ON THE APPROXIMATION OF KOOPMAN SPECTRA FOR MEASURE PRESERVING TRANSFORMATIONS

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Abstract. For the class of continuous, measure-preserving automorphisms on compact metric spaces, a procedure is proposed for constructing a sequence of finite-dimensional approximations to the associated Koopman operator on a Hilbert space. These finite-dimensional approximations are obtained from the so-called “periodic approximation” of the underlying automorphism and take the form of permutation operators. Results are established on how these discretizations approximate the Koopman operator spectrally. Specifically, it is shown that both the spectral measure and the spectral projectors of these permutation operators converge weakly to their infinite dimensional counterparts. Based on this result, a numerical method is derived for computing the spectra of volume-preserving maps on the unit $m$-torus. The discretized Koopman operator can be constructed from solving a bipartite matching problem with $O(\tilde{n}^3 m^2)$ time-complexity, where $\tilde{n}$ denotes the gridsize on each dimension. By exploiting the permutation structure of the discretized Koopman operator, it is further shown that the projections and density functions are computable in $O(m \tilde{n}^m \log \tilde{n})$ operations using the FFT algorithm. Our method is illustrated on several classical examples of automorphisms on the torus that contain either a discrete, continuous, or a mixed spectra. In addition, the spectral properties of the Chirikov standard map are examined using our method.

1. Introduction

The state-space of a measure-preserving dynamical system may be inspected through an analysis of the spectral properties of the associated Koopman operator on a Hilbert space. Within the applied dynamical systems community, a lot of interest has arisen recently in the computation of spectral properties of the Koopman operator (see the review article [6]). On the attractor, these spectral properties are directly related to the statistical properties of a measure-preserving dynamical system. Instead of the individual trajectories, it is exactly these properties that are relevant when it comes down to the comparison of complex dynamics [32]. From an engineering and modeling perspective, one would like to exploit this fact to obtain reduced order models. It was pointed out in [30] that for measure-preserving dynamical systems, the Koopman operator often contains a mixed spectra. The discrete part of the spectrum corresponds to the almost periodic part of the process, whereas the continuous spectrum typically is associated with either chaotic or shear dynamics. In a model reduction application, it would be desirable to use the almost periodic component of the observable dynamics as a reduced order model, while correctly accounting for the remainder contributions with the appropriate noise terms.

A prerequisite to the development of Koopman-based reduced order models are numerical methods which are capable of approximating the spectral decomposition of observables. A lot of work has already been dedicated on this subject. Apart from taking direct harmonic averages [33, 25, 26, 29], most methods rely mostly on the Dynamic Mode Decomposition (DMD) algorithm [37, 39]. Techniques that fall within this class are Extended-DMD [12, 22] as well as Hankel-DMD [4]. Certain spectral convergence results were established for the algorithms. In [4], it was shown that Hankel-DMD converges for systems with ergodic attractors. On the other hand, [22] proved weak-convergence of the eigenfunctions for Extended-DMD. Nevertheless, little is known on how these methods deal with continuous spectrum part of the system.

Numerically, computation of continuous spectra poses a challenge since it involves taking spectral projections of observables onto sets in the complex-plane which are not singleton. One approach is to directly approximate the moments of the spectral measure, and use the Christoffel-Darboux kernel to distinguish the atomic and continuous parts of the spectrum. This route was advocated in [23]. In this paper, we present an alternative viewpoint to the problem which directly takes aim at a specific discretization of the unitary Koopman operator. Our method relies on the concept of “periodic approximation” which originally was introduced by Halmos [16] and later by Lax [24]. The idea is similar to the so-called “Ulam approximation” of the Perron-Frobenius operator [11, 27, 13], but instead of approximating the measure-preserving dynamics by Markov chains (see also [11, 10]), one explicitly enforces a bijection on the state-space partition so that the unitary structure of the

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operator is preserved. In seminal work conducted by Katok and Stepin \cite{71}, it was already shown how these type of approximations are related to certain qualitative properties of the spectra. In this paper, we show how this technique can be used to obtain an actual approximation of the spectral decomposition of the operator.

We point out that the spectrum is approximated only in a weak sense. Indeed, the spectral type of a dynamical system is sensitive to arbitrarily small perturbations, making numerical approximation of the actual operator spectra very challenging. Take for example the map:

\[ T_\epsilon(x) = (x_1, x_2 + \omega + \epsilon \sin(2\pi x_1)), \]

defined on the unit torus. \( T_\epsilon \) has a fully discrete spectrum at \( \epsilon = 0 \), but an absolutely continuous spectrum \cite{51} when \( \epsilon \neq 0 \). Even though the spectral type changed instantaneously, the observable dynamics will not differ very much at finite time-scales under arbitrarily small perturbations. This small change in dynamics is reflected by only small perturbations in the spectral measure of the observable.

1.1. Contributions. The main contribution of this paper is a rigorous discretization of the Koopman operator such that its spectral properties are approximated weakly in the limit. It is shown that despite the difficulties of not knowing the smoothness properties of the underlying spectral projectors, control on the error may still be achieved in an average sense. Our discretization gives rise to a convergent numerical method which is capable of computing the spectral decomposition of the unitary Koopman operator.

We describe how our numerical method can be implemented for volume preserving maps on the \( m \)-torus. The construction of periodic approximations for such kind of maps is a relatively straightforward task and it has been well studied before in the literature under the phrase of “lattice maps” \cite{12,20,14,35}. In our paper, we show that if the map is Lipschitz continuous, a periodic approximation can always be obtained from solving a bipartite matching problem of \( O(n^3m^2) \) time-complexity, where \( n \) denotes the size of the grid in each dimension.

We further show that, once a periodic approximation is achieved, the spectral projections and density plots are computable in \( O(m\ln^m n) \) operations. Here, we heavily exploit the permutation structure of the discretized operator by making use of the FFT algorithm. To illustrate our methodology, we will compute the spectra of some canonical examples which either have a discrete, continuous or mixed spectra. In addition, we analyze the spectral properties of the Chirikov Standard map \cite{5} with our method.

1.2. Paper outline. In order to keep the presentation organized, the paper is split in two parts. In the first part, we cover the theoretical underpinnings of our proposed method by proving spectral convergence for a large class of measure preserving transformations. In the second part, we proceed by applying this general methodology to volume preserving maps on the \( m \)-torus. Numerical examples are provided.

Section 2 describes the outline of the proposed discretization of the Koopman operator. Section 3 elaborates on the notion of periodic approximation. In section 4, we prove operator convergence. Section 6 proves the main spectral results. In section 7, details of the numerical method are presented. In section 8, the performance of the numerical method is analyzed on some canonical examples of Lebesgue measure-preserving transformations on the torus. In section 9, the spectra of the Chirikov standard map is analyzed. Conclusions and future directions of our work are found in section 10.

Part 1. Theory: A finite-dimensional approximation of the Koopman operator with convergent spectral properties

In the first part of the paper, we elaborate on the theory behind the discretization of the Koopman operator using periodic approximations. The theory is discussed under a very general setting. Since the notion of periodic approximation is not as well known as the Ulam approximation in the applied dynamical systems community, we provide some background in places where the more experienced reader would easily stroll through. The main spectral results which are relevant to the numerical computations are found in section 6.

2. Problem formulation

Let \( T : X \rightarrow X \) be a self-map on the compact, norm-induced metric space \( X \subseteq \mathbb{R}^m \). Associate with \( X \) the measure space \((X, \mathcal{M}, \mu)\), where \( \mathcal{M} \) denotes the Borel sigma-algebra and \( \mu \) is an absolutely continuous measure with its support equaling the state-space, i.e. \( \text{supp } \mu = X \). The map \( T \) is assumed to be an invertible measure-preserving transformation such that \( \mu(B) = \mu(T(B)) = \mu(T^{-1}(B)) \) for every \( B \in \mathcal{M} \). In this paper, we refer to such maps as measure-preserving automorphisms.

The Koopman linearization \cite{21} of an automorphisms is performed as follows. Let:

\[ L^2(X,\mathcal{M},\mu) := \{ g : X \rightarrow \mathbb{C} \mid \| g \| < \infty \}, \quad \| g \| := \left( \int_X |g(x)|^2 d\mu(x) \right)^{\frac{1}{2}} \]

denote the space of square-integrable functions on \( X \) with respect to the invariant measure \( \mu \). The Koopman operator \( \mathcal{U} : L^2(X,\mathcal{M},\mu) \rightarrow L^2(X,\mathcal{M},\mu) \) is defined as the composition:

\[ (\mathcal{U}g)(x) := g \circ T(x) \]
where \( g \in L^2(X, \mathcal{M}, \mu) \) is referred to as the observable. The Koopman operator is unitary (i.e. \( U^* = U^{-1} \)), and hence, from the integral form of the spectral theorem \cite{21,28} it follows that the evolution of an observable under (2.1) can be decomposed as:

\[
\mathcal{U}^k g = \int_S e^{i\theta k} dS_\theta g, \quad k \in \mathbb{Z}.
\]

Here, \( S_\theta \) denotes a self-adjoint, projection-valued measure on the Borel sigma-algebra \( \mathcal{B}(\mathbb{S}) \) of the circle \( \mathbb{S} \) parameterized by \( \theta \in [-\pi, \pi) \). The projection-valued measure satisfies the following properties:

(i) For every \( D \in \mathcal{B}(\mathbb{S}) \),

\[
S_D := \int_D dS_\theta
\]

is an orthogonal projector on \( L^2(X, \mathcal{M}, \mu) \).

(ii) \( S_D = 0 \) if \( D = \emptyset \) and \( S_D = I \) if \( D = \mathbb{S} \).

(iii) If \( D_1, D_2 \in \mathcal{B}(\mathbb{S}) \) and \( D_1 \cap D_2 = \emptyset \), then

\[
\langle S_{D_1} g, S_{D_2} h \rangle := \int_X (S_{D_1} g)(x)(S_{D_2} h)(x)d\mu(x) = 0
\]

for every \( g, h \in L^2(X, \mathcal{M}, \mu) \).

(iv) If \( \{D_k\}_{k=1}^\infty \) is a sequence of pairwise disjoint sets in \( \mathcal{B}(\mathbb{S}) \), then

\[
\lim_{m \to \infty} \sum_{k=1}^m S_{D_k} g = S_D g, \quad D := \bigcup_{k=1}^\infty D_k
\]

for every \( g \in L^2(X, \mathcal{M}, \mu) \).

Numerically, our objective is to compute two quantities. Firstly, we wish to obtain a finite-dimensional approximation to the spectral projection:

\[
S_D g = \int_D dS_\theta g
\]

for some given observable \( g \in L^2(X, \mathcal{M}, \mu) \) and interval \( D \subset \mathbb{S} \). Secondly, we wish to obtain an approximating to the spectral density function. In other words, if \( \varphi(\theta) \in \mathcal{D}(\mathbb{S}) \) is some smooth test functions on the circle, then we wish to find an element \( \rho(\theta; g) \in \mathcal{D}^*(\mathbb{S}) \) is the dual space defined as the distributional derivative:

\[
\int_S \varphi(\theta) c(\theta; g) d\theta = -\int_S \varphi(\theta) \rho(\theta; g) d\theta
\]

of the so-called spectral cumulative function on \([-\pi, \pi)\):

\[
c(\theta; g) := \langle S_{[-\pi, \theta]} g, g \rangle.
\]

The approximation of the spectral projections and density functions play a critical role in the applications of model reduction \cite{32,30}. Indeed, on one hand, a low order model is directly obtained from the spectral projections by selecting \( D \) to be a finite union of disjoint intervals which contain a large percentage of the spectral mass. On the other hand, the spectral density functions directly tell us where most of the mass is concentrated for a given observable.

3. The proposed discretization of the Koopman operator

In this section, we introduce our proposed discretization of the Koopman operator.

3.1. Why permutation operators? We motivate our choice of using permutation operators as a means to approximate (2.1) as follows. Just like the Koopman operator, permutation operators are unitary, and therefore, its spectrum is contained on the unit circle. But the Koopman operator also satisfies the following properties:

(i) \( \mathcal{U}(fg) = (\mathcal{U}f)(\mathcal{U}g) \).

(ii) \( \mathcal{U} \) is a positive operator, i.e. \( \mathcal{U}g > 0 \) whenever \( g > 0 \).

(iii) The constant function, i.e. \( g(x) = 1 \) for every \( x \in X \), is an invariant of the operator.

Permutation operators are (finite-dimensional) operators which satisfy the aforementioned properties as well. Overall, they seem to be a natural choice to approximate (2.1). The upcoming results will justify this claim even further.
3.2. The discretization procedure. The following discretization of the Koopman operator is proposed. Consider any sequence of measurable partitions \( \{ \mathcal{P}_n \}_{n=1}^{\infty} \), where \( \mathcal{P}_n := \{ p_{n,1}, p_{n,2}, \ldots, p_{n,q(n)} \} \) such that:

(i) Every partition element \( p_{n,j} \) is compact, connected, and of equal measure, i.e.

\[
\mu(p_{n,j}) = \frac{\mu(X)}{q(n)}, \quad j \in \{1, 2, \ldots, q(n)\}
\]

where \( q : \mathbb{N} \rightarrow \mathbb{N} \) is a strictly, monotonically increasing function. Asking for compactness, one gets that the partition elements intersect. By (3.1), it must follow that these intersections are of zero measure.

(ii) The diameters of the partition elements are bounded by

\[
diam(p_{n,j}) := \sup_{x,y \in p_{n,j}} d(x, y) \leq l(n)
\]

where \( l : \mathbb{N} \rightarrow \mathbb{R} \) is a positive, monotonic function decaying to zero in the limit.

(iii) \( \mathcal{P}_n \) is a refinement of \( \mathcal{P}_m \) whenever \( n > m \). That is, every \( p_{m,j} \in \mathcal{P}_m \) is the union of some partition elements in \( \mathcal{P}_n \).

