Syzygy modules for quasi $k$-Gorenstein rings $^*$ $^\dagger$

Zhaoyong Huang $^\dagger$

Department of Mathematics, Nanjing University, Nanjing 210093, People’s Republic of China

Abstract

Let $\Lambda$ be a quasi $k$-Gorenstein ring. For each $d$th syzygy module $M$ in mod $\Lambda$ (where $0 \leq d \leq k - 1$), we obtain an exact sequence $0 \to B \to M \oplus P \to C \to 0$ in mod $\Lambda$ with the properties that it is dual exact, $P$ is projective, $C$ is a $(d + 1)$st syzygy module of $\text{Ext}^{d+1}_\Lambda(D(M), \Lambda)$ and the right projective dimension of $B^*$ is less than or equal to $d - 1$. We then give some applications of such an exact sequence as follows. (1) We obtain a chain of epimorphisms concerning $M$, and by dualizing it we then get the spherical filtration of Auslander and Bridger for $M^*$. (2) We get Auslander and Bridger’s Approximation Theorem for each reflexive module in mod $\Lambda^{op}$. (3) We show that for any $0 \leq d \leq k - 1$ each $d$th syzygy module in mod $\Lambda$ has an Evans-Griffith representation. As an immediate consequence of (3), we have that, if $\Lambda$ is a commutative noetherian ring with finite self-injective dimension, then for any non-negative integer $d$, each $d$th syzygy module in mod $\Lambda$ has an Evans-Griffith representation, which generalizes an Evans and Griffith’s result to much more general setting.

1. Introduction

Let $\Lambda$ be a left and right noetherian ring and mod $\Lambda$ the category of finitely generated left $\Lambda$-modules.

It is well known that $\Lambda$ possesses rather interesting properties when it satisfies the condition that $\text{gradeExt}^i_\Lambda(N, \Lambda) \geq i$ for any $N \in \text{mod } \Lambda^{op}$ and $1 \leq i \leq k$ (where $k$ is a positive integer). Assume that $\Lambda$ satisfies this grade condition. For any $T$ in mod $\Lambda^{op}$, Auslander and Bridger in [3] Spherical Filtration Theorem 2.37 produced a projective module $Q$ and a filtration $T \oplus Q = T_0 \supseteq T_1 \supseteq \cdots \supseteq T_k$ in mod $\Lambda^{op}$ such that each $T_i/T_{i+1}$ is “spherical” in the sense that the cohomological $\text{Ext}^j_\Lambda(T_i/T_{i+1}, \Lambda) \neq 0$ only if $j = 0$ or $j = i$. They also showed in [3] that under this grade condition the following statements are true: (1) The full subcategory of mod $\Lambda^{op}$ consisting of the modules with projective dimension less than or equal to $k$ is covariantly finite (see [4] for the definition of covariantly finite); (2) A $d$th syzygy module in mod $\Lambda^{op}$ is $d$-torsionfree for any $1 \leq d \leq k$. We remark that the second statement doesn’t hold in general although the converse is always true.

Under the above grade condition, Auslander and Reiten in [5] proved that the right flat dimension of the $i$th term in a minimal injective resolution of $\Lambda$ as a right $\Lambda$-module is less than or equal to $i$ for

$^*$2000 Mathematics Subject Classification: 16E05, 16E30, 16E65, 16P40.
$^\dagger$Keywords: syzygy modules, quasi $k$-Gorenstein rings, duality of spherical filtration, Evans-Griffith representations.
$^\ddagger$E-mail address: huangzy@nju.edu.cn
any $1 \leq i \leq k$; recently, Hoshino and Nishida in [10] proved that the converse also holds. We call a ring quasi $k$-Gorenstein provided it satisfies one of these equivalent conditions. A ring is called quasi Auslander ring if it is quasi $k$-Gorenstein for all $k$.

Recall that $\Lambda$ is called a $k$-Gorenstein ring if the right flat dimension of the $i$th term in a minimal injective resolution of $\Lambda$ as a right $\Lambda$-module is less than or equal to $i - 1$ for any $1 \leq i \leq k$. This notion was introduced by Auslander in [9]. Iwanaga and Sato in [11] called $\Lambda$ an Auslander ring if it is $k$-Gorenstein for all $k$. In [9] Theorem 3.7 Auslander showed that the notion of $k$-Gorenstein rings (and hence that of Auslander rings) is left-right symmetric and that $\Lambda$ is $k$-Gorenstein if and only if the grade of any submodule of $\text{Ext}^i_\Lambda(N, \Lambda)$ for any $N \in \text{mod } \Lambda^\text{op}$ and $1 \leq i \leq k$ is greater than or equal to $i$. However, as already pointed out in [5], the notion of quasi $k$-Gorenstein rings (and hence that of quasi Auslander rings) is not left-right symmetric.

