Kuramoto dilemma alleviated by optimizing connectivity and rationality

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Recently, Antonioni and Cardillo proposed a coevolutionary model based on the intertwining of oscillator synchronization and evolutionary game theory [Phys. Rev. Lett. 118, 238301 (2017)], in which each Kuramoto oscillator can decide whether to interact-or not-with its neighbors, and all oscillators can receive some benefits from the local synchronization but those who choose to interact must pay a cost. Oscillators are allowed to update their strategies according to payoff difference, wherein the strategy of an oscillator who has obtained higher payoff is more likely to be followed. Utilizing this coevolutionary model, we find that the global synchronization level reaches the highest level when the average degree of the underlying interaction network is moderate. We also study how synchronization is affected by the individual rationality in choosing strategy.

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I. INTRODUCTION

Synchronization is the coordination of events to operate a system in unison, which is ubiquitous in natural world and human society. For example, two pendulum clocks suspended side by side swing with the same frequency, a school of fish move in the same direction [1], opinions of different individuals reach a consensus [2], all generators in a power-grid system rotate at the same frequency [3], and so on. Due to the rapid development of network science [4–6], the study of synchronization in complex networks has attracted increasing attention [7, 8]. It has been found that both the network structure [9–11] and the coupling scheme [12–14] play crucial roles in the synchronization processes.

Among many models for synchronization phenomena, the Kuramoto model is the most popular nowadays [15–17], where many oscillators with heterogeneous natural frequencies couple through the sine of their phase differences. When this coupling is strong enough, all oscillators eventually rotate at the same frequency. In its original model, all agents interact with their neighbors all the time. However, the interaction is usually costly in reality. Due to the existence of costly interactions, some agents may choose not to interact with neighbors at some time, which impedes synchronization of the whole system.

How to understand the emergence of synchronization with costly interactions is a challenging issue. Recently, Antonioni and Cardillo incorporated the evolutionary game in the Kuramoto model [18]. They assumed that agents can decide whether to interact-or not-with their neighbors. All agents receive the benefit from the local synchronization but those who choose to interact with their neighbors must pay some cost. Each agent’s payoff is thus given by the net gain of benefit and cost. After each time step, agents are allowed to update their strategies by comparing payoffs with their neighbors. The strategies of agents with higher payoffs are more likely to be imitated. In this sense, this toy model can be named as the evolutionary Kuramoto game (EKG).

The EKG sheds some light on the study of self-organized interactions. Agents in EKG face a dilemma, that is, the interaction with neighbors helps to enhance synchronization of the whole system but the interaction cost weakens the willingness of agents’ participation. So far, how to alleviate this dilemma and enhance the global level of synchronization in EKG are still unclear. In this paper, we find that both too few and too many neighbors would suppress synchronization in EKG. Besides, we have shown that the rationality of agents plays a non-trivial role in EKG.

II. MODEL

Let us consider a network of N nodes and average degree ⟨k⟩. Each node i represents a Kuramoto oscillator and it can decide whether to interact-or not-with its neighbors. We call an agent as a cooperator if it interacts with its neighbors. Those who do not participate in such interaction are called as defectors.

Following the work of Antonioni and Cardillo [18], agent l changes its phase θ_l according to the following equation:

$$\dot{\theta}_l = \omega_l + s_l \lambda \sum_{m=1}^{N} a_{lm} \sin(\theta_m - \theta_l),$$

(1)

where \(\omega_l\) is the natural frequency of \(l\), \(s_l\) is the strategy of \(l\) (\(s_l = 1\) if \(l\) is a cooperator and \(s_l = 0\) if it is a defector), \(\lambda\) is...
the coupling strength, and \( a_{lm} \) is the element of the adjacency matrix (\( a_{lm} = 1 \) if \( l \) and \( m \) is connected, and \( a_{lm} = 0 \) otherwise). In this paper, both the initial phases \( \theta \) and the natural frequencies \( \omega \) of all agents are uniformly distributed over the interval \([-\pi, \pi]\).

