Safe non-smooth black-box optimization with application to policy search

Ilnura Usmanova
Automatic Control Laboratory, ETH Zürich, Switzerland
ILNURAU@CONTROL.EE.ETHZ.CH

Andreas Krause
Machine Learning Institute, ETH Zürich, Switzerland
KRAUSEA@ETHZ.CH

Maryam Kamgarpour
Automatic Control Laboratory, ETH Zürich, Switzerland
MKAMGAR@ETHZ.CH

Abstract

For safety-critical black-box optimization tasks, observations of the constraints and the objective are often noisy and available only for the feasible points. We propose an approach based on log barriers to find a local solution of a non-convex non-smooth black-box optimization problem

\[
\min f^0(x) \text{ subject to } f^i(x) \leq 0, \quad i = 1, \ldots, m,
\]

at the same time, guaranteeing constraint satisfaction while learning with high probability. Our proposed algorithm exploits noisy observations to iteratively improve on an initial safe point until convergence. We derive the convergence rate and prove safety of our algorithm. We demonstrate its performance in an application to an iterative control design problem.

1. Introduction

Motivation

Machine learning algorithms are increasingly being deployed for critical emerging applications such as autonomous driving, personalized medicine and robotics. In such scenarios, safety and reliability of these algorithms is crucial. When the model is unknown, too complex or unreliable, it is common to adopt a black-box bandit setup; our goal is to include safety in these learning techniques.

Related work

In the optimization literature, several constrained optimization algorithms exist guaranteeing feasibility of the iterates given just local information about the constraints. These include bundle methods, Feasible Sequential Quadratic Programming (FSQP) (Jian et al., 2005; Luo et al., 2012; Tang et al., 2014), the Method of Feasible Directions (MFD) (Zoutendijk, 1960) and their variations. However, all these methods require first and/or second order information and do not consider the presence of noise. On the other hand, there are many works on derivative-free optimization, including non-convex and non-smooth problems (Balasubramanian and Ghadimi, 2018; Nesterov and Spokoiny, 2017; Ghadimi and Lan, 2013; Lan, 2013), based on finite difference gradient estimation techniques. However, these methods do not guarantee the feasibility of the points where measurements are taken with respect to unknown constraints. This issue is addressed by interior point methods, where a barrier function is optimized. However, these methods typically require second order information.

Safe learning with zero-th order (bandit) information has been considered in Bayesian Optimization (Berkenkamp et al., 2016; Sui et al., 2015) for non-convex constrained problems. As these works aim at computing a global optimum, they have to solve a complicated and nontrivial non-convex subproblem at each iteration. Moreover, for most common kernel functions, these algorithms require a number of measurements

1. We thank the support of Swiss National Science Foundation, under the grant SNSF 200021_172781, and ERC under the European Union’s Horizon 2020 research and innovation programme grant agreement No 815943.
2. Technical report. Under review by Learning for Dynamics & Control conference (L4DC).

© 2020 I. Usmanova, A. Krause & M. Kamgarpour.
| Problem                                      | Unconstrained & smooth objective | Known linear constraints & smooth objective | Single unknown smooth constraint & smooth objective | Several unknown non-smooth constraints & non-smooth objective |
|----------------------------------------------|----------------------------------|---------------------------------------------|---------------------------------------------------|------------------------------------------------------------|
| Feedback                                    | 2-point bandit feedback          | 2-point bandit feedback                      | 1-point bandit feedback                            | 1-point bandit feedback                                     |
| Safety                                       | Yes (known constraints)          | Yes                                         | Yes                                               | Yes                                                         |
| Optimality condition                         | Stationary point: $E\|\nabla f(x_T)\|_2 \leq \eta$ | $\eta$-stationary point: $\forall u \in D \ E\|\nabla f(x_T) \cdot (x_t - u)\| \leq \eta$ | $\eta$-approximate KKT point                       | $\eta$-approximate KKT point of the smoothed problem       |
| Number of measurements                       | $O\left(\frac{d}{\eta^4}\right)$ or $O\left(\frac{d^2}{\eta^3}\right)$ | $O\left(\frac{d^3}{\eta^2}\right)$ (Balasubramanian and Ghadimi, 2018) | $\tilde{O}\left(\frac{d^3}{\eta^2}\right)$ (Usmanova et al., 2019a) | $\tilde{O}\left(\frac{d^3}{\eta^2}\right)$ (this work)  |

Table 1: Upper $O(\cdot)$ bounds on number of zero-th order oracle calls for non-convex smooth optimization algorithms.

that is exponential on the dimensionality. This makes Safe Bayesian Optimization methods not always applicable to high dimensional problems. Moreover, appropriately choosing a prior distribution and the kernel parameters might not be a trivial task. Gradient based local methods usually do not suffer from these drawbacks.

First order methods in application to barrier functions in the recent past were considered to have exponential runtime bounds due to the bad behavior of any barrier on the boundary of the feasible set. However, in the recent work (Hinder and Ye, 2019) the authors demonstrated that for smooth problems a gradient descent algorithm with adaptive step size on a log barrier function can be tractable, i.e., present attractive polynomial runtime convergence. Motivated by safe learning problems, the recent work (Usmanova et al., 2019b) extended this approach by (Hinder and Ye, 2019) to smooth optimization problems with zero-th order noisy measurements. However, the bound on the number of measurements was valid only in the case of a single smooth constraint function. In this paper, departing from (Usmanova et al., 2019b), we develop a safe algorithm for the non-smooth non-convex constrained optimization problems subject to an arbitrary number of constraint functions. In Table 1, a comparison of our algorithm with the existing methods for unconstrained and constrained zero-th order non-convex optimization is provided. In the first two algorithms a 2-point bandit feedback is assumed, i.e., it is possible to measure at several points with the same noise realization. In our algorithm we assume a more realistic and more complicated setup with changing noise at each measurement.

Safe learning is widely used in application to control of unknown dynamical systems. For example, the work by (Dean et al., 2019) exploited system identification and robust optimal control to learn the safe linear quadratic regulator (LQR) subject to constraints on the state and input trajectories. There are many works aiming safety while learning the optimal policy and dynamics in non-linear control, such as Fisac et al. (2018); Berkenkamp et al. (2016); Gillula and Tomlin (2012). Often Bayesian Optimization approach is used to solve the above problems. However, Bayesian Optimization might not enjoy acceptable scalability with dimensionality, thus limiting its applicability to control. Non-smoothness can also appear in some control problems such as bipedal walking, etc (Ames, 2014). In this paper, we consider an application of our method to safe learning for model-free control. We test our algorithm on the low dimensional example, but theoretically the dependence on the dimensionality is only polynomial and the algorithm can be applied for higher dimensional problems.

**Our contributions** Our main contribution is to propose an algorithm to find an approximate local solution to non-convex non-smooth cost functions subject to non-convex non-smooth constraints, and to establish its convergence rate in expectation in terms of the variance of the noise. Our algorithm is based on the
log barrier gradient descent approach, and only uses zeroth order information, also prove safety with high probability of our algorithm. We derive convergence rate to an approximate stationary point of the smooth approximation of the problem. In the special case where both the cost function and the constraints are smooth, we establish convergence to an approximate KKT point of the initial problem. We validate the performance of our algorithm in application to a simple model-free control problem.

