Finsleroids with three axes in dimension $N = 3$

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Abstract

The Conclusive Theorem has been established to determine the dependence of the three-axes positive-definite Finsleroid metric functions $F$ on the Finsleroid azimuthal angle $\theta$ in the three-dimensional case $N = 3$, provided that the condition of the angle-separation in the involved characteristic functions is implied. The complete set of algebraic and differential equations is derived in all rigor which are necessary and sufficient in order that the function $F$ belong to the class. It proves possible to solve the equations and obtain the explicit dependence of the involved characteristic functions on the angle $\theta$.

Keywords Finsler geometry - Finsler metrics - Metric spaces

Mathematics Subject Classification 53B40 - 53C60
1 Introduction and preliminaries

In the preceding publications [1-3] the method was proposed and used to derive the class of the two-axis pseudo-Finsleroids under the structural pre-assumption of the angle separation in the characteristic functions. The consideration was four-dimensional, $N = 4$, and the signature of Finslerian metric tensor was presumptively $(+-)$. In the present paper the similar method is applied to find the three-axis Finsleroids in the positive-definite three-dimensional case, so that $N = 3$ and the signature is $(+++)$). The treatment as well as the evaluation process are systematic.

The consideration uses the property of separation of dependence with respect to the Finsleroid angles $\theta$ and $\phi$. The evaluation option of the property is highly significant, namely the separation makes it possible to reduce the characteristic partial differential equations to a set of the ordinary differential equations with respect to $\theta$ and $\phi$. As we shall show below, the equations can be solved to give us the explicit algebraic $\theta$-angle representations for all the involved characteristic Finsleroid functions.

The origin of the Finsler Geometry can be traced back to the famous publications [4-6]. During the succeeding decades the geometry has been intensively elaborated in many interesting and important directions (see [7-13]). At present, the Euclidean metric yields the profound geometric base for various scientific theories. The metric is constructed with the help of the quadratic form of tangent vectors to meet the simplest algebraic patterns. To go over the simplest presumption we can naturally apply the methods of the Finsler geometry.

Obviously, the applied ability of the Finslerian methods is proportional to the variety of the particular Finslerian metric functions which can be proposed and well-developed to geometrically underline the possible anisotropic distributions of vectors in the tangent spaces of differentiable manifolds.

It is interesting and constructive to retain the condition of constancy of the curvature of indicatrix, in which case we qualify the Finsler metric function to be of the Finsleroid type in the positive-definite case. The one-axis Finsleroid metric function reveals various interesting properties (see [14,15] and references therein) of the fundamental significance to develop applications. In the present paper, we confine attention to the Finsleroid metrics possessing three axes of anisotropy. Our analysis is systematic, the presentation is concise and complete.

A direction in the tangent space is said to be geometrically distinguished if the direction makes trace in the structure of the Finslerian metric function. We are aiming in the present paper to solve the following problem in the three-dimensional case.

**Problem** Find the Finslerian metric functions $F$ which entail the angle representation (1.3) of the associated Finslerian metric tensor $g$ in the case when the functions $F$ are assigned by three geometrically distinguished axes, provided that the property of separation of the angle dependence is fulfilled.

To this end, we consider a three-dimensional differentiable manifold $M$ and treat the tangent bundle $TM$ as the unification of the tangent spaces $T_x M \subset TM, x \in M$. The tangent vectors $y \in T_x M$ are supported by the points $x \in M$. Let $F$ be a Finslerian metric function, such that $F = F(x,y)$ and the positively first-degree homogeneity with respect to the vector argument $y$ together with sufficient smoothness is implied. The indicatrix $\mathcal{I}_x M \subset T_x M$ in a tangent space $T_x M$ is the surface defined by

$$\mathcal{I}_x M = \{ y \in T_x M : F(x,y) = 1 \}.$$
The definition extends the notion of the Euclidean sphere.

In each tangent space $T_x M$ the Finslerian metric tensor induces a Riemannian metric tensor on the indicatrix (according to the definition (2.1)), so that the indicatrix $\mathcal{I}_x M$ becomes a Riemannian space, to be denoted by $\mathcal{R}^\text{Ind}_x$.

**Definition 1.1** If at a point $x \in M$ of support the indicatrix space $\mathcal{R}^\text{Ind}_x \subset T_x M$ is of constant positive curvature, then the indicatrix body

$$\mathcal{F}_x = \{y \in T_x : F(x,y) \leq 1\} \quad \text{(1.1)}$$

is called the *Finsleroid*.

This definition extends the notion of the Euclidean ball.

If the space $\mathcal{R}^\text{Ind}_x \subset T_x M$ is of constant positive curvature at a point $x \in M$ of support, we call $F$ the *Finsleroid metric function* at this point. If this property fulfills at any point $x \in M$ of the space, we call $F$ the *Finsleroid metric function on $T M$*. We shall denote the function $F$ by $F^{\text{FrD}}$ to emphasize the Finsleroid nature of the function.

The Finslerian metric tensor $g$ constructed from the Finsleroid metric function $F^{\text{FrD}}$ will be called the *Finsleroid metric tensor*.

Accordingly, we shall use the following definition.

**Definition 1.2** The space

$$\mathcal{F}_{(3)} = \{M; TM; y \in TM; F^{\text{FrD}}; g\} \quad \text{(1.2)}$$

is called the three-dimensional *Finsleroid space*.

Since we confine the consideration to the three-dimensional case, the tensorial indices $i, j, ...$ will take on the values (1,2,3). Our consideration will be of local nature, so that we shall often represent the vectors and tensors by means of their local components with respect to the natural frame, in particular $y = \{y^i\}$.

Following the known methods of the Finsler geometry, we construct the tensor $g = g(x, y)$ by means of the derivative components: $g_{ij} = (1/2) \partial^2 F^2 / \partial y^i / \partial y^j$ and $g = \{g_{ij}\}$. We shall use also the tensor $\{h_{ij}\}$ obtained from the expansion $g_{ij} = l_i l_j + h_{ij}$, where $l_i = \partial F / \partial y^i$ are covariant components of the unit vector $l = \{l^i\}$, such that $l^i = y^i / F$; $l_i = g_{ij} l^j$, and $F(x,l) = 1$.

