Artin-Mazur heights and Yobuko heights of proper log smooth schemes of Cartier type, and Hodge-Witt decompositions and Chow groups of quasi-\(F\)-split threefolds

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Abstract

In this article we prove a fundamental inequality between Artin-Mazur heights and Yobuko heights of certain proper log smooth schemes of Cartier type over a fine log scheme whose underlying scheme is the spectrum of a perfect field \(\kappa\) of characteristic \(p > 0\). We also prove that the cohomologies of Witt-sheaves of them are finitely generated \(\mathbb{W}(\kappa)\)-modules if the Yobuko heights of them are finite. As an application, we prove that the \(p\)-primary torsion parts of the Chow groups of codimension 2 of proper smooth threefolds over \(\kappa\) is of finite cotype if the Yobuko heights of them are finite. These are nontrivial generalizations of results in \([JR]\) and \([J]\).

1 Introduction

Let \(\kappa\) be a perfect field of characteristic \(p > 0\). Let \(\pi\) be an algebraic closure of \(\kappa\). Let \(\sigma: \pi \rightarrow \pi\) be the Frobenius automorphism of \(\pi\). Let \(\mathcal{W}\) (resp. \(\mathcal{W}_n\) (\(n \in \mathbb{Z}_{>0}\))) be the Witt ring of \(\kappa\) (resp. the Witt ring of \(\kappa\) of length \(n\)). Let \(Z\) be a proper scheme over \(\kappa\) and let \(q\) be a nonnegative integer. Let \(\Phi^q_{Z/\kappa}\) be the Artin-Mazur group functor of \(Z/\kappa\) in degree \(q\), that is, \(\Phi^q_{Z/\kappa}\) is the following functor:

\[
\Phi^q_{Z/\kappa}(A) := \text{Ker}(H^q_{et}(Z \otimes_{\kappa} A, \mathbb{G}_m) \rightarrow H^q_{et}(Z, \mathbb{G}_m)) \in (\text{Ab})
\]

for artinian local \(\kappa\)-algebras \(A\)'s with residue fields \(\kappa\) ([AM]). If \(\Phi^q_{Z/\kappa}\) is pro-representable by a formal group scheme over \(\kappa\), then we denote the height of \(\Phi^q_{Z/\kappa}\) by \(h^q(\kappa)\). We call \(h^q(Z/\kappa)\) the \(q\)-th Artin-Mazur height of \(Z/\kappa\).

Let \(Y\) be a proper smooth scheme over \(\kappa\). In \([I]\) Illusie has constructed the following slope spectral sequence

\[
E_1^{ij} = H^j(Y, \mathcal{W}\Omega^i_Y) \Rightarrow H_{crys}^{i+j}(Y/W)
\]

by generalizing the slope spectral sequence constructed by Bloch in \([B]\). It is well-known that the \(E_1\)-term \(H^j(Y, \mathcal{W}\Omega^i_Y)\) is not a finitely generated \(\mathcal{W}\)-module in general. For example, if the Artin-Mazur group functor of \(Y/\kappa\) in degree \(j\) is pro-represented by

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a 1-dimensional formal Lie group with infinite height, then \( H^j(Y, W(\mathcal{O}_X)) \otimes_{\mathcal{W}} \mathcal{W}(\mathcal{F}) \simeq \pi_\sigma[[Y]] \), where \( F = 0 \) and \( aV = V a^\sigma \) \((a \in \pi)\) on \( \pi_\sigma[[Y]] \) \((\pi_\sigma \text{ means the last equality})\).

In this article we are interested in the finitely generatedness of \( H^q(Y, W(\mathcal{O}_X)) \) \((q \in \mathbb{N})\) and its remarkable consequences for threefolds over \( \kappa \). Let \( F : Y \to Y \) be the Frobenius endomorphism of \( Y \). In [MR] Mehta and Ramanathan have given the definition of the \( F \)-splitness of \( Y \): \( Y \) is said to be \( F \)-split if the pull-back \( F^*: \mathcal{O}_Y \to F_*\mathcal{O}_Y \) has a section of \( \mathcal{O}_Y \)-modules. In [J] and [JR] Joshi and Rajan have proved the following theorem:

**Theorem 1.1** ([J], [JR]). Assume that \( Y \) is \( F \)-split. Then \( H^q(Y, W(\mathcal{O}_Y)) \) \((q \in \mathbb{N})\) is a finitely generated \( \mathcal{W} \)-module.

As far as we know, no nontrivial generalization of this theorem has been known.

Let \( F : W_n(Y) \to W_n(Y) \) be the Frobenius endomorphism of \( W_n(Y) \). Let \( F^*: W_n(\mathcal{O}_Y) \to F_*(W_n(\mathcal{O}_Y)) \) be the pull-back of \( F \). Recently Yobuko has generalized the notion of the \( F \)-splitness: he has introduced the notion of the quasi-Frobenius splitting height \( h_F(Y) \) of \( Y \) in a highly nontrivial way ([Y1]) as follows.

In [loc. cit.] he has denoted it by \( h^Y(E) \). It is the minimum of positive integers \( n \)'s such that there exists a morphism \( \rho : F_*(W_n(\mathcal{O}_Y)) \to \mathcal{O}_Y \) of \( W_n(\mathcal{O}_Y) \)-modules such that \( \rho \circ F^*: W_n(\mathcal{O}_Y) \to \mathcal{O}_Y \) is the natural projection. (If there does not exist such \( n \), then we set \( h_F(Y) = \infty \).) Because the terminology “quasi-Frobenius splitting height” is too long, we call this height \( Yobuko height \). It seems to us that the Yobuko height is a mysterious invariant of \( Y \). It plays a central role in this article. If \( h_F(Y) < \infty \), then we say that \( Y \) is quasi-\( F \)-split following [Y1].

Let \( s \) be a fine log scheme whose underlying scheme is \( \text{Spec}(\kappa) \). Let \( W(s) \) be the canonical lift of \( s \) over \( \text{Spf}(\mathcal{W}) \). Let \( X/s \) be a proper log smooth scheme of Cartier type. (In this article we do not recall fundamental notions of log geometry in [K1], [K2] and [HK].) Let \( W\Omega^*_X \) be the log de Rham-Witt complex of \( X/s \) and let \( H^{i}_{\text{crys}}(X/W(s)) \) \((q \in \mathbb{N})\) be the log crystalline cohomology of \( X/W(s) \). Following [IR] in the trivial logarithmic case, we say that \( X/s \) is of log Hodge-Witt type if \( H^i(X, W\Omega^*_X) \) is a finitely generated \( \mathcal{W} \)-module for any \( i, j \in \mathbb{N} \). (We do not use a phrase: “\( X/s \) is log Hodge-Witt.”) If \( X/s \) is of log Hodge-Witt type, then the slope spectral sequence

\[
E^{ij}_1 = H^j(X, W\Omega^i_X) \Rightarrow H^{i+j}_{\text{crys}}(X/W(s))
\]

degenerates at \( E_1 \) and there exists the following log Hodge-Witt decomposition for the log crystalline cohomology:

\[
H^q_{\text{crys}}(X/W(s)) = \bigoplus_{i+j=q} H^j(X, W\Omega^i_X) \quad (q \in \mathbb{N})
\]

by the log version of Illusie-Raynaud’s theorem in [IR]. Let \( \mathring{X} \) be the underlying scheme of \( X \). If \( \dim \mathring{X} = 1 \), then \( X/s \) is of log Hodge-Witt type.

The key theorem in this article is the following:

**Theorem 1.2.** Assume that \( \mathring{X} \) is quasi-\( F \)-split. Then \( H^q(X, W(\mathcal{O}_X)) \) \((q \in \mathbb{N})\) is a finitely generated \( \mathcal{W} \)-module.

This is a highly nontrivial generalization of [HK]. To prove this theorem, we prove the following:
Theorem 1.3. Assume that $\bar{X}$ is quasi-$F$-split. Then the dimensions $\dim_\kappa H^q(X, B_n^\Omega^1_{X/s})$ for all $q$’s and all $n$’s are bounded. Here $B_n^\Omega^1_{X/s}$ $(n \in \mathbb{N})$ is a well-known sub $\mathcal{O}_X$-module of $F^*_n(\Omega^1_{X/s})$, where $F: X \to X$ is the Frobenius endomorphism of $X$.

As a corollary of this theorem, we obtain the following:

Corollary 1.4. Set $B_\infty^\Omega^1_{X/s} := \lim_{\longrightarrow} B_n^\Omega^1_{X/s}$. Here we take the inductive limit as abelian sheaves on $\mathcal{O}_X$ and the transition morphisms are the natural inclusion morphisms. Consider $B_\infty^\Omega^1_{X/s}$ as a sheaf of $f^{-1}(\kappa)$-submodules of $\Omega^1_{X/s}$, where $f: X \to s$ is the structural morphism. Assume that $\bar{X}$ is quasi-$F$-split. Then $\dim_\kappa H^q(X, \Omega^1_{X/s}/B_\infty^\Omega^1_{X/s}) \ (q \in \mathbb{N})$ are finite.

This corollary implies the tangent space of the “pro-representable part” of the formal completion of the second Chow group of a proper smooth surface over $\kappa$ due to Stienstra (\cite{St2}) is finite dimensional if it is quasi-$F$-split. See (4.8) in the text for the detailed explanation.

As a corollary of (1.2), we also obtain the following by using the log version of a theorem in \cite{I1} (cf. \cite{L}, \cite{N2}):

Corollary 1.5. Assume that $\bar{X}$ is quasi-$F$-split and that $\dim \bar{X} = 2$. Then $X/s$ is of log Hodge-Witt type.

In \cite{JR} Joshi and Rajan have proved that a proper smooth $F$-split surface over $\kappa$ is ordinary. Hence, by a fundamental theorem in \cite{JR}, it is of Hodge-Witt type. The corollary (1.5) is a generalization of this result in two directions: the logarithmic case and the case where the Yobuko height is finite. (A proper smooth scheme over $\kappa$ with finite Yobuko height is far from being ordinary in general.)

In \cite{J} Joshi has proved the following theorem:

Theorem 1.6 (\cite{J}). Let $Y/\kappa$ be a proper smooth scheme of dimension 3. Then $Y/\kappa$ is of Hodge-Witt type if and only if $H^q(Y, W(\mathcal{O}_Y)) \ (q \in \mathbb{N})$ is a finitely generated $W$-module.

As an immediate corollary of (1.2) and (1.6), we obtain one of the following main results in this article:

Corollary 1.7. Let $Y/\kappa$ be as in (1.6). Assume that $Y$ is quasi-$F$-split. Then $Y/\kappa$ is of Hodge-Witt type.

If we prove the log version of Ekedahl’s duality (12) for dominoes associated to the differential: $d: H^j(X, W\mathcal{O}_X) \to H^{j+1}(X, W\mathcal{O}_X^{i+1})$, then one can obtain the log version of (1.7) as in \cite{J} (6.1)]. We would like to discuss this in a future paper.

In the course of the proof of (1.2), we also obtain the following unexpected result as a bonus:

Theorem 1.8 (Fundamental inequality between Artin-Mazur heights and a Yobuko height). Let $X/s$ be a proper log smooth scheme of Cartier type. Let $q$ be a nonnegative integer. Assume that $H^q(X, \mathcal{O}_X) \simeq \kappa$, that $H^{q+1}(X, \mathcal{O}_X) = 0$ and that the Bockstein operator

$$\beta: H^{q-1}(X, \mathcal{O}_X) \to H^q(X, W_{n-1}(\mathcal{O}_X))$$

is a surjection.
arising from the following exact sequence
\[ 0 \to W_{n-1}(\mathcal{O}_X) \xrightarrow{V} W_n(\mathcal{O}_X) \to \mathcal{O}_X \to 0 \]
is zero for any \( n \in \mathbb{Z}_{\geq 2} \). Here \( V : W_{n-1}(\mathcal{O}_X) \to W_n(\mathcal{O}_X) \) is the Verschiebung morphism. Assume that the functor \( \Phi^q_{X/\kappa} \) is pro-representable. Then
\[ h^d(\tilde{X}/\kappa) \leq h_F(\tilde{X}). \]

In particular, if \( h^q(\tilde{X}/\kappa) = \infty \), then \( h_F(\tilde{X}) = \infty \).

Before we proved this theorem, we had not even imagined that a relation between \( h^q(\tilde{X}/\kappa) \) and \( h_F(\tilde{X}) \) (even \( h^\dim(\tilde{X}/\kappa) \) and \( h_F(\tilde{X}) \)) for a general \( X/s \) as in \( \text{(1.8)} \) exists because the definitions of \( h^q(\tilde{X}/\kappa) \) and \( h_F(\tilde{X}) \) are completely different. After we have proved this theorem, we have been convinced that this theorem is true by the examples stated after \( \text{(1.9)} \) below. The theorem \( \text{(1.8)} \) tells us that the Yobuko height of \( \tilde{X} \) is a upper bound of all Artin-Mazur heights of \( \tilde{X}/\kappa \) under the assumptions in \( \text{(1.8)} \). Because \( H^q(X, \mathcal{W}(\mathcal{O}_X)) \) is isomorphic to the Dieudonné module of \( \Phi^q_{X/\kappa} \) \( \text{(AM)} \), it is a free \( \mathcal{W} \)-module of rank \( h^q(\tilde{X}/\kappa) \) if \( h^q(\tilde{X}/\kappa) < \infty \). \( \text{(1.8)} \) tells us a partial clear reason why \( \text{(1.2)} \) holds.

More generally we would like to ask the following:

**Problem 1.9 (Inequality problem between Artin-Mazur heights and a Yobuko height).** Let \( Z/\kappa \) be a proper geometrically connected scheme. Let \( q \) be a nonnegative integer. If \( h_F(Z) < \infty \), then does the following inequality hold?
\[ \text{rank}_{\mathcal{W}}\{H^q(Z, \mathcal{W}(\mathcal{O}_Z))/(\text{torsion})\} \leq h_F(Z) \]
hold? If \( \dim_{\mathcal{O}_Z} H^q(Z, \mathcal{W}(\mathcal{O}_Z)) = \infty \), then \( h_F(Z) = \infty ? \) Here \( pH^q(Z, \mathcal{W}(\mathcal{O}_Z)) \) is the subgroup of \( p \)-torsion elements of \( H^q(Z, \mathcal{W}(\mathcal{O}_Z)) \).

If the answer for this problem is Yes, if \( \Phi^q_{Z/\kappa} \) is representable and if \( h_F(Z) < \infty \), then \( h^q(Z/\kappa) \leq h_F(Z) \).

In \( \text{[Y]} \) Yobuko has proved an equality \( h^d(Y/\kappa) = h_F(Y) \) for a Calabi-Yau variety \( Y \) over \( \kappa \) of any dimension \( d \). Let \( X/s \) be a proper log smooth scheme of Cartier type.

