SUPEREXPONENTIAL ESTIMATES AND WEIGHTED LOWER BOUNDS FOR THE SQUARE FUNCTION

PAATA IVANISVILI AND SERGEI TREIL

ABSTRACT. We prove the following superexponential distribution inequality: for any integrable $g$ on $[0,1)^d$ with zero average, and any $\lambda > 0$

$$|\{x \in [0,1)^d : g \geq \lambda\}| \leq e^{-\lambda^2/(2^d \|S(g)\|_\infty^2)},$$

where $S(g)$ denotes the classical dyadic square function in $[0,1)^d$. The estimate is sharp when dimension $d$ tends to infinity in the sense that the constant $2^d$ in the denominator cannot be replaced by $C 2^d$ with $0 < C < 1$ independent of $d$ when $d \to \infty$.

For $d = 1$ this is a classical result of Chang–Wilson–Wolff [4]; however, in the case $d > 1$ they work with special square function $S_\infty$, and their result does not imply the estimates for the classical square function.

Using good $\lambda$ inequalities technique we then obtain unweighted and weighted $L^p$ lower bounds for $S$; to get the corresponding good $\lambda$ inequalities we need to modify the classical construction.

We also show how to obtain our superexponential distribution inequality (although with worse constants) from the weighted $L^2$ lower bounds for $S$, obtained in [5].

1. INTRODUCTION

1.1. Setup. On a probability space $(\mathcal{X}, \mathcal{F}, \sigma)$ consider a discrete time atomic filtration, i.e. a sequence of increasing sigma-algebras $\mathcal{F}_n \subset \mathcal{F}$, $n \geq 0$ such that for each $\mathcal{F}_n$ there exists a countable collection $\mathcal{D}_n$ of disjoint sets with the property that every set of $\mathcal{F}_n$ is a union of sets in $\mathcal{D}_n$. We assume that $\mathcal{F}_0 = \{\emptyset, \mathcal{X}\}$ and that the $\sigma$-algebra $\mathcal{F}$ is generated by $\mathcal{F}_n$.

A typical example one should have in mind is the standard dyadic filtration on the unit cube in $Q_0 = [0,1)^d$ in $\mathbb{R}^d$; the collection $\mathcal{D}_n$ in this case is the collection of the dyadic subcubes of $Q_0$ of size $2^{-n}$.

We denote $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$, and we will call the elements of $\mathcal{D}$ atoms or cubes (by analogy with the dyadic filtration). We will often use the notation $|A|$ for $\sigma(A)$ and $dx$ for $d\sigma(x)$.

For a cube $Q \in \mathcal{D}_n$ we denote by $\text{ch} Q$ the collection of its children,

$$\text{ch} Q = \{R \in \mathcal{D}_{n+1} : R \subset Q\}.$$

For $f \in L^1(\mathcal{X})$, and any measurable $A \subset \mathcal{X}$ with $|A| > 0$, we define the average

$$\langle f \rangle_A := \frac{1}{|A|} \int_A f;$$

if $|A| = 0$ we just set $\langle f \rangle_A = 0$.

We say that a filtration is $\alpha$-homogeneous, $0 < \alpha \leq 1/2$, if for any cube $Q$ and for any its child $Q'$

$$|Q'| \geq \alpha |Q|.$$
The standard dyadic filtration in $\mathbb{R}^d$ is clearly $\alpha$-homogeneous with $\alpha = 2^{-d}$.

As a final remark, we can always assume without loss of generality that our probability space $\mathcal{X}$ is the unit interval $[0, 1)$, which gives another justification for the notation for the measure.

1.2. Square functions. Define the expectation operators

$$ E_Q f(x) := \langle f \rangle_Q 1_Q(x), \quad E_n f := \sum_{Q \in D_n} E_Q f, $$

and the martingale difference operators

$$ \Delta_Q := \sum_{R \in \text{ch} Q} E_R - E_Q, \quad \Delta_n := E_{n+1} - E_n = \sum_{Q \in D_n} \Delta_Q. $$

Recall that the classical martingale square function $S$ is defined as

$$ S f := \left( \sum_{n=0}^{\infty} |\Delta_n f|^2 \right)^{1/2} = \left( \sum_{Q \in D} |\Delta_Q f|^2 \right)^{1/2}. $$

One can define other non-classical square functions $S_p$,

$$ S_p f := \left( \sum_{Q \in D} \langle |\Delta_Q f|^p \rangle_Q^{2/p} 1_Q \right)^{1/2} $$

$$ S_\infty f := \left( \sum_{Q \in D} \|\Delta_Q f\|_\infty^2 1_Q \right)^{1/2} $$

In the case of dyadic filtration in $\mathbb{R}$ all the above square functions coincide; for more complicated filtration they are different.

In the case of $\alpha$-homogeneous filtration the square functions $S_p$ are equivalent in the sense of two sided pointwise estimate

$$ S_\infty f(x) \leq C(\alpha, p) S_p f(x) \leq S_\infty f(x). $$

For general filtrations only trivial monotonicity relations hold,

$$ S_p f(x) \leq S_q f(x) \quad p \leq q. $$

1.3. Superexponential estimates of Chang–Wilson–Wolff. If $S_\infty f \in L^\infty$, then the distribution function of $f$ decays superexponentially, i.e. for all $\lambda \geq 0$

$$ \left| \{ x \in \mathcal{X} : f(x) - \langle f \rangle_{\mathcal{X}} \geq \lambda \} \right| \leq e^{-\lambda^2/2\|S_\infty f\|_\infty^2}. $$

This result was stated in [4, Theorem 3.1] for the classical dyadic filtration in $\mathbb{R}^d$ there, but the proof works for an arbitrary atomic filtration.

The superexponential bound (1.3) is the square function analog of the classical Gaussian concentration inequality. The proof of (1.3) presented in [4] is a slight adaptation of the classical Hoeffding’s inequality [7] – well known in probability (see also [1]). Essentially the argument relies on the Hoeffding’s lemma which says that if the random variable $X$ has mean zero and $X$ is bounded almost surely then

$$ \mathbb{E}e^X \leq e^{\|X\|_\infty^2/2}. $$

In case of Chang–Wilson–Wolff’s superexponential bound one uses Hoeffding’s inequality (1.4) with $X$ replaced by $t(f_{n+1} - f_n) 1_Q$ with $|Q| = 2^{-dn}$. 
The superexponential estimate (1.3) allows Wilson [11] (see the first lemma) to obtain weighted $L^p$ estimates for the square function $S\infty f$ in terms of the maximal function $f^*$, see (2.3) below for the definition. Namely for any $0 < p < \infty$ we have
\begin{equation}
\int |f^*|^p w dx \leq \frac{[w]^p_{L^p}}{p} \int (S\infty f)^p w dx.
\end{equation}

In this paper we prove for the $\alpha$-homogeneous filtration the inequalities (1.3) and (1.5) for the classical square function $S$ instead of $S\infty$. For general atomic filtrations (1.5) fails, there is a counterexample for $p = 2$ in [5].

1.4. An example. Let us present a function $f$ on a unit cube $Q_0 = [0,1) \times [0,1)$ such that $Sf \in L^\infty$ but $S\infty f$ is unbounded.

For an interval $I$ let $I_-$ and $I_+$ be its left and right halves respectively. Let $h_I$ denotes the $L^\infty$ normalized Haar function, $h_I = 1_{I_+} - 1_{I_-}$. For a dyadic square $Q = I^1 \times I^2$ define
\begin{align*}
f_Q(x) &:= 1_{I_+}(x_1) h_{I^2}(x_2), & x = (x_1, x_2) \in \mathbb{R}^2.
\end{align*}

Let $Q_k$, $k = 1, 2, \ldots$ be the collection of all squares of the form
\begin{align*}
Q = [0,2^{-k}) \times I,
\end{align*}
where $I$ runs over all dyadic subintervals of $[0,1)$ of length $2^{-k}$, and let $Q := \bigcup_{k \geq 0} Q_k$.

