Chromatic Number, Induced Cycles, and Non-separating Cycles

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Abstract
We study two parameters obtained from the Euler characteristic by replacing the number of faces with that of induced and induced non-separating cycles. By establishing monotonicity of such parameters under certain homomorphism and edge contraction, we obtain new upper bounds on the chromatic number in terms of the number of induced cycles and the Hadwiger number in terms of the number of induced non-separating cycles. As an application, we show that every 3-connected graph with average degree at least $2k$ for some $k \geq 2$ have at least $(k - 1)|V| + Ck^3 \log^{3/2} k$ induced non-separating cycles for some explicit constant $C > 0$. This improves the previous best known lower bound $(k - 1)|V| + 1$, which follows from Tutte’s cycle space theorem. We also give a short proof of this theorem of Tutte.

Keywords Chromatic number · Induced cycles · Hadwiger number · Induced non-separating cycles · Euler characteristic

Mathematics Subject Classification 05C15 · 05C83 · 05C38

1 Introduction
Understanding structure of graphs is often a formidable task. But sometimes, simply counting the number of basic objects such as vertices, edges, and cycles, gives a good understanding on some structural properties of the graphs. This is best illustrated by a classic theorem of Euler, which says that the alternating sum of the number of faces, edges and vertices in a graph $G$ determines its genus. Namely, if a graph $G$ is obtained by a 2-cell division of an orientable surface of genus $g$, then
where $\mathcal{F}(G)$ is the set of all faces in $G$. The quantity in the left hand side is called the Euler characteristic.

The notion of faces depends not only on the graph itself but also on the underlying surface on which the graph is drawn. However, it is well known that face boundaries of a 3-connected planar graphs are precisely given by its induced non-separating cycles. Hence, it is natural to replace the set $\mathcal{F}(G)$ of all faces in the Euler characteristic by some other class of cycles $\mathcal{C}(G)$. Parameters obtained in this way could be used to study different structural properties of graphs other than their genus. Indeed, Vince and Little \cite{8} and Yu \cite{9} used the parameter when $\mathcal{C}(G)$ is a cycle double cover (a collection of cycles such that each edge is contained in exactly two cycles in $\mathcal{C}(G)$) in order to study discrete Jordan curves.

In this paper, we study two parameters when $\mathcal{C}(G)$ is the set $\mathcal{C}(G)$ of all induced cycles and the set $\mathcal{F}(G)$ of all induced non-separating cycles in $G$. Let $\chi(G)$ denote the chromatic number of $G$. Our first main result is the following.

**Theorem 1** Let $G = (V, E)$ be a connected graph and let $\mathcal{C}(G)$ denote the set of all induced cycles in $G$. Then

$$
\left( \frac{\chi(G)}{3} \right) - \left( \frac{\chi(G)}{2} \right) + \left( \frac{\chi(G)}{1} \right) \leq |\mathcal{C}(G)| - |E(G)| + |V(G)|.
$$

Note that Theorem 1 gives an upper bound on $\chi(G)$ in the order of $|\mathcal{C}(G)|^{1/3}$. This is incomparable with the easy bound $\chi(G) \leq 3k + 2$, where $k$ is the maximum number of vertex-disjoint induced cycles in $G$.

Let $h(G)$ denote the Hadwiger number of $G$, the size of maximum complete graph minor in $G$. Our second main result is the following, which gives an upper bound on $h(G)$ in the order of $|\mathcal{F}(G)|^{1/3}$ when $G$ is 3-connected.

**Theorem 2** Let $G = (V, E)$ be a 3-connected graph and let $\mathcal{F}(G)$ denote the set of all induced non-separating cycles in $G$. Then

$$
\left( \frac{h(G)}{3} \right) - \left( \frac{h(G)}{2} \right) + \left( \frac{h(G)}{1} \right) \leq |\mathcal{F}(G)| - |E(G)| + |V(G)|.
$$

We prove Theorems 1 and 2 in Sects. 2 and 3, respectively.

As an application, Theorem 2 gives a new lower bound on the number of induced non-separating cycles in 3-connected graphs. Note that the previous best known lower bound follows from a theorem of Tutte, which states that the cycle space of a 3-connected graph is generated by its induced non-separating cycles \cite{7}. Recall that the rank of the cycle space of a connected graph $G$ equals that of its fundamental group, which is $|E(G)| - |V(G)| + 1$. Hence this gives
\[ |F(G)| \geq |E(G)| - |V(G)| + 1. \] (4)

In fact, this inequality follows directly from Theorem 2, since the left hand side of (3) is at least 1.

Graphs with large average degree are known to have large Hadwiger number due to Kostochka [2]. Namely, for a graph \( G = (V, E) \) with \( |E|/|V| \geq k \) for a fixed integer \( k \geq 2 \), we have \( h(G) \geq k/270\sqrt{\log k} \). Noting that the left hand side of (3) is non-decreasing in \( h(G) \), Theorem 2 yields the following new lower bound on \( |F(G)| \):

**Corollary 3** Let \( G = (V, E) \) be a 3-connected graph with \( |E|/|V| \geq k \) for some integer \( k \geq 2 \). Then we have

\[ |F(G)| \geq (k - 1)|V| + \frac{1}{118098000} \left( k^3 \log^{3/2} k - 1620k^2 \log k + 801900k \log^{1/2} k \right). \] (5)

We remark that this improves the previous best lower bound (4), which reads \( |F(G)| \geq (k - 1)|V| + 1 \).

