AN ORLIK-SOLOMON TYPE ALGEBRA FOR MATROIDS WITH A FIXED LINEAR CLASS OF CIRCUITS

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Abstract. A family $C_L$ of circuits of a matroid $M$ is a linear class if, given a modular pair of circuits in $C_L$, any circuit contained in the union of the pair is also in $C_L$. The pair $(M, C_L)$ can be seen as a matroidal generalization of a biased graph. We introduce and study an Orlik-Solomon type algebra determined by $(M, C_L)$. If $C_L$ is the set of all circuits of $M$ this algebra is the Orlik-Solomon algebra of $M$.

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1. Introduction

Let $\mathcal{A}_C = \{H_1, \ldots, H_n\}$ be a central and essential arrangement of hyperplanes in $\mathbb{C}^d$ (i.e., such that $\bigcap_{H_i \in \mathcal{A}_C} H_i = \{0\}$). The manifold $\mathfrak{M} = \mathbb{C}^d \setminus \bigcup_{H \in \mathcal{A}_C} H$ plays an important role in the Aomoto-Gelfand multivariable theory of hypergeometric functions (see [9] for a recent introduction from the point of view of arrangement theory). There is a rank $d$ matroid $M := M(\mathcal{A}_C)$ on the ground set $[n]$ canonically determined by $\mathcal{A}_C$: a subset $D \subseteq [n]$ is a dependent set of $M$ if and only if there are scalars $\zeta_i \in \mathbb{C}$, $i \in D$, not all nulls, such that $\sum_{i \in D} \zeta_i \theta_{H_i} = 0$, where $\theta_{H_i} \in (\mathbb{C}^d)^*$ denotes a linear form such that $\text{Ker}(\theta_{H_i}) = H_i$.

Let $M$ be a matroid and $M^*$ be its dual. In the following, we suppose that the ground set of $M$ is $[n] := \{1, 2, \ldots, n\}$ and its rank function is denoted by $r_M$. The subscript $M$ in $r_M$ will often be omitted. Let $\mathcal{C} = \mathcal{C}(M)$ be the family of circuits of $M$. Let $K$ be a field and $E = \{e_1, \ldots, e_n\}$ be a finite set of order $n$. Let $\bigoplus_{e \in E} Ke$ be the vector space over $K$ of basis $E$ and $\mathcal{E}$ be the graded exterior algebra $\bigwedge \left( \bigoplus_{e \in E} Ke \right)$, i.e.,

$$\mathcal{E} := \sum_{i=0}^{\infty} \mathcal{E}_i = \mathcal{E}_0(= K) \oplus \mathcal{E}_1(= \bigoplus_{e \in E} Ke) \oplus \cdots \oplus \mathcal{E}_i(= \bigwedge_{e \in E} (\bigoplus_{e \in E} Ke)) \oplus \cdots.$$
For every linearly ordered subset \( X = \{i_1, \ldots, i_m\} \subseteq [n], i_1 < \cdots < i_m \), let \( e_X \) be the monomial \( e_X := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m} \). By definition set \( e_\emptyset = 1 \in K \).

Consider the map \( \partial : E \to E \), extended by linearity from the “differentials”,
\[
\partial e_i = 1 \quad \text{for every } e_i \in E, \quad \partial e_\emptyset = 0 \quad \text{and} \quad \partial e_X = \partial (e_{i_1} \wedge \cdots \wedge e_{i_m}) = \sum (-1)^j e_{i_1} \wedge \cdots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \cdots \wedge e_{i_m}.
\]

The (graded) Orlik-Solomon \( K \)-algebra \( \text{OS}(M) \) of the matroid \( M \) is the quotient \( E / \mathcal{I} \) where \( \mathcal{I} \) denotes the (homogeneous) two-sided ideal of \( E \) generated by the set
\[
\{ \partial e_C : C \in \mathcal{C}(M), |C| > 1 \} \cup \{ e_C : C \in \mathcal{C}(M), |C| = 1 \}
\]
or equivalently by the set
\[
\{ \partial e_C : C \in \mathcal{C}(M), |C| > 1 \} \cup \{ e_C : C \in \mathcal{C}(M) \}.
\]

The de Rham cohomology algebra \( H^\bullet(M^\ast(A_C); K) \) is shown to be isomorphic to the Orlik-Solomon \( K \)-algebra of the matroid \( M^\ast(A_C) \), see [6, 7]. We refer to [5] for a recent discussion on the role of matroid theory in the study of Orlik-Solomon algebras.

