BOUNDARY UNITARY REPRESENTATIONS—RIGHT-ANGLED HYPERBOLIC BUILDINGS

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ABSTRACT. We study the unitary boundary representation of a strongly transitive group acting on a right-angled hyperbolic building. We show its irreducibility. We do so by associating to such a representation a representation of a certain Hecke algebra, which is a deformation of the classical representation of a hyperbolic reflection group. We show that the associated Hecke algebra representation is irreducible.

INTRODUCTION

Considering the group $SL(2, \mathbb{R})$ acting by isometries on the hyperbolic plane and the circle $S^1$ as its boundary, one is led to study various unitary representations of $SL(2, \mathbb{R})$ on the function space $L^2(S^1)$: the so-called “principal series representations.” It is well known that these representations are irreducible and constitute a sizable part of the unitary dual of $SL(2, \mathbb{R})$. This fundamental fact inspired many authors who obtained various generalizations. For example, one may replace the field $\mathbb{R}$ with a non-Archimedean local field, and the group $SL(2)$ with an arbitrary semi-simple algebraic group. Important tools in the study of the representations thus obtained are the associated affine building and the so-called Iwahori-Hecke algebra, see [18], [19], [17]. In another course of generalization, taken in [8] (see also [12], [13], [14], [2]), the authors consider discrete subgroups and show (for example) that the restriction of the principal $SL(2, \mathbb{R})$-representations on $L^2(S^1)$ to the subgroup $W$, generated by the reflections across the sides of a (compact) right angled polygon, is still irreducible.

Our current contribution links the previous two routes: we consider a group acting on a building and its unitary representation on $L^2$ of the boundary. We use the aid of an associated Hecke algebra representation in order to analyze this boundary representation. It turns out that the associated Hecke algebra representation could be seen as a deformation of the unitary representation of $W$ alluded to above. We prove the irreducibility of this deformed representation.
using the tools developed in [1] (see also the recent generalization [15]). Our exact theorem is

**Theorem 1.** Let $X$ be a right-angled (Lobachevsky) hyperbolic building of finite thickness. Let $G$ be a group acting strongly transitively on $X$. The associated unitary representation of $G$ on $L^2(\partial X)$ is irreducible.

As mentioned above, in the course of the proof of Theorem 1 we reduce it to the following theorem, which might be of independent interest.

**Theorem 2.** Let $W$ be the group generated by the reflections across the codimension-1 faces of a compact right-angled polytope in the hyperbolic $n$-space. Fix real parameters $q_s$, indexed by the codimension-1 faces of the polytope and consider the corresponding Iwahori-Hecke algebra $\mathcal{H}$. Then the natural representation of $\mathcal{H}$ on $L^2(S^{n-1})$ is irreducible, provided that for every face $s$, $q_s \geq 1$.

The terms used in the formulations of the above theorems will be explained in the next sections. In particular, the unitary representations considered are the principal series representation with the trivial parameter ($\epsilon = 0$). Unfortunately, the question of whether other principal series representations are irreducible as well remains open. The dimension of the buildings that we deal with is limited: Vinberg proved that compact right-angled hyperbolic polytopes do not exist in dimensions $> 4$ (see [9, Cor. 6.11.7]).

The paper is divided into three parts. In Part I we detail the setting of Theorem 1 and explain its reduction to Theorem 2. Part II describes explicitly the Hecke algebra and its principal series representations. Finally, in Part III, we prove Theorem 2.

**Part I. Representation on the boundary of the building**

1. **Definition of principal series**

In this section we define a series of unitary representations of a hyperbolic building automorphism group on the $L^2$-space of the boundary of the building.

Let $P$ be a bounded polytope with finitely many faces in a hyperbolic space $\mathcal{H}^d$. Suppose that all dihedral angles of $P$ are of the form $\pi/k$, $k \in \mathbb{Z}$. Then the set $S$ of reflections across the codimension-1 faces of $P$ generates a group $W$ acting on $\mathcal{H}^d$, called a *hyperbolic Coxeter group*. The action of $W$ on $\mathcal{H}^d$ is geometric (cocompact and properly discontinuous). The $W$-translates of $P$ have pairwise disjoint interiors and form a tessellation of $\mathcal{H}^d$.

Let $X$ be a building whose Weyl group is a hyperbolic Coxeter group $W$, as above. One can think of $X$ as of a set (of chambers) with a $W$-valued distance function. One can also consider geometric realizations of $X$; one of them will be the hyperbolic realization $|X|$, with chambers isometric to $P$ and apartments isometric to $\mathcal{H}^d$ tessellated by copies of $P$ as above. Codimension-1 faces of $P$ correspond to elements of $S$; this labeling extends consistently to codimension-1 faces in $X$, the label of such face usually called its *type*. In our tessellation of $\mathcal{H}^d$, a codimension-1 face is shared by two chambers; in a building, there should be
more of them, and their number is called the \textit{thickness} of the building along the face. We assume that the thickness is finite and depends only on the type \( s \) of the face, and we denote it \( q_s + 1 \). The \textit{thickness vector} \( q = (q_s)_{s \in S} \) encodes the thickness data. If all \( q_s \) are equal, we interpret \( q \) as their common value; this is the \textit{uniform thickness} case. Yet, they can all be easily re-interpreted to make sense for non-uniform thickness. Comments explaining some details of this re-interpretation will be given in \S 7.

An automorphism of \( X \) is a bijection \( X \to X \) preserving the \( W \)-valued distance. It can be realized geometrically as an isometric map \( |X| \to |X| \); that map preserves types of codimension-1 faces. Finally, we want a group \( G' < G = Aut(X) \) acting strongly transitively on \( X \) (this means transitivity on the set of pairs (chamber \( \in \) apartment)). The existence of a (strongly) transitive action implies that the thickness of the building along a face depends only on the type of that face. The group \( G \) can be equipped with the compact–open topology coming from its action on \( X \). We require \( G' \) to be closed in this topology (anyway, passing to closure preserves strong transitivity). Then both \( G \) and \( G' \) are locally compact, totally discontinuous, second countable, generated by compact subgroups (e.g., by stabilizers of codimension-1 faces of one chamber), unimodular.

We normalize the Haar measure \( \nu \) on \( G \) by requiring that (every) chamber stabilizer has measure 1 (all such stabilizers are conjugate). For a compact-open subgroup \( H \) of \( G \) we denote by \( \nu_H \) the probability Haar measure on \( H \); this measure coincides with a suitably rescaled restriction of \( \nu \) to \( H \).

The metric space \( |X| \) is \textit{CAT}(-1) and has a compact Gromov boundary \( \partial |X| \), which we usually shorten to \( \partial X \) (the set \( X \) itself can be equipped with the gallery distance; then it is quasi-isometric to \( |X| \), and has the same Gromov boundary). The action of \( G \) on \( X \) extends to an action on \( \partial X \). We will define a \( G \)-quasi-invariant measure \( \mu \) on \( \partial X \) and consider the associated family of unitary \( G \) and \( G' \)-representations on \( L^2(\partial X, \mu) \).

We fix an isometric identification of \( \mathcal{H}^d \) (along with \( P \) and the tessellation that it generates) with the the Poincaré disc model \( D^d \). We require that the center 0 of the model corresponds to an interior point of \( P \), also to be denoted 0. The action map of \( W \) onto the orbit of 0 is a quasi-isometry between \( W \) (with the \( S \)-word metric) and \( \mathcal{H}^d \) or \( D^d \); we use it to identify \( \partial W \), \( \partial \mathcal{H}^d \), and \( \partial D^d = S^{d-1} \). Hence, we get a measure \( l \) on \( \partial W \) corresponding to the Lebesgue measure on the sphere. For any chamber \( c \in X \) and any apartment \( A \) containing \( c \) the \( c \)-based folding map identifies \( \partial A \) with \( \partial W \); pulling \( l \) back by this folding we get a measure \( l_c \) on \( \partial A \). The stabilizer \( G_c \) of \( c \) in \( G \) has a probability Haar measure \( \nu_c \). The map \( p^c : G_c \times \partial A \ni (b, x) \mapsto bx \in \partial X \) is surjective by strong transitivity.

**Definition 1.1.**

\[
(1.1) \quad \mu_c = p^c_*(\nu_c \times l_c); \quad \mu = \mu_{c_0} \text{ for some fixed chamber } c_0.
\]
Note that \( \mu_c \) does not depend on the choice of \( A \): any different \( A' \ni c \) is of the form \( b' A \), for some \( b' \in G_c \); then \( (b, x) \mapsto (b(b')^{-1}, b'x) \) transforms one variant of \( v_c \times l_c \) to the other while preserving fibers of \( p^c \). On the other hand, \( \mu_c \) does depend on the choice of \( c \).

**Lemma 1.2.** The measures \( \mu_c, \mu_c' \) are absolutely continuous with respect to each other for \( c, c' \in X \).

**Lemma 1.3.** For any \( g \in G \) and any chamber \( c \in X \) we have \( g_* \mu_c = \mu_{g(c)} \).

In particular, for any \( g \in G \) we get \( g_* \mu = \mu_{g_0} \ll \mu \). Using this fact we now define the principal series of \( G \); Lemmas 1.2 and 1.3 will be proved later.

**Definition 1.4.**

a) Let \( X \) be a hyperbolic building of finite thickness, and let \( G \) be the group of all type-preserving automorphisms of \( X \). Assume that the action of \( G \) on \( X \) is strongly transitive. The principal series of \( G \) is the family \( \rho_c (c \in \mathbb{R}) \) of representations of \( G \) on \( L^2(\partial X, \mu) \) given by

\[
\rho_c(g) f(x) = f(g^{-1}x) \left[ \frac{d(g_* \mu)(x)}{d\mu(x)} \right]^{\frac{1}{2} + i\epsilon}.
\]

b) Let \( G' \) be a closed subgroup of \( G \), still acting strongly transitively on \( X \). The principal series of \( G' \) is the restriction of the above to \( G' \).

**Remark.** In the definition of \( \rho_c \) one could replace \( \mu \) by any measure \( \mu' \) satisfying \( g_* \mu' \ll \mu' \). However, if \( \mu' = h \mu, h \in L^1(\mu) \), then the representation associated to \( \mu' \) isometrically embeds in that corresponding to \( \mu \): the intertwiner is given by

\[
L^2(\mu') \ni f \mapsto h^{\frac{1}{2} + i\epsilon} f \in L^2(\mu).
\]

In particular, if we replace \( c_0 \) by a different chamber we get an equivalent representation.

