The Chromatic Number of Random Regular Graphs

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Abstract. Given any integer \( d \geq 3 \), let \( k \) be the smallest integer such that \( d < 2k \log k \). We prove that with high probability the chromatic number of a random \( d \)-regular graph is \( k \), \( k + 1 \), or \( k + 2 \).

1 Introduction

In [10], Luczak proved that for every real \( d > 0 \) there exists an integer \( k = k(d) \) such that w.h.p.¹ \( \chi(G(n, d/n)) \) is either \( k \) or \( k+1 \). Recently, these two possible values were determined by the first author and Naor [4].

Significantly less is known for random \( d \)-regular graphs \( G_{n,d} \). In [6], Frieze and Luczak extended the results of [9] for \( \chi(G(n, p)) \) to random \( d \)-regular graphs, proving that for all integers \( d > d_0 \), w.h.p.

\[
|\chi(G_{n,d}) - \frac{d}{2 \log d}| = \Theta \left( \frac{d \log \log d}{(\log d)^2} \right).
\]

Here we determine \( \chi(G_{n,d}) \) up to three possible values for all integers. Moreover, for roughly half of all integers we determine \( \chi(G_{n,d}) \) up to two possible values. We first replicate the argument in [10] to prove

Theorem 1. For every integer \( d \), there exists an integer \( k = k(d) \) such that w.h.p. the chromatic number of \( G_{n,d} \) is either \( k \) or \( k + 1 \).

We then use the second moment method to prove the following.

Theorem 2. For every integer \( d \), w.h.p. \( \chi(G_{n,d}) \) is either \( k \), \( k + 1 \), or \( k + 2 \), where \( k \) is the smallest integer such that \( d < 2k \log k \). If, furthermore, \( d > (2k - 1) \log k \), then w.h.p. \( \chi(G_{n,d}) \) is either \( k + 1 \) or \( k + 2 \).

The table below gives the possible values of \( \chi(G_{n,d}) \) for some values of \( d \).

| \( d \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7, 8, 9 \) | \( 10 \) | \( 100 \) | \( 1,000,000 \) |
|---|---|---|---|---|---|---|---|
| \( \chi(G_{n,d}) \) | 3, 4 | 3, 4, 5 | 4, 5 | 4, 5, 6 | 5, 6 | 18, 19, 20 | 46523, 46524 |

¹ Given a sequence of events \( \mathcal{E}_n \), we say that \( \mathcal{E} \) holds with positive probability (w.p.p.) if \( \liminf_{n \to \infty} \Pr[\mathcal{E}_n] > 0 \), and with high probability (w.h.p.) if \( \liminf_{n \to \infty} \Pr[\mathcal{E}_n] = 1 \).
1.1 Preliminaries and outline of the proof

Rather than proving our results for $G_{n,d}$ directly, it will be convenient to work with random $d$-regular multigraphs, in the sense of the configuration model \[5\]: that is, multigraphs $C_{n,d}$ generated by selecting a uniformly random configuration (matching) on $dn$ “vertex copies.” It is well-known that for any fixed integer $d$, a random such multigraph is simple w.p.p. As a result, to prove Theorem 1 we simply establish its assertion for $C_{n,d}$.

To prove Theorem 2 we use the second moment method to show Theorem 3. If $d < 2k \log k$, then w.p.p. $\chi(C_{n,d}) \leq k + 1$.

Proof of Theorem 4. For integer $k$ let $u_k = (2k-1)\log k$ and $c_k = 2k \log k$. Observe that $c_{k-1} < u_k < c_k$. Thus, if $k$ is the smallest integer such that $d < c_k$, then either i) $u_k < d < c_k$ or ii) $u_k-1 < c_k-1 < d \leq u_k < c_k$.

