A counterexample to generalizations of the Milnor-Bloch-Kato conjecture*

By Michael Spieß and Takao Yamazaki

Abstract

We construct an example of a torus $T$ over a field $K$ for which the Galois symbol $K(K; T, T)/nK(K; T, T) \to H^2(K, T[n] \otimes T[n])$ is not injective for some $n$. Here $K(K; T, T)$ is the Milnor $K$-group attached to $T$ introduced by Somekawa. We show also that the motive $M(T \times T)$ gives a counterexample to another generalization of the Milnor-Bloch-Kato conjecture (proposed by Beilinson).

1 Introduction

Let $K$ be a field, $m$ a positive integer and $n$ an integer prime to the characteristic of $K$. The Milnor-Bloch-Kato conjecture asserts that the Galois symbol

$$K_m^M(K)/nK_m^M(K) \to H^m(K, \mathbb{Z}/n\mathbb{Z}(m))$$

from Milnor $K$-groups to Galois cohomology is bijective. Recently, Rost and Voevodsky have announced a proof (special cases have been obtained earlier by Merkurjev-Suslin, Rost and Voevodsky).

In [So], Somekawa has introduced certain generalized Milnor $K$-groups $K(K; A_1, \ldots, A_m)$ attached to semi-abelian varieties $A_1, \ldots, A_m$. If $A_1 = \ldots = A_m = G_m$ is the one-dimensional split torus they agree with the usual $K_m^M(K)$. If $m = 2$, $A_1 = \text{Jac}_X$ and $A_2 = \text{Jac}_Y$ are the Jacobians of smooth, projective and connected curves $X$ and $Y$ over $K$ having a $K$-rational point, then $K(K; A_1, A_2)$ is the kernel of the Albanese map $\text{CH}_0(X \times Y)_{\text{deg}=0} \to \text{Alb}_{X \times Y}(K)$.

Somekawa has defined a Galois symbol

$$K(K; A_1, \ldots, A_m)/nK(K; A_1, \ldots, A_m) \to H^m(K, A_1[n] \otimes \ldots \otimes A_m[n])$$

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and conjectured that it is always injective. In this note we present a counterexample (see section 2). Let us describe it briefly. Let \( L/K \) be a cyclic extension of degree \( n \) and \( \sigma \) a generator of the Galois group \( \text{Gal}(L/K) \). Let \( T \) be the kernel of the norm map \( \text{Res}_{L/K} \mathbb{G}_m \to \mathbb{G}_m \). We show that the norm \( K(L; T, T) \to K(K; T, T) \) induces an isomorphism \( K_2(L; T, T)/(1 - \sigma) \to K_2(K; T, T) \). On the other hand, the corresponding map of Galois cohomology groups \( H^2(L, T[n] \otimes T[n])/(1 - \sigma) \to H^2(K, T[n] \otimes T[n]) \) is neither injective nor surjective (for a suitable choice of \( L/K \)). Note that, since \( T \) is split over \( L \), the Galois symbol \( K_2(L; T, T) \to H^2(L, T[n] \otimes T[n]) \) is bijective. Consequently, \( K_2(K; T, T) \to H^2(K, T[n] \otimes T[n]) \) is in general not injective.

In the section 3 we show that the motive \( M(T \times T) \) gives a counterexample to another generalization of the Milnor-Bloch-Kato conjecture (proposed by Beilinson).

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2 Counterexample to Somekawa’s conjecture

Algebraic groups as Mackey-functors Let \( K \) be a field. For a finite field extension \( L/K \) and commutative algebraic groups \( G \) over \( K \) and \( H \) over \( L \) we denote by \( G_L \) the base change of \( G \) to \( L \) and by \( \text{Res}_{L/K} H \) the Weil restriction of \( H \). The functor \( G \to G_L \) is left and right adjoint to \( H \to \text{Res}_{L/K} H \). In particular there are adjunction homomorphisms \( \iota_{L/K} : G \to \text{Res}_{L/K} G_L \) and \( N_{L/K} : \text{Res}_{L/K} G_L \to G \). When \( L/K \) is a Galois extension, the Galois group \( \text{Gal}(L/K) \) acts canonically on \( \text{Res}_{L/K} G_L \). The following simple result, whose proof will be left to the reader, will be used later.

**Lemma 1** Let \( L/K \) be a cyclic Galois extension of degree \( n \), \( \sigma \) a generator of \( \text{Gal}(L/K) \) and let \( G \) be a commutative algebraic group over \( K \). Let \( G' \) be the kernel of \( N_{L/K} : \text{Res}_{L/K} G_L \to G \) so that \( G_L' \cong G_L^{n-1} \). Then the map

\[
\text{Res}_{L/K}(G_L)^{n-1} \cong \text{Res}_{L/K} G_L' \to \text{Res}_{L/K} G_L
\]

is given on the \( i \)-th summand by \( 1 - \sigma^i \).

