Faster search of clustered marked states with lackadaisical quantum walks

Amit Saha 1,2 · Ritajit Majumdar 3 · Debasri Saha 1 · Amlan Chakrabarti 1 · Susmita Sur-Kolay 3

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Abstract
The nature of discrete-time quantum walks in the presence of multiple marked states can be found in the literature. An exceptional configuration of clustered marked states, which is a variant of multiple marked states, may be defined as a cluster of $k$ marked states arranged in a $\sqrt{k} \times \sqrt{k}$ array within a $\sqrt{N} \times \sqrt{N}$ grid, where $k = n^2$ and $n$ an odd integer. In this article, we establish through numerical simulation that for lackadaisical quantum walks, which is the analogue of a three-state discrete-time quantum walks on a line, the success probability to find a vertex in the marked region of this exceptional configuration is nearly 1 with smaller run-time. We also show that the weights of the self-loop suggested for multiple marked states in the state-of-the-art works are not optimal for this exceptional configuration of clustered marked states. We propose a weight of the self-loop which gives the desired result for this configuration.

Keywords Lackadaisical quantum walks · Multiple marked state · Quantum walks · Clustered-marked states

1 Introduction

As development of quantum computers has progressed significantly in the last decade, quantum algorithms for new problems, which provide potential speedup over their classical counterparts [23], are also being studied enthusiastically. Grover’s algorithm [14] is one of the quantum algorithms that has quadratic speedup for searching a marked location in an unsorted database. Quantum walk [2] is another such quantum
algorithm that can be a prospective candidate for solving search problems [1, 4, 18, 27] with considerable speedup over their classical versions. Quantum walks (QW) have two main variants, namely continuous-time (CTQW)[11] and discrete-time [3, 28] (DTQW). The probability distribution of the particles for both the variants of the quantum walk, spreads quadratically faster in position space compared to the classical random walks [16, 17, 29]. A CTQW evolves under a Hamiltonian which is defined with respect to a graph, and no coin operator is required. A DTQW ideally evolves through a quantum coin operator. In DTQW on a 1D grid (line), a two-state quantum coin operator has been used.

The concept of lazy quantum walks [10], where the walker has equal probability of staying put, was later introduced incorporating a three-state quantum coin operator on a line. This helps to establish a relationship between CTQW and DTQW [10]. A small variation in lazy quantum walks gave birth to a breakthrough algorithm, i.e., lackadaisical quantum walks (LQW) [32], which gives algorithmic speed up over the previous ones [15, 24, 25, 30, 31, 34]. Lackadaisical quantum walks assign a self-loop of weight \( l \) to each vertex (which can be varied as necessary; refer Fig. 1), so that the walker has nonzero probability of staying put.

Wong [33] showed that the lackadaisical quantum walks, having self-loop of weight \( 4/N \) for each vertex, searches a single marked state with a \( \sim 1 \) success probability in \( O(\sqrt{N \log N}) \) steps, thus yielding an \( O(\sqrt{\log N}) \) improvement as compared to the state-of-the-art. Later on, the authors [9, 13, 19, 22, 26] have also studied the advantage of lackadaisical quantum walks for multiple marked states. But, these works have failed to provide a generalized solution for the exceptional configuration of multiple marked states arranged in a \( \sqrt{k} \times \sqrt{k} \) cluster within a \( \sqrt{N} \times \sqrt{N} \) grid using lackadaisical quantum walks.

Here, we propose a weight \( l \) for the self-loop through numerical simulation,

\[
   l \approx \frac{4}{N(k+1)}.
\]

**Fig. 1** A two-dimensional grid of \( N = 5 \times 5 \) vertices with a self-loop of weight \( l \) at each vertex. The boundaries are periodic. A marked vertex is indicated by a red circle [33] (Color figure online)
With this weight, one can attain the success probability $\sim 1$ for finding a vertex in the clustered marked region with run-time lower than the state-of-the-art algorithms because no amplitude amplification is required.

In Sect. 2, we discuss standard quantum walks and lackadaisical quantum walks in a two-dimensional grid briefly. We discuss the works related to our proposed work in Sect. 3. In Sect. 4, we present our proposed methodology for a cluster of marked states on a two-dimension grid using lackadaisical quantum walks. Concluding remarks appear in Sect. 5.

