AN OVERVIEW OF THE KEPLER CONJECTURE

THOMAS C. HALES

1. Introduction and Review

The series of papers in this volume gives a proof of the Kepler conjecture, which asserts that the density of a packing of congruent spheres in three dimensions is never greater than \( \frac{\pi}{\sqrt{18}} \approx 0.74048 \ldots \). This is the oldest problem in discrete geometry and is an important part of Hilbert’s 18th problem. An example of a packing achieving this density is the face-centered cubic packing.

A packing of spheres is an arrangement of nonoverlapping spheres of radius 1 in Euclidean space. Each sphere is determined by its center, so equivalently it is a collection of points in Euclidean space separated by distances of at least 2. The density of a packing is defined as the lim sup of the densities of the partial packings formed by spheres inside a ball with fixed center of radius \( R \). (By taking the lim sup, rather than lim inf as the density, we prove the Kepler conjecture in the strongest possible sense.) Defined as a limit, the density is insensitive to changes in the packing in any bounded region. For example, a finite number of spheres can be removed from the face-centered cubic packing without affecting its density.

Consequently, it is not possible to hope for any strong uniqueness results for packings of optimal density. The uniqueness established by this work is as strong as can be hoped for. It shows that certain local structures (decomposition stars) attached to the face-centered cubic (fcc) and hexagonal-close packings (hcp) are the only structures that maximize a local density function.

Although we do not pursue this point, Conway and Sloane develop a theory of tight packings that is more restrictive than having the greatest possible density [CoSl95]. An open problem is to prove that their list of tight packings in three dimensions is complete.

The face-centered cubic packing appears in Diagram 1.1.
2. Early History, Hariot, and Kepler

The study of the mathematical properties of the face-centered cubic packing can be traced back to a Sanskrit work composed around 499 CE. I quote an extensive passage from the commentary that K. Ploker has made about the formula for the number of balls in triangular piles [Plo00]:

The excerpt below is taken from a Sanskrit work composed around 499 CE, the Āryabhaṭīya of Āryabhaṭa, and the commentary on it written in 629 CE by Bhāskara (I). The work is a compendium of various rules in mathematics and mathematical astronomy, and the results are probably not due the Āryabhaṭa himself but derived from an earlier source: however, this is the oldest source extant for them. (My translation’s from the edition by K. S. Shukla, The Āryabhaṭīya of Āryabhaṭa with the Commentary of Bhāskara I and Somesvarara, New Delhi: Indian National Science Academy 1976; my inclusions are in square brackets. There is a corresponding English translation by Shukla and K. V. Sarma, The Āryabhaṭīya of Āryabhaṭa, New Delhi: Indian National Science Academy 1976. It might be easier to get hold of the earlier English translation by W. E. Clark, The Āryabhaṭīya of Āryabhaṭa, Chicago: University of Chicago Press, 1930.)

 Basically, the rule considers the series in arithmetic progression \( S_i = 1 + 2 + 3 + \ldots + i \) (for whose sum the formula is known) as the number of objects in the \( i \)th layer of a pile with a total of \( n \) layers, and specifies the following two equivalent formulas for the “accumulation of the pile” or \( \sum_{i=1}^{n} S_i \):

\[
\sum_{i=1}^{n} S_i = \frac{n(n+1)(n+2)}{6},
\]

\[
\sum_{i=1}^{n} S_i = \frac{(n+1)^3 - (n+1)}{6}.
\]

What he says is this:

\textit{Āryabhaṭīya}, Gaṇitapāda 21:

For a series [lit. “heap”] with a common difference and first term of 1, the product of three [terms successively] increased by 1 from the total, or else the cube of [the
Bhāskara’s commentary on this verse:

[This] heap [or] series is specified as having one for its common difference and initial term. This same series with one for its common difference and initial term is said to be “heaped up.” “The product of three [terms successively] increased by one from the total” of this so-called heaped-up “series with one for its common difference and initial term”: i.e., the product of three terms, starting from the total and increasing by one. Namely, the total, that plus one, and [that] plus one again. That can be stated as follows: the total, that plus one, and that total plus two. The product of those three divided by 6 is the “solid heap,” the accumulation of the series. Now another method: The cube of the root equal to that [total] plus one is diminished by its root, and divided by 6: thus it follows. “Or else”: [i.e.], the cube of that root plus one, diminished by its own root, divided by 6, is the “solid heap.” Example: Series with 5, 8, and 14 respectively for their total layers: tell me [their] triangular-shaped piles. In order, the totals are 5, 8, 14. Procedure: Total 5. This plus one: 6. This plus one again: 7. Product of those three: 210. This divided by 6 is the accumulation of the series: 35. [He goes on to give the answers for the second two cases, but you doubtless get the picture.] – K. Plofker

