On Miyawaki lifts with respect to Ikeda lifts for $U(2,2)$

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November 25, 2019

Abstract

In this paper, we discuss representation-theoretical Miyawaki lifts for unitary groups in terms of the endoscopic classification. We give an explicit determination of the Miyawaki lifts for $U(1,0)$ and $U(2,1)$ with respect to the Ikeda lifts for $U(2,2)$. This paper contains the result that Ikeda lift for $U(2,2)$ coincides with Hermitian Maass lift.

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1 Introduction

Miyawaki lifts are kinds of lifting of automorphic forms/representations founded by Ikeda. In his original paper [Ike06], Ikeda gave an construction of Siegel modular forms from another Siegel modular forms by using diagonal restrictions of Ikeda lifts as kernel functions. There are some analogues of the result for other groups: the unitary group analogue by Atobe and Kojima [AK18], and the exceptional group analogue by Kim and Yamauchi [KY19]. Recently, Atobe constructed and studied representation-theoretical Miyawaki lifts for Symplectic/Metaplectic groups [Ato17] by using representation-theoretical Ikeda lifts constructed by Ikeda and Yamana [IY15].

In this paper, we discuss representation-theoretical Miyawaki lifts for unitary groups. We reformulate Ikeda lifts and Miyawaki lifts in terms of the endoscopic classification for unitary groups [Mok15, KMSW14] and we determine the correspondence of Miyawaki lift for $U(1,0)$ and $U(2,1)$ in the sense of the reformulation.

Let us describe our results. Let $F$ be a global number field and $E$ be a quadratic extension of $F$. We denote their adeles by $\mathbb{A}_F$ and $\mathbb{A}_E$, respectively. For $A$-parameter $\psi'$ of quasi-split unitary groups, let $\psi'_v$ denote the localization of $\psi'$ at a place $v$ of $F$. Let $S_{\psi'_v}$ denote the component group which corresponds to $\psi'_v$ and let $J_{\psi'_v} : \Pi_{\psi'_v} \hookrightarrow \hat{S}_{\psi'_v}$ denote the canonical injective (we use the formulation in [Art05]) map from local $A$-packet $\Pi_{\psi'_v}$ to the set of characters of $S_{\psi'_v}$ which are trivial on the image of $Z(G(N)_v)^{\text{Gal}(F_v/F)}$. Let $\pi(\psi'_v, \eta)$ denote the element of $\Pi_{\psi'_v}$ which satisfies $J_{\psi'_v}(\pi(\psi'_v, \eta)) = \eta$. 


Let \( \pi \) be a cuspidal representation of \( \mathrm{GL}_2(\mathbb{A}_F) \) with central character \( \epsilon_{E/F} \), which is associated to \( E/F \) by class field theory. Let \( \phi \) denote the base-change of \( \pi \) to \( \mathrm{GL}_2(\mathbb{A}_E) \). Fix a character \( \chi \) of \( F^\times \backslash \mathbb{A}^\times_E \) such that \( \chi|_{\mathbb{A}^\times_E} = \epsilon_{E/F} \) and let \( \phi_\chi = \phi \otimes \chi \circ \det \) and let \( \psi = \phi[2] = \phi \otimes [2] \), where \([n]\) be \( n \)-dimensional irreducible algebraic representation of \( \mathrm{SL}_2(\mathbb{C}) \). Then, \( \phi_\chi \) and \( \psi \) are global discrete A-parameters of \( U(1,1) \) and \( U(2,2) \), respectively.

Let \( \tau = \otimes_v \pi((\phi_\chi)_v, \eta_v) \) be an irreducible (cuspidal) automorphic representation of \( U(1,1)(\mathbb{A}_F) \) with A-parameter \( \phi_\chi \). Then, we define the Ikeda lift \( I_4(\tau) \) of \( \tau \) for \( U(2,2)(\mathbb{A}_F) \) by

\[
I_4(\tau) = \otimes_v \pi(\psi_v, \eta_v)
\]

under the identification \( S_{\psi_v} = S_{(\phi_\chi)_v} \).

Let \( \gamma = \otimes_v \gamma_v \) be an automorphic character of \( U(1,0)(\mathbb{A}_F) \) with A-parameter \( \gamma_v \). We assume that \( \phi \neq \gamma \oplus \gamma^{-1} \). Then, \( M_{\psi}(\gamma) := \phi \oplus \gamma^{-1} \) is a discrete A-parameter of \( U(2,1) \). Let \( \mathbb{H}^2_E \) denote the 2-dimensional split hermitian space over \( E \). Let \( \mathbb{H}^2_E = E_1 \perp E_3 \) be an orthogonal decomposition of \( \mathbb{H}^2_E \) into a 1-dimensional subspace \( E_1 \) and a 3-dimensional subspace \( E_3 \). Then, the isomorphisms \( U(2,2) \simeq U(\mathbb{H}^2_E), U(1,0) \simeq U(V_1), \) and \( U(2,1) \simeq U(V_3) \) induce the inclusion \( U(2,2) \supset U(1,0) \times U(2,1) \). So we define the Miyawaki lift \( M_{I_4(\tau)}(\gamma) \) of \( \gamma \) with respect to \( I_4(\tau) \) as the representation generated by

\[
M_f(\gamma) : g \mapsto \int_{U(1,0)(F)\backslash U(1,0)(\mathbb{A}_F)} f((U(1,0) \times U(2,1))(\mathbb{A}_F)\backslash U(2,1)(\mathbb{A}_F))(h,g)\gamma(h)dh
\]

for all \( f \in I_4(\tau) \). Then our main theorem is as follows.

**Theorem 1.1.** \( M_{I_4(\tau)}(\gamma) \subset L^2_{\text{disc}}(U(2,1)(F)\backslash U(2,1)(\mathbb{A}_F)) \). Moreover,

\[
M_{I_4(\tau)}(\gamma) \simeq \otimes_v \sigma_v,
\]

where \( \sigma_v \) is as follows:

(a) When \( v \) splits over \( E \), \( \sigma_v \) is an irreducible representation of \( \mathrm{GL}_2(\mathbb{F}_v) \) which corresponds to \( M_{\psi}(\gamma)_v \).

(b) When \( v \) is non-archimedean and \( v \) does not split over \( E \), then there is two cases.

(i) If \( \phi_v \) does not contain \( \gamma_v \), under the identification

\[
S_{M_{\psi}(\gamma)_v} = S_{(\phi_\chi)_v} \oplus (\mathbb{Z}/2\mathbb{Z}),
\]

\[
\sigma_v = \pi(M_{\psi}(\gamma)_v, (\eta_v, 1_{\mathbb{Z}/2\mathbb{Z}})).
\]

(ii) If \( \phi_v = \gamma_v \oplus \gamma^{-1}_v \),

\[
\sigma_v = \begin{cases} 
\pi(M_{\psi}(\gamma)_v, (1_{S_{M_{\psi}(\gamma)_v}})) & \text{if } \eta_v = 1_{S_{(\phi_\chi)_v}}; \\
0 & \text{otherwise.}
\end{cases}
\]

(c) When \( v \) is archimedean and does not split over \( E \), then there is two cases. We define \( \chi_\alpha \) for \( \alpha \in \mathbb{Z} \) by \( \chi_\alpha(re^{i\theta}) = e^{i\alpha \theta} (r \in \mathbb{R}_{>0}, \theta \in \mathbb{R}/2\pi\mathbb{Z}) \), and write

\[
\phi_v = | \cdot |^k \chi_k \oplus | \cdot |^{-k} \chi_{-k}, \gamma_v = \chi_{2s},
\]

where \( k \in \mathbb{Z}_{\geq 0}, t \in \mathbb{R} \) and \( s \in \mathbb{Z} \).

(i) If \( t \neq 0 \) or \( k \) is odd or \( |2s| < k \), then, for the identification

\[
S_{M_{\psi}(\gamma)_v} = S_{(\phi_\chi)_v} \oplus (\mathbb{Z}/2\mathbb{Z}),
\]

\[
\sigma_v = \pi(M_{\psi}(\gamma)_v, (\eta_v, 1_{\mathbb{Z}/2\mathbb{Z}})).
\]
Let $F$ be a global field with characteristic 0 and $E$ be a quadratic extension of $F$. Let $c$ be the non-trivial element of $\text{Gal}(E/F)$ and let $(g^c)_{i,j} = (g_{i,j})^c$ for $g \in \text{GL}_N(E)$. We denote the norm and the trace of with respect to $E/F$ by $N$ and $\text{Tr}$, respectively. We fix an element $\kappa$ of $E^\times$ such that $\text{Tr}(\kappa) = 0$ and let $d = \kappa^2 \in F^\times$. We write $F_v$ for the completion of $F$ with respect to a place $v$ of $F$ and $E_v = F_v \otimes_F E$. For each $v$, let $| \cdot |_{F_v}$ be the normalized absolute value of $F_v$ and let $|z|_{F_v} := |z^c|_{F_v}$ for $z \in E_v^\times$. The rings of adeles over $F$ and $E$ are denoted by $A_F$ and $A_E$, respectively. Let $A_{F,\text{fin}}$ (resp. $A_{F,\text{\infty}}$) denote the finite (resp. infinite) part of $A_F$.

For $n \in \mathbb{Z}_{>0}$, let $I_n = \begin{pmatrix} \ddots & 1 \\ \\ 1 & \ddots \\ & & 1 \end{pmatrix} \in \text{GL}_n(F)$ and let

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad J_{2n+1} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

For $p = n$ and $q = n$ or $n - 1$, we define $\text{GU}(p, q)$ and $\text{U}(p, q)$ by

$$\text{GU}(p, q) = \{ g \in \text{Res}_{E/F} \text{GL}_{p+q} | g^cJ_{p+q}g = \nu_{p+q}(g)J_{p+q}, \nu_{p+q}(g) \in \text{GL}_1 \},$$

$$\text{U}(p, q) = \ker(\nu_{p+q}).$$

For convenience, let $\text{GU}(0, 0) = \text{GL}_1, \nu_0 = \text{id}_{\text{GU}(0,0)}$, and $\text{U}(0, 0) = \ker(\nu_0) = \{1\}$. We often denote $\nu_{p+q}$ by $\nu$ for short.

Let $B'_{p+q}$ be the Borel subgroup of $\text{GU}(p, q)$ which consists of upper triangular matrices and let $B_{p+q} = M_{p+q}N_{p+q}$ be its Levi decomposition. For $U(p, q)$, let $B_{p+q} = B'_{p+q} \cap U(p, q) = M_{p+q}N_{p+q}$, similarly.