2 Background

A few preliminaries of quantum random walks are first presented below.

2.1 Discrete time quantum walks in two-dimensions

A quantum random walk consists of a position Hilbert space $H_p$ and a coin Hilbert space $H_c$. A quantum state consists of these two degrees of freedom, $|c\rangle \otimes |v\rangle$ where $|c\rangle \in H_c$ and $|v\rangle \in H_p$. A step in quantum walks is a unitary evolution $U = S.(C \otimes I)$ where $S$ is the shift operator and $C$ is the coin operator, which acts only on the coin Hilbert space $H_c$.

If we consider a $\sqrt{N} \times \sqrt{N}$ grid, then the quantum walk starts in a superposition of states given by

$$|\psi(0)\rangle = \frac{1}{\sqrt{4N}} \left( \sum_{i=1}^{4} |i\rangle \otimes \sum_{x,y=1}^{\sqrt{N}} |x, y\rangle \right)$$

(2)

where (i) each location $(x, y)$ corresponds to a quantum register $|x, y\rangle$ with $x, y \in \{1, 2, \ldots, \sqrt{N}\}$, and (ii) the coin register $|i\rangle$ with $i \in \{\leftarrow, \rightarrow, \uparrow, \downarrow\}$. The most common transformation on the coin register is Grover’s Diffusion transformation given by

$$D = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$  

(3)

The Diffusion operator can also be written as $D = 2|s_D\rangle\langle s_D| - I_4$, where $|s_D\rangle = \frac{1}{\sqrt{4}} \sum_{i=1}^{4} |i\rangle$.

The transformation creates a superposition of the coin states $|i\rangle$, which in turn governs the shift operation. Multiple shift operators have been proposed in the literature [5] out of which, in this paper, we have used the Flip-Flop shift transformation $S$ [6] whose action on the basis states are as follows:

$$|\uparrow, i, j\rangle \rightarrow |\downarrow, i, j + 1\rangle$$

(4)
From Eqs. (4–7), we can infer that the Flip-Flop shift transformation changes the value of the direction register to the opposite after moving to an adjacent position state. It is to be noted that $|\psi(t)\rangle$ is an eigenstate of the operator $U = S . (D \otimes I)$ with eigenvalue $+1$ for any $t \in \mathbb{Z}$. A perturbation is created in the quantum state by applying the coin operator $-I$ instead of $D$ for marked locations. A general quantum walk applies (appropriately for the marked and the unmarked states) this unitary operation $t$ times to create the state $|\psi(t)\rangle$ such that $\langle \psi(t) | \psi(0) \rangle$ becomes close to 0. If $M$ be the set of qubits corresponding to the marked vertices, then the state $|\psi(t)\rangle$ can be expressed as

$$|\psi(t)\rangle = \frac{1}{\sqrt{4}} \left( \sum_{i=1}^{4} |i\rangle \otimes \left( \alpha \sum_{x,y=1; (x,y) \in M}^{\sqrt{N}} |x,y\rangle + \beta \sum_{x,y=1; (x,y) \notin M}^{\sqrt{N}} |x,y\rangle \right) \right)$$

where $|\alpha|^2 + |\beta|^2 = 1$. For the rest of the article, $|\alpha|^2$ is defined as the probability of success, or the probability of finding a vertex in the marked region. It is expected for the LQW algorithm to yield $|\alpha|^2$ close to 1 after $t$ steps (more on the value of $t$ in later sections).

### 2.2 Lackadaisical quantum walks in two-dimensions

In a two-dimensional lackadaisical quantum walk, the degree of freedom of the coin is five-dimensional, i.e., $i \in \{ \leftarrow, \rightarrow, \uparrow, \downarrow, . \}$. The flip-flop transformation conditioned on the $|\cdot\rangle$ coin state is

$$S(\cdot \otimes |i, j\rangle) = \cdot \otimes |i, j\rangle$$

With $l$ self-loops, the coin operator is $D = 2|s_D\rangle\langle s_D| - I_5$, where

$$|s_D\rangle = \frac{1}{\sqrt{4+l}} (|\uparrow\rangle + |\downarrow\rangle + |\leftarrow\rangle + |\rightarrow\rangle + \sqrt{l}|\cdot\rangle)$$

