Is the Cosmological Constant of Topological Origin?

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The observed value of the cosmological constant poses large theoretical problems. We find that topology of the Universe provides a natural source for it. Restricting dynamically an Einstein-Cartan gravity to General Relativity in our observed Universe allows topological invariants to induce an effective cosmological constant from dynamical quintessence-like topological fields. Its evaluation through the boundary of black holes yields a range compatible with the observed value, with uncertainty of three orders of magnitude. In turns, it provides a measurement of the Universe’s isoperimetric constant.

I. INTRODUCTION

The cosmological constant has reappeared at the turn of the century in the toolbox of cosmology with the discovery of fainter than expected distant type Ia supernovae\textsuperscript{3,4} interpreted as cosmic expansion acceleration. This acceleration was confirmed by the combination of cosmic microwave background radiation, clusters and baryon acoustic oscillation observations\textsuperscript{4–7}.

Its simple interpretation as quantum vacuum energy clashes with one of the largest discrepancy of physics: its observed value from the previous cosmic geometry tools yields a value for $\Lambda$ that contradicts the simple evaluation that it should reach order of the Planck scale\textsuperscript{5}. This has been coined the fine tuning problem\textsuperscript{9,10}. Quantum vacuum is ruled by the Planck scale at early time which then should set its initial value, but since $\Lambda$ is a constant, this is the value expected nowadays, far from the hundred order of magnitude lower observed.

In addition, this non-varying value gives rise to the coincidence problem\textsuperscript{11–14} as it also yields a very recent epoch for its emergence as cosmic dominant component: its energy density, set at the initial universe conditions, is unnaturally close to that of present matter and therefore poses some questions\textsuperscript{13,16}. For a review of the questions posed by the cosmological constant, refer to\textsuperscript{5}.

The geometrical approach to gravity through curved spacetime by general relativity (GR) can be generalised to include the possibility of its torsion in Einstein-Cartan (EC) theories\textsuperscript{17,20}, where GR appears as the torsion-less limit, while the curvature-less limit yields the Teleparallel Equivalent to GR case\textsuperscript{21,22}. This work will focus on the GR limit of Einstein-Cartan theories, further restricted by adding the characteristic classes consistent with the topology of the space-time manifold $\mathcal{M}$\textsuperscript{22,23}, i.e. the Euler class $\epsilon(T\mathcal{M}) := C_E$, the Pontryagin class $p_1(T\mathcal{M}) := C_P$ and the Nieh-Yan class (of the Chern-type) $c_2(T\mathcal{M}) := C_N$\textsuperscript{24}. In order for these topological terms to be non trivial, we are lead to assume that the space-time manifold has a boundary $\partial \mathcal{M} \neq \emptyset$. We found it is composed by the hyper-surfaces that define the horizons of black holes in the Universe.

The action is then modified so as to include the appropriate boundary counter terms\textsuperscript{27,28} and such as the field equations are well-posed. The entire procedure will result in the appearance of a cosmological functional $\tilde{\Lambda}$. The dynamical restriction of such torsional theory in $\mathcal{M}$ to GR conditions, in a region $\mathcal{N} \subset \mathcal{M}$ that contains our observed Universe, is achieved by means of the Vielbein-Einstein-Palatini (VEP) formalism and by dynamical systems stability considerations. This allows for the collateral topological effective cosmological constant $\Lambda$.

In Sec. II, we present the inclusion of topological invariants in the Vielbein-Einstein-Palatini action and how their dynamics can recover a GR behaviour. The emergence of an effective topological cosmological constant is presented in Sec. III while its evaluation using black holes occupy Sec. IV before discussing our conclusions in Sec. V.

