A comment on fluctuations and stability limits with application to “superheated” black holes

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Abstract. We point out that, contrary to signs of heat capacities, thermodynamic fluctuations are simply and unequivocally related to onset of instabilities that show up near critical points. Fluctuation theory is then applied to Schwarzschild black holes surrounded by radiation. This shows that slowly evolving black holes along quasi-equilibrium states in cavities greater than $10^6$ Planck length will not evaporate below the critical Hawking limit temperature despite the fact that pure radiation has a much higher entropy.

In a mean field approximation, self-gravitating thermodynamic ensembles have local maxima of entropy or other Massieu functions derived from the entropy by a Legendre transformation, even though the potential energy is not bounded from below (see e.g. Horwitz and Katz 1978). These ensembles are generally not equivalent. This means that the domains of stability of different ensembles related by a Legendre transformation are different. Arguments based on signs of heat capacities and similar thermodynamic coefficients are dubious criteria of stability. Far more secure results are derived from linear series (i.e. equilibrium curves) using Poincaré’s theorem about bifurcations or turning points (Ledoux 1958). One of the most elegant ways of using Poincaré’s method consists in following equilibrium curves of pairs of conjugate parameters: inverse temperature versus energy, angular velocity versus angular momentum, and so on. (Katz 1978, 1979). This is the method used in Kaburaki et al. (1993) for Kerr black holes and in Katz et al. (1993) for Kerr-Newman black holes to find stability limits of stable configurations and degrees of instability in unstable configurations [see Parentani (1994) for the origin of the inequivalence and the degree of stability in canonical ensembles].

Here we want to stress that fluctuation theory is closely related to linear series of conjugate parameters and is indeed applicable to thermodynamic ensembles whether they are equivalent or not. This is readily seen as follows.

Think, for definition, of a Schwarzschild black hole in a sphere filled with photons at the same temperature (Hawking 1976). The total entropy of the system is $S = S(E, V)$, where $E$ is the total energy of hole and radiation and $V$ is the volume of the cavity. Conjugate pairs with respect to $S$ are $(\beta, E)$ and $(P, V)$ with

$$\beta = \frac{\partial S}{\partial E}, \quad P = \frac{1}{\beta} \frac{\partial S}{\partial V}$$ (1)

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Here $\beta^{-1}$ is the temperature, $P$ the pressure (units are $c=\hbar=G=k=1$). Linear series of conjugate variables are the equilibrium curves $\beta(E)$ at fixed $V$ (see figure 1) and $P(V)$ at fixed $E$ (not drawn). Changes of stability in the microcanonical ensemble (i.e. at fixed $E$ and $V$) occur only at the critical point $C$, where there is a vertical tangent ($(\partial \beta/\partial E)_V = \pm\infty$). Positive slopes near $C$ are on a branch of stable configurations, while negative slopes near $C$ on one of unstable configurations. The proof is general and has been given elsewhere but is quite useful to remember and short enough to be reproduced here for the cases in which there is only one fluctuating variable $x$. This is the situation of the Hawking example where $x$ is the partition energy $E_{bh}$, the energy of the black hole and where the left over energy of the radiation is $E_{rad} = E - E_{bh}$.

Let $\tilde{S}(x; E, V)$ be the entropy away from equilibrium. Equilibrium is defined by extremising the entropy with respect to $x$

$$\partial_x \tilde{S} = 0 \tag{2}$$

giving two solutions $x = X(E, V)$ and $x = X_1(E, V)$. Denote $(X, X_1)$ collectively by $X_a$ ($a = \text{nothing or 1}$) and $\tilde{S}(X_a; E, V) = S_a(E, V)$. It is useful to define a temperature away from equilibrium

$$\tilde{\beta} = \frac{\partial E}{\partial \tilde{S}} = \tilde{\beta}(x; E, V). \tag{3}$$

Clearly this function evaluated along the equilibrium configurations $x = X_a$ gives

$$\tilde{\beta}_{x=X_a} = \beta_a = \left(\frac{\partial S_a}{\partial E}\right)_V \tag{4}$$

which is the usual equilibrium inverse temperature. The equilibrium curves $\beta(E)$ and $\beta_1(E)$ are represented in figure 1.