### 1.1 Definitions

In this paper, every graph is finite and simple. Let \( S, H \) be two vertex-disjoint subgraphs in \( G \). We let \( S - V(H) \) denote the subgraph of \( G \) obtained from \( S \) by deleting all vertices of \( H \) and edges incident to them. If \( H \) is the singleton \( (\{v\}, \emptyset) \) for some vertex \( v \) in \( G \), then we denote \( S - v \) for \( S - V(H) \). If \( e \in E(G) \), then we denote by \( H - e \) the subgraph of \( G \) with vertex set \( V(H) \) and edge set \( E(H) \setminus \{e\} \). If two graphs are isomorphic, we write \( G \simeq H \) (see the first paragraph of Sect. 2 for the definition of isomorphism).

If \( H \subseteq G \) and \( v \in V(H) \), then \( N_H(v) \) denotes the set of all neighbors of \( v \) in \( H \) and \( ^\circ_H(v) := |N_H(v)| \). A graph \( G \) is a complete graph if \( uv \in E(G) \) for all distinct vertices \( u, v \) in \( G \). A complete graph with \( n \) vertices is denoted by \( K_n \). A graph \( C \) is called a cycle if it is connected and 2-regular, meaning \( ^\circ_C(v) = 2 \) for all \( v \in V(C) \). We denote by \( C_n \) the cycle of \( n \) vertices. A subgraph \( H \subseteq G \) is induced if \( S \subseteq H \) for any subgraph \( S \subseteq G \) with \( V(S) = V(H) \), and non-separating if either \( G - V(H) \) is connected or empty. For each subset \( V_0 \subseteq V(G) \), we let \( G[V_0] \) denote the induced subgraph of \( G \) with vertex set \( V_0 \). We let \( C(G) \) and \( F(G) \) denote the set of all induced and induced non-separating cycles in \( G \), respectively. Then \( F(G) \) is a subset of \( C(G) \). We denote the parameters in the right hand sides of (2) and (3) by \( \Lambda_C(G) \) and \( \Lambda_F(G) \) when \( G \) is connected and 3-connected, respectively. A graph \( G \) is 3-connected if \( |V(G)| \geq 4 \) and \( G - V(H) \) is connected for all subgraph \( H \) such that \( |V(H)| \leq 2 \).
2 Proof of Theorem 1

In this section, we prove Theorem 1. Given two graphs \( G \) and \( H \), a map \( v : V(G) \to V(H) \) is called a homomorphism if it preserves the adjacency relation, i.e., \( uv \in E(G) \) implies \( v(u)v(v) \in E(H) \). We denote \( v : G \to H \) for a homomorphism \( v : V(G) \to V(H) \). For example, suppose \( H \) is obtained by identifying two nonadjacent vertices \( u, v \) into a new vertex \( w \) and deleting resulting loops and parallel edges. Define \( v : V(G) \to V(H) \) by \( v(x) = x \) if \( x \notin \{u, v\} \) and \( v(x) = w \) otherwise. Then \( v \) is a homomorphism \( G \to H \), which is called a nonedge contraction. We say \( v : G \to H \) is an isomorphism if there exists a homomorphism \( n : H \to G \) such that \( n \circ v : G \to G \) is the identity map on \( V(G) \). A \( n \)-coloring of \( G \) is a homomorphism \( G \to K_n \). The chromatic number of \( G \), denoted by \( \chi(G) \), is the least integer \( n \) such that \( G \) admits a \( n \)-coloring.

We introduce an elementary homomorphism, which will be the basis of our proof of Theorem 1. Fix graphs \( G, G' \) and a vertex \( w \in V(G) \) such that \( \delta(w) \geq 1 \). Let \( B = G[N_G(w)] \) be the induced subgraph of \( G \) on the vertex set \( N_G(w) \). A homomorphism \( v : G \to G' \) is a local homomorphism of \( G \) at \( w \) if \( v \) restricted on \( G - N_G(w) \) is an isomorphism. Define \( [G/w] \) to be the set of graphs

\[
[G/w] = \left\{ G' \mid v : G \to G' \text{ is a local homomorphism of } G \text{ at } w \text{ and } v(B) \text{ is isomorphic to } K_{\chi(B)} \right\}.
\]

In other words, \( [G/w] \) is the set of all graphs obtained by successively identifying pairs of vertices in \( N_G(w) \) such that \( B \) is mapped to the smallest complete graph \( K_{\chi(B)} \) (see Fig. 1). Hence \( [G/w] \) is non-empty.

Recall that \( \Lambda_C(G) \) denotes the parameter in the right hand side of (2) when \( G \) is connected. The key observation in proving Theorem 1 is the following monotonicity of the parameter \( \Lambda_C \) under a local homomorphism \( G \to G' \in [G/w] \), as stated in the following lemma.

