ON \(qp\)-DEFORMATIONS IN STATISTICAL MECHANICS OF
BOSONS IN \(D\) DIMENSIONS

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The Bose distribution for a gas of nonrelativistic free bosons is derived in the framework of \(qp\)-deformed second quantization. Some thermodynamical functions for such a system in \(D\) dimensions are derived. Bose-Einstein condensation is discussed in terms of the parameters \(q\) and \(p\) as well as a parameter \(\nu'\) which characterizes the representation space of the oscillator algebra.

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1 Introduction

The use of quantum groups and quantum algebras is now largely displayed in
theoretical physics. From an elementary point of view, quantum algebras may
be thought of as q- (or qp)-deformations of Lie algebras. Roughly speaking,
there are two types of applications of q-deformations to Physics (cf. Ref. 1).
Applications of type A dot not rely on phenomenology. For example, applications
of type A concern the quantum inverse scattering method, the quantum
Yang-Baxter equation and a true definition of space-time. On the other hand,
applications of type B are entirely phenomenological. Along this vein, we may
mention the use of the two-parameter quantum algebra \( U_{qp}(u_2) \) (a Hopf al-
gebra) to rotational collective dynamics of atomic nuclei.\(^2\) For applications
of type B, we immediately face the problems of the physical interpretation and
the nonuniversality of the deformation parameter(s).

The considerations above can be applied to deformed oscillator algebras.
Indeed, oscillator algebras and quantum algebras may be connected since the q-
(or qp)-deformed creation and annihilation operators, which are defined in an
oscillator algebra, may serve for constructing realizations of q- (or qp)-deformed
Lie algebras.

It is the aim of this lecture to examine the interest of using qp-deformed
creation and annihilation operators in statistical mechanics of bosons in ar-
bitary dimension. In other words, we ask the following question: What are the implications, at the level of the Bose distribution and the Bose-Einstein condensation, of replacing ordinary boson operators by \( q p \)-deformed creation and annihilation operators? Furthermore, is it possible to test a deformed Bose distribution and to find a physical interpretation of the deformation parameter(s)?

This subject was already investigated, mainly in the case of \( q \)-deformed boson operators, by many authors.\(^3\)\(^−\)\(^18\) The case of \( q p \)-deformed boson operators was considered in Refs. 17 and 18. Most of the works in Refs. 3 to 18 use the Fock representation\(^19\)\(^−\)\(^25\) for the deformed oscillator algebra inherent to the considered creation and annihilation operators. In contradiction, the representation introduced by Rideau\(^26\) (see also Ref. 1) for the \( q \)-deformed oscillator algebra is used in Ref. 16.

We shall deal in this paper with both the Fock and the Rideau representations of deformed oscillator algebras. These representations are described in Sec. 2. Then, Sec. 3 is devoted to \( q p \)-deformations of the Bose distribution and its consequence for the Bose-Einstein condensation phenomenon. Finally, some conclusions are presented in Sec. 4. Two appendices close this paper. In particular, Appendix B concerns a \( q p \)-deformed correlation factor for the radiation field.

This work takes its origin in Refs. 9 and 18. It is however to be noticed that some of the developments and the conclusions given here substantially differ from the ones in Refs. 9 and 18.

2 Deformed Oscillator Algebra

Following Refs. 1 and 26, we characterize the algebra \( W_{qpQ} \) by three operators \( a, a^+ \) and \( N \) such that

- the operator \( a^+ \) is the adjoint of \( a \),
- the operator \( N \) is self-adjoint with a discrete nondegenerate spectrum,
- the operators \( a, a^+ \) and \( N \) satisfy the commutation relations

\[
[N, a] = -a, \quad [N, a^+] = a^+, \quad aa^+ - Qa^+a = \frac{1}{q-p}[q^N(q - Q) - p^N(p - Q)]
\]

where the three parameters \( q, p \) and \( Q \) are fixed parameters taken (a priori) in the field of complex numbers.

