Abstract. In this article, we show that the predual \( G_w(U) \) of the weighted space of holomorphic functions has the \( I \)-approximation property if and only if \( E \) has the \( I \)-approximation property, where \( I \) is a suitably chosen operator ideal, and \( w \) is a radial weight defined on a balanced open subset \( U \) of a Banach space \( E \).

Keywords: holomorphic mappings, weighted spaces of holomorphic functions, linearization, approximation property.

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1. Introduction

Approximation property is one of the most fundamental properties in the theory of Banach spaces as it approximates the identity operator by finite rank operators uniformly on compact subsets of the Banach space. This notion appeared for the first time in Banach’s book in 1932 and later a systematic study of this property along with its variants was carried out by A. Grothendieck [16] in 1955, who showed the importance of the approximation property in the structural study of Banach spaces. After the appearance of the work of P. Enflo [15], who constructed an example of a Banach space lacking the approximation property, researches in approximation property gained momentum. Several variants of this property have been introduced and studied extensively; for instance, compact approximation property, weakly compact approximation property etc.. Replacing the ideals of finite rank/compact/weakly compact operators by an arbitrary operator ideal \( I \), S. Berrios and G. Botelho [5] studied the concept of \( I \)-approximation property which means that the identity operator on \( E \) is uniformly approximated by a member of \( I \equiv I(E, E) \) on compact subsets of \( E \). This yields a unification of several variants of the approximation property. In [4], the authors studied the \( I \)-approximation property in spaces of holomorphic functions of bounded type, spaces of weakly uniformly continuous holomorphic functions, spaces of bounded holomorphic functions and thus generalized some of the results obtained in [9, 23]. Approximation properties more general than \( I \)-approximation Property defined corresponding to convex subsets of class of bounded linear operators have also been investigated; see [6] and [22].

In our recent work [17–19], we considered the approximation property and some of its variants for the predual of the weighted spaces of holomorphic functions defined on open subsets of Banach spaces. We continue this study in the present work. We aim at studying some new characterizations of the bounded and compact approximation property besides dealing with the \( I \)-approximation property for the predual of the weighted space of holomorphic mappings.
In Section 2, we give some basic notations, terminology and the results to be used in the sequel. In the next section we show that a Banach space $E$ has the compact approximation property if and only the inclusion map defined on a balanced open subset $U$ of $E$ can be approximated by a weighted holomorphic map with compact range. Also, we obtain some new characterizations of the bounded approximation property in terms of weighted spaces of holomorphic mappings. Finally, in the last section, we prove the main result, namely - a Banach space $E$ has the $I$-approximation property if and only if the predual $B_{w}(U)$ has the $I$-approximation property for a suitably chosen operator ideal $I$ and weight $w$ defined on a balanced open subset $U$ of a Banach space $E$.

2. Preliminaries

Throughout this paper, we shall use the letters $E$ and $F$ to denote the complex Banach spaces and the symbol $E^*$ to denote the topological dual of $E$. We denote by $U$ a non-empty open subset of $E$ and by $U_E$ the open unit ball of $E$. The symbol $B_E^\lambda$ denotes the closed ball of $E$ consisting of the elements with norm $\leq \lambda$. For $\lambda = 1$, $B_1^E = B_E$ is the closed unit ball of $E$. For $m \in \mathbb{N}$, $\mathcal{P}(mE, F)$ denotes the Banach space of all continuous $m$-homogeneous polynomials from $E$ to $F$. A continuous polynomial $P$ is a mapping from $E$ into $F$ which can be represented as a sum $P = P_0 + P_1 + \cdots + P_l$ with $P_m \in \mathcal{P}(mE, F)$ for $m = 0, 1, \ldots, l$. The vector space of all continuous polynomials from $E$ into $F$ is denoted by $\mathcal{P}(E, F)$. A polynomial $P \in \mathcal{P}(mE, F)$ is said to be of finite type if it is of the form

$$P(x) = \sum_{j=1}^{l} \phi_j^m(x)y_j, \ x \in E,$$

where $\phi_j \in E^*$ and $y_j \in F$, $1 \leq j \leq l$. We denote by $\mathcal{P}_f(mE, F)$ the space of finite type polynomials from $E$ into $F$. A polynomial $P \in \mathcal{P}(mE, F)$ is said to be compact if $P(B_E)$ is relatively compact in $F$. We denote by $\mathcal{P}_k(mE, F)$ the space of all compact $m$-homogeneous polynomials. For $m = 1$, $\mathcal{P}_k(E, F) \equiv \mathcal{K}(E; F)$ ($\mathcal{P}_f(E, F) \equiv \mathcal{F}(E; F)$) is the space of all compact (finite) linear operators from $E$ to $F$.

