Recognizing by Spectrum for the Automorphism Groups of Sporadic Simple Groups

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Abstract

The spectrum of a finite group is the set of its element orders, and two groups are said to be isospectral if they have the same spectra. A finite group is said to be recognizable by spectrum, if every finite group isospectral with is isomorphic to G. We prove that if isomorphic to any of the sporadic simple groups , , , , , then is recognizable by spectrum. This finishes the proof of the recognizability by spectrum of the automorphism groups of all sporadic simple groups, except . Furthermore, we show that if is isospectral with , then either is isomorphic to , or is an extension of a group by .

1 Introduction

The spectrum of a finite group is the set of its element orders, and two groups are said to be isospectral if they have the same spectra. A finite group is said to be recognizable by spectrum, if every finite group isospectral with is isomorphic to . We prove that if is isomorphic to any of the sporadic simple groups , , , , , then is recognizable by spectrum. This finishes the proof of the recognizability by spectrum of the automorphism groups of all sporadic simple groups, except . Furthermore, we show that if is isospectral with , then either is isomorphic to , or is an extension of a group by .

Theorem A. If is isomorphic to , , , , then is recognizable by spectrum.

Theorem B. Let be a finite group with . Then, either is isomorphic to , or is an extension of a group by .

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Remark. In the second case of Theorem B, we have not been able to find any group \( G \) with the given properties.

Therefore, Theorem A furnishes the recognizability question for the automorphism groups of all sporadic simple groups except \( J_2 \). In fact, Theorem A and the results of \([12]\) imply the following.

**Corollary.** The automorphism group of every sporadic simple groups, except \( J_2 \), is recognizable by spectrum.

We use the following notation. For a finite group \( G \), denote by \( \pi(G) \) the set of all prime divisors of \( |G| \). In addition, let \( \text{Soc}(G) \) be the socle of \( G \), i.e., the product of all minimal normal subgroups of \( G \). Denote by \( A_n \) and \( S_n \) the alternating and symmetric group on \( n \) letters, respectively.

**Table 1.** The spectrum of automorphism group of some sporadic simple groups.

| \( S \) | \( \mu(\text{Aut}(S)) \) | \( \pi_1(\text{Aut}(S)) \) | \( \pi_2(\text{Aut}(S)) \) |
|-------|----------------|----------------|----------------|
| \( M_{12} \) | 8, 10, 11, 12 | 2, 3, 5 | 11 |
| \( M_{22} \) | 8, 10, 11, 12, 14 | 2, 3, 5, 7 | 11 |
| \( J_2 \) | 10, 14, 15, 24 | 2, 3, 5, 7 | - |
| \( He \) | 16, 17, 20, 24, 28, 30, 42 | 2, 3, 5, 7 | 17 |
| \( M^{cL} \) | 9, 14, 20, 22, 24, 30 | 2, 3, 5, 7, 11 | - |
| \( Suz \) | 13, 16, 18, 21, 22, 24, 28, 30, 40 | 2, 3, 5, 7, 11 | 13 |
| \( O'N \) | 16, 20, 22, 24, 30, 31, 38, 56 | 2, 3, 5, 7, 11, 19 | 31 |

**2 Preliminary Results**

**Lemma 1** Let \( K \) be an Abelian normal subgroup of a group \( G \) and \( \varphi : G \rightarrow \text{Aut}(K) \) be the homomorphism induced by conjugation in \( G \). Suppose that \( Kg \) is an element of order \( m \) in \( G/K \) and \( g^m = 1 \). All elements in the coset \( Kg \) are of order \( m \) if and only if the equality \( f(g^r) = 0 \) holds in the endomorphism ring of \( A \) where \( f(x) = 1 + x + \cdots + x^{m-1} \).

*Proof.* This is the obvious consequence of the well-known equality:

\[
(a^g)^m = g^m \cdot a^{g^m} \cdot a^{g^{m-1}} \cdots a,
\]

where \( a \in K \). □

**Lemma 2** Let \( S = P_1 \times P_2 \times \cdots \times P_t \), where \( P_i \)'s are isomorphic non-Abelian simple groups. Then, there hold

\[
\text{Aut}(S) \cong (\text{Aut}(P_1) \times \text{Aut}(P_2) \times \cdots \times \text{Aut}(P_t)) \rtimes S_t.
\]

In particular, \( |\text{Aut}(S)| = \prod_{i=1}^t |\text{Aut}(P_i)| \cdot t! \).

