Topological strings on toric geometries in the presence of Lagrangian branes

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Abstract We propose expressions for refined open topological string partition function on certain non-compact Calabi Yau 3-folds with topological branes wrapped on the special lagrangian submanifolds. The corresponding web diagrams are partially compact and a lagrangian brane is inserted on one of the external legs. Partial compactification introduces a mass deformation in the corresponding gauge theory. We propose conjectures that equate these open topological string partition functions with the generating function of equivariant indices on certain quiver moduli spaces. To obtain these conjectures we use the identification of topological string partition functions with equivariant indices on the instanton moduli spaces.

1 Introduction

Topological strings [2,26,29] correspond to the subsector of full superstrings spectrum that is invariant under a non-trivial sub algebra of the extended supersymmetry algebra. The observables of this subsector describe various topological properties of the spacetime on which the observables are defined. In physical 4d effective field theory the topological strings with the so-called $A-twist$ generate the following F-term that depends on the vector multiplet moduli

$$\int F_g(W^2)^g$$

where $W^2$ is composed of the Weyl multiplet superfield $W_{\alpha\beta}$ and $F_g$ is the genus $g$ topological string free energy. $F_g$ for $g \geq 2$ describe coefficients in the scattering amplitude of $2g - 2$ graviphotons. The presence of D-branes imply consistent boundary conditions on the world sheet boundaries. In the topological sector the boundary conditions should preserve the BRST symmetry of the world sheet. From the target space (CY) perspective in the A-model the world sheet boundaries are mapped to a particular submanifold $L$ whose dimension is equal to the half that of CY [26] and the restriction of the CY Kähler form $\omega$ to $L$ vanishes. In general, open strings may end on $L$ resulting in the wrapping of an A-model brane on $L$.

The problem of computing the unrefined topological string amplitudes on the toric CY in the presence of both external as well as internal branes was solved by the technique of the topological vertex [1]. On the other hand the refined topological string amplitude in the presence of internal branes turns out to be a subtle problem. Certain surface operators in 4d gauge theory can be realised by wrapping D4-branes on the Lagrangian 3-cycles of the CY 3-fold with the other two directions extending along transverse $\mathbb{R}^2 \subset \mathbb{R}^4$. These two transverse directions correspond to equivariant parameters $\epsilon_1, \epsilon_2$ or $t = e^{-i\epsilon_1}, q = e^{i\epsilon_2}$. The topological branes can
be put on the external non-compact legs or the internal compact legs of the toric diagram associated to the CY. Progress has been made in [23], where authors discuss refinement of holonomies for refined open topological string amplitudes. This was the case for topological branes on the external leg of the toric web diagram. In the presence of internal brane(s), there is a mismatch between the results as computed by using refinement of holonomy prescription and the geometric transition.

In this note we suggest that by utilising the equivalence of topological string amplitudes with equivariant indices [24] on framed moduli spaces, it may be possible to compute the refined open string amplitudes in the presence of internal branes. To this end, the authors [5] proposed a conjecture that equates the refined open string invariants of the special lagrangian branes in toric Calabi-Yau 3-folds, with the Witten index of the supersymmetric quantum mechanical model describing the BPS states attached to the surface defect. The Witten index can be interpreted as the Euler characteristic of the moduli space described by the quantum mechanical model. In other words the conjecture [5] equates the generating function of refined open string invariants of the special lagrangian branes with the instanton partition function in the presence of surface defects. The conjecture was checked to be true to high orders in the asymptotic expansion. According to the geometric engineering argument the $M5$ branes wrapped on a submanifold of the CY give rise to effective five dimensional gauge theories with surface defect.

In the case of partially compactified toric web diagram the refined open topological string amplitude is equated to the generating function of $\chi_y$ genus on the same moduli space [6,15,24]. The Kähler parameter of this new compact direction corresponds to a massive adjoint hyper in the 5d gauge theory [14]. We generalise the results of [5] and state the conjectures in the case of partial compactification of the resolved conifold, $O(-1) \oplus O(-1) \rightarrow \mathbb{P}^1$ and the partial compactification of the total space of the canonical bundle $O(-2,-2)$ of $P^1 \times P^1$. In the fully compactified case we give the expression for the generating function of elliptic genus of the defect moduli space and speculate about its relevance for the refined open topological string amplitude in the presence of internal branes.

In Sect. 2 we restate the conjecture proved in [5], about the equivalence of partition function on quiver moduli space with open Gromov–Witten invariants of certain non-compact Calabi–Yau 3-folds. In Sect. 3 we compute the generating function of $\chi_y$ genus on the quiver moduli space. The expression we obtain is not a polynomial in the mass-deformation parameter $y$. In the next Sect. 3.3 we use analytic continuation to write the $\chi_y$-genus as a polynomial in $y$. In Sect. 3.4 we compute the generating function $\chi_y$-genus for quiver moduli space for rank $r=2$, which corresponds to the refined open topological string partition function on the partially compactified web of the Hirzebruch surface $F_0$. In Sect. 4 we briefly discuss the identification of the Donaldson–Thomas partition function of CY3-fold and K-theory partition function of 5d supersymmetric gauge theory motivated by the geometric engineering argument. This identification was obtained for the unrefined case by making a change of variables. Using certain consistency conditions we propose a generalisation of this change of variables to the refined case. In the last Sect. 5 we given an expression for the elliptic genus on the same moduli space and suggest that it may be related to the open topological string partition function on non-compact Calabi-Yau 3-fold whose corresponding web diagram is fully compactified. The Appendices A, B, C contain the refined topological vertex amplitudes for the remaining preferred directions and the Appendix D contain a summary of the virtual equivariant localisation and fixed point theorems.

2 Quiver model and open topological string amplitudes

A standard D-brane construction of 5d gauge theory with eight supercharges involves [5] $D6$-branes wrapped on the holomorphic curves in a non-compact $K3$ surface $S$.

The type of quiver is defined by the intersection matrix of the configuration of $(-2)$-rational curves after suitable resolution. In the present context only $A_r$-type singularities are considered which are amenable to analysis with toric geometry techniques. So the type IIA vacuum we consider is given by $S \times S^1 \times \mathbb{R}^5 \equiv S \times C^* \times \mathbb{R}^4$. The low energy limit of $D6$-branes wrapped on these rational curves gives rise to 5d quiver gauge theory with eight supercharges. In the more general type IIA supersstring setup we can add $D4$-branes wrapping the special Lagrangian submanifolds, along with $D2$ branes ending on these $D4$-branes. $D2$-branes are wrapped on $(-2)$-rational curves. In other words $D4$-branes serve as defect operators and $D2$ branes correspond to the BPS states bound to the defect operators. The world volume of $D4$-brane is $L \times \mathbb{R}^2$, where $L \subset T \times C^*$ is the special Lagrangian submanifold and $\mathbb{R}^2 \subset \mathbb{R}^4$. By construction a toric Calabi Yau 3-fold $X$ admits a symplectic $U(1)^3$ action and the resulting moment map $\rho : X \rightarrow \mathbb{R}^3$ to the so-called Delzant polytope. The collection of $U(1)^3$ preserving compact and non-compact rational holomorphic curves of $X$ define its toric skeleton. The toric skeleton is mapped by $\rho$ to a trivalent graph $\Delta$ in $\mathbb{R}^3$. The special lagrangian submanifold $L$ under consideration is topologically equivalent to $S^1 \times \mathbb{R}^2$ and is mapped to a half real line which intersects a 1-face of the graph $\Delta$. The external lagrangian cycles intersect the non-compact components of the skeleton, whereas the internal lagrangian cycles intersect compact components. In the web diagrams we always show the branes wrapped on the lagrangian cycles by dashed lines.
The D4-branes are wrapped on the Lagrangian cycles of the CY3-fold, with two directions extending along one of the \( \mathbb{R}^2 \)s of the transverse \( \mathbb{R}^4 \). For the unrefined case the choice of \( \mathbb{R}^2 \) is immaterial. For the refined case the two \( \mathbb{R}^2 \)s inside \( \mathbb{R}^4 \) are rotated by \( q = e^{i \epsilon_1}, t = e^{-i \epsilon_2} \) corresponding to a particular choice of complex structure. Depending on which \( \mathbb{R}^2 \) the D4-brane is extended along, it is called either a q-brane or a t-brane [23]. In the presence of D4-branes the open topological string amplitudes are the generating functions of BPS degeneracies of D2-branes. These D2-branes are wrapped on smooth curves whose boundaries lie on D4-branes. The boundary conditions are necessary for the complete specification of the open string amplitudes and are given by gauge invariant combinations of holonomy operators.

The pure \( SU(r), r \geq 2 \) gauge theories in 5d can be engineered by certain non-compact CY 3-folds. To construct these 3-folds, the total space of \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 \) is orbifolded by the action \( (z_1, z_2) \to (e^{2 \pi i z_1}, e^{-2 \pi i z_2}) \) on the fiber coordinates \( (z_1, z_2) \). The resulting space is singular and can be resolved to a smooth CY 3-fold which contains \( (r-1) \) geometrically ruled surfaces glued together. It is equivalent to the resolved \( A_{r-1} \) fibration over \( \mathbb{P}^1 \). The compact part of the geometry consist of \( r-1 \) Hirzebruch surfaces glued together and the normal geometry of a base \( \mathbb{P}^1 \) in the \( p-1 \)-th and \( p \)-th Hirzebruch surfaces is \( \mathcal{O}(-r+2p-2) \oplus \mathcal{O}(r-2p) \). Formally allowing the value \( r = 1 \) corresponds to the resolved conifold, the total space of \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1 \).

Partially compactifying the web diagrams, which becomes non-planar, changes the CY3-fold to an elliptic CY3-fold which has the structure of the form \( \mathbb{C}^2 / \mathbb{Z}_r \times_f T^2 \). M-theory compactification on this geometry engineers 5d \( \mathcal{N} = 1 \) gauge theory with a single adjoint hypermultiplet [15, 20]. The fields of quantum mechanical system, describing the BPS states bound to the surface operators, arise from low energy modes of D2 − D6, D2 − D4 and D2 − D2 configuration of D-branes in type IIA strings. The field content is summarised as

\[
\begin{align*}
D2 & \rightarrow D6: \text{two } (0, 2) \text{ chiral multiplets} \\
D2 & \rightarrow D4: \text{a } (0, 2) \text{ chiral multiplet and a } (0, 2) \text{ vector multiplet} \\
D2' & \rightarrow D2: \text{two } (0, 2) \text{ chiral multiplets and two } (0, 2) \text{ Fermi multiplets} \\
D2' & \rightarrow D6: \text{a single } (0, 2) \text{ Fermi multiplet.}
\end{align*}
\]

An intricate analysis [5] shows that the moduli space of the supersymmetric vacua of the quantum mechanical model is isomorphic to the data defining ADHM type quiver. In a certain region of the moduli space it is interpreted in terms of certain generalised vector bundles.

