Weak Coherent State Path Integrals

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Abstract

Weak coherent states share many properties of the usual coherent states, but do not admit a resolution of unity expressed in terms of a local integral. They arise e.g. in the case that a group acts on an inadmissible fiducial vector. Motivated by the recent Affine Quantum Gravity Program, the present article studies the path integral representation of the affine weak coherent state matrix elements of the unitary time-evolution operator. Since weak coherent states do not admit a resolution of unity, it is clear that the standard way of constructing a path integral, by time slicing, is predestined to fail. Instead a well-defined path integral with Wiener measure, based on a continuous-time regularization, is used to approach this problem. The dynamics is rigorously established for linear Hamiltonians, and the difficulties presented by more general Hamiltonians are addressed.

I. Introduction

Unlike the standard phase space path integrals constructed by the time slicing method, the path integral with Wiener measure invented by Klauder, Dau-bechies and others uses a continuous-time regularization factor |1|2|3|4|5. This path integral is \( \int \exp\{-i\int [qdp + dG(p, q) + h(p, q)dt]\} d\mu_W^0 \), where \( G \) is an arbitrary \( C^1 \) function and \( h \) is the classical Hamiltonian in a sense
which will be explained later. The pinned Wiener measure $d\mu_W$ is defined with the help of the heat kernel $\int d\mu_W := [\exp\{\nu T \Delta_{LB}\}] (p'', q'', q', p')$. Thus, by way of the Laplace-Beltrami operator $\Delta_{LB}$, a metric is introduced. The formal phase space path integral $N_\nu \int \exp\{-i \int [q\dot{p} + G(p, q) + h(p, q)] dt\} DpDq$ can be given meaning by equating it to the above Wiener measure path integral. Here, $N_\nu$ is a formal normalization constant, and $d\sigma^2$ is the metric mentioned above. The variables $p$ and $q$ in the well-defined Wiener measure path integral are stochastic variables describing Brownian Bridges. The integral $\int q dp$ has to be interpreted as a stochastic integral. The rule adopted here is the Stratonovich mid-point rule $\int q dp := \lim \sum \frac{1}{2} (q_{l+1} + q_l)(p_{l+1} - p_l)$, which guarantees that the ordinary rules of calculus still apply. It was shown, first for the case of a flat and spherical phase space metric [1], then for a hyperbolic metric [2], that the limit of diverging diffusion constant $\nu$ exists for a wide set of quantum Hamiltonians $\mathcal{H}$, including at least all Hamiltonians polynomial in the basic quantum kinematical operators. The limit is equal to the coherent state matrix element $\langle p' q'' | \exp\{-iT\mathcal{H}\} | p' q' \rangle$ of the unitary time-evolution operator and the specific metric determines the coherent states in question: The flat metric is inevitably connected with the coherent states of the Heisenberg-Weyl group (and in the canonical, Cartesian form, it is connected to the canonical coherent states), the spherical metric is associated with the coherent states of the SU(2) group, and the hyperbolic metric leads to the coherent states of the affine group. And with each group comes a set of quantum kinematical operators. Thus, one can say that in these three cases the choice of geometry augmenting the classical phase space manifold determines the quantum kinematical operators uniquely! Furthermore, the classical Hamiltonian that goes with the quantum Hamiltonian $\mathcal{H}$ is given by the lower symbol $h$, implicitly defined by the relation $\mathcal{H} = \int h(p, q) ||pq\rangle \langle pq| d\mu(p, q)$. Here, $d\mu(p, q)$ is the left-invariant group measure of the group which defines the coherent states. This measure is normalized such that $h(p, q) \equiv 1$ leads to $\mathcal{H} = I$, and, thus, provides the usual resolution of unity. Since the Stratononovich rule is used, and since the coherent states merely change labels under canonical (coordinate) transformations, apart from possible phase factors, the path integral $\langle p'' q'' | \exp\{-iT\mathcal{H}\} | p' q' \rangle = \lim_{\nu \to \infty} \int \exp\{-i \int [qdp + dG(p, q) + h(p, q)] dt\} d\mu_W$ is covariant under canonical (coordinate) transformations and the quantiza-

\textsuperscript{1}Other authors call this symbol the upper symbol, since it is involved in an upper bound in the Berezin-Lieb inequalities.
tion is fully geometric in nature \[3-4\]. The foregoing has been extended to arbitrary geometries of the phase space \[6\].

In an attempt to quantize gravity \[7-8\], Klauder was led to consider affine rather than canonical commutation relations for the field operators (the spatial part of the metric and its partner field). In the simplest case of constant fields, the problem reduces to a toy model of just one degree of freedom, namely the affine coherent states. To be more precise, it includes the affine coherent states, which fulfill a fiducial vector admissibility condition \[2-5-9\], but also those states which violate it. These latter states do not resolve unity anymore and, therefore, are called weak coherent states. The Affine Quantum Gravity Program has provided the motivation to raise the question of the existence of path integrals for these weak coherent states.

It is clear that a path integral can not be constructed with weak coherent states in the standard way, since the resolution of unity is the key to the time-slicing approximation. However, the extension of the well-defined path integral with Wiener measure introduced above to the situation of weak coherent states could still be possible, and this is the goal of the present article. Two different methods to extend the Wiener measure path integral will be introduced: the first is based on the spectral decomposition of certain operators and will therefore be called the “spectral approach”. Unfortunately, it is limited to one very special case. The second uses an extra regularization parameter and is consequently called the “regularizing approach”. In both cases, the path integral for zero Hamiltonian is studied first, while the dynamics is introduced as a second step.

II. Weak coherent state path integrals

General definitions

Coherent states are defined by two properties \[10\]:

1) Continuity: The states $|l\rangle$ are a strongly continuous vector-valued function of the label $l$.

2) Resolution of unity: There exists a positive measure $\delta l$ on the label space $\mathcal{L}$ such that the identity operator $\mathbb{1}$ on $\mathcal{H}$ can, upon integration over $\mathcal{L}$, be represented as
\[ \mathbb{I} = \int |l\rangle\langle l| \delta l \]

A more general class of states can be obtained by relaxing the second property:

2') **Completeness**: The family of vectors \( |l\rangle \) is total, i.e., the closed linear span of \( |l\rangle \) is the whole Hilbert space \( \mathfrak{H} \).

States which share the properties 1) and 2') have been named Klauder states \([11]\). They are the disjoint union of the coherent states in the sense above and the weak coherent states, which do not possess a resolution of unity.

**Affine weak coherent states**

The affine group \( (M_+, \circ) \) is the set \( M_+ := \mathbb{R}^+ \times \mathbb{R} \) with the group law

\[ (q, p) \circ (q', p') = (qq' + p - q^{-1}p', p + q^{-1}) \]

and has two nontrivial, inequivalent, irreducible, unitary representations \([2]\) \( U_{\pm}(p, q) = e^{\pm ipQ}e^{-i \ln qD} \), where the generators \( Q > 0 \) and \( D \) obey the affine commutation relation \([Q, D] = iQ\).

