Generalized Macdonald Functions on Fock Tensor Spaces and Duality Formula for Changing Preferred Direction

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Abstract: An explicit formula is obtained for the generalized Macdonald functions on the $N$-fold Fock tensor spaces, calculating a certain matrix element of a composition of several screened vertex operators. As an application, we prove the factorization property of the arbitrary matrix elements of the multi-valent intertwining operator (or refined topological vertex operator) associated with the Ding–Iohara–Miki algebra (DIM algebra) with respect to the generalized Macdonald functions, which was conjectured by Awata, Feigin, Hoshino, Kanai, Yanagida and one of the authors. Our proof is based on the combinatorial and analytic properties of the asymptotic eigenfunctions of the ordinary Macdonald operator of $A$-type, and the Euler transformation formula for Kajihara and Noumi’s multiple basic hypergeometric series. That factorization formula provides us with a reasonable algebraic description of the 5D (K-theoretic) Alday-Gaiotto–Tachikawa (AGT) correspondence, and the interpretation of the invariance under the preferred direction from the point of view of the $SL(2, Z)$ duality of the DIM algebra.

1. Introduction

Let $\mathcal{U}$ be the DIM algebra [DI, Mi]. As for the definition of the DIM algebra, see Definition 2.1. The central object in the present paper is the intertwining operator $\mathcal{V}(x)$ associated with some structure of the $\mathcal{U}$ modules. From the point of view of the geometric engineering, or topological vertex construction for the partition functions for the quantum supersymmetric gauge theories, we regard the $\mathcal{V}(x)$ as the (multi-valent) topological vertex operator. The intertwining operator $\mathcal{V}(z)$ is defined through a certain set of commutation relations with the $\mathcal{U}$-generators. Let $X^{(i)}(z)$’s ($i = 1, \ldots, N$) be the generating currents constructed from the standard Drinfeld current of $\mathcal{U}$ (Definition 2.6), acting on the $N$-fold tensor space $\mathcal{F}_\mu = \otimes_{j=1}^N \mathcal{F}_{\mu_j}$ of the Fock spaces (For the definition of $\otimes$, see Notation 5.7).
Definition 1.1 (Topological vertex). Let $V(x) : \mathcal{F}_u \to \mathcal{F}_v$ be a linear map satisfying the commutation relations

$$\left(1 - \frac{x}{z}\right) X^{(i)}(z) V(x) = \left(1 - \frac{t}{q} \cdot \frac{x}{z}\right) V(x) X^{(i)}(z) \quad (i = 1, \ldots, N), \quad (1.1)$$

and the normalization condition $\langle 0 | V(x) | 0 \rangle = 1$. Here $|0\rangle$ (resp. $\langle 0|$) is the vacuum (resp. dual vacuum) state.

We refer to this operator as the Mukadé operator. Mukadé is a Japanese word which means a centipede. The operator $V(x)$ can be realized by connecting the trivalent intertwining operators as Fig. 1 in Sect. 5, and that figure is the reason why we call $V(x)$ the Mukadé operator. Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ be an $N$-tuple of partitions. Denote by $|K_\lambda\rangle = |K_\lambda(u)\rangle$ the canonical form of the generalized Macdonald function on $\mathcal{F}_u$ (Definition 3.37). The main result of this paper is to prove the following factorization formula conjectured in [AFHKSY1].

Theorem 1.2. We have

$$\langle K_\lambda(v) | V(x) | K_\mu(u) \rangle = \frac{(-\gamma^2)^N e_N(u) x^{\lambda_i}}{(\gamma^2 x)^{\mu_i}} \prod_{i=1}^{N} u_i^{\lambda^{(i)}_j} g_{\lambda^{(i)}_j} \cdot \prod_{i,j=1}^{N} N_{\lambda^{(i)},\mu^{(j)}}(q v_i / t u_j). \quad (1.2)$$

Here, $\gamma = (t/q)^{1/2}$, $e_N(u) = u_1 \cdots u_N$ and $N_{\lambda,\mu}$ is the Nekrasov factor. For the definition of $g_\lambda$, see Definition 3.39.

In our proof, the first important thing is to construct the explicit formula for the generalized Macdonald functions $|K_\lambda\rangle$. We construct this explicit formula by using some screened vertex operators (Theorem 3.26). Next, the operator $V(x)$ can also be realized by a certain fusion of the screened vertex operators. Matrix elements of the composition of the screened vertex operators satisfy a difference equation and correspond to a multiple series $p_n$ (Macdonald functions of A-type, Definition 3.21). The analyticity of matrix elements of screened vertex operators can be clarified by the analyticity of the Macdonald functions $p_n$. The final step of our proof is attributed to a transformation problem with respect to this multiple series $p_n$, and this is proved by the Euler transformation formula for Kajihara and Noumi’s multiple basic hypergeometric series.

We remark that the case $N = 1$ of Theorem 1.2 was proved by using Kajihara-Noumi’s Euler transformation formula and the Pieri rules for the Macdonald polynomials [AFHKSY1]. As for the detail of the proof, unfortunately, it remains unpublished yet [AFHKSY2]. Since the Pieri rules are known only for the standard Macdonald case ($N = 1$), it is totally unclear how we extend that method to the case $N > 1$, which drove the present authors to have the formulation presented in this paper.

This paper is organized as follows. In Sect. 2, we briefly remind the readers of the basic facts about the DIM algebra $\mathcal{F}$ and its module structure on the Fock space $\mathcal{F}$. In Sect. 3, we introduce the generalized Macdonald functions, and they are explicitly constructed by the screened vertex operators (Theorem 3.26). We also discuss the analyticity of the matrix elements of the composition of screened vertex operators (Theorem 3.33). In Sect. 4, we give a proof of the main theorem (1.2). In Sect. 5, the existence of $V(x)$ is proved.
through the explicit construction in terms of the topological vertex (Proposition 5.9). As its application, we can show the invariance under changing the preferred direction of toric diagrams, which is the natural consequence of the dualities in the string theory (Theorem 5.11).

In Appendix A, the Macdonald functions $p_n$ are constructed in terms of intertwining operators. In Appendix B, we give a brief review of the definition of ordinary Macdonald functions and their basic facts. In Appendix C, some useful formulas used in paper are explained. Some proofs in the main text are given in Appendix D. In Appendix E, we revisit the proof of a certain formula for the Kac determinant with respect to $|X_\lambda\rangle$ (Definition 2.9) given in [O]. In particular, we clarify the choice of integral cycles of screening operators. In Appendix F, we give some examples. At last, we present list of notations in “List of Notations” in Appendix F.

**Notations.** A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of non-negative integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$ and containing finitely many nonzero elements. They are identified if their elements without 0 are the same. $\ell(\lambda)$ denotes the length of $\lambda$, i.e., the number of elements in $\lambda$ without 0. For an integer $i > \ell(\lambda)$, we occasionally write $\lambda_i$. Note that it is just 0. The partition in which all elements are 0 is denoted by $\emptyset$. Write $|\lambda| = \sum_{i \geq 1} \lambda_i$ and $n(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i$. Partitions are identified with the Young diagrams. For example, if $\lambda = (4, 4, 2, 1)$, the Young diagram of $\lambda$ is:

```
  1 2 3 4
 5 6 7 8
```

The transpose of $\lambda$ is denoted by $\lambda'$. The coordinate of the box in the $i$-th row and the $j$-th column is denoted by $(i, j)$, say the coordinate of the box $s$ in the above Young diagram is $(2, 3)$. For a coordinate $(i, j) \in \mathbb{Z}^2$, the arm length and the leg length are defined by $a_{\lambda}(i, j) = \lambda_i - j$ and $\ell_{\lambda}(i, j) = \lambda'_j - i$, respectively. $A(\lambda)$ (resp. $R(\lambda)$) is the set of coordinates where we can add (resp. remove) a box. For example, if $\lambda = (3, 3, 1)$, $A(\lambda) = \{(1, 4), (3, 2), (4, 1)\}$ and $R(\lambda) = \{(2, 3), (3, 1)\}$. In this paper, we often use $N$-tuples of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)})$. For an $N$-tuple of partitions $\lambda$, we put $|\lambda| = \sum_{k=1}^N |\lambda^{(k)}|$. Further, we use the following factorials ($q$-Pochhammer symbol) and the theta function:

\begin{align}
(a; q)_\infty &:= \prod_{n=1}^{\infty} (1 - q^{n-1}a), \quad (a; q)_m := \begin{cases} 
\prod_{n=1}^{m} (1 - q^{n-1}a) & (m \geq 1); \\
1 & (m = 0); \\
\prod_{n=1}^{-m} (1 - q^{-n}a)^{-1} & (m \leq -1),
\end{cases} \\
\theta_q(a) &:= (a; q)_\infty(qa^{-1}; q)_\infty. \quad \quad \quad \quad \quad \quad \quad \quad (1.3)
\end{align}
2. Preliminaries

2.1. Ding–Iohara–Miki algebra \( \mathcal{U} \). Recall briefly some basic facts about the DIM algebra \( \mathcal{U} \) [DI,Mi]. Let \( q \) and \( t \) be nonzero complex parameters satisfying \(|q| < 1 \) and \( q^n t^m \neq 1 \) for all integers \( n, m \).

**Definition 2.1.** Let \( \mathcal{U} = \mathcal{U}_{q,t} \) be the unital associative algebra over \( \mathbb{C} \) generated by the Drinfeld currents

\[
x^\pm(z) = \sum_{n \in \mathbb{Z}} x^\pm_n z^{-n}, \quad \psi^\pm(z) = \sum_{\pm n \in \mathbb{Z}_0} \psi^\pm_n z^{-n},
\]

and the invertible central element \( c^{1/2} \), satisfying the following defining relations:

\[
\begin{align*}
\psi^+(z) x^\pm(w) &= g(c^{1/2} w z)^{1/2} x^\pm(w) \psi^+(z), \quad \psi^-(z) x^\pm(w) = g(c^{1/2} w z)^{1/2} x^\pm(w) \psi^-(z), \\
\psi^\pm(z) \psi^\pm(w) &= \psi^\pm(w) \psi^\pm(z), \quad \psi^+(z) \psi^-(w) = \frac{g(c^{1/2} w z)}{g(c^{-1/2} w z)} \psi^-(w) \psi^+(z), \\
[x^\pm(z), x^- (w)] &= \frac{(1-q^\pm)(1-1/t)}{1-q/t} \left( \delta(c^{-1} z/w) \psi^+(c^{1/2} w) - \delta(cz/w) \psi^-(c^{-1/2} w) \right),
\end{align*}
\]

\( G^\pm(z/w) x^\pm(z) x^\pm(w) = G^\pm(z/w) x^\pm(w) x^\pm(z), \) (2.1)

where

\[
g(z) = G^+(z)/G^-(z), \quad G^\pm(z) = (1-q^\pm z)(1-1/t z^\pm) \quad \delta(z) = \sum_{n \in \mathbb{Z}} z^n.
\]

**Fact 2.2.** The \( \mathcal{U} \) admits the (topological) Hopf algebra structure with the Drinfeld coproduct \( \Delta \):

\[
\Delta(c^{1/2}) = c^{1/2} \otimes c^{1/2},
\]

\[
\Delta(x^+(z)) = x^+(z) \otimes 1 + \psi^-(c_{(1)}^{1/2} z) \otimes x^+(c_{(1)} z),
\]

\[
\Delta(x^-(z)) = x^-(c_{(1)} z) \otimes \psi^+(c_{(2)}^{1/2} z) + 1 \otimes x^-(z),
\]

\[
\Delta(\psi^\pm(z)) = \psi^\pm(c_{(2)}^{1/2} z) \otimes \psi^\pm(c_{(1)}^{1/2} z),
\]

where \( c_{(1)}^{1/2} = c^{1/2} \otimes 1 \) and \( c_{(2)}^{1/2} = 1 \otimes c^{1/2} \). We omit the antipode and the counit.

2.2. Fock and Fock tensor modules. A \( \mathcal{U} \)-module is called of level-(\( n, m \)), if the two central elements act as \( c = (t/q)^{n/2} \) and \( (\psi^+ / \psi^-)^{1/2} = (q/t)^{m/2} \). Let \( M \in \mathbb{Z} \). The Fock module \( \mathcal{F} \) of level-(1, \( M \)) is constructed as follows. Let \( \{a_n | n \in \mathbb{Z}\} \) be the Heisenberg algebra with the relation

\[
[a_m, a_n] = m \frac{1 - q^{0|m|}}{1 - q^{-1}} \delta_{m+n,0} a_0,
\]

and act on the Fock space \( \mathcal{F} \) with the vacuum \( |0\rangle \) in the usual way (that is, \( a_n |0\rangle = 0 \) for \( n \in \mathbb{Z}_{>0} \)). \( \mathcal{F}^* \) is the dual space generated by \( |0\rangle \) satisfying \( \langle 0 | a_n = 0 \) (\( n < 0 \)). The basis of \( \mathcal{F} \) (resp. \( \mathcal{F}^* \)) is given by \( |\lambda\rangle = a_{-\lambda_1} a_{-\lambda_2} \cdots |0\rangle \) (resp. \( |\lambda\rangle = |0\rangle \cdots a_{\lambda_2} a_{\lambda_1} \)) with a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \). The bilinear form \( \mathcal{F}^* \otimes \mathcal{F} \to \mathbb{C} \) is given by \( \langle 0 | 0 \rangle = 1 \).
Fact 2.3 ([FHHSY]). Let $u$ be a nonzero complex parameter. The following algebra homomorphism $\rho_u : \mathcal{U} \rightarrow \text{End}(\mathcal{F})$ endows the $\mathcal{U}$-module structure on $\mathcal{F}$:

\[
e^{1/2} \mapsto (t/q)^{1/4}, \quad \phi^+(z) \mapsto uz^{-M}q^{-M/2}t^{M/2}\eta(z), \quad \phi^-(z) \mapsto u^{-1}z^Mq^{M/2}t^{-M/2}\xi(z),
\]

where

\[
\eta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} a_{-n} z^n\right) \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} a_{n} z^{-n}\right),
\]

\[
\xi(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} q^{-n/2}t^{n/2} a_{-n} z^n\right) \exp\left(\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} q^{-n/2}t^{-n/2} a_{n} z^{-n}\right),
\]

\[
\phi^+(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} (1-t^n q^{-n}) q^{n/4}t^{-n/4} a_{n} z^{-n}\right),
\]

\[
\phi^-(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} (1-t^n q^{-n}) q^{n/4}t^{-n/4} a_{-n} z^n\right).
\]

We call $u$ the spectral parameter, and denote this $\mathcal{U}$-module by $\mathcal{F}^{(1,M)}_u$.

We can also define the dual $\mathcal{U}$-module structure $\mathcal{F}^{(1,M)*}_u$ on $\mathcal{F}^*$ through the same $\rho_u$ by regarding its image as in $\text{End}(\mathcal{F}^*)$.

Let $N \in \mathbb{Z}_{\geq 1}$, and let $u = (u_1, \ldots, u_N)$ be an $N$-tuple of nonzero complex parameters. Set

\[
\rho_u^{(N,0)} := (\rho_{u_1} \otimes \rho_{u_2} \otimes \cdots \otimes \rho_{u_N}) \circ \Delta^{(N)},
\]

where $\Delta^{(1)} := \text{id}$, $\Delta^{(N)} := (\text{id} \otimes \cdots \otimes \text{id} \otimes \Delta) \circ \cdots \circ (\text{id} \otimes \Delta) \circ \Delta (N \geq 2)$. (2.7)

The $\rho_u^{(N,0)}$ gives the level-$(N, 0)$ module structure on the $N$-fold tensor space of $\mathcal{F}$. We denote it by $\mathcal{F}^{(1,0)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(1,0)}_{u_N}$. In what follows, we will use the shorthand notation $\mathcal{F}_u = \mathcal{F}^{(1,0)}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(1,0)}_{u_N}$ for simplicity. We also denote $\mathcal{F}^* = \mathcal{F}^{(1,0)*}_{u_1} \otimes \cdots \otimes \mathcal{F}^{(1,0)*}_{u_N}$.

Let $|0\rangle$ (resp. $\langle 0|\rangle$) be the tensor product of the vacuum (resp. dual vacuum) states $|0\rangle^\otimes N$ (resp. $\langle 0|\rangle^\otimes N$) in $\mathcal{F}_u$ (resp. $\mathcal{F}^*_u$). Set

\[
a_n^{(i)} = 1 \otimes \cdots \otimes 1 \otimes a_n \otimes 1 \otimes \cdots \otimes 1,
\]

for simplicity.

Definition 2.4. For $k = 1, 2, \ldots, N$, set

\[
\Lambda^{(i)}(z) := \phi^- (\gamma^{1/2}z) \otimes \cdots \otimes \phi^- (\gamma^{k-3/2}z) \otimes \eta(\gamma^{-k-1}z) \otimes 1 \otimes \cdots \otimes 1.
\]
Fact 2.5 ([FHSSY]). On $\mathcal{F}_u$, we have

$$\rho_{\mathbf{u}}^{(N,0)}(x^+(z)) = \sum_{i=1}^{N} u_i \Lambda^{(i)}(z). \quad (2.10)$$

Definition 2.6. Set $X^{(1)}(z) = \rho_{\mathbf{u}}^{(N,0)}(x^+(z))$. Introduce the set of generators $X^{(k)}_n$ $(k = 1, \ldots, N, n \in \mathbb{Z})$ by performing the fusion of several $X^{(1)}$’s as

$$X^{(k)}(z) = \sum_{n \in \mathbb{Z}} X^{(k)}_n z^{-n} = X^{(1)}(y^{2(1-k)}z)X^{(1)}(y^{2(2-k)}z) \cdots X^{(1)}(z) \in \text{End}(\mathcal{F}_u)[[z^{\pm 1}]]. \quad (2.11)$$

Fact 2.7 ([AFHKSY1]). We have

$$X^{(k)}(z) = \sum_{1 \leq j_1 < \cdots < j_k \leq N} \Lambda^{(j_1)}(z) \cdots \Lambda^{(j_k)}((q/t)^{k-1}z) : u_{j_1} \cdots u_{j_k} :. \quad (2.12)$$

Here, $*: \text{denotes the usual normal ordering in the Heisenberg algebra.}$

Definition 2.8. We denote by $U(N)$ (the completion in the sense of the adic topology, of) the algebra $(X^{(i)}_n|n \in \mathbb{Z}, i = 1, \ldots N)$ in $\text{End}(\mathcal{F}_u)$. Namely, $U(N)$ is the completion of the algebra generated by the set of operators $\left\{X^{(i)}_n\right\}$.

This algebra $U(N)$ can be regarded as the tensor product of the deformed $W_N$ algebra and some Heisenberg algebra [FHSSY].

Definition 2.9. For an $N$-tuple of partitions $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)})$, we define the vectors $|X_\lambda\rangle = |X_\lambda(u)\rangle \in \mathcal{F}_u$ and $(X_\lambda) = \langle X_\lambda(u) | \in \mathcal{F}_u^*$ by

$$\langle X_\lambda(u) | := \langle 0 | \cdots X^{(N)}_{\lambda^{(N)}} X^{(N)}_{\lambda^{(N-1)}} \cdots X^{(2)}_{\lambda^{(2)}} X^{(2)}_{\lambda^{(1)}} \cdots X^{(1)}_{\lambda^{(1)}}. \quad (2.14)$$

In what follows, we omit the spectral parameter $u$, as long as there is no confusion.

Fact 2.10. ([O]) The set $(|X_\lambda\rangle)$ (resp. $(\langle X_\lambda|)$) forms a PBW-type basis of $\mathcal{F}_u$ (resp. $\mathcal{F}_u^*$), if $u_i \neq q^s t^{-r} u_j$ and $u_i \neq 0$ for all $i$, $j$ and $r, s \in \mathbb{Z}$.

In Appendix E, we reproduce the proof (presented in [O]) of Fact 2.10, with detailed treatments concerning the integration cycles.

3. Generalized Macdonald Functions

In [AFHKSY1], a sort of generalization of Macdonald functions are introduced on the level $(N, 0)$ module $\mathcal{F}_u$. In this section, we give a certain explicit formula for the generalized Macdonald functions.
3.1. *Definition of generalized Macdonald functions.* The generalized Macdonald functions are defined as eigenfunctions of $X_0^{(1)}$. In the theory of ordinary Macdonald functions, the fundamental existence theorem follows from the triangulation of Macdonald’s difference operator. In that triangulation, the order of a basis is given by the dominance partial ordering. In the $N = 1$ case, $X_0^{(1)}$ corresponds to Macdonald’s difference operator [AMOS]. In the general $N$ case, the operator $X_0^{(1)}$ can be triangulated by the following partial ordering.

**Definition 3.1.** The partial ordering $^*$ on the set of $N$-tuples of partitions is defined by

$$
\lambda^* > \mu \iff |\lambda| = |\mu|, \sum_{i=k}^{N} |\lambda^{(i)}| \geq \sum_{i=k}^{N} |\mu^{(i)}| \quad (\forall k) \quad \text{and}

(|\lambda^{(1)}|, |\lambda^{(2)}|, \ldots, |\lambda^{(N)}|) \neq (|\mu^{(1)}|, |\mu^{(2)}|, \ldots, |\mu^{(N)}|).
$$

(3.1)

**Remark 3.2.** This partial ordering is defined so that as we move boxes in a right Young diagram to a left one, it gets smaller. On the other hand, as boxes are moved from left to right, it gets larger. Note that if $N$-tuples of partitions $\lambda$ and $\mu$ have the same number of boxes in each Young diagram, then neither $\lambda^* > \mu$ nor $\lambda < \mu$.

**Example 3.3.** If $N = 3$ and the number of boxes is 2, then

$$
(\emptyset, \emptyset, (2)) \quad (\emptyset, \emptyset, (1, 1)) \quad (\emptyset, (1), (1)) \quad ((1), \emptyset, (1)) \quad (\emptyset, (2), \emptyset) \quad (\emptyset, (1, 1), \emptyset) \quad ((1), (1), \emptyset) \quad ((2), \emptyset, \emptyset) \quad ((1, 1), \emptyset, \emptyset).
$$

Here $\lambda \rightarrow \mu$ stands for $\lambda^* > \mu$.

Regarding the products of ordinary Macdonald functions $\prod_{j=1}^{N} P_{\lambda^{(j)}}(a_{-n}^{(j)}) |0\rangle$ as a basis of $\mathcal{F}_\mu$, we can triangulate the operator $X_0^{(1)}$. Here, $P_\lambda(a_{-n})$ is the abbreviation for $P_\lambda(a_{-1}, a_{-2} \ldots)$, which is the Macdonald function obtained by replacing the power sum symmetric functions with generators of the Heisenberg algebra. For the definition and notations of the ordinary Macdonald functions see Appendix B. In Appendix B, we also explain some well-known facts for Macdonald functions.

**Fact 3.4 ([AFO]).** We have

$$
X_0^{(1)} \prod_{j=1}^{N} P_{\lambda^{(j)}}(a_{-n}^{(j)}) |0\rangle = e_\lambda \prod_{j=1}^{N} P_{\lambda^{(j)}}(a_{-n}^{(j)}) |0\rangle + \sum_{\mu < \lambda} \nu_{\lambda, \mu} \prod_{j=1}^{N} P_{\mu^{(j)}}(a_{-n}^{(j)}) |0\rangle,
$$

(3.2)
Similarly, there exists a unique vector $\langle \lambda | P_{\lambda}^{(i)} a_{\lambda}^{(i)} \rangle X_0^{(1)} = \epsilon^*_\lambda \langle \lambda | P_{\lambda}^{(i)} a_{\lambda}^{(i)} \rangle + \sum_{\mu > \lambda} v_{\lambda, \mu}^* \langle \lambda | P_{\lambda}^{(i)} a_{\lambda}^{(i)} \rangle$. (3.3)

Here, $v_{\lambda, \mu}, v_{\lambda, \mu}^* \in \mathbb{C}$. The eigenvalues $\epsilon^\lambda$ and $\epsilon^\lambda_*$ are in the following form:

$$\epsilon^\lambda(u) = \epsilon^\lambda_*(u) = \sum_{k=1}^N u_k e^\lambda(k), \quad \epsilon^\lambda := 1 + (t-1) \sum_{i \geq 1} (q^{\lambda_i} - 1) t^{-i}. \quad (3.4)$$

**Remark 3.5.** In this paper, we assume that the complex parameters $q, t$ and the spectra parameters $u$ are generic in the following sense:

$$\epsilon^\lambda(u) \neq 0 \quad (\forall \lambda); \quad (3.5)$$

$$\epsilon^\lambda(u) \neq \epsilon^\mu(u) \quad (\lambda \neq \mu); \quad (3.6)$$

$$u_i = q^{\mu_n} = u_j \quad (n, m \in \mathbb{Z}, i, j = 1, \ldots, N). \quad (3.7)$$

Under this assumption, the eigenfunctions of $X_0^{(1)}$ can be characterized as follows.

**Fact 3.6 (Existence and Uniqueness [AFO]).** For an $N$-tuple of partitions $\lambda$, there exists a unique vector $\langle P_\lambda \rangle = \langle P_\lambda u \rangle \in \mathcal{F}_u$ such that

- $|P_\lambda(u)\rangle = \prod_{i=1}^N P_{\lambda(i)}(a_{\lambda(i)}^{(i)})|0\rangle \prod_{i=1}^N P_{\mu(i)}(a_{\mu(i)}^{(i)})|0\rangle, \quad u_{\lambda, \mu} \in \mathbb{C}; \quad (3.8)$

- $X_0^{(1)} |P_\lambda(u)\rangle = \epsilon^\lambda(u) |P_\lambda(u)\rangle, \quad \epsilon^\lambda(u) \in \mathbb{C}. \quad (3.9)$

Similarly, there exists a unique vector $\langle P_\lambda \rangle = \langle P_\lambda u \rangle \in \mathcal{F}_u^*$ such that

- $\langle P_\lambda(u) | = \langle 0 \rangle \prod_{i=1}^N P_{\lambda(i)}(a_{\lambda(i)}^{(i)}) + \sum_{\mu > \lambda} u_{\lambda, \mu}^* \langle 0 \rangle \prod_{i=1}^N P_{\lambda(i)}(a_{\lambda(i)}^{(i)}) \langle 0 \rangle, \quad \epsilon^\lambda_*(u) \in \mathbb{C}; \quad (3.10)$

- $\langle P_\lambda(u) | X_0^{(1)} = \epsilon^\lambda_*(u) \langle P_\lambda(u) |, \quad \epsilon^\lambda_*(u) \in \mathbb{C}. \quad (3.11)$

Again, we omit $u$ unless mentioned otherwise.

Note that for integers $n_i$ ($i = 1, \ldots, N$) with $n_i \geq \ell(\lambda(i))$, the eigenvalues can be rewritten as

$$\epsilon^\lambda(u) = (1 - t^{-1}) \sum_{k=1}^N u_i \sum_{k=1}^{n_i} q^{\lambda_i} t^{1-k} + \sum_{k=1}^N u_i t^{-n_i}. \quad (3.12)$$

By this Fact, it is easily seen that the generalized Macdonald functions $|P_\lambda\rangle$ form a basis of $\mathcal{F}_u$. It is also known that the operator $X_0^{(1)}$ is a member of a certain family of infinitely many commutative operators, and $|P_\lambda\rangle$ can be regarded as simultaneous eigenfunctions of them [FHHSY]. Moreover, $|P_\lambda\rangle$ correspond to the torus fixed points in the instanton moduli space, which form a significant basis in the AGT correspondence. Note that this is a $q$-analogue of the AFLT basis [AFLT].
**Fact 3.7** ([AFHKSY1]). \( \langle P_{\lambda} | P_{\mu} \rangle \) and \( \langle P_{\lambda} | \) are orthogonal, and their inner products take the form

\[
\langle P_{\lambda} | P_{\mu} \rangle = \prod_{i=1}^{N} \frac{c'_{\lambda(i)}}{c_{\lambda(i)}} \delta_{\lambda,\mu}.
\]  

(3.13)

Here, we put

\[
c_{\lambda} := \prod_{(i,j) \in \lambda} (1-q\alpha_{j}(i,j) \cdot t\xi_{i}(j,j)+1), \quad c'_{\lambda} := \prod_{(i,j) \in \lambda} (1-q\alpha_{j}(i,j)+1 \cdot t\xi_{i}(j,j)).
\]  

(3.14)

**Definition 3.8.** We introduce \( |Q_{\lambda} \rangle \) such that \( \langle P_{\lambda} | Q_{\mu} \rangle = \delta_{\lambda,\mu} \), i.e., \( |Q_{\lambda} \rangle := \prod_{i=1}^{N} \frac{c'_{\lambda(i)}}{c_{\lambda(i)}} |P_{\lambda} \rangle \).

3.2. **Screening currents and vertex operators.** This subsection and the next are devoted to the construction of an explicit formula for the generalized Macdonald functions \( |Q_{\lambda} \rangle \). Our explicit formula is made up of two ingredients. The one is a certain screened vertex operators on \( \mathcal{F}_u \), and the other is an explicit solution to the eigenfunction problem associated with the Macdonald \( q \)-difference operator of A-type. In this subsection, we introduce the first ingredient, screening currents and some vertex operators. They play an important role also in a realization of our main vertex operator \( \hat{V}(x) \).

**Notation 3.9.** For an \( N \)-tuple of parameters \( \mathbf{u} = (u_1, \ldots, u_N) \), we write

\[
i_{\pm} \cdot \mathbf{u} := (u_1, \ldots, u_{i-1}, t^{\pm 1}u_i, u_{i+1}, \ldots, u_N),
\]

(3.15)

\[
i_{\pm} \cdot \mathbf{u} := (u_1, \ldots, u_{i-1}, tu_i, t^{-1}u_{i+1}, u_{i+2}, \ldots, u_N).
\]

(3.16)

**Definition 3.10.** Define the screening currents \( S^{(i)}(y) : \mathcal{F}_{i^\infty} \rightarrow \mathcal{F}_u \) by the following,

\[
S^{(i)}(z) := 1 \otimes \cdots \otimes 1 \otimes \phi^{sc}(y^{i-1}z) \otimes 1 \otimes \cdots \otimes 1,
\]

(3.17)

with

\[
\phi^{sc}(z) := \exp \left( -\sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} y^{2n} a_n z^n \right) \exp \left( \sum_{n>0} \frac{1}{n} \frac{1-t^{-n}}{1-q^{-n}} y^{2n} a_n z^{-n} \right)
\]

\[
\otimes \exp \left( \sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} y^n a_n z^n \right) \exp \left( -\sum_{n>0} \frac{1}{n} \frac{1-t^{-n}}{1-q^{-n}} y^n a_n z^{-n} \right).
\]

(3.18)

**Remark 3.11.** In this paper, we do not attach zero modes to the screening currents. When discussing commutativity with the algebra \( \mathbb{U}(N) \), we use a function satisfying some difference equation as in the following proposition.

**Proposition 3.12.** Let \( g_i(w) (i = 1, \ldots, N-1) \) be a function which satisfies the difference equation \( g_i(qw) = \frac{u_{i+1}}{u_i} g_i(w) \). Then, the commutation relation between \( S^{(i)}(w)g_i(w) \) and the algebra \( \mathbb{U}(N) \) is a total \( q \)-difference:

\[
\left[ S^{(i)}(w)g_i(w), X_n^{(k)} \right] = (1 - T_{q,w}) \text{ (some operators)} \quad (\forall i,k,n).
\]

(3.19)

Here, \( T_{q,w} \) is the difference operator such that \( T_{q,w} : w \mapsto qw \).
The proof is given in Appendix D.1.

Remark 3.13. The function \( g_t(w) \) can be realized as \( g_t(w) = \frac{\theta_q(t^2u_iw/u_{i+1})}{\theta_q(tw)} \), and integrals of the screening currents commute with \( U(N) \) if the spectral parameters \( u \) are degenerated (See Proposition E.4).

