OPERADS AS DOUBLE FUNCTORS

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Abstract.
It is shown how double categories provide a direct abstract approach to coloured operads; namely, product-preserving normal lax functors $M : (PbC)^{op} \to \text{Cat}$ can be seen as generalized operads, the standard ones arising when $C = \text{Set}_f$. In this context, generalized symmetric monoidal categories are considered, in particular those arising from indexed categories with sums or products.

1. Introduction

The role of double categories as a framework for abstract or formal category theory has been promoted by several authors (notably by Bob Paré and Mike Shulman) and seems by now widely accepted. Indeed, it is well known that many categorical notions and properties can be treated in any “virtual equipment”, playing the role of the double category $\text{Cat}$ of functors and profunctors.

Furthermore, “all” generalized multicategories can be seen as instances of $T$–monoids in some virtual double category $D$, where $T$ is a monad on $D$ (see in particular [Leinster, 2003] and [Cruttwell & Shulman, 2010]). While this approach has the advantage of bringing under the same umbrella many sorts of multicategories, at the same time it may result too general when one is interested in some specific cases. Indeed, the relevant aspects are in a sense hidden in the monad $T$, and the monad itself often looks rather involved or unmanageable. This is the case in particular for one of the most commonly arising sort of multicategories, namely the symmetric ones, also known as (coloured) operads.

The thesis of the present paper is that the theory of double categories is actually more directly involved in summarizing some aspects of (symmetric) multicategories than the above mentioned perspective could suggest. In fact, classical coloured operads (in their non-skeletal form) are simply product-preserving (double lax) functors

$$M : (Pb\text{Set}_f)^{op} \to \text{Set} \quad (1)$$

to the double category of mappings and spans. Thus $(Pb\text{Set}_f)^{op}$, the horizontal dual of the double category of pullback squares in finite sets, is, in a sense, the “double theory” for operads. This can be seen as a generalization of the fact that product-preserving functors

$$M : (Pb\text{Set}_f)^{op} \to \text{SqSet} \quad (2)$$

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are commutative monoids, where $\mathbf{Set}$ can be replaced by any finite-product category $\mathcal{S}$.

By the universal property of the module construction (see [Cruttwell & Shulman, 2010]), which applied to $\mathbf{Set}$ gives $\mathbf{Cat}$, one can equivalently consider, instead of (1), normal functors

$$M : (\mathbb{Pb Set}_f)^\text{op} \to \mathbf{Cat}$$

While the first perspective is more suitable to treat the fibered aspects, this one has technical advantages and is preferred in this work.

Of course, the present approach points toward a notion of generalized operad rather different from the monad-based ones; namely we are led to consider normal product-preserving functors

$$M : (\mathbb{Pb C})^\text{op} \to \mathbf{Cat}$$

(or to $\mathbf{Cat}^{\mathcal{S}}$) as the basic concept. We call such a functor a “DF operad” (on $\mathcal{C}$), where the prefix “DF” stands for “double functor” and will accompany also the various concepts generalized in this context.

Thus, the horizontal part $M_h$ of a DF operad $M$ consists of an indexed category $M : \mathcal{C}^\text{op} \to \mathbf{Cat}$ (giving the $I$-“families” $MI$ which are the domains of the “operations”) such that

$$M(\Sigma I) \cong \Pi(MI)$$

while the vertical part $M_v$ gives the operations themselves and their compositions.

Among DF operads we isolate the “DF (symmetric) monoidal categories” (in which the profunctors $M_v f$ are representable and the associated Beck-Chevalley condition holds) and the “DF fibrations” (in which $M_v f$ is co-represented by $M_h f$).

Typically, DF monoidal categories arise from an indexed categories $M : \mathcal{C}^\text{op} \to \mathbf{Cat}$ satisfying (5) and having indexed sums or products; that is, such that substitution (or reindexing) functors $f^* = M_h f$ have left or right adjoint

$$f ! \dashv f^* \quad ; \quad f^* \dashv f_*$$

satisfying the Beck-Chevalley law. In the case of indexed sums, the corresponding DF monoidal categories are also DF fibrations, since profunctors represented by $f_!$ are also co-represented by $f^*$.