2. Problem statement

Notations and definitions. Let \( \| \cdot \| \) denote the \( l_2 \)-norm on \( \mathbb{R}^d \). A function \( f : \mathbb{R}^d \to \mathbb{R} \) is called \( L \)-Lipschitz continuous if \( |f(x) - f(y)| \leq L \|x - y\| \). It is called \( M \)-smooth if the gradients \( \nabla f(x) \) are \( M \)-Lipschitz continuous, i.e., \( \|\nabla f(x) - \nabla f(y)\|_2 \leq M \|x - y\|_2 \). A random variable \( \xi \) is zero-mean \( \sigma \)-sub-Gaussian if \( \forall \lambda \in \mathbb{R} \mathbb{E} \left[ e^{\lambda \xi} \right] \leq \exp \left( \frac{\lambda^2 \sigma^2}{2} \right) \), which implies that \( \text{Var}[\xi] \leq \sigma^2 \). By \( \mathbb{S}^d \) and \( \mathbb{R}^d \) we denote the unit sphere and the unit ball in \( \mathbb{R}^d \), respectively. We denote the characteristic function of the set \( \mathcal{X} \) by \( \mathbb{I}_\mathcal{X} = \begin{cases} 0, & x \in \mathcal{X} \\ +\infty, & x \notin \mathcal{X} \end{cases} \).

Problem formulation. We consider safe non-convex non-smooth constrained optimization problem

\[
\min_{x \in \mathbb{R}^d} f^0(x) \\
\text{subject to } f^i(x) \leq 0, \ i = 1, \ldots, m,
\]

(1)

where the objective function \( f^0 : \mathbb{R}^d \to \mathbb{R} \) and the constraints \( f^i : \mathbb{R}^d \to \mathbb{R} \) are unknown \( L \)-Lipschitz continuous functions, and can only be accessed at feasible points \( x \). We denote by \( D \) the feasible set \( D := \{ x \in \mathbb{R}^d : f^i(x) \leq 0, i = 1, \ldots, m \} \).

Assumption 1. The set \( D \) has a non-empty interior, and there exists a known starting point \( x_0 \) for which \( f^i(x_0) < 0 \) for \( i = 1, \ldots, m \).

This assumption is common in works on safe learning (Berkenkamp et al., 2016; Sui et al., 2015) or on model-free LQR problems (Fazel et al., 2018; Abbasi-Yadkori et al., 2019).

Information. We assume access to noisy measurements of all cost and constraint function values for any requested feasible point \( x \in D \). In particular, the measurements are given by \( F^i(x, \xi^i) = f^i(x) + \xi^i \forall i = 0, \ldots, m \) with zero-mean sub-Gaussian noise \( \xi^i \). The \( \xi^i \)'s are i.i.d. for different measurements.

Goal. The goal of the algorithm is to find an approximate local optimum, using only noisy information. Moreover, it has to guarantee safety, i.e., constraint satisfaction with high probability for all the points where the measurements are taken. For differentiable non-convex objective and constraints, the notion of local optimality is captured by KKT condition. In this setting, we show that our algorithm converges to an \( \eta \)-approximate KKT point for any \( \eta > 0 \) with constant \( \tau_1, \tau_2 > 0 \):

\[
\begin{align*}
\lambda^i, -f^i(x) & \geq 0, \forall i = 1, \ldots, m \quad (\eta-\text{KKT.1}) \\
\lambda^i(-f^i(x)) & \leq \tau_1 \eta, \forall i = 1, \ldots, m \quad (\eta-\text{KKT.2}) \\
\|\nabla L(x, \lambda)\|_2 & \leq \tau_2 \eta, \quad (\eta-\text{KKT.3})
\end{align*}
\]

where \( L(x, \lambda) \) is the corresponding Lagrangian function. For non-differentiable non-convex objective and constraints, local optimality conditions are less understood. In this case, we show convergence to an approximate KKT point of a corresponding smoothed problem. The smoothing will be described in the approach below.
3. Proposed approach

We propose to construct a log barrier for the smooth approximation of problem (1), and then apply the zero-th order stochastic gradient descent with an adaptive step size to minimize it. To estimate the gradient of the smoothed function we sample points around the current iterate, and take measurements at these points. A measurement is denoted as safe if the point at which it is taken is feasible with high probability.

3.1. Zero-th order gradient estimation.

Our algorithm uses a randomized zero-th order gradient estimator for cost and constraint functions. For a point $x_k$ this the gradient is estimated taking $n_k$ samples uniformly at random on the unit sphere $S^d$.

\[ G^i(x_k, \nu) := \sum_{j=1}^{n_k} \frac{\hat{G}_j^i(x_k, \nu)}{n_k}, \quad \hat{G}_j^i(x_k, \nu) := d \frac{F^i(x_k + \nu s_{kj}, \xi_{kj}^i) - F^i(x_k, \xi_{kj}^i)}{\nu} s_{kj}, \]

where all $\{[\xi_{kj}^i, \xi_{kj}^{i-1}]\}_{j=1,...,n_k}$ are i.i.d. sub-Gaussian random variables, $\nu > 0$ is the sampling radius, $s_{kj}$ are the sampled unit vectors. For the sampling radius $\nu \geq 0$ define the $\nu$-smoothed estimate of the function $f(x)$ by $f_\nu(x) := \mathbb{E}_{s} f(x + \nu b)$, where $b$ is uniformly distributed on the unit ball $B^d$. Then $\mathbb{E}_{s_k, \xi_k} G^i(x_k, \nu) = \nabla f_\nu(x_k)$ (Flaxman et al., 2005; Nesterov and Spokoiny, 2017; Balasubramanian and Ghadimi, 2018). This shows that in expectation we can get a gradient of the smooth estimate $f_\nu(x)$ of non-smooth function $f(x)$ using randomized gradient estimator. Hazan et al. (2016) showed the following properties of $f_\nu(x)$:

1) The gradient $\nabla f_\nu(x)$ is Lipschitz continuous with constant $M_\nu$, satisfying $M_\nu \leq \frac{L}{\nu}$.

2) $\forall x \in \mathbb{R}^d \ |f_\nu(x) - f(x)| \leq \nu L$.

3.2. Smoothed log barrier function.

We address the safe learning problem using the log barriers approach. Define $f^c(x) = \max_{i=1,...,m} f^i(x)$. The logarithmic barrier with parameter $\eta > 0$ of the initial problem with the constraints replaced with $f^c(x)$ above is defined as $B_\eta(x) = f^0(x) - \eta \log(-f^c(x))$. We define the (locally) smoothed barrier function and its gradient using smoothed functions $f^0_\nu(x)$ and $f^c_\nu(x)$ as follows:

\[ B_{\eta, \nu}(x) = f^0_\nu(x) - \eta \log(-f^c_\nu(x)), \]

\[ \nabla B_{\eta, \nu}(x) = \nabla f^0_\nu(x) + \eta \nabla f^c_\nu(x) / -f^c_\nu(x), \]

Our goal is to design an algorithm that converges to a locally optimal point of the smoothed log barrier $B_{\eta, \nu}(x)$, which is basically an unconstrained approximation of constrained smoothed problem: $f^0_\nu(x) + \mathbb{I}_{f^c_\nu(x) \leq 0}$. This will lead $\eta$-approximate KKT conditions hold for the smoothed problem.

