Deformations of Varieties of General Type
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Abstract. We prove that small deformations of a projective variety of general type are also projective varieties of general type, with the same plurigenera.

1. Introduction

Our aim is to prove the following.

Theorem 1. Let $g : X \to S$ be a flat, proper morphism of complex analytic spaces. Fix a point $0 \in S$ and assume that the fiber $X_0$ is projective, of general type, and with canonical singularities. Then there is an open neighborhood $0 \in U \subset S$ such that

(1.1) the plurigenera of $X_s$ are independent of $s \in U$ for every $r$, and
(1.2) the fibers $X_s$ are projective for every $s \in U$.

Here the $r$th plurigenus of $X_s$ is $h^0(Y_s, \omega_{Y_s}^r)$, where $Y_s \to X_s$ is any resolution of $X_s$. By [18, VI.5.2] (see also (10.2)) $X_s$ has canonical singularities, so this is the same as $h^0(X_s, \omega_{X_s}^{[r]})$, where $\omega_{X_s}^{[r]}$ denotes the double dual of the $r$th tensor power $\omega_{X_s}$.

Comments 1.3. Many cases of this have been proved, but I believe that the general result is new, even for $X_0$ smooth and $S$ a disc.

For smooth surfaces proofs are given in [6,15], and for 3-folds with terminal singularities in [8, 12.5.1]. If $g$ is assumed projective, then of course all fibers are projective, and deformation invariance of plurigenera was proved by [20] for $X_0$ smooth, and by [18, Chap.VI] when $X_0$ has canonical singularities. However, frequently $g$ is not projective; see Example 4 for some smooth, 2-dimensional examples. Many projective varieties have deformations that are not projective, not even algebraic in any sense; K3 and elliptic surfaces furnish the best known examples.

In Example 3 we construct a deformation of a projective surface with a quotient singularity and ample canonical class, whose general fibers are non-algebraic, smooth surfaces of Kodaira dimension 0. Thus canonical is likely the largest class of singularities where Theorem 1 holds. See also Example 5 for surfaces with simple elliptic singularities.
The projectivity of $X_0$ is essential in our proof, but (1.1) should hold whenever $X_0$ is a proper algebraic space of general type with canonical singularities. Such results are proved in [19], provided one assumes that either $X_0$ is smooth and all fibers are Moishezon, or almost all fibers are of general type.

Our main technical result says that the Minimal Model Program works for $g : X \to S$. For $\dim X_0 = 2$ and $X_0$ smooth, this goes back to [15]. For $\dim X_0 = 3$ and terminal singularities, this was proved in [8, 12.4.4]. The next result extends these to all dimensions.

**Theorem 2.** Let $g : X \to S$ be a flat, proper morphism of reduced, complex analytic spaces. Fix a point $0 \in S$ and assume that $X_0$ is projective and has canonical singularities. Then every sequence of MMP-steps $X_0 = X_0^0 \to X_0^1 \to X_0^2 \to \cdots$ (see Definition 7) extends to a sequence of MMP-steps $X = X^0 \to X^1 \to X^2 \to \cdots$.

over some open neighborhood $0 \in U \subset S$.

The proof is given in Paragraph 8 when $S$ is a disc $\mathbb{D}$, and in Paragraph 12 in general. The assumption that $X_0$ has canonical singularities is necessary, as shown by semistable 3-fold flips [8]. Extending MMP steps from divisors with canonical singularities is also studied in [1].

If $X_0$ is of general type, then a suitable MMP for $X_0$ terminates with a minimal model $X_0^m$ by [3], which then extends to $g^m : X_0^m \to U$ by Theorem 2. For minimal models of varieties of general type, deformation invariance of plurigenera is easy, leading to a proof of (1.1) in Paragraph 13. This also implies that all fibers are bimeromorphic to a projective variety.

If $X_0$ is smooth, then it is Kähler, and the $X_s$ are also Kähler by [15]. A Kähler variety that is bimeromorphic to an algebraic variety is projective by [16].

