THE $L^2$-NORM OF THE SECOND FUNDAMENTAL FORM OF ISOMETRIC IMMERSIONS INTO A RIEMANNIAN MANIFOLD

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ABSTRACT. We consider critical points of the global squared $L^2$-norms of the second fundamental form and the mean curvature vector of isometric immersions into a fixed background Riemannian manifold under deformations of the immersion. We use the critical points of the former functional to define canonical representatives of a given integer homology class of the background manifold. We study the fibration $S^3 \to Sp(2) \to S^7$ from this point of view, showing that the fibers are the canonical generators of the 3-integer homology of $Sp(2)$ when this Lie group is endowed with a suitable family of left invariant metrics. Complex subvarieties in the standard $\mathbb{P}^n(\mathbb{C})$ are critical points of each of the functionals, and are canonical representatives of their homology classes. We use this result to provide a proof of Kronheimer-Wrowka's theorem on the smallest genus representatives of the homology class of a curve of degree $d$ in $\mathbb{P}^2$, and analyze also the canonical representability of certain homology classes in the product of standard 2-spheres. Finally, we provide examples of background manifolds admitting isotopically equivalent critical points in codimension one for the difference of the two functionals mentioned, of different critical values, which are Riemannian analogs of alternatives to compactification theories that has been offered recently.

1. INTRODUCTION

Consider a fixed background Riemannian manifold $\tilde{M}$ of a given dimension, and let $M$ be an immersed closed submanifold. With the metric induced from that of $\tilde{M}$, $M$ itself becomes a Riemannian manifold. Of all such isometric immersions of $M$ into $\tilde{M}$, we would like to single out the one that is optimal. This is the main theme of our work here.

Before going any further, let us relate the setting just described to the language of contemporary mathematical physics [24]. We liken $\tilde{M}$ to the full world, the bulk in the terminology of physicists. The background metric it carries makes it possible to define the length of curves in the ambient space, and consequently, we can talk of the least action principle for the motion of particles in this world. In this bulk, the isometrically embedded submanifold $M$ is taken to represent a brane. Branes seem to come in two flavours, those bounding portions of the ambient space, or those serving as the boundary locations for open strings. Our setting differs from that in physics in that the ambient space metric is Riemannian, and not Lorentzian. But we dispense with that defect in the analogy, and move along anyway. The question of interest to us can be phrased by asking for the optimal way of isometrically immersing a brane into the bulk.

An extremely worthy effort in this direction can be made if we think of the optimal immersion of a brane as one that extremizes the volume the brane occupies among deformations of the immersion. This of course leads to the study of minimal
submanifolds, which at least in the case of surfaces goes back to Lagrange, or even a bit earlier, to Euler, who studied the catenoid, in the mid 1700s, and which constitutes a subject that have remained an exciting area of research to this date.

Here, we turn to a different criteria in order to measure optimality, again motivated by the development in theoretical physics. Indeed, it seems that the Calabi-Yau manifolds, a particular type of Kähler manifold, are to be the preferred choice for compactifying or curling-up the extra dimensions we do not seem to be able to observe in our bulk world. And though all complex submanifolds of a Kähler manifold are minimal, there are too many minimal submanifolds that are not complex. The said volume criteria does not give a special role to Kähler branes, and in that sense, there might be others that are more appropriate in order to measure the optimality of an immersion, perhaps singling out Kähler branes over other minimal submanifolds.

Since our submanifold $M$ is isometrically immersed into the ambient space $\tilde{M}$, we can measure the length of paths whose end points are in $M$ once we know the immersion. Thus, the issue is strictly about how to place the submanifold $M$ inside the ambient space $\tilde{M}$. Since the metric in $\tilde{M}$ defines canonically a notion of derivative everywhere, the Levi-Civita connection of its metric, we may naturally conceive of an energy associated to an immersion that only depends upon its first order derivatives, and so as not to alter the submanifold itself, make this energy depend upon the normal component of these first derivatives. Thus, we look at the second fundamental form of the immersion, $\alpha$, and declare that the immersion’s energy be measured by its global squared $L^2$-norm, which we shall call $\Pi(M)$. This energy for an immersion resembles the stored-energy function for neo-Hookian materials, and by way of analogy, we may think of the energy of an immersed brane as being determined by the accumulated effect of a tensor of springs that act in the normal directions. In accordance with Hamilton’s principle, an optimal isometric immersion of the submanifold will be one that is a stationary point for $\Pi(M)$ under deformations of the isometric immersion. Since our immersion does not have kinetic energy, this criterion for the optimality of the immersion corresponds exactly to that used to place a static elastodynamic incompressible neo-Hookian material in space, the incompressibility taken here as the analogue of an immersion that is isometric. Notice that in this sense, the functional $\Pi(M)$ is a much higher order energy than that given by the volume of the immersed manifold.

We develop the Euler-Lagrange equations for $\Pi(M)$ under deformations of the isometric immersion of $M$ into $\tilde{M}$. As equations on the coefficients of the second fundamental form of an stationary immersion, they constitute a local nonlinear system of order two, with cubic nonlinearities in the principal curvatures along normal directions. (The local nature of the equations shows at least one defect with the analogy with the motion of incompressible elastic bodies above, which are ruled by pseudodifferential equations instead.) In the case of codimension one, we conclude that all austere manifolds are critical points. Though the set of these is much richer in general, it is likely that the critical points of $\Pi(M)$ characterize austerity in dimensions less or equal than 4. We also prove that all complex submanifolds of complex projective space with the Fubini-Study metric are stationary points, in a sense vindicating the desired preference that we wanted to give to Kähler branes.

For a given homology class of dimension $k$, the embedded minimizers of $\Pi(M)$ within the class are to be considered its canonical representatives. In contrast
with the case where $k = 1$, these minimizers do not have to be totally geodesics, or even exist for that matter. We prove their existence in the case where $\tilde{M} = Sp(2)$ endowed with a suitable left-invariant metric, and the homology class is the generator of $H_3(Sp(2); \mathbb{Z})$. This same background manifold can be used to show the nonexistence of embedded minimizers of $\Pi(M)$ representing the generator of $H_7(Sp(2); \mathbb{Z})$, for the simple reason that this class does not admit embedded representatives.

The cubic nature of the nonlinearity in the Euler-Lagrange equation for $\Pi(M)$ seems to single out Kähler branes of complex dimension less or equal than 3. Exploiting the fact that complex submanifolds are minimal (see, for instance, [15, 31]), we refine our criterion for optimality of the immersion, and take it one step further. The trace of the second fundamental form of an immersion is the mean curvature vector, $H$, whose squared $L^2$-norm we shall call $\Psi(M)$, the Willmore functional [37]. We develop also its Euler-Lagrange equation under deformations of the isometric immersion, and having done so, consider the “energy” functional $\Pi(M) - \Psi(M)$ over the space of deformations of the isometric immersion restricted to represent a fixed integral homology class. In a sense, stationary points of this functional are given by isometric immersions for which the trace-free part of the second fundamental form is as small as possible, paralleling the defining property of Einstein metrics, whose Ricci tensors are traceless. Thus, we look at these stationary points as some sort of Einstein isometric immersions of the brane into the bulk. If $M$ is a canonical isometric immersion of the smallest volume among critical points of $\Pi(M)$ of a fixed critical level, when $M$ itself is minimal, it is then also a critical point of $\Pi(M) - \Psi(M)$, and in fact, that stationary metric on $M$ is a critical point of the total scalar curvature functional in the space of Riemannian metrics on $M$ that can be realized by isometric embeddings into $\tilde{M}$.

Complex submanifolds of complex projective space are examples of such. For when the ambient space $\tilde{M}$ is Kähler, any complex submanifold is an absolute minimum for $\Psi(M)$, and so the said complex submanifolds are critical points of $\Pi(M) - \Psi(M)$ under deformations. Thus, the new functional distinguishes complex submanifolds at least when the bulk background is equal to complex projective space endowed with the Fubini-Study metric. We use this property, and a description of the singularities of minimizing sequences of representatives of a given integral homology class, to prove that among all embedded surfaces in complex projective plane that represent the homology class of a complex curve of fixed degree, the complex ones are the ones with the smallest genus. This is a theorem of Kronheimer and Mwroka [17], after a conjecture commonly attributed to R. Thom.

The energies $\Pi(M)$, $\Pi(M) - \Psi(M)$, and $\Psi(M)$ differ from each other, and this is manifested well in the properties of their stationary points. The differences remains in place when our functionals are compared to others that have been termed Willmore in the literature also [13, 15]. For in the case of codimension one, for instance, Example 3.9 below exhibits a family of Willmore hypersurfaces out of which $\Pi(M)$ singles out as stationary points only those that are Clifford minimal tori.

The choice of $\Pi(M) - \Psi(M)$ as an energy functional for the immersion is very natural, and truly derives from equation (4) below that expresses the intrinsic scalar curvature of $M$ in terms of the scalar curvature of $\tilde{M}$ and additional extrinsic data, an identity that in turn follows from Gauss’ equation. We obtain $\Pi(M) - \Psi(M)$ by
integrating over the immersed manifold the summands in this expression that involve first order covariant derivatives of the connection of the ambient space metric. Thus, the said functional is the averaged global extrinsic first order contribution to the total scalar curvature of the metric on $M$ induced by that of $\tilde{M}$. The remaining unaccounted contribution to the intrinsic total scalar curvature is obtained by averaging the sectional curvatures of the ambient space metric over the submanifold, and in that sense, it is not a contribution that we are at liberty to move by changing the isometric immersion. Its minimal critical points are thus critical points of the total scalar curvature functional over the space of metrics that can be realized by isometric immersions into $\tilde{M}$s, and many of these, are Einstein metrics on $M$, cf. [10].

1.1. Structure of the article. In §2 we introduce the basic notation, and conventions. As we use them later on, we derive expressions for the Ricci tensor and scalar curvature of an immersed manifold in terms on intrinsic and extrinsic data. In §3 we derive the Euler-Lagrange equation for $\Pi(M)$ and $\Psi(M)$ under deformations of the immersion. We write these as a system of $q$ equations, $q$ the codimension of $M$ in $\tilde{M}$, and discuss the case where $M$ is a hypersurface and the background metric is Einstein. We consider also the functional $\Pi(M)$ when its domain is restricted to the set of embedded representatives of a fixed integer homology class $D$ of $\tilde{M}$, and define a canonical representative of $D$ to be a minimum $M$ of smallest volume. The theory is illustrated using the simple Lie group $Sp(2)$ and the generator $D$ of the third homology group $H_3(Sp(2);\mathbb{Z})$. We prove that if $Sp(2)$ is endowed with suitable left-invariant metrics, the canonical representative of $D$ exists and, up to isometric diffeomorphisms, is unique. The special nature of complex submanifolds of complex projective space is presented in §4, where we briefly recall also some relevant facts about the Fubini-Study metric in general, and curves in complex projective plane, both to be used in later portions of our work. We prove that if $(M,g)$ is a complex manifold isometrically immersed into $\mathbb{P}^n(\mathbb{C})$ with its Fubini-Study metric, then $\|\alpha\|^2$ is an intrinsic quantity, and develop various other properties of complex subvarieties in relation to the functional $\Pi(M)$. We continue the analysis of these properties and its relation to canonical representatives of homology classes in §5. In combination with the fact that complex curves minimize $\Psi(M)$, we derive in there an alternative proof of the aforementioned theorem of Kronheimer and Mwroka, avoiding the use of the Seiberg-Witten theory. In this section, we discuss also the canonical representability of the class $(1, 1)$ in the 2-dimensional homology of the product of two spheres. Finally, in §6 we exhibit examples of critical hypersurfaces of $\Pi(M) - \Psi(M)$ of different critical values that are isotopically equivalent to each other in a) $S^{n+1}$, $n \geq 8$, provided with the standard metric; b) the trivial bundle $S^2 \times \mathbb{R}^2$ provided with the Schwarzchild metric; and c) the bundle $T^2S^2$ provided with the Eguchi-Hanson metric. In the last two cases, the second of the exhibited critical hypersurfaces occurs asymptotically at $\infty$.

2. Preliminaries, notations and conventions

Consider a closed Riemannian manifold $(\tilde{M}, \tilde{g})$, and let $M$ be a submanifold of $\tilde{M}$. With the induced metric, $M$ itself becomes a Riemannian manifold. We denote this metric on $M$ by $g$, and proceed to study certain relations between the
geometric quantities of $g$ and $\tilde{g}$ on points of $M \subset \tilde{M}$. We shall denote by $n$ and $\hat{n}$ the dimensions of $M$ and $\tilde{M}$, respectively.

The second fundamental form $\alpha$ of $M$ is defined in terms of the decomposition of the Levi-Civita connection $\nabla^{\tilde{g}}$ into a tangential and normal component \cite{9}. Indeed, under the inclusion map, the pull-back bundle $i^*\tilde{T}M$ can be decomposed as a Whitney sum

$$i^*\tilde{T}M = TM \oplus \nu(M),$$

where $\nu(M)$ is the normal bundle to $M$. Given vector fields $X$ and $Y$ tangent to $M$, we have that

$$\alpha(X,Y) = \pi_{\nu(M)}(\nabla^{\tilde{g}}_XY),$$

where $\pi_{\nu(M)}$ is the orthogonal projection onto the normal bundle.

By uniqueness of the Levi-Civita connection, the orthogonal projection of the ambient space connection onto $TM$ defines $\nabla^{\tilde{g}}_XY$. We have Gauss’ identity

$$(1) \quad \nabla^{\tilde{g}}_XY = \nabla^{\tilde{g}}_X Y + \alpha(X,Y).$$

If we consider a section $N$ of the normal bundle $\nu(M)$, the shape operator is defined by

$$A_N X = -\pi_{\nu(M)}(\nabla^{\tilde{g}}_X N),$$

where in the right side above, $N$ stands for an extension of the original section to a neighborhood of $M$. If $\nabla^{\hat{g}}$ is the connection on $\nu(M)$ induced by $\nabla^{\tilde{g}}$, we have Wiengarten’s identity

$$(2) \quad \nabla^{\hat{g}}_X N = -A_N X + \nabla^{\hat{g}}_X N.$$

The identities \cite{11} and \cite{12} allows us to express the commutator $[\nabla^{\tilde{g}}_X,\nabla^{\tilde{g}}_Y]$ in terms of the commutator $[\nabla^{\hat{g}}_X,\nabla^{\hat{g}}_Y]$, the second fundamental form and the shape operator. We may do so also for $\nabla^{\hat{g}}_{[X,Y]}$. As a consequence, if $Z, W$ are two additional vector fields tangent to $M$, we obtain Gauss’ equation, a fundamental relation between the tangential component of the extrinsic and intrinsic curvature tensors on $M$:

$$(3) \quad g(R^{\tilde{g}}(X,Y)Z,W) = \tilde{g}(R^{\tilde{g}}(X,Y)Z,W) + \tilde{g}(\alpha(X,W),\alpha(Y,Z)) - \tilde{g}(\alpha(X,Z),\alpha(Y,W)).$$

Here, $R^{\tilde{g}}$ stands for the Riemann curvature tensor of the corresponding metric $\tilde{g}$.

The Ricci tensor $r_{\tilde{g}}(X,Y)$ of a Riemannian metric $g$ with curvature tensor $R^{\tilde{g}}$ is defined as the trace of the linear map $L \rightarrow R^{\tilde{g}}(L,X)Y$, while the scalar curvature $s_{\tilde{g}}$ is defined as the metric trace of $r_{\tilde{g}}$. Let $\{e_1, \ldots, e_\hat{n}\}$ be an orthonormal frame for $\tilde{g}$ in a neighborhood of a point in $M$ such that $\{e_1, \ldots, e_n\}$ constitutes an orthonormal frame for $g$ on points of $M$. Letting $H$ be the mean curvature vector, the trace of the second fundamental form, by \cite{3} we obtain that

$$r_{\tilde{g}}(X,Y) = \sum_{i=1}^{n} \tilde{g}(R^{\tilde{g}}(e_i,X)Y,e_i) + \tilde{g}(H,\alpha(X,Y)) - \sum_{i=1}^{n} \tilde{g}(\alpha(e_i,X),\alpha(e_i,Y))$$

$$= r_{\tilde{g}}(X,Y) - \sum_{i=n+1}^{\hat{n}} \tilde{g}(R^{\tilde{g}}(e_i,X)Y,e_i) + \tilde{g}(H,\alpha(X,Y)) - \sum_{i=1}^{n} \tilde{g}(\alpha(e_i,X),\alpha(e_i,Y)),$$

while

$$(4) \quad s_{\tilde{g}} = s_{\tilde{g}} - 2 \sum_{i=1}^{n} \sum_{j=n+1}^{\hat{n}} K_{\tilde{g}}(e_i,e_j) - \sum_{i,j>n} K_{\tilde{g}}(e_i,e_j) + \tilde{g}(H,H) - \tilde{g}(\alpha,\alpha)$$

$$= \sum_{i,j\leq n} K_{\tilde{g}}(e_i,e_j) + \tilde{g}(H,H) - \tilde{g}(\alpha,\alpha).$$
Here \( K_\alpha(e_i, e_j) \) is the \( \hat{g} \)-curvature of the section spanned by the orthonormal vectors \( e_i \) and \( e_j \), and \( \hat{g}(H, H) \) and \( \hat{g}(\alpha, \alpha) \) are the squared-norms of the mean curvature vector \( H \) and the form \( \alpha \), respectively.

In the same manner as Gauss’ equation \( \eqref{eq:gauss} \) describes the tangential component \( R^g(X, Y)Z \) in terms of the intrinsic curvature tensor of \( M \) and the second fundamental form, we may find an analogous type of relation for the normal component \( (R^\hat{g}(X, Y)Z)^\nu \) of this same vector. In so doing, we obtain the following identity, known as Codazzi’s equation:

\[
\nabla_X^\hat{g}\alpha(Y, Z) = (R^\hat{g}(X, Y)Z)^\nu + \nabla_Y^\hat{g}\alpha(X, Z) + A_{\alpha(X, Z)}Y - A_{\alpha(Y, Z)}X \]

\[
\alpha(Y, \nabla_X^\hat{g}Z) - \alpha(X, \nabla_Y^\hat{g}Z) + \alpha([X, Y], Z).
\]

If \( \{e_i\}_{i=1}^n \) is an orthonormal frame of \( TM \) as above that is geodesic at a point \( p \), we derive the following identity

\[
\nabla^{\hat{g}}_e \alpha(e_i, e_j) = (R^{\hat{g}}(e_i, e_i)e_j)^\nu + \nabla^{\hat{g}}_e H + A_H e_i - A_{\alpha(e_i, e_j)}e_j,
\]

which will be used several times below. In writing this expression, we tacitly use the standard summation convention over repeated indexes, a practive to be repeated throughout the article.

3. Variational Formulas

We now develop the Euler-Lagrange equations of the squared \( L^2 \)-norm of the second fundamental form, mean curvature vector, and total extrinsic scalar curvature of a manifold \( M \) isometrically immersed into \((\tilde{M}, \tilde{g})\) under deformations of the immersion. Since the densities of the functionals in question are local operators, in deriving the said equations we may assume that the immersion is in fact a compact embedding. In what follows, the ambient space metric \( \hat{g} \) will be denoted by \( \langle \cdot, \cdot \rangle \), for convenience.

3.1. The second fundamental form. We consider a closed manifold \( M \) of dimension \( n \), and let \( f : M \rightarrow \tilde{M} \) be an isometric immersion into the \( n \)-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\). For the reasons indicated above, we shall assume that \( f(M) = M \rightarrow \tilde{M} \) is a compact embedding. Let \( \alpha \) be the second fundamental form induced by \( f \), and \( \|\alpha\|^2 = \langle \alpha, \alpha \rangle \) be its squared pointwise \( L^2 \)-norm, expression which of course involves the metric on \( M \). We let \( \Pi(M) \) be the integral of \( \|\alpha\|^2 \).

That is to say, if \( d\mu = d\mu_M \) denotes the volume measure on \( f(M) \), then

\[
\Pi(M) = \int_M \|\alpha\|^2 \, d\mu.
\]

Let \( f : (-a, a) \times M \rightarrow \tilde{M} \) be a one parameter family of deformations of \( M \). We set \( M_t = f(t, M) \) for \( t \in (-a, a) \), and have \( M_0 = M \). We would like to compute the \( t \)-derivative of \( \Pi(M_t) = \int_{M_t} \|\alpha_t\|^2 \, d\mu_t \) at \( t = 0 \).