A simple first moment argument (see e.g. [11]) implies that if $d > u_k$ then w.h.p. $\chi(C_{n,d}) > k$. Thus, if $u_k < d < c_k$, then w.h.p. $C_{n,d}$ is non-$k$-colorable while w.p.p. it is $(k+1)$-colorable. Therefore, by Theorem 1 w.h.p. the chromatic number of $C_{n,d}$ (and therefore $G_{n,d}$) is either $k+1$ or $k+2$. In the second case, we cannot eliminate the possibility that $G_{n,d}$ is w.p.p. $k$-colorable, but we do know that it is w.h.p. non-$(k-1)$-colorable. Thus, similarly, it follows that $\chi(G_{n,d})$ is w.h.p. $k$, $k+1$ or $k+2$. \[\square\]

Throughout the rest of the paper, unless we explicitly say otherwise, we are referring to random multigraphs $C_{n,d}$. We will say that a multigraph is $k$-colorable iff the underlying simple graph is $k$-colorable. Also, we will refer to multigraphs and configurations interchangeably using whichever form is most convenient.

2 2-point concentration

In [10], Luczak in fact established two-point concentration for $\chi(G(n, d/n))$ for all $\epsilon > 0$ and $d = O(n^{1/6-\epsilon})$. Here, mimicking his proof, we establish two-point concentration for $\chi(G_{n,d})$ for all $\epsilon > 0$ and $d = O(n^{1/7-\epsilon})$.

Our main technical tool is the following martingale-based concentration inequality for random variables defined on $C_{n,d}$ [12 Thm 2.19]. Given a configuration $C$, we define a switching in $C$ to be the replacement of two pairs $\{e_1, e_2\}$, $\{e_3, e_4\}$ by $\{e_1, e_3\}$, $\{e_2, e_4\}$ or $\{e_1, e_4\}$, $\{e_3, e_2\}$.

**Theorem 4.** Let $X_n$ be a random variable defined on $C_{n,d}$ such that for any configurations $C, C'$ that differ by a switching

$|X_n(C) - X_n(C')| \leq b$,
Given $C$ first inequality in (1) yields an increasing sequence of sets of vertices $u$ such that $k_\text{induced by }s_i \geq i$.

Lemma 1. For any $0 < \epsilon < 1/6$ and $d < n^{1/6-\epsilon}$, w.h.p. every subgraph induced by $s \leq nd^{-3(1+2\epsilon)}$ vertices contains at most $(3/2 - \epsilon)s$ edges.

Lemma 2. For a given function $\omega(n)$, let $k = k(\omega, n, p)$ be the smallest $k$ such that

$$\Pr[\chi(C_{n,d}) \leq k] \geq 1/\omega(n).$$

With probability greater than $1 - 1/\omega(n)$, all but $8\sqrt{nd\log \omega(n)}$ vertices of $C_{n,d}$ can be properly colored using $k$ colors.

Proof. For a multigraph $G$, let $Y_k(G)$ be the minimal size of a set of vertices $S$ for which $G - S$ is $k$-colorable. Clearly, for any $k$ and $G$, switching two edges of $G$ can affect $Y_k(G)$ by at most 4, as a vertex cannot contribute more than itself to $Y_k(G)$. Thus, if $\mu_k = E[Y_k(C_{n,d})]$, Theorem 1 implies

$$\Pr[Y_k \leq \mu_k - \lambda\sqrt{n}] < e^{-\frac{\lambda^2}{16d}} \quad \text{and} \quad \Pr[Y_k \geq \mu_k + \lambda\sqrt{n}] < e^{-\frac{\lambda^2}{16d}}. \quad (1)$$

Define now $u = u(n, p, \omega(n))$ to be the least integer for which $\Pr[\chi(G) \leq u] \geq 1/\omega(n)$. Choosing $\lambda = \lambda(n)$ so as to satisfy $e^{-\lambda^2/(16d)} = 1/\omega(n)$, the first inequality in (1) yields

$$\Pr[Y_u \leq \mu_u - \lambda\sqrt{n}] < 1/\omega(n) \leq \Pr[\chi(G) \leq u] = \Pr[Y_u = 0] \quad .$$

Clearly, if $\Pr[Y_u \leq \mu_u - \lambda\sqrt{n}] < \Pr[Y_u = 0]$ then $\mu_u < \lambda\sqrt{n}$. Thus, the second inequality in (1) implies $\Pr[Y \geq 2\lambda\sqrt{n}] < 1/\omega(n)$ and, by our choice, $\lambda = 4\sqrt{d\log \omega(n)}$.

