On the relation between CT-Groups and NSP-Groups on finite Groups

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Abstract. A subgroup $H$ of a finite group is called $C$-normal if there exist a normal subgroup $N$ of $G$ such that $G=HN$ and $H\cap N$ is subgroup of the core $H$ in $G$. A subgroup $H$ of $G$ is called nearly $S$-permutable in $G$ if for every prime $p$ such that $(p, |H|) = 1$ and for every subgroup $K$ of $G$ containing $H$ the normalizer $N_K(H)$ contains some Sylow $p$-subgroup of $K$. The aim of this paper is to study these concepts and show that there one classes where these concepts coincide and classes for which these concepts are different. Moreover, some examples and properties related to these subgroups are discussed in this paper.

1. Introduction

All considered groups are assumed to be finite. For notations are basic concepts we shall adapt the references[4],[8], and [13]. Several algebraic properties for groups are recognized from the way that specific subgroups are cited in the group. Normality is one of such properties. For instance when $p$-Sylow subgroups are normal in the group then the group is nilpotent. Among the amazing properties of normal subgroups is the following: If $N$ is a normal subgroup of $G$ and $H$ any subgroup of $G$ then $NH=HN$. It was observed that there are subgroups that are not normal but still commute with every subgroup. These subgroups where first introduced by Oystein Ore in 1937 who called them quasinormal and later they were known as permutable. Normality as well as permutability have been studied and generalized by several authors. In this paper we focus on two such generalizations. Mainly, $c$-normal subgroups and nearly $S$-permutable subgroups which were introduced in [4] and in [14] respectively. While, a normal is both $c$-normal and nearly $S$-permutable it will be of interest to have examples of groups (if exists) that are: $c$-normal but not nearly $S$-permutable, and subgroups that are nearly $S$-permutable but not $c$-normal. Analogous examples for the groups in which $c$-normality is transitive and groups in which nearly $S$-permutability is transitive are discussed in this paper.

2. C-normal subgroups and nearly S-permutable

First of all the basic concepts. Recall, that for if $H$ is a subgroup of a group $G$ and $g \in G$ then $H^g = g^{-1}Hg$, and $H_g = \cap \{H^g : g \geq G \}$ is the largest normal subgroup of $G$ contained in $H$.

Lemma 2.1. (see [10]) Let $H$ be a non-empty subset of group $G$, the normalizer of $H$ in $G$ is the set $N_G(H) = \{g \in G : gH = Hg \}$.

Definition 2.2. (see [14]) Let $G$ be a group. A subgroup $H$ of $G$ is $c$-normal in $G$ if there exists a normal subgroup $N$ of $G$ such that $HN=G$ and $H \cap N \leq H_G$. $H$ $c$-norm $G$ denotes $H$ is $c$-normal in $G$.

Next we introduce the concept of nearly $S$-permutable subgroups.
A subgroup $H$ of $G$ is called nearly $S$-permutable in $G$ if for every prime $p$ such that $(p,|H|)=1$ and for every subgroup $K$ of $G$ containing $H$ the normalizer $N_K(H)$ contains some $p$-Sylow subgroup of $K$. We shall write $H \triangleleft_{nsp} G$ to denote that $H$ is nearly $S$-permutable in $G$.

**Remark 2.4.** Let $N$ be a normal subgroup of a group $G$. Then $N^g=g^{-1}Ng=N$. That is $N_g=\cap \{N^g : g \in G\}$, where $N^g=g^{-1}Ng=N$. Moreover, $G\cap N \leq N_g = N$. So, $N$ is $c$-normal in $G$. On the other hand, $N \leq G$ implies $N \leq K$, for any subgroup $K$ of $G$ that contains $N$. Hence, $N_g(N)=G$, and $N_{c}(N)=K$. Therefore, $N \triangleleft_{nsp} G$. So, any normal subgroup is an example of a subgroup that is both $c$-normal and nearly $S$-permutable at the same time.