We denote by \( \mathcal{C}_K \) the category of finite reduced \( K \)-schemes. Thus each object of \( \mathcal{C}_K \) is isomorphic to \( \text{Spec}(E_1 \times \ldots \times E_r) \) where \( E_1, \ldots, E_r/K \) are finite field extensions. A commutative algebraic group \( G \) over \( K \) defines
a Mackey-functor, i.e. a co- and contravariant functor $G : C_K \to \text{Mod}_\mathbb{Z}$ satisfying (i), (ii) below. If $f : X \to Y$ is a morphism we denote by $f_* : G(X) \to G(Y)$ and $f^* : G(Y) \to G(X)$ the homomorphisms induced by co- and contravariant functoriality respectively.

(i) If $X = X_1 \amalg X_2 \in \text{Obj}(C_K)$ then $G(X) = G(X_1) \oplus G(X_2)$.

(ii) If

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
\]

is a cartesian square in $C_K$ then $g^* f_* = (f')_*(g')^*$.

If $K \subseteq E_1 \subseteq E_2$ are two finite field extensions and $f : \text{Spec} E_2 \to \text{Spec} E_1$ the corresponding map in $C_K$ then $f^*$ (resp. $f_*$) is given by $t_{E_2/E_1} : G(E_1) \to G(E_2)$ (resp. $N_{E_2/E_1} : G(E_2) \to G(E_1)$).

**Local symbols** We recall also the notion of a local symbol ([Se] and [So]) for $G$. Let $X \to \text{Spec} K$ is a proper non-singular algebraic curve (note that we do not assume that $X$ is connected). Let $K(X)$ denote the ring of rational functions on $X$ and $|X|$ the set of closed points of $X$. For $P \in |X|$ we denote by $K_P$ the quotient field of the completion $\hat{O}_{X,P}$ of $O_{X,P}$, by $v_P : K_P \to \mathbb{Z} \cup \{\infty\}$ the normalized valuation and by $K(P)$ the residue field of $K_P$. The local symbol at $P$ is a homomorphism $\partial_P : (K_P)^* \otimes G(K(P)) \to G(K(P))$. It is characterized by the following properties:

(i) If $f \in (K_P)^*$ and $g \in G(\hat{O}_{X,P})$ then $\partial_P(f \otimes g) = v_P(f)g(P)$. Here $g(P)$ is the image of $g$ under the canonical map $G(\hat{O}_{X,P}) \to G(K(P))$.

(ii) For $f \in K(X)^*$ and $g \in G(K(X))$ we have $\sum_{P \in |X|} N_{K(P)//K}(\partial_P(f \otimes g)) = 0$.

**Milnor $K$-groups attached to commutative algebraic groups** Let $G_1, \ldots, G_m$ be commutative algebraic groups over $K$. In [So] Somekawa has introduced the Milnor $K$-group $K(K; G_1, \ldots, G_m)$ (actually Somekawa considered only the case of semiabelian varieties though his construction works for arbitrary commutative algebraic groups). It is given as

\[
K(K; G_1, \ldots, G_m) = \left( \bigoplus_X G_1(X) \otimes \ldots \otimes G_m(X) \right) / R
\]
where \( X \) runs through all objects of \( \mathcal{C}_K \) and \( R \) is generated by the following elements:

(R1) If \( f : X \to Y \) is a morphism in \( \mathcal{C}_K \) and if \( x_{i_0} \in G_{i_0}(Y) \) for some \( i_0 \) and \( x_i \in G_i(X) \) for \( i \neq i_0 \), then

\[
x_1 \otimes \ldots \otimes f_*(x_{i_0}) \otimes \ldots \otimes x_m - f^*(x_1) \otimes \ldots \otimes x_{i_0} \otimes \ldots \otimes f^*(x_m) \in R.
\]

(R2) Let \( X \to \text{Spec } K \) be a proper non-singular curve, \( f \in K(X)^* \) and \( g_i \in G_i(K(X)) \). Assume that for each \( P \in |X| \) there exists \( i(P) \) such that \( g_i \in G_i(\hat{\mathcal{O}}_{X,P}) \) for all \( i \neq i(P) \). Then

\[
\sum_{P \in |X|} g_i(P) \otimes \ldots \otimes \partial_P(f \otimes g_{i(P)}) \otimes \ldots \otimes g_m(P) \in R.
\]

For \( X \in \mathcal{C}_K \) and \( x_i \in G_i(X) \) for \( i = 1, \ldots, m \) we write \( \{x_1, \ldots, x_m\}_{X/K} \) for the image of \( x_1 \otimes \ldots \otimes x_m \) in \( K(K; G_1, \ldots, G_m) \) (elements of this form will be referred to as symbols).

A sequence of algebraic groups \( G' \to G \to G'' \) over \( K \) will be called Zariski exact if \( G'(E) \to G(E) \to G''(E) \) is exact for every extension \( E/K \). The proof of the following result is straightforward; hence will be omitted.

