2}_{p,\infty}} \leq \int_0^t \| \partial_\tau u - w_0 \|_{B^{s-2}_{p,\infty}} d\tau
\]
\[
\leq \int_0^t \| \mathcal{P}(u) - \mathcal{P}(u_0) \|_{B^{s-2}_{p,\infty}} d\tau + \int_0^t \| \mathcal{Q}(u) - \mathcal{Q}(u_0) \|_{B^{s-2}_{p,\infty}} d\tau
\]
\[
+ \int_0^t \| u^2 \partial_x u - u_0^2 \partial_x u_0 \|_{B^{s-2}_{p,\infty}} d\tau.
\]

(3.10)

For the term

\[
u^2 \partial_x \rho - u_0^2 \partial_x \rho_0 = (u - u_0)(u + u_0) \partial_x \rho + u_0^2 \partial_x (\rho - \rho_0),
\]
using (4), (2) of Lemma 2.1, Lemma 3.2 and (3.7), one has

\[
\| u^2 \partial_x \rho - u_0^2 \partial_x \rho_0 \|_{B^{s-2}_{p,\infty}} \leq C(\| \rho \|_{B^{s-1}_{p,\infty}} \| u - u_0 \|_{B^{s-1}_{p,\infty}} \| u, u_0 \|_{B^{s}_{p,\infty}} + \| \rho - \rho_0 \|_{B^{s-2}_{p,\infty}} \| u_0 \|_{B^{s}_{p,\infty}}^2)
\]
\[
\leq C\tau.
\]

(3.11)

Similarly,

\[
\rho u \partial_x u - \rho_0 u_0 \partial_x u_0 = (\rho - \rho_0) uu_x + \rho_0(u - u_0)u_x + \rho_0 u_0 \partial_x (u - u_0),
\]
and

\[
\| \rho u \partial_x u - \rho_0 u_0 \partial_x u_0 \|_{B^{s-3}_{p,\infty}} \leq C(\| \rho - \rho_0 \|_{B^{s-2}_{p,\infty}} \| u \|_{B^{s}_{p,\infty}}^2 + \| u - u_0 \|_{B^{s-1}_{p,\infty}} \| u_0 \|_{B^{s}_{p,\infty}}^2 + \| \rho_0 \|_{B^{s-1}_{p,\infty}}^2)
\]
\[
\leq C\tau.
\]

(3.12)

With the aid of (3), (4), (2) of Lemma 2.1, we can find that

\[
\| \mathcal{P}(u) - \mathcal{P}(u_0), \mathcal{Q}(u) - \mathcal{Q}(u_0) \|_{B^{s-2}_{p,\infty}} \leq C(\| u - u_0 \|_{B^{s-1}_{p,\infty}} \| u, u_0 \|_{B^{s}_{p,\infty}}^2 + \| \rho \|_{B^{s-1}_{p,\infty}}^2)
\]
\[
+ C(\| \rho - \rho_0 \|_{B^{s-2}_{p,\infty}} \| \rho \|_{B^{s-1}_{p,\infty}}^2 + \| u_0 \|_{B^{s}_{p,\infty}}^2),
\]
\[
\| u^2 \partial_x u - u_0^2 \partial_x u_0 \|_{B^{s-3}_{p,\infty}} \leq C(\| u - u_0 \|_{B^{s-1}_{p,\infty}} \| u, u_0 \|_{B^{s}_{p,\infty}}^2).
\]

(3.13)

(3.14)

Taking (3.11)-(3.12) into (3.9), (3.13)-(3.14) into (3.10) respectively, we obtain (3.8). Thus we finish the proof of Lemma 3.3.

With Lemma 2.2 and Lemma 3.1-3.3 at hand, we can now give the proof of Theorem 1.1.

