1. The method of zeta-function regularization

Hawking introduced this method\(^1\) as a basic tool for the regularization of infinities in QFT in a curved spacetime\(^2\). The idea is the following\(^3\). One could try to tame Quantum Gravity using the canonical approach, by defining an arrow of time and working on the space-like hypersurfaces perpendicular to it, with equal time commutation relations. Reasons against this: (i) there are many topologies of the space-time manifold that are not a product \(R \times M_3\); (ii) such non-product topologies are sometimes very interesting; (iii) what does it mean ‘equal time’ in the presence of Heisenberg’s uncertainty principle?

One thus turns naturally towards the path-integral approach:

\[
< g_2, \phi_2, S_2 | g_1, \phi_1, S_1 > = \int \mathcal{D}[g, \phi] e^{iI[g, \phi]},
\]

where \(g_j\) denotes the spacetime metric, \(\phi_j\) are matter fields, \(S_j\) general spacetime surfaces \((S_j = M_j \cup \partial M_j)\), \(\mathcal{D}\) a measure over all possible ‘paths’ leading from the \(j = 1\) to the \(j = 2\) values of the intervening magnitudes, and \(I\) is the action:

\[
I = \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-g} d^4 x + \int L_m \sqrt{-g} d^4 x,
\]

\(R\) being the curvature, \(\Lambda\) the cosmological constant, \(g\) the determinant of the metric, and \(L_m\) the Lagrangian of the matter fields. Stationarity of \(I\) under the boundary conditions

\[
\delta g|_{\partial M} = 0, \quad \bar{a} \cdot \delta g|_{\partial M} = 0,
\]

leads to Einstein’s equations:

\[
R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi G T_{ab},
\]

\(T_{ab}\) being the energy-momentum tensor of the matter fields, namely,

\[
T_{ab} = \frac{1}{2\sqrt{-g}} \frac{\delta L_m}{\delta g^{ab}}.
\]

The path-integral formalism provides a way to deal ‘perturbatively’ with QFT in curved spacetime backgrounds\(^3\). First, through a rotation in the complex plane backgrounds\(^3\). If one now adheres to the principle that the Feynman propagator is obtained as the limit for \(\beta \to \infty\) of the thermal propagator, we have shown, some time ago\(^\text{[5]}\), that the usual principal-part prescription in the zeta-function regularization method (to be described below) is actually not needed any more, since it can in fact be replaced by this more general principle.

Next comes the stationary phase approach (also called one-loop, or WKB), for calculating the path integral, which consists in expanding around a fixed background:

\[
g = g_0 + \bar{g}, \quad \phi = \phi_0 + \bar{\phi},
\]

what leads to the following expansion in the Euclidean metric:

\[
\hat{I}[g, \phi] = \hat{I}[g_0, \phi_0] + I_2[\bar{g}, \bar{\phi}] + \cdots
\]

This is most suitably expressed in terms of determinants (for bosonic, resp. fermionic fields) of the kind (here \(A, B\) are the relevant (pseudo-)differential operators in the corresponding Lagrangian):

\[
\Delta_\phi = \text{det} \left( \frac{1}{2\pi \mu^2} A \right)^{-1}, \quad \Delta_\psi = \text{det} \left( \frac{1}{2\pi \mu^2} B \right).
\]
1.1. The zeta function of a ΨDO and its associated determinant

1.1.1. A pseudodifferential operator (ΨDO)

A pseudodifferential operator $A$ of order $m$ on a manifold $M$, is defined through its symbol $a(x, ξ)$, which is a function belonging to the space $S^m (\mathbb{R}^n \times \mathbb{R}^n)$ of $C^\infty$ functions such that for any pair of multi-indexes $α, β$ there exists a constant $C_{α, β}$ so that

$$|\partial_x^α \partial_ξ^β a(x, ξ)| \leq C_{α, β} (1 + |ξ|)^{m-|α|}. \quad (11)$$