*Proof.* See Lemma 2.2 in [24]. □

**Lemma 3** Let \( G \) be a finite group, and let \( N < G \) and \( G/N \) be a Frobenius group with kernel \( F \) and cyclic complement \( C \). If \(|[F], [N]| = 1 \) and \( F \) does not lie in \( NC_G(N)/N \), then \( p|C| \in \omega(G) \) for some prime divisor \( p \) of \( |N| \). Furthermore, if the preimage of \( F \) in \( G \) is a Frobenius group, then

\[
|C| \cdot \prod_{p \in \pi(N)} p \in \omega(G)
\]

*Proof.* See Lemma 1 in [8]. □
3 Recognition of $\text{Aut}(M^eL)$

Let $p$ be a prime. Denote by $S_p$ the set of all finite non-Abelian simple groups whose prime divisors are at most $p$. In this section we deal with the list of finite non-Abelian simple groups in $S_{11}$, determined in [8] (see Table 2 below). In particular, the information from this table implies:

**Lemma 4** If $S \in S_{11}$, then $\{2, 3\} \subseteq \pi(S)$ and $\pi(\text{Out}(S)) \subseteq \{2, 3\}$.

**Table 2.** Finite non-Abelian simple groups $S \in S_{11}$.

| $S$   | Order of $S$ | Out($S$) | $S$   | Order of $S$ | Out($S$) |
|-------|--------------|----------|-------|--------------|----------|
| $A_5$ | $2^2 \cdot 3 \cdot 5$ | 2        | $A_9$ | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 2        |
| $L_2(7)$ | $2^3 \cdot 3 \cdot 7$ | 2        | $M_{22}$ | $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 2        |
| $L_3(9)$ | $2^3 \cdot 3^2 \cdot 5$ | $2^2$    | $J_2$  | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 2        |
| $L_2(11)$ | $2^3 \cdot 3 \cdot 5 \cdot 11$ | 3        | $S_6(2)$ | $2^9 \cdot 3^4 \cdot 5 \cdot 7$ | 1        |
| $A_7$  | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 2        | $U_4(3)$ | $2^7 \cdot 3^6 \cdot 5 \cdot 7$ | $D_8$    |
| $U_3(3)$ | $2^5 \cdot 3^3 \cdot 7$ | 2        | $U_5(2)$ | $2^{10} \cdot 3^5 \cdot 5 \cdot 11$ | 2        |
| $M_{11}$ | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | 1        | $A_{11}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$ | 2        |
| $A_8$  | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 2        | HS     | $2^9 \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 11$ | 2        |
| $L_3(4)$ | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | $D_{12}$ | $S_4(7)$ | $2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$ | $S_3$    |
| $U_4(2)$ | $2^6 \cdot 3^4 \cdot 5$ | 2        | $O_8^-(2)$ | $2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$ | $S_3$    |
| $L_2(49)$ | $2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ | $2^2$    | $A_{12}$ | $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$ | 2        |
| $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 2        | $M^eL$  | $2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$ | 2        |
| $U_3(5)$ | $2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ | $S_3$    | $U_6(2)$ | $2^{15} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11$ | $S_3$    |

**Theorem 1** The automorphism group of $M^eL$ is recognizable by spectrum.

**Proof.** Let $A = \text{Aut}(M^eL)$ and let $G$ be a finite group such that

$$\mu(G) = \mu(A) = \{3^2, 2 \cdot 7, 2^2 \cdot 5, 2 \cdot 11, 2^3 \cdot 3, 2 \cdot 3 \cdot 5\}.$$ 

It is obvious that $s(G) = 1$. The proof of Theorem 1 will be done through a sequence of lemmas.

**Lemma 5** Let $N$ be the maximal normal soluble subgroup of $G$. Then only one of the three primes 5, 7, 11 can divide the order of $N$.

**Proof.** Assume first that $\{5, 7, 11\} \subseteq \pi(N)$. Since every two distinct primes in $\{5, 7, 11\}$ are nonadjacent in $\text{GK}(G)$, the same is true for $\text{GK}(N)$. Now, by Lemma 8 in [6], $N$ is insoluble, which is a contradiction.

