ADAPTIVE EULER-MARUYAMA METHOD FOR
SDES WITH NON-GLOBALLY LIPSCHITZ DRIFT:
PART I, FINITE TIME INTERVAL

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This paper proposes an adaptive timestep construction for an Euler-Maruyama approximation of SDEs with a drift which is not globally Lipschitz. It is proved that if the timestep is bounded appropriately, then over a finite time interval the numerical approximation is stable, and the expected number of timesteps is finite. Furthermore, the order of strong convergence is the same as usual, i.e. order 1/2 for SDEs with a non-uniform globally Lipschitz volatility, and order 1 for Langevin SDEs with unit volatility and a drift with sufficient smoothness. The analysis is supported by numerical experiments for a variety of SDEs.

1. Introduction. In this paper we consider an m-dimensional stochastic differential equation (SDE) driven by a d-dimensional Brownian motion:

\[ dX_t = f(X_t) \, dt + g(X_t) \, dW_t, \]

with a fixed initial value \( X_0 \). The standard theory assumes the drift \( f : \mathbb{R}^m \to \mathbb{R}^m \) and the volatility \( g : \mathbb{R}^m \to \mathbb{R}^{m \times d} \) are both globally Lipschitz. Under this assumption, there is well-established theory on the existence and uniqueness of strong solutions, and the numerical approximation \( \tilde{X}_t \) obtained from the Euler-Maruyama discretisation

\[ \tilde{X}_{(n+1)h} = \tilde{X}_{nh} + f(\tilde{X}_{nh}) \, h + g(\tilde{X}_{nh}) \, \Delta W_n \]

using a uniform timestep of size \( h \) with Brownian increments \( \Delta W_n \), plus a suitable interpolation within each timestep, is known [12] to have a strong error which is \( O(h^{1/2}) \) so that for any \( T, p > 0 \)

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| \tilde{X}_t - X_t \|^p \right] = O(h^{p/2}). \]

The interest in this paper is in other cases in which \( g \) is again globally Lipschitz, but \( f \) is only locally Lipschitz. If, for some \( \alpha, \beta \geq 0 \), \( f \) also satisfies the one-sided growth condition

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\[ \langle x, f(x) \rangle \leq \alpha \|x\|^2 + \beta, \]

where \( \langle \cdot, \cdot \rangle \) denotes an inner product, then it is again possible to prove the existence and uniqueness of strong solutions (see Theorems 2.3.5 and 2.4.1 in [16]). Furthermore (see Lemma 3.2 in [6]), these solutions are stable in the sense that for any \( T, p > 0 \)

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|X_t\|^p \right] < \infty.
\]

The problem is that the numerical approximation given by the uniform timestep Euler-Maruyama discretisation may not be stable. Indeed, for the SDE

\[ dX_t = -X_t^3 \, dt + dW_t, \]

it has been proved [9] that for any \( T > 0 \) and \( p \geq 2 \)

\[
\lim_{h \to 0} \mathbb{E} \left[ \|\hat{X}_T\|^p \right] = \infty.
\]

This behaviour has led to research on numerical methods which achieve strong convergence for these SDEs with a non-globally Lipschitz drift. One key paper in this area is by Higham, Mao & Stuart [6]. First, assuming a locally Lipschitz condition for both the drift and the volatility, they prove that if the uniform timestep Euler-Maruyama discretisation is stable then it also converges strongly. Assuming the drift satisfies a one-sided Lipschitz condition and a polynomial growth condition, they then prove stability and the standard order \( \frac{1}{2} \) strong convergence for two uniform timestep implicit methods, the Split-Step Backward Euler method (SSBE):

\[
\hat{X}_{(n+1)h}^{\ast} = \hat{X}_{nh} + f(\hat{X}_{nh}^\ast) \, h + g(\hat{X}_{nh}^\ast) \Delta W_n.
\]

and the drift-implicit Backward Euler method:

\[
\hat{X}_{(n+1)h} = \hat{X}_{nh} + f(\hat{X}_{(n+1)h}) \, h + g(\hat{X}_{nh}) \Delta W_n.
\]

Mao & Szpruch [19] prove that the implicit \( \theta \)-Euler method

\[
\hat{X}_{(n+1)h} = \hat{X}_{nh} + \theta f(\hat{X}_{(n+1)h}) \, h + (1-\theta)f(\hat{X}_{nh}) \, h + g(\hat{X}_{nh}) \Delta W_n,
\]

converges strongly for \( \frac{1}{2} \leq \theta \leq 1 \) under more general conditions which permit a non-globally Lipschitz volatility.
However, except for some special cases, implicit methods can require significant additional computational costs, especially for multi-dimensional SDEs; therefore, a stable explicit method is desired. Milstein & Tretyakov proposed a general approach which discards approximate paths that cross a sphere with a sufficiently large radius $R$ \cite{milstein21}. However, it is not easy to quantify the errors due to $R$. The explicit tamed Euler method proposed by Hutzenthaler, Jentzen & Kloeden \cite{hutzenthaler10} is

$$\hat{X}_{(n+1)h} = \hat{X}_{nh} + \frac{f(\hat{X}_{nh})}{1 + Ch\|f(\hat{X}_{nh})\|} h + g(\hat{X}_{nh}) \Delta W_n,$$

for some fixed constant $C > 0$. They prove both stability and the standard order $\frac{1}{2}$ strong convergence. This approach has been extended to the tamed Milstein method by Wang & Gan \cite{wang23}, proving order 1 strong convergence for SDEs with commutative noise. Finally, Mao \cite{mao17} proposes a truncated Euler method which has the form

$$\hat{X}_{(n+1)h} = \hat{X}_{nh} + f \left( \min(K\|\hat{X}_{nh}\|^{-1}, 1) \hat{X}_{nh} \right) h + g \left( \min(K\|\hat{X}_{nh}\|^{-1}, 1) \hat{X}_{nh} \right) \Delta W_n.$$

By making $K$ a function of $h$, strong convergence is proved for SDEs satisfying a Khasminskii-type condition which again allows a non-globally Lipschitz volatility; in \cite{mao18} it is proved that the order of convergence is arbitrarily close to $\frac{1}{2}$.

In this paper, we propose instead to use the standard explicit Euler-Maruyama method, but with an adaptive timestep $h_n$ which is a function of the current approximate solution $\hat{X}_{t_n}$. The idea of using an adaptive timestep comes from considering the divergence of the uniform timestep method for the SDE (2). When there is no noise, the requirement for the explicit Euler approximation of the corresponding ODE to have a stable monotonic decay is that its timestep satisfies $h < \hat{X}_{t_n}^{-2}$. An intuitive explanation for the instability of the uniform timestep Euler-Maruyama approximation of the SDE is that there is always a very small probability of a large Brownian increment $\Delta W_n$ which pushes the approximation $\hat{X}_{t_{n+1}}$ into the region $h > 2 \hat{X}_{t_{n+1}}^{-2}$ leading to an oscillatory super-exponential growth. Using an adaptive timestep avoids this problem.

Adaptive timesteps have been used in previous research to improve the accuracy of numerical approximations. Some approaches use local error estimation to decide whether or not to refine the timestep \cite{2,20,13} while others are similar to ours in setting the size of each timestep based on the
current path approximation $\hat{X}_t$ \cite{7, 22}. However, these all assume globally Lipschitz drift and volatility. The papers by Lamba, Mattingly & Stuart \cite{14} and Lemaire \cite{15} are more relevant to the analysis in this paper. They both consider drifts which are not globally Lipschitz, but they assume a dissipative condition which is stronger than the conditions assumed in this paper. Lamba, Mattingly & Stuart \cite{14} prove strong stability but not the order of strong convergence, while Lemaire \cite{15} considers an infinite time interval with a timestep with an upper bound which decreases towards zero over time, and proves convergence of the empirical distribution to the invariant distribution of the SDE.

In this paper we are concerned with strong convergence, not weak convergence, because our interest is in using the numerical approximation as part of a multilevel Monte Carlo (MLMC) computation \cite{3, 4} for which the strong convergence properties are key in establishing the rate of decay of the variance of the multilevel correction. Usually, MLMC is used with a geometric sequence of time grids, with each coarse timestep corresponding to a fixed number of fine timesteps. However, it has been shown that it is not difficult to implement MLMC using the same driving Brownian path for the coarse and fine paths, even when they have no time points in common \cite{5}.

Paper \cite{5} also provides another motivation for this paper, the analysis of Langevin equations with a drift $-\nabla V(X_t)$ where $V(x)$ is a potential function which comes from the modelling of molecular dynamics. \cite{5} considers the FENE (Finitely Extensible Nonlinear Elastic) model which in the case of a molecule with a single bond has a 3D potential $-\mu \log(1 - \|x\|^2)$. Considerations of stability and accuracy lead to the use of a timestep of the form $\delta (1 - \|\hat{X}_n\|)^2 / \max(2\mu, 36)$, for some $0 < \delta \leq 1$. Because of this, we pay particular attention to the case of Langevin equations, and for these we prove first order strong convergence, the same as for the uniform timestep Euler-Maruyama method for globally Lipschitz drifts. Unfortunately our assumptions do not cover the case of the FENE model as we require $-\nabla V(x)$ to be locally Lipschitz on $\mathbb{R}^m$.

