Global Subsonic and Subsonic-Sonic Flows
through Infinitely Long Axially Symmetric Nozzles

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Abstract: In this paper, we establish existence of global subsonic and subsonic-sonic flows through infinitely long axially symmetric nozzles by combining variational method, various elliptic estimates and a compensated compactness method. More precisely, it is shown that there exist global subsonic flows in nozzles for incoming mass flux less than a critical value; moreover, uniformly subsonic flows always approach to uniform flows at far fields when nozzle boundaries tend to be flat at far fields, and flow angles for axially symmetric flows are uniformly bounded away from $\pi/2$; finally, when the incoming mass flux tends to the critical value, subsonic-sonic flows exist globally in nozzles in the weak sense by using angle estimate in conjunction with a compensated compactness framework.

Keywords: Axially symmetric nozzles, Subsonic flows, Subsonic-sonic flows, Stream functions, Compensated compactness

1 Introduction and Main Results

This is a continuation to [13] on the study for subsonic and subsonic-sonic flows in multi-dimensional nozzles. Two dimensional subsonic and subsonic-sonic flows through infinitely long nozzles were studied in detail in [13]. In particular, global smooth subsonic flows and their properties have been established for incoming mass fluxes less than the critical value,
while existence of subsonic-sonic flows is proved for the critical mass flux. However, many of arguments are special to 2-dimensional flows, and it seems difficult to generalize them to the more realistic 3-dimensional flows.

In this paper, we would like to investigate the 3-D flows in nozzles which are infinitely long and axially symmetric.

As far as axially symmetric flows are concerned, one should note the significant result due to Gilbarg, [7], where he showed that if an axially symmetric subsonic nozzle flow approximates to uniform flows at far fields, then the flow speed on the boundary is monotone increasing with respect to the incoming mass flux by a comparison principle, however, existence of such flows is not known. For free boundary problems, in [1], Alt, Caffarelli and Friedman gave a complete study for flows with jet and cavitation by variational methods.

Let us start with 3-D isentropic compressible Euler equations,

\[
\begin{align*}
(\rho u)_x + (\rho v)_y + (\rho w)_z &= 0, \\
(\rho u^2)_x + (\rho uv)_y + (\rho uw)_z + p(\rho)_x &= 0, \\
(\rho uv)_x + (\rho v^2)_y + (\rho vw)_z + p(\rho)_y &= 0, \\
(\rho uw)_x + (\rho vw)_y + (\rho w^2)_z + p(\rho)_z &= 0,
\end{align*}
\]

where \( \rho \) is the density, \((u, v, w)\) is the velocity, and \( p = p(\rho) \) denotes the pressure. In general, it is assumed that \( p'(\rho) > 0 \) for \( \rho > 0 \) and \( p''(\rho) \geq 0 \), where \( c(\rho) = \sqrt{p'(\rho)} \) is called the sound speed. Important examples include polytropic gases and isothermal gases, for polytropic gases, \( p = A\rho^\gamma \) where \( A \) is a constant and \( \gamma \) is the adiabatic constant with \( \gamma > 1 \); and for isothermal gases, \( p = c^2\rho \) with constant sound speed \( c \).

Suppose that the flow is also irrotational, i.e. [5],

\[
\begin{align*}
u_y - v_x &= 0, \\
u_z - w_x &= 0, \\
v_z - w_y &= 0.
\end{align*}
\]

Then it follows from (1) and (2) that the flow satisfies Bernoulli’s law

\[
\frac{q^2}{2} + \int^\rho \frac{p'(\rho)}{\rho} d\rho = C,
\]

where \( q = \sqrt{u^2 + v^2 + w^2} \), and \( C \) is a constant depending on the flow. There are some basic facts about irrotational isentropic steady flows, see [5], which are consequences of
Bernoulli’s law (3). First, $\rho$ is a decreasing function of $q$, attains its maximum at $q = 0$. Second, there is a critical speed $q_c$ such that $q < c_{\text{subsonic}}$ if and only if $q < q_c$. Finally, $\rho q$ is a nonnegative function of $q$, for $q \geq 0$, which is increasing for $q \in (0, q_c)$ and decreasing for $q \geq q_c$, and vanishes at $q = 0$. So $\rho q$ attains its maximum at $q = q_c$, therefore, that the flow is subsonic is equivalent to $\rho q < \rho_c q_c$ and $\rho > \rho_c$. Therefore, we can nondimensionalize the flow as in [3, 13], such that $q_{cr} = 1$, $\rho_{cr} = 1$, and then Bernoulli’s law (3) reduces to