Given a point \( p \in M \), we let \( \{x_1, \ldots, x^n\} \) be a coordinate system of \( M \times (-a, a) \), valid in some neighborhood of \((p, 0)\), such that \( \{x^1, \ldots, x^n\} \) are normal coordinates of \( M \) at \( p \). The induced metric \( g_t \) on \( M_t \) has components \( g_{ij} = \langle e_i, e_j \rangle \), where \( e_i = df(\partial_{x^i}) \), \( i = 1, \ldots, n \), and by assumption, \( g_{ij}(p, 0) = \delta_{ij} \). We set \( T = df(\partial_t) \), the variational vector field of the deformation.
We decompose $T$ into its tangential and normal component, $T = T^t + T^n$. Then the variation of the measure is given by

$$
\frac{d\mu_t}{dt}(p,0) = \left( \text{div}(T^t) - \langle T^n, H \rangle \right) d\mu_0(p,0),
$$

(7)
a well-known expression (see, for instance, [20]) already used extensively in the theory of minimal submanifolds. Since the gradient of a function defined on $M$ is a tangential vector field while the mean curvature is normal, we have that

$$
\hat{\Pi}(M_t) \big|_{t=0} = \frac{d}{dt} \int_{M_t} \|\alpha\|^2 \, d\mu_t \big|_{t=0} = \int_M \left( \frac{d}{dt} \|\alpha\|^2 \big|_{t=0} - \langle \nabla \|\alpha\|^2 + \|\alpha\|^2 H, T \rangle \right) d\mu.
$$

(8)

When computing the variation of the pointwise $L^2$-norm of $\alpha$, it is necessary to use an orthonormal frame along points $(p,t)$ of an integral curve of $T$ for small $t$s. The coordinates $\{x^1, \ldots, x^n\}$ above were chosen so that the coordinate vector fields $\{\partial_{x^i}\}$ are orthonormal and have vanishing $g$-covariant derivatives at $p$. These vector fields commute with $T$. We now take an orthonormal tangent frame $\{e_i\}$ along a sufficiently small neighborhood of the integral curve of $T$ through $p$ that extends the $\partial_{x^i}$s. Thus, $t$ parametrizes the integral curve through $p$, and for each sufficiently small $t$, we have that $\{e_1, \ldots, e_n\}$ is a normal frame of $M_t$ at $(p,t)$. We still have that $[T, e_i] = 0$ and $\nabla_{e_i} e_j = 0$ at $p = (p, 0)$, respectively. With the Einstein summation convention in place, $\|\alpha\|^2$ can be conveniently calculated by $\|\alpha\|^2 = \langle \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle$, as already tacitly done in [1].

By differentiation along the integral curves of $T$, we have that

$$
\frac{d}{dt} \langle \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle \big|_{t=0} = 2\langle \nabla_{T}^{\nabla} \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle + 2\hat{g}^{il} \langle \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle = 2\langle \nabla_{e_i}^{\nabla_{e_i}} e_j, \alpha(e_i, e_j) \rangle + 2\hat{g}^{il} \langle \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle.
$$

(9)

Since $\nabla_{e_i}^{\nabla_{e_i}} e_j$ is tangent to $M_t$, and vanishes at $p$, while $\alpha(e_i, e_j)$ is a normal vector, we have that

$$
\langle \nabla_{e_i}^{\nabla_{e_i}} e_j, \alpha(e_i, e_j) \rangle = T \langle \nabla_{e_i}^{\nabla_{e_i}} e_j, \alpha(e_i, e_j) \rangle - \langle \nabla_{e_i}^{\nabla_{e_i}} e_j, \nabla_{T}^{\nabla} \alpha(e_i, e_j) \rangle = 0.
$$

On the other hand, at $p$ we have that

$$
\langle \nabla_{e_i}^{\nabla_{e_i}} e_j, \alpha(e_i, e_j) \rangle = \langle \nabla_{e_i}^{\nabla_{e_i}} e_j, \alpha(e_i, e_j) \rangle = \langle \nabla_{e_i}^{\nabla_{e_i}} e_j + \nabla_{[T, e_i]}^{\nabla} e_j + R^{\nabla} (T, e_i) e_j, \alpha(e_i, e_j) \rangle = \langle \nabla_{e_i}^{\nabla_{e_i}} e_j + R^{\nabla} (T, e_i) e_j, \alpha(e_i, e_j) \rangle
$$

because $\nabla_{X}^{\nabla} Y$ is tensorial in $X$, and $[T, e_i] p = 0$.

Still computing at $p$, we find that

$$
\langle \nabla_{e_i}^{\nabla_{e_i}} e_j, \alpha(e_i, e_j) \rangle = \langle \nabla_{e_i}^{\nabla_{e_i}} T + \nabla_{e_i}^{\nabla_{e_i}} [T, e_j], \alpha(e_i, e_j) \rangle = \langle \nabla_{e_i}^{\nabla_{e_i}} T, \alpha(e_i, e_j) \rangle + e_i \langle [T, e_j], \alpha(e_i, e_j) \rangle = \langle \nabla_{e_i}^{\nabla_{e_i}} T, \alpha(e_i, e_j) \rangle ,
$$

with the second and third equality being true because we have that $[T, e_i] p = 0$, and with the latter of these making use of the tangentiality of $e_i$ also. Further,

$$
\langle \nabla_{e_i}^{\nabla_{e_i}} T, \alpha(e_i, e_j) \rangle = e_i e_j (T, \alpha(e_i, e_j)) - e_i (T, \nabla_{e_i}^{\nabla} \alpha(e_i, e_j) - e_j (T, \nabla_{e_i}^{\nabla} \alpha(e_i, e_j)) + (T, \nabla_{e_i}^{\nabla} \nabla_{e_i}^{\nabla} \alpha(e_i, e_j) ,
$$

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*Formula [1] implies that $M$ is a critical point of the volume functional under deformations of the isometric immersion if, and only if, $H = 0$. Manifolds with this property are called minimal, as in our Introduction here, §1.
expression whose right hand side is equal to its very last term modulo a differential, and so by Stokes’ theorem, its integral over $M$ coincides with the integral of the said term.

We use the symmetries of the curvature tensor to obtain the identity
\[
\langle R^g(T, e_i) e_j, \alpha(e_i, e_j) \rangle = \langle R^g(\alpha(e_i, e_j), e_i) e_j, T \rangle.
\]

Finally, using the orthonormality of the frame, we conclude that
\[
\dot{g}^{ij} \langle \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle = 2 \langle \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle \alpha(e_i, e_j), T \rangle - (e_i \langle T, e_i \rangle + e_i \langle T, e_i \rangle) \langle \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle.
\]

**Theorem 3.1.** Let $f : (-a, a) \times M \rightarrow \tilde{M}$ be a deformation of an isometrically immersed submanifold $f : M \rightarrow f(M) \hookrightarrow \tilde{M}$ into the Riemannian manifold $(\tilde{M}, \tilde{g})$. We set $M_t = f(t, M)$, and have $M_0 = M$. We let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of $M_t$ for all $t \in (-a, a)$. Then the infinitesimal variation of (8) is given by
\[
\frac{d\Pi(M_t)}{dt} |_{t=0} = \int_M (2\nabla e_i^g \nabla e_j^g \alpha(e_i, e_j) + 2R^g(\alpha(e_i, e_j), e_j) e_i - \|\alpha\|^2 H,T) d\mu + \int_M (4\langle \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle \alpha(e_i, e_i) - \nabla \|\alpha\|^2, T) d\mu - 2 \int_M (e_i \langle T, e_i \rangle + e_i \langle T, e_i \rangle) \langle \alpha(e_i, e_j), \alpha(e_i, e_j) \rangle d\mu.
\]

The isometrically immersed submanifold $f : M \rightarrow f(M) \hookrightarrow \tilde{M}$ satisfies the Euler-Lagrange equation of (10) if, and only if, the normal component of the vector

\[
2\nabla e_i^g \nabla e_j^g \alpha(e_i, e_j) + 2R^g(\alpha(e_i, e_j), e_j) e_i - \|\alpha\|^2 H + 4 \langle \alpha(e_i, e_j), \alpha(e_i, e_i) \rangle \alpha(e_i, e_i)
\]

is identically zero.

**Proof.** The calculations preceding the statement may be used to derive a convenient expression for (9). When this expression is inserted into (8), the variational formula results by applying Stokes’ theorem.

We now compute this variational expression after decomposing the vector field $T$ into its tangential and normal components. Since a tangential deformation merely yields a reparametrization of $M$, and since the vector $\nabla \|\alpha\|^2$ is tangent to this manifold, the Euler-Lagrange equation of (10) is equivalent to the vanishing of the normal component of (11), as stated. \(\square\)

We shall say that $(M, g)$ is canonically placed in $(\tilde{M}, \tilde{g})$ if there exists an isometric embedding $i : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ that is a critical point of (10) under deformations of the embedding. There is no rigidity in how one of these manifolds can be placed inside $\tilde{M}$. We exhibit in [3, 4] a family of isometric embeddings $i_t : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ parametrized by a 7-sphere, such that $i_t(M)$ is canonically placed in $(\tilde{M}, \tilde{g})$ as a totally geodesic submanifold for all values of $t$, and $i_0(M)$ is disjoint from $i_{t'}(M)$ for all $t \neq t'$. Circles can be canonically placed into space forms with the placing corresponding to different critical values, so generally speaking, the same manifold can admit different canonical placements into the ambient background.

### 3.2. The mean curvature

We let $\Psi(M)$ be the integral of the pointwise squared norm of $H$:

\[
\Psi(M) = \int_M \|H\|^2 d\mu.
\]
This functional is generally named after Willmore, who used it to study surfaces in \( \mathbb{R}^3 \). Such a functional was studied earlier by Blaschke and also by Thomsen.

If \( f : (-a,a) \times M \to M \) is a family of isometric deformations of the immersion as above, we now compute the \( t \)-derivative of \( \Psi(M_t) = \int_M \|H_t\|^2 \, d\mu_t \) at \( t = 0 \). Since the ideas are similar to those used above in order to differentiate \( \Theta \), we shall be very brief this time around.

The key to the calculation is given by differentiating along the integral curves of \( T \) to obtain
\[
\frac{d}{dt} \langle H, H \rangle \big|_{t=0} = 2\langle \nabla^g_{\xi_i} \nabla^g_{\xi_j} H + 2\tilde{R}^g(H, e_i)e_i - \|H\|^2 H, T \rangle \, d\mu_t + \int_M \langle 4\langle \alpha(e_i, e_j), H \rangle \alpha(e_i, e_j) - \nabla \|H\|^2, T \rangle \, d\mu_t - 2\int_M \langle e_i(T, e_j) + e_j(T, e_i) \rangle (\alpha(e_i, e_j), H) \alpha(e_i, e_j) \rangle \, d\mu_t.
\]

Then we have the following:

**Theorem 3.2.** Let \( f : (-a,a) \times M \to M \) be a deformation of an isometrically immersed submanifold \( f : M \to f(M) \hookrightarrow \tilde{M} \) into the Riemannian manifold \( (\tilde{M}, \tilde{g}) \). We set \( M_t = f(t, M) \), have \( M_0 = M \), and let \( \{e_1, \ldots, e_n\} \) be an orthonormal frame of \( M_t \) for all \( t \in (-a,a) \). Then the infinitesimal variation of \( \Theta \) is given by
\[
\frac{d\Psi(M_t)}{dt} \bigg|_{t=0} \quad \text{if, and only if, the normal component of the vector}
\]
\[
2\nabla^g_{\xi_i} \nabla^g_{\xi_j} H + 2\tilde{R}^g(H, e_i)e_i - \|H\|^2 H + 4\langle \alpha(e_i, e_j), H \rangle \alpha(e_i, e_j)
\]
is identically zero. \( \square \)

### 3.3. Hypersurfaces

For the case of a hypersurface, we proceed to re-express the Euler-Lagrange equation of Theorems 3.1 & 3.2 as a scalar partial differential equation on the mean curvature function. When \( (\tilde{M}, \tilde{g}) \) is assumed to be Einstein, this scalar equation can be expressed fully in terms of the principal curvatures and the sectional curvatures of all normal sections in the frame.

We write the mean curvature vector in the form \( H = hv \), for some scalar function \( h \) and normal vector \( v \). If \( A \) is the shape operator, Wienerg’s identity (2) implies that \( \langle \alpha(e_i, e_j), N \rangle = \langle A_N(e_i), e_j \rangle \). By Codazzi’s equation (5), we obtain that
\[
\langle \nabla^g_{\xi_i} \nabla^g_{\xi_j} \alpha(e_i, e_j), v \rangle = \langle \nabla^g_{\xi_i} ((\tilde{R}^g(e_j, e_i)e_j)^v + \nabla^g_{\xi_i} \alpha(e_j, e_j) + A_{\alpha(e_j, e_j)} e_i), v \rangle - \langle \nabla^g_{\xi_i} A_{\alpha(e_j, e_j)} e_j, v \rangle
\]
\[
= \langle \nabla^g_{\xi_i} (\tilde{R}^g(e_j, e_i)e_j)^v, v \rangle + \langle \nabla^g_{\xi_i} \nabla^g_{\xi_j} hv, v \rangle + \langle \nabla^g_{\xi_i} A_H e_j, v \rangle - \langle \nabla^g_{\xi_i} A_{\alpha(e_j, e_j)} e_j, v \rangle.
\]

The various terms in this expression can be further developed. We have that
\[
\langle \nabla^g_{\xi_i} \nabla^g_{\xi_j} hv, v \rangle = \langle \nabla^g_{\xi_i} (e_i(h)v + h\nabla^g_{\xi_j} v), v \rangle = \langle (e_i(h)(v))v + 2e_i(h)\nabla^g_{\xi_j} v + h\nabla^g_{\xi_i} \nabla^g_{\xi_j} v, v \rangle
\]
\[
= -\Delta h - h\|\nabla^g_{\xi_i} v\|^2,
\]
and the Weingarten identity (2) yields that

\[ \langle \nabla_{e_i} \tilde{g}^\alpha e_j, \nu \rangle = - \Delta h - h \| A_\nu(e_i) \|^2 - h \| \nabla_{e_i} \nu \|^2 \]

\[ = - \Delta h - h \text{ trace } A_\nu^2, \]

because \( 2\langle \nabla_{e_i} \nu, \nu \rangle = 0 \), and so \( \nabla_{e_i} \nu \) is the zero vector.

We also have that

\[ \langle \nabla_{e_i} A_H(e_i), \nu \rangle = \langle \nabla_{e_i}^\tilde{g} (h A_\nu(e_i)), \nu \rangle = - h \langle A_\nu(e_i), \nabla_{e_i}^\tilde{g} \nu \rangle = h \text{ trace } A_\nu^2. \]

Finally,

\[ \langle \nabla_{e_i} A_\alpha(e_i, e_j) e_j, \nu \rangle = - \langle A_\alpha(e_i, e_j) e_j, \nabla_{e_i}^\tilde{g} \nu \rangle = \langle A_\alpha(e_i, e_j) e_j, A_\nu(e_i) \rangle. \]

We now set \( \alpha(e_i, e_j) = h_{ij}\nu \), and choose the frame so that \( A_\nu \) is diagonal to see that

\[ \langle \nabla_{e_i} A_\alpha(e_i, e_j) e_j, \nu \rangle = h_{ij} \langle A_\nu e_j, A_\nu e_i \rangle = \text{ trace } A_\nu^3. \]

Using these results in (13), we obtain that

\[ \langle \nabla_{e_i}^\tilde{g} \nabla_{e_j}^\tilde{g} \alpha(e_i, e_j), \nu \rangle = \langle \nabla_{e_i}^\tilde{g} (R^\tilde{g}(e_j, e_i) e_j)\nu, \nu \rangle - \Delta h - \text{ trace } A_\nu^3. \]

Once again, by choosing a frame that diagonalizes \( A_\nu \), we can easily see that

\[ \langle \alpha(e_i, e_j), \alpha(e_i, e_i) \rangle \langle \alpha(e_i, e_i), \nu \rangle = \text{ trace } A_\nu^3. \]

We now have the following results. The first states the Euler-Lagrange equation of the functionals (3) and (11) for hypersurfaces in the case when \( (M, \tilde{g}) \) is Einstein. We then draw some consequences.

**Theorem 3.3.** Let \( M \) be a hypersurface of an Einstein manifold \( (\bar{M}, \bar{g}) \). Assume that \( k_1, \ldots, k_n \) are the principal curvatures, with associated orthonormal frame of principal directions \( e_1, \ldots, e_n \). Let \( \nu \) be a normal field along \( M \). Then \( M \) is a critical point of the functional (3) if, and only if,

\[ 2\Delta h = 2(k_1 K_{\tilde{g}}(e_1, \nu) + \cdots + k_n K_{\tilde{g}}(e_n, \nu)) - h \| \alpha \|^2 + 2(k_1^3 + \cdots + k_n^3), \]

and a critical point of the functional (11) if, and only if,

\[ 2\Delta h = 2h(K_{\tilde{g}}(e_1, \nu) + \cdots + K_{\tilde{g}}(e_n, \nu)) + 2h \| \alpha \|^2 - h^3. \]

Notice that we have \( h = k_1 + \cdots + k_n \) and \( \| \alpha \|^2 = k_1^2 + \cdots + k_n^2 \), respectively.

**Proof.** We show that the vanishing of the normal component of (10) yields precisely the first stated critical point equation.

The term \( (\bar{R}^\tilde{g}(e_i, e_j), e_i) e_j \) gives rise to sectional curvatures. Indeed, since \( \alpha(e_i, e_j) = k_i \delta_{ij} \nu \), we have that

\[ \langle \bar{R}^\tilde{g}(\alpha(e_i, e_j), e_j), e_i, \nu \rangle = k_i \langle \bar{R}(\nu, e_i) e_i, \nu \rangle = k_i K_{\tilde{g}}(e_i, \nu). \]

The first term in the right side of (13) is zero. Indeed, \( \langle \nabla_{e_i}^\tilde{g} (\bar{R}^\tilde{g}(e_j, e_i) e_j)^\nu, \nu \rangle = e_i \langle (\bar{R}^\tilde{g}(e_j, e_i) e_j)\nu, \nu \rangle - \langle (\bar{R}^\tilde{g}(e_j, e_i) e_j)^\nu, \nabla_{e_i}^\tilde{g} \nu \rangle = e_i \langle (\bar{R}^\tilde{g}(e_j, e_i) e_j)\nu, \nu \rangle \), because \( \nabla_{e_i}^\tilde{g} \nu \) is tangent to \( M \), and since the metric \( \tilde{g} \) is Einstein, if \( s_{\nu} \) is the scalar curvature, we have that

\[ \langle (\bar{R}^\tilde{g}(e_j, e_i) e_j)^\nu, \nu \rangle = - \langle (\bar{R}^\tilde{g}(e_j, e_i) e_j)\nu, e_j \rangle = - \frac{s_{\nu}}{n} \tilde{g}(e_i, \nu) = 0. \]

The desired equation results by using (15), (13) and (17) in the computation of the normal component of (10).

The vanishing of the normal component of (12) yields the stated critical point equation for (11). In order to derive this, we begin by observing that the normal component of \( \bar{R}^\tilde{g}(H, e_i) e_i \) leads to \( h \sum_i K_{\tilde{g}}(e_i, \nu) \). The highest order derivative
While which simplifies with the lower order cubic term in the right side of (14).

We denote by $S^n_c$ the $n$th dimensional space form of curvature $c$.

**Corollary 3.4.** Let $k_1, \ldots, k_n$ be the principal curvatures of a hypersurface $M$ in $S^n_{c+1}$. Then $M$ is a critical point for the functional (6) if, and only if, its mean curvature function $h$ satisfies the equation

$$2\Delta h = 2ch - h\|\alpha\|^2 + 2(k_1^3 + \cdots + k_n^3),$$

while $M$ is a critical point of the functional (11) if, and only if, its mean curvature function $h$ satisfies the equation

$$2\Delta h = 2cnh + 2h\|\alpha\|^2 - h^3$$

instead.

**Corollary 3.5.** Austere hypersurfaces of a space form $S^n_c$ are critical points of the functional (6), and so they are canonically placed into $S^n_c$. They are absolute minima for the functional (11), functional that is also minimized by any other minimal hypersurface.

**Remark 3.6.** Let $S_r$ be the $r$th symmetric function of the principal curvatures of the hypersurface $M$. For a given real valued function $f$ in $n$ variables, Reilly studied the variation of

$$\int_M f(S_1, \ldots, S_n)\,d\mu$$

under deformations [26]. He deduced the variational formula for the $L^2$-norm of the second fundamental form for hypersurfaces in space forms, obtaining the equation given in Corollary 3.3. Reilly applied this result in the case of $S^3$, where via the identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, he concluded that the only critical points on the 3-sphere whose principal curvatures never change sign are minimal surfaces. This is of course included in our conclusion above, as in dimensions one and two, the notions of austere and minimal submanifolds coincide. We will see next that even if we require that all principal curvatures be constant functions, there are critical hypersurfaces of (6) on the sphere that are not minimal.

**Example 3.7.** Even in codimension one, the class of critical points of (6) is larger than the class of austere hypersurfaces. In order to see that, we use the family of isoparametric hypersurfaces of Nomizu [21], given by $M_{t}^{2n} = \{z \in S^{2n+1} : F(z) = \cos^2 t\}$ for $0 < t < \pi/4$, where $F(z) = (\|x\|^2 - \|y\|^2)^2 + 4(x,y)^2$, $x = x + iy$, $x, y \in \mathbb{R}^{n+1}$. For each $t$, this hypersurface has principal curvatures $k_1 = (1 + \sin 2t)/\cos 2t$ and $k_2 = (-1 + \sin 2t)/\cos 2t$, with multiplicity 1 each, and $k_3 = \tan t$ and $k_4 = -\cot t$ with multiplicity $n - 1$ each, respectively. We thus have that $h = h(t) = k_1(t) + k_2(t) + (n - 1)(k_3(t) + k_4(t))$.

When $n = 2$, $M_{t}^4$ is an austere hypersurface for the value of $t$ given by the root of $h(t) = k_1(t) + k_2(t) + k_3(t) + k_4(t)$ in the interval $(0, \pi/4)$, which coincides with the root of $2h(t) - \|\alpha\|^2 h(t) + 2(k_1^3(t) + k_2^3(t) + k_3^3(t) + k_4^3(t))$, and which is close to
$t = 0.3926990817$. The corresponding critical value of the functional $(6)$ is 12 times the volume of the manifold.

However, for $n \geq 3$, the functions $2h(t) - \|\alpha\|^2h(t) + 2(k_3^2(t) + k_4^2(t) + (n-1)(k_3^2(t) + k_4^2(t)))$ and $h(t)$ do have unique distinct roots on the interval $(0, \pi/4)$. Thus, we have a critical point of $(6)$ that is not a minimal hypersurface. For instance, when $n = 3$, the root of the first function is $t = 0.3775786497$, and the corresponding critical value of $(6)$ is 18.57333958 times the volume. In this case, the function $h(t)$ has a root at $t = 0.4776583091$.

