Abstract. Matrix spherical functions associated to the compact symmetric pair \((\text{SU}(m+2), \text{SU}(2) \times \text{U}(m))\), \(m \geq 2\), having reduced root system of type \(BC_2\), are studied. We consider an irreducible \(K\)-representation \((\pi, V)\) arising from the \(\text{U}(2)\)-part of \(K\), and the induced representation \(\text{Ind}^G_K \pi\) splits multiplicity free. The corresponding spherical functions, i.e. \(\Phi: G \to \text{End}(V)\) satisfying \(\Phi(k_1 g k_2) = \pi(k_1) \Phi(g) \pi(k_2)\) for all \(g \in G\), \(k_1, k_2 \in K\), are studied by studying certain leading terms which involve hypergeometric functions. This is done explicitly using the action of the radial part of the Casimir operator on these functions and their leading terms. To suitably grouped matrix spherical functions we associate two-variable matrix orthogonal polynomials giving a matrix analogue of Koornwinder’s 1970s two-variable orthogonal polynomials, which are Heckman-Opdam polynomials for \(BC_2\). In particular, we find explicit orthogonality relations and the matrix polynomials being eigenfunctions to an explicit second order matrix partial differential operator. The scalar part of the matrix weight is less general than Koornwinder’s weight.

1. Introduction

Spherical functions on compact symmetric spaces and orthogonal polynomials are closely related ever since the work of É. Cartan, see e.g. [6], [10], [38]. The notion of a spherical function taking values in a matrix algebra goes back to the initial introduction of the notion of spherical functions, see e.g. [6, Introduction] and references given there. In case of a matrix spherical function for a compact symmetric space of rank one, there is a connection to matrix orthogonal polynomials. One of the first papers in this direction is Koornwinder’s paper [26] introducing vector valued polynomials, which can be written as matrix orthogonality, see also [19], [20]. The vector polynomials are evaluated in an explicit way in terms of the representations of \(\text{SU}(2)\), see [26, Prop. 3.2]. Another seminal paper making this connection to matrix polynomials explicit is the paper [7] by Grünbaum, Pacharoni and Tirao, where they study the rank one symmetric space \((\text{SU}(3), \text{U}(2))\). The approach of [7] relies on the study of the invariant differential operators on the corresponding homogeneous space. Since then several other approaches have been explored, and many other rank one cases have been studied in detail. For this paper the approach of [19], [20], [21] is the most relevant, see Chapter 13 by Grünbaum, Pacharoni, Tirao in [13] for other approaches and references.
The scalar spherical functions on symmetric spaces have been vastly generalised in the work of Heckman and Opdam, see Heckman’s lecture notes in [9], or Chapter 8 by Heckman and Opdam in [27]. The root multiplicities, i.e. dimensions of root spaces, arising from the symmetric spaces are considered as more general continuous parameters, and the second order partial differential operator extending the radial part of the Casimir operator for the symmetric space plays an important role. A first important step was taken by Koornwinder in the 1970s, who studied several sets of orthogonal polynomials in two variables generalising the spherical functions arising for type $A_2$ and $BC_2$. As a first step for a matrix generalisation, the matrix spherical functions and corresponding matrix orthogonal polynomials need to be considered. For the type $A_n$ this is done in [21], and the purpose of this paper is to study the matrix spherical functions and corresponding matrix orthogonal polynomials for type $BC_2$. A possible next step is then to generalise to more general parameters, one possibility is using shift operators for the classical case of $BC_2$, see Opdam [30, §2], and employ the same shift operator to the matrix case as well. This has been done successfully in the rank one case to go from matrix Chebyshev polynomials to matrix Gegenbauer polynomials, see [22]. We expect that the interpretation can lead to more properties of the corresponding matrix orthogonal polynomials studied in this paper. Moreover, the relation to possible applications in mathematical physics needs to be investigated, see e.g. [35] for more information and references given there.

In this paper we study the matrix spherical functions for the compact symmetric pair $(G, K) = (SU(m+2), S(U(2) \times U(m))$, and we study matrix spherical functions and corresponding matrix orthogonal polynomials as described in Section 1.1 for the case of an irreducible representation of $K$ arising from the $U(2)$-component in $K$. The results of Subsection 1.1 follows [21 Part I], but there are slight variations on this approach, see [31, §9]. Actually, we use the classification of [31] in order to find the right $K$-representations satisfying the multiplicity free Condition 1.1 but [31] gives more possibilities, i.e. also involving other $K$-representations. In this paper we restrict to the $K$-representations arising from the $U(2)$-block in $K$ with a slight assumption on this representation. In this paper we show that instead of studying the more complicated matrix spherical functions we can study more simple leading terms of the matrix spherical functions. The leading terms turn out to be homogeneous polynomials, and homogeneity considerations allow us to prove some results, e.g. on the indecomposability of the corresponding matrix weight and the explicit derivation of the second order matrix partial differential equation. Initially, we study the leading terms for the matrix spherical functions for labels in $B(\mu)$ as defined in Condition 1.2 see Theorem 4.5. Note that to study the matrix spherical functions explicitly we need explicit control over the $K$-intertwiner embedding a specific $K$-representation into a larger irreducible $G$-representation. This is in general hard to do explicitly, but this approach is used successfully in [21] for the symmetric pair corresponding to the group case for type $A$. In this paper we take an alternative approach and we construct the embedding of the specific $K$-representation into a larger tensor product $G$-representation containing the required irreducible $G$-representation as a constituent in the decomposition. Then we have to show that the embedding indeed ‘sees’ the appropriate irreducible $G$-representation. Of course, there are many ways to do this, and in this paper we motivate the choice we
make as follows. Firstly it leads to a leading term whose components are homogeneous polynomials and secondly, the radial part of the Casimir operator on the leading term has a simple expression, see Lemma 6.5. The approach taken is motivated by the preprint [33] by van Pruijssen.

In order to make the connection between the leading terms and the matrix spherical functions explicit we need the action of the radial part of the Casimir operator as an operator acting on matrix valued functions on \( A \). For completeness this action is derived in Appendix A. For the matrix spherical functions corresponding to elements from \( B(\mu) \) as in Condition 1.2 we find an explicit expression in this way in terms of the leading terms, see Proposition 6.4. Then in Section 8 we obtain the leading terms for the general case, and we show that the radial part of the Casimir operator acts in a lower triangular way with respect to the partial ordering. This is analogous to the case for the (scalar) Heckman-Opdam polynomials, see [9, §1.3]. The main result is Theorem 7.1 in which we explicitly give the matrix orthogonality for the corresponding family of two-variable orthogonal polynomials with an explicit matrix weight on a region bounded by two straight lines and a parabola, see Figure 7.1. Theorem 7.1 also states that these matrix polynomials are eigenfunctions of a second order matrix partial differential operator.

We now describe the content of the paper in brief. In the remaining part of the Introduction we briefly recall in Subsection 1.1 the set-up to go from matrix spherical function to matrix orthogonal polynomials, where the number of variables is equal to the rank of the compact symmetric space. This follows [21, Part I]. In Section 2 we briefly describe the structure theory and notation for the compact symmetric pair \((G, K) = (SU(m+2), S(U(2) \times U(m)))\), \( m > 2 \), and we show that the conditions in Subsection 1.1 are satisfied in this case. In Section 3 we develop the building blocks for the leading terms. These are essentially the leading terms in the case of the \( K \)-representation in Subsection 1.1 corresponding to the trivial representation and to the natural representation of the \( U(2) \)-block in \( K \). Building on this we study the leading term for the matrix spherical functions corresponding to \( B(\mu) \) as in Condition 1.2. The leading terms can be fully described in terms of single variable Krawtchouk polynomials, and hence as single variable hypergeometric functions. Next in Section 4 we use the radial part of the Casimir operator to give an explicit expression of the matrix spherical functions corresponding to \( B(\mu) \) in terms of the leading terms. In Section 5 we describe the two-variable matrix weight, and we show that the weight is indecomposable and that its determinant is non vanishing on the interior of the integration region. In Section 7 we describe the two variable matrix orthogonal polynomials, and we describe the corresponding eigenvalue equation involving a second order matrix partial differential operator. We have chosen the coordinates in Theorem 7.1 so that it matches the notation of Koornwinder [24], [25], see also [34]. Theorem 7.1 generalises the results of [24], [25], [34] to the matrix case, but the scalar part of the weight measure in [24], [25], [34] is more general than the one in Theorem 7.1. Theorem 7.1 also contains the case [9, Ch. 5] for this particular symmetric pair (corresponding to the case \( a = 0 \) in the notation of Section 7). In Section 8 we then derive the leading term for the general matrix spherical functions, and we show that the radial part of the Casimir
operator acts in a lower triangular fashion on such a leading term. Finally, in Section 9 we discuss briefly the remaining cases of the \( K \)-representations of this type.

In the course of several proofs we have to manipulate several expressions involving functions in two variables. We have used computer algebra, in particular Maple and Maxima, to check these computations.

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1.1. **General set-up.** In this subsection we recall notations and the necessary results. We follow [21, Part 1], but see also [31, §11], [32]. We consider a compact symmetric pair \((G,K)\) and for its structure theory and results we refer to [10]. For the explicit case \((G,K) = (SU(m+2),SU(2) \times SU(m))\) the structure theory is explicitly given in Section 2.

We label the representations of \( G \), respectively \( K \), by the highest weights \( \lambda^G \), respectively \( \lambda^K \), and such a representation is denoted by \((\pi^G_\lambda, V^G_\lambda), \lambda \in P^G_+ \), and similarly for \( K \). We now fix \( \mu \in P^K_+ \).

In order to apply the general approach of [21] we need to establish three conditions.

**Condition 1.1.** \( \text{Ind}^G_K \pi^K_\mu \) splits multiplicity free.

By Frobenius reciprocity this is equivalent to \([\pi^K_\lambda : \pi^K_\mu] \leq 1\) for all \( \lambda \in P^+_G \), and we put

\[
(1.1) \quad P^+_G(\mu) = \{ \lambda \in P^+_G \mid [\pi^K_\lambda : \pi^K_\mu] = 1 \}.
\]

So, if Condition 1.1 holds, we have

\[
\text{Ind}^G_K \pi^K_\mu = \bigoplus_{\lambda \in P^+_G(\mu)} V^G_\lambda.
\]

For \( \lambda \in P^+_G(\mu) \) we define the corresponding matrix spherical functions

\[
(1.2) \quad \Phi^K_\lambda : G \to \text{End}(V^K_\mu), \quad \Phi^K_\lambda(g) = p \circ \pi^K_\lambda(g) \circ j,
\]

where \( j \in \text{Hom}_K(V^K_\mu, V^K_\lambda) \) is the unitary intertwiner and \( p = j^* \) is the corresponding \( K \)-equivariant orthogonal projection. Then (1.2) is independent of the choice of \( j \) and we have

\[
(1.3) \quad \Phi^K_\lambda(k_1 g k_2) = \pi^K_\mu(k_1) \Phi^K_\lambda(g) \pi^K_\mu(k_2), \quad \forall k_1, k_2 \in K, \forall g \in G.
\]

The space of regular functions \( \Phi : G \to \text{End}(V^K) \) satisfying the left and right \( K \) transformation behaviour as in (1.3) is denoted by \( E^\mu \). Using the Peter-Weyl decomposition we see that \( \{ \Phi^K_\lambda \mid \lambda \in P^+_G(\mu) \} \) forms a linear basis for \( E^\mu \). Then \( E^0 \) is the space of scalar continuous bi-\( K \)-invariant functions, and \( E^\mu \) is an \( E^0 \)-module. Moreover, Schur orthogonality gives

\[
(1.4) \quad \int_G \text{Tr}(\Phi^K_\lambda(g)(\Phi^K_{\lambda'}(g))^*) \, dg = \delta_{\lambda,\lambda'} \frac{(\dim V^K_\mu)^2}{\dim V^K_\lambda}, \quad \lambda, \lambda' \in P^+_G(\mu).
\]

Note that the integrand is a bi-\( K \)-invariant function, so contained in \( E^0 \).
Let $A$ be the abelian subgroup and $M = Z_K(A)$ as in [21, §2]. By the Cartan decomposition, $G = KAK$, and by [13] it suffices to consider

$$\Phi^K_A: A \to \text{End}_M(V^K_\mu),$$

since $\pi^K_\mu(m)\Phi^K_A(a) = \Phi^K_A(\pi^K_\mu(m)a)$ by (1.6). So we need to know the decomposition

$$V^K_\mu|_M \cong \bigoplus_{i=1}^N V^M_{\sigma_i},$$

where $\sigma_i \in P_M^+$ are the corresponding highest weights for $M$. The decomposition (1.6) is again a multiplicity free decomposition, see [32] and also [1, 17].

Note that if the representation $\pi^K_\mu$ induces multiplicity free, then also its dual (or contra-gradient) representation $(\pi^K_\mu)^* = \pi^K_{\mu^*}$ induces multiplicity free, where $\mu^*$ corresponds to the highest weight of the dual representation. Then $P^+_G(\mu^*)$ consists of those $G$-representations for which the dual is in $P^+_G(\mu)$, i.e. $P^+_G(\mu^*) = \{\lambda^* \mid \lambda \in P^+_G(\mu)\}$, where $\lambda^*$ corresponds to the highest weight of the dual of the $G$-representation with highest weight $\lambda$. Then we obtain

$$\Phi^{K^*}_A(\sigma^*) = v^* \Phi^K_A(a^{-1})v, \quad a \in A, \quad v \in V^K_\mu, \quad v^* \in \text{Hom}(V^K_\mu, \mathbb{C}) = V^{K^*}_\mu.$$

Note that if Condition [1.1] holds, then it also holds for the dual $\mu^* \in P^+_K$. Moreover, taking duals gives an involution on the spherical weights $P^+_G(0)$.

**Condition 1.2.** There exists a set of weights $B(\mu) \subset P^+_G$, so that for $\lambda \in P^+_G(\mu)$ there exist unique elements $\nu \in B(\mu)$ and $\lambda_{\text{sph}} \in P^+_G(0)$ with $\lambda = \nu + \lambda_{\text{sph}}$. The restriction map of the torus of $G$ to the torus of $M$ gives a bijection $B(\mu) \cong \{\sigma \in P^+_M \mid |V^K_\mu|_M: V^M_\sigma \leq 1\}$.

Assuming Condition 1.2 is satisfied for $\mu \in P^+_K$, then Condition 1.2 is also satisfied for the dual $K$-representation with highest weight $\mu^*$.

Taking $\mu = 0$, $P^+_G(0)$ corresponds to the spherical weights, and $P^+_G(0) = \bigoplus_{i=1}^n \mathbb{N} \lambda_i$, where $\lambda_1, \cdots, \lambda_n$ are the generators for the spherical weights and $n$ is the rank of the compact symmetric space $(G, K)$. We let $\phi_i = \Phi^0_{\lambda_i}: G \to \mathbb{C}$, which generate the algebra of bi-$K$-invariant polynomials on $G$. For $\lambda = \sum_{i=1}^n d_i \lambda_i \in P^+_G(0)$ we put $|\lambda| = \sum_{i=1}^n d_i$. We use the notation $P_G(\lambda)$ for all the weights occurring in the $G$-representation $\pi^K_\lambda$ of highest weight $\lambda \in P^+_G$, and similarly for other groups.

**Condition 1.3.** For all weights $\nu \in B(\mu)$, for all generators $\lambda_i$ of the spherical weights $P^+_G(0)$ and for all weights $\eta \in P_G(\lambda_i)$ such that $\nu + \eta \in P^+_G(\mu)$, we have by Condition 1.3 a unique $\nu' \in B(\mu)$ such that $\nu + \eta = \nu' + \lambda$ with $\lambda \in P^+_G(0)$. Then $|\lambda| \leq 1$.

Note that if Condition 1.3 holds for $\mu$, then it also holds for the dual $\mu^* \in P^+_K$.

Assuming Conditions 1.1, 1.2 and 1.3 one can show that for $\lambda_{\text{sph}} = \sum_{r=1}^n d_r \lambda_r \in P^+_G(0)$, $d = (d_1, \cdots, d_n) \in \mathbb{N}^n$, there exist unique $n$-variable polynomials $p^\mu_{\nu, \nu', d}$ of total degree
\[ |d| = |\lambda_{\text{sph}}| \text{ so that for } \lambda = \nu + \lambda_{\text{sph}} \in P_{G}^{\mu}(\mu) \text{ and } a \in A \]

\[ \Phi_{\lambda}^{\mu}(a) = \Phi_{\nu + \lambda_{\text{sph}}}^{\mu}(a) = \sum_{r=1}^{N} p_{\nu_{r},\nu_{r}}^{\mu}(\phi_{1}(a), \ldots, \phi_{n}(a)) \Phi_{\nu_{r}}^{\mu}(a) \]

using a slightly different labeling from \cite{21}. Using this expansion in the orthogonality relations \cite{1.4} and reducing the integral for bi-K-invariant functions to an integral over \( A \), see \cite{10} Prop. X.1.19, we find the matrix orthogonality relations

\[ \int_{A} P_{d}(\phi(a))W(\phi(a))(P_{d}(\phi(a)))^{*} |\delta(a)| \, da = c_{d,d'} H_{d}, \]

where \( H_{d} \) is given by \((H_{d})_{i,j} = \delta_{i,j}(\dim V_{\mu}^{K})^{2}/ \dim V_{\nu_{i}+\lambda_{\text{sph}}}^{G}, \phi(a) = (\phi_{1}(a), \ldots, \phi_{n}(a))\),

\[ P_{d}(\phi(a)) = (p_{\nu_{i},\nu_{j}}^{d}(\phi_{1}(a), \ldots, \phi_{n}(a)))_{i,j=1}^{N}, \quad W(\phi(a)) = (\text{Tr}(\Phi_{\nu_{i}}^{\mu}(a)(\Phi_{\nu_{j}}^{\mu}(a))^{*}))_{i,j=1}^{N} \]

and \( c > 0 \) is determined by \( c = \int_{A} |\delta(a)| \, da \) and \( \delta \) is given in \cite{10} Prop. X.1.19, where it is denoted as \( D_{*} \).

2. Structure theory and multiplicity free triples

In this section we specialise to the compact symmetric pair \((G, K) = (\text{SU}(m+2), \text{S}(U(2) \times U(m)))\), \( m > 2 \), for which we study the matrix spherical functions and the related orthogonal polynomials in detail. First we describe the structure theory, see e.g. \cite{10}, needed in order to associate the corresponding orthogonal polynomials in Subsection 2.1. In the remaining part we show that for the explicit \( K \)-representations the conditions of \cite{21} Part I] are satisfied in this case.