The uncertainty product of the irreducible, self-adjoint operators \( Q \) and \( D \) is \( \Delta Q \Delta D \geq \frac{1}{2} \langle Q \rangle \). Setting \( \langle Q \rangle = 1 \) leads to a one-parameter family of minimum uncertainty states given in \( x \)-representation by \([5]\) \( \eta_\beta(x) = N_\beta x^{\beta^{-1}}e^{-\beta x} \) with normalization \( N_\beta = (2\beta)^{\beta/2} \Gamma^{-1/2}(2\beta) \). The affine coherent states are defined as \( |pq\rangle := U_+(p, q)|\eta_\beta\rangle \). The group acts on admissible fiducial vectors, which fulfill \([2, 11, 5]\) \( \langle Q^{-1} \rangle = \int_0^\infty x^{-1}|\eta_\beta(x)|^2 dx < \infty \). Namely these are the states with \( \beta > 1/2 \). Weak coherent states, on the other hand, are generated by the same group action on fiducial vectors with \( 0 < \beta \leq 1/2 \). For the whole parameter range \( 0 < \beta \), the overlap reads

\[ \langle pq|rs \rangle = (qs)^{-\beta}2^{2\beta}[(q^{-1} + s^{-1}) + i\beta^{-1}(p - r)]^{-2\beta} \]

The construction of the affine coherent state path integral with Wiener measure \([2]\) is based on a linear complex polarization condition. For the minimum uncertainty fiducial vectors, \( (Q - 1 + i\beta^{-1}D)|\eta_\beta\rangle = 0 \) holds. Hence, all functions \( \psi(p, q) := \langle pq|\psi \rangle \) are annihilated by the operator \( B = -iq^{-1}\partial_p + 1 + \beta^{-1}q\partial_q \). The same is true for the second-order differential operator

\[ A := \frac{1}{2}B^\dagger B = \frac{1}{2}\{ -\beta^{-1}\partial_qq^2\partial_q - \beta q^{-2}\partial_p^2 - 1 + \beta - 2i\beta q^{-1}\partial_p \} \]
which is a nonnegative, self-adjoint operator with spectrum

\[ \text{spec}(A) = \{ (\beta - \frac{1}{2})^2 - (\beta - \frac{1}{2} - n)^2; n \in \mathbb{N}, n < \beta - \frac{1}{2} \} \cup \{ (\beta - \frac{1}{2})^2, \infty \} \]

For \( \beta > 1/2 \), the operator \( A \) has a discrete eigenvalue 0 and it follows, for \( T > 0 \), that \( \lim_{\nu \to \infty} [e^{-\nu T A}] \delta(p - p') \delta(q - q') |_{p = p', q = q'} = [P_0]_{p''}^{p'} q''^{q'} \), where the expression on the right hand side is the kernel of the projection operator onto the ground state. But this kernel is also given by \((2\pi)^{-1}(1 - \frac{1}{2\beta})\langle p'' q'' | p' q' \rangle \). This is the key part of the construction, since the rest follows by the Feynman-Kac-Stratonovich representation of the kernel of \( e^{-\nu T A} \), which is \( N_\nu e^{-i\int q^2 dt} \frac{1}{2\nu} \int [\beta^{-1} q^2 p^2 + \beta q^{-2} q^2] dt Dp Dq \). As stated in the introduction, this formal expression makes sense as a Wiener measure path integral, and so finally

\[ \langle p'' q'' | p' q' \rangle = \lim_{\nu \to \infty} 2\pi (1 - \frac{1}{2\beta})^{-1} e^{\nu T/2} \int e^{-i\int q dp} d\mu'_W(p, q) \]

which is a well-defined expression\(^2\).

For \( 0 < \beta \leq 1/2 \), i.e., in the weak coherent state case, the operator \( A \) has only a continuous spectrum, and the limit of diverging diffusion constant of the operator \( e^{-\nu T A} \) is zero. Thus, the whole construction outlined above breaks down. To prevent this collapse to a trivial result, two different approaches will be discussed.

A. Spectral approach

The idea in this approach is to determine a \( \nu \)-dependent rescaling factor, such that the limit of diverging diffusion constant will be nontrivial. This was proposed by Klauder \[^5\].

The general case

Let \( X \) be a non-negative self-adjoint operator on a certain Hilbert space and assume zero is in its continuous, but not in its discrete, spectrum. The

\[^2\]The path integral for a non-zero Hamiltonian is constructed in much the same way. The only difference is that \( \nu A \) must be replaced by an operator involving the Hamiltonian \( h \), namely \( \nu A + i h \).
operator $X$ generates a semigroup $e^{-\nu TX}$, which has a spectral representation
\[ e^{-\nu XT} = \int_0^\infty e^{-\nu\lambda T} d\mathbb{E}(\lambda) \quad \text{or} \quad \langle x'' | e^{-\nu TX} | x' \rangle = \int_0^\infty e^{-\nu\lambda T} d\langle x''|\mathbb{E}(\lambda)|x'\rangle. \]

Since only well-behaved potentials will eventually be of interest, the reasonable assumption is made that the measure $d\langle x''|\mathbb{E}(\lambda)|x'\rangle$ has an absolutely continuous, but no singularly continuous part. Then the spectral family can be written as a (weighted) integral over one-dimensional projection operators $\mathbb{E}(\lambda) = \int_{-\infty}^{\lambda} |E\rangle\langle E| dE$. If the generalized eigenstates $|E\rangle$ are $\delta$-orthonormalized, then $\rho(E) = 1$.

The matrix element of $e^{-\nu TX}$ can then be written as
\[ \langle x''|e^{-\nu TX}|x'\rangle = \int_0^\infty e^{-\nu\lambda T} \psi_\lambda(x'')\psi_\lambda^*(x')\rho(\lambda) d\lambda \tag{1} \]
and the $\psi_\lambda$ are continuous in $\lambda$. Moreover, $\rho$ - being part of the measure - is at least right-continuous. For $\delta$-orthonormalized wavefunctions, $\rho(\lambda) \equiv 1$.

The goal is to find the rescaling factor which saves Eq. (1) from becoming trivial in the limit of diverging diffusion constant $\nu$. Since, for very large $\nu$, the factor $e^{-\nu\lambda T}$ suppresses everything but the values for very small $\lambda$, the behavior of $f_{x',x''}(\lambda) := \psi_\lambda(x'')\psi_\lambda^*(x')\rho(\lambda)$ near $\lambda = 0$ is all that matters. To give an example, assume that $f_{x',x''}(\lambda) \propto \lambda^a \text{ for small } \lambda$. Now, the proper rescaling factor can be determined, and in the example it is
\[ \int_0^\infty d\lambda \lambda^a e^{-\nu\lambda T} = \frac{\Gamma(a+1)}{(\nu T)^{a+1}} \tag{2} \]
After rescaling with the inverse one gets $\frac{(\nu T)^{a+1}}{\Gamma(a+1)}\lambda^a e^{-\nu\lambda T} \xrightarrow{\nu \to \infty} \delta(\lambda)$ which represents a $\delta$-function weight on $\lambda = 0$.