The modern mathematical study of spheres and their close packings can be traced to T. Hariot. Hariot’s work – unpublished, unedited, and largely undated – shows a preoccupation with sphere packings. He seems to have first taken an interest in packings at the prompting of Sir Walter Raleigh. At the time, Hariot was Raleigh’s mathematical assistant, and Raleigh gave him the problem of determining formulas for the number of cannonballs in regularly stacked piles. In 1591 he prepared a chart of triangular numbers for Raleigh. Shirley, Hariot’s biographer, writes,

Obviously, this is a quick reference chart prepared for Raleigh to give information on the ground space required for the storage of cannon balls in connection with the stacking of armaments for his marauding vessels. The chart is ingeniously arranged so that it is possible to read directly the number of cannon balls on the ground or in a pyramid pile with triangular, square, or oblong base. All of this Harriot had worked out by the laws of mathematical progression (not as Miss Rukeyser suggests by experiment), as the rough calculations accompanying the chart make clear. It is interesting to note that on adjacent sheets, Harriot moved, as a mathematician naturally would, into the theory of the sums of the squares, and attempted to determine graphically all the possible configurations that discrete particles could assume – a study which led him inevitably to the corpuscular or atomic theory of matter originally deriving from Lucretius and Epicurus. [Shi83,p.242]

Hariot connected sphere packings to Pascal’s triangle long before Pascal introduced the triangle. See Diagram 2.1.
Hariot was the first to distinguish between the face-centered cubic and hexagonal close packings [Mas66,p.52].

Kepler became involved in sphere packings through his correspondence with Hariot in the early years of the 17th century. Kargon writes, in his history of atomism in England,

Hariot’s theory of matter appears to have been virtually that of Democritus, Hero of Alexandria, and, in a large measure, that of Epicurus and Lucretius. According to Hariot the universe is composed of atoms with void space interposed. The atoms themselves are eternal and continuous. Physical properties result from the magnitude, shape, and motion of these atoms, or corpuscles compounded from them.

Probably the most interesting application of Hariot’s atomic theory was in the field of optics. In a letter to Kepler on 2 December 1606 Hariot outlined his views. Why, he asked, when a light ray falls upon the surface of a transparent medium, is it partially reflected and partially refracted? Since by the principle of uniformity, a single point cannot both reflect and transmit light, the answer must lie in the supposition that the ray is resisted by some points and not others.
“A dense diaphanous body, therefore, which to the sense appears to be continuous in all parts, is not actually continuous. But it has corporeal parts which resist the rays, and incorporeal parts vacua which the rays penetrate...”

It was here that Hariot advised Kepler to abstract himself mathematically into an atom in order to enter ‘Nature’s house’. In his reply of 2 August 1607, Kepler declined to follow Hariot, ad atomos et vacua. Kepler preferred to think of the reflection-refraction problem in terms of the union of two opposing qualities – transparency and opacity. Hariot was surprised. “If those assumptions and reasons satisfy you, I am amazed.” [Kar66,p.26]

Despite Kepler’s initial reluctance to adopt an atomic theory, he was eventually swayed, and in 1611 he published an essay that explores the consequences of a theory of matter composed of small spherical particles. Kepler’s essay was the “first recorded step towards a mathematical theory of the genesis of inorganic or organic form” [Why66,p.v].

Kepler’s essay describes the face-centered cubic packing and asserts that “the packing will be the tightest possible, so that in no other arrangement could more pellets be stuffed into the same container.” This assertion has come to be known as the Kepler conjecture. The purpose of this collection of papers is to give a proof of this conjecture.

3. History

The next episode in the history of this problem is a debate between Isaac Newton and David Gregory. Newton and Gregory discussed the question of how many spheres of equal radius can be arranged to touch a given sphere. This is the three-dimensional analogue of the simple fact that in two dimensions six pennies, but no more, can be arranged to touch a central penny. This is the kissing-number problem in \( n \)-dimensions. In three dimensions, Newton said that the maximum was 12 spheres, but Gregory claimed that 13 might be possible.

Newton was correct. In the 19th century, the first papers claiming a proof of the kissing-number problem appeared in [Ben74], [Gun75], [Hop74]. Although some writers cite these papers as a proof, they are hardly rigorous by today’s standards. Another incorrect proof appears in [Boe52]. The first proper proof was obtained by B. L. van der Waerden and Schütte in 1953 [Sch53]. An elementary proof appears in Leech [Lee56]. The influence of van der Waerden, Schütte, and Leech upon the papers in this collection is readily apparent. Although the connection between the Newton-Gregory problem and Kepler’s problem is not obvious, L. Fejes Tóth in 1953, in the first work describing a strategy to prove the Kepler conjecture, made a quantitative version of the Gregory-Newton problem the first step [Fej53].

The two-dimensional analogue of the Kepler conjecture is to show that the honeycomb packing in two dimensions gives the highest density. This result was established in 1892 by Thue, with a second proof appearing in 1910 ([Thu92], [Thu10]). A number of other proofs have appeared since then. Three are particularly notable. Rogers’s proof generalizes to give a bound on the density of packings in any dimension [Rog58]. A proof by L. Fejes Tóth extends to give bounds on the density of packings of convex disks [Fej50]. A third proof, also by L. Fejes Tóth, extends to non-Euclidean geometries [Fej53]. Another early proof appears in [SeM44].