For a place $v$ of $F$, let $\{l_i\}_{i=1}^k \subset \mathbb{Z}_{\geq 0}$ which satisfies $p_0 = p - \sum l_i, q_0 = q - \sum l_i \geq 0$, representations $\tau_i$ of $\text{GL}_{l_i}(E_v)$ and a representation $\pi_0$ of $\text{GU}(p_0, q_0)(F_v)$, let

$$\tau_1 \times \tau_2 \times \cdots \times \tau_k \times \pi_0 = \text{Ind}_{\text{GU}(p_0, q_0)(F_v)}^{\text{GU}(p_0, q_0)(F_v)} \tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_k \boxtimes \pi_0.$$
Here, $P'$ is the standard parabolic subgroup (for $B'_{p+q}$) whose Levi subgroup $M'$ is defined by

$$M' = \begin{pmatrix}
\left( \text{Res}_{E/F} GL_{l_1} \right) & \cdots & \left( \text{Res}_{E/F} GL_{l_k} \right) \\
\text{GU}(p_0, q_0) & * & * \\
\text{GU}(0, 0) & * & * 
\end{pmatrix} \quad (p_0 \neq 0);$$

$$M' = \begin{pmatrix}
\left( \text{Res}_{E/F} GL_{l_1} \right) & \cdots & \left( \text{Res}_{E/F} GL_{l_k} \right) \\
* & * & * \\
* & * & * 
\end{pmatrix} \quad (p_0 = q_0 = 0),$$

and $\text{Ind}$ is the normalized induction. For $U(p, q)$, we define $\tau_1 \times \tau_2 \times \cdots \tau_k \times \tau_0$ similarly.

## 2 Endoscopic classification for quasisplit unitary groups

In this section, we summarize some properties of the endoscopic classification for quasisplit unitary groups proved by Mok [Mok15]. Let

$$G(N) = \{ g \in \text{Res}_{E/F} GL_N \mid \text{^{g^{-1}}I}_N g = I_N \}$$

be the $N$-dimensional quasisplit unitary group with respect to $E/F$. We note that $G(N) \simeq U(N/2, N/2)$ by $g \mapsto \left( \begin{smallmatrix} 1_{N/2} & \kappa_{1/2} \\ \kappa_{1/2}^{-1} & 1_{N/2} \end{smallmatrix} \right) g \left( \begin{smallmatrix} 1_{N/2} & \kappa_{1/2} \\ \kappa_{1/2}^{-1} & 1_{N/2} \end{smallmatrix} \right)^{-1}$ if $N$ is even and $G(N) = U((N + 1)/2, (N - 1)/2)$ if $N$ is odd. For each $v$, let $G(N)_v$ be the algebraic group over $F_v$ which defined by $G(N)_v(R) = G(N)(R)$ for each $F_v$-algebra $R$.

### 2.1 Local parameters

Let $v$ be a place of $F$ and $W_{E_v}$ be the Weil group of $E_v$. We denote by $\Psi^+(G(N)_v)$ the set of (conjugation classes of) local A-parameters of $G(N)_v$. Here, the meaning of local A-parameter is as follows:

**Case 1: $v$ does not split over $E$.** Let $W_{E_v}$ be the Weil group of $E_v$ and fix $w_c \in W_{F_v} \setminus W_{E_v}$. Let

$$L_{E_v} = \begin{cases}
W_{E_v} \times \text{SL}_2(\mathbb{C}) & \text{if } v \text{ is nonarchimedean;} \\
W_{E_v} & \text{if } v \text{ is archimedean.}
\end{cases}$$

Let $\psi$ be an $N$-dimensional semi-simple complex continuous representations of $L_{E_v} \times \text{SL}_2(\mathbb{C})$ of which restriction to $\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ or $\text{SL}_2(\mathbb{C})$ is algebraic. Let $\psi^c$ define by $\psi^c(w) = \psi(w, w w_c^{-1})$ for $w \in L_{E_v} \times \text{SL}_2(\mathbb{C})$. For $b = \pm 1$, we say that $\psi$ is conjugate self-dual with parity $b$ if there is a non-degenerate bilinear form $B : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}$ which satisfies that

$$B(\psi(w)x, \psi^c(w)y) = B(x, y),$$

$$B(y, x) = bB(x, \psi(w_c^2)y)$$

for $x, y \in \mathbb{C}$ and $w \in L_{E_v} \times \text{SL}_2(\mathbb{C})$. Then, we say that $\psi$ is an A-parameter of $G(N)_v$ if $\psi$ is conjugate self-dual with parity $(-1)^{N-1}$.
**Case 2: \( v \) splits over \( E \).** Let \( w, \pi \) be the places of \( E \) on \( v \). We define \( W_{E_v} = W_{E_w} \times W_{E_w} = W_{E_v} \times W_{E_v} \) and let \( L_{E_v} \) define similar to case 1. Then, we say that an \( N \)-dimensional semi-simple complex continuous representations of \( L_{E_v} \times \text{SL}_2(\mathbb{C}) \) which satisfies that \( \psi|_{\text{SL}_2(\mathbb{C})} \) or \( \psi|_{\text{SL}_2(\mathbb{C})} \) is algebraic is an \( A \) parameter of \( G(N)_v \) if \( \psi(w_1, w_2, g_1, g_2) = \psi^v(w_2, w_1, g_1, g_2) \). We often identify \( \psi \) and \( \psi|_{W_{E_v} \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})} \) or \( \psi|_{W_{E_v} \times \text{SL}_2(\mathbb{C})} \).

In each case, we denote

\[
\begin{align*}
\Psi(G(N)_v) & := \{ \psi \in \Psi^+(G(N)_v) \mid \psi(W_{E_v}) \text{ is bounded} \}, \\
\Phi(G(N)_v) & := \{ \psi \in \Psi^+(G(N)_v) \mid \psi|_{\text{SL}_2(\mathbb{C})} \text{ is trivial} \}, \\
\Phi_{\text{bd}}(G(N)_v) & := \Psi(G(N)_v) \cap \Phi(G(N)_v).
\end{align*}
\]

Let \( \psi \) be a semi-simple finite-dimensional complex continuous representations of \( L_{E_v} \times \text{SL}_2(\mathbb{C}) \). Then, there is the decomposition

\[
\psi = \bigoplus \mu_{m_i} \otimes [a_i] \otimes [b_i],
\]

where \( [k] \) is \( k \)-dimensional irreducible algebraic representation of \( \text{SL}_2(\mathbb{C}) \) and \( m_i, a_i, b_i \) are positive integers and \( \mu_{m_i} \) is \( m_i \)-dimensional irreducible representations of \( W_{E_v} \) (if \( v \) is archimedean, then we set \( a_i = 1 \) formally and ignore \( [a_i] \)). We denote \( \mu_{m_i} \otimes [a_i] \otimes [b_i] \) by \( \mu_{m_i}[a_i][b_i] \) for short. In addition, we define \( \phi_{\psi} \) by

\[
\phi_{\psi}(w, g) = \psi(w, g, \text{diag}([w]_{E_v}^{\frac{1}{2}}, |w|_{E_v}^{-\frac{1}{2}})),
\]

where we assume \( | \cdot |_{E_v} \) as a character of \( W_{E_v} \) by \( W_{E_v} \otimes E_v^\times \simeq E_v^\times \).

Let \( \psi \in \Psi^+(G(N)_v) \). If \( v \) does not split over \( E \), let \( S_\psi = \text{Aut}(\psi, B) \) be the group of elements in \( \text{GL}_N(\mathbb{C}) \) which centralize the image of \( \psi \) and preserve \( B \) which is taken as above. If \( v \) splits over \( E \), let \( S_\psi \) be the centralizer of \( \psi(W_{E_v} \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \) or \( \psi(W_{E_v} \times \text{SL}_2(\mathbb{C}) \) in \( \text{GL}_N(\mathbb{C}) \). In each case, let \( S_\psi = \pi_0(S_\psi) \).

We note that if \( v \) splits over \( E \), then \( S_\psi \) is trivial. If not, we can decompose \( \psi \) into a direct sum

\[
\psi = \bigoplus_{i=1}^l m_i \psi_i \oplus (\xi \oplus \xi^*)
\]

with pairwise inequivalent \((-1)^{N-1}\) parity conjugate self-dual irreducible representations \( \psi_i \) of \( L_{E_v} \times \text{SL}_2(\mathbb{C}) \) and multiplicities \( m_i \), representations \( \xi \) and \( \xi^* = (\xi^*)^\vee \) of \( L_{E_v} \times \text{SL}_2(\mathbb{C}) \) which have no conjugate self-dual irreducible subrepresentations with parity \((-1)^{N-1}\). Then,

\[
S_\psi \simeq (\mathbb{Z}/2\mathbb{Z})^l.
\]

### 2.2 The local endoscopic classification

Let us fix a local Whittaker datum \( w_\psi = (B_\psi, \lambda_\psi) \) i.e. a conjugacy class of pair of a Borel subgroup \( B_\psi = M_\psi N_\psi \) of \( G(N)_v \) and a generic character \( \lambda_\psi \) of \( N_\psi(F_v) \). Then, the local endoscopic classification says that each \( \psi \in \Psi(G(N)_v) \) associates a finite multi-set \( \Pi_\psi \) of unitary representations of \( G(N)_v \) of finite length where there exists a canonical injective (if \( v \) is non-archimedean, it is bijective moreover) map

\[
J_\psi = J_{w_\psi, \psi} : \Pi_\psi \to \mathfrak{F}_\psi = \{ \eta \in \mathfrak{S}_\psi \mid \eta(-1_N) = 1 \}.
\]

We remark that if \( v \) is split over \( E \), then \( \Pi_\psi = \{ \tau_\psi \} \), where \( \tau_\psi \) is the irreducible representations of \( \text{GL}_N(F_v) \) which correspond to \( \phi_{\psi} \) by LLC for general linear groups.

For elements of \( \psi \in \Psi^+(G(N)_v) \), we extend the definition of \( \Pi_\psi \). Let \( \psi \in \Psi^+(G(N)_v) \). We decompose \( \psi \) into

\[
\psi = \xi_1 \cdot |E_v^* \oplus \xi_2| \cdot |E_v^* \oplus \cdots \oplus \xi_k| \cdot |E_v^* \oplus \psi_0 \oplus \xi_k^*| \cdot |E_v^* \oplus \cdots \oplus \xi_1^*| \cdot |E_v^*|
\]
with real numbers \( r_i \) which satisfy \( r_1 \geq r_2 \geq \cdots \geq r_k > 0 \), \( \psi_0 \in \Psi(G(N_0)) \) and \( l_i \)-dimensional irreducible representations \( \xi_i \) which satisfy that \( \xi_i(W_{E_v}) \) is bounded \( (N = N_0 + 2 \sum_i l_i) \). Then, we define the multi-set \( \Pi_\psi \) of representations (there is possibility that they are non-unitary) by

\[
\Pi_\psi = \{ \tau_{\phi_{\xi_1}} \times \tau_{\phi_{\xi_2}} \cdots \times \tau_{\phi_{\xi_k}} \times \pi_0 \mid \pi_0 \in \Pi_{\psi_0} \},
\]

where \( \tau_{\phi_{\xi_i}} \) are the irreducible representations of \( GL_{l_i}(E_v) \) which correspond to \( \phi_{\xi_i} \) by LLC for general linear groups. Since \( S_\psi \simeq S_{\psi_0} \) naturally, \( J_\psi \) is defined in an obvious way.