The D2-brane effective action is derived by the dimensional reduction of the field contents of quiver diagram 1. The quiver diagram describes a vector space of supersymmetric flat directions parametrised by the fields \( (A_1, A_2, I, J, B_2, f, g, \sigma_1, \sigma_2) \) as follows

\[
\text{End}(V_1)^{\otimes 2} \oplus \text{Hom}(W, V_1) \oplus \text{Hom}(V_1, W) \oplus \text{End}(V_2) \\
\oplus \text{Hom}(V_1, V_2) \oplus \text{Hom}(V_2, V_1) \\
\oplus u(V_1) \oplus u(V_2)
\]

for hermitian inner product vector spaces \( V_1, V_2 \) and \( W \). The vacuum equations of the D2-branes effective action as given below define a moduli space

\[
\begin{align*}
[A_1, A_1^+] & \rightarrow A_2, A_2^+ & JI^J + f f^J - g^J g & = \xi_1, \\
[B_2, B_2^+] & - f^J f - g^J g & = \xi_2, \\
g A_1 & = 0, & A_1 f & = 0, & g I & = 0, & J f & = 0 \\
[A_1, A_2] & + IJ & = 0, & A_2 f & - f B_2 & = 0, & g A_2 & - B_2 g & = 0, & f g & = 0, \\
& & & [\sigma_1, A_1] & = 0, & [\sigma_1, A_2] & = 0, & [\sigma_2, B_2] & = 0, & \sigma_1 I & = 0, & J \sigma_1 & = 0, & \sigma_1 f & - f \sigma_2 & = 0, & g \sigma_1 & - \sigma_2 g & = 0.
\end{align*}
\]

The moduli space parametrize \( U(V_1) \times U(V_2) \) gauge inequivalent solutions to (4). An important result proven in [5] shows that the moduli space defined by the last set of equations is isomorphic to a different moduli space for generic values of Fayet Illuopolous parameters. The latter moduli space comprises of stable representations of the enhanced ADHM quiver, also described in terms of framed torsion free sheaves on the projective plane. The virtual smoothness in our context means that moduli space of stable representations of the ADHM quiver can be embedded in a smooth variety which is a hyper kähler quotient (Fig. 2).

The enhanced ADHM quiver is given in Fig. 3 along with the relations

\[
\begin{align*}
\alpha_1 \alpha_2 - \alpha_2 \alpha_1 & + \xi \eta, \alpha_1 \phi, \alpha_2 \phi - \phi \beta, \\
\eta \phi, \gamma \xi, \gamma \phi, \gamma \alpha_1, \gamma \alpha_2 & - \beta \gamma.
\end{align*}
\]

A triple of vector spaces \( (V_1, V_2, W) \) is assigned to the vertices \( (e_1, e_2, e_\infty) \) as a representation. Similarly the linear maps \( (A_1, A_2, I, J, B, f, g) \) represent the arrows \( (\alpha_1, \alpha_2, \ldots) \).
The resulting quiver representation is moded out by the relations (5). If we define $(r, n_1, n_2) := (\dim(W), \dim(V_1), \dim(V_2))$, then this triple of positive integers define the numerical type of the quiver. The framing of the enhanced quiver representation corresponds to the existence of an isomorphism $h : W \to C^n$. Moreover a stable\(^1\) representation of type $(r, n_1, n_2)$ implies the existence of a set of parameters $\theta = (\theta_1, \theta_2, \theta_\infty)$ satisfying the relation $n_1 \theta_1 + n_2 \theta_2 + r \theta_\infty$ such that

- Any subrepresentation of type $(0, m_1, m_2)$ satisfies $m_1 \theta_1 + m_2 \theta_2 \leq 0$.
- Any subrepresentation of type $(r, m_1, m_2)$ satisfies $m_1 \theta_1 + m_2 \theta_2 + r \theta_\infty \leq 0$

The following result gives criterion for generic stability conditions:

Given a quiver representation $\mathcal{R}$ of type $(r, n_1, n_2)$ and theta parameters satisfying $\theta_2 > 0, \theta_1 + n_2 \theta_2 < 0$, then the following three statements are equivalent

1. $\mathcal{R}$ is $\theta$-semistable
2. $\mathcal{R}$ is $\theta$-stable
3. (a) $f : V_2 \to V_1$ is injective and $g : V_1 \to V_2$ is identically zero (b) The data $A = (V_1, W, A_1, A_2, I, J)$ satisfies the ADHM stability conditions.

Consider the vector spaces $V_1, V_2, W$ with positive definite dimensions $n_1, n_2, r$ and the direct sum of vector spaces

$$X(r, n_1, n_2) = (\End(V_1) \otimes \End(V_2) \otimes \Hom(V_1, W) \oplus \End(V_2) \otimes \Hom(V_1, V_2) \oplus \Hom(V_1, V_2) \oplus \Hom(V_2, V_1)$$

\[\begin{array}{c}
\End(V_1) \otimes \End(V_2) \\
\rightarrow
\End(V_1) \otimes \Hom(V_1, W) \oplus \End(V_2) \oplus \Hom(V_2, V_1)
\end{array}\]

\[\begin{array}{c}
\End(V_1) \otimes \Hom(V_2, V_1) \oplus \Hom(V_2, W)
\end{array}\]

\[\begin{array}{c}
\rightarrow
\Hom(V_2, V_1)
\end{array}\]

It is clear that $\mathcal{G} = GL(V_1) \times GL(V_2)$ preserves the subset $X_0(r, n_1, n_2) \in X(r, n_1, n_2)$ defined by (4). The space $X_0(r, n_1, n_2)$ parametrizes the representations $\mathcal{R} = (V_1, V_2, W, A_1, I, J, B_2, f, g)$ with two framed representations $R_1, R_2$ being equivalent if the corresponding points in $X_0(r, n_1, n_2)$ belong to the same $GL(V_1) \times GL(V_2)$ orbit. The space $X_0(r, n_1, n_2)$ can be projectivized to a scheme as follows

$$N_0^G(r, n_1, n_2) = X_0(r, n_1, n_2)/\chi \mathcal{G}$$

with the notation $A(X_0(r, n_1, n_2)/\chi \mathcal{G}) := \{ f \in A(X_0(r, n_1, n_2))\}$. There exists an open subscheme $N_0^G(r, n_1, n_2)_0$ which $\mathcal{G}$-orbits are $\chi$-stable.\(^2\) In the parameter space defined by the inequalities $\theta_2 > 0, \theta_1 + n_2 \theta_2 < 0$ the framed representations of the enhanced quiver that satisfy condition (3) can simply be denoted by $\mathcal{N}(r, n_1, n_2)$ by dropping subscripts and superscripts.

The matter couplings in the quantum mechanical model corresponds to the three tautological bundles $V_1, V_2, W$ on the moduli space $\mathcal{N}(r, n_1, n_2)$. These bundles are defined by $\mathcal{W} = \mathcal{O}^\mathcal{N}_{r(n_1, n_2)}, \mathcal{L}_1 = \det(V_1)$ and $\mathcal{L}_2 = \det(V_2)$. Moreover several copies of $\mathcal{L}_1, \mathcal{L}_2$ can be tensored to give rise to the mixed line bundles $\mathcal{L}_{p_1, p_2} = \mathcal{L}\Gamma^{p_1} \otimes \mathcal{L}\Gamma^{p_2}$. The existence of the morphism $s : \mathcal{N}(r, n_1, n_2) \to \mathcal{M}(r, n_1 - n_2)$, where $\mathcal{M}(r, n_1 - n_2)$ is the moduli space of ADHM data of type $(r, n)$, plays a simplifying role in the application of the equivariant fixed point theorems. The tangent space at a point of $\mathcal{N}(r, n_1, n_2)$ is isomorphic to the difference $H^1(\mathcal{C}(\mathcal{R})) - H^2(\mathcal{C}(\mathcal{R}))$ for $\mathcal{R} = (A_1, A_2, I, J, B_2, f)$.\(^2\)

\[^{2}\text{The character function } \chi : \mathcal{G} \to \mathbb{C}_x^\times, \text{ where } \mathcal{G} \text{ acts on the space } X(r, n_1, n_2), \text{ furnishes a definition of } \chi\text{-stability. For this consider the existence of a polynomial } q(x) \text{ on } X(r, n_1, n_2) \text{ such that } q((g_1, g_2)x) = (g_1, g_2)^n q(x) \text{ for positive definite integer } n. \text{ If } q(x_0) \neq 0, x_0 \text{ is called } \chi\text{-semistable. Moreover if } \Delta \subset \mathcal{G} \text{ acts trivially on } X(r, n_1, n_2) \text{ such that } \dim(\mathcal{G}, x_0) = \dim(\mathcal{G}/\Delta) \text{ and the action of } \mathcal{G} \text{ on all such } x_0 \text{ is closed, then } x_0 \text{ is called } \chi\text{-stable.}\]

\[^{1}\text{More precisely } \theta\text{-semistable.}\]
representing a point of $N(r, n_1, n_2)$, where the complex $C(R)$ is defined by with the differentials defined as
\[
d_0(\alpha_1, \alpha_2) = ([\alpha_1, \phi_{\alpha_2}], \alpha_1, -ja_{\alpha_1} + [\alpha_2, j], \phi_{\alpha_2}),
\]
d_1(\alpha_1, \alpha_2, \phi_{\alpha_2}, \phi_{\alpha_2}) = (([\alpha_1, \phi_{\alpha_2}], \alpha_1, \phi_{\alpha_2}) + ja_{\alpha_1} + [\alpha_2, j], \phi_{\alpha_2}, \phi_{\alpha_2}),
\]
d_2(\alpha_1, \alpha_2, c_0, c_2, c_3) = c_1 f + A(c_2 - c_0 c_3 - c_1 c_3)
\]
and the conditions of decoupling the surface operator $\chi$, ADHM quivers. The moduli space, is arranged into a partition function of the quantum mechanics. The supersymmetric ground states are in one to one correspondence with cohomology as free sheaves with second Chern class $t$ and symmetric quantum mechanics. The supersymmetric ground states of this sequence is given by
\[
eq \chi \sum_{i \in \sigma^n} \chi_{i}^{(n)}(\mu, \nu) \sum_{i \in \sigma^n} \chi_{i}^{(n)}(\mu, \nu)
\]
Note that $H^1(C(R))$ parametrises the infinitesimal deformations of $R$ and $H^2(C(R))$ the obstructions to the deformations. Moreover the conditions $H^0(C(R)) = H^3(C(R)) = 0$ imply the stability of a framed representation. The equivariant virtual Euler characteristic of this determinant line bundle is a partition function of the quantum mechanical model.

The application of virtual equivariant localization requires the determination of fixed points of the torus action on the moduli space of the stable representations of the nested ADHM quivers. The moduli space, $N(r, n + d, d)$, for which we compute the Euler characteristic, is a nested Young diagrams satisfying the properties

\[
- \nu \subseteq \mu
- \text{if } (i, j) \in \mu \setminus \nu, \text{ then } (i + 1, j) \notin \mu
\]

Similarly if we denote an ordered sequence of Young diagrams $\mu = (\mu^1, \ldots, \mu^r)$ and $v = (v^1, \ldots, v^r)$ then it is a nested sequence if it is pairwise $(\mu^i, \nu^i)$ nested. The numerical type of this sequence is given by $(|\mu_i|, |\nu_i|)$. For nested sequence the defining algebraic inequalities are described as
\[
0 \leq c^a - e^a \leq 1, \quad 0 \leq c^a_i - e^a_i \leq v^a_{i-1} - v^a_i
\]
where $c^a, e^a$ denote the number of columns of $\mu^a$ and $\nu^a$ respectively and $a = 1, \ldots, r$ and $i \geq 0$. The tangent space to the moduli space at a fixed point $(\mu^a, \nu^a)$ of the torus $T = \mathbb{C}^s \times \mathbb{C}^\times \times (\mathbb{C}^\times)^r$ is regarded as an element of the representation ring of the torus action and is given by the expression
\[
T_a^a N(r, n + d, d) = T_a^a M(r, n)
\]
and the BPS counting function is the Witten index of the superpotential $X$.