$$\frac{q^2}{2} + \int_1^\rho \frac{p'(\rho)}{\rho} d\rho = \frac{1}{2}.$$  

Since $p'(\rho)/\rho > 0$ for $\rho > 0$, so (4) yields a representation of the density

$$\rho = g(q^2),$$

moreover, $g$ is a decreasing function. For example, for polytropic gases, after the nondimensionalization, $p = \rho^\gamma/\gamma$, and (5) is nothing but

$$\rho = g(q^2) = \left(\frac{\gamma + 1 - (\gamma - 1)q^2}{2}\right)^{\frac{1}{\gamma - 1}}.$$  

Furthermore, $\rho$ is a two-valued function of $(\rho q)^2$. Subsonic flows correspond to the branch where $\rho > 1$ if $(\rho q)^2 \in [0, 1)$. Set

$$\rho = H((\rho q)^2)$$

such that $\rho > 1$ if $(\rho q)^2 \in [0, 1)$, therefore, $H$ is a positive decreasing function defined on $[0, 1]$, twice differentiable on $[0, 1)$, and satisfies $H(1) = 1$. Moreover, it follows from (5) and (7) that $(\rho q)^2$ is given in terms of $q^2$ as

$$(\rho q)^2 = G(q^2).$$

Thus

$$g(q^2) = H(G(q^2)).$$

Suppose that the wall of nozzle is impermeable so that

$$(u,v,w) \cdot \vec{n} = 0,$$
where $\vec{n}$ is the outward normal of the solid boundary.

Due to (2), one can introduce a velocity potential $\Phi$ for the flow such that

$$\Phi_x = u, \quad \Phi_y = v, \quad \Phi_z = w.$$ 

Thus the continuity equation becomes

$$\text{div}(g(|\nabla \Phi|^2)\nabla \Phi) = 0.$$ 

Assume now that the nozzle is axi-symmetric as given by

$$D = \{(x, y, z)| 0 \leq \sqrt{y^2 + z^2} < f(x), -\infty < x < \infty\}.$$ 

Consider a smooth flow in the nozzle. Then it follows from continuity equation and (9) that mass fluxes through each section which is transversal to the symmetry axis are the same.

Thus, the problem of finding solutions to smooth flows in a 3-D nozzle reduces to solving the following problem,

$$\begin{cases}
\text{div}(g(|\nabla \Phi|^2)\nabla \Phi) = 0, & \text{in } D, \\
\frac{\partial \Phi}{\partial n} = 0, & \text{on } \partial D, \\
\int_S g(|\nabla \Phi|^2) \frac{\partial \Phi}{\partial \vec{l}} dS = m_0,
\end{cases}$$

where $S$ is the surface transversal to the axis, and $\vec{l}$ is the normal to $S$ which directs to the positive axial direction.

In this paper, it is assumed that there exists $\alpha \in (0, 1)$ such that

$$\|f'\|_{C^\alpha(\mathbb{R})} < \infty, \text{ and } \inf_{\mathbb{R}} f = b > 0.$$ 

Now the main results of this paper can be stated as follows.

**Theorem 1** Suppose that the nozzle boundary satisfies (12). Then there exists a positive constant $\bar{m}$ depending only on $f$ such that if $m_0 < \bar{m}$, there exists an axially symmetric uniformly subsonic flow through the nozzle. More precisely, there exists a smooth solution $\Phi \in C^\infty(D)$ to (11) such that

$$\sup_D |\nabla \Phi| < 1,$$
and

\[ u(x, y, z) = \Phi_x = U(x, r), \quad v(x, y, z) = \Phi_y = V(x, r) \frac{y}{r}, \quad w(x, y, z) = \Phi_z = V(x, r) \frac{z}{r}, \quad (14) \]

where \( r = \sqrt{y^2 + z^2} \) and \( U(x, r), V(x, r) \) are smooth in their arguments, and \( V(x, r) \) vanishes on the symmetry axis.

If the wall of the nozzle tends to be flat at far fields, for example, rescaling if necessary, one may assume that

\[ f(x) \to 1 \text{ as } x \to -\infty, \quad f(x) \to a > 0 \text{ as } x \to \infty. \quad (15) \]

Then the following sharper results hold.