Lemma 2 Let \( m \) be a positive integer and let \( i \in \{1, \ldots, m\} \). Let \( G_1, \ldots, G_m \) be commutative algebraic groups over \( K \) and let \( G_i' \to G_i \to G_i'' \to 1 \) be a Zariski exact sequence of commutative algebraic groups over \( K \). Then the sequence

\[
K(K; G_1, \ldots, G_i', \ldots) \to K(K; G_1, \ldots, G_i, \ldots) \to K(K; G_1, \ldots, G_i'', \ldots) \to 0
\]

is exact as well.

The norm map  Let \( G_1, \ldots, G_m \) be commutative algebraic groups over \( K \) and let \( L/K \) be a finite extension. Set \( K(L; G_1, \ldots, G_m) : = K(L; G_1)_L, \ldots, (G_m)_L) \). Then we have the norm map \( \text{(3)} \)

\[
N_{L/K} : K(L; G_1, \ldots, G_m) \longrightarrow K(K; G_1, \ldots, G_m)
\]

defined on symbols by \( N_{L/K}(\{x_1, \ldots, x_m\}_{X/L}) = \{x_1, \ldots, x_m\}_{X/K} \) for any \( X \in \mathcal{C}_L \) and \( x_i \in G_i(X) \) \((i = 1, \ldots, m)\). We give another interpretation of \( \text{(3)} \) below when \( L/K \) is separable. It is based on the following result.

Lemma 3 Let \( L/K \) be a finite separable extension and let \( i, m \) be positive integers with \( i \leq m \). Let \( G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_m \) be commutative algebraic groups over \( K \) and let \( G_i \) be a commutative algebra group over \( L \). Then, we have an isomorphism

\[
K(K; G_1, \ldots, \text{Res}_{L/K} G_i, \ldots, G_m) \cong K(L; (G_1)_L, \ldots, G_i, \ldots, (G_m)_L).
\]
Proof. To simplify the notation we assume that \( i = m \). We denote by 
\[ \pi^{-1} : C_K \rightarrow C_L \quad \text{and} \quad \pi : C_L \rightarrow C_K \]
the functors 
\[ \pi^{-1}(X \rightarrow \text{Spec} \, K) : = (X \otimes_K L \rightarrow \text{Spec} \, L), \]
\[ \pi(Y \rightarrow \text{Spec} \, L) : = (Y \rightarrow \text{Spec} \, L \rightarrow \text{Spec} \, K). \]
\( \pi \) is left adjoint to \( \pi^{-1} \). For \( X \in C_K \) and \( Y \in C_L \) let 
\[ p_X : X \otimes_K L \rightarrow X, \quad \iota_Y : Y \rightarrow Y \otimes_K L. \]
be the adjunction morphisms. We define homomorphisms 
\[ \phi : K(K;G_1,\ldots,G_{m-1},\text{Res}_{L/K}G_m) \rightarrow K(K;G_1L,\ldots,(G_{m-1})L,G_m), \]
\[ \psi : K(L;G_1L,\ldots,(G_{m-1})L,G_m) \rightarrow K(K;G_1,\ldots,G_{m-1},\text{Res}_{L/K}G_m). \]
as follows. For \( X \in C_K \), \( x_1 \in G_1(X), \ldots,x_{m-1} \in G_{m-1}(X) \) and \( x_m \in G_m(X \otimes_K L) \) we put 
\[ \phi(\{x_1,\ldots,x_m\}_{X/K}) = \{p^*(x_1),\ldots,p^*(x_{m-1}),x_m\}_{(X \otimes_K L)/L}. \]
Conversely, for \( Y \in C_L \) and \( y_1 \in G_1(Y), \ldots,y_m \in G_m(Y) \) let 
\[ \psi(\{y_1,\ldots,y_{m-1},y_m\}_{Y/L}) = \{y_1,\ldots,y_{m-1},\iota_*y_m\}_{Y/K}. \]
One can easily verify that these maps are well-defined and mutually inverse to each other. \( \square \)

Let \( G_1,\ldots,G_m \) be commutative algebraic groups over \( K \) and let \( L/K \) be a finite separable extension. Take any \( i \in \{1,\ldots,m\} \). The map \( N_{L/K} : \text{Res}_{L/K}(G_i)_L \rightarrow G_i \) induces a map 
\[ K(K;G_1,\ldots,\text{Res}_{L/K}(G_i)L,\ldots,G_m) \rightarrow K(K;G_1,\ldots,G_m), \]
and the composition of it with the isomorphism \( \psi \) above coincides with the norm map \( 4 \). When \( L/K \) is a Galois extension, the action of \( \text{Gal}(L/K) \) on \( \text{Res}_{L/K}(G_i)L \) induces its action on 
\[ K(L;G_1,\ldots,G_m) \cong K(K;G_1,\ldots,\text{Res}_{L/K}(G_i)L,\ldots,G_m) \]
and we have \( N_{L/K} \circ \sigma = N_{L/K} \) for all \( \sigma \in \text{Gal}(L/K) \). This action does not depend on the choice of \( i \).

