Quantum Phase Diagrams via Fidelity of Matter-Field Hamiltonians

S. Cordero,* E. Nahmad–Achar, R. López–Peña, and O. Castaños
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, 04510 México DF, Mexico
(Dated: February 10, 2020)

A general procedure is established to calculate the quantum phase diagrams for matter-field Hamiltonian models, via the fidelity concept of quantum information theory. For the generalized Tavis-Cummings and Dicke models of 3-level systems interacting dipolarly with 2 modes of electromagnetic radiation, a quantum phase diagram is explicitly established for \( N_a = 1 \) and \( N_a = 4 \) particles. The quantum phase transitions are classified according to their continuous or discontinuous behavior across the separatrices of the system; a finite-effect characterization of the quantum phase transitions in terms of their continuity and stability is proposed.

I. INTRODUCTION

The analysis of the phase diagram structure of a system of atoms in the presence of a radiation field is important, mainly due to the presence of phase transitions. In contrast to their classical finite-temperature counterparts, quantum systems undergo phase transitions at absolute zero, which connect their static and dynamic behavior, and these take place as the Hamiltonian parameters vary [1]. It is characteristic in these diagrams the distinction of a normal and a collective regime, the difference being the decay rate being proportional to the square of the number of particles \( N_a \) in the latter as opposed to being linear in \( N \) (the expected result for independent atomic emission) [2, 3].

Being quantum optical systems the foundational elements for quantum information and quantum computing, the interest in these studies goes beyond the mere understanding of how they behave into the realm of serious applications, with particular interest in the case of a finite number of particles. Furthermore, systems with a finite number of levels constitute important simplified models for the interaction between matter and radiation which are not only physically realistic, but tractable mathematically [4]. A treatment of the early work on cavity QED with quantized modes demonstrating the reaction of the matter back to the field may be found in [5]. There, the interaction of a 2- or 3-level atom interacting with quantized cavity fields in the rotating wave approximation was studied using the time-dependent Schrödinger equation.

The characterization of quantum phase transitions in terms of the overlap function (fidelity) between two ground states was first described in [6], and the procedure was applied to several Hamiltonian models. The notion of fidelity has been used in the context of information theory as a measure of distinguishability of probability distributions, and applications to the study of phase transitions in condensed matter systems and some of its geometric features have been presented in [7]. For a pedagogical review of the quantum fidelity approach to quantum phase transitions see [8].

In this work we carry out the construction of the ground state energy surfaces for both the generalized Tavis-Cummings model and the generalized Dicke model Hamiltonians. The quantum phase diagrams for the various atomic configurations are built by using the fidelity between neighboring states, and the parity (in terms of the symmetry operators for the system) of the ground states that conform these energy surfaces in different regions is studied. A finite-effect characterization of the quantum phase transitions in terms of their continuity and stability is proposed.

For ease of understanding, explicit examples are given in the case of 3-level atoms in all their configurations interacting with 2 modes of an electromagnetic field, but the work is easily generalized to atoms of any number of levels and to any number of radiation modes.

This paper is organized as follows: Section II describes the Hamiltonian for a system of \( N_a \) atoms of \( n \) levels under dipolar interaction with \( m \) modes of an electromagnetic field, where only one mode promotes transitions between two given atomic levels. The symmetries of the Hamiltonian are calculated, as well as the basis states in terms of direct products of harmonic oscillators for the field and matter sectors, and its dimension. The construction of the ground state energy surfaces for both the generalized Tavis-Cummings model (GTCM) and the generalized Dicke model (GDM) Hamiltonians is presented in section III, and the quantum phase diagrams for the various atomic configurations of 3-level systems are built by using the fidelity between neighboring states. The parity of the ground states that conform these energy surfaces in different regions is also studied. Section IV deals with the characterization of the quantum phase transitions described earlier in terms of their continuity and stability. We end in section V with a summary of the results presented.

II. MODEL

In this work we study a generalized Dicke model (GDM) by considering a system of \( n \)-level atoms interacting dipolarly with \( \ell \) modes of an electromagnetic field...
Here, $H_D$ is the diagonal part in the harmonic oscillator basis states $| n_\alpha, b_k\rangle$ with $\alpha = 1, 2, \ldots, \ell$ and $k = 1, 2, \ldots, n$, given by

$$ H_D = \sum_{s=1}^\ell \Omega_s \nu_s + \sum_{j=1}^n \omega_j A_{jj}, $$

where we use $\hbar = 1$; $\Omega_s$ denotes the field frequency of the $s$-th mode, $\nu_s = a_s^\dagger a_s$ the $s$-th photon number operator with $a_s^\dagger$, $a_s$ the corresponding boson ladder operators, $\omega_j$ the $j$-th atomic level, for which we adopt as convention $\omega_j < \omega_k$ for $j < k$ and fix $\omega_1 = 0$ and $\omega_n = 1$, and $A_{jj}$ is the $j$-th level atomic population operator.