The main idea set forth in this paper is to project observables \( g \in L^2(X, \mathcal{M}, \mu) \) onto a finite-dimensional subspace of indicator functions,

\[
L^2_n(X, \mathcal{M}, \mu) := \left\{ g_n : X \mapsto \mathbb{C} \mid \sum_{j=1}^{q(n)} c_j \chi_{p_{n,j}}(x), \quad c_j \in \mathbb{C} \right\}, \quad \chi_{p_{n,j}}(x) = \begin{cases} 1 & x \in p_{n,j} \\ 0 & x \notin p_{n,j} \end{cases}
\]

by means of a smoothing/averaging operation:

\[
(W_ng)(x) = g_n(x) := \sum_{j=1}^{q(n)} g_{n,j} \chi_{p_{n,j}}(x), \quad g_{n,j} = \frac{q(n)}{\mu(X)} \int_X g(x) \chi_{p_{n,j}}(x) d\mu(x)
\]

and then replace (2.1) by its discrete analogue \( \mathcal{U}_n : L^2_n(X, \mathcal{M}, \mu) \mapsto L^2_n(X, \mathcal{M}, \mu) \) given by

\[
(\mathcal{U}_n g_n)(x) := \sum_{j=1}^{q(n)} g_{n,j} \chi_{T_n^{-1}(p_{n,j})}(x)
\]

where \( T_n : \mathcal{P}_n \mapsto \mathcal{P}_n \) is a discrete map on the partition.

The map \( T_n \) is chosen such that it “mimics” the dynamics of the continuous map \( T \). We impose the condition that \( T_n \) is a bijection. By doing so, we obtain a periodic approximation of the dynamics. Since every partition element is of equal measure (3.1) and since \( T_n \) is a bijection, the resulting discretization is regarded as one which preserves the measure-preserving properties of the original map on the subsigma algebra generated by \( \mathcal{P}_n \), i.e. \( \mu(T_n^{-1}(p_{n,j})) = \mu(p_{n,j}) = \mu(T_n^{-1}(p_{n,j})) \).

The discrete operators \( \{ \mathcal{U}_n \}_{n=1}^{\infty} \) are isomorphic to a sequence of finite-dimensional permutation operators. The spectra for these operators simplify to a pure point spectrum, where the eigenvalues correspond to roots of unity. Let \( v_{n,k} = \sum (v_{n,k})_j \chi_{p_{n,j}} \in L^2_n(X, \mathcal{M}, \mu) \) denote a normalized eigenvector, i.e.

\[
\mathcal{U}_n v_{n,k} = e^{i \theta_{n,k}} v_{n,k}, \quad \| v_{n,k} \| = 1.
\]

The spectral decomposition can be expressed as:

\[
\mathcal{U}_n g_n = \sum_{k=1}^{q(n)} e^{i \theta_{n,k}} S_{n, \theta_{n,k}} g_n
\]

where \( S_{n, \theta_{n,k}} : L^2_n(X, \mathcal{M}, \mu) \mapsto L^2_n(X, \mathcal{M}, \mu) \) denotes the rank-1 self-adjoint projector:

\[
S_{n, \theta_{n,k}} g_n = v_{n,k} \langle v_{n,k}, g_n \rangle = v_{n,k} \left( \int_X e^{i \theta_{n,k}}(x) g_n(x) d\mu \right) = v_{n,k} \left( \frac{\mu(X)}{q(n)} \sum_{j=1}^{q(n)} e^{i \theta_{n,k} j} g_{n,j} \right).
\]

The discrete analogue to the spectral projection (2.3) may then be defined as:

\[
S_{n, D} g_n = \int_D dS_{n, \theta} g_n = \sum_{\theta_{n,k} \in D} S_{n, \theta_{n,k}} g_n
\]

In addition, the discrete analogue to the spectral density function turns out to be:

\[
\rho_n(\theta; g_n) = \sum_{k=1}^{q(n)} \| S_{n, \theta_{n,k}} g_n \| ^2 \delta(\theta - \theta_{n,k}).
\]
3.3. Overview. An overview of the discretization process is given in fig. 1. Our goal is to see whether it is possible to construct a sequence of periodic approximation \( \{ T_n : \mathcal{P}_n \mapsto \mathcal{P}_n \}_{n=1}^{\infty} \) such that the associated discrete operators (3.4) and (3.7) converge to their infinite dimensional counterparts in some meaningful sense.

**Figure 1.** An overview of the discretization process.

### continuous case:

\[
L^2(X, \mathcal{M}, \mu) := \{ g : X \rightarrow \mathbb{C} \mid \| g \| < \infty \}, \quad \| g \| := \left( \int_X |g(x)|^2 \, d\mu \right)^{\frac{1}{2}}
\]

\[
U : L^2(X, \mathcal{M}, \mu) \rightarrow L^2(X, \mathcal{M}, \mu)
\]

\[
(Ug)(x) := g \circ T(x)
\]

\[
Ug = \int_S e^{i\theta} \, dS \theta g
\]

\[
S_D g = \int_D dS \theta g
\]

### discretization of the automorphism:

\[
T_n : \mathcal{P}_n \mapsto \mathcal{P}_n
\]

\( T_n \) is a periodic approximation

### discretization of the observable:

\[
(W_n g)(x) = g_n(x) := \sum_{j=1}^{q(n)} g_{n,j} \chi_{p_n,j}(x)
\]

\[
g_{n,j} := \frac{q(n)}{\mu(p_n,j)} \int_X g(x) \chi_{p_n,j}(x) \, d\mu
\]

\[
S_D g_n = \sum_{\theta_n,k \in D} S_n, \theta_n,k g_n
\]

\[
U_n g_n = \sum_{k=1}^{q(n)} e^{i\theta_n,k} S_n, \theta_n,k g_n
\]

\[
(U_n g_n)(x) := \sum_{j=1}^{q(n)} g_{n,j} \chi_{p_n,j}(x)
\]

### discrete case:

\[
L^2_n(X, \mathcal{M}, \mu) := \{ g_n : X \rightarrow \mathbb{C} \mid \sum_{j=1}^{q(n)} c_j \chi_{p_n,j}(x), \quad c_j \in \mathbb{C} \}, \quad \chi_{p_n,j}(x) = \begin{cases} 1 & x \in p_n,j \\ 0 & x \notin p_n,j \end{cases}
\]

\[
U_n : L^2_n(X, \mathcal{M}, \mu) \rightarrow L^2_n(X, \mathcal{M}, \mu)
\]

4. Periodic approximations

The notion of periodic approximation of a measure-preserving map is not new [16, 17, 24, 34, 18, 36, 40]. It is well-known that the set of measure-preserving automorphisms, which form a group under composition, are densely filled by periodic transformations under various topologies (see e.g. [16]). These properties were useful in proving claims such as whether automorphisms are “generically” ergodic or mixing. Katok and Stepin [18] further showed that the rate at which an automorphism admits a periodic approximation can be directly related to properties of ergodicity, mixing, and entropy. Certain spectral results were also proven in this context. For example, it was shown that if \( T \) admits a cyclic approximation by periodic transformations at a certain rate, then the spectrum of (2.1) is simple. The motivations of these earlier work was different than ours, given that the focus was more on characterizing global properties of automorphisms, rather than formulating a numerical method to approximate its spectral decomposition. Here, the focus will be on the latter, and with that goal in mind, we will prove the following result.
Theorem 4.1 (Existence of a periodic approximation). Let $T : X \mapsto X$ be a continuous, measure-preserving automorphism with the invariant measure $\mu$ absolutely continuous w.r.t. the Lebesgue measure and $\text{supp}(\mu) = X$. If $\{P_n\}_{n=1}^{\infty}$ is a sequence of measurable partitions on $X$ which are refinements and satisfy the conditions (3.1) and (3.2), then there exists a sequence of bijective maps $\{T_n : P_n \mapsto P_n\}_{n=1}^{\infty}$ that periodically approximates $T$ in an asymptotic sense. More specifically, for every fixed $k \in \mathbb{N}$ and compact set $A \in \mathcal{M}$:

\[
\lim_{n \to \infty} \sum_{l=k}^{k+n} d_H(T^l(A), T_n^l(A_n)) = 0
\]

where $d_H(A, B) := \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}$ denotes the Hausdorff metric, and

\[
A_n := \bigcup_{p \in P_n : p \cap A \neq \emptyset} p
\]

is an over-approximation of $A$ by the partition elements of $P_n$.

The proof of this result is postponed to the end of this subsection and will involve two intermediate steps. From a numerical analysis standpoint, theorem 4.1 claims that one can make the finite-time set evolution of $T$ numerically indistinguishable from that of the true map in both the forward and backward direction by choosing $n \in \mathbb{N}$ sufficiently large. In fig. 2 the situation is sketched for one forward iteration of the map.

We note that our formulation of the periodic approximation is more along the lines of that proposed by Lax [24] and not equivalent to the ones proposed by Katok and Stepin [18]. There, the quality of the approximation was phrased in the measure-theoretic context, where the proximity of $T_n$ to $T$ is described in terms of the one-iteration cost [18]:

\[
\sum_{i=1}^{q(n)} \mu(T(p_{n,i}) \Delta T_n(p_{n,i}))
\]

with $\Delta$ denoting the symmetric set difference, i.e. $A \Delta B := (A \setminus B) \cup (B \setminus A)$. It turns out that a specific sequence of maps $\{T_n : P_n \mapsto P_n\}_{n=1}^{\infty}$ may converge in the sense of theorem 4.1 while not converging in the sense of 4.2. Relatively simple examples of such sequences may be constructed. Take for example the map $T(x) = (x + \frac{1}{2}) \mod 1$ on the unit-length circle and choose a partition $P_n = \{p_{n,1}, p_{n,2}, \ldots, p_{n,r^n}\}$ with $p_{n,i} = \left[\frac{i-1}{r^n}, \frac{i}{r^n}\right)$ and $r$ being odd. The mapping:

\[
T_n(p_{n,j}) = p_{n,j^*}, \quad j^* = \begin{cases} j + \left\lfloor \frac{r^n}{2} \right\rfloor & j + \left\lfloor \frac{r^n}{2} \right\rfloor \leq r^n \\ j + \left\lfloor \frac{r^n}{2} \right\rfloor - r^n & j + \left\lfloor \frac{r^n}{2} \right\rfloor > r^n \end{cases}
\]

is the best one can do in terms of the cost 4.2, yet

\[
\sum_{i=1}^{q(n)} \mu(T(p_{n,i}) \Delta T_n(p_{n,i})) = 1, \quad \forall n \in \mathbb{N}.
\]
We also note that the convergence results of the measure-theoretic formulation (see [16]) were stated for for general automorphisms. Here, we restrict ourselves to just automorphisms which are continuous.

**Lemma 4.2.** Let \( T : X \mapsto X \) satisfy the hypothesis stated in theorem 3.1. Then, for any partition \( P_n \) that satisfies the condition (3.1), there exists a bijection \( T_n : P_n \mapsto P_n \) with the property:

\[
T(p_{n,l}) \cap T_n(p_{n,1}) \neq \emptyset \quad \text{and} \quad T^{-1}(p_{n,l}) \cap T_n^{-1}(p_{n,1}) \neq \emptyset, \quad \forall l \in \{1, 2, \ldots, q(n)\}.
\]

Proof. It suffices to show that there exists a map \( T_n : P_n \mapsto P_n \) with the property: \( \mu(T(p_{n,l}) \cap T_n(p_{n,1})) > 0 \) for all \( l \in \{1, 2, \ldots, q(n)\} \), since \( \mu(A \cap B) > 0 \) implies \( A \cap B \neq \emptyset \) for any \( A, B \in \mathcal{M} \), and:

\[
\mu(T(p_{n,l}) \cap T_n(p_{n,1})) = 0, \quad \forall l \in \{1, 2, \ldots, q(n)\} \quad \Rightarrow \quad \mu(T^{-1}(p_{n,l}) \cap T_n^{-1}(p_{n,1})) > 0, \quad \forall l \in \{1, 2, \ldots, q(n)\}.
\]

Otherwise, let \( T_n^{-1}(p_{n,s}) = p_{n,s} \) and \( \mu(T^{-1}(p_{n,k}) \cap T_n^{-1}(p_{n,k})) = 0 \) for some \( k \in \{1, 2, \ldots, q(n)\} \), then \( \mu(T(p_{n,s}) \cap T_n(p_{n,s})) = 0 \) is a contradiction.

Let \( G_n = (P_n, P_n', E) \) denote a bipartite graph where \( P_n' \) is a copy of \( P_n \) and \( (p_{n,k}, p_{n,l}) \in E \) if \( \mu(T(p_{n,k}) \cap p_{n,l}) > 0 \). In order to generate a bijective map so that \( \mu(T(p_{n,j}) \cap T_n(p_{n,1})) > 0 \) for all \( l \in \{1, 2, \ldots, q(n)\} \), we need to uniquely assign every element in \( P_n \) with one in \( P_n' \) using one of the existing edges. We call this new graph \( \tilde{G}_n = (P_n, P_n', \tilde{E}) \) (see fig. 3). To verify that such an assignment is possible, we need to confirm that \( G_n \) admits a perfect matching.

![Figure 3](image-url)

**Figure 3.** Because of the measure-preserving property, the graph \( G_n \) satisfies Hall’s marriage conditions, and hence admits a perfect matching.

To show that this is indeed the case, let \( N_G(B) \subset P_n' \) be the set of all vertices in \( P_n' \) adjacent for some \( B \subset P_n \). By Hall’s marriage theorem (see e.g. [7]), \( G_n \) has a perfect matching if and only if the cardinality \( |B| \leq |N_G(B)| \) for any \( B \subset P_n \). Since \( T \) is a measure-preserving automorphism and because of condition (3.1), it follows that for any \( k \in \{1, 2, \ldots, q(n)\} \), we must have:

\[
\sum_{i=1}^{q(n)} \mu(T(p_{n,k}) \cap p_{n,i}) = \frac{\mu(X)}{q(n)} \quad \forall \mu(T(p_{n,k}) \cap p_{n,i}) \leq \frac{\mu(X)}{q(n)}.
\]

Because of these properties, we deduce that:

\[
|N_G(B)| := \sum_{k \in B} \left( \sum_{l=1}^{q(n)} \mu(T(p_{n,k}) \cap p_{n,l}) \right) 1 \geq \sum_{k \in B} \mu(X) \sum_{l=1}^{q(n)} \mu(T(p_{n,k}) \cap p_{n,l}) = \sum_{k \in B} 1 = |B|
\]

for any arbitrary \( B \subset P_n \). Hence, \( G_n \) has a perfect matching. \( \square \)

We remark that (3.1) plays an essential role in the proof of lemma 4.2 since it is easy to construct counter-examples of non-uniform partitions which fail to satisfy (4.3). e.g. choose a compact set \( A \in \mathcal{M} \) for which \( T(A) \cap A = \emptyset \) and pick a partition wherein \( A \) is divided into \( r \) parts and \( T(A) \) into \( s \) parts, with \( r \neq s \). The value of lemma 4.2 is that it can be used as a means to bound the distance between the forward and inverse images of the partition elements in the Hausdorff metric.
Lemma 4.3. Let \( T : X \mapsto X \) satisfy the hypothesis stated in theorem 4.1 and let \( \{ \mathcal{P}_n \}_{n=1}^{\infty} \) be a sequence of measurable partitions that satisfy both the conditions (3.1) and (3.2). \( T_n : \mathcal{P} \mapsto \mathcal{P}_n \) is a sequence of bijective maps which satisfy the property (4.3) for each \( n \in \mathbb{N} \), then

\[
\lim_{n \to \infty} \sum_{l=-k}^{k} \max_{p \in \mathcal{P}_n} d_H(T^l(p), T^l_n(p)) = 0
\]

for every \( k \in \mathbb{N} \).