**Proof of Lemma 1.3.** Let \( A \) be an apartment containing \( c \). We have a commutative diagram:

\[
\begin{array}{ccc}
G_c \times \partial A & \xrightarrow{t_g \times g} & G_{g(c)} \times \partial (gA) \\
\downarrow p^c & & \downarrow p^{g(c)} \\
\partial X & \xrightarrow{g} & \partial X
\end{array}
\]

where \( t_g \) denotes conjugation by \( g \). Now

\[
g_* (\mu_c) = g_* (p^c_v \times l_c) = p^{g(c)}_v (t_g \times g)_* (p_v \times l_c) = p^{g(c)}_v (v_{g(c)} \times l_{g(c)}) = \mu_{g(c)}.
\]

**Proof of Lemma 1.2.** We may assume that \( c \) and \( c' \) are \( s \)-neighbors for some \( s \in \mathcal{S} \), i.e., that the \( W \)-valued distance from \( c \) to \( c' \) is \( s \) (we get the general statement by applying this special case along a gallery). In the geometric realization these chambers will share a codimension-1 face of type \( s \). For convenience of
notation, we will assume \( c = c_0 \) (the choice of \( c_0 \) was arbitrary). We will also number the \( s \)-neighbors of \( c_0 \) and denote them \( c_1, \ldots, c_q \); we arrange the order of the numbering so that \( c' = c_1 \). (In the non-uniform thickness case, one should read \( q \) as \( q_s \).) Let \( X_i \) be the set of chambers that are closer to \( c_i \) than to any other \( c_j \). By \( \partial X_i \) we denote the set of points in \( \partial X \) that are in the closure of \( X_i \). We put \( \mu_i = \mu_{G_i} \), \( p^i = p^{\mu_i} \), \( l_i = l_{G_i} \), \( G_i = G_{c_i} \). We denote by \( \pi_j \) the \( c_j \)-based folding map \( X \to W \), as well as its extension \( \partial X \to \partial W \). For \( x \in \partial W \) put \( r(x) = \frac{d(s, l)}{\ell}(x) \).

**Lemma 1.5.** Fix an \( i > 0 \). Then

\[
\mu_i|_{\partial X_i} = q(r \circ \pi_0)|_{\partial X_i}.
\]

(The right-hand-side is the measure \( \mu_0|_{\partial X_i} \) multiplied by the function \( r \circ \pi_0 \) and by the number \( q \).)

**Proof.** We choose some apartment \( A \) containing \( c_0 \) and \( c_i \). Then \( \partial A \) will be the support of both \( l_0 \) and \( l_i \). We put \( \partial A_0 = \partial A \cap \partial X_0 \), \( \partial A_i = \partial A \cap \partial X_i \). We will prove the following sequence of equalities:

\[
\mu_i|_{\partial X_i} = p^i_*(\nu_{G_i} \times l_i)|_{\partial X_i} = p^i_*(q \nu_{G_i \cap G_0} \times l_i)|_{\partial X_i}
= p^0_*(q \nu_{G_0 \cap G_i} \times l_i)|_{\partial X_i} = q(r \circ \pi_0|\partial X_i) = q(r \circ \pi_0|\partial X_i).
\]

- The first and last equality are immediate consequences of the definitions of \( \mu_i \) and \( \mu_0 \).
- 2\textsuperscript{nd} equality: We have \( (\cup_{j \neq 0} \partial X_j)/G_0 = \partial A_i \). Moreover, the setwise stabilizer of \( \partial X_i \) in \( G_0 \) is \( G_0 \cap G_i \). Therefore \( \partial X_i/(G_0 \cap G_i) = \partial A_i = \partial X_i/G_i \). It follows that for cosets \( (G_i \cap G_0)b, b \in G_i \), we have \( (G_i \cap G_0)b \partial A_i = \partial X_i \). In particular, for each \( x \in \partial A_i \) we can choose a system \( b_j, j = 1, 2, \ldots, q \) of representatives of right cosets of \( G_i \cap G_0 \) in \( G_i \) such that \( b_j x = x \).

Let us pick a measurable \( U \subseteq \partial X_i \). Fix an \( x \in \partial A_i \) and a system \( b_j \) as above (for this \( x \)). Then

\[
\{ b \in G_i \mid bx \in U \} = \bigcup_j \{ b \in G_i \cap G_0 \mid bx \in U \} b_j.
\]

Since right multiplication by \( b_j \) preserves the Haar measure \( (G_i \text{ is compact, hence unimodular}) \), we have

\[
\nu(\{ b \in G_i \mid bx \in U \}) = \nu(\{ b \in G_i \cap G_0 \mid bx \in U \}).
\]

Integrating this equality over \( \partial A_i \) with respect to \( l_i \) we get that \( p^i_*(\nu|G_i \times l_i) \) and \( p^0_*(\nu|G_0 \cap G_i \times l_i) \) agree on \( U \).

- 3\textsuperscript{rd} equality: The pushed measure is supported on the intersection of the domains of \( p^0 \) and \( p^i \); both maps are defined by the same formula.
- 4\textsuperscript{th} equality: We have \( s_i l = rl \), and (on \( \partial A \) \( \pi_i = s \circ \pi_0 \). In the following calculation the domains of \( \pi_0 \) and \( \pi_i \) are restricted to \( \partial A \):

\[
l_i = (\pi_0^{-1})_* l = ((s \circ \pi_0)^{-1})_* l = (\pi_0^{-1})_* s_* l
= (\pi_0^{-1})_* (rl) = (r \circ \pi_0)(\pi_0^{-1})_* l = (r \circ \pi_0)l_0.
\]
On $G_i \cap G_0$ the measures $\nu|_{G_0}$ and $\nu|_{G_i \cap G_0}$ coincide; $p^0$ maps the remaining part of $G_0$ ($\times \partial A_i$) outside $\partial X_i$. 

To finish the proof of Lemma 1.2 we use principles of symmetry. First

$$
\mu_i|_{\partial X_i} = \frac{1}{q(r \circ \pi_i)} q(r \circ \pi_i) \mu_i|_{\partial X_i} = \frac{1}{q(r \circ \pi_i)} \frac{1}{\mu_0|_{\partial X_0}} (r \circ \pi_i) \mu_0|_{\partial X_0},
$$

(1.10)

the last equalities due to: $\pi_i = s \circ \pi_0$ on $\partial X_0$; $r \circ s = \frac{1}{r}$. Next, for $j \neq 0, i$, we apply (1.10) twice:

$$
\mu_i|_{\partial X_i} = q^{-1}(r \circ \pi_j) \mu_j|_{\partial X_j} = \mu_0|_{\partial X_0}.
$$

(1.11)

We combine (1.5), (1.10), and (1.11) into the following corollary (recall that $\mu = \mu_0$ and $\pi = \pi_0$).

**COROLLARY 1.6.** *Suppose $\phi \in G$ and $\phi c_0 = c_i$ ($i > 0$). Then*

$$
\phi_* \mu = \begin{cases} 
\mu & \text{on } \partial X_i, j \neq 0, i; \\
q(r \circ \pi_i) \mu & \text{on } \partial X_i; \\
q^{-1}(r \circ \pi_i) \mu & \text{on } \partial X_0.
\end{cases}
$$

(1.12)

**REMARK 1.7.** It is possible to deduce from Corollary 1.6 a general formula for $\frac{d\mu_c}{d\mu_{c'}}$ in terms of Busemann functions. In a metric space $(X, \rho)$ one defines the Gromov product $(x|y)_z = \frac{1}{2}(\rho(z, x) + \rho(z, y) - \rho(x, y))$. If the space is hyperbolic and has boundary $\partial X$, we may try to extend the Gromov product by putting $(x|b)_z = \lim_{y \rightarrow b} (x|y)_z$. The limit is taken over a sequence of $y \in X$ converging to $b \in \partial X$; the limit may not exist, or may depend on the choice of the sequence of $y$’s, but in our cases of interest these problems will happen only for $b$ in some zero-measure set. Finally, the Busemann function is $\beta_b^\rho(x,y) = \rho(x,y) - 2(y|b)_x$ (for $x,y \in X, b \in \partial X$). In our case, there are two metrics of interest on the building $X$.

1) Each chamber $c \in X$ has a geometric realization $|c|$ in $|X|$, canonically isometric to the fundamental polytope $P$. Inside $|c|$ there is a copy $0_c$ of the point $0 \in P$. We put $|c - c'| = |0_c - 0_{c'}|$ (the distance from $0_c$ to $0_{c'}$ in $|X|$).

2) The gallery distance: $\ell(c,c')$ is the length of a shortest gallery in $X$ starting at $c$ and ending at $c'$.

Now we can state the formula:

$$
\frac{d\mu_c}{d\mu_{c'}} = e^{-\ell(c,c')} \beta_b^\rho(c',c).
$$

(1.13)

For $c$ adjacent to $c'$ this formula reduces to Corollary 1.6. The general case follows from this special case and the cocycle property of a Busemann function:

$$
\beta_b^\rho(x,z) = \beta_b^\rho(x,y) + \beta_b^\rho(y,z).
$$

(1.14)
As noticed by Garncarek, the right hand side of (1.13) can be rewritten as
\[(1.15) \quad \exp(- (d - 1) \beta_b \beta_b^c (c', c) - \ln(q) \beta_b \beta_b^c (c', c)) = \exp(- (d - 1) \beta_b^{\text{mix}} (c', c)),\]
where \(\beta_b (c', c) = |c' - c| + \frac{\ln q}{d - 1} \ell (c', c)\). This means that the measures \((\mu_c)_{c \in X}\) form a quasi-conformal family with respect to the mix metric. Thus, the results of [15] apply to our setting, yielding (in many cases) a different proof of Theorem 1. Garncarek requires the group to be discrete; his result can be applied to a cocompact lattice in \(G'\). In the right-angled case, such a lattice exists in the automorphism group \(G\) and in many interesting subgroups \(G'\), cf. [6].

This remark holds also in the multi-parameter case. One replaces the gallery metric \(\ell\) by a family \((\ell_s)_{s \in S}\) of pseudo-metrics: \(\ell_s (c, d)\) is the number of \(s\)-type codimension-1 faces traversed by a minimal gallery from \(c\) to \(c'\). Then the mix metric is defined as \(\text{mix} (c', c) = |c' - c| + \sum_{s \in S} \frac{\ln q_s}{d - 1} \ell_s (c', c)\).