**Lemma 2.1** Let \( G = (V, E) \) be a connected graph and fix \( w \in V \). Then for any \( G' \in [G/w] \), we have

\[
\Lambda_C(G') \leq \Lambda_C(G).
\]

![Fig. 1](image-url) An example of local homomorphism \( v : G \to G' \in [G/w] \), which maps \( B = G[N_G(w)] \simeq C_5 \) to \( B' \simeq K_3 \) according to the 3-coloring of \( C_5 \) shown above, while mapping \( G - V(B) \) isomorphically.
We first derive Theorem 1 from Lemma 2.1, and will prove Lemma 2.1 in the rest of this section.

**Proof of Theorem 1** If $G$ is not a complete graph, then we can choose a vertex $w \in V(G)$ such that for any $G' \in [G/w]$, $|V(G')| < |V(G)|$. Hence we may choose a sequence of graphs $G = G_0, G_1, \ldots, G_k$ such that $G_k$ is a complete graph and for all $0 \leq i < k$ we have $G_{i+1} \in [G_i/w_i]$ for some $w_i \in V(G_i)$. Then the composition of homomorphisms $f : G_0 \to G_1 \to \cdots \to G_k$ is a $|V(G_k)|$-coloring on $G$ so we have $\chi(G) \leq |V(G_k)|$. By Lemma 2.1, we have $\Lambda_C(G_k) \leq \Lambda_C(G)$. Note that

$$\Lambda_C(K_{\chi(G)}) = \left( \frac{\chi(G)}{3} \right) - \left( \frac{\chi(G)}{2} \right) + \left( \frac{\chi(G)}{1} \right)$$

is a non-decreasing function in $\chi(G)$. Hence $\Lambda_C(K_{\chi(G)}) \leq \Lambda_C(G_k) \leq \Lambda_C(G)$, as desired. \hfill \Box

In the rest of this section, we prove Lemma 2.1. We begin by describing how induced cycles behave under local homomorphisms.

**Lemma 2.2** Let $G = (V, E)$ be a connected graph with a vertex $w$. Fix a local homomorphism $v : G \to G' \in [G/w]$. Denote $B = G[N_G(w)]$, $\hat{B} = G[N_G(w) \cup \{w\}]$, $H = G - (N_G(w) \cup \{w\})$, $B' = G'[N_G(w)]$, and $\hat{B}' = G'[N_G(w) \cup \{w\}]$. Then we have the following.

(i) There exists an injection $\phi : C(G') \setminus C(\hat{B}') \to C(G) \setminus C(B)$ such that for all $C \in C(G') \setminus C(\hat{B}')$, $v(V(\phi(C))) \setminus \{w\} = V(C)$.

(ii) $|C(G) \setminus C(B)| - |C(G') \setminus C(\hat{B}')| \geq |E(G) \setminus E(B)| - |E(G') \setminus E(\hat{B}')|$. 

**Proof** We first show (i). Recall that $v$ is an isomorphism between $G - V(B)$ and $G' - V(B')$, so we may identify the two graphs by $v$. Fix an induced cycle $C$ in $G'$ that is not contained in $\hat{B}'$. We shall correspond an induced cycle $\phi(C)$ in $G$ such that the property stated at the end of (i) holds. Observe that $C$ does not use the vertex $w$ and contains at most two vertices in $B'$, since otherwise it would be a triangle contained $\hat{B}'$. If $C$ uses no vertex in $B'$, then it is also an induced cycle in $G$ so we define $\phi(C) = C$.

Suppose $C$ uses one vertex, say $z$, in $B'$. Let $x, y$ be the two neighbors of $z$ in $C$. Then $P := C - z \subseteq H$ is an induced path from $x$ to $y$ both in $G'$ and $G$. If there exists a vertex $v$ in $B$ such that $v(v) = z$ and $vx, vy \in E(G)$, then define $\phi(C) = (V(P) \cup \{v\}, E(P) \cup \{vx, vy\})$ (see Fig. 2a). If not, there exist two vertices $u, v$ in $B$ such that $v(u) = v(v) = z$ and $uv \in E(G)$. In this case we define $\phi(C) = (V(P) \cup \{u, v, w\}, E(P) \cup \{ux, vx, wy, wv\})$. Note that $uv \notin E(G)$ since the homomorphism $v$ identifies $u$ and $v$ into the single vertex $z$. Hence $\phi(C)$ indeed is an induced cycle in $G$ (see Fig. 2b).

Lastly, suppose $C$ uses two vertices, say $z_1$ and $z_2$, in $B'$. Let $x, y$ be the neighbors (not necessarily distinct) of $z_1, z_2$ in $C$ respectively. Then $P := C - \{z_1, z_2\} \subseteq H$ is an induced path from $x$ to $y$ both in $G'$ and $G$. Since $v$ is a homomorphism, there exist vertices $v_1, v_2$ in $B$ such that $v(v_1) = z_1$, $v(v_2) = z_2$, and $v_1x, v_2y \in E(G)$. Now
if $v_1v_2 \in E(G)$ we define $\phi(C) = (V(P) \cup \{v_1, v_2\}, E(P) \cup \{v_1x, v_2y, v_1v_2\})$ (see Fig. 3a) and otherwise define $\phi(C) = (V(P) \cup \{v_1, v_2, w\}, E(P) \cup \{v_1x, v_2y, wv_1, wv_2\})$ (see Fig. 3b). This defines the map $\phi : C(G') \setminus C(B') \to C(G) \setminus C(B)$ asserted in (i). From the construction it is clear that the last part of (i) holds. The injectivity of $\phi$ follows from this. This shows (i).

