Redundant poles of the $S$-matrix for the one-dimensional Morse potential

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Abstract We analyze the structure of the scattering matrix, $S(k)$, for the one-dimensional Morse potential. We show that, in addition to a finite number of bound state poles and an infinite number of antibound poles, there exist infinite redundant poles, on the positive imaginary axis, which do not correspond to either of the other types. We explain in detail the role of these redundant poles, in particular when they coincide with the bound poles. This can be solved analytically and exactly. In addition, we obtain wave functions for all these poles and ladder operators connecting them.

1 Introduction

As is well known when we deal with non-relativistic quantum scattering, and under some causality conditions [1], the scattering matrix in the momentum representation, $S(k)$, has an analytic continuation to a meromorphic function on the complex plane. Its isolated singularities are poles, which are classified as bound state poles, in one to one correspondence with the bound states, antibound poles and resonance poles. However, for some types of potentials other kinds of singularities may arise, like branch cuts and redundant poles. The latter do not correspond to physical states and have been studied already long ago [2–4]. This work has been continued by some authors and has inspired a bunch of result in scattering theory either for Hermitian or for non-Hermitian Hamiltonians [5–11].

Very recently, Moroz and Miroshnichenko [12,13] had exhaustively studied the analytic behavior of the scattering matrix $S(k)$ corresponding to the radial Schrödinger equation with potential

$$V(r) = \pm V_0 e^{-r/a},$$

(1.1)

with $V_0 > 0$ and $a > 0$. The authors find a series of redundant poles of $S(k)$.

These results have in part motivated the discussion presented in this paper. We have started with an exactly solvable one-dimensional potential, which is the Morse potential, and study the properties of the scattering matrix, $S(k)$, produced by it. In this case, $S(k)$ can be analytically continued to a meromorphic function on the whole complex plane with an infinite number of simple poles as the only singularities. These poles can be classified into three kinds: (i) A finite number of bound state poles located on the positive imaginary
semi axis. (ii) An infinite number of poles on the negative imaginary semiaxis, which are usually called the virtual or antibound poles. (iii) Finally, an infinite number of simple poles located along the positive imaginary semiaxis. In exceptional cases, some of these poles may coincide with the bound state poles, as we shall see. No resonance poles are present.

Then, we may consider the energies $E_i = (\hbar k_i)^2/(2m)$, which correspond to each of the poles $k_i$, and the eigenvalue problem $H \psi_i(x) = E_i \psi_i(x)$ with suitable boundary conditions. We have obtained the eigenfunctions $\psi_i(x)$ for each of the poles $k_i$. While these eigenfunctions are square integrable for the bound state poles, they are not for all the others. There is also a special point of the spectrum at $E = 0$ that some authors call half-bound state [14].

In previous papers, our group has analyzed ladder operators for exactly solvable models [15–17]. We do the same here. In general, we obtain three independent series of wave functions related by ladder operators. One includes wave functions of both bound and antibound poles. For the wave functions of redundant poles, there exist two independent series and the wave functions inside each of the series are obtained from each other via these ladder operators. We study two exceptional cases, in which bound state and redundant poles may coincide, which produce some unexpected behavior of the ladder operators.

This article is organized as follows: In Sect. 2, we briefly recall the method to solve the one-dimensional Schrödinger equation, so as to obtain the wave functions for bound states and scattering states, with stressing on their asymptotic behavior, which is quite relevant for posterior analysis. In Sect. 3, we obtain the scattering matrix $S(k)$ and its singularities. Section 4 is devoted to the construction and properties of ladder operators. We finish this presentation with some concluding remarks.

2 The Morse potential

We begin with the stationary one-dimensional Schrödinger equation that comes after the Morse potential. This is (we are taking units such that $\hbar = 2m = 1$)

$$\left[ -\frac{d^2}{dx^2} + e^{-2x} - 2(A + 1/2)e^{-x} \right] \psi(x) = E \psi(x), \quad A > 0, \quad -\infty < x < \infty, \quad (2.1)$$

where $E$ is the energy and

$$V_M(x) = e^{-2x} - 2(A + 1/2)e^{-x} \quad (2.2)$$

is the particular form of Morse potential that we will consider along this work, see Fig. 1. Notice that if in Morse potential (2.2) we take $A = -1/2$ and restrict to the positive axis with an infinite barrier at the origin, we get central repulsive potential (1.1) studied in [13]. If, besides the previous modifications, we add a complex displacement $x \rightarrow x + i \pi/2$, we obtain the attractive version of potential (1.1) considered in [12]. Thus, the choice of the Morse potential is particularly interesting in order to study redundant poles.

The exact solvability of this quantum model is based on the fact that Schrödinger equation (2.1) may be transformed into a confluent hypergeometric equation after a sequence of changes of variables. The result is as follows. Let us make the change of variable as well as function in the form

