COMMUTATORS OF SINGULAR INTEGRAL OPERATOR
ON HERZ-TYPE HARDY SPACES WITH
VARIABLE EXponent

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Abstract. Let Ω ∈ L^s(S^{n-1}) for s > 1 be a homogeneous function of
degree zero and b be BMO functions or Lipschitz functions. In this paper,
we obtain some boundedness of the Calderón-Zygmund singular integral
operator T_Ω and its commutator [b, T_Ω] on Herz-type Hardy spaces with
variable exponent.

1. Introduction

The theory of function spaces with variable exponent has been extensively
studied by researchers since the work of Kováčik and Rákosník [7] appeared
in 1991, see [2, 4] and the references therein. In [14], the authors defined the
Herz-type Hardy spaces with variable exponent and gave their atomic charac-
terizations. Moreover, the authors studied the boundedness of some Calderón-
Zygmund integral operators on Herz-type Hardy spaces with variable exponent
in [12] and [13], respectively.

Suppose that S^{n-1} denote the unit sphere in \mathbb{R}^n (n \geq 2) equipped with
normalized Lebesgue measure. Let Ω ∈ L^s(S^{n-1}) for s > 1 be a homogeneous
function of degree zero and

\[ \int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \]

where x' = x/|x| for any x ≠ 0. The Calderón-Zygmund singular integral
operator T_Ω is defined by

\[ T_Ωf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{Ω(x-y)}{|x-y|^n} f(y)dy. \]
Let $b$ be a locally integrable function on $\mathbb{R}^n$. The commutator $[b, T_\Omega]$ generated by the Calderón-Zygmund singular integral operator $T_\Omega$ and $b$ is defined by

$$[b, T_\Omega]f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^n} [b(x) - b(y)]f(y)dy.$$ 

Motivated by [9], [12] and [13], we will prove the boundedness of the Calderón-Zygmund singular integral operator $T_\Omega$ and its commutator $[b, T_\Omega]$ on Herz-type Hardy spaces with variable exponent, where $\Omega \in L^s(S^{n-1})$ for $s > 1$. Our results improve and generalize essentially the results in [12] and [13].

Firstly we recall some standard notations and lemmas in variable $L^p$ spaces.

Given an open set $\Omega \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : \Omega \rightarrow [1, \infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions $f$ on $\Omega$ such that for some $\lambda > 0$,

$$\int_{\Omega} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty.$$ 

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$ 

These spaces are referred to as variable $L^p$ spaces, since they generalized the standard $L^p$ spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$.

The space $L^{p(\cdot)}_{\text{loc}}(\Omega)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(\Omega) := \{ f : f \in L^{p(\cdot)}(E) \text{ for all compact subsets } E \subset \Omega \}.$$ 

Define $\mathcal{P}(\Omega)$ to be the set of $p(\cdot) : \Omega \rightarrow [1, \infty)$ such that

$$p^- = \text{ess inf} \{ p(x) : x \in \Omega \} > 1, \quad p^+ = \text{ess sup} \{ p(x) : x \in \Omega \} < \infty.$$ 

Denote $p'(x) = p(x)/(p(x) - 1)$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot) \in \mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\Omega)$.

In variable $L^p$ spaces there are some important lemmas as follows.

**Lemma 1.1** ([1]). If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies

$$|p(x) - p(y)| \leq \frac{C}{-\log(|x - y|)}, \quad |x - y| \leq 1/2$$

and

$$|p(x) - p(y)| \leq \frac{C}{\log(|x| + e)}, \quad |y| \geq |x|,$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, that is the Hardy-Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. 


Lemma 1.2 ([7]). Let \( p(\cdot) \in \mathcal{P}(\Omega) \). If \( f \in L^{p(\cdot)}(\Omega) \) and \( g \in L^{p'(\cdot)}(\Omega) \), then \( fg \) is integrable on \( \Omega \) and

\[
\int_{\Omega} |f(x)g(x)| \, dx \leq r_p \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)},
\]

where

\[
r_p = 1 + 1/p^- - 1/p^+.
\]

This inequality is named the generalized H"older inequality with respect to the variable \( L^p \) spaces.

Lemma 1.3 ([5]). Let \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \). Then there exists a positive constant \( C \) such that for all balls \( B \) in \( \mathbb{R}^n \) and all measurable subsets \( S \subset B \),

\[
\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C \frac{|B|}{|S|} \quad \text{and} \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_1},
\]

\[
\frac{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq C \left( \frac{|S|}{|B|} \right)^{\delta_2},
\]

where \( \delta_1, \delta_2 \) are constants with \( 0 < \delta_1, \delta_2 < 1 \) and \( \chi_S, \chi_B \) are the characteristic functions of \( S, B \), respectively.

Throughout this paper \( \delta_2 \) is the same as in Lemma 1.2.

Lemma 1.4 ([5]). Suppose \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \). Then there exists a constant \( C > 0 \) such that for all balls \( B \) in \( \mathbb{R}^n \),

\[
\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.
\]

Next we recall the definition of the Herz-type spaces with variable exponent.

Let \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \} \) and \( A_k = B_k \setminus B_{k-1} \) for \( k \in \mathbb{Z} \). Denote \( \mathbb{Z}_+ \) and \( \mathbb{N} \) as the sets of all positive and non-negative integers, \( \chi_k = \chi_{A_k} \) for \( k \in \mathbb{Z} \), \( \bar{\chi}_k = \chi_k \) if \( k \in \mathbb{Z}_+ \) and \( \bar{\chi}_0 = \chi_{B_0} \).

Definition 1.1 ([5]). Let \( \alpha \in \mathbb{R}, 0 < p \leq \infty \) and \( q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \). The homogeneous Herz space with variable exponent \( K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \) is defined by

\[
K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = \{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} < \infty \},
\]

where

\[
\|f\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p}.
\]

The non-homogeneous Herz space with variable exponent \( K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \) is defined by

\[
K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = \{ f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} < \infty \},
\]
where
\[
\|f\|_{K^{\alpha, p}_{q(i)}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^{q(i)}}^p \right\}^{1/p}.
\]

In [14], the authors gave the definition of Herz-type Hardy space with variable exponent \(H^{\alpha, p}_{q(i)}(\mathbb{R}^n)\) and the atomic decomposition characterizations. \(\mathcal{S}(\mathbb{R}^n)\) denotes the space of Schwartz functions, and \(\mathcal{S}'(\mathbb{R}^n)\) denotes the dual space of \(\mathcal{S}(\mathbb{R}^n)\). Let \(G_N(f)(x)\) be the grand maximal function of \(f(x)\) defined by
\[
G_N(f)(x) = \sup_{\phi \in \mathcal{A}_N} \|\phi \ast f(x)\|,
\]
where \(\mathcal{A}_N = \{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|, |\beta| \leq N} |x^\alpha D^\beta \phi(x)| \leq 1 \} \) and \(N > n + 1, \phi \ast \) is the nontangential maximal operator defined by
\[
\phi \ast f(x)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|
\]
with \(\phi_t(x) = t^{-n} \phi(x/t)\).

