Magnetization distribution in the transverse Ising chain with energy flux

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The zero-temperature transverse Ising chain carrying an energy flux $j_E$ is studied with the aim of determining the nonequilibrium distribution functions, $P(M_z)$ and $P(M_x)$, of its transverse and longitudinal magnetizations, respectively. An exact calculation reveals that $P(M_z)$ is a Gaussian both at $j_E = 0$ and $j_E \neq 0$, and the width of the distribution decreases with increasing energy flux. The distribution of the order-parameter fluctuations, $P(M_x)$, is evaluated numerically for spin-chains of up to 20 spins. For the equilibrium case ($j_E = 0$), we find the expected Gaussian fluctuations away from the critical point while the critical order-parameter fluctuations are shown to be non-gaussian with a scaling function $\Phi(x) = \Phi(M_x/\langle M_x \rangle) = \langle M_x \rangle P(M_x)$ strongly dependent on the boundary conditions. When $j_E \neq 0$, the system displays long-range, oscillating correlations but $P(M_x)$ is a Gaussian nevertheless, and the width of the Gaussian decreases with increasing $j_E$. In particular, we find that, at critical transverse field, the width has a $j_E^{-3/8}$ asymptotic in the $j_E \to 0$ limit.

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I. INTRODUCTION

Nonequilibrium steady states (NESS) have been much studied but a description of some generality has not emerged so far. Among the many approaches tried, there are two which continue to receive particular attention. One of them is an attempt to understand the general features of NESS through studies of nonequilibrium phase transitions \cite{1, 2, 3}. The basic assumption here is that the universality displayed by equilibrium phase transitions carries over to critical phenomena in NESS, as well. Thus, by investigating the similarities and differences from equilibrium, one may gain an understanding of the role of various components of the competing dynamics generating the steady state. For example, one may find by observing the universality classes of various nonequilibrium phase transitions that dynamical anisotropies often yield dipole-like effective interactions \cite{4, 5, 6} or that competing non-local dynamics (anomalous diffusion) generates long-range, power-law interactions \cite{7}.

The second approach is more ambitious but less well founded. It concerns the building (approximating, guessing) of the entropy or the distribution function for the nonequilibrium microstates \cite{8, 9, 10}. The idea here is that NESS is often the result of complex nonlinear dynamics which yield distributions that cannot be described by the Boltzmann-Gibbs statistics. The reason e.g. may be that the dynamics generates long-range correlations which results in the loss of additivity of such quantities as the entropy. Furthermore, the dynamics may concentrate the distribution onto a fractal subspace in the long-time limit, a case that again may be problematic in equilibrium statistics. In order to treat such cases, the so called non-extensive statistical mechanics was introduced \cite{8}. It is an approach that takes its name from the non-extensive character of the postulated entropy. It has been much developed \cite{11, 12} as well as criticized (see e.g. \cite{13, 14, 15} for the latest examples) during the last decade. Not surprisingly, the approach had its success in connection with systems which have long-range interactions or display (multi)fractal behavior \cite{16}.

Our work presented below is perhaps best viewed as an attempt to build a bridge between the above approaches. On the one hand, we study the effects of a nonequilibrium constraint on a well investigated phase transition. Namely, we take the transverse Ising chain which has an order-disorder transition as the transverse field ($h$) is varied, and drive it by a field to produce an energy flux, $j_E$ through the system. The resulting steady states have been investigated \cite{17} and it has been found that, in addition to the equilibrium phases, a flux-carrying nonequilibrium phase appears which is distinct by its correlations decaying with distance as a power law. On the other hand, we shall be concerned with the distribution function in the various phases of the above system. More precisely, we shall determine the steady-state distribution functions, $P(M_z)$ and $P(M_x)$ of the $M_z$ (non-ordering field) and $M_x$ (ordering field) components of the macroscopic magnetization in all three phases of the system and at and near its critical point.

