Multiscale mortar mixed finite element methods for the Biot system of poroelasticity

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Abstract

We develop a mixed finite element domain decomposition method on non-matching grids for the Biot system of poroelasticity. A displacement–pressure vector mortar function is introduced on the interfaces and utilized as a Lagrange multiplier to impose weakly continuity of normal stress and normal velocity. The mortar space can be on a coarse scale, resulting in a multiscale approximation. We establish existence, uniqueness, stability, and error estimates for the semi-discrete continuous-in-time formulation under a suitable condition on the richness of the mortar space. We further consider a fully-discrete method based on the backward Euler time discretization and show that the solution of the algebraic system at each time step can be reduced to solving a positive definite interface problem for the composite mortar variable. A multiscale stress–flux basis is constructed, which makes the number of subdomain solves independent of the number of iterations required for the interface problem, and weakly dependent on the number of time steps. We present numerical experiments verifying the theoretical results and illustrating the multiscale capabilities of the method for a heterogeneous benchmark problem.

1 Introduction

In this paper we develop and study a domain decomposition method for the quasistatic Biot system of poroelasticity [14] using mixed finite element subdomain discretizations with non-matching grids along the interfaces. The Biot system models flow of viscous fluids through deformable porous media. The system has a wide range of applications, including in the geosciences, such as earthquakes, landslides, groundwater cleanup, and hydraulic fracturing, as well as in biomedicine, such as arterial flows and biological tissues. The model consists of an equilibrium equation for the solid medium coupled with a mass balance equation for the fluid flow through the medium. Various numerical methods for the Biot system have been developed in the literature, considering two-field displacement–pressure formulations [25,42,50], three-field displacement–pressure–Darcy velocity formulations [32,40,46,47,56,58], three-field displacement–pressure–total pressure formulations [41,43], and four-field stress–displacement–pressure–Darcy velocity mixed formulations [2,57]. In this work we consider the five-field weakly symmetric stress–displacement–rotation–pressure–Darcy velocity formulation [6,38]. The four-field and five-field formulations lead to mixed finite element (MFE) approximations, which exhibit local mass and momentum conservation, accurate normal-continuous Darcy velocity and solid stress, as well as robust and locking-free behavior for a wide range of physical parameters. An additional advantage of the five-field MFE method is that it can be reduced to a positive definite cell-centered scheme for the pressure and displacement only, as it is done in the multipoint stress–multipoint flux MFE method developed in [6], through the use of a vertex quadrature rule and local elimination of some of the variables. We note that our analysis for the weakly symmetric formulation can be carried over to the strongly symmetric formulation found in [57].

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Numerical methods for the Biot system of poroelasticity usually lead to large algebraic systems, due to the coupling of unknowns, as well as the size of the domain and the wide range of scales associated with practical applications. Domain decomposition methods are commonly used for solving large systems resulting from discretizations of partial differential equations, as they lead to parallel and efficient solution algorithms. In this work we focus on non-overlapping domain decomposition methods, where the domain is split into non-overlapping subdomains and the continuity of the solution variables at the subdomain interfaces is enforced through a suitable interface Lagrange multiplier. The global problem can be reduced to solving iteratively an interface problem, involving the solution of smaller subdomain systems at each iteration, which can be performed in parallel. Despite the extensive studies of numerical methods for the Biot system of poroelasticity, there have been relatively few results on domain decomposition methods for this problem and they have been mostly based on the two-field displacement–pressure formulation. To the best of our knowledge, the only paper on domain decomposition for Biot with a mixed formulation is our previous work, which is based on the five-field mixed formulation with weak stress symmetry. Two types of methods are developed in [34]. One is a monolithic domain decomposition method, which involves solving the Biot system on each subdomain. The second is a partitioned method, which splits the Biot system into solving separate elasticity and flow equations, and applies domain decomposition for each of the equations. The developments in [34] are motivated by earlier works on non-overlapping domain decomposition methods for MFE discretizations of Darcy flow and elasticity.

The domain decomposition methods in [34] are limited to subdomain grids that match at the interfaces. In this paper we generalize the work in [34] to enable the use of non-matching subdomain grids through the use of mortar finite elements. This generality provides the flexibility to use different grid resolution in different subdomains, as well as a coarser mortar space, resulting in a multiscale approximation. We refer to the method as a multiscale mortar mixed finite element (MMMFE) method. The MMMFE method has been studied for mixed formulations of scalar elliptic equations in and for weakly-symmetric mixed elasticity in [35]. Following the monolithic domain decomposition method from [34], we utilize a physically heterogeneous Lagrange multiplier vector consisting of interface displacement and pressure variables to impose weakly the continuity of the normal components of stress and velocity, respectively. In contrast to [34], we choose the Lagrange multiplier vector from a space of mortar finite elements defined on a separate interface grid, which allows for handling non-matching subdomain grids through projections from and onto the mortar finite element space. This also allows for the mortar space to be on a coarser scale compared to a finer subdomain grid size. The multiscale capability adds an extra layer of flexibility over the methods from [34].

The main contributions of this paper are as follows. We first consider the semi-discrete continuous-in-time formulation and establish existence, uniqueness, and stability of the MMMFE method for the Biot system, employing the theory of degenerate evolutionary systems of partial differential equations with monotone operators. For the solvability of the associated resolvent problem, under a condition on the richness of the mortar finite element space, we establish an inf-sup condition for the mortar space, as well as inf-sup conditions for the stress and velocity spaces with weak interface continuity of normal components. Next, we establish a priori error estimates for the stress, displacement, rotation, pressure, and Darcy velocity, as well as the displacement and pressure mortar variables in their natural norms. We then consider a fully-discrete method based on the backward Euler time discretization. We show that the solution of the algebraic system at each time step can be reduced to solving a positive definite interface problem for the composite displacement–pressure mortar variable. Motivated by the multiscale flux basis from and the multiscale stress basis from [35], we propose the construction and use of a multiscale stress–flux basis, which makes the number of subdomain solves independent of the number of iterations required for the interface problem. Moreover, since the basis can be reused at each time step, the total number of subdomain solves depends weakly on the number of time steps. This illustrates that the multiscale basis results in a significant reduction of computational cost in the case of time-dependent problems. Finally, we present the results of several numerical tests designed to illustrate the well-posedness,
stability, and accuracy of the proposed MMMFE method. We also consider a test based on data from the Society of Petroleum Engineers SPE10 benchmark, illustrating the multiscale capabilities of the method and the advantages of using a multiscale basis.

The rest of the paper is organized as follows. Section 2 introduces the model problem and its domain decomposition mortar mixed finite element approximation. Various interpolation and projection operators are presented in Section 3, where discrete inf-sup stability bounds are also obtained. Well-posedness of the semi-discrete method is established in Section 4, followed by error analysis in Section 5. In Section 6 we discuss the fully discrete method, the non-overlapping domain decomposition algorithm based on a reduction to an interface problem, and the construction of the multiscale stress–flux basis. Numerical results are reported in Section 7. The paper ends with some concluding remarks in Section 8.

2 Formulation of the method

In this section, we introduce the mathematical model of interest and its mixed finite element approximation. We also develop the framework for the multiscale mortar mixed finite element domain decomposition method. Finally, we introduce the weakly continuous normal stress and velocity spaces and reformulate the MMMFE method in terms of these spaces.

2.1 Mathematical formulation of the model problem

Let \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \) be a simply connected domain. We use the notation \( \mathbb{M}, \mathbb{S} \) and \( \mathbb{N} \) for the spaces of \( d \times d \) matrices, symmetric matrices, and skew-symmetric matrices, respectively, over the field of real numbers. Let \( I \in \mathbb{S} \) denote the \( d \times d \) identity matrix. The partial derivative operator with respect to time, \( \frac{\partial}{\partial t} \), is often abbreviated to \( \partial_t \). \( C \) denotes a generic positive constant that is independent of the discretization parameters \( h \) and \( H \). Throughout the paper, the divergence operator is the usual divergence for vector fields, which produces vector field when applied to matrix field by taking the divergence of each row.

For a set \( G \subset \mathbb{R}^d \), the \( L^2(G) \) inner product and norm are denoted by \( (\cdot, \cdot)_G \) and \( \| \cdot \|_G \), respectively, for scalar, vector, or tensor valued functions. For any real number \( r \), \( \| \cdot \|_{r,G} \) denotes the \( H^r(G) \)-norm. We omit subscript \( G \) if \( G = \Omega \).

Given a vector field \( f \) representing body forces and a source term \( g \), we consider the quasi-static Biot system of poroelasticity [14]:

\[
- \text{div } \sigma(u) = f, \quad \text{in } \Omega \times (0, T], \quad (2.1)
\]

\[
K^{-1}z + \nabla p = 0, \quad \text{in } \Omega \times (0, T], \quad (2.2)
\]

\[
\frac{\partial}{\partial t}(c_0 p + \alpha \text{div } u) + \text{div } z = g, \quad \text{in } \Omega \times (0, T], \quad (2.3)
\]

where \( u \) is the displacement, \( p \) is the fluid pressure, \( z \) is the Darcy velocity, and \( \sigma \) is the poroelastic stress, defined as

\[
\sigma = \sigma_e - \alpha p I. \quad (2.4)
\]

Here \( 0 < \alpha \leq 1 \) is the Biot-Willis constant, and \( \sigma_e \) is the elastic stress satisfying the stress-strain relationship

\[
A \sigma_e = \epsilon(u), \quad \epsilon(u) := \frac{1}{2} (\nabla u + (\nabla u)^T), \quad (2.5)
\]
where \( A \) is the compliance tensor, which is a symmetric, bounded and uniformly positive definite linear operator acting from \( S \to S \), extendible to \( M \to M \). In particular, there exist constants \( 0 < a_\min \leq a_\max < \infty \) such that

\[
\text{for a.e. } x \in \Omega, \quad a_\min \tau \cdot \tau \leq A(x)\tau \cdot \tau \leq a_\max \tau \cdot \tau, \quad \forall \tau \in M.
\]

In the special case of homogeneous and isotropic body, \( A \) is given by,

\[
A\sigma_e = \frac{1}{2\mu} \left( \sigma_e - \frac{\lambda}{2\mu + d\lambda} \text{tr}(\sigma_e)I \right),
\]

where \( \mu > 0 \) and \( \lambda \geq 0 \) are the Lamé coefficients. In this case, \( \sigma_e(u) = 2\mu \epsilon(u) + \lambda \text{div } u I \). Finally, \( c_0 > 0 \) is the mass storativity and \( K \) stands for the conductivity tensor, which equals to the permeability of the media divided by the fluid viscosity. It is spatially dependent, symmetric, and uniformly bounded and positive definite, i.e., for constants \( 0 < k_\min \leq k_\max < \infty \),

\[
\text{for a.e. } x \in \Omega, \quad k_\min \cdot \zeta \cdot \zeta \leq K(x)\zeta \cdot \zeta \leq k_\max \cdot \zeta \cdot \zeta, \quad \forall \zeta \in \mathbb{R}^d.
\]

To close the system, we impose the boundary conditions

\[
\begin{align*}
  u &= 0 & \text{on } \Gamma_D^u \times (0, T], & \sigma n &= 0 & \text{on } \Gamma_D^\sigma \times (0, T], \\
p &= 0 & \text{on } \Gamma_D^p \times (0, T], & z \cdot n &= 0 & \text{on } \Gamma_N^z \times (0, T],
\end{align*}
\]

where \( \Gamma_D^u \cup \Gamma_D^\sigma = \Gamma_D^p \cup \Gamma_N^z = \partial \Omega \) and \( n \) is the outward unit normal vector field on \( \partial \Omega \), along with the initial condition \( p(x, 0) = p_0(x) \) in \( \Omega \). Compatible initial data for the rest of the variables can be obtained from (2.11) and (2.13) at \( t = 0 \). Well posedness analysis for this system can be found in [52].

We consider a five-field mixed variational formulation for (2.11)–(2.15). It uses a rotation Lagrange multiplier \( \gamma := \frac{1}{2} (\nabla u - \nabla u^T) \in \mathbb{N} \) to impose weakly the symmetry of the stress tensor \( \sigma \). The formulation reads: find \((\sigma, u, \gamma, z, p) : [0, T] \to \mathbb{X} \times V \times Q \times Z \times W\) such that \( p(0) = p_0 \) and for a.e. \( t \in (0, T) \),

\[
\begin{align*}
  (A(\sigma + \alpha p I), \tau) + (u, \text{div } \tau) + (\gamma, \tau) &= 0, & \forall \tau \in \mathbb{X}, \\
  (\text{div } \sigma, v) &= - (f, v), & \forall v \in V, \\
  (\sigma, \xi) &= 0, & \forall \xi \in Q, \\
  (K^{-1}z, q) - (p, \text{div } q) &= 0, & \forall q \in Z, \\
  c_0 (\partial_t p, w) + \alpha (\partial_t A(\sigma + \alpha p I), w I) + (\text{div } z, w) &= (g, w), & \forall w \in W,
\end{align*}
\]

where

\[
\begin{align*}
\mathbb{X} &= \{ \tau \in H(\text{div}; \Omega, M) : \tau n = 0 \text{ on } \Gamma_N^\sigma \}, & V &= L^2(\Omega, \mathbb{R}^d), & Q &= L^2(\Omega, \mathbb{N}), \\
Z &= \{ q \in H(\text{div}; \Omega) : q \cdot n = 0 \text{ on } \Gamma_N^z \}, & W &= L^2(\Omega).
\end{align*}
\]

It was shown in [3] that the system (2.11)–(2.15) is well posed.

### 2.2 The semi-discrete multiscale mortar mixed finite element method

Let \( \Omega = \bigcup_{i=1}^N \Omega_i \) be a union of non-overlapping polygonal subdomains. Let \( \Gamma_{i,j} = \partial \Omega_i \cap \partial \Omega_j \), \( \Gamma = \bigcup_{i,j=1}^N \Gamma_{i,j} \), and \( \Gamma_i = \partial \Omega_i \cap \Gamma = \partial \Omega_i \setminus \partial \Omega \). Let \( \mathcal{T}_{h,i} \) be a shape regular simplicial or rectangular finite element partition on \( \Omega_i \) with maximal element diameter \( h \). The partitions are not required to match along the subdomain interfaces. For \( 1 \leq i \leq N \), let \( \mathbb{X}_{h,i} \times V_{h,i} \times Q_{h,i} \times Z_{h,i} \times W_{h,i} \subset \mathbb{X} \times V \times Q \times Z \times W \) be a family of suitable mixed finite element spaces defined on subdomain \( \Omega_i \), where, for a space \( U \) on \( \Omega \), \( U_i = U|_{\Omega_i} \). The elasticity discretizations \( \mathbb{X}_{h,i} \times V_{h,i} \times Q_{h,i} \) can be chosen from any of the stable finite element triplets for
linear elasticity with weakly imposed stress symmetry. Examples of such triplets include $[5, 9, 11, 15, 19, 39]$. These spaces satisfy the inf-sup condition

$$\forall v \in V_{h,i}, \xi \in Q_{h,i}, \quad \|v\|_{\Omega_i} + \|\xi\|_{\Omega_i} \leq C \sup_{0 \neq \tau \in X_{h,i}} \frac{(\text{div} \, \tau, v)_{\Omega_i} + (\tau, \xi)_{\Omega_i}}{\|\tau\|_{\text{div}, \Omega_i}}. \quad (2.16)$$

The flow discretizations $Z_h \times W_h$ can be chosen from any of the stable pressure–velocity pairs of MFE spaces such as the Raviart-Thomas ($RT$) or Brezzi-Douglas-Marini ($BDM$) spaces $[17]$. These spaces satisfy the inf-sup condition

$$\forall w \in W_h, \quad \|w\| \leq C \sup_{0 \neq q \in Z_{h,i}} \frac{(\text{div} \, q, w)_{\Omega_i}}{\|q\|_{\text{div}, \Omega_i}}. \quad (2.17)$$

We define the global finite element spaces on $\Omega$ as follows:

$$X_h = \bigoplus_{1 \leq i \leq N} X_{h,i}, \quad V_h = \bigoplus_{1 \leq i \leq N} V_{h,i}, \quad Q_h = \bigoplus_{1 \leq i \leq N} Q_{h,i}, \quad Z_h = \bigoplus_{1 \leq i \leq N} Z_{h,i}, \quad W_h = \bigoplus_{1 \leq i \leq N} W_{h,i}. \quad (2.18)$$

The spaces $V_h$, $Q_h$, and $W_h$ are equipped with $L^2(\Omega)$-norms. The spaces $X_h$ and $Z_h$ are equipped with the norms

$$\|\tau\|_{X_h}^2 := \|\tau\|^2 + \|\text{div}_h \tau\|^2 \quad \text{and} \quad \|\zeta\|_{Z_h}^2 := \|\zeta\|^2 + \|\text{div}_h \zeta\|^2,$$

where for simplicity we define $\text{div}_h \varphi|_{\Omega_i} := \text{div}(\varphi|_{\Omega_i})$. We note that functions in $X_h$ and $Z_h$ do not have continuity of the normal components across the subdomain interfaces. This discontinuity is addressed using Lagrange multipliers defined on mortar spaces on the interface $\Gamma$, which approximate the traces of the displacement vector and the pressure. The mortar spaces satisfy suitable coarseness conditions, which will be discussed in the later sections. Let $T_{H,i,j}$ be a shape regular finite element partition of $\Gamma_{i,j}$ consisting of simplices or quadrilaterals in $d - 1$ dimensions with maximal element diameter $H$. Let $\Lambda^u_{H,i,j} \subseteq L^2(\Gamma_{i,j}; \mathbb{R}^d)$ and $\Lambda^p_{H,i,j} \subseteq L^2(\Gamma_{i,j})$ be mortar finite element spaces on $\Gamma_{i,j}$ representing the displacement and pressure Lagrange multipliers, respectively. We assume that these mortar spaces contain either continuous or discontinuous piecewise polynomials on $T_{H,i,j}$. The global mortar finite element space on the union of subdomain interfaces $\Gamma$ is defined as

$$\Lambda^u_H = \bigoplus_{1 \leq i < j \leq N} \Lambda^u_{H,i,j}, \quad \Lambda^p_H = \bigoplus_{1 \leq i < j \leq N} \Lambda^p_{H,i,j}. \quad (2.19)$$

The semi-discrete multiscale mortar mixed finite element method for the Biot problem $[2.11]-[2.15]$ is obtained by testing the equations on each subdomain and integrating by parts, which results in interface terms involving the displacement and pressure Lagrange multipliers. The method reads as follows: find $(\sigma_h, u_h, \gamma_h, z_h, p_h, \lambda^u_H, \lambda^p_H) : [0, T] \to X_h \times V_h \times Q_h \times Z_h \times W_h \times \Lambda^u_H \times \Lambda^p_H$ such that for a.e. $t \in (0, T)$,

$$\begin{align*}
(A(\sigma_h + \alpha p_h I), \tau) + (u_h, \text{div}_h \tau) + (\gamma_h, \tau) - \sum_{i=1}^N (\lambda^u_{H,i}, \tau n_i)_{\Gamma_i} &= 0, & \forall \tau \in X_h, \\
(\text{div}_h \sigma_h, v) &= -(f, v), & \forall v \in V_h, \\
(\sigma_h, \xi) &= 0, & \forall \xi \in Q_h, \\
(K^{-1}z_h, \zeta) - (p_h, \text{div}_h \zeta) + \sum_{i=1}^N (\lambda^p_{H,i}, \zeta \cdot n_i)_{\Gamma_i} &= 0, & \forall \zeta \in Z_h, \\
c_0 (\partial_t p_h, w) + \alpha (\partial_t A(\sigma_h + \alpha p_h I), w) + (\text{div}_h z_h, w) &= (g, w), & \forall w \in W_h, \\
\sum_{i=1}^N (\sigma_h n_i, \mu^u)_{\Gamma_i} &= 0, & \forall \mu^u \in \Lambda^u_H, \quad (2.20)
\end{align*}$$
\[ \sum_{i=1}^{N} (z_{h,i} \cdot n_i, \mu^p)_\Gamma_i = 0, \quad \forall \mu^p \in \Lambda_H^p, \quad (2.24) \]

where \( n_i \) is the outward unit normal vector field on \( \Omega_i \). Note that equations (2.23)–(2.24) enforce weak continuity of the normal components of the stress tensor and velocity vector, respectively, across the interface \( \Gamma \).