**Definition 1.2** ([14]). Let \(\alpha \in \mathbb{R}, 0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)\) and \(N > n + 1\).

(i) The homogeneous Herz-type Hardy space with variable exponent \(H^{\alpha, p}_{q(i)}(\mathbb{R}^n)\) is defined by
\[
H^{\alpha, p}_{q(i)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in K^{\alpha, p}_{q(i)}(\mathbb{R}^n) \right\}
\]
and \(\|f\|_{H^{\alpha, p}_{q(i)}(\mathbb{R}^n)} = \|G_N(f)\|_{K^{\alpha, p}_{q(i)}(\mathbb{R}^n)}\).

(ii) The non-homogeneous Herz-type Hardy space with variable exponent \(H^{\alpha, p}_{q(i)}(\mathbb{R}^n)\) is defined by
\[
H^{\alpha, p}_{q(i)}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N(f)(x) \in K^{\alpha, p}_{q(i)}(\mathbb{R}^n) \right\}
\]
and \(\|f\|_{H^{\alpha, p}_{q(i)}(\mathbb{R}^n)} = \|G_N(f)\|_{K^{\alpha, p}_{q(i)}(\mathbb{R}^n)}\).

For \(x \in \mathbb{R}\) we denote by \([x]\) the largest integer less than or equal to \(x\).

**Definition 1.3** ([14]). Let \(n \delta_2 \leq \alpha < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n)\), and non-negative integer \(s \geq [\alpha - n \delta_2] \).

(i) A function \(a(x)\) on \(\mathbb{R}^n\) is said to be a central \((\alpha, q(\cdot))\)-atom, if it satisfies
\begin{enumerate}
\item \(\text{supp}\ a \subset B(0, r) = \{ x \in \mathbb{R}^n : |x| < r \}\).
\item \(\|a\|_{L^{q(i)}(\mathbb{R}^n)} \leq |B(0, r)|^{-\alpha/n}\).
\item \(\int_{\mathbb{R}^n} a(x)x^\beta dx = 0, |\beta| \leq s\).
\end{enumerate}

(ii) A function \(a(x)\) on \(\mathbb{R}^n\) is said to be a central \((\alpha, q(\cdot))\)-atom of restricted type, if it satisfies the conditions (2), (3) above and
\begin{enumerate}
\item \(\text{supp}\ a \subset B(0, r), r \geq 1\).
\end{enumerate}

If \(r = 2^k\) for some \(k \in \mathbb{Z}\) in Definition 1.3, then the corresponding central \((\alpha, q(\cdot))\)-atom is called a dyadic central \((\alpha, q(\cdot))\)-atom.
Lemma 1.5 ([14]). Let $0 < \alpha < \infty$, $0 < p < \infty$ and $q(\cdot) \in B(\mathbb{R}^n)$. Then $f \in H^\alpha_{q(\cdot)}(\mathbb{R}^n)$ (or $H^\alpha_{q(\cdot)}(\mathbb{R}^n)$) if and only if

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k \left( \text{or } \sum_{k=0}^{\infty} \lambda_k a_k \right), \text{ in the sense of } S^\alpha(\mathbb{R}^n),$$

where each $a_k$ is a central $(\alpha, q(\cdot))$-atom (or central $(\alpha, q(\cdot))$-atom of restricted type) with support contained in $B_k$ and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$ (or $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$). Moreover,

$$\|f\|_{H^\alpha_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left( \sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \left( \text{or } \|f\|_{H^\alpha_{q(\cdot)}(\mathbb{R}^n)} \approx \inf \left( \sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p} \right),$$

where the infimum is taken over all above decompositions of $f$.

2. Estimate of the Calderón-Zygmund singular integral operator

A nonnegative locally integrable function $\omega(x)$ on $\mathbb{R}^n$ is said to belong to $A_p(1 < p < \infty)$, if there is a constant $C > 0$ such that

$$\sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x)dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q \omega(x)^{1-p'}dx \right)^{p-1} \leq C < \infty,$$

where $p' = p/(p-1)$.

The weighted $(L^p, L^q)$ boundedness of $T_\Omega$ have been proved by Lu, Ding and Yan [8].

Lemma 2.1 ([8]). Suppose that $\Omega \in L^s(S^{n-1})(s > 1)$ satisfies (1.1). If $\omega \in A_{p'/s'}$, $s' \leq p < \infty$, then there is a constant $C$, independent of $f$, such that

$$\int_{\mathbb{R}^n} |T_\Omega(f)(x)|^p \omega(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx.$$

Lemma 2.2 ([3]). Given a family $\mathcal{F}$ and an open set $E \subset \mathbb{R}^n$, assume that for some $p_0$, $0 < p_0 < \infty$ and for every $\omega \in A_{\infty}$,

$$\int_{E} f(x)^{p_0} \omega(x)dx \leq C_0 \int_{E} g(x)^{p_0} \omega(x)dx, \quad (f, g) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}(E)$ such that $p(\cdot)$ satisfies (1.2) and (1.3) in Lemma 1.1. Then for all $(f, g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(E)$,

$$\|f\|_{L^{p(\cdot)}(E)} \leq C \|g\|_{L^{p(\cdot)}(E)}.$$

Since $A_{p'/s'} \subset A_{\infty}$, by Lemma 2.1 and Lemma 2.2 it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n))$-boundedness of the commutator $T_\Omega$. 

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Before stating our result, let us recall the definition of the $L^s$-Dini condition. We say that satisfies the $L^s$-Dini condition if $\Omega \in L^s(S^{n-1})$ with $s \geq 1$ is homogeneous of degree zero on $\mathbb{R}^n$, and
\[
\int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta < \infty,
\]
where $\omega_s(\delta)$ denotes the integral modulus of continuity of order $s$ of $\Omega$ defined by
\[
\omega_s(\delta) = \sup_{|\rho| < \delta} \left( \int_{S^{n-1}} |\Omega(\rho x') - \Omega(\rho x')|^s dx' \right)^{1/s}
\]
and $\rho$ is a rotation in $\mathbb{R}^n$ and $|\rho| = ||\rho - I||$.