2.1 Holomorphic Euler characteristic, $\chi$, genus and elliptic genus

Consider [4, 9] a rank $d$ vector bundle $E$ on some moduli space $X$. We can form the formal sums of the symmetric product $S_i E$ and the antisymmetric product $\Lambda_i E$ as
\[
\Lambda_i E = \sum_{i=0}^{d} [\Lambda_i E] t^i, \quad S_i E = \sum_{i=0}^{d} [S_i E] t^i
\]

The moduli space $X$ admits a virtual cotangent bundle $\Omega_X = (\Omega_X)^\vee$ and the bundle of $n$-forms $\Omega_X^n = \Lambda^n \Omega_X$. On the scheme $X$ one can consider the perfect obstruction theory $E^\bullet$ resolved to a complex of vector bundles $[E^{-1} \to E^0]$ and a virtual structure sheaf denoted by $O_X^{vir}$. The virtual tangent bundle $T_X^{vir}$ is defined by the class $[E_0] - [E_1]$, where the complex $[E_0 \to E_1]$ is dual to the complex $[E^{-1} \to E^0]$. The difference rank $[E_1] - [E_1]$ defines what is called the virtual dimension of $X$. For a vector bundle $V$ given on $X$ and $[V]^{vir}$ the virtual fundamental class of $X$ as an element of the $(n-m)$-th Chow group of $X$ with rational coefficients, the virtual holomorphic Euler characteristic is defined by
\[
\chi^{vir}(X, V) = \chi(X, V \otimes O_X^{vir})
\]
\[
\chi_{\gamma}(X, V) = \int_{[X]} \left( \sum_{l=1}^{d} e^{x_l} \right) \prod_{i=1}^{n} x_i \left( 1 - y e^{-x_i} \right) \prod_{j=1}^{m} \left( 1 - e^{-u_j} \right) u_j (1 - ye^{-u_j}) \tag{14}
\]

where \(x_1, \ldots, x_n\) denote the Chern roots of \(E_0, u_1, \ldots, u_m\) denote the Chern roots of \(E_1\) and \(v_1, \ldots, v_r\) denote the Chern roots of the vector bundle \(V\). Note that for \(y = 0\), \(\chi_{\gamma}(X, V)\) reduces to the Euler characteristic.

The moduli space \(\mathcal{N}(r, n + d, d')\) is in general non-compact and the cohomology groups \(H^{d, i}\) are not well defined. However due to the toric action, the Atiyah–Singer fixed point theorems can be applied to compute equivariant Euler character. The equivariant Euler character is an element of the quotient field of the partition function on the resolved conifold the defect brane was put on the un-preferred direction and the preferred direction was chosen to be the internal one. Moreover in our case the lagrangian brane is put along an external leg. The open topological string partition function contains both perturbative and non-perturbative contributions, and for \(r = 1\) the refined open topological string partition function on the resolved conifold is given by

\[
Z^{d, \text{open}}(q_1, q_2, \rho_a, T) = Z^{\text{ref}}(q_1, q_2, T) \tag{18}
\]

and for \(r = 2\)

\[
Z^{d, \text{open}}(q_1, q_2, \rho_a, T) = Z^{\text{ref}}(q_1, q_2, \rho_a, T) \tag{19}
\]

where the topological string partition function \(Z^{\text{ref}}\) is normalised by dividing out by the gauge theory perturbative part. We elaborate on the refined topological vertex computation in the next Sect. 3 for the 'compactified' geometries. It is important to note that in computing the open string partition function on the resolved conifold the defect brane was put on the un-preferred direction and the preferred direction was chosen to be the internal one. Moreover in our case the lagrangian brane is put along an external leg. The open topological string partition function contains both perturbative and non-perturbative contributions, and for \(r = 1\) the refined open topological string partition function on the resolved conifold is given by

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and for \(r = 2\)

\[
Z^{d, \text{open}}(q_1, q_2, \rho_a, T) = Z^{\text{ref}}(q_1, q_2, \rho_a, T) \tag{19}
\]

where the topological string partition function \(Z^{\text{ref}}\) is normalised by dividing out by the gauge theory perturbative part. We elaborate on the refined topological vertex computation in the next Sect. 3 for the 'compactified' geometries. It is important to note that in computing the open string partition function on the resolved conifold the defect brane was put on the un-preferred direction and the preferred direction was chosen to be the internal one. Moreover in our case the lagrangian brane is put along an external leg. The open topological string partition function contains both perturbative and non-perturbative contributions, and for \(r = 1\) the refined open topological string partition function on the resolved conifold is given by

\[
Z^{d, \text{open}}(q_1, q_2, \rho_a, T) = Z^{\text{ref}}(q_1, q_2, T) \tag{18}
\]

and for \(r = 2\)

\[
Z^{d, \text{open}}(q_1, q_2, \rho_a, T) = Z^{\text{ref}}(q_1, q_2, \rho_a, T) \tag{19}
\]
perturbative parts of gauge theory. In making the comparison (18) one has to exclude the perturbative part.

3 Generating function of $\chi_y$ genus

Instanton partition functions for gauge theories in the presence of surface defect in 4d, 5d and 6d are the generating functions of Euler characteristics, $\chi_y$-genera and elliptic genera of the moduli space under consideration [24]. For refined topological strings the open string defect amplitude can be written in four ways depending on whether the topological brane extends along $\mathbb{R}_y^2$ or $\mathbb{R}_x^2$ and whether it is put along the external non-compact leg or the internal compact leg [23].

In the M-theory lift of the type IIA topological strings, the equivariant Euler characteristics gets lifted to the $\chi_y$ genus. It is a 5d defect gauge theory compactified on a circle $S^1$. The defect BPS states in M-theory framework are related to M2-brane BPS state counting. Consequently we write down the generating function for the $\chi_y$ genus or 5d defect partition function as

$$Z^{\text{quiver}}_d(q_1, q_2, \rho_b, y, Q)$$

$$\equiv \sum_k Q^k \sum_{|v|=k} \prod_{a,b=1}^{r} \left(1 - yq_1 \rho_b^{-1} q_1 q_2^{j-v-1} \right) \prod_{b=1}^{r} \left(1 - \rho_b \rho_1 q_1^{-1} q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - \rho_1 \rho_1 q_1^{-1} q_2^{j-v-1} \right) \prod_{b=1}^{r} \left(1 - \rho_1 \rho_1 q_1^{-1} q_2^{j-v-1} \right)$$

$$\times \sum_{|a|=v} \prod_{i=1}^{r} \left(1 - yq_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right)$$

$$\times \sum_{|a|=v} \prod_{i=1}^{r} \left(1 - yq_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right)$$

$$\times \sum_{|a|=v} \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right)$$

$$\times \sum_{|a|=v} \prod_{i=1}^{r} \left(1 - yq_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right)$$

$$\times \sum_{|a|=v} \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right)$$

$$\times \sum_{|a|=v} \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right)$$

which for $r = 1$ becomes

$$Z^{5d,\text{quiver}}_d(q_1, q_2, \rho_b, y, Q)$$

$$= \sum_k Q^k \chi_y(M, k, q_1, q_2, \rho_1)$$

$$= \sum_k Q^k \sum_{|v|=k} \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right)$$

$$\times \sum_{|a|=v} \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right) \prod_{i=1}^{r} \left(1 - q_1 q_2^{j-v-1} \right)$$

Here $W_{d,3}(q_1, q_2, \rho_1, y, Q) \equiv \sum_{|a|=v} W_{d,3}(q_1, q_2, \rho_1, y, Q)$ contains the information about the surface defects and the BPS states bound to it.
3.1 Open string/defect brane partition function

\( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1 \): gauging the two external legs

The resolved conifold i.e. the total space of the bundle \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1 \) can be parametrised by the pair of coordinates \((y_1, y_2)\) and \((y_3, y_4)\). These pairs of coordinates define two line bundles on \( \mathbb{P}^1 \) as follows

\[
y_1 = \zeta y_2, \quad y_3 = \zeta^{-1} y_4
\]

for \( \zeta \in \mathbb{P}^1 \). The partial compactification of the corresponding web diagram imposes periodic boundary conditions on the external legs.

Note that in the toric diagram 4 two blue lines indicate the identification of vertical edges, the green line intersects the preferred direction and the red dotted line denotes the lagrangian brane. For our purposes it is more useful to choose the internal line as our preferred direction. With the following definitions of the framing factor \( f_\mu(t, q) \) and the refined topological vertex \( C_{\lambda \mu \nu}(t, q) \) given by [5]

\[
f_\mu(t, q) = (-1)^{\frac{|\mu|}{2} + |\mu|/2 - |\mu|/2} (t^{\mu} q^{-\nu}) s_{\mu/\eta}(t^{-1} q^{-\nu})
\]

\[
C_{\lambda \mu \nu}(t, q) = \left( \frac{t^{\mu} q^{-\nu}}{q^{\mu + \nu} - 1} \right)^{1/2} \prod_{\eta} Z_\eta(t, q)
\]

\[
Z_\eta(t, q) = \prod_{i=1}^{l(\nu)} (1 - q^{k_i - j_i})^{-1}
\]

\[
\rho = \left\{ -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots \right\}, t^{-\rho} q^{-\nu}
\]

\[
= \left\{ t^{1/2} q^{-k_1}, t^{2/2} q^{-k_2}, t^{3/2} q^{-k_3}, \ldots \right\}
\]

the refined amplitude can be written in the following form

\[
Z_{\text{open}}^{\text{ref}} = \sum_{x} s_{\lambda}(x) Z_\lambda(Q_1, Q_2, t, q)
= \sum_{x} s_{\lambda}(x) \sum_{\mu, k} (-Q_1)^{\mu} (-Q_2)^{|\mu|} C_{\mu \lambda \nu}(t, q) C_{\lambda \mu \nu}(t, q)
\]

\[
= \sum_{\nu} (-Q_1)^{\nu} q^{\nu/2} \sum_{i=1}^{\nu} t^{i(1-i)/2} \sum_{\mu} (-Q_1)^{\mu} (Q_2)^{\mu} \prod_{i,j} \left( \frac{1 + (\sqrt{t} \lambda_i)^{-1} q^{-1}}{1 + (\sqrt{t} \lambda_i)^{-1} q^{-1}} \right) \prod_{k} \left( 1 + Q_2 \sqrt{t} \lambda_i \right) q^{-1}
\]

where

\[
P_{\nu}(t^{-\rho}; q, t) = \frac{t^{1/2} q^{1/2}}{q^{\nu} - 1} \prod_{\nu} (1 - t^{(i,j)+1} q^{-l(i,j)})^{-1}
\]

After dividing by the gauge theory perturbative factor \( \prod_{k,l}(1 - Q_2 t^{-\rho} q^{-\rho}) \) and using the following identity [21]

\[
\prod_{k,l}(1 - Q_2 t^{-\rho} q^{-\rho}) = \prod_{k,l}(1 - Q_2 t^{-\rho} q^{-\rho})
\]

\[
Z_{\text{open}}^{\text{ref}} = \sum_{\nu} (-Q_1)^{\nu} q^{\nu/2} \sum_{i=1}^{\nu} t^{i(1-i)/2} \sum_{\mu} (-Q_1)^{\mu} (Q_2)^{\mu} \prod_{i,j} \left( \frac{1 + (\sqrt{t} \lambda_i)^{-1} q^{-1}}{1 + (\sqrt{t} \lambda_i)^{-1} q^{-1}} \right) \prod_{k} \left( 1 + Q_2 \sqrt{t} \lambda_i \right) q^{-1}
\]

Now to extract the contribution of the surface operator we proceed as suggested in [5]. The right hand side of the last equation can be expanded in the basis of symmetric functions, in particular in the basis of monomial symmetric functions \( m_\nu(x_i) \) [25]. By definition, \( m_\nu(0, 0, \ldots) = x_1^n + x_2^n + \cdots \) for any positive integer \( n \). Then we will denote by \( Z_{\text{open}, d}^{\text{ref}} \) the coefficient of \( m_d(0, 0, \ldots) \) in the expansion.