Proof of Theorem 1. The result is trivial for $d = 1, 2$. Given $d \geq 3$, let $k = k(d, n) \geq 3$ be the smallest integer for which the probability that $C_{n,d}$ is $k$-colorable is at least $1/\log \log n$. By Lemma 2, w.h.p. there exists a set of vertices $S$ such that all vertices outside $S$ can be colored using $k$ colors and $|S| < 8\sqrt{nd\log \log \log n} < \sqrt{nd\log n} \equiv s_0$. From $S$, we will construct an increasing sequence of sets of vertices $\{U_i\}$ as follows. $U_0 = S$; for $i \geq 0$, $U_{i+1} = U_i \cup \{w_1, w_2\}$, where $w_1, w_2 \notin U_i$ are adjacent and each of
them has some neighbor in $U_t$. The construction ends, with $U_t$, when no such pair exists.

Observe that the neighborhood of $U_t$ in the rest of the graph, $N(U_t)$, is always an independent set, since otherwise the construction would have gone on. We further claim that w.h.p. the graph induced by the vertices in $U_t$ is $k$-colorable. Thus, using an additional color for $N(U_t)$ yields a $(k + 1)$-coloring of the entire multigraph, concluding the proof.

We will prove that $U_t$ is, in fact, 3-colorable by proving that $|U_t| \leq s_0/\epsilon$. This suffices since by Lemma 1 w.h.p. every subgraph $H$ of $b$ or fewer vertices has average degree less than 2 and hence contains a vertex $v$ with $\text{deg}(v) \leq 2$. Repeatedly invoking Lemma 1 yields an ordering of the vertices in $H$ such that each vertex is adjacent to no more than 2 of its successors. Thus, we can start with the last vertex in the ordering and proceed backwards; there will always be at least one available color for the current vertex. To prove $|U_t| \leq 2s_0 \log n$ we observe that each pair of vertices entering $U_t$ “brings in” with it at least 3 new edges. Therefore, for every $j \geq 0$, $U_j$ has at most $s_0 + 2j$ vertices and at least $3j$ edges.

Thus, by Lemma 1 w.h.p. $t < 3s_0/(4\epsilon)$.

\section{Establishing colorability in two moments}

Let us say that a coloring $\sigma$ is \textit{nearly-balanced} if its color classes differ in size by at most 1, and let $X$ be the number of nearly-balanced $k$-colorings of $C_{n,d}$. Recall that $c_k = 2k \log k$. We will prove that for all $k \geq 3$ and $d < c_{k-1}$ there exist constants $C_1, C_2 > 0$ such that for all sufficiently large $n$ (when $dn$ is even),

$$\mathbb{E}[X] > C_1 n^{-(k-1)/2} k^n \left(1 - \frac{1}{k}\right)^{dn/2},$$

$$\mathbb{E}[X^2] < C_2 n^{-(k-1)} k^{2n} \left(1 - \frac{1}{k}\right)^{dn}. \quad (3)$$

By the Cauchy-Schwartz inequality (see e.g. \cite[Remark 3.1]{7}), we have

$$\Pr[X > 0] > \mathbb{E}[X^2]/\mathbb{E}[X^2] > C_1^2/C_2 > 0,$$

and thus Theorem 3.

To prove (2), (3) we will need to bound certain combinatorial sums up to constant factors. To achieve this we will use the following Laplace-type lemma, which generalizes a series of lemmas in \cite{2,3,4}. Its proof is standard but somewhat tedious, and is relegated to the full paper.
Lemma 3. Let $\ell, m$ be positive integers. Let $y \in \mathbb{Q}^m$, and let $M$ be a $m \times \ell$ matrix of rank $r$ with integer entries whose top row consists entirely of 1's. Let $s, t$ be nonnegative integers, and let $v_i, w_j \in \mathbb{N}^\ell$ for $1 \leq i \leq s$ and $1 \leq j \leq t$, where each $v_i$ and $w_j$ has at least one nonzero component, and where moreover $\sum_{i=1}^s v_i = \sum_{j=1}^t w_j$. Let $f : \mathbb{R}^\ell \to \mathbb{R}$ be a positive twice-differentiable function. For $n \in \mathbb{N}$, define