Next, we will give the corresponding result about the commutator $T_\Omega$ on Herz-type Hardy spaces with variable exponent.

**Theorem 2.1.** Suppose that $0 < \beta \leq 1$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (1.2) and (1.3) in Lemma 1.1 with $\Omega \in L^s(S^{n-1})(s > q^+)$ and satisfies
\[
\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\beta}} d\delta < \infty.
\]
Let $0 < p_1 \leq p_2 < \infty$ and $n\delta_2 \leq \alpha < n\delta_2 + \beta$ (or $0 < \max(n\delta_2, \alpha_2) \leq \alpha_1 < n\delta_2 + \beta$). Then $T_\Omega$ is bounded from $H\hat{K}^{\alpha_1, p_1, q(\cdot)}(\mathbb{R}^n)$ (or $H\hat{K}^{\alpha_2, p_2, q(\cdot)}(\mathbb{R}^n)$) to $\hat{K}^{\alpha_1, p_1, q(\cdot)}(\mathbb{R}^n)$ (or $\hat{K}^{\alpha_2, p_2, q(\cdot)}(\mathbb{R}^n)$).

In the proof of Theorem 2.1, we also need the following lemmas.

**Lemma 2.3** ([2]). Given $E$ and $p(\cdot) \in \mathcal{P}(E)$, let $f : E \times E \to \mathbb{R}$ be a measurable function (with respect to product measure) such that for almost every $y \in E$, $f(\cdot, y) \in L^{p(\cdot)}(E)$. Then
\[
\left\| \int_E f(\cdot, y)dy \right\|_{L^{p(\cdot)}(E)} \leq C \int_E \|f(\cdot, y)\|_{L^{p(\cdot)}(E)} dy.
\]

**Lemma 2.4** ([10]). Define a variable exponent $\tilde{q}(\cdot)$ by $\frac{1}{\tilde{q}(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} (x \in \mathbb{R}^n)$. Then we have
\[
\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|fg\|_{L^{q(\cdot)}(\mathbb{R}^n)}
\]
for all measurable functions $f$ and $g$.

**Lemma 2.5** ([4]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (1.2) and (1.3) in Lemma 1.1. Then
\[
\|xQ\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} \frac{|Q|}{\tilde{q}(x)} & \text{if } |Q| \leq 2^n \text{ and } x \in Q, \\ \frac{|Q|}{p(\cdot)} & \text{if } |Q| \geq 1 \end{cases}
\]
for every cube (or ball) $Q \subset \mathbb{R}^n$, where $p(\infty) = \lim_{x \to \infty} p(x)$. 
Lemma 2.6 ([8]). Suppose that $\Omega$ satisfies the $L^s$-Dini condition ($1 \leq s < \infty$). Then for any $R > 0$ and $x \in \mathbb{R}^n$, when $|y| < R/2$, there is a constant $C > 0$ such that

$$\left( \int_{R < |x| < 2R} \frac{|\Omega(x - y) - \Omega(x)|^s}{|x - y|^n} \right)^{1/s} \leq CR \frac{|y|}{R} + \int_{|y|/2 < |y| < R} \frac{\omega_s(\delta)}{\delta} d\delta.$$

Proof of Theorem 2.1. We only prove the homogeneous case. In [15], the authors proved $K_{\alpha,1,p_2}^\omega(\mathbb{R}^n) \subset K_{\alpha,2,p_2}^\omega(\mathbb{R}^n)$ for $0 < \alpha_2 \leq \alpha_1$. So the non-homogeneous case can be proved in the same way. Let $f \in H K_{\alpha,1,p_1}^\omega(\mathbb{R}^n)$. By Lemma 1.5 we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{H K_{\alpha,1,p_1}^\omega(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of $f$), and $a_j$ is a dyadic central $(\alpha,q(\cdot))$-atom with the support $B_j$. Note that $p_1 \leq p_2$, we have

$$\|T_{\Omega}(f)\|_{K_{\alpha,1,p_2}^\omega(\mathbb{R}^n)} \leq \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \|T_{\Omega}(f)\chi_k\|_{L^p(\mathbb{R}^n)}^{p_1/p_2} \right\}^{p_1/p_2}$$

(2.1)

We first estimate $I_1$. For each $k \in \mathbb{Z}$, $j \leq k - 2$ and a.e. $x \in A_k$, using Lemma 2.3, the Minkowski inequality and the vanishing moments of $a_j$ we have

$$\|T_{\Omega}(a_j)\chi_k\|_{L^p(\mathbb{R}^n)} \leq \int_{B_j} \left\| \frac{\Omega(-\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right\|_{L^p(\mathbb{R}^n)} |a_j(y)| dy.$$

Noting $s > q^+$, we denote $\tilde{q}(\cdot) > 1$ and $\frac{1}{\tilde{q}(x)} = \frac{1}{q(x)} + \frac{1}{2}$. By Lemma 2.4 we have

$$\left\| \frac{\Omega(-\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \chi k(\cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq \left\| \frac{\Omega(\cdot)}{|\cdot|^n} \chi k(\cdot) \right\|_{L^\infty(\mathbb{R}^n)}$$

where

$$\left\| \frac{\Omega(-\cdot - y)}{|\cdot - y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \chi k(\cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq \left\| \frac{\Omega(\cdot)}{|\cdot|^n} \chi k(\cdot) \right\|_{L^\infty(\mathbb{R}^n)}$$
When $|B_k| \leq 2^n$ and $x_k \in B_k$, by Lemma 2.5 we have

$$\|\chi_{B_k}\|_{L^q(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{q} - \frac{1}{p}} \approx \|\chi_{B_k}\|_{L^q(\mathbb{R}^n)} |B_k|^{-\frac{1}{q}}.$$ 

When $|B_k| \geq 1$ we have

$$\|\chi_{B_k}\|_{L^q(\mathbb{R}^n)} \approx |B_k|^{\frac{1}{q} - \frac{1}{p}} \approx \|\chi_{B_k}\|_{L^q(\mathbb{R}^n)} |B_k|^{-\frac{1}{q}}.$$ 

So we obtain $\|\chi_{B_k}\|_{L^q(\mathbb{R}^n)} \approx \|\chi_{B_k}\|_{L^q(\mathbb{R}^n)} |B_k|^{-\frac{1}{q}}$.