The definition of $A$ is given, in the distribution sense, by

$$Af(x) = (2\pi)^{-n} \int e^{i \langle x, ξ \rangle} a(x, ξ) \hat{f}(ξ) \, dξ, \quad (12)$$

where $f$ is a smooth function, $f \in S$ [remember that $S = \{f \in C^\infty(\mathbb{R}^n); \sup_x |x^β \partial_x^α f(x)| < \infty, \forall α, β \in \mathbb{R}^n\}$], $S'$ being the space of tempered distributions and $\hat{f}$ the Fourier transform of $f$. When $a(x, ξ)$ is a polynomial in $ξ$ one gets a differential operator. In general, the order $m$ can be complex. The symbol of a ΨDO has the form

$$a(x, ξ) = a_m(x, ξ) + a_{m-1}(x, ξ) + \cdots + a_0(x, ξ), \quad (13)$$

being $a_k(x, ξ) = b_k(x) ξ^k$.

Pseudodifferential operators are useful tools [7, 8, 9], both in mathematics and in physics. They were crucial for the proof of the uniqueness of the Cauchy problem and also for the proof of the Atiyah-Singer index formula. In quantum field theory they appear in any analytical continuation process (as complex powers of differential operators, like the Laplacian). And they constitute nowadays the basic starting point of any rigorous formulation of quantum field theory through microlocalization, a concept that is considered to be the most important step towards the understanding of linear partial differential equations since the invention of distributions.

1.1.2. The zeta function

Let $A$ a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}$, acting on the space of smooth sections of $E$, an $n$-dimensional vector bundle over $M$, a closed $n$-dimensional manifold. The zeta function $ζ_A$ is defined as

$$ζ_A(s) = \text{tr} \ A^{-s} = \sum_j ζ_j^{-s}, \quad \text{Re} \ s > \frac{n}{m}, \quad (14)$$

where $s_0 = \text{dim} \ M/\text{ord} \ A$ is called the abscissa of convergence of $ζ_A$. Under these conditions, it can be proven that $ζ_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided that the principal symbol of $A$ (that is $a_m(x, ξ)$) admits a spectral cut: $L_θ = \{λ \in \mathbb{C}; \text{Arg} \ λ = θ, θ_1 < θ < θ_2\}$, Spec $A \cap L_θ = \emptyset$ (Agmon-Nirnb erg condition). The definition of $ζ_A(s)$ depends on the position of the cut $L_θ$. The only possible singularities of $ζ_A(s)$ are simple poles at $s_k = (n - k)/m$, $k = 0, 1, 2, \ldots, n - 1, n + 1, \ldots$.

M. Kontsevich and S. Vishik have managed to extend this definition to the case when $m \in \mathbb{C}$ (no spectral cut exists) [10].

1.1.3. The zeta determinant

Let $A$ a ΨDO operator with a spectral decomposition: $\{\varphi_i, λ_i\}_{i \in I}$, where $I$ is some set of indices. The definition of determinant starts by trying to make sense of the product $\prod_{i \in I} λ_i$, which can be easily transformed into a “sum”: $\ln \prod_{i \in I} λ_i = \sum_{i \in I} \ln λ_i$. From the definition of the zeta function of $A$: $ζ_A(s) = \sum_{i \in I} λ_i^{-s}$, by taking the derivative at $s = 0$: $ζ_A'(0) = -\sum_{i \in I} \ln λ_i$, we arrive to the following definition of determinant of $A$: $\text{det}_\mathbb{C} A = \exp [−ζ_A'(0)]$.

A big amount of explicit formulas, useful for the calculation of zeta functions and determinants of operators whose spectrum is known explicitly or implicitly (as roots of a spectral function that incorporates the equation and the boundary conditions, think e.g. of the roots of a Bessel function) are to be found, with all sort of explanations, in our references [11, 12, 7, 8, 9, 13]. Concerning the cases of known spectrum, the most general situations we have been able to consider can be summarized as follows. They are for zeta functions of the form [10]:

$$ζ_1(s) = \sum_{\vec{n} \in \mathbb{Z}^d} [Q(\vec{n}) + A(\vec{n})]^{-s}, \quad (15)$$

