A SHORT PROOF OF LOCAL REGULARITY OF DISTRIBUTIONAL SOLUTIONS OF POISSON’S EQUATION

GIOVANNI DI FRATTA AND ALBERTO FIORENZA

Abstract. We prove a local regularity result for distributional solutions of the Poisson’s equation with \( L^p \) data. We use a very short argument based on Weyl’s lemma and Riesz-Fréchet representation theorem.

1. Introduction

Following the pleasant introduction on the regularity theory of elliptic equations in [15], if \( u \in C^3_0(\Omega) \), \( \Omega \) being an open, bounded set in \( \mathbb{R}^n \), \( n \geq 2 \), then, using integrations by parts and Schwarz’s theorem, and identifying the continuous, compactly supported functions with their corresponding elements in \( L^2(\Omega) \),

\[
\|D^2 u\|^2_{L^2(\Omega)} = \sum_{i,j} \int_{\Omega} \partial_{ij} u \partial_{ij} u \, dx = -\sum_{i,j} \int_{\Omega} \partial_{ij} u \partial_{ij} u \, dx = \sum_{i,j} \int_{\Omega} \partial_{ii} u \partial_{jj} u \, dx = \|\Delta u\|^2_{L^2(\Omega)}.
\]

This means that if \( u \in C^3_0(\Omega) \) solves, in the classical sense, the Poisson’s equation

\[
\Delta u = f, \quad (1)
\]

then the \( L^2 \) norm of the datum \( f \) controls the \( L^2 \) norm of all second derivatives of \( u \). This statement is a typical example of a result in the theory of elliptic regularity, whose main aim is to deduce this kind of results, but under weaker \textit{a priori} hypotheses on the regularity of the solution \( u \).

1.1. The notions of weak, very weak, and distributional solutions. For a given \( f \in L^2(\Omega) \), it is natural to study equation (1) in the \textit{weak sense}. This amounts to interpret (1) as equality between elements of the dual space \( W^{-1,2}(\Omega) \) of the Sobolev space \( W^{1,2}_0(\Omega) \); their images, when tested on every element \( v \in W^{-1,2}_0(\Omega) \), must coincide. If one looks for functions \( u \) in \( W^{1,2}_0(\Omega) \) which satisfy (1) in the weak sense (i.e., for \textit{weak solutions}), the requirement is that

\[
- \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in W^{1,2}_0(\Omega). \quad (2)
\]

Since \( W^{1,2}_0(\Omega) \) is a Hilbert space, by the Riesz-Fréchet representation theorem (see, e.g., [12, p. 118], [3, Theorem 5.5 p. 135]), one gets the existence and uniqueness of the \textit{weak} solution. Note, however, that the weak formulation (2) relies on the apriori assumption that the solution \( u \) has first derivatives with the same integrability property of the datum. For such solutions, one can
prove the $W^{2,2}_{\text{loc}}(\Omega)$ regularity (see, e.g., [15, Theorem 8.2.1]). Also, we recall that under specific assumptions on the regularity of $\Omega$, one can get a better global regularity for $u$, while under regularity assumptions on the datum, one can get a better local regularity result for the solution (see, e.g., [15, Theorem 8.2.2 and Corollary 8.2.1]).

By (2) one gets the following equivalent equation (the equivalence with (2) immediately follows from a standard density argument), where now the test functions $v$ are in $C_0^\infty(\Omega)$:

$$-\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in C_0^\infty(\Omega). \quad (3)$$

When the problem is in this form, one can look for solutions of equation (1) in the space $W^{1,1}_{\text{loc}}(\Omega)$, because the $L^2(\Omega)$ integrability of the gradient is not needed to give sense to the equation. Regularity results when the datum is in $L^p(\Omega)$, $1 < p < \infty$, are classical, and rely upon the well known Calderón-Zygmund inequality from which one can get the $W^{2,p}_{\text{loc}}(\Omega)$ regularity (see, e.g., [15, Theorem 9.2.2, p. 248], or [10, Corollary 9.10, p. 235]). We mention here also the method of difference quotients introduced by Nirenberg (see, e.g., [7, 20], [5, Step 1, p. 121], and [15, Theorem 9.1.2 p.245]).

