PARAMETRIZATION OF RATIONAL MAPS
ON A VARIETY OF GENERAL TYPE,
AND THE FINITENESS THEOREM

LUCIO GUERRA AND GIAN PIETRO PIROLA

(Communicated by Lev Borisov)

Abstract. In a previous paper we provided some update in the treatment of
the finiteness theorem for rational maps of finite degree from a fixed variety to
varieties of general type. In the present paper we present another improvement,
introducing the natural parametrization of maps by means of the space of linear
projections in a suitable projective space, and this leads to some new insight
into the geometry of the finiteness theorem.

Introduction

Let $X$ be an algebraic variety of general type over the complex field. The dom-
ninant rational maps of finite degree $X \to Y$ to varieties of general type, up to
birational isomorphisms $Y \to Y'$, form a finite set. We call this the finiteness
theorem for rational maps on a variety of general type. The proof follows from
the approach of Maehara [7], joined with some recent advances in the theory of
pluricanonical maps due to Hacon and McKernan [4] and to Takayama [8], [9].

In our paper [3], motivated by the wish for some effective estimate for the finite
number of maps in the theorem, we provided some update and refinement in the
treatment of the subject. We brought the rigidity theorem to a general form,
avoiding certain technical restrictions; we pointed out the role of the canonical
volume $\text{vol}(K_X)$ in bounding the rational maps in the finiteness theorem; and we
proposed a new argument leading to a refined version of the theorem.

However, something that was still not satisfactory was the use of a certain bunch
of subvarieties of Chow varieties as a parameter space for rational maps, as in
Maehara’s approach. The most natural and simplest parameter space should be
the space of linear projections in a suitable projective space, already appearing for
instance in the work of Kobayashi and Ochiai [5].

In the present paper we are able to replace the Chow parametrization with the
natural parametrization, and this leads to some new insight into the geometry
of the finiteness theorem. The main result concerns the structure of the special
birational equivalence classes of maps viewed as unions of connected components of
a certain space of linear rational maps; see Theorem 6.1. This has as an immediate
consequence a better refined finiteness theorem; see Theorem 6.2.

Received by the editors October 17, 2011 and, in revised form, March 2, 2012.

2010 Mathematics Subject Classification. Primary 14E05, 14N05.

Key words and phrases. Rational maps, pluricanonical maps, varieties of general type, canonical
volume.
1. Preliminary material

1.1. Results on pluricanonical maps. A recent achievement in the theory of pluricanonical maps is the following theorem of uniform pluricanonical birational embedding, due to Hacon and McKernan [4] and to Takayama [8].

Theorem 1.1. For any dimension $n$ there is some positive integer $r_n$ such that for every $n$-dimensional variety $V$ of general type, the multicanonical divisor $r_nK_V$ defines a birational embedding $V \to V' \subset \mathbb{P}^M$.

A basic tool is the canonical volume of a variety, the invariant arising in the asymptotic theory of divisors; see Lazarsfeld’s book [6]. In terms of the canonical volume we have a bound

(1) $\deg V' \leq \text{vol}(r_nK_V)$;

see [4], Lemma 2.2. Moreover, from elementary geometry we have a bound

(2) $M \leq \deg V' + n - 1$.

Note that the embedded variety $V'$ need not be smooth. Intimately related to the theorem above is the following result, proved in [4] and in [8].

Theorem 1.2. For any dimension $n$ there is some positive number $\epsilon_n$ such that every $n$-dimensional variety $V$ of general type has $\text{vol}(K_V) \geq \epsilon_n$.

For instance, concerning the minimum $r_n$ we know from the classical theory that $r_1 = 3$ and $r_2 = 5$, and a recent result is that $r_3 \leq 73$; while concerning the maximum $\epsilon_n$, it is clear that $\epsilon_1 = 2$ and $\epsilon_2 = 1$ and a recent result is $\epsilon_3 \geq 1/2660$ (see J. A. Chen and M. Chen [1]). Note that [4] and [8] do not give explicit bounds for $r_n$ and $\epsilon_n$ in the theorems above.