In the classical case $\mathcal{C} = \mathbf{Set}_f$, $M_h$ is the standard indexing $\mathcal{A}^I$ for a category $\mathcal{A}$. If $\mathcal{A}$ has finite sums or products, the above mentioned DF monoidal categories are of course the usual cocartesian or cartesian monoidal categories. On the other hand, if the category $\mathcal{A}$ has all small (co)products, we can use $\mathbf{Set}$ in place of $\mathbf{Set}_f$ as the indexing category $\mathcal{C}$ and we still get a (genuinely generalized) DF operad. In this perspective, the role of the Beck-Chevalley condition is to assure that by taking isomorphism classes of a DF monoidal category one gets a (genuinely generalized) DF monoid

$$M : (\mathbb{Pb C})^\text{op} \to \mathbf{Sq Set}$$

(recall (2) and see section 3.1). This makes it precise the idea that also infinite categorical sums or products give a well defined algebraic structure.
The present work originates from [Pisani, 2022], whose emphasis was on the double fibered form of operads (see also [Lambert, 2021]).

2. Some remarks on double categories

We assume the basic notions about double categories and lax functors (see for instance [Paré, 2011], [Paré, 2018] and [Shulman, 2008]). In the present section we fix notations and highlight the facts that will be useful in the sequel; some of them (especially those in section 2.7) seem to be new, or at least not widely known.

2.1. Notations and conventions.

All double categories will be written with the first letter in a blackboard style. Since a double category $A$ is a category in $\text{Cat}$, it has an underlying graph in $\text{Cat}$ with source and target functors:

\[ s : A_1 \rightarrow A_0 \quad ; \quad t : A_1 \rightarrow A_0 \]  

(7)

We adopt the convention that the arrows of $A_0$ are called “horizontal arrows”, while the objects of $A_1$ are called “vertical arrows” or “proarrows” and the arrows of $A_1$ are the “cells” or “squares”. Thus a typical cell $\alpha : f \rightarrow g$ in $A_1$, with $s\alpha = k$ and $t\alpha = l$, can be depicted as

\[ \begin{array}{ccc}
I & \xrightarrow{k} & L \\
\downarrow & \alpha & \downarrow \\
J & \xrightarrow{l} & K
\end{array} \]

(8)

By a double category $A$ we intend a pseudo (or weak) one: the vertical or “external” composition $\circ : A_1 \times_{A_0} A_1 \rightarrow A_1$ is defined up to isomorphisms, as in bicategories. By a functor $F : A \rightarrow B$ we intend a lax one (see [Paré, 2011] or [Shulman, 2008]). A functor is normal if identities are strictly preserved. We denote by $F_h$ and $F_v$ the horizontal and the vertical components of $F$:

\[ \begin{array}{ccc}
I & \xrightarrow{k} & L \\
\downarrow & \alpha & \downarrow \\
J & \xrightarrow{l} & K
\end{array} \quad \xrightarrow{F} \quad \begin{array}{ccc}
FI & \xrightarrow{F \alpha} & FL \\
\downarrow & \quad \downarrow & \downarrow \\
FJ & \xrightarrow{F \alpha} & FK
\end{array} \]

By the opposite double category $A^{\text{op}}$ we mean the horizontal opposite, obtained by taking the opposite of $A_0$ and of $A_1$ (but not reversing $s$ and $t$):

\[ s^{\text{op}} : A_1^{\text{op}} \rightarrow A_0^{\text{op}} \quad ; \quad t^{\text{op}} : A_1^{\text{op}} \rightarrow A_0^{\text{op}} \]
Thus, a cell (8) in \( \mathbb{A} \) corresponds to the following cell in \( \mathbb{A}^{\text{op}} \):

![Diagram](image)

Following Paré, \( \text{Set} = \text{Span Set} \) is the double category of mappings and spans and \( \text{Cat} \) is the double category of functors and profunctors. More generally, for a category \( \mathcal{S} \) with pullbacks, \( \mathbb{C} \text{at} \mathcal{S} \) is the double category of internal categories, functors and profunctors, obtained by applying the module construction to the double category \( \text{Span} \mathcal{S} \). For a category \( \mathcal{C} \), \( \text{Sq} \mathcal{C} \) is the double category of commutative squares in \( \mathcal{C} \), and \( \text{Pb} \mathcal{C} \) is the double category of pullback squares in \( \mathcal{C} \).

2.2. Double categories with products. Following [Paré, 2018] we say that a double category \( \mathbb{A} \) has (finite) products if both \( \mathbb{A}_0 \) and \( \mathbb{A}_1 \) have (finite) products and the source and target functors (7) preserve them.

So, for any pair of proarrows \( f : I \to J \) and \( g : K \to L \) we have a diagram

![Diagram](image)

encompassing a product diagram in \( \mathbb{A}_1 \) along with its source and target diagrams in \( \mathbb{A}_0 \) which also are product diagrams.