A mapping $f : U \to F$ is said to be holomorphic, if for each $\xi \in U$, there exists a ball $B(\xi, r)$ with center at $\xi$ and radius $r > 0$, contained in $U$ and a sequence $\{P_m\}_{m=0}^{\infty}$ of polynomials with $P_m \in \mathcal{P}(mE, F)$, $m \in \mathbb{N}_0$ such that

$$(1) \quad f(x) = \sum_{m=0}^{\infty} P_m(x - \xi) = \sum_{m=0}^{\infty} \frac{1}{m!} \partial^m f(\xi)(x - \xi),$$

where the series converges uniformly for each $x \in B(\xi, r)$. The vector space of all holomorphic mappings from $U$ to $F$ is denoted by $\mathcal{H}(U, F)$ and the compact open topology on $\mathcal{H}(U, F)$, the topology of uniform convergence on compact subsets of $U$, is denoted by $\tau_U$. In case $U = E$, the class $\mathcal{H}(E, F)$ is the space of entire mappings from $E$ into $F$. For $F = \mathbb{C}$, we write $\mathcal{H}(U)$ for $\mathcal{H}(U, \mathbb{C})$.

A subset $A$ of $U$ is said to be $U$-bounded if $A$ is bounded and there exists a neighborhood $V$ of $0$ such that $A + V \subset U$. A mapping $f$ in $\mathcal{H}(U, F)$ is of bounded type if it maps $U$-bounded sets to bounded sets in $F$. The space of holomorphic mappings of bounded type is denoted by $\mathcal{H}_b(U, F)$. The space $\mathcal{H}_b(U, F)$ endowed with the topology $\tau$, the topology of uniform convergence on $U$-bounded sets, is a Fréchet space, cf. [3]. We refer to [3], [13], [14], [21], [24], [25] for notations and background on infinite dimensional holomorphy.
A weight $w$ on $U$ is a continuous and strictly positive function satisfying
\begin{equation}
0 < \inf_{A} w(x) \leq \sup_{A} w(x) < \infty
\end{equation}
for each $U$-bounded set $A$. A weight $w$ defined on an open balanced subset $U$ of $E$ is said to be \textit{radial} if $w(tx) = w(x)$ for all $x \in U$ and $t \in \mathbb{C}$ with $|t| = 1$; and in case of $U = E$ it is said to be \textit{rapidly decreasing} if $\sup_{x \in E} w(x) x^m \leq \infty$ for each $m \in \mathbb{N}_0$. Corresponding to a weight function $w$, the weighted space of holomorphic functions is defined as
\begin{equation}
\mathcal{H}_w(U; F) = \{ f \in \mathcal{H}(U; F) : \| f \|_w = \sup_{x \in U} w(x) \| f(x) \| < \infty \}.
\end{equation}
The space $(\mathcal{H}_w(U; F), \| \cdot \|_w)$ is a Banach space and $B_w$ denotes its closed unit ball. For $F = \mathbb{C}$, we write $\mathcal{H}_w(U) = \mathcal{H}_w(U, \mathbb{C})$. It can be easily seen that the norm topology $\tau_{\| \cdot \|_w}$ on $\mathcal{H}_w(U, F)$ is finer than the topology induced by $\tau_0$, the topology of uniform convergence on compact subsets of $U$. Since the closed unit ball $B_w$ of $\mathcal{H}_w(U)$ is $\tau_0$-compact, the predual of $\mathcal{H}_w(U)$ is given by
\begin{equation}
\mathcal{G}_w(U) = \{ \phi \in \mathcal{H}_w(U)' : \phi|B_w \text{ is } \tau_0 - \text{continuous} \}
\end{equation}
by the Ng Theorem, cf. [26].

\textbf{Proposition 2.1.} Let $w$ be a weight on an open subset $U$ of a Banach space $E$. Then, for each $m \in \mathbb{N}$, the following are equivalent:
\begin{enumerate}[(a)]
\item $\mathcal{P}(mE, F) \subset \mathcal{H}_w(U, F)$ for each Banach space $F$.
\item $\mathcal{P}(mE) \subset \mathcal{H}_w(U)$.
\end{enumerate}

\textbf{Theorem 2.2.} \textbf{(Linearization Theorem)} For an open subset $U$ of a Banach space $E$ and a weight $w$ on $U$, there exists a Banach space $\mathcal{G}_w(U)$ and a mapping $\Delta_w \in \mathcal{H}_w(U, \mathcal{G}_w(U))$ with $\| \Delta_w \|_w \leq 1$ satisfying the following property: for each Banach space $F$ and each mapping $f \in \mathcal{H}_w(U, F)$, there is a unique operator $T_f \in \mathcal{L}(\mathcal{G}_w(U), F)$ such that $T_f \circ \Delta_w = f$. The correspondence $\Psi$ between $\mathcal{H}_w(U, F)$ and $\mathcal{L}(\mathcal{G}_w(U), F)$ given by
\begin{equation}
\Psi(f) = T_f
\end{equation}
is an isometric isomorphism. The space $\mathcal{G}_w(U)$ is uniquely determined upto an isometric isomorphism by these properties.

We write
\begin{equation}
\mathcal{H}_w(U) \otimes F = \{ f \in \mathcal{H}_w(U, F) : f \text{ has finite dimensional range} \}
\end{equation}
and
\begin{equation}
\mathcal{H}_w^c(U, F) = \{ f \in \mathcal{H}_w(U, F) : w f \text{ has relatively compact range} \}.
\end{equation}
The next proposition establishes the interplay between the properties of a mapping $f \in \mathcal{H}_w(U, F)$ and the corresponding operator $T_f \in \mathcal{L}(\mathcal{G}_w(U), F)$.