$$\psi(x) = e^{-\sqrt{-E}x}e^{-e^{-x}}y(z), \quad z = 2e^{-x}. \quad (2.3)$$

Then, we get from (2.1),

$$z y''(z) + \left( 2\sqrt{-E} + 1 - z \right) y'(z) - \left( \sqrt{-E} - A \right) y(z) = 0. \quad (2.4)$$
Now, let us compare (2.4) with the standard form of the confluent hypergeometric function, which is
\[ z y''(z) + (c - z) y'(z) - a y(z) = 0. \] (2.5)
We see that these two equations are identical with the identification
\[ c = 1 + 2\sqrt{-E}, \quad a = -A + \sqrt{-E}. \] (2.6)
As is well known, Eq. (2.5) has two linearly independent solutions in terms of the first kind Kummer function:
\[ y_1(z) = \, _1 F_1(a; c; z), \quad y_2(z) = z^{1-c} \, _1 F_1(a + 1 - c; 2 - c; z). \] (2.7)
Note that,
\[ a' := a + 1 - c = -A - \sqrt{-E}, \quad c' := 2 - c = 1 - 2\sqrt{-E}. \] (2.8)
We remark that for \( c \) integer, only one of these solutions exists, while a second independent solution is the product of a series times a logarithm. Thus, if \( c \) is not an integer, or equivalently, if \( 2\sqrt{-E} \) is not an integer, the general solution of (2.1) has the form
\[ \psi(x) = C_1 \psi_1(x) + C_2 \psi_2(x), \] (2.9)
with
\[ \psi_1(x) = e^{-\sqrt{-E} x} e^{-e^{-x}} \, _1 F_1(-A + \sqrt{-E}; 1 + 2\sqrt{-E}; 2 e^{-x}), \] (2.10)
\[ \psi_2(x) = 2^{-2\sqrt{-E}} e^{-\sqrt{-E} x} e^{-e^{-x}} \, _1 F_1(-A - \sqrt{-E}; 1 - 2\sqrt{-E}; 2 e^{-x}), \] (2.11)
as we may check after a straightforward calculation. Remark that if we change \( \sqrt{-E} \rightarrow -\sqrt{-E} \) in above solutions (2.10)–(2.11), they will be interchanged.

2.1 Bound states

First of all we compute the solutions of the eigenvalue problem associated with the Morse equation which have square-integrable eigenfunctions. One way to obtain those solutions is to analyze the behavior of the eigenfunctions when \( E < 0 \) (see Fig. 1) in the limits \( x \rightarrow \pm \infty \).

- \( x \mapsto +\infty \) In this case, and taking into account that \( _1 F_1(a, c; 0) = 1 \), the asymptotic behavior of general solution (2.9) is,
\[ \psi(x) \approx C_1 e^{-\sqrt{-E} x} + C_2 2^{-2\sqrt{-E}} e^{\sqrt{-E} x}, \quad x \rightarrow \infty. \] (2.12)
Obviously, the second term in the right-hand side of (2.12) grows exponentially for $E < 0$, so that its contribution cannot be square integrable. Consequently, $C_2 = 0$. The first term decays exponentially provided that $E < 0$, which is compatible with square integrable solutions. In consequence, we will keep the solution $ψ(x) = C_1 ψ_1(x)$ and check its behavior at $x \mapsto -∞$.

$•$ $x \mapsto -∞$ We have made the change of variables $z = 2e^{-x}$, so that this limit is equivalent to taking the limit $z \mapsto +∞$. For large values of $z$, the asymptotic form of $1F_1(a, b; z)$ is

$$1F_1(a, c; z) \approx \frac{Γ(c)}{Γ(a)} e^{z} z^{a-c}, \quad z \mapsto +∞,$$  \hspace{1cm} (2.13)

so that

$$ψ_1(x) \approx \frac{Γ(1 + 2\sqrt{-E})}{Γ(-A + \sqrt{-E})} 2^{-(1+A) - \sqrt{-E}} e^{e^{-x} + (1+A)x}, \quad x \mapsto -∞.$$  \hspace{1cm} (2.14)

The conclusion is clear after (2.14). The solution $ψ_1(x)$ cannot be square integrable unless the coefficient in (2.14) vanishes, for which the only possibility is that

$$a = -A + \sqrt{-E} = -n, \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (2.15)

This gives the following possible values for the energy:

$$E_n = -(A - n)^2, \quad n = 0, 1, \ldots, N; \quad N = [A],$$  \hspace{1cm} (2.16)

where $[A]$ denotes the integer part of $A$ less than $A$ (in other words, here $[A] < A$).

The fact that $a$ is now a negative integer shows that the series giving the Kummer function truncates and becomes a polynomial. Thus, we have $N$ bound states, which may be expressed in terms of the truncated Kummer function, or alternatively, in terms of the Laguerre-associated polynomials as:

$$ψ_n(x) = C_n e^{-(A-n)x} e^{-e^{-x}} 1F_1(-n; 1 + 2(A - n); 2e^{-x})$$

$$= C_n e^{-(A-n)x} e^{-e^{-x}} L_n^{2(A-n)}(2e^{-x}),$$  \hspace{1cm} (2.17)

where $C_n$ are normalization constants. Observe that the eigenfunctions $ψ_n(x)$ are products of functions exponentially decreasing as $x \mapsto ±∞$ times a polynomial, hence square integrable. Also note that for bound states $c = 1 + 2(A - n) > 0$, so that the Kummer function in (2.17) is well defined. For values $A \leq 0$, there will be no bound state of the Morse potential.

2.2 Scattering wave functions

The asymptotic analysis of the scattering wave functions may be per se interesting, although our aim lies on its applications in the search of scattering resonances and the scattering matrix. Let us analyze the asymptotic behavior of solutions of (2.1) for positive energy values, $E > 0$. In this point it is better to work in the momentum variable $\sqrt{-E} = ik$ with $k > 0$; then, the explicit form of the two independent solutions of (2.1) is given by

$$ψ_1(x) = e^{-ikx} e^{-e^{-x}} 1F_1(-A + ik; 1 + 2ik; 2e^{-x}),$$  \hspace{1cm} (2.18)

$$ψ_2(x) = e^{ikx} e^{-e^{-x}} 2^{-2ik} 1F_1(-A - ik; 1 - 2ik; 2e^{-x}).$$  \hspace{1cm} (2.19)
In the limit \( x \to +\infty \), the Kummer functions, both in (2.18) and (2.19), go to one, so that the asymptotic form of both solutions for large positive values of \( x \) is given by
\[
\psi_1(x \to +\infty) \approx e^{-ikx}, \quad \psi_2(x \to +\infty) \approx e^{-i2k\log 2 e^{ikx}}. \tag{2.20}
\]

After (2.20), we may consider \( \psi_1(x) \) and \( \psi_2(x) \) as incident (coming from the right) and reflected wave (going back toward the right), respectively.