### 2.3 Clustered marked states in a two-dimensional grid

The notion of clustered marked states was introduced by Nahimovs et al. [20]. In accordance with that paper, we consider the walk on a $\sqrt{N} \times \sqrt{N}$ grid where there are $k$ marked locations arranged in a $\sqrt{k} \times \sqrt{k}$ cluster (Fig. 2). These clustered marked locations can be anywhere in the given grid.
3 Related works

After the success of Grover’s algorithm as a quantum search algorithm, Benioff [8] first showed that if the $N$ data points are arranged in a $\sqrt{N} \times \sqrt{N}$ grid, then the quantum speedup is lost. Since then, research has been carried out to design faster algorithms to search an unsorted database arranged in a two- or higher-dimensional grid. Ambainis et al. [7] proposed an algorithm (referred henceforth as the AKR algorithm) based on quantum random walks with AKR coin. The diffusion operators of the AKR coin for the marked and unmarked states are $-I$ and $D$, respectively, where $D$ is as in Eq. (3). This algorithm can detect a marked state with probability $O\left(\frac{1}{\log N}\right)$ in $O\left(\sqrt{N \log N}\right)$ time. In order to increase the success probability, amplitude amplification is necessary, which has a time complexity of $O\left(\sqrt{\log N}\right)$. This gives an overall running time of the algorithm to be $O\left(\sqrt{N \log N}\right)$. Childs and Goldstone [12, 33] matched this runtime with a CTQW. Ambainis et al. [5] henceforth, proposed another algorithm, which does not require amplitude amplification, and can perform the search in $O\left(\sqrt{N \log N}\right)$ time. Further works [1, 29] have been carried out to study quantum walk algorithms in other graph structures, but in this article, we consider only the two-dimensional grid.

Most of the quantum walk based search algorithms consider one or two marked locations. Nahimovs and Rivosh [21] considered searching for multiple marked states within a $\sqrt{N} \times \sqrt{N}$ grid. They also showed that if $k$ marked states are clustered in a $\sqrt{k} \times \sqrt{k}$ block, then the algorithm of [6] can perform the search in $\Omega\left(\sqrt{N - \sqrt{k}}\right)$ time by using a Grover’s coin, where the diffusion operators for the marked and unmarked states are $D$ and $-D$ respectively. Whereas if the $k$ marked locations are distributed uniformly over the grid, then the algorithm requires $O\left(\frac{N}{k} \log \frac{N}{k}\right)$ time. Nahimovs and Rivosh [20] also showed that if $k$ marked states are grouped in a $\sqrt{k} \times \sqrt{k}$ block, where $k = n^2$ and $n$ is an even number $\geq 2$, then the quantum walk always fails to find any of the marked locations.

In [26], the authors extended the model of [32] to multiple marked states for the first time, arranged in a $\sqrt{k} \times \sqrt{k}$ cluster, where $k = n^2$ and $n$ is an odd number,
and showed by simulation that adjusting the weight of the self-loop to \( \frac{4}{N(k + \lfloor \sqrt{k} \rfloor)} \), the probability of finding a vertex in the marked region increases by \( \sim 0.2 \). The number of steps required is less than that of the quantum walks with no self-loop or Grover’s coin. The authors also showed that adjusting the weight of the self-loop in [32] does not work for multiple mark states. But, the limitation of [26] was that the authors used only \( k = 9 \) and \( N \leq 30 \). Later on, Nahimovs [19], Nahimovs et al. [22] and Giri et al. [13] provided an adjustable weight of the self-loop for multiple marked states, which are \( l = \frac{4(k - \sqrt{k})}{N} \) [19], \( l = \frac{4k}{N} \) [13], respectively, for different grid size \( N \). Albeit, their weights also fail to search a cluster of marked states. In this paper, we provide a generalized solution for this exceptional configuration of clustered marked states, which is thoroughly discussed in the next section.

