Rational curves of degree at most 9
on a general quintic threefold

TRYGVE JOHNSEN and STEVEN L. KLEiman

Abstract. We prove the following form of the Clemens conjecture in low degree. Let \( d \leq 9 \), and let \( F \) be a general quintic threefold in \( \mathbb{P}^4 \). Then (1) the Hilbert scheme of rational, smooth and irreducible curves of degree \( d \) on \( F \) is finite, nonempty, and reduced; moreover, each curve is embedded in \( F \) with normal bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \), and in \( \mathbb{P}^4 \) with maximal rank. (2) On \( F \), there are no rational, singular, reduced and irreducible curves of degree \( d \), except for the 17,601,000 six-nodal plane quintics (found by Vainsencher). (3) On \( F \), there are no connected, reduced and reducible curves of degree \( d \) with rational components.

1. Introduction

Around ten years ago, Clemens posed a conjecture about the rational curves on a general quintic threefold \( F \) in complex \( \mathbb{P}^4 \). Immediately afterwards, S. Katz [16] considered the conjecture in the following form: the Hilbert scheme of rational, smooth and irreducible curves of degree \( d \) on \( F \) is finite, nonempty and reduced; in fact, each curve is embedded with normal bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \). Katz proved this statement for \( d \leq 7 \). Then Clemens [6, p.639] strengthened the conjecture by adding this assertion: all the rational, reduced and irreducible curves on \( F \) are smooth and mutually disjoint. This additional assertion is not completely true; Vainsencher [25, §6.2] proved that \( F \) contains 17,601,000 six-nodal quintic plane curves. By developing Katz’s argument through an extensive case-by-case analysis, we’ll prove a corrected form of Clemens’ stronger conjecture for \( d \leq 9 \). In Section 2, we’ll prove

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Katz’s form of the conjecture. In Section 3, we’ll prove that $F$ contains no singular, rational, reduced and irreducible curves of degree $d$, except for the six-nodal plane quintics. Finally, in Section 4, we’ll prove that $F$ contains no connected, reduced and reducible curves of degree $d$ whose components are all rational.

Consider the incidence scheme $I_d$ that parametrizes the pairs $(C, F)$ where $C$ is a rational, smooth and irreducible curve of degree $d$ in $\mathbb{P}^4$ and where $F$ is a quintic threefold containing $C$. For every $d$, Katz [16, p. 152] proved that his form of Clemens’ conjecture is true if $I_d$ is irreducible and if there exists at least one pair $(C, F) \in I_d$ such that $F$ is smooth and such that $C$, viewed as $\mathbb{P}^1$, is embedded with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. He [16, p. 153] proved the existence of such a pair for every $d$ (Clemens [5, Thm. (1.27), p. 26] had just proved it for infinitely many $d$, and Katz observed that the general case follows via Clemens’ deformation-theoretic argument from an existence result of Mori’s). And Katz proved the irreducibility of $I_d$ for $d \leq 7$ by making the following two observations: (1) the space $M_d$ of all $C$ is irreducible for every $d$; and (2) every fiber of the first projection $I_d \rightarrow M_d$ is, thanks to [11], a projective space of the same dimension for $d \leq 7$. For $d \geq 8$, however, some fibers are bigger than others. Nevertheless, we’ll prove in Section 2 that there aren’t too many big fibers for $d = 8, 9$. Thus we’ll obtain the irreducibility and so Katz’s form of Clemens’ conjecture; see Proposition (2.2) and Theorem (2.1). As a bonus, we’ll obtain Corollary (2.5), which asserts that each $C$ on a general $F$ has some additional important properties; notably, each $C$ is embedded in $\mathbb{P}^4$ with maximal rank, and the restricted twisted sheaf of differentials $\Omega^1_{\mathbb{P}^4}(1)|C$ has a certain splitting type.

Katz’s form of Clemens’ conjecture was recently considered independently by Nijsse [21], by Huybrechts [15], and by Westhoff [27, p. 40]. Nijsse too proved the irreducibility of $I_d$ for $d \leq 9$ by following Katz’s approach, but he handled the curves with big fibers in a rather different manner than we’ll do in Section 2. Huybrechts and Westhoff did not follow Katz’s approach, but used more deformation theory. Each assumed $d \leq 7$, and proved this slightly weaker statement: if the Hilbert scheme of rational, smooth and irreducible curves of degree $d$ on a general $F$ is nonempty, then it’s finite and reduced. Also, each didn’t recover the irreducibility of $I_d$. However, for the other Calabi–Yau threefolds $F$ that are general complete intersections, each was able to prove the preceding statement for correspondingly small $d$, namely, for $d \leq 5$ for $F$ of type $(3,3)$, and for $d \leq 4$ for types $(2,4), (2,2,3)$ and $(2,2,2,2)$. In this connection, Oguiso [22] proved, by modifying Clemens’ argument for quintics, that the Hilbert scheme is, in fact, nonempty for an $F$ of type $(2,4)$.

The above ranges of $d$, established independently by Huybrechts and
by Westhoff, can, almost certainly, be extended somewhat by proceeding instead along the lines in Section 2. The authors have checked the key details, and believe the following ranges come out: \( d \leq 7 \) for types (3,3) and (2,4), and \( d \leq 6 \) for types (2,2,3) and (2,2,2,2). In fact, except for the case \( d = 6 \) and \( F \) of type (2,2,2,2), the incidence scheme \( I_d \) of pairs \((C,F)\) is, almost certainly, irreducible, generically reduced, and of the same dimension as the space \( \mathbf{P} \) of \( F \). In the exceptional case, the authors cannot rule out the presence of a second component, whose \( C \)'s lie on quadric surfaces. In any event, if the second projection \( I_d \to \mathbf{P} \) is surjective, then by Sard's theorem it is étale over a dense open set of \( F \)'s, and so for these \( F \) the Hilbert scheme is finite and reduced.

To prove that a general quintic threefold does not contain curves \( C \) of a given sort, we'll form the incidence scheme of all pairs \((C,F)\) where \( F \) is an arbitrary quintic, and we'll show that this scheme does not dominate the \( \mathbf{P}^{125} \) of all quintics \( F \). In Section 3, we'll treat \( C \)'s that are reduced and irreducible, rational and singular. Here, for convenience, we'll work with parametrized mappings of \( \mathbf{P}^1 \) into \( \mathbf{P}^4 \), not just their images \( C \). Since each \( C \) is the image of every mapping in a four parameter family, we'll need to show that the new incidence scheme is of dimension at most 128. We'll succeed in doing so for all such \( C \) of degree at most 9, with one exception: the six-nodal quintic plane curves. However, Vainsencher [25, §6.2] proved that a general quintic threefold contains 17,601,000 such plane curves. Each is the intersection of the threefold with a sixfold tangent plane. Therefore, we're led to ask if a general quintic threefold also contains sixteen-nodal curves of degree 10, arising from tangent quadric surfaces. The answer is no, see Remark (3.6).

Consider a connected, reduced and reducible curve \( C \) of degree at most 9 with rational components. In Section 4, we'll prove that \( C \) cannot lie on a general quintic threefold. To do so, we'll have a considerable advantage, gained from our work in Sections 2 and 3; namely, we'll be able to assume that \( C \) has two components, each of which is one of the following: a line, a conic, a six-nodal plane quintic, a twisted cubic, or a smooth curve that spans \( \mathbf{P}^4 \) and on which the restriction of \( \Omega_{\mathbf{P}^4}^1(1) \) has a certain splitting type. We'll parametrize the various \( C \) of this sort by parametrizing their components individually using the Hilbert scheme of \( \mathbf{P}^4 \). We'll bound the dimension of the incidence scheme in two steps. First we'll consider those \( C \) whose two components meet in a subscheme of length \( n \), and prove that they form a subset of codimension at least \( n \). Second, we'll bound the number of independent quintic threefolds containing a given \( C \). To do so, we'll need to work harder than ever to prove that \( C \) is 6-regular. Curiously, the regularity of \( C \) seems to depend only on the geometry of its components individually, and not on how they intersect.
Clemens’ full conjecture [6, p.639] has one additional assertion: the number $n_d$ of rational curves of degree $d$, on the general quintic threefold, is divisible by $5^3 \cdot d$. This assertion holds for $d = 1, 2$; indeed, $n_1$ was found by Schubert, and $n_2$ by Katz [16, Thm. 3.1, p.154]. However, Clemens hedged on the factor of $d$, and in fact, $n_3$ is not divisible by 3; indeed, $n_3$ was found by Ellingsrud and Stromme [9] and by Candelas, De la Ossa, Green, and Parkes [3]. The latter four were inspired by considerations of mathematical physics. They predicted values of $n_d$ for $d \leq 10$, and these values are all divisible by $5^3$. Moreover, they gave an algorithm yielding numbers for all $d$. These numbers were shown, by Lian and Yau [19], to be divisible by $5^3$, at least if $d$ isn’t a multiple of 5. On the other hand, Kontsevich [17, §2.2] gave a somewhat different algorithm; although he too was inspired by mathematical physics, his treatment is more algebraic-geometric, and its numbers clearly count both smooth and nodal curves. For the other four types of Calabi–Yau complete intersections of dimension 3, Libgober and Teitelbaum [20] followed in the footsteps of Candelas et al., and developed algorithms. At any rate, all this enumeration has given rise to renewed interest in Clemens’ conjecture in its full form.

Finally, a few words are in order concerning the choice of the ground field and the meaning of the word “general.” As usual, we’ll say that a general quintic threefold in $\mathbf{P}^4$ has a given property if there exists a dense Zariski open subset of $\mathbf{P}^{125}$ whose points represent threefolds with this property. We’ll consider properties that concern curves of a fixed degree, never curves of infinitely many different degrees simultaneously. When the latter is done, it may be necessary to consider the intersection of a countably infinite collection of dense Zariski open subsets. Such an intersection is nonempty if the ground field is uncountable — for example, the complex numbers — and then it is common to call a threefold “generic” if it’s represented by a point of the intersection.

The field of complex numbers is virtually always the ground field of choice in treatments of Clemens’ conjecture. In this paper, however, all our work is algebraic-geometric in nature, and makes sense, although it’s not always valid, in any characteristic. The need for characteristic zero enters into our work in three ways: (i) our use, at the beginning of Section 2, of Katz’s theorem on the existence of rational smooth curves on a general quintic threefold, (ii) our use of the principle of uniform position, or the trisecant lemma, for a singular curve to prove (3-8) via (2-3), and (iii) our use of the Castelnuovo–Halphen bounds on the genera of singular curves, made in the two paragraphs before the statement of Lemma (3.2) and again to prove (3-8). Thus, in positive characteristic, Proposition (2.2) is valid, but possibly the projection $I_d \to \mathbf{P}^{125}$ is not surjective or not separable. So a general quintic threefold con-
tains at most finitely many rational, smooth and irreducible curves of
degree at most 9, but possibly there are none, or if there are some, then
they may move infinitesimally. Also, the threefold may possibly contain
singular, rational, reduced and irreducible curves that are “strange” in
the sense that every tangent line is a trisecant. Therefore, we’ll assume
implicitly from now on that the ground field is algebraically closed and
of characteristic zero.

2. Smooth curves

In this section, we’ll prove the following theorem, which affirms Katz’s
form of Clemens’ conjecture in degree $d$ at most 9. To do so, we’ll
prove Proposition (2.2), which asserts the irreducibility of the incidence
scheme of pairs $(C, F)$ where $C$ is a rational and smooth, reduced and
irreducible curve of degree $d$ in $\mathbb{P}^4$ and where $F$ is a quintic threefold
containing $C$. As a bonus, we’ll obtain Corollary (2.5), which gives
some additional properties possessed by the $C$ on a general $F$. In the
remaining two sections, we will complete our treatment of Clemens’
conjecture by proving two theorems about singular curves.