**Remark 2.1.** The method requires discrete initial data \( p_{h,0} \) and \( \sigma_{h,0} \), which is obtained from the continuous initial data using an elliptic projection. Details are provided in Section 4.1.

### 2.3 Weakly continuous normal stress and velocity formulation

For the purpose of the analysis, we consider an equivalent formulation of (2.18)–(2.24) in the spaces of stress and velocity with weakly continuous normal components. Let

\[ \mathcal{X}_{h,0} = \left\{ \tau \in \mathcal{X}_h : \sum_{i=1}^{N} (\tau n_i, \mu^u)_\Gamma_i = 0, \quad \forall \mu^u \in \Lambda^u_H \right\} \]

and

\[ Z_{h,0} = \left\{ \zeta \in Z_h : \sum_{i=1}^{N} (\zeta \cdot n_i, \mu^p)_\Gamma_i = 0, \quad \forall \mu^p \in \Lambda^p_H \right\}. \]

Then (2.18)–(2.24) can be restated as follows: find \((\sigma_{h}, u_{h}, \gamma_{h}, z_{h}, p_{h}) : [0, T] \to (\mathcal{X}_{h,0}, V_h, Q_h, Z_{h,0}, W_h)\) such that

\[
\begin{align*}
(A(\sigma_{h} + \alpha p_{h} I), \tau) + (u_{h}, \text{div}_{h} \tau)_{\Omega_{i}} + (\gamma_{h}, \tau) &= 0, \quad \forall \tau \in \mathcal{X}_{h,0}, \quad (2.25) \\
(\text{div}_{h} \sigma_{h}, v)_{\Omega_{i}} &= -(f, v), \quad \forall v \in V_{h}, \quad (2.26) \\
(\sigma_{h}, \xi) &= 0, \quad \forall \xi \in Q_{h}, \quad (2.27) \\
(K^{-1}z_{h}, \zeta) - (p_{h}, \text{div}_{h} \zeta)_{\Omega_{i}} &= 0, \quad \forall \zeta \in Z_{h,0}, \quad (2.28) \\
c_0(\partial_t p_{h}, w) + \alpha(\partial_t A(\sigma_{h} + \alpha p_{h} I), w I) + (\text{div}_{h} z_{h}, w)_{\Omega_{i}} &= (g, w), \quad \forall w \in W_{h}. \quad (2.29)
\end{align*}
\]

Note that constructing basis functions for the spaces \( \mathcal{X}_{h,0} \) and \( Z_{h,0} \) is difficult and we use the above formulation only for the sake of analysis.

### 3 Interpolation and projection operators and discrete inf-sup conditions

In this section we discussing various interpolation and projection operators useful in the analysis. We then prove inf-sup stability bounds for the interface jump operators and the weakly continuous stress \( \mathcal{X}_{h,0} \) and velocity \( Z_{h,0} \) spaces under an appropriate condition on the mortar space \( \Lambda_H \).

#### 3.1 Interpolation operators

Let \( Q_{h,i}^p : L^2(\partial \Omega_i, \mathbb{R}^d) \to \mathcal{X}_{h,i} \) and \( Q_{h,i}^p : L^2(\partial \Omega_i) \to Z_{h,i} \cdot n_i \) be the \( L^2 \)-projection operators such that for any \( \phi_u \in L^2(\partial \Omega_i, \mathbb{R}^d) \) and \( \phi_p \in L^2(\partial \Omega_i) \),

\[
\begin{align*}
\langle \phi_u - Q_{h,i}^p \phi_u, \tau n_i \rangle_{\partial \Omega_i} &= 0, \quad \forall \tau \in \mathcal{X}_{h,i}, \quad (3.1) \\
\langle \phi_p - Q_{h,i}^p \phi_p, \zeta \cdot n_i \rangle_{\partial \Omega_i} &= 0, \quad \forall \zeta \in Z_{h,i}. \quad (3.2)
\end{align*}
\]
For any inf-sup stable pair of finite element spaces $X_{h,i} \times V_{h,i}$ with $\text{div} \, X_{h,i} = V_{h,i}$, there exists a mixed canonical interpolant $\hat{\Pi}^\tau : H^1(\Omega,i) \cap X_i \to X_{h,i}$, for any $\tau > 0$, such that for any $\tau \in H^1(\Omega,i) \cap X_i,$

$$
(\text{div}(\hat{\Pi}^\tau \tau - \tau), v)_{\Omega_i} = 0, \quad \forall v \in V_{h,i}, \quad (3.3)
$$
$$
((\hat{\Pi}^\tau \tau - \tau)n_i, \hat{\tau} n_i)_{\Gamma_i} = 0, \quad \forall \hat{\tau} \in X_{h,i}, \quad (3.4)
$$
$$
||\hat{\Pi}^\tau \tau||_{\text{div},\Omega_i} \leq C \left( ||\tau||_{H^1(\Omega_i)} + ||\text{div} \, \tau||_{\Omega_i} \right). \quad (3.5)
$$

Similarly for any inf-sup stable pair $Z_{h,i} \times W_{h,i}$ with $\text{div} \, Z_{h,i} = W_{h,i}$, there exists a mixed canonical interpolant $\Pi^\tau : H^1(\Omega,i) \cap Z_i \to Z_{h,i}$ such that for any $\zeta \in H^1(\Omega,i) \cap Z_i,$

$$
(\text{div}(\Pi^\tau \zeta - \zeta), w)_{\Omega_i} = 0, \quad \forall w \in W_{h,i}, \quad (3.6)
$$
$$
((\Pi^\tau \zeta - \zeta) \cdot n_i, \hat{\zeta} \cdot n_i)_{\Gamma_i} = 0, \quad \forall \hat{\zeta} \in Z_{h,i}, \quad (3.7)
$$
$$
||\Pi^\tau \zeta||_{\text{div},\Omega_i} \leq C \left( ||\zeta||_{H^1(\Omega_i)} + ||\text{div} \, \zeta||_{\Omega_i} \right). \quad (3.8)
$$

Let $P^w_{h,i} : L^2(\Omega_i) \to W_{h,i}$ denote the $L^2$-orthogonal projection such that for any $w \in L^2(\Omega_i),$

$$
\left( P^w_{h,i} w - w, \hat{w} \right)_{\Omega_i} = 0, \quad \forall \hat{w} \in W_{h,i}. \quad (3.9)
$$

Let $P^v_{h,i} : L^2(\Omega_i, \mathbb{R}^d) \to V_{h,i}$ denote the $L^2$-orthogonal projection such that for any $v \in L^2(\Omega_i, \mathbb{R}^d),$

$$
\left( P^v_{h,i} v - v, \hat{v} \right)_{\Omega_i} = 0, \quad \forall \hat{v} \in V_{h,i}. \quad (3.10)
$$

Let $R_{h,i} : L^2(\Omega_i, \mathbb{N}) \to Q_{h,i}$ denote the $L^2$-orthogonal projection such that for any $\xi \in L^2(\Omega_i, \mathbb{N}),$

$$
\left( R_{h,i} \xi - \xi, \hat{\xi} \right)_{\Omega_i} = 0, \quad \forall \hat{\xi} \in Q_{h,i}. \quad (3.11)
$$

We will use an elliptic projection operator onto $X_{h,i}$ \cite{35}. Define $\hat{\Pi}^\sigma : H^1(\Omega,i) \cap X_i \to X_{h,i}$ as the operator that takes $\sigma \in H^1(\Omega,i) \cap X_i$ to its finite element approximation $\hat{\sigma}$ via the solution of the following Neumann problem: for any $\sigma \in H^1(\Omega,i) \cap X_i,$ find $(\hat{\sigma}, \hat{\tau}, \hat{\gamma}) \in X_{h,i} \times V_{h,i} \times Q_{h,i}$ such that

$$
(\hat{\sigma}, \tau)_{\Omega_i} + (\hat{\tau}, \text{div} \, \tau)_{\Omega_i} + (\hat{\gamma}, \tau)_{\Omega_i} = (\sigma, \tau)_{\Omega_i}, \quad \forall \tau \in X^0_{h,i}, \quad (3.12)
$$
$$
(\text{div} \, \hat{\sigma}, v)_{\Omega_i} = (\text{div} \, \sigma, v)_{\Omega_i}, \quad \forall v \in V_{h,i}, \quad (3.13)
$$
$$
(\hat{\sigma}, \xi)_{\Omega_i} = (\sigma, \xi)_{\Omega_i}, \quad \forall \xi \in Q_{h,i}, \quad (3.14)
$$
$$
\hat{\sigma} n_i = (\hat{\Pi}^\sigma \sigma) n_i \text{ on } \partial \Omega_i, \quad (3.15)
$$

where $X^0_{h,i} = \{ \tau \in X_{h,i} : \tau n_i = 0 \text{ on } \partial \Omega_i \}$. More details on the well-posedness and properties of $\hat{\Pi}^\sigma$ can be found in \cite{35}. In particular, the following bounds hold:

$$
||\sigma - \hat{\Pi}^\sigma \sigma||_{\Omega_i} \leq C ||\sigma - \Pi^\sigma \sigma||_{\Omega_i}, \quad \sigma \in H^1(\Omega,i) \cap X_i, \quad (3.16)
$$
$$
||\hat{\Pi}^\sigma \sigma||_{\text{div},\Omega_i} \leq C \left( ||\sigma||_{H^1(\Omega_i)} + ||\text{div} \, \sigma||_{\Omega_i} \right), \quad \sigma \in H^1(\Omega,i) \cap X_i, \quad (3.17)
$$

We also use the Scott-Zhang interpolants (see \cite{51}) $\mathcal{I}^H_H : H^1(\Gamma) \to \Lambda^H_H \cap C(\Gamma)$ and $\mathcal{I}^p_H : H^1(\Gamma) \to \Lambda^p_H \cap C(\Gamma)$, defined to preserve the trace on $\partial \Gamma$ for functions that are zero on $\partial \Gamma$.

Let the finite element spaces $X_{h,i}, V_{h,i}, Q_{h,i}, Z_{h,i}, W_{h,i},$ and $\Lambda_{H,i,j}$ contain polynomials of degree less than or equal to $k \geq 1, l \geq 0, j \geq 0, r \geq 0, s \geq 0,$ and $m \geq 0$, respectively. The operators defined above satisfy the following approximation bounds:

$$
||\psi - \mathcal{I}_H^p \psi||_{t,\Gamma_{i,j}} \leq C H^{m-t} \|\psi\|_{\hat{m},\Gamma_{i,j}}, \quad 0 \leq t \leq 1, \ t \leq \hat{m} \leq m + 1, \quad (3.18)
$$
$$
||\psi - \mathcal{I}_H^p \psi||_{t,\Gamma_{i,j}} \leq C H^{m-t} \|\psi\|_{\hat{m},\Gamma_{i,j}}, \quad 0 \leq t \leq 1, \ t \leq \hat{m} \leq m + 1, \quad (3.19)
$$
\[ \|v - \mathcal{P}_h^u v\|_{\Omega_i} \leq C h^l \|v\|_{\mathcal{I}_i, \Omega_i}, \quad 0 \leq l \leq l + 1, \quad (3.19) \]
\[ \|\zeta - \mathcal{P}_h^p \zeta\|_{\Omega_i} \leq C h^s \|\zeta\|_{\mathcal{S}_i, \Omega_i}, \quad 0 \leq s \leq s + 1, \quad (3.20) \]
\[ \|\zeta - \mathcal{R}_h \zeta\|_{\Omega_i} \leq C h^{\hat{j}} \|\zeta\|_{\mathcal{J}_j, \Omega_i}, \quad 0 \leq \hat{j} \leq j + 1, \quad (3.21) \]
\[ \|\psi - \mathcal{Q}_{h,i} \psi\|_{\Gamma_{i,j}} \leq C h^{k+t} \|\psi\|_{\mathcal{K}_{i,j}, \Omega_i}, \quad 0 \leq \hat{k} \leq k + 1, \quad (3.22) \]
\[ \|\psi - \mathcal{Q}_{h,j} \psi\|_{\Gamma_{i,j}} \leq C h^{r+t} \|\psi\|_{\mathcal{R}_{i,j}, \Omega_i}, \quad 0 \leq \hat{r} \leq r + 1, \quad (3.23) \]
\[ \|\tau - \hat{\Pi} \tau\|_{\Omega_i} \leq C h^k \|\tau\|_{\mathcal{K}_{i}, \Omega_i}, \quad 0 \leq \hat{k} \leq k + 1, \quad (3.24) \]
\[ \|\psi, \zeta\|_{\mathcal{I}_i, \Omega_i}, \quad \|\zeta - \Pi \zeta\|_{\Omega_i} \leq C h^{s} \|\zeta\|_{\mathcal{S}_i, \Omega_i}, \quad 0 \leq \hat{s} \leq s + 1, \quad (3.25) \]

where the functions \(\psi, v, \zeta, \tau,\) and \(\xi\) are taken from the domains of the operators acting on them. Bound (3.16) can be found in [51], bounds (3.18)–(3.22) and (3.25)–(3.26) are standard \(L^2\)-projection approximation bounds [18], and bounds (3.23)–(3.24) can be found in [17,35,49].

We will also use the trace inequalities
\[ \|\psi\|_{\partial \Omega_i} \leq C \|\psi\|_{\mathcal{I}_i, \Omega_i}, \quad t > 0, \quad (3.27) \]
\[ \langle \psi, \tau n \rangle_{\partial \Omega_i} \leq C \|\psi\|_{\mathcal{I}_i, \Omega_i} \|\tau\|_{H(\text{div}; \Omega_i)}, \quad \langle \psi, \zeta \cdot n \rangle_{\partial \Omega_i} \leq C \|\psi\|_{\mathcal{I}_i, \Omega_i} \|\zeta\|_{H(\text{div}; \Omega_i)}, \quad (3.28) \]
which can be found in [30] and [17,49], respectively.

Finally, define the projection operators \(\hat{\Pi}, \Pi, \mathcal{P}_h, \mathcal{P}_h^p, \mathcal{P}_h^u, \mathcal{R}_h, \mathcal{Q}_h^p, \text{ and } \mathcal{Q}_h^p\) on the respective spaces defined in the global domain \(\Omega\) to be the piece-wise application of \(\hat{\Pi}, \Pi, \mathcal{P}_h, \mathcal{P}_h^p, \mathcal{P}_h^u, \mathcal{R}_h, \mathcal{Q}_h^p, \text{ and } \mathcal{Q}_h^p,\) respectively, on subdomains \(\Omega_i\) for \(i = 1, \ldots, N\).

### 3.2 Discrete inf-sup conditions

In this subsection we give inf-sup stability bounds for the mortar space \(\Lambda_H\) and the weakly continuous stress \(\mathcal{X}_{h,0}\) and velocity \(\mathcal{Z}_{h,0}\) spaces under a coarseness condition on the mortar space \(\Lambda_H\).

**Assumption 1.** The mortar space \(\Lambda_H\) is chosen so that there exists a positive constant \(C\) independent of \(H\) and \(h\) such that
\[ \|\mu^*\|_{\Gamma_{i,j}} \leq C \left( \|\mathcal{Q}_{h,i}^* \mu^*\|_{\Gamma_{i,j}} + \|\mathcal{Q}_{h,j}^* \mu^*\|_{\Gamma_{i,j}} \right), \quad \forall \mu^* \in \Lambda_H^*, \quad 1 \leq i < j \leq n, \quad * \in \{p, u\}. \quad (3.29) \]

**Remark 3.1.** Assumption (3.29), which was first introduced in [8], implies that the mortar space \(\Lambda_H\) cannot be too rich compared to the normal traces of the subdomain stress/velocity spaces. In practice, this condition can be satisfied by taking a coarser mortar mesh, see [7,8,44].

**Lemma 3.1** (Pressure mortar inf-sup condition). Under assumption (3.29), there exists a constant \(\beta_D > 0\), independent of \(h\) and \(H\) such that for any \(\mu^p \in \Lambda_H^p,\)
\[ \|\mu^p\|_{\Gamma} \leq \beta_D \sup_{0 \neq \zeta \in Z_h} \sum_{i=1}^{N} \langle \zeta \cdot n_i, \mu^p \rangle_{\Gamma_i} \|\zeta\|_{Z_h}, \quad (3.30) \]

**Proof.** We start with any \(\mu^p \in \Lambda_H^p\) and extend it by zero on \(\partial \Omega\). Let \(\phi_i\) be the solution to the following auxiliary problem
\[ \text{div } \nabla \phi_i = \overline{\mathcal{Q}_{h,i}^p \mu^p}, \quad \text{in } \Omega_i, \quad (3.31) \]
\[ \nabla \phi_i \cdot n_i = \mathcal{Q}_{h,i}^p \mu^p, \quad \text{on } \partial \Omega_i, \quad (3.32) \]

\[ \text{div } \nabla \phi_i = \overline{\mathcal{Q}_{h,i}^p \mu^p}, \quad \text{in } \Omega_i, \quad (3.31) \]
\[ \nabla \phi_i \cdot n_i = \mathcal{Q}_{h,i}^p \mu^p, \quad \text{on } \partial \Omega_i, \quad (3.32) \]
where $Q_{h,i}^p, \mu^p$ denotes the mean value of $Q_{h,i}^p, \mu^p$ on $\partial \Omega_i$. Let $\psi_i = \nabla \phi_i$. The elliptic problem (3.31) is well-posed and its solution satisfies the elliptic regularity bound (3.32).

\[
\| \psi_i \|_{1/2, \Omega_i} + \| \text{div} \psi \|_{\Omega_i} \leq C \| Q_{h,i}^p, \mu^p \|_{\partial \Omega_i},
\]

(3.33)

Take $\zeta_{h,i} = \Pi_i^\# \psi_i \in Z_{h,i}$. Using (3.31), (3.32), and (3.2), we obtain

\[
\langle \zeta_{h,i} \cdot n_i, (\mu^p \rangle_{\Gamma_i} = \langle (\Pi_i^\# \psi_i \cdot n_i, \mu^p) \rangle_{\Gamma_i} = \langle (\Pi_i^\# \psi_i \cdot n_i, Q_{h,i}^p, \mu^p) \rangle_{\Gamma_i} = \langle \psi_i \cdot n_i, Q_{h,i}^p, \mu^p \rangle_{\Gamma_i} = \langle Q_{h,i}^p, \mu^p \rangle_{\Gamma_i} \geq C \| \mu^p \|_{1/2},
\]

(3.34)

where we have used the mortar coarseness assumption (3.29). Next, we note that

\[
\| \zeta_{h,i} \|_{\text{div}, \Omega_i} \leq C \| \mu^p \|_{\Gamma_i},
\]

(3.35)

which follows from the stability of $\Pi_i^\#$ (3.8) with $\epsilon = 1/2$, (3.33), and the stability of $Q_{h,i}^p$.