4 Lackadaisical quantum walks for clustered marked states in a two-dimensional grid

As discussed above, Nahimovs and Rivosh [20] have used quantum walks where the weight on the self-loop is 0, or in other words, is not lackadaisical. They have used Grover’s coin in the AKR algorithm and exhibited that Grover’s coin significantly outperforms the AKR coin with respect to success probability to find a vertex in the marked region in a 2-dimensional grid. Wong [32] has shown that using a weight of \( \frac{4}{N} \) for the self-loop, the probability of finding a single marked state increases with respect to that for non-lackadaisical walks. However, for multiple marked states, this weight provides a probability poorer than non-lackadaisical walks [13, 19, 26]. In this paper, we have proposed a weight of the self-loop, \( l \approx \frac{4}{N(k + 1)} \), for finding clustered marked states which leads to a success probability of nearly 1 in steps fewer than the state-of-the-art models of quantum walks for different grid sizes. The running time and the success probability of the lackadaisical quantum walks completely depends on a weight of the self-loop \( l \). Figure 3 shows the evolution of the probability of finding a vertex in the clustered marked vertices of \( k = 9 \) for the lackadaisical quantum walk on a grid of size \( N = 50 \times 50 \) for various values of \( l \) through simulation. As one can see different values of \( l \) result in different success probabilities and number of steps till the first peak. When \( l = 0 \), i.e., the non-lackadaisical quantum walk, the success probability reaches \( \sim 0.6 \). The success probability becomes \( \sim 1 \) when \( l \approx 0.00015 \), which is \( \frac{4}{N(k + 1)} \). We also observe that with the increase in steps, the success probability is periodic, and the value of the peaks remain the same throughout, which implies that the first peak is the optimum value. The algorithmic presentation of our proposed work is briefly illustrated in Algorithm 1.

Our simulations exhibit that using the weight of the self-loop \( l \approx \frac{4}{N(k + 1)} \), the success probability is close to 1 for different grid sizes ranging from \( 10 \times 10 \) to \( 100 \times 100 \), as given in Table 1. We also observe that error between \(-0.00001\) and
0.0004 is present in the proposed weight for the aforesaid range of grid size. Figure 4 represents the results of our simulations graphically.

In Table 2, we have presented a comparative study of the Grover’s coin and our proposed approach. We have carried out simulations for \( k = 9 \), i.e., nine marked states arranged in a \( 3 \times 3 \) cluster for different grid sizes ranging from \( 10 \times 10 \) to \( 100 \times 100 \). The results clearly show that our proposed approach of lackadaisical quantum walks outperforms Grover’s coin based quantum walk approach both in terms of success probability and run-time. Our lackadaisical quantum walk approach finds a vertex in the cluster marked region with success probability nearly 1 with the number of steps linear in the size of the grid, which may be considered as optimal. For the Grover’s coin based quantum walks, the algorithm needs to be iterated \( \sqrt{\log N} \) times more in order to attain the success probability \( \sim 1 \). For example, in a grid of size 10,000, Grover’s coin based quantum walks attain success probability \( \sim 0.553369 \) after 319 steps. In order to attain success probability \( \sim 1 \) on this grid size, Grover’s coin based quantum walks need 1276 steps, compared to 471 steps required by our proposed coin.

We have compared our work with [13, 19], and the results appear in Tables 3 and 4. In [13, 19], the authors tried to generalize the lackadaisical quantum walk approach for multiple marked states by suggesting the optimal weight for self-loop with \( l = \frac{4(k - \sqrt{k})}{N} \) [19] and \( l = \frac{4k}{N} \) [13]. But, with those weights, their LQW is unable to find a single marked state of this exceptionally configured clustered marked states. In Fig. 5, we show a comparative analysis of our proposed approach versus [13, 19] with respect to the success probability for \( k = 9 \) in different grid sizes from \( 10 \times 10 \) to \( 100 \times 100 \).

Further, we have simulated the result for \( k = 25, 49 \) (i.e., 25 marked states arranged in a \( 5 \times 5 \) and 49 marked states arranged in a \( 7 \times 7 \) cluster), using the same weight of the self-loop \( l \approx \frac{4}{N(k + 1)} \), where error in weight varies between \(-0.000004\) to \(0.000016\) for different grid sizes, which are given in Table 5. Hence, we infer from Tables 1
Algorithm 1: LQW for clustered marked states in a two-dimensional grid

Input: \( N, k \)

// \( N \), the size of the 2-dimensional grid; \( k \), the number of clustered marked states, which can be described as a set of clustered marked vertices \( K(x, y) \) correspond to a quantum register \( |x, y\rangle \) with \( x, y \in [1, \ldots, \sqrt{N}] \).