Let us denote by $\theta = \theta(x,y)$ and $\phi = \phi(x,y)$ the scalars on the manifold $M$ assuming that the functions $\theta(x,y)$ and $\phi(x,y)$ are functionally independent and smooth of at least class $C^2$ with respect to the variable $y$ in each tangent space $T_x$, and also are positively homogeneous of the degree zero with respect to vectors $y \in T_x$. We assume that the Finsler space possesses the following property: two scalars $\theta$ and $\phi$ exist such that the Finslerian metric tensor of the space can be written in the form

$$g_{ij} = l_i l_j + \frac{1}{C} \left( \theta_i \theta_j + \sin^2 \theta \phi_i \phi_j \right) F^2, \quad C = C(x) > 0, \quad \text{(1.3)}$$

where the notation $\{\theta_i = \partial \theta / \partial y^i, \phi_i = \partial \phi / \partial y^i\}$ has been used.

In Sect. 2, the important aspects of the indicatrix geometry required for our Finsleroid study are described.

In Sect. 3, we are inspiring by the attractive idea to construct the Finsleroid space with three axes on a three-dimensional Riemannian space endowed with three vector fields.
Such a Riemannian space is the geometrical background for the theory developed in the present paper. The involved vectors are assumed naturally to be linearly independent at each point of the background manifold. They are also used to define three axes for the Finsleroids. In terms of these axes, we separate the angle dependence of the characteristic functions which enter the functions $F$ in accordance with the Separation Scheme (3.9). The respective notion of the $\mathcal{UZ}$-Finsler space is introduced such that the Finslerian metric function $F$ of the space gets compatible with the Separation Scheme. The interesting concepts of the horizontal section and the vertical section of the Finsleroid are appeared to study.

We also observe the attractive possibility to extend the Euclidean trigonometric functions to obtain the generalized Finsleroid trigonometric functions retaining the Euclidean structure of dependence on the angles $\theta$ and $\phi$. All the components of the $\mathcal{UZ}$-Finsler space tensor $h_{ij}^{\mathcal{UZ}} = g_{ij}^{\mathcal{UZ}} - l_il_j$ are evaluated in the concise and explicit form. The entailed determinant $D^{\mathcal{UZ}} = \det(g_{ij}^{\mathcal{UZ}})$ proves to be the product $D^{\mathcal{UZ}} = D_1D_2$ possessing the property of separation of dependence on $\theta$ and $\phi$, namely $D_1 = D_1(x, \theta)$ and $D_2 = D_2(x, \phi)$.

In Sect. 4, we clarify when the $\mathcal{UZ}$-Finsler space is of the Finsleroid type. To this end we compare the obtained components of the tensor $h_{ij}^{\mathcal{UZ}}$ with the respective components of the Finsleroid-tensor $h_{ij}^{[H,\theta,\phi]}$ written in (2.7). It is surprising enough, that this method which is simple conceptually (although includes rather lengthy calculations) leads to Conclusive Theorem which formulates the general result with the help of two rather simple differential equations (4.3)-(4.4) which we call the $\mathcal{UZ}$-Finsleroid characteristic equations, the $\mathcal{FRD}^{\mathcal{UZ}}$-equations for short. It proves possible to solve the equations to clarify the dependence of the characteristic functions $U$ and $f$ on the angle $\theta$. The solutions are explicitly found in terms of simple algebraic functions (see (4.19) and (4.22)). The $\mathcal{UZ}$-Finsler space is the Finsleroid space at any choice of the dependence of the involved function $Z$ on $t$, provided that the generating function $U$ fulfills the $\mathcal{FRD}^{\mathcal{UZ}}$-equations.

In the last section we emphasize the most important aspects of the developed theory.

2 Differential geometry of Finsleroid indicatrix

The following definition gives rise to significant geometrical ideas and consequences in our analysis.

**Definition 2.1** Formula (1.3) introduces the angle representation of the Finslerian metric tensor, interpreting $\theta$ and $\phi$ as the angle variables.

Indeed, let us denote $u^1 = \theta$ and $u^2 = \phi$, fix a point $x \in M$, and interpret the set $\{u^a\}$ to be a coordinate system on the indicatrix supported by $x \in M$. The indices $a, b, ...$ will be specified on the range (1,2). Parameterize the unit vectors $l = \{l^i\}$ of the tangent space $T_x$ by the help of the coordinates $u^a$ using the respective vector field $t^i = t^i(x, u)$. We get the parametrical representation $t^i = t^i(x, u)$ of unit vectors, which in turn gives rise to the so-called projection factors $t^i_a = \partial t^i / \partial u^a$ (see Sect. V.8 in [7]). With these objects, we can construct the metric tensor $i = \{i_{ab}\}$ on the indicatrix by means of the components $i_{ab} = g_{ij}t^i_at^j_b$. Owing to the identities $l^i_{,a} = 0$ and the expansion $g_{ij} = l_il_j + h_{ij}$, we can write simply

$$i_{ab} = h_{ij}t^i_at^j_b. \quad (2.1)$$

This equality can be inverted with the help of the vectors $u^i_a = \partial u^a / \partial y^i$. Indeed, upon contracting the equality with $u^a_iu^b_j$ and taking into account the Finslerian identity...
\( F^i u^a_j = h^i_j \), we get the inverse representation
\[
    h_{ij} = i_{ab} u^a_i u^b_j F^2. \tag{2.2}
\]
This is a general Finslerian result.

The tensor \( \tilde{i} = \tilde{i}(x, u) \) thus obtained is called the \textit{induced Riemannian metric tensor on indicatrix}, the \textit{indicatrix metric tensor} for short. From Formulas (1.3) and (2.1) we get
\[
    i_{11} = \frac{1}{C}, \quad i_{12} = 0, \quad i_{22} = \frac{1}{C} \sin^2 \theta. \tag{2.3}
\]

Using these components, we can construct the Riemannian curvature tensor \( R^f_{\ abd} = R^f_{\ abd}(x, u) \) and directly obtain the representation
\[
    R^f_{\ abcd} = C(i^{ad} i^{bc} - i^{ac} i^{bd}), \quad R^e_{\ abd} = i^e_{\ ef} R^f_{\ abd}, \tag{2.4}
\]
which just tells us that the indicatrix is a space of the constant curvature \( C \).

