Factorial Powers for Stochastic Optimization

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Abstract

The convergence rates for convex and non-convex optimization methods depend on the choice of a host of constants, including step sizes, Lyapunov function constants and momentum constants. In this work we propose the use of factorial powers as a flexible tool for defining constants that appear in convergence proofs. We list a number of remarkable properties that these sequences enjoy, and show how they can be applied to convergence proofs to simplify or improve the convergence rates of the momentum method, accelerated gradient and the stochastic variance reduced method (SVRG).

1 Introduction

Consider the stochastic optimization problem

$$x_* \in \arg \min_{x \in C} f(x) = \mathbb{E}_\xi [f(x, \xi)],$$

where each $f(x, \xi)$ is convex but potentially non-smooth in $x$ and $C \subset \mathbb{R}^d$ is a bounded convex set. To solve (1) we use an iterative method that at the $k$th iteration samples a stochastic (sub-)gradient $\nabla f(x_k, \xi)$ and uses this gradient to compute a new, and hopefully improved, $x_{k+1}$ iterate. The simplest of such methods is Stochastic Gradient Descent (SGD) with projection:

$$x_{k+1} = \Pi_C (x_k - \eta_k \nabla f(x_k, \xi)),$$

where $\eta_k$ is a sequence of step-sizes. Both variance from the sampling procedure, as well as the non-smoothness of $f$ contribute to the sequence of $x$ iterates not converging directly. The two most commonly used tools to deal with this variance are iterate averaging techniques [Polyak, 1964] and decreasing step-sizes [Robbins and Monro, 1951]. By carefully choosing a sequence of averaging parameters and decreasing step-sizes we can guarantee that the variance of SGD will be kept under control and the method will converge. In this work we focus on an alternative to averaging: momentum. Momentum, which is more commonly thought of as a method for acceleration, can also be used as a replacement for averaging for non-smooth problems, both stochastic and non-stochastic.

Using averaging and momentum to handle variance introduces a new problem: choosing and tuning the additional sequence of parameters. In this work we introduce the use of factorial powers for the averaging, momentum, and step-size parameters. As we will show, the use of factorial powers simplifies and strengthens the convergence rate proofs.

Contributions

1. We introduce factorial powers as a tool for providing tighter or more elegant proofs for the convergence rates of methods using averaging, including dual averaging and Nesterov’s accelerated gradient method.

2. We leverage factorial powers to prove tighter any-time convergence rates for SGD with momentum (SGD+M) in the non-smooth convex and strongly-convex cases.

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3. We describe a novel SVRG variant with inner-loop factorial power momentum, which improves upon the SVRG++ [Allen Zhu and Yuan, 2016] method in both the convex and strongly convex case.

4. We identify and unify a number of existing results in the literature that make use of factorial power averaging, momentum or step-sizes.

2 Factorial Powers

The (rising) factorial powers [Graham et al., 1994] are defined for positive integer $r$ and non-negative integer $k$ via the function:

$$k^r = \prod_{i=0}^{r-1} (k + i).$$  \hfill (3)

Their behavior is similar to the simple powers $k^r$ as $k^r = O(k^r)$, and as we will show, they can typically replace the use of simple powers in proofs. They are closely related to the simplicial polytopic numbers $P_r(k)$ such as the triangular numbers $k(k+1)$ and tetrahedral numbers $\frac{1}{6}k(k+1)(k+2)$, by the relation $P_r(k) = \frac{1}{r!} k^r$. See the left of Figure 1 for contour plots comparing factorial and simple powers.

The advantage of $k^r$ over $k^r$ is that in many cases that arise in proofs, additive, rather than multiplicative operations, are applied to the constants. As we show in Section 3, summation and differencing operations applied to $k^r$ result in other factorial powers, that is, factorial powers are closed under summation and differencing. In contrast, when summing or subtracting simple powers of the form $k^r$, the resulting quantities are polynomials rather than simple powers. It is this closure under summation and differencing that allows us to derive improved convergence rates when choosing step-sizes and momentum parameters based on factorial powers.

Our theory will use a generalization of the factorial powers to non-integer $r$ and integer $k > -r$ using the Gamma function $\Gamma(k) := \int_0^\infty x^{k-1} e^{-x} dx$ so that

$$k^r := \frac{\Gamma(k + r)}{\Gamma(k)},$$  \hfill (4)

with the convention that $0^r = 0$ except for $0^0 = 1$. This is a proper extension because, when $k$ is integer we have that $\Gamma(k) = (k-1)!$ and consequently (4) is equal to (3). This generalized sequence is particularly useful for the values $r = 1/2$ and $r = -1/2$, as they may replace the use of $\sqrt{k}$ and $1/\sqrt{k}$ respectively in proofs.

The factorial powers can be computed efficiently using the log-gamma function to prevent overflow. Using the factorial powers as step sizes or momentum constants adds no computational overhead as they may be computed recursively using simple algebraic operations as we show below, even for the fractional factorial powers which are transcendental numbers generally. The base values for the recursion may be precomputed as constants to avoid the overhead of gamma function evaluations entirely.

Figure 1: (left) Contour plots of the simple powers and the factorial powers. (right) The half-factorial power and associated upper and lower bounds.
Recursion

\[(k + 1)^r = \frac{k + r}{k} k^r\]  
(7)

\[(k + 1)^r = (k + r) (k + 1)^{r-1}\]  
(8)

Summation

\[\sum_{i=0}^{k} i^r = \frac{1}{r + 1} k^{r+1}\]  
(9)

\[\sum_{i=a}^{b} i^r = \frac{1}{r + 1} b^{r+1} - \frac{1}{r + 1} a^{r+1}\]  
(10)

Differences

\[(k + 1)^r - k^r = r (k + 1)^{r-1}\]  
(11)

Ratios

\[\frac{k^{r+q}}{k^r} = (k + r)^q\]  
(12)

Inversion

\[k^{-r} = \frac{1}{(k - r)^r}\]  
(13)

| Table 1: Fundamental Properties of the factorial powers. |

2.1 Notation and Assumptions

We assume throughout that \(f(x, \xi)\) is convex in \(x\). Let \(\nabla f(x, \xi_k)\) denote the subgradient of \(f(x, \xi_k)\) given to the optimization algorithm at step \(k\). Let \(R > 0\) be the radius of the smallest Euclidean-norm ball around the origin that contains the set \(C\). In addition to this assumption, we will use one of the following two sets of assumptions depending on the setting.

Non-smooth functions. We assume \(f(\cdot, \xi)\) is Lipschitz with constant \(G > 0\) for all \(\xi\), that is

\[|f(x, \xi) - f(y, \xi)| \leq G \|x - y\|, \quad \forall x, y \in \mathbb{R}^d.\]  
(5)

Smooth functions. We assume \(\nabla f(\cdot, \xi)\) is Lipschitz with constant \(L > 0\) for all \(\xi\), that is

\[\|\nabla f(x, \xi) - \nabla f(y, \xi)\| \leq L \|x - y\|, \quad \forall x, y \in \mathbb{R}^d,\]  
(6)

and we assume that \(\sigma^2 < \infty\) where \(\sigma^2 = \mathbb{E}_\xi \|\nabla f(x, \xi)\|^2\).

We use the shorthand notation \(\mathbb{E}_\xi \|\cdot\|^2 = \mathbb{E}_\xi \left[\|\cdot\|^2\right]\) and will write \(\mathbb{E}\) instead of \(\mathbb{E}_\xi\) when the conditional context is clear. We defer all proofs to the supplementary material.

3 Properties of Factorial Powers

The factorial powers obey a number of properties which are key for deriving simple and tight convergence proofs, see Table 1. These properties allow for a type of "finite" or "umbral" calculus that uses sums instead of integrals [Graham et al., 1994]. All the proofs of these properties can be found in Section A in the supplementary material.

Often when using telescoping in a proof of convergence, we often need a summation property. For the factorial powers we have the simple formulas (9) and (10). This shows that the factorial powers are closed under summation because on both sides of (10) we have factorial powers. This formula is a discrete analogue of the definite integral \(\int_a^b x^r dx = \frac{1}{r+1} b^{r+1} - \frac{1}{r+1} a^{r+1}\). In contrast, when summing power sequences, we rely on Faulhaber’s formula:

\[\sum_{i=1}^{k} i^r = \frac{k^{r+1}}{r + 1} + \frac{1}{2} k^r + \sum_{i=2}^{r} \frac{B_i}{i!} (r - k + 1)! k^{r-i+1},\]  
(14)
which involves the Bernoulli numbers $B_j := \sum_{i=0}^j \frac{n^i}{i!}$. This is certainly not as simple as (10). Furthermore, to extend (14) to non-integer $r$ complicates matters further [McGown and Parks, 2007]. In contrast the summation properties (9) and (10) also hold for non-integer values.

Another common property used in telescoping arguments is the difference property (11). Once again we have that factorial powers are closed under differencing. In contrast, the simple powers instead require the use of inequalities such as:

$$rx^{r-1} \leq (x + 1)^r - x^r \leq r(x + 1)^r - 1$$

for $r \in (-\infty, 0) \cup [1, \infty)$ or

$$r(x + 1)^{r-1} \leq (x + 1)^r - x^r \leq rx^{r-1}$$

for $r \in (0, 1)$.

Using the above bounds adds slack into the convergence proof and ultimately leads to suboptimal convergence rates.

3.1 Half-Powers

The factorial half-powers $k^{1/2}$ and $k^{-1/2}$ are particularly interesting as they do not easily arise by chance, and so have not to our knowledge appeared in the optimization literature before. We will use them to develop new parameter settings in Theorem 2. The factorial half-powers growth are sandwiched by the standard half-powers as Illustrated in Figure 1

$$\sqrt{k - 1/2} \leq k^{1/2} \leq k^{1/2}, \quad \frac{1}{\sqrt{k - 1/2}} < k^{-1/2} < \frac{1}{\sqrt{k - 1}}. \quad (15)$$

4 From Averaging to Momentum

Here we show that averaging techniques and momentum techniques have a deep connection. We use this connection to motivate the use of factorial power momentum. Our starting point for this is SGD with averaging which can be written using the online updating form

$$x_{k+1} = \Pi_C (x_k - \eta_k \nabla f(x_k, \xi_k)),$$

$$\bar{x}_{k+1} = (1 - \alpha_{k+1}) \bar{x}_k + \alpha_{k+1} x_{k+1}. \quad (16)$$

Now consider the momentum method (SGD+M):

$$m_{k+1} = \beta m_k + (1 - \beta) \nabla f(x_k, \xi_k),$$

$$x_{k+1} = x_k - \alpha_k m_{k+1}, \quad (17)$$

where $\alpha_k$ and $\beta$ are step-size and momentum parameters respectively. At first glance the two methods (16) and SGD+M are not directly related. But as we prove in Theorem 15 in the supplementary material, SGD+M can be re-written in an iterate averaging form given by

$$z_{k+1} = \Pi_C (z_k - \eta_k \nabla f(x_k, \xi_k)),$$

$$\bar{x}_{k+1} = (1 - \alpha_{k+1}) \bar{x}_k + \alpha_{k+1} z_{k+1}, \quad (18)$$

which is formally equivalent to (17) in that the $x_k$ iterates in (17) and (18) are the same when $C = \mathbb{R}^d$ using the mapping $\eta_k = \gamma_k (1 - \beta)$, $\alpha_{k+1} = \alpha_k / \gamma_k$, and $\gamma_k = \sum_{j=0}^k \alpha_j \beta^{j-k-1}$ and $z_0 = x_0$. The $x_k$ update (18) is similar to the moving average in (16), but now the averaging occurs directly on the $x_k$ sequence that the gradient is evaluated on. As we will show, convergence rates of the SGD+M method can be shown for the $x_k$ sequence, with no additional averaging necessary. This method is also known as primal-averaging, and under this name it was explored by Tao et al., 2020 and Taylor and Bach, 2019 without an explicit link to the stochastic momentum method SGD+M.