The rescaling factor can be computed self-consistently, and the general formula reads
\[ \frac{\int_0^\infty e^{-\nu\lambda T} \psi_\lambda(x'')\psi_\lambda^*(x')\rho(\lambda) d\lambda \xrightarrow{\nu \to \infty} \psi_0(x'')\psi_0^*(x')}{\int_0^\infty e^{-\nu\lambda T} \psi_\lambda(0)\psi_\lambda^*(0)\rho(\lambda) d\lambda} \tag{3} \]
The numerator of the last expression, $\psi_0(x'')\psi_0^*(x')$, is the kernel of the desired projection operator onto the ground state, and we have assumed that the denominator is nonzero. The convergence is in a distributional sense (denoted by the symbol $\xrightarrow{\nu}$). If the functional form of $\psi_0(x'')\psi_0^*(x')$ is known to be continuous, then the convergence is pointwise.

3For a singularly continuous measure this would not be possible: $\mu_{sc}(x) = \int_{-\infty}^{x} d\mu_{sc}(y) \neq \int_{-\infty}^{x} (d\mu_{sc}/dy) dy = 0$ since $d\mu_{sc}/dy = 0$ almost everywhere.
Observe, in the example with $f_{x',x''}(\lambda) = \lambda^a$, one must have $a > -1$, or else the rescaling factor would be identically zero (since the integral would be infinity). But, since the rescaling factor can be determined self-consistently, i.e., by the denominator of Eq. (3), which always exists, there is no hidden “trap” to look out for. Moreover, the evaluation of the denominator need not necessarily be at the point $x'' = x' = 0$. It could be at any point $x'' = x' = b$, $b \in \mathbb{R}$, or even $b = \pm \infty$, as long as the function $\psi_\lambda(x)$ is not 0 at $b$. Whatever gives the easiest result is the preferred choice. And the arbitrariness of this choice is not critical: Assume $K$ to be the reproducing kernel of some reproducing kernel Hilbert space, and let $a$ be a positive constant. Then, $aK$ is just as good a reproducing kernel, since the same class of functions arises, only the inner product has to be redefined.

The affine case

The foregoing is now applied to the case of the affine weak coherent states. Unfortunately, $A$ matches the required properties, namely that 0 be in the continuous spectrum, only in the case $\beta = 1/2$! This is true in spite of the fact that $A\langle pq|\psi\rangle = 0$ (for arbitrary $|\psi\rangle$), since an equation $A\psi = \alpha\psi$ need not necessarily imply $\alpha \in \text{spec}(A)$. In fact, the $\psi(p,q) = \langle pq|\psi\rangle$ are not generalized eigenvectors except in the case $\beta = 1/2$ [11]. Consequently, the isolating procedure can only be performed for $\beta = 1/2$, and the general theory above ensures the existence of the weak coherent state path integral.

For the case at hand a connection between the operator $A$ and the one-dimensional Morse operator $H_{\text{Morse}}$ exists [2] and makes the explicit functional form of the generalized eigenfunctions available. With the aid of these, the rescaling factor can be computed explicitly.

The problem to find the eigenfunctions of the operator $A$ is first reduced to a problem on $L^2(\mathbb{R}^+)$ and then to a problem on $L^2(\mathbb{R})$, leading to the Morse operator:

$$A\langle U(p,q)\phi|\psi\rangle = \langle A^*U(p,q)\phi|\psi\rangle$$

$$= \frac{1}{2}\langle[-\beta^{-1}\partial_q q^2 \partial_q - \beta q^{-2} \partial_p^2 - 2i\beta q^{-1} \partial_p + \beta - 1]e^{ipQ}e^{-i\ln qD}\phi|\psi\rangle$$

$$= \frac{1}{2}\beta^{-1}\langle e^{ipQ}e^{-i\ln qD}(D^2 + iD + \beta^2 Q^2 - 2\beta^2 Q + \beta - \beta)(Q^{1/2}\phi')|\psi\rangle$$

$$= \frac{1}{2}\beta^{-1}\langle e^{ipQ}e^{-i\ln qD}Q^{1/2}(D^2 + \beta^2 Q^2 - 2\beta^2 Q + (\beta - 1/2)^2)\phi'|\psi\rangle \quad (4)$$

where $\phi = Q^{1/2}\phi'$. 

7
Under the unitary transformation

\[ (\tilde{U}\psi)(x) = e^{x/2}\psi(e^x) \]  

the operator in braces in the last line of Eq. 4 (called $H$ in [2]) is transformed to the Morse operator:

\[ H_{\text{Morse}} = -\frac{d^2}{dx^2} + \beta^2(e^{2x} - 2e^x) + (\beta - \frac{1}{2})^2 \]  

The eigenfunctions of the Morse operator can be found in [12], and, for $\beta = 1/2$, they are given in momentum representation (and $\delta$-orthonormalized) by

\[ \psi_\lambda(x) = \left(\frac{\sinh(2\pi\lambda)}{\pi}\right)^{1/2} \Gamma(i\lambda)e^{-x/2}W_{1/2,i\lambda}(e^x) \]  

where $W$ is a Whittaker function. With a mass $m = 1/2$, one has the relation $E = \lambda^2$ for energy and momentum, and the $\delta$-orthonormalized eigenfunctions in energy representation are

\[ \psi_E(x) = \left(\frac{\sinh(2\pi\sqrt{E})}{2\pi}\right)^{1/2} \Gamma(i\sqrt{E})e^{-x/2}W_{1/2,i\sqrt{E}}(e^x) \]  

Since the Whittaker function $W_{1/2,0}(z) = e^{-z^2/2}z^{1/2}$, the $x$-dependence of $\psi_{E=0}(x)$ is $e^{-e^x/2}$. Thus, the rescaling factor can best be determined with the choice $x'' = x' = b = -\infty$ where this function is equal to one. For small $E$, the function $f_{-\infty,-\infty}(E) = \psi_E(-\infty)\psi_E'(-\infty)\rho(E) \approx \pi^{-1}E^{-1/2}$ because $\sinh(2\pi\sqrt{E}) \approx 2\pi\sqrt{E}$, $|\Gamma(i\sqrt{E})|^2 \approx 1/E$. Inserting this $E$-dependence into the general formula ($\rho(E) = 1$ because of $\delta$-orthonormalization), one finds the inverse rescaling factor

\[ \int_0^\infty e^{-\nu T} f_{-\infty,-\infty}(E)dE = (\pi\nu T)^{-1/2} \]  

Because of the connection between the “Morse”-level and the original problem [Eqs. (4) and (5)], this is already the proper rescaling factor for the original problem as well.