Notice that Bass showed in [7] that a commutative noetherian ring $\Lambda$ has finite self-injective dimension if and only if $\text{grade} \text{Ext}^i_\Lambda(N, \Lambda) \geq i$ for any $N \in \text{mod } \Lambda$ and $i \geq 1$. So the notion of Auslander rings is in fact a generalization of that of commutative noetherian rings with finite left and right self-injective dimensions.

The discussion in this paper is based on the results mentioned above. For a quasi $k$-Gorenstein ring $\Lambda$ and each $d$th syzygy module in mod $\Lambda$ (where $0 \leq d \leq k - 1$) we obtain here an exact sequence with “nice” properties as follows.

**Theorem** Let $\Lambda$ be a quasi $k$-Gorenstein ring and $M$ a $d$th syzygy module in mod $\Lambda$ (where $d$ is an integer with $0 \leq d \leq k - 1$). Then there is a projective module $P$ in mod $\Lambda$ such that the $d$th syzygy $B$ of $\text{Ext}^{d+1}_\Lambda(D(M), \Lambda)$ (see Section 2 for the definition of $D(M)$) is a submodule of $M \bigoplus P$ and such that the exact sequence

$$0 \to B \to M \bigoplus P \to C \to 0$$

has the following properties:

1. $C$ is a $(d + 1)$st syzygy module.
2. $\text{r.pd}_\Lambda(B^*) \leq d - 1$.
3. The sequence $0 \to B \to M \bigoplus P \to C \to 0$ is dual exact, that is, the induced sequence $0 \to C^* \to M^* \bigoplus P^* \to B^* \to 0$ is exact.

The above theorem is the main result in this paper, we will prove it in Section 3. To prove it, we collect some preliminary results in Section 2. In Section 4 we give some applications of our main theorem. For example, as an application of the theorem, we obtain a chain of epimorphisms concerning a module $M$ in mod $\Lambda$, by dualizing it we then get the spherical filtration of Auslander and Bridger for $M^*$; and furthermore we get Auslander-Bridger’s Approximation Theorem for each
reflexive module in mod $\Lambda^{op}$.

Evans and Griffith in [7] Theorem 2.1 showed that if $\Lambda$ is a commutative noetherian local ring with finite global dimension and contains a field then each non-free $d$th syzygy of rank $d$ has an Evans-Griffith representation. As another application of Theorem above, we show that for a quasi $k$-Gorenstein ring $\Lambda$ and any $0 \leq d \leq k - 1$ each $d$th syzygy module in mod $\Lambda$ has an Evans-Griffith representation; especially, we have that, if $\Lambda$ is a commutative noetherian ring with finite self-injective dimension, then for any non-negative integer $d$, each $d$th syzygy module in mod $\Lambda$ has an Evans-Griffith representation, which generalizes Evans and Griffith’s result in [7] Theorem 2.1 to much more general setting.

2. Preliminaries

In this section, we give some definitions in our terminology and collect some facts which are used in this paper.

Throughout this paper, $\Lambda$ is a left and right noetherian ring, mod $\Lambda$ is the category of finitely generated left $\Lambda$-modules and $\Omega^k$ (mod $\Lambda$) is the full subcategory of mod $\Lambda$ consisting of $k$th syzygy modules. Let $M$ be a module in mod $\Lambda$ (resp. mod $\Lambda^{op}$). We use $\text{l.pd}_\Lambda(M)$ (resp. $\text{r.pd}_\Lambda(M)$) to denote the left (resp. right) projective dimension of $M$. We use $\sigma_M : M \to M^{**}$, defined by $\sigma_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^*$, to denote the canonical evaluation homomorphism. $M$ is called torsionless if $\sigma_M$ is a monomorphism; and $M$ is called reflexive if $\sigma_M$ is an isomorphism. For a non-negative integer $i$, we denote $\text{grade} M \geq i$ if $\text{Ext}^j_\Lambda(M, \Lambda) = 0$ for any $0 \leq j < i$.

Let $M$ be in mod $\Lambda$ (resp. mod $\Lambda^{op}$) and

$$P_1 \to P_0 \to M \to 0$$

a projective resolution of $M$ in mod $\Lambda$ (resp. mod $\Lambda^{op}$). Then we have an exact sequence

$$0 \to M^* \to P_0^* \to P_1^* \to D(M) \to 0$$

in mod $\Lambda^{op}$ (resp. mod $\Lambda$), where $D(M) = \text{Coker}(P_0^* \to P_1^*)$. The following lemma is due to Auslander.

