Majorana fermions in density modulated p-wave superconducting wires

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We study the p-wave superconducting wire with a periodically modulated chemical potential and show that the Majorana edge states are robust against the periodic modulation. We find that the critical amplitude of modulated potential, at which the Majorana edge fermions and topological phase disappear, strongly depends on the phase shifts. For some specific values of the phase shift, the critical amplitude tends to infinity. The existence of Majorana edge fermions in the open chain can be characterized by a topological $Z_2$ invariant of the bulk system, which can be applied to determine the phase boundary between the topologically trivial and nontrivial superconducting phases. We also demonstrate the existence of the zero-energy peak in the spectral function of the topological superconducting phase, which is only sensitive to the open boundary condition but robust against the disorder.

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I. INTRODUCTION

Searching for Majorana fermions (MFs) in condensed matter systems has attracted intensive studies in past years, due to the fundamental interest in exploring the new type of particles fulfilling non-Abelian statistics and the potential applications for the topological quantum computing. Among various proposals for realizing the emergent MFs, the quantum wires with p-wave pairings provide promising candidates for realizing emergent MFs at the ends of wires. As shown in the original work of Kitaev, boundary MFs emerge in the one-dimensional (1D) p-wave superconductor if the system is in a topological phase. Moreover, quantum wires with a strong spin-orbit coupling, or topologically insulating wires, subject to a Zeeman magnetic field and in proximity of a superconductor, are found to exhibit boundary MFs. Particularly, experimental signatures of MFs in hybrid superconductor-semiconductor nanowires have been reported very recently, which stimulates the study of exploring MFs in 1D systems. Schemes of realizing Majorana chains have also been proposed in cold atomic systems, carbon nanotubes, superconducting (SC) circuits, and quantum-dot-superconductor arrays.

As most of theoretical works focus on ideal homogeneous wires with a uniform chemical potential, an important problem is the stability of the MFs under the modulation of density and disorder. In this work, we explore the nonuniform p-wave SC wires with a periodic modulation of the chemical potential, extending the Kitaev’s p-wave SC model, and study the fate of the MFs under the density modulation and disorder. In general, a periodically modulated potential is of benefit to the formation of periodic density waves. If the modulation amplitude is large, one may expect that the SC phase could be destroyed. As the robustness of Majorana mode is protected by the SC gap, the MFs may be unstable in the presence of density modulation and disorder. On the other hand, recent work on the density modulated wires in the absence of SC order parameter indicates the existence of topologically protected edge states. So far, it is unclear whether the Majorana edge states is enhanced or suppressed under the density modulation. The current work will focus on this problem and show the robustness of Majorana edge states against the density modulation and disorder. Particularly, in some parameter regimes, we find that MFs always exist for the arbitrarily strong modulation strength, which may shed light on the design of quantum architectures producing robust MFs.

The rest of paper is organized as follows. We introduce the density modulated p-wave superconductor model and derive the Bogoliubov-de Gennes equations in Section II and demonstrate the zero mode Majorana edge states, which strongly depend on the phase shift, in spectra and wave functions in Section III. In order to confirm the transition, we define a $Z_2$ topological invariant in Section IV. Further we use the definition of the invariant to derive the phase boundaries in Section V. The spectral function is calculated in Section VI with a zero-energy peak detected corresponding to the zero Majorana modes. Section VII gives a summary.

II. MODEL OF DENSITY MODULATED P-WAVE SUPERCONDUCTORS

We consider a typical lattice model of the 1D p-wave superconductor with modulated chemical potentials, which is described by

$$H = \sum_i \left[ (-t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + H.c.) - \mu_i c_i^\dagger c_i \right],$$ \hspace{1cm} (1)

where $c_i^\dagger$ ($c_i$) is the creation (annihilation) operator of fermions at the $i$-th site, $t$ the nearest-neighbor hopping amplitude, $\Delta$ the p-wave SC order parameter and chosen real, and the modulated chemical potential given by

$$\mu_i = V \cos(2\pi i \alpha + \delta)$$ \hspace{1cm} (2)
with $V$ being the strength, $\alpha = p/q$ a rational number ($p$ and $q$ are co-prime integers), and $\delta$ an arbitrary phase shift. The modulated chemical potential can be generated by the bichromatic optical lattice for cold atom systems\textsuperscript{17,23} or through the control of gates for quantum-dot arrays and quantum wires\textsuperscript{20,24}. In this work, we only consider the case with $\alpha$ being a rational number. For the incommensurate case with $\alpha$ being an irrational number, the Hamiltonian (\textsuperscript{1}) with $\Delta = 0$ reduces to the Aubry-Andrè model\textsuperscript{25}, for which the system undergoes a delocalization to localization transition when $V > 2t$. Correspondingly, the SC system with nonzero $\Delta$ and irrational $\alpha$ also undergoes a transition from a topological phase to a topologically trivial localized phase when $V$ exceeds a critical value\textsuperscript{26}.