The verification of the representation (2.4) is a short task. Indeed, constructing the Christoffel symbols \( \tilde{i}_{\ ab} = \tilde{i}_{\ ab}(x, u) \), \( \tilde{i}_{\ ab} = (i_{ae,b} + i_{be,a} - i_{ab,e})/2 \), we obtain the components
\[
    i_{11} = i_{12} = i_{22} = 0, \quad i_{12} = -\sin \theta \cos \theta, \quad i_{21} = \cos \theta/\sin \theta. \tag{2.5}
\]

Evaluating the indicatrix curvature tensor
\[
    \tilde{R}^f_{\ abd} = \Delta_{\ aba} - \Delta_{\ abd} + \tilde{i}^e_{\ ab} \tilde{i}^f_{\ ed} - \tilde{i}^e_{\ ad} \tilde{i}^f_{\ eb}
\]
and then lowering the index to obtain the tensor \( R^e_{\ abd} = i^e_{\ ef} \tilde{R}^f_{\ abd} \), we can observe that
\[
    CR^{1212} = \frac{\partial i^{11}}{\partial u^2} - \frac{\partial i^{12}}{\partial u^1} + \tilde{i}^{e_{21}} i^{f_{21}} e^2 - \tilde{i}^{e_{22}} i^{f_{22}} e_1,
\]
or
\[
    CR^{1212} = -\frac{\partial i^{12}}{\partial u^1} + \tilde{i}^{21} i^{12} = -\sin^2 \theta + \cos^2 \theta - \cos^2 \theta = -\sin^2 \theta.
\]
So the equality \( R_{abcd} = C(i_{ad} i_{bc} - i_{ac} i_{bd}) \) is true.

Thus the following assertion is valid.

**Primary Theorem 2.1** If the Finslerian metric tensor of a Finsler space admits the representation (1.3), then the Finsler space is of the Finsleroid type, namely the indicatrix space \( R^{\Ind}_{\ x} \) is a space of the constant curvature \( C(x) \).

Geometrically, our constructions in the present paper are founded by the following definition.

**Definition 2.2** The space \( \hat{R}^{\Ind}_{\ x} = \{T_x M, \tilde{i}_x, u^a\} \) is called the \textit{Finsleroid indicatrix Riemannian space} supported by the point \( x \in M \), where the indicatrix metric tensor \( \tilde{i}_x \) has the components \( \{i_{11}, i_{12}, i_{22}\} \) listed in (2.3).

We introduce the scalar \( H = H(x) \) which will play the role of the \textit{characteristic Finsleroid parameter} in the Finsleroid space. The geometrical meaning of the scalar is that the square \( H^2 = (H(x))^2 \) taken at a point \( x \) is equal to the value of curvature of the Finsleroid indicatrix constructed in \( T_x \). Accordingly, we make the substitution \( C = H^2 \). The value of \( H \) will be restricted by the condition \( 0 < H \leq 1 \) implied to hold at any admissible
point \( x \in M \). The premised representation (1.3) of the metric tensor can conveniently be written in the form

\[
g_{ij} = l_i l_j + h_{ij}^{[H,\theta,\phi]} \tag{2.6}
\]

with

\[
h_{ij}^{[H,\theta,\phi]} = \frac{1}{H^2} \left( \theta_i \theta_j + \sin^2 \theta \phi_i \phi_j \right) F^2. \tag{2.7}
\]

Because of (2.3), the squared length element \((ds)^2 = i_{ab} du^a du^b\) on the Finsleroid indicatrix reads

\[
(d^2s)_{Fr.} = \frac{1}{H^2} ((d\theta)^2 + \sin^2 \theta (d\phi)^2), \tag{2.8}
\]

which extends the Euclidean representation

\[
(d^2s)_{Euc.} = (d\theta)^2 + \sin^2 \theta (d\phi)^2
\]

by means of the factor \(1/H^2\).

3 Geometrical background and separation of angle dependence

The notion of the Finsleroid space with three axes can naturally be introduced on a three-dimensional Riemannian space. Namely, let \( \mathcal{M}_3^* \) be a three-dimensional differentiable manifold which admits the introduction of three vector fields assuming that the fields are linearly independent at each point \( x \in \mathcal{M}_3^* \). The associated tangent bundle \( T\mathcal{M}_3^* \) is the unification of the respective tangent spaces \( T_x\mathcal{M}_3^* \). Denote by \( i, j, i_{(3)} \) the one-forms which represent the vector fields. We shall apply the localized coordinate representations \( \{i = i_1 y^1, j = j_1 y^2, i_{(3)} = i_{(3)} y^3\} \), where \( i_i = i_i(x), j_j = j_j(x) \), and \( i_{(3)i} = i_{(3)}(x) \), such that \( \{i, j, i_{(3)i}\} \) are the covariant vector fields on \( \mathcal{M}_3^* \) and \( y = \{y^i\} \) denotes the tangent vector: \( y \in T_x\mathcal{M}_3^* \). Below, the short notation \( T_x \) will be used instead of the complete \( T_x\mathcal{M}_3^* \).

Constructing the Riemannian metric tensor \( a = \{a_{ij}(x)\} \) by means of the expansion

\[
a_{ij} = i_i j_j + j_i i_j + i_{(3)i} j_{(3)j}, \tag{3.1}
\]

we get the three-dimensional Riemannian space \( \mathcal{R}_3^* \) according to the following definition.

Definition 3.1 The notion

\[
\mathcal{R}_3^* = \{ \mathcal{M}_3^*, i, j, i_{(3)i}, a \} \tag{3.2}
\]

is called the background three-axes Riemannian space.

The next definition emphasizes the structural role of the introduced vectors \( \{i_i, j_j, i_{(3)i}\} \).

Definition 3.2 The members of the set \( \{i_i, j_j, i_{(3)i}\} \) are called the basis vectors of the space \( \mathcal{R}_3^* \).