The algebra \( W_{qpQ} \) spanned by the deformed \textit{annihilation} (\( a \)), \textit{creation} (\( a^+ \)) and \textit{number} (\( N \)) operators is called a deformed oscillator algebra. In the special cases \( Q = q \) and \( Q = p \), the operators \( a \) and \( a^+ \) are referred to as \( q p \)-boson
operators. They entail the particular cases of the $q$-boson operators as used in mathematics (when $p = 1$) and in physics (when $p = q^{-1}$). Four particular cases are of interest for the applications (see also Refs. 19 to 26).

1) The case $Q = q, p = 1$: This case corresponds to

$$\begin{align*}
[N, a] &= -a, \quad [N, a^+] = a^+, \quad aa^+ - qa^+ a = 1 \\
\end{align*}$$

i.e. to $q$-bosons as used in the mathematical literature (see also Refs. 19 to 21 and 25).

2) The case $Q = q, p = q^{-1}$: This case corresponds to

$$\begin{align*}
[N, a] &= -a, \quad [N, a^+] = a^+, \quad aa^+ - qa^+ a = q^{-N} \\
\end{align*}$$

i.e. to $q$-bosons as used in the physical literature.\textsuperscript{22,23}

3) The case $Q = q^{-1}, p = q^{-1}$: This case corresponds to

$$\begin{align*}
[N, a] &= -a, \quad [N, a^+] = a^+, \quad aa^+ - q^{-1}a^+ a = q^N \\
\end{align*}$$

i.e. to $q$-bosons as used in the physical literature.\textsuperscript{22,23}

4) The case $Q = 0$: This case corresponds to

$$\begin{align*}
[N, a] &= -a, \quad [N, a^+] = a^+, \quad aa^+ = \frac{q^{N+1} - p^{N+1}}{q - p} \\
\end{align*}$$

which admits two interesting subcases corresponding to either $p = 1$ or $p = q^{-1}$.

The nondeformed oscillator algebra corresponds to $(Q = q, p = 1, q \rightarrow 1)$ or $(Q = q, p = q^{-1}, q \rightarrow 1)$ or $(Q = q^{-1}, p = q^{-1}, q \rightarrow 1)$. Note that the cases $(Q = 0, p = 1, q \rightarrow 1)$ and $(Q = 0, p = q^{-1}, q \rightarrow 1)$ correspond to the exotic limiting case where $aa^+ = 1$.

The Hilbertian irreducible representations of the oscillator algebra $W_q \equiv W_{qq^{-1}q}$ were dealt in details by Rideau.\textsuperscript{26} The Hilbertian irreducible representations of the more general algebra $W_{qpQ}$ can be derived equally well. It is enough to say here that the relations

$$\begin{align*}
| n \rangle &= \left( Q^n \lambda_0 + p^{\nu_0} \frac{Q^n - p^n}{q - p} + q^{\nu_0} \frac{q^n - Q^n}{q - p} \right)^\frac{1}{2} | n - 1 \rangle \\
| n \rangle &= \left( Q^{n+1} \lambda_0 + p^{\nu_0} \frac{Q^{n+1} - p^{n+1}}{q - p} + q^{\nu_0} \frac{q^{n+1} - Q^{n+1}}{q - p} \right)^\frac{1}{2} | n + 1 \rangle \\
N | n \rangle &= (\nu_0 + n) | n \rangle \\
\end{align*}$$

provide us with a formal representation of $W_{qpQ}$. We thus obtain an infinity of Hilbertian irreducible representations of $W_{qpQ}$ that may be arranged into two
types: $R(\nu'_0)$ with $n \in \mathbb{N}$ and $S(\lambda_0, \nu_0)$ with $n \in \mathbb{Z}$. We shall deal in this work only with the (Fock) representation $R(0)$ of the algebra $W_{qpq}$ and the (Fock) representation $R(\nu'_0)$, with $\nu'_0 \neq 0$, of the algebra $W_q$. We shall not use the case of the (nonFock) representation $S(\lambda_0, \nu_0)$. This representation which has no limit at $q = p = Q = 1$ is briefly discussed in Appendix A.