A $c$-normal subgroup need not be nearly $S$-permutable. The symmetric group on three letters $S_3$ is the group of smallest order which has a subgroup that is $c$-normal but not nearly $S$-permutable as the following example shows:

**Example 2.5.** Let $S_3$ be the symmetric group on three letters and $P$ any of the 2-Sylow subgroups in $S_3$. If we let $N=A_4$ be the alternating subgroup of $S_3$. Then $PA_4=S_3$ and $P \cap A_4 = 1 \leq P_3 = 1$. Hence, $P$ is $c$-normal in $S_3$. To see that $P$ is not nearly $S$-permutable in $S_3$. Note that $N_{c}(P)=P$. So, if we let $K=S_3 \geq P$ and for $q=3$, the 3-Sylow subgroup of $K$ is $A_4$. Which is not contained in $N_{c}(P)=P$. Therefore, $P$ is not nearly $S$-permutable in $S_3$.

A nearly $S$-permutable subgroup need not be $c$-normal. The faithful semidirect product of the elementary abelian group $E_8$ and the cyclic group $C_4$ of size 4 is the group of smallest which has a nearly $S$-permutable subgroup that is not $c$-normal.

**Example 2.6.** Let $E_8 \cong C_2 \times C_2 \times C_2 = \{(x,y,z): x,y,z \in C_2\}$ be the elementary 2-group which is a vector space over the field $C_2$. Consider $G=E_8 \rtimes C_2$ which has the following presentation:

$$G=\langle x,y,z,a:x^2=y^2=z^2=a^4=e, [x,y]=[y,z]=[x,z]=[x,a]=e, y^a=x\ast y, z^a=y\ast z \rangle.$$ Then $G$ is 2-group of order 32. The definition of nearly $S$-permutability implies that every subgroup in a $p$-group should be nearly $S$-permutable. Hence to find a subgroup in $G$ that is not $c$-normal.

The list of some facts and remarks about the group $G$.

1. $G$ has 50 subgroups 12 of which are normal.

2. From the first Sylow theorem that each subgroup of $G$ of order 16 will be normal. There are three subgroups of order 16 as follows:

$$M_1=\langle x,y,z,a \rangle \cong C_2 \times C_2, M_2=\langle x,y,a \rangle \cong (C_2 \times C_2) \rtimes_{\varphi} C_2,$$

3. From the presentation of the group $G$ that the subgroup $H=\langle y \rangle$ is not normal in $G$, and the order of $|H|=2$. So, the normal core of $G$ in $H$ is the identity subgroup.

For the subgroup $H$ to be subnormal we must find a normal subgroup $N$ of $G$ such that $HN=G$ and $H \cap N \leq H_{G}$. That means, $N \cap H=1$. Hence $|N|=16$. But $y$ belongs to each such normal subgroups. Hence $HN=G$ holds only for $N=G$. But in this case $H \cap N=H \cap G$, hence $H$ is not subgroups in $H_{G}$. Therefore, $H$ is not $c$-normal in $G$.

**3. CT-groups and NSPT-groups**

In general a subgroup on $c$-normal subgroup need not be $c$-normal. For instance, in the alternating group on four letters $A_4$ the 2-Sylow subgroup $P= \langle C_2 \times C_2 \rangle$ is the only proper nontrivial normal subgroup in $A_4$. Therefore, $P$ is $c$-normal in $A_4$. Since, $P$ is abelian then each one of the proper subgroups of order 2 in $P$ will be normal in $P$ and therefore it will be $c$-normal in $P$ but not $c$-normal in $A_4$. Nearly $S$-
permutability, like \(c\)-normality, need not be transitive among subgroups. To see this we consider the following example:

**Example 3.1.** Let \( G = A_4 \) be the alternating group on four letters. Let \( P \) be the 2-Sylow subgroup of \( G \). Then it is clear that \( P \vartriangleleft G \). Now if we let \( H \) be any of the subgroups of order 2 in \( P \). Then, \( H \) is Nearly \( S \)-permutable in \( P \), and \( P \) is nearly \( S \)-permutable in \( G \). But \( H \) is not nearly \( S \)-permutable in \( G \). To see this: let us consider \( H \leq G \), and \( p = 3 \) be a prime number such that \((|H|, p) = 1\). It is well known that \( N_G(H) = P \). Which doesn't contain a 3-Sylow subgroup of \( G \). Therefore \( H \) is not nearly \( S \)-permutable in \( G \).