2.3. Remark. There are also stronger notions of product in a double category; this one is best suited to our needs.

Similarly, if \( \mathbb{A} \) has sums we have diagrams

![Diagram](image)

Clearly, \( \mathbb{A} \) has sums if and only if \( \mathbb{A}^{\text{op}} \) has products.

2.4. Proposition. If \( \mathcal{C} \) has (finite) products, then \( \mathbb{A} = \text{Sq} \mathcal{C} \) also as (finite) products; the same holds true for sums.
2.5. Remark. Actually, the proposition is true for all those double categories $A$ constructed from a bipointed category $\sigma, \tau: 1 \to D$ with $A_0 = C$, $A_1 = CD$ and with $s, t: A_1 \to A_0$ induced by $\sigma$ and $\tau$. If $C$ has (finite) products, then $A$ has (finite) products, since products in $CD$ are computed pointwise. If $D$ is the “free arrow” $\bullet \longrightarrow \bullet$ we get $\text{Sq} C$, while if $D$ is the “free span” $\bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$, we get $\text{Span} C$. Note that while the vertical structure for $\text{Sq} C$ is induced by the pushout (in $\text{Cat}$) of the free arrow with itself, the same is not true for $\text{Span} C$. It would be interesting to find general procedures to construct the vertical structure for $A_1 = CD$ for some sorts of bipointed categories $D$.

2.6. Proposition. If the horizontal component $F_h: A_0 \to C$ of a functor $F: A \to \text{Sq} C$ preserves products, then $F$ itself preserves products.

Proof. The effect of $F$ on a product diagram is

\[
\begin{array}{ccc}
I & \xleftarrow{p_1} & I \times K & \xleftarrow{p_2} & K \\
\downarrow{f} & \downarrow{} & \downarrow{} & \downarrow{} & \downarrow{g} \\
J & \xleftarrow{q_1} & J \times L & \xleftarrow{q_2} & L \\
\end{array}
\]

\[
\begin{array}{ccc}
FI & \xleftarrow{F_h p_1} & F(I \times K) & \xleftarrow{F_h p_2} & FK \\
\downarrow{F_v f} & \downarrow{F_{\pi_1}} & \downarrow{F_{\pi_2}} & \downarrow{F_{v g}} \\
FJ & \xleftarrow{F_{h q_1}} & F(J \times L) & \xleftarrow{F_{h q_2}} & FL \\
\end{array}
\]

By the hypothesis, $F(I \times K) = FI \times FK$ and $F(J \times L) = FJ \times FL$, that is, the horizontal diagrams on the right are products in $C$. Since diagrams in $\text{Sq} C$ are (commutative) diagrams in $C$, the vertical central arrow is forced to be $F_v f \times F_v g$.

2.7. The double category $\mathbb{P}bC$ and Mackey functors. Recall that an extensive category is a category $C$ with finite sums that interact nicely with pullbacks (see
[Carboni et al.]): given a commutative diagram in $\mathcal{C}$

\[
\begin{array}{ccc}
I & \longrightarrow & R \\
\downarrow & & \downarrow \\
J & \overset{i_1}{\longrightarrow} & J + L \overset{i_2}{\longrightarrow} L
\end{array}
\]

where the bottom row is a sum, the top row is also a sum if and only if the two squares are pullbacks.

2.8. Proposition. If $\mathcal{C}$ is extensive, then $\mathbb{Pb}\mathcal{C}$ has sums and the inclusion $\mathbb{Pb}\mathcal{C} \to \mathbb{Sq}\mathcal{C}$ preserves them.

Proof. By the extensivity of $\mathcal{C}$, the following is a sum diagram both in $\mathbb{Sq}\mathcal{C}$ and in $\mathbb{Pb}\mathcal{C}$:

\[
\begin{array}{ccc}
I & \overset{i_1}{\longrightarrow} & I + K \overset{i_2}{\longrightarrow} K \\
\downarrow & & \downarrow \\
J & \overset{f}{\longrightarrow} & J + L \overset{j_2}{\longrightarrow} L \\
\downarrow & & \downarrow \\
J & \overset{j_1}{\longrightarrow} & J + L \overset{j_2}{\longrightarrow} L
\end{array}
\]

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a Mackey functors $\mathcal{C} \to \mathcal{D}$ in the sense of [Lidner, 1976] consists of two functors which coincide on objects

\[
F_! : \mathcal{C} \to \mathcal{D} \quad ; \quad F^* : \mathcal{C}^{\text{op}} \to \mathcal{D}
\]

with $F^*$ preserving finite products and such that for any pullback in $\mathcal{C}$ the corresponding right hand square below commutes:

\[
\begin{array}{ccc}
I & \overset{k}{\longrightarrow} & L \\
\downarrow & & \downarrow \\
J & \overset{l}{\longrightarrow} & K
\end{array} \quad \begin{array}{ccc}
FI & \overset{F^*k}{\leftarrow} & FL \\
\downarrow & & \downarrow \\
FJ & \overset{F^*l}{\leftarrow} & FK
\end{array}
\]

The following proposition is then an easy consequence of the definitions.

