Brane World Dynamics and Conformal Bulk Fields

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Abstract

In the Randall-Sundrum scenario we investigate the dynamics of a spherically symmetric 3-brane world when matter fields are present in the bulk. To analyze the 5-dimensional Einstein equations we employ a global conformal transformation whose factor characterizes the $Z_2$ symmetric warp. We find a new set of exact dynamical collapse solutions which localize gravity in the vicinity of the brane for a stress-energy tensor of conformal weight -4 and a warp factor that depends only on the coordinate of the fifth dimension. Geometries which describe the dynamics of inhomogeneous dust and generalized dark radiation on the brane are shown to belong to this set. The conditions for singular or globally regular behavior and the static marginally bound limits are discussed for these examples. Also explicitly demonstrated is complete consistency with the effective point of view of a 4-dimensional observer who is confined to the brane and makes the same assumptions about the bulk degrees of freedom.

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1 Introduction

Matter fields may be confined to a 3-brane world embedded in a higher dimensional space if gravity propagates away from the brane in the extra dimensions [1, 2]. Remarkably, in this context it is possible to reformulate the hierarchy problem following two alternative paths which admit a fundamental Planck scale in the TeV range. In one approach the extra dimensions need to be compactified to a large finite volume [3]. In the other, the so-called RS1 model, one considers two branes, one of which (the observable brane) must have a negative tension. An exponential hierarchy is generated by a “warp” factor that characterizes the background metric [4-6]. An

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alternative setup involving infinite, non-compact extra dimensions and only positive
tension branes was later proposed in [7].

It is also possible to consider a single brane together with infinite and non-compact
extra dimensions (the so-called RS2 model). The warp localizes gravity in the vicinity
of the brane, allowing for the recovery of 4-dimensional Einstein gravity at low energy
scales [8]-[12]. In this framework the observable universe is a boundary hypersurface
of an infinitely extended $Z_2$ symmetric 5-dimensional anti-de Sitter (AdS) space.
The classical dynamics are defined by the Einstein equations with a negative bulk
cosmological constant $\Lambda_B$, a Dirac delta source representing the brane and a stress-
energy tensor describing other bulk field modes which may exist in the whole AdS
space.

A set of vacuum solutions is given by the metric [9]

$$ ds_5^2 = \frac{l^2}{(|z - z_0| + z_0)^2} (dz^2 + ds_4^2), $$

(1)

where $ds_4^2$ is a 4-dimensional line element generated by a Ricci flat metric. The
parameter $l$ is the AdS radius and is written as $l = 1/\sqrt{\Lambda_B \kappa_5^2/6}$ where $\kappa_5^2 = 8\pi/M_5^3$
with $M_5$ the fundamental 5-dimensional Planck mass. The brane is located at $z = z_0$
and is fine-tuned to have zero cosmological constant, giving $\Lambda_B = -\kappa_5^2 \lambda^2/6$ or $l = 6/(\kappa_5^2 \lambda)$ where $\lambda$ denotes the positive brane tension.

The original RS solution [8] belongs to this class and is obtained when the 4-
dimensional subspace has a Minkowski metric. Another example is the black string
solution [9] which induces the Schwarzschild metric on the brane $^1$. However, the
Kretschmann scalar diverges both at the AdS horizon and at the black string singular-
arity [9] leading to a violation of the cosmic censorship conjecture [13]. This solution
is expected to be unstable near the AdS horizon [14, 15] and is therefore not an
acceptable description of a black hole on the brane. It may decay to a black cylin-
der localized near the brane which is free from naked singularities [9] but, despite
considerable effort, this still remains a conjecture.

While exact solutions interpreted as static black holes localized on a brane have
been found for a 2-brane embedded in a 4-dimensional AdS space [16], a static black
hole localized on a 3-brane is yet to be discovered [17]-[27]. The difficulty in finding a
static solution has led to an interesting conjecture [25] which attempts to relate black
hole solutions localized on the brane in an AdS$_{D+1}$ braneworld, which are found with
the brane boundary conditions, with quantum black holes in $D$ dimensions rather
than classical ones.

Many other 5-dimensional solutions have been determined within the RS brane
world scenario. These include the extensions of the RS geometry to thick branes
[28] and non-fine-tuned branes [29], the 5-dimensional metrics describing Friedmann-
Robertson-Walker (FRW) cosmologies [30, 31] and also other solutions involving the

$^1$This solution was first discussed in a different context in R. C. Myers and M. J. Perry, Ann.
Phys. 172, 304 (1986).
effect of scalar fields in the bulk [32]-[34]. The application of the covariant Gauss-Codazzi (GC) approach [35]-[37] describing the perspective of a 4-dimensional observer restricted to the brane, has permitted the analysis of even more brane world physics [38]-[46]. In addition, several connections have been established between the AdS/CFT correspondence [47] and the RS brane world scenario [11, 25, 48].

In spite of the numerous 5-dimensional exact solutions already found in the RS scenario there are still many effective 4-dimensional metrics which have not been shown to be associated with exact bulk spacetimes. This happens for instance with the inhomogeneous dust and dark radiation dynamics on the brane. While the latter has been deduced within the RS scenario using the covariant GC formulation [44], the former has not yet been shown to have a brane world description. It is thus the purpose of the present work to search for new exact 5-dimensional solutions which might reproduce such effective 4-dimensional brane world dynamics. To attain this objective we consider the generally inhomogeneous dynamics of a spherically symmetric RS 3-brane when field modes other than the Einstein-Hilbert gravity are present in the whole AdS space. Such bulk fields and their influence on the brane world dynamics have been extensively discussed within the RS scenario for example to stabilize the extra dimensions [5, 6, 32, 33] and to analyze the possibility of generating a black hole localized on a 3-brane world [18, 22].

We begin in section 2 with an analysis of the Einstein field equations for the most general 5-dimensional metric consistent with the $Z_2$ symmetry and with spherical symmetry on the brane. We organize the Einstein equations using a global conformal transformation whose factor characterizes the warp of the fifth dimension. If the stress-energy tensor is assumed to have conformal weight $s = -4$, then it is possible to decouple the gravitational dynamics of the bulk matter fields from the 5-dimensional warp. The bulk matter dynamics generate a non-zero pressure along the fifth dimension, which must satisfy a precise equation of state. Thus we obtain a new set of exact dynamical solutions for which gravity is localized near the brane by the conformal warp factor.

In sections 3 and 4 we analyze two examples which belong to this new set of exact 5-dimensional solutions. Both examples permit the use of synchronous coordinates. From the point of view of an observer confined to the brane, they correspond to bulk matter behaving on the brane as pressureless dust and a generalized form of dark radiation. We determine the static marginally bound limits and discuss the conditions for singular or globally regular behavior.

