Deformation of orthosymplectic Lie superalgebra

osp(1|4)

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Abstract. Triangular deformation of the orthosymplectic Lie superalgebra osp(1|4) is defined by chains of twists. Corresponding classical r-matrix is obtained by a contraction procedure from the trigonometric r-matrix. The carrier space of the constant r-matrix is the Borel subalgebra.

1. Introduction

Orthosymplectic Lie superalgebras osp(m|2n) have variety of applications in gauged supergravity, supersymmetric quantum mechanics, integrable $\mathbb{Z}_2$-graded spin chains etc. Similar applications with corresponding changes can have quantum deformations of these superalgebras. In this paper a triangular deformation of the basic Lie superalgebra $g = osp(1|4)$ is given by explicite construction of a universal twist element $F \in U(g) \otimes U(g)$ [1]. It is a natural extension of known triangular deformation of rank 1 Lie superalgebra $osp(1|2)$ [2, 3] along the lines presented in [4] about twists with deformed carrier subspaces. The constructed twist includes all the generators of the Borel subalgebra $B_+ \subset osp(1|4)$. Deformed quantum algebra can be used to define a noncommutative version of the super anti-de Sitter space (s-AdS) generalizing [5–8], deformation of superconformal quantum mechanics [9], deformed spin chains [10] etc.
Deformation of \( osp(1|4) \)

2. Classical \( r \)-matrices

The direction of twist deformation is given by a classical triangular \( r \)-matrix \( r \in g \wedge g \), which is a solution of classical (super) Yang–Baxter equation (cYBE) on \( g \otimes g \otimes g \)

\[
[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = 0,
\]

where in the super-case the commutators and the tensor products are \( \mathbb{Z}_2 \)-graded:

\[
[a, b] = ab - (-1)^{p(a)p(b)}ba,
\]

\[
(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd).
\]

The parity of a homogeneous element \( b \) of a \( \mathbb{Z}_2 \)-graded space \( V = V_0 \oplus V_1 \) is denoted by \( p(b) \). For basic classical Lie superalgebras the classical \( r \)-matrix we are looking for, can be obtained by a contraction procedure from the trigonometric \( r \)-matrix

\[
r(\lambda - \mu) = (r_0 e^{\lambda - \mu} + r_0^{21})/(e^{\lambda - \mu} - 1),
\]

where \( r_0 \) is the super-analog of the Drinfeld–Jimbo constant \( r \)-matrix, the solution to the cYBE,

\[
r_0 = \frac{1}{2} \sum k_i \otimes k_i + \sum_{\alpha \in \Delta_+} e_\alpha \otimes e_{-\alpha}.
\]

Here \( \{k_i\} \) is an orthonormal basis of the Cartan subalgebra and \( \Delta_+ \) is the set of positive (even and odd) roots. The sum

\[
c_2^\otimes := r_0 + r_0^{21} = \sum k_i \otimes k_i + \sum_{\alpha \in \Delta_+} (e_\alpha \otimes e_{-\alpha} + (-1)^{p(\alpha)} e_{-\alpha} \otimes e_\alpha)
\]

is an element of \( \mathcal{U}(g) \otimes \mathcal{U}(g) \) (the tensor Casimir element) invariant with respect to the adjoint action:

\[
[x \otimes 1 + 1 \otimes x, c_2^\otimes] = 0, \quad x \in g.
\]

Take the long root generator \( e_\theta \) and consider adjoint transformation by the group element \( \exp(te_\theta) \):

\[
Ad(\exp(te_\theta))^{\otimes 2}r(\lambda - \mu) = \frac{1}{2} \left( \coth \frac{\lambda - \mu}{2} \cdot c_2^\otimes + \sum_{\alpha \in \Delta_+} (e_\alpha \otimes e_{-\alpha} - (-1)^{p(\alpha)} e_{-\alpha} \otimes e_\alpha) + t \sum_{\alpha \in \Delta_+} e_\alpha \wedge [e_\theta, e_{-\alpha}] \right).
\]

