Irreducible Elements in Metric Lattices

Andreas Lochmann

May 28, 2010

Abstract

We describe a natural generalization of irreducibility in order lattices with arbitrary metrics. We analyse the special cases of valuation metrics and more general metrics for lattices.

This article is mainly based on a part of the author’s doctoral thesis, but answers some additional questions.

1 Introduction

The theory of valuations and metric lattices has been mainly developed and popularized by John von Neumann and Garrett Birkhoff. In the early years of the 1930s, von Neumann worked on a variation of the ergodic hypothesis, and inadvertently competed with George David Birkhoff. Only some years later, his son Garrett Birkhoff pointed von Neumann at the use of lattice theory in Hilbert spaces. He wrote about this in a note of the Bulletin of the AMS in 1958 [Bi2].

John von Neumann’s brilliant mind blazed over lattice theory like a meteor, during a brief period centering around 1935–1937. With the aim of interesting him in lattices, I had called his attention, in 1933–1934, to the fact that the sublattice generated by three subspaces of Hilbert space (or any other vector space) contained 28 subspaces in general, to the analogy between dimension and measure, and to the characterization of projective geometries as irreducible, finite-dimensional, complemented modular lattices.

As soon as the relevance of lattices to linear manifolds in Hilbert space was pointed out, he began to consider how he could use lattices to classify the factors of operator-algebras. One can get some impression of the initial impact of lattice concepts on his thinking about this classification problem by reading the introduction of [...], in which a systematic lattice-theoretic classification of the different possibilities was initiated. [...]

However, von Neumann was not content with considering lattice theory from the point of view of such applications alone. With his keen sense for axiomatics, he quickly also made a series of fundamental contributions to pure lattice theory.

The modular law in its earliest form (as dimension function) appears in two papers from 1936 by Glivenko and von Neumann ([Gl], [vN]). Von Neumann used it (and lattice theory in general) in his paper to define and
study Continuous Geometry (aka. “pointless geometry”), and later applied his knowledge to found Quantum Logic in his Mathematical Foundations of Quantum Mechanics. A later survey about metric posets is [Mi].

The notions of join-irreducibility and join-primeness are fundamental to Lattice Theory, in the same way as the notion of basis is fundamental to Linear Algebra (see [Bi1]). Hence, it seems plausible to ask for an adaptation of join-irreducibility to metric lattices—the author already used this notion in [Lo2] and [Lo1] to decompose Lipschitz functions and deduce a rigidity theorem about Lipschitz function spaces. The aim of this article is to present this new notion of $d$-irreducibility in Section 2 without reference to Lipschitz function spaces. Section 3 repeats the definition of a valuation on a lattice and its connection to metrics, Subsection 3.3 then deduces a characterization of $d$-irreducible elements in valuation lattices. Subsection 3.2 introduces an alternative definition of valuation, which is then generalized in Sections 4 and 5 to include further metrics on lattices, which often are similarly natural but not based on a valuation. Subsection 5.2 finally deals with the closedness of the subset of all $d$-irreducible elements in a lattice and in which sense they are a dense subset of each base.

1.1 Notation

Given an element $p$ of a lattice $L$, denote with $\downarrow p$ its strictly lower set

$$\downarrow p := \{ f \in L : f < p \}.$$

Furthermore, denote with $\wp(A)$ the power set of $A$.

2 Irreducibility Relative to a Metric

Recall the definition of a join-irreducible element $p$ in a lattice $L$:

$$p = f \lor g \Rightarrow p = f \quad \text{or} \quad p = g \quad \forall f, g \in L$$

Let $L$ be equipped with the discrete metric $d_{dis}$. Then the above property is equivalent to the following:

$$d_{dis}(p, f) \land d_{dis}(p, g) \leq d_{dis}(p, f \lor g) \quad \forall f, g \in L$$

In the same sense, $p$ is completely join-irreducible if and only if

$$\bigwedge_{j \in J} d_{dis}(p, f_j) \leq d_{dis}\left(p, \bigvee_{j \in J} f_j\right) \quad \forall (f_j)_{j \in J} \subseteq L, J \neq \emptyset.$$

**Definition 1**

Let $L$ be a lattice with any metric $d$. We call an element $p \in L$ $d$-irreducible if the following holds for all $f, g \in L$:

$$d(p, f) \land d(p, g) \leq d(p, f \lor g)$$

If $L$ is a complete lattice, we call $p$ completely $d$-irreducible, if the following holds for all $(f_j)_{j \in J} \subseteq L$, with $J$ an arbitrary non-empty index set:

$$\bigwedge_{j \in J} d(p, f_j) \leq d\left(p, \bigvee_{j \in J} f_j\right)$$

Denote the subset of $L$ of all completely $d$-irreducible elements with $\operatorname{cmli}(L)$.
Proposition 2 ........................................... 2
Let $L$ be a lattice with any metric $d$. Then each $d$-irreducible element is join-irreducible. However, not every completely $d$-irreducible element necessarily is completely join-irreducible.

Proof Let $p \in L$ be $d$-irreducible and $p = f \lor g$. Then $d(p, f \lor g) = 0$ and hence either $d(p, f) = 0$ or $d(p, g) = 0$ (or both).

For a counter-example to complete join-irreducibility, let $L = [0, 1]$ with standard metric, supremum and infimum. Take $f_n = 1 - 1/n$, $n \in \mathbb{N}^*$, then $p = 1 = \bigvee f_n$, hence $p$ is not completely join-irreducible.