1.2. Bounds for the degree of a rational map. Let $f : X \to Y$ be a rational map of finite degree between varieties of general type. Because of Theorem 1.1, taking the $r_n$-canonical birational models $X'$ and $Y'$ in $\mathbb{P}^M$ (note that $Y'$ lies within the embedding space of $X'$), the map $f$ is identified with a linear rational map $X' \to Y'$, a rational map which is the restriction of a linear projection $\mathbb{P}^M \to \mathbb{P}^M$. For a linear map of finite degree the inequality $\deg f \deg Y' \leq \deg X'$ holds. Using (1) it follows that

(3) $\deg f \leq \deg X' \leq (r_n)^n \text{vol}(K_X)$.

A more precise estimate is as follows. For any rational map of finite degree the inequality $\deg f \text{vol}(K_Y) \leq \text{vol}(K_X)$ holds; see [3], Proposition 3.2. Using Theorem 1.2 it follows that

(4) $\deg f \leq \frac{1}{\epsilon_n} \text{vol}(K_X)$.

This bound is sharp for curves, and in this case it reduces to the usual bound from the Hurwitz formula.

1.3. Families of rational maps. Let $T$ be a smooth variety. If $X \to T$ is a relative scheme over $T$, we denote by $X(t)$ the scheme fibre over $t$ and by $X_t$ the associated reduced scheme.

A family of varieties, parametrized by a smooth variety $T$, is a surjective morphism $X \to T$, with $X$ a variety, such that every scheme fibre $X(t)$ is: (i) irreducible, (ii) generically smooth (in order to be assigned multiplicity one in the
associated algebraic cycle; see Fulton [2], Chap. 10), and (iii) of dimension equal
to the relative dimension of $X$ over $T$, of course. When the structure morphism
is projective or smooth, we speak of a family of projective varieties or a family of
smooth varieties.

A family of rational maps is the datum of a family of varieties $X \to T$ and
a relative scheme $X' \to T$, over the same smooth variety $T$, and a rational map
$f : X \dashrightarrow X'$, commuting with the structural projections, which for every $t \in T$
restricts to a rational map $f_t : X_t \dashrightarrow X'_t$.

1.4. The rigidity theorem. A family of rational maps on a fixed variety $X$ is the
datum of a relative scheme $Y \to T$, with $T$ smooth, and a rational map
$$f : X \times T \dashrightarrow Y,$$
which is a family of rational maps $f_t : X \dashrightarrow Y_t$ in the sense of the previous
definition.

A trivial family is one which is obtained as follows. Let $h : X \dashrightarrow U$ be a
rational map and let $g : T \times U \dashrightarrow Y$ be a birational isomorphism which is a family
of birational isomorphisms $g_t : U \dashrightarrow Y_t$. Then the composite map
$$T \times X \xrightarrow{1 \times h} T \times U \xrightarrow{g} Y$$
is a trivial family, because all maps $g_t \circ h$ are birationally equivalent.

Recall that two dominant rational maps $f : X \dashrightarrow Y$ and $f' : X \dashrightarrow Y'$, defined
on the same variety, are birationally equivalent if there is a birational isomorphism
$g : Y \dashrightarrow Y'$ such that $f' = g \circ f$.

For projective varieties of general type and dominant rational maps of finite
degree, there are results of rigidity.

Theorem 1.3. Let $X$ be a smooth projective variety of general type. Let $T$ be a
smooth variety, let $Y \to T'$ be a family of smooth projective varieties of general type,
and let $f : X \times T \dashrightarrow Y$ be a family of rational maps of finite degree. Then $f$ is a
trivial family, so all maps $f_t$ are birationally equivalent.

The rigidity theorem above was proved by Maehara [7] with some technical
restrictions, and has been brought to the present form in our previous paper [3],
Theorem 2.1. More generally, if the family of image varieties is not known to be a
smooth family, one has the following.

Corollary 1.4. Let $X$ be a projective variety of general type. Let $T$ be a
smooth variety, let $Y \to T$ be a family of projective varieties of general type, and let $f : X \times T \dashrightarrow Y$ be a family of rational maps of finite degree. There is a nonempty
open subset $T'$ of $T$ such that the restriction $f|_{T'} : X \times T' \dashrightarrow Y|_{T'}$ is a trivial
family.