$$S_n = \sum_{\{z \in \mathbb{N}^\ell : M \cdot z = y_n\}} \prod_{i=1}^s (v_i \cdot z)! \prod_{j=1}^t (w_j \cdot z)! f(z/n)^n$$

and define $g : \mathbb{R}^\ell \to \mathbb{R}$ as

$$g(\zeta) = \frac{\prod_{i=1}^s (v_i \cdot \zeta)^{(v_i \cdot \zeta)}}{\prod_{j=1}^t (w_j \cdot \zeta)^{(w_j \cdot \zeta)}} f(\zeta)$$

where $0^0 = 1$. Now suppose that, conditioned on $M \cdot \zeta = y$, $g$ is maximized at some $\zeta^*$ with $\zeta_i^* > 0$ for all $i$, and write $g_{\text{max}} = g(\zeta^*)$. Furthermore, suppose that the matrix of second derivatives $g'' = \partial^2 g / \partial \zeta_i \partial \zeta_j$ is nonsingular at $\zeta^*$.

Then there exist constants $A, B > 0$, such that for any sufficiently large $n$ for which there exist integer solutions $z$ to $M \cdot z = y_n$, we have

$$A \leq \frac{S_n}{n^{-(\ell+s-t-r)/2} g_{\text{max}}^n} \leq B.$$ 

For simplicity, in the proofs of (2) and (3) below we will assume that $n$ is a multiple of $k$, so that nearly–balanced colorings are in fact exactly balanced, with $n/k$ vertices in each color class. The calculations for other values of $n$ differ by at most a multiplicative constant.

4 The first moment

Clearly, all (exactly) balanced $k$-partitions of the $n$ vertices are equally likely to be proper $k$-colorings. Therefore, $E[X]$ is the number of balanced $k$-partitions, $n!/(n/k)^k$, times the probability that a random $d$-regular configuration is properly colored by a fixed balanced $k$-partition.

To estimate this probability we will label the $d$ copies of each vertex, thus giving us $(dn - 1)!!$ distinct configurations, and count the number of such configurations that are properly colored by a fixed balanced $k$-partition. To generate such a configuration we first determine the number of edges between each pair of color classes. Suppose there are $b_{ij}$ edges
between vertices of colors \(i\) and \(j\) for each \(i \neq j\). Then a properly colored configuration can be generated by i) choosing which \(b_{ij}\) of the \(dn/k\) copies in each color class \(i\) are matched with copies in each color class \(j \neq i\), and then ii) choosing one of the \(b_{ij}\) matchings for each unordered pair \(i < j\).

Therefore, the total number of properly colored configurations is

\[
\prod_{i=1}^{k} \frac{(dn/k)!}{\prod_{j \neq i} b_{ij}!} \cdot \prod_{i<j} b_{ij}! = \frac{(dn/k)!^k}{\prod_{i<j} b_{ij}!}.
\]

Summing over all choices of the \(\{b_{ij}\}\) that satisfy the constraints

\[
\forall i : \sum_j b_{ij} = dn/k,
\]

we get

\[
E[X] = \frac{n!}{(n/k)!^k (dn-1)!} \sum_{\{b_{ij}\}} \frac{(dn/k)!^k}{\prod_{i<j} b_{ij}!}
\]

\[
= 2^{dn/2} \cdot \frac{n!}{(n/k)!^k (dn)!} \cdot \sum_{\{b_{ij}\}} \frac{(dn/2)!}{\prod_{i<j} b_{ij}!}.
\]

By Stirling’s approximation \(\sqrt{2\pi n} (n/e)^n < n! < \sqrt{4\pi n} (n/e)^n\) we get

\[
E[X] > D_1 \cdot \frac{2^{dn/2}}{k(d-1)!} \sum_{\{b_{ij}\}} \frac{(dn/2)!}{\prod_{i<j} b_{ij}!},
\]

where \(D_1 = 2^{-(k+1)/2} d(k-1)/2\).