Stable equilibria exist if and only if $x = X_a$ is a configuration at which $\tilde{S}$ is maximum:

$$\left(\frac{\partial^2 \tilde{S}}{\partial x^2}\right)_a < 0. \tag{5}$$

Consider now equation (2) in which $x$ is replaced by $X_a$; (2) reduces then to $0 = 0$ and the following identity holds

$$\left[\frac{\partial(\partial_x \tilde{S})_a}{\partial E}\right]_V = (\partial_x \tilde{\beta})_a + (\partial^2_x \tilde{S})_a \left(\frac{\partial X_a}{\partial E}\right)_V \equiv 0. \tag{6}$$

On the other hand, a further derivative of (4) with respect to $E$ with $V$ kept fixed gives

$$\left(\frac{\partial \beta_a}{\partial E}\right)_V = \left(\frac{\partial^2 E \tilde{S}}{\partial E^2}\right)_a + (\partial_x \tilde{\beta})_a \left(\frac{\partial X_a}{\partial E}\right)_V. \tag{7}$$

Eliminating $\partial X_a/\partial E$ between (6) and (7) gives then

$$\left(\frac{\partial \beta_a}{\partial E}\right)_V = \left(\frac{\partial^2 E \tilde{S}}{\partial E^2}\right)_a - \frac{(\partial_x \tilde{\beta})^2_a}{(\partial^2_x \tilde{S})_a}. \tag{8}$$

Since changes of stability occur when $(\partial^2_x \tilde{S})_a$ goes through zero, they manifest themselves by vertical slopes because for $(\partial^2_x \tilde{S})_a \rightarrow 0$

$$\left(\frac{\partial \beta_a}{\partial E}\right)_V \approx - \frac{(\partial_x \tilde{\beta})^2_a}{(\partial^2_x \tilde{S})_a} \rightarrow \pm\infty \tag{9}$$
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(see figure 1). The point of a vertical slope with coordinates \((E_c, \beta_c)\) is a critical point. Near a critical point the slope is positive on the stable branch \(\beta(E)\) where \((\partial^2_x \tilde{S})_a < 0\) and negative on the unstable branch \(\beta_1(E)\).

Consider now \(\tilde{S}(x; E, V)\) near a point of stable equilibrium, at \(x = X + \Delta x\):

\[
\tilde{S} = S + \frac{1}{2}(\partial^2_x \tilde{S})(\Delta x)^2 + O[(\Delta x)^3]
\]

or, according to (9)

\[
\tilde{S} \approx S - \frac{1}{2} \frac{(\partial_x \tilde{\beta})^2 (\Delta x)^2}{\left(\frac{\partial \beta}{\partial E}\right)_V} = S - \frac{1}{2} \frac{(\Delta \tilde{\beta})^2}{\left(\frac{\partial \beta}{\partial E}\right)_V} \tag{11}
\]

where \(\Delta \tilde{\beta} = \tilde{\beta} - \beta\) and

\[
\left(\frac{\partial \beta}{\partial E}\right)_V > 0. \tag{12}
\]

\(\Delta \tilde{\beta}\) is a deviation \(\tilde{\beta}(x)\) from stable equilibrium values \(\beta(E)\) induced by the fluctuation of \(x\) away from the equilibrium configuration \(X(E, V)\). The probability \(dW\) of such a fluctuation of \(\tilde{\beta}(x)\) in a range \((\tilde{\beta} + \Delta \tilde{\beta}, \tilde{\beta} + \Delta \tilde{\beta} + d\tilde{\beta})\) is proportional to \(\exp(\tilde{S} - S)\) [see Landau and Lifshitz (1980)]. The normalization factor of \(dW\) is easy to find in this quadratic approximation:

\[
dW = \frac{1}{\sqrt{2\pi(\Delta \tilde{\beta})^2}} \exp \left[ -\frac{1}{2} \frac{(\Delta \tilde{\beta})^2}{(\Delta \beta)^2} \right] d\tilde{\beta} \tag{13}
\]

where

\[
(\Delta \beta)^2 = \left(\frac{\partial \beta}{\partial E}\right)_V. \tag{14}
\]