2. Linear class of circuits

Given a family \( \mathcal{C} \) of circuits of a matroid \( M \) set
\[
\mathcal{H}(\mathcal{C}) := \{ H(C) = [n] \setminus C : C \in \mathcal{C}_L \}
\]
be the associated family of hyperplanes of \( M^\ast \). We recall that a pair \( \{X, Y\} \) of subsets of the ground set \([n]\) is a modular pair of \( M([n]) \) if
\[
r(X) + r(Y) = r(X \cup Y) + r(X \cap Y).
\]

**Proposition 2.1.** Let \( \{C_1, C_2\} \) be a pair of circuits of \( M \) and \( \{H(C_1), H(C_2)\} \) be the associated hyperplanes of \( M^\ast \). The following four conditions are equivalent:
\[
\circ \{C_1, C_2\} \text{ is a modular pair of circuits of } M,
\circ \{H(C_1), H(C_2)\} \text{ is a modular pair of hyperplanes of } M^\ast,
\circ r_{M^\ast}(C_1 \cup C_2) = |C_1 \cup C_2| - 2,
\circ r_{M^\ast}(H(C_1) \cap H(C_2)) = r(M^\ast) - 2 = n - r - 2.
\]

**Definition 2.2.** ([10]). We say that the family of circuits \( \mathcal{C}', \mathcal{C}' \subseteq \mathcal{C}(M) \), is a linear class of circuits if, given a modular pair of circuits in \( \mathcal{C}' \), all the circuits contained in the union of the modular pair are also in \( \mathcal{C}' \).

In the following we will always denote by \( \mathcal{C}_L \) a linear class of circuits of the matroid \( M \).

**Definition 2.3.** We say that the family \( \mathcal{H} \) of hyperplanes of \( M \) is a linear class of hyperplanes of \( M \) if, given a modular pair of hyperplanes in \( \mathcal{H} \), all the hyperplanes of \( M \) containing the intersection of the pair are also in \( \mathcal{H} \).

The following corollary is a direct consequence of Proposition 2.1 and Definitions 2.2 and 2.3.
Corollary 2.4. The following two assertions are equivalent:
- The family $C'$ is a linear class of circuits of $M$;
- The set $\mathcal{H}(C')$ is a linear class of hyperplanes of $M^*$.

Remark 2.5. The linear class of hyperplanes $\mathcal{H}(C_L)$ of $M^*$ determines a single-element extension
$$M^*([n]) \overset{\mathcal{H}(C_L)}{\hookrightarrow} N^*([n+1]),$$
where $\{n+1\}$ is in the closure in $N^*([n+1])$ of a hyperplane $H$ of $M^*([n])$, if and only if $H \in \mathcal{H}(C_L)$. Two special cases occur:
- If $C_L = C(M)$ the element $n+1$ is a coloop of $N([n+1])$.
- If $C_L = \emptyset = \mathcal{H}(C_L)$ the element $n+1$ is in general position in $N^*([n+1])$.

In the literature $N([n+1])$ is called the extended lift of $M([n])$ (determined by the linear class of circuits $C_L$).