2.1. Structure theory. From now on we take \( G = \text{SU}(m+2), m > 2, K = \text{S}(U(2) \times U(m)) \) embedded block-diagonally. We view \( U(2) \subset K \) as subgroup as the upper left hand \( 2 \times 2 \)-block of \( K \). The abelian subgroup is \( A = \{ a_{t} = a_{(t_{1},t_{2})} \mid t_{1},t_{2} \in \mathbb{R} \}, \) with

\[ a_{t} = a_{(t_{1},t_{2})} = \begin{pmatrix} \cos t_{1} & 0 & 0 & \cdots & 0 & i \sin t_{1} \\ 0 & \cos t_{2} & 0 & \cdots & 0 & i \sin t_{2} \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ i \sin t_{1} & 0 & 0 & \cdots & 0 & \cos t_{1} \end{pmatrix} \]

where the middle block is the \((m-2) \times (m-2)\)-identity matrix. Then \( M = Z_{K}(A) \) is given by matrices \( m \) which are block diagonal of size \( 2 \times 2, (m-2) \times (m-2), 2 \times 2 \) of the form

\[ m = \begin{pmatrix} D_{1} & 0 & 0 \\ 0 & M_{1} & 0 \\ 0 & 0 & D_{2} \end{pmatrix}, \quad D_{1} = \begin{pmatrix} e^{is_{1}} & 0 \\ 0 & e^{is_{2}} \end{pmatrix}, \quad M_{1} \in U(m-2), \quad D_{2} = \begin{pmatrix} e^{is_{2}} & 0 \\ 0 & e^{is_{1}} \end{pmatrix} \]

with \( \det(m) = 1 \).
As the torus of $G^C$ we take the diagonal elements, and we take this also as the torus of $K^C$. Explicitly,

$$T_{GC} = T_{KC} = \{ \text{diag}(t_1, \cdots, t_{m+2}) \mid t_k \in \mathbb{C}, \prod_{i=1}^{m+2} t_i = 1 \}.$$  

We take the torus of $M^C$ as contained in the torus of $G^C$ and $K^C$,

$$T_{MC} = \{ \text{diag}(t_1, \cdots, t_{m+2}) \mid t_{m+1} = t_2, t_{m+2} = t_1, \prod_{i=1}^{m+2} t_i = 1 \} \subset T_{GC} = T_{KC}.$$  

By $g$, $t$, $m$ and $a$ we denote the corresponding complexified Lie algebras of $G$, $K$, $M$ and $A$. Then the root system $\Delta$ of $g$ is of type $A_{m+1}$, and we denote the standard simple roots $\alpha_i$, $1 \leq i \leq m + 1$. We put $E_i = E_{i,i+1}$, $F_i = E_{i+1,i}$, $H_i = E_{i+1,i+1} - E_{i,i}$, where $E_{i,j}$ is the matrix with all zeroes except the $(i,j)$-th entry. The roots and positive roots are denoted by $Q_G = \bigoplus_{i=1}^{m+1} \mathbb{Z} \alpha_i$ and $Q^+_G = \bigoplus_{i=1}^{m+1} \mathbb{N} \alpha_i$. The partial order $\sigma \preceq \eta$ is $\eta - \sigma \in Q^+_G$.

With this choice of positive roots, we define the fundamental weights for $G$, $K$ and $M$ by

$$\omega_i : T_{GC} = T_{KC} \rightarrow \mathbb{C}, \quad \omega_i(\text{diag}(t_1, \cdots, t_{m+2})) = \prod_{j=1}^{i} t_j, \quad 1 \leq i < m + 2,$$

$$\eta_i : T_{MC} \rightarrow \mathbb{C}, \quad \eta_i(\text{diag}(t_1, t_2, \cdots, t_m, t_2, t_1)) = \prod_{j=1}^{i} t_j, \quad 1 \leq i < m.$$  

Note that $\eta_1, \eta_2$ are characters of $M$. Then we find

$$\omega_i|_{T_{MC}} = \eta_i \quad (1 \leq i < m), \quad \omega_m|_{T_{MC}} = -\eta_2, \quad \omega_{m+1}|_{T_{MC}} = -\eta_1.$$  

Then we have

$$P^+_K = \{ \sum_{i=1}^{m+1} a_i \omega_i \mid a_2 \in \mathbb{Z}, a_i \in \mathbb{N}, i \neq 2 \}, \quad P^+_G = \bigoplus_{i=1}^{m+1} \mathbb{N} \omega_i.$$  

Considering $U(2) \subset K$, we see that the $U(2)$ representations correspond to the elements of $P^+_K$ with $a_j = 0$ for $j \geq 3$.

The reduced root system is of type $BC_2$, and the corresponding reduced Weyl group is generated by $s_1$ and $s_2$, and we put $n_1, n_2 \in N_K(A)$ by

$$n_1 = \begin{pmatrix} J_2 & 0 & 0 \\ 0 & I_{m-2} & 0 \\ 0 & 0 & J_2 \end{pmatrix}, \quad n_2 = \text{diag}(1, -i, 1, \cdots, 1, i, 1)$$

using the notation of Appendix A.1 for the flip $J_2$. Then $n_1 a_t n_1^{-1} = a_{s_1 t}$, $n_2 a_t n_2^{-1} = a_{s_2 t}$ with $s_1 t = (t_2, t_1)$ and $s_2 t = (t_1, -t_2).$
2.2. Multiplicity free triples. The triple \((G, K, \mu)\), \(\mu \in P_K^+\), is a multiplicity free triple if Condition \ref{cond:mu-freeness} is satisfied. Since \((G, K)\) is a symmetric pair, the triple \((G, K, 0)\), where \(\mu = 0\) corresponds to the trivial \(K\)-representation, is a multiplicity free triple. Then we have the spherical weights
\begin{equation}
P_G^+ (0) = \mathbb{N} \lambda_1 \oplus \mathbb{N} \lambda_2, \quad \lambda_1 = \omega_1 + \omega_{m+1}, \quad \lambda_2 = \omega_2 + \omega_m,
\end{equation}
see Krämer \cite{Tabelle 1}. More generally, the multiplicity free triples and the set \(P_G^+ (\mu)\) for a multiplicity free triple \((G, K, \mu)\) are determined by Pezzini and van Pruijssen \cite{31}. We focus on representations of \(K\) that correspond to representations of \(U(2)\subset K\), i.e. we assume \(\mu = a \omega_1 + b \omega_2, \ a \in \mathbb{N}, \ b \in \mathbb{Z}\).

Proposition 2.1. The triple \((G, K, \mu)\), with \(\mu = a \omega_1 + b \omega_2, \ a \in \mathbb{N}, \ b \in \mathbb{Z}\), is multiplicity free. Moreover, \(P_G^+ (\mu) = B(\mu) + P_G^+ (0)\). In case \(b \in \mathbb{N}\) we have
\[ B(\mu) = \{ \nu_i = \nu_i (\mu) | (a - i) \omega_1 + (i + b) \omega_2 + i \omega_{m+1} | 0 \leq i \leq a \}. \]
In case \(b \leq -a\), we have
\[ B(\mu) = \{ \nu_i = \nu_i (\mu) | (a - i) \omega_1 + (-i - b) \omega_m + i \omega_{m+1} | 0 \leq i \leq a \}. \]
and in case \(-a < b < 0\) we have
\[ B(\mu) = \{ \nu_i = \nu_i (\mu) | (a - i) \omega_1 + (-i - b) \omega_m + i \omega_{m+1} | 0 \leq i \leq -b \} \cup \{ \nu_i = \nu_i (\mu) | (a - i) \omega_1 + (b + i) \omega_2 + i \omega_{m+1} | -b \leq i \leq a \}. \]

Remark 2.2. Recall that the \(G\)-representation of highest weight \(\omega_i\) can be realised in the exterior power \(\Lambda^i V\), where \(V = \mathbb{C}^{m+2}\) is the natural \(G\)-representation. It follows that \(\omega_i^* = \omega_{m+2-i}\), and this determines \(\lambda^*\). For the spherical weights, see \cite{23}, \(\lambda_1^* = \lambda_1\) and \(\lambda_2^* = \lambda_2\). The dual of the \(K\)-representation of highest weight \(\mu = a \omega_1 + b \omega_2\) is the \(K\)-representation of highest weight \(\mu^* = a \omega_1 - (a + b) \omega_2\). Indeed, the map \(v \mapsto \langle v, v_\mu \rangle\), with \(v_\mu\) the highest weight vector of \(V^*_\mu\), is the lowest weight vector of \((V^*_\mu)^*\) of weight \(-a \omega_1 - b \omega_2\). Then \(B(\mu^*) = (B(\mu))^*\), and more precisely, in the notation of Proposition \ref{23} \(v_i (\mu^*) = (v_{a-i}(\mu))^*\).

Proof. The proof is a verification using the results and notation of \cite{31}, in particular \cite{31} Table B.2.1. As noted after \cite{31} Def. 9.1 we have \([\pi_\lambda^G : \pi_\mu^K] = 1\) if and only if \((\lambda, -\mu)\) is an element of the so-called extended weight monoid \(\hat{\Gamma}(G/P)\), \(P \subset K^C\) corresponding parabolic subgroup, corresponding to \(G/K\). Now we use \cite{31} Table B.2.1\] to see that the elements of \(\hat{\Gamma}(G/P)\) are nonnegative integral linear combinations of
\[ (\omega_1 + \omega_{m+1}, 0), \ (\omega_1, -\omega_1), \ (\omega_2, -\omega_2), \ (\omega_m, \omega_2), \ (\omega_{m+1}, \omega_2 - \omega_1). \]
We then see that \((\lambda, 0) \in \hat{\Gamma}(G/K)\) if and only if \(\lambda \in P_G^+ (0)\), i.e. \(\lambda\) is a spherical weight. It is now a straightforward calculation to determine the \(\lambda \in P_G^+\) satisfying \((\lambda, a \omega_1 + b \omega_2) \in \hat{\Gamma}(G/K)\). \(\square\)
Note that the representation of $K$ with highest weight $\mu = a\omega_1 + b\omega_2$ has dimension $a + 1$. Denoting the highest weight vector by $v_\mu$ we see that $V^K_\mu$ has an orthogonal basis \( \{ v_k = F^k \cdot v_\mu \mid 0 \leq k \leq a \} \) by considering the representation as a $U(2)$-representation. It follows that, taking $m \in M$ as in (2.2), we have $\pi^K_\mu (m) v_k = e^{i(a+b-k)s_1} e^{i(b+k)s_2} v_k$, so that this corresponds to the $M$-weight $(a-2k) \eta_1 + (b+k) \eta_2$. So

\[
(2.10) \quad V^K_\mu |_M = \bigoplus_{k=0}^a V^M_{\sigma^*_k}, \quad \sigma^*_k = \sigma_k(\mu) = (a-2k) \eta_1 + (b+k) \eta_2
\]
splits multiplicity free into 1-dimensional $M$-representations. Since the $M$-representations are 1-dimensional, we find $\sigma^*_k = -\sigma_k$ and $\sigma_k(\mu') = \sigma_{-k}(\mu')$.

In any of the cases of Proposition 2.1 we have $\nu_i(\mu)|_{T_{\sigma,\eta}} = \sigma_i(\mu)$ using (2.6). This leads to Corollary 2.3.

**Corollary 2.3.** For $\mu = a\omega_1 + b\omega_2 \in P_+^K$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$, Conditions 1.1 and 1.2 are satisfied.

**Proof.** The statement can be obtained by analysing more carefully the extended weight monoid of $[31]$ used in the proof of Proposition 2.1 but it can also be done directly having the $B(\mu)$ at hand. Assume $\lambda \in P_G^+(\mu)$ can be written as $\nu_i + \lambda_{\text{sph}} = \nu_j + \lambda'_{\text{sph}}$, with $\lambda_{\text{sph}} = n_1 \lambda_1 + n_2 \lambda_2$, $\lambda'_{\text{sph}} = m_1 \lambda_1 + m_2 \lambda_2$. Assume first $\mu = a\omega_1 + b\omega_2$ with $b \in \mathbb{N}$, then we have, using Proposition 2.1 and (2.9),

\[
0 = \nu_i + \lambda_{\text{sph}} - \nu_j - \lambda'_{\text{sph}} = (j - i + n_1 - m_1) \omega_1 + (i - j + n_2 - m_2) \omega_2 + (n_2 - m_2) \omega_m + (i - j + n_1 - m_1) \omega_{m+1}.
\]

This gives $n_2 = m_2$, $i = j$ and $n_1 = m_1$, and uniqueness follows. The case $b \leq -a$ follows by duality, and the case $-a < b < 0$ can be proved similarly taking into account the different cases in Proposition 2.1.

The fact that the restriction map gives an isomorphism of $B(\mu)$ and the set of irreducible $M$-modules in $V^K_\mu |_M$ follows from (2.10) and (2.6).

2.3. **Condition 1.3.** In order to be able to apply the general theory as described in Subsection 1.4 we need to check Condition 1.3. Recall that $\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}$ with the convention $\omega_0 = \omega_{m+2} = 0$, which gives

\[
(2.11) \quad \lambda_1 = \sum_{i=1}^{m+1} \alpha_i, \quad \lambda_2 = \alpha_1 + \alpha_{m+1} + 2 \sum_{i=2}^{m} \alpha_i.
\]

Note that for weights $\eta \in P_G(\lambda_i)$, we have $\eta < \lambda_i$, or $\lambda_i - \eta \in Q^+_G$, so that the coefficient of $\alpha_1$ (or $\alpha_{m+1}$) in $\eta$ is less than or equal to 1. Moreover, we see from (2.11) that for $\lambda \in P_G^+(0)$ the degree $|\lambda|$ is equal to the coefficient of $\alpha_1$ (or $\alpha_{m+1}$) in $\lambda$.

**Proposition 2.4.** For $\mu = a\omega_1 + b\omega_2 \in P_+^K$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, Condition 1.3 satisfied.
Proof. We first assume $b \in \mathbb{N}$. Let $\eta = P_G(\lambda_i)$, and assume $\nu_i + \eta = \nu_j + \lambda_{\text{sph}} \in P_G^+(\mu)$ with $\lambda_{\text{sph}} \in P_G^+(0)$. Then
\[
\lambda_{\text{sph}} - \eta = \nu_i - \nu_j = (i - j)(\omega_2 + \omega_{m+1} - \omega_1).
\]
Since $\omega_2 + \omega_{m+1} - \omega_1 = \sum_{k=2}^{m+1} \alpha_k$, we see that in the expansion of simple roots, the coefficient of $\alpha_1$ in $\lambda_{\text{sph}}$ equals the coefficient of $\alpha_1$ in $\eta$, which is less than or equal to 1. Since this coefficient is nonnegative and since $|\lambda_{\text{sph}}|$ is the coefficient of $\alpha_1$, we get that $|\lambda_{\text{sph}}| \leq 1$. This proves Condition 1.3 in case $b \in \mathbb{N}$. By duality it follows for $b \leq -a$.

In case $-a < b < 0$, Proposition 2.1 gives two possible forms for $\nu_i$ and $\nu_j$. In case they have the same form, a similar argument as above proves $|\lambda_{\text{sph}}| \leq 1$. Assume $\nu_i = (a - i)\omega_1 + (-i - b)\omega_m + i\omega_{m+1}$ for $0 \leq i \leq -b$ and $\nu_j = (a - j)\omega_1 + (b + j)\omega_2 + j\omega_{m+1}$ for $-b \leq j \leq a$. Then
\[
\lambda_{\text{sph}} - \eta = \nu_i - \nu_j = (i - j)(\omega_1 - (b + j)\omega_2 - (b + i)\omega_m + (i - j)\omega_{m+1}) \leq -b \leq j \leq a.
\]
and now we use additionally $\omega_1 + \omega_m - \omega_{m+1} = \sum_{k=1}^{m} \alpha_k$. Since $-(b+i) \geq 0$ and $-(b+j) \leq 0$, the coefficient of $\alpha_{m+1}$ in $\lambda_{\text{sph}} - \eta = \nu_i - \nu_j$ is nonpositive. Since the coefficient of $\alpha_{m+1}$ in $\eta$ is at most 1, the coefficient of $\alpha_{m+1}$ in $\lambda_{\text{sph}}$ is at most 1, and thus $|\lambda_{\text{sph}}| \leq 1$. The other situation, with the form of the $\nu_i$ and $\nu_j$ interchanged, can be proved analogously. \qed

Note that we have the following corollary of the proof of Proposition 2.4.

Corollary 2.5. For $\mu = a\omega_1 + b\omega_2 \in P_K^+$, $a \in \mathbb{N}$, $b \in \mathbb{N}$, we have $\nu_i(\mu) > \nu_j(\mu)$ for $i > j$.

A similar statement holds for $b \leq -a$, but not for $-a < b < 0$. Then not all elements can be compared in the partial ordering.

The reduced Weyl group $W = N_K(A)/M$ acts on the $M$-types in $V^K_\mu$. Let $n_w \in N_K(A)$ be a representative of $w$, then (1.3) shows
\[
\Phi^K_\mu(a_{w}) = \Phi^K_\mu(n_w a_{w} n_w^{-1}) = \pi^K_\mu(n_w) \Phi^K_\mu(a_k) \pi^{-1}_\mu(n_w^{-1}) \in \text{End}_M(V^K_\mu).
\]
For $T \in \text{End}_M(V^K_\mu)$, the action $w \cdot T = \pi^K_\mu(n_w) T \pi^K_\mu(n_w^{-1})$ is well-defined, and preserves orthogonal projections, and so it induces an action of $W$ on the $M$-types in $V^K_\mu$. In this case, the decomposition (2.10) splits into 1-dimensional $M$-representations. From (2.8), we see that $s_2 \in W$ acts trivially on the $M$-types, since it commutes with $M$. For $s_1$ we see that it acts on the characters as $s_1 \cdot \eta_1 = \eta_2 - \eta_1$, $s_1 \cdot \eta_2 = \eta_2$ leading to
\[
\text{(2.12)} \quad s_1 \cdot \sigma_k = \sigma_{a-k}, \quad s_2 \cdot \sigma_k = \sigma_k.
\]

3. Special cases

In this section we give the simplest cases of embedding of $K$-representations in tensor products of $G$-representations in order to obtain the leading term. The first case concerns the zonal spherical functions for the weights $\lambda_1$ and $\lambda_2$ generating the spherical weights. This is based on suitable embeddings of the $K$-fixed vector in a two fold tensor product. Next we find the embedding for the fundamental $K$-representation $V^K_{\omega_1}$ in a two fold tensor
product of $G$-representations. This will be used in Section 3 to obtain the leading terms of special matrix spherical functions.