**Lemma 2.1** ([2] Proposition 6.3) Let $M$ and $D(M)$ be as above. Then we have the following exact sequence:

$$0 \to \text{Ext}^1_\Lambda(D(M), \Lambda) \to M \xrightarrow{\sigma_M} M^{**} \to \text{Ext}^2_\Lambda(D(M), \Lambda) \to 0.$$

It is clear that $\text{Ext}^1_\Lambda(D(M), \Lambda) \cong \text{Ext}^{i-2}_\Lambda(M^*, \Lambda)$ for any $i \geq 3$. On the other hand, $\text{Ext}^1_\Lambda(D(M), \Lambda) \cong \text{Ker}\sigma_M$ and $\text{Ext}^2_\Lambda(D(M), \Lambda) \cong \text{Coker}\sigma_M$ by Lemma 2.1. So we get that, although $D(M)$ depends
on the choice of the projective resolution of \( M \), each of \( \text{Ext}^i_\Lambda(\text{D}(M), \Lambda) \) (for any \( i \geq 1 \)) is independent of the choice of the projective resolution of \( M \) and hence is identical up to isomorphisms.

Recall that \( M \) is called \( k \)-torsionfree if \( \text{Ext}^i_\Lambda(\text{D}(M), \Lambda) = 0 \) for any \( 1 \leq i \leq k \) (see [3]). By Lemma 2.1, we have that \( M \) is 1-torsionfree (resp. 2-torsionfree) if and only if it is torsionless (resp. reflexive). We use \( \mathcal{T}^k(\text{mod } \Lambda) \) to denote the full subcategory of \( \text{mod } \Lambda \) consisting of \( k \)-torsionfree modules. It is easy to see that \( \mathcal{T}^k(\text{mod } \Lambda) \subseteq \Omega^k(\text{mod } \Lambda) \). Furthermore, we have the following useful result, which gives some equivalent conditions of \( \mathcal{T}^i(\text{mod } \Lambda) = \Omega^i(\text{mod } \Lambda) \) for any \( 1 \leq i \leq k \).

**Lemma 2.2** For a positive integer \( k \), the following statements are equivalent.

1. \( \text{gradeExt}^{i+1}_\Lambda(M, \Lambda) \geq i \) for any \( M \in \text{mod } \Lambda \) and \( 1 \leq i \leq k - 1 \).
2. \( \Omega^i(\text{mod } \Lambda) = \mathcal{T}^i(\text{mod } \Lambda) \) for any \( 1 \leq i \leq k \).
3. \( \text{gradeExt}^{i+1}_\Lambda(N, \Lambda) \geq i \) for any \( N \in \text{mod } \Lambda^{\text{op}} \) and \( 1 \leq i \leq k - 1 \).
4. \( \Omega^i(\text{mod } \Lambda^{\text{op}}) = \mathcal{T}^i(\text{mod } \Lambda^{\text{op}}) \) for any \( 1 \leq i \leq k \).

**Proof.** The equivalence of (1) and (2) is proved in [3] Proposition 2.26 (or see [5] Proposition 1.6). The other implications are proved in [9] Theorem 2.4. ■

**Corollary 2.3** If \( \Lambda \) is a quasi \( k \)-Gorenstein ring, then \( \Omega^i(\text{mod } \Lambda) = \mathcal{T}^i(\text{mod } \Lambda) \) and \( \Omega^i(\text{mod } \Lambda^{\text{op}}) = \mathcal{T}^i(\text{mod } \Lambda^{\text{op}}) \) for any \( 1 \leq i \leq k + 1 \).

**Proof.** By [5] Proposition 4.2 and Theorem 1.7, we have that \( \text{gradeExt}^{i+1}_\Lambda(M, \Lambda) \geq i \) for any \( M \in \text{mod } \Lambda \) and \( 1 \leq i \leq k \). Now our conclusion follows from Lemma 2.2. ■

3. The Proof of The Theorem

In this section, we will prove the theorem mentioned in Introduction. We proceed in several steps.

