A NON-AUTONOMOUS PREDATOR-PREY MODEL 
WITH INFECTED PREY

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Abstract. A non-constant eco-epidemiological model with SIS-type infectious disease in prey is formulated and investigated, it is assumed that the disease is endemic in prey before the invasion of predator and that predation is more likely on infected prey than on the uninfected. Sufficient conditions for both permanence and extinction of the infected prey, and the necessary conditions for the permanence of the infected prey are established. It is shown that the predation preference to infected prey may even increase the possibility of disease endemic, and that the introduction of new resource for predator could be helpful for it to eradicate the infected prey. Numerical simulations have been performed to verify/extend our analytical results.

1. Introduction. Infectious diseases among predators or preys have potential impacts on the population sizes of interacting predator-prey community (see [13, 15]). Rinderpest was shown to give rise to high mortality in wild animals in Africa at the end of the 19th century; and it was believed that Myxomatosis could be the cause of the enormous decreases in the rabbit population in Australia in the 1950s (see [13]).

For predator-prey relationships, it was found that predator removal decreased the prey population in 54 of the 135 systems examined [26]; in [23], it was suggested that predator removal can lead to a decreased prey population size for a predator-prey system with infected prey. Since the infected prey are less active than those uninfected prey, it is found that they have more chances to be hunted by the predators (see [6, 16]). For detailed instance on predator-prey interaction with disease in prey, we refer to bird-fish community in Salton Sea of California studied in [5, 11], where the fish population infected by a vibrio class of bacteria was

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shown to become weaker and more susceptible to be hunted by piscivorous birds. Similar instances were also found in [24] for wolf that are more likely to attack on heavily infected moose with a lungworm. It is therefore biologically significant for one to consider whether such an infected-prey preference strategy of predation is beneficial or not for the predator itself as well as for the whole interacting system.

Recently, eco-epidemic dynamics have been extensively studied in population biology and larger progress has been achieved, and most of which focus on the constant-environments (see [1, 2, 3, 13, 17, 14, 20, 21, 27, 30, 31, 19]).

As showed in [4, 8, 9, 18, 34], environment usually varies in time or fluctuates periodically due to the seasonal or circadian changes in the surrounding conditions, and thus the model should be time-varying. To our best acknowledge, there are a few literatures devoted to time-varying eco-epidemic models [22, 25, 32]. In [22], the following eco-epidemic model with predation only on the infected prey was proposed and investigated:

\[
\begin{align*}
\frac{ds}{dt} &= A(t) - \beta(t)s - d(t)s, \\
\frac{di}{dt} &= \beta(t)s - c(t)i - \eta(t)i, \\
\frac{dy}{dt} &= y(r(t) - b(t)y) + k(t)\eta(t)i,
\end{align*}
\]

(1)

where the total prey population is composed of two population classes: the class of susceptible prey \(s\) and that of the infected prey, denoted by \(i\); \(A(t)\), \(d(t)\) are the recruited rate and natural death rate of susceptible prey, respectively; \(c(t)\) is the removed rate of \(i\) including natural death rate plus the disease-related death rate; the predator is assumed to grow according to a logistic curve with an intrinsic birth rate \(r(t)\) and predation rate \(\eta(t)\), with \(k(t)\) as the converting rate of prey into predator.

Noting that in model (1), [22] ignored the predation of susceptible prey, as argued in [13, 24], the predator does hunt both the susceptible and infected prey. Similar models were also considered by Bate et al [2]. Thus it is ecologically reasonable to take the predation on uninfected prey into account.

Motivated by the above works of [13, 20, 22], take the moose-wolves system as a biological example[24], we will formulate and study a non-constat predator-prey model with infectious disease in prey, different from model (1), the predation in our model is assumed on both the infected and uninfected prey, however, with preference on the infected, and it is also assumed that growth of predator is completely dependent on hunting over preys, which means our model does not has the positive term \(r(t)\) in equation (1).

Our purpose is to investigate the effects of infected-prey preference strategy of predation on the dynamics, and thus to understand the ecological impacts of such hunting strategy on the predator as well as on the whole predator-prey system. There are many methods to deal with the asymptotical behaviors of the autonomous models, as applied in [1, 2, 3, 13, 20, 27, 30, 31], but they are not suitable to non-autonomous systems. Thus we will develop the methods for asymptotical behaviors of non-autonomous systems engaged in [28, 29, 33]; comparing the model (1) in [22] with no interaction between \(s\) and \(y\), our model is of the strong coupling type with both interactions between \(s\) and \(y\) as well as \(i\) and \(y\). Therefore, it is more mathematically challenging for us.
This paper is organized as follows: we propose our model in the next section and give some preliminaries in Section 3. In Section 4, when the parameters in (2) are time-varying coefficients, we will explore sufficient and necessary conditions on the permanence and extinction of the infected prey. Following Section 4, some interesting numerical simulations will complement our analytical findings. The paper ends with a section of discussions.

2. Model description. We present the following assumptions to build our mathematical model:

- Let $A(t)$ be the recruited rate of the prey population (including newborns and migration) (see [1, 7]) and $d(t)$ be the natural death rate of the prey population. In the absence of disease and capture, the growth rate of the prey population $x(t)$ can be given by

$$\frac{d}{dt} x(t) = A(t) - d(t)x(t).$$

- In the presence of disease, the total prey population $x$ is composed of two population classes: the susceptible prey denoted by $s$, and the infected prey $i$. Here, the infected prey $i$ is removed at the death rate $c(t)$, including natural death rate $d(t)$ and disease-related death rate; moreover, $c(t) \geq d(t)$ for all $t \geq 0$.

- The disease is only spreading among the prey population and is not genetically inherited. The infected population do not recover or become immune. The incidence is assumed to be the bilinear incidence $\beta(t)s(t)i(t)$.

- The predator $y$ has no other resource than prey population; $\eta_1(t)$ and $\eta_2(t)$ are the corresponding predation rates of the susceptible and infected prey, with $0 \leq \eta_1(t) < \eta_2(t)$, for all $t \geq 0$ holds, i.e., the infected prey $i$ is more vulnerable to predation than the uninfected prey $s$; $k_1(t)$ and $k_2(t)$ are the related rates in converting uninfected and infected prey into predator; the predator is confined to be density dependent, as argued by Griffiths et al. in [12], and $b(t)$ is the density dependence rate of the predator population.