2. General nonsense

Suppose that \((V, \rho)\) is a unitary representation of a locally compact group \(G\), and let \(K\) be a compact-open subgroup of \(G\). (The group \(K\) will be the stabilizer of a chamber; in classical cases this would correspond to the Borel or Iwahori subgroup, and not to the maximal compact one.) Then one defines the Hecke algebra \(H(G, K)\) as the convolution algebra of all compactly supported \(K\)-bi-invariant functions on \(G\). Elements of \(H(G, K)\) are continuous (even locally constant), because \(K\) is open. They act on \(V\) by
\[(2.1) \quad \rho (f) v = \int_G f (g) \rho (g) v \, dg.\]
This action preserves the space \(V^K = \{v \in V \mid (\forall k \in K)(\rho (k) v = v)\}\) of \(K\)-invariants due to left \(K\)-invariance of \(f\).

**Proposition 2.1.** Suppose that

1) \(V^K\) is \(G\)-cyclic in \(V\) (i.e., \(V\) is the closure of the linear span of \(\rho (G) V^K\));
2) \(V^K\) is Hecke-irreducible (i.e., \(V^K\) has no non-trivial closed \(H(G, K)\)-invariant subspace).

Then \(V\) is an irreducible \(G\)-representation.

**Proof.** Suppose not. Let \(V = V_0 \oplus V_1\) be a non-trivial orthogonal decomposition into subrepresentations. Then \(V^K = V^K_0 \oplus V^K_1\). The subspaces \(V^K_0\) and \(V^K_1\) are non-zero: if \(V^K_0\) was zero, we would have \(\rho (G) V^K = \rho (G) V^K_1 \subseteq V_1\), contradicting the cyclicity assumption; similarly for \(V^K_1\). But this contradicts Hecke-irreducibility of \(V^K\). \(\square\)

The goal of Part I is to establish \(G\)-cyclicity of \(V^K\) in our context. To do that, we need to recall certain facts about hyperbolic right-angled buildings and their automorphism groups.
3. RIGHT-ANGLED BUILDINGS

A Coxeter group \( (W,S) \) is right-angled if any two elements of \( S \) either commute or span an infinite dihedral subgroup of \( W \). A building is right-angled if its Weyl group is right-angled. A hyperbolic reflection group associated to a polyhedron \( P \) is right-angled if all dihedral angles of \( P \) are \( \pi/2 \).

Morphisms of right-angled buildings are discussed at length in [10, Sec. 4]. Here we just summarize the results. Let \( X \) be a right-angled building. We fix a chamber \( c \). A set \( Y \subseteq X \) is star-like (with respect to \( c \)) if every minimal gallery from \( c \) to any \( y \in Y \) is contained in \( Y \). We choose a well-ordering \( \prec \) on \( X \) such that all initial segments \( X_{<x} \) are star-like. A morphism \( \phi: X \to X \) can be constructed inductively with respect to \( \prec \). We first arbitrarily choose \( \phi(c) \). If \( \phi \) is defined on \( X_{<x} \), we try extend it to \( x \). Two things can happen:

1) (freedom) All minimal galleries from \( x \) to \( c \) are of the form \( (x,y,\ldots,c) \) for some unique \( y \). Then there is choice: if \( s \) is the type of the common face of \( x \) and \( y \), then \( \phi(x) \) can be chosen to be any of the \( s \)-neighbors of \( \phi(y) \).
2) (determinism) There are at least two minimal galleries from \( x \) to \( c \) starting differently: \( (x,y,\ldots,c) \) and \( (x,y',\ldots,c) \) with \( y \neq y' \). Then there is a unique choice of \( \phi(x) \) that extends (the so-far constructed part of) \( \phi \) to a morphism. The values of \( \phi(y), \phi(y') \), and similar values given by other galleries consistently and uniquely determine \( \phi(x) \).

Whether \( x \) falls into 1) or 2) depends only on \( w = \pi_c(x) \in W \). We compare the \( S \)-word lengths \( \ell(w) \) and \( \ell(ws) \): if \( \ell(w) > \ell(ws) \) for just one \( s \in S \), then we are in case 1); otherwise we are in case 2). In [10] the set of all pairs \( (y,s) \) as in case 1) is called the root set of \( X \) and denoted \( R(X) \). To check whether the constructed morphism is an automorphism we have the following criterion: for each \( (y,s) \in R(X) \) the map \( \phi \) restricts to a bijection between the sets of \( s \)-neighbors of \( y \) and of \( \phi(y) \). In particular, if the thickness of \( X \) is type-dependent (given by a thickness vector), then a partial automorphism defined on an initial segment of \( X \) can always be extended to an automorphism.

To make use of the above procedure it is necessary to have well-orderings with star-like initial segments (let us temporarily call them nice). For buildings of finite thickness nice orderings are not hard to come by. Here are some examples.

- Any ordering compatible with gallery distance from \( c \) (i.e., satisfying \( \delta(c,x) < \delta(c,y) \Rightarrow x < y \)) is nice.
- For star-like subsets \( A,B \) of \( X \), we can put a nice ordering on \( A \), then extend it to \( A \cup B \) (so that \( A \) is an initial segment), and then extend it to \( X \) (keeping \( A \cup B \) as an initial segment). We will use this type of ordering for convex sets \( A,B \) containing \( c \).
- Any refinement of a nice ordering of \( W \) is nice. By a refinement we mean an ordering on \( X \) satisfying \( \pi_c(x) < \pi_c(y) \Rightarrow x < y \).

Finally, we need to discuss standard open neighborhoods (cf. [10, Sec. 2]). Let \( X \) be a hyperbolic right-angled building with finite thickness and Weyl group \( W \).
We distinguish a chamber $c_0$ and denote by $\pi$ the $c_0$-based folding map. Pick any wall $H$ in $|W|$. Let $\overline{H}$ is the completed wall, i.e., the closure of $H$ in $|W| \cup \partial W$. Consider the connected components of $(|X| \cup \partial X) \sim \pi^{-1}(\overline{H})$. The ones that do not contain $c_0$ are called standard (open) neighborhoods. Every point $p \in \partial X$ has a basis of open neighborhoods in $|X| \cup \partial X$ consisting of suitably chosen standard neighborhoods. The intersections of sets from this basis with $\partial X$ form a basis of neighborhoods of $p$ in $\partial X$.

Any standard open neighborhood $U$ contains a unique minimal chamber: a chamber $x$ in $U$ with minimal gallery distance to $c_0$. This $x$ satisfies the freedom condition 1). Conversely, any chamber $x$ satisfying 1) is the minimal chamber of a unique standard neighborhood, to be denoted $U(x) \cup \partial U(x)$ ($U(x)$ for the part in $|X|$, $\partial U(x)$ for the part in $\partial X$).

4. Invariants are cyclic

The standing notation throughout the paper is as follows: $X$ is a right-angled hyperbolic building of finite thickness $q$ (either uniform or multi-parameter), with Gromov boundary $\partial X$; $G$ is the type-preserving automorphism group of $X$, equipped with the compact-open topology; $c_0$ is a fixed chamber in $X$ (called “the base chamber”), and $K$ is the stabilizer of $c_0$ in $G$; $G'$ is a closed subgroup of $G$, acting strongly transitively on $X$; we put $K' = G' \cap K$; $(V, \rho_c)$ is a principal series representation of $G$, or its restriction to $G'$.

In this section we prove that $V^K$ is $G'$-cyclic in $V$. We first reduce to the case of $G$ and $K$.

**Lemma 4.1.**

a) $V^K = V^{K'}$. More generally, $V^{G_c} = V^{G'_c}$ for any chamber $c \in X$.

b) $\text{span } \rho_c(G)V^K = \sum_{c \in X} V^{G_c} = \text{span } \rho_c(G')V^{K'}$.

**Proof.** We have $\partial X/K = \partial W = \partial X/K'$, the quotient maps being $\pi$ in both cases. Since $\pi$ is measure preserving, we get $V^K \simeq L^2(\partial W, l) \simeq V^{K'}$ as Hilbert spaces, the isomorphisms given by precomposition with $\pi$.

For $g \in G$ we have $\rho_c(g)V^K = V^gKg^{-1} = V^{g_c}$. To finish the proof of a), let us choose a $g \in G'$, such that $gc_0 = c$; then $V^{G_c} = \rho_c(g)V^K = \rho_c(g)V^{K'} = V^{G'_c}$.

Part b) follows as well:

$$\text{span } \rho_c(G)V^K = \sum_{g \in G} V^{G_{g_0}} = \sum_{c \in X} V^{G_c} = \sum_{c \in X} V^{G'_c}$$

$$= \sum_{g \in G'} V^{G_{g_0}} = \text{span } \rho_c(G')V^{K'}.$$  \hfill $\square$

**Theorem 4.2.** The linear span of $\rho_c(G)V^K$ is dense in $V$.

**Proof.** Assume, by contradiction, that $\text{span } \rho_c(G)V^K = \sum_{c \in X} V^{G_c}$ is not dense in $V$. Then there exists a non-zero function $f \in \left(\sum_{c \in X} V^{G_c}\right)^\perp$. The latter space being a (closed) $G$-subrepresentation, we may average $f$ over a small subgroup of $G$ and still get a non-zero $f$ in the same space; in other words, we may assume
that $f$ is invariant under $G_Y$—the pointwise stabilizer of a sufficiently large finite subset $Y \subseteq X$. We may assume and do assume that $c_0 \in Y$.

It turns out that, for diligently chosen chamber $c \in X$ and open set $U \subseteq \partial X$, the properties of $G_Y$-invariance and $G_c$-invariance are equivalent on $U$. Then, the function $f \chi_U$, if non-zero, will be $G_c$-invariant and not perpendicular to $f$, yielding a contradiction ($\chi_U$ is the characteristic function of $U$). We proceed to the details.

Let the gallery diameter of $Y$ be $M$. Consider the gallery-distance $M$-ball around 1 in $W$, and the finitely many tessellation walls $H'$ in $|W|$ that intersect (the closed geometric realization of) this ball. The union of those walls: $W = \bigcup \{H' \mid \text{gallery distance from 1 to } H' \leq M\}$, is closed in $|W| \cup \partial W$. Also, $W \cap \partial \mathcal{H}^d$ has zero Lebesgue measure—this set is a finite union of zero-measure boundaries of walls. We deduce that $\pi^{-1}(\partial W)$ is closed in $\partial X$ and has $\mu$-measure zero. It follows that there exists $x \in \partial X \sim \pi^{-1}(\partial W)$ such that $f$ has non-zero restriction to every open neighborhood of $x$. Let us pick a standard open neighborhood $U(c) \cup \partial U(c)$ of $x$ in $|X| \cup \partial X$ disjoint from the closed set $\pi^{-1}(W)$.