Meanwhile, by Lemma 2.6 we have

$$\left| \left| \frac{\Omega(-y)}{|-y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right|_{L^q(\mathbb{R}^n)} \left| \left| \chi_k(\cdot) \right| \right|_{L^q(\mathbb{R}^n)} \approx 2^{(k-1)(\frac{n}{2} - n)} \left\{|y| \frac{\omega_k(\delta)}{\delta} \right\} \leq 2^{(k-1)(\frac{n}{2} - n)} \left\{2^{j-k+1} + 2^{j-k+1} \beta \int_0^1 \frac{\omega_k(\delta)}{\delta} d\delta \right\} \leq C2^{(k-1)(\frac{n}{2} - n)} q^{(j-k)\beta}.$$ 

So by Lemma 1.3, Lemma 1.4 and the generalized Hölder inequality we have

$$\|T_{\Omega}(a_j)\chi_k\|_{L^{q'}(\mathbb{R}^n)} = \int_{B_j} \left| \left| \frac{\Omega(-y)}{|-y|^n} - \frac{\Omega(\cdot)}{|\cdot|^n} \right| \right|_{L^q(\mathbb{R}^n)} |\chi_k(\cdot)|_{L^{q'}(\mathbb{R}^n)} |a_j(y)| dy \leq C2^{(k-1)(\frac{n}{2} - n)} q^{(j-k)\beta} \|\chi_{B_k}\|_{L^{q'}(\mathbb{R}^n)} |B_k|^{-\frac{1}{q}} \int_{B_j} |a_j(y)| dy \leq C2^{-kn+(j-k)\beta} \|\chi_{B_k}\|_{L^{q'}(\mathbb{R}^n)} \|a_j\|_{L^q(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q'}(\mathbb{R}^n)} \leq C2^{(j-k)\beta} \|a_j\|_{L^q(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q'}(\mathbb{R}^n)} \leq C2^{-(j+\beta+n\delta_2)}.$$ 

So we have

$$I_1 \leq C \sum_{k=-\infty}^{\infty} 2^{kn\|p\|} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| 2^{-j\alpha + (j-k)(\beta + n\delta_2)} \right)^{p_1} = C \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)(\beta + n\delta_2 - \alpha)} \right)^{p_1}.$$
When $1 < p_1 < \infty$, take $1/p_1 + 1/p_1' = 1$. Since $\beta + n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$I_1 \leq C \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| p_1 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right)$$

$$\times \left( \sum_{j=-\infty}^{k-2} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1'/2} \right)^{p_1/p_1'}$$

$$\leq C \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| p_1 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right)$$

$$= C \sum_{j=-\infty}^{\infty} |\lambda_j| p_1 \left( \sum_{k=j+2}^{\infty} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right)$$

$$\leq C \sum_{j=-\infty}^{\infty} |\lambda_j| p_1 .$$

When $0 < p_1 \leq 1$, we have

$$I_1 \leq C \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| p_1 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \right)$$

$$= C \sum_{j=-\infty}^{\infty} |\lambda_j| p_1 \left( \sum_{k=j+2}^{\infty} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \right)$$

$$\leq C \sum_{j=-\infty}^{\infty} |\lambda_j| p_1 .$$

Next we estimate $I_2$, by the $(L^{q_1}(\mathbb{R}^n), L^{q_1}(\mathbb{R}^n))$-boundedness of the commutator $T_\Omega$ we have

$$I_2 \leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left( \sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1}(\mathbb{R}^n)} \right)^{p_1}$$

$$\leq C \sum_{k=-\infty}^{\infty} \left( \sum_{j=k-1}^{\infty} |\lambda_j| 2^{(k-j)\alpha} \right)^{p_1} .$$

If $0 < p_1 \leq 1$, then we have

$$I_2 \leq C \sum_{j=-\infty}^{\infty} |\lambda_j| p_1 \left( \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \right) \leq C \sum_{j=-\infty}^{\infty} |\lambda_j| p_1 .$$
If \( 1 < p_1 < \infty \), by the Hölder inequality we have

\[
I_2 \leq C \sum_{k = -\infty}^{\infty} \left( \sum_{j=k-1}^{\infty} |\lambda_j|^{p_1 \cdot 2^{(k-j)\alpha p_1}/2} \right) \left( \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p'_1/2} \right)^{p_1/p'_1} \leq C \sum_{j = -\infty}^{\infty} |\lambda_j|^{p_1}.
\]

(2.5)

Thus, by (2.1)-(2.5) we complete the proof of Theorem 2.1. \( \square \)

3. BMO estimate for the commutator of Calderón-Zygmund singular integral operator

Let us first recall that the space \( \text{BMO}(\mathbb{R}^n) \) consists of all locally integrable functions \( f \) such that

\[
\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,
\]

where \( f_Q = \frac{1}{|Q|} \int_Q f(y) \, dy \), the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \) with sides parallel to the coordinate axes and \( |Q| \) denotes the Lebesgue measure of \( Q \).

**Lemma 3.1** ([6]). Let \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), \( k \) be a positive integer and \( B \) be a ball in \( \mathbb{R}^n \). Then we have that for all \( b \in \text{BMO}(\mathbb{R}^n) \) and all \( j, i \in \mathbb{Z} \) with \( j > i \),

\[
\frac{1}{C} \|b\|_*^k \leq \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^k \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|b\|_*^k,
\]

\[
\|(b - b_B)^k \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C (j - i)^k \|b\|_*^k \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)},
\]

where \( B_i = \{ x \in \mathbb{R}^n : |x| \leq 2^i \} \) and \( B_j = \{ x \in \mathbb{R}^n : |x| \leq 2^j \} \).

Let \( b \in \text{BMO}(\mathbb{R}^n) \). The weighted \( (L^p, L^q) \) boundedness of \( [b, T_\Omega] \) have been proved by Lu, Ding and Yan [8].

**Lemma 3.2** ([8]). Suppose that \( \Omega \in L^s(S^{n-1}) \) satisfies (1.1). If \( \omega \in A_{p'/\sigma}', \ 0 < \sigma < \infty \), then there is a constant \( C \), independent of \( f \), such that

\[
\int_{\mathbb{R}^n} |[b, T_\Omega](f)(x)| p_\omega(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)| p_\omega(x) \, dx.
\]

Since \( A_{p'/\sigma}' \subset A_{\infty} \), by Lemma 3.2 and Lemma 2.2 it is easy to get the \( (L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n)) \)-boundedness of the commutator \( [b, T_\Omega] \).