Scaling the spectral parameter \( \lambda \to \varepsilon \lambda, \mu \to \varepsilon \mu \) and \( t \to 2t/\varepsilon \), one gets after contraction

\[
\lim_{\varepsilon \to 0} \varepsilon Ad(\exp(te_\theta/\varepsilon))^{\otimes 2}r(\varepsilon(\lambda - \mu)) = \frac{c_2^\otimes}{(\lambda - \mu)} + t \sum_{\alpha \in \Delta_+} e_\alpha \wedge [e_\theta, e_{-\alpha}].
\]
Due to the Lie algebra nature of the cYBE (1), the resulting expression satisfies this equation as well and both terms satisfy it separately. The second term will be used to deform the universal enveloping algebra $\mathcal{U}(\text{osp}(1|4))$. The Cartan–Weyl basis of the orthosymplectic Lie superalgebra $\text{osp}(1|4)$ has the following generators: $H, J$ (the Cartan subalgebra), $v_\pm, w_\pm$ (the odd root generators), $X_\pm = \pm (v_\pm)^2, Y_\pm = \pm (w_\pm)^2, U_\pm, Z_\pm$. There are two $\text{osp}(1|2)$ subalgebras $\{H, v_\pm, X_\pm\}$ and $\{J, w_\pm, Y_\pm\}$, and the even root generators define the Lie subalgebra $\text{sp}(4) \simeq \text{so}(5)$. Constructing a universal twist element $\mathcal{F}$ we shall use only commutators of the Borel subalgebra generators with subscripts $+$. Some of these commutation relations are

$$[H, X_+] = 2X_+, \quad [H, v_+] = v_+, \quad [H, U_+] = U_+,$$  \hfill (6)

$$[H, Z_+] = Z_+, \quad [Z_+, U_+] = 2X_+, \quad [Z_+, Y_+] = U_+, \quad [Z_+, w_+] = v_+, \quad [v_+, w_+] = U_+.$$

The generators $Z_+, w_+$ correspond to the simple roots. $X_+$ corresponds to the long root $\theta$ and it commutes with all positive root generators and with the generator $J$ of the Cartan subalgebra. The root $\theta$ in (4), (5) is the long one, hence all generators in the constant $r$-matrix (5) correspond to positive roots and one generator belongs to the Cartan subalgebra $[X_\theta, X_{-\theta}] = H$. In the case of $\text{osp}(1|4)$ this constant $r$-matrix is ($X_\theta = X_+$)

$$r = H \wedge X_+ - v_+ \otimes v_+ + Z_+ \wedge U_+.$$

(9)

This is an extension by the odd generator $v_+$ of the classical $r$-matrix defining the extended jordanian twisting of the $\mathcal{U}(\text{so}(5)) \simeq \mathcal{U}(\text{sp}(4))$ [4, 12].

Solution of the cYBE defines a cobracket $\delta$ on the Lie (super) algebra: a map $\delta : g \to g \wedge g$,

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r], \quad x \in g.$$

It is important to point out that the dimension of the kernel of $\delta$ is equal to the number of independent primitive elements in the quantum $\mathcal{U}_\xi(g)$ which is a quantization of the bialgebra $\mathcal{U}(g)$. Proof can be given by transition to the dual Poisson–Lie group using the quantum duality principle [13, 14].

The kernel of cobracket defined by the $r$-matrix (9) is generated by the elements $X_+, J, Y_+, w_+$. It is easy to see that the kernel of the cobracket $\delta$ is a Lie subalgebra $\text{Ker}\delta \subset g$. If there is a triangular $r$-matrix with its carrier in $\text{Ker}\delta$, then the sum of this additional $r$-matrix and the one defined $\delta$, satisfies the cYBE as well. In the case we are interested in there is a super-jordanian $r$-matrix [3]

$$r^{(sj)} = J \wedge Y_+ - w_+ \wedge w_+.$$

(10)
Finally, we shall construct a universal twist corresponding to the $r$-matrix:

$$r = H \wedge X_+ - v_+ \otimes v_+ + Z_+ \wedge U_+ + J \wedge Y_+ - w_+ \otimes w_+.$$  \hfill (11)

(One can add also an abelian term $X_+ \wedge Y_+$, and introduce few independent parameters in front of different constituent $r$-matrices.)

This form of the classical $r$-matrix can be generalized to the case of the universal enveloping algebra of the orthosymplectic Lie superalgebra $osp(1|2n)$. It has (positive) even simple root generators $Z_k$, $k = 1, 2, \ldots, n - 1$, $\alpha_k = \varepsilon_k - \varepsilon_{k+1}$, and one odd simple root generator $v_n$, $\alpha_n = \varepsilon_n$. Other positive root generators are $X_j = v_j^2$, $j = 1, 2, \ldots, n$, $Z_{kj}$ and $U_{kj}$ related to positive roots $\varepsilon_k - \varepsilon_j$ and $\varepsilon_k + \varepsilon_j$, $1 \leq k < j \leq n$ \cite{15}. Commutation relations