As a consistency check, we consider the point of view of an observer confined to the brane in section 5. Using the effective GC formulation we show that if such an observer uses the same description of the field variables, then it indeed leads to identical brane world dynamics. Explicit proofs are provided for dust and dark radiation. We conclude in section 6.
2 5-Dimensional Einstein Equations and Conformal Transformations

Let \((t, r, \theta, \phi, z)\) be a set of comoving coordinates in the 5-dimensional bulk. The most general metric consistent with the \(Z_2\) symmetry in \(z\) and with 4-dimensional spherical symmetry on the brane may be written using a global conformal transformation \([49]\) defined by a \(Z_2\) symmetric warp function \(\Omega = \Omega(t, r, z)\),

\[
\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \tag{2}
\]

The corresponding 5-dimensional line elements are related by

\[
ds_5^2 = \Omega^2 ds_5^2, \tag{3}\]

where

\[
ds_5^2 = dz^2 + ds_4^2 \tag{4}\]

has a 4-dimensional line element which depends on three \(Z_2\) symmetric functions \(A = A(t, r, z), B = B(t, r, z)\) and \(R = R(t, r, z)\) as follows

\[
\tilde{ds}_4^2 = -e^{2A} dt^2 + e^{2B} dr^2 + R^2 d\Omega_2^2. \tag{5}\]

\(R(t, r, z)\) represents the physical radius of the 2-spheres.

When bulk field modes other than the Einstein-Hilbert gravity are present in the 5-dimensional space the dynamical RS action corresponding to \(\tilde{g}_{\mu\nu}\) is given by

\[
\tilde{S} = \int d^4x dz \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2\kappa_5^2} - \Lambda_B - \frac{\lambda}{\sqrt{g_{55}}} \delta(z - z_0) + \tilde{\mathcal{L}}_B \right]. \tag{6}\]

The brane is assumed to be located at \(z = z_0\) and is a fixed point of the \(Z_2\) symmetry of the manifold. The contribution of the bulk fields is defined by the lagrangian \(\tilde{\mathcal{L}}_B\). A Noether variation on the action \([6]\) gives the classical Einstein field equations,

\[
\tilde{G}_\mu^\nu = -\kappa_5^2 \left[ \Lambda_B \delta_\mu^\nu + \lambda \delta(z - z_0) \tilde{g}_\mu^\nu - \tilde{T}_\mu^\nu \right], \tag{7}\]

where the induced metric on the brane is

\[
\tilde{g}_\mu^\nu = \frac{1}{\sqrt{g_{55}}} \left( \delta_\mu^\nu - \delta_5^\nu \delta_5^\mu \right) \tag{8}\]

and the stress-energy tensor associated with the bulk fields is defined by

\[
\tilde{T}_\mu^\nu = \tilde{\mathcal{L}}_B \delta_\mu^\nu - 2 \frac{\delta \tilde{\mathcal{L}}_B}{\delta g^{\mu\alpha}} \tilde{g}^{\alpha\nu} \tag{9}\]

and is conserved in the bulk,

\[
\tilde{\nabla}_\mu \tilde{T}_\mu^\nu = 0. \tag{10}\]
The Einstein field equations (7) are extremely complex when the metric $\tilde{g}_{\mu\nu}$ is considered in its full generality. To be able to solve them we need to introduce simplifying assumptions about the field variables involved in the problem. Let us first assume that under the conformal transformation (2) the bulk stress-energy tensor has conformal weight\(^2\) $s$ \([49]\)

$$\tilde{T}^{\mu\nu} = \Omega^s T^{\mu\nu}. \hspace{1cm} (11)$$

Then substituting Eq. (11) and using the known transformation properties of the other tensors \([49]\) we re-write Eq. (7) as

$$G^{\nu}_{\mu} = -6\Omega^{-2} (\nabla_\mu \Omega) g^{\nu\rho} \nabla_\rho \Omega + 3\Omega^{-1} g^{\nu\rho} \nabla_\rho \nabla_\mu \Omega - 3\Omega^{-1} \delta^\nu_\mu g^{\rho\sigma} \nabla_\rho \nabla_\sigma \Omega - \kappa_5^2 \left[ \Lambda_B \delta^\nu_\mu + \lambda \Omega^{-1} \delta(z - z_0) \gamma^\nu_\mu - \Omega^{s+2} T^{\nu}_{\mu} \right]. \hspace{1cm} (12)$$

Similarly, Eq. (10) also transforms under the conformal transformation. We have

$$\nabla_\mu T^{\mu}_{\nu} + \Omega^{-1} [(s + 7) T^{\mu}_{\nu} \partial_\mu \Omega - T \partial_\nu \Omega] = 0, \hspace{1cm} (13)$$

where $T = T^{\mu}_{\mu}$ is the trace of the bulk stress-energy tensor. If $\tilde{T}^{\nu}_{\mu}$ has conformal weight $s = -4$, it is possible (though not necessary) to separate Eq. (12) as follows

$$G^{\nu}_{\mu} = \kappa_5^2 T^{\nu}_{\mu}, \hspace{1cm} (14)$$

$$6\Omega^{-2} \nabla_\mu \Omega \nabla_\nu \Omega g^{\rho\omega} - 3\Omega^{-1} \nabla_\mu \nabla_\nu \Omega g^{\rho\omega} + 3\Omega^{-1} \nabla_\rho \nabla_\sigma \Omega g^{\rho\sigma} \delta^\nu_\mu = -\kappa_5^2 \Omega^2 \left[ \Lambda_B \delta^\nu_\mu + \lambda \Omega^{-1} \delta(z - z_0) \gamma^\nu_\mu \right]. \hspace{1cm} (15)$$

In Eq. (14) we find the 5-dimensional Einstein equations with fields present in the bulk when the brane and the bulk cosmological constant are absent. It does not depend on the conformal warp factor which is dynamically defined by Eq. (15) and is then the only effect reflecting the existence of the brane or of the bulk cosmological constant in this setting. We emphasize that this is only possible for the special class of bulk fields which have a stress-energy tensor with conformal weight $s = -4$. The decomposition of (7) according to (14) and (15) implies a separation of Eq. (13) because of the Bianchi identity. We must have

$$\nabla_\mu T^{\nu}_{\mu} = 0, \hspace{1cm} (16)$$

$$3T^{\nu}_{\mu} \partial_\mu \Omega - T \partial_\nu \Omega = 0. \hspace{1cm} (17)$$

Note that now $T^{\nu}_{\mu}$ is a conserved tensor field which must satisfy the additional warp constraint equations (17). Since $s \neq -7$ it does not need to be traceless.