To bound the sum in \(\frac{3}{2}\) from below we use Lemma 3. Specifically, \(z\) consists of the variables \(b_{ij}\) with \(i < j\), so \(\ell = k(k-1)/2\). For \(k \geq 3\), the \(k\) constraints \(\frac{4}{3}\) are linearly independent, so representing them as \(M \cdot z = y\) gives a matrix \(M\) of rank \(k\). Moreover, they imply \(\sum_{i<j} b_{ij} = dn/2\), so adding a row of 1’s to the top of \(M\) and setting \(y_1 = d/2\) does not increase its rank. Integer solutions \(z\) exist whenever \(n\) is a multiple of \(k\) and \(dn\) is even. We set \(s = 1\) and \(t = \ell\); the vector \(v_1\) consists of 1’s and the \(w_j\) are the \(\ell\) basis vectors. Finally, \(f(\zeta) = 1\). Thus, \(\ell + s - t - r = -(k-1)\) and

\[
g(\zeta) = \frac{(d/2)^{d/2}}{\prod_{j=1}^{\ell} \zeta_k^{C_k}} \frac{1}{\prod_{j=1}^{\ell} (2\zeta_j/d)^{C_j}} = e^{(d/2)H(2\zeta/d)}
\]

where \(H\) is the entropy function \(H(x) = -\sum x \log x\).
Since $g$ is convex it is maximized when $\zeta_j^* = d/(2\ell)$ for all $1 \leq j \leq \ell$, and $g''$ is nonsingular. Thus, $g_{\text{max}} = (k(k-1)/2)^{d/2}$ implying that for some $A > 0$ and all sufficiently large $n$

\[ E[X] > D_1 \frac{2^{dn/2}}{k^{(d-1)n}} \times \frac{(k(k-1))^{dn/2}}{2} 
\]

\[ = D_1 A n^{-(k-1)/2} k^d \left( 1 - \frac{1}{k} \right)^{dn/2}. \]

Setting $C_1 = D_1 A$ completes the proof.

5 The second moment

Recall that $X$ is the sum over all balanced $k$-partitions of the indicators that each partition is a proper coloring. Therefore, $E[X^2]$ is the sum over all pairs of balanced $k$-partitions of the probability that both partitions properly color a random $d$-regular configuration. Given a pair of partitions $\sigma, \tau$, let us say that a vertex $v$ is in class $(i, j)$ if $\sigma(v) = i$ and $\tau(v) = j$. Also, let $a_{ij}$ denote the number of vertices in each class $(i, j)$. We call $A = (a_{ij})$ the overlap matrix of the pair $\sigma, \tau$. Note that since both $\sigma$ and $\tau$ are balanced

\[ \forall i : \sum_j a_{ij} = \sum_j a_{ji} = n/k. \]

(6)

We will show that for any fixed pair of $k$-partitions, the probability that they both properly color a random $d$-regular configuration depends only on their overlap matrix $A$. Denoting this probability by $q(A)$, since there are $n!/\prod_{ij} a_{ij}$! pairs of partitions giving rise to $A$, we have

\[ E[X^2] = \sum_A \frac{n!}{\prod_{ij} a_{ij}} q(A) \]

(7)

where the sum is over matrices $A$ satisfying (6).

Fixing a pair of partitions $\sigma$ and $\tau$ with overlap matrix $A$, similarly to the first moment, we label the $d$ copies of each vertex thus getting $(dn-1)!$ distinct configurations. To generate configurations properly colored by both $\sigma$ and $\tau$ we first determine the number of edges between each pair of vertex classes. Let us say that there are $b_{ijkl}$ edges connecting vertices in class $(i, j)$ to vertices in class $(k, \ell)$. By definition, $b_{ijkl} = b_{k\ell ij}$, and if both colorings are proper, $b_{ijkl} = 0$ unless $i \neq k$ and $j \neq \ell$. Since the
configuration is \( d \)-regular, we also have

\[
\forall i, j : \sum_{k \neq i, \ell \neq j} b_{ijk\ell} = da_{ij} .
\]  