Definition 3.14. Define the screened vertex operators \( \Phi^{(k)}(x) : \mathcal{F}_{i-k+1,u} \to \mathcal{F}_u \) \((k = 0, 1, \ldots, N - 1)\) by

\[
\Phi^{(0)}(x) = : \exp \left( \sum_{n > 0} \frac{1}{n} \frac{1 - t^n}{1 - q^n} a_n^{(1)} x^{-n} \right) \exp \left( \sum_{n > 0} \frac{1}{n} \frac{1 - \gamma 2^n t^n}{1 - q^n} x^{-n} \right) \right) \times \exp \left( \sum_{n > 0} \frac{1}{n} \frac{1 - \gamma 2^n t^n}{1 - q^n} \sum_{j=2}^N \gamma^{(j-1)n} a_n^{(j)} x^{-n} \right) :,
\]

and as the composition of operators,

\[
\Phi^{(k)}(x) := \prod_{i=1}^k \frac{(q; q)_{\infty}(q/t; q)_{\infty}}{(qu_i/u_{k+1}; q)_{\infty}(qu_{i+1}/tu_i; q)_{\infty}} \cdot \oint_{C} \prod_{i=1}^k \frac{dy_i}{2\pi \sqrt{-1} y_i} \Phi^{(0)}(x) S^{(1)}(y_1) \cdots S^{(k)}(y_k) g(x, y_1, \ldots, y_k)
\]

with an integral kernel

\[
g(x, y_1, \ldots, y_k) = \frac{\theta_q(tu_1y_1/u_{k+1}x)}{\theta_q(ty_1/x)} \prod_{i=1}^{k-1} \frac{\theta_q(tu_{i+1}y_{i+1}/u_{k+1}y_i)}{\theta_q(ty_{i+1}/y_i)}.
\]

Here, the contour of the integration \( C \) is chosen so that \(|t|^{-1} < |y_j/y_i| < |q|\) for \(1 \leq i < j \leq k\), and \(|q/t| < |y_i/x| < 1\) for \( i \geq 1 \).

Remark 3.15. The integration contour is well-defined if \(|t|^{-1} < |q^{N-2}|\). Hence, in what follows, we assume this condition in this paper without Appendix E.

These screened vertex operators can be obtained by combining a certain specialized intertwiner of the DIM algebra and replacing some Jackson integrals with contour ones. In Appendix A, we explain details of this construction and correspondence between Jackson integrals and contour ones.

Remark 3.16. The screened vertex operator \( \Phi^{(k)}(x) \) is normalized such that \( \oint_{C} \frac{dx}{2\pi \sqrt{-1} x} \Phi^{(k)}(x) |0\rangle = |0\rangle \). Indeed, \( \Phi^{(k)}(x) \) can be expanded as

\[
\Phi^{(k)}(y_0) = \oint_{C} \prod_{i=1}^k \frac{dy_i}{2\pi \sqrt{-1} y_i} \sum_{r_1, \ldots, r_k \in \mathbb{Z}} \prod_{i=1}^k \frac{(tu_i/u_{k+1}; q)_{r_i}}{(qu_i/u_{k+1}; q)_{r_i}} (y_i/y_i-1)^{r_i} : \Phi^{(0)}(y_0) S^{(1)}(y_1) \cdots S^{(k)}(y_k) :.
\]
This expansion follows from the operator products (C.7)-(C.15) and Ramanujan’s $1\psi_1$ summation formula (Fact 3.17). By (3.24), it can be shown that the coefficient of $y^r_0$ in the expansion of $\Phi^{(0)}(y_0)\ket{0}$ is given by a finite sum and the constant term with respect to $y_0$ is 1.

**Fact 3.17** ((5.2.1) in [GR]).

$$1\psi_1(a; b; q; z) := \sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n \frac{(q/q)_{\infty}(b/a; q)_{\infty}(az; q)_{\infty}(q/az; q)_{\infty}}{(b/q)_{\infty}(q/a; q)_{\infty}(z; q)_{\infty}(b/az; q)_{\infty}} (|b/a| < z < 1). \quad (3.25)$$

**Remark 3.18.** Note that $g(x, y_1, \ldots, y_k)$ satisfies

$$T_{q, y_i} g(x, y_1, \ldots, y_k) = \frac{u_{i+1}}{u_i} g(x, y_1, \ldots, y_k). \quad (3.26)$$

This corresponds to the difference equation which the function $g_i(z)$ satisfies in Proposition 3.12. Moreover, the screening currents in $\Phi^{(k)}(x)$ commute with the algebra $U(N)$ by taking the integrals. For more details, see the proof of the following lemma.

The screened vertex operators $\Phi^{(k)}(x)$ satisfy the following relation with the algebra $U(N)$.

**Lemma 3.19.** For $k = 0, 1, \ldots, N - 1$ and $r = 1, \ldots, N$,

$$X^{(r)}(z)\Phi^{(k)}(x) - \frac{1 - (q/t)^r z/tx}{1 - z/tx} \Phi^{(k)}(x)X^{(r)}(z) = u_{k+1}(1 - t^{-1})\delta(tx/z)\Psi^{(r)}(x)\Phi^{(k)}(qx)\Psi^{+}(x), \quad (3.27)$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ is the formal delta function, and we defined

$$Y^{(r)}(x) := \sum_{2 \leq i_2 < \cdots < i_r \leq N} : \Lambda^{(i_2)}((q/t)tx) \cdots \Lambda^{(i_r)}((q/t)^{r-1}tx) : u_{i_2} \cdots u_{i_r}, \quad (3.28)$$

$$\Psi^{+}(z) := \exp \left( \sum_{n} \frac{1}{n} (1 - \gamma^{2n}) \sum_{j=1}^{N} \gamma^{(j-1)n} a_n^{(j)} z^{-n} \right) = \prod_{k=0}^{\rho(N)} \frac{1}{\rho^{(N)}(\psi^{+}(\gamma^{-1}t^{-k}z))}. \quad (3.29)$$

*Note that in particular $Y^{(1)}(z) = 1$.*

For the proof, see Appendix D.2

### 3.3. Explicit formula for generalized Macdonald functions

In this subsection, we explain an explicit solution to the eigenfunction equation of the Macdonald $q$-difference operator of A-type. By using its solution, we give an explicit formula for generalized Macdonald functions.

First, we introduce a modified Macdonald $q$-difference operator.
Definition 3.20. Define the operators on $\mathbb{C}[[x_2/x_1, x_3/x_2, \ldots, x_n/x_{n-1}]]$ by

$$D_n^+(s; q, t) := \sum_{k=1}^{n} s_k \prod_{1 \leq \ell < k}^{n} \frac{1 - tx_k/x_\ell}{1 - tx_k/x_\ell} \prod_{k < \ell \leq n}^{n} \frac{1 - x_\ell/x_k}{1 - x_\ell/x_k} T_{q,x_\ell},$$

(3.30)

where $s = (s_1, \ldots, s_n)$ are generic parameters, and $T_{q,x_\ell}$ is the difference operator defined by

$$T_{q,x_\ell} F(x_1, \ldots, x_n) = F(x_1, \ldots, q x_\ell, \ldots, x_n).$$

(3.31)

Though it is essentially equivalent, we also introduce the operator

$$\widetilde{D}_n^+(s; q, t) := \sum_{k=1}^{n} s_k \prod_{1 \leq \ell < k}^{n} \frac{1 - tx_k/x_\ell}{1 - x_k/x_\ell} \prod_{k < \ell \leq n}^{n} \frac{1 - q x_\ell/x_k}{1 - q x_\ell/x_k} T_{q^{-1},x_\ell}^{-1}.$$  

(3.32)

An explicit formula for the eigenfunction of $\widetilde{D}_n^+(s; q, t)$ was conjectured in [S]. Afterward, it was proved, and a solution to the bispectral problem was also given in [NS].

Definition 3.21. Let $x = (x_i)_{1 \leq i \leq n}$ be indeterminates and $s = (s_j)_{1 \leq j \leq n}$ be generic parameters. Define $p_n(x; s|q, t)$ and $f_n(x; s|q, t) \in \mathbb{C}[[x_2/x_1, x_3/x_2, \ldots, x_n/x_{n-1}]]$ by

$$p_n(x; s|q, t) = \sum_{\theta \in M_n} c_n(\theta; s|q, t) \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_i,j},$$

$$f_n(x; s|q, t) = \prod_{1 \leq k < \ell \leq n} (1 - x_\ell/x_k) \cdot p_n(x; s|q^{-1}, t^{-1}).$$

(3.33)

(3.34)

For convenience, we put $f_0 = p_0 = 1$. Here $M_n$ is the set of all $n \times n$ upper triangular matrices with nonnegative integers, in which diagonal elements are 0. $c_n(\theta; s|q, t)$ are coefficients defined by the following recurrence relations

$$c_1(-; s_1, q, t) = 1,$$

$$c_n(\theta_i,j)_{1 \leq i < j \leq n}; (s_j)_{1 \leq i \leq n}|q, t)$$

$$= c_{n-1}(\theta_i,j)_{1 \leq i < j \leq n-1}; (q^{-\theta_i,j} s_j)_{1 \leq i \leq n-1}|q, t) \times d_n(\theta_i,n)_{1 \leq i \leq n-1}; (s_i)_{1 \leq i \leq n}|q, t),$$

(3.35)

$$n = 2, 3, \ldots,$$

with

$$d_n(\theta_i,n)_{1 \leq i \leq n-1}; (s_i)_{1 \leq i \leq n}|q, t)$$

$$= \prod_{i=1}^{n-1} (q/t)^{\theta_i,j} t^{s_j/s_i} q^{\theta_i,j} \prod_{1 \leq i < j \leq n-1} (t^{s_j/s_i} q^{\theta_i,j} / (q^{\theta_i,j} s_j/s_i));$$

(3.36)

From the recurrence relations, we can compute the explicit form of $c_n$ as

$$c_n(\theta_i,j)_{1 \leq i < j \leq n}; (s_i)_{1 \leq i \leq n}|q, t) = \prod_{1 \leq i < j \leq n} (q/t)^{\theta_i,j} t^{s_j/s_i} q^{\theta_i,j} \prod_{1 \leq i < j \leq n} (q^{s_j/s_i} t^{s_j/s_i} q^{\theta_i,j} / (q^{\theta_i,j} s_j/s_i));$$

$$\times \prod_{k=1}^{n} \prod_{1 \leq i < m \leq k} \frac{(q^m s_m/k^{\theta_i,j} q^{s_m/s_i} q^{\theta_i,j}) / (q^{s_m/s_i} t^{s_m/s_i} q^{\theta_i,j}) \prod_{j=1}^{n} (q^{s_j/s_i} t^{s_j/s_i} q^{\theta_i,j})}{(q^{s_m/s_i} t^{s_m/s_i} q^{\theta_i,j}) / (q^{s_m/s_i} t^{s_m/s_i} q^{\theta_i,j}) \prod_{j=1}^{n} (q^{s_j/s_i} t^{s_j/s_i} q^{\theta_i,j})},$$

(3.37)
**Fact 3.22 ([NS, S, BFS]).** The function $p_n(x; s|q, t)$ is an unique formal solution to the eigenfunction equation

$$D_n^1(s; q, t)p_n(x; s|q, t) = (s_1 + \cdots + s_n)p_n(x; s|q, t),$$

up to scalar multiples. Equivalently, $f_n(x; s|q, t)$ is an unique function (up to scalar multiples) such that

$$\tilde{D}_n^1(s; q, t)f_n(x; s|q, t) = (s_1 + \cdots + s_n)f_n(x; s|q, t).$$

**Remark 3.23.** The function $p_n(x; s|q, t)$ and $f_n(x; s|q, t)$ are dual to each other in the sense of Lemma C.2. Furthermore, it is known that the function $p_n(x; s|q, t)$ can also be regarded as a holomorphic function, and its analyticity is clarified in [NS]. We explain its analyticity in Sect. 3.4 (Fact 3.32).

In the beginning of this subsection, we stated to give an explicit formula for generalized Macdonald functions. For that purpose, let us introduce the following notations with respect to the screened vertex operators.

**Notation 3.24.** For an $N$-tuple of nonnegative integers $\mathbf{n} = (n_1, \ldots, n_N)$, we write

$$|\mathbf{n}| := \sum_{s=1}^{N} n_s, \quad [i, k]_n := \sum_{s=1}^{i-1} n_s + k \quad (1 \leq i \leq N, \quad k \leq n_i),$$

$$t^{\pm\mathbf{n}} \cdot \mathbf{u} := (t^{\pm n_1}u_1, \ldots, t^{\pm n_N}u_N).$$

**Definition 3.25.** Let $\mathbf{n} = (n_1, \ldots, n_N)$ be an $N$-tuple of nonnegative integers. Define the operator $V^{(n)}(x_1, \ldots, x_{|\mathbf{n}|}) : F_{t^{-\mathbf{n}} \cdot \mathbf{u}} \to F_{\mathbf{u}}$ by

$$V^{(n)}(x_1, \ldots, x_{|\mathbf{n}|}) = \Phi(0)(x_1) \cdots \Phi(0)(x_{n_1}) \Phi(1)(x_{n_1+1}) \cdots \Phi(1)(x_{n_1+n_2})$$

$$\cdots \Phi(N-1)(x_{N+1}) \cdots \Phi(N-1)(x_{|\mathbf{n}|}).$$

We occasionally write $V^{(n)}(x_1, \ldots, x_{|\mathbf{n}|}) = V^{(n)}(t^{-\mathbf{n}} \cdot \mathbf{u}; x_1, \ldots, x_{|\mathbf{n}|})$ in order that the spectral parameters of representations are clear.

Now, we obtain an explicit formula for the generalized Macdonald functions $|Q_\lambda\rangle$ in terms of the screened vertex operators. The action of the screened vertex operators is controlled by the function $f_n(x; s|q, t)$. Further, we can also obtain an expression of $p_n$ by using $|P_\lambda\rangle$ and the operator $V^{(n)}$. These expressions of $|Q_\lambda\rangle$ and $p_n$ are symmetric to each other in some sense.

**Theorem 3.26.** Let $\mathbf{n} = (n_1, \ldots, n_N)$ be an $N$-tuple of integers satisfying that $n_i \geq \ell(\lambda(i))$ for all $i$. Then under the identification $s[i, k]_n = q^k t^{1-k}u_i \quad (1 \leq k < n_i, \quad i = 1, \ldots, N)$, the followings hold:

$$x^{-\lambda} f_n(x; s|q, t)V^{(n)}(t^{-\mathbf{n}} \cdot \mathbf{u}; x_1, \ldots, x_{|\mathbf{n}|})|0\rangle = \mathcal{R}_\lambda^n(\mathbf{u})|Q_\lambda\rangle,$$

$$x^{-\lambda} \langle P_\lambda| V^{(n)}(t^{-\mathbf{n}} \cdot \mathbf{u}; x_1, \ldots, x_{|\mathbf{n}|})|0\rangle = \mathcal{R}_\lambda^n(\mathbf{u}) p_n|s|q, q/t).$$
Here, we set
\[
x^{-\lambda} := \prod_{i=1}^{N} \prod_{k=1}^{n_i} x_{[i,k]n}^{-\lambda_k^{(i)}}.
\]  
(3.45)

\[\cdots\] \text{x,1 means to take the constant term in \cdots with respect to x, and } \mathcal{R}_x^u(u) \in \mathbb{C}(u_1, \ldots, u_N) \text{ is some coefficient.}

\textbf{Proof.} By Lemma 3.19 and the operator product (C.21) we can get the relation

\[
\begin{align*}
X^{(1)}(z) V^{(n)}(t^{-n} \cdot u : x_1, \ldots, x_{|n|}) &= \prod_{k=1}^{|n|} \frac{1 - qz/t^2 x_k}{1 - z/x_k} \cdot V^{(n)}(x_1, \ldots, x_{|n|}) X^{(1)}(z) \\
+ (1 - t^{-1}) \sum_{i=1}^{n_i} u_i t^{1-k} \delta(t x_{[i,k]}/z) \prod_{1 \leq \ell < [i,k]} \frac{1 - q x_{[i,k]}/(tx_{[i,k]}/x_{[i,k]})}{1 - x_{[i,k]}/x_{[i,k]}} \times T_{q^{-1}, x_{[i,k]}} \left(V^{(n)}(x_1, \ldots, x_{|n|}) \right) \Psi^x(x_{[i,k]}).
\end{align*}
\]  
(3.46)

By taking the constant term of z, we have

\[
\begin{align*}
X^{(1)}_0 V^{(n)}(x_1, \ldots, x_{|n|}) |0\rangle &= V^{(n)}(x_1, \ldots, x_{|n|}) |0\rangle \left(\sum_{i=1}^{N} t^{-n_i} u_i\right) \\
+ (1 - t^{-1}) D^{1}_{|n|} (s_{\lambda_k^{(i)} = 0}; q, q/t) V^{(n)}(x_1, \ldots, x_{|n|}) |0\rangle.
\end{align*}
\]  
(3.47)

Noting the integrals of the total difference vanish, integrating by parts gives the following equality.

\[
\begin{align*}
\left[ x^{-\lambda} f(x; s|q, t) D^{1}_{|n|} (s_{\lambda_k^{(i)} = 0}; q, q/t) V^{(n)}(x_1, \ldots, x_{|n|}) |0\rangle \right]_{x,1} \\
= \left[ \left( D^{1}_{|n|} (s_{\lambda_k^{(i)} = 0}; q, q/t) x^{-\lambda} f(x; s|q, t) \right) V^{(n)}(x_1, \ldots, x_{|n|}) |0\rangle \right]_{x,1} \\
= \left[ \left( x^{-\lambda} D^{1}_{|n|} (s; q, q/t) f(x; s|q, t) \right) V^{(n)}(x_1, \ldots, x_{|n|}) |0\rangle \right]_{x,1} \\
= \left( \sum_{i=1}^{N} \sum_{k=1}^{n_i} s_{[i,k]} \right) \left[ x^{-\lambda} f(x; s|q, q/t) V^{(n)}(x_1, \ldots, x_{|n|}) |0\rangle \right]_{x,1}.
\end{align*}
\]  
(3.48)

Here, in the third equality, we made use of Fact 3.22. This equality identify the LHS of (3.43) with the generalized Macdonald functions up to the normalization.

From (3.47), we can also have

\[
D^{1}_{|n|} (s; q, q/t) x^{-\lambda} \langle P_\lambda | V^{(n)}(x_1, \ldots, x_{|n|}) |0\rangle \\
= (s_1 + \cdots + s_{|n|}) x^{-\lambda} \langle P_\lambda | V^{(n)}(x_1, \ldots, x_{|n|}) |0\rangle.
\]  
(3.49)

Therefore, \(x^{-\lambda} \langle P_\lambda | V^{(n)}(0)\rangle\) is an eigenfunction of \(D^{1}_{|n|}\). This implies (3.44). Since the constant terms in \(p_n\) and \(f_n\) are \(c_1(-; s_1 | q, t) = 1\), it is clear that (3.43) and (3.44) have the same proportionality constant \(\mathcal{R}_x^u(u)\). □
The following proposition gives the explicit form of $\mathcal{R}^{n}_\chi(u)$.

**Proposition 3.27.** The coefficient $\mathcal{R}^{n}_\chi(u)$ is of the form,

$$\mathcal{R}^{n}_\chi(u) = \gamma \sum_{i=1}^{N} (i-1)(\lambda^{(i)}) \prod_{k=2}^{N} \prod_{i=1}^{k-1} \frac{(r^{-n_i+i}u_l/u_k; q)_{-\lambda^{(k)}}}{(q^{-n_i+l-i}u_l/u_k; q)_{-\lambda^{(k)}}}.$$  

(3.50)

The proof of Proposition 3.27 is in Sect. D.3.

### 3.4. Analyticity of matrix elements of $V^{(n)}$.

In this subsection, we describe the fact that the Macdonald functions $p_n$ can be treated as holomorphic functions. Moreover, we show the matrix elements of $V^{(n)}$ can meromorphically extend to the whole $\mathbb{C}^{[n]-1}$, and thus we can deal with the variables $x$ as those in that space, not as formal variables.

**Definition 3.28.** Define the projection for the canonical coordinates $\pi_n : (\mathbb{C}^*)^n \to \mathbb{C}^{n-1}$ such that for each $a = (a_1, \ldots, a_n) \in (\mathbb{C}^*)^n$,

$$\pi_n(a) = (a_2/a_1, \ldots, a_n/a_{n-1}) \in \mathbb{C}^{n-1}.$$  

(3.51)

**Notation 3.29.** In what follows in this subsection, we use the following notations.

- $z := (z_1, \ldots, z_{|n|-1}) = \pi_{|n|}(\langle x_1, \ldots, x_{|n|} \rangle)$,
- $\tilde{z} := (\tilde{z}_1, \ldots, \tilde{z}_{|n+|m|-1}) = \pi_{|n+|m|}(\langle x_1, \ldots, y_1, \ldots \rangle)$,
- $w := (w_1, \ldots, w_{|n|-1}) = \pi_{|n|}(\langle s_1, \ldots, s_{|n|} \rangle)$,
- $\tilde{w} := (\tilde{w}_1, \ldots, \tilde{w}_{|n+|m|-1}) = \pi_{|n+|m|}(\langle s_1, \ldots, s_{|n+|m|} \rangle)$.

**Definition 3.30.** Define the open subset $D_w \subset \mathbb{C}^{n-1}$ by

$$D_w = \{ w = (w_1, \ldots, w_{n-1}) \in \mathbb{C}^{n-1} \mid w_i \cdots w_{j-1} \notin q^{-Z} \cup \{0\}, \ 1 \leq i < j \leq n \}.$$  

(3.52)

so that

$$\pi^{-1}(D_w(r)) = \{ s = (s_1, \ldots, s_n) \in (\mathbb{C}^*)^n \mid s_j/s_i \notin q^{-Z}, \ 1 \leq i < j \leq n \}.$$  

(3.53)

**Definition 3.31.** Define the subsets $U^n_z(r), B^n_z(r) \subset \mathbb{C}^{n-1}$ by

$$U^n_z(r) = \{ z = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} \mid |z_i| < r, \ i = 1, \ldots, n-1 \};$$  

(3.54)

$$B^n_z(r) = \{ z = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} \mid |z_i| \leq r, \ i = 1, \ldots, n-1 \};$$  

(3.55)

so that

$$\pi^{-1}(B^n_z(r)) = \{ x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n \mid |x_j/x_i| \leq r^{-i}, \ 1 \leq i < j \leq n \}.$$  

(3.56)

In order to see the matrix element above can be analytically continued except for its singularities, we make use of the following fact, proved in [NS].
Fact 3.32 ([NS]) Let $\tau$ be a generic complex parameter. For $n = 2, 3, \ldots$, we regard $p_n(x; s|q, \tau)$ as a formal power series in $z = (z_1, \ldots, z_{n-1})$ with coefficients in $O(D^n_w)$, the ring of holomorphic functions on $D^n_w$:

$$p_n(x; s|q, \tau) = \sum_{\theta \in \mathcal{M}_n} c_n(\theta; s|q, \tau) \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_{i,j}} \in O(D^n_w)[[z]]. \quad (3.57)$$

Set $r_0 = |q/\tau|^{n-2}$ if $|q/\tau| \leq 1$, and $r_0 = |\tau/q|^{n-2}$ if $|\tau/q| \geq 1$. Then for any compact subset $K \subset D_w$ and for any $r < r_0$, this series 3.57 is absolutely convergent, uniformly on $B^0(r) \times K$. Hence $p_n(x; s|q, \tau)$ defines a holomorphic function on $U^n_z(r_0) \times D^n_w$.

Then, we have the following theorem.

Theorem 3.33. The operator $V^{(n)}(x_1, \ldots, x_{|n|})$ is well-defined on $\pi^{-1}_{|n|}(U^n_z(r_0))$ with $r_0 = |t^{-1}|$, in the sense that its matrix elements are the holomorphic functions there.

We prepare the following lemma.

Lemma 3.34. Let $u = t^{-n} \cdot v$, $w = t^{-m} \cdot u$ and $n_i \geq \ell(\lambda^{(i)}) \ (\forall i)$. Then

$$x^{-\lambda} \langle P_\lambda | V^{(n)}(v_u; x_1, \ldots, x_{|n|}) V^{(m)}(u_w; y_1, \ldots, y_{|m|}) | 0 \rangle = R_\lambda(v) p_{|n|+|m|}(x, y; s|q, q/t)$$

under the identification

$$s_{i[k]} = q^\nu^{(i)} 1^{1-k} v_i \quad (1 \leq k \leq n_i, i = 1, \ldots, N), \quad (3.59)$$

$$s_{|n|+i[k]} = t^{1-n_i-k} v_i \quad (1 \leq k \leq m_i, i = 1, \ldots, N). \quad (3.60)$$

This lemma can be derived as a corollary of Theorem 3.26 by noting the normalization of the screened vertex operators (Remark 3.16).

Proof of Theorem 3.33. First, by Fact 3.32, the left hand side of (3.58) is a holomorphic function on $U^n_z|n|+|m| (r_0)$. Then, through the pull-back $\pi_{|n|+|m|}$, we regard it as the holomorphic function in $\pi^{-1}_{|n|+|m|}(U^n_z|n|+|m| (r_0))$. Thus, once we fix $y \in (\mathbb{C}^*)^{|m|}$ such that $|y_j/y_i| < r_0^{j-i}$ $(1 \leq i < j \leq |m|)$, it becomes a holomorphic function of $x \in \tilde{\pi}^{-1}_{|n|, y_1}(U^n_z|n| (r_0))$, where

$$\tilde{\pi}_{|n|, y_1}(U^n_z|n| (r_0)) := \{(x_1, \ldots, x_{|n|}) \in (\mathbb{C}^*)^{|n|} \mid |x_j/x_i| < r_0^{j-i} \} \quad (1 \leq i < j \leq |n|), \ |y_1/x_{|n|}| < r_0.$$ 

Second, by multiplying both sides of (3.58) by $y^{-\mu} f_{|m|}(y; s|q, q/t)$ and taking the constant term in $y$, the left hand side of (3.58) becomes $x^{-\lambda} \langle P_\lambda | V^{(n)}(t^{-n} v; x_1, \ldots, x_{|n|}) | Q_\mu \rangle$ with the help of Theorem 3.26. Note that, in this case, taking the constant term means the usual contour integral in $y$, and the contour is chosen to separate $x$ and $y$. This procedure does not affect the holomorphicity in $x$.

At this point, $x^{-\lambda} \langle P_\lambda | V^{(n)}(t^{-n} v; x_1, \ldots, x_{|n|}) | Q_\mu \rangle$ can holomorphically extend to $\pi^{-1}_{|n|}(U^n_z|n| (r_0))$, because we can take arbitrary small $y_1$. As a result, Theorem 3.33 is proved. \(\Box\)
3.5. Integral forms. In this subsection, we define the renormalization of generalized Macdonald functions, which correspond to the integral form of ordinary Macdonald functions up to monomials. In this basis, the main theorem for the matrix elements of a main vertex operator can be written in smarter way than in the basis formed by $|P_\lambda\rangle$, defined above.

**Definition 3.35.** Define

$$ N_{\lambda\mu}(u) := \prod_{(i,j) \in \lambda} \left( 1 - u q^{a_{\mu}(i,j)} f_{\mu}(i,j) + 1 \right) \prod_{(i,j) \in \mu} \left( 1 - u q^{a_{\mu}(i,j) - 1} f_{\mu}(i,j) \right). \quad (3.62) $$

$N_{\lambda\mu}(u)$ is called the Nekrasov factor.

This factor appears in the instanton partition functions in five dimensional $\mathcal{N} = 1$ supersymmetric gauge theories.

**Definition 3.36.** Set

$$ C^{(+)}_{\lambda}(u) := \xi^{(+)}_{\lambda}(u) \times \prod_{1 \leq i < j \leq N} N_{\lambda^{(i)},\lambda^{(j)}}(qu_i/tu_j) \cdot \prod_{k=1}^{N} c_{\lambda^{(k)}}, \quad (3.63) $$

$$ C^{(-)}_{\lambda}(u) := \xi^{(-)}_{\lambda}(u) \times \prod_{1 \leq i < j \leq N} N_{\lambda^{(j)},\lambda^{(i)}}(qu_j/tu_i) \cdot \prod_{k=1}^{N} c_{\lambda^{(k)}}, \quad (3.64) $$

$$ \xi^{(+)}_{\lambda}(u) := \prod_{i=1}^{N} (-1)^{(N-i+1)|\lambda^{(i)}|} u^{(N+i)|\lambda^{(i)}| + \sum_{k=1}^{i} |\lambda^{(k)}|} \cdot \prod_{i=1}^{N} (q/t)^{(1-i)|\lambda^{(i)}|} q^{(i-N)(n(\lambda^{(i)}') + |\lambda^{(i)}|)} f(N-i-1)(n(\lambda^{(i)}) + |\lambda^{(i)}|), \quad (3.65) $$

$$ \xi^{(-)}_{\lambda}(u) := \prod_{i=1}^{N} (-1)^{|\lambda^{(i)}|} u^{(-i+1)|\lambda^{(i)}| + \sum_{k=i}^{N} |\lambda^{(k)}|} \cdot \prod_{i=1}^{N} (q/t)^{(1-i)|\lambda^{(i)}|} q^{(1-i)(n(\lambda^{(i)}') + |\lambda^{(i)}|)} f(i-2)(n(\lambda^{(i)}') + |\lambda^{(i)}|), \quad (3.66) $$

where $c_{\lambda}$ is defined in (3.14) and $n(\lambda) = \sum_{j \geq 1} (j - 1)\lambda_{j}$.

**Definition 3.37.** Define $|K_{\lambda}(u)\rangle \in \mathcal{F}_u$ and $\langle K_{\lambda}(u) | \in \mathcal{F}_u^*$ by

$$ |K_{\lambda}\rangle = |K_{\lambda}(u)\rangle := C^{(+)}_{\lambda}(u) |P_\lambda(u)\rangle, \quad \langle K_{\lambda} | = \langle K_{\lambda}(u) | := C^{(-)}_{\lambda}(u) \langle P_\lambda(u) |. \quad (3.67) $$

This normalization arises from the following conjecture with respect to the expansion coefficient in the PBW-type basis $|X_\lambda\rangle$ and $\langle X_\lambda |$.

**Conjecture 3.38.**

$$ |K_{\lambda}\rangle = \sum_{\mu} \alpha^{(+)}_{\lambda\mu} |X_\mu\rangle, \quad \alpha^{(+)}_{\lambda,(1|\lambda|),\emptyset,...,\emptyset} = 1, \quad (3.68) $$

$$ \langle K_{\lambda} | = \sum_{\mu} \alpha^{(-)}_{\lambda\mu} \langle X_\mu |, \quad \alpha^{(-)}_{\lambda,(1|\lambda|),\emptyset,...,\emptyset} = 1. \quad (3.69) $$
In Appendix F.1, we give examples of the transition matrices $a^{(\pm)}_{\lambda\mu}$.

**Definition 3.39** (Taki’s flaming factors). Define

$$f_{\lambda} := (-1)^{|\lambda|} q^{n(\lambda')} + |\lambda|/2 t^{-n(\lambda)} - |\lambda|/2,$$  

(3.70)

$$g_{\lambda} := q^{n(\lambda')} t^{-n(\lambda)}.$$  

(3.71)

**Proposition 3.40.**

$$\langle K_{\lambda} | K_{\lambda} \rangle = \left( (-1)^N y^2 e_N(u) \right)^{|\lambda|} \prod_{i=1}^N \left( u_i^{\lambda(i)} y^{-2|\lambda(i)|} g_{\lambda(i)} \right)^{(2-N)}$$  

(3.72)

where we put $e_N(u) := \prod_{i=1}^N u_i$.

**Proof.** This follows from

$$c_{\lambda} c_{\lambda}' = (-1)^{|\lambda|} q^{n(\lambda')} + |\lambda| t^{-n(\lambda)} N_{\lambda,\lambda}(1),$$  

(3.73)

$$N_{\lambda,\mu}(y^{-1} x) = N_{\mu,\lambda}(y^{-1} x^{-1} ) \frac{f_{\lambda}}{f_{\mu}}.$$  

(3.74)

$\square$

### 4. Proof of Main Theorem

This section is organized as follows. In Sect. 4.1, before going into the details of the proof of the main theorem, we recall what is our main claim in this paper. In Sect. 4.2, we derive a transformation formula for the multiple basic hypergeometric series, which is one of the key ingredients in the proof in Sect. 4.3 of the main theorem.

**4.1. Statement of main theorem.** First, we define the vertex operator $V(x)$ by relations with the algebra $U(N)$. We call $V(x)$ the Mukadé operator as explained in Introduction.

**Definition 4.1.** Define the linear operator $V(x) = V(x; u) : F_u \to F_v$ by the following relations:

$$\left(1 - \frac{x}{z}\right) X^{(i)}(z) V(x) = \left(1 - (t/q) y^{-1} x \right) X^{(i)}(z), \quad i \in \{1, 2, \ldots, N\}$$  

(4.1)

and $\langle 0 | V(x) | 0 \rangle = 1$.

Then we have the following proposition.