In 1900, Hilbert made the Kepler conjecture part of his 18th problem [Hil01]. Milnor, in his review of Hilbert’s 18th problem, breaks the problem into three parts
1. Is there in \( n \)-dimensional Euclidean Space ... only a finite number of essentially different kinds of groups of motions with a [compact] fundamental region?

2. Whether polyhedra also exist which do not appear as fundamental regions of groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all [Euclidean] space is possible?

3. How can one arrange most densely in space an infinite number of equal solids of given form, e.g. spheres with given radii . . . , that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as great as possible?

Writing of the third part, Milnor states,

For 2-dimensional disks this problem has been solved by Thue and Fejes Tóth, who showed that the expected hexagonal (or honeycomb) packing of circular disks in the plane is the densest possible. However, the corresponding problem in 3 dimensions remains unsolved. This is a scandalous situation since the (presumably) correct answer has been known since the time of Gauss. (Compare Hilbert and Cohn-Vossen.) All that is missing is a proof.

4. The Literature

Past progress toward the Kepler conjecture can be arranged into four categories: (1) bounds on the density, (2) descriptions of classes of packings for which the bound of \( \pi/\sqrt{18} \) is known, (3) convex bodies other than spheres for which the packing density can be determined precisely, (4) strategies of proof.

4.1. Bounds.

Various upper bounds have been established on the density of packings.

0.884 [Bli19],
0.835 [Bli29],
0.828 [Ran47],
0.7797 [Rog58],
0.77844 [Lin86],
0.77836 [Mud88],
0.7731 [Mud93].

Rogers’s is a particularly natural bound. As the dates indicate, it remained the best available bound for many years. His monotonicity lemma and his decomposition of Voronoi cells into simplices have become important elements in the proof of the Kepler conjecture. We give a new proof of Rogers’s bound in “Sphere Packings III.” A function \( \tau \), used throughout this collection, measures the departure of various objects from Rogers’s bound.

Muder’s bounds, although they appear to be rather small improvements of Rogers’s bound, are the first to make use of the full Voronoi cell in the determination of densities. As such, they mark a transition to a greater level of sophistication and difficulty. Muder’s influence on the work in this collection is also apparent.

A sphere packing admits a Voronoi decomposition: around every sphere take the convex region consisting of points closer to that sphere center than to any other sphere center. L. Fejes Tóth’s dodecahedral conjecture asserts that the Voronoi cell of smallest volume is a regular dodecahedron with inradius 1 [Fej42]. The dodecahedral conjecture implies a bound of 0.755 on sphere packings. L. Fejes
Tóth actually gave a complete proof except for one estimate. A footnote in his paper documents the gap. “In the proof, we have relied to some extent solely on intuitive observation [Anschauung].” As L. Fejes Tóth pointed out, that estimate is extraordinarily difficult, and the dodecahedral conjecture has resisted all efforts until now [McL98].

The missing estimate in L. Fejes Tóth’s paper is an explicit form of the Newton-Gregory problem. What is needed is an explicit bound on how close the 13th sphere can come to touching the central sphere. Or more generally, minimize the sum of the distances of the 13 spheres from the central sphere. No satisfactory bounds are known. Boerdijk has a conjecture for the arrangement that minimizes the average distance of the 13 spheres from the central sphere. Van der Waerden has a conjecture for the closest arrangement of 13 spheres in which all spheres have the same distance from the central sphere. Bezdek has shown that the dodecahedral conjecture would follow from weaker bounds than those originally proposed by L. Fejes Tóth [Bez97].

A proof of the dodecahedral conjecture has traditionally been viewed as the first step toward a proof of the Kepler conjecture, and if little progress has been made until now toward a complete solution of the Kepler conjecture, the difficulty of the dodecahedral conjecture is certainly responsible to a large degree.

4.2. Classes of packings.

If the infinite dimensional space of all packings is too unwieldy, we can ask if it is possible to establish the bound $\pi/\sqrt{18}$ for packings with special structures.

If we restrict the problem to packings whose sphere centers are the points of a lattice, the packings are described by a finite number of parameters, and the problem becomes much more accessible. Lagrange proved that the densest lattice packing in two dimensions is the familiar honeycomb arrangement [Lag73]. Gauss proved that the densest lattice packing in three dimensions is the face-centered cubic [Gau31]. In dimensions 4–8, the optimal lattices are described by their root systems, $A_2$, $A_3$, $D_4$, $D_5$, $E_6$, $E_7$, and $E_8$. A. Korkine and G. Zolotareff showed that $D_4$ and $D_5$ are the densest lattice packings in dimensions 4 and 5 ([KoZ73], [KoZ77]). Blichfeldt determined the densest lattice packings in dimensions 6–8 [Bli35]. Beyond dimension 8, there are no proofs of optimality, and yet there are many excellent candidates for the densest lattice packings such as the Leech lattice in dimension 24. For a proof of the existence of optimal lattices, see [Oes90].

