INTRINSIC CAPACITIES ON COMPACT KÄHLER MANIFOLDS

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Abstract. We study fine properties of quasiplurisubharmonic functions on compact Kähler manifolds. We define and study several intrinsic capacities which characterize pluripolar sets and show that locally pluripolar sets are globally "quasi-pluripolar".

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Introduction

Since the fundamental work of Bedford and Taylor [4],[5], several authors have developed a "Pluripotential theory“ in domains of $\mathbb{C}^n$ (or of Stein manifolds). This theory is devoted to the fine study of plurisubharmonic (psh) functions and can be seen as a non-linear generalization of the classical potential theory (in one complex variable), where subharmonic functions and the Laplace operator $\Delta$ are replaced by psh functions and the complex Monge-Ampère operator $(dd^c)^n$. Here $d, d^c$ denote the real differential operators $d := \partial + \overline{\partial}$, $d^c := \frac{i}{2\pi} [\partial - \overline{\partial}]$ so that $dd^c = \frac{i\pi}{\sqrt{-1}} \partial \overline{\partial}$; the normalization being chosen so that the positive measure $(dd^c)^n \log[1 + ||z||^2]_n$ has total mass 1 in $\mathbb{C}^n$. We refer the reader to [3],[9],[28], [29] for a survey of this local theory.

Our aim here is to develop a global Pluripotential theory in the context of compact Kähler manifolds. It follows from the maximum principle that there are no psh functions (except constants) on a compact complex manifold $X$. However there are usually plenty of positive closed currents of bidegree (1, 1) (we refer the reader to [17], chapter 3, for basic facts on positive currents). Given $\omega$ a real closed smooth form of bidegree (1, 1) on $X$, we may consider every positive closed current $\omega'$ of bidegree (1, 1) on $X$ which is cohomologous to $\omega$. When $X$ is Kähler, it follows from the "ddc-lemma" that $\omega'$ can be written as $\omega' = \omega + dd^c \varphi$, where $\varphi$ is a function which is integrable with respect to any smooth volume form on $X$. Such a function $\varphi$ will be called $\omega$-plurisubharmonic ($\omega$-psh for short). It is globally defined on $X$ and locally given as the sum of a psh and a smooth function. We let $PSH(X, \omega)$ denote the set of $\omega$-psh functions. Such functions were introduced by Demailly, who call them quasiplurisubharmonic (qpsh). These are the main objects of study in this article.

There are several motivations to study qpsh functions on compact Kähler manifolds. First of all they arise naturally in complex analytic geometry as positive singular metrics of holomorphic line bundles (see section 4) whose study is central to several questions of complex algebraic geometry. Solving Monge-Ampère equations associated to $\omega$-psh functions has been used to produce metrics with prescribed singularities (see [15]). It is also related to
the existence of canonical metrics in Kähler geometry (see [39]). Important contributions have been made in this direction by Blocki (see [6], [7]) and also by Kolodziej using techniques from local Pluripotential theory (see [30], [31]). Quasiplusharmonic functions have also been used in [23] to define a notion of ω-polynomial convexity and study the fine approximation of positive currents by rational divisors. Last but not least, such functions are of constant use in complex dynamics in several variables (see [19], [20], [24], [25], [35]).

It seems to us appropriate to develop a theory of qpsh functions of its own rather than view these functions as particular cases of the local theory. Although the two theories look quite similar, there are important differences which make the ”compact theory” both simpler and more difficult than the local one. Here are some examples:

- There is no pluriharmonic functions (except constants) on a compact manifold, hence each ω-psh function φ is canonically associated (up to normalization) to its curvature current ωφ := ω + dd^c φ ≥ 0. This yields compactness properties of subsets of PSH(X, ω) (see section 1) which are quite useful (e.g. in complex dynamics, see section 6.2).
- Integration by parts (of constant use in such theories) is quite simple in the compact setting since there is no boundary. As an illustration, we obtain transparent proofs of Chern-Levine-Nirenberg type inequalities (see example 1.8 and section 2). A successful application of this simple observation has been made in complex dynamics in [25].
- On the other hand one loses homogeneity of Monge-Ampère operators in the compact setting. They do have uniformly bounded mass (by Stokes theorem), but there is no performing ”comparison principle”, which is a key tool in the local theory. This is a source of difficulty when, for example, one wishes to solve Monge-Ampère equations on compact manifolds (see [30], [26]).

We shall develop our study in a series of articles. In the present one we define and study several intrinsic capacities which we shall use in our forthcoming articles.

Let us now describe more precisely the contents of the article. In section 1 we define ω-psh functions and gather useful facts about them (especially compactness results such as proposition 1.7). For locally bounded ω-psh functions φ we define the complex Monge-Ampère operator ωφ in section 2. We establish Chern-Levine-Nirenberg inequalities (proposition 2.1) and study the "Monge-Ampère capacity" Capω (definition 2.4). As in the local theory, ω-psh functions are quasicontinuous with respect to Capω (corollary 2.8). The capacity Capω is comparable to the local Monge-Ampère capacity of Bedford and Taylor (proposition 2.10) and moreover enjoys invariance properties (proposition 2.5). In section 3 we define a relative extremal function h^*_E,ω and establish a useful formula (theorem 3.2)

\[ Cap^*_ω(E) = \int_X (-h^*_E,ω) (ω + dd^c h^*_E,ω)^n. \]
This is the global version of the fundamental local formula of Bedford-Taylor [5], \( \text{Cap}(E, \omega) = \int_{\Omega} (dd^c u^n)^n \).

In section 4 we study yet another capacity (the Alexander capacity \( T_\omega \), definition 4.7) which is defined by means of a (global) extremal function (definition 4.1). When \( \omega \) is a Hodge form, it can be defined as well in terms of Tchebychev constants: these are the contents of section 5 (theorem 5.2) where we further give a geometrical interpretation of \( T_\omega \) when \( X = \mathbb{CP}^n \) is the complex projective space and \( \omega \) is the Fubini-Study Kähler form (theorem 5.4), following Alexander’s work [1]. In section 6 we show that locally pluripolar sets can be defined by \( \omega \)-psh functions when \( \omega \) is Kähler: this is our version of a result of Josefson (theorem 6.2). We then give an application in complex dynamics which illustrates how invariance properties of these capacities can be used. Finally in an Appendix we show how to globally regularize \( \omega \)-psh functions, following ideas of Demailly.

This paper lies at the border of Complex Analysis and Complex Geometry. We have tried to make it accessible to mathematicians from both sides. This has of course some consequences for the style of presentation. We have included proofs of some results which may be seen as consequences of results from the local pluripotential theory. We have spent some efforts defining, regularizing and approximating positive singular metrics of holomorphic line bundles, although some of these facts may be considered as classical by complex geometers. Altogether we hope the paper is essentially self-contained.

Our efforts will not be vain if for instance we have convinced specialists of the (local) pluripotential theory that the right point of view in studying the Lelong class \( \mathcal{L}(\mathbb{C}^n) \) of psh functions with logarithmic growth in \( \mathbb{C}^n \) is to consider qpsh functions on the complex projective space \( \mathbb{CP}^n \). We also think this paper should be useful to people working in complex dynamics in several variables where pluripotential theory has become an important tool.

**Warning.** In the whole paper positivity (like e.g. in positive metric and positive current) has to be understood in the weak (french, i.e. non-negativity) sense of currents, except when we talk of a positive line bundle \( L \), in which case it means that \( L \) admits a smooth metric whose curvature is a Kähler form.

### 1. Quasiplurisubharmonic functions

In the sequel, unless otherwise specified, \( L^p \)-norms will always be computed with respect to a fixed volume form on \( X \), which is a compact connected Kähler manifold. Let \( \omega \) be a closed real current of bidegree \((1, 1)\) on \( X \). We assume throughout the article that \( \omega \) has continuous local potentials.

**Definition 1.1.** Set

\[
\text{PSH}(X, \omega) := \{ \varphi \in L^1(X, \mathbb{R} \cup \{-\infty\}) / dd^c \varphi \geq -\omega \text{ and } \varphi \text{ is u.s.c.} \}.
\]

The set \( \text{PSH}(X, \omega) \) is the set of "\( \omega \)-plurisubharmonic" functions.

Observe that \( \text{PSH}(X, \omega) \) is non empty if and only if there exists a positive closed current of bidegree \((1, 1)\) on \( X \) which is cohomologous to \( \omega \). One then says that the cohomology class \([\omega]\) is pseudoeffective. In the sequel we always
assume this property holds. We also always assume that $\omega$ has continuous local potentials. This guarantees that $\omega$-plurisubharmonic functions ($\omega$-psh for short) are upper semi-continuous (u.s.c.), so they are locally hence globally bounded from above. We endow $PSH(X,\omega)$ with the $L^1$-topology. Observe that $PSH(X,\omega)$ is a closed subspace of $L^1(X)$.

**Example 1.2.** The most fundamental example which may serve as a guideline to everything that follows is the case where $X = \mathbb{CP}^n$ is the complex projective space and $\omega = \omega_{FS}$ is the Fubini-Study Kähler form. There is then a 1-to-1 correspondence between $PSH(\mathbb{CP}^n,\omega_{FS})$ and the Lelong class

$$\mathcal{L}(\mathbb{C}^n) := \left\{ \psi \in PSH(\mathbb{C}^n) / \psi(z) \leq \frac{1}{2} \log[1 + |z|^2] + C_\psi \right\}$$

which is given by the natural mapping

$$\psi \in \mathcal{L}(\mathbb{C}^n) \mapsto \varphi(x) = \left\{ \begin{array}{ll}
\psi(x) - \frac{1}{2} \log[1 + |x|^2] & \text{if } x \in \mathbb{C}^n \\
\lim_{y \to x, y \in \mathbb{C}^n} (\psi(y) - \frac{1}{2} \log[1 + |y|^2]) & \text{if } x \in H_\infty,
\end{array} \right.$$

where $H_\infty$ denotes the hyperplane at infinity. One can easily show that this mapping is bicontinuous for the $L^1_{loc}$ topology.

The Lelong class $\mathcal{L}(\mathbb{C}^n)$ of plurisubharmonic functions with logarithmic growth in $\mathbb{C}^n$ has been intensively studied in the last thirty years. It seems to us that the properties of $\mathcal{L}(\mathbb{C}^n)$ are more easily seen when $\mathcal{L}(\mathbb{C}^n)$ is viewed as $PSH(\mathbb{CP}^n,\omega_{FS})$. Further we shall see hereafter that the class $PSH(X,\omega)$ of $\omega$-psh functions enjoys several properties of $\mathcal{L}(\mathbb{C}^n)$ when $\omega$ is Kähler. We start by observing (proposition 1.3.1 & 1.3.2 below) that $PSH(X,\omega)$ and $PSH(X,\omega')$ are comparable if $\omega,\omega'$ are both Kähler.

**Proposition 1.3.**

1) If $\omega_1 \leq \omega_2$ then $PSH(X,\omega_1) \subset PSH(X,\omega_2)$.
2) $\forall A \in \mathbb{R}^+$, $PSH(X, A\omega) = A \cdot PSH(X, \omega)$.
3) If $\omega'$ is cohomologous to $\omega$, $\omega' = \omega + dd^c\chi$, then

$$PSH(X,\omega') = PSH(X,\omega) + \chi.$$

4) If $\varphi, \psi \in PSH(X,\omega)$ then

$$\max(\varphi, \psi), \frac{\varphi + \psi}{2}, \log[e^\varphi + e^\psi] \in PSH(X,\omega)$$

**Proof.** Assertions 1),2),3) follow straightforwardly from the definition. Observe that 1.3.4 says that $PSH(X,\omega)$ is a convex set which is stable under taking maximum and also under the operation $(\varphi, \psi) \mapsto \log[e^\varphi + e^\psi]$. These are all consequences of the corresponding local properties of psh functions. We nevertheless give a proof, in the spirit of this article. That $(\varphi + \psi)/2 \in PSH(X,\omega)$ follows by linearity. The latter assertion is a consequence of the following computation

$$dd^c \log[e^\varphi + e^\psi] = \frac{e^\varphi dd^c \varphi + e^\psi dd^c \psi}{e^\varphi + e^\psi} + \frac{e^{\varphi+\psi} d(\varphi - \psi) \wedge d^c(\varphi - \psi)}{e^\varphi + e^\psi},$$

using that $df \wedge d^c f \geq 0$. This computation makes sense if for instance $\varphi, \psi$ are smooth. The general case follows then by regularizing $\varphi, \psi$ (see Appendix). Finally observe that $\max(\varphi, \psi) = \lim j^{-1} \log[e^{j\varphi} + e^{j\psi}] \in PSH(X,\omega)$. □
It follows from 1.3.3 that \( PSH(X, \omega) \) essentially depends on the cohomology class \([\omega]\). In the same vein we have the following:

**Proposition 1.4.** Let \( \mathcal{T}_{[\omega]}(X) \) denote the set of positive closed currents \( \omega' \) of bidegree \((1, 1)\) on \( X \) which are cohomologous to \( \omega \). Then

\[ PSH(X, \omega) \simeq \mathcal{T}_{[\omega]}(X) \oplus \mathbb{R}. \]

**Proof.** The mapping

\[ \Phi : \varphi \in PSH(X, \omega) \mapsto \omega_{\varphi} := \omega + dd^c \varphi \in \mathcal{T}_{[\omega]}(X) \]

is a continuous affine mapping whose kernel consists of constants mappings: indeed \( \varphi_{\omega} = \varphi_{\psi} \) implies that \( \varphi - \psi \) is pluriharmonic hence constant by the maximum principle. Moreover \( \Phi \) is surjective: if \( \omega' \geq 0 \) is cohomologous to \( \omega \) then \( \omega' = \omega + dd^c \varphi \) for some \( \varphi \in L^1(X, \mathbb{R}) \)-this is the celebrated \( dd^c \)-lemma on Kähler manifolds (see e.g. lemma 8.6, chapter VI in [17]). Thus \( \varphi \) coincides almost everywhere with a function of \( PSH(X, \omega) \) and \( \omega' = \Phi(\varphi) \).

□

**Remark 1.5.** The size of \( PSH(X, \omega) \) is therefore related to that of \( \mathcal{T}_{[\omega]}(X) \) hence only depends on the positivity of the cohomology class \([\omega]\). The more positive \([\omega]\), the bigger \( PSH(X, \omega) \).

When \([\omega]\) is Kähler then \( PSH(X, \omega) \) is large: if e.g. \( \chi \) is any \( C^2 \)-function on \( X \) then \( \varepsilon \chi \in PSH(X, \omega) \) for \( \varepsilon > 0 \) small enough. We will see (theorem 6.2) that \( PSH(X, \omega) \) characterizes locally pluripolar sets when \([\omega]\) is Kähler. It follows from proposition 1.3 that \( PSH(X, \omega) \) and \( PSH(X, \omega') \) have the same "size" if \( \omega \) and \( \omega' \) are both Kähler.

Note on the other hand that \( PSH(X, \omega) \simeq \mathbb{R} \) when \( \omega \) is cohomologous to \([E]\), the current of integration along the exceptional divisor of a smooth blow up. Indeed let \( \pi : X \to \tilde{X} \) be a blow up with smooth center \( Y \), \( \text{codim}_C Y \geq 2 \) (see e.g. chapter 2 of [17] for the definition of blow-ups). Let \( E = \pi^{-1}(Y) \) denote the exceptional divisor and \( \omega = [E] \) be the current of integration along \( E \). If \( \varphi \in PSH(X, [E]) \) then \( dd^c (\varphi \circ \pi^{-1}) \geq 0 \) in \( \tilde{X} \setminus Y \). Since \( \text{codim}_C Y \geq 2 \), \( \varphi \circ \pi^{-1} \) extends trivially through \( Y \) has a global psh function on \( \tilde{X} \). By the maximum principle \( \varphi \circ \pi^{-1} \) is constant hence so is \( \varphi \). Alternatively there is no positive closed current of bidegree \((1, 1)\) on \( X \) which is cohomologous to \([E]\) except \([E]\) itself.

