Kink scattering in a hybrid model

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Abstract

In this work we consider kink-antikink and antikink-kink collisions in a hybrid model. The potential has two topological sectors connecting adjacent minima, so it simulates the $\phi^6$ model. However, in each topological sector, the potential is symmetric around the local maximum and the perturbed static kink solutions have translational and one vibrational mode, so it simulates the $\phi^4$ model. The two-bounce windows for antikink-kink collisions can be explained by the mechanism of Campbell-Schonfeld-Wingate (CSW) resonant transfer of energy between translational to vibrational mode. The high degree of symmetry of the potential leads, for kink-antikink collisions, to a new structure of bounce windows. In particular, depending on the initial velocity, one can have oscillations of the scalar field at the center of mass even for one bounce, or a change of topological sector. We show how a modified CSW mechanism can be used to explain some of the results.

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I. INTRODUCTION

Solitons in nonlinear field theories are solutions with localized energy density that can propagate freely without losing form. The simplest solutions in field theories that attain topological profile are kinks and antikinks in (1,1) spacetime dimensions and can be constructed in theories with one or more scalar fields.

Kink scattering in integrable systems is surprisingly simple, with the solitons gaining at most a phase shift. Some examples of integrable models are: i) KdV equation, connected to the Fermi-Pasta-Ulan problem \[1, 2\] in the continuum limit; ii) nonlinear Schrödinger equation, important for describing nonlinear effects in fiber optics \[3, 4\]; iii) the ubiquitous sine-Gordon equation \[5\], studied among other things in theories describing DNA \[6\] and Josephson junctions \[7–9\].

In nonintegrable models, kink scattering has a complex behavior. For ultrarelativistic velocities and arbitrary potentials, there is an analytical expression for the phase shift \[10\]. The simplest nonintegrable and largely studied is the \(\phi^4\) model \[11–20\]. In that model, for larger initial velocities \(v\) we have inelastic scattering, with the pair of solitons colliding once and separating thereafter. For smaller velocities than a critical one, \(v < v_c\), the kink-antikink forms a composed state named bion that radiates continuously until the complete annihilation of the pair. For smaller velocities with \(v \lesssim v_{crit}\) there are regions in velocity, named two-bounce windows, where the scalar field at the center of mass bounces twice before the final separation of the pair. Stability analysis of the \(\phi^4\) kink leads to a Schrödinger-like equation with two eigenstates: a zero or translational mode, related to the translational invariance of the model and a vibrational mode.

The occurrence of two-bounce windows was explained by Campbell, Schonfeld and Wingate (CSW) \[13\] as a resonance mechanism for the exchange of energy between the translational and the vibrational mode. A counter-example of this mechanism was found for the asymmetric \(\phi^6\) model, where the presence of two-bounce windows is explained not by the (absent) vibrational state of the kink, but due to the presence of vibrational state related to the combined kink-antikink configuration \[21\]. Another counter example is the total suppression of two-bounce windows due to the presence of multiples vibrational states \[22\].

Kink scattering in nonintegrable models is a topic that has been under intense investi-
One can cite polynomial models with one \cite{21, 23–29} and two \cite{30–33} scalar fields, nonpolynomial models \cite{34–39}, vector solitons \cite{40}, multi-kinks \cite{41–44}, interaction with a boundary \cite{45, 46}, models with generalized dynamics \cite{47} and interactions with impurities \cite{48–50}.

Currently there is still no experimental evidence of nonintegrable kink scattering. Recently, using molecular dynamics simulations, it was proposed the realization of nonintegrable $\phi^4$ kinks in buckled graphene nanoribbon \cite{51, 52}. Kinks in such system could be a possibility of experimental verification of negative pressure radiation effects \cite{53, 54}. Ab-initio excited-state dynamics was used for study the photogeneration and time evolution of topological excitations in $\textit{trans}$-polyacetylene. Upon lattice relaxation, the produced pair exhibits a pattern of two-bounce resonance characteristic of nonintegrable dynamics \cite{55}. Then, the investigation of models leading to new patterns of kink scattering in nonintegrable theories could be useful not only to better understand some aspects of nonlinearity, but also to interpret scattering results after ab initio calculations describing these or similar physical systems.

