SCALE INVARIANT ENERGY SMOOTHING ESTIMATES FOR THE SCHRÖDINGER EQUATION WITH SMALL MAGNETIC POTENTIAL

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Abstract. We consider some scale invariant generalizations of the smoothing estimates for the free Schrödinger equation obtained by Kenig, Ponce and Vega in [21, 22]. Applying these estimates and using appropriate commutator estimates, we obtain similar scale invariant smoothing estimates for perturbed Schrödinger equation with small magnetic potential.

1. Introduction

In this work we study smoothing properties of the Schrödinger equation with magnetic potential

$$A = (A_1(t, x), \ldots, A_n(t, x)), \quad x \in \mathbb{R}^n.$$  

Here $A_j(t, x), j = 1, \ldots, n$, are real valued functions, $n \geq 3$ and the corresponding Cauchy problem for Schrödinger equation has the form

$$\begin{cases}
\partial_t u - i\Delta_A u = F, & t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\
u(0, x) = f(x), &
\end{cases}$$

where

$$\Delta_A = \sum_{j=1}^{n} (\partial_{x_j} - iA_j)(\partial_{x_j} - iA_j).$$

The energy type estimates and well–posedness of the Cauchy problem (1.1) in energy space are studied in the works [13] and [14] of Doi.

Since the smoothing properties of this evolution problem are closely connected with suitable resolvent estimates for the solution $U = U(x)$ of the elliptic problem

$$\begin{cases}
\varepsilon U - i\Delta_A U - i\tau U = H, & \varepsilon > 0, \tau > 0, \quad x \in \mathbb{R}^n, H = H(x),
\end{cases}$$

we can use as a starting point the scale invariant smoothing estimate obtained in the works of Kenig, Ponce, Vega [21] and Pertham, Vega [22]. This estimate extends earlier works of Agmon, Hörmander [2] and P. Constantin and J.-C. Saut [8].

The scale invariant estimate for (1.3) with $A = 0$ has the form

$$\||\nabla_x U|| \leq CN(H),$$

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where $C > 0$ is independent of $\varepsilon > 0$, $\tau > 0$,
\[
\|G\|^{2} = \sup_{R > 0} \frac{1}{R} \int_{|x| \leq R} |G(y)|^{2} dy
\]
is the Morrey - Campanato norm, while
\[
N(H) = \sum_{k \in \mathbb{Z}} 2^{k/2} \|H\|_{L^{2}(2^{k-1} \leq |x| \leq 2^{k+1})}.
\]

From this estimate one can use the simple estimate
\[
\sup_{k \in \mathbb{Z}} 2^{-k/2} \|G\|_{L^{2}(2^{k-1} \leq |x| \leq 2^{k+1})} \leq C \|G\|
\]
and derive the following smoothing scale invariant estimate for the solution $u(t, x)$ to (1.1) with $A = 0$ and $f = 0$
\[
(1.5) \quad \int_{\mathbb{R}} \left( \sup_{k \in \mathbb{Z}} 2^{-k/2} \|\nabla_{x} u(t, \cdot)\|_{L^{2}(2^{k-1} \leq |x| \leq 2^{k+1})} \right)^{2} dt \leq C \int_{\mathbb{R}} \left( \sum_{k \in \mathbb{Z}} 2^{k/2} \|F(t, \cdot)\|_{L^{2}(2^{k-1} \leq |x| \leq 2^{k+1})} \right)^{2} dt.
\]

Our purpose in this work is to derive similar scale invariant smoothing estimates for the case of magnetic potential imposing scale invariant smallness assumptions on the magnetic potential $A(x)$. The reason why we treat only small magnetic potential is connected with the necessity to avoid resonances phenomena (see [10], [12], [23], [24], [28], [32], [33]). The absence of eigenvalues of $\Delta_{A}$ with magnetic potential decaying as $(1 + |x|)^{-1-\delta}$ is discussed in [7]. However, even the remarkable result in [7] can not guarantee that $0$ is not an eigenvalue of the Hamiltonian $\Delta_{A}$. The result in [33] shows that even nontrivial smooth compactly supported magnetic field can create resonances.

To avoid possible eigenvalues or resonances of $\Delta_{A}$ we impose the following assumption on the potential $A$.

**Assumption 1.1.** There exists $\varepsilon > 0$, such that we have
\[
(1.6) \quad \max_{1 \leq j \leq n} \sum_{k \in \mathbb{Z}, |\beta| \leq 1} 2^{k(1+|\beta|)} \|D_{x}^{\beta} A_{j}(t, x)\|_{L^{\infty}_{t} L^{\infty}_{|x| \sim 2^{k}}} \leq \varepsilon.
\]

Our main smoothing estimate is the following one.

**Theorem 1.1.** There exists $\varepsilon > 0$ so that for any potential $A(x)$ satisfying (1.6) there exists $C > 0$, so that for any $f \in S(\mathbb{R}^{n})$ and any $F(t, x) \in C_{0}^{\infty}(\mathbb{R} \times (\mathbb{R}^{n} \setminus 0))$ the solution $u(t, x)$ to (1.1) satisfies the estimate
\[
(1.7) \quad \int_{\mathbb{R}} \left( \sup_{k \in \mathbb{Z}} \|x|^{1/2} u(t, \cdot)\|_{\dot{H}_{x}^{-1/2}} \right)^{2} dt \leq C \|f\|_{L^{2}}^{2} + C \int_{\mathbb{R}} \left( \sum_{k \in \mathbb{Z}} \|x|^{1/2} F(t, \cdot)\|_{\dot{H}_{x}^{-1/2}} \right)^{2} dt,
\]
where $\dot{H}_{x}^{s} = \dot{H}^{s}(\mathbb{R}^{n})$ is the classical homogeneous Sobolev space and $|x|^{1/2} = |x|^1{2}\|x^{1/2}Q_{k}(x)\|$ and the Paley - Littlewood partition of unity
\[
(1.8) \quad 1 = \sum_{k \in \mathbb{Z}} Q_{k}(x),
\]
is defined as follows
\[ Q_k(x) = \varphi \left( \frac{|x|}{2^k} \right), \]
where \( \varphi(s) \in C_0^\infty((1/2, 2)) \) is a non-negative function.

The key point to derive this estimate is a suitable scale and time invariant smoothing estimate for the free Schrödinger equation

\[
\begin{aligned}
\partial_t u - i \Delta u &= F, \quad t > 0, \quad x \in \mathbb{R}^n, \\
u(0, x) &= f(x).
\end{aligned}
\]

To be more precise, we introduce the following norms motivated by the statement of the main result in Theorem 1.1. Take

\[ Y = L^2_t(\ell^1_x \hat{H}^{-1/2}), \quad Y' = L^2_t(\ell^2_x \hat{H}^{1/2}), \]

where the spaces \( \ell^k_x B \) for any Banach space \( B \) is introduced in Section 2. Note that \( Y \) is not reflexive (\( (\ell^1_x)' = \ell^\infty_x, \) but \( (\ell^2_x)' \neq \ell_x^{1, 1/2} \)).

Then the estimate of the previous theorem can be rewritten in the form

\[
\| u \|^2_{Y'} \leq C \| F \|^2_{\hat{H}^1} + \| F \|^2_Y,
\]

where here and below

\[
Y = L^2_t(\ell^1_x \hat{H}^{-1/2}), \quad Y' = L^2_t(\ell^2_x \hat{H}^{1/2}).
\]

We shall call \( Y' \) smoothing space.

Then the main point in the proof of Theorem 1.1 is to establish first the following energy smoothing estimate for the case \( A = 0 \).

**Theorem 1.2.** There exists \( C > 0 \), such that for any \( f \in S(\mathbb{R}^n) \) and any \( F(t, x) \in C_0^\infty(\mathbb{R} \times (\mathbb{R}^n \setminus 0)) \) the solution \( u(t, x) \) to (1.9) satisfies the estimate

\[
\| u \|_{L^\infty_t L^2_x} + \| u \|_{Y'} \leq C \| F \|_{L^2_t L^1_x} + C \left( \min_{F = F_1 + F_2} \| F_1 \|_{Y} + \| F_2 \|_{L^1_t L^2_x} \right).
\]

On the basis of the estimate in Theorem 1.2 we shall derive a slightly stronger estimate for the perturbed Schrödinger equation.

**Corollary 1.1.** There exists \( \varepsilon > 0 \) so that for any potential \( A(x) \) satisfying (1.6) there exists \( C > 0 \), so that for any \( f \in S(\mathbb{R}^n) \) and any \( F(t, x) \in C_0^\infty(\mathbb{R} \times (\mathbb{R}^n \setminus 0)) \) the solution \( u(t, x) \) to (1.11) satisfies the estimate

\[
\| u \|_{L^\infty_t L^2_x} + \| u \|_{Y'} \leq C \| F \|_{L^2_t L^1_x} + C \left( \min_{F = F_1 + F_2} \| F_1 \|_{Y} + \| F_2 \|_{L^1_t L^2_x} \right).
\]

As an application we consider the following semilinear Schrödinger equation

\[
\begin{aligned}
\partial_t u - i \Delta_A u &= |V(t, x)| u^p, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\
u(0, x) &= f(x),
\end{aligned}
\]

where \( p > 1 \) and \( V(t, x) \) is a measurable function satisfying the inequality

\[
\sum_{k \in \mathbb{Z}} 2^{ka} \| V(t, x) \|_{L^\infty_{\{|x| \sim 2^k\}}} \leq C < \infty.
\]

Then we have the following global existence result with initial data having small \( L^2 \)-norm only.
Theorem 1.3. Suppose the potential $A(x)$ satisfies (1.6), $V$ obeys (1.15) with $a \in [1, 2)$ and $p = \frac{n + 4}{n + 2a}$. Then there exists $\delta > 0$, so that for any $f \in L^2(\mathbb{R}^n)$ with 
\[ \|f\|_{L^2} \leq \delta \]
the problem (1.14) has a unique global solution
\[ u(t, x) \in C(\mathbb{R}, L^2(\mathbb{R}^n)) \cap Y'. \]

The proof of Theorem 1.2 is based on the estimate (1.5) due to Kenig, Ponce, Vega [21], [22]. In order to have a self contained article we give an alternative proof of this result due to Kenig, Ponce, Vega in Section 9.