Let us now consider Eq. (14). Expanding the Einstein tensor in terms of the metric functions $A$, $B$ and $R$ reveals that its only non-zero off-diagonal elements are

\(^2\)Our assignment of conformal weight depends upon the index position.
\[ G^r_i, G^z_i \text{ and } G^z_r. \] Assuming that \( A = A(t, r), B = B(t, r) \) and \( R = R(t, r) \) sets \( G^z_i, G^z_r \) and the corresponding stress-energy tensor components to zero. Hence, we have

\[ T^z_a = 0, \tag{18} \]

where the latin index represents the 4-dimensional brane coordinates \( t, r, \theta \text{ and } \phi. \) If in addition \( \Omega = \Omega(z) \) then Eq. (15) turns out to be independent of the metric functions \( A, B, R \) and reads

\[ 6\Omega^{-2}(\partial_z \Omega)^2 = -\kappa_5^2\Omega^2\Lambda_B, \]
\[ 3\Omega^{-1}\partial^2_z \Omega = -\kappa_5^2\Omega^2 \left[ \Lambda_B + \lambda \Omega^{-1}\delta(z - z_0) \right]. \tag{19} \]

For definiteness, take as solution of Eq. (19) the RS conformal warp factor

\[ \Omega = \Omega_{RS} \equiv \frac{l}{|z - z_0| + z_0} \tag{20} \]

where \( l = \sqrt{-6/(\Lambda_B\kappa_5^2)} \) and

\[ \Lambda_B + \frac{\kappa_5^2\lambda^2}{6} = 0. \tag{21} \]

Of course, other solutions with warp factors which depend only on the 5-dimensional coordinate \( z \) such as those corresponding to non-fine-tuned branes [29] or thick branes [28] may also be considered (see [22]). Eq. (17) constrains \( T^{\nu}_{\mu} \) to satisfy the equation of state

\[ 2T^z_z = T^c_c. \tag{22} \]

If we consider a diagonal stress-tensor,

\[ T^{\nu}_{\mu} = \text{diag} (-\rho, p_r, p_T, p_T, p_z), \tag{23} \]

where \( \rho, p_r, p_T \) and \( p_z \) denote the bulk matter density and pressures, then Eq. (22) is re-written as

\[ \rho - p_r - 2p_T + 2p_z = 0. \tag{24} \]

Because the metric functions \( A, B \) and \( R \) do not depend on \( z \) Eq. (18) applied to Eq. (16) leads to \( \partial_z p_z = 0 \) and to

\[ \nabla_a T^a_b = 0. \tag{25} \]

Therefore \( \rho, p_r \) and \( p_T \) must also be independent of \( z \). Matter is, however, inhomogeneously distributed along the fifth dimension. The physical energy density, \( \tilde{\rho}(t, r, z) \), and pressures, \( \tilde{p}(t, r, z) \), are related to their counterparts, \( \rho(t, r) \) and \( p(t, r) \), by the scale factor \( \Omega^{-2}(z). \)

Since all the off-diagonal components are zero, Eq. (14) also admits a similar dimensional reduction

\[ G^b_a = \kappa_5^2 T^b_a \]
but now $G_z^z$ is in general non-zero because of the existing pressure $p_z$,

$$G_z^z = \kappa_5^2 p_z. \quad (27)$$

The collapse of the conformal bulk matter is inhomogeneous and defined by Eqs. (25) and (26). As we show in section 5, gravity is localized in the vicinity of the brane. The matter dynamics generates a pressure $p_z$ along the fifth dimension, which must consistently be given by Eqs. (24) and (27). There are no further constraints.

### 3 Dust Dynamics on the Brane

The simplest kind of matter which we may consider is characterized by the equation of state $p_r = p_T = 0$. According to Eq. (24), its density $\rho = \rho_\text{D}$ must generate a pressure $p_z$ along the fifth dimension which satisfies $p_z = -\rho_\text{D}/2$. One can mimic an effective brane cosmological constant, if instead we take

$$\rho = \rho_\text{D} + \frac{\Lambda}{\kappa_5^2}, \quad p_r = p_T = -\frac{\Lambda}{\kappa_5^2}, \quad (28)$$

giving

$$p_z = -\frac{1}{2} \left( \rho_\text{D} + 4 \frac{\Lambda}{\kappa_5^2} \right). \quad (29)$$

Although we will shortly see that $\Lambda$ mimics a cosmological constant on the brane, it should be emphasized that $\Lambda$ is a bulk quantity, distinct from a 4-dimensional cosmological constant. Since the warp factor has already been chosen to be the RS solution (20), in order to find the 5-dimensional metric describing the dynamics we must solve Eqs. (25), (26) and (27) under conditions (28) and (29).

Using Eq. (28) we write Eq. (25) \[50\] as follows

$$\dot{B}\rho_\text{D} = -\dot{\rho}_\text{D} - 2\frac{\dot{R}}{R}\rho_\text{D}, \quad (30)$$

$$A'\rho_\text{D} = 0, \quad (31)$$

where the dot and the prime denote, respectively, partial differentiation with respect to $t$ and $r$. Because of Eq. (31) we have to take the synchronous frame where $A = 0$ to avoid setting $\rho_\text{D}$ to zero. Then the off-diagonal equation is given by

$$G_r^t = \frac{2}{R} \left( \dot{R}' - \dot{B}R' \right) = 0 \quad (32)$$

and has the solution

$$e^B = \frac{R'}{H}, \quad (33)$$
where $H = H(r)$ is an arbitrary positive function of $r$. Introducing Eq. (33) in Eq. (30) we get the required dust density
\[
\rho_D = \frac{2G_N M'}{\kappa^2 R^2 R'},
\]
where $G_N$ is Newton’s gravitational constant and $M = M(r)$ is an arbitrary positive function of $r$ which represents the dust mass inside a shell labelled by $r$. Note that $\rho_D > 0$ is equivalent to $M'/R' > 0$. This implies that the weak, strong and dominant energy conditions [49] are satisfied in 4 and 5 dimensions.