2.9. Proposition. For an extensive category $\mathcal{C}$, the Mackey functors $\mathcal{C} \to \mathcal{D}$ correspond to the finite-product-preserving functors $(\mathbb{Pb}\mathcal{C})^{\text{op}} \to \mathbb{Sq}\mathcal{D}$.

Proof. The correspondence is obviously given by $F_! \leftrightarrow F_!$, and $F^* \leftrightarrow F_h$. By proposition 2.6, preservation of products $\mathcal{C}^{\text{op}} \to \mathcal{D}$ is sufficient to give preservation of products $(\mathbb{Pb}\mathcal{C})^{\text{op}} \to \mathbb{Sq}\mathcal{D}$. □
2.10. **The double category $\mathcal{C}at$.** Recall that $\mathcal{C}at$ is the double category of functors and profunctors. Following [Joyal, 2022] and [Paré, 2011], a profunctor $\Phi : \mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}^{\text{op}} \times \mathcal{D} \to \text{Set}$, rather than $\mathcal{D}^{\text{op}} \times \mathcal{C} \to \text{Set}$ as many authors prefer. One reason for this choice is that profunctors $\mathcal{C} \to \mathcal{D}$ can be seen as categories over the free arrow category $0 \to 1$ (the “barrels” of [Joyal, 2022]), and in this way the domain and the codomain of the profunctor are projected respectively on the domain and the codomain of the free arrow.

A cell in $\mathcal{C}at$

$$
\begin{array}{ccc}
A & \xrightarrow{F} & C \\
\Phi \downarrow & & \downarrow \Psi \\
B & \xleftarrow{G} & D
\end{array}
$$

is a natural transformation $\alpha : \Phi \to \Psi(F-, G-)$, that is a family of mappings

$$
\alpha_{X,Y} : \Phi(X, Y) \to \Psi(FX, GY) \quad ; \quad X \in \mathcal{C}, \ Y \in \mathcal{D}
$$

satisfying the usual conditions. Equivalently, it is a morphism of barrels, that is of categories over the free arrow.

A profunctor $\Phi : \mathcal{C} \to \mathcal{D}$ is “representable” if it isomorphic to $\Phi_F = \mathcal{D}(F-, -)$ for a functor $F : \mathcal{C} \to \mathcal{D}$. As a barrel, $\Phi$ is representable if it is an opfibration over the free arrow. The assignment $F \mapsto \Phi_F$ defines a functor $\text{Sq} \mathcal{C}at \to \mathcal{C}at$. For any square in $\mathcal{C}at$, we get a cell $\alpha$ in $\mathcal{C}at$

$$
\begin{array}{ccc}
A & \xrightarrow{H} & C \\
B & \xleftarrow{L} & D
\end{array}
\begin{array}{ccc}
A & \xrightarrow{H} & C \\
B & \xleftarrow{L} & D
\end{array}
$$

where the $\alpha_{X,Y} : \Phi_F(X, Y) \to \Phi_G(HX, LY)$, that is

$$
\alpha_{X,Y} : B(FX, Y) \to \mathcal{D}(GHX, LY) = \mathcal{D}(LFX, LY)
$$

are given by the action of $L$ on the arrows of $\mathcal{B}$.

2.11. **Proposition.** $\mathcal{C}at$ has products, and the “inclusion” $\text{Sq} \mathcal{C}at \to \mathcal{C}at$ preserves products.

**Proof.** A product diagram in $\mathcal{C}at$

$$
\begin{array}{ccc}
A & \xleftarrow{p_1} & A \times \mathcal{C} & \xrightarrow{p_2} & \mathcal{C} \\
\Phi & \downarrow \pi_1 & \Phi \times \Psi & \downarrow \pi_2 & \Psi \\
B & \xleftarrow{q_1} & B \times \mathcal{D} & \xrightarrow{q_2} & \mathcal{D}
\end{array}
$$
where $\Phi \times \Psi : \mathcal{A} \times \mathcal{C} \to \mathcal{B} \times \mathcal{D}$ is defined by

$$(\Phi \times \Psi)((X, X'), (Y, Y')) = \Phi(X, Y) \times \Psi(X', Y')$$

That this is indeed a product in $\mathcal{C}$at is easily checked directly. Alternatively note that, for $\mathcal{A} = \mathbf{Cat}$, $A_1$ is the category of barrels and the product of $\Phi$ and $\Psi$ over the free arrow corresponds to the above one. Similarly, the fact that $\mathcal{S}q\mathbf{Cat} \to \mathcal{C}$at preserves products is easily checked directly. Alternatively note that the product of two representable profunctors (as barrels which are opfibrations) is again representable, by the product functor. ■