**Theorem 2** Suppose that the wall of the nozzle satisfies both (12) and (15). Then there exists \( \hat{m} > 0 \) such that if \( 0 \leq m_0 < \hat{m} \), there exists a unique axially symmetric uniformly subsonic flow through the nozzle with the properties that

\[ M(m_0) = \sup_{(x, y, z) \in \bar{D}} |\nabla \Phi| < 1, \quad (16) \]

and

\[
\begin{align*}
&\left| (U, V) - \left( \{ G^{-1}(\frac{m_0^2}{\pi^2}) \}^{1/2}, 0 \right) \right| \to 0 \text{ as } x \to -\infty, \\
&\left| (U, V) - \left( \{ G^{-1}(\frac{m_0^2}{\pi^2 a^4}) \}^{1/2}, 0 \right) \right| \to 0 \text{ as } x \to \infty,
\end{align*}
\]

uniformly in \( r \), where \( G \) is defined by (13); moreover, \( M(m_0) \) ranges over \([0, 1)\) as \( m_0 \) varies in \([0, \hat{m})\). Furthermore, if \( 0 < m_0 < \hat{m} \), the axial velocity is always positive in \( \bar{D} \), i.e.

\[ u > 0, \quad (17) \]

and, the flow angle, \( \omega = \arctan \frac{V}{U} \), satisfies

\[ \underline{\omega} \leq \omega \leq \bar{\omega}, \quad (18) \]

where

\[ \underline{\omega} = \min\{ \inf_x \arctan f'(x), 0 \}, \quad \bar{\omega} = \max\{ \sup_x \arctan f'(x), 0 \}. \quad (19) \]
Moreover, for any given $m \in (0, \hat{m})$, there exist a positive constant $\delta = \delta(m) > 0$, such that if $m \in [\underline{m}, \hat{m})$, then

$$q(m) = \inf_{\Omega} |\nabla \Phi| \geq \delta.$$  \hspace{1cm} (20)

We now study the limiting behavior of these subsonic flows in the nozzle when then the cross-section mass fluxes $m_0$ approaches the critical value. In fact, as $m_0 \uparrow \hat{m}$, the corresponding flow fields tend a limit which yields a subsonic-sonic flow in the nozzle.

**Theorem 3** Let $\{m_{0,n}\}$ be any sequence such that $m_{0,n} \to \hat{m}$ as $n \to +\infty$. Denote by $(U_n, V_n)$ the global uniformly subsonic flow corresponding to $m_{0,n}$ as guaranteed by Theorem 2. Then there exists a subsequence, still labelled by $\{(U_n, V_n)\}$ associated with $\{m_{0,n}\}$ such that

$$U_n \to U, \quad V_n \to V,$$  \hspace{1cm} (21)

$$g(q_{n}^2)U_n \to g(q^2)U, \quad g(q_{n}^2)V_n \to g(q^2)V,$$  \hspace{1cm} (22)

where $q_{n}^2 = U_n^2 + V_n^2$, $q^2 = U^2 + V^2$, and $g(q^2)$ is the function defined by (5) through Bernoulli’s law, all the above convergence are almost convergence. Moreover, this limit yields a three dimensional flow with density $\rho(x, y, z) = g(q^2)(x, r)$ and velocity

$$u(x, y, z) = U(x, r), \quad v(x, y, z) = V(x, r)\frac{y}{r}, \quad w(x, y, z) = V(x, r)\frac{z}{r},$$

where $r = \sqrt{y^2 + z^2}$, which satisfies

$$u_y - v_x = 0, \quad v_z - w_y = 0, \quad w_x - u_z = 0 \text{ in } D$$

in the sense of distribution, moreover, for any $\eta \in C^\infty_c(\overline{D})$

$$\iiint_D (\rho u, \rho v, \rho w) \cdot \nabla \eta dx dy dz = 0.$$

This implies that $(u, v, w)$ satisfies boundary condition (4) as the normal trace of the divergence field $(\rho u, \rho v, \rho w)$ on the boundary.