When looking at the functional $(11)$ instead, we can use these isoparametric hypersurfaces also to show that the class of critical points is larger than the class of minimal hypersurfaces. Indeed, for $n = 4$ the function $2n + 2\|\alpha\|^2 - h(t)$ has a root at $t = 0.2153460562$, where $h(t)$ is not zero. Thus, this $M^8$ is a critical hypersurface of $(11)$ in $S^9$. Its critical value is 147.3776409 times the volume. Let us notice that this function is strictly positive for $n$ equal 2 and 3, respectively. □

**Remark 3.8.** In [10], Simons computed the Laplacian of $\alpha$ for a minimal variety of dimension $p$ immersed into an Euclidean $n$-sphere. Using the nonnegativity of the Laplacian, he concluded that for a minimal $p$-variety immersed into the $n$-sphere such that $0 \leq \|\alpha\|^2 \leq p/q$, $q = 2 - 1/(n - p)$, then either $\|\alpha\|^2 = 0$ or $\|\alpha\|^2 = p/q$, respectively. The Nomizu austere hypersurface $M^4 \subset S^5$ above has $\|\alpha\|^2 = 12$, a constant that falls outside the range covered by Simons’ result. This one of the isoparametric families of Nomizu appears as one of the examples studied by Cartan [5]. In [23], Peng and Terng showed that among all of the Cartan’s isoparametric families, the only possible values of $\|\alpha\|^2$ are 0, $n - 1$, $2(n - 1)$, $3(n - 1)$ or $5(n - 1)$, respectively. Their result produces examples of some rather large values of constant $\|\alpha\|^2$ for minimal hypersurfaces of the $n$-sphere. In Example 3.7 we had seen already a rather large value of constant $\|\alpha\|^2$ for a critical hypersurface of $(6)$ in $S^{2n+1}$, $n \geq 3$. □

**Example 3.9.** Since the orbit of a compact connected group of isometries whose volume is extremal among nearby orbits of the same type is a minimal submanifold [13], studying the diagonal action of $SO(3, \mathbb{R})$ in $\mathbb{R}^3 \times \mathbb{R}^3$, we may conclude that $S^2(\sqrt{1/2}) \times S^2(\sqrt{1/2}) \subset S^5$ is a minimal hypersurface. Here, $S^n(r)$ denotes the $n$-sphere of radius $r$, with $S^n(1) = S^n$. The principal curvatures of the said minimal submanifold are 1, 1, −1, and −1, respectively. Thus, this manifold is austere, and consequently, a critical point of $(6)$.

In fact, more is true: of all the products $S^m(\sqrt{n/(m+n)}) \times S^n(\sqrt{m/(m+n)}) \subset S^{m+n+1}$, the only ones that are critical points of $(6)$ must have $m = n$, in which case, they are austere. This follows easily from the explicit calculation of the principal curvatures, which turn out to be $k_1 = \ldots = k_m = \sqrt{(m/n)}$ and $k_{m+1} = \ldots = k_{m+n} = -\sqrt{(n/m)}$, respectively. Cf. [13], §3, and [19], p. 366. All of the alluded to product of spheres are critical points of the functional $\int_M \left(\|\alpha\|^2 - \frac{1}{n}\|H\|^2\right)^{\frac{m+n}{m+n-2}} d\mu$, termed Willmore in these references. The functional $(6)$ distinguishes the symmetric Clifford minimal tori out of these products. These special tori are Einstein manifolds [10]. □
3.4. Submanifolds of arbitrary codimension. We now consider submanifolds of arbitrary codimension $q$, and re-express the Euler-Lagrange equation of Theorems 3.1 & 3.2 as $q$-scalar equations.

For that, we choose an orthonormal frame $\{\nu_1, \nu_2, \ldots, \nu_q\}$ for the normal bundle $\nu(M)$ so that the mean curvature vector is parallel to $\nu_1$.

The higher codimension analogue of (15) along $\nu_1$ is obtained once again by Codazzi’s equation (13), and is given by

$$\langle \nabla \nu_i \alpha(e_i, e_j), \nu_1 \rangle = \langle \nabla \nu_i (R\tilde{g}(e_j, e_i)\nu), \nu_1 \rangle - \Delta h - h\|\nabla \nu_i \nu_1\|^2 - \text{trace } A_{\nu_1} A_{\nu_1}^2$$

(with our summation convention on repeated indexes, the last trace in the right side above is in effect the trace of $A_{\nu_1} \sum_{k=1} A_{\nu_k}^2$). This time around, $\sum_i \nabla \nu_i \nu_1$ is not necessarily the zero vector. The other difference arises in manipulating the expression $(A_{\alpha(e_i, e_j)} e_j, A_{\nu_1}(e_i))$. We now decompose the normal vector $\alpha(e_i, e_j)$ as $\alpha(e_i, e_j) = h_{ij} \nu_k$ instead of the one dimensional expression along $\nu$ used earlier. By our assumptions, $h_{1i} = h$ while $h_{1k} = 0$ for $k \geq 2$, and the cubic trace term in (18) results by choosing frames that diagonalize $A_{\nu_k}$.

Similarly, for any $m$ in the range $1 \leq m \leq q$, we have that

$$\langle \alpha(e_i, e_j) \rangle = \langle \alpha(e_i, e_j) \rangle \alpha(e_i, e_i) \nu_m = \text{trace } A_{\nu_m} A_{\nu_m}^2,$$

and so the vanishing of the component of (10) along $\nu_1$ yields the equation

$$0 = 2\langle R\tilde{g}(\alpha(e_i, e_j), e_i, e_i), \nu_1 \rangle + 2\langle \nabla \nu_i (R\tilde{g}(e_j, e_i)\nu), \nu_1 \rangle - 2\Delta h - 2h\|\nabla \nu_i \nu_1\|^2$$

$$-h\|\alpha\|^2 + 2\text{trace } A_{\nu_i} A_{\nu_k}^2.$$

The equation arising from the vanishing of the component of (10) along $\nu_m$ for $m \geq 2$ follows by a similar argument but is slightly different, the reason being that the trace of $\alpha$ is orthogonal to the said direction. For $H = h\nu_1$, and so

$$\langle \nabla \nu_i \nabla \nu_i H, \nu_m \rangle = 2e_i(h)\langle \nabla \nu_i \nu_1, \nu_m \rangle + he_i\langle \nabla \nu_i \nu_1, \nu_m \rangle - h\text{trace } A_{\nu_1} A_{\nu_m} - h\langle \nabla \nu_i \nu_1, \nabla \nu_i \nu_m \rangle.$$

We also have that

$$\langle \nabla \nu_i A_H(e_i), \nu_m \rangle = h\langle A_{\nu_1}(e_i), A_{\nu_m}(e_i) \rangle = h\text{trace } A_{\nu_1} A_{\nu_m}.$$

And since

$$\langle \nabla \nu_i A_{\alpha(e_i, e_j)} e_j, \nu_m \rangle = -\langle A_{\alpha(e_i, e_j)} e_j, \nabla \nu_i \nu_m \rangle = \langle A_{\alpha(e_i, e_j)} e_j, A_{\nu_m}(e_i) \rangle,$$

choosing frames that diagonalize the $A_{\nu_k}$s, we obtain that

$$\langle \nabla \nu_i A_{\alpha(e_i, e_j)} e_j, \nu_m \rangle = \text{trace } A_{\nu_m} A_{\nu_k}^2.$$

Thus, the vanishing of the component of (10) along $\nu_m$ yields the equation

$$0 = 2\langle R\tilde{g}(\alpha(e_i, e_j), e_j, e_i), e_i, \nu_m \rangle + 2\langle \nabla \nu_i (R\tilde{g}(e_j, e_i)\nu), e_i, \nu_m \rangle + 4e_i(h)\langle \nabla \nu_i \nu_1, \nu_m \rangle$$

$$+ 2he_i(\nabla \nu_i \nu_1, \nu_m) - 2h\langle \nabla \nu_i \nu_1, \nabla \nu_i \nu_m \rangle + 2\text{trace } A_{\nu_m} A_{\nu_k}^2.$$

The vanishing of the various normal components of (12) follow by similar considerations. We simply state the final result. When $q = 1$, the statement that follows displays also the Euler-Lagrange equation of (9) and (11), respectively, in full generality, without the Einstein assumption on $(\tilde{M}, \tilde{g})$ used in Theorem 3.8.

**Theorem 3.10.** Let $M$ be an $n$-manifold of codimension $q$ isometrically immersed in $(\tilde{M}, \tilde{g})$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of $M$. Suppose that $\{\nu_1, \ldots, \nu_q\}$
is an orthonormal frame of the normal bundle of the immersion such that \( H = h \nu_1 \).
Then, \( M \) is a critical point of the functional \( [5] \) if, and only if,
\[
2\Delta h = 2\langle R^g(\alpha(e_i, e_j), e_1) e_1 + \nabla_{e_i}^g (R^g(e_j, e_i) e_j) ^\nu, \nu_1 \rangle - 2h \| \nabla_{e_i}^g \nu_1 \|^2 - h\| \alpha \|^2 + 2\text{trace} A_{\nu_1} A_{\nu_1}^2 ,
\]
and for all \( m \) in the range \( 2 \leq m \leq q \), we have that
\[
0 = 2\langle R^g(\alpha(e_i, e_j), e_j) e_j + \nabla_{e_j}^g (R^g(e_i, e_j) ^\nu, \nu_1) + 2e_i(h) \nabla_{e_i}^g \nu_1, \nu_m \rangle + 2he_i(h) \nabla_{e_i}^g \nu_1, \nu_{m1} + 2h \langle \nabla_{e_i}^g \nu_1, \nabla_{e_i}^g \nu_{m1} \rangle + 2\text{trace} A_{\nu_1} A_{\nu_1}^2 .
\]
\( M \) is a a critical point of the functional \( [11] \) if, and only if,
\[
2\Delta h = 2h(K_{\bar{g}}(e_1, \nu_1) + \cdots + K_{\bar{g}}(e_n, \nu_1)) - 2h \| \nabla_{e_i}^g \nu_1 \|^2 - h^3 + 2h \text{trace} A_{\nu_1}^2 ,
\]
and for all \( m \) in the range \( 2 \leq m \leq q \), we have that
\[
0 = 2h(K_{\bar{g}}(e_1, \nu_1) + \cdots + K_{\bar{g}}(e_n, \nu_1)) + 4e_i(h) \langle \nabla_{e_i}^g \nu_1, \nu_m \rangle + 2he_i(h) \nabla_{e_i}^g \nu_1, \nu_m \rangle - 2h \langle \nabla_{e_i}^g \nu_1, \nabla_{e_i}^g \nu_{m1} \rangle + 2h \text{trace} A_{\nu_1} A_{\nu_1} .
\]
\( \square \)

Remark 3.11. For \( q = 1 \) above, and setting \( \nu = \nu_1 \), we have that
\[
\langle \nabla_{e_i}^g (R^g(e_j, e_i) e_j) ^\nu, \nu \rangle = -e_i(r_{\bar{g}}(e_i, \nu)).
\]
If \((\bar{M}, \bar{g})\) were to be Einstein, cf. with Theorem \( \text{[6]} \) this expression would be identically zero.
\( \square \)

3.5. The extrinsic scalar curvature. We let \( \Theta(M) \) be given by the total extrinsic scalar curvature of the immersion. This is the remaining piece in our analysis of the integral of the various summands in the right side of \( [4] \):
\[
\Theta(M) = \int_M \sum_{1 \leq i, j \leq n} K_{\bar{g}}(e_i, e_j) d\mu .
\]

Theorem 3.12. Let \( M \) be an \( n \)-manifold of codimension \( q \) isometrically immersed in \((\bar{M}, \bar{g})\). Let \( \{e_1, \ldots, e_n\} \) be an orthonormal frame of \( M \). Suppose that \( \{\nu_1, \ldots, \nu_q\} \) is an orthonormal frame of the normal bundle of the immersion such that \( H = h \nu_1 \).
Then, \( M \) is a critical point of the functional \( [22] \) if, and only if,
\[
h \sum_{1 \leq i, j \leq n} K_{\bar{g}}(e_i, e_j) = 0 .
\]
\( \square \)

Proof. Let \( T \) be the variational vector field of the deformation, and \( \{e_1, \ldots, e_n\} \) a normal frame of \( M_t \) at \((p, t)\) for sufficiently small \( t \), as used above. Then the value of \( K_{\bar{g}}(\dot{e}_i, e_j) + K_{\bar{g}}(e_i, \dot{e}_j) \) at \( t = 0 \) is zero because the \( e_j \) component of \( \dot{e}_i \big|_{t=0} \) is minus the \( e_i \) component of \( \dot{e}_j \big|_{t=0} \). The result follows using this fact and \( [7] \) in the differentiation of \( [22] \).
\( \square \)

Corollary 3.13. Let \((M, g)\) be a manifold isometrically embedded into \((\bar{M}, \bar{g})\). If the extrinsic scalar curvature of \( M \) is nowhere zero, \( M \) is a critical point of \( [22] \) if, and only if, it is minimal. In particular, if \( \bar{g} \) has positive or negative sectional curvature, \( M \) is a critical point of \( [22] \) if, and only if, \( M \) is minimal.
The condition on the sectional curvature of \( \tilde{g} \) above can be relaxed somewhat while maintaining the same conclusion. We need to ask that the set of points carrying planar sections be “small” in a sense we leave to the reader to formulate.

For isometric immersions of \( M \) into \( (\tilde{M}, \tilde{g}) \), we may consider the functional

\[
M \mapsto S(M) = \int_M (\|\alpha\|^2 - \|H\|^2) d\mu,
\]

Its critical point equation may be obtained by taking the difference of the critical point equations for the functionals \( \Pi(M) \) and \( \Psi(M) \) given by Theorem 3.10 above. We shall use this functional later on to show the existence of bulk backgrounds with isotopic critical submanifolds of different critical values. These examples exhibit the same flavour of some alternatives theories to compactification currently of interest to physicists [25].

3.6. Canonical homology representatives. Assuming that the topology of \( \tilde{M} \) is nontrivial, we revisit an old question of Steenrod inquiring about the shape that a homology class has. This paraphrases the formulation of Steenrod’s question in the language of D. Sullivan [33].

We assume that \( \tilde{M} \) is a smooth connected oriented manifold, and let \( D \) be a class in \( H^n(\tilde{M}; \mathbb{Z}) \). A closed oriented manifold \( M = M^n \) is said to represent \( D \) if there exists an embedding \( i : M \hookrightarrow \tilde{M} \) such that \( i_*[M] = D \).

We choose and fix a homology class \( D \in H^n(\tilde{M}; \mathbb{Z}) \) in \( \tilde{M} \), and define the space

\[
M_D(\tilde{M}) = \{ M : M \hookrightarrow \tilde{M} \text{ is an isometric embedding, } [M] = D \}.
\]

Since the metric \( \tilde{g} \) on \( \tilde{M} \) is fixed, we drop it altogether from the notation here.

Points in \( M_D(\tilde{M}) \) are manifolds that can have different topology. The stability of the class of smooth embeddings of a compact manifold into a fixed background allows us to introduce a topology on \( M_D(\tilde{M}) \) by topologizing the small deformations of a point in \( M_D(\tilde{M}) \) to define neighborhoods of the point in question.

Remark 3.14. A celebrated result of Thom [35] shows that for any integral homology class \( D \) there exists a nonzero integer \( N \) such that the class \( N \cdot D \) is represented by a submanifold \( M \). We assume here that the class \( D \) is such that \( N \) can be chosen to be 1, and so the space \( M_D(\tilde{M}) \) is nonempty. This nontrivial issue for integer homology classes “evaporates” when using field coefficients; for Thom derives from the result above [35, Corollary II.30] that the homology groups with rational or real coefficients admit bases of systems of elements represented by submanifolds.

He proved that any class in \( H_i(\tilde{M}; \mathbb{Z}), i \leq 6 \), is realizable by a submanifold [34, Theorem II.27, footnote 9 p. 56], and then deduced that for oriented manifolds of dimension at most 9, any integral class can be so realized [35, Corollary II.28, footnote 9].

Thom exhibited an example of a 7 dimensional integer class in a manifold of dimension 14 that is not realizable by a submanifold. Optimal examples of nonrealizable 7-classes on manifolds of dimension 10 had been given in [2]. See Remark 3.22 below.

It is simple to see that in codimension \( q = \tilde{n} - n = 1 \) or 2, there is no obstruction to representing \( D \in H_i(\tilde{M}; \mathbb{Z}) \) by an embedded submanifold, and \( M_D(\tilde{M}) \) will therefore be always nonempty. For instance, if \( q = 2 \), the Poincaré dual of \( D \) will be an element of \( H^2(\tilde{M}; \mathbb{Z}) \), group that is in 1-to-1 correspondence with the group of isomorphism classes of complex line bundles over \( \tilde{M} \) via the first Chern class mapping. The zero set of a generic section of the line bundle that corresponds to
the said Poincaré dual is a smooth submanifold that represents $D$. The structure group of its normal bundle is $\mathbb{SO}(2) \cong \mathbb{U}(1)$.

The *nicest* metric element in the space $\mathcal{M}_D(\tilde{M})$ in (24) should describe the way the homology class $D$ looks like [33]. This problem, thoroughly discussed by Thom via topological techniques, is now revisited by using a variational principle and provided with an alternative geometric answer. We have in mind the definition of a functional on $\mathcal{M}_D(\tilde{M})$ whose minimum —when it can be achieved— defines the nicest element in question. Of course, such an optimization problem will depend upon the functional we choose to use for the purpose.

If $D$ is a one dimensional homology class, its natural optimal representative is the shortest ambient space geodesic loop in the class, a submanifold that when provided with the induced metric has second fundamental form that is identically zero, and so it yields an absolute minimizer of the functional $\Pi$ in (6) defined over the free homotopy class of $D$. (Notice that this is also the natural answer to the question of finding optimal representatives of elements of the fundamental group instead.) However, even in this simple case, the functional $\Pi$ may have other critical points of higher critical value, and these non-minimizer curves would encode additional information on the curvature of the metric in the background manifold. In higher dimension, on the other hand, the curvature itself of the isometrically immersed $(M,g)$ should play more of a role in determining the best representative of $D$, even if the latter were to be defined as a minimizer of $\Pi$ within the class.

The curvature functionals to optimize in order to find canonical representatives of $D$ are plentiful, but the identity (4) suggests a natural choice to make. We start by restricting the domain of (25), and seek critical points of

\[
\mathcal{M}_D(\tilde{M}) \to \mathbb{R} \\
M \mapsto \mathcal{T}(M) = \int_M \tilde{g}(\alpha, \alpha) d\mu_g.
\]

Since the integrand is bounded below and often $D$ has volume minimizer representatives, it is natural to seek critical points of (25) that are minima and have the smallest possible volume among these. Critical points of (25) could have some connected components that bound (which would produce homologically trivial critical points of (6) in their own right), or connected components that cancel each other out partially in homology to produce a representative of $D$. It is then natural to impose the mentioned additional constraint, and seek minimizers of $\Pi$ in $\mathcal{M}_D(\tilde{M})$ that have the smallest volume. A minimum $M$ of smallest volume, should it exist, will be said to be a *canonical representative* of $D$.

Notice that if a canonical representative $M$ of the class $D$ were to be a volume minimizer also, [41] would imply that the metric on $M$ is as close as it can be to an Einstein metric among all the metrics on $M$ induced by the ambient space metric $\tilde{g}$ on $\tilde{M}$.

**Remark 3.15.** It might be natural to restrict the domain of the functional (25) to an open subset of $\mathcal{M}_D(\tilde{M})$ according to the topology of its elements, open subset that encodes additional information about differential invariants of the background manifold $\tilde{M}$. In that case, we might want to redefine a canonical representative of $D$ accordingly.
3.7. An example: the fibration $S^3 \hookrightarrow Sp(2) \cong S^7$. If $\mathbb{H} = \{q = q_0 + q_1i + q_2j + q_3k : (q_0, q_1, q_2, q_3) \in \mathbb{R}^4\}$ is the (skew) field of quaternions, the symplectic group $Sp(n)$ is the group of $n \times n$ matrices $M$ with entries in $\mathbb{H}$ such that $(Mx, My) = x \cdot y\mathbb{H}$. Here, $\langle x, y \rangle = \sum_i x_i y_i$ is the standard quaternionic bilinear form on $\mathbb{H}^n$. $Sp(n)$ is a simple compact Lie group of dimension $2n^2 + n$.

By identifying a quaternion with its corresponding vector of coordinates in $\mathbb{R}^4$, we see that $Sp(1) = S^3$. On the other hand,

$$Sp(2) = \{(q_{ij}) \in M_2(\mathbb{H}) : \sum_{i=1}^{2} q_{ij}q_{ij} = \delta_{ij}, 1 \leq i, j \leq 2\},$$

and its Lie algebra is given by

$$\mathfrak{sp}(2) = \{(q_{ij}) \in M_2(\mathbb{H}) : \bar{q}_{21} = -q_{12}, q_{\alpha\alpha} = -q_{\alpha\alpha} \text{ for } 1 \leq \alpha \leq 2\}.$$

We consider the free action of $Sp(1)$ on $Sp(2)$ defined by letting the quaternion $q \in Sp(1)$ act on $Sp(2)$ according to the rule

$$q \circ (q_{ij}) = \left(\begin{array}{cc} q_{11} & q_{12} \\ q_{21} & q_{22} \end{array}\right) \left(\begin{array}{c} 1 \\ 0 \bar{q} \end{array}\right).$$

The principal fibration of $Sp(2)$ induced by the $\circ$-action has as base $S^7$ with its standard differentiable structure. The projection map $\pi_\circ$ of this fibration is given by

$$\pi_\circ \left(\begin{array}{cc} q_{11} & q_{12} \\ q_{21} & q_{22} \end{array}\right) = \left(\begin{array}{c} q_{11} \\ q_{21} \end{array}\right).$$

The action of $Sp(2)$ on itself by left translations commutes with $\pi_\circ$, and so $Sp(2)$ acts in this manner by bundle morphisms of the fibration $S^3 \hookrightarrow Sp(2) \cong S^7$. We thus obtain an induced action of $Sp(2)$ on $S^7$, which is defined by matrix multiplication when $S^7$ is thought of as the subset of $\mathbb{H}^2$ consisting of column vectors that have unit length.