Lemma 4 Let \( L/K \) be a cyclic Galois extension and let \( \sigma \in \text{Gal}(L/K) \) be a generator. Suppose that for two different \( i \in \{1,\ldots,m\} \) the sequence 
\[ 4 \]
\[ \text{Res}_{L/K}(G_i)_K \xrightarrow{N_{L/K}} G_i \rightarrow 1 \]
is Zariski exact. Then the sequence of abelian groups 
\[ K(L;G_1,\ldots,G_m) \xrightarrow{1-\sigma} K(L;G_1,\ldots,G_m) \xrightarrow{N_{L/K}} K(K;G_1,\ldots,G_m) \rightarrow 0 \]
is exact.
Proof. Suppose that (1) is exact for $i = m - 1, m$. Let $G'_m := \text{Ker}(N_{L/K} : \text{Res}_{L/K}(G_m)_L \to G_m)$. By Lemmas 2 and 3 there are exact sequences

\begin{align*}
(5) \quad & K(K; G_1, \ldots, G'_m) \to K(L; G_1, \ldots, G_m) \xrightarrow{N_{L/K}} K(K; G_1, \ldots, G_m) \to 0 \\
(6) \quad & K(L; G_1, \ldots, G_{m-1}, G'_m) \xrightarrow{N_{L/K}} K(K; G_1, \ldots, G_{m-1}, G'_m) \to 0.
\end{align*}

Since $(G'_m)_L \cong (G_m)_L^{n-1}$ ($n := [L : K]$) we can replace the first group of (6) by $K(L; G_1, \ldots, G_m)^{n-1}$. By Lemma 1 the composite

$$K(L; G_1, \ldots, G_m)^{n-1} \to K(K; G_1, \ldots, G_{m-1}, G'_m) \to K(L; G_1, \ldots, G_m)$$

is given on the $i$-th summand by $1 - \sigma^i$. The assertion follows. $\square$

**Galois symbol** Let $G_1, \ldots, G_m$ be connected commutative algebraic groups over $K$, and let $n$ be an integer prime to the characteristic of $K$. For any finite extension $L/K$, we have a homomorphism $h_L$

$$h_L : K(L; G_1, \ldots, G_m)/n \to H^n(L, G_1[n] \otimes \cdots \otimes G_m[n])$$

called the *Galois symbol*. This is characterized by the following properties.

(i) If $x_i \in G_i(L)$ for $i = 1, \ldots, m$, then $h_L(\{x_1, \ldots, x_m\}_L/L) = (x_1) \cup \ldots \cup (x_m)$. Here we write by $(x_i)$ for the image of $x_i$ in $H^1(L, G_i[n])$ by the connecting homomorphism associated to the exact sequence

$$1 \to G_i[n] \to G_i \xrightarrow{m} G_i \to 1.$$

(ii) If $M/L/K$ is a tower of finite extensions and if $M/L$ is separable (resp. purely inseparable), then the diagram

$$
\begin{array}{ccc}
K(M; G_1, \ldots, G_m)/n & \xrightarrow{h_M} & H^n(M, G_1[n] \otimes \cdots \otimes G_m[n]) \\
\downarrow \quad N_{M/L} & & \downarrow \\
K(L; G_1, \ldots, G_m)/n & \xrightarrow{h_L} & H^n(L, G_1[n] \otimes \cdots \otimes G_m[n])
\end{array}
$$

is commutative, where the right vertical map is the corestriction (resp. the multiplication by $[M : L]$ under the identification $H^n(M, G_1[n] \otimes \cdots \otimes G_m[n]) \cong H^n(L, G_1[n] \otimes \cdots \otimes G_m[n])$).

Property (i) implies in particular that (1) coincides with the usual Galois symbol (1) in the case $G_1 = \ldots = G_m = G_m$. In [52] Remark 1.7, Somekawa conjectured that the Galois symbol associated to semialabelian varieties should be injective.
Galois cohomology of cyclic extensions  Let $L/K$ be a cyclic Galois extension of degree $n$ and let $\sigma$ be a generator of $G := \text{Gal}(L/K)$. For a discrete $G_K$-module $M$, tensoring the short exact sequence of $G$-modules

\begin{equation}
0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \overset{1-\sigma}{\longrightarrow} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0
\end{equation}

with $M$ yields a distinguished triangle

\begin{equation}
M[1] \overset{\alpha}{\longrightarrow} C^\cdot(M) \overset{\beta}{\longrightarrow} M \overset{\gamma}{\longrightarrow} M[2]
\end{equation}

in the derived category $D(G_K)$. Here we denote by $C^\cdot(M)$ the complex $\text{Res}_{L/K} M \overset{1-\sigma}{\longrightarrow} \text{Res}_{L/K} M$ concentrated in degree $-1$ and $0$. The spectral sequence

\begin{equation}
E_1^{p,q} = H^q(K, C^p(M)) \implies E^{p+q} = H^{p+q}(K, C^\cdot(M))
\end{equation}

induces short exact sequences

\begin{equation}
0 \longrightarrow H^q(L, M)_G \longrightarrow H^q(K, C^\cdot(M)) \longrightarrow H^{q+1}(L, M)^G \longrightarrow 0.
\end{equation}