II. MODIFIED VEP RESTRICTED TO GR REGIONS

We focus on the family of actions defined over $\mathcal{M}$ having the following terms:

$$S := S_G \left[ e^c, \omega^a_b, \tilde{\lambda} \right] + S_T \left[ e^c, \omega^a_b, \varphi_j \right] + S_M \left[ \Psi \right],$$

(1)

where $S_G \left[ e^c, \omega^a_b, \tilde{\lambda} \right]$ is the pure gravitational VEP action\textsuperscript{29} plus boundary counter terms, with $e^c$ the vier-
bein frame, \(\omega^a_b\) the total connection and \(\lambda\) the cosmological functional; \(S_T[\psi^c, \omega^a_b, \varphi_j]\) is the topological action that includes each of the characteristic classes \(C_j\) coupled to a field \(\varphi_j\) (\(j = E, P, N\)) plus boundary counter terms; and \(S_M[\Psi]\) is taken as a Dirac action, where \(\Psi\) stands for a collection of mass-less spinor fields.

### A. Field equations

The variation of (1) gives the following field equations for \(\delta e^a, \delta \omega_{ab}, \delta \varphi_j, \delta \psi\), and \(\delta \psi\), respectively:

\[
\Xi_a = -\left[ R_{ab}^{(s)} - \frac{2\lambda}{3} \left| \psi \right|^2 \hat{\nabla}_a^{(s)} \right] \wedge e^b - id\varphi_N \wedge T_a , \tag{2}
\]

\[
\tau_{ab} = d\omega_a \left[ \hat{\nabla}_b^{(s)} + i 2 \left\{ \hat{\varphi}_P R_{ab} - \hat{\varphi}_E R_{ab}^{(s)} \right\} \right] - id\varphi_N \wedge \hat{\nabla}_b^{(s)} , \tag{3}
\]

\[
C_j = -i \frac{\delta \lambda}{\delta \varphi_j} \left| \psi \right|^2 d\mu , \quad j = \{E, P, N\} , \tag{4}
\]

\[
\gamma^a D_a \psi - \frac{\lambda}{4} \bar{\psi} \gamma^a \psi = 0 \quad , \quad D_a \bar{\psi} \gamma^a + \frac{\lambda}{4} \bar{\psi} \gamma^a \psi = 0 \quad , \tag{5}
\]

where \(\tau_{ab}\) is the torsion 2-form, \(T_a\) is the torsion 2-form, \(\Sigma_{ab} := \frac{1}{2} \epsilon^c_{ab} \wedge e^c\) is the Palatini 2-form, \(C_j\) are the characteristic classes, \(d\omega_a\) stands for the exterior covariant derivative, \(D_a\) stands for the gauge covariant derivative and we have defined the quantities:

\[
\hat{\lambda} := \bar{\psi} \lambda \left| \psi \right| \quad ; \quad \left| \psi \right|^2 := \bar{\psi} \psi \quad ; \quad \hat{\varphi}_j := \frac{-\varphi_j}{(4\pi)^2} \tag{6}
\]

\[
\frac{\delta \mathcal{L}_M}{\delta e^a} := \Xi_a = 2Re \left\{ \bar{\psi} \gamma^b D_b \psi \right\} \wedge \hat{\nabla}_a^{(s)} \quad ; \quad i := \sqrt{-1} \nonumber ;
\]

\[
\frac{\delta \mathcal{L}_M}{\delta \omega_{ab}} := \tau_{ab} = \frac{1}{4} \bar{\psi} \sigma_{abc} \psi \wedge e^c \quad ; \quad \sigma^{a_1 \ldots a_k} := \gamma^{a_1} \ldots \gamma^{a_k} \nonumber .
\]

For the interested reader, the technical details will be given in our next paper [33]. We recall that the total connection 1-form \(\omega^a_b\) satisfies the equations:

\[
\omega^a_b = \bar{\omega}^a_b + K^a_b \Rightarrow T^a = K^a_b \wedge e^b , \tag{7}
\]

where \(\bar{\omega}^a_b\) is the torsion-less Levi-Civita connection, while \(\bar{\omega}^a_b\) is the torsion related contortion 1-form.