Output: \( t, \text{success\_prob} \)

// \( t \), the number of lackadaisical quantum walk steps taken to find a vertex in the clustered marked states with \( \text{success\_prob} \), the success probability nearly 1.

// Define \( I \), the weight of each self-loop
\[ I \approx \frac{4}{N(k + 1)}; \]

// Define \( |\psi(0)\rangle \), the initial quantum state
\[ |\psi(0)\rangle = \frac{1}{\sqrt{N(4 + l)}} (\sum_{i=1}^{5} |i\rangle \otimes \sum_{x,y=1}^{\sqrt{N}} |x, y\rangle); \]

// Define \( |i\rangle \), the coin register with \( i \in \{ \leftarrow, \rightarrow, \uparrow, \downarrow, \} \), as the lackadaisical coin operator \( D \)
\[ D = 2|x\rangle\langle x| - I; \]

// Define \( U \), the walk operator
\[ U = S(D \otimes I); \]

// \( S \) is Flip-Flop Shift transformation:
\[ |\uparrow, i, j\rangle \rightarrow |\downarrow, i, j + 1\rangle, \]
\[ |\downarrow, i, j\rangle \rightarrow |\uparrow, i, j - 1\rangle, \]
\[ |\leftarrow, i, j\rangle \rightarrow |\rightarrow, i - 1, j\rangle, \]
\[ |\rightarrow, i, j\rangle \rightarrow |\leftarrow, i + 1, j\rangle, \]
\[ |., i, j\rangle \rightarrow |., i, j\rangle. \]

// Define the clustered marked vertices
\[ K(x, y) \rightarrow -K(x, y), \forall K(x, y); \]

// Initialize the probability and the number of steps
\( \forall K(x, y), \text{prob}(x, y)=0; \)
\( t=0, \text{success\_prob}=0; \)

while \( t \geq 0 \) do

// The evolution of lackadaisical quantum walks
\[ |\psi(t + 1)\rangle = U|\psi(t)\rangle; \]

if \( \forall K(x, y), \text{prob}(x, y) > |\langle \psi | x, y \rangle|^2 \) then

// first global peak is found
break;

else

// increment the number of steps
\( t = t + 1; \)

// Save the current success probability for the next step
\[ \forall K(x, y), \text{prob}(x, y) = |\langle \psi | x, y \rangle|^2; \]

// Calculate the probability of success in finding a vertex in the clustered marked region
\[ \text{success\_prob} = \sum_{|x, y\rangle \in K(x, y)} |\langle \psi | x, y \rangle|^2; \]

end

end

return \( t, \text{success\_prob} \)
Table 1  Success probability and number of steps for proposed LQW with $l \approx \frac{4}{N(k+1)}$ for $k = 9$ and different values of grid size $N$

| Grid size | Weight ($l$) | Success probability | Steps |
|-----------|--------------|---------------------|-------|
| 100       | 0.0044       | 0.874064            | 37    |
| 400       | 0.001        | 0.963963            | 84    |
| 900       | 0.000490     | 0.984277            | 128   |
| 1600      | 0.00025      | 0.986564            | 183   |
| 2500      | 0.00015      | 0.986460            | 229   |
| 3600      | 0.000111     | 0.992471            | 273   |
| 4900      | 0.000082     | 0.993541            | 319   |
| 6400      | 0.000060     | 0.991610            | 368   |
| 8100      | 0.000049     | 0.994785            | 413   |
| 10,000    | 0.000043     | 0.997134            | 471   |

Fig. 4  LQW with $l \approx \frac{4}{N(k+1)}$ for $k = 9$ and different values of grid size $N$: a success probability; and b number of steps of LQW

