COLEMAN INTEGRATION USING THE TANNAKIAN FORMALISM

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1. INTRODUCTION

The theory of Coleman integration has been developed in a number of directions by Coleman [Col82, Col85, CdS88] and later by the author [Bes00a] (a totally different approach was worked out by Colmez [Col98] and by Zarhin [Zar96]). It provides a way of extending the ring of analytic functions on certain rigid analytic domains in such a way that one can often find an essentially unique primitive to a differential form. The guiding principle has always been that \( p \)-adically one can integrate differential forms “locally” but that due to the complete discontinuity of the \( p \)-adic topology naive analytic continuation is impossible and one must use instead “analytic continuation along Frobenius”.

The way in which Coleman does this analytic continuation, using certain polynomials in an endomorphism which is a lift of Frobenius, is admittedly rather strange. It has been our feeling that this tends to put off people from exploring this theory in spite of its many applications. This work grew out of an attempt to make the theory more attractive to a wider mathematical audience by making the procedure of continuation along Frobenius better understood. We later realized that our methods makes it possible to develop the theory from the start in a way which at least to us seems more elegant. In addition, we are able to extend the theory of Coleman iterated integrals, which was only developed for one dimensional spaces, to spaces in arbitrary dimensions.

The idea behind the construction is very simple. An iterated Coleman integral is the element \( y_1 \) in a solution of a system of differential equations

\[
\begin{align*}
  dy_1 &= y_2 \cdot \omega_1 \\
  dy_2 &= y_3 \cdot \omega_2 \\
  &\vdots \\
  dy_{n-1} &= \omega_{n-1} \\
  dy_n &= 0 
\end{align*}
\]

and we may modify the last equation and add one more equation as follows:

\[
\begin{align*}
  dy_{n-1} &= y_n \cdot \omega_{n-1} \\
  dy_n &= 0 
\end{align*}
\]

We do not enter here into the nature of the differential forms \( \omega_i \). Thus, a Coleman iterated integral is a solution of a unipotent system of differential equations. Locally, this system can be solved and we would like to analytically continue this solutions. Here, locally means on a residue class \( U_x \) - the set of points in our rigid space having a common reduction \( x \). According to a Philosophy of Deligne, the collection of all paths between \( x \) and another point \( y \) along which one can do analytic continuation is
a principal homogeneous space for a certain Tannakian fundamental group. There
is an action of Frobenius on this fundamental group and on the space of paths.
Using work of Chiarellotto [Chi98] we are able to show that there is a unique
Frobenius invariant path on this space and thus canonical “analytic continuation
along Frobenius” has been achieved.

Let us get a bit more technical. Let \( K \) be a discrete valuation field with ring of
integers \( \mathcal{V} \) and residue field \( \kappa \) of characteristic \( p \). The basic setup for the theory is
a rigid triple \( (X, Y, P) \) where \( X \) is an open subscheme of the proper \( \kappa \)-subscheme
\( Y \) which is in turn a closed subscheme of a \( p \)-adic formal scheme \( P \) smooth in a
neighborhood of \( X \). To \( P \) one associates its generic fiber, a rigid analytic \( K \)-space,
and in it sits the tube (in the sense of Berthelot) \( |X|_p \) of points reducing to points
of \( X \).

For the development of the theory we restrict to the case where \( \kappa \) is the algebraic
closure of the field \( F_p \) with \( p \) elements. The main output of our constructions
is a certain ring \( A_{\text{Col}}(T) \), where \( T \) is a shorthand for the triple \( (X, Y, P) \).
This ring contains the ring of “overconvergent” rigid analytic functions on \( |X|_p \).
These rings form sheaves with respect to a certain topology induced from the Zariski
topology on \( X \). We can also define modules of differential forms \( \Omega^n_{\text{Col}}(T) \)
from these differentials. The main result of the theory, which is almost trivial once
the machinery has been set up properly, is that this complex of differentials is
exact at the one-forms. This corresponds to Coleman’s theory where one construct
functions by successive integration in such a way that every differential form is
integrable. Note that in Coleman’s theory one is dealing with curves so all forms
are automatically closed.

We can also interpret our Coleman functions as actual functions in the style of
Coleman. We get certain locally analytic functions on the tubes \( |X|_p \). In Coleman’s
theory one gets a bit more since the functions extend to a “wide open space”
containing the tube. This extension becomes complicated in higher dimensions so
we have only carried it out in dimension 1. Indeed, we are able to show that our
theory is identical to Coleman’s theory in the cases he considers.

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Notation: \( K \) is a non-archimedian field of characteristic 0 with ring of integers \( \mathcal{V} \)
and residue field \( \kappa \) of characteristic \( p \). We denote the category of finite dimensional
vector spaces over \( K \) by \( \text{Vec}_K \). All schemes are separated and of finite type over
their respective bases. If \( X \) is a \( \kappa \)-scheme where \( \kappa \) is algebraic over the prime field
\( F_p \), then a Frobenius endomorphism of \( X \) is any morphism obtained in the following
way: Let \( X'/F_{p^r} \) be a scheme such that \( X \cong X' \otimes_{F_{p^r}} \kappa \). The morphism \( \text{Frob}^r \),
where \( \text{Frob} \) is the absolute Frobenius of \( X' \), is an \( F_{p^r} \)-linear endomorphism of \( X' \)
and \( \text{Frob}^r \otimes \text{id} \), considered as a \( \kappa \)-endomorphism of \( X \) via the isomorphism above
is a Frobenius endomorphism of \( X \).
2. Unipotent isocrystals

In this section we review basic facts about connections and crystals, in the rigid and overconvergent settings. Basic sources for these are [Ber96], [CLS99a] and [CLS99b].

Let $V$ be a smooth rigid analytic $K$-space. If $E$ is a coherent sheaf of $O_V$-modules a connection on $E$ is a map
\[ \nabla : E \to E \otimes O_V \Omega^1_V \]
which satisfies the Leibniz formula
\[ \nabla(fe) = e \otimes df + f \cdot \nabla(e) \]
for any $e \in E$ and $f \in O_V$. We will say that the pair $(E, \nabla)$ is a connection on $V$.

A connection $\nabla$ has a curvature $\nabla^2 \in \text{Hom}_{O_V}(E, E \otimes O_V \Omega^2_V)$ given by
\[ \nabla^2(e) = \nabla(\nabla(e)) \]  
(2.1)

A connection is called integrable if $\nabla^2 = 0$. In this case it is well known that $E$ must be locally free. When the connection on $E$ is integrable one obtains a de Rham complex
\[ E \xrightarrow{\nabla} E \otimes O_V \Omega^1_V \xrightarrow{\nabla} E \otimes O_V \Omega^2_V \to \cdots \]  
(2.2)

A map of connections $m : (E, \nabla) \to (E', \nabla')$ is an $O_V$-linear map $m : E \to E'$ such that $\nabla \circ m = (m \otimes \text{id}) \circ \nabla$. This makes the collection of all connections on $V$ into a category. The trivial connection is the connection $1 := (O_V, d)$. We will also call a connection trivial if it is a direct sum of $1$'s. A connection is called unipotent if it is a successive extension of trivial connections.

The following easy result will be the key to many of the constructions to follow:

**Lemma 2.1.** Let $(E, \nabla)$ be a connection with $E$ torsion free and let $f : E \to B$ be a map of $O_V$-modules with $B$ torsion free as well. Let $A \subset E$ be the kernel of $f$.

Construct by induction a sequence of subsheaves:
\[ A_0 = A, \ A_{n+1} = \text{Ker} \left[ A_n \xrightarrow{\nabla} E \otimes O_V \Omega^1_V \to (E \otimes O_V \Omega^1_V)/(A_n \otimes O_V \Omega^1_V) \right] . \]

Let $A_\infty = \cap_{n=0}^\infty A_n$. Then

1. $\nabla$ maps $A_\infty$ to $A_\infty \otimes O_V \Omega^1_V$.
2. The pair $\nabla^{-\infty}A := (A_\infty, \nabla)$ is a connection.
3. Any map of connections to $(E, \nabla)$ whose image lies in $A$ factors through $\nabla^{-\infty}A$.

**Proof.** The first and third assertions are clear. For the second assertion we need to check that $A_\infty$ is coherent. The Leibniz formula implies that the maps defining the $A_n$’s are $O_V$-linear. Therefore, we see by induction that $A_n$, being a kernel of an $O_V$-map of coherent sheaves and using [BGR84, Proposition 9.4.3.2.ii], is coherent.

Let $B_n := E/A_n$. We show by induction on $n$ that $B_n$ is also torsion free. For $n = 0$ this is clear since $B_0 \subset B$. Suppose that $B_n$ is torsion free. From the short exact sequence
\[ 0 \to A_n \to E \to B_n \to 0 \]
and the fact that $\Omega^1_V$ is locally free since $V$ is smooth we find
\[(E \otimes_{O_V} \Omega^1_V)/(A_n \otimes_{O_V} \Omega^1_V) = B_n \otimes_{O_V} \Omega^1_V.\]

It follows that $A_n/A_{n+1} \subset B_n \otimes_{O_V} \Omega^1_V$ and from the induction hypothesis we get that $A_n/A_{n+1}$ is torsion free. The short exact sequence
\[0 \to A_n/A_{n+1} \to B_{n+1} \to B_n \to 0\]
and the induction hypothesis now show that $B_{n+1}$ is an extension of torsion free modules hence also torsion free.

In the course of the proof we showed that $A_n/A_{n+1}$ is torsion free for each $n$. Consider now an open affinoid domain $U \subset V$. It follows by rank consideration that the sequence $A_n|_U$ must stabilize. Indeed, since both $A_n$ and $A_n/A_{n+1}$ are coherent, it follows that their restrictions are associated to the $O_V(U)$-modules $A_n(U)$ and $A_n/A_{n+1}(U)$. By Noether’s normalization lemma [BGR84, Corollary 6.1.2.2] the algebra $O_V(U)$ is finite over some Tate algebra $T_i$ and therefore $\mathbb{L} := O_V(U) \otimes_{T_i} \text{Frac}(T_i)$ is a product of fields. By tensoring with $\mathbb{L}$ and considering the dimensions over the various fields in the product it is clear that $(A_n(U)/A_{n+1}(U)) \otimes_{O_V(U)} \mathbb{L} = 0$ for sufficiently large $n$, which implies, since $A_n(U)/A_{n+1}(U)$ is torsion free, that $A_n(U)/A_{n+1}(U) = 0$. It follows that $A_{n+1}|_U = A_n|_U$ for a sufficiently large $n$ and is thus associated with the module $A_{\infty}(U)$. This proves that $A_{\infty}$ is coherent and completes the proof of the lemma.

From the construction of $\nabla^{-\infty}A$ the following results are clear.

**Lemma 2.2.** In the situation of the previous lemma suppose that $U \subset V$ is an open subspace. Then $\nabla^{-\infty}(A|_U) = (\nabla^{-\infty}A)|_U$.

**Lemma 2.3.** If, in the situation of the previous lemma, we have $y \in A(U)$ and $\nabla y = 0$, then $y \in \nabla^{-\infty}A(U)$.

We may apply the lemmas in particular to the kernel of the curvature. We obtain a connection
\[(2.3) \quad E^{\text{int}} := \nabla^{-\infty} \text{Ker} \nabla^2.\]
The following are immediate consequences of the two lemmas.

**Corollary 2.4.** Every map of connections from an integrable connection to $(E, \nabla)$ factors through $E^{\text{int}}$.

**Corollary 2.5.** If $U \subset V$ is an open subspace and $(E, \nabla)$ is a connection on $V$, then $E^{\text{int}}|_U = (E|_U)^{\text{int}}$.

**Corollary 2.6.** If, in the situation of the previous corollary, we have $y \in E(U)$ and $\nabla y = 0$, then $y \in E^{\text{int}}(U)$.

