THE COHOMOLOGY RING OF UNORDERED
CONFIGURATION SPACES OF THE TORUS

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Abstract. Configuration spaces of points are ubiquitous in mathematics. We compute the following invariants of the configuration space of unordered points on the two dimensional torus:
• the mixed Hodge structure on the cohomology,
• the action of the mapping class group on the cohomology,
• the formality over the rationals,
• the ring structure.

Introduction

We fully describe the cohomology with rational coefficients of the configuration space of unordered points on an elliptic curve (frequently called torus).

Configuration spaces of points are related to physics (state phase of non-colliding particles on a manifold, see [McD75]), robotics (motion planning), knot theory, and topology. Indeed, configuration spaces give invariants for the homeomorphism type of the base space. In the algebraic setting, configuration spaces are open in the moduli spaces of points.

Since the literature is very extensive, we compare our work only with the main results on the (co-)homology of configuration spaces.

The first computation of the cohomology algebra of configuration spaces is due to Arnol’d [Arn69, Arn70] in the case of $\mathbb{R}^2$, see also [CLM76]. This result has been generalized to the configuration space of $\mathbb{R}^n$ by Goresky and Macpherson [GMS83]. Here we approach the problem in the elliptic case using different techniques.

The Betti numbers of configuration spaces can be calculated using the Chevalley–Eilenberg complex: in [BC88] for once-punctured oriented surfaces, in [BCT89] for odd dimensional manifolds and in [DCK17] for surfaces in general. However there is no description of the ring structure; we provide it in the case of elliptic curves.

The Betti numbers $\text{Conf}_n(X)$ are described in the following cases: for $X = \mathbb{P}^2(\mathbb{R})$ [Wan02], for $X$ a sphere [Sal04], for $X = \mathbb{P}^2(\mathbb{C})$ [FT05] and for elliptic curves [Sch16, MCF16].

In this paper we improve the previous results on configuration spaces in an elliptic curve in three ways. We describe:
• the mixed Hodge structure on the cohomology (Theorem 3.2),
• the action of the mapping class group (Theorem 3.2),
• the ring structure (Theorem 4.1).
The formality result over the rationals is proven in Corollary 4.3.

We have computed these invariants using the Kriz model [Kri94, Bib16, Dup15] and the representation theory on it [AAB14, Aza15].
In Section 1 we recall the Kriz model, then (Section 2) we improve the result on the decomposition of the Kriz model into irreducible representations (Theorem 2.7). Descriptions of the mixed Hodge structure and of the action of the mapping class group are obtained in Section 3 by computing the cohomology of the model. Finally, the ring structure is presented in the last section.

1. Definition

Let \( E \) be an elliptic curve and consider, in the Cartesian product, \( E^n \) the configuration space of \( n \) ordered distinct points

\[
C^n(E) \overset{\text{def}}{=} \{ p \in E^n \mid p_i \neq p_j \}.
\]

The symmetric group \( \mathfrak{S}_n \) acts on \( C^n(E) \) permuting the coordinates and the quotient is the configuration space of \( n \) unordered points

\[
\mathcal{U}C^n(E) \overset{\text{def}}{=} C^n(E)/\mathfrak{S}_n.
\]

We also consider the space \( \mathcal{M}^n(E) \), defined by

\[
\mathcal{M}^n(E) \overset{\text{def}}{=} \{ p \in C^n(E) \mid \sum p_i = 0 \}.
\]

Notice that there exists a non canonical isomorphism \( C^n(E) \simeq E \times \mathcal{M}^n(E) \).

1.1. The Kriz model for the cohomology algebra. In this section we recall a model for the cohomology algebra of \( C^n(E) \). The model is a commutative differential bi-graded algebra (d.g.a.) that can be obtain in two different ways: as a specialization of the Kriz model for the configuration spaces or as the second page of the Leray spectral sequence for elliptic arrangements. Our main references for the first approach are [Kri94, AAB14, Aza15] and for the second one are [Dup15, Bib16].

In the following we define the models for the cohomology of \( C^n(E) \) and of \( \mathcal{M}^n(E) \).