The key step to derive the estimate (1.12) from the estimate (1.5) is the following equivalence norm result.

Theorem 1.4. For $n \geq 3, 1 < q < \infty$, for $s \in [-1, 1]$ and $a \in \mathbb{R}$ that satisfy (1.17)
\[ |a| + |s| < \frac{n}{2}, \]
the following norms are equivalent
\[ \left( \sum_{k \in \mathbb{Z}} 2^{qka} \| Q_k |D|^s f\|_{L^2}^q \right)^{1/q}, \]
\[ \left( \sum_{k \in \mathbb{Z}} 2^{qka} \| |D|^s Q_k f\|_{L^2}^q \right)^{1/q}, \]
\[ \left( \sum_{k \in \mathbb{Z}} \| |D|^s |x|^a k^a f\|_{L^2}^q \right)^{1/q}, \]
where $|x|^a_k = |x|^a Q_k(x)$ and the Paley-Littlewood partition of unity $Q_k(x)$ is defined in (1.8). For $q = \infty$ the result is still valid with obvious modification in (1.18).

The main idea to establish the Theorem is similar to the approach developed in [15], [11] and [16] for the case of nonhomogeneous Sobolev spaces and non homogeneous weights. Therefore, we shall make a localization in coordinate space and we shall use the Paley Littlewood partition (1.8). The key point in this approach is to evaluate the norm of the operator of type $Q_k |D|^{-s} Q_m |D|^s$ with $|k - m|$ large enough.

The proof of Theorem 1.4 can be obtained from the estimate for the Cauchy problem with initial data $f = 0$ and the following Theorems (see section 8 for the definition of the spaces $\ell^q_D B$ for any Banach space $B$).

Theorem 1.5. If $q \in [1, 2]$ and $a, s \in \mathbb{R}$ satisfy
\[ \left\{ \begin{array}{l}
|s| \leq 1, \\
|a| + |s| < \frac{n}{2}
\end{array} \right. \]
then
\[ \|f\|_{\ell^q_D \ell^{q*a}_{H'}^s} \leq C \|f\|_{\ell^{q*a}_{H'}^s}. \]
Theorem 1.6. If $q \in [2, \infty]$ and $a, s \in \mathbb{R}$ satisfy (1.19), then

$$\|f\|_{\ell_q^a \dot{H}^s} \leq C\|f\|_{\ell_2^a \dot{H}^s}.$$ 

The plan of the work is the following. The proof of the free smoothing estimate of Theorem 1.2 is given in Section 2. The proof of the main scale invariant smoothing estimate of Theorem 1.1 is done in Section 3. In Section 5 we treat the commutator estimates needed in the proof the equivalence of the norms in Theorem 1.4. Some convolution type inequalities needed in the proofs of Theorem 1.4 are included in Section 6. The concluding steps in the proof of Theorem 1.4 are presented in Section 7. Finally the phase localization and the proofs of Theorems 1.5 and 1.6 are given in the last Section 8. The proof of the estimate due to König, Ponce, Vega is presented in Section 9 for self contained completeness.

2. Weighted Sobolev spaces estimate of the free Schrödinger equation.

Given any Banach space $B \subset D'(\mathbb{R}^n)$ satisfying the property

for any $Q(x) \in C_0^\infty(\mathbb{R}^n)$, $f \in B \Rightarrow Q(x)f \in B,$

we can define for any $q \in [1, \infty]$ and for any $\alpha \in \mathbb{R}$ the space $\ell^q_\alpha B$ as follows

$$\|f\|_{\ell^q_\alpha B} = \left(\sum_{k \in \mathbb{Z}} \|Q_k f\|_B^q 2^{kq\alpha}\right)^{1/q},$$

with obvious modification for $q = \infty$. Note that for any $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we have

$$\|f\|_{\ell^q_\alpha B} < \infty.$$ 

So $\ell^q_\alpha B$ can be defined as the closure of $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with respect to the norm (2.2). An alternative definition is based on the map

$$J : f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset B \rightarrow J_B(f)_k = \|Q_k f\|_B \in \ell^q_\alpha,$$

where $\ell^q_\alpha$ is the space of all sequences $a = (a_k)_{k \in \mathbb{Z}}$ such that

$$\|a\|_{\ell^q_\alpha} = \left(\sum_{k \in \mathbb{Z}} \|a_k\|^q 2^{kq\alpha}\right)^{1/q} < \infty,$$

with obvious modification for $q = \infty$. Then

$$\|f\|_{\ell^q_\alpha B} = \|J_B(f)\|_{\ell^q_\alpha}.$$ 

The space $\ell^q_\alpha B$ is independent of the concrete choice of Paley-Littlewood decomposition

$$\sum_{j \in \mathbb{Z}} Q_j(x) = 1$$

satisfying

$$\{ Q_j(x) \geq 0, \quad \text{supp } Q_j(x) \in \{|x| \sim 2^j\}.$$
A typical example, needed for the smoothing resolvent type estimates, is the case $B = H^s_p$ where $s \in (-1, 1)$, $1 < p < \infty$. For $s > -\frac{2}{p}$ we have $H^s_p(\mathbb{R}^n) \subset D'(\mathbb{R}^n)$ (see [11]) and the norm is defined by
\begin{equation}
\|f\|_{H^s_p} = \|D|^s f\|_{L^p}.
\end{equation}

After this preparation we can turn to the proof of Theorem [12]. Starting with the estimate (1.4), we use Lemma [13] and find
\begin{equation}
\sup_{k \in \mathbb{Z}} \|x|^{-1/2}\nabla u\|_{L^2_t L^2_z} \lesssim \sup_{k \in \mathbb{Z}} \|x|^{-1/2}D|u\|_{L^2_t L^2_z}
\end{equation}
so (1.3) can be rewritten as
\begin{equation}
\sup_{k \in \mathbb{Z}} \|x|^{-1/2}D|u\|_{L^2_t L^2_z} \leq C \left( \sum_{k \in \mathbb{Z}} \|x|^{1/2}F\|_{L^2_t L^2_z} \right).
\end{equation}

Using the fact that $\|D\|^s$ commutes with $\Delta$ one can obtain the following consequence of this estimate
\begin{equation}
\sup_{k \in \mathbb{Z}} \|x|^{-1/2}D|\sigma u\|_{L^2_t L^2_z} \leq C \left( \sum_{k \in \mathbb{Z}} \|x|^{1/2}D|\sigma F\|_{L^2_t L^2_z} \right)
\end{equation}
for any $\sigma \in [0, 1]$. In particular for $\sigma = 1/2$ we get
\begin{equation}
\sup_{k \in \mathbb{Z}} \|x|^{-1/2}D|\sigma u\|_{L^2_t L^2_z} \leq C \left( \sum_{k \in \mathbb{Z}} \|x|^{1/2}D|\sigma F\|_{L^2_t L^2_z} \right)
\end{equation}

To this end, we are in position to apply the result of Proposition [14] and derive that
\begin{equation}
\sup_{k \in \mathbb{Z}} \|x|^{-1/2}D|\sigma u\|_{L^2_t L^2_z} \sim \|u\|_{L^2_t \dot{H}^{-1/2}_x},
\end{equation}
so
\begin{equation}
\sup_{k \in \mathbb{Z}} \|x|^{-1/2}D|\sigma u\|_{L^2_t L^2_z} \sim \sup_{k \in \mathbb{Z}} \|x|^{-1/2}u\|_{L^2_t \dot{H}^{-1/2}_x}.
\end{equation}

In a similar way Proposition [14] implies
\begin{equation}
\sum_{k \in \mathbb{Z}} \|x|^{1/2}D|\sigma F\|_{L^2_t L^2_z} \sim \|\sigma F\|_{L^2_t \dot{H}^{-1/2}_x},
\end{equation}
so
\begin{equation}
\sum_{k \in \mathbb{Z}} \|x|^{1/2}D|\sigma F\|_{L^2_t L^2_z} \sim \sum_{k \in \mathbb{Z}} \|x|^{1/2}F\|_{L^2_t \dot{H}^{-1/2}_x}.
\end{equation}
The estimate (2.11) reads as
\begin{equation}
\sup_{k \in \mathbb{Z}} \|x|^{-1/2}u\|_{L^2_t \dot{H}^{-1/2}_x} \leq C \left( \sum_{k \in \mathbb{Z}} \|x|^{1/2}F\|_{L^2_t \dot{H}^{-1/2}_x} \right)
\end{equation}
or using the notations of this section (see (1.11) and the definition (2.5)) as
\begin{equation}
\left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{Y'} \leq C\|F\|_Y.
\end{equation}

It is easy to derive a similar estimate
\begin{equation}
\left\| \int_t^\infty e^{i(t-s)\Delta} F(s) ds \right\|_{Y'} \leq C\|F\|_Y.
\end{equation}
by the aid of (2.15) and a duality argument for the quadratic form
\[ Q(F, G) = \int \int_{t > s} \langle e^{i(t-s)\Delta} F(s), G(t) \rangle_{L^2(\mathbb{R}^n)} ds dt. \]