For \( \eta \in \text{Im}(J_\psi) \), let \( \pi(\psi, \eta) \) be the element of \( \Pi_\psi \) which satisfies \( J_\psi(\pi(\psi, \eta)) = \eta \), in all cases.

### 2.3 Properties of the local endoscopic classification

We highlight some properties of the local endoscopic classification. Let \( v \) be non-split over \( E \).

- Let \( \psi \in \Psi^+(G(N)_v) \). Then, the all elements of \( \Pi_\psi \) have the same central character \( \omega_\psi \), which corresponds to \( \det \omega_\psi \).

- Let \( \phi \in \Phi_{\text{add}}(G(N)_v) \). Then each element of \( \Pi_\phi \) is a non-zero irreducible tempered representations of \( G(N)(F_v) \). In general, if \( \phi \in \Phi(G(N)_v) \), then each element of \( \Pi_\phi \) has the unique non-zero irreducible quotient. Let \( \text{Irr}(G(N)_v) \) (resp. \( \text{Irr}_{\text{temp}}(G(N)_v) \)) denote the set of irreducible representations (resp. the set of irreducible tempered representations) of \( G(N)(F_v) \). Then,

\[
\text{Irr}(G(N)_v) = \bigsqcup_{\phi \in \Phi(G(N)_v)} \Pi'_\phi
\]

and

\[
\text{Irr}_{\text{temp}}(G(N)_v) = \bigsqcup_{\phi \in \Phi_{\text{add}}(G(N)_v)} \Pi_\phi,
\]

where \( \Pi'_\phi \) for \( \phi \in \Phi(G(N)_v) \) is the set of the irreducible quotients of elements of \( \Pi_\phi \), namely, the L-packet with L-parameter \( \phi \).

- Let \( \psi \in \Psi(G(N)_v) \). Then, for each element \( \pi \) of \( \Pi'_{\phi_\psi} \), there is the unique element of \( \Pi_\psi \) which contains \( \pi \) as subrepresentation and the following diagram commutes:

\[
\begin{array}{ccc}
\Pi_{\phi_\psi} & \xrightarrow{J_{\phi_\psi}} & \widehat{S}_{\phi_\psi} \\
\vee & & \vee \\
\Pi_\psi & \xrightarrow{J_\psi} & \widehat{S}_\psi,
\end{array}
\]

where \( \widehat{S}_{\phi_\psi} \rightarrow \widehat{S}_\psi \) is the map induced by the natural injection \( S_\psi \rightarrow S_{\phi_\psi} \) and \( \Pi_{\phi_\psi} \rightarrow \Pi_\psi \) is the composition of natural maps \( \Pi_{\phi_\psi} \rightarrow \Pi'_{\phi_\psi} \) and \( \Pi'_{\phi_\psi} \rightarrow \Pi_\psi \).

- ([Mok15, §8]) Let \( v \) be non-archimedean and let \( \psi \in \Psi(G(N)_v) \) which factors through \( (L_{F_v}/(\{1\} \times \text{SL}_2(\mathbb{C}))) \times \text{SL}_2(\mathbb{C}) \). We define \( \hat{\phi} \in \Pi_{\text{add}}(G(N)_v) \) by \( \hat{\phi}(w, g_1, g_2) = \psi(w, g_2, g_1) \) for \( (w, g_1, g_2) \in L_{F_v} \times \text{SL}_2(\mathbb{C}) \). Then, the Aubert involution induce the bijection between \( \Pi_{\hat{\phi}} \) and \( \Pi_\psi \) where the following diagram commutes:

\[
\begin{array}{ccc}
\Pi_{\hat{\phi}} & \xrightarrow{J_{\hat{\phi}}} & \widehat{S}_{\hat{\phi}} \\
\vee & & \vee \\
\Pi_\psi & \xrightarrow{J_\psi} & \widehat{S}_\psi,
\end{array}
\]

where \( \widehat{S}_{\hat{\phi}} \leftrightarrow \widehat{S}_\psi \) is the natural isomorphism. Especially, the all elements of \( \Pi_\psi \) are non-zero and irreducible.
• Let $v$ be non-archimedean and $G(N)_v$ be unramified. Let us denote the inertia group of $W_{E_v}$ by $I_{E_v}$. Then, for $\psi \in \Psi(G(N)_v)$ which factors through $(L_{E_v}/(I_{E_v} \times \text{SL}_2(\mathbb{C}))) \times \text{SL}_2(\mathbb{C})$, $J_{\psi}^{-1}(1) (\in \Pi_{\phi_v})$ is the unique unramified representation of $\Pi_\psi$.

• Let $\phi \in \Phi(G(N)_v)$. If $\pi \in \Pi'_\psi$ is \(w_v\)-generic i.e. $\text{Hom}_{\text{Kr}}(\phi, \pi_v) \neq 0$ then $J_\phi(\pi) = 1$. Moreover, if $\phi \in \Phi_{\text{bdd}}(G(N)_v)$ then $\pi \in \Pi'_\phi$ has the unique $w_v$-generic element.

### 2.4 Whittaker data

We provide some additional explanations of Whittaker data. If $N$ is even, there are exact two Whittaker data. Let $\psi_{F_v}$ be a non-trivial additive character of $F_v$ and let $\lambda_{a,\psi_{F_v}}$ be the character of $N_N(F_v)$ defined by

$$
\begin{pmatrix}
1 & \alpha_1 & * & * & * & * & * \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \alpha_{N/2-1} & * & * & * & * & * \\
1 & \alpha_N & * & * & * & * & * \\
1 & -\alpha_N & * & * & * & * & * \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & -\alpha_1 & * & * & * & * & * \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\rightarrow
\psi_{F_v}(a(\text{Tr}_{E/F}(\sum_{i=1}^{N/2-1} \alpha_i) + b))
$$

for $a \in F_v^\times$. Then, one is $(\text{Res}_{E_v/F} B_N, \lambda_{1,\psi_{F_v}})$ and the other is $(\text{Res}_{E_v/F} B_N, \lambda_{a,\psi_{F_v}})$ for $a \in F_v^\times \setminus N_{E/F}(E_v)$. If $N$ is odd, there is exact one Whittaker datum.

### 2.5 Global parameters

We think about a formal commutative sum as follows:

$$\psi = \bigoplus_{i=1}^l k_i \mu_{n_i} \boxtimes [m_i]$$

Here, $k_i, n_i, m_i \in \mathbb{Z}_{\geq 0}$ which satisfy $\sum k_i n_i m_i = N$ and $\mu_{n_i} \boxtimes [m_i]$ (we often supress $\boxtimes$ for short) are pairwise ineivalent pairs of representations where $\mu_{n_i}$ are irreducible cuspidal representations of $\text{GL}_{n_i}(\mathbb{A}_E)$ and $[m_i]$ are $m_i$-dimensional algebraic representations of $\text{SL}_2(\mathbb{C})$. We say that $\psi$ is a global discrete $A$-parameter of $G(N)$ if

• $k_i = 1$ for all $i$

• the Asai L-function $L(s, \mu_{n_i}, \text{As}_{(-1)^{m_i+N}})$ (see [GGP12, §7]) has an simple pole at $s = 1$

We denote by $\Psi_2(G(N))$ the set of global discrete $A$-parameters of $G(N)$.

### 2.6 Localization

For $\psi \in \Psi_2(G(N))$, we can define the localization $\psi_v \in \Psi^+(G(N)_v)$ of $\psi$ at $v$. Namely, if $\psi = \bigoplus_{i=1}^l \mu_{n_i} \boxtimes [m_i]$ then

$$\psi_v = \bigoplus_{i=1}^l \mu_{n_{i,v}} \boxtimes [m_i]$$

where $\mu_{n_{i,v}}$ are representations of $L_{E_v}$ correspond to the $v$-component of $\mu_{n_i}$ by LLC for general linear groups. Moreover, we define $S_\psi = O(1, \mathbb{C})^l \cong (\mathbb{Z}/2\mathbb{Z})^l$, $S_\psi = \pi_0(S_\psi) \cong (\mathbb{Z}/2\mathbb{Z})^l$. Then, there are natural homomorphisms

$$S_\psi \rightarrow S_{\psi_v}$$

and they induce

$$S_\psi \rightarrow S_{\psi_v}.$$
2.7 The global endoscopic classification

Let us fix a global Whittaker datum $\mathbf{w} = (B, \lambda)$ i.e. a $G(N)(F)$-conjugacy class of a pair of a Borel subgroup $B = MN$ of $G(N)$ and a globally generic character $\lambda = \otimes_v \lambda_v$ of $N(\mathbb{A}_F)$. For $\psi \in \Psi_2(G(N))$, we define a multi-set $\Pi_\psi = \otimes_v \Pi_{\psi_v}$ by

$$\Pi_\psi = \otimes_v \Pi_{\psi_v} := \{ \otimes_v \pi_v | \pi_v \in \Pi_{\psi_v}, J_{\psi_v}(\pi_v) = 1 \text{ for almost all } v \}$$

and for $\eta \in \mathfrak{S}_\psi = \{ \eta \in \mathfrak{S}_\psi | \eta(-1, \cdots, -1) = 1 \}$, we define

$$\Pi_\psi(\eta) = \{ \otimes_v \pi_v \in \Pi_\psi | \otimes_v J_{\psi_v}(\pi_v) = \eta \text{ on } \mathfrak{S}_\psi \}.$$

Then, the claim of the endoscopic classification is that when we define

$$L_\psi^2 = \bigoplus_{\pi \in \Pi_\psi(\epsilon_\psi)} \pi$$

for each $\psi \in \Psi_2(G(N))$, $L_{\text{disc}}^2(G(N)(F)\backslash G(N)(\mathbb{A}_F))$ is decomposed into

$$L_{\text{disc}}^2(G(N)(F)\backslash G(N)(\mathbb{A}_F)) = \bigoplus_{\psi \in \Psi_2(G(N))} L_\psi^2,$$

where $\epsilon_\psi$ is the character which is defined by a root number.

By the facts in 2.3, $\Psi_2(G(N))$ is bijective to the set of nearly equivalence classes of irreducible discrete automorphic representations of $G(N)(\mathbb{A}_F)$.