3. DF operads

In this section we study our version of generalized operad, namely “double functor operads” or briefly DF operads. By an operad we intend a coloured operad, that is a symmetric multicategory. In order to introduce the idea, and also because some results will be used in the sequel, we begin by presenting the relative notion of generalized commutative monoid.

3.1. DF monoids. Given an extensive category $\mathcal{C}$ with pullbacks and a category $\mathcal{S}$ with products, a $DF$ monoid (on $\mathcal{C}$ in $\mathcal{S}$) is a product-preserving functor

$$F : (\mathbb{Pb}\mathcal{C})^{op} \to \mathcal{S}q\mathcal{S}$$

By Proposition 2.9, DF monoids correspond to Mackey functors $\mathcal{C} \to \mathcal{S}$ in the sense of [Lidner, 1976], except for the fact that we require preservation of all products, not only of the finite ones. Thus, they consist of functors

$$F_1 : \mathcal{C} \to \mathcal{S} ; \quad F^* : \mathcal{C}^{op} \to \mathcal{S}$$

coinciding on objects, with $F^*$ preserving products and such that for any pullback in $\mathcal{C}$ the corresponding right hand square below commutes:

$$
\begin{array}{c}
I & \xrightarrow{k} & L \\
\downarrow^f & & \downarrow^g \\
J & \xrightarrow{l} & K
\end{array}
\quad
\begin{array}{c}
FI & \xleftarrow{F^*k} & FL \\
\downarrow^{F \downarrow^f} & & \downarrow^{F \downarrow^g} \\
FJ & \xleftarrow{F^*l} & FK
\end{array}
\tag{10}
$$

In order to justify our terminology, we have:

3.2. Proposition. $DF$ monoids on $\mathbf{Set}_f$ in $\mathcal{S}$ coincide with commutative monoids in $\mathcal{S}$. 
PROOF. We give two proofs. The first one appeals to the well-known result in [Lidner, 1976]: if \( C \) is extensive with pullbacks, Mackey functors \( C \to S \) correspond to finite-product-preserving functors \( \text{Span}(C) \to S \). But \( \text{Span}(\text{Set}_f) \) is essentially the category of matrices valued in natural numbers, which is well known to be the Lawvere theory for commutative monoids.

The second proof is more direct and maybe more instructive. We assume for simplicity \( S = \text{Set} \), but the proof can be easily adapted to the more general case. First, since \( F^* : \text{Set}^{op}_f \to \text{Set} \) preserves products and \( \text{Set}^{op}_f \) is the free finitely complete category on the terminal category, \( F^* \) is isomorphic to the family functor on an object \( M \in \text{Set}_f \); thus we can assume \( F^* I = M^I \), and we rewrite the right hand square in (10) as:

\[
\begin{array}{c}
M^I & \xrightarrow{k^*} & M^L \\
\downarrow{f_1} & & \downarrow{g_1} \\
M^J & \xleftarrow{l^*} & M^K
\end{array}
\]

where we put \( f^* = F^* f \) and \( f_1 = F_1 f \). Now, given a family \( x_i, i \in I \) in \( M^I \), we define its “product” \( m_I(x_i) \) as the element \( f_1(x_i) \) of \( M = M^I \), where \( f : I \to 1 \) in \( \text{Set}_f \). The product of a family is stable with respect to bijective reindexing \( k^* \), since the pullback on the left induces a square on the right:

\[
\begin{array}{c}
I & \xleftarrow{k} & L \\
\downarrow{f} & & \downarrow{g} \\
1 & \xleftarrow{id} & 1
\end{array}
\quad
\begin{array}{c}
M^I & \xleftarrow{k^*} & M^L \\
\downarrow{m_1} & & \downarrow{m_L} \\
M & \xleftarrow{id} & M
\end{array}
\]