Equation (3) is a special case of a class of linear equations which can be written in the form

$$-\text{div}(A \nabla u) = f \quad (4)$$

for which it is known ([23]) that even in the case $f \equiv 0$, when $A$ is a matrix function whose entries are locally $L^1$, there exist weak solutions, assumed a priori in $W^{1,1}_{\text{loc}}(\Omega)$, which are not in $W^{1,2}_{\text{loc}}(\Omega)$.

As soon as one assumes that a solution is in $W^{1,2}_{\text{loc}}(\Omega)$, much local regularity can be gained by the celebrated De Giorgi’s theorem (see e.g. [10, Chap. 8] and references therein).

We recall, in passing, that for $\Omega \subset \mathbb{R}^2$, it is possible to prove an existence and uniqueness theorem for weak solutions of (4) (and even for a nonlinear variant) in a space slightly larger than $W^{1,2}_{\text{loc}}(\Omega)$, the so-called grand Sobolev space $W^{1,2}_{\text{loc}}(\Omega)$, when the datum is just in $L^1(\Omega)$ (see [9, Theorem A]). For an excellent survey about solutions of a number of elliptic equations, called very weak because the solutions are assumed a priori in Sobolev spaces with exponents below the natural one, the reader is referred to [11].

One can further weaken the notion of solution, and look for solutions of (1) in the space of regular distributions, that is, among elements of the dual space $\mathcal{D}'(\Omega)$ of $C_0^\infty(\Omega)$ that can be identified with elements of $L^1_{\text{loc}}(\Omega)$. In other words, their images, when computed in every element $\varphi \in C_0^\infty(\Omega)$, must coincide:

$$\int_\Omega u \Delta \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega). \quad (5)$$

Solutions in $L^1_{\text{loc}}(\Omega)$ of equation (5) are called very weak solutions.

A condition on the datum $f$ ensuring existence and uniqueness in $L^1(\Omega)$ has been found in [4, Lemma 1], namely, if $f$ is in the weighted Lebesgue space where the weight is the distance function from the boundary of $\Omega$, there exists a unique solution $u \in L^1(\Omega)$ such that

$$\|u\|_{L^1(\Omega)} \leq \|f \cdot \text{dist}(x, \partial \Omega)\|_{L^1(\Omega)}.$$

Differentiability results for very weak solutions are treated in a number of papers, see, e.g., [6, 21] and references therein (see also [8, Theorem 4.2]). However, all such references gain regularity from data in weighted Lebesgue spaces, where the distance to the boundary is involved in the weight and the domain $\Omega$ has itself some regularity assumptions.
When the datum \( f \) is identically zero, the masterpiece theorem of regularity for very weak solutions has been proved by Hermann Weyl in [28, pp. 415/6]. It dates back to 1940, well before the introduction of Sobolev spaces [18, 16], and is nowadays referred to as Weyl’s Lemma:

**Lemma 1** (H. Weyl, 1940). Let \( \Omega \subset \mathbb{R}^n \) be an open set. Suppose that \( u \in L^1_{\text{loc}}(\Omega) \) and

\[
\int_{\Omega} u \Delta \varphi \, dx = 0 \quad \forall \varphi \in C^\infty_0(\Omega).
\]

Then there exists a unique \( \tilde{u} \in C^\infty(\Omega) \) such that \( \Delta \tilde{u} = 0 \) in \( \Omega \) and \( \tilde{u} = u \) a.e. in \( \Omega \).