**Lemma 4.3.** For every $g \in G_c$ there exists $g' \in G_c \cap G_Y$ such that $g|_{U(c)} = g'|_{U(c)}$.

**Proof of Lemma 4.3.** We choose an apartment $A$ containing $c_0$ and $c$. The map $\pi: A \rightarrow W$ is an isomorphism; we can therefore move all the data (the gallery $M$-ball around 1, the set $W$) from $W$ to $A$. In this proof it will be convenient to consider foldings onto $A$ (rather than $W$). Thus, $\pi$ will denote the folding map $X \rightarrow A$ fixing $c_0$, and $\pi_c$ will fix $c$. Let $H$ be the boundary wall of $\pi(U(c))$. Cut $|A|$ along the walls $H'$ that intersect $H$; let $C$ be the component of $c_0$ in $|A| \sim \bigcup \{H' \mid H' \cap H \neq \emptyset\}$. By the definition of $W$, the gallery $M$-ball around $c_0$ in $A$ is contained in $C$; since $\pi_c$ is a retraction, $Y \subseteq \pi_c^{-1}(C)$. We choose a well-ordering on $X$ starting in $c$, so that: chambers in $U(c)$ form an initial segment; chambers in $\pi_c^{-1}(C)$ form the next segment; all initial segments are star-like. Then start defining the automorphism $g'$ by imposing: $g'|_C = g|_C$ on $\pi_c^{-1}(C)$. This defines a partial automorphism on an initial segment, which extends to an automorphism of $X$.

This lemma implies Theorem 4.2, as follows. The set $\partial U(c)$ is $G_c$-invariant; every $g \in G_c$ preserves $\mu|_{\partial U(c)}$ and $f \chi_{\partial U(c)}$, since so does $g'$ given by the lemma. It follows that $f \chi_{\partial U(c)} \in V^{G_c}$, $(f, f \chi_{\partial U(c)}) = \int_{\partial U(c)} |f|^2 \, d\mu > 0$—contradiction.

**Part II. Right-angled Hecke algebra**

Our goal in this section is to define and discuss the principal series representations of the Hecke algebra on $V^K \cong L^2(\partial W)$. Elements of the Hecke algebra $H(G, K)$ are compactly supported functions on $G$ that are $K$-bi-invariant. In other words, they correspond to finitely supported functions on $G \backslash K$, which, by the Bruhat decomposition, is naturally identified with $W$. Thus, as vector spaces, $H(G, K) \cong \mathbb{C}[W]$. The convolution multiplication in $H(G, K)$ does not correspond to the group algebra multiplication in $\mathbb{C}[W]$, but rather to its deformation described below in (5.1). The isomorphism is explained in [3, Ch. IV, §2,
ex. 24] and in [9, Lemma 19.1.5]. The source [9] proves the result not only for the full automorphism group $G$, but also for its strongly transitive subgroups $G'$. In §5 below we describe the Hecke algebra structure on $C[W]$ associated with an abstract choice of parameter $q$. In §6 we study the representation of $H(G, K)$ on $V^K$. In both sections we consider, only for the simplicity of the discussion, the uniform thickness case $q$. In §7 we will explain how to read through the previous sections while considering the non-uniform thickness case. We will also explain how to define the principal series representations for an arbitrary Hecke algebra, that is, when considering complex parameters $q_s$.

5. Multiplication in right-angled Hecke algebras

In this section $(W, S)$ is a right-angled Coxeter group, not necessarily hyperbolic. The Hecke algebra of $W$ is the space $\mathcal{H} = \bigoplus_{w \in W} C_{w}$ with $C$-bilinear associative multiplication given by

$$e_s e_w = \begin{cases} e_{sw} & \text{if } \ell(sw) > \ell(w) \\ (q-1)e_w + q e_{sw} & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where $w \in W$ and $s \in S$. It is convenient to rescale the generators. As in [16] and [7] we put $T_w = q^{-\ell(w)/2} e_w$. Then

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ T_{sw} + p T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

where $p = \frac{q-1}{\sqrt{q}}$. Elements of the Hecke algebra can be interpreted as functions on $W$:

$$T_w(u) = \begin{cases} 1 & \text{if } u = w \\ 0 & \text{if } u \neq w; \end{cases}$$

this formula extends by linearity to a vector space isomorphism of $\mathcal{H}$ and the space of complex-valued finitely supported functions on $W$. The Hecke multiplication, transported via this isomorphism, can be described as follows.

**Lemma 5.1.**

$$T_s f(u) = \begin{cases} f(sw) & \text{if } \ell(sw) > \ell(u) \\ f(sw) + p f(u) & \text{if } \ell(sw) < \ell(u). \end{cases}$$

To deduce this lemma from (5.2), we use some tools (cf. [11, Sec. 2]). The Hecke algebra $\mathcal{H}$ can be equipped with the scalar product such that $(T_w)_{w \in W}$ form an orthonormal basis. There is also a natural involution: $(\sum_{w \in W} a_w T_w)^* = \sum_{w \in W} \overline{a_w} T_{w^{-1}}$. In particular, $T_s^* = T_s$ and $T_w^* = T_{w^{-1}}$. For all $f, g, h \in \mathcal{H}$ we have $\langle f, gh \rangle = \langle g, f^* h \rangle$ ([11, Prop. 2.1]).

**Proof of Lemma 5.1.** For any $w \in W$, $f \in \mathcal{H}$, we have

$$f(w) = \langle f, T_w \rangle.$$
Hence
\begin{equation}
T_s f (u) = \langle T_s f, T_u \rangle = \langle f, T_s T_u \rangle.
\end{equation}

Now we use (5.2). If \( \ell(su) > \ell(u) \), then (5.6) further equals \( \langle f, T_{su} \rangle = f(su) \). If \( \ell(su) < \ell(u) \), then (5.6) equals \( \langle f, T_{su} + pT_u \rangle = f(su) + pf(u) \).

Our next goal is to establish a similar formula for \( T_w f \) without assuming that \( w \) is a generator.

**Definition 5.2.** For \( A, D \subseteq W \) we denote by \( P(A|D) \) the set of walls in \( W \) separating \( A \) from \( D \). In \( |W| \) a wall divides \( |W| \) into two connected components; this division induces a partition of the set of chambers, hence of \( W \), into two parts. This partition of \( W \) is the combinatorial meaning of “a wall in \( W \)”. We shall often identify walls with the corresponding reflections. In particular, for \( s \in S \) and a wall/reflection \( a \) we denote by \( a^s \) the wall \( s(a) \) the reflection \( sa s \); for sets we put \( A^s = \{ a^s | a \in A \} \). We put a graph structure on \( P(A|D) \): two walls are connected by an edge if and only if they intersect (in \(|W|\)). Note that in a right-angled Coxeter group walls intersect if and only if the corresponding reflections commute. By \( \mathcal{C}(P(A|D)) \) we denote the set of cliques in \( P(A|D) \), including the empty clique. Recall that a clique is any set of vertices pairwise connected by edges. If \( h \in \mathcal{C}(P(A|D)) \), then every two elements of \( h \) commute; the converse is also true: a pairwise commuting family of wall reflections in \( P(A|D) \) is a clique in \( P(A|D) \). We denote by \( h \) the product of all elements of \( h \) (the context will make the meaning clear). Observe that \( h^{-1} = h \).

**Proposition 5.3.** In the Hecke algebra of a right-angled Coxeter group
\begin{equation}
T_w f (u) = \sum_{h \in \mathcal{C}(P(1|u,w))} p^{|h|} f(w^{-1} u h).
\end{equation}

**Proof.** Induction on \( \ell(w) \). The length 1 case is covered by (5.4). Now assume that the formula is true for \( w \) and that \( \ell(sw) > \ell(w) \); we will deduce that the formula holds for \( sw \). There are two cases: either \( \ell(sw) > \ell(u) \), or \( \ell(sw) < \ell(u) \).

Case 1. \( \ell(sw) > \ell(u) \). Then 1, \( w \) and \( u \) are on the same side of the wall \( s \). We have
\begin{equation}
T_{sw} f (u) = T_s(T_w f) (u) = (T_w f) (su) = \sum_{h \in \mathcal{C}(P(1|su,w))} p^{|h|} f(w^{-1} h su)
\end{equation}
while the postulated formula for \( T_{sw} f (u) \) is
\begin{equation}
\sum_{h \in \mathcal{C}(P(1|u,sw))} p^{|h|} f(w^{-1} shu).
\end{equation}
The equality between (5.8) and (5.9) follows from two observations:
(1) \( P(1|u,sw) = P(1|su,w) \), and
(2) if \( c \in P(1|su,w) \), then \( sc = cs \).

To prove them we need a lemma.

**Lemma 5.4.** Let \( s \in S \); suppose \( s \in P(1,g|h) \) and \( c \in P(1|h,g) \). Then \( sc = cs \).
We split the sum in (5.11) into two parts: over which we need to compare with \( g \) (with \( P(1) \), forms of several formulae in this and the next section. We would like to relate it to a very similar formula in a recent preprint pairs. Our formula works for right-angled Coxeter groups, and is, we think, more explicit. We would like to rewrite it as a formula for \( T \) tions, as a preparation for a similar result in the next section. We would like to first sum in (5.10) is equal to the first sub-sum of (5.11). Then we put the second sum in (5.10) is equal to the second sub-sum of (5.11). We deduce that these parts are equal to the two sums in (5.10).

Proof. For a wall \( H \) we denote by \( H^- \) the half-space bounded by \( H \) that contains 1; \( H^+ \) is the other half-space bounded by \( H \). If \( sc \neq cs \), the walls \( s \) and \( c \) do not intersect. Consequently, one of the following holds: \( s^+ \subseteq c^+ \), \( s^- \subseteq c^- \), \( s^+ \subseteq c^- \), \( s^- \subseteq c^+ \). However, \( s \in s^+ \cap c^- \), \( 1 \in s^- \cap c^- \), \( h \in s^+ \cap c^- \), \( g \in s^- \cap c^+ \). Hence, neither inclusion may hold.