Next, consider the trace equation
\[
-G_t^t + G_r^r + 2G_\theta^\theta = -2\frac{\ddot{R}}{R'} - 4\frac{\dot{R}}{R} = \frac{2G_N M'}{R^2 R'} - 2\Lambda.
\]
Integrating twice we find
\[
\dot{R}^2 = \frac{2G_N M}{R} + \frac{\Lambda}{3}R^2 + f,
\]
where $f \equiv f(r)$ is an arbitrary function of $r$ to be interpreted as the energy inside a shell labelled by $r$. Imposing on the initial hypersurface $t = 0$ the condition $R(0, r) = r$ we obtain
\[
\pm t + \psi = \int \frac{dR}{\sqrt{\frac{2G_N M}{R} + \frac{\Lambda}{3}R^2 + f}},
\]
where the signs $+$ or $-$ refer to expansion or collapse and $\psi = \psi(r)$ is given by the evaluation at $t = 0$ of the integral in the r.h.s. Applying the radial equation
\[
G_r^r = -2\frac{\ddot{R}}{R} + \frac{H^2}{R^2} - \frac{1}{R^2} - \frac{\dot{R}^2}{R^2} = -\Lambda
\]
we obtain $H = \sqrt{1 + f}$ and this restricts $f$ to satisfy $f > -1$. Then it is easy to see that Eq. (27),
\[
G_z^z = -\frac{\ddot{R}}{R'} - 2\frac{\dot{R}}{R} + \frac{f - \dot{R}^2}{R^2} + \frac{(f - \dot{R}^2)'}{RR'} = -\frac{G_N M'}{R^2 R'} - 2\Lambda,
\]
is an identity for all $M$ and $f$.

We have thus obtained the 5-dimensional dust collapse solutions
\[
ds_5^2 = \Omega_{RS}^2 \left( dz^2 + ds_4^2 \right),
\]
where the 4-dimensional metric has the LeMaitre-Tolman-Bondi (LTB) [52, 53] form
\[
ds_4^2 = -dt^2 + \frac{R'^2}{1 + f} dr^2 + R^2 d\Omega_2^2,
\]
with the physical radius satisfying Eq. (37).

The marginally bound models (corresponding to $f = 0$) with constant mass function, $M(r) = \text{const.}$, describe static solutions. Indeed, using the standard transformation from the LTB coordinates $(t, r)$ to the curvature coordinates $(T, R)$,

$$T = t + \int dR \sqrt{\frac{2}{3} \frac{\sqrt{R^4 + 2G_NM R}}{R^3 - R + 2G_NM}}.$$  \hspace{1cm} (42)

we obtain an AdS/dS-Schwarzschild black string solution given by Eq. (40) where

$$ds^2_4 = - \left( 1 - \frac{\frac{2G_NM}{R} - \frac{\Lambda}{3} R^2}{\frac{2G_NM}{R} - \frac{\Lambda}{3} R^2 + 2G_NM} \right) dT^2 + 2 \left( 1 - \frac{\frac{2G_NM}{R} - \frac{\Lambda}{3} R^2}{\frac{2G_NM}{R} - \frac{\Lambda}{3} R^2 + 2G_NM} \right)^{-1} dR^2 + R^2 d\Omega^2_2.$$  \hspace{1cm} (43)

In the vacuum we obtain the original RS static solution \[8\] and not the Schwarzschild black string \[9\]. By Birkhoff’s theorem, there are no other static vacuum solutions.

In general the solutions are dynamical and inhomogeneous. An analysis of the potential $V = \frac{\Lambda}{3} R^3 + fR + 2G_NM$ uncovers a rich set of singular and globally regular solutions \[54\].

4 Generalized Dark Radiation Dynamics on the Brane

Let us now consider the possibility of generating on the brane the localized gravitational interaction between a generalized form of inhomogeneous dark radiation and an effective cosmological constant. This system is defined by conformal bulk matter with the equations of state

$$\rho + p_r = 0, \quad p_T + \eta \rho + \frac{\Lambda}{\kappa^2_5} (1 - \eta) = 0,$$  \hspace{1cm} (44)

where $\eta$ is the parameter characterizing the dark radiation model and $\rho$ is given by

$$\rho = \rho_{DR} + \frac{\Lambda}{\kappa^2_5}.$$  \hspace{1cm} (45)

Applying Eq. (24) we find

$$p_z = - (1 + \eta) \rho - \frac{\Lambda}{\kappa^2_5} (1 - \eta).$$  \hspace{1cm} (46)

As before, $\Lambda$ is a bulk quantity and not a 4-dimensional cosmological constant. Nevertheless it will mimic a 4-dimensional cosmological constant on the brane. Thus only for standard dark radiation \[31\] \[44\] with $\eta = -1$ and $\Lambda = 0$, is the fifth dimensional pressure $p_z$ equal to zero. Furthermore, the trace of the stress-energy tensor is

$$T^\mu_\mu = T^a_a + p_z,$$  \hspace{1cm} (47)
and $T^a_a = 2p_z$, so it only vanishes when $\eta = -1$ and $\Lambda = 0$. This implies that only the standard form of dark radiation may be associated with the traceless projected Weyl tensor and so with the 4-dimensional brane world vacuum in the effective Gauss-Codazzi approach \[44\].

After decoupling the RS warp factor the determination of the 5-dimensional metric for the dark radiation system requires the solution of Eqs. (25), (26) and (27) under conditions (44)-(46). Let us start by introducing Eqs. (44) and (45) in Eq. (25). The contribution of $\Lambda$ cancels out and we find the following dark radiation conservation equations

$$\dot{\rho}_{\text{DR}} + 2(1 - \eta)\frac{\dot{R}}{R} \rho_{\text{DR}} = 0 = \rho_{\text{DR}} + 2(1 - \eta)\frac{R'}{R} \rho_{\text{DR}}.$$ (48)

Note that for this generalized dark radiation system we may also safely take the synchronous frame ($A = 0$). Because of the equation of state $\rho + p_r = 0$ the dark radiation has a density defined independently of $A$ as a consequence of which, despite the existing pressures, it admits a synchronous solution. Note as well that this is independent of the relation between $\rho$ and $p_T$. As a consequence the general equations given in \[50\] simplify to give Eq. (48). The corresponding inhomogeneous density solution is given by

$$\rho_{\text{DR}} = \frac{Q_\eta}{\kappa^2} R^{2\eta - 2},$$ (49)

where the constant $Q_\eta$ is the dark radiation tidal charge. If $Q_\eta > 0$ then the dark radiation density is positive. As a consequence the weak, the dominant and the strong energy conditions in 4 dimensions imply, respectively, $\eta \leq 1$, $|\eta| \leq 1$ and $\eta \leq 0$. If these conditions are imposed in 5 dimensions then we find that they lead, respectively, to $\eta \leq 0$, $-2 \leq \eta \leq 0$ and $\eta \leq -1/3$. Of course if $Q_\eta < 0$ then the dark radiation density is negative and all the energy conditions are violated.