The rest of the paper is organised as follows. Section 2 states the main theorems, and proves some minor lemmas. Section 3 has a number of example applications, many from \cite{8}, illustrating how suitable adaptive timestep functions can be defined. It also presents some numerical results comparing the performance of the adaptive Euler-Maruyama method to other methods. Section 4 has the proofs of the three main theorems, and finally Section 5 has some conclusions and discusses the extension to the infinite time interval which will be covered in a future paper.

In this paper we assume the following setting and notation. Let $T > 0$
be a fixed positive real number, and let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with normal filtration \((\mathcal{F}_t)_{t \in [0,T]}\) corresponding to a \(d\)-dimensional standard Brownian motion \(W_t = (W^{(1)}_t, W^{(2)}_t, \ldots, W^{(d)}_t)^T\). We denote the vector norm by \(\|v\| \triangleq (|v_1|^2 + |v_2|^2 + \ldots + |v_m|^2)^{1/2}\), the inner product of vectors \(v\) and \(w\) by \(\langle v, w \rangle \triangleq v_1 w_1 + v_2 w_2 + \ldots + v_m w_m\), for any \(v, w \in \mathbb{R}^m\) and the Frobenius matrix norm by \(\|A\| \triangleq \sqrt{\sum_{i,j} A_{i,j}^2}\) for all \(A \in \mathbb{R}^{m \times d}\).

2. Adaptive algorithm and theoretical results.

2.1. Adaptive Euler-Maruyama method. The adaptive Euler-Maruyama discretisation is

\[
\begin{align*}
t_{n+1} &= t_n + h_n, \\
\hat{X}_{t_{n+1}} &= \hat{X}_{t_n} + f(\hat{X}_{t_n}) h_n + g(\hat{X}_{t_n}) \Delta W_n,
\end{align*}
\]

where \(h_n \triangleq h(\hat{X}_{t_n})\) and \(\Delta W_n \triangleq W_{t_{n+1}} - W_{t_n}\), and there is fixed initial data \(t_0 = 0, \hat{X}_0 = X_0\).

One key point in the analysis is to prove that \(t_n\) increases without bound as \(n\) increases. More specifically, the analysis proves that for any \(T > 0\), almost surely for each path there is an \(N\) such that \(t_N \geq T\).

We use the notation \(\hat{t} \triangleq \max\{t_n : t_n \leq t\}\), \(n_t \triangleq \max\{n : t_n \leq t\}\) for the nearest time point before time \(t\), and its index.

We define the piecewise constant interpolant process \(\bar{X}_t = \hat{X}_{\hat{t}}\) and also define the standard continuous interpolant \([12]\) as

\[
\hat{X}_t = \bar{X}_{\hat{t}} + f(\bar{X}_{\hat{t}})(\hat{t} - \hat{t}) + g(\bar{X}_{\hat{t}})(W_t - W_{\hat{t}}),
\]

so that \(\hat{X}_t\) is the solution of the SDE

\[
(3) \quad d\hat{X}_t = f(\hat{X}_{\hat{t}}) dt + g(\hat{X}_{\hat{t}}) dW_t = f(\bar{X}_t) dt + g(\bar{X}_t) dW_t.
\]

In the following subsections, we state the key results on stability and strong convergence, and related results on the number of timesteps, introducing various assumptions as required for each. The main proofs are deferred to Section 4.

2.2. Stability.

Assumption 1 (Local Lipschitz and linear growth). \(f\) and \(g\) are both locally Lipschitz, so that for any \(R > 0\) there is a constant \(C_R\) such that

\[
\|f(x) - f(y)\| + \|g(x) - g(y)\| \leq C_R \|x - y\|
\]
for all \( x, y \in \mathbb{R}^m \) with \( \|x\|, \|y\| \leq R \). Furthermore, there exist constants \( \alpha, \beta \geq 0 \) such that for all \( x \in \mathbb{R}^m \), \( f \) satisfies the one-sided linear growth condition:

\[
(4) \quad \langle x, f(x) \rangle \leq \alpha \|x\|^2 + \beta,
\]

and \( g \) satisfies the linear growth condition:

\[
(5) \quad \|g(x)\|^2 \leq \alpha \|x\|^2 + \beta.
\]

Together, (4) and (5) imply the monotone condition

\[
\langle x, f(x) \rangle + \frac{1}{2} \|g(x)\|^2 \leq \frac{3}{2}(\alpha \|x\|^2 + \beta),
\]

which is a key assumption in the analysis of Mao & Szpruch [19] and Mao [17] for SDEs with volatilities which are not globally Lipschitz. However, in our analysis we choose to use this slightly stronger assumption, which provides the basis for the following lemma on the stability of the SDE solution.

**Lemma 1 (SDE stability).** If the SDE satisfies Assumption 1, then for all \( p > 0 \)

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|X_t\|^p \right] < \infty.
\]

**Proof.** The proof is given in Lemma 3.2 in [6]; the statement of that lemma makes stronger assumptions on \( f \) and \( g \), corresponding to (8) and (9), but the proof only uses the conditions in Assumption 1.

We now specify the critical assumption about the adaptive timestep.

**Assumption 2 (Adaptive timestep).** The adaptive timestep function \( h : \mathbb{R}^m \to \mathbb{R}^+ \) is continuous and strictly positive, and there exist constants \( \alpha, \beta > 0 \) such that for all \( x \in \mathbb{R}^m \), \( h(x) \) satisfies the inequality

\[
(6) \quad \langle x, f(x) \rangle + \frac{1}{2} h(x) \|f(x)\|^2 \leq \alpha \|x\|^2 + \beta.
\]

Note that if another timestep function \( h^\delta(x) \) is smaller than \( h(x) \), then \( h^\delta(x) \) also satisfies the Assumption 2. Note also that the form of (6), which is motivated by the requirements of the proof of the next theorem, is very similar to (4). Indeed, if (6) is satisfied then (4) is also true for the same values of \( \alpha \) and \( \beta \).
Theorem 1 (Finite time stability). If the SDE satisfies Assumption 1, and the timestep function $h$ satisfies Assumption 2, then $T$ is almost surely attainable (i.e. for $\omega \in \Omega$, $\mathbb{P}(\exists N(\omega) < \infty \text{ s.t. } t_{N(\omega)} \geq T) = 1$) and for all $p > 0$ there exists a constant $C_{p,T}$ which depends solely on $p$, $T$ and the constants $\alpha, \beta$ in Assumption 2, such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|\hat{X}_t\|^p \right] < C_{p,T}.$$ 

Proof. The proof is deferred to Section 4. \qed

To bound the expected number of timesteps, we require an assumption on how quickly $h(x)$ can approach zero as $\|x\| \to \infty$.

Assumption 3 (Timestep lower bound). There exist constants $\xi, \zeta, q > 0$, such that the adaptive timestep function satisfies the inequality

$$h(x) \geq (\xi \|x\|^q + \zeta)^{-1}.$$ 

Given this assumption, we obtain the following lemma.

Lemma 2 (Bounded timestep moments). If the SDE satisfies Assumption 1, and the timestep function $h$ satisfies Assumptions 2 and 3, then for all $p > 0$

$$\mathbb{E} \left[ N_T^p \right] < \infty.$$ 

where $N_T = \min \{n : t_n \geq T\}$ is the number of timesteps required by a path approximation.

Proof. Assumption 3 gives us

$$N_T \leq 1 + T \sup_{0 \leq t \leq T} \frac{1}{h(X_t)} \leq 1 + T \left( \xi \sup_{0 \leq t \leq T} \|\hat{X}_t\|^q + \zeta \right),$$ 

and the result is then an immediate consequence of Theorem 1. \qed

2.3. Strong convergence. Standard strong convergence analysis for an approximation with a uniform timestep $h$ considers the limit $h \to 0$. This clearly needs to be modified when using an adaptive timestep, and we will instead consider a timestep function $h^\delta(x)$ controlled by a scalar parameter $0 < \delta \leq 1$, and consider the limit $\delta \to 0$. 

Given a timestep function $h(x)$ which satisfies Assumptions 2 and 3, ensuring stability as analysed in the previous section, there are two quite natural ways in which we might introduce $\delta$ to define $h^\delta(x)$:

\[
    h^\delta(x) = \delta \min(T, h(x)), \\
    h^\delta(x) = \min(\delta T, h(x)).
\]

The first refines the timestep everywhere, while the latter concentrates the computational effort on reducing the maximum timestep, with $h(x)$ introduced to ensure stability when $\|\hat{X}_t\|$ is large.

In our analysis, we will cover both possibilities by making the following assumption.

**Assumption 4.** The timestep function $h^\delta$, satisfies the inequality

\[
    \delta \min(T, h(x)) \leq h^\delta(x) \leq \min(\delta T, h(x)),
\]

and $h$ satisfies Assumption 2.

Given this assumption, we obtain the following theorem:

**Theorem 2 (Strong convergence).** If the SDE satisfies Assumption 1, and the timestep function $h^\delta$ satisfies Assumption 4, then for all $p > 0$

\[
    \lim_{\delta \to 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|\hat{X}_t - X_t\|^p \right] = 0.
\]

**Proof.** The proof is essentially identical to the uniform timestep Euler-Maruyama analysis in Theorem 2.2 in [6] by Higham, Mao & Stuart.

The only change required by the use of an adaptive timestep is to note that

\[
    \hat{X}_s - \bar{X}_s = f(\bar{X}_s)(s - \delta) + g(\bar{X}_s)(W_s - W_{\delta})
\]

and $s - \delta < \delta T$ and $\mathbb{E} \left[ \|W_s - W_{\delta}\|^2 \mid F_{\delta} \right] = d(s - \delta)$.