The mean square fluctuation \((\Delta \beta)^2\) of \(\tilde{\beta}\) is thus given near the critical point by the positive slope of \(\beta(E, V)\), i.e. where the specific heat is negative. [Note however that in the canonical ensemble, at fixed \(\beta\), the mean square fluctuations of the total energy are given by the slope of \(-E(\beta, V)\) as usual (see (19.6) in Callen 1985).] The real physical mean square fluctuations of temperatures in the bath of photons \((\Delta \beta_{\text{rad}})^2\) and in the black hole \((\Delta \beta_{\text{bh}})^2\) are neither equal to \((\Delta \beta)^2\) nor equal to each other because subsystems have different heat capacities. It is, however, not hard to show that near the critical point

\[
\lim_{x \to X_a} \frac{(\Delta \beta_{\text{rad}})^2}{(\Delta \beta)^2} = \lim_{x \to X_a} \frac{(\Delta \beta_{\text{bh}})^2}{(\Delta \beta)^2} = 1. \tag{15}
\]

Thus \((\Delta \beta)^2\) as given in (14) or (9) represents, to a good degree of approximation, the quadratic fluctuations of temperature that diverge at a critical point. Similar fluctuations of a pressure function \(\tilde{P}(x; E, V) = \beta^{-1}\partial_V \tilde{S}\) might have been considered.

The more general relation between positive slopes and mean square fluctuations of a conjugate variable in linear series near the critical points where a change of stability takes place is obvious: if \(\tilde{\zeta}\) (like \(\tilde{\beta}\)) is the conjugate function of \(Z\) which, like \(E\), is a control parameter with respect to some Massieu function \(\tilde{H}(\tilde{\zeta}; Z)\) (similar to \(\tilde{S}(x; E, V)\)), the equilibrium value of \(\tilde{\zeta}\) is

\[
\zeta = \frac{\partial H}{\partial Z}. \tag{16}
\]
We assume that second derivatives of $\tilde{H}$ are bounded and that, at the critical point, $(\partial_x \tilde{H})_a \neq 0$. If $(\partial_x \tilde{H})_a = 0$, vertical slopes may be transformed into bifurcations. But very small perturbation of $\tilde{H}$ like $\tilde{H} + \varepsilon x E$, $|\varepsilon| \ll |x E/\tilde{H}|$ will easily transform bifurcation into turning points (Thompson and Hut 1973) with almost no change in stability limits.

In these circumstances, near the critical point, the probability of $\tilde{\zeta}$ fluctuating by $\Delta \tilde{\zeta} = \tilde{\zeta} - \zeta_c$ is given by

$$dW = \frac{1}{\sqrt{2\pi(\Delta \zeta)^2}} \exp \left[ -\frac{1}{2} \frac{(\Delta \tilde{\zeta})^2}{(\Delta \zeta)^2} \right] d\tilde{\zeta}, \quad (\Delta \zeta)^2 = \frac{\partial \zeta}{\partial Z} > 0. \quad (17)$$

When there are more variables $x$, we imagine that the $x^i$'s have been taken in such a way that $- (\partial^2_{ij} \tilde{H})_a$ is diagonal, say $\delta_{ij}\lambda_j$. The $\lambda_j$ are called the Poincaré coefficients of stability. If the spectrum of $\lambda_i$'s is non-degenerate (which is always possible to achieve with a small perturbation of $\tilde{H}$), there will be a vertical slope at each point where a change of stability occurs and at the particular point $\partial \zeta/\partial Z$ will be the same approximate value given by (9).

For a simple application of fluctuation theory, we go back to (13) in the case of the Schwarzschild black hole in a cavity for which $\beta(E)$, given in Hawking (1976) and Gibbons and Perry (1978), is

$$E = \frac{\beta}{8\pi} + \frac{\pi^2}{15} V \beta^{-4}, \quad V = \frac{4\pi}{3} L^3 \quad (18)$$

where $L$ is the radius in Planck length unit. Suppose the black hole evolves through a series of quasi-equilibrium configurations by extracting slowly energy out of the box whose size is kept fixed. We know that at the critical point the hole will certainly evaporate. What we want to estimate is the probability per unit time that the black hole evaporates before reaching its stability limit as a result of thermal fluctuations.