Finally, combining (3.34) with (3.35) and defining $\zeta := \zeta_{h,i}$ on $\Omega_i$ completes the proof.

**Lemma 3.2** (Displacement mortar inf-sup condition). Under assumption (3.29), there exists a constant $\beta_E > 0$, independent of $h$ and $H$ such that for any $\mu^u \in \Lambda^u_H$, the following bound holds

\[
\| \mu^u \|_{\Gamma} \leq \beta_E \sup_{0 \neq \tau \in X_h} \frac{\sum_{i=1}^N (\tau n_i, \mu^u)_{\Gamma_i}}{\| \tau \|_{X_h}}.
\]

(3.36)

**Proof.** The proof follows similar arguments as in the proof of Lemma 3.1.

**Lemma 3.3.** Under assumption (3.29), there exists a linear operator $\Pi_0^\# : H^{1/2+\epsilon}(\Omega, \mathbb{M}) \times X \rightarrow X_{h,0}$ for any $\epsilon > 0$, such that for any $\tau \in H^{1/2+\epsilon}(\Omega, \mathbb{M}) \times X$,

\[
\langle \text{div}(\Pi_0^\# \tau - \tau), v \rangle_{\Omega_i} = 0, \quad 1 \leq i \leq N, \quad \forall v \in V_{h,i},
\]

(3.37)

\[
(\Pi_0^\# \tau - \tau, \xi) = 0, \quad \forall \xi \in Q_{h,i},
\]

(3.38)

\[
\| \Pi_0^\# \tau \| \leq C \left( \| \tau \|_{1/2+\epsilon} + \| \text{div} \tau \| \right),
\]

(3.39)

\[
\| \Pi_0^\# \tau - \tau \|_{\Omega_i} \leq C \left( \sum_{i=1}^N h_i \| \tau \|_{\Omega_i} + h_{\tilde{k}} \| \Pi_0^\# \tau \|_{\tilde{k}+\tilde{H}} \right), \quad 0 < \tilde{k} \leq k + 1, \quad 0 < \tilde{h} \leq k + 1,
\]

(3.40)

\[
\| \text{div}(\Pi_0^\# \tau - \tau) \|_{\Omega_i} \leq C h_i \| \text{div} \tau \|_{\Omega_i}, \quad 1 \leq i \leq N, \quad 0 \leq \tilde{h} \leq l + 1.
\]

(3.41)

**Proof.** The proof is based on constructing $\Pi_0^\# \tau_{\partial \Omega_i} = \hat{\Pi}_0^\# (\tau + \delta \tau_i)$, where the correction $\delta \tau_i$ is designed to give weak continuity of the normal components. The proof of (3.37)–(3.40) is given in Lemma 4.6. Bound (3.41) follows from (3.37) and the approximation properties of the $L^2$-projection (18).

**Lemma 3.4.** Under assumption (3.29), there exists a linear operator $\Pi_0^\# : H^{1/2+\epsilon}(\Omega, \mathbb{R}^d) \cap Z \rightarrow Z_{h,0}$ such that for any $\zeta \in H^{1/2+\epsilon}(\Omega, \mathbb{R}^d) \cap Z$,

\[
\langle \text{div}(\Pi_0^\# \zeta - \zeta), w \rangle_{\Omega_i} = 0, \quad 1 \leq i \leq N, \quad \forall w \in W_{h,i},
\]

(3.42)

\[
\| \Pi_0^\# \zeta \|_{Z_h} \leq C \left( \| \zeta \|_{1/2+\epsilon} + \| \text{div} \zeta \| \right),
\]

(3.43)

\[
\| \Pi_0^\# \zeta - \zeta \| \leq C \left( \sum_{i=1}^N h_i \| \zeta \|_{\tilde{\Omega}_i} + h_{\tilde{s}} \| H^{1/2} \zeta \|_{\tilde{s}+\tilde{H}} \right), \quad 0 < \tilde{s} \leq r + 1, \quad 0 < \tilde{r} \leq r + 1,
\]

(3.44)

\[
\| \text{div}(\Pi_0^\# \zeta - \zeta) \|_{\Omega_i} \leq C h_{\tilde{s}} \| \text{div} \zeta \|_{\tilde{s}+\tilde{H}}, \quad 1 \leq i \leq N, \quad 0 \leq \tilde{s} \leq s + 1.
\]

(3.45)

**Proof.** The proof follows from the arguments given in Section 3 and Section 3.
Lemmas 3.3 and 3.4 can be used to show inf-sup stability for to the weakly continuous stress and velocity spaces.

Lemma 3.5. Under assumption (3.29), there exist positive constants $C_E$ and $C_D$ independent of the discretization parameters $h$ and $H$ such that

$$\forall v \in V_h, \xi \in Q_h, \quad \|v\| + \|\xi\| \leq C_E \sup_{0 \neq \tau \in K_{h,0}} \frac{(\text{div}_h \tau, v) + (\tau, \xi)}{\|\tau\|_{X_h}}, \quad (3.46)$$

and

$$\forall w \in W_h, \quad \|w\| \leq C_D \sup_{0 \neq \xi \in W_{h,0}} \frac{(\text{div}_h \xi, w)}{\|\xi\|_{Z_h}}. \quad (3.47)$$

Proof. The proof follows the argument in Fortin’s Lemma \cite[Proposition 2.8]{17}. We present the proof of (3.46). The proof of (3.47) is similar. We first note that the following continuous inf-sup condition holds \cite[Section 2.4.3]{26}:

$$\forall v \in V, \xi \in Q, \quad \|v\| + \|\xi\| \leq \tilde{C}_E \sup_{0 \neq \tau \in H^1(\Omega; M)} \frac{(\text{div} \tau, v) + (\tau, \xi)}{\|\tau\|_1}. \quad (3.48)$$

Using Lemma 3.3 and (3.48), we have, $\forall v \in V_h, \xi \in Q_h$,

$$\sup_{0 \neq \tau \in K_{h,0}} \frac{(\text{div}_h \tau, v) + (\tau, \xi)}{\|\tau\|_{X_h}} \geq \sup_{0 \neq \tau \in H^1(\Omega; M)} \frac{(\text{div} \Pi_0^\ast \tau, v) + (\Pi_0^\ast \tau, \xi)}{\|\Pi_0^\ast \tau\|_{X_h}} = \sup_{0 \neq \tau \in H^1(\Omega; M)} \frac{(\text{div} \tau, v) + (\tau, \xi)}{\|\Pi_0^\ast \tau\|_{X_h}} \geq \frac{1}{C} \sup_{0 \neq \tau \in H^1(\Omega; M)} \frac{(\text{div} \tau, v) + (\tau, \xi)}{\|\tau\|_1} \geq \frac{1}{CC_E} (\|v\| + \|\xi\|),$$

which implies (3.46) with $C_E = C\tilde{C}_E$. \hfill \Box

4 Well-posedness of the semi-discrete multiscale mortar MFE method

In this section we present the well-posedness analysis of the method developed in Section 2.2. We show that the method has a unique solution and establish stability bounds.

4.1 Existence and uniqueness of a solution

We next show the existence of a unique solution to the system of equations (2.18)–(2.24) under the assumption (3.29). We follow closely the proof for the well-posedness of the multipoint flux method for the Biot system given in \cite{6}. We base our proof on the theory for showing the existence of solution to a degenerate parabolic system \cite{53}. In particular, we use \cite[IV, Theorem 6.1(b)]{53} which is stated as follows.

Theorem 4.1. Let the linear, symmetric, and monotone operator $\mathcal{N}$ be given for the real vector space $E$ to its algebraic dual $E^*$, and let $E_b'$ be the Hilbert space which is the dual of $E$ with the seminorm $|x|_b = \sqrt{\mathcal{N}x(x)}$ for $x \in E$. Let $\mathcal{M} \subset E \times E_b'$ be a relation with the domain $D = \{x \in E : \mathcal{M}(x) \neq \emptyset\}$. Assume that $\mathcal{M}$ is monotone and $\text{Range}(\mathcal{N} + \mathcal{M}) = E_b'$. Then for each $x_0 \in D$ and for each $F \in W^{1,1}(0,T; E_b')$, there is a solution $x$ of

$$\frac{d}{dt} \mathcal{N}x(t) + \mathcal{M}x(t) \ni F(t), \quad a.e. \ 0 < t < T,$$

with

$$\mathcal{N}x \in W^{1,\infty}(0,T; E_b'), \quad x(t) \in D, \text{ for all } 0 \leq t \leq T, \text{ and } \mathcal{N}x(0) = \mathcal{N}x_0.$$
Using the above theorem, we now prove that the semi-discrete system (2.18), (2.24) is well-posed. We start by reformulating it to fit the setting of Theorem 4.1. For this purpose, we define operators

$$(A_{σσ}σ_h, τ) = (Aσ_h, τ), \quad (A_{σσ}σ_h, w) = α(Aσ_h, w), \quad (A_{σσ}σ_h, v) = (\text{div}_h σ_h, v),$$

$$(A_{σγ}σ_h, ξ) = (σ_h, ξ), \quad (A_{σλ}σ_h, μ^u) = \sum_{i=1}^N (σ_h n_i, μ^u)_Γ_i, \quad (A_{zz}z_h, ξ) = (K^{-1}z_h, ξ),$$

$$(A_{2p}z_h, w) = -(\text{div}_h z_h, w), \quad (A_{zz}z_h, μ^p) = \sum_{i=1}^N (z_h · n_i, μ^p)_Γ_i,$$

$$(A_{pp}p_h, w) = c_0 (p_h, w) + α^2 (A p_h I, w).$$

In order to fit in the structure of Theorem 4.1, we consider a modified problem where (2.18) is differentiated in time. Introducing the new variables $\dot{u}_h$, $\dot{γ}_h$, and $\dot{λ}_H^u$ representing $\partial_t u_h$, $\partial_t γ_h$, and $\partial_t λ_H^u$, respectively, we differentiate (2.18) in time to get

$$((\partial_t A (σ_h + α p_h I), τ) + (\dot{u}_h, \text{div}_h τ) + (\dot{γ}_h, τ) - \sum_{i=1}^N (\dot{λ}_H^u, τ n_i)_Γ_i = 0, \quad \forall τ \in X_h. \quad (4.1)$$

Using the above definitions of operators we can write the differentiated system (4.1), (2.19)–(2.24) as

$$\frac{d}{dt}N\dot{x}(t) + M\dot{x}(t) = F(t), \quad 0 < t < T, \quad (4.2)$$

where

$$\dot{x} = \begin{pmatrix} σ_h \\ \dot{u}_h \\ \dot{γ}_h \\ z_h \\ \dot{λ}_H^u \\ λ_H^P \end{pmatrix}, \quad N = \begin{pmatrix} A_{σσ} 0 0 0 & A_{σσ}^T 0 \\ 0 0 0 0 & 0 0 \\ 0 0 0 0 & 0 0 \\ 0 0 0 0 & 0 0 \\ 0 0 0 0 & 0 0 \\ 0 0 0 0 & 0 0 \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & A_{σσ}^T & 0 & 0 & 0 & 0 \\ -A_{σu} & 0 & 0 & 0 & 0 & 0 \\ -A_{σγ} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{zz} & 0 & 0 \\ 0 & 0 & -A_{zp} & 0 & 0 & 0 \\ A_{σλ} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ -f \\ 0 \\ 0 \\ g \\ 0 \end{pmatrix}.$$

The space $E$ is $X_h × V_h × Q_h × Z_h × W_h × Λ_H^u × Λ_H^P$. The dual space $E_b'$ is given by $L^2(Ω,M) × 0 × 0 × L^2(Ω) × 0 × 0$ and the condition $F ∈ W^{1,1}(0,T; E_b')$ implies that non-zero source terms can appear only in equations with time derivatives. This means we have to take $f = 0$ in our case. We can fix this issue by considering an auxiliary problem that, for each $t ∈ (0,T]$, solves the system

$$\begin{pmatrix} A_{σσ} & A_{σσ}^T & A_{σσ}^T & 0 & 0 \\ -A_{σu} & 0 & 0 & 0 & 0 \\ -A_{σγ} & 0 & 0 & 0 & 0 \\ A_{σλ} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} σ_h \\ μ^u \\ μ^p \\ λ_H^u \\ λ_H^P \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.3)$$
Such an auxiliary system (4.3) is well-posed and the proof can be found in [35]. Now we can subtract the solution to (4.3) from the original system of equations (2.18)–(2.24) to obtain the modified right-hand side \( F = \left( A_{\xi \sigma} \left( \sigma_h^T - \partial_t \sigma_t^T \right), 0, 0, q - A_{\sigma \rho} \partial_t \sigma_t^T, 0, 0 \right)^T \). Thus, it is enough to analyze (4.2) with \( f = 0 \).

In order to apply Theorem 4.1 for system (4.2), we need to prove the range condition Range(\( N + M \)) = \( E'_0 \) and construct compatible initial data \( \tilde{x}_0 \in D \), i.e., \( M \tilde{x}_0 \in E'_0 \). This is done in the following two lemmas.

**Lemma 4.2.** If assumption (3.29) holds, for the system (4.2) it holds that Range(\( N + M \)) = \( E'_0 \).

**Proof.** The statement of the lemma can be established by proving that the following homogeneous system has only the zero solution: \((\hat{\sigma}_h, \hat{u}_h, \hat{\gamma}_h, \hat{\dot{z}}_h, \hat{\dot{p}}_h, \hat{\lambda}_H) \in X_h \times V_h \times Q_h \times Z_h \times W_h \times \Lambda_H \) such that

\[
(A(\hat{\sigma}_h + \hat{\alpha} \hat{p}_h I), \tau) + (\hat{u}_h, \text{div}_h \tau) + (\hat{\gamma}_h, \tau) - \sum_{i=1}^N \langle \hat{\lambda}_H^n, \tau n_i \rangle_{\Gamma_i} = 0, \quad \forall \tau \in X_h, \tag{4.4}
\]

\[
(\text{div}_h \hat{\sigma}_h, v) = 0, \quad \forall v \in V_h, \tag{4.5}
\]

\[
(\hat{\sigma}_h, \xi) = 0, \quad \forall \xi \in Q_h, \tag{4.6}
\]

\[
(K^{-\frac{1}{2}} \hat{z}_h, \zeta) - (\hat{\dot{p}}_h, \text{div}_h \zeta) + \sum_{i=1}^N \langle \hat{\lambda}^p_i, \zeta \cdot n_i \rangle_{\Gamma_i} = 0, \quad \forall \zeta \in Z_h, \tag{4.7}
\]

\[
c_0 (\partial_t \hat{p}_h, w) + \alpha (A(\hat{\sigma}_h + \hat{\alpha} \hat{p}_h I), w I) + (\text{div}_h \hat{z}_h, w) = 0, \quad \forall w \in W_h, \tag{4.8}
\]

\[
\sum_{i=1}^N \langle \hat{\sigma}_h n_i, \mu^n \rangle_{\Gamma_i} = 0, \quad \forall \mu^n \in \Lambda_H^n, \tag{4.9}
\]

\[
\sum_{i=1}^N \langle \hat{\dot{z}}_h \cdot n_i, \mu^p \rangle_{\Gamma_i} = 0, \quad \forall \mu^p \in \Lambda_H^p. \tag{4.10}
\]

Taking test functions \((\tau, v, \xi, \zeta, w, \mu^n, \mu^p) = (\hat{\sigma}_h, \hat{u}_h, \hat{\gamma}_h, \hat{\dot{z}}_h, \hat{\dot{p}}_h, \hat{\lambda}_H^n, \hat{\lambda}_H^p)\) in the above system and adding the equations together gives \( \| A^\frac{1}{2} (\hat{\sigma}_h + \hat{\alpha} \hat{p}_h I) \|^2 + c_0 \| \hat{\dot{p}}_h \|^2 + \| K^{-\frac{1}{2}} \hat{z}_h \|^2 = 0 \). The coercivity of \( A \), (2.6), and \( K \), (2.8), give \( \hat{\sigma}_h = 0, \hat{\dot{p}}_h = 0, \) and \( \hat{\dot{z}}_h = 0 \). The inf-sup condition with respect to the weakly continuous space \( X_{h,0} \) (3.46) along with (4.4) implies \( \hat{u}_h = 0 \) and \( \hat{\gamma}_h = 0 \). Finally, (3.30) combined with (4.4) implies \( \hat{\lambda}_H^n = 0 \), and (3.36) combined with (4.7) implies \( \hat{\lambda}_H^p = 0 \).

**Lemma 4.3.** Let the assumption (3.29) hold. Given initial data \( p_0 \in H^1(\Omega) \) with \( K \nabla p_0 \in H(\text{div}; \Omega) \), there exists initial data \( \tilde{x}_0 \) for the system (4.2) such that \( M \tilde{x}_0 \in E'_0 \).

**Proof.** We first construct compatible initial data \((\sigma_0, u_0, \gamma_0, z_0, p_0)\) to the continuous system (2.11)–(2.15) from the initial data \( p_0 \) as follows:

1. Solve equations (2.11)–(2.13) using \( p = p_0 \) as given data to obtain \( \sigma_0, u_0, \gamma_0 \).

2. Set \( z_0 = -K \nabla p_0 \) and use integration by parts to show that (2.14) holds.

Next, define \( \tilde{x}_0 = (\sigma_0, u_0, \gamma_0, z_0, p_0, \lambda_H^n, \lambda_H^p) \), where \( \lambda_H^n = u_0 | r \) and \( \lambda_H^p = p_0 | r \). Take the initial data \( x_0 = (\sigma_{h,0}, u_{h,0}, \gamma_{h,0}, z_{h,0}, p_{h,0}, \lambda_{H,0}^n, \lambda_{H,0}^p) \) for the system (2.18)–(2.24) to be the elliptic projection of \( \tilde{x}_0 \):

\[
(\mathcal{N} + \mathcal{M}) x_0 = (\mathcal{N} + \mathcal{M}) \tilde{x}_0, \tag{4.11}
\]

The above problem has a unique solution, due to the argument in the proof of Lemma 4.2. With the reduction of the problem to the case with \( f = 0 \), we have \((\mathcal{N} + \mathcal{M}) \tilde{x}_0 \in \overline{E}_0' \) and, due to (4.11), \( M \tilde{x}_0 = (\mathcal{N} + \mathcal{M}) \tilde{x}_0 - \mathcal{N} x_0 \in E'_0 \). For the differentiated system (4.2) we take the initial data \( \hat{x}_0 \) to be \((\sigma_{h,0}, 0, 0, 0, 0, \lambda_{H,0}^p)\), which also satisfies \( M \hat{x}_0 \in E'_0 \). We note that the initial data \( u_{h,0}, \gamma_{h,0}, \) and \( \lambda_{H,0}^n \) are not required for solving (4.2), but are used to recover the solution to the original problem.
We are now ready to establish that the system \((2.2)\) has a solution using Theorem \(4.1\).