$H^{(s)}_{\text{int}}$ is associated to the dipolar interaction of $s$-mode photons with the $n$-level atoms,

$$ H^{(s)}_{\text{int}} = -\frac{1}{\sqrt{Na}} \sum_{j<k} \mu_{jk}^{(s)} (A_{jk} + A_{kj}) (a_s + a_s^\dagger). $$

Here $\mu_{jk}^{(s)}$ stands for the dipolar coupling strength and the collective atomic operators $A_{jk}$ obey the $U(n)$ algebra, i.e.,

$$ [A_{lm}, A_{kj}] = \delta_{mk} A_{lj} - \delta_{jl} A_{km}. $$

The total number of atoms is determined by the first order Casimir operator

$$ \sum_{k=1}^n A_{kk} = Na I. $$

This model is simplified by considering that transitions between a pair of atomic levels are promoted only by one mode of the electromagnetic field, which is enforced by the condition [9]

$$ \text{If } \mu_{jk}^{(s)} \neq 0 \text{ then } \mu_{jk}^{(s')} = 0 \text{ for } s' \neq s. \ (6) $$

TABLE I. The non-zero dipolar coupling strengths for 3-level atoms interacting with two modes of electromagnetic field determine a particular atomic configuration.

| Configuration | non-zero Dipolar strengths |
|---------------|-----------------------------|
| $\Xi$         | $\mu_{12}^{(1)}, \mu_{23}^{(2)}$ |
| $\Lambda$     | $\mu_{13}^{(1)}, \mu_{23}^{(2)}$ |
| $\nu$         | $\mu_{12}^{(1)}, \mu_{13}^{(2)}$ |

Notice that this condition implies $1 \leq \ell \leq n$ and when $\ell = \ell_0$ with $\ell_0$ the maximum number of permitted atomic transitions, the full system may be divided into two-level sub-systems [10].

We should also note that the interaction term (3) can be divided into two contributions, viz. the rotating and counter-rotating terms; these read for each mode $\Omega_s$ as

$$ H^{(s)}_R = -\frac{1}{\sqrt{Na}} \sum_{j<k} \mu_{jk}^{(s)} (A_{jk} a_s^\dagger + A_{kj} a_s), \ (7) $$

$$ H^{(s)}_C = -\frac{1}{\sqrt{Na}} \sum_{j<k} \mu_{jk}^{(s)} (A_{jk} a_s + A_{kj} a_s^\dagger), \ (8) $$

respectively.

Thus, the Hamiltonian in the rotating wave approximation (RWA), namely the generalized Tavis-Cummings model (GTCM), which preserves the total number of excitations, is obtained by neglecting the counter-rotating terms in (1), i.e.,

$$ H_{\text{rwa}} = H_D + \sum_{s=1}^\ell H^{(s)}_R. \ (9) $$

For $n$-level atoms interacting dipolarly with $\ell$ modes of an electromagnetic field it is well known that there are different atomic configurations. Notice that our convention on the atomic levels allows us to determine a particular atomic configuration by choosing the appropriate dipolar strength $\mu_{jk}^{(s)}$ to vanish. As an example of this, the non-zero dipolar strengths for the atomic configurations of 3-level atoms interacting with two field modes are given in table I.

We have found that it is very useful to use the critical values of the dipolar couplings for 2-level systems [11, 12]

$$ \mu_{jk}^{(s)} := \frac{1}{2} \sqrt{\Omega_s \omega_{jk}}, \quad \omega_{jk} := |\omega_j - \omega_k|, \ (10) $$

FIG. 1. Schematics of an $n$-level atom interacting with $\ell$ modes of electromagnetic field. Transitions between two atomic levels $\omega_j \leftrightarrow \omega_k$, promoted by the mode $\Omega_s$, is indicated with a nonzero value of the dipolar strength $\mu_{jk}$. 

TABLE I. The non-zero dipolar coupling strengths for 3-level atoms interacting with two modes of electromagnetic field determine a particular atomic configuration.
and rewrite the Hamiltonian in terms of dimensionless dipolar strengths \( x_{jk}^{(s)} \) and detuning parameters \( \Delta_{jk}^{(s)} \):

\[
x_{jk}^{(s)} := \frac{\mu_{jk}^{(s)}}{\mu_{jk}^{(s)}}, \quad \Delta_{jk}^{(s)} := \frac{\Omega_s}{\omega_{jk}} - 1.
\]  

(A. Symmetries)

The Hamiltonian (1) preserves the number of particles \( N_a \), and by simple inspection one notes that at least the parity of the total number of excitations is preserved. In order to find the preserved parities of the full Hamiltonian (1) it is convenient to consider the case of the rotating wave approximation (9), because each constant of motion of (9) is related to a parity symmetry of the full Hamiltonian (1). Thus, we propose as a general constant of motion the linear operator

\[
K = \sum_{s} \eta_s \nu_s + \sum_{j=1}^{n} \lambda_j A_{jj},
\]

where the coefficients \( \eta_s \) and \( \lambda_j \) are obtained by imposing the condition \( \{H_{\text{RWA}}, K\} = 0 \). This yields a system of \( R \) independent linear equations given by

\[
\mu_{jk}^{(s)} (\eta_s + \lambda_j - \lambda_k) = 0,
\]

where \( R \) is the rank of the system. We then have \( \zeta_0 = \ell + n - R - 1 \) independent constants of motion, apart from the total number of particles \( N_a \). As an example, Table II shows the operators \( K \) for each atomic configuration in the case of 3-level atoms interacting with two field modes.