2. Graphs and images in a family of maps

Let $f : X \dashrightarrow X'$ be a family of rational maps parametrized by a smooth variety
$T$, as in subsection 1.3. Consider the relative product $X \times_T X'$ and call $p$ and $p'$
the projections to $X$ and $X'$. Assume now that $X \to T$ is a projective morphism.
Thus $p'$ is a closed map. Then define the following:
the closed graph of $f$ in $X \times_T X'$,
$Y$ the closed image of $X$ in $X'$,
$C$ any closed subscheme of $X$ such that $X \setminus C \to T$ is surjective
and $f$ is a regular map $X \setminus C \to Y$,
$E$ the inverse image of $C$ in $\Gamma$.

Note that $p'(\Gamma) = Y$, as $p'$ is a closed map.

A natural question is whether $\Gamma \to T$ is the family of closed graphs for the
given family of maps, more precisely: whether $\Gamma \to T$ is a family of varieties, as in
subsection 1.3, and every reduced fibre $\Gamma_t$ coincides with the closed graph $\Gamma(f_t)$. A
related question is whether $Y \to T$ is the family of closed images $\overline{f_t(X_t)}$, that is:
whether $Y \to T$ is a family of varieties and every reduced fibre $Y_t$ coincides with
the closed image $\overline{f_t(X_t)}$. The following equality of reduced schemes holds:

$$\Gamma_t = \Gamma(f_t) \cup E_t,$$

and from this, applying $p'$, a description of $Y_t$ follows.

**Proposition 2.1.** In the setting above, assume that $T$ is a smooth curve. (1) There
is a nonempty open subset $T'$ of $T$ such that $\Gamma|_{T'} \to T'$ is the family of closed graphs
for the restricted family $f|_{T'}$. (2) Moreover there is a nonempty open subset $T''$ of
$T'$ such that $Y|_{T''} \to T''$ is the family of closed images for the family $f|_{T''}$.

**Proof.** We start with an easy remark. Let $V \to T$ be a surjective morphism of vari-
eties, with irreducible fibres, all of the same dimension. Then there is a nonempty
open subset $T'$ of $T$ such that the restriction $V|_{T'} \to T'$ is a family of varieties.
Now we apply this to the relative varieties $\Gamma$ and $Y$ over the curve $T$. In order
to prove the statement we only need to identify the reduced fibres $\Gamma_t$ and $Y_t$ for
sufficiently general $t$. This is what we do in the following.

(1) First, we show that $\Gamma_t = \Gamma(f_t)$ holds for every $t$ if $E \to T$ is a flat morphism.
Recall that this happens if and only if every irreducible component of $E$ dominates $T$.

Write $\dim X =: n + 1$. We have $\Gamma_t = \Gamma(f_t) \cup E_t$. Remark that $\dim E < n + 1$.
Then $\dim E_t < n$ for every $t$, because of flatness. But all components of $\Gamma_t$ must
have dimension $= n$ for every $t$. Thus $E_t$ is not a component and $\Gamma_t = \Gamma(f_t)$, for
every $t$. In particular, every $\Gamma_t$ is irreducible of dimension $n$.

In the present situation, the statement follows from the remark at the beginning.
In the general case, by generic flatness, we have that $E|_{T'} \to T'$ is flat for some $T'$,
and then, because of the remark, the statement follows.

(2) We know that $Y_t = p'(\Gamma_t)$, and for $t \in T'$ we have from (1) that $\Gamma_t = \Gamma(f_t)$
and hence $Y_t = \overline{f_t(X_t)}$. In particular, every such $Y_t$ is irreducible and necessarily of
dimension $= \dim Y - 1$. Because of the remark above, the statement follows. $\square$

In general, the family of graphs need not exist for the full family of maps, as is
seen later in Remark 5.1.

3. The varieties of general type in a family

Using the technique of extension of differentials, from a special fibre to the total
space of the family, we gave in [3], subsection 1.4, a proof of the assertion that the
property of being a variety of general type is invariant in a 1-dimensional small
deformation, where small refers to the Zariski topology. Here we point out that the
same proof indeed shows a slightly stronger assertion, to the effect that the same property ‘propagates’ from a component of a fibre.