The sought-for weak coherent state path integral for $\beta = 1/2$ and vanishing Hamiltonian is thus

\[ \langle p''q''|p'q'\rangle = \lim_{\nu \to \infty} K_\nu \int e^{-i\int q'dp} d\mu_\nu \]  

with rescaling factor $K_\nu = (\pi\nu T)^{1/2}$.
Introducing dynamics

Since the only case in which the spectral approach worked was $\beta = 1/2$, this value is assumed throughout the remainder of this subsection. Dynamics are introduced by the quantum Hamiltonian $H$, which is a function of the basic kinematical operators $Q$ and $D$. The goal is to represent the propagator $\langle p''q''|\exp\{-iT\mathcal{H}\}|p'q'\rangle$ as a (weak coherent state) path integral. The expression

$$
\langle p''q''|\exp\{-iT\mathcal{H}\}|p'q'\rangle = \lim_{\nu \to \infty} K_{\nu} N_{\nu} \int e^{-i\int [q\dot{p} + h_w(p,q)] dt} e^{-\frac{1}{\nu} \int [\beta^{-1}q^2\dot{p}^2 + \beta q^2 - \beta^{-2}q^2] dt} DpDq
$$

was proposed [5] as the path integral for a class of Hamiltonians which contains at least all Hamiltonians polynomial in $Q$ and $D$. The new symbol $h_w(p,q)$, interpreted as the classical Hamiltonian associated with the quantum Hamiltonian, is implicitly given by

$$
\langle p''q''|H|p'q'\rangle = \lim_{\nu \to \infty} K_{\nu} \int e^{-i\int [q\dot{p} + h_w(p,q)] dt} d\mu_{\nu} W(11)
$$

and will be called the weak symbol.

The whole conjecture is based on the observation that, for a linear Hamiltonian $RQ + SD$, the propagator can be reduced to a mere overlap [5][11]:

$$
\langle p''q''|e^{-i(RQ+SD)T}|p'q'\rangle = \langle p''e^{ST} + R/S \cdot (e^{ST} - 1), q''e^{-ST}|p'q'\rangle (12)
$$

Consequently, the problem is already solved for a linear Hamiltonian, and what remains is to determine the weak symbol associated with $\mathcal{H} = RQ + SD$. According to Eq. (10) the path integral for this Hamiltonian is

$$
\lim_{\nu \to \infty} K_{\nu} \int_{p'q'}^{p''q''} e^{-i\int q dp} d\mu_{\nu} W
$$

Since this is a well-defined functional integral, one can change integration variables...
\[ p(t) \rightarrow p(t)e^{St} + R/S(e^{St} - 1) \]
\[ q(t) \rightarrow q(t)e^{-St} \]

and obtain \( \exp\{-i \int (q(e^{-St})d[p(e^{St} + R/S(e^{St} - 1)]) = \exp\{-i \int [q \, dp + (Rq + Spq)dt] \} \) as the new integrand. The new measure is

\[ \tilde{d}\mu_W = \mathcal{N}_\nu \exp\{- \frac{1}{2\nu} \int [\beta^{-1}(q(e^{-St})^2(p(e^{St} + R/S(e^{St} - 1)))^2 + \beta(q(e^{-St})^{-2}(q(e^{-St}))^2)\, dt] \mathcal{D}[p(e^{St} + R/S(e^{St} - 1)] \mathcal{D}(q(e^{-St})) \]

But, since the measure is actually

\[ \mathcal{D}[p(e^{St} + R/S(e^{St} - 1)] = \lim_{\epsilon \rightarrow 0} \prod_{k=1}^{N} \mathcal{D}[p(t)e^{St}|t_k + \frac{R}{S}(e^{St} - 1)|t_k] \]
\[ = \lim_{\epsilon \rightarrow 0} \prod_{k=1}^{N} \mathcal{D}[p(t)e^{St}|t_k + (pSe^{St} + Re^{St})dt|t_k] \]
\[ = \lim_{\epsilon \rightarrow 0} \prod_{k=1}^{N} \mathcal{D}[p_k e^{Sk\epsilon} + (p_k S e^{Sk\epsilon} + Re^{Sk\epsilon})\epsilon] \]
\[ = \lim_{\epsilon \rightarrow 0} \prod_{k=1}^{N} \mathcal{D}[p_k e^{Sk\epsilon}] = \mathcal{D}p \prod_t e^{St} \]

and analogous \( \mathcal{D}(q(e^{-St}) = \mathcal{D}q \prod_t e^{-St} \), the new measure can be expressed in terms of the old one as

\[ d\tilde{\mu}_W = e^{-\frac{1}{2\nu} \int [\beta^{-1}q^2((Sp^2 + R)^2 + 2(Sp^2 + R)p) + \beta q^{-2}(S^2q^2 - 2Sqq)\, dt] \, d\mu_W} \]
\[ = e^{-\frac{1}{2\nu} \int [\beta^{-1}q^2((Sp^2 + R)^2 + 2(Sp^2 + R)p) + \beta q^{-2}(S^2q^2 - 2Sqq)\, dt] \, d\mu_W} \]

The first equality is again formal and gains meaning by the second line, where the stochastic integrals are understood in the Stratonovich sense, as usual. The change of variables has introduced additional terms in the exponent of the formal expression, which are at most linear in \( \dot{p} \) or \( \dot{q} \), respectively. These terms are not critical since, in the limit of diverging diffusion constant \( \nu \), they will vanish. This means that the total change of the measure disappears in

\[ ^{(\ldots)^*} \text{means the time derivative of the expression in parentheses} \]
the limit. Thus, one can write the path integral with the old measure \( d\mu \) instead of with the new \( d\tilde{\mu} \):

\[
\langle p'' q'' | e^{-(RQ+SD)T} | p' q' \rangle = \lim_{\nu \to \infty} K_{\nu} \int_{p',q'}^{p'',q''} e^{-i\int_{q}^{p} dp + (Rq+Spq)dt} d\mu
\]

Now, the weak symbol can be read off:

\[ h_w(p,q) = Rq + Sqp \]  

The generalization to other Hamiltonians is based on the linearity, completeness, and irreducibility of the basic operators \( Q \) and \( D \) by virtue of which \( \lim_{J \to \infty} \sum_{j=1}^{J} \alpha_j e^{-i(R_j Q+S_j D)} \) weakly converges to any (bounded) operator such as \( e^{-iH_T} \). Thus,

\[
\langle p'' q'' | e^{-iH_T} | p' q' \rangle = \lim_{J \to \infty} \lim_{\nu \to \infty} K_{\nu} \int_{p',q'}^{p'',q''} e^{-i\int_{q}^{p} dp + \sum_{j=1}^{J} \alpha_j e^{-i\int_{Q_j}^{Q_j+S_j} dt} | p' q' \rangle
\]

and the question, on which the next steps depend, is: can the two limits be interchanged? In spite of some effort this question is not yet answered. Assuming that they can, however, one obtains

\[
\langle p'' q'' | e^{-iH_T} | p' q' \rangle = \lim_{\nu \to \infty} K_{\nu} \int_{p',q'}^{p'',q''} e^{-i\int_{Q_j}^{Q_j+S_j} dt} | p' q' \rangle
\]