However, there are families of surfaces with simple elliptic singularities $g : X \to S$ such that $K_{X_0}$ is ample, all fibers are bimeromorphic to an algebraic surface, yet the projective fibers correspond to a countable, dense set on the base; see Example 5.

We use Theorem 14—taken from [14, Thm.2]—to obtain the projectivity of the fibers and complete the proof of Theorem 1 in Paragraph 13.

### 2. Examples and Consequences

The first example shows that Theorem 1 fails very badly for surfaces with non-canonical quotient singularities.

**Example 3.** We give an example of a flat, proper morphism of complex analytic spaces $g : X \to \mathbb{D}$, such that

(3.1) $X_0$ is a projective surface with a quotient singularity and ample canonical class, yet

(3.2) $X_s$ is smooth, non-algebraic, and of Kodaira dimension 0 for very general $s \in \mathbb{D}$. 

Let us start with a K3 surface $Y_0 \subset \mathbb{P}^3$ with a hyperplane section $C_0 \subset Y_0$ that is a rational curve with 3 nodes. We blow up the nodes $Y_0' \to Y_0$ and contract the birational transform of $C_0$ to get a surface $\tau_0 : Y_0' \to X_0$. Let $E_1, E_2, E_3 \subset X_0$ be the images of the 3 exceptional curves of the blow-up.

By explicit computation, we get a quotient singularity of type $\mathbb{C}^2/\mathbb{Z}_3(1,1)$, $(E_i^3) = -\frac{1}{2}$ and $(E_i \cdot E_j) = \frac{1}{2}$ for $i \neq j$. Furthermore, $E := E_1 + E_2 + E_3 \sim K_{X_0}$ and it is ample by the Nakai-Moishezon criterion. (Note that $(E \cdot E_i) = \frac{1}{2}$ and $X_0 \setminus E \cong Y_0 \setminus C_0$ is affine.)

Take now a deformation $Y \to \mathbb{D}$ of $Y_0$ whose very general fibers are non-algebraic K3 surfaces that contain no proper curves. Take 3 sections $B_i \subset Y$ that pass through the 3 nodes of $C_0$. Blow them up and then contract the birational transform of $C_0$; cf. [17]. In general [17] says that the normalization of the resulting central fiber is $X_0$, but in our case the central fiber is isomorphic to $X_0$ since $R^1(\tau_0)_*\mathcal{O}_{Y_0} = 0$. The contraction is an isomorphism on very general fibers since there are no curves to contract. We get $g : X \to \mathbb{D}$ whose central fiber is $X_0$ and all other fibers are K3 surfaces blown up at 3 points.

In general, it is very unclear which complex varieties occur as deformations of projective varieties; see [7] for some of their properties.

**Example 4.** [2] Let $S_0 := (g = 0) \subset \mathbb{P}^3_x$ and $S_1 := (f = 0) \subset \mathbb{P}^3_x$ be surfaces of the same degree. Assume that $S_0$ has only ordinary nodes, $S_1$ is smooth, $\text{Pic}(S_1)$ is generated by the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$ and $S_1$ does not contain any of the singular points of $S_0$. Fix $m \geq 2$ and consider

$$X_m := (g - t^mf = 0) \subset \mathbb{P}^1_x \times \mathbb{A}^1_t.$$  

The singularities are locally analytically of the form $xy + z^2 - t^m = 0$. Thus $X_m$ is locally analytically factorial if $m$ is odd. If $m$ is even then $X_m$ is factorial since the general fiber has Picard number 1, but it is not locally analytically factorial; blowing up $(x = z - t^{m/2} = 0)$ gives a small resolution. Thus we get that

(4.1) $X_m$ is bimeromorphic to a proper, smooth family of projective surfaces iff $m$ is even, but

(4.2) $X_m$ is not bimeromorphic to a smooth, projective family of surfaces.