In this work we investigate kink scattering in a hybrid model with two topological sectors. The model appeared very recently in Ref. \cite{56}, generated through the reconstruction of field theories from reflectionless symmetric quantum mechanical potentials. This result, the constructions of two distinct field theories from the very same quantum mechanical potential, motivate us to further explore the hybrid model, with the aim to identify its specific features under the event of kink collisions. We notice that the presence of the two sectors seems to simulate the $\phi^6$ model studied in the Ref. \cite{21}; however, the stability analysis of kink (and antikink) gives a translational and a vibrational state, as it appears in the $\phi^4$ model. The numerical analysis of antikink-kink scattering shows the expected structure of one-bounce, bion and two-bounce windows explained by the CSW mechanism. However, the analysis of kink-antikink scattering gives unexpected results, such as one-bounce windows and thinner one-bounce windows related to the change of topological sector. We explain our findings of one-bounce windows with the CSW mechanism, adapted for the one-bounce windows.

In the next section we present the model already investigated in Refs. \cite{56, 57}, briefly reviewing its static kink and antikink solutions. In the Sec. III we present our main results and then conclude the work in the Sect. IV.
II. THE MODEL

Let us consider the action

\[ S = \int dtdx \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right), \]

with the potential \[ V(\phi) = \frac{1}{2} \phi^2 (1 - |\phi|)^2. \] (2)

Fig. 1 shows that the potential has three minima in \( \phi = 0 \) and \( \phi = \pm 1 \). Then we have two topological sectors connecting adjacent minima. There is symmetry around the minimum \( \phi = 0 \), so the system engenders symmetric \( \phi = 0 \) and asymmetric \( \phi = \pm 1 \) minima, and can be used to simulate a first-order phase transition, in a way similar to the case of the standard \( \phi^6 \) model. In the current case, however, given a topological sector, the potential is also symmetric around each one of its local maxima, and this induces important differences which will make the stability or Schrödinger-like potential (see below) symmetric, and so well distinct from the stability potential that appears in the \( \phi^6 \) model, which is asymmetric.

The equation of motion is

\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial t^2} - \frac{dV}{d\phi} = 0. \] (3)

Kink static solutions are given by

\[ \phi^{(0,1)}_K(x) = \frac{1}{2} \left( 1 + \tanh \left( \frac{x}{2} \right) \right), \] (4)

\[ \phi^{(-1,0)}_K(x) = \frac{1}{2} \left( -1 + \tanh \left( \frac{x}{2} \right) \right). \] (5)
Corresponding antikink solutions are given by \( \phi_K^{(1,0)}(x) = \phi_K^{(0,1)}(-x) \) and \( \phi_K^{(-1,0)}(x) = \phi_K^{(-1,-1)}(-x) \). Without losing generality we will work in the topological sector connecting the vacua \( \phi = 0 \) and \( \phi = 1 \). Fig. 1b depicts the kink profile \( \phi_K^{(0,1)}(x) \). Perturbing linearly the scalar field around one kink solution \( \phi_K(x) \) as \( \phi(x,t) = \phi_K(x) + \eta(x) \cos(\omega t) \) leads to the Schrödinger-like equation

\[
-\frac{\partial^2 \eta}{\partial x^2} + V_{sch} \eta = \omega^2 \eta.
\]

with the Schrödinger-like potential

\[
V_{sch} = \frac{d^2V}{d\phi^2}.
\]

This stability potential for the kink solution is presented in Fig. 1c. It has two eigenvalues, corresponding to the zero mode (translational mode) and the first bound state (vibrational mode \( \omega_1 = 0.75 \)). We have the same potential \( V_{sch} \) for the antikink \( \phi_K^{(1,0)}(x) \). Moreover, the potentials of perturbations for the coupled kink-antikink and antikink-kink are the same, having also one translational and one vibrational mode. This does not imply the same structure for both scattering processes, as described in the next section.

III. NUMERICAL RESULTS

Here we present our main results of antikink-kink and kink-antikink scattering. We solved the equation of motion with a pseudospectral method on a grid with 2048 nodes with periodic boundary conditions. We fixed \( x = \pm x_0 \) with \( x_0 = 15 \) for the initial symmetric position of the pair and set the grid boundary at \( x = \pm x_{max} \) with \( x_{max} = 200 \). A symplectic method with the Dirichlet condition imposed at the boundaries was also applied to double check our numerical results. We used a 4\(^{th}\) order finite-difference method with spatial step \( \delta x = 0.09 \) and a 6\(^{th}\) order symplectic integrator with time step \( \delta t = 0.04 \).