Although lattice packings are of particular interest because they relate to so many different branches of mathematics, Rogers has conjectured that in sufficiently high dimensions, the densest packings are not lattice packings [Rog64]. In fact, the densest known packings in various dimensions are not lattice packings. The third edition of [CoSl93] gives several examples of nonlattice packings that are denser than any known lattice packings (dimensions 10, 11, 13, 18, 20, 22). The densest packings of typical convex sets in the plane, in the sense of Baire categories, are not lattice packings [Fej95].

Gauss’s theorem on lattice densities has been generalized by A. Bezdek, W. Kuperberg, and E. Makai, Jr. [BKM91]. They showed that packings of parallel strings of spheres never have density greater than $\pi/\sqrt{18}$.

4.3. Other convex bodies.

If the optimal sphere packings are too difficult to determine, we might ask whether the problem can be solved for other convex bodies. To avoid triviali-
ties, we restrict our attention to convex bodies whose packing density is strictly less than 1.

The first convex body in Euclidean 3-space that does not tile for which the packing density was explicitly determined is an infinite cylinder [Bez90]. Here A. Bezdek and W. Kuperberg prove that the optimal density is obtained by arranging the cylinders in parallel columns in the honeycomb arrangement.

In 1993, J. Pach exposed the humbling depth of our ignorance when he issued the challenge to determine the packing density for some bounded convex body that does not tile space [MP93]. (Pach’s question is more revealing than anything I can write on the subject of discrete geometry.) This question was answered by A. Bezdek [Bez94], who determined the packing density of a rhombic dodecahedron that has one corner clipped so that it no longer tiles. The packing density equals the ratio of the volume of the clipped rhombic dodecahedron to the volume of the unclipped rhombic dodecahedron.

4.4. Strategies of proof.

In 1953, L. Fejes Tóth proposed a program to prove the Kepler conjecture [Fej53]. A single Voronoi cell cannot lead to a bound better than the dodecahedral conjecture. L. Fejes Tóth considered weighted averages of the volumes of collections of Voronoi cells. These weighted averages involve up to 13 Voronoi cells. He showed that if a particular weighted average of volumes is greater than the volume of the rhombic dodecahedron, then the Kepler conjecture follows. The Kepler conjecture is an optimization problem in an infinite number of variables. L. Fejes Tóth’s weighted-average argument was the first indication that it might be possible to reduce the Kepler conjecture to a problem in a finite number of variables. Needless to say, calculations involving the weighted averages of the volumes of several Voronoi cells will be significantly more difficult than those involved in establishing the dodecahedral conjecture.

To justify his approach, which limits the number of Voronoi cells to 13, Fejes Tóth needs a preliminary estimate of how close a 13th sphere can come to a central sphere. It is at this point in his formulation of the Kepler conjecture that an explicit version of the Newton-Gregory problem is required. How close can 13 spheres come to a central sphere, as measured by the sum of their distances from the central sphere?

Strictly speaking, neither L. Fejes Tóth’s program nor my own program reduces the Kepler conjecture to a finite number of variables, because if it turned out that one of the optimization problems in finitely many variables had an unexpected global maximum, the program would fail, but the Kepler conjecture would remain intact. In fact, the failure of a program has no implications for the Kepler conjecture. The proof that the Kepler conjecture reduces to a finite number of variables comes only as corollary to the full proof of the Kepler conjecture.

L. Fejes Tóth made another significant suggestion in [Fej64]. He was the first to suggest the use of computers in the Kepler conjecture. After describing his program, he writes,

Thus it seems that the problem can be reduced to the determination of the minimum of a function of a finite number of variables, providing a programme realizable in principle. In view of the intricacy of this function we are far from attempting to determine the exact minimum. But, mindful of the rapid development of our computers, it is imaginable that the minimum may be approximated with
The most widely publicized attempt to prove the Kepler conjecture was that of Wu-Yi Hsiang [Hsi93]. (See also [Hsi93a], [Hsi93b].) Hsiang’s approach can be viewed as a continuation and extension of L. Fejes Tóth’s program. Hsiang’s paper contains major gaps and errors [CoHMS94]. The mathematical arguments against his argument appear in my debate with him in the *Mathematical Intelligencer* ([Hal94], [Hsi95]). There are now many published sources that agree with the central claims of [Hal94] against Hsiang. Conway and Sloane report that the paper “contains serious flaws.” G. Fejes Tóth feels that “the greater part of the work has yet to be done” [Fej95]. K. Bezdek concluded, after an extensive study of Hsiang’s work, “his work is far from being complete and correct in all details” [Bez97]. D. Muder writes, “the community has reached a consensus on it: no one buys it” [Mud97].