Further, we have to derive the estimate
\[ (2.17) \quad \left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L^\infty_t L^2_x} \leq C\| F \|_Y. \]

For the purpose set \( u(t) = \int_0^t e^{i(t-s)\Delta} F(s) ds. \) Then \( u = u(t, x) \) is a solution to
\[ (2.18) \quad \begin{cases} \partial_t u - i\Delta u = F, & t > 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = 0. \end{cases} \]

Multiplying by \( u \) integration over \( \{0 \leq t \leq T, x \in \mathbb{R}^n\} \) we get
\[ \|u(T)\|_{L^2(\mathbb{R}^n)}^2 \leq \int_0^T \langle F(t), u(t) \rangle_{L^2(\mathbb{R}^n)} dt \leq \| F \|_Y \| u \|_{Y'} \cdot \]

Applying (2.15), we arrive at (2.17). In a similar way we get
\[ (2.19) \quad \left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{Y'} \leq C\| F \|_{L^1_t L^2_x}. \]

Finally, it remains to prove
\[ (2.20) \quad \| e^{it\Delta} f \|_{Y'} \leq C\| f \|_{L^2_x}. \]

Consider the operator \( L \) defined by
\[ L : f \in L^2_x \implies e^{it\Delta} f. \]

Our goal is to show that \( L \) is bounded from \( L^2_x \) to \( L^2_t L^\infty_x L^{1/2}_x \). But the continuity of \( L \) from \( L^2_x \) to \( L^2_t L^\infty_x L^{1/2}_x \) follows from the continuity of its (formally) adjoint
\[ L^*f = \int_0^\infty e^{-it\Delta} f(\tau) d\tau, \]
from \( Y \) to \( L^2_x \), which in turns follows from (2.17) and the fact that \( e^{it\Delta} \) is unitary operator in \( L^2_x \).

From (2.13), (2.17), (2.19), (2.20) and standard energy estimate, we get (1.13) and the proof of Theorem 1.2 is completed.

3. Proof of Theorem 1.1

In this section we will prove the Theorem 1.1. so we shall prove the estimate \( (1.7) \), where \( u \) is the solution of the problem \( (1.1) \). First of all we have the identities
\[ \Delta_A u = \sum_{j=1}^n (\partial_{x_j} - iA_j)(\partial_{x_j} - iA_j)u \]
\[ = \Delta u - 2i\nabla \cdot (Au) + Wu, \]
where
\[ W(t, x) = |A(t, x)|^2 - i\nabla \cdot A \]
satisfies
\[ \sum_{k \in \mathbb{Z}} 2^{2k} \| W(t, x) \|_{L_t^\infty L_x^{(|x| \sim 2^k)}} \leq \varepsilon \]
due to (3.6).

So, after a substitution in the equation of (1.1), we obtain
\[
\begin{cases}
  i \partial_t u - \Delta u = -2i \nabla \cdot (Au) + W u + F, & t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\
u(0, x) = f(x).
\end{cases}
\tag{3.2}
\]

First of all, we observe that the term \( W u \), thanks to the smallness assumption (1.1), can be absorbed in the left side of the estimate (1.7). This fact suggests to localize (1.7) and obtain
\[
\begin{cases}
  \partial_t u - i \Delta u = -2i \nabla \cdot (Au) + F, & t \in \mathbb{R}, \quad x \in \mathbb{R}^n \\
u(0, x) = f(x).
\end{cases}
\tag{3.3}
\]

So, using the norms in the spaces \( \ell^p_x \) introduced in (2.2) we apply the estimate (1.7) and obtain
\[
\| u \|_{L_t^2 \ell_x^{\infty, -1/2} H^{1/2}_x} \leq C \| \nabla \cdot (Au) \|_{L_t^2 \ell_x^{1,1/2} H^{-1/2}_x} + C \| F \|_{L_t^2 \ell_x^{1,1/2} H^{-1/2}_x} + C \| f \|_{L^2_x}.
\tag{3.4}
\]

From the equivalent norm estimates in Proposition 7.1 (see the equivalence norms relations in (2.12), (2.13) also) we have
\[
\| \nabla \cdot (Au) \|_{L_t^2 \ell_x^{1,1/2} H^{-1/2}_x} \sim \| Au \|_{L_t^2 \ell_x^{1,1/2} H^{1/2}_x}.
\tag{3.5}
\]

From Proposition 7.2 we have
\[
\| Au \|_{L_t^2 \ell_x^{1,1/2} H^{1/2}_x} \lesssim \| A \|_{L_t^\infty \ell_x^{1,1/2} H^{2n/(n-1)}_x} \| u \|_{L_t^2 \ell_x^{\infty, -1/2} L_x^{2n/(n-1)}} + \| A \|_{L_t^\infty \ell_x^{1,1/2} L_x^{2n/(n-1)}} \| u \|_{L_t^2 \ell_x^{\infty, -1/2} H^{1/2}_x}.
\]

From the Sobolev embedding \( H^{1/2}_x \subset L_x^{2n/(n-1)} \), we obtain
\[
\| u \|_{L_t^2 \ell_x^{\infty, -1/2} L_x^{2n/(n-1)}} \lesssim \| u \|_{L_t^2 \ell_x^{\infty, -1/2} H_x^{1/2}}
\]
while the interpolation inequality of Proposition 7.3 guarantees that
\[
\| A \|_{L_t^\infty \ell_x^{1,1/2} H_x^{2n}} \lesssim \| \nabla A \|_{L_t^\infty \ell_x^{3/2} L_x^{2n}} \| A \|_{L_t^\infty \ell_x^{1,1/2} L_x^{2n}}
\]
so applying the Hőlder inequality
\[
\| g \|_{L^p_x} \lesssim \| g \|_{L_x^{1, n+n/p} L^{\infty}},
\]
we get
\[
\| A \|_{L_t^\infty \ell_x^{1,1/2} H_x^{2n}} \lesssim \| \nabla A \|_{L_t^\infty \ell_x^{1,2} L_x^{\infty}} \| A \|_{L_t^\infty \ell_x^{1,1} L_x^{\infty}} \lesssim \varepsilon^2
\]
due to assumption on \( A \). The above observation implies
\[
\| Au \|_{L_t^2 \ell_x^{1,1/2} H^{1/2}_x} \lesssim \varepsilon \| u \|_{L_t^2 \ell_x^{\infty, -1/2} H^{1/2}_x}.
\]

Using again the estimates (3.4), we obtain
\[
\sum_{k \in \mathbb{Z}} \| x_k^{1/2} u(t, \cdot) \|_{L_t^2 \ell_x^{1/2}} \leq C \| F \|_{L_t^2 \ell_x^{1,1/2} H^{-1/2}_x} + C \| f \|_{L^2_x}.
\tag{3.6}
\]
This concludes the proof of the theorem.
4. Application to the semilinear Schrödinger equation

Turning to the semilinear Schrödinger equation

\[ \partial_t u - i \Delta A u = |Vu|^p, \]

we note that the class of potentials \( V = V(t, x) \), satisfying (1.15), obeys certain rescaling property, thus one can compute the scaling critical regularity

\[ s = \frac{n}{2} - \frac{2 - ap}{p - 1}, \]

and one can expect a well posedness for initial data \( f \in L^2 \) if

\[ p = \frac{n + 4}{n + 2a}. \]

To verify this we shall construct a sequence \( u_k(t, x) \) of functions defined as follows:

\[ u_{-1}(t, x) = 0, \]

then we define the recurrence relation

\[ u_k \to u_{k+1}(t, x) \]

so that

\[ \begin{cases} 
\partial_t u_{k+1} - i \Delta A u_{k+1} = V(t, x) u_k |u_k|^{p-1}, & t \in \mathbb{R}, \ x \in \mathbb{R}^n \\
u_{k+1}(0, x) = f(x). 
\end{cases} \]

The estimate (1.13) suggests to show the convergence of the sequence \( u_k \) in the Banach space

\[ Z = L_t^\infty L_x^2 \cap Y'. \]

The definition of the recurrence relation (4.2) shows that we have to show first the property: the map

\[ u \in Z = L_t^\infty L_x^2 \cap Y' \to V(t, x) u |u|^{p-1} \in L_t^1 L_x^2 + Y \]

is a well defined continuous operator provided \( V \) satisfies (1.15). Our goal is to show

\[ \|u_{k+1}\|_{L_t^\infty L_x^2} + \|u_{k+1}\|_{Y'} \leq C \|f\|_{L^2} + C (\|u_k\|_{L_t^\infty L_x^2} + \|u_k\|_{Y'}^p) \]

or shortly

\[ \|u_{k+1}\|_Z \leq C \|f\|_{L^2} + C \|u_k\|_Z^p. \]

To apply a contraction argument we need also the inequality

\[ \|u_{k+1} - u_k\|_Z \leq C \|u_k - u_{k+1}\|_Z (\|u_k\|_Z + \|u_{k-1}\|_Z)^{p-1}. \]

Combining (4.3) and (4.4), taking \( \|f\|_{L^2} \) sufficiently small, we can show via contraction argument that \( u_k \) converges in \( Z \) to the unique solution of (4.1) with initial data \( u(0) = f \). Since the proofs of (4.3) and (4.4) are similar, we treat (4.3) only. We need actually to verify

\[ \| |Vu|^p \|_Y \leq C \|u\|^p_Z \]

To verify this inequality, we start with the definition of the space \( Y \)

\[ \|g\|_Y \sim \sum_m 2^{m/2} \|D^{1/2} \varphi \left( \frac{\cdot}{2^m} \right) g\|_{L_t^2 L_x^2}. \]
We apply this relation with $g = |Vu|^p$, combined with the Sobolev embedding $H^{1/2}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, with
\begin{equation}
1/q - 1/2 = 1/(2n),
\end{equation}
and get
\begin{equation}
|||Vu|^p||_Y \leq \sum_m 2^{m/2} ||\varphi\left(\frac{\cdot}{2m}\right) Vu||^p_{L^q_{r_1}L^p_{r_2}}.
\end{equation}
We can apply now the interpolation inequality
\begin{equation}
\|\phi\|_{L^q_{r_1}L^p_{r_2}} \leq C \left(\|\phi\|_{L^\infty}^{\theta} \left(\|\phi\|_{L^2}^{1-\theta}\right)\right),
\end{equation}
where $\theta \in (0, 1)$ satisfy the relations
\begin{equation}
\frac{1}{2p} = 1 - \frac{\theta}{2}, \quad \frac{1}{pq} = \frac{\theta}{r_1} + \frac{1 - \theta}{r_2}.
\end{equation}
Hence $p(1 - \theta) = 1$.