The (deformed) Fock representation $R(0)$ of $W_{qpq}$ is obtained from Eq. (6) by taking $\lambda_0 = \nu_0 = 0$ and $n \in \mathbb{N}$. Then, we have

$$a |n\rangle = \sqrt{[[n]]_{qp}} |n-1\rangle,$$

$$a^+ |n\rangle = \sqrt{[[n+1]]_{qp}} |n+1\rangle,$$

$$N |n\rangle = n |n\rangle \quad (7)$$

with $a |0\rangle = 0$. In Eq. (7), we use the notation

$$[[x]]_{qp} = \frac{q^x - p^x}{q - p} \quad (8)$$

It is to be observed that the operator $N$ is a nondeformed operator that coincides with the usual number operator (in the sense that its spectrum is $\mathbb{N}$). Note that

$$[[x]]_{q,p=1} = \frac{q^x - 1}{q - 1}, \quad [[x]]_{q,p=q^{-1}} = \frac{q^x - q^{-x}}{q - q^{-1}} \quad (9)$$

The case of the nondeformed (i.e. usual) Fock representation is reached for $(p = 1, q \to 1)$ or $(p = q^{-1}, q \to 1)$. Finally, the case $(p = 1, q = 0)$ corresponds to the exotic situation where

$$a |n\rangle = |n-1\rangle, \quad a^+ |n\rangle = |n+1\rangle, \quad N |n\rangle = n |n\rangle \quad (10)$$

Going back to the representation $R(0)$ (given by Eq. (6)) of $W_{qpq}$, we observe that

$$a^+ a = [[N]]_{qp}, \quad a a^+ = [[N+1]]_{qp} \quad (11)$$

so that the relation

$$aa^+ - pa^+ a = q^N \quad (12)$$

holds for this representation in addition to the defining relation

$$aa^+ - qa^+ a = p^N \quad (13)$$

Equations (12) and (13) clearly show that the representation $R(0)$ of $W_{qpq}$ exhibits the $q \leftrightarrow p$ symmetry.
The (deformed) Fock representation $R(\nu'_0)$ of $W_\varrho$ is obtained from Eq. (6) by taking $\lambda_0 = 0$, $\nu_0 = \nu'_0 \neq 0$ and $n \in \mathbb{N}$. Then, we have

$$a |n\rangle = q^{-\frac{\nu'_0}{2}} \sqrt{n!}_q |n - 1\rangle$$
$$a^+ |n\rangle = q^{-\frac{\nu'_0}{2}} \sqrt{(n + 1)!}_q |n + 1\rangle$$
$$N |n\rangle = (\nu'_0 + n) |n\rangle$$

with $a |0\rangle = 0$. In Eq. (14), we use the notation

$$[x]_q = [[x]]_{q,p=q^{-1}}$$

and we have $q > 1$ and $\nu'_0 \in \mathbb{R}$. It is to be observed that the operator $N$ does not coincide with the usual number operator; the usual number operator is now $N - \nu'_0$ (in the sense that its spectrum is $\mathbb{N}$). As an important result, we have that if $q \to 1^+$ then $\nu'_0 \to 0$; in other words, we can write

$$\lim_{q \to 1^+} R(\nu'_0) = \text{usual Fock representation}$$

For the representation $R(\nu'_0)$, we have

$$a^+ a = q^{-\nu'_0} [N - \nu'_0]_q, \quad a a^+ = q^{-\nu'_0} [N + 1 - \nu'_0]_q$$

Consequently, we verify that the relations

$$aa^+ - qa^+ a = q^{-N}, \quad aa^+ - q^{-1} a^+ a = q^{N - 2\nu'_0}$$

hold in $W_\varrho$. Equation (18) indicates that the representation $R(\nu'_0)$ of $W_\varrho$ does not present the $q \leftrightarrow q^{-1}$ symmetry.