Based on the above assumptions, our model is given as follows:

$$\begin{align*}
\frac{d}{dt} s(t) &= A(t) - \beta(t)s(t)i(t) - d(t)s(t) - \eta_1(t)y(t)s(t), \\
\frac{d}{dt} i(t) &= \beta(t)s(t)i(t) - c(t)i(t) - \eta_2(t)y(t)i(t), \\
\frac{d}{dt} y(t) &= y(t)(k_1(t)\eta_1(t)s(t) + k_2(t)\eta_2(t)i(t) - b(t)y(t)).
\end{align*}$$

Considering the significance of biology, we only consider the solution of model (2) with initial conditions:

$$s(0) > 0, \ i(0) > 0, \ y(0) > 0, \ \mathcal{E} = \{(s(\vartheta), i(\vartheta), y(\vartheta)) \in R^3_+ : \vartheta > 0\}. \quad (3)$$

3. Preliminaries. For a continuous bounded function $g(t)$ defined on $R^+ = [0, \infty)$, we denote

$$g^\infty = \limsup_{t \to \infty} g(t), \ g_\infty = \liminf_{t \to \infty} g(t).$$

For model (2), throughout this paper we introduce the following conditions.
Lemma 3.2. (see defined on \( R \times b \)) results. By the similar arguments to \( \[ \text{Lemma 3.1.} \] \)

Then, \( \eta \), \( \text{functions } \tilde{t} \), \( A \text{ and } \omega \) \( \text{positive } \omega \), \( \text{s} \), \( \text{positive constants } \omega_1 > 0, \omega_2 > 0 \) and \( \gamma > 0 \) such that

\[
\liminf_{t \to \infty} \int_{t}^{t+\omega_1} A(\theta) d\theta > 0, \quad \liminf_{t \to \infty} \int_{t}^{t+\gamma} \beta(\theta) d\theta > 0,
\]

\[
\liminf_{t \to \infty} \int_{t}^{t+\omega_2} \eta_1(\theta) d\theta > 0, \quad \liminf_{t \to \infty} \int_{t}^{t+\omega_2} \eta_2(\theta) d\theta > 0.
\]

In the following, we state several Lemmas which are useful for the proofs of main results. By the similar arguments to [28, Lemma 2.3] (see also [20, Proposition 1]), we directly have

**Lemma 3.1.** Given that the assumptions \( (A_1), (A_2) \) hold true, then the solution \( (s(t), t(t), y(t)) \) of system (2) with (3) is positive and ultimately bounded for all \( t \geq 0 \).

Consider the following non-autonomous logistic equation

\[
d\frac{dx}{dt}(t) = x(\tilde{r}(t) - b(t)x), \tag{4}
\]

where functions \( \tilde{r}(t) \) and \( b(t) \) are bounded and continuous function defined on \( R^+ \) and \( b(t) \geq 0 \) for all \( t \geq 0 \).

**Lemma 3.2.** (see [22, Lemma 1]) Suppose that there are constants \( \omega_i > 0 \ (i = 1, 2) \) such that

\[
\liminf_{t \to \infty} \int_{t}^{t+\omega_1} \tilde{r}(t) d\theta > 0, \quad \liminf_{t \to \infty} \int_{t}^{t+\omega_2} b(t) d\theta > 0.
\]

Then,

(a) There exist positive constants \( p_2 \) and \( q_2 \) such that for any positive solution \( x(t) \) of (4)

\[
p_2 < \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) < q_2.
\]

(b) Each fixed positive solution \( x^*(t) \) of (4) is globally uniformly attractive.

(c) If \( b_\infty > 0 \), then for any positive solution \( x(t) \) of (4)

\[
\left( \frac{\tilde{r}}{b} \right) \leq \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) < \left( \frac{\tilde{r}}{b} \right)\infty.
\]

(d) If (4) is \( \omega \)-periodic, then (4) has a unique globally uniformly attractive positive \( \omega \)-periodic solution.

Further we consider the following non-autonomous logistic equation

\[
d\frac{dx}{dt}(t) = x(\tilde{r}(t) - b(t)x + e(t)), \tag{5}
\]

where functions \( \tilde{r}(t) \) and \( b(t) \) are given as in (4), function \( e(t) \) is bounded continuous defined on \( R^+ \).

Let \( x(t, t_0, x_0) \) be the solution of (5) with the initial condition \( x(t_0) = x_0 \) and \( x_0(t) \) be some fixed positive solution of (4). We have the following result.
Lemma 3.3. (see [22, Lemma 2]) Suppose that all conditions of Lemma 3.2 hold. Then for any constants \( \epsilon > 0 \) and \( q_2 > 0 \) there exist constants \( \delta = \delta(\epsilon) > 0 \) and \( T = T(\epsilon, q_2) > 0 \) such that for any \( t_0 \in R^+ \) and \( x_0 \in [q_2^{-1}, q_2] \), when \( |e(t)| < \delta \) for all \( t \geq t_0 \), we have
\[
|x(t, t_0, x_0) - x_0| < \epsilon
\]
for all \( t \geq t_0 + T \) holds.

Next, we consider the following non-autonomous linear equation
\[
\frac{dx(t)}{dt} = A(t) - d(t)x, \quad (6)
\]
where \( A(t) \) and \( d(t) \) are bounded continuous defined on \( R^+ \) and \( A(t) \geq 0 \) for all \( t \geq 0 \). We have the following result.

Lemma 3.4. (see [22, Lemma 3]) Suppose that there exist constants \( \omega_i > 0 \) \( (i = 1, 3) \) such that
\[
\liminf_{t \to \infty} \int_t^{t+\omega_1} A(\theta)d\theta > 0, \quad \liminf_{t \to \infty} \int_t^{t+\omega_3} d(\theta)d\theta > 0.
\]
Then,
(a) There exist positive constants \( p_1 \) and \( q_1 \) such that for any positive solution \( x(t) \) of \( (6) \)
\[
p_1 < \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) < q_1.
\]
(b) Each fixed positive solution \( x^*(t) \) of \( (6) \) is globally uniformly attractive.
(c) If \( d_\infty > 0 \), then for any positive solution \( x(t) \) of \( (6) \)
\[
\left( \frac{A}{d} \right)_{\infty} < \liminf_{t \to \infty} x(t) \leq \limsup_{t \to \infty} x(t) < \left( \frac{A}{d} \right)_{\infty}.
\]
(d) If \( (6) \) is \( \omega \)-periodic, then \( (6) \) has a unique globally uniformly attractive positive \( \omega \)-periodic solution.

Further we consider the following non-autonomous linear equation
\[
\frac{dx(t)}{dt} = A(t) - d(t)x + e(t), \quad (7)
\]
where functions \( A(t) \) and \( d(t) \) are given as in \( (6) \), function \( e(t) \) is bounded continuous defined on \( R^+ \). Let \( x(t, t_0, x_0) \) be the solution of \( (7) \) with the initial condition \( x(t_0) = x_0 \) and \( x_0(t) \) be some fixed positive solution of \( (6) \). We have the following result.

Lemma 3.5. (see [22, Lemma 4]) Suppose that there exists a constant \( \gamma > 0 \) such that
\[
\liminf_{t \to \infty} \int_t^{t+\gamma} d(s)ds > 0.
\]
Then for any constants \( \epsilon > 0 \) and \( q_1 > 0 \) there exist constants \( \delta = \delta(\epsilon) > 0 \) and \( T = T(\epsilon, q_1) > 0 \) such that for any \( t_0 \in R^+ \) and \( |x_0| \leq q_1 \), when \( |e(t)| < \delta \) for all \( t \geq t_0 \), we have
\[
|x(t, t_0, x_0) - x_0| < \epsilon
\]
for all \( t \geq t_0 + T \) holds.
4. Main results.