A. Antikink-kink collisions

To solve the equation of motion for antikink-kink scattering we use the following initial conditions

\[
\phi(x,0) = \phi_K(x-x_0,v,0) + \phi_K(x+x_0,-v,0),
\]

\[
\dot{\phi}(x,0) = \dot{\phi}_K(x-x_0,v,0) + \dot{\phi}_K(x+x_0,-v,0),
\]

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Fig. 2: Antikink-kink collisions: $\phi(x, t)$ (upper figures) and $\phi(x = 0, t)$ (lower figures), showing (a) bion state for $v = 0.18$, (b) two-bounces for $v = 0.2$ and (c) one-bounce for $v = 0.27$.

where $\phi_K(x + x_0, v, t)$ means a boost solution for kink. For $v < v_c$ with $v_c = 0.2599$, bion states are achieved, where the scalar field at the center of mass $\phi(0, t)$ changes after the scattering from the initial value $\phi = 0$ to erratic oscillations around the adjacent vacuum $\phi = 1$, as shown in Fig. 2a. After long time emitting scalar radiation, the antikink-kink pair annihilates and the scalar field goes to the vacuum $\phi = 1$. For $v > v_c$ the output is an inelastic scattering between the pair. In this case, $\phi(0, t)$ shows one-bounce (represented by $N_b = 1$) between the vacuum $\phi = 0$ - see Fig. 2c. Also, for some windows in velocities $v \lesssim v_c$, $\phi(0, t)$ presents two-bounce ($N_b = 2$) between the vacuum $\phi = 0$, as shown in Fig. 2b.

Fig. 3 summarizes our main results for the number of bounces as a function of initial velocity. Note the structure of two-bounce windows, whose thickness is reduced as one approaches the limit $v = v_c$ from below. The order of the 2-bounce windows is the number $m$ of oscillations between the bounces. For instance, Fig. 2b shows a plot of $\phi(0, t)$ with $m = 1$, belonging to the first two-bounce window. The CSW mechanism explains the
FIG. 3: Antikink-kink collisions: number of bounces $N_b$ versus initial velocity $v$ showing expected two-bounce windows according to CSW mechanism.

presence of two-bounce windows as a resonant mechanism described by

$$\omega_1 T' = 2\pi m + \theta_1,$$

where $T'$ is the time interval between the bounces and $\theta_1$ is a phase shift. For a detailed analysis of how this equation is verified for the $\phi^4$ model see Ref. [17]. Note also that in between the bounces the scalar field oscillates near to the initial vacuum $\phi = 0$. This is important for comparison with the antikink-kink collisions described in the following.

B. Kink-antikink collisions

In this case the initial conditions are given by

$$\phi(x, 0) = \phi_K(x + x_0, v, 0) + \phi_K(x - x_0, -v, 0) - 1,$$

$$\dot{\phi}(x, 0) = \dot{\phi}_K(x + x_0, v, 0) + \dot{\phi}_K(x - x_0, -v, 0).$$

We analyzed the collisions varying the initial velocity $v$. For $v > v_{\text{crit}}$, with $v_{\text{crit}} = 0.152$, the scalar field gains a phase shift changing the topological sector as $\phi^{(0,1)}_K(x, t) + \phi^{(1,0)}_K(x, t) \rightarrow \phi^{(0,-1)}_K(x, t) + \phi^{(-1,0)}_K(x, t)$, as depicted in Fig. 4a. The scalar field at the center of mass changes abruptly from the vacuum $\phi = 1$ to the vacuum $\phi = 0$, as shown in Fig. 4a. For most velocities $v < v_{\text{crit}}$ we have bion states and the scalar field at the center of mass, initially in the vacuum $\phi = 1$, oscillates erratically after the collision around the vacuum $\phi = 0$ - see Fig. 4b.
FIG. 4: (a) Scalar field $\phi(x,t)$ for the kink-antikink collision at high velocity $v > v_{\text{crit}}$ (here $v = 0.154$). Note the change of phase after the solution. (b) Bion state for $v < v_{\text{crit}}$ (here $v = 0.09$).

FIG. 5: Kink-antikink collisions: scalar field at the center of mass $\phi(x = 0, t)$ versus time for (a) $v = 0.104$, (b) $v = 0.124$ and (c) $v = 0.133$.

For some velocities $v \lesssim v_c$, despite linear perturbations lead to vibrational states for both kink and antikink, there is no evidence of two-bounces like the one described in Fig. 2b for antikink-kink collisions. There we remarked that $\phi(0, t)$ oscillates around the initial vacuum ($\phi = 0$ in Fig. 2b). Here we have windows in which the scalar field, initially in the vacuum $\phi = 1$, bounces once; during the bouncing the scalar field presents a certain number $N$ of oscillations in the other topological sector around $\phi = -1$ - see Figs. 5a-c. Fig. 6(a) shows the distribution of one-bounce windows in a plot of $N$ versus initial velocity. Note from the figure the presence of one-bounce windows with $N$ growing and their thickness decreasing with $v$. In particular, Figs. 5a-c correspond to the first three one-bounce windows from Fig. 6(a).
FIG. 6: Kink-antikink collisions: a) number \( N \) of oscillations during an one-bounce collision versus initial velocity. b) Close to the one-bounce windows there is a series of thinner windows with one-bounce collision followed by a change to the other vacuum state.