**Lemma 4.4.** Let assumptions \((3.29)\) hold. For each \((f, g) \in W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^d)) \times W^{1,\infty}(0, T; L^2(\Omega))\) and \(p_0 \in H^1(\Omega)\) with \(K \nabla p_0 \in H(\text{div}; \Omega)\), the system \((4.2)\) has a solution such that \(\sigma_h(0) = \sigma_{h,0}\) and \(p_h(0) = p_{h,0}\), with the initial data constructed in Lemma \(4.3\). In addition, \(z_h(0) = z_{h,0}\) and \(\lambda_H^p(0) = \lambda_{H,0}^p\).

**Proof.** We first note that the arguments in the proof of Lemma \(4.2\) can be used to show that \(\mathcal{N}\) and \(\mathcal{M}\) are non-negative and therefore, due to linearity, monotone. Using Lemmas \(4.2\) and \(4.3\), an application of Theorem \(4.1\) implies the existence of a solution \(\dot{x} = (\sigma_h, \dot{u}_h, \dot{\gamma}_h, z_h, p_h, \lambda^u_H, \lambda^p_H)\) to \((4.2)\) such that \(\sigma_h(0) = \sigma_{h,0}\) and \(p_h(0) = p_{h,0}\). Next, it is easy to see that \(z_h(0) = z_{h,0}\) by taking \(t \to 0\) in \((2.28)\) and using the fact that \(z_{h,0}\) and \(p_{h,0}\) satisfy \((2.28)\). Finally, taking \(t \to 0\) in \((2.21)\), using that \(z_{h,0}, p_{h,0}\), and \(\lambda_{H,0}^p\) satisfy \((2.21)\), and employing the inf-sup condition \((3.30)\), we conclude that \(\lambda_H^p(0) = \lambda_{H,0}^p\). \(\square\)

Next, we prove the solvability of the original system \((2.18)–(2.24)\).

**Theorem 4.5.** Let assumptions \((3.29)\) hold. For each \((f, g) \in W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^d)) \times W^{1,\infty}(0, T; L^2(\Omega))\) and \(p_0 \in H^1(\Omega)\) with \(K \nabla p_0 \in H(\text{div}; \Omega)\), the system \((2.18)–(2.24)\) has a unique solution such that \(\sigma_h(0) = \sigma_{h,0}\) and \(p_h(0) = p_{h,0}\), where the initial data is constructed in Lemma \(4.3\). In addition, \(u_h(0) = u_{h,0}, \gamma_h(0) = \gamma_{h,0}, z_h(0) = z_{h,0}\) and \(\lambda_H^0(0) = \lambda_{H,0}^0\), and \(\lambda_H^p(0) = \lambda_{H,0}^p\).

**Proof.** Let \((\sigma_h, \dot{u}_h, \dot{\gamma}_h, z_h, p_h, \lambda^u_H, \lambda^p_H)\) be a solution to the differentiated system \((4.1)\), \((2.19)–(2.24)\) obtained in Lemma \(4.4\). For each \(t \in [0, T]\), define

\[
\begin{align*}
\dot{u}_h(t) &= u_{h,0} + \int_0^t \dot{u}_h(s)ds, \\
\dot{\gamma}_h(t) &= \gamma_{h,0} + \int_0^t \dot{\gamma}_h(s)ds, \\
\lambda_H^u(t) &= \lambda_{H,0}^u + \int_0^t \lambda_H^u(s)ds.
\end{align*}
\]

Consider \(x = (\sigma_h, u_h, \gamma_h, z_h, p_h, \lambda_H^0, \lambda_H^p)\). Since equations \((2.19)–(2.24)\) do not involve \(u_h, \gamma_h\) or \(\lambda_H^0\), they still hold for \(x\). It remains to show that \((2.18)\) holds. This follows by integrating \((4.1)\) with respect to time from 0 to any \(t \in (0, T]\) and using \((4.12)\) and the fact that \(\sigma_{h,0}, u_{h,0}, \gamma_{h,0}\), and \(\lambda_{H,0}^p\) are constructed to satisfy \((2.18)\). This completes the proof that the system \((2.18)–(2.24)\) has a solution. Uniqueness of the solution follows from the stability bound presented in the next section. Finally, Lemma \(4.4\) gives us that \(\sigma_h(0) = \sigma_{h,0}\), \(p_h(0) = p_{h,0}\), \(z_h(0) = z_{h,0}\), and \(\lambda_H^p(0) = \lambda_{H,0}^p\), while \((4.12)\) gives us that \(u_h(0) = u_{h,0}, \gamma_h(0) = \gamma_{h,0}\), and \(\lambda_H^u(0) = \lambda_{H,0}^u\). \(\square\)

**4.2 Stability analysis**

In this subsection we give a stability bound for the system \((2.18)–(2.24)\).

**Theorem 4.6.** Under the assumption \((3.29)\), there exists a constant \(C > 0\), independent of \(c_0\) and the discretization parameters \(h\) and \(H\), such that for the solution of \((2.18)–(2.24)\),

\[
\|\sigma_h\|_{L^\infty(0,T;X_h)} + \|u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\gamma_h\|_{L^\infty(0,T;L^2(\Omega))} + \|z_h\|_{L^\infty(0,T;L^2(\Omega))} + \|p_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\lambda_H^u\|_{L^\infty(0,T;L^2(\Gamma))} + \|\lambda_H^p\|_{L^\infty(0,T;L^2(\Gamma))} + \|\dot{\gamma}_h\|_{L^2(0,T;L^2(\Omega))} + \|\dot{\gamma}_h\|_{L^2(0,T;L^2(\Gamma))} + \|\dot{z}_h\|_{L^2(0,T;L^2(\Omega))} + \|p_h\|_{L^2(0,T;L^2(\Omega))} + \|\lambda_H^p\|_{L^2(0,T;L^2(\Gamma))} + \|\lambda_H^u\|_{L^2(0,T;L^2(\Gamma))} + \|\nabla p_0\|_{H(\text{div};\Omega)}
\]

\[
\leq C\left(\|f\|_{H^1(0,T;L^2(\Omega))} + \|g\|_{H^1(0,T;L^2(\Omega))} + \|p_0\|_{H^1(\Omega)} + \|\nabla K p_0\|_{H(\text{div};\Omega)}\right). \tag{4.13}
\]

**Proof.** It is convenient to use the weakly continuous normal stress and velocity formulation \((2.25)–(2.29)\). We differentiate \((2.25)\) in time, combine it with \((2.26)–(2.29)\), and take test functions \((\tau, v, \xi, \zeta, w) = (\sigma_h, \partial_t u_h, \partial_t \gamma_h, z_h, p_h)\) to get

\[
(\partial_t A(\sigma_h + \alpha p_h I), \sigma_h + \alpha p_h I) + c_0 (\partial_t p_h, p_h) + (K^{-1} z_h, z_h) = (f, \partial_t u_h) + (q, p_h).
\]
The above equation can be rewritten as
\[
\frac{1}{2} \partial_t \left( \| A^{\frac{1}{2}} (\sigma_h + \alpha p_h I) \|^2 + c_0 \| p_h \|^2 \right) + \| K^{-\frac{1}{2}} z_h \|^2 = \partial_t (f, u_h) - (\partial_t f, u_h) + (g, p_h). \tag{4.14}
\]
For any \( t \in (0, T) \), we integrate equation (4.14) with respect to time from 0 to \( t \) to get
\[
\frac{1}{2} \left( \| A^{\frac{1}{2}} (\sigma_h + \alpha p_h I) (t) \|^2 + c_0 \| p_h(t) \|^2 \right) + \int_0^t \| K^{-\frac{1}{2}} z_h \|^2 ds = \frac{1}{2} \left( \| A^{\frac{1}{2}} (\sigma_h + \alpha p_h I) (0) \|^2 + c_0 \| p_h(0) \|^2 \right) + \int_0^t \left( (g, p_h) - (\partial_t f, u_h) \right) ds + (f, u_h) (t) - (f, u_h) (0).
\]

Applying the Cauchy-Schwartz and Young’s inequalities, we get, for any \( \epsilon > 0 \),
\[
\| A^{\frac{1}{2}} (\sigma_h + \alpha p_h I) (t) \|^2 + c_0 \| p_h(t) \|^2 + 2 \int_0^t \| K^{-\frac{1}{2}} z_h \|^2 ds \leq \| A^{\frac{1}{2}} (\sigma_h + \alpha p_h I) (0) \|^2 + c_0 \| p_h(0) \|^2 + \epsilon \left( \int_0^t (\| p_h \|^2 + \| u_h \|^2) ds + \| u_h(t) \|^2 \right) + \frac{1}{\epsilon} \left( \int_0^t (\| g \|^2 + \| \partial_t f \|^2) ds + \| f(t) \|^2 \right) + \| f(0) \|^2 + \| u_h(0) \|^2.
\]
A bound for \( \| u_h \| \) and \( \| \gamma_h \| \) follows from the inf-sup condition (3.46) and (2.25):
\[
\| u_h \| + \| \gamma_h \| \leq C_E \sup_{0 \neq \tau \in X_h} \frac{(u_h, \div_h \tau) + (\gamma_h, \tau)}{\| \tau \|_{X_h}} = C_E \sup_{0 \neq \tau \in X_h} \frac{A(\sigma_h + \alpha p_h I, \tau)}{\| \tau \|_{X_h}} \leq C \| \sigma_h + \alpha p_h I \|.
\]
\tag{4.16}

Next, choose test functions \( (\tau, v, \xi) = (\sigma_h, u_h, \gamma_h) \) in (2.25)–(2.27), combine the equations and use the Cauchy-Schwartz and Young’s inequalities to get
\[
\| \sigma_h \|^2 \leq C \left( \| p_h \|^2 + \epsilon \| u_h \|^2 + \frac{1}{\epsilon} \| f \|^2 \right).
\]
Combining the above inequality with (4.16) and taking \( \epsilon \) small enough yields
\[
\int_0^t \left( \| u_h \|^2 + \| \gamma_h \|^2 \right) ds \leq C \int_0^t \left( \| p_h \|^2 + \| f \|^2 \right) ds. \tag{4.17}
\]

Bound for \( \| p_h \| \) can be obtained from the inf-sup condition (3.47) and equation (2.28) as follows:
\[
\| p_h \| \leq C_D \sup_{0 \neq \zeta_h \in Z_h} \frac{\sum_{i=1}^N (\div \zeta_h, \rho_h) \Omega_i}{\| \zeta_h \|_{Z_h}} = C_D \sup_{0 \neq \zeta_h \in Z_h} \frac{(K^{-1} z_h, \zeta_h)}{\| \zeta_h \|_{Z_h}} \leq C \| z_h \|. \tag{4.18}
\]
Further, taking test function \( v = \div \sigma_h \) in (2.20) yields
\[
\| \div \sigma_h \|^2 \leq \| f \|^2. \tag{4.19}
\]
Combining inequalities (4.15)–(4.19) and taking \( \epsilon \) small enough, we obtain
\[
\| (\sigma_h + \alpha p_h I) (t) \|^2 + \| \div \sigma_h (t) \|^2 + \| u_h(t) \|^2 + \| \gamma_h(t) \|^2 + c_0 \| p_h(t) \|^2
\]
\[
+ \int_0^t \left( \| \sigma_h \|^2 + \| \div \sigma_h \|^2 + \| u_h \|^2 + \| \gamma_h \|^2 + \| z_h \|^2 + \| p_h \|^2 \right) ds \leq C \left( \int_0^t \left( \| g \|^2 + \| \partial_t f \|^2 + \| f \|^2 \right) ds + \| f(t) \|^2 + \| \sigma_h(0) \|^2 + \| u_h(0) \|^2 + \| p_h(0) \|^2 + \| f(0) \|^2 \right). \tag{4.20}
\]
Bound on \( \| \text{div}_h z_h \| \).

We continue with deriving a bound for \( \int_0^t \| \text{div}_h z_h \|^2 ds \). In the process we also obtain bounds on \( \| z_h(t) \| \) and \( \| p_h(t) \| \) for all \( t \in (0, T] \), which are independent of \( c_0 \). We start by choosing test function \( w = \text{div}_h z_h \) in (2.29) to obtain

\[
\| \text{div}_h z_h \| \leq C (c_0 \| \partial_t p_h \| + \| \partial_t (\sigma_h + \alpha p_h) \| + \| q \|). \tag{4.21}
\]

To bound the time-derivative terms on the right hand side of (4.21), differentiate equations (2.25)–(2.28) with respect to time and take \( (\tau, v, \xi, \zeta, w) = (\partial_t \sigma_h, \partial_t u_h, \partial_t \gamma_h, z_h, \partial_t p_h) \) in the differentiated equations and equation (2.29). Combining the resulting equations and integrating in time from 0 to \( t \in (0, T] \), we obtain, similarly to equations (4.14)–(4.15),

\[
2 \int_0^t \left( \| \partial_t A^\frac{1}{2} (\sigma_h + \alpha p_h) \|^2 + c_0 \| \partial_t p_h \| ^2 \right) ds + \| K^{-\frac{1}{2}} z_h(t) \|^2 \\
\leq \varepsilon \left( \int_0^t \| \partial_t u_h \|^2 ds + \| p_h(t) \|^2 \right) + \frac{1}{\varepsilon} \left( \int_0^t \| \partial_t f \|^2 ds + \| g(t) \|^2 \right) \\
+ \int_0^t \left( \| p_h \|^2 + \| \partial_t g \|^2 \right) ds + \| K^{-\frac{1}{2}} z_h(0) \|^2 + \| p_h(0) \|^2 + \| g(0) \|^2. \tag{4.22}
\]

To bound \( \| \partial_t u_h \| \), we use the inf-sup condition (3.46) and the time-differentiated (2.25) to obtain

\[
\| \partial_t u_h \| + \| \partial_t \gamma_h \| \leq C \| \partial_t (\sigma_h + \alpha p_h) \|. \tag{4.23}
\]

Combining inequalities (4.18), (4.22), and (4.23) and taking \( \varepsilon \) small enough gives

\[
\int_0^t \left( \| \partial_t (\sigma_h + \alpha p_h) \|^2 + c_0 \| \partial_t p_h \| ^2 + \| \partial_t u_h \|^2 + \| \partial_t \gamma_h \|^2 \right) ds + \| z_h(t) \|^2 + \| p_h(t) \|^2 \\
\leq C \left( \int_0^t \left( \| \partial_t f \|^2 + \| p_h \|^2 + \| \partial_t g \|^2 \right) ds + \| g(t) \|^2 + \| p_h(0) \|^2 + \| g(0) \|^2 \right). \tag{4.24}
\]

Integrating (4.21) with respect to time from 0 to \( t \in (0, T] \) and combining it with (4.24) and (4.20) gives

\[
\| p_h(t) \|^2 + \| z_h(t) \|^2 + \int_0^t \| \text{div}_h z_h \|^2 ds \leq C \left( \int_0^t \left( \| f \|^2 + \| \partial_t f \|^2 + \| g \|^2 + \| \partial_t g \|^2 \right) ds \\
+ \| f(t) \|^2 + \| g(t) \|^2 + \| \sigma_h(0) \|^2 + \| u_h(0) \|^2 + \| p_h(0) \|^2 + \| z_h(0) \|^2 + \| f(0) \|^2 + \| g(0) \|^2 \right) \tag{4.25}
\]

We further note that (4.20) and (4.25) provide bounds on \( \| (\sigma_h + \alpha p_h) (t) \| \) and \( \| p_h(t) \| \), which also gives a bound on \( \| \sigma_h(t) \| \), using

\[
\| \sigma_h \| \leq C \left( \| (\sigma_h + \alpha p_h) (t) \| + \| p_h(t) \| \right). \tag{4.26}
\]

Bounds on \( \| \lambda_H^u \| \) and \( \| \lambda_H^p \| \).

It remains to bound the Lagrange multipliers \( \| \lambda_H^u \| \) and \( \| \lambda_H^p \| \). For this purpose we utilize equations (2.18) and (2.21). Combining the inf-sup bound (3.36) and (2.18) gives

\[
\| \lambda_H^u \|_I \leq \beta_E \sup_{0 \neq \tau \in \mathbb{X}_h} \frac{\sum_{i=1}^N \langle \tau \eta_i, \lambda_H^u \rangle_{\mathbb{X}_h}}{\| \tau \|_{\mathbb{X}_h}} = \beta_E \sup_{0 \neq \tau \in \mathbb{X}_h} \frac{1}{\| \tau \|_{\mathbb{X}_h}} \left( \langle A (\sigma_h + \alpha p_h), \tau \rangle + (u_h, \text{div}_h \tau) + (\gamma_h, \tau) \right) \\
\leq C \left( \| \sigma_h \| + \| p_h \| + \| u_h \| + \| \gamma_h \| \right). \tag{4.27}
\]
Similarly, we bound \( \|\lambda _{H}^{P}\| \) by combining the inf-sup bound (3.30) and (2.21):
\[
\|\lambda _{H}^{P}\|_{\Gamma } \leq \beta _{D} \sup _{0 \neq \zeta \in \mathcal{Z}_{h}} \frac{\sum _{i=1}^{N}(\zeta \cdot n_{i}, \lambda _{H}^{P})_{\Gamma _{i}}}{\|\zeta \|_{\mathcal{Z}_{h}}} = \beta _{D} \sup _{0 \neq \zeta \in \mathcal{Z}_{h}} \frac{1}{\|\zeta \|_{\mathcal{Z}_{h}}} \left( -(K^{-1}z_{h}, \zeta ) + (p_{h}, \text{div}_{h} \zeta ) \right)
\leq C \left( \|z_{h}\| + \|p_{h}\| \right).
\]
(4.28)

Bound on the initial data.

In order to bound the initial data \( \sigma _{h}(0), u_{h}(0), z_{h}(0), \) and \( p_{h}(0) \), recall that the discrete initial data is obtained by taking elliptic projection of the continuous initial data, cf. (4.11). Further note that the continuous initial data is constructed from the original pressure initial data \( p_{0} \) using the procedure described in the proof of Lemma 4.3. Employing the steady-state version of the arguments used in the proof above for the weakly continuous normal stress and velocity formulation of (4.11) gives
\[
\|\sigma _{h}(0)\| + \|u_{h}(0)\| + \|\gamma _{h}(0)\| + \|z_{h}(0)\| + \|p_{h}(0)\| \leq C \left( \|\sigma _{0}\| + \|u_{0}\| + \|\gamma _{0}\| + \|z_{0}\| + \|p_{0}\| \right)
\leq C \left( \|p_{0}\|_{H^{1}(\Omega )} + \|K\nabla p_{0}\|_{H(\text{div};\Omega )} \right).
\]
(4.29)

Finally, bound (4.13) follows by combining inequalities (4.20) and (4.25) - (4.29).

\[ \square \]

5 Error analysis

In this section we establish a combined a priori error estimate for all unknowns in the method.

