A Note on the Category of c-spaces

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Abstract

We prove that the category of c-spaces with continuous maps is not cartesian closed. As a corollary, it follows that the category of locally finitary compact spaces with continuous maps also is not cartesian closed.

Keywords: directed space, c-space, cartesian closed, locally finitary compact

1 Introduction

Many people have been trying to extend domain theory to general topological spaces, see \cite{6,4,7,9,2}. Directed spaces are introduced by Kou and Yu independently \cite{12} in 2014 for generalizing the concept of Scott spaces, which is equivalent to that of $T_0$ monotone determined spaces introduced by Ernè \cite{5}. In the same paper Kou and Yu proved that the category of directed spaces with continuous maps ($\text{DTop}$ for short) is cartesian closed. There are many important directed spaces in domain theory, for instance locally finitary compact spaces are directed spaces; in particular c-spaces and Alexandroff spaces are directed spaces. Since the category of continuous domains is not cartesian closed, and since the position of the category of c-spaces in the category of directed spaces is similar to that of continuous domains in dcpos \cite{3}, a natural question arises: Is the category of c-spaces cartesian closed? In this short note, we answer this question in the negative.

2 Preliminaries

We refer to \cite{8,1,9} for the standard definitions and notations in order theory, topology and domain theory. A partially ordered set $D$ is called a dcpo if every directed subset of $D$ has a supremum in $D$. A upper set $U$ is called a Scott open set if for any directed set $A \subseteq D$, $\bigvee A \in U$ implies $A$ intersects $U$.

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For a topological space $X$, we use $\mathcal{O}(X)$ to denote the lattice of open subsets of $X$. We require that all topological spaces are $T_0$ in this note. Let $X$ be a $T_0$ space, the specializing order $\leq$ is defined as follows: $x \leq y$ if $x$ belongs to the closure of point $y$. A topological space is a $c$-space if for any $x \in X$ and any open neighbourhood $U$ of $x$, there is a point $y \in U$ such that $x \in \text{int}(\uparrow y)$. A space $X$ is locally finitary compact if for any $x \in X$ and its open neighborhood $U$, there is a finite subset $F$ of $U$ such that $x \in \text{int}(\uparrow F)$.

Let $X$ be a $T_0$ space and $\leq$ the specialization order over $X$. A topological space $X$ is called a Scott space if $(X, \leq)$ is a dcpo and the topology on $X$ is equal to the Scott topology on $(X, \leq)$. Every directed set $D$ of $X$ under specialization order can be regarded as a monotone net, we say $D$ converges to $x$ iff for every open neighborhood $U$ of $x$, $D \cap U \neq \emptyset$. We say that $V$ is a directed open set of $X$ if for all directed set $D$ which converges to some point of $V$, then $D \cap V \neq \emptyset$. It is easy to see that every directed open set is an upper set.

**Definition 2.1** [12] Let $X$ be a $T_0$ space. If every directed open set of $X$ is also an open set, then we say that $X$ is a directed space.

There are many important spaces in domain theory which also are directed spaces.

**Example 2.2** (i) Every poset with Scott topology is a directed space.

(ii) All $c$-spaces are directed spaces. In particular, every Alexandroff space is a directed space.

(iii) All locally finitary compact spaces are directed spaces. By the way, every $c$-space is locally finitary compact.

Next we introduce the concept of the exponential object in general category.

**Definition 2.3** Given two objects $X, Y$ in a category $\mathcal{C}$ with binary products, an exponential object, if it exists, is an object $Y^X$ with a morphism $\text{App}: Y^X \times X \rightarrow Y$ such that for every morphism $f: Z \times X \rightarrow Y$, there is a unique morphism $\bar{f}: Z \rightarrow Y^X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
Z \times X & \xrightarrow{f \times \text{id}_X} & Y^X \times X \\
| & \downarrow & |
\end{array}
\quad
\begin{array}{ccc}
y & \xrightarrow{\text{App}} & Y \\
| & \downarrow & |
\end{array}
\quad
\begin{array}{ccc}
f & \xrightarrow{\bar{f}} & Z
\end{array}
$$

The following result describes the underlying set of the exponential object in $\text{Top}$.

**Proposition 2.4** [9] Let $\mathcal{C}$ be any full subcategory of $\text{Top}$ with finite products, and assume that $1 = \{\star\}$ is an object of $\mathcal{C}$. Let $X, Y$ be two objects of $\mathcal{C}$ that have an exponential object $Y^X$ in $\mathcal{C}$.

Then there is a unique homeomorphism $\theta: Y^X \rightarrow [X \rightarrow Y]$, for some unique topology on $[X \rightarrow Y]$ (the set of all continuous functions from $X$ to $Y$), such that $\text{App}(h, x) = \theta(h)(x)$ for all $h \in Y^X, x \in X$.

Moreover, $\bar{f}(z)$ is the image by $\theta^{-1}$ of $f(z, \_)$ for all $f: Z \times X \rightarrow Y, z \in Z$.

Remark: By the above result, we always let the exponential object in $\mathcal{C}$ be the set $[X \rightarrow Y]$ with some unique topology if it exists.