From (4.10) we find
\begin{equation}
\frac{1}{pq} = \frac{\theta}{r_1} + \frac{1 - \theta}{r_2},
\end{equation}
so this relation and (4.7) implies that
\begin{equation}
p = \frac{n + 4}{n + 2a}.
\end{equation}
From (4.8) and (4.9) we get
\begin{equation}
\|Vu|^p\|_Y \leq \|u\|_{L^p_{r_1}L^q_{r_2}} \|x|^{-1} u\|_{L^2_{r_1}L^2_{r_2}},
\end{equation}
so
\begin{equation}
\|Vu|^p\|_Y \leq \|u\|_{L^p_{r_1}L^q_{r_2}} \|u\|_Y.
\end{equation}
and this completes the proof of (4.5).
This completes the proof of Theorem 1.3.

5. ESTIMATE OF THE OPERATOR $Q_k |D|^{-s} Q_m |D|^s$.

Our goal is to compare the norms
\[
\| |D|^{-s} f \|_{L^q_p} = \left( \sum_{k \in \mathbb{Z}} 2^{kqa} \| Q_k |D|^{-s} f \|_{L^q_p}^q \right)^{1/q}
\]
and
\[
\| f \|_{L^q_p} = \left( \sum_{k \in \mathbb{Z}} 2^{kqa} \| |D|^{-s} Q_k f \|_{L^q_p}^q \right)^{1/q}
\]
(see (2.2) or Section 2 for the definition of the spaces $L^{q, \alpha}_x B$, where $B$ is any Banach space such that $B \subset D'(\mathbb{R}^n)$). The key point in the proof that these norms are equivalent is the following estimate of the operator

(5.1) $Q_k |D|^{-s} Q_m |D|^s$ for $|k - m| > 2$.

Lemma 5.1. For any $s \in \mathbb{R}, |s| < 1$, any $p, 1 < p < n$ and any $k, m \in \mathbb{Z}, |k - m| \geq 3$ we have the estimate

(5.2) $\| Q_k |D|^{-s} Q_m |D|^s f \|_{L^p} \leq C \ 2^{t(k, m, s, p)} \| f \|_{L^p},$

where $C = C(s, p)$ independent of $k, m \in \mathbb{Z}$, and

(5.3) $t(k, m, s, p) = \frac{n}{p} + \frac{m}{p'} - \frac{n - (s \lor 0) (k \lor m) - (s \lor 0) (k \land m)}{n}.$

$\frac{1}{p'} = 1 - \frac{1}{p}, \ k \land m = \min(k, m), \ k \lor m = \max(k, m)$.

Proof: We shall prove the Lemma for $s \in \mathbb{C}$ with $\Re s \in [0, 1]$. For the purpose consider the family of operators

(5.4) $T^{i \sigma} = e^{-\sigma^2 Q_k |D|^{-s} Q_m |D|^s}.$

If $\Re z = 0$ then $z = i \sigma, \sigma \in \mathbb{R}$ and

(5.5) $T^{i \sigma} = e^{-\sigma^2 Q_k |D|^{-s} Q_m |D|^s}.$

Applying stationary phase method (in this case simply integration by parts), we see that the operator $Q_k |D|^{-s} Q_m$ has a kernel

$K_{k, m, \sigma}(x, y)$

satisfying

(5.6) $|K_{k, m, \sigma}(x, y)| \leq C Q_k(x) Q_m(y) (1 + \sigma)^{n+1}.$

This estimate implies

(5.7) $\| Q_k |D|^{-s} Q_m g \|_{L^p} \leq C 2^{k\frac{n}{p} + m\frac{n}{p'}} \| g \|_{L^p}.$
Further we apply this inequality with \( g = |D|^\sigma f \) and using the following one (see Theorem 1, Section 2.2 in [30])

\[
\|D|^\sigma f\|_{L^p} \leq C\|f\|_{L^p} (1 + \sigma)^{n+1},
\]

we get

\[
\|T^\sigma(f)\|_{L^p} \leq C \frac{k^{\frac{n}{2}} 2^{m+n_p}}{2^{m(k\vee m)}} e^{-\sigma^2} (1 + \sigma)^{2(n+1)} \|f\|_{L^p} \leq C_1 \frac{k^{\frac{n}{2}} 2^{m+n_p}}{2^{m(k\vee m)}} \|f\|_{L^p}
\]

\( \forall \sigma \in \mathbb{R} \) with \( C_1 \) independent of \( k, m \) and \( \sigma \). If \( z = 1 + i\sigma \), then

\[
T^{1+i\sigma} = e^{1-\sigma^2+2i\sigma} Q_k|D|^{-1-i\sigma} Q_m \nabla |D|^\sigma \nabla |D|
\]

so it is sufficient to estimate the operator

\[
S^\sigma = Q_k|D|^{-1-i\sigma} Q_m \nabla.
\]

Note that

\[
S^\sigma = Q_k(|D|^{-1-i\sigma} \nabla)Q_m - Q_k|D|^{-1-i\sigma} Q'_m,
\]

where \( Q'_m = \nabla Q_m \). The operator \( Q_k(|D|^{-1-i\sigma} \nabla)Q_m \) has kernel \( K'_{k,m} \) satisfying

\[
|K'_{k,m}| \leq C \frac{Q_k(x)Q_m(y)}{2^{m(k\vee m)}} (1 + \sigma)^n+1
\]

and this estimate is verified in the same way as (5.8). The operator \( Q_k|D|^{-1-i\sigma} Q'_m \) has kernel \( K''_{k,m} \) that satisfies the estimate

\[
|K''_{k,m}| \leq C \frac{Q_k(x)Q_m(y)}{2^{m(k\vee m)}} (1 + \sigma)^n+1.
\]

From (5.12) and (5.13) together with (5.8) we find

\[
\|T^{1+i\sigma}(f)\|_{L^p} \leq C \frac{2^{k^{\frac{n}{2}} 2^{m+n_p}}}{2^{n(1-1)(k\vee m)+2m}} \|f\|_{L^p}.
\]

Applying the complex interpolation argument of Stein (see [29]), we get (5.2) for \( 0 < s < 1 \). This complete the proof for \( s \in (0, 1) \).

Next we take \( z = -1 + i\sigma \). Then

\[
T^{1+i\sigma} = e^{1-\sigma^2-2i\sigma} Q_k|D|^\sigma \nabla |D| Q_m \nabla |D|^{-1+i\sigma}.
\]

Then we use the relation

\[
Q_k|D|^\sigma \nabla |D| Q_m \nabla |D|^{-1+i\sigma} = Q_k|D|^\sigma \nabla |D| Q_m \nabla |D|^{-1+i\sigma} + Q_k|D|^\sigma \nabla |D| Q_m \nabla |D|^{-1+i\sigma},
\]

\( \forall \sigma \in \mathbb{R} \) with \( C_1 \) independent of \( k, m \) and \( \sigma \). If \( z = 1 + i\sigma \), then

\[
|K''_{k,m}| \leq C \frac{Q_k(x)Q_m(y)}{2^{m(k\vee m)}} (1 + \sigma)^n+1.
\]
then we obtain

\[ \|Q_k|D|^\sigma \frac{\nabla}{|D|} (\nabla Q_m)|D|^{-1+i\sigma} g\|_{L^p} \leq C \frac{2^{k_m + m^\sigma}}{2n(k^m \nu m)} \|g\|_{L^r}. \]

Taking \( g = |D|^{-1} f \), we get (Hardy-Sobolev)

\[ \|\|D|^{-1} f\|_{L^p} \leq \|f\|_{L^r}, \quad \frac{1}{p} - \frac{1}{r} = \frac{1}{n}. \]

From the fact that \( \frac{1}{r} = \frac{1}{p} - \frac{1}{n} \) we have \( 1 - \frac{1}{r} = \frac{1}{p} - 1 = \frac{1}{p} + \frac{1}{n} \) we arrive at

\[ \|Q_k|D|^\sigma \frac{\nabla}{|D|} (\nabla Q_m)|D|^{-1+i\sigma} f\|_{L^p} \leq C \frac{2^{k_m + m^\sigma}}{2n(k^m \nu m)} \|f\|_{L^p}, \]

provided \( p < n \). Since

\[ \|Q_k|D|^\sigma \frac{\nabla}{|D|} Q_m|D|^{-1+i\sigma} f\|_{L^p} \leq C \frac{2^{k_m + m^\sigma}}{2n(k^m \nu m)} \|f\|_{L^p}, \]

from (5.18) and (5.19) we get

\[ \|T^{1+i\sigma}(f)\|_{L^p} \leq C \frac{2^{k_m + m^\sigma}}{2n(k^m \nu m)} \|f\|_{L^p}. \]

The application of the Stein interpolation argument for \( z \); Re \( z \in [-1, 0] \) combined with the above estimate and (5.3) guarantees that (5.2) is fulfilled for \( s \in (-1, 0) \) and this complete the proof of the Lemma.