We first prove Lemma 3.1 which we use on several occasions.

**Lemma 3.1.** For $i \leq j$ we have

$$V^{G}_{\omega_i} \otimes V^{G}_{\omega_j} \cong \bigoplus_{r=0}^{\min(i,m+2-j)} V^{G}_{\omega_{i-r} + \omega_{j+r}}$$

with the convention $\omega_0 = 0 = \omega_{m+2}$.

**Proof.** Observe that the fundamental weights for the root system of type $A$ are minuscule weights, see e.g. [2, p. 230], and for this case one has the multiplicity free decomposition

$$W_1 \rightarrow \bigoplus_{w \in W/W_1; \omega_i + w \omega_j \in \mathcal{P}^+_G} V^{G}_{\omega_i + w \omega_j},$$

where $W = S_{m+2}$ is the Weyl group for $G$ and $W_{\omega_j} = \{w \in W \mid w \omega_j = \omega_j\} = S_j \times S_{m+2-j}$ is the stabiliser subgroup, see e.g. [14, Prop. 1], [29, Cor. 3.5]. Since $W_{\omega_j}$ is a parabolic subgroup, we can take coset representatives of minimal length [11 §1.10]. Such an element is determined by a sequence $k_1 < k_2 < \cdots < k_j$ of numbers from $\{1, 2, \ldots, m + 2\}$ and defined by $w(j) = k_j$ and extended such that $w$ has minimal length. Using the expression for $\omega_j$ as in [2, Planche I], we get $w \omega_j = \sum_{p=1}^{j} \omega_{k_p} - \omega_{k_{p-1}}$. It remains to determine the choices leading to $\omega_i + w \omega_j \in \mathcal{P}^+_G$. It follows that the sequence $\{k_1, k_2, \ldots, k_j\}$ can have at most one hole. Keeping track of these possibilities yields the result.

### 3.1 Spherical functions on $A$

We first construct explicit generators for the algebra of spherical functions for $(G, K)$. The natural representation $V^{G}_{\omega_1} = V = \mathbb{C}^{m+2}$ of $G$ is equipped with the standard orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_{m+2})$. Recall that $V^{G}_{\omega_j} \cong \Lambda^j V$.

**Lemma 3.2.** $V^{G}_{\omega_1} \otimes V^{G}_{\omega_{m+1}} \cong V^{G}_{\omega_1} \oplus V^{G}_{0}$ and define

$$v_1 = e_1 \otimes e_2 \wedge e_3 \wedge \cdots \wedge e_{m+2} - e_2 \otimes e_1 \wedge e_3 \wedge \cdots \wedge e_{m+2} \in V^{G}_{\omega_1} \otimes V^{G}_{\omega_{m+1}}.$$  

Then $v_1$ is a $K$-invariant vector, i.e. $v_1$ is contained in the 2-dimensional space $(V^{G}_{\omega_1})^K \oplus (V^{G}_{0})^K$ and $v_1$ has a nonzero component in $(V^{G}_{\omega_{1} + \omega_{m+1}})^K = (V^{G}_{\omega_1})^K$.

**Proof.** The tensor product decomposition follows from Lemma 3.1. From [29] we know that $0, \lambda_1 = \omega_1 + \omega_{m+1} \in \mathcal{P}^+_G(0)$, so that $(V^{G}_{\omega_1 + \omega_{m+1}})^K$ and $(V^{G}_{0})^K$ are 1-dimensional. It is a straightforward calculation to check that $v_1$ is a $K$-fixed vector, and the easiest way is to check that $E_i \cdot v_1 = 0$, $i \in \{1, \ldots, m+1\} \setminus \{2\}$ and $H_i \cdot v_1 = 0$, $i \in \{1, 2, \ldots, m+1\}$. Note that $E_2 \cdot v_1 \neq 0$, so that $v$ is not contained in $(V^{G}_{0})^K \cong \mathbb{C}$, and so has a nonzero component in $(V^{G}_{\omega_{1} + \omega_{m+1}})^K$.

Having $v_1$ given explicitly in Lemma 3.2, we can calculate the corresponding matrix entry restricted to $A$ explicitly using $a_k$ in (2.11), and we obtain

$$\langle (\pi^{G}_{\omega_1} \otimes \pi^{G}_{\omega_{m+1}})(a_k^t) v_1, v_1 \rangle = \cos^2 t_1 + \cos^2 t_2.$$
Lemma 3.3. Define $\psi_1: A \to \mathbb{C}$, $\psi_1(a_t) = \cos^2 t_1 + \cos^2 t_2$ and let $\phi_1: A \to \mathbb{C}$ be the spherical function associated to $V^G_{\lambda_1}$, then there exists a positive constant $\xi_1^1$ and a nonnegative constant $\xi_1^0$, so that $\psi_1 = \xi_1^1 \phi_1 + \xi_1^0$ as functions on $A$.

The constants $\xi_1^1$, $\xi_1^0$ can be calculated explicitly, see Lemma 6.2. Moreover, we can also consider the identity as an identity for functions on $G$ by interpreting the matrix entries as functions on $G$.

Proof. Put $(V^G_{\lambda_1})^K = \mathbb{C} \hat{v}$, $\| \hat{v} \| = 1$, and let $V_0^G = \mathbb{C} \hat{v}$, then $\phi_1(a_t) = \langle \pi^G_{\lambda_1}(a_t) \hat{v}, \hat{v} \rangle$. In Lemma 3.2 we see that $v_1 = a \hat{v} + b \hat{v}$ with $0 \neq a \in \mathbb{C}$. Then

$\psi_1(a_t) = \langle (\pi^G_{\omega_1} \otimes \pi^G_{\omega_m+1})(a_t)v, v \rangle = |a|^2 \langle \pi^G_{\lambda_1}(a_t) \hat{v}, \hat{v} \rangle + |b|^2 \langle \pi^G_0(a_t) \hat{v}, \hat{v} \rangle = |a|^2 \phi_1(a_t) + |b|^2.$

In order to find the second spherical function, we proceed similarly.

Lemma 3.4. $V^G_{\omega_1} \otimes V^G_{\omega_m} \cong V^G_{\lambda_2} \oplus V^G_{\lambda_1} \oplus V^G_0$ and define

$v_2 = e_1 \wedge e_2 \wedge e_3 \wedge \cdots \wedge e_{m+2} \in V^G_{\omega_2} \otimes V^G_{\omega_m}.$

Then $v_2$ is a $K$-invariant vector, and $v_2$ has a nonzero component in $(V^G_{\lambda_2})^K$. Moreover,

$\psi_2: A \to \mathbb{C}$, $\psi_2(a_t) = \langle (\pi^G_{\omega_1} \otimes \pi^G_{\omega_m})(a_t)v_2, v_2 \rangle = (\cos t_1)^2 (\cos t_2)^2$

and $\psi_2 = \xi_2^0 \phi_2 + \xi_1^1 \phi_1 + \xi_1^0$, where $\phi_2$ is the spherical function corresponding to $\lambda_2 \in P^+_G(0)$ and the constants $\xi_2^0 > 0$ and $\xi_1^1$ and $\xi_1^0$ are nonnegative.

The proof of Lemma 3.4 follows the line of proof of Lemma 3.2 and Lemma 3.3. It is possible to calculate the constants $\xi_2^0$ explicitly, see Lemma 6.2.

Proof. The tensor product decomposition follows from Lemma 3.1. The $K$-invariance of $v_2$ follows from $E_i \cdot v_2 = 0$, $i \in \{1, \ldots, m+1\} \setminus \{2\}$ and $H_i \cdot v_2 = 0$, $i \in \{1, \ldots, m+1\}$, which follows straightforwardly. Then the matrix entry can be calculated using (2.1), and this gives the statement of the explicit expression for $\psi_2(a_t)$. Since $v_2$ is a linear combination of the $K$-fixed vectors of $V^G_{\lambda_2}$, $V^G_{\lambda_1}$ and $V^G_0$, we find analogously that $\psi_2$ is a linear combination of $\phi_1$, $\phi_2$ and the constant with nonnegative coefficients. Since the function $(\cos t_1)^2 (\cos t_2)^2$ is not a linear combination of $(\cos t_1)^2 + (\cos t_2)^2$ and the constants, the coefficient of $\phi_2$ has to be nonzero.

Remark 3.5. Note that $A \cap M \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and so the spherical functions, satisfying $\phi(ma_t) = \phi(a_t)$ for $m \in A \cap M$, show that the spherical functions in Lemmas 3.2 and 3.4 have to be invariant under $(\cos t_1, \cos t_2) \mapsto (\pm \cos t_1, \pm \cos t_2)$ for all choices of signs.

3.2. The special case $\mu = \omega_1$. From Proposition 2.1 we know that $B(\omega_1)$ consists of $\omega_1$ and $\omega_2 + \omega_m + 1$. The $K$-equivariant map $V^G_{\omega_1} \to V^G_{\omega_1}$ is the standard embedding, which sends the highest weight vector for $K$ to the highest weight vector for $G$ in the natural representation. We need to understand the $K$-equivariant map $V^G_{\omega_1} \to V^G_{\omega_2 + \omega_m + 1}$, and it suffices to understand the $K$-highest weight vector in $V^G_{\omega_2 + \omega_m + 1}$. We proceed as in Subsection 3.1.
Lemma 3.6. \( V_{\omega_2}^G \otimes V_{\omega_{m+1}}^G \cong V_{\omega_2+\omega_{m+1}}^G \oplus V_{\omega_1}^G \) and the vector 
\[
v_0 = e_1 \wedge e_2 \otimes e_1 \wedge e_3 \wedge \cdots \wedge e_{m+2} \in V_{\omega_2}^G \otimes V_{\omega_{m+1}}^G
\]
is a \(K\)-highest weight vector of weight \(\omega_1\). The vector \(v_0\) has a nonzero component in \(V_{\omega_2+\omega_{m+1}}^G\).

It follows from the tensor product decomposition and Proposition 2.1 for \(\mu = \omega_1\) that there is a 2-dimensional space of \(K\)-highest weight vectors of weight \(\omega_1\). It is possible to explicitly write down a linearly independent vector and give the \(K\)-highest weight vectors of weight \(\omega_1\) in \(V_{\omega_2+\omega_{m+1}}^G\) and \(V_{\omega_1}^G\).

Proof. Lemma 3.1 proves the first statement. Note that both representations in the direct sum correspond to \(B(\omega_1) = \{\omega_1, \omega_2 + \omega_{m+1}\}\). It is straightforward to check that \(E_i \cdot v_0 = 0, i \in \{1, \ldots, m+1\} \setminus \{2\}\) and \(H_1 \cdot v_0 = v_0, H_i \cdot v_0 = 0, i \in \{2, \ldots, m+1\}\), so that \(v_0\) is a \(K\)-highest weight vector of weight \(\omega_1\). Note that the \(K\)-highest weight vector of weight \(\omega_1\) in \(V_{\omega_1}^G\) is also a \(G\)-highest weight vector of weight \(\omega_1\), but \(E_2 \cdot v_0 \neq 0\). So the vector \(v_0\) has a nonzero component in \(V_{\omega_2+\omega_{m+1}}^G\). \(\square\)

4. The leading term of matrix spherical functions for \(B(\mu)\)

We focus on the case \(\mu = a\omega_1 + b\omega_2\), with \(b \in \mathbb{N}\), and then discuss the case \(b < 0\) briefly in Section 5. In this case \(\nu_i = (a - i)\omega_1 + i(\omega_2 + \omega_{m+1}) + b\omega_2, 0 \leq i \leq a\), see Proposition 2.1. Instead of trying to determine the \(K\)-equivariant embedding \(V^K_{\mu} \rightarrow V^G_{\nu_i}\), we embed \(V^K_{\mu}\) in a much bigger \(G\)-representation containing \(V^G_{\nu_i}\) in which we can identify a \(K\)-highest weight of weight \(\mu\) that ‘sees’ \(V^G_{\nu_i}\), i.e. has a nonzero component in \(V^G_{\nu_i}\).

Recall that the representation \(V^G_{\omega_{m+1}}\) can be realised in the space of polynomials in variables \((x_1, x_2, \ldots, x_{m+2})\) which are homogeneous of degree \(N\). Its \(G\)-highest weight vector is \(x_1^N\). Now define the tensor product representation with specific element \(u\);

\[
U^G_{\nu_i} = V_{(a-i)\omega_1}^G \otimes \left( V_{\omega_2}^G \otimes V_{\omega_{m+1}}^G \right)^{\otimes i} \otimes \left( V_{\omega_2}^G \right)^{\otimes b}
\]
\[
u = x_1^{a-i} \otimes v_0 \otimes \cdots \otimes v_0 \otimes e_1 \wedge e_2 \otimes \cdots \otimes e_1 \wedge e_2,
\]

where \(v_0\) is as in Lemma 3.6. Then \(u\) is a \(K\)-highest weight vector of weight \(\mu = a\omega_1 + b\omega_2\) by Lemma 3.6, since \(e_1 \wedge e_2 \in V_{\omega_2}^G\) is the \(G\)- and \(K\)-highest weight vector of weight \(\omega_2\). Moreover,

\[
U^G_{\nu_i} = V_{\nu_i}^G \oplus \bigoplus_{\lambda \prec \nu_i} n_\lambda V^G_\lambda,
\]

for certain multiplicities \(n_\lambda\). Since we are only interested in \(\lambda \in P^+_G(\mu)\), we need Lemma 4.1.

Lemma 4.1. Let \(\mu = a\omega_1 + b\omega_2\) with \(b \in \mathbb{N}\). Then \(\{\lambda \in P^+_G(\mu) \mid \lambda \prec \nu_i\} = \{\nu_0, \cdots, \nu_i\}\).
Proof. Using the ideas and identities as in Subsection 2.3 we assume \( \nu_j + n_1 \lambda_1 + n_2 \lambda_2 \geq \nu_i \), \( n_1, n_2 \in \mathbb{N} \). Writing
\[
\nu_i - (\nu_j + n_1 \lambda_1 + n_2 \lambda_2) = (i - j)(-\omega_1 + \omega_2 + \omega_{m+1}) - n_1(\omega_1 + \omega_{m+1}) - n_2(\omega_2 + \omega_m)
\]
we see that this is in \( Q^*_N \) if and only if \( n_1 = n_2 = 0 \) and \( i \geq j \).

Our next objective is to give an explicit expression for the matrix valued spherical function associated to the \( K \)-equivariant embedding \( V^K_\mu \to U^G_{\nu_i} \), which maps the highest weight vector of \( V^K_\mu \) to \( u \). In order to describe the result we need the Krawtchouk polynomials, see e.g. [12, §6.2], [18, §9.11]. The Krawtchouk polynomials are defined as a terminating hypergeometric series and are generated by a generating function:
\[
K_n(x; p, N) = \, _2F_1\left( \begin{array}{c} -n, -x \\ -N \end{array} ; \frac{1}{p} \right), \quad N \in \mathbb{N}, \quad x, n \in \{0, 1, \ldots, N\}
\]
(4.3)
\[
\sum_{n=0}^{N} \binom{N}{n} K_n(x; p, N)t^n = \left(1 - \frac{1-p}{p}t\right)^x (1 + t)^{N-x}.
\]

Note that the Krawtchouk polynomials are self-dual: \( K_n(x; p, N) = K_n(n; p, N) \), and that \( K_0(x; p, N) = 1 = K_n(0; p, N) \).

Proposition 4.2. Let \( \mu = a\omega_1 + b\omega_2, \ b \in \mathbb{N}, \nu_i = (a-i)\omega_1 + i(\omega_2 + \omega_{m+1}) + b\omega_2, \) and \( U^G_{\nu_i} \) the representation defined in (1.1). Then for \( k, l \in \{0, 1, \ldots, a\} \)
\[
\langle \pi_{U^G_{\nu_i}}(a_t) F_1^k \cdot u, F_1^l \cdot u \rangle = \delta_{k,l}\| F_1^k \cdot u \|^2 (\cos t_1)^{a+b-k} (\cos t_2)^{b+2i+k} K_i(k; \frac{\cos^2 t_2}{\cos^2 t_2 - \cos^2 t_1}, a)
\]
with \( a_t \) as in (2.1) and the Krawtchouk polynomials as in (4.3).

Remark 4.3. (i) The fact that we get zero for \( k \neq l \) follows from the fact that spherical functions restricted to \( A \) are \( M \)-intertwiners and the vectors \( F_1^k \cdot u \) correspond to different \( M \)-types for different \( k \). Indeed, \( u \) spans a one-dimensional \( M \)-representation of weight \( \sigma_0(\mu) = a\eta_1 + b\eta_2 \) by (4.1) and Lemma 3.6, and more generally \( F_1^k \cdot u \) corresponds to the one-dimensional \( M \)-representation of weight \( \sigma_k(\mu) = (a - 2k)\eta_1 + (b + k)\eta_2 \), see (2.10) for the \( K \)-representation generated by \( u \).

(ii) For \( k = l \), the right hand side is a polynomial in \( (\cos t_1, \cos t_2) \), and it is a homogeneous polynomial of degree \( a+2b+2i \) in \( (\cos t_1, \cos t_2) \). Note that the degree of homogeneity is independent of \( k \). Indeed, for \( k = l \) the right hand side of Proposition 4.2 equals
\[
\| F_1^k \cdot u \|^2 (\cos t_1)^{a+b-k} \sum_{p=0}^{\min(i,k)} \frac{(-i)_p (-k)_p}{p! (-a)_p} (\cos^2 t_2 - \cos^2 t_1)^p (\cos t_2)^{b+2i+k-2p}
\]
using (4.3) and the standard notation for Pochhammer symbols \((x)_p = \prod_{i=0}^{p-1}(x + i)\), see e.g. [11, 12, 18].