The expression \( \lim_{J \to \infty} \sum_{j=1}^{J} \alpha_j e^{-i\int_{Q_j}^{Q_j+S_j} dt} \) is unfortunately, not of the form \( e^{-i\int q \cdot d\mu + SD dt} \) for a general, local Hamiltonian \( h_w \), e.g. \( e^{-i\int q^2 dt} \) with Hamiltonian \( q^2 \). To produce local Hamiltonians, one would need distributions \( R(t) \) and \( S(t) \) instead of the constants \( R \) and \( S \). Then, taking e.g. \( R(t) = \delta(t-\tau) \), one gets a local expression \( q(\tau) \) and, by forming functions thereof, local Hamiltonians. This was proposed in \[5\]. However, the construction of distributions from piecewise constant functions would require yet another limiting process, and, again, the interchangeability of the limits is questionable.
In the case of a linear Hamiltonian, the weak symbol was shown to be
\( h_w(p, q) = Rq + Spq \). This is exactly what one would expect since the con-
nexion of the basic operators \( Q \) and \( D \) to classical variables is, according
to the weak correspondence principle, \( q \) and \( pq \), respectively. But, the cor-
respondence for a more general Hamiltonian is not immediately clear and
remains to be determined.

B. Regularizing approach

The idea for this second approach is the introduction of an additional regular-
ization factor which will reintroduce a discrete ground state with eigenvalue
zero. Then, the construction of the path integral moves along the same lines
as in the coherent state case (\( \beta > 1/2 \)). The limit to remove the regularization
is taken as the last step.

For large \( q \), the overlap \( \langle pq|p'q' \rangle \) is proportional to \( q^{-\beta} \). Because \( 0 < \beta \leq 1/2 \), a regularization factor which is effective at infinity is required to produce
Hilbert space vectors again. Since, for \( 0 < \beta \leq 1/4 \), \( \int_{-\infty}^{\infty} (c^2 + p^2)^{-2\beta} dp = \infty \)
(where \( c \) is a constant), one must in this case regularize in \( p \), too. For \( 1 <
4\beta < 2 \) this is not required. A regularization in \( p \) will make a regularization
in \( q \) (for small \( q \)) necessary as well [11].

Case \( 1/4 < \beta \leq 1/2 \)

Let

\[ \langle pq|rs \rangle_\varepsilon := N_\varepsilon \langle pq|rs \rangle e^{-(q+s)\varepsilon} \]

be a normalized vector in \( L^2(M_+) \) with normalization constant \( N_\varepsilon \). The
extra factor \( e^{-(q+s)\varepsilon} \) goes to one in the limit \( \varepsilon \to 0 \). For arbitrary \( x \in \mathbb{R} \),
\( y \in \mathbb{R}^+ \), the overlap \( \langle xy|xy \rangle_\varepsilon \) equals \( N_\varepsilon e^{-2y\varepsilon} \). Hence, one can write
\( \langle pq|rs \rangle = \lim_{\varepsilon \to 0} \langle xy|xy \rangle_\varepsilon^{-1} \langle pq|rs \rangle_\varepsilon \) in a self-consistent way without explicitly referring
to the normalization constant. The following notation is used:

\[ \langle xy|xy \rangle_\varepsilon =: c_{\beta, \varepsilon} \]

The new operator \( B_\varepsilon \), which annihilates the modified kernel, is derived
by exploiting analyticity: \( \left[(q^{-1} + s^{-1}) + i\beta^{-1}(p - r)\right]^{-2\beta} =: Y \) is analytic,
so \( \partial(q^{-1} - i\beta^{-1}p)Y = \frac{1}{2}(-q^2 \partial_q + i\beta \partial_p)Y = 0 \). Write \( Y \) as \( e^{g\varepsilon(qs)}\langle pq|rs \rangle_\varepsilon \), and
move $e^{q\varepsilon}(qs)^\beta$ to the left of this operator. Then, $e^{q\varepsilon}(qs)^\beta$ can be cancelled since the expression is everywhere non-zero. The result is the new operator

$$B_\varepsilon = (\beta^{-1}q\partial_q + \beta^{-1}q\varepsilon + 1 - iq^{-1}\partial_p)$$

for which $B_\varepsilon\langle pq|rs\rangle_\varepsilon = 0$. Define $A_\varepsilon := \frac{1}{2}\beta B_\varepsilon^\dagger B_\varepsilon$ then:

$$A_\varepsilon = \frac{1}{2}\beta(-iq^{-1}\partial_p + 1 - \beta^{-1}\partial_q q + \beta^{-1}q\varepsilon)(-iq^{-1}\partial_p + 1 + \beta^{-1}q\partial_q + \beta^{-1}q\varepsilon)$$

$$= \frac{1}{2}\{\beta[-iq^{-1}\partial_p + 1 + \beta^{-1}q\varepsilon]^2 - \beta^{-1}\partial_q q^2\partial_q - 1 - 2\beta^{-1}q\varepsilon\}$$

(20)

$A_\varepsilon$ can be shown to be essentially self-adjoint since the deficiency index equation $((A_\varepsilon^\dagger \pm i)\psi(p,q)) = 0$ has no solution [11]. In a slight abuse of notation the closure of this operator will be denoted by $A_\varepsilon$ as well. It is a self-adjoint, non-negative operator with zero in its discrete spectrum.

The Feynman-Kac-Stratonovich representation of the kernel of the operator $e^{-\nu T A_\varepsilon}$ is (see Appendix B. for the derivation)

$$e^{-\nu A_\varepsilon T} \delta(p-p')\delta(q-q')|_{p=p',q=q'}$$

$$= e^{\nu T/2} \int e^{-i\int (q+\beta^{-1}q^2\varepsilon)dp} e^{\beta^{-1}q\varepsilon dt} d\mu^\nu_W$$

and it follows that

$$\langle p''q''|pq'\rangle = \lim_{\varepsilon \to 0} c^{-1}_{\beta,\varepsilon} \langle p''q''|pq'\rangle_\varepsilon$$

$$= \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} e^{\nu T/2} \int e^{-i\int (q+\beta^{-1}q^2\varepsilon)dp} e^{\beta^{-1}q\varepsilon dt} d\mu^\nu_W$$

(21)

The stochastic processes involved are still Brownian bridges, and, when the stochastic integrals are interpreted in the Stratonovich sense, canonical (coordinate) transformations can be made in the same way as before. Thus, the geometric nature of the quantization is preserved.

**Case 0<\beta\leq 1/4**

For a parameter $\beta \leq 1/4$, a regularization for large $q$ is not enough. It turns out that an additional $p$-regularization will even make a regularization for
small \( q \) necessary (otherwise the overlap would be square integrable, but not in the domain of \( A_\varepsilon \)).