Assume that \( s \) is the log point of \( \kappa \) and that \( \tilde{X} \) is of pure dimension \( d \). If \( X/s \) is a simple normal crossing scheme and if the following three conditions hold, then we have proved an equality \( h^d(\tilde{X}/\kappa) = h_F(\tilde{X}) \) in \( \text{[NY]} \). Here \( \Omega^2_{X/s} \) is the \( d \)-th wedge product of sheaves of logarithmic differential forms on \( X/s \). Yobuko has also proved that \( h^2(\mathcal{O}_X) = h_F(\mathcal{O}) \) for an abelian surface \( Z/\kappa \) (unpublished).

(In particular, in these cases \( X \) (resp. \( Z \)) is \( F \)-split if and only if \( h^d(\tilde{X}/\kappa) = 1 \) (resp. \( h^2(\mathcal{O}_X) = 1 \). This is nontrivial.) In \( \text{[Y2]} \) he has given an example such that \( h^2(Y/\kappa) < h_F(Y) \) for an Enriques surface \( Y \) over \( \kappa \).
Here $\tilde{Y}$ is the K3 cover of $Y$.

By using Ekedahl’s theorem and his remark in [IR], we also obtain the following as a corollary of (1.7):

**Corollary 1.10.** Let the notations be as in (1.7). Then the following hold:

1. The following spectral sequence

   $$(1.10.1) \quad E_1^{ij} = H^i(Y, \mathcal{W}_n \Omega^j_Y) \Rightarrow H_{\text{crys}}^{i+j}(Y/\mathcal{W}_n)$$

   degenerates at $E_2$ for all $n \in \mathbb{N}$.

2. If the operator $F: H^j(Y, \mathcal{W}_n \Omega^j_Y) \to H^j(Y, \mathcal{W}_n \Omega^j_Y)$ ($\forall i, j$) is injective, especially if $H^q_{\text{crys}}(Y/\mathcal{W})$ is torsion-free for $2 \leq q \leq 5$, then the spectral sequence (1.10.1) degenerates at $E_1$ for all $n \in \mathbb{N}$.

In [Y1] Yobuko has proved that the spectral sequence (1.10.1) for the case $n = 1$ degenerates at $E_1$ for a Calabi-Yau variety of any dimension $d$ with finite $d$-th Artin-Mazur height if $d \leq p$ by proving that it has a smooth lift over $\mathcal{W}_2$ and using a famous theorem of Deligne-Illusie ([DI]). In [NY] we have generalized this Yobuko’s theorem for $X/s$ stated after (1.9).

The corollary (1.7) also has an application for the $p$-primary torsion parts of the Chow groups of codimension $2$ of threefolds over $k$ as follows.

Let $Y$ be a proper smooth scheme over $k$. Set $Y_p := Y \times_k \mathfrak{g}$. Let $\mathcal{W}_n \Omega^i_{Y_p, \log} (i \in \mathbb{N})$ be the complex of sheaves of logarithmic parts of $\mathcal{W}_n \Omega^i_{Y_p}$. Let $F: \mathcal{W}_n \Omega^i_{Y_p, \log} \to \mathcal{W}_{n+1} \Omega^i_{Y_p, \log}$ be the induced morphism by the multiplication by $p \times: \mathcal{W}_{n+1} \Omega^i_{Y_p, \log} \to \mathcal{W}_{n+1} \Omega^i_{Y_p, \log}$. Set $H^j(Y_p, \mathbb{Q}_p/\mathbb{Z}_p(i)) := \lim H^{j-i}(Y_p, \mathcal{W}_n \Omega^i_{Y_p, \log})$. Let $CH^r(Y_p)\{p\}$ the $p$-primary torsion parts of the Chow group of codimension $r$ of $Y_p/\mathfrak{g}$. In [J] Joshi has proved the following theorem by using (1.6), Ekedahl’s duality for dominoes and the following injectivity of the $p$-adic Abel-Jacobi map of Gros and Suwa ([GS]):

$$CH^2(Y_p)\{p\} \subseteq H^3(Y_p, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

and their result ([loc. cit., II (3.7)]):

**Theorem 1.11 ([J]).** Let $Y/k$ be a proper smooth scheme of pure dimension $3$. If $H^3(Y_p, \mathcal{O}_{Y_p})$ is a finitely generated $\mathbb{W}(\mathfrak{g})$-module, then $CH^2(Y_p)\{p\}$ is of finite cotype.

In particular, if $Y_p/\mathfrak{g}$ is a $3$-dimensional Calabi-Yau variety with finite third Artin-Mazur height, then $CH^2(Y_p)\{p\}$ is of finite cotype. As far as we know, little had been known about $CH^2(Y_p)\{p\}$ for a $3$-dimensional proper smooth scheme $Y/k$ except his result (cf. [BM] and [MR]).

We obtain the following result as a corollary of (1.6), (1.7) and (1.11):

**Corollary 1.12.** Let $Y/k$ be a proper smooth scheme of pure dimension $3$. Assume that $Y_p$ is quasi-$F$-split. Then $CH^2(Y_p)\{p\}$ is of finite cotype.

| $Y$                  | $h^2(Y/\kappa)$ | $h_F(Y)$ |
|----------------------|-----------------|----------|
| Enriques surface     | $0$             | $h_F(Y)$ |
| classical Enriques   | $0$             | $\infty$ |
| singular Enriques    | $0$             | $1$      |
| supersingular Enriques surface | $0$     | $\infty$ |
The contents of this article are as follows.

In §2 we recall the characterization of the height of the Artin-Mazur formal group of certain proper schemes over $\kappa$. This is a generalization of the characterization for Calabi-Yau varieties over $\kappa$ due to Katsura and Van der Geer (vGK) and this has been proved in a recent preprint [NY]. We also recall a log version of Serre’s exact sequence in [Se], which has been proved in [NY]. Using these results, we have determined the dimensions of cohomologies of sheaves of closed log differential forms of degree 1 in [NY] as in [vGK].

In §3 we generalize the log version of Serre’s exact sequence to the case of higher degrees as in [I1]. For the generalization we recall theory of log de Rham-Witt complexes in [L] and [N2]. In this article we use theory of formal de Rham-Witt complexes in [N2] which makes proofs of log versions of a lot of statements in [I1] simple explicit calculations.

In §4 we prove (1.2) by using the logarithmic version of a key commutative diagram in [Y2]. We also prove (1.8) by using the determination of the dimensions in §2. In this section we also prove (1.10) and (1.12).

In §5 we prove the following theorem:

**Theorem 1.13.** Let $X/s$ be a proper log smooth scheme of Cartier type. Assume that $\Phi^q_{X/\kappa}$ is representable. Let $(\Phi^q_{X/\kappa})^*$ be the Cartier dual of $\Phi^q_{X/\kappa}$. Assume that $h^q(X/\kappa)$ is finite. Assume also that the morphism $F: H^{q+1}(X, W(O_X)) \to H^{q+1}(X, W(O_X))$ is injective. Then

\begin{align}
(1.13.1) \quad h^q(X/\kappa) & \leq \dim_\kappa H^{q-1}(X, \Omega^1_{X/s}) + \dim_\kappa H^q(X, O_X) \\
(1.13.2) \quad \dim((\Phi^q_{X/\kappa})^*) & \leq \dim_\kappa H^{q-1}(X, \Omega^1_{X/s}).
\end{align}

(1.13) is a generalization of the following Katsura and Van der Geer’s results in [vGK2]:

**Proposition 1.14 (vGK2).** Let $Y/\kappa$ be a Calabi-Yau variety of pure dimension $d$. Then $h^d(Y/\kappa) \leq \dim_\kappa H^{d-1}(Y, \Omega^1_{Y/\kappa}) + 1$.

In the theorem (1.13) we need not to assume almost anything: the degree of the Artin-Mazur formal group and the dimension of it are arbitrary and the log variety $X/s$ is very general. We prove this theorem by using theory of log de Rham-Witt complexes in [L] and [N2]; the proof of (1.13) is very different from that in [vGK2]. In the text we prove a more general inequality than (1.13.1).

In §6 we give the definition of an ordinary log scheme at a bidegree. In the same section we also prove that the exotic torsion of the log crystalline cohomology of an $F$-split proper log smooth scheme does not exist. This is a log version of Joshi’s result. We also give concrete examples of $F$-split degenerate log schemes of dimension $\leq 2$.

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**Notation.** For a module $M$ and an element $f \in \text{End}(M)$, $f_*M$ (resp. $M/f$) denotes $\text{Ker}(f : M \to M)$ (resp. $\text{Coker}(f : M \to M)$). We use the same notation for an endomorphism of an abelian sheaf on a topological space.
Convention. We omit the second “log” in the terminology a “log smooth (integral) log scheme”.

2 Recall on results in [NY]

In this section we recall two results in [NY] which are necessary for the proofs of (1.2) and (1.8).

The following is a generalization of Katsura and Van der Geer’s theorem ([vGK, (5.1), (5.2), (16.4)]).

Theorem 2.1 ([NY, (2.3)]). Let $\kappa$ be a perfect field of characteristic $p > 0$. Let $Z$ be a proper scheme over $\kappa$. (We do not assume that $Z$ is smooth over $\kappa$.) Let $q$ be a nonnegative integer. Assume that $H^q(Z, \mathcal{O}_Z) \simeq \kappa$, that $H^{q+1}(Z, \mathcal{O}_Z) = 0$ and that $\Phi^q_{Z/\kappa}$ is pro-representable. Assume also that the Bockstein operator

$$\beta: H^{q-1}(Z, \mathcal{O}_Z) \to H^q(Z, W_{n-1}(\mathcal{O}_Z))$$

arising from the following exact sequence

$$0 \to W_{n-1}(\mathcal{O}_Z) \xrightarrow{V} W_n(\mathcal{O}_Z) \to \mathcal{O}_Z \to 0$$

is zero for any $n \in \mathbb{Z}_{\geq 2}$. Let $V: W_{n-1}(\mathcal{O}_Z) \to W_n(\mathcal{O}_Z)$ be the Verschiebung morphism and let $F: W_n(\mathcal{O}_Z) \to W_n(\mathcal{O}_Z)$ be the induced morphism by the Frobenius endomorphism of $W_n(Z)$. Let $n^q(Z)$ be the minimum of positive integers $n$’s such that the induced morphism

$$F: H^q(Z, W_n(\mathcal{O}_Z)) \to H^q(Z, W_n(\mathcal{O}_Z))$$

by the $F: W_n(\mathcal{O}_Z) \to W_n(\mathcal{O}_Z)$ is not zero. (If $F = 0$ for all $n$, then set $n^q(Z) := \infty$.) Let $h^q(Z/k)$ be the height of the Artin-Mazur formal group $\Phi^q_{Z/\kappa}$ of $Z/k$. Then $h^q(Z/k) = n^q(Z)$.

Proof. See the proofs of [NY] (2.3) (cf. [vGK] (5.1)).

As a corollary of (2.1) we obtain the following (see [vGK] (5.6) and [NY] (2.4) for the proof of (2.2)).

Corollary 2.2 (cf. [vGK] (5.6), [NY] (2.4)). Let the assumptions be as in (2.1). The following equalities hold:

$$(2.2.1) \quad \dim_{\kappa}(\rho H^q(Z, W_n(\mathcal{O}_Z))) = \min\{n, h^q(Z/k) - 1\},$$

$$(2.2.2) \quad \dim_{\kappa}(H^q(Z, W_n(\mathcal{O}_Z))/F) = \min\{n, h^q(Z/k) - 1\}.$$

Next we recall the log version of Serre’s exact sequence in [Se], which has been proved in [NY].

Let $Z$ be a scheme over $\kappa$. Let $F: Z \to Z$ be the absolute Frobenius endomorphism of $Z$. In [Se §7 (18)] Serre has defined the following morphism of abelian sheaves

$$d_n: F^*(\mathcal{O}_Z) \to F^*(\Omega^1_{Z/\kappa})$$
defined by the following formula:

\[ d_n((a_0, \ldots, a_{n-1})) = \sum_{i=0}^{n-1} a_i^{p^{n-1-i}} a_i \quad (a_i \in \mathcal{O}_Z). \]

(In loc. cit. he has denoted \( d_n \) by \( D_n \).) He has remarked that the following formula holds:

\[ d_n((a_0, \ldots, a_{n-1})(b_0, \ldots, b_{n-1})) = b_0^{p^{n-1}} d_n((a_0, \ldots, a_{n-1})) + a_0^{p^{n-1}} d_n((b_0, \ldots, b_{n-1})). \]

Hence the morphism \( d_n : F_*(\mathcal{W}_n(\mathcal{O}_Z)) \to F_n^*(\Omega^1_{Z/\kappa}) \) is a morphism of \( \mathcal{W}_n(\mathcal{O}_Z) \)-modules. (This remark was not given in loc. cit. See also (3.11) below.)

Let \( s \) be as in the Introduction. Let \( F_* : s \to s \) be the Frobenius endomorphism. Let \( Y/s \) be a log smooth scheme of Cartier type. Set \( Y' := Y \times_{s,F_*} s \). Let \( F := F_Y : Y \to Y' \) be the relative Frobenius morphism over \( s \). The log inverse Cartier isomorphism due to Kato is the following isomorphism of sheaves of \( \mathcal{O}_{Y'} \)-modules ([K1] (4.12) (1)):

\[ C^{-1} : \Omega^i_{Y'/s} \xrightarrow{\sim} F_*(\mathcal{H}^i(\Omega^\bullet_{Y/s})). \]

Because \( \tilde{\mathcal{F}} \) is a homeomorphism ([SGA 5] XV Proposition 2 a)), we can identify an abelian sheaf on \( \tilde{Y} \) with an abelian sheaf on \( Y' \). Under this identification, we can also express (2.2.6) as the equality

\[ C^{-1} : \Omega^i_{Y'/s} = \mathcal{H}^i(\Omega^\bullet_{Y/s}) \]

of abelian sheaves. Set \( B_0 \Omega^i_{Y/s} := 0, B_1 \Omega^i_{Y/s} := F_*(\text{Im}(d : \mathcal{O}_Y \to \Omega^1_{Y/s})) \) and \( Z_0 \Omega^i_{Y/s} := \Omega^i_{Y/s} \). We define \( B_n \Omega^i_{Y/s} \) and \( Z_n \Omega^i_{Y/s} \) by the following equalities \((n \geq 1)\):

\[ C^{-1} : B_{n-1} \Omega^i_{Y/s} = B_n \Omega^i_{Y/s}/F_*(B_1 \Omega^i_{Y/s}), \quad C^{-1} : Z_{n-1} \Omega^i_{Y/s} = Z_n \Omega^i_{Y/s}/F_*(B_1 \Omega^i_{Y/s}). \]

Then we have the following inclusions of abelian sheaves:

\[ 0 \subset B_1 \Omega^i_{Y/s} \subset \cdots \subset B_n \Omega^i_{Y/s} \subset B_{n+1} \Omega^i_{Y/s} \subset \cdots \subset Z_n \Omega^i_{Y/s} \subset \cdots \subset Z_1 \Omega^i_{Y/s} \subset \Omega^i_{Y/s}. \]

We can endow \( Z_n \Omega^i_{Y/s} \) and \( B_n \Omega^i_{Y/s} \) with natural \( \mathcal{O}_Y \)-submodule structures of \( F_*(\Omega^i_{Y/s}) \). (These notations \( Z_n \Omega^i_{Y/s} \) and \( B_n \Omega^i_{Y/s} \) are the same as Illusie’s notations in [II] 0 (2.2.2)) in the trivial logarithmic case.)