(8)

To generate a configuration consistent with \( A \) and \( \{b_{ijk\ell}\} \) we now i) choose for each class \((i, j)\), which \( b_{ijk\ell} \) of its \( da_{ij} \) copies are to be matched with copies in each class \((k, \ell)\) with \( k \neq i \) and \( \ell \neq j \), and then ii) choose one of the \( b_{ijk\ell}! \) matchings for each unordered pair of classes \( i < k, j \neq \ell \). Thus,

\[
q(A) = \frac{1}{(dn - 1)!!} \sum_{\{b_{ijk\ell}\}} \left( \prod_{ij} \frac{(da_{ij})!}{(dn)!} \prod_{i<k,j<\ell} b_{ijk\ell}! \right) 
\]

\[
= 2^{dn/2} \prod_{ij} (da_{ij})! \sum_{\{b_{ijk\ell}\}} \frac{(dn/2)!}{(dn)!} \prod_{i<k,j<\ell} b_{ijk\ell}! ,
\]

(9)

where the sum is over the \( \{b_{ijk\ell}\} \) satisfying \( 8 \). Combining \( 9 \) with \( 7 \) gives

\[
E[X^2] = 2^{dn/2} \sum \sum_{a_{ij} b_{ijk\ell}} n! \prod_{ij} a_{ij}! \prod_{ij} (da_{ij})! \prod_{i<k,j<\ell} b_{ijk\ell}! .
\]

(10)

To bound the sum in \( 10 \) from above we use Lemma 3. We let \( z \) consist of the combined set of variables \( \{a_{ij}\} \cup \{b_{ijk\ell} : i < k, j \neq \ell\} \), in which case its dimensionality \( \ell \) (not to be confused with the color \( \ell \)) is \( k^2 + (k(k-1))^2/2 \). We represent the combined system of constraints \( 6, 8 \) as \( M \cdot z = y_n \). The \( k^2 \) constraints \( 8 \) are, clearly, linearly independent while the \( 2k \) constraints \( 6 \) have rank \( 2k - 1 \). Together these imply \( \sum_{ij} a_{ij} = 1 \) and \( \sum_{i<k,j<\ell} b_{ijk\ell} = d/2 \), so adding a row of \( 1 \)'s to the top of \( M \) does not change its rank from \( r = k^2 + 2k - 1 \). Integer solutions \( z \) exist whenever \( n \) is a multiple of \( k \) and \( dn \) is even. Finally, \( f(\zeta) = 2^{d/2}, s = k^2 + 2 \) and \( t = k^2 + 1 + (k(k-1))^2/2, so \ell + s - t - r = -2(k-1) \).

Writing \( \alpha_{ij} \) and \( \beta_{ijk\ell} \) for the components of \( \zeta \) corresponding to \( a_{ij}/n \) and \( b_{ijk\ell}/n \), respectively, we thus have

\[
g(\zeta) = 2^{d/2} \frac{1}{\prod_{ij} \alpha_{ij}^{a_{ij}}} \frac{\prod_{ij} (da_{ij})^{a_{ij}}}{d^d} \frac{(d/2)^{d/2}}{\prod_{i<k,j<\ell} \beta_{ijk\ell}^{b_{ijk\ell}}} 
\]

\[
= \frac{1}{\prod_{ij} \alpha_{ij}^{a_{ij}}} \frac{d^{d/2}}{\prod_{ij} \alpha_{ij}^{a_{ij}}} \prod_{i<k,j<\ell} \beta_{ijk\ell}^{b_{ijk\ell}} .
\]

(11)
In the next section we maximize $g(\zeta)$ over $\zeta \in \mathbb{R}^\ell$ satisfying $M \cdot \zeta = y$. We note that $g''$ is nonsingular at the maximizer we find below, but we relegate the proof of this fact to the full paper.