In the present case, let

\[
\langle pq|rs \rangle_\varepsilon := N_\varepsilon \langle pq|rs \rangle e^{-(q+s)\varepsilon-(q^{-1}+s^{-1})\varepsilon-(p^2+r^2)\varepsilon} \tag{22}
\]

where \( \langle pq|rs \rangle = (qs)^{-\beta}2^{2\beta}[q^{-1}+s^{-1}+i\beta^{-1}(p-r)]^{-2\beta} \) is the (weak coherent state) overlap which is analytic in the complex variable \( z := q^{-1}+i\beta^{-1}p \), apart from the factor \( (qs)^{-\beta} \). One can write the analytic part (previously called \( Y \)) as \( e^{(q+s)\varepsilon+(q^{-1}+s^{-1})\varepsilon+(p^2+r^2)^\varepsilon}(qs)^\beta \langle pq|rs \rangle_\varepsilon \), and let the differential operator

\[
\partial_{q^{-1}-i\beta^{-1}p} = \frac{1}{2}(-q^2\partial_q + i\beta\partial_p)
\]

act on this expression. Using \( \partial_z f = 0 \) (valid for an analytic function), this results in the new operator

\[
B_\varepsilon = \beta^{-1}q\partial_q + 1 + \beta^{-1}q\varepsilon + \beta^{-1}q^{-1}\varepsilon - 2ipq^{-1}\varepsilon - iq^{-1}\partial_p
\]

for which \( B_\varepsilon \langle pq|rs \rangle_\varepsilon = 0 \). As before, define \( A_\varepsilon := \frac{1}{2}B_\varepsilon^\dagger B_\varepsilon \), then

\[
A_\varepsilon = \frac{1}{2}\{\beta(-iq^{-1}\partial_p + 1 + \beta^{-1}q\varepsilon\beta^{-1}q^{-1}\varepsilon)^2 - 2\beta q^{-2}\varepsilon + 4ip\partial_q\varepsilon + 4\beta p^2 q^{-2}\varepsilon^2 - \beta^{-1}\partial_q q^2\partial_q - 1 - 2\beta^{-1}q\varepsilon\} \tag{23}
\]

Instead of trying to solve the deficiency index equation for the “new” \( A_\varepsilon \), one can avoid the question about self-adjointness altogether.

Assume \( A_\varepsilon \) is not self-adjoint. The (sesquilinear) form \( s_\varepsilon(x,y) := \langle x|A_\varepsilon y \rangle \) generated by \( A_\varepsilon \) is closable since \( A_\varepsilon \) is symmetric and bounded below \[13\]. There is a bijection between the set of all (densely defined) closed, below-bounded forms and the set of all self-adjoint, below-bounded operators. Let \( \bar{s}_\varepsilon \) be the closure of the form generated by \( A_\varepsilon \) and \( A_{\bar{s}_\varepsilon} \) be the self-adjoint operator associated with \( \bar{s}_\varepsilon \). Then, \( A_{\bar{s}_\varepsilon} \) preserves the lower bound and is called the Friedrichs’ extension of the operator \( A_\varepsilon \). [It is the unique extension fulfilling \( D(A_{\bar{s}_\varepsilon}) \subset D(\bar{s}_\varepsilon) \) \[13\].]

In a slight abuse of notation \( A_{\bar{s}_\varepsilon} \) will be written as \( A_\varepsilon \). So from now on, \( A_\varepsilon \) denotes the Friedrichs’ extension (which is trivial in the case that \( A_\varepsilon \) is already self adjoint). Then it is clear that \( A_\varepsilon \) is non-negative.

The Feynman-Kac-Stratonovich representation of the kernel of the operator \( \exp\{-\nu T A_\varepsilon\} \) is derived in much the same way as before (see Appendix \[3\])

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\[ e^{\nu T/2} e^{-i\beta^{-1} \epsilon (p'' - p')} \times \int e^{-i \int \left[ (q + \beta^{-1} q^2 \epsilon) \; dp - 2i \beta \epsilon pq^{-2} \; dq \right] + \nu \int \left[ \beta q^{-2} \epsilon + \beta^{-1} q \epsilon \right] dt} \; d\mu_W \]

Partial integration, i.e., \(2\beta \epsilon pq^{-2} dq = -2\beta \epsilon pd(q^{-1}) = -2\beta \epsilon pq^{-1} \langle (p'', q') \rangle + 2\beta \epsilon q^{-1} dp\), leads to

\[ e^{\nu T/2} e^{-i\beta^{-1} \epsilon (p'' - p')} - 2i \beta \epsilon (p'' q'' - 1) \times \int e^{-i \int \left[ (q + \beta^{-1} q^2 \epsilon - 2i \beta \epsilon q^{-2} \epsilon) \; dp + \nu \int \left[ \beta q^{-2} \epsilon + \beta^{-1} q \epsilon \right] dt} \; d\mu_W \]

The phase factors in Eq. (24) are \(\nu\)-independent, so they come outside of the \(\nu\)-limit, where the \(\epsilon\)-limit renders them unity. Finally, one gets:

\[ \langle p'' q'' | p' q' \rangle = \lim_{\epsilon \to 0} \lim_{\nu \to \infty} c_{\beta, \epsilon}^{-1} \langle p'' q'' | p' q' \rangle \epsilon \\
:= \lim_{\epsilon \to 0} \lim_{\nu \to \infty} c_{\beta, \epsilon}^{-1} e^{\nu T/2} \times \int e^{-i \int \left[ (q + \beta^{-1} q^2 \epsilon - 2i \beta \epsilon q^{-2} \epsilon) \; dp + \nu \int \left[ \beta q^{-2} \epsilon + \beta^{-1} q \epsilon \right] dt} \; d\mu_W \]

This is the path integral representation for \(0 < \beta \leq 1/4\).

**Introducing dynamics**

Dynamics is introduced in the same way as for the spectral approach. For a linear Hamiltonian \(\mathcal{H} = RQ + SD\), the problem is already solved as it reduces to an overlap with modified ending points. What remains to do is to write down the path integral. This is straightforward since everything stated previously concerning the measure, etc., remains valid and the formula for \(1/4 < \beta \leq 1/2\) is
reads then, the formula resembles the path integral for coherent states and

\[ \langle p'' q'' | e^{-i(RQ+SD)T} | p' q' \rangle = \langle p'' e^{ST} + R_S(e^{ST} - 1), q'' e^{-ST} | p' q' \rangle \]

\[ = \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} c_{\beta,\varepsilon}^{-1} e^{\nu T/2} \int_{p',q'}^{p'' q''} \exp \{ -i \int (q + \beta^{-1} q^2 \varepsilon) dp \}
+ \nu \int \beta^{-1} q \varepsilon dt \} d\mu_W^\nu \]

\[ = \lim_{\varepsilon \to 0} \lim_{\xi \to \infty} c_{\beta,\varepsilon}^{-1} e^{\nu T/2} \int_{p',q'}^{p'' q''} \exp \{ -i \int (q e^{ST} + \beta^{-1} q^2 e^{-2St} \varepsilon) \}
\times d|pe^{St} + R_S(e^{St} - 1)| + \nu \int \beta^{-1} q e^{-St} \varepsilon dt \} d\mu_W^\nu \]

\[ = \lim_{\varepsilon \to 0} \lim_{\xi \to \infty} c_{\beta,\varepsilon}^{-1} e^{\nu T/2} \int_{p',q'}^{p'' q''} \exp \{ -i \int (q + \beta^{-1} q^2 e^{-St} \varepsilon) \}
+ \int (q + \beta^{-1} q^2 e^{-St} \varepsilon) (Sp + R) dt \] + \nu \int \beta^{-1} q e^{-St} \varepsilon dt \} d\mu_W^\nu \]  (26)