**Lemma 2.7.** If $V$ is an affinoid space and $E$ is a unipotent connection, then the underlying $O_V$-module is free.

**Proof.** This is because on an affinoid space all extensions of $O_V$ by itself are trivial. Indeed, $\text{Ext}^1(O_V, O_V) = H^1(V, O_V) = 0$.

We now recall Berthelot’s notions of tubes and strict neighborhoods and some related terminology from [Bes00b].
Definition 2.8 ([Bes00b, Section 4]). A rigid triple is a triple \((X, Y, P)\) consisting of \(P\), a formal \(p\)-adic \(V\)-scheme, \(Y\) a closed \(\kappa\)-subscheme of \(P\) which is proper over \(\text{Spec}(\kappa)\) and \(X\) an open \(\kappa\)-subscheme of \(Y\) such that \(P\) is smooth in a neighborhood of \(X\).

We will denote a rigid triple by a single letter, usually \(T\). We let \(j\) denote the embedding of \(X\) in \(Y\). When \(T = (X, Y, P)\) is a rigid triple, Berthelot defines the following spaces and notions. To \(P\) corresponds a rigid analytic space \(P_K\), called the generic fiber of \(P\), together with a specialization map \(\text{sp} : P_K \to P\). To any locally open subset \(Z\) of \(P\) (in particular to \(X\) or \(Y\)) corresponds a tube \(|Z|_p\), which is a rigid analytic subspace of \(P_K\) whose underlying set of points is the set \(\text{sp}^{-1}(Z)\). Let \(Z = Y - X\). An admissible open \(U \subset |Y|_p\) is called a strict neighborhood of \(|X|_p\) if \((|U|_p, |Z|_p)\) is an admissible cover of \(|Y|_p\). Let \(V\) be such a strict neighborhood. Berthelot defines a functor \(j^\dagger\) from the category of sheaves on \(V\) to itself by

\[
j^\dagger(F) = \lim_{U} j_U^* F,
\]

where the direct limit is over all \(U\) which are strict neighborhoods of \(|X|_p\) in \(|Y|_p\) contained in \(V\) and \(j_U^*\) is the canonical embedding. We always have \((j^\dagger E)|_{|X|_p} = E|_{|X|_p}\).

A sheaf of \(j^\dagger \mathcal{O}_Y[p]\)-modules on \(|Y|_p\) will be called a \(j^\dagger \mathcal{O}_Y|\cdot|\)-module for short. The notion of coherence for such modules is recalled in [Ber96] just before (2.1.9).

Proposition 2.9 ([Ber96, Proposition (2.1.10)]). Any coherent \(j^\dagger \mathcal{O}_Y[p]\)-module is obtained from a coherent \(\mathcal{O}_V[p]\)-module for some strict neighborhood \(V\) by applying \(j^\dagger\) and any morphism between two such \(j^\dagger \mathcal{O}_Y[p]\)-modules is \(j^\dagger\) of a morphism over a possibly smaller strict neighborhood.

Definition 2.10 ([Bes00b, 4.4]). Rigid triples are made into a category in the following way: If \(T' = (X', Y', P')\) is another rigid triple and \(f : X \to X'\) is a \(\kappa\)-morphism, a map of rigid spaces \(F : U \to Y'|_{p'},\) where \(U\) is a strict neighborhood of \(|X|_p\) in \(|Y|_{p}\), is said to be compatible with \(f\) if its restriction to \(|X|_p\) lands in \(|X'|_{p'}\) and is compatible with \(f\) via the specialization map. A morphism between \(T\) and \(T'\) consists then of a morphism \(f\) together with a germ of a morphism \(F\) compatible with \(f\).

It is proved in [Bes00b, Lemma 4.3] that the inverse image of a strict neighborhood of \(|X'|_{p'}\) in \(|Y'|_{p'}\) under a compatible morphism \(F\) as above is a strict neighborhood of \(|X|_p\) in \(|Y|_{p}\). This easily shows that morphisms of rigid triples can be composed. This also shows that there is a pullback map \(f^*\) from \(j^\dagger \mathcal{O}_Y|\cdot|\)-modules to \(j^\dagger \mathcal{O}_Y[p]\)-modules along a morphism \(f : T \to T'\).

A connection on a \(j^\dagger \mathcal{O}_Y[p]\)-module \(E\) is a map

\[
\nabla : E \to E \otimes_{\mathcal{O}_Y[p]} \mathcal{O}_Y^1|\cdot|_p
\]

satisfying the Leibniz formula. Such a connection is called integrable if its curvature, defined as in (2.1), is 0. The following proposition is part (i) of [Ber96, Proposition (2.2.3)].

Proposition 2.11. If \(E\) is a \(j^\dagger \mathcal{O}_Y[p]\)-module with an integrable connection \(\nabla\), then there exists a strict neighborhood \(V\) of \(|X|_p\) and a connection \((E_0, \nabla_0)\) on \(V\) such that \((E, \nabla) = j^\dagger (E_0, \nabla_0)\).
Let $E$ be a coherent $j^!\mathcal{O}_{Y^!}$-module with a connection $\nabla$ and let $E \to B$ be a map to another $j^!\mathcal{O}_{Y^!}$-module with kernel $A$. Suppose that there exists a strict neighborhood $V$ of $X[p]$, a connection $(E_0, \nabla_0)$ on $V$ and a map of locally free $\mathcal{O}_V$-modules $E_0 \to B_0$ such that $(E, \nabla) = j^!(E_0, \nabla_0)$ and $(E \to B) = j^!(E_0 \to B_0)$. We have $A = j^!A_0$ with $A_0 = \text{Ker}(E_0 \to B_0)$.

**Proposition 2.12.** The connection $\nabla^{-\infty}A := j^!\nabla_0^{-\infty}A_0$ is independent of the choice of $V$, $E_0$, $\nabla_0$ and $B_0$.

**Proof.** Immediate from Lemma 2.2.

We will abuse the standard terminology and call a $j^!\mathcal{O}_{Y^!}$-module $E$ together with an integrable connection an isocrystal on the triple $T$. An isocrystal is called overconvergent if the Taylor expansion map gives an isomorphism on a strict neighborhood of the diagonal. We denote the category of overconvergent isocrystals on $T$ by $\text{Isoc}^1(T)$. The category $\text{Isoc}^1(X, Y, P)$ is independent up to equivalence of $Y$ and $P$ and therefore we may talk of overconvergent isocrystals on $X$. We denote the category of overconvergent isocrystals on $X$ by $\text{Isoc}^1(X, K)$ or simply by $\text{Isoc}^1(X)$ if $K$ is understood. If $f : X' \to X$ is a $\kappa$-morphism there is a pullback map from $\text{Isoc}^1(X)$ to $\text{Isoc}^1(X')$ which is defined in [Ber96] after Definition (2.3.6). If we have rigid triples $T = (X, Y, P)$ and $T' = (X', Y', P')$ and $f$ extends to a map of triples one can show that this pullback map can be realized as the pullback map along this morphism of triples.

An overconvergent isocrystal over $\text{Spec}(\kappa)$ is nothing else than a $K$-vector space, as can easily be seen by choosing the rigid triple $(\text{Spec}(\kappa), \text{Spec}(\kappa), \text{Spf}(V))$. Given a point $x : \text{Spec}(\kappa) \to X$ we can pullback along $x$ to obtain a functor

$$\omega_x = x^* : \text{Isoc}^1(X) \to \mathcal{V}ec_K.$$  

To describe this explicitly it is useful to also consider the rigid triple

$$T_x := (x, x, P).$$

Here of course, the scheme $x$ really means $\text{Spec}(\kappa)$ embedded in $P$ via $x$. In this case the corresponding tube is the so called *residue class* in Coleman’s terminology

$$U_x := ]x[p,$$

and the functor $\omega_x$ is obtained as

$$\omega_x(F, \nabla) = \{v \in F(U_x), \nabla(v) = 0\}$$

If $f : X' \to X$ is a $\kappa$-morphism, $x' \in X'(\kappa)$ and $f(x') = x$, then there is a tautological natural isomorphism

$$\omega_x \cong \omega_x' \circ f^*.$$

An isocrystal on $X$ or on $T$ is called unipotent if it is a successive extension of trivial connections. We denote the category of all unipotent isocrystals on $X$ by $\text{Un}(X, K)$ or $\text{Un}(X)$ if $K$ is understood. It follows from [CLS99a, Proposition 1.2.2] that a unipotent isocrystal is always overconvergent. It follows easily from Proposition 2.9 that if $E$ is a unipotent isocrystal on $(X, Y, P)$ then there exists a strict neighborhood $U$ of $]X[p$ and an integrable unipotent connection on $U$ whose $j^\dagger$ gives $E$. 

Proposition 2.13 ([CLS99a, Proposition I.2.3.2]). The category $\mathcal{U}n(X)$ is closed under extensions, subobjects, quotients, tensor products internal homs duals and inverse images. A point $x \in X(\kappa)$ supplies a fiber functor $\omega_x$ making $\mathcal{U}n(X)$ a neutral Tannakian category.

The cohomology of an overconvergent isocrystal $E$ on $T = (X, Y, P)$ is given by $H^i(Y(P, j)(E \to E \otimes \Omega^1 \to \cdots))$, where the complex in brackets is the de Rham complex of $E$. This cohomology depends only on $X$ hence can be denoted by $H^i(X, E)$.

From now until the end of this section we assume that $X$ is smooth.

Lemma 2.14. Let $U$ be an open dense subset of $X$ and let $E$ be a unipotent isocrystal on $X$. Then the map $H^0(X, E) \to H^0(U, E)$ is an isomorphism.

Proof. This follows if we show that if $D$ is the complement of $U$ then $H^1_D(X, E) = 0$. This is true for the trivial isocrystal by purity [Ber97, Corollaire 5.7] and then follows easily for all unipotent isocrystals by induction. □

Corollary 2.15. In the situation of the previous lemma suppose that $F$ is another unipotent isocrystal. Then we have $\text{Hom}(E, F) \cong \text{Hom}(E|_U, F|_U)$.

Proof. This follows since $\text{Hom}(E, F) = H^0(E \otimes F^*)$ and $E \otimes F^*$ is unipotent by Proposition 2.13. □

Proposition 2.16. Let $U$ be an open subset of $X$. Let $E$ be a unipotent isocrystal on $X$ and let $G$ be a subcrystal of $E|_U$. Then $G$ extends to $X$, i.e., there exists a subcrystal $G'$ of $E$ such that $G'|_U = G$.

Proof. We may assume that $U$ is dense in $X$. Otherwise, since $X$ is smooth we can extend $G$ to the connected components with non-empty intersection with it and extend it as zero in the other components. We use induction on the rank of $E$. We have a short exact sequence

$$0 \to 1 \to E \to E' \to 0$$

with $E'$ unipotent as well. Let $G''$ be the image in $E'|_U$ of $G$. By the induction hypothesis it is the restriction to $U$ of a subcrystal $E''$ of $E'$. Pulling back (2.8) via $E'' \to E'$ we see that we may assume that $G \to E'|_U$ is surjective. Now by rank considerations, either $G = E|_U$, in which case the result is clear, or $G \to E'|_U$ is an isomorphism. In this last case we find a splitting $E' \to E$ over $U$ and by Corollary 2.15 this extends to $X$ giving us our required subcrystal. □

3. Unipotent groups

As explained in the previous section the category $\mathcal{U}n(X)$ of unipotent isocrystals on a smooth scheme $X$ is a Tannakian category and to every point $x \in X(\kappa)$ corresponds a fiber functor $\omega_x$. At this point we do assume we have such a point. We can always achieve that by making an extension of scalars. From the theory of Tannakian categories [DM82] it follows that to $\mathcal{U}n(X)$ corresponds an affine group scheme $G = \pi_1^{\text{rig,un}}(X, x)$. Let $F$ be a Frobenius endomorphism on $X$ fixing $x$ (such an automorphism always exists). It induces an automorphism of $G$, denoted $\phi$. If $y \in X(\kappa)$ is another point of $X$, then there is a principal $G$-homogeneous space $P_{x, y}$ representing the functor $\text{Isom}(\omega_x, \omega_y)$. In particular, an element of $P_{x, y}(K)$ consists of a family of isomorphisms $\lambda_E : \omega_x(E) \to \omega_y(E)$ indexed by the objects of $\mathcal{U}n(X)$ and satisfying the following properties:
1. The map $\lambda_1$ is the identity on $K$.
2. For every two objects $M$ and $N$ of $\mathcal{U}n(X)$ we have $\lambda_M \otimes N = \lambda_M \otimes \lambda_N$.
3. For every map $\alpha : M \to N$ we have $\lambda_N \circ \omega_x(\alpha) = \omega_y(\alpha) \circ \lambda_M$.