The results are surprisingly simple. The distribution functions are Gaussian in the equilibrium phases away from the critical point. This is expected since we have macroscopic quantity and the correlations decay exponentially. The distribution of the non-ordering field remains a Gaussian at the critical point of the equilibrium system, as well. The reason for this is that although the appropriate correlations decay with distance $n$ as a power law but the exponent in the power is large ($1/n^z$), so that the fluctuations $\langle M_x^2 \rangle - \langle M_x \rangle^2$ do not diverge at $h = h_c$. The distribution function of the ordering field becomes nontrivial at $h_c$ and our numerical calculations demonstrate that $P(M_x)$ depends strongly on the bound-
ary conditions taken to be periodic, anti-periodic, and free. The unexpected simplicity is in the current-carrying phase where the energy flux generates long-range correlations decaying as a power law (1/√n) but, nevertheless, the distributions are gaussian. The mathematical reason for this lies in the oscillating character of the correlations which prevents the divergence of the spatial sum of the correlations which in turn are proportional to the fluctuations. Physically, the oscillations in the correlations can be traced to the form of energy flux [see eq. (4) below] which suggest that the consecutive components of the spins are more and more rigidly interconnected as jE is increased and thus fluctuations decrease with increasing jE. This picture will be seen to be valid near the nonequilibrium phase boundaries where the fluctuations as a function of jE can be explicitly calculated.

The above results are presented in the following order. Section II contains a review of the transverse Ising model with energy flux, including the setup of the formalism convenient for calculating the distribution functions. Next (Sec. III), the distribution P(Mz) is calculated exactly. Numerical work on P(Mz) and preliminary analytic work on correlations are presented in Sec. IV followed by concluding remarks in Sec. V.

II. TRANSVERSE ISING MODEL WITH ENERGY FLUX

The transverse Ising chain is one of the simplest system displaying a critical order-disorder transition [13]. It is defined by the Hamiltonian

\[ \hat{H}_t = -J \sum_i \vec{s}_i \cdot \vec{s}_{i+1} - \frac{h}{2} \sum_i s_i^z \]  \hspace{1cm} (1)

where \( \vec{s}_i = \frac{1}{2} \vec{\sigma}_i \) and \( \sigma_i^\alpha \) (\( \alpha = x, y, z \)) denotes the three Pauli matrices at sites \( i = 1, 2, ..., N \) of a d = 1 periodic chain (\( s_{N+1}^z = s_1^z \)), and \( h \) is the transverse field in units of the Ising coupling (\( J = 1 \) is set in the rest of the paper).

The order parameter of this system is \( M_z = \sum_i s_i^z \), and the ground state of this Hamiltonian changes from being disordered \( \langle M_z \rangle = 0 \) for \( h > h_c \) to ordered \( \langle M_z \rangle \neq 0 \) for \( h < h_c \). The transition point \( h = h_c \) is a critical point in the universality class of the two-dimensional Ising model.

In order to drive the above system out of equilibrium, one can add a bulk field that drives the energy flux. The simplest bulk field is the energy flux itself [13] thus, in order to produce a nonequilibrium state of \( \hat{H}_t \) one should find the ground state of the following Hamiltonian

\[ \hat{H} = -\sum_i s_i^z s_{i+1}^z - \frac{h}{2} \sum_i s_i^z - \lambda \hat{J}_E. \]  \hspace{1cm} (2)

Here the driving field \( \lambda \) is again measured in units of \( J \), and the current operator \( \hat{J} \) is the sum of local energy flux given by the following expression

\[ \hat{J}_E = \frac{\hbar}{4} \sum_i (s_i^x s_{i+1}^y - s_i^y s_{i+1}^x). \]  \hspace{1cm} (3)

The driven system defined by (2) and (3) can be solved [15] and one finds that the ground state does not change and the energy flux is zero up to a critical value \( \lambda = \lambda_c(h) \) of the driving field. The ground state expectation value of the energy flux \( \langle \hat{J}_E \rangle / N \) becomes nonzero for \( |\lambda| > \lambda_c \). The resulting phase diagram is depicted in Fig. I.

![FIG. I: Phase diagram of the driven transverse Ising model in the h − |λ| plane where h is the transverse field while λ is the effective field which drives the flux of energy. Pairs of dual-conjugate points are shown by filled squares, circles, and stars; and the line h = 1 is self-dual as discussed in the text. Power-law correlations are present in the nonequilibrium phase, \( J_E \neq 0 \), and on the Ising critical line in the equilibrium phase, \( J_E = 0 \) (h = 1, 0 ≤ |λ| ≤ 2).](image)