In order to diagonalize the quadratic form Hamiltonian, we resort to the Bogoliubov-de Gennes transformation\textsuperscript{27,28} and define a set of new fermion operators,

$$
\eta_n^\dagger = \frac{1}{2} \sum_{i=1}^{L} [(\phi_{n,i} + \psi_{n,i})c_i^\dagger + (\phi_{n,i} - \psi_{n,i})c_i],
$$

where $L$ is the number of lattice sites and $n = 1, \cdots, L$. For convenience, $\phi_{n,i}$ and $\psi_{n,i}$ are chosen to be real due to the reality of all the coefficients in the Hamiltonian (\textsuperscript{1}). In terms of the operators $\eta_n$ and $\eta_n^\dagger$, the Hamiltonian (\textsuperscript{1}) is diagonalized as

$$
H = \sum_{n=1}^{L} \Lambda_n \eta_n^\dagger \eta_n,
$$

where $\Lambda_n$ is the spectrum of the single quasi-particles. From the diagonalization condition, $[\eta_n, H] = \Lambda_n \eta_n$, we can get the following coupled equations:

$$
\Lambda_n \phi_{n,i} = (\Delta - \mu_i)\psi_{n,i+1} - \mu_i \psi_{n,i} - (\Delta + \mu_i)\psi_{n,i-1},
$$

$$
\Lambda_n \psi_{n,i} = -(\Delta + \mu_i)\phi_{n,i+1} + \mu_i \phi_{n,i} - (\Delta + \mu_i)\phi_{n,i-1},
$$

with $i = 1, \cdots, L$. From Eqs. (3), one can prove that if the equations have the solution of $(\phi_{n,i}, \psi_{n,i})$ ($i = 1, \cdots, L$) with a positive eigenvalue $\Lambda_n > 0$, $(\phi_{n,i}, -\psi_{n,i})$ is also the solution with the eigenvalue $-\Lambda_n$, which implies $\eta_n(\Lambda_n) = \eta_n^\dagger(-\Lambda_n)$.

### III. Boundary Majorana Fermions

The solution to Eqs. (3) is related to the boundary condition. The boundary Majorana states can only exist in the system with open boundary conditions (OBC). To seek such a state, we solve Eqs. (3) under OBC of $\psi_{n,L+1} = \psi_{n,0} = \phi_{n,L+1} = \phi_{n,0} = 0$, and the Majorana edge states correspond to the zero mode solution of $\Lambda_n = 0$. For $\Lambda_n = 0$, $\phi_i$ and $\psi_i$ are decoupled and Eqs. (3) can be written in the transfer matrix form

$$
(\phi_{n,i+1}, \phi_{n,i})^T = A_1(\phi_{n,i}, \phi_{n,i-1})^T,
$$

$$
(\psi_{n,i-1}, \psi_{n,i})^T = A_1(\psi_{n,i}, \psi_{n,i+1})^T,
$$

with $A_1 = \left(\begin{array}{cc} -\mu_i & \Delta - \mu_i \\ \Delta + \mu_i & -\mu_i \end{array} \right)$. To understand the boundary Majorana modes, we rewrite the operators $\eta_n$ as

$$
\eta_n^\dagger = \frac{1}{2} \sum_{i=1}^{L} [\phi_{n,i} \gamma_i^A + i \psi_{n,i} \gamma_i^B],
$$

where $\gamma_i^A = c_i^\dagger + c_i$ and $\gamma_i^B = i(c_i - c_i^\dagger)$ are operators of two MFs, which fulfill the relations $(\gamma_i^\beta)^\dagger = \gamma_i^\beta$ and $\{\gamma_i^\alpha, \gamma_{i'}^\beta\} = 2\delta_{ij}\delta_{n\beta}$ with $\alpha$ and $\beta$ taking $A$ or $B$. Coefficients of $\phi_{n,i}$ and $\psi_{n,i}$ in Eq. (4) are just the amplitudes of Majorana operators $\gamma_i^A$ and $\gamma_i^B$, respectively. If there exists a zero mode solution, according to Eqs. (4), we will get a decaying solution for one set of coefficients and a growing one for the other. An example is the special case with $V = 0$ and $\Delta = t$, for which the zero mode solution is given by $(\phi_1, \phi_2, \cdots, \phi_L) = (1, 0, \cdots, 0)$ and $(\psi_1, \cdots, \psi_{L-1}, \psi_L) = (0, \cdots, \phi, 0, 1)$. In this case, $\eta^\dagger(\Lambda = 0) = \frac{1}{2}[\gamma_1^A + i \gamma_1^B]$, which means that the zero mode state is divided into two separated MFs located at the left and right ends, respectively.

