On M-Theory and the Symmetries of Type II
String Effective Actions

Ashok Das
Department of Physics and Astronomy
University of Rochester, Rochester, N.Y. 14627, USA

and

Shibaji Roy
Departamento de Física de Partículas
Universidade de Santiago, E-15706 Santiago de Compostela, Spain

ABSTRACT

We study the “ordinary” Scherk-Schwarz dimensional reduction of the bosonic sector of the low energy effective action of a hypothetical M-theory on $S^1 \times S^1 \cong T^2$. We thus obtain the low energy effective actions of type IIA string theory in both ten and nine space-time dimensions. We point out how to obtain the $O(1, 1)$ invariance of the NS-NS sector of the string effective action correctly in nine dimensions. We dimensionally reduce the type IIB string effective action on $S^1$ and show that the resulting nine dimensional theory can be mapped, purely from the bosonic consideration, exactly to the type IIA theory by an $O(1, 1)$ or Buscher’s T-duality transformations. We then give a dynamical argument, in analogy with that for the type IIB theory in ten dimensions, to show how an S-duality in the type IIA theory can be understood from the underlying nine dimensional theory by compactifying M-theory on a T-dual torus $\tilde{T}^2$.

*E-mail address: roy@gaes.usc.es
I. Introduction:

Recently there has been a remarkable development in our understanding of the non-perturbative behavior of various string theories. One of the most important new ingredients in the dynamics of various string theories is their close relationship [1–3] with the eleven dimensional supergravity theory of Cremmer, Julia and Scherk (CJS) [4, 5]. This latter theory is believed to be the low energy description of an eleven dimensional hypothetical quantum theory which is non-commitally called as M-theory [6] in the literature. It is rather surprising that all the five known string theories including type I, two type II and two heterotic string theories can be obtained by toroidal and orbifold compactifications of this CJS theory [7–9]. Most intriguingly, the type II string theories obtained by toroidal compactification of CJS theory automatically contain the non-perturbative information through the Ramond–Ramond (R-R) fields. The sources for these R-R fields are believed to be extended objects known as Dirichlet(D)-branes [10–13] which should be included in the string theory spectrum as dictated by the underlying U-duality [14] symmetry of the string theory.

In this paper, we study the “ordinary” dimensional reduction [15, 16] of the bosonic sector of N=1, D=11 CJS supergravity theory. In this procedure one splits the original eleven dimensional coordinates (ťx\textsuperscript{\mu}) into D space-time coordinates (x\textsuperscript{\mu}) and d internal coordinates (x\textsuperscript{m}) and demands that the fields and the symmetry transformation laws of the original theory be independent of the internal coordinates. The internal space is usually taken to be compact and in this case we consider it to be S\textsuperscript{1} × S\textsuperscript{1} \cong T\textsuperscript{2}. When we reduce the 11-dimensional supergravity theory first on S\textsuperscript{1} with radius R\textsubscript{1}, we obtain the type IIA theory in space-time dimension ten for small R\textsubscript{1}. Since the radius of the circle is directly proportional to the string coupling constant as measured in terms of the metric of the string theory, the CJS theory represents the non-perturbative limit of the type IIA string theory. Type IIA theory obtained this way, therefore, inherits non-perturbative information from the supergravity theory which shows up in the natural appearance of the R-R fields. Unlike the Neveu-Schwarz–Neveu-Schwarz (NS-NS) gauge fields, the R-R fields do not couple to the ten dimensional dilaton and therefore remain inert under the four dimensional S-duality transformation, a natural consequence of the fact that the R-R charges are carried by the solitonic modes and not by the fundamental string modes [14].

\footnote{Our notations and conventions are described in sec. II.}
We then further reduce the theory on a second $S^1$ with radius $R_2$ and obtain the type IIA string effective action in nine dimensions. It is well-known that the NS-NS sector of a dimensionally reduced string effective action possesses a non-compact global $O(d, d)$ [17] symmetry under which the moduli fields transform in a complicated non-linear way whereas the vector gauge fields transform linearly as the vector representation of $O(d, d)$ with all other fields remaining invariant. So, the type IIA theory in nine dimensions should have a global $O(1, 1)$ invariance in the NS-NS sector. We here show that this is true only if the antisymmetric Kalb-Ramond field also transforms appropriately under the $O(1, 1)$ transformation alongwith the moduli and the vector gauge fields. In fact, we show that a particular combination of the two-form antisymmetric gauge fields and the vector gauge fields remains invariant in order to recover the $O(1, 1)$ invariance of the full NS-NS sector of the nine dimensional action. This result is true in general for $O(d, d)$ invariance of the dimensionally reduced string effective action. We then ask whether the whole type IIA string effective action including the R-R sector is also invariant under $O(1, 1)$ transformation. We find the answer in the negative. However, we point out that the whole action possesses a global $O(2)$ invariance [7] which is kind of trivial since it follows directly from the Lorentz invariance of the CJS theory in eleven dimensions.

It is known that the type IIB supergravity equations of motion [18, 19] can not be obtained from a ten dimensional covariant action. However, if one sets the self-dual five-form field strength to zero, then the equations of motion can be obtained from an action. We take the bosonic sector of this type IIB string effective action which is also known to possess an exact global $SL(2, R)$ invariance [20, 21]. Under this, a complex scalar field formed out of an R-R scalar and the dilaton undergoes a fractional linear transformation whereas the two two-form potentials, one from the NS-NS sector and the other from the R-R sector, transform linearly with the other fields remaining invariant. We reduce this action on $S^1$ and find that the NS-NS sector of the resulting nine-dimensional theory is $O(1, 1)$ invariant as expected if the same combination of the two form gauge field and the vector gauge fields as found for the type IIA case remains invariant. Now, we again ask the question what happens if we make the $O(1, 1)$ transformation on the full bosonic sector of the type IIB theory including the R-R sector. We find that under this transformation type IIB string effective action gets mapped precisely to the previously obtained type IIA string effective action in nine dimensions with some field redefinitions. In obtaining this the fields in the type IIA would have to satisfy certain relations. We note that,
this O(1, 1) transformation reduces precisely to the Buscher’s duality rules [22] for the various components of the metric, the antisymmetric tensor field and the dilaton in ten dimensions. We thus observe, purely from bosonic considerations, that type IIA and type IIB string theories are T-dual to each other. As in type IIA theory, we point out that the nine dimensional type IIB theory also has a global O(2) invariance although it is not obtained from an eleven dimensional theory. At this point, it is natural to ask whether the global SL(2, R) invariance of the ten dimensional type IIB theory remains a symmetry of the nine dimensional theory. Naively, one would expect this to be true since the scalar fields of the original theory remain intact in the process of dimensional reduction. We explicitly show that this naive expectation fails and the nine dimensional theory does not have a manifest SL(2, R) invariance of the action. The SL(2, R) invariance seems special only to D=10 and D=4. Finally, we note that the SL(2, R) S-duality invariance of the type IIB theory in ten dimensions has its origin in eleven dimensions as it could be understood by compactifying CJS theory on $S^1 \times S^1 \cong T^2$. We present here an argument, in complete analogy with the type IIB theory [21, 24], that the type IIA theory in ten dimensions also possesses an SL(2, R) S-duality invariance which can be understood if we compactify the CJS theory on a T-dual torus $\tilde{T}^2$. We donot yet know how this symmetry can be realized at the level of the type IIA action in ten dimension.

The paper is organized as follows. In section II, we study the “ordinary” Scherk-Schwarz dimensional reduction of the bosonic sector of CJS supergravity theory on $S^1 \times S^1 \cong T^2$ and obtain the low energy effective action of the type IIA string theory in ten and nine space-time dimensions. We point out how to correctly recover the noncompact global O(1, 1) invariance of the NS-NS sector of the nine dimensional string effective action. The complete action including the R-R sector has been shown to be invariant under a global O(2) transformation. The reduction on $S^1$ of the bosonic sector of the type IIB string effective action when the five-form field strength is set to zero is presented in section III. By applying an O(1, 1) transformation we show that type IIB action gets mapped to the type IIA action when the fields in type IIA theory satisfy certain conditions. The nine dimensional type IIB theory is shown not to have a manifest SL(2, R) invariance but is invariant under a global O(2) transformation. We also show how to understand an SL(2, R) S-duality invariance in type IIA theory in ten dimensions by considering the compactification of CJS theory on T-dual torus $\tilde{T}^2$. Finally, we present our conclusions.

*This was first observed in ref.[23, 10] from a different point of view.
II. Dimensional Reduction of CJS Theory on $S^1 \times S^1 \simeq T^2$:

In the first part of this section, we perform the “ordinary” Scherk-Schwarz dimensional reduction [15, 16] of the bosonic sector of the N=1, D+d=11 supergravity theory of Cremmer, Julia and Scherk on $S^1$ and fix our notations and conventions. The original D+d coordinates will be split into D=10 space-time coordinates and d=1 internal coordinate. We denote the eleven dimensional fields and coordinates with an ‘inverted hat’, the ten dimensional objects with a ‘hat’ and objects in nine dimensions will be denoted without ‘hat’. The Greek letters ($\lambda, \mu, \ldots$) in the later part of the alphabet will denote the curved space-time indices whereas ($\alpha, \beta, \ldots$) in the beginning of the alphabet will correspond to the flat tangent space indices. Similarly, the latin letters ($m, n, \ldots$) represent the internal indices and ($a, b, \ldots$) will denote the corresponding tangent space indices. The bosonic part of the CJS supergravity action has the form [5],

$$S^{(11)} = \int d^{11} \tilde{x} \tilde{e} \left[ \tilde{R} - \frac{1}{12} \tilde{F}_{\mu \nu \rho \delta} \tilde{F}^{\mu \nu \rho \delta} + \frac{8}{(12)^4} \tilde{e}^{\mu_1 \ldots \mu_{11}} \tilde{F}_{\mu_1 \ldots \mu_4} \tilde{F}_{\mu_5 \ldots \mu_8} \tilde{C}_{\mu_9 \ldots \mu_{11}} \right]$$

(1)

where $\tilde{e} = \det(\tilde{e}_\mu^\alpha)$, $\tilde{e}_\mu^\alpha$ being the elfbein, $\tilde{C}_{\mu \nu \lambda}$ is an antisymmetric three-form gauge field and $\tilde{F}_{\mu \nu \rho \delta}$ is the corresponding field strength. $\tilde{R}$ is the eleven dimensional scalar curvature.