Putting \( g = w \) and \( h = su \) we deduce observation (2) from Lemma 5.4. As for (1), \( P(1|su,w) = P(s|u,sw) = P(1|u,sw) \): the first equality follows from (2); the second is clear if we interpret elements of \( P(1|su,w) \) as walls; the third is true because \( s \) (the only wall separating elements 1 and \( s \)) does not belong to either side.

Case 2. \( \ell(su) < \ell(u) \). We have

\[
T_{sw} f(u) = T_s(T_w f)(u) = T_w f(su) + p(T_w f)(u) = \sum_{h \in P(1|su,w)} p^h f(w^{-1} hsu) + p \sum_{h \in P(1|u,sw)} p^h f(w^{-1} hu),
\]

which we need to compare with

\[
\sum_{k \in P(1|u,sw)} p^k f(w^{-1} sku).
\]

We split the sum in (5.11) into two parts: over \( k \neq s \) and over \( k \equiv s \). We will show that these parts are equal to the two sums in (5.10).

Lemma 5.5. \( \{ c \in P(1|u,sw) \mid cs = sc \} = P(1|u,w) \cup \{ s \} \).

Proof. Suppose \( c \in P(1|u,sw) \), \( cs = sc \) and \( c \neq s \). Then \( c \in P(1|u,sw) \) implies \( c \in P(1|su,w) \); therefore \( c \in P(1|u,w) \). Suppose now that \( c \in P(1|u,w) \). Lemma 5.4 (with \( g = w \), \( h = u \)) implies that \( cs = sc \).

It follows from this lemma that \( h \mapsto k = h \cup \{ s \} \) is a bijection between the sets \( \{ c \in P(1|u,w) \} \) and \( \{ k \in P(1|u,sw) \mid k \equiv s \} \). We have \( #k = #h + 1 \) and \( sk = h \). Thus, the second sum in (5.10) is equal to the second sub-sum of (5.11).

Notice that \( P(1|u,sw)^+ = P(s|su,w) = P(1|su,w) \cup \{ s \} \) (disjoint union). If \( k \neq s \), then we put \( h = k^3 \in P(1|su,w) \). We have \( #k = #h \) and \( sk = hs \). Therefore, the first sum in (5.10) is equal to the first sub-sum of (5.11).

In Proposition 5.3 elements of the Hecke algebra were interpreted as functions, as a preparation for a similar result in the next section. We would like to translate the formula (5.7) into the more usual Hecke algebra language, i.e., to rewrite it as a formula for \( T_w T_u \). This kind of formula has a long history: [19, p. 44], [20, Lemma A.3] discuss the case of affine Weyl groups in terms of BN–pairs. Our formula works for right-angles Coxeter groups, and is, we think, more explicit. We would like to relate it to a very similar formula in a recent preprint of Caspers [7, Lemma 2.7]. In fact, Caspers’ formula inspired us to simplify the forms of several formulae in this and the next section.

Corollary 5.6.

\[
T_w T_u = \sum_{h \in P(1|u,w^{-1})} p^h T_{whu}.
\]
we get the principal series representation of the group algebra of $W$.

The next lemma is somewhat analogous to Lemma 5.1.

Proof. We pick some apartment $A$ containing the base chamber $c_0$ and its $s$-neighbor $c_1$. Let $c_2, \ldots, c_q$ be the other $s$-neighbors of $c_0$. Put $F = f \circ \pi$. Let $y$ be the unique point in $\partial A$ that is mapped to $x$ by $\pi$. Let $K'_i$ be the stabilizer (in $G'$) of the common face of $c_0$ and $c_i$. Under the isomorphism $\mathcal{H} \simeq \mathbb{C}[K'/G'/K']$ the element $e_y$ corresponds to the indicator function of (the set-theoretic difference) $K'_s \setminus K'$. For $i > 0$ choose $s_i \in G'$ that exchanges $c_0$ and $c_i$. Then we have a disjoint union decomposition $K'_s \setminus K' = \bigcup_i s_i K'$.

\[ (s^q f)(x) = \int_{K'_s \setminus K'} \rho_e(g) F(y) \, dg \]

Proof. We put $f = T_v$ in (5.7). We have, using (5.5): $T_u T_v(u) = \langle T_w T_v, T_u \rangle = \langle T_v, T_{w^{-1}} T_u \rangle$; $T_v(w^{-1} hu) = \langle T_v, T_{w^{-1}} hu \rangle$. Plugging into (5.7) we get

\[ \langle T_v, T_{w^{-1}} T_u \rangle = \langle T_v, \sum_{h \in \mathcal{P}(1|u,w)} p^h T_{w^{-1}} hu \rangle, \]

hence

\[ T_{w^{-1}} T_u = \sum_{h \in \mathcal{P}(1|u,w)} p^h T_{w^{-1}} hu. \]

Replacing $w$ by $w^{-1}$ we get (5.12).

\section{Hecke algebra action in principal series representations}

In this section we will describe the Hecke algebra representation on $V^{K'}$ for $V$ in the principal series of $G'$ (the description is the same for $G$ and for $G'$). In this case $V = L^2(\partial X, \mu)$ and $V^{K'} = L^2(\partial X, \mu)^{K'} \simeq L^2(\partial W, l)$, the last isomorphism given by composition with $\pi$. Thus all Hecke representations (for all $q$) are realized on the same Hilbert space $L^2(\partial W, l)$. This includes the case $q = 1$, when we get the principal series representation of the group algebra of $W$. The dependence on $q$ being important for us, we should denote the action of $e_w$ on $f$ by $\rho^q_e(e_w) f$; to shorten this we will write $w^q f$ instead. Similarly, we write $T^q_w$ instead of $\rho^q_e(T_w)$.

Our goal for the remainder of this section is to express the Hecke actions $T^q_w f$ in terms of the Weyl actions $w^1 f$. We start with the case $w = s \in S$. The wall $s$ separates $\partial W$ into two pieces: $\partial s^-$ (on the side of 1) and $\partial s^+$ (on the side of $s$).

The next lemma is somewhat analogous to Lemma 5.1.

**Lemma 6.1.** Suppose $f \in L^2(\partial W, l)$. Then, for $x \in \partial W$:

\[ (T^q_s f)(x) = \begin{cases} q^{-ic}(s^1 f)(x) & \text{for } x \in \partial s^- \\ q^{ic}(s^1 f)(x) + pf(x) & \text{for } x \in \partial s^+ \end{cases} \]

Proof. We pick some apartment $A$ containing the base chamber $c_0$ and its $s$-neighbor $c_1$. Let $c_2, \ldots, c_q$ be the other $s$-neighbors of $c_0$. Put $F = f \circ \pi$. Let $y$ be the unique point in $\partial A$ that is mapped to $x$ by $\pi$. Let $K'_i$ be the stabilizer (in $G'$) of the common face of $c_0$ and $c_i$. Under the isomorphism $\mathcal{H} \simeq \mathbb{C}[K'/G'/K']$ the element $e_y$ corresponds to the indicator function of (the set-theoretic difference) $K'_s \setminus K'$. For $i > 0$ choose $s_i \in G'$ that exchanges $c_0$ and $c_i$. Then we have a disjoint union decomposition $K'_s \setminus K' = \bigcup_i s_i K'$.

\[ (s^q f)(x) = \int_{K'_s \setminus K'} \rho_e(g) F(y) \, dg = \sum_i \int_{K'} F(b^{-1} s_i^{-1} y) \left[ \frac{d(s_i b) + \mu}{d \mu} (y) \right]^{\frac{1}{2} + ic} \, db \]

\[ = \sum_i \int_{K'} F(s_i^{-1} y) \left[ \frac{d(s_i) \mu}{d \mu} (y) \right]^{\frac{1}{2} + ic} \, db \]
The measure \((s_\iota)_* \mu\) has been calculated in Corollary 1.6. Suppose \(x \in \partial s^-\). Then \(F(s_\iota^{-1} y) = f(sx)\) and \(((s_\iota)_* \mu)(y) = \mu_1(y) = q^{-1} r(x) \mu(y)\). Therefore
\[
(6.3) \quad (s^q f)(x) = q f(sx) q^{\frac{1}{2} - i \epsilon} r(x)^{\frac{1}{2} + i \epsilon} = q^{\frac{1}{2} - i \epsilon} r(x)^{\frac{1}{2} + i \epsilon} f(sx) = q^{\frac{1}{2} - i \epsilon} (s^1 f)(x).
\]
Suppose \(x \in \partial s^+\). If \(i > 1\), then \(F(s_\iota^{-1} y) = f(x)\) and \(((s_\iota)_* \mu)(y) = \mu(y)\). If \(i = 1\), then \(F(s_\iota^{-1} y) = f(sx)\) and \(((s_\iota)_* \mu)(y) = \mu_1(y) = q r(x) \mu(y)\). Therefore
\[
(6.4) \quad (s^q f)(x) = (q - 1) f(x) + f(sx) q^{\frac{1}{2} + i \epsilon} r(x)^{\frac{1}{2} + i \epsilon} = (q - 1) f(x) + q^{\frac{1}{2} + i \epsilon} (s^1 f)(x).
\]
Since \(T^q_s = q^{-1/2} s^q\), the lemma follows.

We would like to extend Lemma 6.1 to a formula expressing the Hecke action as a combination of Weyl actions for general \(w \in W\), in analogy to Proposition 5.3. The appropriate statement is given in Theorem 6.2. The argument mimics the proof of Proposition 5.3: we consider the sums indexed by the same cliques and split them into sub-sums in the same way. The differences are that the summands are slightly different and that \(u \in W\) has to be replaced by \(x \in \partial W\) (one can think of \(x\) as of a limit of \(u\)). As before, \(P(A D)\) denotes the set of walls (in \(W\)) separating \(A\) from \(D\), but now \(A\) and \(D\) can be sets of points in \(\partial W\) or elements of \(W\). Recall that there is a graph structure on \(P(A D)\). We will perform summations indexed by cliques in this kind of graphs. Recall that for \(h \in \mathcal{C} P(A D)\) we denote by \(h\) also the product of all reflections corresponding to walls in \(h\). We put \(\tau(w, x) = [\frac{d_{\omega, \mu}(x)}{d_{\omega, \mu}}]^{\frac{1}{2} + i \epsilon}\).