For the commensurate potential with $\alpha = p/q$, $A_i$ is $q$-periodic. One can judge whether the zero mode solution exists by evaluating the two eigenvalues of the product matrix $A = \prod_{i=1}^{L} A_1$. If both of them are either smaller or greater than unity, there is a zero mode solution with MFs located at the ends, otherwise no zero mode solution is available. Solving Eqs. (3), we can get the whole excitation spectrum for the single quasi-particles. To give typical examples of $\alpha = 1/2$ and $1/3$, we show the spectra under OPC (Fig. 1) and the lowest excitation energies, $A_1$, varying with $V$ under PBC and OBC (Fig. 1). It is shown that under OBC, there are zero modes in the...
bulk gap for the whole phase parameter space when \( V \) is small. As \( V \) increases, the excitation gap shrinks; When it exceeds a critical value, the gaps for some \( \delta \)'s close and then reopen with zero modes vanishing, corresponding to a transition from the topological phase to the topologically trivial phase, and the regimes for the existence of the zero modes in the phase parameter space become narrow. However, our numerical results indicate that the zero mode Majorana solutions always exist at some specific values of \( \delta \), such as \( \delta = \pi/2 \) for \( \alpha = 1/2 \) and \( \pi/6 \) for \( \alpha = 1/3 \), no matter how strong the strength \( V \) is (Fig.1 and Fig.2). To understand it, we observe that these \( \delta \)'s have an exotic characteristic, that is, each of them makes the value of \( \cos(2\pi i p/q+\delta) \) zero for one of the sites, \( i \), in a supercell composed of \( q \) sites, independent of the strength of the modulated potential \( V \). The fine tunability of the phase shift \( \delta \) effectively reduces the boundary potential and makes it possible to exist Majorana boundary states even for very large \( V \). This reminds us that we have the exact Majorana edge states for zero chemical potential in non-modulated systems. They have the same origin as the uniform case. From the transfer matrix, we can see the coefficients are bounded to the ends of the open wire if there is a zero modulated potential in a supercell. So these points are very strong to have Majorana boundary states. Their exact expressions can be derived from the boundary conditions in Section XV. For the systems with \( \alpha = 1/2 \) and 1/3 shown in Fig.1 the specific values of \( \delta \) are \( (2m+1)\pi/2 \) for \( \alpha = 1/2 \) and \( (2m+1)\pi/6 \) for \( \alpha = 1/3 \) with \( m \) being an integer. At these specific values \( \delta_n \), the critical value of the phase transition, \( V_c(\Delta, \delta_n) \), which is a function of \( \Delta \) and \( \delta \), tends to infinity. When deviated from these points, as shown in Fig.4, there exists a finite critical value, \( V_c(\Delta, \delta) \), above which there appears a topologically trivial phase without zero mode MFs.

Although we take \( L = 101 \) to give a concrete example in Fig.1 we note that the different length does not affect the existence of the zero mode and the bulk energy shape, however it affects the edge states in the higher excitation gaps. If we select \( L = 102 \), the only difference with the case of \( L = 101 \) is the left or right shifts of the edge states in the higher excitation gaps for \( \alpha = 1/3 \), the regimes with existence of zero modes do not change.

In order to see clearly the differences between the zero mode state and nonzero mode states, we display distributions of \( \phi_i \) and \( \psi_i \) for the zero mode solution and solutions with nonzero eigenvalues of quasi-particles located at the bottom and center of the continuous band (orange squares along the green cut in Fig.3). As shown in Fig.4 for the system with \( \delta = 5\pi/8 \) and \( V = 3 \), the amplitudes of the Majorana operators for the zero mode solution are located at the left and right ends, respectively. As \( \phi_i \) (\( \psi_i \)) decays very quickly away from the left (right) edge, there is no overlap for the Majorana modes of \( \gamma_i^A \) and \( \gamma_i^B \). As a comparison, distributions of \( \phi_i \) and \( \psi_i \) for nonzero modes spread over the whole regime and the corresponding quasiparticle operator can not be split into two separated Majorana operators.