Proof. We will prove this claim using induction. Set \( k = 1 \), if \( T_n : \mathcal{P}_n \mapsto \mathcal{P}_n \) is a bijective map satisfying the property (4.3), then we must have that:

\[
d_H(T(p_{n,1}), T_n(p_{n,1})) \leq \text{diam}(T(p_{n,1})) + \text{diam}(T_n(p_{n,1}))
\]

and

\[
d_H(T^{-1}(p_{n,1}), T_n^{-1}(p_{n,1})) \leq \text{diam}(T^{-1}(p_{n,1})) + \text{diam}(T_n^{-1}(p_{n,1})).
\]

Let \( \epsilon > 0 \) and note that \( T \) has a continuous inverse, since the map is a continuous bijection on a compact metric space \( X \). By compactness, there exist a \( \delta > 0 \) such that:

\[
\text{diam}(p_{n,j}) < \delta \quad \Rightarrow \quad \text{diam}(T(p_{n,j})) < \epsilon/4, \quad \text{diam}(T^{-1}(p_{n,j})) < \epsilon/4
\]

Pick \( n \in \mathbb{N} \) so that \( l(n) < \min \{ \delta, \epsilon/4 \} \) to obtain:

\[
d_H(T^{-1}(p_{n,j}), T_n^{-1}(p_{n,j})) + d_H(T(p_{n,j}), T_n(p_{n,j})) < \frac{\epsilon}{4} + \min \{ \delta, \frac{\epsilon}{4} \} + \frac{\epsilon}{4} + \min \{ \delta, \frac{\epsilon}{4} \} \leq \epsilon
\]

Since \( \epsilon \) and \( p_{n,j} \in \mathcal{P}_n \) are both arbitrary, we have proved (4.4) for the case where \( k = 1 \).

Now to prove the result for \( k > 1 \), we note from the triangle inequality that:

\[
d_H(T^k(p_{n,j}), T^k_n(p_{n,j})) \leq d_H(T(T^{k-1}(p_{n,j})), T(T_n^{k-1}(p_{n,j}))) + d_H(T(T_n^{k-1}(p_{n,j})), T_n(T_n^{k-1}(p_{n,j})))
\]

Using the inductive hypothesis, assume that (4.4) is true for \( k-1 \) so that the distance \( d_H(T^{k-1}(p_{n,j}), T_n^{k-1}(p_{n,j})) \) can be made arbitrarily small by choosing \( n \in \mathbb{N} \) sufficiently large. By uniform continuity of \( T \), there exist a \( N_a \in \mathbb{N} \) sufficiently large, so that:

\[
d_H(T(T^{k-1}(p_{n,j})), T_n(T_n^{k-1}(p_{n,j}))) \leq \frac{\epsilon}{8}
\]

for all \( n > N_a \). Analogously, there exists a \( N_b \in \mathbb{N} \) so that:

\[
d_H(T^{-1}(T^{-k+1}(p_{n,j})), T^{-1}(T_n^{-k+1}(p_{n,j}))) \leq \frac{\epsilon}{8}
\]

for all \( n > N_b \). Setting \( N_c = \max \{ N_a, N_b \} \), we get:

\[
d_H(T^{-k}(p_{n,j}), T_n^{-k}(p_{n,j})) + d_H(T^k(p_{n,j}), T_n^k(p_{n,j})) \leq \frac{\epsilon}{2}
\]

for all \( n > N_c \). Using the inductive hypothesis again, we may then choose a \( N_d \in \mathbb{N} \) so that:

\[
\sum_{l=-k+1}^{k-1} d_H(T^l(p_{n,j}), T_n^l(p_{n,j})) \leq \frac{\epsilon}{2}, \quad \forall n > N_c.
\]

By setting \( N = \max \{ N_c, N_d \} \), we finally obtain:

\[
\sum_{l=-k+1}^{k-1} d_H(T^l(p_{n,j}), T_n^l(p_{n,j})) + d_H(T^{-k}(p_{n,j}), T_n^{-k}(p_{n,j})) + d_H(T^k(p_{n,j}), T_n^k(p_{n,j})) \leq \epsilon
\]

where \( \epsilon > 0 \) and \( p_{n,j} \in \mathcal{P}_n \) were arbitrary. \( \square \)

Note that continuity of \( T \) plays a critical role in the proof of lemma 4.3. Furthermore, observe that \( T \) is assumed to have an absolutely continuous invariant measure with \( \text{supp}(\mu) = X \), which ensures that both conditions (3.1) and (3.2) are satisfied, simultaneously. With lemmas 4.2 and 4.3 we now have the necessary results to complete the proof of theorem 4.1.

Proof of theorem 4.1. Let \( \{ T_n : \mathcal{P}_n \mapsto \mathcal{P}_n \}_{n=1}^{\infty} \) be a sequence of bijective maps which satisfies the property of lemma 4.2 for each \( n \in \mathbb{N} \). Set \( \epsilon > 0 \) and note that \( A_n \) is compact, since it is a finite union of compact sets. By the triangle inequality, we have that:

\[
\sum_{l=-k}^{k} d_H(T^l(A), T_n^l(A_n)) \leq \sum_{l=-k}^{k} d_H(T^l(A), T^l(A_n)) + \sum_{l=-k}^{k} d_H(T^l(A_n), T_n^l(A_n))
\]

We will show that each term above can be made arbitrarily small by selecting \( n \) sufficiently large. From uniform continuity of \( T \) and that \( d_H(A, A_n) \) is monotonically decreasing to 0 as \( n \to \infty \), it follows that there exists a
$N_1 \in \mathbb{N}$ so that the first sum is bounded by $\epsilon/2$ for all $n > N_1$. In order to find also a bound for the second sum, notice that:

$$\sum_{l=-k}^{k} d_H(T^n_l(A_n), T^n_l(A_n)) = \sum_{l=-k}^{k} d_H \left( \bigcup_{p \in P_n, p \cap A \neq \emptyset} T^n_l(p), \bigcup_{p \in P_n, p \cap A \neq \emptyset} T^n_l(p) \right) \leq \sum_{l=-k}^{k} \max_{p \in P_n} d_H(T^n_l(p), T^n_l(p)) \leq \sum_{l=-k}^{k} d_H(T^n_l(p), T^n_l(p))$$

where we have made repetitive use of the Haussdorf property: $d_H(A \cup B, C \cup D) \leq \max\{d_H(A,C), d_H(B,D)\}$ in the first inequality. From Lemma 4.3, it follows that we can choose $n > N_2$ so that the second sum is also bounded by $\epsilon/2$. We finally set $N = \max\{N_1, N_2\}$ to complete the proof.

5. Operator convergence

Next we will prove how the sequence of discrete Koopman operators $\{U_n\}_{n=1}^{\infty}$ converges to the operator defined in (4.1) whenever $\{T_n\}_{n=1}^{\infty}$ is a sequence of periodic approximations converging to the $T$ in the sense of theorem 4.1. The following technical lemma is required.

**Lemma 5.1.** Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of compact sets converging monotonically to the compact set $A$ in the Haussdorf metric (i.e. $\lim_{n \to \infty} d_H(A, A_n) = 0$) and suppose that $\mu(A_n) = \mu(A)$ for every $n \in \mathbb{N}$. Then,

$$\lim_{n \to \infty} \mu(A \Delta A_n) = 0$$

**Proof.** By definition: $\mu(A \Delta A_n) = \mu(A \setminus A_n) + \mu(A_n \setminus A)$, and hence, to prove (5.1) we must show that both of these terms go to zero in the limit. However, since $\mu(A_n) = \mu(A)$, and:

$$\mu(A_n \setminus A) = \mu(A_n) - \mu(A \cap A_n), \quad \mu(A \setminus A_n) = \mu(A) - \mu(A \cap A_n)$$

it follows that $\mu(A \setminus A_n) = \mu(A_n \setminus A)$, and therefore it sufficient to show that either $\mu(A \setminus A_n)$ or $\mu(A_n \setminus A)$ tends to zero.

Consider $\mu(A_n \setminus A)$. Let $A_{\epsilon(n)} := \bigcup_{x \in A} B_\epsilon(n)(x)$ be the $\epsilon(n)$-fattening of $A$ where $\epsilon(n) := d_H(A, A_n)$. Observe that:

$$(A_n \setminus A) \subset (A_{\epsilon(n)} \setminus A) \quad \Rightarrow \quad 0 \leq \mu(A_n \setminus A) \leq \mu(A_{\epsilon(n)} \setminus A)$$

Since $\epsilon(n)$ is a monotonically decreasing sequence, from Theorem 1.19(e) in [35] we know that:

$$\bigcap_{n=1}^{\infty} A_{\epsilon(n)} = A \quad \text{and} \quad \mu(A_{\epsilon(n)}) \to \mu(A) \text{ as } n \to \infty.$$ 

Therefore, $\mu(A_{\epsilon(n)} \setminus A) \to 0$ as $n \to \infty$, leading to the desired result. \qed

In addition to lemma 5.1, we will require knowing some properties on the averaging operation (3.3). The operator (3.3) is an approximation of the identity and therefore $\|g - g_n\| \to 0$ as $n \to \infty$. This property can be verified by first establishing this fact for continuous observables and then employ the fact that $C(X)$ is dense in $L^2(X, M, \mu)$. In addition to being an approximation to the identity, (3.3) is an orthogonal projector which maps observables $g \in L^2(X, M, \mu)$ to their best approximations $g_n \in L^2_m(X, M, \mu)$. This follows from the fact that $W_n$ is idempotent and that $(g - W_n g, W_n g) = 0$.

**Lemma 5.2.** Let $T : X \to X$ satisfy the hypothesis of theorem 4.1 and suppose that $\{T_n : \mathcal{P}_n \to \mathcal{P}_n\}_{n=1}^{\infty}$ is a sequence of discrete maps that periodically approximates $T$ in the sense of (4.1). Define:

$$g = \sum_{j=1}^{q(m)} \epsilon_j \chi_{p_{m,j}} \in L^2_m(X, M, \mu), \quad m \in \mathbb{N}$$

Then:

$$\lim_{n \to \infty} \sum_{l=-k}^{k} \|t^l g - t^l_n g_n\|^2 = 0$$

for every $k \in \mathbb{N}$. 

\[\text{Proof.}\]
Proof. For notational clarity, write $A^{(j)} := p_{m,j}$, $g^{(j)} := \chi_{p_{m,j}}$ and $g^{(j)}_n := \mathcal{W}_n g^{(j)}$. We have:

$$
\sum_{l=-k}^{k} \| \mathcal{U} l g^{(j)} - \mathcal{U}^l g_n \|^2 = \sum_{l=-k}^{k} \left( \sum_{j=1}^{q(m)} c_j (\mathcal{U} l g^{(j)} - \mathcal{U}^l g_n) \right)^2 \\
\leq \sum_{l=-k}^{k} \left( \sum_{j=1}^{q(m)} \| c_j (\mathcal{U} l g^{(j)} - \mathcal{U}^l g_n) \|^2 \right) \\
\leq M \sum_{j=1}^{q(m)} \left( \sum_{l=-k}^{k} \| \mathcal{U} l g^{(j)} - \mathcal{U}^l g_n \|^2 \right), \quad M = \sum_{j=1}^{q(m)} |c_j|^2 \\
\leq q(m) M \max_{j=1, \ldots, q(m)} \left( \sum_{l=-k}^{k} \| \mathcal{U} l g^{(j)} - \mathcal{U}^l g_n \|^2 \right)
$$

Hence, it suffices to show that:

$$
\lim_{n \to \infty} \sum_{l=-k}^{k} \| \mathcal{U} l g^{(j)} - \mathcal{U}^l g_n \|^2 = 0, \quad j = 1, \ldots, m.
$$

Since $\{\mathcal{P}_n\}_{n=1}^{\infty}$ are consecutive refinements, observe that $g^{(j)}_n = g^{(j)}$ for $n \geq m$, which implies:

$$
\sum_{l=-k}^{k} \| \mathcal{U} l g^{(j)} - \mathcal{U}^l g_n \|^2 = \sum_{l=-k}^{k} \| (\mathcal{U} l - \mathcal{U}^l) g^{(j)} \|^2, \quad \text{if } n \geq m \\
= \sum_{l=-k}^{k} \mu(T^{-l}(A^{(j)}) \Delta T^{-l}_n(A^{(j)})).
$$

By theorem 4.1, $\{T^{-l}_n(A^{(j)}))\}_{n=m}^{\infty}$ converges to $T^{-l}(A^{(j)})$ in the Haussdorf metric. Also, since $\mu(T^{-l}(A^{(j)})) = \mu(T^{-l}_n(A^{(j)}))$ for $n \geq m$, it follows from lemma 5.1 that:

$$
\sum_{l=-k}^{k} \mu(T^{-l}(A^{(j)}) \Delta T^{-l}_n(A^{(j)})) \to 0 \quad \text{as } n \to \infty.
$$

□

Theorem 5.3 (Operator convergence). Let $T : X \mapsto X$ satisfy the hypothesis of theorem 4.1 and suppose that $\{T_n : \mathcal{P}_n \mapsto \mathcal{P}_n\}_{n=1}^{\infty}$ is a sequence of discrete maps that periodically approximates $T$ in the sense of (4.1). Then, $\{\mathcal{U}_n : L^2_n(X, \mathcal{M}, \mu) \mapsto L^2_n(X, \mathcal{M}, \mu)\}_{n=1}^{\infty}$ converges to $\mathcal{U}$ in the following sense:

$$
\lim_{n \to \infty} \sum_{l=-k}^{k} \| \mathcal{U} l g - \mathcal{U}^l g_n \|^2 = 0
$$

for every fixed $k \in \mathbb{N}$ and $g \in L^2(X, \mathcal{M}, \mu)$.