Introducing the new variable \( q_\varepsilon := q + \beta^{-1} q^2 e^{-St} \varepsilon \), and the new measure \( d\mu_W^{\nu,\varepsilon} := \exp \{ \nu \int \beta^{-1} q e^{-St} \varepsilon dt \} d\mu_W^\nu \), the complexity of the final expression can be hidden. The new measure is equivalent to the old Wiener measure because the factor \( \exp \{ \nu \int \beta^{-1} q e^{-St} \varepsilon dt \} \) serves as a Radon-Nykodym derivative. Then, the formula resembles the path integral for coherent states and reads

\[ \lim_{\varepsilon \to 0} \lim_{\xi \to \infty} c_{\beta,\varepsilon}^{-1} e^{\nu T/2} \int_{p',q'}^{p'' q''} e^{-i \int q_\varepsilon dp + (Spq_\varepsilon + Rq_\varepsilon) dt} \} d\mu_W^{\nu,\varepsilon} \]  (27)

The \( \varepsilon \)-modified Hamiltonian is given by the weak modified symbol \( h_{w,\varepsilon} := Rq_\varepsilon + Spq_\varepsilon \).

The same procedure for \( 0 < \beta \leq 1/4 \) leads to:

\[ \langle p'' q'' | e^{-i(RQ+SD)T} | p' q' \rangle \]

\[ = \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} c_{\beta,\varepsilon}^{-1} e^{\nu T/2} \int_{p',q'}^{p'' q''} \exp \{ -i \int (q + \beta^{-1} q^2 e^{-St} \varepsilon - 2\beta q^{-1} e^{2St} \varepsilon) dp \}
+ \int (q + \beta^{-1} q^2 e^{-St} \varepsilon - 2\beta q^{-1} e^{2St} \varepsilon) (Sp + R) dt \]
+ \nu \int \beta^{-1} q e^{-St} \varepsilon + \beta q^{-1} e^{2St} \varepsilon dt \} d\mu_W^\nu \]

\[ = \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} c_{\beta,\varepsilon}^{-1} e^{\nu T/2} \int_{p',q'}^{p'' q''} \exp \{ -i \int q_\varepsilon dp + \int (Spq_\varepsilon + Rq_\varepsilon) dt} \} d\mu_W^{\nu,\varepsilon} \]  (28)
Here, the variable $\tilde{q}_\varepsilon := (q + \beta^{-1}q^2e^{-St\varepsilon} - 2\beta q^{-1}e^{2St\varepsilon})$ and the Radon-Nykodym measure $d\tilde{\mu}_W := \exp\{\nu\int(\beta^{-1}qe^{-St\varepsilon} + \beta q^{-2}e^{2St\varepsilon})dt\} \, d\mu_W$ were used. The weak modified symbol is now $h_{w,\varepsilon} = R\tilde{q}_\varepsilon + Sp_\varepsilon$.

The problem of how this can be extended to, say, all polynomial Hamiltonians was already discussed in the spectral approach. Here, on the other hand, there could be a second possibility to proceed. With the discrete ground state artifically reintroduced, it seems possible to construct the path integral in essentially the same way as for zero Hamiltonian. The operator $\nu A_\varepsilon$ has to be replaced by $\nu A_\varepsilon + ih_{w,\varepsilon}$, and the conditions required for the construction will imply restrictions for the functions $h_{w,\varepsilon}$ (see [2] for a guideline to the proof). Observe, that this weak modified symbol does not necessarily have to be the same as the one mentioned in the previous parts of the subsection.

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Appendix A: Feynman-Kac-Stratonovich representation of the operator $A_\varepsilon$

The case $1/4 < \beta \leq 1/2$

The Feynman-Kac-Stratonovich representation of the kernel of the operator $\exp\{-\nu TA_\varepsilon\}$ is derived in the following way:

$$\exp\{-\nu TA_\varepsilon\} \delta(p - p')\delta(q - q')|_{p=p',q=q''}$$

$$= \exp\{-\frac{1}{2}\nu T[\beta(-iq^{-1}\partial_p + 1 + \beta^{-1}q\varepsilon)^2 - \beta^{-1}q^2\partial_q - 1 - 2\beta^{-1}q\varepsilon]\}$$

$$\times \int e^{ix(p-p')-ik(q-q')} \frac{dxdk}{(2\pi)^2} |_{p=p',q=q''}$$

$$= e^{\nu T/2} \lim_{N\to\infty} \exp\{-\frac{1}{2}\nu\delta[\beta(-iq^{-1}\partial_p + 1 + \beta^{-1}q\varepsilon)^2 - 2\beta^{-1}q\varepsilon]\}$$

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\[ \times \exp\left\{ \frac{1}{2} \nu \delta (-\beta^{-1} \partial_q q^2 \partial_q) \right\}^{N} \int e^{ix(p-p')-ik(q-q')} \frac{dxdk}{(2\pi)^2} \bigg|_{p=p'', q=q''} \]

\[ = \lim_{N \to \infty} e^{\nu T/2} \int \exp\left\{ i \sum x_{l+1/2} (p_{l+1} - p_{l}) - ik_{l+1/2} (q_{l+1} - q_{l}) \right\} \times \exp\left\{ -\frac{1}{2} \nu \delta \sum [\beta (q_{l} x_{l+1/2} + 1 + \beta^{-1} q_{l} \varepsilon)^2 - 2 \beta^{-1} q_{l} \varepsilon] \right\} \times \exp\left\{ -\frac{1}{2} \nu \delta \sum \beta^{-1} k_{l+1/2}^2 q_{l}^2 \right\} \prod_{l=0}^{N} \frac{dk_{l+1/2} dq_{l+1/2}}{(2\pi)^2} \prod_{l=1}^{N} dp_{l} dq_{l} \]

\[ =: e^{\nu T/2} \mathcal{N} \int \exp\left\{ i \int (x \dot{p} - k \dot{q}) dt \right\} \times \exp\left\{ -\frac{1}{2} \nu \int \{ \beta (q^{-1} x + 1 + \beta^{-1} q \varepsilon)^2 - 2 \beta^{-1} q \varepsilon + \beta^{-1} k^2 q^2 \} dt \right\} \times \mathcal{D}x \mathcal{D}k \mathcal{D}p \mathcal{D}q \]