The indices of vectors and tensors in the space \( \mathcal{R}_3^* \) will be raised by means of the contravariant tensor \( \{a^{ij}(x)\} \) reciprocal to the covariant tensor \( \{a_{ij}(x)\} \), such that \( a_{in} a^{nj} = \delta_i^j, j^i = a^{in} j_n, \) etc; \( \delta \) stands for the Kronecker symbol.

By means of \( S_x^{(i)} \subset T_x, S_x^{(j)} \subset T_x, \) and \( S_x^{(i_{(3)i})} \subset T_x \) we denote the straight lines which pass through the support center \( O_x \in T_x \) of the tangent space \( T_x \) and respectively include the basis vectors, namely \( i_x \in S_x^{(i)}, j_x \in S_x^{(j)} \), and \( i_{(3)x} \in S_x^{(i_{(3)i})} \). The inclusions
Definition 3.3 The straight line \( S^{(1i)}_x \) is called the vertical Finsleroid axis at point \( x \in \mathcal{M}_3^* \). The straight lines \( S^{(ij)}_x \subset T_x \) and \( S^{(i)}_x \subset T_x \) are called respectively the primary horizontal Finsleroid axis and the secondary horizontal Finsleroid axis at point \( x \in \mathcal{M}_3^* \).

Definition 3.4 The space \( \mathcal{F}_3^* = \{ \mathcal{M}_3^*, i, j, i_{(3)}, a, F, g \} \) (3.3) is called the three-axes Finsleroid space. The entered \( g \) denotes the Finslerian metric tensor derived from the function \( F \).

In this way, we construct the three-dimensional Finsleroid space \( \mathcal{F}_3^* \) on the three-dimensional Riemannian space \( \mathcal{R}_3^* \).

Definition 3.5 The plane \( \mathcal{H}_{xP_3} \subset T_x \) which is orthogonal to the vertical axis \( S^{(1i)}_x \subset T_x \) of the Finsleroid in \( T_x \) and intersects the axis at a point \( P_3 \) is called the horizontal section of the tangent space \( T_x \). The restriction \( \mathcal{F}\mathcal{H}_{xP_3} = \mathcal{F}_x \cap \mathcal{H}_{xP_3} \) is called the horizontal section of the Finsleroid, the \( \mathcal{H} \)-section of the Finsleroid for short; \( \mathcal{F}_x \) is the Finsleroid defined in (1.1).

Definition 3.6 The plane \( \mathcal{V}_{x\phi} \subset T_x \) which includes the vertical axis \( S^{(1i)}_x \subset T_x \) of the Finsleroid in \( T_x \) and corresponds to a fixed angle value \( \phi \) is called the vertical section of \( T_x \). The restriction \( \mathcal{F}\mathcal{V}_{x\phi} = \mathcal{F}_x \cap \mathcal{V}_{x\phi} \) is called the vertical section of the Finsleroid, the \( \mathcal{V} \)-section of the Finsleroid for short.

Definition 3.7 The space \( T_x^* \subset T_x \) which is obtained after deleting the vertical Finsleroid axis, so that \( T_x^* = T_x \setminus S^{(1i)}_x \), is called the axially reduced tangent space. The respective bundle \( T^*\mathcal{M}_3^* \subset T\mathcal{M}_3^* \) is called the axially reduced tangent bundle.

Definition 3.8 The space \( \mathcal{F}^{(s_{(1i)}^3)}_3 = \{ \mathcal{M}_3, T^*\mathcal{M}_3^*, y \in T_x^*\mathcal{M}_3^*, i, j, i_{(3)}, a, F, g \} \) (3.4) is called the axially reduced three-axes Finsleroid space.

Using the basis vectors \( \{ i^j, j^i, i_{(3)}^j \} \), it is convenient to apply the expansion

\[
y^i = i^iy^1 + j^iy^2 + i_{(3)}^iy^3
\]

(3.5)
in terms of the components

\[
\{ y^a \} = \{ y^1, y^2, y^3 \} : \quad y^1 = i, \quad y^2 = j, \quad y^3 = i_{(3)}.
\]

(3.6)

Also, we introduce the notation

\[
z = y^3 \equiv i_{(3)}, \quad c^1 = \frac{y^1}{z}, \quad c^2 = \frac{y^2}{z}, \quad t = \frac{y^1}{y^2} \equiv \frac{c^1}{c^2} \equiv \frac{i}{j},
\]

(3.7)
assuming for definiteness that $z > 0, j > 0, \text{ and } c^2 > 0$.

**Definition 3.9** The function $U = U(x, y)$ which determines the Finslerian metric function $F$ in accordance with the equality $F = iU(3)U$ is called the **primary generating function**.

Henceforth, we assume for the Finslerian metric functions $F = F(x, y)$ of the considered three-axes type the particular structure specified by the condition of **separation of angle dependence**

$$U = \tilde{U}(x, \theta), \quad f = \tilde{f}(x, \theta), \quad Z = \tilde{Z}(x, \phi), \quad t = \tilde{t}(x, \phi)$$

(3.8)

of the **characteristic functions** $\{U, f, Z, t\}$ which enter the functions $F$ in accordance with the following **Separation Scheme**:

$$F = z\tilde{U}(x, f), \quad f = c^2\tilde{Z}(x, t).$$

(3.9)

It follows that

$$\tilde{U}(x, \theta) = \bar{U}(x, \tilde{f}(x, \theta)), \quad \tilde{Z}(x, \phi) = \bar{Z}(x, \tilde{t}(x, \phi)),$$

(3.10)

and

$$\theta = \tilde{\theta}(x, f), \quad \phi = \tilde{\phi}(x, t).$$

(3.11)

Since the functions $U$ and $f$ are homogeneous of degree zero with respect to tangent vectors $\{y^i\}$, it is also convenient to use the representations

$$U = U^{(c)}(x, c^1, c^2), \quad f = f^{(c)}(x, c^1, c^2),$$

(3.12)

where $U^{(c)} = \tilde{U}(x, f^{(c)})$.

**Definition 3.10** A Finsler space is called the **UZ-Finsler space** if the Finslerian metric function $F$ of the space obeys the Separation Scheme (3.9). Such metric functions will be denoted symbolically by $F^{UZ}$.