To prove an order of strong convergence requires new assumptions on $f$ and $g$:

**Assumption 5 (Lipschitz properties).** There exists a constant $\alpha > 0$ such that for all $x, y \in \mathbb{R}^m$, $f$ satisfies the one-sided Lipschitz condition:

\[
    \langle x - y, f(x) - f(y) \rangle \leq \frac{1}{2} \alpha \|x - y\|^2,
\]
and $g$ satisfies the Lipschitz condition:

$$\|g(x) - g(y)\|^2 \leq \frac{1}{2}\alpha \|x - y\|^2.$$  \hfill (9)

In addition, $f$ satisfies the locally polynomial growth Lipschitz condition

$$\|f(x) - f(y)\| \leq (\gamma (\|x\|^q + \|y\|^q) + \mu) \|x - y\|,$$  \hfill (10)

for some $\gamma, \mu, q > 0$.

Note that setting $y = 0$ gives

$$\langle x, f(x) \rangle \leq \frac{1}{2}\alpha \|x\|^2 + \langle x, f(0) \rangle \leq \alpha \|x\|^2 + \frac{1}{2}\alpha^{-1} \|f(0)\|^2,$$

$$\|g(x)\|^2 \leq 2\|g(x) - g(0)\|^2 + 2\|g(0)\|^2 \leq \alpha \|x\|^2 + 2\|g(0)\|^2.$$  

Hence, Assumption 5 implies Assumption 1, with the same $\alpha$ and an appropriate $\beta$.

Also, if the drift and volatility are differentiable, the following assumption is equivalent to Assumption 5, and usually easier to check in practice.

**Assumption 6 (Lipschitz properties).** There exists a constant $\alpha > 0$ such that for all $x, e \in \mathbb{R}^m$ with $\|e\| = 1$, $f$ satisfies the one-sided Lipschitz condition:

$$\langle e, \nabla f(x) e \rangle \leq \frac{1}{2}\alpha$$  \hfill (11)

and $g$ satisfies the Lipschitz condition:

$$\|\nabla g(x)\|^2 \leq \frac{1}{2}\alpha$$  \hfill (12)

and in addition $f$ satisfies the locally polynomial growth Lipschitz condition

$$\|\nabla f(x)\| \leq (2\gamma \|x\|^q + \mu),$$  \hfill (13)

for some $\gamma, \mu, q > 0$.

**Theorem 3 (Strong convergence order).** If the SDE satisfies Assumption 5, and the timestep function $h^\delta$ satisfies Assumption 4, then for all $p > 0$ there exists a constant $C_{p,T}$ such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|\hat{X}_t - X_t\|^p \right] \leq C_{p,T} \delta^{p/2}.$$
Proof. The proof is deferred to Section 4.

Lemma 3 (Number of timesteps). If the SDE satisfies Assumption 5, and the timestep function $h^\delta(x)$ satisfies Assumption 4, with $h(x)$ satisfying Assumption 3, then for all $p > 0$ there exists a constant $c_{p,T}$ such that

$$E[N_T^p] \leq c_{p,T} \delta^{-p},$$

where $N_T$ is again the number of timesteps required by a path approximation.

Proof. Equation (7) and Assumption 3 give

$$N_T \leq 1 + \frac{1}{\delta} \sup_{0 \leq t \leq T} h^\delta(\hat{X}_t) \leq 1 + \frac{\delta^{-1}}{\delta} \sup_{0 \leq t \leq T} \max(h^{-1}(x), T^{-1}) \leq \delta^{-1} T \left( \xi \sup_{0 \leq t \leq T} \|\hat{X}_t\|^p + \zeta + (1 + \delta) T^{-1} \right).$$

The result is then a consequence of Theorem 1 since $h^\delta(x) \leq h(x)$ and therefore $h^\delta(x)$ satisfies the requirements for stability.

The conclusion from Theorem 3 and Lemma 3 is that

$$E \left[ \sup_{0 \leq t \leq T} \|\hat{X}_t - X_t\|^p \right]^{1/p} \leq C_{p,T}^{1/p} c_{1,T}^{1/2} (E[N_T])^{-1/2}$$

which corresponds to order $\frac{1}{2}$ strong convergence when comparing the accuracy to the expected cost.

First order strong convergence is achievable for Langevin SDEs in which $m = d$ and $g$ is the identity matrix $I_m$, but this requires stronger assumptions on the drift $f$.

Assumption 7 (Enhanced Lipschitz properties). There exists a constant $\alpha > 0$ such that for all $x, y \in \mathbb{R}^m$, $f$ satisfies the one-sided Lipschitz condition:

$$\langle x - y, f(x) - f(y) \rangle \leq \frac{1}{2} \alpha \|x - y\|^2.$$

In addition, $f$ is differentiable, and $f$ and $\nabla f$ satisfy the locally polynomial growth Lipschitz condition

$$\|f(x) - f(y)\| + \|\nabla f(x) - \nabla f(y)\| \leq (\gamma (\|x\|^q + \|y\|^q) + \mu) \|x - y\|,$$

for some $\gamma, \mu, q > 0$.  

Lemma 4. If $f$ satisfies Assumption 7 then for any $x, y, v \in \mathbb{R}^m$,
\[
\langle v, f(x) - f(y) \rangle = \langle v, \nabla f(x)(x-y) \rangle + R(x, y, v),
\]
where the remainder term has the bound
\[
|R(x, y, v)| \leq (\gamma (\|x\|^q + \|y\|^q) + \mu) \|v\| \|x-y\|^2.
\]

Proof. If we define the scalar function $u(\lambda)$ for $0 \leq \lambda \leq 1$ by
\[
u(\lambda) = \langle v, f(y + \lambda(x-y)) \rangle,
\]
then $u(\lambda)$ is continuously differentiable, and by the Mean Value Theorem $u(1) - u(0) = u'(\lambda^*)$ for some $0 < \lambda^* < 1$, which implies that
\[
\langle v, f(x) - f(y) \rangle = \langle v, \nabla f(y + \lambda^*(x-y))(x-y) \rangle.
\]
The final result then follows from the Lipschitz property of $\nabla f$.

We now state the theorem on improved strong convergence.

Theorem 4 (Strong convergence for Langevin SDEs). If $m=d$, $g \equiv I_m$, $f$ satisfies Assumption 7, and the timestep function $h^\delta$ satisfies Assumption 4, then for all $T, p \in (0, \infty)$ there exists a constant $C_{p,T}$ such that
\[
E \left[ \sup_{0 \leq t \leq T} \| \hat{X}_t - X_t \|_p^p \right] \leq C_{p,T} \delta^p.
\]

Proof. The proof is deferred to Section 4.

Comment: first order strong convergence can also be achieved for a general $g(x)$ by using an adaptive timestep Milstein discretisation, provided $\nabla g$ satisfies an additional Lipschitz condition. A formal statement and proof of this is omitted as it requires a lengthy extension to the stability analysis. In addition, this numerical approach is only practical in cases in which the commutativity condition is satisfied and therefore there is no need to simulate the Lévy areas which the Milstein method otherwise requires [12].

3. Examples and numerical results. In this section we discuss a number of example SDEs with non-globally Lipschitz drift. In each case we comment on the applicability of the theory and a suitable choice for the adaptive timestep.

We then present numerical results for three testcases which illustrate some key aspects.
3.1. **Scalar SDEs.** In each of the cases to be presented, the drift is of the form

\[ f(x) \approx -c \, \text{sign}(x) \, |x|^q, \quad \text{as} \ |x| \to \infty \]

for some constants \( c > 0, \; q > 1 \). Therefore, as \(|x| \to \infty\), the maximum stable timestep satisfying Assumption 2 corresponds to \( \langle x, f(x) \rangle + \frac{1}{2} h(x)^2 |f(x)|^2 \approx 0 \) and hence \( h(x) \approx 2|x|/|f(x)| \approx 2 \, c^{-1} |x|^{1-q} \). A suitable choice for \( h(x) \) and \( h^\delta(x) \) is therefore

\[ h(x) = \min \left( T, c^{-1} |x|^{1-q} \right), \quad h^\delta(x) = \delta h(x). \]

3.1.1. **Stochastic Ginzburg-Landau equation.** This describes a phase transition from the theory of superconductivity [9, 12].

\[ dX_t = \left( (\eta + \frac{1}{2} \sigma^2) X_t - \lambda X_t^3 \right) dt + \sigma X_t \, dW_t, \]

where \( \eta \geq 0, \; \lambda, \sigma > 0 \). The SDE is usually defined on the domain \( \mathbb{R}^+ \), since \( X_t > 0 \) for all \( t > 0 \), if \( X_0 > 0 \). However, the numerical approximation is not guaranteed to remain strictly positive and the domain can be extended to \( \mathbb{R} \) without any change to the SDE.