\[ = e^{\nu T/2} \mathcal{N} \int \exp\left\{ i \int [(x - q - \beta^{-1} q^2 \varepsilon) \dot{p} - k \dot{q}] \right\} \times \exp\left\{ -\frac{1}{2} \nu \int (\beta q^{-2} x^2 - 2 \beta^{-1} q \varepsilon + \beta^{-1} k^2 q^2) dt \right\} \mathcal{D}x \mathcal{D}k \mathcal{D}p \mathcal{D}q \]

\[ = e^{\nu T/2} \mathcal{N} \int \exp\left\{ -i \int (q + \beta^{-1} q^2 \varepsilon) \dot{p} dt \right\} \times \exp\left\{ \frac{1}{2} \nu \int 2 \beta^{-1} q \varepsilon dt \right\} \exp\left\{ -\frac{1}{2} \nu \int [\beta^{-1} q^2 \dot{p}^2 + \beta q^{-2} \dot{q}^2] dt \right\} \mathcal{D}p \mathcal{D}q \]

with \( \mathcal{N} = T/\delta \). The Lie-Trotter product formula was used to go from the second to the third equality. The indices \( l+1/2 \) and \( l \) serve to emphasize that the temporal lattice points must not coincide for \( x, p \) or \( q, k \), respectively. (This would violate the Heisenberg uncertainty principle.) For the endpoints, the definitions \( p_0 := p' \), \( p_{N+1} := p'' \), \( q_0 := q' \) and \( q_{N+1} := q'' \) were made. Note that \( \exp\left\{ \frac{1}{2} \nu \delta (-\beta^{-1} \partial_q q^2 \partial_q) \right\} \exp\left\{ -ik(q - q') \right\} \approx \exp\left\{ -\frac{1}{2} \nu \delta \beta^{-1} k^2 q^2 \right\} \exp\left\{ -ik(q - q') \right\} \) only to first order in \( \delta \), but that is good enough for the path integral. In the second to last line, \( x \) was substituted by \( x + q - \beta^{-1} q^2 \varepsilon \), and the \( x \)- and \( p \)-integrations were carried out.

**The case** \( 0 < \beta \leq 1/4 \)

The Feynman-Kac-Stratonovich representation of the kernel of the operator \( \exp\{-\nu TA_x\} \) is derived in much the same way as before, and, with the same conventions for notation, it reads
\[
\begin{align*}
&\exp\{-\nu T A_\varepsilon\delta(p - p')\delta(q - q')\}_{p = p'', q = q''} \\
&= \exp\{-\nu T/2[\beta(-iq^{-1}\partial_p + 1 + \beta^{-1}q\varepsilon + \beta^{-1}q^{-1}\varepsilon)^2 - 2\beta q^{-2}\varepsilon \\
&+ 4ip\partial_q\varepsilon + 4\beta p^2 q^{-2}\varepsilon^2 - \beta^{-1}\partial_q q^2 \partial_q - 1 - 2\beta^{-1}q\varepsilon]\} \\
&\times \int \exp\{ix(p - p') - ik(q - q')\} \frac{dxdk}{(2\pi)^2} |_{p = p'', q = q''} \\
&= e^{\nu T/2} \lim_{N \to \infty} \left[ \exp\{-\nu \delta/2 \right. \\
&\left. \beta(-iq^{-1}\partial_p + 1 + \beta^{-1}q\varepsilon + \beta^{-1}q^{-1}\varepsilon)^2 - 2\beta q^{-2}\varepsilon - 2\beta^{-1}q\varepsilon\rangle \\
&\times \exp\{-\nu \delta/2(-i\beta^{-1}\partial_q q^2 \partial_q)\} \exp\{-\nu \delta/2 \cdot 4ip\partial_q\varepsilon\} \\
&\times \exp\{-\nu \delta/2 \cdot 4\beta p^2 q^{-2}\varepsilon^2\} \right]^N \\
&\times \int \exp\{ix(p - p') - ik(q - q')\} \frac{dxdk}{(2\pi)^2} |_{p = p'', q = q''} \\
&= \lim_{N \to \infty} e^{\nu T/2} \int \exp\{i \sum x_{t+1/2}(p_{t+1} - p_t) - ik_{t+1/2}(q_{t+1} - q_t)\} \\
&\times \exp\{-\nu \delta/2 \sum \beta(q_t^{-1}x_{t+1/2} + 1 + \beta^{-1}q_t\varepsilon + \beta^{-1}q_t^{-1}\varepsilon)^2 \\
&- 2\beta q_t^{-2}\varepsilon - 2\beta^{-1}q\varepsilon\} \}
&\times \exp\{-\nu \delta/2 \sum 4ip_t(-ik_{t+1/2})\varepsilon\} \\
&\times \exp\{-\nu \delta/2 \sum 4\beta p_t^2 q_t^{-2}\varepsilon^2\} \prod_{l=0}^{N} \frac{dk_{l+1/2}dx_{l+1/2}}{(2\pi)^2} \prod_{l=1}^{N} dp_idq_idk_i \\
&=: e^{\nu T/2} \mathcal{N} \int \exp\{i \int (\dot{x} - k\dot{q})dt\} \\
&\times \exp\{-\nu/2 \int \{\beta(q^{-1}x + 1 + \beta^{-1}q\varepsilon + \beta^{-1}q^{-1}\varepsilon)^2 - 2\beta q^{-2}\varepsilon - 2\beta^{-1}q\varepsilon \\
&+ \beta^{-1}k^2 q^2 + 4pq\varepsilon + 4\beta p^2 q^{-2}\varepsilon^2\} dt\} \mathcal{D}x \mathcal{D}k \mathcal{D}p \mathcal{D}q \\
&= e^{\nu T/2} \mathcal{N} \int \exp\{i \int [(x - q - \beta^{-1}q^2\varepsilon - \beta^{-1}\varepsilon)\dot{p} - k\dot{q}]dt\} \\
&\times \exp\{-\nu/2 \int (\beta q^{-2}x^2 - 2\beta q^{-2}\varepsilon - 2\beta^{-1}q\varepsilon + \beta^{-1}k^2 q^2 + 4pk\varepsilon \\
&+ 4\beta p^2 q^{-2}\varepsilon^2) dt\} \mathcal{D}x \mathcal{D}k \mathcal{D}p \mathcal{D}q
\end{align*}
\]
\[ = e^{\nu T/2} N \int \exp \left\{ -i \int (q + \beta^{-1}\varepsilon + \beta^{-1}q^2\varepsilon)\dot{p} dt + i \int 2\beta pq^{-2}\varepsilon \dot{q} dt \right\} \]
\[ \times \exp \left\{ \nu/2 \int (2\beta^{-1}q\varepsilon + 2\beta q^{-2}\varepsilon) dt \right\} \]
\[ \times \exp \left\{ -1/(2\nu) \int [\beta^{-1}q^2\dot{p}^2 + \beta q^{-2}\dot{q}^2] dt \right\} \mathcal{D}p\mathcal{D}q \]

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