UNIT CIRCLE ELLIPTIC BETA INTEGRALS

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Abstract. We present some elliptic beta integrals with a base parameter on
the unit circle, together with their basic degenerations.

1. Introduction

The theory of generalized gamma functions has been set up by Barnes [Ba].
A slightly different approach was advocated by Jackson, who considered the basic
 gamma function depending on one base parameter $q$ and the elliptic gamma function
depending (symmetrically) on two bases $p$ and $q$ [J]. For a long time, only the
first of these generalized gamma functions was appreciated in the literature [AAR].
Recently, however, the elliptic gamma function also got appropriate attention after
the work of Ruijsenaars [Ru1], who introduced it in the context of integrable systems
and investigated some of its properties. A further study of the function in question
was conducted by Felder and Varchenko [FV]. A modified elliptic gamma function,
which admits analytic continuation in one of the base parameters, e.g. $q$ onto the
unit circle $|q| = 1$, has been introduced recently by one of us in [S2].

In this paper we study beta type integrals on the unit circle $|q| = 1$ built of
modified elliptic gamma functions, as well as their basic degenerations. The first
exact beta type integration formula involving the conventional elliptic gamma func-
tion was discovered in [S1]. Various multidimensional generalizations of this elliptic
beta integral associated with the $C_N$ and $A_N$ root systems have been investigated
in [DS1, DS2, R, S2]. A general theory of theta hypergeometric integrals on tori
and the Jacobi theta function generalizations of the Meijer function was developed
in [S2]. The beta integrals considered below should be thought of as the $|q| = 1$
counterparts of the integrals in [S1] and [DS1]. Recently, the Bailey’s technique of
deriving identities for series of hypergeometric type [AAR] has been generalized to
the $|q| = 1$ integrals under discussion.

2. Preliminaries:

THE JACOBI THETA FUNCTION AND THE ELLIPTIC GAMMA FUNCTION

The main underlying structural object of this paper is a Jacobi type theta func-
tion defined as

$$\theta(z;p) = (z;p)_\infty(pz^{-1};p)_\infty, \quad (a;p)_\infty = \prod_{n=0}^{\infty} (1 - ap^n),$$

(1)

with $z, p \in \mathbb{C}, |p| < 1$. It satisfies the transformation properties

$$\theta(pz;p) = \theta(z^{-1};p) = -z^{-1}\theta(z;p).$$

(2)
Evidently, \(\theta(z;p) = 0\) for \(z = pn, m \in \mathbb{Z}\), and \(\theta(z;0) = 1 - z\). If we denote \(p = e^{2\pi i \tau}\), \(\text{Im}(\tau) > 0\), then the standard Jacobi \(\theta\)-function \([WW]\) is related to \(\theta(z;p)\) as

\[
\theta_1(u|\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n p^{(2n+1)^2/8} e^{\pi i (2n+1)u} \quad (\text{for } u \in \mathbb{C}).
\]

The complex function \(\theta_1(u|\tau)\) is entire, odd, and doubly quasiperiodic in \(u\)

\[
\begin{align*}
\theta_1(u + 1|\tau) &= -\theta_1(u|\tau), \\
\theta_1(u + \tau|\tau) &= -e^{-\pi i \tau - 2\pi i u} \theta_1(u|\tau).
\end{align*}
\]

The modular \(PSL(2,\mathbb{Z})\)-group,

\[
\tau \rightarrow a\tau + b \quad (ct + d), \quad u \rightarrow \frac{u}{ct + d},
\]

with \(a, b, c, d \in \mathbb{Z}\) and \(ad - bc = 1\), is generated by the two transformations \(\tau \rightarrow \tau + 1, u \rightarrow u\) and \(\tau \rightarrow -\tau^{-1}, u \rightarrow u\tau^{-1}\). Its action on the Jacobi theta function is determined by

\[
\begin{align*}
\theta_1(u|\tau + 1) &= e^{\pi i/4} \theta_1(u|\tau), \\
\theta_1 \left( \frac{u}{\tau} - \frac{1}{\tau} \right) &= -i(\tau)\frac{1}{2}e^{\pi i u^2/\tau} \theta_1(u|\tau).
\end{align*}
\]

(Throughout this paper the sign of the square root is fixed in accordance with the principal branch with the cut chosen on the negative real axis.) From the second of these relations, combined with the modular transformation law for the Dedekind \(\eta\)-function

\[
e^{-\frac{\pi i}{\tau}} (e^{-2\pi i/\tau}; e^{-2\pi i/\tau}) \xi = (-i\tau)^{1/2} e^{-\frac{\pi i}{\tau}} (e^{2\pi i \tau}; e^{2\pi i \tau}) \xi,
\]

one readily deduces a corresponding modular transformation formula for the \(\theta(z;p)\) function

\[
\theta(e^{2\pi i \tau}; e^{-2\pi i \tau}) = -ie^{\pi i (a^2 + b^2 \tau + b + u)} \theta(e^{2\pi i u}; e^{2\pi i \tau}).
\]

The elliptic gamma function \(\Gamma(z;q,p), |q|, |p| < 1\), is related to the above theta function through the difference equations

\[
\Gamma(qz; q,p) = \theta(z; p) \Gamma(z; q,p), \quad \Gamma(pz; q,p) = \theta(z; q) \Gamma(z; q,p).
\]

It is given by the explicit product representation \([Ru1]\)

\[
\Gamma(z; q,p) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}q^{j+1}p^{k+1}}{1 - zq^{j}p^{k}}.
\]

From this representation the following reflection relation is immediate

\[
\Gamma(z; q,p) \Gamma(z^{-1}; q,p) = \frac{1}{\theta(z; p) \theta(z^{-1}; q)}.
\]
3. The modified elliptic gamma function

The modified elliptic gamma function introduced in [S2] is constructed as a product of two elliptic gamma functions of the form in [10], corresponding to two different choices of bases. It is convenient to pass to an additive formulation by introducing three pairwise incommensurate quasiperiods $\omega_1, \omega_2, \omega_3$ and write

$$q = e^{2\pi i \frac{u}{\omega_1}}, \quad p = e^{2\pi i \frac{u}{\omega_2}}, \quad r = e^{2\pi i \frac{u}{\omega_3}},$$

(i.e., $\tau = \omega_3/\omega_2$). The tilded bases $\tilde{q}, \tilde{p},$ and $\tilde{r}$ are the respective modular transformations of $q, p,$ and $r$. For $\text{Im}(\omega_1/\omega_2), \text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) > 0$ (so $|q|, |p|, |r| < 1$), the modified elliptic gamma function is now defined as [S2]

$$G(u; \omega) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i \frac{u}{\omega_1} q^{j+1} p^{k+1}}}{1 - e^{2\pi i \frac{u}{\omega_1} \tilde{q}^{j+1} \tilde{p}^{k+1}}},$$