Next, we will give the corresponding result about the commutator \( [b, T_\Omega] \) on Herz-type Hardy spaces with variable exponent.
Theorem 3.1. Suppose that $b \in \text{BMO}(\mathbb{R}^n)$, $0 < \beta \leq 1$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (1.2) and (1.3) in Lemma 1.1 with $\Omega \in L^s(S^{n-1})(s > q^+)$ and satisfies

$$
\int_0^1 \frac{\omega_s(\delta)}{\delta^{1+\beta}} d\delta < \infty.
$$

Let $0 < p_1 \leq p_2 < \infty$ and $\eta_1, \eta_2 \leq \alpha < n\eta_2 + \beta$ (or $0 < \max(n\eta_2, \alpha_2) \leq \alpha_1 < n\eta_2 + \beta$). Then $[b, T_\Omega]$ is bounded from $\dot{H}^{\alpha,p_1}_{\mathcal{Q}(\cdot)}(\mathbb{R}^n)$ (or $\dot{H}^{\alpha_1,p_1}_{\mathcal{Q}(\cdot)}(\mathbb{R}^n)$) to $\dot{K}^{\alpha,p_2}_{\mathcal{Q}(\cdot)}(\mathbb{R}^n)$ (or $\dot{K}^{\alpha_2,p_2}_{\mathcal{Q}(\cdot)}(\mathbb{R}^n)$).

Proof. Similar to Theorem 2.1, we only prove the homogeneous case. Let $f \in \dot{H}^{\alpha,p_1}_{\mathcal{Q}(\cdot)}(\mathbb{R}^n)$ and $b \in \text{BMO}(\mathbb{R}^n)$. By Lemma 1.5 we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\|f\|_{\dot{H}^{\alpha,p_1}_{\mathcal{Q}(\cdot)}(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of $f$), and $a_j$ is a dyadic central $(\alpha, q(\cdot))$-atom with the support $B_j$. Note that $p_1 \leq p_2$, we have

$$
\| [b, T_\Omega](f) \|_{\dot{K}^{\alpha,p_2}_{\mathcal{Q}(\cdot)}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \| [b, T_\Omega](f) \chi_k \|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_2} \right\}^{p_1/p_2}
\leq \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \| [b, T_\Omega](f) \chi_k \|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p_2}
\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| \| [b, T_\Omega](a_j) \chi_k \|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p_1}
+ C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left( \sum_{j=k-1}^{\infty} |\lambda_j| \| [b, T_\Omega](a_j) \chi_k \|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p_1}
=: J_1 + J_2.

(3.1)

We first estimate $J_1$. For each $k \in \mathbb{Z}$, $j \leq k-2$ and a.e. $x \in A_k$, using Lemma 2.3, the Minkowski inequality and the vanishing moments of $a_j$ we have

$$
\| [b, T_\Omega](a_j) \chi_k \|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq \int_{B_j} \left\| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} \frac{\Omega(\cdot)}{|\cdot|^n} (b(\cdot) - b(y)) \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy
\leq \int_{B_j} \left\| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} \frac{\Omega(\cdot)}{|\cdot|^n} (b(\cdot) - b_{B_j}) \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |a_j(y)| dy
+ \int_{B_j} \left\| \frac{\Omega(\cdot - y)}{|\cdot - y|^n} \frac{\Omega(\cdot)}{|\cdot|^n} \chi_k(\cdot) \right\|_{L^{q(\cdot)}(\mathbb{R}^n)} |b_{B_j} - b(y)||a_j(y)| dy
=: J_{11} + J_{12}.
$$
For $J_{11}$, noting $s > q^+$, we denote $\bar{q}(\cdot) > 1$ and $\frac{1}{\bar{q}(x)} = \frac{1}{q(x)} + \frac{1}{x}$. By Lemma 3.1 and Lemma 2.4 we have

$$\left\| \frac{\Omega(\cdot - y)}{|\cdot - y|^a} - \frac{\Omega(\cdot)}{|\cdot|^a} \right\|_{L^q(\mathbb{R}^n)} b(\cdot - b_{B_s} |\chi_k(\cdot)|_{L^q(\mathbb{R}^n)} \leq C \left\| \frac{\Omega(\cdot - y)}{|\cdot - y|^a} - \frac{\Omega(\cdot)}{|\cdot|^a} \right\|_{L^q(\mathbb{R}^n)} \left( k - j \right) \|b\|_s \|\chi_{B_s} \|_{L^q(\mathbb{R}^n)}.$$

When $|B_k| \leq 2^n$ and $x_k \in B_k$, by Lemma 2.5 we have

$$\|\chi_{B_k} \|_{L^q(\mathbb{R}^n)} \approx | B_k | \frac{n}{n-k} \approx \|\chi_{B_k} \|_{L^q(\mathbb{R}^n)} |B_k|^{-\frac{1}{q}}.$$

When $|B_k| \geq 1$ we have

$$\|\chi_{B_k} \|_{L^q(\mathbb{R}^n)} \approx | B_k | \frac{n}{n-k} \approx \|\chi_{B_k} \|_{L^q(\mathbb{R}^n)} |B_k|^{-\frac{1}{q}}.$$

So we obtain $\|\chi_{B_k} \|_{L^q(\mathbb{R}^n)} \approx \|\chi_{B_k} \|_{L^q(\mathbb{R}^n)} |B_k|^{-\frac{1}{q}}$.

Meanwhile, by Lemma 2.6 we have

$$\left\| \frac{\Omega(\cdot - y)}{|\cdot - y|^a} - \frac{\Omega(\cdot)}{|\cdot|^a} \right\|_{L^q(\mathbb{R}^n)} \chi_k(\cdot) \leq 2^{(k-1)(\frac{n}{2} - n)} \left\{ \frac{|y|}{2^k} + \int_{|y|/2^k}^{\frac{1}{\omega}(\delta)} \frac{\omega_s(\delta)}{\delta} d\delta \right\} \leq 2^{(k-1)(\frac{n}{2} - n)} \left( 2^{j-k+1} + 2^{(j-k+1)q} \int_0^1 \frac{\omega_s(\delta)}{\delta} d\delta \right) \leq C 2^{(k-1)(\frac{n}{2} - n)} 2^{(j-k)\beta}.$$
\[ C_2^{(k-1)(\frac{k}{2}-n)} \| \chi_{B_k} \|_{L^{k}(\mathbb{R}^n)} \leq C_2^{-kn+(j-k)\beta} \| \chi_{B_k} \|_{L^{k}(\mathbb{R}^n)}. \]