As the Hamiltonian in the RWA approximation possesses \( \zeta_0 \) linear independent constant of motion, \( K_1, K_2, \ldots, K_{\zeta_0} \), the full Hamiltonian will preserve \( \zeta_0 \) symmetries defined by the operators

\[
\Pi_\zeta := \exp(i \pi K_\zeta), \quad \zeta = 1, \ldots, \zeta_0.
\]

(B. Bases)

The elements of the basis states are given by a direct product of harmonic oscillators for the field and matter contributions

\[
|\vec{\nu}; \vec{b}\rangle = |\nu_1, \nu_2, \ldots, \nu_l\rangle \otimes |b_1, b_2, \ldots, b_n\rangle,
\]

with \( \nu_s = 0, 1, \ldots \), and positive integer values \( b_k \) which fulfill \( b_1 + b_2 + \cdots + b_n = N_a \), with \( A_{jj'}|\vec{\nu}; \vec{b}\rangle = b_j|\vec{\nu}; \vec{b}\rangle \).

Exploiting the fact that \( K_\zeta \) are constants of motion in the RWA approximation, and using a set of fixed values \( \kappa := \{k_1, k_2, \ldots, k_{\zeta_0}\} \) as indices for the operators \( K_\zeta \), the basis in the RWA approximation may be written as

\[
B_{\text{RWA}}^{(\kappa)} := \{ |\vec{\nu}; \vec{b}\rangle | K_\zeta |\vec{\nu}; \vec{b}\rangle = k_\zeta |\vec{\nu}; \vec{b}\rangle; \zeta = 1, \ldots, \zeta_0 \}.
\]

The dimension of this basis for three-level atoms interacting with two modes of electromagnetic field is given in Appendix A for each atomic configuration. We may see that for special values of \( k_1 \) and \( k_2 \) the dimension of the basis is equal to the degeneracy of a 3-dimensional harmonic oscillator with \( N_a \) quanta.

The matrix representation of the Hamiltonian (1) in the basis \( B_{\text{RWA}}^{(\kappa)} \) yields the matrix \( H_{\text{RWA}} \) in the RWA approximation, because the basis (15) preserves the values of the set \( \kappa \), that is, the matrix elements of the counter-rotating terms with respect to the basis \( B_{\text{RWA}}^{(\kappa)} \) vanish. Therefore, a basis for the full Hamiltonian can be given as the direct sum of the RWA bases.

Let us denote the parity set of each basis by

\[
\sigma = \{e^{i\pi k_1}, e^{i\pi k_2}, e^{i\pi k_3}, \ldots, e^{i\pi k_{\zeta_0}}\} := \text{parity (} \kappa \text{)}
\]

Then, if \( \kappa_0 \) is the parity set of minimum values which \( \kappa \) can take, i.e., \( \sigma := \text{parity (} \kappa_0 \text{)} \), one can construct a basis with fixed parity for the full matrix Hamiltonian by considering the direct sum of bases of the form \( B_{\text{RWA}}^{(\kappa_0 + 2j)} \) as follows

\[
B_\sigma = \bigoplus_{j_1=0}^\infty \bigoplus_{j_2=0}^\infty \cdots \bigoplus_{j_{\zeta_0}=0}^\infty B_{\text{RWA}}^{(\kappa_0 + 2(j_1, \ldots, j_{\zeta_0}))};
\]

the respective matrix Hamiltonian will be denoted by \( H_\sigma \).

Since the basis of the full Hamiltonian is infinite, it is necessary to truncate the basis to upper values for the set \( \kappa_{\text{max}} \). A fidelity criterion to do this truncation was proposed recently [12], which allows us to calculate the ground state of the system to a fixed precision.

III. QUANTUM PHASE DIAGRAM VIA THE FIDELITY

Here we discuss the construction of the ground state energy surfaces for both the GTCM and GDM Hamiltonians. For the GTCM Hamiltonian, the energy surfaces are associated to different sets of values for the constants of motion \( \kappa = \{k_1, k_2, \ldots, k_{\zeta_0}\} \) while for the GDM the energy surfaces are associated to the different sets of parities \( \kappa \). One can discuss both cases by labeling the energy surfaces with \( \sigma_j \), which will represent fixed values of \( \kappa \) for the GTCM while a set of parities of \( \kappa \) for the GDM.

In both cases, the ground state energy surface is obtained by comparing the set of energy surfaces with different values of \( \sigma_j \) and taking the minimum value at each point in parameter space \( (\mu_{ki}) \). For each label \( \sigma_j \) one has a set of energy surfaces \( E_{\sigma_j} = \{E_{\sigma_j}^{(0)}, E_{\sigma_j}^{(1)}, \ldots\} \) as functions of the parameters, with \( E_{\sigma_j}^{(r)} \leq E_{\sigma_j}^{(s)} \) for all \( r < s \).

For each eigenvalue \( E_{\sigma_j}^{(r)} \) there is a corresponding eigenstate \( |\Psi_{\sigma_j}^{(r)}\rangle \) of the Hamiltonian \( H_{\sigma_j} \).

