Universally complete spaces of continuous functions

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Abstract. We characterise Tychonoff spaces $X$ so that $C(X)$ is universally $\sigma$-complete and universally complete, respectively.

Mathematics Subject Classification (2010). Primary 46E05; Secondary 46A40, 54G10.

Keywords. Vector lattices, continuous functions, P-spaces.

1. Introduction

Recently, Mozo Carollo [2] showed, in the context of point-free topology, that the vector lattice $C(X)$ of continuous, real valued functions on a Tychonoff (completely regular $T_1$) space $X$ is universally complete if and only if $X$ is an extremally disconnected P-space. This paper aims to make this result and its proof accessible to those members of the positivity community who, like the author, are less familiar with point-free topology. In so doing, and based on results due to Fremlin [7] and Veksler and Ge˘ıler [15], we obtain a refinement of Mozo Carollo’s result. In particular, we characterise those Tychonoff spaces $X$ for which $C(X)$ is laterally $\sigma$-complete. We also include some remarks on $\sigma$-order continuous duals of spaces $C(X)$ which are universally $\sigma$-complete.

The paper is organised as follows. In Section 2 we introduce definitions and notation used throughout the paper, and recall some results from the literature. Section 3 contains the main results of the paper, namely, characterisations of those Tychonoff spaces $X$ for which $C(X)$ is universally complete and universally $\sigma$-complete, respectively.

2. Preliminaries

Throughout this paper $X$ denotes a Tychonoff space; that is, a completely regular $T_1$ space. $C(X)$ stands for the lattice of all real-valued and continuous functions on $X$. For $u \in C(X)$, $Z(u)$ denotes the zero set of $u$; that is, $Z(u) = u^{-1}[\{0\}]$. The co-zero set of $u$ is $Z^c(u) = X \setminus Z(u)$. The collection
of zero sets in $X$ is denoted $\mathbf{Z}(X)$, while $\mathbf{Z}^c(X)$ consists of all co-zero sets in $X$. For $x \in X$ the collection of open neighbourhoods of $x$ is denoted $\mathcal{N}_x$, and $\mathcal{N}_x^*$ denotes the set of clopen neighbourhoods of $x$. A zero-neighbourhood of $x \in X$ is a set $V \in \mathbf{Z}(X)$ so that $x$ belongs to the interior of $V$. The collection of all zero-neighbourhoods of $x \in X$ is denoted $\mathcal{N}_x^\ast$, and $\mathcal{N}_x^c = \mathcal{N}_x \cap \mathbf{Z}^c(X)$. Observe that $\mathbf{Z}^c(X)$ is a basis for the topology of $X$. Hence for each $x \in X$ and every $V \in \mathcal{N}_x$ there exists $U \in \mathcal{N}_x^c$ so that $U \subseteq V$. Furthermore, for every $V \in \mathcal{N}_x$ there exists $W \in \mathcal{N}_x^\ast$ so that $W \subseteq V$. The standard reference for all of this is \cite{9}.

We write $1$ for the function which is constant one on $X$. More generally, for $A \subseteq X$, the indicator function of $A$ is $1_A$. The constant zero function is $0$.

We recall, see for instance \cite{9}, that $X$ is

(i) \textit{basically disconnected} if the closure of every co-zero set is open;

(ii) \textit{extremally disconnected} if the closure of every open set is open.

Every extremally disconnected space is basically disconnected, but not conversely \cite{9} Problem 4N]. Since $\mathbf{Z}^c(X)$ is a basis for the topology on $X$, every basically disconnected space is zero-dimensional\footnote{The term zero-dimensional should be understood in terms of small inductive dimension \cite[Definition 1.1.1 & Proposition 1.2.1]{6}, as opposed to the Lebesgue covering dimension used in \cite{9}.}; that is, it has a basis consisting of clopen sets. The converse is false. For instance, $\mathbb{Q}$ is zero-dimensional, the set of all open intervals with irrational endpoints forming a basis of clopen sets, but not basically disconnected, since $(0, 1)$ is a co-zero set whose closure is not open.

Each of the properties (i) and (ii) of $X$ corresponds to order-theoretic properties of $C(X)$, see for instance \cite[Theorems 43.2, 43.3, 43.8 & 43.11]{13}. In particular, $X$ is

(i*) basically disconnected if and only if $C(X)$ is Dedekind $\sigma$-complete, if and only if $C(X)$ has the principle projection property;

(ii*) extremally disconnected if and only if $C(X)$ is Dedekind complete, if and only if $C(X)$ has the projection property.