Still, it is completely $d$-irreducible: Any sequence of real numbers $f_n$ with $p = \bigvee f_n$ must converge to $p$ from below, hence $\bigwedge d(p, f_n) = 0$. □

As a consequence, if $L$ is a complemented lattice, join-irreducibility, complete join-irreducibility, $d$-irreducibility, and complete $d$-irreducibility are all equivalent; the irreducible elements are simply those with trivial strictly lower set.

3 Valuations

Definition 3 ............................................. 3
A valuation on a lattice $L$ is a function $v : L \to \mathbb{R}$ which satisfies the modular law

$$v(f) + v(g) = v(f \land g) + v(f \lor g) \quad \forall f, g \in L.$$  

A valuation $v$ on $L$ is called isotone [positive] if for all $f, g \in L$ the relation $f < g$ implies $v(f) \leq v(g)$ $[v(f) < v(g)]$.

If $L$ is totally ordered, then each function $v : L \to \mathbb{R}$ is a valuation. It is isotone [positive] if and only if $v$ is [strictly] monotonically increasing.

Valuations can be used to define metrics on lattices, as the following Lemma demonstrates. It is a part of Theorem X.1 and a note in subsection X.2 of [Bi1], and is proved there. An alternative proof is given in [Lo1].

Lemma 4 .................................................. 4
Let $v$ be an isotone valuation on the distributive lattice $L$. Then

$$d_v(f, g) := v(f \lor g) - v(f \land g)$$

defines a pseudo-metric with the following properties:

1. If there is a least element $0 \in L$, then

$$v(f) = v(0) + d_v(f, 0) \quad \text{for all} \quad f \in L,$$

2. $d_v$ is a metric if and only if $v$ is positive.

We call $d_v$ a valuation (pseudo-)metric. A lattice together with a valuation metric is sometimes called a metric lattice; however, as we will deal with lattices with non-valuation metrics as well (particularly the supremum metric), we should better distinguish between valuation metric lattices and non-valuation metric lattices.
3.1 Examples

Valuations and valuation metrics arise in a multitude of situations:

Example 5
Let \( L = (\mathbb{N}^*, \text{gcd}, \text{lcm}) \). Then each logarithm is a positive valuation on \( L \). The join-irreducible, completely join-irreducible, \( d \)-irreducible and completely \( d \)-irreducible elements are exactly the prime powers.

Example 6
Let \((X, \Sigma, \mu)\) be a probability space. The \( \sigma \)-algebra \( \Sigma \) is a Boolean lattice by union and intersection. Let \( c \in \mathbb{R} \) be arbitrary, then
\[
v(A) := \mu(A) + c
\]
defines an isotone valuation on \( \Sigma \) with \( v(\emptyset) = c \). The valuation \( v \) is positive if and only if there are no null sets in \( X \) other than \( \emptyset \). The distance function \( d_v(A, B) := v(A \cup B) - v(A \cap B) \) is the measure of the symmetric difference \( A \triangle B \) of \( A \) and \( B \), if \( A \triangle B \in \Sigma \). It relates to the Hausdorff distance just as the 1-distance of functions relates to the supremum distance.

Example 7
Let \( V \) be any finite dimensional vector space, and \( L = \text{PG}(V) \) its lattice of subvector spaces, with \( \wedge \) the intersection and \( \vee \) the span (the projective geometry of \( V \)). Then the dimension function is a positive valuation on \( L \). (This is the similarity between dimension and measurement mentioned before in [Bi2].) The join-irreducible, completely join-irreducible, \( d \)-irreducible and completely \( d \)-irreducible elements are exactly the one-dimensional subspaces and the zero-dimensional one.

Example 8
Let \( X \) be a measure space and \( L \) the space of integrable Lipschitz functions of Lipschitz constant \( \leq 1 \). We may apply the Lebesgue integral to gain an isotone valuation on \( L \); as \( f + g = (f \wedge g) + (f \vee g) \) holds pointwise, we conclude
\[
\int f \, d\mu + \int g \, d\mu = \int (f \wedge g) \, d\mu + \int (f \vee g) \, d\mu.
\]
If \( X \) is a Euclidean space, or a discrete space without non-trivial null sets, this valuation is positive, because any non-trivial non-negative Lipschitz function has positive Lebesgue integral. Positivity fails in cases where \( X \) contains an isolated point or continuum of measure zero.

As \( |f - g| = (f \vee g) - (f \wedge g) \) holds pointwise, the valuation metric \( d_v \) equals the \( L^1 \)-distance defined by
\[
d_v(f, g) := \int |f - g| \, d\mu.
\]
Each function \( \Lambda(x, r) : L \to [0, \infty) \) of the form
\[
\Lambda(x, r)(y) := 0 \vee (r - d(x, y))
\]
with \( x \in X \) and \( r \in [0, \infty) \) is join-irreducible, but not necessarily completely join-irreducible. In general, the only \( d \)-irreducible function is the zero function.
The $L^1$-metric can be slightly modified to yield other valuation metrics: Let $\kappa : [0, \infty) \to [0, \infty)$ be a positive valuation (i.e., strictly monotonically increasing), then

$$v_{\mu, \kappa}(f) := \int \kappa(f(x)) \, d\mu(x)$$

is a positive valuation.

### 3.2 Difference Valuations

A nearly equivalent approach to valuations is to use difference valuations:

**Definition 9**

A difference valuation on a distributive lattice $L$ is a function $w : L \times L \to \mathbb{R}$ which satisfies the cut law

$$w(f, g) = w(f, g \lor h) + w(f \land h, g).$$

A difference valuation $w$ is called isotope if its values are non-negative, and positive, if $w(f, g) = 0$ implies $f \leq g$.