**Theorem (2.1)** Let $d \leq 9$, and let $F$ be a general quintic threefold
in $\mathbb{P}^4$. In the Hilbert scheme of $F$, form the open subscheme of rational,
smooth and irreducible curves $C$ of degree $d$. Then this subscheme is
finite, nonempty, and reduced; in fact, each $C$ is embedded in $F$ with
normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

To prove the theorem, we will use the following notation:

(a) $M_d$ will denote the open subscheme of the Hilbert scheme of $\mathbb{P}^4$
parametrizing the rational, smooth and irreducible curves $C$ of
degree $d$;

(b) $\mathbb{P}^{125}$ will denote the projective space parametrizing the quintic
threefolds $F$ in $\mathbb{P}^4$;

(c) $I_d$ will denote the “incidence” subscheme of $M_d \times \mathbb{P}^{125}$ of pairs
$(C, F)$ such that $C \subset F$;

(d) $M_{d,i}$ will denote the locally closed subset of $M_d$ parametrizing
the curves $C$ such that $h^1(\mathcal{I}_C(5)) = i$ where $\mathcal{I}_C$ denotes the ideal
of $C$ in $\mathbb{P}^4$;

(e) $I_{d,i}$ will denote the preimage in $I_d$ of $M_{d,i}$.

By virtue of Katz’s work (see the beginning of Section 1), Theorem (2.1)
is a consequence of the following proposition, whose proof will occupy
most of the rest of this section.

**Proposition (2.2)** For $d \leq 9$, the incidence scheme $I_d$ is irreducible
of dimension 125, and it projects onto $M_d$. 
First, let \( d \) be arbitrary. Recall from [5, pp. 25–6] and [16, p. 152] that \( M_d \) is irreducible of dimension \( 5d + 1 \) and that every component of \( I_d \) has dimension at least 125. On the other hand, consider \( I_d,i \). Its fiber over a curve \( C \) in \( M_d,i \) is a projective space of dimension \( h^0(\mathcal{I}_C(5)) - 1 \).

This dimension can be computed using the standard exact sequence,

\[
0 \to H^0(\mathcal{I}_C(5)) \to H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \to H^0(\mathcal{O}_C(5)) \to H^1(\mathcal{I}_C(5)) \to 0.
\]

Therefore,

\[
\dim I_d,i = \dim M_d,i + (126 - (5d + 1) + i) - 1. \tag{2-1}
\]

If \( d \leq 7 \), then the theorem on p. 492 of [11] yields \( M_d,0 = M_d \), and it follows that \( I_d \) is irreducible of dimension 125. For \( d = 8,9 \) the following lemma implies that \( M_d,0 \) is open and nonempty, that \( \dim I_d,0 = 125 \), and that \( \dim I_d,i < 125 \) for \( i > 0 \); hence, \( I_d,0 \) is irreducible of dimension 125, and its closure is \( I_d \). Thus Proposition (2.2) is proved, given Lemma (2.3).

**Lemma (2.3)** For \( d = 8,9 \), if \( i > 0 \) and if \( M_d,i \) is nonempty, then

\[
\text{cod}(M_d,i, M_d) > i.
\]

To prove the lemma, we begin with some general observations. Fix an arbitrary rational, smooth and irreducible curve \( C \) of any degree \( d \) in \( \mathbb{P}^4 \). First, if \( d \geq 3 \), then \( C \) can lie in no plane because its arithmetic genus \( p_a C \) vanishes. Moreover, if \( d \geq 4 \), then \( C \) can lie in no 2-dimensional quadric cone by [13, Exr. 2.9, p. 384]. Second, if \( C \) lies in a hyperplane \( G \), say with ideal \( \mathcal{I}_C/G \), then

\[
h^1(\mathcal{I}_C(5)) = h^1(\mathcal{I}_C/G(5)). \tag{2-2}
\]

Indeed, this equality results from the exact sequence of twisted ideals,

\[
0 \to \mathcal{I}_G(5) \to \mathcal{I}_C(5) \to \mathcal{I}_C/G(5) \to 0,
\]

because its first term is equal to \( \mathcal{O}_{\mathbb{P}^4}(4) \).

Assume \( d = 8 \). By the theorem on p. 492 of [11], if \( C \) does not lie in a hyperplane, then \( C \) is 6-regular, and so \( h^1(\mathcal{I}_C(5)) = 0 \). Hence, \( M_8 - M_{8,0} \) is contained in the closed set \( N_8 \) of curves in hyperplanes. Clearly,

\[
\text{cod}(N_8, M_8) = (5 \times 8 + 1) - (4 \times 8 + 4) = 8 - 3 = 5.
\]

In particular, \( \text{cod}(M_{8,1}, M_8) > 1 \) as asserted.

Suppose that \( h^1(\mathcal{I}_C(5)) \geq 2 \). Say that \( C \) lies in the hyperplane \( G \). Then \( h^1(\mathcal{I}_C/G(5)) \geq 2 \) by (2-2). Hence, the table on p. 504 of [11] says that \( h^1(\mathcal{I}_C/G(5)) = 5 \). Therefore, if \( M_{8,i} \) is nonempty, then \( i \) is 0, 1, or 5. Now, the table also says that \( C \) lies in a smooth quadric surface \( Q \)
contained in $G$. Since $C$ is smooth, rational, and of degree 8, it is of type $(7,1)$ on $Q$. Hence, $C$ varies in a system of dimension $8 \times 2 - 1$, or 15, on $Q$. Obviously, $Q$ varies in a system of dimension 9 on $G$, and $G$ varies is a system of dimension 4 in $\mathbb{P}^4$. Therefore,

$$\text{cod}(M_{8,5}, M_8) = (5 \times 8 + 1) - (15 + 9 + 4) = 13 > 5.$$  

Thus the lemma holds when $d = 8$.

Assume $d = 9$. There are five possible cases:

1. $C$ lies in no hyperplane, and has no 7-secant line;
2. $C$ lies in some hyperplane $G$, and has no 7-secant line;
3. $C$ lies in no hyperplane, and has a 7-secant line;
4. $C$ lies in some smooth quadric surface $Q$ (and has a 7-secant line);
5. $C$ lies in some hyperplane $G$, but in no smooth quadric surface, and has a 7-secant line.

Here and below, we use the definition given in the middle of p. 501 of [11], and say that a line is $n$-secant to a curve if the two schemes contain a common subscheme of length at least $n$. Now, each of these five cases will be considered in turn.

In Case (1), the theorem on p. 492 of [11] yields that $h^1(I_C(5)) = 0$. In Case (2), Théorème 0.1 on p. 30 of [8] yields the same vanishing. Thus, in either case, $C \in M_{9,0}$.

In Case (3), the table on p. 504 of [11] says that $h^1(I_{C/G}(5)) = 1$. On the other hand, $C$ belongs to the subset of $M_d$ of curves with 7-secant lines, and this subset has codimension at least 8 by Lemma (2.4) below. Thus the curves that fall into Case (3) form a subset of $M_{d,1}$ of codimension at least 8 in $M_d$.

In Case (4), since $C$ is smooth, rational, and of degree 9, it is of type $(8,1)$ on $Q$. So $I_{C/Q}(5) = \mathcal{O}_Q(-3,4)$. So the Künneth formula yields

$$h^1(I_{C/Q}(5)) = 0 \times 0 + 2 \times 5 = 10.$$  

The linear span of $Q$ is a hyperplane $G$. Moreover,

$$h^1(I_{C/Q}(5)) = h^1(I_{C/G}(5)) = h^1(I_C(5));$$

indeed, the second equality is (2-2), and the first can be proved similarly. Now, $C$ varies in a system of dimension 17 on $Q$. Hence, the various $C$ in Case (4) form a subset $S_4$ of $M_{9,10}$, and

$$\text{cod}(S_4, M_9) = (5 \times 9 + 1) - (17 + 9 + 4) = 16 > 10.$$  

In Case (5), by Lemma (2.4) below, the various $C$ in a fixed hyperplane $G$ form a subset of codimension at least 3 in the Hilbert scheme of smooth rational curves of degree 9 in $G$. Since this Hilbert scheme has dimension $4 \times 9$, or 36, and since $G$ varies in a system of dimension 4, the various
C form a subset of $M_g$ of codimension at least $46 - (36 + 4) + 3$, or 9. So it suffices to fix $C$ and to prove that $h^1(\mathcal{I}_C(5)) < 9$, or equivalently by (2-2), that $h^1(\mathcal{I}_{C/G}(5)) < 9$.

Choose in $G$ a plane $H$ that meets $C$ in 9 distinct points, no three of which are collinear. Such an $H$ exists by [1, Lem., p. 109] or by [24, Lem. 1.1, p. 566] (the latter is valid in arbitrary characteristic; the curve may be singular, but must not be strange). Alternatively, such an $H$ exists by Bertini’s theorem and the trisecant lemma (the latter asserts that the trisecants form, at most, a 1-parameter family; it is proved for an arbitrary reduced and irreducible space curve in characteristic 0 in [18, p. 135], and it follows from [13, Prp. 3.8, p. 311] and [13, Thm. 3.9, p. 312] for a smooth, reduced and irreducible space curve in arbitrary characteristic). Let $k \geq 4$. Then the 9 points impose independent conditions on the system of curves of degree $k$ in $H$ by [1, Lem., p. 115]. Hence, in the long exact sequence,

$$H^0(\mathcal{O}_H(k)) \to H^0(\mathcal{O}_{C \cap H}(k)) \to H^1(\mathcal{I}_{C \cap H,H}(k)) \to H^1(\mathcal{O}_H(k)),$$

the first map is surjective. However, the last term vanishes. Therefore,

$$H^1(\mathcal{I}_{C \cap H,H}(k)) = 0.$$

Consequently, the exact sequence of sheaves,

$$0 \to \mathcal{I}_{C/G}(k - 1) \to \mathcal{I}_{C/G}(k) \to \mathcal{I}_{C \cap H,H}(k) \to 0,$$

yields

$$h^1(\mathcal{I}_{C/G}(3)) \geq h^1(\mathcal{I}_{C/G}(4)) \geq h^1(\mathcal{I}_{C/G}(5)) \geq \cdots. \quad (2-3)$$

Consider the standard exact sequence of sheaves,

$$0 \to \mathcal{I}_{C/G}(k) \to \mathcal{O}_G(k) \to \mathcal{O}_C(k) \to 0.$$

Since $H^1(\mathcal{O}_G(k)) = 0$, taking cohomology yields, for $k \geq 0$,

$$h^0(\mathcal{I}_{C/G}(k)) = \binom{k + 3}{3} - (9k + 1) + h^1(\mathcal{I}_{C/G}(k)). \quad (2-4)$$

Now, proceeding by contradiction, assume $h^1(\mathcal{I}_{C/G}(5)) \geq 9$. Then,

$$h^0(\mathcal{I}_{C/G}(3)) \geq 1 \text{ and } h^0(\mathcal{I}_{C/G}(4)) \geq 7.$$

It follows, as will be proved in the next three paragraphs, that

$$h^0(\mathcal{I}_{C/G}(6)) \geq 31. \quad (2-5)$$

Hence, $h^1(\mathcal{I}_{C/G}(6)) \geq 2$. However, the table on p. 504 of [11] says that $h^1(\mathcal{I}_{C/G}(6))$ is equal to 1 if $C$ has an 8-secant line and to 0 if not. Thus there is a contradiction.
Set $H(k) := H^0(\mathcal{I}_C/G(k))$, and view its elements as homogeneous polynomials of degree $k$ in four variables that vanish on $C$. Assume that $H(3)$ contains a nonzero cubic $K$. Then $K$ is irreducible because $C$ lies in no plane, in no quadric cone, and in no smooth quadric surface. Now, as $L$ ranges over the linear forms, the products $KL$ form a 4-dimensional subspace of $H(4)$. Assume that, modulo this subspace, $H(4)$ contains two linearly independent quartics $Q_1$ and $Q_2$. Then $K$ divides no nontrivial linear combination $Q$ of the $Q_i$. So, since $K$ is irreducible, it divides no nonzero product $QT$ where $T$ is a quadric. Let $X$ be the set of all products $QT$, and $Y$ the set of all products $KK'$ where $K'$ is a cubic. Then, therefore, $X \cap Y = \{0\}$.

Each nonzero combination $Q$ is irreducible. Indeed, $Q$ cannot be the product of two quadrics since $C$ lies in no quadric surface. Suppose $Q = K'L$ where $K'$ is a cubic. Then $K'$ is not a multiple of $K$, but $K'$ vanishes on $C$ because $L$ cannot. So $C$ is contained in the complete intersection $C'$ defined by $K$ and $K'$. Hence $C$ is equal to $C'$ because both have degree 9. However, $p_aC' = 10$ by [13, Exr. 7.2(d), p. 54], whereas $p_aC = 0$. Thus each $Q$ is irreducible.

