REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH PERTURBATIONS

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Abstract. This paper deals with a large class of reflected backward stochastic differential equations whose generators arbitrarily depend on a small parameter. The solutions of these equations, named the perturbed equations, are compared in the $L^p$-sense, $p \in [1, 2]$, with the solutions of the appropriate equations of the equal type, independent of a small parameter and named the unperturbed equations. Conditions under which the solution of the unperturbed equation is $L^p$-stable are given. It is shown that for an arbitrary $\eta > 0$ there exists an interval $[t(\eta), T] \subset [0, T]$ on which the $L^p$-difference between the solutions of both the perturbed and unperturbed equations is less than $\eta$.

1. Introduction. It is known that the theory of backward stochastic differential equations (BSDEs for short) was introduced and developed by Pardoux and Peng [24, 28, 25] in the 90s. They introduced the notation of nonlinear BSDE and proved the existence and uniqueness of adapted solutions in their fundamental paper [24]. Furthermore, it was shown in various papers that BSDEs give the probabilistic representation of solutions (at least in the viscosity sense) for a large class of systems of semi-linear parabolic partial differential equations (PDEs) (see [28],[27]). After that, BSDEs are widely used to describe numerous mathematical problems in finance (see [8]), stochastic control and stochastic games (see [11]–[12]), stochastic partial differential equations etc. Consequently, all these applications incited to introduce various types of BSDEs.

Reflected BSDEs have been first introduced in literature by El-Karoui et al. in [7]. For these BSDEs, that is, for the equations of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, 0 \leq t \leq T,$$

$$Y_t \geq L_t, t \leq T \quad \text{and} \quad \int_0^T (Y_s - L_s)dB_s = 0 \quad P-a.s.,$$

(1)

one of the components of the solution is forced to stay above a given barrier/obstacle process $L = \{L_t, t \in [0, T]\}$. The solution is a triple of adapted processes $\{(Y_t, Z_t, K_t), t \in [0, T]\}$ which satisfies Eq. (1).
The process $K = \{K_t, t \in [0, T]\}$ is nondecreasing and its purpose is to push upward the state process $Y = \{Y_t, t \in [0, T]\}$ in order to keep it above the obstacle $L$.

Reflected BSDEs (RBSDEs in short) are connected with a wide range of applications especially the pricing of American options in markets, constrained or not, mixed control, partial differential variational inequalities, real options (see [6], [16], [7], [8], [19] etc. and the references therein). El-Karoui et al. proved in [7] the existence and uniqueness of the solution to Eq. (1) under conditions of square integrability of the data and Lipschitz property for the coefficient $f$. On the one hand, there have been a lot of papers which deal with the issue of the existence and uniqueness results for RBSDEs under weaker assumptions than the ones in [24], and on the other hand, many tried to enlarge the classes of RBSDEs by introducing some new types of those equations.

Gegout-Petit in [10] proposed a class of RBSDEs associated with a multivalued maximal monotone operator defined by the subdifferential of a convex function. Further, Pardoux and Rascanu [26] proved the existence and uniqueness of the solution for RBSDEs on a random (possibly infinite) time interval, involving a sub-differential operator in order to give the probabilistic interpretation for the viscosity solution of some parabolic and elliptic variational inequalities. Then, Ouknine [23] and Bahdali et al. [3], [4] discussed a type of RBSDEs driven by a Brownian motion or the combination of a Brownian motion and a Poisson random measure under Lipschitz conditions, locally Lipschitz conditions, or a monotone conditions on the coefficients. El-Karoui et al. also studied in [7] a type of RBSDEs, where one of the components of solutions is forced to stay above a given barrier, which provided a probabilistic formula for the viscosity solution of an obstacle problem for a parabolic PDE. Since then, there have been many papers on this topic; see, for example, Matoussi [22], Hamadène [13] and Hamadène and Ouknine [14], Lepeltier and Xu [20] and Ren et al. [29, 30]. Each group of authors, had different approaches, for example, Ren et al. in [29] have introduced the notation of generalized RBSDEs and they connected the solution of these equations with the obstacle problem for parabolic PDEs in the case when the data belong only to $L^p$ for some $p \in [1, 2]$.

Recently, Hamdène and Popier in [15] proved that if $\xi, \sup_{t \in [0, T]} (L^T_t)$ and $\int_0^T |f(t, 0, 0)| dt$ belong to $L^p$ for some $p \in [1, 2]$, then RBSDE (1) with one reflecting barrier associated with $(f, \xi, L)$ has a unique solution. Aman gave [1] a similar result for a class of generalized RBSDEs with Lipschitz condition on the coefficients, and he extended these results under non-Lipschitz condition in his paper [2]. The main objective of the present paper is to complete those works and to study Eq. (1) if the terminal condition $\xi$ and generator $f$ are $p$-integrable, $p \in [1, 2]$. The main motivation is that in several applications such as in finance, control, games, PDEs, etc., data are not square integrable.

Perturbed stochastic differential equations, in general, are the topic of permanent interest of many authors, both theoretically and in applications. Stochastic models of complex phenomena under perturbations in analytical mechanics, control theory and population dynamics, for example, can be sometimes compared and approximated by appropriate unperturbed models of a simpler structure. In this way, the problems can be translated on more simple and familiar cases which are easier to solve and investigate. We refer to [9], [18], [32], [33], for example, while an additive type of perturbations for BSDEs is observed in [17]. Note that the study
of perturbed BSDEs is completely different than the ones for forward stochastic differential equations.

Before presenting the main results, we briefly give only the essential notations and definitions which are necessary in the investigation. The initial assumption is that all random variables and processes are defined on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\), where \(\{\mathcal{F}_t, t \in [0, T]\}\) is a natural filtration of a standard \(d\)-dimensional Brownian motion \(B = \{B_t, t \in [0, T]\}\), that is, it is right continuous and complete.

All stochastic processes are defined for \(t \in [0, T]\), where \(T\) is a positive real constant, and they take values in \(\mathbb{R}^n\) for some positive integer \(n\). For any \(k \in \mathbb{N}\) and \(x \in \mathbb{R}^k\), \(|x|\) denotes the Euclidean norm of \(x\).

Further, for any real constant \(p \in [1, 2]\), let us denote that:

(i) \(S^p(\mathbb{R})\) is the set of \(\mathbb{R}\)-valued, adapted and continuous processes \(\{X_t, t \in [0, T]\}\) such that

\[
||X||_{S^p} = E \left[ \sup_{t \in [0, T]} |X_t|^p \right]^{\frac{1}{p}} < \infty.
\]

The space \(S^p(\mathbb{R})\) endowed with the norm \(|| \cdot ||_{S^p}\) is a Banach type.