FIG. 7: Time interval \( T \) of the one-bounce versus number \( N \) of oscillations.

The occurrence of oscillations in the one-bounce collisions can be explained as a mechanism of resonance: initially the pair has its energy in the translational mode; during the oscillations the energy is stored in the vibrational mode. After some oscillations the pair is released provided a relation of the form

\[
\omega_1 T = 2\pi N + \theta_2, \tag{13}
\]
where $T$ is the time interval of the one-bounce and $\theta_2$ is a phase shift. This is similar to the CSW mechanism described by Eq. (10) originally for two-bounce windows, trading i) $T'$ by $T$ and ii) $m$, the number of oscillations between the bounces, by $N$. Fig. 7 shows that indeed there is a linear relation between $T$ and $N$. The measured slope 7.41 is close to the expected value $2\pi/\omega_1 = 7.26$.

Fig. 6b shows that close to the one-bounce windows we found another structure of thinner windows. These thinner windows appear for larger velocities, reducing progressively their thickness and accumulating around the maximum velocity of a given one-bounce window. An example of the peculiar structure of such collisions is depicted in Figs. 8a-c. Initially in the vacuum $\phi = 1$ and after one bounce with a number $N$ of oscillations, the scalar field at the center of mass has another number $n$ of oscillations before tunneling to the other vacuum. The pair $(N, n)$ characterize the $n^{th}$ thinner window near to the $(N - 1)^{th}$ one-bounce window. For example, Figs. 8a-c are characterized, respectively, by the pairs $(2, 1)$, $(2, 2)$, $(2, 3)$. There are collisions belonging to the first, second and third thinner windows, near the first one-bounce window.

IV. CONCLUSIONS

In this work we have investigated kink-antikink ($K\bar{K}$) and antikink-kink ($\bar{K}K$) in a hybrid model. In each one of the two topological sectors the potential is symmetric around the local minima. The equation of motion has symmetric static kink and antikink solutions, and the
FIG. 9: Kink-antikink a) before and b) after collision resulting in a change of topological sector, as $K\bar{K} \rightarrow \bar{K}K$. An impossible antikink-kink c) before and d) after collision with a change of topological sector, as $\bar{K}K \rightarrow K\bar{K}$.

stability analysis for kink and antikink result in a translational and a vibrational mode. We chose to work with initial configurations in the topological sector $(0, 1)/(1, 0)$. Antikink-kink collisions have the usual structure of one-bounce and bion states, with two-bounce windows explained by the CSW mechanism. Kink-antikink collisions result in a different structure, with one-bounce windows that was shown to satisfy an adapted CSW mechanism. Also there are windows substructures characterized by a change of topological sector. This could be represented as $K\bar{K} \rightarrow \bar{K}K$, and the solutions change from the topological sector $(0, 1)$ to the sector $(-1, 0)$ as described in Figs. 9a-b. A similar process for an antikink-kink, $\bar{K}K \rightarrow K\bar{K}$ would mean a changing from the sector $(1, 0)$ to the sector $(-1, 0)$ as described in Figs. 9c-d. This however is not possible since it would demand an infinite amount of energy to change the vacuum in an infinite length interval. Finally, if we had chosen initial configurations in the other topological sector $(0, -1)/(-1, 0)$, the symmetry of the potential guarantees the exchange of behaviors for $K\bar{K}$ and $\bar{K}K$ collisions when compared the results described above. This would mean two-bounce windows for kink-antikink collisions and the possibility of $\bar{K}K \rightarrow K\bar{K}$ with the changing of topological sector, but not for $K\bar{K} \rightarrow \bar{K}K$.

The hybrid model is then different from both the $\phi^4$ and $\phi^6$ models. It is similar to the
\( \phi^6 \) model, in the sense that it engenders two distinct topological sectors, but it is different, because each sector develops the same stability potential, which is symmetric and supports two bound states, the zero or translational mode, and another one, the vibrational mode. In this sense, it is similar to the \( \phi^4 \) model, but the kink-antikink scattering is different and leads to a new structure of bounce windows, which requires a modified mechanism to explain some of the results.

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