Next note that

\[
\ln \left( \prod_{i,j} \left( 1 + y_j t^{-v_j} q^{-1} \right) \right) = \sum_{d=1}^{\infty} \sum_{l=1}^{\infty} (-1)^{l-1} \frac{1}{l} \left( \sum_{k=1}^{\infty} t^{-1} q^{k(1-k)} \right)^l
\]

\[
= \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{l-1} \frac{1}{l} \left( \sum_{k=1}^{\infty} t^{-1} q^{k(1-k)} \right)^l
\]
where \( e_k(y) \), \( k \in \mathbb{Z}_{\geq 0} \) denotes the degree \( k \) elementary symmetric function in the variables \((y_1, y_2, \ldots)\). Recall the fact that for a partition \( \lambda \) and its conjugate partition \( \lambda' \) we have the expansion \([25]\)

\[
e_{\lambda'} = m_{\lambda} + \sum_{\mu} a_{\lambda, \mu} m_{\mu} \tag{29}
\]

where the summation over \( \mu \) is such that \( \mu < \lambda \) and \( a_{\lambda, \mu} \) are non-negative integers. Using the identity (29) it is easy to see that the coefficient of \( m_{d,0,0,\ldots}(x) = x_1^{d_1} + x_2^{d_2} + \cdots \) in the expansion (28) can be obtained by restricting to \( k = 1 \)

\[
e^{\sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \left( -1 \right)^{l-1} \left( t^{-i} q^{(i-1)} e_i(y) \right)^l} = e^{\sum_{i=1}^{\infty} \left( -1 \right)^{l-1} \sum_{d_1}^{\infty} \left( -1 \right)^{l-1} q^{(i-1)} e_i(y) \right)^l} \tag{30}
\]

Defining the quantity \( F_v \) by

\[
F_v(q, t) = \sum_{d_1, d_2, \ldots} q^{d_1-1} t^{-v_1} = \sum_{d_1, d_2, \ldots} q^{d_1-1} t^{-v_1} \tag{31}
\]

we can write

\[
e^{\sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \left( -1 \right)^{l-1} \left( t^{-i} q^{(i-1)} e_i(y) \right)^l} = e^{\sum_{i=1}^{\infty} \left( -1 \right)^{l-1} \left( t^{-i} q^{(i-1)} e_i(y) \right)^l} \tag{32}
\]

From the last expression we find the coefficient of \( m_{d,0,0,\ldots} \) as

\[
\frac{1}{d!} \sum_{\eta=(d_1,2d_2,\ldots)} \prod_{k=1}^{d} \frac{d!}{d_k!} \prod_{k=1}^{d} \left( -1 \right)^{k-1} k F_v(q, t) \tag{33}
\]

where \( \eta = (d_1, 2d_2, \ldots) \) denotes the set of all partitions of \( d \).

Following the same procedure we find \( \hat{Z}_{\text{open}, d} \) as

\[
\hat{Z}_{\text{open}, d} = \sum_{\mu} (-Q_1)^{|\mu|} \frac{q}{t} \frac{q_{\mu}^{(i-1, j)}}{2} P_{\mu} (t^{-\rho}, q; t) P_{\mu} (q^{-\rho}; t, q) \times \prod_{\mu \notin \eta} \left( 1 - 2 q e^{(a(s)k)} + 2 \right) \left( 1 - 2 q e^{(a(s)k) + \frac{1}{2}} \right) \tag{34}
\]

where

\[
F_v(t, q, Q_2) = \sum_{i=1}^{\infty} q^{i-1} t^{-v_1} - Q_2 \sqrt{t} \sum_{i=1}^{\infty} q^{i-1} t^{-v_1} \tag{35}
\]

making a change of variables

\[
q = q_2^{-1}, \quad t = q_1 \tag{36}
\]

The 5d generalisation of (18) states

\[
\hat{Z}_{\text{5d, quiver}} = \hat{Z}_{\text{open}, d} \tag{38}
\]

or explicitly

\[
\sum_{\mu} \hat{Z}_{\text{open}, d} = \hat{Z}_{\text{open}, d} = \sum_{\mu} \hat{Z}_{\text{open}, d} = \hat{Z}_{\text{open}, d} = \sum_{\mu} \hat{Z}_{\text{open}, d}
\]

where \( \text{parameter-identification} \) is described in the Sect. 4. We have to divide by \( Z_{\text{pert}} \) since the refined topological vertex technique computes both perturbative and non-perturbative contributions from the gauge theory point of view, whereas \( W_{\text{open}}(q_1, q_2, y) \) only describes non-perturbative contributions.

### 3.2 Special case of the conjecture

\( v = 0 \)

We know that in the absence of a Lagrangian brane we have to set

\( v = 0 \),

and the conjecture (39) reduces to the special case

\[
\sum_{\mu} \hat{Z}_{\text{open}, d} = \hat{Z}_{\text{open}, d} = \sum_{\mu} \hat{Z}_{\text{open}, d} = \hat{Z}_{\text{open}, d} = \sum_{\mu} \hat{Z}_{\text{open}, d}
\]

where

\[
F_v(t, q, Q_2) = \sum_{i=1}^{\infty} q^{i-1} t^{-v_1} - Q_2 \sqrt{t} \sum_{i=1}^{\infty} q^{i-1} t^{-v_1} \tag{35}
\]

making a change of variables

\[
q = q_2^{-1}, \quad t = q_1 \tag{36}
\]
Using the following definition of the Macdonald symmetric function \( P_v(x; q, t) \)

\[
P_v(t^{-p}; q, t) = t^{\frac{|v|^2}{2}} \prod_{s \in v} (1 - t^{l(s)+1}q^{l(s)})
\]

the mass deformation parameter \([14]\) and does not depend on \( \nu \), the representation of the lagrangian branes. Thus turning \( t = q_1, \quad q = q_2^{-1} \),

\[
Q_2 = \sqrt{\frac{t}{q}}, \quad Q_1 = \sqrt{\frac{t}{Q}}
\]

the identity (41) is satisfied \([21]\) by taking

\[
y = 1 \quad \text{and} \quad q_1 = q_2^{-1} := q
\]

From the 2d quiver quantum mechanical point of view \( y \) is the mass deformation parameter \([14]\) and does not depend on \( \nu \), the representation of the lagrangian branes. Thus turning on \( \nu \) from \( \nu = \emptyset \) to \( \nu \neq \emptyset \) does not change the dependence on the mass parameter \( y \). Note that for \( y = 1 \) and \( q_1 = q_2^{-1} := q \)

\[
W_{v,d}(q_1, q_2, y, \rho_a)
\]

\[
= \sum_{(\mu, \nu)} \left[ \prod_{i=1}^{\mu^a} \prod_{j=1}^{\mu^b} \rho_i q_1^{-i} q_2^{-|\nu|-1} t a,b=1 \right.
\]

\[
\times \left( \prod_{i=1}^{\mu^b+1} \prod_{j=1}^{\mu^b} (1 - y \rho_a \rho_b^{-1} q_1^{j-i} q_2^{j-i}) \prod_{i=1}^{\nu^a+1} (1 - \rho_a \rho_b^{-1} y q_1^{j-i} q_2^{j-i}) \prod_{i=1}^{\nu^b} (1 - \rho_a \rho_b^{-1} y q_1^{j-i} q_2^{j-i}) \right]
\]

\[
= W_{v,d}(q_1, q_2, \rho_a) + O(y^d)
\]

(45)
Noting that

\[
\prod_{i=1}^{x} \prod_{s=1}^{\mu_a} \rho_1 q_1^{1-i} q_2^{-i-s+1} = \sqrt{q_1 q_2^{d+(||y||^2-||\mu||^2)}}
\]

(46)

\[
W_{v,d}(q_1, q_2, y, \rho_a) = \sum_{(a,\nu)} \left[ \frac{1}{|\nu|} \prod_{\nu} \right] \prod_{a,b=1}^{r} \left( \prod_{i=2}^{r^{a+1}} \prod_{j=1}^{r^{b+1}} \prod_{s=1}^{r^{b}} \prod_{t=1}^{r^{b}} \right)
\]

\[
\times \left( \prod_{i=2}^{r^{a+1}} \prod_{j=1}^{r^{b+1}} \prod_{s=1}^{r^{b}} \prod_{t=1}^{r^{b}} \right)
\]

(47)

3.4 Generating function of \(\chi_y\) genus: \(r = 2\)

For this case the partially compactified toric diagram of the total space of the bundle \(O(-2, -2)\) of \(\mathbb{P}^1 \times \mathbb{P}^1\) is given in Fig. 5. Note that the edges which are identified are parallel.

In our computation we choose the horizontal direction as the preferred one and hence the refined topological vertex can be used. An important ingredient in the refined vertex computation is the choice of preferred direction, which should be common to all of the vertices in the web-digram. However if we had chosen the direction along which the lagrangian brane is present as the preferred direction, it is not shared by two of the vertices in the web diagram 5. This requires the introduction of a new refined topological vertex [19].

To avoid the subtleties of the new refined topological vertex we can instead choose to compute refined partition function of the flopped geometry [3,16]. Although for the horizontal preferred direction this is not necessary, we use it to illustrate the procedure. The flopped geometry is obtained from the original geometry by moving in the moduli space defined by the extended Kähler cone of the CY 3-fold under consideration. For instance the geometry defined in (22), has its flopped version defined as

\[
y_1 = \tilde{\xi} y_4, \quad y_2 = \tilde{\xi}^{-1} y_3
\]

(48)

for \(\tilde{\xi} \in \mathbb{P}^1\). The toric Calabi Yau manifolds have the important property that they can be converted to a strip geometry form after appropriate number of blow ups and flop operations [17].

The flopped geometry contains as building blocks partially compactified \(O(-1) \oplus O(-1) \rightarrow \mathbb{P}^1\) s suitably glued. To perform flop transition it is useful to draw the toric diagram in an equivalent way, Fig. 6, which makes the appearance of the building blocks \(O(-1) \oplus O(-1) \rightarrow \mathbb{P}^1\), \(O(-2) \oplus O(0) \rightarrow \mathbb{P}^1\) and \(O(0) \oplus O(-2) \rightarrow \mathbb{P}^1\) manifest.