Therefore, if $Q_1T = Q_2T'$, then $T$ and $T'$ vanish. Hence the products $Q_1T$ and $Q_2T'$ form two 10-dimensional linear spaces that meet only in 0 and that lie in $X$. Therefore, $X$ is a cone of dimension at least 11. On the other hand, $Y$ is a linear space of dimension 20. Since $X$ and $Y$ meet in a single point, any smooth variety containing both of them has dimension at least 31; in particular, their linear span does. Thus (2-5) holds. Thus Lemma (2.3) is proved.

To complete the proof of Proposition (2.2), we have to prove the following lemma.

**Lemma (2.4)** Fix $d \geq 4$ and $s \geq 3$. Fix $b$ with $d > b \geq 3$. In the Hilbert scheme of rational, smooth and irreducible curves of degree $d$ in $\mathbf{P}^s$, form the subset of curves with a $b$-secant line. Then this subset has codimension at least $(s-1)(b-2) - b$.

Indeed, consider the space $M$ of parametrized embeddings of $\mathbf{P}^1$ into $\mathbf{P}^s$ with degree $d$. Let $\Gamma$ in $\mathbf{P}^1 \times \mathbf{P}^s \times M$ be the family of graphs. Then the projection $\Gamma \to \mathbf{P}^1 \times M$ is an isomorphism; so $\Gamma$ is flat over $M$. On the other hand, the projection $\Gamma \to \mathbf{P}^s \times M$ is an embedding, because its fibers over $M$ are embeddings. Hence, this embedding of $\Gamma$ defines a map of $M$ into the Hilbert scheme in question, and the map is obviously surjective. Since codimension can only decrease under pullback along a surjection, we may replace the Hilbert scheme by $M$.

Say that the coordinate tuples on $\mathbf{P}^1$ are $(t, u)$, on $\mathbf{P}^s$ are $(X_0, \ldots, X_s)$,
and on $M$ are $(a_{i,j})$. Then the maps are defined by the $s + 1$ equations,

$$X_i = a_{i,0}t^d + a_{i,1}t^{d-1}u + \cdots + a_{i,d}u^d.$$ 

Fix a divisor $D$ of degree $b$ on $\mathbf{P}^1$. Say $D = P_1 + \cdots + P_b$ and $P_k = (t_k, u_k)$. Then, for each $k$, the above equations become

$$X_i = L_{i,k}(a_{i,0}, a_{i,1}, \ldots, a_{i,d})$$

where the $L_{i,k}$ are linear forms. If $P_k$ is repeated, then replace the $n$th occurrence of $L_{i,k}$ by its $(n - 1)$st derivative.

For each $i$, the forms $L_{i,k}$ for $1 \leq k \leq b$ are independent because $D$ imposes $b$ conditions on the forms of degree $d$ in $(t, u)$. Since the variables $a_{i,j}$ change with $i$, the forms $L_{i,k}$ for all $i, k$ are independent.

Now, consider the condition that $D$ is carried to a length-$b$ subscheme of a line in $\mathbf{P}^s$; it becomes the condition $\text{rank}[L_{i,k}] \leq 2$. Since the $L_{i,k}$ are independent, they may be viewed as defining a linear map from the affine space of $s + 1$ by $d + 1$ matrices onto the affine space of $s + 1$ by $b$ matrices. Hence the condition defines a subset of codimension precisely $(s - 1)(b - 2)$ in $M$. As $D$ varies, the corresponding subsets sweep out a subset of codimension $(s - 1)(b - 2) - c$ where $c \leq b$. Thus the lemma is proved.

**Corollary (2.5)** Let $F$ be a general quintic threefold in $\mathbf{P}^4$, and let $C$ be a rational, smooth and irreducible curve of degree $d \leq 9$ on $F$.

1. Then $C$ is embedded in $\mathbf{P}^4$ with maximal rank.

2. Form the restriction to $C$ of the twisted sheaf of differentials of $\mathbf{P}^4$. Then this locally free sheaf has generic splitting type; namely, if $d = 4n + m$ where $0 \leq m < 4$, then

$$\Omega_{\mathbf{P}^4}^1(1)|_C = \mathcal{O}_C(-n - 1)^m \oplus \mathcal{O}_C(-n)^{4-m}.$$ 

3. If $d \leq 4$, then $C$ is a rational normal curve of degree $d$, and if $4 \leq d \leq 9$, then $C$ spans $\mathbf{P}^4$.

4. If $d = 1$, then $C$ is 1-regular; if $2 \leq d \leq 4$, then $C$ is 2-regular; if $5 \leq d \leq 7$, then $C$ is 3-regular; and if $8 \leq d \leq 9$, then $C$ is 4-regular.

Indeed, first let’s make a general observation. Given an arbitrary proper closed subset $S$ of $M_d$, its preimage in $I_d$ is proper and so has dimension at most 124 thanks to Proposition (2.2). Hence this preimage does not project onto the $\mathbf{P}^{125}$ of quintics. So, since $F$ is general, $C \notin S$.

To prove (2), in $M_d$ form the subset $S$ of curves such that the sheaf in question does not have generic splitting type. Then $S$ is a proper closed subset by a theorem of Verdier’s [26, Thm., p. 139] (see also [23, Thm. 1, p. 181]). So $C \notin S$. Thus (2) holds.

To prove (3), in $M_d$ form the subset $S$ of curves not spanning $\mathbf{P}^4$. Clearly, $\dim S \leq 4d + 4$. If $d \geq 4$, then $4d + 4 < 5d + 1$, and so $S$
is proper. Thus the second assertion of (3) holds. The first assertion follows if \( d = 4 \), and is easy to check directly if \( d \leq 3 \) (see [13, Exr. 3.4, p. 315]).

To prove (1) and (4), recall that, by definition, \( C \) is embedded in \( \mathbb{P}^4 \) with maximal rank if, for all \( k \geq 1 \), the natural map,

\[
H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(k)) \to H^0(C, \mathcal{O}_C(k)),
\]

is injective or surjective (or both). It follows from the standard long exact sequence of cohomology that (1) implies (4).

First, suppose \( d \leq 4 \). Then \( C \) is a rational normal curve of degree \( d \) by (3). So (1) holds because \( C \) is essentially the \( d \)-uple image of \( \mathbb{P}^1 \); alternatively, (1) holds by the theorem on p. 492 of [11]. Then (4) follows; alternatively, (4) holds by Remark (1) on p. 497 of [11].

Finally, suppose \( d \geq 4 \). In \( M_d \) form the subset \( S \) of curves that either don’t span \( \mathbb{P}^4 \) or aren’t of maximal rank. Then \( S \) is proper by the proof of (3) and by the maximal-rank theorem [2, Thm. 1, p. 215]. So \( C \notin S \). Thus (1) holds, and (4) follows. (Curiously, (4) also follows from (2) and from Proposition 1.2 on p. 494 of [11], except when \( d = 6, 7, 9 \). When \( d = 6, 7 \), only 4-regularity comes out this way, and when \( d = 9 \), only 5-regularity does.)

3. Singular curves

In this section, we’ll treat singular curves by proving the following theorem, which complements Theorem (2.1). In the next section, we’ll treat reducible curves.

**Theorem (3.1)** On a general quintic hypersurface in \( \mathbb{P}^4 \), there are no rational and singular, reduced and irreducible curves of degree at most 9, other than the six-nodal plane quintics.

Indeed, let \( M_{d,r,g}^r \) be the space of all parametrized mappings of \( \mathbb{P}^1 \) into \( \mathbb{P}^4 \) of degree \( d \) with these (locally closed) properties: the mappings are birational onto their images \( C \), and each such \( C \) spans an \( r \)-plane and has arithmetic genus \( g \). Note that each \( C \) is the image of every mapping in a four parameter family — namely, the orbit under the parametrized automorphisms of \( \mathbb{P}^1 \) (in fact, the orbit includes every mapping with image \( C \) because of the hypothesis of birationality). Let \( I_{d,r,g}^r \) denote the incidence locus in \( M_{d,r,g}^r \times \mathbb{P}^{125} \); it’s points represent the pairs consisting of a mapping and a quintic such that the image of the former lies in the latter. Except for \( (r, d, g) = (2, 5, 6) \), the theorem asserts, in other words, that for \( r = 2, 3, 4 \) the projection \( I_{d,r,g}^r \to \mathbb{P}^{125} \) is not surjective if \( d \leq 9 \) and \( g \geq 1 \) (if \( g = 0 \), then \( C \) is smooth). Since the projection’s fibers all have dimension at least 4, to prove the that it’s not surjective, we need only establish the bound \( \dim I_{d,r,g}^r < 129 \).
Consider the other projection, $I^{r,g}_d \to M_{d,i}^{r,g}$. Its fiber over a mapping with image $C$ is a projective space of dimension $h^0(I_C(5)) - 1$. Let $M_{d,i}^{r,g}$ denote the (locally closed) subset of $M_{d,i}^{r,g}$ where $h^1(I_C(5)) = i$, and let $I^{r,g}_d$ denote its inverse image in $I^{r,g}_d$. Then, mutatis mutandis, the proof of Equation (2-1) yields

$$\dim I^{r,g}_d \leq \dim M_{d,i}^{r,g} + (126 - (5d + 1 - g) + i) - 1.$$ 

Therefore, to prove the theorem, it suffices to establish the bound,

$$\dim M_{d,i}^{r,g} < 5d + 5 - g - i,$$  \hspace{1cm} (3-1)

unless $r = 2$ and $d = 5$, when a little more analysis is required. The proof proceeds via a case-by-case analysis. The case where $d = 9$ and $r = 3$ is special, and will be handled after the proof of Lemma (3.4) is complete. So, for now, assume either that $d \leq 8$ or that $d = 9$ and $r \neq 3$.

Since $C$ is not a rational normal curve, necessarily $d > r$. Now, $h^1(I_C(5)) = 0$. Indeed, if $C$ lies in a hyperplane $G$, then $h^1(I_C(5))$ is equal to $h^1(I_{C/G}(5))$ by (2-2), whose proof doesn’t require $C$ to be smooth. A similar argument shows further that, if $C$ lies in a plane $H$ contained in $G$, then $h^1(I_{C/G}(5))$ is equal to $h^1(I_{C/H}(5))$; moreover, the latter group always vanishes by Serre’s theorem. Suppose $C$ doesn’t lie in a plane; that is, $r = 3, 4$. Then the desired vanishing holds by the table on p. 504 of [11]. Thus $M_{d,0}^{2,g} = M_{d}^{r,g}$.

Suppose $r = 2$. Given a plane $H$, consider the subset of $M_{d,0}^{2,g}$ of mappings whose image $C$ spans $H$. Obviously, this subset has dimension at most $3(d + 1)$. Now, $H$ varies in a system of dimension $3 \times (5 - 3)$, or 6. So

$$\dim M_{d}^{2,g} \leq 3d + 9.$$ 

On the other hand, $g = (d - 1)(d - 2)/2$. Hence the bound (3-1) holds for $d = 3, 4$. However, for $d = 5$, the two sides of (3-1) are equal; in other words, $I_{5,6}^2$ has dimension 129. Now, the image $C$ of a general mapping in $M_{d}^{2,g}$ has $g$ nodes and no other singularities. Therefore, a general quintic threefold $F$ can contain no rational plane quintic $C$ other than one with six nodes (in fact, Vainsencher [25, §6.2] proved there are 17,601,000 six-nodal $C$ in $F$). Finally, suppose $d \geq 6$. Then $F$ can contain no image $C$. Otherwise, Bezout’s theorem implies that $F$ contains the span $H$ of $C$, so infinitely many lines.