Define $f := \sum_{Q \in \mathcal{Q}} f_Q$. Clearly
\begin{align*}
\Delta_Q f &= \begin{cases} f_Q & Q \in \mathcal{Q}, \\ 0 & Q \notin \mathcal{Q}. \end{cases}
\end{align*}

It is easy to see that $Sf = 1_{Q_0}$, but $S\infty f$ is unbounded (goes to $\infty$ as $x_1 \to 0$) because the squares $Q \in \mathcal{Q}$ have unbounded overlap.

The example illustrates that in higher dimensions one may not expect the bound (1.5) but we will show that this is not the case.

2. Main results

For typographical reasons we will sometimes use the symbol $\mathbb{E} f$ for the average over the whole space $\mathcal{X}$, $\mathbb{E} f := (f)_{\mathcal{X}}$.

The main result of the paper is the following theorem, generalizing the Chang–Wilson–Wolff estimate (1.3) to the standard martingale square function (1.1) in the homogeneous filtration.

**Theorem 2.1.** For an $\alpha$-homogeneous filtration, any $f \in L^1$ with $S(f) \in L^\infty$ we have
\begin{equation}
\mathbb{E} \exp \left( f - S(f) - 2\alpha \right) \leq \exp (\mathbb{E} f).
\end{equation}

The theorem immediately gives the estimate

**Theorem 2.2.** Assume that for an $\alpha$-homogeneous filtration and $f \in L^1$ we have $\|S(f)\|_{L^\infty} < \infty$. Then
\begin{equation*}
\mathbb{E} \exp(f - \mathbb{E} f) \leq e^{\|S(f)\|_{L^\infty}^2/4\alpha}.
\end{equation*}

It follows from Markov’s inequality that
\begin{equation*}
|\{ x \in \mathcal{X} : f(x) - \mathbb{E} f > \lambda \}| = |\{ x \in \mathcal{X} : \exp(t(f(x) - \mathbb{E} f)) > \exp(t\lambda) \}| \leq e^{t^2\|S(f)\|_{L^\infty}^2/4\alpha - t\lambda}
\end{equation*}

Optimizing over all $t$ (the minimum is attained at $t = 2\alpha \lambda/\|Sf\|_{L^\infty}^2$) we obtain the following superexponential bound for the distribution function.
Theorem 2.3. Under assumptions of Theorem 2.2 we have for any \( \lambda \geq 0 \)
\[
\left| \{ x \in \mathcal{X} : f(x) - \mathbb{E}f > \lambda \} \right| \leq e^{-\alpha \lambda^2/\|Sf\|_\infty^2}.
\] (2.2)

Recall that the martingale maximal function \( f^* \) is defined as
\[
f^*(x) := \sup \{ |\mathbb{E}_n f(x)| : n \geq 0 \}.
\] (2.3)

As a corollary of Theorem 2.3 we get the following distribution inequality for \( f^* \), see Section 3.2 for the proof.

Theorem 2.4. Let for an \( \alpha \)-homogeneous filtration and a real-valued function \( f \in L^1 \) we have \( Sf \in L^\infty \) and \( \mathbb{E}f = 0 \). Then
\[
\left| \{ x \in \mathcal{X} : f^*(x) > \lambda \} \right| \leq 2e^{-\alpha \lambda^2/\|Sf\|_\infty^2}.
\] (2.4)

Using good \( \lambda \) inequalities we get from the above Theorem 2.4 the following lower bound for the martingale square function \( S \).

Recall that a weight (i.e. a non-negative integrable function) \( w \) is said to satisfy the martingale \( A_\infty \)-condition if
\[
\sup \left\{ \left( \mathbb{E} \left( w_{\mathcal{Q}} \right)^\ast \mathcal{Q} \right) / \langle w \rangle_\mathcal{Q} \right\} =: [w]_{A_\infty} < \infty;
\] (2.5)

the number \( [w]_{A_\infty} \) is called the \( A_\infty \) characteristic of the weight \( w \).

In this paper we will skip the word martingale, and just say “\( A_\infty \) weight”.

Theorem 2.5. For an \( \alpha \)-homogeneous filtration let \( w \) be an \( A_\infty \) weight. Then for any \( f \in L^1 \) and any \( p, 0 < p < \infty \)
\[
\|f^*\|_{L^p(w)} \leq C(\alpha, p)[w]_{A_\infty}^{1/2} \|Sf\|_{L^p(w)}, \quad C(\alpha, p) = 2^{1/p}2^{(p+4)/2} \alpha^{-3/2}(\ln(2/\alpha))^{1/2}.
\] (2.6)

3. PROOF OF SUPEREXPOSITIONAL ESTIMATES

3.1. Proof of Theorem 2.1. Notice that it is enough to prove Theorem 2.1 for \( f_N := \mathbb{E}_N f \) instead of \( f \) where \( N \) is an arbitrary nonnegative integer. Indeed, since \( \mathbb{E}_N f = \mathbb{E}f \) and
\[
-S(f)^2 \leq -S(f_N)^2,
\] estimate (2.1) for \( f_N \) implies that
\[
\mathbb{E} \exp(f_N - S(f)^2/4\alpha) \leq \exp(\mathbb{E}f)
\]
It follows that the sequence \( g_N := \exp(f_N - S(f)^2/4\alpha) \) is uniformly integrable, for example \( g_N \in L^2 \). Using uniform integrability of \( g_N \) and convergence in measure as \( N \to \infty \) we obtain
\[
\mathbb{E} \exp(f - S(f)^2/4\alpha) \leq \exp(\mathbb{E}f).
\]

In what follows we assume that \( f = f_N \). In this case we have
\[
f_N = f_0 + \sum_{k=0}^{N-1} (f_{k+1} - f_k);
\]
\[
S^2(f_N) = \sum_{k=0}^{N-1} (f_{k+1} - f_k)^2.
\]
We will set \( S(f_0) = 0 \).

Define
\[
U_\alpha(x, y) = e^{x - \frac{y^2}{4\alpha}} \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}_+.
\]
It is enough to show that

\[(3.1) \quad \mathbb{E}U_\alpha(f_N, S(f_N)) \leq \mathbb{E}U_\alpha(f_{N-1}, S(f_{N-1}))\]

Indeed, iterating the inequality (3.1) we will obtain

\[\mathbb{E}U_\alpha(f_N, S(f_N)) \leq \cdots \leq \mathbb{E}U_\alpha(f_0, S(f_0)) = \mathbb{E}U_\alpha(\mathbb{E}f, 0) = \exp(\mathbb{E}f)\]

which proves the theorem.

To prove (3.1) we use the identity \(\mathbb{E} \mathbb{E}_{N-1} = \mathbb{E}\), so we only need to show that

\[\mathbb{E}_{N-1}U_\alpha(f_N, S(f_N)) \leq U_\alpha(f_{N-1}, S(f_{N-1})).\]

The latter estimate simplifies as follows: for each \(Q \in D_{N-1}\) we have

\[\frac{1}{|Q|} \int_Q U_\alpha(f_N, S(f_N)) \leq U_\alpha(f_{N-1}(x), S(f_{N-1})(x)) \quad \text{for all } x \in Q.\]

Since

\[U_\alpha(f_N, S(f_N)) = U_\alpha \left( f_N - (f_N - f_{N-1}), \sqrt{S^2(f_N - f_{N-1})} + (f_N - f_{N-1})^2 \right)\]

we see that \(f_{N-1}\) and \(S(f_{N-1})\) are constant on \(Q\). The difference \(f_N - f_{N-1}\) takes values

\[c_j := \langle f \rangle_{Q_j} - \langle f \rangle_Q \text{ on each } Q_j \in \text{ch } Q, \quad j = 1, \ldots, \#(\text{ch } Q),\]

where \(\#(\text{ch } Q)\) denotes number of children in \(Q\). If we set \(p_j := \frac{|Q_j|}{|Q|} \geq \alpha\) then it follows that \(\sum p_j c_j = 0\).

Thus to prove the theorem we only need to show that for any family of points \(\{c_j\} \subset \mathbb{R}\), positive numbers \(\{p_j\} \subset \mathbb{R}_+\) with \(p_j \geq \alpha, \sum p_j = 1\), and \(\sum p_j c_j = 0\) we have

\[(3.2) \quad \sum_j p_j U_\alpha(x + c_j, \sqrt{y^2 + c_j^2}) \leq U_\alpha(x, y)\]

for all \((x, y) \in \mathbb{R} \times \mathbb{R}_+\).