Notice that if $\gamma_q : [0, 1] \to S^3$ is a smooth curve in $S^3$ that begins at $q \in S^3$ and ends at $1 \in S^3$, the fibration above can be made part of the $t$-parameter family of fibrations $S^3 \to Sp(2) \cong S^7$ defined by the curve of $Sp(1)$-actions

$$q \circ_t (q_{ij}) = \left(\begin{array}{cc} q_{11} & q_{12} \\ q_{21} & q_{22} \end{array}\right) \left(\begin{array}{cc} \tau_q(1-t) \\ 0 \end{array}\right),$$

$t \in [0, 1]$. The fibration corresponding to $t = 1$ projects $(q_{ij})$ onto the point of $S^7$ represented by its second column. It is a sort of rotated version of $\pi_\circ$.

The fibers and base of $\pi_\circ$ play special roles in relation to the geometry defined by certain Riemannian metrics on $Sp(2)$. We let $g_\lambda$ be the metric on the 7-sphere obtained by scaling the vertical space of the Hopf fibration by the factor $\lambda$. Now, as a Lie group of dimension 10, $Sp(2)$ has a 55-dimensional family of distinct left invariant metrics, and of these, there is a 2-dimensional family $g_{\lambda, \mu}$ that descends to $g_\lambda$ on $S^7$ and that induces constant sectional curvature metrics on the $S^3$-fibers of $\pi_\circ$. The metric $g_{\lambda, \mu}$ yields a Riemannian submersion $(Sp(2), g_{\lambda, \mu}) \to (S^7, g_\lambda)$ whose fibers are all isometric 3-spheres. For, since left invariant vector fields are uniquely defined by their values at the identity element of $Sp(2)$, it suffices to describe the metric on $\mathfrak{sp}(2)$. Then $g_{\lambda, \mu}$ is defined by

$$g_{\lambda, \mu} \left(\left(\begin{array}{c} p_1 \\ q_1 \end{array}\right), \left(\begin{array}{c} p_2 \\ q_2 \end{array}\right)\right) = \text{Re}(\lambda \bar{p}_1 p_2 + \bar{q}_1 q_2 + \mu \bar{r}_1 r_2),$$
for $\lambda, \mu$ real positive parameters, and where given any quaternion $x$, $\text{Re} \, x$ denotes its real part.

The norm defined by the metric $g_{\lambda, \mu}$ is given by

$$
\left\| \begin{pmatrix} p & -\bar{q} \\ q & r \end{pmatrix} \right\|_{\lambda, \mu}^2 = \lambda |p|^2 + |q|^2 + \mu |r|^2.
$$

The bi-invariant metric on $Sp(2)$ is $g^{\frac{1}{2}, \frac{1}{2}}$.

We summarize the properties that the family $g_{\lambda, \mu}$ has in the form of a proposition, whose verification should be a simple task for the reader.

**Proposition 3.16.** Let $g_{\lambda, \mu}$ be the family of metrics on $Sp(2)$ defined by (30) and let $g_{\lambda}$ be the induced metric on $S^7$. Then:

1. $\pi_\circ : (Sp(2), g_{\lambda, \mu}) \to (S^7, g_{\lambda})$ is a Riemannian submersion.
2. The fibers of this submersion are canonically placed totally geodesic 3-spheres of constant sectional curvature $1/\mu$.

The family of metrics $g_{\lambda, \mu}$ is invariant under right multiplication by elements of the subgroup $S^3 \times S^3 \subset Sp(2)$, realized as the set of diagonal matrices.

Let $V^o_M$ and $H^o_M$ be the vertical and horizontal spaces of the Riemannian submersion $\pi_\circ$ in (31), respectively. By (28), if $L_M$ stands for the operator given by left translation by $M \in Sp(2)$, we see that

$$(32) \quad V^o_M = L_{M*} \left\{ \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} : p \in \mathbb{H}, \bar{p} = -p \right\}.$$ 

The space $H^o_M$ is the perpendicular of $V^o_M$ in the $g_{\lambda, \mu}$-metric. This computation may be carried at the identity element because the metric is left-invariant. Hence, by (30), we see that

$$(33) \quad H^o_M = L_{M*} \left\{ \begin{pmatrix} p & -\bar{q} \\ q & 0 \end{pmatrix} : p, q \in \mathbb{H}, \bar{p} = -p \right\}.$$ 

We present expressions for $\nabla Z_0 Z_1$ and for the curvature $K_{\lambda, \mu}(Z_0, Z_1)$ of the planar section in $T_M Sp(2)$ spanned by $Z_0$ and $Z_1$. We consider the decompositions

$$Z_0 = T + U, \quad Z_1 = X + V$$

of $Z_0$ and $Z_1$ in terms of their $\pi_\circ$ horizontal and vertical components, respectively. Since $g_{\lambda, \mu}$ is a left-invariant metric, we extend the vectors in a left-invariant manner, to reduce our calculations to one nearby or at the identity, respectively. Hence, by (32) and (33), the vector $Z_0$ is of the form

$$(34) \quad Z_0 = T + U = L_{M*} \left( \begin{pmatrix} p & -\bar{u} \\ u & 0 \end{pmatrix} + L_{M*} \left( \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix} \right) \right)$$

for some quaternions $p, u, r$, with $p$ and $r$ purely imaginary, while the vector $Z_1$ is of the form

$$(35) \quad Z_1 = X + V = L_{M*} \left( \begin{pmatrix} q & -\bar{w} \\ w & 0 \end{pmatrix} + L_{M*} \left( \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \right) \right),$$

for some quaternions $q, w$ and $z$, with $q$ and $z$ purely imaginary also. Though the formulas we provide hold in general, we could assume further that $Z_0$ and $Z_1$ are...
vectors of norm one that are perpendicular to each other, facts encoded into the relations
\[ \lambda |p|^2 + |u|^2 + \mu |r|^2 = 1, \]
\[ \lambda |q|^2 + |w|^2 + \mu |z|^2 = 1, \]
and
\[ \text{Re}(\lambda \bar{p}q + \bar{u}w + \mu \bar{r}z) = 0, \]
respectively.

On any Lie group endowed with a left-invariant metric \( \langle \cdot, \cdot \rangle \), we have that
\[ \langle \nabla_U V, W \rangle = \frac{1}{2} (\langle [U, V], W \rangle - \langle [V, W], U \rangle + \langle [W, U], V \rangle), \]
for \( U, V, \) and \( W \) arbitrary left-invariant fields. The expression for the Levi-Civita connection of \( g_{\lambda, \mu} \) follows from it. Indeed, let us denote by \( V \) and \( H \) the orthogonal projections onto the vertical and horizontal tangent spaces of \( Sp(2) \), respectively. Further, let us denote by \( V' \) and \( H' \) the compositions of \( H \) with the projection onto the tangent to the Hopf fiber and the perpendicular component to it in \( S^7 \), respectively. Then,
\[ \nabla_{Z_0} Z_1 = L_{M^*} \left( \begin{array}{c} \frac{1}{2} (pq - qp - \bar{u}w + \bar{w}u) \\ \lambda uq - \mu Zu - (1 - \lambda) wp + (1 - \mu) rZ \\ \frac{1}{2}(w\bar{u} - w\bar{\bar{u}} + rz - rz) \end{array} \right) \begin{array}{c} \lambda q\bar{u} + \mu \bar{z}u - (1 - \lambda) p\bar{w} - (1 - \mu) \bar{r}w \\ \frac{1}{2}(w\bar{u} - w\bar{\bar{u}} + rz - rz) \end{array} \right) \\
= L_{M^*} \left( \frac{1}{2} \left( \begin{array}{c} Z_{00} Z_{11} + (\lambda - \frac{1}{2}) \left( \begin{array}{cc} 0 & \mu q + \mu wp \\ \mu q + \mu wp & 0 \end{array} \right) - (\mu - \frac{1}{2}) \left( \begin{array}{cc} 0 & \bar{u}z + \bar{w}r \\ \bar{u}z + \bar{w}r & 0 \end{array} \right) \end{array} \right) \\
= L_{M^*} \left( \frac{1}{2} \left( \begin{array}{c} Z_{0} Z_{1} + (\lambda - \frac{1}{2}) \left( [\beta' Z_{0}, V' Z_{1}] + [\beta' Z_{1}, V' Z_{0}] \right) + (\mu - \frac{1}{2}) \left( [\beta Z_{0}, V Z_{1}] + [\beta Z_{1}, V Z_{0}] \right) \end{array} \right) \right) \\
\right) .
\]

The expression for the sectional curvature of \( g_{\lambda, \mu} \) is obtained using the fundamental formulas for Riemannian submersions of O'Neill [22]. We state the result in the form of a theorem.

**Theorem 3.17.** Consider \( Sp(2) \) with the left invariant metric \( g_{\lambda, \mu} \) defined by (50). Let \( Z_0 \) and \( Z_1 \) be left-invariant vectors fields of the form (51) and (55), respectively. Then, the unnormalized curvature \( K_{\lambda, \mu}(Z_0, Z_1) \) of the planar section of \( T_M Sp(2) \) spanned by \( Z_0 \) and \( Z_1 \) is given by
\[ K_{\lambda, \mu}(Z_0, Z_1) = \left( 4 - 3(\lambda + \mu) \right) (|u|^2 |w|^2 - (\text{Re}(\bar{u}w))^2) + \lambda^2 (|q|^2 |w|^2 + |p|^2 |w|^2) + \lambda (|p|^2 |q|^2 - (\text{Re}(p\bar{q}))^2) - 2 \text{Re}(\bar{u}w)(\lambda \text{Re}(p\bar{q}) + \mu \text{Re}(r\bar{z})) + 2(1 - \lambda) \text{Re}(2uwpq - wup\bar{p}) + \mu^2 (|u|^2 |z|^2 + |w|^2 |r|^2) + \mu(|r|^2 |z|^2 - (\text{Re}(r\bar{z}))^2) + 2 \lambda \mu \text{Re}(uq\bar{u} - up\bar{w})z + (wp\bar{w} - wq\bar{u})r + 2 \mu(1 - \mu) \text{Re}(2z\bar{u}z - z\bar{u}r). \]

When \( \lambda = \mu = \frac{1}{2} \), this expression yields
\[ K_{\frac{1}{2}, \frac{1}{2}}(Z_0, Z_1) = \frac{1}{4} ||Z_0, Z_1||^2_{g_{\frac{1}{2}, \frac{1}{2}}}, \]
and except for this one case, the curvature operator \( R_{g_{\lambda, \mu}} \) admits negative eigenvalues. The scalar curvature of the metric is \( s_{g_{\lambda, \mu}} = 2 \left( \frac{4}{3} + 24 - 6(\lambda + \mu) + \frac{3}{\mu} \right) \). For \( (\lambda, \mu) \in (0, 1/2) \times (0, 1/2) \), the sectional curvatures of \( g_{\lambda, \mu} \) are nonnegative. \( \square \)

We may use the explicit expression for the sectional curvature given in Theorem 3.17 to prove that these curvatures are nonnegative for \( (\lambda, \mu) \in (0, 1/2) \times (0, 1/2) \). This method is direct but cumbersome. Fortunately, earlier work of J. Cheeger [6] may be applied to show this nonnegativity also in a more conceptual manner.
Cheeger’s result pertains the study of Riemannian manifolds with isometric group actions that shrink the metric in the direction of the orbits.

**Theorem 3.18.** Let $(\lambda, \mu) \in \mathbb{R}^2_\geq 0$. Then all fibers of $\pi_o : (Sp(2), g_{\lambda, \mu}) \to (S^7, g_\lambda)$ are critical points of the functional (25), of critical value 0.

Let us observe that if we were to intertwine the parameters $\lambda$ and $\mu$ above in Proposition 3.16 we would obtain analogous statements for the fibration $\pi_o$, of (29) instead. The volume of the fibers would then serve as a distinguishing criteria to select the best among these two fibrations and their fibers. We tie this issue with homology considerations next.

The integral homology of $Sp(2)$ is isomorphic to $\mathbb{Z}$ in dimensions 0, 3, 7, and 10. Let $D$ be a positively oriented generator of $H_3(Sp(2); \mathbb{Z})$. We consider the functional (25) in the case of this class. Being a class of dimension 3, $D$ admits realizations by submanifolds of $Sp(2)$ (cf. Remark 3.14). In fact, all fibers of $\pi_o$, $t \in [0, 1]$, realize $D$.

Using the techniques of Sullivan [32] and the calibrations of Harvey & Lawson [14], Tasaki has shown [34] that if $H$ is a closed Lie subgroup of a compact Lie group $G$ that represents a nontrivial real homology class, then the group $G$ carries a left-invariant metric such that the volume of $H$ in the induced metric is less than or equal to the volume of any embedded representative of its homology class. The following result follows by observing that in the case of $Sp(2)$ and the $S^3$-fibers of $\pi_o$ above, any left-invariant metric on $Sp(2)$ for which the horizontal-vertical decomposition for $T_M Sp(2)$ is that given by (32) and (33) works well in the alluded result.

**Theorem 3.19.** Let $D$ be the positively oriented generator of $H_3(Sp(2); \mathbb{Z})$. Assume that $\mu \leq \lambda$. Then in the induced metrics, all fibers of $\pi_o : (Sp(2), g_{\lambda, \mu}) \to (S^7, g_\lambda)$ are volume minimizer elements of $M_D(Sp(2))$, of volume $2\pi^2 \mu^2$ each. □

Hence, under the assumption that $\mu \leq \lambda$, the result above makes of all the fibers of $\pi_o$ candidates for canonical representatives of $D$.

**Theorem 3.20.** Let $D$ be the positively oriented generator of $H_3(Sp(2); \mathbb{Z})$, and let $(\lambda, \mu) \in (0, 1/2] \times (0, 1/2]$ with $\mu \leq \lambda$. Then the functional (25) is bounded below by 0, all fibers of $\pi_o : (Sp(2), g_{\lambda, \mu}) \to (S^7, g_\lambda)$ realize its minimum value 0 and are volume minimizer elements of $M_D(Sp(2))$, and if $(M, g) \in M_D(Sp(2))$ is a volume minimizer 0 of the functional, then $M$ must be isometric to a fiber.

Thus, modulo isometries, the class $D$ has a unique canonical representative, a standard sphere of smallest volume in $M_D(Sp(2))$.

**Proof.** By Proposition 3.16 all fibers are totally geodesic and therefore critical points of (4) and (25), respectively, of critical value 0. The fibers are all diffeomorphic to each other. We show that an isometrically embedded manifold that represents $D$ and is a volume minimizer minimum of (25) must be isometric to a fiber of $\pi_o$.

Let $M \in M_D(Sp(2))$ be totally geodesic and of minimal volume. We ease into the general argument by first considering the case where $\lambda = \mu = \frac{1}{2}$ so that the background metric is the bi-invariant metric on $Sp(2)$. By (37), we see that the Lie bracket of left-invariant vector fields tangent to $M$ is a left-invariant vector field tangent to $M$, and so the restriction of the Lie bracket operation to $M$ gives $M$
the structure of a Lie subgroup of $Sp(2)$. Since the metric on $M$ has nonnegative sectional curvature, $M$ must be one of $S^3$ or $SO(3)$ or $S^1 \times S^1 \times S^1$ as a closed Lie subgroup of $Sp(2)$, and the latter two are ruled out by the fact that $M$ is a homological 3-sphere (the 3-torus can be ruled out also by using the fact that the maximal torus in $Sp(2)$ is of dimension 2; on the other hand, the orthogonal group can be ruled out by the fact that the exponential in $Sp(2)$ of a 3-dimensional Lie subalgebra isomorphic to $su(2)$ is $S^3$; the quotient of this universal cover by $\mathbb{Z}/2$ yields $SO(3) \cong \mathbb{P}^3(\mathbb{R})$). Thus, $M$ is in fact diffeomorphic to a 3-sphere, and since it has minimal volume among representatives of $D$, it must be isometric to an $S^3$ fiber.

In the general case when $Sp(2)$ is given the metric $g_{\lambda,\mu}$, $0 < \mu \leq \lambda \leq 1/2$, the difficulty lies in showing that a totally geodesic submanifold $M$ in $M_D(Sp(2))$ of minimal volume $2\pi \mu^2$ must be a Lie subgroup. We proceed as follows.

For any 3-manifold $(M, g_M)$ isometrically embedded into $(Sp(2), g_{\lambda,\mu})$, by the tubular neighborhood theorem, there exists a neighborhood $O$ of $M$ in $Sp(2)$ that is diffeomorphic to a neighborhood $N_O$ of the zero section of the normal bundle of $M$ in $(Sp(2), g_{\lambda,\mu})$ via the exponential map. The bundle $N_O$ can be endowed with two metrics, the first being just the pull-back of $g_{\lambda,\mu}$ on $O$ under the exponential map, and the second a metric compatible with the normal connection $\nabla^\nu$ of the normal bundle $\nu(O)$ such that the projection $\pi : N_O \to M$ is a Riemannian submersion. We let $A$ be the O’Neill tensor of this submersion.

If $M$ is totally geodesic, the two metrics above on $N_O$ are $C^1$-close and agree with each other to order 1 on the zero section $M \to N_O$ where they are just the background metric $g_{\lambda,\mu}$. Hence, on points of the totally geodesic zero section, the covariant derivatives that these metrics define coincide, and so do tensorial quantities defined in terms of the covariant derivatives and the metric tensor itself.

For any Riemannian submersion, given any pair of horizontal vector fields $X, Y$, we have that $A_X Y = \nabla(\nabla_X Y) = \nabla[X, Y]$. Here, $\nabla$ is the covariant derivative of the Riemannian metric on the total space of the submersion, and $\nabla$ the projection onto the vertical space of the submersion. We apply this to the Riemannian submersion above with O’Neill tensor $A$. Since $M$ is totally geodesic, if we consider left invariant vector fields $X, Y$ in $N_O$ that are tangent to $M$ at one point $p \in M$, the tensorial quantity $(A_X Y)_p = \nabla(\nabla_X Y)_p$ may be computed by taking horizontal extensions $X$ and $Y$ of $X_p$ and $Y_p$, respectively, and finding the projection onto the vertical component of $\nabla_X Y$. But the extensions $X$ and $Y$ are then vector fields tangent to $M$ on points of $M$, and along $M$ we have that $\nabla_X Y = \nabla_X^{g_M} Y$, which is tangent to $M$. Thus, $\nabla[X, Y]_p = 0$ and so $[X, Y]_p$ is tangent to $M$. We conclude that the Lie bracket of left invariant vector fields that are tangent to $M \to N_O$ at $p$ (and so horizontal at $p$) stay horizontal and are tangent to the zero section over it. This confers the tangent space of $M$ at any point with the structure of a Lie subalgebra of $Sp(2)$, and up to diffeomorphism, $M$ must be the image under the group exponential map of a 3-dimensional Lie subalgebra of $sp(2)$. Thus, $M$ is a closed Lie subgroup of $Sp(2)$. The remaining part of the argument is the same as that for the bi-invariant metric given above.

The reasoning above can be extended to yield the following conclusion.

**Corollary 3.21.** Consider the functional (25) for the class $kD$, $k \in \mathbb{Z}$. Its minimum is zero, and modulo isometries, the minimum of smallest volume is uniquely
realized by a set of $k$ fibers if $k \geq 0$, or by $-k$ fibers each with the opposite orientation if $k < 0$. Up to diffeomorphisms, all elements $[M] \in H_3(\mathbb{S}^4; \mathbb{Z})$ admit a unique canonical representative, and the number of connected components of this representative equals the divisibility $\|M/D\| \in \mathbb{Z}_{\geq 0}$ of the class $[M]$. \hfill \square

**Remark 3.22.** If we consider the 7th homology group $H_7(\mathbb{P}(2); \mathbb{Z}) \cong \mathbb{Z}$ instead, its generator $D$ cannot be realized by a submanifold $\mathbb{P}(2)$. As $\dim \mathbb{P}(2) = 10$, this result sharpens Thom's original example of a nonrealizable 7-class in a manifold of dimension 14 (see Remark 3.14). Thus, for this homology class $D$ the space $\mathcal{M}_D(\mathbb{P}(2))$ is empty. Proceeding along similar lines, we may study a problem analogous to the one above by enlarging $\mathcal{M}_D(\mathbb{P}(2))$ to include now all isometric immersed representatives of $D$, which for this 7-class would be a nonempty space. We could attempt then to decide if the infimum of the functional (25) is realized over this extended domain, and perhaps tie up the geometric properties of the minimizers with the nonrepresentability of the class. \hfill \square

4. **Complex submanifolds of complex projective space**

In this section, the ambient space $M$ is assumed to be $\mathbb{P}^n(\mathbb{C})$ provided with the Fubini-Study metric. We show that any of its complex submanifolds is a critical point of (39).