Remark 2.1. Today the results are generalized for non-quasisplit cases by Kaletha, Minguez, Shin and White[14].

3 Ikeda lifts and Miyawaki lifts

In this section, we redefine Ikeda lifts and Miyawaki lifts in terms of the endoscopic classification. We denote by $\epsilon_{E/F} = \otimes_v \epsilon_{E/F_v}$ the character of $\mathbb{A}_E^\times/F^\times$ which corresponds to the extension $E/F$ by the class field theory.

3.1 Ikeda lifts

We fix $n \in \mathbb{Z}_{>1}$. Let $\pi$ be an irreducible cuspidal representation of GL$_2(\mathbb{A}_F)$ with central character $\epsilon_{E/F}^{n-1}$. We denote by $\phi = \phi_\pi$ the (weak) base change lift of $\pi$ to GL$_2(\mathbb{A}_E)$. We fix an automorphic character $\chi$ of $\mathbb{A}_E^\times$ which satisfies that $\chi|_{\mathbb{A}_E^\times} = \epsilon_{E/F}^{n-1}$. Then we can regard $\pi \boxtimes \chi$ as an irreducible cuspidal representation of GU$(1, 1)(\mathbb{A}_F)$ by

$$(\text{GL}_2 \times \text{Res}_{E/F} \text{GL}_1)/\{(a1_2, a^{-1}) \mid a \in \text{GL}_1\} \simeq \text{GU}(1, 1)$$

$$(g, z) \mapsto gz.$$

Let $\phi_\chi = \phi_{\pi, \chi} = \phi \otimes (\chi \circ \text{det})$. Then, the image of the following restriction as functions

$$\pi \boxtimes \chi \rightarrow L^2(\text{U}(1, 1)(F)\backslash \text{U}(1, 1)(\mathbb{A}_F))$$

is an subrepresentation of $L_{\phi_\chi}^2$. So we conclude $\psi = \psi_{\pi, n} = \phi[n]$ is a global discrete $\mathbb{A}$-parameter of U$(n, n)$.

Since $S_{\phi_\chi} \simeq S_\psi(=: S)$ and $S_{(\phi_\chi)_v} \simeq S_{\psi_v}(=: S_v)$ for any $v$, and $\epsilon_{\phi_\chi} = \epsilon_\psi = 1$, we can redefine Ikeda lifts as follows.

Definition 3.1. For an irreducible subrepresentation $\tau = \otimes_v \pi((\phi_\chi)_v, \eta_v)$ of $L_{\phi_\chi}^2$, we define the Ikeda lift $I_n(\tau) = I_n(\tau, \chi)$ of $\tau$ by

$$I_n(\tau) = \otimes_v \pi(\psi, \eta_v).$$
3.2 Types of $\phi$ and $\phi_v$

For convenience, let us classify the parameter $\phi$ by its shape:

**type 1.** $\phi = \mu_2$, where $\mu_2$ is self-dual and conjugate self-dual irreducible cusp representation of $\GL_2(\mathbb{A}_E)$ with parity $(-1)^n$. Then, $S = O(1, \mathbb{C})$.

**type 2.** $\phi = \mu_1 \square \mu_1^{-1}$ where $\mu_1$ is an automorphic character of $\mathbb{A}_E^\times$ which satisfies $\mu_1|_{\mathbb{A}_E^\times} = \epsilon_{E/F}^n$ and $\mu_1 \neq \mu_1^{-1}$. Then, $S = O(1, \mathbb{C})^2$.

Let $v$ be a place of $F$ which does not split over $E$. Let us classify the parameter $\phi_v$ similar to the global one:

**type 1-1.** $\phi_v = \mu_2$, where $\mu_2$ is self-dual and conjugate self-dual irreducible 2-dimensional representation of $W_{F_v}$ with parity $(-1)^n$. Then, $S_v = O(1, \mathbb{C})$.

**type 1-2.** $\phi_v = \mu_1[2]$ where $\mu_1$ is character of $W_{F_v}$ which satisfies $\mu_1|_{W_{F_v}} = \epsilon_{E_v/F_v}^{n-1}$. Then, $S_v = O(1, \mathbb{C})$.

**type 2-1.** $\phi_v = \mu_1 \oplus \mu_1^{-1}$ where $\mu_1$ is an character of $W_{F_v}$ which satisfies $\mu_1|_{W_{F_v}} = \epsilon_{E_v/F_v}^n$ and $\mu_1 \neq \mu_1^{-1}$. Then, $S_v = O(1, \mathbb{C})^2$.

**type 2-2.** $\phi_v = 2\mu_1$ where $\mu_1$ is an character of $W_{F_v}$ which satisfies $\mu_1|_{W_{F_v}} = \epsilon_{E_v/F_v}^n$ and $\mu_1 = \mu_1^{-1}$. Then, $S_v = O(2, \mathbb{C})$.

**type 3.** $\phi_v = \xi \oplus \xi^{-1}$ where $\xi$ is an character of $W_{F_v}$ which satisfies $\xi|_{F_v} \neq \epsilon_{E_v/F_v}^n$ and $\xi^c = \xi$. Then, $S_v = \text{SL}(2, \mathbb{C})$ or $\GL(1, \mathbb{C})$.

Let us make some comments about them:

- If $\phi$ is type 2, then $\phi_v$ is only type 1-1 or 2-2 for any $v$.
- For each archimedean $v$, $\phi_v$ is only type 2-1, 2-2 or 3.

3.3 Some properties of A-packet $\Pi_{\psi_v}$

For the following sections, we show some basic properties of $\Pi_{\psi_v}$.

**Lemma 3.2.** For each non-archimedean place $v$, each element of $\Pi_{\psi_v}$ is irreducible.

**Proof.** We can assume $v$ does not split over $E$. This lemma follows from the facts in 2.4. except that $\phi_v$ is type 1-2. If $\phi_v$ is type 1-2, then the elements of $\Pi_{\psi_v}$ is exactly the the difference of composition factors of

$$\begin{cases}
\text{St}(\mu_1|_{E_v^\times}) \times 1_{U(0,0)} & \text{if } n = 2; \\
\mu_1|_{E_v^\times} \circ \det_{\GL_n(E_v)} \times 1_{U(0,0)} & \text{otherwise}
\end{cases}$$

(we see $\mu_1$ as a character of $E_v^\times$ by $E_v^\times \simeq W_{E_v}^{ab}$) and

$$\begin{cases}
\Pi_{\mu_1[3][1] \oplus \mu_1[1][1]} & \text{if } n = 2; \\
\Pi_{\mu_1[n+1] \oplus \mu_1[n][n-1]} & \text{otherwise}.
\end{cases}$$

(See [Mœg04].) Here, $\text{St}(\mu_1|_{E_v^\times}) = \text{Ind}_{B}^{\GL_n(E_v)}(\mu_1|_{E_v^\times}) \times 1_{U(1)(F_v)}$ where $B$ is the set of upper triangular matrices and $\mu_1$ is defined by $\mu_1(\alpha/\alpha^c) = \mu_1(\alpha)$ for $\alpha/\alpha^c \in U(1)(F_v)$ ($\alpha \in E_v^\times$). It is well known that the length of $\text{St}(\mu_1|_{E_v^\times})$ and $\mu_1|_{E_v^\times} \circ \det_{\GL_n(E_v)}$ are 3 (see [Kon01] and [KS97]).

**Lemma 3.3.** If $n$ is even, the Ikeda lift does not depend on global Whittaker data.
Proof. It immediately follows from \( \# \Pi'_{\psi_v} = 1 \) for each \( v \) and facts in 2.4.

**Lemma 3.4.** Let \( n \) be even. Then, for any non-archimedean \( v \), each element of \( \Pi_{\psi_v} \) is fixed by the actions of \( \text{GU}(n,n)(F_v) \) induced by the conjugation.

**Proof.** If \( v \) splits over \( E \), then it follows from

\[
\text{GU}(n,n)(F_v) = E_v^n \text{U}(n,n)(F_v)
\]

and \( \Pi_{\psi_v} = \Pi'_{\psi_v} \). So we can suppose that \( v \) does not split over \( E \). We fix \( a \in F^\times \setminus N_{E_v/F_v}(E_v^n) \). Then, \( \{ 1_{2n}, \varepsilon_a = ( \begin{smallmatrix} 1 & 0 \\ 0 & a_{1n} \end{smallmatrix} ) \} \) is a complete system of representatives of

\[
\text{GU}(n,n)(F_v)/E_v^n \text{U}(n,n)(F_v) \simeq \{ \pm 1 \}.
\]

Firstly \( \pi(\psi_v, 1_{\mathbb{G}_v}) \) is the the unique quotient of

\[
\tau_{\phi_v} | \det \frac{\varphi_{\pi}}{\tau_{\phi_v}} | \det \frac{\varphi_{\pi}}{\tau_{\phi_v}} \cdots | \det \frac{\varphi_{\pi}}{\tau_{\phi_v}} | \det \frac{1}{2} \simeq 1,
\]

where \( \tau_{\phi_v} \) is the irreducible representation of \( \text{GL}_2(E_v) \) which corresponds to \( \phi_v \). Since \( \varepsilon_a \) acts trivially on the Levi factor \( \text{GL}_2(E_v) \), \( \tau_{\phi_v} | \det \frac{\varphi_{\pi}}{\tau_{\phi_v}} | \det \frac{\varphi_{\pi}}{\tau_{\phi_v}} \cdots | \det \frac{\varphi_{\pi}}{\tau_{\phi_v}} | \det \frac{1}{2} \simeq 1 \) and its unique quotient, so they are fixed by \( \text{GU}(n,n)(F_v) \) as representation.

We take \( \Pi' \in \Pi_{\psi_v} \) in general. Then there exists an irreducible subrepresentation \( \Pi'' \subset L_\psi^n \) which satisfies that \( \Pi'' = \Pi' \). \( \Pi'' \) are automorphic representations of \( \text{U}(n,n)(\mathbb{A}_F) \) and \( \Pi'' \simeq \Pi' \) for almost all places \( v' \). So \( \Pi'' \subset L_\psi^n \) and we conclude \( \Pi'' \in \Pi_{\psi_v} \).

**Remark 3.5.** Lemma 3.4 does not hold if \( n \) is odd. The fact is maybe one reason that Yamana’s construction [Yam19] is not for \( \text{U}(n,n) \) but \( \text{GU}(n,n) \).