It is easy to see that the composite

\begin{equation}
H^{q+1}(K, M) \overset{\alpha}{\longrightarrow} H^q(K, C^\cdot(M)) \longrightarrow H^{q+1}(L, M)^G
\end{equation}

is the restriction and

\begin{equation}
H^q(L, M)_G \longrightarrow H^q(K, C^\cdot(M)) \overset{\beta}{\longrightarrow} H^q(K, M)
\end{equation}

is induced by the corestriction. In particular we have $\gamma(H^q(K, M)) \subseteq \text{Ker}(\text{res} : H^{q+2}(K, M) \rightarrow H^{q+2}(L, M))$ hence

\begin{equation}
n\gamma(H^q(K, M)) = 0.
\end{equation}

For an integer $m$ prime to $\text{char } K$ and $r \in \mathbb{N}$ we write $\mathbb{Z}/m\mathbb{Z}(r) := \mu_m^{\otimes r}$ and

\begin{equation}
H^3(L/K, \mathbb{Z}/m\mathbb{Z}(2)) := \text{Ker}(H^3(K, \mathbb{Z}/m\mathbb{Z}(2)) \overset{\text{res}}{\longrightarrow} H^3(L, \mathbb{Z}/m\mathbb{Z}(2))).
\end{equation}

By restricting $\alpha : H^3(K, \mathbb{Z}/m\mathbb{Z}(2)) \rightarrow H^2(K, C^\cdot(\mathbb{Z}/m\mathbb{Z}(2)))$ to the subgroup $H^3(L/K, \mathbb{Z}/m\mathbb{Z}(2))$ and composing it with the inverse of the first map in (10) we obtain a map

\begin{equation}
H^3(L/K, \mathbb{Z}/m\mathbb{Z}(2)) \rightarrow \text{Ker}(H^2(L, \mathbb{Z}/m\mathbb{Z}(2))_G \overset{\text{cor}}{\longrightarrow} H^2(K, \mathbb{Z}/m\mathbb{Z}(2))).
\end{equation}

**Lemma 5** Assume that $n$ is prime to $\text{char } K$ and $\mu_n^2(\overline{K}) \subseteq K$. Then the homomorphism (12) is injective for $m = n$.  

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Proof. It is enough to show that $\gamma : H^1(K, \mathbb{Z}/n\mathbb{Z}(2)) \to H^3(K, \mathbb{Z}/n\mathbb{Z}(2))$ is zero. Consider the commutative diagram

$$
\begin{align*}
H^1(K, \mathbb{Z}/n\mathbb{Z}(2)) & \longrightarrow H^3(K, \mathbb{Z}/n\mathbb{Z}(2)) \\
\downarrow \gamma & \downarrow \gamma \\
H^3(K, \mathbb{Z}/n\mathbb{Z}(2)) & \longrightarrow H^3(K, \mathbb{Z}/n^2\mathbb{Z}(2))
\end{align*}
$$

induced by the canonical injection $\mathbb{Z}/n\mathbb{Z}(2) \to \mathbb{Z}/n^2\mathbb{Z}(2)$. The assumption $\mu_{n^2}(K) \subset K$ implies that the upper horizontal map can be identified with $K^*/(K^*)^n \to K^*/(K^*)^{n^2}, x(K^*) \mapsto x^n(K^*)^{n^2}$.

In particular the image is contained in $nH^1(K, \mathbb{Z}/n^2\mathbb{Z}(2))$. By (11) it is mapped under $\gamma$ to $n\gamma(H^1(K, \mathbb{Z}/n^2\mathbb{Z}(2))) = 0$. On the other hand it is a simple consequence of the Merkurjev-Suslin theorem [MS] that the lower horizontal map is injective. Hence $\gamma(H^1(K, \mathbb{Z}/n(2))) = 0$. \qed

The counterexample Let $L/K$ be as in the last section and let $T: = \text{Ker}(N_{L/K} : \text{Res}_{L/K} \mathbb{G}_m \to \mathbb{G}_m)$. We make the following assumptions

(13) $n$ is prime to char $K$ and $\mu_{n^2}(K) \subset K$,
(14) $H^3(L/K, \mathbb{Z}/n\mathbb{Z}(2))) \neq 0$.

Proposition 6 The Galois symbol $K(K; T, T)/n \to H^2(K, T[n] \otimes T[n])$ is not injective.