(13)

It satisfies three difference equations

$$G(u + \omega_1; \omega) = \theta(e^{2\pi i \frac{u}{\omega_2}}; p)G(u; \omega),$$

(14a)

$$G(u + \omega_2; \omega) = \theta(e^{2\pi i \frac{u}{\omega_1}}; r)G(u; \omega),$$

(14b)

$$G(u + \omega_3; \omega) = \frac{\theta(e^{2\pi i \frac{u}{\omega_2}}; q)}{\theta(e^{2\pi i \frac{u}{\omega_2}}; \tilde{q}; \tilde{q})}G(u; \omega),$$

(14c)

the latter of which can be rewritten with the aid of modular transformation [8] as

$$G(u + \omega_3; \omega) = e^{-\pi i B_2,2(u;\omega)}G(u; \omega),$$

(14d)

where

$$B_{2,2}(u; \omega) = \frac{u^2}{\omega_1 \omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.$$ Such a system of three difference equations determines the meromorphic function $G(u; \omega)$ up to a multiplicative constant (which is the only meromorphic function with three pairwise incommensurate periods $\omega_{1,2,3}$). Similar to the $\Gamma(z; q, p)$ function, the $G(u; \omega)$ function satisfies a simple reflection equation given by

$$G(u; \omega)G(-u; \omega) = \frac{e^{\pi i B_2,2(u;\omega)}}{\theta(e^{-2\pi i \frac{u}{\omega_2}}; r)\theta(e^{-2\pi i \frac{u}{\omega_2}}; p)}. $$

(15)

If we fix the quasiperiods $\omega_1, \omega_2$ such that $\text{Im}(\omega_1/\omega_2) > 0$, and take $\omega_3$ to infinity in such a way that $\text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) \to +\infty$ (so $p, r \to 0$), then we obtain

$$\lim_{p,r \to 0} \frac{1}{G(u; \omega)} = S(u; \omega_1, \omega_2) = \frac{(e^{2\pi i \omega_2/\omega_1} q)^\infty}{(e^{2\pi i \omega_2/\omega_1} \tilde{q}; \tilde{q})^\infty}.$$ 

(16)

In the modern time, the function $S(u; \omega)$ was introduced by Shintani [Sh]. It is related to the Barnes double gamma function and is called the double sine function [Kn]. Its various properties and applications are discussed, e.g., in [FJLMKLMMNUMLR]. Some $q$-beta integrals expressed in terms of $S(u; \omega)$ were considered in [FKV, PT, SF].

One of the main properties of the double sine function consists of the fact that it can be extended continuously in the quasiperiods $\omega_{1,2}$ from the upper half plane $\text{Im}(\omega_1/\omega_2) > 0$ (so $|q| < 1$) to the positive real axis $\omega_1/\omega_2 > 0$ (so $|q| = 1$), the resulting function still being meromorphic in $u$. A similar situation holds for
the function $G(u; \omega)$, as can be verified by expressing it in terms of the Barnes’ multiple gamma function of the third order $\Gamma_3$. The following theorem provides a convenient representation of $G(u; \omega)$ detailing explicitly its structure when $q$ lies on the unit circle.

**Theorem 1.** Let $\text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) > 0$ and $\text{Im}(\omega_1/\omega_2) \geq 0$. The analytic continuation of $G(u; \omega)$ \[14a\] from the domain $\text{Im}(\omega_1/\omega_2) > 0$ to the boundary $\omega_1/\omega_2 > 0$ is given by the meromorphic function

$$G(u; \omega) = e^{-\pi iP(u) \Gamma(e^{-2\pi i \frac{u}{\omega}}; \tilde{r}, \tilde{p})}, \quad (17a)$$

where $P(u)$ is the following polynomial of the third degree

$$P(u) = \frac{1}{3\omega_1\omega_2\omega_3} \left( u - \frac{1}{2} \sum_{n=1}^{3} \omega_n \right) \left( u^2 - u \sum_{n=1}^{3} \omega_n + \frac{\omega_1\omega_2\omega_3}{2} \sum_{n=1}^{3} \frac{1}{\omega_n} \right). \quad (17b)$$

**Proof.** Let us first assume temporarily that $\text{Im}(\omega_1/\omega_2) > 0$. We denote the right-hand side of $\[17a\]$ by $f(u)$. It is easy to see that

$$\frac{f(u + \omega_1)}{f(u)} = e^{\pi iP(u) - P(u + \omega_1)} \theta(e^{-2\pi i \frac{u}{\omega}}; \tilde{p}) = \theta(e^{2\pi i \frac{u}{\omega}}; p), \quad (18)$$

as a consequence of the modular transformation for theta functions in $[\text{FV}]$. The function $f(u)$ thus satisfies equation $\[14a\]$. By symmetry, it satisfies $\[14b\]$ as well. Analogously, we have that $f(u + \omega_3)/f(u) = e^{\pi iP(u) - P(u + \omega_3)}$, which is seen to coincide with $\[14d\]$. The independence of the quasiperiods $\omega_{1,2,3}$ over $\mathbb{Q}$ now implies that $G(u; \omega)/f(u)$ must be constant. Its value is equal to one, because for $u = (\omega_1 + \omega_2 + \omega_3)/2$ we have that $G(u; \omega) = f(u) = 1$. The theorem then follows upon analytic continuation of the right-hand side of $\[17a\]$ to the positive real axis $\omega_1/\omega_2 > 0$. \hfill $\square$

**Remark 1.** In $[\text{FV}]$, modular transformation properties of the elliptic gamma function were investigated. For $|q|, |p|, |r| < 1$ the equality in $\[17a\]$ follows from one of these modular transformations. Function $\Gamma(z; q, p)$ has a pointwise limit for some particular values of $\omega_1/\omega_2 \in X \subset \mathbb{R}$. $\Gamma_3$, but it does not assume validity of $\[17a\]$ in this regime. Our result consists thus of the observation that, after an appropriate rewriting, this modular transformation provides a representation for the elliptic gamma function that is well defined for all $\omega_1/\omega_2 > 0$ $[\text{FV}]$. 