**Theorem 5.1.** Let \( (\sigma _{h}(t), u_{h}(t), \gamma _{h}(t), z_{h}(t), p_{h}(t), \lambda _{H}^{P}(t), \lambda _{H}^{P}(t)) \in \mathbb{X}_{h} \times \mathcal{V}_{h} \times \mathcal{Q}_{h} \times Z_{h} \times W_{h} \times \Lambda _{H}^{u} \times \Lambda _{H}^{P} \) be the solution to the system of equations (2.18) - (2.21) under the assumption (3.30) for \( t \in [0,T] \), and suppose that the solution of (2.11) - (2.15) is sufficiently smooth. Then there exists a positive constant \( C \), independent of \( h, \) and \( c_{0} \) such that
\[
\|
\sigma - \sigma _{h}\|_{L^{\infty }((0,T);\mathbb{X}_{h})} + \|u - u_{h}\|_{L^{\infty }((0,T);L^{2}(\Omega ))} + \|\gamma - \gamma _{h}\|_{L^{\infty }((0,T);L^{2}(\Omega ))} + \|z - z_{h}\|_{L^{\infty }((0,T);L^{2}(\Omega ))}
\]
\[
+ \|p - p_{h}\|_{L^{\infty }((0,T);L^{2}(\Omega ))} + \|u - \lambda _{H}^{u}\|_{L^{\infty }((0,T);L^{2}(\Gamma ))} + \|p - \lambda _{H}^{P}\|_{L^{\infty }((0,T);L^{2}(\Gamma ))} + \|\sigma - \sigma _{h}\|_{L^{2}(0,T;\mathbb{X}_{h})}
\]
\[
+ \|u - u_{h}\|_{L^{2}(0,T;L^{2}(\Omega ))} + \|\gamma - \gamma _{h}\|_{L^{2}(0,T;L^{2}(\Omega ))} + \|z - z_{h}\|_{L^{2}(0,T;Z_{h})} + \|p - p_{h}\|_{L^{2}(0,T;L^{2}(\Omega ))}
\]
\[
\leq C \left( h^{k_{1}}\|\sigma\|_{H^{1}(0,T;H^{k_{1}+\frac{1}{2}}(\Omega ))} + h^{k_{2}}H^{\frac{1}{2}}\|\sigma\|_{H^{1}(0,T;H^{k_{2}+\frac{1}{2}}(\Omega ))} + h^{k_{3}}\|\text{div}\sigma\|_{L^{\infty }((0,T;H^{k_{3}}(\Omega )))}
\]
\[
+ h^{l_{1}}\|u\|_{L^{2}(0,T;H^{l_{1}}(\Omega ))} + h^{l_{2}}\|u\|_{L^{2}(0,T;H^{l_{2}}(\Omega ))} + h^{l_{3}}\|u\|_{L^{\infty }((0,T;H^{l_{3}}(\Omega )))} + h^{j_{1}}\|\gamma\|_{H^{1}(0,T;H^{j_{1}}(\Omega ))}
\]
\[
+ h^{r_{1}}\|z\|_{H^{1}(0,T;H^{r_{1}}(\Omega ))} + h^{r_{2}}\|z\|_{H^{1}(0,T;H^{r_{2}}(\Omega ))} + h^{s_{1}}\|\text{div}z\|_{L^{2}(0,T;H^{s_{1}}(\Omega ))}
\]
\[
+ h^{s_{2}}\|p\|_{H^{1}(0,T;H^{s_{2}}(\Omega ))} + H^{m_{1}+\frac{1}{2}}\|u\|_{H^{2}(0,T;H^{m_{1}+\frac{1}{2}}(\Omega ))} + H^{m_{2}+\frac{1}{2}}\|p\|_{H^{1}(0,T;H^{m_{2}+\frac{1}{2}}(\Omega ))},
\]
where
\[
0 < k_{1}, k_{2} \leq k + 1, \quad 0 \leq l_{1}, l_{2}, l_{3}, l_{4} \leq l + 1, \quad 0 \leq j_{1} \leq j + 1,
\]
\[
0 < r_{1}, r_{2} \leq r + 1, \quad 0 \leq s_{1}, s_{2} \leq s + 1, \quad 0 \leq m_{1}, m_{2} \leq m + 1.
\]

**Proof.** First, note that the solution to (2.11) - (2.15) satisfies, for \( 1 \leq i \leq N \),
\[
(A(\sigma + \alpha p I), \tau )_{\Omega _{i}} + (u, \text{div}\tau )_{\Omega _{i}} + (\gamma, \tau )_{\Omega _{i}} - \langle u, \tau n_{i}\rangle _{\Gamma _{i}} = 0, \quad \forall \tau \in \mathbb{X}_{i}, \quad (5.1)
\]
\[
(\text{div}\sigma, v)_{\Omega _{i}} = - (f, v)_{\Omega _{i}}, \quad \forall v \in \mathcal{V}_{i}, \quad (5.2)
\]
\[
(\sigma, \xi )_{\Omega _{i}} = 0, \quad \forall \xi \in \mathcal{Q}_{i}, \quad (5.3)
\]
\[
(K^{-1}z, \zeta )_{\Omega _{i}} - (p, \text{div}\zeta )_{\Omega _{i}} + (p, \zeta \cdot n_{i})_{\Gamma _{i}} = 0, \quad \forall \zeta \in Z_{i}, \quad (5.4)
\]
\[ c_0 (\partial_t p, w)_{\Omega_i} + \alpha (\partial_t A(\sigma + \alpha p I), w I)_{\Omega_i} + (\text{div} z, w)_{\Omega_i} = (g, w)_{\Omega_i}, \quad \forall w \in W_i. \quad (5.5) \]

Subtracting the weakly continuous normal stress and velocity system \((2.25)-(2.29)\) from \((5.1)-(5.5)\) gives

\[ (A ((\sigma - \sigma_h) + \alpha (p - p_h) I), \tau) + (u - u_h, \text{div}_h \tau) + (\gamma - \gamma_h, \tau) - \sum_{i=1}^{N} \langle u, \tau n_i \rangle_{\Gamma_i} = 0, \quad \forall \tau \in \mathbb{X}_{h,0}. \quad (6.6) \]

\[ (\text{div}_h (\sigma - \sigma_h), v) = 0, \quad \forall v \in V_h, \quad (5.7) \]

\[ ((\sigma - \sigma_h), \xi) = 0, \quad \forall \xi \in \mathbb{Q}_h, \quad (5.8) \]

\[ (K^{-1} (z - z_h), \zeta) - (p - p_h, \text{div}_h \zeta) + \sum_{i=1}^{N} \langle p, \zeta \cdot n_i \rangle_{\Gamma_i} = 0, \quad \forall \zeta \in Z_{h,0}. \quad (5.9) \]

\[ c_0 (\partial_t (p - p_h), w) + \alpha (\partial_t A((\sigma - \sigma_h) + \alpha (p - p_h) I), w I) + (\text{div}_h (z - z_h), w) = 0, \quad \forall w \in W_h. \quad (5.10) \]

Next, rewrite the above error equations in terms of the approximation errors \(\psi_*\) and discretization errors \(\phi_*\), for \(* \in \{\sigma, u, \gamma, z, p, \lambda^u, \lambda^p\}\) as follows:

\[ \sigma - \sigma_h = (\sigma - \Pi_0^\sigma \sigma) + (\Pi_0^\sigma \sigma - \sigma_h) := \psi_\sigma + \phi_\sigma, \]
\[ u - u_h = (u - \mathcal{P}_h^u u) + (\mathcal{P}_h^u u - u_h) := \psi_u + \phi_u, \]
\[ \gamma - \gamma_h = (\gamma - \mathcal{R}_h \gamma) + (\mathcal{R}_h \gamma - \gamma_h) := \psi_\gamma + \phi_\gamma, \]
\[ z - z_h = (z - \Pi_0^z z) + (\Pi_0^z z - z_h) := \psi_z + \phi_z, \]
\[ p - p_h = (p - \mathcal{P}_h^p p) + (\mathcal{P}_h^p p - p_h) := \psi_p + \phi_p, \]
\[ u - \lambda_H^u = (u - \mathcal{Q}_h^u u) + (\mathcal{Q}_h^u u - \lambda_H^u) := \psi_{\lambda^u} + \phi_{\lambda^u}, \]
\[ p - \lambda_H^p = (p - \mathcal{Q}_h^p p) + (\mathcal{Q}_h^p p - \lambda_H^p) := \psi_{\lambda^p} + \phi_{\lambda^p}. \]

Combining \((5.7)\) with \((3.37)\) gives

\[ \text{div}_h \phi_\sigma = 0, \quad (5.11) \]

and \((5.8)\) combined with \((3.38)\) gives

\[ (\phi_\sigma, \xi) = 0 \quad \text{for} \quad \xi \in \mathbb{Q}_h. \quad (5.12) \]

We rewrite error equation \((5.6)\) as

\[ (A (\phi_\sigma + \alpha \phi_p I), \tau) + (\phi_u, \text{div}_h \tau) + (\phi_\gamma, \tau) = - (A (\psi_\sigma + \alpha \psi_p I), \tau) - (\psi_\gamma, \tau) + \sum_{i=1}^{N} \langle u - \mathcal{I}_H^u u, \tau n_i \rangle_{\Gamma_i}, \quad (5.13) \]

where we have used that \(\sum_{i=1}^{N} \langle \mathcal{I}_H^u u, \tau n_i \rangle_{\Gamma_i} = 0\) for any \(\tau \in \mathbb{X}_{h,0}\). Differentiating the above equation with respect to time gives

\[ (\partial_t A (\phi_\sigma + \alpha \phi_p I), \tau) + (\partial_t \phi_u, \text{div}_h \tau) + (\partial_t \phi_\gamma, \tau) = - (\partial_t A (\psi_\sigma + \alpha \psi_p I), \tau) - (\partial_t \psi_\gamma, \tau) + \sum_{i=1}^{N} \langle \partial_t (u - \mathcal{I}_H^u u), \tau n_i \rangle_{\Gamma_i}. \quad (5.14) \]

Taking \(\tau = \phi_\sigma\) in \((5.14)\) and using \((5.11)\) and \((5.12)\) gives

\[ (\partial_t A (\phi_\sigma + \alpha \phi_p I), \phi_\sigma) = - (\partial_t A (\psi_\sigma + \alpha \psi_p I), \phi_\sigma) - (\partial_t \psi_\gamma, \phi_\sigma) + \sum_{i=1}^{N} \langle \partial_t (u - \mathcal{I}_H^u u), \phi_\sigma n_i \rangle_{\Gamma_i}. \quad (5.15) \]
Error equation (5.10) can be written as
\[ c_0 (\partial_t \phi_p, w) + \alpha (\partial_t A (\psi + \alpha \psi_p I), w I) + (\text{div}_h \phi_z, w) = -\alpha (\partial_t A (\psi + \alpha \psi_p I), w I), \]
where we have used (3.9) and (3.42). Taking \( w = \phi_p \) in (5.16) and combining the resulting equation with (5.15), we obtain
\[ \frac{1}{2} \partial_t \left( \| A^{\frac{1}{2}} (\phi_p + \alpha \psi_p I) \|^2 + c_0 \| \phi_p \|^2 \right) + (\text{div}_h \phi_z, \phi_p) = - (\partial_t A (\psi + \alpha \psi_p I), \phi_p + \alpha \psi_p I) - (\partial_t \psi, \phi_p) + \sum_{i=1}^N \langle \partial_t (u - T_H^I u), \phi_p \rangle_{\Gamma_i}. \]
Error equation (5.9) can be written as
\[ (K^{-1} \phi_z, \zeta) - (\phi_p, \text{div}_h \zeta) = - (K^{-1} \psi_z, \zeta) + \sum_{i=1}^N \langle T_H^p p - p, \zeta \cdot n_i \rangle_{\Gamma_i}, \]
where we have used (3.9) and
\[ \sum_{i=1}^N \langle T_H^p p, \zeta \cdot n_i \rangle_{\Gamma_i} = 0 \quad \forall \zeta \in Z_{h,0}. \]
Taking test function \( \zeta = \phi_z \) in equation (5.18) and combining the resulting equation with (5.17) gives
\[ \frac{1}{2} \partial_t \left( \| A^{\frac{1}{2}} (\phi_p + \alpha \psi_p I) \|^2 + c_0 \| \phi_p \|^2 \right) + \| K^{-\frac{1}{2}} \phi_z \|^2 = - (\partial_t A (\psi + \alpha \psi_p I), \phi_p + \alpha \psi_p I) - (\partial_t \psi, \phi_p) - \sum_{i=1}^N \langle \partial_t (T_H^I u - u), \phi_p \rangle_{\Gamma_i} + \sum_{i=1}^N \langle T_H^p p - p, \phi_z \cdot n_i \rangle_{\Gamma_i}. \]
We bound the first three terms on the right hand side of (5.20) as follows:
\[ | (\partial_t A (\psi + \alpha \psi_p I), \phi_p + \alpha \psi_p I) | + | (\partial_t \psi, \phi_p) | + | (K^{-1} \psi_z, \phi_z) | \]
\[ \leq \| \partial_t A (\psi + \alpha \psi_p I) \| \| \phi_p + \alpha \psi_p I \| + \| \partial_t \psi \| \| \phi_p \| + \| K^{-1} \psi_z \| \| \phi_z \| \]
\[ \leq \frac{C}{\epsilon} (\| \partial_t \psi \|^2 + \| \partial_t \psi_p \|^2 + \| \partial_t \psi \|^2) + \epsilon (\| \phi_p \|^2 + \| \phi_p \|^2 + \| \phi_z \|^2), \]
where we have used Young’s inequality for a some \( \epsilon > 0 \).
Next, we give a bound on the last two boundary terms in the right hand side of equation (5.20). For this, we note that the following bounds hold for any \((\tau, v) \in H(\text{div}; \Omega, M) \times H^1_0(\Omega, \mathbb{R}^d)\) and \((\zeta, w) \in H(\text{div}; \Omega) \times H^1(\Omega)\):
\[ \langle T_H^I v - v, \tau n_i \rangle_{\Gamma_i} = \langle E_i (T_H^I v - v), \tau n_i \rangle_{\partial \Omega_i} \leq C \| E_i (T_H^I v - v) \|_{\frac{1}{2}, \partial \Omega_i} \| \tau \|_{H(\text{div}; \Omega_i)} \leq C \| T_H^I v - v \|_{\frac{1}{2}, \Gamma_i} \| \tau \|_{H(\text{div}; \Omega_i)}, \]
\[ \langle T_H^p w - w, \zeta \cdot n_i \rangle_{\Gamma_i} = \langle E_i (T_H^p \zeta - \zeta), \zeta \cdot n_i \rangle_{\partial \Omega_i} \leq C \| E_i (T_H^p \zeta - \zeta) \|_{\frac{1}{2}, \partial \Omega_i} \| \zeta \|_{H(\text{div}; \Omega_i)} \leq C \| T_H^p \zeta - \zeta \|_{\frac{1}{2}, \Gamma_i} \| \zeta \|_{H(\text{div}; \Omega_i)}, \]
where \( E_i \) denotes the extension by zero from \( \Gamma_i \) to \( \partial \Omega_i \), which is continuous in the \( H^\frac{1}{2} \)-norm for functions that are zero on \( \partial \Gamma_i \), and we have used the trace inequalities in (3.28). Taking \((\tau, v) = (\phi_p, \partial_t u)\) and \((\zeta, w) = (\phi_z, p)\) in (5.22) and (5.23), respectively, and using Young’s inequality gives
\[ \langle \partial_t (T_H^I u - u), \phi_p n_i \rangle_{\Gamma_i} \leq \frac{C}{\epsilon} \| \partial_t (T_H^I u - u) \|_{\frac{1}{2}, \Gamma_i}^2 + \epsilon \| \phi_p \|^2_{\Omega_i}, \]
\begin{equation}
\langle T^p_{\Gamma} p - p, \phi_z \cdot n_i \rangle_{\Gamma_i} \leq \frac{C}{\epsilon} \| T^p_{\Gamma} p - p \|_{\frac{3}{2}, \Gamma_i}^2 + \epsilon (\| \phi_z \|_{L^2_{\Gamma_i}}^2 + \| \text{div} \ \phi_z \|_{L^2_{\Gamma_i}}^2),
\end{equation}
where we also used \((5.11)\). Combining inequalities \((5.20) - (5.24)\) and integrating with respect to time from 0 to \(t \in (0, T]\) gives

\begin{align*}
\| A^\frac{1}{2} \left( \phi_\sigma + \alpha \phi_p I \right) (t) \|^2 &+ c_0 \| \phi_p (t) \|^2 + \int_0^t \| K^{-\frac{1}{2}} \phi_z \|^2 \\
&\leq C \int_0^t \left( \| \partial_t \psi_\sigma \|^2 + \| \partial_t \psi_p \|^2 + \| \psi_\sigma \|^2 + \| \psi_p \|^2 + \| T^u_{\Gamma} \partial_t u - \partial_t u \|_{L^2_{\frac{3}{2}, \Gamma_i}}^2 + \| T^p_{\Gamma} p - p \|_{L^2_{\frac{3}{2}, \Gamma_i}}^2 \right) ds \\
&\quad + \epsilon \int_0^t \left( \| \phi_\sigma \|^2 + \| \phi_p \|^2 + \| \phi_z \|^2 \right) ds + C \int_0^t \| \text{div} \phi_z \|^2 ds + \| A^\frac{1}{2} \left( \phi_\sigma + \alpha \phi_p I \right) (0) \|^2 + c_0 \| \phi_p (0) \|^2.
\end{align*}

(5.26)

Next, bound the errors of the form \(\phi_\star\) for \(\star \in \{\sigma, \gamma, u, p\}\). Using the inf-sup condition \((3.46)\), the error equation \((5.13)\), and \((5.11)\) gives

\begin{align*}
\| \phi_u \| + \| \phi_\gamma \| &\leq C_E \sup_{0 \neq \tau \in \mathcal{X}_h} \frac{(\phi_u, \text{div}_h \tau) + (\phi_\gamma, \tau)}{\| \tau \|_{\mathcal{X}_h}} \\
&= C_E \sup_{0 \neq \tau \in \mathcal{X}_h} \frac{1}{\| \tau \|_{\mathcal{X}_h}} \left( (A (\phi_\sigma + \alpha \phi_p I), \tau) + (A (\psi_\sigma + \alpha \psi_p I), \tau) + (\psi_\gamma, \tau) - \sum_{i=1}^N \left((T^u_{\Gamma_i} u - u, \tau n_i)_{\Gamma_i} \right) \right) \\
&\leq C \left( \| \phi_\sigma + \alpha \phi_p I \| + \| \psi_\sigma \| + \| \psi_\gamma \| + \| \psi_p \| + \| T^u_{\Gamma} u - u \|_{L^2_{\frac{3}{2}, \Gamma_i}} \right).
\end{align*}

(5.27)

where we have used \((5.22)\) with \(v = u\) in the last inequality. The above inequality implies

\begin{equation}
\int_0^t \left( \| \phi_u \|^2 + \| \phi_\gamma \|^2 \right) ds \leq C \int_0^t \left( \| \phi_\sigma \|^2 + \| \phi_p \|^2 + \| \psi_\sigma \|^2 + \| \psi_\gamma \|^2 + \| \psi_p \|^2 + \| T^u_{\Gamma} u - u \|_{L^2_{\frac{3}{2}, \Gamma_i}}^2 \right) ds.
\end{equation}

(5.28)

To bound \(\| \phi_p \|\), we use the inf-sup condition \((3.47)\) and the error equation \((5.18)\) to get

\begin{align*}
\| \phi_p \| &\leq C_D \sup_{0 \neq \xi \in \mathcal{Z}_h} \frac{\sum_{i=1}^N (\text{div} \ \xi, \phi_p)_{\Omega_i}}{\| \xi \|_{\mathcal{Z}_h}} \\
&= C_D \sup_{0 \neq \xi \in \mathcal{Z}_h} \frac{(K^{-1} \phi_z, \xi) + (K^{-1} \psi_z, \xi) - \sum_{i=1}^N (T^p_{\Gamma} p - p, \xi \cdot n_i)_{\Gamma_i}}{\| \xi \|_{\mathcal{Z}_h}} \\
&\leq C \left( \| \psi_z \| + \| \phi_z \| + \| T^p_{\Gamma} p - p \|_{L^2_{\frac{3}{2}, \Gamma_i}} \right).
\end{align*}