Lemma 2.6. Let $N = N([n+1])$ be the extended lift of $M([n])$ determined by the linear class of circuits $C_L, C_L \neq \emptyset, C(M)$. Then $N$ has the family of circuits:
$$\mathcal{C}(N) = \begin{cases} 
C_L \cup C_1 & \text{if } |\cup_{C \in C_L} C| - r_M(\cup_{C \in C_L} C) = n - r - 1; \\
C_L \cup C_1 \cup C_2 & \text{otherwise},
\end{cases}$$
where
$$C_1 := \{ C \cup \{n+1\} : C \in C(M) \setminus C_L \},$$
$$C_2 := \{ C' \cup C'' : C', C'' \text{ is a modular pair of } C(M) \setminus C_L \}.$$

Proof. The matroid $N^*([n+1])$ has the family of hyperplanes:
$$\mathcal{H}(N^*) = \begin{cases} 
\mathcal{H}_0 \cup \mathcal{H}_1 & \text{if } r_{M^*}(\bigcap_{C \in C_L} H(C)) = 1; \\
\mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2 & \text{otherwise},
\end{cases}$$
where
$$\mathcal{H}_0 := \{ H \cup \{n+1\} : H \in \mathcal{H}(C_L) \},$$
$$\mathcal{H}_1 := \{ H(C') : C' \in C(M) \setminus C_L \},$$
$$\mathcal{H}_2 := \{ H' \cap H'' \cup \{n+1\} : H', H'' \text{ is a modular pair of } \mathcal{H}(C(M) \setminus C_L) \}.$$

3. A bias algebra

The pair $(M, C_L)$ can be seen as a matroidal generalization of the pair $(G, C_L)$ (defining a biased graph) where $G$ is a graph and $C_L$ a set of balanced circuits of $G$. A biased graph is a graph together with a linear class of circuits which are called balanced. It is a generalisation of signed and gain graphs which are related to some special class of hyperplane arrangements. In the classical graphic hyperplane arrangements, a hyperplane has equation of the form $x_i = x_j$. In the “signed graphic” arrangements, the equations can be of the form $x_i = \pm x_j$. In the “gain graphic” arrangements, the equations
can be of the form \( x_i = gx_j \) (in the biased case) or of the form \( x_i = x_j + g \) (in the lift case). All these definitions due to T. Zaslavsky are very natural and produce a nice theory \[12\] [13] in connection with graphs, matroids and arrangements. The following bias algebra is close related to the biased graphs (and its matroidal generalizations).

**Definition 3.1.** Let \( C_L \) be a linear class of circuits of the matroid \( M([n]) \) and \( N = N([n + 1]) \) be the extended lift of \( M([n]) \) determined by \( C_L \). Let \( OS(N) \) be the Orlik-Solomon \( K \)-algebra of the matroid \( N \). The bias \( K \)-algebra of the pair \((M, C_L)\), denoted \( Z(M, C_L) \), is the graded quotient of the Orlik-Solomon algebra \( OS(N) \) by the two-sided ideal generated by \( e_{n+1} \), i.e.,

\[
Z(M, C_L) := OS(N)/\langle e_{n+1} \rangle.
\]

**Remark 3.2.** \[11\] This algebra is also known as the Orlik-Solomon algebra of the pointed matroid \( N \), with basepoint \( n + 1 \), see \[5\] Definition 3.2. If \( N \) may be realized by a complex hyperplane arrangement, then \( Z(M, C_L) \) is isomorphic to the cohomology ring of the complement of the decone of this arrangement with respect to the \((n + 1)^{\text{st}}\) hyperplane, \[7\] Corollary 3.57. Two special cases occur when \( M \) itself is realizable and \( C_L \) is either all of \( C(M) \) or the empty set. Indeed, suppose that \( M \) is the matroid associated to a complex hyperplane arrangement \( A \). Then \( Z(M, C(M)) \) is isomorphic to the cohomology of the complement of \( A \) (i.e., the Orlik-Solomon algebra of \( M \)), and \( Z(M, \emptyset) \) is isomorphic to the cohomology of the complement of the affine arrangement attained by translating each of the hyperplanes of \( A \) some distance away from the origin, so that every dependent set will have empty intersection.