3.3. Log barrier gradient estimator.

First, we need to propose a way to estimate $\nabla B_{\eta, \nu}(x_k) = \nabla f^0_\nu(x_k) + \eta \nabla f^c_\nu(x_k) / -f^c_\nu(x_k)$. To estimate $\nabla f^0_\nu(x_k)$ and $\nabla f^c_\nu(x_k)$ we can use $G^0(x_k, \nu)$ and $G^c(x_k, \nu)$ respectively, defined by (2). However, to estimate the denominator we propose to use an upper confidence bound on $-f^c_\nu(x_k)$, constructed as follows. Let $\hat{F}^i_i(x_k) := \frac{\sum_{j=1}^{n_k} F^i(x_k, \xi_{kj}^i)}{n_k} + \sigma \sqrt{n_k \ln \frac{1}{\delta}}$ represent an upper confidence bound on $f^i(x_k)$. We define by
\[ \hat{f}^c(x_k) := \max_{i=1,\ldots,m} \{ \hat{f}^i(x) \} + \nu L \] is an upper confidence bound on \( f^c(x_k) \) and by \( \hat{\alpha}_k := |\hat{f}^c(x_k)| \) a lower confidence bound on \( |f^c(x_k)| \). As shown in Appendix A, \( \mathbb{P} \{ \hat{\alpha}_k \leq \min\{ |f^c(x_k^i)|, |f^c(x_k)| \} \} \geq 1 - \delta \). Then, we propose to estimate \( \nabla B_{\eta,\nu}(x_k) \) by

\[ g_k := G^0(x_k, \nu) + \eta \frac{G^I(x_k, \nu)}{\hat{\alpha}_k}. \] (5)

Next, to define our algorithm we need to make a second assumption.

**Assumption 2** Let \( D' \subset D \) be the subset defined by \( D' = \left\{ x \in \mathbb{R}^d : \hat{f}^c(x) + \eta \leq 0 \right\} \). The norm of the gradient of the constraint function \( \nabla f^c_\nu(x) \) is lower bounded on \( D \setminus D' \) by \( F \), i.e., \( 0 < F \leq \| \nabla f^c_\nu(x) \| \leq L \).

Assumption 2 is needed to demonstrate that close to the boundary the term in the barrier gradient related to the constraints becomes large enough to push the step direction away from the boundary back to the feasible set.

The proposed stochastic zeroth-order algorithm is defined in Algorithm 1 below.

**Algorithm 1** Stochastic Zero-th Order Logarithmic Barrier Method (ZeLoBa)

1: **Input:** \( x_0 \in D \), number of iterations \( K \), \( \eta > 0 \), \( L \), \( F > 0 \), \( C = \frac{F^2}{8L^2 + 4F^2} \), \( \nu = \frac{\tau n}{L} \), sequence \( \{n_k\} \) is defined later;\n2: **while** \( k \leq K \) **do**\n3: Sample \( n_k \) vectors \( s_{kj}, j = 1, \ldots, n_k \) independently from the uniform distribution on \( \mathbb{S}^d \);\n4: Take \( 2n_k \) noisy measurements of each function \( F^i(x_k, \xi_{kj}^i), F^i(x_k + \nu s_{kj}, \xi_{kj}^i) \forall i = 0, \ldots, m \)
5: Compute an estimator \( g_k of \nabla B_{\eta,\nu}(x_k) \) (5);\n6: Compute \( \gamma_k = 1 - \|g_k\| \min \left\{ \frac{\hat{a}_k}{2Lk^2/\nu}, \frac{1}{k^2} \right\} \);
7: \( x_{k+1} = x_k - \gamma_k g_k, \lambda_{k+1} = \frac{\eta}{\lambda_k} \);\n8: Sample \( R \) from a discrete random distribution \( \mathbb{P}\{R = k\} = \frac{\gamma_k \|g_k\|}{\sum_{k=1}^K \gamma_k \|g_k\|} \)
9: **Output:** \( x_R \)

4. **Safety and convergence analysis**

From the algorithm we require the safety of the iterates \( f^c(x_k) \leq 0 \) and the safety of the measurements \( f^c(x_k + \nu s_{kj}) \leq 0 \) with high probability. Also, we require the convergence to the stationary point of the smoothed function in expectation. Here, we show that both properties hold for ZeLoBa algorithm.

4.1. **Safety.**

Given the required accuracy \( \eta \) the smoothing parameter \( \nu \) (which is also the sampling radius) has to be conservative enough to guarantee constraints satisfaction at any measured point \( x_k + \nu s_{kj} \) of ZeLoBa algorithm, where \( \|s_{kj}\| = 1 \). Then, we need to show that the iterates \( x_k \) always keep a sufficient distance from the boundary \(-f^c_\nu(x_k) \geq \Lambda \). We also show how this bound \( \Lambda \) depends on \( \eta \).

To show such keeping distance property, we first need to show how the deviation of \( g_k \) from \( \nabla B_{\eta,\nu}(x_k) \) can be bounded. Define the deviation by \( \zeta_k := g_k - \nabla B_{\eta,\nu}(x_k) \). It is obvious that the deviation \( \zeta_k \) is also dependent on deviations \( \Delta^0_k := G^0(x_k, \nu) - \nabla f^0_\nu(x_k), \Delta^I_k := G^I(x_k, \nu) - \nabla f^I_\nu(x_k) \), thus, we bound them first. We denote \( \Sigma := (d + 1) \sqrt{\ln \frac{1}{\delta} + \ln K (\sigma + L\nu)} \). From the sub-Gaussian property of the noise \( \xi_{kj}^\pm \) and \( L \)-Lipschitz continuity of \( f^i(x), i = 0, \ldots, m \), we have:
Fact 1 For deviations $\Delta^i_k = G^i(x_k, \nu) - \nabla f^i_\nu(x_k)$, $i = 1, c$, in expectation we have $E\|\Delta^i_k\|^2 \leq \frac{\alpha^2}{\eta_k} \left( L^2 + \frac{2n^2}{\nu^2} \right)$.

For all points $x_k$ with $k \leq K$ we have: $P \left\{ \forall k = 1, \ldots, K \| \Delta^i_k \| \leq \frac{\Delta}{\sqrt{\nu} \sqrt{\eta_k}} \right\} \geq 1 - \delta$.

For the proof see Appendix B. Using this result, we can get the following bound on $\zeta_k$:

Fact 2 For deviation $\zeta_k = g_k - \nabla B_\eta,\nu(x_k)$, in expectation we have $E\|\zeta_k\| \leq \frac{(d+1)(c+L\nu)}{\nu \sqrt{\eta_k}} \left( 1 + \frac{2n}{\alpha_k} \right)$.

For all $k \leq K$ for $\zeta_k$ we have $P \left\{ \forall k = 1, \ldots, K \| \zeta_k \| \leq \frac{\Delta}{\sqrt{\nu} \sqrt{\eta_k}} \left( 1 + \frac{2n}{\alpha_k} \right) \right\} \geq 1 - \delta$.

For the proof see Appendix C.

From the above fact, observe that if we keep the distance from the boundary of $\hat{\alpha}_k \geq \Lambda > 0$, we can guarantee the finite bound on the deviation $\zeta_k$. Luckily, the Log Barrier gradient descent approach has this property, together with sufficiently large number of measurements in ZeLoBa ensure this property, as shown in the following Lemma.

Lemma 1 If Assumption 2 holds, then for all iterations $x_k$ of ZeLoBa algorithm with $n_k \geq \frac{64L^2c^2}{\nu^2F^2}$, we have

$P\{\hat{\alpha}_k \geq C\eta\} \geq 1 - \delta$, with $C = \frac{F^2}{8L^2 c^2 + 4F^2}$.