(ii) \(M^p\) is the set of predictable processes \(\{Z_t, t \in [0, T]\}\) with values in \(\mathbb{R}^d\) such that

\[
||X||_{M^p} = E \left[ \left( \int_0^T |Z_t|^2 dt \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} < \infty.
\]

Likewise, \(M^p(\mathbb{R}^n)\) endowed with the norm \(|| \cdot ||_{M^p}\) is a Banach space.

The space \(S^p \times M^p\) will be denoted by \(B^p\).

Let \(\xi\) be an \(\mathbb{R}\)-valued and \(\mathcal{F}_T\)-measurable random variable and let a random function \(f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) be measurable with respect to \(\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)\), where \(\mathcal{P}\) denotes the \(\sigma\)-field of progressive subsets of \([0, T] \times \Omega\). Finally let \(L := \{L_t, t \in [0, T]\}\) be a continuous progressively measurable \(\mathbb{R}\)-valued process.

The following assumptions are introduced for \(\xi, f\) and \(L\):

(H1) \(\xi \in L^p(\Omega)\).

(H2) (i) The process \(\{f(t, 0, 0), t \in [0, T]\}\) satisfies \(E(\int_0^T |f(t, 0, 0)| dt)^p < \infty\);

(ii) there exists a constant \(k > 0\) such that for all \(t \in [0, T], (y, z), (y', z') \in \mathbb{R} \times \mathbb{R}^d\),

\[
|f(t, y, z) - f(t, y', z')| \leq k(|y - y'| + |z - z'|).
\]

(H3) The barrier process \(L\) satisfies:

(i) \(L_T \leq \xi\);

(ii) \(L^+: = L \vee 0 \in S^p(\mathbb{R})\).

The following definitions of the unique solution to Eq. (1), associated with the triple \((\xi, f, L)\), are given in the sequel, as well as the basic existence and uniqueness theorem (see [15]).

Definition 1. The triple \(\{Y_t, Z_t, K_t, t \in [0, T]\}\) is an \(L^p\)-solution to RBSDE (1) with a continuous lower reflecting barrier \(L\), terminal condition \(\xi\) and generator \(f\) if:

1. \(\{Y_t, Z_t, t \in [0, T]\}\) belongs to \(B^p\);
2. \(K = \{K_t, t \in [0, T]\}\) is an adapted continuous nondecreasing process such that \(K_0 = 0\) and \(K_T \in L^p(\Omega)\);
(3) \( Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,ds + K_T - K_t - \int_t^T Z_s\,dB_s \), \( t \in [0,T] \);
(4) \( Y_t \geq L_t, t \in [0,T] \);
(5) \( \int_0^T (Y_s - L_s)\,dK_s = 0 \) \( \text{P-a.s.} \)

**Definition 2.** The triple \( \{(Y_t,Z_t,K_t), t \in [0,T]\} \) is a unique \( L^p \)-solution to RBSDE (1) if for any other solution \( \{(\hat{Y}_t,\hat{Z}_t,\hat{K}_t), t \in [0,T]\} \), the following holds,

\[
\|Y_t - \hat{Y}_t\|_{S^p} = 0, \quad \|Z_t - \hat{Z}_t\|_{M^p} = 0, \quad \|K_t - \hat{K}_t\|_{S^p} = 0. \tag{2}
\]

**Proposition 1** (Hamèdène, Popier [15]). Let \( (H_1) - (H_3) \) hold for \( \xi, f \) and \( L \). Then, RBSDE (1) with one continuous lower reflecting barrier \( L \) associated with \( (\xi, f, L) \) has a unique \( L^p \)-solution, \( p \in ]1,2[ \), i.e. there exists a triple of processes \( \{(Y_t,Z_t,K_t), t \in [0,T]\} \) satisfying (1)-(5) from Definition 1 and (2) from Definition 2.

The following lemma will be essentially used in the sequel.

**Lemma 1** (Hamèdène, Popier [15]). Assume that \( (Y,Z) \in B^p \) is a solution of the equation

\[
Y_t = \xi + \int_t^T f(s,Y_s,Z_s)\,ds + A_T - A_t - \int_t^T Z_s\,dB_s, \quad t \in [0,T],
\]

where:

(i) \( f \) is a function satisfying the previous assumptions;
(ii) The process \( \{A_t, t \in [0,T]\} \) is \( \text{P-a.s.} \) of bounded variation.

Then, for any \( 0 \leq t \leq u \leq T \) it follows that

\[
|Y_u|^p + c(p) \int_t^u |Y_s|^{p-2} |Z_s|^2 \,ds
\leq |Y_u|^p + p \int_t^u |Y_s|^{p-1} \hat{Y}_s \,dA_s + p \int_t^u |Y_s|^{p-1} \hat{Y}_s f(s,Y_s,Z_s)\,ds
- p \int_t^u |Y_s|^{p-1} \hat{Y}_s Z_s\,dB_s,
\]

where \( c(p) = \frac{p-1}{2} \) and \( \hat{Y} = \frac{u}{|y|} 1_{y \neq 0} \).

Problems of perturbed BSDEs can be used in many fields of applications. Clearly, if the state of the model is described by BSDE, any change of the system could be treated as a perturbation of the initial equation. The size of the change could be estimated as the difference between the solutions of the initial equation and the perturbed one. In view of this direction, together with Eq. (1), we study the following generally perturbed RBSDE,

\[
Y_t = \xi^\varepsilon + \int_t^T f^\varepsilon(s,Y_s^\varepsilon,Z_s^\varepsilon,\varepsilon)\,ds + K_T^\varepsilon - K_t^\varepsilon - \int_t^T Z_s^\varepsilon\,dB_s, \quad 0 \leq t \leq T,
\]

\[
Y_t^\varepsilon \geq L_t^\varepsilon, \quad t \leq T \quad \text{and} \quad \int_0^T (Y_s^\varepsilon - L_s^\varepsilon)\,dK_s^\varepsilon = 0, \quad \text{P-a.s.} \tag{3}
\]

where \( \xi^\varepsilon, f^\varepsilon \) and the barrier \( L^\varepsilon \) are defined as \( \xi, f \) and \( L \), respectively, and they depend on a small parameter \( \varepsilon \in (0,1) \). For a given \( (f^\varepsilon, \xi^\varepsilon, L^\varepsilon) \), a triple of adapted processes \( \{(Y_t^\varepsilon, Z_t^\varepsilon, K_t^\varepsilon), t \in [0,T]\} \) is a solution to Eq. (3). In the sequel Eq. (1) is treated as the unperturbed equation, while Eq. (3) is a perturbed one. It is usually expected that the perturbed equation (3) is more complex than the unperturbed one. This fact is a basic motivation for us to introduce conditions guaranteeing the
closeness of the solutions of the perturbed and unperturbed equations in the \(L^p\)-sense, which will be the topic of our study. Moreover, we could expect that the solutions of these equations stay close in the \(L^p\)-sense when \((f, \xi, L)\) and \((f^\varepsilon, \xi^\varepsilon, L^\varepsilon)\) are close in some way.