**Theorem 3.3.** The bias \( K \)-algebra \( Z(M, C_L) \) is independent of the order of the elements of \( M([n]) \), i.e., it is an invariant of the pair \((M, C_L)\). For every linear class \( C_L \), the algebra \( Z(M, C_L) \) is isomorphic to the quotient of the exterior \( K \)-algebra

\[
E := \bigwedge \left( \bigoplus_{i=1}^{n} Ke_i \right)
\]

by the two-sided ideal \( \langle \Im(C_L) \rangle \) generated by the set

\[
\Im(C_L) := \{ \partial e_C : C \in C_L, |C| > 1 \} \cup \{ e_C : C \in C(M) \}.
\]

**Proof.** Since the Orlik-Solomon \( K \)-algebra \( OS(N) \) does not depend of the ordering of the ground set the first part of the theorem follows. The second assertion is a straightforward consequence of Lemma 2.6 □

As the element \( e_{n+1} \) does not appear in the algebra \( Z(M, C_L) \) we will omit it. We remark that the monomial \( e_X, X \subseteq [n] \), in \( Z(M, C_L) \) is different from zero if and only if \( X \) is an independent set of \( M \).

**Corollary 3.4.** The bias \( K \)-algebra \( Z(M, C(M)) \) is the Orlik-Solomon \( K \)-algebra of \( OS(M) \). Furthermore the bias \( K \)-algebra \( Z(M, \emptyset) \) is isomorphic
to the quotient of the exterior algebra $\mathcal{L}^\vee$ by the two-sided ideal generated by the set \(\{e_C : C \in \mathcal{C}(M)\}\).

**Corollary 3.5.** If \(cl_m(n + 1) = n + 1\), the bias $\mathbf{K}$-algebra $\mathcal{Z}(M, \mathcal{C}_L(M))$ is the quotient of the exterior algebra $\mathcal{L}^\vee$ by the two-sided ideal generated by the set

\[
\{\partial e_C : C \in \mathcal{C}_L, |C| > 1\} \cup \{e_C : C \in \mathcal{C}(M)\}.
\]

**Definition 3.6.** Given an independent set $I$, a non-loop element $x \in cl(I) \setminus I$ is said to be $\mathcal{C}_L$-active in $I$ if $C(x, I)$ (i.e., the unique circuit contained in $I \cup x$) is a circuit of the family $\mathcal{C}_L$ and $x$ is the smallest element of $C(x, I)$. An independent set with at least one $\mathcal{C}_L$-active element is said to be $\mathcal{C}_L$-active, and $\mathcal{C}_L$-inactive otherwise. We denote by $a(I)$ the smallest $\mathcal{C}_L$-active element in an active independent set $I$.

**Definition 3.7.** We say that a subset $U \subseteq [n]$ is a $\mathcal{C}_L$-unidependent (set of $M$) if it contains a unique circuit $C(U)$ of $M$, $C(U) \in \mathcal{C}_L$ and $|C(U)| > 1$.

We say that a $\mathcal{C}_L$-unidependent set $U$ is $\mathcal{C}_L$-inactive if the minimal element of $C(U)$, $\min C(U)$, is the the smallest $\mathcal{C}_L$-active element of the independent set $U \setminus \min C(U)$. Otherwise the set $U$ is said $\mathcal{C}_L$-active.

**Definition 3.8.** For every circuit $C \in \mathcal{C}_L$, $|C| > 1$, the set $C \setminus \min(C)$, is said to be a $\mathcal{C}_L$-broken circuit. The family of $\mathcal{C}_L$-inactive independents, denoted $\text{NBC}_{\mathcal{C}_L}$, is the family of independent sets of $M$ not containing a $\mathcal{C}_L$-broken circuit.

Set

\[
\text{nbc}_{\mathcal{C}_L} := \{e_I : I \in \text{NBC}_{\mathcal{C}_L}\}
\]

\[
\text{b}_{\mathfrak{S}(\mathcal{C}_L)} := \{\partial e_U : U \text{ is } \mathcal{C}_L\text{-inactive unidependent}\} \cup \{e_D : D \text{ is dependent}\}.
\]

**Theorem 3.9.** The sets $\text{nbc}_{\mathcal{C}_L}$ and $\text{b}_{\mathfrak{S}(\mathcal{C}_L)}$ are bases, respectively of the bias $\mathbf{K}$-algebra $\mathcal{Z}(M, \mathcal{C}_L)$ and of the ideal $\langle \mathfrak{S}(\mathcal{C}_L) \rangle$.