Substituting Eq. (49) in the Einstein trace equation we obtain

$$-G^t_t + G^r_r + 2G^\theta_\theta = -2\frac{\ddot{R}}{R'} - 4\frac{\dot{R}}{R} = -2\eta Q_\eta R^{2\eta - 2} - 2\Lambda.$$ (50)

After two integrations we find

$$\dot{R}^2 = \frac{Q_\eta}{2\eta + 1} R^{2\eta} + \frac{\Lambda}{3} R^2 + f,$$ (51)

where $f \equiv f(r)$ is the arbitrary function of $r$ interpreted as the energy inside a shell labelled by $r$. Imposing the condition $\dot{R}(0, r) = \dot{r}$ on the initial hypersurface $t = 0$ and integrating Eq. (51) we get

$$\pm t + \psi = \int \frac{dR}{\sqrt{\frac{\Lambda}{3} R^2 + f + \frac{Q_\eta}{2\eta + 1} R^{2\eta}}},$$ (52)

10
Note that we have assumed $\eta \neq -1/2$. For $\eta = -1/2$ we obtain

$$\dot{R}^2 = \frac{Q\sqrt{3}}{R} (1 + \ln R) + \frac{\Lambda}{3} R^2 + f,$$

and then

$$\pm t + \psi = \int \frac{dR}{\sqrt{\frac{\Lambda}{3} R^2 + f + \frac{Q}{R} (1 + \ln R)}}$$

Note as well that if the evolution is to be dominated by $\Lambda$ as $R$ goes to infinity then $\eta$ should satisfy $\eta < 1$. This is true when $Q\eta > 0$ and any one of the energy conditions is satisfied. Applying the radial equation

$$G^r_r = -2 \frac{\ddot{R}}{R'} + \frac{H^2}{R^2} - \frac{1}{R^2} - \frac{\dot{R}^2}{R^2} = -Q\eta R^{2\eta - 2} - \Lambda$$

we again obtain $H = \sqrt{1 + f}$ with $f > -1$. Then Eq. (27),

$$G^z_z = -2 \frac{\ddot{R}}{R'} - 2 \frac{\ddot{R}}{R} + \frac{f - \dot{R}^2}{R^2} + \frac{(f - \dot{R}^2)'}{RR'} = -(1 + \eta) Q\eta R^{2\eta - 2} - 2\Lambda = \kappa^2 p_z,$$

is an identity for all functions $R$ and $f$. Thus we conclude that the metric has the RS-LTB form (40) and (41) with the physical radius given by Eq. (52) for $\eta \neq -1/2$ and by Eq. (54) for $\eta = -1/2$.

### 4.1 Static Limits

Of the dynamical dark radiation models the marginally bound correspond to $f = 0$ and are actually static solutions. Consider first $\eta \neq -1/2$. Transforming from the LTB coordinates $(t, r)$ to the curvature coordinates $(T, R)$ defined by

$$T = t + \int R\frac{\sqrt{\frac{\Lambda}{3} R^2 + \frac{Q\eta}{2\eta + 1} R^{2\eta}}}{R^2 - 1 + \frac{Q\eta}{2\eta + 1} R^{2\eta}}$$

we find new black string solutions given by Eq. (40) with

$$ds^2 = -\left(1 - \frac{Q\eta}{2\eta + 1} R^{2\eta} - \frac{\Lambda}{3} R^2\right) dT^2 + \left(1 - \frac{Q\eta}{2\eta + 1} R^{2\eta} - \frac{\Lambda}{3} R^2\right)^{-1} dR^2 + R^2 d\Omega^2.$$

The 4-dimensional solution (58) for $\eta = -1$ is the inhomogeneous static exterior of a collapsing sphere of homogeneous standard dark radiation [41, 42, 44]. When $\Lambda = 0$ it corresponds to the zero mass limit of the tidal Reissner-Nordström black hole on the brane [38]. The horizons covering the physical singularity at $R_s = 0$ are defined by the transcendental equation

$$1 - \frac{Q\eta}{2\eta + 1} R^{2\eta} - \frac{\Lambda}{3} R^2 = 0.$$
For $\Lambda = 0$ and if the other parameters allowed it there may be an horizon located at

\[ R_h = \left( \frac{2\eta + 1}{Q\eta} \right)^{\frac{1}{2\eta}}. \]  

(60)

If $\Lambda \neq 0$ then in general it is not possible to obtain the exact location of the horizons. The two single exceptions are the models corresponding to $\eta = -1$ and $\eta = 1/2$. For $\eta = -1$ we find the standard dark radiation horizons \[41\] \[44\]. For $\eta = 1/2$ the horizons are given by

\[ R_h = \frac{3Q_{1/2}}{4\Lambda} \left( -1 \pm \sqrt{1 + \frac{16\Lambda}{9Q_{1/2}^2}} \right). \]  

(61)

If $\Lambda < 0$ and $Q_{1/2} > 0$ then we have an inner horizon $R_h^-$ and an outer horizon $R_h^+$. The two horizons merge for $Q_{1/2} = 4\sqrt{-\Lambda}/3$ and for $Q_{1/2} < 4\sqrt{-\Lambda}/3$ the singularity at $R_s = 0$ becomes naked. If $\Lambda > 0$ and $Q_{1/2} > 0$ there is a single horizon at $R = R_h^+$ and for $\Lambda > 0$ and $Q_{1/2} < 0$ the horizon is at $R_h^-$. Note that for $\eta = -1$ the dark radiation with a positive tidal charge satisfies all energy conditions. For $\eta = 1/2$ this is no longer true. Indeed, the strong condition is violated in 4 dimensions and in 5 dimensions none of the energy conditions holds.

For $\eta = -1/2$ we proceed analogously to find a black string given by Eq. \[40\] and

\[ ds_4^2 = -\left[ 1 - \frac{Q_{1/2}}{R} (1 + \ln R) - \frac{\Lambda}{3} R^2 \right] dt^2 + \left[ 1 - \frac{Q_{1/2}}{R} (1 + \ln R) - \frac{\Lambda}{3} R^2 \right]^{-1} dR^2 + R^2 d\Omega_2^2. \]  

(62)

4.2 Exact Dynamical Solutions

Let us now consider the non-marginally bound models corresponding to $f \neq 0$. These are the ones which actually lead to dynamical and inhomogeneous evolutions. In general it is not possible to determine the exact solutions of Eqs. \[52\] and \[51\]. The only exceptions are $\eta = -1$ and $\eta = 1/2$. If $\eta = -1$ we have standard dark radiation and the solutions have already been determined in \[44\]. They are inhomogeneous cosmologies characterized by $\Lambda$, $Q$, $\equiv Q$ and $f$. The corresponding rich structure of physical singularities and regular rebounces was also identified in \[44\]. However, note that for the 5-dimensional solutions defined by Eqs. \[40\], \[41\] and \[52\] the gravitational field is always localized near the brane and the inhomogeneous dark radiation dynamics has been generated on the brane by the bulk fields. They are not a pure vacuum phenomena even if $\eta = -1$ and for such a vacuum gravity is not always confined to the vicinity of the brane \[44\].