Thus, the ground
TABLE II. The operator $K$ Eq. (12) as a function of the three free variables for the different 3-level atomic configurations interacting dipolarly with two modes of radiation, in the RWA approximation. Notice that, for this choice of the free variables, in all cases $\mathbf{K}(0, 0, 1) = \mathbf{A}_{11} + \mathbf{A}_{22} + \mathbf{A}_{33}$ is the first order Casimir operator, which is the number of atoms in the system, and $\mathbf{K}(1, 1, 0)$ is the operator of the total number of excitations.

| Configuration | Constant of motion |
|---------------|--------------------|
| $\Xi$         | $K(\eta_1, \eta_2, \lambda_1) = \eta_1 \nu_1 + \eta_2 \nu_2 + \lambda_1 \mathbf{A}_{11} + (\eta_1 + \lambda_1) \mathbf{A}_{22} + (\eta_1 + \eta_2 + \lambda_1) \mathbf{A}_{33}$ |
| $\Lambda$     | $K(\eta_1, \eta_2, \lambda_1) = \eta_1 \nu_1 + \eta_2 \nu_2 + \lambda_1 \mathbf{A}_{11} + (\eta_1 - \eta_2 + \lambda_1) \mathbf{A}_{22} + (\eta_1 + \lambda_1) \mathbf{A}_{33}$ |
| $V$           | $K(\eta_1, \eta_2, \lambda_1) = \eta_1 \nu_1 + \eta_2 \nu_2 + \lambda_1 \mathbf{A}_{11} + (\eta_1 + \lambda_1) \mathbf{A}_{22} + (\eta_2 + \lambda_1) \mathbf{A}_{33}$ |

State of the system $|\Psi_g\rangle$ has the energy eigenvalue

$$E_g = \min \left\{ \sum_j E_{\sigma_j} \right\} .$$

(16)

Notice that, in particular for the ground state, this energy value is given by $E_g = \min \{E_{\sigma(0)}(0), E_{\sigma(0)}(1), \ldots \}$ since $E_{\sigma(0)}$ is the ground energy value for each label $\sigma_j$. However, one may obtain, in general, the energy surface of excited states by means of

$$E_r = \min \left\{ \sum_j E_{\sigma_j} \right\} \{E_g, E_1, \ldots, E_{r-1}\} ,$$

(17)

where in the set of energy surfaces the lowest $r - 1$ energies $\{E_g, E_1, \ldots, E_{r-1}\}$ have been eliminated. As an example, the first excited state is given by

$$E_1 = \min \left\{ \sum_j E_{\sigma_j} \right\} \{E_g\} .$$

In this work, we focus our study on the ground state, but we want to stress the fact that a similar analysis can be done for excited states.

Quantum Phase Diagram

From the ground energy surface (16), the quantum phase diagram is determined by the set of points where the ground state suffers a sudden change, and this may be detected by means of the calculation of the fidelity concept of quantum information theory, which is defined by

$$F_3(\xi) := |\langle \Psi_g(\xi) | \Psi_g(\xi + \delta) \rangle|^2 ;$$

(18)

this quantity reaches minima at phase transitions because neighboring states there are dissimilar.

In general one finds two types of quantum transitions: (i) When the fidelity reaches the value $F_3(\xi) = 0$ it corresponds to a discontinuous transition because the state changes from a Hilbert subspace to an orthogonal one (preserving or not the parity $\sigma$). (ii) When at a minimum $F_3(\xi) \neq 0$, which corresponds to continuous transitions; in this case the ground state suffers a change but there is overlap with the same Hilbert subspace.

For the GTCM Hamiltonian with a finite number of particles all the transitions are discontinuous because they correspond to different sets of constants of motion, and are therefore orthogonal. However, in the thermodynamic limit [13] many of these different sets collapse, and the transition emerges as a continuous one (or so called second order transition) as expected by the variational solution [14–16].

In what follows we apply the fidelity concept to determine the quantum phase diagrams on the corresponding ground state energy surfaces.

A. 3-Level Atoms

We consider the case of a 3-level system ($n = 3$) interacting with two modes ($\ell = 2$) of radiation. For this case there are three atomic configurations available, called $\Xi$, $\Lambda$, and $V$ after their schematic structures. The Hamiltonian (with $\hbar = 1$ and $\omega_1 = 0$) and the two symmetries for each atomic configuration are:

$\Xi$-configuration.-

$$H_\Xi = \Omega_1 \nu_1 + \Omega_2 \nu_2 + \omega_2 \mathbf{A}_{22} + \omega_3 \mathbf{A}_{33}$$

$$- \frac{\mu_1^{(1)}}{N_a} x_1^{(1)} (A_{12} + A_{21}) \left( a_1 + a_1^\dagger \right)$$

$$- \frac{\mu_2^{(2)}}{N_a} x_2^{(2)} (A_{23} + A_{32}) \left( a_2 + a_2^\dagger \right) ,$$

(19)

and from table II one has two independent symmetries $\Pi_j = \exp(i \pi K_j)$, which we choose to be

$$K_1 = \nu_1 + \nu_2 + A_{22} + 2 A_{33} ,$$

$$K_2 = \nu_2 + A_{33} .$$

(20a)

(20b)

$K_1$ stands for the total number of excitations in the system, and has integer eigenvalues $k_1 = 0, 1, \ldots,$
while \( K_2 \) plays the role of the excitations related to a 2-level subsystem formed by the second and third atomic levels with the mode \( \Omega_2 \), and it takes the values \( k_2 = 0, 1, \ldots, k_1 \).

**A-configuration.**

\[
H_A = \Omega_1 \nu_1 + \Omega_2 \nu_2 + \omega_2 A_{22} + \omega_3 A_{33} \\
- \frac{\mu_1}{\sqrt{N_a}} x_{13}^{(1)} (A_{13} + A_{31}) \left(a_1 + a_1^\dagger\right) \\
- \frac{\mu_2}{\sqrt{N_a}} x_{23}^{(2)} (A_{23} + A_{32}) \left(a_2 + a_2^\dagger\right), \tag{21}
\]

and we chose the symmetry operators

\[
K_1 = \nu_1 + \nu_2 + A_{33}, \tag{22a}
\]

\[
K_2 = \nu_2 + A_{11} + A_{33}. \tag{22b}
\]

Here \( K_1 \) defines the total number of excitations, with integer eigenvalues \( k_1 = 0, 1, \ldots \), and \( K_2 \) can take the values \( k_2 = 0, 1, \ldots, k_1 + N_a \).