\textbf{Proposition 2.3.} Let $U$ be an open subset of a Banach space $E$ and $w$ be a weight on $U$. Then for any Banach space $F$, we have
\begin{enumerate}[(a)]
\item $f \in \mathcal{H}_w(U) \otimes F$ if and only if $T_f \in \mathcal{F}(\mathcal{G}_w(U), F)$.
\item $f \in \mathcal{H}_w^c(U, F)$ if and only if $T_f \in \mathcal{K}(\mathcal{G}_w(U), F)$.
\end{enumerate}
Further, we recall the locally convex topology $\tau_M$ on $H_w(U, F)$ generated by the family
\[
\{p_{\alpha, \bar{A}} : \alpha = (\alpha_j) \in c_0^+, \bar{A} = (A_j), A_j \text{ being finite subset of } U \text{ for each } j \}\end{array}
\]
of semi-norms defined by
\[
p_{\alpha, \bar{A}}(f) = \sup_{j \in \mathbb{N}} (\alpha_j \inf_{x \in A_j} w(x) \sup_{y \in A_j} \|f(y)\|), f \in H_w(U, F).
\]

**Theorem 2.4.** Let $E$ and $F$ be Banach spaces and $w$ be a weight on an open subset of $E$. Then the mapping $\Psi : (H_w(U, F), \tau_M) \rightarrow (L(G_w(U), F), \tau_c)$ is a topological isomorphism.

In case $\Psi$ is restricted to norm bounded subsets of $H_w(U, F)$, we have the following result from [19].

**Theorem 2.5.** Let $E$ and $F$ be Banach spaces and $w$ be a weight on an open subset $U$ of $E$. Then the restriction of the map $\Psi : (H_w(U, F), \tau_c) \rightarrow (L(G_w(U), F), \tau_c)$ on $\| \cdot \|_w$-bounded subsets of $H_w(U, F)$ is a topological isomorphism.

A Banach space $E$ is said to have the approximation property (the compact approximation property) abbreviated as AP(CAP) if for every compact set $K$ of $E$ and $\epsilon > 0$, there exists an operator $T \in F(E, E)$ such that
\[
\sup_{x \in K} \|T(x) - x\| < \epsilon.
\]
If $T \in F(E, E)$ can be chosen with $\|T\| \leq \lambda$ for some $\lambda$, $1 \leq \lambda < \infty$, then $E$ is said to have the $\lambda$-bounded approximation property ($\lambda$-BAP). $E$ has the bounded approximation property (BAP) if $E$ has the $\lambda$-BAP for some $\lambda$, $1 \leq \lambda < \infty$.

The following characterization of the bounded approximation property is due to Grothendieck and proved in [12].

**Theorem 2.6.** For a Banach space $E$ and $1 \leq \lambda < \infty$, the following are equivalent:

(i) $E$ has the $\lambda$-bounded approximation property.

(ii) $B^\lambda_{F(E, F)}(\tau_c) = B_{L(E, F)}$ for every Banach space $F$.

(iii) $B^\lambda_{F(F, E)}(\tau_c) = B_{L(F, E)}$ for every Banach space $F$.

(iv) $B^\lambda_{F(E, E)}(\tau_c) = B_{L(E, E)}$.

Similar to the above characterization of the bounded approximation property, the following characterizations of the compact approximation property is quoted from [9], see also [12].

**Theorem 2.7.** For a Banach space $E$, the following are equivalent:

(i) $E$ has the compact approximation property.

(ii) $K(E, F)(\tau_c) = L(E, F)$ for every Banach space $F$.

(iii) $K(F, E)(\tau_c) = L(F, E)$ for every Banach space $F$.

**Proposition 2.8.** For a Banach space $E$, the following are equivalent:

(i) $E$ has the compact approximation property.

(ii) $P(E, F) = P_k(E, F)$ for every Banach space $F$.

The next characterization of the compact approximation property for a Banach space $E$ in terms of the predual of the weighted spaces has been obtained in [18].
Theorem 2.9. Let \( w \) be a radial weight on a balanced open subset \( U \) of a Banach space \( E \) such that \( \mathcal{P}(mE) \subset \mathcal{H}_w(E) \) for each \( m \in \mathbb{N} \). Then, the following are equivalent:

(i) \( E \) has the compact approximation property.

(ii) \( \mathcal{P}_k(mE, F)_M = \mathcal{H}_w(U, F) \) for each Banach space \( F \) and for each \( m \in \mathbb{N} \).

(iii) \( \mathcal{H}_w(U, F)_M = \mathcal{H}_w(U, F) \) for each Banach space \( F \).

(iv) \( \mathcal{G}_w(U) \) has the compact approximation property.

3. \( \mathcal{H}_w(U, F) \) and the Compact Approximation Property

In this section, we study the compact approximation property for the space \( \mathcal{G}_w(U) \). Let us begin with the following lemma.

Lemma 3.1. For each \( x \in U \), there exists \( \epsilon > 0 \) and a \( U \)-bounded set \( V_{x,\epsilon} \) such that \( x \in V_{x,\epsilon} \subset U \).