Suppose $r = 3$. Then $4 \leq d \leq 8$. Given a hyperplane $G$, consider the subset of $M_{d}^{3,g}$ of mappings whose image $C$ spans $G$. Obviously, this subset has dimension at most $4(d + 1) - 1$, or $4d + 3$, as $g \geq 1$. Now, $G$ varies in a system of dimension 4. So

$$\dim M_{d}^{3,g} \leq 4d + 7.$$
Hence the bound (3-1) holds if \( g < d - 2 \). However, as \( d \leq 8 \), the Castelnuovo–Halphen bounds yield \( g < d - 2 \), except in these seven cases:

\[
(d, g) = (6, 4), \ (7, 6), \ (8, 8), \ (8, 9), \ (7, 5), \ (8, 6), \ (8, 7).
\]

In the first four cases, each \( C \) necessarily lies on a quadric surface. (As is well known, these bounds and exceptions were first proved rigorously by Gruson and Peskine \[10\], Thm. 3.1, p. 49, although they assert the results only for smooth curves. The smoothness enters via an appeal on p. 51 to a preliminary version of Laudal’s \[18\], Cor. 2, p. 147; however, the final version applies to an arbitrary reduced and irreducible curve.) Hence, Lemma (3.2) below yields

\[
\dim M_{d,g}^3 \leq (2d + 12) + 4 = 2d + 16,
\]

because the hyperplanes in \( \mathbf{P}^4 \) form a 4-parameter family. Thus (3-1) holds in those four cases. The remaining three cases are covered by Parts (c) and (d) of Lemma (3.3) below because, if \( g \geq 3 \), then \( C \) has either at least three distinct singularities or at least one singularity with \( \delta \)-invariant at least 2.

Suppose \( r = 4 \). Then \( d \geq 5 \). If \( d = 7, 8, 9 \), then the Castelnuovo–Halphen bounds yield \( g \leq 7 \). (See \[4\], (1.1), p. 27; on pp. 24–5, Ciliberto says that the extension of the bounds to \( \mathbf{P}^r \) for \( r \geq 4 \) is due to Fano and Harris.) Hence the bound (3-1) holds by Lemma (3.4)(a). So suppose \( d \leq 6 \). Then the Castelnuovo–Halphen bounds yield \( g \leq 2 \). If \( g = 1 \), then (3-1) holds by Lemma (3.3)(a). If \( g = 2 \), then \( C \) has either two distinct singularities or one singularity of \( \delta \)-invariant 2; hence, (3-1) holds in the first case by Lemma (3.3)(b), and it holds in the second by Lemma (3.3)(d).

Thus, if \( d \leq 8 \) or if \( d = 9 \) and \( r \neq 3 \), then Theorem (3.1) holds given Lemmas (3.2), (3.3) and (3.4).

**Lemma (3.2)** Fix an integer \( d \), and form the space \( J \) of all pairs \( (\phi, Q) \), where \( \phi \) is a parametrized generic embedding of \( \mathbf{P}^1 \) into \( \mathbf{P}^3 \) and where \( Q \) is a reduced and irreducible quadric surface such that the image \( C \) of \( \phi \) is of degree \( d \) and lies on \( Q \). Then \( J \) is equidimensional of dimension \( 2d + 12 \).

Indeed, the surfaces \( Q \) are parametrized by an open subset \( U \) of \( \mathbf{P}^9 \), and those that are smooth, by a smaller open subset \( V \). The natural action of \( GL(4) \) on \( \mathbf{P}^3 \) induces an action on \( U \) and a compatible action on \( J \). Moreover, \( V \) is an orbit. Hence, every component of \( J|V \) projects onto \( V \). Therefore, to prove the lemma, it suffices to prove (1) that no component of \( J \) projects into \( U \) and (2) that every fiber of \( J|V \to V \) is equidimensional of dimension \( 2d + 3 \). On the other hand,
just to bound the dimension of $J$, which is all that’s needed in the proof of Theorem. (3.1), it suffices to prove (2) and (3), where (3) asserts that every fiber of $J$ over a point of $U - V$ is of dimension at most $2d + 3$.

To prove (1), it clearly suffices to prove this: let $(\phi_0, Q_0)$ be a pair where $Q_0$ is a cone, and let $T$ be the spectrum of the power series ring in one variable $t$; then there exists a map $T \rightarrow J$ which carries the closed point of $T$ to $(\phi_0, Q_0)$ and which carries the generic point to a pair $(\phi', Q')$ where $Q'$ is smooth. Say that the coordinates on $\mathbb{P}^1$ are $(u, v)$ and that those on $\mathbb{P}^3$ are $(X_0, \ldots, X_3)$. Say that

$$\phi_0(u, v) = (\phi_{00}(u, v), \ldots, \phi_{03}(u, v)),$$

where the $\phi_{0j}$ are forms of degree $d$, and that

$$Q_0 : q_0(X_0, \ldots, X_3) = 0$$

where $q_0$ is a form of degree 2. To construct $T \rightarrow J$, it is necessary to construct, for each $i \geq 1$, similar terms,

$$\phi_i(u, v) = (\phi_{i0}(u, v), \ldots, \phi_{i3}(u, v))$$

and $q_i(X_0, \ldots, X_3)$, such that, for each $n \geq 1$,

$$(q_0 + tq_1 + \cdots + t^n q_n)(\phi_0 + t\phi_1 + \cdots + t^n \phi_n) \equiv 0 \mod(t^{n+1}). \quad (3-2)$$

Of course, $q_0(\phi_0) = 0$ because $(\phi_0, Q_0) \in J$.

The proof proceeds by induction on $n$. Suppose that $\phi_i$ and $q_i$ have been constructed for $i < n$ and that, moreover, the $\phi_{ij}$ are linear combinations of the $\phi_{0j}$. In the expansion of the left hand side of (3-2), the coefficient of $t^i$ vanishes for $i < n$ by induction. The coefficient of $t^n$ has the form,

$$((\text{grad} \, q)(\phi_0) \cdot \phi_n + a_n(\phi_0) + q_n(\phi_0)),$$

where the grad $q$ is the gradient and where $a_n$ is a certain quadratic form. Note that $a_1$ vanishes and that each $a_n$ is independent of $\phi_n$ and $q_n$. Now, $GL(4)$ acts transitively on the space $U - V$ of cones. So we may assume that

$$q_0 = X_0^2/2 + X_1X_2.$$

Then the first term of (3-3) becomes

$$\phi_{00}\phi_{n0} + \phi_{02}\phi_{n1} + \phi_{01}\phi_{n2}.$$

Hence, if $n = 1$, then (3-3) vanishes with these choices,

$$q_1 = -(X_0^2 + X_1^2 + X_2^2 + X_0X_3), \quad \phi_{10} = \phi_{00} + \phi_{03}, \quad \phi_{11} = \phi_{02}, \quad \phi_{12} = \phi_{01},$$
and with $\phi_{13}$ arbitrary. If $n \geq 2$, then collect the terms in $a_n(\phi_0)$ so that it acquires the form,

$$\phi_{00} \gamma_0 + \phi_{01} \gamma_1 + \phi_{02} \gamma_2 + c_n \phi_{03}^2,$$

where the $\gamma_k$ are suitable linear combinations of the $\phi_{0j}$ and where $c_n$ is a suitable constant. Then (3-3) vanishes with these choices,

$$q_n = -c_n X_3^2, \quad \phi_{n0} = -\gamma_0, \quad \phi_{n1} = -\gamma_2, \quad \phi_{n2} = -\gamma_1,$$

and with $\phi_{n3}$ arbitrary. The construction is now complete.

Set $\phi' := \sum_{n \geq 0} t^n \phi_n$ and $Q' := \sum_{n \geq 0} t^n q_n = 0$. Then $\phi'$ is a generic embedding because $\phi_0$ is; indeed, $\phi'$ is finite because $\phi_0$ is, and the comorphism of $\phi'$ is generically surjective because that of $\phi_0$ is. By construction, the image of $\phi'$ is of degree $d$ and lies on $Q'$. Finally, $Q'$ is smooth because, if $c := 2(\sum_{n \geq 2} t^n c_n)$, then $\text{grad} \sum_{n \geq 0} t^n c_n$ is equal to

$$(X_0, X_2, X_1, 0) - t(2X_0 + X_3, 2X_1, 2X_2, X_0) - c(0, 0, 0, X_3) = ((1 - 2t)X_0 - tX_3, X_2 - 2tX_1, X_1 - 2tX_2, -tX_0 - cX_3),$$

and the right hand side is clearly nonzero for any power series $X_i(t)$ and $c(t)$, provided at least one $X_i$ is nonzero and $c(t)$ is not $t^2/(1 - 2t)$; the latter restriction can be achieved by an appropriate choice of $\phi_{13}$. Thus (1) holds.

To prove (2), set $Q := P^1 \times P^1$. Then a map $\phi: P^1 \rightarrow Q$ is given by a pair of maps $\phi_i: P^1 \rightarrow P^1$ for $i = 1, 2$. Set $d := \deg \phi$ and $d_i := \deg \phi_i$. Then $d = d_1 + d_2$. Say that the coordinates on $P^1$ are $(u, v)$ and that those on $P^3$ are $(X_0, \ldots, X_3)$. Then a parametrization,

$$(\alpha_i(t, u), \beta_i(t, u)),$$

of each $\phi_i$ yields a parametrization of $\phi$, namely,

$$(\alpha_1(t, u)\beta_1(t, u), \alpha_1(t, u)\beta_2(t, u), \alpha_2(t, u)\beta_1(t, u), \alpha_2(t, u)\beta_2(t, u)).$$

If the parametrization of $\phi_i$ is multiplied by a scalar $v_i$, then the parametrization of $\phi$ is multiplied by $v_1v_2$. Form the space of all pairs of parametrized maps $P^1 \rightarrow P^1$ of bidegree $(d_1, d_2)$; obviously, this space has dimension,

$$2(d_1 + 1) + 2(d_2 + 1) = 2d + 4.$$ 

Now, it is an open condition on a map $\phi: P^1 \rightarrow Q$ to be a generic embedding. Therefore, each fiber of $J|V \rightarrow V$ breaks up into the disjoint union of subspaces, each of dimension $2d + 3$. Thus (2) holds. Finally, (3) was proved by Guest [12, Prop. 2.1, p. 60] via a similar, but a little more complicated, elementary description of the parametrized maps from $P^1$ to the cone with equation $X_2^3 - X_1X_3$. The proof of Lemma (3.2) is now complete.
Lemma (3.3) Fix $d \geq 5$, and let $M_{d,n,\delta}^r$ be the space of parametrized mappings of $\mathbf{P}^1$ into $\mathbf{P}^4$ with these (locally closed) properties: each mapping is birational onto its image $C$, and this $C$ spans an $r$-plane, has degree $d$, and has at least $n$ distinct singularities of which one has $\delta$-invariant at least $\delta$.

(a) Then $\dim(M_{d,1,1}^4) \leq 5d + 3$ and $\dim(M_{d,1,1}^3) \leq 4d + 7$.
(b) Then $\dim(M_{d,2,1}^4) \leq 5d + 1$ and $\dim(M_{d,2,1}^3) \leq 4d + 6$.
(c) Then $\dim(M_{d,3,1}^4) \leq 5d - 1$ and $\dim(M_{d,3,1}^3) \leq 4d + 5$.
(d) Then $\dim(M_{d,1,2}^4) \leq 5d$ and $\dim(M_{d,1,2}^3) \leq 4d + 5$.
(e) Then $\dim(M_{d,2,2}^4) \leq 5d - 2$ and $\dim(M_{d,2,2}^3) \leq 4d + 4$.
(f) Then $\dim(M_{d,1,3}^4) \leq 5d - 2$ and $\dim(M_{d,1,3}^3) \leq 4d + 4$.

Proof. Say that the coordinate tuples on $\mathbf{P}^1$ are $(t, u)$, on $\mathbf{P}^4$ are $(X_0, \ldots, X_4)$, and on $M_{d,n,\delta}^r$ are $(a_{i,j})$. Then the maps are defined by the five equations,

$$X_i = a_{i,0}t^d + a_{i,1}t^{d-1}u + \cdots + a_{i,d}u^d. \quad (3-4)$$

We’ll treat the case $r = 4$ only, as the case $r = 3$ is similar.

To prove (a), consider the condition that two (distinct) points of $\mathbf{P}^1$ are mapped to the same point of $\mathbf{P}^4$. If these points are $(1,0), (0,1)$, and $(1,0,0,0,0)$, then the condition becomes this: $a_{i,0} = 0$ and $a_{i,d} = 0$ for $i = 1, 2, 3, 4$. Hence, the corresponding space of mappings is of dimension $5d - 3$. Now, consider the condition that a point of $\mathbf{P}^1$ is mapped to a cusp on $C$. If these points are $(1,0)$ and $(1,0,0,0,0)$, then the condition becomes this: $a_{i,0} = 0$ and $a_{i,1} = 0$ for $i = 1, 2, 3, 4$. Hence, the corresponding space of mappings is of dimension $5d - 3$ too. Let the first two points vary over $\mathbf{P}^1$, and the third over $\mathbf{P}^4$. Then the corresponding spaces sweep out a space of dimension at most $5d + 3$.