**V-configuration.**

\[
H_V = \Omega_1 \nu_1 + \Omega_2 \nu_2 + \omega_2 A_{22} + \omega_3 A_{33} \\
- \frac{\mu_1}{\sqrt{N_a}} x_{12}^{(1)} (A_{12} + A_{21}) \left(a_1 + a_1^\dagger\right) \\
- \frac{\mu_2}{\sqrt{N_a}} x_{13}^{(2)} (A_{13} + A_{31}) \left(a_2 + a_2^\dagger\right), \tag{23}
\]

and we chose the independent symmetry operators

\[
K_1 = \nu_1 + A_{22}, \tag{24a}
\]

\[
K_2 = \nu_2 + A_{33}. \tag{24b}
\]

\( K_1 \) denotes the number of excitations of a 2-level subsystem formed by the first and second atomic levels interacting with radiation mode \( \Omega_1 \), which takes integer eigenvalues \( k_1 = 0, 1, \ldots \), and \( K_2 \) is the number of excitations of the other 2-level subsystem formed by the first and third atomic levels interacting with mode \( \Omega_2 \), with values \( k_2 = 0, 1, \ldots \). In this case \( K_1 + K_2 \) corresponds to the total number of excitations of the system.

**B. Generalized Tavis-Cummings Model**

In the generalized Tavis-Cummings model the corresponding operators \( K_1 \) and \( K_2 \) are constants of motion. For fixed values of the dimensionless coupling matter-field parameters \( x_1 \) and \( x_2 \) one needs to find the eigenvalues of \( K_1 \) and \( K_2 \) for which the energy surfaces reach their minimum value.

We calculate all the energy surfaces \( \mathcal{E}_\kappa \) with values

\[
\kappa = \{k_1, k_2\},
\]

by considering the integer intervals \( 0 \leq k_1 \leq k_{1\text{max}} \) and \( 0 \leq k_2 \leq k_{2\text{max}} \), where the values \( \{k_{1\text{max}}, k_{2\text{max}}\} \) are those required to describe the ground state energy surface for the maximum values of the dimensionless coupling parameters \( x_1 \) and \( x_2 \) in the region of study of the phase space of the system.

Instead of analyzing \((k_{1\text{max}} + 1)(k_{2\text{max}} + 1)\) surfaces in order to find the two symmetry values \(\{k_1, k_2\}\) that yield the ground state energy surface, we first consider \(\nu_i = 0\), with \(i = 1, 2\), and set equal to zero the corresponding atomic population involved in the transition.

![FIG. 2. (color online) The ground state energy surface per particle as a function of the control parameters \(x_{ij}\) is shown, for the GTCM model and \(N_a = 1\). The values of \(k_1\) and \(k_2\) are given in the colored legends. (a) For the \(Z\)-configuration the parameters are: \(\Omega_1 = 0.25\), \(\Omega_2 = 0.75\), \(\omega_1 = 0\), \(\omega_2 = 0.25\) and \(\omega_3 = 1\). (b) For the \(A\)-configuration, \(\Omega_1 = 1\), \(\Omega_2 = 0.9\), \(\omega_1 = 0\), \(\omega_2 = 0.1\) and \(\omega_3 = 1\). (c) For the \(V\)-configuration, \(\Omega_1 = 0.8\), \(\Omega_2 = 1\), \(\omega_1 = 0\), \(\omega_2 = 0.8\) and \(\omega_3 = 1\). In all cases the energy and frequencies are in units of \(\omega_3\).](image-url)
The procedure just described resembles that to obtain the reduced bases in the GDM [11].

With these constraints we then determine the values that the constants of motion \( (k_1, k_2) \) can take.

Using this rule, we should only calculate the energy surfaces \( E_n \) for:

- \( \kappa = \{ k_1, 0 \}, \{ k_2 + N_a, k_2 \} \) for the \( \Xi \)-configuration,
- \( \kappa = \{ k_1, N_a \}, \{ k_2, k_2 \} \) for the \( \Lambda \)-configuration,
- \( \kappa = \{ k_1, 0 \}, \{ 0, k_2 \} \) for the \( V \)-configuration,

where \( k_1 = 0, 1, \ldots, k_{1\text{max}} \) and \( k_2 = 0, 1, \ldots, k_{2\text{max}} \). The procedure just described resembles that to obtain the reduced bases in the GDM [11].

In figures 2 and 3 the energy surface per particle is shown as a function of the dimensionless dipolar strengths, for each atomic configuration: \( \Xi \), Figs. 2(a) and 3(a); \( \Lambda \), Figs. 2(b) and 3(b); and \( V \), Figs. 2(c) and 3(c). Each surface is colored according to the colored values of \( \{ k_1, k_2 \} \) shown at the right of each plot. Thus the separatrices of the energy surface take place at the loci where a sudden change of color happens, and at these loci the ground state changes from one Hilbert subspace to another.