### 3.4 Miyawaki lifts

Let \( n, \pi \) be as above. Let \( m < n \) be a positive integer and let \( V_1 \) is non-degenerate \( m \)-dimensional hermitian space over \( E \) and \( V_2 = V_1^\perp \perp \mathbb{H}_E^{n-m} \), where \( \mathbb{H}_E \) be the non-degenerate 2-dimensional isotropic hermitian space and \( V_1^\perp \) is the hermitian space which satisfies \( V_1 \perp V_1^\perp = \mathbb{H}_E^n \). Then, \( \text{U}(n,n) \simeq \text{U}(V_1 \perp V_2) \supset \text{U}(V_1) \times \text{U}(V_2) \).

For simplicity, \( \text{U}(V_1) \) and \( \text{U}(V_2) \) are quasisplit. Let \( \psi \) is a discrete \( \text{A} \)-parameter of \( \text{U}(V_1) \). Then, for subrepresentations \( \pi_1 \subset L_\psi^2 \) and \( \pi_2 \subset L_\psi^2 \), we define \( \mathcal{M}_{\psi}((\psi')) \) by

\[
\mathcal{M}_{\psi}((\psi')) = \psi[n-m] \mathbb{H} \psi' \]

and let \( \mathcal{M}_{\pi_1}(\pi_2) \) be the representation of \( \text{U}(V_2)(\mathbb{A}_F) \) generated by

\[
\mathcal{M}_{f_1}(f_2)(g) := \int_{\text{U}(V_1)(F) \setminus \text{U}(V_1)(\mathbb{A}_F)} f_1((\psi_1) \times \psi_2)(g, h) \overline{f_2(h)} dh
\]

for \( f_1 \in \pi_1, f_2 \in \pi_2 \) if it can be defined. For them, the following holds.

**Proposition 3.6.** We assume that \( \mathcal{M}_{\pi_1}(\pi_2) \subset L^2(\text{U}(V_2)(F) \setminus \text{U}(V_2)(\mathbb{A}_F)) \) and \( \mathcal{M}_{\psi}((\psi')) \subset \Psi_2(\text{U}(V_2)) \). Then,

\[
\mathcal{M}_{\pi_1}(\pi_2) \subset L^2_{\mathcal{M}_{\psi}((\psi'))}.
\]

**Proof.** By [AK18, §2] (the proofs of the propositions also works when \( m \) is odd), the nearly equivalence class of \( \mathcal{M}_{\pi_1}(\pi_2) \) is correspond to \( \mathcal{M}_{\psi}((\psi')) \). By [Art11], the intersection of \( \mathcal{M}_{\pi_1}(\pi_2) \) and the continuous spectrum is zero.

In such case, we call \( \mathcal{M}_{\pi_1}(\pi_2) \) the Miyawaki lift of \( \pi_1 \) with respect to \( \pi_2 \). The definition is a representation-theoretical redifnition for [AK18] and the unitary group analogue for [Ato17].

**Remark 3.7.** In above proposition, we only see nearly equivalence classes of automorphic representations. So it is easy to generalize the result for non-quasisplit cases.
4 Hermitian Maass lifts

The aim of this section is to describe Ikeda lifts of degree 2 by theta lifts. We fix $n = 2$ and $\pi, \chi, \phi, \phi_\chi, \psi$ be as §3 and $\psi_F = \otimes_v \psi_{F_v}$ be fixed non-trivial additive character of $k_F/F$. So let $w_{2k} = (B_{2k}, \otimes_v \lambda_1 \phi_{F_v})$ (see §2) and we assume that the maps from local A-packet to characters of component groups are always defined with respect to $w_{2m}$ (for odd unitary groups, they are defined canonically).

Let

\[ S_0 = \begin{pmatrix} 2 & 0 \\ 0 & -2d \end{pmatrix}, \quad S_m = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad (m > 0) \]

and

\[
\begin{align*}
\text{GO}(m + 2, m) &= \{ g \in \text{GL}_{2m+2} \mid \gamma g S_l g = \nu_m'(g) S_m, \nu_m'(g) \in \text{GL}_1 \}, \\
\text{O}(m + 2, m) &= \ker(\nu_m'), \\
\text{GSO}(m + 2, m) &= \{ g \in \text{GO}(m + 2, m) \mid \det g = \nu_m'(g)^{m+1} \}, \\
\text{SO}(m + 2, m) &= \text{O}(m + 2, m) \cap \text{GSO}(m + 2, m).
\end{align*}
\]

We often denote $\nu_m'$ by $\nu$ for short.

Let

\[
B_{m}^{\text{GO}} = \begin{pmatrix} (\text{GL}_1 & * & * & * & * \\ \vdots & * & * & * & * \\ \text{GL}_1 & * & * & * \\ \text{GO}(2, 0) & * & * \\ \text{GO}(2, 0) & * & * & \cdots \end{pmatrix} \subset \text{GO}(m + 2, m)
\]

and

\[
B_{t}^{\text{GO}} = B_{t}^{\text{GO}} \cap \text{O}(m + 2, m), \\
B_{t}^{\text{GSO}} = B_{t}^{\text{GO}} \cap \text{GSO}(m + 2, m), \\
B_{t}^{\text{SO}} = B_{t}^{\text{GO}} \cap \text{SO}(m + 2, m).
\]

For a place $v$ of $F$, $\{i_i\}_{i=1}^k \subset \mathbb{Z}_{>0}$, $m_0 = m - \sum i_i$, representations $\tau_i$ of $\text{GL}_{i_i}(F_v)$, and a representation $\pi_0$ of $\text{GSO}(m_0 + 2, m_0)(F_v)$, let

\[
\tau_1 \times \tau_2 \times \cdots \times \tau_k \times \pi_0 = \text{Ind}_{P^{\text{GSO}}(F_v)}^{\text{GSO}(m+2,m)(F_v)} \tau_1 \boxtimes \tau_2 \boxtimes \cdots \boxtimes \tau_k \boxtimes \pi_0,
\]

where $P^{\text{GSO}}$ is the standard parabolic subgroup (for $B_{m}^{\text{GSO}}$) whose Levi subgroup $M'$ is defined by

\[
M^{\text{GSO}} = \begin{pmatrix} (\text{GL}_{i_1} & \cdots & \text{GL}_{i_k} \\ \text{GU}(m_0 + 2, m_0) & * & \cdots \end{pmatrix}.
\]

For $\text{SO}(m + 2, m)$, we define $\tau_1 \times \tau_2 \times \cdots \times \tau_k \times \pi_0$ similarly.
For each place $v$ of $F$, let
\[
\text{GU}(1, 1)_{F_v}^+ = \ker(\epsilon_{E_v/F_v} \circ \nu) \subset \text{GU}(1, 1)(F_v), \quad \text{GL}^+_{2, F_v} = \text{GU}(1, 1)_{F_v}^+ \cap \text{GL}(2, F_v)
\]
and let
\[
\text{GU}(1, 1)_{A_F}^+ = \text{GU}(1, 1)(A_F) \cap \prod_v \text{GU}(1, 1)_{F_v}^+, \quad \text{GL}^+_{2, A_F} = \text{GU}(1, 1)_{A_F}^+ \cap \text{GL}(2, A_F).
\]

### 4.1 Overview

Let $\tau = \otimes_v \pi((\phi_\chi)_v, \eta_v)$ be a subrepresentation of $L^2_{\phi_\chi}$ (necessarily irreducible and cuspidal) and let $\tau^+ = \tau \boxtimes \chi|_{\text{GL}^+_2(A_F)}$, where $\tau \boxtimes \chi$ is assumed as a representation of $\text{GU}(1, 1)_{A_F}^+$ by $\text{GU}(1, 1)_{A_F}^+ = A^+_E \text{U}(1, 1)(A_F)$.

We see $\tau^+$ as a space of functions on $\text{GL}^+_2(A_F)$, which are left $\text{GL}^+_2(F) = \text{GL}_2(F) \cap \text{GL}^+_2(A_F)$-invariant.

Let $\theta_2(\tau^+)$ be the theta lift of $\tau^+$ to $\text{GO}(4, 2)$. By [Yam14], $\theta_2(\tau^+)$ is non-zero and square integrable modulo center. Since the central character of $\tau^+$ is $\epsilon_{E/F}$, $\theta_2(\tau^+)$ has the trivial central character. So let $\mathcal{H}(\tau^+)$ be the image of the following map
\[
\begin{align*}
\theta_2(\tau^+) &\to L^2(U(2, 2)(F) \setminus U(2, 2)(A_F)) \\
f &\mapsto f \circ i|_{U(2, 2)(A_F)}
\end{align*}
\]
where $i$ be the isomorphism $i : \text{PGU}(2, 2) \simeq \text{PGSO}(4, 2)$. $\mathcal{H}(\tau^+)$ is often called the Hermitian Maass lift of $\tau^+$.

What we going to prove in this section is the following.

**Theorem 4.1.** $\mathcal{H}(\tau^+) = I_2(\tau)$.

### 4.2 Isomorphism $\text{PGSO}(4, 2) \simeq \text{PGU}(2, 2)$ (cf. [Mor14])

Let
\[
V = \{B(x, y, z, w, a, b) | x, y, z, w, a, b \in F\},
\]
where
\[
B(x, y, z, w, a, b) = \begin{pmatrix}
0 & -x & y & -a + b \kappa \\
x & 0 & a + b \kappa & z \\
-\gamma & -a - b \kappa & 0 & w \\
a - b \kappa & -z & -w & 0
\end{pmatrix}.
\]

For $B(x, y, z, w, a, b), B(x', y', z', w', a', b') \in V$, let
\[
(B(x, y, z, w, a, b), B(x', y', z', w', a', b'))_V = \text{Tr}(B(x, y, z, w, a, b)J_4^B(x', y', z', w', a', b')^cJ_4).
\]

Then,
\[
(B(x, y, z, w, a, b), B(x', y', z', w', a', b'))_V = -2(2aa' - 2dbb' + xw' + wx' + yz' + zy')
\]
\[
= -2 \begin{pmatrix} x' & y' & a' & b' & z' & w' \end{pmatrix} S_2 \begin{pmatrix} x \\ y \\ a \\ b \\ z \\ w \end{pmatrix}
\]
so that $\text{GO}(V, (,)_V) \simeq \text{GO}(4, 2)$. Further, the following map
\[
\text{PGU}(2, 2) \to \text{PGSO}(V, (,)_V) \\
g \mapsto [B \mapsto gB'g]
\]

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is well-defined and bijective. We denote the map
\[
PGU(2, 2) \xrightarrow{\sim} PGSO(V, \langle , \rangle_\nu) \xrightarrow{\sim} PGSO(4, 2)
\]
by \(i\).