Below we shall need functional equations satisfied by $S(u; \omega)$

$$\frac{S(u + \omega_1; \omega)}{S(u; \omega)} = \frac{1}{1 - e^{2\pi i \frac{u}{\omega}}}, \quad \frac{S(u + \omega_2; \omega)}{S(u; \omega)} = \frac{1}{1 - e^{2\pi i \frac{u}{\omega}}}, \quad (19)$$

and its asymptotics for $u$ going to infinity. Let us fix the incommensurate quasiperiods $\omega_1$ and $\omega_2$ such that $\text{Im}(\omega_1/\omega_2) > 0$ (so $|q| < 1$). Then it follows from the infinite product representation \[10\] and its modular inverse

$$S(u; \omega_1, \omega_2) = e^{-\pi i B_{2,2}(u; \omega)} \left( e^{-2\pi i u/\omega}; q \right)_\infty$$

that

$$\lim_{\text{Im}(\omega_1/\omega), \text{Im}(\omega_2/\omega) \to +\infty} S(u; \omega) = 1, \quad (21a)$$

and

$$\lim_{\text{Im}(\omega_1/\omega), \text{Im}(\omega_2/\omega) \to -\infty} e^{\pi i B_{2,2}(u; \omega)} S(u; \omega) = 1. \quad (21b)$$
It turns out that the same asymptotics also holds for the boundary domain $\omega_1/\omega_2 > 0$ (so $|q| = 1$), as can be verified by means of an integral representation for $S(u; \omega)$ [KLS, Ru1].

**Remark 2.** If we take the limit $p, r \to 0$ in (17a), then the transition from $G(u; \omega)$ to the double sine function simplifies to

$$
\lim_{\text{Im}(\frac{q}{p}) \to \infty} \left( e^{-\pi i \omega_1 \frac{2n_a e^{-\pi i q}}{1 - e^{-\pi i q}}} \frac{1}{\omega_1} \Gamma(e^{-2\pi i \frac{q}{1 - e^{-\pi i q}}}, e^{-2\pi i \frac{q}{1 - e^{-\pi i q}}}) \right)
$$

$$
= e^{-\pi i \frac{3(2n_a - 1) - \omega_1^2 - \omega_2^2}{4\pi i \omega_1 \omega_2}} S^{-1}(u; \omega_1, \omega_2).
$$

Such a limiting relation was first derived in a different way and in a stronger sense by Ruijsenaars [Ru1].

4. THE ELLIPTIC BETA INTEGRAL ON THE UNIT CIRCLE

We now turn to the elliptic beta integrals. Let $|q|, |p| < 1$ and let $t_n, n = 0, \ldots, 4$, be five complex parameters satisfying the inequalities $|t_n| < 1$ and $|pq| < |A|$, where $A = \prod_{n=0}^{4} t_n$. The elliptic beta integral of [S1] states that

$$
\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\prod_{n=0}^{4} \Gamma(t_n z^\pm; q, p)}{\Gamma(z^\pm, A z^\pm; q, p)} \frac{dz}{z} = \frac{2 \prod_{0 \leq n < m \leq 4} \Gamma(t_n t_m; q, p)}{(q; q)_{\infty} (p; p)_{\infty} \prod_{n=0}^{4} \Gamma(At_n^{-1}; q, p)},
$$

(23)

where $\mathbb{T}$ denotes the positively oriented unit circle. Here we have employed the shorthand notations

$$
\Gamma(z_1, \ldots, z_m; q, p) \equiv \prod_{l=1}^{m} \Gamma(z_i; q, p),
$$

$$
\Gamma(tz^\pm; q, p) \equiv \Gamma(tz, tz^{-1}; q, p), \quad \Gamma(z^\pm; q, p) \equiv \Gamma(z^2, z^{-2}; q, p).
$$

For $p = 0$, the integration formula (23) amounts to an integral explicitly constructed by Rahman in [Rah] through a specialization of the Nassrallah-Rahman integral representation for a very-well-poised basic hypergeometric $9 \varphi_7$ series.

The following theorem provides a modified elliptic beta integral that—unlike (23)—is well defined for $|q| = 1$.

**Theorem 2.** Let $\text{Im}(\omega_1/\omega_2) \geq 0$ and $\text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) > 0$, and let $g_n, n = 0, \ldots, 4$, be five complex parameters subject to the constraints

$$
\text{Im}(g_n/\omega_3) < 0, \quad \text{Im}((A - \omega_1 - \omega_2)/\omega_3) > 0,
$$

with $A = \sum_{n=0}^{4} g_n$. Then the following integration formula holds

$$
\int_{-\omega_3/2}^{\omega_3/2} \prod_{n=0}^{4} G(g_n, u; \omega) \frac{du}{\omega_2} = \kappa \prod_{0 \leq n < m \leq 4} G(g_n + g_m; \omega) \prod_{n=0}^{4} G(A - g_n; \omega),
$$

(24a)

where

$$
\kappa = \frac{-2(q; q)_{\infty} (p; p)_{\infty} (r; r)_{\infty}}{(q, q)_{\infty} (p, p)_{\infty} (r, r)_{\infty}}.
$$

(24b)

Here the integration is meant along the straight line segment connecting $-\omega_3/2$ to $\omega_3/2$ and we employed the shorthand notation $G(a \pm b; \omega) \equiv G(a + b, a - b; \omega)$. 
Proof. We start by substituting relation (17a) into the left-hand side of (24a). This yields
\[ e^{πia} \int_{−ω_3/2}^{ω_3/2} \prod_{n=0}^{4} \frac{Γ(e^{−2πi\frac{2m+u}{ω_3}}; \tilde{r}, \tilde{p})}{Γ(e^{−2πi\frac{2m−u}{ω_3}}; \tilde{r}, \tilde{p})} du, \]
where
\[ a = \frac{2}{3ω_1ω_2ω_3} \left( A^3 - \sum_{n=0}^{4} g_n^3 \right) - \sum_{m=1}^{3} \frac{ω_m}{ω_1ω_2ω_3} \left( A^2 - \sum_{n=0}^{4} g_n^2 \right) + \frac{1}{2} \left( \sum_{m=1}^{3} ω_m \right) \left( \sum_{m=1}^{3} \frac{1}{ω_m} \right). \]
The constraints on the parameters permit to employ formula (23) with the substitutions
\[ z \rightarrow e^{-2πiω_3u}, \quad t_n \rightarrow e^{-2πiω_3g_n}, \quad p \rightarrow e^{-2πiω_3}, \quad q \rightarrow e^{-2πiω_3}, \]
which yields for (25)
\[ \frac{2ω_3ω_1^{-1}e^{πia}}{(r; r)_∞(p; p)_∞} \prod_{0≤n<m≤4} Γ(e^{−2πi\frac{2m+u}{ω_3}}; \tilde{r}, \tilde{p}) \prod_{n=0}^{4} Γ(e^{−2πi\frac{2m−u}{ω_3}}; \tilde{r}, \tilde{p}) = \kappa \prod_{0≤n<m≤4} G(g_n + g_m; \omega) \prod_{n=0}^{4} G(A - g_n; \omega), \]
with
\[ \kappa = \frac{2ω_3e^{πia}}{ω_2(r; r)_∞(p; p)_∞} \left( ∑_{m=1}^{3} ω_m \right) \left( ∑_{m=1}^{3} \frac{1}{ω_m} \right). \]
After applying modular transformation (17) to the infinite products appearing in \( κ \), we obtain
\[ κ = -2 \sqrt{\frac{ω_1}{iω_2}} \frac{e^{πia}}{(r; r)_∞(p; p)_∞}. \]
One more application of (17) allows us to replace the exponential function by a ratio of infinite products, which entails the desired form of \( κ \) in (24b).