Proof sketch: Recall that the estimator $g_k$ defined in (5) consists of three additive parts: $\frac{\eta}{\alpha_k} \nabla f^\nu_\nu(x_k)$, $\nabla f^0_\nu(x_k)$ and noise $\zeta_k$. Using Fact 1 and L-Lipschitz continuity of $f^0_\nu(x)$, we can upper bound the norms of the last two parts $\|\zeta_k\|$ and $\|\nabla f^0_\nu(x_k)\|$. If the norm of the $\nabla f^\nu_\nu(x_k)$ can be separated from 0, i.e., $\|\nabla f^\nu_\nu(x_k)\| \geq F > 0$ (here Assumption 2 is needed) and the coefficient in front of $\nabla f^\nu_\nu(x_k)$ is big enough to guarantee $\langle g_k, \nabla f^\nu_\nu(x_k) \rangle \geq 0$, then the step direction $-g_k$ pushes the method away from the boundary, i.e., $-g_k$ is a locally descent direction of $f^\nu_\nu(x)$. The required lower bound on the coefficient is $\frac{\eta}{\alpha_k} \geq 2C$, that we show in the proof in Appendix D. Hence, if the step size is small enough, then for $\hat{\alpha}_k \leq 2C\eta$, $f^\nu_\nu(x)$ cannot increase, i.e., $\hat{\alpha}_{k+1} \geq \hat{\alpha}_k$ cannot decrease. On the other hand, if $\hat{\alpha}_k$ is bigger than $2C\eta$, then in the next iterate $-f^\nu_\nu(x)$ can not decrease more than twice, and thus $C\eta \leq \hat{\alpha}_{k+1} \leq 2C\eta$. Consequently, $\hat{\alpha}_k \geq C\eta$ for any case. The full proof can be found in Appendix D.

From the above lemma it follows that, $\nu = \frac{C\eta}{L} \leq \frac{\hat{\alpha}_k}{L}$. Thus, we can show that our algorithm is safe:

Proposition 2 Let Assumption 1.2 hold, and $n_k \geq \frac{64L^2c^2}{\nu^2F^2}$. Then, for any $x_{k+1}$ generated by ZeLoBa algorithm with the step size $\gamma_k \leq \frac{\hat{\alpha}_k}{2L\|g_k\|}$, and for any measurement points $x_k + \nu s_k$, given $f^c(x_k) < 0$ we have $P\{f^c(x_k - \gamma_k g_k) \leq 0\} \geq 1 - \frac{\delta}{K}$ and $P\{f^c(x_k + \nu s_k) \leq 0\} \geq 1 - \frac{\delta}{K}$. It follows that since the $x_0$ is feasible due to Assumption 1, then all the points $x_k$ and $x_k + \nu s_k$ are feasible with probability $1 - \delta$.

For the proof see Appendix E.

4.2. Convergence:

Theorem 3 Under Assumptions 1, 2, and if $n_k \geq \frac{64L^2c^2}{\nu^2F^2}$, after $K \geq \frac{1}{\eta}$ iterations of ZeLoBa algorithm we have $E\|\nabla B_\eta,\nu(x_R)\| \leq \eta(1 + C_2 \ln K)$, with $C_1 = \frac{2Ld_I}{C} + \frac{\eta L}{C} (1 + \frac{1}{C})$ and $C_2 = \frac{1}{4C} (1 + \frac{1}{C}) + \frac{L}{2C}$. This implies that for the pair $(x_R, \lambda_R)$ in expectation $\eta$-approximate KKT condition holds with probability $1 - \delta$:

$$\lambda_R - f^\nu_\nu(x_R) \geq 0, \quad (\eta\text{-KKT.1})$$

$$\lambda_R(-f^\nu_\nu(x_R)) \leq 3\eta, \quad (\eta\text{-KKT.2})$$

$$E\|\nabla L(x_R, \lambda_R)\|_2 \leq (C_1 + C_2 \ln K)\eta. \quad (\eta\text{-KKT.3})$$

The total number of measurements required is $N_K = n_k \cdot K = O(\frac{\delta^2}{\eta^2})$. 

6
We consider the application to safe iterative controller design. The control problems with uncertainty in \( \eta \) we can verify that condition \( \eta, \nu \) for all the points the distance of step size \( \gamma_k \) from \( x_k \). Then, using this local smoothness, we bound the improvement in barrier value \( \eta, \nu \) to \( x_k \) from below, sum them up and obtain the bound on the sum of norms of the gradients \( \sum_{k=1}^{K} \frac{\gamma_k}{\|g_k\|} \| \nabla B_{\eta, \nu}(x_k) \| \). After division by \( \sum_{k=1}^{K} \frac{\gamma_k}{\|g_k\|} \), this represents an expectation of \( \| \nabla B_{\eta, \nu}(x) \| \) for \( R \) sampled at Step 8 of ZeLoBa algorithm. Thus, we get the bound on \( \| \nabla B_{\eta, \nu}(x_R) \| \) \( \leq (C_1 + C_2 \ln K) \eta, \nu \), for \( K \geq \frac{1}{\eta, \nu} \). Note that by construction \( \nabla B_{\eta, \nu}(x_R) = \nabla L(x_R, \lambda_R) \) for the smoothed problem. After checking \( \lambda_R (-f_{\nu}^{(i)}(x_R)) \leq 2\eta, \nu \), the statement of Theorem 3 follows directly. The full proof of the theorem is quite technical and can be found in Appendix F.

### 4.3. Connection with initial problem in differentiable case.

Consider now the case when the initial objective and constraints are differentiable. In this section we show that the result obtained in Theorem 3 about local optimality of \( B_{\eta, \nu}(x_R) \) entails satisfaction that the approximate KKT condition for the initial problem (1). We define \( \hat{\lambda}_R \in \mathbb{R}^m \), where

\[
\hat{\lambda}^i_R = \begin{cases} 0, & i \neq \arg \max_i \hat{F}^i(x_R), \\ \frac{\eta}{-F^c(x_R)}, & i = \arg \max_i \hat{F}^i(x_R) \end{cases}
\]

**Proof** We can easily see that condition \( \eta\)-KKT.1 holds with high probability by construction: \( -f_{\nu}^c(x_R) \geq \hat{\lambda}^i_R \geq 0 \). Condition \( \eta\)-KKT.2 holds for all \( i \neq \arg \max_i \hat{F}^i(x_R) \) since \( \hat{\lambda}^i_R \) is just equal to 0. For \( i = \arg \max_i \hat{F}^i(x_R) \), we have \( -\frac{\eta}{-F^c(x_R)} (-f^i(x_R)) \leq \eta, \nu \) \( \leq \eta, \nu \cdot \frac{\sqrt{n} \ln 1/\delta}{\alpha_R} \leq 3\eta \). Last, we can verify that condition \( \eta\)-KKT.3 holds as follows, using \( \| \nabla f_{\nu}^i(x) - \nabla f^i(x) \| \leq \nu Ld \) (Nesterov and Spokoiny, 2017):

\[
\mathbb{E}[L(x_R, \hat{\lambda}_R)] = \mathbb{E}[\nabla f^0(x_R)] + \sum_{i=1}^{m} \hat{\lambda}^i \nabla f^i(x_R)] = \mathbb{E} \left[ \nabla f^0(x_R) + \frac{\eta \nabla f^i(x_R)}{-F^c(x_R)} \right] \\
\leq \mathbb{E} \left[ \nabla f^0(x_R) + \frac{\eta \nabla f^i(x_R)}{-F^c(x_R)} \right] + \| \nabla f^0(x_R) - \nabla f^0(x_R) \| + \eta \left( 1 + \frac{\nu Ld}{\alpha_R} \right) \leq \| \nabla B_{\eta, \nu}(x_R) \| + \eta(d+1) \leq \eta(C_1 \ln K + C_2 + d + 1).
\]