Proof. Let $\epsilon > 0$ and define:

$$
g_m = \sum_{j=1}^{q(m)} c_j X_{p_{m,j}} \in L^2_m(X, \mathcal{M}, \mu), \quad m \in \mathbb{N}.
$$
We perform the following manipulations:

\[
\sum_{l=-k}^{k} \| \mathcal{U}^l g - \mathcal{U}^l g_n \|^2 = \sum_{l=-k}^{k} \| \mathcal{U}^l (g - g_m + g_m) - \mathcal{U}^l (g - g_m + g_m)_n \|^2 \\
\leq \sum_{l=-k}^{k} ( \| g - g_m \| + \| \mathcal{W}_n (g - g_m) \| + \| \mathcal{U}^l g_m - \mathcal{U}^l (g_m)_n \| )^2 \\
\leq \sum_{l=-k}^{k} (2 \| g - g_m \| + \| \mathcal{U}^l g_m - \mathcal{U}^l (g_m)_n \| )^2 \\
\leq \left( \left( \sum_{l=-k}^{k} 4 \| g - g_m \|^2 \right)^{\frac{1}{2}} + \left( \sum_{l=-k}^{k} \| \mathcal{U}^l g_m - \mathcal{U}^l (g_m)_n \|^2 \right)^{\frac{1}{2}} \right)^2 \\
= \left( 2 \| g - g_m \| + \left( \sum_{l=-k}^{k} \| \mathcal{U}^l g_m - \mathcal{U}^l (g_m)_n \|^2 \right)^{\frac{1}{2}} \right)^2
\]

Since (3.3) is approximation of the identity, we may select a \( m \in \mathbb{N} \) so that \( \| g - g_m \| \leq \frac{1}{4} \sqrt{\varepsilon} \). Using lemma 5.2 further establishes that there exists a \( N \in \mathbb{N} \) such that:

\[
\sum_{l=-k}^{k} \| \mathcal{U}^l g_m - \mathcal{U}^l (g_m)_n \| \leq \frac{\varepsilon}{4}, \quad \forall n \geq N.
\]

Overall, we have:

\[
\sum_{l=-k}^{k} \| \mathcal{U}^l g - \mathcal{U}^l g_n \|^2 \leq \varepsilon,
\]

which confirms (5.3). \( \square \)

6. Spectral convergence

In this section, we will analyze how spectral projectors (3.7) converge to (2.3) in the limit.

6.1. An illuminating example. Before we proceed to the general results, let us work out the details of a periodic approximation for a basic example in order to clarify certain subtleties on weak vs. strong convergence of the spectra. In particular, consider the map:

\[
T(x) = (x + \frac{1}{2}) \mod 1, \quad x \in [0, 1),
\]

which is rotation on the circle by a half.

For the partition \( \mathcal{P}_n = \{p_{n,1}, p_{n,2}, \ldots, p_{n,r^n}\} \) with \( p_{n,j} = (\frac{j-1}{r^n}, \frac{j}{r^n}) \), \( j = 1, \ldots, r^n \) for some integer \( r > 1 \), we may define the map:

\[
T_n(p_{n,j}) = p_{n,j^*}, \quad j^* = \begin{cases} 
\lfloor j + \frac{r^n}{2} \rfloor & j + \frac{r^n}{2} \leq r^n \\
\lfloor j + \frac{r^n}{2} \rfloor - r^n & j + \frac{r^n}{2} > r^n
\end{cases}
\]

which clearly is a periodic approximation to the original transformation in the sense of theorem 4.1. The discrete Koopman operator associated with this map is isometric to the circulant matrix. That is, the permutation matrices:

\[
[U_n]_{ij} = \langle \chi_{p_{n,i}}, \chi_{p_{n,j}} \rangle,
\]

are equal to the circulant matrices:

\[
U_n = \begin{bmatrix}
d_1 & d_2 & \cdots & d_3 & d_2 \\
d_2 & d_1 & d_2 & \cdots & d_3 \\
\vdots & d_2 & d_1 & \ddots & \vdots \\
d_2 & \cdots & \cdots & d_2 & d_1 \\
d_2 & \cdots & \cdots & d_2 & d_1
\end{bmatrix}, \quad \text{where } d_{\lfloor \frac{r^n}{2} \rfloor + 1} = 1 \text{ and zero otherwise.}
\]

\(^1\lfloor \cdot \rfloor \) denotes floor the function.
The spectral decomposition of a circulant matrix can be obtained in closed-form using the Discrete Fourier Transform. In our specific case, we have

$$v_{n,k}(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{2\pi i (k-1)(j-1)/n} \chi_{p_{n,j}}(x), \quad \theta_{n,k} = \begin{cases} \frac{(-1)^{n+1}}{2} + \frac{\pi}{r} & r \text{ is even} \\ \frac{(-1)^{n+1} + \pi}{2} - \frac{\pi}{r} & r \text{ is odd} \end{cases}$$

Recall that the spectra of the true (i.e., infinite-dimensional) operator consists of only two eigenvalues located at 1 and $-1$. Yet, from the equations above, we see that this property is maintained for the discrete analogue when $r$ is an even number. For an odd $r$, the eigenvalues of the discrete Koopman operators seem to densely fill up the unit circle as $n \to \infty$.

At first sight, this fragility of the spectrum in the discretization process appear as a serious problem. However, if we weaken our notion of what it means for the discretizations to converge spectrally, this issue can be largely avoided. The ordering of the eigenvalue-eigenvector pairs $\{\theta_{n,k}, v_{n,k}\}_{k=1}^{n}$ in (6.1) provide clues on what approach should be taken. Particularly, notice that the first set of eigenvalues (corresponding to the slower oscillatory modes) are concentrated around the points $\theta = 0$ and $\theta = \pi$. We may then ask ourselves which of the eigenvalue-eigenvector pairs actually play a significant role in the spectral approximation of a specific fixed observable.

The following can be observed.

After application of the smoothing operator (3.3), any square-integrable observable $g \in L^2(B(\mathbb{T}), \mu)$ on the circle (with $\mu$ being the standard Lebesgue measure in this case), can be written in terms of the eigenvectors:

$$g_n(x) = \sum_{k=1}^{n} c_{n,k} v_{n,k}(x),$$

where $v_{n,k} \in L^2(B(\mathbb{T}), \mu)$ are nothing else but the discrete analogues of the Fourier harmonics (see (6.1)). Henceforth, one can show that for any $\epsilon > 0$, there exists an $m = m(\epsilon) \in \mathbb{N}$ and a $N \in \mathbb{N}$ so that:

$$\|g_n\| - \epsilon < \sum_{k \leq m} |c_{n,k}|^2 \leq \|g_n\|, \quad \forall n \geq N.$$  

For this specific $m$ which only depends on $\epsilon$, the set $\{\theta_{n,k} \in S : k \leq m\}$ is at most a $\delta$-distance separated from the eigenfrequencies $\theta = 0$ and $\theta = \pi$, where $\delta > 0$ can be made arbitrarily small by choosing sufficiently large $n \in \mathbb{N}$.

In other words, as $n$ approaches infinity, most of “spectral energy” will get concentrated around the eigenvalue points. From this perspective, the discrete analogues of the Koopman operator do seem to approximate the infinite analogue spectrally. The situation described here is not something which is restricted to only this basic example, but applies more generally.

6.2. Approximation of the spectral projectors. Consider any smooth test function $\varphi \in \mathcal{D}(S)$ on the circle, and define:

$$S_{\varphi}g = \int_S \varphi(\theta) dS_{b}g, \quad S_{n,\varphi}g_n = \sum_{\theta_{n,k} \in S} \varphi(\theta_{n,k}) S_{n,\theta_{n,k}} g_n.$$  

We will prove the following.

**Theorem 6.1.** Let $T : X \mapsto X$ satisfy the hypothesis of theorem 4.1 and suppose that $\{T_n : \mathcal{P}_n \mapsto \mathcal{P}_n\}_{n=1}^{\infty}$ is a sequence of discrete maps that periodically approximates $T$ in the sense of (1.1). For any smooth test function $\varphi \in \mathcal{D}(S)$ and observable $g \in L^2(X, \mathcal{M}, \mu)$, we have:

$$\lim_{n \to \infty} \|S_{\varphi}g - S_{n,\varphi}g_n\| = 0.$$  

**Proof.** Expand the smoothed indicator function $\varphi(\theta)$ by its Fourier series: $\varphi(\theta) = \sum_{l=-\infty}^{\infty} b_{l} e^{il\theta}$, and note that the series is uniformly convergent. We see that:

$$S_{\varphi}g = \int_S \left( \sum_{l=-\infty}^{\infty} b_{l} e^{il\theta} \right) dS_{b}g = \sum_{l=-\infty}^{\infty} b_{l} \left( \int_S e^{il\theta} dS_{b}g \right) = \sum_{l=-\infty}^{\infty} b_{l} \mathcal{U}_{l}^{1} g,$$

where we employed the spectral theorem of unitary operators [2] in the last equality. Similarly, it also holds that:

$$S_{n,\varphi}g_n = \sum_{l=-\infty}^{\infty} b_{l} \mathcal{U}_{n,l}^{1} g_n.$$  

Hence,

$$S_{\varphi}g - S_{n,\varphi}g_n = \sum_{l=-\infty}^{\infty} b_{l} (\mathcal{U}_{l}^{1} g - \mathcal{U}_{n,l}^{1} g_n).$$  

[2] The scaling $1/\sqrt{n\pi}$ here is required for normalization.
Now let $\epsilon > 0$ and choose $k \in \mathbb{N}$ such that:

$$\sum_{|l|>k} |b_l| < \frac{\epsilon}{4\|g\|}$$

This is possible, because the Fourier coefficients of $\varphi(\theta)$ are absolutely summable. Write:

$$\|S_\varphi g - S_{n,\varphi} g_n\| \leq \sum_{l=-\infty}^{\infty} |b_l| \|U^l g - U^l_n g_n\|$$

$$= \sum_{l=-k}^{k} |b_l| \|U^l g - U^l_n g_n\| + \sum_{|l|>k} |b_l| \|U^l g - U^l_n g_n\|$$

$$\leq M \sum_{l=-k}^{k} \|U^l g - U^l_n g_n\| + 2\|g\| \sum_{|l|>k} |b_l|$$

$$\leq M \sum_{l=-k}^{k} \|U^l g - U^l_n g_n\| + \frac{\epsilon}{2}$$

For a fixed $k \in \mathbb{N}$, we can choose by theorem 5.3 there exist an $N \in \mathbb{N}$ so that:

$$\sum_{l=-k}^{k} \|U^l g - U^l_n g_n\| \leq \frac{\epsilon}{2M}, \quad \forall n \geq N.$$

This yields:

$$\|S_\varphi g - S_{n,\varphi} g_n\| \leq \epsilon.$$

□

Remark 6.2. Note that in the proof of theorem 6.1, we explicitly made use of the fact that the discrete operators (3.4) are unitary, which in turn is a consequence of the periodic approximation. Therefore, our arguments would break down if the periodic approximation was replaced with a many-to-one map.

Recall the spectral projectors (2.3) and (3.7), and notice that they may be re-expressed as:

$$S_D g = \int_{\mathbb{S}} \chi_D(\theta) dS_{\theta} g, \quad S_{n,D} g_n = \sum_{\theta_{n,k} \in \mathbb{S}} \chi_D(\theta_{n,k}) S_{n,\theta_{n,k}} g_n,$$

where $\chi_D(\theta)$ is an indicator function on the circle for the interval $D$. To avoid technicalities like having eigenvalues on the boundary of $D$, we will consider a smoothed version of the projectors using the summability kernels [19]. That is, for some $0 < \alpha < 2\pi$, define $\varphi_\alpha : \mathbb{S} \times \mathbb{S} \mapsto \mathbb{R}^+:

$$\varphi_\alpha(x, y) = \begin{cases} \frac{K}{\alpha} \exp \left(\frac{-1}{1 - (\frac{x-y}{\alpha})^2} \right) \frac{d(x,y)}{\alpha} & \text{if } \left|\frac{x-y}{\alpha}\right| < 1 \\ 0 & \text{otherwise} \end{cases},$$

where $d(x, y)$ is the Euclidean metric on $\mathbb{S}$ and $K = (\int_{-1}^{1} \exp(\frac{-1}{1-x^2}) dx)^{-1}$. Now replace the indicator function with:

$$\chi_{D_n}(\theta) = \int_{\mathbb{S}} \varphi_\alpha(\theta, \xi) \chi_D(\xi) d\xi,$$

and define:

$$(6.3) \quad S_{D_n} = \int_{\mathbb{S}} \chi_{D_n}(\theta) dS_{\theta} g, \quad S_{n,D_n} g_n = \sum_{\theta_{n,k} \in \mathbb{S}} \chi_{D_n}(\theta_{n,k}) S_{n,\theta_{n,k}} g_n.$$

We have the following corollary.

Corollary 6.3 (Convergence of spectral projectors). Let $T : X \mapsto X$ satisfy the hypothesis of theorem 4.1 and suppose that $\{T_n : \mathcal{P} \mapsto \mathcal{P}\}_{n=1}^{\infty}$ is a sequence of discrete maps that periodically approximates $T$ in the sense of (4.1). Given any $0 < \alpha < 2\pi$ and interval $D \subset \mathbb{S}$, it follows that:

$$\lim_{n \to \infty} \|S_{D_n} g - S_{n,D_n} g_n\| = 0,$$

where $g \in L^2(X, \mathcal{M}, \mu)$.

Note that the squaring of the norms in theorem 5.3 is immaterial for a finite sum.
6.3. Approximation of the spectral density function. Recall the definitions of the spectral density function \( \rho(\theta; g) \in \mathcal{D}(\mathbb{S}) \), along with its discrete analogue (3.8):

\[
\rho_n(\theta; g_n) = \sum_{k=1}^{g(n)} \| S_{n, \theta_n, k} g_n \|^2 \delta(\theta - \theta_n, k)
\]

To assess the convergence of \( \rho_n(\theta; g_n) \) to \( \rho(\theta; g) \), we again make use of the summability kernels (6.2). The following result may be established.