Thus, for \( a, b \in M \), the element \( m(a, b) \in M \) is not ambiguous and \( m(a, b) = m(b, a) \); the same holds for, say, \( m(a, b, c) \). To show that the operation is associative, consider any three element set, say \( I = \{i, j, k\} \) and any two element set, say \( J = \{r, s\} \). Then, since \( f : I \to 1 \) factors through \( J \) as \( hg \) where \( g : i, j \to r \) and \( g : k \to s \), we have the diagram on the left, where the rows are sums, which corresponds to the diagram on the right:

\[
\begin{array}{c}
S & \xrightarrow{pb} & I \\
\downarrow{r} & & \downarrow{g} & \downarrow{pb} \\
1 & \xrightarrow{h} & J & \xleftarrow{s} & 1 \\
\downarrow{1} & & \downarrow{1} & & \downarrow{1}
\end{array}
\quad
\begin{array}{c}
M^S & \xleftarrow{m} & M^I \\
\downarrow{g_1} & & \downarrow{m_1} \\
M & \xleftarrow{g} & M^J & \xleftarrow{m} & M
\end{array}
\]
Now, since a DF monoid \((\mathbb{Pb} \mathcal{C})^{\text{op}} \to \mathbf{Sq} \mathcal{S}\) preserves products by definition, \(g_t = m \times \text{id}\).

(In general, the product preservation axiom assures that the value of \((-)\); on an arbitrary mapping depends only from its value on the mappings with a terminal codomain.) Since by functoriality \(m g_t = m_I\), we have

\[
m(m(a, b), c) = m((m \times \text{id})(a, b, c)) = (m g_t)(a, b, c) = m_I(a, b, c)
\]

Symmetrically, one has \(m(a, m(b, c)) = m_I(a, b, c)\) as well, which proves associativity. Along the same lines, one can verify the identity laws.

In the other direction, given a commutative monoid \(M\) and a mapping \(f : I \to J\) in \(\mathbf{Set}_f\), it is obvious how to exploit the multiplication of \(M\) in order to obtain a mapping \(f_t : M^I \to M^J\) and it is easy to verify that one so gets a DF monoid.

3.3. DF operads. Given an extensive category \(\mathcal{C}\) with pullbacks and a category \(\mathcal{S}\) with pullbacks, a **DF operad** (on \(\mathcal{C}\) in \(\mathcal{S}\)) is a product-preserving normal functor

\[
M : (\mathbb{Pb} \mathcal{C})^{\text{op}} \to \mathbf{Cat} \mathcal{S}
\]

In the sequel, we just consider the case \(\mathcal{S} = \mathbf{Set}\), but the general case can be easily treated. Thus, a DF operad consists of a functor \(M_h : \mathcal{C}^{\text{op}} \to \mathbf{Cat}\) preserving products and a normal lax functor \(M_v : \mathcal{C} \to \mathbf{Prof}\) coinciding with \(M_h\) on objects and such that for any pullback in \(\mathcal{C}\) there is a corresponding cell in \(\mathbf{Cat}\):

\[
\begin{array}{ccc}
I & \xrightarrow{k} & L \\
\downarrow{f} & \text{pb} & \downarrow{g} \\
J & \xrightarrow{l} & K
\end{array}
\quad
\begin{array}{ccc}
MI & \xleftarrow{M_h k} & ML \\
\downarrow{M_v f} & \alpha & \downarrow{M_v g} \\
MJ & \xleftarrow{M_h l} & MK
\end{array}
\]

The assignment should satisfy the axioms for lax double functors and should also preserve products in the sense of section 2.2. In order to justify our terminology, we are going to show that DF operads on \(\mathbf{Set}_f\) are essentially classical coloured operads (that is symmetric multicategories) in their non-skeletal form. The need for a non-skeletal form of operads was already stressed in [Leinster, 2003], where a version of them (named “fat symmetric multicategories”) is given. We hope to convince the reader that the present version is very natural.

