EQUATIONS FOR SOME NILPOTENT VARIETIES

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To the memory of Bert Kostant

Abstract. Let $O$ be a Richardson nilpotent orbit in a simple Lie algebra $g$ of rank $n$ over $\mathbb{C}$, induced from a Levi subalgebra whose $s$ simple roots are orthogonal, short roots. The main result of the paper is a description of a minimal set of generators of the ideal defining $O$ in $S g^*$. In such cases, the ideal is generated by bases of either one or two copies of the representation whose highest weight is the dominant short root, along with $n - s$ fundamental invariants of $S g^*$. This extends Broer’s result for the subregular nilpotent orbit, which is the case of $s = 1$.

Along the way we give another proof of Broer’s result that $O$ is normal. We also prove a result relating a property of the invariants of $S g^*$ to the following question: when does a copy of the adjoint representation in $S g^*$ belong to the ideal in $S g^*$ generated by another copy of the adjoint representation together with the invariants of $S g^*$?

1. Introduction

1.1. Let $G$ be a connected simple algebraic group over $\mathbb{C}$ with Lie algebra $g$. Let $S g^*$ be the coordinate ring of $g$, $R := (S g^*)^G$ its subring of invariants, and $R_+ \subset R$ the invariants without constant terms. By Chevalley $R$ is a polynomial ring in $n$ generators where $n$ is the rank of $G$. Let $f_1, f_2, \ldots, f_n$ be a set of fundamental invariants of $R$, that is, a set of homogeneous generators of $R$, with $\deg f_1 \leq \ldots \leq \deg f_n$. The degrees $d_1 \leq \cdots \leq d_n$ of these invariants are called the degrees of $G$ and the exponents of $G$ are the numbers $m_i := d_i - 1$ for $1 \leq i \leq n$.

Let $N$ denote the variety of nilpotent elements in $g$ and let $\mathbb{C}[N]$ denote the regular functions on $N$. Since $N$ is closed under scalars, $\mathbb{C}[N]$ is graded and there is a graded surjection $S^i g^* \to \mathbb{C}^i[N]$ for each $i \in \mathbb{N}$. Kostant [Kos63] showed that the ideal in $S g^*$ defining $N$ is $(R_+) = (f_1, \ldots, f_n)$.

Let $\lambda$ denote the variety of nilpotent elements in $g$ and let $\mathbb{C}[N]$ denote the regular functions on $N$. Since $N$ is closed under scalars, $\mathbb{C}[N]$ is graded and there is a graded surjection $S^i g^* \to \mathbb{C}^i[N]$ for each $i \in \mathbb{N}$. Kostant [Kos63] showed that the ideal in $S g^*$ defining $N$ is $(R_+) = (f_1, \ldots, f_n)$.
subgroup $P$ containing $T$, let $X^*(P) \subset X^*(T)$ denote the characters of $P$. Let $W$ be the Weyl group with respect to $T$ and let $s_\alpha$ be the reflection corresponding to $\alpha \in \Phi$.

1.2. The subregular nilpotent orbit $O_{sr}$ is the unique nilpotent orbit in $g$ of dimension equal to $\dim N - 2$. Broer described the ideal defining $O_{sr}$ in $g$.

**Theorem 1.1** (Theorem 4.9 in [Bro93]). The ideal defining $O_{sr}$ in $C[\mathfrak{N}]$ is minimally generated by a basis of the unique copy of $V_{\phi}$ in $C^{\text{ht}(\phi)}[\mathfrak{N}]$.

The ideal defining $O_{sr}$ in $Sg^*$ is minimally generated by $f_1, \ldots, f_{n-1}$, together with a basis for any copy of $V_{\phi}$ which has nonzero image in $C^{\text{ht}(\phi)}[\mathfrak{N}]$.

The main result of this paper is a similar description of the ideal defining $O_{\Theta}$ in $Sg^*$ for certain Richardson orbits studied in [Bro94]. Given $\Theta \subset \Pi$, define $l_\Theta$ to be the corresponding Levi subalgebra containing $h$, and let $p_\Theta$ be the parabolic subalgebra containing both $l_\Theta$ and $b$, with $P_\Theta$ the corresponding subgroup of $G$. Let $n$ denote the nilpotent radical of $b$ and let $n_\Theta$ be the nilradical of $p_\Theta$. So by our choice of $B$, the root spaces of $n$ correspond to the negative roots.

1.3. The generalized exponents $m^\lambda_i$ of $V_\lambda$ are defined by the equation

$$
\sum_{j \geq 0} \dim \text{Hom}_G(V_\lambda, C^j[\mathfrak{N}]) t^j = \sum_{i=1}^k t^{m^\lambda_i},
$$

where by Kostant [Kos63] we have

$$
k = \dim V^T_\lambda
$$

for the number of generalized exponents, where the superscript denotes the $T$-invariants. Also from loc. cit. the generalized exponents of $V_\phi$ are the usual exponents $m_i$ defined above. Indeed, given an invariant $f \in R_+$ and a basis $\{x_i\}$ of $\mathfrak{g}$, the derivatives $\{\frac{\partial f}{\partial x_i}\}$ span a copy of $V_\phi$ in $S\mathfrak{g}^*$, independent of the choice of basis, and moreover, the images of the derivatives of the chosen fundamental invariants $f_i$ are a basis for the $V_\phi$-isotypic component of $C[\mathfrak{N}]$. We are also interested in the generalized exponents for $V_\phi$. Let $r = \dim V^T_\phi$, which equals the number of short roots in $\Pi$ and let $m^\phi_1 \leq \cdots \leq m^\phi_r$ be the generalized exponents for $V_\phi$. Of course, if $g$ is simply-laced these coincide with the usual exponents. In Appendix A.1 and A.2, we recall two ways to determine the $m^\phi_i$ for non-simply-laced types.

When $\Theta$ consist of orthogonal short simple roots, Broer [Bro94] showed that $O_{\Theta}$ is a normal variety and the map $G \times^P n_\Theta \rightarrow O_{\Theta}$ is birational (we give another proof of these facts in §4). Hence by a result of Borho-Kraft (see [Bro94] for the graded version) the analogue of Kostant’s result (1.1) is

$$
dim V^T_\lambda = \dim \left( \text{Hom}_G(V_\lambda, C[O_\Theta]) \right).
$$
The invariants on the left-side of (1.2) are easy to compute for $V_\theta$ and $V_\phi$ since these representations only have roots as non-zero weights. Setting $s = |\Theta|$, the left-side becomes $n - s$ and $r - s$, respectively, since $\Theta$ consists only of short simple roots.

Let $I_\Theta$, respectively $J_\Theta$, be the ideal defining $\overline{O}_\Theta$ in $\mathbb{C}[\mathcal{N}]$, respectively $S_{\mathfrak{g}}^\ast$. Already then we know that there are $s$ independent copies of $V_\theta$ (and of $V_\phi$) in $\mathbb{C}[\mathcal{N}]$ which lie in $I_\Theta$. The main result of the paper is that either one or two copies of $V_\phi$ are needed to generate $I_\Theta$. We need one more definition before stating the result precisely.

1.4. Outside of types $D_n$ and $E_7$, given our assumption on $\Theta$, there is only one orbit $O_\Theta$ for any given value of $s = |\Theta|$. In type $E_7$, there are two orbits with $s = 3$. In type $D_n$, there are two orbits when $s = 2, 3, \ldots, \lceil n/2 \rceil - 1$, with partitions $[2n - 2s - 1, 2s + 1]$ and $[2n - 2s + 1, 2s - 3, 1, 1]$, and three orbits when $n$ is even and $s = n/2$, because of the two very even orbits with partition $[n, n]$, together with the orbit with partition $[n + 1, n - 3, 1, 1]$.

To state the theorem uniformly, we designate two families of orbits among the orbits we are considering. For $e \in O_\Theta$, complete $e$ to an $\mathfrak{sl}_2$-triple $\{e, h, f\}$ with $h \in \mathfrak{h}$ dominant. We assign $O_\Theta$ to the first family if
\begin{equation}
\label{eq:condition1}
m_{r-s+1}^\phi > \phi(h)
\end{equation}
and to the second family, otherwise. It turns out that there are at most two values of $\phi(h)$ for a given $s$ and when there are two values, the smaller one always satisfies the inequality (1.3) and the larger one does not. Also, when there is one orbit for a given $s$, it satisfies the inequality and hence lies in the first family. A calculation shows that the second family consists of $O_\Theta$ with Bala-Carter label $E_6$ in type $E_7$ or with partition $[2n - 2s + 1, 2s - 3, 1, 1]$ for $2 \leq s \leq n/2$ or $[n, n]$ in type $D_n$. See Figure II for some examples. Inequality (1.3) will also be relevant for the proof of the theorem (§5.5).

**Definition 1.2.** For $s \geq 1$, set $m_\Theta$ equal to $m_{r-s+1}^\phi$ or $m_{\lceil r/2 \rceil}^\phi$ according to whether $O_\Theta$ is in the first or second family, respectively.

Our main result, for $\Theta$ consisting of orthogonal short simple roots and $s \geq 1$, is the following.

**Theorem 1.3.** The ideals $I_\Theta$ and $J_\Theta$ are described as follows:

1. The lowest degree copy of $V_\phi$ in $I_\Theta$ occurs in degree $m_\Theta$. Denote such a copy by $V$.
2. For $s \geq 2$, there is a copy $V'$ of $V_\phi$ in $I_\Theta$ in degree $m_{s+2}^\phi$, different from $V$.
3. A basis of $V$ minimally generates $I_\Theta$, except when $O_\Theta$
   \begin{itemize}
   \item has Bala-Carter label $E_6(a_3)$ in type $E_6$; $E_7(a_3)$ or $E_6(a_1)$ in type $E_7$; $E_8(a_3)$ or $E_8(a_4)$ in type $E_8$, or
   \item has partition $[2n - 2s + 1, 2s - 3, 1, 1]$ for $s \geq 3$ in type $D_n$.
   \end{itemize}
   In these cases, a basis of $V'$ is also needed to minimally generate $I_\Theta$.
4. $J_\Theta$ is minimally generated by $n-s$ fundamental invariants and any pre-image of a basis of $V$ and also, in the cases in part (3), of a basis of $V'$. The $n-s$ invariants have degree
Figure 1. The studied nilpotent varieties with second family in red

\[ \begin{array}{ccc}
\{7,1\} & \{9,1\} & \{11,1\} \\
\{5,3\} & \{7,3\} & \{11,3\} \\
\{4,4\}^1 & \{4,4\}^2 & \{9,3\} \\
\{5,5\} & \{7,1,1,1\} & \{9,1,1,1\} \\
\{3,3,1,1\} & \{5,3,1,1\} & \{9,5\} \\
\{6,6\}^1 & \{6,6\}^2 & \{7,3,1,1\} \\
\{7,7\} & \{9,3,1,1\} & \{7,5,1,1\} \\
\end{array} \]

(a) In types $D_4$, $D_5$, $D_6$, and $D_7$

\[ \begin{array}{ccc}
E_7 & E_8 \\
E_6 & E_7(a_1) & E_8(a_1) \\
E_6(a_1) & E_7(a_2) & E_8(a_2) \\
D_5 & E_7(a_3) & E_6 \\
E_6(a_3) & E_6(a_1) & E_8(a_4) \\
\end{array} \]

(b) In types $E_6$, $E_7$, and $E_8$

$d_1, d_2, \ldots, d_{n-s}$ for orbits in the first family and $d_1, d_2, \ldots, \hat{d}_{\lceil \frac{n}{2} \rceil}, \ldots, d_{n-s+1}$ for orbits in the second.