To prove (b), consider the condition that four points of $\mathbf{P}^1$ are mapped 2-to-1 to two points of $\mathbf{P}^4$. If the four are $(1,0), (0,1), (1,1)$ and $(1,v)$ with $v \neq 0, 1$ and if the first two map to $(1,0,0,0,0)$ and the second two map to $(0,0,0,0,1)$, then the condition becomes this: for $i = 1, 2, 3, 4$,

$$a_{i,0} = 0, \quad a_{i,d} = 0, \quad \sum_j a_{i-1,j} = 0, \quad \sum_j a_{i-1,j}v^j = 0. \quad (3-5)$$

These sixteen linear equations are obviously independent. Hence, the corresponding space of mappings is of dimension $5d - 11$. Let the six points vary. Then the corresponding spaces sweep out a space of dimension at most $4 + 4 + 4$ more, or at most $5d + 1$ in all.

Consider the condition that a point of $\mathbf{P}^1$ is mapped to a cusp and that a pair of points is mapped to a node. If the first point is $(1,0)$
and the pair consists of $(1, 1)$ and $(1, v)$ with $v \neq 0, 1$ and if the cusp lies at $(1, 0, 0, 0, 0)$ and the node at $(0, 0, 0, 0, 1)$, then the condition is expressed by sixteen independent linear equations; they are nearly the same as those displayed in (3-5), only $a_{i,d} = 0$ is replaced by $a_{i,1} = 0$. Let the five points vary. Then the corresponding spaces of mappings sweep out a space of dimension at most $5d$. Similarly, the condition that two points of $\mathbf{P}^1$ are mapped to distinct cusps defines a space of mappings of dimension at most $5d - 1$. Thus (b) holds.

In (c), there are essentially two different cases: (i) the three points on $C$ are not collinear, say they are

$$(1, 0, 0, 0, 0), (0, 0, 0, 0, 1), (0, 1, 0, 0, 0);$$

and (ii) the three points are collinear, say they are

$$(1, 0, 0, 0, 0), (0, 0, 0, 0, 1), (a, 0, 0, 0, b)$$

for suitable $a, b$. Suppose first that all three points are nodes of $C$. Say the first is the image of $(1, 0)$ and $(0, 1)$; the second, of $(1, 1)$ and $(1, v)$; and the third of $(1, w)$ and $(1, x)$. Then, as above, the sixteen independent linear equations (3-5) hold. In Case (i), there are these eight additional equations: for $i = 0, 2, 3, 4$,

$$\sum_j a_{i,j}w^j = 0 \text{ and } \sum_j a_{i,j}x^j = 0.$$ 

All twenty-four equations are independent since $d \geq 5$. Hence they define a space of mappings of dimension $5d - 19$. Let the nine points vary. Then the corresponding spaces sweep out a space of dimension at most $6 + 3 \times 4$ more, or at most $5d - 1$ in all. In Case (ii), six of the additional equations hold — namely, those for $i = 1, 2, 3$ — and all twenty-two equations are independent. Hence they define a space of mappings of dimension $5d - 17$. However, the third point in $\mathbf{P}^4$ has only one degree of freedom, not four, given the other two. Hence, as the nine points vary, the corresponding spaces sweep out a space of dimension at most $6 + 2 \times 4 + 1$ more, or at most $5d - 2$ in all. Suppose now that one of the three points on $C$ is a cusp. Then the argument can be modified as in (a) or (b) to yield asserted bound. Now, if $C$ has two cusps (or more), then the asserted bound holds by the argument in (b). Thus (c) holds.

To prove (d), note that a singularity on $C$ with $\delta$-invariant at least 2 falls into one of four cases: it has at least

(i) three branches, or
(ii) two branches, one with a cusp, or
(ii') two branches with the same tangent line, or
(iii) one branch with a higher cusp.
Cases (i) and (ii) can be handled much as in (b). Indeed, consider (i).
If \((1, 0), (0, 1),\) and \((1, 1)\) map to \((1, 0, 0, 0, 0, 0),\) then, for \(i = 1, 2, 3, 4,\)
\[a_{i,0} = 0, \quad a_{i,d} = 0, \quad \sum_j a_{i,j} = 0. \quad (3-6)\]
These twelve linear equations are obviously independent. Hence, the corresponding space of mappings is of dimension \(5d - 7.\)
Let the four points vary. Then the corresponding spaces sweep out a space of dimension at most \(3 + 4\) more, or at most \(5d\) in all.

Consider (ii). If \((1, 0)\) is mapped to a cusp at \((1, 0, 0, 0, 0, 0)\) and if \((0, 1)\) is also mapped to \((1, 0, 0, 0, 0, 0)\), then, for \(i = 1, 2, 3, 4,\)
\[a_{i,0} = 0, \quad a_{i,1} = 0, \quad a_{i,d} = 0. \]
These twelve linear equations are obviously independent. Hence, the corresponding space of mappings is of dimension \(5d - 7.\)
Let the three points vary. Then the corresponding spaces sweep out a space of dimension at most \(2 + 4\) more, or at most \(5d - 1\) in all.

Consider \((\text{ii}')\). If \((1, 0)\) and \((0, 1)\) are mapped to \((1, 0, 0, 0, 0, 0)\), then \(a_{i,0} = 0\) and \(a_{i,d} = 0\) for \(i = 1, 2, 3, 4.\) If also the corresponding tangent lines are equal, then there is a scalar \(r\) such that \(a_{i,d-1} = ra_{i,1}\) for \(i = 0, \ldots, 4.\) Hence, the corresponding space of mappings is of dimension \(5d - 7.\)
Let the three points vary. Then the corresponding spaces sweep out a space of dimension at most \(5d - 1.\)

Finally, consider (iii). Say \((1, 0)\) maps to a higher cusp at \((1, 0, 0, 0, 0, 0)\). Then \(a_{i,0} = 0\) and \(a_{i,1} = 0\) for \(i = 1, 2, 3, 4.\) In addition, if the cusp has multiplicity at least 3, then \(a_{i,2} = 0\) for \(i = 1, 2, 3, 4;\) if not, then no linear combination of the \(X_i\) can vanish to order 3 at \((1, 0)\), and therefore there is a scalar \(r\) such that \(a_{i,3} = ra_{i,2}\) for \(i = 1, \ldots, 4.\) Hence, the corresponding space of mappings is of dimension \(5d - 6.\)
Let the two points vary. Then the corresponding spaces sweep out a space of dimension at most \(1 + 4\) more, or at most \(5d - 1\) in all. The proof of (d) is now complete.

The proofs of (e) and (f) build on the proof of (d). The latter gives more than asserted; namely, in Cases (ii), \((\text{ii}'),\) and (iii), the space of mappings is irreducible of dimension at most \(5d - 1.\) If the dimension is, in fact, at most \(5d - 2,\) then we’re done. Assume not. Then, in each case, a general curve has exactly one singularity of \(\delta\)-invariant exactly 2 and no singularity of greater \(\delta\)-invariant, as we’ll now show.

In Case (ii), proceed as follows. For \(r = 4,\) a general projection from the point \(P := (1, 0, 0, 0, 0, 0)\) gives a smooth curve of degree \(d - 3\) in \(\mathbf{P}^3;\) indeed, simply disregard the common factor of \(tu^2\) in the parametrization of the projected curve. Hence the original curve is smooth outside \(P,\)
and at \(P\) has no other branches than \(t = 0\) and \(u = 0.\) Moreover,
the curves with $\delta$-invariant at least 3 at $P$ and no other branches at $P$ clearly form a proper, closed subset. Indeed, their parametrizations satisfy these additional conditions: if there’s a higher cusp at $u = 0$, then $ka_{i,2} + la_{i,3} = 0$ for $i = 1, \ldots, 4$ and for some fixed $k$, $l$; if there’s a cusp at $t = 0$, then $a_{i,d-1} = 0$ for $i = 1, \ldots, 4$; and if there’s a common tangent at the two branches, then $ka_{i,2} + la_{i,d-1} = 0$ for $i = 1, \ldots, 4$. For $r = 3$, the argument is nearly identical. However, the general projection from $(1,0,0,0)$ yields a plane curve of degree $d - 3$. So it’s not smooth if $d \geq 6$, but has ordinary nodes. Since the parametrization of $X_0$ is arbitrary, the original curve is smooth nevertheless.

In Cases (ii') and (iii), we argue similarly. In these cases, the general parametrizations give curves, which are smooth along $P$, and have no branches passing through $P$, except for $t = 0$ and $u = 0$. Hence the special parametrizations giving a singularity of $\delta$-invariant at least 3 at $P$ form a proper closed subset.

Thus in Parts (e) and (f) the mappings of Cases (ii), (ii') and (iii) form proper closed subspaces. Hence their dimensions are at most $5d - 2$. So, it remains to handle just the mappings of Case (i), those whose image $C$ has at least three branches at one point. (Alternatively, the previous three cases can be handled similarly.) If, say, $(1,0)$, $(0,1)$, and $(1,1)$ map to $(1,0,0,0,0)$, then the twelve equations (3-6) are satisfied.

In (e), each $C$ has an additional singularity. If this singularity lies at $(0,0,0,0,1)$ and if it’s the image of $(w,1)$ and $(x,1)$, then the following eight additional equations are satisfied: for $i = 0, 1, 2, 3$,

$$\sum_j a_{i,j} w^{d-1-j} = 0 \text{ and } \sum_j a_{i,j} x^{d-1-j} = 0.$$ 

All twenty equations are independent, by the theory of generalized Vandermonde determinants, if $w$ and $x$ are general since $d \geq 5$. Hence they define a space of mappings of dimension $5d - 15$. Let the seven points vary. Then the corresponding spaces sweep out a space of dimension at most $5 + 2 \times 4$ more, or at most $5d - 2$ in all. If instead $(0,0,0,0,1)$ is a cusp, and it’s the image of $(w,1)$, then the following eight additional equations are satisfied: for $i = 0, 1, 2, 3$,

$$\sum_j a_{i,j} w^{d-1-j} = 0 \text{ and } \sum_j (d-j)a_{i,j} x^{d-1-j} = 0.$$ 

Again, all twenty equations are independent, but this time there are only six points to vary. Hence the space of maps is of dimension at most $5d - 3$.

In (f), each $C$ has $\delta$-invariant at least 3 at the singular point. Two configurations are possible:

(i) four or more branches,
(ii) three branches whose tangent lines are coplanar.

Consider (i). If also \((v, 1)\) maps to \((1, 0, 0, 0, 0)\), then the following four additional equations are satisfied: for \(i = 1, 2, 3, 4\),

\[
\sum_j a_{i,j} v^{d-j} = 0.
\]

All sixteen equations are independent if \(v \neq 0, 1\). Hence they define a space of mappings of dimension \(5d - 11\). Let the five points vary. Then the corresponding spaces sweep out a space of dimension at most \(4 + 4\) more, or at most \(5d - 3\) in all.

Finally, consider (ii). If \((1, 0), (0, 1),\) and \((1, 1)\) map to \((1, 0, 0, 0, 0)\), then, since their tangent lines are coplanar, the maximal minors of the following matrix vanish:

\[
\begin{pmatrix}
  a_{11} & a_{21} & a_{31} & a_{41} \\
  a_{1,d-1} & a_{2,d-1} & a_{3,d-1} & a_{4,d-1} \\
  c_1 & c_2 & c_3 & c_4
\end{pmatrix}
\]

where \(c_i := \sum_{j=1}^{d-1} (d-j)a_{i,j}\). These condition, together with the twelve equations (3-6), define a space of mappings of dimension \(5d - 9\). Let the four points vary. Then the corresponding spaces sweep out a space of dimension at most \(3 + 4\) more, or at most \(5d - 2\) in all. The proof of Lemma (3.3) is now complete.

**Lemma (3.4)**

(a) If \(d \geq 7\), then \(\dim M_{d,g}^{4}\) \(\leq 5(d+1) - \min(2g, 8)\).

