Non-Anticommutative Deformations of $N=(1,1)$ Supersymmetric Theories

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Abstract

We discuss chirality-preserving nilpotent deformations of four-dimensional $N=(1,1)$ Euclidean harmonic superspace and their implications in $N=(1,1)$ supersymmetric gauge and hypermultiplet theories, basically following [hep-th/0308012] and [hep-th/0405049]. For the SO(4)$\times$SU(2) invariant deformation, we present non-anticommutative Euclidean analogs of the $N=2$ gauge multiplet and hypermultiplet off-shell actions. As a new result, we consider a specific non-anticommutative hypermultiplet model with $N=(1,0)$ supersymmetry. It involves free scalar fields and interacting right-handed spinor fields.

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1 Introduction

In recent years, non-(anti)commutative deformations of supersymmetric field theories received a great deal of attention.

The simplest type of non-commutativity affects the space-time coordinates

\[ x^m \star x^n - x^n \star x^m = i\Theta^{mn} \]  

(1.1)

where \( \Theta^{mn} \) is some constant tensor specifying the deformation. Such non-commutative coordinates arise in the field-theory limit of string theory in a constant \( B \)-field background [1, 2]. For local fields \( f(x) \) and \( g(x) \), this non-commutativity implies the use of the Moyal-Weyl star-product which can be defined via the bi-differential operator \( P \) (Poisson structure)

\[ f \star g = fe^P g, \quad P = \frac{i}{2} \Theta^{mn} \underline{\partial}_m \overrightarrow{\partial}_n. \]  

(1.2)

Moyal-Weyl type deformations of supersymmetric theories in superspace are characterized by a generic Poisson bracket \( APB \) where \( A \) and \( B \) are some superfields and the Poisson operator \( P \) is in general some quadratic form in derivatives with respect to both the even and odd superspace coordinates [3, 4]. Symmetry properties of the operator \( P \) determine unbroken symmetries of the deformed superfield theory: these symmetries are those generators of which commute with \( P \).  

The specific deformed superfield field theories studied so far correspond to some particular degenerate choices of the generic superdifferential Poisson operator \( P \). E.g., the authors of [5] considered the deformations of some theories in harmonic \( N=2 \) superspace [6, 7] corresponding to the standard pure bosonic Poisson structure (1.2).

Deformations of a different kind are the nilpotent or non-anticommutative ones for which the operator \( P \) is bilinear in the proper derivations with respect to Grassmann coordinates. As such one can choose either generators of supersymmetry (Q-deformations), or spinor covariant derivatives (D-deformations). A surge of interest in superfield theories deformed in such a way was triggered by a recent paper [8] where a minimal deformation of the Euclidean \( N=(\frac{1}{2}, \frac{1}{2}) \) superspace was considered. For the chiral \( N=(\frac{1}{2}, \frac{1}{2}) \) coordinates \( (x_m^L, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \) the operator \( P \) defining the relevant star product is given by the simple bracket

\[ APB = -\frac{1}{2}(-1)^{p(A)} C^{\alpha\beta} \partial_\alpha A \partial_\beta B, \quad P = -\frac{1}{2} C^{\alpha\beta} \underline{\partial}_\alpha \overrightarrow{\partial}_\beta, \]  

(1.3)

with \( C^{\alpha\beta} \) being some constant symmetric matrix, \( \partial_\alpha = \partial/\partial \theta^\alpha \), and \( p(A) \) the \( \mathbb{Z}_2 \)-grading. The operator \( P \) defined by (1.3) acts on the \( \theta^\alpha \) coordinates only and retains the \( N=(\frac{1}{2}, 0) \) fraction of the original \( N=(\frac{1}{2}, \frac{1}{2}) \) supersymmetry. It is very important that the corresponding noncommutative product of superfields preserves the chiral and antichiral representations of the \( N = (\frac{1}{2}, \frac{1}{2}) \) supersymmetry. Like the bosonic deformation (1.1), (1.2),

\[ ^1 \text{In general, this criterion should be applied in a weak sense, i.e. for the commutator sandwiched between the superfields } A \text{ and } B. \]
this purely fermionic deformation also originates from string theory, as discussed in [8] and [9]-[12].

Deformations of the $N=2$ superfield theories along similar lines were discussed in [13]. In this contribution we shall focus on the harmonic-superspace formalism of the nilpotently deformed Euclidean $N=(1,1)$ theories, basically following Refs. [14, 15, 16] (see also [17, 18]).

The Grassmann harmonic analyticity is the key notion of the off-shell superfield description of $N=2$ supersymmetric field theories in four dimensions [6, 7] where it plays the role analogous to chirality in $N=1$ superfield theories. In particular, the analytic gauge and hypermultiplet superfields are the building-blocks of off-shell interactions, and the harmonic analytic superspace formalism is indispensable for quantum supergraph calculations. By construction, the nilpotent Q-deformations (and some special D-deformations) of $N=(1,1)$ Euclidean superspace preserve this harmonic G-analyticity [14, 15]. Yet, the chirality also plays the important role in $N=2$ and $N=(1,1)$ supersymmetric gauge theories, so the deformations which we shall consider preserve as well both chiralities.

In Section 2 we review the nilpotent Q-deformations of the Euclidean chiral $N=(1,1)$ superspace and analyze the role of the standard conjugation or an alternative pseudoconjugation in Euclidean $N=(1,1)$ supersymmetric theories. The corresponding bi-differential operator $P$ preserves chirality and anti-chirality, and half of the original $N=(1,1)$ supersymmetry ($N=(1,0)$ supersymmetry). For special choices, however, $N=(1,\frac{1}{2})$ supersymmetry or the whole automorphism group $SO(4)\times SU(2)$ can be retained.

Section 3 is devoted to the chirality-preserving $SO(4)\times SU(2)$ invariant deformation of the gauge $N=(1,1)$ theories in the harmonic superspace. This singlet deformation breaks half of supersymmetries and gives rise to some additional interactions of the scalar field $\bar{\phi}$ of the $N = (1,1)$ gauge multiplet with the remaining components of the latter [16].

Non-anticommutative interactions of the Grassmann-analytic hypermultiplets are considered in Section 4. Formally these interactions resemble those considered in the bose-deformed harmonic superspace of [5], however, the component contents of these two theories are entirely different. As a new explicit example, we analyze in some detail the simplest hypermultiplet self-interaction which vanishes in the anticommutative-superspace limit. In the component action of this model, the scalar fields do not interact with fermions, and only some specific fermionic self-interaction is present, with two derivatives on fermions. The solvable equation for the right-handed fermions contains the nonlinear source constructed from the left-handed ones which are free.