**Theorem 3.1.** Let $T$ be a smooth irreducible curve, let $Y$ be a variety, and let $Y \to T$ be a projective morphism. Assume that some fibre $Y_a$ has an irreducible component $Z$ which is a variety of general type and that the restriction $Y \setminus Y_a \to T \setminus \{a\}$ is a family of varieties, as in subsection 1.3. Then there is a nonempty open subset $T'$ of $T$ such that $Y_t$ is a variety of general type for $t \in T'$.

**Proof.** Let $V \to Y$ be a resolution of singularities such that the strict transform $Z'$ of $Z$ is smooth. So $Z'$ is of general type, and $\dim H^0(Z', mK_{Z'}) \geq cm^n$ for $m \gg 0$. Denote by $\pi$ the composite map $V \to Y \to T$. Since $V \to T$ is generically smooth, and since $Z \to T$ is generically a family of varieties, restricting to some neighborhood of $a$, we may assume that for every $t \neq a$ the induced map $V_t \to Y_t$ is a resolution of singularities. As the general $V_t$ is irreducible, it follows that every $V_t$ is connected, by the Zariski connectedness theorem.

The extension theorem of Takayama [9] applies and gives us that there is a surjective restriction homomorphism

$$\pi_*\mathcal{O}_V(mK_V) \otimes k(a) \to H^0(Z', mK_{Z'}).$$

The image $\pi_*\mathcal{O}_V(mK_V)$ is a torsion free coherent sheaf on the smooth curve $T$; hence it is a locally free sheaf. So the dimension of $\pi_*\mathcal{O}_V(mK_V) \otimes k(t)$ is constant. For $t = a$ this dimension is $\geq cm^n$ for $m \gg 0$, by what we have seen above.

For $t \neq a$, since $mK_V|_{V_t} = mK_{V_t}$, one has the restriction homomorphism

$$\pi_*\mathcal{O}_V(mK_V) \otimes k(t) \to H^0(V_t, \mathcal{O}_{V_t}(mK_V|_{V_t})) = H^0(V_t, mK_{V_t}),$$

and in a smaller neighborhood of $a$ we may assume that this is an isomorphism for $t \neq a$. It follows that $\dim H^0(V_t, mK_{V_t}) \geq cm^n$ for $m \gg 0$; hence $Y_t$ is of general type. This holds for every $t$ in a neighborhood of $a$. 

4. **Rigidity and limits**

Another key point in our treatment is a result about limit maps in a generically trivial family of maps. The result that we give here is only slightly more general than the one in our previous paper, and the proof given here is more apparent.

Let $X$ be a projective variety. Let $T$ be a smooth irreducible curve, let $Y \to T$ be a projective morphism, and let $f : T \times X \dasharrow Y$ be a family of rational maps on $X$, as in subsection 1.4. Assume that for every $t \in T$ the rational map $f_t : X \dasharrow \overline{f_t(X)}$ is of finite degree $k$.

Assume moreover that the family is *generically trivial*, as in Corollary 1.4. That is, there is a nonempty open subset $T'$ of $T$ such that the restriction $f|_{T'}$ is obtained as

$$T' \times X \xrightarrow{1 \times h} T' \times U \xrightarrow{g} Y|_{T'},$$

where $h : X \dasharrow U$ is a fixed dominant rational map and where $g$ is a birational isomorphism which restricts to a birational isomorphism $g_t : U \dasharrow Y_t$ for every $t \in T'$. Then $f_t = g_t \circ h$ for $t \in T'$, so all these maps are birationally equivalent, of degree $\deg(f_t) = k = \deg(h)$.

**Proposition 4.1.** Assume that $f : T \times X \dasharrow Y$ is a family of rational maps of constant degree $\deg(f_t) = k$, and assume that the family is generically trivial, as in the setting above. Then all maps $f_t$ are in the same birational equivalence class.
Proof. Let \( a \in T \) be any point, and let us prove that \( f_a \) is in the birational equivalence class of every \( f_t \) with \( t \in T' \).