Outside of type $D_n$ with $n$ even, there is a unique choice for $V$ and for $V'$ in the statement of theorem. In type $D_n$ with $n$ even, where $n = r$, there are two equal exponents: $m_\frac{r}{2} = m_\frac{r+1}{2} = n - 1$, so we will give the precise choice for $V$ when $m_{\Theta} = n - 1$ in §5. The choice of $V'$ will be unique except when $s = \frac{n}{2} + 1$, in which $V$ and $V'$ can be any choices so that $V + V'$ equals the $V_\Theta$-isotypic space in $\mathbb{C}^{n-1}[N]$.

1.5. The proof works by induction on $s$, with the base case of $s = 0$ due to Kostant. The $s = 1$ case is Broer's result, with the same proof. The general case relies on various cohomological statements established in §2. To find a minimal set of generators, we prove a result, Proposition 4.2, that makes use of recent results related to flat bases of invariants from [DCP15].
In type $A_n$ the result, with a different proof, is due to Weyman [Wey02] (see also [Wey89]). Here, $O_\Theta$ depends only $s = |\Theta|$ and the possible orbits have partition type $[n+1-s, s]$ with $s < \frac{n}{2}$. Let $X$ be a generic matrix of $\mathfrak{gl}_{n+1}$. That is, $X = (x_{ij})$, where the $x_{ij}$ are $(n+1)^2$ variables. Then for an integer $k \geq 1$, the entries of any matrix power $X^k$ of $X$ span a copy of the adjoint representation of $g \subset \mathfrak{gl}_{n+1}$ and a copy of the trivial representation in $S^k g^\ast$. Weyman’s result is that $J_\Theta$ is minimally generated by the entries of $X^{n+1-s}$ and the fundamental invariants $\text{tr}(X^i)$ for $2 \leq i \leq n-s$, given by the traces of these powers. Note that the entries of $X^{n+1-s}$ already contain $\text{tr}(X^{n+1-s})$ in their span, so this agrees with the statement of the theorem. It is easy enough to see that each entry of $X^{n+1-s}$ vanishes on $O_\Theta$ since $M^{n+1-s} = 0$ for $M \in O_\Theta$, so the main part is to show that these entries are enough to generate $I_\Theta$.

Our theorem has a similar interpretation in the other classical Lie algebras using the standard matrix representations ($\S 5$). We can also find a matrix interpretation in the exceptional groups using the smallest non-trivial irreducible representation of $g$ ($\S 5$ and Appendix $B$); this is useful for the applications considered by the first author in [Joh17]. After establishing the needed cohomological statements in $\S 2$, we use them to locate a sufficient set of generators of $I_\Theta$ in $\S 3$. In $\S 4$ we prove the results related to invariants to locate a minimal set of generators of $J_\Theta$, and then give the explicit descriptions in each case in $\S 5$, completing the details of the proof of Theorem 1.3. In $\S 6$ we find the defining equations for some additional nilpotent varieties in the non-simply-laced types that occur by folding a simply-laced $g$. In $\S 7$ we determine which $\Theta$ have a $P_\Theta$-covariant of weight $\phi$; this relies on some direct calculations in $\S 8$. In Appendices $A$ and $B$, we record some results about finding explicit invariants.

2. Cohomological Statements

Let $\Omega$ be a set of orthogonal short simple roots. The proof of the main theorem makes use of results about the cohomology groups

$$H^i(G/P_\Omega, S^\bullet \mathfrak{n}_\Omega^\ast \otimes \mathbb{C}_\lambda) \simeq H^i(G/B, S^\bullet \mathfrak{n}_\Omega^\ast \otimes \mathbb{C}_\lambda),$$

where $\lambda \in X^\ast(P_\Omega)$ and $\mathbb{C}_\lambda$ is the corresponding one-dimensional representation of $P_\Omega$. We refer to Jantzen [Jan04] for definitions. Note that $\lambda \in X^\ast(P_\Omega)$ means $\langle \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in \Omega$. To simplify notation, for $m \in \mathbb{Z}$, we follow Broer and write

$$H^i_\Omega(\lambda)[-m]$$

to refer to the graded $G$-module, with grading by $j \in \mathbb{Z}$,

$$\bigoplus_{j \in \mathbb{Z}} H^i(G/B, S^{j-m} \mathfrak{n}_\Omega^\ast \otimes \mathbb{C}_\lambda).$$

The starting point of the proof of the theorem is that the Springer resolution $G \times^P \mathfrak{n} \to \mathcal{N}$ is birational and $\mathcal{N}$ is normal so that $\mathbb{C}^i[\mathcal{N}]$ is isomorphic as $G$-module to $H^0(G/B, S^i \mathfrak{n}^\ast)$. Then,
as in Broer’s proof, we show that $\overline{O}_\Omega$ is cut out from some $\overline{O}_\Omega$ with $|\Omega| = s - 1$ by an ideal whose graded $G$-module structure equals that of $H^i(\phi)[-m]$, for some degree shift $m$.

We need to show at various stages that the higher cohomology of some of these modules vanishes. One case that is known for general $P_\Omega$ is Theorem 2.2 in [Bro94]: Let $\lambda \in X^*(P_\Omega)$ be dominant, then

$$H^i(\lambda) = 0 \text{ for } i > 0,$$

where we leave off the graded shift when $m = 0$.

By a sequence of cohomological moves, we can sometimes prove vanishing for mildly non-dominant $\lambda \in X^*(P_\Omega)$. A full account for the case of $\Omega = \emptyset$ is given by [Bro93], Theorem 2.4. The basic move, which we call the $A_1$-move, goes back to Demazure [Dem76] and is the basis for the $\Omega = \emptyset$ result in [Bro93]. A general $A_k$-move is the subject of [Som05] and this result gets used to prove the normality of nilpotent varieties in type $E_6$ [Som03] and of the very even nilpotent varieties in type $D_n$ with $n$ even [Som03].

We now summarize the three moves needed in this paper. For simplicity, we assume $\lambda \in X^*(P_\Omega)$, that is, $\langle \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in \Omega$ and, as above, that $\Omega$ consists of orthogonal simple short roots, but these results hold more generally.

**Proposition 2.1.** Let $\beta \in \Pi$ with $\langle \lambda, \beta^\vee \rangle = -1$.

1. If $\beta$ is orthogonal to all roots in $\Omega$, then $s_\beta(\lambda) = \lambda + \beta$ and

$$H^i(\lambda) \simeq H^i(\lambda + \beta)[-1] \text{ for all } i \geq 0 \quad (A_1 \text{ move}).$$

2. Let $\beta_1 \in \Omega$ be such that $\beta_1, \beta$ determine an $A_2$-subsystem and such that $\langle \lambda, \beta_1^\vee \rangle = 0$. If $\beta_1$ and $\beta$ are both orthogonal to all roots in $\Omega \setminus \{\beta_1\}$, then $s_{\beta_1} s_{\beta}(\lambda) = \lambda + \beta + \beta_1$ and

$$H^i(\lambda) \simeq H^i(\lambda + \beta + \beta_1)[-1] \text{ for all } i \geq 0 \quad (A_2 \text{ move}),$$

where $\Omega' := s_{\beta_1} s_{\beta}(\Omega) = (\Omega \setminus \{\beta_1\}) \cup \{\beta\}$.

3. Let $\beta_1, \beta_2 \in \Omega$ be such that $\beta_1, \beta, \beta_2$ determine an $A_3$-subsystem and such that $\langle \lambda, \beta_2^\vee \rangle = 0$. If $\beta_1, \beta, \beta_2$ are each orthogonal to all roots in $\Omega \setminus \{\beta_1, \beta_2\}$, then $s_{\beta_1} s_{\beta_2} s_{\beta}(\lambda) = \lambda + \beta_1 + 2\beta + \beta_2$ and

$$H^i(\lambda) \simeq H^i(\lambda + \beta_1 + 2\beta + \beta_2)[-2] \text{ for all } i \geq 0 \quad (A_3 \text{ move}).$$

There is also an $A_2$-move, as in (2), when instead $\langle \lambda, \beta^\vee \rangle = 0$. Namely,

$$H^i(\lambda) \simeq H^i(\phi)[-m] \text{ for all } i \geq 0,$$

This is useful to change between associated parabolic subgroups.

The main result of this section is the following.

**Theorem 2.2.** Let $\lambda \in X^*(P_\Omega)$ be a short positive root. Then for some $m \in \mathbb{N}$

$$H^i(\lambda) \simeq H^i(\phi)[-m] \text{ for all } i \geq 0,$$
where $\Omega' = w(\Omega)$ and $\phi = w(\lambda)$ for some $w \in W$. In particular, $\Omega'$ consists of orthogonal short simple roots and $\phi \in X^*(P_{\Omega'})$.

**Proof.** If $\lambda \neq \phi$, then there exists $\beta \in \Pi$ such that $\langle \lambda, \beta' \rangle = -1$ since $\lambda$ is short. Clearly, $\beta \notin \Omega$. Also $g$ is simple and not of type $A_1$ since $\lambda \neq \phi$, so $\beta$ is connected in the Dynkin diagram to at least one other and at most three other simple roots, denoted $\beta_1, \ldots, \beta_k$ ($1 \leq k \leq 3$). We will treat the three cases separately:

1. $k = 3$. Then $\Pi$ contains a $D_4$ subsystem with $\beta$ as the central node and $\beta_1, \beta_2, \beta_3$ as the outer nodes. If $\beta_i \in \Omega$ for all $i$, then $\lambda$ is $W$-conjugate to $\lambda + 2(\beta_1 + 2\beta + \beta_2 + \beta_3)$. Since $\lambda$ and $\gamma := \beta_1 + 2\beta + \beta_2 + \beta_3$ are both short roots (since, e.g., $\beta_1$ is short), results on root strings show this is impossible unless $\lambda = -\gamma$ since such a root string can only consist of two roots. But the latter is ruled out since $\lambda$ is positive.

   If say $\beta_1, \beta_2 \in \Omega$ and $\beta_3 \notin \Omega$, then we can use the $A_3$-move, with the $A_3$ comprising $\beta_1, \beta_2$ since any simple root connected to this $A_3$ cannot be in $\Omega$. Similarly, if, say, only $\beta_1 \in \Omega$ we can use the $A_2$-move and otherwise the $A_1$-move.

2. $k = 2$. Then $\beta_1, \beta, \beta_2$ form a connected subsystem of rank 3. First, we show that if $\beta_1 \in \Omega$ or $\beta_2 \in \Omega$, then $\beta$ is also short. Assume $\beta_1 \in \Omega$ and $\beta$ is long. Then
   
   $$s_\beta s_{\beta_1} \beta(\lambda) = \lambda + r(\beta + \beta_1)$$

   is a root with $r = 2$ or $r = 3$. Again by results on root string this is impossible if $\lambda$ is a positive root since $\lambda$ and $\beta + \beta_1$ are both short.

   Consequently, either $\beta$ is long and neither $\beta_1$ nor $\beta_2$ is in $\Omega$ and so we can use the $A_1$-move. Or, $\beta_1$ and/or $\beta_2$ belong to $\Omega$ and $\beta$ is short and then we can use the $A_3$-move or $A_2$-move depending on whether or not both are in $\Omega$.

3. $k = 1$. This case is the same as the previous one.

In all cases, we get

$$H_i^\lambda(\lambda) \approx H_i^{\lambda'}(\lambda')[-m']$$

for some positive integer $m'$ and moreover $\Omega' = x(\Omega)$ and $\lambda' = x(\lambda)$ for some $x \in W$. In particular, $\lambda' \in X^*(P_{\Omega'})$. Since $\text{ht}(\lambda') > \text{ht}(\lambda)$, the result follows by induction on height. □

**Corollary 2.3.** Suppose there exists $\beta \in \Pi$ with $\beta$ short and $\phi \in X^*(P_{\Omega \cup \{\beta\}})$. Then there exists $m \in \mathbb{N}$ and $w \in W$ with

$$H_i^\lambda(\phi + \beta) \approx H_i^{\lambda'}(\mu)[-m]$$

for all $i \geq 0$, where $\mu = w(\phi + \beta)$ is dominant and $\Omega' = w(\Omega)$.