For the following, let us see the images of some elements of GU(2, 2):
\[
i \begin{pmatrix} 1 & \lambda \\ 1 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 1 \\ 1 & \lambda \end{pmatrix}, \quad \lambda \in GL_1,
\]
\[
i \begin{pmatrix} \alpha & \beta \\ (\beta^-)^{-1} & (\alpha^-)^{-1} \end{pmatrix} = \begin{pmatrix} N(\alpha \beta) & N(\alpha) \\ \alpha \beta & N(\beta) \end{pmatrix}, \quad \alpha, \beta \in \text{Res}_{E/F} GL_1,
\]
\[
i \begin{pmatrix} \alpha & A \\ (\alpha^-)^{-1} \det A \end{pmatrix} = \begin{pmatrix} (\text{Res}_{E/F}(\alpha) \det A)^{-1} A & \alpha \end{pmatrix}, \quad \alpha \in \text{Res}_{E/F} GL_1, A \in GL_2.
\]

Henceforth, we often identify \(\alpha = a + b\kappa \in \text{Res}_{E/F} GL_1\) and \((\begin{smallmatrix} a & d \\ b & a \end{smallmatrix}) \in GSO(2, 0)\) as above.

### 4.3 Theta lifts

To avoid confusion, we write the formulas of the Weil representations what we use explicitly. Let \(\psi_E = \otimes_v \psi_{E_v} = \otimes_v \psi_{F_v} \circ \text{Tr}_{E/F}\). For \(k, m \in \mathbb{Z}_{\geq 0}\), let
\[
R_k = \{(g, h) \in GU(1, 1) \times GU(k + 1, k) \mid \nu(g) = \nu(h)\}
\]
and
\[
R'_m = \{(g, h) \in GL_2 \times GO(m + 2, m) \mid \nu(g) = \nu(h)\}.
\]

#### Unitary group case.
Firstly, we define local things. Let \(v\) be a place of \(F\) and let \(\rho\) be a character of \(E_v^\times\) which satisfies that \(\rho|_{F_v^\times} = \epsilon_{E_v/F_v}\). We define the Weil representation \(\omega_v^{(k, \rho)}\) of \(U(1, 1)(F_v) \times U(k + 1, k)(F_v)\) on \(S(E_v^{2k+2})\) by
\[
\omega_v^{(k, \rho)}(1, h) \varphi(x) = \varphi(h^{-1}x), \quad h \in U(k + 1, k)(F_v),
\]
\[
\omega_v^{(k, \rho)} \left( \begin{pmatrix} a & b \\ 0 & (a^{-1})^* \end{pmatrix} , 1 \right) \varphi(x) = \rho(a) |a|_{E_v}^{(2k+1)/2} \varphi(a^c x), \quad a \in E_v^\times,
\]
\[
\omega_v^{(k, \rho)} \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} , 1 \right) \varphi(x) = \varphi(x) \psi_{E_v} \left( \frac{1}{2} b^c J_{2k+1} x \right), \quad b \in F_v,
\]
\[
\omega_v^{(k, \rho)} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , 1 \right) \varphi(x) = \gamma_v^{-1} \int_{E_v^{2k+1}} \varphi(-y) \psi_{E_v} \left( \frac{x^c J_{2k+1} y}{2} \right) dy
\]
for \(\varphi \in S(E_v^{2k+2})\) and \(x \in E_v^{2k+2}\). Here, \(dy\) is the normalized Haar measure and \(\gamma_v\) is the Weil constant. Further, we define \(L\) by
\[
L(h) \varphi(x) = |\nu(h)|_{E_v^{(2k+1)/2}} \varphi(h^{-1} x) \quad h \in GU(k + 1, k)(F_v),
\]
and we extend $\omega^{(k,\rho)}_v$ on $R_k(F_v)$ by
$$\omega^{(k,\rho)}_v(g, h) = \omega^{(k,\rho)}_v(g \left( \begin{smallmatrix} 1 & 0 \\ 0 & \nu(h)^{-1} \end{smallmatrix} \right), 1) \circ L(h)$$
for $(g, h) \in R_k(F_v)$ (see [Rob96]). Let $\Omega^{(k,\rho)}_v = \text{ind}_{R_k(F_v)}^{G_k(U(k+1, k)(F_v))} \omega^{(k,\rho)}_v$, where ind is compact induction.

If $\sigma'$ is an irreducible admissible representation of $U(k+1, k)(F_v)$ (resp. $GU(k+1, k)(F_v)$), the maximal $\sigma'$-isotypic quotient of $\omega^{(k,\rho)}_v$ (resp. $\Omega^{(k,\rho)}_v$) has the form
$$\Theta^\rho(\sigma') \boxtimes \sigma'$$
where $\Theta^\rho(\sigma')$ is some smooth representation of $U(1, 1)(F_v)$ (resp. $GU(1, 1)(F_v)$). Let $\theta^\rho(\sigma')$ be the maximal semisimple quotient of $\Theta^\rho(\sigma')$. It is known that $\Theta^\rho(\sigma')$ has finite length and $\theta^\rho(\sigma')$ is irreducible if $\Theta^\rho(\sigma') \neq 0$. We note that the central character of $\Theta^\rho(\sigma')$ is $\chi^\sigma(\sigma')$ defined by $\chi^\sigma(\sigma') = \chi^\sigma(\sigma')^{-1}$ for $\alpha \in E^\times_v$. Moreover, if $\sigma'$ be an irreducible admissible representation of $GU(k+1, k)(F_v)$, then $\Theta^\rho(\sigma') = \chi^\sigma(\sigma')$ and $\theta^\rho(\sigma') = \chi^\sigma(\sigma')$ by the same way of [GGS90, Lemma3.1], where $\sigma'_0 = \sigma'|_{U(1, 1)(F_v)}$.

Secondly, we define global things. Let $\rho = \otimes_v \rho_v$ be an automorphic character of $\mathbb{A}_F^\times$ which satisfies that $\rho|_{A_F} = \epsilon_E/F$. Let $\omega_v^{(k,\rho)} = \otimes_v \omega_v^{(k,\rho_v)}$. For $\varphi \in \mathcal{S}(E_v^{2k+1})$ and a cuspid form $f$ of $GU(k+1, k)(A_F)$, let
$$\theta^\rho_v(g, h) = \sum_{\gamma \in E_v^{2k+1}} \omega^{(k,\rho)}_v(g, h) \varphi(\gamma)$$
and
$$\theta^\rho_v(f)(g') = \int_{U(k+1, k)(F_v)/U(k+1, k)(A_F)} \theta^\rho_v(g', h_0 h') f(h_0 h') dh_0$$
g' $\in GU(1, 1)(A_F)$, where $h' \in GU(k+1, k)(A_F)$ satisfies $\nu(h') = \nu(g')$ (we note that the integral does not depends on the choice of $h'$). For an irreducible cuspidal representation $\sigma'$ of $GU(k+1, k)(A_F)$, let $\theta^\rho(\sigma')$ be the representation of $GU(1, 1)(A_F)$ generated by $\{\theta^\rho_v(f) \mid \varphi \in \mathcal{S}(E_v^{k+1}), f \in \sigma'\}$.

**Symplectic-orthogonal case.** Let $v$ be a place of $F$. We define the Weil representation $\omega_v^m$ of $SL_2(F_v) \times O(m+2, m)(F_v)$ on $S(F_v^{2m+2})$ by

$$\omega_v^m(1, h) \varphi(x) = \varphi(h^{-1} x), \quad h \in O(m+2, m)(F_v),$$

$$\omega_v^m(\left( \begin{array}{cc} a & b \\ 0 & a^{-1} \end{array} \right), 1) \varphi(x) = \epsilon_{E_v/F_v}(a)a_v^{m+1} \varphi(ax), \quad a \in F_v^\times,$$

$$\omega_v^m(\left( \begin{array}{cc} b & 0 \\ 1 & 0 \end{array} \right), 1) \varphi(x) = \varphi(x) \psi_{F_v}(\frac{1}{2} b \bar{x} s_m x), \quad b \in F_v,$$

$$\omega_v^m(\left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right), 1) \varphi(x) = \gamma^{-1} \int_{F_v^{m+2}} \varphi(-y') \psi_{F_v}(\bar{x} s_m y') dy'$$

For $\varphi \in \mathcal{S}(F_v^{2m+2})$ and $x \in F_v^{2m+2}$. Here, $dy'$ is the normalized Haar measure. We extend $\omega_v^m$ for $R'_m(F_v)$ and define $\Omega_v^m$ in the same way as unitary group case. If $\tau'$ is an irreducible admissible representation of $SL_2(F_v)$ (resp. $GL_2(F_v)$), the maximal $\tau'$-isotypic quotient of $\omega_v^m$ (resp. $\Omega_v^m$) has the form
$$\Theta^m(\tau') \boxtimes \tau'$$
where $\Theta^m(\tau')$ is some smooth representation of $SL_2(F_v)$ (resp. $GL_2(F_v)$). Let $\theta^m(\tau')$ be the maximal semisimple quotient of $\Theta^m(\tau')$. We note that the central character of $\Theta^m(\tau')$ is $\chi^m(\tau') \epsilon_{E_v/F_v}(1)$ (resp. $\chi^m(\tau') \epsilon_{E_v/F_v}(1)$), where $\chi^m(\tau')$ is the central character of $\tau'$.

Let $\omega^m = \otimes_v \omega_v^m$. For $\varphi \in \mathcal{S}(A_F^{2m+2})$ and a cuspid form $f$ of $GL_2(F_v)$, let
$$\theta^m_v(g, h) = \sum_{\gamma \in F_v^{2m+2}} \omega^m(g, h) \varphi(\gamma)$$
and
$$\theta^m_v(f)(g') = \int_{GL_2(F_v)} \theta^m_v(g', h_0 h') f(h_0 h') dh_0$$
g' $\in GL_2(F_v)$, where $h' \in GL_2(F_v)$ satisfies $\nu(h') = \nu(g')$ (we note that the integral does not depends on the choice of $h'$). For an irreducible cuspidal representation $\tau'$ of $GL_2(F_v)$, let $\theta^m(\tau')$ be the representation of $GL_2(F_v)$ generated by $\{\theta^m_v(f) \mid \varphi \in \mathcal{S}(F_v^{2m+2}), f \in \tau'\}$.
Let $\tau_\mathbb{P}$ proposiiton 4.2. representations. For each finite place $v$ of $F$, let $\omega_v$ be a prime element of $F_v$ and $q_v$ be the cardinality of residue field of $F_v$. Then, we can realize $\tau_v$ unramified. Then, we can realize $\tau_v$ characters $\chi_v$. If $v$ is unramified over $E_v$, we can take $\chi_1$ and $\chi_2$ as $\chi_1 = q_v^{-1}$ and $\chi_2 = \chi_1$. By using the result of [Ral82] for $\tau_{v|SL_2(F_v)}$ and 4.2.

