Hyperchaotic attractors of three-dimensional maps and scenarios of their appearance

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Abstract

We study bifurcation mechanisms of the appearance of hyperchaotic attractors in three-dimensional maps. We consider, in some sense, the simplest cases when such attractors are homoclinic, i.e. they contain only one saddle fixed point and entirely its unstable manifold. We assume that this manifold is two-dimensional, which gives, formally, a possibility to obtain two positive Lyapunov exponents for typical orbits on the attractor (hyperchaos). For realization of this possibility, we propose several bifurcation scenarios of the onset of homoclinic hyperchaos that include cascades of both supercritical period-doubling bifurcations with saddle periodic orbits and supercritical Neimark-Sacker bifurcations with stable periodic orbits, as well as various combinations of these cascades. In the paper, these scenarios are illustrated by an example of three-dimensional Mirá map.

Keywords. Hyperchaotic attractor, homoclinic orbit, three-dimensional map.
1 Introduction

This paper is devoted to the study of bifurcation scenarios leading to the appearance of hyperchaotic attractors in three-dimensional map

\[
\begin{align*}
\bar{x} &= y, \\
\bar{y} &= z, \\
\bar{z} &= M_1 + Bx + M_2z - y^2.
\end{align*}
\]

(1)

where \(x, y,\) and \(z\) are phase variables; \(M_1, M_2,\) and \(B\) are parameters. This map is the well-known “homoclinic map” which appears as the normal form of first return maps near homoclinic and heteroclinic tangencies of multidimensional systems \([1, 2, 3, 4]\). Note that map (1) has the constant Jacobian \(B\). When \(B = 0\), it becomes effectively two-dimensional map of the form \(\bar{y} = z, \quad \bar{z} = M_1 + M_2z - y^2\). It is the well-known two-dimensional endomorphism introduced and studied by C. Míra yet in 60s \([5]\), see also \([6]\). Therefore, we call map (1) the three-dimensional Míra map.

Recall, that the term “hyperchaotic attractor” was introduced by Rössler in \([7]\) for strange attractors with (at least) two directions of exponential instability. In order to be hyperchaotic, an attractor should contain nontrivial hyperbolic subsets whose unstable manifold have dimension \(\geq 2\). If these subsets occupy a large part of the attractor, then typical orbits of the attractor have (at least) two positive Lyapunov exponents (LE), and this fact is usually established numerically.

As a consequence of the above, the study of hyperchaotic dynamics in map (1) looks very promising. Also, one of the reasons is that, as was shown in the paper \([8]\), in a certain region of parameters (region \(\text{SH}(1,2)\) in Fig. 8) there is a nontrivial hyperbolic set with the simplest structure – the three-dimensional Smale horseshoe – in which all its unstable manifolds are two-dimensional. Hereinafter, we will prescribe a type \((n, m)\) for hyperbolic periodic orbits with \(n\)-dimensional stable and \(m\)-dimensional unstable invariant manifolds (where types \((n, 0)\) and \((0, m)\) relate, respectively, to the stable and completely unstable periodic orbits). Then, we can denote the above horseshoe as the Smale horseshoe of type \((1,2)\). Accordingly, in the parameter space along pathways from the region where the map has a simple attractor (stable fixed point) to the region where there is such horseshoe, one can earn hyperchaotic strange attractors. How it can be done and which bifurcations occur along such pathways are the main questions of the current paper.

At first, let us recall some well-known facts on strange attractors in map (1). When \(B = 0\), the corresponding two-dimensional Míra endomorphism has, in certain regions of the \((M_1, M_2)\)-parameter plane, hyperchaotic attractors, the so-called snap-back repellers \([9]\), consisting mainly of completely unstable orbits (with two positive Lyapunov exponents). As was shown yet in \([10]\), for small \(B\), these attractors are transformed into hyperchaotic attractors for the three-dimensional diffeomorphism (1) that keeps a tendency of exponential expansion along two directions. When \(B\) is not too small (and \(|B| < 1\)), map (1) can demonstrate other types of hyperchaotic attractors, in particular, the so-called discrete Shilnikov homoclinic attractors containing a saddle-focus fixed point with two-dimensional unstable invariant manifold \([11, 13, 14]\). These attractors are of special interest since they are often observed in multidimensional systems including those from applications.\(^1\) In the current paper, we study bifurcation mechanisms of the

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\(^1\)In particular, such hyperchaotic attractors were found in the models of gas bubbles’ dynamics \([31]\).
appearance of hyperchaotic attractors for both cases of small and not very small values of the Jacobian $B$.

In the paper, we propose two types of bifurcation scenarios leading to the appearance of hyperchaotic attractors in one-parameter families of three-dimensional maps. In turn, both these scenarios can be divided into two complementary parts. The first part is phenomenological, it outlines the main stages of the formation of the so-called discrete homoclinic attractors that, by definition [11, 13], contain only one saddle fixed point and entirely its unstable invariant manifold. In the case under consideration, the fixed point is either a saddle-focus of type $(1,2)$ or a saddle with two negative unstable multipliers. This phenomenological part includes only a few of local and global bifurcations leading from a simple attractor (stable fixed point) to a homoclinic attractor containing nontrivial hyperbolic subsets with two-dimensional unstable manifolds.

These subsets form a “skeleton” for an expected hyperchaotic attractor. However, as often happens, this attractor can not be hyperchaotic, especially for initial stages of its formation. For the appearance of “genuine” hyperchaos, it is necessary that:

- orbits with two directions of instability occupy the most part of an attractor.

This problem is the essence of the second, empirical (experimental or numerical), part of the scenarios. This part is much more complicated and variable, because, unlike the first one, it includes infinite sequences of various bifurcations. These sequences contain, in particular, bifurcations of periodic orbits (stable and saddle of type $(2,1)$) transforming them into saddle periodic orbits of type $(1,2)$, giving a way to get two positive Lyapunov exponents on the attractor.

The phenomenological part of scenarios outlines, in general terms, routes to the onset of homoclinic attractors containing a fixed point $O$ of type $(1,2)$. Together with the unstable manifold, such attractor also contains all homoclinic orbits to the fixed point, i.e. such orbits in which the stable and unstable manifolds of the fixed point intersect. In general case, these intersections are transverse. By the Smale-Shilnikov theorem [18, 19] this implies that the attractor must contain nontrivial hyperbolic subsets whose unstable invariant manifolds are two-dimensional. However, this fact does not automatically lead to hyperchaos of the attractor, since, as it happens in most cases, this attractor is not hyperbolic and may contain also saddle periodic orbits of type $(2,1)$ (and even periodic sinks) which together can make a greater contribution to the calculation of the averaged Lyapunov exponents.

In particular, this is true for discrete Shilnikov attractors containing a saddle-focus fixed point of type $(1,2)$. Such attractors were introduced and partially studied in the papers [11, 13, 14], where a phenomenological scenario of their appearance was also described. The main stages of this scenario are schematically represented in Fig. 6. They are as follows:

(i) the supercritical Neimark-Sacker bifurcation with a stable fixed point $O$ after which this point becomes saddle-focus of type $(1,2)$ and a stable invariant curve $L$ appears in its neighborhood (Fig. 6a);
(ii) the formation of Shilnikov whirlpool tightening almost all orbits from some absorbing domain (Fig. 6c);

(iii) the emergence of transversal intersection between one-dimensional stable and two-dimensional unstable invariant manifolds of the saddle-focus fixed point $O$.

In Section 2.2 we discuss hyperchaotic properties of discrete Shilnikov attractors and study a class of interior bifurcations that create and support the spiral hyperchaos.

We consider also another type of discrete homoclinic attractors which contain a saddle fixed point with a pair of negative unstable multipliers. We show how such attractors can appear in one-parameter families of three-dimensional maps. The key steps of the corresponding phenomenological scenario are as follows:

(i) the supercritical period-doubling bifurcation with a stable fixed point $O$ after which this point becomes saddle of type (2,1), see Fig. 2b;

(ii) the supercritical period-doubling bifurcation with the saddle fixed point $O$ after which it becomes saddle of the desired type (1,2), see Fig. 2c;

(iii) the emergence of transversal homoclinic intersection between stable and unstable manifolds of $O$, see Fig. 3.

Since the proposed new scenario looks as an extension of the well-known scenario of the onset of the Hénon attractor, we call the proposed attractors hyperchaotic Hénon-like attractors.

In more detail, both phenomenological scenarios are considered in Section 2 where we also supplement them by the description of the second, empirical, part responded for the appearance of two positive Lyapunov exponents in numerical experiments. This part is motivated, in a sense, by the papers [20, 21, 22] where it was shown (on examples of two-dimensional endomorphism and coupled flow systems) that chaotic attractors can transform to hyperchaotic ones as a result of absorption of periodic orbits with two-dimensional unstable invariant manifolds. We suggest two new bifurcation mechanisms leading to such transformations. The first one is associated with the appearance of infinite cascades of period-doubling bifurcations with saddle periodic orbits of type (2,1) transforming them into orbits of type (1,2). The second mechanism is related to the direct formation of saddle-focus periodic orbits of type (1,2) via Neimark-Sacker bifurcations with stable periodic orbits. In both cases, the absorption of hyperchaotic saddles (of types (1,2)) by the attractor happens via homoclinic bifurcations.

In the second part of the paper (Sections 4–6), we apply the proposed scenarios for studying mechanisms of the appearance of hyperchaotic attractors in one-parameter families of the three-dimensional Mirá map (1). We consider cases of small and not very small values of the Jacobian $B$. In both cases we start from a stable fixed point $O_+$ which appears via a saddle-node bifurcation together with a saddle fixed point $O_-$ of type (2,1) and change parameters towards the region SH(1,2), see Figure 8, where the Smale horseshoe of type (1,2) exists.

In three-dimensional maps, another type of Hénon-like attractors is also possible. It appears as a result of a sequence of period-doubling bifurcations with a stable invariant curve, and was called as quasiperiodic Hénon-like attractor [59, 41, 42] by analogy with Hénon-like attractors.
The first bifurcation on this pathway is always the supercritical Neimark-Sacker bifurcation after which the point $O_+$ becomes saddle-focus of type (1,2) and a stable invariant curve $L$ is born in its neighborhood. Further, this curve breaks down giving, at first, a rise of a certain chaotic attractor (with only one positive Lyapunov exponent) and, then, this attractor becomes hyperchaotic. The study of accompanying bifurcations is, in fact, the main aim of this part of the paper.