For the \( \Xi \)-configuration in Fig. 2(a) the normal region was determined by the values \( \{ k_1, k_2 \} = \{ 0, 0 \} \) (black region) where the cavity is in the vacuum state and the atom is in its lower atomic level. The collective region is divided into two collective sub-regions: the \( S_{12} \) sub-region where the mode \( \Omega_2 \) dominates, is characterized by the set of values \( \{ k_1, 0 \} \) and yields values for the basis states with \( \nu_2 = 0 \) and \( b_3 = 0 \), hence there are only transitions between the atomic levels \( \omega_1 \rightleftharpoons \omega_2 \); and the \( S_{23} \) sub-region with dominant mode \( \Omega_2 \), characterized by the set of values \( \{ k_2 + N_a, k_2 \} \) and transitions \( \omega_2 \rightleftharpoons \omega_5 \). Notice that states with non-zero contribution of the mode \( \Omega_1 \) should satisfy \( N_a = \nu_1 + b_2 + b_3 = 1 \).

For the \( \Lambda \)-configuration in Fig. 2(b) the normal region was determined by the values \( \{ k_1, k_2 \} = \{ 0, 1 \} \) (black region) while the collective region is divided into two sub-regions, namely \( S_{13} \) and \( S_{23} \) where the modes \( \Omega_1 \) and \( \Omega_2 \) dominate the atomic transitions respectively. Here, the sub-region \( S_{13} \) is characterized by \( \{ k_1, 1 \} \) (the states with non-zero contribution of the mode \( \Omega_2 \) satisfy \( N_a = \nu_2 + b_1 + b_3 = 1 \)), and the sub-region \( S_{23} \) by the values \( \{ k_2, k_2 \} \) (the states with non-zero contribution of the mode \( \Omega_1 \) fulfill \( \nu_1 - b_1 = 0 \)).

The special case \( \{ k_1, k_2 \} = \{ 1, 1 \} \) [dark green in the color legend of figure 2(b)] corresponds to linear combinations of the states \( |1, 0, 1, 0, 0, 0 \rangle, |0, 0, 0, 0, 1 \rangle \), and \( |0, 1, 0, 1, 0 \rangle \), for all the values shown of \( x_{13} \) and \( x_{23} \). Superpositions of the first two states lie in the region \( S_{13} \), and those of the last two states in the region \( S_{23} \). The state \( |0, 0, 0, 0, 1 \rangle \) lies in both regions, and has no contributions of any kind of photons. These regions are not shown divided by a separatrix since the figure is drawn according to values of \( \{ k_1, k_2 \} = \{ 1, 1 \} \). The transition is made explicit when we draw the probability that the ground state is in one of the first or third states above, as in figure 4. In this last figure, the dashed (blue) line gives the probability for \( |1, 0, 1, 0, 0 \rangle \) to be the ground state, the continuous (green) line that of \( |0, 1, 0, 1, 0 \rangle \), and the dot-dashed (orange) line that of the state \( |0, 0, 0, 0, 1 \rangle \). We see that in different regions we have the superpositions mentioned above with higher probabilities.

For the \( V \)-configuration Fig. 2(c), the normal region is that determined by the values \( \{ k_1, k_2 \} = \{ 0, 0 \} \) (black region) while the collective regions are \( S_{12} \) with values for \( \{ k_1, 0 \} \) and \( S_{13} \) with values for \( \{ 0, k_2 \} \). This figure exhibits clearly the polychromatic behavior obtained in the variational solution [10].

Similar results are obtained when the number of particles increases as shown in figure 3 for the three atomic configurations with \( N_a = 4 \) particles. Notice that the shape of the normal region for the \( \Lambda \)-configuration becomes more rectangular when the number of particles grows, as expected due to the variational solution.
FIG. 4. (color online) Probability that the ground state is $|1, 0, 1, 0, 0\rangle$ (dashed blue line), $|0, 1, 0, 1, 0\rangle$ (continuous green line), or $|0, 0, 0, 0, 1\rangle$ (dot-dashed orange line), as a function of the dipolar strengths for the atomic configuration $\Lambda$ and parameter values are given in figure 2, for $x_{13} = 3$.

C. Generalized Dicke Model

In a similar way to the GTCM model, figures 5 and 6 show the energy surface per particle as function of the dimensionless coupling strengths for GDM model. Here the separatrix is obtained by means of the fidelity concept in quantum information.

In figure 5(a), we show for the $\Xi$-configuration that the ground state energy surface is constituted by states with two parities: $ee$ (even-even), and $oe$ (odd-even). The normal region is in the $ee$ sector. The quantum separatrix is formed by three kinds of lines: The dashed line separates the $ee$- and $oe$- energy surfaces, hence there are discontinuous quantum transitions along this line. The continuous line on the $ee$-energy surface provides continuous transitions from one ground state to another of the same parity. The dotted lines on the $oe$-energy surface divide solutions where one of the photon modes $\Omega_1$ or $\Omega_2$ dominates; across these lines, for small values of the control parameters where there are two lines the transitions are continuous, while where these lines coalesce one has discontinuous transitions. This is in agreement with the variational calculation [10].

For the $\Lambda$-configuration we find that the ground state energy surface is constituted by regions where the parities $eo$, $ee$, and $oo$ are preserved [cf. Fig. 5(b)]. The normal region is divided into two sub-regions where states with parity $eo$ or $ee$ dominate. Similar to the case of the dashed lines in Fig. 5(a), these divide the three regions where parity is preserved. The continuous line divides the normal and collective regions of $ee$ and $oe$ parity, while the dotted lines divide the $oo$-surface where states exhibit only one photon mode in the solution. Across these dotted lines we have continuous transitions for small values of $x_{13}$ and $x_{23}$, and discontinuous transitions for large values of the coupling parameters (where the lines coalesce).