2The singlet Q-deformation of U(1) gauge theory was independently considered in [18].
2 Deformations of N=(1,1) Euclidean chiral superspace

The Euclidean N=(1,1) superspace has as its automorphisms the Euclidean space spinor group Spin(4) ~ SU(2)_L × SU(2)_R and the R-symmetry group SU(2)×O(1,1) properly acting on the coordinates x^m, θ^α_k, ̄θ^β_k. We prefer to use the chiral coordinates z_L ≡ (x^m_L, θ^α_k, ̄θ^β_k) to parametrize this superspace. These Euclidean coordinates z_L are real with respect to the standard conjugation
\[ \tilde{\theta}_k^α = \varepsilon^{kj} \varepsilon_{αβ} \theta_j^β, \quad \tilde{\bar{\theta}}^k_δ = -\varepsilon_{kj} \varepsilon_{δα} \bar{\theta}_j^α, \quad \tilde{A} \tilde{B} = \tilde{B} \tilde{A}. \]

This conjugation squares to identity on any object, and with respect to it the N=(1,1) superspace has the real dimension (4|8). However, if we wish to treat the N=(1,1) superspace as a real subspace of the N=(1,1) superspace (like N = 1 supersubspace in the standard Minkowski N = 2,4D superspace), e.g. in order to be able to make reductions to the theories considered in [8], we cannot limit ourselves merely to this standard conjugation. Indeed, the Euclidean N=(1,2,1) superspace cannot be real with respect to the complex conjugation: two independent SU(2) spinor coordinates have the real dimension 8 which coincides with the Grassmann dimension of the whole N=(1,1) superspace.

The alternative SU(2)-breaking pseudoconjugation in the same Euclidean N=(1,1) superspace was considered in [14]:
\[ (θ_k^α)^* = ε_αβ θ_k^β, \quad (\tilde{θ}_δ^k)^* = \varepsilon_{δα} \tilde{θ}_k^α, \quad (x_L^m)^* = x_L^m, \quad (AB)^* = B^* A^*. \]

The existence of this pseudoconjugation does not impose any further restriction on the N=(1,1) superspace which has the same dimension (4|8) as with respect to the complex conjugation. Clearly, with respect to this pseudoconjugation, θ^α_k and ̄θ^β_k are ‘real’, so they form an N=(1,2,1) subspace of the ‘real’ dimension (4|4) in N=(1,1) superspace (such subspaces can be singled out in a few different ways). The standard conjugation (2.1) and the pseudoconjugation (2.2) act differently on the objects transforming by non-trivial representations of the R-symmetry SU(2). The map * squares to −1 on the Grassmann coordinates and the associated spinor fields, and to +1 on any bosonic monomial or field. On the singlets of SU(2), both maps act as the standard complex conjugation. In particular, the invariant actions are real with respect to both * and ~, despite the fact that the component fields may have different properties under these (pseudo)conjugations.

After this digression, let us come back to our main subject, Q-deformations of N=(1,1) theories. In chiral coordinates, the simplest Poisson structure operator is
\[ P = -\frac{1}{2} C_{ik}^{αβ} \overleftarrow{Q}_α^i \overleftarrow{Q}_β^k = -\frac{1}{2} C_{ik}^{αβ} \overleftarrow{\partial}_α^i \overleftarrow{\partial}_β^k \]

Some ambiguities of generalized conjugations in Grassmann algebras (C-antilinear maps with squares equal to ±1) were discussed in [20].
and the Poisson bracket for two superfields $A$ and $B$ is defined as

$$APB = -\frac{1}{2}(-1)^{p(A)}(\partial^k_\alpha A)C^\alpha_\beta (\partial^j_\beta B) = -(-1)^{p(A)p(B)}BPA . \tag{2.4}$$

Here, $C^\alpha_\beta = C^\beta_\alpha$ are some constants, $p(A)$ is the $\mathbb{Z}_2$-grading, and the partial spinor derivatives act as

$$\partial^k_\alpha \theta^\beta = \delta^k_\alpha \delta^\beta \quad \text{and} \quad \partial^k_\alpha \tilde{\theta}^\beta = \delta^k_\alpha \delta^\beta . \tag{2.5}$$

By definition, the bracket (2.4) preserves both chirality and anti-chirality and does not touch $SU(2)_R$ acting on dotted indices. Generically, it breaks half of the original $N=(1,1)$ supersymmetry since the generators $\tilde{Q}_{\alpha k}$ do not commute with the operator $P$. We demand $P$ to be real, i.e. invariant under some antilinear map in the algebra of superfields. The two possible (pseudo)conjugations lead to different conditions on the constants $C^\alpha_\beta$. The constant deformation matrix can be split into two irreducible parts,

$$C^\alpha_\beta = C^\alpha_\beta_{(kj)} + 2\varepsilon^\alpha_\beta \varepsilon_{kj}I , \tag{2.6}$$

where $I$ is a real parameter. The second, singlet part preserves the full $SO(4)\times SU(2)$ symmetry:

$$P_s = -I \hat{Q}^k_\alpha \hat{Q}^s_\alpha , \quad AP_s B = -I(-1)^{p(A)}Q^k_\alpha AQ^s_\alpha B . \tag{2.7}$$

Given the operator (2.4), the Moyal product of two superfields reads

$$A \star B = A e^PB = AB + APB + \frac{1}{2}AP^2B + \frac{1}{6}AP^3B + \frac{1}{24}AP^4B \tag{2.8}$$

where the identity $P^5 = 0$ was used. This star product preserves both chirality and antichirality and breaks $N=(0,1)$ supersymmetry. In the approach with the star product only free actions preserve all supersymmetries while interactions get deformed and they are not invariant under the $N=(0,1)$ supersymmetry transformations.

The (3,3) part $C^\alpha_\beta_{(kl)}$ of the deformation matrix breaks the R-symmetry $SU(2)$, so we should choose one of the alternative reality conditions to define the minimal form of the matrix $C^\alpha_\beta_{(kl)}$. The minimal representation of this (3,3) part has the following form:

$$C^\alpha_\beta_{(12)} = C^\alpha_\beta , \quad C^\alpha_\beta_{(11)} = C^\alpha_\beta_{(22)} = 0 ,$$

$$AP_{C}B = -\frac{1}{2}(-1)^{p(A)}C^\alpha_\beta (Q^1_\alpha AQ^2_\beta B + Q^2_\alpha AQ^1_\beta B) , \tag{2.9}$$

if we assume that $C^\alpha_\beta$ is real with respect to the $\sim$ conjugation, $\sim C^\alpha_\beta = C_{\alpha\beta}^{(ik)}$.