Next we show (ii). It suffices to show that the number of induced cycles in $C(G) \setminus C(B)$ that are not in the image of $\phi$ is at least the right hand side of the inequality in (ii). Let $R$ the set of “wedges” in $G - w$ that becomes a $K_2$ by $v$, i.e.,

$$R = \{W \mid V(W) = \{x, y, z\}, \{x, z\} \subseteq V(B), y \in V(H), \text{ and } v(x) = v(z)\}$$

Note that for each $W \in R$, $\hat{W}$ is an induced $C_4$ in $G$ whose image under $v$ is a length 2 path in $G'$. By the property of $\phi$, these 4-cycles are not in the image of $\phi$. On the other hand, each such wedge $W \in R$ corresponds to a pair of edges in $E(G) \setminus E(B)$ that are identified by $v$. Thus $|R|$ equals to the right hand side of (ii). This shows the assertion.

The following proposition will be useful in the proof of Theorem 1.
**Proposition 2.3** Suppose Lemma 2.1 holds for every pair \((G, w)\) of connected graph \(G\) and a vertex \(w \in V(G)\) whenever \(|V(G)| \leq n\) for some integer \(n \geq 1\). Then for any graph \(H\) not necessarily connected and \(|V(H)| < n\), we have
\[
|C(H)| \geq |C(K_{\chi(H)})|.
\]

**Proof** We first observe that for any connected graph \(S\) such that \(|V(S)| < n\), we have
\[
|C(S)| \geq |C(S')|
\]
for any \(u \in V(S)\) and for all \(S' \in [S/u]\). Indeed, the assumption implies
\[
|C(S)| - |C(S')| \geq (|E(S)| - |E(S')|) - (|V(S)| - |V(S')|) \geq 0,
\]
where the second inequality follows since the homomorphism \(S \to S' \in [S/u]\) identifies more edges than vertices; for, if two non-adjacent vertices \(x\) and \(y\) in \(N_S(u)\) are identified, then the two edges \(ux\) and \(vy\) are also identified.

Now, let \(H_1, \ldots, H_k\) be the components of \(H\). For each \(H_i\), choose a sequence of graphs \(H_i = H_i(0), H_i(1), \ldots, H_i(r_i) = H_i'\) such that \(H_i'\) is a complete graph and we have \(H_i(j + 1) \in [H_i(j)/w_i(j)]\) for some \(w_i(j) \in V(H_i(j))\) for all \(0 \leq j < r_i\). Since \(|V(H_i)| \leq |V(H)| < n\), one can apply (11) so that
\[
|C(H_i)| \geq |C(H_i')| \quad 1 \leq i \leq k.
\]

Next, embed all complete graphs \(H_i'\) into a maximal one, say \(H'_s\), by some homomorphism. Note that
\[
|C(H)| = \sum_{i=1}^k |C(H_i)| \geq \max_{1 \leq i \leq k} (|C(H_i')|) = |C(H'_s)|.
\]

On the other hand, the composition of all homomorphisms we have used gives a coloring \(H \to H'_s\). By the minimality of \(\chi(H)\), we have \(|V(H'_s)| \geq \chi(H)\). Since \(H'_s\) is a complete graph, this yields \(|C(H'_s)| \geq |C(K_{\chi(H)})|\). Then the assertion follows from (14).

Now we are ready to give a proof of Lemma 2.1.

**Proof of Lemma 2.1** We use induction on \(|V|\). We may assume \(|V| > 1\) since otherwise the assertion is trivial. For the induction step, let \(n \geq 2\) and suppose for any pair \((G, w)\) of connected graph \(G\) and a vertex \(w \in V(G)\) that the assertion holds whenever \(|V(G)| < n\). We use the same notation as in the statement of Lemma 2.2. Denote \(\Delta C := |C(G)| - |C(G')|\), \(\Delta E := |E(G)| - |E(G')|\), and \(\Delta V := |V(G)| - |V(G')|\).

Since \(B' \cong K_{\chi(B)}\) and \(w \notin B\), by the induction hypothesis and Proposition 2.3, we have
\begin{equation}
|C(B)| \geq |C(B')|.
\end{equation}

It is easy to see that \(|C(\hat{B})| = |C(B)| + |E(B)|\) and similarly for \(\hat{B}'\). Hence we have

\begin{equation}
|C(\hat{B})| - |C(\hat{B}')| = |C(B)| - |C(B')| + |E(B)| - |E(B')| \geq |E(B)| - |E(B')|.
\end{equation}

Then note that

\begin{equation}
|E(\hat{B})| - |E(\hat{B}')| = |E(B)| - |E(B')| + |V(B)| - |V(B')| = |E(B)| - |E(B')| + \Delta V.
\end{equation}