The following (2.3) is a log version of a generalization of Serre’s result in [II].

**Proposition 2.3 ([NY] (3.5), (3.6)).** Let \( n \) be a positive integer. Denote the following composite morphism

\[ F_*(\mathcal{W}_n(\mathcal{O}_Y)) \xrightarrow{d_n} F_n^*(\Omega^1_{Y/\kappa}) \to F_n^*(\Omega^1_{Y/s}) \]

by \( d_n \) again. Then the following fold:

(1) The morphism \( d_n \) factors through \( B_n \Omega^1_{Y/s} \) and the following sequence

\[ 0 \to \mathcal{W}_n(\mathcal{O}_Y) \xrightarrow{F} F_*(\mathcal{W}_n(\mathcal{O}_Y)) \xrightarrow{d_n} B_n \Omega^1_{Y/s} \to 0 \]
is exact. Here we denote the morphism $W_n(F^*) = W_n(F_{Y/s}^*)$ (resp. $F_*(W_n(O_Y)) \to B_n\Omega^1_{Y/s}$) by $F$ (resp. $d_n$) again by abuse of notation. Consequently $d_n$ induces the following isomorphism of $W_n(O_Y)$-modules:

\[(2.3.2)\quad F_*(W_n(O_Y))/W_n(O_Y) \sim \to B_n\Omega^1_{Y/s}.\]

(2) Let $R: W_n(O_Y) \to W_{n-1}(O_Y)$ be the projection. Then the following diagram

\[(2.3.3)\quad F_*(W_n(O_Y)) \xrightarrow{d_n} B_n\Omega^1_{Y/S_0} \quad \quad F_*(R) \quad \quad \downarrow \quad \quad \quad \downarrow C \quad \quad \quad \quad F_*(W_{n-1}(O_Y)) \xrightarrow{d_{n-1}} B_{n-1}\Omega^1_{Y/S_0}\]

is commutative.

**Proof.** See the proof of [NY (3.5)].

**Definition 2.4.** We call the exact sequence \[(2.3.1; n)\] of $W_n(O_Y)$-modules the log Serre exact sequence of $Y/s$ in level $n$.

**Corollary 2.5** ([NY (3.7)]). Let the assumptions be as in (2.1) for $Z := \bar{Y}$. Then $H^q(Y, W_n(O_Y))/F = H^q(Y, B_n\Omega^1_{Y/s})$. Consequently

\[(2.5.1)\quad \dim_h H^q(Y, B_n\Omega^1_{Y/s}) = \min\{n, h^q(\bar{Y}/\kappa) - 1\}.\]

**Proof.** See the proof of [NY (3.7)]; it is easy to derive this from (2.2) and (2.3).

### 3 A generalization of the log Serre exact sequence

In this section we recall theory of formal de Rham-Witt complexes in [N2] and generalize the log version of Serre’s exact sequence \[(2.3.1; n)\] by using this theory. That is, we prove that the following sequence

\[(3.0.1)\quad W_{n+1}\Omega^i_{Y/s} \xrightarrow{F} F_{W_n(Y)}(W_n\Omega^i_{Y/s}) \xrightarrow{F_{W_n(Y)}(F^{n-1}d)} B_n\Omega^{i+1}_{Y/s} \to 0 \quad (i \in \mathbb{N})\]

is exact in the category of $W_{n+1}(O_Y)$-modules for a log smooth scheme $Y/s$ of Cartier type. This generalization is a log version of Illusie’s generalization of Serre’s exact sequence in [H], though he has considered the exactness in [loc. cit.] only in the category of abelian sheaves not in the category of modules of the Witt sheaves of structure sheaves. If the reader wants to know only the proofs of the results (1.2) and (1.8) in the Introduction, he can skip this section. However we shall use (3.5.17) below for the proof of (1.13.1) and use several results in this section for results in the book [N3].

Let $\kappa$ be a perfect field of characteristic $p > 0$. Let $W$ be the Witt ring of $\kappa$. Let $\sigma$ be the Frobenius automorphism of $W$.

Let $(T, W)$ be a ringed topos: $T$ is a topos and $W$ is the constant sheaf in $T$ defined by the Witt ring $W$. Let $\Omega^*$ be a bounded complex of torsion-free $W$-modules and let $\phi: \Omega^* \to \Omega^*$ be a $\sigma$-linear morphism of complexes of $W$-modules. Let $p$ be a prime number. Set $\Omega^*_1 := \Omega^*/p\Omega^*$. We assume that the following conditions

\[(3.0.2) \sim (3.0.6)\) hold:
(3.0.2) $\Omega^i = 0$ for $i < 0$.

(3.0.3) $\Omega^i$ ($\forall i \in \mathbb{N}$) is $p$-torsion-free and $p$-adically complete $\mathcal{W}$-modules.

(3.0.4) $\phi(\Omega^i) \subset \{ \omega \in p^i\Omega^i \mid d\omega \in p^{i+1}\Omega^{i+1} \} \ (\forall i \in \mathbb{N})$.

(3.0.5) There exists a $\sigma$-linear isomorphism $C^{-1}: \Omega^i_1 \sim \mathcal{H}^i(\Omega^i_1) \ (\forall i \in \mathbb{N})$.

(3.0.6) A composite morphism $(\text{mod } p) \circ p^{-i}\phi: \Omega^i \rightarrow \Omega^i \rightarrow \Omega^i_1$ factors through $\ker(d: \Omega^i_1 \rightarrow \Omega^{i+1}_1)$ and the following diagram is commutative:

\[
\begin{array}{ccc}
\Omega^i & \xrightarrow{\text{mod } p} & \Omega^i_1 \\
\downarrow{p^{-i}\phi} & & \downarrow{C^{-1}} \\
\Omega^i & \xrightarrow{\text{mod } p} & \mathcal{H}^i(\Omega^i_1)
\end{array}
\]

First we recall the following:

**Proposition 3.1 ([N2 (6.4)])**. Let $i$ (resp. $n$) be a non-negative (resp. positive) integer. Set

\[
Z^i_n := \{ \omega \in \Omega^i \mid d\omega \in p^n\Omega^{i+1} \}, \quad B^i_n := p^n\Omega^i + d\Omega^{i-1} \quad \text{and} \quad \mathfrak{M}_n\Omega^i := Z^i_n/B^i_n.
\]

Then the morphism $\phi: \Omega^i \rightarrow \Omega^i$ induces the following isomorphism of sheaves of $\mathcal{W}$-modules:

\[
(3.1.1) \quad \mathfrak{M}_n\Omega^i \xrightarrow{\phi} p^iZ^i_{n+1}/(p^{i+n}Z^i_1 + p^{i-1}dZ^{i-1}_1).
\]

By (3.0.4) we have the following isomorphism

\[
(3.1.2) \quad \mathfrak{M}_1\Omega^i = \mathcal{H}^i(\Omega^i_1) \xrightarrow{C^{-1}} \Omega^i_1.
\]

Recall the following morphisms

\[
F: \mathfrak{M}_{n+1}\Omega^i \rightarrow \mathfrak{M}_n\Omega^i, \quad V: \mathfrak{M}_n\Omega^i \rightarrow \mathfrak{M}_{n+1}\Omega^i,
\]

\[
d: \mathfrak{M}_n\Omega^i \rightarrow \mathfrak{M}_n\Omega^{i+1}, \quad \pi: \mathfrak{M}_{n+1}\Omega^i \rightarrow \mathfrak{M}_n\Omega^i
\]

of sheaves of $\mathcal{W}$-modules in $\mathcal{T}$ as follows: $F$ (resp. $V$) is a morphism induced by $\text{id}: \Omega^i \rightarrow \Omega^i$ (resp. $p \times \text{id}: \Omega^i \rightarrow \Omega^i$); $d$ is a morphism induced by $p^{-n}d: Z^i_n \rightarrow \Omega^{i+1}$; $\pi$ is the following composite surjective morphism (cf. [HK (4.2)]):

\[
(3.1.3) \quad \mathfrak{M}_{n+1}\Omega^i = Z^i_{n+1}/B^i_{n+1} \xrightarrow{\text{prog}^{-1}} p^iZ^i_{n+1}/p^iB^i_{n+1} \xrightarrow{\text{proj}} p^iZ^i_{n+1}/(p^{i+n}Z^i_1 + p^{i-1}dZ^{i-1}_1) \xrightarrow{\phi} Z^i_n/B^i_n = \mathfrak{M}_n\Omega^i.
\]

Then the following formulas hold:

\[
(3.1.4) \quad d^2 = 0, \quad FdV = d, \quad FV = VF = p, \quad F\pi = \pi F, \quad \pi d = \pi d, \quad V\pi = \pi V.
\]
Lemma 3.2 ([N2, (6.7)]). Let \( * \) be a positive integer \( n \) or nothing. Set
\[
\mathfrak{M}_n^* = \lim_{\leftarrow \pi} \mathfrak{M}_n^*.
\]
Then there exists a natural \( \mathcal{W}_* \)-module structure on \( \mathfrak{M}_n^* \) (see the explanation after \( \textnormal{(3.9)} \) below).

We have called \( \mathfrak{M}_n^* \) (resp. \( \mathfrak{M}_n^* \)) the formal de Rham-Witt complex (resp. formal de Rham-Witt complex of length \( n \)) of \( (\mathcal{M}^*, \phi, C^{-1}) \).

In [N2 (6.16.2), (6.16.3)] we have defined the following \( \kappa \)-submodules of \( \Omega^i_1 \) inductively for \( n \in \mathbb{N} \):
\[
Z_0 \Omega^i_1 := \Omega^i_1, \quad Z_1 \Omega^i_1 := \operatorname{Ker}(d: \Omega^i_1 \rightarrow \Omega^{i+1}_1), \quad Z_{n+1} \Omega^i_1/B\Omega^i_1 \xrightarrow{\sim} Z_0 \Omega^i_1.
\]
\[
B_0 \Omega^i_1 := 0, \quad B_1 \Omega^i_1 := B\Omega^i_1 := \operatorname{Im}(d: \Omega^{i-1}_1 \rightarrow \Omega^i_1), \quad B_{n+1} \Omega^i_1/B\Omega^i_1 \xrightarrow{\sim} B_n \Omega^i_1.
\]
Here we consider \( Z_0 \Omega^i_1 \) and \( B_n \Omega^i_1 \) as subsheaves of \( \kappa \)-modules of \( \sigma^*(\Omega^i_1) \).

Lemma 3.3 ([N2, (6.17)]). (1) \( Z_0 \Omega^i_1 = (Z_n^i + p\Omega^i)/p\Omega^i \) \( (n \in \mathbb{Z}_{>0}) \).
(2) \( B_n \Omega^i_1 = (p^{-(n-1)}dZ_{n-1}^i + p\Omega^i)/p\Omega^i \) \( (n \in \mathbb{Z}_{>0}) \).

In the proof of [N2 (6.17)] we have proved that the following morphisms
\[
C^{-1} = p^{-i} \phi: (Z_n^i + p\Omega^i)/p\Omega^i \rightarrow (Z_{n+1}^i + p\Omega^i)/(p\Omega^i + d\Omega^{i-1}) \quad (n \in \mathbb{Z}_{>0})
\]
and
\[
C^{-1} = p^{-i} \phi: (p^{-(n-1)}dZ_{n-1}^i + p\Omega^i)/p\Omega^i \rightarrow (p^{-n}dZ_{n-1}^i + p\Omega^i)/(p\Omega^i + d\Omega^{i-1}) \quad (n \in \mathbb{Z}_{>0})
\]
are isomorphisms of \( \kappa \)-modules. Set \( Z_n \mathfrak{W}_1 \Omega^i := Z_{n+1} \Omega^i_1/B\Omega^i_1 \) and \( B_n \mathfrak{W}_1 \Omega^i := B_{n+1} \Omega^i_1/B\Omega^i_1 \). These are \( \kappa \)-submodules of \( \sigma^*(\mathfrak{W}_1 \Omega^i) \).

Let us consider the following composite morphisms of \( \kappa \)-modules:
\[
C: Z_n \mathfrak{W}_1 \Omega^i = Z_{n+1} \Omega^i_1/B\Omega^i_1 \xrightarrow{\textnormal{proj}} Z_{n+1} \Omega^i_1/B_2 \Omega^i_1 \xrightarrow{(C^{-1})^{-1}} Z_n \Omega^i_1/B\Omega^i_1 = Z_{n-1} \mathfrak{W}_1 \Omega^i \quad (n \geq 0),
\]
(3.3.4)
\[
C: B_n \mathfrak{W}_1 \Omega^i = B_{n+1} \Omega^i_1/B\Omega^i_1 \xrightarrow{\textnormal{proj}} B_{n+1} \Omega^i_1/B_2 \Omega^i_1 \xrightarrow{(C^{-1})^{-1}} B_n \Omega^i_1/B\Omega^i_1 = B_{n-1} \mathfrak{W}_1 \Omega^i \quad (n \geq 1).
\]

The following (3) is a formal generalization of \( \textnormal{(2.3.1.77)} \) (cf. [11] I (3.11)]).

Proposition 3.4 (cf. [11] I (3.11)]. (1) The morphism \( F^n: \mathfrak{M}_{n+1} \Omega^i \rightarrow \mathfrak{M}_1 \Omega^i \) induces the following isomorphism of abelian sheaves in \( \mathcal{T} \):
\[
\mathfrak{M}_{n+1} \Omega^i/V\mathfrak{M}_n \Omega^i \sim Z_n \mathfrak{W}_1 \Omega^i.
\]
(2) The following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{M}_{n+1}\Omega^i & \xrightarrow{F^n} & Z_n\mathfrak{M}_1\Omega^i \\
\pi & \downarrow & \downarrow c \\
\mathfrak{M}_n\Omega^i & \xrightarrow{F^{n-1}} & Z_{n-1}\mathfrak{M}_1\Omega^i.
\end{array}
\]

(3.4.2)

(3) The morphism \( F^{n-1}d: \mathfrak{M}_n\Omega^i \rightarrow \mathfrak{M}_1\Omega^{i+1} \) induces the following isomorphism of abelian sheaves in \( T \):

\[
\mathfrak{M}_n\Omega^i / F\mathfrak{M}_{n+1}\Omega^i \cong B_n\mathfrak{M}_1\Omega^{i+1}.
\]

(3.4.3)

(4) The following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{M}_{n+1}\Omega^i & \xrightarrow{F^nd} & B_{n+1}\mathfrak{M}_1\Omega_1^{i+1} \\
\pi & \downarrow & \downarrow c \\
\mathfrak{M}_n\Omega^i & \xrightarrow{F^{n-1}d} & B_n\mathfrak{M}_1\Omega_1^{i+1}.
\end{array}
\]

(3.4.4)

Proof. (1): By (3.3) (1) it suffices to prove that the following morphism

\[
\text{proj.}: Z_{n+1}^i \rightarrow (Z_{n+1}^i + p\Omega^i)/p\Omega^i
\]

is surjective and its kernel is equal to \( pZ_n^i \). The surjectivity is obvious. Let \( \omega \) be a local section of \( Z_{n+1}^i \). Assume that \( \omega = p\eta \) with \( \eta \in \Omega^i \). Then \( p^{n+1}\Omega^i \ni d\omega = pd\eta \).