Our convention for the signature of the tangent space Lorentz metric is $(-, +, +, \ldots)$. The scalar curvature is defined as

$$\tilde{R} = \tilde{e}^{\mu \alpha} \tilde{e}^{\nu \beta} \tilde{R}_{\mu \nu \alpha \beta}$$

(2)

where

$$\tilde{R}_{\mu \nu \alpha \beta} = \partial_\mu \tilde{\omega}_{\nu \alpha \beta} + \tilde{\omega}_{\mu \alpha}^{\; \lambda} \tilde{\omega}_{\nu \gamma \beta} - (\tilde{\mu} \leftrightarrow \tilde{\nu})$$

(3)

Our convention for the spin connection is

$$\tilde{\omega}_{\alpha \beta \gamma} = -\tilde{\Omega}_{\alpha \beta, \gamma} + \tilde{\Omega}_{\beta \gamma, \alpha} - \tilde{\Omega}_{\gamma \alpha, \beta}$$

(4)

with

$$\tilde{\Omega}_{\alpha \beta, \gamma} = \frac{1}{2} \left( \tilde{e}_\alpha^\mu \partial_\beta \tilde{e}_\gamma^\mu - \tilde{e}_\beta^\mu \partial_\alpha \tilde{e}_\gamma^\mu \right)$$

(5)

We note that $\tilde{\Omega}_{\alpha \beta, \gamma}$ is antisymmetric in its first two indices whereas $\tilde{\omega}_{\alpha \beta \gamma}$ is antisymmetric in its last two indices. The field content of the theory (1) is just an elfbein $\tilde{e}_\mu^\alpha$ and a
totally antisymmetric three-form potential \( \tilde{C}_{\mu \nu \rho} \). The scalar curvature term in (1) will be simplified by using the following identity which is valid in any space-time dimensions:

\[
\int d^D x \varepsilon \Lambda(x) R = \int d^D x \varepsilon \Lambda(x) \left[ \omega_{\alpha \beta \gamma} \omega^{\alpha \beta \gamma} + \omega_{\alpha \gamma}^{\alpha} \omega_{\beta}^{\gamma} + 2 e^{\mu \alpha} (\partial_{\mu} \log \Lambda) \omega_{\beta \alpha}^{\beta} \right] \quad (6)
\]

Where \( \Lambda(x) \) is an arbitrary function of \( x \).

In the “ordinary” dimensional reduction the field variables will be taken as independent of the internal coordinates. Using SO(1, 10) Lorentz invariance of the eleven dimensional theory the elfbein is usually taken in the triangular form [5] as given below:

\[
\tilde{e}_{\mu}^{\alpha} = \left( \begin{array}{c} e_\mu^\alpha \\ 0 \\ e_m^a \end{array} \right) \quad \text{and} \quad \tilde{e}_{\alpha}^{\mu} = \left( \begin{array}{c} e_\alpha^\mu \\ 0 \\ -e_a^\alpha A_{\mu}^a \end{array} \right) \quad (7)
\]

With this convention the metric and its inverse then take the following form:

\[
\tilde{g}_{\mu \nu} = \left( \begin{array}{cc} g_{\mu \nu} + A_{\mu}^m A_{\nu}^m & A_{\mu}^m \\ A_{\nu}^m & g_{mn} \end{array} \right) \quad \text{and} \quad \tilde{g}^{\mu \nu} = \left( \begin{array}{cc} g^{\mu \nu} & A^{\mu} A^{\nu} \\ -A^{\mu} & g^{mn} \end{array} \right) \quad (8)
\]

We have not used any accent to denote the fields on the right hand side because we will take these structures of the vielbein and the metric for both ten and nine dimensions. Here \( e_\mu^\alpha \) and \( e_m^a \) are respectively the D(space-time) and d(internal) vielbeins. \( A_{\mu}^m \) are the d vector gauge fields which result from the dimensional reduction. In this convention, the non-vanishing components of the spin-connections are:

\[
\begin{align*}
\omega_{\alpha \beta \gamma} \\
\omega_{\alpha \beta} &= \frac{1}{2} e_\alpha^\mu e_\beta^\nu e_m^a F_{\mu \nu}^m = -\omega_{\alpha \beta} \\
\omega_{\alpha \beta a} &= \frac{1}{2} \left[ e_a^m e_\alpha^\mu \partial_{\mu} e_{bm} - e_b^m e_\alpha^\mu \partial_{\mu} e_{am} \right] \\
\omega_a^{\alpha \beta} &= -\frac{1}{2} e_a^m e_\beta^n e_\alpha^\mu \partial_{\mu} g_{mn}
\end{align*} \quad (9)
\]

where \( F_{\mu \nu}^m = \partial_{\mu} A_{\nu}^m - \partial_{\nu} A_{\mu}^m \). Using (6) with \( \Lambda(x) = 1 \), (7) and (9) it is a straightforward exercise to verify the following relation for the dimensional reduction from 11 \( \rightarrow \) D dimensions (note that this relation is valid in general for the dimensional reduction from any D+d \( \rightarrow \) D dimensions).

\[
\int d^{11} \varepsilon \tilde{\varepsilon} \tilde{R} \rightarrow \int d^D x \sqrt{-g} \Delta \left[ R - \frac{1}{4} g_{mn} F_{\mu \nu}^m F^{\mu \nu} \\
+ \frac{1}{4} g^{\mu \nu} \partial_{\mu} g_{mn} \partial_{\nu} g^{mn} + g^{\mu \nu} \partial_{\mu} \log \Delta \partial_{\nu} \log \Delta \right] \quad (10)
\]
where \( g = \det g_{\mu \nu}, \Delta^2 = \det g_{mn}, g_{\mu \nu} \) and \( R \) being the D dimensional metric and the scalar curvature respectively. In (10) we have also set the integral over the internal coordinates on a compact manifold to unity. Since we want to reduce the theory (1) just by one dimension we choose \( m = n = 10 \) and

\[
\Delta^2 = \det g_{mn} = g_{10,10}
\]

\[
\hat{A}_{\hat{\mu}}^{10} \equiv \hat{A}_{\hat{\mu}}^{(1)}
\]

Using this \( g_{10,10} \) and \( \hat{A}_{\hat{\mu}}^{10} \) in eq.(10) and performing the dimensional reductions of the other terms in (1) we obtain,

\[
S^{(10)} = \int d^{10} \hat{x} \hat{e} \left[ \Delta \hat{R} - \frac{1}{4} \Delta^3 \hat{F}_{\hat{\mu} \hat{\nu}} \hat{F}^{\hat{\mu} \hat{\nu}} - \frac{1}{3} \Delta^{-1} \hat{H}_{\hat{\mu} \hat{\nu} \hat{\rho}} \hat{H}^{\hat{\mu} \hat{\nu} \hat{\rho}} - \frac{1}{12} \Delta \hat{F}_{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \hat{F}^{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} \right]
\]

where we have defined,

\[
\hat{B}_{\hat{\mu} \hat{\nu}}^{(1)} = \hat{C}_{\hat{\mu} \hat{\nu} 10} = \hat{C}_{\hat{\mu} \hat{\nu} 10}
\]

\[
\hat{H}_{\hat{\mu} \hat{\nu} \hat{\rho}} = \partial_{\hat{\mu}} \hat{B}_{\hat{\nu} \hat{\rho}}^{(1)} + \text{cyc. in } \hat{\mu} \hat{\nu} \hat{\rho}
\]

\[
\hat{F}_{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} = \partial_{\hat{\mu}} \hat{C}_{\hat{\nu} \hat{\rho} \hat{\sigma}} - \partial_{\hat{\nu}} \hat{C}_{\hat{\rho} \hat{\sigma} \hat{\mu}} + \partial_{\hat{\rho}} \hat{C}_{\hat{\sigma} \hat{\mu} \hat{\nu}} - \partial_{\hat{\sigma}} \hat{C}_{\hat{\mu} \hat{\nu} \hat{\rho}}
\]

(13)

We notice in (12) that the scalar curvature term and the \((\hat{H}^{(1)})^2\) term have different powers of \( \Delta \). In order to reproduce the NS-NS sector of the type IIA string effective action correctly, they should have the same power of \( \Delta \) which could then be identified with the usual dilaton coupling. We notice that this could be done with the rescaling of the ten-dimensional metric by

\[
\hat{g}_{\hat{\mu} \hat{\nu}} \rightarrow \Delta^{-1} \hat{g}_{\hat{\mu} \hat{\nu}}
\]

With this rescaling we find that both the terms are multiplied by \( \Delta^{-3} \). So, in order to produce the correct dilaton coupling we set

\[
\Delta^{-3} = e^{-2\hat{\phi}}
\]

or, \( \Delta = e^{2\hat{\phi}} \)

(15)
We also note that for a general rescaling of the metric of the form \( g_{\mu\nu} \to e^{\alpha \phi(x)} g_{\mu\nu} \), in \( D \) dimensions, the scalar curvature changes as

\[
R \to e^{-\alpha \phi} \left[ R - \alpha (D - 1) g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{\alpha^2}{4} (D - 1) (D - 2) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right]
\]  

(16)

So, under this rescaling \( \sqrt{-g} R \) changes as

\[
\sqrt{-g} R \to \sqrt{-g} e^{\frac{1}{2}(D-2)\alpha \phi} \left[ R - \alpha (D - 1) g^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{\alpha^2}{4} (D - 1) (D - 2) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right]
\]

Using (16), (15) and (14) in (12), we obtain the type IIA string effective action in \( D=10 \) in the form,

\[
S^{(10)}_{\text{IIA}} = \int d^{10} \hat{x} \sqrt{-\hat{g}} \left[ e^{-2\hat{\phi}} \left( \hat{R} + 4 \partial_{\hat{\mu}} \hat{\phi} \partial^{\hat{\mu}} \hat{\phi} - \frac{1}{3} \hat{H}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{H}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}} \right) - \frac{1}{4} \hat{F}^{(1)}_{\hat{\mu}\hat{\nu}} \hat{F}^{(1)}_{\hat{\mu}\hat{\nu}} - \frac{1}{12} \hat{F}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{F}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\rho}} + \frac{8}{(12)^4} \sqrt{-g} \left( 3 \hat{F}_{\hat{\mu}_1...\hat{\mu}_4} \hat{F}_{\hat{\mu}_5...\hat{\mu}_8} \hat{F}^{(1)}_{\hat{\mu}_9\hat{\mu}_{10}} - 8 \hat{F}_{\hat{\mu}_1...\hat{\mu}_4} \hat{H}^{(1)}_{\hat{\mu}_5\hat{\mu}_6\hat{\mu}_7} \hat{C}_{\hat{\mu}_8\hat{\mu}_9\hat{\mu}_{10}} \right) \right]
\]

(17)

We here note that the first three terms in (17) with the usual dilaton coupling represent the NS-NS sector and is common to all string theories. The fourth and the fifth terms belong to the R-R sector whereas the last term has a mixing between the NS-NS and R-R sectors. The last three terms, however, do not couple to the dilaton and encode the non-perturbative information inherited from CJS theory in eleven dimensions. Also, we note from (15) that as the expectation value of \( \Delta \) represents the radius of compactification, \( R_1 \), of the circle \( S^1 \), it is related to the string coupling constant as \( R_1 \sim e^{\frac{1}{2} \hat{\phi}} \sim \lambda \). Because of the Weyl scaling (14), the radius as measured by the string metric would be \( R_1^s \sim e^{\frac{1}{2}\hat{\phi}} e^{\frac{1}{2}\hat{\phi}} \sim \lambda \). So, as \( R_1^s \to \infty \), \( \lambda \to \infty \) and therefore, CJS theory represents the non-perturbative limit of type IIA theory [1].