3.4. Proposition. **DF operads on \(\mathbf{Set}_f\) give the non-skeletal version of operads.**

**Proof.** (Sketch.) Let \(\mathbf{N}\) be the usual skeleton of \(\mathbf{Set}_f\), with objects \(n = \{1, 2, \ldots, n\}\), and suppose that a bijection is \(b_I : I \to n\) is given, for every \(I \in \mathbf{Set}_f\). In one direction, given a classical operad \(M\) we get a DF operad \(\overline{M}\) in the following way: \(\overline{MI}\) is \(M_0^I\) where \(M_0\) is the underlying category of \(M\), and for a mapping \(l : J \to K\) in \(\mathbf{Set}_f\), \(\overline{M_h l} : \overline{MK} \to \overline{MJ}\) is the standard reindexing functor. If \(f : I \to J\) and \(J\) is terminal in \(\mathbf{Set}_f\), we get the
profunctor $\Phi_f = \overline{M}_v f : M_0' \to M$ by posing $\Phi_f(X; A) = M(A_1, \ldots, A_n; A)$ (where we use $b_I$ to reindex the $I$-family $X = A_i, i \in I$). For a general $f : I \to J$ in $\text{Set}_f$, we get $\Phi_f = \overline{M}_v f : M_0' \to M_0'$ by posing $\Phi_f(X; Y) = \Pi_j \Phi_{f_j}(X_j; B_j)$, where $Y = B_j, j \in J$, and $f_j$ and $X_j$ are obtained by restricting $f$ and $X$ to $f^{-1}j \subseteq I$. Laxity cells $\Phi_g \Phi_f \to \Phi_{gf}$ are of course given by composition in $M$ and, for a pullback square in $\text{Set}_f$, the corresponding cell $\alpha$ in (11) is obtained with the aid of the action of bijective mappings $n \to n$ on the arrows of $M$, mediated by the $b_I$.

In the other direction, given a DF operad $M$ on $\text{Set}_f$, we define a classical operad $\overline{M}$ which has $\overline{M} = M_1$ as objects set and with $\overline{M}(A_1, \ldots, A_n; A) = (M_v f)(X, A)$, where $X = A_i, i \in n$ and $f : n \to 1$. Composition in $\overline{M}$ is given by the laxity cells and actions of bijective mappings $k : n \to n$ is given by the cell $\alpha$ below.

![Diagram](image)

The fact that DF operads $M : (\text{Pb Set}_f)^\text{op} \to \text{Cat}$ preserve products, assures that value of $M_v$ on arbitrary mappings $f : I \to J$ is determined by their value on mappings with a terminal codomain. Indeed, $f = \Sigma_j f_j$ in $\text{Pb Set}_f$, so that $f = \Pi_j f_j$ in $(\text{Pb Set}_f)^\text{op}$ and $M_v f = \Pi_j M_v f_j$ in $\text{Cat}$. It follows that the DF operads $M$ and $(\overline{M})$ are essentially the same. This complete our sketch of the proof.

3.5. Remark. Note that among the laws implicit in the lax functoriality of $M$, there is the one regarding composition of a family of arrows with an arrow acted over by a bijective mapping $k : n \to n$: composing along the vertical left side of the right hand diagram below gives the same result as composing along the vertical right side and then acting on the composition by the bijective mapping $t$. Such a $t$, obtained by composing the pullbacks on the left, is exactly the mapping which requires a rather involved explicit description in the usual definitions of symmetric multicategories.
3.6. Remark. Commutative monoids (in $\text{Set}$) can be seen as discrete symmetric multicategories $M$, that are those such that the functor $M \to 1$ is a discrete opfibration of multicategories. Explicitly, this means that for any $A_1, \ldots, A_n \in M$ there is exactly one arrow $A_1, \ldots, A_n \to A$ out of them, giving their product $A \in M$. Translating this particular case in the present framework, we note that the categories $MI$ are then discrete and the profunctors $\Phi_f$ are given, for $f : I \to J$, $X = (A_i)_{i \in I}$ and $Y = (B_j)_{j \in J}$, by $\Phi_f(X, Y) = 1$ if $B_j$ is the product (in the above sense) of $(A_i)_{i \in f^{-1}j}$, for any $j \in J$, and $\Phi_f(X, Y) = 0$ otherwise. In other terms, the $M_v f = \Phi_f$ are in fact mappings assigning to any $X \in MI = M^I$ the unique $Y$ for which $\Phi_f(X, Y) = 1$. Furthermore, for this particular sort of categories (namely the discrete ones) and profunctors (which are represented by mappings), the cells in $\text{Cat}$ are cells in $\text{SqSet}$.

Thus, as a particular case of proposition 3.4, we find again the characterization of commutative monoids given in section 3.1.

3.7. Remark. As one may expect, the natural notion of morphism of lax functors $F, G : A \to B$, which is a double natural transformation $F \to G$ (see [Paré, 2011] or [Shulman, 2008]), gives the right notion of morphism of DF operads $M, N : (\mathbb{P}bC)^{\text{op}} \to \text{Cat}$, so that we have the category $\text{Op}_C$. A sharper form of proposition 3.4 should then yield an equivalence between the category of classical operads and $\text{Op}_{\text{Set}_f}$, as in the analogous result in [Leinster, 2003] on fat symmetric multicategories.