(5.29)

where we have used \((5.23)\) with \(w = p\) to obtain the last inequality. The above inequality yields

\begin{equation}
\int_0^t \| \phi_p \|^2 ds \leq C \int_0^t \left( \| \psi_z \|^2 + \| \phi_z \|^2 + \| T^p_{\Gamma} p - p \|_{L^2_{\frac{3}{2}, \Gamma_i}}^2 \right) ds.
\end{equation}

(5.30)

To bound the term \(\int_0^t \| \phi_\sigma \|^2 ds\), which appears on the right-hand side of \((5.28)\), we take \(\tau = \phi_\sigma\) in \((5.13)\) and \(\xi = \phi_\gamma\) in \((5.8)\), and use \((5.11) - (5.12)\) to get

\begin{align*}
\| A^\frac{1}{2} \phi_\sigma \|^2 &= -\left( A^\frac{1}{2} \alpha \phi_p I, \phi_\sigma \right) - (A (\psi_\sigma + \alpha \psi_p I), \phi_\sigma) - (\psi_\gamma, \phi_\sigma) - \sum_{i=1}^N (T^u_{\Gamma_i} u - u, \phi_\sigma n_i)_{\Gamma_i} + (\psi_\gamma, \phi_\gamma) \\
&\leq C \left( \| \phi_p \| + \| \psi_\sigma \| + \| \psi_p \| + \| \psi_\gamma \| + \| T^u_{\Gamma} u - u \|_{L^2_{\frac{3}{2}, \Gamma_i}} \| \phi_\sigma \| + \| \psi_\sigma \| \| \phi_\gamma \| \right)
\end{align*}
\[
\leq \frac{C}{\epsilon} \left( \|\phi_p\|^2 + \|\psi_\sigma\|^2 + \|\psi_p\|^2 + \|\psi_z\|^2 + \|I_H^t u - u\|_{L^2(R)}^2 \right) + \epsilon \left( \|\phi_\sigma\|^2 + \|\phi_\gamma\|^2 \right),
\]  
(5.31)

where we have used (5.22), (5.11), and Young’s inequality. Integrating (5.31) with respect to time from 0 to \(t \in (0, T)\), and taking \(\epsilon\) small enough, we get

\[
\int_0^t \|\phi_\sigma\|^2 ds \leq C \int_0^t \left( \|\phi_p\|^2 + \|\psi_\sigma\|^2 + \|\psi_p\|^2 + \|\psi_z\|^2 + \|I_H^t u - u\|_{L^2(R)}^2 \right) ds + \epsilon \int_0^t \|\phi_\gamma\|^2 ds.
\]  
(5.32)

Combining (5.26)–(5.32) and (5.11), and taking \(\epsilon\) small enough gives

\[
\|\phi_\sigma + \alpha \phi_I\|^2 + \|\phi_u\|^2 + \|\phi_\sigma\|^2 + c_0 \|\phi_p\|^2 + \|\text{div}_h \phi_\sigma\|^2
\]

\[
+ \int_0^t \left( \|\phi_\sigma\|^2 + \|\phi_u\|^2 + \|\phi_\gamma\|^2 + \|\phi_z\|^2 + \|\phi_p\|^2 + \|\text{div}_h \phi_\sigma\|^2 \right) ds
\]

\[
\leq C \left( \int_0^t \left( \|\partial_t \psi_\sigma\|^2 + \|\partial_t \psi_p\|^2 + \|\partial_t \psi_z\|^2 + \|\psi_\sigma\|^2 + \|\psi_p\|^2 + \|\psi_z\|^2 + \|\phi_\sigma\|^2 \right) ds
\]

\[
+ \|\psi_\sigma\|^2 + \|\psi_p\|^2 + \|\psi_z\|^2 + \|I_H^t u - u\|^2_{L^2(R)} + \|I_H^t p - p\|^2_{L^2(R)}
\]

\[
+ \int_0^t \left( \|I_H^t \partial_t u - \partial_t u\|^2_{L^2(R)} + \|I_H^t u - u\|^2_{L^2(R)} + \|I_H^t p - p\|^2_{L^2(R)} \right) ds + \|\phi_\sigma(0)\|^2 + \|\phi_p(0)\|^2 \right).  
\]  
(5.33)

**Bound on \(\|\text{div}_h \phi_\sigma\|\).**

Next, we obtain a \(L^2(0, T)\) bound on the error in \(\text{div}_h z_h\), as well as bounds on the error in \(\|z_h(t)\|\) and \(\|p_h(t)\|\) for all \(t \in (0, T]\), which are independent of \(c_0\). We start by taking \(w = \phi_z\) in (5.10), to get

\[
\|\text{div}_h \phi_z\|^2 = - (c_0 \partial_t \phi_p, \text{div}_h \phi_z) - (c_0 \partial_t \psi_p, \text{div}_h \phi_z) - \alpha (\partial_t A(\phi_\sigma + \alpha \phi_I), (\text{div}_h \phi_z))
\]

\[
- \alpha (\partial_t A(\psi_\sigma + \alpha \psi_p), (\text{div}_h \phi_z)) - (\psi_z, \text{div}_h \phi_z)
\]

\[
= - (c_0 \partial_t \phi_p, \text{div}_h \phi_z) - \alpha (\partial_t A(\phi_\sigma + \alpha \phi_I), (\text{div}_h \phi_z)) - \alpha (\partial_t A(\psi_\sigma + \alpha \psi_p), (\text{div}_h \phi_z)),
\]

where the last equality follows from (3.9) and (3.42). The above inequality implies

\[
\int_0^T \|\text{div}_h \phi_z\|^2 ds \leq C \int_0^T \left( c_0 \|\partial_t \phi_p\|^2 + \|\partial_t (\phi_\sigma + \alpha \phi_I)\|^2 + \|\psi_p\|^2 + \|\psi_z\|^2 + \|\phi_\sigma\|^2 \right) ds.  
\]  
(5.34)

In order to bound \(c_0 \|\partial_t \phi_p\|^2\) and \(\|\partial_t (\phi_\sigma + \alpha \phi_I)\|^2\), we differentiate in time (5.12) and (5.18), combine them with (5.14) and (5.16), and take test functions \(\tau = \partial_t \phi_\sigma, \xi = \partial_t \phi_\gamma, \zeta = \phi_z\), and \(w = \partial_t \phi_p\) to get the following time differentiated version of (5.20):

\[
\|\partial_t A \frac{1}{\alpha} (\phi_\sigma + \alpha \phi_I)\|^2 + c_0 \|\partial_t \phi_p\|^2 + \frac{1}{2} \partial_t \|K^{-\frac{1}{2}} \phi_z\|^2
\]

\[
= - (\partial_t A(\psi_\sigma + \alpha \psi_p)), \partial_t (\phi_\sigma + \alpha \phi_I)) - (\partial_t \psi_\gamma, \partial_t (\phi_\sigma + \alpha \phi_I)) - \alpha (\partial_t K^{-1} \phi_z, \phi_z)
\]

\[
- \sum_{i=1}^N (\partial_t (I_H^t u - u), \partial_t \phi_\sigma n_i)_{\Gamma_i} + \sum_{i=1}^N (\partial_t I_H^t p - \partial_t p, \phi_z \cdot n_i)_{\Gamma_i},
\]  
(5.35)

where we have used the fact that \((\partial_t \psi_\gamma, \partial_t \phi_\sigma) = (\partial_t \psi_\gamma, \partial_t (\phi_\sigma + \alpha \phi_I))\).

Using the Cauchy-Schwarz and Young’s inequalities for the first three terms on the right in (5.35) with \(\epsilon > 0\) and taking \(\epsilon\) small enough results in

\[
\|\partial_t (\phi_\sigma + \alpha \phi_I)\|^2 + c_0 \|\partial_t \phi_p\|^2 + \|\partial_t \phi_z\|^2 \leq C \left( \|\partial_t \psi_\sigma\|^2 + \|\partial_t \psi_p\|^2 + \|\partial_t \psi_\zeta\|^2 + \|\partial_t \psi_z\|^2 + \epsilon \|\phi_z\|^2 \right)
\]
To bound $\langle \partial_t (I_H^u u - u), \partial_t \phi_n \rangle_{\Gamma_1}$, we use integration by parts to rewrite it as

$$
\langle \partial_t (I_H^u u - u), \partial_t \phi_n \rangle_{\Gamma_1} = \frac{\partial}{\partial t} \langle (I_H^u u - u, \phi_n) \rangle_{\Gamma_1} - \langle \partial^2_t (I_H^u u - u), \phi_n \rangle_{\Gamma_1}.
$$

(5.37)

To bound the last term on the right in (5.37), we take $(\tau, v) = (\phi_n, \partial^2_t u)$ in (5.22) and use (5.11) to get

$$
\left| \langle \partial^2_t (I_H^u u - u), \phi_n \rangle_{\Gamma_1} \right| \leq C \parallel I_H^u \partial^2_t u - \partial^2_t u \parallel_{L^2(\Omega)} \parallel \phi_n \parallel_{H^1(\Omega)} \leq \frac{C}{\epsilon} \parallel I_H^u \partial^2_t u - \partial^2_t u \parallel_{L^2(\Omega)}^2 + \epsilon \parallel \phi_n \parallel_{L^2(\Omega)}^2.
$$

(5.38)

To bound the term $\langle I_H^p \partial p - \partial_t p, \phi_z \cdot n_i \rangle_{\Gamma_1}$, in (5.36), we take $(\zeta, w) = (\phi_z, \partial_t p)$ in (5.23) to get

$$
\left| \langle I_H^p \partial p - \partial_t p, \phi_z \cdot n_i \rangle_{\Gamma_1} \right| \leq C \parallel I_H^p \partial p - \partial_t p \parallel_{L^2(\Omega)}^2 \parallel \phi_z \parallel_{H^1(\Gamma)} \leq \frac{C}{\epsilon} \parallel I_H^p \partial p - \partial_t p \parallel_{L^2(\Omega)}^2 + \epsilon \parallel \phi_z \parallel_{H^1(\Gamma)}^2.
$$

(5.39)

Combining (5.36)–(5.39), integrating with respect to time from 0 to $t \in (0, T)$, and using (5.25) for the first term on the right in (5.37), we obtain

$$
\parallel \phi_z \parallel^2 + \int_0^t \parallel \partial_t (\phi_{\alpha} + \alpha \phi_p) \parallel^2 + c_0 \parallel \partial_t \phi_p \parallel^2 \, ds
$$

$$
\leq C \int_0^t \left( \parallel \partial_t \phi_{\alpha} \parallel^2 + \parallel \partial_t \phi_p \parallel^2 + \parallel \partial_t \psi_{\alpha} \parallel^2 + \parallel \partial_t \psi_p \parallel^2 + \parallel \partial_t \psi_{\alpha} \parallel^2 + \parallel \psi_p \parallel^2 + \parallel \psi_{\alpha} \parallel^2 \right)
$$

$$
+ C \parallel I_H^p \partial p - \partial_t p \parallel_{L^2(\Gamma)}^2 + \epsilon \left( \int_0^t \parallel \phi_{\alpha} \parallel^2 + \parallel \phi_{\alpha} \parallel^2 + \parallel \partial_t \phi_{\alpha} \parallel^2 + \parallel \partial_t \phi_{\alpha} \parallel^2 \right)
$$

$$
+ C \left( \parallel \phi_z(0) \parallel^2 + \parallel \phi_z(0) \parallel^2 + \parallel \partial_t \phi_z(0) \parallel_{L^2(\Gamma)}^2 \right).
$$

(5.40)

Combining (5.34) and (5.40) and taking $\epsilon$ small enough implies

$$
\parallel \phi_z \parallel^2 + \int_0^t \parallel \partial_t \phi_z \parallel^2 \, ds
$$

$$
\leq C \int_0^t \left( \parallel \partial_t \psi_{\alpha} \parallel^2 + \parallel \partial_t \psi_p \parallel^2 + \parallel \partial_t \psi_{\alpha} \parallel^2 + \parallel \psi_p \parallel^2 + \parallel \psi_{\alpha} \parallel^2 \right)
$$

$$
+ C \parallel I_H^p \partial p - \partial_t p \parallel_{L^2(\Gamma)}^2 + \epsilon \left( \int_0^t \parallel \phi_{\alpha} \parallel^2 + \parallel \phi_{\alpha} \parallel^2 + \parallel \partial_t \phi_{\alpha} \parallel^2 + \parallel \partial_t \phi_{\alpha} \parallel^2 \right)
$$

$$
+ C \left( \parallel \phi_z(0) \parallel^2 + \parallel \phi_z(0) \parallel^2 + \parallel \partial_t \phi_z(0) \parallel_{L^2(\Gamma)}^2 \right) + C \left( \parallel \phi_z(0) \parallel^2 + \parallel \phi_z(0) \parallel^2 + \parallel \partial_t \phi_z(0) \parallel_{L^2(\Gamma)}^2 \right).
$$

(5.41)

Finally, combining (5.41) with (5.33) taking $\epsilon$ small enough, and using (5.29) and the inequality

$$
\parallel \phi_{\alpha} \parallel \leq C \left( \parallel \phi_{\alpha} + \alpha \phi_p \parallel + \parallel \phi_p \parallel \right),
$$

we arrive at

$$
\parallel \phi_{\alpha}(t) \parallel^2_{X_h} + \parallel \phi_{\alpha}(t) \parallel^2 + \parallel \phi_{\gamma}(t) \parallel^2 + \parallel \phi_z(t) \parallel^2 + \parallel \phi_p(t) \parallel^2
$$

$$
+ \int_0^t \left( \parallel \phi_{\alpha} \parallel^2_{X_h} + \parallel \phi_{\alpha} \parallel^2 + \parallel \phi_{\alpha} \parallel^2_{X_h} + \parallel \phi_{\alpha} \parallel^2 \right) \, ds
$$

$$
\leq C \int_0^t \left( \parallel \partial_t \phi_{\alpha} \parallel^2 + \parallel \partial_t \phi_p \parallel^2 + \parallel \partial_t \psi_{\alpha} \parallel^2 + \parallel \psi_{\alpha} \parallel^2 \right) + \parallel \psi_{\gamma} \parallel^2 + \parallel \psi_p \parallel^2 + \parallel \psi_{\gamma} \parallel^2 + \parallel \psi_p \parallel^2 \, ds
$$

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\[ + \int_0^t \left( \| T^p_H u - u \|^2_{\mathcal{H}^1} + \| \partial_t (T^p_H u - u) \|^2_{\mathcal{H}^1} + \| \partial^2_t (T^p_H u - u) \|^2_{\mathcal{H}^1} + \right. \\
\left. + \| T^p_H p - p \|^2_{\mathcal{H}^1} + \| \partial_t (T^p_H p - p) \|^2_{\mathcal{H}^1} \right) ds + \| \psi_\sigma(t) \|^2 + \| \psi_\psi(t) \|^2 + \| \psi_\gamma(t) \|^2 + \| \psi_\tau(t) \|^2 \\
+ \| (T^p_H u - u)(t) \|^2_{\mathcal{H}^1} + \| \partial_t (T^p_H u - u)(t) \|^2_{\mathcal{H}^1} + \| (T^p_H p - p)(t) \|^2_{\mathcal{H}^1} \\
+ \| \phi_\sigma(0) \|^2 + \| \phi_\psi(0) \|^2 + \| \phi_\gamma(0) \|^2 + \| \partial_t (T^p_H u - u)(0) \|^2_{\mathcal{H}^1} \right). \] (5.42)

**Bound on \( \| \phi_\lambda^u \|_\Gamma \) and \( \| \phi_\lambda^p \|_\Gamma \).**

In order to bound the error in \( \| \lambda_H^u \|_\Gamma \), we take the difference between equations (5.1) and (2.18) to get

\[ (A((\sigma - \sigma_h) + \alpha (p - p_h) I), \tau) + (u - u_h, \text{div}_h \tau) + (\gamma - \gamma_h, \tau) = \sum_{i=1}^N (u - \lambda_H^u, \tau n_i)_{\Gamma_i}, \quad \forall \tau \in X_h. \]

We can split the error terms in the above equation and use (3.1) to rewrite it as

\[ \sum_{i=1}^N \langle \phi_\lambda^u, \tau n_i \rangle_{\Gamma_i} = (A(\phi_\sigma + \alpha \phi_\psi), \tau) + (A(\psi_\sigma + \alpha \psi_\psi), \tau) + (\phi_u, \text{div}_h \tau) \]

\[ + (\psi_u, \text{div}_h \tau) + (\phi_\gamma, \tau) + (\psi_\gamma, \tau), \quad \forall \tau \in X_h. \]

The inf-sup stability bound (3.36) combined with the above equation implies

\[ \| \phi_\lambda^u \|_\Gamma \leq \beta_E \sup_{0 \neq \tau \in X_h} \frac{\sum_{i=1}^N \langle \tau n_i, \phi_\lambda^u \rangle_{\Gamma_i}}{\| \tau \|_{X_h}} = \beta_E \sup_{0 \neq \tau \in X_h} \frac{1}{\| \tau \|_{X_h}} \left( (A(\phi_\sigma + \alpha \phi_\psi), \tau) + (A(\psi_\sigma + \alpha \psi_\psi), \tau) + (\phi_u, \text{div}_h \tau) + (\psi_u, \text{div}_h \tau) + (\phi_\gamma, \tau) + (\psi_\gamma, \tau) \right) \]

\[ \leq C(\| \phi_\sigma \| + \| \phi_\psi \| + \| \phi_u \| + \| \phi_\gamma \| + \| \psi_\sigma \| + \| \psi_\psi \| + \| \psi_\gamma \| + \| \psi_\tau \|). \] (5.43)

To bound the error in \( \| \lambda_H^p \|_\Gamma \) we take the difference between (5.4) and (2.21) and use (3.2) to obtain

\[ (K^{-1} \phi_z, \zeta) + (K^{-1} \psi_z, \zeta) - (\phi_\psi, \text{div}_h \zeta) - (\psi_\psi, \text{div}_h \zeta) = \sum_{i=1}^N -\langle \phi_\lambda^p, \zeta \cdot n_i \rangle_{\Gamma_i}, \quad \forall \zeta \in Z_h. \]

The inf-sup stability bound (3.30) combined with the above equation implies

\[ \| \phi_\lambda^p \|_\Gamma \leq \beta_D \sup_{0 \neq \zeta \in Z_h} \frac{\sum_{i=1}^N \langle \zeta \cdot n_i, \phi_\lambda^p \rangle_{\Gamma_i}}{\| \zeta \|_{Z_h}} \]

\[ = \beta_D \sup_{0 \neq \zeta \in Z_h} \frac{(K^{-1} \phi_z, \zeta) + (K^{-1} \psi_z, \zeta) - \sum_{i=1}^N (\phi_\psi, \text{div} \zeta)_{\Omega_i} - \sum_{i=1}^N (\psi_\psi, \text{div} \zeta)_{\Omega_i}}{\| \zeta \|_{Z_h}} \]

\[ \leq C(\| \phi_z \| + \| \phi_p \| + \| \psi_z \| + \| \psi_p \|). \] (5.44)

**Bound on the initial errors.**

In order to bound the initial errors \( \| \phi_\sigma(0) \|, \| \phi_\psi(0) \|, \) and \( \| \phi_\gamma(0) \| \) that appear in (5.42), we recall that we obtain the discrete initial data from the elliptic projection of the continuous initial data, cf. (4.11). Following the arguments similar to the ones used to arrive at (4.29), we get

\[ \| \phi_\sigma(0) \| + \| \phi_\psi(0) \| + \| \phi_\gamma(0) \| + \| \phi_u(0) \| \leq C(\| \psi_\sigma(0) \| + \| \psi_\psi(0) \| + \| \psi_\gamma(0) \| + \| \psi_z(0) \| + \| \psi_u(0) \|). \] (5.45)
To bound terms \(\|I_H^nv - v\|_{1/2,\Gamma}\) and \(\|I_H^nw - w\|_{1/2,\Gamma}\) that appear in (5.42), we use (3.16)–(3.17) and (3.27) to obtain
\[
\|I_H^nv - v\|_{1/2,\Gamma} \leq CH^{\hat{m} - 1/2}\|v\|_{\hat{m} + 1/2,\Omega}, \quad 1/2 \leq \hat{m} \leq m + 1, \quad (5.46)
\]
\[
\|I_H^nw - w\|_{1/2,\Gamma} \leq CH^{\hat{m} - 1/2}\|w\|_{\hat{m} + 1/2,\Omega}, \quad 1/2 \leq \hat{m} \leq m + 1. \quad (5.47)
\]

Finally, the assertion of the theorem follows by combining bounds (5.42)–(5.47) with the approximation results (3.18)–(3.22), (3.40)–(3.41) and (3.44)–(3.45).