Proof. Since \( U \) is open, there exists \( \epsilon > 0 \) such that \( x + 2\epsilon B_E \subset U \). Define \( V_{x,\epsilon} = x + \epsilon B_E \). Then \( V_{x,\epsilon} \) is \( U \)-bounded and \( x \in V_{x,\epsilon} \subset U \). \( \square \)

Let us recall from [1] that a map \( f \in \mathcal{H}(U, F) \) is said to be compact if for each \( x \in U \), there exists a neighborhood \( V_x \) of \( x \) such that \( V_x \subset U \) and \( f(V_x) \) is relatively compact in \( F \). We denote by \( \mathcal{H}_k(U, F) \) the space of all compact holomorphic mappings from \( U \) to \( F \). A characterization of compact holomorphic mappings in terms of Taylor series coefficients was obtained in [1] by R. Aron and M. Schottenloher. This was further generalized by E. Caliskan and P. Rueda [11] for \( \xi \)-balanced domains (an open set \( U \) is said to be \( \xi \)-balanced if \( (1 - \mu)\xi + \mu x \in U \) for all \( x \in U \) and \( \mu \in \mathbb{C} \) with \( |\mu| \leq 1 \) ) in locally convex spaces of which a particular case is quoted below.

Proposition 3.2. Let \( U \) be a balanced open subset of a Banach space \( E \) and \( f \in \mathcal{H}(U, F) \). Then the following are equivalent:

(a) \( f \in \mathcal{H}_k(U, F) \).

(b) \( P_m f(0) \in \mathcal{P}_k(mE, F) \) for all \( m \in \mathbb{N} \).

(c) \( P_m f(x) \in \mathcal{P}_k(mE, F) \) for all \( m \in \mathbb{N} \) and \( x \in U \).

Relating \( \mathcal{H}_w(U, F) \) with \( \mathcal{H}_k(U, F) \), we prove the following inclusion result.

Proposition 3.3. Let \( U \) be an open subset of a Banach space \( E \) and \( w \) be a weight defined on \( U \). Then \( \mathcal{H}_w(U, F) \subset \mathcal{H}_k(U, F) \) for each Banach space \( F \).

Proof. Let \( f \in \mathcal{H}_w(U, F) \). Then \( T_f \in \mathcal{K}(\mathcal{G}_w(U), F) \) by Proposition 2.3 (b). Fix \( x \in U \) arbitrarily. Then by Lemma 3.1, there exists \( \epsilon > 0 \) such that

\[
V_{x,\epsilon} = x + \epsilon B_E \subset U,
\]

where \( V_{x,\epsilon} \) is \( U \)-bounded. By Theorem 2.2 \( f(V_{x,\epsilon}) = T_f \circ \Delta_w(V_{x,\epsilon}) \). Since \( \Delta_w \in \mathcal{H}_w(U, \mathcal{G}_w(U)) \subset \mathcal{H}_b(U, \mathcal{G}_w(U)) \), \( \Delta_w(V_{x,\epsilon}) \) is bounded in \( \mathcal{G}_w(U) \). Thus \( f(V_{x,\epsilon}) \) is relatively compact in \( F \). Consequently, \( f \in \mathcal{H}_k(U, F) \). \( \square \)

Remark 3.4. Note that the reverse implication in Proposition 3.3 does not hold even in the particular case of \( w \equiv 1 \), cf. [23, Example 3.2].

Theorem 3.5. Let \( w \) be a radial weight on a balanced open subset \( U \) of a Banach space \( E \) such that \( \mathcal{P}(E) \subset \mathcal{H}_w(U) \). Then the following assertions are equivalent:
(a) \( E \) has the CAP.
(b) \( \overline{H_w(U, F)}^{\tau_M} = H_w(U, F) \) for every Banach space \( F \).
(c) \( H_w(U, F) \subset \overline{H_w(U, F)}^{\tau_M} \) for every Banach space \( F \).

Proof. (a) \( \implies \) (b) is the same as (i) \( \implies \) (iii) of Theorem 2.9.

(b) \( \implies \) (c): Since \( \tau_c \leq \tau_M \), \( \overline{H_w(U, F)}^{\tau_M} \subset \overline{H_w(U, F)}^{\tau_c} \) for each Banach space \( F \). Thus the implication holds.

(c) \( \implies \) (a): Let \( m \in \mathbb{N} \), define \( Q_m : (H(U, F), \tau_c) \to \mathcal{P}(mE, F) \) as \( Q_m(f) = P_m f(0), f \in H(U, F) \).

Now, \( Q_m \) is a continuous projection for each \( m \in \mathbb{N} \), by [14, Proposition 3.22]. Therefore \( Q_m(f_\alpha) \overset{\tau_M}{\to} Q_m(P) \), that is, \( P_m f_\alpha(0) \overset{\tau_M}{\to} P \), where \( P_m f_\alpha(0) \in \mathcal{P}(mE, F) \) as \( f_\alpha \in H_w(U, F) \) and \( H_w(U, F) \subset H_k(U, F) \). Thus \( \overline{\mathcal{P}_k(mE, F)}^{\tau_c} = \mathcal{P}(mE, F) \). Hence \( E \) has the CAP by Proposition 2.8.

\[ \square \]

**Theorem 3.6.** Let \( w \) be a radial weight on a balanced open subset \( U \) of a Banach space \( E \) such that \( \mathcal{P}(E) \subset H_w(U) \). Then the following assertions are equivalent:

(a) \( E \) has the CAP.
(b) \( I_U \in \overline{H_w(U, E)}^{\tau_M} \), where \( I_U : U \to E \) is the inclusion mapping.