The drift and volatility satisfy Assumptions 1 and 5, and therefore all of the theory is applicable, with a suitable choice for \( h^\delta(x) \), based on (16) and (17), being

\[ h^\delta(x) = \delta \min \left( T, \lambda^{-1} x^{-2} \right). \]

3.1.2. **Stochastic Verhulst equation.** This is a model for a population with competition between individuals [9].

\[ dX_t = \left( (\eta + \frac{1}{2} \sigma^2) X_t - \lambda X_t^2 \right) dt + \sigma X_t \, dW_t, \]

where \( \eta, \lambda, \sigma > 0 \). The SDE is defined on the domain \( \mathbb{R}^+ \), but can be extended to \( \mathbb{R} \) by modifying it to

\[ dX_t = \left( (\eta + \frac{1}{2} \sigma^2) X_t - \lambda |X_t| X_t \right) dt + \sigma X_t \, dW_t, \]

so that the drift is positive in the limit \( x \to -\infty \).

The drift and volatility then satisfy Assumptions 1 and 5, and therefore all of the theory is applicable, with a suitable choice for \( h^\delta(x) \), based on (16) and (17), being

\[ h^\delta(x) = \delta \min \left( T, \lambda^{-1} |x|^{-1} \right). \]
3.2. Multi-dimensional SDEs. With multi-dimensional SDEs there are two cases of particular interest. For SDEs with a drift which, for some $\beta > 0$ and sufficiently large $\|x\|$, satisfies the condition

$$\langle x, f(x) \rangle \leq -\beta \|x\| \|f(x)\|,$$

one can take $\langle x, f(x) \rangle + \frac{1}{2} h(x) |f(x)|^2 \approx 0$ and therefore a suitable definition of $h(x)$ for large $\|x\|$ is

$$h(x) = \min(T, \|x\|/\|f(x)\|).$$

For SDEs with a drift which does not satisfy the condition, but for which $\|f(x)\| \to \infty$ as $\|x\| \to \infty$, an alternative choice for large $\|x\|$ is to use

$$(18) \quad h(x) = \min(T, \gamma \|x\|^2/\|f(x)\|^2),$$

for some $\gamma > 0$. The difficulty in this case is choosing the best value for $\gamma$, taking into account both accuracy and cost.

3.2.1. Stochastic van der Pol oscillator. This describes state oscillation [8].

$$\begin{align*}
\mathrm{d}X_t^{(1)} &= X_t^{(2)} \, \mathrm{d}t \\
\mathrm{d}X_t^{(2)} &= \left(\alpha \left(\mu - (X_t^{(1)})^2\right) X_t^{(2)} - \delta X_t^{(1)}\right) \, \mathrm{d}t + \beta \, \mathrm{d}W_t
\end{align*}$$

where $\alpha, \mu, \delta, \beta > 0$. It can be put in the standard form by defining

$$f(x) \equiv \begin{pmatrix} \alpha \mu x_2^2 - \delta x_1^2 \\ \alpha (\mu - x_1^2) x_2 - \delta x_1 \end{pmatrix}, \quad g(x) \equiv \begin{pmatrix} 0 \\ \beta \end{pmatrix}.$$  

It follows that

$$\langle x, f(x) \rangle = -\alpha x_1^2 x_2^2 + \alpha \mu x_2^2 + (1-\delta) x_1 x_2 \leq \left(\alpha \mu + \frac{1}{2} (1-\delta)\right) \|x\|^2.$$  

Therefore the drift and volatility satisfy Assumption 1 and the numerical approximations will be stable if the maximum timestep is defined by (18).

However, it can be verified that $\langle e, \nabla f(x) e \rangle$ is not uniformly bounded for an arbitrary $e$ such that $\|e\| = 1$, and therefore the drift does not satisfy the one-sided Lipschitz condition. Hence the stability and strong convergence theory in this paper is applicable, but not the theorems on the order of convergence. Nevertheless, numerical experiments exhibit first order strong convergence, which is consistent with the fact that the volatility in uniform, so it seems there remains a gap here in the theory.
3.2.2. \textit{Stochastic Lorenz equation.} This is a three-dimensional system modelling convection rolls in the atmosphere [8].

\[
\begin{align*}
\frac{dX_t^{(1)}}{dt} &= \left(\alpha_1 X_t^{(2)} - \alpha_1 X_t^{(1)}\right) dt + \beta_1 X_t^{(1)} dW_t^{(1)} \\
\frac{dX_t^{(2)}}{dt} &= \left(\alpha_2 X_t^{(1)} - X_t^{(2)} - X_t^{(1)} X_t^{(3)}\right) dt + \beta_2 X_t^{(2)} dW_t^{(2)} \\
\frac{dX_t^{(3)}}{dt} &= \left(X_t^{(1)} X_t^{(2)} - \alpha_3 X_t^{(3)}\right) dt + \beta_3 X_t^{(3)} dW_t^{(3)}
\end{align*}
\]

where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 > 0$, and so we have:

\[
\begin{align*}
f(x) &\equiv \begin{pmatrix} \alpha_1 (x_2 - x_1) \\
\alpha_2 x_1 - x_2 - x_1 x_3 \\
x_1 x_2 - \alpha_3 x_3 \end{pmatrix}, \quad g(x) &\equiv \begin{pmatrix} \beta_1 x_1 & 0 & 0 \\
0 & \beta_2 x_2 & 0 \\
0 & 0 & \beta_3 x_3 \end{pmatrix}
\end{align*}
\]

The diffusion coefficient is globally Lipschitz, and since $\langle x, f(x) \rangle$ consists solely of quadratic terms, the drift satisfies the one-sided linear growth condition. Noting that $\|f\|^2 \approx x_1^2 (x_2^2 + x_3^2) < \|x\|^4$ as $\|x\| \to \infty$, an appropriate maximum timestep is

\[
h(x) = \min(T, \gamma \|x\|^{-2}),
\]

for any $\gamma > 0$. However, the drift does not satisfy the one-sided Lipschitz condition, and therefore the theory on the order of strong convergence is not applicable.

3.2.3. \textit{Langevin equation.} The multi-dimensional Langevin equation is

\[
\frac{dX_t}{dt} = -\nabla V(X_t) dt + dW_t.
\]

In molecular dynamics applications, $V(x)$ represent the potential energy of a molecule, while in other applications $V = -\frac{1}{2} \log \pi + \text{const}$ where $\pi : \mathbb{R}^m \to \mathbb{R}^+$ is an invariant measure. $V$ is usually defined on $\mathbb{R}^m$, infinitely differentiable, and satisfies all of the assumptions in this paper so the theory is fully applicable, leading to order 1 strong convergence.

3.2.4. \textit{FENE model.} The FENE (Finitely Extensible Nonlinear Elastic) model is a Langevin equation describing the motion of a long-chained polymer in a liquid [1, 5]. The unusual feature of the FENE model is that the potential $V(x)$ becomes infinite for finite values of $x$. In the simplest case of a molecule with a single bond, $x$ is three-dimensional and $V(x)$ takes the form $V(x) = -\log(1-\|x\|^2)$. The SDE is defined on $\|x\| < 1$, with the
drift term ensure that $\|X_t\| < 1$ for all $t > 0$. Also, it can be verified that $\langle x, f(x) \rangle \leq 0$.

Because the SDE is not defined on all of $\mathbb{R}^3$, the theory in this paper is not applicable. However, it was one of the original motivations for the analysis in this paper, since it seems natural to use an adaptive timestep, taking smaller timestep as $\|\hat{X}_t\|$ approaches 1, to maintain good accuracy, as the drift varies so rapidly near the boundary, and to greatly reduce the possibility of needing to clamp the computed solution to prevent it from crossing a numerical boundary at radius $1-\delta$ for some $\delta \ll 1$ [5]. Numerical results indicate that the order of strong convergence is very close to 1.

3.3. Numerical results. The numerical tests include three testcases from [10] plus one new test which provides some motivation for the research in this paper.

3.3.1. Testcase 1. The first scalar testcase taken from [10] is

$$dX_t = -X_t^5 \, dt + X_t \, dW_t, \quad X_0 = 1,$$

with $T = 1$. The three methods tested are the Tamed Euler scheme, with $C = 1$, the implicit Euler scheme, and the new Euler scheme with adaptive timestep

$$h_\delta(x) = \delta \frac{\max(1, |x|)}{\max(1, |f(x)|)}.$$
Figure 1 shows the the root-mean-square error plotted against the average timestep. The plot on the left shows the error in the terminal time, while the plot on the right shows the error in the maximum magnitude of the solution. The error in each case is computed by comparing the numerical solution to a second solution with a timestep, or $\delta$, which is 4 times smaller.

When looking at the error in the final solution, all 3 methods have similar accuracy with $\frac{3}{2}$ order strong convergence. However, as reported in [10], the cost of the implicit method per timestep is much higher. The plot of the error in the maximum magnitude shows that the new method is slightly more accurate, presumably because it uses smaller timesteps when the solution is large. The plot was included to show that comparisons between numerical methods depend on the choice of accuracy measure being used.

3.3.2. Testcase 2. The second scalar testcase taken from [10] is

$$dX_t = (X_t - X_t^3) \, dt + X_t \, dW_t, \quad X_0 = 1,$$

with $T = 1$. The results in Figure 2 are similar to the first testcase.

3.3.3. Testcase 3. The third testcase taken from [10] is 10-dimensional,

$$dX_t = (X_t - \|X_t\|^2 \, X_t) \, dt + dW_t, \quad X_0 = 0,$$

with $T = 1$. The results in the left-hand plot in Figure 3 show that the error in the final value exhibits order 1 strong convergence using all 3 methods, as expected.
3.3.4. Testcase 4. The final testcase is for the 3-dimensional FENE SDE discussed previously,

\[ \text{d}X_t = -\frac{X_t}{1-\|X_t\|^2} \text{d}t + \text{d}W_t, \quad X_0 = 0, \]

with \( T = 1 \). As commented on previously, this SDE is not covered by the theory in this paper, but it is a motivation for the research because it is natural to use an adaptive timestep of the form

\[ h^\delta(x) = \frac{\delta}{4}(1-\|x\|^2) \]

to reduce the timestep when \( \|\hat{X}_t\| \) approaches the maximum radius.