**5. Experiments**

We consider the application to safe iterative controller design. The control problems with uncertainty in measurements are usually complicated by the fact that the noise propagates through time horizon. This happens because the noise enters the closed-loop controller, and hence recursively arises in all the time steps. In such a case, the noise model in the cost gets more complicated than just sub-Gaussian noise. For this reason, we assume that the system dynamics are not directly affected by noise, but that the constraint and cost measurements are. This simple noise model can be used to describe uncertain navigation tasks, as the one that follows. Consider the basic unicycle dynamics \( \dot{x} = v \cos \theta, \dot{y} = v \sin \theta, \dot{\theta} = \omega \). Here the states \( q = [x, y, \theta] \)
describe the spatial coordinates $x, y$ and the direction angle $\theta$. The control inputs $u = [v, \omega]$ describe the velocity of the unicycle $v$ and the velocity in the angle change $\omega$. The discrete equivalent approximation is (Nino-Suarez et al., 2006; Adinandra et al., 2012):

$$q_{t+1} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = q_t + \begin{bmatrix} 2v_t + \gamma(\omega_t) \cos(\theta_t + \frac{dt}{2}\omega_t) \\ 2u_t + \gamma(\omega_t) \sin(\theta_t + \frac{dt}{2}\omega_t) \\ dt\omega_t \end{bmatrix},$$

where $\gamma(\omega_t) = \begin{cases} \sin(\frac{dt}{2}\omega_t), & \omega_t \neq 0 \\ \frac{dt}{2}, & \omega_t = 0 \end{cases}$. Linear feedback law $u_{t+1} = Uq_t, U \in \mathbb{R}^{3 \times 2}$. We aim to find the closed loop linear controller: $u_t = U(q_{t-1} - q_B), U \in \mathbb{R}^{3 \times 2}$, $U$ is the optimizing parameter. The goal is to lead the vehicle from a starting point $q_A$ to a goal destination $q_B$ while avoiding collisions with high-probability. We define the cost as follows: $f^0(U) = \min_{U \in \mathbb{R}^{3 \times 2}} \frac{1}{T} \sum_{t=1}^{T} \|q_t(U) - q_B\|^2$. In the finite horizon problem the constraints are formulated such that the dynamics trajectory not to violate the ball shaped obstacle placed at $(x_C, y_C)^T$ with radius 1. We are present with the following constrained optimization problem:

$$\min_{U \in \mathbb{R}^{3 \times 2}} \frac{1}{T} \sum_{t=1}^{T} \|q_t(U) - q_B\|^2$$

subject to $f^c(U) = 1 - \| (x_t(U), y_t(U))^T - (x_C, y_C)^T \|^2 \leq 0$, $t = 1, \ldots, T$.

Then $f^c(U) = \max_{t=1, \ldots, T} 1 - \| (x_t(U), y_t(U))^T - (x_C, y_C)^T \|^2 \leq 0$. Consider an above optimization problem with $(x_C, y_C) = (0, 0), q_B = (2, 0, \pi), q_A = (-2, 0, \pi), T = 10$, and assume that the cost and the constraints can be measured only by noisy sensors. Then, we address this problem using the ZeLoBa algorithm. Starting from a safe control policy: $U_0 = \begin{bmatrix} 0.01 & -1.5 & 0.5 \\ 0.1 & 0.2 & 0.5 \end{bmatrix}$, the ZeLoBa algorithm iteratively improves it. In the figures below we demonstrate the achieved results of made 20 trials of the stochastic algorithm. Trajectory that is generated by $U_0$ controller is demonstrated in Figure 2. The final controller performance plot of one of the stochastic algorithm trial is in Figure 3. We use $\nu_k = \min\{\eta, \hat{\alpha}_k\}$ for safety, $n_k = d + 1, N_K = 3500.$

**Figure 1:** a) Average convergence and minimal distance from the obstacle
**Figure 2:** b) Control trajectory with $U_0$
**Figure 3:** c) Final control trajectory obtained by ZeLoBa with $\eta = 0.001.$
References

Yasin Abbasi-Yadkori, Nevena Lazic, and Csaba Szepesvari. Model-free linear quadratic control via reduction to expert prediction. In The 22nd International Conference on Artificial Intelligence and Statistics, pages 3108–3117, 2019.

Sisdarmanto Adinandra, Erik Schreurs, and Henk Nijmeijer. A practical model predictive control for a group of unicycle mobile robots. IFAC Proceedings Volumes, 45(17):472–477, 2012.

Aaron D Ames. Human-inspired control of bipedal walking robots. IEEE Transactions on Automatic Control, 59(5):1115–1130, 2014.

Krishnakumar Balasubramanian and Saeed Ghadimi. Zeroth-order (non)-convex stochastic optimization via conditional gradient and gradient updates. In Advances in Neural Information Processing Systems, pages 3455–3464, 2018.

Felix Berkenkamp, Andreas Krause, and Angela P Schoellig. Bayesian optimization with safety constraints: safe and automatic parameter tuning in robotics. arXiv preprint arXiv:1602.04450, 2016.

Sarah Dean, Stephen Tu, Nikolai Matni, and Benjamin Recht. Safely learning to control the constrained linear quadratic regulator. In 2019 American Control Conference (ACC), pages 5582–5588. IEEE, 2019.

Maryam Fazel, Rong Ge, Sham Kakade, and Mehran Mesbahi. Global convergence of policy gradient methods for the linear quadratic regulator. In International Conference on Machine Learning, pages 1466–1475, 2018.

Jaime F Fisac, Anayo K Akametalu, Melanie N Zeilinger, Shahab Kaynama, Jeremy Gillula, and Claire J Tomlin. A general safety framework for learning-based control in uncertain robotic systems. IEEE Transactions on Automatic Control, 2018.

Abraham D Flaxman, Adam Tauman Kalai, and H Brendan McMahan. Online convex optimization in the bandit setting: gradient descent without a gradient. In Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms, pages 385–394. Society for Industrial and Applied Mathematics, 2005.

Saeed Ghadimi and Guanghui Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization, 23(4):2341–2368, 2013.

Jeremy H Gillula and Claire J Tomlin. Guaranteed safe online learning via reachability: tracking a ground target using a quadrotor. In 2012 IEEE International Conference on Robotics and Automation, pages 2723–2730. IEEE, 2012.

Elad Hazan, Kfir Yehuda Levy, and Shai Shalev-Shwartz. On graduated optimization for stochastic non-convex problems. In International conference on machine learning, pages 1833–1841, 2016.

Oliver Hinder and Yinyu Ye. A polynomial time log barrier method for problems with nonconvex constraints. arXiv preprint: https://arxiv.org/pdf/1807.00404.pdf, 2019.

Jin-Bao Jian, Chun-Ming Tang, Qing-Jie Hu, and Hai-Yan Zheng. A feasible descent sqp algorithm for general constrained optimization without strict complementarity. Journal of Computational and Applied Mathematics, 180(2):391–412, 2005.
Guanghui Lan. The complexity of large-scale convex programming under a linear optimization oracle. *arXiv preprint arXiv:1309.5550*, 2013.