**IV. \( Z_2 \) TOPOLOGICAL INVARIANT**

The presence or absence of zero mode MFs is determined by the \( Z_2 \) topological class of the bulk superconductor. As no boundary zero mode solution is available for the system with periodic boundary conditions (PBC), we can choose a \( Z_2 \) topological invariant (or ‘Majorana number’) to characterize the topological nature of the bulk system. In order to define such a topological in-
variant, we shall consider the system with PBC. For the periodic system with \( \alpha = p/q \), it is convenient to use the Fourier transform, \( c_i = c_{ij} = \sqrt{N} \sum_k c_{s,k} e^{ikq} \) with \( i = s + (l - 1)q \), which transforms the Hamiltonian \( H \) into

\[
H_k = -\sum_{s=1}^{q-1} \left( -tc^\dagger_{s,k} c_{s+1,k} + \Delta c_{s,k} c_{s+1,-k} \right) - tc^\dagger_{q,k} c_{1,k} e^{ikq} + \Delta c_{q,k} c_{1,-k} e^{-ikq} + H.c.
\]

(7)

where \( s = 1, \ldots, q \) is the index of inner sites in a supercell, \( l = 1, \ldots, L/q \) the index of the \( l \)-th supercell, and \( k \) the momentum defined in the reduced Brillouin zone of \([0, 2\pi/q]\). Then we define a set of new operators:

\[
\gamma_{2s-1} = c_{s,k} + c_{s,-k} \gamma_{2s} = \left(c_{s,k} - c_{s,-k}\right)/i,
\]

with the anticommutation relations \( \{\gamma^\dagger_m(k), \gamma_n(k')\} = 2\delta_{mn}\delta_{k,k'} \) and \( \gamma^\dagger_m(k) = \gamma_m(-k) \). \( \gamma_m(0) = \gamma_m(\pi/q) \) are just Majorana operators due to \( \gamma^\dagger_m(0) = \gamma_m(0) \) and \( \gamma^\dagger_m(\pi/q) = \gamma_m(-\pi/q) = \gamma_m(\pi/q) \). In the basis of the new operators, we can transform the Hamiltonian into the form:

\[
H = \frac{i}{4} \sum_k \sum_{m,n} B_{m,n}(k) \gamma_m(-k) \gamma_n(k),
\]

with \( B_{2s-1,2s} = -B_{2s-1,2s-1} = -\mu_s \) for \( s = 1, \ldots, q \), \( B_{2s-1,2s+1} = \Delta - t \), \( B_{2s+1,2s+1} = -2\Delta - t \), \( B_{1,2q} = -B^*_{2q,1} = -(\Delta + t)e^{-ikq} \), and \( B_{2q,2q-1} = -(\Delta - t)e^{-ikq} \).

The parameters \( B_{m,n}(k) \) form a \( 2q \times 2q \) matrix \( B(k) \), and here only \( B(0) \) and \( B(\pi/q) \) are skew-symmetric. Following Kitaev, we can calculate the \( Z_2 \) topological invariant defined as:

\[
M = \text{sgn}[\text{Pf}(B(0))] \text{sgn}[\text{Pf}(B(\pi/q))],
\]

(10)

where \( \text{Pf}(X) = \frac{1}{N!} \sum \text{sgn}(P) X_{P_1 P_2} \cdots X_{P_{N-1} P_N} \) is the Pfaffian of the skew-symmetric matrix \( X \) with \( P \) standing for a permutation of \( 2N \) elements of \( X \) and \( \text{sgn}(P) \) the corresponding sign of the permutation. According to the definition, generally we have \( M = \pm 1 \) with \( M = 1 \) corresponding to a \( Z_2 \)-topologically trivial phase and \( M = -1 \) to a \( Z_2 \)-topologically non-trivial phase. In Fig. 3 we display the \( Z_2 \) topological invariant versus the phase shift, \( \delta \), for systems with the same parameters as in Fig. 1. Comparing Fig. 3 with Fig. 1, we see the exact correspondence between the presence (absence) of zero mode MFs and \(-1 (+1)\) value of the \( Z_2 \) topological invariant. Especially, \( M \) takes the value of 0 (blue square dots in Fig. 3(b)) at \( V = 2 \), which is just the critical point \( V_c \) of the phase transition from a \( Z_2 \)-topologically non-trivial phase to a topologically trivial phase for the corresponding \( \delta \). In principle, we can always determine \( V_c(\Delta, \delta) \) through the condition of \( M = 0 \) and give the whole phase diagram.