**Proof.** We will show the two statements at the same time by proving that both sets are spanning and that they have the correct size. Let $I$ be an independent set of $M$. If $I$ is $\mathcal{C}_L$-active then we have

\[
e_I = \sum_{x \in C(a(I), I) \setminus a(I)} \zeta_x e_{I \cup a(I) \setminus x},
\]

where $\zeta(x) \in \{-1, 1\}$. This is an expression for $e_I$ with respect to lexicographically smaller $e_X$ where $X$ is an independent of $M$ and $|X| = |I|$. By induction, we get that the set $\text{nbc}_{\mathcal{C}_L}$ is a generator of the graded algebra $\mathcal{Z}(M, \mathcal{C}_L)$.

Let $U$ be a $\mathcal{C}_L$-unidependent set of $M$. Suppose that $U$ is $\mathcal{C}_L$-active and let $a = \min C(U)$ and set $I := C(U) \setminus a$. Note that $\{C(U), C(a(I), I)\}$
is a modular pair of circuits of $C_L$, so every circuit contained in the cycle $C(U) \cup C(a(I), I)$ is in $C_L$. From the definition of the map $\partial$ we know that
\[
\partial e_U = \sum_{x \in C(U) \setminus a} \epsilon_x \partial e_{U \setminus a(I) \setminus x},
\]
where $\epsilon_x \in \{-1, 1\}$. This is an expression for $\partial e_U$ with respect to lexicographically smaller $\partial e_X$, where $X$ is a $C_L$-undependent and $|U| = |X|$. By induction, we get that the set $b_{3(C_L)}$ is a generator of $\langle 3(C_L) \rangle$. By the definition of $Z(M, C_L)$, we know that $\dim(Z(M, C_L)) + \dim(\langle b_{3(C_L)} \rangle) = \dim(E) = 2^n$.

Given a subset $X$ of $[n]$, it is either dependent or independent $C_L$-active or independent $C_L$-inactive. To every independent $C_L$-active independent set $I$ corresponds uniquely the unidependent $C_L$-inactive $I \cup a(I)$. We have then that
\[
|\text{nb}_C(M)| + |b_{3(C_L)}| = 2^n.
\]
\[\square\]

We define the deletion and contraction operation for an arbitrary subset of circuits $C' \subseteq C(M)$ setting:
\[
C' \setminus x := \{C \in C' : x \notin C\}
\]
and
\[
C'/x := \begin{cases} 
C' \setminus x & \text{if } x \text{ is a loop of } M, \\
\{C \setminus x : x \in C \in C'\} \sqcup \{C \in C' : x \notin cl_M(C)\} & \text{otherwise.}
\end{cases}
\]

From the preceding definition, we can see that given a circuit $C$ of $C'/x$, where $x$ is a non-loop of $M$, there exists a unique circuit $\tilde{C} \in C'$ such that
\[
\tilde{C} := \begin{cases} 
C \cup x & \text{if } x \in cl_M(C), \\
C & \text{otherwise.}
\end{cases}
\]

**Proposition 3.10.** Let $M$ be a matroid and $C_L$ be a linear class of circuits of $M$. For an element $x$ of the matroid, the circuit sets $C_L \setminus x$ and $C_L/x$ are linear classes of $M \setminus x$ and $M/x$, respectively.

**Proof.** The statement for the deletion is clear. If $x$ is a loop the result is also clear for the contraction. Suppose that $x$ is a non-loop of $M$. If $Y \subseteq X$ are sets such that $r_M(X) = r_M(Y) + 1$ then we have
\[
r_{M/x}(X \setminus x) = r_{M/x}(Y \setminus x) + \epsilon, \; \epsilon \in \{0, 1\}.
\]
So, if $\{C_1, C_2\}$ is a modular pair of circuits of $C_L/x$, $\{\tilde{C}_1, \tilde{C}_2\}$ is also a modular pair of circuits of $C_L$. We see also from Equation (3.2) that if $C \subseteq C_1 \cup C_2$ is a circuit of $M/x$ then $\tilde{C} \subseteq \tilde{C}_1 \cup \tilde{C}_2$, so $\tilde{C} \in C_L$ and necessarily $C \in C_L/x$.\[\square\]
Definition 3.11. For a pair \((M, \mathcal{C}_L)\) and an element \(x\) of \(M\), we define the deletion and the contraction of the pair \((M, \mathcal{C}_L)\) by:

\[
(M, \mathcal{C}_L) \setminus x := (M \setminus x, \mathcal{C}_L \setminus x)
\]

and

\[
(M, \mathcal{C}_L)/x := (M/x, \mathcal{C}_L/x).
\]

As a corollary of Theorem 3.3 we have:

Proposition 3.12. For every element \(x\) of \(M\), there is a unique monomorphism of vector spaces,

\[
i_x : Z(M, \mathcal{C}_L) \setminus x \to Z(M, \mathcal{C}_L),
\]

such that, for every independent set \(I\) of \(M \setminus x\), we have \(i_x(e_I) = e_I\).

Proposition 3.13. For every non-loop element \(x\) of \(M\), there is a unique epimorphism of vector spaces,

\[
p_x : Z(M, \mathcal{C}_L) \to Z(M, \mathcal{C}_L)/x,
\]

such that, for every subset \(I = \{i_1, \ldots, i_\ell\} \subseteq [n]\),

\[
p_x e_I := \begin{cases} e_{I \setminus x} & \text{if } x \in I, \\ \pm e_{I \setminus y} & \text{if } \exists y \in I \text{ such that } \{x, y\} \in \mathcal{C}_L, \\ 0 & \text{otherwise.} \end{cases}
\]

More precisely the value of the coefficient \(\pm 1\) in the second case is the sign of the permutation obtained by replacing \(y\) by \(x\) in \(I\).

Proof. From Theorem 3.3 it is enough to prove that the map \(p_x\) is well determined, i.e., for all \(\mathcal{C}_L\)-unidependent \(U = (i_1, \ldots, i_m)\) set of \(M\), we have

\[
p_x \partial e_U = 0 \in \mathbb{Z}(\mathcal{C}_L/x).
\]

We can also suppose that \(x\) is the last element \(n\). Note that if \(n \in U\) then \(U \setminus n\) is a \(\mathcal{C}_L/n\)-unidependent set of \(M/n\). If \(n \notin U\) but there is \(y \in U\) and \(\{n, y\} \in \mathcal{C}_L\), we know that \(e_U = \pm e_{U \setminus y,n}\) in \(Z(M, \mathcal{C}_L)\). Suppose that \(n \notin U\) and that there does not exist \(y \in U\) such that \(\{n, y\} \in \mathcal{C}_L\). Then it is clear that \(p_n \partial e_U = 0\). Suppose that \(n \in U\). It is easy to see that

\[
\pm p_n \partial e_U = \sum_{j=1}^{m-1} e_{U \setminus \{j, n\}} = 0.
\]

Finally, if an independent set \(I\) of \(M\) contains an element \(y\) such that \(\{x, y\}\) is a circuit in \(\mathcal{C}_L\), we know that there is a scalar \(\chi(I; x, y) \in \{-1, 1\}\) such that \(e_I = \chi(I; x, y) e_{I \setminus y,x}\). More precisely the value of \(\chi(I; x, y) \in \{-1, 1\}\) is the sign of the permutation obtained by replacing \(y\) by \(x\) in \(I\). 

Theorem 3.14. Let \(M\) be a loop free matroid and \(\mathcal{C}_L\) be a linear class of circuits of \(M\). For every element \(x\) of \(M\), there is a splitting short exact sequence of vector spaces