By means of decomposition \(7\), the total curvature 2-form can then be written as:

\[
R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = \hat{R}^a_b + \Theta^a_b , \tag{8}
\]

where \(\hat{R}^a_b\) is the torsion-less part of the curvature, formally equivalent to that of a GR curvature 2-form, while the term \(\Theta^a_b\) concentrates all the contributions from the torsion related quantities.

### B. GR restriction

The form of the field equations (2 - 5) plus the decomposition of the curvature (3), suggests a contortion 1-form \(K^a_b\):

\[
K^a_b = \frac{i}{3} \epsilon^{abc} \mathcal{L}_c (\varphi_N) e^d , \tag{9}
\]

where \(\mathcal{L}_a(\cdot)\) is the Lie derivative along the vector \(e_a\). We are able to solve the field equations (2 - 5), inserting Eq. (3) back and assuming a functional behavior of the couplings \(\varphi_j\) of the form \(d\varphi_j := \varphi_N^* d\varphi_j\) for \(j = E, P\) where \(d\varphi_j\) proceeds from the field equations.

The dynamical system (2) and (3) drives \(\varphi_N\) to be a slowly varying function in a region \(\mathcal{N}\), rendering it torsion-less and thus GR-like.

Ensuring the coherence of the set of topological restrictions (1) and imposing dynamical stability we obtain a cosmological functional \(\lambda\) of the form:

\[
\lambda = \frac{4\Lambda}{\left| \psi \right|^2} u_{\varphi_N} \quad \text{where} \quad u_{\varphi_N} := \exp \left( -\frac{\sqrt{3}}{4} |\varphi_N| \right) \tag{10}
\]

where \(\Lambda\) is now a constant. Consequently, this implies that the kinetic term associated to the zero-form \(\varphi_N\) should behave as a Lyapunov function. Leading to the following expression:

\[
\left| \frac{d\varphi_N}{4} \right|^2 = \Lambda \left\{ 1 - u_{\varphi_N} \right\} ,
\]

which can be used to characterize the GR-like region \(\mathcal{N}\) of the space-time manifold \(\mathcal{M}\), via the condition \(d\varphi_N = 0\), as equivalent to \(\varphi_N = 0\), while \(\varphi_N\) satisfies a non trivial boundary value problem valid for the non-GR domains.

Here we want to contrast our philosophy with that of more traditional field theoretic approaches and put forward three remarks: the introduction of topology does modify the traditional set up of the simple quantum vacuum energy interpretation, our approach contrasts fundamentally with some studies of dark energy in non-trivial topologies [e.g. 31, 32], and with field theoretic emerging gravity paths. In the first case the containing spacetime considered does not present a boundary as in our model, but the order of magnitude of the changes are not expected to bridge the traditional vacuum energy to observation gap. In the second case, our topology is simply connected as indicated by the 0-puncture Betti number \(b_0 = 1\) (see Sec. [33]) and thus in agreement with current observational expectations on topology. Although

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1 Originally the field equations suppose the presence of a fermion potential \(V(\Psi)\). However, since this potential is not involved in the final calculation, we prefer to omit it out of clarity.

2 We are using the notation \(A^a_{bcd} := \frac{1}{2} \epsilon^{abcde} A_{bcd}^e\) for the Lie dual acting over any \(A^{cd} \in \Omega(M)\) with two spin indices \(c, d\) and the notation \(\ast A_{ab}\) for the hodge dual of the form \(A_{ab}\), which is defined over the vierbein basis as \(\ast (e^{a_1} \wedge \cdots \wedge e^{a_n}) := (\ast e^{a_1} \wedge \cdots \wedge e^{a_n}) := \frac{1}{(4\pi)^2} \frac{\epsilon^{a_1 \ldots a_n}}{(1 - n)} \wedge \cdots \wedge (1 - n) e\) and it extends to the entire space \(\Omega(M)\) by linearity.
the third case presents similarities to our approach, our key component is the cosmological functional \( \lambda \) which, through the field equations \([11]\) is the source of topology for our space-time manifold, as well as a way to ensure minimum topological requirements such as smoothness, simple connected-ness and orientation, while emerging gravity approaches such as Refs. \([33, 36]\) build topology from iterative processes.