In the second part of this section, we consider the dimensional reduction of the type IIA theory, eq.(17), on a second \( S^1 \). In this case we follow exactly the same procedure as before and take the zehnbein and the ten dimensional metric in the same form as given in (7) and (8). We take the ninth component of the metric as \( g_{99} = \chi \) and the vector gauge field which originates in the dimensional reduction of the ten dimensional metric \( \hat{g}_{\hat{\mu}\hat{\nu}} \) is denoted as \( A^{(2)}_{\mu} \). In order to reduce the scalar curvature and the dilaton term in (17) we note the following relation for the dimensional reduction from 10 → D dimensions (this
relation is also valid for the reduction from any D+d → D,
\[
\int d^{10} x \sqrt{-\hat{g} e^{-2\hat{\phi}}} \left( \hat{R} + 4 \partial_{\hat{\mu}} \hat{\phi} \partial^{\hat{\mu}} \hat{\phi} \right)
\]
\[
\rightarrow \int d^{D} x \sqrt{-g e^{-2\phi_D}} \left( R + 4 \partial_{\mu} \phi_D \partial^{\mu} \phi_D - \frac{1}{4} g_{\mu\nu} F_{\mu\nu}^m F^{m\mu\nu} + \frac{1}{4} \partial_{\mu} g_{\mu\nu} \partial^{\mu} g^{\mu\nu} \right)
\]
(18)

where D is any dimension lower than ten. Note that we have here used the identity (6) with \( \Lambda = e^{-2\phi} \). The dilaton fields \( \hat{\phi} \) and \( \phi_D \) are related as,
\[
\hat{\phi} = \phi_D + \frac{1}{4} \log (\det g_{mn})
\]
(19)

For \( m = n = 9 \) and \( A^9_{\mu} \equiv A_{\mu}^{(2)} \), we obtain from (18) the reduced form of the scalar curvature and the dilaton term as (We will denote the nine dimensional dilaton as \( \phi \)),
\[
\int d^{9} x \sqrt{-g e^{-2\phi}} \left( R + 4 \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{4} \partial_{\mu} \log \chi \partial^{\mu} \log \chi - \frac{1}{4} \chi F_{\mu\nu}^{(2)} F^{(2)\mu\nu} \right)
\]
(20)

We then write below the reduced form of the other terms in (17) separately,

3rd term:
\[
\int d^{10} x \sqrt{-\hat{g}} \left( -\frac{1}{3} \right) \hat{H}_{\mu\nu\rho}^{(1)} \hat{H}^{(1)\mu\nu\rho}
\]
\[
\rightarrow \int d^{9} x \sqrt{-g} \left[ \frac{1}{4} \chi \frac{1}{2} \left( F_{\mu\nu}^{(1)} + a F_{\mu\nu}^{(2)} \right) \left( F^{(1)\mu\nu} + a F^{(2)\mu\nu} \right) - \frac{1}{2} \chi^{-\frac{1}{2}} \partial_{\mu} a \partial^{\mu} a \right]
\]
(22)

4th term:
\[
\int d^{10} x \sqrt{-\hat{g}} \left( -\frac{1}{12} \right) \hat{F}_{\mu\nu\rho\sigma} \hat{F}^{\mu\nu\rho\sigma}
\]
\[
\rightarrow \int d^{9} x \sqrt{-g} \left[ -\frac{1}{12} \chi \frac{1}{2} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} - \frac{1}{3} \chi^{-\frac{1}{2}} \left( H_{\mu\nu\rho}^{(2)} - a H_{\mu\nu\rho}^{(1)} \right) \left( H^{(2)\mu\nu\rho} - a H^{(1)\mu\nu\rho} \right) \right]
\]
(23)

5th term:
\[
\int d^{10} x \sqrt{-\hat{g}} \left( -\frac{1}{12} \right) \hat{F}_{\mu\nu\rho\sigma}\hat{F}^{\mu\nu\rho\sigma}
\]
\[
\rightarrow \int d^{9} x \sqrt{-g} \left[ -\frac{1}{12} \chi \frac{1}{2} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} - \frac{1}{3} \chi^{-\frac{1}{2}} \left( H_{\mu\nu\rho}^{(2)} - a H_{\mu\nu\rho}^{(1)} \right) \left( H^{(2)\mu\nu\rho} - a H^{(1)\mu\nu\rho} \right) \right]
\]
(23)

last term:
\[
\int d^{10} x \left[ \frac{8}{(12)^4} \hat{F}_{\mu_1...\mu_{10}} \left( 3 \hat{F}_{\mu_1...\mu_8} \hat{F}_{\mu_5...\mu_8} B_{\mu_9}^{(1)} - 8 \hat{F}_{\mu_1...\mu_4} \hat{H}_{\mu_5...\mu_7}^{(1)} \hat{C}_{\mu_8\mu_9} \right) \right]
\]
\[
\rightarrow \int d^{9} x \left[ \frac{3}{(12)^4} \left[ F_{\mu_1...\mu_9} F_{\mu_5...\mu_8} A_{\mu_9}^{(3)} - 4 F_{\mu_1...\mu_4} \epsilon^{ij} H_{\mu_5...\mu_7}^{(i)} B_{\mu_9}^{(j)} \right.ight.
\]
\[
+ 4 F_{\mu_1...\mu_4} H_{\mu_5...\mu_8} A_{\mu_9}^{(3)} + 2 \epsilon^{ij} H_{\mu_1...\mu_4} H_{\mu_5...\mu_7}^{(i)} C_{\mu_8\mu_9}
\]
\[
+ 4 F_{\mu_1...\mu_4} F_{\mu_5...\mu_8} C_{\mu_9} + 6 \epsilon^{ij} H_{\mu_1...\mu_4} H_{\mu_5...\mu_7}^{(i)} A_{\mu_9}^{(3)} B_{\mu_8\mu_9}^{(k)}
\]
\[
+ 12 F_{\mu_1...\mu_4} F_{\mu_5...\mu_8} A_{\mu_9}^{(3)} B_{\mu_8\mu_9}^{(k)}] \right]
\]
(24)
We have expressed the reduced form of this term in terms of the indexed fields for later convenience, where \(i, j, k = 1, 2\) and \(\epsilon^{12} = -\epsilon^{21} = 1\). Our definitions and the dimensionally reduced form of the gauge fields as well as their field strengths are listed below:

\[
\hat{g}_{\mu\nu} \rightarrow \begin{cases} 
\hat{g}_{99} = g_{99} = \chi \\
\hat{g}_{\mu 9} = g_{\mu 9} = \chi A^{(2)}_{\mu} \\
\hat{g}_{\mu\nu} = g_{\mu\nu} + \chi A^{(2)}_{\mu} A^{(2)}_{\nu} 
\end{cases}
\] (25)

\[
\hat{\phi} = \phi + \frac{1}{4} \log \chi
\] (26)

\[
\hat{A}^{(1)}_{\mu} \rightarrow \begin{cases} 
A_{9} = \hat{A}_{9} = a \\
A^{(1)}_{\mu} = A^{(1)}_{\mu} - a A^{(2)}_{\mu} 
\end{cases}
\] (27)

\[
\hat{B}^{(1)}_{\mu\nu} \rightarrow \begin{cases} 
B^{(1)}_{\mu 9} = \hat{B}^{(1)}_{\mu 9} = A^{(3)}_{\mu} \\
B^{(1)}_{\mu\nu} = \hat{B}^{(1)}_{\mu\nu} + A^{(2)}_{\mu} A^{(3)}_{\nu} - A^{(2)}_{\nu} A^{(3)}_{\mu} 
\end{cases}
\] (28)

\[
\hat{C}^{(1)}_{\mu\nu\lambda} \rightarrow \begin{cases} 
C^{(1)}_{\mu 9} = \hat{C}^{(1)}_{\mu 9} = B^{(2)}_{\mu\nu} - a B^{(1)}_{\mu\nu} - \left( A^{(1)}_{\mu} A^{(3)}_{\nu} - A^{(1)}_{\nu} A^{(3)}_{\mu} \right) \\
C^{(1)}_{\mu\nu\lambda} = \hat{C}^{(1)}_{\mu\nu\lambda} - \left( A^{(2)}_{\mu} \hat{C}^{(1)}_{\nu\lambda 9} + \text{cyc. in } \mu\nu\lambda \right) 
\end{cases}
\] (29)

The field strength associated with various gauge fields are given below:

\[
F^{(2)}_{\mu\nu} = \partial_{\mu} A^{(2)}_{\nu} - \partial_{\nu} A^{(2)}_{\mu}
\] (30)

\[
F^{(3)}_{\mu\nu} = \partial_{\mu} A^{(3)}_{\nu} - \partial_{\nu} A^{(3)}_{\mu}
\] (31)

\[
F_{\mu 9} = \partial_{\mu} a
\] (32)

\[
F_{\mu\nu} = F^{(1)}_{\mu\nu} + a F^{(2)}_{\mu\nu}
\] (33)

\[
H^{(1)}_{\mu 9} = \hat{H}^{(1)}_{\mu 9} = F^{(3)}_{\mu\nu}
\] (34)

\[
H^{(1)}_{\mu\nu\lambda} = \partial_{\mu} B^{(1)}_{\nu\lambda} - F^{(2)}_{\mu\nu} A^{(3)}_{\lambda} + \text{cyc. in } \mu\nu\lambda \\
= \partial_{\mu} B^{(1)}_{\nu\lambda} - \frac{1}{2} \left( F^{(2)}_{\mu\nu} A^{(3)}_{\lambda} + F^{(2)}_{\mu\nu} A^{(3)}_{\lambda} \right) + \text{cyc. in } \mu\nu\lambda
\] (35)

where we have defined,

\[
B^{(1)}_{\mu\nu} = B^{(1)}_{\mu\nu} - \frac{1}{2} \left( A^{(2)}_{\mu} A^{(3)}_{\nu} - A^{(2)}_{\nu} A^{(3)}_{\mu} \right)
\] (36)