So by Lemma 3.1 and the generalized Hölder inequality we have

\begin{equation}
J_{12} = \int_{B_j} \left\| \frac{\Omega(-y)}{1-|y|^n} - \frac{\Omega(\cdot)}{\|n\|_{L^1(\mathbb{R}^n)}} \chi_k(\cdot) \right\|_{L^k(\mathbb{R}^n)} |b_{B_j} - b(y)| |a_j(y)| \, dy
\end{equation}

\begin{align*}
&\leq C_2^{-kn+(j-k)\beta} \| \chi_{B_k} \|_{L^{k}(\mathbb{R}^n)} \int_{B_j} |b_{B_j} - b(y)| |a_j(y)| \, dy \\
&\leq C_2^{-kn+(j-k)\beta} \| \chi_{B_k} \|_{L^{k}(\mathbb{R}^n)} \| (b_{B_j} - b) \chi_{B_j} \|_{L^{k}(\mathbb{R}^n)} \| a_j \|_{L^1(\mathbb{R}^n)} \\
&\leq C \| b \| \| 2^{-kn+(j-k)\beta} \| \chi_{B_k} \|_{L^{k}(\mathbb{R}^n)} \| a_j \|_{L^{k}(\mathbb{R}^n)} \| B_j \|_{L^{k}(\mathbb{R}^n)}. \\
&\leq C(k - j)2^{-j\alpha+(j-k)(\beta+n\delta_2)} \| b \|_*. \\
\end{align*}

So we have

\begin{align*}
J_1 &\leq C \| b \| \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| (k - j)2^{-j\alpha+(j-k)(\beta+n\delta_2)} \right)^{p_1} \\
&= C \| b \| \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| (k - j)2^{(j-k)(\beta+n\delta_2-\alpha)} \right)^{p_1}. \\
\end{align*}

When \( 1 < p_1 < \infty \), take \( 1/p_1 + 1/p_1' = 1 \). Since \( \beta + n\delta_2 - \alpha > 0 \), by the Hölder inequality we have

\begin{align*}
J_1 &\leq C \| b \| \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right)^{p_1/p_1'} \\
&\times \left( \sum_{j=-\infty}^{k-2} (k - j)^{p_1/2} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right)^{p_1/p_1'} \\
&\leq C \| b \| \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j| 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right)^{p_1/p_1'} \\
&= C \| b \| \sum_{j=-\infty}^{\infty} |\lambda_j| p_1 \left( \sum_{k=j+2}^{\infty} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1/2} \right). \\
\end{align*}
When \(0 < p_1 \leq 1\), we have
\[
J_1 \leq C\|b\|_{\dot{p}_1}^{p_1} \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} (k-j)^{p_1} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \right)
\]
\[
= C\|b\|_{\dot{p}_1}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left( \sum_{k=j+2}^{\infty} (k-j)^{p_1} 2^{(j-k)(\beta+n\delta_2-\alpha)p_1} \right)
\]
\[
\leq C\|b\|_{\dot{p}_1}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\]

Next we estimate \(J_2\), by the \((L^{q_1}(\mathbb{R}^n), L^{q_1}(\mathbb{R}^n))\)-boundedness of the commutator \([b, T]\) we have
\[
J_2 \leq C\|b\|_{\dot{p}_1}^{p_1} \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left( \sum_{j=k-1}^{\infty} |\lambda_j|\|a_j\|_{L^{q_1}(\mathbb{R}^n)} \right)^{p_1}
\]
\[
\leq C\|b\|_{\dot{p}_1}^{p_1} \sum_{j=-\infty}^{\infty} \left( \sum_{k=1}^{j+1} 2^{(k-j)\alpha p_1} \right)^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\]

If \(0 < p_1 \leq 1\), then we have
\[
J_2 \leq C\|b\|_{\dot{p}_1}^{p_1} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{j+1} |\lambda_j|^{p_1} 2^{(k-j)\alpha p_1} \leq C\|b\|_{\dot{p}_1}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\]

If \(1 < p_1 < \infty\), by the Hölder inequality we have
\[
J_2 \leq C\|b\|_{\dot{p}_1}^{p_1} \sum_{k=-\infty}^{\infty} \left( \sum_{j=k-1}^{\infty} |\lambda_j|^{p_1} 2^{(k-j)\alpha p_1/2} \right)^{p_1/p_1'} \left( \sum_{j=k-1}^{\infty} 2^{(j-k)\alpha p_1/2} \right)^{p_1'/p_1'}
\]
\[
\leq C\|b\|_{\dot{p}_1}^{p_1} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\]

Thus, by (3.1), (3.4)-(3.7) we complete the proof of Theorem 3.1. \(\square\)

4. Lipschitz estimate for the commutator of Calderón-Zygmund singular integral operator

For \(0 < \gamma \leq 1\), the Lipschitz space \(\text{Lip}_\gamma(\mathbb{R}^n)\) is defined as
\[
\text{Lip}_\gamma(\mathbb{R}^n) = \left\{ f : \|f\|_{\text{Lip}_\gamma} = \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^{\gamma}} < \infty \right\}.
\]
Let $b \in \text{Lip}_\gamma(\mathbb{R}^n)$. It is easy to know that $\| [b, T] \| \leq C \| b \|_{\text{Lip}_\gamma} \| T_{\Omega, \gamma} \|$, where

$$T_{\Omega, \gamma} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^{n-\gamma}} f(y) \, dy.$$ 

In [11], the authors proved that $T_{\Omega, \gamma}$ is bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$ for $1/q_1(x) - 1/q_2(x) = \gamma/n$ and $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying conditions (1.2) and (1.3) in Lemma 1.1 with $q_1^+ < n/\gamma$. So we can get the following theorem.

**Theorem 4.1.** Suppose that $b \in \text{Lip}_\gamma(\mathbb{R}^n)$ with $0 < \gamma \leq 1$. If $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (1.2) and (1.3) in Lemma 1.1 with $q_1^+ < n/\gamma$, $1/q_1(x) - 1/q_2(x) = \gamma/n$, $\Omega \in L^s(S^{n-1})(s > q_2^+)$ with $1 \leq s' < q_1^-$. Then $[b, T]$ is bounded from $L^{p_1}(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$.

Next, we will give the Lipschitz estimate about the commutator $[b, T]$ on Herz-type Hardy spaces with variable exponent.