\[
0 \to Z(M, \mathcal{C}_L) \setminus x \xrightarrow{i_x} Z(M, \mathcal{C}_L) \xrightarrow{p_x} Z(M, \mathcal{C}_L)/x \to 0.
\]
Proof. From the definitions we know that $p_x \circ i_x$, is the null map so $\text{Im}(i_x) \subseteq \text{Ker}(p_x)$. We will prove the equality $\text{dim}(\text{Ker}(p_n)) = \text{dim}(\text{Im}(i_n))$. By a re-ordering of the elements of $[n]$ we can suppose that $x = n$. The minimal $C_L/n$-broken circuits of $M$ are the minimal sets $X$ such that either $X$ or $X \cup \{n\}$ is a $C_L$-broken circuit of $M$ (see [1] Proposition 3.2.e). Then

$$\text{NBC}_{C_L/n} = \{X : X \subseteq [n - 1] \text{ and } X \cup \{n\} \in \text{NBC}_{C_L}\}$$

and we have

$$(3.5) \quad \text{NBC}_{C_L} = \text{NBC}_{C_L/n} \cup \{I \cup n : I \in \text{NBC}_{C_L/n}\}.$$  

So $\text{dim}(\text{Ker}(p_n)) = \text{dim}(\text{Im}(i_n))$. There is a morphism of vector spaces

$$p_n^{-1} : Z(M, C_L)/n \to Z(M, C_L),$$

where, for every $I \in \text{NBC}_{C_L/n}$, we have $p_n^{-1}e_I := e_{I \cup n}$. It is clear that $p_n \circ p_n^{-1}$ is the identity map. From Equation (3.5) we conclude that the exact sequence (3.4) splits. $\square$

Remark 3.15. A large class of algebras, the so called $\chi$-algebras (see [4] for more details), contain the Orlik-Solomon, Orlik-Terao [8] (associated to vectorial matroids) and Cordovil algebras [3] (associated to oriented matroids). Following the same ideas it is possible to generalize the definition of the bias algebras and obtain a class of bias $\chi$-algebras, determined by a pair $(M, C_L)$, and that contain all the mentioned algebras.

Similarly to [4], we now construct, making use of iterated contractions, the dual basis $\text{NBC}_{nbc}^*$ of the standard basis $\text{NBC}_{nbc}$. Let $Z(M, C_L)_h$ be the subspace of $Z(M, C_L)$ generated by the set

$$\{e_X : X \text{ is an independent set of } M \text{ and } |X| = h\}.$$  

We associate to the (linearly ordered) independent set $I = (i_1, \ldots, i_h)$ of $M$ the linear form on $Z(M, C_L)_h$, $p_I : Z(M, C_L)_h \to K$,

$$(3.6) \quad p_I := p_{e_{i_1}} \circ p_{e_{i_2}} \circ \cdots \circ p_{e_{i_h}}.$$  

We also associate to the linearly ordered independent $I = (i_1, \ldots, i_j)$ the flag of its final independent subsets, defined by

$$\{I_t : I_t = (i_t, \ldots, i_j), 1 \leq t \leq j\}.$$  

Proposition 3.16. Let $I = (i_1, \ldots, i_h)$ and $J = (j_1, \ldots, j_h)$ be two linearly ordered independents of $M$, then we have $p_I(e_J) \neq 0$ if and only if there is a permutation $\tau \in \mathfrak{S}_h$ such that for every $1 \leq t \leq h$, $j_{\tau(t)} \in \text{cl}(I_t)$ and $C(j_{\tau(t)}, I_t) \in C_L$. When the permutation $\tau$ exists, it is unique and we have $p_I(e_J) = \text{sgn}(\tau)$. In particular we have $p_I(e_I) = 1$ for any independent set $I$.

Proof. The first equivalence is very easy to prove in both directions. To obtain the expression of $p_I(e_J)$ we just need to iterate $h$ times the formula of contraction of Proposition [4] 2 With the definition of the permutation $\tau$ we know that $p_I(e_{\tau(1)} \wedge \cdots \wedge e_{\tau(h)}) = 1$. By the antisymmetric of the wedge
Theorem 3.17. The set \( \{ p_I : I \in \text{NBC}_{CL} \} \) is the dual basis of the standard basis \( \text{nbc}_{CL} \) of \( Z(M, CL) \).