The choice of the * pseudoconjugation (2.2) is compatible with the decomposition of $N=(1,1)$ into two $N=(\frac{3}{2}, \frac{3}{2})$ superalgebras. Therefore, it allows one to choose a degenerate deformation

$$P(Q^2) = -\frac{1}{2}C(\tilde{Q}^2_1 \tilde{Q}^2_2 + \tilde{Q}^2_2 \tilde{Q}^2_1) . \tag{2.10}$$

which does not involve $Q^1_\alpha$ and contains the real parameter $C$. In this case, only $\tilde{Q}_{\alpha 2}$ are broken, but not the supercharges $\tilde{Q}_{\alpha 1}$. Hence, the deformation $P(Q^2)$ preserves the larger fraction $N=(1, \frac{1}{2})$ of the original $N=(1,1)$ supersymmetry.
It is of course possible to consider more general deformations affecting both the chiral and anti-chiral sectors. E.g. one can take the anticommuting set of pseudoreal generators $Q^{2}_{\alpha}, \bar{Q}_{\dot{\alpha}1}$ and construct the real deformation operator $\hat{P}$ and the corresponding bracket for even superfields $A$ and $B$ as

$$A\hat{P}B = -C^{\alpha\beta}Q^{2}_{\alpha}AQ^{2}_{\beta}B - B^{\alpha\dot{\alpha}}(Q^{2}_{\alpha}A\bar{Q}_{\dot{\alpha}1}B + \bar{Q}_{\dot{\alpha}1}AQ^{2}_{\alpha}B) - \bar{C}^{\dot{\alpha}\dot{\beta}}\bar{Q}_{\dot{\alpha}1}A\bar{Q}_{\dot{\beta}1}B.$$ (2.11)

It is evident that this deformation operator defines an associative star-product and it commutes with all spinor derivatives $D^{k}_{\alpha}, \bar{D}^{\dot{k}}_{\dot{\alpha}}$, as well as with 4 generators of supersymmetry $Q^{2}_{\alpha}, \bar{Q}_{\dot{\alpha}1}$. Hence it breaks half of supersymmetry and preserves both chiralities.

3 Chirality-preserving singlet deformations of $N=\text{(1,1)}$ harmonic superspace

Harmonic superspace with noncommutative bosonic coordinates $x^{m}_{A}$ has been discussed in [5]. This deformation yields nonlocal theories but preserves the whole $N=2$ supersymmetry. The nilpotent D-deformations of Euclidean $N=(1,1)$ superspace also preserving the full amount of supersymmetry were considered in [13]. Within the harmonic superspace formalism, a special case of such deformations, the singlet one preserving the SO(4)×SU(2) symmetry, one of two chiralities and harmonic analyticity, was addressed in [14, 15]. In particular, in [15] $N=(1,1)$ gauge theory with such D-deformation was studied (see also a recent preprint [22]). Further in this contribution we shall not discuss this type of nilpotent deformations. Instead, we shall concentrate on the supersymmetry-breaking singlet nilpotent Q-deformation associated with the operator $P_{s}$ (2.7). We shall essentially use the Euclidean version of the harmonic superspace approach, following refs. [14, 16].

The basic concepts of the harmonic superspace approach in its Euclidean variant coincide, up to a few minor distinctions, with those of the standard (Minkowski) $N=2, D=4$ harmonic superspace as collected in the book [7]. In both versions, the key ingredient is the SU(2)/U(1) harmonics $u_{i}^{\pm}$, $u^{+}u^{-} = 1$, where SU(2) is the R-symmetry group. The chiral-analytic coordinates $Z_{C} = (x^{m}_{L}, \theta^{+\alpha}, \tilde{\theta}^{+\dot{\alpha}}, u_{i}^{\pm})$ in the $N=(1,1)$ harmonic superspace are related to the analytic coordinates via the shift of the bosonic coordinate

$$x^{m}_{A} = x^{m}_{L} - 2i(\sigma^{m})^{\alpha\dot{\alpha}}\theta^{+\alpha}\tilde{\theta}^{+\dot{\alpha}}, \quad \theta^{\pm\alpha} = \theta^{\alpha\dot{\alpha}}u_{i}^{\pm}, \quad \tilde{\theta}^{\pm\dot{\alpha}} = \tilde{\theta}^{\alpha\dot{\alpha}}u_{i}^{\pm}.$$ (3.1)

The (pseudo)conjugations (2.1) and (2.2) can be extended to the harmonics and the coordinates of the harmonic superspace [14]. These two (pseudo)conjugations act identically on invariants and harmonic superfields, e.g. $(A^{k}B_{k})^{*} = (\tilde{A}^{k}\tilde{B}_{k})$ or $(q^{+})^{*} = \tilde{q}^{+}$, but they differ when acting on harmonics or R-spinor component fields, e.g. $(A_{k})^{*} \neq \tilde{A}_{k}$. An important invariant pseudoreal subspace is the analytic Euclidean harmonic superspace, parametrized by the coordinates

$$(x^{m}_{A}, \theta^{+\alpha}, \tilde{\theta}^{+\dot{\alpha}}, u_{k}^{\pm}) \equiv (\zeta, u).$$ (3.2)
The supersymmetry-preserving spinor and harmonic derivatives in different coordinate bases are defined in \[14, 15, 16\]. A Grassmann-analytic \((G\)-analytic) superfield \(\Phi = \Phi(\zeta, u)\) is defined by the constraints

\[
\begin{align*}
D_+^\alpha \Phi(\zeta, \theta^-, \bar{\theta}^-, u) &= D_+^\alpha \Phi(\zeta, \theta^-, \bar{\theta}^-, u) = 0. 
\end{align*}
\]

(3.3)

It is important that the chirality-preserving operator \(P\) (2.4) also preserves Grassmann analyticity:

\[
\begin{align*}
\{P, (D_+^\alpha, \bar{D}_+^\dot{\alpha})\} &= 0.
\end{align*}
\]

(3.4)

In what follows it will be convenient to deal with harmonic projections of the \(N=(1, 1)\) supersymmetry generators

\[
\begin{align*}
Q^k_\alpha = u^+ k Q^-_\alpha - u^- k Q^+_\alpha, \quad Q_{\dot{k} \dot{\alpha}} = u^+_k \bar{Q}^-_{\dot{k} \dot{\alpha}} - u^-_k \bar{Q}^+_{\dot{k} \dot{\alpha}}.
\end{align*}
\]

(3.5)

For instance, in the chiral-analytic coordinates we have

\[
\begin{align*}
Q^+_{\alpha} &= \partial^-_{\alpha}, \quad Q^-_{\alpha} = -\partial^+_{\alpha}
\end{align*}
\]