Proof. We put \( a_t(r, s) = \exp(sE_1) a_t \exp(rF_1) \), then using the unitarity of the representation \( U^{G}_{\nu_i} \) we obtain
\[
\langle \pi_{U^{G}_{\nu_i}}(a_t) F_t^{k} \cdot u, F_t^{l} \cdot u \rangle = \frac{\partial^k}{\partial r^k} \bigg|_{r=0} \frac{\partial^l}{\partial s^l} \langle \pi_{U^{G}_{\nu_i}}(a_t(r, s)) u, u \rangle.
\]
Now
\[
a_t(r, s) = \begin{pmatrix} A & 0 & B \\ 0 & I & 0 \\ C & 0 & D \end{pmatrix}, \quad A = \begin{pmatrix} \cos t_1 + rs \cos t_2 & s \cos t_2 \\ r \cos t_2 & \cos t_2 \end{pmatrix},
\]
\[
B = \begin{pmatrix} si \sin t_2 & i \sin t_1 \\ i \sin t_2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} ri \sin t_2 & i \sin t_2 \\ i \sin t_1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \cos t_2 & 0 \\ 0 & \cos t_1 \end{pmatrix}
\]
and we can calculate the action of \( a_t(r, s) \) on each of the factors in \( u \in U^{G}_{\nu_i} \). We get
\[
\langle a_t(r, s) \cdot x_1^{a-i}, x_1^{a-i} \rangle = (\cos t_1 + rs \cos t_2)^{a-i} \langle x_1^{a-i}, x_1^{a-i} \rangle
\]
\[
\langle a_t(r, s) \cdot v_0, v_0 \rangle = \cos t_1 \cos t_2 (\cos t_2 + rs \cos t_1)^{a-i} \langle v_0, v_0 \rangle
\]
\[
\langle a_t(r, s) \cdot e_1 \wedge e_2, e_1 \wedge e_2 \rangle = \cos t_1 \cos t_2 \langle e_1 \wedge e_2, e_1 \wedge e_2 \rangle
\]
and this gives
\[
\langle \pi_{U^{G}_{\nu_i}}(a_t(r, s)) u, u \rangle = (\cos t_1 + rs \cos t_2)^{a-i} (\cos t_2 + rs \cos t_1)^{i} (\cos t_1 \cos t_2)^{b+i} \langle u, u \rangle.
\]
The first two factors can be expanded in terms of Krawtchouk polynomials using the generating function of (4.3), and this gives
\[
\frac{\langle \pi_{U^{G}_{\nu_i}}(a_t(r, s)) u, u \rangle}{\langle u, u \rangle} = (\cos t_1)^{a-b} (\cos t_2)^{b+2i} \sum_{n=0}^{a} \binom{a}{n} K_n(i; \frac{\cos^2 t_2}{\cos^2 t_2 - \cos^2 t_1}, a) \left( rs \frac{\cos t_2}{\cos t_1} \right)^n
\]
Now the statement of the proposition follows using (4.3). \( \square \)

Remark 4.4. Note that the right hand side of (4.5) is a polynomial of the product \( rs \). This follows from (4.4) being zero for \( k \neq l \), and this follows from the fact that \( a_t \) commutes with \( M \) and \( F^k \cdot u \) and \( F^l \cdot u \) realise different one-dimensional \( M \)-representations for \( k \neq l \), cf. Remark 4.3(ii).

We can now collect the results of this section into Theorem 4.5.

Theorem 4.5. Let \( \mu = a \omega_1 + b \omega_2 \), \( a, b \in \mathbb{N} \), and let \( \nu_i = (a-i) \omega_1 + i (\omega_2 + \omega_{m+1}) + b \omega_2 \in B(\mu), i \in \{0, \ldots, a\} \). Let \( v_\mu \) be the highest weight vector of \( V^{K}_{\mu} \), and define \( j: V^{K}_{\mu} \to U^{G}_{\nu_i} \) to be the \( K \)-equivariant map sending \( v_\mu \mapsto u \). Then
\[
Q_{v_\mu} : G \to \text{End}(V^{K}_{\mu}), \quad g \mapsto j^* \circ \pi_{U^{G}_{\nu_i}}(g) \circ j
\]
is a matrix spherical function and restricted to $A$ we have

$$Q^\mu_{\nu}(a_t)\left(F_1^k \cdot v_\mu\right) = q^\mu_{\nu,\sigma_k}(a_t) F_1^k \cdot v_\mu,$$

Moreover, as matrix spherical functions on $G$ we have

$$Q^\mu_{\nu} = \sum_{r=0}^i a_r^{\mu} \Phi^\mu_{\nu_r}, \quad a_r^{\mu} \in \mathbb{C}, \quad a_r^{\mu} \neq 0.$$

So we see that the transition of the elements $\Phi^\mu_{\nu_r}, i \in \{0, \ldots, a\}$ to $Q^\mu_{\nu}, i \in \{0, \ldots, a\}$ is given by a triangular matrix with nonzero diagonal entries. Hence, $(Q^\mu_{\nu})_{i=0}^a$ and $(\Phi^\mu_{\nu})_{i=0}^a$ span the same space of matrix spherical functions, from which the matrix part $W$ of the weight as in (1.9) can be obtained.

**Corollary 4.6.** For $i \in \{0, \ldots, a\}$ we have $\Phi^\mu_{\nu_r} = \sum_{r=0}^i d_r^{\mu} Q^\mu_{\nu_r}$ with $d_r^{\mu} \in \mathbb{C}$ and $d_r^{\mu} \neq 0$.

**Proof of Theorem 4.5.** The first statement follows from the general set-up in Subsection 1.1 and Proposition 4.2. For the last statement we recall that $\{\Phi^\mu_{\nu_r} \mid \lambda \in P^+_0(\mu)\}$ form a basis for the matrix spherical functions, see Subsection 1.1. By (4.2) and Lemma 4.1 we find that the only matrix spherical functions of type $\mu$ occurring in $U^G_{\nu_i}$ are $\Phi^\mu_{\nu_r}, r \in \{0, \ldots, i\}$. It remains to show that $a_r^{\mu} \neq 0$.

In case $r = 0$ we have $Q^\mu_{\nu_0} = \Phi^\mu_{\nu_0}$ since both are the identity in $\text{End}(V^K_\mu)$ for the identity in $G$, so $a^0_0 = 1$. Assume that $a_r^{\mu} \neq 0$ for $i \in \{0, \ldots, r - 1\}, 1 \leq r \leq a$, and $a_r^{\mu} = 0$. We show that this leads to a contradiction. Indeed, then $Q^\mu_{\nu_r}$ can be expressed in terms of $\Phi^\mu_{\nu_j}, j < r$, which in turn can be expressed in terms of $Q^\mu_{\nu_j}, j < r$. Hence, there is a nontrivial linear dependence between the matrix spherical functions $\sum_{j=0}^r c_j Q^\mu_{\nu_j} = 0$. Evaluating at $a_t$, acting on the $K$-highest weight vector $v_\mu \in V^K_\mu$ and taking inner products with $v_\mu$ and using the first part of the theorem, i.e. Proposition 4.2, we get a nontrivial linear dependence of the form

$$\sum_{j=0}^r c_j (\cos t_1)^{a+b}(\cos t_2)^{b+2j} = 0, \quad \forall t_1, t_2.$$

This is the required contradiction. \hfill \Box

5. The matrix weight

We keep $\mu = a \omega_1 + b \omega_2$ with $a, b \in \mathbb{N}$ fixed. Then we identify $\text{End}_M(V^K_\mu) \cong \mathbb{C}^{a+1}$ by Schur’s Lemma and (2.10) and we set

$$\phi^\mu_{\lambda,\sigma_k}(a_t): A \to \mathbb{C}, \quad \Phi^\mu_{\lambda}(a_t)|_{V^K_\mu} = \phi^\mu_{\lambda,\sigma_k}(a_t) \text{Id}_{V^K_\mu} \quad (5.1)$$
for \( \lambda \in P^*_G(\mu) \), \( k \in \{0, \ldots, a\} \). Note that \( W \)-invariance leads to, see (2.12),
\[
(5.2) \quad \Phi^\mu_{\lambda, \sigma_k}(s_1 a_t) = \Phi^\mu_{\lambda, \sigma_{a-k}}(a_t), \quad \Phi^\mu_{\lambda, \sigma_k}(s_2 a_t) = \Phi^\mu_{\lambda, \sigma_k}(a_t)
\]
and similarly for \( q^\mu_{\lambda, \sigma_k}(a_t) \) because of Theorem [1.5]. The nontrivial action for \( q^\mu_{\lambda, \sigma_k}(a_t) \) corresponds to Pfaff’s transformation formula for \( G_{\lambda, \sigma} \).

Proposition 5.1. Moreover, \( \Phi(5.4) \) is invertible. Upon defining the matrices \( \Phi_0 \) and \( Q_0 \) on \( A \) by \( (\Phi_0)_{i,j} = \Phi^\mu_{\lambda, \sigma_j} \) and \( (Q_0)_{i,j} = q^\mu_{\lambda, \sigma_j} \), we see that Corollary 4.6 can be rephrased as \( \Phi_0 = L Q_0 \), and we calculate \( L \) explicitly in Proposition [6.7]. Moreover, \( \Phi_0(s_1 a_t) = \Phi_0(a_t) J \), where \( J_{i,j} = 1 \) if \( i + j = a \) and \( J_{i,j} = 0 \) otherwise, and similarly \( Q_0(s_1 a_t) = Q_0(a_t) J \) by (5.2).

As a function on \( A \) we see that the matrix weight \( W \) in (1.9) can be written as \( \Phi_0 \Phi_0^* \), for which each matrix entry is a polynomial in \( (\phi_1, \phi_2) \). Note that the weight \( W \) is a matrix function on \( A \) which is invariant for the action of the reduced Weyl group. We switch from the matrix weight \( W \) on \( A \) to the matrix weight \( S = Q_0(Q_0)^* \), so that \( W = LSL^* \) as functions on \( A \) for the constant lower triangular matrix \( L \). Note that \( S \) as matrix function on \( A \) is invariant for the action of the reduced Weyl group. Note that \( S \) is a polynomial in \((\psi_1, \psi_2)\) and we have for the matrix entries \( S_{i,j} \) of the weight \( S \)
\[
(5.3) \quad S_{i,j}(\psi_1(a_t), \psi_2(a_t)) = \sum_{k=0}^a q^\mu_{\nu, \sigma_k}(a_t)q^\mu_{\nu, \sigma_k}(a_t) = \sum_{k=0}^a \left( \cos t_1 \right)^{2a+2b-2k} \left( \cos t_2 \right)^{2b+2k-2i+2j} K_k(i; \cos^2 t_2 - \cos^2 t_1, a) K_k(j; \cos^2 t_1 - \cos^2 t_2, a)
\]
and by this expression we see that \( S_{i,j}(\psi_1(a_t), \psi_2(a_t)) \) is a homogeneous polynomial in \( (\cos t_1, \cos t_2) \) of degree \( 2a + 4b + 2i + 2j \). The simplest non-scalar cases for \( a = 1 \) and \( a = 2 \) give the following expressions for \( S(\psi_1, \psi_2) \)
\[
(5.4) \quad \psi_2^b \left( \psi_1 2 \psi_2 \right)^2 \quad \text{and} \quad \psi_2^b \left( \frac{3}{2} \psi_1 \psi_2 \right)^2 \psi_2 + \psi_2^b \left( \frac{3}{4} \psi_1 \psi_2 \right)^2 \psi_2 \psi_2^b \left( \psi_2^2 - \psi_2 \right)
\]
Note that in (5.4) the matrix part of \( S \) is determined by \( b \), and the \( b \)-dependence is only in the scalar part \( \psi_2^b \). This follows in general from (5.3).

Proposition 5.1. The matrix weight \( S \) is indecomposable, i.e.
\[
\mathcal{A} = \{ T \in M_{a+1}(\mathbb{C}) \mid TS(\psi_1(a_t), \psi_2(a_t)) = S(\psi_1(a_t), \psi_2(a_t))T^*, \forall t_1, t_2 \} = \mathbb{R} \text{Id},
\]
\[
\mathcal{A}' = \{ T \in M_{a+1}(\mathbb{C}) \mid TS(\psi_1(a_t), \psi_2(a_t)) = S(\psi_1(a_t), \psi_2(a_t))T, \forall t_1, t_2 \} = \mathbb{C} \text{Id}.
\]

Remark 5.2. These notions of indecomposability of the matrix weight for multivariable weights have not yet been introduced, but it follows the definition of the single variable case [23, 36], which can be generalised directly. Note that \( \mathcal{A}' \), which is denoted \( A \) in [23], is a \( * \)-algebra, and \( \mathcal{A} \) is a real vector space. The corresponding vector spaces for the weight \( W = LSL^* \) are then also trivial, which follows directly for \( \mathcal{A} \) and the invertibility of \( L \). For \( \mathcal{A}' \) this follows from [23 Thm 2.3].
Proof. Recall that the degree of $S_{i,j}$ as a homogeneous polynomial in $(\cos t_1, \cos t_2)$ is $2a + 4b + 2i + 2j$. Assume $T \in \mathcal{A}'$ so that $ST = TS$. We consider the $(i,j)$th entry:

$$\sum_{k=0}^{a} S_{i,k}(\cos t_1, \cos t_2) T_{k,j} = \sum_{r=0}^{a} T_{i,r} S_{r,j}(\cos t_1, \cos t_2), \quad \forall t_1, t_2.$$ 

Consider this a polynomial identity in $(\cos t_1, \cos t_2)$ and consider the total degree of both sides. Assume that $i < j$, then we see that $T_{i,r} = 0$ for $r > a + i - j$. Taking $j = a$, we see that $T_{i,r} = 0$ for $r > i$. So $T$ is lower triangular. A similar deduction for $i > j$ shows that $T$ is upper triangular, and so $T$ is diagonal. Then we obtain $S_{i,j}(\cos t_1, \cos t_2) T_{j,j} = T_{i,i} S_{i,j}(\cos t_1, \cos t_2)$, and since $S_{i,j}(\cos t_1, \cos t_2)$ is a nonzero function, we find $T_{i,i} = T_{j,j}$. So $T$ is a multiple of the identity.

Assume $T \in \mathcal{A}$ so that $TS = ST^*$. We consider the $(i,j)$th entry:

$$\sum_{k=0}^{a} S_{i,k}(\cos t_1, \cos t_2) T_{j,k} = \sum_{r=0}^{a} T_{i,r} S_{r,j}(\cos t_1, \cos t_2), \quad \forall t_1, t_2.$$ 

Arguing as in the previous case, we see that $i < j$ leads to $T$ being lower triangular. This gives $\sum_{k=0}^{j} S_{i,k}(\cos t_1, \cos t_2) T_{j,k} = \sum_{r=0}^{j} T_{i,r} S_{r,j}(\cos t_1, \cos t_2)$. Considering the homogeneous part of highest degree $2a + 4b + 2i + 2j$ gives $T_{j,j} = T_{i,i}$, so that each diagonal entry is equal to the same real number. Next comparing the homogeneous part of the same degree leads to $S_{i,k}(\cos t_1, \cos t_2) T_{j,k} = T_{i,k+i-j} S_{k+i-j,j}(\cos t_1, \cos t_2)$, so that in case $j > i$ we get $T_{j,k} = 0$ for $0 \leq k < j - i$. Taking $i = 0$ shows that $T$ is upper triangular. Hence, $T$ is a real multiple of the identity. 

Next we calculate the determinant of $S$. For this it suffices to calculate the determinant of $Q_0$ for which we use the orthogonality properties of the Krawtchouk polynomials. Recall e.g. [12 §6.2], [13 §9.11], using the notation of [13], the orthogonality relations

$$(5.5) \quad \sum_{x=0}^{N} w(x; p, N) K_n(x; p, N) K_m(x; p, N) = \delta_{m,n} h(n; p, N),$$

$$w(x; p, N) = \binom{N}{x} p^x (1 - p)^{N-x}, \quad h(n; p, N) = \frac{(-1)^n n!}{(-N)_n} \left(1 - \frac{p}{N}\right)^n,$$

which is a positive finite discrete measure for $0 < p < 1$. Rewriting shows that the matrix $B = \left( \frac{w(x; p, N)}{\sqrt{h(n; p, N)}} K_n(x; p, N) \right)_{n,x=0}^{N}$ is an orthogonal matrix, so of determinant $\pm 1$. Writing $B$ as product of a diagonal matrix times the matrix with entries the Krawtchouk polynomials times a diagonal matrix, and introducing additional parameters gives

$$(5.6) \quad \det \left( t^n s^k K_n(x; p, N) \right)_{n,x=0}^{N} = \pm (st)^{\frac{1}{2}N(N+1)} \left( \prod_{n=0}^{N} h(n; p, N) \right)^{\frac{1}{2}} \left( \prod_{x=0}^{N} w(x; p, N) \right)^{-\frac{1}{2}}.$$
Proposition 5.3. For \( \mu = a\omega_1 + b\omega_2, \ a, b \in \mathbb{N}, \ a_4 \in A \) we have
\[
\det(S(a_4)) = \left(\prod_{n=0}^{a} \left(\begin{array}{c} a \\ n \end{array}\right)\right)^{-2} (\cos t_1 \cos t_2)^{2b(a+1)} (\cos t_1 \cos t_2 (\cos^2 t_1 - \cos^2 t_2))^{a(a+1)}
\]

Proof. With \( Q_0(a_4)_{i,j} = q_{\nu_i,\sigma_j}(a_4), \ 0 \leq i, j \leq a, \) expressed in Theorem 4.5 in terms of Krawtchouk polynomials we take out the terms independent of \( i, j, \) and next we apply (5.6) to get
\[
\det(Q_0(a_4)) = \pm (\cos^{a+b} t_1 \cos^{b} t_2)^{a+1} \left(\frac{\cos t_2}{\cos t_1}\right)^{\frac{1}{2}a(a+1)} (\cos^2 t_2)^{\frac{1}{2}a(a+1)} \left(\prod_{n=0}^{a} \left(\begin{array}{c} a \\ n \end{array}\right)\right)^{-1} \
\times \left(\frac{1-p}{p}\right)^{\frac{1}{2}a(a+1)} (1-p)^{\frac{1}{2}a(a+1)}
\]
with \( p = \frac{\cos^2 t_0}{\cos^2 t_2 - \cos^2 t_1} \) as in Proposition 4.2 using that \( \left(\begin{array}{c} -a \\ -1 \end{array}\right)^m = \binom{a}{m} \). Here we assume for the time being that \( 0 < p < 1 \) so that all square roots are well-defined. Simplifying gives
\[
\det(Q_0(a_4)) = \pm \left(\prod_{n=0}^{a} \left(\begin{array}{c} a \\ n \end{array}\right)\right)^{-1} (\cos t_1 \cos t_2)^{b(a+1)} (\cos t_1 \cos t_2 (\cos^2 t_1 - \cos^2 t_2))^{\frac{1}{2}a(a+1)}
\]
and this proves the statement for \( 0 < p < 1 \). Since we know all entries of \( S \) are polynomial in \( (\cos t_1, \cos t_2) \), cf. Remark 4.3(ii), the determinant of \( S \) is polynomial in \( (\cos t_1, \cos t_2) \) and the result holds for all \( a_4 \). \( \Box \)

Remark 5.4. Now by the results of Subsection 4.1 and [10, Prop. X.1.19] we have (1.9) involving the matrix weight \( W \), hence \( S \). In this case \( \delta: A \to \mathbb{R} \) is given by
\[
\delta(a_4) = (\sin t_1)^{2(m-2)} (\sin t_2)^{2(m-2)} \sin(2t_1) \sin(2t_2) \sin^2(t_1 + t_2) \sin^2(t_1 - t_2) = 4(\sin t_1)^{2m-3} (\sin t_2)^{2m-3} \cos t_1 \cos t_2 (\cos^2 t_1 - \cos^2 t_2)^2,
\]
see [10, §X.5] using Appendix A. In particular, from Proposition 5.3 and (5.7) we see that \( \det(S(a_4)) = 0 \) implies \( \delta(a_4) = 0 \).