Given a valuation $v$ on a distributive lattice $L$,

$$w(f, g) := v(f) - v(f \land g)$$

defines a difference valuation, as one can easily check. The cut law follows from the modular equality and vice versa—it has been dubbed “cut law” because of its appearance when applied to sets in a Venn diagram, see Figure 1. The difference valuation is isotope/positive if and only if the valuation $v$ is isotope/positive. If $L$ admits a least element 0, each [isotope/positive] difference valuation $w$ in turn defines an [isotope/positive] valuation $v$ by

$$v(f) := w(f, 0) + c$$

for any $c \in \mathbb{R}$, and any valuation of $L$ with difference valuation $w$ is of this form. Finally, the distance function $d$ of a valuation can be equally well expressed as

$$d(f, g) = w(f, g) + w(g, f).$$

If the lattice $L$ is complemented, $w(f, g)$ equals $v(f \setminus g)$. Difference valuations are easier to use in cases where a lattice is not complemented, as they can be used as substitutes for the relative complement operation in calculations with metrics. For example, the proof of Lemma 4 can be seen by a simple application of Venn diagrams (see Figure 2); for details and further examples to deduce metric inequalities in order lattices see [Lo1].

### 3.3 $d$-Irreducible Elements

As a triviality, in the definition of a join-irreducible element,

$$p = f \lor g \Rightarrow p = f \text{ or } p = g \ \forall f, g \in L,$$

the elements $f$ and $g$ may be chosen to be $\in \downarrow p \subseteq L$. This accounts for $d$-irreducible elements as well, but is less trivial.
Lemma 10 ........................................ 10
Let L be a distributive lattice, and d a positive valuation metric on L.
p ∈ L is d-irreducible if and only if
\[
d(p, f) \land d(p, g) \leq d(p, f \lor g)
\]
holds for all f, g ∈ \uparrow p. In this case, “≤” can be replaced by “=”. If L is completely distributive, then the analog holds for complete d-irreducibility as well.

Proof Let f, g ∈ L be arbitrary and p ∈ L as above. Then holds:
\[
d(f \lor g, p) = w(p, f \lor g) + w(f \lor g, p)
\geq w(p, f \lor g) + (w(f, p) + w(g, p))
= d((f \lor g) \land p, p) + (w(f, p) + w(g, p))
\geq (d(f \lor p, p) \land d(g \lor p, p)) + (w(f, p) + w(g, p))
= (w(p, f) \land w(p, g)) + w(f, p) + w(g, p)
\geq (w(p, f) + w(f, p)) \land (w(p, g) + w(g, p))
= d(f, p) \land d(g, p)
\]
For equality, note that \( d(p, f) \geq d(p, f \lor g) \) is obvious because \( f \leq f \lor g \leq p \); same holds for \( g \) and thus

\[
d(p, f) \land d(p, g) \geq d(p, f \lor g).
\]

There is a characterization of join-irreducibility of an element \( p \in L \) in terms of its strictly lower set \( \downarrow p \): \( p \) is join-irreducible if and only if for each \( f, g \in \downarrow p \) holds \( f \lor g \in \downarrow p \), i.e. if and only if \( \downarrow p \) is join-closed. Analogously, \( p \) is a completely join-irreducible element of a complete lattice \( L \) if and only if \( \downarrow p \) is join-complete (i.e. each supremum of elements of \( \downarrow p \) again is contained in \( \downarrow p \)). For valuation metrics, there is a similar characterization of \( d \)-irreducibility:

**Theorem 11**

Let \( L \) be a distributive lattice, and \( d \) a positive valuation metric on \( L \). An element \( p \in L \) is \( d \)-irreducible if and only if the strictly lower set \( \downarrow p \) is totally ordered.

**Proof** “\( \Rightarrow \)”: Let \( f, g \in \downarrow p \) be arbitrary.

\[
d(f \lor g, p) = d(f, p) \land d(g, p)
\]

\[
= w(p, f) \land w(p, g)
\]

\[
= (w(g, f) + w(p, f \lor g)) \land (w(f, g) + w(p, g \lor f))
\]

\[
= (w(g, f) \land w(f, g)) + w(p, g \lor f)
\]

\[
= (w(g, f) \land w(f, g)) + d(f \lor g, p)
\]

and hence \( w(g, f) \land w(f, g) = 0 \), thus one of them is zero, and we have either \( f \leq g \) or \( g \leq f \).

“\( \Leftarrow \)”: Let \( f, g \in \downarrow p \) be arbitrary (see Lemma [11] why we may restrict to \( \downarrow p \)). As \( \downarrow p \) is totally ordered, \( f \lor g \) is \( f \) or \( g \), and hence the condition for \( d \)-irreducibility is trivial.

□

Theorem 11 shows that \( d \)-irreducibility does not depend on the concrete choice of a valuation metric for the lattice \( L \). This result resembles an earlier connection found in Lipschitz function spaces: If \( L \) is the space of bounded non-negative Lipschitz functions of a metric space \( X \) with Lipschitz constant \( \leq 1 \) with pointwise supremum and infimum and supremum metric \( d_\infty \), then the completely \( d_\infty \)-irreducible elements are exactly those functions of the form

\[
\Lambda(x, r) : L \to [0, \infty)
\]

\[
y \mapsto (r - d_X(x, y)) \lor 0
\]

with \( x \in L \) and \( r \in [0, \infty) \) (see Example [8], [10], [11]). Although the supremum metric \( d_\infty \) is not a valuation metric, but an intervaluation metric (see Definition [13]), its completely \( d_\infty \)-irreducible elements are fully determined without any reference to the chosen metric on \( L \). One might even get rid of the metric on \( X \) by referring to minimal functions with a given function value at a single point.
4 Ultravaluations

One advantage of the definition of difference valuations in Subsection 3.2 is the following alternative to valuations in lattices, which adds further examples to our list of metrics on lattices and is easily described in terms of a variant of Definition 9.