\begin{align*}
[H_k, X_k] &= 2X_k, \quad [H_k, v_k] = v_k, \quad (12) \\
[Z_{kj}, U_{kj}] &= 2X_k, \quad [v_k, v_j] = U_{kj}, \quad (13) \\
[Z_{kj}, v_j] &= v_k, \quad [H_k, Z_{kj}] = Z_{kj}, \quad [H_k, U_{kj}] = U_{kj} \quad (14)
\end{align*}

are used to prove that

$$r = \sum_{k=1}^{n} a_k(H_k \wedge X_k + \sum_{j>k}^{n} Z_{kj} \wedge U_{kj} - v_k \otimes v_k)$$  \hfill (15)

satisfies the cYBE. Similar to the symplectic algebra $sp(2n)$ \cite{18} the sum of $r$-matrices (15) corresponds to the sequence of injections

$$osp(1|2) \subset osp(1|4) \subset \ldots \subset osp(1|2n).$$

3. Universal twist element of $B_+ \subset osp(1|4)$

Construction of an explicit form of the twist corresponding to the triangular $r$-matrix (9) will be realized according to the recipe known as chains of twists \cite{16,17}. The classical $r$-matrix is decomposed into few terms which generate factors of the universal twist. This decomposition consists of the jordanian $r$-matrix

$$r^{(j)} = H \otimes X_+ - X_+ \otimes H := H \wedge X_+,$$  \hfill (16)

extended jordanian $r^{(ej)}$ and super-jordanian $r$-matrices

$$r^{(ej)} = r^{(j)} + Z_+ \wedge U_+, \quad r^{(s)} = r^{(j)} - v_+ \otimes v_+, \quad (17)$$

Twist elements corresponding to $r^{(ej)}$ and $r^{(s)}$ are known \cite{16,3}, and they are represented in a factorized form

$$F^{(ej)} = F^{(e)} F^{(j)}, \quad F^{(s)} = F^{(s)} F^{(j)},$$  \hfill (18)
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where

$$F^{(j)} = \exp(H \otimes \sigma), \quad F^{(e)} = \exp\left(\frac{1}{2}Z_+ \otimes U_+e^{-\sigma}\right),$$

$$F^{(s)} = \exp(-v_+ \otimes v_+ f(\sigma \otimes 1, 1 \otimes \sigma)), \quad \sigma = \frac{1}{2}\ln(1 + X_+).$$

The function $f$ is symmetric. It can be presented as the series expansion $\sum f_n$, or through the factorization of $F^{(s)}$ as

$$F^{(s)} = (1 - (v_+ \otimes v_+)(f_1(\sigma) \otimes f_1(\sigma)))F^{(e)}, \quad f_1(\sigma) = (e^\sigma + 1)^{-1}$$

with an appropriate coboundary twist

$$F^{(e)} = (u \otimes u)\Delta(u^{-1}), \quad u = \left(\frac{1}{2}(e^\sigma + 1)\right)^{\frac{1}{2}}.$$

Due to the commutativity of $X_+$ or $\sigma$ and $v_+$ with $Z_+, U_+$, the twisting by $F^{(s)}$ after $F^{(j)}$ does not change the coproducts of $Z_+$ and $U_+$, and vice versa. Hence, one can arrange these twists to form an extended superjordanian twist

$$F^{(esj)} = F^{(s)}F^{(e)}F^{(j)} = F^{(e)}F^{(s)}F^{(j)},$$

corresponding to the $r$-matrix.

To take into account further extension of the $r$-matrix by a super-jordanian term of the second $\text{osp}(1|2)$ subalgebra, we have to use a deformed carrier space transforming the generators $Y_+, w_+$ according to

The generators $v_+, U_+$ and $\sigma(X_+) = \frac{1}{2}\ln(1 + X_+)$ are entering into this transformation. After the twist their coproducts are

$$\Delta^{(esj)}(\sigma) := F^{(esj)}\Delta(\sigma)(F^{(esj)})^{-1} = \Delta_0(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma,$$

$$\Delta^{(esj)}(v_+) = \Delta^{(ej)}(v_+) = v_+ \otimes 1 + e^\sigma \otimes v_+,$$

$$\Delta^{(esj)}(U_+) = \Delta^{(ej)}(U_+) = U_+ \otimes e^\sigma + e^{2\sigma} \otimes U_+.$$

The twisted coproducts of the Borel subalgebra generators $J, Y_+, w_+$ of the second $\text{osp}(1|2)$ are

$$\Delta^{(esj)}(J) = F^{(esj)}\Delta(J)(F^{(esj)})^{-1} = \Delta_0(J) = J \otimes 1 + 1 \otimes J,$$

$$\Delta^{(esj)}(Y_+) = \Delta_0(Y_+) + \frac{1}{2}U_+ \otimes U_+e^{-\sigma} + \frac{1}{4}X_+ \otimes U_+^2e^{-2\sigma},$$

and we get a rather lengthy expression for $\Delta^{(esj)}(w_+)$.