It follows from previous proposition that any set of "normalized" \( \omega \)-psh functions is in 1-to-1 correspondence with \( \mathcal{T}_{[\omega]}(X) \) which is compact for the weak topology of currents. This is the key to several results to follow: normalized \( \omega \)-psh functions form a compact family in \( L^1(X) \).

**Proposition 1.6.** Let \( (\varphi_j) \in PSH(X, \omega)^\mathbb{N} \).

1) If \( (\varphi_j) \) is uniformly bounded from above on \( X \), then either \( \varphi_j \) converges uniformly to \( -\infty \) on \( X \) or the sequence \( (\varphi_j) \) is relatively compact in \( L^1(X) \).

2) If \( \varphi_j \to \varphi \) in \( L^1(X) \), then \( \varphi \) coincides almost everywhere with a unique function \( \varphi^* \in PSH(X, \omega) \). Moreover

\[ \sup_X \varphi^* = \lim_{j \to +\infty} \sup_X \varphi_j. \]

3) In particular if \( \varphi_j \) is decreasing, then either \( \varphi_j \to -\infty \) or \( \varphi = \lim \varphi_j \in PSH(X, \omega) \). Similarly, if \( \varphi_j \) is increasing and uniformly bounded from
above then \( \varphi := (\lim \varphi_j)^* \in \text{PSH}(X, \omega) \), where \( ^* \) denotes the upper-semi-continuous regularization.

**Proof.** This is a straightforward consequence of the analogous local result for sequences of psh functions. We refer the reader to [17], chapter 1, for a proof. Note that 1.6.2 is a special case of a celebrated lemma attributed to Hartogs. \( \square \)

The next result is quite useful (see [42], [43] for a systematic use).

**Proposition 1.7.** The family

\[
\mathcal{F}_0 := \{ \varphi \in \text{PSH}(X, \omega) / \sup_X \varphi = 0 \}
\]

is a compact subset of \( \text{PSH}(X, \omega) \).

If \( \mu \) is a probability measure such that \( \text{PSH}(X, \omega) \subset L^1(\mu) \) then

\[
\mathcal{F}_\mu := \{ \varphi \in \text{PSH}(X, \omega) / \int_X \varphi d\mu = 0 \}
\]

is a relatively compact subset of \( \text{PSH}(X, \omega) \). In particular there exists \( C_\mu \) such that \( \forall \varphi \in \text{PSH}(X, \omega) \),

\[
-C_\mu + \sup_X \varphi \leq \int_X \varphi d\mu \leq \sup_X \varphi.
\]

**Proof.** It follows straightforwardly from proposition 1.6.1 that \( \mathcal{F}_0 \) is a relatively compact subset of \( \text{PSH}(X, \omega) \). Moreover \( \mathcal{F}_0 \) is closed by Hartogs lemma (1.6.2).

Let \( (\varphi_j) \in \mathcal{F}_\mu \). Then \( \psi_j := \varphi_j - \sup_X \varphi_j \in \mathcal{F}_0 \) which is relatively compact. Assume first \( \mu \) is smooth. Then \( (\int_X \psi_j d\mu) \) is bounded: this is because if \( \psi_{jk} \to \psi \in L^1(X) \) then \( \psi_{jk} \mu \to \psi \mu \) in the weak sense of (negative) measures hence \( \int_X \psi_{jk} d\mu \to \int_X \psi d\mu > -\infty \). Now \( \int_X \psi_j d\mu = \int_X \varphi_j d\mu - \int_X \sup_X \varphi_j d\mu = - \sup_X \varphi_j \) thus \( \sup_X \varphi \) is bounded and we can apply the previous proposition to conclude that \( (\varphi_j) \) is relatively compact (it cannot converge uniformly to \( -\infty \) since \( \int_X \varphi_j d\mu = 0 \)).

When \( \mu \) is not smooth, it only remains to prove that \( (\int_X \psi_j d\mu) \) is bounded. Assume on the contrary that \( \int_X \psi_j d\mu \to -\infty \). Extracting a subsequence if necessary we can assume \( \int_X \psi_j d\mu \leq -2^i \). Set \( \psi = \sum_{j \geq 1} 2^{-j} \psi_j \). This is a decreasing sequence of \( \omega \)-psh functions, hence \( \psi \in \text{PSH}(X, \omega) \) or \( \psi \equiv -\infty \). Now it follows from the previous discussion that \( \int_X \psi dV \geq -C \) if \( dV \) denotes some smooth probability measure on \( X \). Thus \( \int_X \psi dV > -\infty \) hence \( \psi \in \text{PSH}(X, \omega) \). We obtain a contradiction since by the Monotone convergence theorem, \( \int_X \psi d\mu = \sum_{j \geq 1} 2^{-j} \int_X \psi_j d\mu = -\infty \). \( \square \)

**Example 1.8.** It was part of our definition 1.1 that \( \omega \)-psh functions are integrable with respect to a fixed volume form. Therefore \( \text{PSH}(X, \omega) \subset L^1(\mu) \) for every smooth probability measure \( \mu \) on \( X \). More generally if \( \mu \) is a probability measure on \( X \) such that

\[
\mu = \Theta + dd^c(S),
\]

where \( \Theta \) is smooth and \( S \) is a **positive** current of bidimension \( (1,1) \) on \( X \), then \( \text{PSH}(X, \omega) \subset L^1(\mu) \) for any smooth \( \omega \). Indeed let \( \varphi \in \text{PSH}(X, \omega) \),
\( \varphi \leq 0 \). If \( \varphi \) is smooth, it follows from Stokes theorem that
\[
0 \leq \int_X (\varphi) d\mu = \int_X (\varphi) \Theta + \int_X (\varphi) dd^c S \\
\leq C_\Theta ||\varphi||_{L^1} + \int_X S \wedge (-dd^c \varphi) \\
\leq C_\Theta ||\varphi||_{L^1} + \int_X S \wedge \omega < +\infty,
\]
where the last inequality follows from \( S \geq 0 \) and \(-dd^c \varphi \leq \omega\). The general case follows by regularizing \( \varphi \) (see Appendix).

Probability measures satisfying (1) naturally arise in complex dynamics (see [25]). Observe also that Monge-Ampère measures arising from the local theory of Bedford and Taylor [5] do satisfy (1): if \( u \) is psh and locally bounded near e.g. the unit ball \( B \) of \( \mathbb{C}^n \), we can extend it to \( \mathbb{C}^n \) as a global psh function with logarithmic growth considering
\[
U(z) := \begin{cases} 
\max(u(z), A \log^+ |z| - \sup_B |u| - 1) & \text{if } z \in B \\
A \log^+ |z| - \sup_B |u| - 1 & \text{if } z \in (1 + \epsilon)B \setminus B \end{cases}
\]
where \( \log^+ |z| := \max(\log |z|, 0) \) and with \( A \) large enough. We assume \( A = 1 \) for simplicity. Now \( \varphi := U - \frac{1}{2} \log(1 + |z|^2) + C \) extends as a bounded function in \( PSH(\mathbb{C}^n, \omega) \), where \( \omega \) is the Fubini-Study Kähler form on \( \mathbb{C}^n \), so \( \varphi \geq 0 \) if \( C > 0 \) is large enough. To conclude note that, setting \( \omega^\varphi := \omega + dd^c \varphi \geq 0 \), we get \( \omega^\varphi_n = (dd^c u)^n \) in \( B \) and
\[
\omega^\varphi_n = \omega^n + dd^c S, \text{ where } S = \varphi \sum_{j=0}^{n-1} \omega^\varphi_j \wedge \omega^{n-1-j} \geq 0.
\]
The Monge-Ampère operator \( \omega^\varphi_n \) will be defined in the next section.

**Example 1.9.** If \( \mu \) is a probability measure on \( X = \mathbb{C}^n \) and \( \omega \) denotes as before the Fubini-Study Kähler form, then
\[
\varphi_\mu(x) := \int_{\mathbb{C}^n} \log \left( \frac{||x \wedge y||}{||x|| \cdot ||y||} \right) d\mu(y)
\]
defines a \( \omega \)-psh function on \( \mathbb{C}^n \). Such functions have been considered by Molzon, Shiffman and Sibony [34], [33] in order to define capacities on \( \mathbb{C}^n \). However they do not characterize pluripolar sets.

2. Monge-Ampère capacity

We assume in this section that \( \omega \) is a Kähler form on \( X \). Let \( T \) be a positive closed current of bidegree \((p,p)\) on \( X \), \( 0 \leq p \leq n = \dim_{\mathbb{C}} X \). It can be thought of as a closed differential form of bidegree \((p,p)\) with measure coefficients whose total variation is controlled by
\[
||T|| := \int_X T \wedge \omega^{n-p}.
\]
We refer the reader to chapter 3 of [17] for basic properties of positive currents. Given \( \varphi \in PSH(X, \omega) \) we write \( \varphi \in L^1(T) \) if \( \varphi \) is integrable with respect to each (measure) coefficient of \( T \). This is equivalent to \( \varphi \) being
integrable with respect to the trace measure $T \wedge \omega^{n-p}$. In this case the current $\varphi T$ is well defined, hence so is

$$\omega_\varphi \wedge T := \omega \wedge T + dd^c(\varphi T).$$

This is again a positive closed current on $X$, of bidegree $(p+1, p+1)$. Indeed positivity is a local property which is stable under taking limits. One can locally regularize $\varphi$ and approximate $\omega_\varphi \wedge T$ by the currents $\omega_{\varphi_\varepsilon} \wedge T$ which are positive since $\omega_{\varphi_\varepsilon}$ are smooth positive forms.

When $\varphi \in PSH(X, \omega) \cap L^\infty(X)$ then $\varphi \in L^1(T)$ for any positive closed current $(p, p)$ of bidegree $(p, p)$. One can thus inductively define $\omega_\varphi \wedge T$, $1 \leq j \leq n - p$, for $\varphi \in PSH(X, \omega) \cap L^\infty(X)$. For $T = 0$ and $j = n$ one obtains the complex Monge-Ampère operator, $\omega_\varphi^n$. It follows from the local theory that the operator $\varphi \mapsto \omega_\varphi^n$ is continuous under monotone sequences (see [5]). The proof of these continuity properties is simpler in our compact setting. We refer the reader to [26] where this is proved in a more general global context.

**Proposition 2.1** (Chern-Levine-Nirenberg inequalities). Let $T$ be a positive closed current of bidegree $(p, p)$ on $X$ and $\varphi \in PSH(X, \omega) \cap L^\infty(X)$.

Then $||\omega_\varphi \wedge T|| = ||T||$. Moreover if $\psi \in PSH(X, \omega) \cap L^1(T)$, then $\psi \in L^1(T \wedge \omega_\varphi)$ and

$$||\psi||_{L^1(T \wedge \omega_\varphi)} \leq ||\psi||_{L^1(T)} + [2 \sup X \psi + \sup X \varphi - \inf X \varphi]||T||.$$

**Proof.** By Stokes theorem, $\int_X dd^c \varphi \wedge T \wedge \omega^{n-p-1} = 0$, hence

$$||\omega_\varphi \wedge T|| := \int_X \omega_\varphi \wedge T \wedge \omega^{n-p-1} = \int_X T \wedge \omega^{n-p} := ||T||.$$

Consider now $\psi \in L^1(T)$. Since $T$ has measure coefficients, this simply means that $\psi$ is integrable with respect to the total variation of these measures. Assume first $\psi \leq 0$, $\varphi \geq 0$ and $\varphi, \psi$ are smooth. Then

$$||\psi||_{L^1(T \wedge \omega_\varphi)} := \int_X (\psi) T \wedge \omega_\varphi \wedge \omega^{n-p-1} = ||\psi||_{L^1(T)} + \int_X (\psi) T \wedge dd^c \varphi \wedge \omega^{n-p-1}.$$

Now it follows from Stokes theorem that

$$\int_X (-\psi) T \wedge dd^c \varphi \wedge \omega^{n-p-1} = \int_X \varphi T \wedge (-dd^c \psi) \wedge \omega^{n-p-1}$$

$$\leq \int_X \varphi T \wedge \omega^{n-p} \leq \sup X \varphi \int_X T \wedge \omega^{n-p},$$

where the forelast inequality follows from $\varphi T \wedge \omega^{n-p} \geq 0$ and $-dd^c \psi \leq \omega$. This yields

$$||\psi||_{L^1(T \wedge \omega_\varphi)} \leq ||\psi||_{L^1(T)} + \sup X \varphi ||T||.$$

The general case follows by regularizing $\varphi, \psi$, observing that $\omega_\varphi = \omega_{\varphi'}$ where $\varphi' = \varphi - \inf X \varphi \geq 0$, and decomposing $\psi = \psi' + \sup X \psi$ with $\psi' = \psi - \sup X \psi \leq 0$. □

**Remark 2.2.** The fact that the $L^1$-norm of $\psi$ with respect to the probability measure $T \wedge \omega_\varphi \wedge \omega^{n-p-1}$ is controlled by its $L^1$-norm with respect to $T \wedge \omega^{n-p}$...
is similar to the phenomenon already encountered in example 1.8: one can write
\[ T \wedge \omega_\psi \wedge \omega^{n-p-1} = T \wedge \omega^{n-p} + dd^c S, \quad S = (\varphi - \inf X)T \wedge \omega^{n-p-1} \geq 0. \]

This type of estimates is usually referred to as "Chern-Levine-Nirenberg inequalities", in reference to [10] where simpler-but fundamental- \( L^\infty \)-estimates were established (with \( \psi = \text{constant} \)). Estimates involving the \( L^1 \)-norm of \( \psi \) were first proved in the local context by Cegrell [8] and Demailly [14].

A straightforward induction yields the following

**Corollary 2.3.** Let \( \psi, \varphi \in \text{PSH}(X, \omega) \) with \( 0 \leq \varphi \leq 1 \). Then
\[ 0 \leq \int_X |\psi|\omega^n \leq \int_X |\psi|\omega^n + n[1 + 2 \sup X] \int_X \omega^n. \]

Following Bedford-Taylor [5] and Kolodziej [31] we introduce the following Monge-Ampère capacity.

**Definition 2.4.** Let \( K \) be a Borel subset of \( X \). We set
\[ \text{Cap}_\omega(K) := \sup \left\{ \int_K \omega^n_\varphi / \varphi \in \text{PSH}(X, \omega), 0 \leq \varphi \leq 1 \right\}. \]

Note that this definition only makes sense when the cohomology class \( [\omega] \) is big, i.e. when \( [\omega]^n > 0 \), and when it admits locally bounded potentials \( \varphi \). This implies, by a regularization result of Demailly [14], that \( [\omega] \) is big and nef. To simplify we actually assume that \( \omega \) (hence \( [\omega] \)) is Kähler.

**Proposition 2.5.**
1) If \( K \subset K' \subset X \) are Borel subsets then
\[ \text{Vol}_\omega(K) := \int_K \omega^n \leq \text{Cap}_\omega(K) \leq \text{Cap}_\omega(K') \leq \text{Cap}_\omega(X) = \text{Vol}_\omega(X). \]

2) If \( K_j \) are Borel subsets of \( X \) then \( \text{Cap}_\omega(\cup K_j) \leq \sum \text{Cap}_\omega(K_j) \). More over \( \text{Cap}_\omega(\cup K_j) \) if \( K_j \subset K_{j+1} \).

3) If \( \omega_1 \leq \omega_2 \) then \( \text{Cap}_{\omega_1}(\cdot) \leq \text{Cap}_{\omega_2}(\cdot) \). For all \( A \geq 1 \), \( \text{Cap}_\omega(\cdot) \leq \text{Cap}_{A\omega}(\cdot) \leq A^A \text{Cap}_\omega(\cdot) \). In particular if \( \omega, \omega' \) are two Kähler forms then there exists \( C \geq 1 \) such that
\[ \frac{1}{C} \text{Cap}_\omega(\cdot) \leq \text{Cap}_{\omega'}(\cdot) \leq C \cdot \text{Cap}_\omega(\cdot). \]

4) If \( f : X \to X \) is holomorphic then for all Borel subset \( K \) of \( X \),
\[ \text{Cap}_\omega(f(K)) \leq \text{Cap}_{f^*\omega}(K). \]

In particular \( \text{Vol}_\omega(f(K)) = \text{Cap}_\omega(K) \) for every \( \omega \)-isometry \( f \).