It is not difficult to extend the result of Lemma 5.1 for \( |k - m| \leq 3 \). Note that a formal calculus of \( t(k, m, s, p) \) for \( |k - m| \leq 3 \) in (5.3) gives \( 2^{4(k, m, s, p)} \sim 1 \). To verify

\[ \|Q_k|D|^{-s} Q_m|D|^s f\|_{L^p} \leq C \|f\|_{L^p}, \]

for \( |s| < 1, 1 < p < n \), it is sufficient to use a scale argument and to show (5.22) for \( k = m = 0 \) so we shall verify the inequality

\[ \|Q_0|D|^{-s} Q_0|D|^s f\|_{L^p} \leq C \|f\|_{L^p}. \]

Here we can use an interpolation argument as in the proof of Lemma 5.1 Then we have to show that \( L_\sigma = Q_0|D|^{-1+i\sigma} Q_0\nabla \) is \( L^p \)-bounded. But

\[ L_\sigma = Q_0(|D|^{-1+i\sigma} \nabla)Q_0 + Q_0|D|^i\sigma |D|^{-1}(\nabla Q_0). \]

Since \( |D|^{-1}\nabla \) is operator of order 0 it is \( L^p \)-bounded and \( |D|^i\sigma \) is also \( L^p \)-bounded, we see that \( Q_0(|D|^{-1+i\sigma} \nabla)Q_0 \) is \( L^p \)-bounded. From the property

\[ |D|^{-1} : L^r \to L^p, \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{n}, \]

and

\[ \nabla Q_0 : L^p \to L^r, \]

we see that \( |D|^{-1}(\nabla Q_0) : L^p \to L^p \) so \( L_\sigma \) is \( L^p \)-bounded. This observation and a Stein interpolation argument implies (5.22) for \( s \in (0, 1) \). To cover the case \( s \in (-1, 0) \) we have to show that

\[ L_\sigma' = Q_0|D|^{-s} \frac{\nabla}{|D|} \nabla Q_0|D|^{-1+i\sigma}, \]

is \( L^p \)-bounded.
Lemma 5.2. For any \( q \in \mathbb{R}, |s| < 1 \) any \( p, 1 < p < n \) there exists a constant \( C = C(s,p,n) > 0 \) so that for any \( k, m \in \mathbb{Z} \), and for \( f \in S(\mathbb{R}^n) \) we have
\[
\|Q_k|D|^{-s}Q_m|D|^s f\|_{L^p} \leq C 2^{t(k,m,s,p)} \|f\|_{L^p},
\]
where \( t(k,m,s,p) \) is defined in (5.3).

Finally we use a duality argument and find:

Lemma 5.3. For any \( s \in \mathbb{R} \) such that \(|s| < 1\) we have for any \( p, (\frac{n}{n-1} < p < n) \) there exists a constant \( C = C(s,p,n) > 0 \) so that for \( f \in S(\mathbb{R}^n) \) we have
\[
\|Q_k|D|^{-s}Q_m|D|^s f\|_{L^p} \leq C 2^{t(k,m,s,p)} \|f\|_{L^p},
\]
where \( t(k,m,s,p) \) is defined in (5.3).

**Proof:** For any \( f, g \in S(\mathbb{R}^n) \) we have
\[
|\langle g, |D|^{-s}Q_k|D|^s Q_m f \rangle| = |\langle Q_m|D|^{-s}Q_k|D|^s g, f \rangle| \leq \|f\|_{L^p} \|Q_m|D|^{-s}Q_k|D|^s g\|_{L^{p'}}
\]
Applying for \( p' \) the estimate of (5.2), we find
\[
\|Q_m|D|^{-s}Q_k|D|^s Q_m g\|_{L^{p'}} \leq 2^{t(m,k,s,p')} \|g\|_{L^p},
\]
where
\[
t(m,k,s,p') = k \frac{n}{p} + m \frac{n}{p'} - (n - (s \lor 0))(k \lor m) - (s \lor 0)(k \land m) = t(k,m,s,p).
\]
This complete the proof. \( \square \)

6. Discrete Estimates.

Consider the operator
\[
T : a = \{a_k\}_{k \in \mathbb{Z}} \mapsto Ta = b_m = \sum_{k \in \{k-m \geq 4\}} t_{k,m} a_k,
\]
where
\[
t_{k,m} = 2^{m\lambda} 2^{\mu k} 2^{-\beta(m \lor k)}, \quad m \lor k = \max(m,k),
\]
\[
\lambda > 0, \mu > 0, \quad \beta = \lambda + \mu.
\]

**Lemma 6.1.** If \( \lambda, \mu > 0 \), and \( \beta = \lambda + \mu \), then the operator
\[
T : \ell^q \to \ell^q
\]
is bounded for any \( q \in [1, \infty] \).
Proof: First we consider the cases \( q = \infty \) and \( q = 1 \), then we apply the interpolation argument. We represent \( Ta \) as

\[
Ta = T_1a + T_2a,
\]

where

\[
(T_1a)_m = \sum_{k=m+1}^{\infty} t_{k,m}a_k,
\]

\[
(T_2a)_m = \sum_{k=-\infty}^{m} t_{k,m}a_k.
\]

From (6.2) we find for \( T_1a \)

\[
\|T_1a\|_{\infty} \leq C \sup_{m \in \mathbb{Z}} \left( \sum_{k=m+1}^{\infty} t_{k,m} \right) \|a\|_{\infty} \leq \left( \sum_{k=m+1}^{\infty} 2^m \lambda^2 k^\mu \right) \|a\|_{\infty},
\]

so

\[
\|T_1a\|_{\infty} \leq C \|a\|_{\infty}.
\]

From (6.2), for \( T_2a \) we have the following estimate

\[
\|T_2a\|_{\infty} \leq C \sup_{m \in \mathbb{Z}} \left( \sum_{k=-\infty}^{m} 2^m \lambda^2 k^\mu \right) \|a\|_{\infty} = C \sup_{m \in \mathbb{Z}} \left( \sum_{k=-\infty}^{m} 2^m \lambda^2 k^\mu \right) \|a\|_{\infty} = C \|a\|_{\infty}.
\]

This estimate and (6.8) imply

\[
\|T_a\|_{\infty} \leq C \|a\|_{\infty}.
\]

For \( q = 1 \) we have

\[
\|T_1a\|_1 \leq C \sup_{k \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}, m \geq k} t_{k,m} \right) \|a\|_1 \leq \sum_{m=k}^{\infty} 2^m \lambda^2 k^\mu \|a\|_1 = C \sup_{k \in \mathbb{Z}} 2^k \left( \sum_{m=k}^{\infty} 2^{-m\mu} \right) \|a\|_1 \leq 2C \|a\|_1.
\]
In a similar way we estimate $T_2a$,
\[
\|T_2a\|_1 \leq C\sup_{k \in \mathbb{Z}} \left( \sum_{m=-\infty}^{k-1} 2^{m\lambda}2^{k\lambda}2^{-\beta k} \right) \|a\|_1 = \\
C\sup_{k \in \mathbb{Z}} \left( \sum_{m=-\infty}^{k-1} 2^{m\lambda}2^{-\beta} \|a\|_1 \right) \leq C\|a\|_1.
\] (6.12)
Thus we get
\[
\|Ta\|_1 \leq C\|a\|_1,
\] (6.13)
and this completes the proof of the Lemma.

It is easy to obtain the corresponding weighted version of Lemma 6.1 in terms of weighted $\ell^q$ spaces
\[
\ell^{q,\alpha} = \{a = (a_k)_{k \in \mathbb{Z}}; \sum_k 2^{kq\alpha} |a_k|^q < \infty\}. 
\] (6.14)
For the purpose consider the operator
\[
J^\alpha : a \rightarrow b = J^\alpha a, 
\]
defined as follows
\[
b_k = 2^{k\alpha}a_k, 
\] (6.15)
we have the two following Lemmas.

**Lemma 6.2.** The application $J^\alpha : \ell^q \rightarrow \ell^{q,\alpha}$ is an isomorphism for any $\alpha \in \mathbb{R}$ and any $q \in [1, \infty]$.

**Lemma 6.3.** If $\sigma, \nu, \lambda, \mu$ are real numbers such that
\[
\left\{ \begin{array}{l}
\lambda + \sigma > 0, \\
\mu - \nu > 0,
\end{array} \right.
\] (6.16)
then for $\beta = \lambda + \sigma + \mu - \nu$ we have
\[
T : \ell^{q,\sigma} \rightarrow \ell^{q,\nu}
\]
where $T$ is defined by (6.1) and (6.2).

**Proof:** Let
\[
\tilde{T} : J_\sigma T J_\nu^{-1}.
\]
Then Lemma 6.2 guarantees that $T : \ell^{q,\sigma} \rightarrow \ell^{q,\nu}$ if and only if $\tilde{T} : \ell^q \rightarrow \ell^q$. Note that $\tilde{T}$ by
\[
t_{m,k} = 2^{m(\lambda + \sigma)2^{k(\mu - \nu)}2^{-\beta(m\vee k)}}. 
\] (6.17)
So applying Lemma 6.1 with $\tilde{\lambda} = \lambda + \sigma$ and $\tilde{\mu} = \mu - \nu$, we complete the proof.