Let $p, q$ and $r$ be distinct primes in $\{5, 7, 11\}$ given in arbitrary order. Assume that two of them, for definiteness $p$ and $q$, divide $|N|$, whereas $r$ does not. Consider a Hall $\{p, q\}$-subgroup $T$ in $N$. By Frattini argument, $G = N\text{GK}(T)$. Therefore, the normalizer $N_G(T)$ contains an element of order $r$, which acts fixed-point-freely on $T$. Thompson Theorem implies that $T$ is nilpotent. Hence $p \cdot q \in \omega(T) \subseteq \omega(G)$, which is a contradiction. $\square$

**Lemma 6** There exists a finite simple group $S \in S_{11}$ such that $S \leq G/N \leq \text{Aut}(S)$.

**Proof.** Let $\overline{G} = G/N$. Then $S = \text{Soc}(\overline{G}) = P_1 \times P_2 \times \cdots \times P_k$, where $P_i$ are non-Abelian simple groups and $\overline{G} \leq \text{Aut}(S)$. It is obvious that $P_i \in S_{11}$, so we need only to prove that $k = 1$.

Suppose $k \geq 2$. By Lemma 4, $|P_i|$ is divisible by 3. By Lemma 5, there exists a prime $p \in \{7, 11\}$ which divides the order of $\overline{G}$. It is clear that neither 7 nor 11 can divide $|S|$, since otherwise $7 \cdot 3^2$ or $11 \cdot 3 \in \omega(S) \subseteq \omega(G)$, which is a contradiction. Hence $\pi(S) = \pi(P_1) = \{2, 3, 5\}$. Thus, $p$ divides the order of $\text{Out}(S)$. But $\text{Out}(S) = \text{Out}(S_1) \times \cdots \times \text{Out}(S_m)$, where the groups $S_j$ are the direct products...
of those $P_i$ which are isomorphic and $S \cong S_1 \times \cdots \times S_m$. Therefore, for some $j$, $p$ divides the order of an outer automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. By Lemma 4, the order of $\text{Out}(P_i)$ is not divisible by $p$. By Lemma 2, $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^t \cdot t!$. Therefore, $t \geq p$ and $S_j$ admits an automorphism of order $p$ which centralizes the element $(a, a, \ldots, a)$ of $S_j$ of order 3, where $a$ is any element of $P_i$ of order 3. Thus $3 \cdot p \in \omega(G)$, a contradiction. □

Lemma 7 $S \cong M^cL$.

Proof. We consider all possibilities for the group $S$ consecutively.

1. $S$ is isomorphic to $A_5$, $L_2(7)$, $A_6 \cong L_2(9)$, $L_2(8)$, $U_3(3)$ or $U_4(2)$. Since the order of $\text{Out}(S)$ is not divisible by 5, 7, 11 and only one of these primes divides the order of $S$, we have a contradiction by Lemma 5.

2. $S$ is isomorphic to $L_2(49)$ or $S_4(7)$. In this case $25 \in \omega(S) \setminus \omega(G)$; a contradiction.

3. $S$ is isomorphic to $A_n$, $n = 10, 11$ or 12. Since $3 \cdot 7 \in \omega(S) \setminus \omega(G)$, we have a contradiction.

4. $S$ is isomorphic to $U_5(2)$ or $U_6(2)$. Since $18 \in \omega(S) \setminus \omega(G)$, which is a contradiction.

5. $S$ is isomorphic to $L_3(4)$, $U_3(5)$, $J_2$, $S_6(2)$, $U_4(3)$, $O_{n}^+(2)$ or $A_n$, $n = 7, 8$ or 9. Since 11 does not divide the order of $\text{Aut}(S)$, we have $11 \in \pi(N)$. On the other hand, each of these groups contains a Frobenius group of order 21. By Lemma 3, there exists an element of order $11 \cdot 3$ in $G$; a contradiction.

6. $S$ is isomorphic to $L_2(11)$, $M_{11}$ or $M_{12}$. Since 7 does not divide the order of $\text{Aut}(S)$, we have $7 \in \pi(N)$. Moreover, each of these groups contains a Frobenius subgroup of order 55. Now, by Lemma 3, there exists an element of order $7 \cdot 5$ in $G$; a contradiction.