Factorial powers play a key role in the choice of the momentum parameters $\alpha_{k+1}$, and the resulting convergence rate of (16). Standard (equal-weighted) averaging given by

$$\bar{x}_k := \frac{1}{k+1} \sum_{i=0}^k x_i$$

or

$$\bar{x}_k := \left(1 - \frac{1}{k+1}\right) \bar{x}_{k-1} + \frac{1}{k+1} x_k. \quad (19)$$

results in a sequence that “forgets the past” at a rate of $1/k$. Indeed, if we choose an arbitrary initial point $x_0$ (or at least without any special insight), to converge to the solution we must “forget” $x_0$. To forget $x_0$ faster, we can use a weighted average that puts more weight on recent iterates. We propose the use of the factorial powers to define a family of such weights that allows us to tune how fast we forget the past. In particular, we propose the use of momentum constants as described in the following proposition.
We can also use factorial power momentum with \( r > -1 \). Again when the factorial power step sizes are used instead, this inequality is replaced by the equality

\[
\text{Proposition 1. } \forall x_k \in \mathbb{R}^n, \text{ for } k = 1, \ldots, \text{ be a sequence of iterates, and let } r > -1 \text{ be a real number. For } k \geq 0, \text{ the factorial power average}
\]

\[
x_{k+1} = \left(1 - \frac{r + 1}{k + r + 1}\right)x_k + \frac{r + 1}{k + r + 1}x_{k+1}.
\]

\[ (21) \]

Shamir and Zhang [2013] introduced the polynomial-decay averaging \[ (21) \] for averaged SGD under the restriction that integer \( r > 0 \). Proposition [1] extends the result to non-integer values with a range of \( r > -1 \). Next we use factorial power averaging to get state-of-the-art convergence results for SGD+M.

4.1 Applying factorial powers

The any-time convergence of SGD+M is a good case study for the application of the half-factorial powers.

**Theorem 2.** Let \( f(x, \xi) \) be \( G \)-Lipschitz and convex in \( x \). The projected SGD+M method \[ (18) \] with \( \eta_k = \sqrt{1/2R}(k + 1)^{-1/2} \) and \( c_{k+1} = 1/(k + 1) \) satisfies after \( n \) steps:

\[
\mathbb{E} [f(x_n) - f(x_*)] \leq \sqrt{2RG} (n + 2)^{-1/2}.
\]

This result is strictly tighter than the \( \sqrt{2RG/\sqrt{n+1}} \) convergence rate that arises from the use of square-root sequences (see Theorem [21] in the appendix) as used by Tao et al. [2020]. The use of half-factorial powers also yields more direct proofs, as inequalities are replaced with equalities in many places. For instance, when \( \eta_k = \eta/\sqrt{k+1} \), a bound of the following form arises in the proof:

\[
\sqrt{k+1} - \sqrt{k} \leq \frac{1}{2\sqrt{k}}.
\]

If factorial power step sizes \( \eta_k = \eta(k + 1)^{-1/2} \) are used instead, then this bounding operation is replaced with an equality that we call the inverse difference property:

\[
\frac{1}{(k + 1)^{-1/2}} = \frac{1}{k^{-1/2}} = \frac{1}{2k^{1/2}}.
\]

The standard proof also requires summing the step sizes, requiring another bounding operation

\[
\sum_{i=0}^{k} \frac{1}{(i+1)^{-1/2}} \leq 2\sqrt{k+1}.
\]

Again when the factorial power step sizes are used instead, this inequality is replaced by the equality

\[
\sum_{i=0}^{k} (i + 1)^{-1/2} = 2(k + 1)^{1/2}.
\]

We can also use factorial power momentum with \( r = 3 \) to show that SGD+M converges at a rate of \( O(1/n) \) for strongly-convex non-smooth problems in the following theorem.

**Theorem 3.** Let \( f(x, \xi) \) be \( G \)-Lipschitz and \( \mu \)-strongly convex in \( x \). The projected SGD+M method \[ (17) \] with \( \eta_k = \frac{1}{\mu(k+1)} \) and \( c_{k+1} = \frac{4}{k+4} \) satisfies after \( n \) steps:

\[
\mathbb{E} [f(x_n) - f(x_*)] \leq \frac{2G^2}{\mu} (n + 2)^{-1} = \frac{2G^2}{\mu(n+1)}.
\]

This \( O(1/n) \) rate of convergence is the fastest possible in this setting [Agarwal et al., 2009]. This rate of convergence has better constants than that established by using a different momentum scheme in Tao et al. [2020]. Higher order averaging is also necessary to obtain this rate for the averaged SGD method, as established by Lacoste-Julien et al. [2012] and Shamir and Zhang [2013], however in that case only \( r = 1 \) averaging is necessary to obtain the same rate.
Algorithm 1 Our proposed SVRG+M method

Initialize: $x^0_{m_0-1} = x^0_{m_0-1} = x_0$

\[
\text{for } s = 1, 2, \ldots \text{ do} \quad \triangleright \text{outer-loop} \\
\tilde{x}^s = x^s_{m_0-1}, \quad x^0 = x^0_{m_0-1}, \quad z^0 = z^0_{m_0-1} \\
\nabla f(\tilde{x}^{s}) = \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\tilde{x}^{s}) \\
\text{for } t = 0, 1, \ldots, m_s - 1 \text{ do} \quad \triangleright \text{inner-loop} \\
\quad \text{Sample } j \text{ uniformly at random} \\
\quad g^s_t = \nabla f_j(x^s_t) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\tilde{x}^{s}) \\
\quad z^s_{t+1} = z^s_t - \eta g^s_t \\
\quad x^s_{t+1} = (1 - c_{t+1}) x^s_t + c_{t+1} z^s_{t+1} \quad \triangleright \text{Iterate Averaging} \\
\text{end for} \\
\text{end for}
\]

5 From Momentum to Acceleration

A higher order $r$ for the factorial powers is useful when the goal is to achieve convergence rates of the order $O(1/n^{r+1})$. Methods using equal weighted $r = 0$ momentum can not achieve convergence rates faster than $O(1/n)$, since that is the rate that they “forget” the initial conditions. To see this, note that in a sum $1/(n+1)\sum_{i=0}^{n} z_i$, the $z_0$ value decays at a rate of $O(1/n)$. When using the order $r$ factorial power for averaging (20), the initial conditions are forgotten at a rate of $O(1/n^{r+1})$. The need for $r = 1$ averaging arises in a natural way when developing accelerated optimization methods for non-strongly convex optimization, where the best known rates are of the order $O(1/n^2)$ obtained by Nesterov’s method. As with the SGD+M method, Nesterov’s method can also be written in an equivalent iterate averaging form [Auslender and Teboulle, 2006]:

\[
y_k = (1 - c_{k+1}) x_k + c_{k+1} z_k \\
z_{k+1} = z_k - \rho_k \nabla f(y_k) \\
x_{k+1} = (1 - c_{k+1}) x_k + c_{k+1} z_{k+1}, \\
\]  

(22)

where $\rho_k$ are the step sizes, and initially $z_0 = x_0$. In this formulation of Nesterov’s method we can see that the $x_k$ sequence uses iterate averaging of the form (18). To achieve accelerated rates with this method, the standard approach is to use $\rho_k = 1/(Lc_{k+1})$ and to choose momentum constants $c_k$ that satisfy the inequality

\[c_k^2 - c_{k-1}^{-1} \leq c_k^{-2}.
\]

This inequality is satisfied with equality when using the following recursive formula:

\[c_{k+1}^{-1} = \frac{1}{2} \left(1 + \sqrt{1 + 4c_k^{-1}}\right),
\]

but the opaque nature and lack of closed form for this sequence is unsatisfying. Remarkably, the sequence $c_{k+1} = 2/(k + 2)$ also satisfies this inequality, as pointed out by Teng [2008], which is a simple application of $r = 1$ factorial power momentum. We show in the supplementary material how using factorial powers together with the iterate averaging form of momentum gives a simple proof of convergence for this method, using same proof technique and Lyapunov function as for the regular momentum method. By leveraging the properties of factorial powers, the proof follows straightforwardly with no “magic” steps.

**Theorem 4.** Let $x_k$ be given by (22). Let $f(x, \xi)$ be $L$-smooth and convex. If we set $c_k = 2/(k + 2)$ and $\rho_k = (k + 1)/(2L)$ then

\[f(x_n) - f(x_*) \leq \frac{2L}{n^2} \|x_0 - x_*\|^2.
\]

(23)

This matches the rate given by Beck and Teboulle [2009] asymptotically, and is faster than the rate given by Nesterov’s estimate sequence approach [Nesterov, 2013] by a constant factor.

6 Variance Reduction with Momentum

Since factorial power momentum has clear advantages in situations where averaging of the iterates is otherwise used, we further explore a problem where averaging is necessary and significantly
complicates the formulation: the stochastic variance-reduced gradient method (SVRG). The SVRG method [Johnson and Zhang, 2013] is a double loop method, where the iterations in the inner loop resemble SGD steps, but with an additional additive variance reducing correction. In each other loop, the average of the iterates from the inner loop are used to form a new “snapshot” point. We propose the SVRG+M method (Algorithm 1). This method modifies the improved SVRG++ formulation of [Allen Zhu and Yuan, 2016] to further include the use of iterate averaging style momentum. See Algorithm 1.

Our formulation has a number of advantages over existing schemes. In terms of simplicity, it includes no resetting operations, so the \( x \) and \( z \) sequences start each outer loop at the values from the end of the previous one. Additionally, the snapshot \( \hat{x} \) is up-to-date, in the sense that it matches the final output point \( x \) from the previous step, rather than being set to an average of points as in SVRG/SVRG++.

The non-strongly convex case is an application of non-integer factorial power momentum. Using a large step size \( \eta = 1/6L \) we show in Theorem 5 that the optimal momentum parameters \( c_k \) correspond to a \((k+1)^{3/2}\) factorial power averaging of the iterates. The strongly convex case in

Theorem 6 uses fixed momentum (i.e. an exponential moving average), since no rising factorial sequence can give linear convergence rates. Both of these rates improve the constants non-trivially over the SVRG++ method.

**Theorem 5.** (non-strongly convex case) Let \( f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \) where each \( f_i \) is \( L \)-smooth and convex. By setting \( c_t = 1/(t+1)^{2/3}, \) \( \eta = 1/6L, \) and \( m_s = 2m_{s-1} \) in Algorithm 1, we have that

\[
\mathbb{E} [f(x_{m_s}^S) - f(x_*)] \leq \frac{1}{2S} [f(x_0) - f(x_*)] + \frac{9L \|x_0 - x_*\|^2}{2S m_0}.
\]

The non-strongly convex convergence rate is linear in the number of epochs, however each epoch is twice as long as the previous one, resulting in an overall \( 1/t \) rate.