Since \( \Omega^i \) is torsion free, \( d\eta \in p^n\Omega^i \). Hence the kernel of the morphism (3.4.5) is \( pZ_n^i \).

(2): The diagram (3.4.2) is equal to

\[
\begin{array}{ccc}
\mathfrak{M}_{n+1}\Omega^i & \xrightarrow{\text{proj}} & Z_{n+1}\Omega_1^i / B_1\Omega_1^i \\
\pi & \downarrow & \downarrow c^\sim \\
\mathfrak{M}_n\Omega^i & \xrightarrow{\text{proj}} & Z_n\Omega_1^i / B_1\Omega_1^i \cong Z_n\Omega_1^i / B_1\Omega_1^i.
\end{array}
\]

(3.4.6)

In [N2 (6.18.2)] we have already proved that this is commutative.

(3): By (3.3) (2) it suffices to prove that the following morphism

\[
p^{-n}d: Z_n^i \rightarrow (p^{-n}dZ_n^i + p\Omega^{i+1})/p\Omega^{i+1} \quad (n \in \mathbb{Z}_{>0})
\]

is surjective and its kernel is equal to \( Z_{n+1}^i \). This is obvious.

(4): It suffices to prove that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{M}_{n+1}\Omega^i & \xrightarrow{p^{-(n+1)d}} & \mathfrak{M}_{n+1}\Omega^{i+1} \\
\pi & \downarrow & \downarrow c^{-1} \cong \\
\mathfrak{M}_n\Omega^i & \xrightarrow{p^{-n}d} & \mathfrak{M}_n\Omega^{i+1} \rightarrow Z_n\Omega_1^{i+1} / B_1\Omega_1^{i+1}.
\end{array}
\]

(3.4.7)

Consider sections \([\omega] \in \mathfrak{M}_{n+1}\Omega^i \) (\( \omega \in Z_{n+1}^i \)) and \([\eta] \in \mathfrak{M}_n\Omega^i \) (\( \eta \in Z_n^i \)) such that \( p^i\omega - \phi(\eta) \in p^{i+1}Z_n^i + p^{i-1}dZ_1^{i-1} \). Then \( \pi([\omega]) = [\eta] \) by the definition of \( \pi \).

We also have the following equalities:

\[
p^{-(n+1)}d(\omega - p^{-i}\phi(\eta)) = p^{-(n+1)}d\omega - p^{-(i+1)}\phi(p^{-n}d\eta)
\]

(3.4.8)
and

\begin{equation}(3.4.9)\end{equation}
\[p^{-(n+1)}d(p^nZ_1^i + p^{-1}dz_1^{i-1}) = p^{-1}dz_1^i.\]

By \([3.3]\) (2), this sheaf \(p\) is contained in \(B_2\Omega_1\). Hence, by \([3.4.8]\) and \([3.4.9]\), the right hand side on \([3.4.8]\) is equal to zero in \(Z_{n+1}\Omega_1^i/B_2\Omega_1^i\). This implies that the diagram \([3.4.7]\) is commutative.

**Proposition 3.5** \([N2, (6.14)]\) (cf. \([I1, I (3.31)]\), \([L, p. 258])\). Let \(n > r\) be two positive integers. Then the following sequence is exact:

\begin{equation}(3.5.1; r, n)\end{equation}
\[0 \rightarrow \mathfrak{W}_{n-r}\Omega_1^{-1}/F^r\mathfrak{W}_{n}\Omega_1^{-1} \xrightarrow{dV} \mathfrak{W}_n\Omega_1^i/V^r\mathfrak{W}_{n-r}\Omega_1^i \rightarrow \mathfrak{W}_r\Omega_1^i \rightarrow 0.\]

Consequently the following sequence is exact:

\begin{equation}(3.5.1; r)\end{equation}
\[0 \rightarrow \mathfrak{W}_n\Omega_1^{-1}/F^r\mathfrak{W}_n\Omega_1^{-1} \xrightarrow{dV} \mathfrak{W}_n\Omega_1^i/V^r\mathfrak{W}_n\Omega_1^i \rightarrow \mathfrak{W}_r\Omega_1^i \rightarrow 0.\]

**Theorem 3.6** \([N2, (6.15)]\) (cf. \([I1, I (3.31)])\). Let \(r \) be a non-negative integer. Let \(\text{Fil}^r\) be the canonical filtration on \(\mathfrak{W}_n\Omega_1^i\) : \(\text{Fil}^r\mathfrak{W}_n\Omega_1^i := \ker(\mathfrak{W}_n\Omega_1^i \rightarrow \mathfrak{W}_r\Omega_1^i)\). Then the following formula holds:

\begin{equation}(3.6.1)\end{equation}
\[\text{Fil}^r\mathfrak{W}_n\Omega_1^i = V^r\mathfrak{W}_n\Omega_1^i + dV^r\mathfrak{W}_n\Omega_1^i.\]

**Corollary 3.7** \([N2, (6.6)]\) (cf. \([I1, I (3.31)]\), \([IR, II (1.1.1)]\), \([L, (2.16)]\)). Let \(R\) be the Cartier-Dieudonné-Raynaud algebra over \(\kappa\). Let \(n \) be a positive integer. Set \(R_n := R/(V^nR + dV^nR)\). The canonical morphism

\begin{equation}(3.7.1)\end{equation}
\[R_n \otimes_R \mathfrak{W}_n^\bullet \rightarrow \mathfrak{W}_n^\bullet\]

is an isomorphism.

**Proposition 3.8** \([N2, (6.23)]\) (cf. \([I1, I (3.21.1.5)]\), \([L, (1.20)])\). Let \(n \) be a non-negative integer. Then \(d^{-1}(p^n\mathfrak{W}_n^i) = F^n\mathfrak{W}_n^i\).

**Theorem 3.9** \([N2, (6.24)]\) (cf. \([IR, II (1.2)]\), \([L, (2.17)])\). The isomorphism \([3.7.1]\) induces the following isomorphism in \(D^b(T, A_n[d])\):

\begin{equation}(3.9.1)\end{equation}
\[R_n \otimes_R \mathfrak{W}_n^\bullet \sim\mathfrak{W}_n^\bullet.\]

Let \(Z\) be a scheme of characteristic \(p > 0\). Let \(\mathcal{W}_n(\mathcal{O}_Z)^i\) be the obverse Witt sheaf of \(Z\) denoted by \(W_n(\mathcal{O}_Z)^i\) in \([N2, \S 7]\). Let \(\mathcal{B}\) be a \(p\)-torsion free quasi-coherent sheaf of commutative rings with unit elements in \(\mathcal{Z}_{zar}\) with a surjective morphism \(\mathcal{B} \rightarrow \mathcal{O}_Z\) of sheaves of rings in \(\mathcal{Z}_{zar}\). Assume that \(\text{Ker}(\mathcal{B} \rightarrow \mathcal{O}_Z) = p\mathcal{B}\) and that each \(\Omega_i^i\) \((i \in \mathbb{N})\) is a quasi-coherent \(\mathcal{B}\)-module. Then we can endow \(\mathfrak{W}_n\Omega_n^i\) with a natural \(\mathcal{W}_n(\mathcal{O}_Z)^i\)-module structure (cf. \([IR, III (1.5)])\): for a local section \(c := (c_0, \ldots, c_{n-1})\) \((c_i \in \mathcal{O}_Z (0 \leq i \leq n - 1))\) and a local section \(\omega\) of \(Z_n^i\), we define \(c \cdot [\omega]\) as follows: \(c \cdot [\omega] = [(\sum_{j=0}^{n-1} p^j c_j \cdot \omega)]\), where \(c_j \in \mathcal{B}/p^n\mathcal{B}\) is a lift of \(c_j\).

We can easily check that \(\mathfrak{W}_n\Omega_n^i\) is a quasi-coherent \(\mathcal{W}_n(\mathcal{O}_Z)^i\)-module and that the morphisms \(\pi: \mathfrak{W}_{n+1}\Omega_n^i \rightarrow \mathfrak{W}_n\Omega_n^i\) is a morphism of \(\mathcal{W}_n(\mathcal{O}_Z)^i\)-modules. We can easily check that \(\mathfrak{W}_n\Omega_n^i\) is a quasi-coherent \(\mathcal{W}_n(\mathcal{O}_Z)^i\)-module and that the morphism \(\pi: \mathfrak{W}_{n+1}\Omega_n^i \rightarrow \mathfrak{W}_n\Omega_n^i\) is a morphism of \(\mathcal{W}_n(\mathcal{O}_Z)^i\)-modules. We consider \(Z_n\mathfrak{W}_1\Omega^i\) and \(B_n\mathfrak{W}_1\Omega^i\) as \(\mathcal{O}_Z\)-submodules of \(F_n^a(\mathfrak{W}_1)^i\).
Proposition 3.10. Let $F_{W_n(Z)} : W_n(Z) \to W_n(Z)$ be the Frobenius endomorphism of $W_n(Z)$. Then the following hold:

1. The following exact sequence

\[
\begin{align*}
(3.10.1) \quad F_{W_n(Z)*}(\mathcal{M}_n\Omega^i) & \xrightarrow{V} \mathcal{M}_{n+1}\Omega^i \xrightarrow{F^n} Z_n\mathcal{M}_1\Omega^i \to 0 \\
\end{align*}
\]

obtained by \(\text{[3.4.1]}\) is an exact sequence of $W_{n+1}(\mathcal{O}_Z)'$-modules.

2. The following exact sequence

\[
\begin{align*}
(3.10.2) \quad \mathcal{M}_{n+1}\Omega^i & \xrightarrow{F} F_{W_n(Z)*}(\mathcal{M}_n\Omega^i) \xrightarrow{F_{W_n(Z)*}(F^{n-1}d)} B_n\mathcal{M}_1\Omega^{i+1} \to 0 \\
\end{align*}
\]

obtained by \(\text{[3.4.3]}\) is an exact sequence of $W_{n+1}(\mathcal{O}_Z)'$-modules.

Proof. Set $c(n + 1) := (c_0, \ldots, c_n) \in W_{n+1}(\mathcal{O}_Z)$ ($c_i \in \mathcal{O}_Z$ ($0 \leq i \leq n$)). Set also $c(n) := (c_0, \ldots, c_{n-1}) \in W_n(\mathcal{O}_Z)$. Let $\overline{c}_j \in B/p^{n+1}B$ be a lift of $c_j$. For a local section $\omega$ of $\mathcal{M}_m\Omega^i$ ($m = n, n + 1$), let $\overline{\omega} \in Z_m^i$ be a representative of $\omega$ and let $[\overline{\omega}]_m = \omega$ be the class of $\overline{\omega}$ in $\mathcal{M}_m\Omega^i$. We use the similar notation for $[\overline{\omega}]_l$ for $l \leq m$.

1. Let $\omega$ be a local section of $F_{W_n(Z)*}(\mathcal{M}_n\Omega^i)$. Then

\[
(3.10.3) \quad c(n + 1) \cdot V(\omega) = \sum_{j=0}^{n} p^j \overline{c}_j^{n+1-j} p\overline{\omega}|_{n+1} = [\sum_{j=0}^{n-1} p^j (\overline{c}_j^p)^{p^{n-j}} \overline{\omega}]|_{n+1} = [\sum_{j=0}^{n-1} p^j (\overline{c}_j^p)^{p^{n-j}} \overline{\omega}]|_{n+1} = V(c(n) \cdot \omega).
\]

This formula shows that $V$ is a morphism of $W_{n+1}(\mathcal{O}_Z)'$-modules.

Let $\omega$ be a local section of $\mathcal{M}_{n+1}\Omega^i$. Then

\[
(3.10.4) \quad F^n(c(n + 1) \cdot \omega) = \sum_{j=0}^{n} p^j \overline{c}_j^{p^{n+1-j}} \overline{\omega}|_1 = [(\overline{c}_0^p)^{p^n} \omega]|_1 = c_0 \cdot [\omega]_1.
\]

This formula shows that $F^n : \mathcal{M}_{n+1}\Omega^i \to Z_n\mathcal{M}_1\Omega^i$ is a morphism of $W_{n+1}(\mathcal{O}_Z)'$-modules.

2. Let $\omega$ be a local section of $\mathcal{M}_{n+1}\Omega^i$. Then

\[
(3.10.5) \quad F(c(n + 1) \cdot \omega) = \sum_{j=0}^{n} p^j \overline{c}_j^{p^{n+1-j}} \overline{\omega}|_n = \sum_{j=0}^{n-1} p^j (\overline{c}_j^p)^{p^{n-j}} \overline{\omega}|_n = c(n) \cdot F(\omega).
\]

Let $\omega$ be a local section of $F_{W_n(Z)*}(\mathcal{M}_n\Omega^i)$. Then

\[
(3.10.6) \quad F^{n-1}d(c(n) \cdot \omega) = [p^{-n}d(\sum_{j=0}^{n-1} p^j (\overline{c}_j^p) \overline{\omega})]|_1 = [\sum_{j=0}^{n-1} \overline{c}_j^{p^{n+1-j}-1} \overline{d} \overline{c}_j \wedge \overline{\omega} + \sum_{j=0}^{n-1} p^j \overline{c}_j^{p^{n+1-j}} p^{-n} d \overline{\omega}]|_1
\]

\[
= [(\sum_{j=0}^{n-1} p^j \overline{c}_j^{p^{n+1-j}}) p^{-n} d \overline{\omega}]|_1 = [(\overline{c}_0^p)^{p^n} p^{-n} d \overline{\omega}]|_1 = c_0 \cdot F^{n-1}d \omega.
\]

This formula shows that $F_{W_n(Z)*}(F^{n-1}d) : F_{W_n(Z)*}(\mathcal{M}_n\Omega^i) \to B_n\mathcal{M}_1\Omega^{i+1}$ is a morphism of $W_n(\mathcal{O}_Z)'$-modules. \qed
Remark 3.11. (1) In [11 I (3.11)] we can find a corresponding statement to [3.10]. However, the \( W_{x+1}(O_Z)^j \)-module structures were not considered in [loc. cit.]: in [loc. cit.] only exact sequences of abelian sheaves have been considered. However the well-known relation “\( xy = V(Fxy) \)” implies that \( V \) in (3.10.1) is compatible with the \( W_{x+1}(O_Z) \)-structures.

(2) Let \( Z/s \) be a fine log scheme. The proposition (3.10) is important because several properties of the de Rham-Witt sheaf \( W_n \Omega^j_Z \) (\( j \in \mathbb{N} \)) are obtained by properties of \( Z_n \Omega^j_Z \) or \( B_n \Omega^j_Z +1 \) \([N2]\).

Definition 3.12. We call the exact sequences (3.10.1) and (3.10.2) of \( W_{x+1}(O_Z) \)-modules the log Illusie exact sequence of \((\Omega^*, \phi)\) in level \( n \).