Let \( \Lambda \) be the external algebra over \( \mathbb{Q} \) with generators

\[
\{ x_i, y_i, \omega_{i,j} \}_{1 \leq i < j \leq n}.
\]

We set the degree of each \( x_i \) and \( y_i \) equal to \((1, 0)\) and the degree of \( \omega_{i,j} \) equal to \((0, 1)\). Define the differential \( d: \Lambda \to \Lambda \) of bi-degree \((2, -1)\) on generators as follows: \( dx_i = 0 \) and \( dy_i = 0 \) for \( i = 1, \ldots, n \) and

\[
d \omega_{i,j} \overset{\text{def}}{=} (x_i - x_j)(y_i - y_j).
\]

For the sake of notation we set \( \omega_{i,j} := \omega_{j,i} \) for \( i > j \).

We define the d.g.a. \( A^{\bullet, \bullet} \) as the quotient of \( \Lambda \) by the following relations:

\[
(x_i - x_j)\omega_{i,j} = 0 \quad \text{and} \quad (y_i - y_j)\omega_{i,j} = 0,
\]

\[
\omega_{i,j}\omega_{j,k} - \omega_{i,j}\omega_{k,i} + \omega_{j,k}\omega_{k,i} = 0.
\]

Notice that the ideal is preserved by the differential map, thus the differential \( d: A^{\bullet, \bullet} \to A^{\bullet, \bullet} \) is well defined.

Remark 1.1. The model \( A^{\bullet, \bullet} \) coincides with the Kriz model \( E^{\bullet}_d \) introduced in [Kri94] up to shifting the degrees, i.e.

\[
A^{p,q} \simeq E^{p+q}_q.
\]
In order to study the cohomology of \( A^{••} \) we need to introduce the elements 
\[ u_{i,j} = x_i - x_j, \ v_{i,j} = y_i - y_j \text{ and } \gamma = \sum_{i=1}^{n} x_i, \ \overline{\gamma} = \sum_{i=1}^{n} y_i \in A^{1,0}. \]

We define the d.g.a. \( B^{••} \) as the subalgebra of \( A^{••} \) generated by \( u_{i,j}, v_{i,j} \) and \( \omega_{i,j} \) for \( 1 \leq i < j \leq n \). Let \( D^{•,0} \) be the subalgebra of \( A^{••} \) generated by \( \gamma \) and \( \overline{\gamma} \) endowed with the zero differential map. Notice that
\[
A^{••} \simeq B^{••} \otimes_{\mathbb{Q}} D^{•,0}
\] (1)
as differential algebras and that \( D^{•,0} \) is the cohomology ring of the elliptic curve \( E \).

The mixed Hodge structure on the cohomology of algebraic varieties defines a bi-gradation compatible with the algebra structure (see [Del75, p.81]).

The following result is a particular case of [Bib16, Theorem 3.3] and of [Dup15, Theorem 1.2].

**Theorem 1.2.** The cohomology algebra of \( C^n(E) \) (or of \( M^n(E) \)) with rational coefficients is isomorphic to the cohomology of the d.g.a. \( A^{••} \) (respectively of \( B^{••} \)). Moreover, the \( n^2 \)-sheeted covering
\[
E \times M^n(E) \to C^n(E)
\]
\[
(q,p) \mapsto (p_i + q_{i=1,...,n})
\]
induces the isomorphism of eq. (1).

2. **Representation theory on the Križ model**

Now we study the action of the symmetric group \( S_n \) and of \( SL_2(\mathbb{Q}) \) on the algebras \( A^{••} \) and \( B^{••} \). Those actions are given by a geometric action on \( C^n(E) \).

2.1. **Definition of the actions.** Consider the action of \( S_n \) on \( C^n(E) \) defined by
\[
\sigma^{-1} \cdot (p_1, \ldots, p_n) = (p_{\sigma(1)}, \ldots, p_{\sigma(n)})
\]
for all \( \sigma \in S_n \). This induces an action on \( A^{••} \) and on \( B^{••} \) defined by
\[
\sigma^{-1} \cdot (x_{i}) = x_{\sigma(i)},
\]
\[
\sigma^{-1} \cdot (y_{i}) = y_{\sigma(i)},
\]
\[
\sigma^{-1} \cdot (\omega_{i,j}) = \omega_{\sigma(i),\sigma(j)}
\]
for all \( 1 \leq i < j \leq n \) and all \( \sigma \in S_n \).