4.1. The prey subsystem. First, we consider system (2) without the predator, that is, its prey subsystem, which takes the form as follows:

\[
\begin{align*}
\frac{ds(t)}{dt} &= A(t) - \beta(t)s(t)i(t) - d(t)s(t), \\
\frac{di(t)}{dt} &= \beta(t)s(t)i(t) - c(t)i(t) - \eta_1(t)y(t)i(t), \\
\frac{dy(t)}{dt} &= y(t)(k_2(t)\eta_1(t)i(t) - b(t)y(t)).
\end{align*}
\]

(8)

with the initial conditions

\[
s(0) > 0, \quad i(0) > 0, \quad \mathcal{L}_1 = \{(s(\vartheta), i(\vartheta)) \in \mathbb{R}^{2+} : \vartheta > 0\}.
\]

(9)

It is obvious that the solution \((s(t), i(t))\) of model (8) with initial condition (9) is positive, and that system (8) can be analyzed in the following feasible region

\[
\mathcal{L}_2 = \{(s(t), i(t)) \in \mathbb{R}^{2+} : 0 < s + i < \left(\frac{A}{d}\right)^\infty, i \geq 0, s > 0\}.
\]

Let the function

\[
\ell(t, x) = \beta(t)x - c(t)
\]

(10)

and let \(x^*(t)\) be some fixed solution of equation \(s'(t) = A(t) - d(t)s(t)\) with initial value \(x^*(0) > 0\). Using the Theorems 3.1, 3.2 and 4.1 in [35] about an SIRVS epidemic model, we directly have the following results

**Corollary 1.** Suppose that assumptions \((A_1), (A_2)\) hold and there exists a \(\lambda > 0\) such that

\[
R_0 := \liminf_{t \to \infty} \int_t^{t+\lambda} \ell(\sigma, x^*(\sigma))d\sigma > 0.
\]

(11)

Then system (8) are permanent.

Throughout the rest of this paper, we assume that the conditions for Corollary 1 always hold true for system (2), i.e., we assume that the susceptible prey and the infected prey coexist permanently before the invasion of predator.

4.2. Model (2) with the predation only over the infected prey. In this subsection, we investigate the model (2) without predation over the susceptible prey, i.e., \(\eta_1(t) \equiv 0\), as the extreme case of predator’s preference toward the infected prey, it takes the form as follows:

\[
\begin{align*}
\frac{ds(t)}{dt} &= A(t) - \beta(t)s(t)i(t) - d(t)s(t), \\
\frac{di(t)}{dt} &= \beta(t)s(t)i(t) - c(t)i(t) - \eta_2(t)y(t)i(t), \\
\frac{dy(t)}{dt} &= y(t)(k_2(t)\eta_2(t)i(t) - b(t)y(t)).
\end{align*}
\]

(12)

Model (12) is similar to (1) considered in [22], but (12) does not has the positive term \(r(t)\) in equation (1). For model (12), we have:

**Theorem 4.1.** System (12) with the initial conditions (3) is permanent if it satisfies the conditions for Corollary 1.
Proof. By Lemma 3.1, the solution \((s(t), i(t), y(t))\) to system (12) is positive and ultimately bounded for all \(t \geq 0\), thus it suffices for us to prove that every solution has positive eventually upper bound, thus there exists a positive constant \(q_y\) such that
\[
\limsup_{t \to \infty} y(t) \leq q_y.
\]

First, using the similar arguments to [29, Lemma 2.5], we are able to show that there exists a positive constant \(p_{s1}\) such that
\[
\liminf_{t \to \infty} s(t) \geq p_{s1}.
\]

Second, noting the conditions for Corollary 1 hold true, thus using the similar arguments to [29, Proposition 1], one can prove that there exists some constant \(p_{i1} > 0\) such that
\[
\limsup_{t \to \infty} i(t) \geq p_{i1}.
\]

In fact, if (13) does not hold, i.e., for any arbitrarily small constant \(\varepsilon > 0\), one always has \(\limsup_{t \to \infty} i(t) < \varepsilon\), by the first and third equations of (12) and using Lemma 3.4, we can get \(\limsup_{t \to \infty} s(t) \geq x^*(t) - \varepsilon \cdot c_0\) and \(\limsup_{t \to \infty} y(t) \leq \left(\frac{k_2\eta_2}{b}\right)\varepsilon\), respectively, where \(c_0\) is some positive constant, which guarantees that the inequality \(\liminf_{t \to \infty} \int_{t}^{t+\lambda} \ell(\sigma, x^*(\sigma))d\sigma > 0\) holds true for every sufficiently small \(\varepsilon\), where \(\ell(\cdot)\) is defined in (10). By the second equation of (12), we get
\[
\frac{di(t)}{dt} = \beta(t)s(t)i(t) - c(t)i(t) - \eta_2(t)y(t)i(t)
\geq \left[\beta(t)(x^*(t) - \varepsilon \cdot c_0) - c(t) - \eta_2(t)\left(\frac{k_2\eta_2}{b}\right)\varepsilon\right]i(t)
\geq \left[\liminf_{t \to \infty} [t, x^*(t) - \varepsilon \cdot c_0] - \eta_2(t)\left(\frac{k_2\eta_2}{b}\right)\varepsilon\right]i(t).
\]

By \(\liminf_{t \to \infty} \int_{t}^{t+\lambda} \ell(\sigma, x^*(\sigma))d\sigma > 0\) and \((A_2)\), we have \(\liminf_{t \to \infty} \int_{t}^{t+\lambda} \ell(\sigma, x^*(\sigma) - \varepsilon \cdot c_0)d\sigma > 0\) and \(\liminf_{t \to \infty} \int_{t}^{t+\omega_2} \eta_2(\theta)\left(\frac{k_2\eta_2}{b}\right)\varepsilon d\theta > 0\) holds true, noting Lemma 3.2, inevitably there exist positive constant \(p_i\) such that \(\liminf_{t \to \infty} i(t) \geq p_i\), a contradiction to \(\limsup_{t \to \infty} i(t) < \varepsilon\) for any arbitrarily small constant \(\varepsilon\), this proves (13).

Next, we prove the existence of positive lower bound of \(i(t)\). Indeed, we can follow the similar methods engaged in Step 3 of [29, Proposition 1], by which we are able to obtain that there exists some constant \(p_{i2} > 0\) with \(p_{i2} < p_{i1}\) such that
\[
\liminf_{t \to \infty} i(t) \geq p_{i2}.
\]

With (15), noticing Lemma 3.2 and using the third equation of (12), one can easily follow the uniform persistence of \(y(t)\), this completes the proof of Theorem 4.1. \(\Box\)

4.3. The permanence of susceptible prey and predator in system (2). On the permanence of species \(s\) and \(y\), we have the following result.