Here we note a hitherto unnoticed property of the Hamiltonian (2). Namely, the duality properties of the transverse Ising model [14] have an appropriate generalization to the full nonequilibrium phase diagram of \( \hat{H} \). Indeed, let us denote the action of duality transformation by \( s_i^\alpha \rightarrow s_i^{\alpha^*} \) and define the transformation as it is done for the Ising model in a transverse field

\[ s_i^{z^*} = 2s_i^x s_{i+1}^x, \quad s_i^{x^*} s_{i+1}^{x^*} = \frac{1}{2} s_i^z. \]  \hspace{1cm} (4)

It can be verified then that the Hamiltonian \( \hat{H}[h, \lambda, \{ s_i^\alpha \}] \) characterized by the two couplings \( (h, \lambda) \) transforms into an identical Hamiltonian with couplings \( (h, \lambda)^* = (h^{-1}, -\lambda h) \)

\[ \hat{H}[h, \lambda, \{ s_i^\alpha \}] = h\hat{H}[h^{-1}, -\lambda h, \{ s_i^{\alpha^*} \}]. \]  \hspace{1cm} (5)

Hence the duality transformation leaves the whole \( |\lambda| = \lambda_c \) curve globally invariant and, furthermore, it leaves the \( h = 1 \) line pointwise invariant (examples of dual-conjugate points are given in Fig. I). In order to keep
the formulas simple, from now on we shall restrict our
analysis to $\lambda \geq 0$, that is to $j_E \geq 0$.

The self-dual $h = 1$ line is expected to display special
properties. For example, quantities such as $\lambda_c/\lambda$, or the
wavevectors $q_{\pm}$, where the excitation spectrum is gapless,
are left invariant by the duality transformation. Furthermore,
the functional form of various physical quantities (dispersion,
energy flux, fluctuations) considerably simplify on this line
and thus the knowledge of selfduality helps locating limits where
exact calculation can be carried out.

Our main goal is to calculate the distribution functions
$P(M_z)$ and $P(M_x)$ in various regions in the above phase
diagram. These functions are defined as

$$P(M_\alpha) = \langle \delta(M_\alpha - \sum_i s_i^\alpha) \rangle$$

(6)

where brackets denote the ground-state expectation value
(note that we have omitted the index $\alpha$ in $P_\alpha(\cdot)$, i.e. the
argument of the function defines which distribution is
considered).

There are two parts to our calculations. The function
$P(M_x)$ is evaluated numerically by diagonalizing $\hat{H}$
for chains containing up to $N = 18 - 20$ spins, while $P(M_z)$
is found analytically. The exact calculation is possible
because $M_z = \sum_q (c_q^+ c_q - 1/2)$ is a quadratic form in
the fermion operators $(c_q^+, c_q)$ in which the Hamiltonian
$\hat{H} = \hat{H}_I - \lambda_E$ is quadratic as well. Thus the calculation
of the generating function

$$G(s) = \langle e^{-sM_z} \rangle$$

(7)

becomes a problem of evaluating Gaussian integrals. We
begin with this part of the problem.

### A. Formalism

The Hamiltonian $\hat{H}$ can be diagonalized by first
introducing creation-annihilation operators, then employing
the Jordan-Wigner transformation to transform them into
fermion operators $(c_q, c_q^+)$ and, finally, using
a Bogoljubov transformation on the $c_q$ and $c_{-q}$
components of the Fourier transforms of $c_q$-s. The calculation
of $P(M_z)$ becomes relatively simple if after using
the Jordan-Wigner transformation one passes to a path-
integral formulation (see e.g. for a pedagogical
account). One then finds that the system is described by
the following quadratic action

$$S[\bar{c}, c] = \int \frac{d\omega}{2\pi} \int \frac{dq}{2\pi} \left[ \frac{1}{2} \langle \bar{c}_q(\omega) c_{-q}(\omega) \rangle B_{q,\omega} \left( \bar{c}_{-q}(\omega) c_q(\omega) \right) \right]$$

(8)

where the Grassman fields $\bar{c}_q(\omega)$ and $c_q(\omega)$ are related to
$c_q^+$ and $c_q$ correspondingly, while the scattering matrix
$B_{q,\omega}$ has the following inverse

$$B_{q,\omega}^{-1} = \begin{pmatrix} \langle \bar{c}_{-q}(\omega) c_q(\omega) \rangle & \langle \bar{c}_{-q}(\omega) c_{-q}(\omega) \rangle \\ \langle c_q(\omega) c_{-q}(\omega) \rangle & \langle c_q(\omega) c_q(\omega) \rangle \end{pmatrix}.$$  