Thus by Lemma 2.2 (ii), we have

\begin{equation}
\Delta C = (|C(\hat{B})| - |C(\hat{B}')|) + \left(|C(G) \setminus C(\hat{B})| - |C(G') \setminus C(\hat{B}')|\right) \geq |E(B)| - |E(B')| + |E(G) \setminus E(\hat{B})| - |E(G') \setminus E(\hat{B}')|
\end{equation}

\begin{equation}
= \Delta E - \Delta V,
\end{equation}

as desired. This shows the assertion. \(\square\)

3 Proof of Theorem 2

In this section, we prove Theorem 2. Given a graph \(G\) and an edge \(e = uv \in E(G)\), let \(G/e\) be the graph obtained by identifying vertices \(u\) and \(v\) into a single vertex \(v_e\), where we delete any multiple edges and loops so that the resulting graphs are always simple. This operation \(G \rightarrow G/e\) is called edge contraction. We say a graph \(H\) is a minor of \(G\) if it can be obtained by a sequence of edge contractions from a subgraph \(H' \subseteq G\). We write \(H \leq_m G\) if \(H\) is a minor of \(G\). The Hadwiger number of \(G\), denoted \(h(G)\), is the maximum number \(n\) such that \(K_n \leq_m G\).

Let \(\phi_e : V(G) \rightarrow V(G/e)\) be the vertex map induced by the edge contraction \(G \rightarrow G/e\). Note that this gives a homomorphism \(G - e \rightarrow G/uv\), which identifies the non-adjacent vertices \(u, v\) in \(G - e\) into \(v_e\) and deletes any resulting loops and parallel edges. For any subgraph \(H' \subseteq G\), we denote by \(H/e\) the subgraph of \(G/e\) with vertex set \(\phi_e(V(H))\) and edge set \(\{\phi_e(x)\phi_e(y) \in E(G/e) \mid xy \in E(H)\setminus\{e\}\}\). Note that this agrees with the usual edge contraction when \(e \in E(H)\). If \(u \in V(H)\) and \(v \notin V(H)\), then \(H/e\) equals the graph obtained from \(H\) by renaming \(u\) as \(v_e\). If both \(u\) and \(v\) are not in \(H\), then \(H/e = H\).

Recall that \(\Lambda_F(G)\) denotes the parameter in the right hand side of (3) when \(G\) is 3-connected. The following is a key observation in proving Theorem 2.

**Lemma 3.1** Let \(G = (V, E)\) be a 3-connected graph and let \(e \in E\) be such that \(G/e\) is 3-connected. Then
\[ \Lambda_F(G/e) \leq \Lambda_F(G). \] (23)

We prove Lemma 3.1 later. In order to prove Theorem 2, we need the existence of an edge \( e \in E(G) \) such that \( G/e \) remains 3-connected and \( h(G/e) = h(G) \), provided \( G \) is not a complete graph. This is given by Seymour’s celebrated “Splitter theorem”, which is stated below in a simple form.

**Theorem 3.2** (Seymour [4]) Let \( H \leq mG \) where both \( H \) and \( G \) are 3-connected and \( |V(G)| > 4 \). If \( |V(H)| < |V(G)| \), then there exists an edge \( e \in E(G) \) such that \( G/e \) is 3-connected and contains \( H \) as a minor.

**Proof of Theorem 2** Let \( G = (V, E) \) be a 3-connected graph. We use induction on \(|V|\). Let \( K \) be the largest complete graph that is a minor of \( G \), so \( K = K_{h(G)} \). Recall that \( h(G) \geq 4 \), since \( G \) is 3-connected if and only if there is a sequence of edge contractions mapping \( G \) to \( K_4 \) (see Tutte [6]). Noting that
\[ \Lambda_F(K_{h(G)}) = \left( \frac{h(G)}{3} \right) - \left( \frac{h(G)}{2} \right) + \left( \frac{h(G)}{1} \right), \]
we may assume that \( G \) is not a complete graph. Then \(|V| > |V(K)|\), and also \(|V| > 4\) because \( G \) is 3-connected. For the induction step, we may assume the assertion holds for 3-connected graphs with less than \(|V|\) vertices. By Theorem 3.2, there exists an edge \( e \) in \( G \) such that \( G/e \) is 3-connected and still contains \( K \) as a minor. Thus \( K \leq mG/e \leq mG \), which yields \( h(G/e) = h(G) \). Then by the induction hypothesis and Lemma 3.1, we obtain
\[ \Lambda_F(K) = \Lambda_F(K_{h(G/e)}) \leq \Lambda_F(G/e) \leq \Lambda_F(G), \]
as desired. \( \square \)

Next, we prove Lemma 3.1. To this end, we first need to understand how induced non-separating cycles in a 3-connected graph behave when contracting an edge \( e \) such that the resulting graph is still 3-connected.

Let \( G \) be a graph and fix an edge \( e = uv \in E(G) \). For each \( C \in F(G/e) \) such that \( v_e \in V(C), C - v_e \) is an induced path both in \( G \) and \( G/e \). We denote by \( C * e \) the subgraph of \( G \) with vertex set \( V(C - v_e) \cup \{u,v\} \) and edge set \( E(C - v_e) \cup \{uv\} \cup \{xy \in E(G) \mid x \in \{u,v\}, y \in (NG(u) \cup NG(v)) \cap V(C)\} \). Note that \( C * e \) contains the edge \( e \) and it is the largest subgraph (w.r.t. inclusion) in \( G \) such that \( (C * e)/e = C \). Moreover, observe that \( C * e \) is one of the four types in Fig. 4 up to isomorphism, depending on the degrees of vertices \( u \) and \( v \) in \( C * e \).