**Proof.** That \( \text{Vol}_\omega(\cdot) \leq \text{Cap}_\omega(\cdot) \) is a straightforward consequence of the definition (since \( \omega_0 = \omega \)). It then follows from Stokes theorem that \( \int_X \omega^n_\varphi = \int_X \omega^n \) for every \( \varphi \in \text{PSH}(X, \omega) \cap L^\infty(X) \), thus \( \text{Vol}_\omega(X) = \text{Cap}_\omega(X) \).

Property 2) is a straightforward consequence of the definitions.

If \( \omega_1 \leq \omega_2 \) then \( \text{PSH}(X, \omega_1) \subset \text{PSH}(X, \omega_2) \) hence \( \text{Cap}_{\omega_1}(\cdot) \leq \text{Cap}_{\omega_2}(\cdot) \).

Fix \( A \geq 1 \). If \( \psi \in \text{PSH}(X, A\omega) \) is such that \( 0 \leq \psi \leq 1 \) then \( \psi/A \in \text{PSH}(X, \omega) \) with \( 0 \leq \psi/A \leq 1/A \leq 1 \). Moreover \( (A\omega + dd^c\psi)^n = A^n(\omega + dd^c(\psi/A))^n \). This shows \( \text{Cap}_{A\omega}(\cdot) \leq A^A \text{Cap}_\omega(\cdot) \).
In particular if $\omega, \omega'$ are both Kähler then $A^{-1} \omega \leq \omega' \leq A \omega$ for some constant $A \geq 1$, hence $C^{-1} \text{Cap}_\omega(\cdot) \leq \text{Cap}_{\omega'}(\cdot) \leq C \cdot \text{Cap}_{\omega'}(\cdot)$ with $C = A^n$.

It remains to prove 4). It follows from the change of variables formula that if $\varphi \in PSH(X, \omega)$ with $0 \leq \varphi \leq 1$ then
\[
\int_{f(K)} \omega^n \leq \int_K f^* \omega^n = \int_K (f^* \omega + dd^c(\varphi \circ f))^n \leq \text{Cap}_{f^* \omega}(K)
\]
since $\varphi \circ f \in PSH(X, f^* \omega)$ with $0 \leq \varphi \circ f \leq 1$. We infer $\text{Cap}_\omega(f(K)) \leq \text{Cap}_{f^* \omega}(K)$. When $f$ is a $\omega$-isometry, i.e. $f \in \text{Aut}(X)$ with $f^* \omega = \omega$, then the mapping $\varphi \mapsto \varphi \circ f$ is an isomorphism of $\{u \in PSH(X, \omega) / 0 \leq u \leq 1\}$, whence $\text{Cap}_\omega(f(K)) = \text{Cap}_{f^* \omega}(K) = \text{Cap}_\omega(K)$.

Let $PSH^-(X, \omega)$ denote the set of negative $\omega$-psh functions. A set is said to be $PSH(X, \omega)$-polar if it is included in the $-\infty$ locus of some function $\psi \in PSH(X, \omega)$, $\psi \not\equiv -\infty$. As we shall soon see, the sets of zero Monge-Ampère capacity are precisely the $PSH(X, \omega)$-polar sets. We start by establishing the following:

**Proposition 2.6.** If $P$ is a $PSH(X, \omega)$-polar set, then $\text{Cap}_\omega(P) = 0$. More precisely, if $\psi \in PSH^-(X, \omega)$ then
\[
\text{Cap}_\omega(\psi < -t) \leq \frac{1}{t} \left[ \int_X (-\psi) \omega^n + n\text{Vol}_\omega(X) \right], \forall t > 0.
\]

**Proof.** Fix $\varphi \in PSH(X, \omega)$ such that $0 \leq \varphi \leq 1$. Fix $t > 0$ and set $K_t = \{x \in X / \psi(x) < -t\}$. By Chebyshev’s inequality,
\[
\int_{K_t} \omega^n \leq \int_X (-\psi/t) \omega^n \leq \frac{1}{t} \left[ \int_X (-\psi) \omega^n + n\text{Vol}_\omega(X) \right],
\]
where the last inequality follows from corollary 2.3. Taking supremum over all $\varphi$'s yields the claim. \qed

Observe that the previous proposition says that $\text{Cap}_\omega^*(P) = 0$, where $\text{Cap}_\omega^*(E) := \inf\{\text{Cap}_\omega(G) / G \text{ open with } E \subset G\}$, is the outer capacity associate to $\text{Cap}_\omega$.

Our aim is now to show that $\omega$-psh functions are quasicontinuous with respect to $\text{Cap}_\omega$ (corollary 2.8). We first need to show that decreasing sequences of $\omega$-psh functions converge "in capacity".

**Proposition 2.7.** Let $\psi, \psi_j \in PSH(X, \omega) \cap L^\infty(X)$ such that $(\psi_j)$ decreases to $\psi$. Then for each $\delta > 0$,
\[
\text{Cap}_\omega(\{\psi_j > \psi + \delta\}) \to 0.
\]

**Proof.** We can assume w.l.o.g. that $\text{Vol}_\omega(X) = 1$ and $0 \leq \psi_j - \psi \leq 1$. Fix $\delta > 0$ and $\varphi \in PSH(X, \omega)$, $0 \leq \varphi \leq 1$. By Chebyshev inequality, it suffices to control $\int_X (\psi_j - \psi) \omega^n$ uniformly in $\varphi$. It follows from Stokes theorem that
\[
\int_X (\psi_j - \psi) \omega^n = \int_X (\psi_j - \psi) \omega \wedge \omega^{n-1}_\varphi - \int_X d(\psi_j - \psi) \wedge d^c \varphi \wedge \omega^{n-1}_\varphi.
\]
Now by Cauchy-Schwartz inequality,
\[
\left| \int_X df_j \wedge d^c \varphi \wedge \omega^{n-1} \right| \leq \left( \int_X df_j \wedge f_j \wedge \omega^{n-1}_\varphi \right)^{1/2} \cdot \left( \int_X d^c \varphi \wedge f_j \wedge \omega^{n-1}_\varphi \right)^{1/2},
\]
where we set \( f_j := \psi_j - \psi \geq 0 \). Moreover
\[
\int_X d\varphi \wedge d^c \varphi \wedge \omega^n_{\varphi} = \int_X \varphi(-dd^c \varphi) \wedge \omega^n_{\varphi} \leq \int_X \varphi \omega \wedge \omega^n_{\varphi} \leq 1,
\]
since \( \varphi \omega \wedge \omega^n_{\varphi} \geq 0 \), \( -dd^c \varphi \leq \omega \) and \( \varphi \leq 1 \). Similarly
\[
\int_X df_j \wedge d^c f_j \wedge \omega^n_{\varphi} = \int_X -f_jdd^c f_j \wedge \omega^n_{\varphi} \leq \int_X f_j \omega \wedge \omega^n_{\varphi}.
\]
Altogether this yields
\[
\int_X (\psi_j - \psi) \omega^n_{\varphi} \leq \int_X (\psi_j - \psi) \omega \wedge \omega^n_{\varphi} + \left( \int_X (\psi_j - \psi) \omega \wedge \omega^n_{\varphi} \right)^{1/2} \leq \sqrt{2} \left( \int_X (\psi_j - \psi)(\omega + \omega \wedge \omega^n_{\varphi}) \right)^{1/2},
\]
where the last inequality follows from the elementary inequalities \( 0 \leq a \leq \sqrt{a} \leq 1 \) and \( \sqrt{a + b} \leq \sqrt{2} \sqrt{a + b} \).

Going on replacing at each step a term \( \omega_{\varphi} \) by \( \omega + \omega \psi \), we end up with
\[
\int_X (\psi_j - \psi) \omega^n_{\varphi} \leq 2 \left( \int_X (\psi_j - \psi)(\omega + \omega \psi)^n \right)^{1/2}.
\]
The majorant being independent of \( \varphi \) and converging to 0 as \( j \to +\infty \) (by dominated convergence theorem), this completes the proof.

**Corollary 2.8** (Quasicontinuity). Let \( \varphi \in PSH(X, \omega) \). For each \( \varepsilon > 0 \) there exists an open subset \( O_\varepsilon \) of \( X \) such that \( \text{Cap}_\omega(O_\varepsilon) < \varepsilon \) and \( \varphi \) is continuous on \( X \setminus O_\varepsilon \).

**Proof.** For \( t > 0 \) large enough, the set \( O_1 = \{ \varphi < -t \} \) has capacity \( < \varepsilon/2 \) by proposition 2.6. Working in \( X \setminus O_1 \) we can thus replace \( \varphi \) by \( \varphi_t = \max(\varphi, -t) \) which is bounded on \( X \). Regularizing \( \varphi \) (see Appendix), we can find a sequence \( \psi_j \) of smooth \( A \omega \)-psh functions which decrease to \( \varphi_t \) on \( X \), for some \( A \geq 1 \). By proposition 2.7, the set \( O_j = \{ \psi_{k_j} > \varphi_t + 1/j \} \) has capacity \( < \varepsilon 2^{-j-1} \) if \( k_j \) is large enough. Now \( \psi_{k_j} \) uniformly converges to \( \varphi = \varphi_t \) on \( X \setminus O_\varepsilon \), \( O_\varepsilon = \cup_{j \geq 1} O_j \), so \( \varphi \) is continuous on \( X \setminus O_\varepsilon \) and \( \text{Cap}_\omega(O_\varepsilon) \leq \varepsilon \). \( \Box \)

**Example 2.9.** The capacity \( \text{Cap}_\omega(\cdot) \) does not distinguish between "big sets". Assume indeed there exists an ample divisor \( D \) such that \( |D| \sim k\omega \), \( k \in \mathbb{N} \). Then there exists \( \varphi \in PSH(X, \omega) \) such that \( dd^c \varphi = k^{-1}|D| - \omega \). Note that \( \varphi \in \mathcal{C}^\infty(X \setminus D), \ v^\varphi \in \mathcal{C}^0(X) \) and \( \{ \varphi = -\infty \} = D \). Replacing \( \varphi \) by \( \varphi - \sup_X \varphi \) if necessary, we may assume \( \sup_X \varphi = 0 \). Consider \( \varphi_c = \max(\varphi, -c) \in PSH(X, \omega) \cap \mathcal{C}^0(X) \). Then \( \varphi_c \equiv \varphi \) outside some neighborhood \( V_c = \{ \varphi < -c \} \) of \( D \). Since \( 0 \leq 1 + \varphi_c \leq 1 \) and \( \omega_1 + \varphi_1 = \omega_\varphi = 0 \) in \( X \setminus V_1 \), we get
\[
\text{Cap}_\omega(X) = \int_X (\omega_1 + \varphi_1)^n = \int_{V_1} (\omega_1 + \varphi_1)^n \leq \text{Cap}_\omega(V_1),
\]
hence \( \text{Cap}_\omega(V_1) = \text{Cap}_\omega(X) \).

As a concrete example take \( X = \mathbb{C}P^n \) and \( \omega = \omega_{FS}, D \) being some hyper-plane \( H_\infty " \)at infinity" \( (k = 1) \). Set \( \varphi[z : t] = \log|t| - \frac{1}{2} \log(||z||^2 + |t|^2) \) where
z denotes the euclidean coordinates in \( \mathbb{C}^n = \mathbb{C}P^n \setminus H_{\infty} \) and \( H_{\infty} = (t = 0) \). Observe that \( \sup_{\mathbb{C}P^n} \varphi = 0 \). One then computes

\[
\mathbb{C}P^n \setminus V_1 = \left\{ z \in \mathbb{C}^n \mid |z| \leq \sqrt{e^2 - 1} \right\}.
\]

Thus the capacity of the complement of any euclidean ball of radius smaller than \( \sqrt{e^2 - 1} \) equals 1.

The definition of \( \text{Cap}_\omega \) mimics the definition of the relative Monge-Ampère capacity introduced by Bedford and Taylor in [5]. Fix \( \mathcal{U} = \{ \mathcal{U}_\alpha \} \) a finite covering of \( X \) by strictly pseudoconvex open subsets of \( X \), \( \mathcal{U}_\alpha = \{ x \in X / \varrho_\alpha(x) < 0 \} \), where \( \varrho_\alpha \) is a strictly psh smooth function defined in a neighborhood of \( \overline{\mathcal{U}_\alpha} \). Fix \( \delta > 0 \) such that \( \mathcal{U}_\delta = \{ \mathcal{U}_\alpha^\delta \} \) is still a covering of \( X \), where \( \mathcal{U}_\alpha^\delta = \{ x \in X / \varrho_\alpha(x) < -\delta \} \). For a Borel subset \( K \) of \( X \), we set

\[
\text{Cap}_{BT}(K) := \sum_\alpha \text{Cap}_{BT}(K \cap \mathcal{U}_\alpha^\delta, \mathcal{U}_\alpha),
\]

where

\[
\text{Cap}_{BT}(E, \Omega) := \sup \left\{ \int_E (dd^c u)^n / u \in PSH(\Omega), 0 \leq u \leq 1 \right\}
\]

is the capacity studied by Bedford and Taylor. The next proposition is due to Kolodziej [31]. We include a slightly different proof.

**Proposition 2.10.** There exists \( C \geq 1 \) such that

\[
\frac{1}{C} \text{Cap}_\omega(\cdot) \leq \text{Cap}_{BT}(\cdot) \leq C \cdot \text{Cap}_\omega(\cdot).
\]

**Proof.** Let \( E \) be a Borel subset of \( X \). Since \( \text{Cap}_\omega(E \cap \mathcal{U}_\alpha^\delta) \leq \text{Cap}_\omega(E) \leq \sum_\alpha \text{Cap}_\omega(E \cap \mathcal{U}_\alpha^\delta) \), it is sufficient to show that if \( \Omega = \{ x \in X / \varrho(x) < 0 \} \) is a smooth hyperconvex subset of \( X \), then there exists \( C \geq 1 \) such that for all \( E \subset \Omega_\delta \),

\[
\frac{1}{C} \text{Cap}_\omega(E) \leq \text{Cap}_{BT}(E, \Omega) \leq C \cdot \text{Cap}_\omega(E),
\]

where \( \Omega_\delta = \{ x \in X / \varrho(x) < -\delta \} \).

It is an easy and well known fact in the local theory that the capacities \( \text{Cap}(\cdot, \Omega) \) and \( \text{Cap}(\cdot, \Omega') \) are comparable when \( \Omega' \subset \Omega \) (see e.g. theorem 6.5 in [14]). Therefore we can assume (passing to a finer covering if necessary) that \( \omega = dd^c \psi \) near \( \overline{\Omega} \). Fix \( C_1 > 0 \) such that \( -C_1 \leq \psi \leq C_1 \) on \( \Omega \). Fix \( \varphi \in PSH(X, \omega) \) such that \( 0 \leq \varphi \leq 1 \) on \( X \) and set \( u = (2C_1)^{-1}(\varphi + \psi + C_1) \). Then \( u \in PSH(\Omega) \) and \( 0 \leq u \leq 1 \), hence

\[
\int_E \omega^n = (2C_1)^n \int_E (dd^c u)^n \leq (2C_1)^n \text{Cap}_{BT}(E, \Omega),
\]

which yields \( \text{Cap}_\omega(E) \leq (2C_1)^n \text{Cap}_{BT}(E, \Omega) \). Observe that we have not used here that \( \omega \) is Kähler.