Zhijun Luo, Zhibin Zhu, and Guohua Chen. Modifying feasible sqp method for inequality constrained optimization. In *International Conference on Information Computing and Applications*, pages 323–330. Springer, 2012.

Yurii Nesterov and Vladimir Spokoiny. Random gradient-free minimization of convex functions. *Foundations of Computational Mathematics*, 17(2):527–566, 2017.

PA Nino-Suarez, E Aranda-Bricaire, and M Velasco-Villa. Discrete-time sliding mode path-tracking control for a wheeled mobile robot. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 3052–3057. IEEE, 2006.

Yanan Sui, Alkis Gotovos, Joel Burdick, and Andreas Krause. Safe exploration for optimization with gaussian processes. In *International Conference on Machine Learning*, pages 997–1005, 2015.

Chun-ming Tang, Shuai Liu, Jin-bao Jian, and Jian-ling Li. A feasible sqp-gs algorithm for nonconvex, nonsmooth constrained optimization. *Numerical Algorithms*, 65(1):1–22, 2014.

Ilnura Usmanova, Andreas Krause, and Maryam Kamgarpour. Log barriers for safe non-convex black-box optimization. *arXiv preprint https://arxiv.org/abs/1912.09478*, 2019a.

Ilnura Usmanova, Andreas Krause, and Maryam Kamgarpour. Safe convex learning under uncertain constraints. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 2106–2114, 2019b.

Guus Zoutendijk. *Methods of feasible directions: a study in linear and non-linear programming*. Elsevier, 1960.
Appendix

A: Proof of lower confidence bound $\hat{c}_k$

**Proof** We denote by $\hat{F}^i(x_k) := \frac{\sum_{i=1}^{n_k} F^i(x_k + s_{kj}^i \nu)}{n_k} = f^i(x_k) + \frac{\sum_{j=1}^{n_k} \xi_{kj}^i}{n_k}$. Then $\hat{F}^i(x_k) - f^i(x_k)$ is zero-mean $\frac{\alpha}{n_k}$-sub-Gaussian. Then, it is easy to see that for any $\delta \in (0,1)$ we have $\forall i = 1, \ldots, m$

$$\Pr \left\{ |\hat{F}^i(x_k) - f^i(x_k)| \leq \frac{\sigma \sqrt{\ln \frac{1}{\delta}}}{\sqrt{n_k}} \right\} \geq 1 - \delta.$$ 

Thus, for $\hat{F}^i(x_k) = \frac{\sum_{i=1}^{n_k} F^i(x_k + s_{kj}^i \nu)}{n_k} + \frac{\sigma}{\sqrt{n_k}} \sqrt{\ln \frac{1}{\delta}}$ it holds that $\Pr \{ f^i(x_k) \leq \hat{F}^i(x_k) \} \geq 1 - \bar{\delta}$.

Next, recall that $f_{\nu}^i(x_k) = \max_{i=1,\ldots,m} f_i^i(x_k)$, thus, $\Pr \{ f_{\nu}^i(x_k) \leq \max_{i=1,\ldots,m} \hat{F}^i(x_k) \} \geq 1 - \bar{\delta}$, since this condition can be violated only if all of the upper bounds are violated at the same time, and they are independent.

Next, note from the properties of *smoothed* functions (Flaxman et al., 2005; Hazan et al., 2016) that $|f_{\nu}^i(x_k) - f^c(x_k)| \leq \nu L$, hence, $\max \{ f_{\nu}^i(x_k), f^c(x_k) \} \leq f^c(x_k) + \nu L$, and thus

$$\Pr \left\{ \max \{ f_{\nu}^i(x_k), f^c(x_k) \} \leq \hat{F}^c(x_k) + \nu L \right\} \leq 1 - \bar{\delta}.$$

It follows directly that $\Pr \{ \hat{c}_k \leq \min \{ |f_{\nu}^i(x_k)|, |f^c(x_k)| \} \} \geq 1 - \bar{\delta}$. \hfill \qed

B: Proof of Fact 1

**Proof** Denote $\bar{\delta} = \delta / K$. Recall that

$$\Delta_k^i = \frac{1}{n_k} \sum_{j=1}^{n_k} \left( d \frac{f^i(x_k + \nu s_{kj}) - f^i(x_k)}{\nu} s_{kj} - \nabla f_{\nu}^i(x_k) \right) + \frac{1}{n_k} \sum_{j=1}^{n_k} d \frac{\xi_{kj}^i s_{kj} - \bar{\xi}_{kj}^{i-} s_{kj}}{\nu}.$$

Note that from Lipschitzness of $f^i(x)$ we have boundness $\| d \frac{f^i(x_k + \nu s_{kj}) - f^i(x_k)}{\nu} s_{kj} - \nabla f_{\nu}^i(x_k) \| \leq (d+1)L$, i.e., it is $(d+1)L$-sub-Gaussian. Also, $\| s_{kj} \| = 1$. Hence, we get

$$\text{Var} \Delta_k^i = E \| \Delta_k^i \|^2 \leq \frac{(d+1)^2 L^2}{n_k} + d^2 E_{\xi_{kj}} \sum_{j=1}^{n_k} \frac{(\xi_{kj}^+)^2 \| s_{kj} \|^2 + (\xi_{kj}^-)^2 \| s_{kj} \|^2}{\nu^2 n_k^2} \leq \frac{(d+1)^2}{n_k} \left( L^2 + \frac{2\sigma^2}{\nu^2} \right).$$

For any $z$ which is $Z$-sub-Gaussian it holds that $\forall \bar{\delta} \in [0,1] \Pr \left\{ \| z \| \leq Z \sqrt{\ln \frac{1}{\bar{\delta}}} \right\} \geq 1 - \bar{\delta}$. Thus, it is easy to see that $\Pr \left\{ \| \Delta_k^i \| \leq \frac{(d+1)^2}{\sqrt{n_k}} \left( L^2 \ln \frac{1}{\delta} + \frac{\sigma^2 \ln \frac{1}{\nu}}{\nu} \right) \right\} \geq 1 - \bar{\delta}$. Using the probability of union of the events $A_i$ inequality $1 - \Pr \{ \cup A_i \} \geq 1 - \sum \Pr \{ A_i \}$, we get the result holding for $K$ points with probability $1 - K \bar{\delta} = 1 - \delta$. For $\| \Delta_k^i \|$ the proof is similar, just $\| \xi_{kj}^{i+} \| \leq \min_{i=1,\ldots,m} \| \xi_{kj}^{i-} \|$. \hfill \qed
C: Proof of Fact 2

Proof

\[ ||\zeta_k|| = \|g_k - \nabla B_{\eta, \nu}(x_k)\| = \left\| G^0(x_k, \nu, \xi^0_k, s_k) - \nabla f^0_\nu(x_k) + \eta \left( \frac{G^c(x_k, \nu, s_k, \xi_k)}{\|F^c(x_k)\|} - \frac{\nabla f^c_\nu(x_k)}{\|f^c_\nu(x_k)\|} \right) \right\| \]
\[ \leq \|\Delta^0_k\| + \eta \left\| \frac{G^c(x_k, \nu, s_k, \xi_k)}{\|F^c(x_k)\|} - \frac{\nabla f^c_\nu(x_k)}{\|f^c_\nu(x_k)\|} \right\| \leq \|\Delta^0_k\| + \eta \frac{\|\nabla f^c_\nu(x_k)\|}{\alpha_k^2} \left( \hat{\alpha}_k - \alpha_k \right) \]
\[ \leq \frac{\Sigma}{\nu\sqrt{n_k}} \left( 1 + \frac{\eta}{\alpha_k} \right) + \frac{\eta L \sigma \sqrt{\ln \frac{1}{\delta}}}{\alpha_k^2} \leq \frac{1}{\nu\sqrt{n_k}} \left( \Sigma + \frac{\Sigma \eta}{\alpha_k} + \frac{\sigma \sqrt{\ln \frac{1}{\delta}}}{\alpha_k^2} \right) \leq \frac{\Sigma}{\nu\sqrt{n_k}} \left( 1 + \frac{2\eta}{\alpha_k} \right). \]