Let \( x, y \in V(C - v_e) \) such that \( ux, vy \in E(G) \) (see Fig. 4). Define the following subgraphs of \( G \):
\[ C_{uv} := (V(C - v_e) \cup \{u,v\}, E(C - v_e) \cup \{ux, vy, uv\}) \] (25)
Contradiction, that both $G$ is nonempty since $G/\mathcal{C}_0 f$ is separating. So we may assume $G$ in $\mathcal{C}_0$. For (ii), suppose $V_{\mathcal{C}u} \subseteq \mathcal{C}_0 u \subseteq \mathcal{C}_0 e$. Then the following holds.

(i) If $\circ_{C+C}(u) = \circ_{C+C}(v) = 2$, then $C_{uv} \subseteq F(G)$.

(ii) If $\circ_{G+C}(u) = 3$ and $\circ_{C+C}(v) < 3$, then $C_{uv} \subseteq F(G)$.

(iii) If $\circ_{C+C}(u) = \circ_{C+C}(v) = 3$, then either $C_{uv}$ or $C_v$ is in $F(G)$.

Proof For (i), suppose $\circ_{C+C}(u) = \circ_{C+C}(v) = 2$ (see Fig. 4a). Then $C * e = C_{uv} \subseteq C(G)$ and $G - V(C * e) = (G/e) - V(C)$ is connected since $C \subseteq F(G/e)$. Hence $C_{uv} \subseteq F(G)$. For (ii), suppose $\circ_{C+C}(u) = 3$ and $\circ_{C+C}(v) < 3$ (see Fig. 4b, c). Then $C_{uv} \subseteq C(G)$. Since $G - V(C * e) = G/e - V(C)$ is connected and $V(C * e) = V(C_{uv}) \cup \{v\}$, $G - V(C_{uv})$ is connected if $v$ is adjacent to some vertex in $G - V(C * e)$. If not, since $C * e$ is an induced subgraph in $G$, it follows that $\circ_{G}(v) < 3$. But this contradicts 3-connectivity of $G$. Hence $C_{uv} \subseteq F(G)$.

For (iii), note that both $C_u$ and $C_v$ are induced cycles in $G$. Suppose, for a contradiction, that both $C_1$ and $C_2$ are separating. Let $B = G - V(C * e) \subseteq G$, which is nonempty since $G \neq K_4$. If $u$ is adjacent to $V(B)$ in $G$, then $C_2$ is non-separating. So we may assume $u$ is not adjacent to $V(B)$. Similarly, we may assume $v$ is not adjacent to $V(B)$. Let $L = \{u, v\} \subseteq G$ and let $x, y$ be the two neighbors of $v_e$ in $C$, which are also the endpoints of the induced path $C - v_e$ in $G$. Then $G - \{x, y\}$ is not connected, where the nonempty subgraphs $L$ and $B$ in $G$ belong to different components. But this contradicts the 3-connectivity of $G$. This shows (iii).

Based on the previous proposition, it is now straightforward to construct an injection $F(G/e) \rightarrow F(G)$ when both $G$ and $G/e$ are 3-connected.

Proposition 3.4 Let $G = (V, E)$ and $e = uv \in E$ be such that both $G$ and $G/e$ are 3-connected. Then there exists an injection $\psi : F(G/e) \rightarrow F(G)$ such that $\psi(C)/e = C$ for all $C \subseteq F(G/e)$.

Proof Let $C \subseteq F(G/e)$. If $C$ does not use $v_e$, then $C \subseteq G$ and $(G - V(C))/e = G/e - V(C)$ so $C \subseteq F(G)$. Hence we define $\psi(C) = C$. Otherwise, we let $\psi(C)$ be one of $C_u$, $C_v$, or $C_{uv}$ (defined above Proposition 3.3), according to Proposition 3.3.
so that \( \psi(C) \in F(G) \). If both \( C_u \) and \( C_v \) are in \( F(G) \) in the case of Proposition 3.3 (iii), then define \( \psi(C) \) to be one of them arbitrarily. In all cases, it is easy to see that \( \psi(C)/e = C \). This shows the assertion. □

Now we are ready to prove Lemma 3.1.