We note that usually, before the destruction, the curve $L$ becomes resonant: a pair of period-q stable and saddle (2,1) orbits appears on it (inside the corresponding Arnold tongue); in this case the resonant curve is the closure of unstable manifold of the period-q saddle orbit.

In the paper we consider only the cases of the so-called strong resonances 1:3 and 1:4 which are the most difficult and exiting. Besides, their influence on the organization of corresponding bifurcation diagrams is the most visible (see e.g. Lyapunov diagrams for the map under consideration in Fig. 10 where codimension-2 points giving rise to the Arnold tongues with the strong resonances 1:3 and 1:4 are denoted by R$_3$ and R$_4$, respectively). In the case of two-dimensional diffeomorphisms, the nongenerated 1:3 resonance leads usually to the global instability [23], the corresponding invariant curve breaks down without the appearance of stable element of dynamics. However, this is no longer true for three-dimensional maps with not small values of the Jacobian. In particular, as is shown in [11, 13], regular and chaotic attractors can appear in map (1) after the destruction of the curve $L$ near the 1:3 resonance. In this paper, we show that these attractors can be even hyperchaotic. However, they occupy very thin regions in the parameter space, and can appear only for specific (not small) values of the Jacobian $B$, see red-colored regions below the point R$_3$ in Fig. 10d.

As for the 1:4 resonance, we show that, unlike the 1:3 resonance, it responses for the emergence of the most visible Arnold tongue with stable dynamics. Inside this tongue, the resonant period-4 stable and saddle (2,1) orbits undergo numerous bifurcations. Moreover, a type of these bifurcations essentially depends on values of the Jacobian $B$.

For small values of $B$, these bifurcations include a cascade of period-doubling bifurcations resulting in a 4-component Hénon-like attractor containing infinitely many saddle orbits of type (2,1), see Fig. 13. In their turn, these orbits, as well as the resonant period-4 saddle orbit, also undergo cascades of period-doubling bifurcations leading to the formation of hyperchaotic attractor containing infinitely many periodic saddle orbits of type (1,2), see Section 4 for more detail.

The stable period-4 orbits can also undergo the supercritical Neimark-Sacker bifurcation followed by the creation of a 4-component discrete Shilnikov-like attractor. This scenario of the transition to hyperchaos is more typical for the cases of not small values of the parameter $B$ ($B \in [0.3, 0.6]$, see Fig 10b). Independently of this, the resonant period-4 saddle orbit (of type (1,2)), as in the case of small $B$, undergoes an infinite cascade of period-doubling bifurcations leading to the creation of nontrivial hyperbolic subset of type (1,2). Then, a new type of hyperchaotic attractor can appear when this subset or its parts are absorbed by the above mentioned 4-component Shilnikov attractor. After such absorption, the 4-component attractor transforms into a one-component attractor containing both period-4 saddle-focus and saddle orbits of type (1,2), see Section 5 for more detail.

It is important to note, that in map (1) the boundary of the absorbing domain for attractors developed from the point $O_+$ is formed by the two-dimensional stable invariant
manifold $W^s$ of the second fixed point $O_-$. Therefore, when, finally, these attractors collide with $W^s(O_-)$, almost all orbits escape to infinity. We also note, that before this crisis the resulting strange attractor can become of Shilnikov type due to the absorption of the fixed point $O_+$ which, in all cases, is the saddle-focus of type (1,2).

In the last part of the paper (Section 3), we demonstrate another scenario of the destruction of the curve $L$. We show that for not small values of the Jacobian (e.g. for $B = 0.7$) this curve can undergo a quite long sequence of period-doubling bifurcations resulting in the formation of a chaotic attractor with one positive and one very close to zero Lyapunov exponents. We discuss this phenomenon and give some simple explanation for it.

2 Scenarios of the appearance of hyperchaotic attractors

In this section, we describe scenarios (both phenomenological and empirical parts) that lead to the appearance of hyperchaotic attractors in three-dimensional maps. Let us consider a one-parameter family of three-dimensional maps $\bar{x} = F(x, \varepsilon)$ depending on a parameter $\varepsilon$. In the presented below scenarios, we start from a fixed point $O$ which is asymptotically stable and belongs to some absorbing domain $D_a(O)$, see Fig. 2a. Finally, as a result of a series of codimension one bifurcations, we obtain a homoclinic attractor containing the same point $O$ which, after several bifurcations, becomes a saddle or saddle-focus with the two-dimensional unstable invariant manifold. We also describe an empirical part of scenarios due to which most periodic saddle orbits inside an attractor gets the two-dimensional unstable invariant manifolds.

2.1 Hyperchaotic Hénon-like attractor

The hyperchaotic Hénon-like attractor is a homoclinic attractor containing a saddle fixed point with a pair of negative unstable multipliers. It can appear as a result of the scenario that can be viewed, in a sense, as a generalization of the well-known scenario of the birth of Hénon attractor. Therefore, let us recall, firstly, some details on the Hénon-like attractors and scenarios of their appearance.

2.1.1 Some details about Hénon-like attractors

The Hénon attractor is a homoclinic attractor of the two-dimensional Hénon map [24]. It contains the saddle fixed point $O$ with a negative unstable multiplier and is formed after the Feigenbaum cascade of period-doubling bifurcations [25, 26, 27] followed by a cascade of heteroclinic “band-merging” bifurcations [28]. Let us denote the stable and unstable multipliers of the point $O$ by $\lambda$ and $\gamma$. Then, for the Hénon attractor the following conditions always hold: $\gamma < -1$ (unstable multiplier is negative) and $|\gamma \lambda| < 1$ (area-contracting condition). Depending on the sign of the stable multiplier $\lambda$, Hénon attractors can be of two types: if $-1 < \lambda < 0$ the attractor is orientable, see the example of its phase portrait in Fig. 1a; and if $0 < \lambda < 1$ it is nonorientable, see the phase portrait in Fig. 1b; (note that exactly this attractor was discovered and studied by M. Hénon in
If the Hénon attractor exists it also contains the unstable manifold $W^u$ of $O$ and homoclinic points $h_i$ to $O$, i.e., such points where $W^u(O)$ intersects with the stable manifold $W^s(O)$.

Moreover, as other homoclinic attractors, the Hénon attractor can be viewed as the closure of the unstable manifold $W^u(O)$. Figures 1b and 1d show the schematic representation (skeleton) of orientable and nonorientable Hénon attractors, respectively. Let us briefly describe their structure.

The unstable invariant manifold $W^u(O)$ is divided by the point $O$ into two connected components – separatrices $W^{u+}$ and $W^{u-}$. Since the unstable multiplier $\gamma$ of $O$ is negative ($\gamma < -1$), the separatrices $W^{u+}$ and $W^{u-}$ are invariant under $F^2$ and such that $F(W^{u+}) = W^{u-}$ and $F(W^{u-}) = W^{u+}$. This implies that points of $W^u$ jump under iterations of $F$ alternately from one separatrix to another.

The stable manifold $W^s(O)$ is also one-dimensional and it is divided by the point $O$ into two separatrices $W^{s+}$ and $W^{s-}$. For the orientable Hénon attractor, the stable multiplier $\lambda$ is negative and thus, as for the unstable manifold, $F(W^{s+}) = W^{s-}$ and $F(W^{s-}) = W^{s+}$.

Let $h_1$ be an intersection point of $W^{u+}$ with $W^{s+}$. Then, $h_2$ is an intersection point of $W^{u-}$ with $W^{s-}$, since $F(W^{u+}) = W^{u-}$ and $F(W^{s+}) = W^{s-}$; $h_3$ is again an intersection point of $W^{u+}$ with $W^{s+}$, etc. Correspondingly, the points $h_1, h_2, \ldots$ are homoclinic points of some homoclinic to $O$ orbit. Here, the points with odd indices $h_1, h_3, \ldots$ belong to the separatrix $W^{s+}$, while the points with even indices $h_2, h_4, \ldots$ belong to $W^{s-}$, see

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\[24\] The question of the existence of Hénon attractor is very delicate. For sufficiently small positive values of the Jacobian of the Hénon map Benedicks and Carleson proved in [66] that the set of parameter values corresponding to the chaotic attractor is a Cantor set with the positive Lebesgue measure. Moreover, the set of parameter values corresponding to stable periodic orbits is dense in any neighborhood of parameters with the chaotic attractor. For sufficiently large values of the Jacobian (very close to the classical Hénon’s parameters) an analogous result was obtained in [67] by means of computer-assisted proof methods. Thus, for a concrete parameters, one can never be sure whether the chaotic attractor is observed or it just a transient chaos, after which orbits will tend to some stable periodic orbit with an extremely narrow absorbing domain.
For the nonorientable Hénon attractor $\lambda$ is positive and, thus, all homoclinic points $h_1, h_2, ...$ belong to one separatrix (e.g. $W^{s+}$) jumping from one unstable separatrix to another, see Fig. 1b. In both orientable and nonorientable cases, the homoclinic Hénon attractor is exactly the closure of the unstable manifold $W^u(O)$.

It is worth noting that homoclinic attractors with the same structure, further Hénon-like attractors, appear in many two-dimensional (and also multidimensional) maps, as well as in Poincaré maps for various multidimensional systems of differential equations.

### 2.1.2 Phenomenological part of the scenario of hyperchaotic Hénon-like attractor appearance

We start with the phenomenological part of the scenario leading to the appearance of hyperchaotic Hénon-like attractor purposely skipping some details (accompanying bifurcations) which will be given in the framework of empirical part of the scenario in Sec. 2.1.3.