D: Proof of Lemma 1

Proof Assume that for some \( x_k \) we have \(-f^c_\nu(x_k) \leq \frac{\nu^2}{2L^3+\|\zeta_k\|L^2} \eta \). Note that then \( \hat{\alpha}_k \leq \frac{\nu^2}{2L^3+\|\zeta_k\|L^2} \eta \). By Assumption 2, at that point \( F \leq \|\nabla f^c_\nu(x_k)\| \leq L \). Then the step direction \(-g_k = \nabla B_{\eta, \nu}(x_k) + \zeta_k \) is such that the smoothed constraint \( f^c_\nu(x) \) decreases along it \(-g_k, \nabla f^c_\nu(x_k) \geq 0 \). This is because:

\[ \langle -g_k, \nabla f^c_\nu(x_k) \rangle = \langle \nabla B_{\eta, \nu} + \zeta_k, \nabla f^c_\nu(x_k) \rangle = \langle \nabla f^0_\nu(x_k) + \eta \frac{\nabla f^c_\nu(x_k)}{\alpha_k} + \zeta_k, \nabla f^c_\nu(x_k) \rangle \]
\[ \geq \langle \nabla f^0_\nu(x_k), \nabla f^c_\nu(x_k) \rangle + \frac{\|\nabla f^c_\nu(x_k)\|^2}{\sqrt{2F^2/(L^2+\|\zeta_k\|L)}} - \|\zeta_k\|L \geq L^2 + \|\zeta_k\|L - L^2 - \|\zeta_k\|L \geq 0. \]

If the step size \( \gamma_k \) is small enough, it guarantees the decrease of \( f^c_\nu(x) \). Without loss of generality, assume \( F \leq L/2 \) and \( F \leq \sqrt{2} \) (if not, we can always replace \( F \leftarrow F/2 \) everywhere.) Then

\[ \|g_k\| \geq \frac{\eta \|\nabla f^c_\nu(x_k)\|}{\sqrt{2F^2/(L^2+\|\zeta_k\|L)} \eta} - \|\nabla f^c_\nu(x_k)\| - \|\zeta_k\| \geq \frac{(L + \|\zeta_k\|L)}{F} - (L + \|\zeta_k\|) \left( \frac{L}{F} - 1 \right) \geq 2L. \]

Then we obtain that \( \gamma_k \leq \frac{1}{2M_k} = \frac{\nu}{4F} \) since \( \gamma_k \leq \frac{\hat{\alpha}_k}{\|g_k\|k^{2/3}} \leq \frac{C\eta}{\|g_k\|k^{2/3}} = \frac{\nu}{4F} \). Thus, indeed for \( x_k \) such that \( \hat{\alpha}_k \leq \frac{F^2}{(L^2+\|\zeta_k\|L^2)} \eta \) we have \( \gamma_k \leq \frac{1}{2M_k} \), thus, in the the next step \( f^c_\nu(x_{k+1}) < f^c_\nu(x_k) \). On the other hand, if \(-f^c_\nu(x_k) \geq \frac{F^2}{2(L^2+\|\zeta_k\|L^2)} \eta \) for one step it cannot decrease more than twice \(-f^c_\nu(x_{k+1}) \geq \frac{F^2}{2(L^2+\|\zeta_k\|L^2)} \eta \). Hence, for \( x_k \) generated by ZeLoBa we can guarantee \(-f^c_\nu(x_k) \geq \frac{F^2}{2(L^2+\|\zeta_k\|L^2)} \eta \geq C\eta \) for some \( C \). Then, combining Lemma 1 and the previous inequality we require \( C \leq \frac{\nu F^2}{2(L^2+\|\zeta_k\|L^2)} \), i.e., \( C \leq \frac{F^2 - 4L^2(\frac{1+\eta}{\sqrt{\eta^2}})}{2(\sqrt{\eta^2} + \frac{L^2}{\sqrt{\eta^2}})} \).

If \( \sqrt{n_k} \geq \frac{8\Sigma C\eta}{\nu F^2} \), then taking \( C = \frac{F^2}{L^2 + F^2/2} \) is enough. Thus \( \hat{\alpha}_k \geq C\eta - \frac{\Sigma}{\sqrt{n_k}} - \nu L/2 = C\eta/2 - \frac{\Sigma}{\sqrt{n_k}} \geq C\eta/2 - \frac{\eta C^2 F^2}{8} \geq \frac{1}{4} C\eta = \hat{C}\eta. \)

---

12
E: Proof of Proposition 2

**Proof** If \( n_k \geq \frac{64L^2\gamma_k^2}{\nu^2} \), then based on Lemma 1 for \( \nu = C\eta \) it holds that \( \|x_k - (x_k + \nu s_{k,j})\| \leq \frac{-f^i(x_k)}{L} \).

Similarly, from the step size \( \gamma_k \) definition \( \|x_k - x_{k+1}\| \leq \frac{-f^i(x_k)}{2L} \).

For any point \( y \in \mathbb{R}^d \) satisfying \( \|x_k - y\| \leq \frac{\min_i -f^i(x_k)}{2L} \) we have \( \forall i = 1, \ldots, m \),

\[
    f^i(y) \leq f^i(x_k) + \langle \nabla f^i(x_k), y - x_k \rangle \leq f^i(x_k) + L\|y - x_k\| \leq f^i(x_k) + L\frac{-f^i(x_k)}{L} \leq 0.
\]

Moreover, if it satisfies \( \|x_k - y\| \leq \frac{\min_i -f^i(x_k)}{2L} \) we have \( \forall i = 1, \ldots, m \),

\[
    f^i(y) \leq f^i(x_k) + L\frac{-f^i(x_k)}{2L} \leq \frac{1}{2} f^i(x_k) \leq 0.
\]

The statement of the proposition follows from this directly. 