All three methods are clamped so that they do not exceed a radius of \( r_{\text{max}} = 1 - 10^{-10} \); if the new computed value \( \hat{X}_{t_{n+1}} \) exceeds this radius then it is replaced by \( (r_{\text{max}}/\|\hat{X}_{t_{n+1}}\|)\hat{X}_{t_{n+1}} \).

The numerical results in the right-hand plot in Figure 3 show that the new scheme is considerably more accurate than either of the others, confirming that an adaptive timestep is desirable in this situation in which the drift varies enormously as \( \|\hat{X}_t\| \) approaches the maximum radius.
4. Proofs. This section has the proofs of the three main theorems in this paper, one on stability, and two on the order of strong convergence.

4.1. Theorem 1.

Proof. The proof proceeds in four steps. First, we introduce a constant $K$ to modify our discretisation scheme. Second, we derive an upper bound for $\|\hat{X}_t^K\|_p$. Third, we show that the moments $\mathbb{E}[\sup_{0 \leq t \leq T} \|\hat{X}_t^K\|_p]$ are each bounded by a constant $C_{p,T}$ which depends on $p$ and $T$ but is independent of $K$. Finally, we reach the desired conclusion by taking the limit $K \to \infty$ and using the Monotone Convergence theorem.

The proof is given for $p \geq 4$; the result for $0 \leq p < 4$ follows from Hölder’s inequality.

**Step 1: K-Scheme definition**

For any $K > \|X_0\|$, we modify our discretisation scheme to:

$$\hat{X}_{t_{n+1}}^K = P_K \left( \hat{X}_{t_n}^K + f(\hat{X}_{t_n}^K) h_n + g(\hat{X}_{t_n}^K) \Delta W_n \right),$$

where $P_K(Y) \triangleq \min(1, K/\|Y\|) Y$ and therefore $\|\hat{X}_{t_n}^K\| \leq K, \forall n$. The piecewise constant approximation for intermediate times is again $\hat{X}_t^K = \hat{X}_{\hat{t}}^K$, and the continuous approximation is

$$\hat{X}_t^K = P_K \left( \hat{X}_{\hat{t}}^K + f(\hat{X}_{\hat{t}}^K) (t-\hat{t}) + g(\hat{X}_{\hat{t}}^K) (W_t-W_{\hat{t}}) \right).$$

Since $h(x)$ is continuous and strictly positive, it follows that $h^K_{\min} \triangleq \inf_{\|x\| \leq K} h(x) > 0$.

This strictly positive lower bound for the timesteps implies that $T$ is attainable.

**Step 2: pth-moment of K-Scheme solution**

If we define $\phi(x) \triangleq x + h(x)f(x)$, then (20) gives

$$\|\hat{X}_{t_{n+1}}^K\|^2 \leq \|\hat{X}_{t_n}^K\|^2 + 2 h_n \left( \langle \hat{X}_{t_n}^K, f(\hat{X}_{t_n}^K) \rangle + \frac{1}{2} h_n \|f(\hat{X}_{t_n}^K)\|^2 \right) + 2 \langle \phi(\hat{X}_{t_n}^K), g(\hat{X}_{t_n}^K) \Delta W_n \rangle + \|g(\hat{X}_{t_n}^K) \Delta W_n\|^2$$

Using condition (6) for $h(x)$ then gives

$$\|\hat{X}_{t_{n+1}}^K\|^2 \leq \|\hat{X}_{t_n}^K\|^2 + 2 \alpha \|\hat{X}_{t_n}^K\|^2 h_n + 2 \beta h_n + 2 \langle \phi(\hat{X}_{t_n}^K), g(\hat{X}_{t_n}^K) \Delta W_n \rangle + \|g(\hat{X}_{t_n}^K) \Delta W_n\|^2.$$ (21)
Similarly, for the partial timestep from $t$ to $t$, since $(t-t) \leq h_n$,
\begin{equation}
(\hat{X}_t^K, f(\hat{X}_t^K)) + \frac{1}{2} (t-t) ||f(\hat{X}_t^K)||^2 \leq \alpha ||\hat{X}_t^K||^2 + \beta,
\end{equation}
and therefore we obtain
\begin{align}
||\hat{X}_t^K||^2 &\leq ||\hat{X}_0^K||^2 + 2\alpha ||\hat{X}_t^K||^2(t-t) + 2\beta (t-t) \\
&+ 2 \langle \hat{X}_t^K + f(\hat{X}_t^K)(t-t), g(\hat{X}_t^K)(W_t-W_\lambda)) \\
&+ ||g(\hat{X}_t^K)(W_t-W_\lambda)||^2.
\end{align}
(23)
Summing (21) over multiple timesteps and then adding (23) gives
\begin{align}
||\hat{X}_t^K||^2 &\leq ||X_0||^2 + 2\alpha \left( \sum_{k=0}^{n_t-1} ||\hat{X}_{t_k}^K||^2 h_k + ||\hat{X}_t^K||^2(t-t) \right) + 2\beta t \\
&+ 2 \sum_{k=0}^{n_t-1} \langle \phi(\hat{X}_{t_k}^K), g(\hat{X}_{t_k}^K)\Delta W_k \rangle + \sum_{k=0}^{n_t-1} ||g(\hat{X}_{t_k}^K)\Delta W_k||^2 \\
&+ 2 \langle \hat{X}_t^K + f(\hat{X}_t^K)(t-t), g(\hat{X}_t^K)(W_t-W_\lambda) \\
&+ ||g(\hat{X}_t^K)(W_t-W_\lambda)||^2.
\end{align}
Re-writing the first summation as a Riemann integral, and the second as an Itô integral, raising both sides to the power $p/2$ and using Jensen’s inequality, we obtain
\begin{align}
||\hat{X}_t^K||^p &\leq 7^{p/2-1} \left\{ ||X_0||^p + \left( 2\alpha \int_0^t ||\hat{X}_s^K||^2 ds \right)^{p/2} + (2\beta t)^{p/2} \\
&+ \left| \int_0^t \langle \phi(\hat{X}_s^K), g(\hat{X}_s^K) dW_s \rangle \right|^{p/2} + \left( \sum_{k=0}^{n_t-1} ||g(\hat{X}_{t_k}^K)\Delta W_k||^2 \right)^{p/2} \\
&+ \left| 2 \langle \hat{X}_t^K + f(\hat{X}_t^K)(t-t), g(\hat{X}_t^K)(W_t-W_\lambda) \rangle \right|^{p/2} \\
&+ ||g(\hat{X}_t^K)(W_t-W_\lambda)||^p \right\}. 
\end{align}
(24)
Step 3: Expected supremum of $p$th-moment of K-Scheme
For any $0 \leq t \leq T$ we take the supremum on both sides of inequality (24) and then take the expectation to obtain
\begin{align}
\mathbb{E} \left[ \sup_{0 \leq s \leq t} ||\hat{X}_s^K||^p \right] &\leq 7^{p/2-1} (I_1 + I_2 + I_3 + I_4 + I_5),
\end{align}
where

\[ I_1 = \|X_0\|^p + \mathbb{E} \left[ \left( 2\alpha \int_0^t \|\bar{X}_s^K\|^2 \, ds \right)^{p/2} \right] + (2\beta t)^{p/2}, \]

\[ I_2 = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| 2 \int_0^s \langle \phi(\bar{X}_u^K), g(\bar{X}_u^K) \rangle \, dW_u \right|^{p/2} \right], \]

\[ I_3 = \mathbb{E} \left[ \left( \sum_{k=0}^{n_t-1} \|g(\bar{X}_{kT}) \Delta W_k\|^2 \right)^{p/2} \right], \]

\[ I_4 = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| 2(\bar{X}_s^K + f(\bar{X}_s^K)(s-s), g(\bar{X}_s^K)(W_s-W_s)) \right|^{p/2} \right], \]

\[ I_5 = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|g(\bar{X}_s^K)(W_s-W_s)\|^p \right]. \]

We now consider \( I_1, I_2, I_3, I_4, I_5 \) in turn. Using Jensen’s inequality, we obtain

\[ I_1 \leq \|X_0\|^p + (2\alpha)^{p/2} T^{p/2-1} \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|\bar{X}_u^K\|^p \right] \, ds + (2\beta T)^{p/2}. \]

For \( I_2 \), we begin by noting that up to condition (6), for \( u < t \) we have

\[ \|\phi(\bar{X}_u^K)\|^2 = \|\bar{X}_u^K\|^2 + 2h(\bar{X}_u^K) \left( \langle \bar{X}_u^K, f(\bar{X}_u^K) \rangle + \frac{1}{2} h(\bar{X}_u^K)\|f(\bar{X}_u^K)\|^2 \right) \leq \|\bar{X}_u^K\|^2 + 2h(\bar{X}_u^K) (\alpha\|\bar{X}_u^K\|^2 + \beta) \leq (1 + 2\alpha T)\|\bar{X}_u^K\|^2 + 2\beta T, \]

and hence by Jensen’s inequality

\[ \|\phi(\bar{X}_u^K)\|^{p/2} \leq 2^{p/2-1} \left( 1 + 2\alpha T \right)^{p/4} \|\bar{X}_u^K\|^{p/2} + (2\beta T)^{p/4}. \]