Table 2  Number of steps and success probability of the LQW algorithm with the Grover’s coin versus proposed coin, for $k = 9$ in different values of grid size $N$

| Grid size | Grover’s coin | Proposed coin |
|-----------|---------------|---------------|
|           | Success probability | Steps w/o amplification | Success probability | Steps |
| 100       | 0.129953       | 23            | 0.874064       | 37    |
| 400       | 0.611614       | 57            | 0.963963       | 84    |
| 900       | 0.601515       | 83            | 0.984277       | 128   |
| 1600      | 0.592153       | 122           | 0.986564       | 183   |
| 2500      | 0.584134       | 154           | 0.986460       | 229   |
| 3600      | 0.570086       | 186           | 0.992471       | 273   |
| 4900      | 0.569388       | 213           | 0.993541       | 319   |
| 6400      | 0.563390       | 255           | 0.991610       | 368   |
| 8100      | 0.557533       | 287           | 0.994785       | 413   |
| 10,000    | 0.553369       | 319           | 0.997134       | 471   |
Table 3  Success probability and number of steps taken by the proposed LQW for \( k = 9 \) and different values of grid size \( N \) with (i) \( l = \frac{4(k - \sqrt{k})}{N} \) [19] and (ii) the proposed \( l \approx \frac{4}{N(k + 1)} \)

| Grid size | Weight [19] | Success probability | Steps | Proposed weight | Success probability | Steps |
|-----------|-------------|---------------------|-------|-----------------|---------------------|-------|
| 100       | 0.24        | 0.203058            | 74    | 0.0044          | 0.874064            | 37    |
| 400       | 0.060000    | 0.025015            | 31    | 0.001           | 0.963963            | 84    |
| 900       | 0.026667    | 0.076135            | 130   | 0.000490        | 0.984277            | 128   |
| 1600      | 0.015       | 0.041122            | 313   | 0.00025         | 0.986564            | 183   |
| 2500      | 0.0096      | 0.053291            | 393   | 0.00015         | 0.986460            | 229   |
| 3600      | 0.006667    | 0.069063            | 267   | 0.000111        | 0.992471            | 273   |
| 4900      | 0.004898    | 0.002277            | 253   | 0.000082        | 0.993541            | 319   |
| 6400      | 0.00375     | 0.066896            | 787   | 0.000060        | 0.991610            | 368   |
| 8100      | 0.002963    | 0.052395            | 727   | 0.000049        | 0.994785            | 413   |
| 10,000    | 0.0024      | 0.000764            | 2149  | 0.000043        | 0.997134            | 471   |

and 5 that using the weight of the self-loop \( l \) as \( \approx \frac{4}{N(k + 1)} \) for \( k = 9, 25, 49 \) on the different grid sizes that have been considered, yields the best results. These numerical simulation results provide a strong evidence toward generalization with respect to higher grid sizes and higher values of \( k \). Albeit due to the memory constraint, the formal proof remains as a future scope of work. Grover’s coin-based quantum walk [20] or existing lackadaisical quantum walk [13, 19] approaches fail to find any of the marked states if \( k > 9 \) for this exceptional configuration. But, according to the trend shown in Table 5, although our algorithm can achieve a success probability \( \sim 1 \), the number of steps for marked states \( k > 25 \) grows significantly. We plan to explore the reason for such an increase beyond \( k > 25 \), and propose a suitable loop weight to reduce the number of steps.
Fig. 5  Comparative analysis of success probability of the proposed LQW for $k = 9$ and different values of grid size $N$ with (i) the proposed $l = \frac{4}{N(k + 1)}$ (blue), (ii) $l = \frac{4(k - \sqrt{k})}{N}$ [19] (red), and (iii) $l = \frac{4k}{N}$ [13] (grey) (Color figure online)

Table 5  Success probability and the number of steps taken by the proposed LQW with $l \approx \frac{4}{N(k + 1)}$ for $k = 25$ and $k = 49$ in different values of grid size $N$

| Marked-states size | Grid size | Weight ($l$) | Success probability | Steps |
|--------------------|-----------|--------------|---------------------|-------|
| $\sqrt{25} \times \sqrt{25}$ | 400       | 0.000400     | 0.946912            | 1135  |
|                     | 900       | 0.000166     | 0.894135            | 4529  |
|                     | 1600      | 0.000097     | 0.935753            | 6009  |
|                     | 2500      | 0.000062     | 0.991250            | 18,557 |
| $\sqrt{49} \times \sqrt{49}$ | 400       | 0.000204     | 0.896939            | 234,022 |

This proposed work has been simulated on Python 3.7, with processor Intel(R) Core(TM) i5-6300U CPU 2.40 GHz, 2.50 GHz, RAM 8.00 GB, and 64-bit windows operating system. For higher grid sizes with more marked states, higher configuration of the system with respect to both processor and memory are likely to be required.