**Proof.** Consider the subsystem of simple roots orthogonal to $\phi$. Then $\beta$ and $\Omega$ belong to the simple roots of this subsystem and satisfy the hypothesis of the previous theorem with $\lambda = \beta$. Hence working only in the irreducible component of the subsystem containing $\beta$, we have $H_i^\lambda(\phi + \beta) \approx H_i^{\lambda'}(\phi + \nu)[-m]$ where $m \in \mathbb{N}$ and $\nu$ is a dominant short root for the subsystem. Then $\mu := \phi + \nu$ is dominant since $\nu$ is a short root and so has inner product at worst $-1$.
at the simple coroots not orthogonal to \( \phi \). The statements about \( w \) follow from the previous theorem. \( \square \)

3. Finding sufficient generators

As before, \( s = |\Theta| \) where \( \Theta \) is a set of orthogonal short simple roots. Suppose \( s \geq 1 \). Pick an element \( \alpha \in \Theta \) and set \( \Omega := \Theta \setminus \{ \alpha \} \). Let \( I_{\Theta,\alpha} \) be the ideal of \( O_\Theta \) in \( \mathbb{C}[\overline{O}_\Omega] \).

**Proposition 3.1.** \( I_{\Theta,\alpha} \) is generated by a basis of a copy of \( V_\phi \) in \( \mathbb{C}[\overline{O}_\Omega] \).

**Proof.** Restricting linear functions on \( n_\Omega \) to \( n_\Theta \) gives the short exact sequence of \( B \)-modules

\[
0 \to C_\alpha \to n_\Omega^* \to n_\Theta^* \to 0
\]

that has Koszul resolution

\[
0 \to S^{\bullet-1}_\alpha \otimes C_\alpha \to S^\bullet n_\Omega^* \to S^\bullet n_\Theta^* \to 0,
\]

which in turn gives a long exact sequence, which simplifies to

\[
0 \to H^0_\Omega(\alpha)[-1] \to H^0_\Omega(0) \to H^0_\Theta(0) \to H^1_\Omega(\alpha)[-1] \to 0
\]

since \( H^1_\Omega(0) = 0 \) by (2.1) for \( \lambda = 0 \). By Theorem 2.2 there exists \( w \in W \) with \( \Omega' := w(\Omega) \) and a positive integer \( m \) such that

\[
H^i_\Omega(\alpha)[-1] \simeq H^i_\Omega'(\phi)[-m] \text{ for all } i \geq 0.
\]

The latter vanishes for \( i > 0 \) by (2.1) for \( \lambda = \phi \), yielding the exact sequence

\[
0 \to H^0_\Omega'(\phi)[-m] \to H^0_\Omega(0) \to H^0_\Theta(0) \to 0.
\]

Hence the natural map

\[
H^0_\Omega(0) \to H^0_\Theta(0)
\]

is surjective, and so also, by induction on \( s \), the map

\[
H^0(0) \to H^0_\Theta(0)
\]

is surjective.

This implies (see [Jan04]) that \( H^0_\Theta(0) \simeq \mathbb{C}[\overline{O}_\Theta] \) and \( H^0_\Omega(0) \simeq \mathbb{C}[\overline{O}_\Omega] \) since \( H^0(0) \simeq \mathbb{C}[N] \), and also incidentally that \( \overline{O}_\Theta \) is normal, giving a variant of the proof given in [Bro94]. We conclude from (3.2) that \( H^0_\Omega(\phi)[-m] \simeq I_{\Theta,\alpha} \) as \( G \)-modules.

Next, consider the sequence of restrictions

\[
Sg^* \to Sn^* \to Sn^*_{\Omega'}
\]

and then the sequence

\[
Sg^* \otimes C_\phi \to Sn^* \otimes C_\phi \to Sn^*_{\Omega'} \otimes C_\phi.
\]

Taking global sections over \( G/B \) and using that

\[
H^0(G/B, Sg^* \otimes C_\phi) \simeq Sg^* \otimes H^0(G/B, C_\phi) \simeq Sg^* \otimes V_\phi,
\]
we get maps of \( G \times S^*g \)-modules

\[ S^*g \otimes V_\phi \to H^0_\phi(\phi) \to H^0_{\Omega'}(\phi). \]

By Broer [Bro93, Proposition 2.6] the first map is surjective since \( \phi \) is dominant.

To get the surjectivity of the second map, we repeat the first part of the proof, but now with respect to \( \Omega' \) and \( \Omega'' := \Omega' \setminus \{\beta\} \) for some choice of \( \beta \in \Omega' \). Then (3.1) with \( \alpha \) replaced by \( \beta \) and tensoring with \( C_\phi \) yields the exact sequence

\[ 0 \to H^0_{\Omega'}(\phi + \beta)[-1] \to H^0_{\Omega'}(\phi) \to H^0_{\Omega'}(\phi + \beta)[-1] \to 0. \]

Then Corollary 2.3 and (2.1) imply the \( H^1 \) term vanishes and thus \( H^0_{\Omega'}(\phi) \to H^0_{\Omega'}(\phi) \) is surjective. Finally by induction on \(|\Omega'|\) we deduce that \( S^*g \otimes V_\phi \to H^0_{\Omega'}(\phi) \) is surjective.

Thus the copy of \( V_\phi \) in \( I_{\Theta, \alpha} \) in degree \( m \) generates \( I_{\Theta, \alpha} \simeq H^0_{\Omega'}(\phi)[-m] \). \( \square \)

**Corollary 3.2** (see also [Bro94]). The variety \( \overline{\Theta} \) is normal and the map \( G \times P \to \overline{\Theta} \) is birational.

**Proof.** As mentioned in the proof, the normality result follows from the surjectivity of \( H^0(0) \to H^0_\Theta(0) \). This also implies the birationality statement. \( \square \)

**Corollary 3.3.** The ideal \( I_{\Theta} \) defining \( \overline{\Theta} \) in \( N \) is generated by \( s \) independent copies of \( V_\phi \).

**Proof.** This follows from Proposition 3.1 and induction on \( s \). \( \square \)

As noted in §1, there are exactly \( s \) independent copies of \( V_\phi \) in \( I_{\Theta} \). Another way to see this is that each \( H^0_k(\phi)[-m] \) for \( S \subset \Theta \) contains a single copy of \( V_\phi \), using Kostant’s multiplicity formula and the vanishing of \( H^i_k(\phi)[-m] \) for \( i > 0 \).

### 4. Invariants

By Corollary 3.3 we know \( s \) copies of \( V_\phi \) will generate \( I_{\Theta} \). In this section we show that at most two copies of \( V_\phi \) are needed and moreover that \( n - s \) fundamental generators are further needed to minimally generate \( J_{\Theta} \). It turns out that the question of when one copy of \( V_\phi \) or \( V_\theta \) lies in an ideal in \( S^*g \) generated by another copy is related to a property of invariants from De Concini, Papi, Procesi [DCPP13], which is related to flat bases of invariants [SYS80]. On the other hand, the classical types of \( A_n \), \( C_n \) and most cases in \( D_n \) could also be resolved using Appendix A, as we explain later.

Pick a basis of \( \{x_i\} \) with \( 1 \leq i \leq N \) of \( g \) and a dual basis \( \{y_i\} \) with respect to the Killing form \((\cdot, \cdot)\). When needed, we identify \( g \simeq g^* \) using the Killing form. Let \( p \) and \( q \) be two homogeneous invariants of degree \( a + 1 \) and \( b + 1 \), respectively. Then (1) \( p \circ q := \sum_i \frac{\partial p}{\partial x_i} \frac{\partial q}{\partial y_i} \) is again an invariant, homogeneous of degree \( a + b \); and (2) the span of the \( \frac{\partial p}{\partial x_i} \) (or the \( \frac{\partial q}{\partial y_i} \)) gives a copy of the adjoint representation in \( S^*g \). We write \([p] \) for this copy.
Lemma 4.1. The polynomials

$$w_j := \sum_{i=1}^{N} \frac{\partial^2 p}{\partial x_j \partial x_i} \frac{\partial q}{\partial y_i},$$

with $1 \leq j \leq N$, if nonzero, span a copy of the adjoint representation in $Sg^*$. 

Proof. Let $\pi \in \text{Hom}(g \otimes g, S^2g)$ be the defining homomorphism of $S^2g$. Then $\pi$ can be thought of as element of $\text{Hom}(g, S^2g \otimes g^*) \simeq \text{Hom}(g \otimes g, S^2g)$. Then the image of $\pi(g)$ under the map $S^2g \otimes g^* \simeq S^2g^* \otimes g^* \rightarrow S^{a+b-3}g^*$, determined by $p$ and $q$, has basis given by the $w_j$. □

Following [DCPP13], a homogeneous element $g \in R_+$ is called a generator if it is not an element of the ideal $R^2_+$ in $R$ and we write $p \equiv q$ if $p - q \in R^2_+$ for $p, q \in R$. We use $\overline{U}$ for the image in $\mathbb{C}[N]$ of a subset $U \subset Sg^*$.

Proposition 4.2. Let $f_k$ and $f_l$ be two fundamental invariants. Let $U = [f_k]$ and $V = [f_l]$. Define $\mathcal{I} = (\overline{V})$, an ideal in $\mathbb{C}^m[N]$. Define $\mathcal{J} = (V, \{f_i \mid d_i < d_k\})$, an ideal in $Sg^*$.

The following are equivalent:

1. There exists a generator $p$ such that $p \circ f_l \equiv f_k$.
2. $\overline{U} \cap \mathcal{I} = \{0\}$.
3. $f_k \in \mathcal{J}$.

For any of the equivalent statements to hold, it is necessary by (1) for $m_k - m_l + 2 = d_k - d_l + 2$ to be a degree of $g$ since this quantity is $\deg(p)$ and $p$ is a generator.

Proof. (1) $\Rightarrow$ (2). Suppose there exists a generator $p$ with $p \circ f_l \equiv f_k$. Then $\deg(p) = m_k - m_l + 2$. Let

$$w_j = \sum_{i=1}^{N} \frac{\partial^2 p}{\partial x_j \partial x_i} \frac{\partial f_l}{\partial y_i},$$

which by Lemma 4.1 is a basis for a copy of the adjoint representation in $S^{m_k}g^*$. Now

$$\sum_{j=1}^{N} x_j w_j = \sum_{j=1}^{N} x_j \left( \sum_{i=1}^{N} \frac{\partial^2 p}{\partial x_j \partial x_i} \frac{\partial f_l}{\partial y_i} \right)$$



(4.1) \begin{align*}
&= \sum_{i=1}^{N} \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j} \left( \frac{\partial p}{\partial x_i} \right) \frac{\partial f_l}{\partial y_i} = \sum_{i=1}^{N} \left( (m_k - m_l + 1) \frac{\partial p}{\partial x_i} \right) \frac{\partial f_l}{\partial y_i} = (m_k - m_l + 1) p \circ f_l, \\
&\text{by Euler's formula since the } \frac{\partial p}{\partial x_j} \text{ are homogeneous of degree } m_k - m_l + 1.
\end{align*}

Next, from Kostant’s fundamental description [Kos63] of $Sg^*$ as $R \otimes H$, where $H \subset Sg^*$ is a graded subspace isomorphic as $G$-module to $\mathbb{C}[N]$, and the fact that the $[f_j]$ are a basis for the $V_\theta$-isotypic component in $\mathbb{C}[N]$, we know that every homogeneous copy $T$ of the adjoint representation in $Sg^*$ is the span of elements

(4.2) \begin{align*}
v_i := \sum_{j=1}^{n} r_j \frac{\partial f_j}{\partial x_i}
\end{align*}
for \( r_1, \ldots, r_n \in R \) homogeneous (and independent of \( i \)). The choice of \( r_j \) is unique up to a scalar that is independent of \( j \). In particular, \( T \) lies in \((R_+)^n \), the ideal in \( Sg^* \) generated by \( R_+ \) if and only if all \( r_j \) are of positive degree. Since \( \sum x_i v_i = \sum_j d_j r_j f_j \), we deduce that \( T \subset (R_+) \) if and only if there exists a basis of \( v_i' \) of \( T \) with \( \sum x_i v_i' \in R_+^2 \). Since there is only one copy of the trivial representation in \( g^* \otimes g \simeq g \otimes g \), any such invariant \( \sum x_i v_i' \) is well-defined up to a scalar. From \( p \circ f_t \equiv f_k \), we conclude from (4.1) and the fact that \( f_k \) is a generator that at least one \( w_j \), and hence all \( w_j \), are not contained in \((R_+) \), and moreover the span of the images of the \( w_j \) in \( \mathbb{C}^{m_k[N]} \) coincides with that of both \([p \circ f_t]\) and \( U \). Since clearly each \( w_j \in I \), this concludes the proof of this implication.