(3.6)

where \(\partial_{\pm\alpha} = \partial/\partial \theta^{\pm\alpha}\). In these coordinates, different terms in the product (2.8) with the singlet \(Q\)-deformation operator \(P_s\) are explicitly expressed as

\[
\begin{align*}
AP_s B &= I(-1)^{p(A)}(\partial^-_{\alpha} A \partial_+^\alpha B + \partial_+^\alpha A \partial^-_{\alpha} B), \\
\frac{1}{2} AP_s^2 B &= -\frac{I^2}{4}(\partial_+^\alpha)^2 A(\partial^-_{\alpha})^2 B - \frac{I^2}{4}(\partial^-_{\alpha})^2 A(\partial_+^\alpha)^2 B + I^2 \partial^+_{\alpha\beta} \partial_+^\beta A \partial_+^\alpha \partial^-_{\alpha} B, \\
\frac{1}{6} AP_s^3 B &= \frac{I^3}{4}(-1)^{p(A)} \partial^-_{\alpha}(\partial_+^\alpha)^2 A \partial^+_{\alpha}(\partial^-_{\alpha})^2 B + \frac{I^3}{4}(-1)^{p(A)} \partial^+_{\alpha}(\partial^-_{\alpha})^2 A \partial^-_{\alpha}(\partial_+^\alpha)^2 B, \\
\frac{1}{24} AP_s^4 B &= \frac{I^4}{16}(\partial_+^\alpha)^2 (\partial^-_{\alpha})^2 A(\partial_+^\alpha)^2 (\partial^-_{\alpha})^2 B.
\end{align*}
\]

(3.7)

Note that the last two terms vanish for the analytic superfields.

Now we turn to some details of the deformed \(N=(1, 1)\) gauge theory in harmonic superspace. It largely mimics the harmonic superspace formulation of non-abelian \(N = 2\) gauge theory in 4D Minkowski space [7].

The basic superfield of the \(N = (1, 1)\) gauge theory is the analytic anti-Hermitian potential \(V^{++}\) with the values in the algebra of the gauge group which we choose to be \(U(n)\). The gauge transformation of the \(U(n)\) gauge potential \(V^{++}\) reads

\[
\delta_{\Lambda} V^{++} = D^{++} \Lambda + [V^{++}, \Lambda],
\]

(3.8)

where \(\Lambda\) is an anti-Hermitian analytic gauge parameter and \(D^{++}\), in the chiral-analytic basis, is

\[
D^{++} = \partial^{++} + \theta^+\alpha \partial^-_{\alpha} + \bar{\theta}^+\dot{\alpha} \partial^-_{\dot{\alpha}}, \quad \partial^{++} = u^+ \frac{\partial}{\partial u^{-i}}.
\]

(3.9)
In the Wess-Zumino (WZ) gauge we shall use the expansion of the potential in $\bar{\phi}^{+\dot{\alpha}}$

$$V^{++}_{WZ} = \bar{\phi}^{++} + \bar{\phi}^{+\dot{\alpha}}V^{+\dot{\alpha}} + (\bar{\phi}^+)^2V,$$

$$\bar{\phi}^{++}(x_\Lambda, \theta^+, u) = (\theta^+)^2\bar{\phi}, \quad V^{+\dot{\alpha}}(x_\Lambda, \theta^+, u) = 2\theta^+ A^{\dot{\alpha}}_\Lambda + 4(\theta^+)^2\Psi^{-\dot{\alpha}},$$

$$V(x_\Lambda, \theta^+, u) = \phi + 4\theta^+ A^{\dot{\alpha}}_\Lambda + 3(\theta^+)^2D^{-\dot{\alpha}}$$

(3.10)

where $\Psi^{-\dot{\alpha}} = u_k\Psi^k_\alpha, \bar{\Psi}^{-\dot{\alpha}} = u_k\bar{\Psi}^k_\dot{\alpha}, D^{-\dot{\alpha}} = u_k\partial^k D^{\dot{\alpha}}$ and all component fields are functions of $x^m_\Lambda$.

For what follows it will be convenient to rewrite the expression for the WZ-potential in the chiral-analytic basis, using the relation (3.2)

$$V^{++}_{WZ}(Z_C, u) = v^{++}(z_C, u) + \bar{\phi}^{++}(z_C, u) + (\bar{\phi}^+)^2v(z_C, u)$$

(3.11)

where the chiral superfunctions depend on the coordinates $x^m_L, \theta^{+\alpha}, \theta^{--\alpha}$ and $u^\pm$ only

$$v^{++}(z_C, u) = (\theta^+)2\bar{\phi}(x_L), \quad v^{+\dot{\alpha}}(z_C, u) = V^{+\dot{\alpha}}(x_L, \theta^+, u) - 2i\theta^{-\alpha}\partial^{\dot{\alpha}}\bar{\phi}^{++}(x_L, \theta^+, u)$$

$$= -2\theta^+ A^{\dot{\alpha}} + 4(\theta^+)^2u_k\bar{\Psi}^{ak} + 2i\theta^{-\alpha}(\theta^+)^2\partial^{\alpha\dot{\alpha}}\bar{\phi},$$

$$v(z_C, u) = V(x_L, \theta^+, u) + i\theta^{-\alpha}\partial^{\dot{\alpha}}V^{+\dot{\alpha}}(x_L, \theta^+, u) - (\theta^-)^2\Box\bar{\phi}^{++}(x_L, \theta^+, u)$$

$$= \phi + 4\theta^+ A^{\dot{\alpha}} + 3(\theta^+)^2D^{-\dot{\alpha}} - 2i(\theta^+\theta^-)\partial_m A_m + \theta^+ \sigma_{mn} \theta^- F_{mn}$$

$$+ 4i\theta^{-\alpha}(\theta^+)^2\partial_{\alpha\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}} - (\theta^-)^2(\theta^+)2\Box\bar{\phi}. \quad (3.12)$$

Here all component fields (after separating the harmonic dependence) are functions of $x^m_L$.

Now we specialize to the simplest case of the U(1) gauge group. The corresponding $P_\alpha$-deformed gauge and $N=(1,0)$ supersymmetry transformations of the component fields can be readily found [16]. They are given, respectively, by

$$\delta_a \phi = -8IA_m \partial_m a, \quad \delta_a \bar{\phi} = 0, \quad \delta_a A_m = (1 + 4I\bar{\phi})\partial_m a,$$

$$\delta_a \Psi^k_\alpha = -4I\bar{\Psi}^{ak} \partial_{a\dot{\alpha}} a, \quad \delta_a \bar{\Psi}^\dot{\alpha}_k = 0, \quad \delta_a D^{kl} = 0 \quad (3.13)$$

and

$$\delta_c \phi = 2\epsilon^{ak} \Psi^k_\alpha, \quad \delta_c \bar{\phi} = 0, \quad \delta_c A_m = \epsilon^{ak}(\sigma_m)_{a\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}}_k,$$

$$\delta_c \bar{\Psi}^\dot{\alpha}_k = -\epsilon_{a\dot{\alpha}} D^{kl} + \frac{1}{2}(1 + 4I\bar{\phi})(\sigma_{mn}\epsilon^k_\alpha)F_{mn} - 4i\epsilon^k_\alpha A_m \partial_m \bar{\phi},$$

$$\delta_c D^{kl} = i\partial_m [(\epsilon^k \sigma_m \bar{\Psi}^{l} + \epsilon^l \sigma_m \bar{\Psi}^{k})(1 + 4I\bar{\phi})] \quad (3.14)$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$.