**Proposition 4.2.** The $V(x)$ exists uniquely.

**Proof.** If the operator $V(x)$ exists, the uniqueness is clear by definition of $V(w)$ and the fact that the vectors $|X_{\lambda}\rangle$ form a basis (Fact 2.10). (See also Appendix F.2) The existence of $V(w)$ will be shown late in the Proposition 5.9. $\square$
Remark 4.3. The realization of $\mathcal{V}(x)$ given in Sect. 5 shows that $\mathcal{V}(x)$ satisfies the relation with the algebra $\mathcal{U}$ introduced in [AFHKSY1, Definition 3.12] after some simple renormalization. However, if we adopt its relation as definition of the vertex operator, the uniqueness is nontrivial.

In Appendix F.2, we demonstrate the calculation of the matrix elements of $\mathcal{V}(x)$ by using the defining relation (4.1). The matrix elements with respect to generalized Macdonald functions are factorized and reproduce the Nekrasov factor:

**Theorem 4.4.**

$$
\langle K_{\lambda}(v)|\mathcal{V}(x)|K_{\mu}(u)\rangle = \left((-\gamma^2)^N e_{N}(u)x\right)^{|\lambda|} \prod_{i=1}^{N} \left(\nu_{i}^{\lambda(i)} g_{\mu(i)}\right)^{N-1} \cdot \prod_{i,j=1}^{N} N_{\lambda(i),\mu(j)}(qv_{i}/tu_{j}).
$$

(4.2)

As a direct result of this theorem, we obtain an algebraic description of the 5D (K-theoretic) AGT correspondence [AGT,AY1,AY2]. Theorem 4.4 makes us possible to calculate the multi-point functions. In particular, the two-point function, which can be regarded as $q$-analogue of the four-point conformal block in 2D CFT [1], corresponds to the Nekrasov partition function of the 5-dimensional $\mathcal{N} = 1 U(N)$ gauge theory with $2N$ fundamental matters [AK1,AK2]. (See also Remark 3.14, 3.15 in [AFHKSY1] for more detail.):

$$
\langle 0|\mathcal{V}(w; z_{1})\mathcal{V}(v; z_{2})|0\rangle = \sum_{\lambda} \langle 0|\mathcal{V}(w; z_{1})|K_{\lambda}\rangle \langle K_{\lambda}|\mathcal{V}(v; z_{2})|0\rangle \langle K_{\lambda}|K_{\lambda}\rangle = \sum_{\lambda} \left(e_{N}(u)z_{2}\right)^{|\lambda|} \prod_{i,j=1}^{N} \frac{N_{\lambda(i),\mu(j)}(qv_{i}/tu_{j})N_{\lambda(j),\mu(i)}(qv_{j}/tu_{i})}{N_{\lambda(i),\mu(j)}(qv_{i}/tu_{j})}.
$$

(4.3)

This way of description originates in the work in the 4D case by Alba, Fateev, Litvinov and Tarnopolsky [AFLT]. Moreover, the analyses in the 4D case using $SH^c$ algebra and geometric representation are given in [KMZ] and [N1], respectively. Negut gives also the geometric understanding of 5D (K-theoretic) version [N2]. In what follows in this section, we prove Theorem 4.4 by some properties of Macdonald functions and a certain multiple basic hypergeometric series.

4.2. Transformation formula. First of all, we show a certain transformation formula (Proposition 4.8) for multiple series with respect to Macdonald functions $p_{n}(x; s|q, t)$, which plays an essential role in our proof of the main theorem. That formula is based on the Euler transformation of Kajihara and Noumi’s multiple basic hypergeometric series, which is defined as follows.

---

1 As $X_{\nu}^{(1)}|0\rangle = (u_{1} + \cdots + u_{N})|0\rangle$, the vectors $|0\rangle$ and $|0\rangle$ are not the ordinary vacuum states for the algebra $\mathcal{U}(\bar{N})$ but highest weight vectors of highest weight $u$. Hence, the function $\langle 0|\mathcal{V}(x_{1})\cdots\mathcal{V}(x_{n})|0\rangle$ can be regarded as $q$-analogue of the $(n + 2)$-point conformal block.
**Definition 4.5.** Define

$$
\phi_{m,n}(a_1, \ldots, a_m \mid b_1, \ldots, b_n, u) = \sum_{\mu \in \mathbb{Z}_m^m} u \sum_{i=1}^m \mu_i \phi_{m,n}(a_1, \ldots, a_m \mid b_1 y_1, \ldots, b_n y_n)
$$

with

$$
\phi_{m,n}(a_1, \ldots, a_m \mid b_1, \ldots, b_n) = \prod_{i<j} q^{\mu_i x_i - \mu_j x_j} \prod_{i,j} (a_j x_i / x_j ; q)_{\mu_i} \prod_{i,k} (b_k x_i ; q)_{\mu_i}.
$$

Let us also mention that the elliptic analogue is studied in [KN]. Kajihara and Noumi gave the following $q$-Euler transformation formula.

**Fact 4.6 ([K, KN]).**

$$
\phi_{m,n}(a_1, \ldots, a_m \mid b_1 y_1, \ldots, b_n y_n ; u) = (a_1 \cdots a_m b_1 \cdots b_n u / e^n ; q)_{\infty}
$$

$$
\times \phi_{n,m}(c/b_1, \ldots, c/b_n \mid x y_1, \ldots, x y_n \mid c x_1, \ldots, c x_m ; a_1 \cdots a_m b_1 \cdots b_n u / e^n).
$$

We prepare the following notation.

**Definition 4.7.** For nonnegative integers $n, m$ and $\mu = (\mu_i)_{1 \leq i \leq m} \in \mathbb{Z}^m$, we introduce

$$
N_{\mu}^{n,m}(s_1, \ldots, s_{n+m}) := \prod_{k=1}^{n+m} \left( \prod_{i=1}^{n+m} (q s_{n+k} / t s_i ; q)_{\mu_i} \right) \prod_{1 \leq i < j \leq m} (t q^{-\mu_i} s_{n+j} / s_{n+i} ; q)_{\mu_j}.
$$

By using Fact 4.6, we can show the following formula which transforms the Macdonald function $p_{n+m}(x ; s | q, t)$ to another series that contains $p_m$ as inner summation.

**Proposition 4.8.** Let $s_i$ $(i = 1, \ldots, n + m)$ be generic complex parameters and $|t| > |q|^{-(n-2)}$. We put $x_i = x t^{n-i}$ for $i = 1, \ldots, n$ and $x_{n+k} = y_k$ for $k = 1, \ldots, m$. Then

$$
\prod_{k=1}^{n+m} \frac{(q t^{-1} x ; q)_{\infty}}{(t q/x ; q)_{\infty}} \cdot p_{n+m}(x_1, \ldots, x_{n+m} ; s_1, \ldots, s_{n+m} | q, t)
$$

$$
= \prod_{i=1}^{n} \frac{(q/t ; q)_{\infty}}{(q/t^i ; q)_{\infty}} \cdot \prod_{1 \leq i < j \leq n} \frac{(q s_j / t s_i ; q)_{\infty}}{(q s_j / s_i ; q)_{\infty}}
$$

$$
\times \sum_{\mu \in \mathbb{Z}_m^m} N_{\mu}^{n,m}(s_1, \ldots, s_{n+m}) p_m(y_1, \ldots, y_m ; q^{\mu_1} s_{n+1}, \ldots, q^{\mu_n} s_{n+m} | q, t) \prod_{k=1}^{n+m} (t y_k / x)^{\mu_k}.
$$

**Remark 4.9.** Throughout this subsection, $s_i$ are treated as generic parameters with $s_i \neq 0$ and $s_i \neq q^{r_1} t^{r_2} s_j$ $(r_1 \in \mathbb{Z}, r_2 = 0, \pm 1, \forall i, j)$. The above transformation formula holds even if $s_i$ are not specialized as (3.59) and (3.60).

In particular, if $m = 0$, this proposition is nothing but the following specialization formula.
Fact 4.10 ([NS]). Let $|t| > |q|^{-(n-2)}$. Then

$$p_n(x; s|q,t)|_{x_j \rightarrow t^{-n-i}} = \prod_{i=1}^{n} \frac{(q/t; q)_\infty}{(q; q)_\infty} \cdot \prod_{1 \leq i < t \leq n} \frac{(qs_j/t s_i; q)_\infty}{(qs_j/s_i; q)_\infty}. \quad (4.9)$$

For the proof of Proposition 4.8, we prepare two lemmas. They are tools for attributing our transformation formula (4.8) to the one of the multiple basic hypergeometric series. The proofs of these lemmas are by direct calculation of factorials (See Appendix D.4 and D.5).

Lemma 4.11. Let $\sigma = (\sigma_k)_{1 \leq k \leq m-1} \in \mathbb{Z}^{m-1}$ and $\theta = (\theta_i)_{1 \leq i \leq n+m-1} \in \mathbb{Z}^{n+m-1}$. Then under the transformation $\rho_k = \sigma_k - \theta_{n+k}$, we have

$$\prod_{1 \leq i < j \leq n} \frac{(qq^{-\theta_j} s_j / t q^{-\theta_i} s_i; q)_\infty}{(qs_j / ts_i; q)_\infty} \cdot \lim_{h \rightarrow 1} \tilde{N}^{n,m}_{\rho} (h; s_1, \ldots, s_{n+m-1})$$

$$= \prod_{1 \leq i < j \leq n} \frac{(qs_j / ts_i; q)_\infty}{(qs_j / s_i; q)_\infty} \cdot \prod_{i=1}^{n} q^{\theta_i/t - \theta_i} \sum_{m=1}^{n-1} q^{\theta_i / t - \theta_i}.$$ \quad (4.10)

Here, $h$ is a parameter which goes to 1 in the limit, satisfying $q^m t^2 h \neq 1 (\forall r_1, r_2 \in \mathbb{Z})$. $\tilde{N}^{n,m}_{\mu}$ is defined by

$$\tilde{N}^{n,m}_{\mu} (h; s_1, \ldots, s_{n+m}) := \prod_{k=1}^{m} \left( \prod_{j=1}^{n+k} \frac{(qs_{n+k} / ts_i; q)_{\mu_k}}{(hq s_{n+k} / s_i; q)_{\mu_k}} \right) \cdot \prod_{1 \leq i < j \leq m} \frac{(t q^{-\mu_i s_{n+j} / s_{n+i}}; q)_{\mu_j}}{(q^{-\mu_i s_{n+j} / s_{n+i}}; q)_{\mu_j}}.$$ \quad (4.11)

Remark 4.12. If $\rho_k < 0$ and $\rho_k + \theta_{n+k} > 0$ for some $k$, then $\tilde{N}^{n,m}_{\rho} = 0$ and $\phi^{n,m-1}_{\rho}$ diverges in the limit $h \rightarrow 1$. However, (4.10) converges as a result. In order to avoid the problem with respect to this divergence, we inserted the parameter $h$.

Lemma 4.13. Let $\rho = (\rho_k)_{1 \leq k \leq m-1} \in \mathbb{Z}^{m-1}$ and $\nu = (\nu_k)_{1 \leq k \leq m} \in \mathbb{Z}^{m}$. Then

$$\lim_{h \rightarrow 1} \tilde{N}^{n,m}_{\rho} (h; s_1, \ldots, s_{n+m-1})$$

$$\times \phi^{n,m}_{\nu} \left( \begin{array}{c} t \\ \rho_1 s_{n+1} \ldots, q / t \\ \rho_{n+m-1} s_{n+m-1}, h / ts_{n+1}, \ldots, h / ts_{n+m-1} \end{array} \right)$$

$$= N^{n,m}_{\mu} (s_1, \ldots, s_{n+m}) \cdot d_m ((\theta_1); q, t)$$ \quad (4.12)

under the transformation

$$\rho_k = \mu_k - \theta_k \quad (k = 1, \ldots, m-1),$$ \quad (4.13)

$$\nu_k = \theta_k \quad (k = 1, \ldots, m-1),$$ \quad (4.14)

$$\nu_m = \mu_m.$$ \quad (4.15)
Now, we prove Proposition 4.8 by using these lemmas and the \( q \)-Euler transformation formula.

**Proof of Proposition 4.8.** The proof is done by induction on \( m \). If \( m = 0 \), (4.8) follows from Fact 4.10. By assumption that it holds for \( m - 1 \), it can be shown that the left hand side of (4.8) is

\[
\sum_{\theta \in \Z_{>0}^{n+m-1}} \frac{(q y_m / t^n x; q)_\infty}{(t y_m / x; q)_\infty} d_{n+m}^{\theta}(\theta, s | q, t) \prod_{i=1}^{n+m-1} (x_{n+m}/x_i)^{\theta_i} 
\]

\[
\times \prod_{k=1}^{m-1} \frac{(q y_k / t^n x; q)_\infty}{(t y_k / x; q)_\infty} \cdot p_{n+m-1}(x_1, \ldots, x_{n+m-1}; q^{-\theta_1} s_1, \ldots, q^{-\theta_{n+m-1}} s_{n+m-1} | q, t) 
\]

\[
= \sum_{\theta \in \Z_{>0}^{n+m-1}} \prod_{i=1}^{n} \frac{(q / t; q)_\infty}{(q / t^i; q)_\infty} \cdot \frac{(q y_m / t^n x; q)_\infty}{(t y_m / x; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(q q^{-\theta_i} s_j / t q^{-\theta_i} s_i; q)_\infty}{(q q^{-\theta_j} s_i / q^{-\theta_i} s_i; q)_\infty} 
\]

\[
\times d_{n+m}^{\theta}(\theta, s | q, t) N_{\sigma}^{n,m-1}(q^{-\theta_1} s_1, \ldots, q^{-\theta_{n+m}} s_{n+m}) p_{m-1}((y_i); (q \sigma_i q^{-\theta_i} s_{n+i}) | q, t) 
\]

\[
\times \prod_{k=1}^{m-1} (t y_k / x)^{\sigma_k}. \tag{4.16}
\]

By the contribution of the factor \( \prod_{k=1}^{m-1} 1/(q; q)_{\sigma_k} \) in \( N_{\sigma}^{n,m-1} \), we can extend the range which \( \theta \) and \( \sigma \) ran over to

\[
\theta \in \Z_{>0}^{n+m-1}, \quad \sigma \in \Z^{m-1}. \tag{4.17}
\]

Under this range, by applying Lemma 4.11, we can rewrite (4.16) as

\[
\lim_{h \to 1} \sum_{\theta \in \Z_{>0}^{n+m-1}} \prod_{i=1}^{n} \frac{(q / t; q)_\infty}{(q / t^i; q)_\infty} \cdot \frac{(q y_m / t^n x; q)_\infty}{(t y_m / x; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(q q^{-\theta_i} s_j / t q^{-\theta_i} s_i; q)_\infty}{(q q^{-\theta_j} s_i / q^{-\theta_i} s_i; q)_\infty} 
\]

\[
\times N_{\rho}^{n,m-1}(h; s_1, \ldots, s_{n+m-1}) p_{m-1}((y; (q^\rho_i s_{n+i}) | q, t) \prod_{k=1}^{m-1} (t y_k / x)^{\rho_k} 
\]

\[
\times q_{\theta}^{n+m-1,m} \left( \frac{t}{h s_1}, \ldots, \frac{t}{h s_{n+m-1}} \big | q q^{\rho_1} s_{n+1} / t, \ldots, q q^{\rho_{m-1}} s_{n+m-1} / t, ts_{n+m} \right) 
\]

\[
\times \prod_{i=1}^{n} (q y_m / t^n x)^{\theta_i}. \tag{4.18}
\]

By using the Euler transformation formula for the multiple hypergeometric series (Fact 4.6), (4.18) can be written as

\[
\lim_{h \to 1} \sum_{\varepsilon \in \Z_{>0}^{m-1}} \prod_{i=1}^{n} \frac{(q / t; q)_\infty}{(q / t^i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(q s_j / s_i; q)_\infty}{(q s_j / s_i; q)_\infty} 
\]

\[
\times \prod_{i=1}^{n} (q y_m / t^n x)^{\theta_i}.
\]
4.3. Proof of Theorem 4.4. We are now in a position to show Theorem 4.4. This subsection is devoted to its proof.

Since the defining relation of $\mathcal{V}(w)$ and the expansion coefficients of $|K_\lambda\rangle$ in the basis of $|X_\lambda\rangle$ are written by rational functions of $q, t, u_i$ and $v_i$, the matrix elements $\langle K_\lambda | \mathcal{V}(z) | K_\mu \rangle$ determine the unique rational function of them by Proposition 4.2. Therefore, it suffices to prove (4.2) in the case $v_i = t^m u_i$ ($i = 1, \ldots, N$) for all sufficiently large $n_i \in \mathbb{Z}$, i.e., $n_i \geq \ell(\lambda^{(i)})$ by analytic continuation.

**Step 1: realization of $\mathcal{V}(x)$** Firstly, we give a realization of $\mathcal{V}(x)$ in the case $v = t^n \cdot u$ as follows.

**Definition 4.14.** Let $|t| > |q|^{-(n-2)}$. Define $\tilde{\mathcal{V}}(n)(x) = \tilde{\mathcal{V}}(n)(v; x) : \mathcal{F}_u \rightarrow \mathcal{F}_v$ with $u = t^{-n} \cdot v$ by

$$\tilde{\mathcal{V}}(n)(x) = \lim_{x \rightarrow t^{-n} \cdot x} \prod_{1 \leq i < j \leq |n|} \left( \frac{(t x_j / x_i; q)_{\infty}}{(q x_j / t x_i; q)_{\infty}} \right) \cdot V(n)(x_1, \ldots, x_{|n|}) A_{(|n|)}^{-1}(x),$$

where

$$A_{(r)}(x) = \exp \left( \sum_{n > 0} \frac{(1 - (q / t)^r)(1 - t^{(1-r)n}) t^{2r}}{n(1 - q^n)(1 - t^{-n})} \sum_{i = 1}^{N} \gamma^{(i-1)n} a_n^{(i)} x^{-n} \right).$$

**Remark 4.15.** Note that $\sum_{i = 1}^{N} a_n^{(i)}$ is the boson corresponding to the Cartan part $\Delta^{(N)}(\psi^+(z))$.

**Proposition 4.16.** $\tilde{\mathcal{V}}(n)(x)$ is well-defined on $\mathbb{C}^*$, i.e., its arbitrary matrix elements are holomorphic functions there.
Before the proof, we prepare the following fact. This fact tells us the duality of the Macdonald functions under exchanging \( t \) and \( q/t \).

**Fact 4.17** ([NS]). The formal series \( p_n(x; s|q, t) \) with the leading coefficient 1 satisfies the symmetry relation

\[
p_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_\infty}{(q x_j/t x_i; q)_\infty} \cdot p_n(x; s|q/t, t).
\]

(4.23)

**Proof of Proposition 4.16.** We introduce the following product of currents,

\[
\tilde{\mathcal{A}}(r)(x_1, \ldots, x_r) := \prod_{k=1}^r \exp \left( \sum_{n>0} \frac{(1 - (q/t)^r) t^{2r}}{n(1 - q^n)} \sum_{i=1}^N y^{(i-1)n} a_n^{(i)} x_k^{-n} \right).
\]

(4.24)

We have the following equality.

\[
\prod_{1 \leq i < j \leq |n|} \frac{(tx_j/x_i; q)_\infty}{(q x_j/t x_i; q)_\infty} \cdot \prod_{1 \leq i < j \leq |m|} \frac{(ty_j/y_i; q)_\infty}{(q y_j/t y_i; q)_\infty} \\
\times x^{-\lambda} \langle P_{\lambda} | V^{(n)}(x_1, \ldots, x_{|n|}) \tilde{\mathcal{A}}_{\{n\}}^{-1}(x_1, \ldots, x_{|n|}) V^{(m)}(\nu; y_1, \ldots, y_{|m|}) | 0 \rangle \\
= \prod_{1 \leq i < j \leq |n|+|m|} \frac{(tx_j/x_i; q)_\infty}{(q x_j/t x_i; q)_\infty} \cdot \mathcal{R}_{\lambda}(v) p_{|n|+|m|}((x, y); s|q, t) \\
= \mathcal{R}_{\lambda}(v) p_{|n|+|m|}((x, y); s|q, t).
\]

(4.25)

Here, for brevity of notation, we set \( x_{|n|+i} = y_i \), and used Fact 4.17 and Lemma 3.34.

Then, by the same argument as the proof of Theorem 3.33, we can show the matrix elements of

\[
\prod_{1 \leq i < j \leq |n|} \frac{(tx_j/x_i; q)_\infty}{(q x_j/t x_i; q)_\infty} \cdot V^{(n)}(x_1, \ldots, x_{|n|}) \tilde{\mathcal{A}}_{\{n\}}^{-1}(x_1, \ldots, x_{|n|})
\]

are the holomorphic functions on \( \pi_{|n|}^{-1}(U^{|n|}_z(\tilde{r}_0)) \) with \( \tilde{r}_0 = |q/t|^{-|n|-2} \). Note that in this case, \( y^{-\mu} f_{|m|}(y; s|q, t) \) is multiplied before the integration in \( y \). Under the assumption \( \left| t \right| > |q|^{-(n-2)}, |t^{-1}| < \tilde{r}_0 \), and thus we can safely take the limit \( x_i \to t^{-|n|-i} x \). This limit ends with the operator \( \tilde{V}^{(n)} \), and this completes the proof. \( \square \)

This operator \( \tilde{V}^{(n)}(x) \) is a realization of \( V \) in the case \( v = t^n \cdot u \). This follows from the following relation which are the essentially same as (4.1).

**Proposition 4.18.** For \( r = 1, \ldots, N \), the \( \tilde{V}^{(n)}(\nu; u; x) \) satisfies

\[
(1 - t^{-|n|} \frac{x}{z}) X^{(r)}(z) \tilde{V}^{(n)}(\nu; u; x) = (q/t)^r \left( 1 - (t/q)^r t^{-|n|} \frac{x}{z} \right) \tilde{V}^{(n)}(\nu; u; x) X^{(r)}(z).
\]

(4.26)
Proof. By Lemma 3.19, we can get

\[
X^{(r)}(z) V^{(n)}(\frac{v}{u}; x_1, \ldots, x_{|n|}) = \prod_{k=1}^{|n|} \frac{1 - (q/t)^r z / tx_k}{1 - z / tx_k} \cdot V^{(n)}(x_1, \ldots, x_{|n|}) X^{(r)}(z)
\]

\[
+ (1 - t^{-1}) \sum_{i=1}^N \sum_{k=1}^{n_i} v_i t^{1-k} \delta(t x_{[i,k]n} / z) \prod_{1 \leq \ell < [i,k]n} \frac{1 - (q/t)^r x_{[i,k]n} / x_\ell}{1 - x_{[i,k]n} / x_\ell}
\]

\[
\times \cdot \prod_{|i,k|n < \ell \leq |n|} \frac{1 - tx_\ell / qx_{[i,k]n}}{1 - x_\ell / qx_{[i,k]n}} \tilde{U}^{(n)}_{[i,k]n} (x_1, \ldots, x_{|n|}) \Psi^+ (x_{[i,k]n}),
\]

(4.27)

where \( \tilde{U}^{(n)}_{[i,k]n} \) is the operator obtained by replacing \( \Phi^{(i)}(x_{[i,k]n}) \) in \( V^{(n)} \) with \( Y^{(r)}(x_{[i,k]n}) \)

\[
\Phi^{(i)}(q x_{[i,k]n}), \text{ i.e.,}
\]

\[
\tilde{U}^{(n)}_{[i,k]n} (x_1, \ldots, x_{|n|}) := \Phi^{(0)}(x_1) \cdots \Phi^{(i)}(x_{[i,k]n} - 1) Y^{(r)}(x_{[i,k]n}) \Phi^{(i)}(q x_{[i,k]n})
\]

\[
\Phi^{(i)}(x_{[i,k]n+1}) \cdots \Phi^{(N-1)}(x_{|n|}).
\]

(4.28)

Because of the \( q \)-difference of the operator \( \Phi^{(i)}(x_{[i,k]n}) \) in \( \tilde{U}^{(n)}_{[i,k]n} \), if \( [i,k]n \neq 1 \), the product of \( \tilde{U}^{(n)}_{[i,k]n} \) and \( \prod (tx_{j,i}; q)_\infty / (qx_J / tx_i; q)_\infty \) vanishes under the principal specialization, i.e.,

\[
\lim_{x_i \to t^{n-i} x} \prod_{1 \leq j \leq |n|} \frac{(tx_{j,i}; q)_\infty}{(qx_J / tx_i; q)_\infty} \tilde{U}^{(n)}_{[i,k]n} (x_1, \ldots, x_{|n|}) = 0 \quad ([i,k]n \neq 1).
\]

(4.29)

Furthermore, by the operator product (C.22), we have

\[
A_{(s)}(x) X^{(r)}(z) = \prod_{k=1}^{r-1} \frac{1 - t^{-k} z / x}{1 - t^{-k} (q/t)^r z / x} \cdot X^{(r)}(z) A_{(s)}(x).
\]

(4.30)

By (4.27), (4.29) and (4.30), it can be shown that

\[
X^{(r)}(z) \tilde{V}^{(n)}(x) = \frac{1 - (q/t)^r z / t^{n|x}_x}{1 - z / t^{n|x}_x} \tilde{V}^{(n)}(x) X^{(r)}(z)
\]

(4.31)

\[
+ (1 - t^{-1}) v_1 t^{1-k} \lim_{x_i \to t^{n-i} x} \delta(t x_i / z) \prod_{1 < \ell \leq |n|} \frac{1 - tx_\ell / qx_\ell}{1 - x_\ell / x_\ell}
\]

\[
\times \prod_{1 \leq i < j \leq |n|} \frac{(tx_{j,i}; q)_\infty}{(qx_J / tx_i; q)_\infty} \cdot \tilde{U}^{(n)}_1 \Psi^+ (x_1).
\]

(4.32)

By multiplying the both hand sides by \( (1 - t^{-|n|} z / x) \), we have

\[
\left( 1 - t^{-|n|} \frac{z}{x} \right) X^{(r)}(z) \tilde{V}^{(n)}(\frac{v}{u}; x) = \left( 1 - (q/t)^r t^{-|n|} \frac{z}{x} \right) \tilde{V}^{(n)}(\frac{v}{u}; x) X^{(r)}(z).
\]

(4.33)

After some simple calculation, we obtain Proposition 4.18. \( \Box \)
Step 2: evaluation of the matrix elements of $\widetilde{V}^{(n)}(x)$ Next, we evaluate the matrix elements of $\widetilde{V}^{(n)}(x)$. Let $s' := (s'_i)_{1 \leq i \leq |m|}$ with
\begin{equation}
\label{eq:434}
s'_{i[k]} := q^{\mu_k} t^{1-n_i-k} v_i \quad (1 \leq k \leq m_i, \ i = 1, \ldots, N)
\end{equation}
and $s = (s_i)_{1 \leq i \leq |n|+|m|}$ be the same one given in (3.59), (3.60), i.e.,
\begin{equation}
\label{eq:435}
s_{i[k]} = q^{\mu_k} t^{1-k} v_i \quad (1 \leq k \leq n_i, \ i = 1, \ldots, N),
\end{equation}
\begin{equation}
\label{eq:436}
s_{i[k]} = t^{1-n_i-k} v_i \quad (1 \leq k \leq m_i, \ i = 1, \ldots, N).
\end{equation}
Further, we write $x_{|n|+i} = y_i$. By the explicit formula for $|Q_\lambda\rangle$ (Theorem 4.4) and Lemma 3.34, we have
\begin{equation}
\label{eq:437}
\langle P_\lambda \mid \widetilde{V}^{(n)}(x) \mid Q_\mu \rangle = \frac{1}{\mathcal{R}_\mu^R(u)} \left[ y^{-\mu} f_m(y; s', q, q/t) \left( P_\lambda \mid \widetilde{V}^{(m)}(x) V^{(m)}(y_1, \ldots, y_m) \mid 0 \right) \right]_{y_1, 1} = \frac{\mathcal{R}_\lambda^R(v)}{\mathcal{R}_\mu^R(u)} \prod_{i=1}^{\frac{|n|}{1 < j \leq |n|}} \frac{(q s_j / ts_i ; q)_{\infty}}{(q s_j / s_i ; q)_{\infty}} \prod_{1 \leq i \leq m} \frac{(q y_j / ty_i ; q)_{\infty}}{(t y_i / y_j ; q)_{\infty}} \times \prod_{k=1}^{\frac{|m|}{1 \leq k \leq |m|}} \frac{(q y_k / t^{\infty}x ; q)_{\infty}}{(t x / q)_{\infty}} \cdot p_{|n|+|m|}(x)_{1 \leq i \leq |n|+|m|} \cdot s_{i} \cdot t_{i} \right]_{y_1, 1}.
\end{equation}
By virtue of the transformation formula of Proposition 4.8, the Macdonald function $p_{|n|+|m|}$ in (4.37) can be transformed to the summation of $p_{|m|}$ as
\begin{equation}
\label{eq:438}
\frac{\mathcal{R}_\lambda^R(v)}{\mathcal{R}_\mu^R(u)} \prod_{i=1}^{\frac{|n|}{1 < j \leq |n|}} \frac{(q / t_i ; q)_{\infty}}{(q / t_j ; q)_{\infty}} \prod_{1 \leq i \leq m} \frac{(q s_j / ts_i ; q)_{\infty}}{(q s_j / s_i ; q)_{\infty}} \cdot \sum_{v \in Z^{\frac{|m|}{1 \leq i \leq |m|}}} N_{v, |m|} (s_1, \ldots, s_{|n|+|m|}) p_{|m|}(y) \left( q^{v_i} s_{|n|+i} \right) \prod_{k=1}^{\frac{|m|}{1 \leq k \leq |m|}} \frac{(q y_k / t^{\infty}x)_{\infty}}{(t x / q)_{\infty}}
\end{equation}
Here, we used Fact 4.17. Note that since $s_{|n|+i,k} = t^{1-n_i-k} v_i$, if $\nu$ does not satisfy $v_i \geq v_{i,2} \geq \cdots \geq v_{i,m_i}$, i.e., cannot be regarded as an $N$-tuple of partitions, then $N_{\nu, |m|} = 0$. By Lemma C.2 and
\begin{equation}
\label{eq:439}
x^\lambda \bigg|_{x \rightarrow t^{\infty} x} = x^{\lambda} |t^{(n-1)|\lambda|} | t^{\sum_{k=1}^{\frac{|m|}{1 \leq i \leq |m|}} n_k}
\end{equation}
we obtain
\[
(P_\lambda | \tilde{V}^{(n)}(x)| Q_\mu) = x^{\lambda_k - |\mu_j|} \prod_{i=1}^{N} t^{-n(\lambda^{(i)}) - |\lambda^{(i)}|} \sum_{k=1}^{n} n_k
\]
\[
\times \frac{\mathcal{R}_\lambda^n(v)}{\mathcal{R}_\mu^n(u)} \prod_{i=1}^{n} \frac{(q/t; q)_\infty}{(q/t_i; q)_\infty} \prod_{1 \leq i < j \leq |n|} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty} \cdot N_{[\mu]|m} (s_1, \ldots, s|n|+|m|).
\]

Here, we put
\[
[\mu]^m = ([\mu]_i^m)_{1 \leq i \leq |m|} := (\mu_1^{(1)}, \ldots, \mu_{m_1}^{(1)}, \mu_1^{(2)}, \ldots, \mu_{m_2}^{(2)}, \ldots, \mu_1^{(N)}, \ldots, \mu_{m_N}^{(N)}).
\]

Moreover, (4.39) leads to
\[
\frac{(K_\lambda | \tilde{V}^{(n)}(\chi)| K_\mu)}{(0 | \tilde{V}^{(n)}(\chi)| 0)} = C_\lambda^{(-)} C_\mu^{(+)} \prod_{i=1}^{N} \prod_{1 \leq i \leq |n_k|} \frac{C_{\lambda^{(i)}}'}{C_{\mu^{(i)}}'} \cdot x^{\lambda_k - |\mu_j|} \prod_{i=1}^{N} t^{-n(\lambda^{(i)}) - |\lambda^{(i)}|} \sum_{k=1}^{n} n_k
\]
\[
\times \frac{\mathcal{R}_\lambda^n(v)}{\mathcal{R}_\mu^n(u)} \prod_{1 \leq i < j \leq |n|} \frac{(qs_j/ts_i; q)|\lambda^{(i)}| - |\lambda^{(j)}|}{(qs_j/s_i; q)|\lambda^{(i)}| - |\lambda^{(j)}|} \cdot N_{[\mu]|m} (s_1, \ldots, s|n|+|m|).
\]

Note that
\[
\prod_{1 \leq i < j \leq |n|} \frac{(qs_j/ts_i; q)|\lambda^{(i)}| - |\lambda^{(j)}|}{(qs_j/s_i; q)|\lambda^{(i)}| - |\lambda^{(j)}|} = \prod_{k=1}^{N} \prod_{1 \leq i < j \leq |n_k|} \frac{(q^{(k)}_{\lambda^{(i)} - \lambda^{(j)} + 1} - j + i; q)_{-\lambda^{(i)} + \lambda^{(j)}}}{(q^{(k)}_{\lambda^{(i)} - \lambda^{(j)} + 1} - j + i; q)_{-\lambda^{(i)} + \lambda^{(j)}}}
\]
\[
\times \prod_{1 \leq k < l \leq N} \prod_{1 \leq i \leq |n_k|} \frac{(q^{(k)}_{\lambda^{(i)} - \lambda^{(j)} + 1} - j + i; q)_{-\lambda^{(i)} + \lambda^{(j)}}}{(q^{(k)}_{\lambda^{(i)} - \lambda^{(j)} + 1} - j + i; q)_{-\lambda^{(i)} + \lambda^{(j)}}}.
\]

Finally, it can be shown that the expression (4.42) coincides with the Nekrasov factors
\[
\frac{(K_\lambda | \tilde{V}^{(n)}(\chi)| K_\mu)}{(0 | \tilde{V}^{(n)}(\chi)| 0)} = (-1)^N e_N(v)x^{\lambda_k} \left( (t/q)t^{|n|} \right)^{-|\mu_j|} \prod_{i=1}^{N} v_i^{-N(1)-|\lambda^{(i)}|} ((q/t)u_i)^{|\mu^{(i)}|} g_{\lambda^{(i)}} g_{\mu^{(i)}}
\]
\[
\times \prod_{i=1}^{N} (q/t)^{N(1)-|\mu^{(i)}|-\sum_{k=1}^{n} |\mu^{(k)}|} N_{\lambda^{(i)}, \mu^{(j)}} (t^{n_j} v_i/v_j).
\]

The coincidence between (4.42) and (4.44) can be proved by induction on \(\lambda\) and \(\mu\). We give some formulas to show this coincidence in Appendix C.4. The difference between (4.1) and (4.26) can be modified by the transformation
\[
x \rightarrow t^{|n|} x, \quad u_i \rightarrow (q/t) u_i \quad (i = 1, \ldots, N).
\]
Noting that the renormalization constant (3.63) are also modified, we can see that the equation (4.44) shows Theorem 4.4 in the case \( v = t^u \cdot u \). Therefore, by analytic continuation, Theorem 4.4 holds for the general case.