**Theorem 6.2.** Let \(f \in L^2(\partial W, l)\) and let \(\epsilon = 0\). Then
\[
(6.5) \quad (T^q_w f)(x) = \sum_{h \in \mathcal{P}(1|x, w)} \mu^h((hw)^{\frac{1}{2}} f)(x)
\]

**Proof.** We proceed by induction on \(\ell(w)\), the first step being given by Lemma 6.1.

Case 1: \(1\), \(w\) and \(x\) are all on the same side of the wall \(s\) (the wall separating 1 from \(s\)). Then
\[
T^q_s (T^q_w f)(x) = \tau(s, x) (T^q_w f)(sx)
= \tau(s, x) \sum_{h \in \mathcal{P}(1|x, sx, w)} \mu^h((hw)^{\frac{1}{2}} f)(sx)
= \tau(s, x) \sum_{h \in \mathcal{P}(1|x, sx, w)} \mu^h f(hw, sx) f(w^{-1} h^{-1} sx)
= \sum_{h \in \mathcal{P}(1|x, sx, w)} \mu^h \tau(shw, x) f(w^{-1} hsx),
\]
while the right hand side of the formula for \((T^q_{sw} f)(x)\) that we are trying to prove is
\[
(6.7) \quad \sum_{h \in \mathcal{P}(1|x, sw)} \mu^h((hsw)^{\frac{1}{2}} f)(x) = \sum_{h \in \mathcal{P}(1|x, sw)} \tau(hsw, x) p^h f(w^{-1} shx).
\]
Now, exactly as in the proof of Proposition 5.3, we observe that \(P(1|x, sw) = P(1|sx, w)\) and that if \(h \in \mathcal{C} P(1|sx, w)\) then \(hs = sh\).
Case 2: The wall $s$ separates $1$ and $w$ from $x$.

\[
T^d_s(T^d_w f)(x) = \tau(s, x)(T^d_w f)(sx) + p(T^d_w f)(x)
\]

\[
= \tau(s, x) \sum_{h \in \mathcal{P}(1|sx, w)} p^h((hw)^1 f)(sx) + p \sum_{h \in \mathcal{P}(1|x, w)} p^h((hw)^1 f)(x)
\]

\[
(6.8) \quad = \tau(s, x) \sum_{h \in \mathcal{P}(1|sx, w)} p^h(\tau(hw, sx) f(w^{-1} h sx))
\]

\[
\quad + p \sum_{h \in \mathcal{P}(1|x, w)} p^h(\tau(hw, x) f(w^{-1} h x))
\]

while the postulated expression for $(T^d_{sw} f)(x)$ is

\[
(6.9) \quad \sum_{k \in \mathcal{P}(1|x, sw)} p^{sk}(ksw)^1 f(x) = \sum_{k \in \mathcal{P}(1|x, sw)} p^{sk} \tau(ksw, x) f(w^{-1} sk x).
\]

We consider the same bijection between the summands of (6.8) and those of (6.9) as when we compared (5.10) with (5.11) (x taking the role of w). In the case $s \not\in k \in \mathcal{P}(1|x, sw)$ the corresponding $h = k^q$; to compare the summands we observe that $\tau(s, x) \tau(hw, sx) = \tau(sw, x) = \tau(ksw, x)$.

By the same argument, just keeping track of the extra factor with imaginary exponent, one can prove a similar formula for $\epsilon \neq 0$:

\[
(6.10) \quad (T^d_w f)(x) = \sum_{h \in \mathcal{P}(1|x, w)} p^h q^{-i\epsilon \beta^q_x(1, hw)}(\tau(hw)^1 f)(x),
\]

where $\beta^q_x$ is the Busemann function defined in Remark 1.7. We would like to re-formulate Theorem 6.2 in an $x$-free way. We need the following lemma.

**Lemma 6.3.** Let $h$ be a collection of pairwise intersecting tessellation walls in $|W|$. Then $\cap h$—the intersection of the walls in $h$—is non-empty. Moreover, $\cap h$ is a totally geodesic subspace of $\mathcal{H}^d$ of codimension equal to the cardinality of $h$. The product of reflections in the elements of $h$ is the reflection in $\cap h$.

**Proof.** Since $W$ is right-angled, reflections in the walls belonging to $h$ pairwise commute. Therefore, they generate a finite abelian group of isometries of $\mathcal{H}^d$. This group has a fixed point (cf. [4, Cor. II.2.8]); that point belongs to $\cap h$. Inspecting the local picture at that point we get the remaining claims of the lemma.

Recall that for a wall $H$ we have defined $H^+$ as the set of all $w \in W$ that are separated from $1$ by $H$. For a collection $h$ of pairwise intersecting walls (in symbols: $\cap h \neq \emptyset$) we put $h^+ = \cap |H^+| H \in h$. We denote by $\partial h^+$ the set of all $x \in \partial W$ that are separated from $1$ by each $H \in h$. (For $h = \emptyset$ the natural conventions are: $\cap h \neq \emptyset$, $h^+ = W$, $\partial h^+ = \partial W$.) The characteristic (indicator) function of a set $U$ will be denoted $\chi_U$. We now re-state Lemma 6.1 and Theorem 6.2.

**Corollary 6.4.** Let $f \in L^2(\partial W, l)$, $s \in S$, $w \in W$, and let $\epsilon = 0$. Then

\[
(6.11) \quad T^d_s f = s^1 f + p \chi_{\delta s^+} \cdot f
\]
and

\[ T^q_w f = \sum_{h \cup h' = w, h' \neq w} p^{\#h} \chi_{\partial h'} \cdot (hw)^1 f. \]  

7. The general setting of right-angled Hecke algebras and its principal series representations

Consider now the case where \( q \) is not constant, but is \( s \)-dependent. The formulae of Sections 5 and 6 still hold—with special reading. In the formulae where \( e_s \) or \( T_s \) is involved, \( q \) is \( q_s \) and \( p \) is \( p_s = \frac{q_s - 1}{\sqrt{q_s}} \) (these are (5.1), (5.2), (6.1), and their direct applications). Proofs of other formulae are inductive, with steps carried out one generator at a time. In powers like \( p^{\#h} \) or \( q^{\ell(w)} \), the base \( p \) or \( q \) is a tuple of numbers \( (p_s)_{s \in S} \) or \( (q_s)_{s \in S} \). The exponent \( \#h \) or \( \ell(w) \) always counts the walls in a certain set; the walls have types which are elements of \( S \), thus each exponent can be transformed to a multi-index. In the case of \( \ell(w) \) this is done as follows: \( \ell(w) \) is the number of walls separating 1 and \( w \); it can be read as \( (\ell_s(w))_{s \in S} \), where \( \ell_s(w) \) is the number of walls of type \( s \) that separate 1 and \( w \). Then \( q^{\ell(w)} = \prod_{s \in S} q_s^{\ell_s(w)} \).

Next we want to suggest another interpretation of Section 6. We consider the algebra \( \mathcal{H} \) defined in Section 5, for a given set of complex parameters \( q_s \) and interpretation of the formulae as described above. One may now read equation (6.1) in Lemma 6.1 (taking \( q = q_s \)) as a definition of a representation of the generators of \( \mathcal{H} \) as operators on \( L^2(\partial W) \). It is easy to check by direct calculation that these operators satisfy \( (s^q)^2 = (q - 1)s^q + qI \). One then reads Theorem 6.2 and its proof as stating that the above gives a well defined representation of \( \mathcal{H} \).

Part III. Hecke irreducibility

Our goal in this part is to show that the representation of \( \mathcal{H} \) on \( L^2(\partial W) \) described in equation (6.5) is irreducible. Recall that this will also prove the irreducibility of the representation \( \rho \) of \( G' \) considered in Part I. Our proof is a modification of the proof given in [1]. The main tool which allows us to use [1] almost verbatim is the following pointwise inequality between functions on \( \partial W \):

\[ w^1 \leq T^q_w 1 = q^{-\ell(w)/2} w^q 1 \leq C w^1 1. \]

This inequality is established in Section 9. In Section 10 we explain how to use it to prove Hecke irreducibility.

The Hecke algebra representation on \( L^2(\partial W) \) is given by equation (6.5). As explained in Section 7, this formula makes sense for all complex values of \( q \), not just for positive integers. Our proof of irreducibility will work for all real \( q_s \geq 1 \). We assume this is the case from now on. For readability purposes we adopt the conventions explained in Section 7, writing our formulae as if \( q_s \) was a unique parameter \( q \).

The notation for this part is as follows. By 1 we denote the constant function with value 1 on \( \partial W \), but we also use 1 for the unit element of the group \( W \). We identify elements of \( W \) with points in \( \mathcal{H}^{d} \) via the orbit map \( w \mapsto w0 \) (hence, 1
often stands for the point 0). Distance of $x$ and $y$ in $\mathcal{H}^d$ is denoted $|x − y|$; in particular, $|w − w'|$ is the hyperbolic distance from $w0$ to $w0’. We shorten $|w − 1|$ to $|w|$. Gromov products with unspecified base-point will have base-point 1. The space $\mathcal{H}^d$ is Gromov hyperbolic; the hyperbolicity constant (as in [5, Def. 2.1.6] or in [4, Def. III.H.1.20]) will be denoted $\delta$. The length-parameterized geodesic from $a$ to $b$ will be denoted $\gamma^b_a$, we shorten $\gamma^b_0$ to $\gamma^b$—thus $\gamma^w = \gamma^w_0$. The ball of center $m$ and radius $r$ in a metric space $M$ is $M(m, r)$. The closed $r$-neighborhood of a set $A$ will be denoted $A[r]$. By $h$ we denote a finite set of pairwise intersecting tessellation walls in $\mathcal{H}^d$, as well as the composition of reflection across these walls (the group $W$ is right-angled, hence the order of composition is not important). The intersection of the walls in $h$ is a totally geodesic subspace $\cap h$. The map $h$ is an isometry of $\mathcal{H}^d$; in fact, it is the orthogonal reflection in $\cap h$ (Lemma 6.3).

8. INDIVIDUAL ESTIMATES

The goal of this section is to establish estimates for $((hw)^11)(z) = \tau(hw, z)$. Throughout this section we assume that $h$ is a collection of pairwise perpendicular tessellation walls, $w \in h^+$ and $z \in \partial h^+$. Recall $\tau(hw, z) = \exp(−\eta \beta_z(1, hw)/2)$, where $\eta = d − 1$ is the dimension of the boundary $\partial W$ (cf. [1, p. 52]). The Busemann function $\beta_z(1, hw) = \lim_{z→-z}(|1 − hw| − 2(|x|hw))$, the limit being taken over points $x \in \mathcal{H}^d$ converging to $z$.