$X$ is a P-space \cite{5} if the intersection of countably many open sets in $X$ is open. Equivalently, $X$ is a P-space if $Z(u)$ is open (hence clopen) for every $u \in C(X)$. Clearly, every discrete space is a P-space, but the converse is false, see \cite{9} Problem 4N]. In fact, there exists a P-space without any isolated points \cite{9} Problem 13P]. Evidently, every P-space is basically disconnected (in particular, every $Z \in \mathbf{Z}(X)$ is open), but not conversely, see \cite{9} Problem 4M].

The following basic lemma may well be known, but we have not found it in the literature. We include the simple proof for the sake of completeness.

\textbf{Lemma 2.1.} Let $X$ be zero-dimensional. Then the following statements are equivalent.

(i) $X$ is a P-space.
(ii) The intersection of countably many clopen sets is clopen.
(iii) The union of countably many clopen sets is clopen.

**Proof.** By definition, (i) implies (ii) and (iii), and, (ii) and (iii) are equivalent. It therefore suffices to show that (ii) implies (i).

Assume that (ii) is true. For each \( n \in \mathbb{N} \) let \( U_n \) be an open subset of \( X \). Let \( U = \bigcap \{ U_n : n \in \mathbb{N} \} \). If \( U = \emptyset \) we are done, so assume that \( U \neq \emptyset \). Fix any \( x \in U \). Since \( X \) is zero-dimensional, there exists for each \( n \in \mathbb{N} \) a set \( V_n \in \mathcal{N}_x^* \) so that \( V_n \subseteq U_n \). Let \( V = \bigcap_{n \in \mathbb{N}} V_n \). Then \( x \in V \subseteq U \) and, by assumption, \( V \) is clopen, hence open. Therefore \( U \) is open so that \( X \) is a P-space. \( \square \)

We recall, for later use, the following results of Fremlin \[7] and Veksler and Geiler \[15], respectively; see also \[1\].

**Theorem 2.2.** Let \( L \) be a Dedekind complete vector lattice. Then the following statements are equivalent.

(i) \( L \) is universally complete.
(ii) \( L \) is universally \( \sigma \)-complete and has a weak order unit.

**Theorem 2.3.** Let \( L \) be an Archimedean vector lattice. The following statements are true.

(i) If \( L \) is laterally complete then \( L \) has the projection property.
(ii) If \( L \) is laterally \( \sigma \)-complete then \( L \) has the principle projection property.

3. Universally complete \( C(X) \)

We begin this section with a characterisation of those \( X \) for which \( C(X) \) is universally \( \sigma \)-complete.

**Theorem 3.1.** The following statements are equivalent.

(i) \( C(X) \) is laterally \( \sigma \)-complete.
(ii) \( C(X) \) is universally \( \sigma \)-complete.
(iii) \( X \) is a P-space.

**Proof.** Assume that \( C(X) \) is laterally \( \sigma \)-complete. It follows from Theorem 2.3(ii) that \( C(X) \) has the principle projection property. Therefore \( C(X) \) is Dedekind \( \sigma \)-complete, hence universally \( \sigma \)-complete. Conversely, if \( C(X) \) is universally \( \sigma \)-complete then, by definition, it is laterally \( \sigma \)-complete. Hence (i) and (ii) are equivalent.

Assume that \( C(X) \) is laterally \( \sigma \)-complete. Then \( C(X) \) has the principle projection property so that \( X \) is basically disconnected, hence zero-dimensional. We show that \( X \) is a P-space.

By Lemma 2.1 it suffices to show that the intersection of countably many clopen subsets of \( X \) is clopen. Assume that \( U_k \subseteq X \) is clopen for each \( k \in \mathbb{N} \), and let \( U = \bigcap \{ U_k : k \in \mathbb{N} \} \). We claim that \( U \) is clopen.

Let \( V_0 = X \), \( V_1 = U_1 \) and, for each natural number \( n > 1 \), let \( V_n = U_1 \cap \ldots \cap U_n \). Then \( V_n \) is clopen for each \( n \in \mathbb{N} \), \( U = \bigcap \{ V_n : n \in \mathbb{N} \} \)
and $V_{n+1} \subseteq V_n$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $W_n = V_{n-1} \setminus V_n$. Then each $W_n$ is clopen and $W_n \cap W_m = \emptyset$ whenever $n \neq m$. Moreover, $\bigcup\{W_n : n \in \mathbb{N}\} = X \setminus U$. Indeed, the inclusion $\bigcup\{W_n : n \in \mathbb{N}\} \subseteq X \setminus U$ is immediate. For the reverse inclusion, consider some $x \in X \setminus U$. There exists $n \in \mathbb{N}$ so that $x \in X \setminus V_n$. Let $n_0 = \min\{n \in \mathbb{N} : x \in X \setminus V_n\}$. Then, since $V_0 = X$, $x \in V_{n-1} \setminus V_n = W_n$. Hence $x \in \bigcup\{W_n : n \in \mathbb{N}\}$.