**Example 5.** Let $E \subset \mathbb{P}^2$ be a smooth cubic and take $r$ general lines $L_i \subset \mathbb{P}^2$. To get $S_0$, blow up all singular points of $E + \sum L_i$ and then contract the birational transform of $E + \sum L_i$. A somewhat tedious computation shows that $K_{S_0}$ is ample for $r \geq 6$. It has 1 simple elliptic singularity (coming from $E$) and $r$ quotient singularities (coming from the $L_i$).

Deform this example by moving the $3r$ points $E \cap \sum L_i$ into general position $p_i^1, \ldots, p_i^{3r} \in E$ and the points $L_i \cap L_j$ into general position on $\mathbb{P}^2$. Blow up these points and then contract the birational transform of $E$ to get the surfaces $S_t$. It has only 1 simple elliptic singularity (coming from $E$).

We get a flat family of surfaces with central fiber $S_0$ and general fibers $S_t$. Let $L$ denote the restriction of the line class on $\mathbb{P}^2$ to $E$.

It is easy to see that such a surface $S_t$ is non-projective if the $p_i^t$ and $L$ are linearly independent in $\text{Pic}(E)$. Thus $S_t$ is not projective for very general $t$ and has Kodaira dimension 0.
The next result is the scheme-theoretic version of Theorem 1. Ideally it should be proved by the same argument. However, some of the references we use, especially [18], are worked out for analytic spaces, not for general schemes. So for now we proceed in a somewhat roundabout way.

**Corollary 6.** Let $S$ be a noetherian, excellent scheme over a field of characteristic 0. Let $g : X \to S$ be a flat, proper algebraic space. Fix a point $0 \in S$ and assume that $X_0$ is projective, of general type and with canonical singularities. Then there is an open neighborhood $0 \in S^0 \subset S$ such that, for every $s \in S^0$,

\begin{equation}
(6.1) \text{the plurigenera } h^0(X_s, \omega^{[r]}_{X_s}) \text{ are independent of } s \text{ for every } r, \text{ and }
\end{equation}

\begin{equation}
(6.2) \text{the fiber } X_s \text{ is projective.}
\end{equation}

**Proof.** A proper algebraic space $Y$ over a field $k$ is projective iff $Y_K$ is projective over $K$ for some field extension $K \supset k$. Noetherian induction then shows that it is enough to prove the claims for the generic points of the completions (at the point $0 \in S$) of irreducible subvarieties $0 \in T \subset S$. Since the defining equations of $\hat{T}$ and of $X \times_S \hat{T}$ involve only countably many coefficients, we may assume that the residue field is $C$.

Consider now the local universal deformation space $\text{Def}(X_0)$ of $X_0$ in the complex analytic category; see [4]. It is the germ of a complex analytic space and there is a complex analytic universal family $G : X \to \text{Def}(X_0)$. Since a deformation over an Artin scheme is automatically complex analytic, we see that the formal completion $\hat{G} : \hat{X} \to \hat{\text{Def}}(X_0)$ is the universal formal deformation of $X_0$. In particular, $X \times_S \hat{T}$ is the pull-back of $\hat{G} : \hat{X} \to \hat{\text{Def}}(X_0)$ by a morphism $\hat{T} \to \hat{\text{Def}}(X_0)$. Thus Theorem 1 implies both claims. \qed

### 3. Relative MMP

See [9] for a general introduction to the minimal model program.

**Definition 7.** (*MMP-steps and their extensions*) Let $X \to S$ be a proper morphism of complex analytic spaces with irreducible fibers. Assume that $K_{X/S}$ is $\mathbb{Q}$-Cartier. By an **MMP-step** for $X$ over $S$ we mean a diagram

\begin{equation}
\begin{array}{ccc}
X & \xrightarrow{\pi} & X^+ \\
\phi \downarrow & & \phi^+ \\
Z & \nearrow &
\end{array}
\end{equation}

where all morphisms are bimeromorphic and proper over $S$, $-K_{X/S}$ is ample over $Z$, $K_{X^+/S}$ is ample over $Z$ and $\phi^+$ is small (that is, without exceptional divisors).