5. Experiments with other Decompositions

The results of my early efforts to prove the Kepler conjecture are published in [Hal92], [Hal93]. A decomposition dual to the Voronoi decomposition is the Delaunay decomposition. The papers show that the Delaunay decomposition also leads to an optimization problem in a finite number of variables that implies the Kepler conjecture. This approach led to the first proof that the face-centered cubic and hexagonal-close packings are locally optimal in the appropriate sense. Also, the optimization problem was studied numerically, and this gave the first direct numerical evidence of the truth of the Kepler conjecture. This method also shows how the Delaunay and Voronoi decompositions can be superimposed in a way that gives slightly better bounds than either does individually.

Unfortunately, the approach based on Delaunay decomposition also rapidly runs into enormously difficult technical complications. There was little hope of solving the problem using rigorous methods. I set my program aside until 1994, when I began again with renewed energy. I proposed a new decomposition that is a hybrid of the Voronoi and Delaunay decompositions. Since then, this hybrid decomposition has passed through a long series of refinements. The most important stages of that development are described in “Sphere Packings I” and “Sphere Packings II.” Its final form was worked out in collaboration with S. Ferguson in “A Formulation of the Kepler conjecture.”

6. Complexity

Why is this a difficult problem? There are many ways to answer this question.

This is an optimization problem in an infinite number of variables. In many respects, the central problem has been to formulate a good finite dimensional approximation to the density of a packing. Beyond this, there remains an extremely difficult problem in global optimization, involving nearly 150 variables. We recall that even very simple classes of nonlinear optimization problems, such as quadratic optimization problems, are NP-hard [HoPT95]. A general highly nonlinear program of this size is regarded by most researchers as hopeless (at least as far as rigorous methods are concerned).

There is a considerable literature on many closely related nonlinear optimization problems (the Tammes problem, circle packings, covering problems, the Leonard-Jones potential, Coulombic energy minimization of point particles, and so forth).
Many of our expectations about nonlattice packings are formed by the extensive experimental data that have been published on these problems. The literature leads one to expect a rich abundance of critical points, and yet it leaves one with a certain skepticism about the possibility of establishing general results rigorously.

The extensive survey of circle packings in [Mel97] gives a broad overview of the progress and limits of the subject. Problems involving a few circles can be trivial to solve. Problems involving several circles in the plane can be solved with sufficient ingenuity. With the aid of computers, various problems involving a few more circles can be treated by rigorous methods. Beyond that, numerical methods give approximations but no rigorous solutions. Melissen's account of the 20-year quest for the best separated arrangement of 10 points in a unit square is particularly revealing of the complexities of the subject.

Kepler's problem has a particularly rich collection of (numerical) local maxima that come uncomfortably close to the global maximum [Hal92]. These local maxima explain in part why a large number (around 5000) of planar maps are generated as part of the proof of the conjecture. Each planar map leads to a separate nonlinear optimization problem.

7. Contents of Papers

There are two papers that form part of the proof of the Kepler conjecture that precede this collection.

“Sphere Packings I” defines a decomposition of space according to a hybrid of the Delaunay simplices and Voronoi decomposition. We will not go into the computational difficulties that arise with the pure Delaunay and pure Voronoi decompositions. Suffice it to say that the technical problems surrounding either of the pure strategies are immense.

The parts of the hybrid decomposition that lie around a given sphere center form a star-shaped region called the decomposition star. Just as a bound on the volume of Voronoi cells leads to a bound on the density of a packing, a bound on a function called the score on the space of decomposition stars leads to a bound on the density. A sufficiently good bound on the score \( \frac{8}{\pi T} \) leads to the desired bound \( \frac{\pi}{\sqrt{18}} \) on the density of a packing.

The aim is to show that the bound \( \frac{8}{\pi T} \) holds for every decomposition star. With each decomposition star is associated a planar map that describes its combinatorial structure. “Sphere Packings I” proves the bound \( \frac{8}{\pi T} \) for every decomposition star whose planar map is a triangulation.

“Sphere Packings I” also lays the foundation for other papers by listing classical formulas for dihedral angles, solid angles, volumes of Voronoi cells, and so forth. It uses interval arithmetic to prove various inequalities by computer.

“Sphere Packings II” proves a local optimality result. The decomposition stars of the face-centered cubic and hexagonal close packings score \( \frac{8}{\pi T} \). If they are deformed, the score drops below \( \frac{8}{\pi T} \). “Sphere Packings II” proves local optimality in the strong sense that among the planar graphs that occur in the fcc packing and the hcp, the only decomposition stars that score \( \frac{8}{\pi T} \) are those of the fcc and hcp. That is, they are the only global maxima on the connected components of the space of decomposition stars to which they belong.

The contributions to this collection begin with the paper “A Formulation of the Kepler Conjecture.” It brings the structure of the decomposition star and the
scoring functions into final form. The papers in the series were written over a five-year period. As our investigations progressed, we found that it was necessary to make some adjustments. However, we had no desire to start over, abandoning the results of “Sphere Packings I” and “Sphere Packings II.” “A Formulation” gives a new decomposition of space. The Delaunay simplices are replaced with simplices that approximate them (quasi-regular tetrahedra and quarters). Voronoi cells are replaced as well with slightly modified objects called V-cells. The scoring function is also adjusted. “A Formulation” shows that all of the main theorems from “Sphere Packings I” and “Sphere Packings II” can be easily recovered in this new context with a few simple lemmas. It also lays the foundations for much of “Sphere Packings III,” “Sphere Packings IV,” and “Sphere Packings V.”