**Theorem 6.** (strongly convex case) Let \( \frac{1}{n} \sum_{i=1}^{n} f_i(x) \) where each \( f_i \) is \( L \)-smooth and \( \mu \)-strongly convex. Let \( \kappa = L/\mu. \) By setting \( m_s = 6\kappa, c_k = \frac{3}{3+\kappa}, \) and \( \eta_k = 1/(10L) \) in Algorithm 1, we have that

\[
\mathbb{E} [f(\hat{x}_S) - f(x_*)] \leq \left( \frac{6}{10} \right)^S \left[ f(x_0) - f(x_*) + \frac{3}{4\mu} \|x_0 - x_*\|^2 \right].
\]

7 Further Applications

Classical (non-stochastic) dual averaging uses updates of the form [Nesterov, 2009]:

\[
s_{k+1} = s_k + \nabla f(x_k), \quad x_{k+1} = \arg \min_x \left\{ \langle s_{k+1}, x \rangle + \hat{\beta}_{k+1} \frac{\gamma}{2} \|x - x_0\|^2 \right\},
\]

where sequence \( \hat{\beta}_k \) is defined recursively with \( \hat{\beta}_0 = \hat{\beta}_1 = 1, \) and \( \hat{\beta}_{k+1} = \hat{\beta}_{k+1} + 1/\hat{\beta}_{k+1}. \) This sequence grows approximately following the square root, as \( \sqrt{2k+1} \leq \hat{\beta}_{k+1} \leq \frac{1}{1+\sqrt{2k}} + \sqrt{2k+1} \) for \( k \geq 1, \) and obeys a kind of summation property \( \sum_{i=0}^{k} \frac{1}{\beta_i} = \hat{\beta}_{k+1}. \) Nesterov’s sequence has the disadvantage of not having a simple closed form, but it otherwise provides tighter bounds than using \( \beta_k = \sqrt{k+1}. \) The bound on the duality gap is given by:

\[
\max_{x, \|x\| \leq R} \left\{ \frac{1}{n+1} \sum_{i=0}^{n} \langle \nabla f(x_i), x_i - x \rangle \right\} \leq \left( \frac{\sqrt{2}}{1+\sqrt{3}} \frac{1}{n+1} + \frac{2}{\sqrt{n+1}} \right) RG.
\]

The factorial powers obey a similar summation relation, and they have the advantage of an explicit closed form.

**Theorem 7.** after \( n \) steps of the dual averaging method with \( \hat{\beta}_k = 1/(k+1)^{-1/2} \) and \( \gamma = G/R:

\[
\max_{x, \|x\| \leq R} \left\{ \frac{1}{n+1} \sum_{i=0}^{n} \langle \nabla f(x_i), x_i - x \rangle \right\} \leq 2RG(n+2)^{-1/2} < \frac{2RG}{\sqrt{n+1}}.
\]
The dual averaging method described above has been further combined with iterate averaging to give

\[ s_{k+1} = s_k + \lambda_k \nabla f(x_k), \]

\[ z_{k+1} = \arg \min_x \left\{ \langle s_{k+1}, x \rangle + \frac{\gamma}{2} \|x - x_0\|^2 \right\}, \]

\[ x_{k+1} = (1 - c_{k+1}) x_k + c_{k+1} z_{k+1}. \]

This is the “Subgradient Method with Double Simple Averaging” from Nesterov and Shikhman [2015], although “Dual Averaging with Momentum” would be a better name, given the link between momentum methods and iterate averaging. This method enjoys the same last-iterate convergence results that we have shown for SGD when \( 1/(k+1) \) momentum is used, compared to the average-iterate convergence of dual averaging.

Factorial power step size schemes have also arisen for the conditional gradient method:

\[ p_{k+1} = \arg \min_{p \in C} \langle p, \nabla f(x_k) \rangle, \quad x_{k+1} = (1 - c_{k+1}) x_k + c_{k+1} p_{k+1}. \]

For this method the most natural step sizes satisfy the following recurrence (“open loop” step sizes)

\[ c_{k+1} = c_k - \frac{1}{2} c_k^2, \]

which Dunn and Harshbarger [1978] note may be replaced with \( c_{k+1} = 1/(k+1) \). Another approach that more closely approximates the open-loop steps is the factorial power weighting \( c_{k+1} = 2/(k+2) \) as used in Jaggi [2013] and Bach [2015].

8 Experiments

For our experiments we compared the performance of factorial power momentum on a strongly-convex but non-smooth machine learning problem: regularized multi-class support vector machines. We consider 4 problems from the LIBSVM repository: GLASS, PROTEIN, USPS, and VOWEL. We used batch-size 1 and the step sizes recommended by the theory for both SGD with \( r = 1 \) averaging, as well as SGD with factorial power momentum as we developed in Theorem 3. We induced strong convexity by using weight decay of strength 0.001. The median as well as interquartile range bars from 40 runs are shown. Since our theory suggests \( r = 3 \), we tested \( r = 0, 1, 3, 5 \) to verify that \( r = 3 \) is the best choice. The results are shown in Figure 2. We see that when using factorial power momentum, using \( r = 0, 1 \) is worse than \( r = 3 \), and using \( r = 5 \) is no better than \( r = 3 \), so the results agree with our theory. The momentum method also performs a little better than SGD with post-hoc averaging, however it does appear to be substantially more variable between runs, as the interquartile range shows. We provide further experiments covering the SVRG+M method in the supplementary material.

9 Conclusion

Factorial powers are a flexible and broadly applicable tool for establishing tight convergence rates as well as simplifying proofs. As we have shown, they have broad applicability both for stochastic optimization and beyond.
**Broader Impact Statement**

Besides providing a theoretical framework for the analysis of existing algorithms, this work also introduces new optimization algorithms which may lead to faster training of machine learning models, which benefits the machine learning community in terms of reduced model training time, and may benefit the environment due to reduced emissions from training. Our algorithms do not introduce any additional ethics, bias or societal considerations when used in place of existing optimization methods.

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A Proof of Properties of Factorial Powers

**Proposition 8.** Recursive properties:

\[(k + 1)^r = \frac{k + r}{k} k^r, \quad (k + 1)^r = (k + r) (k + 1)^{r-1}. \quad (24)\]

*Proof.* Using the definition directly:

\[(k + 1)^r = \frac{\Gamma(k + r + 1)}{\Gamma(k + 1)} = \frac{\Gamma(k + r) (k + r)}{\Gamma(k) k} = \frac{k + r}{k} k^r \]

and

\[(k + 1)^r = \frac{\Gamma(k + 1 + r)}{\Gamma(k + 1)} = \frac{\Gamma(k + r)(k + r)}{\Gamma(k + 1)} = (k + r) (k + 1)^{r-1} \]

\[\square\]

**Proposition 9.** Difference property:

\[(k + 1)^r - k^r = r(k + 1)^{r-1}. \quad (26)\]

*Proof.* We apply the recursive property in \(k\), then in \(r\):

\[(k + 1)^r - k^r = \frac{k + r}{k} k^r - k^r = \frac{r}{k} k^r \]

\[= \frac{1}{1 + \frac{r}{k}} (k + 1)^r \]

\[= r \frac{1}{k + r} (k + 1)^r \]

\[= r (k + 1)^{r-1}. \]

\[\square\]

**Proposition 10.** Ratio property:

\[\frac{k^{r+q}}{k^r} = (k + r)^q, \]

*Proof.*

\[\frac{k^{r+q}}{k^r} = \frac{\Gamma(k + r + q)}{\Gamma(k)} = \frac{\Gamma(k + r + q)}{\Gamma(k + r)} = (k + r)^q. \]

\[\square\]
Proposition 11. Summation property, for integer $b \geq a \geq 0$:

$$\sum_{i=a}^{b} i^r = \frac{1}{r+1} b^{r+1} - \frac{1}{r+1} a^{r+1}.$$  

Proof. This property is a direct consequence of telescoping the difference property. \qed

Proposition 12. Inverse difference property:

$$\frac{1}{(k + 1)^{-1/2}} - \frac{1}{k^{-1/2}} = \frac{1}{2} \frac{1}{k^{1/2}}. \tag{27}$$

Proof. We apply the inverse property followed by the difference property then the inverse property again:

$$\frac{1}{(k + 1)^{-1/2}} - \frac{1}{k^{-1/2}} = \left( k + 1 - \frac{1}{2} \right)^{1/2} - \left( k - \frac{1}{2} \right)^{1/2}$$

$$= \frac{1}{2} (k + 1/2)^{-1/2}$$

$$= \frac{1}{2} \frac{1}{k^{1/2}}. \quad \text{ \qedsymbol}$$

Lemma 13. Let $r \geq 0$ and $j \geq 0$. Then consider the sequence:

$$c_k = \frac{r + 1}{k + j + r},$$

then:

$$\frac{1 - c_k}{c_k} (k + j)^r = \frac{1}{c_{k-1}} (k + j - 1)^r.$$  

Proof. Simplifying:

$$\left( \frac{1}{c_k} - 1 \right) (k + j)^r = \left( \frac{k + j + r}{r + 1} - 1 \right) (k + j)^r$$

$$= \left( \frac{k + j + r - r - 1}{r + 1} \right) (k + j)^r$$

$$= \frac{k + j - 1}{r + 1} (k + j)^r$$

$$= \frac{k + j + r - 1}{r + 1} \cdot \frac{k + j - 1}{k + j + r - 1} (k + j)^r$$

$$= \frac{1}{c_{k-1}} \frac{k + j - 1}{k + j + r - 1} (k + j)^r.$$  

Now applying the recursion property Eq. (7) gives:

$$\frac{k + j - 1}{k + j + r - 1} (k + j)^r = (k + j - 1)^r,$$

giving the result. \qedsymbol
B Proof of Proposition 1

Proposition 14. Let $z_k \in \mathbb{R}^n$ for $k = 0, \ldots$ be a sequence of points, and let $r > -1$ be a real number. Define $\bar{x}$ as the origin. The moving average:

$$\bar{x}_k = (1 - c_k) \bar{x}_{k-1} + c_k z_k,$$

$$c_k = \frac{r + 1}{k + r + 1},$$

is equivalent to the factorial power weighted average:

$$\bar{x}_k = \frac{r + 1}{(k + 1)^{r+1}} \sum_{i=0}^{k} (i + 1)^r z_i.$$

Proof. We show by induction. For the base case, consider $k = 0$. Then:

$$\bar{x}_0 = (1 - c_0) \bar{x}_{-1} + c_0 z_0$$

$$= \left(1 - \frac{r + 1}{r + 1}\right) \bar{x}_{-1} + \frac{r + 1}{r + 1} z_0$$

$$= z_0.$$

Likewise, using the summation property (9) we have that

$$\bar{x}_0 = \frac{r + 1}{1^{r+1}} \sum_{i=0}^{0} (i + 1)^r z_i$$

$$= \frac{(r + 1) 1^r}{1^{r+1}} z_0$$

$$= z_1.$$

We have used the recursive property $(k + 1)^r = (k + r) (k + 1)^{r-1}$ to simplify.

For the inductive case, consider $k \geq 1$ and suppose that $\bar{x}_{k-1} = \frac{r + 1}{k^{r+1}} \sum_{i=0}^{k-1} (i + 1)^r z_i$. We may write the update as

$$\bar{x}_k = \frac{r + 1}{(k + 1)^{r+1}} \sum_{i=0}^{k} (i + 1)^r z_i,$$

$$= \frac{r + 1}{(k + 1)^{r+1}} \sum_{i=1}^{k-1} (i + 1)^r z_i + (r + 1) \frac{(k + 1)^r}{(k + 1)^{r+1}} z_k$$

$$= \frac{k^{r+1}}{(k + 1)^{r+1}} \frac{r + 1}{k^{r+1}} \sum_{i=1}^{k-1} (i + 1)^r z_i + (r + 1) \frac{(k + 1)^r}{(k + 1)^{r+1}} z_k$$

$$= \frac{k^{r+1}}{(k + 1)^{r+1}} \bar{x}_{k-1} + (r + 1) \frac{(k + 1)^r}{(k + 1)^{r+1}} z_k,$$

where in the last line we used the induction hypothesis. To show the equivalence to the moving average form $\bar{x}_k = (1 - c_k) \bar{x}_{k-1} + c_k z_k$, we just need to show that:

$$c_k = \frac{(k + 1)^r}{(k + 1)^{r+1}}$$

and

$$1 - c_k = \frac{k^{r+1}}{(k + 1)^{r+1}}.$$

where $c_k = \frac{r + 1}{k^{r+1}}$. These two identities follow from applying the recursive properties, Eq. (8):
For the other term we use Eq. (7):

\[
\frac{kr+1}{(k+1)^r+1} = \frac{kr+1}{k} \frac{k}{k+r+1} = 1 - \frac{r+1}{k+r+1} = 1 - c_k.
\]

\[\square\]