F: Proof of Theorem 3

**Proof** Recall that \( \hat{\gamma}_k = \min \{ \frac{\gamma_k}{2L^2k/\nu^2}, \frac{1}{2L^2} \} \). In the paper (Hinder and Ye, 2019) the authors have shown that \( L_2(x_k) = M \left( 1 + \sum_{i=1}^m \frac{2\gamma_k}{-f^i(x_k)} \right) + \sum_{i=1}^m \frac{4\gamma_k L}{-f^i(x_k)^2} \) represents a "local" Lipschitz constant of \( \nabla B_{\eta,\nu}(x) \) at the point \( x_k \). If we replace all \( f^i(x) \) by single smooth constant \( f^i_0(x) \). Then its gradient is locally \( L_{2,\nu} \) Lipschitz continuous, i.e., the gradient of the smoothed barrier has the local Lipschitz constant:

\[
    L_{2,\nu}(x_k) \leq M_{\nu} \left( 1 + \frac{2\gamma_k}{\alpha_k} \right) + \frac{4L^2\eta}{\alpha_k}. \tag{7}
\]

Let \( \hat{\gamma}_k = \gamma_k \|g_k\| \). Then, at each iteration of Algorithm 1 we have

\[
    B_{\eta,\nu}(x_k) - B_{\eta,\nu}(x_{k+1}) \geq -\gamma_k \langle \nabla B_{\eta,\nu}(x_k), g_k \rangle - \frac{1}{2} L_{2,\nu}(x_k) \gamma_k^2 \|g_k\|^2 = [\text{Taylor's theorem and local smoothness}]
\]

\[
    = \gamma_k \langle \zeta_k, g_k \rangle + \gamma_k \|g_k\|^2 - \frac{1}{2} L_{2,\nu}(x_k) \gamma_k^2 \|g_k\|^2 \geq \|g_k\| \hat{\gamma}_k - \left( M_{\nu} + \frac{\eta M_{\nu}}{\alpha_k} + \frac{4\eta L^2}{\alpha_k} \right) \hat{\gamma}_k^2
\]

\[
    \geq \|\nabla B_{\eta,\nu}(x_k)\| \|g_k\| - 2\|\zeta_k\| \hat{\gamma}_k - \left( M_{\nu} + \frac{\eta M_{\nu}}{\alpha_k} + \frac{4\eta L^2}{\alpha_k} \right) \hat{\gamma}_k^2.
\]

Hence, we can derive the following bound using telescopic sum

\[
    B_{\eta,\nu}(x_0) - \sum_{k=0}^K (B_{\eta,\nu}(x_k) - B_{\eta,\nu}(x_{k+1})) \geq \sum_{k=0}^K \|\nabla B_{\eta,\nu}(x_k)\| \|g_k\| - \sum_{k=0}^K 2\|\zeta_k\| \hat{\gamma}_k - \sum_{k=0}^K \frac{M_{\nu}\hat{\gamma}_k^2}{2} - \sum_{k=0}^K \frac{M_{\nu}\eta\hat{\gamma}_k^2}{2\alpha_k}.
\]

We define \( D_f = B_{\eta,\nu}(x_0) - B_{\eta,\nu}(x_0^*) \). Then

\[
    \sum_{k=0}^K \|\nabla B_{\eta,\nu}(x_k)\| \|g_k\| \leq D_f + \sum_{k=0}^K 2\|\zeta_k\| \hat{\gamma}_k + \sum_{k=0}^K \frac{M_{\nu}\hat{\gamma}_k^2}{2} + \sum_{k=0}^K \frac{M_{\nu}\eta\hat{\gamma}_k^2}{2\alpha_k} + \sum_{k=0}^K \frac{4\eta L^2\hat{\gamma}_k^2}{2\alpha_k}.
\]
Hence, if $R$ is a random variable such that $P\{R = k\} = \frac{\gamma_k}{\sum_{k=1}^{K} \gamma_k}$, then

$$\mathbb{E}\|\nabla B_{\theta,\nu}(x_R)\| \leq \frac{1}{\sum_{k=1}^{K} \gamma_k} \left( D_f + \sum_{k=0}^{K} 2\|\zeta_k\|\gamma_k + \sum_{k=0}^{K} \frac{M_{\nu}\gamma_k^2}{2} + \sum_{k=0}^{K} \frac{M_{\nu}\eta\gamma_k^2}{2\alpha_k} + \sum_{k=0}^{K} \frac{4\eta L^2\gamma_k^2}{2\alpha_k^2} \right).$$

Note that $\gamma_k = \frac{1}{2L} \min \left\{ \frac{\alpha_k}{k^{2/5}}, \frac{1}{k^{1/5}} \right\}$, since both terms in $\min \left\{ \frac{\alpha_k}{k^{2/5}}, \frac{1}{k^{1/5}} \right\}$ are smaller than 1 (without loss of generality assume $f^c(x) \geq -1$, otherwise can scale), consequently both of them are smaller than their product. Consequently, $\sum_{k=1}^{K} \gamma_k \geq K^{2/5}\eta C/2L = \eta K^{2/5}C/2L$. Then $\frac{1}{\sum_{k=1}^{K} \gamma_k} \leq \frac{2L}{C K^{2/5}\eta}$.

$$\mathbb{E}\|\nabla B_{\theta,\nu}(x_R)\| \leq \frac{D_f}{\sum_{k=0}^{\hat{K}} \gamma_k} + \frac{2}{\sum_{k=1}^{K} \gamma_k} \sum_{k=0}^{\hat{K}} 2\|\zeta_k\|\gamma_k + \frac{\sum_{k=1}^{K} \frac{M_{\nu}\gamma_k^2}{2} (1 + \frac{\eta}{\alpha_k})}{\sum_{k=0}^{\hat{K}} \gamma_k} + \frac{\sum_{k=0}^{K} \frac{4\eta L^2\gamma_k^2}{2\alpha_k^2}}{\sum_{k=0}^{\hat{K}} \gamma_k}$$

$$\leq \frac{2LD_f}{C\eta K^{2/5}} + 2\max_{k \leq \hat{K}} \mathbb{E}\|\zeta_k\| + \frac{2L^2 \sum_{k=1}^{K} \frac{1/L^2}{2k^{2/5}} (1 + \frac{1}{\sqrt{C}})}{\nu C\eta K^{2/5}} + \frac{L\eta \sum_{k=1}^{K} \frac{1}{2k^{4/5}}}{C\eta K^{2/5}}$$

$$\leq \frac{2LD_f}{C\eta K^{2/5}} + 2\max_{k \leq \hat{K}} \mathbb{E}\|\zeta_k\| + \frac{L\eta K^{1/5}}{4C^2\eta^2 K^{2/5}} + \frac{L\eta K^{1/5}}{2C^2\eta K^{2/5}}.$$

If $K \geq \frac{1}{\eta^4}, \nu \geq \frac{C_0}{L}$, and $n_k \geq \frac{64L^2\nu^2}{\nu^2 L^2}$, then $\mathbb{E}\|\nabla B_{\theta,\nu}(x_R)\| \leq \eta(C_1 + C_2 \ln K)$,

with $C_1 = \frac{2LD_f}{C} + \frac{\eta L}{C} (1 + \frac{1}{\sqrt{C}})$ and $C_2 = \frac{1}{4C} (1 + \frac{1}{\sqrt{C}}) + \frac{L}{2C}$.

Next, consider the pair $(x_R, \lambda_R) = (x_R, \frac{\nu}{\alpha_R})$. Now we check the approximate KKT conditions:

$\eta$-KKT.1 holds with high probability by construction $\lambda_R \geq 0, -f^c(x_R) \geq \tilde{\alpha}_R \geq 0$.

$\eta$-KKT.2 holds since:

$$\frac{\eta}{\tilde{\alpha}_R} (-f^c(x_R)) \leq \eta + \frac{\tilde{\alpha}_R - f^c(x_R)}{\tilde{\alpha}_R} \leq \eta + \eta \frac{\sqrt{\ln 1/\delta} / \sqrt{n_R} + \nu L}{\alpha_R} \leq \eta + \eta \frac{2\nu L}{\alpha_R} \leq 3\eta.$$

$\eta$-KKT.3 in expectation by definition $\mathbb{E}\|\nabla f^c(x_R) + \lambda \nabla f^c(x_R)\| = \mathbb{E}\|\nabla B_{\theta,\nu}(x_R)\| \leq (C_1 + C_2 \ln K)\eta$, which was just proved above.