The dynamics for $\eta = 1/2$ is actually similar to that of standard dark radiation \[44\]. Indeed, the solutions may also be organized by $\Lambda$ and by the functions $Y$ and $\beta$
now defined as
\[ Y = R + \frac{3Q_{1/2}}{4\Lambda}, \quad \beta = \frac{3}{\Lambda} \left( \frac{3Q_{1/2}^2}{16\Lambda} - f \right). \tag{63} \]

To illustrate consider \( \Lambda > 0 \) and allow \( Q_{1/2} \) to be a real parameter as \( Q \). If \( \beta > 0 \) then \(-1 < f < 3Q_{1/2}^2/(16\Lambda) \) and so the solutions are
\[ \left| R + \frac{3Q_{1/2}}{4\Lambda} \right| = \sqrt{\beta} \cosh \left[ \pm \sqrt{\frac{\Lambda}{3}} t + \cosh^{-1} \left( \frac{r + \frac{3Q_{1/2}}{4\Lambda}}{\sqrt{\beta}} \right) \right]. \tag{64} \]

If \( \beta < 0 \) then \( f > 3Q_{1/2}^2/(16\Lambda) \) and we obtain
\[ R + \frac{3Q_{1/2}}{4\Lambda} = \sqrt{-\beta} \sinh \left[ \pm \sqrt{\frac{\Lambda}{3}} t + \sinh^{-1} \left( \frac{r + \frac{3Q_{1/2}}{4\Lambda}}{\sqrt{-\beta}} \right) \right]. \tag{65} \]

If \( \beta = 0 \) then \( f = 3Q_{1/2}^2/(16\Lambda) \) and we get an homogeneous solution
\[ \left| R + \frac{3Q_{1/2}}{4\Lambda} \right| = \left| r + \frac{3Q_{1/2}}{4\Lambda} \right| \exp \left( \pm \sqrt{\frac{\Lambda}{3}} t \right). \tag{66} \]

Clearly, solutions (64) and (65) are intrinsically dependent on \( r \) and so correspond to inhomogeneous cosmologies which cannot be reduced to the standard homogeneous dS or Robertson-Walker spaces by any coordinate transformation.

### 4.3 Singularities and Regular Rebounces

To analyze the space of solutions of the dark radiation models let us first take \( \eta \neq -1/2 \). Then we consider Eq. (51) written as
\[ R^\sigma \dot{R}^2 = V, \tag{67} \]
where the potential \( V \) is
\[ V = V(R,r) = \frac{Q_\eta}{2\eta + 1} R^{2\eta + \sigma} + \frac{\Lambda}{3} R^{2+\sigma} + f R^\sigma, \tag{68} \]
and the parameter \( \sigma \geq 0 \) is only non-zero when \( \eta \) is negative. For example if \( \eta = -1 \) then \( \sigma = -2\eta = 2 \). Again in general it is not possible to study this potential exactly. The same is true for \( \eta = -1/2 \) which involves a logarithm of \( R \). Only for \( \eta = -1 \) and \( \eta = 1/2 \) is such exact analysis possible. For \( \eta = -1 \) a rich structure of singularities and regular rebounces was found and discussed in [44]. A similar structure of solutions may now be shown to exist for \( \eta = 1/2 \). To illustrate consider \( \Lambda > 0 \). The potential is written as
\[ V = \frac{\Lambda}{3} R^2 + \frac{Q_{1/2}}{2} R + f = \frac{\Lambda}{3} \left( Y^2 - \beta \right). \tag{69} \]
Then as for $\eta = -1$ there are at most two regular rebounce epochs and a phase of continuous accelerated expansion to infinity.

For $\beta < 0$ it is clear that $V > 0$ for all values of $R \geq 0$. It satisfies $V(0, r) = f$ with $f > 3Q_{1/2}^2/(16\Lambda)$ and it grows to infinity with $R$ as $\Lambda R^2$. The dark radiation shells may either expand forever or collapse to the singularity after a proper time $t = t_s(r)$ given by

$$t_s(r) = \sqrt{\frac{3}{\Lambda}} \left[ \sinh^{-1} \left( \frac{r + \frac{3Q_{1/2}}{4\Lambda}}{\sqrt{-\beta}} \right) - \sinh^{-1} \left( \frac{3Q_{1/2}}{4\Lambda \sqrt{-\beta}} \right) \right]. \quad (70)$$

For $\beta > 0$ and independently of $Q_{1/2}$ the configuration $-1 < f < 0$ leads to globally regular solutions with a single rebounce epoch at $R = R_s$ where

$$R_s = -\frac{3Q_{1/2}}{4\Lambda} + \sqrt{\beta}. \quad (71)$$

Indeed, this is the minimum radius a collapsing dark radiation shell can have and it is reached after the time $t = t_s(r)$ where

$$t_s(r) = \sqrt{\frac{3}{\Lambda}} \cosh^{-1} \left( \frac{r + \frac{3Q_{1/2}}{4\Lambda}}{\sqrt{\beta}} \right). \quad (72)$$

At this point the shells reverse their motion and expand continuously with ever increasing speed.

If $\beta > 0$ and $0 < f < 3Q_{1/2}^2/(16\Lambda)$ then for $Q_{1/2} > 0$ there are no rebounce points in the allowed dynamical region $R \geq 0$. The time to reach the singularity is now given by

$$t_s(r) = \sqrt{\frac{3}{\Lambda}} \left[ \cosh^{-1} \left( \frac{r + \frac{3Q_{1/2}}{4\Lambda}}{\sqrt{\beta}} \right) - \cosh^{-1} \left( \frac{3Q_{1/2}}{4\Lambda \sqrt{\beta}} \right) \right]. \quad (73)$$

On the other hand for $Q_{1/2} < 0$ there are two rebounce epochs at $R = R_{s\pm}$ with

$$R_{s\pm} = -\frac{3Q_{1/2}}{4\Lambda} \pm \sqrt{\beta}. \quad (74)$$

Since $V(0, r) = f > 0$ a singularity also forms at $R_s = 0$. The phase space of allowed dynamics is divided in two disconnected regions separated by the forbidden interval $R_{s-} < R < R_{s+}$ where the potential is negative. For $0 \leq R \leq R_{s-}$ the dark radiation shells may expand to a maximum radius $R = R_{s-}$ in the time $t_{s-} = t_s$ where

$$t_s(r) = \sqrt{\frac{3}{\Lambda}} \cosh^{-1} \left( \frac{r + \frac{3Q_{1/2}}{4\Lambda}}{\sqrt{\beta}} \right). \quad (75)$$

At this rebounce epoch the shells start to fall towards the singularity which is reached after the time

$$t_s(r) = \sqrt{\frac{3}{\Lambda}} \cosh^{-1} \left( \frac{|3Q_{1/2}|}{4\Lambda \sqrt{\beta}} \right). \quad (76)$$
If \( R \geq R_{*+} \) then there is a collapsing phase to the minimum radius \( R = R_{*+} \) taking the time \( t_{*+} = t_{*} \) followed by reversal and subsequent accelerated continuous expansion. The singularity at \( R_{s} = 0 \) does not form and so the solutions are globally regular.