**Definition 8.1.** For $w \in h^+$ we put

\begin{equation}
(w | h) = |w| − |w − hw|/2.
\end{equation}

This quantity will be investigated (and the notation explained) in the next section. The estimate for $\tau(hw, z)$ that we need is as follows.

**Proposition 8.2.** Suppose that $h$ is a collection of pairwise perpendicular tessellation walls. Assume that $w \in h^+$, $z \in \partial h^+$. Then

\begin{equation}
\tau(hw, z) \leq e^{\eta \delta} \exp(\eta \min(|z|w), (|w| \cap h)) \exp(−\eta |w|/2).
\end{equation}

This estimate is equivalent to

\begin{equation}
\beta_z(1, hw) \geq |w| − 2 \min(|z|w), (|w| \cap h)) + 2\delta,
\end{equation}

which is the limit (as $x → z$) of the following pair of inequalities:

**Proposition 8.3.** Suppose $w, x \in h^+$, Then

a) $2(|x|hw) − |hw| \leq 2(|x|w) − |w|;$

b) $2(|x|hw) − |hw| \leq 2(|w| \cap h) − |w| + 2\delta.$

Our main tool will be the following:

**Lemma 8.4.** Let $h$ be a collection of pairwise perpendicular hyperplanes in $\mathcal{H}^d$. Suppose that two points $a, b \in \mathcal{H}^d$ are not separated by any of these hyperplanes. Then $|a − b| \leq |a − hb|.$
Proof. The geodesic segment $[a, hb]$ intersects every hyperplane of $h$. Let the intersection points be $p_1, \ldots, p_j$, numbered from $a$ towards $hb$, and let $h_1$ be the hyperplane passing through $p_1$. Then $[a, p_1] \cup h_1 [p_1, p_2] \cup h_1 h_2 [p_2, p_3] \cup \ldots \cup h_1 [p_j, hb]$ is a piecewise geodesic path from $a$ to $b$ of length $|a - hb|$. □

Proof of Prop. 8.3.a). We apply Lemma 8.4 to $h$ and the points $x, w \in h^+$ to get $|x - w| \leq |x - hw|$. Then

$$2(x|hw) - |hw| = |x| - |x - hw| \leq |x| - |x - w| = 2(x|w) - |w|.$$ □

Proof of Prop. 8.3.b). Expanding the Gromov product and Def. 8.1 we see that the required inequality is equivalent to

$$|x| - |x - hw| \leq |w| - |w - hw| + 2\delta,$$

which we rewrite as

$$|x - 1| + |w - hw| \leq |w - 1| + |x - hw| + 2\delta.$$ (8.6)

A basic property of quadrangles in hyperbolic spaces ([5, Sec. 2.4.1]) applied to $(x, w, 1, hw)$ yields

$$|x - 1| + |w - hw| \leq \max(|x - w| + |1 - hw|, |x - hw| + |w - 1|) + 2\delta.$$ (8.7)

By Lemma 8.4 we get $|x - w| \leq |x - hw|$, $|1 - hw| \leq |1 - w|$. Now (8.6) follows from (8.7).

9. Hecke action estimate

We begin with an explanation of $(w| \cap h)$, and then show the pointwise inequality $T^w_1 \leq Cw^1$. Informally, if $w \in h^+$, then the geodesic from $1$ to $w$ gets close to $\cap h$ at time $|h|/2$ and stays near $\cap h$ till time $(w| \cap h) = |w| - |w - hw|/2.$

**Lemma 9.1.**

$$(w| \cap h) \geq 0.$$

**Proof.** By Lemma 8.4 we have $|1 - hw| \leq |1 - w|$. Using this and the triangle inequality we get

$$2(w| \cap h) = 2|w| - |w - hw| = |w - 1| + |1 - w| - |w - hw| \geq |w - 1| + |1 - hw| - |w - hw| \geq 0.$$ (9.1)

**Lemma 9.2.** If $x \in h^+$, then $(x|h) \geq |h|/2$. If $z \in \partial h^+$, then $(z|h) \geq |h|/2$.

**Proof.** By Lemma 8.4, we have $|x| = |x - 1| \geq |x - h|$, hence $2(x|h) = |x| + |h| - |x - h| \geq |h|$. The boundary version is obtained by passing to limits. □

**Proposition 9.3.** Suppose that $w \in h^+$. Then $\gamma^w(|h|/2, (w| \cap h))$ is contained in the (closed) $\delta$-neighborhood $\cap h[\delta]$ of $\cap h$.

**Proof.** Since $\cap h[\delta]$ is convex, it suffices to show that $\gamma^w(|h|/2, \gamma^w((w| \cap h)) \in \cap h[\delta]$. By Lemma 9.2, $(w|h) \geq |h|/2$. This implies that $|\gamma^w(|h|/2) - \gamma^h(|h|/2)| \leq \delta$ (cf. [5, Def. 1.2.2]). Now $\gamma^h(|h|/2)$ is the midpoint of $[1, h]$, and it belongs to $\cap h$. □
We can now repeat the argument with the points 1 and \( w \) interchanged. The point \( hw \) plays the role of the point \( h \), and the distance from \( w \) to \( \cap h \) is \( |w - hw|/2 \). We get \( (1|hw)_w \geq |w - hw|/2 \), hence \( |\gamma_w^1((w - hw)/2) - \gamma_w^{hw}((w - hw)/2)| \leq \delta \). Now \( \gamma_w((w - hw)/2) = \gamma^w((w \cap h)) \), while \( \gamma_w^{hw}((w - hw)/2) \) is the midpoint of \([w, hw]\) and belongs to \( \cap h \).

**Corollary 9.4.** Let \( w \in h^+ \), \( z \in \partial h^+ \). Then \( \gamma^w(\min((z|w), (w|\cap h))) \in h[2\delta] \).

**Proof.** If the minimum is \((w|\cap h)\), the claim follows directly from Prop. 9.3.

Suppose then that \((z|w) < (w|\cap h)\). By hyperbolicity ([5, Def. 2.1.6]) and Lemma 9.2 we have

\[
(9.2) \quad (z|w) \geq \min((z|h), (w|h)) - \delta \geq |h|/2 - \delta.
\]

Therefore \((z|w) \in [|h|/2 - \delta, (w|\cap h)]\), and

\[
\gamma^w((z|w)) \in \gamma^w([|h|/2 - \delta, (w|\cap h)]) \subseteq \gamma^w([|h|/2, (w|\cap h)]|\delta)
\]

\[
(9.3) \quad \subseteq \cap h[\delta]|\delta = \cap h[2\delta].
\]

To summarize our discussion, we put \( s(h) = \min((z|w), (w|\cap h)) \). (We suppress the dependence on \( z \) and \( w \) since these will be fixed.) Then we put together Cor. 9.4 and Prop. 8.2:

**Corollary 9.5.** Let \( h \) be such that \( w \in h^+ \), \( z \in \partial h^+ \). Then

\[
(9.4) \quad \tau(hw, z) \leq e^{\eta\delta} \exp(\eta s(h)) \exp(-\eta|w|/2).
\]

Furthermore, \( \cap h \) intersects the closed ball of radius \( 2\delta \) centered at \( \gamma^w(s(h)) \).

To proceed, we need an estimate of the number of \( h \) with a given \( s(h) \).

**Proposition 9.6.** There exists a constant \( M \) (depending only on the tessellation) such that for any \( z \) and \( w \), and any (closed) interval \( I \) of length \( \leq 1 \) contained in \([0, (z|w)]\), the number of \( h \) satisfying \( w \in h^+ \), \( z \in \partial h^+ \), \( s(h) \in I \) is at most \( M \).

**Proof.** For any \( h \) as in the statement the set \( \cap h \) intersects \( (\gamma^w(I))[2\delta] \) (by Corollary 9.5). The latter set is contained in \( \mathcal{H}^d(\gamma^w(\max(I)), 1 + 3\delta) \). Any tessellation chamber intersecting this ball is contained in \( S = \mathcal{H}^d(\gamma^w(\max(I)), 1 + 3\delta + \text{diam}(P)) \), where \( P \) is a chamber. Now all chambers have the same volume, and the volume of a ball is finite and depends only on the radius. Therefore the number of chambers contained in \( S \) is uniformly bounded. Consequently, the total number of faces of these chambers is also uniformly bounded, and the latter number is not smaller than the number of \( h \)'s that we are after. \( \square \)

Note that \( s(h) \) always belongs to \([0, (z|w)]\). Indeed, \((w|\cap h) \geq 0\) by Lemma 9.1, while a Gromov product is non-negative by the triangle inequality.

**Proposition 9.7.** There exists a constant \( C \) depending only on the tessellation and on the parameter \( q \), such that for any \( w \in W \) we have a pointwise inequality

\[
(9.7) \quad w^1 \leq T_w^q \leq q^{-\ell(w)/2}w^q \leq Cw^1.
\]
\textbf{Proof.} By Lemma 6.3 and Corollary 6.4 we have

\begin{equation}
T_w^q f = w^1 f + \sum_{j=1}^{d} p_j \sum_{h \neq \varnothing, w \in h^+} \chi_{\partial h^+} \cdot (hw)^1 f.
\end{equation}

In particular, for a \( z \in \partial W \),

\begin{equation}
(T_w^q 1)(z) = (w^1 1)(z) + \sum_{j=1}^{d} p_j \sum_{h \neq \varnothing, w \in h^+, z \in \partial h^+} \tau(hw, z).
\end{equation}

Let \( Q = \max\{p^j | j = 1, \ldots, d\} \). In the calculation below \( h \) always satisfies \( h \neq \varnothing, \cap h \neq \varnothing, w \in h^+ \), \( z \in \partial h^+ \). We only state explicitly the extra conditions. We use Cor. 9.5.

\begin{equation}
T_w^q 1(z) \leq w^1 1(z) + Q \sum_h \tau(hw, z)
\end{equation}

\begin{equation}
\leq w^1 1(z) + Q \sum_h e^{2\delta e^{\|s(h)\|} e^{-\|hw\|/2}}
\end{equation}

\begin{equation}
\leq w^1 1(z) + Q e^{2\delta e^{-\|hw\|/2}} \sum_h e^{\|s(h)\|}
\end{equation}

We now focus on the sum:

\begin{equation}
\sum_h e^{\|s(h)\|} \leq \sum_{i=0}^{\|z(w)\|} \sum_{h|s(h)\in ([\|z(w)\|-(i+1),[\|z(w)\|]-i]} e^{\|s(h)\|}
\end{equation}

\begin{equation}
\leq \sum_{i=0}^{\|z(w)\|} |\{h | s(h) \in ([\|z(w)\|-(i+1),[\|z(w)\|]-i])|\} e^{\|z(w)\|-(i)}
\end{equation}

\begin{equation}
\leq e^{\|z(w)\|} \sum_{i=0}^{\|z(w)\|} M e^{-\|z(w)\|} \leq M e^{\|z(w)\|} \sum_{i=0}^{\infty} e^{-\|z(w)\|}
\end{equation}

\begin{equation}
\leq \frac{M}{1-e^{-\|z(w)\|}}
\end{equation}

Recalling that \( w^1 1(z) = \tau(w, z) = e^{\|z(w)\|-[w]/2} \) and putting \( C = 1 + e^{2\delta Q M \frac{1}{1-e^{-\|z(w)\|}}} \) we obtain the claim.