7. $S$ is isomorphic to HS or $M_{22}$. Since $\text{Aut}(S)$ does not contain an element of order 9, therefore, $|N|$ is divisible by 3. On the other hand, $S$ contains a Frobenius subgroup of order $8 \cdot 7$, and so $3 \cdot 7$ belongs to $\omega(G)$, which is impossible.

Thus $S \cong M^cL$. □

Lemma 8 $N = 1$.

Proof. Suppose $N \neq 1$. As in Lemma 3.4 of [21], we may assume that $N$ is a nontrivial elementary Abelian $p$-group where $p = 2, 3, 5, 7$ or 11. Since $S$ contains a Frobenius subgroup of order $8 \cdot 7$, Lemma 3 implies that $p \neq 3, 5, 7, 11$.

Let $p = 2$. If $M^cL$ acts trivially on $N$, then $G$ contains an element of order 18 which is not the case. If $M^cL$ acts faithfully, then $G$ contains an element of order 16 since $M^cL$ contains a Frobenius group $3^2 : 8$ which lies in $3^6 : M_{10}$ of $M^cL$. □

Lemma 9 $G \cong \text{Aut}(M^cL)$.

Proof. By Lemmas 6, 7, 8, $G \cong M^cL$ or $G \cong \text{Aut}(M^cL)$, and since $20 \in \omega(G) \setminus \omega(M^cL)$, we deduce that $G \cong \text{Aut}(M^cL)$. This proves both the lemma and the theorem. □

4 Recognition of $\text{Aut}(M_{12})$, $\text{Aut}(M_{22})$, $\text{Aut}(He)$, $\text{Aut}(Suz)$, $\text{Aut}(O'N)$

Theorem 2 If $S$ is isomorphic to one of the sporadic simple groups $M_{12}$, $M_{22}$, $He$, $Suz$ or $O'N$, then $\text{Aut}(S)$ is recognizable by spectrum.
Proof. Let $G$ be a finite group and $\omega(G) = \omega(\text{Aut}(S))$. By Theorems 2 and 3 in [12], $G$ has a normal 2-subgroup $N$ such that $G/N \cong S$. Now we prove that $N = 1$. Suppose false. One can assume that $N$ is elementary Abelian and absolutely irreducible as $S$-module. If $S$ is isomorphic to $He$, $Su_2$ or $O'N$, then the table of 2-Brauer characters of $S$ shows that $G$ contains an element of order $2 \cdot 17$, $2 \cdot 13$ or $2 \cdot 31$, respectively, which is a contradiction. If $S$ is isomorphic to $M_{12}$ or $M_{22}$, then since $\text{Aut}(S)$ does not contain an element of order $2 \cdot 11$, the table of Brauer characters shows that $N$ as $S$-module has dimension 10, but then the Atlas of finite group representations and Lemma 1 show that some pre-image in $G$ of an element contained in the class $8A$ of $S$ is of order $16 \notin \omega(\text{Aut}(S))$. Thus, $N = 1$ and $G \cong \text{Aut}(S)$. \end{proof}

5 On Recognition of $\text{Aut}(J_2)$

**Theorem 3** Let $G$ be a group such that $\omega(G) = \omega(\text{Aut}(J_2))$. Then, either $G \cong \text{Aut}(J_2)$, or $G$ has a normal 2-subgroup $N$ such that $G/N \cong \Dot{A}_8$.

**Proof.** Let $A = \text{Aut}(J_2)$ and let $G$ be a finite group with

$$\mu(G) = \mu(A) = \{2 \cdot 5, 2 \cdot 7, 3 \cdot 5, 2^3 \cdot 3\}.$$ 

Evidently, $s(G) = 1$. We divide the proof into a number of separate lemmas. Let $N$ be the maximal normal soluble subgroup of $G$.

**Lemma 10** There are some finite simple groups $S \in S_7$ with $7 \in \pi(S)$ such that $S \leq G/N \leq \text{Aut}(S)$.

**Proof.** Let $\overline{G} = G/N$. Then $S = \text{Soc}(\overline{G}) = P_1 \times P_2 \times \cdots \times P_k$, where $P_i$ are non-Abelian simple groups and $\overline{G} \leq \text{Aut}(S)$. It is clear that $P_i \in S_7$. Moreover, by Lemma 4, $|P_i|$ is divisible by 3 and $\pi(\text{Out}(P_i)) \subseteq \{2, 3\}$.