As for the Hénon attractor, the first step in the framework of scenario is the supercritical period-doubling bifurcation occurring with the stable fixed point $O$ at $\varepsilon = \varepsilon_{PD}$. After this bifurcation, the point $O$ becomes saddle of $(2,1)$-type and a stable period-2 orbit $(p_1, p_2)$ appears in its neighborhood, see Fig. 2b. The saddle point $O$ has the following set of multipliers: $\gamma < -1, -1 < \lambda_1 < 0$ and $|\lambda_2| < 1$. The unstable invariant manifold $W^u(O)$, separated by the point $O$ into two separatrices $W^{u+}$ and $W^{u-}$, is a segment with endpoints $p_1$ and $p_2$. Since this manifold corresponds to the negative multiplier $\gamma < -1$, we have a semi-local symmetry between the pair of separatrices $W^{u+}$ and $W^{u-}$ ($F(W^{u+}) = W^{u-}$ and $F(W^{u-}) = W^{u+}$). The stable invariant manifold $W^s$ corresponds to a pair of real multipliers $\lambda_1$ and $\lambda_2$. Suppose, $|\lambda_1| > |\lambda_2|$, i.e. $\lambda_1$ corresponds to the leading direction $W^{ls}$, and $\lambda_2$ – to the strong stable manifold $W^{ss}$ touching at $O$ the eigenvector of $\lambda_2$. Then, $\lambda_2 > 0$ if the map $F$ is orientable and $\lambda_2 < 0$, otherwise. Further, we consider only orientable maps. The nonorientable cases will be considered in future papers.
Remark 1 The first step in the framework of this scenario is the same as for the creation of the so-called discrete Lorenz and figure-eight homoclinic attractors \[12, 13, 14\]. Both these attractors are remarkable since they can be pseudohyperbolic \[15, 16, 17\]. Pseudohyperbolicity is a weak version of hyperbolicity. Chaotic dynamics of pseudohyperbolic attractors persist under small perturbations (as for hyperbolic attractors), despite the possible occurrence of homoclinic tangencies inside them. Note that the saddle fixed point belonging to both these attractors should be area-expanding, which means that \(\gamma \lambda_1 > 1\).

The next principal bifurcation in the framework of the scenario is the supercritical period-doubling bifurcation with the saddle fixed point \(O\) at \(\varepsilon = \varepsilon^{PD}\). After this bifurcation, the point \(O\) becomes saddle of \((1,2)\)-type a period-2 saddle orbit \((s_1, s_2)\) of \((2,1)\)-type appears in its neighborhood, see Fig. 2c. Both unstable multipliers \(\gamma\) and \(\lambda_1\) of the point \(O\) are negative at \(\varepsilon > \varepsilon^{PD}\), the stable multiplier \(\lambda_2\), since we consider the orientable case, is positive, i.e. the following conditions on the multipliers are met here

\[
\gamma < \lambda_1 < -1, \quad 0 < \lambda_2 < 1.
\]

A restriction \(F_u\) of the initial map \(F\) into the local unstable manifold \(W^u_{loc}(O)\) has a fixed point \(\tilde{O} = O \cap W^u_{loc}\) which is the unstable node with a pair of multipliers \(\gamma < \lambda_1 < -1\). Thus, in \(W^u_{loc}\), there are a strong unstable invariant manifold \(W^{uu}\) touching at \(\tilde{O}\) the eigenvector corresponding to the unstable multiplier \(\gamma\), and a leading unstable direction \(W^{lu}\) corresponding to the multiplier \(\lambda_1\). Also note that the curves \(W^{uu}\) and \(W^{lu}\) divide \(W^u_{loc}\) into four fragments \(\Pi_1, \Pi_2, \Pi_3,\) and \(\Pi_4\) and, since both multipliers of \(\tilde{O}\) are negative, \(F_u(\Pi_1) = \Pi_3, \quad F_u(\Pi_3) = \Pi_1, \quad F_u(\Pi_2) = \Pi_3,\) and \(F_u(\Pi_4) = \Pi_2\). All orbits in \(W^u_{loc}\), except those that belong to \(W^{uu}\), tend in backward time to the node \(\tilde{O}\) along the smooth cubic parabola-like curves. These curves touch the leading direction \(W^{lu}\) at \(\tilde{O}\), see Fig. 3a. If \(h_1 \in \Pi_1\) is one of such points, then each its odd (even) iteration under \(F_u^{-1}\) tends to \(\tilde{O}\) along right (left) branch of this parabola-like curve staying in \(\Pi_3\) \((\Pi_1)\).

It is important to note that since, by the moment, only orbits belonging to \(W^s(O)\) tend to the saddle fixed point \(O\), for other orbits there is no mechanism to return into the neighborhood of this point. Such mechanism appears at \(\varepsilon = \varepsilon^H\) together with the
emergence of a homoclinic orbit to $O$, see Fig. 3b. This orbit appears when the stable manifold $W^s(O)$ separated by the point $O$ into two separatrices $W^{s+}$ and $W^{s-}$ starts to intersect with the unstable manifold $W^u(O)$. Suppose that $h_1$ is the first intersection point of $W^{s+}$ with $W^u$ at $\Pi_1$, see Fig. 3a. Then (since $F^{-1}(W^{s+}) = W^{s+}$ and $F_u(\Pi_1) = \Pi_3$), $h_2 = F^{-1}(h_1)$ is the intersection point of the same stable separatrix $W^{s+}$ with $W^u$ in $\Pi_3$; $h_3 = F^{-1}(h_2)$ is again an intersection point of $W^{s+}$ with $W^u$ in $\Pi_1$, etc. Correspondingly, the points $h_1, h_2, \ldots$ are homoclinic points of some homoclinic to $O$ orbit, see Fig. 3b.

At $\varepsilon > \varepsilon_H$, this homoclinic orbit gives a nontrivial hyperbolic subset of (1,2)-type. If there are no other attractors in the absorbing domain $D_a(O)$, we obtain a homoclinic attractor containing the saddle point $O$ with a pair of negative unstable multipliers. If the amount of saddle orbits of (1,2)-type inside the attractor is greater than the amount of saddle orbits of (2,1)-type, a pair of Lyapunov exponents becomes positive and we observe hyperchaos in numerical experiments. As other types of homoclinic attractors, this attractor is formed by the closure of unstable manifold of the fixed point $O$. We call attractors of such type hyperchaotic Hénon-like attractors.

In the following section, we show how hyperchaotic Hénon-like attractors can naturally appear in multidimensional systems demonstrating cascades of period-doubling bifurcations. In Sec. 2.2.2 we generalize the empirical part of this scenario to attractors developed from stable periodic orbits occurring inside Arnold tongues. In Section 4 we demonstrate its implementation for periodic resonant orbits occurring inside Arnold tongues of the three-dimensional Miřá map (1).

### 2.1.3 Empirical part of the scenario of hyperchaotic Hénon-like attractor appearance

Let us present the empirical part of the scenario and show how the described above attractor can appear in multidimensional systems demonstrating transition to chaos via cascades of period-doubling bifurcation. Suppose that, at $\varepsilon = \varepsilon_H$, a Hénon-like attractor appears after a cascade of period-doubling bifurcations followed by a cascade of heteroclinic band-merging bifurcations, see Fig. 4. The schematic representation (skeleton) of this attractor is presented in Fig. 5a (left panel). Here, as in the two-dimensional case, the unstable multiplier of the point $O$ is negative ($\gamma < -1$) and the stable ones are real. Depending on the signs of stable multipliers $\lambda_1$ and $\lambda_2$ homoclinic structures for Hénon-like attractors can be of four possible types (two in orientable and two in nonorientable cases). Here, we consider only one of orientable cases, see the location of multipliers in the top-right insert of Fig. 5a. Note that this attractor is a three-dimensional generalization of the classical orientable Hénon attractor.

Let us explain the structure of its skeleton. In the three-dimensional case the stable invariant manifold $W^s(O)$ is two-dimensional here. A restricted map $F_s$ of the initial map $F$ into $W^s_{loc}(O)$ has a fixed point $\bar{O} = O \cap W^s_{loc}$ which is a stable node with a pair of multipliers $\lambda_1$ and $\lambda_2$.

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4We note that the described hyperchaotic attractor is not hyperbolic. Together with periodic saddle orbits of (1,2)-type it also contains periodic saddle orbits of a (2,1)-type and homoclinic (heteroclinic) tangencies between invariant manifolds of various saddles. Moreover, it is possible to show that such attractor also contains heterodimensional cycles connecting these saddles [68, 69, 70]. Such cycles robustly persist under small perturbation of the system guaranteeing the presence of a nontrivial hyperbolic subset of (1,2)-type.
Figure 4: Schematic representation of the scenario of hyperchaotic Hénon-like attractor appearance as a result of a cascade of period-doubling bifurcations with a stable fixed point \( O \) followed by cascades of period-doubling bifurcations with periodic saddle orbits of \((2,1)\)-type.

Figure 5: Illustration of an empirical part of the scenario of hyperchaotic Hénon-like attractor appearance via cascades of period-doubling bifurcation. (a) Skeleton of Hénon-like attractor emerging after a cascade of period-doubling bifurcations followed by cascade of band-fusion bifurcations; (b) skeleton after a period-doubling bifurcation with the saddle fixed point \( O \).

According to the condition \( |\lambda_1| > |\lambda_2| \), in \( W^s_{loc} \) there are the strong stable invariant manifold \( W^{ss} \) touching at \( \hat{O} \) the eigenvector corresponding \( \lambda_2 > 0 \) and the leading direction \( W^{ls} \) corresponding to \( \lambda_1 < 0 \). Also note that the curve \( W^{ss} \) divides \( W^s_{loc} \) into two parts \( \Pi_1 \) and \( \Pi_2 \), and, since the leading multiplier is negative \((\lambda_1 < 0)\), \( F_s(\Pi_1) = \Pi_2 \) and \( F_s(\Pi_2) = \Pi_1 \). All orbits in \( W^s_{loc} \), except those that belong to \( W^{ss} \) tend to the node \( \hat{O} \) along the parabola-like curves touching leading direction \( W^{ls} \), see the right panel in Fig. 5a. If we take some point \( h_1 \) belonging to one of such curves \( p_c \) and consider its images under \( F_s \), we obtain an orbit \( h_1, h_2, ... \) tending to \( \hat{O} \) on one (e.g. right) side of \( W^{ls} \) and jumping from one branch of \( p_c \) to another after each iteration.

Now, it is easy to imagine the geometrical skeleton of the Hénon-like attractor presented in Fig. 5a. Suppose \( h_1 \) is the intersection point of \( W^{u+} \) with \( W^s \) at its upper part \( \Pi_1 \). Then, (since \( F(W^{u+}) = W^{u-} \) and \( F_s(\Pi_1) = \Pi_2 \)) \( h_2 = F(h_1) \) is the intersection point of \( W^{u-} \) with \( W^s \) in its bottom part \( \Pi_2 \); \( h_3 = F(h_2) \) is again an intersection point of \( W^{u+} \)
with $W^{s+}$ in $\Pi_1$, etc.