We may assume that \( U \) is a normal variety. Recall that for a rational map of varieties over a base curve, from a normal variety to a variety which is proper over the base, the exceptional locus is of codimension \( \geq 2 \), by the valuative criterion of properness for instance. It follows that \( g : T \times U \dashrightarrow Y \) restricts to a rational map \( g_a : U \dashrightarrow Y_a \).

Since \( f = g \circ (1 \times h) \) holds as an equality of rational maps \( T \times X \dashrightarrow Y \), there is equality of restrictions \( f_a = g_a \circ h \). Also, since \( \deg(f_a) = k = \deg(h) \), \( \deg(g_a) = 1 \) and \( f_a \) is birationally equivalent to \( h \) and to every \( f_t \). \( \square \)

5. Linear rational maps

Let \( \mathbb{P}^m = \mathbb{P}(V^{m+1}) \) and let \( X \subseteq \mathbb{P}^m \) be a nondegenerate subvariety, of dimension \( n \). The space of linear maps \( \mathbb{P}^m \dashrightarrow \mathbb{P}^m \) is the projective space
\[
\mathbb{P}^N = \mathbb{P}(\text{End}(V)) \quad \text{with} \quad N = (m+1)^2 - 1.
\]
We denote by \( \alpha = \vec{\ell} \) a point in \( \mathbb{P}^N \) and by \( x = \sigma \) a point in \( \mathbb{P}^m \). The evaluation homomorphism \( (\ell, v) \mapsto \ell(v) \) determines a rational map
\[
\mathbb{P}^N \times X \dashrightarrow \mathbb{P}^m,
\]
and this is the family of linear rational maps \( \alpha : X \dashrightarrow \mathbb{P}^m \). We denote by \( \overline{\alpha(X)} \) the closed image and by \( \Gamma(\alpha) \) the closed graph of the map \( \alpha \).

The subscheme \( C \subset \mathbb{P}^N \times X \) defined by \( \ell(v) = 0 \) is the exceptional locus of the rational map above. Consider the projection \( C \to \mathbb{P}^N \). The fibre \( C_\alpha \) is the trace in \( X \) of the center of the linear projection \( \alpha : \mathbb{P}^m \dashrightarrow \mathbb{P}^m \).

Remark 5.1. The subscheme \( \Gamma \subset \mathbb{P}^N \times X \times \mathbb{P}^m \) defined by \( \ell(v) \wedge w = 0 \) is the closed graph of the rational map above. Clearly \( \Gamma \) contains \( C \times \mathbb{P}^m \). The projection \( \Gamma \to \mathbb{P}^N \) does not define the family of graphs. The fibre is given by \( \Gamma_\alpha = \Gamma(\alpha) \cup C_\alpha \times \mathbb{P}^m \). It is clear, just looking at dimensions, that \( \Gamma_\alpha = \Gamma(\alpha) \) if and only if \( C_\alpha = \emptyset \).

In \( \mathbb{P}^N \) define the following subsets:
\[
R \quad \text{the subset of all } \alpha \text{ such that } \overline{\alpha(X)} \text{ is of finite degree},
\]
\[
R_k \quad \text{the subset of all } \alpha \in R \text{ with } \deg(\alpha) = k,
\]
for every integer \( k > 0 \).

**Proposition 5.2.** (1) \( R \) is an open subset. (2) \( R_k \) is a constructible subset for every \( k > 0 \).

**Proof.** (1) In \( (\mathbb{P}^N \times X) \setminus C \) let \( U \) be the subset of pairs \((\alpha, x)\) such that
\[
dim_x \alpha^{-1}(\overline{\alpha(X)}) = 0.
\]
It is an open subset. In \( \mathbb{P}^N \) the image of \( U \) coincides with \( R \). In fact, if \( \alpha \) admits some point \( x \in X \setminus C_\alpha \) which is isolated in its fibre, then its general fibre is of dimension 0. As the projection \( \mathbb{P}^N \times X \to \mathbb{P}^N \) is an open map, \( R \) is open in \( \mathbb{P}^N \).