### III. EFFECTIVE TOPOLOGICAL COSMOLOGICAL CONSTANT

If the topological numbers are calculated we obtain:

\[
 n_P := \Re \int M C_P = 0 ; \quad n_N := \Re \int M C_N = 0 , \tag{11}
\]

while the Euler number yields:

\[
 Z \ni n_E := \Re \int M C_E = -\frac{16\Lambda^2}{3(4\pi)^2} \| u^2_{\hat{\varphi}_N} \|^2_{L^2} , \tag{12}
\]

which is finite by topology. We can also write \( n_E \) using its representation as an alternate series of the Betti numbers \( b_i (M) \) (\( i = 0, \ldots, 4 \)):

\[
 n_E = \sum_{j=0}^{4} (-1)^j b_j (M) ,
\]

as well as using Poincaré duality, i.e \( b_i = b_{4-i} \) \([24]\), to write:

\[
 n_E = 2b_0 - 2b_3 + b_2 = -2k_E^2 b_3 . \tag{13}
\]

In Eq. \((13)\), we recall that the \( i \)-th Betti number measures the number of \( i \)-th punctures in a manifold. As \( M \) (and consequently \( N \)) is assumed to be simply connected, i.e. the manifold is composed by only one connected component, therefore we get \( b_0 = 1 \). Since \( n_P \) turns out to be null in \( M \), the Hirzebruch signature theorem implies that \( b_2 = 2b \), where \( b^+ = b^- = b \) and \( b^+, b^- \) are the dimensions of maximal positive \( \mathcal{H}^+ \) and negative \( \mathcal{H}^- \) subspaces for the form in \( H^2 (M; \mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^- \), respectively. There seems to be no clear evidence of physical objects that can be interpreted as strict 2-punctures in our Universe, and that would appear for instance as a naked line of singularities. The cosmic censorship conjecture \([37]\) would prescribe it to be zero. However, we do not discard their presence but assume its associated number \( b \) should be finite. Finally, in our observable Universe, causality effectively defines horizon 3-hypersurfaces around space-time singularities in \( M \). Those hyper-surfaces effectively act as 3-punctures of \( M \), which total number is \( b_3 \). The clearest example of these singularities are BHs, which observations indicate there should be a large number. Therefore we assume \( b_3 \) to be large compared with \( b \), thus the latter is negligible compared with the former. This justifies our factoring in Eq. \((13)\) with \( k_E \) verifying:

\[
 \mathbb{R} \ni k_E^2 := 1 - \frac{1 + b}{b_3} \lesssim 1 . \tag{14}
\]

Therefore, from Eq. \((12)\) the theoretical results follows:

\[
 \Lambda_T^2 = \frac{3(4\pi)^2 k_E^2 b_3}{8 \| u^2_{\hat{\varphi}_N} \|^2_{L^2}} \Rightarrow \Lambda_T \approx \frac{2\pi k_E C}{(\frac{2}{3} \text{Vol} (\partial M))^2} , \tag{15}
\]

where we have used the estimate:

\[
 0 \leq \| u^2_{\hat{\varphi}_N} \|^2_{L^2} \simeq \frac{\text{Vol} (\partial N)}{C^2} < \infty , \tag{16}
\]

where \( C^2 \) is the Cheeger or isoperimetric constant \([38]\) with dimensions of \([\text{length}]\).

### IV. \( \Lambda_T \) EVALUATION FROM BLACK HOLES

The topological cosmological constant can be evaluated from interpreting expression \((15)\). We decompose this evaluation into three main elements that needs separate estimations: the value of the constants \( k_E, C \) and of the average spacetime boundary volume \( \langle \text{Vol} (\partial M) \rangle \).