The paper is organized in the following way: In Section 2, we formulate the problem of perturbed RBSDEs and we give a preliminary result the \(L^p\)-difference between the state processes \(Y^\varepsilon\) and \(Y\), which will be needed in the sequel. In Section 3, we discuss a stability problem, that is, we give conditions under which the \(L^p\)-difference between the solutions of both the perturbed and unperturbed equations tends to zero as \(\varepsilon\) tends to zero. However, from the modeling point of view, it is usually important to study the closeness between the state processes \(\xi\).

2. Formulation of the problem and preliminary results. According to the previous discussion, we introduce the following assumptions:

(A0) For the final conditions \(\xi^\varepsilon, \xi \in L^p(\Omega)\), there exists a non-random function \(\alpha_0(\varepsilon), \varepsilon \in (0, 1)\), such that

\[E|\xi^\varepsilon - \xi|^p \leq \alpha_0(\varepsilon).\]

(A1) For the generators \(f^\varepsilon\) and \(f\), there exists a non-random function \(\alpha_1(\varepsilon), \varepsilon \in (0, 1)\), such that

\[
\sup_{(t,y,z) \in [0,T] \times B^p} |f^\varepsilon(t,y,z,\varepsilon) - f(t,y,\varepsilon)| \leq \alpha_1(\varepsilon) \ a.s.
\]

(A2) For the barrier processes \(L^\varepsilon\) and \(L\), there exists a non-random function \(\alpha_2(\varepsilon), \varepsilon \in (0, 1)\), such that

\[E \sup_{t \in [0,T]} |L^\varepsilon_t - L_t|^p \leq \alpha_2(\varepsilon).
\]

The following proposition is an auxiliary result which will be needed in the sequel.

**Proposition 2.** Let \(p \in [1,2]\) and let \(\{(Y_t, Z_t, K_t), t \in [0,T]\}\) and \(\{(Y^\varepsilon_t, Z^\varepsilon_t, K^\varepsilon_t), t \in [0,T]\}\) be the solutions to Eqs. (1) and (3), respectively. Let also \((\xi, f, L)\) and \((\xi^\varepsilon, f^\varepsilon, L^\varepsilon)\) satisfy assumptions (A0) – (A3) and conditions (H1) – (H3). Then,

\[E|Y^\varepsilon_t - Y_t|^p \leq C_1 e^{c_1(T-t)}, \quad t \in [0,T],
\]

where \(c_1 = p - 1 + pk + \frac{\alpha_2^p}{p-1}\) and \(C_1 = \alpha_0(\varepsilon) + \alpha_0^p(\varepsilon)T + \alpha_2^{p-1}(\varepsilon)(E|K_T|^p)^{\frac{1}{p}}\).

**Proof.** Let us denote for \(t \in [0,T]\) that

\[
\hat{\xi} = \xi^\varepsilon - \xi, \quad \hat{Y}_t = Y^\varepsilon_t - Y_t, \quad \hat{Z}_t = Z^\varepsilon_t - Z_t, \quad \hat{K}_t = K^\varepsilon_t - K_t.
\]

If we subtract Eqs. (1) and (3), we obtain

\[
\hat{Y}_t = \hat{\xi} + \int_t^T (f^\varepsilon(s,Y^\varepsilon_s,Z^\varepsilon_s,\varepsilon) - f(s,Y_s,Z_s)) ds + \hat{K}_T - \hat{K}_t - \int_t^T \hat{Z}_s dB_s, \quad t \in [0,T].
\]
Applying Lemma 1 on $|\hat{Y}_i|^p$, we have

$$|\hat{Y}_i|^p + c(p) \int_t^T |\hat{Y}_s|^{p-2} 1_{\hat{Y}_s \neq 0} |\hat{Z}_s|^2 \, ds \quad (6)$$

where $I_i(t), i = 1, 2, 3, 4$ are the appropriate integrals. In order to estimate $I_1(t)$, we apply the elementary inequality $a^{p-1}b \leq \frac{p-1}{p}a^p + \frac{1}{p}b^p, a, b \geq 0$ and assumption (A1). Then,

$$I_1(t) = p \int_t^T |\hat{Y}_s|^{p-1} |f^\varepsilon(s, Y_s, Z_s, \varepsilon) - f(s, Y_s, Z_s)| \, ds \quad (7)$$

$$\leq (p-1) \int_t^T |\hat{Y}_s|^p \, ds + \int_t^T |f^\varepsilon(s, Y_s, Z_s, \varepsilon) - f(s, Y_s, Z_s)|^p \, ds.$$

$$\leq (p-1) \int_t^T |\hat{Y}_s|^p \, ds + \alpha^p_1(\varepsilon)(T-t).$$

In order to estimate $I_2(t)$, we use the elementary inequality $2ab \leq \frac{a^2}{2} + 2b^2$,

$$I_2(t) = 2p \sqrt{\frac{2}{p(p-1)}} k \int_t^T |\hat{Y}_s|^{p-1} \hat{Z}_s \, ds \quad (8)$$

where $c(p) = p(p-1)/2$.