Before we prove the theorems, there are a few remarks in order.
**Remark 1** In contrast to two dimensional plane flows, three dimensional flows are much more complicated. Indeed, some of the key arguments in [13] cannot be applied to three dimensional case directly. Even for irrotational steady axially symmetric subsonic flows, there are some difficulties near the symmetry axis, see (29). Therefore, it seems difficult to show the existence of subsonic flows by fixed point argument as in plane flows in [13]. Fortunately, for axisymmetric flows, equation (29) has a variational structure, which is one of the key points to show the existence of subsonic solutions.

**Remark 2** It should be noted that one cannot adapt the analysis of [1] directly to study the properties of the subsonic flow in Theorem 2 since for jet flow, the pressure is prescribed on the jet surface, so the flow speed is known by Bernoulli’s law, thus it is easier to see whether the flow is subsonic and whether it approaches to uniform flows at far fields.

**Remark 3** In all the theorems in this chapter, we require only $C^{1,\alpha}$ smoothness of $f$. Similar to the proofs given in this paper, one can prove all results in [13] under the condition that nozzle boundaries are $C^{1,\alpha}$ instead of $C^{2,\alpha}_{loc}$. Furthermore, it is only required that $f$ itself tends to constants at far fields instead of its higher derivatives, which improves the results in [13].

**Remark 4** Theorem 2 provides the existence of flows studied by Gilbarg in [7]. Moreover, applying the comparison principle obtained by Gilbarg in [7], the maximum speed of flows obtained in Theorem 2 is monotone increasing with respect to incoming mass flux.

**Remark 5** There are some fragmentary descriptions of some phenomena on the axially symmetric subsonic flows past a body, for the reference, please refer to [3], [9], [8]. For applications of the theory of compensated compactness to two dimensional transonic and subsonic-sonic flows, please see [12], [4], [13].

The rest of the paper is organized as follows: in Section 2, we derive the governing equation and boundary conditions for axially symmetric irrotational flows. In Section 3, we adapt the variational method used in [1] to prove Theorem 1. Subsequently, in Section
4, we prove that subsonic flows will approach uniform flows at far fields when the nozzle boundaries tend to be flat at far fields, which will yield the existence of the critical value for incoming mass fluxes. In Section 5, positivity of axial velocity and uniform estimates for flow angles for axially symmetric flows are proved. In last section, Section 6, we use a compensated compactness framework to show the existence of weak subsonic-sonic flows.

2 Axially Symmetric Flows

In this section, we will derive the governing equations and boundary conditions for axially symmetric irrotational flows in cylindrical coordinates and in terms of stream functions.

In the cylindrical coordinates \((x, r, \theta)\), let the fluid density and velocity be \(\rho(x, r, \theta)\) and \((U(x, r, \theta), V(x, r, \theta), W(x, r, \theta))\), where \(U, V,\) and \(W\) are axial velocity, radial velocity and swirl velocity respectively. Then \((x, y, z), \rho,\) and \((u, v, w)\) satisfy

\[
\begin{align*}
x &= x, \quad y = r \cos \theta, \quad z = r \sin \theta; \\
\rho(x, y, z) &= \rho(x, r, \theta), \quad u(x, y, z) = U(x, r, \theta); \\
v(x, y, z) &= V(x, r, \theta) \cos \theta + W(x, r, \theta)(-\sin \theta), \\
w(x, y, z) &= V(x, r, \theta) \sin \theta + W(x, r, \theta) \cos \theta.
\end{align*}
\]

It should be noted that for axi-symmetric flows, one has

\[
U(x, r, \theta) = U(x, r), \quad V(x, r, \theta) = V(x, r), \quad W(x, r, \theta) = W(x, r).
\]

Since the flow is also assumed to be irrotational, one has

\[
v_z - w_y = -\frac{(rW)_r}{r} = 0,
\]

this implies that

\[
W = \frac{c(x)}{r}.
\]

Thus \(W \equiv 0\) since \(W\) is bounded near \(r = 0\). Therefore, for axially symmetric irrotational flows, one has

\[
u = U(x, r), \quad v = V(x, r)\frac{y}{r}, \quad w = V(x, r)\frac{z}{r}, \quad \text{and} \quad \rho = \rho(x, r),
\]

(23)
where \( r = \sqrt{y^2 + z^2} \). Then the continuity equation reduces to

\[
(rpU)_r + (rpV)_x = 0. \tag{24}
\]

Moreover, the irrotational condition \(2\) changes to

\[
U_r - V_x = 0. \tag{25}
\]

Bernoulli’s law \(4\) is still of the same form with \( q = \sqrt{U^2 + V^2} \).