6 A tight relaxation

Maximizing $g(\zeta)$ over $\zeta \in \mathbb{R}^\ell$ satisfying $M \cdot \zeta = y$ is greatly complicated by the constraints

$$\forall i, j : \sum_{k \neq i, \ell \neq j} \beta_{ijkl} = d\alpha_{ij}. \tag{12}$$

To overcome this issue we i) reformulate $g(\zeta)$ and ii) relax the constraints, in a manner such that the maximum value remains unchanged while the optimization becomes much easier.

The relaxation amounts to replacing the $k^2$ constraints (12) with their sum divided by 2, i.e., with the single constraint

$$\sum_{i < k, j \neq \ell} \beta_{ijkl} = d/2. \tag{13}$$

But attempting to maximize (11) under this single constraint is, in fact, a bad idea since the new maximum is much greater. Instead, we maximize the following equivalent form of $g(\zeta)$

$$g(\zeta) = \frac{1}{\prod_{ij} \alpha_{ij}} \frac{d^{d/2} \prod_{ij} \alpha_{ij} \sum_{k \neq i, \ell \neq j} \beta_{ijkl}}{\prod_{i < k, j \neq \ell} \beta_{ijkl}^{d/2}}, \tag{14}$$

derived by using (12) to substitute for the exponents $d\alpha_{ij}$ in the numerator of (11). This turns out to be enough to drive the maximizer back to the subspace $M \cdot \zeta = y$.

Specifically, let us hold $\{\alpha_{ij}\}$ fixed and maximize $g(\zeta)$ with respect to $\{\beta_{ijkl}\}$ using the method of Lagrange multipliers. Since log $g$ is monotonically increasing in $g$, it is convenient to maximize log $g$ instead. If $\lambda$ is the Lagrange multiplier corresponding to the constraint (13), we have for all $i < k, j \neq \ell$:

$$\lambda = \frac{\partial}{\partial \beta_{ijkl}} \log g(\zeta) = \frac{\partial}{\partial \beta_{ijkl}} \left( \beta_{ijkl} \log(\alpha_{ij}\alpha_{kl}) - \beta_{ijkl} \log \beta_{ijkl} \right)$$

$$= \log \alpha_{ij} + \log \alpha_{kl} - \log \beta_{ijkl} - 1$$
and so
\[ \forall i < k, j \neq l : \beta_{ijkl} = C \alpha_{ij} \alpha_{kl}, \quad \text{where } C = e^{-\lambda - 1}. \quad (15) \]

Clearly, such \( \beta_{ijkl} \) also satisfy the original constraints (12), and therefore the upper bound we obtain from this relaxation is in fact tight.

To solve for \( C \) we sum (15) and use (13), getting
\[ \frac{2}{C} \sum_{i < k, j \neq \ell} \beta_{ijkl} = \frac{d}{C} \sum_{i \neq k, j \neq \ell} \alpha_{ij} \alpha_{kl} = 1 - \frac{2}{k} + \sum_{ij} \alpha_{ij}^2 = p. \]

Thus \( C = d/p \) and (15) becomes
\[ \forall i < k, j \neq l : \beta_{ijkl} = \frac{d \alpha_{ij} \alpha_{kl}}{p} \quad (16) \]

Observe that \( p = p(\{a_{ij}\}) \) is the probability that a single edge whose endpoints are chosen uniformly at random is properly colored by both \( \sigma \) and \( \tau \), if the overlap matrix is \( a_{ij} = \alpha_{ij} n \). Moreover, the values for the \( b_{ijkl} \) are exactly what we would obtain, in expectation, if we chose from among the \( (\binom{n}{2}) \) edges with replacement, rejecting those improperly colored by \( \sigma \) or \( \tau \), until we had \( dn/2 \) edges—in other words, if our graph model was \( G(n, m) \) with replacement, rather than \( G_{n,d} \).