In the limit \( x \to -\infty \), we have to take into account relation (2.13), so as to obtain the following asymptotic forms:
\[
\begin{align*}
\psi_1(x \to -\infty) &\approx \frac{\Gamma(1+2ik)}{\Gamma(-A+i k)} 2^{-(1+A)-ik} e^{2e^{-x}+(1+A)x}, \\
\psi_2(x \to -\infty) &\approx \frac{\Gamma(1-2ik)}{\Gamma(-A-i k)} 2^{-(1+A)-ik} e^{2e^{-x}+(1+A)x}. \tag{2.21}
\end{align*}
\]

Observe that, in the limit \( x \to -\infty \), both solutions diverge, although their functional form is similar, save for the multiplicative constants. This makes it possible to choose a linear combination of both solutions so that its asymptotic behavior for large negative values is equal to zero. This arrives to a relation between the coefficients \( C_i, i = 1, 2 \) in (2.9) as
\[
C_2 = -\frac{\Gamma(-A-i k) \Gamma(1+2ik)}{\Gamma(-A+i k) \Gamma(1-2ik)} C_1. \tag{2.22}
\]

Thus, we have chosen solutions with the property \( \psi(x \to -\infty) = 0 \). Due to the form of the Morse potential, this is physically reasonable.

After the above considerations, we may consider \( \psi_1(x) \) and \( e^{2ik\log 2} \psi_2(x) \) as the incoming and the outgoing wave functions, respectively, which justifies the following the change of notation: \( \psi_1(x) := \psi_{\text{IN}}(x) \) and \( e^{2ik\log 2} \psi_2(x) := \psi_{\text{OUT}}(x) \). In this notation, general solution (2.9) becomes
\[
\psi(x) = C_{\text{IN}} \psi_{\text{IN}}(x) + C_{\text{OUT}} \psi_{\text{OUT}}(x), \tag{2.23}
\]
which after (2.22) gives the scattering matrix \( S(k) = C_{\text{OUT}}/C_{\text{IN}} \) with the form:
\[
S(k) = -\frac{\Gamma(-A-i k) \Gamma(1+2ik)}{\Gamma(-A+i k) \Gamma(1-2ik)} e^{-2ik\log 2}. \tag{2.24}
\]

Notice that for \( k \) real, \( S(k) \) has modulus one. The phase shift in the momentum representation is given by
\[
S(k) = e^{i2\delta(k)} \implies \delta_A(k) = -\frac{i}{2} \log \left( -\frac{\Gamma(-A-i k) \Gamma(1+2ik)}{\Gamma(-A+i k) \Gamma(1-2ik)} \right) - k \log 2. \tag{2.25}
\]

In the next section, we discuss the analytic properties of the function \( S(k) \).

3 The poles of the scattering matrix \( S(k) \)

The function \( S(k) \) admits an analytic continuation for \( k \) complex, which has some singularities, which in our case are simple poles. One of the main objectives of the present paper is the determination and the analysis of properties of these poles, which can be done after Eq. (2.24). These poles are of four different kinds \([1,12,18]\):

- Each of the bound states yields to a simple pole on the positive imaginary semiaxis. In the present model, only a finite number of such poles exists.
Fig. 2  Phase shifts ($\delta(k)$, left) and its derivative ($\delta'(k)$, right) for different values of $A$: $A = 1.5$ (blue, continuous), $A = 1.9$ (green, dashed) and $A = 2.1$ (red, dotted)

- Simple poles on the negative part of the imaginary semiaxis are associated with virtual states, also called antibound states. Only those antibound states, located at values of $ik$ with $k$ near zero, may have observable effects.
- Pairs of poles on the lower half-plane, symmetrically located with respect to the negative semiaxis, correspond to scattering resonance states, in other words to quantum unstable states. These poles are often infinite in number, and in principle, they may have arbitrary multiplicity. Poles symmetrically placed with respect to the imaginary axis have the same multiplicity. However, poles with multiplicities higher than two are not computable, in general. Our model does not have resonance poles.
- In addition, poles on the positive imaginary semiaxis, which do not correspond to bound states, may exist. These are called redundant poles of the $S$-matrix. We shall show that the model under study has redundant poles and study some of their properties.

After the properties of the $\Gamma(z)$ function, we have to look for the singularities in the numerator of (2.24); nevertheless, we have to check whether or not these poles coincide with poles in the denominator, what may lead to regular points. One of the characteristics of the present model is the absence of resonance poles. The absence of resonances is a consequence of the form of function (2.25), although it can be surmised after the behavior of the phase shift $\delta(k)$. Resonances are usually identified by a sudden and abrupt change of $\delta'(k)$ at given energies [18] as well as a sharp maximum of $\delta'(x)$, something that is not observed here, as we may see after Fig. 2 (in which we have also represented the derivative $\delta'(k) = d\delta(k)/dk$.

The singularities of $S(k)$ are simple poles on the imaginary axis. Then, we have three possibilities depending on the value of the parameter $A$ of the Morse potential, for which we have split its discussion into the next three subsections.