Due to \(24\), one can introduce a stream function \( \psi(x,r) \) such that

\[
\psi_r = rpU, \quad \psi_x = -rpV. \tag{26}
\]

Then Bernoulli’s law \(4\) becomes to

\[
\frac{1}{2\rho} \left| \frac{\nabla \psi}{r} \right|^2 + \int_1^\rho \frac{p'(\rho)}{\rho} \, d\rho = \frac{1}{2}. \tag{27}
\]

Therefore, it follows from \(7\) that \( \rho \) can be represented as

\[
\rho = H\left(\frac{\left| \nabla \psi \right|}{r} \right)^2, \tag{28}
\]

so the irrotationality \(25\) changes to

\[
\text{div} \left( H\left(\frac{\left| \nabla \psi \right|}{r} \right)^2 \right)^{-1} \frac{\nabla \psi}{r} = 0. \tag{29}
\]

The no-flow boundary condition \(9\) on the nozzle wall becomes

\[
(U, V) \cdot \vec{N} = 0, \tag{30}
\]

where \( \vec{N} \) is the normal of the curve \( r = f(x) \). It follows from \(30\) that \( \psi \) is a constant in each connected component of the solid boundaries.

Note that for smooth axisymmetric flows in the nozzle, it follows from \(26\) that \( \psi \) is a constant on the symmetry axis. Thus \( r = 0 \) is a streamline.

Since the flow is axially symmetric, one may consider only symmetric part of the domain. Let

\[
\Omega = \{(x, r) \mid 0 < r < f(x), -\infty < x < \infty \} \tag{31}
\]
with boundaries

$$T_1 = \{(x,r) | r = 0, -\infty < x < \infty \}, \quad T_2 = \{(x,r) | r = f(x), -\infty < x < \infty \}. \quad (32)$$

For convenience, we denote by $D_0$ the three dimensional domain induced by $\Omega$,

$$D_0 = \{(x,y,z) | 0 < \sqrt{y^2 + z^2} < f(x), -\infty < x < \infty \}. \quad (33)$$

Then, to study the 3-dimensional problem, (11), for axisymmetric flows, one may first study the following 2-dimensional problem

$$\begin{aligned}
\text{div} \left( \left( H \left( \frac{\nabla \psi}{r} \right)^2 \right)^{-1} \frac{\nabla \psi}{r} \right) &= 0, \quad \text{in } \Omega, \\
\psi &= 0, \quad \text{on } T_1, \\
\psi &= m = \frac{m_0}{2\pi}, \quad \text{on } T_2.
\end{aligned} \quad (34)$$

### 3 Subsonic Flows Associated with Small Incoming Mass Flux

This section is mainly devoted to the proof of Theorem 1. Our approach is motivated strongly by the important work [1] by Alt, Caffarelli and Friedman. The proof can be divided into 10 steps.

Step 1. Subsonic truncation and shielding singularity. By direct calculations, it is easy to find that the derivative of function $H(s)$ goes to negative infinity as $s \to 1$. To control the ellipticity and avoid singularity of $H'$, one may truncate $H$ as follows

$$\tilde{H}(s) = \begin{cases} 
H(s), & \text{if } 0 \leq s < \tilde{m}^2, \\
H \left( (\tilde{m} + 1)^2 \right), & \text{if } s \geq (\tilde{m} + 1)^2,
\end{cases} \quad (35)$$

where $\tilde{m} < 1$ is a constant to be determined, and $\tilde{H}$ is a smooth decreasing function. Set

$$q^2 = s/\tilde{H}^2(s). \quad (36)$$

Since $\tilde{H}^2(s) - 2\tilde{H}\tilde{H}'(s)s > 0$, we can represent $s$ as a function of $q^2$, $s = \tilde{G}(q^2)$. Obviously, $\tilde{G}$ is an increasing function. Define $\rho = \tilde{g}(q^2)$ as

$$\tilde{g}(q^2) = \tilde{H}(\tilde{G}(q^2)). \quad (37)$$
Then it is easy to check that
\[
\Lambda \geq \tilde{g} + 2q^2 \frac{d\tilde{g}}{dq^2} = \frac{\tilde{H}(\tilde{G}(q^2))}{\tilde{H}(G(q^2)) - 2H'(G(q^2))\tilde{G}(q^2)} \geq \nu
\] (38)
for some positive real numbers \(\Lambda\) and \(\nu > 0\) which depend on \(\tilde{H}\).