C  The iterate Averaging reformulation of Momentum

Theorem 15. Consider the Momentum method

\[
m_{k+1} = \beta m_k + (1 - \beta) \nabla f(x_k, \xi_k), \quad x_{k+1} = x_k - \alpha_k m_{k+1},
\]

where \(m_0 = 0\) is equivalent to the iterate averaging form,

\[
z_{k+1} = z_k - \eta_k \nabla f(x_k, \xi_k), \quad x_{k+1} = (1 - c_{k+1}) x_k + c_{k+1} z_{k+1},
\]

in the sense that the \(x\) iterates are the same, if

\[
\eta_k = \gamma_k (1 - \beta), \quad c_{k+1} = \gamma_k, \quad \text{and} \quad \gamma_k = \sum_{i=0}^{k} \frac{\alpha_i}{\beta^{k-i+1}}.
\]

Proof. Note that from (30) we can write \(\gamma_k\) as the recurrence

\[
\beta \gamma_k = \gamma_{k-1} - \alpha_k.
\]

Indeed, expanding this recurrence gives (30). Assume the iterates \(x_k\) are given by (28). Let

\[
z_{k+1} = x_k - \gamma_k m_{k+1}.
\]

It follows that

\[
z_{k+1} = x_k - \gamma_k m_{k+1}
\]

\[
\begin{align*}
(28) & \quad (x_{k-1} - \alpha_{k-1} m_k) - \gamma_k (\beta m_k + (1 - \beta) \nabla f(x_k, \xi_k)) \\
(29) & \quad x_{k-1} - (\alpha_k + \beta \gamma_k) m_k - \gamma_k (1 - \beta) \nabla f(x_k, \xi_k) \\
(30) & \quad x_{k-1} - \gamma_{k-1} m_k - \gamma_k (1 - \beta) \nabla f(x_k, \xi_k) \\
(32) & \quad z_k - \eta_k \nabla f(x_k, \xi_k).
\end{align*}
\]

Furthermore

\[
x_{k+1} = x_k - \alpha_k m_{k+1}
\]

\[
\begin{align*}
(28) & \quad x_k - \frac{\alpha_k}{\gamma_k} (x_k - z_{k+1}) \\
(30) & \quad (1 - c_{k+1}) x_k + c_{k+1} z_{k+1}.
\end{align*}
\]

\[\square\]
D The iterate Averaging reformulation of Heavy Ball

Theorem 16. Consider the stochastic Heavy Ball method
\[ x_{k+1} = x_k - \alpha_k \nabla f(x_k, \xi_k) + \beta_k (x_k - x_{k-1}) \]
with the convention that \( x_{-1} = x_0 \) is equivalent to the iterate averaging form,
\[ z_{k+1} = z_k - \eta_k \nabla f(x_k, \xi_k), \] \[ x_{k+1} = (1 - c_k) x_k + c_k z_{k+1}, \] (33)
in the sense that the \( x \) iterates are the same, if
\[ \alpha_k = \eta_k c_{k+1}, \quad \beta_k = c_k + 1 \frac{1 - c_k}{c_k}, \quad \text{and} \quad z_0 = x_0 \]

Proof. Consider \( k \geq 1 \). Substituting the \( z_{k+1} \) equation into the \( x_{k+1} \) equation of the iterate averaging form.
\[ x_{k+1} = (1 - c_k) x_k + c_k z_k \] the iterate averaging form:
\[ z_k = \frac{1}{c_k} x_k - \frac{1}{c_k} (1 - c_k) x_{k-1}, \]
we get:
\[ x_{k+1} = (1 - c_k) x_k + c_k \left( \frac{1}{c_k} x_k - \frac{1}{c_k} (1 - c_k) x_{k-1} - \eta_k \nabla f(x_k, \xi_k) \right) \]
\[ = x_k - \eta_k c_k \nabla f(x_k, \xi_k) + c_k \left( -x_k + \frac{1}{c_k} x_k - \left( \frac{1}{c_k} - 1 \right) x_{k-1} \right) \]
\[ = x_k - \eta_k c_k \nabla f(x_k, \xi_k) + c_k \left( \frac{1}{c_k} - 1 \right) (x_k - x_{k-1}) \]
Equating terms gives the result. For the base case, when \( k = 0 \) you have for the heavy ball method:
\[ x_1 = x_0 - \alpha_0 \nabla f(x_0, \xi_0) + \beta_0 (x_0 - x_{-1}) = x_0 - \alpha_0 \nabla f(x_0, \xi_0) \]
and for the iterate averaging form:
\[ z_1 = x_0 - \eta_0 \nabla f(x_0, \xi_0), \] \[ x_1 = (1 - c_1) x_0 + c_1 z_1, \]
therefore:
\[ x_1 = (1 - c_1) x_0 + c_1 (x_0 - \eta_0 \nabla f(x_0, \xi_0)) \]
\[ = x_0 - \eta_0 c_1 \nabla f(x_0, \xi_0), \]
so they are equivalent in the \( k = 0 \) case. \( \Box \)

E Convergence Theorems for (projected) SGD+M

Theorem 17. For the projected SGD+M method:
\[ x_k = (1 - c_k) x_{k-1} + c_k z_k, \] \[ z_{k+1} = \Pi_C (z_k - \eta_k \nabla f(x_k, \xi_k)), \] (34)
where \( c_k \leq 1 \). If each \( f(\cdot, \xi) \) is convex and \( G \)-Lipschitz then
\[ \| z_{k+1} - x^* \|^2 \leq \| z_k - x^* \|^2 + \eta_k^2 G^2 \]
\[ -2 \frac{1}{c_k} \eta_k [f(x_k, \xi_k) - f(x^*, \xi_k)] + 2 \left( \frac{1}{c_k} - 1 \right) \eta_k [f(x_{k-1}, \xi_k) - f(x^*, \xi_k)]. \]
Proof. We start with \( z_{k+1} \) instead of the usual expansion in terms of \( x_{k+1} \):
\[
||z_{k+1} - x_*||^2 = ||II_C(z_k - \eta_k \nabla f(x_k, \xi_k)) - II_C(x_*)||^2 \\
\leq ||z_k - \eta_k \nabla f(x_k, \xi_k) - x_*||^2 \\
= ||z_k - x_*||^2 - 2\eta_k \langle \nabla f(x_k, \xi_k), z_k - x_* \rangle + \eta_k^2 G^2 \\
= ||z_k - x_*||^2 - 2\eta_k \left( \nabla f(x_k, \xi_k), x_k - \left( \frac{1}{c_k} - 1 \right) (x_{k-1} - x_k) - x_* \right) + \eta_k^2 G^2 \\
= ||z_k - x_*||^2 + \eta_k^2 G^2 \\
- 2\eta_k \langle \nabla f(x_k, \xi_k), x_k - x_* \rangle - 2\eta_k \left( \frac{1}{c_k} - 1 \right) \langle \nabla f(x_k, \xi_k), x_k - x_{k-1} \rangle
\]

Using the following two convexity inequalities
\[
\langle \nabla f(x_k, \xi_k), x_k - x_{k-1} \rangle \leq f(x_k, \xi_k) - f(x_{k-1}, \xi_k) \\
(\nabla f(x_k, \xi_k), x_* - x_k) \leq f(x_*, \xi_k) - f(x_k, \xi_k)
\]
combined with \((1/c_k - 1) \geq 0\) gives
\[
||z_{k+1} - x_*||^2 \leq ||z_k - x_*||^2 + \eta_k^2 G^2 \\
- 2\eta_k [f(x_k, \xi_k) - f(x_*, \xi_k)] - 2\left( \frac{1}{c_k} - 1 \right) \eta_k [f(x_k, \xi_k) - f(x_{k-1}, \xi_k)].
\]

Now rearranging further:
\[
||z_{k+1} - x_*||^2 \leq ||z_k - x_*||^2 + \eta_k^2 G^2 \\
- 2\frac{1}{c_k} \eta_k [f(x_k, \xi_k) - f(x_*, \xi_k)] + 2\left( \frac{1}{c_k} - 1 \right) \eta_k [f(x_{k-1}, \xi_k) - f(x_*, \xi_k)].
\]

Taking expectations with respect to \( \xi_k \) and applying Equation \ref{eq:35} gives the result. \( \square \)

**Corollary 18.** Consider the Lyapunov function:
\[
A_k = ||z_k - x_*||^2 + \frac{2}{c_k} \eta_{k-1} [f(x_{k-1}) - f(x_*)]
\]
If for \( k \geq 2 \),
\[
\left( \frac{1}{c_k} - 1 \right) \eta_k \leq \frac{1}{c_k - 1} \eta_{k-1},
\]
and for \( k = 1 \) we have \((1/c_1 - 1) \eta_1 \leq 0\), then SGD+M steps satisfy the following relation for \( k \geq 1 \).
\[
\mathbb{E}[\xi_k] [A_{k+1}] \leq A_k + \eta_k^2 G^2,
\]
when each \( f(\cdot, \xi) \) is convex and \( G \)-Lipschitz.

**Corollary 19.** Let \( \mathbb{E}[\cdot] \) denote the expectation with respect to all \( \xi_i \), with \( i \leq n \). Suppose that the constraint set \( C \) is contained in an \( R \)-ball around the origin. Then telescoping and applying the law of total expectation gives:
\[
\mathbb{E} ||z_{n+1} - x_*||^2 + \frac{2}{c_n} \eta_n \mathbb{E} [f(x_n) - f(x_*)] \leq R^2 + \sum_{i=0}^{n} \eta_i^2 G^2
\]

**E.1 Proof of Theorem 2** Any-time convergence with factorial power step sizes

**Theorem 20.** Consider the projected SGD+M method Eq. \ref{eq:18} when \( \eta_k = \sqrt{1/2} \frac{R}{G} (k + 1)^{-1/2} \) and \( c_k = 1/(k + 1) \), when each \( f(x, \xi) \) is \( G \)-Lipschitz, convex and the constraint set \( C \) is contained within an \( R \)-ball around \( x_0 \), then:
\[
\mathbb{E} [f(x_n) - f(x_*)] \leq \sqrt{2} RG (n + 2)^{-1/2} \leq \frac{\sqrt{2} RG}{\sqrt{n + 1}}.
\]
\textbf{Proof.} Consider Theorem 17 in expectation conditioned on $\xi_k$:
\[
\mathbb{E} \|z_{k+1} - x_*\|^2 \leq \|z_k - x_*\|^2 + \eta^2 k G^2 \\
- 2 \frac{1}{c_k} \eta_k (f(x_k) - f(x_*)) + 2 \left( \frac{1}{c_k} - 1 \right) \eta_k (f(x_{k-1}) - f(x_*)).
\]

We will use a step size $\eta_k = \eta (k + 1)^{-1/2}$ for some constant $\eta$, and multiply this expression by $1/(k + 1)^{-1/2}$:
\[
\frac{1}{(k + 1)^{-1/2}} \mathbb{E} \|z_{k+1} - x_*\|^2 \leq \frac{1}{(k + 1)^{-1/2}} \|z_k - x_*\|^2 + (k + 1)^{-1/2} \eta^2 G^2 \\
- 2 \frac{1}{c_k} \eta (f(x_k) - f(x_*)) + 2 \left( \frac{1}{c_k} - 1 \right) \eta (f(x_{k-1}) - f(x_*)).
\]

(37)

Now we prove the result by induction. First consider the base case $k = 0$. Since $P_{-1/2}(1) \leq 1$ we have that
\[
\frac{1}{1 - 1/2} \|z_0 - x_*\|^2 = \frac{1}{\sqrt{\pi}} \|z_0 - x_*\|^2 \leq \frac{1}{\sqrt{\pi}} R^2.
\]

Consequently taking $k = 0$ in (37) gives
\[
\frac{1}{(k + 1)^{-1/2}} \mathbb{E} \|z_1 - x_*\|^2 \leq \frac{1}{(k + 1)^{-1/2}} \|z_0 - x_*\|^2 + (k + 1)^{-1/2} \eta^2 G^2 - 2 \frac{1}{c_0} \eta (f(x_0) - f(x_*)) \\
+ 2 \left( \frac{1}{c_0} - 1 \right) \eta (f(x_{0-1}) - f(x_*)) \\
\leq \frac{1}{\sqrt{\pi}} R + (k + 1)^{-1/2} \eta^2 G^2 - 2 \eta (f(x_0) - f(x_*)).
\]