Inequality (3.2) follows from the lemma below.

**Lemma 3.1.** For all \(p_j \geq \alpha \in (0, 1/2]\) with \(\sum_j p_j = 1\), and any \(c_j \in \mathbb{R}\) with \(\sum_j p_j c_j = 0\) we have

\[\sum_j p_j e^{c_j - \frac{c_j^2}{4\alpha}} \leq 1.\]

**Proof.** Consider \(f(c) := e^{c - \frac{c^2}{4\alpha}}\). Notice that

\[f''(c) = \frac{f(c)}{4\alpha^2} \left((c - 2\alpha)^2 - 2\alpha\right).\]

Therefore, \(f(c)\) is concave on \(I_\alpha := [2\alpha - \sqrt{2\alpha}, 2\alpha + \sqrt{2\alpha}]\), and convex on \(I_\alpha^\pm = (2\alpha + \sqrt{2\alpha}, \infty)\) and \(I_\alpha^- = (-\infty, 2\alpha - \sqrt{2\alpha})\). Consider our family of points \(\{c_j\} \subset \mathbb{R}\). Without loss of generality we can assume that if there are points from \(\{c_j\}\) which belong to \(I_\alpha\) then they are equal to each other. Indeed, suppose there are points \(c_1, c_2 \in I_\alpha\) such that \(c_1 \neq c_2\). Consider new pairs \(c_1^* = c_2^* = \frac{p_1 c_1 + p_2 c_2}{p_1 + p_2}\). Clearly \(c_1^*, c_2^* \in I_\alpha\), and by concavity we have

\[p_1 f(c_1) + p_2 f(c_2) \leq p_1 f(c_1^*) + p_2 f(c_2^*).\]

Since \(p_1 c_1 + p_2 c_2 = p_1 c_1^* + p_2 c_2^*\) we see that by replacing the points \(c_1, c_1\) with \(c_1^*, c_2^*\) we only increase the value of \(p_1 f(c_1) + p_2 f(c_2)\). Thus all points from \(I_\alpha\) should coincide. In other words we can assume that we have only one point in \(I_\alpha\) with big weight \(p_{m_1} + \ldots + p_{m_2}\).

Next, consider those points \(c_j\) which belong to \(I_\alpha^+\). We claim that if there are such points then we can assume that there is only one such point. Indeed, let \(c_1, c_2 \in I_\alpha^+\) be such that
\[ c_1 < c_2. \] Pick any number \( t > 0 \) such that \( c_1 - t \in I_\alpha^+ \). Consider \( c_1^* = c_1 - t \) and \( c_2^* = c_2 + tp_1/p_2 \). Clearly \( c_1^* < c_1 < c_2 < c_2^* \), and \( p_1c_1 + p_2c_2 = p_1c_1^* + p_2c_2^* \). It follows from convexity that
\[
p_1f(c_1) + p_2f(c_2) \leq p_1f(c_1^*) + p_2f(c_2^*).
\]

Thus moving points \( c_1, c_2 \) apart from each other we only increase the value \( p_1f(c_1) + p_2f(c_2) \). Therefore at some moment \( c_1 = 2\alpha + \sqrt{2\alpha} \in I_\alpha \).

In a similar way we can assume that if there are points in \( I_\alpha^- \) then there is only one such point.

Next, suppose we have a point \( c_1 \) which belongs to \( I_\alpha^- \) and \( c_m \in I_\alpha^+ \). We claim that by moving \( c_1 \) and \( c_m \) close to each other and keeping the quantity \( p_1c_1 + p_mc_m \) the same we will only increase the value \( p_1f(c_1) + p_mf(c_m) \). Indeed, the claim follows from the observation that the function \( f(c) \) is symmetric with respect to the point \( c = 2\alpha \), and \( f(c) \) is decreasing for \( c \geq 2\alpha \). Thus, by moving points \( c_1, c_m \) close to each other we will reach the position when one of the points \( c_1, c_m \) belong to \( I_\alpha \).

Finally to prove the lemma we need to consider only 3 cases: 1) when we have only one point \( c_1 \) which belongs to \( I_\alpha \); 2) When we have two points \( c_1, c_2 \) with \( c_1 \in I_\alpha \), and \( c_2 \in I_\alpha^+ \); 3) When we have \( c_1, c_2 \) with \( c_1 \in I_\alpha \) and \( c_2 \in I_\alpha^- \).

In the first case the condition \( \sum_j p_jf_j = 0 \) implies that \( c_1 = 0 \), and therefore the lemma is trivial.

In the second case since \( p_1c_1 + p_2c_2 = 0 \) we can assume that \( c_1 < 0 \) and \( c_2 > 2\alpha + \sqrt{2\alpha} \). Moving points \( c_1, c_2 \) close to each other and keeping the equality \( p_1c_1 + p_2c_2 = 0 \) we will only increase the value \( p_1f(c_1) + p_2f(c_2) \). Using this procedure we can reach the position when \( c_2 \in I_\alpha \) which reduces to the first case.

In the third case we need to prove that if \( p_1 + p_2 = 1, p_1, p_2 \geq 0, c_1p_1 + c_2p_2 = 0, p_1, p_2 \geq \alpha, \alpha \leq 1/2, c_2 \leq 2\alpha - \sqrt{2\alpha}, c_1 \in I_\alpha \) we have
\[
p_1e^{c_1} + p_2e^{c_2} \leq 1.
\]

We can also assume that \( p_2 = \alpha \). Indeed, if \( p_2 > \alpha \) then we can decompose \( p_2 = \alpha + p^+ \), and think that we have two points coinciding at \( c_2 \) with weights \( \alpha \) and \( p^+ \). Then repeating the previous discussion, i.e., moving them apart from each other, we will arrive to the conclusion that \( p_2 = \alpha \).

Without loss of generality we can assume that \( 2\alpha \geq c_1 \geq 0 \). Indeed, \( c_1 < 0 \) would contradict to the assumption \( p_1c_1 + p_2c_2 = 0 \). If \( c_1 \) were greater than \( 2\alpha \) then we would move points \( c_2, c_1 \) close to each other keeping the condition \( p_1c_1 + p_2c_2 = 0 \) and increasing the value \( p_1f(c_1) + p_2f(c_2) \) (we remind that \( f \) is decreasing on \((2\alpha, \infty)\) and increasing on \((-\infty, 2\alpha)\)). By making change of variables \( a_j := 1 - \frac{c_j}{2\alpha} \) we need to show that
\[
(1 - \alpha)e^{(1-a_1^2)\alpha} + \alpha e^{(1-a_2^2)\alpha} \leq 1
\]
provided that \( (1 - \alpha)a_1 + \alpha a_2 = 1, \) and \( 0 \leq a_1 \leq 1, \frac{1}{\sqrt{2\alpha}} \leq a_2 \) (the assumption \( a_1 \leq 1 \) comes from the fact that \( (1 - \alpha)a_1 + \alpha a_2 = 1 \)). In this case (3.3) follows from the following Proposition 3.2 where we set \( n := 1/\alpha \geq 2, a := a_1 \) and \( a_2 = n - (n - 1)a \).

Thus, the lemma, and so Theorem 2.1 are proved modulo Proposition 3.2 below. \qed

**Proposition 3.2.** For any \( n \geq 2 \) we have
\[
\max_{a \in [0, 1]} (n - 1)e^{\frac{1-a^2}{n}} + e^{\frac{1-a(1-a) + \alpha}{n}} \leq n.
\]
Proof. Consider

\[ f(a) := (n - 1)e^{\frac{1-a^2}{n}} + e^{\frac{1-(n(1-a)+a)^2}{n}} - n \] for \( a \in [0, 1] \).