Continuing with the other field strengths,

\[
F^{(2)}_{\mu\nu\lambda 9} = \hat{F}^{(2)}_{\mu\nu\lambda 9} = H^{(2)}_{\mu\nu\lambda} - a H^{(1)}_{\mu\nu\lambda}
\] (37)

where \(H^{(2)}_{\mu\nu\lambda}\) is defined as

\[
H^{(2)}_{\mu\nu\lambda} = \partial_{\mu} B^{(2)}_{\nu\lambda} + F^{(3)}_{\mu\nu} A^{(1)}_{\lambda} + \text{cyc. in } \mu\nu\lambda \\
= \partial_{\mu} B^{(2)}_{\nu\lambda} + \frac{1}{2} \left( F^{(1)}_{\mu\nu} A^{(3)}_{\lambda} + F^{(3)}_{\mu\nu} A^{(1)}_{\lambda} \right) + \text{cyc. in } \mu\nu\lambda
\] (38)
where,

\[ \bar{B}_{\mu\nu}^{(2)} \equiv B_{\mu\nu}^{(2)} - \frac{1}{2} \left( A_{\mu}^{(1)} A_{\nu}^{(3)} - A_{\nu}^{(1)} A_{\mu}^{(3)} \right) \]  

and finally,

\[ F_{\mu\nu\rho} = \partial_\mu \bar{C}_{\nu\lambda\rho} - \partial_\nu \bar{C}_{\mu\lambda\rho} + \partial_\lambda \bar{C}_{\mu\nu\rho} - \partial_\rho \bar{C}_{\mu\nu\lambda} + \left[ F_{\mu\nu}^{(i)} \bar{B}_\rho^{(i)} + F_{\nu\lambda}^{(i)} \bar{B}_\rho^{(i)} - \frac{1}{2} \epsilon^{ij} F_{\mu\nu}^{(3)} A_{\lambda}^{(i)} A_{\rho}^{(j)} - \frac{1}{2} \epsilon^{ij} F_{\nu\lambda}^{(3)} A_{\mu}^{(i)} A_{\rho}^{(j)} + \text{cyc. in } \nu\lambda\rho \right] \]  

where,

\[ \bar{C}_{\mu\nu\lambda} = C_{\mu\nu\lambda} + \left[ \frac{1}{2} \epsilon^{ij} A_{\mu}^{(i)} A_{\nu}^{(j)} A_{\lambda}^{(3)} + \text{cyc. in } \mu\nu\lambda \right] \]

Note from (25) that the radius of the circle of compactification is given by the expectation value of the field \( \chi^{\frac{1}{2}} \) as measured in string metric and will be used later. Also, we have introduced both \( B_{\mu\nu} \) and \( \bar{B}_{\mu\nu} \), because we will see that it is \( B_{\mu\nu} \) which will remain invariant under the global O(1, 1) transformation and not \( \bar{B}_{\mu\nu} \). Under the global O(2) transformations also, \( \bar{B}_{\mu\nu} \)'s transform in a nice way as opposed to \( B_{\mu\nu} \)'s. The particular forms of the field strengths are chosen for later convenience.

The complete reduced form of the type IIA string effective action in nine dimensions can be written using (20)–(24) as follows:

\[ S_{\text{IIA}}^{(9)} = \int d^9 x \sqrt{-g} \left[ e^{-2\phi} \left( R + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \partial_\mu \log \chi \partial^\mu \log \chi - \frac{1}{4} \chi F_{\mu\nu}^{(2)} F^{(2)\mu\nu} 
- \frac{1}{4} \chi^{-1} F_{\mu\nu}^{(3)} F^{(3)\mu\nu} - \frac{1}{12} H_{\mu\nu\lambda}^{(1)} H^{(1)\mu\nu\lambda} \right) - \frac{1}{2} \chi^{-\frac{1}{2}} \partial_\mu a \partial^\mu a 
- \frac{1}{12} \chi^{-\frac{1}{2}} \left( H_{\mu\nu\lambda}^{(2)} - a H_{\mu\nu\lambda}^{(1)} \right) \left( H_{\mu\nu\lambda}^{(2)} - a H_{\mu\nu\lambda}^{(1)} \right) 
- \frac{1}{4} \chi^{\frac{1}{2}} \left( F_{\mu\nu}^{(1)} + a F_{\mu\nu}^{(2)} \right) \left( F_{\mu\nu}^{(1)} + a F_{\mu\nu}^{(2)} \right) - \frac{1}{12} \chi^{\frac{1}{2}} F_{\mu\nu\lambda} F^{\mu\nu\lambda} \rho 
+ \frac{1}{\sqrt{-g}} \frac{1}{2(12)^3} \left( F_{\mu_1...\mu_9}^{(3)} A_{\mu_1...\mu_9}^{(3)} - 4 F_{\mu_1...\mu_4}^{(3)} A_{\mu_5...\mu_9}^{(3)} + 4 F_{\mu_1...\mu_4}^{(3)} A_{\mu_5...\mu_9}^{(3)} - 4 F_{\mu_1...\mu_4}^{(3)} A_{\mu_5...\mu_9}^{(3)} + 4 F_{\mu_1...\mu_4}^{(3)} A_{\mu_5...\mu_9}^{(3)} 
+ 4 F_{\mu_1...\mu_4}^{(3)} A_{\mu_5...\mu_9}^{(3)} + 6 \epsilon^{ij} H_{\mu_1\mu_2\mu_3}^{(i)} H_{\mu_4\mu_5\mu_6}^{(j)} \bar{C}_{\mu_7\mu_8\mu_9} 
+ 12 F_{\mu_1...\mu_4}^{(3)} A_{\mu_5...\mu_9}^{(3)} \right) \right] \]  

In writing down (42) we have rescaled \( A_{\mu}^{(3)} \rightarrow \frac{1}{2} A_{\mu}^{(3)} \), \( B_{\mu}^{(1)} \rightarrow \frac{1}{2} B_{\mu}^{(1)} \), \( B_{\mu}^{(2)} \rightarrow \frac{1}{2} B_{\mu}^{(2)} \) and \( C_{\mu\nu\lambda} \rightarrow \frac{1}{2} C_{\mu\nu\lambda} \) to recast the action in the standard form. This action is also obtained in ref.[7] with different definitions and conventions of the reduced fields. But, note that we do not agree with the reduction of the topological term eq.(47) in ref.[7]. The first
six terms in (42) represent the NS-NS sector which couples to nine dimensional dilaton $\phi$, the next four terms which do not couple to the dilaton encode the non-perturbative information and represent the R-R sector whereas the last term gives a mixing between the NS-NS and the R-R sectors. We first concentrate on the NS-NS sector of the action (42) and show that it is O(1, 1) invariant. We here follow closely the notation adopted in the paper of Maharana and Schwarz [15]. Since the nine dimensional dilaton and the metric remain invariant under O(1, 1) transformation the first two terms are invariant. The third, fourth and fifth terms can be written as

$$-\frac{1}{4} \partial_\mu \log \chi \partial^\mu \log \chi - \frac{1}{4} \chi F^{(2)}_{\mu\nu} F^{(2)\mu\nu} - \frac{1}{4} \chi^{-1} F^{(3)}_{\mu\nu} F^{(3)\mu\nu}$$

$$= \frac{1}{8} \text{tr} \partial_\mu M \partial^\mu M^{-1} - \frac{1}{4} F^T_{\mu\nu} M^{-1} F^{\mu\nu}$$

(43)

where

$$M = \left( \begin{array}{cc} \chi^{-1} & 0 \\ 0 & \chi \end{array} \right) \quad \text{and} \quad F_{\mu\nu} = \left( \begin{array}{c} F^{(2)}_{\mu\nu} \\ F^{(3)}_{\mu\nu} \end{array} \right)$$

(44)

Here $M$ is an O(1, 1) matrix, since $M^T \eta M = \eta$ with $\eta = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$. So, (43) is invariant under a global O(1, 1) transformation $M \rightarrow \Omega M \Omega^T$, where $\Omega^T \eta \Omega = \eta$, if $A_\mu = \left( \begin{array}{c} A^{(2)}_\mu \\ A^{(3)}_\mu \end{array} \right)$ transforms as $A_\mu \rightarrow \Omega A_\mu$. The form of the O(1, 1) transformation matrix $\Omega$ is:

$$\Omega = \left( \begin{array}{cc} 0 & \lambda \\ \lambda^{-1} & 0 \end{array} \right)$$

(45)

where $\lambda$ is a constant parameter. So, we find the O(1, 1) transformation of the moduli $\chi$ and the vector gauge fields as

$$\tilde{\chi} = \lambda^{-2} \chi^{-1}$$

$$\tilde{A}^{(2)}_\mu = \lambda A^{(3)}_\mu$$

$$\tilde{A}^{(3)}_\mu = \lambda^{-1} A^{(2)}_\mu$$

(46)

We note that (43) is indeed invariant under the transformations in (46). Next we find that $(H^{(1)})^2$ term in (42) is also invariant under O(1, 1) transformation since from eq.(35) we observe that $H_{\mu\nu\lambda}^{(1)}$ can be expressed as

$$H_{\mu\nu\lambda}^{(1)} = \partial_\mu \tilde{B}^{(1)}_{\nu\lambda} - \frac{1}{2} A^T_{\mu} \eta F_{\nu\lambda} + \text{cyc. in } \mu \nu \lambda$$

(47)

and this is invariant if we require that $\tilde{B}^{(1)}_{\mu\nu} = B^{(1)}_{\mu\nu} - \frac{1}{2} \left( A^{(2)}_\mu A^{(3)}_\nu - A^{(2)}_\nu A^{(3)}_\mu \right)$ does not transform. The second term in (47) is invariant because $\Omega^T \eta \Omega = \eta$. This shows that it
is $B_{\mu\nu}^{(1)}$ and not $B_{\mu\nu}^{(1)}$ which should remain invariant under $O(1, 1)$ (as noted incorrectly in ref.[17]) for the $O(1, 1)$ invariance of the NS-NS sector. In fact, $B_{\mu\nu}^{(1)}$ in the present case transforms under (46) as,

$$\tilde{B}_{\mu\nu}^{(1)} = B_{\mu\nu}^{(1)} - \left( A_{\mu}^{(2)} A_{\nu}^{(3)} - A_{\nu}^{(2)} A_{\mu}^{(3)} \right)$$

When both $A_{\mu}^{(2)}$ and $A_{\mu}^{(3)}$ are chosen to be zero, it is only then $B_{\mu\nu}^{(1)}$ would remain invariant. We will see in section III that the transformation (46) corresponds to Buscher’s duality transformation for a special value of $\lambda$. We also note that our result here is true for general ‘d’ dimensional reduction and in order to obtain the $O(d, d)$ invariance in that case, $B_{\mu\nu}^{(1)}$ should transform as,

$$\tilde{B}_{\mu\nu}^{(1)} = B_{\mu\nu}^{(1)} - A_{\mu}^{T} \epsilon A_{\nu}$$

where $\epsilon = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $A_{\mu} = \begin{pmatrix} A_{\mu}^{(2)} \\ A_{\mu}^{(3)} \end{pmatrix}$ with $I$ representing the $d$-dimensional identity matrix and $m = 1, 2, \ldots, d$. It is clear that the full action (42) does not have the global non-compact $O(1, 1)$ symmetry under (46). We will, however, come back to this question later in section III.