6. RADIAL PART OF THE CASIMIR OPERATOR

In order to obtain precise information on the matrix spherical functions in its relation to the matrix functions \( Q^m_{\mu} \) in Theorem 4.5 and Corollary 4.6 we use the Casimir operator. Since the Casimir operator acts as a multiple of the identity in a representation \( \pi^G_\lambda \) with scalar \( c_\lambda = \langle \lambda, \lambda \rangle + 2\langle \lambda, \rho \rangle \), where \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \), see [10, Prop. 5.28], we have
\[
R^\mu(\Omega)\Phi^\mu_\lambda|_A = c_\lambda \Phi^\mu_\lambda|_A,
\]
where \( R^\mu(\Omega) \) is the radial part of the Casimir operator as in Appendix A. For convenience, the explicit expression of \( R^\mu(\Omega) \) is derived in Appendix A. The functions \( \Phi^\mu_\lambda|_A \) are eigenfunctions of a much larger class of differential operators arising from a subalgebra of the universal enveloping algebra [3, Ch. 9], but we only use the Casimir operator. The eigenvalues play an important role in order to distinguish the eigenfunctions.
Lemma 6.1. Let $\lambda_1, \lambda_2 \in P_G^+$ with $\lambda_1 < \lambda_2$ and $\lambda_1 \neq \lambda_2$, then $c_{\lambda_1} < c_{\lambda_2}$.

Proof. Rewrite $c_{\lambda} = (\lambda + \rho, \lambda + \rho) - (\rho, \rho)$, then
c_{\lambda_2} - c_{\lambda_1} = (\lambda_2 + \rho, \lambda_2 + \rho) - (\lambda_1 + \rho, \lambda_1 + \rho) = (\lambda_1 + \lambda_2 + 2\rho, \lambda_2 - \lambda_1)
and since $\lambda_1 + \lambda_2 + 2\rho$ is in the interior of the positive Weyl chamber and $\lambda_2 - \lambda_1 \in Q_G^+$, the right hand side is positive. □

As a first application we calculate the constants in Lemma 3.3 and Lemma 3.4.

Lemma 6.2. With the notation of Section 3 we have as functions on $A$

$$\psi_1 = \frac{2m\phi_1 + 4}{m + 2}, \quad \psi_2 = \frac{m - 1}{m + 2} \phi_2 + \frac{2(m + 1)}{(m + 2)^2} \phi_1 + \frac{2m + 1}{m^2(m + 2)}$$

Remark 6.3. The relation is invertible;

$$\phi_1 = \frac{(m + 2)\psi_1 - 4}{m}, \quad \phi_2 = \frac{m(m + 1)\psi_2 - (m + 1)\psi_1 + 2}{m(m - 1)}$$

Proof. In case $\mu = 0$, $R^0(\Omega)$ is an explicit second order partial differential operator, see (A.5), where all terms involving $\pi^K_\mu$ are set to zero. Put $f_1(t_1, t_2) = \psi_1(a_t) = \cos^2 t_1 + \cos^2 t_2$, then by a trigonometric calculation (or using computer algebra) $R^0(\Omega)f_1 = (2m + 4)f_1 - 8$.

Since

$$c_{\lambda_1} = \langle \omega_1 + \omega_{m+1}, \sum_{n=1}^{m+1} \alpha_n \rangle + 2 \sum_{1 \leq i < j \leq m+2} \langle \omega_1 + \omega_{m+1}, \sum_{p=1}^{j-1} \alpha_p \rangle = 2m + 4$$

using (2.11), we get that $-8a = c_{\lambda_1}b$ when writing $\phi_1 = a\psi_1 + b$ using Lemma 3.3 and $R^0(\Omega)\phi_1 = c_{\lambda_1}\phi_1$. Evaluating at the identity, using $\phi_1(e) = 1, \psi_1(e) = 2$, fixes the constant.

For $\psi_2$, put $f_2(t_1, t_2) = \psi(a_t) = \cos^2 t_1 - \cos^2 t_2$. Then we find $R^0(\Omega)f_2 = (4m + 4)f_2 - 2f_1$. In this case $c_{\lambda_2} = 4m + 4$, and we identify the expansion by considering the eigenvalue equation and the evaluation at $e$ using the first result as well. □

Next we go back to the situation of $\mu = a\omega_1 + b\omega_2$ with $a, b \in \mathbb{N}$. The basis $(F^k_1 \cdot v_\mu)^g_k=0, v_\mu$ being the highest weight vector of $V^K_\mu$, gives the $M$-decomposition, and

$$\pi^K_\mu(E_{1,1})F^k_1 \cdot v_\mu = (a + b - k)F^k_1 \cdot v_\mu, \quad \pi^K_\mu(E_{2,2})F^k_1 \cdot v_\mu = (b + k)F^k_1 \cdot v_\mu,$$

and we see that almost all actions of the Lie algebra in the expression for the radial part of the Casimir operator $R^\mu(\Omega)$ of (A.5) commute with the action of $M$. Only for the third line of the expression for $R^\mu_m(\Omega)$ corresponding to the middle roots of $\text{BC}_2$ we get a nontrivial interaction of the $M$-types. Put $G_k = \langle G(\cdot)F^k_1 \cdot v_\mu, F^k_1 \cdot v_\mu \rangle : A \to \mathbb{C}$ for the scalar action of $G : A \to \text{End}_M(V^K_\mu)$ on $V^K_\mu \subset V^K_\mu$, then we can rewrite the radial part of the Casimir operator, see Appendix A.3 as

$$\frac{1}{2} \sum_{p=1}^{2} \frac{\partial^2 G_k}{\partial t_p^2} + (R^\mu_m(\Omega)G)_k + (R^\mu_m(\Omega)G)_k + (R^\mu_m(\Omega)G)_k,$$

(6.3)
where the respective parts are given by

\[ (R^\mu(\Omega_m)G)_k = \frac{1}{2(m+2)} (m(a+b-k)^2 - 4(a+b-k)(b+k) + m(b+k)^2)G_k \]

for the action corresponding to \( \Omega_m \), and the term for the short roots is equal to

\[ (R^\mu(\Omega)G)_k = -(m-2) \sum_{i=1}^2 \frac{\cos t_i \partial G_k}{\sin t_i \partial t_i} \]

and the term for the middle roots gives

\[
\begin{align*}
(R^\mu(\Omega)G)_k &= -\frac{\cos(t_1 + t_2)}{\sin(t_1 + t_2)} \left( \frac{\partial G_k}{\partial t_1} + \frac{\partial G_k}{\partial t_2} \right) - \frac{\cos(t_1 - t_2)}{\sin(t_1 - t_2)} \left( \frac{\partial G_k}{\partial t_1} - \frac{\partial G_k}{\partial t_2} \right) \\
&\quad - \left( \frac{\cos(t_1 + t_2)}{\sin^2(t_1 + t_2)} + \frac{\cos(t_1 - t_2)}{\sin^2(t_1 - t_2)} \right) \left( (k+1)(a-k)G_{k+1} + k(a-k+1)G_{k-1} \right) \\
&\quad + \left( \frac{1}{\sin^2(t_1 + t_2)} + \frac{1}{\sin^2(t_1 - t_2)} \right) \left( ((k+1)(a-k) + k(a-k+1))G_k \right)
\end{align*}
\]

and the term for the long roots simplifies to

\[ (R^\mu(\Omega)G)_k = -\sum_{i=1}^2 \frac{\cos(2t_i)}{\sin(2t_i)} \frac{\partial G_k}{\partial t_i} + \frac{(a+b-k)^2}{2\cos^2 t_1} G_k + \frac{(b+k)^2}{2\cos^2 t_2} G_k. \]

Having described the radial part of the Casimir operator explicitly, we can use the action to make the constants in Theorem 4.5 and Corollary 4.6 explicit.

**Proposition 6.4.** As functions \( A \rightarrow \text{End}_M(V^K) \) we have

\[ \Phi^\mu_{i_r} = \frac{(m+b+i)_i}{(m)_i} \sum_{\nu_i} \frac{(-i)_{i-r}(-i-b)_{i-r}}{(i-r)!(1-m-2i-b)_{i-r}} Q^\mu_{i_r} \]

The key ingredient in the proof of Proposition 6.4 is the action of the radial part of the Casimir operator on the functions \( Q^\mu_{i_r} : A \rightarrow \text{End}_M(V^K) \).

**Lemma 6.5.** For \( i \in \{0, \ldots, a\} \) and \( Q^\mu_{i_r} \) as in Theorem 4.5 we have

\[ R^\mu(\Omega)Q^\mu_{i_r} = c_{i_r}Q^\mu_{i_r} - 2i(b+i)Q^\mu_{i_r-1} \]

where \( c_{i_r} \) is the eigenvalue of \( \Phi^\mu_{i_r} \) for \( R^\mu(\Omega) \);

\[ c_{i_r} = 2i^2 + 2i(b+m) + (m+1)a + 2mb + \frac{1}{m+2}((m+1)a^2 + 2mb(a+b)). \]

**Proof.** Note that \( R^\mu(\Omega)\Phi^\mu_{i_r} = c_{i_r}\Phi^\mu_{i_r} \) with \( c_{i_r} = \langle \nu_i, \nu_i \rangle + 2\langle \nu_i, \rho \rangle \), and the explicit value of \( c_{i_r} \) follows by a calculation. This shows that \( c_{i_r} < c_{i_r+1} \). This also follows more generally from Corollary 2.8 and Lemma 6.1. Since the transition of the basis of \( (\Phi^\mu_{i_r})_{i=0}^{a} \) to the basis \( (Q^\mu_{i_r})_{i=0}^{a} \) is triangular, we find \( R^\mu(\Omega)Q^\mu_{i_r} = c_{i_r}Q^\mu_{i_r} + \sum_{r=0}^{i-1} C_rQ^\mu_{i_r} \), for certain constants \( C_r \).
These constants can be determined considering the action on $V_{σ_0}^M$ of this identity using $q_{σ_i}^μ(a_t) = (\cos t_1)^{a+b}(\cos t_2)^{b+2i}$ and

\[(6.4) \quad q_{σ_i}^μ(a_t) = \frac{a-i}{a}(\cos t_1)^{a+b-1}(\cos t_2)^{b+2i+1} + \frac{i}{a}(\cos t_1)^{a+b+1}(\cos t_2)^{b+2i-1},\]

where we use $K_1(x;p,N) = 1 - \frac{x}{pN}$ for the Krawtchouk polynomials, see Theorem 4.3.

Using this we find by a trigonometric calculation (using computer algebra)

\[(R^μ(Ω)Q^μ_{σ_i})_0 = c_{σ_i}(\cos t_1)^{a+b}(\cos t_2)^{b+2i} - 2i(b+i)(\cos t_1)^{a+b}(\cos t_2)^{b+2i-2}.\]

The right hand side is $c_{σ_i}q_{σ_i,σ_0}(a_t) - 2i(b+i)q_{σ_i-1,σ_0}(a_t)$, so that $C_{r-1} = -2i(b+i)$ and $C_r = 0$ for $r < i - 1$ since the functions $q_{σ_i,σ_k}^μ$ are independent for $i \in \{0, \ldots, a\}$. □

**Remark 6.6.** The fact that the right hand side of Lemma 6.5 consists of just two matrix leading terms makes it possible to derive many explicit results for the matrix spherical functions. This is one of the main motivations to consider these specific leading terms.

**Proof of Proposition 6.4.** Apply $R^μ(Ω)$ to Corollary 4.6 using that the $Φ^μ_{σ_i}$ are eigenfunctions for $R^μ(Ω)$, Lemma 6.5 and that the $Q^μ_{σ_i}$ are linearly independent to find the recursion $d^i_i c_{σ_i} = d^{i+1}_i c_{σ_i} - d^{i+1}_r 2(r+1)(b+r+1)$ for $r < i$. Using the value for $c_{σ_i}$ as in Lemma 6.5 we obtain

\[d^i_i (i-r)(b+m+r+i) = d^{i+1}_r (r+1)(b+r+1) \implies d^i_i = \frac{(-i)_{i-r}(-i-b)_{i-r}}{(i-r)! (1-m-2i-b)_{i-r}} d^{i+1}_r\]

by iteration and it remains to determine $d^i_i$. Evaluating at the identity element $e \in A$ and using that $Q^μ_{σ_i}$ and $Φ^μ_{σ_i}$ are the identity at $e$, we find

\[\frac{1}{d^i_i} = \sum_{r=0}^{i} \frac{(-i)_{i-r}(-i-b)_{i-r}}{(i-r)! (1-m-2i-b)_{i-r}} = 2F1\left(\begin{array}{c} -i, -i-b \\ 1-m-2i-b \end{array} ; 1\right) = \frac{(1-m-i)_i}{(1-m-2i-b)_i}\]

by the Chu-Vandermonde summation, see e.g. [11 Cor. 2.2.3], [12 §1.4]. Simplifying $d^i_i$ gives the result. □

As a next step we translate the Proposition 6.4 into the transition for the matrix weight $W$ and $S$. Recall the matrix functions $Φ_0$ and $Q_0$ as defined in §5.

**Proposition 6.7.** We have $Φ_0 = LQ_0$ with the constant lower triangular matrix $L$ given by $L_{i,j} = 0$ for $j > i$ and

\[L_{i,j} = (-1)^{i+j} \binom{i}{j} \frac{(m+b+i)_i}{(m)_i} \frac{(b+j+1)_{i-j}}{(m+i+j+b)_{i-j}}, \quad 0 \leq j \leq i \leq a\]

and its inverse is the lower triangular matrix given by $(L^{-1})_{i,j} = 0$ for $j > i$ and

\[(L^{-1})_{i,j} = \binom{i}{j} \frac{(m)_i}{(m+b+j)_j} \frac{(b+j+1)_{i-j}}{(m+2j+b-1)_{i-j}}, \quad 0 \leq j \leq i \leq a.\]
Proof. Recall from Section 5 and Proposition 6.4 that as functions on $A$ we have

$$(\Phi_0)_{i,k} = \phi_{\nu_i,\sigma_k}^\nu = (\Phi_i^\nu)_{k} = \sum_{r=0}^{i} d_r^i (Q_{\nu_r}^i)_{k} = \sum_{r=0}^{i} d_r^i q_{\nu_r,\sigma_k}^i = \sum_{r=0}^{a} L_{i,r} (Q_0)_{r,k}$$

with $L_{i,r} = d_r^i$ for $i \leq r$ and $L_{i,r} = 0$ for $i > r$. Rewriting gives the matrix $L$.

To show that $L^{-1}$ is as given we need to show the nontrivial case; for $j \leq i$ we have to show $\sum_{r=j}^{i} L_{i,r}(L^{-1})_{r,j} = \delta_{i,j}$. Taking out the $r$-independent parts, we see that this equivalent to showing

$$\delta_{i,j} = \sum_{r=j}^{i} (-1)^{i+r} \binom{i}{r} \frac{(b + r + 1)_{i-r}}{(m + i + r + b)_{i-r}} \frac{(b + j + 1)_{r-j}}{(m + 2j + b - 1)_{r-j}}.$$

The right hand side can be rewritten as

$$\frac{(b + i + j)_{i-j}}{(m + i + j + b - 1)_{i-j}} \binom{i}{j} (-1)^{i+j} \sum_{k=0}^{i-j} \binom{j-i}{k} \frac{(m + i + j + b - 1)_{k}}{(m + 2j + b)_{k}}$$

and the sum is a terminating $\text{$_2F_1$}$-series at 1, which can be evaluated by the Chu-Vandermonde summation, see e.g. [11] Cor. 2.2.3, [12], §1.4, as $\frac{1_{i+j-i}}{(m+2j+b)_{i-j}}$ so that the numerator gives 0 unless $i = j$, in which case we find 1. \hfill \Box

7. MATRIX ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

In Section 5 we have established the matrix weight for the polynomials, and in this section we establish some more properties for these matrix orthogonal polynomials in two variables with $B_{2\nu}$-symmetry. In particular, we make the orthogonality relations more explicit. Moreover, we derive the matrix partial differential operator to which these matrix polynomials are eigenfunction.