The mapping class group \( MCG(E) \) acts naturally on \( C^n(E) \) and on \( UC^n(E) \).

**Remark 2.1.** The action of the mapping class group \( MCG(E) \) on the cohomology \( H^1(E;\mathbb{Z}) \) factor through the symplectic modular group \( Sp_2(\mathbb{Z}) = SL_2(\mathbb{Z}) \) since it preserve the intersection form. The surjectivity of the map \( MCG(E) \to SL_2(\mathbb{Z}) \) follows from the study of Dehn twists. This action extends to the rationals: \( SL_2(\mathbb{Q}) \) acts on \( H^1(E;\mathbb{Q}) \).

The action of \( SL_2(\mathbb{Q}) \) on \( H^1(E;\mathbb{Q}) \) extend to \( A^{••} \) (and on \( B^{••} \)), defined as follows. We denote the irreducible representation of \( SL_2(\mathbb{Q}) \) of dimension \( k+1 \) by \( \nabla_k \). The group \( SL_2(\mathbb{Q}) \) acts trivially on \( \omega_{i,j} \) for all \( 1 \leq i < j \leq n \) and, for all \( i \leq n \), as the irreducible representation \( \nabla_1 \) on the two dimensional subspace generated by \( x_i \) and \( y_i \).
The maximal torus \( h : \mathbb{C}^* \hookrightarrow \text{SL}_2(\mathbb{Q}) \), given by the diagonal matrices \( M_i = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \), have a nice action on generators:
\[
\begin{align*}
h(M_i) & \cdot x_i = tx_i, \\
h(M_i) & \cdot y_i = t^{-1}y_i, \\
h(M_i) & \cdot \omega_{i,j} = \omega_{i,j},
\end{align*}
\]
for all \( 1 \leq i < j \leq n \).

**Remark 2.2.** Since the actions of \( \mathfrak{S}_n \) and of \( \text{SL}_2(\mathbb{Q}) \) commute, then \( A^{**}, B^{**} \) and \( D^{*0} \) become \( \mathfrak{S}_n \times \text{SL}_2(\mathbb{Q}) \)-modules.

### 2.2. Decomposition into \( \mathfrak{S}_n \)-representations

We recall a result of [AAB14, Theorem 3.15] on the decomposition of \( A^{**} \) into \( \mathfrak{S}_n \)-modules. The notations used here follow the ones in [AAB14].

Let \( L_\ast = (\lambda_1, \ldots, \lambda_t) \) be a partition of \( n \). We mark all blocks with labels in \( \{1, x, y, xy\} \), an ordered basis of \( H^*(E) \). The order is \( 1 \prec x \prec y \prec xy \).

**Definition 2.3.** A *marked partition* \( (L_\ast, H_\ast) \) is a partition \( L_\ast \vdash n \) together with marks \( H_\ast = (h_1, \ldots, h_t), h_i \in \{1, x, y, xy\} \) such that: if \( \lambda_i = \lambda_{i+1} \) then \( h_i \geq h_{i+1} \).

For any partition \( L_\ast \vdash n \) consider the cycles \( c_i = (L_{i-1} + 1, L_{i-1} + 2, \ldots, L_i) \) for \( L_i = \sum_{j=1}^t \lambda_j \). Let \( C_{L_\ast} \) be the subgroup of \( \mathfrak{S}_n \) generated by the disjoint cycles \( c_i \) for \( i = 1, \ldots, t \).

Consider a marked partition \( (L_\ast, H_\ast) \), for any pair of blocks with same length and same mark \( (\lambda_i = \lambda_j, h_i = h_j) \) define the permutation
\[
\sigma_{i,j} = (L_{i-1} + 1, L_{j-1} + 1)(L_{i-1} + 2, L_{j-1} + 2) \cdots (L_i, L_j).
\]

Let \( N_{L_\ast, H_\ast} \) be the subgroup of the normalizer of \( C_{L_\ast} \) generated by the permutations \( \sigma_{i,j} \) for each pair of blocks with same length and same mark. This group is isomorphic to a product of symmetric groups. Call \( Z_{L_\ast, H_\ast} \) the semidirect product \( C_{L_\ast} \rtimes N_{L_\ast, H_\ast} \).