Let us now consider the formal limit \( p, r \to 0 \) of the integral (24b). To this end, we fix the quasiperiods \( ω_{1, 2} \) such that \( \operatorname{Im}(ω_1/ω_2) ≥ 0 \) and \( \operatorname{Re}(ω_1/ω_2) > 0 \), and we furthermore pick \( ω_3 = itω_2 \) with \( t > 0 \). For \( t \to +∞ \) the integral of Theorem 2 then degenerates formally to
\[ \int \frac{S(±2u, A; u; ω)}{πω_2} S(g_n + u; ω) du = -2 \frac{(q; q)_∞}{(q; q)_∞} \prod_{n=0}^{4} S(A - g_n; ω) S(g_n + g_m; ω), \]
where the integration is along the line \( L \equiv iω_2R \), and with parameters subject to the constraints \( \operatorname{Re}(g_n/ω_2) > 0 \) and \( \operatorname{Re}(A - g_n/ω_2) < 1 \). This integral was deduced by a similar formal limit from the standard elliptic beta integral (23) and rigorously proved by Stokman in [St], where it was referred to as the hyperbolic Nassrallah-Rahman integral.

Remark 3. The generalized gamma function \( [z; τ]_∞ \) used in [St] coincides with the double sine function (16) upon setting \( z = (ω_2 - u)/ω_2 \) and \( τ = -ω_2/ω_1 \).
Remark 4. To further elucidate the intimate relation between the elliptic beta integrals and the 

- elliptic function of $u$

- Remark

if, for the displacement $\omega_1$, the ratio of its integrands $\Delta(u + \omega_1)/\Delta(u)$ constitutes an elliptic function of $u$ with periods $\omega_2, \omega_3$ (say). Now, by the change of variables

$$z = e^{\frac{2\pi i}{\omega_2}}, \quad t_n = e^{\frac{2\pi i}{\omega_2}a_n}, \quad A = e^{\frac{2\pi i}{\omega_2}A_n},$$

the integral (28) passes into the additive form

$$\int_{-\omega_2/2}^{\omega_2/2} \Delta(u) \frac{du}{\omega_2} = \frac{2 \prod_{0 \leq n < m \leq 4} \Gamma(e^{2\pi i \frac{a_n a_m}{\omega_2}}; q, p)}{(q; q)_\infty (p; p)_\infty \prod_{a=0}^{4} \Gamma(e^{2\pi i \frac{A_n}{\omega_2}}; q, p)}, \quad (27a)$$

with the integrand given by

$$\Delta(u) = \prod_{n=0}^{4} \frac{\Gamma(e^{2\pi i \frac{a_n}{\omega_2}}; q, p)}{\Gamma(e^{4\pi i \frac{n}{\omega_2}}; q, p)}, \quad (27b)$$

where $\Gamma(e^{a+b}; q, p) \equiv \Gamma(e^{a}, e^{b}; q, p)$. We then have that

$$\frac{\Delta(u + \omega_1)}{\Delta(u)} = e^{2\pi i \frac{\omega_1}{\omega_2}} \frac{\theta(e^{4\pi i \frac{a_1}{\omega_2}}; q, p)}{\theta(e^{4\pi i \frac{a_2}{\omega_2}}; q, p)} \prod_{m=0}^{4} \frac{\theta(e^{2\pi i \frac{A_m}{\omega_2}}; q, p)}{\theta(e^{2\pi i \frac{A_m}{\omega_2}}; q, p)}, \quad (28)$$

which is seen to be an elliptic function of $u$ with the periods $\omega_2$ and $\omega_3$. With the aid of the difference equation (27) for the modified gamma function, it is not difficult to verify that the integrand of modified elliptic beta integral (24a) also provides a solution to (27). Hence both integrals (28) and (27) are elliptic hypergeometric integrals with integrands satisfying (28). Whereas the solution of (27) originating from integral (24a) is well-defined only for $|q| < 1$ (or for $|q| > 1$ upon performing the inversion $q \to q^{-1}$ in the elliptic gamma function [S2]), the modified integral (24a) corresponds to a solution that extends analytically from the regime $|q| < 1$ to the unit circle $|q| = 1$. In this sense the modified integral (24a) may be seen as a $|q| = 1$ analog of the original elliptic beta integral (24).

5. Multiple integrals

The following integral constitutes a multidimensional generalization of the elliptic beta integral in (24a)

$$\frac{1}{(2\pi i)^N} \int_{-\infty}^{\infty} \prod_{1 \leq j < k \leq N} \frac{\Gamma(tz_j z_j^*; q, p)}{\Gamma(z_j z_j^*; q, p)} \prod_{j=1}^{N} \frac{\Gamma(t z_j^2; q, p)}{\Gamma(z_j^2; q, p)} \prod_{m=0}^{4} \frac{\Gamma(t_n z_j^2; q, p)}{\Gamma(z_j^2; q, p)} dz_1 \ldots dz_N \prod_{m=0}^{4} \frac{\Gamma(t_n z_j^2; q, p)}{\Gamma(z_j^2; q, p)}, \quad (29)$$

where $B \equiv \prod_{n=0}^{t_n}$, $\Gamma(z t_j x_j^*; q, p) \equiv \Gamma(t z x_j, t z x_j^*; q, p)$, and with parameters subject to the constraints $|t|, |z|, |t_n| < 1$ and $|pq| < |B|$. This multiple elliptic beta integral was first formulated as a conjecture in [DS1]. Next, it was shown in [DS2] that the conjecture in question follows from a vanishing hypothesis for a related multiparameter elliptic beta integral. Recently, a complete proof of the integral in (24a) was found by Rains [R].
The following theorem provides the corresponding multidimensional generalization of the modified elliptic beta integral in Theorem 2.