**Proof of Theorem.** The case \( d = 0 \): Put \( C = \text{Im} \sigma_M \) and \( B = \text{Ext}^1_\Lambda(\text{D}(M), \Lambda) \). Then we have an exact sequence in \( \text{mod } \Lambda \):

\[
0 \to B \to M \to C \to 0.
\]

Since \( C \) is a submodule of \( M^{**} \), \( C \) is torsionless and \( C \in \Omega^1(\text{mod } \Lambda) \). On the other hand, \( \text{gradeExt}^1_\Lambda(\text{D}(M), \Lambda) \geq 1 \) by assumption, that is, \( B^* = 0 \), so the obtained exact sequence is desired.

The case \( d = 1 \): Assume that

\[
0 \to B \xrightarrow{f} P \xrightarrow{g} \text{Ext}^2_\Lambda(\text{D}(M), \Lambda) \to 0
\]

is an exact sequence in \( \text{mod } \Lambda \) with \( P \) projective. Consider the following pull-back diagram with the middle row splitting:
Because $M^{**}$ is a dual, $M^{**}$ is a second syzygy. Since \( \text{gradeExt}^2_A(D(M), \Lambda) \geq 2 \) by assumption, \( P^* \xrightarrow{f^*} B^* \) is an isomorphism and $B^*$ is projective. We know from [1] Proposition 20.14 that $\sigma^*_M$ is epic, so, by applying the functor $\text{Hom}_A(\cdot, \Lambda)$ to the above diagram, we get the following commutative diagram with exact columns and rows:

\[
\begin{array}{cccc}
0 & \rightarrow & \text{Ext}_A^2(D(M), \Lambda) & \rightarrow \\
\downarrow & & \downarrow & \\
0 & \rightarrow & [\text{Ext}_A^2(D(M), \Lambda)]^* (= 0) & \rightarrow \\
\downarrow & & \downarrow & \\
0 & \rightarrow & P^* & \rightarrow \\
\downarrow & & \downarrow & \\
B^* & \rightarrow & M^{**} & \rightarrow \\
\downarrow & & \downarrow & \\
0 & & \rightarrow & M \\
\end{array}
\]

It is easy to see that \( \text{Coker}(M^{***} \rightarrow P^* \bigoplus M^*) = B^* \). Then the middle column in the former diagram:

\[
0 \rightarrow B \rightarrow M \bigoplus P \rightarrow M^{**} \rightarrow 0
\]

is desired.

**The case \( d \geq 2 \):** Let

\[
\cdots \rightarrow P_i \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M^* \rightarrow 0
\]

(1)

be a projective resolution of $M^*$ in mod $\Lambda^{op}$. If $d = 2$, then $M$ is reflexive by Lemma 2.2. So we have an exact sequence:
$0 \to M(\cong M^{**}) \to P_0^* \to P_1^* \to \cdots \to P_{d-2}^* \to P_{d-1}^* \to N \to 0$

where $N = \text{Coker}(P_0^* \to P_1^*)$. Now suppose $d \geq 3$. Since $M$ is a $d$th syzygy module, it follows from Lemma 2.2 that $M$ is $d$-torsionfree and so $\text{Ext}_\Lambda^i(M^*, \Lambda) = 0$ for any $1 \leq i \leq d - 2$. From this fact and the exact sequence (1) we yield the following exact sequence:

$$0 \to M(\cong M^{**}) \to P_0^* \to P_1^* \to \cdots \to P_{d-2}^* \to P_{d-1}^* \to N \to 0 \quad (2)$$

where $N = \text{Coker}(P_{d-2}^* \to P_{d-1}^*)$. So for any $d \geq 2$ we have an exact sequence of the form (2).

By Lemma 2.1 we get easily the following exact sequence:

$$0 \to \text{Ext}^{d+1}_\Lambda(D(M), \Lambda) \to N \xrightarrow{\sigma_N} N^{**} \to \text{Ext}^{d+2}_\Lambda(D(M), \Lambda) \to 0.$$

Write $K = \text{Ext}^{d+1}_\Lambda(D(M), \Lambda)$ and $Y = \text{Im} \sigma_N$. Let $U^*$ and $V^*$ be the projective resolutions of $K$ and $Y$, respectively. Then there is a projective module $P'$ in mod $\Lambda$ such that we have the following commutative diagram with exact columns and rows:

```
0 → 0 → 0
0 → B → M ⊕ P' → C' → 0
0 → U_{d-1} → U_{d-1} ⊕ V_{d-1} → V_{d-1} → 0
... → ... → ...
0 → U_0 → U_0 ⊕ V_0 → V_0 → 0
0 → K → N → Y → 0
0 → 0 → 0
```

where $\overline{M}$ is the greatest direct summand of $M$ without projective summands.