Proof. Let $\sigma$ be a generator of $G: = \text{Gal}(L/K)$. The exact sequence

$$
1 \longrightarrow \mathbb{G}_m \longrightarrow \text{Res}_{L/K} \mathbb{G}_m \longrightarrow \text{Res}_{L/K} \mathbb{G}_m \longrightarrow 1
$$

yields two short exact sequences

(15) $1 \longrightarrow \mathbb{G}_m \longrightarrow \text{Res}_{L/K} \mathbb{G}_m \longrightarrow T \longrightarrow 1$,
(16) $1 \longrightarrow T \longrightarrow \text{Res}_{L/K} \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1$.

Correspondingly, (8) induces two short exact sequences

(17) $0 \to \mathbb{Z} \to \mathbb{Z}[G] \to X \to 0, \quad 0 \to X \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$

where $X$ denotes the cocharacter group of $T$. Note that the sequence (15) is Zariski exact by Hilbert 90. Since the map $\text{Res}_{L/K} \mathbb{G}_m \to T$ factors through...
Res\(_L/K\) \(\mathbb{G}_m\) \(\rightarrow\) Res\(_L/K\) \(T\) \(\rightarrow\) \(T\) the sequence Res\(_L/K\) \(T\) \(\rightarrow\) \(T\) \(\rightarrow\) 1 is Zariski exact as well. By Lemma 4 the upper horizontal map in the diagram

\[
\begin{array}{ccc}
(K(L; T, T)/n)_G & \xrightarrow{N_{L/K}} & K(K; T, T)/n \\
\downarrow & & \downarrow \\
H^2(L, T[n] \otimes T[n])_G & \xrightarrow{\text{cor}} & H^2(K, T[n] \otimes T[n])
\end{array}
\]

is an isomorphism. The vertical maps are Galois symbols. Since \(T_L\) is a split torus the left vertical map is an isomorphism by the Merkurjev-Suslin theorem [MS]. Thus to finish the proof it remains to show that the lower vertical arrow is not injective. Note that \(T[n] \cong \mathbb{Z}/n\mathbb{Z}(1) \otimes X\). Hence the assertion follows from Lemma 5 and Lemma 7 below.

**Lemma 7** There exists homomorphisms of \(G\)-modules \(e : \mathbb{Z} \rightarrow X \otimes_{\mathbb{Z}} X\) and \(f : X \otimes_{\mathbb{Z}} X \rightarrow \mathbb{Z}\) such that \(f \circ e : \mathbb{Z} \rightarrow \mathbb{Z}\) is multiplication by \(n - 1\).

**Proof.** For a \(G\)-module \(M\) we write \(M^\vee\) for the \(G\)-module \(\text{Hom}(M, \mathbb{Z})\).

Let \((\ , \ ) : \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \rightarrow \mathbb{Z}\) be the symmetric pairing given by

\[
(g, g') = \begin{cases} 
1 & \text{if } g = g', \\
0 & \text{if } g \neq g'.
\end{cases}
\]

It yields an isomorphism \(\mathbb{Z}[G] \rightarrow \mathbb{Z}[G]^\vee\). For a submodule \(M \subseteq \mathbb{Z}[G]\) let

\[M^\perp = \{x \in \mathbb{Z}[G] \mid (x, m) = 0 \ \forall \ m \in M\}.\]

Then we have \(X^\perp = \mathbb{Z}S\) and \((\mathbb{Z}S)^\perp = X\) where \(S = \sum_{i=0}^{n-1} \sigma^i\). Thus \((18)\) yields an isomorphism \(X \cong (\mathbb{Z}[G]/\mathbb{Z}S)^\vee\). By \((17)\) we have \(\mathbb{Z}[G]/\mathbb{Z}S \cong X\), hence

\[X \otimes_{\mathbb{Z}} X \cong X \otimes_{\mathbb{Z}} X^\vee \cong \text{Hom}(X, X)\]

Thus it suffices to prove the assertion for \(\text{Hom}(X, X)\). Obviously, for the two maps \(e : \mathbb{Z} \rightarrow \text{Hom}(X, X), m \mapsto m \text{id}_X\) and \(f : \text{Hom}(X, X) \rightarrow \mathbb{Z}, \tau \mapsto \text{Tr}(\tau)\)

we have \(f \circ e = \text{rank}(X) = n - 1\). \(\square\)

**Remark 8** It is easy to construct examples where the assumptions (13) and (14) above are satisfied. For instance if \(K\) is a 2-local field satisfying property (13) and \(L/K\) is any cyclic extension of degree \(n\) then (14) holds by [Ka].

### 3 Counterexample to a conjecture of Beilinson

We first introduce some notation and recall a few facts from [Vo1] and [MVW]. Let \(K\) be a field of characteristic zero. Let \(\text{Cor}_K\) denote the additive category of finite correspondences ([MVW], 1.1). The objects of \(\text{Cor}_K\)
are smooth separated \(K\)-schemes of finite type and for \(X,Y \in \text{Obj}(\text{Cor}_K)\) the group of morphisms \(\text{Cor}_K(X,Y)\) is the free abelian group generated by integral closed subschemes \(W\) of \(X \times Y\) which are finite and surjective over \(X\). Let \(D^{-}(\text{Shv}_{\text{Nis}}(\text{Cor}_K))\) (resp. \(D^{-}(\text{Shv}_{\text{et}}(\text{Cor}_K))\)) denote the derived category of complexes of Nisnevich (resp. étale) sheaves with transfer bounded from above.