3.1 General case: $A$ is neither integer nor half-integer

For $A > 0$ here are two series of simple poles that have a different role (see Fig. 3, left).
(a) **First series: bound–antibound poles.** These poles are located at the points:

$$-A - ik_1 = -n_1 \implies k_1 = i(A - n_1), \quad n_1 = 0, 1, 2, \ldots.$$ (3.1)

About these series of poles, we may say the following.

(a.i) Assume they are of form (3.1) with $(A - n_1) > 0$. We have a finite number of poles of this type, $k_1(n_1)$, for $n_1 = 0, \ldots, \lfloor A \rfloor$, on the positive imaginary axis. They are in one-to-one correspondence with bound states of negative energies (recall (2.16) of...
section 2.1)

\[ E(n_1) = k_1^2(n_1) = -(A - n_1)^2. \]

(a.ii) The other possibility is that being \( k_1 \) of form (3.1), one has that \((A - n_1) < 0\). Then, there are infinite values of poles, \( k_1(n_1) \), for \( n_1 = [A] + 1, [A] + 2, \ldots \), which are located on the negative imaginary axis. These are the so-called antibound or virtual state poles [19].

The regular wave function, which corresponds to the energies \( E(n_1) \), can be trivially obtained from (2.19) and takes the form:

\[
\psi_{n_1}(x) = e^{-(A-n_1)(x-2\log 2)} e^{-e^{-x}} \, _1F_1(-n_1; 1 + 2(A - n_1); 2e^{-x}). \tag{3.2}
\]

We can see that for \( n_1 \) positive integer and \( A - n_1 > 0 \), \( \psi_{n_1}(x) \) is identified with the solution \( \psi_1 \) in (2.17) and it is square integrable. Thus it corresponds to a bound state. For \( n_1 \) positive integer and \( A - n_1 < 0 \), \( \psi_{n_1}(x) \) is identified with \( \psi_2 \) in (2.20), and it is not square integrable and satisfies pure outgoing conditions. Thus, it corresponds to a so-called antibound (or virtual) state.

In the case \( A < 0 \), there will be no bound poles, while the infinite series of antibound poles will be at the negative values starting at \( A: k_1 = i(A - n_1), \ n_1 = 0, 1, 2, \ldots \)

(b) **Second series: redundant poles**. In addition to the previous series of bound–antibound poles, we have the so-called redundant poles, which are located at the following points on the imaginary axis:

\[ 1 + 2ik_2 = -n_2 \implies k_2 = \frac{i}{2} (1 + n_2), \quad n_2 = 0, 1, 2, \ldots \tag{3.3} \]

By reasons that will be clarified soon, we split this series into two infinite subseries depending on whether \( n_2 \) is even or odd, as follows:

(b.i) Even series: \( k_2 = \frac{i}{2} (1 + 2n_2), \ n_2 = 0, 1, 2, \ldots \)

(b.ii) Odd series: \( k_2 = i(1 + n_2), \ n_2 = 0, 1, 2, \ldots \)

Contrarily to the series of bound–antibound poles, the values \( k(n_2) \) of the redundant poles are independent of the parameter \( A \), as it is clear from (3.3). For \( A \) neither integer nor half-integer, there is no overlapping between redundant poles and bound–antibound poles (see Fig. 3, left). Then, the regular wave functions for the energies of the redundant poles \( E(n_2) \) have the following form:

\[
\psi_{n_2}(x) = e^{-\frac{n_2+1}{2}(x-2\log 2)} e^{-e^{-x}} \, _1F_1 \left( -A + \frac{1}{2} (n_2 + 1); n_2 + 2; 2e^{-x} \right). \tag{3.4}
\]

We discard the second solution, since it has a logarithmic divergence. Thus, in the general case (\( A \) neither integer nor half-odd), we can say that the redundant poles do not have to do with physical properties of the states, and they only affect the mathematical character of the general solution: One of the independent solutions can be chosen hypergeometric, but the second one must be logarithmic function. As solutions (3.4) are neither square integrable nor they satisfy pure outgoing conditions, this could be the origin of the term “redundant” for such poles [12, 13]. Next, we analyze the situation for the other values of \( A \).

### 3.2 \( A \) is an integer number, \( A = N \)

If \( A \) is either integer or half-integer, an overlapping between one subseries of redundant poles and the bound–antibound poles occurs (see Fig. 3, center and right). For \( A \) integer, this means
that the finite number of bound poles \((a.i)\) will coincide with some redundant poles of the odd series \((b.ii)\). This introduces some modifications on the map of poles, which may be summarized as follows (we will assume the most interesting case \(A = N > 0\), at the end of this subsection we will briefly comment the situation \(A = N \leq 0\)):

(a) The series of bound–antibound poles collapses and only the finite bound poles survive. These poles are located at the positive imaginary points

\[
k_1 = i(N - n_1), \quad n_1 = 0, 1, 2, \ldots, N - 1.
\]

This has important consequences: Since \(N - n_1 > 0\), we have exactly \(N\) bound state poles, or equivalently, \(N\) bound states with energies \(E(n_1) = k_1^2(n_1)\), \(n_1 = 0, 1, 2, \ldots N - 1\), and no antibound poles. The regular wave function for the bound states has the same form of generic case (3.2) (up to normalization):

\[
\psi_{n_1}(x) = e^{-(N-n_1)(x-2\log 2)} e^{-e^{-x}} \binom{1}{1 + 2(N - n_1); 2e^{-x}}, \quad (3.6)
\]

for \(n_1 = 0, \ldots, N - 1\).