Proof. Pick two elements \( e_I \) and \( e_J \) in \( \text{nbc}_{CL} \), \( |I| = |J| = h \). We just need to prove that \( p_I(e_J) = \delta_{IJ} \) (the Kronecker delta). From the preceding proposition we already have that \( p_I(e_I) = 1 \). Suppose for a contradiction that there exists a permutation \( \tau \) such that \( j_{\tau(t)} \in \text{cl}(I_t) \) and \( C(j_{\tau(t)}, I_t) \in CL \) for every \( 1 \leq t \leq h \). Suppose that \( j_{\tau(m+1)} = i_{m+1}, \ldots, j_{\tau(h)} = i_h \) and \( i_m \neq j_{\tau(m)} \). Then there is a circuit \( C \subseteq CL \) such that

\[
i_m, j_{\tau(m)} \in C \subseteq \{ i_m, j_{\tau(m)}, i_{m+1}, i_{m+2}, \ldots, i_h \}.
\]

If \( j_{\tau(m)} < i_m \) [resp. \( i_m < j_{\tau(m)} \)] we conclude that \( I \notin \text{NBC}_{CL} \) [resp. \( J \notin \text{NBC}_{CL} \)], a contradiction. \( \square \)

The following corollary is an extension of results of [2], [3] and [4].

Corollary 3.18. Let \( J = \{ j_1, \ldots, j_t \} \) be an independent set of \( M \) such that the expansion of \( e_J \) in \( \text{nbc}_{CL} \) is \( e_J = \sum I \in \text{nbc}_{CL} \xi(I, J)e_I \). Then the following are equivalent:

\begin{itemize}
  \item \( \xi(I, J) \neq 0 \),
  \item there exists a permutation \( \tau \) such that \( e_{\tau(t)} \in \text{cl}(I_t) \) and \( C(e_{\tau(t)}, I_t) \in CL \) for every \( 1 \leq t \leq h \). Moreover, in the case where \( \xi(I, J) \neq 0 \) we have \( \xi(I, J) = \text{sgn}(\tau) \).
\end{itemize}

\( \square \)

References

[1] Brylawski, T.: The broken-circuit complex. \textit{Trans. Amer. Math. Soc.} \textbf{234} (1977), no. 2, 417–433.
[2] Cordovil, R., and Etienne, G.: A note on the Orlik-Solomon algebra. \textit{European J. of Combin.} \textbf{22} (2001), 165–170.
[3] Cordovil, R.: A commutative algebra for oriented matroids. \textit{Discrete and Computational Geometry} \textbf{27} (2002), 73–84.
[4] Cordovil, R. and Forge, D.: Diagonal bases in Orlik-Solomon type algebras. \textit{Annals of Combinatorics} \textbf{7} (2003), 247–257.
[5] Falk, Michael J.: Combinatorial and algebraic structure in Orlik–Solomon algebras. \textit{European J. Combin.} \textbf{22}, (2001), no. 5, 687–698.
[6] Orlik, Peter; Solomon, Louis: Combinatorics and topology of complements of hyperplanes. \textit{Invent. Math.} \textbf{56} (1980), no. 2, 167–189.
[7] Orlik, Peter; Terao, Hiroaki: Arrangements of Hyperplanes. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 300. \textit{Springer-Verlag}, Berlin, 1992.
[8] Orlik, Peter; Terao, Hiroaki: Commutative algebras for arrangements. \textit{Nagoya Math. J.} \textbf{134} (1994), 65–73.
[9] Orlik, Peter; Terao, Hiroaki: Arrangements and hypergeometric integrals. MSJ Memoirs \textbf{9}, \textit{Mathematical Society of Japan}, Tokyo, 2001.
[10] Tutte, W. T.: Lectures on matroids. \textit{Res. Nat. Bur. Standards Sect. B} \textbf{69B} (1965), 1–47.
[11] Proudfoot, Nicholas, private communication.
[12] Zaslavsky, T.: Biased graphs. I. Bias, balance, and gains. J. Combin. Theory Ser. B 47 (1989), 32–52.
[13] Zaslavsky, T.: Biased graphs. II. The three matroids. J. Combin. Theory Ser. B 51 (1991), 46–72.

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