Theorem 4.2. Suppose that assumptions \((A_1), (A_2)\) hold. Then \(s(t)\) and \(y(t)\) in system (2) are permanent.
Proof. By Lemma 3.1, the solution \( (s(t), i(t), y(t)) \) to system (2) is ultimately bounded for all \( t \geq 0 \), i.e., there exist constants \( q_1 > 0 \) and \( T_1 > 0 \), such that
\[
\max \{ s(t), i(t), y(t) \} \leq q_1, \quad (16)
\]
for all \( t \geq T_1 \) holds. Thus it suffices for us to prove that \( s(t) \) and \( y(t) \) have positive eventually lower bound.

First we prove the permanence of \( s(t) \). Noting the boundedness of \( i(t) \) and condition \( (A_1) \), by a similar application of comparison theorem to [29, Lemma 2.5] in the first equation of (2), it follows from Lemma 3.4 that there exist constants \( p_{1s} := [A/(d + q_1 \beta + q_1 \eta_1)]_{\infty} > 0 \) and \( T_2 \geq T_1 \), with \( p_{1s} \) independent of any positive solution of model (2), such that
\[
s(t) \geq p_{1s} > 0, \quad (17)
\]
for all \( t \geq T_2 \) holds.

Next, we prove the uniform persistence of \( y(t) \). From the first and the second equations of (2), noticing (16), \( (A_1) \) and \( (A_2) \), we obtain
\[
\frac{d(s(t) + i(t))}{dt} \geq A(t) - [c(t) + \eta_2(t) q_1](s(t) + i(t)), \quad (18)
\]
for all \( t \geq T_1 \) holds. By Lemma 3.4, it follows that there exist constants \( p_1 := [A/(c + \eta_2 q_1)]_{\infty} \) and \( T_3 \geq T_2 \), such that
\[
s(t) + i(t) \geq p_1 > 0, \quad (19)
\]
for all \( t \geq T_3 \) holds. For \( y(t) \), by (19) and the third equation of model (2), we directly have
\[
\frac{dy}{dt} \geq y(t)[\tilde{k}_\infty \eta_1(t) p_1 - b(t) y(t)], \quad (20)
\]
for all \( t \geq T_3 \) holds, with \( \tilde{k}_\infty = \min \{ k_1_{\infty}, k_2_{\infty} \} > 0 \). By Lemma 3.2 and assumption \( (A_2) \), there exist positive constants \( p_{1y} := [\tilde{k}_\infty \eta_1 p_1 / b]_{\infty} \) and \( T_4 > T_3 \), such that
\[
y(t) \geq p_{1y} > 0, \quad (21)
\]
for all \( t \geq T_4 \) holds.

This completes the proof of Theorem 4.2. \( \square \)

4.4. The permanence and extinction of infected prey in system (2).

Theorem 4.3. Suppose that assumptions \( (A_1), (A_2) \) hold and there exists a constant \( \lambda > 0 \) such that
\[
\liminf_{t \to \infty} \lambda^{-1} \int_t^{t+\lambda} (\beta(\theta)s^*(\theta) - c(\theta) - \eta_1(\theta)\bar{y}(\theta))d\theta > 0, \quad (22)
\]
then the infected prey in model (2) is permanent, where \( s^*(t) \) is some fixed positive solution of the following non-autonomous linear equation
\[
\frac{ds(t)}{dt} = A(t) - (d(t) + q_1 \eta_1(t)) s(t), \quad (23)
\]
and \( \bar{y}(t) \) is some fixed positive solution of the following non-autonomous logistic equation
\[
\frac{dy(t)}{dt} = y(t) \left[ q_{si} \cdot \eta_2(t) \cdot \max \{ k_1(t), k_2(t) \} - b(t) y(t) \right], \quad (24)
\]
with \( q_{si} := [A/(d + \eta_1 p_{1y})]_{\infty} \), and \( p_{1y}, q_1 \) are defined above.
Proof. Let \((s(t), i(t), y(t))\) be any positive solution of model (2). Firstly, from condition (22) there are positive constants \(\epsilon_1, \epsilon_2\) and \(T_0\) such that for all \(t \geq T_0\)

\[
\int_t^{t+\lambda} (\beta(\theta)(s^*(\theta) - \epsilon_1) - c(\theta) - \eta_2(\theta)(\bar{y}(\theta) + \epsilon_1))d\theta > \epsilon_2. \tag{25}
\]

Consider the following auxiliary equation

\[
\frac{du}{dt} = A(t) - [d(t) + q_1\eta_1(t)]u - \beta(t)q_1\alpha, \tag{26}
\]

\(\alpha > 0\) is a some constant. From Lemma 3.5, for \(\epsilon_1\) and \(q_1\) given in above there are positive constants \(\delta_1 = \delta_1(\epsilon_1)\) and \(H_1 = H_1(\epsilon_1, q_1)\) such that for any \(t_0 \in R^+\) and \(u_0 \in [0, q_1]\), when \(\beta(t)q_1\alpha < \delta_1\) for all \(t \geq t_0\), we have

\[
|u(t, t_0, u_0) - s^*(t)| < \epsilon_1 \text{ for all } t \geq t_0 + H_1, \tag{27}
\]

where \(u(t, t_0, u_0)\) is the solution of (26) with the initial condition \(u(t_0) = u_0\).

Further consider the following auxiliary equation

\[
\frac{dv(t)}{dt} = v(t)[q_2\eta_2(t)\max\{k_1(t), k_2(t)\} - b(t)v(t) + \alpha\eta_2(t)\max\{k_1(t), k_2(t)\}}. \tag{28}
\]

From Lemma 3.3, for \(\epsilon_1\) and \(q_1\) are given in above there are positive constants \(\delta_2 = \delta_2(\epsilon_1)\) and \(H_2 = H_2(\epsilon_1, q_1)\) such that for any \(t_0 \in R^+\) and \(v_0 \in [q_1^{-1}, q_1]\), when \(\alpha\eta_2(t)\max\{k_1(t), k_2(t)\} < \delta_2\) for all \(t \geq t_0\), we have

\[
|v(t, t_0, v_0) - \bar{y}(t)| < \epsilon_1 \text{ for all } t \geq t_0 + H_2, \tag{29}
\]

where \(v(t, t_0, v_0)\) is the solution of (28) with the initial condition \(v(t_0) = v_0\).

Choose the constant as follows

\[
\alpha_0 = \min \left\{\frac{\delta_1}{\beta^\infty q_1 + 1}, \frac{\delta_2}{\eta_2^\infty \max\{k_1^\infty, k_2^\infty\} + 1}\right\}. \tag{30}
\]

We first prove \(\limsup\limits_{t \to \infty} i(t) > \alpha_0\). Assume the inverse, i.e., there exists a positive solution \((s(t), i(t), y(t))\) of model (2) such that \(\limsup\limits_{t \to \infty} i(t) < \alpha_0\). Then from the first equation of (2), we have

\[
\frac{ds(t)}{dt} \geq A(t) - [d(t) + q_1\eta_1(t)]s(t) - \beta(t)q_1\alpha_0
\]

holds true for large \(t\). Using comparison theorem and equation (27), we obtain \(s(t) \geq s^*(t) - \epsilon_1\) for all large \(t\).