(9)

Here the correlators are given by

$$\langle c_q(\omega) c_{-q}(\omega) \rangle = \langle \bar{c}_q(\omega) c_{-q}(\omega) \rangle^* =$$

$$\frac{1}{2\Lambda_q} \left[ \frac{h}{\omega - \Lambda_q^+} - \frac{h}{\omega - \Lambda_q^-} \right]$$

(10)

and

$$\langle \bar{c}_{-q}(\omega) c_q(\omega) \rangle = \langle \bar{c}_{-q}(\omega) c_{-q}(\omega) \rangle^* =$$

$$\frac{i\sin q}{2\Lambda_q} \left[ \frac{1}{\omega - \Lambda_q^+} - \frac{1}{\omega - \Lambda_q^-} \right]$$

(11)

where $\Lambda_q$ and $\Lambda_q^\pm$ are the dispersion relations for $\hat{H}_I$
and $\hat{H}$, respectively.

$$\Lambda_q = \frac{1}{2} \sqrt{1 + h^2 + 2h \cos q}$$

(12)

$$\Lambda_q^\pm = \pm \Lambda_q + \frac{\lambda h}{4} \sin q.$$  

(13)

In order to return to the time variables, we must first
study the two branches $\Lambda_q^\pm$ of the spectrum.

### B. Spectrum and energy flux

Flux of energy is present in the system only above a
critical drive $\lambda > \lambda_c$. (see Fig. where

$$\lambda_c(h) = \begin{cases} 2 & \text{if } h \geq 1 \\ \frac{1}{h} & \text{if } h \leq 1 \end{cases} $$

(14)

Indeed, it is not hard to see that

$$\Lambda_q^+(\lambda \leq \lambda_c) > 0 ; \quad \Lambda_q^-(\lambda \leq \lambda_c) < 0.$$  

(15)

Hence the ground state is unchanged with respect to the
pure Ising model ground state as long as the driving field
does not exceed $\lambda_c(h)$. This means that all observables
will assume their Ising model values and no energy cur-
rent will be flowing through the chain.

However, for $\lambda \geq \lambda_c$ one may see that $\Lambda_q^+$ (resp. $\Lambda_q^-$)
changes sign over the interval $I_2 = [q_-, q_+)$ (resp. $I_4 = [-q_+, -q_-]$). The explicit expression of the wave vectors
$q_{\pm}$ is deduced from

$$\cos q_{\pm} = \frac{-4 \pm \sqrt{(\lambda^2 h^2 - 4)(\lambda^2 - 4)}}{\lambda^2 h}, \quad -\pi \leq q_{\pm} \leq \pi.$$  

(16)

Beyond the critical drive, the excitation spectrum gives
rise to a ground state which breaks the left-right sym-
metry and, indeed, it may be verified that a nonzero
energy flux is present in the chain with the explicit form of the flux is given by

\[ j_E = \frac{\hbar}{4\pi\lambda^2} \sqrt{\left(\lambda^2 - \frac{4}{\hbar^2}\right)\left(\lambda^2 - 4\right)}. \tag{17} \]

For later applications we specify the \( j_E = j_E(\lambda) \) function in the vicinity of the critical drive, as \( \lambda \to \lambda^c \),

\[ j_E \approx \begin{cases} \sqrt{\frac{4}{\hbar^2}} \sqrt{\lambda - \frac{1}{\pi}} & \text{for } h > 1, \lambda_c = 2 \\ \frac{1}{\sqrt{\pi}} (\lambda - 2) & \text{for } h = 1, \lambda_c = 2 \\ \frac{h^{3/2} \sqrt{\lambda - \frac{1}{n}}}{4\pi} & \text{for } h < 1, \lambda_c = \frac{2}{h} \end{cases} \tag{18} \]

Besides, as they will naturally arise in the upcoming discussion, we further define here intervals \( I_1 = [-\pi, q_+], \) \( I_3 = [q_+, -q_+] \) and \( I_5 = [-q_-, \pi] \) which are complementary to \( I_2 \) and \( I_4 \) in \( [-\pi, \pi] \).