**Proof of Lemma 3.1** Let \( G = (V, E) \) be a 3-connected graph and let \( e \in E \) be such that \( G/e \) is 3-connected. Let \( T_e(G) \) be the set of all triangles in \( G \) using the edge \( e \). It is easy to observe that

\[
|E(G)| - |E(G/e)| = |T_e(G)| + 1. \tag{28}
\]

On the other hand, note that every triangle in \( T_e(G) \) is non-separating since \( G/e \) is 3-connected. It follows that the image \( \im(\psi) \) of \( \psi \) is disjoint from \( T_e(G) \). Indeed, if there exists \( C \in F(G/e) \) such that \( \psi(C) = T \) for some \( T \in T_e(G) \), then \( C = \psi(C)/e = T/e \cong K_2 \), a contradiction. This gives us

\[
|F(G)| - |\im(\psi)| \geq |T_e(G)|. \tag{29}
\]

Then we have

\[
\Lambda_F(G) - \Lambda_F(G/e) = (|F(G)| - |F(G/e)|) - (|E(G)| - |E(G/e)|) + 1 \tag{30}
\]

\[
= (|F(G)| - |\im(\psi)|) - (|T_e(G)| + 1) + 1 \geq 0, \tag{31}
\]

as desired. This shows the assertion. □

We conclude this section by giving a short proof of Tutte’s cycle space theorem, which is stated as Theorem 3.5. We will use the injection \( \psi : F(G/e) \to F(G) \) that we constructed in proving Theorem 2. Given a simple graph \( G \), its **edge space**, denoted \( E(G) \), is a free \( \mathbb{Z}_2 \)-module with basis \( \{1(e) : e \in E(G)\} \), where \( 1(e) \) is the indicator function of the edge \( e \). For any subgraph \( H \subseteq G \), denote \( 1(H) := \sum_{e \in E(H)} 1(e) \). The submodule of \( E(G) \) generated by the elements in \( \{1(C) : C \subseteq G \text{ is a cycle}\} \) is called the **cycle space** of \( G \), which we denote by \( \mathcal{C}(G) \). Denote by \( \mathcal{F}(G) \) the submodule of \( \mathcal{C}(G) \) generated by the elements in \( \{1(F) : F \in F(G)\} \). Below we show that, when \( G \) is 3-connected, \( \mathcal{F}(G) \) in fact spans the entire cycle space.

**Theorem 3.5** (Tutte [7]) For any 3-connected graph \( G \), we have \( \mathcal{C}(G) = \mathcal{F}(G) \).

**Proof** In this proof, all summations are taken modulo 2. We may assume \( G \) is not a complete graph and \( |V| \geq 5 \) since otherwise the assertion is clear. Suppose, for a contradiction, that there exists a counterexample \( G = (V, E) \) with \( |V(G)| \) minimal. Let \( e = uv \) be an edge in \( G \) such that \( G/e \) is 3-connected (the existence of such “contractible” edge follows from Theorem 3.2, but a short proof of which is given in [1, Lemma 3.2.1]). Then \( \mathcal{C}(G/e) = \mathcal{F}(G/e) \) and \( T_e(G) \subseteq F(G) \), where \( T_e(G) \) denotes the set of triangles in \( G \) using the edge \( e \).

Note that for any subgraph \( H \subseteq G \), \( 1(H) \in \mathcal{C}(G) \) if and only if \( H \) is the union of some edge-disjoint cycles in \( G \). Denote by \( W_e(H) \) the set of all vertices \( w \in V(H) \) such that \( uw, vw \in E(H) \) if \( u, v \in V(H) \), and \( \emptyset \) otherwise. For any distinct vertices
Choose any $x \subseteq G$ such that $1(x) \in \mathcal{C}(G) \setminus \mathcal{F}(G)$. Let $\tilde{x}$ be any subgraph in $G$ such that

$$1(\tilde{x}) = 1(x) + \sum_{w \in W_e(x)} 1(T_{uvw}).$$

(32)

Then observe that $W_e(\tilde{x}) = \emptyset$ and $1(\tilde{x}) \in \mathcal{C}(G) \setminus \mathcal{F}(G)$. Clearly $W_e(x) = \emptyset$ implies $W_e(\tilde{x}) = \emptyset$. If $W_e(x) \neq \emptyset$, then for each $w \in W_e(x)$, adding $1(T_{uvw})$ to $1(x)$ cancels out the indicators $1(uw)$ and $1(vw)$ from $1(x)$ and adds $1(\tilde{x})$. This shows $W_e(\tilde{x}) = \emptyset$. Furthermore, recall that $T_e(G) \subseteq F(G)$. This implies $1(\tilde{x}) \in \mathcal{C}(G) \setminus \mathcal{F}(G)$ since otherwise

$$1(x) = \left(1(x) + \sum_{w \in W_e(x)} 1(T_{uvw}) \right) + \sum_{w \in W_e(x)} 1(T_{uvw}) \in \mathcal{F}(G),$$

(33)

which is a contradiction.

Next, rename $\tilde{x}$ as $x$ so that $1(x) \in \mathcal{C}(G) \setminus \mathcal{F}(G)$ and $W_e(x) = \emptyset$. Note that for any $H \subseteq G$ with $1(H) \in \mathcal{C}(G)$, $1(H/e) \in \mathcal{C}(G/e)$ if and only if $W_e(H) = \emptyset$. Hence $1(x/e) \in \mathcal{C}(G/e) = \mathcal{F}(G/e)$, so there exist $F_1, \ldots, F_n \in F(G/e)$ such that $1(x/e) = \sum_{i=1}^n 1(F_i)$. We may choose such $x \subseteq G$ so that $n \geq 1$ is as small as possible.