There is an obvious composition map

\[(3.1) \quad P_{x,y} \times P_{y,z} \to P_{x,z} .\]

If $f : X \to Y$ is a morphism with $f(x') = x$ and $f(y') = y$, there is an induced map $f_\ast : P_{x',y'} \to P_{x,y}$. On $K$-points this map sends the collection of maps $\lambda_E$ to the collection of maps

$$f_\ast(\lambda)_E := \lambda_{f^*E} : \omega_z(E) = \omega_{z'}(f^*E) \to \omega_{y'}(f^*E) = \omega_y(E) .$$

Suppose our Frobenius endomorphism fixes both $x$ and $y$. The action of Frobenius then induces an automorphism of $P_{x,y}$, denoted $\varphi$, compatible with the $G$-action in the sense that $\varphi(ga) = \phi(g)\varphi(a)$ for $g \in G$ and $x \in P$. The goal of this section is to prove the following theorem.

**Theorem 3.1.** The map $G \to G$ given by $g \mapsto g^{-1}\phi(g)$ is an isomorphism.

**Corollary 3.2.** For any $x, y \in X(k)$ there is a unique $a_{x,y} \in P_{x,y}(K)$ fixed by the automorphism $\varphi$ induced by any Frobenius endomorphism $F$ fixing both $x$ and $y$. Furthermore, The composition map (3.1) sends $(a_{x,y}, a_{y,z})$ to $a_{x,z}$.

**Proof.** Any two Frobenius endomorphisms have a common power so that it is sufficient to prove the result for a single Frobenius endomorphism. We first show uniqueness. In fact we show this for a point of $P_{x,y}$ in an arbitrary extension field $L$. Suppose $a$ and $ga$ are both fixed by $\varphi$ with $g \in G(L)$. Then we have

$$ga = \varphi(ga) = \phi(g)\varphi(a) = \phi(g)a .$$

Therefore, $g = \phi(g)$ so $g^{-1}\phi(g) = e = e^{-1}\phi(e)$ so the theorem implies $g = e$. We now show existence. For some possibly huge field $L/K$, which we may assume Galois by further extension, we have a point $a_0 \in P_{x,y}(L)$. For the point $a = ga_0$ to be a fixed point we need

$$ga_0 = \varphi(ga_0) = \phi(g)\varphi(a_0) ,$$

or $g^{-1}\phi(g)\varphi(a_0) = a_0$. Let $h \in G(L)$ be the unique element such that $h\varphi(a_0) = a_0$. By the theorem we may solve $g^{-1}\phi(g) = h$ and obtain our fixed point $a = a_{x,y}$.

Since the fixed point $a$ is unique, we have $\sigma(a) = a$ for every Galois automorphism, which shows that $a$ is already defined over $K$. The behavior of these points under composition is again clear from uniqueness.

We can reformulate the above result in much more elementary terms using the residue classes of (2.6).

**Corollary 3.3.** For every unipotent overconvergent isocrystal $E$ on a rigid triple $(X, Z, P)$, for every two points $x, y \in X(k)$ and for every $v_x \in E(U_x)$ with $\nabla(v_x) = 0$ there is a unique way to associate $v_y \in E(U_y)$ with $\nabla(v_y) = 0$ such that the following properties are satisfied:

1. The association $v_x \mapsto v_y$ is $K$-linear.
2. For the trivial isocrystal it sends $0$ to $1$.
3. It is functorial in the following sense: If $f : E \to E'$ is a morphism of isocrystals then $f(v_y)$ is associated with $f(v_x)$. 

4. It is compatible with tensor products in the following sense: If $v'_x$ is a horizontal section of the isocrystal $E'$ on $U_x$ associated with the horizontal section $v'_y \in E'(U_y)$, then the section $v_x \otimes v'_x \in E \otimes E'(U_x)$ is associated with $v_y \otimes v'_y \in E \otimes E'(U_y)$.

5. It is compatible with Frobenius in the following sense: Let $F : X \to X$ be a Frobenius endomorphism that fixes the points $x$ and $y$. Then we have the commutative diagram

$$
\begin{array}{ccc}
\omega_x(E) & \longrightarrow & \omega_y(E) \\
\downarrow & & \downarrow \\
\omega_x(F^*(E)) & \longrightarrow & \omega_y(F^*(E))
\end{array}
$$

with the vertical isomorphisms (2.7).

In addition this association satisfies that if $v_y$ is associated with $v_x$ and $v_z$ is associated with $v_y$, then $v_z$ is associated with $v_x$.

Proof. This is just a reformulation of corollary 3.2. □

The last property above is valid for any morphism, not just a Frobenius endomorphism, and in still greater generality.

**Proposition 3.4.** Let $f : X \to Y$ be a morphism and suppose $f(x') = x$ and $f(y') = y$. Let $E \in \mathcal{U}(Y)$. Then we have the commutative diagram

$$
\begin{array}{ccc}
\omega_x(E) & \longrightarrow & \omega_y(E) \\
\downarrow & & \downarrow \\
\omega_x(f^*(E)) & \longrightarrow & \omega_y(f^*(E))
\end{array}
$$

Proof. In the diagram above, the horizontal maps are induced by the elements $a_{x,y} \in P_{x,y}$ and $a_{x',y'} \in P_{x',y'}$ in the top and bottom row respectively. The vertical maps are given in (2.7) For appropriate choice of Frobenius endomorphisms $F_X$ and $F_Y$ on $X$ and $Y$ respectively, inducing maps $\varphi_X$ and $\varphi_Y$ on the corresponding principal spaces, we have $F_Y \circ f = f \circ F_X$. It follows from this that $f_*(a_{x',y'})$ is $\varphi_Y$ invariant and therefore must be equal to $a_{x,y}$. Spelling this out gives the result. □

**Definition 3.5.** We call the association $v_x \mapsto v_y$ analytic continuation along Frobenius from $x$ to $y$.

To begin the proof of the theorem we need to recall the work of Chiarellotto [Chi98]. Chiarellotto considers the completed enveloping algebra $U = \hat{U}(\text{Lie} G)$ of the Lie algebra of the group $G$ together with its augmentation ideal $\mathfrak{a}$. We summarize the results of loc. cit. that we need in the following theorem.

**Theorem 3.6.**

1. The $K$-algebra $U$ is complete with respect to the $\mathfrak{a}$-adic topology.

2. The quotients $U_n = U/\mathfrak{a}^n$ are finite dimensional over $K$.

3. The category $\text{Rep}_K(G)$ of algebraic representations of $G$ over $K$ is equivalent to the category $\text{Mod}_U$ of $U$-modules which are finite dimensional $K$-vector spaces and for which the action of $U$ is continuous when they are provided with the discrete topology. In particular, the category $\text{Mod}_U$ is Tannakian.
and the functor \( \omega : \text{Mod}_U \to \text{Vec}_K \) sending a module to its underlying vector space is a fiber functor.

4. There is an automorphism \( \phi : U \to U \) of the augmented algebra \( U \) such that under the equivalence \( \text{Mod}_U \cong \text{Rep}_K(G) \cong \text{Un}(X) \) the operations of twisting a \( U \)-module by \( \phi \), twisting a \( G \)-representation by \( \phi \) and the pullback \( F^* \) via \( F \) correspond.

5. The \( \phi \)-module \( a^n/a^{n+1} \) is mixed with negative weights, i.e., \( \phi \) has on it eigenvalues which are Weil numbers with strictly negative weights.

Proof. Part 1 is [Chi98, Lemma II.2.4]. 2. and 5. both follow from the fact, proved in the course of proving Proposition II.3.3.1 there, that there is a surjective morphism, Frobenius equivariant from its construction, \( H_{\text{rig}}^1(X/K)^{\otimes (-n)} \to a^n/a^{n+1} \), and it is well known that \( H_{\text{rig}}^1(X/K) \) is mixed with positive weights.

Part 3 is stated in [Chi98, II, §2] and proved in the appendix to [CLS99b]. Finally, 4. is discussed in section II.3 of [Chi98]. \( \Box \)

We denote by \( \pi_{n,m} : U_n \to U_m \) the obvious projection. Notice that the completeness with respect to the \( a \)-adic topology means that \( U = \varprojlim_n U_n \). We denote by \( U \hat{\otimes} U \) the completed tensor product which can be defined by

\[
U \hat{\otimes} U := \lim_n U_n \otimes U_n.
\]

The next result is fairly obvious and probably well known, giving \( U \) part of the structure of a coalgebra.

**Lemma 3.7.** There exists a canonical algebra homomorphism \( \Delta : U \to U \hat{\otimes} U \) such that the tensor product in \( \text{Mod}_U \), which we denote by \( \boxtimes \), is given in the following way: If \( M \) and \( N \) are two \( U \)-modules, then \( M \boxtimes N \) has the tensor product over \( K \) as an underlying vector space and the \( U \)-action is the composition of \( \Delta \) with the obvious \( U \hat{\otimes} U \)-action.

Proof. That \( M \boxtimes N \) has the underlying \( K \)-vector space structure \( M \otimes_K N \) is clear from the fact that \( \omega \) is a tensor functor. By Theorem 3.6.2 we have \( U_n \in \text{Mod}_U \), hence \( U_n \boxtimes U_n \in \text{Mod}_U \). Write the action of \( U \) on objects of \( \text{Mod}_U \) as \( (u, m) \mapsto u \circ m \). Let \( u \in U \). Then we define a sequence \( \Delta_n(u) := u \circ (1 \otimes 1) \in U_n \otimes U_n \), where \( 1 \otimes 1 \in U_n \otimes U_n \). It is easy to see that the \( \Delta_n(u) \) define an element \( \Delta(u) \in U \hat{\otimes} U \) and that \( \Delta \) so defined is \( K \)-linear. Suppose now that \( M \) and \( N \) belong to \( \text{Mod}_U \). The \( U \)-action factors through some \( U_n \). For each \( x \in M \) and \( y \in N \) there are maps of \( U \)-modules, \( u \mapsto ux \) and \( u \mapsto uy \), from \( U_n \) to \( M \) and \( N \) respectively. Their tensor product sends \( 1 \otimes 1 \) to \( x \otimes y \) hence \( \Delta_n(u) = u \circ (1 \otimes 1) \) to \( u \circ (x \otimes y) \) so \( u \circ (x \otimes y) = \Delta_n(u)(x \otimes y) = \Delta(u)(x \otimes y) \). This is the last assertion of the lemma from which the fact that \( \Delta \) is a ring homomorphism follows easily. Indeed, we have

\[
\Delta_n(uv) = (uv) \circ (1 \otimes 1) = u \circ (v \circ (1 \otimes 1)) = \Delta_n(u)\Delta_n(v)(1 \otimes 1) = \Delta_n(u)\Delta_n(v).
\]

\( \Box \)

Now let \( L \) be a commutative \( K \)-algebra. We have \( U_L := U \hat{\otimes} L = \varprojlim_n U_n \otimes_K L \). The maps \( \epsilon \) and \( \Delta \) extend to \( \epsilon_L : U_L \to L \) and \( \Delta_L : U_L \to U_L \hat{\otimes} U_L \).