To treat the singularity in the coefficients of the equation (29) as \(r \to 0\), one may shield the singularity by first solving the following problem

\[
\begin{align*}
\text{div} \left( \left( \tilde{H} \left( \left| \nabla \psi \right|_r \right)^2 \right)^{-1} \nabla \psi \right) &= 0, \quad \text{in } \Omega, \\
\psi &= 0, \quad \text{on } T_1, \\
\psi &= m, \quad \text{on } T_2.
\end{align*}
\] (39)

Step 2. Variational problem. The problem (39) is a boundary value problem for an elliptic equation in a unbounded domain, therefore, we use a series of Dirichlet problems in bounded domains to approximate it. Thus consider first the following problem

\[
\begin{align*}
\text{div} \left( \left( \tilde{H} \left( \left| \nabla \psi \right|_r \right)^2 \right)^{-1} \nabla \psi \right) &= 0, \quad \text{in } \Omega_L, \\
\psi &= \frac{r^2}{f^2(x)}m, \quad \text{on } \partial \Omega_L,
\end{align*}
\] (40)

where \(\Omega_L = \{(x,r)| (x,r) \in \Omega, |x| < L\}\). The problem (40), can be solved by a variational method. The existence of solution to problem (40) is equivalent to find minimizer \(\psi_L^\delta \in \mathcal{A}_L = \{\phi| \phi \in W^{1,2}(\Omega_L), \phi - \frac{r^2}{f^2(x)}m \in W_0^{1,2}(\Omega_L)\}\) for the following minimization problem

\[
\mathcal{J}_L(\psi_L^\delta) = \inf_{\phi \in \mathcal{A}_L} \mathcal{J}_L(\phi),
\] (41)

where

\[
\mathcal{J}_L(\phi) = \int_{\Omega_L} F \left( \left| \nabla \phi \left( \frac{r}{r + \delta} \right) \right|^2 \right) (r + \delta) dx dr,
\] (42)

and \(F\) is defined by

\[
F(s) = \int_0^s (\tilde{H}(t))^{-1} dt.
\] (43)

Since \(\tilde{H}\) is a smooth decreasing function, therefore, \(\mathcal{F}(p_1, p_2, r) = F \left( \left| \frac{(p_1, p_2)}{r + \delta} \right|^2 \right) (r + \delta)\) is a convex function of \(p = (p_1, p_2)\), using the standard theory in calculus of variations, for
example, Theorem 2 and Theorem 3 in Section 8.2 in [6], the problem (41) has a unique solution since the functional $J_L$ is also coercive.

Step 3. Estimates for minimizers. For each $L$, there exists a unique solution $\psi^\delta_L$ of the problem (41). Each minimizer $\psi^\delta_L$ is a weak solution to the Dirichlet problem of the Euler-Lagrange equation, (40). Then by a weak maximum principle for the problem (40), see Theorem 8.1 in [10], one gets

$$0 \leq \psi^\delta_L \leq m \text{ in } \Omega_L.$$  \hspace{1cm} (44)

Using Caccioppoli’s inequality, both in interior and on the boundary, and Theorem 6.5 and Theorem 6.8 in [11], one can obtain

$$\|\nabla \psi^\delta_k\|_{L^2(\Omega_L)} \leq C(L, \|f'\|_{C^0}, \delta, m), \quad \forall k > 2L.$$  \hspace{1cm} (45)

Then Hölder estimates for the gradient of minimizers to the functional (42), and Theorem 8.6 in [11], imply that there exists $\alpha_1 \in (0, \alpha)$ such that

$$\|\psi^\delta_k\|_{C^{1, \alpha_1}(\Omega_L)} \leq C(L, \alpha_1, \|f'\|_{C^0}, \delta, m), \quad \forall k > 2L.$$  \hspace{1cm} (46)

Moreover, the interior Schauder estimate, Theorem 10.18 in [11], shows that for any $\Sigma \subset \subset \Omega_L$, it holds that

$$\|\psi^\delta_k\|_{C^{2, \alpha_1}(\Sigma)} \leq C(\Sigma, L, \alpha_1, \delta, m), \quad \forall k > L.$$  \hspace{1cm} (47)