If $X$ is $\mathbb{Q}$-factorial and the relative Picard number of $X/Z$ is 1, then there are 2 possible MMP steps:

- **Divisorial:** $\phi$ contracts a single divisor and $\phi^+$ is the identity.
- **Flipping:** both $\phi$ and $\phi^+$ are small.

However, in general there is a more complicated possibility:

- **Mixed:** $\phi$ contracts (possibly several) divisors and $\phi^+$ is small.
For our applications we only need to know that, by \cite[3.52]{9}, \(X^+\) exists iff \(\ominus_{r \geq 0} \omega_{Z/S}^{[r]}\) (which is equal to \(\ominus_{r \geq 0} \phi_* \omega_X^{[r]}\)) is a finitely generated sheaf of \(\mathcal{O}_Z\)-algebras, and then

\[
X^+ = \text{Proj}_Z \ominus_{r \geq 0} \omega_{Z/S}^{[r]}.
\]

(7.2)

We index a sequence of MMP-steps by setting \(X^0 := X\) and \(X^{i+1} := (X^i)^+\).

Fix a point \(s \in S\) and let \(X_s\) denote the fiber over \(S\). We say that a sequence of MMP-steps (over \(S\)) \(X^0 \to X^1 \to X^2 \to \cdots\) extends a sequence of MMP-steps (over \(s\)) \(X^0_s \to X^1_s \to X^2_s \to \cdots\) if, for every \(i\),

\[
\begin{align*}
X^i_s \xrightarrow{\pi_s^i} X^{i+1}_s \\
\phi^i_s \backsimeq (\phi^i_s)^+ \\
Z^i_s
\end{align*}
\]

is the fiber over \(s\) of

\[
\begin{align*}
X^i \xrightarrow{\pi^i} X^{i+1} \\
\phi^i \backsimeq (\phi^i)^+ \\
Z^i
\end{align*}
\]

(7.3)

8 (Proof of Theorem 2 for \(S = \mathbb{D}\), the disc). Since MMP-steps preserve canonical singularities, by induction it is enough to prove the claim for one MMP step. So we drop the upper index \(i\) and identify \(K_X/\mathbb{D}\) with \(K_X\).

Let \(\phi : X_0 \to Z_0\) be an extremal contraction. By \cite{17}, it extends to a contraction \(\phi : X \to Z\), where \(Z\) is flat over \(\mathbb{D}\) with central fiber \(Z_0\) since \(R^1(\phi_* \mathcal{O}_{X_0}) = 0\). Note that \(K_X\) is \(\mathbb{Q}\)-Cartier by (10.1), and \(\phi\) is projective since \(-K_X\) is \(\phi\)-ample.

If \(\phi_0\) is a divisorial contraction, then \(K_{Z_0}\) is \(\mathbb{Q}\)-Cartier, and so is \(K_Z\) by (10.1). Thus \(X^+ = Z\).

If \(\phi_0\) is a flipping or mixed contraction, then \(K_Z\) is not \(\mathbb{Q}\)-Cartier. By (7.2),

\[
X^+ = \text{Proj}_Z \ominus_{r \geq 0} \omega_{Z}^{[r]},
\]

(8.1)

provided \(\ominus_{r \geq 0} \omega_{Z}^{[r]}\) is a finitely generated sheaf of \(\mathcal{O}_Z\)-algebras. (We have identified \(\omega_Z\) with \(\omega_{Z/\mathbb{D}}\).)

Functoriality works better if we twist by the line bundle \(\mathcal{O}_Z(Z_0)\) and write it as

\[
X^+ = \text{Proj}_Z \ominus_{r \geq 0} \omega_{Z}^{[r]}(r_{Z_0}).
\]

Let \(\tau : Y \to X\) be a projective resolution of \(X\) (that is, \(\tau\) is projective) such that \(Y_0\), the bimeromorphic transform of \(X_0\), is also smooth. Set \(g := \phi \circ \tau\).