The ratio of the Euler number to the number of three punctures, \( k_E = \sqrt{-\frac{\Lambda_T}{2b_3}} \) seen in Eq. \((13)\) is considered as the ratio of the topological Euler number to the number of black holes (BHs) contained in the manifold. Indeed as we can carve out the causally disconnected BH event horizons interiors from the rest of spacetime, we consider its boundary to consist in the sum of those hypersurfaces, and thus to correspond to the 3-punctures measured by the third Betti number \( b_3 = \mathcal{N}_{BH} \). For any sensible spacetime, \( b_3 \) should dominate the other terms in Eq. \((13)\) we can evaluate \( k_E \approx 1 \).

The isoperimetric or Cheeger constant, \( C \), gives the ratio of the spacetime 4-volume to its boundary hyper-surface 3-volume. Although it is generally unknown, it was calculated in some specific cases \([39, 40]\), and is thus considered to be of order \( \sim 10^1 \).

Once we have decided to identify the boundary of the spacetime to the horizons of its BHs, the evaluation of the topological cosmological constant relies on the estimation of the average spacetime BH horizon volume \( \langle \text{Vol} (\partial M) \rangle \). This requires to make some assumptions.

#### A. Average BH boundary volume estimation

To compute the volume of the average BH in our Universe, the following assumptions are made: (a) as the Universe’s total boundary is taken to be the sum of all BHs horizons, that hypersurface is assumed to be
given by BHs equivalent Schwarzschild horizons. This neglects Kerr Horizons deformations and the different horizons shapes taken at the moment of BH mergers, assuming each BH can be approximated by an isolated Schwarzschild horizon. (b) the Universe’s BH distribution is assumed to be represented by our past lightcone BHs observations and present knowledge of that distribution is sufficient for the calculation, giving the boundary volume as its first moment. (c) the distribution’s average volume per BH in the Universe is well approximated by the volume of a BH starting with the average BH mass. (d) the volume of a BH of given initial mass is approximated to proceed from an almost instant creation with a mass picked in the observed lightcone distribution followed by a long Hawking radiation phase evaporation.

The computation of our Universe’s average BH volume requires to evaluate, from observed BH distribution, the average BH mass of the Universe, to get the BH horizon volume of a fixed given mass, so as to put them together in an evaluation of the Universe’s boundary volume.

1. Average BH mass evaluation

We chose to evaluate the BH mass distribution based on the observations and computations of Refs [41, 44]. From them, we have extracted averages and standard deviations from the expected peaks in the BH distributions around stellar mass BHs, Primordial BHs (PBHs), Intermediate mass BHs (IMBHs) and Super Massive BHs (SMBHs). Estimating the average mass from a geometrical weighted average, we also used a conservative treatment of errors, and obtained

\[
\langle M_{BH} \rangle \sim 10^{4.04^{+0.49}_{-0.41}} M_\odot, \quad \text{(17)}
\]

which is slightly below our evaluation of the IMBH average and above the dominating PBH peak average estimate.

2. BH volume of a given mass

For a given BH mass \( M \), the BH formation phase is neglected since its dynamical time is expected to be considerably much less than the Hawking radiation evaporation time. The horizon volume is therefore estimated considering the BH appears at creation with initial mass \( M \) and evaporates through Hawking radiation, until the complete BH evaporation, evaluated considering the mass-energy loss [e.g. 43, 46]. As previously mentioned, we consider that mass to be ascribed to a simple isolated Schwarzschild BH. The calculation of such volume yields

\[
\text{Vol}_{BH} (M) = 1.96 \times 10^{87} \left( \frac{M}{M_\odot} \right)^5 \text{m}^3. \quad \text{(18)}
\]

3. Universe’s total boundary volume

Following our assumptions above, the resulting boundary volume of the Universe can be evaluated by the product of the total number of BHs with the average BH volume, given by introducing the average BH mass estimate (17) into the BH boundary volume evaluation (18), so we obtain

\[
\text{Vol} (\partial \mathcal{M}) \sim N_{BH} 10^{40.75^{+2.5}_{-1.1}} \text{m}^3. \quad \text{(19)}
\]

B. Evaluating the topological cosmological constant

Now that we have the estimate of the BH boundary volume (19), we can finally input it into the model result (15) to get \( \Lambda_T \) as a function of the Cheeger and \( k_E \) constants

\[
\Lambda_T \approx 10^{-52.9^{+1.5}_{-1.3}} k_E C \approx 10^{-52.9^{+1.5}_{-1.3}} C, \quad \text{(20)}
\]

since we previously argued that \( k_E \approx 1 \), only the Cheeger constant remains.