To show the proposition, what we should do is to determine explicit the theta correspondence for unramified representations. Let $v$ be a place of $F$ which splits or unramified over $E$ and where $\tau_v$, $\chi_v$ and $\theta(\tau_v^+)_{|GSO(4,2)(F_v)}$ are unramified. Then, we can realize $\tau_v$ as a subquotient of $\chi_v \chi_1 \times 1_{U(0,0)}(F_v)$ for some unramified character $\chi'$ of $E_v$. Similarly, we can realize $\theta(\tau_v^+)_{|GSO(4,2)(F_v)}$ as a subquotient of $\chi_1 \times \chi_2 \times E_v$. Then, the following holds.

**Lemma 4.3.** $\theta(\tau_v^+)_{|GSO(4,2)(F_v)} \circ i$ is a subquotient of $\chi_1 \times E_v \times E_v \times E_v$.

**Proof.** If $v$ is unramified over $E$, we can take $\chi_1$ and $\chi_2$ as $\chi_1 = q_v^{-1}$ and $\chi_2 = \chi_1$. By using the result of [Ral82] for $\tau_{v|SL_2(F_v)}$ and 4.2.

Even if $v$ is split over $E$, it is essentially the same (see [Ich04]).

Proposition 4.2 follows from 4.3, immediately.

### 4.5 Determination of local components

Let $v$ be a place of $F$. For $\tau' \in \Pi(\phi_v)$, we define $\tau'^+ = \tau' \boxtimes \chi_v|_{GL_2,v}$. Then, in this subsection, we show the following.

**Theorem 4.4.** Let $\pi((\phi_v,\eta')) \in \Pi(\phi_v)$. Then,

$$\theta^2(\pi((\phi_v,\eta'))^+)|_{U(2,2)(F_v)} = \pi(\psi_v,\eta').$$

We make some remarks:

- Any subrepresentation of $\theta^2(\pi((\phi_v,\eta'))^+)|_{U(2,2)(F_v)}$ is a subrepresentation of some elements of $\Pi(\phi_v)$ by global discussion.

- By Lemma 3.4., the theorem holds if $\Pi(\phi_v)$ is singleton i.e. $v$ splits over $E$ or $\phi_v$ is type 1-1, 1-2, or 3 (in 3.2).

So we can assume that $\phi_v$ is type 2-1 or 2-2. Then, since $\widehat{S}(\phi_v) \simeq \times S(\phi_v) \simeq \{\pm 1\}$, let $\tau_{\pm 1} = \pi((\phi_v,\pm 1)$ for short.
Non-archimedean cases. Let \( v \) be a non-archimedean place of \( F \) which is non-split over \( E \) and we assume \( \phi_v \) is type 2-1 or 2-2.

Lemma 4.5. Let \( \phi_v = \mu_1 \oplus \mu_1^{-1} \) (if \( \mu_1 = \mu_1^{-1} \) then \( \phi_v \) is type 2-2). Then, \( \theta^0(\tau_{+1}^+) \neq 0 \).

Proof. By [GI16],
\[
\tau_{+1} \boxtimes \chi_v = \theta^{\chi_v \mu_1^{-1}}(\mu_1).
\]
So the lemma follows from the following see-saw:
\[
\begin{align*}
\text{GO}(2,0) & \quad \text{GU}(1,1)^+ \\
\text{GU}(1,0) & \quad \text{GL}_2^+.
\end{align*}
\]

Corollary 4.6. For \( v \) as above, Theorem 4.4 holds.

Proof. It is sufficient to show that \( \theta^2(\tau_{+1}^+)|_{U(2,2)(F_v)} \neq \pi(\psi_v, 1) \) and \( \theta^2(\tau_{+1}^+)|_{U(2,2)(F_v)} \neq \theta^2(\tau_{-1}^+)|_{U(2,2)(F_v)} \). By lemma 4.5 and the conservation relation [SZ15], \( \theta^1(\tau_{+1}^+) = 0 \) and \( \theta^2(\tau_{-1}^+) \neq 0 \) holds.

First, let \( \phi_v \) is type 2-1. Then \( \Pi(\phi_v) \) is supercuspidal L-packet. So \( \theta^2(\tau_{+1}^+) \) is supercuspidal and \( \theta^2(\tau_{-1}^+) \) is not. Since The element of \( \Pi(\phi_v) \) is not supercuspidal, so Theorem 4.4 holds.

Second, let \( \phi_v \) is type 2-2. Then, \( \{ \tau_{+1}^0 = \tau_{-1}^0|_{SL_2(F_v)} \} \) are the L-packet of \( SL_2(F_v) \) with L-parameter \( \phi_0 = 2e_{E_v/F_v} \oplus 1 \). By [GGS09],
\[
\theta^2(\tau_{+1}^0)|_{SO(4,2)(F_v)} = \theta^2(\tau_{-1}^0)|_{SO(4,2)(F_v)}.
\]

By [AG17], \( \theta^2(\tau_{-1}^0)|_{SO(4,2)(F_v)} \) is the unique quotient of
\[
\text{St}(|X|^{\frac{1}{2}}) \ltimes 1_{SO(2,0)},
\]
where \( \text{St}(X') = \text{Ind}_{B(F_v)}^{GL_2(F_v)}(X'| \cdot |X'|^{\frac{1}{2}} X'|) \) for a character \( X' \) of \( F_v^* \) (\( B = \{ * * \} \subset GL_2 \)). By the adjointness, \( \theta^2(\tau_{+1}^0)|_{GSO(4,2)(F_v)} \) is the unique quotient of
\[
\text{St}(|X|^{\frac{1}{2}}) \ltimes \cdot |X|^{\frac{1}{2}} X
\]
for some character \( X \) of \( E_v^* \) which satisfies \( X|_{E_v^*} = 1 \). Since
\[
\text{St}(|X|^{\frac{1}{2}}) \ltimes \cdot |X|^{\frac{1}{2}} X = \cdot |X|^{\frac{1}{2}} E_v X \ltimes (\text{St}(1) \boxtimes X)
\]
as representation of \( GU(2,2)(F_v) \), so \( \theta^2(\tau_{+1}^0)|_{U(2,2)(F_v)} \) is the unique quotient of
\[
\cdot |X|^{\frac{1}{2}} E_v X \ltimes ((X| \cdot |X|^{\frac{1}{2}} X_v \ltimes 1) X \circ \det)
\]
where \( \hat{X}(\alpha/\alpha^c) = X(\alpha) \) for \( \alpha \in E_v^* \). So the L-parameter of \( \theta^2(\tau_{+1}^0)|_{U(2,2)(F_v)} \) is \( X| \cdot |X|^{\frac{1}{2}} \oplus X(2) \oplus X*| \cdot |X|^{\frac{1}{2}} \). So
\[
\theta^2(\tau_{+1}^0)|_{U(2,2)(F_v)} \neq \pi(\psi_v, 1).
\]

Similarly, \( \theta_2(\tau_{+1}^0)|_{U(2,2)(F_v)} \) has L-parameter \( 2Y| \cdot |Y|^{\frac{1}{2}} \oplus 2Y*| \cdot |Y|^{\frac{1}{2}} \) for some \( Y \). Especially, \( \theta^2(\tau_{+1}^0)|_{U(2,2)(F_v)} \neq \theta^2(\tau_{-1}^0)|_{U(2,2)(F_v)} \) and Theorem 4.4 holds.
5 Proof of the main theorem

Let \( \pi, \chi, \phi, \phi_\lambda, \psi, \psi_F \) be as \( \S 4 \). Let \( \gamma \) be an automorphic character of \( U(1,0)(\mathbb{A}_F) \) with \( \Lambda \)-parameter \( \tilde{\gamma} \) which satisfies that \( \phi \neq \tilde{\gamma} \cdot \tilde{\gamma}^{-1} \). Let \( M_{\psi}(\tilde{\gamma}) = \phi \circ \tilde{\gamma}^{-1} \) and \( M_{I_4}(\gamma) \) is the Miyawaki lift of \( \gamma \) with respect to \( I_4(\tau) \) and the following diagonal embedding

\[
U(1,0) \times U(2,1) \hookrightarrow U(2,2)
\]

\[
(\alpha, h) \mapsto M \begin{pmatrix} \alpha & \varepsilon \end{pmatrix} M^{-1}
\]

where

\[
M = \begin{pmatrix} 0 & -\kappa & 0 & 0 \\
\kappa/2 & 0 & \kappa/2 & 0 \\
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}
\]

We note that in fact \( M_{I_4}(\gamma) \) does not depend on the embedding because \( I_4(\tau) \) is fixed by conjugation of \( \text{GU}(2,2) \) and each two diagonal embeddings of \( U(1,0) \times U(2,1) \) on \( U(2,2) \) are conjugate in \( \text{GU}(2,2) \).

The aim of this section is to show the following theorem. It is the essential part of the main theorem.

**Theorem 5.1.** \( M_{I_4}(\gamma) \subset L^2_{\mathcal{M}_\psi(\tau)} \). Moreover, any irreducible subrepresentation \( \sigma \) of \( M_{I_4}(\gamma) \) satisfies that \( \theta^{\kappa,\gamma^{-1}}(\sigma_v \otimes \gamma_v \circ \det) = \tau_v \) for all \( v \).

The last of the main theorem, namely the determination of \( M_{\mathcal{M}_\psi}(\tau) \), follows from [GI16](\( v \) is non-archimedean and \( E_v/F_v \) is quadratic extension), [Pan00](\( v \) is archimedean and \( E_v/F_v \) is quadratic extension), [Min08](\( v \) is non-archimedean and \( E_v = F_v \oplus F_v \)), and [AB95, Møeg89](\( v \) is archimedean and \( E_v = F_v \oplus F_v \)), straightforward. So we never discuss about that any more.

5.1 Square integrability

Firstly, we show the square-integrability of \( M_{I_4}(\gamma) \).

Let \( \mathcal{F} = \theta^\psi_\tau(f) \circ i_{U(2,2)(\mathbb{A}_F)} \in I_4(\tau) \) (\( f \in \tau^+, \varphi \in \mathcal{S}(\mathbb{A}_F^0) \)). Let \( P_0 = M_0U_0 = B_4, P_1 = M_1U_1, P_2 = M_2U_2 \) be the standard parabolic subgroups whose Levi subgroups \( M_i \) are isomorphic to \( \text{Res}_{E/F} \text{GL}_1 \times \text{Res}_{E/F} \text{GL}_2, \text{Res}_{E/F} \text{GL}_2, \text{GL}_1 \times U(1,1), \) respectively. Let \( \mathcal{F}_i \) define

\[
\mathcal{F}_i(g) = \int_{U_i(F) \backslash U_i(\mathbb{A}_F)} \mathcal{F}(ug) du
\]

and

\[
s\mathcal{F} = \mathcal{F} - \mathcal{F}_2 = \mathcal{F}_1 + \mathcal{F}_0.
\]

Then, \( \mathcal{F} = s\mathcal{F} + (\mathcal{F}_1 - \mathcal{F}_0) + \mathcal{F}_2 \). Since \( s\mathcal{F} \) is bounded, so all that is lefts are to evaluate the contribution of \( (\mathcal{F}_1 - \mathcal{F}_0) \) and \( \mathcal{F}_2 \).