5 Conclusion

In this paper, we have studied the application of quantum random walks on an $\sqrt{N} \times \sqrt{N}$ grid, where the marked states are arranged in a $\sqrt{k} \times \sqrt{k}$ cluster and $k = n^2$ for all values of $n \geq 1$. Our results are completely based on numerical simulations which show that using lackadasical quantum walks, where the weight of the self-loop is $l \approx \frac{4}{N(k + 1)}$ (correct up to 6 decimal places), the success probability to find a vertex in the marked region becomes close to 1 in time less than that of quantum walks with no self-loop for any odd value of $n \geq 1$, where the marked states are arranged in a $\sqrt{k} \times \sqrt{k}$ cluster and $k = n^2$. Furthermore, we have showed by simulation that the
weights of [13, 19] fail to find any of the mark states in this exceptional configuration. Lastly, we have showed that our proposed weight is also optimal for higher values of $k$, whereas Grover’s coin fails to find any of the marked states when $k > 9$.

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**Data availability**  Our manuscript has no associated data.

**References**

1. Aharonov, D., Ambainis, A., Kempe, J., Vazirani, U.: Quantum walks on graphs. In: Proceedings of the Thirty-Third Annual ACM Symposium on Theory of Computing, STOC ’01, pp. 50–59. Association for Computing Machinery, New York, NY, USA (2001). https://doi.org/10.1145/380752.380758
2. Aharonov, Y., Davidovich, L., Zagury, N.: Quantum random walks. Phys. Rev. A 48, 1687–1690 (1993). https://doi.org/10.1103/PhysRevA.48.1687
3. Ambainis, A.: Quantum walk algorithm for element distinctness. In: 45th Annual IEEE Symposium on Foundations of Computer Science, pp. 22–31 (2004). https://doi.org/10.1109/FOCS.2004.54
4. Ambainis, A.: Quantum walks and their algorithmic applications. Int. J. Quantum Inf. 1 (2004). https://doi.org/10.1142/S0219749903000383
5. Ambainis, A., Backurs, A., Nahimovs, N., Ozols, R., Rivosh, A.: Search by quantum walks on two-dimensional grid without amplitude amplification. In: TQC, vol. 7582 (2011). https://doi.org/10.1007/978-3-642-35656-8_7
6. Ambainis, A., Kempe, J., Rivosh, A.: Coins make quantum walks faster. In: Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA’05, pp. 1099–1108. Society for Industrial and Applied Mathematics, USA (2005)
7. Ambainis, A., Rivosh, A.: Quantum walks with multiple or moving marked locations. In: Geffert, V., Karhumäki, J., Bertoni, A., Preneel, B., Návrat, P., Bieliková, M. (eds.) SOFSEM 2008: Theory and Practice of Computer Science, pp. 485–496. Springer, Berlin (2008)
8. Benioff, P.: Space searches with a quantum robot (2000). https://doi.org/10.1090/comm/305/05212
9. de Carvalho, J.H.A., de Souza, L.S., Neto, F.M.d.P., Ferreira, T.A.E.: On applying the lackadaisical quantum walk algorithm to search for multiple solutions on grids (2021). https://doi.org/10.48550/ARXIV.2106.06274
10. Childs, A.M.: On the relationship between continuous-and discrete-time quantum walk. Commun. Math. Phys. 294(2), 581–603 (2009). https://doi.org/10.1007/s00220-009-0930-1
11. Childs, A.M., Cleve, R., Deotto, E., Farhi, E., Gutmann, S., Spielman, D.A.: Exponential algorithmic speedup by a quantum walk. In: Proceedings of the Thirty-Fifth ACM Symposium on Theory of Computing-STOC’03 (2003). https://doi.org/10.1145/780542.780552
12. Childs, A.M., Goldstone, J.: Spatial search and the Dirac equation. Phys. Rev. A 70(4), 042312 (2004). https://doi.org/10.1103/physreva.70.042312
13. Giri, P.R., Korepin, V.: Lackadaisical quantum walk for spatial search. Mod. Phys. Lett. A 35(08), 2050043 (2019). https://doi.org/10.1142/s0217732320500431