C. Inverse of the scattering matrix in the absence or presence of energy flux

When no current flows through the system, \( j_E = 0 \), the equal-time transform of \((B_{q,\omega})^{-1}\), denoted by \( B_q^{-1} \), reads:

\[ B_q^{-1} = \left( \begin{array}{cc} \frac{1}{2} \sin q & \frac{1}{2} \cos q + \lambda q \\ \frac{1}{2} \cos q + \lambda q & \frac{1}{2} \sin q \end{array} \right) \tag{19} \]

For \( j_E \neq 0 \), on the other hand, \( B_q^{-1} \) is given by Eq. \( (19) \) only for \( q \in I_1 \cup I_3 \cup I_5 \). If \( q \in I_2 \cup I_4 \), its expression is changed to

\[ I_2: \quad B_q^{-1} = \left( \begin{array}{cc} 0 & 0 \\ -1 & 0 \end{array} \right), \quad I_4: \quad B_q^{-1} = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \tag{20} \]

Having the expressions for \( B_q^{-1} \), we can start the calculation of the distribution function \( P(M_z) \).

III. DISTRIBUTION FUNCTION FOR THE TRANSVERSE MAGNETIZATION

A. Calculation of the generating function and its moments

The generating function \( G(s) \) of \( P(M_z) \) can be expressed through fermionic operators as

\[ G(s) = \langle e^{-sM_z} \rangle = e^{N\hat{z}} \langle e^{-\sum q \hat{c}_q^\dagger \hat{c}_q} \rangle. \tag{21} \]

After normal ordering \( e^{-sc_q^\dagger \hat{c}_q} \) and using the Grassman fields, we are left with evaluating the following expression

\[ G(s) = e^{N\hat{z}} \langle e^{-\tilde{s} \sum q \hat{c}_q^\dagger \hat{c}_q} \rangle, \quad \tilde{s} = 1 - e^{-s} \tag{22} \]

and the fields in the exponentials are evaluated at some fixed time. In order to evaluate \( G(s) \), we recall the following result for Grassmann integrals \[ \Pi \]: the vacuum expectation value of observables of the form \( \langle e^{-\sum q \hat{c}_q^\dagger \hat{c}_q} \rangle \) can be obtained as:

\[ \langle e^{-\sum q \hat{c}_q^\dagger \hat{c}_q} \rangle = \Pi \sqrt{\det \left( \begin{array}{cc} -\sqrt{z_q} & -q \left( B_q^{-1}\right)_{11} \\ -1 + z_q \left( B_q^{-1}\right)_{21} & \sqrt{z_q} \left( B_q^{-1}\right)_{22} \end{array} \right)} \tag{23} \]

As one can see \( G(s) \) is a special case of eq. \( (23) \) and it can be evaluated by using the appropriate expressions \( (19) \) or \( (20) \) for \( B_q^{-1} \).

If no current flows, that is for \( \lambda \leq \lambda_c \), we find that \( \ln G \) (the cumulant generating function) is given by

\[ \ln G(s) = \frac{N}{2} \int_{-\pi}^{\pi} \frac{dq}{2\pi} \ln \left[ (1 - n_q) e^{s} + n_q e^{-s} \right] \tag{24} \]

where

\[ n_q = \frac{(h + \cos q + 2\Lambda_q)}{(4\Lambda_q)} \tag{25} \]

One can verify that the normalization condition \( G(0) = 1 \) is satisfied, and one can also recover the well-known result found by Pfuetz \[ \text{(17)} \] for the magnetization

\[ -\frac{\partial \ln G(s)}{\partial s} \bigg|_0 = \langle M_z \rangle = N \int_{-\pi}^{\pi} \frac{dq}{2\pi} \left( n_q - \frac{1}{2} \right). \tag{26} \]

As we shall see below, \( P(M_z) \) is a Gaussian thus, in addition to \( \langle M_z \rangle \), the variance of the transverse magnetization \( \sigma^2(h) = \langle M_z^2 \rangle - \langle M_z \rangle^2 \) will characterize the distribution. It can be obtained from the second derivative of \( \ln G(s) \) as

\[ \sigma^2(h) = 2 \int_{-\pi}^{\pi} \frac{dq}{2\pi} n_q (1 - n_q) \tag{27} \]