**Proposition 3.8.** Let \( G' \) be the functor from commutative \( K \)-algebras to groups given by

\[
G'(L) = \{ u \in U_L | \epsilon_L(u) = 1, \Delta_L(u) = u \otimes u \},
\]
which is a group with the multiplication induced by the algebra multiplication on $U_L$. Then there is a natural isomorphism $G \cong G'$

Proof. By Theorem 3.6.3 the group $G$ is the group corresponding to the Tannakian category $\text{Mod}_U$ together with the fiber functor $\omega$ sending a module to its underlying vector space. We recall that an element of $G(L)$ consists of a family of maps $(\lambda_M)$, where $M$ runs over the objects of $\text{Mod}_U$. For each $M$, $\lambda_M : \omega(M) \otimes_K L \to \omega(M) \otimes_K L$ is an automorphism. Such a family should satisfy the following properties:

1. The map $\lambda_1$ is the identity on $L$, where $1$ is the unit object of $\text{Mod}_U$, i.e., the $U$-module $U/\mathfrak{a}$.
2. For any two objects $M$ and $N$ we have $\lambda_{MN} = \lambda_M \otimes_L \lambda_N$.
3. For every map $\alpha : M \to N$ we have $\lambda_N \circ (\omega(\alpha) \otimes 1) = (\omega(\alpha) \otimes 1) \circ \lambda_M$.

It is clear that $G'(L)$ is a group. Let $M \in \text{Mod}_U$. Clearly $U_L$ acts on $M \otimes_K L$. An element $u \in G'(L)$ therefore defines an automorphism $\lambda_M$ of $M \otimes_K L$. It is immediately checked that the family $(\lambda_M)$ defines an element of $G(L)$. This gives a map $G'(L) \to G(L)$. The inverse map is given as follows: Let $(\lambda_M)$ be an element of $G(L)$. We define $u_n = \lambda_{U_n}(1) \in U_n \otimes_K L$. The elements $u_n$ are compatible under the maps $\pi_{n,m} \otimes 1$ since the identity elements are. They thus define an element $u \in U_L$. The inverse map will map $(\lambda_M)$ to $u$ and this map is well defined once we show that $u \in G'(L)$. That $\epsilon_L(u) = 1$ follows immediately from $\lambda_1 = 1$. We next show that for any $M \in \text{Mod}_U$ the automorphism $\lambda_M$ is given by multiplication by $u$. Suppose $M$ is in fact a $U_n$-module. Let $x \in M$ and define a map $\alpha_x : U_n \to M$ by $\alpha_x(v) = vx$. This is clearly a map of $U$-modules. Then

$$\lambda_M(x \otimes 1) = \lambda_M \circ (\alpha_x \otimes 1)(1) = (\alpha_x \otimes 1) \circ \lambda_{U_n}(1) = (\alpha_x \otimes 1) \circ u_n = u_n \cdot (x \otimes 1) = u \cdot (x \otimes 1).$$

By $L$-linearity this extends to give the claim. This shows that the composition $G(L) \to G'(L) \to G(L)$ is the identity. It remains to check that $\Delta(u) = u \otimes u$ and for that it suffices to check that $\Delta_n(u_n) = u_n \otimes u_n$. But this is now clear from property 2 of the family $\lambda_M$ and from the construction of $\Delta_n$. 

We henceforth identify $G$ with $G'$. By Theorem 3.6.4 twisting the $U$-action by $\phi$ is a tensor functor, from which it follows via Lemma 3.7 that $\Delta \circ \phi = \phi \circ \phi \circ \Delta$. This immediately shows that the obvious action of $\phi$ on $U$ induces an action of $\phi$ on $G'$. In fact, via the identification of $G$ with $G'$ this action is exactly the action of $\phi$ on $G$, as can be straightforwardly checked after noting that this last action is given by sending the collection $(\lambda_M)$ to the collection $(\mu_M)$ where $\mu_M = \lambda_{F^*M}$ ($F^* M$ has the same underlying vector space as $M$) and using Theorem 3.6.4 again.

Proof of Theorem 3.1. We need to show that $g \mapsto g^{-1} \phi(g)$ induces a bijection on $L$-points for any commutative $K$-algebra $L$. We first claim that for any $a \in U_L$ with $\epsilon_L(a) = 1$ there is a unique $x \in U_L$ with $\epsilon_L(x) = 1$ such that $\phi(x) = xa$. We show by induction the same for $x_n \in U_n \otimes L$. Then by uniqueness these solutions glue to give the unique $x$. For $n = 1$ there is nothing to prove. Suppose we already found $x_{n-1}$. Let $B = \{x \in U_n \otimes L | \pi_{n,n-1}(x) = x_{n-1}\}$. This is an affine space for $C \otimes_K L$ where $C := a^n - 1/a^n$. We have a map $T : B \to C \otimes_K L$ given by $T(x) = \phi(x) - xa$. Since $\epsilon_L(a) = 1$ we find for $y \in C \otimes_K L$ that $T(y + x) = (S \otimes 1)(y) + T(x)$ with $S : C \to C$ given by $S(y) = \phi(y) - y$. It follows from Theorem 3.6.5 that $S$ is invertible. This immediately implies that $T$ is invertible and in particular is a unique solution $x_n$ to $T(x_n) = 0$. 


It remains to prove that if \( a \in G(L) \), then the unique \( x \) so constructed is also in \( G(L) \). To do this we repeat the above argument with \( U_L \) replaced by \( U_L \otimes U_L \), \( \phi \) replaced by \( \phi \otimes \phi \) and \( \epsilon_L \) replaced by \( \epsilon_L \cdot \epsilon_L \). The same argument works by using (3.2) and the fact that the successive quotients \( U_n \otimes U_n / U_n - 1 \otimes U_n - 1 \) still have only negative weights. We need to prove that \( \Delta(x) = x \otimes x \). We have

\[
\Delta(x)(a \otimes a) = \Delta(x)\Delta(a) = \Delta(xa) = \Delta(\phi(x)) = (\phi \otimes \phi)\Delta(x).
\]

On the other hand

\[
(x \otimes x)(a \otimes a) = (xa) \otimes (xa) = \phi(x) \otimes \phi(x) = (\phi \otimes \phi)(x \otimes x).
\]

We see that \( \Delta(x) \) and \( x \otimes x \) are both solutions to the same equation which, since \( (\epsilon_L \cdot \epsilon_L)(a \otimes a) = 1 \) has a unique solution, and they are therefore equal.

\( \square \)

4. Coleman functions

Starting from this section we assume that \( \kappa \) is the algebraic closure of \( \mathbb{F}_p \). This restriction is imposed by wanting \( \kappa \) to be algebraic over \( \mathbb{F}_p \) and also algebraically closed. The first requirement seems essential because we are using Frobenius endomorphisms. The second restriction is not essential but it makes some things more pleasant.

Let \( T = (X, Y, P) \) be a rigid triple as in section 2. As we did in section 3 we denote, for a point \( x \in X \) (a point will always mean a closed point), the corresponding tube \( [x] \cdot p \) by \( U_x \). For a locally free \( j^1 \mathcal{O}_{Y^1} \)-module \( \mathcal{F} \) we define \( \mathcal{F}(T) = \Gamma([Y^1, P, \mathcal{F}] \). We also denote \( A(T) = \Gamma([Y^1, j^1 \mathcal{O}_{Y^1}] \) and \( \Omega^1(T) = \Gamma([Y^1, j^1 \Omega^1_1] \).

**Definition 4.1.** Let \( \mathcal{F} \) be a locally free \( j^1 \mathcal{O}_{Y^1} \)-module. The category \( A_{\text{abs}}(T, \mathcal{F}) \) of abstract Coleman functions on \( T \) with values in \( \mathcal{F} \) is defined as follows: Its objects are triples \( (M, s, y) \) where

- \( M = (M, \nabla) \) is a unipotent isocrystal on \( T \).
- \( s \in \text{Hom}(M, \mathcal{F}) \).
- \( y \) is a collection of sections, \( \{ y_x \in M(U_x), \ x \in X \} \), with \( \nabla(y_x) = 0 \), which correspond to each other via “analytic continuation along Frobenius” as in Definition 3.5.

A homomorphism \( f \) between \( (M, s, y) \) and \( (M', s', y') \) is a morphism of isocrystals \( f : M \to M' \) such that \( f^*(s') = s \) and \( f(y_x) = y'_x \) for any \( x \in X \).

Note that the condition \( f(y_x) = y'_x \) need only be checked at one point and then holds for all other points using Corollary 3.3.3. The direct sum of two abstract Coleman functions is given by

\[
(M_1, s_1, y_1) + (M_2, s_2, y_2) = (M_1 \bigoplus M_2, s_1 + s_2, y_1 \oplus y_2).
\]

**Definition 4.2.** Given two locally free \( j^1 \mathcal{O}_{Y^1} \)-modules \( \mathcal{F} \) and \( \mathcal{G} \) we define a tensor product functor \( \otimes : A_{\text{abs}}(U, \mathcal{F}) \times A_{\text{abs}}(U, \mathcal{G}) \to A_{\text{abs}}(U, \mathcal{F} \otimes \mathcal{G}) \) by the formula

\[
(M_1, s_1, y_1) \otimes (M_2, s_2, y_2) := (M_1 \bigotimes M_2, s_1 \otimes s_2, y_1 \otimes y_2).
\]

**Definition 4.3.** Let \( T \) and \( \mathcal{F} \) be as above. The collection \( A_{\text{Col}}(T, \mathcal{F}) \) of Coleman functions on \( T \) with values in \( \mathcal{F} \) is the collection of connected components of the category \( A_{\text{abs}}(T, \mathcal{F}) \). The Coleman function corresponding to the triple \( (E, s, y) \in A_{\text{Col}}(T, \mathcal{F}) \) is...
A_{abs}(T, F)$ we denote by $[E, s, y]$. In particular we define Coleman functions and forms and Coleman forms with values in a $j^!\mathcal{O}_{Y\mathcal{T}}$-module,

\[
\begin{align*}
A_{\text{Col}}(T) & := A_{\text{Col}}(T, j^!\mathcal{O}_{Y\mathcal{T}}), \\
\Omega^1_{\text{Col}}(T) & := A_{\text{Col}}(T, j^!\Omega^1_{Y\mathcal{T}}), \\
\Omega^i_{\text{Col}}(T, F) & := A_{\text{Col}}(T, F \otimes j^!\mathcal{O}_{Y\mathcal{T}}),
\end{align*}
\]

It will follow from Proposition 4.12 below that the collection of Coleman functions is in fact a set. The following proposition is formal and is left for the reader.

**Proposition 4.4.** The direct sum of abstract Coleman functions gives $A_{\text{Col}}(T, F)$ the structure of an abelian group and it is in fact a $K$-vector space with multiplication by a scalar $\alpha$ given by multiplying (say) the third component with $\alpha$. If $G$ is another locally free $j^!\mathcal{O}_{Y\mathcal{T}}$-module then the tensor product induces a bilinear map $A_{\text{Col}}(T, F) \times A_{\text{Col}}(T, G) \to A_{\text{Col}}(T, F \otimes G)$. In particular, the set $A_{\text{Col}}(T)$ is a commutative $K$-algebra.

**Definition 4.5.** Let $f : F \to G$ be a map of $j^!\mathcal{O}_{Y\mathcal{T}}$-modules. Then we obtain a functor $f^* : A_{\text{abs}}(T, F) \to A_{\text{abs}}(T, G)$ via $f^*(M, s, y) = (M, f(s), y)$, and therefore a map $f^* : A_{\text{Col}}(T, F) \to A_{\text{Col}}(T, G)$.