Finally, we point out as also noted in ref.[7], that the whole action (42) has an obvious global $O(2)$ invariance since it is obtained from an eleven dimensional theory. In the process of dimensional reduction the original Lorentz group $SO(1, 10)$ gets split up into $SO(1, 8) \times O(2)$. So, the resulting nine dimensional theory inherits a global $O(2)$ invariance. Indeed one can check that the action (42) is invariant under the following $O(2)$ transformations:

$$
\begin{align*}
\delta a & = -\theta \left( 1 + a^2 - \chi \frac{1}{2} e^{-2\phi} \right) \\
\delta e^\phi & = \frac{1}{2} \theta a e^\phi \\
\delta \chi & = -\theta \chi a \\
\delta A_{\mu}^{(3)} & = 0 \\
\delta A_{\mu}^{(i)} & = \theta \epsilon^{ij} A_{\mu}^{(j)} \\
\delta g_{\mu\nu} & = \theta g_{\mu\nu} a \\
\delta \tilde{B}_{\mu}^{(i)} & = \theta \epsilon^{ij} \tilde{B}_{\mu}^{(j)} \\
\delta H_{\mu\nu\lambda}^{(i)} & = \theta \epsilon^{ij} H_{\mu\nu\lambda}^{(j)} \\
\delta C_{\mu\nu\lambda} & = \delta \tilde{C}_{\mu\nu\lambda} = 0 \\
\delta F_{\mu\nu\lambda\rho} & = 0
\end{align*}
$$

where $\theta$ is a constant infinitesimal parameter. In the above the indices $i, j = 1, 2$ with $\epsilon^{12} = -\epsilon^{21} = 1$. Also, we note that although the string frame metric transforms under the $O(2)$ transformation, the Einstein frame metric in nine dimensions $\tilde{g}_{\mu\nu} = e^{-\frac{1}{4} \phi} g_{\mu\nu}$ does not. We have written down our definitions of various fields with explicit $O(2)$ indices ‘$(1)$’ and ‘$(2)$’. It is just a straightforward exercise to verify the invariance of the action (42) under the transformations (50). Note that the topological term and the $F^2$ term are individually invariant under the $O(2)$ transformation. It should be pointed out that under
the O(2) transformation it is $B_{\mu\nu}^{(i)}$ which has a nice transformation property and not $B_{\mu\nu}^{(i)}$. Also it is much easier to verify the invariance of the action (42) in the Einstein frame since the Einstein metric remains inert under the O(2) transformation. Many interesting consequences with a finite version of this O(2) symmetry has been pointed out in ref.[7].

III. Reduction of Type IIB Theory on $S^1$ and Duality Symmetries:

It is well-known that the equations of motion of the $N = 2$, $D = 10$ chiral supergravity theory can not be obtained from a covariant action in ten dimensions. The bosonic sector of this theory contains a four-form gauge potential whose field-strength is self-dual. If one sets this five-form field strength to zero, then it is known that the equations of motion can be derived from a covariant action. We, therefore, consider this action which is the low energy effective action of the type IIB string theory. The action has the following form:

$$S_{\text{IIB}}^{(10)} = \int d^{10}\tilde{x} \sqrt{-\tilde{g}_B} \left[ e^{-2\hat{\phi}} \left( \hat{R}_B + 4\partial_\mu \hat{\phi}_B \partial^\mu \hat{\phi}_B - \frac{1}{12} \hat{h}_{\mu\rho\lambda}^{(1)} \hat{h}_{(1)}^{(1)\mu\rho\lambda} \right) - \frac{1}{2} \partial_\mu \hat{\psi} \partial^\mu \hat{\psi} - \frac{1}{12} \left( e^{-2\hat{\phi}} \hat{h}_{\mu\rho\lambda}^{(1)} \right) \left( \hat{h}_{(2)}^{(2)\mu\rho\lambda} + \hat{\psi} \hat{h}_{\mu\rho\lambda}^{(1)} \right) \right]$$ (51)

We have denoted the metric, the scalar curvature and the dilaton with a subscript ‘$B$’ and the field strength with small letters to distinguish them from the corresponding fields in the type IIA theory. The type IIB theory (as also common to all other string theories) is known to contain a metric, a dilaton and a two-form gauge field $\hat{b}_{\mu\nu}^{(1)}$ (with field strength $\hat{h}_{(1)}^{(1)\mu\rho\lambda}$) in the NS-NS sector which is represented by the first three terms in (51) with the usual dilaton coupling. The R-R sector consists of a scalar $\hat{\psi}$, a two-form gauge field $\hat{b}_{\mu\nu}^{(2)}$ (with field strength $\hat{h}_{(2)}^{(2)\mu\rho\lambda}$) and a four-form gauge field whose field strength has been set to zero. The R-R sector does not couple to the dilaton as is clear from the last two terms in (51) and contains the non-perturbative information of type IIB string theory. It is known that the action (51) possesses a global SL(2, R) invariance [20, 21]. This symmetry can be better understood in the Einstein frame since the Einstein metric remains inert under the SL(2, R) transformation. So, in order to show this symmetry we first express the action in the Einstein metric $\hat{g}_{B,\bar{\mu}\bar{\nu}} = e^{-\frac{1}{2}\hat{\phi}} \hat{g}_{B,\bar{\mu}\bar{\nu}}$ as follows:

$$\hat{S}_{\text{IIB}}^{(10)} = \int d^{10}\tilde{x} \sqrt{-\hat{g}_B} \left[ \tilde{R}_B - \frac{1}{2} \partial_\mu \hat{\phi}_B \partial^\mu \hat{\phi}_B - \frac{1}{2} e^{2\hat{\phi}_B} \partial_\mu \hat{\psi} \partial^\mu \hat{\psi} - \frac{1}{12} \left( e^{-\hat{\phi}_B} \hat{h}_{\mu\rho\lambda}^{(1)} \hat{h}_{(1)}^{(1)\mu\rho\lambda} + e^{\hat{\phi}_B} \left( \hat{h}_{(2)}^{(2)\mu\rho\lambda} + \hat{\psi} \hat{h}_{\mu\rho\lambda}^{(1)} \right) \left( \hat{h}_{(2)}^{(2)\mu\rho\lambda} + \hat{\psi} \hat{h}_{\mu\rho\lambda}^{(1)} \right) \right) \right]$$ (52)
This action can now be expressed in a manifestly SL(2, R) invariant form as given below,

$$S^{(10)}_{\text{IIB}} = \int d^{10}x \sqrt{-\hat{g}_{B}} \left[ \hat{R}_{B} + \frac{1}{4} \text{tr} \partial_\mu S^{-1} \partial^\mu S - \frac{1}{12} \hat{\mathcal{H}}^T_{\mu \rho \lambda} \hat{S} \hat{\mathcal{H}}_{\mu \rho \lambda} \right]$$

where

$$S = \begin{pmatrix} \hat{\psi}^2 e^{\hat{\phi}_{\hat{B}}} + e^{-\hat{\phi}_{\hat{B}}} \hat{\psi} e^{\hat{\phi}_{\hat{B}}} \\ \hat{\psi} e^{\hat{\phi}_{\hat{B}}} \\ e^{\hat{\phi}_{\hat{B}}} \end{pmatrix}$$

represents an SL(2, R) matrix and \( \hat{\mathcal{H}} = \begin{pmatrix} \hat{\mu}^{(1)} \\ \hat{\mu}^{(2)} \end{pmatrix} \). Under a global SL(2, R) transformation \( S \to \Lambda S \Lambda^T \) and \( \left( \begin{pmatrix} \hat{\mu}^{(1)} \\ \hat{\mu}^{(2)} \end{pmatrix} \right) \to B_{\tilde{\mu} \tilde{\nu}} \to (\Lambda^{-1})^T B_{\mu \nu}, \) the action (53) is easily seen to be invariant. With these transformations the complex scalar field \( \hat{\rho} = (\hat{\psi} + ie^{-\hat{\phi}_{\hat{B}}} \) undergone a fractional linear transformation whereas the two two-form potentials transform linearly. In particular, choosing \( \Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( \hat{\psi} = 0, \) the string coupling constant transforms to its inverse showing a strong-weak coupling duality in the theory. We will point out later that this symmetry has its origin in the eleven dimensional CJS theory compactifying on a torus \( T^2. \)