**Example 2.4.** Let \( (L_\ast, H_\ast) \) be the marked partition \( L_\ast = (2, 2, 2, 1, 1) \vdash 8 \) and \( H_\ast = (x, x, 1, x, x) \). The subgroup \( C_{L_\ast} \simeq (\mathbb{Z}/2\mathbb{Z})^3 \) of \( \mathfrak{S}_8 \) is generated by \( (1, 2), (3, 4) \) and \( (5, 6) \). The subgroup \( N_{L_\ast, H_\ast} \simeq \mathfrak{S}_2 \times \mathfrak{S}_2 \) is generated by \( (1, 3)(2, 4) \) and \( (7, 8) \). Finally, \( Z_{L_\ast, H_\ast} \) is a group isomorphic to \( (\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_2) \times \mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_2 \).

We define the following one-dimensional representations. Let \( \varphi_n \) be a non-trivial character of the cyclic group and \( \varphi_{L_\ast} \), the character of \( C_{L_\ast} \simeq \mathbb{Z}/\lambda_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/\lambda_t \mathbb{Z} \) given by
\[
\varphi_{L_\ast} \overset{\text{def}}{=} \text{sgn}_n |C_{L_\ast}| \cdot (\varphi_{\lambda_1} \boxtimes \cdots \boxtimes \varphi_{\lambda_t}).
\]

Recall that the degree \( \text{deg} \) of \( 1, x, y, xy \) are respectively \( 0, 1, 1, 2 \). Let \( \alpha_{L_\ast, H_\ast} \) be the one dimensional representation of \( N_{L_\ast, H_\ast} \simeq \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_t} \) defined on generators by
\[
\alpha_{L_\ast, H_\ast} (\sigma_{i,j}) \overset{\text{def}}{=} (-1)^{\lambda_i + \text{deg}(h_i)} + 1.
\]

Set \( \xi_{L_\ast, H_\ast} \) to be the one dimensional representation \( \varphi_{L_\ast} \boxtimes \alpha_{L_\ast, H_\ast} \) of \( Z_{L_\ast, H_\ast} \).

We define \( |L_\ast| = n - t \) for a partition \( L_\ast = (\lambda_1, \ldots, \lambda_t) \) of \( n \) and for a mark \( H_\ast \) the numbers \( |H_\ast| = \sum_{i=1}^t \text{deg}(h_i) \) and \( \|H_\ast\| = |\{ i \mid h_i = x \}| - |\{ i \mid h_i = y \}| \).
Theorem 2.5 ([AAB14, Theorem 3.15]). The algebra $A^{••}$ decomposes as $S_n$-representation in the following way:

$$A^{p,q} = \bigoplus_{|L_*|=q, |H_*|=p} \text{Ind}_{\mathfrak{sl}_2 \times H_*}^{S_n} \xi_{L_*}, H_*.$$

Example 2.6. Consider the marked partition $(L_*, H_*)$ of Example 2.4, the characters are shown in the following table.

| $(1,2)$ | $(3,4)$ | $(5,6)$ | $(1,3)(2,4)$ | $(7,8)$ |
|---|---|---|---|---|
| $\varphi$ | $-1$ | $-1$ | $-1$ | $1$ |
| $\alpha$ | $-1$ | $-1$ | $-1$ | $1$ |
| $\xi$ | $-1$ | $-1$ | $-1$ | $1$ |

2.3. Decomposition into $S_n \times SL_2(\mathbb{Q})$-representations. We will use the decomposition of Theorem 2.5 to obtain a decomposition of $A^{••}$ into $S_n \times SL_2(\mathbb{Q})$-modules.

Recall that the representation $V_k$ of $SL_2(\mathbb{Q})$, once restricted to a representation of the maximal torus $\mathbb{C}^*$, splits as

$$V_k = \bigoplus_{a=0}^k V(2a - k),$$

where $V(b)$ is the subspace of vectors $v$ such that $h(M_t) \cdot v = t^b v$.