Let $w_n = n1_{W_n}$, $n \in \mathbb{N}$, and $F = \{w_n : n \in \mathbb{N}\}$. Then $F \subseteq C(X)^+$ and the $w_n$ are mutually disjoint. Therefore, since $C(X)$ is universally $\sigma$-complete, $w = \sup F$ exists in $C(X)$.

Fix $x \in U$. There exists $V \in \mathcal{N}_x$ so that $w(y) < w(x) + 1$ for all $y \in V$. Fix a natural number $N_0 \geq w(x) + 1$. Then, for all $n \geq N_0$ and $y \in W_n$, $w(y) \geq w_n(y) = n \geq N_0 \geq w(x) + 1$ so that $y \notin V$. Therefore $V \cap W_n = \emptyset$ for all $n \geq N_0$. Let $W = V_{N_0} \cap V$. Then $W \in \mathcal{N}_x$ and, since $W_n \cap V_{N_0} = \emptyset$ for all $n < N_0$, $W \cap W_n = \emptyset$ for all $n \in \mathbb{N}$. Therefore $W \subseteq X \setminus \bigcup\{W_n : n \in \mathbb{N}\} = U$.

This shows that $U$ is open, and, since each $U_k$ is closed, $U$ is also closed, hence clopen. By Lemma 2.3, $X$ is a P-space. Hence (i) implies (iii).

Assume that $X$ is a P-space. Consider a countable set $F$ of mutually disjoint elements of $C(X)^+$. We observe that for each $x \in X$ there is at most one $u \in F$ so that $u(x) > 0$. Hence the function

$$w : X \ni x \mapsto \sup\{u(x) : u \in F\} \in \mathbb{R}^+$$

is well defined. We claim that $w \in C(X)$ so that $w = \sup F$ in $C(X)$.

Fix $x \in X$. Assume that $w(x) > 0$. Then there exists $u \in F$ and $V \in \mathcal{N}_x$ so that $u(y) = w(y) > 0$ for all $y \in V$. Hence $w$ is continuous at $x$. Suppose $w(x) = 0$. Then $u(x) = 0$ for all $u \in F$. Since $X$ is a P-space there exists for each $u \in F$ a $V_u \in \mathcal{N}_x$ so that $u(y) = 0$ for every $y \in V_u$. The set $V = \cap\{V_u : u \in F\}$ is an open neighbourhood of $x$, and $w(y) = 0$ for all $y \in V$. Hence $w$ is continuous at $x$. Thus $w$ is continuous at every $x \in X$, hence on $X$. Therefore $C(X)$ is laterally $\sigma$-complete. Hence (iii) implies (i). \qed

Mozo Carollo’s characterisation of those $X$ for which $C(X)$ is universally complete now follows easily.

**Corollary 3.2.** The following statements are equivalent.

(i) $C(X)$ is laterally complete.

(ii) $C(X)$ is universally complete.

(iii) $X$ is an extremally disconnected P-space.

**Proof.** Assume that $C(X)$ is laterally complete. By Theorem 2.3 (i), $C(X)$ has the projection property and is therefore Dedekind complete, hence universally complete. Conversely, if $C(X)$ is universally complete, then it is laterally complete. Therefore (i) and (ii) are equivalent.

Assume that $C(X)$ is universally complete. Then, since $C(X)$ is Dedekind complete, $X$ is extremally disconnected, and by Theorem 3.1 $X$ is a P-space.
Suppose that $C(X)$ is an extremally disconnected P-space. Then $C(X)$ is Dedekind complete and, by Theorem 3.1, laterally $\sigma$-complete. Since $C(X)$ has a weak order unit, it is universally complete by Theorem 2.2. □

Remark 3.3. Isbell [11] showed that if $X$ is an extremally disconnected P-space, and $X$ has non-measurable cardinal, then $X$ is discrete. It is consistent with ZFC that every cardinal is non-measurable.

Corollary 3.4. The following statements are equivalent.

(i) $C(X)$ is laterally $\sigma$-complete.

(ii) $C(X)$ is universally $\sigma$-complete.