The proof given by Weyl in [28] is elementary and clever. Modern rephrasing of the proof can be found in classical textbooks (see, e.g., [15, Corollary 1.2.1], [5, Theorem 4.7], [24, Appendix, n.2], [27]). A beautiful note devoted entirely on this result and its development is the paper by Strook [26], where Weyl’s lemma is stated under the weaker assumption that \( u \in D'(\Omega) \). Indeed, one can go still further, and write (5) (in fact, (1)) in the form

\[
\langle u, \Delta \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in C^\infty_0(\Omega).
\]

Any solution of equation (6), is called a distributional solution of the Poisson’s equation (1). The statement proved therein is the following (see also [22, 13, 31] for a more general result, valid for a broader class of differential operators).

**Lemma 2** (Weyl’s lemma in \( D'(\Omega) \)). Let \( \Omega \subset \mathbb{R}^n \) be an open set. Suppose that \( u \in D'(\Omega) \) satisfies \( \Delta u = f \in C^\infty(\Omega) \) in the sense of (6). Then \( u \in C^\infty(\Omega) \).

The proof in [26] is very short and elegant. For our purposes, however, it is sufficient the particular case \( f \equiv 0 \), for which the proof in [26] further simplifies.

### 1.2. Contributions of present work.

In this note, we are interested in regularity results for distributional solutions of (1), i.e., solutions satisfying (6). We prove a local regularity result for distributional solutions of the Poisson’s equation with \( L^p \) data. We use a concise argument based on Weyl’s lemma and Riesz-Frèchet representation theorem. As a byproduct, we get the following classical result on very weak solutions

**Theorem 1.** If \( f \in L^1_{\text{loc}}(\Omega) \), then any solution \( u \in L^2_{\text{loc}}(\Omega) \) of \( \Delta u = f \) (i.e., satisfying (5)) belongs to \( W^{2,2}_{\text{loc}}(\Omega) \).

Theorem 1, known since 1965 (see [1, Theorem 6.2 p. 58] for a more general result, proved for uniformly elliptic operators with Lipschitz continuous coefficients), is also quoted in the Brezis book [3, Remark 25 p. 306], where it is claimed the delicateness of the proof of interior regularity of very weak solutions, based on estimates for the difference quotient operator (see [1, Def. 3.3 p. 42]). In [2, Section 3 p. 92] the reader can find a modern proof, valid for a wide class of operators, which uses a precise estimate by Hörmander in combination with a spectral representation for hypoelliptic operators. We quote also [29, Theorem 1.3], where for general operators with locally Lipschitz continuous coefficients, in the case \( f \equiv 0 \), it is shown that very weak solutions in \( L^1_{\text{loc}}(\Omega) \) are in fact in \( W^{2,p}_{\text{loc}}(\Omega) \) for every \( p \in [1, \infty) \); in [30, Proposition 1.1], the same authors, for general operators having locally Lipschitz continuous coefficients, in the case \( f \in L^p_{\text{loc}}(\Omega), 1 < p < \infty \), get that very weak solutions in \( L^1_{\text{loc}}(\Omega) \) are in fact in \( W^{2,p}_{\text{loc}}(\Omega) \).

For other results of regularity for very weak solutions of the Poisson’s equation, see, e.g., [17, Section 7.2 p. 223] and [19, Section 4.1 p. 198]. In particular, we mention here that, following
Hilbert, one can ask whether a solution, being a distribution, is analytic in the case where the right-hand side \( f \) is analytic: the answer is positive for equation (4) when \( A \) is analytic, see [14].

2. **Regularity of very weak solutions of Poisson’s equation in the \( L^2 \)-setting**

The main ingredient is stated in the following result which, remarkably, is essentially based on Weyl’s lemma.