### 3 Bose Distribution and Bose-Einstein Condensation

Let us consider a gas of nonrelativistic free bosons. Its Hamiltonian (in a second-quantized form) reads

$$H := \sum_k H_k, \quad H_k := (E_k - \mu)\nu_k$$

In Eq. (19), $\mu$ is the chemical potential while $E_k$ and $\nu_k$ are the kinetic energy of a boson and the number operator for the bosons, in the $k$ mode, respectively. The Bose factor for the $k$ mode is then

$$f_k := \frac{1}{Z} \text{tr}(e^{-\beta H} a_k^+ a_k)$$
where

\[ Z := \text{tr} \left( e^{-\beta H} \right) \]  

is the partition function and \( \beta = (k_B T)^{-1} \) the reciprocal temperature.

It is essential to note that Eqs. (19)-(21) have the same form as in the nondeformed case. They correspond to the basic formulas for the statistical mechanics of bosons. Here, we do not deform the basic formulas. The trace and the exponential in (20) and (21) are nondeformed functions. The deformation is introduced via the use of annihilation (\( a_k \)) and creation (\( a_k^+ \)) operators that satisfy \( qp \)-deformed commutation relations. The form chosen for \( H_k \) in Eq. (19) is nothing but the common one where the usual number operator \( \nu_k \) takes its eigenvalues in \( \mathbb{N} \). The deformation cannot enter the theory at this level by replacing \( \nu_k \) by \( a_k^+ a_k \) in (19) since the operator \( a_k^+ a_k \) has not its eigenvalues in \( \mathbb{N} \). On the contrary, we must keep \( a_k^+ a_k \) rather than \( \nu_k \) in (20). Indeed, should we use \( \nu_k \) in place of \( a_k^+ a_k \) in (20), we would get a nondeformed value of the Bose factor. As a conclusion, Eqs. (19)-(21) constitute a reasonable starting point for a (minimal) deformation of the Bose distribution.

We are now in a position to derive several types of deformed Bose distributions. The deformation is explicitly introduced by using either the representation \( R(0) \) of \( W_{qpq} \) (see Eq. (7)) or the representation \( R(\nu'_0) \), with \( \nu'_0 \neq 0 \), of \( W_q \) (see Eq. (14)). The two representations can be treated (in a formal way) on the same footing by considering (cf. Eqs. (7) and (14)) the representation

\[
\begin{align*}
  a_k |n\rangle &= p_{\nu_k}^{\frac{1}{2}} \sqrt{[n]_{qp}} |n - 1\rangle \\
  a_k^+ |n\rangle &= p_{\nu_k}^{\frac{1}{2}} \sqrt{[n+1]_{qp}} |n + 1\rangle \\
  N_k |n\rangle &= (\nu'_0 + n) |n\rangle
\end{align*}
\]

of the algebra \( W_{qpq} \). We assume (mathematical hypothesis): \( \{a_k, a_k^+, N_k\} \) and \( \{a_j, a_j^+, N_j\} \) are two commuting sets when \( j \neq k \). Furthermore, we take (physical hypothesis):

\[ \nu_k := N_k - \nu'_0 \]  

As a trivial result, we have

\[ Z = \prod_k \frac{1}{1 - e^{-\eta}} \]  

where

\[ \eta := \beta (E_k - \mu) \]
The partition function $Z$ is thus independent of the deformation parameters $q$ and $p$. Then, the Bose factor $f_k$ can be written

$$f_k = p^\nu_0 (1 - e^{-\eta}) \text{tr}(e^{-\eta(N_k - \nu_0)}[N_k - \nu_0])_{qp}$$

which can be shown to converge in each of the following five cases:

1. $(q, p) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $0 < q < e^{-\beta\mu}$ and $0 < p < e^{-\beta\mu}$
2. $(q, p) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $p = q^{-1}$ and $e^{\beta\mu} < q < e^{-\beta\mu}$
3. $(q, p) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $p = 1$ and $0 < q < e^{-\beta\mu}$
4. $(q, p) \in \mathbb{C} \times \mathbb{C}$ with $p = \bar{q}$ and $0 < |q| < e^{-\beta\mu}$
5. $(q, p) \in S^1 \times S^1$ with $p = \bar{q} = q^{-1}$.