4.1. **The Fubini-Study metric.** We realize $\mathbb{P}^n(\mathbb{C})$ as the quotient $\mathbb{S}^{2n+1}/\mathbb{S}^1$, where the action of $\mathbb{S}^1$ on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ is by complex scalar multiplication. The Fubini-Study metric $\tilde{g}$ on it may be defined by the Kähler form of the metric, which on an affine chart $U$ with holomorphic coordinates $z = (z^1, \ldots, z^n)$ is given by

$$\omega_U = \frac{i}{2} \partial \bar{\partial} \log (1 + z^1 \bar{z}^1 + \cdots + z^n \bar{z}^n) = \frac{i}{2} \frac{\delta_{\alpha \beta} (1 + \sum z^\gamma \bar{z}^\gamma) - z^\beta \bar{z}^\alpha}{(1 + \sum z^\gamma \bar{z}^\gamma)^2} dz^\alpha \wedge d\bar{z}^\beta.$$ 

These locally defined forms give rise to a globally defined form $\omega$ that is $J$-invariant, closed, and positive.

The Ricci form and scalar curvature of the Fubini-Study metric are given by $\rho_{\tilde{g}} = 2(n+1)\omega$ and $s_{\tilde{g}} = 4n(n+1)$, respectively. The first Chern class of complex projective space is given by the positive class $c_1(\mathbb{P}^n(\mathbb{C})) = ((n+1)/\pi) [\omega]$.

Let $\bar{g}$ be the standard metric on $\mathbb{S}^{2n+1}$. With the Fubini-Study metric $g$ as defined above on the base, we obtain a Riemannian submersion $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{P}^n(\mathbb{C})$. Let $\{u,v\}$ be an orthonormal basis for a planar section at a point $p$ of $\mathbb{P}^n(\mathbb{C})$, and let $\{\bar{u}, \bar{v}\}$ denote the horizontal lift of this basis to a point $\bar{p}$ of the fiber over $p$. Then we have (22) that

$$K_g(u,v) = 1 + 3|\bar{g}(\bar{u}, \bar{v})|^2,$$

and so, the sectional curvature of the Fubini-Study metric ranges in the interval $[1, 4]$, with the maximum attained by holomorphic or antiholomorphic sections, that is to say, sections spanned by $u, v$ where $v = \pm Ju$, while the minimum is attained by sections spanned by $u, v$ where $Jv$ is orthogonal to $u$ also. In fact, if $R^g$ is the (0,3)-curvature tensor of $g$, we have that

$$R^g(v,u)u = \cos(t)v_0 + 4 \sin(t)Ju,$$

where

$$v = \cos(t)v_0 + \sin(t)Ju$$

is the decomposition of $v$ with $v_0$ orthogonal to both, $u$ and $Ju$.\

\[\text{PROOF.}\]
4.2. Complex submanifolds of complex projective space.

**Proposition 4.1.** For any complex Riemannian manifold \((M, g)\) isometrically immersed in \(\mathbb{P}^n(\mathbb{C})\), the function \(\|\alpha\|^2\) is an intrinsic invariant, that is to say, it does not depend upon the immersion.

**Proof.** Let \(m\) be the complex dimension of \(M\). We use (4) and (39) to conclude that

\[
(41) \quad s_g = 4n(n + 1) - 2(n - m)(4 + 2(n + m - 1)) - \|\alpha\|^2 = 4m(m + 1) - \|\alpha\|^2.
\]

Since the scalar curvature is intrinsic, the result follows. \(\square\)

**Remark 4.2.** If the background manifold were to be the space form \(S^m_\delta\), or any of its quotients by a discrete subgroup, then for any isometrically immersed submanifold \((M, g)\) the quantity \(\|\alpha\|^2 - \|H\|^2\) would be intrinsic also. This would follow by (41) using the same argument above, all the sectional curvatures now being the constant \(c\). \(\square\)

**Theorem 4.3.** Any complex submanifold of \(\mathbb{P}^n(\mathbb{C})\) is a critical point of \(\|\alpha\|^2\).

**Proof.** We use the critical point equations as described in Theorem 4.1. By the minimality of every complex submanifold of a Kähler manifold, our task reduces to verifying that for any \(m, 1 \leq m \leq q\), we have that

\[
(42) \quad \langle R^\delta(\alpha(e_i, e_j), e_j)e_i, \nu_m \rangle + \langle \nabla^\delta_e R^\delta(e_j, e_i)e_j, e_i\rangle + \text{trace } A_\nu_m A^2_{\nu_k} = 0,
\]

which we do by verifying that each of the three summands on the left is identically zero.

We decompose \(\alpha(e_i, e_j)\) as \(\alpha(e_i, e_j) = \langle A_\nu e_i, e_j \rangle \nu_l\). Let us consider a fixed index \(l\) in the range \(1 \leq l \leq q\). We then choose a basis \(\{e_i\}\) that diagonalizes \(A_\nu\). Since \(\nu_l\) is orthogonal to both, \(e_i\) and \(Je_i\), by (10) we obtain that

\[
\langle A_\nu e_i, e_j \rangle \langle R^\delta(\nu_l, e_j)e_i, \nu_m \rangle = \langle A_\nu e_i, e_i \rangle \langle \nu_l, \nu_m \rangle = 0,
\]

where in deriving the last equality in the case when \(l = m\), we use the property that complex submanifolds of a Kähler manifold are austere. Thus,

\[
\langle R^\delta(\alpha(e_i, e_j), e_j)e_i, \nu_m \rangle = \langle A_\nu e_i, e_i \rangle \langle R^\delta(\nu_l, e_j)e_i, \nu_m \rangle = 0.
\]

In dealing with the middle summand, it is convenient to assume that the normal fields \(\{\nu_1, \ldots, \nu_q\}\) has been chosen so that \(\langle \nabla^\delta e_i a_i \nu_l \rangle = 0\). We then use (10) once again to conclude that

\[
\langle \nabla^\delta_e R^\delta(e_j, e_i)e_j, \nu_m \rangle = e_i \langle R^\delta(e_j, e_i)e_j, \nu_m \rangle = 0,
\]

because, for each \(i \neq j\), the decomposition of \(e_i\) as \(\cos(t)(e_i)_0 + 4 \sin(t)Je_i\), with \((e_i)_0\) orthogonal to both, \(e_j\) and \(Je_j\), produces a tangent vector to the complex submanifold.

Since for any normal field \(\nu\) we have that \(A_\nu J = -JA_\nu\), any of the cubic traces will have to be zero also. For in a neighborhood of a fixed but arbitrary point \(p \in M\), we may choose an oriented local normal frame for \(TM\) of the form \(\{\nu_1, \nu_{1'}, \ldots, \nu_{d}, J\nu_{d}\}\), and obtain that

\[
\text{trace } A_{\nu_m} A_{\nu_k} = \langle (A_{\nu_m} A^2_{\nu_k} \nu_1, \nu_1) + (A_{\nu_m} A^2_{\nu_k} J\nu_1, J\nu_1) \rangle
\]

\[
= \langle (A_{\nu_m} A^2_{\nu_k} \nu_1, \nu_1) - (JA_{\nu_m} A^2_{\nu_k} \nu_1, J\nu_1) \rangle = 0.
\]
and Vianian metric. In general, if V the set of fixed points of complex conjugation in V the inclusion map, then critical value follows by integration of the expression (41) in Proposition 4.1.

regarded as a real n-dimensional submanifold of Pn. And for any two complex m-submanifolds in Pn, their volumes [ω]m/m! and cup product c1 ∪ [ω]m−1 coincide (c1 being the first Chern class). Since 4π/(n − 1)! times the latter is the total scalar curvature, the statement about the critical value follows by integration of the expression (11) in Proposition 4.1.

We denote by Pn(R) the real projective n-space endowed with its natural Riemannian metric. In general, if V is a real nonsingular variety, we denote by V(R) and V(C) its set of real and complex points, respectively. We identify V(R) with the set of fixed points of complex conjugation in V(C). In this manner, Pn(R) is regarded as a real n-dimensional submanifold of Pn(C). If i : Pn(R) → Pn(C) is the inclusion map, then V(R) ⊂ Pn(R) → V(C) ⊂ Pn(C).

This completes the proof.

Remark 4.4. The argument used above to show the vanishing of trace AνmAνk holds for complex submanifolds of an arbitrary background Kähler manifold. It is in the vanishing of the other two terms that we used properties specific to the Fubini-Study metric on Pn(C). It would be of interest to see the extent to which the stated result holds over arbitrary Kähler manifolds. A simple case to consider is that of the product Pn1(C) × · · · × Pnk(C), where the argument above can be adapted to prove that

**Theorem 4.3.** Any complex submanifold of the product Pn1(C) × · · · × Pnk(C) is a critical point of (6).

Using Corollary 3.13, we derive a more general result:

**Theorem 4.5.** Let (M, J, g) be a Kähler manifold, and M be a complex submanifold provided with the induced Kähler metric g. If the extrinsic scalar curvature of M is nowhere zero, then M is a critical point of (25) within its homology class.

*Proof.* The total scalar curvature of (M, g) is a topological invariant, and (M, g) is a critical point of (22) and (11).

Remark 4.6. There are critical points of (6) in Pn(C) that are not complex submanifolds. For instance, the projection of the minimal torus S1(1/3) × S1(1/3) × S1(1/3) ⊂ S3 onto P2(C) is a totally real minimal flat torus T2 for which ∥α∥2 = 2. In affine holomorphic coordinates (z1, z2), T2 = {(z1, z2) : |z1| = |z2| = 1}. We may see that T2 is a critical point of (6) in P2(C) through a direct argument, showing that the summands in (42) are all zero. Clearly, this torus is homologically trivial in P2(C).

Suppose that in the general setting we take the background manifold to be Kähler, say (M, J, g). Let D ∈ Hn(M; Z). Since complex submanifolds minimize the volume within their homology class, then we know that if the minimum of (25) were to be realized by a complex manifold, such a critical point would be a critical point of (6), (11), and of (28), separately. In the case of Pn(C), by Theorem 4.3 we know that smooth algebraic subvarieties that represent elements of H∗(Pn(C); Z) are critical points of (6), and consequently, they are critical points of (28) and (25) also. We shall analyze later on if algebraic curves are, in fact, canonical representatives of their homology classes in P2(C). For now, we have the following:

**Theorem 4.7.** All complex submanifolds in M_D(Pn(C)) are critical points of (6), (28) and (25), and they all have the same critical value.

*Proof.* By Theorem 4.3 complex submanifolds are critical points of these functionals. And for any two complex m-submanifolds in M_D(Pn(C)), their volumes [ω]m/m! and cup product c1 ∪ [ω]m−1 coincide (c1 being the first Chern class). Since 4π/(n − 1)! times the latter is the total scalar curvature, the statement about the critical value follows by integration of the expression (11) in Proposition 4.1.
Corollary 4.8. Any $k$-dimensional real subvariety $V = V^k$ of $\mathbb{P}^n(\mathbb{R})$ is a critical point of $[\Omega]$. If we assume further that both $V(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{R})$ are oriented, with the orientations compatible with each other, and if we let $D = [V(\mathbb{R})]$ in $H_k(\mathbb{P}^n(\mathbb{R}); \mathbb{Z})$, then all real varieties in $M_D(\mathbb{P}^n(\mathbb{R}))$ are critical points of (23) and (24), and they all have the same critical value.

Proof. Real projective space is a totally geodesic submanifold of complex projective space.

4.3. Algebraic curves in $\mathbb{C}P^2$. A curve $S_d$ of degree $d$ in $\mathbb{P}^2(\mathbb{C})$ is, up to diffeomorphisms, the Riemann surface given by the zeroes of the polynomial $p_d(Z_0, Z_1, Z_2) = Z_0^d + Z_1^d + Z_2^d$.

$$S_d = \{ Z = [Z_0 : Z_1 : Z_2] \in \mathbb{C}P^2 : p_d(Z) = 0 \}.$$ 

A complex projective line $H$ in $\mathbb{P}^2(\mathbb{C})$ defines a generator $h = [H]$ of the second integral homology group $H_2(\mathbb{P}^2(\mathbb{C}); \mathbb{Z})$, and $S_d$ represents the integer homology element $dh$. Since the total Chern class of $\mathbb{P}^2(\mathbb{C})$ is given by $c(\mathbb{P}^2(\mathbb{C})) = (1 + \hat{h})^3$, where $\hat{h} = [\omega]/\pi$ is the generator of $H^2(\mathbb{P}^2(\mathbb{C}); \mathbb{Z})$ dual to the homology element $h$ above, if $i : S_d \hookrightarrow \mathbb{P}^2(\mathbb{C})$ is the inclusion map and $x = i^*\tilde{h}$, we have that $c_1(S_d) = (3 - d)x$.

Proposition 4.9. The $L^2$-norm of the second fundamental form of a curve of degree $d$ in $\mathbb{P}^2(\mathbb{C})$ is given by

$$\Pi(S_d) = 4\pi d(d - 1).$$

Proof. In this dimension, the Fubini-Study metric has scalar curvature 24. Since the total scalar curvature of a Kähler metric with Kähler class $\Omega$ is $4\pi c_1 \cdot \Omega^{n-1}/(n-1)!$, by (11) and (39) we see that

$$4\pi c_1(S_d) \cdot [S_d] = 4\pi (3 - d)d = 8\pi d - \int \|\alpha\|^2 d\mu,$$

and the result follows. 

We may apply Gauss-Bonnet to rewrite $c_1(S_d) \cdot [S_d]$ in terms of the Euler characteristic of $S_d$. Using the value of $\Pi(S_d)$ computed above, (11) then yields that the genus $g_{S_d}$ of $S_d$ is given by

$$g_{S_d} = \frac{(d - 1)(d - 2)}{2}.$$ 

This genus of $S_d$ can be had via an argument independent of the one above, using either the adjunction or Hurwitz’s formula also.

We consider the integrand $\sum_{1 \leq i, j \leq n} K_{ij}(e_i, e_j)$ of the functional $\Theta$ in (22) when $M$ is a (real) surface $S$ in $\mathbb{P}^2$ representing a homology class $D$. We identify $S \in M_D(\mathbb{P}^2(\mathbb{C}))$ with its image under the embedding, and let $\{e_1, e_2\}$ and $\{\nu_1, \nu_2\}$ be orthonormal frames for its tangent and normal bundles, respectively. Thus, $\{e_1, e_2, \nu_1, \nu_2\}$ is an orthonormal frame for the tangent bundle of the projective plane on points of $S$. The function

$$2K_{ij}(e_1, e_2).$$

on $S$ has a maximum value 8 at a point $p \in S$ if the frame $\{e_1, e_2, \nu_1, \nu_2\}$ at this point is such that $Je_1 = \pm e_2$ and $J\nu_1 = \pm \nu_2$. If the frame is positively oriented, this condition is equivalent to saying that the frame is holomorphic. On the other
hand, the minimum of the function is 2, and this is achieved at points where the frame \(\{e_1, e_2, \nu_1, \nu_2\}\) is purely real. Indeed, by (39), we have that
\[
2K(e_1, e_2) = 2 + 6|\tilde{g}(\tilde{e}_1, J\tilde{e}_2)|^2,
\]
where \(\tilde{g}\) is the standard metric on \(S^5\), and \(\tilde{e}_i\) is the horizontal lift of \(e_i\) in the Riemannian submersion \(S^1 \rightarrow S^5 \rightarrow \mathbb{P}^2(\mathbb{C})\) to a point on the fiber over \(p\). The statements made above follow readily from this.

We have the following:

**Theorem 4.10.** If \(d \neq 3\), there are no Lagrangean elements in \(M_D(\mathbb{P}^2(\mathbb{C}))\).

**Proof.** Let \(c_1\) be the pullback to \(M\) of the first Chern class of \(\mathbb{P}^2(\mathbb{C})\). Then (cf. with the proof of Proposition 4.9) we have that
\[
c_1 \cdot [M] = (3 - d)d.
\]

**Theorem 4.11.** If \(f_t : S_d \rightarrow \mathbb{P}^2(\mathbb{C})\) is a small deformation of an algebraic curve \(S_d\) in \(\mathbb{P}^2(\mathbb{C})\) of degree \(d\). Then
\[
\Theta(f_t(S_d)) = 2 \int_{f_t(S_d)} K(e_1, e_2)d\mu \geq 8\pi d,
\]
\[
S(f_t(S_d)) = \int_{f_t(S_d)} (\|a\|^2 - \|H\|^2)d\mu \geq 4\pi d(d - 1),
\]
and
\[
\Theta(f_t(S_d)) - S(f_t(S_d)) = 4\pi d(3 - d).
\]

**Proof.** The extrinsic scalar curvature of any algebraic variety \(M\) of dimension \(m\) in \(\mathbb{P}^n(\mathbb{C})\) is given by
\[
\Theta(M) = 4m(m + 1) \int_M d\mu,
\]
so its second variation under small deformations of the variety is \(4m(m + 1)\)-times the second variation of the volume form. By Proposition 3.2.2, Theorem 3.5.1, the latter is given by a nonnegative bilinear form on the normal bundle of \(M\) whose kernel is the space of globally defined holomorphic cross sections of the normal bundle. Thus, under small variations of \(M\), \(\Theta(M)\) is either deformed into a variety isotopic to \(M\) with the same extrinsic scalar curvature or into a submanifold homologous to \(M\) of larger extrinsic scalar curvature. The result now follows by the Gauss-Bonnet theorem, since (4) implies that
\[
\int_{f_t(S_d)} s_g d\mu_g = \Theta(f_t(S_d)) - S(f_t(S_d)),
\]
and the Euler characteristic of any embedded surface stays constant under small perturbations of the surface. \(\square\)

We shall improve the small deformation result above, and show that it holds for deformations in \(M_{[S_d]}\):

5. Oriented Surfaces in \(S^2 \times S^2\) and \(\mathbb{P}^2(\mathbb{C})\)

5.1. The diagonal in the product \(S^2 \times S^2\). Let us examine the case of \(M = S^2 \times S^2\) provided with the product of the standard metrics and complex structures on the factors. The nontrivial integer homology groups are \(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}\) and \(\mathbb{Z}\) in dimensions 0, 2 and 4, respectively. Let \(A = [S^2 \times \{q\}]\) and \(B = \{p\} \times S^2\). Then \(\{A, B\}\) generates \(H_2(M; \mathbb{Z})\). We consider the problem of finding a canonical representative of \(D = A + B\).
The union of the submanifolds $S^2 \times \{q\}$ and $\{p\} \times S^2$ represents the class $D$. These submanifolds intersect at the point $p \times q$, and the intersection is modeled in complex coordinates $\{z^1, z^2\}$ by the set $L = \{(z^1, z^2) \in \mathbb{C}^2 : z^1 z^2 = 0, |z^1|^2 + |z^2|^2 \leq 1\}$.

The singularity of this representative can be resolved by removing the pair $(B, L)$ in $\mathbb{C}^2$, where $B$ is the $z = (z^1, z^2)$-ball of radius 1 centered at the origin, and replacing it by the pair $(B, L')$, where $L'$ is a perturbation of $L = \{(z^1, z^2) : z^1 z^2 = \varepsilon, |z^1|^2 + |z^2|^2 \leq 1, 0 < \varepsilon \ll 1\}$ such that $\partial L = \partial L' \subset \partial B$ (see [1] p. 38-39). This procedure produces a smooth surface $M$ in $S^2 \times S^2$ that represents $D$, $D = [M]$, because the sets $L$ and $L'$ are homologous to each other in $(B, \partial B)$. The surface $M$ is not a complex curve.

The surface $M$ is homologous to the diagonal $\mathcal{D} = \{(p, p) : p \in S^2\}$, surface that represents the class $\mathcal{D}$ also. The diagonal $\mathcal{D}$ is a critical point of (6), and being a Kähler submanifold of the ambient Kähler manifold, it is a critical point of (11) also of minimal volume among all the elements in $\mathcal{M}_D(S^2 \times S^2)$.

**Theorem 5.1.** Let the product $S^2 \times S^2$ be provided with its standard metric and complex structure, and let us consider the homology class $D = [S^2 \times \{q\}] + \{p\} \times S^2 \in H_2(S^2 \times S^2; \mathbb{Z})$. Let $\mathcal{D}$ be the diagonal in $S^2 \times S^2$. Then $\mathcal{D}$ is a critical point of (6), (11), (23), and (24), respectively, in each case, of critical value zero, and modulo isometries, $\mathcal{D}$ is the canonical representative of $D$.

**Proof.** By Theorem 4.3, $\mathcal{D}$ is a critical point of (6). In fact, $\mathcal{D}$ is a totally geodesic Kähler submanifold of $S^2 \times S^2$ of scalar curvature 1 that is a volume minimizer element of $\mathcal{M}_D(S^2 \times S^2)$, of volume $8\pi$. We now prove that on $\mathcal{M}_D(S^2 \times S^2)$, the functional (24) is optimally bounded below by zero, and that a minimum of smallest volume is a totally geodesic sphere of constant scalar curvature 1.

Let $M$ be an oriented connected Riemannian surface $(M, g)$ isometrically embedded into $(S^2 \times S^2, \hat{g})$ that minimizes (24) and have the smallest volume among minimizers. Then $M$ is a totally geodesic surface in $(S^2 \times S^2, \hat{g})$ that is connected and has volume $8\pi$.

Let $X, Y$ be an orthonormal basis of the tangent space $T_p M$ at $p \in M$. We have that $K_{\hat{g}}(X, Y) = |X_1 \wedge Y_1|^2 + |X_2 \wedge Y_2|^2$, where $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ are the decompositions of $X$ and $Y$ into components tangential to the factors, respectively.