The local order parameter for a pair of nodes \( l \) and \( m \), \( r_{lm} \), is calculated by

\[
r_{lm} = e^{i(\theta_l + \theta_m)t}/2.
\]

The benefit of agent \( l \) is defined as its local order parameter, which is determined by

\[
r_l = \frac{\sum_{m=1}^{N} a_{lm} r_{lm}}{\sum_{m=1}^{N} a_{lm}}.
\]

The cost of an agent \( l \) is defined as the absolute value of the angular acceleration, i.e.,

\[
c_l = |\dot{\theta}_l(t) - \dot{\theta}_l(t - \epsilon)|,
\]

where \( \epsilon \) is the step length used to compute Eq. \( \theta(t) \) with the fourth-order Runge-Kutta method. Note that the cost for a defector is zero since its frequency is fixed at the natural frequency. Finally, the payoff of an agent is defined as

\[
P_l = r_l - \alpha \frac{c_l}{2\pi},
\]

where the cost is divided by \( 2\pi \) to make it commensurable with the benefit and a scalar \( \alpha \) is named as the relative cost.

Initially, cooperators and defectors are randomly distributed in a network with equal probability. All agents synchronously update their strategies at discrete time steps \( t_k \) \( (t_{k+1} - t_k = \epsilon) \) according to the famous Fermi rule \([19]\). Specifically, an agent \( l \) randomly selects one of its neighbors, \( m \), and adopts its strategy with a probability given by

\[
W(s_l \leftarrow s_m) = \frac{1}{1 + \exp[-\beta(P_m - P_l)]},
\]

where the parameter \( \beta \) \( (>0) \) characterizes noise to permit irrational choices (henceforth we call \( \beta \) as the rationality parameter). Agents become more rational, i.e., have greater probability to follow the strategies of neighbors who have obtained higher payoffs, as the rationality parameter \( \beta \) increases.

After a sufficiently long transient time, one can calculate the average level of cooperation \( \langle C \rangle \) and the global synchronization \( \langle r_G \rangle \). Here \( \langle C \rangle \) is defined as the fraction of cooperators in the steady state. The global order parameter \( r_G \) is calculated by

\[
r_G e^{i\Psi} = \frac{1}{N} \sum_{m=1}^{N} e^{i\theta_m},
\]

where \( i \) is the imaginary unit and \( \Psi \) is the average phase of the system.

FIG. 1: (Color online) The average level of cooperation (synchronization) \( \langle C \rangle \) \( \langle r_G \rangle \) as a function of (a) the relative cost \( \alpha \) and (b) the coupling strength \( \lambda \), respectively. The average degree of the network \( \langle k \rangle = 8 \) and the rationality parameter \( \beta = 1 \). For (a), the coupling strength \( \lambda = 1 \). For (b), the relative cost \( \alpha = 0.1 \).

III. ANALYSES

Without loss of generality, we carry out our study of the model in Erdős-Rényi (ER) random graphs \([20]\) with size \( N = 2000 \). We set the step length \( \epsilon = 0.01 \) in this paper (we have checked that qualitative results keep unchanged for a wide range of \( \epsilon \)). In all simulations, we first wait 90000 Monte Carlo time steps to let the system attain steady state, and then run another 10000 Monte Carlo time steps to calculate the average level of cooperation (synchronization) \( \langle C \rangle \) \( \langle r_G \rangle \). Finally, each data presented below results from an average over 200 independent realizations.

A. Effects of the relative cost \( \alpha \) and the coupling strength \( \lambda \)

We first briefly review the effects of the relative cost \( \alpha \) and the coupling strength \( \lambda \) on the emergence of synchronization and cooperation, which reproduce the findings reported in \([18]\). Figure \( \theta \) shows the average level of cooperation (synchronization) \( \langle C \rangle \) \( \langle r_G \rangle \) as a function of the rela-
FIG. 2: (Color online) The average level of cooperation $\langle C \rangle$ as a function of the average degree $\langle k \rangle$ for different values of (a) the relative cost $\alpha$, (b) the coupling strength $\lambda$ and (c) the rationality parameter $\beta$, respectively. The average level of synchronization $\langle r_G \rangle$ as a function of the average degree $\langle k \rangle$ for different values of (d) the relative cost $\alpha$, (e) the coupling strength $\lambda$ and (f) the rationality parameter $\beta$, respectively. For (a) and (d), the coupling strength $\lambda = 1$ and the rationality parameter $\beta = 1$. For (b) and (e), the relative cost $\alpha = 0.1$ and the rationality parameter $\beta = 1$. For (c) and (f), the relative cost $\alpha = 0.2$ and the coupling strength $\lambda = 1$. 