(2) \( \Rightarrow \) (3). Suppose \( \overline{U} \subset I \). Then certainly \( U \subset \overline{U} \). Hence \( f_k = \frac{1}{d_k} \sum_j x_j \frac{\partial f_k}{\partial x_j} \) by Euler’s formula and so \( f_k \in \overline{J} \).

(3) \( \Rightarrow \) (1). There is a graded surjection

\[ Sg^*[−m_l] \otimes g \to (V), \]

where \((V)\) refers to the ideal in \( Sg^* \), and so any invariant in \((V)\) comes from an element in \((Sg^*[−m_l] \otimes g)^G \). But \((Sg^*[−m_l] \otimes g)^G \simeq \text{Hom}(g, Sg^*[−m_l]) \), and thus any invariant in \((V)\) takes the form \( \sum v_i \frac{\partial f_k}{\partial y_i} \) with some \( v_i \) as in (4.2). Since \( f_k \in \overline{J} \), it follows that \( f_k \equiv \sum i v_i \frac{\partial f_k}{\partial y_i} \) for some such \( v_i \). Since \( f_k \) is a generator, \( \sum i v_i \frac{\partial f_k}{\partial y_i} \) is therefore a generator, and thus at least one of the \( r_j \) in the expansion of the \( v_i \)’s must be degree 0. Hence \( f_k \equiv \sum i \frac{\partial p}{\partial x_i} \frac{\partial f_k}{\partial y_i} \) for \( p \) a linear combination of the chosen fundamental invariants. In particular, \( p \) is a generator.

It is known from [DCP15] when condition (1) in the Proposition holds; it can be checked by restricting to a Cartan subalgebra of \( g \). To state the result, we need to take the matrix representation of \( \frak{so}_{2n} \) and a generic matrix \( X \in \frak{so}_{2n} \). Then a set of fundamental invariants of \( \frak{so}_{2n} \) are given by the Pfaffian \( \text{Pf}(X) \) and \( \text{tr}(X^{2i}) \) for \( 1 \leq i \leq n − 1 \).

**Theorem 4.3** ([DCP15]). Condition (1) in the Proposition is equivalent to

\[ d_k − d_l + 2 \text{ is equal to a degree of } G, \]

except when \( G \) has type \( D_n \) with \( n \) even and one of the following is true:

- If \( f_t \equiv c \cdot \text{Pf}(X) \) for some \( c \), then condition (1) holds only when \( d_k = 2n − 2 \).
- If \( d_k = n \) and \( f_k \neq c \cdot \text{tr}(X^n) \) for all scalars \( c \), then condition (1) holds only when \( d_l = 2 \).

In particular, condition (1) always holds when \( k = n \), that is, \( f_k = f_n \) has maximal degree \( d_n \), equal to the Coxeter number. For types \( A_n, B_n, C_n \) and most cases in \( D_n \), it is straightforward to show that \([f_k] \subset ([f_t])\) in \( \mathbb{C}[N] \) if and only if \( d_k \geq d_l \) and hence even deduce Theorem 4.3 from Proposition 4.2, rather than the other way around (see [3A]). In the others cases we use Proposition 4.2 and Theorem 1.3 to study such ideals:

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1That is, \( X = \sum_{i=1}^{N} x_i v_i \), where the \( v_i \) are a \( \mathbb{C} \)-basis of \( \frak{so}_{2n}(\mathbb{C}) \) and the \( x_i \) are formal variables.
Proposition 4.4. An ideal in \( \mathbb{C}[\mathcal{N}] \) generated by multiple copies of \( V_\theta \) and \( V_\phi \) is minimally generated by bases of at most two such copies. Let \( I \subset \mathbb{C}[\mathcal{N}] \) be an ideal of the above type. Then the preimage of \( I \) in \( Sg^* \) is minimally generated by at most two copies of \( V_\theta \) and/or \( V_\phi \) together with \( n - t \) fundamental invariants, where \( t \) is number of independent copies of \( V_\theta \) in \( I \).

Proof. By Proposition [A.1], the first statement for non-simply-laced groups follows from the statement for the simply-laced ones. We now check the first statement in all possible cases in the simply-laced types. In type \( A_n \) any such ideal is generated by its copy in lowest degree, either by the direct matrix argument (\( [A.3] \)) or the fact that \( d_k - d_l + 2 \) is always a degree. In type \( D_n \), by Theorem [4.3], any such ideal is generated by the restriction of the entries of \( X^{2i+1} \) for some \( i \), or the restriction of the derivatives of \( Pf(X) \), or both. When \( n \) is even, there are also the ideals generated by the restriction of the derivatives of \( \text{tr}(X^n) + cPf(X) \) for any \( c \neq 0 \), alone or together with the restriction of the entries of \( X^{2i+1} \) for some \( i \) with \( 2i + 1 < n - 1 \). In the exceptional types the copies of \( V_\theta \) in \( \mathbb{C}[\mathcal{N}] \) occur in unique degrees and we label them by the corresponding degree (that is, by an exponent). In type \( E_6 \), the only such ideals not generated by a single copy are \( (V_4, V_5) \), \( (V_5, V_7) \) and \( (V_7, V_8) \). In type \( E_7 \), they are \( (V_5, V_7), (V_7, V_9) \), \( (V_9, V_{11}) \), \( (V_{11}, V_{13}) \). In type \( E_8 \), they are \( (V_7, V_{11}), (V_{11}, V_{13}), (V_{13}, V_{17}), (V_{17}, V_{19}), (V_{19}, V_{23}) \). This is shown by checking whether or not (\( [4.3] \)) holds.

The statement about minimal generators in \( Sg^* \) is clear from Proposition [4.2] when there is a single copy of \( V_\theta \) generating \( I \). There is also a version of Proposition [4.2] where \( V = [f_l] \) with \( \epsilon(f_l) = -f_l \) (see \( [A.1] \)) is replaced by its restriction to \( g_0 \), yielding a copy of \( V_\phi \). Then the implication (3) implies (1) still holds, where the derivatives in (1) are with respect to a basis of \( g_0 \). So the result holds also when there is a single copy of \( V_\phi \) from Proposition [A.1] and the simply-laced case. In the remaining cases, where the ideal is generated by two representations, say \( V_j \) and \( V_i \) with \( j \geq i \), we check that any fundamental invariant of degree at least \( j + 1 \) is already contained in either \( (V_i) \) or \( (V_j) \). The result follows. \( \square \)

Remark 4.5. Analogous statements for Propositions [4.2] and [4.4] hold for the reflection representation \( V_\theta^T \) and the irreducible representation \( V_\phi^T \) of the Weyl group \( W \).

5. Proof of Theorem [1.3] and Explicit Generators

The classical cases can mostly be handled using the material from Appendix [A]. We take the standard matrix representation of each classical \( g \) and let \( X \) be a generic matrix. From Appendix [A], in types \( A, C, D \), the span of the entries of \( X^{m_\phi} \), restricted to \( \mathbb{C}[\mathcal{N}] \), give distinct copies of \( V_\phi \) and outside of type \( D \), these are all the copies. We now check by hand which ones vanish on \( O_\Theta \) (see also [Ric87] or [5.3]). The results for \( I_\Theta \) follow from the following results for \( J_\Theta \). Assume \( s \geq 1 \).

5.1. Type \( A_n \). \( O_\Theta \) has partition type \([n + 1 - s, s] \). By Appendix [A.3] the \( s \) copies of \( V_\phi \) in \( I_\Theta \) come from the span of the entries of \( X^{n+1-j} \) for \( 1 \leq j \leq s \). Hence, a minimal generating set
for \( J_\Theta \) is given by the entries of \( X^{n+1-s} \) and \( \text{tr}(X^2), \ldots, \text{tr}(X^{n-s}) \), and we recover Weyman’s result. These orbits are in the first family and the theorem follows since \( m^{\phi}_{r-s+1} = n + 1 - s \).

5.2. Type \( D_n \).

5.2.1. \( O_\Theta \) has partition type \([2n - 2s - 1, 2s + 1]\) for \( 1 \leq s < \frac{n}{2} \). By Appendix A.5 the \( s \) copies of \( V_\phi \) in \( I_\Theta \) come from the span of the entries of \( X^{2n-2j-1} \) for \( 1 \leq j \leq s \). Hence, a minimal generating set for \( J_\Theta \) is given by a basis chosen from the entries of \( X^{2n-2s-1} \) and \( \text{tr}(X^2), \text{tr}(X^4), \ldots, \text{tr}(X^{2n-2s-2}) \), and \( \text{Pf}(X) \). These orbits are in the first family and the theorem follows since \( m^{\phi}_{r-s+1} = 2n - 2s - 1 \).

5.2.2. \( O_\Theta \) has partition type \([2n - 2s + 1, 2s - 3, 1, 1]\) for \( 2 \leq s \leq \frac{n}{2} + 1 \). The entries of \( X^{2n-2j-1} \) vanish on \( O_\Theta \) if and only if \( 1 \leq j \leq s - 1 \), so we are missing one copy of \( V_\phi \); by Appendix A.5 or Proposition 4.4, it must lie in degree \( n \), otherwise the entries of \( X^{2n-2s-1} \) would lie in \( I_\Theta \). Indeed, the derivatives of \( \text{Pf}(X) \) with respect to a basis of \( \mathfrak{g} \) vanish on \( M \in \mathfrak{so}_{2n} \) whenever \( \text{rank}(M) \leq 2n - 4 \) (see [Joh17]). Hence, by Proposition 4.4, for \( s \geq 3 \), a minimal generating set for \( J_\Theta \) is given by a basis of the entries of \( X^{2n-2s+1} \), the derivatives of \( \text{Pf}(X) \), and \( \text{tr}(X^2), \text{tr}(X^4), \ldots, \text{tr}(X^{2n-2s}) \). For \( s = 2 \), a minimal generating set for \( J_\Theta \) is given by the derivatives of \( \text{Pf}(X) \) and \( \text{tr}(X^2), \text{tr}(X^4), \ldots, \text{tr}(X^{2n-2s}) \). These orbits are in the second family except for when \( s = \frac{n}{2} + 1 \). The theorem follows since \( m_\Theta = n - 1 \) and \( m_{r-s+2} = 2n - 2s + 1 \) in all cases.

5.2.3. \( O_\Theta \) has partition type \([n, n]\) with \( n \) even. There are two orbits with partition type \([n, n]\) and \( s = \frac{n}{2} \). We first show that the derivatives of \( \frac{1}{2n} \text{tr}(X^n) + c \text{Pf}(X) \) vanish on \( O_\Theta \) for either \( c = 1 \) or \( c = -1 \).