“Sphere Packings V” treats the decomposition stars that have a particular planar map, the pentagonal prisms. This arrangement of spheres has a long history. Boerdijk was the first to realize its importance, and used it to produce a counterexample to a conjecture of L. Fejes Tóth about an explicit form of the Gregory-Newton problem [Boe52]. This arrangement shows up in various other places, such as [SHDC95] and [Hsi93]. This arrangement gives a counterexample to my earliest expectations, that a pure Delaunay decomposition should lead directly to a finite dimensional formulation of the Kepler conjecture. This arrangement is either a counterexample or comes extremely close to being a counterexample in the various formulations of the Kepler conjecture that we have considered. It had to be singled out for special and careful attention. The purpose of “Sphere Packings V” is to show that this particular type of decomposition star has score under 8 pt.

“Sphere Packings III,” “Sphere Packings IV,” and a final paper, “The Kepler Conjecture,” treat all the remaining possibilities. “Sphere Packings III” treats all planar maps in which every face is a triangle or quadrilateral (except for Boerdijk’s pentagonal prism, treated in “Sphere Packings V”). “Sphere Packings IV” gives preliminary results on the general planar map. The main result shows that the planar map, with a few explicit exceptions, divides the unit sphere into polygons. Each polygon is at most a octagon. The results of “Sphere Packings IV” are particularly technical, because unlike the other cases, we cannot assume that the decomposition stars have any particular structure.

The preliminaries in IV permit a classification of all planar maps that are relevant for the proof of the Kepler conjecture. There are about 5000 cases that arise. (This classification was done by computer.) After reducing this list to under 100 cases by general linear programming methods, the final cases are eliminated case by case. With the elimination of the last case, the Kepler conjecture is proved.

8. Computers

As this project has progressed, the computer has replaced conventional mathematical arguments more and more, until now nearly every aspect of the proof relies on computer verifications. Many assertions in these papers are results of computer calculations. To make the proof of Kepler’s conjecture more accessible, I have posted extensive resources [Hal98].

Computers are used in various significant ways. They will be mentioned briefly here, and then developed more thoroughly elsewhere in the collection, especially in the final paper.

1. Proof of inequalities by interval arithmetic. “Sphere Packings I” describes a
method of proving various inequalities in a small number of variables by computer by interval arithmetic.

2. Combinatorics. A computer program classifies all of the planar maps that are relevant to the Kepler conjecture.

3. Linear programming bounds. Many of the nonlinear optimization problems for the scores of decomposition stars are replaced by linear problems that dominate the original score. They are solved by linear programming methods by computer. A typical problem has between 100 and 200 variables and 1000 and 2000 constraints. Nearly 100,000 such problems enter into the proof.

4. Branch and bound methods. When linear programming methods do not give sufficiently good bounds, they have been combined with branch and bound methods from global optimization.

5. Numerical optimization. The exploration of the problem has been substantially aided by nonlinear optimization and symbolic math packages.

6. Organization of output. The organization of the few gigabytes of code and data that enter into the proof is in itself a nontrivial undertaking.

9. Acknowledgments

I am indebted to G. Fejes Tóth’s survey of sphere packings in the preparation of this overview [Fej97]. For a much more comprehensive introduction to the literature on sphere packings, I refer the reader to that survey and to standard references on sphere packings such as [CoSi93], [PaA95], [Goo97], [Rog64], [Fej64], and [Fej72].

A detailed strategy of the proof was explained in lectures I gave at Mount Holyoke and Budapest during the summer of 1996 [Hal96]. See also the 1996 preprint, “Recent Progress on the Kepler Conjecture,” available from [Hal98].

I owe the success of this project to a significant degree to S. Ferguson. His thesis solves a major step of the program. He has been highly involved in various other steps of the solution as well. He returned to Ann Arbor during the final three months of the project to verify many of the interval-based inequalities appearing in the appendices of “Sphere Packings IV” and “The Kepler Conjecture.” It is a pleasure to express my debt to him.

Sean McLaughlin has been involved in this project during the past year through his fundamental work on the dodecahedral conjecture. By detecting many of my mistakes, by clarifying my arguments, and in many other ways, he has made an important contribution.

I thank S. Karni, J. Mikhail, J. Song, D. J. Muder, N. J. A. Sloane, W. Casselman, T. Jarvis, P. Sally, E. Carlson, and S. Chang for their contributions to this project. I express particular thanks to L. Fejes Tóth for the inspiration he provided during the course of this research. This project received the generous institutional support from the University of Chicago math department, the the Institute for Advanced Study, the journal Discrete and Computational Geometry, the School of Engineering at the University of Michigan (CAEN), and the National Science Foundation. Software (cfsqp)* for testing nonlinear inequalities was provided by the Institute for Systems Research at the University of Maryland.