$$ζ_2(s) = \sum_{\vec{n} \in \mathbb{N}^d} A(\vec{n})^{-s}, \quad (16)$$

where $Q$ is a quadratic non-negative form and $A$ a general affine form (they give rise to Epstein and Barnes zeta functions, respectively). I have also obtained explicit results (given in terms of asymptotic series) for the much more involved cases where the summation indices are ‘interchanged’, namely:

$$ζ_3(s) = \sum_{\vec{n} \in \mathbb{N}^d} [Q(\vec{n}) + A(\vec{n})]^{-s}, \quad (17)$$

$$ζ_4(s) = \sum_{\vec{n} \in \mathbb{Z}^d} A(\vec{n})^{-s}. \quad (18)$$

2. Four-fermion models in curved spacetimes

Four-fermion models [11, 12] —usually considered in the $1/N$ expansion—are interesting due to the fact that they provide the opportunity to carry out an explicit, analytical study of composite bound states and dynamical chiral symmetry breaking. At the same time, these
Some uses of $\zeta$–regularization in quantum gravity and cosmology

3

theories —and specially their renormalizable 2d and 3d variants— exhibit specific properties which are similar to the basic behaviors of some realistic models of particle physics. Moreover, this class of theories can be used for the description of the standard model (SM) itself, or of some particle physics phenomena in the SM.

Having in mind the applications of four-fermion models to the early universe and, in particular, the chiral symmetry phase transitions that take place under the action of the external gravitational field, there has been some activity in the study of four-fermion models in curved spacetime. The effective potential of composite fermions in curved spacetime has been calculated in different dimensions and dynamical chiral symmetry breaking, fermionic mass generation and curvature-induced phase transitions have been investigated in full detail. However, in most of these cases only the linear curvature terms of the effective potential had been taken into account. But it turns out in practice that it is often necessary to consider precisely the strong curvature effects to dynamical symmetry breaking. In fact we could see that going beyond the linear-curvature approximation could lead to qualitatively different results.

With the help of the zeta-function regularization techniques, we investigated some 2d and 3d four-fermion models which were renormalizable—in the $1/N$ expansion— in a maximally symmetric constant-curvature space (either of positive or of negative curvature). The renormalized effective potential was found for any value of the curvature and the possibility of dynamical symmetry breaking in a curved spacetime was carefully explored. Furthermore, the phase structure of the theory was also described in detail.

One should go to the original papers for details [4]. In particular, we calculated the effective potential of composite fermions in the Gross-Neveu model, in the spaces $S^2$ and $H^2$. The phase diagram in $S^2$ was constructed and it was shown that for any value of the coupling constant there exists a curvature above which chiral symmetry is restored. For the case of $H^2$, we showed that chiral symmetry is always broken. The asymptotic expansions of the effective potential were given explicitly, both for small and for strong curvature. The three-dimensional case was then studied. We considered two different four-fermion models: one which exhibits a continuous U(2) symmetry and another where we concentrated ourselves on two discrete symmetries which happened never to be simultaneously broken. We studied explicitly the dynamical P and Z$_2$ symmetry breaking pattern in $H^3$ and $S^3$.

We started with the discussion of the Gross-Neveu model in the de Sitter space. This model, although rather simple in its conception, displays a quite rich structure, similar to that of realistic four-dimensional theories—as renormalizability, asymptotic freedom and dynamical chiral symmetry breaking. Some discussions of chiral symmetry restoration in the Gross-Neveu model for different external conditions (such as an electromagnetic field, non-zero temperature or a change of the fermionic number density) had appeared in the past (the influence of kink-antikink configurations on the phase transitions was described too).