Then, we suppose that the saddle point $O$ of (2,1)-type undergoes a cascade of period-doubling bifurcations, see the schematic phase portrait after the first period-doubling bifurcation in Fig. 5b. After each such bifurcation the periodic orbit gets the desirable two-dimensional unstable manifold and a double-period orbit of (2,1)-type appears in its neighborhood. We, then, suppose that the period-$2^n$ saddle orbits of (2,1)-type emerging after the corresponding period-doubling bifurcations, as well as all other saddles of (2,1)-type, also undergo full cascades of period-doubling bifurcation after which all these orbits become of (1,2)-type. As a result we obtain a multi-component hyperchaotic attractor.

Further, the multi-component hyperchaotic attractor goes through a cascade of heteroclinic band-merging bifurcations. These bifurcations are similar, in a sense, to those ones which lead to the pairwise merger of components of chaotic attractor arising after the full cascade of period-doubling bifurcations within the second part of the scenario resulting in the birth of Hénon-like attractors. After each such bifurcation the number of components decreases by 2 times.

The final bifurcation here, resulting in the merger of two last components, leads to the emergence of homoclinic intersections between $W^s(O)$ and $W^u(O)$. As a result, the hyperchaotic Hénon-like attractor occurs at $\varepsilon = \varepsilon^H$. Note, that this intersection persists at some parameter region. However, the hyperchaotic Hénon-like attractor (as the classical Hénon attractor) is not pseudohyperbolic (robust). It contains the saddle fixed point $O$ with the two-dimensional unstable invariant manifold and (as we have an attractor) with such a saddle index $\rho = \lambda_1\lambda_2$, that $|\rho| < 1$. As was shown in [2, 3, 29], stable periodic orbits can appear in this case under arbitrarily small perturbations due to homoclinic tangencies.

2.2 Discrete Shilnikov attractor

As shown at the beginning of this section, a period-doubling bifurcation can be the first step in the framework of onset of Hénon-like and hyperchaotic Hénon-like attractors. A natural question arises here. Which homoclinic attractors can appear if the first codimension one bifurcation is a Neimark-Sacker bifurcation? The answer to this question was given in [11] (see also [13], and [14]) where a scenario of the appearance of the so-called discrete Shilnikov attractor containing a saddle-focus fixed point of (1,2)-type was proposed. This scenario goes back to the paper by Shilnikov [30] where a similar scenario was proposed for one-parameter families of three-dimensional flows. Since this scenario can play an important role, in the development of hyperchaos [31], let us briefly describe it.

2.2.1 Phenomenological part of the scenario of discrete Shilnikov attractor appearance

As in the pervious case, at $\varepsilon < \varepsilon_1$, the fixed point $O$ is stable but focal (a pair of its multipliers is complex-conjugate), see Fig. 6a. Suppose that at $\varepsilon = \varepsilon_1$ it undergoes the supercritical Neimark-Sacker bifurcation. As a result, this fixed point becomes a saddle-focus of (1,2)-type, and a stable invariant curve $L$ is born in its neighborhood, see Fig. 6b. Note that after the birth, this curve has a nodal type: the two-dimensional unstable invariant manifold $W^u$ of $O$ is a disc with an edge on $L$. Then, at $\varepsilon = \varepsilon_2$, the stable
curve becomes focal, and, as a result, $W^u(O)$ starts to wind on it forming the so-called Shilnikov whirlpool, see Fig. 6c. All orbits from the absorbing domain $D_a(O)$ (except the stable separatrix $W^{-s}(O)$) are tightened by this whirlpool. With further increase in $\varepsilon$, the sizes of whirlpool is increased, and finally, at $\varepsilon = \varepsilon_3$, the stable separatrix $W^{s-}(O)$ touches $W^u(O)$. As a result, at some interval $\varepsilon_3 < \varepsilon < \varepsilon_4$, the fixed point $O$ has a transversal homoclinic structure. By Smale and Shilnikov [18, 19], in the neighborhood of the transversal intersection $W^u(O) \cap W^{s+}(O)$ there exist countably many periodic orbits of the same type as the fixed point $O$ which is a saddle-focus of (1,2)-type.

Figure 6: Illustration towards the scenario of onset of a discrete Shilnikov attractor containing a saddle-focus fixed point of (1,2)-type. This figure is taken from [14].

If the stable invariant curve $L$ breaks down (by Afraimovich-Shilnikov [32] or due to some other scenario [33, 34, 35, 36]) giving chaotic attractor (torus-chaos), then, on some subinterval inside $\varepsilon \in (\varepsilon_3, \varepsilon_4)$, it can contain the fixed point $O$ together with the nontrivial hyperbolic set of (1,2)-type and become hyperchaotic.

Remark 2 In the framework of the described scenario we suppose that the fixed point $O$ undergoes the supercritical Neimark-Sacker bifurcation. However, it can undergo the
subcritical Neimark-Sacker bifurcation i.e. the point $O$ can sharply lose the stability due to the merger with a saddle invariant curve existing in a neighborhood of $O$. In this case, a discrete Shilnikov attractor can appear sharply. Similar scenario (but for a three-dimensional flow system) was observed e.g. in [37].

2.2.2 Empirical part of the scenario of hyperchaotic discrete Shilnikov attractor appearance

The empirical part of the scenario for the emergence of Shilnikov attractors describes concrete mechanisms of chaotization of the invariant curve $L$ (Fig. 7a) and the absorption of the saddle-focus fixed point $O$ by the resulting torus-chaos attractor. Let us describe possible scenarios leading to the emergency of hyperchaotic attractors on the base of $L$. Usually, before the destruction, an invariant curve becomes resonant: a pair of stable and saddle period-$q$ orbits $O_q$ and $S_q$ appears on it as a result of the saddle-node bifurcation. In this case, the curve is the closure of the unstable one-dimensional invariant manifold of $S_q$, see Fig. 7b. Note that the stable two-dimensional invariant manifold of $S_q$ forms the boundary of absorbing domain for the stable point $O_q$ or attractors developing from it in accordance with one of two following scenarios.

\begin{itemize}
  \item [(a)] $O_q$ saddle-focus (1,2)
  \item [(b)] $O_q$ saddle (2,1)
  \item [(c)] $O_q$ undergoes the supercritical period-doubling bifurcation
  \item [(d)] $O_q$ undergoes the supercritical Neimark-Sacker bifurcation
\end{itemize}

**Figure 7:** Schematic representation for two possible mechanisms of the destruction of the curve $L$: (a) $L$ is a nonresonant (ergodic) invariant curve; (b) $L$ is a resonant curve (a pair of stable and saddle period-$q$ orbits $O_q$ and $S_q$ appear on it via the saddle-node bifurcation); (c) $O_q$ undergoes the supercritical period-doubling bifurcation (e) $O_q$ undergoes the supercritical Neimark-Sacker bifurcation.
1. The point $O_q$ undergoes the supercritical period-doubling bifurcation after which it becomes saddle of (2,1)-type and a stable period-2$q$ orbit $O_{2q}$ is born in its neighborhood, see Fig. 7c, i.e. in this case we have a hypothetical possibility to obtain the hyperchaotic Hénon-like attractor on the base of the resonant orbit $O_q$ (see Sec. 2.1.3 for more detail). However, we note that the cascade of period-doubling bifurcations with the stable orbit can terminate at some moment by a Neimark-Sacker bifurcation after which the resulting periodic orbit gets the two-dimensional unstable manifold immediately, and a stable multi-component invariant curve is born. In its turn, this curve, before the destruction, also becomes resonant, for it one of two cases under consideration is applicable, and so on.

2. The point $O_q$ undergoes the supercritical Neimark-Sacker bifurcation after which it becomes saddle-focal of (1,2)-type and a stable $q$-component invariant curve $L_q$ is born in its neighborhood, see Fig. 7d. So, here we have possibility to obtain a $q$-component Shilnikov attractor. It is important to note, that the curve $L_q$, before the destruction also becomes resonant, for it one of two cases under consideration is applicable, and so on.

Concerning the resonant saddle orbit $S_q$ (as well as other resonant saddle orbits arising in the framework of the described above sequence), it undergoes the cascade of period-doubling bifurcations resulting in a one-component nontrivial hyperbolic set of (1,2)-type $SH_q(1,2)$ (the same as with the saddle fixed point in the framework of the scenario described in Sec. 2.1.3). Touching the one-dimensional stable manifolds of $SH_q(1,2)$ the attractor developed from $O_q$ undergoes crisis and collides into a one-component attractor which, after that, contains $SH_q(1,2)$.

Note that, before becoming resonant, the curve $L$, in some cases, can go through a sufficiently large sequence of period-doubling bifurcations. After each such bifurcation a curve becomes saddle, and a stable double-round invariant curve emerges in its neighborhood, see e.g. Fig. 7e. Such transition to chaotic attractors (including Shilnikov ones), is observed quite often (see e.g. [38, 39, 40, 41, 42, 43, 44, 45, 46, 47]) and can lead to the birth of “flow-like” chaotic attractors possessing one positive and one very close to zero Lyapunov exponent (for flow systems chaos with additional close to zero Lyapunov exponent in the spectrum is observed in this case). In Sec. 6 we demonstrate and study this phenomenon in more detail.

3 Three-dimensional Mirá map: main bifurcations and dynamical regimes

In this section we study main bifurcations and discuss the most interesting dynamical regimes in the three-dimensional Mirá map (1). Recall, that this map has a constant Jacobian, $J = B$. Various types of attractors are possible only when the map is dissipative, i.e. when $|B| < 1$. In this paper we consider only orientation preserving (orientable) maps, when $B$ is positive ($0 < B < 1$).