Remark 5.1. The above theorem implies that for sufficiently smooth solution variables, the error in using our method is of \(O(h^{k+1} + h^{l+1} + h^{j+1} + h^{r+1} + h^{s+1} + h^{m+1/2})\). Assuming we use inf-sup stable pairs of FE spaces containing polynomials of degree \(j = s\), and \(k = r\), and \(l \leq k\), we could choose \(H = O(h^{l+1})\) to get a total error bound of order \(O(h^{l+1})\). For example, for the choice of \(l = 0\) and \(m = 1\), we could choose \(H = O(h^{3/2})\) and for \(l = 0\) and \(m = 2\), we could choose \(H = O(h^{5/2})\) to obtain a total convergence rate of \(O(h)\). We will demonstrate the results for different choices of \(H(h)\) in the numerical results section.

6 Non-overlapping domain decomposition algorithm

In this section, we discuss the implementation of the multiscale mortar mixed finite element method using a non-overlapping domain decomposition method. First, we present a fully discrete version of the system (2.18)–(2.24) using backward Euler time discretization. Then we describe the reduction of the algebraic system at each time step to a mortar interface problem, which can be solved using an iterative solver like GMRES. Finally, we discuss the use of a multiscale basis to increase the efficiency of the method.

6.1 Time discretization

For time discretization, we use the backward Euler method. Let \(\{t_n\}_{n=0}^{N_T}; t_n = n\Delta t, \Delta t = T/N_T\), be a uniform partition of \((0, T)\). We discretize a related formulation to the system (2.18)–(2.24), in which the constitutive elasticity equation (2.18) is differentiated in time. The reason for this is that this approach results in a positive definite interface problem; details can be found in [31]. We introduce the variables \(\dot{u}_h = \partial_t u_h, \dot{\gamma}_h = \partial_t \gamma_h\) and \(\lambda_h = \partial_t \lambda_H\) representing the time derivatives of the displacement, rotation, and displacement-Lagrange multiplier, respectively. In addition, in order to make more clear the incorporation of boundary conditions in the domain decomposition algorithm, we present the method for non-homogeneous Dirichlet boundary conditions
\[
u = g_u \text{ on } \Gamma_D, \quad p = g_p \text{ on } \Gamma_D^p.
\]
The fully discrete multiscale mortar MFE method reads as follows: for \(0 \leq n \leq N_T - 1\) and \(1 \leq i \leq N\), find \((\sigma_{h,i}^{n+1}, \dot{u}_{h,i}^{n+1}, \dot{\gamma}_{h,i}^{n+1}, \lambda_{h,i}^{n+1}, \lambda_{H,i}^{n+1}) \in \mathbb{X}_{h,i} \times \mathbb{V}_{h,i} \times \mathbb{Q}_{h,i} \times \mathbb{Z}_{h,i} \times \mathbb{A}_H \times \mathbb{A}_H^p\) such that:
\[
\begin{align*}
(A(\sigma_{h,i}^{n+1}, \alpha p_{h,i}^{n+1}), \tau)_{\Omega_i} + \Delta t (\dot{u}_{h,i}^{n+1}, \text{div } \tau)_{\Omega_i} + \Delta t (\dot{\gamma}_{h,i}^{n+1}, \tau)_{\Omega_i} = & \quad \Delta t (\lambda_{H,i}^{n+1}, \tau n_i)_{\Gamma_i} + \Delta t (\partial_t g_{u,i}^{n+1}, \tau n_i)_{\partial \Omega_i \cap \Gamma_D^p} + (\lambda_{H,i}^{n+1}, \tau)_{\Omega_i}, \quad \forall \tau \in \mathbb{X}_{h,i}, \\
(\text{div } \sigma_{h,i}^{n+1}, v)_{\Omega_i} = & \quad -(f^{n+1}, v)_{\Omega_i}, \quad \forall v \in \mathbb{V}_{h,i}, \\
(\sigma_{h,i}^{n+1}, \xi)_{\Omega_i} = & \quad 0, \quad \forall \xi \in \mathbb{Q}_{h,i}, \\
(K^{-1} \dot{z}_{h,i}^{n+1}, \zeta)_{\Omega_i} - (\dot{p}_{h,i}^{n+1}, \text{div } \zeta)_{\Omega_i} = & \quad -(\lambda_H^{p,n+1}, \zeta \cdot n_i)_{\Gamma_i} - (g_{p,i}^{n+1}, \zeta \cdot n_i)_{\partial \Omega_i \cap \Gamma_D^p}, \quad \forall \zeta \in \mathbb{Z}_{h,i}.
\end{align*}
\]
\[
\begin{align*}
&c_0 \left( p_{h,i}^{n+1}, w \right)_{\Omega_i} + \alpha \left( A(\sigma_{h,i}^{n+1} + \alpha p_{h,i}^{n+1} I), w I \right)_{\Omega_i} + \Delta t \left( \text{div} \, z_{h,i}^{n+1}, w \right)_{\Omega_i}, \\
&= c_0 \left( p_{h,i}^n, w \right)_{\Omega_i} + \alpha \left( A(\sigma_{h,i}^n + \alpha p_{h,i}^n I), w I \right)_{\Omega_i} + \Delta t \left( g^{n+1}, w \right)_{\Omega_i}, \\
&\forall w \in W_{h,i}, \quad (6.5)
\end{align*}
\]
\[
\sum_{i=1}^{N} \langle \sigma_{h,i}^{n+1} n_i, \mu^{u} \rangle_{\Gamma_i} = 0,
\forall \mu^{u} \in \Lambda_{H}^{u}, \quad (6.6)
\]
\[
\sum_{i=1}^{N} \langle z_{h,i}^{n+1} \cdot n_i, \mu^{p} \rangle_{\Gamma_i} = 0,
\forall \mu^{p} \in \Lambda_{H}^{p}. \quad (6.7)
\]

The original variables can be recovered using
\[
\begin{align*}
&u_{h}^{n} = u_{h}^{0} + \Delta t \sum_{k=1}^{n} \dot{u}_{h,k}, \quad \gamma_{h} = \gamma_{h}^{0} + \Delta t \sum_{k=1}^{n} \dot{\gamma}_{h,k}, \quad \lambda_{h}^{u,n} = \lambda_{h}^{u,0} + \Delta t \sum_{k=1}^{n} \dot{\lambda}_{h,k}^{u}. \quad (6.8)
\end{align*}
\]

### 6.2 Reduction to an interface problem

We solve the system resulting from [6.1]–[6.7] at each time step by reducing it to an interface problem for the mortar variables. To simplify the notation, define
\[
\lambda_{H} = \left( \frac{\lambda_{H}^{u}}{\lambda_{H}^{p}} \right), \quad \Lambda_{H} = \left( \frac{\Lambda_{H}^{u}}{\Lambda_{H}^{p}} \right),
\]
and let \( \lambda_{H,i} \) and \( \Lambda_{H,i} \) denote the restrictions of \( \lambda_{H} \) and \( \Lambda_{H} \) to \( \Gamma_i \), respectively. We introduce two sets of complementary subdomain problems.

The first set of problems reads as follows: for \( 1 \leq i \leq N \), find \( (\bar{\sigma}_{h,i}^{n+1}, \bar{u}_{h,i}^{n+1}, \bar{\gamma}_{h,i}^{n+1}, \bar{z}_{h,i}^{n+1}, \bar{p}_{h,i}^{n+1}) \in X_{h,i} \times V_{h,i} \times Q_{h,i} \times Z_{h,i} \times W_{h,i} \) such that
\[
\begin{align*}
&\left( A(\bar{\sigma}_{h,i}^{n+1} + \alpha \bar{p}_{h,i}^{n+1} I), \tau \right)_{\Omega_i} + \Delta t \left( \bar{u}_{h,i}^{n+1}, \text{div} \, \tau \right)_{\Omega_i} + \Delta t \left( \bar{\gamma}_{h,i}^{n+1}, \tau \right)_{\Omega_i} \\
&= \Delta t \left( \partial_{g} \bar{u}_{h,i}^{n+1}, \tau n_i \right)_{\partial \Omega_i \cap \Gamma_D} + \left( A(\sigma_{h,i}^{n+1} + \alpha p_{h,i}^{n+1} I), \tau \right)_{\Omega_i}, \quad \forall \tau \in X_{h,i}, \quad (6.9)
\end{align*}
\]
\[
\begin{align*}
&\left( \text{div} \, \bar{\sigma}_{h,i}^{n+1}, v \right)_{\Omega_i} = - \left( f^{n+1}, v \right)_{\Omega_i}, \\
&\left( \bar{\sigma}_{h,i}^{n+1}, \xi \right)_{\Omega_i} = 0, \\
&\left( K^{-1} \bar{z}_{h,i}^{n+1}, \zeta \right)_{\Omega_i} - \left( \bar{p}_{h,i}^{n+1}, \text{div} \, \zeta \right)_{\Omega_i} = \left( g_{p}^{n+1}, \zeta \cdot n_i \right)_{\partial \Omega_i \cap \Gamma_D}, \quad \forall \zeta \in Z_{h,i}, \quad (6.12)
\end{align*}
\]
\[
\begin{align*}
&c_0 \left( \bar{p}_{h,i}^{n+1}, w \right)_{\Omega_i} + \alpha \left( A(\sigma_{h,i}^{n+1} + \alpha p_{h,i}^{n+1} I), w I \right)_{\Omega_i} + \Delta t \left( \text{div} \, \bar{z}_{h,i}^{n+1}, w \right)_{\Omega_i} \\
&= \Delta t \left( g^{n+1}, w \right)_{\Omega_i} + c_0 \left( p_{h,i}^{n}, w \right)_{\Omega_i} + \alpha \left( A(\sigma_{h,i}^{n} + \alpha p_{h,i}^{n} I), w I \right)_{\Omega_i}, \quad \forall w \in W_{h,i}. \quad (6.13)
\end{align*}
\]

Note that these subdomain problems have zero Dirichlet data on the subdomain interfaces, the true source terms \( f \) and \( g \) and outside boundary conditions \( g_u \) and \( g_p \), and previous time step data \( \sigma_{h,i}^{n} \) and \( p_{h,i}^{n} \).

The second set of equations reads as follows: given \( \lambda_{H} \in \Lambda_{H} \), for \( 1 \leq i \leq N \), find \( (\sigma_{h,i}^{n+1}(\lambda_{H,i}), \bar{u}_{h,i}^{n+1}(\lambda_{H,i}), \bar{\gamma}_{h,i}^{n+1}(\lambda_{H,i}), \bar{z}_{h,i}^{n+1}(\lambda_{H,i}), \bar{p}_{h,i}^{n+1}(\lambda_{H,i})) \in X_{h,i} \times V_{h,i} \times Q_{h,i} \times Z_{h,i} \times W_{h,i} \) such that:
\[
\begin{align*}
&\left( A(\sigma_{h,i}^{n+1}(\lambda_{H,i}) + \alpha \bar{p}_{h,i}^{n+1}(\lambda_{H,i}) I), \tau \right)_{\Omega_i} + \Delta t \left( \bar{u}_{h,i}^{n+1}(\lambda_{H,i}), \text{div} \, \tau \right)_{\Omega_i} \\
&+ \Delta t \left( \bar{\gamma}_{h,i}^{n+1}(\lambda_{H,i}), \tau \right)_{\Omega_i} = \Delta t \left( \lambda_{H,i}^{u}, \tau n_i \right)_{\Gamma_i}, \quad \forall \tau \in X_{h,i}, \quad (6.14)
\end{align*}
\]
\[
\begin{align*}
&\left( \text{div} \, \sigma_{h,i}^{n+1}(\lambda_{H,i}), v \right)_{\Omega_i} = 0, \quad \forall v \in V_{h,i}. \quad (6.15)
\end{align*}
\]
\begin{equation}
\left(\sigma_{h,i}^{n+1}(\lambda_{H,i}), \xi\right)_{\Omega_i} = 0, \quad \forall \xi \in Q_{h,i}, \tag{6.16}
\end{equation}
\begin{equation}
\left(K^{-1} z_{h,i}^{n+1}(\lambda_{H,i}), \xi\right)_{\Omega_i} - \left(p_{h,i}^{n+1}(\lambda_{H,i}), \text{div} \, \xi\right)_{\Omega_i} = -\left(\lambda^p_{H,i}; \zeta \cdot n_i\right)_{\Gamma_i}, \quad \forall \zeta \in Z_{h,i}, \tag{6.17}
\end{equation}
\begin{equation}
c_0 \left(p_{h,i}^{n+1}(\lambda_{H,i}), w\right) + \alpha \left(A(\sigma_{h,i}^{n+1}(\lambda_{H,i}) + \alpha p_{h,i}^{n+1}(\lambda_{H,i})I), w\right)_{\Omega_i} + \Delta t \left(\text{div} \, z_{h,i}^{n+1}(\lambda_{H})\right) = 0, \quad \forall w \in W_{h,i}. \tag{6.18}
\end{equation}

Note that these problems have \(\lambda_{H,i}\) as Dirichlet boundary data on the interfaces \(\Gamma\), zero source terms, zero boundary data on the outside boundary \(\partial \Omega\), and zero data from the previous time step.

Define the bilinear forms \(a^{n+1}_{H,i} : \Lambda_{H,i} \times \Lambda_{H,i} \to \mathbb{R}\), \(\forall i \in \{1, \ldots, N\}\), \(a^{n+1}_{H,i} : \Lambda_{H,i} \to \mathbb{R}\), and the linear functional \(g^{n+1}_H : \Lambda_H \to \mathbb{R}\) for all \(0 \leq n \leq N_T - 1\) by
\begin{equation}
a^{n+1}_{H,i}(\lambda_{H,i}, \mu_i) = \langle \sigma_{h,i}^{n+1}(\lambda_{H,i}) n_i, \mu_i \rangle_{\Gamma_i} - \langle z_{h,i}^{n+1}(\lambda_{H,i}) \cdot n_i, \mu_i \rangle_{\Gamma_i}, \quad a^{n+1}_H(\lambda, \mu) = \sum_{i=1}^N a^{n+1}_{H,i}(\lambda_{H,i}, \mu_i), \tag{6.19}
\end{equation}

It follows from (6.6)–(6.7) that the solution to the global problem (6.1)–(6.7) is equivalent to solving the following interface problem for \(\lambda^{n+1}_H \in \Lambda_H\):
\begin{equation}
a^{n+1}_H(\lambda^{n+1}_H, \mu) = g^{n+1}_H(\mu), \quad \forall \mu \in \Lambda_H, \tag{6.19}
\end{equation}

and setting
\begin{align*}
\sigma_{h,i}^{n+1} &= \sigma_{h,i}^{n+1}(\lambda^{n+1}_H) + \tilde{\sigma}_{h,i}^{n+1}, \\
u_{h,i}^{n+1} &= \nu_{h,i}^{n+1}(\lambda^{n+1}_H) + \tilde{\nu}_{h,i}^{n+1}, \\
\gamma_{h,i}^{n+1} &= \gamma_{h,i}^{n+1}(\lambda^{n+1}_H) + \tilde{\gamma}_{h,i}^{n+1}, \\
z_{h,i}^{n+1} &= z_{h,i}^{n+1}(\lambda^{n+1}_H) + \tilde{z}_{h,i}^{n+1}, \\
p_{h,i}^{n+1} &= p_{h,i}^{n+1}(\lambda^{n+1}_H) + \tilde{p}_{h,i}^{n+1}.
\end{align*}

### 6.3 Solution of the Interface Problem

We introduce the linear operators \(A^{n+1}_{H,i} : \Lambda_{H,i} \to \Lambda'_{H,i}\), for \(1 \leq i \leq N\), and \(A^{n+1}_H : \Lambda_H \to \Lambda'_H\) such that for any \(\lambda_H \in \Lambda_H\),
\begin{equation}
\langle A^{n+1}_{H,i}\lambda_{H,i}, \mu_i \rangle = a^{n+1}_{H,i}(\lambda_{H,i}, \mu_i), \quad \forall \mu_i \in \Lambda_{H,i}, \quad \langle A^{n+1}_H\lambda_H, \mu \rangle = \sum_{i=1}^N \langle A^{n+1}_{H,i}\lambda_{H,i}, \mu_i \rangle, \quad \forall \mu \in \Lambda_H.
\end{equation}

We also define the functional \(G^{n+1}_H \in \Lambda'_H\) such that
\begin{equation}
\langle G^{n+1}_H, \mu \rangle = g^{n+1}_H(\mu), \quad \forall \mu \in \Lambda_{H,i}.
\end{equation}

The interface problem (6.19) can now be reformulated as finding \(\lambda^{n+1}_H \in \Lambda_H\) such that
\begin{equation}
A^{n+1}_H\lambda^{n+1}_H = G^{n+1}_H. \tag{6.20}
\end{equation}

Consider the \(L^2\)-orthogonal projections \(Q^{u,T}_{h,i} : \mathbb{X}_{h,i} n_i \to \Lambda'_u\) and \(Q^{p,T}_{h,i} : Z_{h,i} \cdot n_i \to \Lambda'_p\), which are the adjoint operators of \(Q^{u}_{h,i}\) and \(Q^{p}_{h,i}\), respectively, introduced in (3.1)–(3.2). Using this notation, we have
\begin{equation}
A^{n+1}_{H,i}\lambda_{H,i} = \left( Q^{u,T}_{h,i} \sigma_{h,i}^{n+1}(\lambda_{H,i}) n_i \right) \left( Q^{p,T}_{h,i} z_{h,i}^{n+1}(\lambda_{H,i}) \cdot n_i \right), \quad i = 1, \ldots, N. \tag{6.21}
\end{equation}
is positive definite. The proof can be easily extended to the current setting using mortar variable. Conse-

quently, we use GMRES to solve the interface problem \((6.20)\). The action of the interface operator \(A_H^{n+1}\) required at each GMRES iteration is computed using the steps described in Algorithm 1.