Finally, I wish to give my special thanks to Kerri Smith, who has been my greatest source of support and encouragement through it all.

* www.isr.umd.edu/Labs/CACSE/FSQP/fsqp.html
References

[Ben74] Bender, C., Bestimmung der grössten Anzahl gleich grosser Kugeln, welche sich auf eine Kugel von demselben Radius, wie die übrigen, auflegen lassen, *Archiv Math. Physik* 56 (1874), 302–306.

[Bez90] A. Bezdek and W. Kuperberg, Maximum density space packing with congruent circular cylinders of infinite length, *Mathematica* 37 (1990), 74–80.

[BKM91] A. Bezdek, W. Kuperberg, and E. Makai Jr., Maximum density space packing with parallel strings of balls, *DCG* 6 (1991), 227–283.

[Bez94] A. Bezdek, A remark on the packing density in the 3-space in *Intuitive Geometry*, ed. K. Böröczky and G. Fejes Tóth, *Colloquia Math. Soc. János Bolyai* 63, North-Holland (1994), 17–22.

[Bez97] K. Bezdek, Isoperimetric inequalities and the dodecahedral conjecture, *Internat. J. Math.* 8, no. 6 (1997), 759–780.

[Bli19] H. F. Blichfeldt, Report on the theory of the geometry of numbers, *Bull. AMS*, 25 (1919), 449–453.

[Bli29] H. F. Blichfeldt, The minimum value of quadratic forms and the closest packing of spheres, *Math. Annalen* 101 (1929), 605–608.

[Bli35] H. F. Blichfeldt, The minimum values of positive quadratic forms in six, seven and eight variables, *Math. Zeit.* 39 (1935), 1–15.

[Boe52] Boerdijk, A. H. Some remarks concerning close-packing of equal Spheres, *Philips Res. Rep.* 7 (1952), 303–313.

[CoHMS94] J. H. Conway, T. C. Hales, D. J. Muder, and N. J. A. Sloane, On the Kepler conjecture, *Math. Intelligencer* 16, no. 2 (1994), 5.

[CoSl95] J. H. Conway, N. J. A. Sloane, What are all the best sphere packings in low dimensions? *DCG* 13 (1995), 383–403.

[CoSl93] J. H. Conway and N. J. A. Sloane, Sphere packings, lattices and groups, second edition, Springer-Verlag, New York, 1993 (third edition, to appear).

[Fej93] G. Fejes Tóth and W. Kuperberg, Recent results in the theory of packing and covering, in *New trends in discrete and computational geometry*, ed. J. Pach, Springer 1993, 251–279.

[Fej95] G. Fejes Tóth, Review of [Hsi93], *Math. Review* 95g#52032, 1995.

[Fej95b] G. Fejes Tóth, Densest packings of typical convex sets are not lattice-like, *DCG*, 14 (1995), 1–8.
[Fej97] G. Fejes Tóth, Recent progress on packing and covering, preprint.

[Fej72] L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, second edition, Springer-Verlag, Berlin New York, 1972.

[Fej64] L. Fejes Tóth, Regular figures, Pergamon Press, Oxford London New York, 1964.

[Fej42] L. Fejes Tóth, Über die dichteste Kugellagerung, *Math. Zeit.* 48 (1942–1943), 676–684.

[Fej50] L. Fejes Tóth, Some packing and covering theorems, *Acta Scientiarum Mathematicarum (Széchenyi)* 12/A, 62–67.

[Fej53] L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, Springer, Berlin, first edition, 1953.

[Gau31] C. F. Gauss, Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seber, *Göttingische gelehrte Anzeigen*, 1831 Juli 9, also published in *J. reine angew. Math.* 20 (1840), 312–320, and *Werke*, vol. 2, Königliche Gesellschaft der Wissenschaften, Göttingen, 1876, 188–196.

[Goo97] J. E. Goodman and J. O’Rourke, Handbook of discrete and computational geometry, CRC, Boca Raton and New York, 1997.

[Gun75] S. Günther, *Ein stereometrisches Problem*, Archiv der Math. Physik 57 (1875), 209–215.

[Hal92] T. C. Hales, The sphere packing problem, *J. Computational Applied Math.* 44 (1992), 41–76.

[Hal93] T. C. Hales, Remarks on the density of sphere packings in three dimensions, *Combinatorica* 13 (1993), 181–187.

[Hal94] T. C. Hales, The status of the Kepler conjecture, *Math. Intelligencer* 16, no. 3, (1994), 47–58.

[Hal96] T. C. Hales, http://www.math.lsa.umich.edu/~hales/holyoke.html

[Hal98] T. C. Hales, http://www.math.lsa.umich.edu/~hales/packings.html

[Hil01] D. Hilbert, Mathematische Probleme, *Archiv Math. Physik* 1 (1901), 44–63, also in *Proc. Sym. Pure Math.* 28 (1976), 1–34.