The hardest part of the proof is Nakayama’s theorem (9) which gives a surjection

\[
\ominus_{r \geq 0} g_* \omega_Y^{[r]}(r_{Y_0}) \twoheadrightarrow \ominus_{r \geq 0} (g_0)_* \omega_{Y_0}^{[r]}.
\]

(8.2)

Since \(X_0\) has canonical singularities \(\tau_* \omega_{Y_0}^{[r]} = \omega_{X_0}^{[r]}\), and hence \(g_* \omega_Y^{[r]} = \omega_{Z_0}^{[r]}\). We also have a natural inclusion \(g_* \omega_Y^{[r]}(r_{Y_0}) \hookrightarrow \omega_{Z}^{[r]}(r_{Z_0})\). Thus pushing forward (8.2) we get a surjection

\[
\ominus_{r \geq 0} g_* \omega_Y^{[r]}(r_{Y_0}) \twoheadrightarrow \ominus_{r \geq 0} \omega_{Z}^{[r]}(r_{Z_0}) \twoheadrightarrow \ominus_{r \geq 0} \omega_{Z_0}^{[r]}.
\]

(8.3)

Note that \(\ominus_{r \geq 0} \omega_{Z_0}^{[r]}\) is a finitely generated sheaf of \(\mathcal{O}_{Z_0}\)-algebras, defining the MMP-step of \(X_0 \to Z_0\).
Now (11) says that $\oplus_{r \geq 0} \omega^r_Z(rZ_0)$ is also a finitely generated sheaf of $\mathcal{O}_Z$-algebras, at least in some neighborhood of the compact $Z_0$. \hfill \square

Next we discuss various results used in the proof.

**Theorem 9.** [18, VI.3.8] Let $\pi : Y \to S$ be a projective, bimeromorphic morphism of analytic spaces, $Y$ smooth and $S$ normal. Let $D \subseteq Y$ be a smooth, non-exceptional divisor. Then the restriction map

$$\pi_*\omega^n_Y(mD) \to \pi_*\omega^n_D$$

is surjective for $m \geq 1$. \hfill \square

This is a special case of [18, VI.3.8] applied with $\Delta = 0$ and $L = K_Y + D$.

**Warning.** The assumptions of [18, VI.3.8] are a little hard to find. They are outlined 11 pages earlier in [18, VI.2.2]. It talks about varieties, which usually suggest algebraic varieties, but [18, p.231, line 13] explicitly states that the proofs work with analytic spaces; see also [18, p.14]. (The statements of [18] allow for a boundary $\Delta$. However, $K_Y + D + \Delta$ should be $\mathbb{Q}$-linearly equivalent to a $\mathbb{Z}$-divisor and $[\Delta] = 0$ is assumed on [18, p.231]. There seem to be few cases when both of these can be satisfied.)

**Lemma 10.** [18, VI.5.2] Let $g : X \to S$ be a flat morphism of complex analytic spaces. Assume that $X_0$ has a canonical singularity at a point $x \in X_0$. Then there is an open neighborhood $x \in X^* \subset X$ such that

1. $K_{X^*/S}$ is $\mathbb{Q}$-Cartier, and
2. all fibers of $g|_{X^*} : X^* \to S$ have canonical singularities.

**Proof.** (1) is proved in [10, 3.2.2]; see also [11, 12.7] and [13, 2.8]. The harder part is (2), proved in [18, VI.5.2]. \hfill \square

**Remark 10.3.** If $S$ is smooth then $X^*$ has canonical singularities. By induction, it is enough to prove this when $S = \mathbb{D}$. Then the proof of [18, VI.5.2] shows that even the pair $(X^*, X_0 \cap X^*)$ has canonical singularities.

**Lemma 11.** Let $\pi : X \to S$ be a proper morphism of normal, complex spaces. Let $L$ be a line bundle on $X$ and $W \subseteq S$ a Zariski closed subset. Assume that $\mathcal{O}_W \otimes_S (\oplus_{r \geq 0} \pi_* L^r)$ is a finitely generated sheaf of $\mathcal{O}_W$-algebras.