Recall that if $(F, \nabla_F)$ and $(M, \nabla_M)$ are two connections, then the induced connection on $\text{Hom}(M, F)$ is given by

\[
(\nabla_{\text{Hom}(M, F)} f)(m) = \nabla_F(f(m)) - f \otimes \text{id}_M \cdot \nabla_M(m).
\]

**Definition 4.6.** Let $(F, \nabla_F)$ be a connection and let $(M, \nabla_M)$ be a unipotent isocrystal on $T$. Using the de Rham differentials of $\text{Hom}(M, F)$ we obtain the de Rham differentials on Coleman forms with values in $F$, $\nabla_F : \Omega^i_{\text{Col}}(T, F) \to \Omega^{i+1}_{\text{Col}}(T, F)$ given by $\nabla_F [M, s, y] = [M, \nabla_{\text{Hom}(M, F)} s, y]$. If $\nabla_F$ is integrable, then the de Rham differentials on Coleman forms give a complex,

\[
A_{\text{Col}}(T, F) \to \Omega^1_{\text{Col}}(T, F) \to \Omega^2_{\text{Col}}(T, F) \to \cdots,
\]

called the Coleman de Rham complex of $(F, \nabla_F)$. In particular, for the trivial connection we obtain the Coleman de Rham complex of $T$.

**Definition 4.7.** Let $f : T' \to T$ be a map of triples. Then there is a pullback functor $f^* : A_{\text{abs}}(T, F) \to A_{\text{abs}}(T', f^* F)$ given by $f^*(E, s, y) = (f^* E, f^* s, f^* y)$. This is well defined since Proposition 3.4 shows that the pullbacks of sections corresponding under analytic continuation on $T$ give sections on $T'$ with the same property. This functor induces a $K$-linear map $f^* : A_{\text{Col}}(T, F) \to A_{\text{Col}}(T', f^* F)$. Using Definition 4.5 we immediately obtain a ring homomorphism $f^* : A_{\text{Col}}(T) \to A_{\text{Col}}(T')$ and maps $f^* \Omega^i_{\text{Col}}(T) \to \Omega^i_{\text{Col}}(T)$ compatible with the differentials of the Coleman de Rham complex.

If we wish to interpret Coleman functions as actual functions, in the style of Coleman, we can do the following.

**Lemma 4.8.** Suppose $X = x$ is a point. Let $[M, s, y] \in A_{\text{Col}}(T, F)$. Then the section

\[
\theta([M, s, y]) := s(y_x) \in F(U_x)
\]
depends only on the Coleman function and not on the particular representing abstract Coleman function.
Definition 4.9. The space of locally analytic functions on $T$ with values in $\mathcal{F}$ is defined to be the product

$$A_{\text{loc}}(T, \mathcal{F}) := \prod_{x \in X} \mathcal{F}(U_x).$$

Note that $A_{\text{loc}}(T, j^! \mathcal{O}_Y)$ is naturally a ring.

Definition 4.10. The map $\theta : A_{\text{Col}}(T, \mathcal{F}) \rightarrow A_{\text{loc}}(T, \mathcal{F})$ is defined by

$$f \rightarrow \prod_{x \in X} \theta(x^* f).$$

Proposition 4.11. The map $\theta$ is $K$-linear. For $\mathcal{F} = j^! \mathcal{O}_Y$, it is a ring homomorphism. Given a connection $\nabla$ on $\mathcal{F}$ there is an obvious de Rham differential $\nabla : \Omega^1_{\text{loc}}(T, \mathcal{F}) \rightarrow \Omega^{1+1}_{\text{loc}}(T, \mathcal{F})$ and this differential is compatible with the de Rham differential on Coleman functions via $\theta$.

Proof. The only point which is not completely clear is the compatibility of the differentials on $A_{\text{Col}}$ and $A_{\text{loc}}$. This follows, for example on functions, from the fact that the sections $y_x$ are horizontal, hence for an isocrystal $M$ and a section $s \in \text{Hom}(M, \mathcal{F})$ we have $\nabla_{\mathcal{F}}(s(y_x)) = (\nabla_{\text{Hom}(M, \mathcal{F})}(s))(y_x)$.

Proposition 4.12. For any $z \in X$ the composition $A_{\text{Col}}(T, \mathcal{F}) \xrightarrow{z^*} A_{\text{Col}}(T_z, \mathcal{F}) \xrightarrow{\theta} \mathcal{F}(U_z)$ is injective. In particular, the pullback map $z^* : A_{\text{Col}}(T, \mathcal{F}) \rightarrow A_{\text{Col}}(T_z, \mathcal{F})$ is injective.

Proof. Suppose $s(y_z) = 0 \in \mathcal{F}(U_z)$. Let $M_s = \nabla^{-\infty} \ker(s)$. Since $y_z$ is horizontal we have by Lemma 2.3 that $y_z \in M_s(U_z)$. Applying Corollary 3.3.3 to the inclusion $M_s \subset M$ shows that $y_x \in M_s(U_x)$ for any $x \in X$. It follows that $(M_s, 0, y)$ is an abstract Coleman function and the inclusion $M_s \hookrightarrow M$ defines a morphism of abstract Coleman functions. On the other hand the 0 map provides a map from $(M_s, 0, y)$ to $(0, 0, 0)$ showing that $[M, s, y] = [M_s, 0, y] = 0$.

Corollary 4.13. The uniqueness principle holds for Coleman functions, i.e., if $\theta(f)$ vanishes on an open subset of $|X|_p$, then $f = 0$ and therefore $\theta(f)$ vanishes identically.

Proof. The open subset has a non-zero open intersection with at least one $U_x$, it follows that $\theta(f)$ vanishes on $U_x$ and by the proposition we have $f = 0$.

Corollary 4.14. The kernel of $d$ on $A_{\text{Col}}(T)$ is $K$.

Proof. If $f \in A_{\text{Col}}(T)$ and $df = 0$ then $\theta(f)$ is locally constant. Suppose its value on some $U_x$ is $c$. Then $f$ and $c$ coincide on $U_x$ hence $f = c$ by Corollary 4.13.

Theorem 4.15. Suppose $(F, \nabla_{F})$ is a unipotent isocrystal on $T$. Then the Coleman de Rham complex of $(F, \nabla_{F})$ is exact at the one forms. In particular, the sequence $A_{\text{Col}}(T) \rightarrow \Omega^1_{\text{Col}}(T) \rightarrow \Omega^2_{\text{Col}}(T)$ is exact.

Proof. Let $[E, \omega, y] \in \Omega^1_{\text{Col}}(T, F)$ so $\omega \in \text{Hom}(E, F \otimes \Omega^1)$. Define the connection $M$ as follows. As a $j^! \mathcal{O}_Y$-module $M = E \oplus F$ and $\nabla_M(e, f) = (\nabla_{E}(e), \nabla_{F}(f) - \omega(e))$. We let $\pi_1$ and $\pi_2$ be the projections on $E$ and $F$ respectively. Note that $\pi_1$ is horizontal. The connection $M$ is an extension of unipotent isocrystals and is therefore unipotent. However, it may not be integrable. We consider $N = M^{\text{int}}$. Choose some closed point $x_0 \in X$. Since $[E, \omega, y]$ maps to 0 in $A_{\text{Col}}(T, F \otimes \Omega^2)$
it follows that $\nabla_F(\omega(yx_0)) = 0$. The de Rham complex of $\nabla_F$ restricted to $U_{x_0}$ is exact. Therefore we can choose $g \in F(U_{x_0})$ such that $\nabla_F(g) = \omega(yx_0)$. Then $m_{x_0} := (yx_0, g) \in M(U_{x_0})$ and $\nabla_M(m_{x_0}) = 0$. By Corollary 2.6 we have $m_{x_0} \in N(U_{x_0})$. We can analytically continue $m_{x_0}$ to a collection $m = \{m_x\}$. We claim that the abstract Coleman function $[N, \pi_2, m]$ satisfies $\nabla_F[N, \pi_2, m] = [E, \omega, y]$. Indeed, $\nabla_F[N, \pi_2, m] = [N, \nabla_{\text{Hom}(M,F)}(\pi_2), m]$ and

$$\nabla_{\text{Hom}(M,F)}(\pi_2)(e, f) = \nabla_F(\pi_2(e, f)) - \pi_2(\nabla_M(e, f))$$

$$= \nabla_F(f) - (\nabla_F(f) - \omega(e)) = \omega(e)$$

so $\nabla_{\text{Hom}(M,F)}(\pi_2) = \omega \circ \pi_1$. It follows that the restriction of $\pi_1$ to $N$ induces a morphism $\nabla_F(N, \pi_2, m) \to (E, \omega, y)$ and our claim follows and with it the theorem. 

\[\square\]

**Definition 4.16.** An abstract Coleman function $(M, s, y)$ is called minimal if the following two conditions are satisfied:

1. If $N \subset M$ is a subcrystal and $y_x \in N(U_x)$ for one (hence all) $x \in X$, then $N = M$.
2. There is no non-zero subcrystal of $M$ contained in $\text{Ker} \ s$.

**Lemma 4.17.** Any abstract Coleman function $(M, s, y)$ has a minimal subquotient.

**Proof.** If there is a subcrystal $N$ such that $y_x \in N(U_x)$ then $(N, s|_N, y)$ is a subobject of $(M, s, y)$. If $N \subset \text{Ker} \ s$ then the abstract Coleman function $(M/N, s, y \mod N)$ is a quotient object of $(M, s, y)$. Repeating this we get by rank considerations to a minimal subquotient. 

\[\square\]

**Proposition 4.18.** A minimal abstract Coleman function representing a given Coleman function is unique up to a unique isomorphism.

**Proof.** Suppose that $(M_i, s_i, y_i)$, $i = 1, 2$, represent the same Coleman function. The Coleman function $[M_1 \oplus M_2, s_1 - s_2, y_1 \oplus y_2]$ is therefore 0 and it follows that for any $x \in X$, $(y_1)_x + (y_2)_x \in \text{Ker}(s_1 - s_2)$. Let $N = \nabla^\sim \text{Ker}(s_1 - s_2)$. Then by Lemma 2.3 we have for any $x \in X$ that $(y_1)_x + (y_2)_x \in N(U_x)$. Let $\pi_i$ be the projection from $N$ to $M_i$. We claim that both $\pi_1$ and $\pi_2$ are isomorphisms. To show that $\pi_1$ in injective we notice that $\text{Ker} \pi_1 \iso M_2$ is injective and $\pi_2(\text{Ker} \pi_1)$ is a subcrystal of $M_2$ contained in $\text{Ker} \ s_2$ hence must be 0 by minimality. On the other hand, $\pi_1(N)$ is a subcrystal of $M_1$ and $(y_1)_x \in \pi_1(N)(U_x)$ for any $x \in X$ hence again by minimality $\pi_1(N) = M_1$. The same arguments apply by symmetry to $\pi_2$. It is now clear that the projections induce isomorphisms of abstract Coleman functions $\pi_i : (N, s_i, y_1) \to (M_i, s_i, y_i)$. But $s_1 = s_2$ on $N$ and therefore $(M_1, s_1, y_1) \iso (M_2, s_2, y_2)$. Let $\alpha_1$ and $\alpha_2$ be two maps of $(M_1, s_1, y_1)$ into $(M_2, s_2, y_2)$. Then $\text{Ker}(\alpha_1 - \alpha_2)$ is a subcrystal of $M_1$ containing $y_1$ and therefore must be equal to $M_1$, hence $\alpha_1 = \alpha_2$. 

\[\square\]

Let $T = (X, Y, P)$ be a rigid triple. If $U \subset X$ is open, then the triple

$$(4.1) \quad T_U := (U, Y, P)$$

is also a rigid triple.

**Lemma 4.19.** Let $U$ be an open subset of $X$ and let $(M, s, y)$ be a minimal abstract Coleman function on $T$. Then its restriction to $T_U$ is also minimal.