To recover the singularity later by taking the limit $\delta \to 0^+$, we need a more precise estimate than (44). Set

$$\bar{\psi} = \frac{(r + \delta)^2}{b^2} \psi,$$  \hspace{1cm} (48)

where $b$ is defined in (12). Then it is easy to check that $\bar{\psi}$ satisfies the equation

$$\text{div} \left( \left( \bar{\psi} \right) \right) = 0.$$  \hspace{1cm} (49)

Because $\Omega_L$ satisfies a uniform exterior cone condition, $\psi^\delta_L \in C^0(\overline{\Omega_L})$ by Theorem 8.29 in [10]. Moreover, by (47), $\psi^\delta_L \in C^{2, \alpha_1}(\Omega_L)$. Therefore, both $\psi^\delta_L$ and $\bar{\psi}$ satisfy the equation
in (40) on $\Omega_L$ in the classical sense. Obviously, $\bar{\psi} \geq \psi_{L}^{\delta}$ on $\partial \Omega_L$. Thus, it follows from a comparison principle, Theorem 10.1 in [10], that

$$\psi_{L}^{\delta} \leq \bar{\psi} \quad \text{in } \Omega_L. \quad (50)$$

Step 4. Existence of solutions to (39). By a diagonal process and Arzela-Ascoli lemma, it follows from (46) that there exists a sequence $\{n_k\}$ such that

$$\psi_{n_k}^{\delta} \chi_{\Omega_{n_k}} \rightarrow \psi^{\delta} \quad \text{in } C^{1,\mu}(\Omega_L) \quad \text{for } \forall L > 0$$

with $0 < \mu < \alpha_1$. Therefore, $\psi^{\delta}$ is a weak solution to the problem (39). Then it follows from (17) that $\psi^{\delta} \in C^{2,\mu}(\Omega_L), \forall L > 0$.

Step 5. Recover singular coefficients. Due to (50), we have

$$\psi^{\delta}(x, r) \leq \frac{m}{b^2} (r + \delta)^2.$$ 

Therefore, $\psi^{\delta} \rightarrow 0$ on $T_1$. Moreover, for $\forall \varepsilon > 0$, on each set $\Omega_{L,\varepsilon} = \{(x, r) | |x| < L, \varepsilon < r < f(x)\}$, it follows from Caccioppoli’s inequality and Hölder gradient estimate in a similar way as for (45) and (46), that $\psi^{\delta}$ satisfies the following estimate

$$\|\psi^{\delta}\|_{C^{1,\alpha_1}(\Omega_{L,\varepsilon})} \leq C(L, \varepsilon, \|f\|_{C^\alpha}, m). \quad (51)$$

Due to a diagonal process and Arzela-Ascoli Lemma again, there exists a subsequence $\{\delta_k\}$ such that

$$\psi^{\delta_k} \rightarrow \psi \quad \text{in } C^{1,\mu}(\Omega_{L,\varepsilon}) \quad \text{for each } L > 0, \varepsilon > 0.$$

In particular,

$$\psi^{\delta_k} \rightarrow \psi \quad \text{pointwise in } \Omega. \quad (52)$$

Moreover, $\psi \in C^{1,\mu}$ solves the problem

$$\begin{cases}
\text{div} \left( \bar{H} \left( \frac{\nabla \psi}{r} \right)^2 \right)^{-1} \nabla \psi = 0, & \text{in } \Omega, \\
\psi = \frac{r^2}{f^2(x)} m, & \text{on } \partial \Omega,
\end{cases} \quad (53)$$

weakly and satisfies

$$0 \leq \psi \leq \frac{m}{b^2} r^2. \quad (54)$$
It follows from the standard bootstrap arguments that $\psi \in C^{2,\mu}(\Omega)$.

Step 6. Subsonic estimate near the symmetry axis. In this step, our aim is to show that
\[
\left| \frac{\nabla \psi (x, r)}{r} \right| \leq Cm
\]  
for $0 < r < \frac{b}{2}$. To do this, we note an important observation due to [1], that if $r_0 < b/2$, then
\[
\psi_0(x, r) = \frac{1}{t^2} \psi(x_0 + tx, r_0 + tr), \quad t = \frac{r_0}{2},
\]  
satisfies
\[
\text{div} \left( \left( \bar{H} \left( \frac{|\nabla \psi_0|^2}{2 + r} \right) \right)^{-1} \frac{\nabla \psi_0}{2 + r} \right) = 0 \quad \text{in} \ B_1((0, 0)).
\]
It follows from (54) that
\[
0 \leq \psi_0 \leq Cm \quad \text{in} \ B_1((0, 0)).
\]
Therefore, by Moser’s iteration, Theorem 8.18 in [10], one can get
\[
|\nabla \psi_0| \leq Cm \quad \text{in} \ B_{\frac{1}{2}}((0, 0)).
\]
In particular,
\[
\left| \frac{\nabla \psi (x_0, r_0)}{r} \right| = |\nabla \psi_0(0, 0)| \leq Cm.
\]  
(57)