**Theorem 3.** Let \( \text{Im}(\omega_1/\omega_2) \geq 0 \) and \( \text{Im}(\omega_3/\omega_1), \text{Im}(\omega_3/\omega_2) > 0 \), and let \( g, g_n, n = 0, \ldots, 4 \), be six complex parameters subject to the constraints

\[
\text{Im}(g/\omega_3), \text{Im}(g_n/\omega_3) < 0, \quad \text{Im}((B - \omega_1 - \omega_2)/\omega_3) > 0,
\]

with \( B \equiv (2N - 2)g + \sum_{n=0}^{4} g_n \). Then

\[
\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{1 \leq j < k \leq N} \frac{G(g \pm u_j \pm u_k; \omega)}{G(\pm u_j \pm u_k; \omega)} \prod_{j=1}^{N} \frac{\prod_{n=0}^{4} G(g_n \pm u_j; \omega)}{G(\pm 2u_j, B \pm u_j; \omega)} \prod_{n=0}^{4} G((1-j)g + g_n + g_m; \omega),
\]

with \( \kappa \) given by (24b) and \( G(c \pm a \pm b; \omega) = G(c + a + b, c + a - b, c - a + b, c - a - b; \omega) \).

**Proof.** The proof is analogous to that of Theorem 2. Specifically, after substituting (17a) into the left-hand side of (30) and application of the multiple beta integral in (29), we arrive at the integration formula stated in the theorem upon expressing the resulting evaluation in terms of the modified gamma function \( G(u; \omega) \). To infer the correctness of the value of the proportionality constant \( \kappa^N N! \), one observes that the dependence on \( g \) in the factors originating from the exponential multipliers cancels out. The expression for the proportionality constant then follows from the fact that for \( g \to 0 \) integral (30) reduces to the \( N \)-th power of the elliptic beta integral (24a).

**Remark 5.** In [DS2, S2] various other types of multiple elliptic beta integrals were formulated. These can be extended to the unit circle \(|q| = 1\) in a similar fashion.

For \( p = 0 \), elliptic beta integral (29) reduces to a Gustafson’s multiple integral corresponding to the Nassrallah-Rahman type generalization of the Selberg integral (24a). The following theorem provides a corresponding multiple analog of the integral in (29). The integration formula in question can be obtained formally from the modified elliptic beta integral (30) with \( \omega_1/\omega_2 > 0 \) by taking the limit \( p, r \to 0 \) in the manner explained below Theorem 2.

**Theorem 4.** Let \( \omega_1, \omega_2 \) be quasiperiods such that \( \text{Im}(\omega_1/\omega_2) \geq 0 \) and \( \text{Re}(\omega_1/\omega_2) > 0 \). Furthermore, let \( g, g_n, n = 0, \ldots, 4 \), be parameters subject to the constraints \( \text{Re}(g/\omega_1), \text{Re}(g/\omega_2), \text{Re}(g_n/\omega_2) > 0 \) and \( \text{Re}((B - \omega_1)/\omega_2) < 1 \) (with \( B \) as in Theorem 2). Then

\[
\int_{\mathbb{R}^N} \Delta(\mathbf{u}; g) \frac{du_1}{\omega_2} \cdots \frac{du_N}{\omega_2} = N(\mathbf{g}),
\]

where \( \mathbb{L} = i\omega_2 \mathbb{R} \),

\[
\Delta(\mathbf{u}; g) = \prod_{1 \leq j < k \leq N} \frac{S(\pm u_j \pm u_k; \omega)}{S(\pm \pm u_j \pm u_k; \omega)} \prod_{j=1}^{N} \frac{\prod_{n=0}^{4} S(g_n \pm u_j; \omega)}{S(\pm 2u_j, B \pm u_j; \omega)} \prod_{n=0}^{4} S((1-j)g + g_n + g_m; \omega),
\]

and

\[
N(\mathbf{g}) = (-2)^N N! \prod_{j=1}^{N} \frac{S(\pm u_j; \omega)}{S(\pm \pm u_j; \omega)} \prod_{n=0}^{4} \frac{\prod_{m=0}^{4} S((1-j)g + B - g_n; \omega)}{S((j-1)g + g_n + g_m; \omega)},
\]
Through a parameter specialization, Gustafson’s multiple integral of the Nassrallah-Rahman type can be reduced to a multiple Askey-Wilson integral first derived in [Gu1]. The corresponding degeneration of Theorem 4 reads as follows.

**Theorem 5.** Let $\omega_1, \omega_2$ be quasiperiods such that $\text{Im}(\omega_1/\omega_2) \geq 0$ and $\text{Re}(\omega_1/\omega_2) > 0$, and let $g, m, n = 0, \ldots, 3,$ be parameters subject to the constraints $\text{Re}(g/\omega_1)$, $\text{Re}(g/\omega_2), \text{Re}(g/m/\omega_2) > 0$ and $\text{Re}((B - \omega_1)/\omega_1) < 1$ with $B \equiv (2N - 2)g + \sum_{n=0}^{3} m_n g_n$. Then

\[
\int_{L_N} \prod_{1 \leq j < k \leq N} \frac{S(\pm u_j \pm u_k; \omega)}{S(g \pm u_j \pm u_k; \omega)} \prod_{j=1}^{N} \frac{S(\pm 2u_j; \omega)}{S(g \pm u_j; \omega)} \frac{du_1}{\omega_2} \cdots \frac{du_N}{\omega_2} = (-2)^N g^{N} (q; q)_{\infty}^{N} \prod_{j=1}^{N} S(g; \omega) S((1 - j)g + B; \omega) \prod_{0 \leq n < m \leq 3} S(g_n + g_m; \omega),
\]

(32)

For $N = 1$, the integral in Theorem 5 reduces to a single variable Askey-Wilson type integral

\[
\int_{L} \prod_{j=1}^{3} S(g_n \pm u; \omega) \frac{du}{\omega} = -2 q^{(\tilde{Q}; q)_{\infty}} (q; q)_{\infty} \prod_{0 \leq n < m \leq 3} S(g_n + g_m; \omega),
\]

(33)

which was established by Ruijsenaars [Ru3] and Stokman [St].

Formally, the integral of Theorem 5 follows from that of Theorem 4 with $\omega_1/\omega_2 > 0$ upon setting $g_4 \to g_4 + i\omega_2 t$ and performing the limit $t \to +\infty$. However, it is not a simple matter to upgrade such formal limiting relations between Theorem 3 and Theorems 4, 5 to rigorous proofs of the latter integration formulas. A direct proof of Theorems 4 and 5 modelled after Rains’ proof of the multiple elliptic beta integral [29], is given in Section 6 below. As it was communicated to us by Stokman after finishing this paper, Theorem 5 can be proved by a multivariable generalization of the method of [St] as well.

**6. Proof of Theorems 4 and 5**

We first detail the proof of Theorem 4 and then indicate some modifications so as to incorporate Theorem 5.