A slight generalization of Lemma 6.1 can be obtained for the case when $\lambda, \mu, \beta$ are vectors in $\mathbb{R}^2$, that is
\[
\left\{ \begin{array}{l}
\lambda = (\lambda_1, \lambda_2) \\
\mu = (\mu_1, \mu_2) \\
\beta = (\beta_1, \beta_2).
\end{array} \right.
\]
Then (6.18)
\[
\begin{align*}
T a &= b, \quad \text{where} \quad b_m = \sum_{k \in \mathbb{Z}^2} t_{mk} a_k, \quad m \in \mathbb{Z}^2, \\
a &= (a)_k. 
\end{align*}
\]
where
\[
t_{m,k} = 2\sum_{j=-1}^{\infty} m_j \lambda_j + k_j \mu_j - \beta_j(m_j \vee k_j).
\]
The assumption (6.20) can be replaced again by the following one
\[
\lambda_j > 0, \mu_j > 0, \quad j = 1, 2.
\]
**Lemma 6.4.** If \(\lambda, \mu, \beta \in \mathbb{R}^2\) satisfy \(\beta_j = \lambda_j + \mu_j, i = 1, 2\) and (6.17) then
\[
T : \ell^{q_1}_{k_1} \ell^{q_2}_{k_2} \to \ell^{q_1}_{k_1} \ell^{q_2}_{k_2},
\]
is bounded for \(q = (q_1, q_2), 1 \leq q_j \leq \infty\).

**Remark 6.1.** Given any sequence \(a = \{a_{k_1 k_2}\}_{k_1, k_2} \subset \mathbb{Z}\) we can consider the norm
\[
\|a\|_{\ell^{q_1}_{k_1} \ell^{q_2}_{k_2}} = \left( \sum_{k_1, k_2} \left( \sum_{k_2} |a_{k_1 k_2}|^{q_2} \right)^{q_1/q_2} \right)^{1/q_1},
\]
( with obvious modification if \(q_1 = \infty\) or \(q_2 = \infty\) ), and the corresponding Banach space \(\ell^{q_1}_{k_1} \ell^{q_2}_{k_2}\). Note that
\[
\ell^{q_1}_{k_1} \ell^{q_2}_{k_2} \neq \ell^{q_2}_{k_2} \ell^{q_1}_{k_1},
\]
but the assertion of Lemma 6.4 is still true if we replace \(\ell^{q_1}_{k_1} \ell^{q_2}_{k_2}\) by \(\ell^{q_2}_{k_2} \ell^{q_1}_{k_1}\). The corresponding generalization of Lemma 6.4 is the following,

**Lemma 6.5.** If \(\sigma, \nu, \lambda, \mu \in \mathbb{R}^2\) satisfy
\[
\left\{
\begin{align*}
\lambda_j + \sigma_j > 0, \\
\mu_j - \nu_j > 0,
\end{align*}
\right.
\]
then for \(\beta_j = \lambda_j + \sigma_j + \mu_j - \nu_j, j = 1, 2\) the operator \(T\) defined by (6.15) and (6.18) is in \(B(\ell^{q_1}_{k_1} \ell^{q_2}_{k_2}, \ell^\infty_{k_1} \ell^\infty_{k_2})\).

7. **Space localization.**

Given any Banach space \(B \subset D'(\mathbb{R}^n)\) satisfying the property
\[
(7.1) \quad \text{for any } Q(x) \in C_0^\infty(\mathbb{R}^n), \ f \in B \Rightarrow Q(x) f \in B,
\]
we can define for any \(p \in [1, \infty]\) and for any \(a \in \mathbb{R}\) the space \(\ell^{\alpha, a}_x B\) as follows
\[
(7.2) \quad \|f\|_{\ell^{\alpha, a}_x B} = \left( \sum_{k \in \mathbb{Z}} 2^{k \alpha a} \|Q_k f\|_B^q \right)^{1/q},
\]
with obvious modification for \(q = \infty\). Note that for any \(f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})\) we have
\[
\|f\|_{\ell^{\alpha, a}_x B} < \infty.
\]
So \(\ell^{\alpha, a}_x B\) can be defined as the closure of \(C_0^\infty(\mathbb{R}^n \setminus \{0\})\) with respect to the norm (7.2). An alternative definition is based on the map
\[
(7.3) \quad J : f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset B \rightarrow J_B(f)_k = \|Q_k f\|_B \in \ell^{\alpha, a},
\]
where $\ell^{q,\alpha}$ is the space of all sequences $a = (a_k)_{k \in \mathbb{Z}}$ such that

$$\|a\|_{\ell^{q,\alpha}} = \left( \sum_{k \in \mathbb{Z}} |a_k|^q 2^{kq\alpha} \right)^{1/q},$$

with obvious modification for $p = \infty$. Then

$$\|f\|_{\ell^{q,\alpha}B} = \|J_B(f)\|_{\ell^{q,\alpha}}.$$

The space $\ell^{q,\alpha}B$ is independent of the concrete choice of Paley-Littlewood decomposition

$$\sum_{j \in \mathbb{Z}} Q_j(x) = 1$$

satisfying

$$\begin{cases}
Q_j(x) \geq 0, \\
\text{supp } Q_j(x) \in \{|x| \sim 2^j\}.
\end{cases}$$

A typical example is the case $B = \dot{H}^s_x$ where $s \in (-1, 1), 1 < p < \infty$. For $s > -\frac{n}{p}$ we have $\dot{H}^s_x(\mathbb{R}^n) \subset D'$ (see [11]) and the norm is defined by

$$\|f\|_{\dot{H}^s_x} = \|D^s f\|_{L^p}.$$

Our next goal is to show the equivalence of the norms

$$\|D^{-s} f\|_{\ell^{q,\alpha}_L^p} = \left( \sum_{k \in \mathbb{Z}} 2^{kq} \|Q_k D^{-s} f\|_{L^p}^q \right)^{1/q}$$

and

$$\|f\|_{\ell^{q,\alpha}_{\dot{H}^s_x}^p} = \left( \sum_{k \in \mathbb{Z}} 2^{kq} \|D^{-s} Q_k f\|_{L^p}^q \right)^{1/q}.$$}

The proof of the equivalence norm Theorem 1.4 is a direct consequence (taking $p = 2$) of the following estimates.

**Proposition 7.1.** For $p \in (n/(n-1), n), q \in [1, \infty]$, for $s \in [-1, 1]$ and $\alpha \in \mathbb{R}$ that satisfy

$$|a| + |s| < \min \left( \frac{n}{p}, \frac{n}{p'} \right)$$

one can find a constant $C = C(n, s, p, q, \alpha) > 0$ so that

$$C^{-1} \|D^{-s} f\|_{\ell^{q,\alpha}_L^p} \leq \|f\|_{\ell^{q,\alpha}_{\dot{H}^s_x}^p} \leq C \|D^{-s} f\|_{\ell^{q,\alpha}_L^p}.$$

**Proof:** The left inequality in (7.10) is equivalent to

$$\left( \sum_{k \in \mathbb{Z}} 2^{kq} \|Q_k D^{-s} f\|_{L^p}^q \right)^{1/q} \leq C \left( \sum_{k \in \mathbb{Z}} 2^{kq} \|D^{-s} Q_k f\|_{L^p}^q \right)^{1/q}.$$

Indeed, given any integers $k, m \in \mathbb{Z}$ with $|k - m| > 2$ we have the identity

$$Q_k |D|^{-s} Q_m f = Q_k |D|^{-s} Q_m |D|^{-s} \tilde{Q}_m f.$$
where \( \tilde{Q}_m = \frac{1}{4}(Q_{m-1} + Q_m + Q_{m+1}) \) is another Paley-Littlewood partition of unity such that \( \tilde{Q}_m(s) = 1 \) for \( s \in \text{supp } Q_m \). To verify it is sufficient to show that

\[
(7.13) \quad \left( \sum_{k \in \mathbb{Z}} 2^{kqa} \|Q_k|D|^{-s}f\|_{L^p}^q \right)^{1/q} \leq C \left( \sum_{m \in \mathbb{Z}} 2^{mqa} \|D|^{-s}\tilde{Q}_m f\|_{L^p}^q \right)^{1/q}.
\]

From the estimate of Lemma 5.1 we have

\[
(7.14) \quad \|Q_k|D|^{-s}Q_m |D|^s f\|_{L^p} \leq C 2^{t(k,m,s,p)} \|f\|_{L^p},
\]

where \( t(k,m,s,p) \) is defined in 5.3 Applying the above estimate with \( g = |D|^{-s}\tilde{Q}_m f \) together with Lemma 6.3 we complete the proof of (7.13).