**Lemma 3.** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set, and let \( u \in \mathcal{D}'(\Omega) \). Then

\[
\Delta u \in W^{-1,2}(\Omega) \implies u \in W^{1,2}_{\text{loc}}(\Omega).
\]

**Proof.** Since \( \Delta u \in W^{-1,2}(\Omega) \), by Riesz representation theorem, there exists \( v \in W^{1,2}_0(\Omega) \) such that \( \Delta v = \Delta u \) in \( \mathcal{D}'(\Omega) \). In particular, \( \Delta (u - v) = 0 \) in \( \mathcal{D}'(\Omega) \). By Weyl’s lemma (Lemma 2 used with \( f \equiv 0 \)), we know that \( u - v \in C^\infty(\Omega) \). Hence \( u = (u - v) + v \in C^\infty(\Omega) + W^{1,2}_0(\Omega) \subseteq W^{1,2}_{\text{loc}}(\Omega) \).

We remark that solutions of Dirichlet problems by Hilbert spaces methods are a classic matter for weak solutions, see, e.g., [12, p. 117]. Again, for weak solutions, we quote [25, Lemma 2.1 p.48], where from the assumption of being locally in a Sobolev space, the authors get a better local regularity, still in Sobolev spaces.

**Theorem 2.** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set, and let \( u \in \mathcal{D}'(\Omega) \). If \( \Delta u \in L^2_{\text{loc}}(\Omega) \), then \( \nabla u \in W^{1,2}_{\text{loc}}(\Omega) \).

If, in addition, \( u \in L^2_{\text{loc}}(\Omega) \), then \( u \in W^{2,2}_{\text{loc}}(\Omega) \).

**Proof.** Due to the local character of the result, we can assume \( \Delta u \in L^2(\Omega) \). Therefore, it is sufficient to note that if \( \Delta u = f \) with \( f \in L^2(\Omega) \) then, for any distributional partial derivative of \( u \), we have \( \Delta(\nabla u) = \nabla f \) with \( \nabla f \in W^{-1,2}(\Omega) \). By the previous lemma, we get \( \nabla u \in W^{1,2}_{\text{loc}}(\Omega) \).

Thus, \( u \in W^{2,2}_{\text{loc}}(\Omega) \) if we assume \( u \in L^2_{\text{loc}}(\Omega) \).

3. **Regularity of very weak solutions of Poisson’s Equation in the \( L^p \)-setting**

We point out that the same argument shows that if \( f \in L^p(\Omega), 1 < p < \infty \), then \( u \in W^{2,2}_{\text{loc}}(\Omega) \). Precisely, the following result holds:

**Theorem 3.** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set, and let \( u \in \mathcal{D}'(\Omega) \). If \( \Delta u \in L^p(\Omega) \), then \( \nabla u \in W^{1,p}_{\text{loc}}(\Omega) \).

If, in addition, \( u \in L^p_{\text{loc}}(\Omega) \), then \( u \in W^{2,2}_{\text{loc}}(\Omega) \).

**Proof.** Indeed (see, e.g., [25, pp. 10-11]), if \( 1/p + 1/q = 1 \) and \( F \in W^{-1,q'}(\Omega) \) then there exists a function \( u_F \in W^{1,p}(\Omega) \) such that

\[
- \int_\Omega \nabla u_F \cdot \nabla \varphi = \langle F, \varphi \rangle
\]

for every \( \varphi \in W^{1,q}_{0}(\Omega) \). Note that this can be considered as the \( q \)-exponent version of the Riesz representation theorem. Now, (8) implies that for any \( F \in W^{-1,q'}(\Omega) \) there exists a distribution in \( v_F \in W^{1,p}(\Omega) \) such that \( \Delta v_F = F \) in \( \mathcal{D}'(\Omega) \). After that, assume that \( f \in L^p(\Omega) \) and \( u \in \mathcal{D}'(\Omega) \) is a distributional solution of (6). Then \( \nabla u \) satisfies \( \Delta(\nabla u) = \nabla f \) with \( \nabla f \in W^{-1,q'}(\Omega) \). Therefore, as in Lemma 3, \( \nabla u \in W^{1,p}_{\text{loc}}(\Omega) \), and we conclude.
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