Since $Y$ is a submodule of $N^{**}$, $Y$ is torsionless and hence it is in $\Omega^1(\text{mod } \Lambda)$. So $C'$ is in $\Omega^{d+1}(\text{mod } \Lambda)$. On the other hand, $\text{grade } K = \text{grade} \text{Ext}^{d+1}_\Lambda(D(M), \Lambda) \geq d + 1$ by assumption, so we get an exact sequence $0 \to U_0^* \to \cdots \to U_{d-1}^* \to B^* \to 0$ and hence $r.pd_\Lambda(B^*) \leq d - 1$. It is trivial that every
homomorphism $f : B \rightarrow \Lambda$ may extends to a homomorphism $g : U_{d-1} \rightarrow \Lambda$, so $f$ may extends to a homomorphism $h : M \oplus P' \rightarrow \Lambda$ and hence the sequence $0 \rightarrow C'' \rightarrow M^* \oplus P'' \rightarrow B^* \rightarrow 0$ is exact.

Put $M = M \oplus P''$, $P = P' \oplus P''$ and $C = C' \oplus P''$. Then from the exact sequence $0 \rightarrow B \rightarrow M \oplus P' \rightarrow C' \rightarrow 0$ we yield the following exact sequence:

$$0 \rightarrow B \rightarrow M \oplus P \rightarrow C \rightarrow 0,$$

which is desired. ■

4. Applications

In this section we will give some applications of the main theorem. We fir st have the following result.

**Corollary 4.1** If $\Lambda$ is a quasi Auslander ring, then for any non-negative integer $d$ and $M$ in $\Omega^d(\text{mod } \Lambda)$, there is a projective module $P$ in mod $\Lambda$ such that the $d$th syzygy $B$ of $\text{Ext}^{d+1}_\Lambda(D(M), \Lambda)$ is a submodule of $M \oplus P$ and such that the exact sequence $0 \rightarrow B \rightarrow M \oplus P \rightarrow C \rightarrow 0$ has the following properties:

1. $C \in \Omega^{d+1}(\text{mod } \Lambda)$.
2. $r.pd_\Lambda(B^*) \leq d - 1$.
3. The sequence $0 \rightarrow B \rightarrow M \oplus P \rightarrow C \rightarrow 0$ is dual exact.

**Proposition 4.2** (The duality of spherical filtration) Let $\Lambda$ be a quasi $k$-Gorenstein ring. Then, for each $M$ in mod $\Lambda$, there is a projective module $P$ in mod $\Lambda$ and a chain of epimorphisms:

$$M \oplus P = M_0 \twoheadrightarrow M_1 \twoheadrightarrow \cdots \twoheadrightarrow M_{k-1} \twoheadrightarrow M_k,$$

such that

1. $B_d = \text{Ker}(M_d \rightarrow M_{d+1})$ is a $d$th syzygy of $\text{Ext}^{d+1}_\Lambda(D(M), \Lambda)$ (or equivalently, $B_d$ is a $d$th syzygy of $\text{Ext}^{d-1}_\Lambda(M^*, \Lambda)$ if $d \geq 2$) for any $0 \leq d \leq k - 1$.
2. $M_d \in \Omega^d(\text{mod } \Lambda)$ for any $0 \leq d \leq k$.
3. $r.pd_\Lambda(B^*_d) \leq d - 1$ for any $0 \leq d \leq k - 1$.
4. Each exact sequence $0 \rightarrow B_d \rightarrow M_d \rightarrow M_{d+1} \rightarrow 0$ is dual exact for any $0 \leq d \leq k - 1$.

**Proof.** We proceed by employing induction with successive applications of Theorem in Introduction.

First, by Theorem and its proof, we have an exact sequence in mod $\Lambda$:

$$0 \rightarrow B_0 \rightarrow M \rightarrow C_1 \rightarrow 0$$ (3)
with the properties that it is dual exact, \( B_0 = \text{Ext}^1_A(D(M), \Lambda) \) and \( C_1 = \text{Im} \sigma_M \). Notice that \( B_0^* = 0 \) and \( \text{Im} \sigma_M \) is torsionless, then by Lemma 2.1 we have that \( \text{Ext}^2_A(D(C_1), \Lambda) = \text{Ext}^2_A(D(\text{Im} \sigma_M), \Lambda) \cong (\text{Im} \sigma_M)^*/\text{Im} \sigma_M \cong M^{**}/\text{Im} \sigma_M \cong \text{Ext}^2_A(D(M), \Lambda) \).