(b) The even subseries of redundant poles will conserve the same values, but the odd subseries will have a quite different behavior.

(b.i) Even series: \(k_2 = \frac{i}{2} (2n_2 + 1), n_2 = 0, 1, 2, \ldots\). There is nothing new concerning the even series of redundant poles, which go as in the general case (\(A\) neither integer nor half-integer). Associated with these poles there is only one regular wave function of hypergeometric type, which is

\[
\psi_{n_2}(x) = e^{-\frac{2n_2+1}{2}(x-2\log 2)} e^{-e^{-x}} \binom{1}{1/2 (2n_2 + 1); 2n_2 + 2; 2e^{-x}}. \quad (3.7)
\]

(b.ii) Odd series: \(k_2 = i(1 + n_2), n_2 = 0, 1, 2, \ldots, N - 1\). Here, only the odd redundant poles which exactly match with the bound poles remain. Since the corresponding hypergeometric solutions are square integrable, we can affirm that no odd redundant poles exist in this case (Fig. 3, center).

In summary, if the value of \(A\) is a natural number \(N\), the bound poles remain (and coincide with some odd redundant poles). The antibound poles and the rest of odd redundant poles disappear (these two types of poles compensate in some way). The even redundant poles are not affected and have the same values as in the general case.

In the special case \(A = N \leq 0\), there will be no bound nor antibound poles, and from the odd redundant poles there will survive only the first ones located at \(k = in, n = 1, 2, \ldots, N\).

3.3 \(A\) is a half-odd number, \(A = (N - \frac{1}{2})\)

Now, the map of poles is similar to the precedent case with \(A\) integer (see Fig. 3, right), although we may observe some minor still noteworthy differences, as seen from the following analysis.

(a) In the series of bound–antibound poles only the bound poles remain. They are located at the points

\[
k_1 = i(N - n_1 - \frac{1}{2}), \quad n_1 = 0, \ldots, N - 1.
\]

All the antibound poles disappear. The regular square-integrable wave functions for these bound states have the form

\[
\psi_{n_1}(x) = e^{-(N-n_1-1/2)(x-2\log 2)} e^{-e^{-x}} \binom{1}{1; 2(N - n_1); 2e^{-x}}. \quad (3.8)
\]
(b) The two subseries of redundant poles are

(b.i) Even series: \( k_2 = i (n_2 + \frac{1}{2}) \), \( n_2 = 0, 1, 2, \ldots, N - 1 \). The series is finite and redundant poles coincide with the bound poles. In this case the rest of poles of the even subseries are suppressed.

(b.ii) Odd series: \( k_2 = i (n_2 + 1) \), \( n_2 = 0, 1, 2, \ldots \). This subseries of redundant poles remain unaltered with respect to the general case.

In Fig. 4 it is shown the poles of the scattering matrix for the last two special cases, where bound and some redundant poles coincide: \( A \) integer and \( A \) half-integer. The case of \( A \) being a negative half-integer has similar consequences as in the negative integer case above mentioned.

3.4 Comments on the redundant poles

There are two natural questions concerning the “redundant” poles.

- The first one is about the origin of such poles. They seem not physical since, in the generic case, they do not depend on the values of the parameter \( A \) of the potential. Although redundant poles are in the positive imaginary \( k \)-axis, they cannot be identified as bound states since the corresponding regular wave functions are not square integrable. In fact, the values \( k \) of redundant poles correspond to values of the parameter \( c \) (see (2.8)) where the Kummer function is not well defined. This has to do with the fact that the scattering matrix \( S(k) \) was defined in terms of incoming and outgoing wave functions \( \psi_{IN}, \psi_{OUT} \), but these functions were identified with the two independent solutions \( \psi_1 \) and \( \psi_2 \) given in (2.18)–(2.19) in terms of Kummer functions. Therefore, it is reasonable that \( S(k) \) fails to be meaningful when the hypergeometric functions \( \psi_1 \) or \( \psi_2 \) cease to span the general solution.

- The second question raised is about the behavior of the poles of \( S(k) \) for the special cases \( A = N \) and \( A = N - 1/2 \). In this point, we must recall that for each pole of \( S(k) \) in the imaginary \( k \)-axis there is a zero of \( S(k) \) at the complex conjugate point \( k^* \). Take, for instance, the case \( A = N \) that can be seen as a limit of the general case when \( A \to N \). In this limit we get the finite bound poles

\[
k_1(n_1) = i(N - n_1), \quad n_1 = 0, 1, \ldots, N - 1.
\]

The antibound poles are the continuation of this sequence in the negative imaginary \( k \) axis:

\[
k_1(n'_1) = -in'_1, \quad n'_1 = 0, 1, \ldots, \infty.
\]

While the even redundant subseries of poles are positioned at the positive imaginary \( k \) axis:

\[
k_2(n_2) = in_2, \quad n_2 = 0, 1, \ldots, \infty.
\]

It is clear that the even redundant poles \( k_2(n_2) \) and the antibound poles \( k_1(n'_1) \) are complex conjugate of each other. In the limit, this leads to a collapse because at each of these points the \( S(k) \) matrix has both a simple zero and a simple pole and they become regular (nonzero) points. So, all of them, antibound negative poles and positive even redundant poles disappear. However, the finite bound poles \( k_1(n_1) \) are conserved since at these particular points there is a confluence of two simple poles and one simple zero, giving rise to a resulting simple pole. The complex conjugate finite points will be simple zeros of the scattering matrix.
We conclude here the analysis on the bound, antibound and redundant poles. In the next section, we shall construct ladder operators connecting wave functions corresponding to such poles.