In addition, the linear growth condition (5) gives

\[ \|g(\bar{X}_u^K)\|^{p/2} \leq 2^{p/2-1} \left( \alpha^{p/4} \|\bar{X}_u^K\|^{p/2} + \beta^{p/4} \right), \]

and combining the last two equation, there exists a constant \( c_{p,T} \) depending on \( p \) and \( T \), in addition to \( \alpha, \beta \), such that

\[ \|\phi(\bar{X}_u^K)^T g(\bar{X}_u^K)\|^{p/2} \leq c_{p,T} (\|\bar{X}_u^K\|^p + 1). \]
Then, by the Burkholder-Davis-Gundy inequality, there is a constant $C_p$ such that

\[
I_2 \leq C_p 2^{p/2} E \left[ \left( \int_0^t \| \phi(\tilde{X}_u^K) T g(\tilde{X}_u^K) \|^2 \, du \right)^{p/4} \right] \\
\leq C_p 2^{p/2} T^{p/4-1} E \left[ \int_0^t \| \phi(\tilde{X}_u^K) T g(\tilde{X}_u^K) \|^{p/2} \, du \right] \\
\leq c_p,T C_p 2^{p/2} T^{p/4-1} \left( \int_0^t E \left[ \sup_{0 \leq u \leq s} \| \tilde{X}_u^K \|^p \right] \, ds + T \right).
\]

For $I_3$, we start by observing that by standard results there exists a constant $c_p$ which depends solely on $p$ such that for any $t_k \leq s < t_{k+1}$,

\[
(25) \quad E \left[ \sup_{t_k \leq u \leq s} \| W_u - W_{t_k} \|^p \mid \mathcal{F}_{t_k} \right] = c_p (s - \bar{s})^{p/2}.
\]

One variant of Jensen’s inequality, when $h_k, u_k$ are both positive and $p \geq 1$, is

\[
\left( \sum_k h_k u_k \right)^p \leq \left( \sum_k h_k \right)^{p-1} \sum_k h_k u_k^p.
\]

Using this, and (25) with $s \equiv t_{k+1}$ so that $s - \bar{s} = h_k$,

\[
I_3 \leq T^{p/2-1} E \left[ \sum_{k=0}^{n_t-1} h_k \| g(\tilde{X}_{t_k}^K) \|^p \frac{\| \Delta W_k \|^p}{h_k^{p/2}} \right] \\
\leq T^{p/2-1} c_p E \left[ \int_0^t \| g(\tilde{X}_s^K) \|^p \, ds \right].
\]

Using condition (5), and Jensen’s inequality, we then obtain

\[
I_3 \leq (2 T)^{p/2-1} c_p \left( \alpha^{p/2} \int_0^t E \left[ \sup_{0 \leq u \leq s} \| \tilde{X}_u^K \|^p \right] \, ds + \beta^{p/2} T \right).
\]

For $I_4$, using (22) and following the same argument as for $I_2$, there exists a constant $c_{p,T}$ depending on both $p$ and $T$ such that

\[
\| \tilde{X}_s^K + f(\tilde{X}_s^K)(s - \bar{s}) \|^p \| g(\tilde{X}_s^K) \|^{p/2} \leq c_{p,T} \left( \| \tilde{X}_s^K \|^p + 1 \right).
\]
Therefore, again using (25),

\[ I_4 \leq 2^{p/2} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| (\hat{X}_s^K + f(\hat{X}_s^K)(s-\Delta), g(\hat{X}_s^K)(W_s - W_\Delta)) \right|^{p/2} \right] \]

\[ \leq c_{p,T} 2^{p/2} \mathbb{E} \left[ \sum_{k=0}^{n_{t-1}} (\|\hat{X}_{t_k}^K\|^{p+1}) \sup_{t_k \leq s \leq t_{k+1}} \| (W_s - W_\Delta) \|^{p/2} \right. \]

\[ + (\|\hat{X}_t^K\|^{p+1}) \sup_{0 \leq s \leq t} \| (W_s - W_\Delta) \|^{p/2} \right] \]

\[ \leq c_{p/2} c_{p,T} 2^{p/2} T^{p/4-1} \mathbb{E} \left[ \sum_{k=0}^{n_{t-1}} (\|\hat{X}_{t_k}^K\|^{p+1} h_k + (\|\hat{X}_t^K\|^{p+1} (t-\Delta) \right] \]

\[ \leq c_{p/2} c_{p,T} 2^{p/2} T^{p/4-1} \left( \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|\hat{X}_u^K\| \right] ds + T \right). \]

Similarly, using the same definition for \( c_p \), we have

\[ I_5 \leq c_p (2T)^{p/2-1} \left( \alpha^{p/2} \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|\hat{X}_u^K\| \right] ds + \beta^{p/2} T \right). \]

Collecting together the bounds for \( I_1, I_2, I_3, I_4, I_5 \), we conclude that there exist constants \( C^1_{p,T} \) and \( C^2_{p,T} \) such that

\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|\hat{X}_s^K\|^p \right] \leq C^1_{p,T} + C^2_{p,T} \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|\hat{X}_u^K\| \right] ds, \]

and Grönwall’s inequality gives the result

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|\hat{X}_t^K\|^p \right] \leq C^1_{p,T} \exp(C^2_{p,T} T) \triangleq C_{p,T} < \infty. \]

**Step 4: Expected supremum of \( p \)-th-moment of \( \hat{X}_t \)**

For any \( \omega \in \Omega \), \( \hat{X}_t = \hat{X}_t^K \) for all \( 0 \leq t \leq T \) if, and only if, \( \sup_{0 \leq t \leq T} \| \hat{X}_t \| \leq K \). Therefore, by the Markov inequality,

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \| \hat{X}_t \| < K \right) = \mathbb{P} \left( \sup_{0 \leq t \leq T} \| \hat{X}_t^K \| < K \right) \geq 1 - \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| \hat{X}_t^K \|^4 \right]/K^4 \rightarrow 1 \]

as \( K \rightarrow \infty \). Hence, almost surely, \( \sup_{0 \leq t \leq T} \| \hat{X}_t \| < \infty \) and \( T \) is attainable. Also,

\[ \lim_{K \rightarrow \infty} \sup_{0 \leq t \leq T} \| \hat{X}_t^K (\omega) \| = \sup_{0 \leq t \leq T} \| \hat{X}_t (\omega) \| \]
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and for $0 < K_1 \leq K_2,$

$$\sup_{0 \leq t \leq T} \| \hat{X}^K_t(\omega) \| \leq \sup_{0 \leq t \leq T} \| \hat{X}^K_t(\omega) \| \leq \sup_{0 \leq t \leq T} \| \hat{X}_t(\omega) \|.$$ 

Therefore, by the Monotone Convergence Theorem,

$$E \left[ \sup_{0 \leq t \leq T} \| \hat{X}_t \|^p \right] = \lim_{K \to \infty} E \left[ \sup_{0 \leq t \leq T} \| \hat{X}^K_t \|^p \right] \leq C_{p,T}.$$ 

4.2. Theorem 3.

**Proof.** The approach which is followed is to bound the approximation error $e_t \triangleq \hat{X}_t - X_t$ by terms which depend on either $\hat{X}_s - \bar{X}_s$ or $e_s,$ and then use local analysis within each timestep to bound the former, and Grönwall’s inequality to handle the latter.

The proof is again given for $p \geq 4;$ the result for $0 \leq p < 4$ follows from Hölder’s inequality.

We start by combining the original SDE with (3) to obtain

$$de_t = (f(\bar{X}_t) - f(X_t)) \, dt + (g(\bar{X}_t) - g(X_t)) \, dW_t,$$

and then by Itô’s formula, together with $e_0 = 0,$ we get

$$\|e_t\|^2 \leq 2 \int_0^t \langle e_s, f(\bar{X}_s) - f(X_s) \rangle \, ds - 2 \int_0^t \langle e_s, f(\hat{X}_s) - f(\bar{X}_s) \rangle \, ds$$

$$+ \int_0^t \|g(\bar{X}_s) - g(X_s)\|^2 \, ds + 2 \int_0^t \langle e_s, (g(\bar{X}_s) - g(X_s)) \, dW_s \rangle.$$

Using the conditions in Assumption 5, (8) implies that

$$\langle e_s, f(\bar{X}_s) - f(X_s) \rangle \leq \frac{1}{2} \alpha \|e_s\|^2,$$