(b) If \(d \geq 9\), then \(\dim M_{d,g}^{3}\) \(\leq 4(d+1) - \min(g, 5)\).

**Proof.** For \(g \leq 3\), the assertions follow from Lemma (3.3). In the remaining cases, the proof is similar to that of this lemma, and the same notation will be used. For \(g \geq 4\), there are five cases of configurations of minimal singularities on \(C\), corresponding to the partitions of 4:

(i) four singularities with \(\delta\)-invariant 1,

(ii) one singularity with \(\delta\)-invariant 2 and two with \(\delta\)-invariant 1,

(iii) two singularities with \(\delta\)-invariant 2,

(iv) one singularity with \(\delta\)-invariant 3 and one with \(\delta\)-invariant 1,

(v) one singularity with \(\delta\)-invariant 4.

Each case will now be considered in turn.

In Case (i), the proof is similar to the proofs of (a), (b) and (c) of Lemma (3.3). Indeed, first fix the locations of the four singularities in \(\mathbb{P}^4\) (resp., in \(\mathbb{P}^3\)) and the locations of their preimages in \(\mathbb{P}^1\). Doing so imposes \(4 \times 8\) (resp., \(4 \times 6\)) independent linear conditions on the coefficients \(a_{i,j}\) in the equations (3-4) because \(d \geq 7\). If the four singularities are nodes, then, when the locations of the twelve points are varied, the corresponding linear spaces of mappings sweep out a space of dimension.
at most $8 + 4 \times 4$ more (resp., $8 + 4 \times 3$ more), or dimension at most $5d - 3$ (resp., $4d$) in all. If $n$ of the singularities are cusps, then the dimension is $n$ more. Thus the assertions hold in Case (i).

In Cases (ii) and (iii), the proofs are similar to those of (d) and (e) of Lemma (3.3). First consider a singularity with $\delta$-invariant 2. Fix its type. Fix its location and the locations of its preimages. Recall that doing so imposes independent linear conditions on the coefficients $a_{i,j}$ in the equations (3-4). If there are three preimages, then, when the locations of the four points are varied, the corresponding linear spaces of mappings sweep out a space of dimension at most $5d$ (resp., $4d + 1$). If there are fewer preimages, then the dimension is smaller. Now, it is not hard to see similarly that, because $d \geq 7$, imposing a second singularity with $\delta$-invariant 2 decreases the dimension by at least 5 more. Thus the assertions hold in Case (iii). On the other hand, it is not hard to see that imposing two additional singularities with $\delta$-invariant 1 decreases the dimension by at least 4 (resp., 2). Thus the assertions hold in Case (ii).

In Case (iv), repeat the argument proving (f) of Lemma (3.3), and note that imposing an additional singularity with $\delta$-invariant 1 decreases the dimension by at least 2 (resp., 1).

In Case (v), the proof is similar to the proofs of (d) and (f) of Lemma (3.3). There are five subcases, according to the number of branches. If the number is 5, then the space of mappings has dimension at most $5(d+1) - 5 \times 4 + (5+4)$, or $5d - 6$ (resp., $4(d+1) - 5 \times 3 + (5+3)$, or $4d - 3$). If the number is 4, then the space of mappings has dimension at most $5(d+1) - 4 \times 4 + (4+4)$, or $5d - 3$ (resp., $4(d+1) - 4 \times 3 + (4+3)$, or $4d - 1$); in fact, the dimension is less because the four tangent lines must span a 3-space, but the crude bound is sufficient. If the number is 3, then, by the proof of (d) of Lemma (3.3), the space of mappings with coplanar tangent lines has dimension at most $5d - 2$ (resp., $4d$); however, since $\delta = 4$, two of the lines must coincide or one branch must be cuspidal, and so the space of mappings in question must have dimension at most $5d - 3$ (resp., $4d - 1$). Finally, if the number is 2 or 1, then, by the proof of (d) of Lemma (3.3), the space of mappings such that $\delta \geq 2$ has dimension at most $5d - 1$ (resp., $4d$); pursuing the same line of reasoning further, it is easy to see that the subspace of mappings such that $\delta \geq 3$ has dimensions at most $5d - 3$ (resp., $4d - 1$).

The proof of (a) is now complete. Moreover, in all cases of (b) considered so far, the space of mappings has dimension at most $4d - 1$, except in the case of four singularities with $\delta$-invariant 1, where the dimension is $4d$. However, the reasoning in that case shows that, in the case of five singularities with $\delta$-invariant 1, the dimension is at most $4d - 1$ because $d \geq 9$. The proof of (b) is now complete. Thus Lemma (3.4) is proved.
To complete the proof of Theorem (3.1), it remains to establish the bound (3-1) for \( r = 3 \) and \( d = 9 \), namely, the bound,

\[
\dim M^{3,g}_{9,i} < 50 - (g + i).
\] (3-7)

Consider the image \( C \) of a mapping in \( M^{3,g}_{9} \). Then \( i = h^1(I_C(5)) \), and as was noted before, the latter \( h^1 \) is equal to \( h^1(I_{C/G}(5)) \), where \( G \) denotes the hyperplane spanned by \( C \); thus,

\[ i = h^1(I_{C/G}(5)). \]

The proof of (2-1) yields, in the present case, that, for \( k \geq 0 \),

\[
h^0(I_{C/G}(k)) = \binom{k + 3}{3} - (9k + 1 - g + h^1(O_C(k))) + h^1(I_{C/G}(k)).
\]

However, \( h^1(O_C(k)) = 0 \) if \( 9k > 2g - 2 \), because then \( \deg(\omega_C(-k)) < 0 \) where \( \omega_C \) is the dualizing sheaf, which is torsion free of rank 1. Since \( d = 9 \), the Castelnuovo–Hilben bounds yield \( g \leq 12 \). Hence, in particular,

\[
g + h^1(I_{C/G}(3)) = h^0(I_{C/G}(3)) + 8.
\] (3-8)

Now, the proof of (2-3) does not require \( C \) to be smooth. Therefore,

\[
g + h^1(I_{C/G}(5)) \leq h^0(I_{C/G}(3)) + 8.
\] (3-9)

Suppose that \( C \) lies in a quadric surface \( Q \). Then \( Q \) is irreducible as \( C \) lies in no plane. If \( C \) lies in a second surface \( K \) in \( G \), then \( C \) lies in \( Q \cap K \). So, if \( \deg K \leq 4 \), then \( K \supseteq Q \) by Bezout’s theorem since \( d = 9 \) and \( Q \) is irreducible. Hence, if \( \deg K = 3 \), then \( K = Q + H \) where \( H \) is a hyperplane. Therefore, \( h^0(I_{C/G}(3)) = 4 \). Hence, (3-8) yields \( g + h^1(I_{C/G}(5)) \leq 12 \). So the right hand side of (3-7) is at least 38. On the other hand, Lemma (3.2) implies that the various \( C \) in question correspond to a subspace \( J \) of \( M^{3,g}_{9} \) of dimension 30. Consequently, we may assume that \( C \) lies in no quadric.

Suppose that \( g + h^1(I_{C/G}(5)) \geq 10 \). Then (3-8) implies that \( C \) lies on two different cubics, \( K \) and \( G \) say. They are irreducible because \( C \) lies in no quadric. Hence their complete intersection is a curve of degree 9 containing \( C \), so equal to \( C \). Therefore, \( g = 10 \) by [13, Exr. 7.2(d), p. 54] for example. Moreover, \( h^1(I_{C/G}(5)) \) vanishes; indeed, in the exact sequence of twisted ideals,

\[
0 \rightarrow I_{K/G}(5) \rightarrow I_{C/G}(5) \rightarrow I_{C/K}(5) \rightarrow 0,
\]

the first term is equal to \( O_G(2) \) and the third term is equal to \( O_K(2) \), and so both terms have vanishing \( h^1 \). Thus \( C \) is among the curves treated in the next paragraph.
Suppose that $g \leq 10$ and $h^1(I_{C/G}(5)) = 0$. Then the right hand side of (3-7) is equal to $50 - g$, so is at least 40. On the other hand, Lemma (3.4)(b) implies that $M^{3,g}$ has dimension at most 39.

Finally, suppose that $g < 10$ and $h^1(I_{C/G}(5)) \neq 0$. Since $d = 9$, the latter condition implies that $C$ has a 7-secant line by virtue of Theorem 0.1 on p. 30 of [8] (although the statement in print does not make it completely clear, the theorem does indeed apply to any singular, reduced and irreducible curve, according to a personal communication from its author on 950705). Since $g < 10$, by the second paragraph above, $g + i < 10$; hence, the right hand side of (3-7) is at least 41. Hence (3-7) holds by the following lemma, and once it’s proved, the proof of Theorem (3.1) will be complete.

**Lemma (3.5)** The mappings $\phi : \mathbb{P}^1 \to \mathbb{P}^3$ of degree $d$ at least 8 whose images $C$ are singular and have a 7-secant line form a space of dimension at most $4d$.

**Proof.** The proof is similar to the last part of that of Lemma (2.4). Fix a divisor $D$ of degree 7 on $\mathbb{P}^1$, and fix a line $L$ in $\mathbb{P}^3$. Consider the mappings $\phi$ such that $\phi^{-1}L \supset D$. It is not hard to see that the $\phi$ form a linear space of dimension $4d - 10$. The corresponding images $C$ need not have $L$ as a 7-secant because the scheme $C \cap L$ might have length less than 7. However, every $C$ with $L$ as a 7-secant arises in this way. Now, let $D$ and $L$ vary. Then the corresponding linear spaces sweep out an irreducible space of dimension at most $7 + 4$ more, or $4d + 1$ in all. Since a general point represents an embedding (see [7, Prop. 4.4, p. 80] for instance), the points representing mappings with singular images form a subset of smaller dimension. Thus the assertion holds.

**Remark (3.6)** Even though a general quintic threefold does contain many six-nodal quintic plane curves, it does not contain any rational, reduced and irreducible, decic curve that lies on a quadric surface (possibly singular). Indeed, such a curve $C$ would be the complete intersection of the quadric and the quintic, since a general quintic contains no quadric surface. It follows that $C$ has arithmetic genus 16 and that $h^1(I_C(5))$ vanishes. Hence, by the reasoning that led to the bound (3-1), it suffices to prove that the various possible $C$ form a space of dimension at most $5 \times 10 + 5 - 16 - 0$, or 39. However, by Lemma (3.2), this space has dimension at most $2 \times 10 + 12 + 4$, or 36, and the proof is complete.

### 4. Reducible curves

In this section we will prove the following theorem, which completes our treatment of Clemens’ conjecture in degree at most 9. The theorem says, in other words, that on a general quintic threefold there is no pair
of intersecting rational, reduced and irreducible curves whose degrees
total at most 9.

**Theorem (4.1)** On a general quintic threefold in $\mathbb{P}^4$, there is no connected, reducible and reduced curve of degree at most 9 whose com-
ponents are rational.

Indeed, suppose there is such a curve $C$. Obviously, we may assume $C$
has two components. Consider one of them. By Theorem (3.1), either
it’s a six-nodal plane quintic or it’s smooth. If it’s smooth, then, by
Corollary (2.5)(3), either it’s a rational normal curve or it spans $\mathbb{P}^4$,
and by Corollary (2.5)(2), the restricted twisted sheaf of differentials of
$\mathbb{P}^4$ has generic splitting type. We are now going to prove that, in fact,
there can be no such $C$.

Let $M'_a$ be the open subscheme of the Hilbert scheme of $\mathbb{P}^4$ parametriz-
ing the smooth irreducible curves of degree $a$ that are rational normal
curves if $a \leq 4$ and that span $\mathbb{P}^4$ if $a \geq 4$; in addition, assume that the
restriction of $\Omega^1_{\mathbb{P}^4}(1)$ has generic splitting type. Let $N_5$ be the scheme
parametrizing the six-nodal plane quintics in a variable plane in $\mathbb{P}^4$. Let
$R_{a,b,n}$ (resp., $S_{a,n}$) be the locally closed subset of $M'_a \times M'_b$ (resp., of
$M'_a \times N_5$) of pairs $(A, B)$ such that $A \cap B$ is exactly of length $n$. Let
$I_{a,b,n}$ (resp., $J_{a,n}$) be the locally closed subset of $R_{a,b,n} \times \mathbb{P}^{125}$ (resp.,
of $S_{a,n} \times \mathbb{P}^{125}$) of triples $(A, B, F)$ such that $A \subset F$ and $B \subset F$. The
$F$ that contain a plane form a proper closed subset of $\mathbb{P}^{125}$. Form its
complement, and the latter’s preimage in $J_{a,n}$. Then replace $S_{a,n}$ by
the image of that preimage, and replace $J_{a,n}$ by the preimage of the new
$S_{a,n}$. Then, given any pair $(A, B)$ in $S_{a,n}$, there is an $F$ that contains
both $A$ and $B$, but not any plane. It suffices now to prove that $I_{a,b,n}$
(resp., $J_{a,n}$) has dimension at most 124 for $a + b \leq 9$ (resp., $a \leq 4$) and
$n \geq 1$.