Map (1) has up to two fixed points $O_+(x_+, y_+, z_+)$ and $O_-(x_-, y_-, z_-)$ with coordinates

$$x_\pm = y_\pm = z_\pm = \frac{M_2 + B - 1}{2} \pm \sqrt{\frac{(1 - B - M_2)^2}{4} + M_1}.$$ (2)
As one can see, these points exist only when the root expression is non-negative. Points $O_{\pm}$ are born under the saddle-node bifurcation (when one multiplier is equal to 1) on the surface

$$SN : M_1 = -\frac{(1 - B - M_2)^2}{4}. \quad (3)$$

A period-doubling bifurcation occurs when a multiplier is equal to -1 on the surface

$$PD : M_1 = \frac{3(B + M_2)^2 + 2(B + M_2) - 1}{4}. \quad (4)$$

The third codimension one bifurcation appears when a pair of multipliers becomes equal to $e^{\pm i\phi}, \phi \in (0, \pi)$. This case corresponds to a Neimark–Sacker bifurcation which appears on the surface

$$NS : M_1 = \frac{(2 - M_2 - B + M_2B - B^2)^2 - (1 - B - M_2)^2}{4}, \quad |M_2 - B| < 2. \quad (5)$$

Note that relations (3)-(5) define in the three-dimensional parameter space $(M_1, M_2, B)$ the region of stability of the point $O_+$. Figure 8 shows several slices of this space for various values of the parameter $B$. In this figure the region of stability of $O_+$ is colored in blue.

![Figure 8: Main bifurcations curves for the map (1) for various values of the parameter $B$. $SN, PD,$ and $NS$ are the saddle-node, period-doubling, and Neimark-Sacker bifurcation curves. The region of stability of the point $O_+$ is colored in blue. The region SH(1,2) corresponds to the existence of a nontrivial hyperbolic subset of (1,2)-type.](image)

In Fig. 8 we also plot the region SH(1,2) inside which the nonwandering set of map (1) is a hyperchaotic hyperbolic set consisting of saddle periodic orbits of (1,2)-type. Further, we also call this set Smale horseshoe of (1,2)-type. According to the theorem of Gonchenko and Li (th. 2 in [8]), this region is bounded by a curve

$$M_1 = \left(\tilde{\rho} + \sqrt{\tilde{\rho}^2 + \frac{1}{4}}\right)^2 - (1 - M_2 - B) \left(\tilde{\rho} + \sqrt{\tilde{\rho}^2 + \frac{1}{4}}\right),$$

where

$$\tilde{\rho} = \frac{3 + 5(|B| + |M_2|)}{3 + 4(|B| + |M_2|)} \left(1 + |B| + |M_2|\right).$$
In this paper we study evolution of attractors along pathways form the region of stability of the fixed point $O_+$ towards the region $\text{SH}(1,2)$.

As shown below, all interesting dynamical regimes in map (1) for $0 < B < 1$ are associated with bifurcations of the fixed point $O_+$. Thus, it is convenient to shift this point to the origin which gives the following representation for map (1)

\[
\begin{aligned}
\bar{x} &= y \\
\bar{y} &= z \\
\bar{z} &= Bx + Cy + Az - y^2.
\end{aligned}
\] (6)

The main difference between maps (1) and (6) is that in the last map both fixed points $O_+$ and $O_-$ always exist. The point $O_+$ becomes stable here under a transcritical saddle-node bifurcation (but note via a saddle-node bifurcations as it is for map (1)) on a plane

\[
\text{TR} : C = 1 - A - B.
\] (7)

As in map (1), other boundaries of stability of this point are determined by a period-doubling bifurcation on a plane

\[
\text{PD}_1 : C = 1 + A + B
\] (8)

and a Neimark–Sacker bifurcation on a surface

\[
\text{NS}_1 : C = B^2 - AB - 1, \quad -2 < A - B < 2,
\] (9)

see bifurcation diagram for $B = 0.5$ in Fig. 9. Respectively, relations (7)–(9) define in the three-dimensional parameter space $(A, B, C)$ the region of stability of the fixed point $O_+$ bounded by the transcritical saddle-node (TR), period-doubling (PD$_1$), and Neimark-Sacker (NS$_1$) bifurcation curves.

In Figure 10 we show diagrams of Lyapunov exponents (Lyapunov diagrams) for various values of the Jacobian $B$ ($B \in \{0.1, 0.3, 0.5, 0.7\}$). For these diagrams we use the following color coding: blue – for periodic regimes (all Lyapunov exponents are negative), green – for quasiperiodic regimes (the maximal Lyapunov exponent vanishes, while all other exponents are negative), yellow – for chaotic attractors (with only one positive Lyapunov exponent), and red color – for hyperchaotic attractors (with two positive Lyapunov exponents).

Phase portraits of various attractors (for different parameters) are shown in Figure 11. In Fig. 11a we demonstrate the stable invariant curve which appears after the supercritical Neimark-Sacker bifurcations with the fixed point $O_+$; in Fig. 11b – the stable period-4 orbit which emerges as a resonance on this curve; in Fig. 11c – the 4-component Hénon-like attractor; in Fig. 11d – the hyperchaotic Hénon-like attractor on the base of period-4 saddle orbit, in Fig. 11e – the hyperchaotic 4-component Shilnikov attractor; and in Fig. 11f – the hyperchaotic Shilnikov attractor containing the fixed point $O_+$.

As one can see in Fig. 10 hyperchaotic attractors occupy vast regions under the NS$_1$-curve of the supercritical Neimark-Sacker bifurcation. After this bifurcation, the

\[5\] On these Lyapunov diagrams we plot only regimes associated with the fixed point $O_+$. Above the curve TR, corresponding to a transcritical saddle-node bifurcation, the point $O_-$ becomes stable (this point is swapped by the stability with $O_+$). However, we do not color the corresponding part of $(A, C)$-parameter plane.
Figure 9: Main bifurcations curves for map (6) for $B = 0.5$. The curves $TR$, $PD_1$, and $NS_1$ correspond to the transcritical saddle-node, period-doubling, and supercritical Neimark-Sacker bifurcations. The region of stability of the point $O_+$ is colored in pink.

Point $O_+$ becomes saddle-focus of (1,2)-type and a stable invariant curve $L$ is born in its neighborhood, see e.g. Fig. 11a. Depending on the rotation number $\rho$, this curve can be resonant (if $\rho$ is rational, i.e. $\rho = p/q$) or ergodic (if $\rho$ is irrational). In the parameter plane, regions corresponding to the ergodic stable curve alternate with Arnold tongues originating from the curve of Neimark-Sacker bifurcation where $\rho = p/q$. Inside Arnold tongues (close enough to the curve $NS_1$) the stable invariant curve is resonant, see e.g. Fig. 11b. It is formed by the closure of the unstable invariant manifold of the period-$q$ saddle orbit emerging together with the stable one on the boundaries of the corresponding Arnold tongue.

The widest tongues correspond to the resonances with small $q$. Among them, the so-called strong resonances 1:3 and 1:4 are the most interesting and important. A tongue corresponding to the 1:3 resonance originates from a point $(A, C) = (B - 1, B - 1)$. It gives quite thin regions with stable dynamics, see Fig. 10. However, inside it the emergence of chaotic and, even, hyperchaotic attractors is possible, see more detail in Section 6. On the contrary, a tongue corresponding to the 1:4 resonance gives the vastest area with stable dynamics (see again Lyapunov diagrams in Fig. 10). It originates from a point $(A, C) = (B, -1)$. The stable period-4 orbit inside this tongue can give rise to different types of chaotic and, even, hyperchaotic homoclinic attractors containing this orbit. If the period-4 orbit undergoes a period-doubling bifurcation, we can obtain Hénon-like attractors, see Fig. 11c, and, finally, hyperchaotic Hénon-like attractors, see Fig. 11d. If the period-4 orbit undergoes the supercritical Neimark-Sacker bifurcations, giving a 4-component stable invariant curve, we, then, can observe a 4-component hyperchaotic Shilnikov attractor, see Fig. 11e, and, finally, hyperchaotic Shilnikov attractor containing...
the saddle-focus fixed point $O_+$. It is important to note that similar attractors and transitions to them are observed also in other Arnold tongues. In the framework of this paper, we study bifurcations associated only with the strong resonances 1:3 and 1:4. In particular, we show that hyperchaotic attractors on the base of the period-4 resonant orbit appear in accordance with the scenarios presented in Sec. 2.
Hyperchaos in map (6) via cascades of period-doubling bifurcations with periodic saddle orbits

In this section, we study one of the possible mechanisms of the appearance of hyperchaotic attractors in map (6) with sufficiently small values of the Jacobian $B$. Let us fix $B = 0.1$. Lyapunov diagram for this case is presented in Fig. 10b. Figure 12 shows a zoomed region of this diagram near Arnold tongue for the 1:4 resonance, and the corresponding bifurcation diagram obtained with help of MatContM package [48, 49, 50]. We denote the stable period-4 resonant orbit by $P_4 = (p_1, p_2, p_3, p_4)$, and the corresponding saddle orbit (of (2,1)-type) by $S_4 = (s_1, s_2, s_3, s_4)$. Let us, first, explain the organization of bifurcation curves associated with the 1:4 resonance and, then, study bifurcations along one-parameter pathways from the stable period-4 orbit $P_4$ to hyperchaotic attractors in the bottom part of this diagram.

The left and right boundaries of the stability region of $P_4$ are formed by a pair of saddle-node bifurcation cures $SN_4$ originating from the codimension 2 point $R_4$ where $O_+$ has a pair $e^{\pm i\pi/2}$ of multipliers. The bottom boundary of this region consists of three fragments: PD$^4$, NS$^4$, and, again, PD$^4$. The large left and right fragments correspond to the supercritical period-doubling bifurcation, and the small middle fragment corresponds to the supercritical Neimark-Sacker bifurcation. We note that the curve PD$^4$ touches
the curve $\text{SN}^4$ in a point $\text{ff}^4$ corresponding to a codimension 2 fold-flip bifurcation, where the point $P^4$ has the pair $(+1, -1)$ of multipliers. As is known from [51] (see also [52]), depending on coefficients of the corresponding normal form, this bifurcation can be of four possible types. Our studies show that here we observe the fourth case from [51]. Namely, above this point, on the curve $\text{PD}^4$, the period-doubling bifurcation occurs with the stable periodic orbit $P^4$, while below this point, on the fragment $\text{pd}^4$, period-doubling bifurcation occurs with the saddle period-4 orbit $S^4$. Respectively, the period-doubling bifurcation transforms the stable orbit $P^4$ to a saddle of (2,1)-type and the saddle orbit $S^4$ to a saddle of (1,2)-type.