In order to estimate $I_3(t)$, we will define a mapping $(x, a) \rightarrow \tilde{\theta}(x, a) = |x - a|^{p-2} 1_{x \neq a}(x-a), (x, a) \in \mathbb{R} \times \mathbb{R}$. Clearly, the function $x \rightarrow \tilde{\theta}(x, a)$ is non-decreasing, while the function $a \rightarrow \tilde{\theta}(x, a)$ is non-increasing. If we denote that $\tilde{L}_s = L_s^x - L_s$ and since $Y_s^x \geq L_s^x, Y_s \geq L_s$, then

$$I_3(t) = p \int_t^T |\tilde{Y}_s|^{p-1} \hat{Z}_s \, ds \quad (9)$$

$$= p \int_t^T |\tilde{Y}_s|^{p-1} \hat{Z}_s \, ds - p \int_t^T |\tilde{Y}_s|^{p-1} \hat{Z}_s \, ds.$$
Substituting (7), (8) and (9) in (6) yields

\[
\alpha \leq p \int_t^T |L_s^\varepsilon - L_s|^{p-2} I_{(L_s^\varepsilon - L_s, \neq 0)} |L_s^\varepsilon - L_s| dK_s
\]

\[
= p \int_t^T \bar{\theta}(L_s^\varepsilon, Y_s) dK_s
\]

\[
\leq p \int_t^T |L_s^\varepsilon - L_s|^{p-2} I_{(L_s^\varepsilon - L_s, \neq 0)} |L_s^\varepsilon - L_s| dK_s
\]

\[
- p \int_t^T |L_s^\varepsilon - L_s|^{p-2} I_{(L_s^\varepsilon - L_s, \neq 0)} |L_s^\varepsilon - L_s| dK_s
\]

\[
= p \int_t^T \tilde{L}_s|^{p-2} I_{(\tilde{L}_s, \neq 0)} \tilde{L}_s dK_s
\]

\[
- p \int_t^T |\tilde{L}_s|^{p-2} I_{(\tilde{L}_s, \neq 0)} \tilde{L}_s dK_s
\]

Substituting (7), (8) and (9) in (6) yields

\[
|\hat{Y}_t|^p + \frac{c(p)}{2} \int_t^T |\hat{Y}_s|^{p-2} I_{(\hat{Y}_s, \neq 0)} |\hat{Z}_s|^2 ds
\]

\[
\leq |\hat{L}|^p + \left( p - 1 + pk + \frac{pk^2}{p-1} \right) \int_t^T |\hat{Y}_s|^p ds + p \int_t^T |\tilde{L}_s|^{p-1} d(\tilde{K}_s)
\]

Taking expectation on (10) and by applying the previous elementary inequality, we derive that

\[
E|\hat{Y}_t|^p \leq E|\hat{Y}_t|^p + \frac{c(p)}{2} E \int_t^T |\hat{Y}_s|^{p-2} I_{(\hat{Y}_s, \neq 0)} |\hat{Z}_s|^2 ds
\]

\[
\leq \alpha_0(\varepsilon) + \left( p - 1 + pk + \frac{pk^2}{p-1} \right) \int_t^T E|\hat{Y}_s|^p ds
\]

\[
+ p \left( E \sup_{s \in [0,T]} |\tilde{L}_s|^p \right)^{\frac{p-1}{p}} (E|\tilde{K}_T|^p)^{\frac{1}{p}} + \alpha_1^p(\varepsilon)(T-t)
\]

\[
\leq \alpha_0(\varepsilon) + \alpha_2^p(\varepsilon)(T-t) + \alpha_2^p(\varepsilon)(E|\tilde{K}_T|^p)^{\frac{1}{p}}
\]

\[
+ \left( p - 1 + pk + \frac{pk^2}{p-1} \right) \int_t^T E|\hat{Y}_s|^p ds.
\]

Likewise, since \( K_T, K_T^2 \in L^p(\Omega) \), then \( E|\tilde{K}_T|^p < \infty \). Finally, the estimate (4) holds straightforwardly by applying the following version of the Gronwall-Bellman inequality (5, Theorem 1.5): Let \( u(t) \) be a continuous function in \([a, b]\), \( f(t) \) be Riemann integrable function in \([a, b]\) and \( c = const > 0 \). If \( u(t) \leq f(t) + c \int_t^b u(s) ds \), \( t \in [a, b] \), then \( u(t) \leq f(t) + c \int_a^b f(s)e^{c(s-t)} ds \), \( t \in [a, b] \). The proof is now complete. ⧫

3. The \( L^p \) -stability. In order to estimate the \( L^p \)-difference between the solutions to Eqs. (1) and (3), we need the assertion referring to the \( L^p \)-stability of the solution to Eq. (1).

**Theorem 1.** Let all the conditions of Proposition 2 be satisfied and let the functions \( \alpha_0(\varepsilon), \alpha_1(\varepsilon), \alpha_2(\varepsilon) \) tend to zero as \( \varepsilon \) tends to zero, uniformly in \( t \in [0, T] \). Then,
From (6), (11), (16) and (17), we find that

\[ E \sup_{t \in [0,T]} |Y_t^\varepsilon - Y_t| \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad (12) \]

\[ E \left( \int_0^T |Z_s^\varepsilon - Z_s|^2 \, ds \right)^{\frac{p}{2}} \rightarrow 0, \quad \varepsilon \rightarrow 0, \quad (13) \]

\[ E \sup_{t \in [0,T]} E|K_t^\varepsilon - K_t|^2 \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (14) \]

**Proof.** Let us denote that

\[ \phi(\varepsilon) = \max \left\{ \alpha_0(\varepsilon), \alpha_1(\varepsilon), \alpha_2(\varepsilon) \right\}. \quad (15) \]

From Proposition 2, we have that \( C_1 \leq \phi(\varepsilon) \tilde{C} \), where \( \tilde{C} = 1 + T + (E|\tilde{K}_T|^2)^{\frac{p}{2}} \) and, therefore,

\[ E|\tilde{Y}_t^\varepsilon|^p \leq \phi(\varepsilon) \tilde{C} e^{c_1(T - t)}, \quad t \in [0,T]. \]

Since \( \phi(\varepsilon) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \), then for every \( t_0 \in [0,T] \),

\[ \sup_{t \in [t_0,T]} E|\tilde{Y}_t^\varepsilon|^p \leq \phi(\varepsilon) \tilde{C} e^{c_1(T - t_0)} \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (16) \]

In order to estimate the \( L^p \)-closeness between the processes \( Z_t \) and \( Z_t^\varepsilon(t) \), as well as \( K_t \) and \( K_t^\varepsilon(T) \), we need estimate \( E \sup_{t \in [0,T]} |\tilde{Y}_t^\varepsilon| \), that is to estimate \( I_4(t) \). By applying the Burkholder-Davis-Gundy inequality \([21]\) and Young inequality, \( u^\alpha v^{1-\alpha} \leq \alpha u + (1 - \alpha)v, \quad v \geq 0, \quad \alpha \in [0,1] \), we have

\[ E \sup_{t \in [t_0,T]} I_4(t) = E \sup_{t \in [t_0,T]} \left( -p \int_t^T |\tilde{Y}_s|^{p-1} sgn(\tilde{Y}_s) \tilde{Z}_s \, dB_s \right) \]

\[ = E \sup_{t \in [t_0,T]} \left( p \int_{t_0}^T |\tilde{Y}_s|^{p-1} sgn(\tilde{Y}_s) \tilde{Z}_s \, dB_s \right) \]

\[ \leq 4\sqrt{2p} E \left( \int_{t_0}^T |\tilde{Y}_s|^{2p-2} |\tilde{Z}_s|^2 \, ds \right)^{\frac{1}{2}} \]

\[ \leq 4\sqrt{2p} E \left( \sup_{t \in [t_0,T]} |\tilde{Y}_t^\varepsilon| \int_{t_0}^T |\tilde{Y}_s|^{p-2} |\tilde{Z}_s|^2 \, ds \right)^{\frac{1}{2}} \]

\[ \leq \frac{1}{2} E \sup_{t \in [t_0,T]} |\tilde{Y}_t^\varepsilon|^p + 16p^2 E \int_{t_0}^T |\tilde{Y}_s|^{p-2} |\tilde{Z}_s|^2 \, ds. \]