1. Can \( \Lambda_T \) be the observed cosmological constant?

Converting the latest Planck observations [47] in the appropriate units, we compute \( \Lambda_O = 10^{-51.08^{+0.01}_{-0.01}} \text{m}^{-2} \). The isoperimetric constant has been evaluated for a 4-manifold with null sectional curvature, Ref. [39, 40] and the value \( C = 11.8 \) was obtained. Approximating the Universe’s value with it,

\[
\Lambda_T \approx 10^{-51.8^{+1.5}_{-1.3}} \text{m}^{-2}, \quad \text{(21)}
\]

and thus our \( \Lambda_T \) estimate is compatible with the observed \( \Lambda_O \). Given that \( C \sim O(10) \), we argue that the topology of the Universe is a serious candidate for the cosmological constant origin, and this giving naturally its low value and avoiding the cosmological constant fine tuning problem.

2. The Universe isoperimetric constant can be measured

Although the previous evaluation gives the correct observed cosmological constant, the actual value of \( C \) for our Universe remains undetermined. If we assume \( \Lambda_T = \Lambda_O \), Eq. (20) can be used to measure, through the average BH mass and volume estimates of this work and the current cosmological constant observations [47], the value of the Universe’s isoperimetric constant which contributes to the determination of the topology of the Universe. We obtain

\[
C = 10^{1.82^{+1.31}_{-1.51}}, \quad \text{(22)}
\]

which gives a reasonable value compared to expectations.
V. CONCLUSIONS

Introducing the topological invariants \[24, 28, 48\] in the gravitation theory as Lagrange multipliers induces an effective cosmological constant. The extra degrees of freedom and restrictions of the topological invariants in an Einstein-Cartan gravity \[17, 20\] are handled by a reasonable ansatz. They allow to obtain a GR-like behaviour dynamically, driven by the invariants coupling zero forms that can be considered as effective dark energy fields. Indeed, we argue that their dynamics constrain the value of the expansion acceleration to agree with the BH boundary of the Universe. The induced effective topological cosmological constant in the GR-like theory is produced by the Euler invariant. It depends on the Betti numbers of the spacetime \[24, 19\], the isoperimetric constant \[38\] and the volume of the manifold boundary. The interpretations of \(\Lambda\) in terms of number and surface of BHs in the Universe allows to evaluate it from current estimates of the BH distribution of the Universe \[41, 44\]. We found, with some reasonable assumptions on the isoperimetric constant, that it is compatible with current cosmological constant observations \[4, 5, 17\], escaping the cosmological constant fine tuning problem \[8, 10\]. Our BH volume evaluation being based on some BH distribution estimations that rely on gravitational waves and BH population knowledge, future improvements in those domains, both experimental and theoretical, are expected to allow narrowing on the topological cosmological constant estimation, increasing the testability of the approach compared with the Universe’s acceleration or geometrical observations of \(\Lambda_0\). We conjecture that the remaining unknown isoperimetric constant could be independently obtained from the development of an emerging geometry approach \[12, 30\] to the dynamic theory of the manifold topology, with the potential to perhaps clarify the cosmological constant coincidence problem \[11, 14\] as well from topological considerations.

ACKNOWLEDGEMENTS

The authors wish to thank M. Fontanini and E. Huguet for very useful discussions, as well as O. Bertolami for interesting perspectives. The work of M. Le D. has been supported by Lanzhou University starting fund and PNPD/CAPES20132029. M. Le D. also wishes to acknowledge IFT/UNESP for hosting the beginning of this project.

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