It is worth noting that very close to the point $\text{ff}^4$, there is other important codimension 2 bifurcation, significantly contributing to the organization of the bifurcation diagram. Namely, on the curve $\text{pd}^4$, we observe a point of generalized period-doubling bifurcation which gives rise to a curve of a saddle-node bifurcation $\text{SN}^8$ corresponding to the birth of a pair of stable and saddle period-8 orbits $P^8$ and $S^8$. A fragment of the upper boundary of the existence region for $P^8$ is formed by the curve $\text{PD}^4$. The bottom boundary of this region consists (as for the orbit $P^4$) of three fragments: $\text{PD}^8$, $\text{NS}^8$, and $\text{PD}^8$. Similar to the case of period-4 orbit $P^4$, the period-doubling bifurcation curve $\text{PD}^8$ touches the curve $\text{SN}^8$ at a fold-flip bifurcation point $\text{ff}^8$. The bottom branch of this curve ($\text{pd}^8$) corresponds to the period-doubling bifurcation with the saddle orbit $S^8$. In its turn, close to the point $\text{ff}^8$, on the curve $\text{pd}^8$, we again observe the generalized period-doubling bifurcation giving rise to a saddle-node bifurcation curve $\text{SN}^{16}$ and so on.

We believe that an infinite sequence of codimension 2 fold-flip points $\text{ff}^1$, $\text{ff}^8$, $\text{ff}^{16}$, . . . , accumulates to some point $\text{ff}^\infty$ belonging to the boundary between periodic and chaotic
This cascade of the fold-flip bifurcations gives rise to a pair of cascades of period-doubling bifurcations: the first – with the stable periodic orbit $P^4$ (on the curves $PD_4$, $PD_8$, $PD_{16}$, . . .) and the second – with the saddle periodic orbit $S^4$ (on the curves $pd^4$, $pd^8$, $pd_{16}$, . . .). In its turn, the periodic saddle orbits $P^4$ and $P^8$, occurring after the corresponding period-doubling bifurcations, also undergo a cascade of period-doubling bifurcations. Respectively, the curves of first such period-doubling bifurcations reside below the Neimark-Sacker bifurcation curves $NS^4$, $NS^8$ and coincide with the same curves $PD^4$ and $PD^8$ (green dashed lines in Fig. 12b). We note that the type of period-doubling bifurcation along the curves $PD^4$ and $PD^8$ is changed at the codimension two points where these curves interest with the curves $NS^4$ and $NS^8$, respectively.

Thus, along pathways transverse to the lines $PD^n$ and $pd^n$ one can expect the implementation of the scenario presented in Section 2.1.3. Let us confirm it by studying bifurcations along a one-parameter pathway $AB$ with fixed $A = 0$. The results of corresponding analysis are shown in Figure 13.

First, the stable orbit $P^4$ undergoes a cascade of period-doubling bifurcations, see phase portrait after the first two steps (slightly below the curve $PD^{16}$ in Fig. 13b. The orbit $P^4$ becomes saddle of (2,1)-type after the first period-doubling bifurcation. Then, with further decrease in $C$, a cascade of heteroclinic band-merging bifurcations occurs, and, as a result, a four-component Hénon-like attractor containing $P^4$ appears, see Fig. 13b. The corresponding bifurcation tree is presented in Fig. 13h. In Fig. 13e depicting the graph of distance between the attractor and $p_1$ component of $P^4$, one can see that the saddle orbit $P^4$ starts to belong to the attractor (which means the Hénon-like attractor occurrence) at $C = C_{4\text{HA}} \approx -1.52$.

The next step in the framework of hyperchaotic attractors development is a period-doubling bifurcation with the saddle orbit $P^4$ occurring on the curve $PD_4$ (at $C = C_{PD_4} \approx -1.616$). After this bifurcation, $P^4$ becomes saddle of (1,2)-type and a period-8 saddle orbit of (2,1)-type appears in its neighborhood, see the red-colored continuation tree for the $p_1$ component of $P^4$ in Fig. 13h. As it can be seen in Figs. 13e and Fig. 13h, at $C < C_{PD_4}$ the point $p_1$ also belongs to the attractor. Moreover, as is shown in the graph of Lyapunov exponents presented in Fig. 13h, the attractor becomes hyperchaotic near the mentioned above period-doubling bifurcation. The resulting hyperchaotic attractor at $C = -1.62$ is presented in Fig. 13h.

Note that before the period-doubling bifurcation (at $C > C_{PD_4}$) with the saddle orbit $P^4$, the four-component Hénon-like attractor collides into the one-component attractor. It happens due to the boundary crisis: the unstable invariant manifold $W^u(P^4)$, forming the attractor, begins to intersect with the two-dimensional stable invariant manifold of the period-16 saddle orbit forming the boundary of its absorbing domain. (The corresponding heteroclinic bifurcation occurs at $C = C_{4\text{cr}} \approx -1.58$, see the jump of distance between the attractor and $s_1$ in Fig. 14a). This period-16 saddle orbit appears after two period-doubling bifurcations (happened on the curves $pd^4$ and $pd^8$, see Fig. 12b) with the saddle orbit $S^4$ which, as was mentioned above, goes through the full cascade of period-doubling bifurcations (see also the red-colored tree in Fig. 14b). With a further decrease in $C$, at $C = C_{4\text{ab}} \approx -1.65$, the attractor absorbs the saddle orbit $S^4$ of type (1,2), see again Fig. 14a.

It is worth noting that the hyperchaotic attractor presented in Fig. 13h contains the orbits $P^4$ and $S^4$ but does not contain the fixed point $O_+$ which becomes a saddle-focus of (1,2)-type at $C < -0.99$. With a further decrease in $C$, the distance between the
attractor and this fixed point decreases and, finally, at $C = C_{Sh} \approx -1.66$ it vanishes, see Fig. 13f. As a result, a hyperchaotic Shilnikov attractor appears, see Fig. 13d.

5 Hyperchaos in map (6) via onset of multicomponent Shilnikov attractors

In this section we study scenarios of the appearance of hyperchaotic attractors in map (6) with not very small values of the Jacobian $B$. Let us fix $B = 0.5$. Lyapunov diagram for
Figure 14: Diagrams illustrating bifurcations of the period-4 saddle orbit $S^4 = (s_1, s_2, s_3, s_4)$ which appears together with $P^4$ via a saddle-node bifurcation on the curve $SN^4$: (a) the graph of distance between the attractor and $s_1$; (b) bifurcation tree depicting dependency of $x$-coordinate for each four iteration of the map on parameter $C$ with the superimposed continuation tree for $s_1$.

In this case, hyperchaotic attractors occurs here after the destruction of the stable invariant curve $L$. However another mechanisms of the destruction of $L$ is more typical in this case.

Figure 15 shows an enlarged fragment of Lyapunov diagram near the Arnold tongue for the 1:4 resonance and the corresponding bifurcation diagram where we use the same denotations for periodic orbits and bifurcation curves associated with this resonance as in Sec. 4.

Figure 15: (a) $B = 0.5$. The fragment of Lyapunov diagram near the Arnold tongue corresponding to the 1:4 resonance, and (b) the corresponding bifurcation diagram on which we use the same denotations for bifurcation curves and points as in the diagram in Fig. 12.

As in the previous case, here we also observe the cascade of fold-flip bifurcation (ff, ff, ff, ff).
ff, . . . ) which gives rise to the pair of cascades of period-doubling bifurcations with the stable period-4 orbit $P^4$ (the curves $PD^4$, $PD^8$, . . .) and with the saddle period-4 orbit $S^4$ (the curves $pd^4$, $pd^8$, . . .). The generalized period-doubling bifurcations occurring on the curves $PD^4$, $PD^8$, . . . near the corresponding fold-flip points generate a cascade of saddle-node bifurcation curves $SN^4$, $SN^8$, . . . A pair of period-$(4 \cdot 2^n)$ stable and saddle orbits appears at the intersection of each such curve.

However, unlike the case of small values of the Jacobian $B$, the curves $pd^4$, $pd^8$, . . . are located above the curves $PD^4$, $PD^8$, . . ., compare Figures 12 and 15b. The study of transformation of bifurcation diagrams from the case of small values of $B$ (in particular, starting with $B = 0$) to the case of not small values of $B$ seems very interesting and promising problem for future research.

As in the case of small $B$, the region of stability for the period-4 orbit $P^4$ is bounded from below by the period-doubling bifurcation curve $PD^4$, Neimark-Sacker bifurcation curve $NS^4$, and again period-doubling bifurcation curve $PD^4$. However, the fragment $NS^4$ is much wider here comparing with the case of small $B$. Moreover, the curves $NS^4$ and $PD^4$ are organized in such a way, that most of pathways from $P^4$ to hyperchaos passes through one of the Neimark-Sacker bifurcation curves, see Fig. 15.

Further, let us fix $A = 0$ (together with $B = 0.5$) and study bifurcations along a pathway $CD$ from the stable period-4 orbit $P^4$ to hyperchaotic attractors. The results of corresponding one-parameter analysis are shown in Figure 16.

In contrast to the cases of small $B$ (see Sec. 4), the stable period-4 orbit $P^4$ undergoes here the supercritical Neimark-Sacker bifurcation. As a result, a stable 4-component invariant curve is born in the neighborhood of $P^4$ while this orbit becomes a saddle-focus of (1,2)-type. With decreasing $C$, this 4-component curve breaks down and a 4-component torus-chaos attractor (with only one positive Lyapunov exponent, see the graph of Lyapunov exponents in Fig. 16g) appears, see Fig. 16a.

Then, according to the scenario described in [31, 53] (see also Sec. 2.2), this torus-chaos attractor absorbs the saddle-focus orbit $P^4$ and, as a result, a 4-component hyperchaotic Shilnikov attractor appears, Fig. 16b. This attractor is homoclinic, it contains $P^4$, entirely its unstable invariant manifold $W^u$, and homoclinic points belonging to the intersection $W^u(P^4) \cap W^s(P^4)$. As it can be seen from Fig. 16b, depicting the distance between the attractor and a $p_4$-component of $P^4$, the attractor starts to contain $P^4$ at $C = C_{4sh} \approx -1.762$. Moreover, $P^4$ belongs to the attractor on a quite large interval of parameter $C$. Fig. 17a depicting the phase portrait of attractor and the stable one-dimensional manifold $W^s(P^4)$ shows that $W^u(P^4)$ and $W^s(P^4)$ intersect transversally on this interval. By Smale and Shilnikov [18, 19], the emergence of such an intersection implies a countable many saddle-focus orbits with the two-dimensional unstable manifold inside the attractor.