The nonpolynomial superfield action of the Q-deformed gauge theory has been given in [14] as an integral over the full superspace in the chiral coordinates, by analogy with the undeformed $N=2$ superfield action [21]. It was shown in [16] that the $P_\alpha$-deformed U(1) gauge action can be conveniently rewritten as the integral over the chiral superspace

$$S^{(I)} = \frac{1}{4} \int d^4x_L d^4\theta A^2 \quad (3.15)$$
where $\mathcal{A}(x_L, \theta^+, \theta^-, u)$ is the deformed chiral superfield strength. The latter appears as the lowest component in the $\tilde{\theta}^{\pm \alpha}$ expansion of the covariantly chiral superfield strength $\mathcal{W}$:

$$\mathcal{W} \equiv -\frac{1}{4}(\tilde{D}^+)^2 V^{--} = \mathcal{A} + \tilde{\theta}^{+ \dot{\alpha}} \tau^{--} + (\tilde{\theta}^+)^2 \tau^{-2}$$  \hspace{1cm} (3.16)

and the action (3.15) can be rewritten as

$$S^{(I)} = \frac{1}{4} \int d^4x_L d^4\theta \mathcal{W}^2.$$  \hspace{1cm} (3.17)

It can be shown that the remaining two components in (3.16) do not contribute to (3.17).

The composite harmonic connection $V^{--}$ is connected with the basic potential $V^{++}$ via the deformed harmonic zero curvature equation [14]

$$D^{++}V^{--} - D^{--}V^{++} + [V^{++}, V^{--}] = 0$$  \hspace{1cm} (3.18)

where, in the chiral-analytic basis,

$$D^{--} = \partial^{--} + \theta^{-\alpha} \partial_{+\alpha} + \tilde{\theta}^{-\dot{\alpha}} \partial_{+\dot{\alpha}}, \quad \partial^{--} = u^{-i} \frac{\partial}{\partial u^{+i}}.$$

As a consequence of (3.18), the chiral superfield $\mathcal{A}$ satisfies the homogeneous harmonic equation

$$[\partial^{++} + (1 + 4I\bar{\phi})\theta^{+\alpha} \partial_{-\alpha}] \mathcal{A} = 0$$  \hspace{1cm} (3.19)

and some additional nonlinear inhomogeneous equation [16]:

$$[\partial^{++} + (1 + 4I\bar{\phi})\theta^{+\alpha} \partial_{-\alpha}] \varphi^{--} + 2(\mathcal{A} - v) - I (\varphi^+ \varphi^- - \varphi^- \varphi^-) + \frac{I^3}{4} \partial_{-\alpha}^2 \varphi^+ \partial_{+\alpha} (\varphi^-)^2 = 0$$  \hspace{1cm} (3.20)

where $\varphi^{--}$ and $\varphi^{--}$ are the proper chiral coefficients of the expansion of $V^{--}$ in $\tilde{\theta}^{\pm \alpha}$. They can be calculated in terms of the component fields.

The undeformed chiral U(1) superfield strength has the following component field content

$$W_0(x_L, \theta^+, \theta^-, u) = \varphi + 2\theta^+ \psi^- - 2\theta^- \psi^+ + (\theta^+)^2 d^-$$

$$-2(\theta^+ \theta^-) d^{+-} + (\theta^-)^2 d^{++} + (\theta^- \sigma_{mn} \theta^+) f_{mn} + 2i[(\theta^-)^2 \sigma_m \partial_m \psi^+] + (\theta^+)^2 \varphi^- + (\theta^-)^2 \varphi^+ - (\theta^+)^2 (\theta^-)^2 \square \bar{\phi}$$  \hspace{1cm} (3.21)

where $f_{mn} = \partial_m a_n - a_m \partial_n$, $\psi^\pm = \psi^i(x_L)u^\pm_i$, $d^{+-} = u^+_k u^-_l d^{kl}(x_L)$, etc. This superfield obeys the free harmonic equation $D^{++}W_0 = 0$ and transforms under $N = (1,0)$ supersymmetry as

$$\delta_w W_0 = (\epsilon^{-\alpha} \partial_{-\alpha} + \epsilon^{+\alpha} \partial_{+\alpha}) W_0.$$  \hspace{1cm} (3.22)

It is rather straightforward to show that $\mathcal{A}$ can be constructed as a nonlinear transformation of the undeformed U(1) superfield strength $W_0$

$$\mathcal{A}(x_L, \theta^+, \theta^-, u) = (1 + 4I\bar{\phi})^2 W_0(x_L, \theta^+, (1 + 4I\bar{\phi})^{-1} \theta^-, u).$$  \hspace{1cm} (3.23)
The nonlinear relations between the undeformed and deformed U(1) component fields following from (3.23) are

\[
\begin{align*}
\varphi &= (1 + 4I\tilde{\phi})^{-2}[\phi + 4I(1 + 4I\tilde{\phi})^{-1}(A_m^2 + 4I^2(\partial_m\tilde{\phi})^2)], \\
a_m &= (1 + 4I\tilde{\phi})^{-1}A_m, \quad \bar{\psi}_a^k = (1 + 4I\tilde{\phi})^{-1}\bar{\Psi}_a^k, \\
\psi_a^k &= (1 + 4I\tilde{\phi})^{-2}[\Psi_a^k + 4I(1 + 4I\tilde{\phi})^{-1}A_{a\alpha}\bar{\Psi}_{\alpha k}], \\
d^{kl} &= (1 + 4I\tilde{\phi})^{-2}[\mathcal{D}^{kl} + 8I\bar{\Psi}_{a\alpha}\bar{\Psi}_{a\alpha}].
\end{align*}
\]  

(3.24)

The \(N=(1, 0)\) supersymmetry transformation of the deformed chiral superfield is given by

\[
\delta_\alpha A = [(1 + 4I\tilde{\phi})\epsilon^{-\alpha}\partial_{-\alpha} + \epsilon^{+\alpha}\partial_{+\alpha}]A.
\]

(3.25)

The deformed U(1) gauge superfield action can be expressed in terms of the abelian undeformed objects up to a total spinor derivative in the integrand

\[
S(\theta) = \frac{1}{4} \int d^4x_L d^4\theta A^2 = \frac{1}{4} \int d^4x_L d^4\theta (1 + 4I\tilde{\phi})^2W^2_0.
\]  

(3.26)