The latter two cases, although acceptable from the mathematical point of view, do not lead to physically acceptable statistical distributions.

As a formal result, Eq. (26) yields

$$f_k = p^\nu_0 \left( \frac{q - 1}{q - p e^{-\eta} - q} + \frac{p - 1}{p - q e^{-\eta} - p} \right)$$

or alternatively

$$f_k = p^\nu_0 \frac{e^{-\eta} - 1}{(e^{-\eta} - q)(e^{-\eta} - p)}$$

In the limiting situation where $q \to 1$ and $p \to 1$, we recover the classical Bose distribution

$$\Phi_k = \frac{1}{e^{\eta} - 1}$$

The distribution $f_k$ can be rewritten as

$$f_k = p^\nu_0 \left[ \frac{q - 1}{q - p \Phi_k + (1 - q)\Phi_k} + \frac{p - 1}{p - q \Phi_k + (1 - p)\Phi_k} \right]$$

in term of $\Phi_k$.

For the purpose of practical applications, it is useful to know the development of the distribution $f_k$ in integer series. In this respect, we have

$$f_k = p^\nu_0 \sum_{j=0}^{\infty} e^{\eta(j+1)} \left( \frac{q - 1}{q - p} g_j^q + \frac{p - 1}{p - q^p} g_j^p \right)$$

which reduces to the expansion $\sum_{j=1}^{\infty} e^{-\eta j}$ of the ordinary Bose factor when $q \to 1$ and $p \to 1$.

In the thermodynamical limit, the energy spectrum of the system of bosons may be considered as a continuum. Thus, $f_k$ is replaced by the $qp$-dependent
factor \( f(\varepsilon) \) which is \( f_k \) with \( E_k = \varepsilon \). Therefore, in physical applications, we have to consider integrals of the type

\[
J_s := \int_0^\infty \varepsilon^s f(\varepsilon) d\varepsilon
\]  

(32)

By using the development (31), Eq. (32) leads to

\[
J_s = \Gamma(s+1)(k_B T)^{s+1}\sigma(s+1)_{qp}, \quad s > -1
\]  

(33)

where

\[
\sigma(s+1)_{qp} = p^D \sum_{j=0}^\infty \frac{e^{s\mu(j+1)}}{(j+1)^{s+1}}([j+1]_{qp} - [j]_{qp})
\]  

(34)

In Eq. (33), \( \Gamma \) is the Euler integral of the second type. In the case where \( \mu = 0 \), we have \( \sigma(s+1)_{qp} \rightarrow \zeta(s+1) \) for \( q \rightarrow 1 \) and \( p \rightarrow 1 \), where \( \zeta(s+1) \) is the Riemann Zeta series (that converges for \( s > 0 \)).

We now derive some thermodynamical quantities, in \( D \) dimensions, for a free gas of \( N(D) \) bosons contained in a volume \( V(D) \). The density

\[
\rho(D) := \frac{N(D)}{V(D)}
\]  

(35)

is given by

\[
\rho(D) = N_0(D)\Gamma\left(\frac{D}{2}\right)(k_B T)^{\frac{D}{2}}\sigma\left(\frac{D}{2}\right)_{qp}
\]  

(36)

where

\[
N_0(D) := \frac{1}{2(2\pi)^{\frac{D}{2}}} \frac{D}{\Gamma\left(\frac{D}{2} + 1\right)} g^{\frac{D}{2}} m^{\frac{D}{2}} \hbar^D
\]  

(37)

In Eq. (37), \( g \) is the degree of (spin) degeneracy and \( m \) the mass of a boson. The total energy can then be calculated to be

\[
E(D) = N_0(D)V(D)\Gamma\left(\frac{D}{2} + 1\right)(k_B T)^{\frac{D}{2} + 1}\sigma\left(\frac{D}{2} + 1\right)_{qp}
\]  

(38)