**Theorem 6.4** (Approximation of the spectral density function). Let:

\[
\rho_{\alpha}(\theta; g) := \int_{\mathbb{S}} \varphi_{\alpha}(\theta, \xi) \rho(\xi; g) d\xi, \quad \rho_{\alpha,n}(\theta; g_n) := \int_{\mathbb{S}} \varphi_{\alpha}(\theta, \xi) \rho_n(\xi; g_n) d\xi.
\]

Then:

\[
\lim_{n \to \infty} \rho_{\alpha,n}(\theta; g_n) = \rho_{\alpha}(\theta; g), \quad \text{uniformly.}
\]

**Proof.** To prove uniform convergence, we will establish that: (i) \( \rho_{\alpha}(\theta; g) - \rho_{\alpha,n}(\theta; g_n) \) forms an equicontinuous family, and (ii) \( \rho_{\alpha,n}(\theta; g_n) \) converges to \( \rho_{\alpha}(\theta; g) \) in the \( L^2 \)-norm. Uniform convergence of \( \rho_{\alpha,n}(\theta; g_n) \) to \( \rho_{\alpha}(\theta; g) \) is an immediate consequence of these facts. Indeed, if this wasn’t the case, there would exist an \( \epsilon > 0 \) and a subsequence \( n_k \) such that \( |\rho_{\alpha}(\theta; g) - \rho_{\alpha,n_k}(\theta; g_{n_k})| \geq \epsilon \) for all \( k \in \mathbb{N} \). But by equicontinuity, we may choose a \( \delta > 0 \) such that:

\[
d(\rho_{\alpha}(\phi; g) - \rho_{\alpha,n}(\phi; g_n), \rho_{\alpha}(\theta; g) - \rho_{\alpha,n}(\theta; g_n)) < \epsilon/2,
\]

whenever \( d(\phi, \theta) < \delta \). This leads to a contradiction to (ii) as \( ||\rho_{\alpha}(\cdot; g) - \rho_{\alpha,n_k}(\cdot; g_{n_k})||_2 \geq \epsilon/2\sqrt{\delta} \). What follows next is a derivation of the claims (i) and (ii):

(i) To show that \( \rho_{\alpha}(\theta; g) - \rho_{\alpha,n}(\theta; g_n) \) is an equicontinuous family, we will confirm that its derivative \( \rho'_{\alpha,n}(\theta; g_n) \) is uniformly bounded. Consider the Fourier expansion of \( \rho_{\alpha}(\theta; g) - \rho_{\alpha,n}(\theta; g_n) \):

\[
\rho_{\alpha}(\theta; g) - \rho_{\alpha,n}(\theta; g_n) = \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} b_n(l; g) e^{il\theta}, \quad b_n(l; g) := \int_{\mathbb{S}} e^{-il\theta} (\rho_{\alpha}(\theta; g) - \rho_{\alpha,n}(\theta; g_n)) d\theta.
\]

According to the spectral theorem of unitary operators [2], we have by construction that:

\[
a(l; g) := \int_{\mathbb{S}} e^{-il\theta} \rho(\theta; g) d\theta = \langle g, U_l g \rangle, \quad a_n(l; g_n) := \int_{\mathbb{S}} e^{-il\theta} \rho_n(\theta; g_n) d\theta = \langle g, U_l^n g_n \rangle, \quad l \in \mathbb{Z}.
\]

The functions \( \rho_{\alpha}(\theta; g) \) and \( \rho_{\alpha,n}(\theta; g_n) \) are defined as convolutions with a \( C^\infty \) smooth function, and recognizing that convolutions imply pointwise multiplication in Fourier domain, we obtain:

\[
b_n(l; g) = da_n(l; g) - a_n(l; g_n),
\]

where:

\[
d_{a}(l) := \int_{\mathbb{S}} e^{-il\theta} \varphi_{\alpha}(\theta, 0) d\theta \quad \text{and} \quad |d_{a}(l)| \leq \frac{C_{\alpha}}{1 + |l|^N} \text{ for every } N \in \mathbb{N}.
\]

Now examining the derivative \( \rho'_{\alpha,n}(\theta; g_n) \) more closely, we see that:

\[
|\rho'_{\alpha,n}(\theta; g_n)| = \left| \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} ilb_n(l; g) e^{il\theta} \right| \leq \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} |l||b_n(l; g)| \leq \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} |l||d_{a_n}(l) - a_n(l; g_n)| \leq \frac{C_{\alpha}}{1 + |l|^N},
\]

which is a convergent sum for \( N \geq 3 \). Notice that summability is possible because the constant \( C_{\alpha} \) only depends on \( \alpha \) and is independent of \( n \).

(ii) To show that \( \rho_{\alpha,n}(\omega; g_n) \) converges to \( \rho_{\alpha}(\omega; g) \) in the \( L^2 \)-norm, we will use Parseval’s identity to confirm that the sum \( \sum_{l \in \mathbb{Z}} |b_n(l; g)|^2 \) can be made arbitrarily small. At first, note that:

\[
a(l; g) - a_n(l; g_n) = \langle g, U_l g - U_l^n g_n \rangle - \langle g - g_n, U_l^n g_n \rangle.
\]
By the triangle inequality and Cauchy-Schwarz, we obtain:

\[ |a(l; g) - a_n(l; g_n)| \leq \|g\| \|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\| + \|g - g_n\| \|\mathcal{U}^l_{a_n} g_n\| \]
\[ \leq \|g\| (\|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\| + \|g - g_n\|). \]

Let \( \epsilon > 0 \) and choose \( k \in \mathbb{N} \) such that:

\[ (6.5) \quad \sum_{|l| > k} |d_\alpha(l)|^2 \leq \epsilon. \]

This is always possible, because \( \varphi_\alpha(\theta, 0) \) is a \( C^\infty \) smooth function, and therefore also square-integrable.

The following upper bound can be established:

\[ \sum_{l = -\infty}^{\infty} |b_n(l; g)|^2 = \sum_{l = -\infty}^{\infty} |d_\alpha(l)|^2 |a(l; g) - a_n(l; g_n)|^2 \]
\[ \leq \sum_{l = -\infty}^{\infty} |d_\alpha(l)|^2 (\|g\| (\|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\| + \|g - g_n\|))^2 \]
\[ \leq \|g\|^2 \sum_{l = -\infty}^{\infty} |d_\alpha(l)|^2 (\|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\|^2 + 2 \|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\| \|g - g_n\| + \|g - g_n\|^2) \]
\[ = \|g\|^2 \sum_{|l| = -k}^{k} |d_\alpha(l)|^2 \left(\|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\|^2 + 2 \|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\| \|g - g_n\| + \|g - g_n\|^2\right) \]
\[ + |g|^2 \sum_{|l| > k} |d_\alpha(l)| \left(\|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\|^2 + 2 \|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\| \|g - g_n\| + \|g - g_n\|^2\right) \]
\[ \leq \|g\|^2 \max_{-k \leq l \leq k} |d_\alpha(l)|^2 \sum_{l = -k}^{k} \|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\|^2 + 16 |g|^2 \sum_{|l| > k} |d_\alpha(l)|^2 \]
\[ + |g|^2 \|g - g_n\| \sum_{l = -k}^{k} |d_\alpha(l)|^2 (2 \|\mathcal{U}^l g - \mathcal{U}^l_{a_n} g_n\| + \|g - g_n\|). \]

Now apply (6.5) and theorem 5.3 to complete the proof. \( \square \)

Part 2. Applications: A convergent numerical method for volume-preserving maps on the \( m \)-torus

The second part of the paper is devoted to how spectral projections (2.3) and density functions (2.4) are computed in practice. We will limit ourselves to volume-preserving maps on the \( m \)-torus. That is, \( X = \mathbb{T}^m \) and is equipped with the geodesic metric:

\[ (6.6) \quad d(x, y) := \max_{k \in \{1, \ldots, m\}} \min\{|x_i - y_i|, 1 - x_i - y_i\}. \]

From here onwards, \( T : \mathbb{T}^m \to \mathbb{T}^m \) refers to an invertible, continuous Lebesgue measure-preserving transformation on the unit \( m \)-torus. In other words, \( \mu \) the Lebesgue measure is lebesgue measure and \( \mu(B) = \mu(T(B)) = \mu(T^{-1}(B)) \) for every \( B \in \mathcal{B}(\mathbb{T}^m) \), where \( \mathcal{B}(\mathbb{T}^m) \) denotes the Borel sigma algebra on \( \mathbb{T}^m \).

7. DETAILS OF THE NUMERICAL METHOD

The numerical method can be broken-down into three steps. The first step is the actual construction of the periodic approximation \( T_n : \mathcal{P}_n \to \mathcal{P}_n \). The second step is obtaining a discrete representation of the observable \( g_n \). The third step is the evaluation of the spectral projections and density functions. What follows next is a detailed exposition for each of these three steps.

7.1. Step 1: Construction of the periodic approximation. A periodic approximation can be obtained either through analytic means or by explicit construction. We will first discuss the general algorithm for constructing periodic approximations.
7.1.1. Discretization of volume-preserving maps. Following the principles in part I, call:
\[ P_{n,j} := \left[ \frac{j_1 - 1}{n}, \frac{j_1}{n} \right) \times \left[ \frac{j_2 - 1}{n}, \frac{j_2}{n} \right) \times \cdots \times \left[ \frac{j_m - 1}{n}, \frac{j_m}{n} \right) \]
where:
\[ \tilde{n} = Cn^a, \quad C, a \in \mathbb{N}, \]
and \( j = (j_1, j_2, \ldots, j_m) \) is a multi-index with \( j_i \in \{1, \ldots, \tilde{n}\} \). To simplify the notation in some cases, it is useful to alternate between the multi-index \( j \) and single-index \( j \in \{1, 2, \ldots, \tilde{n}^m\} \) through a lexicographic ordering whenever it is convenient. Consider a partitioning of the unit \( m \)-torus (with \( \mu(\mathbb{T}^m) = 1 \)) into the \( m \)-cubes:
\[ P_n = \{ p_{n,j} \}_{j=1}^{q(n)}, \quad q(n) = \tilde{n}^m \]
and note that this set is isomorphic to \( \mathbb{N}^m :\{j \in \mathbb{N}^m : 0 < j_i \leq \tilde{n}, i = 1, \ldots, m \} \).

The sequence of partitions \( \{P_n\}_{n=1}^{\infty} \) of the \( m \)-torus form a collection of refinements which satisfy the properties:
\[ \mu(p_{n,j}) = \frac{1}{\tilde{n}^m} \quad \text{and} \quad \text{diam}(p_{n,j}) = \frac{\sqrt{m}}{\tilde{n}}, \quad j \in \{1, 2, \ldots, \tilde{n}^m\}, \]
where \( \tilde{n} \) is defined as in (7.2). It is clear from these properties that \( \{P_n\}_{n=1}^{\infty} \) is a family of equal-measure partitions which meet the conditions: \( \mu(p_{n,j}) \to 0 \) and \( \text{diam}(p_{n,j}) \to 0 \) as \( n \to \infty \).

7.1.2. General algorithm for constructing periodic approximations. The construction algorithm which we propose here is a slight variant to the method proposed in [20]. Although, here we will show that the construction is fast if the underlying map is Lipschitz continuous.

The bipartite matching problem. A bipartite graph \( G = (X, Y, E) \) is a graph where every edge \( e \in E \) has one vertex in \( X \) and one in \( Y \). A matching \( M \subseteq E \) is a subset of edges where no two edges share a common vertex (in \( X \) or \( Y \)). The goal of the bipartite matching problem is to find a maximum cardinality matching, i.e. one with largest number of edges possible. A matching is called perfect if all vertices are matched. A perfect matching is possible if, and only if, the bipartite graph satisfies the so-called Hall’s marriage condition:

\[ \text{for every } B \subseteq X \text{ implies } |B| \leq |N_G(B)| \]
where \( N_G(B) \subseteq Y \) is the set of all vertices adjacent to some vertex in \( A \) (see e.g. [7]).

Bipartite matching problems belong to the class of combinatorial problems for which well-established polynomial-time algorithms exist, e.g. the Ford-Fulkerson algorithm may be used to find a maximum cardinality matching in \( \mathcal{O}(|V||E|) \) operations, whereas the Hopcroft-Karp algorithm does it in \( \mathcal{O}(\sqrt{|V||E|}) \) (see e.g. [4][9]).

The algorithm. In section 3.2 we showed that the solution to the bipartite matching problem of the graph \( G_n = (P_n, \mathcal{P}_n, E_n) \), \((p_{n,s}, p_{n,t}) \in E_n \) if \( \mu(T(p_{n,s}) \cap p_{n,t}) > 0 \)
has a perfect matching. This fact played a key role in proving theorem 3.11. In practice, it is not advisable to construct this graph explicitly since it requires the computation of set images.

Fortunately, a periodic approximation may be obtained from solving a related bipartite matching problem for which set computation are not necessary. Consider the partition \( \mathcal{P}_n \) and associate to each partition element \( p_{n,j} \in \mathcal{P}_n \) a representative point:
\[ x_{n,j} = \psi_n(p_{n,j}) := \left( \frac{j_1 - \frac{1}{2}}{n}, \ldots, \frac{j_m - \frac{1}{2}}{n} \right) \in p_{n,j}. \]
Define:
\[ \varphi_n(x) = \left( \frac{n x_1 + 1}{2}, \ldots, \frac{n x_m + 1}{2} \right), \]
to be the function that maps the representative points onto the lattice \( \mathbb{N}^m :\{j \in \mathbb{N}^m : 0 < j_i \leq \tilde{n}, i = 1, \ldots, m \} \), i.e. \( j = \varphi_n(x_{n,j}) \). If \( \lfloor \cdot \rfloor \) (\( \lceil \cdot \rceil \)) denote the floor (resp. ceil) function, we may define the following set-valued map:
\[ F^{(i)}(x) = \{ j \in \mathbb{N}^m : \lfloor \varphi_n(x_{i,j}) \rfloor - t + 1 \leq j_i \leq \lceil \varphi_n(x_{i,j}) \rceil + t - 1, \quad i = 1, \ldots, m \} \]
to introduce the family of neighborhood graphs:
\[ \hat{G}^{(t)}_n := (P_n, \mathcal{P}_n, \hat{E}^{(t)}_n), \quad (p_{n,s}, p_{n,t}) \in \hat{E}^{(t)}_n \quad \text{if} \quad \hat{t} \in F^{(i)}(T(x_{n,s})) \]
The proposal is to find a maximum cardinality matching for the bipartite graph \( \hat{G}^{(t)}_n \). Notice that \( \hat{E}^{(t)}_n \subseteq \hat{E}^{(s)}_n \) whenever \( s \geq t \). Hence, if no perfect matching is obtained for some value of \( t \), repeated dilations of the graph eventually will. The algorithm is summarized below.