Theorem 2.7. The algebra $A^{••}$ decomposes as $S_n \times SL_2(\mathbb{Q})$-representation in the following way:

$$A^{p,q} \cong \bigoplus_{a=0}^p (A_a^p \otimes A_{a+2}^q) \boxtimes V_a,$$

where

$$A_a^p \overset{\text{def}}{=} \bigoplus_{|L_*|=q, |H_*|=p, \|H_*\|=a} \text{Ind}_{\mathfrak{sl}_2 \times H_*}^{S_n} \xi_{L_*}, H_*.$$

Notice that $A_a^p$ is zero if $a \not\equiv p \mod 2$, or if $a < p$, or if $a > n - q - p$. Observe also that the virtual representation $A_a^p \otimes A_{a+2}^q$ is effective.

Proof of Theorem 2.7. Observe that the maximal torus of $SL_2(\mathbb{Q})$ acts on $A_a^p$ with weight $a$, thus by Theorem 2.5 we have

$$A^{p,q} = \bigoplus_{a=-p}^p A_a^{p,q}.$$

Using (2) we obtain the thesis. \qed

Define the $S_n$-invariant subalgebra of $A^{••}$ by $UA^{••}$ and of $B^{••}$ by $UB^{••}$. Obviously we have $UA^{••} = UB^{••} \otimes \mathbb{Q} D^*.$

We use the previous calculation to compute $UA^{••}$.

Corollary 2.8. For $q > p + 1$ we have $UA^{p,q} = 0$. 
Proof. Let $I_n$ be the trivial representation of $S_n$. We use Theorem 2.3 to show that
\[
\langle I_n, A^{p,q}\rangle_{\mathfrak{S}_n} = 0
\]
for $q > p + 1$. Indeed, it is enough to prove that
\[
\langle I_n, \text{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{L,\mathbb{H}}^n} \xi_{L,\mathbb{H}}\rangle_{\mathfrak{S}_n} = 0
\]
for all $(L, H)$ with $|L| = q$ and $|H| = p$. By Frobenius reciprocity we have
\[
\langle I_n, \text{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{L,\mathbb{H}}^n} \xi_{L,\mathbb{H}}\rangle_{\mathfrak{S}_n} = \langle \text{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{L,\mathbb{H}}^n} I_n, \xi_{L,\mathbb{H}}\rangle_{Z_{L,\mathbb{H}}^n}
\]
Since the representations in the right hand side are one-dimensional the value of
\[
\langle I_n, \text{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{L,\mathbb{H}}^n} \xi_{L,\mathbb{H}}\rangle_{\mathfrak{S}_n}
\]
is non zero if and only if $\xi_{L,\mathbb{H}} = \mathbb{1}$.

By definition $\xi_{L,\mathbb{H}} = \mathbb{1}$ is equivalent to $\varphi_{L} = \mathbb{1}$ and $\alpha_{L,\mathbb{H}} = \mathbb{1}$. From the fact that $\varphi_k = \text{sgn}_k$ only for $k = 1, 2$, $\psi_k = \mathbb{1}$ if and only if $\lambda_i = 1, 2$ for all $i = 1, \ldots, t$. The condition $\alpha_{L,\mathbb{H}} = \mathbb{1}$ implies that the only marked blocks of $(L, H)$ that appears more than once are the ones with $\lambda_i = 2$ and $\deg(h_i) = 1$ or the ones with $\lambda_i = 1$ and $\deg(h_i) \neq 1$.

Consequently, $\langle I_n, \text{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{L,\mathbb{H}}^n} \xi_{L,\mathbb{H}}\rangle_{\mathfrak{S}_n} \neq 0$ only if $L = (2^i, 1^{n-2q})$ and the degree of $h_i$ is 1 for $i < q$. By hypothesis we have $|H| = p < q - 1$, thus $\langle I_n, \text{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{L,\mathbb{H}}^n} \xi_{L,\mathbb{H}}\rangle_{\mathfrak{S}_n} = 0$. \hfill \Box

Corollary 2.9. For $q > p + 1$ we have $UB^{p,q} = 0$. \hfill \Box

3. The additivity structure of the cohomology

We compute the cohomology with rational coefficients of the unordered configuration spaces of $n$ points, taking care of the Mixed Hodge structure and of the action of $SL_2(\mathbb{Q})$. The integral cohomology groups are known only for small $n$ in [Nap03, Table 2], where a cellular decomposition of ordered configuration spaces is given.

In this section, we use the calculation of the Betti numbers of $\mathcal{U}C^n(E)$ to determine the Hodge polynomial in the Grothendieck ring of $SL_2(\mathbb{Q})$.