Then every compact subset $W' \subseteq W$ has an open neighborhood $W' \subseteq U \subseteq S$ such that $\mathcal{O}_U \otimes_S (\oplus_{r \geq 0} \pi_* L^r)$ is a finitely generated sheaf of $\mathcal{O}_U$-algebras.

**Proof.** The question is local on $S$, so we may as well assume that $W$ is a single point. We may also assume that $\mathcal{O}_W \otimes_S (\oplus_{r \geq 0} \pi_* L^r)$ is generated by $\pi_* L$. After suitable blow-ups we are reduced to the case when the base locus of $L$ is a Cartier divisor $D$. By passing to a smaller neighborhood, we may assume that every irreducible component of $D$ intersects $\pi^{-1}(W)$. By the Nakayama lemma, the base locus of $L^r$ is a subscheme of $rD$ that agrees with it along $rD \cap \pi^{-1}(W)$. Thus $rD$ is the the base locus of $L^r$ for every $r$. We may thus replace $L$ by $L(-D)$ and assume that $L$ is globally generated.

Thus $L$ defines a morphism $X \to \text{Proj}_S \oplus_{r \geq 0} \pi_* L^r$, let $\pi' : X' \to S$ be its Stein factorization. Then $L$ is the pull-back of a line bundle $L'$ that is ample on $X' \to S$ and $\oplus_{r \geq 0} \pi'_* L^r = \oplus_{r \geq 0} \pi'_* L''$ is finitely generated. \hfill \square
12 (Proof of Theorem 2 for general $S$). As in Paragraph 8, it is enough to prove the claim for one MMP step, so let $\phi_0 : X_0 \to Z_0$ be an extremal contraction and $\phi : X \to Z$ its extension. As before, $Z$ is flat over $S$ with central fiber $Z_0$.

We claim that, for every $r$,

\begin{align}
\omega_{Z/S}^{[r]} & \text{ is flat over } S, \quad (12.1) \\
\omega_{Z/S}^{[r]}|_{Z_0} & \cong \omega_{Z_0}^{[r]}, \quad (12.2)
\end{align}

In the language of [12] or [13, Chap.9], this says that $\omega_{Z/S}^{[r]}$ is its own relative hull. There is an issue with precise references here, since [13, Chap.9] is written in the algebraic setting. However, [13, 9.72] considers hulls over the spectra of complete local rings. Thus we get that there is a unique largest subscheme $\hat{S}^u \subset \hat{S}$ (the formal completion of $S$ at $0$) such that (1–2) hold after base change to $\hat{S}^u$.

By Paragraph 8 we know that (1–2) hold after base change to any disc $D \to S$, which implies that $\hat{S}^u = \hat{S}$. That is, (1–2) hold for $\hat{S}$. Since both properties are invariant under formal completion, we are done.

Now we know that

$$X^+ := \text{Proj}_Z \bigoplus_{r \geq 0} \omega_{Z/S}^{[r]}, \quad (12.3)$$

is flat over $S$ and its central fiber is $X_0^+$. Thus it gives the required extension of the flip of $X_0 \to Z_0$. \hfill \Box

4. Proof of Theorem 1

We give a proof using only the $S = D$ case of Theorem 2.

13. Fix $r \geq 2$ and assume first that $S = D$. Since $X_0$ is of general type, a suitable MMP for $X_0$ ends with a minimal model $X_0^m$, and, by Theorem 2, $X_0 \dashrightarrow X_0^m$ extends to a fiberwise bimeromorphic map $X \dashrightarrow X^m$. We have $g^m : X^m \to D$. (From now on, we replace $D$ with a smaller disc whenever necessary.) Since $K_{X^m_0}$ is nef and big, the higher cohomology groups of $\omega_{X_0}^{[r]}$ vanish for $r \geq 2$. Thus $s \mapsto H^0(X^m_s, \omega^{[r]}_{X^m_s})$ is locally constant at the origin.