Lemma 12

Let $L$ be a distributive lattice, and let $w : L \times L \to [0, \infty)$ be a map which satisfies

1. $w(f, g) = 0$ whenever $f \leq g$, and
2. $w(f, g) = w(f \land h, g) \lor w(f, g \lor h)$ $\forall f, g, h \in L$.

We call $w$ a difference ultravaluation, or just ultravaluation. Define

$$d_w(f, g) := w(f, g) \lor w(g, f).$$

Then $d_w$ is a pseudo-ultrametric. $d_w$ is an ultrametric if and only if $w(f, g) = 0 \Rightarrow f \leq g$ holds.

Proof

To get from difference valuations to ultravaluations, we just replaced all occurrences of “+” by “∨”. As both operations are associative and commutative, we can transfer most proofs of valuations just by replacing “+” by “∨”, this includes the proof of the triangle inequality:

$$d_w(f, g) = w(f, g \lor h) \lor w(f, g \land h) \lor w(g, h \land f) \leq w(f, h).$$

$\Rightarrow d_w(f, g) \leq w(f, g) \lor w(g, h) \lor w(h, f) = d_w(f, h) \lor d_w(h, g)$

On the other hand, contrary to the valuation case, the property $d_w(f, f) = 0$ does not follow from property (2) – we have to conclude it from (1).

Assume $w(f, g) = 0 \Rightarrow f \leq g$ holds. Let $d_w(f, g) = 0$. This implies $w(f, g) = 0$ and $w(g, f) = 0$, and hence $f \leq g$, $g \leq f$, and $f = g$. Now assume $d_w$ is a metric, $f \not\leq g$, and $w(f, g) = 0$. Then

$$w(f, f \land g) = w(f, f \lor g) \lor w(f, g) = 0.$$ 

Due to $f \not\leq g$, we have $f \neq f \land g$, hence

$$0 < d_w(f, f \land g) = w(f, f \land g) \lor w(f \land g, f) = w(f \land g, f).$$

But $f \land g \leq f$, contradiction. □

4.1 Examples

Example 13

Let $X$ be any set, $\kappa : X \to [0, \infty)$ arbitrary and fixed, and $L$ a lattice of subsets of $X$. For $A, B \in L$ consider

$$w(A, B) := 0 \lor \sup_{x \in A \land B} \kappa(x).$$

$w$ defines an ultravaluation.

Choose $\kappa$ to be a positive constant, then the ultrametric resulting from $w$ will be the discrete metric on $L$. 8
Example 14
Let $X$ be any metric space and $\text{Lip}_0 X$ its lattice of bounded Lipschitz functions $\Lambda(x, r)$ of Lipschitz constant $\leq 1$. Besides its Stone representation, we want to provide another, more intuitive representation of the space $\text{Lip}_0 X$ by a lattice of sets, using its hypograph (see "epigraph" in [Ro]).

\[ \text{hyp} : \text{Lip} X \rightarrow \wp (X \times [0, \infty)) \]
\[ f \mapsto \{(x, r) : f(x) \leq r \}. \]

(Im hyp, $\cap$, $\cup$) obviously is isomorphic to $(\text{Lip}_0 X, \wedge, \vee)$ as a lattice; however, they are not yet isomorphic as complete lattices: Infinite unions of the closed sets in Im hyp are not closed in general – we have to use the union with closure "∪" instead of the traditional union. (Alternatively, we could identify subsets of $X \times [0, \infty)$ with the same closure.)

We now apply Example 13. The most canonical $\kappa$ would be $\kappa = \pi_2$, the projection onto $[0, \infty)$. The corresponding ultrametric on $L$ is

\[ d_\kappa(f, g) = 0 \lor \sup \{f(x) \lor g(x) \mid x \in X \text{ such that } f(x) \neq g(x)\}. \]

We shall call this metric the "peak metric" on $\text{Lip} X$.

Another possible choice for $\kappa$ is as follows: Choose a basepoint $x_0 \in X$ and $\kappa(x, r) := d_X(x, x_0)$. Then $d_\kappa$ will describe the greatest distance from $x_0$ at which $f$ and $g$ still differ. Finally, $\kappa(x, r) := \exp(-d_X(x, x_0))$ will describe the least distance from $x_0$ at which $f$ and $g$ differ. We will call the first case the "outer basepoint metric" and the second case the "inner basepoint metric".

An application of the lower basepoint metric is as follows: Given a free group $F$ with neutral element $x_0$, identify each normal subgroup $N \subseteq F$ with its characteristic function on $F$. These are 1-Lipschitz functions in the canonical word metric of $F$. $d_\kappa$ then defines a topology on $\text{Lip} F$, which restricts to the Cayley topology ([MR], V.10) on the subset of normal subgroups.

The $\Lambda$-functions defined in Example 8 are exactly the $d$-irreducible functions of the peak metric. The $d$-irreducible functions of the outer basepoint metric are those functions $\Lambda(x, r)$ with $x \neq x_0$, the inner basepoint metric doesn't admit any non-trivial $d$-irreducible function in general. Finally, none of these three metrics admits a non-trivial completely $d$-irreducible function.

Lemma 15
Let $X$ be finite, and let $L$ be a lattice of subsets of $X$. Then any ultraposition on $L$ is of the form of Example 13.