5. Refined Topological Vertex and Changing Preferred Direction

In [AFS], the intertwiners among the \( \mathcal{U} \)-modules are introduced, and their matrix elements are identical to the refined topological vertex of [IKV]. In this section, we compute the matrix elements of the ladder diagrams, which are obtained by gluing the intertwiners. The result shows the invariance under changing the preferred directions of the diagrams, which is a natural consequence of the S-duality in the string theory.

5.1. Trivalent intertwiner and refined topological vertex. We introduced the \( \mathcal{F}^{(1,M)} \)-module in Sec 2.2. In order to introduce the intertwiners, we need the other one, referred to as the \( \mathcal{F}^{(0,1)}_v \).

**Fact 5.1** ([FT,FFJMM]). Let \( u \) be an indeterminate. We can endow a \( \mathcal{U} \)-module structure to \( \mathcal{F} \) by setting

\[
\begin{align*}
    c^{1/2} P_\lambda &= P_\lambda, \\
    x^+(z) P_\lambda &= \sum_{i=1}^{\ell(\lambda)+1} A^+_{\lambda,i} \delta(q^{\lambda_i} t^{-i+1} u/z) P_{\lambda+i}, \\
    x^-(z) P_\lambda &= q^{1/2} t^{-1/2} \sum_{i=1}^{\ell(\lambda)} A^-_{\lambda,i} \delta(q^{\lambda_i-1} t^{-i+1} u/z) P_{\lambda-1}, \\
    \psi^+(z) P_\lambda &= q^{1/2} t^{-1/2} B^+_{\lambda}(u/z) P_\lambda, \\
    \psi^-(z) P_\lambda &= q^{-1/2} t^{1/2} B^-_{\lambda}(z/u) P_\lambda,
\end{align*}
\]

with

\[
\begin{align*}
    A^+_{\lambda,i} &= (1-t) \prod_{j=1}^{i-1} \left(1 - q^{\lambda_i - \lambda_j} t^{-i+j+1}(1 - q^{\lambda_i - \lambda_j+1} t^{-i+j-1}) \right), \\
    A^-_{\lambda,i} &= (1-t^{-1}) \prod_{j=i+1}^{\ell(\lambda)} \left(1 - q^{\lambda_i - \lambda_j} t^{-j+i+1}(1 - q^{\lambda_i - \lambda_j+1} t^{-j+i-1}) \right), \\
    B^+_{\lambda}(z) &= \frac{1 - q^{\lambda_1-1} t z}{1 - q^{\lambda_1 z}} \prod_{i=1}^{\infty} \left(1 - q^{\lambda_i} t^{-i} z\right)(1 - q^{\lambda_i+1} t^{-i+1} z), \\
    B^-_{\lambda}(z) &= \frac{1 - q^{\lambda_1+1} t^{-1} z}{1 - q^{-\lambda_1 z}} \prod_{i=1}^{\infty} \left(1 - q^{-\lambda_i} t z\right)(1 - q^{-\lambda_i+1} t^{-1} z).
\end{align*}
\]

We denote this module as \( \mathcal{F}^{(0,1)}_v \)-module.

Then, we introduce the intertwiners among the modules defined above.
Fact 5.2 ([AFS]). When \( w = -uv \), there exists a unique intertwiner which satisfies
\[
\Phi \begin{bmatrix} (1, M + 1), -uv \\ (0, 1), v; (1, M), u \end{bmatrix} : \mathcal{F}_v^{(0,1)} \otimes \mathcal{F}_u^{(1,M)} \rightarrow \mathcal{F}_w^{(1,M+1)};
\]
\[
a \Phi = \Phi \Delta(a) \quad (\forall a \in \mathcal{U}) \tag{5.10}
\]
and the normalization condition \( \langle 0 \Phi(1 \otimes |0\rangle) = 1 \). Moreover, its component \( \Phi_\lambda \), defined by
\[
\Phi_\lambda(\alpha) = \Phi(P_\lambda \otimes \alpha) \quad (\forall P_\lambda \otimes \alpha \in \mathcal{F}_v^{(0,1)} \otimes \mathcal{F}_u^{(1,M)}), \tag{5.11}
\]
has the following realization,
\[
\Phi_\lambda \begin{bmatrix} (1, M + 1), -uv \\ (0, 1), v; (1, M), u \end{bmatrix} = \hat{\iota}(\lambda, u, v, M) \hat{\Phi}_\lambda(v), \tag{5.12}
\]
where
\[
\hat{\iota}(\lambda, u, v, M) = (-uv)^{\frac{\text{dim}}{2}}(-u)^{-(M+1)|\lambda|} f_\lambda^{M-1} q^{n(\lambda')}/c_\lambda,
\]
\[
\hat{\Phi}_\lambda(v) =: \Phi_\theta(v) \eta_\lambda(v) :,
\]
\[
\Phi_\theta(v) = \exp\left( -\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1-q^n} a_n v^n \right) \exp\left( -\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^n} a_n v^{-n} \right), \tag{5.13}
\]
\[
\eta_\lambda(v) =: \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} \eta(q^{j-1} t^{-i+1} v) :.
\]
Similarly, the following intertwiner exists uniquely,
\[
\Phi^* \begin{bmatrix} (1, M), u; (0, 1), v \\ (1, M + 1), -uv \end{bmatrix} : \mathcal{F}_w^{(1,M+1)} \otimes \mathcal{F}_v^{(0,1)} \rightarrow \mathcal{F}_u^{(1,M)} \otimes \mathcal{F}_v^{(1,M+1)},
\]
\[
\Delta(a) \Phi^* = \Phi^* a \quad (\forall a \in \mathcal{U}), \tag{5.14}
\]
with normalization \( \Phi^*(|0\rangle) = |0\rangle \otimes 1 + \cdots \), and its component, defined by
\[
\Phi^*(\alpha) = \sum_\lambda \Phi^*_\lambda(\alpha) \otimes Q_\lambda \quad (\forall \alpha \in \mathcal{F}_w^{(1,M+1)}), \tag{5.15}
\]
is realized by
\[
\Phi^*_\lambda \begin{bmatrix} (1, M), v; (0, 1), u \\ (1, M + 1), -uv \end{bmatrix} = \hat{\iota}^*(\lambda, u, v, M) \hat{\Phi}^*_\lambda(u), \tag{5.16}
\]
where
\[
\hat{\iota}^*(\lambda, u, v, M) = (q^{-1} v)^{-\frac{\text{dim}}{2}}(-u)^{M|\lambda|} f_\lambda^{-M} q^{n(\lambda')}/c_\lambda,
\]
\[
\hat{\Phi}^*_\lambda(u) =: \Phi^*_\theta(u) \hat{\xi}_\lambda(u) :,
\]
\[
\Phi^*_\theta(u) = \exp\left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{1-q^n} q^{-n/2} t^{n/2} a_n u^n \right) \exp\left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^n} q^{-n/2} t^{n/2} a_n u^{-n} \right), 
\]
\[
\hat{\xi}_\lambda(u) =: \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} \xi(q^{j-1} t^{-i+1} u) :. \tag{5.17}
\]
Notation 5.3. In what follows, we mainly consider the \( M = 0 \) case, and we introduce the simplified notations for the intertwiners.

\[
\Phi[u, v] := \Phi \begin{bmatrix} (1, 1), -vu \\ (0, 1), v; (1, 0), u \end{bmatrix}, \quad \Phi^*[u, v] := \Phi^* \begin{bmatrix} (1, 0), u; (0, 1), v \\ (1, 1), -vu \end{bmatrix}, \quad (5.18)
\]

and their components,

\[
\Phi_\lambda[u, v] := \Phi_\lambda \begin{bmatrix} (1, 1), -vu \\ (0, 1), v; (1, 0), u \end{bmatrix}, \quad \Phi^*_\lambda[u, v] := \Phi^*_\lambda \begin{bmatrix} (1, 0), u; (0, 1), v \\ (1, 1), -vu \end{bmatrix}, \quad (5.19)
\]

We also assign the trivalent diagrams to each intertwiner as follows. The arrows stand for the \( F^{(0,1)} \)-modules, and we refer to this direction as the preferred direction, following the terminology of the refined topological vertex in [IKV].

These intertwiners can be identified with the refined topological vertex, invented in [IKV]. To see this, we define the refined topological vertex.

Definition 5.4. The refined topological vertex is defined by

\[
C^{(IKV)}_{\lambda, \mu, \nu}(t, q) = \left( \frac{q}{t} \right)^{[\mu\nu]} \frac{\kappa(\mu)\kappa(\nu)}{2} \sum_{\eta} \left( \frac{q}{t} \right)^{\eta[\lambda \mu \nu]} s_{\lambda'/\eta}(t^\rho q^{-\nu})s_{\mu/\eta}(t^{-\nu'} q^{-\rho}), \quad (5.20)
\]

where \( c_\lambda \) is defined in (3.14), \( [\lambda] = \sum_i \lambda_i^2, \) \( \rho = (-1/2, -3/2, -5/2, \ldots) \) and \( \kappa(\lambda) = \sum_i \lambda_i^2 \lambda_i + 1 - 2i \).

The following fact shows the intertwiners can be regarded as the refined topological vertex.

Fact 5.5. ([AFS])

\[
\frac{1}{\langle P_\lambda, P_\lambda \rangle_{q,t}} \langle S_{\mu}, \Phi_\lambda \begin{bmatrix} (1, M + 1), -vu \\ (0, 1), v; (1, M), u \end{bmatrix} \vert s_\mu \rangle \quad (5.21)
\]

\[
= \left( \frac{q^{-1/2} u}{-v} \right)^{\vert \lambda \vert} f_\lambda^{-M} \cdot (-q^{-1/2} v)^{-\vert \nu \vert} f_\nu \cdot (t^{-1/2} v)^{\vert \mu \vert} \cdot (-1)^{\vert \mu \vert + \vert \nu \vert + \vert \lambda \vert} C^{(IKV)}_{\mu, \nu, \lambda'}(t, q),
\]

\[
\langle S_\nu, \Phi^*_\lambda \begin{bmatrix} (1, M), v; (0, 1), u \\ (1, M + 1), -vu \end{bmatrix} \vert s_\mu \rangle \quad (5.22)
\]

\[
= \left( \frac{-u}{q^{-1/2} v} \right)^{\vert \lambda \vert} f_\lambda^M \cdot (-q^{-1/2} u)^{\vert \nu \vert} f_\nu^1 \cdot (t^{-1/2} u)^{-\vert \mu \vert} \cdot C^{(IKV)}_{\mu', \nu, \lambda'}(t, q).
\]
5.2. Changing preferred direction of ladder diagrams. In this subsection, we will show the matrix elements of the following ladder diagrams are identical up to some monomial factors.

**Notation 5.6.** We introduce some notations \( w_i \) and \( w'_i \) related to the spectral parameters \( u, v \) of the modules.

\[
\begin{align*}
    w_1 & = w, \\
    w'_i & = \frac{u_i}{v_i}, \quad \text{for } i = 1, \ldots, N, \\
    w_{i+1} & = w'_i, \quad \text{for } i = 1, \ldots, N - 1.
\end{align*}
\]  

**Notation 5.7.** For integers \( n \leq m \), we write

\[
\widehat{\otimes}_{i=n}^{m} A_i := A_n \otimes \cdots \otimes A_m. 
\]

**Definition 5.8.** Define the map \( T^H(u, v; w) \) by the following composition

\[
\begin{align*}
    \mathcal{F}^{(0,1)}_{v_1} & \otimes \cdots \otimes \mathcal{F}^{(0,1)}_{v_{N-1}} \otimes \mathcal{F}^{(1)}_{v_N} \otimes |0\rangle \\
    \xrightarrow{id \otimes \cdots \otimes \Phi^*} & \mathcal{F}^{(0,1)}_{v_1} \otimes \cdots \otimes \mathcal{F}^{(1,0)}_{w_N} \otimes \mathcal{F}^{(1)}_{u_N} \\
    \xrightarrow{id \otimes \cdots \otimes id} & \mathcal{F}^{(1,1)}_{v_1} \otimes \cdots \otimes \mathcal{F}^{(1,0)}_{w_N} \otimes \mathcal{F}^{(0,1)}_{u_N} \\
    \xrightarrow{id \otimes \cdots \otimes \Phi^* \otimes id} & \mathcal{F}^{(1,1)}_{v_1} \otimes \cdots \otimes \mathcal{F}^{(1,1)}_{w_N} \\
    \xrightarrow{\Phi^* \otimes \cdots \otimes id} & |0\rangle \otimes \mathcal{F}^{(0,1)}_{v_1} \otimes \cdots \otimes \mathcal{F}^{(0,1)}_{u_N}.
\end{align*}
\]

Here, \( \cdots \otimes |0\rangle \) and \( |0\rangle \otimes \cdots \) mean taking the vacuum expectation value at the level (1,0) modules. For simplicity, we introduce the following notation,

\[
T^H(u, v; w) = \langle 0 | \Phi^*[w_1, u_1] \Phi'[w_1', v_1] \Phi*[w_2, u_2] \Phi'[w_2', v_2] \cdots \Phi*[w_N, u_N] \Phi[w'_N, v_N] |0\rangle.
\]  

(5.25)
We denote its matrix elements by
\[
T^H_{\lambda, \mu}(u, v; w) = \langle 0 | \Phi^*_{\mu(1)}[w_1, u_1] \Phi_{\lambda(1)}[w'_1, v_1] \Phi^*_{\mu(2)}[w_2, u_2] \Phi_{\lambda(2)}[w'_2, v_2] \cdots 
\cdots \Phi^*_{\mu(N)}[w_N, u_N] \Phi_{\lambda(N)}[w'_N, v_N] | 0 \rangle.
\] (5.26)

Also, define the vertex operator $T^V(u, v; w) : \mathcal{F}_u \to \mathcal{F}_v$ by
\[
T^V(u, v; w) := T(u, v; w) / \langle 0 | T(u, v; w) | 0 \rangle,
\] (5.27)
with
\[
T(u, v; w) := \sum_{v^{(1)}, \ldots, v^{(N-1)}} \prod_{i=1}^{N-1} \frac{c_{v^{(i)}}}{c'_v} \Phi^*_{\nu^{(i)}}[v_1, w'_1] \Phi_{\lambda}[u_1, w_1] 
\otimes \Phi^*_{\mu^{(2)}}[v_2, w'_2] \Phi_{\lambda^{(1)}}[u_2, w_2] \otimes \cdots \otimes \Phi^*_{\nu^{(N-1)}}[v_N, w'_N] \Phi_{\lambda^{(N-1)}}[u_N, w_N],
\] (5.28)
and its matrix elements are denoted by
\[
T^V_{\lambda, \mu}(u, v; w) = \langle P_\lambda | T^V(u, v; w) | P_\mu \rangle.
\] (5.29)

The operator $T^V$ guarantees the existence of the Mukadé operator, defined in Definition 4.1, and the following proposition completes the proof of Proposition 4.2.

**Proposition 5.9.** For arbitrary $i \in \{1, 2, \ldots, N\}$, the operator $T^V(u, v; w)$ satisfies
\[
\left(1 - \frac{w}{z}\right) X^{(i)}(z) T^V(u, v; w) = \gamma^{-i} \left(1 - \gamma^{2i} \frac{w}{z}\right) T^V(u, v; w) X^{(i)}(z).
\] (5.30)

**Proof.** The proof is done by the direct calculation. For more details, see Sec 5.3. □

**Remark 5.10.** The factor $\gamma^{-i}$ in the right hand side of (5.30) can be compensated by redefining the spectral parameters as $\gamma^{-1} u_i$. Under this redefinition, the matrix elements of $T^V(u, v; w)$ agree with those of $V(w)$.

**Theorem 5.11.** The following equality between the two matrix elements holds:
\[
T^H_{\lambda, \mu}(u, v; w) \sim T^V_{\lambda, \mu}(u, v; w).
\] (5.31)

Here, $\sim$ means the both sides are identical up to some monomial factor and $\mathcal{G}$-factors $G(z) := \prod_{i,j=0}^\infty (1 - z q^i t^{-j})$, which appear from the normal orderings among $\Phi_{\nu}$’s and $\Phi^*_{\nu}$’s.

**Proof.** We can show, up to the monomial factors and $\mathcal{G}$-factors, the both sides are equal to
\[
\prod_{i,j=1}^N N_{\lambda^{(i)}, \mu^{(j)}}(v_i/\gamma u_j) 
\prod_{1 \leq i < j \leq N} N_{\mu^{(i)}, \mu^{(j)}}(qu_i/tu_j) 
\prod_{k=1}^N c_{\mu^{(k)}} \prod_{1 \leq i < j \leq N} N_{\lambda^{(i)}, \lambda^{(j)}}(qv_j/tv_i) \prod_{k=1}^N c_{\lambda^{(k)}}.
\] (5.32)

For the right hand side, it is obvious from Theorem 4.4 and Definition 3.37. For the left hand side, it can be shown by a direct computation, using some formulas in Appendix C. □

**Remark 5.12.** This proposition implies the invariance under changing preferred directions of the refined topological vertex.
5.3. Proof of Proposition 5.9. First, we give the proof for \( X^{(1)}(z) \). For \( X^{(k)}(z) \) with \( k > 1 \), the strategy of the proof is similar to that of \( k = 1 \) case, and thus we just show the sketch of the proof.

\( k = 1 \)

case. In order to simplify the notation, we rewrite the operator \( T^V(\mathbf{u}, \mathbf{v}; w) \) as

\[
T^V(\mathbf{u}, \mathbf{v}; w) = \sum_{\nu(1), \ldots, \nu(N-1)} C_{\nu} : \Phi_\theta(w_1) \Phi_{\nu(1)}^*(w'_1) : \otimes : \Phi_\nu(w_2) \Phi_{\nu(2)}^*(w'_2) : \otimes \\
\cdots \otimes : \Phi_{\nu(N-1)}(w_N) \Phi_{\nu(N)}^*(w'_N) := \sum_{\nu(1), \ldots, \nu(N-1)} T^V_{\nu}(\mathbf{u}, \mathbf{v}; w),
\]

(5.33)

with

\[
C_{\nu} := \prod_{i=1}^{N} \frac{C(\nu(i))}{C_{\nu(i)}} G(w_i/\gamma w'_i)^{-1} \\
\times \hat{\iota}(\nu^{(i-1)}, u_i, w_i, 0) \hat{\iota}(\nu^{(i)}, w'_i, v_i, 0) N_{\nu^{(i-1)},0}(w_i/\gamma w'_i) / \langle 0 | T^V(\mathbf{u}, \mathbf{v}; w) | 0 \rangle,
\]

(5.34)

which appear from the normal orderings. Here, we put \( \nu^{(0)} = \nu^{(N)} = 0 \) for convenience. We also omit the parameters \( \mathbf{u}, \mathbf{v} \) in \( T^V(\mathbf{u}, \mathbf{v}; w) \) in what follows.

We first compute the commutation relations between \( \Lambda(j), j = 1, \ldots, N \) and \( T^V(w) \). Putting \( z_j = \gamma^{-1} z \), we have

\[
(w_j/\gamma w'_j) v_j \Lambda^{(j)}(z) T^V_{\nu}(w) = \frac{1-z/y^2 w_1}{1-z/w_1} T^V_{\nu}(w) u_j \Lambda^{(j)}(z) \\
= -u_j \left( \text{delta functions from } A(v^{(j-1)}) \right) \\
- u_j \sum_{y \in R(v^{(j)})} \delta(y z_j/w'_j, x_j) C_{y} \prod_{x \in R(v^{(j-1)})} 1 - x/y x_j \prod_{x \in A(v^{(j-1)})} 1 - x/y^2 x_j \\
\times \cdots \otimes : \eta(y^{-1} x_j, x_j) \Phi_{\nu^{(j-1)}}^*(w_j) : \otimes : \Phi_{\nu^{(j)}}(w_j) : \otimes \cdots \\
= -u_j \left( \text{delta functions from } A(v^{(j-1)}) \right) \\
- u_j \sum_{y \in R(v^{(j)})} \delta(y z_j/w'_j, x_j) C_{y} \prod_{x \in R(v^{(j-1)})} 1 - x/y x_j \prod_{x \in A(v^{(j-1)})} 1 - x/y^2 x_j \\
\times \cdots \otimes : \eta(y^{-1/2} x_j, x_j) \Phi_{\nu^{(j-1)}}^*(w_j) : \otimes : \Phi_{\nu^{(j)}}(w_j) : \otimes \cdots ,
\]

(5.35)

where \( A(v) \) and \( R(v) \) is defined in Sect. 1. Through the computation, we used the equality

\[
\eta(y^{-1} u) = : \varphi^-(y^{-1/2} u) \xi^{-1}(u) : .
\]

(5.36)

Now, for \( \nu^{(j)} \neq \emptyset \), fix a \( y \in R(v^{(j)}) \) and pick up the corresponding term from the above equation. Then, we rewrite \( \nu^{(j)} - y \) as \( \nu^y \), and express the term using \( \nu^y \). This procedure ends with the following expression:

\[
- u_j \delta(y z_j/w'_j, x_j) C_{\nu} \prod_{x \in R(v^{(j-1)})} 1 - x/y x_j \prod_{x \in A(v^{(j-1)})} 1 - x/y^2 x_j \\
\times \cdots \otimes : \varphi^-(y^{-1/2} x_j, x_j) \Phi_{\nu^{(j-1)}}(w_j) \Phi_{\nu^{(j-1)}}^*(w'_j) : \otimes : \Phi_{\nu^{(j)}}(w_{j+1}) \Phi_{\nu^{(j+1)}}^*(w_{j+1}) : \otimes \cdots
\]

\[
- u_j \delta(y z_j/w'_j, x_j) C_{\nu} \prod_{x \in R(v^{(j-1)})} 1 - x/y x_j \prod_{x \in A(v^{(j-1)})} 1 - x/y^2 x_j \\
\times \cdots \otimes : \varphi^-(y^{-1/2} x_j, x_j) \Phi_{\nu^{(j-1)}}(w_j) \Phi_{\nu^{(j-1)}}^*(w'_j) : \otimes : \Phi_{\nu^{(j)}}(w_{j+1}) \Phi_{\nu^{(j+1)}}^*(w_{j+1}) : \otimes \cdots
\]
We use the following simplified notation, case.

\[
\mathcal{A}_{34} = M. \text{ Fukuda, Y. Ohkubo, J. Shiraishi}
\]

We multiply the both sides by 1

\[ X^n \]

is multiplied, and we obtain the expected commutation relation between

\[ V \]

\[ T \]

\[
\Lambda_1(\cdots, v_{(j-1)}(w_j) \hat{\Phi}_{v_{(j+1)}}(w_{j+1}) : \otimes : \hat{\Phi}_{v_{(j+1)}}(w_{j+1}) \otimes \cdots .
\]

\[
(5.37)
\]

While taking summation over \( v \), these terms are cancelled by the terms which appear from the commutation relation between \( \Lambda^{(j+1)} \) and \( T^V_{\nu}(w) \) with \( \nu^{(j+1)} = \nu^{(j)} \),

\[
(w_{j+1}/\gamma w_{j+1}) u_{j+1} \Lambda^{(j+1)}(z) T^V_{\nu}(w) - \frac{1 - z/\gamma^2 w_1}{1 - z/w_1} T^V_{\nu}(w) u_{j+1} \Lambda^{(j+1)}(z)
\]

\[
= -u_{j+1} \left( \text{delta functions from} R(\nu^{(j+1)})' \right)
\]

\[
- u_{j+1} \sum_{y \in A(\nu^{(j)})'} \delta(z_{j+1}/w_{j+1} \gamma \chi_{j}) \frac{1 - \gamma^2 \chi_{j}/\chi_{x}}{\prod_{x \in A(\nu^{(j)})', x \neq y} (1 - \chi_{y}/\chi_{x})} \times \prod_{x \in R(\nu^{(j+1)})'} (1 - \gamma \chi_{j}/\chi_{x}) \frac{1 - \gamma^2 \chi_{j}/\chi_{x}}{\prod_{x \in R(\nu^{(j+1)})'} (1 - \gamma \chi_{j}/\chi_{x})} \times \cdots \times \delta(\gamma^{1/2} w_{j+1} \gamma \chi_{j}) \hat{\Phi}_{\nu_{(j-1)}}(w_j) \times \hat{\Phi}_{\nu_{(j+1)}}(w_{j+1}) : \otimes : \hat{\Phi}_{\nu_{(j+1)}}(w_{j+1}) : \otimes \cdots .
\]

\[
(5.38)
\]

While summing up for \( j \) from 1 to \( N \), the delta function which is not cancelled by the above mechanism exists at \( A(\nu^{(0)}) = A(\theta) \). This delta function vanishes when we multiply the both sides by \( 1 - w_1/z \), and we finally obtain the main claim of Proposition 5.9 for the \( k = 1 \) case.

\( k > 1 \)

case. We use the following simplified notation,

\[
\Lambda^{(i_1, \ldots, i_k)}(z) := \Lambda^{(i_1)}(z) \cdots \Lambda^{(i_k)}(\gamma^2(1-k)z) : .
\]

\[
(5.39)
\]

We can compute the commutation relations between \( \Lambda^{(i_1, \ldots, i_k)}(z) \) and \( T^V_{\nu}(w) \) as,

\[
\prod_{j=1}^k (w_j/\gamma w_{j+1}) u_1 \cdots u_k \Lambda^{(i_1, \ldots, i_k)}(z) T^V_{\nu}(w) - \frac{1 - z/\gamma^2 w_1}{1 - z/w_1} T^V_{\nu}(w) u_1 \cdots u_k \Lambda^{(i_1, \ldots, i_k)}(z)
\]

\[
= -u_1 \cdots u_k \left( \text{delta functions from} A(\nu^{(i_1)}), R(\nu^{(i_2)}), A(\nu^{(i_2-1)}), R(\nu^{(i_2)}), \right.
\]

\[
\left. \cdots, A(\nu^{(i_k-1)}) \text{ and} R(\nu^{(i_k)}). \right)
\]

\[
(5.40)
\]

The delta functions related to \( R(\nu^{(i)}) \) (for \( j = 1, \ldots, k \)) cancel those related to \( A(\nu^{(i)}) \) which appear in the commutation relations between \( \Lambda^{(i_1, \ldots, i_k-1)}(z) \) and \( T^V_{\nu}(w) \). This sequence of the cancellation begins when \( i_j = i_{j-1} + 1 \) and terminates when \( i_j = i_{j+1} - 1 \), because in those cases, the poles and zeros related to \( \nu^{(i_j)} \) cancel each other, and no delta functions related to them appear.

As a result, the only delta function which survives the above cancellation mechanism is that related to \( A(\nu^{(0)}) \). Similarly to the \( k = 1 \) case, it vanishes when \( 1 - w_1/z \) is multiplied, and we obtain the expected commutation relation between \( X^{(k)}(z) \) and \( T^V_{\nu}(w) \).
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Appendix A. Construction of Macdonald Symmetric Functions in terms of Topological Vertex

We can obtain Macdonald functions as matrix elements of some compositions of the intertwining operators. The similar derivation and its supersymmetric version are given in [Z]. Set

\[
\tilde{T}_j(x) = \tilde{T}_j(u; x) := \prod_{k=1}^{N} \mathcal{G}(u_k/\gamma v_k) \cdot \langle 0 | \mathcal{T}^V(v, u; x) | 0 \rangle \cdot \mathcal{T}^V(v, u; x) \mid_{v_k \rightarrow \gamma^{-1} - \delta_{k,1} u_k, \quad (1 \leq k \leq N)}
\]

(A.1)

where the overall factor is chosen for the later convenience. Though the prefactor consisting of the \( G \)-factors gives zeros under the restriction of \( v \), but the \( G \)-factors which appear from the normal orderings among \( \Phi \)'s and \( \Phi^* \)'s in \( \mathcal{T}^V \), cancel those zeros, and thus the operator is well-defined. By operator products (C.26) and Lemma C.1, it can be seen that the Young diagrams are restricted to only one row when gluing intertwiners over the vertical representations (Fig. 2), i.e.,

\[
\tilde{T}_j(x) = \sum_{0 \leq m_1 \leq m_2 \leq \ldots \leq m_{i-1} < \infty} \gamma^{-m_{i-1}} \prod_{k=1}^{i-1} \mathcal{G}(c_{(m_k)}^t \gamma u_k) \frac{q\gamma \gamma^k}{\gamma^k \gamma^{-1}} (\mathcal{G}(\gamma u_k))^{m_k} f_{(m_k)}^{-1} \\
\times N_{\Phi, (m_{i-1}), (m_2), (m_1)}(1) \\
\times \left( \hat{\Phi}^*_y((\gamma^{-1} x) \hat{\Phi}^{(y)}(x)) : \bigotimes_{k=2}^{i-1} \hat{\Phi}^*_y((\gamma^{-k} x) \hat{\Phi}^{(y)}(\gamma^{-k+1} x)) : \bigotimes_{k=1}^{N} \hat{\Phi}^*_y((\gamma^{-k} x) \hat{\Phi}^{(y)}(\gamma^{-k+1} x)) \right)
\]

(A.2)

where we put \( m_0 = 0 \). Here, \( \bigotimes \) is introduced in Notation 5.7. Then, we have \( \tilde{T}_j(x) = \Phi^{(0)}(t^{-1} x). \) Note that the operators \( \Phi^{(k)}(x) \) (\( k = 0, \ldots, N - 1 \)) in this section slightly differ from those in Sec. 3.2. In this section, \( \Phi^{(k)}(x) \) is a map \( \mathcal{F}_{t^{-1} \gamma^{-k+1}} \mathcal{F}_u \rightarrow \mathcal{F}_u \) with \( \mathcal{F}_u = (\gamma^{-1} u_1, \ldots, \gamma^{-1} u_N) \), that is, the spectral parameter differs by the factor \( \gamma^{-1} \).

We obtain the expression of Macdonald functions in terms of intertwiners of the DIM algebra. The following proposition says the vacuum expectation value of Fig. 3 gives the Macdonald function of the \( A \) type.