Algorithm 7.1. To obtain a periodic approximation, do the following:
1. Initialize \( t = 1 \).

\footnote{Note that the partition elements \( p_{n,s} \in \mathcal{P}_n \) are interpreted here as vertices in a bipartite graph.}
2. Find the maximum cardinality matching of $\hat{G}_n^{(t)}$.
3. If the matching is perfect, assign matching to be $T_n$. Otherwise set $t \leftarrow t + 1$ and repeat step 2.

**Theorem 7.1 (Algorithmic correctness).** Algorithm 7.1 terminates in a finite number of steps and yields a map $T_n : \mathcal{P}_n \rightarrow \mathcal{P}_n$ with the desired asymptotic property described in (4.1).

**Proof.** For large enough $t$, the graph $\hat{G}_n^{(t)}$ eventually becomes the complete graph. Hence, the algorithm must terminates in a finite number of steps. However, we must show that the algorithm terminates way before that, yielding a map with the asymptotic property (4.1).

In [12], it was proven that for any Lebesgue measure-preserving map $T : \mathbb{T}^d \rightarrow \mathbb{T}^d$, there exists a periodic map for which:

\[
\sup_{p_n,j \in \mathcal{P}_n} \inf_{x \in \mathbb{T}^d} \max \{d(x, \psi_n(p_n,j)), d(T(x), \psi_n(T_n(p_n,j)))\} \leq \frac{1}{2n},
\]

where $d(\cdot, \cdot)$ refers to the metric defined in (6.6). By virtue of this fact, if one is able to construct a bipartite graph $\hat{G}_n = (\mathcal{P}_n, \mathcal{P}_n, \tilde{E}_n)$ for which:

\[
(p_n, \hat{p}_n) \in \tilde{E}_n \text{ if } \inf_{x \in \mathcal{P}_n} d(T(x), \psi_n(p_n, \hat{p}_n)) \leq \frac{1}{2n},
\]

then this bipartite graph, or any super-graph, should admit a perfect matching. Indeed, if we would assume the contrary, then this would imply that one is required to construct a periodic approximation $T_n$ for which $d(T(x), \psi_n(T_n(p_n,j))) \geq \frac{1}{2n}$, an immediate contradiction of (7.8).

This fact has implications on the graph $\hat{G}_n^{(t)}$. As soon as $t$ hits a value for which $\hat{E}_n \supseteq \tilde{E}_n$, the graph $\hat{G}_n^{(t)}$ is guaranteed to have a perfect matching (although it may occur even before that). Notice that by compactness of the $m$-torus, $T$ admits a modulus of continuity $\omega : [0, \infty) \rightarrow [0, \infty)$ with $d(T(x), T(y)) \leq \omega(d(x, y))$. An upper bound on the value of $t$ for which $\hat{G}_n^{(t)}$ becomes a supergraph of $\hat{G}_n$ is given by:

\[
t^* = \lceil \overline{n}/(1/\bar{n}) + 1/2 \rceil \leq (\bar{n} + 1) \left(\omega(1/\bar{n}) + \frac{1}{2\bar{n}}\right).
\]

That is, the algorithm should terminate for $t \leq t^*$.

By assuming that the algorithm terminates at $t^*$, a worst-case error bound can be established for any map generated from the bipartite graphs. We see that:

\[
\sup_{p_n,j \in \mathcal{P}_n} d(\psi_n(T_n(p_n,j)), T(\psi_n(p_n,j))) \leq \frac{t^*}{\bar{n}} \leq \frac{\bar{n} + 1}{\bar{n}} \left(\omega(1/\bar{n}) + \frac{1}{2\bar{n}}\right).
\]

Next, using the inequality:

\[
d_H(T(p_n,j), T_n(p_n,j)) \leq \text{diam}(T_n(p_n,j)) + d(\psi_n(T_n(p_n,j)), T(\psi_n(p_n,j))) + \text{diam}(T(p_n,j)),
\]

it follows that:

\[
\lim_{n \rightarrow \infty} d_H(T(p_n,j), T_n(p_n,j)) \leq \lim_{n \rightarrow \infty} \frac{1}{\bar{n}} + \frac{\bar{n} + 1}{\bar{n}} \left(\omega(1/\bar{n}) + \frac{1}{2\bar{n}}\right) + \omega(1/\bar{n}) = 0.
\]

From here onwards, one can proceed in the same fashion as the proofs of he proofs of lemma 4.3 and theorem 4.1 to show that $\{T_n\}_{n=1}^\infty$ indeed satisfies the convergence property (4.1). \qed

**Complexity of Algorithm 7.1.** Since the number of edges in $\hat{G}_n^{(t)}$ is proportional to the number of nodes for small $t$-values, the Hopcroft-Karp algorithm will solve a matching problem in $O(n^{3/2})$ operations. If additionally, the map $T$ is assumed to be Lipschitz, i.e. $\omega(d(x, y)) = Kd(x, y)$, the algorithm terminates before $t^* = \lceil K + \frac{1}{2} \rceil$, i.e. independently of the discretization level $\bar{n}$. Hence, $O(n^{3/2})$ is also the overall time-complexity of algorithm.

We note that explicit storage of the periodic map is $O(\bar{n}^m)$ in complexity. Ideally, it is desirable to obtain a formulaic expression for $T_n$.

### 7.1.3. Analytic constructions of periodic approximations.

With the boxed partition of the $m$-torus [7.1], a subclass of maps may be periodically approximated analytically through simple algebraic manipulations (see e.g. [14]). In addition to some purely mathematical transformations, this also includes maps which arise in physical problems. In fact, any perturbed Hamiltonian twist map like Chirikov’s Standard Map [8], or certain kinds of volume preserving maps such as Feingold’s ABC map [15], belong to this category.

A key observation is that these Lebesgue measure-preserving transformations consists of compositions of the following basic operations:

1. A signed permutation:

\[
T(x) = \begin{bmatrix}
\xi(1)x_{\sigma(1)} \\
\vdots \\
\xi(m)x_{\sigma(m)}
\end{bmatrix}, \text{ with } \sigma(\cdot) \text{ a permutation and } \xi(\cdot) \in \{-1, 1\}.
\]

(7.9)
which is periodically approximated by:

$$T_n(p_{n,j}) = \psi_n^{-1} \circ \varphi_n^{-1} \circ \left( \begin{bmatrix} \xi(1) & \vdots & \xi(m) \end{bmatrix} \right) \mod n \circ \varphi_n \circ \psi_n(p_{n,j}).$$

2. A translation:

$$T(x) = \begin{bmatrix} x_1 + \omega_1 \\ \vdots \\ x_m + \omega_m \end{bmatrix} \mod 1,$$

which is periodically approximated by:

$$T_n(p_{n,j}) = \psi_n^{-1} \circ \varphi_n^{-1} \circ \left( \begin{bmatrix} \lfloor \varphi_n(\omega_1) \rfloor \\ \vdots \\ \lfloor \varphi_n(\omega_m) \rfloor \end{bmatrix} \right) \mod n \circ \varphi_n \circ \psi_n(p_{n,j}).$$

where $\lfloor \cdot \rfloor$ denotes the nearest-integer function.

3. A shear:

$$T(x) = \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ x_{i+1} + f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \end{bmatrix},$$

which is periodically approximated by:

$$T_n(p_{n,j}) = \psi_n^{-1} \circ \varphi_n^{-1} \circ \left( \begin{bmatrix} j_1 + \lfloor \varphi_n(f(j_1), \ldots, \varphi_n^{-1}(j_{i-1}), \varphi_n^{-1}(j_{i+1}), \ldots, \varphi_n^{-1}(j_m)) \rfloor \\ \vdots \\ j_m \end{bmatrix} \right) \mod n \circ \varphi_n \circ \psi_n(p_{n,j}).$$

Any map which is expressible as a finite composition of these operations can be periodically approximated through approximation of each its component. That is, the $T(x) = T_n \circ \cdots \circ T_1(x)$ is approximated by $T_n(x) = T_{n_1} \circ \cdots \circ T_{n_k}(x)$. It is not hard to show that this would lead to a sequence of approximations that satisfy (4.1). Examples of maps which may be approximated in this manner are:

- Arnold’s Cat map [5]:

$$T(x) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mod 1,$$

which is the composition of a permutation and shear map:

$$T(x) = T_2 \circ T_1 \circ T_2 \circ T_1(x),$$

where: $T_1(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $T_2(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

- Anzai’s example [3] of a skew product transformation:

$$T(x) = \begin{bmatrix} x_1 + \gamma \\ x_1 + x_2 \end{bmatrix} \mod 1,$$

which is the composition of a shear map and a translation:

$$T(x) = T_2 \circ T_1(x),$$

where: $T_1(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $T_2(x) = \begin{bmatrix} x_1 + \gamma \\ x_2 \end{bmatrix}$.

- Chirikov Standard map [8]:

$$T(x) = \begin{bmatrix} x_1 + x_2 + K \sin(2\pi x_1) \\ x_2 + K \sin(2\pi x_1) \end{bmatrix} \mod 1,$$

which is the composition of two distinct shear maps in orthogonal directions:

$$T(x) = T_2 \circ T_1(x),$$

where: $T_1(x) = \begin{bmatrix} x_1 \\ x_2 + K \sin(2\pi x_1) \end{bmatrix} \mod 1$, $T_2(x) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mod 1.$
• The ABC-map of Feingold [15] which is the composition of 3 shear maps:

\[ T(x) = T_3 \circ T_2 \circ T_1(x), \]

where:

\[ T_1(x) = \begin{bmatrix} x_1 + A \sin(2\pi x_1) + C \cos(2\pi x_3) \\ x_2 \\ x_3 \end{bmatrix}, \quad T_2(x) = \begin{bmatrix} x_1 \\ x_2 + B \sin(2\pi x_1) + A \cos(2\pi x_3) \\ x_3 \end{bmatrix}, \]

\[ T_3(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + C \sin(2\pi x_2) + B \cos(2\pi x_1) \end{bmatrix}. \]

7.2. Step 2: obtain a discrete representation of the observable. The averaging operator (3.3) is used to obtain a discrete representation to the observable. The operator (3.3) is an orthogonal projector which maps observables in \( L^2(\mathbb{T}^m, \mathcal{B}(\mathbb{T}^m), \mu) \) onto their best approximations in \( L^\infty(\mathbb{T}^m, \mathcal{B}(\mathbb{T}^m), \mu) \). In practice, it makes no sense to explicitly evaluate this projection since we are mostly in the spectra of well-behaved functions. A discrete representation of the observable may also be obtained by simply sampling the function at the representative points (7.4). The averaging operator (3.3) is replaced with:

\[ \langle W_{\mu} g \rangle (x) = \bar{g}_n(x) := \sum_{j=1}^{q(n)} \delta(x_{n,j})\chi_{p_{n,j}}(x). \]

Indeed, for observables that are continuous, it is not hard to show that \( \|g_n - \bar{g}_n\| \to 0 \) as \( n \to \infty \).

7.3. Step 3: Computing spectral projection and density function. The discrete Koopman operator (3.4) is isomorphic to the permutation matrix \( U_n \in \mathbb{R}^{q(n) \times q(n)} \) by the similarity transformation:

\[ [U_n]_{ij} := \langle \chi_{p_{n,i}}, U_n \chi_{p_{n,j}} \rangle = \begin{cases} 1 & \text{if } T(p_{n,j}) = p_{n,i} \\ 0 & \text{otherwise} \end{cases}. \]

This property is exploited to evaluate the spectral projections (3.7) and density functions (??) very efficiently.

7.3.1. The eigen-decomposition of a permutation matrix. Permutation matrices admit an analytical expression for their eigen-decomposition. A critical part of this expression is the so-called cycle decomposition which decouples the permutation matrix into its cycles. More specifically, associated to every permutation \( U_n \in \mathbb{R}^{q(n) \times q(n)} \) one can find a different permutation matrix \( P_n \in \mathbb{R}^{q(n) \times q(n)} \) such that by a similarity transformation, we have:

\[ P_n^T U_n P_n = C_n \]

where \( C_n \) is the circulant matrix:

\[ C_n = \begin{bmatrix} C_n^{(1)} \\ \vdots \\ C_n^{(s)} \end{bmatrix}, \quad C_n^{(k)} := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n^{(k)} \times n^{(k)}}. \]

The decomposition of (7.15) into its cyclic subspaces is obtained as follows. Initialize \( U_{n,1} := U_n \) and call:

\[ P_n^{(1)} := \begin{bmatrix} U_{n,1} e_1 & \cdots & U_{n,1}^{(s)} e_1 & e_{n^{(s)+1}} & \cdots & e_{q(n)} \end{bmatrix}, \quad \text{with } U_{n,1}^{(1)} e_1 = e_1. \]

A similarity transformation yields:

\[ U_{n,2} := P_n^{(1)T} U_{n,1} P_n^{(1)} = \begin{bmatrix} C_n^{(1)} \\ \tilde{U}_{n,1} \end{bmatrix}. \]

Repeating this process for all \( s \) cycles should result in \( U_{n,s} = C_n \) and \( P_n = P_n^{(1)} \cdots P_n^{(s)}. \)