For the $V$-configuration one finds that the ground energy surface has only states with $ee$ parity [cf. Fig. 5(c)]. The separatrix exhibits continuous transitions from the normal to the collective regions for small values of the dipolar couplings, while for large values of these the transitions from $S_{12}$ to $S_{13}$ are discontinuous (again where the dotted lines coalesce).

In contrast to the solution for $N_a = 1$, we find that for $N_a = 4$ the ground state energy surface has only $ee$ parity states for the three atomic configurations. These are shown in Fig. 6. Their separatrices present both kinds of transition. For the $\Xi$ configuration the transition is continuous from the normal to the $S_{12}$ regions, for the $\Lambda$ case the transition is continuous from the normal to the $S_{13}$ regions, and for the $V$ configuration it is continuous.
from the normal to both superradiant regions. All other crossings are discontinuous.

It is important to stress that the parities refer to those of the constants of motion chosen to work with from table II, for each atomic configuration. The parity of the ground state will also change with the number of particles \( N \). In our case, with the parameters given in figure 2 and the symmetries given by the operators Eqs. (20), (22) and (24), we find that for an even number of particles the ground state energy surface possesses an \( ee \) parity for the \( \Xi \)- and \( \Lambda \)-configurations, while for an odd number of particles it presents different parities. In contrast, for the \( V \)-configuration the ground state energy surface is formed by states with an \( ee \) parity independently of the number of particles.

**IV. QUANTUM PHASE TRANSITIONS**

In order to characterize the types of transitions in the quantum phase diagram, we consider, without loss of generality, the case of a single particle in the \( \Lambda \)-configuration, since in this case the phase diagram presents all kinds of transitions [cf. Fig. 5(b)].

We use the Ehrenfest classification [17] and consider the lowest derivatives for which the ground state energy surface remains continuous, but this requires necessarily numerical calculations and these progressively lose precision. On the other hand, by means of the Hellmann-Feynman theorem [18–20] one may evaluate the first derivative to the same numerical precision as the calculation of the ground state energy surface. In figure 7 we plot the derivative of the ground state energy surface with respect to the control parameters \( x_{ij} \): for \( x_{13} \) in Fig. 7(a) one may observe that around the normal region there is a discontinuity along the separatrix due to a change of parity [see dashed line of figure 5(b)]; also one can notice that the region \( S_{23} \) has a small dependence on \( x_{13} \) (its corresponding derivative is close to zero). For \( x_{23} \) in Fig. 7(b) a discontinuity in the derivative appears.
FIG. 8. (color online). Above: Density matrices at the points A and B in the figure below, across a discontinuous transition in the space of parameters corresponding to the phase diagram of the Λ configuration for $N_a = 1$. For these regions we have $\text{Tr} \, AB = 0$.

between the normal and $S_{13}$ regions; the latter has also a small dependence in the control parameter $x_{23}$. The discontinuity on the derivative indicates a first order transition. This occurs because the ground state changes from one subspace to another orthogonal to it. However, one may also see in Fig. 7(b) that this discontinuity vanishes for large values of the control parameters.

The dependence on $x_{jk}$ of the energy surface reflects the fact that the collective region is divided into monochromatic regions where a sub-system with a single mode determines the bulk of the ground state, and then the derivative of the ground energy surface clearly marks the boundary between these regions.

Since the Ehrenfest classification does not permit us to characterize the kind of transition present, due to the possible loss of precision in the numerical calculations, one may catalogue them only as continuous or discontinuous, and distinguish these by using a test based on the fidelity between neighboring states Eq. (18). This classification, however, determines a discontinuous transition only when $F_\delta(\xi) = 0$ since this condition is met when the subspace where the ground state at point $\xi$ lies is orthogonal to the subspace where the ground state at point $\xi + \delta$ lies.

For a finite number of particles, we will encounter situations in which the fidelity is close to zero (but not zero), and diminishes as $N_a$ increases, corresponding to the case $F(\xi) = 0$ in the limit $N_a \to \infty$ and thus to a discontinuous transition. On the other hand, there are situations with $F(\xi) \neq 0$ and it either remains different from zero as $N_a$ increases, or reaches zero in the large $N_a$ limit; the former will be stable-continuous transitions, while the latter unstable-continuous transitions.

One may visually see this situation by plotting the density matrices of the states at two points lying on different sides of a separatrix. Figure 8 generates a plot that gives a visual representation of the values of elements in a matrix. We show here the density matrices of ground states at points A and B, indicated in the bottom panel corresponding to the phase diagram of the Λ configuration for $N_a = 1$. Across this separatrix a change of parity occurs (dashed line), and hence a discontinuous transition occurs. Darker-colored areas in the density matrices indicate values $|\rho_{ij}| \geq 10^{-3}$, while white regions correspond to $|\rho_{ij}| < 10^{-16}$; so the colored region provides the bulk of the ground state. Clearly the matrices are orthogonal; indeed, we have $\text{Tr} \, AB = 0$.

Now, for the case of a continuous transition with $F_\delta(\xi) \neq 0$, the phase diagram of the Λ configuration for a single particle figure 5(b) shows the stable-continuous transitions indicated by continuous lines, while the unstable-continuous with short-dotted lines.