The dependence of $\langle C \rangle$ and $\langle r_G \rangle$ on the coupling strength $\lambda$ is non-monotonic. For $\beta = 1$, the lower limit $\lambda_{c1}$ is determined by the network structure as 

$$\lambda_{c1} = \lambda_{MF} \frac{\langle k \rangle}{\langle k^2 \rangle},$$

where $\lambda_{MF}$ ($\lambda_{MF}=4$ in this paper) is the critical value of the coupling strength in the mean field case [21]. Below $\lambda_{c1}$, oscillators rotate incoherently even they are all cooperators. 

From Eqs. (1) and (4), one can note that the interaction cost is positively related to the coupling strength $\lambda$. Too large value of $\lambda$ tremendously reduces the ratio of benefit to cost, leading to the extinction of cooperators and an incoherent state of the system. According to [18], cooperators can survive when

$$\sqrt{2 + 2\sin(\varepsilon \lambda)} - \sqrt{2 \varepsilon \lambda} > \alpha. \tag{9}$$

To satisfy the above equation, the coupling strength must be below a certain value $\lambda_{c2}$.

We want to emphasize that the critical values of phase transitions are accurate for large system size. But in this paper, we do not focus on the phase transitions and hence use a small system.

B. Effects of the average degree $\langle k \rangle$

Figure 2 shows the average level of cooperation $\langle C \rangle$ (synchronization $\langle r_G \rangle$) as a function of the average degree $\langle k \rangle$ for
FIG. 3: (Color online) (a) The average level of cooperation $\langle C \rangle$ and (b) the average level of synchronization $\langle r_C \rangle$ as a function of the average degree $\langle k \rangle$ for WS and BA networks. The coupling strength $\lambda = 1$, the relative cost $\alpha = 0.07$ and the rationality parameter $\beta = 1$. In WS networks, the rewiring probability is set to be 0.2.

FIG. 4: (Color online) The average level of synchronization $\langle r_C \rangle$ as a function of the rationality parameter $\beta$ for different values of (a) the relative cost $\alpha$ and (b) the coupling strength $\lambda$, respectively. The average degree of the network is $\langle k \rangle = 8$. For (a), the coupling strength $\lambda = 1$. For (b), the relative cost $\alpha = 0.2$.

different values of the relative cost $\alpha$, the coupling strength $\lambda$ and the rationality parameter $\beta$. From Fig. 2(a), c), we observe that the cooperation level $\langle C \rangle$ keep almost unchanged for small values of $\langle k \rangle$ and then decreases to zero as $\langle k \rangle$ continually increases.

In the original Kuramoto model, the increase of the average degree facilitates the attainment of synchronization [22]. However, the dependence of $\langle r_C \rangle$ on $\langle k \rangle$ in the EKG displays a nonmonotonic behavior. As shown in Fig. 2(d, f), there exists an optimal value of $\langle k \rangle$, at which $\langle r_C \rangle$ is maximized. From Fig. 2(d) and 2(e), we observe that the optimal value of $\langle k \rangle$ tends to decrease as the relative cost $\alpha$ or the coupling strength $\lambda$ increases. For $\alpha = 0.07, 0.1$ and $0.2$, the optimal value of $\langle k \rangle$ is 18, 12 and 7 respectively (see Fig. 2(f)). For $\lambda = 0.7, 1$ and 1.5, the optimal value of $\langle k \rangle$ is 20, 12 and 8 respectively (see Fig. 2(e)). However, the optimal value of $\langle k \rangle$ appears to be a non-monotonic function of the rationality parameter $\beta$. For $\beta = 0.2, 1$ and 10, the optimal value of $\langle k \rangle$ is 9, 8 and 9 respectively, as shown in Fig. 2(f).