Let us for the moment assume that the quasiperiods $\omega_1, \omega_2$ are incommensurate over $\mathbb{Q}$. The double sine function $S(u; \omega)$ then has simple zeros located at the points $u = -\omega_1 N - \omega_2 N$ and simple poles at $u = \omega_1 (1 + N) + \omega_2 (1 + N)$. Therefore, the integrand $\Delta(u; g)$ in (31) has poles at the points

\[
\pm u_j = -B + \omega_1 (1 + N) + \omega_2 (1 + N), \quad g + \omega_1 N + \omega_2 N, \quad g - u_k + \omega_1 N + \omega_2 N, \quad g + u_k + \omega_1 N + \omega_2 N, \quad k = 1, \ldots, N, \quad k \neq j,
\]

where $j = 1, \ldots, N$.

Combination with the asymptotics in Eqs. (21a), (21b) reveals that the quotients $S(u; \omega)/S(g + u; \omega)$ and $S(2u; B + u; \omega)/\prod_{n=0}^{3} S(g_n + u)$ are smooth and bounded on the complex line $L = \omega_2 S$. Indeed, for $u = ix \omega_2, x \in \mathbb{R}$ we stay away from poles and we have that

\[
\frac{S(i \omega_2 x; \omega)}{S(g + i \omega_2 x; \omega)} = \begin{cases}
O(1) & \text{for } x \to +\infty \\
O(e^{-2\pi x g/\omega_1}) & \text{for } x \to -\infty
\end{cases}
\]
\[
\frac{S(2i\omega_2 x, B + i\omega_2 x; \omega)}{\prod_{n=0}^1 S(g_n + i\omega_2 x; \omega)} = \begin{cases} 
O(1) & \text{for } x \to +\infty \\
O(e^{2\pi x(2(N-1)g/\omega_1 + 1 + \omega_2/\omega_1)}) & \text{for } x \to -\infty 
\end{cases}
\]

It thus follows that the integrand \(\Delta(u; g)\) is smooth and exponentially decaying at infinity on the integration domain \(\mathbb{L}^N\). Hence, the integral in Eq. 33 converges.

To infer the validity of the integration formula we distinguish three parameters \(g_0, g_1, g_2\) and factorize the integrand as \(\Delta(u; g) = \Delta_+(u)\Delta_-(u)\) with

\[
\Delta_+(u) = \prod_{1 \leq j < k \leq N} \frac{S(u_j \pm u_k; \omega)}{S(g + u_j \pm u_k; \omega)} \prod_{j=1}^N \frac{S(2u_j, B - u_j, \omega_1 + C - u_j; \omega)}{S(\omega_1 + C + u_j; \omega) \prod_{n=0}^1 S(g_n + u_j; \omega)},
\]

where \(C = (N - 1)g + g_0 + g_1 + g_2\) and \(\Delta_-(u) = \Delta_-(u_1, \ldots, u_n)\). Similarly, we introduce the shifted functions

\[
\Delta_+(u)\Delta_-(u) = \Delta(u; g, g_0 + \frac{\omega_1}{2}, g_1 + \frac{\omega_1}{2}, g_2 + \frac{\omega_1}{2}, g_3 - \frac{\omega_1}{2}, g_4 - \frac{\omega_1}{2}).
\]

Let us for the moment assume that the parameters are such that the shifted parameters (as well as the ones obtained after shifts by \(\pm \omega_2/2\)) also belong to the domain indicated in the theorem. We then have the following equality

\[
\int_{\mathbb{L}^N} \Delta_+(u_1 + \frac{\omega_1}{2}, \ldots, u_N + \frac{\omega_1}{2})\Delta_-(u) \, du_1 \cdots du_N = \int_{\mathbb{L}^N} \Delta_+(u)\Delta_-(u_1 - \frac{\omega_1}{2}, \ldots, u_N - \frac{\omega_1}{2}) \, du_1 \cdots du_N,
\]

which follows by shifting the integration contours \(L\) on the left-hand side by \(-\omega_1/2\).

Notice that such shifts are permitted by the Cauchy theorem due to the absence of poles in the strip between \(L\) and \(\mathbb{L} + \omega_1/2\) combined with the exponential decay at infinity. Indeed, the quotient \(S(u; \omega)/S(g + u; \omega)\) is holomorphic on the strip \(\{u = s\omega_1 + i\omega_2 \mid 0 \leq s \leq 1, -\infty < x < \infty\}\) and the quotient \(S(2u, B + u, C + u; \omega)/\left(S(C + \frac{\omega_1}{2} + u, g_3 - \frac{\omega_1}{2} + u, g_4 - \frac{\omega_1}{2} + u; \omega) \prod_{n=0}^1 S(g_n + \frac{\omega_1}{2} + u; \omega)\right)\) is holomorphic on the strip \(\{u = s\omega_1 + i\omega_2 \mid 0 \leq s \leq 1/2, -\infty < x < \infty\}\).

By performing sign flips of the form \(u_j \to -u_j\) in the integration variables and summing over all \(2^N\) possible ways, we obtain from 35 that

\[
\int_{\mathbb{L}^N} \rho(u; g)\Delta_+(u)\Delta_-(u) \, du_1 \cdots du_N = \int_{\mathbb{L}^N} \bar{\rho}(u; g)\Delta_+(u)\Delta_-(u) \, du_1 \cdots du_N,
\]
with
\[
\rho(u; g) = \sum_{\nu_j = \pm 1} \frac{\Delta_+ (\nu_1 u_1 + \frac{\omega_1}{2}, \ldots, \nu_N u_N + \frac{\omega_N}{2})}{\Delta_+ (\nu_1 u_1, \ldots, \nu_N u_N)},
\]
\[
\tilde{\rho}(u; g) = \sum_{\nu_j = \pm 1} \frac{\Delta_- (\nu_1 u_1 - \frac{\omega_1}{2}, \ldots, \nu_N u_N - \frac{\omega_N}{2})}{\Delta_- (\nu_1 u_1, \ldots, \nu_N u_N)}.
\]