The fiber of $I_{a,b,n}$ (resp., $J_{a,n}$) over a pair $(A, B)$ is a projective space
of dimension $h^0(\mathcal{I}_C(5)) - 1$, where $C$ is the reduced curve $A \cup B$ and $\mathcal{I}_C$ is its ideal in $\mathbb{P}^4$. Hence, mutatis mutandis, the proof of Equation (2-1)
yields

$$
dim I_{a,b,n} \leq \dim R_{a,b,n} + 125 - \min_C \{ h^0(\mathcal{O}_C(5)) - h^1(\mathcal{I}_C(5)) \}
$$

(resp., $\dim J_{a,n} \leq \dim S_{a,n} + 125 - \min_C \{ h^0(\mathcal{O}_C(5)) - h^1(\mathcal{I}_C(5)) \}$).

To bound the term $h^0(\mathcal{O}_C(5))$, consider the exact sequence,

$$0 \rightarrow \mathcal{I}_{B/C} \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_B \rightarrow 0$$

where $\mathcal{I}_{B/C}$ is the ideal of $B$ in $C$. Now, $\mathcal{I}_{B/C}$ is equal to the ideal $\mathcal{I}_{D/A}$
where $D := A \cap B$. Hence,

$$h^0(\mathcal{O}_C(5)) \geq \chi(\mathcal{O}_C(5)) = \chi(\mathcal{I}_{D/A}(5)) + \chi(\mathcal{O}_B(5)).$$
Now, $A$ and $B$ are smooth, rational and of degrees $a$ and $b$ (resp., $B$ is a six-nodal plane quintic), and $D$ is of length $n$. Hence
\[ h^0(\mathcal{O}_C(5)) \geq 5(a + b) + 2 - n \quad \text{(resp., } h^0(\mathcal{O}_C(5)) \geq 5a + 21 - n). \]

Theorem (4.1) is now a consequence of the following two lemmas.

**Lemma (4.2)** With the notation as above, for $a + b \leq 9$ (resp., $a \leq 4$) and $n \geq 1$,
\[ \dim R_{a,b,n} \leq 5(a + b) + 1 - n \quad \text{(resp., } \dim S_{a,n} \leq 5a + 20 - n). \]

**Lemma (4.3)** With the notation as above, for $a + b \leq 9$ (resp., $a \leq 4$) and $n \geq 1$,
\[ h^1(I_C(5)) = 0. \]

To prove Lemma (4.2), first consider a pair $(A, B)$ in $R_{a,b,n}$. Fix $B$ momentarily, and let $A$ vary in the fiber $\Phi_B$ of $R_{a,b,n}$ over $B$. We may assume that $a \leq b$, whence $a \leq 4$ and $b \leq 9 - a$. We’ll consider each of the four values of $a$ in turn. First, suppose $a = 1$. Then $A$ is an $n$-secant line of $B$. If $n = 1$, then $\dim \Phi_B = 4$ since the lines through a fixed point of $B$ form a $\mathbb{P}^3$. Now, $M'_b$ has dimension $5b + 1$. So $R_{1,b,1}$ has dimension $5b + 5$, the asserted bound. If $n = 2$, then $\dim \Phi_B = 2$ since a bisecant is determined by its “end-points.” So $R_{1,b,1}$ has dimension $5b + 3$, one less than the asserted bound. Suppose $n \geq 3$. Then $b \geq 4$ since neither a conic nor a twisted cubic has even a trisecant. So $B$ spans $\mathbb{P}^4$. Hence $n \leq b - 2$; indeed, if $H$ is the hyperplane spanned by an $n$-secant line of $A$ and two general points of $B$, then $H \cap B$ has length $b$. Therefore, Lemma (2.4) implies that the various $B$ with an $n$-secant line $A$ form a space of dimension at most $5b - 2n + 7$. Now, $\dim \Phi_B \leq 2$ since $n \geq 2$. So $R_{1,b,n}$ has dimension at most $5b - 2n + 9$, which is at most the asserted bound of $5b - n + 6$ since $n \geq 3$. Thus Lemma (4.2) holds when $a = 1$.

Suppose $a = 2$. Then $A$ is an $n$-secant conic of $B$, and we argue much as above. If $n = 1$, then $\dim \Phi_B = 9$ since there is a $\mathbb{P}^4$ of conics that pass through a fixed point $P$ of $B$ and that lie in a fixed plane and since the planes through $P$ form a 4-dimensional space. So $R_{2,b,1}$ has dimension $5b + 10$, the asserted bound. If $n = 2$, then $\dim \Phi_B = 7$ since there is a $\mathbb{P}^3$ of conics that contain a fixed length-2 subscheme $W$ of $B$ and that lie a fixed plane and since the planes through $W$ form a $\mathbb{P}^2$. So $R_{2,b,2}$ has dimension $5b + 8$, one less than the asserted bound. Suppose $n = 3$. Then $\dim \Phi_B \leq 5$ because the length-3 subschemes $W$ of $B$ form a threefold, and the conics through $W$ form a surface when $W$ determines a plane. So $R_{2,b,3}$ has dimension at most $5b + 6$, two less than the asserted bound for $n = 3$. Since the same reasoning applies to both $R_{2,b,4}$ and $R_{2,b,5}$, the latter have dimension at most $5b + 6$, the asserted
bound for \( n = 5 \). Suppose \( n = 5 \) and \( b \geq 4 \). Then \( \dim \Phi_B \leq 4 \). Indeed, there is at most one conic through a given length-5 scheme \( W \), and if there is one, then \( W \) must be planar. Now, the length-5 subschemes \( W \) of \( B \) are parametrized by its fifth symmetric product, which is irreducible, and a general such \( W \) is not planar since \( B \) spans \( \mathbb{P}^4 \) as \( b \geq 4 \). Thus \( \dim \Phi_B \leq 4 \). Hence \( \dim R_{2, b, 5} \leq 5b + 5 \). Since the same reasoning applies to \( R_{2, b, 6} \), the latter has dimension at most \( 5b + 5 \), the asserted bound for \( n = 6 \) if \( b \geq 4 \).

It now suffices to show that, if \( n \geq 6 \), then \( n = 6 \) and \( b = 7 \). Recall that \( a = 2 \leq b \). Denote the plane of \( A \) by \( G \). If \( b = 2 \), then \( B \) is a conic, say with plane \( H \). If \( H = G \), then \( n = 4 \). If not, set \( L := H \cap G \). Then \( A \cap B \) is a subscheme of \( A \cap L \), and the latter has length at most \( 2 \); so \( n \leq 2 \). If \( b = 3 \), then \( B \) is a twisted cubic, say spanning the hyperplane \( H \). If \( H \supset G \), then \( G \cap B \) has length 3, but it contains \( A \cap B \). If \( H \nsubseteq G \), then \( H \cap A \) has length 2, but it contains \( A \cap B \); so \( n \leq 2 \). Finally, if \( b \geq 4 \), then \( B \) spans the ambient \( \mathbb{P}^4 \). Let \( H \) be the hyperplane spanned \( G \) and a general point of \( B \). Then \( H \cap B \) has length \( b \). Hence \( n \leq b - 1 \), but \( b \leq 9 - a \) and \( a = 2 \). Thus \( n \leq 6 \), and if \( n = 6 \), then \( b = 7 \), as required. Thus Lemma (4.2) holds when \( a = 2 \).

Suppose \( a = 3 \). Then \( A \) is a twisted cubic, \( n \)-secant to \( B \), and \( 3 \leq b \leq 6 \). Let \( G \) and \( H \) be the linear spans of \( A \) and \( B \). Suppose \( b = 3 \). Then \( B \) is a twisted cubic too. So both \( G \) and \( H \) are hyperplanes. If they are distinct, then \( n \leq 3 \), because \( G \cap H \) is plane, so meets \( A \) is a scheme of length 3, and it contains \( A \cap B \). If \( G = H \), then \( n \leq 6 \), because \( A \) lies in a quadric surface \( Q \) in \( H \) not containing \( B \), and \( Q \cap B \) has length 6. Suppose \( b \geq 4 \). Then \( B \) spans \( \mathbb{P}^4 \). So \( G \) meets \( B \) in a subscheme of length \( b \), and it contains \( A \cap B \). Hence \( n \leq b \leq 6 \). Now, the fiber \( \Phi_B \) consists of all the twisted cubics \( A \) that meet \( B \) in a length-\( n \) subscheme \( W \). So \( \Phi_B \) has dimension at most \( 16 - 2n \) for \( n \leq 4 \), because the various \( W \) form an \( n \)-fold, the various hyperplanes \( G \) containing a fixed \( W \) form a \( \mathbb{P}^{4-n} \), and the various \( A \) in a fixed \( G \) containing a fixed \( W \) form a space of dimension at most \( 12 - 2n \) by Lemma (4.4) below. Hence \( R_{3, b, n} \) has dimension at most \( 5b + 17 - 2n \), or \( n - 1 \) less than the asserted bound of \( 5b + 16 - n \) for \( n \leq 4 \). Since \( R_{3, b, n} \) contains \( R_{3, b, 4} \) for \( n \geq 4 \), the latter has dimension at most \( 5b + 9 \), which is less than the asserted bound also for \( n = 5, 6 \). Thus Lemma (4.2) holds when \( a = 3 \).

Suppose \( a = 4 \). Then \( b = 4 \) or \( b = 5 \). So \( A \) is a rational normal curve in \( \mathbb{P}^4 \), and \( B \) spans \( \mathbb{P}^4 \) too. Hence \( n \leq 2b \), because \( A \) lies in a quadric hypersurface \( Q \) not containing \( B \), and \( Q \cap B \) has length \( 2b \). Now, the fiber \( \Phi_B \) consists of all the \( A \) that meet \( B \) in a length-\( n \) subscheme \( W \). So \( \Phi_B \) has dimension at most \( 21 - 2n \) for \( n \leq 6 \), because the various \( W \) form an \( n \)-fold, and the various \( A \) containing a fixed \( W \) form a space of dimension at most \( 21 - 3n \) by Lemma (4.4) below. Hence \( R_{4, b, n} \) has
dimension at most $22 + 5b - 2n$, or $n - 1$ less than the asserted bound of $21 + 5b - n$, for $n \leq 6$. Since the same reasoning as for $R_{4,b,6}$ applies to $R_{4,b,n}$ for $n \geq 4$, the latter has dimension at most $10 + 5b$, which is at most the asserted bound also for $7 \leq n \leq 2b$. Thus Lemma (4.2) holds when $a = 4$. The first assertion of Lemma (4.2) is now proved.

To complete the proof of Lemma (4.2), consider a pair $(A, B)$ in $S_{a,n}$. Then $A$ is a rational normal curve of degree $a$ where $1 \leq a \leq 4$, and $B$ is a six-nodal plane quintic, say with plane $H$. Then $H \not\ni A$, because there is a quintic hypersurface $F$ that contains $A$ and $B$, but not $H$, so meets $H$ in $B$. Fix $B$ momentarily, and let $A$ vary in the fiber $\Phi_B$ of $S_{a,n}$ over $B$. First, suppose $a = 1$, or $A$ is an $n$-secant line of $B$. Now, $A \not\subset H$, so $n = 1$. Hence $\dim \Phi_B = 4$ since the lines through a fixed point of $B$ form a $\mathbb{P}^3$. Therefore $S_{1,1}$ has dimension 24, which is the asserted bound. Thus Lemma (4.2) holds when $a = 1$.