By (10.2) $X_s$ and $X^m_s$ both have canonical singularities, so they have the same plurigenera. Therefore $s \mapsto H^0(X_s, \omega^{[r]}_{X_s})$ is also locally constant at the origin. By Serre duality, the deformation invariance of $H^0(X_s, \omega_{X_s})$ is equivalent to the deformation invariance of $H^n(X_s, \mathcal{O}_{X_s})$. In fact, all the $H^i(X_s, \mathcal{O}_{X_s})$ are deformation invariant. For this the key idea is in [5], which treats deformations of varieties with normal crossing singularities. The method works for varieties with canonical (even log canonical) singularities; this is worked out in [13, Sec.2.5].

For arbitrary $S$, note that $s \mapsto H^0(X_s, \omega^{[r]}_{X_s})$ is a constructible function on $S$, thus locally constant at $0 \in S$ if it is locally constant on every disc $D \to S$. Once $s \mapsto H^0(X_s, \omega^{[r]}_{X_s})$ is locally constant at $0 \in S$, Grauert’s theorem guarantees that $g_* \omega^{[r]}_{X/S}$ is locally free at $0 \in S$ and commutes with base changes.

In principle it could happen that for each $r$ we need a smaller and smaller neighborhood, but the same neighborhood works for all $r \geq 1$ by Lemma 11.
Thus the plurigenera are deformation invariant, all fibers are of general type, and $g$ is fiberwise bimeromorphic to the relative canonical model

$$X^c := \text{Proj}_S \oplus_{r \geq 0} g^m_* \omega^{[r]}_{X^m/S},$$

which is projective over $S$. The projectivity of all fibers now follows from the more precise Theorem 14. □

The following is a special case of [14, Thm.2].

**Theorem 14.** Let $g : X \to S$ be a flat, proper morphism of complex analytic spaces whose fibers have rational singularities only. Assume that $g$ is bimeromorphic to a projective morphism $g^p : X^p \to S$, and $X_0$ is projective for some $0 \in S$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $S = \bigcup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i \text{ is projective.}$$

□

5. Open Problems

For deformations of varieties of general type, the following should be true.

**Conjecture 15.** Let $X_0$ be a projective variety of general type with canonical singularities. Then its universal deformation space $\text{Def}(X_0)$ has a representative $X \to S$ where $S$ is a scheme of finite type and $X$ is an algebraic space.

For varieties of non-general type, the following is likely true [19, 1.10].

**Conjecture 16.** Let $g : X \to S$ be a flat, proper morphism of complex analytic spaces. Assume that $X_0$ is projective and with canonical singularities. Then the plurigenera $h^0(X_s, \omega^{[r]}_{X_s})$ are independent of $s \in S$ for every $r$, in some neighborhood of $0 \in S$.

**Comments.** One can try to follow the proof of Theorem 1. If $X_0$ is not of general type, we run into several difficulties in relative dimensions $\geq 4$. MMP is not know to terminate and even if we get a minimal model, abundance is not known. If we have a good minimal model, then we run into the following.

**Conjecture 17.** Let $X$ be a complex space and $g : X \to S$ a flat, proper morphism. Assume that $X_0$ is projective, has canonical singularities and $\omega^{[r]}_{X_0}$ is globally generated for some $r > 0$. Then the plurigenera are locally constant at $0 \in S$.

**Comments.** More generally, the same may hold if $X_0$ is Moishezon (that is, bimeromorphic to a projective variety), Kähler or in Fujiki’s class $C$ (that is, bimeromorphic to a compact Kähler manifold; see [21] for an introduction).

A positive answer is known in many cases. [8, 12.5.5] proves this if $X_0$ is projective and has terminal singularities. However, the proof works for the Moishezon and class $C$ cases as well.

The projective case with canonical singularities is discussed in [18, VI.3.15–16]; I believe that the projectivity assumption is very much built into the proof given there; see [18, VI.3.11].
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