Proof. (a) \( \implies \) (b): Let \( E \) has the CAP. Then by Theorem 3.5 ((a) \( \implies \) (b)) \( \overline{H_w(U, F)}^{\tau_M} = H_w(U, F) \). Since \( I_U \in \mathcal{P}(mE, E) \) and \( \mathcal{P}(mE, E) \subset H_w(U, E) \), \( I_U \in \overline{H_w(U, E)}^{\tau_M} \).

(b) \( \implies \) (a): By (b), there exist a net \( (f_\alpha)_{\alpha \in \Lambda} \subset H_w(U, E) \) such that \( f_\alpha \overset{\tau_M}{\to} I_U \). By Theorem 2.4 \( T_{f_\alpha} \overset{\tau_c}{\to} T_{I_U} \equiv T \).

Define \( S = d^1 \Delta_w(0) \). Then \( S \in \mathcal{L}(E, \mathcal{G}_w(U)) \) and, by the Cauchy integral formula

\[
S(t) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{\Delta_w(\xi t)}{\xi^2} d\xi, \quad t \in E,
\]

where \( r > 0 \) is chosen such that \( \{\xi t : |\xi| \leq r\} \subset U \). Note that \( T \circ \Delta_w = I_E \), where \( I_E : E \to E \) is the identity operator on \( E \). Therefore

\[
T \circ S(t) = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{t}{\xi} d\xi = t, \quad \text{for all } t \in E.
\]

Now, \( T_{f_\alpha} \circ S \overset{\tau_c}{\to} T \circ S \) gives \( T_{f_\alpha} \circ S \overset{\tau_c}{\to} I_E \). Since \( T_{f_\alpha} \in \mathcal{K}(\mathcal{G}_w(U), E) \) by Proposition 2.8(b), \( T_{f_\alpha} \circ S \in \mathcal{K}(E, E) \) for each \( \alpha \in \Lambda \) and (a) follows.

\[ \square \]

The next result of this section deals with a characterization of the bounded approximation property in terms of finite rank weighted holomorphic mappings.

**Theorem 3.7.** Let \( w \) be a bounded weight on the open unit ball \( U_E \) of a Banach space \( E \).

Then the following assertions are equivalent:

(a) \( E \) has the BAP.
(b) \( \overline{B_{H_w(U_E)}^{\tau_c} \otimes E} = B_{H_w(U_E; E)} \) for some \( \lambda, 1 \leq \lambda < \infty \).
(c) \( I_{U_E} \in \overline{B_{H_w(U_E)}^{\tau_c} \otimes E} \) for some \( \lambda', 1 \leq \lambda' < \infty \).
Definition 4.1. For each Banach space $B \in L_{\mathcal{H}_w(U_E, E)}$ such that $T_B$ is continuous by Theorem 2.1. (i) $\iff$ (ii).

Proposition 3.9. Let $I$ be the open unit ball of a Banach space $E$ and $1 \leq \lambda < \infty$. Then for each Banach space $F$, the following assertions are equivalent:

(a) $F$ has the $\lambda$-BAP.
(b) $E \in \mathcal{H}_w(U_E, E)$ for each open subset $V \subset F$.
(c) $I_{U_E} \in \mathcal{B}_w^\lambda(U_E) \otimes E$.

Proof. Since $\|P\| = 1$ when $U = U_E$, cf. [23] Proposition 2.3(b)], the implications (a) $\implies$ (b) $\implies$ (c) are proved in [10] as (c) $\implies$ (d) $\implies$ (e) of Proposition 3.

(c) $\implies$ (a) is a consequence of (c) $\implies$ (a) of Theorem 3.7 by taking $w \equiv 1$.

4. $\mathcal{H}_w(U, F)$ and $\mathcal{I}$-approximation property

In this section we study $\mathcal{I}$-approximation property for the predual of a weighted space of holomorphic functions. A Banach space $E$ is said to have $\mathcal{I}$-approximation property if the identity operator on $E$ is uniformly approximated by a member of $\mathcal{I}(E, E)$ on compact subsets of $E$.

Let us first recall the multilinear/polynomal ideal generalization of an operator ideal from [7, 8].

Definition 4.1. Let $\mathcal{I}$, $\mathcal{I}_1$, $\mathcal{I}_2$, ..., $\mathcal{I}_m$ be operator ideals.

(a) A mapping $A \in \mathcal{L}(E_1, E_2, \ldots, E_m; F)$ is said to be of type $\mathcal{L}[\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m]$ if there exist Banach spaces $G_1, G_2, \ldots, G_m$ and operators $u_j \in \mathcal{I}_j(E_j, G_j)$, $j = 1, 2, \ldots, m$, and a mapping $B \in \mathcal{L}(G_1, G_2, \ldots, G_m; F)$ such that $A = B \circ (u_1, u_2, \ldots, u_m)$. For $\mathcal{I} = \mathcal{I}_1 = \mathcal{I}_2 = \cdots = \mathcal{I}_m$, we write $\mathcal{L}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m) = \mathcal{L}[\mathcal{I}]$.
(b) **Composition Ideal of Multilinear Operators**: A mapping \(A \in \mathcal{L}(E_1, E_2, \ldots, E_m; F)\) belongs to \(\mathcal{I} \circ \mathcal{L}\) if there are Banach space \(G\), a mapping \(B \in \mathcal{L}(E_1, E_2, \ldots, E_m; G)\) and \(u \in \mathcal{I}(G, F)\) such that \(A = u \circ B\).