**Theorem 4.2.** Suppose that $b \in \text{Lip}_\gamma(\mathbb{R}^n)$ with $0 < \gamma \leq 1$. If $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (1.2) and (1.3) in Lemma 1.1 with $q_1^+ < n/\gamma$, $1/q_1(x) - 1/q_2(x) = \gamma/n$, $\Omega \in L^s(S^{n-1})(s > q_2^+)$ with $1 \leq s' < q_1^-$ and satisfies

$$\int_0^1 \frac{\omega_\gamma(\delta)}{\delta^{n+\gamma}} \, d\delta < \infty.$$ 

Let $0 < p_1 \leq p_2 < \infty$ and $n\delta_2 \leq \alpha < n\delta_2 + \gamma$ (or $0 < \max(n\delta_2, \alpha_2) \leq \alpha_1 < n\delta_2 + \gamma$). Then $[b, T]$ maps $HK_0^\alpha(\mathbb{R}^n)$ continuously into $HK_0^\alpha(\mathbb{R}^n)$.

**Proof.** Similar to Theorem 2.1, it suffices to prove homogeneous case. Let $f \in HK_0^\alpha(\mathbb{R}^n)$ and $b \in \text{Lip}_\gamma(\mathbb{R}^n)$. By Lemma 1.5 we get $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $\| f \|_{HK_0^\alpha(\mathbb{R}^n)} \approx \inf(\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$ (the infimum is taken over above decompositions of $f$), and $a_j$ is a dyadic central $(\alpha, q_1(\cdot))$-atom with the support $B_j$. We have

$$\| [b, T] f \|_{HK_0^{\alpha, p_2}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \| [b, T] f \|_{L^{p_2}(\mathbb{R}^n)} \right\}^{p_1/p_2} \leq \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \| [b, T] f \|^{p_1}_{L^{p_2}(\mathbb{R}^n)}$$

$$\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left( \sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1}\| [b, T] f \|^{p_1}_{L^{p_2}(\mathbb{R}^n)} \right)^{p_1} + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left( \sum_{j=k}^{\infty} |\lambda_j|^{p_1}\| [b, T] f \|^{p_1}_{L^{p_2}(\mathbb{R}^n)} \right)^{p_1} =: U_1 + U_2.$$
We first estimate $U_1$. For each $k \in \mathbb{Z}$, $j \leq k - 2$ and a.e. $x \in A_k$, using Lemma 2.3, the Minkowski inequality and the vanishing moments of $a_j$ we have

$$
\| [b, T_0] (a_j)_\chi \|_{L^{q_2}(\mathbb{R}^n)} 
\leq \int_{B_j} \left\| \frac{\Omega(-y)}{|-y|^n} \frac{\Omega(k)}{|k|^n} (b(\cdot) - b(0)) \chi_k(\cdot) \right\|_{L^{q_2}(\mathbb{R}^n)} |a_j(y)| \, dy 
\leq \int_{B_j} \left\| \frac{\Omega(-y)}{|-y|^n} \frac{\Omega(k)}{|k|^n} |b(\cdot) - b(0)| \chi_k(\cdot) \right\|_{L^{q_2}(\mathbb{R}^n)} |a_j(y)| \, dy 
+ \int_{B_j} \left\| \frac{\Omega(-y)}{|-y|^n} \frac{\Omega(k)}{|k|^n} \chi_k(\cdot) \right\|_{L^{q_2}(\mathbb{R}^n)} |b(0) - b(y)| \|a_j(y)\| \, dy 
= U_{11} + U_{12}.
$$

For $U_{11}$, noting $s > q_2^+$, we denote $\tilde{q}_2(\cdot) > 1$ and $\frac{1}{\tilde{q}_2(y)} = \frac{1}{q_2(y)} + \frac{1}{s}$. By Lemma 2.4 we have

$$
\left\| \frac{\Omega(-y)}{|-y|^n} \frac{\Omega(k)}{|k|^n} |b(\cdot) - b(0)| \chi_k(\cdot) \right\|_{L^{q_2}(\mathbb{R}^n)} 
\leq \left\| \frac{\Omega(-y)}{|-y|^n} \frac{\Omega(k)}{|k|^n} \chi_k(\cdot) \right\|_{L^s(\mathbb{R}^n)} \left\| |b(\cdot) - b(0)| \chi_k(\cdot) \right\|_{L^{q_2}(\mathbb{R}^n)} 
\leq C \|b\|_{\text{Lip}, 2k^\gamma} \left\| \frac{\Omega(-y)}{|-y|^n} \frac{\Omega(k)}{|k|^n} \right\|_{L^s(\mathbb{R}^n)} \|\chi B_k\|_{L^{q_2}(\mathbb{R}^n)}.
$$

When $|B_k| \leq 2^n$ and $x_k \in B_k$, by Lemma 2.5 we have

$$
\|\chi B_k\|_{L^{q_2}(\mathbb{R}^n)} \approx |B_k|^{\frac{\tilde{q}_2^-}{2q_2^-}} \approx \|\chi B_k\|_{L^{q_2}(\mathbb{R}^n)} |B_k|^{-\frac{s}{q_2^n} - \frac{1}{s}}.
$$

When $|B_k| \geq 1$ we have

$$
\|\chi B_k\|_{L^{q_2}(\mathbb{R}^n)} \approx |B_k|^{\frac{\tilde{q}_2^-}{2q_2^-}} \approx \|\chi B_k\|_{L^{q_2}(\mathbb{R}^n)} |B_k|^{-\frac{s}{q_2^n} - \frac{1}{s}}.
$$

So we obtain

$$
\|\chi B_k\|_{L^{q_2}(\mathbb{R}^n)} \approx \|\chi B_k\|_{L^{q_2}(\mathbb{R}^n)} |B_k|^{-\frac{s}{q_2^n} - \frac{1}{s}}.
$$