![Fig. 5](image1.png)

**Fig. 5** Partially compactified toric diagram of total space of the bundle \(O(-2, -2)\) of \(\mathbb{P}^1 \times \mathbb{P}^1\)

![Fig. 6](image2.png)

**Fig. 6** Equivalent to toric diagram in Fig. 5
Finally performing the flop on $\mathbb{P}^1$ whose normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, results in the web diagram given in Fig. 7.

The kähler parameters of the pre-flopped geometry to the flopped geometry are related as

$$Q_1 = Q'_1 Q'_\rho, \quad Q_2 = Q'_2 Q'_\rho, \quad \tilde{Q}_1 = \tilde{Q}'_1 Q'_\rho, \quad \tilde{Q}_2 = \tilde{Q}'_2 Q'_\rho,$$

$$Q_\rho = Q'^{-1}_\rho.$$  \hfill (49)

The crucial result that allows for the use of flop transition to compute the open topological string amplitude is the flop invariance of topological string computations by refined vertex technique [22,27,28]. If we denote the CY3-folds corresponding to web diagrams in Figs. 5 and 7 as $X_1, X_2$, the flop invariance implies

$$Z_{X_2}^{\text{refined, open}}(Q_1, Q'_2, Q'_1, Q'_\rho, q, t; x_i)$$

$$= Z_{X_1}^{\text{refined, open}}(Q'_1 Q'_\rho, Q'_2 Q'_\rho, Q_\rho, q, t; x_i)$$  \hfill (50)

We get the following refined amplitude for the flopped geometry in Fig. 7 using the refined topological vertex formalism

$$Z^{\text{ref}}_{X_2} (Q_1, Q_2, Q'_1, Q'_2, Q'_\rho, q, t; x)$$

$$= \sum_{\text{all indices}} \left(-Q_1\right)^{\tilde{m}_1} \left(-Q_2\right)^{\tilde{m}_2} \left(-Q'_1\right)^{\tilde{m}'_1} \left(-Q'_2\right)^{\tilde{m}'_2} \left(-Q'_\rho\right)^{\tilde{m}'_\rho}$$

$$\times C_{\tilde{m}_1 \tilde{m}_2 \tilde{m}'_1 \tilde{m}'_2 \tilde{m}'_\rho}(q, t) C_{\rho \rho \rho}^{\tilde{m}_1 \tilde{m}'_1 \tilde{m}'_\rho}(q, t) C_{\rho \rho \rho}^{\tilde{m}_2 \tilde{m}'_2 \tilde{m}'_\rho}(q, t) C_{\rho \rho \rho}^{\tilde{m}'_1 \tilde{m}'_2 \tilde{m}'_\rho}(q, t)$$

\hfill (51)

Using the expression for the refined topological vertex (23) and using the following skew Schur function identities repeatedly

$$\sum_{\lambda} s_{\lambda / a}(x)s_{\lambda / b}(y) = \prod_{i,j} \left(1 - x_i y_j \right)^{-1} \sum_{\eta} s_{\eta / a}(x)s_{\eta / b}(y)$$

\hfill (52)

we get the expression

$$Z^{\text{ref}}_{\text{open}, X_2} (Q_1, Q_2, Q'_1, Q'_2, Q'_\rho, q, t; x)$$

$$= \sum_{\tilde{m}_1 \tilde{m}_2} \left(-Q_1\right)^{\tilde{m}_1} \left(-Q_2\right)^{\tilde{m}_2} Z_{\tilde{m}_1} (q, t) Z_{\tilde{m}_2} (q, t) Z_{\tilde{m}'_1} (q, t) Z_{\tilde{m}'_2} (q, t) \times \prod_{i,j} \left(1 - Q_i q^{-\tilde{m}_1 - \rho i} q^{-\tilde{m}'_1 - \rho j}\right) \times \prod_{i,j} \left(1 - Q_j q^{-\tilde{m}_2 - \rho i} q^{-\tilde{m}'_2 - \rho j}\right) \times \prod_{i} \left(1 - Q_q q^{-\tilde{m}_1 - \rho i} q^{-\tilde{m}'_1 - \rho i}\right) \times \prod_{i} \left(1 - Q_q q^{-\tilde{m}_2 - \rho i} q^{-\tilde{m}'_2 - \rho i}\right) \times \prod_{i} \left(1 - Q_q q^{-\tilde{m}_1 - \rho i} q^{-\tilde{m}'_1 - \rho i}\right)^{-1} \times \prod_{i} \left(1 - Q_q q^{-\tilde{m}_2 - \rho i} q^{-\tilde{m}'_2 - \rho i}\right)^{-1}$$

\hfill (53)

Since the refined topological vertex formalism gives both perturbative and non-perturbative parts from gauge theory viewpoint, we have to normalise it (53) to exclude the perturbative part. The normalised partition function turns out to be

$$Z^{\text{ref}}_{\text{open}, X_2} (Q_1, Q_2, Q'_1, Q'_2, Q'_\rho, q, t; x)$$

$$= \sum_{\tilde{m}_1 \tilde{m}_2} \left(-Q_1\right)^{\tilde{m}_1} \left(-Q_2\right)^{\tilde{m}_2} Z_{\tilde{m}_1} (q, t) Z_{\tilde{m}_2} (q, t) Z_{\tilde{m}'_1} (q, t) Z_{\tilde{m}'_2} (q, t) \times \prod_{i,j} \left(1 - Q_i q^{-\tilde{m}_1 - \rho i} q^{-\tilde{m}'_1 - \rho j}\right) \times \prod_{i,j} \left(1 - Q_j q^{-\tilde{m}_2 - \rho i} q^{-\tilde{m}'_2 - \rho j}\right) \times \prod_{i} \left(1 - Q_q q^{-\tilde{m}_1 - \rho i} q^{-\tilde{m}'_1 - \rho i}\right) \times \prod_{i} \left(1 - Q_q q^{-\tilde{m}_2 - \rho i} q^{-\tilde{m}'_2 - \rho i}\right) \times \prod_{i} \left(1 - Q_q q^{-\tilde{m}_1 - \rho i} q^{-\tilde{m}'_1 - \rho i}\right)^{-1} \times \prod_{i} \left(1 - Q_q q^{-\tilde{m}_2 - \rho i} q^{-\tilde{m}'_2 - \rho i}\right)^{-1}$$

\hfill (54)

Note that since for $\tilde{Q}_1 = \tilde{Q}_2 = Q_\rho$ the exponent of $Q_\rho$ counts the instanton number, the gauge theory perturbative part is extracted by the limit $Q_\rho \to 0$. 

\[\square\] Springer
\[
\prod \left( 1 - Q_1 Q_2 Q_\rho t^{-\tilde{\nu}_1^{-\rho_1}} q^{-\tilde{\mu}_1^{-\rho_1}} \right) \\
\prod \left( 1 - Q_1 Q_2 Q_\rho t^{-\tilde{\nu}_2^{-\rho_2}} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\prod \left( 1 + x_j \sqrt{\frac{q}{T}} t^{-\rho_i} q^{-\tilde{\mu}_i} \right) \prod \left( 1 + Q_2 Q_\rho_q t^{-\rho_i} q^{-\tilde{\mu}_i} \right)^{-1} \\
\prod \left( 1 - Q_2 Q_\rho_q t^{-\rho_i} q^{-\tilde{\mu}_i} \right)^{-1} \\
\prod \left( 1 - Q_1 Q_2 Q_\rho_q t^{-\rho_i} q^{-\tilde{\mu}_i} \right)
\] 
(54)

A crucial step is to write the normalised partition function in terms of the Kähler parameters of the pre-flopped geometry (5) using (49):

\[
\hat{Z}^{ref}_{open, X_1} (Q_1', Q_2', Q_1', Q_2', Q_\rho, q, t; x) \\
= \sum_{\tilde{\mu_1}, \tilde{\mu_2}} (-Q_1' Q_2')^{-\tilde{\mu}_1^{-\rho_1}} (-Q_2' Q_\rho')^{-\tilde{\mu}_2^{-\rho_2}} \hat{Z}_{\tilde{\mu}_1}
\times (q, t) \hat{Z}_{\tilde{\mu}_2} (q, t) \hat{Z}_{\tilde{\mu}_1} (q, t)
\times \sqrt{\frac{Q_1' - 1}{Q_2' - 1}} q^{-\mu_1^{-\rho_1} t^{-\tilde{\mu}_1^{-\rho_1}}} \\
\times \prod \left( 1 - Q_1' Q_2' q^{-\mu_1^{-\rho_1} t^{-\tilde{\mu}_1^{-\rho_1}}} \right) \\
\times \prod \left( 1 - (Q_2')^{-1} q^{-\tilde{\mu}_1^{-\rho_1} t^{-\tilde{\mu}_1^{-\rho_1}}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right)
\] 
(55)

It is important to note that the last expression has to be expanded in terms of \( Q_\rho' \) instead of \((Q_\rho')^{-1}\) to prove its equivalence to (54). Note that for particular values

\[
Q_1' Q_\rho' = \sqrt{\frac{T}{q}},
\]
\[
Q_2' Q_\rho' = \sqrt{\frac{T}{q}}
\]
(56)

the expression (55) reduces to

\[
\hat{Z}^{ref}_{open, X_1} (Q_1', Q_2', Q_1', Q_2', Q_\rho, q, t; x) \\
= \sum_{\tilde{\mu_1}, \tilde{\mu_2}} (-Q_1' Q_2')^{-\tilde{\mu}_1^{-\rho_1}} (-Q_2' Q_\rho')^{-\tilde{\mu}_2^{-\rho_2}} \sqrt{\frac{Q_1' - 1}{Q_2' - 1}} q^{-\mu_1^{-\rho_1} t^{-\tilde{\mu}_1^{-\rho_1}}}
\times \prod \left( 1 - Q_1' Q_2' q^{-\mu_1^{-\rho_1} t^{-\tilde{\mu}_1^{-\rho_1}}} \right) \\
\times \prod \left( 1 - (Q_2')^{-1} q^{-\tilde{\mu}_1^{-\rho_1} t^{-\tilde{\mu}_1^{-\rho_1}}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - Q_2' Q_\rho' t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right)
\] 
(57)

Moreover for the geometric engineering of pure SU(2) with zero Chern–Simons coefficient for web diagram in Fig. 5, as in our case, we should impose the restrictions

\[
Q_1' = Q_2' = Q_b
\]
\[
Q_1' = Q_2' = Q_f
\]
(58)

Restricting to the identification of parameters given in (58), corresponding to pure SU(2) gauge theory, we get

\[
\hat{Z}^{ref}_{open} (Q_b, Q_f, Q_m, q, t; x) \\
= \sum_{\tilde{\mu_1}, \tilde{\mu_2}} (-Q_b Q_m^{-1})^{-\tilde{\mu}_1^{-\rho_1}} (-Q_b Q_m^{-1})^{-\tilde{\mu}_2^{-\rho_2}} \sqrt{\frac{Q_b - 1}{Q_m - 1}} q^{-\mu_1^{-\rho_1} t^{-\tilde{\mu}_1^{-\rho_1}}}
\times \prod \left( 1 - Q_f Q_m t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - (Q_m)^{-1} q^{-\tilde{\mu}_1^{-\rho_1} t^{-\tilde{\mu}_1^{-\rho_1}}} \right) \\
\times \prod \left( 1 - Q_f Q_m t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - (Q_m)^{-1} q^{-\tilde{\mu}_1^{-\rho_1} t^{-\tilde{\mu}_1^{-\rho_1}}} \right) \\
\times \prod \left( 1 - Q_f Q_m t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right) \\
\times \prod \left( 1 - Q_f Q_m t^{-\rho_i} q^{-\tilde{\mu}_2^{-\rho_2}} \right)
\] 
(59)
Similar to (57), choosing \( Q_f Q_m = \sqrt{z} \) results in the simplified expression

\[
Z^\text{open} (Q_b, \ Q_f, \ Q_m, \ q, \ t; x) |_{Q_a = \sqrt{z}}^\text{ref} = \sum_{\tilde{\mu}_1, \tilde{\mu}_2} (-Q_b Q_m)^{|\tilde{\mu}_1| + |\tilde{\mu}_2|} \sqrt{q|\tilde{\mu}_1|^2 + |\tilde{\mu}_2|^2}
\]

\[
\sqrt{q|\tilde{\mu}_1|^2 + |\tilde{\mu}_2|^2}
\]