Lemma 3.13. Assume that \( \mathfrak{M}_1 \Omega^i \) is an \( O_Z \)-module of finite type and that \( F_Z \) is a finite morphism. Assume that \( Z \) is a noetherian scheme. Then \( Z_n \mathfrak{M}_1 \Omega^i \) and \( B_n \mathfrak{M}_1 \Omega^i \) are coherent \( O_Z \)-modules.

Proof. By the assumption, \( F_Z^j(\mathfrak{M}_1 \Omega^i) \) is a coherent \( O_Z \)-module. Hence \( Z_n \mathfrak{M}_1 \Omega^i \) and \( B_n \mathfrak{M}_1 \Omega^i \) are coherent \( O_Z \)-modules. \( \Box \)

Proposition 3.14 \([N2] (6.12) (2)\)]. Let \( F_Z : Z \to Z \) be the Frobenius endomorphism of \( Z \). Assume that

\[
C^{-1} : \Omega^i_1 \iso \mathfrak{M}_1 \Omega^i = \mathcal{H}^i(F_Z, (\Omega^*_i))
\]

is an isomorphism of \( O_Z \)-modules. If \( \Omega^i_1 \) (\( j = i - 1, i \)) is an \( O_Z \)-module of finite type and if \( F_Z \) is finite, then \( \mathfrak{M}_n \Omega^i \) is a \( W_n(O_Z)^j \)-module of finite type.

Let \( s \) be a fine log scheme whose underlying scheme is \( \text{Spec}(\kappa) \). If \( Z \) is a underlying scheme of a log smooth scheme \( Y \) of Cartier type over \( s \), then \( W_n(O_Z)^j \) is a \( W_n(O_Y)^j \)-module, \([N2] (7.5)\], where \( W_n(O_Y)^j \) is a reverse Witt sheaf of \( Y/s \) in the sense of [loc. cit.]. By this identification, \( \mathfrak{M}_n \Omega^i \) is a quasi-coherent \( W_n(O_Y)^j \)-module.

Proposition 3.15 \([N2] (6.27) (1)\)]. Let \( W_n(s) \) and \( W(s) \) be the canonical lifts of \( s \) over \( W_n \) and \( W \), respectively. Let \( Y \) be a log smooth scheme of Cartier type over \( s \). Let \( Y/W(s) \) be a formally log smooth lift of \( Y/s \) with a lift \( \Phi : Y \to Y \) of the Frobenius endomorphism of \( Y \). Set \( Y_n := Y \otimes_W W_n \) (\( n \in \mathbb{Z}_{>0} \)). \( \Omega_n^* \) be the log de Rham complex of \( Y_n/W_n(s) \). Set \( \Omega^* := \lim_n \Omega_n^* \). Let \( C^{-1} : \Omega^i_1 \iso \mathcal{H}^i(\Omega^*_i) \) be the log inverse Cartier isomorphism \([K1] (4.12) (1)\]). Then \( (\Omega^*, \Phi^*, C^{-1}) \) satisfies the conditions (2.1.3) \( \sim \) (2.1.7) for \( T = (Y_{\text{zar}}, W) \).

Corollary 3.16. Let \( Y \) be a log smooth scheme of Cartier type over \( s \). Let \( W_n \Omega^*_Y \) (\( *=n \) or nothing) be the log de Rham-Witt complex of \( Y/s \). Then the statements in this section with the replacement of \( \mathfrak{M}_n \Omega^i \) by \( W_n \Omega^*_Y \) hold.

The following proposition and the following corollary tells us that the former is a generalization of [2.3]:

Proposition 3.17. Let \( (W_n \Omega^*_Y)^j \) be the obverse log de Rham-Witt complex of \( Y/W_n(s) \) defined in \([N2] \ S7\] and denoted by \( (W_n \Omega^*_Y)^j \) in [loc. cit.]. Let \( C^{-n} : (W_n \Omega^*_Y)^j \iso W_n \Omega^*_Y \) be the isomorphism of Raynaud algebras over \( \kappa \) defined in \([N2] (7.0.5)\]. \([N2] (7.5)\] we have proved that this is an isomorphism. Then the following diagram of \( W_{n+1}(O_Y)^j \)-modules and \( W_{n+1}(O_Y) \)-modules is commutative:

\[
\begin{align*}
(W_n \Omega^*_Y)^j/F(W_n+1 \Omega^*_Y)^j & \xrightarrow{F^n-1d_c} B_n \Omega^*_Y^{j+1} \\
C^{-n} & \cong & \cong & \cong \\
W_n \Omega^*_Y/FW_{n+1} \Omega^*_Y & \xrightarrow{F^n-1d_c} B_n W_n \Omega^*_Y^{j+1}.
\end{align*}
\]
Proof. This immediately follows from the comparison isomorphism $C^{-n} : (W_n \Omega^i_Y)' \sim \sim W_n \Omega^i_Y$ (N2 (7.5)) which is compatible with $d$'s and $F$'s. $\square$

**Corollary 3.18.** The upper horizontal isomorphism $F^{n-1}d: (W_n \Omega^i_Y)' / F(W_{n+1} \Omega^i_Y)' \sim B_n \Omega^{i+1}_Y$ in $\text{3.17,1}$ for the case $i = 0$ is equal to the isomorphism $\text{2.3.2}$.

*Proof.* By the construction of the morphism $s_n$ in $\text{N2 (7.0.5)}$, the composite morphism $C^{-1} \circ F^{n-1}d$ is equal to the following morphism

$$W_n(\mathcal{O}_Y)' / F W_{n+1}(\mathcal{O}_Y)' \ni (a_0, \ldots, a_{n-1}) \mapsto \sum_{i=0}^{n-1} a_i^{\alpha - 1 - i} da_i \in B_n W_1 \Omega^i_Y$$

(see HK p. 251 for the definition of the morphism $\delta$). Because $C^{-1}(a_i^{\alpha - 1 - i} da_i) = a_i^{\alpha - 1 - i} da_i$, the upper horizontal isomorphism is equal to the morphism $d_n$. $\square$

## 4 Finiteness of cohomologies of log Hodge-Witt sheaves

In this section we prove the main results (1.2) and (1.8).

Let $s$ be as in the Introduction. Let $X/s$ be a proper log smooth scheme of Cartier type.

The following is a log version of IR II (2.2), (3.1).

**Theorem 4.1.** The $E_2$-terms of the slope spectral sequence

$$E^{ij}_1 = H^j(X, W \Omega^i_X) \Rightarrow H^i_{\text{crys}}(X/W(s))$$

are finitely generated $W$-modules.

*Proof.* By using (3.9) and (3.16), the proof is the same as that of IR II (2.2), (3.1). $\square$

**Theorem 4.2.** Let $q$ and $i$ be integers. Assume that $\dim (H^{q-1}(X, B_n \Omega^{i+1}_X))$ is bounded for all $n$. Then the differential $d: H^q(X, W \Omega^i_X) \rightarrow H^q(X, W \Omega^{i+1}_X)$ is zero.

Consequently

$$H^q(X, W \Omega^i_X)/dH^q(X, W \Omega^{i+1}_X)$$

is a finitely generated $W$-module.

*Proof.* (cf. the proof of IR (5.1)) Recall that

$$F H^q(X, W \Omega^i_X) := \text{Ker}(F: H^q(X, W \Omega^i_X) \rightarrow H^q(X, W \Omega^i_X)).$$

By the log version of IR II (3.8) it suffices to prove that $\dim (F H^q(X, W \Omega^i_X)) < \infty$. By the exact sequence

$$0 \rightarrow W \Omega^i_X \xrightarrow{F} W \Omega^i_X \rightarrow W \Omega^i_X / F \rightarrow 0,$$

we have the following surjection

$$H^{q-1}(X, W \Omega^i_X / FW \Omega^i_X) \rightarrow F H^q(X, W \Omega^i_X).$$

It suffices to prove that $H^{q-1}(X, W \Omega^i_X / FW \Omega^i_X)$ is a finitely generated $W$-module. Because $H^{q-1}(X, W \Omega^i_X / FW \Omega^i_X) = \lim_{\rightarrow n} H^{q-1}(X, W_n \Omega^i_X / FW_{n+1} \Omega^i_X)$, it suffices to
prove that $H^{q-1}(X, \mathcal{W}_n \Omega_X^1 / \mathcal{W}_n \Omega_X^1)$'s are finite dimensional $k$-vector spaces of bounded dimensions for all $n$'s. By (3.3) and (3.13),

$$H^{q-1}(X, \mathcal{W}_n \Omega_X^1 / \mathcal{W}_n \Omega_X^1) = H^{q-1}(X, B_n \mathcal{W}_1 \Omega_X^1) \simeq H^{q-1}(X, B_n \Omega_X^1).$$

Hence $H^{q-1}(X, \mathcal{W}_n \Omega_X^1 / \mathcal{W}_n \Omega_X^1)$'s are finite dimensional $k$-vector spaces of bounded dimensions for all $n$'s by the assumption. Now we see that $H^q(X, \mathcal{W}_X \Omega_X^{-1}) = E^n$ is finitely generated by (3.1).

**Theorem 4.3.** Assume that $X$ is quasi-$F$-split. Then $\dim_X H^q(X, B_n \Omega_X^1)$ is bounded for all $n$ and for all $q$.

**Proof.** (cf. JR (2.4.1)) Let $n$ be a positive integer. Push out the exact sequence (2.3.1) for the case $Y = X$ by the morphism $F^{n-1}: \mathcal{W}_n(\mathcal{O}_X) \rightarrow \mathcal{O}_X$. Then we have the following exact sequence of $\mathcal{O}_X$-modules:

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_n \rightarrow B_n \Omega_X^{-1} \rightarrow 0,$$

where $\mathcal{E}_n := \mathcal{O}_X \oplus \mathcal{W}_n(\mathcal{O}_X) \cdot F_* \mathcal{W}_n(\mathcal{O}_X))$. Set $h := h_F(X) < \infty$. For $n = h$, the exact sequence (4.3.1) is split. It is easy to check that the exact sequence (4.3.1) is split for $n \geq h$ (NY (8.2) (2))). Hence

$$H^q(X, \mathcal{E}_n) = H^q(X, \mathcal{O}_X) \oplus H^q(X, B_n \Omega_X^{-1})$$

for $n \geq h$. In particular,

$$\dim_X H^q(X, \mathcal{E}_n) = \dim_X H^q(X, \mathcal{O}_X) + \dim_X H^q(X, B_n \Omega_X^{-1}).$$

Following Y2 (4.1), (4.2) in the trivial logarithmic case, consider the following diagram

\[
\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{E}_n & \rightarrow & B_n \Omega_X^{-1} & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & \mathcal{O}_X & \hookrightarrow & \mathcal{E}_1 = F_* (\mathcal{O}_X) & \overline{\rightarrow} & B_1 \Omega_X^{-1} & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & & & & & & 0 & & 0 \\
\end{array}
\]

of $\mathcal{O}_X$-modules. Here we have used the commutative diagram (2.3.3). Using the snake lemma, we obtain the following exact sequence:

$$0 \rightarrow F_* (B_n \Omega_X^{-1}) \rightarrow \mathcal{E}_n \rightarrow F_* (\mathcal{O}_X) \rightarrow 0.$$

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $H^q(X, F_* (\mathcal{F})) = H^q(X, \mathcal{F})$ with $k$-module structure obtained by the Frobenius automorphism $\sigma$ of $k$ since $\overline{F}$ is finite. Hence, by (4.3.5),

$$\dim_X H^q(X, \mathcal{E}_n) \leq \dim_X H^q(X, B_n \Omega_X^{-1}) + \dim_X H^q(X, \mathcal{O}_X).$$
The equality (4.3.3) and the inequality (4.3.6) imply that
(4.3.7) \( \dim_{\kappa} H^q(X, B_n \Omega^1_{X/s}) \leq \dim_{\kappa} H^q(X, B_{n-1} \Omega^1_{X/s}) \).
This implies that
(4.3.8) \( \dim_{\kappa} H^q(X, B_n \Omega^1_{X/s}) \leq \max\{\dim_{\kappa} H^q(X, B_m \Omega^1_{X/s}) \mid 1 \leq m \leq h, 0 \leq q \leq \dim \hat{X}\} \).

Corollary 4.4. (1.2) holds.

Proof. This follows from (4.2) and (4.3).

Remark 4.5. To prove (1.2), we have followed the argument in the proof of [3] (5.1). However Serre has proved the following in [Se, p. 510, Corollaire 1]: \( H^q(X, \mathcal{W}(\mathcal{O}_X)) \) is a finitely generated \( \mathcal{W} \)-module if and only if the dimension of \( \lim_{n \to \infty} H^q(X, \mathcal{W}_n(\mathcal{O}_X)/\mathcal{W}_{n-1}(\mathcal{O}_X)) \) over \( \kappa \) is finite. It is clear that, if the dimension of \( H^q(X, \mathcal{W}_n(\mathcal{O}_X)/\mathcal{W}_{n-1}(\mathcal{O}_X)) \) over \( \kappa \) is bounded for \( n \)'s, then the dimension of \( \lim_{n \to \infty} H^q(X, \mathcal{W}_n(\mathcal{O}_X)/\mathcal{W}_{n-1}(\mathcal{O}_X)) \) is finite. If one would like to prove only (1.2), only this Serre's result, the log Serre exact sequence (4.3) are enough.

Corollary 4.6. Set \( H^q := H^q(X, \mathcal{W}(\mathcal{O}_X)) \) and \( V H^q := \ker(V : H^q \to H^q) \) and \( p H^q := \ker(p : H^q \to H^q) \). Then the subvector space \( V H^q \) in \( p H^q \) has finite codimension in \( H^q \).

Proof. Consider the following exact sequence
\[
0 \to V H^q \to p H^q \to \tilde{p} H^q \to 0.
\]
Since the morphism \( V : \tilde{p} H^q / V H^q \to F H^q \) is injective and \( \dim_{\kappa}(F H^q) < \infty \), we have the desired inequality \( \dim_{\kappa}(\tilde{p} H^q / V H^q) < \infty \).

Corollary 4.7. The following hold:

(1) (4.4) holds.
(2) (4.5) holds.
(3) (4.7) holds.