Let $\psi : F(G/e) \to F(G)$ be the injection asserted in the statement of Proposition 3.4. We claim that there exists a (possibly empty) subset $T_0 \subseteq T_e(G)$ and $x' \subseteq G$ such that

$$1(x) + 1(\psi(F_n)) + \sum_{T \in T_0} 1(T) = 1(x')$$

(34)

and $W_e(x') = \emptyset$. To see the assertion follows from the claim, note that (34) yields $1(x') \in \mathcal{C}(G)$. Moreover, since $W_e(x') = \emptyset$ it holds that

$$1(x'/e) = 1(x/e) + 1(\psi(F_n)/e) = \left(\sum_{i=1}^n 1(F_i)\right) + 1(F_n) = \sum_{i=1}^{n-1} 1(F_i).$$

(35)

Then the minimality of $n$ implies $1(x') \in \mathcal{F}(G)$. Since $T_e(G) \subseteq F(G)$, all indicators in (34) except $1(x)$ belong to $\mathcal{F}(G)$. This implies $1(x) \in \mathcal{F}(G)$, a contradiction.

To show the claim, first let $\beta \subseteq G$ be such that $1(\beta) = 1(x) + 1(\psi(F_n))$ with $|V(\beta)|$ minimal. If $W_e(\beta) = \emptyset$, then the claim holds for $T_0 = \emptyset$. Otherwise, suppose there exists a vertex $w \in W_e(\beta)$, so $uw, vw \in E(\beta)$. Since $W_e(x) = \emptyset$, either $uw$ or $vw$ is in $E(\psi(F_n))$. So $v_e \in V(F_n)$. Let $x, y \in V(F_n)$ be the two neighbors of $v_e$ in $F_n \subseteq G/e$. Then $x, y \in V(G)$, and $w$ must be either $x$ or $y$ since $F_n$ is induced in $G/e$. This shows $W_e(\beta) \subseteq \{x, y\}$. Without loss of generality, we assume $w \in W_e(\beta)$. If $W_e(\beta) = \{x\}$, then the claim holds with $T_0 = \{T_{uxx}\}$. Otherwise, $W_e(\beta) = \{x, y\}$ and the claim holds with $T_0 = \{T_{uxx}, T_{uyv}\}$. This shows the assertion. 

\(\square\)
4 Concluding Remarks

For a 3-connected planar graph $G$, the parameter $\Lambda_F(G)$ agrees with the Euler characteristic, so $\Lambda_F(G) = 2$. So if Theorem 1 could be shown for the parameter $\Lambda_F$ instead of $\Lambda_C$, this would imply $\chi(G) \leq 4$ for all 3-connected planar $G$. This is equivalent to the famous four-color theorem. Proving such a result would require a “chain theorem” for homomorphisms, playing a similar role of the Splitter theorem for edge contractions. Indeed, Kriesell [3] showed that 3-connected non-complete graphs always have a contractible nonedge, i.e., a pair of non-adjacent vertices $u, v$ such that identifying them yields a 3-connected graph. However, it seems that cycles behave less nicely under nonedge contractions than under edge contractions, as identifying two nonadjacent vertices could make a non-separating induced path into a non-separating induced cycle.

In Lemma 3.1 and Theorem 2, we have shown that if $K$ is a complete graph minor of a 3-connected graph $G$, then $\Lambda_F(K) \leq \Lambda_F(G)$. Does this hold for any 3-connected minor $H \leq mG$? In fact, Lemma 3.1 and a similar argument in the proof of Theorem 2 shows that $\Lambda_F(H) \leq \Lambda_F(G)$ whenever $H$ is a 3-connected minor of $G$ that can be obtained by a sequence of edge contractions. For general minors involving edge deletions, one would naturally try to show a monotonicity of $\Lambda_F$ under edge deletions, which is similar to Lemma 3.1. Namely, this reads that for every 3-connected non-complete graph $G$, there exists an edge $e$ such that $G - e$ is 3-connected and

$$\Lambda_F(G - e) \leq \Lambda_F(G).$$

(36)

Note that (36) is equivalent to $|F(G - e)| \leq |F(G)| - 1$.

Indeed, Thomassen [5] showed that in a 3-connected graph, there are at least two induced non-separating cycles using any fixed edge $e$, just like the facial cycles of a planar graph do. So at least one element of $F(G)$ is deleted in $G - e$. However, it could be that many non-separating cycles in $G$ have “chord” $e$ so that deletion of which makes all of them induced. More specifically, let $G$ be a graph with a vertex $w$ such that $w$ is adjacent to all vertices in $H$, where $H$ is a 2-connected graph consisting of $n$ internally vertex-disjoint induced paths $P_1, \ldots, P_n$ from a vertex $s$ to a vertex $t$. Suppose that $e = st \in E(G)$ and all of $P_i$’s have length $\geq 2$. Note that every induced cycle in $H$ and $H - e$ is non-separating in $G$ and $G - e$, respectively. Now $H$ has exactly $n$ induced cycles, whereas $H - e$ has $n(n - 1)/2$ of them. Consequently, $|F(G)| = n + |E(H)|$ but $|F(G - e)| = n(n - 1)/2 + |E(H)| - 1$. Therefore $\Lambda_F(G - e) > \Lambda_F(G)$ when $n \geq 4$.

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