If \( \beta = 0 \) then \( f = 3Q_{1/2}^{2}/(16\Lambda) \). For \( Q_{1/2} < 0 \) there is one rebounce point at

\[
R_{*} = -\frac{3Q_{1/2}}{4\Lambda}.
\]

In this case \( V(0, r) = f > 0 \) and then a singularity also forms at \( R_{s} = 0 \). There is no forbidden region in phase space but the point at \( R_{*} \) turns out to be a regular fixed point which divides two distinct dynamical regions. Indeed if a shell starts at \( R = R_{*} \) then it will not move for all times. If initially \( R < R_{*} \) then either the shell expands towards \( R_{*} \) or it collapses to the singularity. The time to get to the singularity is finite,

\[
t_{*}(r) = \frac{3}{\Lambda} \ln \left( \frac{3Q_{1/2}}{4\Lambda r + 3Q_{1/2}} \right),
\]

but the time to expand to \( R_{*} \) is infinite. If initially \( R > R_{*} \) then the collapsing dark radiation shells also take an infinite time to reach \( R_{*} \). If \( Q_{1/2} > 0 \) there are no real rebounce epochs and the collapsing dark radiation simply falls to the singularity at \( R_{s} = 0 \). The collision proper time is

\[
t_{*}(r) = -\frac{3}{\Lambda} \ln \left( \frac{3Q_{1/2}}{4\Lambda r + 3Q_{1/2}} \right).
\]

5 Gauss-Codazzi Equations and the Localization of Gravity on the Brane

In the special conformal setting we have defined the RS warp solution (20) has been factored out of the 5-dimensional problem. This is then reduced to the resolution of Eqs. (25) and (26) with the fifth dimensional pressure, \( p_{z} \), satisfying conditions (24) and (27). A set of effective 4-dimensional brane world geometries are generated and two examples are the inhomogenous dust and dark radiation dynamics. These effective 4-dimensional metrics should be deduced by an observer confined to the brane which makes the same assumptions about the bulk degrees of freedom. Moreover, in this case the 4-dimensional observer should also agree about the localization of gravity in the vicinity of the brane. Let us now show that the 5-dimensional approach developed in this work is in this sense consistent with the effective Gauss-Codazzi formulation [35]-[37].

Consider then the Gauss-Codazzi approach for the RS brane world scenario and assume that the matter degrees of freedom which exist on the brane only originate
in matter modes present in the AdS bulk space. Then the effective 4-dimensional Einstein equations are given by

$$G_{\nu}^{\mu} = \frac{2\kappa_5^2}{3} \left[ T_\alpha^\beta q_\alpha^\beta + \left( T_\alpha^\beta n^\alpha n_\beta - \frac{1}{4} T_\alpha^\alpha \right) q_\mu^\nu \right] + \mathcal{K}_\alpha^\alpha \mathcal{K}_\mu^\nu \nonumber$$

$$- \mathcal{K}_\mu^\alpha \mathcal{K}_\nu^\alpha - \frac{1}{2} q_\mu^\nu \left( \kappa^2 - \mathcal{K}_\alpha^\beta \mathcal{K}_\alpha^\nu \right) - \mathcal{E}_\mu^\nu, \quad (80)$$

where $G_{\mu}^\nu = G_\alpha^\beta q_\mu^\alpha q_\beta^\nu$, $n^\mu = \delta_z^\mu$ is the unit normal to the brane, $q_\mu^\nu = \delta^\nu_\mu - n_\mu n^\nu$ is the tensor which projects orthogonally to $n^\mu$,

$$T_\mu^\nu = -\Lambda_B \delta_\mu^\nu + T_\mu^\nu \quad (81)$$

is the stress-energy tensor,

$$\mathcal{K}_\mu^\nu = \lim_{z \to z_0^+} K_\mu^\nu = -\frac{\kappa_5^2 \lambda}{6} q_\mu^\nu \quad (82)$$

is the extrinsic curvature and

$$\mathcal{E}_\mu^\nu = \lim_{z \to z_0^+} C_{\rho \sigma \beta} n^\alpha n^\beta q_\mu^\rho q_\sigma^\nu \quad (83)$$

is the traceless projection of the 5-dimensional Weyl tensor.

Substituting Eqs. (81) and (82) in Eq. (80) we find

$$G_{\mu}^\nu = -\frac{\kappa_5^2}{2} \left( \Lambda_B + \frac{\kappa_5^2 \lambda^2}{6} \right) \delta_\mu^\nu + \frac{2\kappa_5^2}{3} \left[ T_\alpha^\beta q_\mu^\alpha q_\beta^\nu + \left( T_\alpha^\beta n^\alpha n_\beta - \frac{1}{4} T_\alpha^\alpha \right) q_\mu^\nu \right] - \mathcal{E}_\mu^\nu. \quad (84)$$

Applying the covariant derivative it is clear that in general the projected Weyl tensor is not conserved because of the fields present in the bulk.

In the effective 4-dimensional point of view the metric $g_{\mu \nu}$ is defined by Eqs. (4) and (5). Then we obtain $\mathcal{E}_z^\mu = 0$. If in addition the RS identity (21) is assumed to be satisfied then Eq. (84) becomes Eq. (26) and so the 4-dimensional observer does find the same dynamics on the brane.

Moreover, if $T_\alpha^b$ is conserved as in Eq. (25) then the projected Weyl tensor must satisfy

$$\nabla_a \mathcal{E}_b^a = \frac{\kappa_5^2}{6} \nabla_b T_z^b. \quad (86)$$

If it is verified that

$$\mathcal{E}_a^b = \frac{\kappa_5^2}{3} \left( -T_a^b + \frac{1}{2} T_z^z \delta_a^b \right) \quad (87)$$

then Eq. (85) becomes Eq. (26) and so the 4-dimensional observer does find the same dynamics on the brane.
This may be explicitly checked for the dust and dark radiation systems. First determine $E_{\mu}^\nu$ using the alternative Eqs. (83) and (87). For dust both lead to the same result

$$ E_t^t = \frac{\kappa_5^2}{4} \rho_o, \quad E_r^r = E_\theta^\theta = E_\phi^\phi = -\frac{\kappa_5^2}{12} \rho_o. $$

(88)

Thus Eq. (87) is indeed verified and then it is easy to see that Eq. (85) reduces to the 4-dimensional Einstein equations for dust and a cosmological constant

$$ G_t^t = -\kappa_5^2 \rho_o - \Lambda, \quad G_r^r = G_\theta^\theta = G_\phi^\phi = -\Lambda. $$