Let \( e \in \mathcal{O} := O_\Theta \) and put \( e \) in an \( \mathfrak{so}_2 \)-subalgebra \( \mathfrak{s} \) spanned by the triple \( \{e, h, f\} \). Identifying \( \mathfrak{g} \) with the Lie algebra of skew-symmetric matrices \( \mathfrak{so}_{2n} \subset \mathfrak{gl}_{2n} \), we can consider the matrices \( s(x) := e + xf^{n-1} \) with \( x \) an indeterminate. Since \( n \) is even and \( f \) is skew-symmetric, it follows that \( f^{n-1} \) is skew-symmetric and thus \( s(x) \in \mathfrak{so}_{2n} \) for each \( x \in \mathbb{C} \). Now \( \mathbb{C}^{2n} \), viewed as the defining representation for \( \mathfrak{so}_{2n} \), decomposes under the restriction to \( \mathfrak{s} \) as the direct sum of two copies of the irreducible \( n \)-dimensional representation \( U_n \) of \( \mathfrak{s} \). Working with respect to the basis of \( \mathbb{C}^{2n} \) coming from the standard basis of \( U_n \), it is easy to see that \( s(1) \) has two repeated diagonal blocks with rational entries and that \( s(x)^n \) is a multiple \( \lambda \) of the identity matrix, with \( \lambda = ax \) for some nonzero rational number \( a \). The first property means that \( \det(s(1)) \) is positive rational, and the second, that \( \det(s(x)^n) = (ax)^{2n} \). It follows that \( \det(s(x)) = (ax)^2 \) since the determinant is continuous in \( x \) and thus \( \text{Pf}(s(x)) = \pm 2n \lambda x \) since the Pfaffian of a skew-symmetric matrix squares to its determinant. At the same time, \( \text{tr}(s(x)^n) = 2nax \). So let \( X = s(x) \). Then the derivative at \( x \) of \( \frac{1}{2n} \text{tr}(X^n) + c \text{Pf}(X) \) must vanish on \( \mathcal{O} \) for either \( c = 1 \) or \( c = -1 \).

Next, it is clear that the entries of \( X^{2n-2j-1} \) vanish on \( \mathcal{O} \) if and only if \( 1 \leq j < s \). As in \( \S 5.2.2 \) this means the one missing copy of \( V_\phi \) is in degree \( n \), which we just located. By Proposition 4.4, \( \mathcal{O} \) a minimal generating set for \( J_\Theta \) for the two very even orbits is given by a basis of the derivatives
of \( \frac{1}{n} \text{tr}(X^n) \pm \text{Pf}(X) \) and \( \text{tr}(X^2), \text{tr}(X^4), \ldots, \text{tr}(X^n) \). The theorem follows since \( m_\Theta = n - 1 \) and \( m_{r-s+2} = n + 1 \).

5.3. **Type \( B_n \).** The subregular orbit is the only case with \( s \geq 1 \), already handled by Broer. Since \( r = 1 \) there is a unique copy of \( V_\phi \) in \( \mathbb{C}[\mathcal{N}] \), and is located in degree \( n \). It must therefore cut out the subregular orbit since \( s = 1 \). By Appendix A.3 it is obtained by restricting the derivatives of \( \text{Pf}(X) \) to the nilcone in \( \mathfrak{so}_{2n+1} \), where \( X \) is a generic matrix of \( \mathfrak{so}_{2n+2} \). The minimal set of invariants needed in \( J_\Theta \) are just \( \text{tr}(X^2), \text{tr}(X^4), \ldots, \text{tr}(X^{2n-2}) \) by Proposition 4.4 or Broer’s simpler argument.

5.4. **Type \( C_n \).** \( \mathcal{O}_\Theta \) has partition type \([2n-2s, 2s]\) for \( 1 \leq s < \frac{n}{2} \). By Appendix A.4 the \( s \) copies of \( V_\phi \) in \( I_\Theta \) come from the span of the entries of \( X^{2n-2j} \) for \( 1 \leq j \leq s \). Hence a minimal generating set for \( J_\Theta \) is given by a basis of the entries of \( X^{2n-2s} \) and \( \text{tr}(X^2), \text{tr}(X^4), \ldots, \text{tr}(X^{2n-2s-2}) \).

5.5. **Exceptional Types.** We deduce the results from Corollary 3.3 and Proposition 4.4 and find explicit generators using Appendix A. We can still manage to avoid calculating the degree shift \( m \) in Theorem 2.2 with a simple method to determine almost all copies of \( V_\phi \) that vanish on \( \mathcal{O}_\Theta \); this method would also work for classical types.

As in the introduction \( \{e, h, f\} \) is an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \) with \( h \in \mathfrak{h} \) dominant. Kostant’s observation [Kos63] for \( e \) regular carries over more generally (see [Ree98]) and was used by Richardson [Ric87, Proposition 2.2] for \( V_\theta \) in the same way as we do here. That is, for a highest weight representation \( V_\lambda \) and its linear dual \( V_\lambda^* \), recall that \( V_\lambda^{G_e} \), where \( G_e \) is the centralizer of \( e \) in \( G \), carries a grading via the action of \( \frac{1}{2}h \) and this graded space is isomorphic to

\[
\bigoplus_i \text{Hom}_G(V_\lambda^*, \mathbb{C}[\mathcal{O}_e]).
\]

Suppose a copy \( V \) of \( V_\lambda^* \) lies in degree \( m_i^\lambda \) in \( \mathbb{C}[\mathcal{N}] \). We have

\[
(5.1) \quad m_i^\lambda > \frac{1}{2} \lambda(h) \implies V \text{ vanishes on } \mathcal{O}_e.
\]

Hence by the definition (5.1), for all orbits in the first family, including in the classical types, there are \( s \) distinct generalized exponents for \( \lambda = \phi \) satisfying the inequality in (5.1), namely, \( m_r^\phi \geq m_{r-1}^\phi \geq \cdots \geq m_{r-s+1}^\phi \). This allows us to locate all copies of \( V_\phi \simeq V_\phi^* \) in \( I_\Theta \) and to show that the copy in lowest degree occurs in degree \( m_{r-s+1}^\phi \).

As in Appendix B, let \( U \) be a non-trivial irreducible representation of \( \mathfrak{g} \) of minimal dimension and embed \( \mathfrak{g} \subset \mathfrak{gl}_N \) for a choice of basis of \( U \). Then taking a generic matrix \( X \in \mathfrak{g} \) the polynomials \( f_i := \text{tr}(X^{d_i}) \) are a set of fundamental invariants. The derivatives of these invariants along \( \mathfrak{g} \) determine \( n \) copies of \( V_\phi \), which are independent on restriction to \( \mathcal{N} \). We label these copies of \( V_\Theta \) as \( V_{\phi_1}, \ldots, V_{\phi_n} \) since the degrees are distinct in the exceptional groups.
5.6. **Type** $E_6$. Every orbit is in the first family so (5.1) finds all $s$ copies of $V_\theta$ in $I_\Theta$. Hence Proposition 4.4 is enough to complete the proof. Here and below we take a basis of the listed copies of $V_\theta$ to obtain minimal generators for $J_\Theta$.

$E_6(a_1)$: $V_{11}$ and $f_1, f_2, f_3, f_4, f_5$; $D_5$: $V_8$ and $f_1, f_2, f_3, f_4$; $E_6(a_3)$: $V_7, V_8$ and $f_1, f_2, f_3$.

5.7. **Type** $F_4$ and $G_2$. The subregular orbit is the only case with $s \geq 1$ and was treated by Broer. Both orbits satisfy (1.3) and so the copy of $V_\phi$ in degree $m_r^\phi$ is the unique copy in $I_\Theta$. In $F_4$ this $V_\phi$ is obtained by restricting the adjoint representation $V_8$ from $E_6$ to $F_4$ by §A.1. In $G_2$ the $V_\phi$ is obtained by restricting the unique $V_\phi$ for $B_3$ in degree 3.

5.8. **Type** $E_7$. Except for the $E_6$ orbit, all orbits are in the first family so (5.1) finds all $s$ copies in $I_\Theta$. For the $E_6$ orbit, however, $s = 3$ but $\frac{1}{6}\theta(h) = 11$, which implies only that $V_{13}$ and $V_{17}$ are in $I_\Theta$. The ideal $I_\Theta$ cannot coincide with the ideal for $E_7(a_3)$, which contains $V_{11}$. Hence by Proposition 4.4 the only other ideal of the desired kind containing 3 copies of $V_\theta$ is $(V_9)$.

$E_7(a_1)$: $V_{17}$ and $f_1, \ldots, f_6$; $E_7(a_2)$: $V_{13}$ and $f_1, \ldots, f_5$

$E_7(a_3)$: $V_{11}, V_{13}$ and $f_1, f_2, f_3, f_4$ $E_6$: $V_9$ and $f_1, f_2, f_3, f_5$

$E_6(a_1)$: $V_9, V_{11}$ and $f_1, f_2, f_3$.

5.9. **Type** $E_8$. Every orbit is in the first family.

$E_8(a_1)$: $V_{17}$ and $f_1, \ldots, f_7$; $E_8(a_2)$: $V_{23}$ and $f_1, \ldots, f_6$

$E_8(a_3)$: $V_{19}, V_{23}$ and $f_1, \ldots, f_5$; $E_8(a_4)$: $V_{17}, V_{19}$ and $f_1, f_2, f_3, f_4$

6. **Other orbits in non-simply-laced cases**

Using §A.1 and Proposition 4.4 we can find equations for those nilpotent varieties in a non-simply-laced Lie algebra that lie in one of the orbits $O_\Theta$ under the embedding $\mathfrak{g}_0$ into $\mathfrak{g}$. That these equations generate the ideal $J$ defining the nilpotent variety requires cohomological results along the lines of Theorem 2.2 that will appear elsewhere.

6.1. $[2n-2s+1, 2s-1, 1]$ in **Type** $B_n$ for $s \geq 1$. From the $D_{n+1}$ result for $[2n-2s+1, 2s-1, 1, 1]$, the ideal $J$ is minimally generated by a basis of the copy of $V_\phi$ in degree $n$, and when $s \geq 2$ a basis of the entries of $X^{2n-2s+1}$ (a copy of $V_9$), and $\text{tr}(X^2), \ldots, \text{tr}(X^{2n-2s})$.

6.2. $[n, n]$ in **Type** $C_n$, $n$ odd. From the $A_{2n-1}$ result for $[n, n]$, the ideal $J$ is generated by a basis of the entries of $X^n$ (a copy of $V_\theta$) and $\text{tr}(X^2), \ldots, \text{tr}(X^{n-1})$.

6.3. $F_4(a_2)$ in **Type** $F_4$. From the $E_6(a_3)$ case in $E_6$, the ideal $J$ is minimally generated by a copy of $V_\theta$ in degree 7, a copy of $V_\phi$ in degree 8, and fundamental invariants in degrees 2 and 6.
7. Covariants

By running the proof of Theorem 2.2 backward in any given case, we obtain the following identity for various sets $\Omega$ of orthogonal short simple roots with $\phi \in X^*(P_\Omega)$:

\begin{equation}
H_{\Omega}^i(-\phi)[m] \simeq H_{\Omega'}^i(-\alpha)[1] \text{ for all } i \geq 0,
\end{equation}

for some positive integer $m$ and $\alpha$ a simple root orthogonal to the simple roots in $\Omega'$ and where $\Omega'$ and $\Omega$ are conjugate by an element in $W$. Next by [Dem76], we get

\begin{equation}
H_{\Omega'}^0(-\alpha)[1] \simeq H_{\Omega'}^0(0).
\end{equation}

Also the steps of the proof can be repeated with $\lambda = 0$, and using (2.2), to obtain $H_{\Omega'}^0(0) \simeq H_{\Omega}(0)$. Hence

\begin{equation}
H_{\Omega}^0(-\phi)[m] \simeq H_{\Omega}^0(0)
\end{equation}

and unraveling the notation for the lowest degree term in the grading

\[ H^0(G/P, S^m(n_\Omega^*) \otimes C_{-\phi}) \simeq H^0(G/P, C) \simeq C \]

where $P = P_\Omega$. It follows that there exists a $P$-equivariant polynomial $\sigma : n_\Omega \to C_{\phi}$ of homogeneous degree $m$. Such an $\sigma$ is called a $P$-covariant. Any such covariant is unique and its zero set determines a well-defined orbit of codimension two in $\overline{O_\Omega}$. By Theorem 2.2 and then (5.1), the degree $m$ of $\phi$ satisfies $m \leq \frac{1}{2}\phi(h)$ where $\{e, h, f\}$ is a triple for $e \in O_\Omega$ and $h \in h$ is dominant. But it turns that equality often holds in our cases. Consider the subspace $j = \oplus_{i \geq 2}g_i$ defined by the action of $\text{ad}(h)$.