3.8. Remark. In the case $C = \text{Set}_f$, exponentiable DF operads $M : (\mathbb{P}bC)^{\text{op}} \to \text{Cat}$ are those such that $M$ is a pseudo functor, that is laxity cells are isos. Indeed, these are the promonoidal symmetric multicategories, which coincide with the exponentiable ones (see [Pisani, 2014]). It seems likely that this characterization of exponentiability holds for any base $C$.

3.9. DF monoidal categories. A DF operad $M : (\mathbb{P}bC)^{\text{op}} \to \text{Cat}$ is a DF monoidal category if the following conditions are satisfied:

1. the profunctors $\Phi_f = M_v f : MI \to MJ$ are representable:
   
   for any $f : I \to J$ in $C$, there is a functor $f : MI \to MJ$ such that
   
   $\Phi_f(X, Y) \cong MJ(f, X, Y)$

2. the Beck-Chevalley condition holds:

   for any pullback square in $C$, as the left hand one below, the square on the right commutes up to isomorphisms:

\[
\begin{array}{ccc}
I & \xrightarrow{k} & L \\
\downarrow f & & \downarrow g \\
J & \xrightarrow{l} & K
\end{array}
\quad
\begin{array}{ccc}
MI & \xrightarrow{k^*} & ML \\
\downarrow f & & \downarrow g \\
MJ & \xrightarrow{l^*} & MK
\end{array}
\]
Of course, for $\mathcal{C} = \textbf{Set}_f$, we get the usual symmetric monoidal categories, given in the universal form of representable symmetric multicategories (see [Hermida, 2000], [Leinster, 2003] and [Pisani, 2014]).

In particular, $f_!$ can be given by sums or products for the horizontal indexed category $I \mapsto MI; f \mapsto f^* = M_hf$ as mentioned in the introduction, yielding cartesian or cocommutative DF monoidal categories.

3.10. **Proposition.** By taking isomorphism classes of a DF monoidal category $M$, one gets a DF monoid $\mid M\mid : (\mathcal{PbC})^{op} \to \textbf{SqSet}$. Namely, $\mid M\mid I = \mid MI\mid$, $\mid M_hf\mid = \mid f^*\mid$ and $\mid M_vf\mid = \mid f_!\mid$

**Proof.** The Beck-Chevalley condition (12) ensures that the right hand square

\[
\begin{array}{ccc}
I & \xrightarrow{k} & L \\
\downarrow{f} & & \downarrow{g} \\
J & \xrightarrow{t} & K
\end{array}
\quad
\begin{array}{ccc}
\mid MI\mid & \xrightarrow{[k]^*} & \mid ML\mid \\
\downarrow{[f_!]} & & \downarrow{[g_!]} \\
\mid MJ\mid & \xrightarrow{[t]^*} & \mid MK\mid
\end{array}
\]

commutes in $\textbf{Set}$.

3.11. **Remark.** By proposition 3.4, DF monoids in $\textbf{Cat}$

$M : (\mathcal{PbSet}_f)^{op} \to \textbf{SqCat}$

correspond to commutative monoids in $\textbf{Cat}$, that is strict symmetric monoidal categories. It would be nice if general symmetric monoidal categories could be captured by functors

$M' : (\mathcal{PbSet}_f)^{op} \to \textbf{Sq}^\ast\textbf{Cat}$

where $\textbf{Sq}^\ast\textbf{Cat}$ is some kind of weak or pseudo version of $\textbf{SqCat}$. For instance, one could check whether something like the subcategory $\mathcal{Q}^\ast\textbf{Cat}$ of the of the double category $\mathcal{Q}\textbf{Cat}$ of quintets in $\textbf{Cat}$, formed by those cells which are natural isomorphisms, serves the purpose. In this case, one could simply say that DF monoidal categories are those DF operads $M : (\mathcal{PbC})^{op} \to \textbf{Cat}$ which factors through a functor $M' : (\mathcal{PbC})^{op} \to \textbf{Sq}^\ast\textbf{Cat}$.

3.12. **DF Fibrations.** A DF fibration is a DF operad $M : (\mathcal{PbC})^{op} \to \textbf{Cat}$ such that the profunctors $\Phi_f = M_vf : MI \to MJ$ are co-represented by $f^* = M_hf$, for any $f : I \to J$ in $\mathcal{C}$;

$\Phi_f(X,Y) \cong MI(X,f^*Y)$

Thus, as expected, a DF fibration is determined by its horizontal component, that is the indexed category

$I \mapsto MI; f \mapsto f^*$

The vertical component, which as a normal lax functor $\mathcal{C} \to \textbf{Prof}$ corresponds to a category over $\mathcal{C}$, embodies the associated fibration.
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