The onset of the 4-component Shilnikov attractor is not the final step in the framework of the development of hyperchaotic dynamics along the pathway $CD$. With a further decrease in $C$ (at $C = C_{4cr} \approx -1.789$), this attractor collides into the one-component attractor, see Fig. 16c. As in the case described in Sec. 4, this collision happens due to the boundary crisis, see the explosive growth of the attractor size in the bifurcation tree presented in Fig. 16h and the jump of distance between the attractor and $s_1$-component of $S^4$ in Fig. 16i (green-colored graph). However, in this case, the period-4 saddle orbit $S^4$ undergoes before the crisis of attractor a complete cascade of period-doubling bifurcations (on the curves $pd^4$, $pd^8$, . . . in Fig. 15a), see a part of the continuation tree for the $s_1$-component of $S^4$ in Fig. 16h (red-colored graph). As a result, a non-attractive hyperbolic
Figure 16: Graphs illustrating the onset of hyperchaotic attractors along the pathway CD: \((B = 0.5, A = 0.5)\). (a)–(d) phase portraits of attractors: (a) \(C = -1.745\) – four-component torus-chaos attractor; (b) \(C = -1.78\) – four-component hyperchaotic Shilnikov attractor containing the saddle-focus period-4 orbit \(P^4\) (LE: \(\lambda_1 = 0.036, \lambda_2 = 0.0187,\) and \(\lambda_3 = -0.748\)); (c) \(C = -1.79\) – four-component attractor transforms to the one-component attractor (LE: \(\lambda_1 = 0.072, \lambda_2 = 0.0315,\) and \(\lambda_3 = -0.797\)); (d) \(C = -1.83\) – hyperchaotic Shilnikov attractor containing the saddle-focus fixed point \(O_+\) (LE: \(\lambda_1 = 0.131, \lambda_2 = 0.0346,\) and \(\lambda_3 = -0.858\)). (e) the graph of distance between the attractor and the \(p_1\)-component of \(P^4\). (f) the graphs of distance between the attractor and \(O_+\) (in blue color), and the attractor and the \(s_1\)-component of \(S^4\) (in green color). (g) the graph of LE on parameter \(C\). (h) bifurcation trees depicting dependency of \(x\)-coordinate for each four iteration of the map on parameter \(C\) (in black color) and continuation of \(s_1\) (in red color).

The one-component hyperchaotic attractor presented in Fig. 16 contains both periodic saddle orbits \(P^4\) and \(S^4\) which have the two-dimensional unstable manifold but does not contain the saddle-focus fixed point \(O_+\). With a further decrease in \(C\), the distance

set with the two-dimensional unstable manifolds appears on the base of this orbit. At \(C = C_{4cr}\), the 4-component attractor touches the stable manifolds of this set and, at \(C < C_{4cr}\), merges with it into the one-component attractor. Note that this collision is also clearly visible in the graph of Lyapunov exponents presented in Fig. 16, where one can observe the jump of the second Lyapunov exponent (the so-called jump of hyperchaoticity [54]) at \(C = C_{4cr}\) associated with this collision.
Figure 17: Phase portraits of (a) one component of the hyperchaotic 4-component Shilnikov attractor presented in Fig. 16c, and (b) the hyperchaotic Shilnikov attractor containing \( O_+ \) presented in Fig. 16d. Green-colored curve in both panels is that branch of the stable invariant manifold \( W^s \) which forms the homoclinic intersection with the unstable manifold \( W^u \) forming an attractor.

between the attractor and this point decreases and finally, at \( C = C_{ab} \approx -1.820 \), it vanishes, see blue-colored graph in Fig. 16c. As a result, the hyperchaotic Shilnikov attractor containing the point \( O_+ \) appears, see Fig. 16d. It contains infinitely many saddle-focus, as well as saddle periodic orbits with the two-dimensional unstable manifolds. Figure 17b shows the transversal homoclinic structure for this attractor, additionally confirming the inclusion of \( O_+ \) to it.

Further decreasing \( C \) leads to the destruction of the homoclinic Shilnikov attractor which happens via a boundary crisis: the unstable manifold \( W^u(O_+) \) forming the attractor intersects with the stable two-dimensional manifold \( W^s(O_-) \) bounding its absorbing domain.

We would like to note that a similar transition to hyperchaos is observed also along many other pathways from the stable fixed point \( O_+ \) to hyperchaotic attractors. Passing through other Arnold tongues, corresponding e.g. to a \( p/q \)-resonance, one can observe a transition from a stable period-\( q \) orbit to a Shilnikov attractor containing this orbit which becomes a saddle-focus of (1,2)-type after the corresponding Neimark-Sacker bifurcation. Then, the \( q \)-component Shilnikov attractor collides into a one-component attractor which, with a further decrease in \( C \), can absorb saddle-focus fixed point \( O_+ \) before the crisis.

However, it is not the case for transition near the strong 1:3 resonance. We do not observe a stable period-3 orbit inside the corresponding tongue despite the possible existence (e.g. for \( B = 0.7 \)) of hyperchaotic attractors below it, see Fig. 16c. The enlarged fragment of the corresponding Lyapunov diagram is presented in Figure 18a. Let us briefly describe main stages of the development of hyperchaotic attractors along a vertical pathway passing through \( A = -0.44 \). The stable invariant curve \( L \) deforms near the 1:3 resonance, see Fig. 18b. Then, a higher periodic resonance occurs inside the corresponding Arnold tongue, see Fig. 18c. With a further decrease in \( C \), a multi-component invariant curve, appears from this periodic orbit under the supercritical Neimark-Sacker bifurcation. Soon, the multi-round curve breaks down giving torus-chaos attractor, see
Fig. 18d. Finally, this strange attractor becomes hyperchaotic, see Fig. 18e.

![Figure 18: (a) The fragment of Lyapunov diagram near the 1:3 resonance, $B = 0.7$. (b)-(e) phase portraits of attractors along a pathway with fixed $A = -0.44$: (b) $C = -0.36$, the stable invariant curve $L$ deforms near the 1:3; (c) higher periodic resonance appears inside the corresponding small Arnold tongue; (d) $C = -0.41$, torus-chaos attractor (LE: $\lambda_1 = 0.017$, $\lambda_2 = -0.011$, and $\lambda_3 = -0.363$) (e) $C = -0.44$, hyperchaotic attractor (LE: $\lambda_1 = 0.015$, $\lambda_2 = 0.004$, and $\lambda_3 = -0.376$).]

6 Various types of discrete Shilnikov attractors

In Sections 4 and 5 we studied the development of homoclinic attractors in map (6) for both cases of small and not very small values of the Jacobian $B$. In both cases the evolution of attractors is terminated via the boundary crisis of a corresponding homoclinic attractor. This attractor contains the fixed point $O_+$, which, after the supercritical Neimark-Sacker bifurcation, becomes saddle-focus of type (1,2). Correspondingly, before the crisis we observed a hyperchaotic discrete Shilnikov attractor.

However, not in all cases the appearance of discrete Shilnikov attractors implies hyperchaos. It can lead to chaos with only one positive Lyapunov exponent. Such phenomenon was observed, for example, in nonholonomic models of Celtic stone [44] and Chaplygin top [43], in models of identical globally coupled oscillators [45] and in other systems. In this section we show that the same phenomenon can be observed in the map under consideration, i.e., depending on values of the Jacobian, discrete Shilnikov attractors in the three-dimensional Mirá map can have either two or only one positive Lyapunov exponent. Also we give an explanation of this phenomenon.

Figure 19 shows a pair of $(A, C)$-parameter diagrams for map (6) with $B = 0.5$. A panel (a) is the enlarged fragment of the right-bottom part of the Lyapunov diagram presented in Fig. 10c, and a panel (b) is the corresponding part of a distance diagram depicting the distance between an attractor and the fixed point $O_+$. For the Lyapunov diagram we use the same color coding as in Sec. 3 complementing it with gray color.
corresponding to chaotic attractors with close to zero second Lyapunov exponent ($|\lambda_2| < 0.003$). Black color in the distance diagram corresponds to a small distance between the attractor and $O_+$ (the distance less than 0.001), see the legend to the right of the panel (b). The top-right region in Figs. 19 corresponds to the stability region of the point $O_+$, homoclinic attractors containing the saddle-focus fixed point $O_+$ of type (1,2) appear in the bottom-center region, colored in black in the panel (b). As it is clearly seen in the Lyapunov diagram from the panel (a), it can be either hyperchaotic or not.

Let us further fix $A = 1.33$ and study the onset of strange attractors along the pathway EF: $(A = 1.33, B = 0.5)$. The results of corresponding bifurcation analysis are shown in Figure 20. At the beginning, the stable fixed point $O_+$ is a unique attractor of the map. Then, at the curve $NS$, this point undergoes the supercritical Neimark-Sacker bifurcation after which $O_+$ becomes saddle-focus of type (1,2) and the stable invariant curve $L$ appears in its neighborhood, see Fig. 20a. Then, this curve undergoes two period-doubling bifurcations. After the first period-doubling this curve becomes saddle and a stable doubled (2-round) invariant curve $L^2$ appears in its neighborhood. In its turn, the curve $L^2$ also undergoes a period-doubling bifurcation, it becomes saddle and a 4-round stable invariant curves appears, see Fig. 20b. With further decrease in $C$, the 4-round invariant curve breaks down, and, as a result, a torus-chaos attractor is born, see Fig. 20c.

The graph of distance between the attractor and the fixed point $O_+$, presented in Fig. 20f, shows that the attractor starts to contain $O_+$ at $C \approx -1.882$. Figure 21a showing the transverse homoclinic structure for $O_+$ additionally confirms the inclusion of $O_+$ to the attractor. As it can be seen from the graph of Lyapunov exponents (Fig. 20f), the attractor becomes hyperchaotic on the interval $C \in (-1.89, -1.88)$. However, since
the second Lyapunov exponent is slightly positive \(0 < \lambda_2 < 0.006\), hyperchaos here is very “weak”.

Moreover, the second Lyapunov exponent \(\lambda_2\) can vanish for some open large regions of parameters adjacent to regions with homoclinic attractors. To see it, just superimpose the black-colored region in the bottom-center of Fig. 20b with the corresponding part of Lyapunov diagram presented in the panel (a).