Notice that

\[ f'(a) = 2(n - 1) \left( e^{\frac{1-(n(1-a)+a)^2}{n}}(n(1-a) + a) - ae^{\frac{1-a^2}{n}} \right). \]

Therefore \( f(1) = f'(1) = 0 \) and \( f''(1) = -\frac{2(n-1)(n-2)}{n} < 0 \). Also notice that

\[ f(0) = (n - 1)e^{1/n} + e^{(1-n^2)/n} - n < 0. \]

Indeed, we rewrite (3.4) as \( 1 - \frac{1}{n} + \frac{e^{-n}}{n} < e^{-1/n} \), and we use estimate \( e^{-1/n} \geq 1 - 1/n + 1/2n^2 - 1/6n^3 \). Then our claimed inequality would follow from \( (6e^{-n^2} - 3n + 1)/(6n^3) < 0 \) which is true because \( 6e^{-n^2} \leq 24e^{-2} < 4 \leq 3n - 1 \) for \( n \geq 2 \) (notice that \( e^2 > 6 \)).

Thus the only interesting case is what happens with \( f(a_0) \) at critical points \( a_0 \), i.e., \( f'(a_0) = 0 \). Consider \( f'(a_0) = 0 \). The equation can be simplified as follows

\[ e^{\frac{1-(n(1-a_0)+a_0)^2}{n}} = \frac{a_0}{n(1-a_0) + a_0} e^{\frac{1-a_0^2}{n}}. \]

Substituting into the expression for \( f(a) \) we see that it would be sufficient to show

\[ (n - 1)e^{\frac{1-a_0^2}{n}} + \frac{a_0}{n(1-a_0) + a_0} e^{\frac{1-a_0^2}{n}} - n \leq 0. \]

Simplifying further it is enough to show the following

\[ 1 - \frac{1-a_0}{a_0 + n(1-a_0)} \leq e^{\frac{a_0^2 - 1}{n}}. \]

We will show that this inequality holds in fact for all \( a_0 \in [0, 1] \). We estimate the right hand side by \( e^{-x} \geq 1 - x + x^2/2 - x^3/6 \). Therefore it would suffice to show that

\[ 0 \leq \frac{1-a_0}{a_0 + n(1-a_0)} + \frac{a_0^2 - 1}{n} + \frac{(a_0^2 - 1)^2}{2n^2} + \frac{(a_0^2 - 1)^3}{6n^3}. \]

If we denote \( w = 1 - a^2 \in [0, 1] \) the latter inequality simplifies as follows

\[
\begin{align*}
-\frac{w^2}{6n^3} + \frac{w}{2n^2} - \frac{1}{n} + \frac{1}{nw + 1 - w + \sqrt{1-w}} & \\
& \geq -\frac{w^2}{6n^3} + \frac{w}{2n^2} - \frac{1}{n} + \frac{1}{nw + 1 - w + 1 - w/2} \\
& = 12(1-w)n^3 + 6(w-1)(w+4)n^2 + (-2w^3 - 9w^2 + 12w)n + 3w^3 - 4w^2 \\
& = 6n^3(w(2n-3) + 4)
\end{align*}
\]

The second derivative in \( n \) of the numerator of the last expression is \( 12(1-w)(6n-4-w) \) which is positive for \( n \geq 2 \). So the numerator is convex in \( n \), and we will estimate from below by its tangent line at point \( n = 2 \). The tangent line has the expression

\[ w^2(2-w) + (-2w^3 + 15w^2 - 60w + 48)(n-2) \]

The first term is positive since \( w \in [0, 1] \). The coefficient in front of \( (n-2) \) is decreasing in \( w \), and attains its minimal value when \( w = 1 \) which is positive. \( \square \)
3.2. **Proof of Theorem 2.4.** Proof of this theorem is well-known and simple: we present it here only for the readers’ convenience.

Given $\lambda > 0$ let $\mathcal{R} = \mathcal{R}\lambda \subset \mathcal{D}$ be the collection of maximal cubes such that

$$|\langle f \rangle_Q| > \lambda.$$ 

Define function $\tilde{f}$ by replacing $f$ on cubes $R \in \mathcal{R}$ by $\langle f \rangle_R$, i.e.

$$\tilde{f} = f - \sum_{R \in \mathcal{R}} (f - \langle f \rangle_R) 1_R.$$ 

It is easy to see that

$$\{x \in \mathcal{X} : f^*(x) > \lambda\} = \{x \in \mathcal{X} : |\tilde{f}(x)| > \lambda\} = \bigcup_{R \in \mathcal{R}} R,$$

and that $S\tilde{f}(x) \leq Sf(x) \text{ a.e.}$ Applying Theorem 2.3 to $f$ and $-f$ we get (2.4). \hfill \Box

4. **Weighted estimates**

4.1. **Unweighted and weighted good $\lambda$ inequalities.** The following lemma was proved in [4, Corollary 3.1] (with $1/2$ instead of $\alpha^3/(1 - 2\alpha + 2\alpha^2)$) for the square function $S_{\infty}$. The proof for the classical square function $S$ requires some modifications, so for the reader’s convenience we present it here.

**Lemma 4.1.** For an $\alpha$-homogeneous filtration and $f \in L^1$, and $\lambda > 0$, let $Q = Q(\lambda)$ be the collection of maximal cubes $Q \in \mathcal{D}$ such that $|\langle f \rangle_Q| > \lambda$.

Then for each $Q \in \mathcal{Q}$ we have the following good $\lambda$ inequality

$$|\{x \in Q : f^*(x) > 2\lambda, Sf(x) \leq \epsilon \lambda\}|$$

$$\leq 2 \exp \left( -\frac{\alpha^3(1 - \epsilon)^2}{(1 - 2\alpha + 2\alpha^2)\epsilon^2} \right) |Q|$$

**Proof.** Consider the family $Q$ of maximal cubes $Q \in \mathcal{D}$ such that

$$|\langle f \rangle_Q| > \lambda.$$ 

For each $Q \in \mathcal{Q}$ consider the set

$$E_Q := \{x \in Q : Sf(x) \leq \epsilon \lambda, f^*(x) > 2\lambda\}.$$ 

Note that if $E_Q \neq \emptyset$, then $|\langle f \rangle_Q| \leq (1 + \epsilon)\lambda$: indeed, if $\tilde{Q}$ is the parent of $Q$, then by the construction $|\langle f \rangle_{\tilde{Q}}| \leq \lambda$ and for $x \in E_Q$

$$|\langle f \rangle_{\tilde{Q}} - \langle f \rangle_Q| \leq Sf(x) \leq \epsilon \lambda.$$ 

The inequality $|\langle f \rangle_Q| \leq (1 + \epsilon)\lambda$ implies that if we define $f_1 := (f - \langle f \rangle_Q) 1_Q$,

$$E_Q \subset E^1_Q := \{x \in Q : f_1^*(x) > (1 - \epsilon)\lambda, Sf_1(x) \leq \epsilon \lambda\}.$$ 

Indeed, the inequality $1_Q(x)Sf(x) \geq Sf_1(x)$ is trivial. Next, if $f^*(x) > 2\lambda$ for $x \in Q$ then it follows that there exists $\tilde{Q} \subset Q, \tilde{Q} \in \mathcal{D}$ with $x \in \tilde{Q}$ such that $|\langle f \rangle_{\tilde{Q}}| > 2\lambda$, and therefore

$$f_1^*(x) \geq |\langle f \rangle_{\tilde{Q}} - \langle f \rangle_Q| \geq |\langle f \rangle_{\tilde{Q}}| - |\langle f \rangle_Q| > (1 - \epsilon)\lambda.$$
Let $\mathcal{R}$ be the collection of maximal cubes $R \subset Q$ such that $Sf_1(x) \geq \varepsilon \lambda$ everywhere on $R$. The set $E_Q$ does not intersect cubes $R \in \mathcal{R}$, so if we define $f_2$ as

$$(4.2) \quad f_2 := \sum_{K \in \mathcal{D}(Q) \setminus \bigcup_{R \in \mathcal{R}} \mathcal{D}(R)} \Delta_K f,$$

where $\mathcal{D}(Q) := \{ K \in \mathcal{D} : K \subset Q \}$, then

$$E_Q^1 = E_Q^2 := \{ x \in Q : f_2^2(x) > (1 - \varepsilon)\lambda, Sf_2(x) \leq \varepsilon \lambda \}.$$

Up to this moment the proof was exactly as the proof of [4, Corollary 3.1]. If one uses the square function $S_{\infty}$ (or $S_p$) instead of $S$, on can conclude, that because the term $\|\Delta_K f\|_\infty^2 1_K$ is constant on $K$, then by the construction $S_{\infty}f_2(x) \leq \varepsilon \lambda$ on $Q$, and then use the corresponding superexponential estimates [4, Theorem 3.1] (Theorem 2.4 in this paper).