The study of the Gross-Neveu model in an external gravitational field had been performed using the Schwinger method. Unfortunately, the generalization of the Schwinger procedure to curved spacetime is not free from ambiguities and this is why the previous results included some mistake, which we managed to correct, by using a rigorous mathematical treatment of the fermionic propagator in (constant curvature) spacetime. Our starting point was the action

$$S = \int d^2x \sqrt{-g} \left[ \bar{\psi} i \gamma^\mu \partial_\mu \psi + \frac{\lambda}{2N} (\bar{\psi} \psi)^2 \right],$$

(19)

with $N$ the number of fermions, $\lambda$ the coupling constant, $\gamma^\mu(x) = \gamma^a e^a_\mu(x)$, with $\gamma^a$ the ordinary Dirac matrix in flat space, and $\nabla_\mu$ the covariant derivative. By introducing the auxiliary field $\sigma$, this could be rewritten as

$$S = \int d^2x \sqrt{-g} \left[ \bar{\psi} i \gamma^\mu \partial_\mu \psi - \frac{N}{2\lambda} \sigma^2 - \sigma \bar{\psi} \psi \right],$$

(20)

with $\sigma = -\frac{1}{N} \bar{\psi} \psi$. Furthermore, we went to Euclidean space.

There is no place for further details. Using the powerful zeta-function method, the renormalized effective potential was found for any value of the curvature and its asymptotic expansion was given explicitly, both for small and for strong curvature. The influence of gravity on the dynamical symmetry breaking pattern of some U(2) flavor-like and discrete symmetries was described in detail. The phase diagram in $S^2$ was constructed and it was shown that, for any value of the coupling constant, a curvature existed above which chiral symmetry was restored. For the case of $H^2$, it was always broken. In three dimensions, in the case of positive curvature, $S^3$, it was seen that curvature could induce a second-order phase transition. For $H^3$ the configuration given by the auxiliary fields equated to zero was not a solution of the gap equation. The physical relevance of the results was discussed. In particular, we could see explicitly that the effect of a negative curvature is similar to that of the presence of a magnetic field.

For the two-dimensional Gross-Neveu model on $S^2$—where the chiral symmetry is a discrete one—we showed the possibility of chiral symmetry breaking and of fermion mass generation. Note that the curvature of a two-dimensional de Sitter space acts here as some external parameter (like temperature) which induces the chiral symmetry phase transition. In this sense, and owing to the fact that we treated curved spacetime exactly, the de Sitter space could not be considered to be some fluctuation over flat spacetime.
In the absence of matter our action for action know dilaton models, e.g. the bosonic string effective interact with the dilaton via 2D Dirac fermions (notice that we chose the matter to much more difficult case in which the action contained Natrix \( \psi \) It thus included a dilaton field, \( \Phi \), correspondingly. We obtained the covariant effective action was of the following form renormalizable model of 2D gravity with matter. Its action corresponding to a very general multiplicatively discussed already. We obtained the covariant effective action of dilaton, scalars and Majorana spinors was discussed too. The one-loop renormalization of quantum dilaton gravity with Majorana spinors was discussed too. The conditions of multiplicative renormalizability were specified and some examples of multiplicatively renormalizable dilaton potentials were explicitly obtained, as starting point to discuss 2D quantum dilaton-fermion cosmology. There is no place here to describe the usefulness of the zeta-function regularization procedure in all these developments and the reader is addressed to the original references, as quoted above (for a review of recent developments in this direction see [19], and for related result with possible cosmological application [20]).

3. Uses in dilaton gravity

Two-dimensional dilaton gravity interacting with a four-fermion model and scalars was investigated in a series of papers of our collaboration (see, e.g., [14, 17, 18] to mention only a few). The one-loop covariant effective action for 2D dilaton gravity with Majorana spinors (including the four-fermion interaction) was obtained, and the technical problems which appeared in any attempt at generalizing such calculations to the case of the most general four-fermion model described by Dirac fermions were properly discussed. A solution to these problems was found, based on its reduction to the Majorana spinor case. The general covariant effective action for 2D dilaton gravity with the four-fermion model described by Dirac spinors was given. The one-loop renormalization of dilaton gravity with Majorana spinors was carried out and the specific conditions for multiplicative renormalizability were found. A comparison with the same theory but with a classical gravitational field was also done.