From the first and the second equations of (2), noticing (16), (A1) and (A2), we obtain

\[
\frac{d(s(t) + i(t))}{dt} \leq A(t) - [d(t) + \eta_1(t)p_1y][s(t) + i(t)] \tag{30}
\]

holds true for large \(t\). Thus by Lemma 3.4, we have \(\limsup\limits_{t \to \infty}(s(t) + i(t)) \leq q_{si}\). Hence, using the third equation of model (2) and noting \(\eta_1(t) \leq \eta_2(t)\) for all \(t \geq 0\), we have

\[
\frac{dy(t)}{dt} \leq y(t)[q_{si} \cdot \eta_2(t)\max\{k_1(t), k_2(t)\} + \alpha_0\eta_2(t)\max\{k_1(t), k_2(t)\} - b(t)y(t)]
\]

holds true for all large \(t\).

By the comparison theorem and (29) we obtain \(y(t) \leq \bar{y}(t) + \epsilon_1\) for all large \(t\).

Therefore, by the above inequalities on \(s(t), y(t)\), we have

\[
\frac{di(t)}{dt} \geq i(t)[\beta(t)(s^*(t) - \epsilon_1) - c(t) - \eta_2(t)(\bar{y}(t) + \epsilon_1)]
\]
the case of for all
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when to consider Case 2. For this case, there exist two large constants t and
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, integrating the second equation of model
for any t
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≥
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(\bar{\theta})\] has positive lower bound \alpha_0.
Next, we will discuss the following two possible cases:
Case 1. \( i(t) \geq \alpha_0 \) for all large t;
Case 2. \( i(t) \) oscillates about \alpha_0 for all large t.
Case 1 suffices to show that \( i(t) \) is permanent. Therefore, we now only need to consider Case 2. For this case, there exist two large constants \( t_1 \) and \( t_2 \) with \( t_1, t_2 \geq T_4 \) and
\[ i(t_1) = i(t_2) = \alpha_0, i(t) < \alpha_0 \text{ for all } t \in (t_1, t_2). \] (31)
We further consider two constants
\[ h_1 = \sup_{t \geq 0} \{ c(t) + \eta_2(t)q_2 \}, \quad H = \max \{ H_1, H_2 \}, \]
and
\[ h_2 = \sup_{t \geq 0} \{ \beta(t)(s^*(t) - \epsilon_1) + c(t) + \eta_2(t)(\bar{y}(t) + \epsilon_1) \}. \]
If \( t_2 - t_1 \leq H \), for any \( t \in [t_1, t_2] \), integrating the second equation of model (2) from \( t_1 \) to \( t \), we can obtain
\[ i(t) = i(t_1) \exp \int_{t_1}^{t} \left[ \beta(\theta)s(\theta) - c(\theta) - \eta_2(\theta)y(\theta) \right] d\theta \]
\[ \geq i(t_1) \exp \int_{t_1}^{t} \left[ -c(\theta) - \eta_2(\theta)q_2 \right] d\theta \]
\[ \geq \alpha_0 \exp(-h_1 H) \]
for any \( t \in [t_1, t_2] \), when \( t \leq t_1 + H \), then according to the above discussion on the case of \( t_2 - t_1 \leq H \), we have \( i(t) \geq \alpha_0 \exp(-h_1 H) \). Particularly, we also obtain \( i(t_1 + H) \geq \alpha_0 \exp(-h_1 H) \).
If \( t_2 - t_1 > H \), then a similar argument as in the proofs of above inequalities (27) and (29) we can obtain
\[ s(t) \geq s^*(t) - \epsilon_1, \quad y(t) \leq \bar{y}(t) + \epsilon_1 \text{ for all } t \in [t + H, t_2], \] (33)
when \( t > t_1 + H \), then we choose an integer \( p \geq 0 \) such that \( t \in [t_1 + H + p\lambda, t_1 + H + (p + 1)\lambda] \), integrating the second equation of model (2) from \( t_1 + H \) to \( t \), by (25) and (33), we can obtain
\[ i(t) = i(t_1 + H)e^{\int_{t_1 + H}^{t_2} \beta(\theta)s(\theta) - c(\theta) - \eta_2(\theta)y(\theta) d\theta} \]
\[ \geq \alpha_0 e^{(-h_1 H) e^{\int_{t_1 + H}^{t_1 + H + p \lambda} \beta(\theta) (s^*(\theta) - \epsilon_1) - c(\theta) - \eta_2(\theta)(\bar{y}(\theta) + \epsilon_1) d\theta}} \]
\[ \geq \alpha_0 e^{(-h_1 H) e^{\int_{t_1 + H + p \lambda}^{t_1 + H + p \lambda} \beta(\theta) (s^*(\theta) - \epsilon_1) - c(\theta) - \eta_2(\theta)(\bar{y}(\theta) + \epsilon_1) d\theta}} \]
\[ \geq \alpha_0 e^{(-h_1 H) e^{\int_{t_1 + H + p \lambda}^{t_1 + H + p \lambda} \beta(\theta) (s^*(\theta) - \epsilon_1) - c(\theta) - \eta_2(\theta)(\bar{y}(\theta) + \epsilon_1) d\theta}} \]
\[ \geq \alpha_0 e^{(-h_1 H) e^{(-h_2 \lambda)}}. \]
Set $p_{1i} = \alpha_0 \exp\{-(h_1 H + h_2 \lambda)\}$, then from above discussions we finally obtain
\[ i(t) \geq p_{1i}, \text{ for all } t \in [t_1, t_2]. \]

This completes the proof of Theorem 4.3.

**Remark 1.** We see that $\beta(t)s^*(t) - c(t) - \eta_2(t)\bar{y}(t)$ is the available maximum growth rate of the infected prey at any time $t$. The left hand of inequality (22) indicates inferior limit of the available maximum growth rate of the infected prey in the mean on interval $[t, t + \lambda]$ for some constant $\lambda > 0$. Therefore, Theorem 4.3 shows that the infected prey must be permanent when the inferior limit is positive.

**Remark 2.** If we assume that all coefficients $A(t), \beta(t), d(t), c(t), \eta_1(t), \eta_2(t), b(t)$ and $k_1(t), k_2(t)$ are constant, then system (2) degenerates to the following autonomous system
\[
\begin{align*}
\frac{ds(t)}{dt} &= A - \beta s(t)i(t) - ds(t) - \eta_1 y(t)s(t), \\
\frac{di(t)}{dt} &= \beta s(t)i(t) - ci(t) - \eta_2 y(t)i(t), \\
\frac{dy(t)}{dt} &= y(t)(k_1\eta_1 s(t) - by(t)) + k_2\eta_2 i(t)y(t).
\end{align*}
\]  
(34)

Evidently, from Theorem 4.3, we have if $\beta \tilde{s}^* - c - \eta_2 \tilde{y} > 0$, then the infected prey in model (34) is permanent, where $\tilde{s}^*$ and $\tilde{y}$ correspond to constant solutions of autonomous equations for (23) and (24), respectively.