(38)

Inductive case: consider the case $k \geq 1$. To facilitate telescoping we want $\frac{1}{k^{-1/2}} \|z_k - x_*\|^2$ on the right, so to this end we rewrite:
\[
\frac{1}{(k + 1)^{-1/2}} \|z_k - x_*\|^2 = \frac{1}{k^{-1/2}} \|z_k - x_*\|^2 + \left( \frac{1}{k^{-1/2}} - \frac{1}{(k + 1)^{-1/2}} \right) \|z_k - z_*\|^2 \\
\leq \frac{1}{k^{-1/2}} \|z_k - z_*\|^2 + \left( \frac{1}{k^{-1/2}} - \frac{1}{(k + 1)^{-1/2}} \right) R^2.
\]

Now since $k \geq 1$ we can apply the inverse difference property:
\[
\frac{1}{(k + 1)^{-1/2}} - \frac{1}{k^{-1/2}} = \frac{1}{2 k^{1/2}}.
\]

which gives:
\[
\frac{1}{(k + 1)^{-1/2}} \mathbb{E} \|z_{k+1} - x_*\|^2 \leq \frac{1}{k^{-1/2}} \|z_k - x_*\|^2 + \frac{1}{2 k^{1/2}} R^2 + (k + 1)^{-1/2} \eta^2 G^2 \\
- 2 \frac{1}{c_k} \eta (f(x_k) - f(x_*)) + 2 \left( \frac{1}{c_k} - 1 \right) \eta (f(x_{k-1}) - f(x_*)).
\]

Since $c_k = 1/(k + 1)$ and $\frac{1}{k^{1/2}} = (k + 1)^{-1/2}$ we have that
\[
\frac{1}{(k + 1)^{-1/2}} \mathbb{E} \|z_{k+1} - x_*\|^2 \leq \frac{1}{k^{-1/2}} \|z_k - x_*\|^2 + \frac{1}{2} \left( k + \frac{1}{2} \right)^{-1/2} R^2 + (k + 1)^{-1/2} \eta^2 G^2 \\
- 2(k + 1) \eta (f(x_k) - f(x_*)) + 2k \eta (f(x_{k-1}) - f(x_*)).
\]

(39)
Now taking expectation and adding up both sides of (39) from 1 to n and using telescopic cancellation gives
\[
\frac{1}{(n + 1)^{1/2}} \mathbb{E} \|z_{n+1} - x_*\|^2 \leq \frac{1}{(n + 1)^{1/2}} \mathbb{E} \|z_1 - z_*\|^2
\]
\[
+ \frac{1}{2} R^2 \sum_{i=1}^{n} (i + 1) \frac{1}{2} - \sum_{i=1}^{n} (i + 1) \frac{1}{2} \eta^2 G^2
\]
\[
+ 2\eta (f(x_0) - f(x_*)) - 2(n + 1) \eta \mathbb{E}[f(x_n) - f(x_*)].
\]
Now using the base case (38) we have that
\[
\frac{1}{(n + 1)^{1/2}} \mathbb{E} \|z_{n+1} - x_*\|^2
\]
\[
\leq R^2 + 1 - \frac{1}{\eta^2 G^2} - 2\eta (f(x_0) - f(x_*)) - 2(n + 1) \eta \mathbb{E}[f(x_n) - f(x_*)]
\]
\[
+ \frac{1}{2} R^2 \sum_{i=1}^{n} (i + 1) \frac{1}{2} - \sum_{i=1}^{n} (i + 1) \frac{1}{2} \eta^2 G^2 + 2\eta (f(x_0) - f(x_*))
\]
\[
= \frac{1}{\sqrt{\pi}} R^2 + \frac{1}{2} R^2 \sum_{i=1}^{n} (i + 1) \frac{1}{2} - \sum_{i=1}^{n} (i + 1) \frac{1}{2} \eta^2 G^2 - 2(n + 1) \eta \mathbb{E}[f(x_n) - f(x_*)].
\]
(40)

Using the summation property Eq. (9) we have:
\[
\sum_{i=1}^{n} (i + 1) \frac{1}{2} = 2(n + 1/2)^{1/2} - 2(3/2)^{1/2}
\]
\[
= 2(n + 1)^{1/2} - \frac{4}{\sqrt{\pi}}
\]
\[
\leq 2(n + 1)^{1/2} - \frac{4}{\sqrt{\pi}}
\]
and:
\[
\sum_{i=1}^{n} (i + 1) \frac{1}{2} = 2(n + 1)^{1/2}.
\]

So after dividing by 2(n + 1)\eta:
\[
\mathbb{E} [f(x_n) - f(x_*)] \leq \frac{1}{2} \left( \frac{1}{\eta} R^2 + 2\eta G^2 \right) \frac{2(n + 1)^{1/2}}{n + 1}
\]

We now use the ratio property on:
\[
\frac{(n + 1)^{1/2}}{n + 1} = \frac{(n + 1)^{1-1/2}}{(n + 1)^{1/2}} = (n + 2)^{-1/2},
\]

and solve for the best step size \( \eta \), which is \( \eta = \sqrt{1/2G} \) giving:
\[
\mathbb{E} [f(x_n) - f(x_*)] \leq \sqrt{2} R G (n + 2)^{-1/2}
\]
\[
< \frac{\sqrt{2} R G}{\sqrt{n + 1}}
\]

E.2 Any-time convergence with standard step sizes:

Theorem 21. Let \( f(x, \xi) \) be \( G \)-Lipschitz and convex for every \( \xi \). When \( \eta_k = \frac{R}{G\sqrt{2(k+1)}} \) and \( c_k = \frac{1}{k+1} \) in the projected \( SGD+\Pi \) method [18] we have that
\[
\mathbb{E} [f(x_n) - f(x_*)] \leq \frac{\sqrt{2} RG}{\sqrt{n + 1}}.
\]
(41)
Proof. We use \( \eta_k = \eta / \sqrt{k + 1} \) and \( c_k = \frac{1}{k + 1} \) in the result from Theorem 17, taking expectation and multiplying both sides by \( \sqrt{k + 1} \) gives

\[
\sqrt{k + 1} \mathbb{E} \left[ \|z_{k+1} - x_*\|^2 \right] \leq \sqrt{k + 1} \|z_k - x_*\|^2 + \frac{1}{\sqrt{k + 1}} \eta^2 G^2 - 2(k + 1)\eta \mathbb{E} [f(x_k) - f(x_*)] + 2k\eta \mathbb{E} [f(x_{k-1}) - f(x*)]. \tag{42}
\]

For \( k = 0 \) the above gives

\[
\mathbb{E} \|z_1 - x_*\|^2 \leq R^2 + \eta^2 G^2 - 2\eta \mathbb{E} [f(x_0) - f(x*)]. \tag{43}
\]

For \( k \geq 1 \), from concavity of the square root function

\[
\sqrt{k + 1} - \sqrt{k} \leq \frac{1}{2\sqrt{k}}, \tag{44}
\]

we have that

\[
\sqrt{k + 1} \|z_k - x_*\|^2 \leq \left( \sqrt{k + 1} + \frac{1}{2\sqrt{k}} \right) \|z_k - x_*\|^2 \leq \sqrt{k} \|z_k - x_*\|^2 + \frac{1}{2\sqrt{k}} R^2.
\]

Plugging the above into (42) gives

\[
\sqrt{k + 1} \mathbb{E} \|z_{k+1} - x_*\|^2 \leq \sqrt{k} \|z_k - x_*\|^2 + \left( \frac{1}{2\sqrt{k}} \right) R^2 + \frac{1}{\sqrt{k + 1}} \eta^2 G^2 - 2(k + 1)\eta \mathbb{E} [f(x_k) - f(x_*)] + 2k\eta \mathbb{E} [f(x_{k-1}) - f(x*)].
\]

Now we telescope for \( 1 \) to \( n \) giving:

\[
\sqrt{n + 1} \mathbb{E} \|z_{n+1} - x_*\|^2 \leq \|z_1 - x_*\|^2 + \sum_{i=1}^{n} \left( \frac{1}{2\sqrt{i}} \right) R^2 + \sum_{i=1}^{n} \frac{1}{\sqrt{i + 1}} \eta^2 G^2 - 2(n + 1)\eta \mathbb{E} [f(x_n) - f(x_*)] + 2\eta \mathbb{E} [f(x_0) - f(x*)].
\]

Using the base case (43) we have that

\[
\sqrt{n + 1} \mathbb{E} \|z_{n+1} - x_*\|^2 \leq R^2 + \sum_{i=1}^{n} \left( \frac{1}{2\sqrt{i}} \right) R^2 + \sum_{i=0}^{n} \frac{1}{\sqrt{i + 1}} \eta^2 G^2 - 2(n + 1)\eta \mathbb{E} [f(x_n) - f(x_*)].
\]

Now using the integral bounds

\[
\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2(\sqrt{n} - 1),
\]

\[
\sum_{i=0}^{n} \frac{1}{\sqrt{i + 1}} \leq 2\sqrt{n + 1},
\]

and re-arranging gives

\[
2(n + 1)\eta \mathbb{E} [f(x_n) - f(x_*)] \leq \sqrt{n} R^2 + \frac{1}{\sqrt{n + 1}} \eta G^2 \leq \sqrt{n} R^2 + 2\sqrt{n + 1} \mathbb{E} \|z_{n+1} - x_*\|^2.
\]

Dividing through by \( 2(n + 1)\eta \) gives

\[
2(n + 1)\eta \mathbb{E} [f(x_n) - f(x_*)] \leq \frac{\sqrt{n}}{2(n + 1)\eta} R^2 + \frac{1}{\sqrt{n + 1}} \eta G^2 \leq \frac{1}{\sqrt{n + 1}} \left( \frac{1}{2\eta} R^2 + \eta G^2 \right).
\]

Minimizing the above in \( \eta \) gives \( \eta = R / (\sqrt{2}G) \) which gives (44) and concludes the proof. \( \square \)
F Strongly convex convergence

Consider again the SGD+M method with a projection step given by
\[
    z_{k+1} = \Pi_C \left( z_k - \eta_k \nabla f(x_k, \xi_k) \right),
\]
\[
    x_{k+1} = (1 - c_{k+1}) x_k + c_{k+1} z_{k+1}.    
\]

Lemma 22. For \(\lambda_{k+1} = \frac{k+2}{2}\) and \(c_{k+1} = \frac{4}{k+4}\):
\[
    A_{k+1} : = \|x_{k+1} - x_* + \lambda_{k+1} (x_{k+1} - x_k)\|^2 = \|2z_{k+1} - x_k - x_*\|^2
\]

Proof. The relation follows from substitution of the known relations:
\[
    A_{k+1} = \|x_{k+1} - x_* + \lambda_{k+1} (x_{k+1} - x_k)\|^2
    = \| (\lambda_{k+1} + 1) x_{k+1} - \lambda_{k+1} x_k - x_* \|^2
    = \| (\lambda_{k+1} + 1) (1 - c_{k+1}) x_k + c_{k+1} z_{k+1} - \lambda_{k+1} x_k - x_* \|^2
    = \| (\lambda_{k+1} + 1) (1 - c_{k+1}) x_k + c_{k+1} z_{k+1} + \left[ (\lambda_{k+1} + 1) (1 - c_{k+1}) - \lambda_{k+1} \right] x_k - x_* \|^2
    = \| (\lambda_{k+1} + 1) (1 - c_{k+1}) z_{k+1} + \left[ (\lambda_{k+1} - \lambda_{k+1} c_{k+1} + 1 - c_{k+1}) x_k - \lambda_{k+1} x_k \right] - x_* \|^2
    = \| (\lambda_{k+1} + 1) c_{k+1} z_{k+1} + \left[ (1 - (\lambda_{k+1} + 1) c_{k+1}) x_k \right] - x_* \|^2. 
\]