For the reverse inequality we consider \( \chi \in C^\infty(X) \) such that \( \chi \equiv 0 \) in \( X \setminus \Omega \) and \( \chi < 0 \) in \( \Omega \). Replacing \( \chi \) by \( \varepsilon \chi \) if necessary, we can assume \( \chi \in PSH(X, \omega) \). This is because \( \omega \) is Kähler (and this is the only place
where we shall use this crucial assumption). Fix now \( \varepsilon > 0 \) so small that \( \chi \leq -\varepsilon \) on \( \Omega_\delta \). Let now \( u \in PSH(\Omega) \) be such that \( 0 \leq u \leq 1 \) on \( \Omega \). Consider

\[
\varphi(x) = \begin{cases} 
\frac{u - \psi + C_1}{2 + 2C_1} & \text{in } \Omega_\delta \\
\max \left( \frac{u - \psi + C_1}{2 + 2C_1}, \frac{2}{\varepsilon} \chi(x) + 1 \right) & \text{in } \Omega \setminus \Omega_\delta \\
1 & \text{in } X \setminus \Omega \n
\end{cases}
\]

Observe that \( 0 \leq u' := (u - \psi + C_1)/(2 + 2C_1) \leq (1 + 2C_1)/(2 + 2C_1) < 1 \) in \( \Omega \). Therefore \( \varphi \in PSH(X, \frac{2}{\varepsilon} \omega) \) since \( \frac{2}{\varepsilon} \chi(x) + 1 \leq -1 < u' \) in \( \Omega_\delta \), while \( \frac{2}{\varepsilon} \chi(x) + 1 \equiv 1 > u' \) on \( \partial \Omega \). Note also that \( 0 \leq \varphi \leq 1 \) thus for \( E \subset \Omega_\delta \),

\[
\frac{1}{(2 + 2C_1)^n} \int_E (dd^c u)^n = \int_E \left( \frac{\omega}{2 + 2C_1} + dd^c \varphi \right)^n \leq \int_E \left( \frac{2}{\varepsilon} \omega + dd^c \varphi \right)^n \leq \text{Cap}_{2\omega/\varepsilon}(E) \leq \left( \frac{2}{\varepsilon} \right)^n \text{Cap}_\omega(E)
\]

hence \( \text{Cap}_{BT}(E, \Omega) \leq 4^n (1 + C_1)^n \varepsilon^{-n} \text{Cap}_\omega(E) \).

Since locally pluripolar sets are precisely the sets of zero relative capacity [5], we obtain the following

**Corollary 2.11.** \( \text{Cap}_\Omega^*(P) = 0 \iff \text{Cap}_{BT}^*(P) = 0 \iff P \) is locally pluripolar.

We shall show later on that locally pluripolar sets are \( PSH(X, \omega) \)-polar when \( \omega \) is Kähler (see theorem 6.2).

The following two results are direct consequences of the corresponding results of Bedford and Taylor [4], [5].

**Theorem 2.12** (Dirichlet Problem). Let \( \varphi \in PSH(X, \omega) \cap L^\infty(X) \). Let \( B \) be a small ball in \( X \). Then there exists \( \hat{\varphi} \in PSH(X, \omega) \) such that \( \hat{\varphi} = \varphi \) in \( X \setminus B \), \( \varphi \geq \hat{\varphi} \) and \( (\omega_{\hat{\varphi}})^n = 0 \) in \( B \). Moreover if \( \varphi_1 \leq \varphi_2 \) then \( \hat{\varphi}_1 \leq \hat{\varphi}_2 \).

**Theorem 2.13** (Comparison principle). Let \( \varphi, \psi \in PSH(X, \omega) \cap L^\infty(X) \). Then

\[
\int_{\{\varphi < \psi\}} \omega^n_{\varphi} \leq \int_{\{\varphi < \psi\}} \omega^n_{\psi}.
\]

**3. The relative extremal function**

We now introduce a substitute for the relative extremal function which has revealed so useful in the local theory [5]. Namely if \( E \) is a Borel subset of \( X \), we set

\[
h_{E, \omega}(x) := \sup \left\{ \varphi(x) \mid \varphi \in PSH(X, \omega), \varphi \leq 0 \text{ and } \varphi|_E \leq -1 \right\}.
\]

We let \( h_{E, \omega}^* \) denote its upper-semi-continuous regularization, which we call the relative \( \omega \)-plurisubharmonic extremal function of the subset \( E \subset X \). It enjoys several natural properties; we list some of them below. The proofs follow from standard arguments together with theorem 6.2:

- The function \( h_{E, \omega}^* \) is \( \omega \)-psh. It satisfies \( -1 \leq h_{E, \omega}^* \leq 0 \) on \( X \) and \( h_{E, \omega}^* = -1 \) on \( E \setminus P \), where \( P \) is pluripolar.
- If \( E \subset X \) and \( P \subset X \) is pluripolar, then \( h_{E \setminus P}^* \equiv h_{E, \omega}^* \); in particular \( h_{P}^* \equiv 0 \).
- If \( (E_j) \) increases towards \( E \subset X \), then \( h_{E_j}^* \) decreases towards \( h_{E}^* \).
• If \((K_j)\) is a sequence of compact subsets decreasing towards \(K\), then \(h_{K_j}^*\) increases (a.e.) towards \(h_K^*\).

As in the local theory, the complex Monge-Ampère of the relative extremal function of a subset \(E \subset X\) vanishes outside \(E\), except perhaps on the set \(\{h_E^* = 0\}\) which, in the local theory, lies in the boundary of the domain.

**Proposition 3.1.** When the open set \(\Omega_E := \{x \in X / h_{E,\omega}^*(x) < 0\}\) is non-empty, then

\[
(\omega_{h_{E,\omega}^*})^n = (\omega + dd^c h_{E,\omega}^*)^n = 0 \text{ in } \Omega_E \setminus \overline{E}.
\]

**Proof.** Assume that \(\Omega_E := \{h_{E,\omega}^* < 0\}\) is non-empty. It follows from Choquet’s lemma that there exists an increasing sequence \(\varphi_j\) of \(\omega\)-psh functions such that \(\varphi_j = -1\) on \(E\), \(\varphi_j \leq 0\) on \(X\), and \(h_{E,\omega}^*(\varphi) = (\lim \varphi_j)\). Let \(a \in \Omega_E \setminus \overline{E}\) and fix a small ball \(B \subset \Omega_E\) centered at point \(a\). Let \(\widehat{\varphi}_j = (\varphi_j)_B\) denote the functions obtained by applying theorem 2.12, so that \((\omega_{\widehat{\varphi}_j})^n = 0\) in \(B\). If \(B\) is chosen small enough, then \(\widehat{\varphi}_j < 0\) in \(B\), hence \(\widehat{\varphi}_j \leq 0\) on \(X\), while \(\widehat{\varphi}_j = \varphi_j = -1\) on \(E\). This can be seen by showing that \(0_B \to 0\) as the radius of the ball \(B\) shrinks to 0. Therefore \(\lim \int \omega_{\widehat{\varphi}_j} = h_{E,\omega}^*, \text{ hence } (\omega_{h_{E,\omega}^*})^n \equiv 0\) in a neighborhood of \(a\), which prove our claim. \(\square\)

We now establish an important result which expresses the capacity in terms of the relative extremal function for any subset. It will show in particular that the set function \(\text{Cap}_\omega^*\) is a capacity in the sense of Choquet which is outer regular. For simplicity, we write \(h_E\) for \(h_{E,\omega}^*\).

**Theorem 3.2.** Let \(E \subset X\) be any Borel subset, then

\[
\text{Cap}_\omega^*(E) = \int_X (\omega_{h_E^*})^n. \tag{\dagger}
\]

The Monge-Ampère capacity satisfies the following continuity properties:

1) If \((E_j)_{j \geq 0}\) is an increasing sequence of arbitrary subsets of \(X\) and 
\(E := \cup_{j \geq 0} E_j\) then

\[
\text{Cap}_\omega^*(E) = \lim_{j \to +\infty} \text{Cap}_\omega^*(E_j).
\]

2) If \((K_j)_{j \geq 0}\) is a decreasing sequence of compact subsets of \(X\) and 
\(K := \cap_{j \geq 0} K_j\) then

\[
\text{Cap}_\omega(K) = \text{Cap}_\omega^*(K) = \lim_{j \to +\infty} \text{Cap}_\omega(K_j).
\]

In particular \(\text{Cap}_\omega^*(\cdot)\) is an outer regular Choquet capacity on \(X\).

**Proof.** We first establish (\dagger) when \(E = K \subset X\) is compact. Observe that in the definition of the capacity, it is enough to restrict ourselves to \(\omega\)-psh functions \(\varphi\) such that \(-1 < \varphi < 0\). Let \(\varphi\) be such a function. The pluripolar set \(N := \{h_K < h_K^*\}\) is of measure 0 for the measure \(\omega_\varphi^n\). Since \(-1 < \varphi\), we have \(K \subset N \cup \{h_K^* < \varphi\}\), hence the comparison principle yields

\[
\int_K \omega_\varphi^n \leq \int_{\{h_K^* < \varphi\}} \omega_\varphi^n \leq \int_{\{h_K < \varphi\}} \omega_{h_K^*}^n.
\]
This shows \( \text{Cap}_\omega(K) \leq \int_{\Omega_K} \omega^n_{h_K} \), since \( \{ h_K^* < \varphi \} \subset \Omega_K \). It follows from proposition 3.1 that
\[
\text{Cap}_\omega(K) \leq \int_{\Omega_K} (\omega^n_{h_K}) = \int_K (\omega^n_{h_K})^n,
\]
whence equality.

Recall that \( K \cap \{ h_K^* > -1 \} \subset \{ h_K < h_K^* \} \) is of measure 0 for \( \omega^n_{h_K^*} \), so
\[
\text{Cap}_\omega(K) = \int_K (\omega^n_{h_K}) = \int_K (-h_{h_K}^*)(\omega^n_{h_K}) = \int_X (-h_{h_K}^*)(\omega^n_{h_K})^n,
\]
where the last equality follows from the fact that the equilibrium measure \( (\omega^n_{h_K})^n \) is supported on \( K \cup \{ h_K = 0 \} \).

We assume now that \( E = G \subset X \) is an open subset. Let \((K_j)\) be an exhaustive sequence of compact subsets of \( G \) which increases to \( G \). Since \( h_{K_j}^* \uparrow h_G \) on \( X \), it follows from classical convergence results (see [40]) that
\[
(-h_{K_j}^*)(\omega^n_{h_{K_j}^*}) \rightarrow (-h_G^*)(\omega^n_{h_G})
\]
in the weak sense of measures on \( X \), therefore
\[
\int_X (-h_G^*)(\omega^n_{h_G}) = \lim_{j \to +\infty} \int_X (-h_{K_j}^*)(\omega^n_{h_{K_j}^*}) = \lim_{j \to +\infty} \text{Cap}_\omega(K_j) = \text{Cap}_\omega(G),
\]
This proves \((\dagger)\) when \( E \subset X \) is an open subset.

Finally let \( E \subset X \) be any subset. By definition of the outer capacity, there is a sequence of open subsets \((O_j)_{j \geq 1}\) of \( X \) containing \( E \) such that
\[
\text{Cap}_\omega^*(E) = \lim_{j \to +\infty} \text{Cap}_\omega(O_j).
\]
We can assume w.l.o.g. that the sequence \((O_j)_{j \geq 1}\) is decreasing.

By a classical topological lemma of Choquet, there exists an increasing sequence \((u_j)_{j \geq 1}\) negative \( \omega \)-psh functions on \( X \) s.t. \( u_j = -1 \) on \( E \) with \( u_j \uparrow h_E^* \) almost everywhere on \( X \). We set for each \( j \in \mathbb{N}, G_j := O_j \cap \{ u_j < -1 + 1/j \} \). Then \((G_j)\) is a decreasing sequence of open subsets of \( X \) such that \( E \subset G_j \subset O_j \) and \( u_j - 1/j \leq h_{G_j} \leq h_E \), so \( h_{G_j} \uparrow h_E^* \) almost everywhere on \( X \). We infer \((h_{G_j})(\omega^n_{h_{G_j}^*}) \rightarrow (h_E^*)(\omega^n_{h_E^*})\) in the weak sense of measures on \( X \), thus
\[
\int_X (-h_E^*)(\omega^n_{h_E^*}) = \lim_{j \to +\infty} \int_X (-h_{G_j}^*)(\omega^n_{h_{G_j}^*}).
\]
On the other hand we have by construction \( \text{Cap}_\omega^*(E) \leq \lim_{j \to +\infty} \text{Cap}_\omega^*(G_j) \leq \lim_{j \to +\infty} \text{Cap}_\omega^*(O_j) = \text{Cap}_\omega^*(E) \). Therefore using \((\dagger)\) for open subsets, we get
\[
\int_X (-h_E^*)(\omega^n_{h_E^*}) = \text{Cap}_\omega^*(E).
\]

Observe that 1) follows straightforwardly from this formula. Indeed if \((E_j)\) increases towards \( E \), then \( h_{E_j}^* \) decreases towards \( h_E^* \), hence \((h_{E_j}^*)(\omega^n_{h_{E_j}^*}) \rightarrow (h_E^*)(\omega^n_{h_E^*})\), so that
\[
\text{Cap}_\omega^*(E) = \int_X (-h_{E_j}^*)(\omega^n_{h_{E_j}^*}) = \lim_{j \to +\infty} \int_X (-h_{E_j}^*)(\omega^n_{h_{E_j}^*}) = \lim \text{Cap}_\omega^*(E_j).
\]

It remains to prove 2). Let \((K_j)\) be a decreasing sequence of compact subsets of \( X \) which converges to \( K \). We claim that \( h_{K_j}^* \uparrow h_K^* \) almost everywhere on \( X \). Indeed, the extremal function \( h_{K_j}^* \) increases almost everywhere to a \( \omega \)-psh function \( h \) such that \( h \leq h_K^* \) on \( X \). We want to prove that \( h_K \leq h \).
on $X$. Let $u \in \text{PSH}(X, \omega)$ such that $u \leq 0$ on $X$ and $u|_{K} \leq -1$. Fix $\varepsilon > 0$ and consider the open subset $G_{\varepsilon} := \{u < -1 + \varepsilon\}$. Then $K \subset G_{\varepsilon}$, thus $K_j \subset G_{\varepsilon}$ for $j$ large enough. This yields $u - \varepsilon \leq h_{K_j}$ for $j$ large enough, hence $u \leq h$ on $X$. Therefore $h_{K} \leq h$ on $X$ as claimed.

Since $(-h_{K_j})\omega^{n}_{h_{K_j}}$ converges weakly to $(-h_{K})\omega^{n}_{h_{K}}$ on $X$, we infer

$$\text{Cap}_{\omega}(K) = \int_{X} (-h_{K})\omega^{n}_{h_{K}} = \lim_{j \to +\infty} \int_{X} (-h_{K_j})\omega^{n}_{h_{K_j}} = \lim_{j \to +\infty} \text{Cap}_{\omega}(K_{j}).$$

From this last property, taking a decreasing sequence $(K_{j})_{j \geq 0}$ of compact subsets such that $K = \cap_{j \geq 0} K_{j}$ and $K_{j+1} \subset K_{j}$ for any $j \in \mathbb{N}$ we obtain $\text{Cap}_{\omega}(K) = \lim_{j \to +\infty} \text{Cap}_{\omega}(K_{j}) = \text{Cap}_{\omega}(K)$. \hfill \Box

**Corollary 3.3.** For any subset $P \subset X$, we have $\text{Cap}^{*}_{\omega}(P) = 0 \iff h_{P}^{*} \equiv 0$.

4. **Alexander capacity**

We now introduce another capacity which is defined by means of a global extremal function. It is closely related to the projective capacity introduced by Alexander in [1]. We assume throughout this section that $\omega$ is a closed real current on $X$ with continuous local potentials.