In the case of very small $\langle k \rangle$, the influence of neighbors is so weak that oscillators are inclined to rotate at their own frequencies. According to Eq. 3, we easily obtain $\langle k \rangle / \langle k^2 \rangle = 1/(\langle k \rangle + 1)$ for ER random graphs. Thus the critical coupling strength arousing the onset of a coherent state increases as the average degree $\langle k \rangle$ decreases, indicating that one must strengthen the coupling in order to reach the global synchronization for small $\langle k \rangle$. In fact, small $\langle k \rangle$ not only hinders synchronization in Kuramoto model but also impedes consensus in opinion dynamics [21]. On the other hand, in the case of very large $\langle k \rangle$, the system becomes close to the well-mixed scenario. Due to the costly interaction, the payoffs of cooperators are lower than those of defectors when $\langle k \rangle$ is very large. In this case, the extinction of cooperators is inevitable. The full defection will lead to an incoherent state of the system since all agents do not interact with neighbors and rotate with their natural frequencies. The extinction of cooperators for the case of large $\langle k \rangle$ has also been found in the prisoner’s dilemma game [23]. Combining the discussion of the two limits of $\langle k \rangle$, highest synchronization level should be realized for some intermediate values of $\langle k \rangle$.

In the above studies, we use ER random graphs. In fact, qualitative results remain unchanged for other kinds of networks including Watts-Strogatz (WS) small-world networks [24] and Barabási-Albert (BA) scale-free networks [25]. From Fig. 3 one can see that for WS or BA networks, the cooperation level $\langle C \rangle$ decreases as the average degree $\langle k \rangle$ increases while the synchronization level $\langle r_C \rangle$ is maximized at a moderate value of $\langle k \rangle$.

C. Effects of the rationality parameter $\beta$

In the previous work [18], the rationality parameter $\beta$ is set to be 1. However, studies on other evolutionary games have shown that, the cooperation level is largely affected by the rationality parameter [24–29]. Figure 4 shows the average
level of synchronization \(\langle r_G \rangle\) as a function of the rationality parameter \(\beta\) for different values of the relative cost \(\alpha\) and the coupling strength \(\lambda\). One can see that for small values of \(\alpha\) or \(\lambda\) (e.g., \(\alpha = 0.1\) or \(\lambda = 0.7\)), \(\langle r_G \rangle\) increases with \(\beta\). On the other hand, for large values of \(\alpha\) or \(\lambda\) (e.g., \(\alpha = 1.5\) or \(\lambda = 10\)), \(\langle r_G \rangle\) decreases as \(\beta\) increases. For moderate values of \(\alpha\) and \(\lambda\), we notice that \(\langle r_G \rangle\) is minimized at the middle values of \(\beta\). The dependence of the average cooperation level \(\langle C \rangle\) on \(\beta\) also obeys the above rule (results are not shown here).

In Fig. 5 we plot a color coded map of \(\langle r_G \rangle\) in the parameter plane \((\beta, \alpha)\) by setting \(\langle k \rangle = 8\) and \(\lambda = 1\). We find that, for \(0.25 < \alpha < 1.4\), \(\langle r_G \rangle\) is minimized at the middle values of \(\beta\). For \(\alpha > 1.4\), \(\langle r_G \rangle\) decreases as \(\beta\) increases.

It has been known that the formation of clusters plays an important role in the evolutionary games [30, 31]. A cooperator (defector) cluster is a connected component fully composed of cooperators (defectors). For small values of \(\alpha\) or \(\lambda\), the interaction cost is much smaller than the benefits due to mutual synchronization, leading to high payoffs of agents inside cooperator clusters. The payoffs of agents inside defector clusters are usually low since they change phases independently (no local synchronization is expected). Thus for small values of \(\alpha\) or \(\lambda\), the expansion of cooperator clusters becomes easier as agents are more rational (i.e., the increase of \(\beta\)). On the other hand, for very large values of \(\alpha\) or \(\lambda\), the interaction cost is so large that cooperators cannot form stable clusters to resist the invasion of defectors. In this case, the decrease of \(\beta\) can reduce the probability that a cooperator is replaced by a defector. Note that \(\beta = 0\) corresponds to a random strategy update, which is conceptually similar to a neutral drift of the voter model [32]. For \(\beta = 0\), the system has the equal probability to terminate in the full cooperation or the full defection, leading to the average cooperation level \(\langle C \rangle = 0.5\).