It is interesting to note that the fluctuations in \( M_z \) are independent of the magnetic field in the ordered phase

\[ \sigma^2(h) = \begin{cases} 1/4 & \text{for } |h| \leq 1 \\ 1/(4h^2) & \text{for } |h| > 1 \end{cases} \tag{28} \]

In the presence of nonzero energy flux (\( \lambda > \lambda_c \)), the generating function is more complicated only because of the limits of integration in \( (24) \):

\[ \ln G(s) = \frac{N}{2} \int_{I_1 \cup I_3 \cup I_5} \frac{dq}{2\pi} \ln \left[ (1 - n_q) e^{s} + n_q e^{-s} \right] \tag{29} \]

Accordingly, the first and second cumulants of the magnetization are given by

\[ \langle M_z \rangle = \frac{N}{2} \int_{I_1 \cup I_3 \cup I_5} \frac{dq}{2\pi} \frac{h + \cos q}{2\Lambda_q} \tag{30} \]
\[ \sigma^2(h) = \frac{1}{2} \int_{I_1 \cup I_2 \cup I_3} dq \frac{\sin^2 q}{2\pi (1 + h^2 + 2h \cos q)} \]  
(31)

where we have used eq. (25) to write out the integrands explicitly. Note that we added a \( \lambda \) subscript to \( G \), \( \langle M_z \rangle \) and \( \sigma^2 \) in order to indicate that these quantities do depend on \( \lambda \) for \( \lambda > \lambda_c \).

**B. The transverse magnetization distribution is a Gaussian**

In order to show that the transverse magnetization distribution is a Gaussian, let us consider the \( n^{th} \) cumulant \( \langle M^n_z \rangle_c \) which is the coefficient in front of \( (-s)^n/n! \) in the expansion of \( \ln G(s) \). The latter coefficient is \( N \) times an integral of a polynomial in \( n_q \), both in the current carrying and current free phases and, furthermore, \( n_q \) is a nonsingular, strictly positive function of \( q \) (note that \( n_q \) is finite even at \( h = 1 \)). Hence each cumulant depends linearly on \( N \). In particular, as we have seen, the variance of \( M_z \) denoted by \( N\sigma^2 \) and \( \sigma^2 \) (in Eq. (31) or (27)) is finite.

It follows from the linear \( N \) dependence of the cumulants that

\[ \frac{\langle M^n_z \rangle_c}{\langle M_z \rangle_c^{n/2}} \sim \frac{N}{N_{n/2}} \sim N^{(2-n)/2} \]  
(32)

thus the above ratio goes to zero for all \( n > 2 \) as \( N \to \infty \). We may therefore conclude that the limiting form of the distribution function of the transverse magnetization is a Gaussian of variance \( N\sigma^2 \):

\[ P(M_z) = \frac{1}{\sqrt{2\pi N\sigma^2}} e^{-(M_z - \langle M_z \rangle)^2/2N\sigma^2} \]  
(33)

The above result applies over the whole phase diagram and it is useful to check the numerical procedure employed in Section IV by evaluating \( P(M_z) \) for finite-size systems. As can be seen in Fig. 2, there is a convergence to the limiting form with increasing \( N \) and, furthermore, the nearly Gaussian fluctuations of \( M_z \) are observed already at small sizes (\( N = 16 - 20 \)). It is remarkable that the deviations from the Gaussian are very close to those of an \( N \) step random walk with a drift determined from a correspondence between the left (right) moves and the up (down) spins generating the average \( \langle M_z \rangle \).

**C. Width of the Gaussian**

Recalling the expression of the average and the variance \( \langle M_z \rangle \) we calculate them in the limit of vanishing flux (\( \lambda \to \lambda_c^+ \)). Let us define

\[ \delta M_z = \langle M_z \rangle(h, \lambda) - \langle M_z \rangle(h), \quad \delta \sigma^2 = \sigma^2(h, \lambda) - \sigma^2(h) \]  
(34)

![FIG. 2: Distribution function \( P(M_z) \) for the longitudinal magnetization \( M_z \) on the critical line \( h = h_c = 1, \lambda \leq 2 \). Results for periodic boundary conditions are displayed. The solid line is the asymptotic Gaussian while the dashed line is the displacement distribution of a 20-step random walk of steplength 1/2 having a drift generating an average displacement equal to \( \langle M_z \rangle \).](image)