The specific heat at constant volume easily follows from \( C_V = \frac{\partial E}{\partial T} \). We obtain

\[
C_V(D) = \frac{D}{2} N(D)k_B \left[ \left(\frac{D}{2} + 1\right)\frac{\sigma\left(\frac{D}{2} + 1\right)_{qp}}{\sigma\left(\frac{D}{2}\right)_{qp}} - \frac{D}{2} \frac{\sigma\left(\frac{D}{2}\right)_{qp}}{\sigma\left(\frac{D}{2} - 1\right)_{qp}} \right]
\]  

(39)
Furthermore, the entropy is
\[ S(D) = N(D)k_B \left[ \frac{D}{2} + 1 \right] \frac{\sigma(D/2 + 1)_{qp}}{\sigma(D/2)_{qp}} - \frac{\mu}{k_B T} \] (40)

Note that the state equation for the gas of bosons is
\[ p(D) = 2 \frac{E(D)}{V(D)} \] (41)
so that the pressure \( p(D) \) assumes the same form as in the nondeformed case.

Finally, other thermodynamical quantities may be determined in a simple manner by using the thermodynamic potential
\[ \Omega(D) = -\frac{2}{D} N_0(D) V(D) J_D = -\frac{2}{D} E(D) \] (42)
where \( J_D \) is given by (33) and (34).

We now examine the condensation of a system of bosons in \( D \) dimensions. The corresponding density for such a system is given by (36) which can be rewritten as
\[ \rho(D) = N_0(D) J_D^{-1} \] (43)
as a function of the integral \( J_D^{-1} \).

As in the classical case (i.e. \( q = p = 1 \)), we have to define a procedure for generating the critical Bose temperature \( T_c(D) \). For this purpose, we follow Ref. 9 by taking
\[ \beta \mu = \beta \mu_1 - \ln q, \quad q > 1 \] (44)
where \( \mu_1 \) is a negative constant corresponding to the chemical potential in the classical case. This \( q \)-dependence of the chemical potential \( \mu \) makes it possible to have a good behaviour of the distribution \( f_k \) when \( \mu_1 \to 0^- \) only for the two situations \( p = q^{-1} \) and \( p = 1 \). Therefore, in the remaining part of this section, the parameter \( p \) stands for either \( p = q^{-1} \) or \( p = 1 \). The Bose-Einstein condensation is then obtained by introducing \( \mu_1 = 0 \) in Eq. (43). Thus, we find that the critical temperature below which we obtain Bose-Einstein condensation is given by
\[ T_c(D)_{qp} = p^{-\frac{q}{q-1}} \frac{1}{k_B} \left[ \frac{\rho(D)}{N_0(D) \Gamma(D/2) \sigma_0(D/2)_{qp}} \right] \] (45)
where
\[ \sigma_0(D/2)_{qp} := \frac{1}{q + 1} \sum_{j=0}^{\infty} \frac{1 + q^{-2j-1}}{(j + 1)^2} \] for \( p = q^{-1} \) and \( 1 < q < e^{-\beta \mu} \) (46)
We thus end up with three models: a two-parameter model $M_1$ with $(\nu'_0 \in \mathbb{R}^*, p = q^{-1}, 1 < q < e^{-\beta\mu})$ and two one-parameter models, viz. $M_2$ with $(\nu'_0 = 0, p = q^{-1}, 1 < q < e^{-\beta\mu})$ and $M_3$ with $(\nu'_0 = 0, p = 1, 1 < q < e^{-\beta\mu})$. For each of these models, the Bose-Einstein condensation phenomenon takes place only when the dimension $D$ is greater than 2, exactly as in the classical case. (The series in Eqs. (46) and (47) converge for $D \geq 3$.) The models $M_1$, $M_2$ and $M_3$ can be easily tested for $D = 3$ by comparing for each model the theoretical and experimental temperatures $T_c(3)$ for $^4$He superfluid in phase II. For the models $M_2$ and $M_3$, we obtain

$$T_c(3)_{qp} > T_c(3)_{\text{classical}} > T_c(3)_{\text{exp}} \sim 2.17 \text{ K}$$

(48)