To understand the nonmonotonic behavior appearing in the moderate region of \(\alpha\) and \(\lambda\), we study the time evolution of the cooperation level \(\langle C(t) \rangle\) (synchronization level \(\langle r_G(t) \rangle\)) for different values of the rationality parameter \(\beta\). From Fig. 6(a), we see that, for \(\beta = 4\), \(\langle C(t) \rangle\) decreases to 0 as time evolves, meanwhile, \(\langle r_G(t) \rangle\) keeps close to 0 but slightly peaks at about \(t = 100\). For \(\beta = 20\), \(\langle C(t) \rangle\) initially decreases and then increases to 1, meanwhile, \(\langle r_G(t) \rangle\) gradually increases to about 0.9 (see Fig. 6(b)). From Fig. 6 one can find that cooperator clusters continuously shrink and finally disappear in the case of \(\beta = 4\). However, for a large value of \(\beta\) (e.g., \(\beta = 20\)), cooperator clusters can survive after the initial invasion of defectors. Note that the change of strategies happens on the border that separates clusters of cooperators and defectors. For a smaller value of \(\beta\), cooperators along the border are more likely to be replaced by neighboring defectors, leading to the instability of cooperator clusters. However, for a very large value of \(\beta\), cooperators along the border cannot be invaded once their payoffs are higher than those of neighboring defectors. Cooperators inside a stable cluster can gain more and more benefit from the mutual synchronization and the interaction cost continually decreases as time evolves. From Fig. 6(b), we observe that \(\langle r_G(t) \rangle\) rapidly increases from 0.05 to 0.6 while \(\langle C(t) \rangle\) keeps around 0.6 during \(200 < t < 500\). In this period, cooperators get more and more synchronized, leading to the latter growth of cooperator clusters. Summarizing, for moderate interaction cost, sufficient high extent of rationality helps cooperators to survive and form tiny clusters to compete with defectors in the initial stage, and then enables them to strike back, leading to the high level of cooperation and synchronization in the later stage. Moderate rationality is, however, lack of the ability to stabilize the cluster of cooperators, resulting in the lowest level of cooperation and synchronization. It was worth noting that such nonmonotonic phenomenon, i.e., the cooperation level reaches the minimum at a moderate value of the rationality parameter, has also been observed in the spatial public goods game [33] and snowdrift.
FIG. 7: (Color online) Time series of the average payoff of cooperators \( \langle P_C(t) \rangle \) and defectors \( \langle P_D(t) \rangle \). The inset shows the average cost of cooperators as time evolves. The average degree of the network \( k = 8 \), the relative cost \( \alpha = 0.4 \), the coupling strength \( \lambda = 1 \) and the rationality parameter \( \beta = 20 \).

Figure 7 shows time series of the average payoff of cooperators \( \langle P_C(t) \rangle \) and defectors \( \langle P_D(t) \rangle \) for the case of \( \beta = 20 \). One can see that initially \( \langle P_C(t) \rangle \) is lower than \( \langle P_D(t) \rangle \). When \( t > 10 \), \( \langle P_C(t) \rangle \) gradually exceeds \( \langle P_D(t) \rangle \). From the inset of Fig. 6, we observe that the average cost of cooperators \( \langle c_C(t) \rangle \) decreases as time evolves. In the late stage of evolution, the system is almost occupied by cooperators and only a few defectors survive. The phases of these cooperators are almost the same while defectors change their phases independently. Thus the payoff of a defector fluctuates greatly in the late stage of evolution. A defector gain a high (low) payoff when its phase is the same (different) with those of cooperators.

IV. CONCLUSIONS

In conclusion, we have studied how the average interaction degree and the extent of rationality affect the coevolution of cooperation and synchronization. Our main findings can be summarized as follows. (i) The cooperation level is positively correlated with the synchronization level, that is, more cooperators will promote synchronization. (ii) Both of the cooperation and synchronization levels decrease as the relative cost increases. (iii) Both of the cooperation and synchronization levels are maximized in the middle range of the coupling strength. (iv) The cooperation level decreases as the average degree of the network increases. But the synchronization level is maximized at a moderate value of the average degree. (v) The extent of rationality plays a nontrivial role in the coupling dynamics. For small (large) values of the relative cost or the coupling strength, both the cooperation and synchronization levels increase (decrease) as the rationality parameter increases. For moderate values of the relative cost and the coupling strength, however, both the cooperation and synchronization levels are minimized at a middle value of the rationality parameter.

Our results offer a deeper understanding of the interplay of synchronization dynamics and game theory. There remains a number of open questions in EKG. For example, how the clustering coefficient and degree correlations of the underlying interaction network affect the synchronization? What would happen if we use other rules of strategy updating in EKG? We hope our work can stimulate more researchers into the study of coevolution of synchronization and cooperation.

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