(c) **Composition Polynomial Ideal**: A polynomial \(P \in \mathcal{P}^m(E, F)\) belongs to \(\mathcal{I} \circ \mathcal{P}\) if there exist a Banach space \(G\) and \(Q \in \mathcal{P}^m(G, F)\) and an operator \(u \in \mathcal{I}(E, G)\) such that \(P = u \circ Q\). In this case we write \(P \in \mathcal{I} \circ \mathcal{P}^m(E, F)\).

The following generalizations for \(\mathcal{I}\)-approximation property are quoted from [5].

**Theorem 4.2.** For a Banach space \(E\), the following are equivalent:

(i) \(E\) has the \(\mathcal{I}\)-approximation property.

(ii) \(\overline{\mathcal{I}(E, F)}^e = \mathcal{L}(E, F)\) for every Banach space \(F\).

(iii) \(\overline{\mathcal{I}(F, E)}^e = \mathcal{L}(F, E)\) for every Banach space \(F\).

Observe that the above theorem is a generalization of Theorem 2.7.

**Proposition 4.3.** Let \(\mathcal{I}\) be an operator ideal and \(E\) be a Banach space with \(\mathcal{I}\)-AP. Then every complemented subspace of \(E\) has \(\mathcal{I}\)-AP.

The following characterization of \(\mathcal{I}\)-AP in terms of \(m\)-homogeneous polynomials is proved in [5].

**Theorem 4.4.** Let \(E\) be a Banach space and \(\mathcal{I}\) be an operator ideal such that \(\mathcal{L}[\mathcal{I}] \subset \mathcal{I} \circ \mathcal{L}\). Then \(E\) has \(\mathcal{I}\)-AP if and only if \(\mathcal{P}^m(E, F) \subset \overline{\mathcal{I} \circ \mathcal{P}^m(E, F)}^e\) for every \(n \in \mathbb{N}\) and Banach space \(F\).

Let us define \(\mathcal{I} \circ \mathcal{H}_w(U, F) = \{ f \in \mathcal{H}(U, F) : f = S \circ g \text{ for some Banach space } G, S \in \mathcal{I}(G, F) \text{ and } g \in \mathcal{H}_w(U, G) \}\). Generalizing Proposition 2.3 for an arbitrary operator ideal \(\mathcal{I}\), we have

**Theorem 4.5.** Let \(f \in \mathcal{H}_w(U, F)\) and \(\mathcal{I}\) be an operator ideal. Then \(f \in \mathcal{I} \circ \mathcal{H}_w(U, F)\) if and only if \(T_f \in \mathcal{I} \mathcal{H}_w(U, F)\).

**Proof.** Suppose \(f \in \mathcal{I} \circ \mathcal{H}_w(U, F)\). Then there exist a Banach space \(G\), an operator \(S \in \mathcal{I}(G, F)\) and a map \(g \in \mathcal{H}_w(U, G)\) such that \(f = S \circ g\). Note that

\[ T_f(\delta_x) = f(x) = S \circ g(x) = S \circ T_g(\delta_x). \]

Since \(\text{span}\{\Delta_w(x) = \delta_x : x \in U\}\) is dense in \(\mathcal{G}_w(U)\), cf. [4, Lemma 7], \(T_f = S \circ T_g \in \mathcal{I} \mathcal{H}_w(U, F)\).

Conversely, assume that \(T_f \in \mathcal{I} \mathcal{H}_w(U, F)\) for \(f \in \mathcal{H}_w(U, F)\). Since \(\Delta_w \in \mathcal{H}(U, \mathcal{G}_w(U))\) and \(f = T_f \circ \Delta_w\), \(f \in \mathcal{I} \circ \mathcal{H}_w(U, F)\).

Next, we show that each mapping in \(\mathcal{I} \circ \mathcal{P}\) admits a factorization in terms of weighted holomorphic mappings.

**Proposition 4.6.** Let \(\mathcal{I}\) be an operator ideal and \(P : E \rightarrow F\) be a continuous polynomial such that \(P = P_0 + P_1 + P_2 + \cdots + P_n\) with \(P_l \in \mathcal{I} \circ \mathcal{P}^l(E, F)\) for each \(0 \leq l \leq n\). Then \(P \in \mathcal{I} \circ \mathcal{H}_w(U, F)\).

**Proof.** Though the proof of this result is analogous to the proof of [2, Theorem 2.4], we outline the same for the sake of convenience.
Since \( P = P_0 + P_1 + P_2 + \cdots + P_n \) with \( P_l \in \mathcal{I} \circ \mathcal{P}(E, F) \) for each \( 0 \leq l \leq n \), there are Banach spaces \( G_0, G_1, G_2, \ldots, G_n, Q_l \in \mathcal{P}(E, G) \) and \( A_l \in \mathcal{I}(G_l, F) \) such that \( P_l = A_l \circ Q_l \) for each \( 0 \leq l \leq n \).