**Proof of Theorem 1.1** By the definition of the Besov norm, we have

\[
\| \rho - \rho_0 \|_{B^{s-1}_{p,\infty}} \geq 2^{n(s-1)} \| \Delta_n (\rho - \rho_0) \|_{L^p}
\]
\[
= 2^{n(s-1)} \| \Delta_n (\tilde{\rho} + t v_0) \|_{L^p}
\]
\[
\geq t2^{n(s-1)} \| \Delta_n (u_0^2 \partial_x \rho_0 + \rho_0 u_0 \partial_x u_0) \|_{L^p} - 2^{n(s-1)} \| \Delta_n \tilde{\rho} \|_{L^p}
\]
\[
\geq t2^{n(s-1)} \| \Delta_n (u_0^2 \partial_x \rho_0) \|_{L^p} - t2^{n(s-1)} \| \Delta_n (\rho_0 u_0 \partial_x u_0) \|_{L^p} - 2^{n(s-1)} \| \Delta_n \tilde{\rho} \|_{L^p}
\]
\[
\geq t2^{n(s-1)} \| \Delta_n (u_0^2 \partial_x \rho_0) \|_{L^p} - Ct \| \rho_0 u_0 \partial_x u_0 \|_{B^{s-1}_{p,\infty}} - C2^{2n} \| \tilde{\rho} \|_{B^{s-1}_{p,\infty}}.
\]

(3.15)

Since

\[
\Delta_n (u_0^2 \partial_x \rho_0) = \Delta_n (u_0^2 \partial_x \rho_0) - u_0^2 \partial_x \Delta_n \rho_0 + u_0^2 \partial_x \Delta_n \rho_0
\]
\[
= [\Delta_n, u_0^2 \partial_x] \rho_0 + u_0^2 \partial_x \Delta_n \rho_0,
\]
making full use of Lemma 2.2, (1) of Lemma 2.1 and (3.2), we infer that
\[
\|2^{n(s-1)}[\Delta_n, u_0^2 \partial_x] \rho_0\|_{L^\infty} \leq C\|\partial_x (u_0^2)\|_{L^\infty}\|\rho_0\|_{B^{s-1}_{p,\infty}} + C\|\partial_x \rho_0\|_{L^\infty}\|u_0^2\|_{B^{s-1}_{p,\infty}} \\
\leq C\|u_0\|_{B^s_{p,\infty}}^2 \|\rho_0\|_{B^{s-1}_{p,\infty}} \leq C;
\]

\[
\|\rho_0 u_0 \partial_x u_0\|_{B^{s-1}_{p,\infty}} \leq C\|\rho_0\|_{B^{s-1}_{p,\infty}} \|u_0\|_{B^s_{p,\infty}}^2 \leq C.
\]
Taking above estimates into (3.15), we get
\[
\|\rho - \rho_0\|_{B^{s-1}_{p,\infty}} \geq t2^{n(s-1)}\|u_0^2 \partial_x \Delta_n \rho_0\|_{L^p} - C_1 t - C_2 2^{2n} \|\rho\|_{B^{s-3}_{p,\infty}}.
\]
using Lemma 3.1 and Lemma 3.3 yield that
\[
\|\rho - \rho_0\|_{B^{s-1}_{p,\infty}} \geq C_3 t 2^n - C_1 t - C_2 2^{2n} t^2.
\]
Choosing \( n \) large enough such that \( C_3 2^n > 2C_1 \), then we have
\[
\|\rho - \rho_0\|_{B^{s-1}_{p,\infty}} \geq \frac{C_3 t 2^n}{2} - C_2 2^{2n} t^2.
\]
As time \( t \) tends to zero, picking \( t 2^n \approx \delta < \frac{C_3}{4C_2} \), we obtain
\[
\|\rho - \rho_0\|_{B^{s-1}_{p,\infty}} \geq \frac{C_3}{2} \delta - C_2 \delta^2 \geq \frac{C_3}{4} \delta.
\]
Similarly, we have
\[
\|u - u_0\|_{B^{s-1}_{p,\infty}} \geq \frac{C_4}{4} \delta.
\]
This completes the proof of Theorem 1.1.

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