(60)

The topological string expression (59) is to be compared with the quiver moduli partition function (following (21) and (45)) given as follows

\[
Z^\text{d} (q_1, q_2, \rho_a, y, Q) = \sum_k Q^k \sum_{|v| = k} \prod_{(i, j) \in \omega} (1 - y q^1)^{Q}_1 (1 - y q^{\nu}_1)^{Q}_2 \prod_{(i, j) \in \omega} (1 - y q^1)^{Q}_1 (1 - y q^{\nu}_1)^{Q}_2 \]

\[
\times \sum_{(\mu, \nu)} (\prod_{a=1}^{2} \prod_{i=1}^{m_a} \prod_{s=1}^{\nu} \rho_a q_1^{i-1} q_2^{-i-s+1}) \prod_{a, b = 1}^{2} \prod_{i=1}^{m_a} \prod_{s=1}^{\nu} (1 - \rho_a - y q_1^{i-1} q_2^{-i-s+1})
\]

\[
\times (\prod_{j=1}^{n_a} \prod_{i=1}^{m_b} \prod_{s=1}^{\nu} (1 - \rho_b q_1^{j-1} q_2^{-j-s+1})) \prod_{j=1}^{n_b} \prod_{i=1}^{m_b} \prod_{s=1}^{\nu} (1 - \rho_b q_1^{j-1} q_2^{-j-s+1})
\]

\[
\times (\prod_{j=1}^{n_a} \prod_{i=1}^{m_b} \prod_{r=0}^{\nu} (1 - \rho_a q_1^{j-1} q_2^{-j-s-1+r})) (1 - \rho_a q_1^{j-1} q_2^{-j-s-1+r})
\]

\[
Z^\text{d} (q_1, q_2, \rho_a, y, Q) |_{y = 1} = \sum_k Q^k \sum_{|v| = k} \sum_{(\mu, \nu)} (\prod_{a=1}^{2} \prod_{i=1}^{m_a} \prod_{s=1}^{\nu} \rho_a q_1^{i-1} q_2^{-i-s+1})
\]

(62)

\[
Z^\text{d} (q_1, q_2, \rho_a, y, Q) = \text{parameter-identification} Z^\text{open, d} (Q_f, Q_b, Q_m, q, t)
\]

(64)

Note that after parameter identification \( q = q_2^{-1}, t = q_1, Q_f = \rho_1^{-1} \rho_2 \), the decompactification limit \(-log(Q_m) \to \infty \) leads to the result (5.3) proved in [5]

\[
Z^\text{d} (q_1, q_2, \rho_a, \rho_b, Q) = \tilde{Z}^\text{open, d} (q_1, q_2, \rho_1^{-1} \rho_2, Q)
\]

(65)

4 Relations between the parameters in the conjecture

In this section we describe consistency conditions that lead to the complete identification of parameters from the two sides of the conjectures (39, 64). We will also formulate the conjecture in a more general form that contains information about the holonomy observables, denoted by \( x \), parametrising the lagrangian branes. For the unrefined case in the absence of lagrangian branes on the external toric legs see [6, 8]. Note the following facts:

- In the decompactification limit \(-log(Q_1) \to \infty \) the 4d version of (39) imposes the following identifications [5]

\[
t = q_1, \quad q = q_2^{-1}, \quad Q_2 = T \sqrt{q_1 q_2}
\]

(63)

- By considering the special case \( v = \emptyset, y = 1 \) in (39) one must choose \( Q_1 = \sqrt{z} \) for the identity to hold. Since from 2d sigma model point of view \( y \) is the mass parameter, this identification of parameters does not change for \( v \neq \emptyset \).

- As we consider cases \( r \geq 2 \) the characters \( \rho_a \), are related to the Kähler parameters \(-log(Q_f), -log(Q_b)\) of the fiber and base directions.

Given these constraints, the generalisation of (39) to the rank = 2 case will then be given by
D4-D2 branes. This moduli space modulo certain equivalence relations was shown to be isomorphic to the nested Hilbert scheme of points on \( \mathbb{C}^2 \). This scheme, denoted by \( \mathcal{N}(\gamma) \), depends on an ordered sequence \( \gamma = \{m_i\}_{0 \leq i \leq k} \) of positive integers and parametrize a sequence of ideals sheaves \( 0 \subset I_0 \subset I_{k-1} \subset \ldots \subset I_k \) of zero dimensional subschemes \( Z_k \subset \mathbb{C}^2 \) and the corresponding topological data \( \chi(\mathcal{O}_{Z_k}) = \sum_{j=0}^{i} \gamma_j \) for \( 0 \leq i \leq l \).

The character valued partition function is given by the equivariant \( \chi \), genus of a bundle \( V \) on \( \mathcal{N}(\gamma) \). The bundle is identical to the bundle \( \mathcal{L}(\rho) \) described earlier and given as

\[
\mathcal{V}(\gamma) \equiv \eta^* \left( \mathcal{V}_{g,p} \right) \simeq \eta^* \left( T^* \mathcal{H}^\otimes g \otimes \det \mathcal{V} \right)
\]

(66)

The existence of the morphism \( \eta : \mathcal{N}(\gamma) \to \mathcal{H}' \) to the Hilbert scheme of \( r \) points on \( \mathbb{C}^2 \) makes it possible to apply the equivariant localization. The fixed points are given by the monomial ideals that are in one-to-one correspondence with the partitions of \( r \).

More interesting is the appearance of the modified Kostka–Macdonald coefficients. It can be explained by the existence of a map from the nested Hilbert scheme

\[
\rho : \mathcal{N}(1, 1, \ldots, 1) \to \mathcal{H}'
\]

to isospectral Hilbert scheme discussed in [13]. The web of maps is shown below in the Fig. 8 as a commutative diagram. With this commutative diagram in mind, it was shown that there exists two very important pushforward maps

\[
\rho^\#_{red} \circ \mathcal{O}_{\mathcal{N}(\gamma)} = \mathcal{O}_{\mathcal{H}^\#_{red}},
\]

\[
\pi^\#_{red} \circ \mathcal{O}_{\mathcal{H}^\#_{red}} \simeq (\pi^\#_{red} \circ \mathcal{O}_{\mathcal{H}^\#_{red}})^{\mathcal{S}_r} = \mathcal{P}^\#_{r}
\]

(67)

Next we summarise a sequence of arguments that fixes the notation of this section and which lead to the expansion of the topological string partition function in terms of the modified Macdonal polynomials.

(a) since \( \mathcal{N}(1, 1, \ldots, 1) \) is reduced, this implies the existence of the morphism \( \rho^\#_{red} : \mathcal{N}(1, 1, \ldots, 1) \to \mathcal{H}^\#_{red} \).

(b) \( \rho^\#_{red} \circ \mathcal{O}_{\mathcal{N}(1,1,\ldots,1)} \equiv \mathcal{O}_{\mathcal{H}^\#_{red}} \).

(c) the pushforward map \( \pi^\#_{red} \circ \mathcal{H}^\#_{red} \) is a vector bundle \( \mathcal{P} \) on the Hilbert scheme and is isomorphic to the pushforward map \( \eta^* \mathcal{O}_{\mathcal{N}(1,1,\ldots,1)} \).

(d) consider the stabiliser \( S_{\gamma} \subset S_r \) of the partition \( \gamma \) with \( S_r \) the symmetric group of order \( r \). This shows that \( \mathcal{H}' \) furnishes a representation of \( S_r \) by the restriction map, \( S_{\gamma} \times \mathcal{H}' \to \mathcal{H}' \).

(e) \( \mathcal{H}' \) denotes the quotient of \( \mathcal{H}' \) by \( S_r \).

(f) there exists a morphism \( \rho' : \mathcal{N}(\gamma) \to \mathcal{H}' \) which is also true for the corresponding reduced schemes \( \rho^\#_{red} : \mathcal{N}(\gamma) \to \mathcal{H}^\#_{red} \).

As a consequence of the Eqs. (67) in the \( \mathcal{T} \)-equivariant framework we have

\[
\chi^T_r \left( \mathcal{N}(\gamma), \eta^\# \mathcal{V}_{g,p} \right) = \chi^T_r \left( \mathcal{H}', \left( \mathcal{P}^\#_{\mathcal{S}_r}, \mathcal{S}_r \right) \chi \mathcal{V}_{g,p} \right) = \sum_{\mu} \chi^T_{\mathcal{H},\mu}(q_1, q_2, y) \mathcal{H}_r(q_1, q_2, y)
\]

where \( \mathcal{H}_r \) denotes an unordered partition of \( r \) determined by the sequence \( \gamma \). \( K_{\lambda, \gamma}(q_1, q_2) \) are the Kostka numbers and \( K_{\lambda, \gamma}(q_1, q_2) \) are the modified Kostka–Macdonald coefficients and in the second last equality equivariant localization was used. The character valued (K-theoretic) partition function is then given by the generating function of the \( \chi \) genus

\[
\mathcal{Z}_K(q_1, q_2; y, x) = \sum_{\mu} \chi^T_{\mathcal{H},\mu}(q_1, q_2, y) \mathcal{H}_r(q_2, q_1; x)
\]

(70)

For the moduli space of this section, i.e. \( \mathcal{H}'(\mathbb{C}^2) \) the equivariant localization yields

\[
\Omega^\#_{\mathcal{S}_r}(q_1, q_2, y) = \prod_{\mu}(q_1^{\ell(\mu)} q_2^{a(\mu)})^{g-1-p} \times \prod_{\mu}(1 - y q_1^{\ell(\mu)} q_2^{a(\mu) + 1})^p (1 - y q_1^{\ell(\mu)} q_2^{a(\mu)})^p
\]

(71)

with \( a(\mu) \) and \( \ell(\mu) \) defined as the arm length and the leg length of a box in the Young diagram corresponding to \( \mu \). Therefore

\[
\mathcal{Z}_K(q_1, q_2, y, x) = \sum_{\mu} \prod_{\mu}(q_1^{\ell(\mu)} q_2^{a(\mu)})^{g-1-p} \times \prod_{\mu} \left( \frac{1 - y q_1^{\ell(\mu)} q_2^{a(\mu) + 1}}{1 - q_1^{\ell(\mu)} q_2^{a(\mu)}} \right)^p \times \mathcal{H}_r(q_2, q_1; x)
\]

(72)
The quantity \( Z^{top,\;open}(q, x, y) \) can be independently computed using the topological vertex formalism. It turns out [8] e.g. for \( x = \{Q, 0, 0, \ldots\} \), to be equal to the right hand side of (73) for all allowed values of \( g \) and \( \rho \)

\[
1 + \sum_{\mu \neq 0 \in \mu} \prod_{\mu \neq 0 \in \mu} \left( \frac{q y^\mu (1-y)^{\mu_1} \cdots (1-y)^{\mu_r}}{(1-y)^{\mu_1} \cdots (1-y)^{\mu_r}} \right)^2 \times (1-y)^{\mu_1} (1-y)^{\mu_2} \cdots (1-y)^{\mu_r} Q^{|\mu|}
\]

\[
= \sum_{\mu \neq 0 \in \mu} \left( y^\mu (1-y)^{\mu_1} \cdots (1-y)^{\mu_r} \right)^2 \times (1-y)^{\mu_1} (1-y)^{\mu_2} \cdots (1-y)^{\mu_r} Q^{|\mu|}
\]

where \( \kappa(\mu) = \sum_{\square \in \mu} (i(\square) - j(\square)) \).