Using the redefinitions of the deformed fields (3.24), one can obtain the component Lagrangian of the deformed U(1) gauge theory as \(L^{(i)} = (1 + 4I\tilde{\phi})^2L_0\) where \(L_0\) is the free undeformed Lagrangian

\[
L_0 = -\frac{1}{2} \varphi \Box \tilde{\phi} + \frac{1}{4}(f_{mn}^2 + \frac{1}{2}\varepsilon_{mnrs}f_{mn}f_{rs}) - i\bar{\psi}_a^k\partial_{a\alpha}\bar{\Psi}_{\alpha k} + \frac{1}{4}(d^{kl})^2.
\]  

(3.27)

It is obvious that the scalar, fermionic and auxiliary terms in the action can be given the form of the free kinetic terms by properly rescaling the fields \(\varphi, \psi_a^k\) and \(d^{kl}\). However, the nonlinear interaction of the fields \(\tilde{\phi}\) and \(f_{mn}\),

\[
\frac{1}{4}(1 + 4I\tilde{\phi})^2(f_{mn}^2 + \frac{1}{2}\varepsilon_{mnrs}f_{mn}f_{rs}),
\]

cannot be removed by any field redefinition.

Now let us shortly discuss how the above generalizes to the nonabelian U(\(n\)) case (\(n \geq 2\)). We use the WZ-gauge for the U(\(n\)) potential (3.10), and the corresponding deformed component gauge transformations are

\[
\begin{align*}
\delta_\alpha \phi &= -i[a, \tilde{\phi}], \quad \delta_\alpha \bar{\Psi}_a^k = -i[a, \bar{\Psi}_a^k], \quad \delta_\alpha \mathcal{D}^{kl} = -i[a, \mathcal{D}^{kl}], \\
\delta_\alpha A_m &= \partial_m a + i[A_m, a] + 2I\{\tilde{\phi}, \partial_m a\}, \\
\delta_\alpha \phi &= -i[a, \phi] - 4I\{\tilde{\phi}, \partial_m a\} - 4I^2[\tilde{\phi}, a], \\
\delta_\alpha \Psi_a^k &= -i[a, \Psi_a^k] - 2I\sigma_{a\alpha} \{\bar{\Psi}_{\alpha k}, \partial_m a\}.
\end{align*}
\]

(3.29)

The \(P_s\)-deformed U(\(n\)) chiral gauge superfield \(A\) satisfies the following equation:

\[
D^{++}A + I\theta^\alpha \{\phi, \partial_{-\alpha} A\} + (\theta^+)^2 [\tilde{\phi}, A] + I^2[\tilde{\phi}, (\partial_-)^2 A] = 0
\]

(3.30)
where $\bar{\phi}$ is the Hermitian matrix scalar field. It is convenient to define the following matrix operator:

$$L = 1 + 2I\{\bar{\phi}, \}_{\alpha},$$

then the first two terms in eq.(3.30) can be rewritten as $(\partial^{+} + L\theta^{+}\partial_{-})A$. The undeformed harmonic chiral $U(n)$ superfield $A$ has the following component expansion

$$A = \varphi + 2\theta^{+}\psi^{+} - 2\theta^{-}\psi^{+} + (\theta^{+})^{2}d^{-} + (\theta^{+}\theta^{-})\{[\varphi, \bar{\phi}] - 2d^{+}\} - (\theta^{+})^{2}d^{+}$$

$$+ (\theta^{+}\sigma_{mn}\theta^{-})f_{mn} + 2(\theta^{+})^{2}\theta^{-}\{i\xi^{+} - [\bar{\phi}, \psi^{+}]\} + 2i(\theta^{+})^{2}\theta^{-}\xi^{-}$$

$$- (\theta^{+})^{2}(\theta^{-})^{2}(p + [\bar{\phi}, d^{+}])$$

(3.32)

where all the component fields are $n \times n$ matrices and the following short-hand notation is used:

$$\nabla_{m} = \partial_{m} + i[a_{m}, \ ] , \ f_{mn} = \partial_{m}a_{n} - \partial_{n}a_{m} + i[a_{m}, a_{n}],$$

$$\xi^{k}_{\alpha} = (\sigma_{m})_{\alpha\dot{\alpha}}\nabla_{m}\bar{\psi}^{\dot{\alpha}k}, \ \ p = \nabla^{2}\bar{\phi} + \{\bar{\psi}^{\dot{\alpha}k}, \bar{\psi}_{\dot{\alpha}k}\} + \frac{1}{2}([\bar{\phi}, [\bar{\phi}, \varphi]]). \ (3.33)$$

The deformed chiral $U(n)$ superfield can be written as a sum of two $N=(1, 0)$ covariant objects

$$\mathcal{A}(x_{L}, \theta^{+}, \theta^{-}, u) = [L^{2} + L(1 - L)(\theta^{-}\partial_{-}) - \frac{1}{4}(1 - L)^{2}(\theta^{-})^{2}(\partial_{-})^{2}]\mathcal{A}(x_{L}, \theta^{+}, \theta^{-}, u)$$

$$- 4I^{2}\hat{A}(x_{L}, \theta^{+}, u)$$

(3.34)

where $A$ is the undeformed $U(n)$ superfield (3.32), and the $\bar{\phi}$-dependent matrix operator $L$ (3.31) commutes with $\theta^{\pm\alpha}$ and $\partial_{-\alpha}$ and acts on all matrix quantities standing to the right. The second part $\hat{A}$ is a traceless chiral-analytic $N=(1, 0)$ superfield

$$\hat{A}(x_{L}, \theta^{+}, u) = \hat{p} - [\bar{\phi}, d^{+}] + 2\theta^{+}\alpha\{i[\bar{\phi}, \xi^{-}_{\alpha}] - [\bar{\phi}, [\bar{\phi}, \psi^{-}_{\alpha}]]\}$$

$$+ (\theta^{+})^{2}[\bar{\phi}, [\bar{\phi}, d^{-}]], \ \ \hat{p} = p - \frac{1}{n}\text{Tr}p. \ (3.35)$$

Both parts of $\mathcal{A}$ are thus expressed in terms of the undeformed field components of the superfield $A$ (3.32).