From (6), (11), (16) and (17), we find that

\[ E \sup_{t \in [t_0,T]} I_4(t) \leq \frac{1}{2} E \sup_{t \in [t_0,T]} |\tilde{Y}_t^\varepsilon|^p + \frac{32p^2}{c(p)} \phi(\varepsilon) \tilde{C} e^{c_1(T - t_0)}. \quad (18) \]

Now, from (10) and the previous estimates, we derive that

\[ E \sup_{t \in [t_0,T]} |\tilde{Y}_t^\varepsilon|^p \leq \frac{1}{2} E \sup_{t \in [t_0,T]} |\tilde{Y}_t^\varepsilon|^p + \frac{32p^2}{c(p)} \phi(\varepsilon) \tilde{C} e^{c_1(T - t_0)} \]

\[ + c_1 \int_{t_0}^T E|\tilde{Y}_s|^p \, ds + \phi(\varepsilon) \tilde{C}. \]
Hence,

\[
E \sup_{t \in [t_0, T]} |\hat{Y}_t|^p \leq 2\phi(\varepsilon)C \left( e^{\varepsilon(T-t_0)} \left( 1 + \frac{32p^2}{c(p)} \right) + 1 \right) \equiv \phi(\varepsilon)A_1(t_0),
\]

(19)

where \( A_1(t_0) \) is a generic positive constant. Since \( \phi(\varepsilon) \to \varepsilon \to 0 \), then \( E \sup_{t \in [t_0, T]} |\hat{Y}_t|^p \to 0 \), as \( \varepsilon \to 0 \). Therefore, (12) holds if we take \( t_0 = 0 \).

Now we can estimate the other two parts.

For every \( i \in \{0, 1, 2, \ldots\} \) and arbitrary \( t_0 \in [0, T] \), let us define stopping times

\[
\tau_i = \inf \left\{ t \in [0, T], \int_{t_0}^{t} ||\hat{Z}_s||^2 \, ds \geq i \right\} \wedge T.
\]

Clearly, \( \tau_i \uparrow T \) a.s. when \( i \to \infty \). If we apply the Ito formula to \( e^{k\tau}|\hat{Y}_i|^2 \), \( t \in [t_0, \tau_i] \), we find that

\[
|\hat{Y}_{\tau_i}|^2 + \int_{t_0}^{\tau_i} e^{k\tau_s}|\hat{Z}_s|^2 \, ds
\]

\[
= e^{k\tau_s}|\hat{Y}_{\tau_s}|^2 + \int_{t_0}^{\tau_i} e^{k\tau_s} Y_s^2 \left[ 2\left( f^\varepsilon(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon) - f(s, Y_s, Z_s) \right) - kY_s \right] \, ds
\]

\[
+ 2 \int_{t_0}^{\tau_i} e^{k\tau_s} \hat{Y}_s \, d\hat{K}_s - 2 \int_{t_0}^{\tau_i} e^{k\tau_s} \hat{Z}_s \, dB_s
\]

\[
\equiv e^{k\tau_s}|\hat{Y}_{\tau_s}|^2 + J_1 + J_2 - 2 \int_{t_0}^{\tau_i} e^{k\tau_s} \hat{Z}_s \, dB_s,
\]

(20)

where estimates \( J_1 \) and \( J_2 \) are the appropriate integrals. To estimate \( J_1 \), we see for \( \lambda_1, \lambda_2 > 0 \) that

\[
J_1 = 2 \int_{t_0}^{\tau_i} e^{k\tau_s} \hat{Y}_s \left( f^\varepsilon(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon) - f(s, Y_s, Z_s) \right) \, ds - k \int_{t_0}^{\tau_i} e^{k\tau_s} \hat{Y}_s^2 \, ds
\]

\[
= 2 \int_{t_0}^{\tau_i} e^{k\tau_s} \hat{Y}_s \left( f^\varepsilon(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon) - f(s, Y_s^\varepsilon, Z_s) \right) \, ds
\]

\[
+ 2 \int_{t_0}^{\tau_i} e^{k\tau_s} \hat{Y}_s \left( f(s, Y_s^\varepsilon, Z_s^\varepsilon) - f(s, Y_s, Z_s) \right) \, ds - k \int_{t_0}^{\tau_i} e^{k\tau_s} \hat{Y}_s^2 \, ds
\]

\[
\leq 2 \sup_{s \in [t_0, \tau_i]} e^{k\tau_s}|\hat{Y}_s|^2 + \lambda_1 |T - t_0| \alpha^2(\varepsilon)
\]

\[
+ \left( \frac{k}{\lambda_2} + k \right) \int_{t_0}^{\tau_i} e^{k\tau_s} |\hat{Y}_s|^2 \, ds + k\lambda_2 \int_{t_0}^{\tau_i} e^{k\tau_s} |\hat{Z}_s|^2 \, ds.
\]

(21)

Similary, for \( \lambda_3 > 0 \),

\[
J_2 = 2 \int_{t_0}^{\tau_i} e^{k\tau_s} \hat{Y}_s \, d\hat{K}_s \leq 2 \sup_{s \in [t_0, \tau_i]} e^{k\tau_s}|\hat{Y}_s| \int_{t_0}^{\tau_i} \, d\hat{K}_s
\]

\[
\leq \frac{1}{\lambda_3} \sup_{s \in [t_0, \tau_i]} e^{2k\tau_s}|\hat{Y}_s|^2 + \lambda_3 \left( \int_{t_0}^{\tau_i} \, d\hat{K}_s \right)^2
\]

\[
= \frac{1}{\lambda_3} \sup_{s \in [t_0, \tau_i]} e^{2k\tau_s}|\hat{Y}_s|^2 + \lambda_3 (\hat{K}_{\tau_i} - \hat{K}_{t_0})^2.
\]