Proof. (1): Set \( d := \dim \hat{X} \). By the following exact sequence
\[
0 \to B_{\infty} \Omega^1_{X/s} \to \Omega^1_{X/s} \to \Omega^1_{X/s}/B_{\infty} \Omega^1_{X/s} \to 0,
\]
we obtain the following exact sequence
\[
0 \to H^0(X, B_{\infty} \Omega^1_{X/s}) \to H^0(X, \Omega^1_{X/s}) \to H^0(X, \Omega^1_{X/s}/B_{\infty} \Omega^1_{X/s}) \to \cdots
\]
\[
\to H^d(X, B_{\infty} \Omega^1_{X/s}) \to H^d(X, \Omega^1_{X/s}) \to H^d(X, \Omega^1_{X/s}/B_{\infty} \Omega^1_{X/s}) \to 0.
\]
Hence it suffices to prove that \( H^d(X, B_{\infty} \Omega^1_{X/s}) \) \( (q \in \mathbb{N}) \) is finite dimensional. Consider \( B_{\infty} \Omega^1_{X/s} \) as a sheaf of \( f^{-1}(\kappa) \)-submodules of \( \Omega^1_{X/s} \), where \( f : X \to s \) is the structural morphism. We denote this resulting sheaf by \( \sigma_{\infty}^n(B_{\infty} \Omega^1_{X/s}) \). Because \( H^q(X, B_{\infty} \Omega^1_{X/s}) = \lim_{n \to \infty} H^q(X, \sigma_{\infty}^n(B_{\infty} \Omega^1_{X/s})) \) and because \( \dim_{\kappa} H^q(X, \sigma_{\infty}^n(B_{\infty} \Omega^1_{X/s})) = \dim_{\kappa} H^q(X, B_{\infty} \Omega^1_{X/s}) \), we see that \( \dim_{\kappa} H^q(X, B_{\infty} \Omega^1_{X/s}) < \infty \) by (4.3).

(2): The proof is the same as that of [1] II (3.14).

(3): In [3] (6.1) Joshi has proved that \( Y/\kappa \) is of Hodge-Witt type if and only if \( H^i(Y, \mathcal{W}(\mathcal{O}_Y)) \) \( (i, j \in \mathbb{N}) \) is a finitely generated \( \mathcal{W} \)-module. Hence (2) follows from (4.3).
Remark 4.8. ([4.4]) is very important in Bloch-Stienstra’s theory in ([St1], [St2]).

Let us recall their theory.

Let $Y$ be a proper smooth scheme over $\kappa$. Let $K_{i,Y}$ ($i \in \mathbb{Z}_{\geq 1}$) be the sheafification of the following presheaf of abelian groups on $Y$: $U \mapsto K_i(\Gamma(U, \mathcal{O}_Y))$, where $U$ is an open subscheme of $Y$ and $K_i$ means the $i$-th Quillen’s $K$-group.

Let us consider the following inductive system

$$
\mathcal{W}_1(\mathcal{O}_Y) \xrightarrow{V} \mathcal{W}_2(\mathcal{O}_Y) \xrightarrow{V} \cdots \xrightarrow{V} \mathcal{W}_n(\mathcal{O}_Y) \xrightarrow{V} \mathcal{W}_{n+1}^-(\mathcal{O}_Y) \xrightarrow{V} \cdots.
$$

and set $\mathcal{W}(\mathcal{O}_Y) := \varinjlim_n \mathcal{W}_n(\mathcal{O}_Y)$. For $m, n \in \mathbb{Z}_{\geq 1}$, we consider a morphism $\partial_n : \mathcal{W}(\mathcal{O}_Y) \to \mathcal{W}_n^0(\mathcal{O}_Y)$ defined by the following

$$
\partial_n|_{\mathcal{W}_n(\mathcal{O}_Y)} := \begin{cases} 
-dV^{n-m} & \text{if } m \geq n, \\
-F^{m-n}d & \text{if } m \leq n.
\end{cases}
$$

Here $\mathcal{W}_n^0(\mathcal{O}_Y)$ is the de Rham-Witt sheaf defined in [H]. (Note that the de Rham-Witt sheaf is isomorphic to the sheaf of $p$-typical curves defined in [B] ([H] 1, 5)). Then the projection $\mathcal{W}_{n+1}^-\rightarrow \mathcal{W}_n^0(\mathcal{O}_Y)$ induces the surjective morphism $\partial_{n+1}^\circ \mathcal{W}(\mathcal{O}_Y) \to \partial_n\mathcal{W}(\mathcal{O}_Y)$. Set $\mathcal{W}_n^0(\mathcal{O}_Y) := \lim_n \mathcal{W}_n^0(\mathcal{O}_Y)$ and $\partial\mathcal{W}(\mathcal{O}_Y) := \text{Ker}(\mathcal{W}_n^0(\mathcal{O}_Y) \to \mathcal{W}_1^0(\mathcal{O}_Y)/\partial\mathcal{W}(\mathcal{O}_Y))$. Let $D(\kappa)$ be the Dieudonné ring of $\kappa$. It is well-known that $\mathcal{W}_n^0(\mathcal{O}_Y)$ and $\partial\mathcal{W}(\mathcal{O}_Y)$ are sheaves of left $D(\kappa)$-module and $\partial\mathcal{W}(\mathcal{O}_Y)$ is a subsheaf of left $D(\kappa)$-modules of $\mathcal{W}_n^0(\mathcal{O}_Y)$. By replacing the roles of $F$ and $V$, $\partial\mathcal{W}(\mathcal{O}_Y)$ is a subsheaf of right $D(\kappa)$-modules of $\mathcal{W}_n^0(\mathcal{O}_Y)$.

Let $A$ be an artinian $\kappa$-algebra. Set $\hat{\mathcal{C}}K_i(A) := \text{Ker}(K_i(A[x]) \to K_i(A))$. Let $F_m$ and $V_m$ ($m \in \mathbb{Z}_{\geq 1}$) be the standard operators on $\hat{\mathcal{C}}K_i(A)$ induced by those on $K_i(A[x])$. Set $e := \sum_{(m,p)=1} \mu(m) V_m F_m$. Set $T\hat{\mathcal{C}}K_i(A) := e\hat{\mathcal{C}}K_i(A)$ and $\hat{W}(A) := T\hat{\mathcal{C}}K_1(A)$. The last group is a left $D(\kappa)$-module. The following functor arises in Bloch-Stienstra’s theory:

$$
B^0_{\mathcal{C}Y/\kappa}(A) := H^q(Y, \mathcal{W}_n^0(\mathcal{O}_Y)/\partial\mathcal{W}(\mathcal{O}_Y) \otimes_{D(\kappa)} \hat{W}(A)) \in (\text{Ab})
$$

for artinian local $\kappa$-algebras $A$’s with residue fields $\kappa$. Assume that $Y/\kappa$ is proper and smooth. Then the tangent space $T(B^0_{\mathcal{C}Y/\kappa}(A))$ of this functor is equal to $H^q(Y, \mathcal{W}_n^0(\mathcal{O}_Y)/B_{\kappa}(\mathcal{O}_Y))$ ([St1] IV (3.16)). Hence we obtain the following theorem by ([3.4]):

**Theorem 4.9.** Let the notations be as above. Assume that $Y$ is quasi-$F$-split. Then $T(B^0_{\mathcal{C}Y/\kappa}(A))$ is a finite dimensional $\kappa$-vector space.

In a future article, we would like to discuss the structure of the formal completion of the second Chow group $\hat{\mathcal{C}}H^2_{\mathcal{C}Y/\kappa}$ for a quasi-$F$-split proper smooth scheme over $\kappa$ of dimension $\leq 3$ in more detail.

**Corollary 4.10.** ([3.8]) holds.

**Proof.** Assume that $h_F(\hat{X}) < \infty$, By ([2.5.1] and [4.3.7]),

$$
\min\{n, h^0(\hat{X}/\kappa) - 1\} \leq \min\{n - 1, h^0(\hat{X}/\kappa) - 1\}
$$

for $n \geq h_F(\hat{X})$. This implies that $h^0(\hat{X}/\kappa) \leq h_F(\hat{X})$. If $h_F(\hat{X}) = \infty$, then there is nothing to prove. 

\[\square\]
Remark 4.11. It seemed to me at first that the proof of (4.10) was considerably mysterious because we do not consider $\text{Ext}^1_X(B_n \Omega^1_{X/s}, \mathcal{O}_X)$ at all nor do not use the log Serre duality of Tsuji ([15, (2.21)]); the decomposition (4.3.2) enables us to obtain (4.10).

Example 4.12. Let $Y/\kappa$ be a $K3$-surface with finite second Artin-Mazur height. Then $H^2(Y, \mathcal{W}(\mathcal{O}_Y))$ is a finitely generated $\mathcal{W}$-module. (In [11, p. 653] Illusie has already proved that they are of Hodge-Witt type.)

More generally, let $Y/\kappa$ be a $d$-dimensional Calabi-Yau variety with finite $d$-th Artin-Mazur height $h$. Then $H^d(Y, \mathcal{W}(\mathcal{O}_Y)) \cong \mathcal{W}^d$. Because $H^0(Y, \mathcal{W}(\mathcal{O}_Y)) = \mathcal{W}$ and $H^q(Y, \mathcal{W}(\mathcal{O}_Y)) = 0$ ($q \neq 0, d$), $H^q(Y, \mathcal{W}(\mathcal{O}_Y))$ is a finitely generated $\mathcal{W}$-module for any $q$. Consequently, if $d = 3$, then $Y/\kappa$ is of Hodge-Witt type by [1] (6.1). In particular,

$$H^q_{\text{crys}}(Y/\mathcal{W}) = \bigoplus_{i+j=q} H^j(Y, \mathcal{W} \Omega^i_Y) \quad (q \in \mathbb{N})$$

The following is a generalization of [1] (11.4):

Corollary 4.13. (1.12) holds.

Proof. By (4.7) (3) in the case $\kappa$ is an algebraically closed field, the induced morphism by the derivative $H^j(Y, \mathcal{W}(\Omega^i_{Y/\kappa})) \rightarrow H^j(Y, \mathcal{W}(\Omega^i_{Y/\kappa}))$ $(i, j \in \mathbb{N})$ is zero. In particular, the induced morphism by the derivative $H^2(Y, \mathcal{W}(\Omega^1_{Y/\kappa}) \rightarrow H^2(Y, \mathcal{W}(\Omega^2_{Y/\kappa}))$ is zero. Hence (4.13) follows from [CS, III (4.7)].

We recall the following theorem due to Yobuko.

Theorem 4.14 ([1, (3.5)]). Let $Y$ be a Calabi-Yau variety of pure dimension $d$. Then $h^d(Y/\kappa) = h_F(Y)$.

In fact we generalized this theorem in [NY]:

Theorem 4.15 ([NY, (10.1)]). Let $X$ be a proper log smooth, integral and saturated log scheme over $s$ of pure dimension $d$. Assume that $X/s$ is of Cartier type and of vertical type. Assume also that the following three conditions hold:

(a) $H^{d-1}(X, \mathcal{O}_X) = 0$ if $d \geq 2$,
(b) $H^{d-2}(X, \mathcal{O}_X) = 0$ if $d \geq 3$,
(c) $\Omega^d_{X/s} \cong \mathcal{O}_X$.

Then $h_F(X) = h^d(X/\kappa)$.

Remark 4.16. Let $X/s$ be as in (4.15).

(1) By (4.15) we see that $h^d(\hat{X}/\kappa)$ is independent of the choice of the structural morphism $\hat{X} \rightarrow \kappa$; it depends only on $\hat{X}$.

(2) Let the notations be in (4.15). By the equality $h_F(\hat{X}/\kappa) = h^d(\hat{X}/\kappa)$, $X$ is $F$-split if and only if $h^d(\hat{X}/\kappa) = 1$.

Corollary 4.17. (1.10) holds.

Proof. This follows from (1.7) (3) and [IR, IV (4.7), (4.8)].

The following is a generalization of [1] (11.3).
Corollary 4.18. Let the notations and the assumptions be as in (4.15). Assume that the log structures of $s$ and $X$ are trivial, that is, $X$ is a proper smooth scheme over $\kappa$. Assume that $h_F(\check{X}/\kappa) < \infty$. Let $q \geq 2$ be an integer and assume that $X$ is of pure dimension $2q - 1$. Let $l \neq p$ be a prime number. Let $A^q(X_\infty)$ be the subgroup of $\text{CH}^q(X_\infty)$ generated by cycles which are algebraically equivalent to 0. Let $A^q(X_\infty)[l]$ be the $l$-primary torsion part of $A^q(X_\infty)$. Then the following restriction of the $l$-adic Abel-Jacobi map of Bloch to $A^q(X_\infty)[l]$ is not surjective.

\[
A^q(X_\infty)[l] \to H^{2q-1}(X_\infty \otimes_{\mathbb{Q}_l} \mathbb{Z}_l(q))
\]

Proof. By the equality $h_F(\check{X}/\kappa) = h_F(\check{X}/\kappa)$, the Dieudonné module of the Artin-Mazur formal group $\Phi_{2q-1}(\check{X}/\kappa)$ is a free $\mathcal{W}$-module of finite rank $1 \leq h_F(\check{X}/\kappa) < \infty$. Because this module is isomorphic to $H^{2q-1}(X, \mathcal{W}(O_X)), H^{2q-1}(X, \mathcal{W}(O_X)) \otimes_\mathbb{W} K_0 \neq 0$. Hence the slopes of $H^{2q-1}(X, \mathcal{W}(O_X)) \otimes_\mathbb{W} K_0$ is not contained in $[q-1, q]$ since $q-1 \geq 1$. By [Su, (3.4)] (as in [J, (11.2)]), the $l$-adic Abel-Jacobi map of Bloch (4.18.1) is not surjective. We can generalize the Yobuko height as follows.

Let $i$ be a nonnegative integer. Then we have the following exact sequences

\[
(4.18.2) \quad 0 \to \mathcal{W}_n \mathcal{O}_X^i \to \mathcal{W}_n \Omega^n_{X/s} \overset{\Phi_{n-1}^i}{\to} B_n \Omega^{i+1}_{X/s} \to 0 \quad (n \in \mathbb{Z}_{\geq 0})
\]

of $\mathcal{W}_n(O_X)$-modules. Consider the push-out of the exact sequence (4.18.2) by the projection $\mathcal{W}_n \mathcal{O}_X^i \to \mathcal{W}_2 \Omega^n_X$ and let

\[
(4.18.3) \quad 0 \to \mathcal{W}_2 \Omega^i_X \to \mathcal{E}_n^i \to B_n \Omega^{i+1}_{X/s} \to 0 \quad (n \in \mathbb{Z}_{> 0}).
\]

be the resulting exact sequence of $O_X$-modules. We say that $X/s$ has height $h < \infty$ at $i$ if (4.18.3) is split for all $n \geq 0$. If (4.18.3) is not split, then we set $h = \infty$. (Note that, by (3.4), if (4.18.3) is split for some $n \in \mathbb{Z}_{\geq 0}$, then (4.18.3) is split for any $m \geq n$.) We denote $h$ by $h_F^i(X/s)$. It is easy to check that $h_F^i(X/s) = h_F(\check{X})$.