By (11) and the Gauss-Bonnet theorem, we obtain that

$$0 = \int_M \|\alpha\|^2 d\mu = \int_M (\|\alpha\|^2 - \|H\|^2) d\mu = 2 \int_M K_{\hat{g}}(X, Y) d\mu - 4\pi \chi(M),$$

and since the sectional curvature of the ambient space metric is nonnegative, $M$ can be either a torus or a sphere only. Let us view the product $S^2 \times S^2$ as the total space of the Riemannian submersion given by the projection onto one of the factors. If $M$ were a torus, then $K_{\hat{g}}(X, Y) = 0$ and at every $p \in M$ we could choose $\{X, Y\}$ with $X$ tangent to the base and $Y$ tangent to the fiber. Using the exponential map, we would then conclude that $M = S^1 \times S^1 \subset S^2 \times S^2$, which contradicts the fact that $M$ represents a nontrivial class in homology. So $M$ must be a topological sphere, $\chi(M) = 2$, and

$$0 = \int_M \|\alpha\|^2 d\mu = \int_M (\|\alpha\|^2 - \|H\|^2) d\mu = 2 \int_M (|X_1 \wedge Y_1|^2 + |X_2 \wedge Y_2|^2) d\mu - 8\pi.$$

Since horizontal geodesics stay horizontal, at no $p \in M$ it is possible to have $T_p M$ horizontal or vertical. Otherwise, $M$ would be either a horizontal or a fiber sphere,
respectively, neither one of which represents the homology class $D$. Therefore, $M$ is the graph of a diffeomorphism $f : S^2 \to S^2$, and by the transivity of the action of the group of isometries of the standard metric on $S^2$, we conclude that $K_{\bar{g}}(X, Y) = |X_1 \wedge Y_1|^2 + |X_2 \wedge Y_2|^2$ is a constant function on $M$, the constant $\frac{1}{2}$, that $f$ is an isometry of the sphere with its standard metric, and that $M$ (the graph of $f$) is isometric to $D$ with its induced metric.

The canonical representative $\mathcal{D}$ of $D$ above is a rigid complex submanifold of $S^2 \times S^2$ of zero genus in $\mathcal{M}_D(S^2 \times S^2)$. The immersed representative $S^2 \times \{p\} \cup \{q\} \times S^2$ is an absolute minimizer of $\Pi$ also. Its local desingularizations at the intersection point produces a one parameter family of embedded spheres that represents $D$ and have a value of $\Pi$ close to but larger than $8\pi$, which when subtracting the total scalar curvature of the representative yields a value close to but larger than 0. Thus, the functional $\mathcal{I}$ can be extended from $\mathcal{M}_D(S^2 \times S^2)$ to the space of immersed representatives of $D$, and on this extended domain, there are singular current points of minimal mass that are absolute minimizers of the extension. The canonical representative $\mathcal{D}$ is a smooth current that lies in $\mathcal{M}(D)$, has $\Pi = 0$, and so it minimizes the extension also, is a volume minimizer among representatives of $D$.

**Remark 5.2.** If $J'$ is any almost complex structure on $S^2 \times S^2$ tamed by $\omega_1 + \omega_2$ (for instance, the $J$ in Theorem 5.1, Theorem 2.4.C in [12] ensures the existence of a connected regular $J'$ curve into $S^2 \times S^2$ that represents $A + B$ and has genus 1, and the collection $M(J')_{1,1}$ of all such curves form a smooth manifold of dimension 6. So for $J' = J$ these curves are “symplectic canonical representatives” of $A + B$, but none of them are metric canonical representative in the sense described by the Theorem above.

The class $2D = 2(A + B)$ does not admit an embedded sphere representative [16]. This new class may be represented by the union of two horizontal and two fiber vertical spheres, respectively, with this immersed representative an absolute minimizer of $\Pi$. Desingularizing at the four intersection points while preserving the homology class, we obtain an embedded noncomplex torus representative of $2D$ for which $\Pi$ is close to but larger than $16\pi$. Thus, when extended to the space of immersed representatives, the value of $\Pi$ for the elements of the embedded family of representatives whose limit is the immersed one are larger than the value value zero of $\Pi$ for the limit. The class $2D$ admits an 8-parameter family of complex tori that represent it, each one of which is a volume minimizer within the class, of volume $4\pi$. We conjecture that these tori are the canonical representatives of $2D$, their second fundamental forms all having global $L^2$-norm equal to $16\pi$. This would show that we can find singular currents that are absolute minima of the extension of $\mathcal{I}$ to the space of immersed representatives of the class $2D$, which minimize the volume also. The gap between the functional values at this singular currents and the values at the smooth tori representatives would measure the minimum number of self-intersection points that immersed smooth representatives could have. The conjectured behaviour is reminiscent of one parameter families of mappings that fail to be Fredholm just at a discrete set of parameters, where the mappings have finite dimensional kernel and cokernel but not closed range, and the index jumps [27]. The minimum value of $\Pi$ on embedded manifolds representer an integral homology class jumps when the integral class changes.
5.2. Complex curves in $\mathbb{P}^2(\mathbb{C})$. In Theorem 4.7 we have shown that all complex submanifolds in $\mathcal{M}_D(\mathbb{P}^n(\mathbb{C}))$ are critical points of the functional $\mathcal{J}$ in [25] of the same critical value, and by their minimality, also critical points of the domain restricted functional $\mathcal{J}_{D}(\mathbb{P}^n(\mathbb{C}))$ in [23], all associated with the same critical value. It is then natural to ask if they are the minimum. We study this optimization problem when $n = 2$ and $D = [S_d] \in H_2(\mathbb{P}^2(\mathbb{C}); \mathbb{Z})$ next. This will yield our first examples of infimums of $\mathcal{J}$ over $\mathcal{M}_{[S_d]}(\mathbb{P}^2(\mathbb{C}))$ that are no longer realized by totally geodesic absolute minimizers of the functional. Thus, we are forced to study precompactness properties of minimizing sequences.

Consider a minimizing sequence $\{M_n\} \subset \mathcal{M}_{[S_d]}(\mathbb{P}^2(\mathbb{C}))$:

$$\mathcal{J}(M_n) \to t = \inf_{M \in \mathcal{M}_{[S_d]}} \int_M \|\alpha\|^2 d\mu_M.$$ 

We may assume that:

- **M1.** All the $M_n$s are connected, and $t$ is at most equal to $\mathcal{J}(S_d) = 4\pi d(d - 1)$.
- **M2.** The $M_n$s can be taken to be of genus $g_{M_n}$ and areas $\mu_n$ satisfying the bounds

$$g_{M_n} \leq \frac{(d - 2)(d - 1)}{2} + \frac{3}{4} d,$$

and

$$\pi d \leq \mu_n \leq 2\pi(d(d - 1) + 2).$$

Indeed, by (41) and the bounds for the sectional curvature of the the Fubini-Study metric in $\mathbb{P}^2(\mathbb{C})$, we have that

$$\int_M \|\alpha\|^2 d\mu = \int_M (2K_g(e_1, e_2) + \|H\|^2) d\mu - 4\pi \chi(M) \geq 2\mu(M) - 4\pi \chi(M),$$

which by M1 implies the upper bounds for $g_{M_n}$ and $\mu_n$, respectively. On the other hand, $\mu(S_d) = \pi d$ is the absolute minimizer of the area functional in $\mathcal{M}_{[S_d]}(\mathbb{P}^2(\mathbb{C}))$.

**M3.** By possibly passing to a subsequence, we may them assume that the minimizing sequence $\{M_n\} \subset \mathcal{M}_{[S_d]}(\mathbb{P}^2(\mathbb{C}))$ is given by a sequence of isometric embeddings

$$f_n : (\Sigma, g_n) \to f_n(\Sigma) = M_n \hookrightarrow (\mathbb{P}^2(\mathbb{C}), g),$$

of a fixed connected surface $\Sigma$, whose genus $g_\Sigma$ satisfies (43), and where the areas $\mu_{g_n}(\Sigma)$ satisfy (44). Notice that if $M = M_n = M_1 \neq M_2$ where $M_2$ bounds some 3-manifold in $\mathbb{P}^2(\mathbb{C})$, then $M_1$ must be homologous to $S_d$. We shall prove that $M_1$ is in effect homeomorphic to $S_g$ and that $(M_n, g_n)$ converges to a complex curve of degree $d$ endowed with the metric induced by the Fubini-Study metric on $\mathbb{P}^2(\mathbb{C})$.

**M4.** We have the $L^1$ bound

$$\left| \int_M s_g d\mu_g \right| \leq \int_M |s_g| d\mu_g \leq \Theta(M) + \mathcal{J}(M),$$

for the scalar curvature of all the $M_n$s in the minimizing sequence. This follows by (41), and the the Cauchy-Schwarz and Young inequalities, from which we can derive the pointwise estimate

$$|s_g| \leq 2K_g(e_1, e_2) + \|\alpha\|^2$$

In Theorem 4.7 we have shown that all complex curves in $\mathcal{M}_D(\mathbb{P}^n(\mathbb{C}))$ are critical points of the functional $\mathcal{J}$ in [25] of the same critical value, and by their minimality, also critical points of the domain restricted functional $\mathcal{J}_{D}(\mathbb{P}^n(\mathbb{C}))$ in [23], all associated with the same critical value. It is then natural to ask if they are the minimum. We study this optimization problem when $n = 2$ and $D = [S_d] \in H_2(\mathbb{P}^2(\mathbb{C}); \mathbb{Z})$ next. This will yield our first examples of infimums of $\mathcal{J}$ over $\mathcal{M}_{[S_d]}(\mathbb{P}^2(\mathbb{C}))$ that are no longer realized by totally geodesic absolute minimizers of the functional. Thus, we are forced to study precompactness properties of minimizing sequences.

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$$\pi d \leq \mu_n \leq 2\pi(d(d - 1) + 2).$$

Indeed, by (41) and the bounds for the sectional curvature of the the Fubini-Study metric in $\mathbb{P}^2(\mathbb{C})$, we have that

$$\int_M \|\alpha\|^2 d\mu = \int_M (2K_g(e_1, e_2) + \|H\|^2) d\mu - 4\pi \chi(M) \geq 2\mu(M) - 4\pi \chi(M),$$

which by M1 implies the upper bounds for $g_{M_n}$ and $\mu_n$, respectively. On the other hand, $\mu(S_d) = \pi d$ is the absolute minimizer of the area functional in $\mathcal{M}_{[S_d]}(\mathbb{P}^2(\mathbb{C}))$.

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$$f_n : (\Sigma, g_n) \to f_n(\Sigma) = M_n \hookrightarrow (\mathbb{P}^2(\mathbb{C}), g),$$

of a fixed connected surface $\Sigma$, whose genus $g_\Sigma$ satisfies (43), and where the areas $\mu_{g_n}(\Sigma)$ satisfy (44). Notice that if $M = M_n = M_1 \neq M_2$ where $M_2$ bounds some 3-manifold in $\mathbb{P}^2(\mathbb{C})$, then $M_1$ must be homologous to $S_d$. We shall prove that $M_1$ is in effect homeomorphic to $S_g$ and that $(M_n, g_n)$ converges to a complex curve of degree $d$ endowed with the metric induced by the Fubini-Study metric on $\mathbb{P}^2(\mathbb{C})$.

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$$\left| \int_M s_g d\mu_g \right| \leq \int_M |s_g| d\mu_g \leq \Theta(M) + \mathcal{J}(M),$$

for the scalar curvature of all the $M_n$s in the minimizing sequence. This follows by (41), and the the Cauchy-Schwarz and Young inequalities, from which we can derive the pointwise estimate

$$|s_g| \leq 2K_g(e_1, e_2) + \|\alpha\|^2$$
for the scalar curvature of any \( M \in \mathcal{M}_{[S^4]}(\mathbb{P}^2(\mathbb{C})) \). Then (15) follows by integration over \( M \).

M5. The minimizing sequence of isometric embeddings (15) can be taken to be given by conformal mappings with the scalar curvature of the metrics uniformly bounded in \( L^2(\Sigma, d\mu_{g_n}) \).

**Proof.** We consider the conformal class defined by \( g_n \) on \( \Sigma \), and its almost complex structure \( J_n \). For dimensional reasons, the triple \((\Sigma, g_n, J_n)\) is Kähler. On such a manifold, there exists a conformal deformation \( c_n : \Sigma \rightarrow \Sigma \) to an extremal metric \( c_n^* g_n \) of constant scalar curvature \( s_{g_n} = 4\pi \chi(\Sigma)/\mu_{g_n}(\Sigma) \) [4]. This extremal metric minimizes the \( L^2 \) norm of the scalar curvature of metrics on the given conformal class. The result follows, as the areas \( \mu_{g_n}(\Sigma) \) satisfy (44). \( \square \)

Given a Riemannian surface \((M,g)\) isometrically embedded into \( \mathbb{P}^2(\mathbb{C}) \), we let

\[
\mathcal{E}_g(M) = \int_M \|\alpha\|^4 d\mu_g
\]

denote the “energy” of \( M \). The statement above makes it natural to assume, which we do, that the sequence \( \{\mathcal{E}_{g_n}(M_n)\} \) is uniformly bounded.

Let us consider a minimizing sequence \( \{(M_n, g_n)\} \) satisfying conditions M1-M5. In the conformal class of each element \((M_n, g_n)\) in this sequence, we shall consider a sequence of metrics \( \{g_{c_m}^n\} \) that minimizes the energy \( \mathcal{E}_g(M_n) \). Notice that the conformal invariance of \( \mathcal{T}(M_n) \) and \( \mu_n \), and the Cauchy-Schwarz inequality imply that

\[
\mathcal{E}_g(M_n) \geq b_n := \frac{T^2(M_n)}{\mu_{g_n}}
\]

for any metric \( g \) conformally equivalent to \( g_n \), with equality if, and only if, \( \|\alpha\|^2 \) is constant. We call \( b_n \) the energy barrier of the conformal class of \( g_n \). We study the formation of singularities of \( \{g_{c_m}^n\} \) as \( \mathcal{E}_{g_n}(M_n) \rightarrow b_n \), and then study how these singularities evolve as the barrier \( b_n \) varies with \( g_n \) so that \( \mathcal{T}(M_n) \rightarrow t \). These two steps are analogous to those in the study of strongly extremal metrics [28, 30, 29], where we first study the existence of extremal metrics within a Kähler class, and then study how these metrics change as the value of the energy of one such varies as a function of the class towards the global minimum of the energy within the base of the Kähler cone.

5.2.1. **Precompactness properties of the energy in a fixed conformal class.** Given a Riemannian surface \((M,g)\) isometrically embedded into \( \mathbb{P}^2(\mathbb{C}) \), we study the problem of minimizing \( \mathcal{E}_g(M) \) within the conformal class of \( g \). In studying the formation of singularities of energy minimizing sequences, we follow closely the work of X.X. Chen [8], with the appropriate modifications required by our problem.

For any \((M,g) \in \mathcal{M}_{[S^4]}(\mathbb{P}^2(\mathbb{C}))\) that passes through a point \( p \) in the projective plane, we define \( D(p) \) to be

\[
D(p) = \{ q \in M : \text{dist}(q,p) \leq \iota \},
\]

where \( \iota = \frac{1}{2}\text{inj rad}_M \). If \( r < \iota \), we denote by \( D_r(p) \) the disk of radius \( r \) centered at \( p \). We use conformal coordinates \( z = x + iy \) on \( D(p) \), and let \( g_0 = dx^2 + dy^2 \) be the standard flat metric with Laplacian \( \Delta_0 \) and area form \( d\mu_0 \). Then the metric \( g \)
in $D(p)$ is of the form $g = e^{\varphi}g_0$ for some real valued function $\varphi$, thus identifying $g$ with $\varphi$, a fact that we use freely below. We have the relation

$$s_g = e^{-\varphi} \Delta_0 \varphi,$$

where $s_g$ is the scalar curvature of $g$.

Given constants $C_1$, $C_2$, we define

$$M^{C_1:C_2} = \{ \tilde{g} : \tilde{g} = e^{\varphi}g, \mu_\tilde{g}(M) \leq C_1, E_\tilde{g}(M) \leq C_2 \}.$$

If $\tilde{g} \in M^{C_1:C_2}$ and $U$ is any subdomain in $D(p)$, we define the local area, density and energy over $U$ by

$$\mu_\varphi(U) = \int_U e^{\varphi} d\mu_0, \quad t_\varphi(U) = \int_U \|\alpha\|^2 e^{\varphi} d\mu_0, \quad e_\varphi(U) = \int_U \|\alpha\|^4 e^{\varphi} d\mu_0,$$

and the local conformal invariant

$$\theta_\varphi(U) = \int_U 2K_\varphi(e_1,e_2) e^{\varphi} d\mu_0,$$

respectively. We have that

$$2\mu_\varphi(U) \leq \theta(U) \leq 8\mu_\varphi(U).$$

Let $g_n$ be an energy minimizing sequence of metrics in the conformal class of $g$. We say that $p$ is a bubble point for $g_n$ if there exist constants $a$ and $e$ such that, for any $r$ with $0 < r < \iota$, we have that

$$\liminf_n \mu_{\varphi_n}(D_r(p)) \geq a > 0, \quad \liminf_n e_{\varphi_n}(D_r(p)) \geq e \geq 0,$$

respectively. The largest constants $a_p$ and $e_p$ for which this holds are the concentration weights of area and energy at the bubble point $p$.

The local area and energy function at $p$ are defined by the expressions

$$A_p(r) = \limsup_n \mu_{\varphi_n}(D_r(p)), \quad E_p(r) = \limsup_n e_{\varphi_n}(D_r(p)),$$

and the local concentration of these quantities at $p$ are defined as the limits

$$A_p = \lim_{r \to 0} A_p(r), \quad E_p = \lim_{r \to 0} E_p(r).$$

We say that $p$ is a basic bubble point if $A_p > 0$, and $E_p \geq 0$. Passing to a subsequence if necessary, we see that every basic bubble point is a bubble point.

The following isoperimetric inequality theorem plays for us the same essential role it has in the work of Chen [8]. We state it here for completeness, and refer the reader to [3] for details and historical references about it. This result will be used freely below.

**Theorem 5.3.** [3] Let $g$ be a metric in an Euclidean disk $D$ whose scalar curvature $s_g$ is in $L^1(D,d\mu_g)$. Then given any relatively compact disk $D'$ in $D$, we have that

$$\int_{D'} |s_g| d\mu_g \geq 4\pi - \frac{|\partial D'|^2}{\mu_g(D')}.$$

We now have the following key result, the analogue of [8 Lemma 2]. It implies the existence of at most finitely many bubble points for sequences in $M^{C_1:C_2}$.

**Lemma 5.4.** Let $\{g_n\} \in M^{C_1:C_2}$ be an energy minimizing sequence in the conformal class of the metric $g$ on $M$. If $p$ is a bubble point for the $g_n$s, then

$$A_p(64A_p + E_p) \geq 8\pi^2.$$
Proof. Given \( \varepsilon > 0 \) sufficiently small, since \( A_p(r) \) is a monotonically increasing function, we can choose a small enough radius \( i_0 \) such that
\[
A_p \leq A_p(i_0) = \limsup_n \mu_{\varphi_n}(D_{i_0}(p)) < \left(1 + \frac{\varepsilon}{2}\right) A_p ,
\]
and then choose \( n \) large enough so that
\[
\mu_{\varphi_n}(D_{i_0}(p)) < (1 + \varepsilon) A_p .
\]

For any \( i_1 \) such that \( 0 < i_1 < i_0 \), we consider Chen's waist concentration function \( l_p(i_1, i_0) \):
\[
l_p(i_1, i_0) = \liminf_n \min_{i_1 \leq r \leq i_0} |\partial D_r(p)|_{g_n} .
\]
By [3, Lemma 1], we have that \( \lim_{r \to 0} l_p(i, i_0) = 0 \). Thus, we can choose \( i_1 < i_0 \) small enough such that \( l_p(i_1, i_0) < \varepsilon \), an integer \( N = N(\varepsilon) \) such that
\[
\min_{i_1 \leq r \leq i_0} |\partial D_r(p)|_{g_n} < 2\varepsilon \quad \text{for all} \quad n \geq N ,
\]
and \( r_n \in [i_1, i_0] \) such that
\[
|\partial D_{r_n}(p)|_{g_n} < 3\varepsilon .
\]
It follows that
\[
A_p \leq \mu_{\varphi_n}(D_{i_1}(p)) \leq \mu_{\varphi_n}(D_{r_n}(p)) \leq (1 + \varepsilon) A_p ,
\]
and, by Theorem [5,3] that
\[
\int_{D_{r_n}(p)} |s_{g_n}| d\mu_{g_n} \geq 4\pi - \frac{|\partial D_{r_n}(p)|_{g_n}^2}{\mu_{\varphi_n}(D_{r_n}(p))} > 4\pi - \frac{9\varepsilon^2}{A_p} > 0 .
\]

By integration of the pointwise estimate \([47]\), and the Cauchy-Schwarz inequality, we conclude that
\[
\left( \int_{D_{r_n}(p)} |s_{g_n}| d\mu_{g_n} \right)^2 \leq (\theta_{\varphi_n}(D_{r_n}) + t_{\varphi_n}(D_{r_n}))^2 \leq 2(64\mu_{\varphi_n}^2(D_{r_n}) + \mu_{\varphi_n}(D_{r_n})e_{\varphi_n}(D_{r_n})) ,
\]
and so we have that
\[
\left( \frac{4\pi - \frac{9\varepsilon^2}{A_p}}{(1 + \varepsilon)A_p} \right)^2 \leq \left( \frac{\mu_{\varphi_n}(D_{r_n}(p))}{\mu_{\varphi_n}(D_{r_n}(p))} \right)^2 \leq 2(64\mu_{\varphi_n}^2(D_{r_n}) + e_{\varphi_n}(D_{r_n})) .
\]
Since \( r_n \leq i_0 \), we conclude that
\[
\left( \frac{4\pi - \frac{9\varepsilon^2}{A_p}}{(1 + \varepsilon)A_p} \right)^2 \leq 2(64\mu_{\varphi_n}(D_{i_0}) + e_{\varphi_n}(D_{i_0})) , \quad n \geq N ,
\]
which proves that
\[
2 \liminf_n (64\mu_{\varphi_n}(D_{i_0}(p)) + e_{\varphi_n}(D_{i_0}(p))) > \left( \frac{4\pi - \frac{9\varepsilon^2}{A_p}}{(1 + \varepsilon)A_p} \right)^2 .
\]
If we then let \( \varepsilon \to 0 \), we conclude that
\[
\liminf_n (64\mu_{\varphi_n}(D_{i_0}(p)) + e_{\varphi_n}(D_{i_0}(p))) \geq \frac{8\pi^2}{A_p} ,
\]
and so
\[
64A_p(i_0) + E_p(i_0) \geq \frac{8\pi^2}{A_p} .
\]
The desired results follows by computing the limit as $i_0 \to 0$.