**Proof.** Easy from Proposition 2.16. 

\[\square\]
Definition 4.20. Let $F$ be a $j^!\mathcal{O}_Y$-module. The association $U \mapsto A_{\text{Col}}(T_U, F)$ defines a presheaf on the Zariski site of $X$. We denote this presheaf by $\mathcal{O}_{\text{Col}}(T, F)$.

Proposition 4.21. The presheaf $\mathcal{O}_{\text{Col}}(T, F)$ is a sheaf.

Proof. We know that for any two opens $U \subset V$ with $U$ not empty the restriction $A_{\text{Col}}(T_V, F) \to A_{\text{Col}}(T_U, F)$ is injective by Proposition 4.12. To prove the sheaf property it therefore suffices to show that if $\{U_i\}$ is an open covering of $U$ and we have $f_i \in A_{\text{Col}}(T_{U_i}, F)$ such that $f_i$ and $f_j$ coincide on $T_{ij} := T_{U_i \cap U_j}$ then there is an $f \in A_{\text{Col}}(T_U, F)$ restricting to the $f_i$. Suppose $f_i$ has a minimal representation $(M_i, s_i, y_i)$. For each pair $i$ and $j$ the restrictions of $(M_i, s_i, y_i)$ and $(M_j, s_j, y_j)$ to $T_{ij}$ are minimal by Lemma 4.19 and represent the same function by assumption. It follows from Proposition 4.18 that there is a unique isomorphism $\alpha_{ij} : M_i|_{T_{ij}} \to M_j|_{T_{ij}}$ carrying $y_i$ to $y_j$ and $s_j$ to $s_i$. By the uniqueness these isomorphisms match well when $i$ and $j$ vary and therefore allow us to glue the $M_i$ to an isocrystal $M$ on $T_U$. It is also clear than that the $s_i$ glue together to $s \in \text{Hom}(M, F)$ and that the $y_i$ taken together supply a well defined horizontal section $y_x \in M(U_x)$ for any $x \in U$. Thus we obtain our required Coleman function.$\square$

5. Comparison with Coleman’s theory

In this section we would like to show that our theory generalizes the theory of iterated integrals due to Coleman [Col82, CdS88]. In Coleman’s theory the functions are built in a recursive process of integration starting from holomorphic forms. To show that our theory gives the same result we begin by showing that in the relevant case our theory admits a similar recursive description.

Definition 5.1. Let $X \subset Y$ be an open immersion of $\mathcal{V}$-schemes such that $X$ is smooth and $Y$ is complete. We associate to the pair $(X, Y)$ the triple $T_{(X, Y)} := (X \otimes_Y \kappa, Y \otimes_Y \kappa, \hat{Y})$, where $\hat{Y}$ is the $p$-adic completion of $Y$. We will call such a rigid triple tight. An affine rigid triple is a tight rigid triple $T_{(X, Y)}$ with $X$ affine.

Lemma 5.2. If $T$ is affine and $E$ is a unipotent isocrystal on $T$, then its underlying $j^!\mathcal{O}_Y$-module is free.

Proof. If $(X, Y, P)$ is affine then there is a basis of strict neighborhoods of $|X|_p$ in $|Y|_p$ which are affinoid (compare the proof of [Ber97, Proposition 1.10]). The result therefore follows from Lemma 2.7.$\square$

We now show that in the affine case our theory admits a recursive description.

Proposition 5.3. Suppose $T$ is affine. Let $A_{\text{Col}, n}(T)$ be defined recursively as follows: Let $A_{\text{Col}, 1}(T) = A(T)$ and let $A_{\text{Col}, n+1}(T)$ be the product inside $A_{\text{Col}}(T)$ of $A(T)$ with $\{f \in A_{\text{Col}}(T), df \in \Omega^1(T) \cdot A_{\text{Col}, n-1}(T)\}$. Then $A_{\text{Col}}(T) = \cup_n A_{\text{Col}, n}(T)$.

We will in fact prove a stronger result that will be needed in the next section.

Definition 5.4. For any locally free $j^!\mathcal{O}_Y$-module $F$ we define the subspace of Coleman functions of degree at most $n$ with values in $F$ on $T$, denoted $A_{\text{Col}, n}(T, F)$ to be the subspace of all Coleman functions $[E, s, y]$ where $E$ has a filtration $E = F^0 \supset F^1 \supset \cdots \supset F^n \supset F^{n+1} = 0$ (i.e., of length $n + 1$) by sub isocrystals and where the graded pieces are trivial connections (recall that we called a connection trivial if it is a direct sum of one dimensional trivial connections).
It is clear that $A_{\text{Col},n}(T,F)$ is an $A(T)$-submodule of $A_{\text{Col}}(T,F)$ and that $A_{\text{Col}}(T,F) = \bigcup_n A_{\text{Col},n}(T,F)$.

**Proposition 5.5.** Suppose $T$ is affine. Then $A_{\text{Col},n}(T,F) = A_{\text{Col},n}(T) \cdot F(T)$, where $A_{\text{Col},n}(T)$ has been defined in Proposition 5.3. In particular, $A_{\text{Col},n}(T)$ is the same as $A_{\text{Col},n}(T,\jmath^!\mathcal{O}_{\mathcal{Y}[\lambda]})$ as defined here.

**Proof.** Suppose $[E,s,y]$ is a Coleman function in a representation of degree at most $n$. Since $T$ is affine the underlying $j^!\mathcal{O}_{\mathcal{Y}[\lambda]}$-module of $E$ is free by Lemma 5.2. We may therefore write $s = \sum g_ir_i$ with $g_i \in \text{Hom}(E,\jmath^!\mathcal{O}_{\mathcal{Y}[\lambda]})$ and $r_i \in F(T)$. It thus suffices to prove the proposition for $F = \jmath^!\mathcal{O}_{\mathcal{Y}[\lambda]}$, i.e., prove that $A_{\text{Col},n}(T) = A_{\text{Col},n}(T,\jmath^!\mathcal{O}_{\mathcal{Y}[\lambda]})$. It follows from the proof of Theorem 4.15 that a closed form in $A_{\text{Col},n}(T,\Omega^1)$ has an integral in $A_{\text{Col},n+1}(T,\jmath^!\mathcal{O}_{\mathcal{Y}[\lambda]})$ and so it is clear by induction that $A_{\text{Col},n}(T) \subset A_{\text{Col},n}(T,\jmath^!\mathcal{O}_{\mathcal{Y}[\lambda]})$. For the other direction, suppose that $[E,s,y]$ is a Coleman function and that $E$ can be written in a short exact sequence $0 \to E_1 \to E \xrightarrow{\pi_2} E_2 \to 0$ where $E_1$ is trivial and $E_2$ has a filtration as in $E$ but of length $n$. We may further find a splitting $\pi_1 : E \to E_1$, which need not be compatible with the connection. Since $s$ can be written as $s_i \circ \pi_1 + s_2 \circ \pi_2$ with $s_i \in \text{Hom}(E_1,\jmath^!\mathcal{O}_{\mathcal{Y}[\lambda]})$ it suffices to prove the result with $s = s_i \circ \pi_1$ for $i = 1,2$. We first notice that $[E,s_2 \circ \pi_2, y] = [E,s_2,\pi_2(y)]$ (since $\pi_2$ is horizontal) so it even belongs to $A_{\text{Col},n-1}(T)$ by the induction hypothesis. Suppose that $s_1 \in \text{Hom}_T(E_1,1)$. Then $\nabla^*(s_1 \circ \pi_1)$ vanishes on $E_1$ and therefore equals $\omega_2 \circ \pi_2$ for some $\omega_2 \in \text{Hom}(E_2,\jmath^!\mathcal{O}_{\mathcal{Y}[\lambda]})$. It follows that $d[E,s_1 \circ \pi_1, y] = [E,\nabla^*(s_1 \circ \pi_1), y] = [E_2,\omega,\pi_2(y)]$. This belongs to $A_{\text{Col},n-1}(T,\jmath^!\mathcal{O}_{\mathcal{Y}[\lambda]})$ hence by the induction hypothesis to $\Omega^1(T) : A_{\text{Col},n-1}(T)$. Note that in the $n = 0$ case we have $E_2 = 0$ and the same argument implies simply that $d[E,s_1 \circ \pi_1, y] = 0$ hence that $[E,s_1 \circ \pi_1, y]$ is a constant. Since $E_1$ is trivial a general $s_1$ can be written as a combination of $s_1$'s of the type that was already considered with coefficients in $A(T)$. This completes the induction step and also shows the case $n = 0$.

**Remark 5.6.** It is easily seen that a unipotent isocrystal $E$ with underlying free $j^!\mathcal{O}_{\mathcal{Y}[\lambda]}$-module is isomorphic to an isocrystal of the form $M_B$ defined as follows: The underlying module is $(j^!\mathcal{O}_{\mathcal{Y}[\lambda]})^n$. The connection depends on the $n \times n$ upper triangular matrix $B$ with entries in $\Omega^1(T)$ and diagonal entries 0 and is given by $\nabla(x_1,\ldots,x_n) = (dx_1,\ldots,dx_n) + (x_1,\ldots,x_n) \cdot B$. For such an isocrystal, having a filtration as above of length $n + 1$ means that $B$ is block upper triangular with $n + 1$ blocks (the blocks are zero matrices).

We now briefly recall Coleman’s theory, using the description given in [Bes00c, Section 2] and some of the notation that was established before. We first make the general remark that Coleman allows $K$ to be any complete subfield of $\mathbb{C}_p$ and there is no assumption that $K$ is discretely valued. The only essential difficulty in extending the general theory to this case is that the Frobenius behavior of rigid cohomology (indeed, even finite dimensionality) is not known in this case. This is the same limitation as in Coleman’s theory. In cases where one knows this behavior we believe the other details can be extended, though we have not checked that in detail.

We consider a pair $(\mathcal{X},\mathcal{Y})$ as in Definition 5.1 where $\mathcal{Y}$ is assumed in addition to be a smooth projective and surjective scheme of relative dimension 1 over $\mathcal{V}$. Let $T = (X,Y,P)$ be the associated rigid triple as above. Then $Y - X = \{e_1,\ldots,e_n\}$, a
finite set of points. A “basic wide open” $U$, in Coleman’s terminology (see [CdS88, 2.1]), is simply a strict neighborhood of $X[p]$ in $Y[p]$ and its “underlying affinoid” is just $X[p]$. The spaces of functions and forms which we called $A(U)$ and $\Omega^1(U)$ in [Bes00c] are what we called here $A(T)$ and $\Omega^1(T)$. An endomorphism $\phi : T \to T$ lifting a Frobenius endomorphisms $X \to X$ is what we called in loc. cit., following Coleman, a Frobenius endomorphism of $U$. A theorem of Coleman [CdS88, Theorem 2.2.] guarantees such a Frobenius endomorphism always exists.

For a sufficiently small $U$ we can set theoretically decompose $U$ into a disjoint union of opens sets $U_x$ over $x \in Y(\kappa)$. For $x \in X$ these sets are the usual residue classes ones defined in (2.6), which we now call residue discs, and are each isomorphic to the open unit disc $\{|z| < 1\}$, while for $x \in \{e_1, \ldots, e_n\}$ these are the intersections of $U$ with the usual discs and are isomorphic to an open annulus $\{r < |z| < 1\}$.

The differential $d : A(U_x) \to \Omega^1(U_x)$ is surjective when $U_x$ is a disc. On the other hand, when $U_x$ is an annulus there is no integral to $dz/z$. To integrate it one needs to introduce a logarithm, and for this one chooses a branch of the $p$-adic logarithm and define $A_{\log}(U_x)$ to be $A(U_x)$ if $U_x$ is a disc and to be the polynomial ring in the function $\log(z)$ over $A(U_x)$ if $U_x$ is an annulus with local parameter $z$ (the choice of the local parameter $z$ does not matter). For either a disc or an annulus $U_x$ with a local parameter $z$ we set $\Omega^1_{\log}(U_x) := A_{\log}(U_x)dz$.