But for our square function we have no control on how big $Sf_2$ is on cubes $R \in \mathcal{R}$! Therefore, to apply Theorem 2.4 we need first to modify the function $f_2$.

Let $R \in \mathcal{R}$ and let $\hat{R}$ be its parent. If we are lucky, and $\hat{R}$ has only one child $R$ that belongs to $\mathcal{R}$, we can estimate $|\Delta_{\hat{R}} f|$ on $R$. Namely, for $x \in \hat{R} \setminus R$

$$|\Delta_{\hat{R}} f(x)| \leq Sf_2(x) \leq \varepsilon \lambda,$$

and since $\Delta_{\hat{R}} f$ has zero average we can estimate that for $x \in R$

$$|\Delta_{\hat{R}} f| \leq \frac{1 - \alpha}{\alpha} \varepsilon \lambda.$$

But if we are not so lucky, and $\hat{R}$ has more than one child that belongs to $\mathcal{R}$, then we cannot estimate the value of $\Delta_{\hat{R}} f$ on such children. If $R_k$ are all the children of $\hat{R}$ that belong to $\mathcal{R}$, we can definitely estimate the average,

$$\left| \langle \Delta_{\hat{R}} f \rangle \cup_{k \in \mathcal{R}_k} \right| \leq \frac{1 - \alpha}{\alpha} \varepsilon \lambda,$$

but because of possible cancellations, the values on $R_k$ can be arbitrarily large.

So let us change the martingale difference $\Delta_{\hat{R}} f = \Delta_{\hat{R}} f_1 = \Delta_{\hat{R}} f_2$ by replacing its values on the cubes $R_k$ by the average $\langle \Delta_{\hat{R}} f \rangle \cup_{k \in \mathcal{R}_k}$. The resulting function will still have zero average, so it is also a martingale difference, let us call it $f_{\hat{R}}$.

So, if we define the function $f_3$ by replacing in (4.2) the martingale differences $\Delta_{\hat{R}} f$ by $f_{\hat{R}}$ for the parents $\hat{R}$ of cubes $R \in \mathcal{R}$, then outside of cubes $R \in \mathcal{R}$ we have $f_2 = f_3$ and $Sf_2 = Sf_3$; note also that $f_{\hat{R}} = \Delta_{\hat{R}} f_3$. Therefore,

$$E_Q^2 \subset E_Q^3 := \{ x \in Q : f_3^3(x) > (1 - \varepsilon)\lambda, Sf_3(x) \leq \varepsilon \lambda \}.$$

But $Sf_3$ can be estimated,

$$\|Sf_3(x)\|_\infty^2 \leq (\varepsilon \lambda)^2 + (\varepsilon \lambda)^2 \frac{(1 - \alpha)^2}{\alpha^2} = (\varepsilon \lambda)^2 \frac{1 - 2\alpha + 2\alpha^2}{\alpha^2},$$

and we can apply the superexponential distributional estimates from Theorem 2.4 to get that

$$|E_Q| \leq |E_Q^3| \leq 2 \exp \left( - \frac{\alpha^3 (1 - \varepsilon)^2 \lambda^2}{(1 - 2\alpha + 2\alpha^2) \varepsilon^2 \lambda^2} \right) |Q| \quad \Box$$
Suppose that for an \( \alpha \)-homogeneous filtration and \( w \in L^1 \) we will denote

\[
w(E) := \int_E w(x) \, dx
\]

**Lemma 4.2.** Suppose that for an \( \alpha \)-homogeneous filtration and \( w \in L^1 \), \( w \geq 0 \) we have for a cube \( Q_0 \in \mathcal{D} \)

\[
\langle w^* \rangle_{Q_0} \leq A(w)_{Q_0}.
\]

Let \( E \subset Q_0 \) be a union of some cubes in \( \mathcal{D}(Q_0) \). Then

\[
w(E) \leq 2A \frac{\ln(2/\alpha)}{\ln(|Q_0|/|E|)} w(Q_0)
\]

**Proof.** Define \( E_0 := E \) and

\[
E_k := \{ x \in \mathcal{X} : (1_E)^*(x) > (\alpha/2)^k \}, \quad k = 1, 2, \ldots, \lfloor \log_{2/\alpha}(|Q_0|/|E|) \rfloor := N_\alpha.
\]

Let \( E_k \) be the collection of maximal cubes \( Q \subset E_k \). It is easy to see that for any \( Q \in \mathcal{E}_k \),

\[
\sum_{R \in \mathcal{E}_{k-1}, R \subset Q} |R| \leq \frac{1}{2} |Q|.
\]

Let \( f := w 1_{E_k} \). Since for every cube \( Q \in \mathcal{E}_k \)

\[
|f^*(x)| \geq |Q|^{-1} \sum_{K \in \mathcal{E}_k, K \subset Q} w(K), \quad \forall x \in Q,
\]

we conclude summing over all \( Q \in \mathcal{E}_k \) and using (4.4) that

\[
\int_{E_k \setminus E_{k-1}} f^*(x) \, dx \geq \frac{1}{2} w(E),
\]

and clearly \( \int_E f^*(x) \, dx \geq w(E) \). Therefore

\[
(1 + N_\alpha/2) w(E) \leq \int_{Q_0} f^* \, dx \leq \int_{Q_0} w^* \, dx \leq A \int_{Q_0} w \, dx = Aw(Q_0).
\]

Noticing that

\[
(1 + N_\alpha/2) \geq (1 + N_\alpha)/2 \geq \frac{1}{2} \log_{2/\alpha}(|Q_0|/|E|) = \frac{\ln(|Q_0|/|E|)}{2 \ln(2/\alpha)}
\]

concludes the proof. \( \square \)

**4.2. Proof of Theorem 2.5.** The standard good \( \lambda \) inequalities reasoning implies that if

\[
w(\{ x \in \mathcal{X} : f^*(x) > 2\lambda, S f(x) \leq \varepsilon \lambda \}) \leq 2^{-(p+1)} w(\{ x \in \mathcal{X} : f^* > \lambda \}),
\]

then

\[
\| f^* \|_{L^p(w)} \leq \varepsilon^{-1/2} 2^{(p+1)/p} \| S f \|_{L^p(w)}
\]

Recall that \( Q \) is the collection of maximal cubes \( Q \in \mathcal{D} \) such that \( |\langle f \rangle_Q| > \lambda \). For \( Q \in \mathcal{Q} \) denote

\[
E_Q := \{ x \in Q : f^*(x) > 2\lambda, S f(x) \leq \varepsilon \lambda \}.
\]

To prove (4.5) it is sufficient to show that for all \( Q \in \mathcal{Q} \)

\[
w(\{ x \in Q : f^*(x) > 2\lambda, S f(x) \leq \varepsilon \lambda \}) \leq 2^{-(p+1)} w(Q);
\]
taking the sum over all $Q \in \mathcal{Q}$ give us (4.5).