\[\text{FIG. 4: } Z_2 \text{ topological invariant, } M, \text{ vs the phase shift, } \delta, \text{ for system with } t = \Delta = 1, \alpha = \frac{1}{2} ((a)-(c)) \text{ and } \frac{1}{3} ((d)-(f)) \text{ under PBC.}\]

\[\text{FIG. 5: Phase diagrams expanded by } V, \Delta, \text{ and } \delta \text{ for the cases of (a) } \alpha = \frac{1}{2} \text{ and (b) } \alpha = \frac{1}{3}. \text{ Here we set } t = 1. \text{ The curved surfaces are the phase boundaries above which is } Z_2 \text{-topologically trivial while below which is } Z_2 \text{-topologically non-trivial.} \]

\[\text{V. PHASE DIAGRAM}\]

Without loss of generality, we choose \( \Delta, V \geq 0 \) in the following discussion. From the definition of the matrix \( B(k) \), we know that for \( \alpha = 1/2, \text{ Pf}(B(0)) = -V^2 \cos^2 \delta - 4t^2 \) and \( \text{Pf}(B(\pi/2)) = -V^2 \cos^2 \delta + 4\Delta^2 \). Here due to \( \text{Pf}(B(0)) < 0 \), we just need to set \( \text{Pf}(B(\pi/2)) = 0 \) to get the phase boundary condition, that is, \( V^2 \cos^2 \delta = 4\Delta^2 \), or equivalently \( V | \cos \delta | = 2\Delta \) (Fig. 3(a)). Likewise, for \( \alpha = 1/3, \text{ Pf}(B(0)) = -\frac{1}{3} \cos 3\delta - 2(t^2 + 3\Delta^2) \) and \( \text{Pf}(B(\pi/3)) = -\frac{2}{3} \cos 3\delta + 2(t^2 + 3\Delta^2) \). The phase boundary condition can be also got easily as \( V^3 | \cos 3\delta | = 8t(t^2 + 3\Delta^2) \) (Fig. 3(b)). From Fig. 3 we see clearly again that there exist some specific points \( \delta_s \), at which the system is always \( Z_2 \)-topologically non-trivial for arbi-
trary $V$. On the other hand, the critical value, $V_c(\Delta, \delta)$, increases with the increase of the SC pairing amplitude $\Delta$.

For the general case of $\alpha = p/q$, we can infer the phase boundary condition by analyzing the expression of Pfaffian. The non-permutated term, $B_{12}B_{34} \cdots B_{2q-1,2q}$, gives $(-1)^q \prod_{s=1}^q \mu_s$. Due to the sparsity of the matrix $B$, we see that there are only three kinds of permutations which contribute non-zero terms: 1. $B_{2s-1,2s}B_{2s+1,2s+2} \rightarrow B_{2s-1,2s+2}B_{2s,2s+1}$ or $B_{1,2}B_{2q,1-2q} \rightarrow B_{1,2q}B_{2,2q-1}$, which makes a replacement of $\mu_s \mu_{s+1}$ or $\mu_s \mu_{q}$ in the non-permutated term by $(\Delta^2 - t^2)^q$ or $(\Delta^2 - t^2)e^{-2ikq}$, respectively; 2. $B_{2s-1,2s} \rightarrow B_{2s,2s+1}$ ($s = 1, \ldots, q - 1$) and $B_{2q-1,2q} \rightarrow B_{1,2q}$, which gives $-(t + \Delta)^q e^{-ikq}$; 3. $B_{2s-1,2s} \rightarrow B_{2s-1,2s+2}$ ($s = 1, \ldots, q - 1$) and $B_{2q-1,2q} \rightarrow B_{2,2q-1}$, which gives $-(t - \Delta)^q e^{-ikq}$. These permutations are independent of each other. Our numerical study shows that when $k = 0$ and $\pi/q$, the sum of all terms generated by the first kind of permutations is zero except the full-permutated ones (if there is) with any $-\mu_s$ left. So the first kind contributes a term of $0$ for odd $q$'s or $(\Delta^2 - t^2)^q [\Delta^2 - t^2 e^{-2ikq}]$ for odd $q$'s. Thus we have $\text{Pf}(B(0)) = -\prod_{s=1}^q \mu_s - [(t + \Delta)^q + (t - \Delta)^q]$ for odd $q$'s or $\prod_{s=1}^q \mu_s - [(t + \Delta)^q/2 - (t - \Delta)^q/2]$ for even $q$'s, and $\text{Pf}(B(\pi/q)) = -\prod_{s=1}^q \mu_s + [(t + \Delta)^q + (t - \Delta)^q]$ for odd $q$'s or $\prod_{s=1}^q \mu_s + [(t + \Delta)^q/2 - (t - \Delta)^q/2]$ for even $q$'s. Therefore, we can get the general formula for the boundaries: For $q$ odd,