Simplification of \(\rho(u; g)\) reveals that
\[
\rho(u; g) = \sum_{\nu_j = \pm 1} \prod_{1 \leq j < k \leq N} \frac{1-t \nu_j \nu_k}{1-\nu_j \nu_k} \frac{(1-t^{N-1} t_0 t_1 t_2 z_j^{-\nu_j}) \prod_{j=0}^2 (1-t^2 z_j^{\nu_j})}{1-z_j^{\nu_j}} \]
\[
= \prod_{j=1}^N (1-t^{j-1} t_0 t_1) (1-t^{j-1} t_0 t_2) (1-t^{j-1} t_1 t_2),
\]
(38)

with \(t = e^{2\pi i g/\omega_2}, t_n = e^{2\pi in g/\omega_2}, z_k = e^{2\pi i u_k/\omega_2}\). The summand in \(\tilde{\rho}(u; g)\) is invariant under permutations of \(z_k\) and inversions \(z_k \rightarrow z_k^{-1}\). The product of \(\rho(u; g)\) and the factor
\[
\prod_{1 \leq j < k \leq N} \frac{(1-z_j z_k) (1-z_j z_k^{-1})}{z_j z_k}
\]
\[
\prod_{j=1}^N 1-z_j^{\nu_j}
\]
yields a Laurent polynomial in \(z_j, j = 1, \ldots, N\), which is antisymmetric with respect to both transformations (separate permutations of \(z_j\) and inversions \(z_k \rightarrow z_k^{-1}\)). Any such polynomial is proportional to the multiplicative factor given above. The constant of proportionality is found after setting \(z_j = t_0 t_1 t_2\), which leaves only one term in the sum (with all \(\nu_j = 1\)) giving the right-hand side expression in \(\tilde{\rho}(u; g)\).

In the same way, one obtains for \(\tilde{\rho}(u; g)\) that replacing in relation \(\tilde{\rho}(u; g)\) by \(t_3, t_4 q^{-1/2}, t^{N-1} t_0 t_1 t_2 q^{1/2}\),
\[
\tilde{\rho}(u; g) = \sum_{\nu_j = \pm 1} \prod_{1 \leq j < k \leq N} \frac{1-t \nu_j \nu_k}{1-\nu_j \nu_k} \frac{(1-t^2 q^{-1/2} z_j^{\nu_j}) (1-t q^{-1/2} z_j^{\nu_j})}{1-z_j^{\nu_j}} \]
\[
\times (1-t^{N-1} t_0 t_1 t_2 q^{1/2} z_j^{\nu_j}) (1-B q^{-1/2} z_j^{\nu_j})
\]
\[
= \prod_{j=1}^N (1-t^{j-1} t_3 t_4/q) (1-t^{j-1} B/t_3) (1-t^{j-1} B/t_4),
\]
where \(B = e^{2\pi i B/\omega_2}\). The derived expressions demonstrate that the functions in question are constant in the integration variables \(u\), that is \(\rho(u; g) = \rho(g)\) and \(\tilde{\rho}(u; g) = \tilde{\rho}(g)\). Hence, we can pull the corresponding factors out of the integrals and rewrite \(\rho(u; g)\) as
\[
\int_{L_N} \Delta_+(u) \Delta_-(u) \, du_1 \cdots du_N = \frac{\tilde{\rho}(g)}{\rho(g)} \int_{L_N} \tilde{\Delta}_+(u) \tilde{\Delta}_-(u) \, du_1 \cdots du_N,
\]
whence
\[
\frac{N(g)}{N(g_0 + \frac{\omega_1}{2}, g_1 + \frac{\omega_1}{2}, g_2 + \frac{\omega_1}{2}, g_3 - \frac{\omega_1}{2}, g_4 - \frac{\omega_1}{2})} = \frac{\tilde{\rho}(g)}{\rho(g)}.
\]
(39)
As a result, we deduce that the ratio of the left- and right-hand sides of (31a) is invariant with respect to the shifts \( g_{0,1,2} \to g_{0,1,2} + \omega_1/2, \ g_{3,4} \to g_{3,4} - \omega_1/2 \), and, by symmetry, any permutation of indices of the parameters. The double sine function \( S(u/\omega) \) and, so, the integrand in (31a) and the integral’s value \( N(g) \) are symmetric with respect to the permutation of \( \omega_1 \) and \( \omega_2 \) \cite{KLS}. The contour of integration \( L \) breaks the symmetry between \( \omega_1,2 \), but the transformations used in (31a) and (39) are purely algebraic and do not depend on the contour of integration. Therefore, the ratio of interest is invariant with respect to the shifts \( g_{0,1,2} \to g_{0,1,2} + \omega_2/2, \ g_{3,4} \to g_{3,4} - \omega_2/2 \) as well (with all permutations of indices of parameters).

By analyticity, without changing the integral’s value we can replace the contour of integration \( L \) by any other contour embracing the same set of poles. For an appropriately deformed contour, we can establish invariance of the ratio under the shifts \( g_n \to g_n + k\omega_1/2 + m\omega_2/2 \), for arbitrary \( k, m \in \mathbb{Z} \). Taking \( \omega_1, \omega_2 > 0 \), we can choose a subset of these points with a limiting point in the parameter space for which we can choose \( L \) as the integration contour. Moreover, making sequential \( \omega_1,2/2 \) shifts in different directions we can escape large intermediate deformations of the integration contour (such an argument is similar to the one given in \cite{SI}). Therefore, the ratio of the left- and right-hand sides of (31a) is equal to a function of \( \omega_{1,2} \) and \( g \), which we denote as \( f(\omega_1, \omega_2, g) \).

In order to see that \( f(\omega_1, \omega_2, g) \) actually equals to one, it is necessary to use an analog of the residue formula derived in \cite{DST}. Namely, we take one of the parameters, say, \( g_0 \) such that one pole from the half plane \( \text{Re}(u/\omega_2) > 0 \) crosses the contour of integration \( L \). A similar move takes place for the pole at \( u = -g_0 \) from the \( \text{Re}(u/\omega_2) < 0 \) half plane (due to the reflection invariance). We keep intact all other poles in the half planes to the left or right of \( L \). This is possible to do by an appropriate choice of \( g, g_1, \ldots, g_4, \) and \( \omega_{1,2} \). It is not difficult to see that the residues of these crossing poles taken, say, over the variable \( u_N \) have poles at the points \( u_k = \pm(g_0 + g), k = 1, \ldots, N - 1 \), (instead of \( u_k = \pm g_0 \)) and they are still located to the left or right of \( L \) for \( \text{Re}((g_0 + g)/\omega_2) < 0 \). Similar shifts of the poles occur each time we calculate the residues. Therefore we denote \( \rho_k = g_0 + (k - 1)g \) and impose the restrictions

\[
\text{Re} \left( \frac{\rho_k}{\omega_2} \right) < 0, \quad k = 1, \ldots, N, \quad \text{Re} \left( \frac{g_0 + \omega_1}{\omega_2} \right), \text{Re} \left( \frac{g_0 + \omega_2}{\omega_2} \right) > 0.
\]