By Lemma 4.2 the above inequality (4.6) holds if

$$2^{-(p+1)} \geq 2A \frac{\ln(2/\alpha)}{\ln(|Q|/|E_Q|)},$$

and estimating $|Q|/|E_Q|$ using Lemma 4.1 we see that (4.7) holds if

$$A 2^{p+2} \ln(2/\alpha) \leq \alpha^3 (1 - \varepsilon)^2 \left(1 - 2\alpha + 2\alpha^2\right)\varepsilon^2 - \ln 2.$$

If we put

$$\varepsilon^{-1} = A^{1/2} 2^{(p+3)/2} \alpha^{-3/2} (\ln(2/\alpha))^{1/2},$$

the inequality (4.8) is satisfied, and the theorem is proved. □

4.3. Unweighted $L^p$ estimates for the square function. The standard technique of good $\lambda$ inequalities also allows one to get the unweighted $L^p$ estimates for the square function. Note that the estimate behaves as $C p^{1/2}$ as $p \to \infty$

**Theorem 4.3.** For the classical square function $S$ and any $p \in (0, \infty)$

$$\|f^*\|_{L^p} \leq 4 \cdot 2^{1/p} \alpha^{-3/2} (p + 2)^{1/2} \|Sf\|_{L^p}.$$

**Proof.** The standard good $\lambda$ inequalities reasoning implies that if

$$|\{x \in X : f^*(x) > 2\lambda, Sf(x) \leq \varepsilon \lambda\}| \leq 2^{-(p+1)} |\{x \in X : f^* > \lambda\}|,$$

then

$$\|f^*\|_{L^p} \leq \varepsilon^{-1} 2^{(p+1)/p} \|Sf\|_{L^p}.$$

Lemma 4.1 shows that (4.10) holds if $\varepsilon$ satisfies

$$\frac{\alpha^3 (1 - \varepsilon)^2}{(1 - 2\alpha + 2\alpha^2)\varepsilon^2} \geq p + 2.$$

Since $\varepsilon^{-1} = 2\alpha^{-3/2} (p + 2)^{1/2}$ satisfies the above inequality, (4.11) immediately gives (4.9). □

4.4. Comparison with the estimates from [5]. It was proved in [5, Theorem 3.4] that for the $n$-adic filtration, i.e. for a filtration where each atom $Q$ has exactly $n$ children of equal measure, the following estimate holds for all $f$ with $\langle f \rangle_X = 0$:

$$\|f\|_{L^2(w)} \leq n[w]^{1/2} A_{\infty}^{\text{sem}} \|Sf\|_{L^2(w)}.$$

Here $X = [0, 1)$ (as we discussed it above in the introduction, we can always assume that without loss of generality), and $A_{\infty}^{\text{sem}}$ means the *semiclassical* $A_{\infty}$-condition,

$$\sup_{Q \in D} \langle M_Q w \rangle_Q / \langle w \rangle_Q =: [w]_{A_{\infty}^{\text{sem}}} < \infty,$$

where $M_Q$ is the localized classical Hardy–Littlewood maximal function, one the real line

$$M_Q f(x) = \sup \{\|f\|_I : I \subset Q, I \ni x\},$$

and the supremum is taken over all intervals $I \subset Q$. We assumed here, that our probability space is the unit interval $[0, 1)$, which as it was discussed above we can do without loss of generality.
It can be seen from the proof of [5, Theorem 3.4] that for an \( \alpha \)-homogeneous filtration

\[
\|f\|_{L^2(w)} \lesssim \alpha^{-1} [w]_{A^\infty}^{1/2} \|Sf\|_{L^2(w)},
\]

for any \( f \) with \( \langle f \rangle_X = 0 \).

It was shown in [5, Theorem 3.3] that for a general non-homogeneous filtration the \( A^\infty \) condition on the weight \( w \) is not sufficient for the estimate

\[
\|f\|_{L^2(w)} \leq C \|Sf\|_{L^2(w)}.
\]

If one considers the (stronger) classical \( A^\infty \) condition on the real line; in the classical \( A^\infty \) condition on the line the supremum in (4.12) is taken over all intervals \( Q \), not only \( Q \in D \).

Therefore, a dependence on \( \alpha \) that blows out when \( \alpha \to 0 \) must be present in the estimates (4.14), (2.6).

It can be shown that

\[
[w]_{A^\infty} \lesssim \alpha^{-3/2} [w]_{A^\infty}; \tag{4.15}
\]

the proof is essentially given in the proof of [2, Chapter 3, Theorem 3.8]; an obvious adaptation gives a proof of (4.15).

Therefore, the estimate (4.13) implies the following estimate in terms of the martingale \( A^\infty \) characteristic \( [w]_{A^\infty} \):

\[
\|f\|_{L^2(w)} \lesssim \alpha^{-3/2} [w]_{A^\infty}^{1/2} \|Sf\|_{L^2(w)}. \tag{4.16}
\]

Note that this estimate has the same exponent at \( \alpha \) as the estimate (2.6); the estimate (2.6) has an extra factor \( (\ln(2/\alpha))^{1/2} \), but it deals with a bigger function \( f^\ast \) instead of \( f \). So, we can say that the asymptotic in \( \alpha \) in (4.14) and (4.16) are essentially equivalent.

But we do not know how to get using the superexponential estimate (2.3) a better exponent at \( \alpha \) in (2.6) by replacing \( [w]_{A^\infty} \) by the bigger \( A^\infty \) characteristic \( [w]_{A^\infty} \).

5. From weighted estimates to the superexponential distribution inequalities.

A standard and well-known to experts reasoning allows one to obtain from the proved in [5] estimate

\[
\|f\|_{L^2(w)} \leq C [w]_{A^\infty}^{1/2} \|f\|_{L^2(w)}, \tag{4.14}
\]

above, the superexponential distribution inequalities (2.4) (with appropriate parameters).

For the reader’s convenience we summarize the known results here.

5.1. Preliminary lemmas. Recall that a weight \( w \) on the real line (or its subinterval) satisfies the (classical) \( A_1 \) condition if

\[
\sup_{x \in \mathbb{R}} \frac{Mw(x)}{w(x)} := [w]_{A_1} < \infty; \tag{5.1}
\]

here \( M \) is the classical Hardy–Littlewood maximal function (4.13).

It is well-known and trivial to see that the \( A_1 \) condition is stronger than \( A^\infty \) condition considered in this paper and that the \( A_1 \) characteristic \( [w]_{A_1} \) majorates all the \( A^\infty \) characteristics.

Recall that the norm of the Hardy–Littlewood maximal operator \( M \) in \( L^p(\mathbb{R}) \), \( 1 < p \leq \infty \), is estimated by \( C_M p^\prime \), where \( C_M \) is an absolute constant and \( p^\prime \) is the dual Hölder exponent, \( 1/p + 1/p^\prime = 1 \).
**Lemma 5.1.** Let measurable functions $f$ and $g$ on $\mathbb{R}$ (or on an interval $I \subset \mathbb{R}$) be such that for all $A_1$ weights

\begin{equation}
\|f\|_{L^2(w)} \leq B[w]_{A_1}^{\kappa} \|g\|_{L^2(w)},
\end{equation}

for some $B < \infty$, $\kappa > 0$.

Then for all $p \geq 2$ we have in the non-weighted $L^p$

\begin{equation}
\|f\|_{L^p} \leq \sqrt{2} C_M B p^{\kappa} \|g\|_{L^p}.
\end{equation}

**Proof.** The proof is the standard Rubio de Francia extrapolation argument, as it is presented on pp. 356–357 of [6]. For $p \geq 2$ and $r = p/2$

\[ \|f\|_{L^r}^2 = \|f^2\|_{L^r} = \sup \left\{ \int_X |f|^2 \phi : \phi \geq 0, \|\phi\|_{L^r} = 1 \right\}. \]

Take $\phi \in L^r$, $\|\phi\|_{L^r} = 1$. Let $M^1 \phi = M \phi$, $M^2 \phi = M(M \phi)$, $M^{n+1} \phi = M(M^n \phi)$. Define the weight $w$ by

\[ w := \phi + \sum_{n=1}^{\infty} (C_M p)^{-n} M^n \phi. \]

Since $\|M^1 \phi\|_{L^r} \leq C_M r = C_M p/2$, we can conclude that

\[ (C_M p)^{-1} \|M^1 \phi\|_{L^r} \leq 2^{-1} \|\phi\|_{L^r}, \]

\[ (C_M p)^{-n-1} \|M^{n+1} \phi\|_{L^r} \leq 2^{-1} (C_M p)^{-n} \|M^n \phi\|_{L^r} = 2^{-(n+1)} \|\phi\|_{L^r}, \quad n \geq 1, \]

so

\begin{equation}
\|w\|_{L^r} \leq \|\phi\|_{L^r} \sum_{k=0}^{\infty} 2^{-n} = 2 \|\phi\|_{L^r},
\end{equation}

and therefore $w$ is finite a.e.