[Hop74] Hoppe R. *Bemerkung der Redaktion*, Math. Physik 56 (1874), 307-312.

[HoPT95] R. Horst, P.M. Pardalos, N.V. Thoai, *Introduction to Global Optimization*, Kluwer, 1995.
[Hsi93] W.-Y. Hsiang, On the sphere packing problem and the proof of Kepler’s conjecture, Internat. J. Math 93 (1993), 739-831.

[Hsi93a] W.-Y. Hsiang, On the sphere packing problem and the proof of Kepler’s conjecture, in *Differential geometry and topology* (Alghero, 1992), World Scientific, River Edge, NJ, 1993, 117–127.

[Hsi93b] W.-Y. Hsiang, The geometry of spheres, in *Differential geometry* (Shanghai, 1991), World Scientific, River Edge, NJ, 1993, 92-107.

[Hsi95] W.-Y. Hsiang, A rejoinder to T. C. Hales’s article “The status of the Kepler conjecture,” *Math. Intelligencer* 17, no. 1, (1995), 35–42.

[Kar66] R. Kargon, Atomism in England from Hariot to Newton, Oxford, 1966.

[Kep66] J. Kepler, The Six-cornered snowflake, Oxford Clarendon Press, Oxford, 1966, forward by L. L. Whyte.

[KoZ73] A. Korkine and G. Zolotareff, Sur les formes quadratiques, *Math. Annalen* 6 (1873), 366–389.

[KoZ77] A. Korkine and G. Zolotareff, Sur les formes quadratiques positives, *Math. Annalen* 11 (1877), 242–292.

[Lag73] J. L. Lagrange, Recherches d’arithmétique, *Nov. Mem. Acad. Roy. Sc. Bell Lettres Berlin* 1773, in *Œuvres*, vol. 3, 693–758.

[Lee56] J. Leech, The Problem of the Thirteen Spheres, *The Mathematical Gazette*, Feb 1956, 22–23.

[Lin86] J. H. Lindsey II, Sphere packing in $\mathbb{R}^3$, *Mathematika* 33 (1986), 137–147.

[Mas66] B. J. Mason, On the shapes of snow crystals, in [Kep66].

[McL98] S. McLaughlin, A proof of the dodecahedral conjecture, preprint.

[Mel97] J. B. M. Melissen, Packing and covering with circles, Ph.D. dissertation, Univ. Utrecht, Dec. 1997.

[Mil76] J. Milnor, Hilbert’s problem 18: on crystallographic groups, fundamental domains, and on sphere packings, in *Mathematical developments arising from Hilbert problems*, *Proc. Symp. Pure Math.*, vol 28, 491–506, AMS, 1976.

[MP93] W. Moser, J. Pach, Research problems in discrete geometry, DIMACS Technical Report, 93032, 1993.

[Mu88] D. J. Muder, Putting the best face on a Voronoi polyhedron, *Proc. London Math. Soc.* (3) 56 (1988), 329–348.
[Mud93] D. J. Muder A New Bound on the Local Density of Sphere Packings, *Discrete and Comp. Geom.* 10 (1993), 351–375.

[Mud97] D. J. Muder, letter, in *Fermat’s enigma*, by S. Singh, Walker, New York, 1997.

[Oes90] J. Oesterlé, Empilements de sphères, Séminaire Bourbaki, vol. 1989/90, Astérisque (1990), No. 189–190 exp. no. 727, 375–397.

[PaA95] J. Pach, P.K. Agarwal, *Combinatorial geometry*, John Wiley, New York 1995.

[Plo00] K. Plofker, private communication, January 2000.

[Ran47] R. A. Rankin, *Annals of Math.* 48 (1947), 228–229.

[Rog58] C. A. Rogers, The packing of equal spheres, *Proc. London Math. Soc.* (3) 8 (1958), 609–620.

[Rog64] C. A. Rogers, *Packing and covering*, Cambridge University Press, Cambridge, 1964.

[Sch53] K. Schütte and B.L. van der Waerden, Das Problem der dreizehn Kugeln, *Math. Annalen* 125, (1953), 325–334.

[SeM44] B. Segre and K. Mahler, On the densest packing of circles, *Amer. Math Monthly* (1944), 261–270.

[Shi83] J. W. Shirley, *Thomas Harriot: a biography*, Oxford, 1983.

[SHDC95] N. J. A. Sloane, R. H. Hardin, T. D. S. Duff, J. H. Conway, Minimal-energy clusters of hard spheres, *DCG* 14, no. 3, (1995), 237–259.

[Thu92] A. Thue, Om nogle geometrisk taltheoretiske Theoremer, *Forandlingern- eved de Skandinaviske Naturforskeres* 14 (1892), 352–353.

[Thu10] A. Thue, Über die dichteste Zusammenstellung von kongruenten Kreisen in der Ebene, *Christinim Vid. Selsk. Skr.* 1 (1910), 1–9.

[Why66] L. L. Whyte, forward to [Kep66].