For the components of the unit vector $l^i = y^i/F$ we obtain the following angle representations:

$$l^3 = \frac{1}{\tilde{U}(x, \theta)}, \quad l^2 = \frac{\tilde{f}(x, \theta)}{\tilde{U}(x, \theta)} \frac{1}{\tilde{Z}(x, \phi)}, \quad l^1 = \tilde{t}(x, \phi) \equiv \frac{\tilde{f}(x, \theta)}{\tilde{U}(x, \theta)} \frac{\tilde{t}(x, \phi)}{\tilde{Z}(x, \phi)},$$

(3.13)

It is instructive to compare the introduced functions with their Euclidean precursors:

$$\tilde{U}^{\text{Euclidean}} = \frac{1}{\cos \theta}, \quad \tilde{f}^{\text{Euclidean}} = \tan \theta, \quad \tilde{Z}^{\text{Euclidean}} = \frac{1}{\sin \phi}, \quad \tilde{t}^{\text{Euclidean}} = \frac{1}{\tan \phi},$$

and

$$l^{3;\text{Euclidean}} = \cos \theta, \quad l^{2;\text{Euclidean}} = \sin \theta \sin \phi, \quad l^{1;\text{Euclidean}} = \sin \theta \cos \phi.$$

It can be said that our method consists in extending the Euclidean trigonometric functions to obtain the generalized **Finsleroid trigonometric functions**:

$$\cos(x, \theta) = \frac{1}{\bar{U}(x, \theta)}, \quad \sin(x, \theta) = \frac{\tilde{f}(x, \theta)}{\bar{U}(x, \theta)}, \quad \tan(x, \theta) = \tilde{f}(x, \theta),$$

(3.14)
and

\[
\cos(x, \phi) = \frac{\dot{t}(x, \phi)}{\dot{Z}(x, \phi)}, \quad \sin(x, \phi) = \frac{1}{\dot{Z}(x, \phi)}, \quad \tan(x, \phi) = \frac{1}{\dot{t}(x, \phi)},
\]

(3.15)

retaining the Euclidean structure of dependence on the angles \(\theta\) and \(\phi\):

\[
l^{3,\text{Finsleroid}} = \cos(x, \theta), \quad l^{2,\text{Finsleroid}} = \sin(x, \theta) \sin(x, \phi),
\]

(3.16)

and

\[
l^{1,\text{Finsleroid}} = \sin(x, \theta) \cos(x, \phi).
\]

(3.17)

With this trigonometric extension, it is convenient to introduce the \textit{radius} \(R_{x}^{P_{3}}\) of the \textit{horizontal section} \(F_{x}^{P_{3}}\) of the Finsleroid:

\[
R_{x}^{P_{3}} = \frac{\dot{f}}{U} \bigg|_{P_{3}} \equiv \sin(x, \theta^{P_{3}}),
\]

(3.18)

where \(\theta^{P_{3}}\) corresponds to \(P_{3}\). This \(R_{x}^{P_{3}}\) can conveniently be juxtaposed with the value

\[
z_{x}^{P_{3}} = \frac{1}{U} \bigg|_{P_{3}} \equiv \cos(x, \theta^{P_{3}}).
\]

(3.19)

The conditions (3.8)-(3.11) specify the dependence of the characteristic functions on tangent vectors \(\{y^i\}\), and therefore the form of derivatives of the functions with respect to \(y^i\). In particular, we obtain the expansions

\[
l_i = z_i U + z U_i, \quad U_i = \dot{U}_i \theta_i, \quad \theta_i = \dot{\theta}_i f_i, \quad \theta_{ij} = \dot{\theta}_{ij} f_i f_j + \ddot{\theta}_{ij},
\]

(3.20)

\textit{etc.}, where the subscripts \(\theta\) and \(f\) mean differentiations; \(l_i = \partial F/\partial y^i, z_i = \partial z/\partial y^i, U_i = \partial U/\partial y^i, f_i = \partial f/\partial y^i, \theta_i = \partial \theta/\partial y^i\), and \(\theta_{ij} = \partial \theta_i/\partial y^j\).

Let us use the notation

\[
U_f = \frac{\partial \dot{U}}{\partial f}, \quad U_{ff} = \frac{\partial^2 \dot{U}}{\partial f \partial f}, \quad U_1 = \frac{\partial U^{(c)}}{\partial c^1}, \quad U_2 = \frac{\partial U^{(c)}}{\partial c^2}, \quad f_1 = \frac{\partial f^{(c)}}{\partial c^1}, \quad f_2 = \frac{\partial f^{(c)}}{\partial c^2},
\]

such that

\[
\frac{\partial f}{\partial y^3} = -\frac{1}{z} f, \quad \frac{\partial f}{\partial y^1} = \frac{1}{z} f_1, \quad \frac{\partial f}{\partial y^2} = \frac{1}{z} f_2.
\]

(3.22)

In many instances it is convenient to use the equalities \(U_1 = U_f f_1\) and \(U_2 = U_f f_2\). The identity

\[
f = c^1 f_1 + c^2 f_2
\]

(3.21)

holds fine due to the homogeneity implied. Evaluating the covariant unit vector components \(l_i = \partial F/\partial y^i = l_1 i_i + l_2 j_i + l_3 i_{[3]} i\) yields

\[
l_1 = U_1, \quad l_2 = U_2, \quad l_3 = U - U_f f
\]

(3.22)

[notice that \(l_3 = U - U_f (f_1 c^1 + f_2 c^2)\)].
With this preparation, all the derivatives \( l_{ij} = \partial l_i / \partial y^j \) can be found: at first, we arrive at
\[
y^3 l_{33} = U_{ff} f^2, \quad y^3 l_{31} = -U_{ff} f f_1, \quad y^3 l_{32} = -U_{ff} f f_2,
\]
and after that we can use \( U_1 = U f f_1 \equiv l_1 \) and conclude that \( y^3 l_{11} = U_{ff} f f_1 f_1 + U f f_1 f_{11} \), etc. By following this method, we find all the components \( h^{UZ}_{ij} = F l_{ij} \) of the tensor \( h^{UZ} \) and establish the validity of the following assertion.