Different approaches to the quantization of 2D dilaton gravities (mainly, string inspired models) had been discussed already. We obtained the covariant effective action corresponding to a very general multiplicatively renormalizable model of 2D gravity with matter. Its action was of the following form

\[
S = -\int d^2x \sqrt{g} \left[ \frac{1}{2} Z(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + C(\Phi) R \right. \\
\left. -\frac{i}{2} g(\Phi) \bar{\psi}_a \gamma^\lambda \partial_\lambda \psi_a + b(\Phi) \left( \bar{\psi}_a N_{ab} \psi_b \right)^2 \right. \\
\left. -\frac{1}{2} f(\Phi) g^{\mu\nu} \partial_\mu \chi_i \partial_\nu \chi_i + V(\Phi, \chi) \right].
\] (21)

It thus included a dilaton field, \( \Phi \), \( n \) Majorana fermions \( \psi_a \) interacting quartically via a symmetric constant matrix \( N_{ab} \), and \( m \) real scalars \( \chi_i \). We also considered the much more difficult case in which the action contained 2D Dirac fermions (notice that we chose the matter to interact with the dilaton via arbitrary functions).

This action described and generalized many well-known dilaton models, e.g. the bosonic string effective action

\[
Z(\Phi) = 8 e^{-2\Phi}, \quad C(\Phi) = e^{-2\Phi}, \quad V(\Phi) = 4 \lambda^2 e^{-2\Phi},
\]

\[
g(\Phi) = b(\Phi) = 0, \quad f(\Phi) = 1.
\] (22)

In the absence of matter our action for

\[
Z = 0, \quad C(\Phi) = \Phi, \quad V(\Phi) = \Lambda \Phi,
\] (23)

coincided with the Jackiw-Teitelboim action. One could also add gauge fields to the matter sector.

We constructed the covariant effective action of the theory (21), studied its one-loop renormalization and discussed some thereby connected issues. We described, in full detail, the calculation of the one-loop covariant effective action in 2D dilaton gravity with Majorana spinors, what was the first example of such a kind of calculation in two dimensions, as has been recognized latter by many authors in their mentions to our paper. The inclusion of scalars was also discussed there. We proceeded with the computation of the covariant effective action of dilaton, scalars and Majorana spinors for quantum systems in classical spacetime. We discussed the technical problems which appeared in the derivation of the covariant effective action in 2D dilaton gravity with Dirac fermions: the solution of these problems was found, via reduction of the system to the case of the theory of quantum dilaton gravity with Majorana spinors. The one-loop renormalization of quantum dilaton gravity with Majorana spinors was discussed too. The conditions of multiplicative renormalizability were specified and some examples of multiplicatively renormalizable dilaton potentials were explicitly obtained, as starting point to discuss 2D quantum dilaton-fermion cosmology.

4. Cosmological uses

4.1. The zero point energy

If \( H \) is now the Hamiltonian corresponding to a physical, quantum system, the zero point energy is given by

\[
<0|H|0>.
\] (24)

where \( |0> \) is the vacuum state. In general, after normal ordering we’ll have:

\[
H = \left( n + \frac{1}{2} \right) \lambda_n a_n a_n^\dagger,
\] (25)

and this yields for the vacuum energy:

\[
<0|H|0> = \frac{\hbar c}{2} \sum_n \lambda_n.
\] (26)

(I won’t normally keep track of the \( \hbar \)'s and \( c \)'s that will be set equal to 1.) The physical meaning of this energy was the object of a very long controversy, involving many first-rate physicists, until the late Hendrik Casimir gave the explanation (over fifty years ago) that is widely accepted nowadays, and that’s the reason
why the zero-point energy is usually associated with his name.

The expression above acquires a very important meaning as soon as one compares different settings, e.g., one where some sort of boundary conditions are imposed to the vacuum (e.g., a pair of parallel plates, infinitely conducting, in the vacuum corresponding to the electromagnetic field) with another situation where the boundary conditions (the plates) are absent (they have been sent to infinity). The difference yields a physically observable energy.