**Remark 6.1.** The solution algorithm for the multiscale mortar MFE method has the performance advantage over the similar method for matching grids discussed in [34] that a coarse mortar mesh could be used to obtain a smaller interface problem due to the reduction in the mortar degrees of freedom. Moreover, as discussed in Theorem 5.7 and Remark 5.7, optimal order accuracy on the fine scale can be maintained with a suitable choice of the mortar space polynomial degree.

### 6.4 Implementation with multiscale stress–flux basis

As noted in Remark 6.1, a coarser mortar mesh can lead to a smaller interface problem, but even in that case the number of subdomain solves of the type \((6.14)-(6.18)\) is directly proportional to both the number of time steps and the number of GMRES iterations at each time step. Following [24,35], we propose the construction and use of a multiscale stress–flux basis (MSB), which makes the number of subdomain solves independent of the number of GMRES iterations required for the interface problem and the number of time steps.

Let \(\{\beta_{H,i}^k\}_{k=0}^{N_{H,i}}\) be a basis for \(\Lambda_{H,i}\), where \(N_{H,i}\) denotes the number of degrees of freedom associated with the finite element space \(\Lambda_{H,i}\). We calculate and store the action of the interface operator of the form

\[
A_{H,i} \beta_{H,i}^k = Q_{h,i}^T \begin{pmatrix} \sigma_{h,i}^k(\beta_{H,i}^k) n_i \\ -z_{h,i}^k(\beta_{H,i}^k) \cdot n_i \end{pmatrix}, \quad k = 1, \ldots, N_{H,i},
\]

(6.22)

where \(\sigma_{h,i}^k(\beta_{H,i}^k)\) and \(z_{h,i}^k(\beta_{H,i}^k)\) are obtained by solving \((6.14)-(6.18)\) with \(\beta_{H,i}^k\) as the Dirichlet boundary data. A detailed description of the construction of the multiscale basis elements \(\{\phi_{H,i}^k\}_{k=0}^{N_{H,i}}\) with \(\phi_{H,i}^k = A_{H,i} \beta_{H,i}^k\) is given in Algorithm 2. see [24,35] for similar constructions. We use the notation \(Q_{h,i}^p = (Q_{h,i}^u, Q_{h,i}^p)\).

For any \(\lambda_{H,i} \in \Lambda_{H,i}\), consider the mortar basis decomposition, \(\lambda_{H,i} = \sum_{k=0}^{N_{H,i}} \lambda_{k,i} \phi_{H,i}^k\). Using the multiscale basis, Algorithm 1 for computing the action of the interface operator \(A_H^{n+1} \lambda_H\) can be replaced by computing a linear combination of the multiscale basis as follows:

\[
A_{H,i} \lambda_{H,i} = \sum_{k=0}^{N_{H,i}} \lambda_{k,i} A_{H,i} \beta_{H,i}^k = \sum_{k=0}^{N_{H,i}} \lambda_{k,i} \phi_{H,i}^k.
\]

(6.23)
Algorithm 2: Construction of a multiscale stress–flux basis $\phi^k_{H,i} = A_{H,i}^k \beta^k_{H,i}$

for $k = 1, \ldots, N_{H,i}$:

1. Project $\beta^k_{H,i}$ onto the subdomain boundary space: $\beta^k_{H,i} \rightarrow Q_{h,i}^k \beta^k_{H,i}$.

2. Solve the system (6.14)–(6.18) using the projected function $Q_{h,i}^k \beta^k_{H,i}$ as Dirichlet boundary data, to obtain $\sigma^*_{h,i}(\beta^k_{H,i})$ and $z^*_{h,i}(\beta^k_{H,i})$.

3. Project the solution variables to the mortar space to obtain $\phi^k_{H,i} = Q_{u,T}^H \sigma^*_{h,i}(\beta^k_{H,i}) n_i - Q_{p,T}^H z^*_{h,i}(\beta^k_{H,i}) \cdot n_i$.

end for

Remark 6.2. The multiscale stress–flux basis is computed and saved once and can be reused over all time steps and all GMRES iterations, which gains a significant performance advantage in the case of time-dependent problems like the one we consider. We illustrate the efficiency of using the multiscale stress–flux basis in Example 2 in the numerical section.

7 Numerical Results

In this section, we report the results of several numerical tests designed to illustrate the well-posedness, stability, and convergence of the multiscale mortar non-overlapping domain decomposition method for the Biot system of poroelasticity that we have developed. We further discuss the computational efficiency of the method, including the advantage of using a multiscale basis. The numerical schemes are implemented using the finite element package deal.II [4,12].

We use the finite element triplet $X_h \times V_h \times Q_h = (BDM_1)^2 \times (Q_0)^2 \times Q_0$ [9,11] for elasticity and the finite element pair $Z_h \times W_h = BDM_1 \times Q_0$ [17] for Darcy on rectangular meshes. Here BDM_1 stands for the lowest-order Brezzi-Douglas-Marini space [17] and $Q_k$ denotes polynomials of degree $k$ in each variable. For the mortar spaces, $\Lambda^u_H$ is taken to be $(DQ_m)^2$, and $\Lambda^p_H$ is taken to be $DQ_m$ with $m = 1$ or 2, where $DQ_k$ represents the discontinuous finite element spaces containing polynomials of degree $k$, which lives on the subdomain interfaces. The polynomial degrees of the finite element spaces used in the numerical examples are given in Table 1. For solving the interface problem, we use non-restarted unpreconditioned GMRES with a tolerance $10^{-6}$ on the relative residual $r_k/r_0$ as the stopping criteria.

Table 1: Degree of polynomials associated with FEM spaces used for numerical experiments.

| $X_h$ : $k$ | $V_h$ : $l$ | $Q_h$ : $j$ | $Z_h$ : $r$ | $W_h$ : $s$ | $\Lambda_H$ : $m$ |
|-------------|-------------|-------------|-------------|-------------|----------------|
| 1           | 0           | 0           | 1           | 0           | 1 or 2 |

In Example 1, we test the stability, convergence, and efficiency of the multiscale mortar MFE method using linear ($m = 1$) or quadratic ($m = 2$) mortar spaces by solving the problem with a known solution on successively refined meshes. In Example 2, we apply the multiscale mortar MFE method to solve a benchmark problem with a highly heterogeneous medium. We compare the efficiency of the multiscale versus fine scale methods and study the computational advantage of constructing the multiscale stress–flux basis (MSB) discussed in Section 6.4.

7.1 Example 1: convergence rates

In this example we test the solvability, stability, and convergence of the multiscale mortar MFE method. The global computational domain $\Omega$ is taken to be the unit square $(0,1)^2$. We consider the following
Table 1: Physical and Numerical Parameters

| Parameter                      | Value |
|-------------------------------|-------|
| Permeability tensor ($K$)     | $I$   |
| Lame coefficient ($\mu$)      | 100.0 |
| Lame coefficient ($\lambda$)  | 100.0 |
| Mass storativity ($c_0$)      | 1.0, $10^{-3}$ |
| Biot-Willis constant ($\alpha$)| 1.0   |
| Time step ($\Delta t$)        | $10^{-3}, 10^{-4}$ |
| Number of time steps          | 100   |

The physical and numerical parameters are given in Figure 1 (left). Using this information, we derive the analytical solution

$$p = \exp(t)(\sin(\pi x)\cos(\pi y) + 10), \quad u = \exp(t) \left( \frac{x^3y^4 + x^2 + \sin((1 - x)(1 - y))\cos(1 - y)}{(1 - x)^4(1 - y)^3 + (1 - y)^2 + \cos(xy)\sin(x)} \right).$$

The plots in the case of quadratic mortar space look similar. The plots demonstrate the efficacy of the method in enforcing continuity of the solution variables across non-matching subdomain interfaces, using weakly coarse mortar spaces.
Table 2: Example 1, convergence for linear mortar \((m = 1)\) with \(H = 2h\), \(\Delta t = 10^{-4}\) and \(c_0 = 1.0\).

| \(h\) | \# GMRES | \(\|\sigma - \sigma_h\|_{L^\infty(L^2)}\) | \(\|\text{div} (\sigma - \sigma_h)\|_{L^\infty(L^2)}\) | \(\|\gamma - \gamma_h\|_{L^\infty(L^2)}\) | \(\|u - u_h\|_{L^\infty(L^2)}\) |
|-----|-----------|-------------------|-------------------|-------------------|-------------------|
| 1/4 | 16 rate   | 1.23e+01          | 6.09e-01          | 1.39e+00          | 5.78e-01          |
| 1/8 | 28 -0.81  | 3.24e-02          | 1.92              | 3.11e-01          | 0.97              |
| 1/16| 46 -0.72  | 8.20e-03          | 1.98              | 1.56e-01          | 0.99              |
| 1/32| 73 -0.67  | 2.08e-03          | 1.98              | 7.82e-02          | 1.00              |
| 1/64| 122 -0.74 | 5.39e-04          | 1.94              | 3.91e-02          | 1.00              |

Table 3: Example 1, convergence for quadratic mortar \((m = 2)\) with \(H = \sqrt{h}\), \(\Delta t = 10^{-4}\) and \(c_0 = 1.0\).

| \(h\) | \# GMRES | \(\|\sigma - \sigma_h\|_{L^\infty(L^2)}\) | \(\|\text{div} (\sigma - \sigma_h)\|_{L^\infty(L^2)}\) | \(\|\gamma - \gamma_h\|_{L^\infty(L^2)}\) | \(\|u - u_h\|_{L^\infty(L^2)}\) |
|-----|-----------|-------------------|-------------------|-------------------|-------------------|
| 1/4 | 22 rate   | 1.26e-01          | 6.09e-01          | 1.39e+00          | 5.79e-01          |
| 1/16| 40 -0.43  | 8.25e-03          | 1.97              | 1.56e-01          | 0.98              |
| 1/64| 65 -0.35  | 5.62e-04          | 1.93              | 3.91e-02          | 1.00              |

degree, without a loss in accuracy. Tables 4 and 5 indicate that the stability and convergence rates are not affected by smaller values of \(c_0\), which is consistent with the theoretical bounds established in the previous sections.

### 7.2 Example 2: heterogeneous medium

In this example, we demonstrate the performance of the multiscale mortar method in a practical application with highly heterogeneous medium. First, we compare the accuracy and efficiency of the multiscale method with \(H > h\) to a fine scale method with \(H = h\). We then study the computational advantage of using a multiscale stress–flux basis. We use the porosity and the permeability data from the Society of Petroleum Engineers 10th Comparative Solution Project (SPE10)\(^1\). The data are given on a \(60 \times 220\) grid covering the rectangular region \((0, 60) \times (0, 220)\). We decompose the global domain into \(3 \times 5\) subdomains consisting of identical rectangular blocks. The Young’s modulus is obtained from the porosity field data using the relation \(E = 10^2 \left(1 - \frac{\phi}{c}\right)^{2.1}\), where \(c = 0.5\) refers to the porosity at which the Young’s modulus vanishes, see [37] for details. The input fields are presented in Figure 3. The problem parameters and boundary conditions are given in Table 6 and the source terms are set to zero. These conditions describe a flow from left to right, driven by the gradient in the pressure. We use a compatible initial condition for pressure, \(p_0 = 1 - x\). To obtain discrete initial data, we set \(p^0_h\) to be the \(L^2\)-projection of \(p_0\) onto \(W_h\) and solve a mixed elasticity domain decomposition problem at \(t = 0\) to obtain \(\sigma^0_h\), \(u^0_h\), \(\gamma^0_h\), and \(\lambda^2_h\).

We use a global \(60 \times 220\) grid and solve the problem using both fine scale \((H = h)\) and coarse \((H > h)\) mortar spaces. For the coarse mortar case, we use both linear \((m = 1)\) and quadratic \((m = 2)\) mortars with one and two mortar elements per subdomain interface. The comparison of the computed solution

\(^1\)https://www.spe.org/web/csp/datasets/set02.htm
Table 4: Example 1, convergence for linear mortar with $H = 2h$, $\Delta t = 10^{-4}$ and $c_0 = 10^{-3}$.

| $h$  | # GMRES | $\|\sigma - \sigma_h\|_{L^\infty(L^2)}$ | $\|\text{div}(\sigma - \sigma_h)\|_{L^\infty(L^2)}$ | $\|\gamma - \gamma_h\|_{L^\infty(L^2)}$ | $\|u - u_h\|_{L^\infty(L^2)}$ |
|------|---------|---------------------------------|-----------------|----------------|----------------|
| 1/4  | 16      | 1.25e-01                        | 6.09e-01        | 1.39e+00       | 5.78e-01       |
| 1/8  | 29      | -0.86                           | 3.30e-02        | 0.97           | 5.67e-01       |
| 1/16 | 50      | -0.79                           | 8.34e-03        | 0.99           | 1.46e-01       |
| 1/32 | 87      | -0.80                           | 2.09e-03        | 1.00           | 7.31e-02       |
| 1/64 | 157     | -0.85                           | 5.38e-04        | 1.00           | 3.65e-02       |

Table 5: Example 1, convergence for quadratic mortar with $H = \sqrt{h}$, $\Delta t = 10^{-4}$ and $c_0 = 10^{-3}$.

| $h$  | # GMRES | $\|\sigma - \sigma_h\|_{L^\infty(L^2)}$ | $\|\text{div}(\sigma - \sigma_h)\|_{L^\infty(L^2)}$ | $\|\gamma - \gamma_h\|_{L^\infty(L^2)}$ | $\|u - u_h\|_{L^\infty(L^2)}$ |
|------|---------|---------------------------------|----------------|----------------|----------------|
| 1/4  | 23      | 1.25e-01                        | 6.09e-01        | 1.39e+00       | 5.79e-01       |
| 1/16 | 41      | -0.41                           | 8.39e-03        | 0.98           | 1.46e-01       |
| 1/64 | 72      | -0.41                           | 5.61e-04        | 1.00           | 3.65e-02       |

using different choices of mortars is given in Figures 4-6. The solution variables are very similar for all five cases, indicating that the multiscale mortar method obtains comparable accuracy to the fine scale discretization, even in the case of the coarsest mortar grid with one linear mortar per interface. On the other hand, the computational cost of the multiscale mortar method is smaller than the fine scale method. This is evident from Table 7, where the number of GMRES iterations and subdomain solves are reported, noting that the number of subdomain solves dominates the computational complexity of the method. We observe that the multiscale mortar method requires much fewer number of GMRES iterations and solves compared to fine scale method. As a result, the multiscale mortar method obtains comparable accuracy to the fine scale method at a significantly reduced computational cost.

We further test the effect of using a multiscale stress–flux basis (MSB) on the computational cost of the method. If MSB is not used, the number of subdomain solves equals the total #GMRES iterations across all time steps and number of time steps, where the last term comes from two extra solves required to solve the system (6.9)−(6.18) initially and recovering the final solution after the interface GMRES converges. On the other hand, in the case of using MSB, total number of subdomain solves equals the $\dim(\Lambda_{H,i}) + 2 \times$ number of time steps. Note that the first term in the cost in the case of no-MSB is directly proportional to the number of GMRES iterations and time steps, while the corresponding term in the case of MSB method is independent of both the number of GMRES iterations and the number of time steps, since the MSB can be reused over all time steps. Therefore the computational efficiency of the MSB for steady-state problems due to the independence of the global number of mortar degrees of freedom is further magnified by the number of subdomain solves in the case of time-dependent problems. These conclusions are illustrated in the last two columns of Table 7, where we observe that the number of solves in the case of no MSB is at least an order of magnitude larger than the MSB case. We conclude that the construction of MSB is an excellent tool to make the multiscale mortar method even more efficient than it already is.

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Figure 2: Example 1, computed solution at the final time step using a linear mortar on non-matching subdomain grids, \( h = 1/32 \), \( \Delta t = 10^{-3} \) and \( c_0 = 1.0 \); top: \( x \)-stress (left), \( y \)-stress (middle), displacement (right); bottom: rotation (left), velocity (middle), pressure (right).

compared to the fine scale method discussed in [34].

| Parameter                  | Value | Boundary | \( \sigma \) | \( u \) | \( p \) | \( z \) |
|----------------------------|-------|----------|-------------|--------|--------|--------|
| Mass storativity \( (c_0) \) | 1.0   | Left     | \( \sigma n = -\alpha n \) | -1     | -      | -      |
| Biot-Willis constant \( (\alpha) \) | 1.0   | Bottom   | \( \sigma n = 0 \) | -      | -      | \( z \cdot n = 0 \) |
| Time step \( (\Delta t) \)    | \( 10^{-3} \) | Right    | -           | 0      | 0      | -      |
| Total time \( (T) \)        | 0.1   | Top      | \( \sigma n = 0 \) | -      | -      | \( z \cdot n = 0 \) |

8 Conclusions

We presented a multiscale mortar mixed finite element method for the Biot system of poroelasticity in a five-field fully mixed formulation. The method allows for non-matching subdomain grids at the interfaces, using a composite mortar Lagrange multiplier space that approximates the displacement and pressure on a (possibly coarse) mortar interface grid to impose weakly stress and flux continuity. We established the well-posedness of the method and carried out a multiscale a priori error analysis. The results are robust in the limit of small storativity coefficient. We further presented a non-overlapping domain decomposition algorithm based on a Schur complement reduction of the global system to a (coarse scale) mortar interface problem, which is solved with a Krylov space iterative method. Each iteration requires solving Dirichlet type subdomain problems, which can be performed in parallel. A series of numerical tests illustrates the stability and convergence properties of the method, as well as its computational efficiency. We observed,
Table 7: Example 2, #GMRES iterations and maximum number of subdomain solves.

| mortar                  | Average #GMRES | Total #GMRES | Total #Solves |
|-------------------------|----------------|--------------|---------------|
| linear fine scale       | 343            | 34375        | 34575         |
| 1 linear per interface  | 41             | 4149         | 4349          |
| 1 quadratic per interface | 61         | 6184         | 6384          |
| 2 linear per interface  | 80             | 8010         | 8210          |
| 2 quadratic per interface | 123        | 12302        | 12502         |

both theoretically and numerically, that fine scale order convergence can be obtained even for a coarse mortar mesh with a suitable choice of the mortar polynomial degree. An application of the method to a highly heterogeneous benchmark problem illustrates that the multiscale mortar method can achieve comparable accuracy to the fine scale method at a highly reduced computational cost. Moreover, the use of a pre-computed multiscale stress–flux basis further increases the efficiency, making the computational cost independent of the global number of interface degrees of freedom and weakly dependent on the number of time steps.

Several extensions of the presented work are possible. These include combining the multiscale mortar techniques developed here with splitting methods for the Biot system of poroelasticity studied, e.g., in [1,3,16,34,54,59], as well as asynchronous and adaptive time stepping using space-time [13,31,33], parallel-in-time [16], a posteriori error estimation [1,2], and multirate techniques.

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Figure 4: Example 2, pressure (color) and velocity (arrows); from left to right: fine scale, single linear mortar per interface, two linear mortars per interface, single quadratic mortar per interface (left), two quadratic mortars per interface.

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Figure 5: Example 2, velocity magnitude; from left to right: fine scale, single linear mortar per interface, two linear mortars per interface, single quadratic mortar per interface (left), two quadratic mortars per interface.

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Figure 6: Example 2, displacement vector (arrows) and its magnitude; from left to right: fine scale, single linear mortar per interface, two linear mortars per interface, single quadratic mortar per interface (left), two quadratic mortars per interface.

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