Proof For $x \in X$ and $A, B \subseteq X$ define

\[ \kappa(x) := \inf \{w(C, D) : C, D \in L \text{ with } x \in C, x \notin D\} \]
and

\[ w'(A, B) := 0 \lor \sup_{y \in A \setminus B} \kappa(y). \]

Assume $w'(A, B) > w(A, B)$. Then there is $y \in A \setminus B$ with $\kappa(y) \geq w(A, B)$, but this cannot happen, as one may choose $C = A$ and $D = B$. Hence, assume $w'(A, B) < w(A, B)$. Then for all $y \in A \setminus B$ there should be $C, D \in L$ with $y \in C \setminus D$ and $w(C, D) < w(A, B)$. As

\[ w(C, D) \geq w(C \wedge A, D \vee B), \]
we might choose without loss of generality \( C \subseteq A \) and \( D \supseteq B \), as choosing \( C \cap A \) instead of \( C \) and \( D \cup B \) instead of \( D \) further decreases \( w(C, D) \).

The cut law now yields

\[
w(A, B) = w(C \land D, B) \lor w(C, D) \lor w(A \lor D, B \lor C) \lor w(A, C \lor D).
\]

As \( w(C, D) < w(A, B) \) by assumption, we find that at least one of \( (C \cap D) \setminus B \), \( (A \cup D) \setminus (B \cup C) \), and \( A \setminus (C \cup D) \) must be non-empty. Choose \( y' \) out of their union and repeat the above argument for the now smaller subset. We get an infinite sequence of different elements from \( X \), which is a contradiction because \( X \) is finite. \( \Box \)

**Example 16**

Not all ultravaluations are of the kind of Example 13. Let \( X \) be any metric space and \( L \) the lattice of subsets of \( X \). Define \( w(A, B) \) to be the Hausdorff dimension of \( A \setminus B \in L \) plus 1, and 0 if \( A \setminus B = \emptyset \). Then \( w \) is an ultravaluation and \( d_w \) an ultrametric.

The \( d \)-irreducible subsets and the completely \( d \)-irreducible subsets are exactly the join-irreducible subsets, namely those with one or zero elements, because \( L \) is complemented.

Comparing Examples 7 and 16, one should note that the join operation in the former is the span, but in the latter is the union. Thus, the first example gives rise to a valuation, the second one to an ultravaluation.

### 4.2 \( d \)-Irreducible Elements

Lemma 10 can be easily adapted to the case of ultravaluations by replacing all remaining “+” by “\( \lor \)”. Indeed, Lemma 10 holds in an even broader generalization, what we will demonstrate in Lemma 20.

When following the proof of Theorem 11 for ultravaluation metrics (remember that join-irreducibility is \( d_{\text{dis}} \)-irreducibility for the discrete metric \( d_{\text{dis}} \), which is an ultravaluation metric), one ends up with the following inequality:

\[
d(f, g) \leq d(p, f \lor g)
\]

for all \( d \)-irreducible elements \( p \) and all \( f, g \in \downarrow p \). If \( L \) contains a least element \( 0 \in L \), we conclude as special case

\[
d(0, g) \leq d(g, p) \quad \forall g \in \downarrow p.
\]

One would hope that there is a similar characterization of \( d \)-irreducible elements in the ultravaluation case as it is in the valuation case. Starting from the case of the discrete metric, one would ask whether join-irreducibility is exactly this characterization, i.e. whether all join-irreducible elements are \( d \)-irreducible for any ultravaluation metric \( d \). This, however, is wrong.

**Example 17**

We refer to Example 13. Let \( X = \{1, 2, 3\} \subseteq \mathbb{Z} \) and let \( \kappa \) be the identity. Let \( L \) be the lattice \( \{\emptyset, \{2\}, \{3\}, \{2, 3\}, X\} \) of subsets of \( X \). Then \( X \in L \) is join-irreducible (because it is the only set containing 1), but not \( d \)-irreducible: \( d(X, \{2\}) = 3, d(X, \{3\}) = 2 \) and \( d(X, \{2, 3\}) = 1 \). In particular, this example shows that \( d \)-irreducibility depends on the concrete choice of \( \kappa \), respectively on the choice of the ultravaluation.
Figure 3: We refer to Example 13. Let $L$ be the lattice of sets spanned by the shown sets of natural numbers and let $\kappa$ be the identity. A set $A$ is not $d$-irreducible, if and only if there are subsets $B$ and $C \in L$ of $A$, such that both $B$ and $C$ contain at least one number each, which is larger than any of the remaining numbers in $A \setminus (B \cup C)$. Which of the shown subsets are $d$-irreducible?

**Question** Is there a nice criterion to decide whether all join-irreducible elements in an ultravaluation metric lattice are $d$-irreducible?

Lemma 15 characterizes all finite ultravaluation lattices. However, finding the $d$-irreducible subsets in a finite ultravaluation lattice can still be non-trivial. We demonstrate this by restating the problem as a puzzle in Figure 3 and leave it to the reader to find any patterns.

5 Intervaluations and Topological Aspects

We now present a generalized notion of valuation which includes normal valuations and ultravaluations. In addition, this notion of intervaluations also includes the supremum metric of function spaces, just as the $L^1$-metric was found to be a valuation in Example 8.