(22)
Likewise,

\[
(\tilde{K}_{\tau_i} - \tilde{K}_{t_0})^2 \\
= \left(\tilde{Y}_{\tau_i} - \tilde{Y}_{t_0} - \int_{t_0}^{\tau_i} (f^e(s,Y_s^e,Z_s^e,\varepsilon) - f(s,Y_s,Z_s)) \, ds \right. \\
\left. + \int_{t_0}^{\tau_i} \tilde{Z}_s \, dB_s \right)^2 \\
\leq 4 \left[ |\tilde{Y}_{\tau_i}|^2 + |\tilde{Y}_{t_0}|^2 + \left| \int_{t_0}^{\tau_i} (f^e(s,Y_s^e,Z_s^e,\varepsilon) - f(s,Y_s,Z_s)) \, ds \right|^2 \\
+ \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 \, dB_s \right]^2 \\
\leq 4 \left[ |\tilde{Y}_{\tau_i}|^2 + |\tilde{Y}_{t_0}|^2 + (T - t_0) \int_{t_0}^{\tau_i} |f^e(s,Y_s^e,Z_s^e,\varepsilon) - f(s,Y_s,Z_s)|^2 \, ds \\
+ \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 \, dB_s \right]^2 \\
\leq 4 \left[ |\tilde{Y}_{\tau_i}|^2 + |\tilde{Y}_{t_0}|^2 + 2(T - t_0)^2 \alpha_1^2(\varepsilon) + 4k^2(T - t_0) \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 \, ds \\
+ 4k^2(T - t_0) \int_{t_0}^{\tau_i} |\tilde{Y}_s|^2 \, ds + \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 \, dB_s \right]^2. \tag{23}
\]

Substituting (21), (22) and (23) in (20) yields

\[
(1 - 4\lambda_3)|\tilde{Y}_{\tau_i}|^2 + (1 - k\lambda_2) \int_{t_0}^{\tau_i} e^{ks}|\tilde{Z}_s|^2 \, ds - 16\lambda_3 k^2(T - t_0) \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 \, ds \\
\leq (e^{k\tau_i} + 4\lambda_3)|\tilde{Y}_{\tau_i}|^2 + \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_3} \right) \sup_{s \in [t_0,\tau_i]} e^{2ks}|\tilde{Y}_s|^2 \\
+ 4\lambda_3 \left( \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 \, ds \right)^2 - 2 \int_{t_0}^{\tau_i} e^{ks}\tilde{Y}_s \tilde{Z}_s \, dB_s \\
+ \left( \frac{k}{\lambda_2} + k + 16k^2(T - t_0)\lambda_3 \right) \int_{t_0}^{\tau_i} e^{ks}|\tilde{Z}_s|^2 \, ds \\
+ (\lambda_1 + 8\lambda_3)(T - t_0)^2 \alpha_1^2(\varepsilon).
\]

Hence,

\[
[1 - k\lambda_2 - 16k^2(T - t_0)\lambda_3] \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 \, ds \\
\leq \left( e^{k\tau_i} + 4\lambda_3 + \frac{1}{\lambda_1} + \frac{1}{\lambda_3} \right) \sup_{s \in [t_0,\tau_i]} e^{2ks}|\tilde{Y}_s|^2 \\
+ 4\lambda_3 \left( \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 \, ds \right)^2 - 2 \int_{t_0}^{\tau_i} e^{ks}\tilde{Y}_s \tilde{Z}_s \, dB_s \\
+ \left( \frac{k}{\lambda_2} + k + 16k^2(T - t_0)\lambda_3 \right) \int_{t_0}^{\tau_i} e^{ks}|\tilde{Z}_s|^2 \, ds \\
+ (\lambda_1 + 8\lambda_3)(T - t_0)^2 \alpha_1^2(\varepsilon). \tag{24}
\]
The last inequality can be written as
\[
c_2(t_0) \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 ds \leq c_3(t_0) \sup_{s \in [t_0, \tau_i]} |\tilde{Y}_s|^2 + 4 \lambda_3 \left( \int_{t_0}^{\tau_i} \tilde{Z}_s dB_s \right)^2 + 2 \left( \int_{t_0}^{\tau_i} e^{k_s} \tilde{Y}_s \tilde{Z}_s dB_s \right)^2 + (\lambda_1 + 8 \lambda_3)(T - t_0)^2 \alpha_1(\varepsilon),
\]
where
\[
c_2(t_0) = 1 - k \lambda_2 - 16 k^2 (T - t_0) \lambda_3,
\]
\[
c_3(t_0) = \left[ e^{kT} + 4 \lambda_3 + \frac{1}{\lambda_1} + \frac{1}{\lambda_3} + \frac{k}{\lambda_2} + k + 16 k^2 (T - t_0) \lambda_3 \right] (T - t_0) e^{2 k T}.
\]
By applying the inequality \((\sum_{i=1}^{m} a_i)^k \leq (m^{k-1} \lor 1) \sum_{i=1}^{m} a_i^k, a_i \geq 0, k \geq 0\) on (25) and by taking expectation, we obtain
\[
c_2^\frac{p}{2} (t_0) E \left( \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 ds \right)^\frac{p}{2} \leq c_3^\frac{p}{2} (t_0) E \sup_{s \in [t_0, \tau_i]} |\tilde{Y}_s|^p + 4^\frac{p}{2} \lambda_3^\frac{p}{2} E \left( \int_{t_0}^{\tau_i} |\tilde{Y}_s|^2 |\tilde{Z}_s|^2 ds \right)^\frac{p}{2} + 2^\frac{p}{2} E \left( \int_{t_0}^{\tau_i} e^{k_s} \tilde{Y}_s \tilde{Z}_s dB_s \right)^2 + (\lambda_1 + 8 \lambda_3)^\frac{p}{2} (T - t_0)^p \phi(\varepsilon).
\]
We can now estimate the second and third terms on the last relation by applying the Burkholder–Davis–Gundy inequality,
\[
E \left( \int_{t_0}^{\tau_i} \tilde{Z}_s dB_s \right)^p \leq C_p E \left( \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 ds \right)^\frac{p}{2},
\]
\[
2^\frac{p}{2} E \left( \int_{t_0}^{\tau_i} e^{k_s} \tilde{Y}_s \tilde{Z}_s dB_s \right)^\frac{p}{2} \leq C_2^\frac{p}{2} 2^\frac{p}{2} e^{k \tau_i} E \left[ \left( \int_{t_0}^{\tau_i} |\tilde{Y}_s|^2 |\tilde{Z}_s|^2 ds \right)^\frac{p}{2} \right].
\]
where \(\lambda_4 > 0, c_4 = \frac{1}{\lambda_4} C_2^\frac{p}{2} 2^p e^{pkT}, C_p = (32/p)^p/2\) and \(C_2^\frac{p}{2} = (64/p)^p/4\) are the universal constants. From the previous estimates and (26), we find that
\[
c_2^\frac{p}{2} (t_0) E \left( \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 ds \right)^\frac{p}{2} \leq c_3^\frac{p}{2} (t_0) E \sup_{s \in [t_0, \tau_i]} |\tilde{Y}_s|^p + 4^\frac{p}{2} \lambda_3^\frac{p}{2} C_p E \left( \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 ds \right)^\frac{p}{2} + c_4 E \sup_{s \in [t_0, \tau_i]} |\tilde{Y}_s|^p + \lambda_4 E \left( \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 ds \right)^\frac{p}{2} + (\lambda_1 + 8 \lambda_3)^\frac{p}{2} (T - t_0)^p \phi(\varepsilon),
\]
that is
\[
\left( c_2^\frac{p}{2} (t_0) - 4^\frac{p}{2} \lambda_3^\frac{p}{2} C_p - \lambda_4 \right) E \left( \int_{t_0}^{\tau_i} |\tilde{Z}_s|^2 ds \right)^\frac{p}{2} \leq (c_3^\frac{p}{2} (t_0) + c_4) E \sup_{s \in [t_0, \tau_i]} |\tilde{Y}_s|^p + (\lambda_1 + 8 \lambda_3)^\frac{p}{2} (T - t_0)^p \phi(\varepsilon).
\]
The constants $\lambda_2, \lambda_3$ and $\lambda_4$ can be chosen such that $c_2^2(t_0) - 4\frac{\epsilon}{5}^2\lambda_3^2 C_p - \lambda_4 > 0$, for example, we can take $\lambda_2 = \frac{1}{2k}, \lambda_3 = \frac{1}{4[16k^2(T-t_0) + 4C_p^2]}, \lambda_4 = \frac{1}{2}$. Then, we find from (19) and (27) that
\[
E \left( \int_{t_0}^{T} |\tilde{Z}_s|^2 \, ds \right)^{\frac{p}{2}} \leq \frac{(c_2^2(t_0) + c_4)A_1(t_0) + (8\lambda_3 + \lambda_4)^{\frac{p}{2}}(T-t_0)^2\phi(\epsilon)}{c_2^2(t_0) - 4\frac{\epsilon}{5}^2\lambda_3^2 C_p - \lambda_4} \equiv A_2(t_0) \phi(\epsilon),
\]
(28)
where $A_2(t_0)$ is a positive generic constant. By the Fatou Lemma,
\[
E \left( \int_{t_0}^{T} |\tilde{Z}_s|^2 \, ds \right)^{\frac{p}{2}} \to 0, \quad \epsilon \to 0.
\]
Hence, (13) holds if we take $t_0 = 0$.