4.1. **Global extremal functions.**

**Definition 4.1.** Let $K$ be a Borel subset of $X$. We set

$$V_{K,\omega} := \sup \{\varphi(x) / \varphi \in \text{PSH}(X, \omega), \varphi \leq 0 \text{ on } K\}.$$ 

This definition mimics the definition of the so-called "Siciak’s extremal function" usually defined for Borel subset of $X = \mathbb{CP}^{n}$ that are bounded in $\mathbb{C}^{n} = X \setminus H_{\infty}$, where $H_{\infty}$ denotes some hyperplane at infinity. This function was introduced and studied by Siciak in [37],[38] (see also [41]). One can indeed check that this definition coincides with the classical one if one chooses $\omega = [H_{\infty}]$ to be the current of integration along the hyperplane $H_{\infty}$. Similarly one could consider the case where $\omega = [D]$ is the current of integration along a positive divisor $D$ on $X$ and let $D$ play the role of infinity. This approach has been used by some authors working in Arakelov geometry to define capacities on projective varieties (see [32],[11] and references therein). However this forces them to consider only compact subsets of $X \setminus D$ and leads to less intrinsic notions of capacities.

In this article we always assume that the currents $\omega$ involved admit continuous potentials. This insures that the Monge-Ampère operator $\omega^{n}_{\varphi}$ is well-defined on extremal functions $V_{K,\omega}$. If $\varphi \in L^{1}(X)$, we shall denote by $\varphi^{*}$ its upper-semi-continuous regularization.

**Theorem 4.2.** Let $K$ be a Borel subset of $X$.

1) $K$ is $\text{PSH}(X,\omega)$-polar iff $\sup_{X} V_{K,\omega}^{*} = +\infty$ iff $V_{K,\omega}^{*} \equiv +\infty$.

2) If $K$ is not $\text{PSH}(X,\omega)$-polar then $V_{K,\omega}^{*} \in \text{PSH}(X,\omega)$ and satisfies $V_{K,\omega}^{*} \equiv 0$ in the interior of $K$, $(\omega V_{K,\omega}^{*})^{n} = 0$ in $X \setminus \overline{K}$ and

$$\int_{K} (\omega V_{K,\omega}^{*})^{n} = \int_{X} \omega^{n} = \text{Vol}_{\omega}(X).$$
Proof. Assume \( \sup_X V_{K,\omega}^* = +\infty \). By a lemma of Choquet (see lemma 4.23 in [17], chapter 1), we can find an increasing sequence of functions \( \varphi_j \in PSH(X,\omega) \) such that \( \varphi_j = 0 \) on \( K \) and \( V_{K,\omega}^* = (\lim \nearrow \varphi_j)^* \). Extracting a subsequence if necessary, we can assume \( \sup_X \varphi_j \geq 2^j \). Set \( \psi_j = \varphi_j - \sup_X \varphi_j \). These functions belong to \( \mathcal{F}_0 \) which is a compact subfamily of \( PSH(X,\omega) \) (corollary 1.7). Recall that if \( \mu \) is a smooth volume form on \( X \) then there exists \( C_\mu \) such that \( \int \psi d\mu \geq -C_\mu \) for all \( j \). Set \( \psi := \sum_{j\geq 1} 2^{-j} \psi_j \). Then \( \psi \in PSH(X,\omega) \) as a decreasing limit of functions in \( PSH(X,\omega) \) with \( \int_X \psi d\mu \geq -C_\mu > -\infty \). Now for every \( x \in K \) we get \( \psi(x) = -\sum_{j\geq 1} 2^{-j} \sup_X \varphi_j = -\infty \) hence \( K \subset \{ \psi = -\infty \} \), i.e. \( K \) is \( PSH(X,\omega) \)-polar.

Conversely assume \( K \) is \( PSH(X,\omega) \)-polar, \( K \subset \{ \psi = -\infty \} \) for some \( \psi \in PSH(X,\omega) \). Then for all \( c \in \mathbb{R}, \psi+c \in PSH(X,\omega) \) and \( \psi+c \leq 0 \) on \( K \). Therefore \( V_{K,\omega} \geq \psi + c, \forall c \in \mathbb{R} \). This yields \( V_{K,\omega} = +\infty \) on \( X \setminus \{ \psi = -\infty \} \), hence \( V_{K,\omega}^* \equiv +\infty \) on \( X \) since \( \{ \psi = -\infty \} \) has zero volume. We have thus shown the following circle of implications: \( K \) is \( PSH(X,\omega) \)-polar \( \Rightarrow \) there exists \( \psi \) such that \( V_{K,\omega}^* \equiv +\infty \Rightarrow \sup_X V_{K,\omega}^* = +\infty \Rightarrow K \) is \( PSH(X,\omega) \)-polar.

Assume now that \( K \) is not \( PSH(X,\omega) \)-polar. Then \( V_{K,\omega}^* \in PSH(X,\omega) \) (see proposition 1.6.2) and clearly satisfies \( V_{K,\omega}^* = 0 \) in the interior of \( K \). If we show that \( (\omega V_{K,\omega}^*)^n = 0 \) in \( X \setminus K \) then

\[
\int_K (\omega V_{K,\omega}^*)^n = \int_X (\omega V_{K,\omega}^*)^n = \int_X \omega^n,
\]

as follows from Stokes theorem. Let \( \varphi_j \in PSH(X,\omega) \) be an increasing sequence such that \( \varphi_j = 0 \) on \( K \) and \( V_{K,\omega}^* = (\lim \nearrow \varphi_j)^* \). Fix \( B \) a small ball in \( X \setminus K \). Let \( \tilde{\varphi}_j \) be the solution of the Dirichlet problem with boundary values \( \varphi_j \). Then \( \tilde{\varphi}_j \in PSH(X,\omega), \tilde{\varphi}_j = \varphi_j \) in \( X \setminus B \) (in particular \( \tilde{\varphi}_j = 0 \) on \( K \) hence \( \tilde{\varphi}_j \leq V_{K,\omega} \)) and the sequence \( (\tilde{\varphi}_j) \) is again increasing (theorem 2.12). Since \( (\omega \tilde{\varphi}_j)^n = 0 \) in \( B \) and \( (\lim \nearrow \tilde{\varphi}_j) = V_{K,\omega}^* \), it follows from the continuity of the complex Monge-Ampère on increasing sequences that \( (\omega V_{K,\omega}^*)^n = 0 \) in \( B \). As \( B \) was an arbitrarily small ball in \( X \setminus K \) we infer \( (\omega V_{K,\omega}^*)^n = 0 \) in \( X \setminus K \).

The following corollary has to be related to proposition 1.7.

**Corollary 4.3.** Let \( K \) be a Borel subset of \( X \) and set

\[
\mathcal{F}_K := \{ \varphi \in PSH(X,\omega) / \sup_K \varphi = 0 \}.
\]

Then \( \mathcal{F}_K \) is relatively compact iff \( K \) is not \( PSH(X,\omega) \)-polar.

**Proof.** Observe that \( V_{K,\omega}(x) = \sup \{ \varphi(x) / \varphi \in \mathcal{F}_K \} \). Thus if \( \mathcal{F}_K \) is relatively compact then it is uniformly bounded from above, hence \( \sup_X V_{K,\omega} < +\infty \), i.e. \( K \) is not \( PSH(X,\omega) \)-polar.

Assume conversely that \( K \) is not \( PSH(X,\omega) \)-polar. Let \( (\varphi_j) \in \mathcal{F}_K^N \). Then \( \varphi_j \leq V_{K,\omega} \leq \sup_X V_{K,\omega} < +\infty \) hence \( (\varphi_j) \) is uniformly bounded from above. It follows from proposition 1.6 that \( (\varphi_j) \) is relatively compact. Indeed it can not converge uniformly to \( -\infty \) since \( \sup_K \varphi_j = 0 \) (see proposition 1.6). \( \square \)
Proposition 4.4. Let $K$ be a Borel subset of $X$.

1) If $K' \subset K$ then $V_{x,\omega} \leq V_{K,\omega} \leq V_{K',\omega}$ and $\sup x V_{x,\omega} = 0$. Furthermore $V_{x,\omega} \equiv 0$ when $\omega \geq 0$.

2) If $\omega_1 \leq \omega_2$ then $V_{K,\omega_1} \leq V_{K,\omega_2}$.

3) For all $A > 0$, $V_{K,A,\omega} = A \cdot V_{K,\omega}$.

4) If $\omega' = \omega + dd^c \chi$ then
$$-\chi + \inf_{x} \chi + V_{K,\omega} \leq V_{K,\omega'} \leq V_{K,\omega} + \sup_{x} \chi - \chi.$$

5) If $f : X \to X$ is holomorphic then
$$V_{f(K),\omega} \circ f \leq V_{K,f^*\omega}.$$ 

In particular if $f$ is a $\omega$-isometry then $V_{f(K),\omega} = V_{K,\omega}$.

Proof. That $K \mapsto V_{K,\omega}$ is decreasing follows straightforwardly from the definition. Observe that $0 \in \text{PSH}(X,\omega)$ when $\omega \geq 0$, hence $V_{x,\omega} \equiv 0$ in this case. When $\omega$ is smooth (but not positive), considering $V_{x,\omega}^*$ will be a useful way of constructing a positive closed current $\omega V_{x,\omega}^*$ with minimal singularities (see section 5).

Assertions 2,3,4 are simple consequences of proposition 1.3. The last assertion results from the following observation: if $\varphi \in \text{PSH}(X,\omega)$ is such that $\varphi \leq 0$ on $f(K)$, then $\varphi \circ f$ belongs to $\text{PSH}(X,f^*\omega)$ and satisfies $\varphi \circ f \leq 0$ on $K$.

Example 4.5. Assume $X = \mathbb{C}^n$, $\omega$ is the Fubini-Study Kähler form and let $B_R$ denote the euclidean ball centered at the origin and of radius $R$ in $\mathbb{C}^n \subset \mathbb{C}^n$. Then for $x \in \mathbb{C}^n$,
$$V_{B_R,\omega}(x) = \max \left( \log \frac{|x|}{R} + \frac{1}{2} \log [1 + R^2] - \frac{1}{2} \log [1 + ||x||^2]; 0 \right).$$

Indeed set $\psi_R := \max \left( \frac{1}{2} \log [1 + ||x||^2], \varphi_R \right)$, where $\varphi_R = \frac{1}{2} \log [1 + R^2] + \log \frac{|x|}{R}$. Recall that the usual Siciak’s extremal function of $B_R$ is $\log^+ \frac{|x|}{R}$. Therefore $\frac{1}{2} \log [1 + ||x||^2] \leq \varphi_R = \psi_R$ for $||x|| \geq R$. On the other hand if $||x|| < R$ then $1 + ||x||^2 > (1 + R^2) \frac{|x|^2}{R^2}$ hence $\frac{1}{2} \log [1 + ||x||^2] > \varphi_R$ in $B_R$.

Now let $u \in \text{PSH}(\mathbb{C}^n,\omega)$ such that $u \leq 0$ in $B_R$. Then $v = u + \frac{1}{2} \log [1 + ||x||^2] \in \mathcal{L}(\mathbb{C}^n)$. Since $v \leq \frac{1}{2} \log [1 + R^2]$ in $B_R$ we infer $v \leq \frac{1}{2} \log [1 + R^2] + \log^+ \frac{|x|}{R} = \psi_R$ in $\mathbb{C}^n \setminus B_R$. Moreover $v \leq \frac{1}{2} \log [1 + ||x||^2] = \psi_R$ in $B_R$ hence $v \leq \psi_R$ in $\mathbb{C}^n$. This shows $V_{B_R,\omega} = \psi_R - \frac{1}{2} \log [1 + ||x||^2]$ on $\mathbb{C}^n$.

Proposition 4.6.

1) If $E$ is an open subset, then $V_{E} = V_{E}^*$.

2) Let $E$ be a Borel subset and $P$ a $\text{PSH}(X,\omega)$-polar set. Then $V_{E \cup P}^* \equiv V_{E}^*$.

3) Let $(E_j)$ be an increasing sequence of Borel subsets and set $E = \bigcup E_j$. Then $V_{E,\omega} = \lim \bigvee V_{E_j,\omega}$ if $\omega$ is Kähler.

4) Let $K_j$ be a decreasing sequence of compact subsets of $X$ and set $K = \bigcap K_j$. Then $V_{K_j,\omega} \not\sim V_{K,\omega}$, hence $V_{K_j,\omega} \not\sim V_{K,\omega}$ a.e.
5) Fix $E \subset X$ a non-pluripolar set. Then there exists $G_j$ a decreasing sequence of open subsets, $E \subset G_j$, such that $V^*_E = \lim V^*_{G_j}$.

Proof. We write here $V^*_E$ for $V_{E,\omega}$ since $\omega$ is fixed and no confusion can arise.

Let $E$ be an open subset of $X$. Observe that $V^*_E \leq 0$ on $E$, hence $V^*_E \leq 0$ on $E$ which is open. Therefore $V^*_E \leq V^*_E$, whence equality. This proves 1).

Let $w \in \text{PSH}(X, \omega)$, $w \leq 0$, and fix $P \subset \{w = -\infty\}$. Fix $E$ a Borel subset of $X$. Clearly $V_{E,P} \leq V^*_E$ hence $V_{E,P} \leq V^*_E$. Conversely let $\varphi \in \text{PSH}(X, \omega)$ be such that $\varphi \leq 0$ on $E$. Then $\forall \varepsilon > 0$, $\psi_\varepsilon := (1 - \varepsilon)\varphi + \varepsilon w \in \text{PSH}(X, \omega)$ satisfies $\psi_\varepsilon \leq 0$ on $E \cup P$, hence $\psi_\varepsilon \leq V_{E,P}$. Letting $\varepsilon \to 0$ we infer $\varphi \leq V_{E,P}$ on $X \setminus P$, hence $\varphi \leq V^*_E$ on $X$. Thus $V^*_E \leq V_{E,P}$.

Let $E_j$ be an increasing sequence of subsets of $X$ and set $E = \bigcup_{j \geq 0} E_j$. Let $v := \lim \bigcap_{j \geq 0} V^*_{E_j}$ (the limit is decreasing by 4.4.1). If $E$ is $\text{PSH}(X, \omega)$-polar then so are all the $E_j$'s, hence $V^*_E \equiv +\infty = \lim V^*_{E_j}$. So let us assume $E$ is not $\text{PSH}(X, \omega)$-polar. Then $v \in \text{PSH}(X, \omega)$ since $v \geq V^*_E \not\equiv -\infty$ (see proposition 1.6.3). Observe that $v = 0$ on the set $E \setminus N$, where $N = \bigcup_{j \geq 0} \{E_j \leq V^*_{E_j}\}$. The latter is called a negligible set. It follows from the local theory [5] together with theorem 6.2 that $N$ is $\text{PSH}(X, \omega)$-polar. Therefore $V^*_E \leq v \leq V^*_E$ by 2).

Let $K_j$ be a decreasing sequence of compact subsets and set $K = \bigcap_j K_j$. Clearly $\lim \uparrow V^*_{K_j} \leq V^*_K$. Fix $\varepsilon > 0$ and let $\varphi \in \text{PSH}(X, \omega)$ be such that $\varphi \leq 0$ on $K$. Then $\{\varphi < \varepsilon\}$ is an open set which contains all $K_j$'s, for $j \geq j_0$ large enough. Thus $\varphi - \varepsilon \leq 0$ on $K_j$, hence $\varphi - \varepsilon \leq \lim \uparrow V^*_{K_j}$. Taking the supremum over all such $\varphi$'s and letting $\varepsilon \to 0$ yields the reverse inequality $V^*_K \leq \lim \uparrow V^*_{K_j}$. The conclusion on the convergence of the upper semi-continuous regularizations follows now from proposition 1.6.

It remains to prove 5). By Choquet's lemma, there exists an increasing sequence $\varphi_j \in \text{PSH}(X, \omega)$ such that $\varphi_j \leq 0$ on $E$ and $V^*_E = (\sup_j \varphi_j)^*$. Set $G_j := \{\varphi_j < 1/j\}$. This defines a decreasing sequence of open subsets containing $E$. Observe that $\varphi_j - 1/j \leq V_{G_j} \leq V^*_E$, hence $\lim \varphi_j \leq \lim V_{G_j} \leq V^*_E$. Therefore $V^*_E = \lim V^*_{G_j}$.