Observe that $H^\bullet(\mathcal{U}C^n(E)) = H^\bullet(\mathcal{C}^n(E))^{\mathfrak{S}_n}$ by the Transfer Theorem. Define the series
\[
T(u, v) = \frac{1 + u^3v^4}{(1 - uv^2)^2} = 1 + 2u^2v^3 + u^3v^4 + 3u^4v^6 + 2u^5v^7 + \ldots
\]
and let $T_n(u, v)$ be its truncation at degree $n$ in the variable $u$.

The computation of the Betti numbers of unordered configuration space of $n$ points in an elliptic curve was done simultaneously by [DCK17], [MCF16], and [Sch16] in different generality. We point to the last reference because [Sch16, Theorem] fits exactly our generality.

Theorem 3.1. The Poincaré polynomial of $\mathcal{U}C^n(E)$ is $(1 + t)^2T_{n-1}(t, 1)$.

We use the notation $V u^k v^h$ to denote a vector space $V$ in degree $k$ with an Hodge structure of weight $h$. We prove a stronger version of Theorem 3.1.

Theorem 3.2. The Hodge polynomial of $\mathcal{U}C^n(E)$ with coefficients in the Grothendieck ring of $SL_2(\mathbb{Q})$ is
\[
(V_0 + V_1 uv + V_0 u^2v^2) \left( \sum_{i=0}^{n-1} V_i u^{2i}v^{3i} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} V_{i-1} u^{2i+1}v^{3i+1} \right)
\]
and the ordinary Hodge polynomial is \((1 + uv)^2T_{n-1}(u, v)\).

Figure 1 represent the module \(H^{**}(UB, d)\) that correspond to the right factor of eq. [4].

The stable cohomology of \(\{\mathcal{UC}^n(E)\}_n\) is \(\lim\) \(H^k(\mathcal{UC}^n(E); \mathbb{Q})\), see \cite{Chu12, EW15, CEF15} for the definition and some properties. However, in our case the stable cohomology in degree \(k\) coincides with \(H^k(\mathcal{UC}^n(E); \mathbb{Q})\) for any \(n \geq k + 1\).

**Remark 3.3.** The Hodge polynomial of stable cohomology in the Grothendieck ring of \(SL_2(\mathbb{Q})\) is
\[
\Lambda^* \mathcal{V}_1 u v \cdot S^* \mathcal{V}_1 u^2 v^3 \cdot (1 + u^3 v^4).
\]

### 3.1. Some elements in cohomology.

**Definition 3.4.** Let \(\alpha, \overline{\alpha} \in A^{1,1}, \beta \in A^{1,2}\) be the elements
\[
\alpha \overset{\text{def}}{=} \sum_{i, k, h} (x_i - x_k) \omega_{k, h} \\
\overline{\alpha} \overset{\text{def}}{=} \sum_{i, k, h} (y_i - y_k) \omega_{k, h} \\
\beta \overset{\text{def}}{=} \sum_{i, j, k, h} (3x_i - x_j - 2x_k)(y_j - y_k) \omega_{k, h}
\]

where the sum is taken over different indices \(i, j, k, h\) with \(k < h\).

Notice that the elements \(\alpha\) and \(\overline{\alpha}\) are defined only for \(n > 2\) and \(\beta\) for \(n > 3\). Remember that \(\gamma, \overline{\gamma} \in D^{1}\) were already defined as \(\sum_i x_i\) and \(\sum_i y_i\).

**Lemma 3.5.** The element \(\alpha\) belongs to \(UB^{1,1}\), is non-zero, and \(d \alpha = 0\).

**Proof.** First observe that \(\alpha = \sum_{i, k, h} u_{i, k} \omega_{k, h} \in B^{1,1}\). For all \(\sigma \in S_n\) we have \(\sigma \alpha = \alpha\). Since the elements \(\omega_{i, j}\) are linearly independent it is enough to observe
that the coefficient of $\omega_{1,2}$ is non-zero. Finally, we compute $d\alpha$:

$$d\alpha = \sum_{i,k<h} x_i d\omega_{k,h} - x_k d\omega_{k,h}$$

$$= \sum_{i,k<h} x_i(x_k - x_h)(y_k - y_h) + x_k x_h(y_k - y_h)$$

$$= \sum_{i,k,h} x_i x_k y_k - x_i x_h y_k + x_k x_h y_k$$

$$= -\sum_{i,k,h} x_i x_h y_k = 0,$$

where all sums are taken over three distinct indexes. \hfill \Box

**Lemma 3.6.** The element $\beta$ belongs to $UB^{1,2}$, is non-zero, and $d\beta = 0$.

**Proof.** Observe that

$$\beta = \sum_{i,j,k<h} (u_{i,j} + 2u_{i,k})v_{j,k}\omega_{k,h} \in B^{1,2}$$

and that $\sigma \beta = \beta$ for all $\sigma \in S_n$ by the relation $v_{j,k}\omega_{k,h} = v_{j,h}\omega_{k,h}$. The coefficient of $\omega_{1,2}$ is non-zero and so $\beta \neq 0$. Using the computation in the proof of Lemma 3.5 we can observe that $d\left( \sum_{i,j,h<k} 3x_i(y_j - y_k)\omega_{k,h} \right) = 0$. The claim $d\beta = 0$ follows from:

$$d(\beta) = d\left( \sum_{j,k,h} (x_j + 2x_k)(y_j - y_k)\omega_{k,h} \right)$$

$$= \sum_{j,k<h} x_j y_j d\omega_{k,h} + x_j y_k(x_k - x_h)y_h - 2x_k y_j x_h(y_k - y_h) - 2x_k y_k x_h y_h$$

$$= \sum_{j,k,h} x_j y_j x_k y_k - x_i x_j x_h y_k + x_j y_k x_k y_h - 2x_k y_j x_h y_k - x_k y_k x_h y_h$$

$$= 0 \hfill \Box$$

**Lemma 3.7.** For $n > 2q$ the element $\alpha^q$ is non-zero.

**Proof.** Let us rewrite $\alpha$ as $\alpha = \sum_{i,k<h} x_i \omega_{k,h} - (n-2) \sum_{k<h} x_k \omega_{k,h}$. We show that the coefficient of the monomial $x_1 \omega_{1,2} x_3 \omega_{3,4} \cdots x_{2q-1} \omega_{2q-1,2q}$ (defined for $n \geq 2q$) is non-zero for $n > 2q$. This coefficient is

$$a_q = q! \sum_{\sigma \in S_q} sgn(\sigma)(2-n)^{\text{Fix}\sigma}2^{|\sigma|-|\text{Fix}\sigma|}.$$  

We claim that

$$\sum_{\sigma \in S_q} sgn(\sigma)x^{\text{Fix}\sigma} = (x - 1)^{k-1}(x + q - 1),$$

since both sides are the determinant of the matrix

$$\begin{pmatrix}
  x & 1 & \cdots & 1 \\
  1 & x & \cdots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & \cdots & x
\end{pmatrix}.$$  

Thus $a_q = (-1)^q q! n^{q-1}(n - 2q)$ that is non-zero for $n > 2q$. \hfill \Box
Lemma 3.8. For \( n > 2q + 1 \) the element \( \alpha^{q-1} \beta \) is non-zero.

Proof. Let us rewrite \( \beta \) as

\[
\beta = \sum_{i,j,k<h} x_i y_j \omega_{k,h} - 2(n-3) \sum_{i,k<h} (x_i y_k + x_k y_i) \omega_{k,h} - (n-3) \sum_{i,k<h} x_i y_i \omega_{k,h} + 2(n-2)(n-3) \sum_{k<h} x_k y_k \omega_{k,h}.
\]

Let \( b_q \) be the coefficient in \( \alpha^{q-1} \beta \) of the monomial

\[
x_1 \omega_{1,2} x_3 \omega_{3,4} \cdots x_{2q-1} \omega_{2q-1,2q} y_{2q+1}.
\]

This monomial is defined for \( n \geq 2q + 1 \) and we will show that \( b_q \neq 0 \) for \( n > 2q + 1 \). The number \( b_k \) coincides with the coefficient of the same monomial in the product

\[
\alpha^{q-1} \left( \sum_{i,j,k<h} x_i y_j \omega_{k,h} - 2(n-3) \sum_{i,k<h} x_k y_i \omega_{k,h} \right).
\]