Similar to the case of the ultravaluation, we first recognize the possibility to replace “+” in the definition of a difference valuation by any commutative and associative binary operation. But this alone will not suffice to encompass the supremum metric, we have to weaken the main property of a difference evaluation as well:

**Definition 18**

An intervaluation on a distributive lattice $(L, \land, \lor)$ is a map $w : L \to [0, \infty)$ together with a commutative and associative binary operation $\circ_w : [0, \infty) \times [0, \infty) \to [0, \infty)$, such that the following properties hold:

1. $\circ_w 0 = 0 \quad \circ_w r = r$
2. $r \circ_w t \leq (r + s) \circ_w (t + u) \leq (r \circ_w t) + (s \circ_w u)$
3. $r \lor s \leq r \circ_w s$ (follows from (1) and (2))
4. $f \leq g \Rightarrow w(f, g) = 0$
Proposition 19

An intervaluation \( w \) is positive if

\[
\forall f, g \in L \quad w(f, g) = 0 \Rightarrow f \leq g.
\]

**Proof** (1) We choose \( h = f \) or \( h = g \) in both modular inequalities:

\[
\begin{align*}
0 \circ_w w(\cdot, \cdot) \leq w(\cdot, \cdot) & \leq 0 + w(\cdot, \cdot) \\
w(f, g) \circ_w 0 & \leq w(f, g) & \leq w(f, g) + 0 \\
\text{and} \quad d_w(f \lor g, g) & = w(f \lor g, g) \circ_w 0 = w(f, g).
\end{align*}
\]

(2) From the definition we see \( d_w(f, g) \geq 0 \) and \( d_w(f, f) = 0 \) for all \( f, g \in L \). As \( \circ_w \) is commutative, \( d_w \) is symmetric.

\[
d_w(f, g) = w(f, g) \circ_w w(g, f)
\]

\[
\leq (w(f \land h, g) + w(f, g \lor h)) \circ_w (w(g \land h, f) + w(g, f \lor h))
\]

\[
\leq (w(h, g) + w(f, h)) \circ_w (w(h, f) + w(g, h))
\]

\[
= (w(f, h) + w(h, g)) \circ_w (w(h, f) + w(g, h))
\]

\[
\leq (w(f, h) \circ_w w(h, f)) + (w(h, g) \circ_w w(g, h))
\]

\[
= d_w(f, h) + d_w(h, g)
\]

(3, “\( \Rightarrow \)”) Assume \( 0 = w(f, g) = w(f, f \land g) \). Then \( d_w(f, f \land g) = 0 + 0 = 0 \). As \( d_w \) is a metric, we have \( f = f \land g \), so \( f \leq g \).

(3, “\( \Leftarrow \)”) \( d_w(f, g) = 0 \) implies \( w(f, g) = 0 \) and \( w(g, f) = 0 \), hence \( f \leq g \leq f \), and \( f = g \). \( \Box \)

We now show the generalization of Lemma 10 for intervaluations, which we already announced in subsection 12.

Lemma 20

Let \( L \) be a distributive lattice, and \( d \) a positive intervaluation metric on \( L \). \( p \in L \) is \( d \)-irreducible if and only if

\[
d(p, f) \land d(p, g) \leq d(p, f \lor g)
\]

holds for all \( f, g \in \downarrow p \). In this case, “\( \leq \)” can be replaced by “\( \Leftarrow \)”.

If \( L \) is completely distributive, then the analog holds for complete \( d \)-irreducibility as well.
Proof. Let \( f, g \in L \) be arbitrary and \( p \in L \) as above. Then holds:

\[
d(f \lor g, p) = w(p, f \lor g) \circ w w(f \lor g, p) \\
\geq w(p, f \lor g) \circ w \left( w(f, p) \circ w w(g, p) \right) \\
= d((f \lor g) \land p, p) \circ w \left( w(f, p) \circ w w(g, p) \right) \\
\geq \left( d(f \lor p, p) \lor d(g \lor p, p) \right) \circ w \left( w(f, p) \circ w w(g, p) \right) \\
= \left( w(p, f) \land w(p, g) \right) \circ w \left( w(f, p) \circ w w(g, p) \right) \\
\geq \left( w(p, f) \circ w w(f, p) \land \left( w(p, g) \circ w w(g, p) \right) \right) \\
= d(f, p) \land d(g, p)
\]

(1: definition, 2: by left modular inequality, 3: definition, 4: hypothesis, 5: definition, 6: by cases and monotony of \( \circ w \) (property (2) in Definition 15), 7: definition). Each step holds in the infinite case as well. \( \square \)

5.1 Examples

Example 21

There are several possible choices for the commutative and associative binary operation \( \circ w \) in Definition 15. Choosing addition leads directly to the definition of valuations. The next important choice is the maximum operation: Properties (1) and (3) are obviously fulfilled, the left side of (2) as well. (2.right) needs some short consideration: As \( + \) distributes over \( \lor \), the right-hand side equals

\[
(r \lor t) + (s \lor u) = (r + s) \lor (r + u) \lor (t + s) \lor (t + u)
\]

which is greater or equal \((r + s) \lor (t + u)\) for all \( r, s, t, u \in [0, \infty) \).

Each norm \( \| \cdot \| \) on \( \mathbb{R}^2 \) with certain normalization properties qualifies as an operation \( \circ w \) via \( r \circ w s := \| (r, s) \| \). This accounts for the \( \ell^p \)-norms:

\[
r \circ_p s := \| (r, s) \|_p := \sqrt[p]{r^p + s^p}
\]

for \( p \in [1, \infty) \). Again, properties (1), (2.left) and (3) of Definition 15 are trivial. Property (2.right) is the triangle inequality of the \( \ell^p \)-norms (i.e. a special case of the Minkowski inequality \( WT \)).

Given any metric \( d \) on \( L \) we may define \( w_d(f, g) := d(f \lor g, g) \) and deduce \( \circ_w \) from \( d(f, g) = w_d(f, g) \circ w w_d(g, f) \). The operation \( \circ_w \) must be commutative due to the symmetry of \( d_w \). From the remaining properties of Definition 15 property (4) follows directly from \( d(g, g) = 0 \), while the rest is less obvious.