Next, we dimensionally reduce the action (51) on \( S^1. \) Since we have studied the dimensional reduction of type IIA theory in detail in section II, we here give the results. The reduced form of the action is given below:

$$S^{(9)}_{\text{IIB}} = \int d^9x \sqrt{-\hat{g}_{B}} \left[ e^{-2\hat{\phi}_{\hat{B}}} \left( R_{B} + 4\partial_\mu \phi_{\hat{B}} \partial^\mu \phi_{\hat{B}} - \frac{1}{4} \partial_\mu \log \chi_{B} \partial^\mu \log \chi_{B} \right. \right.$$  
\[ 
-\frac{1}{4} \chi_{B} f^{(3)}_{\mu \nu \rho} f^{(3)}_{\mu \nu \rho} - \frac{1}{4} \chi_{B}^{-1} f^{(1)}_{\mu \nu} f^{(1)}_{\mu \nu} - \frac{1}{12} h_{\mu \nu \lambda}^{(1)} h_{\mu \nu \lambda}^{(1)} \bigg) - \frac{1}{2} \chi_{B}^{\frac{1}{2}} \partial_\mu \psi \partial^\mu \psi \bigg] 
\left] 
- \frac{1}{4} \chi_{B}^{\frac{1}{2}} \left( h^{(2)}_{\mu \nu \lambda} + \psi h^{(1)}_{\mu \nu \lambda} \right) \left( h^{(2)}_{\mu \nu \lambda} + \psi h^{(1)}_{\mu \nu \lambda} \right) \right.$$

$$\frac{1}{4} \chi_{B}^{\frac{1}{2}} \left( f^{(2)}_{\mu \nu} + \psi f^{(1)}_{\mu \nu} \right) \left( f^{(2)}_{\mu \nu} + \psi f^{(1)}_{\mu \nu} \right) \bigg]$$

where our definitions of fields and the corresponding field strengths are:

$$\hat{g}_{B \tilde{\mu} \tilde{\nu}} \longrightarrow \left\{ \begin{array}{l} \hat{g}_{B,99} = g_{B,99} = \chi_{B} \\
\hat{g}_{B,\mu 9} = g_{B,\mu 9} = \chi_{B} a^{(3)}_{\mu 9} \\
\hat{g}_{B,\mu \nu} = g_{B,\mu \nu} + \chi_{B} a^{(3)}_{\mu \nu} \end{array} \right. \quad (56)$$

$$\hat{\phi}_{B} = \phi_{B} + \frac{1}{4} \log \chi_{B} \quad (57)$$

$$\hat{b}^{(1)}_{\mu \nu} \longrightarrow \left\{ \begin{array}{l} \hat{b}^{(1)}_{\mu 9} = \hat{b}^{(1)}_{\mu 9} = a^{(1)}_{\mu 9} \\
\hat{b}^{(1)}_{\mu \nu} = \hat{b}^{(1)}_{\mu \nu} + a^{(3)}_{\mu \nu} a^{(1)}_{\mu \nu} - a^{(3)}_{\mu \nu} a^{(1)}_{\mu \nu} \end{array} \right. \quad (58)$$

$$\hat{b}^{(2)}_{\mu \nu} \longrightarrow \left\{ \begin{array}{l} \hat{b}^{(2)}_{\mu 9} = \hat{b}^{(2)}_{\mu 9} = a^{(2)}_{\mu 9} \\
\hat{b}^{(2)}_{\mu \nu} = \hat{b}^{(2)}_{\mu \nu} + a^{(3)}_{\mu \nu} a^{(2)}_{\mu \nu} - a^{(3)}_{\mu \nu} a^{(2)}_{\mu \nu} \end{array} \right. \quad (59)$$
\[ \hat{\psi} = \psi \]  

The corresponding field strengths are:

\[ f^{(3)}_{\mu \nu} = \partial_\mu a^{(3)}_\nu - \partial_\nu a^{(3)}_\mu \]  

\[ h^{(1)}_{\mu \nu \theta} = \hat{h}^{(1)}_{\mu \nu \theta} = f^{(1)}_{\mu \nu} = \partial_\mu a^{(1)}_\nu - \partial_\nu a^{(1)}_\mu \]  

\[ h^{(1)}_{\mu \nu \lambda} = \partial_\mu b^{(1)}_{\nu \lambda} - f^{(3)}_{\mu \nu} a^{(1)}_\lambda + \text{cyc. in } \mu \nu \lambda \]  

\[ \equiv \partial_\mu \bar{b}^{(1)}_{\nu \lambda} - \frac{1}{2} \left( f^{(3)}_{\mu \nu} a^{(1)}_\lambda + f^{(1)}_{\mu \nu} a^{(3)}_\lambda \right) + \text{cyc. in } \mu \nu \lambda \]  

where we have defined

\[ \bar{b}^{(1)}_{\mu \nu} \equiv b^{(1)}_{\mu \nu} - \frac{1}{2} \left( a^{(3)}_\mu a^{(1)}_\nu - a^{(3)}_\nu a^{(1)}_\mu \right) \]  

and finally,

\[ h^{(2)}_{\mu \nu \theta} = \hat{h}^{(2)}_{\mu \nu \theta} = f^{(2)}_{\mu \nu} = \partial_\mu a^{(2)}_\nu - \partial_\nu a^{(2)}_\mu \]  

\[ h^{(2)}_{\mu \nu \lambda} = \partial_\mu b^{(2)}_{\nu \lambda} - f^{(3)}_{\mu \nu} a^{(2)}_\lambda + \text{cyc. in } \mu \nu \lambda \]  

\[ \equiv \partial_\mu \bar{b}^{(2)}_{\nu \lambda} - \frac{1}{2} \left( f^{(3)}_{\mu \nu} a^{(2)}_\lambda + f^{(2)}_{\mu \nu} a^{(3)}_\lambda \right) + \text{cyc. in } \mu \nu \lambda \]  

with

\[ \bar{b}^{(2)}_{\mu \nu} \equiv b^{(2)}_{\mu \nu} - \frac{1}{2} \left( a^{(3)}_\mu a^{(2)}_\nu - a^{(3)}_\nu a^{(2)}_\mu \right) \]  

We again notice here that the NS-NS sector of the reduced action (55) represented by the first six terms has a non-compact global O(1, 1) symmetry of the form \( M \to \Omega M \Omega^T \) and \( \mathcal{A}_\mu \to \Omega \mathcal{A}_\mu \) where,

\[ M = \begin{pmatrix} \chi_B^{-1} & 0 \\ 0 & \chi_B \end{pmatrix} \quad \text{and} \quad \mathcal{A}_\mu = \begin{pmatrix} a^{(3)}_\mu \\ a^{(1)}_\mu \end{pmatrix} \]  

Note that \( M \) is an O(1, 1) matrix satisfying \( M^T \eta M = \eta \) with \( \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). \( \Omega \) is the O(1, 1) matrix of transformation which has the form \( \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix} \). Under the O(1, 1) transformation, the moduli field \( \chi_B \) and the vector gauge fields transform as

\[ \bar{\chi}_B = \lambda^{-2} \chi_B^{-1} \]

\[ \bar{a}^{(3)}_\mu = \lambda a^{(1)}_\mu \]

\[ \bar{a}^{(1)}_\mu = \lambda^{-1} a^{(3)}_\mu \]
Also, note that in order to recover the O(1, 1) invariance of the NS-NS sector $\bar{b}^{(1)}_{\mu\nu}$ does not transform whereas $b^{(1)}_{\mu\nu}$ transforms as,

$$\bar{b}^{(1)}_{\mu\nu} = b^{(1)}_{\mu\nu} - \left( a^{(3)}_{\mu} a^{(1)}_{\nu} - a^{(3)}_{\nu} a^{(1)}_{\mu} \right)$$

(70)

The metric $g_{B,\mu\nu}$ and the dilaton $\phi_B$ do not transform under the O(1, 1) transformation.

As noted before for the type IIA theory, we again notice that the full action (55) does not remain invariant under the O(1, 1) transformation but changes to

$$\tilde{S}_{IIB}^{(9)} = \int d^9x \sqrt{-g_B} \left[ e^{-2\phi_B} \left( R_B + 4\partial_\mu \phi_B \partial^\mu \phi_B - \frac{1}{4} \partial_\mu \log \chi_B \partial^{\mu} \log \chi_B - \frac{1}{4} \chi_B f^{(3)}_{\mu\nu} f^{(3)}_{\mu\nu} - \frac{1}{4} \lambda^{-1} \chi_B \partial_\mu \partial^\mu \psi \partial^\mu \psi \right) - \frac{1}{12} h^{(1)}_{\mu\nu\lambda} h^{(1)}_{\mu\nu\lambda} \right]$$

(71)

We have assumed here, that like $h^{(1)}_{\mu\nu\lambda}$, the other field strength $h^{(2)}_{\mu\nu\lambda}$ also does not transform under under O(1, 1) transformation. This, however, means that the R-R two-form potential $b^{(2)}_{\mu\nu}$(also $\bar{b}^{(2)}_{\mu\nu}$) transforms in a non-trivial non-local way under the duality transformation. Also, the scalar $\psi$ is assumed to remain inert under the O(1, 1) transformation.

Now comparing this T-dual action (71) and the nine dimensional type IIA action (42), we find that they precisely match if the fields in the type IIA theory satisfy the following relations:

$$F_{\mu\nu\rho\lambda} = 0$$

(72)

$$\epsilon^{\mu_1...\mu_9} \epsilon^{ij} H^{(i)}_{\mu_1\mu_2\mu_3} H^{(j)}_{\mu_4\mu_5\mu_6} \left( \tilde{C}^{(k)}_{\mu_7\mu_8\mu_9} + 3 A^{(k)}_{\mu_7} \tilde{B}^{(k)}_{\mu_8\mu_9} \right) = 0$$

(73)

and we make the following field identifications:

$$g_{B,\mu\nu} \equiv g_{\mu\nu}$$

$$\phi_B \equiv \phi$$

$$\lambda^{2} \chi_B \equiv \chi$$

$$\psi \equiv -a$$

$$\lambda^{-1} a^{(3)}_{\mu} \equiv - A^{(2)}_{\mu}$$

$$\lambda a^{(1)}_{\mu} \equiv A^{(3)}_{\mu}$$

$$a^{(2)}_{\mu} \equiv A^{(1)}_{\mu}$$

$$h^{(1)}_{\mu\nu\lambda} \equiv H^{(1)}_{\mu\nu\lambda}$$

$$h^{(2)}_{\mu\nu\lambda} \equiv H^{(2)}_{\mu\nu\lambda}$$

(74)

Here eq.(72) simply says that the dimensionally reduced four form field strength in type IIA theory has to be zero, whereas eq.(73) says that the dual of the combination $\bar{C}^{(i)}_{\mu\nu\lambda} + \left( A^{(i)}_{\mu} \tilde{B}^{(i)}_{\nu\lambda} + \text{cyc. in } \mu\nu\lambda \right)$ is transverse to the $H$'s. Note also that the identification (74) is consistent in the sense that the gauge fields $a^{(3)}_{\mu}$ and $A^{(2)}_{\mu}$ appear from the dimensional
reduction of the ten dimensional metric in type IIB theory and IIA theory respectively. Also $a_\mu^{(1)}$ and $A_\mu^{(3)}$ appear from the reduction of the two-form potential in ten dimensions. The other fields in these two theories have completely different origin and so, there is no contradiction in identifying them in nine dimensions. We would like to point out here that both the antisymmetric two-form potentials (NS-NS and R-R) in the two theories are related to each other in non-local way. We have thus shown, purely from the bosonic consideration, that type IIA and type IIB theories are T-dual to each other.