First, we claim that the order of $\overline{G}$ is divisible by 7. Suppose false. Then, 7 divides $|N|$ and also $\pi(S) = \pi(P_1) = \{2, 3, 5\}$. Thus, $P_i$ can only be isomorphic to $\Dot{A}_5$, $\Dot{A}_6$ or $U_4(2)$. In any case, $\overline{G}$ contains the Frobenius subgroup $\Dot{A}_4$ of order 4 · 3, and by Lemma 3, $G$ has an element of order 21, which is a contradiction. Therefore, $|\overline{G}|$ is divisible by 7.

Next, we will show that the order of $S$ is also divisible by 7. Suppose not. In this case, again, $P_i$ can only be isomorphic to $\Dot{A}_5$, $\Dot{A}_6$ or $U_4(2)$. By previous paragraph, 7 divides the order of $\text{Out}(S)$. But $\text{Out}(S) = \text{Out}(S_1) \times \cdots \times \text{Out}(S_m)$, where the groups $S_j$ are the direct products of those $P_i$ which are isomorphic and $S \cong S_1 \times \cdots \times S_m$. Therefore, for some $j$, 7 divides the order of an outer automorphism group of a direct product $S_j$ of $t$ isomorphic simple groups $P_i$. By Lemma 4, $\text{Out}(P_i)$ is not divisible by $p$. By Lemma 2, $|\text{Aut}(S_j)| = |\text{Aut}(P_i)|^t \cdot t!$. Therefore, $t \geq 7$ and $S_j$ admits an automorphism of order 7 which centralizes the element $(a, a, \ldots, a)$ of $S_j$ of order 3, where $a$ is any element of $P_i$ of order 3. Thus $3 \cdot 7 \in \omega(G)$, a contradiction.

Finally, it is easy to see that $k = 1$, since otherwise $7 \cdot 3 \in \omega(S) \subseteq \omega(G)$; a contradiction. The lemma is proved. \end{proof}

Below, we list the simple groups $S \in S_7$ satisfy $7 \in \pi(S)$:

\begin{itemize}
  \item $\Dot{A}_n, n = 7, 8, 9, 10; J_2$;
  \item $L_2(7)$, $L_2(8)$, $L_2(49)$, $L_3(4)$, $U_3(3)$, $U_3(5)$, $U_4(3)$, $S_6(2)$, $S_4(7)$, $O_8^+(2)$.
\end{itemize}

**Lemma 11** $S \cong J_2$ or $\Dot{A}_8$.

**Proof.** Since $G$ has no element of order 9, 20 or 25, $S$ can not be isomorphic to $\Dot{A}_9$, $\Dot{A}_{10}$, $L_2(8)$, $U_4(3)$, $S_6(2)$, $O_8^+(2)$, $L_2(49)$, $U_5(3)$, or $S_4(7)$. Therefore, $S$ can only be

\begin{itemize}
  \item $\Dot{A}_n, n = 7, 8; J_2$;
  \item $L_2(7)$, $L_3(4)$ or $U_3(3)$.
\end{itemize}

We consider above possibilities for the group $S$ consecutively.

First, Suppose that $S$ is isomorphic to $\Dot{A}_7$. Then, since $\text{Aut}(\Dot{A}_7) = S_7$ does not contain an element of order 15 and $G/N \leq \text{Aut}(\Dot{A}_7)$, hence $3 \in \pi(N)$ or $5 \in \pi(N)$. On the other hand, since $S$
contains Frobenius subgroups $4 : 3$ and $3^2 : 4$, it follows by Lemma 3 that $9 \in \omega(G)$ or $20 \in \omega(G)$, which is a contradiction.

Next, we assume that $S$ is isomorphic to $L_2(7)$ or $U_3(3)$. Since $S$ contains a Frobenius subgroup of order $4 \cdot 3$, for any prime $p$ of the order of $N$, by Lemma 3, $G$ has an element of order $3 \cdot p$. This means that $N$ is a $\{2,5\}$-group. Moreover, since $15, 24 \in \omega(G) \setminus \omega(\text{Aut}(S))$, it follows that $|N| = 2^\alpha \cdot 5^\beta$ where $\alpha, \beta \in \mathbb{N}$.