The differential $A_{\log}(U_x) \to \Omega^1_{\log}(U_x)$ is surjective also for the annuli, as one easily discovers by doing integration by parts of polynomials in logs with power series coefficients. Then one defines locally analytic functions and one forms on $U$ by

$$A_{\text{loc}}(U) := \prod_x A_{\log}(U_x), \quad \Omega^1_{\text{loc}}(U) := \prod_x \Omega^1_{\log}(U_x).$$

There is an obvious differential $d : A_{\text{loc}}(U) \to \Omega^1_{\text{loc}}(U)$, which is clearly surjective.

Coleman’s idea is now as follows: One constructs a certain subspace $M(U)$ of $A_{\text{loc}}(U)$, containing $A(U)$, which we call the space of Coleman functions, and a vector space map (integration), which we denote by $\int$ or by $\omega \mapsto F_\omega$, from $W(U) := M(U) \cdot \Omega^1(U)$ (product taking place inside $\Omega^1_{\text{loc}}(U)$) to $M(U)/K \cdot 1$. The map $\int$ is characterized by three properties:

1. It is a primitive for the differential in the sense that $dF_\omega = \omega$.
2. It is Frobenius equivariant in the sense that $\int(\phi^* \omega) = \phi^* \int(\omega)$.
3. If $g \in A(U)$, then $F_{dg} = g + K$.

The construction relies on a simple principle: If $\int$ has already been defined on some space $W$, and $\omega \in \Omega^1_{\text{loc}}(U)$ is such that there is a polynomial $P(t)$ with $K$-coefficients such that $P(\phi(t))\omega = \eta \in W$, then the conditions on the integral force the equality $P(\phi^*)(F_\omega) = F_\eta + \text{Const}$. When $P$ has no roots of unity as roots this condition fixes $F_\omega$ up to a constant. Starting with $W_0(U) = dA(U)$ one finds a unique way of integrating all $\omega \in W_1(U) = \Omega^1(U)$. One defines recursively $M_{i+1}(U) := A(U) \cdot \int(W_i(U))$ and $W_{i+1}(U) := M_{i+1}(U) \cdot \Omega^1(U) = (\int(W_i(U))) \cdot \Omega^1(U)$ and checks that the principle above permits extending $\int$ uniquely to $W_{i+1}(U)$. Finally one sets $M(U) = \bigcup_i M_i(U)$. Then clearly $W(U) = \bigcup_i W_i(U)$. The entire theory turns out to be independent of the choice of $\phi$.

**Theorem 5.7.** In the situation above there exists a ring isomorphism $\tilde{\theta} : A_{\text{Col}}(T) \to M(U)$ and an isomorphism $\tilde{\theta} : \Omega^1_{\text{Col}}(T) \to W(U)$ compatible with the isomorphism
on functions such that for $f \in A_{\text{Col}}(T)$ or $f \in \Omega^1_{\text{Col}}(T)$ we have $\hat{\theta}(f)|_{X[p]} = \theta(f)$. These isomorphisms are also compatible with the differential.

Proof. To define $\hat{\theta}$ we would like to extend the definition of the fiber functors $\omega_x$ to $x \in Y - X$. This can be done as follows: Suppose $N \in \mathcal{U}n(T)$. Then it is $j^! N$ for some $N$ defined on some strict neighborhood of $]X[p]$ in $]Y[p]$. For each $x \in Y - X$ this neighborhood contains an annulus around $x$ isomorphic to $A_r = \{r < |z| < 1\}$ for some $r$. Over $A_r$ the underlying module to $N$ trivializes by Lemma 2.7. Thus it is isomorphic to some $M_B$ as in the proof of Proposition 5.3. Since $d : A_{\log}(U_x) \to \Omega^1_{\log}(U_x)$ is surjective it follows easily that $\nabla_N$ has a full set of solutions on $\tilde{N}(U_x) \otimes_{A(U_x)} A_{\log}(U_x)$ and the functor that sends $N$ to the set of solutions is the required fiber functor. Therefore, given a Coleman function $f = [N, s, y]$, the collection $y$ extends to give $y_x \in \tilde{N}(U_x) \otimes_{A(U_x)} A_{\log}(U_x)$ and applying the section $s$ we get an element in $A_{\log}(U_x)$. Thus, the definition of $\theta$ extends immediately to provide the ring homomorphism $\hat{\theta}$ extending $\theta$ into $A_{\log}(U)$ and a similar map on differential forms. These maps are injective because they extends $\theta$ which is already injective by Proposition 4.12. The maps $\hat{\theta}$ on functions and forms are clearly compatible with differentials. It therefore suffices to prove that the image of $\hat{\theta}$ is exactly $M(U)$ (a similar argument applies to differential forms). We claim that in fact $\theta(A_{\text{Col},n}(T)) = M_n(U)$. By the way these spaces are constructed it is easy to see that using induction we only need to prove the following claim: If $\theta(A_{\text{Col},n-1}(T)) = M_{n-1}(U)$ and $f \in A_{\text{Col},n}(T)$, then $\theta(f) = \int \hat{\theta}(df) + \text{Const}$. This follows by the fact that $f$ is uniquely determined by the condition of being $\phi$-equivariant and the fact that $\hat{\theta}(df) \mapsto \hat{\theta}(f) + \text{Const}$ is $\phi$-equivariant since $\phi$ is an endomorphism of $T$.

6. The $p$-adic $\tilde{\partial}$

In this section we present a construction of what we call the $p$-adic $\tilde{\partial}$ operator on Coleman forms “of first order”. This construction has its roots in a result of Coleman and de Shalit to be recalled below. The justification for the title of $\tilde{\partial}$ mainly comes from the application to $p$-adic Arakelov theory [Bes01].

Let $U$ be a basic wide open in a curve over $\mathbb{C}_p$. Then Coleman and de Shalit [CdS88, Lemma 2.4.4] prove the following result.

**Proposition 6.1.** If $\omega_i, \eta_i \in \Omega^1(U)$, $i = 1, \ldots, n$, $\omega \in \Omega^1(U)$ and the $\eta_i$ are independent in $H^1_{\text{dR}}(U)$, then a relation $\sum \omega_i \cdot \int \eta_i + \omega = 0$ implies that $\omega_i = \omega = 0$ for all $i$.

We remark that the result of Coleman and de Shalit is more general than the one we presented because it deals with integration in an arbitrary logarithmic crystal.

**Corollary 6.2.** There exists a well defined map $\hat{\partial} : W_2(U) \to H^1_{\text{dR}}(U) \otimes \Omega^1(U)$ sending $\Theta = \sum \omega_i \cdot \int \eta_i$ to $\hat{\partial}(\Theta) = \sum [\eta_i] \otimes \omega_i$.

We call $\hat{\partial}$ the $p$-adic $\tilde{\partial}$ operator. We want to generalize the construction in terms of our new definition of Coleman functions. As is obvious from the corollary, our $\tilde{\partial}$ should be defined on Coleman forms of degree at most 1. As it turns out, there is no reason to restrict to forms.

Let $\mathcal{F}$ be a locally free $j^! \mathcal{O}_{\mathbb{C}_p}$-module on $T$. A Coleman function of degree at most 1 on $T$ with values in $\mathcal{F}$ is given in some representation by the following data which we encapsulate in the triple $(\mathcal{E}, s, y)$:
1. An isocrystal $E$ sitting in the short exact sequence $\mathcal{E}$:

$$0 \to E_1 \to E \to E_2 \to 0,$$

such that $E_1$ and $E_2$ are trivial.

2. A homomorphism $s \in \text{Hom}(E, F)$.

3. A compatible system of horizontal section $y_x \in E(U_x)$ for any $x \in X$.

We perform the following construction: The projection of the $y_x$ give a compatible system of horizontal sections of $E_2$. Since $E_2$ is trivial this system comes from a global horizontal section $y_2$ of $E_2$. The isocrystal $E$ gives an extension class $[E] \in \text{Ext}_T^1(E_2, E_1)$. The horizontal section $y$ is an element of $\text{Hom}_\nabla(j^1\mathcal{O}|_Y|, E_2)$. We can pullback the extension $[E]$ via $y_2$ to obtain $[E] \circ y_2 \in \text{Ext}_T^1(j^1\mathcal{O}|_Y|, E_1)$. The homomorphism $s$ restricts to $s_1 \in \text{Hom}(E_1, F)$.

Proposition 6.5. We now show that the definition of $\partial$ depends only on the underlying Coleman function $[E, s, y]$ and the resulting map $\partial: A_{\text{Col}, 1}(T, \mathcal{F}) \to H^1_{\text{rig}}(X) \otimes \mathcal{F}(T)$ is linear.

Proof. First we claim that $\partial(E, s, y)$ depends only on $(E, s, y)$. Indeed, if we are given two short exact sequences for the same $E$, $\mathcal{E}$: $0 \to E_1 \to E \to E_2 \to 0$ and $\mathcal{E}'$: $0 \to E'_1 \to E \to E'_2 \to 0$, we may consider $F$ which is the limit of the diagram $1 \xrightarrow{(y_2, y_2')} E_2 \oplus E'_2 \leftarrow E$. There is then a map $F \to E$ which extends to a map of abstract Coleman functions and a map of short exact sequences with both $\mathcal{E}$ and $\mathcal{E}'$. Lemma 6.3 therefore proves the claim. We have seen (Lemma 4.17) that an abstract Coleman function has a minimal subquotient. Since subquotients extend to maps of short exact sequences for an appropriate choice of filtration on the subquotient we now see that an abstract Coleman function has the same $\partial$ as its minimal representative, hence $\partial$ depends only on the underlying Coleman function. The linearity is now straightforward.

We now show that the definition of $\partial$ above indeed coincides (up to sign) with the one we have given for curves using the work of Coleman and de Shalit.

Proposition 6.5. For an affine $T$ the operator $\partial$ defined in the previous proposition sends $F \cdot f$ with $F \in A_{\text{Col}, 1}(T)$, $df \in \Omega^1(T)$ and $f \in \mathcal{F}(T)$ to $-[df] \otimes f \in H^1_{\text{rig}}(X) \otimes \mathcal{F}(T)$. 

Therefore, \( T \) is a rigid triple (4.1) \( U \) component. The result is now clear.

**Proposition 6.6.** When \( T \) is tight we have \( \text{Ker} \tilde{\partial} = \mathcal{F}(T) \). When \( T \) is affine we have a short exact sequence

\[
0 \to \mathcal{F}(T) \to A_{\text{Col},1}(T, \mathcal{F}) \xrightarrow{\tilde{\partial}} H^1_{\text{rig}}(X) \otimes \mathcal{F}(T) \to 0
\]

**Proof.** By covering a tight situation with affine ones it suffices to prove the second statement. Suppose \( T \) is affine. The surjectivity of \( \tilde{\partial} \) is clear from Proposition 6.5. To prove the exactness suppose \( f \in A^1_{\text{Col},1}(T, \mathcal{F}) \) is in the kernel of \( \tilde{\partial} \). It follows from Proposition 5.5 that \( f \) can be written as \( \sum f_i \int \eta_i \) and by Proposition 6.5 we have then \( \tilde{\partial} f = - \sum f_i \otimes [\eta_i] \). We may assume that the \( f_i \) are independent over \( K \). It now follows that \( [\eta_i] = 0 \) for each \( i \), hence that \( \eta_i = dg_i \) with \( g_i \in A(T) \) and therefore \( \int \eta_i = g_i \) for an appropriate choice of \( g_i \). Thus, \( f = \sum f_i g_i \in \mathcal{F}(T) \).  