To verify the right inequality in (7.11) it sufficient to show

\[
(7.15) \quad \left( \sum_{k \in \mathbb{Z}} 2^{kqa} \|D|^{-s}Q_k f\|_{L^p}^q \right)^{1/q} \leq C \left( \sum_{m \in \mathbb{Z}} 2^{mqa} \|Q_k|D|^{-s}\tilde{Q}_m f\|_{L^p}^q \right)^{1/q}.
\]

To this end we use the relation

\[
|D|^{-s}Q_k f = |D|^{-s}Q_k|D|^s f = \sum_{m \in \mathbb{Z}} |D|^{-s}Q_k D|s Q_m|D|^{-s}.
\]

From Lemma 5.3 we have

\[
(7.16) \quad \|Q_k|D|^{-s}Q_m |D|^s f\|_{L^p} \leq C 2^{t(k,m,s,p)} \|f\|_{L^p},
\]

where \( t(k,m,s,p) \) is defined in 5.3 so applying Lemma 6.3 we obtain (7.15) and complete the proof of the Proposition.

\[\square\]

**Proposition 7.2.** For \( p \in (n/(n-1), n), q \in [1, \infty], \) \( s \in [0, 1] \) and \( a > 0 \) that satisfy one can find a constant \( C = C(n, s, p, q, a) > 0 \) so that

\[
(7.17) \quad \|fg\|_{L^p} \leq C \left( \|f\|_{L^{a_1}_{p_1}} \|g\|_{L^{a_2}_{p_2}} \|f\|_{L^{a_3}_{p_3}} \|g\|_{L^{a_4}_{p_4}} \right),
\]

provided \( a_1, a_2, a_3, a_4 \geq 0 \) and \( 1 \leq p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4 \leq \infty \) satisfy

\[
a_1 + a_2 = a_3 + a_4 = a, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}.
\]

**Proof:** The proof uses the previous Proposition and the standard multiplicative Sobolev inequality

\[
\|fg\|_{H^s_p} \leq C \left( \|f\|_{H^s_{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{H^s_{p_4}} \right),
\]

so we omit the details.

\[\square\]

Using the interpolation property

\[
(H^s_{p_1}, H^s_{p_2})_\theta = H^s_p, \quad s = (1-\theta)s_1 + \theta s_2, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{\theta}{p_2},
\]

we arrive at
Proposition 7.3. For \( p \in (n/(n-1), n), q \in [1, \infty], \) for \( s \in (0, 1) \) and \( a > 0 \) that satisfy (7.9) one can find a constant \( C = C(n, s, p, q, a) > 0 \) so that

\[
\| f \|_{L^q,0} \leq C \left( \| f \|_{L^{q_1},a} H^s \right)^{1-\theta} \left( \| f \|_{L^{q_2},0} \right)^{\theta},
\]

provided \( a_1, a_2 \geq 0 \) and \( 1 < p_1, p_2, q_1, q_2 < \infty \) satisfy

\[
a = a_1(1-\theta) + a_2\theta, s = 1-\theta, \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.
\]

Remark 7.1. Note that, using the norms introduced in (7.2), we can the estimate (1.12) can be written as

\[
\| u \|_{L^2,0} \leq C \| f \|_{L^2} + C \| F \|_{L^2}, \quad 1 < p_1, p_2, q_1, q_2 < \infty.
\]

8. Phase localization

Given any Banach space \( B \subset D'(\mathbb{R}^n) \) satisfying the property

\[
\text{for any } P(\xi) \in C^\infty_0(\mathbb{R}^n), \ f \in B \Rightarrow P(D)f \in B,
\]

we can define for any \( r \in [1, \infty] \) and for any \( s \in \mathbb{R} \) the space \( \ell^r,s D_B \) as follows

\[
\| f \|_{\ell^r,s D_B} = \left( \sum_{k \in \mathbb{Z}} \| P_k(D)f \|_{B}^r 2^{kr} \right)^{1/r},
\]

with obvious modification for \( r = \infty \). Here \( \{P_k(\xi)\} \) is a Paley-Littlewood decomposition.

Our goal is to find some concrete examples of Banach spaces \( B \) satisfying the embedding

\[
B \subset \ell^r,0 D_B.
\]

Therefore we look for estimate of type

\[
\left( \sum_{k \in \mathbb{Z}} \| P_k(D)f \|_{B}^r \right)^{1/r} \leq C \| f \|_{B}.
\]

A typical example for Banach space \( B \) satisfying \( X \) is \( B = L^p \) with \( 1 < p \leq 2 \), so

\[
\left( \sum_{k \in \mathbb{Z}} \| P_k(D)f \|_{L^p}^2 \right)^{1/2} \leq C \| f \|_{L^p}, \quad 1 < p \leq 2.
\]

Having in mind that spaces \( \ell^r,s D_B \) are natural candidate for estimate of type \( X \), we shall verify that the conditions

\[
\begin{cases}
1 \leq q \leq 2, \ p = 2 \\
|a| + |s| < \frac{n}{2},
\end{cases}
\]

imply \( X \). More precisely we have

Lemma 8.1. If \( q \in [1, 2] \) and \( a, s \in \mathbb{R} \) satisfy

\[
\begin{cases}
|s| \leq 1, \\
|a| + |s| < \frac{n}{2},
\end{cases}
\]

imply \( X \). More precisely we have
then

\begin{equation}
\|f\|_{\tilde{H}^{s,a}_{12}} \leq C\|f\|_{\tilde{H}^{s,a}_{12}}.
\end{equation}

**Proof:** For any \( f \in S(\mathbb{R}^n) \) we have (for any \( r, q \in (1, \infty) \))

\begin{equation}
\|f\|_{\tilde{H}^{s,a}_{12}} \leq \left( \sum_{k_1 \in \mathbb{Z}} \left( \sum_{k_2 \in \mathbb{Z}} 2^{k_1 q a} \|D^s P_{k_2}(D)|^q f|^q_{L^2} \right) \right)^{1/2}
\end{equation}

where here and below we use the discrete norm in \( \tilde{H}^{s,a}_{12} \) introduced in (6.22). In the second equivalence relation we have used Proposition 7.1. Further we have

\begin{equation}
Q_{k_1} P_{k_2}(D) f = \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} Q_{k_1} P_{k_2}(D)Q_{m_1} P_{m_2}(D)\tilde{P}_{m_2}(D)\tilde{Q}_{m_1} f.
\end{equation}

It is not difficult, using again integration by parts argument, to see that for \( |k_1 - m_1| \geq 3 \) we have

\begin{equation}
\|Q_{k_1} P_{k_2}(D)Q_{m_1} g\|_{L^2} \leq C \frac{2^{(k_1 + m_1)n/2}}{2(k_1 \vee m_1)n} \|g\|_{L^2}
\end{equation}

so

\begin{equation}
\|Q_{k_1} P_{k_2}(D)Q_{m_1} P_{m_2}(D)f\|_{L^2} \leq C \frac{2^{(k_1 + m_1)n/2}}{2(k_1 \vee m_1)n} \|f\|_{L^2}.
\end{equation}

In a similar way, using the same integration by parts argument, we find for \( |k_2 - m_2| \geq 3 \)

\begin{equation}
\|P_{k_2}(D)Q_{m_1} P_{m_2}(D)g\|_{L^2} \leq C \frac{2^{(k_2 + m_2)n/2}}{2(k_2 \vee m_2)n} \|g\|_{L^2},
\end{equation}

so

\begin{equation}
\|Q_{k_1} P_{k_2}(D)Q_{m_1} P_{m_2}(D)f\|_{L^2} \leq C \frac{2^{(k_2 + m_2)n/2}}{2(k_2 \vee m_2)n} \|f\|_{L^2}.
\end{equation}

An interpolation between (8.12) and (8.14) gives

\begin{equation}
\|Q_{k_1} P_{k_2}(D)Q_{m_1} P_{m_2}(D)f\|_{L^2} \leq C t_{k,m}^{(\theta)} \|f\|_{L^2},
\end{equation}

where \( k = (k_1, k_2) \in \mathbb{Z}^2, m = (m_1, m_2) \in \mathbb{Z}^2, \theta \in [0, 1] \) will be chosen later on and

\begin{equation}
t_{k,m}^{(\theta)} = \frac{2^{(k_1 + m_1)\theta n/2}}{2(k_1 \vee m_1)\theta n} \frac{2^{(k_2 + m_2)(1-\theta)n/2}}{2(k_2 \vee m_2)(1-\theta)n}.
\end{equation}

If \( a, s \in \mathbb{R} \) satisfy

\begin{equation}|a| + |s| < \frac{n}{2},
\end{equation}

then we can choose \( \theta \in [0, 1] \) so that

\begin{equation}
\begin{cases}
|a| \leq \frac{n}{2} (1 - \theta), \\
|s| < \frac{n}{4} \theta.
\end{cases}
\end{equation}
Using the argument of the proof of Lemma 5.2, we see that (8.14) is fulfilled without the restrictions $|k_1 - m_1| \geq 3, |k_2 - m_2| \geq 3$. Applying Lemma 6.5, we get

\begin{equation}
\|Q_{k_1}P_{k_2}(D)f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}} \leq C\|\tilde{P}_{m_2}(D)\tilde{Q}_{m_1}f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}}.
\end{equation}

For $1 \leq q \leq 2$, we have the inequality

\begin{equation}
\|\tilde{P}_{m_2}(D)\tilde{Q}_{m_1}f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}} \leq \|\tilde{P}_{m_2}(D)\tilde{Q}_{m_1}f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}}
\end{equation}

and from relation

\begin{equation}
\|\tilde{P}_{m_2}(D)g\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}} \cong \|g\|_{H^s}
\end{equation}

we get

\begin{equation}
\begin{cases}
\|Q_{k_1}P_{k_2}(D)f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}} \leq C\|\tilde{P}_{m_2}(D)\tilde{Q}_{m_1}f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}} \cong \|f\|_{H^s}
\end{cases}
\end{equation}

This inequalities and (8.23) imply

\begin{equation}
\|f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}H^s} \leq C\|f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}H^s}.
\end{equation}

This completes the proof. \(\square\)

Further, we obtain in a similar way the following.

**Lemma 8.2.** If $q \in [2, \infty]$ and $a, s \in \mathbb{R}$ satisfy

\begin{equation}
\begin{cases}
|s| \leq 1, \\
|a| + |s| < \frac{n}{2}
\end{cases}
\end{equation}

then

\begin{equation}
\|f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}H^s} \leq C\|f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}H^s}.
\end{equation}

In a similar way we can verify the following.