The following two lemmas of X. Chen [8] hold in our context. We provide a detailed proof for the second one of these only. Notice that the validity of the first requires only to have upper bounds control on the sequence of areas of the $M_n$s.

**Lemma 5.5.** [Lemma 3] Let $\{g_n\} \in \mathcal{M}^{C_1,C_2}$ be an energy minimizing sequence in the conformal class of the metric $g$ on $M$. Then,

$$\liminf_n \min_{0 \leq i_1 \leq r \leq i_0} \frac{1}{i_0^2} \int_0^{2\pi} \varphi_n(r \cos \theta, r \sin \theta) d\theta$$

is bounded above for any $[i_1, i_0]$.

**Lemma 5.6.** [Lemma 5] Let $\{g_n\} \in \mathcal{M}^{C_1,C_2}$ be an energy minimizing sequence in the conformal class of the metric $g$ on $M$. If $p$ is not a basic bubble point for the $g_n$s (that is to say, if $A_p = 0$) and $i_0 > 0$ is sufficiently small, there exists a constant $C > 0$ and an integer $N$ such that for $0 < \alpha < 2$ and $r \leq i_0$, we have that

$$\frac{1}{i_0^\alpha} \mu_{\varphi_n}(D_r(p)) < C$$

for all $n \geq N$.

**Proof.** We can choose $i_0$ sufficiently small such that $2A_p(i_0)(64A_p(i_0) + E_p(i_0)) < (2\pi(2 - \alpha))^2$. Given this choice, let $C$ be a constant such that

$$\frac{\mu_{\varphi_n}(D_{i_0}(p))}{i_0^\alpha} < C \quad \text{for all } n.$$

The desired statement holds for this value of $C$.

Indeed, let us consider the function $f_n(r) = \mu_{\varphi_n}(D_r(p)) - Cr^\alpha$. Assume that for all $n$ we have $i_n < i_0$ such that $f_n(i_n) > 0$. Then there must exist $r_n \in (i_n, i_0)$ such that $f_n(r_n) = 0$ and $f'_n(r_n) < 0$:

$$\mu_{\varphi_n}(D_{r_n}(p)) = Cr_n^\alpha, \quad \int_0^{2\pi} e^{\varphi_n} r_n d\theta < C\alpha r_n^{\alpha - 1}.$$

We then conclude that $|\partial D_{r_n}(p)|^2 < 2\pi\alpha Cr_n^\alpha$ because

$$\left(\int_0^{2\pi} e^{\frac{2}{n} \varphi_n} r_n d\theta\right)^2 < \left(\int_0^{2\pi} e^{\varphi_n} r_n d\theta\right) \int_0^{2\pi} r_n d\theta < 2\pi Cr_n^\alpha,$$

and by Theorem [5.3] we see that

$$\int_{D_{r_n}} |s_{g_n}| d\mu_{\varphi_n} > 4\pi - \frac{|\partial D_{r_n}(p)|^2}{\mu_{\varphi_n}(D_{r_n}(p))} > 4\pi - \frac{2\pi C\alpha r_n^\alpha}{Cr_n^\alpha} = 2\pi(2 - \alpha).$$

This contradicts the fact that since $r_n \leq i_0$, by (17), the square of the left side of this expression is bounded above by $2A_p(i_0)(64A_p(i_0) + E_p(i_0))$, and this quantity is bounded above by $(2\pi(2 - \alpha))^2$.

The following results will be used in combination with Lemma [5.4] to show that sequences in $\mathcal{M}^{C_1,C_2}$ have at most finitely many bubble points.

**Lemma 5.7.** Let $\{g_n\} \in \mathcal{M}^{C_1,C_2}$ be an energy minimizing sequence in the conformal class of the metric $g$ on $M$. If $p$ is not a basic bubble point for the $g_n$s, there exists a small neighborhood $\mathcal{O}_p$ of $p$ and a positive constant $C$ such that

$$\sup_n \sup_{q \in \mathcal{O}_p} \varphi_n(q) < C.$$
Proof. If the statement were false, we could modify the sequence of metrics slightly so that \( \varphi_n(p) \to \infty \). We contradict this fact.

Let us choose a small enough disk \( D_{i_0}(p) \) so that Lemma 5.6 applies for some constant \( C \) and integer \( N \). Consider the function \([i_1,i_0] \ni r \mapsto \varphi_n(r \cos \theta, r \sin \theta)\), and for simplicity, set \( q_n(\theta) = (r \cos \theta, r \sin \theta) \). Then we have that

\[
\int_0^{2\pi} (i_0 \partial_r \varphi_n(q_{i_0}(\theta))) d\theta - \int_0^{2\pi} (i_1 \partial_r \varphi_n(q_{i_1}(\theta))) d\theta = \int_{i_0}^{i_1} \int_0^{2\pi} (\partial_r (r \partial_r \varphi_n(r \theta))) d\theta dr
\]

\[
\leq \int_{i_0}^{i_1} \int_0^{2\pi} (r \Delta_\theta \varphi_n d\theta dr)
\]

\[
\leq \int_{i_0}^{i_1} \int_0^{2\pi} |s_{g_n}| d\mu_{g_n},
\]

\[
\leq 2(64\mu_{g_n}^2(A_{i_0,i_1}^p) + \mu_{g_n}(A_{i_0,i_1}^p)E_{g_n}(A_{i_0,i_1}^p))^{\frac{1}{2}},
\]

where \( A_{i_0,i_1}^p \) is the annular region \( D_{i_0}(p) \setminus D_{i_1}(p) \). In obtaining the last two inequalities we have used (45) and (17), respectively. Passing to the limit as \( i_1 \to 0 \), and setting \( r = i_0 \), we obtain that

\[
\int_0^{2\pi} (r \partial_r \varphi_n(q_r(\theta))) d\theta \leq \sqrt{2} \mu_{g_n}^\frac{1}{2}(D_r(p)) (64\mu_{g_n}^2(D_r(p)) + \mu_{g_n}(D_r(p))e_{g_n}(D_r(p)))^{\frac{1}{2}}
\]

\[
\leq \sqrt{2} \mu_{g_n}^\frac{1}{2}(D_r(p)) (64C_1^2 + C_1C_2)^{\frac{1}{2}}.
\]

Thus, if

\[
\psi_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_n(r \cos \theta, r \sin \theta) d\theta,
\]

we have that

\[
| \psi_n(r) - \psi_n(0) | \leq \frac{1}{2\pi} \int_0^r \int_0^{2\pi} u \partial_u \varphi_n d\theta \left| \frac{du}{u} \right| \leq \frac{1}{2\pi} \int_0^r \left( 2(64C_1^2 + C_1C_2) \right)^{\frac{1}{2}} \mu_{g_n}^\frac{1}{2}(D_u(p)) \frac{du}{u}.
\]

By Lemma 5.5 there exists a subsequence of the \( \psi_n \)’s that has bounded limit as \( n \to \infty \). We relabel this subsequence as \( \psi_n \) itself. By Lemma 5.6 the singularity of \( \mu_{g_n}(D_u(p)) \) at \( u = 0 \) is integrable, and the integral is uniformly bounded. These facts yield a contradiction since \( \psi_n(0) = \varphi_n(0) \to \infty \).

\[\square\]

**Lemma 5.8.** Let \( \{g_n\} \in \mathcal{M}^{C_1,C_2} \) be an energy minimizing sequence in the conformal class of the metric \( g \) on \( M \). Suppose that \( C_3 \) is a constant such that

\[
\sup_n \sup_{q \in D(p)} \varphi_n(q) < C_3.
\]

Then for any relatively compact subdomain \( \Omega \subset D(p) \), there exists a constant \( C \) and a constant \( \beta = \beta(C_1,C_2,C_3) \) such that

\[
\sup_{\Omega} \varphi_n < \beta \inf_{\Omega} \varphi_n + C.
\]

**Proof.** By the upper bound on the supremum of \( \varphi_n \) and the \( L^2 \)-bound on the scalar curvature, it follows that \( \varphi_n \in L^2(D(p)) \) relative to the flat metric, with a uniform bound on the \( H^2 \)-norm. If we write \( \varphi_n = u_n + v_n \) with \( \Delta_n \varphi_n = \Delta_n u_n \) in the interior of \( D(p) \), and \( \varphi_n |_{\partial D(p)} = v_n |_{\partial D(p)} \), by elliptic regularity for the Laplacian we see that \( u_n \in H^2(D(p)) \) uniformly, and by the Sobolev embedding theorem, its modulo is bounded by a constant \( C \) independent of \( n \). It follows that the harmonic
function \( v_n = \phi_n - u_n \) is uniformly bounded above, and so by Schauder estimates, given any relatively compact subdomain \( \Omega \subset D(p) \), there exist a constant \( C \) and a constant \( \beta = \beta(C_1, C_2, C_3) \) in \((0, 1)\) such that

\[
\sup_{\Omega} (C - v_n) \leq \frac{1}{\beta} \inf_{\Omega} (C - v_n).
\]

The desired result follows. \( \Box \)

**Proposition 5.9.** Let \( \{g_n = e^{\phi_n} g\} \in \mathcal{M}^{C_1, C_2} \) be an energy minimizing sequence in the conformal class of the metric \( g \) on \( M \). Then for any subsequence of \( g_n \), the embedded disk \( D(p) \) carries at most \( (64 C_1^2 + C_2^2)/(2 \sqrt{2} \pi) \) bubble points, and there exists a subsequence that has at most finitely many bubble points and no additional basic bubble points in \( D(p) \).

**Proof.** By Lemma 5.4, a basic bubble point requires the concentration of a definite amount of area at the point, and so, there can be only finitely many of them. Suppose that \( p_1, \ldots, p_k \) is an enumeration of all of these points. We have that

\[
\sum_{j=1}^{k} A_{p_j} \leq \int_{D(p)} e^{\phi_n} d\mu_0 \leq C_1,
\]

and, by the estimate in Lemma 5.4, that

\[
\frac{8\pi^2}{A_{p_j}} \leq 64 A_{p_j} + E_{p_j}.
\]

Then,

\[
2(64C_1 + C_2) \geq 2 \sum_{j=1}^{k} (64 A_{p_j} + E_{p_j}) \geq \sum_{j=1}^{k} \frac{16\pi^2}{A_{p_j}} \geq \frac{16k^2 \pi^2}{\sum_{j=1}^{k} A_{p_j}} \geq \frac{16k^2 \pi^2}{C_1},
\]

and the desired upper bound for \( k \) follows.

If the sequence \( g_n \) has \( l \) distinct bubble points, and an additional basic bubble point \( p \), by passing to a subsequence we make of \( p \) a bubble point, and the subsequence has now an additional bubble. This process must terminate in finitely many steps, or else we would exceed the bound above for \( k \).

We let \( \mathcal{T}(D(p)) = H^2(D(p)) \cup \{\phi_{-\infty}\} \), where \( H^2(D(p)) \) is the Sobolev space of order two on the domain \( D(p) \) relative to the flat metric, and \( \phi_{-\infty} = -\infty \) is the function that gives rise to the trivial tensor \( g_{-\infty} = e^{\phi_{-\infty}} g = 0 \). We can now state the two key consequences of these results in our context. Their proof can be carried out exactly as in the proofs of [8, Theorems 1, 3], using the lemmas above instead of their analogues in [8].

**Theorem 5.10.** Let \( \{g_n \} \in \mathcal{M}^{C_1, C_2} \) be an energy minimizing sequence in the conformal class of the metric \( g \) on \( M \). Let \( D(p) \) be an embedded disk in \( M \) centered at \( p \) as above where the metric \( g_n \)'s can be represented as \( g_n = e^{\phi_n} g_0 \), \( g_0 = dx^2 + dy^2 \)
the standard flat metric in conformal coordinates on $D(p)$. Then there exists a subsequence $\{g_{n_j}\}$ with at most finitely many bubble points $p_1, \ldots, p_k$ in $D(p)$, $1 \leq k \leq (64C_1^2 + C_1C_2)^{1/2}/(2\sqrt{2}\pi)$, and a metric $\tilde{g} = e^{\tilde{\varphi}}g_0$, $\tilde{\varphi} \in \mathcal{F}_{loc}(D(p) \setminus \{p_1, \ldots, p_k\})$, such that

$$\varphi_{n_j} \to \tilde{\varphi} \text{ in } \mathcal{F}_{loc}(D(p) \setminus \{p_1, \ldots, p_k\}),$$

and if the area and energy concentrations at $p_i$ are $A_{p_i}$ and $E_{p_i}$, respectively, we have that

$$\lim_j \mu_{\varphi_{n_j}}(D(p)) = \mu_{\tilde{\varphi}}(D(p) \setminus \{p_1, \ldots, p_k\}) + \sum_{i=1}^{k} A_{p_i},$$

and

$$\lim_j e_{\varphi_{n_j}}(D(p)) = e_{\tilde{\varphi}}(D(p) \setminus \{p_1, \ldots, p_k\}) + \sum_{i=1}^{k} E_{p_i},$$

respectively. We obtain a subsequence of metrics $\{g_{n_j}\}$ with a finite number of bubble points $\{q_1, \ldots, q_k\} \subset M$, $k \leq (64C_1^2 + C_1C_2)^{1/2}/2\sqrt{2}\pi$, and a metric $\tilde{g} = e^{\tilde{\varphi}}g$ such that $g_{n_j} \to \tilde{g}$ in $\mathcal{F}_{loc}(M \setminus \{q_1, \ldots, q_k\})$, and we have the relations

$$\lim_j \mu_{\varphi_{n_j}}(M) = \mu_{\tilde{\varphi}}(M \setminus \{q_1, \ldots, q_k\}) + \sum_{i=1}^{k} A_{q_i},$$

and

$$\lim_j e_{\varphi_{n_j}}(M) = e_{\tilde{\varphi}}(M \setminus \{q_1, \ldots, q_k\}) + \sum_{i=1}^{k} E_{q_i}.$$

**Theorem 5.11.** Let $\{g_n\} \in \mathcal{M}^{C_1,C_2}$ be a sequence of energy minimizing metrics on the disk $D(p)$ centered at $p$, where the metric $g_n$ can be represented as $g_n = e^{\varphi_n}g_0$, $g_0 = dx^2 + dy^2$ the standard flat metric in conformal coordinates $z = x + iy$ on $D(p)$. Suppose that $p$ is the only bubble point in $D(p)$ for the $g_n$s, that the area and energy concentrations at $p$ are $A_p$ and $E_p$, respectively, and that there exists $\tilde{\varphi}$ such that $\varphi_n \to \tilde{\varphi}$ in $\mathcal{F}_{loc}(D \setminus \{p\})$. Then we can choose a sequence $\{\varepsilon_n \to 0\}$ so that the renormalized uniformized factors $\varphi_n = \varphi_n(\varepsilon_n(x,y)) + \ln \varepsilon_n$ admit a subsequence $\{\phi_{n_j}\}$ with finitely many bubble points $q_1, \ldots, q_k$, $1 \leq k \leq (64C_1^2 + C_1C_2)^{1/2}/(2\sqrt{2}\pi)$, and a conformal factor $\tilde{\phi} \in \mathcal{F}(S^2 \setminus \{\infty, q_1, \ldots, q_k\})$ such that

$$\phi_{n_j} \to \tilde{\phi} \text{ in } \mathcal{F}_{loc}(S^2 \setminus \{\infty, q_1, \ldots, q_k\}),$$

and we have the relations

$$A_p \geq \mu_{\tilde{\phi}}(S^2 \setminus \{q_1, \ldots, q_k\}) + \sum_{i=1}^{k} A_{q_i},$$

and

$$E_p \geq E_{\tilde{\phi}}(S^2 \setminus \{q_1, \ldots, q_k\}) + \sum_{i=1}^{k} E_{q_i},$$

respectively. The renormalize sequence of metrics are of the form $\tilde{g}_n(z) = g_n(\varepsilon_n z + z(p_n))$ where $p_n \to p$, $p_n \in D(p)$ a point where the supremum of the area of $g_n$ in $D(p)$ occurs.
We may now prove a thin-thick decomposition for the limit of any energy minimizing sequence in the conformal class of \((M_n, g_n)\). This is the analogue of the Cheeger-Gromov theorem [7] proven under a weaker integral condition on the curvature tensors.

**Theorem 5.12.** Let \((M_n, g_n)\) be any element of a \(\mathcal{F}\)-minimizing sequence \(\{(M_n, g_n)\}\) satisfying conditions M1-M5. We set \(C_1 = 2\pi(d(d - 1) + 2)\), let \(C_2\) be any uniform upper bound of all the \(E_{\epsilon_n}(M_n)\)s, and consider any locally convergent sequence of energy minimizing metrics \(\{g_m^n\}\) in the conformal class of \(g_n\). Given any \(\epsilon > 0\), there exist integers \(N_{\text{thin}}\) and \(N_{\text{thick}}\), depending only on \(\epsilon\) and \((64C_1^2 + C_1C_2)^{\frac{1}{2}}(2\sqrt{2}\pi)^{-1}\), and a decomposition of \((M_n, g_m^n)\) into thick and thin components

\[M_n = \bigcup_{j=1}^{N_{\text{thick}}} M_j^{\text{thick}} \cup \bigcup_{j=1}^{N_{\text{thin}}} M_j^{\text{thin}},\]

such that

1. Each \((M_j^{\text{thick}}, g_m^n|_{M_j^{\text{thick}}})\) converges in \(H^2_{\text{loc}}\) to a metric on \(S^2\) with a finite number of small discs deleted whose sizes can be made as small as possible.
2. Each \(M_j^{\text{thick}}\) is connected, but no any two of them forms a connected subset of \(M_n \setminus \bigcup_{j=1}^{N_{\text{thick}}} M_j^{\text{thin}}\).
3. Each \((M_j^{\text{thin}}, g_m^n|_{M_j^{\text{thin}}})\) is homeomorphic to a cylinder \(\mathbb{S}^1 \times (a, b)\), and the length of any loop \(\mathbb{S}^1 \times p\), \(a < p < b\) is strictly less than \(\epsilon\).

**Corollary 5.13.** Exactly \(d\) of the thick components \(M_j^{\text{thick}}\) are homeomorphic to a homologically nontrivial sphere in \(\mathbb{P}^2(\mathbb{C})\) with a finite number of points deleted, and all of the remaining thick components as well as all of the thin components bound in \(\mathbb{P}^2(\mathbb{C})\).

**Proof.** We can shrink each of the thin components to produce in the limit a homology representative of \([S_d]\) in \(\mathbb{P}^2(\mathbb{C})\) consisting of 2-spheres any two of which intersect at a point. For dimensional reasons, these spheres can be made to intersect transversally. The result follows. \(\square\)

In the renormalization process of Theorem 5.11, a neck is created for each of the bubbles of the initial one. The length of a circle in these necks is bounded above by \(\epsilon\), and as \(n\) increases to \(\infty\), the conformal distance between the bounding circles approaches \(\infty\) while part of the interior of the neck collapses into a line. The collapsing occurs while either the scalar curvature or the diameter of the neck remain bounded, and the size of the neck can be made arbitrarily small by letting the lengths of the bounding circles go to zero.

**Corollary 5.14.** There exists an energy minimizing sequence \(\{g_m^n\}\) in the conformal class of \(g_n\) such that \((M_n, g_m^n)\) converges to a transversal intersection of \(d\) complex lines each with its induced Fubini-Study metric, and a collection of 2-spheres intersecting transversally each of which bounds in \(\mathbb{P}^2(\mathbb{C})\), where the energy of the limit metric over the portion of the manifold that bounds is equal to the barrier \(b_n\) of the metric \(g_n\).

**Proof.** We deform conformally each of the \(d\) nonbounding spheres into complex lines, pushing their initial excess area through the necks onto the bounding spheres. Complex lines are totally geodesic submanifolds of \(\mathbb{P}^2(\mathbb{C})\), and so they contribute nothing to the energy, which must then come from the remaining portion of the limiting manifold. But the limiting manifold realizes the infimum of the energy, which is given by the barrier \(b_n\) of the conformal class of \((M_n, g_n)\). \(\square\)
5.3. Precompactness properties of $\mathcal{I}$ over $\mathcal{M}_d([S_d])$. We now study the asymptotics of the functional $\mathcal{I}$ over representatives of $[S_d] \in H_2(\mathbb{P}^2(\mathbb{C}); \mathbb{Z})$ as these bubble off into the transverse intersections of $d$ complex lines that absolutely minimize the energy functional over all conformal classes. For that, first we look in further detail at the resolution of singularities of the latter introduced in §5.1.