We now show that the O(1, 1) or the T-duality transformation (69) is nothing but the Buscher’s duality transformation [25] of the various components of the metric, antisymmetric tensor field and the dilaton in ten dimensions for a particular value of $\lambda$. Since the following relations are valid for the common NS-NS sector of any string theory we will omit the index ‘$B$’. The various components of the metric, the antisymmetric tensor field and the dilaton in ten and nine dimensions can be seen from eqs.(56), (58) and (57) to be related as,

$$
\begin{align*}
\hat{g}_{99} & = g_{99} = \chi \\
\hat{g}_{\mu 9} & = g_{99} a_\mu^{(3)} = \chi a_\mu^{(3)} \\
\hat{g}_{\mu \nu} & = g_{\mu \nu} + \chi a_\mu^{(3)} a_\nu^{(3)} \\
\hat{b}_{\mu 9}^{(1)} & = b_{\mu 9}^{(1)} = a_\mu^{(1)} \\
\hat{b}_{\mu \nu}^{(1)} & = b_{\mu \nu}^{(1)} - \frac{1}{2} \left( a_\mu^{(3)} a_\nu^{(1)} - a_\nu^{(3)} a_\mu^{(1)} \right) \\
\hat{\phi} & = \phi + \frac{1}{4} \log \chi = \phi + \frac{1}{4} \log g_{99}
\end{align*}
$$

(75)

We have expressed $\hat{b}_{\mu \nu}^{(1)}$ in terms of $\hat{b}_{\mu 9}^{(1)}$ because we know that it does not transform under O(1, 1) transformation (69). This is crucial to obtain the Buscher’s duality rule [22] correctly. Under the duality transformation, they transform to

$$
\begin{align*}
\tilde{g}_{99} & = \lambda^{-2} \chi^{-1} = \frac{1}{\lambda^2 g_{99}} \\
\tilde{g}_{\mu 9} & = \lambda^{-1} \chi^{-1} a_\mu^{(1)} = \frac{\hat{b}_{\mu 9}^{(1)}}{\lambda \hat{g}_{99}} \\
\tilde{g}_{\mu \nu} & = \hat{g}_{\mu \nu} - \chi a_\mu^{(3)} a_\nu^{(3)} + \chi^{-1} a_\mu^{(1)} a_\nu^{(1)} \\
& = \hat{g}_{\mu \nu} - \frac{1}{\hat{g}_{99}} \left( \hat{g}_{\mu 9} \hat{g}_{\nu 9} - \hat{b}_{\mu 9}^{(1)} \hat{b}_{\nu 9}^{(1)} \right) \\
\tilde{b}_{\mu 9}^{(1)} & = \lambda^{-1} a_\mu^{(3)} = \frac{1}{\lambda} \frac{\hat{g}_{\mu 9}}{\hat{g}_{99}}
\end{align*}
$$

(76) (77) (78) (79)
We point out that these O(1, 1) transformations (76)–(81) match precisely with Buscher's rule for \( \lambda = -1 \). For generic \( \lambda \), however, we do not get any new information from (76)–(81) since it is clear that \( \lambda \) can be scaled away in all the expressions except the dilaton transformation rule in (81) by scaling the ninth coordinate \( x^9 \rightarrow \lambda x^9 \). In (81), this scaling does not completely absorb the \( \lambda \) term, but, since \( \lambda \) is a constant, gives a constant shift in the dilaton which can be absorbed into the gravitational constant in front of the string effective action not displayed explicitly.

We would also like to comment that since we have mapped the type IIB theory in nine dimensions exactly to the type IIA theory by an O(1, 1) or T-duality transformation by field identifications, the type IIB theory also possesses a global O(2) invariance in nine dimensions very much like the type IIA theory. The complete O(2) transformations in the type IIB theory are given by

\[
\begin{align*}
\delta \psi &= \theta_B \left( 1 + \psi^2 - \chi_B^{-1} e^{-2\phi_B} \right) \\
\delta e^{\phi_B} &= -\frac{1}{4} \theta_B \psi e^{\phi_B} \\
\delta \chi_B &= -\theta_B \chi_B \psi \\
\delta a^{(3)}_{\mu} &= 0
\end{align*}
\]

\[
\begin{align*}
\delta a^{(i)}_{\mu} &= \theta_B \epsilon^{ij} a^{(j)}_{\mu} \\
\delta g_{B,\mu\nu} &= -\theta_B g_{B,\mu\nu} \psi \\
\delta \check{b}^{(i)}_{\mu\nu} &= \theta_B \epsilon^{ij} \check{b}^{(j)}_{\mu\nu} \\
\delta h^{(i)}_{\mu\nu\lambda} &= \theta_B \epsilon^{ij} h^{(j)}_{\mu\nu\lambda}
\end{align*}
\]  

(82)

Here \( \theta_B \) is a constant infinitesimal parameter and \( i, j = 1, 2 \) with \( \epsilon^{12} = -\epsilon^{21} = 1 \). One can check indeed that the action (55) is invariant under this transformation although type IIB theory is not obtained from the dimensional reduction of an eleven dimensional theory.

It is quite natural to expect that the type IIB theory in nine dimensions (55) should have a manifest global SL(2, R) invariance like its parent theory in ten dimensions. We say it is natural, because the matrix \( S \) eq. (54) that was constructed to show the SL(2, R) invariance of the action only involves the scalar fields which remain intact under dimensional reduction. We, however, show that this expectation fails primarily because the matrix \( S \) in (54) is manifestly dependent on the number of the space-time dimensions.

To show this it is enough to consider a simpler case when \( \chi_B = 1 \), i.e. at the self-dual point. (The general case can also be worked out quite easily.) We first rewrite the action (55) in the Einstein frame since the Einstein metric remains inert under SL(2, R) transformation.
With the Weyl scaling \( \bar{g}_{B,\mu\nu} = e^{-\frac{4}{7}\phi_B} g_{B,\mu\nu} \) the action takes the form:

\[
\tilde{S}^{(9)}_{IIB} = \int d^9 x \sqrt{-\bar{g}_B} \left[ \bar{R}_B - \frac{4}{7} \partial_{\mu} \phi_B \partial^\mu \phi_B - \frac{1}{2} e^{2\phi_B} \partial_{\mu} \psi \partial^\mu \psi \\
- \frac{1}{4} e^{\frac{4}{7}\phi_B} \left( f^{(3)}(\mu \nu \sigma) f^{(3)}(\mu \nu \sigma) + f^{(1)}(\mu \nu) f^{(1)}(\mu \nu) \right) - \frac{1}{12} e^{\frac{8}{7}\phi_B} h^{(1)}_{\mu \nu \lambda} h^{(1)}_{\mu \nu \lambda} \\
- \frac{1}{12} e^{\frac{6}{7}\phi_B} \left( h^{(2)}_{\mu \nu \lambda} + \psi h^{(1)}_{\mu \nu \lambda} \right) \left( h^{(2)}_{\mu \nu \lambda} + \psi h^{(1)}_{\mu \nu \lambda} \right) \\
- \frac{1}{4} e^{\frac{10}{7}\phi_B} \left( f^{(2)}(\mu \nu) + \psi f^{(1)}(\mu \nu) \right) \left( f^{(2)}(\mu \nu) + \psi f^{(1)}(\mu \nu) \right) \right]
\]  

(83)

We note that in order to reproduce the dilaton as well as the scalar term in the action correctly in any dimension, we should choose the matrix \( S \) to have the following form: (It is worth emphasizing here that the dependence on \( D \), alternately, could be transferred to the exponential which is equivalent to rescaling the dilaton. In fact, this is the way one gets SL(2, R) invariance in four dimensional string theory.)

\[
S = e^{\phi_B} \left( \frac{D-2}{8} \psi^2 + e^{-2\phi_B} \psi \left( \frac{D-2}{8} \right)^{\frac{1}{2}} 1 \right)
\]  

(84)

so that

\[
\frac{2}{D-2} \text{tr} \partial_{\mu} S^{-1} \partial^\mu S = -4 \frac{4}{D-2} \partial_{\mu} \phi_B \partial^\mu \phi_B - \frac{1}{2} e^{2\phi_B} \partial_{\mu} \psi \partial^\mu \psi
\]  

(85)

In particular, in ten dimensions (85) correctly reproduces the scalar terms and it matches with the form of \( S \) given in eq.(54). Also, for nine dimensions it produces the scalar terms in (83) correctly. But it is the explicit \( D \)-dependence in the matrix \( S \) which spoils the manifest SL(2, R) invariance of the other terms in the action. For example, if we want to produce \( h_{\mu \nu \lambda} \) terms we choose

\[
\mathcal{H}_{\mu \nu \lambda} = \left( e^{-\frac{4}{7}\phi_B} h^{(1)}_{\mu \nu \lambda} \right) \left( e^{-\frac{8}{7}\phi_B} h^{(2)}_{\mu \nu \lambda} \right)
\]  

(86)

then,

\[
\mathcal{H}^{T}_{\mu \nu \lambda} \mathcal{H}^{\mu \nu \lambda} = e^{-\frac{4}{7}\phi_B} h^{(1)}_{\mu \nu \lambda} h^{(1)}_{\mu \nu \lambda} \\
+ e^{\frac{8}{7}\phi_B} \left( h^{(2)}_{\mu \nu \lambda} + \psi \left( \frac{D-2}{8} \right)^{\frac{1}{2}} h^{(1)}_{\mu \nu \lambda} \right) \left( h^{(2)}_{\mu \nu \lambda} + \psi \left( \frac{D-2}{8} \right)^{\frac{1}{2}} h^{(1)}_{\mu \nu \lambda} \right)
\]  

(87)

So, for \( D=9 \) we almost get the \( h_{\mu \nu \lambda} \) term in (83) except the factor \( \left( \frac{D-2}{8} \right)^{\frac{1}{2}} \). Note that the exponential factor can always be adjusted properly by scaling the other fields in an
appropriate way. For $D=10$, it produces the $h_{\mu \nu \lambda}$ term correctly because in that case $D-2 = 1$. But for other $D$ we will not get $h_{\mu \nu \lambda}$ term correctly since the factor cannot be scaled away. Thus, we conclude that eventhough the ten dimensional type IIB theory has a manifest SL(2, R) invariance, this manifest symmetry gets spoiled by dimensional reduction.

It can also be checked in a similar fashion that the type IIA string effective action in nine dimensions, eq.(42), does not have a manifest SL(2, R) invariance even when the conditions (72) and (73) are satisfied. It is also not expected since the parent theory eq.(17) does not have this symmetry in any manifest way at least in terms of the variables in which the action is written. We, however, give here a dynamical argument to show that like in type IIB theory in ten dimensions, type IIA theory also should have an SL(2, R) S-duality invariance in ten dimensions.