(2) In \( \mathbb{P}^N \times X^{\times k} \) let \( U_k \) be the subset of sequences \((\alpha, x_1, \ldots, x_k) =: (\alpha, \vec{x})\) such that every \((\alpha, x_i)\) belongs to \( U \) and \( \alpha(x_1) = \cdots = \alpha(x_k) \), while in the sequence \((x_1, \ldots, x_k)\) there is no coincidence. For every \( \alpha \in R \) denote by \( U_k(\alpha) \) the fibre of \( U_k \) over \( \alpha \). Let \( V_k \) be the subset such that \( \dim_{(\alpha, \vec{x})} U_k(\alpha) = n \). This is a
locally closed subset in \( \mathbb{P}^N \times X^k \). In \( \mathbb{P}^N \) the image \( V_k' \) of \( V_k \) is the locus of \( \alpha \in R \) with \( \deg \alpha \geq k \). In fact, if \( \alpha \) admits some sequence \((x_1, \ldots, x_k)\) such that \( \dim(\mathbb{P}) U_k(\alpha) = n \), as the projection \( U_k(\alpha) \to X \) has 0-dimensional fibres, then \( U_k(\alpha) \) dominates \( X \), and hence for a general point \( x_1 \) the fibre of \( \alpha \) contains at least \( k \) distinct points \( x_1, \ldots, x_k \). It follows that \( R_k \) coincides with \( V_k' \setminus V_{k+1}' \). \( \square \)

6. Refined finiteness theorem

Let \( X \) be a smooth projective variety of general type, of dimension \( n \). Let \( X' \subset \mathbb{P}^M \) be the image of \( X \) in the \( r_n \)-canonical birational embedding; see Theorem 1.1. Here \( M = h^0(X, r_nK_X) - 1 \) is bounded above in \([2]\). Every rational map of finite degree \( f : X \to Y \) to a smooth projective variety of general type, taking the \( r_n \)-canonical model \( Y' \subset \mathbb{P}^M \), gives rise to a linear rational map \( \alpha : X' \to \mathbb{P}^M \) with \( \overline{\alpha(X')} = Y' \).

In this natural way the set of birational equivalence classes of rational maps of finite degree from \( X \) to varieties of general type is injected into the set of birational equivalence classes of linear rational maps of finite degree from \( X' \) to \( \mathbb{P}^M \). Our main result is concerned with the geometric structure of these special equivalence classes.

**Theorem 6.1.** Let \( X \) be a smooth projective variety of general type. A birational equivalence class of rational maps of degree \( k \) from \( X \) to smooth projective varieties of general type forms a union of connected components of \( R_k \).

**Proof.** Let \( \alpha \in R_k \) be such that \( \overline{\alpha(X)} \) is of general type. Let \( T \) be a smooth irreducible curve with a morphism \( T \to R_k \), which we write as \( t \mapsto \alpha_t \), and with some point \( a \in T \) such that \( a \mapsto \alpha \). We claim that all maps \( \alpha_t \) are birationally isomorphic to \( \alpha \).

Consider the rational map \( T \times X' \to T \times \mathbb{P}^M \) which represents the family of maps \( \alpha_t \). Let \( Y \) be its closed image in \( T \times \mathbb{P}^M \). There is a nonempty open subset \( T' \) of \( T \) such that \( Y|_{T'} \to T' \) is the family of closed images by Proposition 2.1.

The fibre \( Y_a \) contains \( \overline{\alpha(X')} \), a variety of general type. It follows from Theorem 3.1 that, shrinking \( T' \) if necessary, we may assume that for every \( t \in T' \) the variety \( \alpha_t(X') \) is of general type.

Then it follows from Corollary 1.4 to the rigidity theorem that, shrinking \( T' \) again, we may assume that the restriction \( T' \times X' \to Y|_{T'} \) is a trivial family. Then it follows from Proposition 1.1 that all maps \( \alpha_t \) with \( t \in T \) are birationally equivalent, as we claimed.