It is well-known that the Discrete Fourier Transform (DFT) matrix diagonalizes any circulant matrix. In particular, for \( C_n^{(k)} \in \mathbb{R}^{n^{(k)} \times n^{(k)}} \) the decomposition reads:

\[ C_n^{(k)} = (\text{DFT})_{n^{(k)}} A_{n^{(k)}} (\text{DFT})_{n^{(k)}}, \]
May proceed to compute an approximation of the spectral density function (6.4):

\[ (\text{DFT})_{n(k)} = \frac{1}{\sqrt{n(k)}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n(k)-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n(k)-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n(k)-1)} & \omega^{2(n(k)-1)} & \cdots & \omega^{(n(k)-1)(n(k)-1)} \end{bmatrix}, \quad \Lambda_n = \begin{bmatrix} \omega^0 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \omega^{n(k)-1} \end{bmatrix}, \]

and \( \omega = \exp\left(\frac{2\pi i}{n(k)}\right) \). By combining (7.16) with (7.17), the following explicit expression for the eigen-decomposition of (7.15) is obtained:

\[ (7.18) \quad U_n = V_n \Lambda_n V_n^*, \]

where:

\[ V_n = P_n \begin{bmatrix} (\text{DFT})^*_{n(1)} \\ \vdots \\ (\text{DFT})^*_{n(n)} \end{bmatrix}, \quad \Lambda_n = \begin{bmatrix} \Lambda_{n(1)} \\ \vdots \\ \Lambda_{n(n)} \end{bmatrix}. \]

### 7.3.2. Step 3a: Computing the spectral projection

With the help of (7.18) one may also obtain a closed-form expression for the vector representation of (3.7). Indeed, if \( g_n \in \mathbb{C}^{n(n)} \) and \( S_{n,D} \in \mathbb{C}^{q(n) \times q(n)} \) respectively denote:

\[ [g_n]_j = \langle \chi_{p_{n,j}}; g_n \rangle, \quad [S_{n,D}]_{ij} := \langle \chi_{p_{n,i}}; S_{n,D} \chi_{p_{n,j}} \rangle, \]

then it is straightforward to show that (3.7) is reduced to the formula:

\[ (7.19) \quad S_{n,D} g_n = V_n \mathbf{1}_D (\Lambda_n) V_n^* g_n, \quad \text{where:} \quad \mathbf{1}_D (e^{i\theta}) = \begin{cases} 1 \quad \text{if } \theta \mod 2\pi \in D \\ 0 \quad \text{otherwise} \end{cases}. \]

The evaluation of (7.19) will involve finding the cycle decomposition (7.16), which in turn requires traversing the cycles of the permutation matrix \( U_n \in \mathbb{R}^{n(n) \times q(n)} \). This requirement is equivalent to traversing the trajectories of the discrete map \( T_n : P_n \mapsto P_n \). We arrive to the following algorithm.

**Algorithm 7.2.** To compute the spectral projection, do the following:

1. Initialize a vector \( f \in \mathbb{R}^{q(n)} \) of all zeros and set \( j = 1 \) and \( k = 1 \).
2. Given some \( j \in \{1, \ldots, q(n)\} \), if \( |f_j| = 1 \) move on to step 4 directly. Otherwise, traverse the trajectories of the discrete map until it has completed a cycle, i.e.

\[ p_{n,j(l)} = T^n_{n}(p_{n,j(1)}), \quad l = 1, 2, \ldots, n(k), \quad \text{where:} \quad j(1) = j \quad \text{and} \quad p_{n,j(l)} = T^n_{n}(p_{n,j(1)}). \]

To demarcate visited partition elements, set \( |f_j| = 1 \) for \( l = 1, 2, \ldots, n(k) \).

3. Denote by \( \omega = \exp\left(\frac{2\pi i}{n(n)}\right) \). Apply the FFT and inverse-FFT algorithms to evaluate:

\[ [S_{n,D} g_n]_{l(t)} = \frac{1}{n} \sum_{\omega^{(l-1)(t-1)} \in D} \omega^{(l-1)(t-1)} \left( \sum_{r=1}^{q(n)} \omega^{(t-1)(r-1)} g(\psi(p_{n,j(r)})) \right) \]

for \( l = 1, 2, \ldots, n(k) \). Increment \( k \leftarrow k + 1 \).

4. Move on to the next partition element \( p_{n,j} \in P_n \), i.e. \( j \leftarrow j + 1 \), and repeat step 2 until all partition elements have been visited.

**Complexity of Algorithm 7.2.** The most dominant part of the computation is the application of the FFT and inverse-FFT algorithms on the cycles. Hence, the time-complexity of the algorithm is \( \mathcal{O}\sum_{l=1}^{n(k)} n(l)(\log n(l)) \). In the worst-case scenario, it may occur that the permutation (7.15) consist of one single large cycle. In that case, we obtain the conservative estimate \( \mathcal{O}(dn^d \log n) \). We note that the evaluation of the spectral projections may be further accelerated using sparse FFT techniques. This is notably true for projections wherein the interval \( D \subset S \) is small, leading to a highly rank-deficient \( S_{n,D} \in \mathbb{C}^{q(n) \times q(n)} \).

### 7.3.3. Step 3b: Computing the spectral density function

By modifying step 3 of the previous algorithm, one may proceed to compute an approximation of the spectral density function (6.4):

\[ \rho_{\alpha,n}(\theta; g_n) := \int_S \varphi_{\alpha}(\xi, \theta) \rho_{\alpha}(\xi; g_n) d\xi, \]

where:

\[ \varphi_{\alpha}(x, y) = \begin{cases} \frac{\alpha}{\pi} \exp\left(\frac{-1}{\pi(\alpha^2 - \alpha^2)}\right) & \frac{d(x,y)}{\alpha^2} < 1 \\ 0 & \text{otherwise} \end{cases}, \quad \rho_{\alpha}(\theta; g_n) = \sum_{k=1}^{q(n)} \| S_{n,\theta_{n,k}} g_n \|_2^2 \delta(\theta - \theta_{n,k}). \]

If (6.4) needs to evaluated at some collection of points \( \{\theta_j\}_{j=1}^{R} \subset S \), the algorithm would look as follows:
Algorithm 7.3. To evaluate the spectral density function on \( \{ \theta_j \}^R_{j=1} \subset \mathbb{S} \), do the following:
1. Initialize a vector \( f \in \mathbb{R}^{q(n)} \) of all zeros and also initialize \( \rho_{a,n}(\theta_j; g_n) = 0 \) for \( r = 1, \ldots, R \). Furthermore, set \( j = 1 \) and \( k = 1 \).
2. Given some \( j \in \{ 1, \ldots, q(n) \} \), if \( |f|_j = 1 \) move on to step 4 directly. Otherwise, traverse the trajectories of the discrete map until it has completed a cycle, i.e. 
\[
p_{n,j}(l) = T^l_n(p_{n,j(1)}), \quad l = 1, 2, \ldots, n(k), \quad \text{where: } j(1) = j \quad \text{and} \quad p_{n,j(1)} = T^{n(k)}_n(p_{n,j(1)}).
\]
To demarcate visited partition elements, set \( |f|_j(l) = 1 \) for \( l = 1, 2, \ldots, n(k) \).
3. Denote by \( \omega = \exp(\frac{2\pi i}{m} j) \). Apply the FFT algorithms to update:
\[
\rho_{a,n}(\theta_j; g_n) \leftarrow \frac{K}{na} \sum \frac{1}{a'd(2\pi(n-1)/a)} \left\{ \exp \left( \frac{-1}{1 - \left( d \frac{2\pi (t-1) \theta_j}{n} \right)^a} \right) \left| \sum_{l=1}^{n(k)} \omega^{(t-1)l} g(p_{n,j(l)}) \right|^2 \right\} \quad \text{for } r = 1, 2, \ldots, R.
\]
Increment \( k \leftarrow k + 1 \).
4. Move on to the next partition element \( p_{n,j} \in \mathcal{P}_n \), i.e. \( j \leftarrow j + 1 \), and repeat step 2 until all partition elements have been visited.

Complexity of Algorithm [7.3] Since \( \alpha > 0 \) is typically small, the most dominant part of the computation will again be the application of the FFT algorithm in step 3. Assuming that \( R \ll n(k) \), the time-complexity of the algorithm will be \( O(\sum_{t=1}^{n(k)} n(t) \log n(t)) \), reducing to \( O(dn^4 \log n) \) in the worst-case scenario of a cyclic permutation.

8. Canonical examples

In this section we analyze the behavior of our numerical method on three canonical examples of Lebesgue measure-preserving transformations on the m-torus: the translation map \( \Psi \), Arnold’s cat map \( \Phi \), and Anzai’s skew-product transformation \( \tilde{\Phi} \). The considered maps permit an easy closed form solution for both the spectral projectors and density functions, hence making them a good testing bed to analyze the performance of the numerical method. Furthermore, the examples cover the full range of possible “spectral scenario”s that one may encounter in practice: the first map has a fully discrete spectrum, the second one has a fully continuous spectrum, and the third has a mixed spectrum.

8.1. Translation on the m-torus. Consider a translation on the torus \( \Psi \):
\[
T(x) = \begin{bmatrix} x_1 + \omega_1 \\ \vdots \\ x_m + \omega_m \end{bmatrix} \mod 1.
\]
This is the type of map one would typically obtain from a Poincare section of an integrable Hamiltonian expressed in action-angle coordinates. Denote by \( \mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m \) and \( \omega = (\omega_1, \ldots, \omega_m) \in \mathbb{R}^m \). The map \( \Psi \) is ergodic if and only if \( k \cdot \omega \notin \mathbb{Z} \) for all \( k \in \mathbb{Z}^m \) and \( k 
eq 0 \).

8.1.1. Singular and regular subspaces. The Koopman operator associated with \( \Psi \) has a fully discrete spectrum. Moreover, the Fourier basis elements turn out to be eigenfunctions for the operator, i.e.
\[
U e^{2\pi i(k \cdot x)} = e^{2\pi i(k \cdot \omega)} e^{2\pi i(k \cdot x)}.
\]
The singular and regular subspaces are respectively:
\[
H_s = \text{span}\{e^{2\pi i(k \cdot x)} : k \in \mathbb{Z}^m\}, \quad H_r = \emptyset.
\]

8.1.2. Spectral density function. The spectral density function for observables in the form:
\[
g = \sum_{k \in \mathbb{Z}^m} a_k e^{2\pi i(k \cdot x)} \in L^m(T^m, \mathcal{B}(T^m), \mu),
\]
is given by the expression:
\[
\rho(\theta; g) = \sum_{k \in \mathbb{Z}^m} |a_k|^2 \delta(\theta - 2\pi (k \cdot \omega) \mod 2\pi).
\]
Noteworthy to mention is that the spectra of the operator for the map \( \Psi \) can switch from isolated to dense with arbitrarily small perturbations to \( \omega \in \mathbb{R}^m \).

5 Expect for the trivial eigenvalue at 1
8.1.3. Spectral projectors. Given an interval \( D = [a, b) \subset \mathbb{S} \), we have the following straightforward expression for the spectral projectors:

\[
S_D g = \sum_{ \frac{2\pi (k \cdot \omega)}{2\pi k \in D} \mod 2\pi} a_k e^{2\pi i (k \cdot x)}.
\]

8.1.4. Numerical results. In fig. 4, we plot the spectral density function (2.4) for the case when \( d = 1, \omega_1 = 1/3 \), and:

\[
g(x) = \frac{\sin(4\pi x)}{1 + \cos^2(2\pi x) + \sin(7\pi x)}.
\]

As the partitions are refined, it is evident from fig. 4 that the energy becomes progressively more concentrated around the eigenvalues of the true spectra at \( e^{i \frac{2\pi (k-1)}{6}} \), with \( k = 1, 2, 3 \).

![Figure 4](image-url)

**Figure 4.** Approximations of the spectral density function (2.4) for the translation map (7.10) with \( m = 1 \) and \( \omega_1 = 1/3 \). Spectra is approximated for the observable (8.2) with \( \alpha = 2\pi/500 \). The black curve is approximation and the red curve is the true density.

![Figure 5](image-url)

**Figure 5.** Results obtained for the translation map (7.10) with \( m = 2 \) and \( (\omega_1, \omega_2) = (\frac{1}{2}, \frac{1}{3}) \). Computations were performed for the observable (8.3) at a spatial discretization of \( \tilde{n} = 1000 \).
In fig. 5 we examine the case when \( d = 2 \) and \( (\omega_1, \omega_2) = (\frac{1}{2}, \frac{1}{3}) \). The trajectories of this map are all 6-periodic, and eigenvalues are also located 6th roots of unity (i.e. \( e^{\frac{2\pi ik-1}{6}} \), with \( k = 1, 2, \ldots, 6 \)). Shown are spectral projections (3.7) of the observable:

\[
g(x) = \sin(2\pi x_1) \cos(2\pi x_2) + \sin(\pi x_2) + \frac{1}{\sin^4(\pi x_1)} + 1
\]

around narrow intervals of these eigenvalues.

8.2. Arnold’s cat map. Consider the area-preserving map \( T : \mathbb{T}^2 \mapsto \mathbb{T}^2 \), defined in (7.12):

\[
T(x) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mod 1.
\]

Arnold’s cat map (7.12) is a well-known example of a Anosov diffeomorphism, in which the dynamics are locally characterizable by expansive and contractive directions. The map (7.12) has positive Kolmogorov entropy, is strongly mixing, and therefore also ergodic.