Because of the taken constraints upon \( g, \omega_{1,2} \), we get a simpler residue formula than the one derived in \cite{DST}:

\[
\int_{L_d} \Delta(u; g) \frac{du_1}{\omega_2} \cdots \frac{du_N}{\omega_2} = \sum_{m=0}^{N} 2^m m! \binom{N}{m} \int_{L_{N-m}} \mu_m(u) \frac{du_1}{\omega_2} \cdots \frac{du_{N-m}}{\omega_2}, \tag{40}
\]

where \( L_d \) is a deformation of the contour \( L \) such that it separates the same sets of poles as \( L \) did before we started to change \( g_0 \). The factor \( 2^m \) emerges because the residues appear in pairs and their values coincide (due to the \( u_k \to -u_k \) reflection invariance of the integrand and different orientation of the contours encircling poles to the left and right of \( L \)). The factors \( m! \) and \( \binom{N}{m} \) count the number of orderings of \( m \) cycles and the number of ways to pick up these cycles out of \( N \) possibilities.
The residue functions have the form \( \mu_0(u) = \Delta(u; g) \) and for \( m > 0 \)
\[
\mu_m(u) = \kappa_m \delta_{m,N-m}(u) \Delta_{N-m}(u; g),
\]
where \( \Delta_{N-m}(u; g) \) is obtained from integrand if we replace in it \( N \) by \( N - m \) but keep \( B = (2N - 2)g + \sum_{n=0}^4 g_n \) unchanged. Other coefficients are
\[
\kappa_m = (-1)^m \frac{(\tilde{q}; q)_m}{(q; q)_m} \prod_{1 \leq j < k \leq m} \frac{S(\pm \rho_k - \rho_j; \omega)}{S(g \pm \rho_k - \rho_j; \omega)} \prod_{l=1}^{m} S(-2\rho_m, B + \rho_l; \omega) \prod_{n=1}^{4} S(g_n \pm \rho_l; \omega),
\]
and
\[
\delta_{m,N-m}(u) = \prod_{1 \leq k \leq m} \frac{S(\pm \rho_k \pm u_k; \omega)}{S(g \pm \rho_k \pm u_k; \omega)}.
\]
The expressions for \( \mu_m(u) \) are derived by induction. Indeed, the form of \( \mu_1(u) \) is easily established after taking into account the relation
\[
\lim_{u \to \pm g_0} \frac{u \mp g_0}{S(g_0 \mp u; \omega)} = \pm \frac{\omega_2}{2\pi i} \frac{(\tilde{q}; q)_\infty}{(q; q)_\infty},
\]
and the fact that the contours encircling the corresponding poles are oriented clockwise for the upper signs and anticlockwise for the lower signs (this gives the total minus sign in \( \kappa_1 \)).

Suppose that \( \mu_m \) is given by \( \kappa_1 \) for some \( m > 1 \). In order to find \( \mu_{m+1} \) it is necessary to compute the residues for poles located at \( u_{N-m} = \pm \rho_{m+1} \). A simple computation shows that, indeed,
\[
\int_{c_m} \mu_m(u) \frac{du_{N-m}}{\omega_2} = \mu_{m+1}(u),
\]
where \( c_m \) is a small size clockwise orientated contour encircling the pole at \( u_{N-m} = \rho_{m+1} \).

By analyticity, our deformations of the parameters and of the contour of integration do not change the integral value and, therefore, the right-hand side sum in \( \kappa_1 \) equals to \( f(\omega_1, \omega_2, g) N(g) \). We now divide both sides of this equality by \( N(g) \) and take the limit \( g_4 \to -g_0 - (N-1)g \). For \( m < N \), the coefficients \( \kappa_m(g) \), which can be represented in the form
\[
\kappa_m = (-1)^m \frac{(\tilde{q}; q)_m}{(q; q)_m} \prod_{1 \leq j < k \leq m} \frac{S(g, (2 - m - l)g - 2g_0; \omega)}{S(lg; \omega)} \prod_{r=1}^{4} S(2g_0 + \sum_{r=1}^{4} g_r + (2n + l - 3)g, \sum_{r=1}^{4} g_r + (2n - l - 1)g; \omega) \prod_{r=1}^{4} S(g_r + g_0 + (l - 1)g, g_r - g_0 - (l - 1)g; \omega),
\]
do not contain diverging factors in this limit and the integrals, which they are multiplied by, remain bounded. Therefore, only the term with \( m = N \) survives and, by simple computation, we obtain
\[
\lim_{g_4 \to -g_0 - (N-1)g} \frac{2^N N!\kappa_N}{N(g)} = 1,
\]
which means that \( f(\omega_1, \omega_2, g) = 1 \). After proving equality in the taken restricted region of parameters (where the parameters shifted by \( \pm \omega_{1,2}/2 \) satisfy the needed constraints and \( \omega_{1,2} > 0 \)), we can analytically extend it to the values
of $\omega_{1,2}$ and parameters $g, g_n$ in the domain indicated in the formulation of the theorem. Theorem 4 is thus proved.

We now turn to the Askey-Wilson type integral (31a). Its convergence conditions essentially differ from the previous case. Indeed, we have

$$S(2i\omega_2x; \omega) \prod_{n=0}^{3} S(g_n + i\omega_2x; \omega) = \begin{cases} O(1) & \text{for } x \to +\infty \\ O(e^{2\pi(1+\omega_2/\omega_1 - \sum_{n=0}^{3} g_n/\omega_1)}) & \text{for } x \to -\infty \end{cases}$$

Combining together these limiting relations with the asymptotics for the ratio $S(i\omega_2x; \omega)/S(g + i\omega_2x; \omega)$, we see that the integrand remains bounded in the integration domain $L^N$ and decays exponentially fast on its infinities if we take $\text{Re}((B - \omega_2)/\omega_1) < 1$, where $B = (2N - 2)g + \sum_{n=0}^{3} g_n$.

Invariance of the ratio of left- and right-hand sides of equality (31a) under the specified parameter shifts relied only on algebraic manipulations with the integrand. Therefore we can repeat them for the limiting expression of the integrand appearing after taking the limit $\text{Im}(g_4/\omega_2), \text{Im}(g_4/\omega_1) \to +\infty$ (or $t_4 \to 0$). This simplifies the integrand for (31a) to the one for (32). Therefore, limiting analogs of equalities (34)–(39) show that the ratio of the left- and right-hand sides of (32) do not depend on the shifts in the parameter space $g_{0.1.2} \to g_{0.1.2} + \omega_{1.2}/2$, $g_3 \to g_3 - \omega_{1.2}/2$ and the ones obtained by permutation of indices. Using, again, an analytical continuation and the appropriately simplified version of the residue calculus, we see that the ratio of interest is actually equal to one. As a result, we establish validity of integral (32) as well.

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