Define \( G = G_0 \times G_1 \times G_2 \times \cdots \times G_n, \) \( Q : E \to G \) and \( A : G \to F \) by \( Q(x) = (Q_0(x), Q_1(x), \ldots, Q_n(x)) \) and \( A((y_0, y_1, y_2, \ldots, y_n)) = A_0(y_0) + A_1(y_1) + \cdots + A_n(y_n) \). Then \( A \in \mathcal{I}(G, F) \) and \( Q \in \mathcal{P}(E, G) \).

Note that \( A \circ Q(x) = A_0 \circ Q_0(x) + A_1 \circ Q_1(x) + A_2 \circ Q_2(x) + \cdots + A_n \circ Q_n(x) = P_0(x) + P_1(x) + P_2(x) + \cdots + P_n(x) = P(x) \) for all \( x \in E \), where \( Q \in \mathcal{P}(E, G) \subset \mathcal{H}_w(U, G) \). Thus \( P \in \mathcal{I} \circ \mathcal{H}_w(U, F) \).

Using the above proposition, we obtain our main result.

**Theorem 4.7.** Let \( w \) be a radial weight on a balanced open subset \( U \) of a Banach space \( E \) such that \( \mathcal{H}_w(U, F) \) contains all polynomials and \( \mathcal{I} \) be an operator ideal such that \( \mathcal{L}[I] \subset \mathcal{I} \circ \mathcal{L} \). Then the following assertions are equivalent:

(a) \( E \) has \( \mathcal{I} \)-AP.
(b) \( \mathcal{P}(mE, F) = \mathcal{I} \circ \mathcal{P}(mE, F) \) for each \( m \in \mathbb{N} \) and each Banach space \( F \).
(c) \( \mathcal{H}_w(U, F) = \mathcal{I} \circ \mathcal{H}_w(U, F) \) for each Banach space \( F \).
(d) \( \mathcal{G}_w(U) \) has the \( \mathcal{I} \)-AP.
(e) \( \mathcal{I}_U \in \mathcal{I} \circ \mathcal{H}_w(U, F) \) for each Banach space \( F \).

**Proof.** (a) \( \Leftrightarrow \) (b) follows from [3] Theorem 4.2(a) \( \Leftrightarrow \) (f).

(b) \( \Rightarrow \) (c): Let \( p \) be a \( \tau_M \)-continuous semi-norm on \( \mathcal{H}_w(U, F) \) and \( f \in \mathcal{H}_w(U, F) \). Since \( \mathcal{P}(E, F) \) is \( \tau_M \)-dense in \( \mathcal{H}_w(U, F) \), cf. [17] Proposition 4.6, there exists \( P \in \mathcal{P}(E, F) \) such that

\[
(3) \quad p(f - P) < \frac{\epsilon}{2}.
\]

Let \( P = P_0 + P_1 + \cdots + P_l \), \( P_m \in \mathcal{P}(mE, F) \) for each \( 0 \leq m \leq l \). Then by (b), there exists \( Q_m \in \mathcal{I} \circ \mathcal{P}(mE, F) \) such that

\[
(4) \quad p(Q_m - P_m) < \frac{\epsilon}{2(m + 1)}
\]

for each \( 0 \leq m \leq l \).

Define \( Q = Q_0 + Q_1 + \cdots + Q_l \). By Proposition 4.6, \( Q \in \mathcal{I} \circ \mathcal{H}_w(U, F) \) and

\[
(5) \quad p(P - Q) = p\left( \sum_{m=0}^{l} P_m - \sum_{m=0}^{l} Q_m \right) < \frac{\epsilon}{2}
\]

by (4). Hence by (3) and (5), we have

\[
p(Q - f) < \epsilon.
\]

Thus (c) follows.

(c) \( \Rightarrow \) (d): Since \( \Delta_w \in \mathcal{H}_w(U, \mathcal{G}_w(U)) \), \( \Delta_w \in \mathcal{I} \circ \mathcal{H}_w(U, \mathcal{G}_w(U)) \) by taking \( F = \mathcal{G}_w(U) \) in (c). Thus there exists a net \( (f_\alpha) \subset \mathcal{I} \circ \mathcal{H}_w(U, \mathcal{G}_w(U)) \) such that \( f_\alpha \overset{\tau_M}{\to} \Delta_w \). By Theorem 2.4, we get

\[
T_{f_\alpha} \overset{\tau_M}{\to} T_{\Delta_w}.
\]

Note that \( T_{\Delta_w} = I_{\mathcal{G}_w(U)} \) and \( T_{f_\alpha} \in \mathcal{I}(\mathcal{G}_w(U), \mathcal{G}_w(U)) \) by Theorem 4.5. Thus \( I_{\mathcal{G}_w(U)} \in \mathcal{I}(\mathcal{G}_w(U), \mathcal{G}_w(U)) \) and hence (d) follows.
$(d) \implies (a)$: Since $E$ is complemented in $G_w(U)$, cf. [17, Proposition 3.6], $(a)$ follows from $(d)$ and Proposition 4.3.