Example 22

The standard metric on \([0, \infty)\) is an intervaluation metric with

\[
w(r, s) := 0 \lor (r - s).
\]

However, one may freely choose \( \circ_w \) to be addition or maximum. To prove the cut law for both choices, it suffices to show

\[
0 \lor (r - s) = \left( 0 \lor (r - (s \lor t)) \right) + \left( 0 \lor ((r \land t) - s) \right).
\]

For this, we make use of \( a + b = (a \land b) + (a \lor b) \) with \( a = r \land s \) and \( b = r \lor t \), then add \( r \) to both sides, rearrange and apply \( x - (x \land y) = 0 \lor (x - y) \).
Example 23

Let \((X, \mu)\) be a measure space, \(p \in (1, \infty)\) arbitrary, and \(L\) the lattice of \(L^p\)-integrable non-negative Lipschitz functions of Lipschitz constant \(\leq 1\). Define

\[
\max (r, s) := (r^p + s^p)^{1/p}, \\
\min (f, g) := \sqrt[p]{\int |f - (f \wedge g)|^p \, d\mu}.
\]

As \(|r - (r \wedge s)|^p + |s - (r \wedge s)|^p = |r - s|^p\) for all \(r, s \in [0, \infty)\), the corresponding \((pseudo-)metric\) is just the \(L^p\)-metric

\[
d_p(f, g) = \sqrt[p]{\int |f - g|^p \, d\mu}.
\]

Properties (1)-(3) of Definition 18 follow from Example 21, (4) is trivial. The left cut law can be shown by pointwise analysis and case distinction \((h \leq g \text{ vs. } h > g)\), the right cut law follows from Example 22 and the Minkowski inequality. \(d_p\) might be a pseudo-metric, depending on \(\mu\).

Example 24

Here is a minimal example for a non-intervaluation metric: Take \(L = \{a, b, c\}\) with \(a < b < c\), and \(d(a, c) = 1\), \(d(a, b) = 2\), \(d(b, c) = 3\). Then \(w(c, a) = 1\), although \(w(c \wedge b, a) = 2\) and \(w(c, a \lor b) = 3\), which both contradict the cut law and Proposition 19.1, no matter what \(\circ w\) is.

Example 25

The Lipschitz constant provides a much more interesting example for a non-intervalulation metric. Let \(X\) be an arbitrary true metric space, and \(L\) a complete lattice of functions \(f : X \to \mathbb{R}\) with bounded Lipschitz constant. The Lipschitz constant of a function \(f \in L\) and the corresponding pseudo-metric are given by

\[
\text{LC}(f) := \sup_{x, y \in X} \frac{|f(x) - f(y)|}{d(x, y)}, \\
\text{d}_{\text{LC}}(f, g) := \text{LC}(f - g).
\]

They are used by \(Wv\) as ingredient to the utilized norm, called Lipschitz norm, which is defined as \(\|f\|_L := \|f\|_\infty \vee \text{LC}(f)\). However, neither defines an intervaluation: Although Weaver shows in his Proposition 1.5.5 that \(\text{LC}\) fulfills a modular inequality for ultravaluations

\[
\text{LC}(f \lor g) \lor \text{LC}(f \land g) \leq \text{LC}(f) \lor \text{LC}(g)
\]

the inverse inequality is wrong, as there is no bound to \(\text{LC}(f)\) by any combination of \(\text{LC}(f \land g)\) and \(\text{LC}(f \lor g)\). To see this, consider the two-point-space \(X = \{a, b\}\) of diameter \(l < 1\), and the Lipschitz-functions \(f = (0, l)\) and \(g = (l, 0)\). Then \(\text{LC}(f) = \|f\|_L = 1\), but \(\text{LC}(f \wedge g) = \text{LC}(f \vee g) = 0\) and \(\|\cdot\|_L = l\) in both cases.

Correspondingly, the cut law is explicitly violated by \(\text{d}_{\text{LC}}\), as one can see when \(f\) and \(g\) are two different constant functions, and \(h\) crosses them both.
We now concentrate on the special case of the supremum metric.

**Proposition 26**

Let $Z$ be a distributive lattice with intervaluation metric $d$ (with corresponding $w_d$ and $\circ_d$), with $r \circ_d s = r \lor s$ for all $r, s \in [0, \infty)$. Let $X$ be an arbitrary space, and $L$ a complete lattice of functions $f : X \to Z$ with pointwise infima and suprema. If

$$w_\infty(f, g) := \bigvee_{x \in X} w_d(f(x), g(x))$$

is bounded, it defines an intervaluation metric on $L$ with $r \circ_\infty s = r \lor s$ for all $r, s \in [0, \infty)$, which equals the supremum metric $d_\infty$.

**Proof**
The left inequality of the cut law is trivial. For the right side we have to use that a supremum of sums is less than or equal to a sum of suprema, which in turn follows from complete distributivity:

$$\bigvee_{x \in X} w_d(f x, g x) \leq \bigvee_{x \in X} w_d(f x, (g \lor h)(x)) + w_d((f \land h)(x), g x)$$

$$\leq \bigvee_{x \in X} w_d(f x, (g \lor h)(x)) + \bigvee_{x \in X} w_d((f \land h)(x), g x)$$

□

**Corollary 27**

Let $X$ be any metric space. The supremum metric $d_\infty$ is an intervaluation metric on the space $\text{Lip}_0 X$ of bounded, non-negative Lipschitz functions on $X$ with Lipschitz-constant $\leq 1$.

**Proof**

Lip $X$ is a complete lattice, as one can easily check. We find $r \circ_{d_\infty} s = r \lor s$ and

$$w_{d_\infty}(f, g) = \bigvee_{x \in X} |f(x) - (f \land g)(x)| = 0 \lor \bigvee_{x \in X} (f(x) - g(x)),$$

which is the intervaluation metric of Proposition 26 applied to Example 22.