14. Grover, L.K.: A fast quantum mechanical algorithm for database search. In: Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing, STOC’96, pp. 212–219. Association for Computing Machinery, New York, NY, USA (1996). https://doi.org/10.1145/237814.237866
15. Hoyer, P., Yu, Z.: Analysis of lackadaisical quantum walks. Quantum Inf. Comput. 20, 1137–1152 (2020)
16. Kempe, J.: Quantum random walks: an introductory overview. Contemp. Phys. 44(4), 307–327 (2003). https://doi.org/10.1080/00107151031000110776
17. Kendon, V.M.: A random walk approach to quantum algorithms. Philos. Trans. R. Soc. A: Math. Phys. Eng. Sci. 364(1849), 3407–3422 (2006). https://doi.org/10.1098/rsta.2006.1901
18. Magniez, F., Santha, M., Szegedy, M.: Quantum algorithms for the triangle problem. SIAM J. Comput. 37(2), 413–424 (2007). https://doi.org/10.1137/050643684
19. Nahimovs, N.: Lackadaisical quantum walks with multiple marked vertices. In: SOFSEM (2019)
20. Nahimovs, N., Rivosh, A.: Exceptional configurations of quantum walks with Grover’s coin. In: Kofroň, J., Vojnar, T. (eds.) Mathematical and Engineering Methods in Computer Science, pp. 79–92. Springer International Publishing, Cham (2016)
21. Nahimovs, N., Rivosh, A.: Quantum walks on two-dimensional grids with multiple marked locations. In: Freivalds, R.M., Engels, G., Catania, B. (eds.) SOFSEM 2016: Theory and Practice of Computer Science, pp. 381–391. Springer, Berlin (2016)
22. Nahimovs, N., Santos, R.A.M.: Lackadaisical quantum walks on 2d grids with multiple marked vertices. J. Phys. A: Math. Theor. 54(41), 415301 (2021). https://doi.org/10.1088/1751-8121/ac21e3
23. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge University Press (2010). https://doi.org/10.1017/CBO9780511976667
24. Rhodes, M.L., Wong, T.G.: Search by lackadaisical quantum walks with nonhomogeneous weights. Phys. Rev. A 100(4), 042303 (2019). https://doi.org/10.1103/physreva.100.042303
25. Rhodes, M.L., Wong, T.G.: Search on vertex-transitive graphs by lackadaisical quantum walk. Quantum Inf. Process. 19, 1–16 (2020)
26. Saha, A., Majumdar, R., Saha, D., Chakrabarti, A., Sur-Kolay, S.: Search of clustered marked states with lackadaisical quantum walks (2018). arXiv:1804.01446
27. Shenvi, N., Kempe, J., Whaley, K.B.: Quantum random-walk search algorithm. Phys. Rev. A 67(5), 052307 (2003). https://doi.org/10.1103/physreva.67.052307
28. Tregenna, B., Flanagan, W., Maile, R., Kendon, V.: Controlling discrete quantum walks: coins and initial states. New J. Phys. 5, 83 (2003). https://doi.org/10.1088/1367-2630/5/1/383
29. Venegas-Andraca, S.E.: Quantum walks: a comprehensive review. Quantum Inf. Process. 11(5), 1015–1106 (2012). https://doi.org/10.1007/s11128-012-0432-5
30. Wang, H., Zhou, J., Wu, J., Yi, X.: Adjustable self-loop on discrete-time quantum walk and its application in spatial search (2017). arXiv:1707.00601
31. Wang, K., Wu, N., Xu, P., Song, F.: One-dimensional lackadaisical quantum walks. J. Phys. A: Math. Theor. 50(50), 505303 (2017). https://doi.org/10.1088/1751-8121/aa9235
32. Wong, T.G.: Grover search with lackadaisical quantum walks. J. Phys. A: Math. Theor. 48(43), 435304 (2015). https://doi.org/10.1088/1751-8113/48/43/435304
33. Wong, T.G.: Spatial search by continuous-time quantum walk with multiple marked vertices. Quantum Inf. Process. 15(4), 1411–1443 (2016). https://doi.org/10.1007/s11128-015-1239-y
34. Wong, T.G.: Faster search by lackadaisical quantum walk. Quantum Inf. Process. 17(3), 052307 (2018). https://doi.org/10.1007/s11128-018-1840-y

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