For \( y = 1 \) it was indicated that the following identity is crucial for proving the last conjectural equality

\[
\sum_{\square \in \mu} (d(\square) - a(\square)) = \sum_{\square \in \mu} (j(\square) - i(\square))
\]

(75)

4.2 Refined topological strings

The main purpose of this work is to give the generalisations of these conjectural identities for the refined topological string case.

\[
Z^{5d,\;instanton}(q_1, q_2, \rho_1, \rho_2, \gamma, Q, x) = \sum_{k} \prod_{x_i} (1-y)^{\mu_1} (1-y)^{\mu_2} \cdots (1-y)^{\mu_r} Q^{|\mu|}
\]

(76)

To substantiate the conjecture (73) for the refined topological strings in the rank \( r = 1, 2 \) cases we have to find a map between the chemical potentials from the two sides. A natural generalisation for spacetime equivariant parameters is

\[
q_1 = t y^{-1}, \quad q_2 = q^{-1} y^{-1}, \quad Q_2 = T \sqrt{\frac{t}{q}} y^{-1}, \quad Q_1 = \sqrt{\frac{t}{q}} y^{-1}, \quad Q_{f_i} = \rho_i \rho_{f_i}^{-1}
\]

(77)

Making this change of variables conjecturally identifies the 5d Nekrasov partition function with the refined open topological string partition function

\[
Z^{ref,\;open}(q, t, Q_1, Q_2; Q_{f_i}; x) = Z^{5d,\;instanton}(q, Q_2, \rho_2, Q_1; x)
\]

(78)
4.3 Lagrangian branes along the un-preferred vs preferred direction: Schur polynomials vs Macdonald polynomials

The Schur polynomials and the Macdonald polynomials are two of the symmetric functions bases in which we can expand the refined open topological string partition functions. The choice of the Schur polynomials corresponds [23] to the lagrangian branes present on the unpreferred leg of the toric diagram, whereas the Macdonald polynomials basis corresponds to the lagrangian brane on the preferred leg of the toric diagram. Interestingly the modified Macdonald polynomials can be expanded in terms of both the Schur functions and the Macdonald polynomials as follows [6,12,12]

\[
\tilde{H}_\mu(x; q, t) = \sum \tilde{K}_{\lambda\mu}(q, t) s_\lambda(x)
\]

\[
\tilde{H}_\mu(x; q, t) = t^{-\sum_{\mu(i)}(\mu(i)-1)} J_\mu \left[ \frac{x}{1-t^{-1}} ; q, t^{-1} \right]
= t^{-\sum_{\mu(i)}(\mu(i)-1)} \prod_{s \in D(\lambda)} (1 - q^{a(s)} t^{l(s)+1}) P_\lambda(x; q, t)
\]

where \( \tilde{K}_{\lambda\mu}(q, t) \) are the modified Kostka coefficients and have interesting combinatorial properties, \( a(s) \) and \( l(s) \) denote the arm length and the leg length of the square \( s \) respectively, \( J_\mu(x; q, t) \) is defined as the integral form of the modified Macdonald polynomials

\[
J_\mu(\frac{x}{1-t^{-1}} ; q, t^{-1}) := \prod_{s \in D(\lambda)} (1 - q^{a(s)} t^{l(s)+1}) P_\lambda(x; q, t)
\]

and

\[
\tilde{K}_{\lambda\mu}(q, t) = \frac{x}{1-t^{-1}} ; q, t^{-1}
\]

A crucial point is that the choice of the preferred or unpreferred direction for the placement of the lagrangian branes corresponds to the expansion of the modified Macdonald polynomials in the Macdonald polynomials basis or the Schur functions basis.

5 Elliptic genus: a speculation for the refined open string invariants of special lagrangian branes for fully compactified web

In the same vein as the purported equality of the Donaldson–Thomas partition function of the CY3-fold and the K-theory partition function of the framed quiver moduli space given in the last section, we give an expression for the generating function of the elliptic genus of the same framed moduli space of section 2 and propose that

The Donaldson–Thomas partition functions of the CY3 – folds (Fig. 10)

\[
= \text{The generating function of the elliptic genus given in (87, 88)}
\]

In the presence of vector bundles on quiver moduli space \( \mathcal{N}(r, k + d, d) \), the natural generalisation of the \( \chi_y \)-genus is the so-called elliptic genus. The elliptic genus contains topological information about the vector bundles and can be arranged as a generating series of cohomology groups of the vector bundles. To define it, consider a vector bundle \( V \) on \( X \) and define the formal product

\[
E(V) = \oplus_{n \geq 1} (\Lambda_{-y^n} V^\vee \otimes \Lambda_{-y^{-1}} V^n \otimes S_q^n(V \oplus V^\vee))
\]

(83)

It is interesting to note that this formal product is the vector bundle analogue of the Jacobi triple product formula[10,11]. The elliptic genus \( \chi_{\text{elliptic}}(X, V; y, q) \) is defined by its relation to the chi-y genus as

\[
\chi_{\text{elliptic}}(X, V; y, q) := y^{-\text{rk}TX/2} \chi_{\text{y}}(X, E(T_X) \otimes V)
\]

(84)

Then using Riemann-Roch theorem the elliptic genus \( \chi_{\text{elliptic}}(X, V; y, q) \) can be expressed by

\[
\chi_{\text{elliptic}}(X, V; y, q) = \int_X y^{-\text{rk}TX/2} \text{ch}(\Lambda_{-y} T_X^\vee) \text{ch}(E(T_X)) \text{td}(T_X) \text{ch}(V)
= \int_X \left( \sum e^{\text{ch}(E(T_X))} \prod_{i=1}^{n} \theta(a_i T_X, \theta(t_i)) \right)
\times \prod_{i=1}^{m} \theta(\frac{a_i T_X}{\theta(t_i)}, \theta(t_i))
\]

(85)

where \( q = e^{2\pi i \tau}, y = e^{2\pi i m} \) and \( \theta(\tau, \tau) = q^{1/8} y^{1/2-y^{-1/2}} \prod_{i=1}^{n} (1-q^i)(1-q^{-i}y)(1-q^{-1}y^{-1}) \). For the moduli space \( \mathcal{N}(r, n + d, d) \) one has to generalise the above definitions to include virtual schemes allowing an equivariant torus action and use equivariant localisation. We follow the fixed point formulas given in these references to write down the final expressions for \( \chi_y \)-genus and elliptic genus of \( \mathcal{N}(r, n + d, d) \). Note that \( T_X \) for \( X = \mathcal{N}(r, n + d, d) \) is given in (11). For a quick review of the fixed point formulae see appendix (1).
The computation of the refined open topological string amplitude corresponding to the Euler characteristic and \( \chi_r \) genus was relatively simple in the sense that the Lagrangian brane was put on the external leg of the toric diagram. So it was a topological amplitude in the presence of the external branes. In the case of a totally compactified web diagram, all the legs are essentially internal. The instanton partition function of the six dimensional theory with surface defect can be interpreted as the generating function of the elliptic genus

\[
Z_{r,d}^{\text{ref, quiver}} = \sum_k Q_\rho^k \chi_{\text{ell}}(\mathcal{N}(r, k + d, d), q_1, q_2, y, \rho_\sigma, Q_\sigma)
\]

(86)

For the values of \( r \geq 2 \) the compactified web diagram is given in the Fig. 10, [18]. This figure is the web diagram of a CY3-fold described as a resolved \( A_{r-1} \) fibration over \( \mathbb{P}^1 \).

We can also see the diagram as obtained by gluing \( r - 1 \) Hirzebruch surfaces. The local geometry of the intersection between \( i \)-th and \( i \)-th Hirzebruch surfaces is given by the bundle \( \mathcal{O}(-r + 2i) \oplus \mathcal{O}(r - 2i) \rightarrow \mathbb{P}^1 \) for \( i = 1, 2, \ldots, r \).

The corresponding elliptic genus for the quiver moduli \( \mathcal{N}(r, k + d, d) \) for \( r \geq 2 \) is given by
**Fig. 10** Fully compactified web diagram of the total space of $A_{r-1}$ fibration over $\mathbb{P}^1$
6 Conclusions

We gave expressions for the $\chi_y$ genus and elliptic genus for a quiver moduli space $N(r, n + d, d)$ described by stable representations of an enhanced ADHM quiver of type $(n+d,d,r)$. Then a conjecture is formulated that equates generating function of $\chi_y$ genus for $r = 1$ and $r = 2$ with the open topological string partition functions on CY 3-folds given by partially compactified resolved conifold and partially compactified total space of the bundle $O(-2, -2)$ of $\mathbb{P}^1 \times \mathbb{P}^1$. Finally it is suggested that the generating function of the elliptic genus may be related to the open topological string partition function on these CY 3-folds whose corresponding web diagrams are fully compactified. To discuss the conjectures for $SU(3)$ and higher rank 5d mass deformed gauge theories, it turns out to be necessary to deal with what are called shifted web diagrams [3]. This is a work in progress.

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Appendix A: $Z_{\text{open}}^\text{ref}(Q_f, Q_m, q; t; x)$ on the partially compactified geometry of section (3.4): Alternate expression

In Sect. 3.4 we computed the topological string partition function on the flop of this geometry and then analytically continued the partition function to the pre-flopped geometry. In this appendix we compute the partition function using the refined topological vertex without flopping the geometry. For this case the partially compactified toric diagram of the total space of the bundle $O(-2, -2)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ is given below.