Suppose $a = 2$. Then $A$ is an $n$-secant conic of $B$. Denote the plane of $A$ by $G$. Then $G \not\ni H$ because $H \not\ni A$. If $G \cap H$ is a line $L$, then $A \cap B$ is contained in $A \cap L$, which has length 2; hence, $n \leq 2$. If $G \cap H$ is a point, then clearly $n = 1$. If $n = 1$, then $\dim \Phi_B = 9$ since, given $P \in B$, there is a $\mathbb{P}^4$ of planes $G$ through $P$, and there is a $\mathbb{P}^4$ of conics $A$ that lie in a fixed $G$ and pass through $P$. Hence $S_{2,1}$ has dimension 29, which is the asserted bound. If $n = 2$, then $\dim \Phi_B = 7$ since the length-2 subschemes $W$ of $B$ form a surface, the planes $G$ through a fixed $W$ form a $\mathbb{P}^2$, and there is a $\mathbb{P}^3$ of conics $A$ that lie in a fixed $G$ and pass through $W$. Hence $S_{2,2}$ has dimension 27, which is one less than the asserted bound. Thus Lemma (4.2) holds when $a = 2$.

Suppose $a = 3$. Then $A$ is a twisted cubic, $n$-secant to $B$. Denote the hyperplane of $A$ by $G$. If $G \supset H$, then $A \cap B$ is contained in $A \cap H$, which has length 3; hence $n \leq 3$. If $G \cap H$ is a line $L$, then $A \cap L$ has length 2; hence, $n \leq 2$. Now, the fiber $\Phi_B$ has dimension at most $16 - 2n$ for $n \leq 3$, because the length-$n$ subschemes $W$ of $B$ form an $n$-fold, the various hyperplanes $G$ containing a fixed $W$ form a $\mathbb{P}^{4-n}$, and the various $A$ in a fixed $G$ containing a fixed $W$ form a space of dimension at most $12 - 2n$ by Lemma (4.4) below. Hence $S_{3,n}$ has dimension $36 - 2n$, which is at most the asserted bound for $n = 1, 2, 3$. Thus Lemma (4.2) holds when $a = 3$.

Suppose $a = 4$. Then $A$ is a rational normal quartic, $n$-secant to $B$. Moreover, $n \leq 3$; indeed, the plane $H$ of $B$ can intersect $A$ in a subscheme of length at most 3 since $H$ and a general point of $B$ span a hyperplane, which meets $A$ in a subscheme of length 4. Now, the fiber $\Phi_B$ has dimension at most $21 - 2n$, because the length-$n$ subschemes $W$ of $B$ form an $n$-fold, and the various $A$ containing a fixed $W$ form a space of dimension at most $21 - 3n$ by Lemma (4.4) below. Hence $S_{4,n}$ has dimension $41 - 2n$, which is at most the asserted bound for $n = 1, 2, 3$. 

Thus Lemma (4.2) holds when $a = 4$. The proof of Lemma (4.2) is complete, given the following lemma.

**Lemma (4.4)** In the space of all rational normal curves of degree $a$ in $\mathbb{P}^n$, those that contain a given subscheme of length $n$ form a subset of codimension $(a - 1)n$ for $n \leq a + 2$.

**Proof.** Consider the incidence scheme $I$ of pairs $(C, W)$ where $C$ is a rational normal curve and $W$ is a length-$n$ subscheme of $C$. The fiber of $I$ over a fixed $C$ is isomorphic to $\mathbb{P}^n$, and the various $C$ form an irreducible scheme of dimension $(a + 1)^2 - 4$; hence, $I$ is irreducible of dimension $(a + 1)^2 - 4 + n$. Consider the projection of $I$ onto the space of $W$, which is a smooth open subscheme $U$ of dimension $an$ in the Hilbert scheme of $\mathbb{P}^a$. This projection is equivariant for the natural action of $GL(a + 1)$. Since $n \leq a + 2$, those pairs $(C, W)$ such that $W$ is concentrated at a point $P$ form a closed orbit; it is isomorphic to the incidence space of pairs $(C, P)$. This orbit maps onto a closed orbit in $U$, which has dimension $(a - 1)n + 1$ and lies in the closure of every orbit of $U$. Hence the fiber over such a $W$ has codimension $(a - 1)n$. Therefore, the fiber over an arbitrary $W$ has codimension $(a - 1)n$ by lower semi-continuity of codimension. Thus Lemma (4.4) is proved.

To prove Lemma (4.3), first suppose that $B$ is smooth. Say $B$ spans an $r$-plane. Suppose $a = 1$. Then $b \leq 8$. Hence $C$ is $(3+b-r)$-regular by [11, Rmk. (1), p. 497], and so 6-regular if $b \leq 7$. Suppose $b = 8$. Then the proof of [11, Rmk. (1)] yields even that $C$ is 5-regular. Indeed, by [11, Prp. (1.2), p. 494], it suffices to show that, on the normalization of $C$, which is the disjoint union of $A$ and $B$, there is an invertible sheaf $\mathcal{A}$ such that $h^0(\mathcal{A})$ is 5 and $H^1(\mathcal{A})$ vanishes, where $\mathcal{M}$ is the pullback of $\Omega^1_{\mathbb{P}^4}(1)$. By the definition of $R_{a,b,n}$,

$$\Omega^1_{\mathbb{P}^4}(1)|A \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^3,$$

and

$$\Omega^1_{\mathbb{P}^4}(1)|B \cong \mathcal{O}_{\mathbb{P}^1}(-2)^4.$$

So, if we define $\mathcal{A}$ as the structure sheaf on $A$ and as $\mathcal{O}_{\mathbb{P}^1}(3)$ on $B$, then $\mathcal{A}$ has the desired two properties. Thus Lemma (4.3) holds when $a = 1$.

Suppose that $a \geq 2$. Then $b \leq 7$. We may assume that $a \leq b$. Then $a \leq 4$. So $A$ is a rational normal curve. Hence $C$ is $(4+b-r)$-regular by [11, Rmk. (1), p. 497], and so 6-regular if $b \leq 6$. Suppose $b = 7$, so $a = 2$. Then, much as above, the proof of [11, Rmk. (1)] yields that $C$ is 6-regular. Indeed,

$$\Omega^1_{\mathbb{P}^4}(1)|A \cong \mathcal{O}_{\mathbb{P}^1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}^2,$$

and

$$\Omega^1_{\mathbb{P}^4}(1)|B \cong \mathcal{O}_{\mathbb{P}^1}(-2)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

So, if we define $\mathcal{A}$ as $\mathcal{O}_{\mathbb{P}^1}(1)$ on $A$ and as $\mathcal{O}_{\mathbb{P}^1}(3)$ on $B$, then $\mathcal{A}$ has the desired two properties. Thus Lemma (4.3) holds when $B$ too is smooth.
Suppose that $B$ is a six-nodal plane quintic. Let $K$ be its plane. By hypothesis, there is a quintic threefold $F$ that contains both $A$ and $B$, but not $K$. Hence $F \cap K = B$ by Bezout’s theorem. Therefore,

$$A \cap K \subset A \cap F \cap K \subset B.$$ 

Now, $A$ is a rational normal curve of degree $a \leq 4$. Hence, $A$ is smooth and 2-regular, so 4-regular. Therefore, $C$ is 5-regular by the following lemma, and after it is proved, the proofs of Lemma (4.3) and of Theorem (4.1) will be complete.

**Lemma (4.5)** Let $A$ and $B$ be two reduced curves in $\mathbf{P}^4$, and $C$ their union. Assume that $A$ and $B$ have no common components, that $B$ spans a plane $K$, and that $A$ is smooth along $A \cap K$. If $A$ is $(m - 1)$-regular where $m \geq \deg B$, then $C$ is $m$-regular.

**Proof.** First, we’ll use Hirschowitz’s “mèthode d’Horace” (see [14]). Let $H$ be a general hyperplane containing $K$, and set $D := C \cap H$. Then there is a natural exact sequence of sheaves,

$$0 \to \mathcal{I}_A/\mathbf{P}^4(-1) \to \mathcal{I}_C/\mathbf{P}^4 \xrightarrow{u} \mathcal{I}_D/H \to 0$$

because

$$\text{Ker } u = \mathcal{I}_C/\mathbf{P}^4 \cap \mathcal{I}_H/\mathbf{P}^4 = \mathcal{I}_A/\mathbf{P}^4 \cap \mathcal{I}_B/\mathbf{P}^4 \cap \mathcal{I}_H/\mathbf{P}^4$$

$$= \mathcal{I}_A/\mathbf{P}^4 \cap \mathcal{I}_H/\mathbf{P}^4 = \mathcal{I}_A/\mathbf{P}^4(-1);$$

the last equation holds because $A$ is reduced and because $H$, being general, contains no component of $A$. By hypothesis, $A$ is $(m - 1)$-regular. Therefore, $C$ will be $m$-regular if $\mathcal{I}_D/H$ is so.

Set $A' := A \cap H$. Then $A'$ is the disjoint union of two finite closed subschemes: the part $S$ of $A'$ with support off $K$, and the part $T$ with support on $K$. Correspondingly, there is a natural exact sequence of sheaves,

$$0 \to \mathcal{I}_{A'}/H \to \mathcal{I}_S/H \oplus \mathcal{I}_T/H \to \mathcal{O}_H \to 0$$

By hypothesis, $\mathcal{I}_A/\mathbf{P}^4$ is $(m - 1)$-regular; whence, so is $\mathcal{I}_{A'}/H$. On the other hand, $\mathcal{O}_H$ is 0-regular, so $(m - 1)$-regular. Therefore, $\mathcal{I}_S/H$ is $(m - 1)$-regular.

Since $A$ is smooth along $A \cap K$, the latter is a locally principal subscheme of $A$. Hence, since $H$ is general containing $K$, the schemes $A \cap K$ and $T$ coincide. By hypothesis, $A \cap K$ is a subscheme of $B$. Hence is $T$ also. It follows that the scheme $D$ is the disjoint union of $S$ and $B$. Indeed, this statement is clearly true off $T$. So work locally at an arbitrary point $P$ of $T$. Since $D \supset B \supset T$, we have

$$(\mathcal{I}_C/\mathbf{P}^4, P + \mathcal{I}_H/\mathbf{P}^4, P) \subset \mathcal{I}_B/\mathbf{P}^4, P \subset (\mathcal{I}_A/\mathbf{P}^4, P + \mathcal{I}_H/\mathbf{P}^4, P),$$  

(4-1)
and it suffices to check that the first inclusion is an equality. Given \( \beta \in \mathcal{I}_{B/P^4} \), write \( \beta = \alpha + \gamma \) where \( \alpha \in \mathcal{I}_{A/P^4} \) and \( \gamma \in \mathcal{I}_{H/P^4} \). Then

\[
\alpha = \beta - \gamma \in \mathcal{I}_{A/P^4} \cap \mathcal{I}_{B/P^4} = \mathcal{I}_{C/P^4}.
\]

Hence \( \beta \) lies in the left hand term of (4-1), as required. Thus \( D = S \cup B \).

Therefore, \( D \cap K = B \) because \( K \) is disjoint from \( S \) and contains \( B \).

Again we’ll use Hirschowitz’s “mèthode d’Horace.” Much as above, there is a natural exact sequence of sheaves,

\[
0 \rightarrow \mathcal{I}_{S/H}(-1) \rightarrow \mathcal{I}_{D/H} \xrightarrow{v} \mathcal{I}_{B/K} \rightarrow 0,
\]

because \( D \cap K = B \) and because

\[
\text{Ker } v = \mathcal{I}_{D/H} \cap \mathcal{I}_{K/H} = \mathcal{I}_{S/H} \cap \mathcal{I}_{B/H} \cap \mathcal{I}_{K/H} \]
\[
= \mathcal{I}_{S/H} \cap \mathcal{I}_{K/H} = \mathcal{I}_{S/H}(-1);
\]

the last equation holds because \( K \) is disjoint from \( S \). Now, it was proved above that \( \mathcal{I}_{S/H} \) is \((m-1)\)-regular. On the other hand, \( \mathcal{I}_{B/K} \) is equal to \( \mathcal{O}_K(-b) \) where \( b := \deg B \), and the latter sheaf is \( m \)-regular since \( m \geq b \). Therefore, \( \mathcal{I}_{D/H} \) is \( m \)-regular. Consequently, \( C \) is \( m \)-regular by the first paragraph, and the proof is complete.

5. References

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