Firstly, the Haar measure on $A$ is $dt_1 dt_2$ on $[-\pi, \pi] \times [-\pi, \pi]$ and using the invariance under the sign changes, we can reduce to the integral over $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$. Using (5.7) we find for the normalising constant in (1.9)

$$(7.1) \quad \frac{1}{c} = \int_{A} |\delta(a)| da = 4^2 \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} |\delta(a_1)| dt_1 dt_2 = \frac{32}{m^2(m^2 - 1)}.$$

In order to make the connection to the $B_{2\nu}$-case as originally introduced by Koornwinder [24], [25], see also [34], [30], we make an affine change of variable $\psi_1 = \frac{1}{4}x_1 + 1$, $\psi_2 = \frac{1}{4}x_2 + \frac{1}{4}x_1 + \frac{1}{4}$, or, in terms of $t_1$ and $t_2$, $x_1 = \cos(2t_1) + \cos(2t_2)$, $x_2 = \cos(2t_1) \cos(2t_2)$. Then the map sending $(t_1, t_2) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ to $(x_1, x_2)$ is a 2 : 1-mapping onto the region bounded by the parabola $x_1^2 = 4x_2$ and the lines $x_2 = x_1 - 1$, $x_2 = 1 - x_1$, see Figure 7.1. This is exactly the region of integration for the polynomials studied in [24], [25], [34]. For $d = (d_1, d_2) \in \mathbb{N}^2$ we define matrix polynomials $R_d$ of size $(a + 1) \times (a + 1)$ of degree $d$ by

$$(7.2) \quad R_d(x_1, x_2) = P_d(\phi_1, \phi_2)L$$
where we use the notation for $P_d$ as in Subsection 1.1 the affine transformation from $(x_1, x_2)$ to $(\psi_1, \psi_2)$ as given above and the affine transformation from Lemma 6.2 and with $L$ as in Proposition 6.7. Finally, we define the matrix weight

$$S^a(x_1, x_2) = S(\psi_1, \psi_2)$$

with $S(\psi_1(a), \psi_2(a))$ defined in (5.3) for the case $a \in \mathbb{N}, b = 0$, and using the coordinate change of $(\psi_1, \psi_2)$ to $(x_1, x_2)$. In case $a = 1$ we obtain

$$S^1(x_1, x_2) = \begin{pmatrix} \frac{1}{2}x_1 + 1 & \frac{1}{4}(x_1 + x_2 + 1) \\ \frac{1}{2}(x_1 + x_2 + 1) & \frac{1}{4}(\frac{1}{2}x_1 + 1)(x_1 + x_2 + 1) \end{pmatrix},$$

and for $a = 2$ we obtain that $S^2(x_1, x_2)$ equals

$$\begin{pmatrix} \frac{3}{8}(\frac{1}{2}x_1 + 1)^2 - \frac{x_1+x_2+1}{4} & \frac{3}{8}(\frac{1}{2}x_1 + 1)(x_1 + x_2 + 1) & \frac{3}{16}(x_1 + x_2 + 1) \\ \frac{3}{8}(\frac{1}{2}x_1 + 1)(x_1 + x_2 + 1) & \frac{3}{16}(x_1 + x_2 + 1)(x_1 + x_2 + 1)^2 & \frac{3}{32}(\frac{1}{2}x_1 + 1)^2((\frac{1}{2}x_1 + 1)^2 - \frac{x_1+x_2+1}{8}) \\ \frac{3}{16}(x_1 + x_2 + 1)^2 & \frac{3}{32}(\frac{1}{2}x_1 + 1)(x_1 + x_2 + 1)^2 & \frac{3}{16}(\frac{1}{2}x_1 + 1)^2((\frac{1}{2}x_1 + 1)^2 - \frac{x_1+x_2+1}{8}) \end{pmatrix}.$$ 

These examples follow from (5.4) taking $b = 0$ and $\psi_1 = \frac{1}{2}x_1 + 1$, $\psi_2 = \frac{1}{2}x_2 + \frac{1}{4}x_1 + \frac{1}{4}.$

**Theorem 7.1.** The matrix polynomials $R_d$ defined by (7.2) are orthogonal on the region of integration $I$ as in Figure 7.1 and

$$\int_I R_d(x_1, x_2) S^a(x_1, x_2) (R_{d'}(x_1, x_2))^t (1 - x_1 + x_2)^{-2(1 + x_1 + x_2)t} dx_1 dx_2$$

$$= \delta_d d^2 2^{2m+2b-10} m^2 (m^2 - 1) H_d$$

where $S^a$ is positive definite on $I$ with positive determinant on the interior of $I$. Moreover, the weight function is indecomposable. Here the matrix $H_d$ is a diagonal matrix with $(H_d)_{k,k} = (a + 1)^2 / \dim V^{G}_{\delta + d_1 \lambda_1 + d_2 \lambda_2}$.

Moreover, the polynomials $R_d$ are eigenfunctions to a second order matrix partial differential operator:

$$R_d R^0(\Omega) - R_d C^a + R_d (\Lambda_0 + S) = \Lambda_d R_d$$
where \( \Lambda_d = \text{diag}(c_{r_1+d_1, \lambda_1}^{a_1} \cdots c_{r_1+d_2, \lambda_2}^{a_2}) \), \( d = (d_1, d_2) \in \mathbb{N}^2 \), \( \Lambda_0 = \Lambda_{(0,0)} \), and \( S \) is the lower triangular matrix with one nonzero subdiagonal with \( S_{r,r-1} = -2r(b+r) \). The operator \( R^0(\Omega) \) is the second order partial differential operator acting from the right as the identity times the classical partial differential operator

\[
(2x_1^2 - 4x_2 - 4) \frac{\partial^2}{\partial x_1^2} + (-2x_1^2 + 4x_2^2 + 4x_2) \frac{\partial^2}{\partial x_2^2} + 4x_1(x_2 - 1) \frac{\partial^2}{\partial x_1 \partial x_2} + 2((m+2)x_1 + 2m - 4) \frac{\partial}{\partial x_1} + 2((m-2)x_1 + 2 + (2m+2)x_2) \frac{\partial}{\partial x_2}
\]

and \( C^\mu \) is the first order matrix differential operator \( \frac{\partial}{\partial x_1} C^\mu_1 + \frac{\partial}{\partial x_2} C^\mu_2 \) where \( C^\mu_1 \) and \( C^\mu_2 \) are tridiagonal polynomial matrices of degree 1 given by

\[
(C^\mu_1)_{r,r} = 2((a+b) + r)x_1 - 2(b-2r), \quad (C^\mu_2)_{r,r} = 2((b+r)(2x_2 - x_1) + a(x_2 + 1)),
\]

\[
(C^\mu_1)_{r,r-1} = -(C^\mu_1)_{r,r-1} = -r(x_1 + x_2 + 1), \quad (C^\mu_2)_{r,r+1} = -(C^\mu_2)_{r,r+1} = -4(a-r).
\]

Moreover, the matrix partial differential operator is symmetric with respect to the matrix weight.

Remark 7.2. Note that the scalar part of the weight in Theorem 7.1 is the weight considered by Koornwinder \cite{24,25} and Sprinkhuizen-Kuyper \cite{34} for the special case \( \alpha = m - 2 \), \( \beta = b \), \( \gamma = \frac{1}{2} \). Similarly, in the case \( \mu = 0 \), i.e. \( a = b = 0 \), the partial differential operator reduces to the partial differential operator studied in \cite{24,25,34} up to a scalar multiple for these choices of parameters. The case \( a = 0 \), \( b \in \mathbb{N} \), gives the case of a non-trivial character of \( K \), and this corresponds to Heckman [9, Ch. 5].

Note that in the scalar case the 2-variable orthogonal polynomials can be expressed in terms of Jacobi polynomials \cite{25,34,34,34,34} Lemma 3.1. It is not clear if in this case we also have an explicit expression of \( R_d(x_1, x_2) \) in terms of matrix Jacobi polynomials of a single variable.

Proof of the orthogonality in Theorem 7.1. Observe that the Jacobian for the change of \( (t_1, t_2) \) to \( (x_1, x_2) \) is given by

\[
32 \left| \sin(t_1) \sin(t_2) \cos(t_1) \cos(t_2) \left( \cos^2(t_1) - \cos^2(t_2) \right) \right|
\]

and \( \sin^2(t_1) \sin^2(t_2) = \frac{1}{4}(1 - x_1 + x_2) \). (\( \cos^2(t_1) - \cos^2(t_2) \))^2 = \( \frac{1}{4}(x_1^2 - 4x_2) \). Keeping track of the constants involved, the statements on the orthogonality follow from \( 1.2 \) and from Section 5 in particular Proposition 5.3 and Remark 5.4 and \( S = Q_0 Q_0^* \) being positive. □

In order to prove the statement of Theorem 7.1 concerning the partial differential operator, we need to be able to rewrite the eigenvalue equation of the radial part of the Casimir operator \( R^\mu(\Omega) \) acting on the eigenvector \( \Phi^\mu_k|A \) in terms of an operator acting on the polynomials \( R_d \). For this we need to conjugate \( R^\mu(\Omega) \) with the matrix function \( Q_0 \), see \cite{21} §3.2. We collect the technical result in Lemma 7.3.

Lemma 7.3. We have for \( i = 1, 2 \) as matrix valued functions on \( A \)

\[
\frac{\partial \psi_i}{\partial t_1} \frac{\partial Q_0}{\partial t_1} + \frac{\partial \psi_i}{\partial t_2} \frac{\partial Q_0}{\partial t_2} = C_i(\psi_1, \psi_2) Q_0
\]
where we consider the functions as functions of \((t_1, t_2)\) by evaluating at \(a_t \in A\). Here \(C_i(\psi_1, \psi_2)\) is a matrix polynomial in \((\psi_1, \psi_2)\) of total degree at most 1, where the non-zero entries are explicitly given by

\[
C_1(\psi_1, \psi_2)_{r,r} = 2(a + 2b + 2r - (a + b + r)\psi_1),
\]

\[
C_2(\psi_1, \psi_2)_{r,r} = 2((b + r)\psi_1 - (a + 2b + 2r)\psi_2),
\]

\[
C_1(\psi_1, \psi_2)_{r,r-1} = C_2(\psi_1, \psi_2)_{r,r-1} = 2r\psi_2, \quad C_1(\psi_1, \psi_2)_{r,r+1} = C_2(\psi_1, \psi_2)_{r,r+1} = 2(a - r).
\]

Note that the tridiagonal matrices coincide on the off-diagonal entries.

There are analogues of Lemma 7.3 with \(Q_0\) replaced by \(\Phi_0\) and \(\psi_i\) replaced by \(\phi_i\) or \(x_i\), see also the first paragraph of the proof. However, in general it is hard to calculate the right hand side explicitly. In this case we can do the explicit calculation because of the homogeneity properties of the entries of \(Q_0\) and \(\psi_1, \psi_2\).

**Proof.** Lemma 3.9 of [21] implies that

\[
\frac{\partial \phi_1}{\partial t_1} \frac{\partial \Phi_0}{\partial t_1} + \frac{\partial \phi_1}{\partial t_2} \frac{\partial \Phi_0}{\partial t_2} = C_i'(\phi_1, \phi_2)\Phi_0
\]

for a matrix polynomial \(C_i'\) in \((\phi_1, \phi_2)\) of maximal total degree 1, where we use the adjoint of [21] Lemma 3.9. Using \(\Phi_0 = LQ_0\) and the affine transformation of \((\phi_1, \phi_2)\) to \((\psi_1, \psi_2)\) given in Lemma 6.2 proves the general statement of the lemma, and it remains to determine the polynomials \(C_i\).

Take \(i = 1\) and consider the \((r,s)\)-entry of the left hand side of the identity. Since \((Q_0)_{r,s}(a_t) = q^{a_t}_{r,s}(a_t)\) is a homogeneous polynomial of degree \(a + 2b + 2r\) in \((\cos t_1, \cos t_2)\), we see by an explicit calculation that

\[
(7.4) \quad \frac{\partial \psi_1}{\partial t_1} \frac{\partial (Q_0)_{r,s}}{\partial t_1} + \frac{\partial \psi_1}{\partial t_2} \frac{\partial (Q_0)_{r,s}}{\partial t_2} = 2(a + 2b + 2r)(Q_0)_{r,s} + E_{r,s}
\]

where \(E_{r,s}\) is a homogeneous polynomial of degree \(a + 2b + 2r + 2\) in \((\cos t_1, \cos t_2)\). Since \(\psi_1\), respectively \(\psi_2\), is homogeneous of degree 2, respectively 4, and the fact that \(C_1(\psi_1, \psi_2)\) is of degree at most 1, we have \(E_{r,s} = a_r \psi_1(Q_0)_{r,s} + b_r \psi_2(Q_0)_{r-1,s} + c_r(Q_0)_{r+1,s}\) for coefficients \(a_r, b_r\) and \(c_r\). So we see that \(C_1(\psi_1, \psi_2)\) is a tridiagonal matrix, and we determine the coefficients. For this we take \(s = 0\) and recall from Theorem 4.5 that \((Q_0)_{r,0}(a_t) = q^{a_t}_{r,0}(a_t) = (\cos t_1)^{a+b}(\cos t_2)^{b+2r}\). So that

\[
\frac{\partial \psi_1}{\partial t_1} \frac{\partial (Q_0)_{r,0}}{\partial t_1} + \frac{\partial \psi_1}{\partial t_2} \frac{\partial (Q_0)_{r,0}}{\partial t_2} = 2(a + b + 2r)(Q_0)_{r,0} - 2(a + b)(\cos t_1)^{a+b+2}(\cos t_2)^{b+2r} - 2(b + 2r)(\cos t_1)^{a+b}(\cos t_2)^{b+2r+2}
\]

and comparing with the explicit form of \(E_{r,s}\) we get \(a_r + b_r = -2(a + b)\) and \(a_r + c_r = -2(b + 2r)\). Writing \(b_r\) and \(c_r\) in terms of \(a_r\), we now take \(s = 1\) in (7.4) and we use the explicit expression (6.3) in order to obtain by a calculation (using computer algebra) that \(a_r = -2a - 2b - 2r\). This gives the expression for \(C_1(\psi_1, \psi_2)\).
In case \( i = 2 \) we proceed similarly and we get
\[
\frac{\partial \psi_2}{\partial t_1} (Q_0)_{r,s} \frac{\partial (Q_0)_{r,s}}{\partial t_1} + \frac{\partial \psi_2}{\partial t_2} (Q_0)_{r,s} = -2(a + 2b + 2r)\psi_2(Q_0)_{r,s} + E_{r,s}
\]
where again \( E_{r,s} \) is a homogeneous polynomial of degree \( a + 2b + 2r + 2 \) in \( (\cos t_1, \cos t_2) \) and hence of the form \( E_{r,s} = a_r \psi_1(Q_0)_{r,s} + b_r \psi_2(Q_0)_{r-1,s} + c_r(Q_0)_{r+1,s} \) as before. So also \( C_2(\psi_1, \psi_2) \) is tridiagonal. Taking \( s = 0 \) in (7.3) we find by a calculation that \( a_r + b_r = 2(b + 2r) \) and \( a_r + c_r = 2(a + b) \) in this case. Eliminating \( b_r \) and \( c_r \) in terms of \( a_r \) and now taking \( s = 1 \) in (7.5) and using the explicit form (6.4), we find by a calculation (using computer algebra) \( a_r = 2b + 2r \). This gives the expression for \( C_2(\psi_1, \psi_2) \).

In order to derive the partial differential operator of Theorem 7.1, we observe that in this case we can rewrite (1.8) as
\[
(7.6) \quad \Phi^{\mu}_{\nu+\lambda_{\text{ph}}} (a_t) = \sum_{r=0}^{a} q^{\mu}_{\nu,\nu+\nu}(\psi_1(a_t), \psi_2(a_t)) Q^\mu_{\nu+\nu}(a_t)
\]
using Theorem 4.3 and Remark 6.3. Note that \( q^{\mu}_{\nu,\nu+\nu} \) is a polynomial of total degree \( |\mathbf{d}| \). Note that \( q^{\mu}_{\nu,\nu+\nu} \) are entries of the matrix polynomials \( R^\mu(\Omega) \) up to a change of coordinates. Since \( \Phi^{\mu}_{\nu+\lambda_{\text{ph}}} (a_t) \) is an eigenvector of the radial part \( R^\mu(\Omega) \) of the Casimir operator, we need to derive the action \( R^\mu(\Omega) \) on \( f(\psi_1(a_t), \psi_2(a_t)) Q^\mu_{\nu}(a_t) \) for \( f \) a 2-variable scalar function. It can be checked from (6.3) that, cf. proof of [21] Lemma 3.9,
\[
(7.7) \quad R^\mu(\Omega) (f(\psi_1, \psi_2) Q^\mu_{\nu}) = f(\psi_1, \psi_2) (R^\mu(\Omega) Q^\mu_{\nu}) + (R^0(\Omega) f(\psi_1, \psi_2)) Q^\mu_{\nu} - 2 \sum_{p=1}^{2} \frac{\partial f}{\partial t_p} \frac{\partial Q^\mu_{\nu}}{\partial t_p}
\]
and the first term follows from Lemma 6.5 and the last term can be dealt with using Lemma 7.3 and the chain rule. So we can rewrite \( R^\mu(\Omega) (f(\psi_1, \psi_2) Q^\mu_{\nu}) \) completely in terms of \( Q^\mu_{\nu} \)'s. So we get an eigenvalue equation for the \( q^{\mu}_{\nu,\nu+\nu} \)’s, which is
\[
(7.8) \quad c_{\nu+\lambda_{\text{ph}}} \sum_{r=0}^{a} q^{\mu}_{\nu,\nu+\nu}(\psi_1, \psi_2) Q^\mu_{\nu+\nu} = \sum_{r=0}^{a} (R^0(\Omega) q^{\mu}_{\nu,\nu+\nu}(\psi_1, \psi_2)) Q^\mu_{\nu+\nu} + \sum_{r=0}^{a} \left( c_r Q^\nu_{\nu+\nu} - 2r(b + r)Q^\nu_{\nu+1} \right) - 2 \sum_{p=1}^{2} \sum_{r,u=0}^{a} \frac{\partial q^{\mu}_{\nu,\nu+\nu}}{\partial \psi_p} (\psi_1, \psi_2) C_p(\psi_1, \psi_2) Q^\mu_{\nu+\nu}
\]
where \( \lambda_{\text{ph}} = d_1 \lambda_1 + d_2 \lambda_2 \), \( \mathbf{d} = (d_1, d_2) \in \mathbb{N}^2 \).