**Proposition A.1.**

\[
\langle 0 | \tilde{T}_1(u; x_1) \tilde{T}_2(x_2) \cdots \tilde{T}_N(x_N) | 0 \rangle \propto p_N(x; u|q, q/t).
\]

(A.3)
Fig. 2. The operator $\tilde{T}_i(x)$. (⇒ stands for a simplification of the diagram for convenience)

Fig. 3. The diagram for the Macdonald function $p_N(x; u|q, q/t)$

Proof. The operators $\hat{\Phi}_m(z)$ and $\hat{\Phi}^*_m(z)$ can be decomposed as

$$\hat{\Phi}_m(z) = :\Phi_\theta(t^{-1}z)A(q^mz) :, \quad \hat{\Phi}^*_m(z) = :\Phi^*_\theta(t^{-1}_{m}z)A^*(q^mz) :,$$  \hspace{1cm} (A.4)
As in the proof of Proposition 3.12 (Appendix D.1),

\[ A(z) = \exp \left( - \sum_{n>0} \frac{1-t^{-n}}{n(1-q^n)} a_{-n} z^n \right) \exp \left( \sum_{n>0} \frac{1-t^n}{n(1-q^n)} a_n z^{-n} \right), \]  

(A.5) 

\[ A^*(z) = \exp \left( \sum_{n>0} \frac{1-t^{-n}}{n(1-q^n)} \gamma^n a_{-n} z^n \right) \exp \left( - \sum_{n>0} \frac{1-t^n}{n(1-q^n)} \gamma^n a_n z^{-n} \right). \]  

(A.6) 

Then the tensor product of \( A(z) \) and \( A^*(z) \) corresponds to the screening current (Definition 3.10):

\[ A^*(z) \otimes A(z) = \phi^S_c (\gamma^{-1} t^{-1} z). \]  

(A.7) 

Therefore, we have

\[ \tilde{T}_i(u; x) = \left( \frac{q/t; q}{q; q} \right)_\infty \times \sum_{0 \leq m_1 \leq m_2 \leq \cdots \leq m_i < \infty} \Phi^{(0)}(t^{-1}x) \tilde{S}^{(1)}(q^{m_1}x) \]  

(A.8) 

\[ \cdots \tilde{S}^{(i-1)}(q^{m_{i-1}}x) \prod_{k=1}^{i-1} (u_{k+1}/u_k)^{m_k}. \] 

Here, we put

\[ \tilde{S}^{(k)}(z) = S^{(k)}(\gamma^{-2k} t^{-1} z). \]  

(A.9) 

As in the proof of Proposition 3.12 (Appendix D.1), \( X^{(1)}(z) \) commutes with \( \tilde{S}^{(k)}(w) \) up to \( q \)-difference:

\[ \left[ X^{(1)}(z), \tilde{S}^{(k)}(w) \right] = (t-1)(u_{k+1}T_{q,w} - u_k) \left( \delta \left( \frac{\gamma^{-2k} w}{q z} \right) : \Lambda^{(k)}(\gamma^{-2k} w/q) \tilde{S}^{(k)}(w) : \right). \]  

(A.10) 

Further, by the property of the operator products

\[ \tilde{S}^{(k-1)}(x) \Lambda^{(k)}(\gamma^{-2k} x/q) = \Phi^{(0)}(t^{-1}x) \Lambda^{(1)}(\gamma^{-2} x/q) = 0, \]  

(A.11) 

we obtain

\[ \Phi^{(0)}(t^{-1}x) \cdot \left[ X^{(1)}_0, \sum_{m=0}^{\infty} \tilde{S}^{(1)}(q^m x)(u_2/u_1)^{m} \right] = 0, \]  

(A.12) 

\[ \tilde{S}^{(k-1)}(x) \cdot \left[ X^{(1)}_0, \sum_{m=0}^{\infty} \tilde{S}^{(k)}(q^m x)(u_{k+1}/u_k)^{m} \right] = 0 \quad (k \geq 2). \]  

(A.13) 

This leads to

\[ X^{(1)}(z) \tilde{T}_i(u; w) - \gamma \frac{1 - qz/tw}{1 - zw} \tilde{T}_i(u; w) X^{(1)}(z) = u_i (1 - t^{-1}) \tilde{T}_i(u; qw) \Psi^+(t^{-1} w) \delta(w/z). \]  

(A.14)
By this relation, we get

\[
\langle 0 | X^{(1)}(z) \tilde{T}_1(x_1) \cdots \tilde{T}_N(x_N) | 0 \rangle = \gamma^N \prod_{k=1}^N \frac{1 - qz/t x_k}{1 - z/x_k} \\
\cdot \langle 0 | \tilde{T}_1(x_1) \cdots \tilde{T}_N(x_N) X^{(1)}(z) | 0 \rangle + (1 - t^{-1}) \sum_{i=1}^N \delta(x_i/z) u_i \prod_{k=1}^{i-1} \frac{1 - q x_i/t x_k}{1 - x_i/x_k}
\]

\[
\prod_{k=i+1}^N \frac{1 - t x_k/q x_i}{1 - x_k/x_i} T_{q,x_i} \langle 0 | \tilde{T}_1(x_1) \cdots \tilde{T}_N(x_N) | 0 \rangle .
\]

(A.15)

Thus, the considered matrix elements are eigenfunctions of the difference operator \( D_N^1 \):

\[
D_N^1(u; q, q/t) \langle 0 | \tilde{T}_1(x_1) \cdots \tilde{T}_N(x_N) | 0 \rangle = (u_1 + \cdots + u_N) \langle 0 | \tilde{T}_1(x_1) \cdots \tilde{T}_N(x_N) | 0 \rangle .
\]

(A.16)

\( \square \)

Remark A.2. By using the expression (A.8), it is shown that the operator \( \tilde{T}_1(x) \) also represents the screened vertex operator \( \Phi^{(i-1)}(x) \). Whereas \( \tilde{T}_i(x) \) is written by a sort of Jackson integrals, \( \Phi^{(i-1)}(x) \) is introduced by counter integrals. In order to justify the equivalence of the two expressions, see the following argument.

We first note the following deformation of the formal series,

\[
\sum_{m \in \mathbb{Z}_{\geq 0}} \delta(q^m z) \alpha^m = \sum_{m \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}} (q^m z)^k \alpha^m
\]

\[
= \sum_{k \in \mathbb{Z}} \frac{1}{1 - q^k z^k}
\]

\[
= \sum_{k \in \mathbb{Z}} \frac{\theta_q(z) \theta_q(\alpha z)}{(q; q)_\infty (q^{-1}; q)_\infty} \theta_q(z) \quad \text{(Contour integral)}.
\]

(A.17)

Of course, this series does not converge for arbitrary \( q \) and \( \alpha \). However, by the following reason, we can choose some domains for \( q \) and \( \alpha \) so that the matrix elements of \( \tilde{T}_i(x) \) converge there, and thus they are identical to those of \( \Phi^{(i-1)}(x) \). We discuss the example of the \( i = 2 \) case. The matrix elements of \( \tilde{T}_2(x) \) are of the form,

\[
\langle P_\lambda | \tilde{T}_2(x) | Q_\mu \rangle \sim \int \frac{d w}{w} \sum_{m \geq 0} \delta(q^m x/w) \alpha^m \langle P_\lambda | \Phi^{(0)}(x) \tilde{S}^{(1)}(w) | Q_\mu \rangle
\]

\[
\sim \int \frac{d w}{w} \sum_{m \geq 0} \sum_{k \in \mathbb{Z}} \delta(q^m x/w)^k \alpha^m \sum_{n \geq 0} \frac{(t; q)_n}{(q; q)_n} (w/x)^n
\]

\[
\times \langle P_\lambda | : \Phi^{(0)}(x) \tilde{S}^{(1)}(w) : | Q_\mu \rangle ,
\]

(A.18)

where \( \alpha = u_2/u_1 \), and \( \sim \) means the both sides are equal up to some overall factors. The factor \( \langle P_\lambda | : \Phi^{(0)}(x) \tilde{S}^{(1)}(w) : | Q_\mu \rangle \) is some finite sum of the monomial of the form \( x^a w^b \) with the maximums and minimums of \( a \)'s and \( b \)'s depending on \( \lambda \) and \( \mu \). Then, by taking the constant term in \( w \) in the above, the running index \( k \) is fixed as \( k = n + b \geq b \). Therefore, when we choose the \( q \) and \( \alpha \) so that

\[
|\alpha q^{\min(b)}| \ll 1 ,
\]

(A.19)
the deformation (A.17) is justified by putting \( k < \min(b) \) terms zero, and thus

\[
\langle P_\lambda | \tilde{T}_2(x) | Q_\mu \rangle \sim \int \frac{dw}{w} \sum_{m \geq 0} \sum_{k \in \mathbb{Z}} (q^m x/w)^k \alpha^m \sum_{n \geq 0} \frac{(t; q)_n}{(q; q)_n} (w/x)^n \times \langle P_\lambda | \Phi^{(0)}(x) \tilde{S}^{(1)}(w) : | Q_\mu \rangle
\]

\[
\sim \int \frac{dw \theta_q(ax/w)}{w \theta_q(x/w)} \langle P_\lambda | \Phi^{(0)}(x) \tilde{S}^{(1)}(w) | Q_\mu \rangle \sim \langle P_\lambda | \Phi^{(1)}(x) | Q_\mu \rangle.
\]

This discussion can extend to the general \( i \) case, and the equivalence between the Jackson integral and contour integral is justified.

### Appendix B. Basic Facts for Ordinary Macdonald Functions

In this section, we give the definition and basic facts for ordinary Macdonald functions. Let \( \Lambda_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n} \) be the ring of symmetric polynomials, and \( \Lambda = \lim_{\Lambda_n} \Lambda_n \) be the projective limit in the category of graded rings, i.e., the ring of symmetric functions. \( m_\lambda \) denotes the monomial symmetric function. We denote the power sum symmetric functions by \( p_n = p_n(x) = \sum_{i=1}^n x_i^n \). For a partition \( \lambda \), set \( p_\lambda = \prod_{i \geq 1} p_{\lambda_i} \). Macdonald functions are defined as orthogonal functions with respect to the following scalar product.

**Definition B.1.** Define the bilinear form \( \langle -, - \rangle_{q,t} : \Lambda \otimes \Lambda \rightarrow \mathbb{C} \) by

\[
\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} 1 - q^{\lambda_i} 1 - t^{\lambda_i}, \quad z_\lambda := \prod_{i \geq 1} i^{m_i} \cdot m_i!,
\]

(B.1)

where \( m_i \) is the number of entries in \( \lambda \) equal to \( i \).

**Fact B.2.** ([Ma]) There exists an unique function \( P_\lambda \in \Lambda \) such that

\[
P_\lambda = m_\lambda + \sum_{\mu < \lambda} \alpha_{\lambda,\mu} m_{\mu} \quad (\alpha_{\lambda,\mu} \in \mathbb{C});
\]

(B.2)

\[
\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad (\lambda \neq \mu).
\]

(B.3)

The symmetric functions \( P_\lambda \) are called Macdonald symmetric functions. It is known that

\[
\langle P_\lambda, P_\lambda \rangle_{q,t} = c'_\lambda/c_\lambda.
\]

(B.4)

Here, \( c_\lambda \) and \( c'_\lambda \) are defined in (3.14). Set \( Q_\lambda := \langle P_\lambda, P_\lambda \rangle_{q,t}^{-1} P_\lambda \), so that \( \langle P_\lambda, Q_\lambda \rangle_{q,t} = 1 \).

In this paper, we regard power sum symmetric functions as variables of Macdonald functions and write \( P_\lambda = P_\lambda(p_n) \), \( Q_\lambda = Q_\lambda(p_n) \). These are abbreviation for \( P_\lambda(p_1, p_2, \ldots) \) and \( Q_\lambda(p_1, p_2, \ldots) \). We often substitute the generators \( a_n \) in the Heisenberg algebra into Macdonald functions as \( P_\lambda(a_{-n}) \). Note that this substitution preserves the properties of Macdonald functions as follows. The space of symmetric functions \( \Lambda \) and the Fock space \( \mathcal{F} \) are isomorphic as graded vector spaces, and they can be identified by

\[
\Lambda \rightarrow \mathcal{F}, \quad p_\lambda \mapsto |a_\lambda\rangle.
\]

(B.5)

Further, the bilinear form on the Fock spaces preserves the structure of the scalar product \( \langle -, - \rangle_{q,t} \) under this identification, i.e.,

\[
\langle p_\lambda, p_\mu \rangle = \langle a_\lambda, a_\mu \rangle.
\]

(B.6)

We describe some facts for Macdonald functions that are used in this paper.
Kernel function. The following function $\Pi(x, y; q, t)$ is called the kernel function:

$$
\Pi(x, y; q, t) := \prod_{n \geq 1} \exp \left( \frac{1 - t^n}{1 - q^n} p_n(x)p_n(y) \right).
$$

(B.7)

This function $\Pi(x, y; q, t)$ can be expanded in terms of dual bases parametrized by partitions as follows.

**Fact B.3.** ([Ma]) Let $u_\lambda, v_\lambda \in \Lambda$ be homogeneous symmetric functions of level $|\lambda|$ and $\{u_\lambda\}, \{v_\lambda\}$ form $\mathbb{C}$-bases on $\Lambda$. Then, the followings are equivalent:

- $\langle u_\lambda, v_\lambda \rangle_{q, t} = \delta_{\lambda, \mu}$ for all $\lambda, \mu$; 
- $\Pi(x, y; q, t) = \sum_{\lambda} u_\lambda(x)v_\lambda(y)$.

(B.8) (B.9)

Note that especially, we have

$$
\Pi(x, y; q, t) = \sum_{\lambda} P_\lambda(p_n(x))Q_\lambda(p_n(y)).
$$

(B.10)

Pieri formula. Define the symmetric function $g_r \in \Lambda$ to be the expansion coefficient of $y^r$ in the following series:

$$
\exp \left( \sum_{n \geq 1} \frac{1 - t^n}{n} \frac{1 - q^n}{1 - q^n} y^n \right) = \sum_{r \geq 0} g_r y^r.
$$

(B.11)

For a partition $\lambda$ and a coordinate $s$, we write

$$
b_\lambda(s) = \begin{cases} 
1 - q^a_{\lambda}(s) r_{\lambda}(s) + 1, & s \in \lambda; \\
1 - q^a_{\lambda}(s) r_{\lambda}(s), & \text{otherwise.}
\end{cases}
$$

(B.12)

**Fact B.4.** ([Ma]) We have the Pieri rules:

$$
g_r Q_\mu = \sum_{\lambda} \prod_{s \in R_{\lambda/\mu} - C_{\lambda/\mu}} b_\mu(s) b_\lambda(s) \cdot Q_\lambda.
$$

(B.13)

Here, the summation is over the partitions $\lambda$ such that $\lambda/\mu$ is a horizontal $r$-strip, i.e., $\lambda/\mu$ has at most one box in each column. $R_{\lambda/\mu}$ (resp. $C_{\lambda/\mu}$) is the union of the rows (resp. columns) that intersect $\lambda - \mu$.

Note that by this formula, we can see that $Q_\mu (\mu \not\geq \lambda)$ does not appear in the expansion of the product $\prod_{i \geq 1} g_\lambda$ in the basis of Macdonald functions.
Another scalar product. There is another scalar product for which Macdonald functions $P_\lambda$’s are pairwise orthogonal. Let $L_r = \mathbb{C}[x_1^{\pm1}, \ldots, x_r^{\pm1}]$ be the $\mathbb{C}$-algebra of Laurent polynomials in $r$ variables. For $f(x_1, \ldots, x_r) \in L_r$, we put $\bar{f} = f(x_1^{-1}, \ldots, x_r^{-1})$.

**Definition B.5.** The scalar product $(\cdot, \cdot)'_r$ on $L_r$ is defined by

$$
(f, g)'_r : = \oint \prod_{i=1}^{r} \frac{dx_i}{2\pi \sqrt{-1}} f \bar{g} \Delta(x),
$$

where

$$
\Delta(x) := \prod_{i < j} (x_i/x_j; q)_{\infty} (t x_i/x_j; q)_{\infty}.
$$

**Fact B.6.** ([Ma]) Let $P_\lambda^{(r)} = P_\lambda|_{x_i \to 0 (i > r)}$, $Q_\mu^{(r)} = Q_\mu|_{x_i \to 0 (i > r)} \in \Lambda_r$ be the Macdonald symmetric polynomials in $r$ variables. Then

$$
\langle P_\lambda^{(r)}, Q_\mu^{(r)} \rangle'_r = \delta_{\lambda, \mu} \langle 1, 1 \rangle'_r \prod_{(i, j) \in \lambda} \frac{1 - q^{i-j}t^r-i+1}{1 - q^i t^{r-i}},
$$

where $\ell(\lambda), \ell(\mu) \leq r$.

**Appendix C. Some Useful Formulas**

**C.1 List of operator products.** In this subsection, we list some formulas for the normal ordering among the various operators appeared in the main text. We have

$$
\Lambda^{(i)}(z) S^{(i)}(w) = \frac{1 - t^2 w/z}{1 - t w/z} : \Lambda^{(i)}(z) S^{(i)}(w) :,
$$

(C.1)

$$
\Lambda^{(i+1)}(z) S^{(i)}(w) = \frac{1 - w/z}{1 - t w/z} : \Lambda^{(i+1)}(z) S^{(i)}(w) :,
$$

(C.2)

$$
\Lambda^{(j)}(z) S^{(i)}(w) =: \Lambda^{(j)}(z) S^{(i)}(w) : \text{ for } j < i \text{ and } j > i + 1,
$$

(C.3)

$$
S^{(i)}(w) \Lambda^{(i)}(z) = \frac{1 - q z/t^2 w}{1 - q z/t w} : S^{(i)}(w) \Lambda^{(i)}(z) :,
$$

(C.4)

$$
S^{(i)}(w) \Lambda^{(i+1)}(z) = \frac{1 - z/w}{1 - z/t w} : S^{(i)}(w) \Lambda^{(i+1)}(z) :,
$$

(C.5)

$$
S^{(i)}(w) \Lambda^{(j)}(z) =: S^{(i)}(w) \Lambda^{(j)}(z) : \text{ for } j < i \text{ and } j > i + 1,
$$

(C.6)

$$
\Phi^{(0)}(z) S^{(1)}(w) = \frac{(t w/z; q)_{\infty}}{(w/z; q)_{\infty}} : \Phi^{(0)}(z) S^{(1)}(w) :,
$$

(C.7)

$$
\Phi^{(0)}(z) S^{(i)}(w) =: \Phi^{(0)}(z) S^{(i)}(w) : (i \geq 2),
$$

(C.8)

$$
S^{(1)}(z) \Phi^{(0)}(w) = \frac{(q w/z; q)_{\infty}}{(q w/tz; q)_{\infty}} : S^{(1)}(z) \Phi^{(0)}(w) :,
$$

(C.9)

$$
S^{(i)}(z) \Phi^{(0)}(w) =: S^{(i)}(z) \Phi^{(0)}(w) : (i \geq 2),
$$

(C.10)

$$
\Phi^{(0)}(z) \Phi^{(0)}(w) = \frac{(q w/tz; q)_{\infty}}{(t w/z; q)_{\infty}} : \Phi^{(0)}(z) \Phi^{(0)}(w) :,
$$

(C.11)
\[ S^{(i)}(z)S^{(i)}(w) = (1 - w/z) \left( \frac{qw/tz; q}{zw/tz; q} \right) \infty : S^{(i)}(z)S^{(i)}(w) : , \quad (C.12) \]

\[ S^{(i)}(z)S^{(i+1)}(w) = \frac{(tw/z; q) \infty}{(zw/tz; q) \infty} : S^{(i)}(z)S^{(i+1)}(w) : , \quad (C.13) \]

\[ S^{(i+1)}(z)S^{(i)}(w) = \frac{(qw/tz; q) \infty}{(tw/z; q) \infty} : S^{(i+1)}(z)S^{(i)}(w) : (\forall i), \quad (C.14) \]

\[ S^{(i)}(z)S^{(j)}(w) =: S^{(i)}(z)S^{(j)}(w) : \text{ for } |i - j| > 2, \quad (C.15) \]

\[ \Lambda^{(1)}(z) \Phi^{(0)}(x) = \frac{1 - x/z}{1 - tx/z} : \Lambda^{(1)}(z) \Phi^{(0)}(x) :, \quad (C.16) \]

\[ \Phi^{(0)}(x) \Lambda^{(1)}(z) = \frac{1 - z/x}{1 - qz/r^2 x} : \Phi^{(0)}(x) \Lambda^{(1)}(z) :, \quad (C.17) \]

\[ \Lambda^{(i)}(z) \Phi^{(0)}(x) =: \Lambda^{(i)}(z) \Phi^{(0)}(x) :, \quad (C.18) \]

\[ \Phi^{(0)}(x) \Lambda^{(i)}(z) = \frac{1 - z/t x}{1 - q z/r^2 x} : \Phi^{(0)}(x) \Lambda^{(i)}(z) : (i > 1), \quad (C.19) \]

\[ \Psi^+(z)S^{(i)}(w) = S^{(i)}(w)\Psi^+(z) =: \Psi^+(z)S^{(i)}(w) : (\forall i), \quad (C.20) \]

\[ \Psi^+(z)\Phi^{(0)}(w) = \frac{1 - tw/z}{1 - w/z} : \Psi^+(z)\Phi^{(0)}(w) :, \quad (C.21) \]

\[ A_{(s)}(x) \Lambda^{(i)}(z) = \prod_{k=1}^{r-1} \frac{1 - t^{-k} z/x}{1 - t^{-k-1} q z/x} : \Lambda^{(i)}(z) A_{(s)}(x) :, \quad (C.22) \]

\[ A^{(r)}(x)\Phi^{(0)}(y) = \prod_{k=0}^{r-2} \left( \frac{(t^{-k} q y/t x; q) \infty}{(t^{-k} y/x; q) \infty} \right) : A^{(r)}(x)\Phi^{(0)}(y) :, \quad (C.23) \]

\[ A^{(r)}(x)S^{(i)}(y) =: A^{(r)}(x)S^{(i)}(y) :, \quad (C.24) \]

and with \( G(z) = \prod_{i,j=0}^{\infty} (1 - qz/r^{i+j}) \),

\[ \Phi_\lambda(v_i) \Phi^*_\mu(u_j) = G(u_j/\gamma v_i) - N_{\mu,\lambda}(u_j/\gamma v_i) : \Phi_\lambda(v_i) \Phi^*_\mu(u_j) :, \quad (C.25) \]

\[ \Phi^*_\mu(u_j) \Phi_\lambda(v_i) = G(v_i/\gamma u_j) - N_{\mu,\lambda}(v_i/\gamma u_j) : \Phi^*_\mu(u_j) \Phi_\lambda(v_i) :, \quad (C.26) \]

\[ \Phi^{(i)}_\lambda(v_i) \Phi^{(j)}_\lambda(v_j) = \frac{G(v_j/\gamma^2 v_i) - N^{(i)}_{\lambda}(v_j/\gamma^2 v_i)}{N^{(i)}_{\lambda}(v_j/\gamma^2 v_i)} : \Phi^{(i)}_\lambda(v_i) \Phi^{(j)}_\lambda(v_j) :, \quad (C.27) \]

\[ \Phi^{(i)}_{\mu}(u_i) \Phi^{(j)}_\mu(u_j) = \frac{G(u_j/u_i) - N^{(i)}_{\mu}(u_j/u_i)}{N^{(i)}_{\mu}(u_j/u_i)} : \Phi^{(i)}_{\mu}(u_i) \Phi^{(j)}_\mu(u_j) :, \quad (C.28) \]

\[ \text{C.2 On Nekrasov factors.} \quad \text{For a partitions } \lambda \text{ and non-negative integers } r, s \in \mathbb{Z}_{\geq 0}, \text{ define } B_{r,s}(\lambda) \text{ to be the partition obtained by removing the 1st to } s \text{-th rows and the 1st to } r \text{-th columns, i.e.,} \]

\[ B_{r,s}(\lambda) := (P(\lambda_{s+i} - r))_{i \geq 1}, \quad P(n) = \begin{cases} n, & n \geq 0; \\ 0, & n < 0. \end{cases} \quad (C.29) \]

For example, if \( \lambda = (5, 5, 4, 4, 4, 1, 1) \), then \( B_{2,1}(\lambda) = (3, 2, 2, 2) \). We have the following vanishing condition for the Nekrasov factor.
Lemma C.1. If \( m \geq 0 \) and \( n \leq 0 \), then
\[
N_{\lambda, \mu}(q^m t^n) \neq 0 \iff \mu \supset B_{-n,m}(\lambda).
\] (C.30)

If \( m \leq -1 \) and \( n \geq 1 \), then
\[
N_{\lambda, \mu}(q^m t^n) \neq 0 \iff \lambda \supset B_{n-1, -m-1}(\mu).
\] (C.31)

Note that in particular, \( N_{n, \mu}(t) \neq 0 \) if and only if \( \mu = (m) \) for some \( m \in \mathbb{Z}_{\geq 0} \).

C.3 Duality of functions \( p_n(x; s|q, t) \) and \( f_n(x; s|q, t) \).

Lemma C.2. Let \( \lambda \) and \( \mu \) satisfy \( \ell(\lambda^{(i)}) \), \( \ell(\mu^{(i)}) \) \leq n_i.

\[
\left[ x^{-\lambda} \lambda \mu f_{|n|}(x; s(\lambda)|q, q/t) \right]_{x,1} \delta_{\lambda, \mu}.
\] (C.32)

Here, \( s(\lambda) = (s_j(\lambda))_{1 \leq j \leq |n|} \), \( s_{i,k}|n\) = \( q^{\lambda^{(i)}_k} t^{1-k} u_i \).

Proof. We denote the LHS by \( F(\lambda|\mu) \). Inserting the Macdonald operator \( D^1_n(s; q, q/t) \) and integrating by parts give
\[
\left[ x^{-\lambda} \lambda(f_{|n|}(x; s(\lambda)|q, q/t) D^1_n(\tilde{s}; q, q/t) x^{\mu} p_{|n|}(x; s(\mu)|q, q/t) \right]_{x,1} = \left[ D^1_n(\tilde{s}; q, q/t) x^{-\lambda} f_{|n|}(x; s(\lambda)|q, q/t) x^{\mu} p_{|n|}(x; s(\mu)|q, q/t) \right]_{x,1},
\]
where \( \tilde{s}_{i,k} = u_i t^{1-k} \). The LHS gives \( \epsilon_\mu F(\lambda|\mu) \), and the RHS gives \( \epsilon_\lambda F(\lambda|\mu) \). Thus we can show \( F(\lambda|\mu) = C(\lambda) \delta_{\lambda, \mu} \) with the coefficient \( C(\lambda) = F(\lambda|\lambda) \). We can show \( C(\lambda) = 1 \) because both \( f_{|n|}(x; s(\lambda)|q, q/t) \) and \( p_{|n|}(x; s(\mu)|q, q/t) \) are in \( \mathbb{C} \) \( s _{i,k} |n] \[ x_2 / x_1, x_3 / x_2, \ldots, x_n / x_{n-1} ] \) and so is their product. \( \square \)

C.4 Some formulas to prove (4.44). The coincidence between (4.42) and (4.44) can be identified with the following equation.

Proposition C.3.

\[
\frac{R_n^n}{R_{\mu}^n} N_{|\mu|^m} s_1, \ldots, s_{|n|^m} \prod_{1 \leq i < j \leq |n|} \frac{(q s_j / t s_i) |\lambda_i - |\lambda_j |}{(q s_j / s_i) |\lambda_i - |\lambda_j |} = N \prod_{i=1}^N \epsilon^{-(\lambda^{(i)}|\mu^{(i)})(\lambda^{(i)}|\mu^{(i)})(t^{n_i})-(N-i)|\mu^{(i)}|}
\]
\[
\times (\lambda^{(i)}-|\mu^{(i)}|) \sum_{1 \leq i < j \leq N} (v_i / v_j)^{-(\lambda^{(i)}+|\mu^{(i)}|)(t^{2n_i}+|\lambda^{(i)}| q^{(i)(i)})}
\]
\[
\times \prod_{i=1}^{N-N} \left( \frac{f_{u^{(i)}}}{f^{u^{(i)}}} \right)^{N-i} \prod_{1 \leq i < j \leq N} (v_i / v_j)^{-(\lambda^{(i)}+|\mu^{(i)}|)(t^{n_j}+v_i / v_j)}
\]
\[
\times \prod_{i=1}^{N} \left( \frac{1}{c^{u^{(i)}}_{\mu^{(i)}}} \right) \prod_{1 \leq i < j \leq N} N_{\lambda^{(i)} \mu^{(i)}} (q v_j / t v_i) \prod_{1 \leq i < j \leq N} N_{\mu^{(i)} \mu^{(i)}} (q t^{-n_i+n_j} v_i / v_j).
\] (C.34)
Lemma C.4. With $n = \ell(\lambda)$ and $m = \ell(\mu)$, the following holds:

\[
\frac{N_{\lambda\mu}(t^n)}{c_{\lambda}c_{\mu}'} = (-t/q)^{|\mu|}r^n(\mu)-2n(\lambda)-|\lambda|(t^n)|\lambda|+|\mu|q^{-n(\mu')}
\times \prod_{1 \leq i < j \leq n} (qs_j/qs_i)_{\lambda_i-\lambda_j} \prod_{i=1}^{m} (q/t)_{\mu_i} \prod_{j=1}^{n\mu'-j} \prod_{i=1}^{m\mu'-j} (qs_{j+n}/qs_{i})_{\mu_j} \prod_{1 \leq i < j \leq m} (qs_i/qs_j)_{\mu_j},
\]

where

\[
s_i = q^{\lambda_i}t^{1-i} \quad (i = 1, \ldots, n), \quad s_{n+i} = t^{1-n-i} \quad (i = 1, \ldots, m), \quad \text{and} \quad \sigma_i = q^{\mu_i}t^{1-i}.
\]

The equality that we obtain by removing the factors of this type from the both sides of (C.34), can be shown by using following relations.