(iii) $X$ is a P-space.

(iv) $C(X)$ is a von Neumann regular ring.

(v) $C(X)$ is $z$-regular.

Remark 3.5. Recall that a space $X$ is called realcompact if for every Tychonoff space $Y$ containing $X$ as a proper dense subspace, the map $C(Y) \ni f \mapsto f|_X \in C(X)$ is not onto; that is, $X$ is not C-embedded in $Y$, see [5, page 214]

If $X$ is a realcompact P-space, then $C(X)^\sim$ has a peculiar structure. Indeed, due to a result of Fremlin [7, Proposition 1.15], every $\varphi \in C(X)^\sim$ is a finite linear combination of linear lattice homomorphisms form $C(X)$ into $\mathbb{R}$. Xiong [16] showed that every such homomorphism is a positive scalar multiple of a point evaluation. Hence

$$C(X)^\sim = \text{span}\{\delta_x : x \in X\} = c_{00}(X).$$

However, each $\delta_x$ is $\sigma$-order continuous. Indeed, consider a decreasing sequence $(u_n)$ in $C(X)^+$ so that $\inf\{u_n(x) : n \in \mathbb{N}\} > 0$ for some $x \in X$. Then there exists a real number $\epsilon > 0$ so that for every $n \in \mathbb{N}$ there exists $V_n \in \mathcal{N}_x$ such that $u_n(y) > \epsilon$ for every $y \in V_n$. Since $X$ is a P-space, $V = \bigcap\{V_n : n \in \mathbb{N}\}$ is open. Therefore there exists $v \in C(X)$ so that $0 < v \leq \epsilon 1$ and $v(y) = 0$ for $y \in X \setminus V$. Since $u_n(y) > \epsilon$ for all $y \in V$ and $n \in \mathbb{N}$ it follows that $0 \leq v \leq u_n$ for all $n \in \mathbb{N}$; hence $u_n$ does not decrease to $0$ in $C(X)$. This shows that $\delta_x \in C(X)^\sim$.

Combining all of the above, we see that

$$C(X)^\sim = \text{span}\{\delta_x : x \in X\} = c_{00}(X) = C(X)^\sim.$$

Remark 3.6. The condition that $\delta_x \in C(X)^\sim$ for all $x \in X$ does not imply that $X$ is a P-space. In fact, this property characterises the so called almost-P-spaces introduced by Veksler [14], see also [12]. A space $X$ is an almost-P-space if the nonempty intersection of countably many open sets has nonempty

\[\text{span}\{\delta_x : x \in X\} = c_{00}(X) = C(X)^\sim.\]

\footnote{For every $u \in C(X)$ there exists $v \in C(X)$ so that $u = vu^2$.}

\footnote{Every proper prime $z$-ideal in $C(X)$ is a minimal prime $z$-ideal, see [3, 4] for details.}

\footnote{Realcompact spaces were introduced by Hewitt [10] under the name “Q-spaces”, and defined as follows: $X$ is a Q-space if every free maximal ring ideal in $C(X)$ is hyper-real. See for instance [12 Problem 8A no. 1] for the equivalence of our definition and Hewitt’s.
interior; equivalently, every \( Z \in \mathcal{Z}(X) \) has nonempty interior. Thus every P-space is an almost-P-space, but not conversely, see [12].

De Pagter and Huijsmans [4] showed that \( C(X) \) has the \( \sigma \)-order continuity property if and only if \( X \) is an almost-P-space. Hence, if \( X \) is an almost-P-space, then \( \delta_x \in C(X)_c \) for every \( x \in X \). For the converse, suppose that \( X \) is not an almost-P-space. Then there exists \( u \in C(X)^+ \) so that \( Z(u) \) has empty interior. For each \( n \in \mathbb{N} \), let \( u_n = 1 \wedge (nu) \). Then \( (u_n) \) is increasing and bounded above by 1. Let \( v \in C(X) \) be an upper bound for \( (u_n) \). If \( x \in X \setminus Z(u) \) then \( \sup\{u_n(x) : n \in \mathbb{N}\} = 1 \) so that \( v(x) \geq 1 \). Since \( Z(u) \) has empty interior and \( v \) is continuous, it follows that \( v(x) \geq 1 \) for all \( x \in X \). Therefore \( u_n \uparrow 1 \) in \( C(X) \). But if \( x \in Z(u) \), then \( \delta_x(u_n) = u_n(x) = 0 \) for every \( n \in \mathbb{N} \) so that \( \delta_x \notin C(X)_c \).

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