It is easy to see that $Mw \leq C_M pw$, so $w$ is an $A_1$ weight with $[w]_{A_1} \leq C_M p$ (we need the a.e. boundedness of $w$ to avoid the case $w \equiv +\infty$). Continuing with the argument, we get

\[ \int_X |f|^2 \varphi dx \leq \int_X |f|^2 w dx \leq B[w]_{A_1}^{2\kappa} \|g\|^2_{L^2(w)} \leq B^2 (C_M p)^{2\kappa} \|g\|^2_{L^2(w)}. \]

Estimating

\[ \|g\|^2_{L^2(w)} = \int_X |g|^2 w dx \leq \|g\|^2_{L^r} \|w\|_{L^r} = \|g\|^2_{L^p} \|w\|_{L^r} \leq 2 \|g\|^2_{L^p}, \]

the last inequality holds because of (5.4)) we get that

\[ \int_X |f|^2 \varphi dx \leq 2B^2 (C_M p)^{2\kappa} \|g\|^2_{L^p}, \]

and taking supremum over all $\phi \geq 0$, $\|\phi\|_{L^r} = 1$ we get the conclusion of the lemma. \qed

**Lemma 5.2.** Let $\kappa > 0$ and let $f \in L^1(\mathcal{X})$, $g \in L^\infty(\mathcal{X})$, $|\mathcal{X}| = 1$ be such that for all $p \geq 1/\kappa$

\begin{equation}
\|f\|_{L^p} \leq B p^{\kappa} \|g\|_{L^\infty}.
\end{equation}

Then for $\gamma = (2e)^{-1} \kappa B^{-1/\kappa} \|g\|_{L^\infty}^{-1/\kappa}$

\begin{equation}
\int e^{\gamma |f|^{1/\kappa}} \leq C < \infty
\end{equation}
where

\begin{equation}
C = 1 + \sum_{n=1}^{\infty} \frac{(n/(2e))^n}{n!} < \infty
\end{equation}

Remark. The series in (5.7) trivially converges by the ratio test.

Proof of Lemma 5.2. Representing \(e^{\gamma |f|^{1/\kappa}}\) as a power series and using (5.5) we get

\[
\int e^{\gamma |f|^{1/\kappa}} dx = 1 + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \|f\|^{n/\kappa}_{L^{n/\kappa}} \leq 1 + \sum_{n=1}^{\infty} \frac{\gamma^n B^{n/\kappa} \left( \frac{N}{\kappa} \right)^n}{n!} \|g\|^{n/\kappa}_{L^{\infty}}
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{(n/(2e))^n}{n!} = C < \infty;
\]

as it was mentioned in the remark above, the convergence of the series follows easily from the ratio test. \(\square\)

Remark 5.3. As one can see from the proof of Lemma 5.2 it is sufficient to assume that (5.5) holds only for all sufficiently large \(p\). Indeed, suppose the estimate \(\|f\|^{\ell/\kappa}_{L^{\ell/\kappa}} \leq B^{\ell/\kappa} \|g\|^{\ell/\kappa}_{L^{\infty}}\) holds only for \(n \geq N\). But by monotonicity for \(1 \leq \ell \leq N\) we have

\[
\|f\|^{\ell/\kappa}_{L^{\ell/\kappa}} \leq 1 + \|f\|^{N/\kappa}_{L^{N/\kappa}} \leq 1 + B^{N/\kappa} \|g\|^{N/\kappa}_{L^{\infty}},
\]

so one can estimate

\[
\int e^{\gamma |f|^{1/\kappa}} dx \leq e^{\gamma} + N \left( \frac{N/(2e))^N}{N!} + \sum_{n=N+1}^{\infty} \frac{(n/(2e))^n}{n!} \right) =: C_1 < \infty.
\]

5.2. Obtaining superexponential estimates from (4.14). Taking \(\kappa = 1/2\) and applying lemma 5.1 to the estimate (4.14) we get that for \(f \in L^1(\mathcal{X})\) with \(\langle f \rangle_{\mathcal{X}} = 0\) and \(p \geq 2\)

\begin{equation}
\|f^*\|_{L^p} \lesssim \alpha^{-1} p^{1/2} \|Sf\|_{L^p},
\end{equation}

Lemma 5.2 then gives us

\begin{equation}
\int_{\mathcal{X}} e^{\alpha x^2 |f|^2/\|Sf\|_{L^\infty}^2} dx \leq C < \infty.
\end{equation}

This inequality in turn implies the superexponential distribution inequality

\begin{equation}
|\{x \in \mathcal{X} : |f(x)| > \lambda\}| \leq Ce^{-\alpha x^2 \lambda^2/\|Sf\|_{L^\infty}^2}.
\end{equation}

Remark 5.4. Estimate (5.8) gives us better dependence on the homogeneity parameter \(\alpha\) then the estimate (4.9) that we got from the superexponential estimate (2.4); the drawback of (5.8) is that it holds only for \(p \geq 2\), versus \(0 < p < \infty\) in (4.9).

However, the superexponential estimate (5.10) we got from there gives us worse dependence on \(\alpha\) then the estimates (2.2), (2.4).
5.3. A discussion. It is a remarkable result of Wang [9] that for any $3 \leq p < \infty$ and any conditionally symmetric martingale $f$ we have the sharp bound

$$\|f\|_p \leq R_p\|Sf\|_p, \quad \mathbb{E}f = 0,$$  \hspace{1cm} (5.11)

where $R_p$ is the rightmost zero of the Hermite function $H_p(x)$. Here $H_p(x)$ is the solution of the Hermite differential equation

$$H''_p - xH'_p + pH_p = 0$$

which grows relatively slowly at infinity, i.e., $H_p(x) = x^p + o(x^p)$ as $x \to \infty$. It is known that $R_p = O(\sqrt{p})$ as $p \to \infty$. In our context “conditionally symmetric” martingale means that for any dyadic cube $Q$ the martingale differences $\Delta_Q f$ and $-\Delta_Q f$ have the same distribution function.

Clearly that for the standard dyadic filtration on $[0,1)^d$ not all martingales are conditionally symmetric. In fact, for such a filtration any $f \in L^1$ defines a conditionally symmetric martingale if and only if $d = 1$.

Thus we obtain that if $f$ is conditionally symmetric martingale with $\mathbb{E}f = 0$ and $\|Sf\|_\infty < 0$, then estimate (5.11), combined with the facts $R_p = O(\sqrt{p})$, $\|Sf\|_p \leq \|Sf\|_\infty$, and Lemma 5.2 implies the superexponential bound

$$\{|x \in \mathcal{X} : |f(x)| > \lambda\} \leq Ce^{-\lambda^2/\|Sf\|_\infty^2}.$$  \hspace{1cm} (5.12)

Finally we remark that for the classical dyadic square function on $[0,1)^d$ there is no estimate of the type (5.11), i.e.,

$$\|f\|_p \leq C\sqrt{p}\|Sf\|_p, \quad \mathbb{E}f = 0$$

with $C$ independent of $d$, and $p > N$ for some positive $N$. Indeed, otherwise this would give us superexponential bound (5.12) independent of dimension $d$, but we will see in the next section that the estimate

$$\{|x \in [0,1)^d : f(x) > \lambda\} \leq e^{-\lambda^2/(2d\|Sf\|_\infty^2)}$$

is sharp when dimension $d$ goes to infinity, see Theorem 6.1.