The building block for $r = 2$ corresponding locally to $O(-2) \oplus O(0) \rightarrow \mathbb{P}^1$ is given by

\[
Z_{\text{open}}^\text{ref}(Q_f, Q_m, q; t; x) = \sum (-Q_m)^{|\mu|} (-Q_f)^{|\nu|} \tilde{Z}_{V_2}(q, t) \tilde{f}_{V_2}(q, t) \tilde{Z}_{V_1}(q, t) \tilde{f}_{V_1}(q, t)
\]

with framing factor $f_{\mu}(t, q)$ and refined topological vertex $C_{\mu \nu}(t, q)$ given by

\[
f_{\mu}(t, q) = (-1)^{|\mu|} t \frac{|\mu|}{|\mu|} q^{\frac{|\mu|}{2}} q^{-|\mu|/2} \tilde{Z}_{V_1}(q, t) \sum_{\eta} \left( \frac{q \eta}{t} \frac{|\mu| - |\eta|}{2} \right)^{\frac{|\mu|}{2}}
\]

\[
C_{\mu \nu}(t, q) = \left( \frac{t}{q} \right)^{|\mu|} q^{\frac{|\mu|}{2}} \tilde{Z}_{V_1}(q, t) \sum_{\eta} \left( \frac{q \eta}{t} \frac{|\mu| - |\eta|}{2} \right)^{\frac{|\mu|}{2}}
\]

The gluing of the local geometries $O(-2) \oplus O(0) \rightarrow \mathbb{P}^1$, $O(0) \oplus O(-2) \rightarrow \mathbb{P}^1$ taking into account the normal-to-the-base directions as well as the compactification of the external leg without brace corresponds to the following amplitude

\[
Z_{\text{open}}^\text{ref}(Q_f, Q_b, Q_m, q; t; x) = \sum (-Q_m)^{|\mu|} (-Q_b)^{|\nu|} \tilde{Z}_{V_2}(q, t) \tilde{f}_{V_2}(q, t) \tilde{Z}_{V_1}(q, t) \tilde{f}_{V_1}(q, t) \tilde{Z}_{V_1}(q, t) \tilde{f}_{V_1}(q, t)
\]

Using the following skew Schur function identities repeatedly

\[
\sum_{\lambda} s_{\lambda}(y) s_{\lambda}(x) = \prod_{i,j} (1 - x_i y_j)^{-1} \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y)
\]

we get the following expression

\[
Z_{\text{open}}^\text{ref}(Q_f, Q_b, Q_m, q; t; x) = \sum (-Q_b)^{|\nu|} (-Q_m)^{|\mu|} \left( \frac{q^{1/2} + q^{-1/2}}{2} \right)^{|\mu|} \left( \frac{q^{1/2} + q^{-1/2}}{2} \right)^{|\nu|} \tilde{Z}_{V_2}(q, t) \tilde{f}_{V_2}(q, t) \sum_{\eta} \left( \frac{q \eta}{t} \frac{|\mu| - |\eta|}{2} \right)^{\frac{|\mu|}{2}}
\]

\[
\times \left( \frac{t}{q} \right)^{|\mu|} q^{\frac{|\mu|}{2}} \tilde{Z}_{V_1}(q, t) \sum_{\eta} \left( \frac{q \eta}{t} \frac{|\mu| - |\eta|}{2} \right)^{\frac{|\mu|}{2}}
\]

\[
\times \left( \frac{t}{q} \right)^{|\mu|} q^{\frac{|\mu|}{2}} \tilde{Z}_{V_1}(q, t) \sum_{\eta} \left( \frac{q \eta}{t} \frac{|\mu| - |\eta|}{2} \right)^{\frac{|\mu|}{2}}
\]

\[
\times \left( \frac{t}{q} \right)^{|\mu|} q^{\frac{|\mu|}{2}} \tilde{Z}_{V_1}(q, t) \sum_{\eta} \left( \frac{q \eta}{t} \frac{|\mu| - |\eta|}{2} \right)^{\frac{|\mu|}{2}}
\]

\[
\times \left( \frac{t}{q} \right)^{|\mu|} q^{\frac{|\mu|}{2}} \tilde{Z}_{V_1}(q, t) \sum_{\eta} \left( \frac{q \eta}{t} \frac{|\mu| - |\eta|}{2} \right)^{\frac{|\mu|}{2}}
\]

\[
\times \left( \frac{t}{q} \right)^{|\mu|} q^{\frac{|\mu|}{2}} \tilde{Z}_{V_1}(q, t) \sum_{\eta} \left( \frac{q \eta}{t} \frac{|\mu| - |\eta|}{2} \right)^{\frac{|\mu|}{2}}
\]
is given by

\[
\nu(t, s) \prod_{i,j} \left[ \left( 1 + Q \frac{Q_{1,\mu_1}}{q} e^{-\nu_1 t - \nu_2 s} x_{i,j} \right)^{-1} \right]^{-1} \prod_{i,j} \left[ \left( 1 + Q \frac{Q_{2,\mu_2}}{q} e^{-\nu_1 t - \nu_2 s} x_{i,j} \right)^{-1} \right]^{-1}
\]

(93)

The gauge theory instanton part of the partition function that is relevant for the comparison with quiver partition function is given by

\[
Z_{\text{open}}^{\text{ref, v}}(Q_f, Q_b, Q_m, q, t; x) = \sum_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6} (-Q_{1,\mu_1} (-Q_{2,\mu_2} (-Q_{3,\mu_3} (-Q_{4,\mu_4} (-Q_{5,\mu_5} (-Q_{6,\mu_6}))))
\]

(94)

Appendix B: \(Z_{\text{open}}^{\text{ref}}(Q_b, Q_f, Q_m, q, t; x)\): on the geometry (7): preferred direction vertical

\[
Z_{\text{open, X}_2}^{\text{ref, v}}(Q_1, Q_2, Q_3, Q_4, q, t) = \sum_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6} (-Q_{1,\mu_1} (-Q_{2,\mu_2} (-Q_{3,\mu_3} (-Q_{4,\mu_4} (-Q_{5,\mu_5} (-Q_{6,\mu_6}))))
\]

(95)

Using the refined topological vertex definition (23) and the identities in (52) we get the following expression for the refined open partition function

\[
Z_{\text{open, X}_2}^{\text{ref, v}}(Q_1, Q_2, Q_3, Q_4, q, t) = \sum_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6} (-Q_{1,\mu_1} (-Q_{2,\mu_2} (-Q_{3,\mu_3} (-Q_{4,\mu_4} (-Q_{5,\mu_5} (-Q_{6,\mu_6}))))
\]

(96)

Appendix C: \(Z_{\text{open}}^{\text{ref, v}}(Q_b, Q_f, Q_m, q, t; x)\): on the geometry (7): preferred direction diagonal

\[
Z_{\text{open, X}_2}^{\text{ref, v}}(Q_1, Q_2, Q_3, Q_4, q, t) = \sum_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8} (-Q_{1,\mu_1} (-Q_{2,\mu_2} (-Q_{3,\mu_3} (-Q_{4,\mu_4} (-Q_{5,\mu_5} (-Q_{6,\mu_6}))))
\]

(97)

Using the refined topological vertex definition (23) and the identities in (52) we get the following expression for the refined open partition function

\[
Z_{\text{open, X}_2}^{\text{ref, v}}(Q_1, Q_2, Q_3, Q_4, q, t) = \sum_{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8} (-Q_{1,\mu_1} (-Q_{2,\mu_2} (-Q_{3,\mu_3} (-Q_{4,\mu_4} (-Q_{5,\mu_5} (-Q_{6,\mu_6}))))
\]

(98)
Appendix D: Fixed point formulae for the virtual Euler characteristics, the virtual $\chi_y$ genus and the virtual elliptic genus

In this appendix we summarise the treatment of the virtual localisation as discussed in [9]. Consider a scheme $X$ that is equivariantly embedded into a nonsingular scheme $Y$ globally. There is a $\mathbb{C}^\ast$ action on the later. Let $Y^f$ denote the nonsingular fixed point locus in $Y$ and let $X^f$ be defined by $X^f = X \cap Y^f$. Moreover $Y^f$ can be written as a union of irreducible components as $= \cup_i Y_i$ and correspondingly $X^f = \cup_i X_i$. We will denote by $[X_i]$ the virtual fundamental class, by $\mathcal{O}_{X}^{vir}$ the virtual structure sheaf and by $N_{vir}$ the virtual normal bundle of $X_i$. For the vector bundle $V$ defined on $X^f$ the virtual pushforward $p_{\ast}^{vir}(\cdot)$ is related to the pushforward $p_{\ast}(\cdot)$ by

$$ p_{\ast}^{vir}(V) = p_{\ast}(ch(V).td(T_{X}^{vir}) \cap [X]^{vir}) \quad (99) $$

Consider an equivariant lift of $V$ denoted by $\tilde{V}$. We consider the restriction of $\tilde{V}$ to $X_i$ denoted by $\tilde{V}_i$ and a projection $p_i : X_i \to pt$. Here and below we denote by ch the equivariant Chern character, by td the equivariant Todd genus, by $\sum_{i=0}^{\infty} (-1)^i \Lambda_i B$ the action of $\Lambda_{-1}$ on the vector bundle $B$ and by Eu the equivariant Euler class. The virtual Riemann–Roch theorem gives

$$ \chi^{vir}(X, V) = \int_{[X]^{vir}} ch(V).td(T_{X}^{vir}) $$

$$ \quad = p_{\ast}(ch(\tilde{V}).td(T_{X}^{vir}) \cap [X]^{vir})|_{\epsilon=0} \quad (100) $$

where the parameter $\epsilon$ is related to the equivariant lift $\tilde{V}$. Then by applying the localisation formula we get

$$ p_{\ast}(ch(\tilde{V}).td(T_{X}^{vir}) \cap [X]^{vir}) $$

$$ \quad = \sum_i p_{\ast}(ch(\tilde{V}_i).td(T_{X_i}^{vir}) \cap [X_i]^{vir}) $$

$$ \quad = \sum_i p_{\ast}(td(T_{X_i}^{vir}).ch(\tilde{V}_i/\Lambda_{-1}(N_{vir})^{\gamma}) \cap [X_i]^{vir}) \quad (101) $$

Using the following identities

$$ td(T_{X_i}^{vir} | X_i) = td(T_{X_i}^{vir})td(N_{vir}^{\gamma}) $$

$$ td(N_{vir}^{\gamma}) = Eu(N_{vir}^{\gamma}/\Lambda_{-1}(N_{vir})^{\gamma}) \quad (102) $$

and the eq.(99) we get

$$ p_{\ast}(ch(\tilde{V}).td(T_{X}^{vir}) \cap [X]^{vir}) = \sum_i p_{\ast}^{vir}(\tilde{V}_i/\Lambda_{-1}(N_{vir})^{\gamma}) \quad (103) $$

By definition

$$ \chi^{vir}(X, \tilde{V}, \epsilon) := \sum_i p_{\ast}^{vir}(\tilde{V}_i/\Lambda_{-1}(N_{vir})^{\gamma}) \quad (104) $$

Moreover the eq.(103) implies that $\chi_{\ast}^{vir}(X, V, \epsilon) \in \mathcal{O}[\epsilon]$ and $\chi_{\ast}^{vir}(X, V, \epsilon) = \chi_{\ast}^{vir}(X, \tilde{V}, 0)$. Similar manipulations lead to the fixed point formulae for the $\chi$-y genus and the $\text{elliptic}$ genus.

$$ \chi_{\ast}^{vir}(X, V) = (\sum_i \chi_{\ast}^{vir}(X_i, \tilde{V}_i \otimes \Lambda_{-1}(N_{vir})^{\gamma}/\Lambda_{-1}(N_{vir})^{\gamma})|_{\epsilon=0} $$

(105)

If we define $n_i = rank(N_{vir}^{\gamma})$ then

$$ E\text{ll}_{\ast}^{vir}(X, z, \tau) $$

$$ = (\sum_i y^{n_i/2}E\text{ll}_{\ast}^{vir}(X_i, E(N_{vir})\Lambda_{-1}(N_{vir})^{\gamma}/\Lambda_{-1}(N_{vir})^{\gamma}, z, \tau)|_{\epsilon=0} $$

(106)

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