The above quantities exhibit singular behavior as one enters the current-carrying phase. For the magnetization we find

\[ \delta M_z = -N \int_{-\infty}^{+\infty} dq \frac{h + \cos q}{\Lambda_q} \]  
(35)

\[ \delta M_z = \left\{ \begin{array}{ll} \frac{1}{\sqrt{\pi}} (1 - h^{-2})(\lambda - \lambda_c)^{1/2} & \text{for } h > 1, \lambda_c = 2 \\
\frac{1}{\sqrt{\pi}} (\lambda - \lambda_c) & \text{for } h = 1, \lambda_c = 2 \\
\frac{1}{\sqrt{\pi}} N^{1/2} h^{3/2} (\lambda - \lambda_c)^{3/2} & \text{for } h < 1, \lambda_c = \frac{2}{N} \end{array} \right. \]  
(36)

and for the variance

\[ h \geq 1, \lambda_c = 2, \quad \delta \sigma_z^2 \simeq -\frac{1}{\pi h^2} \sqrt{\lambda - \lambda_c} \]  
\[ h \leq 1, \lambda_c = \frac{2}{N}, \quad \delta \sigma_z^2 \simeq -\frac{1}{\pi h} \sqrt{\lambda - \lambda_c} \]  
(37)

Using (18) we find, as \( j^E \to 0 \),

\[ h \neq 1, \quad \delta \sigma_z^2 \simeq -j^E \]  
\[ h = 1, \quad \delta \sigma_z^2 \simeq -j^{1/2} \]  
(38)

As can be seen, the variance of the transverse magnetization is smaller in the current carrying phase than in the current free phase. This supports the view that imposing a current stiffens the system, and thus decreases fluctuations.

**IV. DISTRIBUTION FUNCTION FOR THE LONGITUDINAL MAGNETIZATION**

The exact evaluation of \( P(M_z) \) appears to be a non-trivial task and we have been able to calculate it only
numerically for finite size chains. Since the expression for $P(M_x)$ is a ground-state expectation value, we had to find the ground-state wave function and, due to the sparseness of the Hamiltonian matrix, the Lanczos algorithm could be used effectively. Since the ground state wave function is needed with precision, we were able to accomplish this task for chain lengths of up to $N = 20$ with the results displayed on Figs. 3–5.

A. Equilibrium distribution

In the equilibrium system ($j_E = 0$), the correlation length is infinite only at the critical point. Thus one expects $P(M_x)$ to be a Gaussian for $h > 1$, a sum of two Gaussians for $h < 1$, and a nontrivial distribution emerges only at criticality ($h = h_c = 1$). This is indeed what we observe, apart from the finite size effects showing up in nongaussian corrections close to $h = 1$. At the critical point itself, $P(M_x)$ shows fast convergence to the asymptotic form as can be seen on Fig. 3 where the $N ≥ 16$ points appear to have settled on the asymptotic curve. This means that the $N$ dependence of $P(M_x)$ is almost all in the scaling variable $M_x/\sqrt{\langle M_x^2 \rangle}$, a remarkable feature that has been observed in a series of equilibrium- and nonequilibrium critical states. Note also that the finite-size effects show up mainly in the large $M_x/\sqrt{\langle M_x^2 \rangle}$ region. This is in accord with the general observation that the large-argument region of the scaling function is related to the long-wavelength properties of the system.

![Fig. 3: Scaling function for the distribution $P(M_x)$ of the longitudinal magnetization $M_x$ on the critical line $h = h_c = 1$, $\lambda ≤ 2$, for periodic boundary conditions. In order to demonstrate the smallness of finite-size effects, the small systems $(N = 8, 10, 12)$ are displayed by full symbols while the larger systems $(N = 16–20)$ are all shown by a single empty symbol.](image)

![Fig. 4: Distribution function $P(M_x)$ for the longitudinal magnetization $M_x$ at the critical line $h = h_c = 1$, $\lambda ≤ 2$. Results for periodic, free, and antiperiodic boundary conditions are displayed for system sizes $N = 16 – 20$.](image)

B. Nonequilibrium distribution

In the nonequilibrium case ($j_E \neq 0$), we find that similarly to $M_z$, the fluctuations of the longitudinal magnetization $M_x$ are also Gaussian (Fig. 5). Remarkably, the finite size, non-gaussian corrections are small even near the $h = 1$, $\lambda = 2$ point.