Finally, we assume that $S$ is isomorphic to $L_3(4)$. Since $S$ contains Frobenius subgroups of orders $7 \cdot 3$ and $16 \cdot 5$, and $G$ does not contain an element of order $9, 25$ or $35$, $N$ is a 2-group. But then $G/N$ and so $\text{Aut}(L_3(4))$ contains an element of order $15$, which is a contradiction. □

**Lemma 12** If $S \cong A_8$, then $N$ is a 2-subgroup and $G/N \cong A_8$.

**Proof.** By Lemma 10, $G/N \cong A_8$ or $G/N \cong \text{Aut}(A_8) = S_8$. First of all, since $S$ contains Frobenius subgroups $4 : 3$ and $3^2 : 4$, it follows by Lemma 3 that $N$ is a 2-group. Moreover, the case when $G/N \cong S_8$ is impossible. In contrary case, let $V$ be a $G$-chief factor of $N = O_2(G)$. By [3], $\dim(V) = 6, 8$ or $40$, since in other cases $C_G(x)$, where $x$ is an element of order $15$, contains an involution and $G$ should contain an element of order $30$ which is not the case. If $\dim(V) = 6$ or $40$, then, using GAP and the corresponding information from [23], it is easy to calculate that some element $a$ of order $10$ in $G/N$ induces in $V$ by conjugation a linear transformation $t$ such that $1 + t + t^2 + \cdots + t^9 \neq 0$ and hence, by Lemma 1, $G$ contains an element of order $20$ which is impossible. Thus, $\dim(V) = 8$ and hence an element $x \in G/N$ from the class $3A$ acts on $N$ fixed-point-freely. Let $L \cong L_2(4)$ be a subgroup of $G/N$ containing $x$. By Higman’s theorem about action of $L_2(2^m)$ on a 2-group with an element of order $3$ acting fixed-point-freely (see [2, Theorem 8.2]), $N$ is elementary Abelian, hence $G$ cannot contain an element of order $24$. This contradiction proves the lemma. □

**Lemma 13** If $S \cong J_2$, then $G \cong \text{Aut}(J_2)$.

**Proof.** First of all, we prove that $N$ is a non-trivial 2-group. To prove this, we observe that:

1. If $H/N \cong J_2$ with $\omega(H) \subseteq \omega(\text{Aut}(J_2))$ and $N$ is a non-trivial $p$-group, then $p = 2$ and $C_H(N) \subseteq N$.

**Proof.** Suppose that $p \neq 2$. Then $p = 3, 5$ or $7$ and one can assume that $N$ is elementary Abelian $p$-group. Suppose first that $p = 7$. Then the table of 7-Brauer characters of $J_2$ (see [14]) shows that an element from the class $3a$ in $J_2$ centralizes an element of order 3 in $N$ and hence $H$ contains an element of order 21 which is not the case. If $p = 3$ or $p = 5$, then the corresponding table of $p$-Brauer characters shows that $H$ contains an element of order $7 \cdot p$ which is not the case. Thus $p = 2$. If $C_H(N) \nsubseteq N$, then $C_H(N)N = H$ and hence $C_H(N)$ contains an element of order 30 which is not the case. The lemma is proved. □

2. If $H/N \cong J_2$ and $\omega(H) \subseteq \omega(\text{Aut}(J_2))$, then $N$ is a 2-group.

**Proof.** Suppose false. If $H$ is a minimal counter-example, then, by part (1) and Frattini argument, $N$ is an extension of a non-trivial elementary Abelian $p$-group $P$ of odd order by a non-trivial 2-group and $C_H(P) = P$.

It is obvious that $H/N$ acts on the center $Z$ of 2-group $N/P$. If $H/N$ acts trivially on $Z$ then $H/P$ contains an element of order $30$ which is not the case. Thus $H/N$ acts faithfully, and hence $ZS$, where $S$ is a subgroup of order 7 in $H/N$, contains a Frobenius subgroup $[Z,S] \cdot S$. Then the full pre-image of this group contains an element of order $7 \cdot p$, which is not the case. □

Therefore, $N$ is a 2-group and by Lemma 10, $G/N \cong J_2$ or $G/N \cong \text{Aut}(J_2)$. If $G/N \cong J_2$, then $N \neq 1$, and since $G$ cannot contain an element of order 30, the table of 2-Brauer characters shows that an element of order 7 from $G$ acts on $N$ fixed-point-freely and hence $G$ cannot contain an element of order $14 \in \omega(\text{Aut}(J_2))$. Thus $G/N \cong \text{Aut}(J_2)$. 