**Proposition 7.1.** Suppose $j \subset n_\Omega$, then a $P$-covariant $\sigma$ with weight $\phi$ has degree $\frac{1}{2}\phi(h)$.

**Proof.** Consider the cocharacter $\chi : C^* \to T$ with $T \subset P$, determined by $h$. By our convention that $B$ corresponds to the negative roots, and the hypothesis on $j$, we can choose the $sl_2$-triple so that $f \in n_\Omega$. Then for $\xi \in C^*$, we can evaluate $\chi(\xi) \cdot \sigma(f)$ in two ways. First, $\chi(\xi) \cdot \sigma(f) = \xi^{\phi(h)} \sigma(f)$ since $\chi(\xi) \in T$ and $\sigma(f) \in C_{\phi}$. Second, $\chi(\xi) \cdot f = \xi^{-2}f$, so $\chi(\xi) \cdot \sigma(f) = \sigma(\chi(\xi)^{-1} \cdot f) = \sigma(\xi^2 f)$. It follows that the degree of $\sigma$ as a function in $Sn_{\Omega}^*$ is $\frac{1}{2}\phi(h)$. \qed

We want to find all $\Omega$, consisting of orthogonal simple short roots with $\phi \in X^*(P)$, that can have a $P$-covariant of weight $\phi$. It turns out that these are exactly the cases obtained in the manner above, i.e., by reversing Theorem 2.2. Note that such an $\Omega$ cannot have the maximal value of $|\Omega|$ for $g$ since there exists a short simple root orthogonal to all elements in such an $\Omega$. Moreover, we find for each orbit $O_\Theta$ with $|\Theta|$ not maximal that there exists $\Omega$ with $O_\Theta = O_\Omega$ having a $P$-covariant of weight $\phi$ satisfying Proposition 7.1. The result is

**Proposition 7.2.** There exists a $P$-covariant with weight $\phi$ for $P = P_\Omega$ with $\phi \in X^*(P)$, except in the following cases:

1. Type $B_n$ with $\Omega = \{\alpha_n\}$.
2. Type $C_n$ with $\alpha_1 \in \Omega$. 
(3) Type $D_n$ with $\{\alpha_1, \alpha_{n-1}, \alpha_n\} \subset \Omega$.

(4) Type $F_4$ with $\Omega = \{\alpha_3\}$.

(5) Type $E_6$ with $\Omega = \{\alpha_1, \alpha_4, \alpha_6\}$.

(6) Type $E_7$ with $\Omega = \{\alpha_3, \alpha_5, \alpha_7\}$ or $\{\alpha_2, \alpha_3, \alpha_5, \alpha_7\}$.

(7) Type $E_8$ with $\{\alpha_2, \alpha_5, \alpha_7\} \subset \Omega$.

Proof. The obstacle to (7.1) holding, using the proof of Theorem 2.2, is encountering a $D_4$ subsystem of $\Pi$ with the roots corresponding to the end nodes lying in $\Omega$ or a $C_2$ subsystem with short root lying in $\Omega$. This implies, for example, that if $\mathfrak{g}$ is simply-laced and $|\Omega| \leq 2$ or $\mathfrak{g}$ is of type $A_n$, then we never encounter such a subsystem and therefore a covariant always exists.

Now, let $\Pi_\phi$ be the simple roots orthogonal to $\phi$ and $l_\phi$ the Levi subalgebra they determine. We need only consider the equivalence classes of $\Omega \subset \Pi_\phi$ under the action of $W(l_\phi)$, since if one $\Omega$ works in a class, so will all the others using (2.2). By the proof of the main theorem, there is at least one successful $\Omega$ for each orbit $O_\Theta$ with $|\Theta| = |\Omega| + 1$. In fact, it turns out that each equivalence class exactly corresponds to a unique way to cut out an $O_\Theta$ from an $\overline{O_\Omega}$. We now proceed through each case.

In $D_n$, the type of $l_\phi$ is $D_1 \times D_{n-2}$ and so there are generally going to be four equivalence classes of $\Omega$ for any given $t = |\Omega|$, depending on whether $\alpha_1 \in \Omega$ or not, and both $\alpha_{n-1}$ and $\alpha_n$ are in $\Omega$ or not. There are also the extra cases arising from when $\Omega$ determines a very even orbit in the factor of $D_{n-2} \subset l_\phi$. Lemmas 8.4 and 8.6 handle the cases where $\alpha_1 \not\in \Omega$. These are the cases such that some $\Omega$ in the equivalence class satisfies Proposition 7.1. Next, Lemma 8.5 and (8.1) handle the cases with $\alpha_1 \in \Omega$ and $\alpha_{n-1}$ and $\alpha_n$ are not both in $\Omega$.

In $E_6$ there is only one equivalence class when $|\Omega| \leq 2$ since $l_\phi$ has type $A_5$ and each of these will therefore be covered by Proposition 7.1. The remaining case, when $|\Omega| = 3$, is an exceptional case.

In $E_7$, there are several equivalence classes since $l_\phi$ has type $D_6$. When $t = |\Omega| \leq 2$ we know (7.1) holds since we do not encounter a $D_4$. But there is one case for $t = 2$ not covered by Proposition 7.1 (see Lemma 8.7). This case arises because there are two different orbits $O_\Theta$ with $|\Theta| = 3$. Next, there are three equivalence classes when $t = 3$. Two of them must work since we can obtain $E_8(a_4)$ from either $E_6$ or $E_8(a_3)$. The calculations are carried out in Lemmas 8.8 and 8.9 and there is an $\Omega$ in both cases covered by Proposition 7.1. The remaining equivalence class for $t = 3$ is an exceptional case.

In $E_8$, there is an extra equivalence class when $t = 3$ since $l_\phi$ has type $E_7$. It is covered by Proposition 7.1 (see Lemma 8.8).

The $C_n$ cases are all covered in Lemma 8.3 and the type $A_n$ cases come for free (or from Lemma 8.2). We have now covered all cases that are not exceptional cases.

To complete the proof, we check by hand that the other cases encounter a $D_4$ or $C_2$ if we carry out the steps in Theorem 2.2 and this turns out to be enough to show that $\phi$ is not a $P$-covariant, but $2\phi$ is, using [Som16, §2.1]. The fact that this holds is related to the fact that...
for all the exceptional cases, which include the cases where $|\Omega|$ is maximal, each such $O_{\Omega}$ has $G$-equivariant fundamental group isomorphic to $S_2$ (for $G$ adjoint).

\[\square\]

**Remark 7.3.** For the exceptional cases listed in the proposition, it is possible to show that $H^I_{\Omega}(\phi)[-d]$ equals the $G$-module of sections of the non-trivial $G$-equivariant line bundle on $O$ when $G$ is adjoint. Here, $2d$ is the degree of the covariant for $2\phi$. When $n_\Theta = j$, we also get that $d = \frac{1}{2}\phi(h)$ as in Proposition [5.4]. When $\phi = \theta$, the value of $d$ is no longer an exponent of $g$, but it can be described as an exponent for a hyperplane arrangement (see [ST97], [Bro99]). For the $E_8$ maximal case, for example, $d = 14$.

**Remark 7.4.** Propositions [7.1] and [7.2] explain why $\frac{1}{2}\phi(h)$ is often a generalized exponent for $V_\phi$. Since the value of $\phi(h)$ must weakly decrease as we move down the partial order on nilpotent orbits, it is perhaps not surprising that (5.1) ends up determining the copies of $V_\phi$ in $I_\Theta$ for orbits in the first family.

**Example 7.5.** Consider the subregular orbit in $D_n$. Here $\Theta$ is a single simple root and when $\Theta \neq \{\alpha_2\}$, then there is a $P$-covariant $\sigma$. When $\Theta = \{\alpha_1\}$, the degree of $\sigma$ is $n - 1$, while in all other cases $\sigma$ has degree $2n - 5$. The latter cases include $\Theta = \{\alpha_{n-2}\}$, which yields $n_\Theta = j$ and so $2n - 5 = \frac{1}{2}\phi(h)$. The two different situations correspond to the two different orbits contained in $\Sigma_{\Theta}$. The first is the orbit $[2n-3, 1, 1, 1]$ and the second is $[2n-5, 5]$ (when $n \geq 5$). The orbits are seen here as $G$-saturation of the zero set of each $\sigma$.

**Remark 7.6.** The first part of Proposition 2.4 in [Bro94] and Proposition 7.2 give a way to avoid Corollary 2.3 and the inductive steps in the proof of Proposition 3.1 that show that the ideal determined by $H_{\Theta}(\phi)$ is generated by $V_\phi$. On the other hand, it seems the part of Proposition 2.4 in *loc. cit.* related to $\text{ht}_P$ is incorrect as the above example shows. In particular, the result assigned to Richardson is not true: for example, for the subregular orbit $O$ in $D_4$ and $\Theta = \{\alpha_1\}$, then $O$ does not meet $n_1$.

8. **Direct calculations**

8.1. **Helpful lemma.** Let $m$ be a standard Levi subalgebra of $g$ of type $A_k$. Let $\Pi_m = \{\beta_1, \ldots, \beta_k\}$ be the simple roots of $m$ with $\beta_1$ and $\beta_k$ the two extreme vertices of the Dynkin diagram of $m$. Let $m'$ be the standard Levi subalgebra containing $m$ whose simple roots $\Pi_{m'}$ consist of $\Pi_m$ and all simple roots adjacent to some simple root in $\Pi_m$. Let $\Omega \subset \Pi$ be a set of orthogonal simple roots.

**Lemma 8.1.** Assume that $\Omega \cap \Pi_{m'} \subset \{\beta_2, \beta_3, \ldots, \beta_k\}$. Let $t = |\Omega \cap \Pi_{m'}|$.

Given $\lambda \in X^+(P_{\beta_2, \ldots, \beta_k})$ such that

$$\langle \lambda, \beta_1' \rangle = -1,$$

we have

$$H^i_{\Omega}(\lambda) \simeq H^i_{\Omega'}(\lambda + \beta_1 + \beta_2 + \ldots + \beta_k)[-k + t],$$

for all $i \geq 0$ and for any $\Omega' \subset \Pi$ satisfying:
Next, since \( \langle \lambda + \beta_1, \beta_2 \rangle = 0 \) and \( \sum_{i=1}^{m} \alpha_i \) is the dimension of the subalgebra \( \mathfrak{m} \), the following result holds: 

(1) \( \Omega' \cap \Pi \mathfrak{m}' \subset \{ \beta_1, \beta_2, \ldots, \beta_{k-1} \} \) consists of orthogonal simple roots;

(2) \( \Omega' \cap (\Pi \setminus \Pi \mathfrak{m}') = \Omega \cap (\Pi \setminus \Pi \mathfrak{m}') \); and

(3) \( |\Omega| = |\Omega'| \).