**Lemma 5.2.** \( \int_{U(1,0)(F) \backslash U(1,0)(\mathbb{A}_F)} \mathcal{F}_2(M \begin{pmatrix} \alpha & \varepsilon \\
\alpha & \varepsilon \end{pmatrix} M^{-1})\gamma(\alpha)d\alpha = 0 \) for all \( \alpha \in U(2,1)(\mathbb{A}_F) \).

**Proof.** For \( \alpha \in U(1,0)(\mathbb{A}_F) \),

\[
M \begin{pmatrix} \alpha & \varepsilon \\
\alpha & \varepsilon \end{pmatrix} M^{-1} = \begin{pmatrix} 1 & (1 + \alpha)/2 & -\kappa(1 - \alpha)/4 \\
-(1 - \alpha)/\kappa & (1 + \alpha)/2 \\
0 & 0 & 1 \end{pmatrix}
\]
so the above integral is well-defined. Further, for $\beta \in \mathbb{A}_F^n$ such that $\beta/\beta^c = \alpha$,

$$i \left( M \begin{pmatrix} \alpha & 1 \\ 1 & 1 \end{pmatrix} M^{-1} \right) = \begin{pmatrix} a & -db/2 \\ -2b & a \end{pmatrix} \beta \begin{pmatrix} a & db/2 \\ 2b & a \end{pmatrix}.$$

Here, we identify $\beta = a + bk \in \mathbb{A}_F^n$ and $(a, b) \in \text{GSO}(2, 0)(\mathbb{A}_F)$. It is sufficient to show for $h = 1$. Then, by tower property $[\text{Ral}84]$,

$$F_2 \left( M \begin{pmatrix} \alpha & 1 \\ 1 & 1 \end{pmatrix} \right) \cdot \theta_2 (i(u)i(M \begin{pmatrix} \alpha & 1 \\ 1 & 1 \end{pmatrix} M^{-1})du$$

$$= \int_{\text{SL}_2(F) \setminus \text{SL}_2(\mathbb{A}_F)} \sum \gamma \omega_0(g_0g_\beta, \beta) \left( \phi \left( \frac{R}{*} \right) \right) (\gamma)dRF(g_0g_\beta)dR_0$$

where $g_\beta = \left( \begin{array}{cc} 1 & \beta^c \\ \beta & 1 \end{array} \right)$. If $\theta^0(\tau^+) = 0$, the lemma holds. If $\theta^0(\tau^+) \neq 0$, then $\phi$ is type-1 and $\theta^0(\tau^+)\langle \text{GSO}(2, 0)(\mathbb{A}_F) = \mu_1 \oplus \mu_1^{-1}$ by lemma 4.4. Since $\gamma(\alpha) = \gamma(\beta)$ and we assumed $\gamma \neq \mu_1^\pm 1$, so the lemma holds in general.

**Lemma 5.3.** $(F_1 - F_0)(U(1,0)(\mathbb{A}_F) \times U(2,1)(\mathbb{A}_F))$ is bounded.

**Proof.** Let $S_3 = \omega_3A_3K_3$ be a Siegel set (see [MW95]) of $U(2,1)(\mathbb{A}_F)$, where $\omega_3$ be a sufficiently big compact subset of $B_3(\mathbb{A}_F)$ and $A_3$ is the image of $\{ \text{diag}(a, 1, a^{-1}) \mid a \in \mathbb{R}_0^+, a > a_0 \}$ diagonal embedding on $U(2,1)(\mathbb{A}_F)$ for sufficiently small $a_0 > 0$, and $K_3$ is a compact subgroup of $U(2,1)(\mathbb{A}_F)$ which satisfies $B_3(\mathbb{A}_F)K_3 = U(2,1)(\mathbb{A}_F)$. Let $\omega_1$ is a compact set of $U(1,0)(\mathbb{A}_F)$ which satisfies that $\omega_1U(1,0)(F) = U(1,0)(\mathbb{A}_F)$. Then, there are a Siegel set $S_4 = \omega_4A_4K_4$ and a finite set $\{ i \}$ of $U(2,2)(\mathbb{A}_F)$ which satisfies $M(\omega_1, \omega_3)M^{-1} \subset \omega_4$, $M(\omega_1, \omega_3)M^{-1} \subset A_4$, and $MK_3M^{-1} \subset \cup_i K_d i$. Further, there are the finite set $\{ \phi_i \}$ of automorphic forms on $\text{GL}_2(\mathbb{A}_E)$ with some central characters which satisfies

$$|(F_1 - F_0)(pak)| < \text{const. } \sum |s\phi_i(pa)|_{\text{Res}_{E/F} \text{GL}_2}$$

for $p \in \omega_4$, $a \in A_4$, and $k \in \cup_i K_d i$, where $s\phi(g) = \phi(g) - \int_{\text{Res}_{E/F} \phi((\ast) \ast)} du$. Especially, for $g \in \text{diag}(\omega_1, S_3)$, there are some $a > a_0$, $p \in \{ (\ast, \ast) \} \subset \text{GL}_2(\mathbb{A}_E)$,

$$|(F_1 - F_0)(MgM^{-1})| < \text{const. } \sum |s\phi_i(pa)\left( a^{1/2} \begin{pmatrix} a^{-1/2} & \ast \\ \ast & a \end{pmatrix} \right)|.$$

Since $p$ struggles some compact set and $s\phi_i$ are rapidly decreasing, so the lemma holds.

So $M_{\beta}(\gamma)$ is bounded. Especially, the following holds.

**Corollary 5.4.** $M_{\beta}(\gamma) \subset L^2_{\text{Res}_{E/F}}$.  

**5.2 See-Saw**

Secondly, we describe $M_{\beta}(\gamma)$ by global theta lifts.
Lemma 5.5. Let
\[ \Gamma \] be an irreducible subrepresentation of \( L^2_{M_u(\gamma)} \). We note that \( \sigma \) is cuspidal.

Proposition 5.6. \( \sigma \subset \mathcal{M}_{I_4}(\tau)(\gamma) \iff \chi_{\gamma}^{-1} \left( \sigma \otimes \gamma \circ \det \right) = \tau \)

Proof. Let \( \Gamma \) be a automorphic character of \( \mathbb{A}_F^\times \) which satisfies that \( \Gamma|_{U(1,0)(\mathbb{A}_F)} = \gamma \) and \( \Sigma = \sigma \boxtimes \Gamma^{-1} \) be the extension of \( \sigma \) on \( GU(2,1)(\mathbb{A}_F) \). Then, for \( f \in \sigma, \varphi \in S(\mathbb{A}_F^\times) \) and \( f' \in \tau, \)

\[
\int_{GU(2,1)(\mathbb{A}_F) \backslash GU(2,1)(\mathbb{A}_F)} \chi_{\varphi}^{-1}(f \otimes \gamma \circ \det)(g) \frac{dg}{M^{-1}} = \int_{GU(2,1)(\mathbb{A}_F) \backslash GU(2,1)(\mathbb{A}_F)} \chi_{\varphi}^{-1}(f \otimes \gamma \circ \det)(g) \frac{dg}{M^{-1}} \]

by see-saw identity.
5.3 Reduction to local theta lifts

Lastly, we reduce the non-vanishing of global theta lifts to local theta lifts by using Yamana’s result [Yam14].

Proposition 5.7. For an irreducible subrepresentation $\sigma'$ of $L^2_{\mathcal{M}_V(\gamma) \otimes \gamma_{d}^{\text{det}}}$, if $\theta^{\kappa_{V_0}^{-1}}(\sigma'_v) \neq 0$ for all $v$ then $\theta^{\kappa_{V_0}^{-1}}(\sigma') \neq 0$.

Proof. By [Yam14], and the functional equation

$$L(\sigma', s) = \epsilon(\sigma', s)L(\sigma'^{\vee}, 1 - s),$$

of doubling L-functions, what we need to show is $L(\sigma'^{\vee}, s)$ has pole at $s = 1$ and has no pole at $s = 3/2, 2, 5/2, \ldots$. Let $S_{\text{fin}}(\text{resp. } S_{\infty})$ be the set of finite(resp. infinite) places of $F$ which satisfies $E_v \neq F_v \oplus F_v$ and $\sigma'_v$ is not unramified. Then, we have

$$L(\sigma'^{\vee}, s) = L^{GJ}(\phi \otimes \gamma^{-1} \circ \text{det}, s)\zeta_E(s) \prod_{v \in S_{\text{fin}}} L(\sigma'^{\vee}_v, s)(1 - q_{E_v}^{s}) \prod_{v \in S_{\infty}} L^{GJ}(\phi_v \otimes \gamma_{v}^{1}, s)2(2\pi)^{-s}\Gamma(s),$$

where $L^{GJ}$ is Godement-Jaquet L-function, $\zeta_E(s)$ is the complete Dedekind Zeta function of $E$, and $q_{E_v}$ is the cardinality of the residue field of $E_v$. It is known that

- $L^{GJ}(\phi \otimes \gamma^{-1} \circ \text{det}, s)$ is holomorphic on $\text{Re}(s) > 1$ and non-zero at $s = 1$ ([JS76]) since $\phi \neq \gamma \boxtimes \gamma^{-1}$,
- $\zeta_E(s)$ is holomorphic on $\text{Re}(s) > 1$ and has simple pole at $s = 1$,
- for $v \in S_{\text{fin}} \sqcup S_{\infty}$, $L(\sigma'^{\vee}_v, s)$ is meromorphic function with no zeros,
- for $v \in S_{\text{fin}}$, $L^{GJ}(\phi_v \otimes \gamma_{v}^{1}, s)$ is finite product of some factors $(1 - q_{E_v}^{s})^{-1}$ where $|\text{Ref}| \leq 1/2$,
- for $v \in S_{\infty}$, $L^{GJ}(\phi_v \otimes \gamma_{v}^{1}, s)$ is finite product of some factors $2(2\pi)^{-s(s+\lambda+k/2)}\Gamma(s + \lambda + k/2)$ where $|\text{Ref}| \leq 1/2$ and $k \in \mathbb{Z}_{\geq 0}$.

So, $L(\sigma'^{\vee}, s)$ has a pole at $s = 1$ and has no pole at $s = 3/2, 2, 5/2, \ldots$.

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