3.5 A remark concerning $E = 0$

We examine the solutions to the eigenvalue problem associated with Schrödinger equation (2.1) with $E = 0$, or equivalently, $k = 0$. Obviously, we only need the determination of the eigenfunctions, which must have form (2.18) or (2.19) indistinctly, as both functions coincide as $k = 0$. Consequently, for this particular value there must exist another linearly independent solution for this eigenvalue problem, which comes to be the product of a series times a logarithm. Due to its logarithmic divergency, we discard this second solution, exactly as we did in previous cases. It remains a solution as product of an exponential times a Kummer function,

$$\psi(x) = e^{-e^{-x}} \mathcal{F}_1(-A; 1; 2e^{-x}).$$

(3.9)
The behavior of wave function (3.9) depends on the values of $A$. If $A$ were a non-negative integer, $A = 0, 1, 2, \ldots$, then (3.8) goes to zero in the limit $x \rightarrow -\infty$ and is bounded everywhere. Some authors call semibounded these types of states [14,20]. For other values of $A > 0$, function (3.8) shows a strong divergence as $x \rightarrow -\infty$, so that it may not have a physical meaning.

4 Ladder operators

Let us go back to Schrödinger equation (2.1), in which we restrict ourselves to negative energies $E = -\epsilon^2$. Then, multiply (2.1) to the left by $e^{2x}$. After a simple terms rearrangement, (2.1) becomes

\[
h_\epsilon \psi_\epsilon(x) \equiv \left[ -\epsilon^2 \frac{d^2}{dx^2} + 2(A + 1/2) \epsilon^2 x - 2(A + 1/2) \epsilon^2 x^2 \right] \psi_\epsilon(x) = -\psi_\epsilon(x),
\]

where $\psi_\epsilon(x)$ is the eigenfunction of the Hamiltonian $H$ with eigenvalue $-\epsilon^2$ and the meaning of $h_\epsilon$ is obvious. Next, let us define two first-order differential operators $A_\epsilon^+$ and $A_\epsilon^-$, having the following form:

\[
A_\epsilon^+ := -e^x \frac{d}{dx} + \beta e^x + \gamma, \quad A_\epsilon^- := e^x \frac{d}{dx} + \beta' e^x + \gamma',
\]

such that (4.1) becomes factorized:

\[
h_\epsilon \psi_\epsilon(x) = \left[ A_\epsilon^+ A_\epsilon^- + D_\epsilon \right] \psi_\epsilon(x) = -\psi_\epsilon(x).
\]

Here, $\beta, \beta', \gamma, \gamma'$ are constants to be determined and $D_\epsilon$ is independent of $x$, although it may depend on $\epsilon$. After some straightforward calculations, we obtain

\[
A_\epsilon^+ = -e^x \frac{d}{dx} + (1 + \epsilon) e^x - \frac{1 + 2A}{2\epsilon + 1}, \quad A_\epsilon^- = e^x \frac{d}{dx} + \epsilon e^x - \frac{1 + 2A}{2\epsilon + 1}, \quad D_\epsilon = -\frac{(1 + 2A)^2}{(1 + 2\epsilon)^2}.
\]

Notice that for negative energy, $\epsilon$ is a real number, so that care must be taken with denominators. We may write two different expressions for Eq. (4.1) in terms of ladder operators (4.4):

\[
(A_\epsilon^+ A_\epsilon^- + D_\epsilon) \psi_\epsilon(x) = (A_{\epsilon-1}^- A_{\epsilon-1}^+ + D_{\epsilon-1}) \psi_\epsilon(x) = h_\epsilon \psi_\epsilon(x) = -\psi_\epsilon(x).
\]
Then, we can determine a simple intertwining relation between consecutive operators $h_\epsilon$ and $h_{\epsilon-1}$ through the ladder operators. Let us write

$$h_\epsilon = A^-_{\epsilon-1}A^+_{\epsilon-1} + D_{\epsilon-1}, \quad h_{\epsilon-1} = A^+_{\epsilon-1}A^-_{\epsilon-1} + D_{\epsilon-1}. \quad (4.6)$$

Then,

$$A^+_{\epsilon-1}h_\epsilon = h_{\epsilon-1}A^+_{\epsilon-1}, \quad h_\epsilon A^-_{\epsilon-1} = A^-_{\epsilon-1}h_{\epsilon-1}, \quad (4.7)$$

where the first equation in (4.7) is the result of multiplying the first equation in (4.6) to the left by $A^+_{\epsilon-1}$, while the second equation in (4.7) comes from multiplication to the right of (4.6) by $A^-_{\epsilon-1}$. In both cases, we also have to take into account the second relation in (4.6).

Relations (4.7) have the following important consequence:

$$A^+_{\epsilon-1}\psi_\epsilon(x) \propto \psi_{\epsilon-1}, \quad A^-_{\epsilon-1}\psi_{\epsilon-1}(x) \propto \psi_\epsilon, \quad (4.8)$$

where the symbol $\propto$ means “equal save for a multiplicative constant,” which is not essential for our purposes [21].

It is quite important to remark that the signs plus and minus do not correspond necessarily to creation and annihilation operators, respectively. This depends on the relation between $\epsilon$ and the quantum number $n$, $i = 1, 2$ which will fix the order of eigenfunctions. Next, we will study the application of the ladder operators to wave functions for different series of $S(k)$ poles as described in the previous section.