It is left to estimate the difference between the processes $K$ and $K^\varepsilon$. From (5), we have
\[
\tilde{K}_t = \tilde{\xi} - \tilde{Y}_t + \int_{t}^{T} (f^\varepsilon(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon) - f(s, Y_s, Z_s)) \, ds - \int_{t}^{T} \tilde{Z}_s \, dB_s + \tilde{K}_T.
\]
In view of (19) and (28), we derive that
\[
E \sup_{t \in [t_0,T]} |\tilde{K}_t|^p \leq 5^{p-1} \left\{ E|\xi|^p + E \sup_{t \in [t_0,T]} |\tilde{Y}_t|^p + E \left| \int_{t_0}^{T} (f^\varepsilon(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon) - f(s, Y_s, Z_s)) \, ds \right|^p \right. \\
+ E \sup_{t \in [t_0,T]} \left| \int_{t}^{T} \tilde{Z}_s \, dB_s \right|^p + E|\tilde{K}_T|^p \right\} \\
\leq 5^{p-1} \left\{ \phi(\epsilon) + A_1(t_0)\phi(\epsilon) + (T-t_0)^{\frac{p}{2}} 2^{\frac{p}{2}} 2^{\frac{p}{2}} \left[ 2^{\frac{p}{2}} k^p E \left( \int_{t_0}^{T} [\tilde{Y}_t]^2 + [\tilde{Z}_t]^2 \right) \right]^{\frac{p}{2}} \\
+ E \left( \int_{t_0}^{T} |f^\varepsilon(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon) - f(s, Y_s, Z_s)|^2 \, ds \right)^{\frac{p}{2}} \right\} \\
+ C_p A_2(t_0) \phi(\epsilon) + E|\tilde{K}_T|^p \right\} \\
\leq 5^{p-1} \left\{ 1 + A_1(t_0) + (T-t_0)^{\frac{p}{2}} 2^{\frac{p}{2}} 2^{\frac{p}{2}} \left[ 2^{p-1} k^p [(T-t_0)^{\frac{p}{2}} 2^{p-1} k^p \tilde{C} \varepsilon^{2}(T-t_0) + A_2(t_0)] \\
+ (T-t_0)^{\frac{p}{2}} \right] + C_p A_2(t_0) \right\} \phi(\epsilon) + 5^{p-1} E|\tilde{K}_T|^p.
\]
Since
\[
\tilde{K}_T = \tilde{Y}_0 - \tilde{\xi} - \int_{0}^{T} (f^\varepsilon(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon) - f(s, Y_s, Z_s)) \, ds + \int_{0}^{T} \tilde{Z}_s \, dB_s,
\]
in accordance with the last estimate, we have that
\[ E|\hat{K}_T|^p \leq 4^{p-1} \left[ E|\hat{Y}_0|^p + E|\hat{\xi}|^p + E \int_0^T (f^\varepsilon(s, Y_s^\varepsilon, Z_s^\varepsilon, \varepsilon) - f(s, Y_s, Z_s)) \, ds \right]^p 
+ E \int_0^T \hat{Z}_s \, dB_s \] 
\leq 4^{p-1} \left[ \overline{C} e^{\gamma T} + T^{\frac{p}{2}} \left( 2^{p-1} k^p (T^{\frac{p}{2}} \overline{C} e^{\gamma T} + A_2(0)) + T \right) + A_2(0) \right] \phi(\varepsilon).