The category of effective motivic complexes \(\text{DM}^{\text{eff},-}_{\text{Nis}}(K)\) (resp. étale effective motivic complexes \(\text{DM}^{\text{eff},-}_{\text{et}}(K)\)) is the full subcategory of \(D^{-}(\text{Shv}_{\text{Nis}}(\text{Cor}_K))\) (resp. \(D^{-}(\text{Shv}_{\text{et}}(\text{Cor}_K))\)) which consists of complexes \(C^*\) with homotopy invariant cohomology sheaves \(H^i(C^*)\) for all \(i\) (see [Vo1], 3.1 or [MVW], 14.1, resp. 9.2). \(\text{DM}^{\text{eff},-}_{\text{Nis}}(K)\) and \(\text{DM}^{\text{eff},-}_{\text{et}}(K)\) are triangulated tensor categories. They are equipped with the t-structure induced from the standard t-structure on \(D^{-}(\text{Shv}_{\text{Nis}}(\text{Cor}_K))\) (resp. \(D^{-}(\text{Shv}_{\text{et}}(\text{Cor}_K))\)). There is a covariant functor \(M: \text{Cor}_K \rightarrow \text{DM}^{\text{eff},-}_{\text{Nis}}(K), X \mapsto M(X)\) and we have \(M(X \times Y) = M(X) \otimes M(Y)\). There is also the "change of topology" functor \(\alpha^*: \text{DM}^{\text{eff},-}_{\text{Nis}}(K) \rightarrow \text{DM}^{\text{eff},-}_{\text{et}}(K)\). It is a tensor functor which admits a right adjoint \(R\alpha^*: \text{DM}^{\text{eff},-}_{\text{et}}(K) \rightarrow \text{DM}^{\text{eff},-}_{\text{Nis}}(K)\).

Beilinson [Be] has proposed the following generalization of the Milnor-Bloch-Kato conjecture: For any smooth affine \(K\)-scheme \(X\) the adjunction morphism \(M(X) \rightarrow R\alpha^*M(X)\) induces an isomorphism on cohomology in degrees \(\leq 0\), i.e. the map

\[
\alpha_X : M(X) \longrightarrow t_{\leq 0} R\alpha^* M(X)
\]

is an isomorphism in \(\text{DM}^{\text{eff},-}_{\text{Nis}}(K)\).

If \(X = (\mathbb{G}_m)^d = \mathbb{G}_m \times \cdots \times \mathbb{G}_m\) (\(d\)-fold product of \(\mathbb{G}_m\)) we have \(M(X) \cong (\mathbb{Z} \oplus \mathbb{Z}(1)[1])^d\). Thus \(\alpha_X\) is an isomorphism if and only if

\[
Z(n) \longrightarrow t_{\leq n} R\alpha^* Z(n)
\]

is an isomorphism for all \(n \leq d\). It is known (compare [SV]) that the Milnor-Bloch-Kato conjecture is equivalent to the assertion that \((20)\) is an isomorphism for all \(n \geq 0\).

Let \(L/K\) be a separable quadratic extension and let \(T: = \text{Ker}(N_{L/K} : \text{Res}_{L/K} \mathbb{G}_m \rightarrow \mathbb{G}_m)\). We shall show that \((19)\) is in general not an isomorphism for \(X = T^n\) for \(n \geq 2\). By ([HK], 7.3) there exists a canonical decomposition \(M(T) = \mathbb{Z} \oplus \mathbb{Z}(L/K,1)[1]\) where \(\mathbb{Z}(L/K,1)\) is the cone of the morphism \(\mathbb{Z}(1) \rightarrow \text{Res}_{L/K} \mathbb{Z}(1)\).

Remarks 9 (a) Here is a more explicit description of the motive \(\mathbb{Z}(L/K,1)\). The torus \(T\) defines a homotopy invariant étale (hence Nisnevich) sheaf with transfer and therefore an element of \(\text{DM}^{\text{eff},-}_{\text{Nis}}(K)\). We have

\[
\mathbb{Z}(L/K,1) \cong T[-1].
\]
This can be deduced from the corresponding statement for $G_m$ ([MVW], 4.1) and the exactness of \([15]\) (as a sequence in $\text{Shv}_{\text{Nis}}(\text{Cor}_K)$).