4.2. Alexander capacity.

Definition 4.7. Let $K$ be a Borel subset of $X$. We set

$$T_\omega(K) := \exp(-\sup_X V^*_K).$$

This capacity characterizes again $\text{PSH}(X, \omega)$-polar sets:

Proposition 4.8. Let $P$ be a Borel subset. Then $T_\omega(P) = 0$ iff $P$ is $\text{PSH}(X, \omega)$-polar. Moreover if $\varphi \in \text{PSH}(X, \omega)$ then

$$T_\omega(\varphi < -t) \leq C_\varphi \exp(-t), \forall t \in \mathbb{R},$$

where $C_\varphi = \exp(-\sup_X \varphi)$.

Proof. The first assertion follows from theorem 4.2. Let $\varphi \in \text{PSH}(X, \omega)$, $t \in \mathbb{R}$ and set $K_t = \{\varphi < -t\}$. Then $\varphi + t \leq 0$ on $K_t$ hence $\varphi + t \leq V^*_{K_t,\omega}$. We infer $\sup_X \varphi + t \leq \sup_X V^*_{K_t,\omega}$ which yields $T_\omega(K_t) \leq \exp(-\sup_X \varphi \exp(-t))$. □
The following proposition is an immediate consequence of proposition 4.4. It shows that capacities $T_\omega$, $T_{\omega'}$ are comparable if $\omega, \omega'$ are both Kähler. Further they enjoy nice invariance properties.

**Proposition 4.9.**

1) For all Borel subsets $K' \subset K \subset X$, $T_\omega(K') \leq T_\omega(K) \leq T_\omega(X) = 1$.

2) If $\omega_1 \leq \omega_2$ then $T_{\omega_1}(\cdot) \geq T_{\omega_2}(\cdot)$. For all $A > 0$, $T_{A\omega}(\cdot) = [T_\omega(\cdot)]^A$. In particular if $\omega$ and $\omega'$ are both Kähler then there exists $C \geq 1$ such that

$$[T_\omega(\cdot)]^C \leq T_{\omega'}(\cdot) \leq [T_\omega(\cdot)]^{1/C}.$$  

3) If $\omega' = \omega + dd^c \chi$ then

$$\frac{1}{C} T_\omega(\cdot) \leq T_{\omega'}(\cdot) \leq C \cdot T_\omega(\cdot),$$

where $C = \exp(\sup_X \chi - \inf_X \chi) \geq 1$.

4) If $f : X \to X$ is a holomorphic map then $T_{f^* \omega}(\cdot) \leq T_\omega \circ f(\cdot)$. In particular if $f$ is a $\omega$-isometry then $T_\omega \circ f = T_\omega$.

**Remark 4.10.** Following Zeriahi [43] one can prove that for all $\alpha < 2/\nu(X,\omega)$ there exists $C_\alpha > 0$ such that

$$\Vol(\cdot) \leq C_\alpha T_\omega(\cdot)^\alpha,$$

where $\nu(X,\omega) = \sup \{ \nu(\varphi, x) / \varphi \in PSH(X,\omega), x \in X \}$ and $\nu(\varphi, x)$ denotes the Lelong number of $\varphi$ at point $x$. In particular it follows from proposition 4.8 that $\forall \varphi \in PSH(X,\omega)$ with $\sup_X \varphi = 0$,

$$\Vol(\varphi < -t) \leq C_\alpha \exp(-\alpha t), \forall t \in \mathbb{R}.$$  

Such inequalities are quite useful in complex dynamics [22],[24] and in the study of the complex Monge-Ampère operator [30].

**Example 4.11.** Assume $X = \mathbb{CP}^n$, $\omega$ is the Fubini-Study Kähler form and $B_R$ is the euclidean ball centered at the origin and of radius $R$ in a chart $\mathbb{C}^n \subset \mathbb{CP}^n$. We have explicitly computed the extremal function in this case (example 4.5). This yields

$$T_\omega(B_R) = \frac{R}{\sqrt{1 + R^2}}.$$  

Observe that $T_\omega(B_R) \sim R$ as $R \to 0$. This shows the optimality of the rate of decreasing in proposition 4.8.

The capacity $T_\omega$ in example 4.11 has to be related to the capacity $T_{\mathbb{B}^n}$ which measures compact subsets of the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$. It is defined as follows: given $K$ a Borel subset of $\mathbb{C}^n$, $T_{\mathbb{B}^n}(K) := \exp(-\sup_{\mathbb{B}^n} L_K)$, where

$$L_K(z) = \sup \{ v(z) / v \in \mathcal{L}(\mathbb{C}^n), \sup_K v \leq 0 \}$$

is the Siciak’s extremal function of $K$ and $\mathcal{L}(\mathbb{C}^n)$ denotes the Lelong class of psh functions with logarithmic growth in $\mathbb{C}^n$ (see example 1.2). Let $\omega = \omega_{FS}$ denote the Fubini-Study Kähler form on $\mathbb{CP}^n$. One easily checks that

$$V_{K,\omega} - \log \sqrt{2} \leq L_K - \frac{1}{2} \log[1 + |z|^2] \leq V_{K,\omega} \text{ in } \mathbb{C}^n.$$
We infer straightforwardly \( \sup_{B^n} L_K \leq \log \sqrt{2} + \sup_{\mathbb{P}^n} V_{K,\omega} \) hence \( T_{B^n}(K) \geq 2^{-1/2} T_{\omega}(K) \). We also have a reverse inequality. Indeed set \( R = \mathbb{R}^n \). Then \( \sup \varphi \leq \sup_{B^n} \varphi + C_1 \), where \( C_1 = \sup_{\mathbb{P}^n} V_{B^n,\omega} = \log \sqrt{2} \). Therefore
\[
\sup_{\mathbb{P}^n} V_{K,\omega} \leq \sup_{B^n} V_{K,\omega} + \log \sqrt{2} \leq \sup_{B^n} L_K + \log 2,
\]
which yields
\[
\frac{1}{2} T_{\omega}(K) \leq T_{B^n}(K) \leq 2 T_{\omega}(K).
\]

**Example 4.12.** Assume again \( X = \mathbb{C}^n \) and \( \omega \) is the Fubini-Study Kähler form. Consider the totally real subspace \( \mathbb{R}^n \) of points with real coordinates (the closure of \( \mathbb{R}^n \subset \mathbb{C}^n \) in \( \mathbb{P}^n \)). Then
\[
\frac{1}{2(1 + \sqrt{2})} \leq T_{\omega}(\mathbb{R}^n) \leq 1.
\]
Indeed set \( B_{\mathbb{R}^n} := \mathbb{R}^n \cap B^n \). It follows from the discussion above that
\[
T_{\omega}(\mathbb{R}^n) \geq \frac{1}{2} T_{B^n}(B_{\mathbb{R}^n}).
\]
Now there is an explicit formula for \( L^*_{B_{\mathbb{R}^n}} \) (Lundin’s formula, see [29]),
\[
L^*_{B_{\mathbb{R}^n}}(z) = \sup \{\log^+ |h(z, \xi)|/||\xi|| = 1\}, \quad z \in \mathbb{C}^n,
\]
where \( h(\xi) = \zeta + \sqrt{\xi^2 - 1} \). A simple computation yields \( |h(\xi)| \leq \log ||z|| + \sqrt{|z|^2 + 1} \) for \( \zeta = z, \xi > \) with \( ||\xi|| = 1 \). We infer
\[
L_{B_{\mathbb{R}^n}}(z) \leq \log \left[ ||z|| + \sqrt{|z|^2 + 1} \right] \leq \log [1 + \sqrt{2}] \quad \text{in} \quad \mathbb{B}^n,
\]
which yields the desired inequality.

Observe that the minorant is independent of the dimension \( n \). This has been used recently in complex dynamics by Dinh and Sibony [20].

**Remark 4.13.** It follows from proposition 4.6 that \( T_{\omega} \) is a generalized capacity in the sense of Choquet which is outer regular.

5. **Tchebychev constants**

In this section we consider the case where \( \omega \) is (smooth and) represents the first Chern class of a holomorphic line bundle \( L \) on \( X \).

Recall that a holomorphic line bundle \( L \) on \( X \) is a family of complex lines \( \{L_x\}_{x \in X} \) together with a structure of complex manifold of dimension \( 1 + \dim_{\mathbb{C}} X \) such that the projection map \( \pi : L \to X \) taking \( L_x \) on \( x \) is holomorphic. Moreover one can always locally trivialize \( L \): there exists an open covering \( \{U_\alpha\} \) of \( X \) and biholomorphisms \( \Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C} \) which take \( L_x = \pi^{-1}(x) \) isomorphically onto \( \{x\} \times \mathbb{C} \). The line bundle \( L \) is then uniquely (i.e. up to isomorphism) determined by its transition functions \( g_{\alpha\beta} \in O^*(U_{\alpha\beta}), U_{\alpha\beta} := U_\alpha \cap U_\beta, \) where
\[
g_{\alpha\beta} := (\Phi_\alpha \circ \Phi_\beta^{-1})(1)\times_\mathbb{C}.
\]
Note that the \( g_{\alpha\beta} \)’s satisfy the cocycle condition \( g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g^{-1}_{\gamma\alpha} \equiv 1 \), hence define a class \( \{g_{\alpha\beta}\} \in H^1(X, O^*) \). The first Chern class of \( L \) is the image \( c_1(L) \in H^2(X, \mathbb{Z}) \) of \( \{g_{\alpha\beta}\} \) under the mapping \( c_1 : H^1(X, O^*) \to H^2(X, \mathbb{Z}) \) induced by the exponential short exact sequence \( 0 \to \mathbb{Z} \to O \to O^* \to 0 \).
We let \( \Gamma(X, L) \) denote the set of holomorphic sections of \( L \) on \( X \): \( s \in \Gamma(X, L) \) is a collection \( s = \{ s_\alpha \} \) of holomorphic functions \( s_\alpha \) on \( U_\alpha \) satisfying the compatibility condition \( s_\alpha = g_{\alpha \beta} s_\beta \) on \( U_{\alpha \beta} \). Similarly a (singular) metric \( \psi \) of \( L \) on \( X \) is a collection \( \psi = \{ \psi_\alpha \} \) of functions \( \psi_\alpha \in L^1(U_\alpha) \) satisfying \( \psi_\alpha = \psi_\beta + \log |g_{\alpha \beta}| \) in \( U_{\alpha \beta} \). The metric is said to be smooth if the \( \psi_\alpha \)'s are \( C^\infty \)-smooth functions. A smooth metric always exists. The metric \( \psi \) is said to be positive if the \( \psi_\alpha \)'s are psh functions. In particular if \( s = \{ s_\alpha \} \) is a holomorphic section of \( L \) on \( X \), then \( \psi = \{ \psi_\alpha := \log |s_\alpha| \} \) is a positive (singular) metric of \( L \) on \( X \). Note that we make here a slight abuse of terminology: differential geometers usually call “metric” the non-negative (usually smooth and non-vanishing) quantities \( e^{-\psi} = \{ e^{-\psi_\alpha} \} \).

Given a (singular) metric \( \psi = \{ \psi_\alpha \} \) of \( L \) on \( X \), we consider its curvature \( \Theta_\psi := dd^c \psi_\alpha \) in \( U_\alpha \). This yields a globally well defined real closed current on \( X \) since \( dd^c \log |g_{\alpha \beta}| = 0 \) in \( U_{\alpha \beta} \). It is a standard consequence of de Rham’s isomorphism that this current represents the image of the first Chern class of \( L \) under the mapping \( i : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}) \) (induced by the inclusion \( i : \mathbb{Z} \to \mathbb{R} \)). The line bundle \( L \) is said to be pseudoeffective (resp. positive) if it admits a (singular) positive metric (resp. a smooth metric whose curvature is a Kähler form).

Fix \( h = \{ h_\alpha \} \) a smooth metric of \( L \) on \( X \) and set \( \omega := \Theta_h \). Then \( \text{PSH}(X, \omega) \) is in 1-to-1 correspondence with the set of positive singular metrics of \( L \) on \( X \). Indeed if \( \psi \) is such a metric then \( \varphi := \psi - h \) is globally well defined on \( X \) and such that \( dd^c \varphi \geq -\omega \). Conversely if \( \varphi \in \text{PSH}(X, \omega) \) then \( \psi = \{ \psi_\alpha := \varphi + h_\alpha \} \) defines a positive singular metric of \( L \) on \( X \). We can thus rephrase the pseudoeffectivity property as follows:

\( L \) is pseudoeffective \( \iff \text{PSH}(X, \omega) \neq \emptyset \).

Given \( L \) a pseudoeffective line bundle, it is interesting to know whether \( L \) admits a positive metric which is less singular than any another. This notion has been introduced in \([18]\) and happens to be related to very special extremal functions:

**Proposition 5.1.** Let \( (L, h) \) be a pseudoeffective line bundle on \( X \) equipped with a smooth metric \( h \). Set \( \omega := \Theta_h \). Then

\[
\h_{\min} := h + V^*_X,\omega
\]

is a positive singular metric of \( L \) on \( X \) with “minimal singularities”. More precisely if \( \psi \) is a positive singular metric of \( L \) on \( X \), then there exists a constant \( C_\psi \) such that \( \psi \leq \h_{\min} + C_\psi \).

**Proof.** Let \( \psi \) be a positive singular metric of \( L \) on \( X \). Then \( \psi - h \) is a globally well defined \( \omega \)-psh function. It is u.s.c. hence bounded from above on \( X \): we let \( C_\psi \) denotes its maximum. Then \( \psi - h - C_\psi \leq 0 \) on \( X \), hence \( \psi - h \leq V^*_X,\omega + C_\psi \), which yields \( \psi \leq \h_{\min} + C_\psi \). \( \square \)

In the sequel we assume \( L \) is positive and \( h \) has been chosen so that \( \omega := \Theta_h \) is a Kähler form. For \( s \in \Gamma(X, L^N) \), we let \( ||s||_{\text{NH}} \) denote the norm of \( s \) computed with respect to the metric \( Nh \): it is defined in \( U_\alpha \) by \( ||s||_{\text{NH}} := ||s_\alpha|| |e^{-Nh}|_\alpha \). The definition is independent of \( \alpha \) thanks to the compatibility conditions.
For a given Borel subset $K$ of $X$, we define its Tchebychev constants

$$M_{N\omega}(K) := \inf \left\{ \sup_K \|s\|_{N\omega} / s \in \Gamma(X, L^N), \sup_X \|s\|_{N\omega} = 1 \right\}.$$ 

Note that an obvious rescaling argument shows that $M_{d\omega}$ remains unchanged if we replace $h$ by $h + C$ so that it really depends on $\omega = \Theta_h$ rather than on $h$. Consider

$$T'_\omega(K) := \inf_{N \geq 1} [M_{N\omega}(K)]^{1/N}.$$ 

**Theorem 5.2.** Let $K$ be a compact subset of $X$. Then

$$T_\omega(K) = T'_\omega(K).$$

**Proof.** The core of the proof consists in showing that

$$V_{K,\omega}(x) = \sup \left\{ \frac{1}{N} \log \|s\|_{N\omega}(x) / N \geq 1, s \in \Gamma(X, L^N) \text{ and } \sup_K \|s\|_{N\omega} \leq 1 \right\}.$$ 

Note that for any of the sections $s$ involved in the supremum, $\varphi := N^{-1} \log \|s\|_{N\omega}$ belongs to $PSH(X, \omega)$ and satisfies $\varphi \leq 0$ on $K$. Therefore $\varphi \leq V_{K,\omega}$.

Conversely fix $x_0 \in X$ and $a < V_{K,\omega}(x_0)$. Fix $\varphi \in PSH(X, \omega)$ such that $\sup_K \varphi \leq 0$ and $\varphi(x_0) > a$. Regularizing $\varphi$ (see Appendix) and translating, we can assume $\varphi \in PSH(X, \omega) \cap C^\infty(X)$, $\sup_K \varphi < 0$ and $\varphi(x_0) > a$. Fix $\varepsilon > 0$. Let $B = B(x_0, r)$ be a small ball on which $\varphi > a$. We choose $B$ so small that the oscillation of $h$ is smaller than $\varepsilon$ on $B$. Let $\chi$ be a test function with compact support in $B$ and such that $\chi \equiv 1$ in $B(x_0, r/2)$. We can assume w.l.o.g. that $B \subset U_{\alpha_0}$ for some $\alpha_0$ but $B \cap U_\beta = \emptyset$ for all $\beta \neq \alpha_0$. This insures that $\chi$ is a smooth section of $L^N$ for all $N \geq 1$.