(10) implies that

$$\left| \langle e_s, f(\bar{X}_s) - f(\hat{X}_s) \rangle \right| \leq \|e_s\| \, L(\hat{X}_s, \bar{X}_s) \|\bar{X}_s - \hat{X}_s\|$$

$$\leq \frac{1}{2} \|e_s\|^2 + \frac{1}{2} L(\hat{X}_s, \bar{X}_s)^2 \|\bar{X}_s - \hat{X}_s\|^2$$

where $L(x,y) \triangleq \gamma(\|x\|^q + \|y\|^q) + \mu,$ and (9) gives

$$\|g(\bar{X}_s) - g(X_s)\|^2 \leq \frac{1}{2} \alpha \|\bar{X}_s - X_s\|^2 \leq \alpha \|e_s\|^2 + \alpha \|\bar{X}_s - \hat{X}_s\|^2.$$
Hence,
\[
\|e_t\|^2 \leq (2\alpha+1) \int_0^t \|e_s\|^2 \, ds + \int_0^t \left( L(\hat{X}_s, \bar{X}_s)^2 + \alpha \right) \|\hat{X}_s - \bar{X}_s\|^2 \, ds
\]
\[
+ 2 \int_0^t \langle e_s, (g(\hat{X}_s) - g(X_s)) \rangle \, dW_s.
\]
and then by Jensen’s inequality we obtain
\[
\|e_t\|^p \leq (3T)^{p/2-1}(2\alpha+1)^{p/2} \int_0^t \|e_s\|^p \, ds
\]
\[
+ (3T)^{p/2-1} \int_0^t \left( L(\hat{X}_s, \bar{X}_s)^2 + \alpha \right)^{p/2} \|\hat{X}_s - \bar{X}_s\|^p \, ds
\]
\[
+ 3^{p/2-1}2^{p/2} \left| \int_0^t \langle e_s, (g(\hat{X}_s) - g(X_s)) \rangle \, dW_s \right|^{p/2}.
\]
Taking the supremum of each side, and then the expectation yields
\[
E \left[ \sup_{0 \leq s \leq t} \|e_s\|^p \right] \leq (3T)^{p/2-1}(2\alpha+1)^{p/2} \int_0^t E \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] \, ds
\]
\[
+ (3T)^{p/2-1} \int_0^t E \left[ \left( L(\hat{X}_s, \bar{X}_s)^2 + \alpha \right)^{p/2} \|\hat{X}_s - \bar{X}_s\|^p \right] \, ds
\]
\[
+ 3^{p/2-1}2^{p/2} E \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \langle e_u, (g(\hat{X}_u) - g(X_u)) \rangle \, dW_u \right|^{p/2} \right].
\]
By the Hölder inequality,
\[
E \left[ \left( L(\hat{X}_s, \bar{X}_s)^2 + \alpha \right)^{p/2} \|\hat{X}_s - \bar{X}_s\|^p \right]
\]
\[
\leq \left( E \left[ \left( L(\hat{X}_s, \bar{X}_s)^2 + \alpha \right)^p \right] E \left[ \|\hat{X}_s - \bar{X}_s\|^{2p} \right] \right)^{1/2},
\]
and $E \left[ \left( L(\hat{X}_s, \bar{X}_s)^2 + \alpha \right)^p \right]$ is uniformly bounded on $[0, T]$ due to the stability property in Theorem 1.

In addition, by the Burkholder-Davis-Gundy inequality (which gives the constant $C_p$ which depends only on $p$) followed by Jensen’s inequality plus
the Lipschitz condition for \( g \), we obtain

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \langle e_u, (g(\hat{X}_u) - g(X_u)) \rangle \mathrm{d}W_u \right|^{p/2} \right] \\
\leq C_p \mathbb{E} \left[ \left( \int_0^t \|e_s\|^2 \|g(\hat{X}_s) - g(X_s)\|^2 \mathrm{d}s \right)^{p/4} \right].
\]

\[
\leq C_p \, T^{p/4-1} \left( \frac{1}{2} \alpha \right)^{p/4} E \left[ \int_0^t \|e_s\|^{p/2} \|\dot{X}_s - X_s\|^{p/2} \mathrm{d}s \right].
\]

\[
\leq C_p \, T^{p/4-1} \left( \frac{1}{2} \alpha \right)^{p/4} E \left[ \int_0^t \left( \frac{1}{2} \|e_s\|^p + \frac{1}{2} \|\dot{X}_s - X_s\|^p \right) \mathrm{d}s \right].
\]

Hence, using \( \mathbb{E} \left[ \|\dot{X}_s - X_s\|^p \right] \leq (\mathbb{E} \left[ \|\dot{X}_s - X_s\|^{2p} \right])^{1/2} \), there are constants \( C_{p,T}, C_{p,T}^2 \) such that

\[
(26) \quad \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^p \right] \leq C_{p,T}^1 \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] \mathrm{d}s + C_{p,T}^2 \int_0^t \left( \mathbb{E} \left[ \|\dot{X}_s - X_s\|^{2p} \right] \right)^{1/2} \mathrm{d}s.
\]

For any \( s \in [0, T] \), \( \dot{X}_s - X_s = f(\hat{X}_s)(s - \bar{s}) + g(\hat{X}_s)(W_s - W_{\bar{s}}) \), and hence, by a combination of Jensen and Hölder inequalities, we get

\[
\mathbb{E} \left[ \|\dot{X}_s - X_s\|^p \right] \leq 2^{2p-1} \left( \mathbb{E} \left[ \|f(\hat{X}_s)\|^{4p} \right] \mathbb{E} \left[ (s - \bar{s})^{4p} \right] \right)^{1/2} + 2^{2p-1} \left( \mathbb{E} \left[ \|g(\hat{X}_s)\|^{4p} \right] \mathbb{E} \left[ \|W_s - W_{\bar{s}}\|^{4p} \right] \right)^{1/2}.
\]

\( \mathbb{E} \|f(\hat{X}_s)\|^{4p} \) and \( \mathbb{E} \|g(\hat{X}_s)\|^{4p} \) are both uniformly bounded on \([0, T]\) due to stability and the polynomial bounds on the growth of \( f \) and \( g \). Furthermore, we have \( \mathbb{E} [(s - \bar{s})^{4p}] \leq (\delta T)^{4p} \leq \delta^p T^{4p} \), and by standard results there is a constant \( c_p \) such that \( \mathbb{E} \|W_s - W_{\bar{s}}\|^{4p} = \mathbb{E} [\mathbb{E} \|W_s - W_{\bar{s}}\|^{4p} \mid F_{\bar{s}}] \leq c_p (\delta T)^{2p} \). Hence, there exists a constant \( C_{p,T}^3 > 0 \) such that \( \mathbb{E} \left[ \|\dot{X}_s - X_s\|^{2p} \right] \leq C_{p,T}^3 \delta^p \), and therefore equation (26) gives us

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^p \right] \leq C_{p,T}^1 \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] \mathrm{d}s + C_{p,T}^2 \sqrt{T} \sqrt{C_{p,T}^3 T} \delta^{p/2},
\]

and Grönwall’s inequality then provides the final result. $\square$
4.3. Theorem 4.

Proof. The error $e_t \triangleq \hat{X}_t - X_t$ satisfies the SDE $de_t = (f(\hat{X}_t) - f(X_t)) \, dt$ and hence
\[
\|e_t\|^2 = 2 \int_0^t \langle e_s, f(\hat{X}_s) - f(X_s) \rangle \, ds - 2 \int_0^t \langle e_s, f(\hat{X}_s) - f(\tilde{X}_s) \rangle \, ds \leq \alpha \int_0^t \|e_s\|^2 \, ds - 2 \int_0^t \langle e_s, f(\hat{X}_s) - f(\tilde{X}_s) \rangle \, ds,
\]
due to the one-sided Lipschitz condition (8), so therefore
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^p \right] \leq \alpha^{p/2} (2T)^{p/2-1} \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] \, ds + 2^{p-1} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \int_0^s \langle e_u, f(\hat{X}_u) - f(\tilde{X}_u) \rangle \, du \right]^{p/2}.
\]