**Theorem 4.4.** Suppose that assumptions $(A_1), (A_2)$ hold. Then the necessary condition for the persistence of the infected prey in model (2) is that there exists a constant $\lambda > 0$ such that (22) holds, that is,
\[
\liminf_{t \to \infty} \lambda^{-1} \int_{t}^{t+\lambda} (\beta(\theta)s^*(\theta) - c(\theta) - \eta_2(\theta)\bar{y}(\theta))d\theta > 0,
\]
where, $s^*(t)$ fixed positive solution of the following linear equation
\[
\frac{ds(t)}{dt} = A(t) - (d(t) + \eta_1(t)q_2)s(t),
\]
(35)
and $\bar{y}(t)$ be some fixed positive solution of the following non-autonomous logistic equation
\[
\frac{dy(t)}{dt} = y(t)[k_1(\eta_1(t)p_{1s} - b(t)y(t))],
\]
(36)
p_{1s} = \left(\frac{A}{d + q_1(1 + \eta_2q_2)}\right)_{\infty}, \ q_2 := \left(\frac{k_1\eta_1 + k_2q_2q_2}{b}\right)_{\infty} \text{ are positive lower and upper bound of susceptible prey and predator, respectively.}

**Proof.** If the conclusion is not true, then for any constant $\lambda > 0$ we have
\[
\liminf_{t \to \infty} \int_{t}^{t+\lambda} (\beta(\theta)s^*(\theta) - c(\theta) - \eta_2(\theta)\bar{y}(\theta))d\theta \leq 0.
\]  
(37)
Let $(s(t), i(t), y(t))$ be some positive solution of model (2). Since $\lim\inf_{t \to \infty} i(t) > 0$, there is a constant $\zeta > 0$ such that
\[
i(t) \geq \zeta,
\]
(38)
for all $t \geq 0$ holds. From (17), there exists a constant $T_3 > 0$ such that for any $t \geq T_3$ we have $s(t) \geq p_{1s}$. 

\[\]
Let $V(t) = s(t) - s^*(t)$, then for any $t \geq T_3$ we have
\[
\frac{dV(t)}{dt} = -\beta(t)is - (d(t) + y(t)\eta_1(t))s + (t)s^*(t)
\leq -\beta(t)p_1s - (d(t) + p_2\eta_1(t))V(t).
\]
Hence,
\[
V(t) \leq V(T_2)e^{-\int_{T_2}^{t} (d(\theta) + p_2\eta_1(\theta))d\theta} - p_1s \int_{T_2}^{t} \beta(\theta)e^{-\int_{\tau}^{t} (d(\tau) + p_2\eta_1(\tau))d\tau}d\theta.
\]
From
\[
\liminf_{t \to \infty} \int_{t}^{t+\gamma} \beta(\theta)d\theta > 0
\]
and
\[
\liminf_{t \to \infty} \int_{t}^{t+\min\{\omega_3, \omega_6\}} (d(\theta) + p_2\eta_1(\theta))d\theta > 0,
\]
we can obtain
\[
\lim_{t \to \infty} \exp \left( -\int_{T_2}^{t} (d(\theta) + p_2\eta_1(\theta))d\theta \right) = 0,
\]
and
\[
\liminf_{t \to \infty} \int_{T_2}^{t} \beta(\theta) \exp \left( -\int_{t}^{t+\gamma} (d(\tau) + p_2\eta_1(\tau))d\tau \right) d\theta > 0.
\]
Hence, there exist constants $\epsilon > 0$ and $T_3 > T_3$ such that
\[
V(t) = s(t) - s^*(t) \leq -\epsilon \text{ for all } t \geq T_4. \quad (39)
\]
From the third equation of model (2) and $s(t) \geq p_1s$ for $t \geq T_3$, we have
\[
\frac{dy(t)}{dt} \geq y(t) [k_1(t)\eta_1(t)p_1s - b(t)y(t)],
\]
for all $t \geq T_3$. According to Lemma 3.2, we obtain that there exists a constant $T_5 \geq T_3$ such that
\[
y(t) \geq y(t) - \epsilon. \quad (40)
\]
for all $t \geq T_5$. From (39) and (40), we can obtain
\[
\frac{di(t)}{dt} \geq i(t)(\beta(t)s(t) - c(t) - \eta_2(t)y(t)) - i(t)(\beta(t)(s^*(t) - \epsilon) - c(t) - \eta_2(t)(y(t) - \epsilon)), \quad (41)
\]
for any $t \geq T_6 := \max\{T_4, T_5\}$. From $\liminf_{t \to \infty} \int_{t}^{t+\gamma} \beta(\theta)d\theta > 0$, we obtain that there exist $\omega > 0$ and $T_7 > T_6$ such that for any $t \geq T_7$ we have $\int_{t}^{t+\gamma} \beta(\theta)d\theta > \frac{1}{2}\omega$. Let the constant $p > 0$ be given such that $q_1 \exp \left(-\frac{p}{2}\omega\right) < \zeta$. For this constant $p > 0$, we further choose an integer $K_p > 0$ such that
\[
\frac{1}{2}K_p\omega - \beta^p\gamma \geq p,
\]
let $\lambda = K_p\gamma$, for any $t \geq T_7$,
\[
\int_{t}^{t+\gamma} \beta(\theta)d\theta = \left( \int_{t}^{t+\gamma} + \int_{t+(K_p-1)\gamma}^{t+K_p\gamma} + \int_{t+K_p\gamma}^{t+\lambda} \right) \beta(\theta)d\theta \geq \frac{1}{2}K_p\omega - \beta^p\gamma \geq p. \quad (42)
\]
From (37) and (42), we have
\[
\liminf_{t \to \infty} \int_t^{t+\lambda} (\beta(\theta)(s^*(\theta) - \epsilon) - c(\theta) - \eta_2(\theta)y(\theta))d\theta \leq -\frac{1}{2}p\epsilon,
\]
hence, there is \(t^* > T_5\) such that
\[
\liminf_{t \to \infty} \int_t^{t+\lambda} (\beta(\theta)(s^*(\theta) - \epsilon) - c(\theta) - \eta_2(\theta)y(\theta))d\theta \leq -\frac{1}{2}p\epsilon.
\]
Integrating (41) from \(t^*\) to \(t^* + \lambda\), we obtain
\[
i(t^* + \lambda) \\
\leq i(t^*) \exp \int_{t^*}^{t^*+\lambda} (\beta(\theta)(s^*(\theta) - \epsilon) - c(\theta) - \eta_2(\theta)y(\theta))d\theta \\
\leq q_1 \exp \left(-\frac{1}{2}p\epsilon\right) < \zeta.
\]
This leads to a contradiction with (38). This completes the proof of Theorem 4.4. \(\square\)

**Theorem 4.5.** Suppose that assumptions (A1), (A2) hold and there exists a constant \(\lambda^* > 0\) such that
\[
\limsup_{t \to \infty} \frac{1}{\lambda^*} \int_t^{t+\lambda^*} (\beta(\theta)s_0(\theta) - c(\theta) - \eta_2(\theta)y(\theta))d\theta \leq 0, \quad (43)
\]
then the infected prey in model (2) is extinct. Here \(s_0(t), y(t)\) be some fixed positive solutions of (6) and (36), respectively.