6. Sharpness of Theorems 2.1, 2.3

6.1. Sharpness of theorem 2.1. We remark that the constant $\alpha$ in the left hand side of (2.1) is sharp when $\alpha = 1/2$, in this case even Theorem 2.2 is sharp (see the proof of Lemma 3.3 in [8]). It is also true asymptotically as $\alpha \to 0$, in the sense that $\alpha$ in (2.1) cannot be replaced by $C\alpha$ with $C > 1$ independent of $\alpha$ and the estimate being true for all $\alpha > 0$. Indeed, this is true even for the square function $S$ in the $n$-adic filtration (here $\alpha = 1/n$). To verify the sharpness of the estimate for the case $n \to \infty$, which corresponds to $\alpha \to 0$, assume the contrary that we have

$$\int_{Q_0} \exp \left( f - C\frac{2dS(f)^2}{4} \right) \leq \exp \left( \int_{Q_0} f \right)$$

for some $C$ with $0 < C < 1$ and $d$ sufficiently large. Consider the dyadic cubes $Q \in \mathcal{D}_1$ of the first generation, i.e., $|Q| = 2^{-d}$. We remind that $Q_0 = [0,1)^d$. In total we have $2^d$ such cubes. Take any $f$ such that $f = 2^{1-d}$ on $2^d - 1$ such cubes, and $f = 2(2^d - 1)$ on the remaining last cube. Clearly

$$\int_{[0,1]^d} f = 2^{1-d}(2^d - 1)2^{-d} + 2(2^d - 1)2^{-d} = 0.$$
Inequality (6.1) is equivalent to showing that
\[(1 - \alpha) \exp (2\alpha - C\alpha) + \alpha \exp \left(2(\alpha - 1) - C\frac{(\alpha - 1)^2}{\alpha}\right) \leq 1,\]
where \(\alpha = 2^{-d}\). By Taylor’s formula the left hand side of (6.2) at point \(\alpha = 0\) can be written as \(1 + (1 - C)\alpha + O(\alpha^2)\). Therefore we see that the condition \(C \geq 1\) is necessary for the inequality to hold.

Our next remark is that for each fixed \(\alpha \in (0,1/2)\) one can replace the constant \(\alpha\) in the denominator in exponent of the left hand side of (2.1) by a better constant \(C(\alpha)\) with \(C(\alpha) \geq \alpha\). Repeating our arguments without any changes one can show that the largest constant \(C = C(\alpha)\) one can have in the inequality
\[\mathbb{E} \exp \left(f - \frac{S(f)^2}{4C}\right) \leq \exp (\mathbb{E}f)\]
coincides with the largest number \(C\) which satisfies the inequality
\[p_1 e^{c_1 - \frac{c^2}{p_1^2}} + p_2 e^{c_2 - \frac{c^2}{p_2^2}} \leq 1,\]
for all \(c_1, c_2 \in \mathbb{R}\), any \(p_1, p_2 \geq \alpha\) with \(p_1 + p_2 = 1\) and \(p_1 c_1 + p_2 c_2 = 0\). The best possible such constant \(C = C(\alpha)\) can be found, but it does not have a nice form, and we did not find it useful for the practical purposes. However, we just showed that \(C(\alpha) \geq \alpha\) and the latter estimate interpolates in a sharp way the endpoint cases for the range \(\alpha \in (0,1/2]\).

6.2. Sharpness of Theorem 2.3.

**Theorem 6.1.** Theorem 2.3 is sharp when \(\alpha \to 0\), meaning that \(\alpha\) in the exponent in the right hand side of (2.2) cannot be replaced by \(c\alpha\) with \(c > 1\).

More precisely, given \(c > 1\), for all sufficiently small \(\alpha\) and for all \(\lambda > \lambda(\alpha)\) there exists an \(\alpha\)-homogeneous filtration and a function \(f\), \(\|Sf\|_{\infty} = 1\) such that
\[|\{x \in X : f(x) > \lambda\}| > e^{-c\alpha \lambda^2}\]
As a corollary we can see that the superexponential estimate like (2.2) fails for non-homogeneous filtrations.

**Proof of Theorem 6.1.** Let us first describe the construction for \(\alpha = 1/n\), \(n > 2\), the construction for general \(\alpha\) will be the same with obvious modifications. So, let \(\alpha = 1/n\). Consider the standard \(n\)-adic filtration.

Any set \(E \in \mathcal{F}_k\) can be split into two disjoint sets \(E^\alpha, E^\beta \in \mathcal{F}_{k+1}\) such that
\[\lfloor E^\alpha \rfloor = \alpha \lfloor E \rfloor, \quad \lfloor E^\beta \rfloor = (1 - \alpha) \lfloor E \rfloor;\]
the set \(E^\alpha\) can be constructed, for example, by picking one of the intervals \(I \in \mathcal{D}_{k+1}\) out of each interval \(J \in \mathcal{D}_k\) comprising \(E\), and the set \(E^\beta\) will be the rest.

We start with \(E_0 = X\), and split it as above into the sets \(E_0^\alpha, E_0^\beta \in \mathcal{F}_1\), and denote \(E_1 := E_0^\beta\). We then repeat the same procedure for \(E_1\), and inductively construct sets \(E_k \in \mathcal{F}_k\) by defining \(E_{k+1} := E_k^\beta\).

The same construction works for general \(\alpha\). We start with \(X\), which is the only element of \(\mathcal{D}_0\), and construct the sets \(\mathcal{D}_k\) inductively. Namely, to get \(\mathcal{D}_{k+1}\) we split each \(I \in \mathcal{D}_k\) into finitely many intervals \(I_j\) such that \(|I_j| = \alpha |I|\) and \(|I_j| \geq \alpha |I|\) for \(j \geq 2\). Clearly, that in this situation we can split a set \(E \in \mathcal{F}_k\) into the sets \(E^\alpha, E^\beta \in \mathcal{F}_{k+1}\) as above, and construct inductively the sets \(E_k\).
For now let $\alpha$ be fixed. Define the functions
\[ d_k := \frac{\beta_k^{\alpha} - 1}{\alpha} - \frac{1}{\alpha} E_k^{\alpha}. \]
Clearly the functions $d_k$ are $F_{k+1}$-measurable and $E_k d_k = 0$, so they are martingale differences.

In what follows we will use the notation $f \sim g$ for the sharp asymptotic equivalence of $f$ and $g$, meaning $\lim(f/g) = 1$; when it is not evident from the context, the direction for the limit will always be specified.

Take $N > 1/\alpha^2$. Then an easy observation shows that for $g_N := \sum_{k=0}^{N-1} d_k$
\[ \|Sg_N\|_{\infty} \sim (N - 1 + (1 - \alpha)^2/\alpha^2)^{1/2} \sim N^{1/2} \quad \text{as} \ N \to \infty \]
and that
\[ g_N(x) \geq N \quad \text{on a set of measure } (1 - \alpha)^N \]

For sufficiently large $\lambda > 0$ define $N(\lambda)$ to be the smallest integer $N$ such that $N/\|Sg_N\|_{\infty} \geq \lambda$; clearly
\[ N(\lambda) \sim \lambda^2, \quad N(\lambda)/\|Sg_{N(\lambda)}\|_{\infty} \sim \lambda \quad \text{as} \ \lambda \to \infty. \]
Define $f = f_\lambda := g_{N(\lambda)}/\|Sg_{N(\lambda)}\|_{\infty}$. Then $\|Sf_\lambda\|_{\infty} = 1$ and $f_\lambda(x) \geq \lambda$ on a set of measure
\[ (1 - \alpha)^N(\lambda) \sim (1 - \alpha)\lambda^2 = e^{\lambda^2 \ln(1 - \alpha)} \quad \text{as} \ \lambda \to \infty. \]
Noticing that $\ln(1 - \alpha) \sim -\alpha$ as $\alpha \to 0$ concludes the proof.

Namely, for $c > 1$ take $c_1, c < c_1 < 1$. Since $\lim_{\alpha \to 0^+} \alpha^{-1} \ln(1 - \alpha) = -1$ we conclude that for all sufficiently small $\alpha$, $0 < \alpha \leq \alpha_0(c_1)$
\[ \ln(1 - \alpha) \geq -c_1 \alpha. \]
So, using the estimate (6.4) we see that for a fixed sufficiently small $\alpha$ as above, the estimate (6.3) holds for all sufficiently large $\lambda$, i.e. for all $\lambda > \Lambda(c, \alpha)$. \hfill \Box

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Department of Mathematics, Princeton University; MSRI; UC Irvine, USA
E-mail address: paatai@princeton.edu (P. Ivanisvili)

Department of Mathematics, Brown University, Providence, RI 02912, USA
E-mail address: treil@math.brown.edu (S. Treil)