Let $\psi_1$ be a smooth positive metric of $L^{N_1} \otimes K^*_X$ on $X$ (this is possible if $N_1$ is chosen large enough since $L$ is positive). Let $\psi_2$ be a positive metric of $L^{N_2}$ on $X$ which is smooth in $X \setminus \{x_0\}$ and with Lelong number $\nu(\psi_2, x_0) \geq n = \dim C_X$ (this is again possible if $N_2$ is large enough, since $L$ is ample). Observe that $\overline{\partial} \chi$ is a smooth $\overline{\partial}$-closed $(0, 1)$-form with values in $L^N$ (for all $N \geq 1$). Alternatively it is a smooth $\overline{\partial}$-closed $(n, 1)$-form with values in $L^N \otimes K^*_X$. Applying Hörmander’s $L^2$-estimates (see e.g. [15], chapter VIII) with weight $\psi_N := (N - N_1 - N_2)(\varphi + h) + \psi_1 + \psi_2$, we find a smooth section $f$ of $L^N$ such that $\overline{\partial}f = \overline{\partial}\chi$ and

$$\int_X |f|^2 e^{-2(N-N_1-N_2)(\varphi+h)-2\psi_1-2\psi_2} dV_\omega \leq C_1 \int_X |\overline{\partial}\chi|^2 e^{-2\psi_N} dV_\omega.$$ 

Note that $\overline{\partial}\chi$ has support in $B \setminus B(x_0, r/2)$ where $\psi_N$ is smooth so that both integrals are finite. Since $\nu(\psi_2, x_0) \geq n$, this forces $f(x_0) = 0$. The second integral is actually bounded from above by $C_2 e^{-2N(a-\varepsilon)}$, where $C_2$ is independent of $N$, since $-\varphi < -a$ on $B$ and the oscillation of $h$ is smaller than $\varepsilon$ on $B$. Therefore $s := \chi - f \in \Gamma(X, L^N)$ satisfies $s(x_0) = 1$ and

$$\int_X |s|^2 e^{-2N(\varphi+h)} dV_\omega \leq C_3 e^{-2N(a-\varepsilon)},$$

where $C_3$ is independent of $N$. Now $\varphi < 0$ in a neighborhood of $K$, so the mean-value inequality applied to the subharmonic functions $|s_\alpha|^2$ yields for
all \( x \) in \( K \),
\[
|s|^2 e^{-2Nh}(x) \leq C_0 \int_{B(x, \delta)} |s|^2(y) e^{-2N[|\varphi+h|(y)]} e^{2N[h(y) - h(x) + \varphi(y)]} d\lambda(y)
\]
\[
\leq C_0 e^{-2N(a-\varepsilon)}
\]
if \( \delta \) is so small that \( \sup_{B(x, \delta)} \varphi > 0 \) is bigger than the oscillation of \( h \) on \( B(x, \delta) \). Therefore \( S := C_{4}^{-1/2} e^{N(a-\varepsilon)} \) satisfies \( \sup_K ||S||_{N_{h}} \leq 1 \) and \( N^{-1} \log ||S||_{N_{h}}(x_0) \geq a - \varepsilon - \frac{\log C_4}{2N} \). Letting \( N \to +\infty, \varepsilon \to 0 \) and \( a \to V_{K, \omega}(x_0) \) completes the proof of the equality.

To conclude observe that by rescaling one gets
\[
- \log T_{\omega}(K) = \sup_X V_{K, \omega}
\]
\[
= \sup \left\{ \frac{1}{N} \sup_X \log ||S||_{N_{h}} / N \geq 1, S \in \Gamma(X, L^N) \text{ and } \sup_K ||S||_{N_{h}} = 1 \right\}
\]
\[
= \sup \left\{ \frac{1}{N} \sup_X \log ||S||_{N_{h}} / N \geq 1, S \in \Gamma(X, L^N) \text{ and } \sup_X ||S||_{N_{h}} = 1 \right\}
\]
\[
= - \log T'_{\omega}(K).
\]

\( \Box \)

**Projective capacity.** We assume here that \( X = \mathbb{CP}^n \) is the complex projective space and \( \omega = \omega_{FS} \) is the Fubini-Study Kähler form. We give in this context a geometrical interpretation of the capacity \( T_{\omega} \). This will shed some light on the notion of projective capacity introduced by Alexander [1].

Let \( \tau : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n \) denote the canonical projection map. We let \( \mathbb{B}^{n+1} \) denote the unit ball in \( \mathbb{C}^{n+1} \). Recall that the polynomially convex hull \( \hat{F} \) of a compact set \( F \) of \( \mathbb{C}^{n+1} \) is defined as \( \hat{F} := \{ x \in \mathbb{C}^{n+1} / |P(x)| \leq \sup_F |P|, \forall P \text{ polynomial} \} \).

The following result gives an interesting interpretation of the capacity \( T_{\omega} \).

**Theorem 5.3.** Let \( K \) be a compact subset of \( \mathbb{CP}^n \). Then
\[
T_{\omega}(K) = \sup \{ r > 0 / r \mathbb{B}^{n+1} \subset \hat{K}_0 \},
\]
where \( K_0 = \tau^{-1}(K) \cap \partial \mathbb{B}^{n+1} \).

**Proof.** Let \( K, K_0 \) be as in the theorem. Observe that \( K_0 \) is a circled subset of \( \partial \mathbb{B}^{n+1} \); if \( z \in K_0 \) then \( e^{i\theta}z \in K_0 \), \( \forall \theta \in [0, 2\pi] \). For such compacts, the polynomial hull \( \hat{K}_0 \) coincides with the "homogeneous polynomial hull",
\[
\hat{K}_0^h := \{ x \in \mathbb{C}^{n+1} / |P(x)| \leq \sup_F |P|, \forall P \text{ homogeneous polynomial} \}.
\]

Indeed one inclusion \( \hat{K}_0 \subset \hat{K}_0^h \) is clear, so assume \( z_0 \in \hat{K}_0^h \). Let \( P = \sum_{j=0}^d P_j \) be a polynomial of degree \( d \) decomposed into its homogenous components. Observe that \( P_j(x) = (2\pi)^{-1} \int_0^{2\pi} P(e^{i\theta}x)e^{-ij\theta}d\theta \). Therefore \( \sup_{K_0} |P_j| \leq \sup_{K_0} |P| \) since \( K_0 \) is circled. Fix \( t \in [0, 1] \). Then
\[
|P(tz_0)| \leq \sum_{j=0}^d t^j|P_j(z_0)| \leq \frac{1}{1-t} \sup_{K_0} |P|.
\]
We infer \( tz_0 \in \hat{K}_0 \). Letting \( t \to 1^- \) and using that \( K_0 \) is closed we get \( z_0 \in \hat{K}_0 \), whence \( \hat{K}_0 = \hat{K}_0^h \).

Fix now \( z \in \mathbb{C}^{n+1} \) such that \( \|z\| \leq T_\omega(K) \). Let \( P \) be a homogeneous polynomial of degree \( d \). Then

\[
|P(z)| = \|z\|^d \left| \frac{z}{\|z\|} \right| \leq T_\omega(K)^d \sup_{\partial \mathbb{B}^{n+1}} |P|
\]

Now set \( \psi(z) = d^{-1} \log |P(z)| - \log \|z\| \) and \( \varphi = \psi - \sup_K \psi \). Then \( \varphi \in PSH(X, \omega) \) with \( \sup_K \varphi \leq 0 \) hence \( \varphi \leq V_K, \omega \). Therefore

\[
T_\omega(K)^d \leq \exp(-d \sup_{\mathbb{C}^{n+1}} \varphi) = \frac{\sup_{\partial \mathbb{B}^{n+1}} |P|}{\sup_{\partial \mathbb{B}^{n+1}} |P|}.
\]

Together with (2) this yields \( |P(z)| \leq \sup_K |P| \) hence \( z \in \hat{K}_0^h = \hat{K}_0 \). Thus \( K_0 \) contains the ball centered at the origin of radius \( T_\omega(K) \).

Conversely since \( T_\omega(K) = T_\omega'(K) \) (theorem 4.1), one can find homogenous polynomials \( P_j \) of degree \( d_j \) such that \( \sup_{\partial \mathbb{B}^{n+1}} |P_j|^{-1/d_j} \sup_{K_0} |P_j|^{1/d_j} \to T_\omega(K) \). Assume \( r \mathbb{B}^{n+1} \subset \hat{K}_0 \). Then

\[
r^{d_j} \sup_{\partial \mathbb{B}^{n+1}} |P_j| = \sup_{r \mathbb{B}^{n+1}} |P_j| \leq \sup_K |P_j|
\]

yields \( r \leq T_\omega(K) \). \( \square \)

**Remark 5.4.** Sibony and Wong [36] have been first in showing that if a compact subset \( K \) of \( \mathbb{C}P^n \) is large enough then the polynomial hull of \( K \) contains a full neighborhood of the origin in \( \mathbb{C}^{n+1} \). They used the (complicated) notion of \( \Gamma \)-capacity. Their approach has been simplified by Alexander [1] who introduced a projective capacity which is comparable to \( T_\omega \) (see theorem 4.4 in [1]). The proof given above is essentially Alexander’s (see also theorem 4.3 in [38]).

This result has been used recently in complex dynamics (see [19], [25]).

**Further capacities.** In our definition of Chebyshev constants we have normalized holomorphic sections \( s \in \Gamma(X, L^N) \) by requiring \( \sup_X \|s\|\|N_h = 1 \). Given \( \mu \) a probability measure such that \( PSH(X, \omega) \subset L^1(\mu) \) and \( A \in \mathbb{R} \), we could as well consider

\[
M_{N, \omega}^{\mu, A}(K) := \left\{ \sup_K \|s\||N_h / s \in \Gamma(X, L^N), \int_X \log \|s\||N_h d\mu = A \right\}.
\]

This normalization has the following pleasant property: if \( s \in \Gamma(X, L^N) \) and \( s' \in \Gamma(X, L^{N'}) \) are so normalized then \( s \cdot s' \in \Gamma(X, L^{N+N'}) \) again satisfies

\[
\int_X \log \|ss'\||N_{N+N'}d\mu = A.
\]

We infer \( M_{N+N', \omega}^{\mu, A} \leq M_{N, \omega}^{\mu, A} \cdot M_{N', \omega}^{\mu, A} \) so that

\[
T_\omega^{\mu, A}(K) := \inf_{N \geq 1} \left[ M_{N, \omega}^{\mu, A}(K) \right]^{1/N} = \lim_{N \to +\infty} \left[ M_{N, \omega}^{\mu, A}(K) \right]^{1/N}.
\]

This yields a whole family of capacities which are all comparable to \( T_\omega \) thanks to proposition 1.7: there exists \( C = C(\mu, A) \geq 1 \) such that

\[
\frac{1}{C} T_\omega(\cdot) \leq T_\omega^{\mu, A}(\cdot) \leq C T_\omega(\cdot).
\]

The projective capacity of Alexander [1] is precisely \( T_\omega^{\mu, A} \) for \( X = \mathbb{C}P^n, \omega = \omega_{FS}, \mu = \omega^n \) and \( A = \int_{\mathbb{C}P^n} (\log |z_n| - \log \|\langle z_0, \ldots, z_n \rangle\|) \omega^n(\{z\}) \).
6. Comparison of capacities and applications

6.1. Josefson’s theorem. In this section we assume that $\omega$ is Kähler and normalized by $Vol_\omega(X) = 1$. We first prove inequalities relating $T_\omega$ and $Cap_\omega$. Then we prove (theorem 6.2) a quantitative version of Josefson’s theorem that every locally pluripolar set is actually $PSH(X,\omega)$-polar. In the local theory this result is due to El Mir [21]. We follow the approach of Alexander-Taylor [2].

**Proposition 6.1.** There exists $A > 0$ s.t. for all compact subsets $K$ of $X$,

$$\exp \left[ -\frac{A}{Cap_\omega(K)} \right] \leq T_\omega(K) \leq e \cdot \exp \left[ -\frac{1}{Cap_\omega(K)^{1/n}} \right].$$

**Proof.** Set $M_K = \sup_X V_{K,\omega}$. If $M_K = +\infty$ then $K$ is $PSH(X,\omega)$-polar (theorem 4.2) and there is nothing to prove: $T_\omega(K) = Cap_\omega(K) = 0$. So we assume in the sequel $M_K < +\infty$ hence $V_{K,\omega} \in PSH(X,\omega)$. If $M_K \geq 1$ then $u_K := M_K^{-1}V_{K,\omega} \in PSH(X,\omega)$ with $0 \leq u_K \leq 1$ on $X$. Since $\omega_{V_{K,\omega}} \leq M_K\omega_{u_K}$, we get

$$\frac{1}{M_K^2} = \frac{1}{M_K^2} \int_K (\omega_{V_{K,\omega}})^n \leq \int_K (\omega_{u_K})^n \leq Cap_\omega(K)$$

whence $T_\omega(K) \leq \exp(-Cap_\omega(K)^{-1/n})$.

If $0 \leq M_K \leq 1$ then $0 \leq V_{K,\omega} \leq 1$ hence $V_{K,\omega} - 1$ coincides with the relative extremal function $h_{K,\omega}$ (see proposition 2.14). We infer

$$1 = \int_K (\omega_{V_{K,\omega}})^n \leq Cap_\omega(K) \leq Cap_\omega(X) = 1,$$

while $T_\omega(K) \leq T_\omega(X) = 1$. Thus in both cases $T_\omega(K) \leq e\cdot\exp(Cap_\omega(K)^{-1/n})$.

We now prove the reverse inequality. We can assume $M_K \geq 1$, otherwise it is sufficient to adjust the value of $A$. Let $\varphi \in PSH(X,\omega)$ be such that $\varphi \leq 0$ on $K$. Then $\varphi \leq M_K$ on $X$, hence $w := M_K^{-1}(\varphi - M_K) \in PSH(X,\omega)$ satisfies $\sup_X w \leq 0$ and $w \leq -1$ on $K$. We infer $w \leq h_{K,\omega}^*$, hence

$$w_K := \frac{V_{K,\omega} - M_K}{M_K} \leq h_{K,\omega}^* \leq 0.$$

Now $\sup_X (V_{K,\omega}^* - M_K) = 0$, so it follows from proposition 1.7 that $\int_X |V_{K,\omega}^* - M_K|^\omega \leq C_1$ for some constant $C_1 > 0$ independent of $K$. We infer

$$Cap_\omega(K) = \int_K (\omega_{h_{K,\omega}^*})^n \leq \int_X [-h_{K,\omega}^*](\omega_{h_{K,\omega}^*})^n \leq \frac{1}{M_K} \int_X (V_{K,\omega}^* - M_K)(\omega_{h_{K,\omega}^*})^n \leq \frac{C_2}{M_K},$$

using corollary 2.3 and the fact that $h_{K,\omega}^* = -1$ on $K$, except perhaps on a pluripolar set which has zero $(\omega_{h_{K,\omega}^*})^n$-measure. This yields the desired inequality.

It follows from the previous proposition and corollary 2.8 that $\omega$-psh functions are quasicontinuous with respect to the capacity $T_\omega$.

**Theorem 6.2.** Locally pluripolar sets are $PSH(X,\omega)$-polar.
Proposition 6.3. There exists $0 < \alpha < 1$ such that for all Borel subsets $K$ of $X$, for all $j \in \mathbb{N}$,

$$[\alpha T_\omega(K)]^{\lambda j} \leq T_\omega(f^j(K)).$$
Proof. This follows straightforwardly from proposition 4.9:
\[ T_\omega(f^j(K)) \geq T_{(f^j)^*\omega}(K) = [T_{\lambda^{-j}f^j} \circ \omega](K)]^{\lambda^j} \geq [\alpha T_\omega(K)]^{\lambda^j}, \]
where the first two inequalities follow from 4.9.4 and 4.9.2 and last one follows from 4.9.3 and the fact that \( \lambda^{-j}(f^j)^* \omega = \omega + dd^c g_j \), where \( g_j \) is uniformly bounded.