**Proof.** From (A2), there exist constants \(\eta > 0\) and \(T_0 > 0\) such that
\[
\int_t^{t+\gamma} \beta(\theta)d\theta \geq \eta, \text{ for all } t \geq T_0.
\]
For any constant \(0 < \epsilon < 1\), we set \(\epsilon_0 = \min \left\{ \frac{\lambda^* \eta}{2\gamma}, \frac{1}{2}p\epsilon \right\}\). If (43) holds, then there exist constants \(\delta > 0\) and \(T_1 > T_0\) such that
\[
\int_t^{t+\lambda^*} (\beta(\theta)(s_0(\theta) + \delta) - c(\theta) - \eta_2(\theta)(y(\theta) - \delta))d\theta \leq \epsilon_0,
\]
for all \(t \geq T_1\). Choose an integer \(n_0\) satisfying \(\frac{2\lambda^*}{\lambda} \leq n_0 \leq \frac{2\lambda^*}{\lambda} + 1\). Define \(\lambda_0 = n_0\lambda^*\), then we have
\[
\int_t^{t+\lambda_0} (\beta(\theta)(s_0(\theta) + \delta) - c(\theta) - \eta_2(\theta)(y(\theta) - \delta) - \beta(\theta)\epsilon)d\theta \leq \int_t^{t+\lambda_0} \beta(\theta)\epsilon d\theta - \int_t^{t+2\gamma} \beta(\theta)\epsilon d\theta \\
\leq n_0\epsilon_0 - 2\eta\epsilon \leq -\frac{1}{2}p\epsilon.
\]
According to system (2), we have
\[
\frac{d(s(t) + i(t))}{dt} \leq A(t) - d(t)(s(t) + i(t)),
\]
for all \( t \geq T_1 \). By Lemma 3.4, there exists a constant \( T_2 \geq T_1 \) such that
\[
s(t) + i(t) \leq s_0(t) + \delta
\]
for all \( t \geq T_2 \). By the third equation of system (2), we obtain
\[
\frac{dy(t)}{dt} \geq y(t)(k_1(t)\eta_1(t)p_{1s} - b(t)y(t)),
\]
for all \( t \geq T_1 \). From Lemma 3.2, there exists a constant \( T_3 \geq T_1 \) such that \( y(t) \geq y(t) - \delta \) for all \( t \geq T_3 \).

Set \( T = \max\{T_2, T_3\} \), then for any \( t \geq T \), we have
\[
\frac{di(t)}{dt} = i(t)(\beta(t)s(t) - c(t) - \eta(t)y(t)) \\
\leq i(t)[\beta(t)(s_0(t) + \delta - i(t)) - c(t) - \eta(t)(y(t) - \delta)].
\]
(45)

For any constant \( \epsilon \in (0, 1) \), if \( i(t) \geq \epsilon \) for all \( t \geq T \), then integrating (45) from \( T \) to \( t \), we can obtain
\[
i(t) \leq i(T) \exp \int_{T}^{t} [\beta(\theta)(s_0(\theta) + \delta) - c(\theta) - \eta_2(\theta)(y(\theta) - \delta) - \beta(\theta)\epsilon]d\theta.
\]

From (44), it follows that \( i(t) \to 0 \) as \( t \to \infty \). This reduces to a contradiction with \( i(t) \geq \epsilon \). Hence, there exists a \( t_1 \geq T \) such that \( i(t_1) < \epsilon \). Denote
\[
h = \sup_{t \geq T}\{\beta(t)(s_0(t) + \delta) + c(t) + \eta(t)(y(t) - \delta) + \beta(t)\}.
\]

Finally, we will prove
\[
i(t) \leq \epsilon \exp(h\lambda_0), \quad \text{for all } t \geq t_1.
\]
(46)

for all \( t \geq t_1 \). If it is not true, then there exists a \( t_2 > t_1 \) such that \( i(t_2) > \epsilon \exp(h\lambda_0) \). From \( i(t_1) < \epsilon \), there exists a \( t_3 \in (t_1, t_2) \) such that \( i(t_3) = \epsilon \) and \( i(t) > \epsilon \) for all \( t \in (t_3, t_2) \). Let \( p \geq 0 \) be an integer such that \( t_2 \in (t_3 + p\lambda_0, t_3 + (p + 1)\lambda_0) \), then integrating (45) from \( t_3 \) to \( t_2 \), we have
\[
\epsilon \exp(h\lambda_0) < i(t_2),
\]
i(t_2) \leq i(t_3) \exp \int_{t_3}^{t_2} [\beta(t)(s_0(t) + \delta) - c(t) - \eta_2(t)(y(t) - \delta) - \beta(t)\epsilon]dt,
\]
i(t_3) \exp \int_{t_3}^{t_2} [\beta(t)(s_0(t) + \delta) - c(t) - \eta_2(t)(y(t) - \delta) - \beta(t)\epsilon]dt \leq \epsilon \exp(h\lambda_0),
\]
which means a contradiction. Hence, we must have (46) holds true. From the arbitrariness of \( \epsilon \), we finally obtain \( i(t) \to 0 \) as \( t \to \infty \).

This completes the proof of Theorem 4.5. \( \square \)

**Remark 3.** From the detailed expressions of \( s_0 \) and \( y \) defined in Theorem 4.5, for inequality (43), one can also easily have: the available maximum growth rate \( \beta(t)s_0(t) - c(t) - \eta_1(t)y(t) \) of the infected prey at any time \( t \) decrease with the increase of \( \eta_1(t) \). Thus [22, inequality (35)] implies the establishment of inequality (43).

**Corollary 2.** [22, inequality (35)] implies the establishment of inequality (43) iff \( \eta_{1\infty} > \frac{b^\infty Y_0}{k_{1\infty}p_{1s}} \) holds true.
Proof. By (36), we get that $y > \frac{k_1 \eta_1 \rho_{P1s}}{k_1 \rho_{P1s}}$, thus when $\eta_1 > \frac{b^* y_0}{k_1 \rho_{P1s}}$, we can obtain

$$\eta_1 > \frac{b^* y_0}{k_1 \rho_{P1s}} \iff \frac{k_1 \eta_1 \rho_{P1s}}{k_1 \rho_{P1s}} > y_0 \iff y > y_0.$$ 

Further, we can get

$$\beta(t)s_0(t) - c(t) - \eta_2(t)y(t) < \beta(t)s_0(t) - c(t) - \eta(t)y_0(t),$$

where $\eta(t) = \eta_2(t)$ is hunting rate on infected prey, $p_{1s}$ is positive lower bound of susceptible prey, and therefore [22, inequality (35)] implies the establishment of inequality (43) (where function $y_0(t)$ correspond function $y_0(t)$ in [22, inequality (16)]).