Lemma C.5. Under the $\lambda_i \to \lambda_i + 1$ or $\mu_i \to \mu_i + 1$, we also have the following induction relations,

\[
\frac{N_{\lambda+1\mu}(u)}{N_{\lambda,\mu}(u)} = (1 - ut^{\ell(\mu)}\chi_x) \frac{\ell(\mu)}{1 - u^{\mu}}t^{1-\chi_x}, \quad \text{(C.36)}
\]

\[
\frac{N_{\lambda,\mu+1}(u)}{N_{\lambda,\mu}(u)} = (1 - t^{1-\ell(\lambda)}u/q\chi_y) \frac{\ell(\lambda)}{1 - u^{\lambda}}t^{1-\chi_y}, \quad \text{(C.37)}
\]

where

\[
\chi_x = q^{\lambda_i}t^{1-i}, \quad \chi_y = q^{\mu_i}t^{1-i}.
\]

Notation C.6. In what follows in this proof, we set

\[
s[i,k]_m = q^{\lambda(i)k}t^{1-k}v_i \quad (1 \leq k \leq n_i, \ i = 1, \ldots, N), \quad \text{(C.39)}
\]

\[
s[n+i,k]_m = t^{1-n_i-k}v_i \quad (1 \leq k \leq m_i, \ i = 1, \ldots, N), \quad \text{(C.40)}
\]

\[
\sigma[i,k]_m = q^{\mu(i)k}t^{1-k}v_i \quad (1 \leq k \leq m_i, \ i = 1, \ldots, N). \quad \text{(C.41)}
\]

Definition C.7. We set

\[
\hat{N}[n,\mu](s) := \prod_{i=1}^{N} (2\sum_{j=1}^{i-1} n_j + n_i - |n|) |\lambda(i)|
\times \prod_{l=1}^{N} \prod_{k=1}^{m_i} \prod_{j=1}^{n_j} (qs_{i,k}m_{i}t/qs_{j,i}m_{i}; q)_{\mu_k} \prod_{j=1}^{m_i} \prod_{l=1}^{m_j} (qs_{i,k}m_{i}/qs_{j,i}m_{i}; q)_{\mu_k}
\times \prod_{1 \leq k < l \leq N} \prod_{1 \leq i \leq m_i} (qt^{-n_{i+k}}/q^{\sigma[i,j]_m/\sigma[i,j]_m}); q)_{\mu_j}^{(l)}
\times \prod_{1 \leq k < l \leq N} \prod_{1 \leq j \leq m_j} (qs_{i,j}m_{i}/qs_{i,j}m_{i}; q)_{-\lambda_j^{(l)}+h_j^{(k)}}
\times \prod_{1 \leq k < l \leq N} \prod_{1 \leq j \leq n_i} (qs_{i,j}m_{i}/qs_{i,j}m_{i}; q)_{-\lambda_j^{(l)}+h_j^{(k)}}. \quad \text{(C.42)}
\]
Lemma C.8. Under $\lambda_k^{(i)} \to \lambda_k^{(i)} + 1$, that is, $s[i,k]_n \to q s[i,k]_n$, we have

$$\gamma - \sum_{j=1}^{N} (j-1)(\gamma^{(i)} + \delta_{j,i}) \frac{R^n_{\lambda + 1}^{(i)}(v)}{R^n_{\lambda}^{(i)}(v)} = \prod_{j=1}^{i-1} \frac{1 - q^{-\gamma^{(i)} t^{-n_j + 1}} / \nu_{ij}}{1 - q^{-\gamma^{(i)} t^{-n_j + k}} / \nu_{ij}},$$

with $\nu_{ij} = v_i / v_j$, and

$$\frac{\hat{N}[n|m]_{[\mu]}(\ldots, q s[i,k]_n, \ldots)}{\hat{N}[n|m]_{[\mu]}(s)} = \prod_{l=1}^{i-1} \frac{1 - t^{-n_l - m_l} v_l / s[i,k]_n}{1 - t^{-n_l} v_l / s[i,k]_n} \prod_{l=i+1}^{N} \frac{1 - t^{-n_l - m_l} v_l / s[i,k]_n}{1 - t^{-n_l} v_l / s[i,k]_n}
\times \left( \prod_{j=1}^{i-1} \prod_{1 \leq l \leq n_j} \frac{1 - q^{\mu_{j} s[i,j]_m / s[i,k]_n}}{1 - q^{\mu_{j} s[i,j]_m / t s[i,k]_n}} \right)
\times \left( \prod_{j=i+1}^{N} \prod_{1 \leq l \leq n_j} \frac{1 - t s[i,j]_m / s[i,j]_l}{1 - s[i,k]_m / s[i,j]_l} \right).$$

Lemma C.9. Similarly, under $\mu_k^{(i)} \to \mu_k^{(i)} + 1$, that is, $\sigma[i,k]_n \to q \sigma[i,k]_n$, we have

$$\frac{\hat{N}[n|m]_{[\mu + 1^{(i)}]}(s)}{\hat{N}[n|m]_{[\mu]}(s)} = \prod_{j=1, j \neq l}^{N} \prod_{l=1}^{n_j} \frac{1 - q^{\mu_{k}^{(i)} + 1} s[i,k]_m / s[i,l]_n}{1 - q^{\mu_{k}^{(i)} + 1} s[i,k]_m / s[i,l]_n}
\times \prod_{j=1}^{i-1} \left( \frac{1 - q^{\mu_{k}^{(i)} + 1} t^{-k_l - n_j + n_j} v_{ij}}{1 - q^{\mu_{k}^{(i)} + 1} t^{-n_j + n_j} v_{ij}} \prod_{l=1}^{m_j} \frac{1 - t^{-n_j + n_l} \sigma[i,j]_m / t \sigma[i,k]_n}{1 - t^{-n_j + n_l} \sigma[i,j]_m / \sigma[i,k]_n} \right)
\times \prod_{j=i+1}^{N} \left( \frac{1 - t^{-n_j + n_j} v_{ij} \chi_y}{1 - q t^{-n_j + n_j} t^{m_j} v_{ij} \chi_y} \prod_{l=1}^{m_j} \frac{1 - t^{-n_j + n_l} \sigma[i,k]_m / \sigma[i,j]_m}{1 - t^{-n_j + n_l} \sigma[i,k]_m / t \sigma[i,j]_m} \right).$$

Combining these identities, we complete the proof of (C.34).

Appendix D. Some Proofs of Lemmas and Propositions

D.1 Proof of Proposition 3.12. By the operator product formulas (C.1)–(C.6), the operator $\Lambda^{(j)}(z)$ with $j \neq i, i+1$ does not contribute in the commutation relation, and it can be shown that

$$\left[ : \Lambda^{(i)}(z) \Lambda^{(i+1)}(\gamma^{-2}z) :, S^{(i)}(w) \right] = 0. \quad (D.1)$$

Hence, it is enough to consider the relation only with $u_i \Lambda^{(i)}(z) + u_{i+1} \Lambda^{(i+1)}(z)$. 
We have
\[ \Lambda^{(i)}(z)S^{(i)}(w) - S^{(i)}(w)\Lambda^{(i)}(z) = (1 - t)\delta \left( \frac{tw}{qz} \right) : \Lambda^{(i)}(tw/q)S^{(i)}(w) : , \] (D.2)
and
\[ \Lambda^{(i+1)}(z)S^{(i)}(w) - S^{(i)}(w)\Lambda^{(i+1)}(z) = (1 - t^{-1})\delta \left( \frac{tw}{z} \right) : \Lambda^{(i+1)}(tw)S^{(i)}(w) : = (1 - t^{-1})\delta \left( \frac{tw}{z} \right) : \Lambda^{(i)}(tw)S^{(i)}(wq) : . \] (D.3)

Therefore, by the property \( g_i(qz) = \frac{u_{i+1}}{iu_i} g_i(z) \) with respect to the \( q \)-difference, we obtain
\[ \left( u_i\Lambda^{(i)}(z) + u_{i+1}\Lambda^{(i+1)}(z) \right) S^{(i)}(w)g_i(w) - S^{(i)}(w) \left( tu_i\Lambda^{(i)}(z) + t^{-1}u_{i+1}\Lambda^{(i+1)}(z) \right) g_i(w) = (t - 1)u_i(T_q, w - 1)\delta \left( \frac{tw}{qz} \right) : \Lambda^{(i)}(tw/q)S^{(i)}(w) : g(w). \] (D.4)

\[ \square \]

D.2 Proof of Lemma 3.19. First, we show the relation for \( k = 0 \). In this proof, we write
\[ \Lambda^{(i_1, \ldots, i_r)}(z) =: \Lambda^{(i_1)}(z) \cdots \Lambda^{(i_r)}((q/t)^{r-1}z) : \] (D.5)
for \( i_1 < \cdots < i_r \). By the operator products (C.16)–(C.19) and the relation \( \Phi^{(0)}(w)\Lambda^{(1)}(tw) := \Phi^{(0)}(qw)\Psi^+(w) \), it can be shown that if \( i_1 = 1 \),
\[ \Lambda^{(i_1, \ldots, i_r)}(z)\Phi^{(0)}(x) - t^{-1} \frac{1 - (q/t)^rz/tx}{1 - z/tx} \Phi^{(0)}(x)\Lambda^{(i_1, \ldots, i_r)}(z) = (1 - t^{-1})\delta(tx/z) : \Lambda^{(i_2)}((q/t)tx) \cdots \Lambda^{(i_r)}((q/t)^{r-1}tx)\Phi^{(0)}(qx)\Psi^+(x) : . \] (D.6)
If \( i_1 \geq 2 \),
\[ \Lambda^{(i_1, \ldots, i_r)}(z)\Phi^{(0)}(x) - \frac{1 - (q/t)^{r-1}z/tx}{1 - z/tx} \Phi^{(0)}(x)\Lambda^{(i_1, \ldots, i_r)}(z) = 0. \] (D.7)

Thus, we obtain the relation in the case \( k = 0 \):
\[ X^{(r)}(z)\Phi^{(0)}(x) - \frac{1 - (q/t)^rz/tx}{1 - z/tx} \Phi^{(0)}(x)X^{(r)}(z) = u_1(1 - t^{-1})\delta(tx/z)Y^{(r)}(x)\Phi^{(0)}(qx)\Psi^+(x). \] (D.8)

Applying the screening operators to this relation from the right side, we have the case \( k \neq 0 \). Indeed, \( \Psi^+(x) \) commutes with \( S^{(i)}(y) \):
\[ \Psi^+(z)S^{(i)}(y) = S^{(i)}(y)\Psi^+(z). \] (D.9)
Noting that we have
\[ T_{q,x}g(x, y_1, \ldots, y_k) = \frac{u_1}{u_{k+1}} g(x, y_1, \ldots, y_k), \quad (D.10) \]
and by virtue of (D.9) and commutativity of the screening operators, we can establish the relation for general \( k \).

We can show the commutativity of the screening operators as follows. First, it is clear that
\[ \Phi^{(0)}(x) \cdot \left[ X^{(r)}(z), S^{(1)}(y_1) \cdots S^{(k)}(y_k)g(x, y_1, \ldots, y_k) \right] \]
\[ = \sum_{i=1}^{k} \Phi^{(0)}(x)S^{(1)}(y_1) \cdots \left[ X^{(r)}(z), S^{(i)}(y_i) \right] \cdots S^{(k)}(y_k)g(x, y_1, \ldots, y_k). \quad (D.11) \]

By calculating the commutation relation as in the proof of Proposition 3.12, the RHS of (D.11) consists of terms as
\[ (1 - T_{q,y_i}) \delta \left( \frac{ty_{i+2}^q}{q^2} \right) \Phi^{(0)}(x)S^{(1)}(y_1) \cdots \]
\[ \cdots : \Lambda^{(j_1, \ldots, j_{i+1}, j_{i+2}, \ldots, j_r)}(ty_{i+2}^q / q) S^{(i)}(y_i) : \cdots S^{(k)}(y_k)g(x, y_1, \ldots, y_k). \quad (D.12) \]

Note that \( j_{i+2} \geq i + 2 \). Let us investigate the positions of poles in \( y_i \). Combining the \( \theta \)-functions containing \( y_i \) in \( g(x, y_1, \ldots, y_k) \) and the operator products among screening operators and \( \Phi^{(0)}(x) \), there appears the factor
\[ \frac{1}{\theta_q(ty_i / y_i - 1) \theta_q(ty_i + 1 / y_i)} \frac{1}{(ty_i / y_i - 1) \theta_q(ty_i + 1 / y_i)} \]
\[ = \frac{1}{(qy_i - y_i / ty_i) \infty (qy_i / ty_i + 1) \infty} \frac{1}{(y_i / y_i - 1) \theta_q(ty_i + 1 / y_i) \infty}, \quad y_0 := x. \quad (D.13) \]

From \( \Phi^{(0)}(x) \) and \( \Lambda^{(i)}(ty_i / q) \) in \( \Lambda^{(j_1, \ldots, j_{i+1}, j_{i+2}, \ldots, j_r)} \), we have
\[ \frac{1 - t^2y_i / qx}{1 - y_i / tx}. \quad (D.14) \]

Noticing that for \( i \geq 2 \), the operator product of \( S^{(i-1)}(y_{i-1}) \) and \( \Lambda^{(i)}(ty_i / q) \) are
\[ S^{(i-1)}(y_{i-1}) \Lambda^{(i)}(ty_i / q) = \frac{1 - ty_i / qy_{i-1}}{1 - y_i / qy_{i-1}} : S^{(i-1)}(y_{i-1}) \Lambda^{(i)}(ty_i / q) :, \quad (D.15) \]

we have the following set of poles of (D.11) in \( y_i \):
\[ y_i = tx, \quad (D.16) \]
\[ y_i = t - q^{2+n} y_{i-1}, \quad y_i = q^{n} y_{i+1}, \quad (D.17) \]
\[ y_i = t q^{-1-n} y_{i+1} \quad (i \geq 1, n = 0, 1, 2, \ldots), \quad (D.18) \]

and
\[ y_i = q^{-n+1} y_{i-1} \quad \text{for} \ i \geq 2, \quad (D.19) \]
\[ y_i = q^{-n} x \quad \text{for} \ i = 1 \quad (n = 0, 1, 2 \ldots). \quad (D.20) \]
If $r = 1$, it gives us the all poles. In general, this list does not exhaust the possible poles. In case $r \geq 2$, from $S^{(j)}(y_j)$ and $\Lambda^{(jm)}$ in $\Lambda^{(j_1, \ldots, j_{\ell-1}, j_{\ell+2}, \ldots, j_r)}$ with $m \neq \ell + 1$, we have extra poles. From the operator product formulas for them, we have

$$\prod_{m=1}^{\ell} \frac{1 - t^{-1}(q/t)^{-\ell+m-1}y_i/y_{jm}}{1 - (q/t)^{-\ell+m-1}y_i/y_{jm}} \frac{1 - (q/t)^{-\ell+m-2}y_i/y_{jm-1}}{1 - q^{-1}(q/t)^{-\ell+m-1}y_i/y_{jm-1}} \times \prod_{m=\ell+2}^r \frac{1 - t(q/t)^{\ell-m+1}y_{jm}/y_i}{1 - (q/t)^{\ell-m+1}y_{jm}/y_i} \frac{1 - (q/t)^{\ell-m+2}y_{jm-1}/y_i}{1 - q(q/t)^{\ell-m+1}y_{jm-1}/y_i}, \quad (D.21)$$

In addition, from $\Phi^{(0)}(x)$ and $\Lambda^{(jm)}$, the following factor arises:

$$\prod_{m \neq \ell+1} 1 - t^{-1}(q/t)^{-\ell+m-1}y_i/x \quad \frac{1 - t q^{-1}(q/t)^{-\ell+m-1}y_i}{1 - q^{-1}(q/t)^{-\ell+m-1}y_i}. \quad (D.22)$$

Summarizing these, we can show that the poles in $y_i$ are in the following positions (Though not all following points are poles, all poles should be in the followings or (D.16)-(D.20)). For $i \geq 1$,

$$y_i = (q/t)^{-n-1}y_{j+1}, \quad y_i = q(q/t)^{-n-1}y_j \quad (j > i), \quad (D.23)$$

$$y_i = q(q/t)^{-n}x, \quad (n = 0, 1, 2 \ldots), \quad (D.24)$$

and for $i \geq 2$,

$$y_i = (q/t)^{n+1}y_j \quad (1 \leq j < i), \quad y_i = q(q/t)^{n+1}y_{j-1} \quad (2 \leq j < i), \quad (D.25)$$

$$y_i = q(q/t)^{n+1}x \quad (n = 0, 1, 2 \ldots). \quad (D.26)$$

For the given integration contour, the poles (D.17), (D.25) and (D.26) are in the disk $\{z; |z| < |qy_i|\}$. On the other hand, the poles (D.16), (D.18), (D.19), (D.20), (D.23) and (D.24) are in $\{z; |z| > |y_i|\}$. Therefore, the change of variable $y_i \rightarrow qy_i$ is not affected by these poles, and the commutation relation (D.11) becomes zero after the integrals. □

**D.3 Proof of Proposition 3.27.** By taking the constant terms of $V^{(n)}(x_1, \ldots, x_{|m|})|0\rangle$ with respect to $x_i$, the proportional constant $R^{\lambda}_{\lambda}$ is calculated as the expansion coefficient in front of $|Q_\lambda\rangle$ in the basis of generalized Macdonald functions. We first consider only the operators that contain the creation operators $a_{-n}^{(N)}$’s with respect to the $N$-th Fock space. That is, we take the constant terms of

$$\prod_{i=1}^{n_N} y_i^{-(N)} \times \Phi^{(N-1)}(y_1, 0) \cdots \Phi^{(N-1)}(y_{n_N}, 0)|0\rangle$$

$$= \prod_{i=1}^{n_N} y_i^{-(N)} \oint \prod_{1 \leq i \leq n_N \leq N} \frac{dy_i, m}{2\pi \sqrt{-1} y_i, m} \sum_{(r_i, m) \in \tilde{M}} \prod_{i=1}^{n_N} \prod_{m=1}^{N-1} R_{r_i, m}(a_{-1}^{(N)})(y_i, m) y_i, m \quad (D.27)$$

$$\times \Phi^{(0)}(y_1, 0)S^{(1)}(y_1, 1) \cdots S^{(N-1)}(y_{1}, N-1) : \cdots \times \Phi^{(0)}(y_{n_N}, 0)S^{(1)}(y_{n_N}, 1) \cdots S^{(N-1)}(y_{n_N}, N-1) : |0\rangle.$$
Here, we used the expansion formula (3.24), and \( \tilde{M} = \text{Mat}(n_N, N - 1; \mathbb{Z}) \) is the set of \( n_N \times (N - 1) \) matrices with integral entries. We denote \( x_{i[N, i]a} \) by \( y_{i,0} \) in this proof. Further we set

\[
\alpha_{i,j}^{(k)} = t^{-n_i+i} \frac{u_l}{u_k},
\]

(D.28)

\[
R_r(\alpha) = \frac{(\alpha; q)_r}{(q^{\alpha/\ell}; q)_r}.
\]

(D.29)

Let \( C(z) = \sum_{k \geq 0} C_k z^k \), \( \tilde{C}(z) = \sum_{k \geq 0} \tilde{C}_k z^k \) and \( C^{(\pm)}(z) = \sum_{k \geq 0} C_k^{(\pm)} z^k \) be the formal power series defined by

\[
C(z) = \frac{(qz/\ell; q)_\infty}{(\ell z; q)_\infty}, \quad \tilde{C}(z) = (1 - z) \frac{(qz/\ell; q)_\infty}{(z; q)_\infty}, \quad C^{(+)}(z) = \frac{(qz; q)_\infty}{(qz/\ell; q)_\infty}, \quad C^{(-)}(z) = \frac{(qz; q)_\infty}{(qz/\ell; q)_\infty}.
\]

(D.30)

(D.31)

These series correspond to the operator product formulas among \( \Phi^{(0)} \) and \( S^{(i)} \)'s. Moreover, we write

\[
E_i^{(m)}(e) = - \sum_{s=1}^{N} e_s^{(m)} + \sum_{s=1}^{N} e_s^{(m)},
\]

(D.32)

\[
K_i^{(m)}(k, \ell) = - \sum_{s=1}^{N} \ell_s^{(m-1)} + \sum_{s=1}^{N} k_s^{(m)},
\]

(D.33)

\[
L_i^{(m)}(k, \ell) = - \sum_{s=1}^{N} k_s^{(m)} + \sum_{s=1}^{N} \ell_s^{(m-1)},
\]

(D.34)

for \( e = (e_{i,j}^{(m)})_{0 \leq m \leq N-1, i, j} \), \( k = (k_{i,j}^{(m)})_{0 \leq m \leq N-1, i, j} \) and \( \ell = (\ell_{i,j}^{(m)})_{0 \leq m \leq N-2, i, j} \). Here \( M_n \) is the set of strictly upper triangular \( n \times n \) matrices with nonnegative integral entries. With these notations, (D.27) can be rewritten as

\[
\prod_{i=1}^{n_N} y_i^{(-i)}_{1,0} \cdot \oint_{1 \leq i \leq n_N, 1 \leq m \leq N-1} \frac{dy_{i,m}}{2\pi \sqrt{-1} y_{i,m}} \sum_{(r_{i,m}) \in \tilde{M}} \prod_{i=1}^{n_N} \prod_{m=1}^{N-1} R_{r_{i,m}}(\alpha_i^{(N)})(\frac{y_{i,m}}{y_{i,m-1}})^{r_{i,m}}
\]

\[
\times \prod_{1 \leq i < j \leq n_N} C_{j,0}(y_{j,0})^{N-1} \prod_{m=1}^{N-1} C^{(-)}(y_{j,m-1}) \tilde{C}(y_{j,m}) C^{(+)}(y_{j,m})
\]

\[
\times \Phi^{(0)}(y_{i,0}) S^{(1)}(y_{i,1}) \cdots S^{(N-1)}(y_{i,N-1}) : \{0\}
\]

\[
= \oint_{1 \leq i \leq n_N, 1 \leq m \leq N-1} \frac{dy_{i,m}}{2\pi \sqrt{-1} y_{i,m}} \sum_{(r_{i,m}) \in \tilde{M}} \prod_{i=1}^{n_N} \prod_{m=1}^{N-1} \frac{y_i^{(-i)}_{1,0} + E_i^{(0)}(e) + K_i^{(1)}(k, \ell) - r_{i,1} + a_i^{(0)}}{1 - \lambda_j^{(N)} + e_{i,\ell,k} M_{N-1}}
\]
\[ \times \prod_{m=1}^{N-2} \prod_{n=1}^{n_N} E_i^{(m)}(e) + k_i^{(m+1)}(k, \ell) + L_i^{(m)}(k, \ell) \rightarrow r_{i,m+1} + r_{i,m} + a_i^{(m)} \]

\[ \times \prod_{i=1}^{n_N} R_i^{(N-1)}(e) + L_i^{(N-1)}(k, \ell) + r_{i,N-1} + a_i^{(N-1)} \]

\[ \times \prod_{i=1}^{n_N} \prod_{m=1}^{N-1} R_{r_i,m}^{(N)}(\alpha_i^{(N)}) \cdot \prod_{1 \leq i < j \leq n_N} C_{\xi_{i,j}}(0) \prod_{m=1}^{N-1} C_{\xi_{i,j}}^{(-)}(m) C_{\xi_{i,j}}^{(+)}(m-1) \]

\[ \times : \prod_{i=1}^{n_N} \Phi_0^{(0)}(0) S^{(1)}(0) \cdots S^{(N-1)}(0, -a_i^{(N-1)} : |0) \cdot \]

Here, \( \hat{M}_{\geq 0} = \text{Mat}(n_N, N - 1; \mathbb{Z}_{\geq 0}) \), \( \Phi_0^{(0)}(z) = \sum_{n \in \mathbb{Z}} \Phi_n^{(0)} z^{-n} \), and \( S^{(i)}(z) = \sum_{n \in \mathbb{Z}} S_n^{(i)} z^{-n} \). Since the integral gives us the constant terms in \( y_{i,m} \), we have

\[ r_{i,N-1} = -E_i^{(N-1)} - L_i^{(N-1)} - a_i^{(N-1)} \]

\[ r_{i,N-2} = -E_i^{(N-1)} - L_i^{(N-2)} - K_i^{(N-1)} - a_i^{(N-2)} + r_{i,N-1} \]

\[ = -E_i^{(N-1)} - E_i^{(N-2)} - L_i^{(N-1)} - L_i^{(N-2)} - K_i^{(N-1)} - a_i^{(N-1)} - a_i^{(N-2)} \]

\[ \ldots \]

\[ r_{i,1} = - \sum_{m=1}^{N-1} \left( E_i^{(m)} + L_i^{(m)} + K_i^{(m)} + a_i^{(m)} \right) + K_i^{(1)} \]

and

\[ a_i^{(0)} + E_i^{(0)} + \sum_{m=1}^{N-1} \left( E_i^{(m)} + L_i^{(m)} + K_i^{(m)} + a_i^{(m)} \right) - \lambda_i^{(N)} = 0. \]

Since \( \sum_{i=1}^{n_N} E_i^{(m)} = 0 \) and \( \sum_{i=1}^{n_N} (L_i^{(m)} + K_i^{(m)}) = 0 \), it is shown that

\[ |\lambda^{(N)}| - \sum_{i=1}^{n_N} a_i^{(N-1)} = \sum_{m=0}^{N-2} \sum_{i=1}^{n_N} a_i^{(m)} \geq 0. \]

Therefore, \( \sum_{i=1}^{n_N} a_i^{(N-1)} \) takes its maximum value \( |\lambda^{(N)}| \) when \( a_i^{(m)} = 0 \) for all \( i \) and \( m \leq N - 2 \). Since only the operators : \( \prod_{i=1}^{n_N} S_{a_i^{(N-1)}} \) : have the creation operators acting on the \( N \)-th Fock space, it is clear that the maximum degree in the \( N \)-th Fock component is \( |\lambda^{(N)}| \).

Before taking expansion coefficients in the basis of generalized Macdonald functions, we investigate the one in the basis of products of ordinary Macdonald functions \( \prod_{i=1}^{N} Q_{\mu_i}^{(i)}(a_i^{(i)} : |0) \). Consider the terms of level \( |\lambda^{(N)}| \) with respect to the \( N \)-th Fock space. Then, \( a_i^{(N-1)} \) satisfies

\[ a_i^{(N-1)} = \lambda_i^{(N)} - E_i^{(0)} - \sum_{m=1}^{N-1} \left( E_i^{(m)} + L_i^{(m)} + K_i^{(m)} \right). \]
Furthermore, by the form of \( E_i^{(m)} \), \( K_i^{(m)} \) and \( L_i^{(m)} \), it can be seen that only the following vectors appear:

\[
: \prod_{i=1}^{n_N} S_{(N-1)}^{(i)} : |0\rangle, \quad \mu \geq \lambda^{(N)}. \tag{D.42}
\]

By the Pieri formula (Fact B.4), we can write the terms of level \( |\lambda^{(N)}| \) with respect to the \( N \)-th Fock space as

\[
: \prod_{i=1}^{n_N} S_{(N-1)}^{(i)} : |0\rangle / \left\{ a_{(N-1)}^{(N)} |n, \mu\rangle \right\} = \gamma^{(N-1)\mu} \prod_{i=1}^{n_N} g^{(N)}_{\mu_i} |0\rangle,
\]

\[
= \gamma^{(N-1)\mu} Q_{\mu} (a_{-n}^{(N)}) |0\rangle + \sum_{\rho > \mu} (\text{const.}) Q_{\rho} (a_{-n}^{(N)}) |0\rangle.
\]

(D.43)

Here, \( g^{(N)}_{\mu} \) is defined by

\[
\sum_{n \geq 0} z^n g^{(N)}_{\mu} = : \exp \left( \sum_{n > 0} \frac{1 - q^n}{1 - q^n a_{(N)}^{(N)}} \right) :. \tag{D.44}
\]

Therefore, there appears \( Q_{\lambda^{(N)}} (a_{-n}^{(N)}) |0\rangle \) only in the case that \( \mu = \lambda^{(N)} \) on (D.42), i.e., the case that

\[
e^{(m)}_{i,j} = \ell^{(m)}_{i,j} = k^{(m)}_{i,j} = 0 \tag{D.45}
\]

for all \( i, j, m \). Then

\[
r_{i,m} = -a_{i}^{(N-1)} = -\lambda^{(N)}_{i}. \tag{D.46}
\]

for all \( i, m \).

From the above discussion, we have

\[
\bigg. \left. x^{-\lambda} V^{(n)} (x_1, \ldots, x_{[m]}) |0\rangle \right| \text{const. of } x_i = \prod_{k=1}^{N-1} \prod_{i=1}^{n_k} x_i^{(k)} |0\rangle \left\{ \gamma^{(N-1)\mu^{(N)}} \prod_{i=1}^{n_N} \prod_{m=1}^{N-1} R_{-\lambda^{(N)}} (a_{(N)}^{(N-1)}) Q_{\lambda^{(N)}} (a_{-n}^{(N)}) |0\rangle + \sum_{\mu^{(N)} > \lambda^{(N)}} (\text{const.}) Q_{\mu^{(N)}} (a_{-n}^{(N)}) |0\rangle + O \left( \frac{|\mu^{(N)}|}{|\lambda^{(N)}|} \right) \right. \left. \text{and } |\mu^{(N)}| = |\lambda^{(N)}| \right) \bigg. \text{const. of } x_i \bigg. \right).
\]

(D.47)

Here \( O \ (P) \) expresses the terms \( \prod_{i=1}^{N} Q_{\mu^{(i)\mu}} (a_{-n}^{(i)}) |0\rangle \) with \( \mu \) satisfying the proposition \( P \). By repeating the similar argument \( N - 1 \) times, we obtain

\[
ex^{-\lambda} V^{(n)} (x_1, \ldots, x_{[m]}) |0\rangle \left| \text{const. of } x_i \right. = \left. \left( \gamma^{\sum_{i=1}^{N} (i-1)\mu^{(i)}} \prod_{k=2}^{N} \prod_{i=1}^{n_k} \prod_{m=1}^{k-1} R_{-\lambda^{(k)}} (a_{(k-1)}^{(k)}) \right) \prod_{i=1}^{N} Q_{\lambda^{(i)}} (a_{-n}^{(i)}) |0\rangle \right. \%
\]
\[ + \sum_{\mu^{(j)} \geq \lambda^{(j)}(\forall j), \mu^{(k)} \neq \lambda^{(k)}} \text{(const.)} \prod_{i=1}^{N} Q_{\mu^{(i)}}(a_{-n}^{(i)} | 0) + O \left( \mu^{\ast} < \lambda \right). \tag{D.48} \]

The existence theorem of generalized Macdonald functions (Fact 3.6) shows that the coefficient in front of \( Q_\lambda \) in the basis of generalized Macdonald functions is the same as the one in front of \( \prod_{i=1}^{N} Q_{\lambda^{(i)}}(a_{-n}^{(i)} | 0) \) in (D.48). \( \square \)

**D.4 Proof of Lemma 4.11.** First, it can be shown that

\[
\prod_{1 \leq i < j \leq n+m} \frac{(q^{-\theta_j} q s_j / t s_i; q)_{\theta_i}}{(q^{-\theta_i} q s_j / q^{-\theta_j} s_i; q)_{\theta_i}} = \prod_{1 \leq i < j \leq n} \frac{(q s_j / t s_i; q)_{\infty}}{(q^{-\theta_j} q s_j / q^{-\theta_i} s_i; q)_{\infty}} \prod_{1 \leq i < j \leq n} \frac{(q s_j / s_i; q)_{\theta_i}}{(q s_j / t s_i; q)_{\theta_i}}. \tag{D.49}
\]

and

\[
\prod_{1 \leq i < j \leq n} \frac{(q s_j / t s_i; q)_{\infty}}{(q^{-\theta_j} q s_j / q^{-\theta_i} s_i; q)_{\infty}} = \prod_{1 \leq i < j \leq n} \frac{(q s_j / s_i; q)_{\theta_i}}{(q s_j / t s_i; q)_{\theta_i}}. \tag{D.50}
\]

By (D.49) and (D.50), we have

\[
\prod_{1 \leq i < j \leq n} \frac{(q s_j / t s_i; q)_{\infty}}{(q^{-\theta_j} q s_j / q^{-\theta_i} s_i; q)_{\infty}} \cdot d_{n+m}(\theta; s | q, t) = \prod_{1 \leq i < j \leq n} \frac{(q s_j / s_i; q)_{\theta_i}}{(q s_j / t s_i; q)_{\theta_i}}. \tag{D.51}
\]

Furthermore, we can see

\[
\prod_{k=1}^{m-1} \prod_{i=1}^{n+k} \frac{(q^{-\theta_{n+k}} s_{n+k} / q^{-\theta_i} s_i; q)_{\sigma_k}}{(q^{-\theta_{n+k}} s_{n+k} / q^{-\theta_i} s_i; q)_{\sigma_i}} = \lim_{h \to 1} \prod_{k=1}^{m-1} \prod_{i=1}^{n+k} \frac{(q^{\rho_k} s_{n+k} / t s_i; q)_{\theta_i}}{(h q^{\rho_k} s_{n+k} / s_i; q)_{\theta_i}} \frac{(q s_{n+k} / s_i; q)_{\rho_k}}{(h q s_{n+k} / s_i; q)_{\rho_k}} \frac{(q s_{n+k} / s_i; q)_{\theta_i}}{(q s_{n+k} / t s_i; q)_{\theta_i}}. \tag{D.52}
\]

and

\[
\prod_{1 \leq i < j \leq m-1} \frac{(q^{-\sigma_k} q^{-\theta_{n+k}} s_{n+k} / q^{-\theta_i} s_i; q)_{\sigma_i}}{(q^{-\sigma_k} q^{-\theta_{n+k}} s_{n+k} / q^{-\theta_i} s_i; q)_{\sigma_i}} = \prod_{1 \leq i < j \leq m-1} \frac{(q s_{n+k} / s_{n+k} + i ; q)_{\rho_i}}{(q^{-\rho_k} s_{n+k} / s_{n+k} + i ; q)_{\rho_i}} \frac{(q s_{n+k} / s_{n+k} + i ; q)_{\theta_i}}{(q^{-\rho_k} s_{n+k} / s_{n+k} + i ; q)_{\theta_i}} \times t^{\theta_{n+k}}. \tag{D.53}
\]

Combining (D.51), (D.52) and (D.53) yields Lemma 4.11. \( \square \)
D.5 Proof of Lemma 4.13. Set for short

\[ A = \prod_{k=1}^{m-1} \frac{q^{\nu_k + \rho_k} s_{n+k} - q^{\nu_m} s_{n+m}}{q^{\rho_k} s_{n+k} - s_{n+k}} \times \prod_{k=1}^{m-1} \frac{(t; q)_v (q q^{\rho_k} s_{n+k}/t s_{n+m}; q) v_k (t q^{-\rho_k} s_{n+m}/s_{n+k}; q) v_m}{(q; q)_v (q q^{\rho_k} s_{n+k}/s_{n+k}; q) v_k (q q^{-\rho_k} s_{n+m}/s_{n+k}; q) v_m}, \]  
\hspace{0.5cm} (D.54)

\[ B = \prod_{1 \leq k < l \leq m-1} \frac{q^{\nu_k + \rho_k} s_{n+k} - q^{\nu_l + \rho_l} s_{n+l}}{q^{\rho_k} s_{n+k} - q^{\rho_l} s_{n+l}} \cdot \prod_{k \neq l} \frac{(t q^{\rho_k - \rho_l} s_{n+k}/s_{n+l}; q) v_k}{(q q^{\rho_k - \rho_l} s_{n+k}/s_{n+l}; q) v_k}, \]  
\hspace{0.5cm} (D.55)

\[ C = \prod_{k=1}^{m-1} \prod_{i=1}^{n+m-1} \frac{h q^{\rho_k} s_{n+k}/t s_i; q) v_k}{(h q^{\rho_k} s_{n+k}/s_i; q) v_k}, \]  
\hspace{0.5cm} (D.56)

\[ D = (q/t; q)_v \prod_{i=1}^{n+m-1} \frac{(h qs_{n+m}/t s_i; q) v_m}{(hqs_{n+m}/s_i; q) v_m}. \]  
\hspace{0.5cm} (D.57)

Then \( \phi_{\nu}^{m, n+m-1} = ABCD \). First, we can get

\[ A = \prod_{k=1}^{m-1} (q/t)^{\theta_k} (t; q)_{\theta_k} (t q^{-\mu_k + \mu_m} s_{n+m}/s_{n+k}; q)_{\theta_k} (q q^{-\mu_k + \mu_m} s_{n+m}/s_{n+k}; q)_{\theta_k} \times \prod_{k=1}^{m-1} (t q^{-\mu_k} s_{n+m}/s_{n+k}; q)_{\mu_m} / (q^{-\mu_k} s_{n+m}/s_{n+k}; q)_{\mu_m}, \]  
\hspace{0.5cm} (D.58)

The first product in (D.58) reproduces the factors in \( d_m((\theta_i); (q^{\mu_i} s_{n+i})\mid q, t) \), i.e., the first product in (3.36). Next, we have

\[ \lim_{h \to 1} \tilde{N}_{\rho}^{n, m-1}(h; s_1, \ldots, s_{n+m-1}) C = N_{\mu}^{n, m-1}(s_1, \ldots, s_{n+m-1}) E, \]  
\hspace{0.5cm} (D.59)

where

\[ E = \prod_{1 \leq k < l \leq m-1} t^{-\theta_k} (t q^{-\mu_k + \mu_l} s_{n+l}/s_{n+k}; q)_{\theta_k - \theta_l} / (q^{-\mu_k + \mu_l} s_{n+l}/s_{n+k}; q)_{\theta_k - \theta_l}. \]  
\hspace{0.5cm} (D.60)

\( BE \) reproduces the factors in \( d_m((\theta_i); (q^{\mu_i} s_{n+i})\mid q, t) \):

\[ BE = \prod_{1 \leq i < j \leq m} \frac{(t q^{-\mu_k + \mu_l} s_{n+l}/s_{n+k}; q)_{\theta_k} (q q^{-\mu_k + \mu_l - \theta_i} s_{n+l}/t s_{n+k}; q)_{\theta_k}}{(q q^{-\mu_k + \mu_l - \theta_i} s_{n+l}/s_{n+k}; q)_{\theta_k} (q q^{-\mu_k + \mu_l - \theta_i} s_{n+l}/s_{n+k}; q)_{\theta_k}}, \]  
\hspace{0.5cm} (D.61)

This corresponds to the second product in (3.36). The product of \( N_{\mu}^{n, m-1}, D \) and the remaining factor in (D.58) is

\[ N_{\mu}^{n, m-1}(s_1, \ldots, s_{n+m-1}) \cdot \prod_{k=1}^{m-1} (t q^{-\mu_k} s_{n+m}/s_{n+k}; q)_{\mu_m} \cdot \lim_{h \to 1} D = N_{\mu}^{n, m}(s_1, \ldots, s_{n+m}). \]  
\hspace{0.5cm} (D.62)

Therefore, Lemma 4.13 follows.
Appendix E. Kac Determinant Revisited

The formula for the Kac determinant with respect to the vectors $|X_k\rangle$ has been discussed in [O]. That shows the fact that $|X_k\rangle$ form a basis on the Fock space (Fact 2.10). For the sake of reader’s convenience, we revisit the proof, clarifying the choice of the integral cycles. Here, we construct the $q$-invariance cycles by using the elliptic theta function.