(89)

For the dark radiation system we also conclude that both Eqs. (83) and (87) lead to the same result

$$ E_t^t = E_r^r = \frac{\kappa_5^2}{6} (1 - \eta) \rho_{DR}, \quad E_\theta^\theta = E_\phi^\phi = -\frac{\kappa_5^2}{6} (1 - \eta) \rho_{DR} $$

(90)

so that Eq. (87) is satisfied. Then Eq. (85) reduces to the 4-dimensional Einstein equations for dark radiation and a cosmological constant

$$ G_t^t = G_r^r = -\kappa_5^2 \rho_{DR} - \Lambda, \quad G_\theta^\theta = G_\phi^\phi = \kappa_5^2 \rho_{DR} - \Lambda. $$

(91)

In the 5-dimensional picture it is clear that whatever the effective 4-dimensional solution to be considered, the conformal RS warp factor ensures that gravity is always localized in the vicinity of the brane. Let us now show that the observer confined to the brane may reach the same conclusion. In the covariant Gauss-Codazzi approach the tidal acceleration away from the brane is defined by

$$ a_T = -\lim_{z \to z_0^+} R_{\mu \nu \alpha \beta} n^\nu u^\mu n^\alpha n^\beta, $$

(92)

where $u^\mu = \delta^\mu_0$ is the extension off the brane of the 4-velocity field which satisfies $u^\mu n_\mu = 0$ and $u^\mu u_\mu = -1$. For the gravitational field to be localized near the brane $a_T$ must be negative. Consider the identity

$$ R_{\mu \nu \alpha \beta} = C_{\mu \nu \alpha \beta} + \frac{2}{3} \left\{ g_{\mu [\alpha} R_{\beta] \nu} + g_{\nu [\beta} R_{\alpha] \mu} \right\} - \frac{1}{6} R g_{\mu [\alpha} g_{\beta] \nu}, $$

(93)

where the square brackets denote anti-symmetrization. Introducing the 5-dimensional Einstein equation

$$ G^\nu_\mu = \kappa_5^2 T^\nu_\mu $$

(94)

we find for the tidal acceleration the following expression

$$ a_T = \frac{\kappa_5^2 \Lambda_B}{6} - \frac{\kappa_5^2}{3} \left( T^\alpha_\alpha u^\alpha u_\beta - T^\beta_z + \frac{T_\alpha^\alpha}{2} \right) - E^\beta_\alpha u^\alpha u_\beta. $$

(95)

If the contributions of $T^\nu_\mu$ and $E^\nu_\mu$ cancel each other,

$$ E^\beta_\alpha u^\alpha u_\beta = -\frac{\kappa_5^2}{3} \left( T^\beta_\alpha u^\alpha u_\beta - T^\beta_z + \frac{T_\alpha^\alpha}{2} \right), $$

(96)
then the tidal acceleration is given by

\[ a_T = \frac{\kappa_5^2 \Lambda_B}{6}, \]  

(97)

which is indeed always negative in the RS 5-dimensional AdS space.

This may also be explicitly confirmed for the dust and dark radiation systems. For dust Eqs. (28) and (29) imply

\[ a_T = \frac{\kappa_5^2 \Lambda_B}{6} - \frac{\kappa_5^2}{4} \rho_D - \mathcal{E}_\alpha^\beta u^\alpha u^\beta. \]  

(98)

Then using Eq. (88) it is easy to see that Eq. (96) is indeed verified. In the vacuum the solutions are conformally flat and so we also obtain the same result. Note, however, that using the effective Gauss-Codazzi approach it is possible to find a much richer non-conformally flat vacuum dynamics which also admits a brane cosmological constant [44]. Then it turns out that gravity is not always confined to the vicinity of the brane [44]. On the other hand for dark radiation generated on the brane by the conformal bulk fields Eqs. (44)-(46) imply

\[ a_T = \frac{\kappa_5^2 \Lambda_B}{6} - \frac{\kappa_5^2}{3} (1 - \eta) \rho_{DR} - \mathcal{E}_\alpha^\beta u^\alpha u^\beta. \]  

(99)

Once more Eq. (96) is verified as may be checked introducing Eq. (90). Then the gravitational field is always bound to the vicinity of the brane.

6 Conclusions

Any theory in which our universe is viewed as a brane must reproduce the large scale predictions of 4-dimensional general relativity, in particular the gravitational collapse of matter, on it. In the bulk, this matter could be localized about the brane or extended. In this work, we have analyzed the dynamical collapse of extended matter in a $Z_2$ symmetric, 5-dimensional world in which a single spherically symmetric, positive tension, gravity confining brane exists. We have applied a global conformal transformation to give a clarifying organization to the Einstein equations for the most general 5-dimensional metric consistent with the $Z_2$ symmetry and with the spherical symmetry on the brane. Assuming that the bulk stress-energy tensor has conformal weight $s = -4$ and that only the conformal warp factor depends on the coordinate of the fifth dimension, we showed that the bulk matter dynamics on the brane produces a pressure along the fifth dimension which is required to satisfy a well defined equation of state.

With this analysis we have discovered a new class of exact dynamical solutions for which the conformal warp factor localizes gravity in the vicinity of the brane. We have considered the two specific examples of 5-dimensional geometries which define
on the brane the gravitational dynamics of inhomogeneous dust and generalized dark radiation in the presence of an effective brane cosmological constant, also generated by bulk matter. For these examples we have discussed the static marginally bound limits and the conditions defining the solutions as singular or as globally regular.

Finally, we have also analyzed the point of view of an observer confined to the brane to show that an identical description of the variables in the problem consistently leads to the same localized brane world dynamics.

The bulk matter we consider is extended and not localized near the brane because the conformal weight of the stress-energy tensor is \( s = -4 \). Its density and pressures depend on the coordinate of the fifth dimension and diverge on the horizon. This problem is also encountered by the black string solution \([9]\), where the Kretschmann invariant is shown to diverge on the AdS horizon. It can be solved if we use the first RS model in which there are two branes with opposite tensions present and we live on the brane with negative tension. Alternatively, one can look for solutions in a single brane model that respect the localization of gravity while incorporating a simultaneous localization of matter about the brane. This would amount to looking for solutions in which the conformal metric is \( g_{\mu\nu}(t, r, z) \), with \( g_{\mu\nu}(t, r, z = z_0) = g_{\mu\nu}(t, r) \) where \( g_{\mu\nu}(t, r) \) corresponds to one of the class of solutions we have presented plus (possibly) quantum corrections \([25]\) (see \([22]\) for a similar approach regarding static black hole solutions). This will be examined in future work.

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