□

### 5.2 Topological Aspects

We finally take a look at the subset $\text{cmli}(L)$ of all completely $d$-irreducible elements of a complete lattice $L$ with intervaluation metric $d$.

**Proposition 28**

Let $L$ be a complete lattice with intervaluation metric $d$, and let $L$ be metrically complete. Then $\text{cmli}(L)$ is topologically closed.

**Proof**

Let $(p_n) \subseteq \text{cmli}(L)$, $n \in \mathbb{N}^*$ be some sequence of completely $d$-irreducible elements converging to $p \in L$, and $(f_j)_j \in J$ any non-empty family
in $L$. Then for any $n \in \mathbb{N}^*$ holds
\[
\begin{align*}
d \left( p, \bigvee_{j} f_j \right) & \geq d \left( p_n, \bigvee_{j} f_j \right) - d(p, p_n) \\
& \geq \bigwedge d(p_n, f_j) - d(p, p_n) \\
& \geq \bigwedge (d(p, f_j) - d(p, p_n)) - d(p, p_n) \\
& \geq \bigwedge d(p, f_j) - 2d(p, p_n) \to 0
\end{align*}
\]
i.e. the element $p$ is completely $d$-irreducible.

Definition 29 ...................................................... 29
Let $L$ be a lattice with metric $d$, $R \geq 0$ arbitrary. We define an $R$-base of $L$ to be a subset $B \subseteq L$ such that for any $f \in L$ there is $(b_j)_{j \in J} \subseteq B$, $J$ an arbitrary non-empty index set, such that $d(f, \bigvee_{j \in J} b_j) \leq R$. A base simply is a 0-base.

Proposition 30 ...................................................... 30
Consider an $R$-base $B$ of a complete lattice $L$ with intervaluation metric $d$, $R \geq 0$. Then for each $\delta > 0$, $\text{cmli}(L)$ is in the $(R + \delta)$-ball around $B$. In particular, if $R = 0$, $\text{cmli}(L)$ lies in the metrical closure of $B$.

Proof Let $p \in \text{cmli}(L)$ be arbitrary. As $B$ is an $R$-base, there are $b_j \in B$, $j \in J \neq \emptyset$, such that
\[
\begin{align*}
d \left( p, \bigvee_{j \in J} b_j \right) & \leq R.
\end{align*}
\]
From Definition 4 we infer that there is a sequence $(c_k) \subseteq B$, $k \in K \subseteq J$ whose distances to $p$ converge to $R$. If $R = 0$, the sequence $(c_j)$ metrically converges to $p$. □

Propositions 28 and 30 might help in identifying all completely $d$-irreducible elements of a concretely given lattice.

Example 31 ...................................................... 31
It is easy to see that, if $B$ is a base, and $b \in B$ not a join-irreducible element, then $B \setminus \{b\}$ is a base as well (if $b = f \lor g$, $f$ and $g$ are joins of elements of $B$, and as $f, g < b$, $b$ is not part of these joins). Using the Lemma of Zorn, it is possible to deduce that the subset of all join-irreducible elements constitutes a base for any sufficiently nice lattice.

Unfortunately, this is not the case with $d$-irreducible elements: Let $L'$ be the completely distributive complete lattice $[0, 3] \times [0, 2]$ with component-wise supremum and infimum, and with supremum metric. Then consider the sublattice $L \subseteq L'$ formed by the five elements
\[
L := \{(0, 0), (1, 0), (0, 1), (1, 1), (2, 2)\}.
\]
We find $\text{cmli}(L) = \{(0, 0), (1, 0), (0, 1)\}$, as $(1, 1) = (1, 0) \lor (0, 1)$. $p = (2, 2)$ is join-irreducible in this lattice, but not $d$-irreducible: Take $f_1 = (1, 0)$, $f_2 = (0, 1)$, then $\bigwedge d(p, f_j) = 2$, but $d(p, \bigvee f_j) = 1$. Nevertheless, $(2, 2)$ must be part of any 0-base of $L$. 16
References

[Bi1] G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications Vol. XXV, 2nd ed. (1948) and 3rd ed. (1960)

[Bi2] G. Birkhoff, *Von Neumann and Lattice Theory*, Bull. Amer. Math. Soc. 64, Nr 3, Part 2 (1958) 50–56, http://www.ams.org/bull/1958-64-03/S0002-9904-1958-10192-5/ S0002-9904-1958-10192-5.pdf

[dH] P. de la Harpe, *Topics in Geometric Group Theory*, The University of Chicago Press (2000)

[Gl] V. Glivenko, *Géométrie des systèmes de chosn normées*, Am. Jour. of Math. 58 (1936) 799–828

[Lo1] A. Lochmann, *Rough Isometries of Order Lattices and Groups*, Niedersächsische Staats- und Universitätsbibliothek, Doctoral Thesis, http://webdoc.sub.gwdg.de/diss/2009/lochmann/

[Lo2] A. Lochmann, *Rough Isometries of Lipschitz Function Spaces*, preprint at http://arxiv.org/abs/0710.1109

[Mn] B. Monjardet, *Metrics on partially ordered sets — a survey*, Discrete Mathematics 35 (1981) 173–184

[Ro] R. T. Rockafellar, *Convex Analysis*, Princeton University Press (1970)

[vN] J. von Neumann, *Lectures on continuous geometries*, Princeton 1936-1937 (2 vols.), in particular chapter XVII

[Wr] D. Werner, *Funktionalanalysis*, Springer (2005)

[Wv] N. Weaver, *Lipschitz Algebras*, World Scientific (1999)

Georg-August-Universität Göttingen, Germany
eMail lochmann@uni-math.gwdg.de