Therefore, the models $M_2$ and $M_3$ give results which are worse than the one given by the classical model (corresponding to $q = p = 1$) for which we have

$$T_c(3)_{\text{classical}} \equiv T_c(3)_{q=1,p=1} = \frac{2\pi\hbar^2}{mk_B} \left[ \frac{\rho(3)}{2.612g} \right]^{\frac{3}{2}}$$

(49)

Unlike the models $M_2$ and $M_3$, for the model $M_1$, it is possible to find couples $(\nu'_0, q)$ or which $T_c(3)_{q,p=q-1}$ is in agreement with the experimental value. As a point of fact, our test rules out the models $M_2$ and $M_3$.

4 Conclusions

Among the various $q\rho$-deformations of the Bose factor studied in this work, only the ones corresponding to $p = q^{-1}$ or $p = 1$ yield acceptable statistical distributions for bosons. However, there is no $q$-deformation, with $\nu'_0 = 0$, of the Bose factor that leads to a correct value of the Bose-Einstein transition temperature. This has been tested for $^4$He but we note that $T_c(3)_{qp} > T_c(3)_{\text{classical}}$ holds in general. The latter result generally applies to other $q$-deformations, available in the literature, based on the use of the representation $R(0)$ of the oscillator algebra $W_q$.

The model $M_1$ deserves more optimistic concluding remarks. As a matter of fact, for this model, corresponding to $(\nu'_0 < 0, p = q^{-1}, 1 < q < e^{-\beta\mu})$, it is possible to decrease the critical temperature $T_c(D)_{qp}$ and to get Bose-Einstein condensation temperatures in accordance with experiment. However, as a drawback, the model $M_1$ depends on two parameters $\nu'_0$ and $q$. The parameter
$q$ brings a translation factor $\beta^{-1}\ln q$ for the chemical potential since Eq. (44) gives

$$\mu = \mu_1 - \frac{\ln q}{\beta} = \mu_1 - k_B T \ln q$$

(50)

that clearly shows that the importance of the $q$-deformation increases with the temperature. The parameter $\nu_0'$ is more difficult to interpret. This parameter is essential, via the factor $q^{2\nu_0'}$ in Eq. (45), for decreasing the critical temperature. In this respect, it might be of importance for describing some interactions between bosons.

To close this paper, it is worth noting that alternative choices, non-linear in $N$, are possible for the Hamiltonian $H$ in Eq. (19). In this direction, we may mention, in the case $p = q^{-1}$, the works of Refs. 4, 5, 10, 15, and 16. We hope to return on these matters in the future, especially on the difficult problem of the interpretation of the parameter $\nu_0'$.

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**Appendix A**

We deal in this appendix with the representation $S(\lambda_0, \nu_0)$ of the oscillator algebra $W_q$ with $0 < q < 1$. This representation follows from

$$a |n\rangle = (q^n \lambda_0 + q^{-\nu_0} |n\rangle_q)^{\frac{1}{2}} |n-1\rangle$$

$$a^+ |n\rangle = (q^{n+1} \lambda_0 + q^{-\nu_0} |n+1\rangle_q)^{\frac{1}{2}} |n+1\rangle$$

$$N |n\rangle = (\nu_0 + n) |n\rangle$$

(51)

where $n \in \mathbb{Z}$ and the real parameters $q$, $\lambda_0$ and $\nu_0$ satisfy the constraint equation

$$q^{-\nu_0} \frac{q}{1-q} < \lambda_0 < q^{-\nu_0} \frac{1}{1-q}$$

(52)

A characteristic of this representation is that the spectrum of the operator $N$ is not bounded from below. It is thus hardly feasible to connect $N$ to the usual number operator. Another important characteristic concerns the absence of a limit of $S(\lambda_0, \nu_0)$ when $q \to 1^-$. In fact, the generic representation $S(\lambda_0, \nu_0)$
disappears at \( q = 1 \) since Eq. (52) becomes meaningless for \( q \to 1^- \). The fact that representation \( S(\lambda_0, \nu_0) \) has no limit at \( q = 1 \) is probably the reason why this representation has never been used (to the best of our knowledge) in physical applications.