**Proof.** The proof is by induction on \( k \). If \( k = 1 \), then \( t = 0 \) and the \( A_1 \)-move gives the results with \( \Omega' = \Omega \). Consider general \( k \geq 2 \).

If \( \beta_2 \notin \Omega \), then the \( A_1 \)-move gives \( H^1_{\Omega}(\lambda) \simeq H^1_{\Omega'}(\lambda + \beta_1)[1] \). Now \( \langle \lambda + \beta_1, \beta_2 \rangle = -1 \) and \( \lambda + \beta_1 + \beta_2 \in X^*(P_{\{\beta_3, \ldots, \beta_k\}}) \) and so we can apply induction to the Levi subalgebra of type \( A_{k-1} \) with simple roots \( \beta_2, \ldots, \beta_k \) to get 

\[
H^1_{\Omega}(\lambda + \beta_1)[1] \simeq H^1_{\Omega'}(\lambda + \beta_1 + \beta_2 + \ldots + \beta_k)[-1 - (k-1) + t]
\]

for any \( \Omega' \) as in the statement since if needed we can use (2.2) repeatedly to ensure that \( \Omega' \) includes \( \beta_1 \).

If \( \beta_2 \in \Omega \), then the \( A_2 \)-move gives 

\[
H^1_{\Omega}(\lambda) \simeq H^1_{\Omega'}(\lambda + \beta_1 + \beta_2)[-1],
\]

where \( \Omega' \) and \( \Omega \) are the same, except that \( \Omega' \) includes \( \beta_1 \) instead of \( \beta_2 \). Now \( \langle \lambda + \beta_1 + \beta_2, \beta_3 \rangle = -1 \) and \( \lambda + \beta_1 + \beta_2 \in X^*(P_{\{\beta_3, \ldots, \beta_k\}}) \) and so we can apply induction to the Levi subalgebra of type \( A_{k-2} \) with simple roots \( \beta_3, \ldots, \beta_k \) to get 

\[
H^1_{\Omega}(\lambda + \beta_1 + \beta_2)[-1] \simeq H^1_{\Omega'}(\lambda + \beta_1 + \beta_2 + \beta_3 + \ldots + \beta_k)[-1 - (k-2) + (t-1)]
\]

for any \( \Omega' \) as in the statement since if needed we can use (2.2) to permute around the roots of \( \Omega' \) to include \( \beta_1 \) and \( \beta_2 \). In either case, the result follows. \( \square \)

### 8.2. Type \( A_n \)

Let \( \Omega = \{ \alpha_3, \ldots, \alpha_{2t-1}, \alpha_{2t+1} \} \). One use of Lemma 8.1 gives

**Lemma 8.2.** For all \( i \geq 0 \), \( H^i_{\Omega}(\alpha_1)[1] \simeq H^i_{\{\alpha_2, \alpha_4, \ldots, \alpha_{2t-2}, \alpha_{2t}\}}(\phi)[-n + t] \).

### 8.3. Type \( C_n \)

Let \( \Omega = \{ \alpha_3, \ldots, \alpha_{2t-1}, \alpha_{2t+1} \} \) with \( 2t + 1 < n \).

**Lemma 8.3.** For all \( i \geq 0 \), we have \( H^i_{\Omega}(\alpha_1)[1] \simeq H^i_{\Omega}(\phi)[-2n + 2 + 2t] \).

**Proof.** Here, \( \phi = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \alpha_n \). Use Lemma 8.1 applied to the Levi subalgebra \( \mathfrak{m} \) with simple roots \( \{ \alpha_2, \ldots, \alpha_{n-1} \} \) and \( \lambda = \alpha_1 \) to get 

\[
H^i_{\Omega}(\alpha_1)[1] \simeq H^i_{\{\alpha_2, \alpha_4, \ldots, \alpha_{2t-2}, \alpha_{2t}\}}(\sum_{i=1}^{n-1} \alpha_i)[-1 - (n-2) + t].
\]

Next, since \( \langle \sum_{i=1}^{n-1} \alpha_i, \alpha_n \rangle = -1 \), one use of the \( A_1 \) move yields that the latter is isomorphic to 

\[
H^i_{\{\alpha_2, \alpha_4, \ldots, \alpha_{2t-2}, \alpha_{2t}\}}(\sum_{i=1}^{n} \alpha_i)[-n + t].
\]

Another use of Lemma 8.1 applied to \( \mathfrak{m} \), with the ordering of the roots reverse and with \( \lambda = \sum_{i=1}^{n} \alpha_i \), gives the isomorphism with \( H^i_{\Omega}(\phi)[-2n + 2 + 2t] \). \( \square \)

### 8.4. Type \( D_n \)
8.4.1. Let $\Omega = \{\alpha_3, \ldots, \alpha_{2t-1}, \alpha_{2t+1}\}$. Let $\Omega' = -w_0(\Omega)$, which is different from $\Omega$ only in the case when $n = 2t + 2$. The proof of the next lemma is similar to the previous ones.

**Lemma 8.4.** For all $i \leq 0$, we have $H^i_{\Omega}(\alpha_1)[-1] \simeq H^i_{\Omega'}(\phi)[-2n - 3 + 2t]$.

In the case $n = 2t + 2$, the proof also works when the roles of $\Omega$ and $\Omega'$ are interchanged.

8.4.2. Now let $\Omega = \{\alpha_{n-2t+1}, \alpha_{n-2t+3}, \ldots, \alpha_{n-3}, \alpha_{n-1}\}$ with $t \geq 1$.

**Lemma 8.5.** For all $i \geq 0$, we have $H^i_{\Omega}(\alpha_n)[-1] \simeq H^i_{\{\alpha_1, \alpha_3, \ldots, \alpha_{2t-3}, \alpha_{2t-1}\}}(\phi)[-n + 1]$.

**Proof.** When $t \geq 2$, the first $t - 1$ moves are type $A_3$, giving

$$H^i_{\Omega}(\alpha_n)[-1] \simeq H^i_{\Omega}(\alpha_{n-2t+1} + 2\alpha_{n-2t+2} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)[-2t + 1].$$

Then $n - 2t$ moves of type $A_2$ are used to add the remaining roots to get the isomorphism with

$$H^i_{\{\alpha_1, \alpha_{n-2t+3}, \ldots, \alpha_{n-3}, \alpha_{n-1}\}}(\phi)[-n + 1].$$

We can then change the parabolic by using $[2,2]$ to shift over the simple roots to get the isomorphism in the statement of the lemma. □

When $n = 2t$, we also get a symmetric variant. Let $\Omega = \{\alpha_1, \alpha_3, \ldots, \alpha_{n-3}\} \cup \{\alpha_n\}$. Then

$$(8.1) \quad H^i_{\Omega}(\alpha_{n-1})[-1] \simeq H^i_{\Omega}(\phi)[-n + 1].$$

8.4.3. Now let $\Omega = \{\alpha_{n-2t+3}, \alpha_{n-2t+5}, \ldots, \alpha_{n-3}, \alpha_{n-1}, \alpha_n\}$ with $t \geq 2$. Similar to the previous cases,

**Lemma 8.6.** For all $i \geq 0$, we have $H^i_{\Omega}(\alpha_1)[-1] \simeq H^i_{\Omega}(\phi)[-2n + 1 + 2t]$.

8.5. **Others cases where $t = 2$ or $t = 3$.**

**Lemma 8.7.** In $E_7$, for all $i \geq 0$, we have

$$H^i_{\{\alpha_2, \alpha_3\}}(\alpha_7)[-1] \simeq H^i_{\{\alpha_2, \alpha_3\}}(\phi)[-9].$$

**Lemma 8.8.** In $E_7$ and $E_8$, for $m = 9, 17$, respectively, for all $i \geq 0$, we have

$$H^i_{\{\alpha_3, \alpha_5, \alpha_7\}}(\alpha_2)[-1] \simeq H^i_{\{\alpha_2, \alpha_3, \alpha_5\}}(\phi)[-m].$$

**Lemma 8.9.** In $E_7$, for all $i \geq 0$, we have

$$H^i_{\{\alpha_2, \alpha_5, \alpha_7\}}(\alpha_3)[-1] \simeq H^i_{\{\alpha_2, \alpha_5, \alpha_7\}}(\phi)[-11].$$
APPENDIX A. Restricting to subalgebras

A.1. Non-simply laced Lie algebras. We state some results, likely already known, about the relationship between a simple non-simply-laced \( g_0 \), i.e., \( g_0 \) of type \( B_n, C_n, G_2, F_4 \), and its associated simple simply-laced Lie algebra \( g \), i.e., \( D_{n+1}, A_{2n-1}, D_4, E_6 \), respectively. As is well-known, \( g_0 \) arises as the invariant space of an automorphism \( \epsilon : g \to g \), preserving \( h \). Moreover, \( \epsilon \) can be taken to be induced from a diagram automorphism of \( \Pi \), also denoted \( \epsilon \), and we can pick root vectors \( e_\alpha \) for \( \alpha \in \Pi \) such that \( \epsilon(e_\alpha) = e_{\epsilon(\alpha)} \). Also \( \epsilon \) has order equal to 2 or 3. See [OV90] for these results. Let \( \Pi_I = \{ \alpha \in \Pi \mid \epsilon(\alpha) = \alpha \} \) and \( \Pi_s = \Pi \setminus \Pi_I \). The key property about \( \epsilon \) is that for all \( \alpha \in \Pi_s \), we have that \( \epsilon(\alpha) \) and \( \alpha \) are orthogonal. Assume for simplicity of statement that \( d = 2 \), so that \( g = g_0 \oplus g_1 \) is the eigenspace decomposition under \( \epsilon \) for eigenvalues 1 and \(-1\), respectively.

Then for any such \( \epsilon \), it is an exercise to show (in a case-free manner) that \( g_0 \) is a Lie algebra with two root lengths, the non-zero weights of the \( g_0 \)-representation on \( g_1 \) consist of the short roots of \( g_0 \) and the zero weight space of the \( g_0 \)-representation on \( g_1 \) is exactly \( h_1 \), and that, in fact,

\[
(A.1) \quad g \simeq g_0 \oplus V_\phi
\]
as \( g_0 \)-module, where \( \phi \) now refers to the dominant short root of \( g_0 \). Moreover, every root \( \beta \) of \( g \) restricted to \( h_0 \) is a root \( \bar{\beta} \) of \( g_0 \) and \( \text{ht}(\beta) = \text{ht}(\bar{\beta}) \) where the latter is computed using the induced simple roots of \( g_0 \). See Proposition 8.3 in [Kac85].

Pick an \( \epsilon \)-stable set of fundamental invariants \( f_1, \ldots, f_n \) for \( g \). Let \( F_0 = \{ f_i \mid \epsilon(f_i) = f_i \} \) and \( F_1 = \{ f_i \mid \epsilon(f_i) = -f_i \} \).

**Proposition A.1.** The following statements hold:

1. The restriction to \( g_0 \) of the elements in \( F_0 \) gives a set of fundamental invariants for \( g_0 \).
2. The Jacobian of the \( f_i \)'s remains nonzero upon restriction to the nilcone \( N_0 \) of \( g_0 \). The derivatives of \( f_i \) span a copy of \( V_\theta \) (resp., \( V_\phi \)) in \( \mathbb{C}[N_0] \) when \( f_i \in F_0 \) (resp., \( f_i \in F_1 \)).
3. The exponents of \( g_0 \) are the \( \deg(f_i) - 1 \) for \( f_i \in F_0 \).
4. The generalized exponents of \( V_\phi \) are the \( \deg(f_i) - 1 \) for \( f_i \in F_1 \).