**Proposition 3.1** The tensor \( h^{UZ}_{ij} \) of the \( UZ \)-Finsler space can be given by the following components:
\[
h^{UZ}_{33} = U_1 f^2, \quad h^{UZ}_{31} = -U_1 f f_1, \quad h^{UZ}_{32} = -U_1 f f_2,
\]
and
\[
h^{UZ}_{11} = U_1 f_1 f_1 + U_2 f_1 f_{11}, \quad h^{UZ}_{12} = U_1 f_1 f_2 + U_2 f_1 f_{12}, \quad h^{UZ}_{22} = U_1 f_2 f_2 + U_2 f_2 f_{22},
\]
where \( U_1 = U U_{ff} \) and \( U_2 = U U_f \).

In turn, the explicit list of these representations determines explicitly all the components
\[
g^{UZ}_{ij} = h^{UZ}_{ij} + l_{ij},
\]
where \( g^{UZ} \) is the Finslerian metric tensor of the \( UZ \)-Finsler space.

Using the list, the attentive calculations of the determinant
\[
D^{UZ} = \det(g^{UZ}_{ij})
\]
lead to the following significant result.

**Proposition 3.2** The determinant \( D^{UZ} \) of the Finslerian metric tensor \( g^{UZ} \) of the \( UZ \)-Finsler space is the product
\[
D^{UZ} = D_1 D_2
\]
with the factors
\[
D_1 = U^4 U_{ff} U_f \frac{1}{f}, \quad D_2 = Z^3 Z_{tt}
\]
which possess the property of separation of dependence on angles \( \theta \) and \( \phi \), namely
\[
D_1 = D_1(x, \theta), \quad D_2 = D_2(x, \phi).
\]

**4 The \( UZ \)-Finsler space of Finsleroid type**

The representations (3.23) indicate that subjecting \( \theta \) to the nonlinear differential equation
\[
\dot{\theta}_f \ddot{\theta}_f = H^2 \frac{1}{U} U_{ff}
\]
is necessary and sufficient in order that the Finsleroid equalities
\[
h_{3c} = h^{(H, \theta, \phi)}_{3c} \equiv F^2 \frac{1}{H^2} \theta_3 \theta_c, \quad c = 1, 2, 3,
\]
hold, where \( \theta_c = \dot{\theta}_f \partial f / \partial y^c \equiv \partial \theta / \partial y^c \). The vanishing \( \partial \phi / \partial y^3 = 0 \) has been taken into account.
Thus we come to the equality

\[ h_{ij} = \frac{1}{H^2} \left( \theta_i \theta_j + H^2 c^2 \frac{1}{U} U_f Z t_i t_j \right) F^2, \]

where \( t_i = \partial t/\partial y^i \). Since \( f = c^2 Z \), we can write

\[ h_{ij} = \frac{1}{H^2} \left( \theta_i \theta_j + H^2 A_1 A_2 t_i t_j \right) F^2, \quad A_1 = \frac{1}{U} U_f, \quad A_2 = \frac{1}{Z} Z t_t. \] (4.2)

This representation gives rise to the validity of the following assertions.

**Conclusive Theorem** The \( \mathcal{UZ} \)-Finsler space is the Finsleroid space if and only if the following two nonlinear differential equations are fulfilled:

\[ \ddot{\theta}_f \dot{\theta}_f = H^2 \frac{1}{U} U_f f, \quad \frac{1}{U} U_f f = T \sin^2 \theta, \] (4.3)

with

\[ T = C \frac{1}{H^2}, \quad T = T(x), \quad C = C(x), \quad H = H(x). \] (4.4)

At the same time, the condition of constancy of the indicatrix curvature of the \( \mathcal{UZ} \)-Finsler space does not impose any equation on the dependence of the function \( Z = Z(x, t) \) on the argument \( t \), as well as of the function \( \hat{Z} = \hat{Z}(x, \phi) \) on \( \phi \). To comply with the constancy of the indicatrix curvature, the function \( \phi = \phi(x, t) \) must obey the equality

\[ \phi t \phi t = C \frac{Z_{tt}}{Z}, \] (4.5)

that is, the function must be given by the integral

\[ \phi = \int \sqrt{C \frac{Z_{tt}}{Z}} \, dt. \] (4.6)

The involved function \( Z \) can depend on the argument \( t \) in an arbitrary way.

**Remark 4.1** The \( \mathcal{UZ} \)-Finsler space is the Finsleroid space at any choice of the dependence of \( Z \) on \( t \), provided that the generating function \( U \) fulfills the equations (4.3). This arbitrariness of the function \( Z = Z(x, t) \) is rather unexpected phenomenon, and even a suspicious statement for the first glance. However, the explanation of the phenomenon can be formulated in simple words. Namely, the condition of constancy of the indicatrix curvature of the \( \mathcal{UZ} \)-Finsler space gives rise to the similar equations \( \ddot{\theta}_f \dot{\theta}_f = H^2 U_f f / U \) and \( \phi t \phi t = C Z_{tt} / Z \). For the function \( U \) the respective representation (2.7) of the tensor \( h_{ij}^{\{H, \theta, \phi\}} \) gives the second equation \( U_f f / U = T \sin^2 \theta \) because of the presence of the function \( \sin^2 \theta \) in the right-hand part of the representation. No functions of \( \phi \) enters the right-hand part, - whence no second equation can be obtained for the dependence of the function \( \hat{Z} \) on \( \phi \).

**Remark 4.2** Owing to the assumed Separation Scheme, the structure of dependence of the functions \( A_1 \) and \( A_2 \) on tangent vectors is principally different, namely we have \( A_1 = A_1(x, f) \) and \( A_2 = A_2(x, t) \) (see (4.2)). Therefore, to get the Finsleroid representation

\[ h_{ij}^{\mathcal{UZ}} = h_{ij}^{\{H, \theta, \phi\}} \] (4.7)
with $h_{ij}^{(H; \theta, \phi)} = (1/H^2) (\theta, \theta_j + \sin^2 \theta \phi_i \phi_j) F^2$ (see (2.7)), where $\phi = \phi(x, t)$ and $\phi_i = \phi_t t_i$, the equations $(1/U)U_f f = T \sin^2 \theta$ and $\phi_t \phi_t - C(1/Z)Z_{tt}$ with $T = T(x)$, $C = C(x)$, and $T = C/H^2$ must be fulfilled.