The Gaussian result is somewhat surprising since the flux generates power-law correlations in $\langle s_{\alpha}^1 s_{\alpha}^1 \rangle$ decaying with distance as $1/\sqrt{n}$. Thus one may presume that the current carrying states are effectively critical. This is not so, however, since the correlations are oscillating and the fluctuations $\langle M_x^2 \rangle$ (given by the integral of the correlations) are finite. Actually $\langle M_x^2 \rangle$ is decreasing with increasing $j_E$ (see below) and thus we see here another example of fluxes generating power-law correlations but only the strongly non-gaussian character of the distributions but also the fact that scaling functions do vary with changing the boundary conditions. The boundary condition dependence of the critical scaling functions is known.
nevertheless making the system more rigid.

The decrease of $\langle M_z^2 \rangle$ with increasing $j/c$ can be seen in the numerical studies of the finite spin chains. On the selfdual line ($h = 1$), however, this can be shown analytically in the limit of vanishing flux ($\lambda \to \lambda_c = 2$) where one finds

$$\langle M_z^2 \rangle \propto N \kappa^{-3/8}.$$  \hspace{1cm} (39)

The derivation of the above result is possible because the ($h = 1$, $\lambda = \lambda_c$) point is a critical point with infinite correlation length. Approaching this point along the $h = 1$, $\lambda \to \lambda^* = 1$, line, one can observe from numerical studies [13] that the wavelength $\kappa^{-1}$ of the oscillation of the correlation function $C_x(r) = \langle s_i s_{i+r} \rangle \propto \cos \kappa r$ diverges as $\kappa^{-1} \propto (\lambda - \lambda_c)^{-1/2}$. This diverging wavelength allows one to take a continuum limit and to establish [3] to all orders in perturbation theory, that the order-parameter correlations possess the following scaling form

$$C_x(r) \approx A \frac{1}{r^{1/4}} F(\kappa r).$$  \hspace{1cm} (40)

with the small argument limit of the scaling function explicitly given by $F(x) = 1 - x^2/2 + x^4/16 + O(x^6)$.

The derivation of the above results follows the ideas [31, 32, 33, 34] used for the calculation of the order-parameter fluctuations in the equilibrium transverse Ising model. Namely, the correlations are expressed as a Pfaffian of a block Toeplitz matrix constructed of $2 \times 2$ matrices. Then, in the equilibrium case, the analysis of the Toeplitz matrices in the asymptotic limit of $r \to \infty$, and $h \to 1$ with $r(h - 1)$ kept fixed yields $C_x(r) \sim r^{-1/4} F(r(h - 1))$. A similar asymptotic analysis in the current-carrying phase, using the scaling limit $r \to \infty$, and $\kappa \sim (\lambda - \lambda_c)^{1/2} \to 0$ with $\kappa r$ kept finite, results in eq. (40). The derivation is rather technical and we shall present it, together with the analysis of other scaling limits, in a separate publication [35].

Once the correlations are known, the fluctuations can be calculated from

$$\langle M_z^2 \rangle = N [1 + 2 \sum_{r \geq 1} C_x(r)] \propto N \int_0^{+\infty} dr C_x(r)$$  \hspace{1cm} (41)

where changing the sum into integral is again allowed because of the diverging characteristic length scale $\kappa^{-1}$. Using now the scaling form (40) we find that

$$\langle M_z^2 \rangle \propto N \kappa^{-3/4} \propto N \kappa^{-3/4}.$$  \hspace{1cm} (42)

The above expression demonstrates the decrease of fluctuations with increasing flux and it also tells us how the Gaussian distribution crosses over to the nontrivial shape observed at the critical point.

V. FINAL REMARKS

Returning to the problems discussed in the Introduction, we can see that the connection between NESS and critical states in terms of (universal) distribution functions is not straightforward. NESS is generated by fluxes, and fluxes may or may not generate long-range correlations. There are numerous examples [1, 2, 3] where the fluctuations to the presence of the flux. Driven diffusive systems [4] provide other examples [36] where the fluctuations decrease while the fluxes induce power-law correlations. Thus we should conclude that, in general, the power-law correlations generated by global fluxes cannot be the source of possible universality of nonequilibrium distribution functions. It remains, however, an intriguing question whether the weak long-range interactions supported by global fluxes can underlie a kind of ”weak” universality classification of distributions in NESS.

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