Next, by Theorem and its proof, we have an exact sequence in \( \text{mod} \, \Lambda \):

\[
0 \to B_1 \to C_1 \bigoplus P_1 \to C_2 \to 0
\]

with the properties that it is dual exact, \( C_2 = C_1^{**} \), \( P_1 \) is projective, \( B_1 \) is a first syzygy of \( \text{Ext}^2_A(D(C_1), \Lambda) \cong (\text{Ext}^2_A(D(M), \Lambda)), \) \( B_1^* \) is projective and \( C_2 \in \Omega^2(\text{mod} \, \Lambda) \). Then we have that \( C_2 \oplus B_1^* \cong C_1^* \oplus P_1^* \cong M^* \oplus P_1^* \) and \( \text{Ext}^1_A(C_2, \Lambda) \cong \text{Ext}^1_A(M^*, \Lambda) \) for any \( i \geq 1 \).

Now suppose that \( k \geq 3 \) and for any \( 0 \leq d \leq k - 2 \) there is an exact sequence in \( \text{mod} \, \Lambda \):

\[
0 \to B_d \to C_d \bigoplus P_d \to C_{d+1} \to 0
\]

with the properties that it is dual exact, \( P_d \) is projective, \( B_d \) is a \( d \)th syzygy of \( \text{Ext}^d_A(D(C_d), \Lambda) \), \( r.p.d_A B_d \leq d - 1 \), \( C_0 = M \) and \( C_{d+1} \in \Omega^{d+1}(\text{mod} \, \Lambda) \). Then we have that \( \text{Ext}^{k-2}_A(C_{k-1}, \Lambda) \cong \text{Ext}^{k-2}_A(C_{k-2}, \Lambda) \cong \cdots \cong \text{Ext}^{k-2}_A(C_2, \Lambda) \cong \text{Ext}^{k-2}_A(M^*, \Lambda) \).

By Theorem, there is a projective module \( P_{k-1} \) and an exact sequence in \( \text{mod} \, \Lambda \):

\[
0 \to B_{k-1} \to C_{k-1} \bigoplus P_{k-1} \to C_k \to 0,
\]

such that

1. \( B_{k-1} \) is a \((k - 1)\)st syzygy of \( \text{Ext}^k_A(D(C_{k-1}), \Lambda) \); or equivalently, \( B_{k-1} \) is a \((k - 1)\)st syzygy of \( \text{Ext}^{k-2}_A(C_{k-1}, \Lambda) \cong \text{Ext}^k_A(D(M^*, \Lambda) \cong \text{Ext}^k_A(D(M), \Lambda)) \).
2. \( C_k \in \Omega^k(\text{mod} \, \Lambda) \).
3. \( r.p.d_A (B_{k-1}^*) \leq k - 2 \).
4. The induced sequence \( 0 \to C_k^* \to C_{k-1}^* \bigoplus P_{k-1}^* \to B_{k-1}^* \to 0 \) is exact.

Put \( P = \bigoplus_{i=1}^{k-1} P_i, M_0 = M \bigoplus P, M_i = C_i \bigoplus (\bigoplus_{j=i}^{k-1} P_j) \) for any \( 1 \leq i \leq k - 1 \) and \( M_k = C_k \).

Then we get our conclusion. ■

Let \( T \in \text{mod} \, \Lambda^{op} \). We remark that if one takes a chain of epimorphisms of \( T^* \) as in Proposition 4.2 and dualizes it, one then obtains the spherical filtration of Auslander and Bridger for \( T^{**} \). Notice that a module \( T \) in \( \text{mod} \, \Lambda^{op} \) is reflexive if and only if it is isomorphic to \( T^{**} \). So we in fact obtain the spherical filtration of Auslander and Bridger for each reflexive module in \( \text{mod} \, \Lambda^{op} \), and thus we may regard Proposition 4.2 as a duality of the spherical filtration of Auslander and Bridger.

As a corollary of Proposition 4.2, we get Auslander-Bridger’s Approximation Theorem (see [3] Theorem 2.41) for \( T^{**} \) (or for \( T \) if \( T \) is reflexive) as follows.
Corollary 4.3 Let $\Lambda$ be a quasi $k$-Gorenstein ring. Then, for any $T \in \text{mod } \Lambda^{op}$, there are a projective module $P$ and an exact sequence in $\text{mod } \Lambda^{op}$:

$$0 \to X \to T^{**} \bigoplus P \to Y \to 0$$

such that

(1) It is dual exact.

(2) $\text{r.pd}_\Lambda Y \leq k - 2$.