Now using:
\[
    (\lambda_{k+1} + 1) c_{k+1} = \left( \frac{k + 2}{2} + 1 \right) \frac{4}{k + 4} = \frac{k + 4}{2} \frac{4}{k + 4} = 2,
\]
we have:
\[
    A_{k+1} = \|2z_{k+1} - x_k - x_*\|^2. 
\]
\(\square\)

F.1 Proof of Theorem 23

Theorem 23. Let \(f(x, \xi)\) be \(G\)-Lipschitz and \(\mu\)--strongly convex in \(x\) for every \(\xi\). The projected SGD+M method (17) with \(\eta_k = \frac{1}{\mu(k+1)}\) and \(c_{k+1} = \frac{4}{k+4}\) satisfies
\[
    E \left[ f(x_n) - f(x_*) \right] \leq \frac{2G^2}{\mu(n+1)}.
\]

Proof. We will define a few constants to reduce notational clutter. Let
\[
    \rho_k = \frac{k - 1}{k + 1}, \text{ and } \lambda_{k+1} = \frac{k + 2}{2}.
\]

We will first apply the contraction property of the projection operator (using the fact that \(x_k\) and \(x_*\) are always within the constraint set) so that
\[
    A_{k+1} = \|2z_{k+1} - x_k - x_*\|^2
    = 4 \left\| \Pi_C \left( z_k - \eta_k \nabla f(x_k, \xi_k) \right) - \left( \frac{1}{2} x_k + \frac{1}{2} x_* \right) \right\|^2
    = 4 \left\| \Pi_C \left( z_k - \eta_k \nabla f(x_k, \xi_k) \right) - \Pi_C \left( \frac{1}{2} x_k + \frac{1}{2} x_* \right) \right\|^2
    \leq \|2z_k - 2\eta_k \nabla f(x_k, \xi_k) - x_k - x_*\|^2. 
\]
Now we use $z_k = \frac{1}{c_k} x_k - \left(\frac{1}{c_k} - 1\right) x_{k-1}$:

$$A_{k+1} \leq \left\| \frac{2}{c_k} x_k - 2 \left(\frac{1}{c_k} - 1\right) x_{k-1} - 2\eta_k \nabla f(x_k, \xi_k) - x_* \right\|^2$$

$$= \left\| 2 \left(\frac{1}{c_k} - 1\right) x_k - 2 \left(\frac{1}{c_k} - 1\right) x_{k-1} + x_k - 2\eta_k \nabla f(x_k, \xi_k) - x_* \right\|^2$$

$$= \left\| x_k - 2\eta_k \nabla f(x_k, \xi_k) - x_* \right\|^2 + 4 \left(\frac{1}{c_k} - 1\right)^2 \left\| x_k - x_{k-1} \right\|^2$$

$$+ 4 \left(\frac{1}{c_k} - 1\right) \langle x_k - x_{k-1}, x_k - x_* \rangle - 4\eta_k \left(\frac{1}{c_k} - 1\right) \langle \nabla f(x_k, \xi_k), x_k - x_* \rangle.$$

Now from Lemma 22 $A_k = \left\| x_k - x_* + \lambda_k (x_k - x_{k-1}) \right\|^2$ so:

$$4 \left(\frac{1}{c_k} - 1\right) \langle x_k - x_{k-1}, x_k - x_* \rangle = 2 \left(\frac{1}{\lambda_k} - 1\right) A_k - \frac{2}{\lambda_k} \left(\frac{1}{c_k} - 1\right) \left\| x_k - x_* \right\|^2 - 2\lambda_k \left(\frac{1}{c_k} - 1\right) \left\| (x_k - x_{k-1}) \right\|^2. \quad (45)$$

Notice that:

$$2 \left(\frac{1}{\lambda_k} \left(\frac{1}{c_k} - 1\right) \right) = 2 \frac{2}{k+1} \left(\frac{k+3}{4} - 1\right)$$

$$= \frac{1}{k+1} (k + 3 - 4)$$

$$= \frac{k-1}{k+1} = \rho_k.$$

So we have:

$$A_{k+1} = \left\| x_k - 2\eta_k \nabla f(x_k, \xi_k) - x_* \right\|^2 + \left(4 \left(\frac{1}{c_k} - 1\right) - 2\lambda_k \right) \left(\frac{1}{c_k} - 1\right) \left\| x_k - x_{k-1} \right\|^2$$

$$= \rho_k A_k - \rho_k \left\| x_k - x_* \right\|^2 - 8\eta_k \left(\frac{1}{c_k} - 1\right) \langle \nabla f(x_k, \xi_k), x_k - x_{k-1} \rangle.$$

Now note that:

$$4 \left(\frac{1}{c_k} - 1\right) - 2\lambda_k = 4 \left(\frac{k+3}{4} - 1\right) - 2 \frac{k+1}{2}$$

$$= (k-1) - 2 \frac{k+1}{2} \leq 0.$$

Further expanding $\left\| x_k - 2\eta_k \nabla f(x_k, \xi_k) - x_* \right\|^2$ and rearranging then gives

$$A_{k+1} = \rho_k A_k + (1 - \rho_k) \left\| x_k - x_* \right\|^2 + 4\eta_k^2 \left\| \nabla f(x_k, \xi_k) \right\|^2$$

$$= -4\eta_k \langle \nabla f(x_k, \xi_k), x_k - x_* \rangle - 8\eta_k \left(\frac{1}{c_k} - 1\right) \langle \nabla f(x_k, \xi_k), x_k - x_{k-1} \rangle.$$

We now apply the two inequalities:

$$- \langle \nabla f(x_k, \xi_k), x_k - x_* \rangle \leq - \left[ f(x_k, \xi_k) - f(x_*, \xi_k) \right] - \frac{\mu}{2} \left\| x_k - x_* \right\|^2,$$

$$- \langle \nabla f(x_k, \xi_k), x_k - x_{k-1} \rangle \leq f(x_{k-1}, \xi_k) - f(x_k, \xi_k),$$

which gives:

$$A_{k+1} = \rho_k A_k + (1 - \rho_k - 2\mu\eta_k) \left\| x_k - x_* \right\|^2 + 4\eta_k^2 \left\| \nabla f(x_k, \xi_k) \right\|^2$$

$$= -4\eta_k \left[ f(x_k, \xi_k) - f(x_*, \xi_k) \right] + 8\eta_k \left(\frac{1}{c_k} - 1\right) \left[ f(x_{k-1}, \xi_k) - f(x_k, \xi_k) \right].$$
Taking expectations and using $\mathbb{E}_x \| \nabla f (x_k, \xi_k) \|^2 \leq G^2$ gives:

$$
\mathbb{E} A_{k+1} = \rho_k A_k + (1 - \rho_k - 2\mu \eta_k) \| x_k - x_* \|^2 + 4\eta_k^2 G^2
$$

$$
= -4\eta_k \left[ f(x_k) - f(x_*) \right] + 8\eta_k \left( \frac{1}{c_k} - 1 \right) \left[ f(x_{k-1}) - f(x_k) \right].
$$

Further grouping of function value terms gives:

$$
\mathbb{E} A_{k+1} = \rho_k A_k + (1 - \rho_k - 2\mu \eta_k) \| x_k - x_* \|^2 + 4\eta_k^2 G^2
$$

$$
= - \left( 8\eta_k \left( \frac{1}{c_k} - 1 \right) + 4\eta_k \right) \left[ f(x_k) - f(x_*) \right] + 8\eta_k \left( \frac{1}{c_k} - 1 \right) \left[ f(x_{k-1}) - f(x_*) \right].
$$

Now we simplify constants, recalling that $\rho_k = \frac{k-1}{k+1}$ and $c_k = \frac{4}{k+3}$:

$$
8\eta_k \left( \frac{1}{c_k} - 1 \right) = 2 \frac{4}{\mu(k+1)} \left( \frac{k+3}{4} - 1 \right)
$$

$$
= \frac{2}{\mu} \frac{1}{k+1} (k-1)
$$

$$
= \rho_k \frac{2}{\mu},
$$

using this we have:

$$
8\eta_k \left( \frac{1}{c_k} - 1 \right) + 4\eta_k = \frac{2}{\mu} \frac{k-1}{k+1} + 4 \frac{1}{\mu(k+1)}
$$

$$
= \frac{2}{\mu} \frac{k-1+2}{k+1}
$$

$$
= \frac{2}{\mu}.
$$

Also note that:

$$
1 - \rho_k - 2\mu \eta_k = 1 - \frac{k-1}{k+1} - \frac{2\mu}{\mu(k+1)}
$$

$$
= 1 - \frac{k+1-2}{k+1} - \frac{2}{k+1}
$$

$$
= 0.
$$

So we have:

$$
\mathbb{E} A_{k+1} + \frac{2}{\mu} \left[ f(x_k) - f(x_*) \right] = \rho_k \left[ A_k + \frac{2}{\mu} f(x_{k-1}) - f(x_*) \right] + 4\eta_k^2 G^2.
$$

Based on the form of this equation, we have a Lyapunov function

$$
B_{k+1} = A_{k+1} + \frac{2}{\mu} \left[ f(x_k) - f(x_*) \right],
$$

then:

$$
\mathbb{E} B_{k+1} \leq \rho_k B_k + 4\eta_k^2 G^2,
$$

with $\rho_k$ descent plus noise. To finish the proof, we multiply by $k(k+1)$ and simplify the last term:

$$
(k+1) k \mathbb{E}[B_{k+1}] \leq k (k-1) B_k + \frac{4}{\mu^2} G^2.
$$

We now telescope from $k = 1$ to $n$, using the law of total expectation:

$$
(n+1) n \mathbb{E}[B_{n+1}] \leq \frac{4n}{\mu^2} G^2,
$$

$$
\therefore \mathbb{E} \left[ f(x_n) - f(x_*) \right] \leq \frac{2G^2}{\mu(n+1)}.
$$

$\square$
G Accelerated method

Consider the following iterate averaging form of Nesterov’s method:

\[ y_k = (1 - \theta_k) x_k + \theta_k z_k \]
\[ z_{k+1} = z_k - \frac{(k + 1)}{\gamma L} \nabla f(y_k) \]
\[ x_{k+1} = (1 - \theta_k) x_k + \theta_k z_{k+1} \]

with \( z_0 = x_0 \). Note the following two key relations, that can be derived by rearranging the above relations:

\[ z_k = y_k - \left( \frac{1}{\theta_k} - 1 \right) (x_k - y_k) \]
\[ x_{k+1} - y_k = \theta_k (z_{k+1} - z_k) \]

**Lemma 24.** Each step of Nesterov’s accelerated method obeys:

\[ -f(y_k) \leq -f(x_{k+1}) - 2L \left( \frac{\gamma}{(k + 1)^2} - \frac{1}{(k + 2)^2} \right) \| z_{k+1} - z_k \|^2. \]

**Proof.** We start with the Lipschitz smoothness upper bound:

\[ f(x_{k+1}) \leq f(y_k) + \langle \nabla f(y_k), x_{k+1} - y_k \rangle + \frac{L}{2} \| x_{k+1} - y_k \|^2, \]

\[ \therefore -f(y_k) \leq -f(x_{k+1}) + \langle \nabla f(y_k), x_{k+1} - y_k \rangle + \frac{L}{2} \| x_{k+1} - y_k \|^2. \]

Now using \( x_{k+1} - y_k = \theta_k (z_{k+1} - z_k) \) and \( \nabla f(y_k) = -L \gamma / (k + 1) (z_{k+1} - z_k) \), so:

\[ \therefore -f(y_k) \leq -f(x_{k+1}) - \frac{L \gamma}{\alpha (k + 1)} \langle (z_{k+1} - z_k), \theta_k (z_{k+1} - z_k) \rangle + \frac{L}{2} \| \theta_k (z_{k+1} - z_k) \|^2. \]