In general the sums appearing here are all divergent. They give rise to the most primitive, but physically meaningful, examples of zeta function regularization one can think of. In fact, according to the definitions above:

$$<0|H|0> = \frac{1}{2} \zeta_H(-1).$$  

(27)

It is important to notice that the zero-point energy is something one always has to keep in mind when considering any sort of quantum effect. Its contribution can be in some cases negligible, even by several orders of magnitude (as seems to be the case with sonoluminescence effects), but it can be of a few percent (as in some laser cavity effects), or even of some 10−30% as in the case of several wetting phenomena of alcali surfaces by Helium. Not to speak of the specifically devised experiments, where it may account for the full result.

In the case of the calculation of the value of the cosmological constant, it is immediate to see from the expressions considered before that:

$$<0|T_{\mu\nu}|0> = \frac{\Lambda}{8\pi G} + \frac{1}{2V} \sum_n \lambda_n,$$

(28)

where $V$ is the volume of the space manifold and the second term as a whole is the vacuum energy density corresponding to the quantum field (or fields) we are considering. Unless the first term (the cosmological constant), the vacuum energy density is not a constant (it goes as $a^{-4}$, $a$ being a typical cosmological length). However, this does not prevent the mixing of the two contributions when one considers, e.g., ‘the presently observed value of the cosmological constant’. What we have calculated is the second contribution for a scalar field of very low mass.

4.2. A simple model

Consider the space-time to be of one of the following types: $\mathbb{R} \times \mathbb{T}^p \times \mathbb{T}^q$, $\mathbb{R} \times \mathbb{T}^p \times \mathbb{S}^q$, . . . , which are actually plausible models for the space-time topology. A (nowadays) free scalar field pervading the universe will satisfy

$$(-\Box + \xi R + M^2)\phi = 0,$$

(29)

restricted by the appropriate boundary conditions (e.g., periodic, in the first case considered). We shall call $\rho_{\phi}$ the contribution to $\rho_{\nu}$ from this field.

$$\rho_{\phi} = \frac{1}{2V} \sum_i \frac{\lambda_i}{\mu} = \frac{1}{2V} \sum_k \frac{1}{\mu} (k^2 + M^2)^{1/2},$$

(30)

where the sums $\sum_i$ and $\sum_k$ are generalized ones (most common case: a multidimensional series together with a multidimensional integral) and $\mu$ is the usual mass-dimensional parameter to render the eigenvalues adimensional (we take $\hbar = c = 1$ and shall insert the dimensionfull units only at the end of the calculation). The mass $M$ of the field will be here considered to be arbitrarily small and will be kept different from zero. This is nice, both for computational reasons as well as for physical ones, since a very tiny mass for the field can never be excluded.

After going through some lengthy calculations that use the power of zeta-regularization, we reach the conclusion that coincidence with the observational value for the cosmological constant is obtained for the contribution of a massless scalar field, $\rho_{\phi}$, for $p$ large compactified dimensions and $q = p + 1$ small compactified dimensions, $p = 0, \ldots, 3$, and this for values of the small compactification length, $b$, of the order of 100 to 1000 times the Planck length $l_P$ (what is actually a very reasonable conclusion, according also to other approaches). To be noticed is the fact that full agreement is obtained only for cases where there is exactly one small compactified dimension in excess of the number of large compactified dimensions. $p$ and $q$ refer to the compactified dimensions only, but there may be other, non-compactified dimensions (exactly 3−$p$ in the case of the ‘large’ ones). In particular, the cases of pure spherical compactification and of mixed toroidal (for small magnitudes) and spherical (for big ones) compactification can be treated in this way and yield results in the same order of magnitude range. Both these cases correspond to (observational) isotropic spatial geometries. Also to be remarked again is the non-triviality of these calculations, when carried out exactly, what is apparent from the use of the generalized Chowla-Selberg formula. Simple power counting is absolutely unable to provide the correct order of magnitude of the results.

The most precise fits with the observational value of the cosmological constant are obtained for $b$ between $b = 100l_p$ and $b = 1000l_p$, with (1,2) and (2,3) compactified (large,small) dimensions, respectively. There is in fact no tuning of a ‘free parameter’ here. All them correspond to a marginally closed universe, in full agreement too with other completely independent analysis of the observational data [21, 22, 23, 24].

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