So we reach the conclusion. Every irreducible curve through \( \alpha \in R_k \) is the image of a smooth irreducible curve \( T \) as above and therefore is fully contained in the birational equivalence class of \( \alpha \). Therefore every connected curve through \( \alpha \in R_k \) is fully contained in the birational equivalence class of \( \alpha \). Since \( R_k \) is constructible, by Proposition 5.2 this means that the connected component of \( \alpha \) in \( R_k \) is contained in the birational equivalence class of \( \alpha \). \( \square \)

The space \( R \) admits the stratification \( \bigsqcup R_k \), where the degree \( k \) is bounded above in \([3]\) in terms of the function \( r_n \) or in \([4]\) in terms of the function \( \epsilon_n \). As an immediate consequence of the previous result we obtain the following refined version of the finiteness theorem, which improves our previous result \([2]\, \text{Theorem} \, 4.3)\).
Theorem 6.2. Let $X$ be a smooth projective variety of general type. The number of birational equivalence classes of rational maps of finite degree from $X$ to smooth projective varieties of general type is bounded above by the number of connected components of strata in the stratification $R = \bigsqcup R_k$.

We showed in [3] that the finite number of classes of maps in the finiteness theorem has an upper bound of the form $B(n, v)$, where $n = \dim(X)$ and $v = \text{vol}(K_X)$, and that such a function $B$ can be explicitly computed in terms of the function $r_n$. This is obtained by means of rather cumbersome computations with the complexity of a certain bunch of subvarieties of Chow varieties that was used as a parameter space for rational maps. We believe that an analogous computation working with the much simpler parametrization established in the present paper will lead to a simpler procedure and to a better result for the function $B$.

Acknowledgements

The first author was partially supported by Finanziamento Ricerca di Base 2008 Università Perugia. The second author was partially supported by INdAM (GN-SAGA) and FAR 2010 (PV): “Varietà algebriche, calcolo algebrico, grafi orientati e topologici”.

References

[1] J. A. Chen, M. Chen. Explicit birational geometry of 3-folds of general type, II. J. Differential Geom. 86 (2010), 237–271. MR2772551 (2012b:14020)
[2] William Fulton, Intersection theory, Ergebnisse der Mathematik und ihrer Grenzgebiete. (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984. MR732620 (85k:14004)
[3] Lucio Guerra and Gian Pietro Pirola, On the finiteness theorem for rational maps on a variety of general type, Collect. Math. 60 (2009), no. 3, 261–276, DOI 10.1007/BF03191371. MR2560970 (2010m:14016)
[4] Christopher D. Hacon and James McKernan, Boundedness of pluricanonical maps of varieties of general type, Invent. Math. 166 (2006), no. 1, 1–25, DOI 10.1007/s00222-006-0504-1. MR2242631 (2007e:14022)
[5] Shoshichi Kobayashi and Takushiro Ochiai, Meromorphic mappings onto compact complex spaces of general type, Invent. Math. 31 (1975), no. 1, 7–16. MR0402127 (53 #5948)
[6] R. Lazarsfeld. Positivity in algebraic geometry. I. Ergeb. Math. Grenzgebiete. (3), vol. 48, Springer-Verlag, Berlin, 2004. MR2095471 (2005k:14001a)
[7] Kazuhiro Maehara, A finiteness property of varieties of general type, Math. Ann. 262 (1983), no. 1, 101–123, DOI 10.1007/BF01474173. MR690010 (85e:14015)
[8] Shigeharu Takayama, Pluricanonical systems on algebraic varieties of general type, Invent. Math. 165 (2006), no. 3, 551–587, DOI 10.1007/s00222-006-0503-2. MR2242627 (2007m:14014)
[9] Shigeharu Takayama, On the invariance and the lower semi-continuity of plurigenera of algebraic varieties, J. Algebraic Geom. 16 (2007), no. 1, 1–18, DOI 10.1090/S1056-3911-06-00455-3. MR2257317 (2007i:32021)

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI PERUGIA, VIA VANVITELLI 1, 06123 PERUGIA, ITALIA
E-mail address: lucio.guerra@unipg.it

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PAVIA, VIA FERRATA 1, 27100 PAVIA, ITALIA
E-mail address: gianpietro.pirola@unipv.it