Appendix B

We discuss here the consequence of a \( qp \)-deformation of the correlation function \( g^{(2)} \) of order two associated to the radiation field. In the nondeformed case, we know that \( g^{(2)} \) takes two values, viz. \( g^{(2)} = 1 \) for a coherent monomode radiation and \( g^{(2)} = 2 \) for a chaotic monomode radiation. An interesting question arises: Is it possible to interpolate between the latter two values by replacing the ordinary boson operators by \( qp \)-deformed boson operators? In this appendix, we limit ourselves to the representation \( R(0) \) of the oscillator algebra \( W_{qpq} \).

Our basic hypothesis is to describe the radiation field by an assembly of photons (\( \mu = 0 \)) with the Hamiltonian

\[
h := \sum_k h_k, \quad h_k := \hbar \omega_k \nu_k
\]

The corresponding Bose statistical distribution for the \( k \) mode is then given by (28) where \( \eta \) is replaced by \( \xi := \beta \hbar \omega_k \). In the case of \( qp \)-bosons, we adopt the definition

\[
g^{(2)} := \frac{\langle a^+ a^+ a a \rangle}{\langle a^+ a \rangle^2}
\]

where \( a \) and \( a^+ \) stand for the annihilation and creation operators for the \( k \) mode. The abbreviation \( \langle X \rangle \) in (54) denotes the mean statistical value \( Z^{-1} \text{tr} (e^{-\beta h} X) \) for an operator \( X \). Equation (54) can be developed as

\[
g^{(2)} = p^{-1} \langle (a^+ a)^2 \rangle - (qp)^{-1} \langle q^N a^+ a \rangle \langle a^+ a \rangle^2
\]

Of course, \( \langle a^+ a \rangle = f_k \) as given by (28) with \( \eta \equiv \xi \). In addition, the other average values in (55) can be calculated to be

\[
\langle (a^+ a)^2 \rangle = \frac{(e^\xi - 1)(e^\xi + qp)}{(e^\xi - q^2)(e^\xi - qp)(e^\xi - p^2)}
\]

and

\[
\langle q^N a^+ a \rangle = q \frac{e^\xi - 1}{(e^\xi - q^2)(e^\xi - qp)}
\]
Finally, we obtain
\[ g^{(2)} = (q + p) \frac{1}{e^\xi - 1} \frac{(e^\xi - q)^2(e^\xi - p)^2}{(e^\xi - q^2)(e^\xi - q)(e^\xi - p^2)} \] (58)
with convergence conditions that parallel the ones for \( f_k \) (with \(-2\beta\mu\) replaced by \(\xi\)). [For instance, when \((q,p) \in \mathbb{R}^+ \times \mathbb{R}^+\), we must have \(0 < q < e^{\xi^2}\) and \(0 < p < e^{\xi^2}\).] The qp-deformed factor \(g^{(2)}\) depends on the parameters \(q\) and \(p\) in a symmetrical manner (\(q \leftrightarrow p\) symmetry). It also presents a dependence on the energy \(\hbar \omega_k\) of the \(k\) mode and on the temperature \(T\). In the limiting case where \(q \to 1\) and \(p \to 1\), we get \(g^{(2)} = 2\) that turns out to be the value for the chaotic monomode radiation of the black body. Let us now examine the cases of low temperatures and high energies. In these cases, \(e^\xi\) is the dominating term in each of the differences occurring in Eq. (58). Therefore, we have
\[ g^{(2)} \sim q + p \] (59)
at low temperature or high energy. Equation (58) shows that we can interpolate in a continuous way from \(g^{(2)} = 1\) (coherent phase) to \(g^{(2)} = 2\) (chaotic phase). It is thus possible to reach the value \(g^{(2)} = 1\) without employing coherent states. It should be observed that we can even obtain either \(g^{(2)} > 2\) or \(g^{(2)} < 1\) from Eq. (58). The situation where \(g^{(2)} < 1\) may be interesting for describing antibunching effects of the light field.

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