**Theorem 2.5** [12] The category of directed spaces and continuous maps is cartesian closed.

Next, we build a relationship between directed spaces and Scott spaces, which will be used later.

**Definition 2.6** Let $X$ be a $T_0$ space. If $X$ with the specialization order is a dcpo and every open set of $X$ is Scott open in $(X, \leq)$. Then we say that $X$ is a $d$-space.

**Lemma 2.7** A directed space is a Scott space iff it is a $d$-space.

**Proof.** We only need to show the “if” part. Let $X$ be a $d$-space and a directed space, obviously every open set of $X$ is Scott open of $(X, \leq)$ since $X$ is a $d$-space. Now take any Scott open set $U$ of $(X, \leq)$ and
for any directed set $D$ converges to some point $x$ of $U$. Assume that $D \cap U = \emptyset$, then $b = \bigvee_D D \notin U$. It follows that $x \in X \setminus \downarrow b$. Because $D$ converges to $x$ and $X \setminus \downarrow b$ is open in $X$, there is some $d \in D$ such that $d \in X \setminus \downarrow b$, a contradiction. Hence the assumption is wrong. It means that $U$ is a directed open set of $X$. Since $X$ is a directed space, the topology on $X$ is exactly the Scott topology on $(X, \leq)$. \hfill $\Box$

We list some results about separate continuity and joint continuity.

**Theorem 2.8** [11] Let $E$ be a $T_0$ space. The following conditions are equivalent:

(i) $E$ is locally finitary compact.

(ii) For all $T_0$ space $X$, if a map from $X \times E$ is separately continuous, then it is jointly continuous.

**Corollary 2.9** Let $X$ be a c-space and $Y$ a $T_0$ space. For any $T_0$ space $Z$, a map $f: X \times Y \to Z$ is continuous (i.e. jointly continuous) iff it is separately continuous.

### 3 The category of c-spaces

We now prove our main result.

**Theorem 3.1** The category of c-spaces with continuous maps (CS for short) is not cartesian closed.

**Proof.** Let $Z^-$ be the set of non-positive integers with the Scott topology. Assume CS is a ccc. It is easy to see that the topological product $X \times Y$ is the categorical product because $X \times Y$ is a c-space. Since CS is cartesian closed, there exists exponential topology $\tau$ on $[Z^- \to Z^-]$, we denote by $[Z^- \to Z^-]_\tau$. Then for any c-space $Y$ and any map $f: Y \times Z^- \to Z^-$, $f$ is continuous iff $\tilde{f}: Y \to [Z^- \to Z^-]_\tau$ is continuous.

**Claim 1:** The specialization order on $[Z^- \to Z^-]_\tau$ is equal to the pointwise order.

For any $g_1, g_2 \in [Z^- \to Z^-]_\tau$ with $g_1 \leq g_2$ ($g_1 \neq g_2$), take $Y = S$ with Scott topology. A map $\theta: S \to [Z^- \to Z^-]_\tau$ is defined as $\theta(1) = g_2, \theta(0) = g_1$. Then $\theta$ is continuous. Hence $\hat{\theta}: S \times Z^- \to Z^-$ is continuous. It follows that

$$g_1(x) = \hat{\theta}(0, x) \leq \hat{\theta}(1, x) = g_2(x)$$

for any $x \in X$.

For any $g_1, g_2 \in [Z^- \to Z^-]_\tau$ with $g_1 \leq g_2$, consider a continuous map $f: S \times Z^- \to Z^-$ which is defined as $f(0, x) = g_1(x), f(1, x) = g_2(x) \forall x \in X$. It follows that the transpose map $\tilde{f}$ is continuous hence monotone, which implies that

$$g_1 = \tilde{f}(0) \leq \tilde{f}(1) = g_2.$$

**Claim 2:** $[Z^- \to Z^-]_\tau$ is a d-space.

We only need to show that every directed subfamily $(g_i)_{i \in I}$ of $[Z^- \to Z^-]_\tau$ converges to its supremum $g = \bigvee_{i \in I} g_i$. Let $Y$ be the set $I \cup \{\infty\}$ with the topology generated by $\{\uparrow_i \cup \{\infty\} : i \in I\}$. Obviously $Y$ is a c-space. Consider a map $f: Y \times Z^- \to Z^-$ which is defined as $f(\infty, x) = g(x), f(i, x) = g_i(x)$. It is easy to see that $f$ is continuous since $f$ is continuous iff it is separately continuous by 2.9. It follows that $f: Y \to [Z^- \to Z^-]_\tau$ is continuous, and so $(g_i = f(i))_i$ converges to $f(\infty) = g$.

Therefore $\tau$ is just the Scott topology on $[Z^- \to Z^-]$. But from [10] we know that $[Z^- \to Z^-]$ is not a continuous domain, hence it is not a c-space, a contradiction. \hfill $\Box$

**Theorem 3.2** [8] A meet continuous dcpo is a continuous dcpo iff it is a quasicontinuous dcpo.

Notice that $[Z^- \to Z^-]$ is a meet continuous semilattice which is not continuous, hence it is not a quasicontinuous dcpo. Then we have the following result.

**Corollary 3.3** The category of locally finitary compact spaces with continuous maps is not cartesian closed.
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