**Definition 4.1** The equations (4.3)-(4.4) formulated in the above Conclusive Theorem will be called the $UZ$-Finsleroid characteristic equations, the $FRD_UZ$-equations for short.

**Proposition 4.1** The dependence of the characteristic functions $U = \tilde{U}(x, \theta)$ and $f = \tilde{f}(x, \theta)$ of the $UZ$-Finsleroid space on the argument $\theta$ can explicitly be found upon integrating the $FRD_U^UZ$-equations (4.3)-(4.4), namely the respective representations are given by Formulas (4.19) and (4.22).

Let us arrive at the solutions. To this end we differentiate the second $FRD_U^UZ$-equation $(1/U)U_f f = T \sin^2 \theta$ with respect to $f$, whereupon using the first $FRD_U^UZ$-equation $\ddot{\theta} \ddot{f} = H^2 (1/U)U_{ff}$, which yields

$$-rac{1}{f} T \sin^2 \theta T \sin^2 \theta + \frac{1}{H^2} (\ddot{\theta} f)^2 f + \frac{1}{f} T \sin^2 \theta = 2 \ddot{\theta} f \sin \theta \cos \theta.$$ 

This equation can be reduced to

$$-T \sin^4 \theta + \frac{1}{TH^2} (\ddot{\theta} f)^2 + \sin^2 \theta (\sin^2 \theta + \cos^2 \theta) = 2 f \ddot{\theta} f \sin \theta \cos \theta.$$ 

Noting $TH^2 = C$, we arrive at the quadratic equation

$$\left( \frac{1}{\sqrt{C}} \ddot{\theta} f - \sqrt{C} \sin \theta \cos \theta \right)^2 = (T - 1) \sin^4 \theta + (C - 1) \sin^2 \theta \cos^2 \theta.$$ 

Inserting

$$\ddot{\theta} f = C R_{17} \sin \theta,$$

we obtain the equation

$$C \left( R_{17} - \cos \theta \right)^2 = (T - 1) \sin^2 \theta + (C - 1) \cos^2 \theta$$

from which the function $R_{17}$ can be found, namely

$$R_{17} = \cos \theta + \sqrt{L_{17}},$$

where

$$L_{17} = \left( \frac{1}{H^2} - \frac{1}{C} \right) \sin^2 \theta + \left( 1 - \frac{1}{C} \right) \cos^2 \theta.$$ 

The last function can conveniently be written in the form

$$L_{17} = P_1 + H_1 \sin^2 \theta \equiv H_2 - H_1 \cos^2 \theta,$$

where

$$P_1 = 1 - \frac{1}{C}, \quad H_1 = \frac{1}{H^2} - 1, \quad H_2 = \frac{1}{H^2} - \frac{1}{C}.$$ 

The argument dependence of this function is of the type $L_{17} = L_{17}(x, \theta)$. 
In order to find the dependence of the functions \( U \) and \( f \) on the argument \( \theta \) we introduce the functions

\[
I = \exp \left( -\sqrt{H_1} \arcsin \left( \frac{\sqrt{H_1}}{H_2} \cos \theta \right) \right) \quad (4.13)
\]

and

\[
Y = \left( \frac{1}{\sqrt{H_2} \sin \theta} \left( \sqrt{L_{17}} + \sqrt{P_1} \cos \theta \right) \right)^2 \equiv \frac{\sqrt{P_1} \cos \theta + \sqrt{L_{17}}}{-\sqrt{P_1} \cos \theta + \sqrt{L_{17}}} \quad (4.14)
\]

together with

\[
Y = \frac{Y_+}{Y_-} \equiv \left( \frac{Y_+}{\sin \theta} \right)^2, \quad (4.15)
\]

where

\[
Y_+ = \sqrt{\sin^2 \theta + a \cos^2 \theta + \sqrt{a} \cos \theta}, \quad Y_- = \sqrt{\sin^2 \theta + a \cos^2 \theta - \sqrt{a} \cos \theta} \quad (4.16)
\]

with \( a = P_1/H_2 \). The function

\[
Y_1 = Y_1 \sqrt{P_1} \quad (4.17)
\]

enters various significant Finsleroid representations.

With

\[
\frac{1}{U} U f = T \sin^2 \theta, \quad \theta f = C R_{17} \sin \theta, \quad T = C \frac{1}{H^2},
\]

we come to the following differential equation for the generating function \( U = \bar{U} \):

\[
\frac{1}{U} \bar{U}_{\theta} = \frac{1}{H^2 R_{17}} \sin \theta. \quad (4.18)
\]

Integrating yields the explicit representation

\[
\bar{U} = C_{22} \frac{1}{R_{17}} I, \quad C_{22} = C_{22}(x), \quad (4.19)
\]

which can readily be verified.

The last formulas entail

\[
\left( \frac{1}{U} \right)_{\theta} = -\frac{1}{C_{22}} \frac{1}{H^2 I} \frac{1}{\sin \theta}. \quad (4.20)
\]

Also, we should solve the equation (4.8) to determine the dependence of \( f = \tilde{f}(x, \theta) \) on \( \theta \). To this end we write the equation as

\[
(ln \, \tilde{f})_{\theta} = \frac{1}{C R_{17} \sin \theta}. \quad (4.21)
\]

On the other hand,

\[
\left( \ln \left( Y_1 \frac{1}{R_{17}} \sin \theta \right) \right)_{\theta} = \frac{1}{C R_{17} \sin \theta}. \quad (4.21)
\]
We observe that the equality
\[ (\ln \tilde{f})_\theta = \left( \ln \left( \frac{\sin \theta}{R_{17}} \right) \right)_\theta \]
holds. Therefore, the sought solution reads
\[ \tilde{f} = C_{33} \frac{\sin \theta}{R_{17}} Y_1, \quad C_{33} = C_{33}(x). \quad (4.22) \]
Thus Proposition 4.1 is valid.

With the help of (4.21) we can obtain
\[ \tilde{f}_\theta = C_{33} \frac{1}{C(R_{17})^2} Y_1. \quad (4.23) \]

It follows also that
\[ \frac{1}{U} \tilde{f} = C_{33} \frac{1}{C_{22} I} Y_1 \sin \theta \]
is the radius of the horizontal section (cf. (3.18)).