Note that \( \theta_k^2 = \frac{4}{(k + 2)^2} \) so:

\[ -f(y_k) \leq -f(x_{k+1}) - \frac{L}{k + 1} \frac{2}{(k + 2)} \| z_{k+1} - z_k \|^2 + \frac{L}{2} \frac{4}{(k + 2)^2} \| z_{k+1} - z_k \|^2. \]

Grouping terms gives the lemma. \( \square \)

**Proof of Theorem**

**Theorem 25.** Using \( \gamma = 1/2 \) the iterate averaging form of Nesterov’s method obeys

\[ f(x_n) - f(x_*) \leq \frac{2L}{n^2} \| x_0 - x_* \|^2 \]

**Proof.** We start by expanding a distance to solution term:

\[ \| z_{k+1} - x_* \|^2 = \| z_k - x_* - (z_k - z_{k+1}) \|^2 \]
\[ = \| z_k - x_* \|^2 - 2(k + 1) \frac{1}{\gamma L} \langle \nabla f(y_k), z_k - x_* \rangle + \| z_{k+1} - z_k \|^2. \]

Simplifying the inner product term:

\[ -2(k + 1) \frac{1}{\gamma L} \langle \nabla f(y_k), z_k - x_* \rangle = -2(k + 1) \frac{1}{\gamma L} \langle \nabla f(y_k), y_k - \left( \frac{1}{\theta_k} - 1 \right) (x_k - y_k) - x_* \rangle \]
\[ = -2(k + 1) \frac{1}{\gamma L} \langle \nabla f(y_k), y_k - x_* \rangle \]
\[ - 2(k + 1) \frac{1}{\gamma L} \left( \frac{1}{\theta_k} - 1 \right) \langle \nabla f(y_k), y_k - x_* \rangle \]

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Then we apply the inequalities:

\[ -\langle \nabla f(y_k), y_k - x_* \rangle \leq f(x_*) - f(y_k), \]
\[ -\langle \nabla f(y_k), y_k - x_k \rangle \leq f(x_k) - f(y_k). \]

So we have

\[ \|z_{k+1} - x_*\|^2 \leq \|z_k - x_*\|^2 + \|z_{k+1} - z_k\|^2 - 2(k + 1) \frac{1}{\gamma L} [f(y_k) - f(x_*)] + 2(k + 1) \frac{1}{\gamma L} [f(x_k) - f(y_k)]. \]

Now rearranging the function value terms:

\[ \|z_{k+1} - x_*\|^2 \leq \|z_k - x_*\|^2 + \|z_{k+1} - z_k\|^2 - (k + 1) \frac{1}{\gamma L} [f(x_k) - f(x_*)] + (k + 1) \frac{1}{\gamma L} [f(x_k) - f(x_*)]. \]

Now we use Lemma 24 on \(-f(y_k)\):

\[ -(k + 1) \frac{1}{\gamma L} f(y_k) \leq -(k + 1) \frac{1}{\gamma L} f(x_{k+1}) - 2 \left( 1 - \frac{k + 1}{\gamma L} \right) \|z_{k+1} - z_k\|^2, \]

giving:

\[ \|z_{k+1} - x_*\|^2 \leq \|z_k - x_*\|^2 + \left( 1 - 2 \left( 1 - \frac{k + 1}{\gamma L} \right) \right) \|z_{k+1} - z_k\|^2 - (k + 1) \frac{1}{\gamma L} [f(x_{k+1}) - f(x_*)] + (k + 1) \frac{1}{\gamma L} [f(x_k) - f(x_*)]. \]

When \( \gamma = 2 \) then \(-2 \left( 1 - \frac{k + 1}{2} \right) \leq -1 \) so

\[ \|z_{k+1} - x_*\|^2 + (k + 1) \frac{1}{\gamma L} [f(x_{k+1}) - f(x_*)] \leq \|z_k - x_*\|^2 + (k + 1) \frac{1}{\gamma L} [f(x_k) - f(x_*)]. \]

Now we apply Lemma 13 to give a telescopic sum:

\[ \|z_{k+1} - x_*\|^2 + (k + 1) \frac{1}{\gamma L} [f(x_{k+1}) - f(x_*)] \leq \|z_k - x_*\|^2 + k \frac{1}{\gamma L} [f(x_k) - f(x_*)]. \]

After telescoping:

\[ f(x_n) - f(x_*) \leq \frac{2L}{n^2} \|x_0 - x_*\|^2. \]

\[ \square \]

H \quad SVRG+M

Lemma 26. \cite{Johnson and Zhang, 2013} The following bound holds for \( g_t^* \) at each step:

\[ \mathbb{E} \|g_t^*\|^2 \leq 4L \|f(x_t^*) - f(x_*)\| + 4L \|f(\tilde{x}^{s-1}) - f(x_*)\|. \]

H.1 Proof of Theorem 5 (Convex Case)

Theorem 27. At the end of epoch \( S \), when using \( r = 1/2 \) factorial power momentum given by

\[ c_t = \frac{1/2 + 1}{t + 1/2 + 1}, \]

and step size \( \eta = \frac{1}{m^*} \), the expected function value is bounded by:

\[ \mathbb{E} [f(x_m^S) - f(x_*)] \leq \frac{1}{2^S} [f(x_0) - f(x_*)] + \frac{9L \|x_0 - x_*\|^2}{2^S m_0}. \]
Proof. We start in the same fashion as for non-variance reduced momentum methods:

\[ \mathbb{E} \| z_{t+1}^s - x_*^s \|^2 = \mathbb{E} \| z_t^s - \eta g_t^s - x_*^s \|^2 \]

\[ = \| z_t^s - x_*^s \|^2 - 2\eta \langle \nabla f(x_t^s), z_t^s - x_*^s \rangle + \eta^2 \mathbb{E} \| g_t^s \|^2 \]

\[ = \| z_t^s - x_*^s \|^2 - 2\eta \left( \langle \nabla f(x_t^s), x_t^s - \left( \frac{1}{c_t} - 1 \right) (x_{t-1}^s - x_t^s) \rangle \right) + \eta^2 \mathbb{E} \| g_t^s \|^2 \]

\[ - 2\eta \langle \nabla f(x_t^s), x_t^s - x_*^s \rangle - 2\eta \left( \frac{1}{c_t} - 1 \right) \langle \nabla f(x_t^s), x_t^s - x_{t-1}^s \rangle. \]

Using the following two convexity inequalities

\[ \langle \nabla f(x_t^s), x_*^s - x_t^s \rangle \leq f(x_*^s) - f(x_t^s) \]

\[ \langle \nabla f(x_t^s), x_{t-1}^s - x_t^s \rangle \leq f(x_{t-1}^s) - f(x_t^s) \]

combined with \((1/c_t - 1) \geq 0\) gives

\[ \mathbb{E} \| z_{t+1}^s - x_*^s \|^2 \leq \| z_t^s - x_*^s \|^2 + \eta^2 \mathbb{E} \| g_t^s \|^2 \]

\[ - 2\eta \left( f(x_t^s) - f(x_*^s) \right) - 2\eta \left( \frac{1}{c_t} - 1 \right) \left( f(x_t^s) - f(x_{t-1}^s) \right). \]

Now rearranging further:

\[ \mathbb{E} \| z_{t+1}^s - x_*^s \|^2 \leq \| z_t^s - x_*^s \|^2 + \eta^2 \mathbb{E} \| g_t^s \|^2 \]

\[ - 2\eta \left( \frac{1}{c_t} - 1 \right) \left( f(x_t^s) - f(x_{t-1}^s) \right) + 2\eta \left( \frac{1}{c_t} - 1 \right) \left( f(x_t^s) - f(x_*^s) \right). \]

Now using Lemma [26]

\[ \mathbb{E} \| z_{t+1}^s - x_*^s \|^2 \leq \| z_t^s - x_*^s \|^2 + 4\eta^2 \mathbb{E} \left( f(\hat{x}^{s-1}) - f(x_*) \right) \]

\[ - 2\eta \left( \frac{1}{c_t} - 2\eta L \right) \left( f(x_t^s) - f(x_*^s) \right) + 2\eta \left( \frac{1}{c_t} - 1 \right) \left( f(x_t^s) - f(x_{t-1}^s) \right). \]

Now for the purposes of telescoping, define \( \lambda_t = p(t+1) \), we want

\[ \frac{1}{c_t} - 2\eta L = p(t + 1) \]

\[ \frac{1}{c_t} - 1 = pt \]

These equations are satisfied for \( p = 1 - 2L\eta = \frac{2}{3} \), when \( \eta = \frac{1}{6L} \) and:

\[ c_t = \frac{1}{pt+1} = \frac{1/2 + 1}{t + 1/2 + 1} \]

This corresponds to \( r = 1/2 \) factorial power momentum. So we have:

\[ \mathbb{E} \| z_{t+1}^s - x_*^s \|^2 \leq \| z_t^s - x_*^s \|^2 + \frac{1}{9L} \left( f(\hat{x}^{s-1}) - f(x_*) \right) \]

\[ - \frac{2}{9L} \left( t + 1 \right) \left( f(x_t^s) - f(x_*^s) \right) + \frac{2}{9L} \left( f(x_{t-1}^s) - f(x_*^s) \right). \]

We now telescope from \( t = 0 \) to \( t = m_s - 1 \), using the law of total expectation (i.e. \( \mathbb{E} [\mathbb{E} [X|Y]] = \mathbb{E} [X] \)), so that this expectation is unconditional:

\[ \mathbb{E} \| z_{m_s - 1}^s - x_*^s \|^2 \leq \| z_0^s - x_*^s \|^2 + \frac{m_s}{9L} \left( f(\hat{x}^{s-1}) - f(x_*^s) \right) \]

\[ - \frac{2m_s}{9L} \left( f(x_t^s) - f(x_*^s) \right). \]
Which we can write as:

\[
\frac{9L}{2m_s} \mathbb{E} \left\| z_{m_s-1}^s - x_* \right\|^2 + [f(x_t^s) - f(x_*)] \leq \frac{9L}{2m_s} \left\| z_0^s - z_* \right\|^2 + \frac{1}{2} \left[ f\left(\tilde{x}_{m_s-1}^s\right) - f\left(x_*\right)\right]
\]

Noting that the choice \( z_0^s = z_{m_s-1}^{s-1} \) and \( m_s = 2m_{s-1} \) gives:

\[
\frac{\left\| z_0^s - x_* \right\|^2}{m_s} = \frac{1}{2} \frac{\left\| z_{m_s-1}^{s-1} - x_* \right\|^2}{m_{s-1}}
\]

So we may form the Lyapunov function:

\[
B^s = \frac{9L}{2m_s} \mathbb{E} \left\| z_{m_s-1}^s - x_* \right\|^2 + [f(x_t^s) - f(x_*)]
\]

which gives the simple relation:

\[
\mathbb{E} [B^s] \leq \frac{1}{2} \mathbb{E} [B^{s-1}].
\]

So after \( S \) epochs we have:

\[
\mathbb{E} [B^S] \leq 2^{-S} B^O.
\]

and so:

\[
\mathbb{E} [f(x_{m_s-1}^s) - f(x_*)] \leq \frac{1}{2^S} \left[ f(0) - f(x_*) \right] + \frac{9L \left\| x_0 - x_* \right\|^2}{2^S m_0} \]

\[\square\]

### H.2 Proof of Theorem (Strongly Convex Case)

**Theorem 28.** When each \( f_i \) is strongly convex with constant \( \mu \), we may use \( m, c, \eta \) constants that don’t depend on the step. In particular, after epoch \( s \), when \( m = 6k \) and \( c = \frac{5}{3 \mu + \gamma} \), and \( \eta = 1/(10L) \):

\[
\mathbb{E} [B^s] \leq \frac{6}{10} B^{s-1},
\]

where:

\[
B^s = \mathbb{E} [f(x^s) - f(x_*)] + \frac{3}{4} \mu \left\| x_{m_s}^s - x_* + \lambda \left( x_{m_s}^s - x_{m_s-1}^s \right) \right\|^2.
\]