Meanwhile, by Lemma 2.6 we have

$$
\left\| \frac{\Omega(-y)}{|-y|^n} \frac{\Omega(k)}{|k|^n} \chi_k(\cdot) \right\|_{L^s(\mathbb{R}^n)} 
\leq 2^{(k-1)(\frac{\tilde{q}_2^-}{2q_2^-} - 1)} \left\{ \frac{|y|}{2^k} + \int_{|y|/2^k}^{1} \frac{\omega_k(\delta)}{\delta} \, d\delta \right\} 
\leq 2^{(k-1)(\frac{\tilde{q}_2^-}{2q_2^-} - 1)} \left( 2^{-k+1} + 2^{(j-k+1)\gamma} \int_0^1 \frac{\omega_k(\delta)}{\delta} \, d\delta \right) 
\leq C 2^{(k-1)(\frac{\tilde{q}_2^-}{2q_2^-} - 1)} 2^{(j-k)\gamma}.
$$
So by the generalized Hölder inequality we have

$$U_{11} = \int_{B_1} \left\| \frac{\Omega(\cdot - y)}{\cdot - y^n} - \frac{\Omega(\cdot)}{\cdot - y^n} \right\|_{L^{2q}(\mathbb{R}^n)} \left| b(\cdot) - b(0) \right| |\chi_k(\cdot)| \left| a_j(y) \right| dy$$

$$\leq C \|b\|_{Lip_p} 2^{k\gamma} 2^{(k-1)\left(\frac{n}{2} - n\right)} 2^{(j-k)\gamma} \|\chi_{B_k}\|_{L^{n_1}(\mathbb{R}^n)} |B_k|^{-\frac{1}{2} - \frac{n}{p}} \int_{B_1} |a_j(y)| dy$$

$$\leq C \|b\|_{Lip_p} 2^{-kn + (j-k)\gamma} \|\chi_{B_k}\|_{L^{n_1}(\mathbb{R}^n)} \|a_j\|_{L^{n_1}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{n_1}(\mathbb{R}^n)}.$$ 

For $U_{12}$, similar to the method of $U_{11}$ we have

$$U_{12} = \int_{B_1} \left\| \frac{\Omega(\cdot - y)}{\cdot - y^n} - \frac{\Omega(\cdot)}{\cdot - y^n} \right\|_{L^{2q}(\mathbb{R}^n)} \left| b(0) - b(y) \right| |\chi_k(\cdot)| \left| a_j(y) \right| dy$$

$$\leq C 2^{-kn + (j-k)\gamma - k\gamma} \|\chi_{B_k}\|_{L^{n_1}(\mathbb{R}^n)} \int_{B_1} \left| b(0) - b(y) \right| |a_j(y)| dy$$

$$\leq C \|b\|_{Lip_p} 2^{-kn + 2(j-k)\gamma} \|\chi_{B_k}\|_{L^{n_1}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{n_1}(\mathbb{R}^n)} \|a_j\|_{L^{n_1}(\mathbb{R}^n)}$$

$$\leq C \|b\|_{Lip_p} 2^{-kn + (j-k)\gamma} \|\chi_{B_k}\|_{L^{n_1}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{n_1}(\mathbb{R}^n)} \|a_j\|_{L^{n_1}(\mathbb{R}^n)}.$$ 

By (4.2), (4.3), Lemma 1.3 and Lemma 1.4 we have

$$\| [h, T_{\Omega_j}] (a_j) \chi_k \|_{L^{2q}(\mathbb{R}^n)}$$

$$\leq C \|b\|_{Lip_p} 2^{-kn + (j-k)\gamma} \|\chi_{B_k}\|_{L^{n_1}(\mathbb{R}^n)} \|a_j\|_{L^{n_1}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{n_1}(\mathbb{R}^n)}$$

$$\leq C \|b\|_{Lip_p} 2^{(j-k)\gamma} \|a_j\|_{L^{n_1}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{n_1}(\mathbb{R}^n)}.$$ 

$$\leq C 2^{-j\alpha + (j-k)(\gamma + n\delta_2)} \|b\|_{Lip_p}.$$ 

So we have

$$U_1 \leq C \|b\|_{Lip_p} \sum_{k=-\infty}^{\infty} 2^{k\alpha + p_1} \left( \sum_{j=-\infty}^{p_1} \left| \lambda_j \right| 2^{-j\alpha + (j-k)(\gamma + n\delta_2)} \right)^{p_1}.$$
When $1 < p_1 < \infty$, take $1/p_1 + 1/p_1' = 1$. Since $\gamma + n\delta_2 - \alpha > 0$, by the Hölder inequality we have

$$U_1 \leq C\|b\|_{\text{Lip}_p}^p \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\gamma + n\delta_2 - \alpha)p_1/2} \right)^{p_1/p_1'}$$

(4.4)

$$\leq C\|b\|_{\text{Lip}_p}^p \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} 2^{(j-k)(\gamma + n\delta_2 - \alpha)p_1/2} \right)^{p_1/p_1'}$$

$$\leq C\|b\|_{\text{Lip}_p}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left( \sum_{k=j+2}^{\infty} 2^{(j-k)(\gamma + n\delta_2 - \alpha)p_1/2} \right)$$

$$\leq C\|b\|_{\text{Lip}_p}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.$$

When $0 < p_1 \leq 1$, we have

$$U_1 \leq C\|b\|_{\text{Lip}_p}^p \sum_{k=-\infty}^{\infty} \left( \sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{(j-k)(\gamma + n\delta_2 - \alpha)p_1} \right)$$

(4.5)

$$\leq C\|b\|_{\text{Lip}_p}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left( \sum_{k=j+2}^{\infty} 2^{(j-k)(\gamma + n\delta_2 - \alpha)p_1} \right)$$

$$\leq C\|b\|_{\text{Lip}_p}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.$$

Next we estimate $U_2$, by the $(L^{q_1}((\mathbb{R}^n)), L^{q_2}((\mathbb{R}^n)))$-boundedness of the commutator $[b, T_\alpha]$ we have

$$U_2 \leq C\|b\|_{\text{Lip}_p}^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left( \sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^{q_1}(\mathbb{R}^n)} \right)^{p_1}$$

$$\leq C\|b\|_{\text{Lip}_p}^p \sum_{k=-\infty}^{\infty} \left( \sum_{j=k-1}^{\infty} |\lambda_j| 2^{(j-k)\alpha} \right)^{p_1}.$$
If $0 < p_1 \leq 1$, then we have
\begin{equation}
U_2 \leq C\|b\|_{L^{p_1}_{\text{Lip}}}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \leq C\|b\|_{L^{p_1}_{\text{Lip}}}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{equation}

If $1 < p_1 < \infty$, by the Hölder inequality we have
\begin{equation}
U_2 \leq C\|b\|_{L^{p_1}_{\text{Lip}}}^p \sum_{k=-\infty}^{\infty} \left( \sum_{j=k-1}^{\infty} |\lambda_j|^{p_1} 2^{(k-j)\alpha p_1/2} \right)^{p_1/p'_1} \left( \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p'_1/2} \right)^{p_1/p'_1} \leq C\|b\|_{L^{p_1}_{\text{Lip}}}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{equation}

Thus, by (4.1), (4.4)-(4.7) we complete the proof of Theorem 4.1. □

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