In the multi-parameter case, the index \( j \) in formulae (9.5), (9.6), and in the definition of \( Q \), should be read as a multi-index (with 0-1 components and total degree \( \leq d \)).

10. \textbf{THE BADER-MUCHNIK ARGUMENT}

Our goal now is to prove irreducibility of the representation \( \rho_0 \) of the Hecke algebra \( \mathcal{H} = H(G, K) \) on the space \( V^K \). The argument follows very closely an irreducibility argument in [1]. For easier comparison, we adjust our notation to match [1]. Thus, the boundary \( \partial W \) will be denoted \( B \), and the measure \( l \) will be called \( \nu \). We also put \( H = V^K = L^2(B, \nu) \). The base-point 0 will be denoted \( p \). We will no longer use \( p \) in the role of \((q - 1)/\sqrt{q}\). Also, we will use \( w^q \) rather than \( T_w^q \); that will enable us to use \( T \) for a different operator, as in [1]. The
characteristic (indicator) function of a set $U$ is $\chi_U$. Convergence of operators means weak convergence.

Let $\Gamma$ be a torsion-free finite-index subgroup of $W$. For $q = 1$, the Hecke representation reduces to a group algebra representation of $W$; this can further be restricted to $\Gamma$. It is shown in [1] that that representation of $\Gamma$ is irreducible. This is achieved by showing that the von Neumann algebra generated by the representation operators is the whole $\text{End}(H)$. More precisely, for a measurable $U \subseteq B$ with zero-measure boundary the operators

$$T_{t}^{xU} = \frac{1}{|S_t|} \sum_{\gamma \in S_t} \chi_U(z(\gamma)) \gamma^1$$

are shown to converge to $\langle \cdot, 1 \rangle \chi_U$ as $t \to \infty$. We explain the notation: $z(\gamma)$ is a shorthand for $z^{yp}$—the limit point of the geodesic ray from $p$ through $\gamma p$. The set $S_t$ is a spherical layer of $\Gamma$:

$$S_t = \{ \gamma \in \Gamma | |p - \gamma p| \in (t - R, t + R) \}$$

for $R = \text{diam}(\Gamma \sim \mathcal{H}^d)$. In our setting, a completely analogous result is true:

**Lemma 10.1.** For any $U \subseteq B$ with zero-measure boundary, $\langle \cdot, 1 \rangle \chi_U$ is a limit point (as $t \to \infty$) of the operators

$$q_{T_{t}}^{xU} = \frac{1}{|S_t|} \sum_{\gamma \in S_t} \chi_U(z(\gamma)) \gamma^q.$$  

**Proof.** First, we wish to argue that (for fixed $q$ and $U$, and varying $t$) the operators $q_{T_{t}}^{xU}$ have uniformly bounded norms on $L^2(B)$.

These operators map non-negative functions to non-negative functions. Moreover, for every non-negative $f \in L^2(B)$ we have $q_{T_{t}}^{xU} f \leq q_{T_{t}}^{1} f$, so that it is enough to establish a uniform bound for $\|q_{T_{t}}^{1}\|_{L^\infty \to L^\infty}$. This is done by first estimating

$$\|q_{T_{t}}^{1}\|_{L^\infty \to L^\infty} = \|q_{T_{t}}^{1}\|_{L^\infty} = \left\| \frac{1}{|S_t|} \sum_{\gamma \in S_t} \gamma^{q} \right\|_{L^\infty}.$$  

From the pointwise estimate (Prop. 9.7) we get $q^{-\ell(\gamma)/2} \|\gamma^{q} 1\|_{L^\infty} \leq C \|\gamma^{1} 1\|_{L^\infty}$, $q^{-\ell(\gamma)/2} \|\gamma^{q} 1, 1\| \geq \|\gamma^{1} 1, 1\|$, hence

$$\|q_{T_{t}}^{1}\|_{L^\infty \to L^\infty} \leq C \left\| \frac{1}{|S_t|} \sum_{\gamma \in S_t} \gamma^{q} \right\|_{L^\infty}.$$  

The latter is uniformly bounded by Prop. 4.5 of [1] (the notation there is $\lambda^{yp} = \gamma^{1} 1$). Next, we observe that $q_{T_{t}}^{1}$ are self adjoint: for a generator $s$ of $W$ we have $(s^q)^* = s^q$ (as we see from (6.11)), hence for $w \in W$ we get $(w^q)^* = (w^{-1})^q$; the set $S_t$ is invariant under taking inverses, because $|w| = |w^{-1}|$. Consequently, the uniform $L^\infty$–operator norm bound yields a uniform $L^{1}$–operator norm bound. Finally, Riesz–Thorin interpolation gives a uniform $L^{2}$–operator norm bound.

The Banach–Alaoglu theorem implies that the family $q_{T_{t}}^{xU}$ has a limit point as $t \to \infty$. Let $q_{T_{t}}^{xU}$ be such a point.
Second, we will show that \( q T_\infty^U = \langle \cdot, 1 \rangle \chi_U \).
As in [1], for \( U \subseteq B \) and \( a > 0 \) we denote by \( U(a) \) the \( e^{-a} \)-neighborhood of \( U \) in \( B \).

**Claim 10.2** (cf. Lemma 5.2 of [1]). Let \( V \subseteq B, a > 0 \). There is a constant \( C_0 \) such that for every \( \gamma \in \Gamma \) satisfying \( z(\gamma) \not\in V(a) \) we have

\[
\frac{\langle \gamma^q 1, \chi_V \rangle}{\langle \gamma^q 1, 1 \rangle} \leq \frac{C_0 e^a}{|\gamma|}.
\]

**Proof.** For \( q = 1 \), the claim is a part of Lemma 5.2 in [1]. The general case follows from this special case, because \( q^{-\ell(\gamma)/2} \gamma^q 1 \leq C \gamma^1 1, \ q^{-\ell(\gamma)/2} \langle \gamma^q 1, 1 \rangle \geq \langle \gamma^1 1, 1 \rangle \).

**Claim 10.3** (cf. Prop. 5.1 of [1]). Assume we are given a family of elements \( \psi_t \in \mathbb{R} \Gamma \) with non-negative coefficients, where \( t \) is a real positive parameter. Suppose that for every \( t \) we have \( \| \psi_t \|_{L^1} \leq 1 \), and that for every \( \gamma \in \Gamma \) we have that \( \lim_{t \to \infty} \psi_t(\gamma) = 0 \). Then, for every measurable \( V \subseteq B \) and every \( a > 0 \),

\[
\limsup_{t \to \infty} \sum_{\gamma \in \Gamma} \psi_t(\gamma) \frac{\langle \gamma^q 1, \chi_V \rangle}{\langle \gamma^q 1, 1 \rangle} \leq \limsup_{t \to \infty} \sum_{\gamma \in \Gamma} \psi_t(\gamma) \chi_{V(a)}(z(\gamma))
\]

**Proof.** The proof is as in [1, p. 59], with \( \rho(\gamma) \) replaced with \( \gamma^q \) everywhere. An outline is as follows. The sum on the left hand side is divided into three parts:

1) \( |\gamma| < t_0 \) — this part goes to 0 since \( \psi_t \to 0 \) pointwise;
2) \( |\gamma| > t_0 \) and \( z(\gamma) \in V(a) \) — this is bounded by the right hand side;
3) \( |\gamma| > t_0 \) and \( z(\gamma) \not\in V(a) \) — this is negligible by Claim 10.2.

Specializing to \( \psi_t = \frac{1}{|S_t|} \sum_{\gamma \in S_t} \chi_U(z(\gamma)) \gamma \) we get

**Claim 10.4** (cf. Cor. 5.3 of [1]). For measurable \( U, V \subseteq B \) that are positive distance apart

\[
\lim_{t \to \infty} \langle q T_t^U 1, \chi_V \rangle = 0.
\]

An easy consequence is

**Claim 10.5** (implicit in [1]). For measurable \( U, V \subseteq B \) that are positive distance apart, and a measurable \( W \subseteq B \),

\[
\lim_{t \to \infty} \langle q T_t^U \chi_W, \chi_V \rangle = 0.
\]

**Proof.** \( 0 \leq \langle q T_t^U \chi_W, \chi_V \rangle \leq \langle q T_t^U 1, \chi_V \rangle \to 0 \).

Specializing Claim 10.3 to \( \psi_t = \frac{1}{|S_t|} \sum_{\gamma \in S_t} \chi_U(z(\gamma)) \gamma^{-1} \) we get

**Claim 10.6** (cf. Cor. 5.4 of [1]). For measurable \( U, V \subseteq B \) and every \( a > 0 \)

\[
\limsup_{t \to \infty} \langle q T_t^U \chi_V, 1 \rangle \leq \limsup_{t \to \infty} \frac{1}{|S_t|} \sum_{\gamma \in S_t} \chi_U(z(\gamma^{-1})) \chi_{V(a)}(z(\gamma)).
\]
Adding them up deduce (for $q$

Proof. Sketched above. For other details see [1, p. 60].

Thus the von Neumann algebra $VN^q$ generated by $\{\gamma^q \mid \gamma \in \Gamma\}$ contains all operators $\langle \cdot, \cdot \rangle_U$ (assuming $\nu(\partial U) = 0$). Hence, it also contains all $\langle \cdot, \chi_U \rangle \chi_U$, as this is $\langle \cdot, \chi_U \rangle \circ \langle \cdot, \chi_U \rangle^*$ (again, assuming $\nu(\partial U) = \nu(\partial V) = 0$). Now [1, Lemma B.3] implies $VN^q = \text{End}(H)$. Hecke irreducibility of $H$ follows by Schur’s Lemma.

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