**Proof.** We can use the same proof technique as we applied in the non-variance reduced case to deduce the following 1-step bound:

\[
\mathbb{E} A_{t+1}^s \leq (1 - \rho - \mu \nu) \left\| x_t^s - x_* \right\|^2
\]

\[
+ \rho A_t + 4L \nu^2 \left[ f(\tilde{x}_{m_s-1}^s) - f(x_*) \right]
\]

\[
- 2\nu \left( 1 + \rho \lambda - 2L \nu \right) \left[ f(x_t^s) - f(x_*) \right] + 2\rho \lambda \nu \left[ f(x_{m_s-1}^s) - f(x_*) \right]
\]

Where:

\[
\rho = \frac{(\lambda + 1) \beta}{\lambda},
\]

\[
\nu = (\lambda + 1) \alpha,
\]

We need \( 1 - \rho - \mu \nu \leq 0 \), which suggests for step sizes of the form \( \nu = 1/(qL) \),

\[
\rho = 1 - \mu \nu = 1 - \frac{1}{qk}
\]

Now in order to see \( \rho \) decrease in function value each step, we will require:

\[-2\nu \left( 1 + \rho \lambda - 2L \nu \right) \leq -2\nu,
\]

so solving at equality gives:

\[
1 + \rho \lambda - 2L \nu = \lambda,
\]

\[
\therefore 1 - 2L \nu = (1 - \rho) \lambda,
\]

26
\[ \lambda = \frac{1 - 2/q}{1/q \kappa} = (q - 2) \kappa \]

This gives:
\[ 2\lambda \nu = 2 \frac{(q - 2) \kappa}{q} \frac{1}{qL} = \frac{2}{\mu} \left(1 - \frac{2}{q}\right) \]

Making these substitutions, our one-step bound can be written as:
\[
E A^s_{m+1} + \frac{2}{\mu} \left(1 - \frac{2}{q}\right) [f(x^s_m) - f(x^s_0)] \leq \rho^m A^s_0 + \rho^m \frac{2}{\mu} \left(1 - \frac{2}{q}\right) [f(\hat{x}^{s-1}) - f(x^s_0)]
\]
\[
+ \frac{4}{q^2L} [f(\hat{x}^{s-1}) - f(x^s_0)].
\]

We can now telescope using the sum of a geometric series \(\sum_{i=0}^{k-1} \rho^j = \frac{1 - \rho^k}{1 - \rho}\) and the law of total expectation to give:
\[
E A^s_{m+1} + \frac{2}{\mu} \left(1 - \frac{2}{q}\right) [f(x^s_m) - f(x^s_0)] \leq \rho^m A^s_0 + \rho^m \frac{2}{\mu} \left(1 - \frac{2}{q}\right) [f(\hat{x}^{s-1}) - f(x^s_0)]
\]
\[
+ \frac{1 - \rho^m}{1 - \rho} \frac{4}{q^2L} [f(\hat{x}^{s-1}) - f(x^s_0)].
\]

These expectations are now unconditional. Now multiplying by \(\mu/2\), simplifying with \(1 - \rho = \frac{1}{q\kappa}\) gives:
\[
\frac{\mu}{2} E A^s_{m+1} + \left(1 - \frac{2}{q}\right) [f(x^s_m) - f(x^s_0)]
\]
\[
\leq \rho^m \frac{\mu}{2} A^s_0 + \left(\rho^m \left(1 - \frac{2}{q}\right) + \frac{2}{q} (1 - \rho^m)\right) [f(\hat{x}^{s-1}) - f(x^s_0)]
\]

Dividing by \(\left(1 - \frac{2}{q}\right)\):
\[
\frac{\mu}{2q - 2} E A^s_{m+1} + [f(x^s_m) - f(x^s_0)]
\]
\[
\leq \rho^m \frac{\mu}{2q - 2} A^s_0 + \left(\rho^m + \frac{2}{q - 2} (1 - \rho^m)\right) [f(\hat{x}^{s-1}) - f(x^s_0)]
\]

Now we can try \(q = 6\) for instance, giving
\[
\rho^m + \frac{2}{q - 2} (1 - \rho^m) = \rho^m + \frac{1}{2} (1 - \rho^m) = \frac{1}{2} \rho^m + \frac{1}{2}
\]

Then if we use \(m = 6\kappa\) we get \(\rho^m \leq \exp(-1) \leq 2/5\) for \(m = 6\) to give:
\[
\frac{3}{4} \mu E A^s_{m+1} + [f(x^s_m) - f(x^s_0)] \leq \frac{6}{10} \left[\frac{3}{4} \mu A^s_0 + [f(\hat{x}^{s-1}) - f(x^s_0)]\right]
\]

Then we may determine the momentum and step size constants \(\alpha, \beta\):
\[
\beta = \frac{\lambda}{\lambda + 1} \rho = \frac{(6 - 2) \kappa}{(6 - 2) \kappa + 1} \left(1 - \frac{1}{6\kappa}\right)
\]
\[
= \frac{4\kappa}{4\kappa + 1} \left(\frac{6\kappa - 1}{6\kappa}\right)
\]
\[
= 2 \frac{6\kappa - 1}{3(4\kappa + 1)}
\]
\[
= 4\kappa - 2/3
\]
\[
= \frac{4\kappa + 1}{4\kappa + 1}
\]
\[
= 1 - \frac{5/3}{4\kappa + 1}
\]
\[ \alpha = \frac{\nu}{\lambda + 1} = \frac{1}{6L} \frac{1}{4\kappa + 1}. \]

To write in iterate averaging form, we have \( \beta = 1 - c \) and
\[ c = \frac{5}{3} \frac{1}{4\kappa + 1}. \]

from \( \alpha_k = \eta c \) we get for \( \eta \):
\[ \eta = \frac{\frac{1}{6L} \frac{1}{4\kappa + 1}}{\frac{5}{3} \frac{1}{4\kappa + 1}} = \frac{1}{10L}. \]

\[ \lambda \]

\section{Dual averaging}

Let:
\[ \delta_n = \max_{x, \|x\| \leq R} \left\{ \sum_{i=0}^{n} \langle \nabla f(x_i), x_i - x \rangle \right\}. \]

\textbf{Theorem 29.} after \( n \) steps of the dual averaging method with \( \hat{\beta}_k \) given by Nesterov’s recursive sequence, and using \( \gamma = \frac{G}{\sqrt{2R}} \):
\[ \frac{1}{k+1} \delta_{k+1} \leq \left( \frac{\sqrt{2}}{(1 + \sqrt{3}) (k + 1)} + \frac{2}{\sqrt{k+1}} \right) RG. \]

\textbf{Proof.} \cite{Nesterov2009} establishes the following bound:
\[ \delta_k \leq \gamma \hat{\beta}_{k+1} R^2 + \frac{1}{2} G^2 \sum_{i=0}^{k} \frac{1}{\hat{\beta}_i}. \]

The \( \hat{\beta}_i \) sequence given in \cite{Nesterov2009} satisfies \( \sum_{i=0}^{k} \frac{1}{\hat{\beta}_i} = \hat{\beta}_{k+1} \) and \( \hat{\beta}_{k+1} \leq \frac{1}{1 + \sqrt{3}} + \sqrt{2k + 1} \) so we have:
\[ \delta_k \leq \left( \gamma R^2 + \frac{1}{2} G^2 \frac{1}{\gamma} \right) \left( \frac{1}{1 + \sqrt{3}} + \sqrt{2k + 1} \right) . \]

The optimal step size is \( \gamma = \frac{G}{\sqrt{2R}} \) So:
\[ \delta_k \leq \left( \frac{1}{\sqrt{2}} RG + \frac{1}{\sqrt{2}} RG \right) \left( \frac{1}{1 + \sqrt{3}} + \sqrt{2k + 1} \right) \]
\[ \delta_k \leq RG \left( \frac{\sqrt{2}}{1 + \sqrt{3}} + \sqrt{4k + 2} \right) . \]

Using the concavity of the square-root function:
\[ \sqrt{4k + 2} \leq \sqrt{4k + 4} + \frac{1}{2} \left[ 4k + 2 - 4k - 4 \right] \]
\[ = \sqrt{4k + 4} - \frac{1}{\sqrt{4k + 4}} \]

We need to normalize this quantity by \( 1/(k + 1) \), so we have:
\[ \frac{\sqrt{4k + 2}}{k + 1} \leq \frac{\sqrt{4k + 4}}{k + 1} - \frac{1}{2(k + 1)^{3/2}} \]
\[ \leq \frac{2}{\sqrt{k + 1}} . \]

Therefore the bound on the normalization of \( \delta \) is:
\[ \frac{1}{k + 1} \delta_{k+1} \leq \left( \frac{\sqrt{2}}{(1 + \sqrt{3}) (k + 1)} + \frac{2}{\sqrt{k+1}} \right) RG. \]
Factorial power

Theorem 30. after $n$ steps of the dual averaging method with $\hat{\beta}_k = 1/(k+1)^{-1/2}$ and $\gamma = G/R$

$$\frac{1}{k+1} \delta_{k+1} \leq 2RG(n + 2)^{-1/2} < \frac{2RG}{\sqrt{n + 1}}.$$ 

Proof. Recall the bound:

$$\delta_k \leq \gamma \hat{\beta}_{k+1} R^2 + \frac{1}{2} G^2 \frac{1}{\gamma} \sum_{i=0}^{k} \frac{1}{\hat{\beta}_i}.$$ 

We use $\hat{\beta}_i = 1/(i+1)^{-1/2}$ the sum is:

$$\sum_{i=0}^{k} \frac{1}{\hat{\beta}_i} = \frac{1}{1-1/2} (k + 1)^{1/2} - \frac{1}{1-1/2} (1)^{1/2}.$$ 

Recall also that:

$$\hat{\beta}_{k+1} = \frac{1}{(k + 2)^{-1/2}} = (k + 2)^{1/2}.$$ 

So:

$$\delta_k \leq \gamma R^2 (k + 3/2)^{1/2} + G^2 \left((k + 1)^{1/2} - 2 (1)^{1/2}\right).$$ 

Using step size $\gamma = G/R$

$$\delta_k \leq RG (k + 3/2)^{1/2} + RG \left((k + 1)^{1/2} - 2 (1)^{1/2}\right)$$ 

$$= RG \left((k + 3/2)^{1/2} + (k + 1)^{1/2} - 2 (1)^{1/2}\right)$$ 

$$\leq 2RG (k + 1)^{1/2}.$$ 

Now to normalize by $1/(k+1)$ we use:

$$\frac{(k + 1)^{1/2+r}}{(k + 1)^r} = (k + 1 + r)^q,$$

with $r = 1$ and $q = -1/2$, so that:

$$\frac{(k + 1)^{1/2}}{k + 1} = (k + 2)^{-1/2}.$$ 

We further use $(k + 2)^{-1/2} < (k + 1)^{-1/2}$, giving:

$$\frac{1}{k+1} \delta_k < \frac{2RG}{\sqrt{k + 1}}$$ 

$\square$


Figure 3: SVRG+M training loss convergence

J SVRG+M Experiments

We compared the SVRG+M method against SVRG both with the $r = 1/2$ momentum suggested by the theory as well as equal weighted momentum. We used the same test setup as for our SGD+M experiments, except without the addition of weight decay in order to test the non-strongly convex convergence. Since the selection of step-size is less clear in the non-strongly convex case, here we used a step-size sweep on a power-of-2 grid, and we reported the results of the best step-size for each method. As shown in Figure 3, SVRG+M is faster on two of the test problems and slower on two. The flat momentum variant is a little slower than $r = 1/2$ momentum, however not significantly so.