If \( N \neq 1 \), then one can assume that \( N \) is elementary Abelian and \( N \) is an absolutely irreducible \( \text{Aut}(J_2) \)-module over a field of characteristic 2. Since \( G \) contains an element of order 30, the table of 2-Brauer characters of \( \text{Aut}(J_2) \) shows that \( N \) is of dimension 12. Using Lemma 1 and the information on \( \text{Aut}(J_2) \) from [23], it is easy to check that some pre-image in \( G \) of an element contained in the class 10A of \( \text{Aut}(J_2) \) is of order 20 \( \notin \omega(\text{Aut}(J_2)) \). □

References

[1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[2] G. Higman, *Odd characterizations of finite simple groups*, (Lecture notes, University of Michigan, 1968).

[3] C. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An atlas of Brauer characters*, Clarendon Press, Oxford, 1995.

[4] A. S. Kondrat´ev, *On prime graph components of finite simple groups*, Math. USSR Sbornik, 180(6)(1989), 787-797.

[5] H. Li and W. J. Shi, *A characterization of some sporadic simple groups*, Chinese Ann. Math. 14A(2)(1993), 144-151. (in Chinese)

[6] M. S. Lucido and A. R. Moghaddamfar, *Groups with complete prime graph connected components*, J. Group Theory, 7(3)(2004), 373-384.

[7] V. D. Mazurov, *The set of orders of elements in a finite group*, Algebra and Logic, 33(1)(1994), 49-56.

[8] V. D. Mazurov, *Characterizations of finite groups by sets of element orders*, Algebra and Logic, 36(1)(1997), 23-32.

[9] V. D. Mazurov and W. J. Shi, *A note to the characterization of sporadic simple groups*, Alg. colloq. 5(3)(1998), 285-288.

[10] V. D. Mazurov, *Recognition of finite simple groups \( S_4(q) \) by their element orders*, Algebra and Logic, 41(2)(2002), 93-110.

[11] V. D. Mazurov, *Characterization of groups by arithmetic properties*, Algebra Colloquium, 11(1)(2004), 129-140.

[12] A. R. Moghaddamfar, A. R. Zokayi and M. R. Darafsheh, *On characterizability of the automorphism groups of sporadic simple groups by their element orders*, Acta Mathematica Sinica, 20(4)(2004), 653-662.

[13] C. E. Praeger and W. J. Shi, *A characterization of some alternating and symmetric groups*, Comm. Algebra, 22(5)(1994), 1507-1530.

[14] M. Sch¨onert et al., *GAP- Groups, Algorithms and Programming*, (2004), http://www-gap.mcs.st-and.ac.uk/

[15] W. J. Shi, *A characterestic property of \( J_1 \) and \( PSL_2(2^n) \)*, Adv. in Math. 16(1987), 397-401. (in Chinese)

[16] W. J. Shi, *A characterestic property of Mathieu groups*, Chinese Ann. Math. 9A(5)(1988) 575-580. (in Chinese)

[17] W. J. Shi, *A characterestic property of Conway simple group \( Co_2 \)*, Chinese J. Math. 9(2)(1989), 171-172, (in Chinese)
[18] W. J. Shi, *A characterization of the Higman-Sims simple group*, Houston J. Math. 16(1990), 597-602.

[19] W. J. Shi, *The characterization of the sporadic simple groups by their element orders*, Algebra Colloq. 1(2)(1994), 159-166.

[20] W. J. Shi and H. Li, *A characteristic property of $M_{12}$ and $PSU(6,2)$*, Acta Math. Sinica 32(6)(1989), 758-764. (in Chinese)

[21] A. V. Vasilev, *On recognition of all finite nonabelian simple groups with orders having prime divisors at most 13*, Sib. Math. J., 46(2)(2005), 246-253.

[22] J. S. Williams, *Prime graph components of finite groups*, J. of Algebra, 69(2)(1981), 487-513.

[23] R. A. Wilson, *ATLAS of finite group representations*, http://web.mat.bham.ac.uk/atlas/

[24] A. V. Zavarnitsine, *Recognition of alternating group of degrees $r+1$ and $r+2$ for primes $r$ and the group of degree 16 by their element order sets*, Algebra and Logic, 39(6)(2000), 370-377.

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