**Lemma 7.4.** Let \( Q^\mu_{\mathbf{d}} = Q^\mu_{\mathbf{d}}(\psi_1, \psi_2) \) be the matrix polynomial defined by \( (Q^\mu_{\mathbf{d}})_{i,j}(\psi_1, \psi_2) = q^{\mu}_{i,j;\mathbf{d}}(\psi_1, \psi_2) \) using (1.6), then
\[
Q^\mu_{\mathbf{d}} R^0(\Omega) - \frac{\partial Q^\mu_{\mathbf{d}}}{\partial \psi_1} C_1(\psi_1, \psi_2) - \frac{\partial Q^\mu_{\mathbf{d}}}{\partial \psi_2} C_2(\psi_1, \psi_2) + Q^\mu_{\mathbf{d}}(\Lambda_0 + S) = \Lambda_{\mathbf{d}} Q^\mu_{\mathbf{d}}
\]
where \( \Lambda_{\mathbf{d}}, \Lambda_0, \) and \( S \) are as in Theorem 7.1, \( C_i(\psi_1, \psi_2), i = 1, 2, \) are the matrix polynomials of at most degree 1, see Lemma 7.3. Moreover, \( R^0(\Omega) \) is a matrix second order partial differential operator in \( (\psi_1, \psi_2) \) acting entrywise, considered as acting from the right.
Remark 7.5. Note that the radial part $R^0(\Omega)$ acts as a matrix differential operator when considered as multiplied by the identity. This has to be rewritten as differential operator with respect to the variables $(\psi_1, \psi_2)$, which can be done since the spherical functions are polynomials in $(\phi_1, \phi_2)$, hence in $(\psi_1, \psi_2)$, see Vretare [37]. For convenience we write down the terms of $R^0(\Omega) f$, where $f$ is a scalar polynomial in $(\psi_1, \psi_2)$. Then $R^0(\Omega_m)$ is zero, and $-\frac{1}{2} \sum_{p=1}^2 \frac{\partial^2}{\partial \psi_p^2}$ in (6.3) corresponds to

$$(2\psi_1 - 2) \frac{\partial f}{\partial \psi_1} + (4\psi_2 - \psi_1) \frac{\partial f}{\partial \psi_2} + (2\psi_1^2 - 2\psi_1 - 4\psi_2) \frac{\partial^2 f}{\partial \psi_1^2} + (4\psi_2^2 - 2\psi_1 \psi_2) \frac{\partial^2 f}{\partial \psi_1 \partial \psi_2} + (4\psi_1 \psi_2 - 8\psi_2) \frac{\partial^2 f}{\partial \psi_1 \partial \psi_2}.$$  

$R^0_{s}(\Omega) f$ corresponds to $2(m - 2)\psi_1 \frac{\partial f}{\partial \psi_1} + 4(m - 2)\psi_2 \frac{\partial f}{\partial \psi_2}$, and $R^0_{s}(\Omega) f$ corresponds to $(2\psi_1 - 2) \frac{\partial f}{\partial \psi_1} + (4\psi_2 - \psi_1) \frac{\partial f}{\partial \psi_2}$ and $R^0_{m}(\Omega) f$ corresponds to $(4\psi_1 - 4) \frac{\partial f}{\partial \psi_1} + 4\psi_2 \frac{\partial f}{\partial \psi_2}$.

Proof. Write (7.8) in matrix notation, then we obtain the result of the lemma multiplied by the matrix function $Q_0$ from the right. Since $Q_0$ is generically invertible, see the proof of Proposition 5.3, the lemma follows.

Proof of the partial differential equation in Theorem 7.1. Comparing (7.6) with (7.2) and (1.8), we see that $R_A$ and $Q_A$ are the same up to the change of coordinates from $(\psi_1, \psi_2)$ for $Q_A$ to $(x_1, x_2)$ for $R_A$. Note that $x_1 = 2\psi_1 - 2$, $x_2 = 4\psi_1 - 2\psi_1 + 1$, making this affine change of coordinates gives the expression of $R^0(\Omega)$ in the $(x_1, x_2)$-coordinates as given in Theorem 7.1. It remains to make the change of coordinates in the other terms involving the first order differentials, which is straightforward.

Note that the radial part $R^0(\Omega)$ of the Casimir operator is symmetric with respect to the inner product $\langle \Phi, \Psi \rangle = \frac{1}{2} \int_A \text{Tr} (\Phi(a)(\Psi(a))^*) |\delta(a)| d\alpha$ for matrix spherical functions $\Phi, \Psi$, and the results given in Subsection 1.1. Since the second order matrix partial differential operator is obtained by conjugation by $Q_0$, we obtain the symmetry.

8. The leading term of $\Phi_A^\mu$

In Section 4 we have introduced the leading term $Q_\nu^\mu$ of the matrix spherical functions for $\Phi_\nu^\mu$ for $\nu \in B(\mu)$. Using these results we can determine the leading term $Q_\lambda^\mu$ of the matrix spherical functions for $\Phi_\lambda^\mu$ for $\lambda \in P_+^G(\mu)$. We do this by introducing the leading term from an embedding of $V_\mu^K$ in a large tensor product representation, similarly to the construction in Section 4. We then show by using the radial part of the Casimir operator, that this is indeed a leading term by establishing the lower triangularity of the radial part of the Casimir operator on these functions.

Assume as before $\mu = a\omega_1 + b\omega_2$ with $a, b \in \mathbb{N}$ and we take $\lambda \in P_+^G(\mu)$. By Condition 1.2 we can write $\lambda = \nu_1 + d_1 \lambda_1 + d_2 \lambda_2$ with $\nu_1 \in B(\mu)$, $d_1, d_2 \in \mathbb{N}$. Generalising the construction of $\psi_1, \psi_2$ and $Q_\nu^\mu$, as in Section 5 and 6, we define the tensor product representation and an explicit element by

$$W_\lambda^G = (V_{\omega_1}^G \otimes V_{\omega_{m+1}}^G)^{\otimes d_1} \otimes (V_{\omega_2}^G \otimes V_{\omega_m}^G)^{\otimes d_2} \otimes U_{\nu_1}^G, \quad w = v_1^{\otimes d_1} \otimes v_2^{\otimes d_2} \otimes u \in W_\lambda^G.$$
using the notation of Lemmas 3.2 and 3.3 and (4.1). Using the results of Sections 3 and 4 we see that \( w \) is a \( K \)-highest weight vector of highest weight \( \mu \) in \( W^G_\mu \). So we get a \( K \)-intertwiner \( j: V^K_\mu \to W^G_\mu \) mapping the highest weight vector \( v_\mu \in V^K_\mu \) to \( w \).

**Proposition 8.1.** Define the matrix spherical function \( Q^\mu_\lambda: G \to \text{End}(V^K_\mu) \) by \( Q^\mu_\lambda(g) = j^* \circ \pi^G_\lambda(\mu) \circ j) \), then

\[
Q^\mu_\lambda(a_t) = (\psi_1(a_t))^{d_1} (\psi_2(a_t))^{d_2} Q^\mu_{\nu_t}(a_t)
\]

and \( Q^\mu_\lambda|_A = \sum_{\lambda' < \lambda, \lambda' \in P^G_\mu(\mu)} a_{\lambda'} \Phi^\mu_{\lambda'}|_A \) for constants \( a_{\lambda'} \).

Note in particular, that the action of \( Q^\mu_\lambda(a_t) \) on the one-dimensional constituent \( V^M_{\sigma_k} \) in \( V^K_\mu \) is given by

\[(8.1) \quad (\psi_1(a_t))^{d_1} (\psi_2(a_t))^{d_2} q^\mu_{\nu_t, \sigma_k}(a_t)\]

which is a homogeneous polynomial in \((\cos t_1, \cos t_2)\) of degree \(2d_1 + 4d_2 + 2a + 4b + 2i\), see Remark [4.3] and Theorem [4.5].

**Proof.** As noted, \( w \) is a highest weight vector for the action of \( K \) of highest weight \( \mu \), so by construction \( Q^\mu_\lambda \) is a matrix spherical function, and by Subsection 3.4 it is a linear combination of \( \Phi^\mu_\lambda \) for \( \lambda \in P^G_\mu(\mu) \) by the Peter-Weyl theorem. Since we have the decomposition \( W^G_\lambda = \bigoplus_{\lambda' \prec \lambda} n_{\lambda'} V^G_{\lambda'} \), with \( n_\lambda = 1 \), by repeated application of e.g. [29] Lemma (3.1)], the expression for \( Q^\mu_\lambda|_A \) follows.

For the proof of the explicit expression, we use the notation as in the proof of Proposition 4.2. Since the matrix entry of \( a_t(r, s) \) acting on \( v_i \) and taking inner product with \( v_i \) is \( \psi_i(a_t) \) for \( i = 1, 2 \) by Lemmas 3.2 and 3.3 we find the result from Theorem 4.5.

In order to understand the decomposition of \( Q^\mu_\lambda \) of Proposition 8.1 we calculate the action of the radial part of the Casimir operator on \( Q^\mu_\lambda \) as a function on \( A \). Recall (7.6), and take \( f \) a polynomial in \((\psi_1, \psi_2)\), then this leads to Proposition 8.2. Note that Proposition 8.2 generalises Lemma 6.5, but Lemma 6.5 is used in the proof of Proposition 8.2.

**Proposition 8.2.** We have as functions on \( A \),

\[
R^\mu(\Omega)Q^\mu_\lambda = c_\lambda Q^\mu_\lambda + \sum_{\lambda' < \lambda, \lambda' \in P^G_\mu(\mu)} b_{\lambda'} Q^\mu_{\lambda'}
\]

**Corollary 8.3.** In Proposition 8.1 we have \( a_\lambda \neq 0 \), so that there exists constants \( b_{\lambda'} \) with

\[
\Phi^\mu_\lambda = \sum_{\lambda' \prec \lambda} b_{\lambda'} Q^\mu_{\lambda'}, \quad b_{\lambda} \neq 0.
\]

The statement of Corollary 8.3 motivates to call the matrix spherical function \( Q^\mu_\lambda \) the leading term of \( \Phi^\mu_\lambda \).

**Proof.** In case \( a_\lambda = 0 \) in Proposition 8.1 we have \( Q^\mu_\lambda \) in the span of \( \Phi^\mu_{\lambda'} \) for \( \lambda' \prec \lambda \) which is an invariant space for the radial part of the Casimir \( R^\mu(\Omega) \) with eigenvalues \( c_{\lambda'} \). By Lemma 6.1 the eigenvalue \( c_\lambda \) is not contained in this set, but Proposition 8.2 and Proposition 8.1
applied to $\lambda' < \lambda$ shows that the eigenvalue $c_\lambda$ has to occur, since $R^\mu(\Omega)$ acts in a lower triangular way on the $Q_\lambda^\mu$'s. This is the required contradiction.

So this means that we can invert the relation of Proposition 8.1 giving the stated expansion. □

**Proof of Proposition 8.2.** Put $f(\psi_1, \psi_2) = \psi_1^{d_1}\psi_2^{d_2}$, then the first term on the right hand side of (7.7) follows from Lemma 6.5. For the second term we have by a calculation

\[
R^0(\mu)f = 2(d_1^2 + d_1(1 + 2d_2 + m) + 2d_2^2 + 2md_2)\psi_1^{d_1}\psi_2^{d_2} - 2d_2\psi_1^{d_1+1}\psi_2^{d_2-1}
\]

\[-2d_1(d_1 + 4d_2 + 3)d_2\psi_1^{d_1-1}\psi_2^{d_2} - 4d_1(d_1 - 1)d_2\psi_1^{d_1-2}\psi_2^{d_2+1},
\]

which follows from the explicit expression of the radial part of the Casimir operator, $R^0(\Omega)$, in the $(\psi_1, \psi_2)$-coordinates, see Remark 7.5. For the final term $-\sum_{\rho=1}^{2} \frac{\partial f}{\partial \psi_s}(C_s(\psi, \psi_2)Q_0)_{\mu,k}$ we consider the action on the constituent $V_\lambda^M$ in $V_\mu^K$, and we find

\[-\sum_{s=1}^{2} \frac{\partial f}{\partial \psi_s}(C_s(\psi, \psi_2)Q_0)_{\mu,k}
\]

using the chain rule and Lemma 7.3. By the explicit expression of Lemma 7.3 this term gives

\[
-c_{\nu_1} + 2(d_1^2 + d_1(1 + 2d_2 + m) + 2d_2^2 + 2md_2) + (a + b + i)d_1 + 2(a + b + 2i)d_2.
\]

Write $\lambda = \nu + \lambda_{\text{sph}}$, with $\lambda_{\text{sph}} = d_1\lambda_1 + d_2\lambda_2$, then the eigenvalue $c_\lambda$ can be written as

\[c_\lambda = c_{\nu_1} + \langle \lambda_{\text{sph}}, \lambda_{\text{sph}} \rangle + 2\langle \lambda_{\text{sph}}, \nu_1 + \rho \rangle.
\]

Since $\langle \lambda_{\text{sph}}, \lambda_{\text{sph}} \rangle = 2d_1^2 + 4d_1d_2 + 4d_2^2$, and $\langle \lambda_1, \nu_1 + \rho \rangle = a + b + i + m + 1$, $\langle \lambda_2, \nu_1 + \rho \rangle = a + b + 2i + 2m$, we see that the coefficient of $Q_\lambda^\mu$ in $R^\mu(\Omega)Q_\lambda^\mu$ is $c_\lambda$. □

Note that the proof of Proposition 8.2 actually gives a complete expression for the action of $R^\mu(\Omega)$ on $Q_\lambda^\mu$. For completeness we list in Table 8.1 the $\lambda' < \lambda$ for which $Q_\lambda^\mu$ occurs with a non-zero $b_{\lambda'}$ whose explicit value is listed as well.

Note that indeed all $\lambda'$ satisfy $\lambda' \in P_+^G(\mu)$ and $\lambda' < \lambda$, which can be checked using the results of Section 2

9. THE CASE $\mu = a\omega_1 + b\omega_2$ WITH $b$ NEGATIVE

In general, we obtain from (1.7) and $\sigma_k(\mu^*) = \sigma_{a-k}(\mu)$ for $\mu = a\omega_1 + b\omega_2$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$, see Subsection 2.2, and $a_{-1}^{-1} = a_{-t}$,

\[q^\mu_{\mu_{\rho\{\mu^*\},\sigma_\epsilon(\mu^*)}(a_t) = q^\mu_{\rho_{\mu_{\rho\epsilon\mu}(\mu),\sigma_{a-k}(\mu)}(a_{-t})}
\]
extending to the notation (5.1) to more general $\mu$ and stressing the dependence on $\mu$ and $\mu^*$ in the corresponding weights. So for the corresponding $\Phi_0^\mu$, we obtain

\begin{equation}
\Phi_0^\mu(a_t) = J\Phi_0^\mu(a_{-t})J, \quad J_{i,j} = \delta_{i+j,a}, \quad 0 \leq i, j \leq a.
\end{equation}

Applying (1.7) to (1.8) using that $\lambda^*_\text{sph} = \lambda_\text{sph}$ and that spherical functions satisfy $\phi(a_t) = \phi(a_{-t})$ we obtain for the matrix polynomials $P_\lambda^\mu(\phi_1, \phi_2) = F_\lambda(\phi_1, \phi_2)$ introduced in (1.9)

\begin{equation}
P_\lambda^\mu(\phi_1, \phi_2) = JP_\lambda^\mu(\phi_1, \phi_2)J.
\end{equation}

The weight function satisfies $W^\mu(\phi_1, \phi_2) = JW^\mu(\phi_1, \phi_2)J$ as follows from (9.1), so that we see that the matrix polynomials for $\mu = a\omega_1 + b\omega_2$ and $\mu^* = a\omega_1 - (a+b)\omega_2$ are essentially the same. So this covers the case $b \leq -a$.

It remains to consider the case $-a < b < 0$ with $a \in \mathbb{N}$, $b \in \mathbb{Z}$, and using duality we can restrict to the case $-\frac{1}{2}a \leq b < 0$. However, in this case we cannot extend the method established for the case $b \geq 0$ easily, due to the fact that the bottom splits into two parts. The results for each of these parts cannot be easily related to each other.

**Remark 9.1.** The case that $\mu^* = \mu$, i.e. $a \in 2\mathbb{N}$ and $b = -\frac{1}{2}a$ or $\mu = 2c\omega_1 - c\omega_2$ for $c \in \mathbb{N}$, is exhibiting different behaviour. Assume $c \geq 1$, we see that the corresponding space $A$ and $A^\prime$ as in Proposition 5.1 for the matrix weight $W$, see Remark 5.2, are no longer trivial, since $A^\prime$ and $A$ both contain $J$. Calculations for small values of $c$ in $\mu = 2c\omega_1 - c\omega_2$ indicate that we may expect $A^\prime = \mathbb{C}J \oplus \mathbb{C}1d$ and $A = \mathbb{R}J \oplus \mathbb{R}1d$ with $A^\prime$ and $A$ defined as in Proposition 5.1.

Note that in the study of matrix orthogonal polynomials of a single variable related to $(SU(2) \times SU(2), \text{diag})$ the weight is also reducible, see [19, Prop. 6.4, Thm. 6.5]. In that case the algebra $A^\prime$ is also two-dimensional with a similarly defined non-trivial element. So we see that self-duality of the $K$-representations in these cases leads to reducibility of the weight for the corresponding matrix orthogonal polynomials. The precise relation requires more attention in general.