4.1 Bound–antibound series

In this series, according to (3.1), we have $\epsilon(n_1) = A - n_1$ and $k(n_1) = i \epsilon(n_1)$. Since we are discussing each series separately, it is convenient to drop the subindex in $n_{1,2}$ and write $n$ instead for simplicity. In terms of this index $n$, the operators $A^\pm_n$ are:

$$A^+_{n} = -\epsilon^x \frac{d}{dx} + (A - n + 1)\epsilon^x - \frac{1 + 2A}{2(A - n) + 1}, \quad (4.9)$$

$$A^-_{n} = \epsilon^x \frac{d}{dx} + (A - n)\epsilon^x - \frac{1 + 2A}{2(A - n) + 1}, \quad (4.10)$$

$$D_{n} = -\frac{(1 + 2A)^2}{(1 + 2(A - n))^2}. \quad (4.11)$$

Since in terms of $n$, $\epsilon(n) = A - n$, we have that $\epsilon(n \pm 1) = \epsilon(n) \mp 1$ and we conclude that

$$A^+_{n+1}\psi_n(x) \propto \psi_{n+1}, \quad A^-_{n+1}\psi_{n+1}(x) \propto \psi_n. \quad (4.12)$$

Therefore, when the operators $A^-_{n}$ and $A^+_{n}$ act on bound or antibound states, they behave as lowering and raising operators, respectively, with respect to the index $n$. Here, three situations are possible:

- **General case.** This means that $A$ is neither integer nor half-integer. The ground state wave function, $\psi_0(x)$, is annihilated by $A^+_0 \cdot A^-_0 \psi_0(x) = 0$. Then, we obtain the wave functions of the $|A| + 1$ bound states and afterward the infinite number of antibound states just by applying the operator $A^\pm_n$, $n = 1, 2, \ldots$ and using the first relation in (4.12). Therefore, ladder operators (4.9)–(4.10) connect all the bound and antibound wave functions.
- **$A$ is a positive integer: $A \in \mathbb{N}$.** Then, $\epsilon(n) = A - n$ with $n = 0, 1, \ldots, N - 1$. The wave functions for bound states are $\psi_0(x), \ldots, \psi_{N-1}(x)$. In this case we will add the semibound state $\psi_N(x)$, corresponding to the value zero of the energy (there is no antibound poles). The corresponding wave functions are shown in (3.6). The first bound
state also obeys the equation $A_0^- \psi_0(x) = 0$. By applying the operator $A_n^+, n = 1, 2, \ldots$ we get the next bound states and the semibound state $\psi_N(x)$. However, in this case there is no antibound poles, therefore what happens if we continue applying $A_n^+$? Here, notice that there is another form to write the same set of the wave functions of bound states by setting $\epsilon(n) = N - n$, this time with $n = N, N + 1, \ldots, 2N$, which gives

$$\tilde{\psi}_n(x) = e^{(N-n)(x-2 \log 2)} e^{-e^{-x}} \frac{1}{2} F_1(N - n; 1 - 2(N - n); 2e^{-x}). \quad (4.13)$$

It is easy to show that $\psi_{N-n} = \tilde{\psi}_{N+n}, n = 0, \ldots, N$. Let us consider the following finite sequence of repeated wave functions:

$$\{\psi_0, \ldots, \psi_{N-1}, \psi_N, \tilde{\psi}_{N+1}, \ldots, \tilde{\psi}_{2N}\}. \quad (4.14)$$

We can drop now the tildes in (4.14) for simplicity, as there is no risk of confusion. Then, the ladder operators $A_n^\pm$ act in a natural way on the wave functions of sequence (4.14), so as to give:

$$A_{n+1}^+ \psi_{n+1}(x) \propto \psi_{n+1}, \quad A_{n+1}^- \psi_{n+1}(x) \propto \psi_{n}, \quad n = 0, \ldots 2N - 1$$

$$A_{2N+1}^+ \psi_{2N}(x) = 0, \quad A_{0}^- \psi_0(x) = 0. \quad (4.15)$$

Therefore, if $A = N \in \mathbb{N}$, the ladder operators will exclusively connect all the bound wave functions $\psi_n, n = 0, 1, \ldots, N - 1$ and the semibound $\psi_N(x)$.

- If $A$ is a positive half-integer, the relation between eigenfunctions and ladder operators is similar as in the previous case with $A$ integer. The ladder operators connect the wave functions of the bound states. Note that in this case the wave function for the value $k = 0$ is not included in the sequence of wave functions.

### 4.2 Redundant pole series

Now, we analyze the action of the ladder operators on eigenfunctions (3.4) with eigenvalues $\epsilon = \frac{1}{2} (n + 1)$, where we have settled $n = n_2$ for simplicity, corresponding to the redundant poles of $S(k)$. Here, we proceed exactly as in the previous cases, replacing the wave functions for the bound–antibound wave function series by the wave functions corresponding to the redundant pole series. Nevertheless, we have a peculiarity here: Ladder operators connect wave functions for even and odd redundant poles separately. Consequently, we analyze each subseries independently.