Assume that $h_F(\check{X}/s) < \infty$. Then, by the same proof as that of (4.13), $H^q(X, B_n \Omega^{i+1}_{X/s})$ is bounded for all $n$. Indeed, we have the following commutative diagram of exact sequences

\[
\begin{array}{cccccc}
0 & \to & \mathcal{W} \mathcal{O}^i_X & \to & \mathcal{E}_n^i & \to & B_n \Omega^{i+1}_{X/s} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow c^{n-1} \\
0 & \to & \mathcal{W} \mathcal{O}^i_X & \to & \mathcal{E}_n^i = F_*(\Omega^i_{X/s}) & \overset{d}{\to} & B_1 \Omega^{i+1}_{X/s} & \to & 0
\end{array}
\]

of $O_X$-modules and the following exact sequence

\[
(4.18.5) \quad 0 \to B_{n-1} \Omega^{i+1}_{X/s} \to \mathcal{E}_n^i \to F_*(\Omega^i_{X/s}) \to 0
\]

of $O_X$-modules. Hence, by the same proof as that of (4.13), we obtain the following inequality

\[
\dim_k H^q(X, B_n \Omega^{i+1}_{X/s}) \leq \dim_k H^q(X, B_{n-1} \Omega^{i+1}_{X/s}) + \dim_k H^q(X, \Omega^i_{X/s}) - \dim_k H^q(X, \mathcal{W} \mathcal{O}^i_X)
\]

21
for \( n \geq h \). In this way, we see that \( H^q(X, B_n \Omega^{i+1}_X) \) is bounded for all \( n \). By (1.2) the differential \( H^q(X, \mathcal{W} \Omega^i_X) \to H^q(X, \mathcal{W} \Omega^{i+1}_X) \) is zero. Consequently \( H^q(X, \mathcal{W} \Omega^i_X)/dH^q(X, \mathcal{W} \Omega^{i-1}_X) \) is a finitely generated \( \mathcal{W} \)-module.

If the log structures of \( s \) and \( X \) are trivial and if \( h^1_p(X/\kappa) < \infty \), then the differential \( H^2(X, \mathcal{W} \Omega^i_X) \to H^2(X, \mathcal{W} \Omega^{i+1}_X) \) is zero. Hence, by [GS] II (3.7) \( CH^2(X, p) \) is of finite cotype. Hence we obtain the following:

**Theorem 4.19.** Let \( X/\kappa \) be a proper smooth scheme. If \( h^1_p(X/\kappa) < \infty \), then \( CH^2(X, p) \) is of finite cotype.

### 5 Upper bounds of heights of Artin-Mazur formal groups

Let \( X/s \) be as in the beginning of the previous section. In this section we give a upper bound of the height of the Artin-Mazur formal group \( \Phi^q_X/\kappa \) (\( q \in \mathbb{N} \)) by using the dimensions of log Hodge cohomologies of \( X/\kappa \). This is a very broad generalization of Katsura and Van der Geer’s upper bound for the Artin-Mazur formal group of a Calabi-Yau variety over \( \kappa \). To give the upper bound, we use ([GK2], (2.4)) and ([4]), (3.17) in [3]. In this section we are influenced by the argument in [11] II (4.1)∼(4.6).

**Theorem 5.1.** Let \( q \) and \( i \) be nonnegative integers. Assume that the operator

\[
F : H^{q+1}(X, \mathcal{W} \Omega^i_X) \to H^{q+1}(X, \mathcal{W} \Omega^i_X)
\]

is injective. Furthermore, assume that the operator

\[
dV : H^q(X, \mathcal{W} \Omega^i_X) \to H^q(X, \mathcal{W} \Omega^{i+1}_X)
\]

is zero for \( j = i - 1 \) and \( j = i \). Then there exists the following diagram

\[
\begin{array}{c}
0 \longrightarrow H^q(X, \mathcal{W} \Omega^i_X)/V \xrightarrow{F} H^q(X, \mathcal{W} \Omega^i_X)/p \xrightarrow{\text{proj}} H^q(X, \mathcal{W} \Omega^i_X)/F \longrightarrow 0 \\
\downarrow \ 
\downarrow \ 
\downarrow
\\
H^q(X, \Omega^i_X/V) \xrightarrow{\sim} H^q(X, \Omega^i_X/F) \xrightarrow{\uparrow}
\\
H^q(X, \Omega^i_{X/s}) \longrightarrow H^q(X, \Omega^{i+1}_{X/s})
\end{array}
\]

where the morphism \( H^q(X, \Omega^{i+1}_{X/s}) \to H^q(X, \mathcal{W} \Omega^i_X/F) \) is constructed in the proof of this theorem and it is surjective.

**Proof.** By the first assumption, we have the following exact sequence

\[
(5.1.2) \quad 0 \longrightarrow H^q(X, \mathcal{W} \Omega^i_X)/V \xrightarrow{F} H^q(X, \mathcal{W} \Omega^i_X)/p \longrightarrow H^q(X, \mathcal{W} \Omega^i_X)/V \longrightarrow 0.
\]

By (3.5.17) for the case \( r = 1 \) and (3.10), we have the following exact sequence

\[
(5.1.3) \quad 0 \longrightarrow \mathcal{W} \Omega^i_X/F \xrightarrow{dV} \mathcal{W} \Omega^{i+1}_X/V \longrightarrow \Omega^{i+1}_{X/s} \longrightarrow 0.
\]
Hence we have the following exact sequence:

\[(5.1.4) \quad \cdots \to H^{q-1}(X, \Omega^{i+1}_{X_j}) \to H^{q}(X, W\Omega^{i}_X/F) \xrightarrow{dV} H^{q}(X, W\Omega^{i+1}_X/V) \to H^{q}(X, \Omega^{i+1}_{X_j}) \to \cdots.\]

By the exact sequence (4.2.1) and the first assumption, we have the following isomorphism

\[(5.1.5) \quad H^{q}(X, W\Omega^{i}_X/F) \cong H^{q}(X, W\Omega^{i}_X).\]

By the exact sequence

\[0 \to W\Omega^{i+1}_X \xrightarrow{V} W\Omega^{i+1}_X \to W\Omega^{i+1}_X/V \to 0,\]

we have the following injection

\[(5.1.6) \quad H^{q}(X, W\Omega^{i+1}_X/V) \subset H^{q}(X, W\Omega^{i+1}_X/V).\]

Since the following diagram

\[
\begin{array}{ccc}
H^q(X, W\Omega^i_X/F) & \xrightarrow{dV} & H^q(X, W\Omega^i_X) \\
\downarrow{dV} & & \downarrow{dV} \\
H^q(X, W\Omega^{i+1}_X/V) & \to & H^q(X, W\Omega^{i+1}_X/V)
\end{array}
\]

is commutative, the morphism

\[dV : H^q(X, W\Omega^i_X/F) \to H^q(X, W\Omega^{i+1}_X/V)\]

is zero. Hence we see that the morphism

\[H^{q-1}(X, \Omega^{i+1}_{X_j}) \to H^{q}(X, W\Omega^{i}_X/F) = H^{q}(X, W\Omega^{i}_X)/F\]

is surjective by considering the case \(j = i\) in (5.1.4) and that the morphism

\[H^{q}(X, W\Omega^{i}_X/V) \to H^{q}(X, \Omega^{i+1}_{X_j})\]

is injective by considering the case \(j = i - 1\) in (5.1.4). We have proved (5.1).

**Corollary 5.2.** Let the assumptions be as in (5.1). Let \(G^{i q}\) be the \(p\)-divisible group whose Cartier module is \(H^{q}(X, W\Omega^{i}_X)\). Let \((G^{i q})^*\) be the Cartier dual of \(G^{i q}\). Let \(h(G^{i q})\) be the height of \(G^{i q}\). Then

\[(5.2.1) \quad \dim G^{i q} \leq \dim_{\kappa} H^{q}(X, \Omega^{i}_{X_j}).\]

\[(5.2.2) \quad \dim(G^{i q})^* \leq \dim_{\kappa} H^{q-1}(X, \Omega^{i+1}_{X_j}).\]

and

\[(5.2.3) \quad h(G^{i q}) \leq \dim_{\kappa} H^{q-1}(X, \Omega^{i+1}_{X_j}) + \dim_{\kappa} H^{q}(X, \Omega^{i}_{X_j}).\]

**Proof.** Because \(\dim G^{i q} = \dim_{\kappa}(H^{q}(X, W\Omega^{i}_X)/V)\), \(\dim(G^{i q})^* = \dim_{\kappa}(H^{q}(X, W\Omega^{i}_X)/F)\), and \(h(G^{i q}) = \dim_{\kappa} H^{q}(X, W\Omega^{i}_X)/p\), we obtain the inequality (5.2.1), (5.2.2) and (5.2.3), respectively, by (5.1).
The following is a generalization of [vGK2 (2.3)]; our assumption is much weaker than that of [loc. cit.]:

**Corollary 5.3.** ([1,3]) holds.

**Proof.** By the assumption, $H^q(X, \mathcal{W}(\mathcal{O}_X))$ is a free $\mathcal{W}$-module of finite rank $h^q(X/\kappa)$. The induced morphism $d$: $H^q(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^q(X, \mathcal{W}\Omega^n_X)$ by the derivative $d$: $\mathcal{W}(\mathcal{O}_X) \rightarrow \mathcal{W}\Omega^n_X$ is zero by [NY1 (2.5)] or the log version of [IR II (3.8)]. Now (1.13.1) follows from (5.2.3).

**Example 5.4.** Let $X/s$ be a log Calabi-Yau variety of pure dimension $d$. Assume that $h^d(X/\kappa)$ is finite. Then $h^d(X/\kappa) \leq H^{d-1}(X, \Omega^1_{X/s}) + 1$. This is a log version of [vGK2 (2.4)]. As in the trivial logarithmic case, we say that $X/s$ is rigid if $H^{d-1}(X, \Omega^1_{X/s}) = 0$. (This is equivalent to the vanishing of $H^1(X, \Omega^d_{X/s})$ by the log Serre duality of Tsuji ([Ts (2.21)]).) Consequently the height of a rigid log Calabi-Yau variety is 1 or $\infty$.

**Theorem 5.5.** Let $q$ and $i$ be nonnegative integers. Let the assumptions be as in (5.1). Furthermore, assume that the operator

$$V: H^q(X, \mathcal{W}\Omega^i_X) \rightarrow H^q(X, \mathcal{W}\Omega^i_X)$$

is injective. Then there exists the following diagram (5.5.1)

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H^q(X, \mathcal{W}\Omega^i_X)/F & \rightarrow & H^q(X, \mathcal{W}\Omega^i_X)/p & \rightarrow & H^q(X, \mathcal{W}\Omega^i_X)/V & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow &\
H^q(X, \mathcal{W}\Omega^i_X)/F & \rightarrow & H^q(X, \mathcal{W}\Omega^i_X)/p & \rightarrow & H^q(X, \mathcal{W}\Omega^i_X)/V & \rightarrow & H^q(X, \mathcal{W}\Omega^i_X)/V
\end{array}
$$

**Proof.** By the new assumption, we have the following exact sequence (5.5.2)

$$0 \rightarrow H^q(X, \mathcal{W}\Omega^i_X)/F \rightarrow H^q(X, \mathcal{W}\Omega^i_X)/p \rightarrow H^q(X, \mathcal{W}\Omega^i_X)/V \rightarrow 0.$$

The rest of the proof is the same as that of (5.1).

The following is a log version of [II II (4.6)].

**Corollary 5.6 (cf. [II II (4.6)])**. Let $q$ be a nonnegative integer. For any $i$ and $j$ such that $i + j = q$, assume that the operators

$$V: H^i(X, \mathcal{W}\Omega^j_X) \rightarrow H^i(X, \mathcal{W}\Omega^j_X),$$

$$F: H^{i+1}(X, \mathcal{W}\Omega^j_X) \rightarrow H^{i+1}(X, \mathcal{W}\Omega^j_X)$$

are injective and that the operator

$$dV: H^i(X, \mathcal{W}\Omega^{j+1}_X) \rightarrow H^q(X, \mathcal{W}\Omega^{j+1}_X)$$

is zero. Then there exists the following exact sequence (5.6.1)

$$0 \rightarrow H^0(X, \mathcal{W}\Omega^i_X)/p \rightarrow H^0(X, \mathcal{W}\Omega^i_X)/p \rightarrow H^1(X, \mathcal{W}\Omega^i_X)/p \rightarrow H^1(X, \mathcal{W}\Omega^i_X)/p \rightarrow H^1(X, \Omega^i_{X/s}) \rightarrow H^2(X, \Omega_{X/s}^i - 2) \rightarrow \cdots \rightarrow H^{q-1}(X, \Omega^i_{X/s}) \rightarrow H^q(X, \mathcal{W}(\mathcal{O}_X))/p \rightarrow H^q(X, \mathcal{O}_X) \rightarrow 0.$$
Proof. (5.6) follows from (5.5).

Remark 5.7. (1) We leave the log version of [II] II (4.5) to the reader.

(2) In the trivial log case, the assumptions in (5.6) are slightly weaker than those in [II] II (4.6).

6 Ordinary log schemes and $F$-split log schemes

In this section we give the definition of the ordinarity at a bidegree for a proper log smooth scheme of Cartier type. We also generalize results in [I] and [JR] for $F$-split log schemes over $s$. We also prove that the nontrivial exotic torsions of log crystalline cohomologies of $F$-split proper log smooth schemes do not exist. This is a log version of Joshi’s result ([I]). We also give the criterion of the $F$-splitness for certain log schemes.

Let the notations be as in the previous section.

Definition 6.1 (cf. [BK, (7.2)], [IR, IV (4.12), (4.13)], [L, §4]). Let $q$ be a nonnegative integer.

(1) We say that $X/s$ is ordinary at $(0,q)$ if $H^q(X, B\Omega^1_{X/s}) = 0$.

(2) We say that $X/s$ is ordinary at $(0,* )$ if $H^q(X, B\Omega^1_{X/s}) = 0$ for any $q \in \mathbb{N}$.

Proposition 6.2. Let $q$ be a nonnegative integer. Then the following are equivalent:

(1) $X/s$ is ordinary at $(0,q)$.

(2) For any positive integer $n$, $H^q(X, B_n \Omega^1_{X/s}) = 0$.

(3) For any positive integer $n$, $H^q(X, W_n(\mathcal{O}_X))/F = 0 = F H^{q+1}(X, W_n(\mathcal{O}_X))$.

(4) $H^q(X, \mathcal{O}_X)/F = 0 = F H^{q+1}(X, \mathcal{O}_X)$.

Proof. (1) $\implies$ (2): Recall the right vertical exact sequence in (4.3.4):

\[
0 \to F_*(B_{n-1}\Omega^1_{X/s}) \to B_n\Omega^1_{X/s} \to B_1\Omega^1_{X/s} \to 0.
\]

By noting that $H^q(X, F_*(B_{n-1}\Omega^1_{X/s})) \simeq H^q(X, B_{n-1}\Omega^1_{X/s})$ and that $H^q(X, B_1\Omega^1_{X/s}) \simeq H^q(X, B_0\Omega^1_{X/s})$ and using induction on $n$, we obtain the implication (1) $\implies$ (2).

(2) $\implies$ (3): By (2.3.1) we have the following exact sequence:

\[
0 \to H^q(X, W_n(\mathcal{O}_X))/F \to H^q(X, B_n\Omega^1_{X/s}) \to F H^{q+1}(X, W_n(\mathcal{O}_X)) \to 0.
\]

Hence we obtain the implication (2) $\implies$ (3).

(3) $\implies$ (4): This is obvious.

(4) $\implies$ (1): By (6.2.2) for the case $n = 1$, we have the following exact sequence:

\[
0 \to H^q(X, \mathcal{O}_X)/F \to H^q(X, B_1\Omega^1_{X/s}) \to F H^{q+1}(X, \mathcal{O}_X) \to 0.
\]

Hence we obtain the implication (4) $\implies$ (1).