In the framework of the described scenario, the saddle invariant curves \(L\), as well as the 2-round saddle curve \(L^2\), are included to the attractor. In nonresonant case, an invariant curve has one positive, one negative, and one zero Lyapunov exponents. The inclusion of such curve e.g. \(L\) to an attractor implies the appearance of transverse homoclinic intersection between the stable and unstable manifolds of this curve and, by Shilnikov theorem [55, 56], the existence of normally hyperbolic set [57] in the neighborhood of such homoclinic intersections. In particular, this set contains countably many invariant curves of the same nature as \(L\). If the most of this curves are nonresonant, than the central subspace in the decomposition of this set is neutral, which means that there are no expansions and no contractions in this subspace.

We assume that the vanishing of the second Lyapunov exponent on the observable chaotic attractors is explained by the inclusion of such sets to these attractors. In the case
under consideration, these sets make a greater contribution to dynamics in comparison with other hyperbolic subsets.

Finally, in order to support this assumption, we consider one more case of the creation of Shilnikov attractor for $B = 0.7$, see the corresponding Lyapunov diagram in Fig. 10d and its enlarged fragment near the right-bottom corner in Fig. 22a. The corresponding distance diagram is presented in Fig. 22b. From these figures, one can see that there are two regions where discrete Shilnikov attractors appear. However, none of these regions admit hyperchaotic attractors, see the panel (a). Moreover, hyperchaotic attractors are not observable in this case at all.

Let us consider a pathway GH: $(B = 0.7, A = 1.55)$ and study bifurcations leading to the birth of homoclinic attractors in the left-bottom part of the diagrams presented in Fig. 22. The results of corresponding one-parameter bifurcation analysis are shown in Fig. 23. In this case, the scenario of the appearance of discrete Shilnikov attractors is almost the same as it was in the previous case, cf. Figs. 23a–23f with Figs. 20a–20f. The main difference between these two cases is that the stable invariant curve $L$ undergoes here a quite long sequence of period-doubling bifurcations before the destruction of the resulting multiround stable invariant curve.

Fig. 23g shows the corresponding bifurcation tree which we compute using the following scheme. We construct a Poincaré-like maps on the plane $(x, z)$ introducing a cross-section-like box $|y| < 0.001$ in the phase space of the map. If a point of the attractor falls into this box we take its $x$-coordinate and plot on the graph $x(C)$. For each value of the parameter $C$ we plot 3000 points. The enlarged fragment of this bifurcation tree is shown in Fig. 23h. It is important to note that the observed bifurcation tree looks like the well-known Feigenbaum tree accompanying the creation of Hénon-like
Figure 22: \((A, C)\)-parameter diagrams for map (6) at \(B = 0.7\). (a) Diagram of Lyapunov exponents, the color scheme is the same as in Fig. 19b. (b) Diagram of the Euclidean distance between an attractor and the fixed point \(O_+\); in black colored regions this distance is less than 0.001. Top-right region corresponds to the stability region of \(O_+\); homoclinic attractor containing the saddle-focus point \(O_+\) of type \((1,2)\) appears in the bottom regions, colored in black in panel (b). All chaotic attractors observed here have one positive and one close to zero Lyapunov exponents.

... attractors. As it is known, such attractors often appear after a cascade of period-doubling bifurcations \([25, 26, 27]\) followed by a cascade of heteroclinic band-fusion bifurcations \([28]\) which result in the absorption of periodic saddle orbits emerging after the corresponding period-doubling bifurcations. By the same manner, a sequence of period-doubling bifurcations with a stable invariant curves leads to the appearance of multicomponent chaotic attractor (on the corresponding two-dimensional Poincaré-like map) which, then, transforms to the one-component attractor via a sequence of heteroclinic bifurcations resulting in the absorption of saddle invariant curves emerging after the corresponding period-doubling bifurcations. The resulting one-component attractor were called in \([59, 41, 42]\) as quasiperiodic Hénon-like attractor. It has one positive and one close to zero Lyapunov exponents.

Finally, the chaotic attractor absorbs the saddle-focus fixed point \(O_+\) of type \((1,2)\), and a discrete Shilnikov attractor appears, see Fig. 23d. The corresponding homoclinic structure for \(O_+\) is shown in Fig. 21b. Note that this attractor contains both infinitely many saddle invariant curves, as well as saddle-focus periodic orbits of type \((1,2)\). However, the contribution of saddle invariant curves to chaotic dynamics in this case, is much greater than the contribution of saddle periodic orbits, and, thus, the attractor has one positive and one very close to zero Lyapunov exponents.

In future papers, we will show that such long sequences of period-doubling bifurcations with stable invariant curves, as well as the appearance of chaotic attractors with close zero Lyapunov exponents in map (6) and in other three-dimensional maps (including Poincaré maps for four-dimensional flows), can be explained by the “flow nature” of the map, when orbits of a chaotic attractor of this three-dimensional map can be well approximated by a time-shift map of trajectories of some three-dimensional flow.
Figure 23: The evolution of attractors along the pathway \( \mathcal{GH} : (B = 0.7, A = 1.55) \). (a)–(d) phase portraits of attractors: (a) \( C = -1.7 \) – the stable invariant curve \( L \); (b) \( C = -1.9 \) – doubled invariant curve; (c) \( C = -1.96 \) – torus-chaos attractor (LE: \( \lambda_1 = 0.0348, \lambda_2 = 0.0001 \), and \( \lambda_3 = -0.392 \)); (d) \( C = -2.03 \) – discrete Shilnikov attractor (LE: \( \lambda_1 = 0.107, \lambda_2 \approx 0 \), and \( \lambda_3 = -0.463 \)). (e) The graph of LE \( \lambda_1, \lambda_2 \), and \( \lambda_3 \) on parameter \( C \). (f) The graph of Euclidean distance between the attractor and the fixed point \( O_+ \). (g) Bifurcation tree \( \chi(C) \) computing with help of a Poincaré-like map on the plane \((x, z)\) (using the cross-section-like box \( |y| < 0.001 \)) and (f) its enlarged fragment.

In particular, we will show that map (6) near discrete Shilnikov attractors with close to zero Lyapunov exponent, can be good approximated by a flow normal form of the bifurcation of triple degenerated fixed point with multipliers \((1, 1, 1)\). As is known, cascades of period-doubling bifurcations with the subsequent appearance of Shilnikov spiral attractors is a typical scenario of the appearance of chaotic attractors in three-dimensional
flows, see e.g. \[60, 61, 62, 63, 64\].

7 Discussion

We have proposed bifurcation scenarios leading from a stable fixed point to a hyperchaotic attractor in a one-parameter families of three-dimensional maps and applied them for studying hyperchaotic attractors in the homoclinic three-dimensional Mirá map (1). These scenarios consist of two parts. The first, phenomenological, part describes a few main bifurcations resulting to the appearance of a homoclinic attractor containing saddle fixed point of type (1,2). The second, empirical, part describes accompanying bifurcations after which most of periodic orbits inside an attractor gets two-dimensional unstable manifolds.

We have shown that for map (1) phenomenological part is the extension of the well-known Shilnikov scenario \[30\] to the case of three-dimensional maps \[11, 13, 14\]. In the framework of this scenario, a stable fixed point \(O_+\) undergoes the supercritical Neimark-Sacker and a stable invariant curve \(L\) appears in its neighborhood. Then, this curve breaks down giving a torus-chaos attractor with the isolated saddle-focus point \(O_+\) of type (1,2). The final part of the scenario is the inclusion (absorption) of this point by an attractor. The resulting discrete Shilnikov attractor contains this point.

The empirical part of the scenario describes mechanisms of the destruction of the curve \(L\). Usually, before the destruction, this curve becomes resonant. We have shown that depending on values of parameters the corresponding resonant orbits can give rise to hyperchaotic periodic orbits of type (1,2) in two different ways:

(i) the stable resonant orbit undergoes a cascade of period-doubling bifurcations. (This cascade can be interrupted by the supercritical Neimark-Sacker bifurcation. In this case, see (ii)). In their turn, the resulting periodic saddle orbits of type (2,1), as well as the resonant saddle orbits of type (2,1) undergo a cascade of period-doubling bifurcations transforming the corresponding periodic orbits to saddles of type (1,2).

(ii) the stable resonant orbit undergoes the supercritical Neimark-Sacker bifurcation transforming this orbit to a saddle-focus of type (1,2) and a stable multicomponent invariant curve appears. Then, this invariant curve (as the curve \(L\)) becomes resonant, (i) or (ii) is implemented and so on. The saddle resonant orbit, as well as in the first way, undergoes a cascade of period-doubling bifurcations.

In both cases, an attractor absorbs the sets of periodic saddle orbits of type (1,2) and becomes hyperchaotic.

In the paper, we have also proposed a new phenomenological scenario leading to the creation of the so-called hyperchaotic Hénon-like attractor containing a saddle fixed point with a pair of negative unstable multipliers, However, we have not observed its implementation in map (1).

Finally, we would like to note one important property of such maps. One of its fixed point is always in the origin and its eigenvalues depend only on the parameter \(A, B,\) and \(C\) \[14\]. In Figure 24 we show an extended bifurcation diagram for this fixed point (when \(B = 0.1\)) above the Lyapunov diagram. Such extended diagram was called a saddle chart in \[14\].

In this diagram one can see four regions with possible hyperchaotic homoclinic attractors: the region I – with a discrete Shilnikov attractor, the region II – with a hyperchaotic
Hénon-like attractor, the region III – with a hyperchaotic attractor containing a fixed point with a pair of positive unstable multipliers and the region IV – with a hyperchaotic attractor containing a fixed point with a pair of real unstable multipliers with different signs. Varying nonlinear terms in the map we can expect new types of hyperchaotic homoclinic attractors. This problem looks very promising, especially for the region IV, inside which, as well as inside the region I, by Gonchenko and Li theorem [8], the nonwandering set of the map is a Smale horseshoe of type (1,2). Thus, one can expect the existence of hyperchaotic attractors in IV for quite general families of three-dimensional Míra-like maps.

Figure 24: Saddle chart [14] superimposed with the Lyapunov diagram for map (6), \( B = 0.1 \). In regions \( SH_{1,2}(1,2) \) the nonwandering set of the map is nontrivial hyperbolic of type (1,2). Hyperchaotic homoclinic attractors of different types are possible in region I–IV.

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