The $N=(1, 0)$ supersymmetry transformation of $\mathcal{A}$ has the following form:

$$\delta_{\epsilon}\mathcal{A} = 2(\epsilon^{-}\theta^{+})[\bar{\phi}, \mathcal{A}] + L\epsilon^{-}\alpha\partial_{-\alpha}A + \epsilon^{+\alpha}\partial_{+\alpha}A. \ (3.36)$$

It is worth noting that the undeformed anti-self-duality equation in the $N=(1, 1)$ supersymmetric $U(n)$ gauge theory [23, 24] can be written in the pure chiral superfield form as

$$A = 0, \ (3.37)$$

which, as follows from (3.32), amounts to the following set of matrix component equations

$$f_{mn}(\sigma_{mn})_{\alpha}^{\dot{\alpha}} = 0, \ \ \varphi = \psi^{k} = d^{kl} = 0,$$

$$(\sigma_{m})_{\alpha\dot{\alpha}}(\partial_{m}\bar{\psi}^{\dot{\alpha}k} + i[a_{m}, \bar{\psi}^{\dot{\alpha}k}]) = 0, \ \ \nabla^{2}\bar{\phi} + \{\bar{\psi}^{\dot{\alpha}k}, \bar{\psi}_{\dot{\alpha}k}\} = 0. \ (3.38)$$
These anti-self-dual $U(n)$ solutions preserve only the $N=(1,0)$ supersymmetry, so it is natural that the same undeformed solutions survive in the $I$-deformed $U(n)$ gauge theory

$$A = 0 \iff \mathcal{A} = 0.$$  \hspace{1cm} (3.39)

The $I$-deformed $U(n)$ gauge theory component action can be directly obtained from the superfield chiral action

$$S_n = \frac{1}{4} \int d^4x d^4\theta \text{Tr} \mathcal{A}^2 = \int d^4x d^4\theta \text{Tr} \left\{ \frac{1}{4} (L\mathcal{A})^2 - 2I^2 \mathcal{A}\mathcal{A} \right\},$$  \hspace{1cm} (3.40)

using relations (3.32) and (3.35). In the limit $I \to 0$ the first term yields the action of the undeformed $U(n)$ gauge theory. The non-standard second term contains higher derivative terms, in particular $I^2(\Box \phi)^2$, which can hopefully be removed by a redefinition of the scalar field $\varphi$ (so far we have checked this only for the bilinear free part of the total action).

4 Interactions of hypermultiplets in deformed harmonic superspace

The free $q^+$ hypermultiplet actions of ordinary harmonic theory [7] are not deformed in the non-anticommutative superspace:

$$S_0(q^+) = \frac{1}{2} \int du d\zeta^{-4} q^+_a \star D^{++} q^{+a} = \frac{1}{2} \int du d\zeta^{-4} q^+_a D^{++} q^{+a}.$$  \hspace{1cm} (4.1)

Here $d\zeta^{-4} = d^4x_A(D^-)^4$ and the additional ‘Pauli-Gürsey’ SU(2)$_P$ indices $a, b = 1, 2$ were introduced: $q^{+a} = \varepsilon^{ab} q^{+}_b = (\tilde{q}^+, q^+)$. Let us consider the $\hat{\theta}^{+\dot{\alpha}}$-expansion of the superfield doublet $q^{+a}$ in the analytic basis

$$q^{+a} = c^{+a} + \tilde{\theta}^{+\dot{\alpha}} \kappa^{\dot{\alpha}a} + (\hat{\theta}^{+})^2 b^{-a},$$

$$D^{++} q^{+a} = \partial^{++} c^{+a} + \tilde{\theta}^{+\dot{\alpha}} (\partial^{++} \kappa^{\dot{\alpha}a} + 2i\theta^{+\dot{\alpha}} \partial\kappa^{\dot{\alpha}a} c^{+a})$$

$$+ (\hat{\theta}^{+})^2 (\partial^{++} b^{-a} + i\theta^{+\dot{\alpha}} \partial\kappa^{\dot{\alpha}a} c^{+a})$$  \hspace{1cm} (4.2)

where

$$c^{+a} = f^a + \theta^{+a} \rho^a, \quad (\hat{\theta}^{+})^2 g^a, \quad \kappa^{\dot{\alpha}a} = \chi^{\dot{\alpha}a} + \theta^{+\dot{\alpha}} \rho^{\dot{\alpha}a} + (\theta^{+})^2 \Sigma^{a\dot{\alpha}},$$

$$b^{-a} = h^a + \theta^{+a} \Sigma^a + (\theta^{+})^2 X^a$$  \hspace{1cm} (4.3)

and, for brevity, the U(1) charges of the component fields $f^a, g^a, h^a, \ldots$ are suppressed. The component fields are functions of $x^m_A$ and harmonics. The chiral representation of the free action (i.e., with the integration over $\hat{\theta}^{+\dot{\alpha}}$ manifestly performed) reads

$$S_0(q^+) = - \int du d^4x_A d^2\theta^+ \left[ \frac{1}{2} b^{-a} \partial^{++} c^{+a} + \frac{1}{2} c^{+a} \partial^{++} b^{-a} + \frac{1}{4} \kappa^{\dot{\alpha}a} \partial^{++} \kappa^{\dot{\alpha}a} \right.$$ \hspace{1cm} (4.4)

$$+ \frac{1}{2} \theta^{+\alpha} (c^{+a} \partial\kappa^{\dot{\alpha}a} \kappa^{\dot{\alpha}a} - \kappa^{\dot{\alpha}a} \partial\kappa^{\dot{\alpha}a} c^{+a}) \right].$$
The non-anticommutativity shows up in the hypermultiplet self-interactions. If we prefer to work in the manifestly SU(2)$_P$ covariant formalism, it is convenient to define two independent combinations:

$$\{q^{+a}, q^{+b}\}_\ast, \quad [q^{+a}, q^{+b}]_\ast = 2q^{+a}P_s q^{+b} = \varepsilon^{ab}C^{++}.$$  \hspace{1cm} (4.5)

The square of the first superfield contracted with some SU(2)$_P$-breaking constant parameter $C_{(ab)}$ gives a non-anticommutative generalization of the self-interaction $[q^{+a}q^{+b}C_{(ab)}]^2$ which yields the familiar Taub-NUT hyper-Kähler metric on the bosonic target space [7]. Leaving this generalization for the future study, we shall consider a simpler example of the deformed self-interaction constructed out of the second combination in (4.5) and vanishing in the anticommutative limit $I \to 0$

$$S_\nu(q^+) = -\frac{\nu}{4} \int du \, d\zeta^{-4} \, C^{++} \ast C^{++} = -\frac{\nu}{4} \int du \, d\zeta^{-4} \, C^{++} = -\frac{\nu}{4} \int du \, d\zeta^{-4} \, C^{++}$$  \hspace{1cm} (4.6)

where $\nu$ is a coupling constant and the overall sign was chosen for further convenience. Note that this superfield interaction is nilpotent, $(C^{++})^2 \sim (\bar{\theta}^+)^2$, and preserves both SU(2)$_P$ and the R-symmetry SU(2) which acts on harmonics.