With further manipulation, we obtain that \( b_q \) is the coefficient of the above monomial in the expression

\[
3\alpha^q y_{2q+1} + n\alpha^{q-1} \sum_{k<h} x_k \omega_{k,h} y_{2q+1}.
\]

Using the computation in the proof of Lemma 3.7, we obtain

\[
b_q = 3(-1)^q q!n^{q-1}(n-2q) + nq(-1)^q (q-1)!n^{q-2}(n-2q+2) = 2(-1)^q q!n^{q-1}(n-2q-1).
\]

□

Proof of Theorem 3.2. It is enough to prove that the Hodge polynomial of \( UB \) in the Grothendieck ring of \( SL_2(\mathbb{Q}) \) is

\[
\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathcal{V}_i u^{2i} v^{3i} + \sum_{i=1}^{\lfloor \frac{q-1}{2} \rfloor} \mathcal{V}_{i-1} u^{2i+1} v^{3i+1}.
\]

4. THE COHOMOLOGY RING

In this section we determine the cup product structure in the cohomology of \( UC^n(E) \) and we prove the formality result.

In the following we consider graded algebras with an action of \( SL_2(\mathbb{Q}) \) and all the ideal are taken to be \( SL_2(\mathbb{Q}) \)-stable.

Theorem 4.1. The cohomology ring of \( C^n(E) \) is isomorphic to

\[
\Lambda^* \mathcal{V}_1 \otimes S^* \mathcal{V}_1 [b]/(a^{n+1}, a^{\frac{n+1}{2}} b, b^2),
\]

where \( a \) is an element in \( V(1) \subset S^1 \mathcal{V}_1 \) and \( b \) an indeterminate \( SL_2(\mathbb{Q}) \)-invariant of degree 3.
Proof. It is enough to prove that \( H^*(UB) \simeq S^*V_1 / (a^2, b^2) \). Define the morphism \( \varphi : S^*V_1 / (a^2, b^2) \to H^*(UB) \) that sends \( a, b \) to \( \alpha, \beta \) respectively. It is well defined because \( H^k(UB) = 0 \) for \( k \geq n \) and \( \beta^2 = 0 \) since it has odd degree. We show that \( \varphi \) is a morphism of algebras. Observe that the multiplication map \( H^2(UB)^{\otimes k} \to H^{2k}(UB) \) for \( n > 2k \) is non-zero by Lemma 3.7, thus it coincides with
\[
H^2(UB)^{\otimes k} \simeq V_1^{\otimes k} \to S^k V_1 \simeq \cong H^{2k}(UB).
\]
For \( n > 2k + 1 \) by Lemma 3.8 the multiplication map \( H^{2k-2}(UB) \otimes H^3(UB) \to H^{2k+1}(UB) \) is non-zero. Moreover, it is an isomorphism
\[
H^{2k-2}(UB) \otimes H^3(UB) \simeq V_{2k-2} \otimes V_0 \simeq V_{2k-2} \simeq H^{2k+1}(UB).
\]
The map \( \varphi \) is surjective since \( H^*(UB) \) is generated by \( \alpha^i \) and \( \alpha^i \beta \) as \( SL_2(\mathbb{Q}) \)-module by Theorem 3.2. A dimensional reasoning shows the injectivity of \( \varphi \).

**Corollary 4.2.** The cohomology \( H^*(\mathcal{U}C^n(E)) \) is generated as algebra in degrees one, two and three.

**Proof.** A minimal set of generators is given by \( \alpha, \overline{\alpha}, \beta, \gamma, \gamma, \overline{\gamma} \).

**Corollary 4.3.** The space \( \mathcal{U}C^n(E) \) is formal on the rationals.

**Proof.** We prove that \( (UB, d) \) is formal. Consider the subalgebra \( K^{**} \) of \( UB^{**} \) generated by \( \alpha, \overline{\alpha}, \beta \) endowed with the zero differential. It is concentrate in degrees \( (i, i) \) and \( (i, i+1) \) because \( \beta^2 = 0 \). Since \( K \cap \text{Im} d = 0 \) (Corollary 2.9), the algebra \( (UB, d) \) is formal. As a consequence \( UA \) is formal. The space \( \mathcal{U}C^n(E) \) is formal since our model \( (UA, d) \) is equivalent to the Sullivan model.

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