From (14) and (18)

$$\overline{(\Delta \beta)^2} = \left( \frac{\partial \beta}{\partial E} \right)_V = \frac{8\pi}{\left[ 1 - \left( \frac{\beta}{\beta_1} \right)^5 \right]} \quad (19)$$

in which

$$\beta_c = \left( \frac{128\pi^4}{45} \right)^{\frac{1}{7}} L^\frac{6}{7} \approx 3.1 L^\frac{6}{7}. \quad (20)$$

The probability of complete evaporation of the black hole is essentially the probability of the system fluctuating from $\beta(E)$ to $\beta_1(E)$. Indeed, as soon as $\tilde{\beta}$ is slightly bigger than $\beta_1$, the probability of going back to $\beta$ is negligible for $S_{\text{rad}} - S_1 \gg S - S_1$. Therefore, to estimate $dW$, let us take for a typical value of $\Delta \tilde{\beta}$ the following one $\Delta \tilde{\beta} = \beta - \beta_c$ (taking $\beta - \beta_1$ would give an underestimate of that probability because the quadratic approximation is no longer valid over such an interval). With $|(\beta - \beta_c)/\beta_c| \ll 1$ and with (18) and (19), $dW$ as given in (13) becomes

$$dW \left( \frac{\Delta \tilde{\beta}}{\beta_c}, L \right) \approx 0.2 \left( \frac{\Delta \tilde{\beta}}{\beta_c} \right)^{\frac{1}{2}} \exp \left[ -0.9 \left( \frac{\Delta \tilde{\beta}}{\beta_c} \right)^3 L^\frac{6}{7} \right] d\tilde{\beta}. \quad (21)$$

Hence, for a given value of $\Delta \tilde{\beta}/\beta_c$, the probability of evaporation depends only on the size of the box. For instance, say, $\Delta \tilde{\beta}/\beta_c = 0.1$ and $L = 10^3$, $dW = 10^{-3} d\tilde{\beta}$ but for $L = 10^4$, $dW = 8 \cdot 10^{-28} d\tilde{\beta}$, i.e.
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$dW$ drops drastically for $L > 10^3$. If $\Delta \beta/\beta_c = 0.01$ and $L = 10^6$, $dW = 6 \cdot 10^{-9} d\beta$ but for $L = 10^7$, $dW \simeq 2 \cdot 10^{-105} d\beta$.

Having found $dW$, we may now estimate the rate of evaporation. When the equilibrium configuration evolves slowly from P to C (see figure 1), the characteristic time scale of the return to equilibrium after a jump in energy of $\Delta x$ is of the order of $\Delta t \approx \beta^4 \Delta x$ (see Zurek 1980). Since the characteristic time $t_c$ for a jump from point P at $\beta(E)$ to point Q at $\beta_1(E)$ is also proportional to $\beta^4$, it follows that the probability of evaporation per unit time is of the order of $dW/\beta^4 (\beta_c - \beta)$ (see Piran and Wald 1982 for a similar argument).

What the slope of $\beta(E)$ shows therefore is that for $L > 10^6$, fluctuations of temperature leading to complete evaporation are totally negligible and the evolution of the black hole proceeds down almost to point C before evaporation takes place. The bigger the box, the closer to C. This holds in spite of the fact that near C, configurations are metastable and evaporated black holes (i.e. pure radiation) have considerably more entropy: $S_{rad} - S_{bh+rad} \approx L^{6/5} > 10^7$.

Let us add that for $L > 10^6$, backreaction and quantum gravity are totally negligible effects (York 1985) and that the radiation is far from being general relativistic near the critical point (negligible self-gravity — see Parentani et al. 1994). Thus, Hawking’s approximation for $S$ on which our approximations are based is perfectly good. Instead, for smaller boxes, the thermodynamic analysis probably loses its validity because the quantum fluctuations become important since one approaches the Planck temperature at point C.

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Note added in proof. Since this work has been submitted we have done further work (Parentani et al. 1994) on the inequivalence of ensembles when long-range forces are present. A general analysis of the behavior of the fluctuations is presented.

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Figure caption

**Figure 1.** Qualitative features of Hawking’s $\beta(E)$, $\beta_1(E)$ and of $\beta_{\text{rad}}(E)$. $\Delta \tilde{\beta}(P)$ is the fluctuation around a point of equilibrium $P$. The critical fluctuation which will cause the evaporation of black hole are of the order of $\beta(P) - \beta_1(Q)$. The vertical line $E = E_B(V)$ is that for which $S = S_{\text{rad}}(E)$. For $E < E_B$ the entropy $S(E)$ is smaller than $S_{\text{rad}}(E)$ at fixed volume $V$. Thus, any point $P$ between $C$ and $B$ represents a superheated metastable (in fact a stable – see the text) black hole in equilibrium with radiation. Correspondingly, the portion of $E > E_B$ on the curve of $\beta_{\text{rad}}(E)$ expresses Gibbons and Perry’s (1978) superheated radiation state.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9412038v1