One can easily calculate the chiral components of the composite superfields

$$C^{++} = q^{+a}_a P_s q^{+a} = -4iI\bar{\theta}^{+\dot{\alpha}} \partial_{a\dot{\alpha}} q^{+a}_a \partial_\dot{\alpha} q^{+a} = -4iI\bar{\theta}^{+\dot{\alpha}} \partial_{a\dot{\alpha}} q^{+a}_a \partial_\dot{\alpha} q^{+a} + 2I(\bar{\theta}^+)^2(\partial_{a\dot{\alpha}}^\dot{\alpha} \partial_\dot{\alpha}^a c^{+a} - \partial_{a\dot{\alpha}} c^{+a} \partial_\dot{\alpha}^a c^{+a}) \equiv (C^{++})^2 = -8I^2(\bar{\theta}^+)^2 B^+ \dot{B}^+, \quad B^+ \dot{c}^{+a} = \partial^\dot{\alpha} c^{+a} \partial_{+\alpha} c^{+a}.$$ \hspace{1cm} (4.7)

The deformed interaction (4.6) contains superfields $c^{+a}$ only

$$S_\nu(q^+) = 2\nu I^2 \int du \, d^4 x_A \, d^2 \bar{\theta}^+ B^+ \dot{B}^+ \dot{\alpha}$$  \hspace{1cm} (4.8)

The total superfield action $S_0(q^+) + S_\nu(q^+)$ yields the hypermultiplet equation of motion

$$D^{++} q^{+a} = \nu q^{+a} P_s q^{+b} = J^{(+3)a}(q^+)$$  \hspace{1cm} (4.9)

where $J^{(+3)a}(q^+)$ is the nonlinear nilpotent source. After performing the $\theta$-integration, the total action contains an infinite number of auxiliary fields coming from the harmonic expansions of the components in (4.3). These auxiliary fields can be eliminated using the appropriate non-dynamical equations collected in the $\theta^+, \bar{\theta}^+$ expansion of (4.9).

The non-dynamical equations of motion for $c^{+a}$ and $\kappa^{\dot{\alpha}a}$ have the following form:

$$\partial^+ c^{+a} = 0, \quad \partial^+ \kappa^\alpha_a = -2i\theta^+ \partial_{a\dot{\alpha}} c^{+a} = 0.$$  \hspace{1cm} (4.10)
In components, the solution to these equations is given by
\[
c^{+a} = u^k f^{ak}(x) + \theta^{+a} \rho_a^a(x), \quad \kappa^a_\alpha = \chi_0^a(x) + 2i u_k^\alpha \theta^{+a} \partial_{\alpha a} f^{ak}(x).
\] (4.11)

The last equation, for the chiral component \( b^{-a} \), also follows from eq. (4.9)
\[
\partial^{++} b^{-a} + i \theta^{+a} \partial_{\alpha a} \kappa^{\dot{\alpha} a} = -4 \nu I^2 [\partial_{\alpha a} c^{+a} \partial^{\dot{\beta} \dot{a}} c^{+b} \partial_{+ \beta c} c^b + \partial_m c^{+a} \partial_m c^b + (\partial) c^{+a} c^b]
\]
and is solved by
\[
b^{-a} = -4 \nu I^2 u^+_k [\partial_{\alpha a} f^{a k}(\partial^{\dot{\beta} \dot{a}} \rho^{ab})\rho_{\beta b} + \rho^a_\beta \partial_{\alpha a} \chi^\dot{a}_a + \nu I^2 (\partial_{\alpha a} \rho_{\gamma a}) \rho^{\alpha a} (\partial^{\dot{\beta} \dot{a}} \rho^{b\gamma}) \rho_{\beta b}].
\] (4.13)

Eliminating the auxiliary component fields from the action \( S_0 + S_\nu \) by substitution (4.11), one obtains the physical action of this model. It contains the standard free kinetic terms for the physical bosonic and fermionic fields, as well as some fermionic self-interaction with two derivatives:
\[
S = \int d^4 x [\frac{1}{2} \partial_m f^{ak} \partial_m f_{ak} + \frac{i}{2} \rho_{ab} \partial_{\alpha a} \chi^\dot{a}_a - \nu I^2 (\partial_{\alpha a} \rho_{\gamma a}) \rho^{\alpha a} (\partial^{\dot{\beta} \dot{a}} \rho^{b\gamma}) \rho_{\beta b}].
\] (4.14)

The scalar field \( f^{ak} \) and the left-handed spinor field \( \rho_a^a \) satisfy the free massless equations in this model
\[
\square f^{ak} = 0, \quad \partial_{\alpha a} \rho^{\alpha a} = 0.
\] (4.15)

At the same time, the equation for the right-handed spinor field \( \chi_0^a \) contains the nonlinear spinor source depending on the left-handed spinor field
\[
i \partial_{\gamma a} \chi^\dot{a}_a = -4 \nu I^2 [\rho_{\beta b}(\partial_{\alpha a} \rho_{\gamma a})(\partial^{\dot{\beta} \dot{a}} \rho^{b\gamma}) + \rho_{\beta a}(\partial_{\gamma \dot{a}} \rho_{ab})(\partial^{\dot{\beta} \dot{a}} \rho^{b\gamma}) + \rho_{\gamma b}(\partial_{\dot{a} a} \rho_\alpha) (\partial^{\dot{\beta} \dot{a}} \rho^{b\gamma}) + \rho_{\beta a}(\partial_{\gamma \dot{a}} \rho_{ab})(\partial^{\dot{\beta} \dot{a}} \rho^{b\gamma}) - \nu I^2 J_{aa} \rho(x)].
\] (4.16)

Note that the last two terms in \( J_{aa} \) are vanishing on the mass-shell of the free fields \( \rho_a^a \). The exact classical solution for \( \chi^\dot{a} \) is a sum of the free right-handed fermion \( \chi_0^a \) and the inhomogeneous solution with the above nilpotent spinor source:
\[
\chi^\dot{a} = \chi_0^a + i \nu I^2 \int d^4 y \partial_{x^\dot{a}} D^0(x - y) J_{\alpha a}^a \rho(y),
\] (4.17)
\[
\partial_{\alpha a} \chi_0^a = 0, \quad \square_x D^0(x - y) = \delta^4(x - y).
\]

Thus the considered model is exactly soluble at the classical level.

The component form of some other nilpotently deformed \( q^+ \) self-interactions and the deformed hypermultiplet interactions with the analytic gauge superfield \( V^{++} \) will be studied elsewhere.
5 Conclusions

In this contribution, basically following refs. [14, 15, 16], we briefly reviewed recent results on the nilpotent non-anticommutative deformations of Euclidean $N=(1,1)$ superspace, with the main emphasis on the structure of the singlet $Q$-deformation of $N = (1, 1)$ gauge theories. This deformation breaks half of $N = (1, 1)$ supersymmetry, but preserves $O(4)$ and $SU(2)$ automorphism symmetries, as well as both chiralities and harmonic Grassmann analyticity. We also considered a simple new example of the $Q$-deformed hypermultiplet action, with the self-interaction vanishing in the anticommutative limit. This model is exactly solvable at the classical level.

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