(3) The homomorphism $T^{**} \bigoplus P \to Y$ induces isomorphisms $\text{Ext}^i_{\Lambda}(Y, \Lambda) \cong \text{Ext}^i_{\Lambda}(T^{**}, \Lambda)$ for any $1 \leq i \leq k - 2$.

(4) If $T^{**} \to H$ is a homomorphism with $\text{r.pd}_\Lambda H \leq k - 2$, then the above exact sequence induces an isomorphism $\text{Hom}_{\Lambda}(Y, H) \cong \text{Hom}_{\Lambda}(T^{**}, H)$.

Proof. Let $T$ be in $\text{mod } \Lambda^{op}$. Then $T^*$ is in $\text{mod } \Lambda$. By Proposition 4.2, there are a projective module $P$ and exact sequences in $\text{mod } \Lambda^{op}$:

$$0 \to M_2^* \to M_1^* (\cong T^{**} \bigoplus P) \to B_1^* \to 0$$

and

$$0 \to M_3^* \to M_2^* \to B_2^* \to 0$$

with $M_2 \in \Omega^2(\text{mod } \Lambda)$, $M_3 \in \Omega^3(\text{mod } \Lambda)$, $B_1^*$ projective and $\text{r.pd}_\Lambda B_2^* \leq 1$.

Consider the following push-out diagram:

From the bottom row in the above diagram we know that $\text{r.pd}_\Lambda Y_1 \leq 1$. Notice that the first column $0 \to M_3^* \to M_2^* \to B_2^* \to 0$ is dual exact and the middle row $0 \to M_2^* \to M_1^* \to B_1^* \to 0$ splits, then it is easy to verify that the middle column $0 \to M_3^* \to M_1^* \to Y_1 \to 0$ is dual exact.

We then consider the following push-out diagram:
where the middle row is the middle column in the former diagram, the exactness of the first column follows from Proposition 4.2, $M_4 \in \Omega^4(\mod \Lambda)$ and $\text{r.pd}_\Lambda B_3 \leq 2$. From the bottom row in the above diagram we know that $\text{r.pd}_\Lambda Y_2 \leq 2$. Notice that both the middle row and the first column in above diagram are dual exact, we then get the following exact commutative diagram:

So $M_1^* \to M_4^*$ is epic and hence the middle column in the above diagram is dual exact.

Continuing this process, we finally get an exact sequence in $\mod \Lambda^{op}$:

$$0 \to X \to T^{**} \bigoplus P(\cong M_1^*) \to Y \to 0 \quad (4)$$

which is dual exact, where $X = M_k^*$ (where $M_k \in \Omega^k(\mod \Lambda)$) and $\text{r.pd}_\Lambda Y \leq k - 2$.

Since $M_k \in \Omega^k(\mod \Lambda)$, $\text{Ext}_\Lambda^i(X, \Lambda) = \text{Ext}_\Lambda^i(M_k^*, \Lambda) = 0$ for any $1 \leq i \leq k - 2$. So $\text{Ext}_\Lambda^i(Y, \Lambda) \cong \text{Ext}_\Lambda^i(T^{**}, \Lambda)$ for any $2 \leq i \leq k - 2$. On the other hand, from the fact that $\text{Ext}_\Lambda^1(M_k^*, \Lambda) = 0$ and the dual exactness of the sequence (4) we have that $\text{Ext}_\Lambda^1(Y, \Lambda) \cong \text{Ext}_\Lambda^1(T^{**}, \Lambda)$ and thus $\text{Ext}_\Lambda^i(Y, \Lambda) \cong \text{Ext}_\Lambda^i(T^{**}, \Lambda)$ for any $1 \leq i \leq k - 2$. So, if $T^{**} \to H$ is a homomorphism with $\text{r.pd}_\Lambda H \leq k - 2$, it then follows from [3] Lemma 2.42 that the exact sequence (4) induces an isomorphism $\text{Hom}_\Lambda(Y, H) \cong \text{Hom}_\Lambda(T^{**}, H)$. We are done.

Let $\Lambda$ be a commutative noetherian ring and let $n$ be a non-negative integer and $M$ in $\Omega^n(\mod \Lambda)$. An Evans-Griffith representation of $M$ is an exact sequence in $\mod \Lambda$:

$$0 \to S \to B \to M \to 0$$
where $B$ is an $n$th syzygy of $\text{Ext}_\Lambda^{n+1}(D(M), \Lambda)$ and $S$ is in $\Omega^{n+2}(\text{mod } \Lambda)$ (c.f. [7]). In the case $\Lambda$ is not necessarily commutative we also call such an exact sequence an Evans-Griffith representation of $M$.