Remark 4. If we assume that all coefficients $A(t)$, $\beta(t)$, $d(t)$, $c(t)$, $\eta_1(t)$, $\eta_2(t)$, $b(t)$, $k_1(t)$, $k_2(t)$ are constant, then system (2) degenerates to the autonomous system (34).

Evidently, from Theorem 4.5, we have if $\beta \tilde{s}_0 - c - \eta_2 \tilde{y} \leq 0$, then the infected prey in model (34) is extinct (where $\tilde{s}_0$ and $\tilde{y}$ correspond to constant solutions of autonomous equations (6) and (36), respectively).

Remark 5. If the predator grows according to a logistic curve with an intrinsic birth rate $r(t)$, then condition (43) in Theorem 4.5 takes the form as

$$\limsup_{t \to \infty} \frac{1}{\lambda^*} \int_t^{t+\lambda^*} (\beta(\theta)s_0(\theta) - c(\theta) - \eta_2(\theta)\tilde{y}(\theta))d\theta \leq 0, \quad (47)$$

then the infected prey in model (2) is extinct.

Where $\tilde{y}(t)$ be some fixed positive solution of the following non-autonomous logistic equation

$$\frac{dy(t)}{dt} = y(t)[r(t) + k_1(t)\eta_1(t)p_{1s} - b(t)y(t)], \quad (48)$$

$p_{1s}$, $q_2$ are positive lower and under bound of susceptible prey and predator, respectively. Contrast (48) and (36), we get $y < \tilde{y}$, then (47) implies the establishment of inequality (43).

5. Numerical simulations. In this section, we perform simulations to confirm and extend our analytical results of the eco-epidemic model (2). We use Matlab’s ode 45s routine to solve the ODE model (2).

First, we increase $\eta_1(t)$ (the capture rate over susceptible prey) while fixing the other parameters, (see FIG. 1). More specifically, we set the same parameter values as those in [22]: $A(t) = 30 + 5\sin t$, $\beta(t) = 0.2\cos t + 5$, $d(t) = 4 + \cos t$, $k_1(t) = k_2(t) = 0.3\cos t + 0.6$, $b(t) = 6 + 2\sin t$, $c(t) = 10 + \cos t$ and $\eta_2(t) = 5 + \cos t$.

FIG. 1 (a) shows that if the predator only hunts the infected prey, then the prey and predator coexist permanently, which confirms Theorem 4.1: On the other hand, with the increase of the capture rate over susceptible prey $\eta_1(t)$ may result in the extinction of the infected prey, as shown in FIG. 1 (d), this verifies Theorem 4.5: moreover, considering the density of the predator, it follows from FIG. 1 (a)-(d) that the increase of $\eta_1(t)$ may enlarge the density of the predator $y(t)$.

Second, we use FIG. 2 to show the impacts of non-periodic oscillation on the dynamical behaviors of model (2), here we increase $\eta_1(t)$ while set the other parameters: $A(t) = 30 + 5\sin(\sqrt{t})$, $c(t) = 10 + \cos(\sqrt{t})$, $\beta(t) = 0.2\cos(\sqrt{t}) + 5$, $d(t) = 4 + \cos(\sqrt{t}) + 5$. 
Figure 1. Solutions of system (2) with different hunting rate $\eta_1(t)$ on susceptible prey $s$. Here (a): $\eta_1(t) = 0$; (b): $\eta_1(t) = 1.5 + \cos t$; (c): $\eta_1(t) = 3 + \cos t$; (d): $\eta_1(t) = 4.5 + \cos t$.

For model (49), we set $\eta_1(t) = \cos t + 1$, and all the other parameters except $r(t)$ are same as those for FIG. 1, and we let $r(t)$ vary.

FIG. 3 (b) shows that introduction of the external resources $r(t) \neq 0$ of the predator may enlarge its density, but there are differences in the effects of the prey: increase the density of the susceptible prey $s$ as well as decrease that of the
A NON-AUTONOMOUS PREDATOR-PREY MODEL WITH INFECTED PREY

Figure 2. Solutions of system (2) with different hunting rate $\eta_1(t)$ on susceptible prey $s$. (a): $\eta_1(t) = 0$; (b): $\eta_1(t) = 1.5 + \cos(\sqrt{t})$; (c): $\eta_1(t) = 3 + \cos(\sqrt{t})$; (d): $\eta_1(t) = 4.5 + \cos(\sqrt{t})$.

infected prey $i$, and as shown in FIG. 3 (c), a sufficiently large $r(t)$ may drive $i$ into elimination and destroy the permanence of system (49); it’s worth noting that for the susceptible prey $s$, the impacts of $r(t)$ are complicated, as shown from FIG. 3 (a), (b): $s$ may even benefit from the increase of $r(t)$. This supports Remark 5.

6. Discussion. In this paper, we have formulated a non-autonomous eco-epidemic model with infected prey, with the assumptions that the disease is endemic in prey without the invasion of predator and that predation is more likely on infected prey than on the uninfected. We have developed a rigorous analysis of the model by applying the comparison theory of differential equations and the oscillation theory: If the predation is only on the infected prey, Theorem 4.1 shows that all the population in the system are always permanent; if the predation is on both the susceptible and the infected prey, Theorem 4.2 shows that the susceptible prey and predator in the model are always permanent; sufficient conditions on the permanence and extinction of the infected prey are also obtained in Theorems 4.3-4.5. Our analysis yields that the upper and lower growth rate of the infected prey in the mean on interval $[t, t + \lambda^*]$ for some constants $\lambda^* > 0$ (see (43) and (22)) correspond to related sufficient conditions for the extinction and permanence of the infected prey.

Mathematically, we overcome the challenging from the stronger coupling in our model (2) than that in model (1) considered by [22], our results extend [22, Theorem 1] in Theorem 4.1 to the case $r(t) \equiv 0$. Our results Theorems 4.3-4.5 enrich...
the corresponding [22, Theorems 2-4] with the applications to the model (2) with predation on both the susceptible and the infected prey.

Biologically, our above analytic results show that the predation preference to infected prey may increase the possibility of disease endemic among preys. As the predation is completely on the infected prey, the disease will always keep endemic among preys, however, as the preference is gradually weaken, the disease may be eliminated among preys. In addition FIG. 2 also show that a more “balancing” predation strategies over the susceptible and infected prey could be helpful for predator to reach its maximum density, which is consistent with the corresponding conclusions in [20] for autonomous eco-epidemic model.

In this paper, the problem on how to obtain the predator’s optimal hunting strategy for its maxima density is still open, which we leave as our future work.

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