(b) Let $A_1, \ldots, A_n$ be semi-abelian varieties over $K$. It should be possible to identify the generalized Milnor $K$-group $K(K; A_1, \ldots, A_n)$ with a Hom-group in $\text{DM}^{\text{eff}}_{\text{Nis}}(K)$. For that we view $A_1, \ldots, A_n$ again as elements in $\text{Shv}_{\text{Nis}}(\text{Cor}_K)$. Then we expect that

\[ K(K; A_1, \ldots, A_n) \cong \text{Hom}_{\text{DM}^{\text{eff}}_{\text{Nis}}(K)}(\mathbb{Z}, A_1 \otimes \cdots \otimes A_n). \]

If $A_1 = \ldots = A_n = G_m$ this is proved in ([MVW], lecture 5) and it is likely that the proof given there can be adapted to the case of arbitrary semi-abelian varieties.

For $p, q \geq 0$ and $n = p + q$ we define

\[ Z(L/K, p, q) := Z(L/K, 1)^{\otimes p} \otimes Z(q) \]

and denote by $C(p, q)$ the cone of $Z(L/K, p, q) \rightarrow t_{\leq n} R^\alpha \alpha^* Z(L/K, p, q)$. Note that $Z(L/K, p, q)[n]$ is a direct summand of $M(T^p \times (G_m)^q)$. We also put $C(n) := C(0, n)$. We have

\[ C(n) \cong (t_{\geq n+1} R^\alpha \mathbb{Q}/\mathbb{Z}(n))[−1] \]

This follows from the Milnor-Bloch-Kato conjecture (in fact for our purpose we need \([21]\) only after localization at the prime 2 where it follows from the Milnor conjecture \([V62]\)).

Tensoring $Z(1) \rightarrow \text{Res}_{L/K} Z(1) \rightarrow Z(L/K, 1) \rightarrow Z(1)[1]$ with $Z(L/K, p−1, q)$ (for $p \geq 1, q \geq 0$) yields a distinguished triangle

\[ Z(L/K, p−1, q+1) \rightarrow \text{Res}_{L/K} Z(n) \rightarrow Z(L/K, p, q) \rightarrow Z(L/K, p−1, q+1)[1] \]

hence also a triangle

\[ C(p−1, q+1) \rightarrow \text{Res}_{L/K} C(n) \rightarrow C(p, q) \rightarrow C(p−1, q+1)[1]. \]

The following Lemma follows easily by induction on $q$ using \([21]\) and \([22]\).

**Lemma 10** Let $p \geq 1, q \geq 0$ and $n = p + q$. Then we have $H^k(C(p, q)) = 0$ for $k < q + 2$ and

\[ H^{q+2}(C(p, q))(K) \cong H^{n+1}(L/K, \mathbb{Q}/\mathbb{Z}(n)) \]

where $H^{n+1}(L/K, \mathbb{Q}/\mathbb{Z}(n)) = \text{Ker}(H^{n+1}(K, \mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\text{res}} H^{n+1}(L, \mathbb{Q}/\mathbb{Z}(n)))$.

\[ \text{This remark has been communicated to us by B. Kahn.} \]
Since \([L : K] = 2\) we have
\[
H^{n+1}(L/K, \mathbb{Q}/\mathbb{Z}(n)) \cong H^{n+1}(L/K, \mathbb{Q}_2/\mathbb{Z}_2(n)) \cong H^{n+1}(L/K, \mathbb{Z}/2\mathbb{Z}(n)) \\
\cong H^{n+1}(L/K, \mathbb{Z}/2\mathbb{Z})
\]
(the second isomorphism is a consequence of the Milnor conjecture). Now the following Proposition follows by applying Lemma 10 for \((p, q) = (2, 0)\) and \((n, 0)\).

**Proposition 11**

(a) There exists a short exact sequence
\[
0 \rightarrow H^0(M(T \times T))(K) \rightarrow R^0\alpha_*\alpha^*M(T \times T)(K) \rightarrow H^3(L/K, \mathbb{Z}/2\mathbb{Z}) \rightarrow 0
\]
In particular if \(H^3(L/K, \mathbb{Z}/2\mathbb{Z}) \neq 0\) then (19) is not an isomorphism for \(X = T \times T\).

(b) More generally let \(n\) be an integer \(\geq 2\) and assume that \(H^{n+1}(L/K, \mathbb{Z}/2\mathbb{Z}) \neq 0\). Then the map (19) is not an isomorphism for \(X = T^n\). More precisely either the map
\[
H^{2-n}(M(X)) \rightarrow R^{2-n}\alpha_*\alpha^*M(X)
\]
is not surjective or
\[
H^{3-n}(M(X)) \rightarrow R^{3-n}\alpha_*\alpha^*M(X)
\]
is not injective.

An \(n\)-local field \(K\) of characteristic 0 provides an example where the above assumption holds. In fact by [Ka] we have \(H^{n+1}(L/K, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\) for such fields.

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Michael Spieß  
Fakultät für Mathematik  
Universität Bielefeld  
Postfach 100131  
D-33501 Bielefeld, Germany  
mspiess@math.uni-bielefeld.de

Takao Yamazaki  
Mathematical Institute  
Tohoku University  
Aoba  
Sendai 980-8578, Japan  
ytakao@math.tohoku.ac.jp