**Proposition 4.4** Let $\Lambda$ be a quasi $k$-Gorenstein ring. Then, for any $0 \leq d \leq k - 1$, each module in $\Omega^d(\text{mod } \Lambda)$ has an Evans-Griffith representation.

**Proof.** Let $M$ be in $\Omega^d(\text{mod } \Lambda)$. By Theorem there is a projective module $P$ and an exact sequence in mod $\Lambda$:

$$0 \to B \overset{\alpha}{\to} M \bigoplus P \overset{\beta}{\to} C \to 0$$

satisfying the properties that $B$ is a $d$th syzygy of $\text{Ext}_\Lambda^{d+1}(D(M), \Lambda)$ and $C \in \Omega^{d+1}(\text{mod } \Lambda)$.

Let $\gamma$ be the composition: $B \overset{\alpha}{\to} M \bigoplus P \overset{(1,0)}{\to} M$, that is, $\gamma = (1,0)\alpha$. Suppose that $Q \overset{\delta}{\to} M \to 0$ is exact in mod $\Lambda$ with $Q$ projective. Then we have the following commutative diagram with exact columns and rows:

$$
\begin{array}{ccccccc}
0 & \to & B & \overset{\alpha}{\to} & M & \bigoplus & P & \overset{\beta}{\to} & C & \to & 0 \\
| & | & | & | & | & | & | & | & | & | \\
0 & \to & S & \to & P & \bigoplus & Q & \to & C & \to & 0 \\
| & | & | & | & | & | & | & | & | & | \\
0 & \to & B & \bigoplus & Q & \overset{(\alpha, \gamma)}{\to} & M & \bigoplus & P & \bigoplus & Q \overset{(\beta, 0)}{\to} & C & \to & 0 \\
| & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & &
On the other hand, by [6] we have that a commutative noetherian ring \( \Lambda \) has finite self-injective dimension if and only if \( \text{gradeExt}_\Lambda^i(N, \Lambda) \geq i \) for any \( N \in \text{mod} \Lambda \) and \( i \geq 1 \). So, by Corollary 4.5, we immediately have the following

**Corollary 4.6** If \( \Lambda \) is a commutative noetherian with finite self-injective dimension, then for any non-negative integer \( d \), each module in \( \Omega^d(\text{mod} \Lambda) \) has an Evans-Griffith representation.

Observe that a special instance of Corollary 4.6 was already considered by Evans and Griffith in [7] Theorem 2.1. They claimed that if \( \Lambda \) is a commutative noetherian local ring with finite global dimension and contains a field then each non-free \( d \)th syzygy of rank \( d \) has an Evans-Griffith representation. Corollary 4.6 generalizes this result to much more general setting.

**Acknowledgements** The author was partially supported by Specialized Research Fund for the Doctoral Program of Higher Education. This paper was finished during a visit of the author to Okayama University from January to June, 2004. The author is grateful to Prof. Yuji Yoshino for his kind hospitality.

**References**

[1] F.W. Anderson and K.R. Fuller, Rings and Categories of modules, 2nd ed, Graduate Texts in Mathematics **13**, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[2] M. Auslander, *Coherent functors*, in Proceedings of the Conference on Categorial Algebra, La Jolla, 1965, Springer-Verlag, Berlin, 1966, pp. 189–231.

[3] M. Auslander and M. Bridger, *Stable module theory*, Memoirs Amer. Math. Soc. **94**, American Mathematical Society, Providence, Rhode Island, 1969.

[4] M. Auslander and I. Reiten, *Applications to contravariantly finite subcategories*, Adv. Math. **86**(1991), 111–152.

[5] M. Auslander and I. Reiten, *Syzygy modules for noetherian rings*, J. Algebra **183**(1996), 167–185.

[6] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82**(1963), 8–28.

[7] E. G. Evans and P. Griffith, *Syzygies of critical rank*, Quart. J. Math. Oxford **35**(1984), 393–402.

[8] R. M. Fossum, P. A. Griffith and I. Reiten, Trivial Extensions of Abelian Categories, Lecture Notes in Mathematics **456**, Springer-Verlag, Berlin-Heidelberg-New York, 1975.
[9] Z. Y. Huang, *On the grade of modules over noetherian rings*, Preprint (2003).

[10] M. Hoshino and K. Nishida, *A generalization of the Auslander formula*, Preprint (2003).

[11] Y. Iwanaga and H. Sato, *On Auslander’s n-Gorenstein rings*, J. Pure and Appl. Algebra 106 (1996), 61–76.