**Definition E.1.** Let $1 \leq k \leq N-1$ and $u_k = q^s t^{-r} u_{k+1} \ (r, s \in \mathbb{Z}_{>0})$. Define the vector $|\chi^{(k)}_{r,s}\rangle \in \mathcal{F}_u$ by the integral

$$|\chi^{(k)}_{r,s}\rangle := \oint \frac{dz}{z} S^{(k)}(z_1) \cdots S^{(k)}(z_r) \prod_{i=1}^r \frac{\theta_q(t^{2i} u_k z_i / u_{k+1})}{\theta_q(t z_i)} |\emptyset\rangle. \tag{E.1}$$

Here and hereafter, we use the shorthand notation

$$\oint \frac{dz}{z} := \oint_{T} \prod_{i=1}^r \frac{dz_i}{2\pi \sqrt{-1} z_i}, \tag{E.2}$$

where the cycle is the $r$-dimensional torus $T : |z_1| = \cdots = |z_r| = 1$. Note that $u$ is the spectral parameter of the codomain of $S^{(k)}(z_1)$.

**Proposition E.2.** The vector $|\chi^{(k)}_{r,s}\rangle$ does not vanish. In particular, this is of level $rs$.

Let us prepare a lemma with respect to the symmetrization of theta functions. Set

$$\hat{F}_{r,s}(z_1, \ldots, z_r) := \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} \prod_{i=1}^r \frac{\theta_q(q t^{i-1} z_\sigma(i))}{\theta_q(t z_\sigma(i))} \cdot \prod_{1 \leq i < j \leq r} t^{-1} \frac{\theta_q(t z_\sigma(i)/z_\sigma(j))}{\theta_q(t^{-1} z_\sigma(i)/z_\sigma(j))}. \tag{E.3}$$

**Lemma E.3.**

$$\hat{F}_{r,s}(z_1, \ldots, z_r) = \frac{1}{r!} \prod_{i=2}^r \frac{\theta_q(t^i)}{\theta_q(t)} \cdot \prod_{1 \leq i < j \leq r} t^i \frac{\theta_q(z_i / z_j)}{\theta_q(t z_i / z_j)} \cdot \prod_{i=1}^r \theta_q(q^s t z_i). \tag{E.4}$$

As for the proof of this lemma, see the proof of Lemma 4 in [JLMP].

**Proof of Proposition E.2.** First, by using the operator products of screening currents and Lemma E.3, we have

$$|\chi^{(k)}_{r,s}\rangle = \oint \frac{dz}{z} \Delta(z) \prod_{1 \leq i < j \leq r} \frac{\theta_q(t z_i / z_j)}{\theta_q(z_i / z_j)} \cdot \prod_{i=1}^r \frac{\theta_q(q^{s} t^{2i-1} z_i)}{\theta_q(t z_i)} : S^{(k)}(z_1) \cdots S^{(k)}(z_r) : |\emptyset\rangle$$

$$= \oint \frac{dz}{z} \Delta(z) \prod_{1 \leq i < j \leq r} \frac{\theta_q(t z_i / z_j)}{\theta_q(z_i / z_j)} \cdot \prod_{i=1}^r \frac{1}{\theta_q(t z_i)} \hat{F}_{r,s}(z_1, \ldots, z_r) : S^{(k)}(z_1) \cdots S^{(k)}(z_r) : |\emptyset\rangle$$

$$= \frac{1}{r!} \prod_{i=2}^r \frac{\theta_q(t^i)}{\theta_q(t)} \oint \frac{dz}{z} \Delta(z) q^s z^{s(1-s)} \prod_{i=1}^r (t z_i)^{-s} : \prod_{i=1}^r S^{(k)}(z_i) : |\emptyset\rangle. \tag{E.5}$$

Here, $\Delta(z)$ is defined in (B.15). Note that $: \prod_{i=1}^r S^{(k)}(z_i) : |\emptyset\rangle$ can be regarded as the kernel function for the Macdonald functions. Hence, it is expanded in terms of the
Macdonald functions (See Fact B.3). Note that $\prod_{i=1}^r z_i^{-s}$ is the Macdonald polynomial with a rectangular Young diagram in $r$ variables. Therefore, (E.5) can be written as

$$q^{rs(1-s)/2} T^{-rs} \prod_{i=2}^r \frac{\theta_q(t^i)}{\theta_q(t)} \cdot \sum_{\lambda} P_{\lambda}(\alpha^{(k)}_{-n}) |0\rangle \langle P^{(r)}_{s}, Q^{(r)}_{\lambda}(z)\rangle_r$$

$$= q^{rs(1-s)/2} T^{-rs} \prod_{i=2}^r \frac{\theta_q(t^i)}{\theta_q(t)} \cdot P_{s}^{(r)}(\alpha^{(k)}_{-n}) |0\rangle \langle P^{(r)}_{s}, Q^{(r)}_{s}(z)\rangle_r,$$  

(E.6)

where $P^{(r)}_{\lambda}(z)$ and $Q^{(r)}_{\lambda}(z)$ denote the Macdonald polynomials in $r$ variables,

$$\alpha^{(k)}_{-n} := \gamma^k n (\gamma a^{(k)}_{-n} + a^{(k+1)}_{-n}),$$  

(E.7)

and $\langle,\rangle'$ denotes the scalar product defined in Appendix B. Since the Macdonald polynomials are pairwise orthogonal for $\langle,\rangle'$ and the inner product $\langle P^{(r)}_{s}, Q^{(r)}_{s}\rangle'_r$ can be evaluated (Fact B.6) to be nonvanishing. Thus, $|\chi_{r,s}^{(k)}\rangle \neq 0$ and of level $rs$. \hfill \Box

We show the following commutativity with the algebra $U(N)$. The proof is similar to the case corresponding to the Minimal model, given in [JLMP].

**Proposition E.4.** Let $r, s \in \mathbb{Z}_{\geq 0}$ and $k, j \in \{1, \ldots, r\}$. Further we assume $|t| < |q|$. Then

$$\left[ X^{(j)}(z), \int \frac{dw}{w} S^{(k)}(w_1) \cdots S^{(k)}(w_r) \prod_{i=1}^r \frac{\theta_q(t^{2i} w_{u_{ik+1}})}{\theta_q(t w_i)} \right] = 0,$$

where the spectral parameter of the codomain of $S^{(k)}(w_1)$ is $u$ with $u_k = q^s t^{-r} u_{k+1}$.

**Proof.** As in the proof of Proposition 3.12, it suffices to consider the relation only with $\Lambda^{(k)}(z) + \Lambda^{(k+1)}(z)$. By (D.4), we have

$$\left[ \Lambda^{(k)}(z) + \Lambda^{(k+1)}(z), \int \frac{dw}{w} S^{(k)}(w_1) \cdots S^{(k)}(w_r) \cdot \prod_{i=1}^r \frac{\theta_q(t^{2i} w_{u_{ik+1}})}{\theta_q(t w_i)} \right]$$

$$= \sum_{m=1}^r \int \frac{dw}{w} (t - 1) t^{m-1} u_k (T_{q,w_m} - 1) \delta \left( \frac{tw_m}{q z} \right) \Delta(w)$$

$$\times \prod_{1 \leq i < j \leq r} \frac{\theta_q(t w_i/w_j)}{\theta_q(w_i/w_j)} \cdot \prod_{i=1}^r \frac{\theta_q(q^s t^{-r} w_i)}{\theta_q(t w_i)} \times \prod_{i=1}^{m-1} \frac{1 - t^{-1} w_m/w_i}{1 - w_m/w_i}$$

(E.9)

By symmetrizing the variables $w_i$'s, we have

$$\frac{1}{r!} \sum_{m=1}^r \sum_{\sigma \in \mathfrak{S}_r} \int \frac{dw}{w} (t - 1) u_k (T_{q,w_{\sigma(m)}} - 1) \delta \left( \frac{tw_{\sigma(m)}}{q z} \right) \Delta(w).$$
By Lemma E.3, this can be rewritten as

\[
\frac{1}{r!} \sum_{l=1}^{r} \oint \frac{dw}{w} (t - 1) u_k (T_q, w_l - 1) \delta \left( \frac{tw_l}{q^s z} \right) \Delta(w)
\]

\times \prod_{i=2}^{r} \frac{\theta_q (t^i)}{\theta_q (t)} \cdot \prod_{i=1}^{r} q^{\frac{s(t+1)}{2}} (z_{i})^{-s} \cdot \prod_{i \neq l} \frac{1 - tw_i/w_l}{1 - w_i/w_l} : \Lambda^{(k)} (tw_l/q) \prod_{i=1}^{r} S^{(k)} (z_i) :.
\] (E.11)

In this expression, we have poles in \( w_l \) at \( w_l = 0, w_l = q^n tw_l \) and \( w_l = q^{-n+1} t^{-1} w_l \) \((i \neq l, n = 1, 2, \ldots)\). Since \(|t| < |q|\), they do not change the integral while we \( q \)-shift the cycle as \( w_l \to q w_l \). Therefore, the integral (E.11) is zero. □

Proposition E.2 and Proposition E.4 show the existence of the singular vectors of the algebra \( \mathfrak{U}(N) \).

**Corollary E.5.** The vector \( \chi_{r,s}^{(k)} \) is a singular vector of level \( rs \), i.e.,

\[
X^{(i)}_n \chi_{r,s}^{(k)} = 0
\] (E.12)

for all \( n > 0 \) and \( i = 1, \ldots, N \).

We revisit the proof of the following formula for the Kac determinant \( \det_n := \det \left( [X_\lambda | X_\mu] \right)_{\lambda, \mu \vdash n} \).
Proposition E.6. We have
\[
\det_n = \prod_{\lambda \vdash n} b_{\lambda}^{(t^1)}(q)b_{\lambda}^{(t^1)}(t^{-1}) \\
\times \prod_{1 \leq r, s \leq n, r \neq s} \left( (u_1 u_2 \cdots u_N)^2 \prod_{1 \leq i < j \leq N} (u_i - q^{s} t^{-r} u_j)(u_i - q^{r} t^{-s} u_j) \right)^{P^{(N)}(n-rs)}.
\]  
(E.13)

Here \(b_{\lambda}^{(q)} \) := \( \prod_{i \geq 1} \prod_{k=1}^{m_i} (1 - q^{k}) \), \(b_{\lambda}^{(t)}(q) := \prod_{i \geq 1} \prod_{k=1}^{m_i} (-1 + q^{k})\). \(P^{(N)}(n)\) is the number of \(N\)-tuples of Young diagrams of size \(n\), i.e., \#\{\(\lambda = (\lambda^{(1)}, \ldots, \lambda^{(N)})\) \mid |\(\lambda| = n\}.\) In particular, if \(N = 1\),
\[
\det_n = \prod_{\lambda \vdash n} b_{\lambda}(q)b_{\lambda}^{(t^1)}(t^{-1}) \times u_{1}^{2 \sum_{\lambda \vdash n} \ell(\lambda)}.
\]  
(E.14)

Proof. The inner product \(\langle X_{\lambda}^{(i)}|X_{\lambda}^{(j)} \rangle\) can be calculated by commutation relations of \(X_{n}^{(i)}\). The parameters \(u_{1}, \ldots, u_{N}\) arise from the eigenvalues of \(X_{0}^{(i)}\). Therefore, it can be seen that \(\langle X_{\lambda}^{(i)}|X_{\lambda} \rangle\) is a polynomial in \(\sum_{j=1}^{n} u_{ji} \)
\[
m_{(1)}(u_{1}, \ldots, u_{N}) = \sum_{j_1 < \cdots < j_i} u_{j_1} \cdots u_{j_i}
\]  
(E.15)

over \(\mathbb{Q}(q^{1}, q^{\frac{1}{2}}, t^{1})\), and thus so does the \(\det_n\). Define the action of the symmetric group \(\mathfrak{S}_{N}\) (the Weyl group of type \(A_{N-1}\)) on polynomials in \(u_{j}\) in the usual way. Since \(m_{(1)}(u_{1}, \ldots, u_{N})\) is invariant with respect to this action, \(\det_n\) is also invariant, i.e., a symmetric polynomial in \(u_{j}\).

Furthermore, let us introduce the new parameters \(u_{i}^{'}\) and \(u_{i}^{''}\) by
\[
\prod_{i=1}^{N} u_{i}^{'} = 1, \quad u_{i} = u_{i}^{'}u_{i}^{''}.
\]  
(E.16)

Then \(\langle X_{\lambda}^{(i)}|X_{\mu} \rangle\) can be decomposed as
\[
\langle X_{\lambda}^{(i)}|X_{\mu} \rangle = (u_{i}^{'})^{N} \sum_{k=1}^{N} k(\ell(\lambda^{(k)}) + \ell(\mu^{(k)})) \times (\text{polynomial in } u_{i}^{'}).\]
\]  
(E.17)

Therefore, \(\det_n\) can be written as
\[
\det_n = (u_{i}^{'})^{2} \sum_{\lambda \vdash n} \sum_{k=1}^{N} k(\ell(\lambda)) \times F(u_{1}^{',} \ldots, u_{N}^{'}),
\]  
(E.18)

where \(F(u_{1}^{',} \ldots, u_{N}^{'})\) is some polynomial in \(u_{i}^{'}\). Note that the maximum degree of \(F(u_{1}^{',} \ldots, u_{N}^{'})\) with respect to each \(u_{i}^{'}\) is \(2 \sum_{\lambda \vdash n} \sum_{k=1}^{N} \ell(\lambda^{(k)})\).

By Corollary E.5, it can be seen that for \(r, s \in \mathbb{Z}_{>0}\) with \(rs \leq n\), the Kac determinant \(\det_n\) has the factors
\[
(u_{k} - q^{s} t^{-r} u_{k+1})^{P^{(N)}(n-rs)} = (u_{i}^{''}(u_{k}^{'} - q^{s} t^{-r} u_{k+1}^{'}))^{P^{(N)}(n-rs)}
\]  
(E.19)
in the usual way. By the $\mathcal{S}_N$ invariance, $\det_n$ has also the factor

$$(u_i - q^{\pm s} t^r u_j)^{P^N(n-r)S} = \left(u''(u_i' - q^{\pm s} t^r u_j')\right)^{P^N(n-r)S} \quad (E.20)$$

for $i \neq j$. Noticing the degree of $F(u'_1, \ldots, u'_N)$, we can see that

$$\det_n = g_{N,n}(q,t) \times (u'')^{\sum_{i=1}^N n_i} \prod_{1 \leq i < j \leq N} \prod_{1 \leq r < s \leq n} \left((u_i' - q^s t^{-r} u_j')(u_i' - q^{-r} t^s u_j')\right)^{P^N(n-r)S}$$

$$= g_{N,n}(q,t) \prod_{1 \leq r < s \leq n} \left((u_1u_2 \cdots u_N)^2 \prod_{1 \leq i < j \leq N} (u_i - q^s t^{-r} u_j)(u_i - q^{-r} t^s u_j)\right)^{P^N(n-r)S}. \quad (E.21)$$

Here, $g_{N,n}(q,t) \in \mathbb{Q}(q^{\frac{1}{2}}, t^{\frac{1}{2}})$. Thus, we obtained the vanishing loci of the Kac determinant $\det_n$. The prefactor $g_{N,n}(q,t)$ has been evaluated in [O]. □

As a corollary of Proposition E.6, Fact 2.10 follows.

Appendix F. Examples

F.1 Examples of $|K_x\rangle$. We present examples of the transition matrix $\alpha_{\lambda,\mu}^{(\pm)}$ from $|K_x\rangle$ to the PBW-type basis $|X_\lambda\rangle$.

Examples of $\alpha_{\lambda,\mu}^{(+)}$ in the case $N = 1$:

| $\lambda$ \ $\mu$ | (1) | (2) | (1, 1) |
|-----------------|---|---|---|
| (1)             | 1 | 0 | 0 |
| (2)             | 0 | (q-1)u_1 | 1 |
| (1, 1)          | 0 | (q-t-1)u_1 | 1 |

$\alpha_{\lambda,\mu}^{(+)}$ in the case $N = 2$:

| $\lambda$ \ $\mu$ | (\emptyset, (1)) | ((1), \emptyset) |
|-----------------|----------------|------------------|
| (\emptyset, (1)) | -1/t^2u_1 | 1 |
| ((1), \emptyset) | -1/t^2u_1 | 1 |
Examples of $\alpha_{\lambda, \mu}^{(-)}$ in the case $N = 2$:

$$
\begin{array}{c|cccc}
\lambda \setminus \mu & (\emptyset, (2)) & (\emptyset, (1^2)) & ((1), (1)) & ((2), \emptyset) \\
\hline
(\emptyset, (2)) & \frac{(q-1)(-u_1+u_1q^t+u_1^2q^t^2u_2)}{qtu_1} & \frac{t^2}{q^2u_1^2} & \frac{t}{q^2u_1} & \frac{1}{u_1} \\
(\emptyset, (1, 1)) & \frac{(q-1)(-tq_1+tu_1-u_1-u_2)}{q^2u_2} & \frac{t^3}{q^2u_2^2} & \frac{t}{q^2u_2} & \frac{1}{q^2u_2} \\
((1), (1)) & \frac{(q-1)(t^{-1})(q_1+tu_1-tu_1-u_2)}{qu_1} & \frac{t^2}{q^2u_1^2} & \frac{t}{q^2u_1} & \frac{1}{u_1} \\
((2), \emptyset) & \frac{(q-1)(tu_1-qu_1^2+tu_1^2u_2)}{tu_1} & \frac{t^3}{qu_1^2} & \frac{t}{qu_1} & \frac{1}{u_1} \\
((1, 1), \emptyset) & \frac{(q-1)(-u_1+u_2tu_1-u_2)}{tu_2} & \frac{t^3}{qu_1^2} & \frac{t}{qu_1} & \frac{1}{u_1} \\
\end{array}
$$

Examples of $\alpha_{\lambda, \mu}^{(+)}$ in the case $N = 3$:

$$
\begin{array}{c|cccc}
\lambda \setminus \mu & (\emptyset, (1, 1)) & (\emptyset, (1), \emptyset) & (\emptyset, (1), \emptyset) & (1, \emptyset) \\
\hline
(\emptyset, (1, 1)) & \frac{t^2}{q^2u_1^2} & \frac{t}{qu_1} & \frac{1}{u_1} \\
(\emptyset, (1), \emptyset) & \frac{t^2}{q^2u_1^2} & \frac{t}{qu_1} & \frac{1}{u_1} \\
((1), \emptyset) & \frac{t^2}{q^2u_1^2} & \frac{t}{qu_1} & \frac{1}{u_1} \\
\end{array}
$$

Examples of $\alpha_{\lambda, \mu}^{(-)}$ in the case $N = 3$:

$$
\begin{array}{c|cccc}
\lambda \setminus \mu & ((1), \emptyset) & ((1), \emptyset) & ((1), \emptyset) & ((1), \emptyset) \\
\hline
(1, \emptyset) & 1 & \frac{1}{u_1} & \frac{1}{u_1^2} \\
(\emptyset, (1)) & 1 & \frac{1}{u_2} & \frac{1}{u_2^2} \\
((1, 1), \emptyset) & 1 & \frac{1}{u_3} & \frac{1}{u_3^2} \\
\end{array}
$$
F.2 Examples of matrix elements \( \langle X_\lambda | \mathcal{V}(w) | X_\mu \rangle \). In this section, we demonstrate how to calculate the matrix elements \( \langle X_\lambda | \mathcal{V}(w) | X_\mu \rangle \) by the defining relation of \( \mathcal{V}(w) \). Let us first explain it in general case. If \( \lambda \neq (\emptyset, \ldots, \emptyset) \), let \( j = \min\{i \mid \lambda^{(i)} \neq \emptyset \} \). The defining relation gives

\[
\langle X_\lambda | \mathcal{V}(w) | X_\mu \rangle = w \left\langle X_{(\emptyset, \ldots, (\lambda_1^{(j)} \lambda_2^{(j)} \ldots, \lambda_{(j+1)}^{(j)} \ldots, \lambda^{(N)})} \right| X_{(\lambda_1^{(j)} - 1)}^{(j)} | \mathcal{V}(w) \rangle | X_\mu \rangle \\
+ \left\langle X_{(\emptyset, \ldots, (\lambda_2^{(j)} \lambda_3^{(j)} \ldots, \lambda_{(j+1)}^{(j)} \ldots, \lambda^{(N)})} \right| \mathcal{V}(w) \left( X_{(\lambda_1^{(j)})}^{(j)} - (q/t)^j w X_{(\lambda_1^{(j)})}^{(j)} \right) | X_\mu^{(j)} \rangle.
\] 

(F.9)

The first term can be rewritten by matrix elements \( \langle X_v | \mathcal{V}(w) | X_\mu \rangle \) satisfying \( |v| = |\lambda| - 1 \) (particularly, in the case \( \lambda_1^{(j)} - 1 < \lambda_2^{(j)} \)), and the second term can be expanded by vectors \( |X_\rho \rangle \) of level \( |\rho| = |\mu| - \lambda_1^{(j)} \) or \( |\mu| - \lambda_1^{(j)} + 1 \). (If \( |\mu| - \lambda_1^{(j)} \), \( |\mu| - \lambda_1^{(j)} + 1 < 0 \), it means just 0 vectors.) If \( \lambda = (\emptyset, \ldots, \emptyset) \), moving negative modes \( X_n^{(l)} \) to the left side of \( \mathcal{V}(w) \) by the defining relation, we have the expression in terms of Young diagrams of smaller size. In this way, the matrix elements \( \langle X_v | \mathcal{V}(w) | X_\mu \rangle \) can be inductively and uniquely determined.

The followings are examples.

In \( N = 1 \) case First, by the defining relation of \( \mathcal{V}(w) \), we have

\[
\langle X_0^{(1)} | \mathcal{V}(w) | X_0 \rangle = w \langle 0 | X_0^{(1)} \mathcal{V}(w) | 0 \rangle + \langle 0 | \mathcal{V}(w) \left( X_0^{(1)} - (t/q)wX_0^{(1)} \right) | 0 \rangle.
\]

(F.10)

By \( X_0^{(1)} | 0 \rangle = u_1 | 0 \rangle \), \( \langle 0 | X_0^{(1)} = v_1 \langle 0 | \) and \( X_1^{(1)} | 0 \rangle = 0 \), we get

\[
\langle X_0^{(1)} | \mathcal{V}(w) | X_0 \rangle = wv_1 - (t/q)wu_1.
\]

(F.11)

Similarly, it is easily seen that

\[
\langle X_0^{(1)} | \mathcal{V}(w) | X_1^{(1)} \rangle = (q/t)w^{-1} \langle 0 | \mathcal{V}(w)X_0^{(1)} | 0 \rangle \\
- (q/t)w^{-1} \langle 0 | \left( X_0^{(1)} - (t/q)wX_0^{(1)} \right) \mathcal{V}(w) | 0 \rangle \\
= (q/t)w^{-1}u_1 - (q/t)w^{-1}v_1.
\]

(M.12)

Moreover, we have

\[
\langle X_0^{(1)} | \mathcal{V}(w) | X_1^{(1)} \rangle = w \langle 0 | X_0^{(1)} \mathcal{V}(w)X_0^{(1)} | 0 \rangle + \langle 0 | \mathcal{V}(w) \left( X_1^{(1)} - (t/q)wX_0^{(1)} \right) X_0^{(1)} | 0 \rangle.
\]

(F.13)

By the direct calculation of the free field expression, we have

\[
X_0^{(1)} | X_0^{(1)} \rangle = (t^{-1} - qt^{-1} + q)u_1 \langle X_0^{(1)} \rangle,
\]

(F.14)

\[
X_1^{(1)} | X_0^{(1)} \rangle = -(1 - q)(1 - t^{-1})u_1^2 | 0 \rangle.
\]

(F.15)

By (F.14) and (F.15), we obtain

\[
\langle X_0^{(1)} | \mathcal{V}(w) | X_0^{(1)} \rangle = wv_1 \langle X_0^{(1)} | \mathcal{V}(w) | X_0^{(1)} \rangle - (1 - q)(1 - t^{-1})u_1^2 \\
- (t/q)w(t^{-1} - qt^{-1} + q)u_1 \langle X_0^{(1)} | \mathcal{V}(w) | X_0^{(1)} \rangle.
\]

(F.16)
The matrix element \( \langle X_0 | \mathcal{V}(w) | X_{(1)} \rangle \) is already calculated. Since \( |X_{(1)}\rangle = |K_{(1)}\rangle \) and \( \langle X_{(1)} | = \langle K_{(1)} | \) in this particular case, \( \langle X_{(1)} | \mathcal{V}(w) | X_{(1)} \rangle \) is factorized and corresponds to the Nekrasov factor (the right hand side of (4.2)):

\[
\langle X_{(1)} | \mathcal{V}(w) | X_{(1)} \rangle = \langle K_{(1)} | \mathcal{V}(w) | K_{(1)} \rangle = \frac{(qv_1 - u_1)(tu_1 - v_1)}{t}.
\]

(4.17)

(4.11), (4.12) and (4.17) are the simplest examples of our main theorem (Theorem 4.4).

We list other cases. Let us first prepare the formula for the action of the algebra \( U(N) \) on the PBW-type basis \( |X_{\lambda}\rangle \).

\[
X_0^{(1)} \left( \frac{|X_{(2)}\rangle}{|X_{(1,1)}\rangle} \right) = \left( \frac{(q^2-q^2q+q^2-2q+q+1)u_1}{(q-1)^2(q-1)^2u_1^2} \right) \left( \frac{(q-1)(t-1)}{t^2} \right) \left( \frac{|X_{(2)}\rangle}{|X_{(1,1)}\rangle} \right).
\]

(4.18)

\[
X_1^{(1)} \left( \frac{|X_{(2)}\rangle}{|X_{(1,1)}\rangle} \right) = \left( \frac{(q-1)(t-1)(q^2+q+1)u_1}{(q-1)(t-1)(q^2+q+1)u_1^2} \right) |X_{(1)}\rangle,
\]

(4.19)

\[
\langle X_{(1)} | X_0^{(1)} \rangle = (t^{-1} - qt^{-1} + q)v_1 \langle X_{(1)} |,
\]

(4.20)

\[
\langle X_{(1,1)} | X_0^{(1)} \rangle = \frac{(q - 1)^2q(t - 1)^2v_1^2}{t^3} \langle X_{(2)} | + \frac{v_1(qt - q + 1)^2}{t^2} \langle X_{(1,1)} |.
\]

(4.21)

By using these relation, the matrix elements for larger Young diagrams are inductively determined as follows:

\[
\langle X_{(2)} | \mathcal{V}(w) | X_{(1)} \rangle = w \langle X_{(1)} | \mathcal{V}(w) | X_{(1)} \rangle,
\]

(4.22)

\[
\langle X_{(1,1)} | \mathcal{V}(w) | X_{(1)} \rangle = wv_1(t^{-1} - qt^{-1} + q) \langle X_{(1)} | \mathcal{V}(w) | X_{(1)} \rangle
- (t/q)uw_1 \langle X_{(1)} | \mathcal{V}(w) | X_{(1)} \rangle,
\]

(4.23)

\[
\langle X_{(3)} | \mathcal{V}(w) | X_{(1)} \rangle = w \langle X_{(2)} | \mathcal{V}(w) | X_{(1)} \rangle,
\]

(4.24)

\[
\langle X_{(2,1)} | \mathcal{V}(w) | X_{(1)} \rangle = w \langle X_{(1,1)} | \mathcal{V}(w) | X_{(1)} \rangle,
\]

(4.25)

\[
\langle X_{(1,1,1)} | \mathcal{V}(w) | X_{(1)} \rangle = \frac{(q - 1)^2q(t - 1)^2v_1^2}{t^3} \langle X_{(2)} | \mathcal{V}(w) | X_{(1)} \rangle
+ w \frac{v_1(qt - q + 1)^2}{t^2} \langle X_{(1,1)} | \mathcal{V}(w) | X_{(1)} \rangle
+ (t/q)uw_1 \langle X_{(1,1)} | \mathcal{V}(w) | X_{(1)} \rangle,
\]

(4.26)

\[
\langle X_{(2)} | \mathcal{V}(w) | X_{(2)} \rangle = (q/t)w^{-1} \langle X_{(2)} | \mathcal{V}(w) | X_{(1)} \rangle,
\]

(4.27)

\[
\langle X_{(2)} | \mathcal{V}(w) | X_{(1,1)} \rangle = (q/t)w^{-1}(t^{-1} - qt^{-1} + q)u_1 \langle X_{(2)} | \mathcal{V}(w) | X_{(1)} \rangle
- (q/t)uw_1 \langle X_{(2)} | \mathcal{V}(w) | X_{(1)} \rangle,
\]

(4.28)

\[
\langle X_{(2)} | \mathcal{V}(w) | X_{(3)} \rangle = (q/t)w^{-1} \langle X_{(2)} | \mathcal{V}(w) | X_{(2)} \rangle,
\]

(4.29)

\[
\langle X_{(2,1)} | \mathcal{V}(w) | X_{(1,1)} \rangle = (q/t)w^{-1} \langle X_{(2,1)} | \mathcal{V}(w) | X_{(1,1)} \rangle,
\]

(4.30)

\[
\langle X_{(1,1,1)} | \mathcal{V}(w) | X_{(2)} \rangle = (q/t)w^{-1} \frac{(q - 1)^2q(t - 1)^2u_1^2}{t^3} \langle X_{(2)} | \mathcal{V}(w) | X_{(2)} \rangle
+ (q/t)w^{-1} \frac{(qt - q + 1)^2u_1}{t^2} \langle X_{(1,1,1)} | \mathcal{V}(w) | X_{(1,1)} \rangle
- (q/t)uw^{-1} \langle X_{(2)} | \mathcal{V}(w) | X_{(1,1)} \rangle.
\]

(4.31)
By combining these matrix elements and the examples of transition matrices $\omega_{\alpha,\beta}^{(\pm)}$ from $|K_\lambda\rangle$ to $|X_\mu\rangle$ in the last subsection, we can check Theorem 4.4.