The Finsleroid trigonometric functions (see (3.14)) admit the following explicit representations:
\[ \text{Sin}(x, \theta) = \frac{C_{33}}{C_{22} I} Y_1 \sin \theta, \quad \text{Cos}(x, \theta) = \frac{1}{C_{22} R_{17}} Y_1 \]
\[ \quad \text{(4.25)} \]

If we compare two formulas \((\tilde{\theta}_f f)^2 = H^2(1/U)U_{ff} f^2 \) and \( \tilde{\theta}_f f = C R_{17} \sin \theta \) (see (4.3) and (4.8)), we can obtain the equality
\[ \frac{1}{U} U_{ff} f^2 = \frac{1}{H^2} C^2(R_{17})^2 \sin^2 \theta. \quad (4.26) \]

Owing to \( U = C_{22}(1/R_{17})I \) (see (4.19)), we can also propose the representation
\[ U U_{ff} = \frac{1}{H^2} C^2(C_{22})^2 I^2 \sin^2 \theta \]
\[ \quad \text{(4.27)} \]
with very simple dependence on the variable \( \theta \) in the right-hand part.

Taking into account the equalities
\[ f = C_{33} \frac{\sin \theta}{R_{17}} Y_1, \quad U = C_{22} \frac{1}{R_{17}} I \]
(see (4.19) and (4.22)), we can conclude that
\[ \frac{1}{U} U_{ff} = \frac{1}{H^2(C_{33})^2(Y_1)^2} C^2(R_{17})^4 \]
and
\[ U^3 U_{ff} = \frac{C^2(C_{22})^4}{H^2(C_{33})^2 (Y_1)^2} I^4. \quad (4.28) \]

The representations (4.26)-(4.28) help us to evaluate the determinant \( D = \det(g_{ij}) \) of the Finsleroid metric tensor \( g_{ij} = g_{ij}(x, y) \). Namely, we can elucidate the dependence
of the factor $D_1$ on the argument $\theta$ in the determinant $D = \det(g_{ij}) = D_1D_2$ (see (3.26)). To this end, the representations
\[
\frac{1}{U} U_{ff} f^2 \frac{1}{H^2} C^2(R_{17})^2 \sin^2 \theta, \quad \frac{1}{U} U_{ff} = T \sin^2 \theta,
\]
(see (4.26) and (4.3)) can conveniently be applied in (3.27), which yields
\[
D_1 = U^6 \frac{1}{H^2} C^2(R_{17})^2 \sin^2 \theta T \sin^2 \theta \frac{1}{f^4}.
\]
Inserting here the functions
\[
U = C_{22} \frac{1}{R_{17}} I, \quad f = C_{33} \frac{\sin \theta}{R_{17}} Y_1
\]
(see (4.19) and (4.22)) we get the following result:
\[
D_1 = C_{11} I^6 \frac{1}{(Y_1)^4}, \quad C_{11} = C_{11}(x), \quad C_{11} = \frac{1}{H^2} C^2 T \frac{(C_{22})^6}{(C_{33})^4}, \quad (4.29)
\]

5 Conclusions

The established Conclusive Theorem yields the exhaustive answer to the question “What is the form of the ODE-s which describe the azimuthal $\theta$-angle dependence of the three-axes Finsleroid characteristic functions provided that the Separation Scheme is used?”

At the same time, the derived ODE-s don’t impose any restriction on the dependence of the characteristic functions on the Finsleroid polar angle $\phi$. The dependence can be introduced or specified by means of additional geometrical conditions.

The clarification of the regularity properties of the obtained solutions requires the systematic and attentive study.

References

[1] Asanov,G.S.:Pseudo-Finsleroid metric function of spatially anisotropic relativistic type (2015). [arXiv:1512.02268]
[2] Asanov,G.S.:Pseudo-Finsleroid metrics with two axes, European Journal of Mathematics. 3(4), 1076-1097 (2017). DOI 10.1007/s40879-017-0160-6
[3] Asanov,G.S.:Two-axes pseudo-Finsleroid metrics: general overview and angle-regular solution (2017). [arXiv:1709.02683]
[4] Cartan,E.:Les Espaces de Finsler. Actualités Scientifiques et Industrielles, vol. 79. Hermann, Paris (1934)
[5] Berwald,L.:"Über Finslerische und verwandte Räume, Cas. Mat. Fys. 64, 1-16 (1935)
[6] Busemann,L.:The Geometry of Finsler Spaces, Bull. Amer. Math. Soc. 56, 5-15 (1950)
[7] Rund,H.:The Differential Geometry of Finsler Spaces. Die Grundlehren der Mathematischen Wissenschaften, Band, vol. 101. Springer, Berlin, 1959
[8] Horvath,J.I.:New geometrical methods of the theory of physical fields, Nuov. Cim. 9 (1958) 444-496.
[9] Ingarden,R.S.:On physical applications of Finsler Geometry, Contemporary Mathematics 196 (1996) 213-223.
[10] Asanov,G.S.:Finsler Geometry, Relativity and Gauge Theories, D. Reidel Publ. Comp., Dordrecht 1985.
[11] Bao, D., Chern, S.-S., Shen, Z.: An Introduction to Riemann-Finsler Geometry, Graduate Texts in Mathematics, vol. 200. Springer, New York (2000)

[12] Matveev, V.S., Papadopoulos, A., Rademacher, H.-B., Sabau, S.V. (Eds.): The Topical Issue “Finsler Geometry: New Methods and Perspectives” European Journal of Mathematics (2017) 3: 763-766. DOI 10.1007/s40879-017-0195-8

[13] European Journal of Mathematics, December 2017, Issue 4, Pages 763-1273. Special Issue: Finsler Geometry, New Methods and Perspectives

[14] Asanov, G.S.: Finsler connection properties generated by the two-vector angle developed on the indicatrix-inhomogeneous level, Publ. Math. Debrecen 82(1) (2013) 125-155

[15] Vincze, Cs.: On Asanov’s Finsleroid-Finsler metrics as the solutions of a conformal rigidity problem, Differential Geometry and its Applications 53 148-168 (2017)