We consider an immersed complex singular representative $\Sigma$ of $[S_d]$ whose singularities are modeled in local complex coordinates $(z_1, z_2)$ by

$$z_1 z_2 = 0.$$ 

If $z_1 = w_1 + iw_2$ and $z_2 = w_1 - iw_2$, then $z_1 z_2 = w_1^2 + w_2^2$, and so the equation above is just the zero locus of the quadratic complex function $q(w) = w_1^2 + w_2^2$. We regularize the singularity by considering

$$z_1 z_2 = q(w) = w_1^2 + w_2^2 = \varepsilon$$

for $0 < \varepsilon \ll 1$, and apply the surgery procedure discussed in §5.1 to the linked discs $L = \{ z = (z^1, z^2) : z^1 z^2 = 0, |z| < r \}$ to obtain a representative $\Sigma'$ of the class $[\Sigma]$ that is also smooth in a neighborhood of $z = 0$.

Notice that if $w_j = x_j + iy_j$, $j = 1, 2$, $x_j, y_j$ real, then the singular fiber $q(w) = 0$ intersects the Euclidean sphere $S^3_r = \{ w = (w_1, w_2) : |w| = r \}$ in the $r/\sqrt{2}$ Euclidean sphere bundle of the cotangent space $T^*S^1_{r/\sqrt{2}}$ of the Euclidean circle $S^1_{r/\sqrt{2}}$, as we must have the relations

$$\begin{align*}
\sum_{j=1}^2 x_j^2 - y_j^2 &= 0, \\
\sum_{j=1}^2 x_j y_j &= 0, \\
\sum_{j=1}^2 x_j^2 + y_j^2 &= r^2.
\end{align*}$$

Thus, the two loops $L = \{ z_1 z_2 = 0 \} \cap S^3_r$ are linked, and have linking number 1, and the desingularization procedure replaces the intersecting discs $\{ z_1 z_2 = 0 : |z| \leq r \}$ by the minimal genus Seifert surface for the Hopf link, an annulus with boundary $L$.

Suppose now that $\Sigma$ has singularities of the type above relative to an affine system $(z_1, z_2)$ of local coordinates in $\mathbb{P}^2(\mathbb{C})$. We look at the jump in the extrinsic scalar curvature of $\Sigma$ arising from regularizing the discs $D_r = B_r(0) \cap \{ z_1 z_2 = 0 \} \subset \Sigma$ to $D_{r,\varepsilon} = B_r(0) \cap \{ z_1 z_2 = \varepsilon \}$ according to the procedure above. We denote by $\Sigma_\varepsilon$ the embedded representative of $[\Sigma]$ so obtained. We study the quantities

$$\Theta_r = \int_{D_r} 2K(e_1, e_2) d\mu_g = 8 \int_{D_r} d\mu_g |_{D_r},$$

and

$$\Theta_{r,\varepsilon} = \int_{D_{r,\varepsilon}} 2K(e_1, e_2) d\mu_g = 8 \int_{D_{r,\varepsilon}} d\mu_g |_{D_{r,\varepsilon}},$$

respectively. We denote by $\Theta(\Sigma)$ and $\Theta(\Sigma_\varepsilon)$ the total extrinsic scalar curvatures of the minimal current $\Sigma$ and the embedded submanifold $\Sigma_\varepsilon$, respectively.

The Fubini-Study metric restricted to either one of the discs in $D_r$ is given by

$$\omega = \frac{i}{2(1 + |z|^2)^2} dz \wedge d\bar{z},$$
while its restriction to $D_{r,ε}$ is given by

$$\omega_ε = \frac{i}{2(1 + |z|^2 + \frac{ε^2}{|z|^2})^2} \left(1 + 4 \frac{ε^2}{|z|^2} + \frac{ε^2}{|z|^4}\right) dz \wedge d\bar{z}.$$ 

We then have that

$$\Theta_r = 8 \int_{D_r} d\mu_g = 8\pi \frac{2π^2}{2 + r^2},$$

and

$$\Theta_{r,ε} = 8 \int_{D_{r,ε}} \frac{i}{2(1 + |z|^2 + \frac{ε^2}{|z|^2})^2} \left(1 + 4 \frac{ε^2}{|z|^2} + \frac{ε^2}{|z|^4}\right) dz \wedge d\bar{z} = 8\pi + 8\pi \frac{π^2}{1 + r^2} + o(ε),$$

respectively.

**Lemma 5.15.** For the embedded representative $Σ_ε$ of $[Σ]$, we have that

$$\lim_{ε \to 0} \mathcal{T}(Σ_ε) = 4πd(d - 1), \quad \lim_{ε \to 0} \Psi(Σ_ε) = 0, \quad \lim_{ε \to 0} \Theta(Σ_ε) = Θ(Σ) = 8\pi d.$$

**Proof.** The minimal current $Σ$ is the weak limit of the embedded current $Σ_ε$, and so $Ψ(Σ_ε) \to 0$ from above as $ε \to 0$. Letting $r \to \infty$ in the asymptotic expansions found above, we see that $Θ(Σ_ε) - Θ(Σ)$ approaches zero as $ε \to 0$.

The statement about $\mathcal{T}(Σ_ε)$ then follows from the identity

$$\mathcal{T}(M) = Θ(M) + Ψ(M) - \int_M s_g d\mu_g.$$ 

We just need to observe that the regularization $Σ_ε$ of $Σ$ at each of its $d$-intersection points results into a surface of genus $\left(\frac{n - 1}{2}\right) = \frac{(d-1)(d-2)}{2}$, and so applying the Gauss-Bonet theorem, we obtain that

$$\mathcal{T}(M_ε) = Θ(M_ε) + Ψ(M_ε) - 8\pi \left(1 - \frac{(d - 1)(d - 2)}{2}\right) \to 4\pi d(d - 1).$$

This finishes the proof. □

We now have the following:

**Theorem 5.16.** Over $\mathcal{M}_{[S_d]}(\mathbb{P}^2(\mathbb{C}))$, the functionals $[25]$ is globally bounded below,

$$\mathcal{M}_{[S_d]}(\mathbb{P}^2(\mathbb{C})) \ni Σ \mapsto \mathcal{T}(Σ) = \int_Σ \tilde{q}(α, α)d\mu_g ≥ 4πd(d - 1),$$

and the lower bound is achieved if, and only if, $Σ = S_d$ is an algebraic curve of degree $d$.

**Proof.** Consider a minimizing sequence $(M_n, g_n)$ satisfying M1-M5. By M3, the topology of all the $M_n$s is fixed and equal to the topology of a fixed surface $Σ$, and by Theorem 5.12, we get a decomposition of $(M_n, g_n) = (f_n(Σ), g_n)$ as a union of a fixed number of topological spheres tubed together by cylinders, with exactly $d$ of the spheres nontrivial in homology. It follows that $t$ cannot strictly less than $4πd(d - 1)$. For it if were, then then from some sufficiently large $N$ on, the barrier $b_n = \mathcal{T}(M_n)/μ_{g_n}$ would be strictly less than the barrier of a complex curve of degree $d$, which cannot be by Corollary 5.14 and Lemma 5.15. Since each of the bounding spheres can be collapsed into a cylinder with negligible contribution to the energy and to $\mathcal{T}$ as $n \to ∞$, at $\mathcal{T}(M_n) \to t$, the areas $μ_{g_n}$ must converge to $πd$, the area of a complex curve of degree $d$ in $\mathbb{P}^2(\mathbb{C})$. It follows then that if $(M_n, g_n)$
develops singularities as $n \to \infty$, the value of $\mathcal{T}(M_n)$ becomes infinitesimally close to $4\pi d(d-1)$ but not quite exactly that value. On the other hand, by Proposition 4.9, the value of $\mathcal{T}$ for any complex curve of degree $d$ is exactly $4\pi d(d-1)$. Thus, the infimum $t$ is achieved, and the minimizing sequence $(M_n, g_n)$ either converges to a curve of degree $d$ in $\mathbb{P}^2(\mathbb{C})$, or it degenerates into an intersection of $d$-complex lines union perhaps additional line segments tying two of these lines together, segments that could be homotoped away to a point. \hfill \Box

**Theorem 5.17.** (Kronheimer-Mrowka [17]) Let $\Sigma$ be a connected oriented smooth embedded surface in $\mathbb{C}\mathbb{P}^2$ that represents the homology class of an algebraic curve of degree $d$. If $g_\Sigma$ is its genus, we have that

$$g_\Sigma \geq \frac{(d-1)(d-2)}{2},$$

and the equality is achieved if $\Sigma$ is a complex curve in $\mathbb{C}\mathbb{P}^2$.

**Proof.** If there were an embedded representative of $S_d$ of genus less than $\left(\frac{n-1}{2}\right)$, then by minimizing the energy in the conformal class of such a representative, and then collapsing the spheres that bound produced by Theorem 5.12, we would obtain a representative of the class $[S_d]$ with a barrier strictly less than the barrier of a complex curve of degree $d$, which cannot be. \hfill \Box

6. **Homotopic submanifolds of varying critical levels for $S_4$**

In their alternative to compactification, Randall and Sundrum [25] consider a three brane in a five dimensional bulk space, and quantize the system and treat the non-normalizable modes by introducing a second brane at a certain distance from the first. Brane directions and time yield space-time slices. The combined branes are to be placed at the boundaries of a fifth dimension. The action function they consider is the sum of suitable multiples of the volume and total scalar curvature in space-time, together with boundary contributions from the branes. Then they find a warped product solution to Einstein’s equation on the five dimensional bulk that carries two space-time slices that are invariant under the Poincaré group, the effective action at each of these being different from each other. In this section we use the functional (23) to exhibit examples with the same flavour as that exhibited by the work of Randall and Sundrum.

Indeed, we study critical points of (23) that are localized in a particular direction, and thus that two different critical points can be homotopically deformed into each other through embedded submanifolds. In our examples, the fixed backgrounds are Riemannian manifolds of dimension four, but we can take their metric product with a fifth time direction. After doing so, the said critical points multiplied by time become the analogues of the brane-time slices of Randall and Sundrum, and the different critical values correspond to these slices' different effective actions. In each of our last two examples here, the second critical point occurs asymptotically at $\infty$ and is flat. The “nonasymptotic” critical points of all of our examples are neither critical points of (6) nor of (11) separately.

By Theorem 3.10, the critical point equation of (23) in the codimension one case is given by

$$2k_i K_\bar{g}(e_i, \nu) - 2h \sum_i K_\bar{g}(e_i, \nu) - 2e_i(r_\bar{g}(e_i, \nu)) + 2 \text{trace} A_{\nu}^3 - 3 \|\alpha\|^2 h + h^3 = 0,$$
where $\nu$ is a normal field along $M \subset \tilde{M}, \{e_1, \ldots, e_n\}$ is the orthonormal frame of principal directions with associated principal curvatures $k_1, \ldots, k_n$, and $r_3$ is the Ricci curvature tensor of $\tilde{g}$. If the background manifold is Einstein, this critical point equation simplifies to

$$2 \sum_i k_i K_{\tilde{g}}(e_i, \nu) - 2h \sum_i K_{\tilde{g}}(e_i, \nu) + 2\text{trace}_{\tilde{g}} A^3 - 3\|\alpha\|^2 h + h^3 = 0. \tag{51}$$

In our examples with two isotopic solutions to this equation, the localization occurs in the normal direction.

6.1. Cartan’s isoparametric families. Let us reconsider the isoparametric families $M_t^{2n} \subset \mathbb{S}^{2n+1}$ of Example [3.7] (re)discovered by Nomizu [21]. The manifold $M_t^{2n}$ has principal curvatures $k_1(t)$ and $k_2(t)$ each with multiplicity one, and $k_3(t)$ and $k_4(t)$ each with multiplicity $(n-1)$. Thus, the mean curvature function is given by $h(t) = k_1(t) + k_2(t) + (n-1)(k_3(t) + k_4(t))$, while $a(t) = \text{trace} A^3 = k_1^2(t) + k_2^2(t) + (n-1)(k_3^2(t) + k_4^2(t))$, and $M_t^{2n}$ is a critical point of (23) if, and only if, $t \in (0, \pi/4)$ is a root of the equation

$$2(1-2n)h(t) + 2a(t) - 3h(t)((k_1^2(t) + k_2^2(t) + (n-1)(k_3^2(t) + k_4^2(t)) + h^3(t) = 0. \tag{52}$$

For $n = 1$ this equation has no solution on $(0, \pi/4)$. For $n = 2$, there is only one solution in the indicated range, and the corresponding critical submanifold $M_t$ is the austere hypersurface of $\mathbb{S}^3$ already pointed out in Example [3.7]. The critical value is 12 times the volume of $M_t$. For $n = 3$, there is also exactly one solution in the said range, $t = 0.5268183350$, but the corresponding critical point $M_t^3$ is not a minimal hypersurface of $\mathbb{S}^7$. The critical value is $19.71086118$ times the volume of $M_t$.

For $n \geq 4$ the equation above has two solutions in the range $(0, \pi/4)$, and their critical values are real numbers of opposite signs. For instance, for $n = 4$ (so the background bulk is of dimension 10, space and time together), one root is $t = 0.1830436696$, and the corresponding critical value for the hypersurface $M_t^8 \subset \mathbb{S}^9$ is $-131.2969104$ times the volume. The other root occurs at $t = 0.5770248421$, and the critical value of the associated $M_t^8$ is $27.29691039$ times the volume. Each of these two critical points $M_t^8$ inherits a metric of positive scalar curvature.

Remark 6.1. The example above is unsatisfactory in that the homology classes represented by the exhibited critical submanifolds are trivial. Examples otherwise would require a background with nontrivial homology.

6.2. The Schwarzschild background. The Schwarzschild metric is a Ricci flat metric on the total space of the trivial bundle $\mathbb{S}^2 \times \mathbb{R}^2 \to \mathbb{S}^2$. It can be cast as a doubly warped product on $(-\infty, \infty) \times S^1 \times S^1$,

$$\tilde{g} = dr^2 + \psi^2(r)ds^2_1 + \psi^2(r)ds^2_2,$$

with suitable asymptotics at $r = 0$ and $r = \infty$, respectively. We have that

$$\psi^2 = 1 - \frac{\beta}{\psi}, \quad \psi = \frac{1}{2\beta} \varphi,$$

where $\psi(0) = \beta > 0$. The solution to the equation satisfied by $\psi$ exists for all $r \in (-\infty, \infty)$; it is an even function of $r$, and $\psi(r)/|r| \to 1$ as $r \to \pm\infty$.

We take $\nu = \partial_r$, the gradient of the distance function $r$, and $\{e_1, e_2, e_3\}$ an orthonormal set with $e_1$ tangent to the $S^1$ factor, and $e_2$ and $e_3$ tangent to the $S^2$
factor. The principal curvatures of the $S^1 \times S^2$ hypersurfaces defined by setting \( r \) to a constant are

\[
k_1 = -\frac{\dot{\varphi}}{\varphi}, \quad k_2 = k_3 = -\frac{\dot{\psi}}{\psi}.
\]

We have that

\[
\tilde{K}_{\tilde{g}}(e_1, \nu) = -\dddot{\varphi}/\varphi, \quad \tilde{K}_{\tilde{g}}(e_2, \nu) = K_{\tilde{g}}(e_3, \nu) = -\dddot{\psi}/\psi.
\]

Notice that \( \sum_i K_{\tilde{g}}(e_i, \nu) = 0 \) follows on the basis of these expressions for the sectional curvatures, and (52), a fact consistent with the Ricci flatness of the metric.

The critical hypersurfaces slices \( r = r_0 \) for (23) are determined by the roots of the function

\[
C = 2 \left( \frac{\dddot{\varphi}}{\varphi} \varphi + 2 \frac{\dddot{\psi}}{\psi} \psi \right) - 6 \frac{\dot{\psi}}{\varphi} \left( \frac{\dddot{\psi}}{\psi} \psi \right)^2.
\]

This is an odd function of \( r \) that is positive to the immediate right of 0. Using the asymptotic behaviour of \( \psi \), we see that it becomes negative at some point, and so it must have a zero \( r_0 > 0 \), which yields a critical slice \( S^1 \times S^2 \) for the functional (23) with metric \( \varphi^2(r_0) ds^2_1 + \sigma^2(r_0) ds^2_2 \), the \( S^1 \)-factor being flat while the \( S^2 \)-factor is Einstein of constant \((-\dddot{\psi}/\psi + (1 - \dot{\psi}^2)/\psi^2 - \dot{\psi}\dddot{\psi}/\psi^2) |_{r=r_0} \). Thus, this critical slice is provided with a geometric structure modeled by \( \mathbb{R} \times S^2 \), one of the eight such structures in Thurston’s geometrization program.

Further, as \( r \to \infty \), the function \( C \) increases up to 0, and we obtain asymptotically a second critical point for (23) that looks like \( S^1 \times \mathbb{R}^2 \) with the flat product metric, and where the metric in the first factor is \( (2/\beta)^2 ds^2_1 \). The geometric model of this critical point is \( \mathbb{R}^3 \).

The critical hypersurface at small distance is highly curved in comparison with the critical point at large distance, which is a scalar flat space. This behaviour is reminiscent of some of the features of \textit{spacetime foam}, which is taken to be the building block of the universe in some theories of gravity. Notice that our functional (23) is just a linear combination of the total scalar curvature and a functional equivalent (in the sense that it is bounded above and below by a suitable multiple of it) to the volume form of the submanifold, with the latter being exactly a constant times the volume in the case of hypersurfaces defined by setting \( r \) equal to a constant.

By the Ricci flatness of the background metric, (41) implies that the scalar curvature of the critical slice determined by \( r = r_0 \) is given by the value of the function

\[
S = 2 \frac{\dddot{\psi}}{\psi} \left( \frac{\dddot{\psi}}{\psi} + 2 \frac{\dddot{\varphi}}{\varphi} \right)
\]

at \( r = r_0 \). Thus, the first of the critical points has positive scalar curvature. The asymptotic critical point at \( \infty \) has zero scalar curvature. In fact, \( S(r) \approx (2/r)(1/r + \beta/r^2) \) as \( r \to \infty \).

6.3. The Eguchi-Hanson background. The Eguchi-Hanson metric is a family of scaled Ricci flat metric on the total space of the tangent bundle \( T S^2 \to S^2 \) to the 2-sphere. If we use the coframe \( \{ \sigma^1, \sigma^2, \sigma^3 \} \) that identifies \( S^3 \) as the Lie group \( SU(2) \), the metric can be written as a doubly warped product

\[
\tilde{g} = dr^2 + \varphi^2(r)(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2,
\]
where
\[ \dot{\varphi} = \psi, \quad \dot{\varphi}^2 = 1 - k\varphi^{-4}, \]
subject to the boundary conditions \( \varphi(0) = k^\frac{1}{4}, \dot{\varphi}(0) = 0, \psi(0) = 0, \) and \( \ddot{\psi}(0) = 2. \)

The parameter \( k \) can be changed via scaling.

We take \( \nu = \partial_r \), the gradient of the distance function \( r \), and \( \{ e_1, e_2, e_3 \} \) dual to \( \{ \varphi \psi \sigma^1, \varphi \sigma^2, \varphi \sigma^3 \} \). The principal curvatures of the hypersurfaces defined by setting \( r \) to a constant are given by
\[
\kappa_1 = -\frac{\partial_r (\varphi \psi)}{\varphi \psi}, \quad \kappa_2 = \kappa_3 = -\frac{\partial_r \varphi}{\varphi},
\]
while
\[
K_{\tilde{g}}(e_1, \nu) = -\frac{\partial_r^2 (\varphi \psi)}{\varphi \psi}, \quad K_{\tilde{g}}(e_2, \nu) = K_{\tilde{g}}(e_3, \nu) = -\frac{\partial_r^2 \varphi}{\varphi}.
\]

The critical hypersurfaces slices \( r = r_0 \) for (23) are determined then by the roots of the function
\[
C = 2 \left( \frac{\partial_r (\varphi \psi)}{\varphi \psi} \frac{\partial_r^2 (\varphi \psi)}{\varphi \psi} + 2 \frac{\partial_r \varphi}{\varphi} \frac{\partial_r^2 \varphi}{\varphi} \right) - 6 \frac{\partial_r (\varphi \psi)}{\varphi \psi} \left( \frac{\partial_r \varphi}{\varphi} \right)^2.
\]

This function is positive to the immediate right of 0, and by the asymptotic behaviour of \( \psi \), it becomes negative at some point, and increases to 0 as \( r \to \infty \). So the function must have a a zero \( r_0 > 0 \), yielding a critical slice for the functional (23) with metric \( \varphi \psi^2(r_0)(\varphi \psi^2(r_0)(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2) \), a rescaled Berger sphere, and an asymptotic critical slice at \( r = \infty \). The asymptotic critical point is an \( \mathbb{P}^3(\mathbb{R}) \). The cone on it as a base has the asymptotic flat metric \( r^2((\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2) \), and the metric on the asymptotic critical slice is induced by it. As in the previous example, the critical hypersurface at small distance is highly curved, while the critical point at large distance is a scalar flat space.

The scalar curvature of a critical slice in the metric induced by the background Eguchi-Hanson metric is given by the value of the function
\[
S = 2 \frac{\partial_r \varphi}{\varphi} \left( \frac{\partial_r \varphi}{\varphi} + 2 \frac{\partial_r (\varphi \psi)}{\varphi \psi} \right)
\]
at \( r = r_0 \). Thus, the first of the critical points has positive scalar curvature, while the asymptotic critical point at \( \infty \) has zero scalar curvature, as already observed. In fact, \( S(r) \approx \frac{1}{2r}(3r + 4k/r^3) \) as \( r \to \infty \).

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