Within a single timestep, $\hat{X}_s - \bar{X}_s = f(\bar{X}_s)(s-\bar{s}) + (W_s - \bar{W}_s)$, and therefore Lemma 4 gives
\[
\langle e_s, f(\hat{X}_s) - f(\bar{X}_s) \rangle = \langle e_s, \nabla f(\bar{X}_s)(\hat{X}_s - \bar{X}_s) \rangle + R_s = \langle e_s, (s-\bar{s}) \nabla f(\bar{X}_s)f(\tilde{X}_s) \rangle + R_s + \langle (e_s - e_{\bar{s}}), \nabla f(\bar{X}_s)(W_s - \bar{W}_s) \rangle + \langle e_{\bar{s}}, \nabla f(\bar{X}_s)(W_s - \bar{W}_s) \rangle,
\]
where $|R_s| \leq \left( \gamma (\|\tilde{X}_s\|^q + \|\bar{X}_s\|^q) + \mu \right) \|e_s\| \|\hat{X}_s - \bar{X}_s\|^2$. Hence,
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \langle e_u, f(\hat{X}_u) - f(\bar{X}_u) \rangle \, du \right|^{p/2} \right] \leq 4^{p/2-1} (I_1 + I_2 + I_3 + I_4),
\]
where
\[
I_1 = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \langle e_u, (u-\bar{u}) \nabla f(\bar{X}_u)f(\tilde{X}_u) \rangle \, du \right|^{p/2} \right],
I_2 = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s R_u \, du \right|^{p/2} \right],
I_3 = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \langle (e_u - e_{\bar{u}}), \nabla f(\bar{X}_u)(W_u - \bar{W}_u) \rangle \, du \right|^{p/2} \right],
I_4 = \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \langle e_{\bar{u}}, \nabla f(\bar{X}_u)(W_u - \bar{W}_u) \rangle \, du \right|^{p/2} \right].
\]
We now bound $I_1, I_2, I_3, I_4$ in turn. Noting that $s - \bar{s} \leq \delta T$,
\[
I_1 \leq T^{p/2-1} \int_0^t \mathbb{E} \left[ \|e_s\|^{p/2} (\delta T)^{p/2} \|f(\bar{X}_s)\|^{p/2} \|\nabla f(\bar{X}_s)\|^{p/2} \right] ds
\]
\[
\leq \frac{1}{2} T^{p/2-1} \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] ds
\]
\[
+ \frac{1}{2} T^{p/2-1} (\delta T)^p \int_0^t \mathbb{E} \left[ \|f(\bar{X}_s)\|^p \|\nabla f(\bar{X}_s)\|^p \right] ds.
\]
The last integral is finite because of stability and the polynomial bounds on the growth of both $f$ and $\nabla f$, and hence there is a constant $C_{\bar{p},T}^1$ such that
\[
I_1 \leq \frac{1}{2} T^{p/2-1} \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] ds + C_{\bar{p},T}^1 \delta^p.
\]
Similarly, using the Hölder inequality,
\[
I_2 \leq T^{p/2-1} \int_0^t \mathbb{E} \left[ \|e_s\|^{p/2} \gamma (\|\bar{X}_s\|^q + \|\bar{X}_s\|^q + \mu)^{p/2} \|\bar{X}_s - \bar{X}_s\|^p \right] ds
\]
\[
\leq \frac{1}{2} T^{p/2-1} \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] ds
\]
\[
+ \frac{1}{2} T^{p/2-1} \int_0^t \mathbb{E} \left[ \gamma (\|\bar{X}_s\|^q + \|\bar{X}_s\|^q + \mu)^{2p} \right] \mathbb{E} \left[ \|\bar{X}_s - \bar{X}_s\|^{4p} \right]^{1/2} ds,
\]
and hence, using stability and bounds on $\mathbb{E} \left[ \|\bar{X}_s - \bar{X}_s\|^{4p} \right]$ from the proof of Theorem 3, there is a constant $C_{\bar{p},T}^2$ such that
\[
I_2 \leq \frac{1}{2} T^{p/2-1} \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] ds + C_{\bar{p},T}^2 \delta^p.
\]
For the next term, $I_3$, we start by bounding $\|e_s - e_\Delta\|$. Since
\[
e_s - e_\Delta = \int_\Delta^s (f(\bar{X}_u) - f(X_u)) du,
\]
by Jensen’s inequality and Assumption 7 it follows that
\[
\|e_s - e_\Delta\|^p \leq (\delta T)^{p-1} \int_\Delta^s \|f(\bar{X}_u) - f(X_u)\|^p du
\]
\[
\leq (2 \delta T)^{p-1} \int_\Delta^s L^p(\bar{X}_u, X_u) \left( \|e_u\|^p + \|\bar{X}_u - \bar{X}_u\|^p \right) du,
\]
where \( L(\bar{X}_u, X_u) \equiv \gamma(\|\bar{X}_u\|^k + \|X_u\|^k) + \mu \). We again have an \( O(\delta^{p/2}) \) bound for \( E[\|\bar{X}_s - X_s\|^p] \), while Theorem 3 proves that there is a constant \( c_{p,T} \) such that

\[
E[\|e_s\|^p] \leq c_{p,T} \delta^{p/2}.
\]

Combining these, and using the Hölder inequality and the finite bound for \( E[L^p(\bar{X}_u, X_u)] \) for all \( p \geq 2 \), due to the usual stability results, we find that there is a different constant \( c_{p,T} \) such that

\[
E[\|e_s - e_\bar{s}\|^p] \leq c_{p,T} \delta^{3p/2}.
\]

Now,

\[
I_3 \leq T^{p/2-1} \int_0^t E\left[ \|e_s - e_\bar{s}\|^p/2 \|\nabla f(\bar{X}_s)\|^p/2 \|W_s - W_\bar{s}\|^{p/2} \right] ds,
\]

so using the Hölder inequality and the usual stability bounds, we conclude that there is a constant \( C_{p,T}^3 \) such that

\[
I_3 \leq C_{p,T}^3 \delta^p.
\]

Lastly, we consider \( I_4 \). For the timestep \([t_n, t_{n+1}]\), we have

\[
d((t-t_n)(W_t - W_{t_n})) = (W_t - W_{t_n}) \, dt + (t-t_{n+1}) \, dW_t
\]

and therefore, integrating by parts within each timestep,

\[
\int_0^s \langle e_u, \nabla f(\bar{X}_u)(W_u - W_\bar{u}) \rangle \, du
\]

\[
= \int_0^s (\bar{u} - u) \langle e_u, \nabla f(\bar{X}_u) \, dW_u \rangle - (\bar{s} - s) \langle e_\bar{s}, \nabla f(\bar{X}_s)(W_s - W_\bar{s}) \rangle
\]

where \( \bar{u} = \min\{t_n : t_n > u\} = t_{n+1} \). Hence, \( I_4 \leq 2^{p/2-1}(I_{41} + I_{42}) \) where

\[
I_{41} = E\left[ \sup_{0 \leq s \leq t} \left( \int_0^s (\bar{u} - u) \langle e_u, \nabla f(\bar{X}_u) \, dW_u \rangle \right)^{p/2} \right],
\]

\[
I_{42} = E\left[ \sup_{0 \leq s \leq t} \left( (\bar{s} - s) \langle e_\bar{s}, \nabla f(\bar{X}_s)(W_s - W_\bar{s}) \rangle \right)^{p/2} \right].
\]

By the Burkholder-Davis-Gundy inequality,

\[
I_{41} \leq C_p E\left[ \left( \int_0^t (\bar{s} - s)^2 \|e_\bar{s}\|^2 \|\nabla f(\bar{X}_s)\|^2 \, ds \right)^{p/4} \right]
\]

\[
\leq C_p T^{3p/4-1} E\left[ \int_0^t \|e_\bar{s}\|^{p/2} \delta^{p/2} \|\nabla f(\bar{X}_s)\|^{p/2} \, ds \right]
\]

\[
\leq \frac{1}{2} C_p T^{3p/4-1} E\left[ \int_0^t \left( \sup_{0 \leq u \leq s} \|e_u\|^p + \delta^p \|\nabla f(\bar{X}_s)\|^p \right) \, ds \right]
\]
with $\mathbb{E}[\|\nabla f(\bar{X}_s)\|^p]$ uniformly bounded on $[0, T]$ so that there is a constant $C_{p,T}^{41}$ such that

$$I_{41} \leq \frac{1}{2} C_{p,T}^{3p/4-1} \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] ds + C_{p,T}^{41} \delta^p.$$  

Turning to $I_{42}$, Young’s inequality and Hölder’s inequality give

$$I_{42} \leq \frac{1}{2\xi} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^p \right] + \frac{\xi}{2} (2\delta T)^p \left( \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|\nabla f\|^2p \right] \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|W_s\|^2p \right] \right)^{1/2}$$

for any $\xi > 0$, and hence there is a constant $C_{p,T}^{42}$ such that

$$I_{42} \leq \frac{1}{2\xi} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^p \right] + \xi C_{p,T}^{42} \delta^p.$$

Returning to (27), and inserting the bounds for $I_1$, $I_2$, $I_3$, $I_4$, $I_{41}$, and $I_{42}$, with $\xi = 2^{3p/2-4}$, gives

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^p \right] \leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|e_s\|^p \right] + C_{p,T}^5 \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} \|e_u\|^p \right] ds + C_{p,T}^6 \delta^p,$$

for certain constants $C_{p,T}^5, C_{p,T}^6$. Rearranging and using Grönwall’s inequality we obtain the final conclusion that there exists a constant $C_{p,T}$ such that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|e_t\|^p \right] \leq C_{p,T} \delta^p.$$

5. Conclusions and future work. The central conclusion from this paper is that by using an adaptive timestep it is possible to make the Euler-Maruyama approximation stable for SDEs with a globally Lipschitz volatility and a drift which is not globally Lipschitz but is locally Lipschitz and satisfies a one-sided linear growth condition. If the drift also satisfies a one-sided Lipschitz condition then the order of strong convergence is $1/2$, when looking at the accuracy versus the expected cost of each path. For the important class of Langevin equations with unit volatility, the order of strong convergence is 1.

The numerical experiments suggest that in some applications the new method may not be significantly better than the tamed Euler-Maruyama method proposed and analysed by Hutzenthaler, Jentzen & Kloeden [10], but in others it is shown to be superior.
One direction for extension of the theory is to SDEs with a volatility which is not globally Lipschitz, but instead satisfies the Khasminskii-type condition used by Mao & Szpruch [17, 19]. Another is to extend the analysis to Milstein approximations, which are particularly important when the SDE is scalar or satisfies the commutativity condition which means that the Milstein approximation does not require the simulation of Lévy areas. Another possibility is to use a Lyapunov function $V(x)$ in place of $\|x\|^2$ in the stability analysis; this might enable one to prove stability and convergence for a larger set of SDEs. For SDEs such as the stochastic van der Pol oscillator and the stochastic Lorenz equation, if we could prove exponential integrability using the approach of Hutzenthaler, Jentzen & Wang [11] then it may be possible to prove the order of strong convergence using a local one-sided Lipschitz condition.

A future paper will address a different challenge, extending the analysis to ergodic SDEs over an infinite time interval. As well as proving a slightly different stability result with a bound which is uniform in time, the convergence analysis will show that under certain conditions the error bound is also uniformly bounded in time. This is in contrast to the analysis in this paper in which the bound increases exponentially with time.

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