$$\prod_{s=1}^q \mu_s = (t + \Delta)^q + (t - \Delta)^q,$$

(11)

and for $q$ even,

$$-\prod_{s=1}^q \mu_s = (t + \Delta)^q + (t - \Delta)^q + 2(\Delta^2 - t^2)^q/2$$

(12)

if $\prod_{s=1}^q \mu_s < 0$, or

$$\prod_{s=1}^q \mu_s = (t + \Delta)^q + (t - \Delta)^q - 2(\Delta^2 - t^2)^q/2$$

(13)

if $\prod_{s=1}^q \mu_s > 0$. The special cases of $\alpha = 1/2$ and $1/3$ can be derived from these general formulas.

The above results show that the phase boundary formula is different for $q$ being even or odd. We note that the even-odd effect comes from the cosine form of the modulated potential, the values of which have a mirror symmetry with respect to zero value if we choose even number of sites in one period. That does not affect the fact of the existence of these special $\delta$, where the critical amplitude $V_c$ tends to infinity. Only the number of these points are different, i.e., $q$ for even cases and $2q$ for odd cases.

\section*{VI. SPECTRAL FUNCTION}

As the Majorana edge states correspond to the zero mode solution protected by the presence of an energy gap, one would expect a zero-energy peak appearing in the corresponding spectral function. The momentum-resolved spectral function with momentum $k$ and energy $\omega$ ($\hbar = 1$) is defined as $A(k, \omega) = \langle \mu | e^{-iHt} | \mu \rangle / \Delta$, where $G_r(k, \omega) = \int_{-\infty}^{\infty} G_r(k, t) e^{i\omega t - \delta^2 r^2} dt$ and $G_r(t) = -i\delta(t) \langle \mu | G \{c_k(t), c_k^2(0)\} | \mu \rangle$ is the single particle ( retarded) Green function with $|\mu\rangle$ being the ground state of the system, and $c_k(t) = e^{iHt} c_k e^{-iHt}$ is the fermionic annihilation operator in the momentum space with $c_k = \frac{1}{\sqrt{2}} \sum_j c_j e^{-ikj}$. And the corresponding total spectral function is $A(\omega) = \sum_k A(k, \omega)$. As shown in Fig. (a) and (b), an obvious zero-energy peak is observed for the system with OBC. On the contrary, we find no zero-energy peak in the system with PBC.

The Majorana edge states are expected to be immune to local perturbations. To explore the effect of disorder, here we add a random on-site potential, $H_0 = \sum_i W_i c_i^\dagger c_i$, to the Hamiltonian (1), which usually leads to Anderson localizations, where $W_i$ is uniformly distributed in the range of $[-W/2, W/2]$ with $W$ being the disorder strength. Fig. (c) shows that energy gaps
between the zero mode and the excitation modes are smeared when the disorder strength is comparable to the excitation gap. Even in the presence of strong disorder, for example, $W = 10$, the peak still exists, although its relative height becomes shorter as well. Meanwhile, the bulk energy bands are broadened by the disorder, and other peaks are not stable under disorder as the zero peak. Finally, they will be smeared by the disorder with the increase in disorder strength. The robustness of the zero peak against local perturbations provides the possibility to detect it in experiment even affected by environment disorder.

VII. SUMMARY

In summary, we find that in the density modulated p-wave SC wires, by tuning the phase shift, the zero mode Majorana edge states emerge in some intervals, and some of them are strong enough, independent of the strength of the periodic density modulation. The appearance of the MFs demonstrate the $Z_2$ topological nature. After calculating the $Z_2$ topological invariant, we get the phase diagram for the transition from topologically nontrivial phases to trivial phases. By this model, we supply a good platform to have possible schemes of searching for MFs. At last, we also give an evidence in the spectral function where a zero-energy peak appears under OBC, even subject to disorder. The spectral function is possible to be experimentally detected by the photoemission spectroscopy.

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