It has been observed by Aspinwall [24] that the strong-weak coupling duality (which we have interchangably called an SL(2, R) S-duality) symmetry in type IIB string theory in ten dimensions has its origin in the underlying eleven dimensional theory by compactifying it on $S^1 \times S^1 \approx T^2$. Since we have also studied the dimensional reduction of eleven dimensional CJS theory, we can reformulate his argument in terms of the fields we have defined in section II. Note from (15), that when we compactified the CJS theory on $S^1$ to obtain type IIA theory, the radius of the circle of compactification was found to be related to the ten dimensional dilaton by

$$R_1 \sim e^{\frac{2}{3} \phi}$$

(88)

So, the type IIA string coupling constant is given by

$$\lambda_A = e^{\frac{2}{3} \phi} \sim R_1^3$$

(89)

When this type IIA theory was further compactified on second $S^1$, the radius of the circle was chosen to be $\chi^{\frac{1}{2}}$ (see eq.(25)). This radius is measured in string metric. Because, we had to rescale the ten dimensional metric (see eq. (14)) to obtain type IIA theory, so the radius of the second circle in terms of the original metric would be given as

$$R_2 \sim e^{-\frac{1}{3} \phi} \chi^{\frac{1}{2}}$$

(90)

So, writing in terms of the nine dimensional fields we get

$$R_1 \sim e^{\frac{2}{3} \phi} \chi^{\frac{1}{2}}$$

$$R_2 \sim e^{-\frac{1}{3} \phi} \chi^{\frac{1}{2}}$$

(91)
Since the string coupling constant of the type IIA theory \( \lambda_A = e^{(\phi + \frac{1}{4} \log \chi)} \) changes to that of type IIB theory under the T-duality transformation \( \tilde{\chi} = \chi^{-1} \) (we have set \( \lambda = -1 \) as this produces the Buscher’s duality transformation), we have

\[
\lambda_B = e^{(\phi - \frac{1}{4} \log \chi)} \sim \frac{R_1}{R_2} \quad (92)
\]

We point out that the nine dimensional type IIA theory should have a symmetry if we interchange \( R_1 \) and \( R_2 \). In other words, we should have the same theory if we compactify CJS theory first on \( S^1 \) with radius \( R_1 \) and then on the other \( S^1 \) with radius \( R_2 \) or first on \( S^1 \) with radius \( R_2 \) and then on the other \( S^1 \) with radius \( R_1 \). But this symmetry of the nine dimensional theory has a dramatic consequence for the type IIB theory in ten dimensions since it takes the string coupling constant to its inverse. Thus we find a strong-weak coupling duality symmetry in type IIB theory as a consequence of \( R_1 \leftrightarrow R_2 \) symmetry of type IIA theory in nine dimensions. In fact, by taking into account the angle between \( R_1 \) and \( R_2 \), the SL(2, R) symmetry group of the type IIB theory can be identified with the modular group of the torus \( T^2 \cong S^1 \times S^1 \) as also observed in ref.[21].

We now compute the T-dual of \( R_1 \) and \( R_2 \) as given by,

\[
\tilde{R}_1 \sim e^{\frac{2}{3} \phi \chi^{-\frac{1}{6}}} \sim \left( \frac{R_1}{R_2} \right)^{\frac{2}{3}} \sim \lambda_B^{\frac{2}{3}} \quad (93)
\]

\[
\tilde{R}_2 \sim e^{-\frac{1}{3} \phi \chi^{-\frac{1}{12}}} \sim \left( \frac{R_1}{R_2} \right)^{\frac{2}{3}} R_1^{-\frac{2}{3}} \sim \frac{\lambda_B^{\frac{2}{3}}}{\lambda_A} \quad (94)
\]

Using (93) and (94) we find

\[
\lambda_A \sim \frac{\tilde{R}_1}{\tilde{R}_2} \quad (95)
\]

Since the nine dimensional theory should have an exchange symmetry \( \tilde{R}_1 \leftrightarrow \tilde{R}_2 \), we thus find a strong-weak coupling duality symmetry also in type IIA theory in ten dimensions. This symmetry has its origin in the eleven dimensional CJS theory by compactifying it on T-dual torus \( \tilde{T}^2 \) with radii \( \tilde{R}_1 = \left( \frac{R_1}{R_2} \right)^{\frac{2}{3}} \) and \( \tilde{R}_2 = \left( \frac{R_1}{R_2} \right)^{\frac{2}{3}} R_1^{-\frac{2}{3}} \). It remains an interesting puzzle to obtain this symmetry of the type IIA action directly in ten dimensions.

\textbf{IV. Conclusions:}

We have performed the “ordinary” Scherk-Schwarz dimensional reduction of the bosonic sector of the low energy effective action (CJS theory) of a hypothetical M-theory
on $S^1 \times S^1 \cong T^2$. In this way we have obtained the low energy effective actions of type IIA string theory in both ten and nine space-time dimensions. Type IIA string effective actions obtained this way not only contain the usual NS-NS gauge fields but also contain the R-R gauge fields which do not couple to the dilaton and encode the non-perturbative information inherited from the eleven dimensional CJS theory. Since the string coupling constant in ten dimensions is directly proportional to the radius of compactification of the circle, M-theory describes the non-perturbative limit of type IIA string theory. The NS-NS sector of the nine dimensional type IIA string effective action has a noncompact global $O(1, 1)$ invariance. We have pointed out how to recover this symmetry correctly under which the moduli, the vector gauge fields as well as the antisymmetric Kalb-Ramond field transform in a nontrivial way. The full action, including the R-R sector in nine dimensions does not possess this symmetry, however, it has a global $O(2)$ invariance as a consequence of the Lorentz symmetry of the original eleven dimensional theory. We then considered the type IIB string effective action containing the R-R fields when the self-dual five-form field strength is set to zero. This action is known to possess an SL(2, R) duality invariance. We performed the dimensional reduction of this action on $S^1$. The NS-NS sector of the resulting nine dimensional theory also has a noncompact global $O(1, 1)$ invariance, but much like the type IIA theory in nine dimensions, the full action does not have this symmetry. However, we have shown that under this $O(1, 1)$ transformation the complete bosonic sector of the type IIB theory in nine dimensions reduces to the type IIA action with proper field identifications. In order to obtain this, the fields in the type IIA theory would have to satisfy the relations given in eqs.(72) and (73). This, therefore, shows purely from the bosonic considerations that the type IIA and type IIB theories are T-dual to each other. We then showed that for a particular value of the parameter of the $O(1, 1)$ transformation, it reduces precisely to the Buscher’s duality rules for the various components of the metric, the antisymmetric tensor field and the dilaton in ten dimensions. We have pointed out that the type IIB theory in nine dimensions has a global $O(2)$ invariance although it is not obtained from an eleven dimensional theory. The original manifest SL(2, R) invariance of the type IIB theory in ten dimensions have been shown to be spoiled by the dimensional reduction and the nine dimensional theory does not have this symmetry in any manifest way. We then presented a dynamical argument to show that the type IIA theory in ten dimensions also has an SL(2, R) S-duality invariance. This can be understood by compactifying CJS theory on T-dual torus $\tilde{T}^2$. It would be
very interesting to realize this symmetry in a manifest way at the level of type IIA action in ten dimensions.

Acknowledgements:

We would like to thank E. Bergshoeff for discussions on some points. A.D. would like to thank the members of the Departamento de Fisica de Particulas for hospitality and S.R. would like to thank A. V. Ramallo and J. M. Sanchez de Santos for discussions. This work was supported in part by U.S.D.O.E. Grant No. DE-FG-02-91ER40685. The work of S.R. has been supported in part by a fellowship from the Spanish Ministry of Education (MEC).

References:

1. E. Witten, Nucl. Phys. B443 (1995) 85.
2. E. Witten, Some Comments On String Dynamics, preprint IASSNS-HEP-63, [hep-th/9507121].
3. C. M. Hull, String Dynamics at Strong Coupling, preprint QMW-95-50, [hep-th/9512181].
4. E. Cremmer, B. Julia and J. Scherk, Phys. Lett. B76 (1978), 409.
5. E. Cremmer and B. Julia, Nucl. Phys. B159 (1979) 141.
6. E. Witten as quoted in J. H. Schwarz, Phys. Lett. B367 (1996) 97.
7. E. Bergshoeff, C. Hull and T. Ortin, Nucl. Phys. B451 (1995) 547.
8. J. Polchinski and E. Witten, Evidence for Heterotic–Type I String Duality, preprint IASSNS-HEP-95-81, NSF-ITP-95-135, [hep-th/9510163].
9. P. Horava and E. Witten, Heterotic and Type I String Dynamics from Eleven Dimensions, preprint IASSNS-HEP-95-86, PUPT-1571, [hep-th/9510209].
10. J. Dai, R. G. Leigh and J. Polchinski, Mod. Phys. Lett. A4 (1989) 2073.
11. R. G. Leigh, Mod. Phys. Lett. A4 (1989) 2767.

12. J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724.

13. P. K. Townsend, *D-Branes from M-Branes*, preprint R/95/59, hep-th/9512062.

14. C. M. Hull and P. K. Townsend, Nucl. Phys. B438 (1995) 109.

15. J. Scherk and J. H. Schwarz, Nucl. Phys. B153 (1979) 61.

16. E. Cremmer, in *Supergravity 1981*, eds. S. Ferrara and J. G. Taylor (Cambridge University Press, 1982).

17. J. Maharana and J. H. Schwarz, Nucl. Phys. B390 (1993) 3.

18. J. H. Schwarz, Nucl. Phys. B226 (1983) 269.

19. P. Howe and P. West, Nucl. Phys. B238 (1984) 181.

20. C. M. Hull, Phys. Lett. B357 (1995) 545.

21. J. H. Schwarz, Phys. Lett. B360 (1995) 13.

22. T. Buscher, Phys. Lett. B194 (1987) 59; Phys. Lett. B201 (1988) 466.

23. M. Dine, P. Huet and N. Seiberg, Nucl. Phys. B322 (1989) 301.

24. P. Aspinwall, *Some Relationship Between Dualities in String Theory*, preprint CLNS-95/1359, hep-th/9508154.

25. E. Bergshoeff, B. Janssen and T. Ortin, *Solution Generating Transformations and the String Effective Action*, preprint UG-1/95, QMW-PH-95-1, hep-th/9506156.