**Lemma 8.3.** If $q \in [1, 2]$ and $a, s \in \mathbb{R}$ satisfy

\begin{equation}
\begin{cases}
|s| \leq 1, \\
|a| + |s| < \frac{n}{2}
\end{cases}
\end{equation}

and $R$ is a pseudo differential operator with convolution type symbol homogeneous of degree 0, then

\begin{equation}
\|Rf\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}H^s} \leq C\|f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}H^s}.
\end{equation}

By using a duality argument one can relax the assumptions on $q$ and obtain the following.

**Lemma 8.4.** If $q \in [1, \infty]$ and $a, s \in \mathbb{R}$ satisfy

\begin{equation}
\begin{cases}
|s| \leq 1, \\
|a| + |s| < \frac{n}{2}
\end{cases}
\end{equation}

and $R$ is a pseudo differential operator with convolution type symbol homogeneous of degree 0, then

\begin{equation}
\|Rf\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}H^s} \leq C\|f\|_{L^2_{\ell_2^0 \ell_1^{0,\alpha}}H^s}.
\end{equation}
9. Appendix: The Kenig, Ponce, Vega estimate (1.5) for the free Schrödinger equation.

In this section we shall recall the basic scale invariant smoothing estimate due to Kenig, Ponce, Vega.

One possible proof of the Kenig, Ponce, Vega estimate (1.5) is based on the following lemmas:

**Lemma 9.1.** For any \( u \in S(\mathbb{R}^n) \) we have

\[
\|u(x_1, x')\|_{L^2_t L^2_x} \leq C \left( \sum_{k \in \mathbb{Z}} \| x_1^{-1/2} u(x) \|_{L^2_x} \right),
\]

where \( x = (x_1, x') \) with \( x_1 \in \mathbb{R} \) and \( x' \in \mathbb{R}^{n-1} \).

**Proof.** We can consider the case of \( n = 1 \), since a similar argument works for \( n > 1 \). Let \( u \in S(\mathbb{R}) \), we have

\[
\|u(x)\|_{L^1} \leq C \left( \sum_{k \in \mathbb{Z}} Q_k(x) u(x) \right). \quad (9.2)
\]

From the Cauchy-Schwartz inequality and the fact that for the functions \( Q_k(x) \) we have \( \text{supp} Q_k(x) \subset \{ 2^{k-1} \leq |x| \leq 2^{k+1} \} \), we obtain

\[
\left\| \sum_{k \in \mathbb{Z}} Q_k(x) u(x) \right\|_{L^2} \leq C \left( \sum_{k \in \mathbb{Z}} \| x^{1/2}_k u(x) \|_{L^2} \right), \quad (9.3)
\]

so

\[
\|u(x)\|_{L^1} \leq C \left( \sum_{k \in \mathbb{Z}} \| x^{1/2}_k u(x) \|_{L^2} \right). \quad (9.4)
\]

\( \square \)

Similarly, we have

**Lemma 9.2.** For any \( u \in S(\mathbb{R}^n) \) and any \( \forall \in \mathbb{R}^n \) we have

\[
\sup_{k \in \mathbb{Z}} \| x^{1/2}_k u(x) \|_{L^2_{t, x}} \leq C \| u(x_1, x') \|_{L^\infty_t L^2_{x, x'}}, \quad (9.5)
\]

where \( x = (x_1, x') \) with \( x_1 \in \mathbb{R} \) and \( x' \in \mathbb{R}^{n-1} \).

The key point in the proof of (9.5) is to establish the estimate

\[
\| \partial_1 u(t, x_1, x') \|_{L^\infty_t L^2_{x, x'}} \leq C \| F(t, x_1, x') \|_{L^1_t L^2_{x, x'}}, \quad (9.6)
\]

We shall show now that this estimate completes the proof of (9.5).

From (9.5), (9.1) and (9.3) we get

\[
\sup_{k \in \mathbb{Z}} \| x^{1/2}_k \partial_1 u(t, x) \|_{L^2_{t, x}} \leq C \left( \sum_{k \in \mathbb{Z}} \| x^{1/2}_k F(t, x) \|_{L^2_{t, x}} \right). \quad (9.7)
\]

Using the fact that the Schrödinger equation in (1.5) and the norm in the right side of (9.7) are invariant under the action of the group of rotations \( SO(n) \), we obtain

\[
\sup_{k \in \mathbb{Z}} \| x^{1/2}_k \partial_j u(t, x) \|_{L^2_{t, x}} \leq C \left( \sum_{k \in \mathbb{Z}} \| x^{1/2}_k F(t, x) \|_{L^2_{t, x}} \right), \quad \forall j = 1, \ldots, n. \quad (9.8)
\]
To prove \ref{lem:9.1} we need of the following two lemmas.

**Lemma 9.3.** There exists a constant $C \geq 0$ so that for all $\lambda \in \mathbb{C}$ and all $v \in S(\mathbb{R})$ we have
\begin{equation}
\|v(x)\|_{L^\infty} \leq C \left\| \left( \frac{d}{dx} - \lambda \right) v(x) \right\|_{L^1}.
\end{equation}
\[9.9\]

**Proof.** Suppose that $\text{Re} \lambda \leq 0$. Let $w(x) = (d/dx - \lambda)v(x) \in S(\mathbb{R})$. Then
\begin{equation}
v(x) = \int_{-\infty}^{\infty} e^{\lambda(x-y)}w(y)dy.
\end{equation}
\[9.10\]
Thus
\begin{equation}
\|v(x)\|_{L^\infty} \leq C \|w(x)\|_{L^1}.
\end{equation}
\[9.11\]
The estimate \ref{lem:9.1} follows for $\text{Re} \lambda \leq 0$. A similar argument works for $\text{Re} \lambda \geq 0$. In fact, we have $w(x) = (d/dx - \lambda)v(x)$ so
\begin{equation}
v(x) = \int_{-\infty}^{\infty} e^{\lambda(x-y)}w(y)dy.
\end{equation}
\[9.12\]
and we obtain, as in the previous case,
\begin{equation}
\|v(x)\|_{L^\infty} \leq C \|w(x)\|_{L^1}.
\end{equation}
\[9.13\]

The estimate \ref{lem:9.1} follows from the following.

**Lemma 9.4.** For $n \geq 1$ there exists a constant $C \geq 0$ so that for all $\lambda \in \mathbb{C}$ and all $v \in S(\mathbb{R}^n)$ we have
\begin{equation}
\|\partial_1 v(x_1, x')\|_{L^\infty_{x_1} L^2_{x'}} \leq C \|(-\Delta - \lambda) v(x_1, x')\|_{L^1_{x_1} L^2_{x'}}.
\end{equation}
\[9.14\]
where $x = (x_1, x')$ and $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$.

**Proof.** Consider first the case $n = 1$. Let be $\lambda = -\mu^2$. Then by lemma \ref{lem:9.3}
\begin{equation}
\left\| \frac{d}{dx} v(x) \right\|_{L^\infty} \leq \frac{1}{2} \left\| \left( \frac{d}{dx} - \mu \right) v(x) \right\|_{L^\infty} + \frac{1}{2} \left\| \left( \frac{d}{dx} + \mu \right) v(x) \right\|_{L^\infty} \leq C \left\| -\mu^2 v(x) \right\|_{L^1}.
\end{equation}
\[9.15\]
Consider now the case $n > 1$. Given any $v \in S(\mathbb{R}^n)$ we denote by
\[\bar{v}(x_1, k'), \quad k' = (k_2, \ldots, k_n)\]
its partial Fourier transform with respect to $x'$, i.e.
\begin{equation}
\bar{v}(x_1, k') = (2\pi)^{-n-1} \int e^{-ik'x'} v(x_1, x')dx'.
\end{equation}
\[9.16\]
Using the one-dimensional result \ref{lem:9.3}, for each fixed $k'$, we obtain
\begin{equation}
|\partial_1 \bar{v}(x_1, k')|^2 \leq C \int |(-\partial_1^2 + |k'|^2 - \lambda)\bar{v}|^2 dx_1.
\end{equation}
\[9.17\]
Integrating with respect to $k'$ and using the Plancherel identity, we derive
\begin{equation}
\|\partial_1 v(x_1, x')\|_{L^\infty_{x_1} L^2_{x'}} \leq C \|(-\Delta - \lambda) v(x_1, x')\|_{L^1_{x_1} L^2_{x'}}.
\end{equation}
\[9.18\]
This completes the proof. \qed
In fact the basic idea of the proof of (9.6) is to compute the Fourier transform with respect to the temporal variable of the equation and obtain

$$-i\Delta \hat{u}(\lambda, x) - i\lambda \hat{u}(\lambda, x) = \tilde{F}(\lambda, x).$$

Now if we split $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and indicate with $v_{\lambda}(x_1, x') = \hat{u}(\lambda, x)$, we can apply the Lemma and have

$$\|\partial_1 v_{\lambda}(x_1, x')\|_{L_{x_1}^\infty L_{x'}^2} \leq C \|(-\Delta - \lambda) v_{\lambda}(x_1, x')\|_{L_{x_1}^1 L_{x'}^2}.$$  

(9.20)

This estimate with the equation (9.19) give the following other one

$$\|\partial_1 \hat{u}(\lambda, x_1, x')\|_{L_{x_1}^\infty L_{x'}^2} \leq C \|(-\Delta - \lambda) \hat{u}(\lambda, x_1, x')\|_{L_{x_1}^1 L_{x'}^2} \leq$$

$$\leq \|\tilde{F}(\lambda, x_1, x')\|_{L_{x_1}^1 L_{x'}^2}.$$  

(9.21)

The application of Plancherel’s theorem in the temporal variable gives the estimate (9.6) and this completes the proof of (1.5).

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