**Appendix A. The radial part of the Casimir operator**

In general the determination of the radial part of an operator arising from a suitable element in the universal enveloping algebra is due to Harish-Chandra in unpublished papers.

| $\lambda'$ | $b_{\lambda'}$ |
|------------|---------------|
| $(d_1-1)\lambda_1 + d_2\lambda_2 + \nu_i$ | $-2d_1(d_1 + 4d_2 + 3) - 2d_1(a + 2b + 2i)$ |
| $(d_1-2)\lambda_1 + (d_2 + 1)\lambda_2 + \nu_i$ | $-2d_1(d_1 - 1)$ |
| $(d_1+1)\lambda_1 + (d_2 - 1)\lambda_2 + \nu_i$ | $-2d_2 - 2d_2(b + i)$ |
| $d_1\lambda_1 + d_2\lambda_2 + \nu_{i-1}$ | $-2i(b + i) - 2id_2$ |
| $(d_1-1)\lambda_1 + (d_2 + 1)\lambda_2 + \nu_{i-1}$ | $-2id_1$ |
| $(d_1-1)\lambda_1 + d_2\lambda_2 + \nu_{i+1}$ | $-2(a - i)d_1$ |
| $d_1\lambda_1 + (d_2 - 1)\lambda_2 + \nu_{i+1}$ | $-2(a - i)d_2$ |

**Table 8.1.** Table for the remaining coefficients in Proposition 5.2.
$\beta \in R$ & $\dim g_\beta$ & $\alpha \in \Delta$ with $\alpha|_{\alpha'} = \beta$
\hline
$f_1 - f_2$ & 2 & $\varepsilon_1 - \varepsilon_2, \varepsilon_{m+1} - \varepsilon_{m+2}$
$f_1 + f_2$ & 2 & $\varepsilon_i - \varepsilon_{m+i}, i = 1, 2$
$2f_i, 1 \leq i \leq 2$ & 1 & $\varepsilon_i - \varepsilon_{m+3-i}$
$f_i, 1 \leq i \leq 2$ & 2 & $\varepsilon_i - \varepsilon_{2+j}, \varepsilon_{2+j} - \varepsilon_{m+3-i}, 1 \leq j \leq m - 2$
$f_2 - f_1$ & 2 & $\varepsilon_2 - \varepsilon_1, \varepsilon_{m+2} - \varepsilon_{m+1}$
$-f_1 - f_2$ & 2 & $\varepsilon_{m+i} - \varepsilon_i, i = 1, 2$
$-2f_i, 1 \leq i \leq 2$ & 1 & $\varepsilon_{m+3-i} - \varepsilon_i$
$-f_i, 1 \leq i \leq 2$ & 2 & $\varepsilon_{2+j} - \varepsilon_i, \varepsilon_{m+3-i} - \varepsilon_{2+j}, 1 \leq j \leq m - 2$
\hline

Figure A.1. The restricted root system of type BC$_2$.

from 1960, see [8]. The result is mainly used for representations of noncompact Lie groups, see [15, Ch. VIII], [38, Ch. 9]. In this case we need to do this for the compact setting, and we derive the explicit expression from the Casimir element in the centre of $U(g)$. For this we follow Casselman and Miličić [3].

A.1. Structure theory. In order to calculate the radial part of the Casimir operator following [3], we note that $K = G^\theta$ with $\theta(g) = JgJ$, $J = \text{diag}(-1, -1, 1, \cdots, 1)$. In order to do the calculation we conjugate to the maximally split case. So we take

\begin{equation}
(A.1) \quad J' = \begin{pmatrix} 0 & 0 & J_2 \\ 0 & I_{m-2} & 0 \\ J_2 & 0 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} 1/\sqrt{2} & I_2 & 1/\sqrt{2} \\ 0 & I_{m-2} & 0 \\ -1/\sqrt{2} & I_2 & 1/\sqrt{2} \end{pmatrix} \in SU(m + 2), \quad u^*J'u = J,
\end{equation}

where $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\theta'(g) = J'gJ'$, so that $u\theta(g)u^* = \theta'(u^*gu)$ and $K' = G^{\theta'} = uKu^*$. We use the same notation for the involutions $\theta$ and $\theta'$ for the complexified Lie algebras.

Now $g = \mathfrak{so}(m + 2, \mathbb{C})$ has the root system $\Delta = \{\varepsilon_i - \varepsilon_j\}_{1 \leq i \neq j \leq m+2}$, $g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} g_\alpha$, where $\mathfrak{h}$ is the Cartan subalgebra consisting of the diagonal elements in $g$. The matrix $E_{i, j}$ spans $g_{\varepsilon_i - \varepsilon_j}$, see Subsection 2.1. Then $a' = uau^*$ consists of diagonal matrices $X = \text{diag}(d_1, d_2, 0 \cdots, 0, -d_2, -d_1)$, and we let $f_i(X) = d_i, i = 1, 2$. Then the reduced root system $R$ is of type BC$_2$ and the identification is given in Figure A.1. Then the positive roots of $\Delta$ and $R$ correspond to each other. Moreover, $m' = umu^* = m$. With $A' = uAu^*$, and $a'_4 = uau^* = \text{diag}(e^{it_1}, e^{it_2}, 1 \cdots, 1, e^{-it_2}, e^{-it_1})$ we have $M' = Z_K(A') = uZ_K(A)u^* = M$. Let $n_1f_1 + n_2f_2$ be the character of $A$ sending $a'_4 \mapsto e^{i(n_1t_1 + n_2t_2)}$.

Then the root space decomposition for the action of $A'$ is given by

$$g = a' \oplus m' \oplus \bigoplus_{\beta \in R} g_\beta, \quad g_\beta = \bigoplus_{\alpha \in \Delta, \alpha|_{\alpha'} = \beta} g_\alpha$$

where, for $\alpha = \varepsilon_i - \varepsilon_j \in \Delta$, $g_\alpha = \mathbb{C}Y_\alpha$ with $Y_\alpha = E_{i, j}$, where we use the same notation $\beta$ for the corresponding derivative $\alpha' \mapsto \varepsilon_i - \varepsilon_j \mapsto \varepsilon_{i'} - \varepsilon_{j'}$. Note that $\theta'$ gives an action on $\Delta$ by $\theta'(\alpha)(H) = \alpha(\theta'(H))$ for $H \in \mathfrak{h}$. Then $-\theta'$ is an involution of $\{\alpha \in \Delta \mid \alpha|_{\alpha'} = \beta\}$ for $\beta \in R$. 

A.2. **Casimir element.** The Killing form on \( g \) is given by \( B(X,Y) = \text{Tr}(XY) \) up to a positive multiple, and the Casimir element \( \Omega = \sum_i X_i X_i^* \in Z(U(g)) \) where \( \{X_i\}_i \) is a basis for \( g \) and \( \{X_i^*\}_i \) its dual basis with respect to \( B \). Put \( H_i = E_{i,i} - E_{m+i,m+3-i}, i = 1, 2 \), as the basis for \( \mathfrak{a}' \), then \( H_i^* = \frac{1}{2} H_i \) and note that \( E_{i,j}^* = E_{j,i} \) for \( i \neq j \), or \( Y_i^* = Y_j \). Observe that \( B|_{m \times m} \) is non-degenerate, and let \( \Omega_m \) be the corresponding Casimir element. So we get

\[
(A.2) \quad \Omega = \Omega_m + \frac{1}{2} \sum_{i=1}^2 H_i^2 + \sum_{\beta \in R^+} \sum_{\substack{\alpha \in \Delta^+ \setminus \{0\} \atop \alpha|_{\mathfrak{a}'} = \beta}} (Y_{\alpha} Y_{-\alpha} + Y_{-\alpha} Y_{\alpha}).
\]

Now we want to rewrite (A.2) following [3] §2. So let \( a \in A'^{\text{reg}} \), i.e. \( \beta(a) \neq \pm 1 \) for all \( \beta \in R^+ \). Define \( X^a = \text{Ad}(a^{-1})X \), \( X \in U(g) \), and let \( \alpha \in \Delta \) with \( \alpha|_{\mathfrak{a}'} = \beta \). Then, see [3 Lemma 2.2],

\[
(A.3) \quad X_{\alpha} = Y_{\alpha} + \theta' Y_{\alpha} = Y_{\alpha} + Y_{\theta' \alpha} \in \mathfrak{t}', \quad Y_{\alpha} = \frac{\beta(a)}{1 - \beta(a)^2} (X_{\alpha}^a - \beta(a)X_{\alpha}).
\]

In order to obtain the infinitesimal Cartan decomposition of the Casimir element \( \Gamma_{\alpha}^{-1}(\Omega) \), see [3 Theorem 2.1], we need to write \( \Omega \) as a sum of elements of the form \( X^a HY \) with \( X, Y \in U(\mathfrak{t}') \), \( H \in U(\mathfrak{a}') \). Note that the first two terms in (A.2) are of the right form. Using (A.3) we see that

\[
\sum_{\substack{\alpha \in \Delta^+ \setminus \{0\} \atop \alpha|_{\mathfrak{a}'} = \beta}} (Y_{\alpha} Y_{-\alpha} + Y_{-\alpha} Y_{\alpha}) = \frac{-1}{(\beta(a) - \beta(a)^{-1})^2} \sum_{\substack{\alpha \in \Delta^+ \setminus \{0\} \atop \alpha|_{\mathfrak{a}'} = \beta}} \left( X_{\alpha}^a X_{-\alpha}^a + X_{-\alpha}^a X_{\alpha} + X_{\alpha} X_{-\alpha} + X_{-\alpha} X_{\alpha} \right)
\]

\[
- \beta(a)^{-1} X_{\alpha}^a X_{-\alpha} - \beta(a) X_{-\alpha}^a X_{\alpha} - \beta(a) X_{\alpha} X_{-\alpha} - \beta(a)^{-1} X_{-\alpha} X_{\alpha}).
\]

Next observe \( \sum_{\alpha \in \Delta^+ \setminus \{0\} \atop \alpha|_{\mathfrak{a}'} = \beta} X_{\alpha}^a X_{\alpha} = \sum_{\alpha \in \Delta^+ \setminus \{0\} \atop \alpha|_{\mathfrak{a}'} = \beta} X_{\alpha}^a X_{-\alpha}^a = \sum_{\alpha \in \Delta^+ \setminus \{0\} \atop \alpha|_{\mathfrak{a}'} = \beta} X_{\alpha}^a X_{\alpha} \) using the involution \( -\theta' \) and \( X_{\alpha} = X_{\theta' \alpha} \). Similarly, we can take other terms together. Then only the last two terms are not yet of the right form.

**Lemma A.1.** For \( \alpha \in \Delta^+ \) with \( \alpha|_{\mathfrak{a}'} = \beta \) we have

\[
[X_{\alpha}^a, X_{-\alpha}] + [X_{\theta' \alpha}^a, X_{\theta' \alpha}] = (\beta(a)^{-1} - \beta(a))(H_{\alpha} + H_{-\theta' \alpha}) \in \mathfrak{a}'
\]

where \( H_{e_{i-j}} = E_{i,i} - E_{j,j} \).

**Proof.** Using (A.3) we rewrite the commutators in terms of the \( Y_{\alpha} \)'s. The mixed terms cancel and we are left with

\[
[X_{\alpha}^a, X_{-\alpha}] + [X_{\theta' \alpha}^a, X_{\theta' \alpha}] = (\beta(a)^{-1} - \beta(a))[Y_{\alpha}, Y_{-\alpha}] + (\beta(a) - \beta(a)^{-1})[Y_{\theta' \alpha}, Y_{-\theta' \alpha}]
\]

in terms of commutators of the \( Y_{\alpha} \)'s. Since the right hand side is in \( \mathfrak{h} \) and in the \(-1\)-eigenspace of \( \theta' \) we see that is contained in \( \mathfrak{a}' \). \( \square \)
Using this in the expression for the Casimir element leads to the infinitesimal Cartan decomposition for $\Omega$:

\begin{equation}
\Omega = \Omega_m + \frac{1}{2} \sum_{i=1}^{2} H_i^2 + \frac{1}{2} \sum_{\beta \in R^+} \beta(a) + \beta(a)^{-1} \dim g_\beta H_\beta + 2 \sum_{\beta \in R^+} \left( \frac{\beta(a) + \beta(a)^{-1}}{\beta(a) - \beta(a)^{-1}} \right)^2 \sum_{\alpha \in \Delta^+} X_\alpha X_{-\alpha} - 2 \sum_{\beta \in R^+} \left( \frac{1}{\beta(a) - \beta(a)^{-1}} \right)^2 \sum_{\alpha \in \Delta^+} X_\alpha X_{-\alpha} + X_\alpha X_{-\alpha},
\end{equation}

where $H_\beta = n_1 H_1 + n_2 H_2$ for $\beta = n_1 f_1 + n_2 f_2$.

A.3. The left invariant differential operator corresponding to the Casimir element. Let $F: G \rightarrow \text{End}(V^K_{\mu})$, where $V^K_{\mu}$ is the same representation space as $V^K$, and the action is given by $\pi^K_{\mu}(k') = \pi^K_{\mu}(a^* k' a)$, $k' \in K'$. We assume $F$ satisfies $F(k'_1 g k'_2) = \pi^K_{\mu}(k'_1) F(g) \pi^K_{\mu}(k'_2)$, so that $F$ is determined by its restriction to $A'$ and, since $M' = M$, we have $F: A' \rightarrow \text{End}_M(V^K_{\mu})$. Now the action of $\Omega$ as a left invariant operator satisfies $(\Omega \cdot F)|_{A'} = R(\Omega) \cdot (F|_{A'})$, where $R(\Omega)$ is the radial part of the Casimir element. In the decomposition (A.4), $\Omega_m$ acts as a scalar on each $M$-type by Schur’s Lemma. So the action of $\Omega_m$ on $F|_{A'}$ is by multiplying by a diagonal constant matrix. The second term acts as a second order differential operator, and the third term as a first order differential operator by observing that, after putting $f(t_1, t_2) = F(a'_k)$, we have $i H_p \cdot f = \frac{\partial f}{\partial t_p}$. The action of the differential operators do not involve the $M$-type. Then $X_\alpha X_{-\alpha} \cdot (F|_{A'}) = \pi^K_{\mu}(X_\alpha)(F|_{A'}) \pi^K_{\mu}(X_{-\alpha})$, and similarly $X_\alpha X_{-\alpha} \cdot (F|_{A'}) = \pi^K_{\mu}(X_\alpha X_{-\alpha})(F|_{A'})$ and $X_\alpha X_{-\alpha} \cdot (F|_{A'}) = (F|_{A'}) \pi^K_{\mu}(X_\alpha X_{-\alpha})$, see [3], where we use the same notation for the representation of the Lie algebra. In order to calculate these terms, we restrict to the $K$-representation of highest weight $\mu = a \omega_1 + b \omega_2$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$. We can then read off $X_\alpha$ using Figure [A.4] and next see which entry of $u^* X_\alpha u$ is in the upper left $2 \times 2$-block. Finally, we conjugate back and we find the following expression for the radial part of the Casimir operator for a function $F: A \rightarrow \text{End}_M(V^K_{\mu})$ for $\mu = a \omega_1 + b \omega_2$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$, where $G(t_1, t_2) = F(a_k)$:

\begin{equation}
(R^\mu(\Omega) G)(t_1, t_2) = R^\mu(\Omega_m) G(t_1, t_2) - \frac{1}{2} \sum_{p=1}^{2} \frac{\partial^2 G}{\partial t_p^2}(t_1, t_2) + (R^\mu_\alpha(\Omega) G)(t_1, t_2) + (R^\mu_{\alpha\alpha}(\Omega) G)(t_1, t_2) + (R^\mu_{\alpha\beta}(\Omega) G)(t_1, t_2),
\end{equation}

where the action is split according to the short, middle and long roots of $BC_2$. We obtain

\begin{equation}
(R^\mu_\alpha(\Omega) G)(t_1, t_2) = -(m - 2) \sum_{i=1}^{2} \frac{\cos t_i}{\sin t_i} \frac{\partial G}{\partial t_i}(t_1, t_2)
\end{equation}

since for the short roots $f_i$ the element $u^* X_\alpha u$ is not contained in the upper left $2 \times 2$-block, and so the last three terms in (A.4) do not contribute for the short roots. So the operator
\( R^\mu_\sigma(\Omega) \) is independent of the \( K \)-representation \( \pi^K_\mu \). For the middle roots \( f_1 \pm f_2 \) we get that the operator \( R^\mu_m(\Omega) \) is defined by \((R^\mu_m(\Omega)G)(t_1, t_2) = \)

\[- \frac{\cos(t_1 + t_2)}{\sin(t_1 + t_2)} \left( \frac{\partial G}{\partial t_1}(t_1, t_2) + \frac{\partial G}{\partial t_2}(t_1, t_2) \right) - \frac{\cos(t_1 - t_2)}{\sin(t_1 - t_2)} \left( \frac{\partial G}{\partial t_1}(t_1, t_2) - \frac{\partial G}{\partial t_2}(t_1, t_2) \right) \]

\[- \left( \frac{\cos(t_1 + t_2)}{\sin^2(t_1 + t_2)} + \frac{\cos(t_1 - t_2)}{\sin^2(t_1 - t_2)} \right) \left( \pi^K_\mu(E_1)G(t_1, t_2)\pi^K_\mu(E_1) + \pi^K_\mu(F_1)G(t_1, t_2)\pi^K_\mu(F_1) \right) + \frac{1}{2} \left( \frac{1}{\sin^2(t_1 + t_2)} + \frac{1}{\sin^2(t_1 - t_2)} \right) \left( \pi^K_\mu(E_1F_1 + F_1E_1)G(t_1, t_2) + G(t_1, t_2)\pi^K_\mu(E_1F_1 + F_1E_1) \right) \]

and for the long roots \( 2f_i \) we get

\[(R^\mu_m(\Omega)G)(t_1, t_2) = - \sum_{i=1}^{2} \frac{\cos(2t_i)}{\sin(2t_i)} \frac{\partial G}{\partial t_i}(t_1, t_2) - \sum_{i=1}^{2} \frac{\cos(2t_i)}{\sin^2(2t_i)} \pi^K_\mu(E_{i,i})G(t_1, t_2)\pi^K_\mu(E_{i,i}) + \frac{1}{2} \sum_{i=1}^{2} \frac{1}{\sin^2(2t_i)} \left( \pi^K_\mu(E_{i,i})^2G(t_1, t_2) + G(t_1, t_2)\pi^K_\mu(E_{i,i})^2 \right). \]

In order to describe the action of \( \Omega_m \) we only need the action on the 1-dimensional \( M \)-representation \( V^{\mu}_{\sigma_k} \) occurring in \( V^K_\mu \), see (1.6). Let \( M_1 = E_{1,1} + E_{m+2,m+2} - \frac{2}{m+2} \sum_{r=3}^{m} E_{r,r} \) and \( M_2 = E_{2,2} + E_{m+1,m+1} - \frac{2}{m+2} \sum_{r=3}^{m} E_{r,r} \), then the \( M_i \)’s are orthogonal to the \((m - 2) \times (m - 2)\)-block of \( M \), so that we only need to take the action of \( M_1 \) and \( M_2 \) into account. Note that \( M_1 \), respectively \( M_2 \), acts as \( a + b - k \), respectively \( b + k \) on \( V^{\mu}_{\sigma_k} \). Since \( M_1^* = \frac{m+2}{2} M_1 - \frac{1}{m+2} M_2 \) and \( M_2^* = \frac{m+2}{2} M_2 - \frac{1}{m+2} M_1 \), this gives

\[ R^\mu_m(\Omega_m)|_{V^{\mu}_{\sigma_k}} = \frac{1}{2(m+2)}(m(a + b - k)^2 - 4(a + b - k)(b + k) + m(b + k)^2), \]

for \( k \in \{0, \ldots, a\} \).

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