**Corollary 6.4.** Let \( \varphi \in \text{PSH}(X, \omega) \). Then the sequence \( (\lambda^{-j}\varphi \circ f^j) \) is relatively compact in \( L^1(\mathbb{C}P^n) \).

**Proof.** Set \( \varphi_j = \lambda^{-j}\varphi \circ f^j \). Observe that \( \varphi_j \) is uniformly bounded from above and that \( \varphi_j + g_j \in \text{PSH}(\mathbb{C}P^n, \omega) \). It follows from proposition 1.6 that either \( \varphi_j \) converges uniformly towards \( -\infty \) or it is relatively compact in \( L^1(\mathbb{C}P^n) \).

It is sufficient to show that for \( A > 0 \) large enough, \( \lim_{j \to \infty} T_\omega(\varphi_j < -A) < T_\omega(X) = 1 \). Observe that \( f^j(\varphi_j < -A) = \{ \varphi < -A\lambda^j \} \). Therefore
\[ [\alpha T_\omega(\varphi_j < -A)]^{\lambda^j} \leq T_\omega(\varphi < -A\lambda^j) \leq C \exp(-A\lambda^j), \]
where the last inequality follows from proposition 4.8. We infer
\[ \lim_{j \to \infty} T_\omega(\varphi_j < -A) \leq \frac{1}{\alpha} \exp(-A) < 1 \]
for \( A > -\log \alpha \) large enough. \( \square \)

**Corollary 6.5.** There exists \( C > 0 \) such that for all Borel subset \( K \) of \( \mathbb{C}P^n \) and for all \( j \in \mathbb{N} \),
\[ \text{Vol}_\omega(f^j(K)) \geq \exp\left(-\frac{C\lambda^j}{\text{Vol}_\omega(K)}\right). \]

In other words the volume of a given set can not decrease too fast under iteration. Such volume estimates are used in complex dynamics to prove fine convergence results towards the Green current \( T_f \) (see [22], [24]). One may hope that dynamical capacity estimates will allow to establish convergence results in higher codimension.

**Proof.** By the change of variables formula one gets
\[ \text{Vol}_\omega(f^j K) = \int_{f^j K} \omega^n \geq \frac{1}{d_l} \int_K (f^j)^* \omega^n = \frac{1}{d_l} \int_K |J_{FS}(f^j)|^2 \omega^n, \]
where \( d_l = \lambda^n \) denotes the topological degree of \( f \) and \( J_{FS}(f) \) stands for the jacobian of \( f \) with respect to the Fubini-Study volume form. Observe that \( \log |J_{FS}(f)| = u - v \) is a difference of two qps functions \( u, v \in \text{PSH}(X, \omega) \) for some \( A = A(\lambda, f) \). Moreover by the chain rule,
\[ \frac{1}{\lambda^j} \log |J_{FS}(f^j)| = \sum_{i=0}^{j-1} \frac{1}{\lambda^j} \log |J_{FS}(f) \circ f^i|. \]
Since \( \lambda^{-1} \log |J_{FS}(f) \circ f^i| \) is relatively compact in \( L^1(\mathbb{C}P^n) \) (previous corollary), the concavity of the log yields
\[ \frac{1}{\text{Vol}_\omega(K)} \int_K |J_{FS}(f^j)|^2 \omega^n \geq \exp\left(\frac{2\lambda^j}{\text{Vol}_\omega(K)} \int_K \frac{1}{\lambda^j} \log |J_{FS}(f)| |\omega^n\right) \]
\[ \geq \exp\left(-\frac{C_1 \lambda^j}{\text{Vol}_\omega(K)}\right), \]
The conclusion follows by observing that $\alpha \exp(-x/\alpha) \geq \exp(-2x/\alpha)$, for all $\alpha > 0$ and all $x \geq 1/e$. $\square$

7. Appendix: Regularization of qpsh functions

It is well-known that every psh function $\varphi$ can be locally regularized, i.e. one can find locally a sequence $\varphi_j$ of smooth psh functions which decrease towards $\varphi$ (see e.g. [17], chapter 1). Similarly one can always locally regularize $\omega$-psh functions. It is interesting to know whether one can also globally regularize $\omega$-psh functions.

When $X$ is a complex homogeneous manifold (i.e. when $\text{Aut}(X)$ acts transitively on $X$), it is possible to approximate any $\omega$-psh function by a decreasing sequence of smooth $\omega$-psh functions (see [23], [27]). In general however there is a loss of positivity: it will be possible to approximate $\varphi \in \text{PSH}(X, \omega)$ by a decreasing sequence of smooth functions $\varphi_j$ but the curvature forms $dd^c \varphi_j$ will have to be more negative than $-\omega$. How negative depends on the positivity of the cohomology class $[\omega]$.

Consider e.g. $\pi : X \to \mathbb{P}^2$ the blow up of $\mathbb{P}^2$ at point $p$, $E = \pi^{-1}(p)$ the exceptional divisor and let $\omega = [E]$ be the current of integration along $E$. Then $\text{PSH}(X, \omega) \simeq \mathbb{R}$ (see Remark 1.5) so every psh function has logarithmic singularities along $E$, hence is not smooth. Alternatively $E$ has self-intersection $-1$ so its cohomology class cannot be represented by smooth non-negative forms, not even by smooth forms with (very) small negativity.

Following Demailly’s fundamental work [12], [14], [18] (to cite a few) we show herebelow that regularization with no loss of positivity is possible when $\omega$ is a Hodge form (i.e. a Kähler form with integer class). This yields a ”simple” regularization process when $X$ is projective. We would like to mention that Demailly has produced over the last twenty years much finer regularization results. We nevertheless think it is worth including a proof, since it is far less technical than Demailly’s more general results (although our proof heavily relies on his ideas). We thank P.Eyssidieux for his helpful contribution regarding that matter.

**Theorem 7.1.** Let $L \to X$ be a positive holomorphic line bundle equipped with a smooth strictly positive metric $h$, and set $\omega := \Theta_h > 0$.

Then for every $\varphi \in \text{PSH}(X, \omega)$, there exists a sequence $\varphi_j \in \text{PSH}(X, \omega) \cap C^\infty(X)$ such that $\varphi_j$ decreases towards $\varphi$.

**Proof.** Let $\varphi \in \text{PSH}(X, \omega)$. We can assume w.l.o.g. that $\varphi \leq 0$ on $X$. Let $\psi = \{\psi_\alpha := \varphi + h_\alpha \in \text{PSH}(U_\alpha)\}$ denote the associated (singular) positive metric of $L$ on $X$, where $\{U_\alpha\}$ denotes an open cover of $X$ trivializing $L$ (see section 4).

**Step 1.** We consider the following Bergman spaces

$$\mathcal{H}_{j,j_0} := \left\{ s \in \Gamma(X, L^j) / \int_X |s|^2 e^{-2h_{j,j_0}} dV_\omega < +\infty \right\},$$

where $h_{j,j_0} = (j - j_0) \psi + j_0 h$, $j_0$ a fixed large integer (to be specified later).

Let $\sigma_1^{(j,j_0)}, \ldots, \sigma_s^{(j,j_0)}$ be an orthonormal basis of $\mathcal{H}_{j,j_0}$ and set

$$\psi_{j,j_0} := \frac{1}{2j} \log \left[ \sum_{s=1}^s |\sigma_s^{(j,j_0)}|^2 \right] = \frac{1}{2j} \sup_{s \in B_{j,j_0}} \log |s|^2,$$
where $B_{j,j_0}$ denotes the unit ball of radius 1 centered at 0 in $H_{j,j_0}$. Clearly $\psi_{j,j_0}$ defines a positive (singular) metric of $L$ on $X$, equivalently $\varphi_{j,j_0} := \psi_{j,j_0} - h \in PSH(X,\omega)$. If $x \in U_\alpha$ and $s = \{s_\alpha\} \in H_{j,j_0}$, then $|s_\alpha|^2$ is subharmonic in $U_\alpha$ hence

$$|s_\alpha(x)|^2 \leq \frac{C_1}{r^{2n}} \int_{B(x,r)} |s_\alpha(x)|^2 \leq \frac{C_2}{r^{2n}} e^{2s_{B(x,r)} h_{j,j_0}} \int_X |s|^2 e^{-2h_{j,j_0}} dV_\omega,$$

where $r > 0$ is so small that $B(x,r) \subset U_\alpha$. We infer

$$\varphi_{j,j_0}(x) \leq (1 - j_0/j) \sup_{B(x,r)} \varphi + \frac{C_3 - n \log r}{j}.$$ (3)

There is also a reverse inequality which uses a deep extension result of Ohsawa-Takegoshi-Manivel (see [16]): there exists $j_0 \in \mathbb{N}$ and $C_4 > 0$ large enough so that $\forall x \in X, \forall j \in \mathbb{N}$, there exists $s \in \Gamma(X, L^2)$ with

$$\int_X |s|^2 e^{-2h_{j,j_0}} dV_\omega \leq C_4 |s(x)|^2 e^{-2h_{j,j_0}(x)}.$$ Choose s so that the right hand side is equal to 1, hence $s \in B_{j,j_0}$. Then

$$\psi_{j,j_0}(x) \geq \frac{1}{2j} \log |s(x)|^2 = \left(1 - \frac{j_0}{j}\right) \psi(x) + \frac{j}{j_0} h(x) - \frac{\log C_4}{2j}.$$ We infer

$$\varphi_{j,j_0}(x) \geq \left(1 - \frac{j_0}{j}\right) \varphi(x) - \frac{\log C_4}{2j} \geq \varphi(x) - \frac{\log C_4}{2j},$$

(4) since $\varphi \leq 0$ on $X$. It follows from (3) and (4) that $\varphi_j \to \varphi$ in $L^1(X)$.

**Step 2.** We now show, following [18] that $(\varphi_{j,j_0})_j$ is almost subadditive. Let $s \in \Gamma(X, L^{j_1+j_2})$ with

$$\int_X |s|^2 e^{-2h_{j_1+j_2,j_0}} dV_\omega \leq 1.$$ We may view $s$ as the restriction to the diagonal $\Delta$ of $X \times X$ of a section $S \in \Gamma(X \times X, L_1^{j_1} \otimes L_2^{j_2})$, where $L_i = \pi_i^* L$ and $\pi : X \times X \to X$ denotes the projection onto the $i^{th}$ factor, $i = 1, 2$. Consider the Bergman spaces

$$\mathcal{H}_{j_1,j_2,j_0} := \left\{ S \in \Gamma(X \times X, L_1^{j_1} \otimes L_2^{j_2}) / \int_{X \times X} |S|^2 e^{-2h_{j_1,j_0/2}}(x) - 2h_{j_2,j_0/2}(y) dV_{\omega_1}(x) dV_{\omega_2}(y) < +\infty \right\},$$

where $\omega_i = \pi_i^* \omega$. It follows from the Ohsawa-Takegoshi-Manivel $L^2$-extension theorem [14] that there exists $S \in \Gamma(X \times X, L_1^{j_1} \otimes L_2^{j_2})$ such that $S|_{\Delta} = s$ and

$$\int_{X \times X} |S|^2 e^{-2h_{j_1,j_0/2} - 2h_{j_2,j_0/2}} dV_{\omega_1} dV_{\omega_2} \leq C_5 \int_X |s|^2 e^{-2h_{j_1+j_2,j_0}} dV_\omega \leq C_5,$$

where $C_5$ only depends on the dimension $n = \dim_{\mathbb{C}} X$. Observe that $\{\sigma_{l_1}^{(j_1,j_0/2)}(x) \cdot \sigma_{l_2}^{(j_1,j_0/2)}(y)\}_{l_1,l_2}$ forms an orthonormal basis of $\mathcal{H}_{j_1,j_2,j_0}$, thus

$$S(x,y) = \sum_{l_1,l_2} c_{l_1,l_2} \sigma_{l_1}^{(j_1,j_0/2)}(x) \sigma_{l_2}^{(j_2,j_0/2)}(y).$$
with \( \sum |c_1,j_2|^2 \leq C_5 \). It follows therefore from Cauchy-Schwarz inequality that

\[
\| s(x) \|^2 = \| S(x,x) \|^2 \leq C_5 \sum_{i_1} |\sigma_{j_1,j_0}(x)|^2 \sum_{i_2} |\sigma_{j_2,j_0}(y)|^2,
\]

which yields

\[
\varphi_{j_1+j_2,j_0} \leq \frac{\log C_5}{2(j_1+j_2)} + \frac{j_1}{j_1+j_2}\varphi_{j_1,j_0} + \frac{j_2}{j_1+j_2}\varphi_{j_2,j_0}.
\]

Note finally that \( \varphi_{j,j_0} \leq \varphi_{j,j_0} \) since \( \varphi = \psi - h \leq 0 \), therefore \( \hat{\varphi}_j := \varphi_{2j,j_0} + 2^{-j-2} \log C_5 \) is decreasing.

**Step 3.** It remains to make \( \hat{\varphi}_j \) smooth. Indeed it has all the other required properties: it is decreasing and by Step 1 we have for all \( x \in X \),

\[
(5)\quad \varphi(x) \leq \hat{\varphi}_j(x) \leq (1 - j_0 2^{-j}) \sup_{B(x,r)} \varphi + \frac{C_6 - n \log r}{2^j},
\]

so that \( \hat{\varphi}_j \to \varphi \). Let \( \sigma_{j_1}^{(2)} \), \( \ldots \), \( \sigma_{N,j}^{(2)} \in \Gamma(X,L^{2^j}) \) be such that \( (\sigma_i^{(2)})_l \) is a basis of \( \Gamma(X,L^{2^j}) \) and set

\[
\varphi_j := \frac{1}{2^{j+1}} \log \left[ \sum_{l=1}^{s_{j_1}} |\sigma_i^{(2)}|^2 + \varepsilon_j \sum_{l=1+s_{j_2}}^{N,j} |\sigma_i^{(2)}|^2 \right] + \frac{\log C_5}{2^{j+2}} - h.
\]

Clearly \( \varphi_j \in \text{PSH}(X,\omega) \). Moreover \( \varphi_j \in \mathcal{C}^\infty(X) \) because \( L^{2^j} \) is very ample if \( j \) is large enough (hence we can find, for every \( x \in X \), a holomorphic section of \( L^{2^j} \) on \( X \) which does not vanish at \( x \)). Finally we can choose \( \varepsilon_j > 0 \) that decrease so fast to zero that \( (\varphi_j) \) is still decreasing and converges to \( \varphi \). \( \square \)

**Corollary 7.2.** Let \( \omega \) be a Kähler form on a projective algebraic manifold \( X \). Then there exists \( A \geq 1 \) such that for every \( \varphi \in \text{PSH}(X,\omega) \), we can find \( \varphi_j \in \text{PSH}(X,\omega_j) \cap \mathcal{C}^\infty(X) \) which decrease towards \( \varphi \).

**Proof.** Let \( \varphi \in \text{PSH}(X,\omega) \). Since \( X \) is projective, we can find a Hodge form \( \omega' \). Then \( C^{-1} \omega' \leq \omega \leq C \omega' \) for some constant \( C \geq 1 \). Since \( \text{PSH}(X,\omega) \subset \text{PSH}(X,C \omega') \), it follows from the previous theorem that we can find \( \varphi_j \in \text{PSH}(X,\omega_j) \cap \mathcal{C}^\infty(X) \) that decrease towards \( \varphi \). Now the result follows from \( \text{PSH}(X,\omega_j) \subset \text{PSH}(X,\omega_j) \) with \( A = C^2 \).

**Remark 7.3.** When \( X \) is merely Kähler, the above result still holds but the proof is far more intricate. We refer the reader to Demailly’s papers for a proof.

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