Finite group in which \( c\)-normality transitive is called \( CT\)-group. If nearly \( S \)-permutability is transitive in \( G \) then \( G \) is called \( NSPT\)-group. These two concepts were introduced in [2] and [3] respectively.

**Theorem 3.2.** Let \( P \) be sylow \( p \)-subgroub in \( G \). If \( P \) is nearly \( S \)-permutable in \( G \), then \( P \) is normal in \( G \).

**Proof:** The theorem holds trivially for \( p \)-groups. Now let \( p \in \text{syl}_p(G) \) and assume that \( p \) is nearly \( S \)-permutable in \( G \).

Let \( q \) be any prime number such that \( P \cap |G| \) and \( q \neq p \). Now \( G = K \) and \( P \leq G \) and \((q,p) = 1\) hence if \( Q \in \text{syl}_q(G) \) implies \( Q \leq N_G(P) \). On the other hand (by lemma 1.1) \( P \leq N_G(P) \). Hence, \( N_G(P) \) contains all sylow \( p \)-subgroups of \( G \). But \( G = (P : p \in \text{syl}_p(G)) \). Hence \( G \leq N_G(P) \) implies \( G = N_G(P) \) and \( P \vartriangleleft G \).

**Question 3.3.** If the sylow \( p \)-subgroubs of \( G \) is \( C \)-normal in \( G \) does it imply that \( P \) normal in \( G \)?

**Answer:** \( C \)-normality for a sylow \( p \)-subgroub doesn't imply normality as the following example shows:

**Example 3.4.** The sylow 2-subgroub \( P = \langle (12) \rangle \) of the symmetric group \( S_3 \) is \( C \)-normal since \( PA_3 = S_3 \) and \( P \cap A_3 = 1 \leq P \). Hence \( P \) is \( C \)-normal, but \( P \) is not normal in \( S_3 \).

**Problem 3.5.** Is there any relation between \( CT\)-groups and \( NSPT\)-groups in general. Give an example (if exists) of a \( CT\)-group that is not \( NSPT\)-group . Give an example (if exists) of a \( NSPT\)-group that is not \( CT\)-group.

**4. Conclusion**

In this Remark 2.4 that \( c\)-normal subgroups and nearly \( S \)-permutable subgroups could be the same. In general, neither \( c\)-normality implies nearly \( S \)-permutability nor nearly \( S \)-permutability implies \( c\)-normality as shown in examples[2.5 and 2.6] respectivly.

**5. Suggested Problems**

In order to formulate our problems we shall need the following concepts:

**Definition 5.1.** (see [6,Definition 1.2]) Let \( H \) be a subgroup of a group \( G \). Then we say that \( H \) is nearly \( S \)-permutably embedded in \( G \) if \( G \) has a subgroup \( T \) and an \( S \)-permutably embedded subgroup \( C \leq H \) such that \( HT = G \) and \( T \cap H \leq C \).

**Problem 5.2.** Is it true that: if a subgroup \( H \) is nearly \( S \)-permutably embedded in \( G \) then \( H \) is nearly \( S \)-permutable? What about the converse?
Definition 5.3. (see [7]) Let \( H \) be a subgroup of a group \( G \). Then \( H \) is \( SQ \)-supplemented in \( G \) if \( G \) has a subgroup \( T \) and an \( S \)-permutably embedded subgroup \( C \leq H \) such that \( HT=G \) and \( T \cap H \leq C \).

Problem 5.4. Is it true that: if a subgroup \( H \) is \( SQ \)-supplemented in \( G \) then \( H \) is nearly \( S \)-permutable? What about the converse?

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7. References

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