**In $N=2$ case** If there is only one box in all Young diagrams, it is clear that

\[
\langle X_{(1,1)} | \mathcal{V}(w) | X_{(1,1)} \rangle = w \langle X_{(1)} | \mathcal{V}(w) | X_{(1)} \rangle -(t/q)w \frac{(q-1)(t-1)u_1^2}{t} , \tag{F.32}
\]

\[
\langle X_{(1,1)} | \mathcal{V}(w) | X_{(1)} \rangle = w(t^{-1}-qt^{-1}+q)v_1 \langle X_{(1)} | \mathcal{V}(w) | X_{(1)} \rangle + \frac{(q-1)(t-1)u_1^2}{t} \langle X_{(1)} | \mathcal{V}(w) | X_{0} \rangle - (t/q)w(t^{-1}-qt^{-1}+q)u_1 \langle X_{(1)} | \mathcal{V}(w) | X_{(1)} \rangle . \tag{F.33}
\]

As in the $N=1$ case, we prepare the formula for the action of $X^{(i)}$ on $X_{(1)}$.

\[
\left( \begin{array}{c} \langle X_{(1,1)} | v \rangle \\ \langle X_{(0,1)} | v \rangle \end{array} \right) X^{(i)} = \left( \begin{array}{cc} \chi_{1,1}^{(i)} & \chi_{1,2}^{(i)} \\ \chi_{2,1}^{(i)} & \chi_{2,2}^{(i)} \end{array} \right) \left( \begin{array}{c} \langle X_{(1,1)} | v \rangle \\ \langle X_{(0,1)} | v \rangle \end{array} \right) , \tag{F.38}
\]

where

\[
\chi_{1,1}^{(1)} = \frac{(t-1)v_1(-qv_1 + qt v_1 + v_1 + tv_2)}{t} , \tag{F.39}
\]

\[
\chi_{1,2}^{(1)} = \frac{(t-1)v_2(-qv_1 + qt v_1 + v_1 + tv_2)}{qt} , \tag{F.40}
\]

\[
\chi_{2,1}^{(1)} = \frac{(t-1)v_1 v_2(-qv_1 + qt v_1 + v_1 + tv_2)}{t} , \tag{F.41}
\]

\[
\chi_{2,2}^{(1)} = \frac{(t-1)v_1 v_2(t v_2 q^2 - v_1 q^2 + tv_2 q^2 + tv_2 q^2 - t^2 v_1 q + tv_1 q - v_2 q + v_2 q + t^2 - 1)}{t q} , \tag{F.42}
\]

\[
\chi_{1,1}^{(2)} = \frac{(t-1)v_1 v_2(q^2 t v_1 + q^2 t v_2 - q^2 v_1 - q^2 v_2 - qt v_1 - qt v_2 - v_1 v_1 + q v_1 + q v_2 + t^2 - 1)}{t^2} , \tag{F.43}
\]

\[
\chi_{1,2}^{(2)} = \frac{(t-1)v_1 v_2(q t v_1 + q t v_2 - q v_1 - q v_2 - v_1 v_1 + q v_2 + t^2)}{t^2} , \tag{F.44}
\]

\[
\chi_{2,1}^{(2)} = \frac{(t-1)v_1^2 v_2^2(q^2 t - q^2 + q t^2 - 2 q t + q + t)}{t^2} , \tag{F.45}
\]

\[
\chi_{2,2}^{(2)} = \frac{(t-1)v_1^2 v_2^2(q^2 t - q^2 + q t^2 - 2 q t + q + t)}{t^2} . \tag{F.46}
\]

By these, the matrix elements can be written as

\[
\langle X_{(2,0)} | \mathcal{V}(w) | X_{0,0} \rangle = w \langle X_{(1,0)} | \mathcal{V}(w) | X_{0,0} \rangle , \tag{F.47}
\]
\begin{align*}
\langle X_{(1,1),\theta} | V(w) | X_{\theta,\vartheta} \rangle &= w X_{1,1}^{(1)} \langle X_{(1),\theta} | V(w) | X_{\theta,\vartheta} \rangle + w X_{1,1}^{(1)} \langle X_{\theta,(1)} | V(w) | X_{\theta,\vartheta} \rangle \\
&- w(t/q)(u_1 + u_2) \langle X_{(1),\theta} | V(w) | X_{\theta,\vartheta} \rangle, \quad (F.48) \\
\langle X_{(1),(1)} | V(w) | X_{\theta,\vartheta} \rangle &= w X_{2,1}^{(1)} \langle X_{(1),\theta} | V(w) | X_{\theta,\vartheta} \rangle + w X_{2,2}^{(1)} \langle X_{(1),(1)} | V(w) | X_{\theta,\vartheta} \rangle \\
&- w(t/q)(u_1 + u_2) \langle X_{\theta,(1)} | V(w) | X_{\theta,\vartheta} \rangle, \quad (F.49) \\
\langle X_{\theta,(2)} | V(w) | X_{\theta,\vartheta} \rangle &= w \langle X_{\theta,(1)} | V(w) | X_{\theta,\vartheta} \rangle, \quad (F.50) \\
\langle X_{\theta,(1,1)} | V(w) | X_{\theta,\vartheta} \rangle &= w X_{2,1}^{(2)} \langle X_{(1),\theta} | V(w) | X_{\theta,\vartheta} \rangle + w X_{2,2}^{(2)} \langle X_{(1),(1)} | V(w) | X_{\theta,\vartheta} \rangle \\
&- w(t/q)^2(u_1 u_2) \langle X_{\theta,(1)} | V(w) | X_{\theta,\vartheta} \rangle. \quad (F.51)
\end{align*}

Furthermore, the direct calculation gives
\begin{align*}
X_0^{(i)} \begin{pmatrix} |X_{(1),\theta}\rangle \\ |X_{\theta,(1)}\rangle \end{pmatrix} &= \begin{pmatrix} k_{1,1}^{(i)} & k_{1,2}^{(i)} \\ k_{2,1}^{(i)} & k_{2,2}^{(i)} \end{pmatrix} \begin{pmatrix} |X_{(1),\theta}\rangle \\ |X_{\theta,(1)}\rangle \end{pmatrix}, \quad (F.52) \\
X_1^{(i)} \begin{pmatrix} |X_{(1),\theta}\rangle \\ |X_{\theta,(1)}\rangle \end{pmatrix} &= \begin{pmatrix} \zeta_1^{(i)} \\ \zeta_2^{(i)} \end{pmatrix} |X_{\theta,\vartheta}\rangle. \quad (F.53)
\end{align*}

Here,
\begin{align*}
k_{1,1}^{(1)} &= q^{-1}t^{-2}(t-1)(q^2 tu_1^2 + q^2 tu_2^2 + q^2 tu_1 u_2 - q^2 u_1^2 - q^2 u_2^2 - q^2 u_1 u_2 - qt^2 u_2^2) \\
&- qt^2 u_1 u_2 + qt u_2^2 + 2qt u_1 u_2 + qu_1^2 + qu_2^2 + qu_1 u_2 - tu_2^2 - tu_1 u_2), \quad (F.54) \\
k_1^{(1)} &= \frac{(t-1)u_2(qtu_2 - qu_2 + tu_1 + u_2)}{t^2 \sqrt{\frac{q}{t}}}, \quad (F.55) \\
k_{2,1}^{(1)} &= -t^{-3}(t-1)u_1 u_2(-q^2 tu_1 - q^2 tu_2 + q^2 u_1 + q^2 u_2 + qt^2 u_2) \\
&- qu_2 - qu_1 - qu_2 - t^2 u_2 + tu_2), \quad (F.56) \\
k_{2,2}^{(1)} &= \frac{(t-1)u_1 u_2 \sqrt{\frac{q}{t}}(qtu_2 - qu_2 + tu_1 + u_2)}{t^2}, \quad (F.57) \\
k_{1,1}^{(2)} &= q^{-1}t^{-3}(t-1)u_1 u_2(q^3 tu_1 + q^3 tu_2 \\
&- q^3 u_1 - q^3 u_2 - q^3 tu_1 - q^2 tu_2 + q^2 u_1 + q^2 u_2 + qt^2 u_1 + qt^2 u_2 - t^3 u_2), \quad (F.58) \\
k_{1,2}^{(2)} &= \frac{(t-1)u_1 u_2 \sqrt{\frac{q}{t}}(qtu_1 + q^2 tu_2 - q^2 u_1 - q^2 u_2 - qtu_1 - qtu_2 + qu_1 + qu_2 + t^2 u_2)}{t^3 \sqrt{\frac{q}{t}}}, \quad (F.59) \\
k_{2,1}^{(2)} &= \frac{q(t-1)u_1^2 u_2^2 (q^2 t - q^2 + qt^2 - 2qt + q + t)}{t^4}, \quad (F.60) \\
k_{2,2}^{(2)} &= \frac{(t-1)u_1^2 u_2^2 \sqrt{\frac{q}{t}} (q^2 t - q^2 + qt^2 - 2qt + q + t)}{t^4}, \quad (F.61) \\
\zeta_1^{(1)} &= \frac{(q-1)(t-1)(qu_1 + qu_2 u_1 + qu_2 - tu_2 u_1)}{qt}, \quad (F.62) \\
\zeta_1^{(2)} &= \frac{(q-1)q(t-1)u_1 u_2 (u_1 + u_2)}{t^2}, \quad (F.63) \\
\zeta_2^{(1)} &= \frac{(q-1)(t-1)u_1 u_2 (u_1 + u_2)}{t^2}, \quad (F.64) \\
\zeta_2^{(2)} &= \frac{(q-1)(t-1)u_1^2 u_2^2 (q + t)}{t^2}. \quad (F.65)
Thus, we obtain

\[
\langle X_{(1),\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle = w(v_1 + v_2 + \kappa_1^{(1)} \langle X_{\emptyset,\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle + \kappa_1^{(1)} \langle X_{\emptyset,\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle - w(t/q)\xi_1^{(1)}, \tag{F.68}
\]

\[
\langle X_{(1),\emptyset}|\mathcal{V}(w)|X_{\emptyset,(1)}\rangle = w(v_1 + v_2 + \kappa_1^{(1)} \langle X_{\emptyset,\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle + \kappa_1^{(1)} \langle X_{\emptyset,\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle - w(t/q)\xi_1^{(1)}, \tag{F.69}
\]

\[
\langle X_{\emptyset,(1)}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle = wv_1v_2 + \kappa_1^{(2)} \langle X_{\emptyset,\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle + \kappa_1^{(2)} \langle X_{\emptyset,\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle - w(t/q)^2\xi_1^{(2)}, \tag{F.70}
\]

\[
\langle X_{\emptyset,(1)}|\mathcal{V}(w)|X_{\emptyset,(1)}\rangle = wv_1v_2 + \kappa_1^{(2)} \langle X_{\emptyset,\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle + \kappa_1^{(2)} \langle X_{\emptyset,\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset}\rangle - w(t/q)^2\xi_2^{(2)}. \tag{F.71}
\]

In $N = 3$ case Also in the representations of higher level, we can calculate the matrix elements similarly:

\[
\langle X_{(1),\emptyset,\emptyset}|\mathcal{V}(w)|X_{\emptyset,\emptyset,\emptyset}\rangle = w(v_1 + v_2 + v_3) - w(t/q)(u_1 + u_2 + u_3), \tag{F.72}
\]

\[
\langle X_{\emptyset,(1),\emptyset}|\mathcal{V}(w)|X_{\emptyset,\emptyset,\emptyset}\rangle = w(v_1v_2 + v_1v_3 + v_2v_3) - w(t/q)^2(u_1u_2 + u_1u_3 + u_2u_3), \tag{F.73}
\]

\[
\langle X_{\emptyset,\emptyset,(1)}|\mathcal{V}(w)|X_{\emptyset,\emptyset,\emptyset}\rangle = wv_1v_2v_3 - w(t/q)^3u_1u_2u_3, \tag{F.74}
\]

\[
\langle X_{\emptyset,\emptyset,\emptyset}|\mathcal{V}(w)|X_{(1),\emptyset,\emptyset}\rangle = (q/t)w^{-1}(u_1 + u_2 + u_3) - (q/t)w^{-1}(v_1 + v_2 + v_3), \tag{F.75}
\]

\[
\langle X_{\emptyset,\emptyset,\emptyset}|\mathcal{V}(w)|X_{\emptyset,\emptyset,(1)}\rangle = (q/t)^2w^{-1}(u_1u_2 + u_1u_3 + u_2u_3)
- (q/t)^2w^{-1}(v_1v_2 + v_1v_3 + v_2v_3), \tag{F.76}
\]

\[
\langle X_{\emptyset,\emptyset,\emptyset}|\mathcal{V}(w)|X_{\emptyset,\emptyset,(1)}\rangle = (q/t)^3w^{-1}u_1u_2u_3 - (q/t)^3w^{-1}v_1v_2v_3. \tag{F.77}
\]

**List of Notations**

**General notations.**

- $N$, fixed positive number
- $\gamma = (t/q)^{1/2}$,
- $u = (u_1, \ldots, u_N)$, spectral parameters of the $N$-fold Fock tensor spaces
- $t^{\pm i} \cdot u := (u_1, \ldots, u_{i-1}, t^{\pm 1}u_i, u_{i+1}, \ldots, u_N)$,
- $t^{\pm i} \cdot u := (u_1, \ldots, u_{i-1}, tu_i, t^{-1}u_{i+1}, u_{i+2}, \ldots, u_N)$,
- $t^{\pm n} \cdot u := (t^{\pm n_1}u_1, \ldots, t^{\pm n_N}u_N)$,
- $n = (n_1, \ldots, n_N)$, $n_i$ stands for the number of the $\Phi^{(i)}$'s in $V^{(n)}(x)$
- $|n| := \sum_{i=1}^{N} n_i$, the total number of the $\Phi^{(i)}$'s in $V^{(n)}(x)$
\[ [i, k] = [i, k]_n := \sum_{s=1}^{i-1} n_s + k, \]

\[ m = (m_1, \ldots, m_N), \quad m_i \text{ is the number of the } \Phi(i)^{\prime} \text{s in } |Q_\lambda \rangle \]

\[ |m| := \sum_{i=1}^{N} m_i, \quad \text{the total number of the } \Phi(i)^{\prime} \text{s in } |Q_\lambda \rangle \]

\[ [i, k]_m := \sum_{s=1}^{i-1} m_s + k, \]

\[ s = (s_i), \quad \text{generic parameter in Macdonald functions} \]

Especially, in some propositions in Sect. 3, and in Sect. 4, they are specialized as

\[ s_{[i, k]_n} = q^{1-k} t^{1-k} v_i \quad (1 \leq k \leq n_i, \ i = 1, \ldots, N), \]

\[ s_{[n+i, k]_m} = t^{1-n_i-k} v_i \quad (1 \leq k \leq m_i, \ i = 1, \ldots, N), \]

\[ [\lambda]^n = ([\lambda]_i^n)_{1 \leq i \leq |\lambda|} := (\lambda_1^{(1)}, \ldots, \lambda_{n_1}^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_{n_2}^{(2)}, \ldots, \lambda_1^{(N)}, \ldots, \lambda_n^{(N)}), \]

\[ \bigotimes_{i=1}^{m} A_i := A_n \otimes \cdots \otimes A_m, \]

\[ n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i, \]

\[ g_\lambda = q^{n(\lambda')} t^{-n(\lambda)}, \]

\[ G(z) = \prod_{i,j=0}^{\infty} (1 - z q^i t^j). \]

The Nekrasov factor

\[ N_{\lambda, \mu}(u) = \prod_{(i,j) \in \lambda} \left( 1 - u q^{a_\lambda(i,j)} t^{\ell_\mu(i,j)+1} \right) \prod_{(i,j) \in \mu} \left( 1 - u q^{-a_\mu(i,j)-1} t^{-\ell_\lambda(i,j)} \right). \]

Taki’s Flaming factor

\[ f_\lambda = (-1)^{|\lambda|} q^{n(\lambda')+|\lambda|/2} t^{-n(\lambda)-|\lambda|/2}. \]

Algebras and representations.

\( U \), \ the Ding-Iohara-Miki algebra (Definition 2.1)

\( x^\pm(z), \psi^\pm(z) \), \ the Drinfeld currents of the DIM algebra

\( \mathcal{F}^{(1,M)}_u \), \ the level-(1, M) (horizontal) representations (Fact 2.3)

\( \eta(z), \xi(z), \varphi^\pm(z) \), \ the currents in the horizontal Fock representations

\( a_n \), \ generators of the Heisenberg algebra (eq.(2.4))

\[ a_n^{(i)} = 1 \otimes \cdots \otimes 1 \otimes a_n \otimes 1 \otimes \cdots \otimes 1, \]
\( \mathcal{F}_{v}^{(0,1)} \), the level-(0, 1) (vertical) representations (Fact 5.1)

\( \Phi(x), \Phi^*(x), \Phi_\lambda(x), \Phi_\lambda^*(x) \), the intertwining operator of the DIM algebra (Fact 5.2)

\( U(N) \), the algebra generated by \( X_n^{(i)} \)'s (Definition 2.8)

\[ X^{(k)}(z) = \sum_{1 \leq j_1 < \cdots < j_k \leq N} : \Lambda^{(j_1)}(z) \cdots \Lambda^{(j_k)}((q/t)^k z) : u_{j_2} \cdots u_{j_k}, \quad (\text{Definition 2.4}) \]

\( \Lambda^{(i)} := \phi^-(y^{1/2} \zeta) \otimes \cdots \otimes \phi^-(y^{i-3/2} \zeta) \otimes \eta(y^{i-1} \zeta) \otimes 1 \otimes \cdots \otimes 1, \)

\( Y^{(r)}(x) := \sum_{2 \leq i_2 < \cdots < i_r \leq N} : \Lambda^{(i_2)}((q/t)x) \cdots \Lambda^{(i_r)}((q/t)^{r-1} x) : u_{i_2} \cdots u_{i_r}, \)

\( \Lambda^{(i_1, \ldots, i_k)}(z) := \Lambda^{(i_1)}(z) \cdots \Lambda^{(i_k)}(y^{2(1-k)} z) : \)

PBW-type bases (Definition 2.9)

\[ |X_k\rangle = X^{(1)}_{-\lambda_1} X^{(2)}_{-\lambda_2} \cdots X^{(N)}_{-\lambda_N} X^{(1)}_{\lambda_1} X^{(2)}_{\lambda_2} \cdots X^{(N)}_{\lambda_N} \cdots |\rangle \),

\[ \langle X_k| = \langle 0| X^{(N)}_{-\lambda_N} X^{(1)}_{\lambda_1} X^{(2)}_{\lambda_2} \cdots X^{(1)}_{-\lambda_1} X^{(2)}_{-\lambda_2} \cdots X^{(N)}_{-\lambda_N} \cdots . \]

Vertex operators associated with generalized Macdonald Functions.

Screening currents (Definition 3.10)

\[ S^{(i)}(z) := 1 \otimes \cdots \otimes 1 \otimes \phi^sc(y^{i-1} z) \otimes 1 \otimes \cdots \otimes 1, \quad i = 1, \ldots, N - 1, \]

\[ \phi^sc(z) := \exp \left( - \sum_{n > 0} \frac{1}{n-1} \frac{1-t^n}{1-q^n} y^{2n} a_{-n} z^n \right) \exp \left( \sum_{n > 0} \frac{1-t^n}{n-1} \frac{1-q^n}{y^{-n} a_{-n} z^{-n}} \right) \]

\[ \otimes \exp \left( \sum_{n > 0} \frac{1-t^n}{n} \frac{1-q^n}{y^n a_{-n} z^n} \right) \exp \left( - \sum_{n > 0} \frac{1-t^n}{n} \frac{1-q^n}{y^{-n} a_{-n} z^{-n}} \right) . \]

Shifted screening currents

\[ \tilde{S}^{(k)}(z) = S^{(k)}(y^{-2k-1} z) . \]

Screened vertex operators (Definition 3.14)

\[ \Phi^{(0)}(x) := \exp \left( \sum_{n > 0} \frac{1}{n-1} \frac{1-t^n}{1-q^n} a_{-n} u^n \right) \exp \left( \sum_{n > 0} \frac{1}{n-1} \frac{1-q^n}{t^{-n} a_{-n} x^{-n}} \right) \]

\[ \times \exp \left( \sum_{n > 0} \frac{1}{n} \frac{1-y^{2n}}{1-q^n} \sum_{j=2}^{N} \frac{1}{j-1} \frac{y^{(j-1)n}}{a_{-n} x^{-n}} \right), \]

\[ \Phi^{(k)}(x) := \prod_{i=1}^{k} \left( \frac{q^y_1}{u_1} ; q_1 \right) \left( \frac{q^y_{k+1}}{u_{k+1}} ; q_1 \right) \int_{C} \prod_{i=1}^{k} \frac{dy_i}{2 \pi \sqrt{-1} y_i} \Phi^{(0)}(x) \]

\[ \times \, S^{(1)}(y_1) \cdots S^{(k)}(y_k) g(x, y_1, \ldots, y_k), \]
\[
g(x, y_1, \ldots, y_k) := \frac{\theta_q(tu_1 y_1/u_{k+1} x)}{\theta_q(ty_1/x)} \prod_{i=1}^{k-1} \frac{\theta_q(tu_{i+1} y_{i+1}/u_{k+1} y_i)}{\theta_q(ty_{i+1}/y_i)}.
\]

Cartan operator arising from the commutation relation between \(\Phi^{(k)}(x)\) and \(X^{(i)}(z)\) (Lemma 3.19)

\[
\Psi^+(z) := \exp \left( \sum \frac{1}{n} (1 - \gamma^{2n}) \sum_{j=1}^{N} \gamma^{(j-1)n} a_n^{(j)} z^{-n} \right).
\]

Composition of screened vertex operators (Definition 3.25)

\[
V^{(n)}(x_1, \ldots, x_{|n|}) = \Phi^{(0)}(x_1) \cdots \Phi^{(0)}(x_{n_1}) \Phi^{(1)}(x_{n_1+1}) \cdots \Phi^{(1)}(x_{n_1+n_2}) \cdots \cdots \Phi^{(N-1)}(x_{|N-1|n}) \cdots \Phi^{(N-1)}(x_{|N|n}).
\]

Symmetric functions and vectors in the \(N\)-fold tensor Fock spaces.

\(p_\lambda\), the power sum symmetric function

\(m_\lambda\), the monomial symmetric functions

\(P_\lambda, Q_\lambda\), the ordinary Macdonald functions

\(P_\lambda(a^{(i)}_{-n}), Q_\lambda(a^{(i)}_{-n})\) Macdonald functions obtained by replacing

the power sum symmetric function \(p_n\) by the boson \(a^{(i)}_{-n}\)

\(P_\lambda^{(r)}, Q_\lambda^{(r)}\), the ordinary Macdonald polynomials in \(r\) variables

Generalized Macdonald functions (eigenfunctions of \(X^{(1)}_0\)) (Theorem 3.26)

\[
\langle |P_\lambda\rangle, \langle P_\lambda | \rangle := \prod_{i=1}^{N} \frac{c_\lambda^{(i)}}{c_\lambda^{(i)}} |P_\lambda\rangle,
\]

\[
c_\lambda = \prod_{(i,j) \in \lambda} \left( 1 - q^{a_{(i,j)} - 1} \ell_\lambda(i,j)+1 \right),
\]

\[
c'_\lambda = \prod_{(i,j) \in \lambda} \left( 1 - q^{a_{(i,j)}+1} \ell_\lambda(i,j) \right).
\]

Integral form (Definition 3.37)

\[
\langle |K_\lambda\rangle, \langle K_\lambda | \rangle := \prod_{i=1}^{N} \frac{c_\lambda^{(i)}(-)}{c_\lambda^{(i)}(+)} |P_\lambda\rangle,
\]

\[
C^{(+)}_\lambda := \xi^{(+)}_\lambda \times \prod_{1 \leq i < j \leq N} N^{(i,j)}_{\lambda(i), \lambda(j)} (qu_i/tu_j) \prod_{k=1}^{N} c_\lambda^{(k)},
\]

\[
C^{(-)}_\lambda := \xi^{(-)}_\lambda \times \prod_{1 \leq i < j \leq N} N^{(i,j)}_{\lambda(i), \lambda(j)} (qu_j/tu_i) \prod_{k=1}^{N} c_\lambda^{(k)}.
\]
\[ \xi_{\lambda}^{(+)} := \prod_{i=1}^{N} (-1)^{(N-i+1)|\lambda^{(i)}|} u_i^{(-N+i)|\lambda^{(i)}|+\sum_{k=1}^{i} |\lambda^{(k)}|} \]
\[ \times \prod_{i=1}^{N} (q/t)^{\left(\frac{i-i}{t} \right)|\lambda^{(i)}|} (i-N) \left( n(\lambda^{(i)}) + |\lambda^{(i)}| \right) \frac{1}{t} (N-i-1) \left( n(\lambda^{(i)}) + |\lambda^{(i)}| \right), \]
\[ \xi_{\lambda}^{(-)} := \prod_{i=1}^{N} (-1)^{i|\lambda^{(i)}|} u_i^{(-i+1)|\lambda^{(i)}|+\sum_{k=1}^{N} |\lambda^{(k)}|} \]
\[ \times \prod_{i=1}^{N} (q/t)^{\left(\frac{i-i}{t} \right)|\lambda^{(i)}|} (1-i) \left( n(\lambda^{(i)}) + |\lambda^{(i)}| \right) \frac{1}{t} (i-2) \left( n(\lambda^{(i)}) + |\lambda^{(i)}| \right). \]

Factor arising from the application of Ramanujan’s \( 1 \psi_1 \) summation formula (Proposition 3.27)

\[ R_N^\mu(u) = \gamma \sum_{i=1}^{N} (i-1)|\lambda^{(i)}| \prod_{k=2}^{N} \prod_{i=1}^{k-1} \frac{(t^{-n_i+i} u_i/u_k; q)_{-i}^{(k)}}{(q t^{-n_i+i-1} u_i/u_k; q)_{-i}^{(k)}}. \]

Factors in the Macdonald functions and hypergeometric series.

\[ d_n((\theta_i)_{1 \leq i \leq n-1}; (s_i)_{1 \leq i \leq n}|q, t) = \]
\[ = \prod_{i=1}^{n-1} (q/t)^{\theta_i} (t; q)_{\theta_i} \left( ts_i/s_i; q \right)_{\theta_i} \prod_{1 \leq i < j \leq n-1} \frac{(ts_j/s_j; q)_{\theta_i} (q^{-\theta_j} q s_j/t s_i; q)_{\theta_i}}{(q s_j/s_j; q)_{\theta_i} (q^{-\theta_i} s_j/s_i; q)_{\theta_i}}, \]
\[ c_n((\theta_i, j)_{1 \leq i < j \leq n-1}; (s_i)_{1 \leq i \leq n}|q, t) = \]
\[ = \prod_{1 \leq i < j \leq n-1} (\theta_i, j)_{1 \leq i < j \leq n-1-1}; (q^{-\theta_i} s_i)_{1 \leq i \leq n}|q, t), \]
\[ \times d_n((\theta_i, n)_{1 \leq i \leq n-1}; (s_i)_{1 \leq i \leq n}|q, t). \]

The Macdonald functions (Definition 3.21)

\[ p_n(x; s|q, t) = \sum_{\theta \in M_n} c_n(\theta; s|q, t) \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_i, j}, \]
\[ f_n(x; s|q, t) = \prod_{1 \leq k < \ell \leq n} (1 - x_k/x_k) \cdot \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_i, j}. \]

Kajihara and Noumi’s multiple basic hypergeometric series (Definition 4.5)

\[ \phi^m_n(a_1, \ldots, a_m; b_1, \ldots, b_n; x_1, \ldots, x_m; c_1, \ldots, c_n, u) = \sum_{\mu \in \mathbb{Z}^{\geq 0}} u^{\sum_{i=1}^{m} \mu_i} \phi^m_n(a_1, \ldots, a_m; b_1, \ldots, b_n; x_1, \ldots, x_m; c_1, \ldots, c_n; u). \]
Factors arising in the transformation formula from $p_{n+m}$ to another series containing $p_m$ as inner summation (Definition 4.7)

$$
N_{\mu}^{n,m}(s_1, \ldots, s_{n+m}) := \prod_{k=1}^{m} \left( \prod_{i=1}^{n+k} \frac{(qs_{n+k}/ts_i; q)_{\mu_k}}{(qs_{n+k}/s_i; q)_{\mu_k}} \right) \prod_{1 \leq i < j \leq m} \frac{(t q^{-\mu_i} s_{n+j}/s_{n+i}; q)_{\mu_j}}{(q^{-\mu_i} s_{n+j}/s_{n+i}; q)_{\mu_j}},
$$

$$
\tilde{N}_{\mu}^{n,m}(h; s_1, \ldots, s_{n+m}) := \prod_{k=1}^{m} \left( \prod_{i=1}^{n+k} \frac{(qs_{n+k}/ts_i; q)_{\mu_k}}{(hq_{s_{n+k}/s_i}; q)_{\mu_k}} \right) \prod_{1 \leq i < j \leq m} \frac{(t q^{-\mu_i} s_{n+j}/s_{n+i}; q)_{\mu_j}}{(q^{-\mu_i} s_{n+j}/s_{n+i}; q)_{\mu_j}},
$$

where $n, m$ are nonnegative integers and $\mu = (\mu_i)_{1 \leq i \leq m} \in \mathbb{Z}^m$.

**Mukadé operator and its relatives.** The defining relation of the vertex operator $\mathcal{V}(x) : \mathcal{F}_u \rightarrow \mathcal{F}_v$ (Definition 4.1)

$$
\left(1 - x\frac{1}{z}\right) X^{(i)}(z) \mathcal{V}(x) = \left(1 - (t/q)^{i} x\frac{1}{z}\right) \mathcal{V}(x) X^{(i)}(z) \quad i \in \{1, 2, \ldots, N\}.
$$

Realization of $\mathcal{V}(x)$ for the special case $u_i = t^{-n_i} u_j$ (Definition 4.14)

$$
\tilde{V}^{(n)}(x) = \lim_{x_i \rightarrow t^{|n|-i} x} \prod_{1 \leq i < j \leq |n|} \frac{(tx_j/x_i; q)_{\infty}}{(q x_j/t x_i; q)_{\infty}} \mathcal{V}^{(n)}(x_1, \ldots, x_{|n|}) A_{(|n|)}^{-1}(x),
$$

$$
A_{(r)}(x) = \exp \left( \sum_{n > 0} \frac{(1 - (q/t)^r)(1 - t (1-r)n)_{r}}{n(1-q^n)(1-t^n)} \sum_{i=1}^{N} \gamma^{(i-1)n}a^{(i)}_n x^{-n} \right).
$$

Mukadé operators connected toward vertical and horizontal directions (Definition 5.8)

$$
\mathcal{T}^{V}(u, v; w), \quad \mathcal{T}^{H}(u, v; w), \quad \mathcal{T}^{H}_{\lambda, \mu}(u, v; w), \quad \mathcal{T}^{V}_{\lambda, \mu}(u, v; w).
$$

Mukadé operator specialized so that Young diagrams are restricted to only one row (Eq. A.8)

$$
\mathcal{T}_{\lambda}(x) = \mathcal{T}_{\lambda}(u; x).
$$

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