Remark 6.3. ([BK (1.4)], (resp. [NY (1.3)])) Let $X/s$ be an abelian variety (resp. $K3$-surface) over $\kappa$. Then $X/s$ is ordinary if and only if it is ordinary at $(0,1)$ (resp. $(0,2)$).

The following is not included in [BK (7.3)], [IR, IV (4.13)] and [L (4.1)].
Proposition 6.4 (cf. [BK (7.3)], [IR IV (4.13)], [L (4.1)]). Let \( q \) be a non-negative integer. Then the following are equivalent:

1. \( X/s \) is ordinary at \((0, \ast)\).
2. For any positive integer \( n \) and for any \( q \in \mathbb{N} \), \( H^q(X, B_n \Omega^1_{X/s}) = 0 \).
3. For a positive integer \( n \) and for any \( q \in \mathbb{N} \), \( H^q(X, B_n \Omega^1_{X/s}) = 0 \).
4. For any positive integer \( n \) and any \( q \in \mathbb{N} \), the operator \( F : H^q(X, W_n(O_X)) \rightarrow H^q(X, W_n(O_X)) \) is bijective.
5. For a positive integer \( n \) and any \( q \in \mathbb{N} \), the operator \( F : H^q(X, W_n(O_X)) \rightarrow H^q(X, W_n(O_X)) \) is bijective.
6. For any \( q \in \mathbb{N} \), the operator \( F : H^q(X, W_n(O_X)) \rightarrow H^q(X, W_n(O_X)) \) is bijective.
7. For any \( q \in \mathbb{N} \), the natural morphism \( H^q(X, Z/p^n) \otimes_{\mathbb{Z}/p^n} W_n(p) \rightarrow H^q(X, W_n(O_X)) \) is an isomorphism.
8. For any \( q \in \mathbb{N} \), the natural morphism \( H^q(X, W_n(O_X)) \rightarrow H^q(X, W_n(O_X)) \) is an isomorphism.

Proof. The implications \((1) \implies (2) \implies (3) \), \((2) \iff (4) \), \((3) \iff (5) \) and \((1) \iff (6) \) immediately follows from [6.2] or obvious. Hence it suffices to prove the implications \((3) \implies (1) \), \((6) \iff (7) \iff (8) \) and \((8) \iff (9) \). Assume that \((3) \) holds. Let \( n \) be a positive integer in \((3) \). By [6.2.1] we have the following exact sequence of abelian groups:

\[
\begin{array}{c}
H^q(X, B_n \Omega^1_{X/s}) \rightarrow H^q(X, B_n \Omega^1_{X/s}) \\
H^q(X, B_n \Omega^1_{X/s}) \rightarrow H^q(X, B_n \Omega^1_{X/s}) \\
H^q(X, B_n \Omega^1_{X/s}) \rightarrow H^q(X, B_n \Omega^1_{X/s})
\end{array}
\]

Hence \((6.3.1)\)

\[
H^q(X, B_n \Omega^1_{X/s}) = H^q(X, B_n \Omega^1_{X/s}) \quad (\forall q).
\]

On the other hand, we have the following exact sequence of abelian sheaves by the definition of \( B_n \Omega^1_{X/s} \):

\[
\begin{array}{c}
0 \rightarrow B_n \Omega^1_{X/s} \rightarrow B_n \Omega^1_{X/s} \\
B_n \Omega^1_{X/s} \rightarrow B_n \Omega^1_{X/s} \\
B_n \Omega^1_{X/s} \rightarrow B_n \Omega^1_{X/s} \rightarrow 0.
\end{array}
\]

Taking the long exact sequence of [6.3.3] and using the assumption, we have the following equality

\[
H^q(X, B_n \Omega^1_{X/s}) = H^q(X, B_n \Omega^1_{X/s}) \quad (\forall q).
\]
Here we have identified abelian sheaves on $\hat{X}$ with those on $X'$. By [L (1.13)] the sheaf $B_m\Omega^1_{X/s}$ ($m \in \mathbb{N}$) is a locally free sheaf of $\mathcal{O}_X$-modules of finite rank and it commutes with the base changes of $s$. Hence $B_m\Omega^1_{X/s} = \kappa \otimes_{\sigma,\kappa} B_m\Omega^1_{X/s} \simeq B_m\Omega^1_{X/s}$, where $\sigma$ is the Frobenius automorphism of $\kappa$. Hence we have the following equality by (6.3.4):

$$H^q(X, B_{n-1}\Omega^1_{X/s}) = H^{q+1}(X, B_1\Omega^1_{X/s}) \quad (\forall q).$$

By (6.3.5) and (6.3.6) we have the following equality:

$$H^q(X, B_1\Omega^1_{X/s}) = H^{q+2}(X, B_1\Omega^1_{X/s}) \quad (\forall q).$$

If $q > \dim X$, $H^q(X, B_1\Omega^1_{X/s}) = 0$. Hence $H^q(X, B_1\Omega^1_{X/s}) = 0$ for any $q \in \mathbb{N}$. We have proved the implication (3) $\implies$ (1).

By the following exact sequence

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{G}_a^{1-F} \mathbb{G}_a \rightarrow 0$$

on $(X_{\mathbb{F}})_{et}$ and using the surjectivity of the morphism $1 - F : H^q(X_{\mathbb{F}}, \mathcal{O}_{X_{\mathbb{F}}}) \rightarrow H^q(X_{\mathbb{F}}, \mathcal{O}_{X_{\mathbb{F}}})$ ([II II (5.3)]), we have the following exact sequence

$$0 \rightarrow H^q_{et}(X_{\mathbb{F}}, \mathbb{F}_p) \rightarrow H^q(X_{\mathbb{F}}, \mathcal{O}_{X_{\mathbb{F}}}) \rightarrow H^q(X_{\mathbb{F}}, \mathcal{O}_{X_{\mathbb{F}}}) \rightarrow 0.$$

The implications (6) $\iff$ (7) $\iff$ (8) are special cases of [CL (3.3)] and [MS §2].

The implication (8) $\implies$ (9) follows from the following commutative diagram of exact sequences on $(X_{\mathbb{F}})_{et}$ and the induction on $n$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/p^{n-1} & \longrightarrow & \mathbb{Z}/p^n & \longrightarrow & \mathbb{Z}/p & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | \\
0 & \longrightarrow & \mathcal{W}_{n-1}(\mathbb{F}_p) & \overset{V}{\longrightarrow} & \mathcal{W}_n(\mathbb{F}_p) & \longrightarrow & \mathbb{F}_p & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | \\
0 & \longrightarrow & \mathcal{W}_{n-1}(\mathcal{O}_{X_{\mathbb{F}}}) & \overset{V}{\longrightarrow} & \mathcal{W}_n(\mathcal{O}_{X_{\mathbb{F}}}) & \longrightarrow & \mathcal{O}_{X_{\mathbb{F}}} & \longrightarrow & 0.
\end{array}
$$

The implication (9) $\implies$ (8) is obvious.

We have completed the proof of (6.4). \qed

Remark 6.5. I do not know whether the statement with the replacement of “any positive integer $n” in (9) by “a positive integer $n” is equivalent to (1).

Remark 6.6. As in [MS Conjecture 1.1], one can conjecture the following:

Let $X$ be a proper smooth scheme over a field of characteristic zero. Let $X'$ be a proper flat model of $X$ over a $\mathbb{Z}$-algebra $A$ of finite type. Then there exists a dense set of closed points $T \subset \text{Spec}(A)$ such that $X'/t$ is ordinary at $(0, \ast)$ for every $t \in T$.

Proposition 6.7. The following hold:

1. Let $q$ be a nonnegative integer. Assume that $X/s$ is ordinary at $(0, q - 1)$. Then the submodule of $p$-torsions of $H^q(X, \mathcal{W}(\mathcal{O}_X))$ is equal to that of $V$-torsions of $H^q(X, \mathcal{W}(\mathcal{O}_X))$.

2. Assume that $X$ is $F$-split. Then $X/s$ is ordinary at $(0, \ast)$.
Proof. (1): Since \( F H^q(X, W(O_X)) = \lim_{n \to \infty} F H^q(X, W_n(O_X)) \), \( F H^q(X, W(O_X)) = 0 \) by (5.2). Since \( FV = p \), we immediately obtain (1).

(2): The proof of (2) is the same as that of \( \text{JR} \) (2.4.1) by using the log version of Serre’s exact sequence (2.3.1) for the case \( n = 1 \).

Let \( q \) be a nonnegative integer. Let \( H^q_{\text{crys}}(X/W(s)) \) be the log crystalline cohomology of \( X/W(s) \) [K1]. Next we discuss exotic torsions in \( H^q_{\text{crys}}(X/W(s)) \) as in [J].

Let \( q \) be a nonnegative integer. Set \( Q^q := \text{Im}(H^q_{\text{crys}}(X/W(s)))_{\text{tor}} \to H^q(X, W(O_X)) \). As in [II] II (6.7.3)], we define the module \( H^q_{\text{crys}}(X/W(s))_{\text{e}} \) of exotic torsions in \( H^q_{\text{crys}}(X/W(s)) \) as the following quotient

\[ H^q_{\text{crys}}(X/W(s))_{\text{e}} := Q^q/(H^q(X, W(O_X)))_{\text{tor}} \cap Q_q. \]

(In [loc. cit.] only the case \( q = 2 \) has been considered.)

The following is a log version of a generalization of [J] (7.3).

**Proposition 6.8.** Assume that \( X/s \) is ordinary at \((0, q)\). Then \( H^q_{\text{crys}}(X/W(s))_{\text{e}} = 0 \).

**Proof.** This follows from (6.7) (1). \( \square \)

**Corollary 6.9.** Assume that \( \hat{X} \) is \( F \)-split. Then \( H^q_{\text{crys}}(X/W(s))_{\text{e}} = 0 \) \((q \in \mathbb{N})\).

**Proof.** This follows from (6.7) (2) and (6.8). \( \square \)

The following is a log version of [JR] (2.4.2) with slightly weaker assumption. Our proof is slightly more immediate than the proof in [loc. cit.].

**Proposition 6.10.** Assume that, \( X/s \) is of vertical type, and that \( \Omega^d_{X/s} \) is trivial and that \( X/s \) is ordinary at \((0, d - 1)\). Then \( \hat{X} \) is \( F \)-split.

**Proof.** By using the log Serre duality of Tsuji ([TS] (2.21)) and using the ordinarity at \((0, d - 1)\), we have the following equalities:

\[ \text{Ext}_X^1(B_1 \Omega^1_{X/s}, O_X) = \text{Ext}_X^1(B_1 \Omega^1_{X/s}, \Omega^d_{X/s}) = H^{d-1}(X, B_1 \Omega^1_{X/s})^* = 0. \]

Here \(*\) means the dual vector space. \( \square \)

If \( \text{dim} \hat{X} \leq 2 \), we can give explicit examples easily for an \( F \)-split proper degenerate log variety by the classification of lower dimensional proper smooth varieties.

**Proposition 6.11.** Assume that \( s \) is the log point of \( \text{Spec}(\kappa) \). Let \( X \) be a proper log Calabi-Yau variety over \( s \). Assume that \( \hat{X}/\kappa \) is not smooth. Then the following hold:

(1) Assume that \( \hat{X} \) is of pure dimension 1. (In this case we say that \( X/s \) is a log elliptic curve.) Then \( \hat{X} \) is \( F \)-split.

(2) Assume that \( \hat{X} \) is of pure dimension 2. (In [N1], in this case, we have said that \( X/s \) is a log K3-surface.) If \( \hat{X} \) is of Type II ([N1 §3]), then \( \hat{X} \) is \( F \)-split if and only if the isomorphic double elliptic curve of \( \hat{X} \) is ordinary. If \( \hat{X} \) is of Type III ([loc. cit.]), then \( \hat{X} \) is \( F \)-split.

(3) Let the notations be as in (2). If \( \hat{X} \) is of Type II ([N1 §3]) and if the isomorphic double elliptic curve of \( X \) is supersingular, then \( h_F(X/\kappa) = 2 \).
Proof. Set \(d := \dim \hat{X}\). By (4.15), \(\hat{X}\) is \(F\)-split if and only if \(h^d(\hat{X}/\kappa) = 1\). Let \(\hat{X}^{(i)} (i \in \mathbb{Z}_{\geq 0})\) be the disjoint union of the \((i + 1)\)-fold intersections of the different irreducible components of \(\hat{X}\). Then, by [RS Theorem 1], we have the following spectral sequence

\[
E^{ij}_{\ast} = H^j(\hat{X}^{(i)}_{\ast} \otimes \mathcal{O}_{\hat{X}^{(i)}_{\ast}}) \Rightarrow H^i(X, \mathcal{O}_X) \quad (6.10.1)
\]

obtained by the following exact sequence

\[
0 \to \mathcal{W}(\mathcal{O}_X) \to \mathcal{W}(\mathcal{O}_{\hat{X}^{(0)}_{\ast}}) \to \mathcal{W}(\mathcal{O}_{\hat{X}^{(1)}_{\ast}}) \to \cdots \quad (6.10.2)
\]

Let \(D(\Phi^q_{\mathcal{X}/\kappa}) (q \in \mathbb{N}_{\geq 1})\) be the Dieudonné module of \(\Phi^q_{\mathcal{X}/\kappa}\). Then \(D(\Phi^q_{\mathcal{X}/\kappa}) = H^q(X, \mathcal{W}(\mathcal{O}_X))\) ([AM]).

(1): By the easier proof than that of [N1 (3.4)], it is easy to see that \(\hat{X}\) is an \(n\)-gon \((n \geq 2)\). By (6.10.1) we easily see that

\[
D(\Phi^1_{\mathcal{X}/\kappa}) = H^1(X, \mathcal{W}(\mathcal{O}_X)) = \text{Coker}(H^0(\hat{X}^{(0)}_{\ast} \otimes \mathcal{O}_{\hat{X}^{(0)}_{\ast}}) \to H^0(\hat{X}^{(1)}_{\ast} \otimes \mathcal{O}_{\hat{X}^{(1)}_{\ast}}))) = \mathcal{W}.
\]

Hence \(h^1(\hat{X}/\kappa) = 1\). By (4.15) we obtain (1).

(2): By the criterion of [N2 (5.4)] and (4.15), we obtain (2).

(3) Let \(E\) be the double elliptic curve over \(\kappa\). By (6.10.1) we easily see that

\[
D(\Phi^2_{\mathcal{X}/\kappa}) = H^1(X, \mathcal{W}(\mathcal{O}_X)) = H^1(E, \mathcal{W}(\mathcal{O}_E)).
\]

Hence \(h^2(\hat{X}/\kappa) = 2\). By (4.15) we obtain (3).

\(\square\)

Remark 6.12. In (6.11) the existence of log structures are not necessary for the conclusions of (6.11): if \(Y/\kappa\) is a combinatorial \(K3\)-surface of Type II or III, then the conclusions of (6.11) hold.

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