$(c) \implies (e)$: Since $I_{U} \in \mathcal{P}(m^E,F) \subset \mathcal{H}_w(U,F)$, $(e)$ follows from $(c)$.

$(e) \implies (a)$: As $I_{U} \in I \circ \mathcal{H}_w(U,E)^\tau_{M}$, there exists a net $(f_\alpha) \subset I \circ \mathcal{H}_w(U,E)$ such that $f_\alpha \xrightarrow{\tau_{M}} I_{U}$. Also, by Theorem 2.4 $T_{f_\alpha} \xrightarrow{T_{I_{U}}} T_{I_{U}}$. Then proceeding as in the proof of Theorem 3.6 $T_{f_\alpha} \circ S \xrightarrow{T_{E}} I_{E}$. As $T_{f_\alpha} \in I(\mathcal{G}_w(U),E)$ for each $\alpha$ in view of Theorem 4.5 $E$ has the $I$–AP by Theorem 4.2.

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References

[1] R. Aron and M. Schottenloher, Compact holomorphic mappings on Banach spaces and the approximation property, J. Funct. Anal., 21, 1976, 7–30.
[2] R. Aron, G. Botelho, D. Pellegrino and P. Rueda, Holomorphic mappings associated to composition ideals of polynomials. Rend. Lincei Mat. Appl. 21, 261-274 (2010)
[3] J. A. Barroso, Introduction to Holomorphy, North-Holland Math. Studies, 106, North-Holland, Amsterdam, 1985.
[4] M. J. Beltran, Linearization of weighted (LB)-spaces of entire functions on Banach spaces. Rev. R. Acad. Cien. Exactas Fis. Nat., Ser. A Mat., RACSAM 106, 1 (2012), 275-286.
[5] S. Berrios and G. Botelho, Approximation properties determined by operator ideals and approximability of homogeneous polynomials and holomorphic functions. Stud. Math. 208, 97-116 (2012).
[6] A. Bhar and M. Gupta, A note on generalized approximation property, J. Funct. Spaces Appl., Article ID 325141, 6 pp, 2013.
[7] G. Botelho, Ideals of polynomials generated by weakly compact operators, Note Mat. 25 (2005/2006), 69–102.
[8] G. Botelho, D. Pellegrino and P. Rueda, On composition ideals of multilinear mappings and homogeneous polynomials, Publ. Res. Inst. Math. Sci. 43 (2007), 1139-1155.
[9] E. Caliskan, Bounded holomorphic mappings and the compact approximation property in Banach spaces, Port. Math. (N.S.) 61 (2004), 25-33.
[10] E. Caliskan, The bounded approximation property for the predual of the space of bounded holomorphic mappings, Studia Math. 177(3) (2006), 225–233.
[11] E. Caliskan and P. Rueda, The compact approximation property for spaces of holomorphic mappings on Fréchet spaces, Rev. Mat. Complut. 34, 185–201 (2021).
[12] P.G. Casazza, Approximation properties, In: Handbook of the Geometry of Banach spaces, Vol. 1, W.B. Johnson and J. Lindenstrauss (eds.), North Holland, Amsterdam, 2001, 271-316.
[13] S. Dineen, Complex Analysis in Local Convex Spaces, North-Holland Math. Studies, vol. 57, North-Holland, Amsterdam, 1981.
[14] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, Springer-Verlag, London, 1999.
[15] P. Enflo, A counterexample to the approximation problem in Banach spaces, Acta Math., 130, 1973, 309–317.
[16] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc. No. 16, 1955.
[17] M. Gupta and D. Baweja, Weighted Spaces of Holomorphic Functions on Banach Spaces and The Approximation Property, Extracta Math., 31(2), 2016, 123-144.
[18] M. Gupta and D. Baweja, The Compact Approximation Property for Weighted Spaces of Holomorphic Mappings” Advances in Real and Complex Analysis with Applications, the Birkhäuser (Springer) book series “Trends in Mathematics”, Eds. M. Ruzhansky et. al., 2017.
[19] M. Gupta and D. Baweja, The Bounded Approximation Property for the Weighted Spaces of Holomorphic Mappings on Banach Spaces, Glasgow Math. J., 60(2), 307-320, 2018.
[20] E. Jorda, Weighted vector-valued holomorphic functions on Banach spaces Abstract and Applied Analysis. Vol. 2013. Hindawi Publishing Corporation, 2013.
[21] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. I, Ergeb. Math. Grenzgeb., Bd. 92, Springer, Berlin, 1977.
[22] A. Lissitsin, K. Mikkor and E. Oja, Approximation properties determined by spaces of operators and approximability in operator topologies, Illinois J. Math., 52(2), 2008, 563–582.
[23] J. Mujica, Linearization of bounded holomorphic mappings on Banach spaces. Trans. Amer. Math. Soc. 324, 2 (1991), 867-887.
[24] J. Mujica, Complex Analysis in Banach Spaces, North-Holland Math. Studies,
[25] L. Nachbin, Topology on Spaces of Holomorphic Mappings, Springer-Verlag, New York, 1969.
[26] K. F. Ng, On a theorem of Dixmier, Math. Scand. 29 (1971), 279 - 280.

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