Substituting the values (16) in (14) and applying (13) yields the following upper bound on \( g(\zeta) \):
\[
g(\zeta) \leq \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2}}{(d/p)^{\sum_{i < k, j \neq \ell} \beta_{ijkl} \prod_{i < k, j \neq \ell} \alpha_{ij} \alpha_{kl}}} \frac{d^{d/2}}{(d/p)^{\sum_{i \neq k, j \neq \ell} \alpha_{ij} \alpha_{kl}}} \frac{\prod_{ij} \alpha_{ij}^{\alpha_{ij} \alpha_{ij} \alpha_{kl}}} {\prod_{i \neq k, j \neq \ell} \alpha_{ij}^{\alpha_{ijkl}}} \frac{d^{d/2}}{(d/p)^{\sum_{i \neq k, j \neq \ell} \alpha_{ij} \alpha_{kl}}} \\
= \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2}}{(d/p)^{d/2}} \left( \frac{\prod_{ij} \alpha_{ij}^{\alpha_{ij} \alpha_{ijkl}}} {\prod_{i \neq k, j \neq \ell} \alpha_{ij}^{\alpha_{ijkl}}} \right) \frac{d^{d/2}}{(d/p)^{d/2}} \\
= \frac{p^{d/2}}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \equiv g_{G(n,m)}(\{\alpha_{ij}\}) .
\]

In [4, Thm 5], Achlioptas and Naor showed that for \( d < c_{k-1} \) the function \( g_{G(n,m)} \) is maximized when \( \alpha_{ij} = 1/k^2 \) for all \( i, j \). In this case \( p = (1 - 1/k)^2 \), implying
\[
g_{\text{max}} \leq k^2 p^{d/2} = k^2 \left( 1 - \frac{1}{k} \right)^d
\]
and, therefore, that for some constant $C_2$ and sufficiently large $n$

$$\mathbb{E}[X^2] \leq C_2 n^{-(k-1)} k^{2n} \left(1 - \frac{1}{k}\right)^{dn}.$$

7 Directions for further work

A sharp threshold for regular graphs. It has long been conjectured that for every $k > 2$, there exists a critical constant $c_k$ such that a random graph $G(n, m = cn)$ is w.h.p. $k$-colorable if $c < c_k$ and w.h.p. non-$k$-colorable if $c > c_k$. It is reasonable to conjecture that the same is true for random regular graphs, i.e. that for all $k > 2$, there exists a critical integer $d_k$ such that a random graph $G_{n,d}$ is w.h.p. $k$-colorable if $d \leq d_k$ and w.h.p. non-$k$-colorable if $d > d_k$. If this is true, our results imply that for $d$ in “good” intervals $(u_k, c_k)$ w.h.p. the chromatic number of $G_{n,d}$ is precisely $k + 1$, while for $d$ in “bad” intervals $(c_{k-1}, u_k)$ the chromatic number is w.h.p. either $k$ or $k + 1$.

Improving the second moment bound. Our proof establishes that if $X, Y$ are the numbers of balanced $k$-colorings of $G_{n,d}$ and $G(n, m = dn/2)$, respectively, then $\mathbb{E}[X^2]/\mathbb{E}[X^2] = \Theta(\mathbb{E}[Y^2]/\mathbb{E}[Y^2])$. Therefore, any improvement on the upper bound for $\mathbb{E}[Y^2]$ given in [4] would immediately give an improved positive-probability $k$-colorability result for $G_{n,d}$.

In particular, Moore has conjectured that the function $g_{G(n,m)}$ is maximized by matrices with a certain form. If true, this immediately gives an improved lower bound, $c^*_k$, for $k$-colorability satisfying $c^*_{k-1} \rightarrow u_k - 1$. This would shrink the union of the “bad” intervals to a set of measure 0, with each such interval containing precisely one integer $d$ for each $k \geq k_0$.

3-colorability of random regular graphs. It is easy to show that a random 6-regular graph is w.h.p. non-3-colorable. On the other hand, in [1] the authors showed that 4-regular graphs are w.p.p. 3-colorable. Based on considerations from statistical physics, Krzakala, Pagnani and Weigt [8] have conjectured that a random 5-regular graph is w.h.p. 3-colorable. The authors (unpublished) have shown that applying the second moment method to the number of balanced 3-colorings cannot establish this fact (even with positive probability).

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