The deformed generator $\tilde{Y}_+$ with the primitive coproduct is given by the adjoint transformation

$$\tilde{Y}_+ = Ad \exp(-Z_+U_+\sigma/2X_+) Y_+ = Y_+ - \frac{1}{4}U_+^2e^{-2\sigma}. \quad (23)$$
Hence, the second jordanian factor is similar to the \(sp(4) \simeq so(5)\) case \([4]\) and has the form \(\exp(J \otimes \sigma(\tilde{Y}_+))\). Although, the deformed coproduct of the odd generator \(\tilde{w}_+\) is rather cumbersome after the transformation \((23)\) the coproduct of \(\tilde{w}_+\)

\[
\tilde{w}_+ = Ad \exp(-Z_+U_+\sigma/2X_+)w_+ = w_+ - \frac{1}{2}v_+U_+\frac{e^{-\sigma}}{e^\sigma + 1}
\]

is primitive as well. Hence, one can add to \((22)\) an additional factor \(\mathcal{F}^{(sj2)}\) corresponding to the second \(osp(1|2)\)

\[
\mathcal{F}^{(sj2)} = \exp(-\tilde{w}_+ \otimes \tilde{w}_+ f(\tilde{\sigma} \otimes 1, 1 \otimes \tilde{\sigma})) \exp(J \otimes \sigma(\tilde{Y}_+)).
\]

where \(\tilde{\sigma} := \sigma(\tilde{Y}_+) = \frac{1}{2} \ln(1 + \tilde{Y}_+)\).

A universal twist with the carrier space including all generators of the Borel subalgebra \(B_+ \subset osp(1|4)\), is given by the product

\[
\mathcal{F} = \mathcal{F}^{(sj2)} \mathcal{F}^{(esj)}
\]

where the super-jordanian twist \(\mathcal{F}^{(sj2)}\) \([3]\) is constructed using the generators \(J, \tilde{Y}_+, \tilde{w}_+\) as in \((18), (20)\). The universal \(R\)-matrix \([1]\) of the twisted \(\mathcal{U}(osp(1|4))\) is

\[
\mathcal{R} = \mathcal{F}_{21} \mathcal{F}^{-1}.
\]

It is natural to conjecture that a similar chain of twists can be constructed for \(\mathcal{U}(osp(1|2n))\) starting with the \(r\)-matrix \([15]\) as it was in the case of \(\mathcal{U}(sp(2n))\) and \(\mathcal{U}(so(2n + 1))\) \([18, 19]\). This would extend in some sense an analogy between algebras \(so(2n + 1)\) and \(osp(1|2n)\) \([20]\).

One can introduce more convenient set of generators of the twisted Borel subalgebra \(B_+ \subset osp(1|4)\) using the FRT-formalism \([21]\), and the upper triangular form of the generators of \(B_+\) in the defining \(5 \times 5\) irreducible representation \(\rho\). The new generators are the entries of the universal \(L\)-operator: \(L = (\rho \otimes id)\mathcal{R}\). Using the symmetric grading \((0, 0, 1, 0, 0)\) of the rows and columns one gets

\[
L = \begin{pmatrix}
T^{-1} & u & V & z & h \\
0 & K^{-1} & W & j & \tilde{z} \\
0 & 0 & 1 & \tilde{W} & \tilde{V} \\
0 & 0 & 0 & K & \tilde{u} \\
0 & 0 & 0 & 0 & T
\end{pmatrix}.
\]

The coproduct of these generators is given by the matrix product of the \(L\)-operators. The commutation relations follow from the \(RTT\)-relation: \(RL_1L_2 = L_2L_1R\) \([21]\), taking into account extra signs in the \(\mathbb{Z}_2\)-graded tensor product \([11]\). The \(R\)-matrix can be obtained using also a super-version of the \(r^3 = 0\) theorem: \(R = \exp(\eta r_\rho)\), where \(r_\rho = (\rho \otimes \rho)r\). Similarly to the twisting of \(osp(1|2)\) \([3]\) one can express the new generators in terms of the initial ones.
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