In figure 9 the density matrices of two points around a stable-continuous transition are plotted. One may observe that the bulk of the ground state grows smoothly as the control parameter moves from point A to point B; in fact for these states one finds a fidelity close to one $\text{Tr} \, AB \sim 0.9$. On the other hand, figure 10 shows the situation for an unstable-continuous transition: the density matrices make clear that the bulk of the ground state suffers a very significant change between neighboring states. In this case we find a fidelity close to zero $\text{Tr} \, AB \sim 0.2$, and this is an indicator that abrupt changes occur on observables, particularly an study of the Wigner quasi-probability distribution function exhibits these abrupt changes [21].
FIG. 10. (color online) Above: Density matrices at the points A and B in the figure below, across an unstable-continuous transition in the space of parameters corresponding to the phase diagram of the Λ configuration for $N_a = 1$. For these states we have $\text{Tr} \ A \ B \sim 0.2$.

V. CONCLUSIONS

In this work we have exhibited a polychromatic behavior in the phase diagram of the quantum solution of a system of 3-level atoms interacting with two modes of electromagnetic field, in the generalized Tavis-Cumming model (RWA approximation) and in the Dicke model. That is, the phase diagram which divides the normal and collective regions shows that the collective region divides itself into subregions where the bulk of the ground state is dominated by a sub-system with a single mode. This is in agreement with the variational solution found in [10].

For the Tavis-Cumming model we found that the polychromatic behavior appears even for a single atom, and the phase space is formed by an infinite number of discontinuous transitions at which the constants of motion change their values Figs. 2 and 3.

Effects of the finite number of particles in the GDM model are visually exemplified by comparing the phase diagrams for $N_a = 1$ and $N_a = 4$ particles in Figs. 5 and 6, respectively. In fact, we found for 3-level atoms that for an even number of particles the ground energy state does not suffer a change in parity across the phase diagram, while for an odd number of particles changes of parity appear for the Ξ- and Λ-configurations. We showed also that the derivative of the ground energy provides visually the collective sub-regions where a single photon contributes highly to the ground state, and that a finer characterization of the kind of transition may be given, by using the density matrices, in terms of their continuity and stability.

Finally, although in this work we have given numerical results for the case of 3-level atoms, the variational result shows that this polychromatic behavior is valid for general systems involving $n$-levels and $\ell$ modes of electromagnetic field.

ACKNOWLEDGMENTS

This work was partially supported by DGAPA-UNAM under projects IN101619 and IN100120.

Appendix A: Dimensions of the RWA basis

In order to find the subspace degeneracy in the RWA approximation, for the different atomic configurations Ξ, Λ, and V, we use the expression for the degeneracy of an $N$-dimensional Harmonic oscillator with $n$ quanta excitations

$$g_N(n) := \frac{(n + N - 1)!}{n!(N-1)!}.$$  \hspace{1cm} (A1)

This yields the following results, for fixed values of $k_1$ and $k_2$:

$$D_\Xi(k_1, k_2) = \begin{cases} 
  g_3(k_1 - k_2) & N_a \geq k_1 - k_2 \ \& \ k_2 \geq \frac{k_1}{2} \\
  g_3(k_1) - g_3(k_1 - k_2 - 1) - 2g_3(k_2 - 1) & N_a \geq k_1 - k_2 \ \& \ k_2 < \frac{k_1}{2}, \\
  g_3(N_a) - g_3(N_a - k_2 - 1) & N_a > k_2 \ \& \ N_a < k_1 - k_2 \\
  g_3(N_a) & N_a \leq k_2 \ \& \ N_a < k_1 - k_2 
\end{cases},$$ \hspace{1cm} (A2)

$$D_\Lambda(k_1, k_2) = \begin{cases} 
  g_3(k_2) & k_2 < k_1 \ \& \ N_a \geq k_2 \\
  g_3(k_1) & k_2 \geq k_1 \ \& \ N_a \geq k_2 \\
  g_3(N_a + k_1 - k_2) & k_2 \geq k_1 \ \& \ N_a < k_2 \\
  g_3(N_a) & k_2 < k_1 \ \& \ N_a \leq k_1 \ \& \ N_a < k_2
\end{cases}. \hspace{1cm} (A3)$$
\[ D_V(k_1, k_2) = \begin{cases} g_2(k_1) g_2(k_2) & N_a \geq k_1 + k_2 \\
g_2(k_2) g_2(N_a) - g_3(k_2 - 1) & N_a \leq \frac{k_1 + k_2}{2} & k_2 < N_a \| \\
k_1 + k_2 > N_a > \frac{k_1 + k_2}{2} & N_a < k_1 \| \\
g_2(k_1) g_2(N_a) - g_3(k_1 - 1) & N_a \leq \frac{k_1 + k_2}{2} & k_1 < N_a \|
\end{cases} \\
\begin{cases} 1 + g_2(k_1 - 1) g_2(k_2 - 1) + g_2(N_a - 1) g_2(k_1 + k_2 - 1) - g_3(k_1 + k_2 - 2) - g_3(N_a - 2) & k_1 + k_2 > N_a > \frac{k_1 + k_2}{2} & N_a \geq k_1 & N_a \geq k_2 \\
g_3(N_a) & k_2 \geq N_a + 1 & k_1 \geq N_a + 1 \end{cases}. \]