8.2.1. Singular and regular subspaces. The Koopman operator associated with Arnold’s cat map is known to have a “Lebesgue spectrum” (see e.g. [5]). A Lebesgue spectrum implies the existence of an orthonormal basis for \( L^2(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), \mu) \) consisting of the constant function and \( \{\varphi_{s,t}\} \in \mathcal{L} \in \mathbb{Z} \), such that:

\[
\mathcal{U}^l \varphi_{s,t} = \varphi_{s+t l}, \quad l \in \mathbb{Z}.
\]

It follows from such a basis that the eigenvalue at \( \lambda = 1 \) is simple, i.e. consisting only the constant function, with the remaining part of the spectrum absolutely continuous. In the case of the Cat map, the basis \( \{\varphi_{s,t}\} \in \mathcal{L} \in \mathbb{Z} \) is just a specific re-ordering of the Fourier elements \( \{e^{2\pi ik(l_1x_1+l_2x_2)}\} \in \mathbb{Z}^2 \). Indeed, by applying the Koopman operator (2.1) onto a Fourier element, we observe the transformation:

\[
\mathcal{U} e^{2\pi i(k_1 x_1+k_2 x_2)} = e^{2\pi i(k_1 x_1+k_2 x_2)}, \quad k'_1 = 2k_1 + k_2, \quad k'_2 = k_1 + k_2, \quad (k_1, k_2) \in \mathbb{Z}^2.
\]

Partitioning \( \mathbb{Z}^2 \) into the orbits of (8.4), i.e. \( \mathbb{Z}^2 = \xi_0 \cup \xi_1 \cup \xi_2 \cup \xi_3 \ldots \), we notice that all orbits, except for \( \xi_0 = \{(0, 0)\} \), consists of countable number of elements. The singular and regular subspaces are respectively:

\[
H_s = \text{span}\{1\}, \quad H_r = \text{span}\{e^{i2\pi(k_1 x_1+k_2 x_2)} : (k_1, k_2) \neq 0\}.
\]

8.2.2. Spectral density function. The spectral density function (6.4) of a generic observable is found by solving the trigonometric moment problem [2]:

\[
d_l := \langle \mathcal{U}^l \varphi, g \rangle = \int_{\mathbb{T}^2} (\mathcal{U}^l \varphi)(x)g(x) \, d\mu = \int_{\mathbb{S}} e^{il\theta} \rho(\theta; g) \, d\theta, \quad l \in \mathbb{Z}.
\]

If the observable consists of a single Fourier element, we note: (i) \((k_1, k_2) = (0, 0)\) implies \(d_l = 1, \forall l \in \mathbb{Z}\) which leads to \(\rho(\theta; 1) = \delta(\theta)\) (i.e. Dirac delta distribution), (ii) \((k_1, k_2) \neq (0, 0)\) implies \(d_l = 0, \forall l \neq 0\) and \(d_0 = 1\), leading to \(\rho(\theta; e^{i2\pi(k_1 x_1+k_2 x_2)}) = \frac{1}{\delta} \chi_D\) (i.e. a uniform density). By virtue of these properties, the spectral density function (6.4) of a generic observable can be expressed as:

\[
\rho(\theta; g) = |a_0|^2 \delta(\theta) + \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} \left( \sum_{s \in \mathbb{N}} \sum_{t \in \mathbb{Z}} a_{s,t}^* a_{s,t} \right) e^{il\theta}.
\]

8.2.3. Spectral projectors. For an interval \( D = [c - \delta, c + \delta] \subset \mathbb{S} \), one can use the Fourier series expansions of the indicator function:

\[
\chi_D = \sum_{l \in \mathbb{Z}} b_l(D) e^{il\theta}, \quad b_l(D) := \int_{\mathbb{S}} e^{-il\theta} \chi_D(\theta) \, d\theta = \frac{1}{2\pi} \left\{ e^{-icl} (e^{-i\delta l} - e^{i\delta l}) \right\} \quad l \neq 0
\]

\[
= \frac{\delta}{\delta} \quad l = 0
\]

to obtain an approximation of the spectral projector with the help of functional calculus. Using the Lebesgue basis in (8.5), the following expression for the spectral projector can be obtained:

\[
S_D g = \nu(D) + \sum_{l \in \mathbb{Z}} b_l(D) \mathcal{U}^l g = a_0 \nu(D) + \sum_{l \in \mathbb{Z}} b_l(D) \left( \sum_{s \in \mathbb{N}} \sum_{t \in \mathbb{Z}} a_{s,t} \varphi_{s,t+l} \right)
\]

where the singular measure \( \nu(D) = 1 \) whenever \( 0 \in D \). The frequency content of the projection increases drastically as the width of the interval is shrunk. Assuming \( a_0 = 0 \), and \( ||g|| = 1 \), we see that:

\[
\|S_D g\| = \sum_{l \in \mathbb{Z}} \frac{|b_l(D)|^2}{2\delta} \left( \sum_{s \in \mathbb{N}} \sum_{t \in \mathbb{Z}} a_{s,t}^* a_{s,t+l} \right) \quad \frac{|b_l(D)|^2}{2\delta} = \frac{2 - 2 \cos(l\delta)}{(2\pi\delta)^2}, \quad l \neq 0.
\]
The Lebesgue basis functions \( \varphi_{s,t} \) become increasingly oscillatory for larger values of \( t \). Reducing the width of the interval places more weight on the higher frequency components. As \( \delta \to 0 \), the frequency content in fact becomes unbounded.

### 8.2.4. Numerical results

For observables that consist of only a finite number of Fourier components, a closed-form expression of the spectral density function can be obtained. For example:

\[
\begin{align*}
g_1(x) &= e^{2\pi i (2x_1 + x_2)} \\
g_2(x) &= e^{2\pi i (2x_1 + x_2)} + \frac{1}{2} e^{2\pi i (5x_1 + 3x_2)} \\
g_3(x) &= e^{2\pi i (2x_1 + x_2)} + \frac{1}{2} e^{2\pi i (5x_1 + 3x_2)} + \frac{1}{4} e^{2\pi i (13x_1 + 8x_2)}
\end{align*}
\]

\[
\begin{align*}
\rho(\theta; g_1) &= \frac{1}{2\pi} \\
\rho(\theta; g_2) &= \frac{1}{2\pi} \left( \frac{5}{4} + \cos \theta \right) \\
\rho(\theta; g_3) &= \frac{1}{2\pi} \left( \frac{21}{16} + \frac{10}{8} \cos \theta + \frac{1}{2} \cos 2\theta \right)
\end{align*}
\]

In fig. 6, the spectral densities of these observables are approximated with (6.4) using the proposed method. Clearly, the result indicate that better approximations are obtained by increasing the discretization level. In fig. 7, the spectral projections for the first observable are plotted. As expected, shrinkage of the interval leads to more noisy figures.

![Figures showing approximations of the spectral density function](image)

**Figure 6.** Approximations of the spectral density function of the cat map (7.12) for the observables in (8.8). The spectral resolution is set to \( \alpha = 2\pi/500 \). The black curves denote the approximations and the red curve is the true density.
8.3. **Anzai’s skew-product transformation.** One can use skew-products of dynamical systems to construct examples of maps that have mixed spectra [3]. An example of such a transformation is the map (7.13):

\[
T(x_1, x_2) = (x_1 + \gamma, x_1 + x_2) \mod 1,
\]

for which \(\gamma \in [0, 1)\). The map (7.13) is the composition of a translation and a shear. The transformation is ergodic whenever \(\gamma \in \mathbb{R}\) is irrational.

### 8.3.1. Singular and regular subspaces.

The mixed spectrum of the operator is recognized by examining the cyclic subspaces generated by the Fourier basis elements. Essentially, Fourier elements that solely depend on the coordinate \(x_1\) behave as if the dynamics of the map are that of a pure translation, whereas the remaining Fourier elements do observe a shear and belong to a cyclic subspace of infinite length. The singular and regular subspaces of the operator are respectively given by:

\[
H_s = \text{span}\{\sigma_t := e^{2\pi i (tx_1)}, \quad t \in \mathbb{Z}\}
\]

and

\[
H_r = \text{span}\{\varphi_{s,t,w} := e^{2\pi i ((t+sw)x_1+sx_2)}, \quad s, w \in \mathbb{Z}, t \in \mathbb{N} \text{ with } s \neq 0, t < |s|\}.
\]

### 8.3.2. Spectral density function.

Application of the Koopman operator yields:

\[
U_t \sigma_t = e^{2\pi i \gamma t} \sigma_t \quad \text{and} \quad U_t \varphi_{s,t,w} = e^{2\pi i \left(s \left(w + \frac{l-1}{2}\right)\right) \gamma} \varphi_{s,t,w+t}.
\]

Solving the trigonometric moment problem for observables expressed in the form:

\[
g = \sum_{t \in \mathbb{Z}} a_t \sigma_t + \sum_{s \in \mathbb{Z}, s \neq 0} \sum_{t \in \mathbb{N}, t < |s|} \sum_{w \in \mathbb{Z}} c_{s,t,w} \varphi_{s,t,w} \in L^2(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), \mu),
\]

yield the density function:

\[
\rho(\theta; g) = \sum_{t \in \mathbb{Z}} |a_t|^2 \delta(\theta - 2\pi t \mod 2\pi) + \frac{1}{2\pi} \sum_{t \in \mathbb{Z}} \left( \sum_{s \in \mathbb{Z}, s \neq 0} \sum_{t \in \mathbb{N}, t < |s|} \left( \sum_{w \in \mathbb{Z}} e^{-2\pi i \left(s \left(w + \frac{l-1}{2}\right)\right) \gamma} c_{s,t,w}^* c_{s,t,w+t} \right) \right) e^{i\theta}. 
\]

### 8.3.3. Spectral projectors.

For the interval \(D = [c - \delta, c + \delta) \subset \mathbb{S}\), an expression for the spectral projection is given by:

\[
S_D g = \sum_{2\pi t \gamma \mod 2\pi \in D} a_t \sigma_t + \sum_{t \in \mathbb{Z}} b_t(D) \left( \sum_{s \in \mathbb{Z}, s \neq 0} \sum_{t \in \mathbb{N}, t < |s|} \sum_{w \in \mathbb{Z}} e^{2\pi i \left(s \left(w + \frac{l-1}{2}\right)\right) \gamma} c_{s,t,w}^* \varphi_{s,t,w+t} \right),
\]

where the coefficients \(b_t(D)\) are defined as in (8.7).
8.3.4. Numerical results. Now set $\gamma = 1/3$ and consider the observable:

$$g(x_1, x_2) = \frac{1}{20} e^{2\pi i x_1} + \frac{1}{20} e^{4\pi i x_1} + \frac{1}{5} e^{6\pi i x_1} + e^{2\pi i x_2} + \frac{1}{2} e^{2(\pi i(x_1 + x_2))}$$

The spectral decomposition is given by:

$$\rho(\theta; g) = \frac{1}{400} \left( \delta(\theta) + \delta(\theta - \frac{2\pi}{3}) \right) + \frac{1}{25} \delta(\theta - \frac{4\pi}{3}) + \frac{1}{2\pi} \left( \frac{5}{4} + \cos \theta \right)$$

In fig. 8(a) the spectral density function is plotted for different discretization levels. Convergence of the spectra is observed by refinement of the grid. In fig. 8(b) spectral projections of the observable are shown for small intervals centered around the eigenvalues $e^{\frac{2\pi i}{3} k}$, with $k = 1, 2, 3$. Note that the exact eigenfunctions are not recovered due to the presence of continuous spectra which is interleaved in the projection. In fig. 8(c) projections are shown in a region where only continuous spectrum is present.

Figure 8. Results obtained for Anzai’s skew product transformation (7.13) with $\gamma = 1/3$ and observable (8.11). Spectral projections were computed at a discretization level $\tilde{n} = 2000$.

9. Case study: the Chirikov standard map

In this section we apply our method to the family of area-preserving maps introduced by Chirikov (7.14):

$$T(x) = \begin{bmatrix} x_1 + x_2 + K \sin(2\pi x_1) \\ x_2 + K \sin(2\pi x_1) \end{bmatrix} \mod 1$$

Unlike the examples of the previous section, finding an explicit expression of the spectra is highly non-trivial (except for the case when $K = 0$). In figs. 9 to 13 the spectral properties of the standard map are examined.
for $K$-values ranging between 0 and 0.35. In fig. 9, approximations of the spectral density functions (6.4) are plotted for the observable:

\begin{equation}
  g(x) = e^{i4\pi x_1} + e^{i3\pi x_1} + 0.01 e^{i2\pi x_2}
\end{equation}

It can be seen that sharp peaks form at locations other than eigen-frequency $\theta = 0$. These peaks illustrate that the purely continuous spectra disintegrate, with the rise of discrete spectra for the operator. Eigenfunctions of (2.1) may be recovered from the spectral projections by means of centering the projection on narrow intervals around the respective eigenfrequency. In figs. 10 to 13, this is done for respectively $\theta = 0, \pi, 2\pi/3, \pi/2$. The eigenfunction at $\theta = 0$ yields an invariant partition of the state-space, the eigenfunctions of the other frequencies $\theta = \pi, 2\pi/3, \pi/2$ provide periodic partitions of period 2,3 and 4, respectively (see also [25, 26]).

![Figure 9. Evolution of the spectral density function of the Chirikov map (7.14) for the observable (9.1). Results for $n = 2000$ and $\alpha = 2\pi/500$.](image)

10. Conclusions & future work

In this paper, we introduced a procedure for discretizing the unitary Koopman operator associated with an invertible, measure-preserving transformation on a compact metric space. The method relies on the construction of a periodic approximation of the dynamics, and thereby, approximates the action of the Koopman operator with a permutation. By doing so, one preserves the unitary nature of the underlying operator, which in turn, enables one to approximate the spectral properties of the operator in a weak sense. The results do not involve any assumptions on the spectral type, and therefore effectively handles systems with continuous or mixed spectra. The discretization procedure is constructive, and a convergent numerical method is formulated to compute the spectral decomposition for Lebesgue measure-preserving transformations on the $m$-torus. The general method involves solving a bipartite matching problem to obtain the periodic approximation. One then effectively traverses the cycles of this periodic approximation to compute the spectral projections and density functions of observables. Our method is closely related to taking harmonic averages of observable traces [23]. In fact, one effectively computes the harmonic averages of the discrete periodic dynamical system using the FFT algorithm.

It would be nice to derive some explicit convergence rate results for the current algorithm. Additionally, it would be interesting to investigate whether a similar discretization procedure can be followed for dissipative dynamics, in which the periodic approximation is replaced with a many-to-one map. Generalizations of the procedure to flows will be covered in a upcoming paper.

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Figure 10. Spectral projections computed for the Chirikov map (7.14) at the interval $D = [-0.02, 0.02]$ with $\tilde{n} = 2000$. The depicted projection approximates the eigenfunctions at $\theta = 0$, which generates an invariant partition of the state-space.

Figure 11. Spectral projections computed for the Chirikov map (7.14) at the interval $D = [\pi - 0.02, \pi + 0.02]$ with $\tilde{n} = 2000$. The depicted projection approximates the eigenfunctions at $\theta = \pi$, which generates an period-2 partition of the state-space.
Figure 12. Spectral projections computed for the Chirikov map \((7.14)\) at the interval \(D = [2\pi/3 - 0.02, 2\pi/3 + 0.02]\) with \(\tilde{n} = 2000\). The depicted projection approximates the eigenfunctions at \(\theta = 2\pi/3\), which generates an period-3 partition of the state-space.

Figure 13. Spectral projections computed for the Chirikov map \((7.14)\) at the interval \(D = [\pi/2 - 0.02, \pi/2 + 0.02]\) with \(\tilde{n} = 2000\). The depicted projection approximates the eigenfunctions at \(\theta = 2\pi/3\), which generates an period-3 partition of the state-space.