Consider a rigid triple \( T = (X, Y, P) \). Recall that for an open \( U \subset X \) we have a rigid triple (4.1) \( T_U = (U, Y, P) \). Let \( \hat{H}^0_{\mathcal{F}} \) be the presheaf on the Zariski site of \( X \) given by \( U \mapsto H^1_{\text{rig}}(U) \otimes \mathcal{F}(T_U) \). It turns out that \( \hat{H}^0_{\mathcal{F}} \) is not a sheaf and that there is an interesting obstruction for gluing. Suppose \( T \) is a tight rigid triple arising from a pair \((X', Y')\). Let \( \{V_i\} \) be an affine covering of \( X' \). Letting \( U_i = V_i \otimes_Y \kappa \) we obtain a covering \( U = \{U_i\} \) of \( X \). Let \( \hat{H}(U, \bullet) \) be the \( \check{\text{Cech}} \) cohomology with respect to the covering \( U \). We shorthand \( \hat{H}(U, \mathcal{F}) \) for the \( \check{\text{Cech}} \) cohomology of the presheaf \( U \to \mathcal{F}(T_U) \).

**Definition 6.7.** The map \( \Psi : \hat{H}^0(U, \hat{H}^0_{\mathcal{F}}) \to \hat{H}^1(U, \mathcal{F}) \) is the natural map derived from the exact sequences of (6.1)

\[
0 \to \mathcal{F}(T_{U_i}) \to A_{\text{Col},1}(T_{U_i}, \mathcal{F}) \xrightarrow{\tilde{\partial}} H^1_{\text{rig}}(X) \otimes \mathcal{F}(T_{U_i}) \to 0.
\]

Explicitly, the map \( \Psi \) is given as follows: For a collection \( \{\Omega_i \subset H^0_{\mathcal{F}}(U_i)\} \) with \( \Omega_i = \Omega_j \) on \( U_{ij} := U_i \cap U_j \), we choose \( f_i \in A_{\text{Col},1}(T_{U_i}, \mathcal{F}) \) with \( \tilde{\partial}(f_i) = \Omega_i \). The differences \( f_{ij} = f_i - f_j \mid_{T_{U_{ij}}} \) satisfy \( \tilde{\partial}(f_{ij}) = 0 \) hence by Proposition 6.6 \( f_{ij} \in \mathcal{F}(T_{U_{ij}}) \). It is immediate to check that the collection \( \{f_{ij}\} \) is a cocycle and gives a well defined cohomology class in \( \hat{H}^1(U, \mathcal{F}) \) which is \( \Psi(\{\Omega_i\}) \).

By [Ber97, 1.2(ii) and Proposition 1.10] we have an isomorphism between \( H^1_{\text{rig}}(X) \) and the hyper \( \check{\text{Cech}} \) cohomology of \( U \) with values in the presheaf of complexes \( U \to \Omega^\bullet(T_U) \). In particular we obtain a map \( H^1_{\text{rig}}(X) \to \hat{H}^1(U, A) \), where \( A \) stands for the presheaf \( U \to A(T_U) \).
Proposition 6.8. We have the following commutative diagram.

\[
\begin{array}{ccc}
H^1_{\text{rig}}(X) \otimes \mathcal{F}(T) & \longrightarrow & \hat{H}^0(U, H^2_{T,\mathcal{F}}) \\
\downarrow & & \downarrow \Psi \\
\hat{H}^1(U, A) \otimes \mathcal{F}(T) & \longrightarrow & \hat{H}^1(U, \mathcal{F})
\end{array}
\]

Proof. We can take an element of \(H^1_{\text{rig}}(X)\) and represent it as a hyper-cocycle with respect to the covering \(U\), \(\beta = \{\{\eta_i\}, \{g_{ij}\}\}\) where \(\eta_i \in \Omega^1(T_{U_i})\), \(g_{ij} \in A(T_{U_{ij}})\), the \(g_{ij}\) form a one-cocycle, \(d\eta_i = 0\) and we have \(dg_{ij} = \eta_i - \eta_j\) on \(T_{U_{ij}}\). If \(f \in \mathcal{F}(T)\), then the image of \(\beta \otimes f\) in \(\hat{H}^0(U, H^2_{T,\mathcal{F}})\) is given by the collection \(\Omega_i = \eta_i \otimes f|_{T_{U_i}}\). To compute the image under \(\Psi\) we, following the procedure described after Definition 6.7, choose lifts \(f_i = (\int \eta_i) \cdot f \in A_{\text{Col}}(T_{U_i}, \mathcal{F})\) and consider the cocycle resulting from the differences \(f_i - f_j = (\int \eta_i - \int \eta_j) \cdot f\). But \(d(g_{ij} - (\int \eta_i - \int \eta_j)) = 0\) so each of the functions \(g_{ij} - (\int \eta_i - \int \eta_j)\) is constant on \(T_{U_{ij}}\). Since the Čech cohomology of \(U\) with constant coefficients is trivial we can always arrange to fix the integrals \(\int \eta_i\) in such a way that \(g_{ij} = \int \eta_i - \int \eta_j\) so that the application of \(\Psi\) gives \(\{g_{ij}f\}\). This is clearly the same as going along the diagram first down and then right. \(\square\)

Corollary 6.9. An element \(\alpha \in \hat{H}^0(U, H^2_{T,\mathcal{F}})\) comes from \(H^1_{\text{rig}}(X) \otimes \mathcal{F}(T)\) if and only if \(\Psi(\alpha)\) is in the image of \(H^1_{\text{rig}}(X) \otimes \mathcal{F}(T) \rightarrow \hat{H}^1(U, A) \otimes \mathcal{F}(T) \rightarrow \hat{H}^1(U, \mathcal{F})\).

Proof. The only if part is clear. For the if part we notice that we may modify \(\alpha\) by elements from \(H^1_{\text{rig}}(X) \otimes \mathcal{F}(T)\) so that we may assume \(\Psi(\alpha) = 0\). But then it follows from the definition of \(\Psi\) and the fact that \(A_{\text{Col}}\) is a sheaf that \(\alpha\) comes from an element \(f \in A_{\text{Col}}(T, \mathcal{F})\) via restriction to the \(T_{U_i}\) and taking \(\partial\) and therefore \(\alpha\) comes from \(\partial(f) \in H^1_{\text{rig}}(X) \otimes \mathcal{F}(T)\). \(\square\)

Remark 6.10. There is an interesting example to the corollary above that will be used in \(p\)-adic Arakelov theory: Suppose \(T\) comes from a pair \((X, \mathcal{Y})\) and \(X = \mathcal{Y}\) is a relative curve and that \(\mathcal{F}\) is the sheaf \(\Omega^1\). Then the composed map \(H^1_{\text{rig}}(X) \otimes \mathcal{F}(T) \rightarrow \hat{H}^1(U, A) \otimes \mathcal{F}(T) \rightarrow \hat{H}^1(U, \mathcal{F})\) is easily seen to be the same as the cup product \(H^1_{\text{ir}}(X_K/K) \otimes F^1H^1_{\text{dR}}(X_K/K) \rightarrow H^2_{\text{dR}}(X_K/K)\). By Poincaré duality this map can be non surjective only if the genus of the generic fiber is 0. Thus, in positive genus the map \(H^1_{\text{rig}}(X) \otimes \mathcal{F}(T) \rightarrow \hat{H}^0(U, H^2_{T,\mathcal{F}})\) is surjective.

7. Application: Coleman iterated integrals

As an example of Coleman functions in more than one variable, we want to discuss Coleman iterated integrals as functions of the two ends simultaneously. The use of this for computations of syntomic regulators of fields is sketched in [BdJ01].

Consider a tight rigid triple \(T = (X, \mathcal{Y}, P)\) of dimension 1 and forms \(\omega_1, \omega_2, \ldots, \omega_n \in \Omega^1(T)\). Let \(S\) and \(z\) be two points in \(]X[p). The iterated integrals,

\[
f_k(S, z) := \int_S^z \omega_1 \circ \omega_2 \circ \cdots \circ \omega_k,
\]
are defined recursively, just as in the complex case, by

\[ f_1(S, z) = \int_S^z \omega_1(t), \quad f_k(S, z) = \int_S^z f_{k-1}(S, t)\omega_k(t) . \]

The integration at each step is Coleman integration in the variable \( z \). Here we considered the point \( S \) as fixed. We now want to consider the functions \( f_k(S, z) \) as a function of \( S \) as well. Let \( T \times T \) be the triple \((X \times X, Y \times Y, P \times P)\).

**Proposition 7.1.** The functions \( f_k(S, z) \) are Coleman functions in two variables on \( T \times T \).

This is clear for \( f_1 \), indeed, let \( F \) be a Coleman integral of \( \omega_1 \). Then \( f_1(S, z) = F(z) - F(S) \) which is clearly a Coleman function. To continue, we want to know, at least formally, the partial derivatives of \( f_k \) with respect to \( S \) and \( z \). Suppose we can write \( \omega_1(t) = g_i(t)dt \). Clearly,

\[ \frac{\partial}{\partial z} f_k(S, z) = f_{k-1}(S, z)g_k(z) = (\int_S^z \omega_1 \circ \omega_2 \circ \cdots \circ \omega_{k-1})g_k(z) . \]

On the other hand, we have

\[ \frac{\partial}{\partial S} f_k(S, z) = -f_{k-1}(S, S) \cdot g_k(S) + \int_z^S \left( \frac{\partial}{\partial S} f_{k-1}(S, t) \right) \omega_k(t) . \]

The first term is 0. We can therefore repeat the computation expressing the result in terms of \( f_{k-2} \) and so on. The process ends when we get to \( \frac{\partial}{\partial S} f_1(S, z) = -g_1(S) \).

Therefore

\[ \frac{\partial}{\partial S} f_k(S, z) = -g_1(S) \int_z^S \omega_2 \circ \omega_3 \circ \cdots \circ \omega_k . \]

To conclude, we would expect that

\[ df_k = \Omega_k := (\int_S^z \omega_1 \circ \omega_2 \circ \cdots \circ \omega_{k-1}) \cdot \omega_k(z) - (\int_S^z \omega_2 \circ \omega_3 \circ \cdots \circ \omega_k) \cdot \omega_1(S) . \]

**Proof of Proposition 7.1.** We prove by induction on the number of differential forms the more precise statement saying that \( \int_S^z \omega_1 \circ \omega_2 \circ \cdots \circ \omega_k \) is a Coleman function of \( S \) and \( z \) and that its differential is indeed \( \Omega_k \) as defined above. Suppose that we proved this for at most \( k-1 \) differential forms. By the induction hypothesis it is clear that \( \Omega_k \) is a Coleman differential form and we easily compute that

\[
\begin{align*}
\Omega_k &= d\left(\int_S^z \omega_1 \circ \omega_2 \circ \cdots \circ \omega_{k-1}\right) \cdot \omega_k(z) - d\left(\int_S^z \omega_2 \circ \omega_3 \circ \cdots \circ \omega_k\right) \cdot \omega_1(S) \\
&= -\left(\int_S^z \omega_2 \circ \omega_3 \circ \cdots \circ \omega_{k-1}\right) \cdot \omega_1(S) \cdot \omega_k(z) \\
&- \left(\int_S^z \omega_2 \circ \omega_3 \circ \cdots \circ \omega_{k-1}\right) \cdot \omega_k(z) \cdot \omega_1(S) = 0
\end{align*}
\]

It follows that \( \Omega \) can be integrated. Let \( F(S, z) \) be its integral. It is immediate to see that the restriction of \( \Omega_k \) to the diagonal \( S = z \) is 0. By functoriality \( F \) is constant on the diagonal and we may therefore assume that \( F(S, S) = 0 \). But then for fixed \( S \) this last equality together with the fact that \( dF(S, z) = (\int_S^z \omega_1 \circ \omega_2 \circ \cdots \circ \omega_{k-1}) \cdot \omega_k(z) \) implies that \( F(S, z) = f_k(S, z) \).\[\square\]
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