**Proof.** First, \( g_0 \) contains a regular nilpotent element of \( g \), namely \( \sum_{\alpha \in \Pi} e_\alpha \). Since the \( f_i \) are fundamental invariants for \( g \), their Jacobian is nonzero when evaluated at any regular nilpotent element of \( g \). Hence, the \( n \) vectors of derivatives of the \( f_i \)'s are linearly independent on restriction to the nilcone of \( g_0 \). Now if \( \epsilon(f_i) = f_i \), the derivatives with respect to vectors in \( g_1 \) cannot be \( \epsilon \)-invariant and so vanish as functions on \( g_0 \). That means the derivatives with respect to vectors in \( g_0 \) cannot vanish and so span a copy of the adjoint representation of \( g_0 \). Similarly for \( \epsilon(f_i) = -f_i \). Hence by the linear independence of the vector of derivatives of the \( f_i \)'s on the the nilcone of \( g_0 \) and the fact that \( \dim V_\theta^T + \dim V_\phi^T = \dim g^T \), we have accounted for all the generalized exponents of \( V_\theta \) and \( \dim V_\phi \) by Kostant’s result ([I.1]). \( \square \)
A.2. Kostant-Shapiro formula for exponents. The Kostant-Shapiro formula states [Kos59] that the exponents for \( \mathfrak{g} \) are obtained as the dual partition \( \mu \) of the partition of \( |\Phi^+| \) given by

\[
\#\{\alpha \in \Phi^+ \mid \text{ht}(\alpha) = j\}
\]

for \( j = 1, 2, \ldots, \text{ht}(\theta) \).

We recall that the proof arises by taking a regular nilpotent \( e \) and one of its \( \mathfrak{sl}_2 \)-subalgebras \( \mathfrak{s} \) with standard basis \( e, h, f \) with \( h \in \mathfrak{h} \) dominant. Since \( \alpha(h) = 2 \) for \( \alpha \) simple, the value \( \text{ht}(\alpha) \) coincides with \( \frac{1}{2} \alpha(h) \) for any root \( \alpha \). Since \( h \) is regular and the values \( \alpha(h) \) are even for each root \( \alpha \), the representation of \( \mathfrak{s} \) on \( \mathfrak{g} \) has \( n = \dim(\mathfrak{h}) \) irreducible constituents. Moreover, the centralizer \( \mathfrak{g}_e \) of \( e \) consists of extermal weight vectors and hence the grading of the \( n \)-dimensional space \( \mathfrak{g}_e \) by \( \frac{1}{2} \alpha(h) \) coincides with the values of \( \mu \).

Since in this case \( G_e \) is connected, we have \( \mathfrak{g}^{G_e} = \mathfrak{g}^e \) and the former has dimension \( n \) since the moment map \( T^*G/B \to \mathcal{N} \) is birational, hence so does the latter. But then \( \mathfrak{g}_e = \mathfrak{g}^e \) for dimension reasons and since \( e \in \mathfrak{g}_e \). Finally, the discussion in §5.3 and the normality of \( \mathcal{N} \) give that the generalized exponents for \( \mathfrak{g} = V_0 \) are given by the values of \( \mu \).

The same proof applies to \( V_\phi \) with \( \Phi \) replaced by \( \Phi_s \), the short roots of \( \mathfrak{g}_0 \), since these are the nonzero weights of \( V_\phi \), they correspond to one-dimensional weight spaces, and the argument above and from §4.3 shows that the kernel of \( \text{ad}(e) \) on \( V_\phi \) coincides with \( V_{\phi e}^{G_e} = V_{\phi e}^e \) since all spaces have dimension \( r = \dim V_{\phi e}^T \). We first learned of this result from [Ion04], where it is proved in a different way. Ion also credits Stembridge and Bazlov.

**Proposition A.2** (Theorem 4.5 in [Ion04], [Vis06]). Let \( \Phi_s^+ \) denote the short positive roots of \( \mathfrak{g}_0 \). Then the dual partition of the partition of \( |\Phi_s^+| \) given by

\[
\#\{\alpha \in \Phi_s^+ \mid \text{ht}(\alpha) = j\}
\]

for \( j = 1, 2, \ldots, \text{ht}(\phi) \) is equal to the generalized exponents \( m_1^\phi \leq m_2^\phi \leq \cdots \leq m_r^\phi \) of \( V_\phi \).

**Example A.3.** In type \( F_4 \), there are 2 short roots of each height 1, 2, 3, and 4, and 1 short root of each height 5, 6, 7, and 8. Therefore, the generalized exponents for \( \phi \) are 4 and 8. The usual exponents are 1, 5, 7, 11 using the heights for all the positive roots. Combining with Proposition A.1, we know how the involution of \( E_6 \) that fixes \( F_4 \) acts on a set of \( e \)-stable fundamental invariants for \( E_6 \).

A.3. **Type \( A_n \).** For a generic matrix \( X \in \mathfrak{g}(n+1) \), the entries of \( X^i \) afford a representation isomorphic to \( V_0 \oplus \mathbb{C} \) for \( \mathfrak{g} \simeq \mathfrak{s}l_{n+1} \). These entries are not all zero on the nilcone of \( \mathfrak{g} \) when \( 1 \leq i \leq n \) since the regular element has Jordan form with one block of size \( n+1 \). Consequently, they give the \( n \) independent copies of \( V_0 \) in \( \mathbb{C}[\mathcal{N}] \). And since the derivatives of the \( \text{tr}(X^{i+1}) \) and the entries of \( X^i \) span the same space in \( S\mathfrak{g}^* \), the fundamental invariants can be taken to be \( \text{tr}(X^2) \ldots \text{tr}(X^{n+1}) \) since a set of invariant functions are fundamental invariants if and only if the vector of their derivatives are linearly independent when restricted to the \( \mathcal{N} \). Now, the matrix equation \( X^j = X^{j-i}X^i \) for \( j \geq i \) shows that the ideal in \( S\mathfrak{g}^* \) generated by the entries of \( X^i \) contains the entries of \( X^j \) for \( j \geq i \). Moreover, an ideal in \( S\mathfrak{g}^* \) generated by various copies
of $V_\theta$, non-vanishing on $\mathcal{N}$, and a set of fundamental invariants is minimally generated by the entries of $X^i$ for some $i$ and $\text{tr}(X^2), \ldots, \text{tr}(X^{i-1})$.

A.4. Type $C_n$. Let $X$ be a generic matrix of $\mathfrak{sp}_{2n} \subset \mathfrak{gl}_{2n}$. Since eigenvalues of matrices in $\mathfrak{sp}_{2n}$ come in pairs of opposite sign, $\text{tr}(X^i)$ vanishes on $\mathfrak{sp}_{2n}$ when $i$ is odd; hence, Proposition A.4 and A.3 imply that $\text{tr}(X^2), \text{tr}(X^4), \ldots, \text{tr}(X^{2n})$ are a set of fundamental invariants.

Proposition A.4. We have

1. The entries of $X^{2i-1}$ restrict to $\mathbb{C}[N_0]$ to give a nonzero copy of $V_\theta$ for $i = 1, 2, \ldots, n$.
2. The entries of $X^{2i}$ restrict to $\mathbb{C}[N_0]$ to give a nonzero copy of $V_\theta$ for $i = 1, \ldots, n-1$.
3. Any ideal in $\mathfrak{S}g^*_N$ generated by various copies of $V_\theta$ or $V_\theta$ (non-vanishing on $N_0$) and a set of fundamental invariants is minimally generated by a basis of the entries of $X^j$ for some $j < 2n$ and $\text{tr}(X^2), \text{tr}(X^4), \ldots, \text{tr}(X^{2i})$ with $2i < j$.

The last part follows from the matrix equation in A.3 together with the key fact that the entries of $X^i$, restricted to $N_0$, span a single irreducible representation.

A.5. Types $B_n$ and $D_n$. An analogous story to that in A.4 works for $g \simeq \mathfrak{so}_N \subset \mathfrak{sl}_N$ and we get a version of Proposition A.4. We omit the proof since it is straightforward (and probably already known). Here, $\mathfrak{sl}_N$ decomposes as $V_0 \oplus V_{2\omega_1}$ as $\mathfrak{so}_N$-module. As is well-known, the Pfaffian $\text{Pf}(X)$ is a generator for $\mathfrak{so}_N$ when $N$ is even.

Proposition A.5. Let $X$ be a generic matrix of $\mathfrak{so}_N \subset \mathfrak{sl}_N$. Then,

1. The functions $\text{tr}(X^2), \text{tr}(X^4), \ldots, \text{tr}(X^{2n-2})$, together with $\text{tr}(X^{2n})$ (resp. $\text{Pf}(X)$), are complete set of fundamental invariants for $B_n$ (resp. $D_n$).
2. The restriction of the entries of $X^{2i-1}$ to $\mathbb{C}[N]$ gives nonzero copies of $V_\theta$ when $i = 1, 2, \ldots, n-1$ for $D_n$ and when $i = 1, 2, \ldots, n$ for $B_n$.
3. The restriction of the entries of $X^{2i}$ to $\mathbb{C}[N]$ gives all the nonzero copies of $V_{2\omega_1}$ in $\mathbb{C}[N]$ when $i = 1, 2, \ldots, n-1$ for $D_n$ and when $i = 1, 2, \ldots, n$ for $B_n$.
4. Any ideal in $\mathfrak{S}g^*_N$ generated by the entries of $X^k$ for various $k$ is minimally generated by a basis of the entries of $X^j$ for some $j$.

Finally, consider $g_0 \simeq \mathfrak{so}_{2n+1} \subset g \simeq \mathfrak{so}_{2n+2}$ and let $X$ be a generic matrix of $\mathfrak{so}_{2n+2}$. Since $\epsilon(\text{Pf}(X)) = -\text{Pf}(X)$ and $\epsilon(\text{tr}(X^{2i})) = \text{tr}(X^{2i})$, then by Proposition A.3 the derivatives of $\text{Pf}(X)$ along $g_0$ give the unique copy of $V_\theta$ in $\mathbb{C}[N_0]$, in degree $n$.

Appendix B. Explicit examples of invariants

Theorem B.1. Let $\phi : g \rightarrow \mathfrak{gl}(V_\lambda)$ be a non-trivial highest weight representation of $g$ of minimal dimension. Let $d$ be a degree for $g$. Let $X$ be the generic matrix in $\mathfrak{gl}(V_\lambda)$ with respect to some basis of $V_\lambda$. Then the restriction of $\text{tr}(X^d)$ to $g$ is a generator of $R$. Moreover, if $g$ is not of type $D_{2k}$, then this gives a complete set of fundamental invariants for $g$. 

Proof. We checked the result using Magma (see [Joh17]) by restricting \( \text{tr}(X^d) \) to the Kostant-Slodowy slice of \( \mathfrak{g} \) to the nilcone and using Kostant’s slice result [Kos63]. Namely, we observe that the restriction to the slice of \( \text{tr}(X^d) \) contains a linear term. The last statement follows since all the degrees are distinct in these cases. \( \square \)

Remark B.2. Normally to find a set of fundamental invariants we have to find a complete set, restrict to the Cartan subalgebra, and compute the Jacobian (see [Lee74]). The method above has the advantage of being able to check one invariant at a time. Thus, for example, we can check that \( \text{tr}(X^8) \) and \( \text{tr}(X^{12}) \) are generators for \( R \) in \( E_8 \), allowing perhaps a simpler starting point to the computer calculations in [DCPP15]. At the same time, this choice of invariants when restricted to the \( \mathfrak{h} \) seems more natural than that in [Lee74]. For instance in \( E_8 \), we have that \( \sum_{\alpha \in \Phi^+} \alpha^{d_i} \) for \( 1 \leq i \leq 8 \) is a set of fundamental invariants for \( W \).

Remark B.3. The first author [Joh17] used this observation about invariants and Broer’s description of the ideal defining the closure of the subregular nilpotent orbit \( \mathcal{O}_{sr} \) to explicitly describe a generic singularity of \( \overline{\mathcal{O}_{sr}} \). This gives another way to obtain the result in [FJLS17, §5.6].

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