- **Even series.** We define a new label $m$ so that $n = 2m$ and, then, $\epsilon = \frac{1}{2} (n + 1) = m + 1/2$. In terms of this new label $m$, we have the following explicit expressions for the ladder operators:

$$A_m^+ = -e^x \frac{d}{dx} + (m + \frac{3}{2})e^x - \frac{1 + 2A}{2(m + 1)},$$

$$A_m^- = e^x \frac{d}{dx} + (m + \frac{1}{2})e^x - \frac{1 + 2A}{2(m + 1)},$$

$$D_m = \frac{(1 + 2A)^2}{4(m+1)^2}, \quad m = 0, 1, 2 \ldots \quad (4.16)$$

The wave functions corresponding to these even redundant poles are given by

$$\psi_m(x) = e^{(2m+1)(ln(2)-\frac{1}{2})} e^{-e^{-x}} \frac{1}{2} F_1(-A + (m + 1/2); 2m + 2; 2e^{-x}), \quad m \geq 0. \quad (4.17)$$
As we have done for the bound–antibound series, we may consider \( \tilde{\psi}_m \) wave functions labeled with negative values of \( m \) and defined for \( m \leq -1 \) as

\[
\tilde{\psi}_m(x) = e^{-(2m+1)(Ln(2)-\frac{1}{2})} e^{-e^{-x}} F_1(-A - (m + 1/2); -2m; 2e^{-x}). \tag{4.18}
\]

It is easy to check that

\[
\tilde{\psi}_m = \tilde{\psi}_{-m-1}, \quad m = 0, 1, 2, \ldots, \tag{4.19}
\]

expressions that define a double sequence of wave functions given by

\[
\{\ldots, \psi_1, \psi_0, \tilde{\psi}_{-1}, \tilde{\psi}_{-2}, \ldots\}. \tag{4.20}
\]

As before, we drop the tildes in (4.20), so as to write the action of the ladder operators in a compact form, which should not cause any confusion. This action can be written as

\[
A_{n-1}^+ \psi_n(x) \propto \psi_{n-1}, \quad A_{n-1}^- \psi_{n-1}(x) \propto \psi_n, \quad n \in \mathbb{Z}, \tag{4.21}
\]

where \( \mathbb{Z} \) is the set of integer numbers, either positive or negative. Here one should have taken into account that \( \epsilon(m \pm 1) = \epsilon(m) \pm 1 \). Also observe another peculiarity for these series: There exists no fundamental or ground state to be annihilated by any of the ladder operators.

- **Odd series.** Now, the new label \( m \) should be given by \( n = 2m + 1 \), so that \( \epsilon = \frac{1}{2} (n + 1) = m + 1 \). For completeness, we include the explicit form of the ladder operators in terms of \( m \) for the odd series:

\[
A_m^+ = -e^x \frac{d}{dx} + (m + 2) e^x - \frac{1 + 2A}{2m + 3},
\]

\[
A_m^- = e^x \frac{d}{dx} + (m + 1) e^x - \frac{1 + 2A}{2m + 3},
\]

\[
D_m = -\frac{(1 + 2A)^2}{(2m + 3)^2}. \tag{4.22}
\]

The non-normalized wave function labeled by the index \( m \) is now,

\[
\psi_m(x) = e^{(2m+2)(Ln(2)-\frac{1}{2})} e^{-e^{-x}} F_1(-A + m + 1; 2m + 3; 2e^{-x}). \tag{4.23}
\]

The remainder of the discussion goes exactly as in the even series, so that we omit it.

We have completed the discussion on the ladder operators for all series of eigenfunctions.

5 Concluding remarks

We have studied the analytic properties of the scattering matrix \( S(k) \) for the one-dimensional Morse potential which is in the list of one-dimensional potentials [22] producing exactly solvable Schrödinger equations. We have shown that all poles are simple and lie on the imaginary axis, which implies the absence of resonances.

All poles of the scattering matrix are simple, and their classification and behavior depend on the parameter \( A \) of the potential, see Eq. (2.1). In terms of this classification, there are three groups that we consider separately: Either \( A \) is integer, or half-integer or neither of both.

When \( A \) is neither integer nor half-integer, we have obtained two independent series of poles, the former is the set of bound–antibound poles. Bound poles lie on the positive imaginary semiaxis and are in one-to-one correspondence with bound states, while antibound poles
lie on the negative imaginary axis and are infinite in number. The second one is the list of redundant poles, located along the positive imaginary axis. For the values of $A$ considered, redundant poles are independent of bound state poles and the only information they bring us is the absence of a basis of solutions made of hypergeometric functions. See illustration in Fig. 4.

When $A$ is an integer or a half-integer number, we have observed an anomalous behavior in the sense that some of the redundant poles and the bound state poles overlap. This is a much more interesting situation. This coincidence is a kind of interference that eliminates an infinite set of the redundant subseries (where the coincidence applies), as well as the infinite set of antibound states. The finite number of bound poles is at the same time redundant, i.e., the second independent solution is not hypergeometric.

All these poles are located at imaginary momentum values $ik$ with energies $E = (\hbar^2 k^2)/(2m)$. These energies may be looked as eigenvalues of the Hamiltonian, and we obtain their corresponding eigenfunctions as solutions of the Schrödinger equation. Only those eigenfunctions in correspondence with bound state poles are square integrable.

Following previous works by our group [15,16,23], we have constructed ladder operators that connect the eigenfunctions in correspondence with the $S(k)$ poles. We have shown that, in the most general case, there exist three different chains of eigenfunctions connected by the ladder operators: The eigenfunctions for the bound–antibound poles belong to the first chain. Here, there is a ground state, which plays the same role as the ground state in the harmonic oscillator. We also analyze the exceptional cases in which $A$ is either integer or half-integer, where the results are similar with some variations. In these exceptional cases the ladder operators connect the finite sequence of bound states in a double way. This behavior reminds us of the $su(2)$ Lie algebra corresponding to integer ($A$ integer) or half-integer ($A$ half-integer) spin.

Then, there are two other chains of eigenfunctions of redundant poles, depending on the even or odd character of a label, which are mutually independent. Here there are some peculiarities, such as the non-existence of a ground state. Creation and annihilation connect doubly infinite series of eigenfunctions.

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