Hence, it follows from (29) that there exists a constant \( A_3(t_0) > 0 \) such that
\[ E \sup_{t \in [0, T]} |\hat{K}_t|^p \leq A_3(t_0) \phi(\varepsilon) \to 0, \quad \varepsilon \to 0. \quad (30) \]
Then, the estimate (14) holds if we take \( t_0 = 0 \), which completes the proof. \( \square \)

4. The \( L^p \)-closeness on an interval. In view of Theorem 1, one can deduce that the state processes \( Y_t^\varepsilon \) and \( Y_t \) and the control processes \( Z_t^\varepsilon \) and \( Z_t \) could be arbitrarily close for \( \varepsilon \) sufficiently small. However, from the modeling point of view, it is usually important to study the closeness between \( Y_t^\varepsilon \) and \( Y_t \) not on the whole time interval \([0, T]\) but only near to the terminal values \( \xi^\varepsilon \) and \( \xi \). In accordance with this requirement, for some permissible \( \eta > 0 \) and \( \varepsilon \) sufficiently small, we will find \( \hat{t}(\eta) = \hat{t} \in [0, T] \) so that the rate of the closeness between \( Y_t^\varepsilon \) and \( Y_t \) does not exceed \( \eta \) on \([\hat{t}, T]\). Likewise, we will estimate the closeness between the control processes \( Z_t^\varepsilon \) and \( Z_t \) on \([\hat{t}, T]\). In other words, the following assertion holds

**Theorem 2.** Let all the conditions of Theorem 1 hold, function \( \phi(\varepsilon), \varepsilon \in (0, 1) \) be continuous and monotone increasing, defined with
\[ \phi(\varepsilon) = \max \left\{ \alpha_0(\varepsilon), \alpha_1(\varepsilon), p \alpha_2^p(\varepsilon) \right\} \]
and let constants \( c_1 = p - 1 + pk + \frac{p k^2}{p - 1} \) and \( \overline{C} = 1 + T + (E|\hat{K}_T|^p)^{\frac{1}{p}} \) be given. Then, for an arbitrary constant \( \eta > 0 \) and \( \varepsilon \in (0, \Phi^{-1}(\eta)] \), there exists \( \hat{t} \in [0, T] \), where
\[ \hat{t} = \max \left\{ 0, T - \frac{1}{c_1} \ln \frac{\eta}{\Phi(\varepsilon) \overline{C}} \right\}, \]
such that
\[ \sup_{t \in [\hat{t}, T]} E|Y_t^\varepsilon - Y_t|^p \leq \eta, \quad (31) \]
\[ E \left( \int_{\hat{t}}^T |\hat{Z}_s|^2 \, ds \right)^{\frac{p}{2}} \leq A_2(\hat{t}) \phi(\varepsilon), \quad (32) \]
\[ E \sup_{t \in [\hat{t}, T]} |\hat{K}_t|^p \leq A_3(\hat{t}) \phi(\varepsilon), \quad (33) \]
and \( A_2(\hat{t}) \) and \( A_3(\hat{t}) \) are constants defined in (28) and (30), respectively.

**Proof.** Let us introduce the function \( S(\varepsilon, T - t), t \in [0, T] \), such that
\[ S(\varepsilon, T - t) = \phi(\varepsilon) \overline{C} e^{\gamma(T - t)}. \]
For an arbitrary \( \eta > 0 \), it must be
\[ S(\varepsilon, 0) \leq \eta \leq S(\varepsilon, T), \]
that is,

\[ \phi(\varepsilon) \tilde{C} \leq \eta \leq \phi(\varepsilon) \tilde{C} e^{c_1 T}. \]

Since \( \phi(\varepsilon) \) decreases if \( \varepsilon \) decreases, it follows that

\[ \varepsilon_1 = \phi^{-1}\left( \frac{\eta}{\tilde{C} e^{c_1 T}} \right) \leq \varepsilon \leq \phi^{-1}\left( \frac{\eta}{\tilde{C}} \right) = \varepsilon_2, \]

where \( \phi^{-1} \) is the inverse function of \( \phi \). For every \( \varepsilon \in [\varepsilon_1, \varepsilon_2] \), it is now easy to determine \( \hat{t} \) from the relation \( S(\varepsilon, T - \hat{t}) = \eta \), that is,

\[ \hat{t} = \frac{1}{c_1} \ln \frac{\eta}{\phi(\varepsilon) \tilde{C}}. \]

If \( \varepsilon \in (0, \varepsilon_1) \), then \( \eta > S(\varepsilon, T) \). If \( \varepsilon \in (0, \varepsilon_2) \), let us take

\[ \bar{t} = \max\{0, \hat{t}\} = \max \left\{ 0, T - \frac{1}{c_1} \ln \frac{\eta}{\phi(\varepsilon) \tilde{C}} \right\}. \]

Hence, for every \( \varepsilon \in (0, \varepsilon_2) \), it is easy to see that

\[ \mathbb{E} \sup_{t \in [0, T]} |Y_{\varepsilon}^\tau - Y_{\varepsilon}|^p \leq S(\varepsilon, T - \bar{t}) = \eta. \]

Clearly, \( \bar{t} \uparrow T \) as \( \varepsilon \uparrow \varepsilon_2 \) and \( \bar{t} \downarrow 0 \) as \( \varepsilon \downarrow \varepsilon_1 \), that is, \( \bar{t} \downarrow 0 \) as \( \varepsilon \downarrow 0 \).

Especially, in the case of additive perturbations, the generator of Eq. (3) is

\[ f^\varepsilon(t, y, z, \varepsilon) = f(t, y, z) + \alpha_1(t, y, z, \varepsilon), \]

where \( \alpha_1 \) is defined as the function \( f \) and depends on a small parameter \( \varepsilon \). Then, assumption \( A1 \) reduces so that there exists a non-random function \( \hat{\alpha}_1(\varepsilon) \) such that

\[ \sup_{(t, y, z) \in [0, T] \times B^p} |\alpha_1(t, y, z, \varepsilon)| \leq \hat{\alpha}_1(\varepsilon) \text{ a.s.} \]

The barrier process is also additively perturbed, i.e.

\[ L_t^\varepsilon = L_t + l_t^\varepsilon, \]

where \( l_t^\varepsilon \) is defined as the barrier \( L_t \). Assumption \( A2 \) becomes

\[ \mathbb{E} \sup_{t \in [0, T]} |l_t^\varepsilon|^p \leq \hat{\alpha}_2(\varepsilon), \]

where \( \hat{\alpha}_2(\varepsilon) \) is a non-random function. Under these conditions, all the previous assertions referring to a general type of perturbations hold.

Note that the study in the present paper could be extended to RBSDEs whose coefficients satisfy the monotonicity condition instead of the Lipschitz one (see [31], for instance). However, it requires a new additional investigation and could be the topic of some forthcoming studies.

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