Step 7. Subsonic estimate away from the symmetry axis. In this step, we derive the estimate (53) for $r > b/4$. For any given $(x_0, r_0) \in \Omega_{\infty, b/4} = \{(x, r)|(x, r) \in \Omega, r > b/4\}$, noting that all the coefficients in the equation (53) are bounded on $B_{b/8}((x_0, r_0)) \cap \Omega_{\infty, b/4}$, moreover,
\[
0 \leq \psi \leq m
\]
due to (54) and (52), one can derive by Moser’s iteration that
\[
|\nabla \psi(x_0, r_0)| \leq Cm.
\]
Therefore,
\[
\left| \frac{\nabla \psi (x_0, r_0)}{r} \right| \leq Cm,
\]  
(58)
since $r_0 > b/4$. 

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Step 8. Uniform Hölder continuity of velocity field near the symmetry axis. It follows from Step 6 and Step 7 that the flow is subsonic except on the axis when the incoming mass flux is sufficiently small. To show that the flow is subsonic globally, we first need to show that the velocity field is well-defined along the symmetry axis. In fact, we have the following stronger results.

**Lemma 4** Let \( \psi \) be a solution to problem (53) satisfying
\[
\left| \frac{\nabla \psi}{r} \right| \leq C \quad \text{in } \Omega.
\]
(59)

Then \( \frac{\nabla \psi}{r} \) is uniformly Hölder continuous up to the symmetry axis, moreover,
\[
\lim_{(x,r) \to (x_0,0)} \frac{\psi_x}{r}(x,r) = 0 \text{ for any } x_0 \in (-\infty, \infty).
\]
(60)

More precisely, there exists \( \beta \in (0, 1) \) such that
\[
\left[ \frac{\nabla \psi}{r} \right]_{C^\beta((l_1, l_2) \times (0, h_0))} \leq C(|l_2 - l_1|)
\]
holds for \( 0 < h_0 \leq \frac{b}{4} \) and any real numbers \( l_1 < l_2 \).

**Proof:** Since \( \psi \in C^{2,\mu}(\Omega) \cap C^{1,\mu}(\Omega \cup T_2) \) satisfies the equation in (53) and admits the bound (59), therefore, the axially symmetric potential
\[
\varphi(x, r) = \int_{(0,f(0))}^{(x,r)} \left( \tilde{H} \left( \left| \frac{\nabla \psi}{r} \right|^2 \right) \right)^{-1} \frac{\psi_r}{r} \, dx - \left( \tilde{H} \left( \left| \frac{\nabla \psi}{r} \right|^2 \right) \right)^{-1} \frac{\psi_x}{r} \, dr
\]
(62)
is well-defined and path independent except on the symmetry axis \( \{r = 0\} \). Moreover,
\[
\varphi_x = \left( \tilde{H} \left( \left| \frac{\nabla \psi}{r} \right|^2 \right) \right)^{-1} \frac{\psi_r}{r}, \quad \varphi_r = - \left( \tilde{H} \left( \left| \frac{\nabla \psi}{r} \right|^2 \right) \right)^{-1} \frac{\psi_x}{r}.
\]
(63)

Therefore, by (59),
\[
|\nabla \varphi| \leq C \text{ in } \Omega = \{0 < r < f(x)\},
\]
(64)
which implies that \( \varphi \) can be extended to \( \bar{\Omega} \) as
\[
\varphi(x_0, 0) = \lim_{(x,r) \in \Omega, (x,r) \to (x_0,0)} \varphi(x, r), \quad \forall x_0 \in (-\infty, \infty).
\]
Note that the axially symmetric potential \( \varphi \) induces a 3-D potential function

\[
\Phi(x, y, z) = \varphi(x, \sqrt{y^2 + z^2})
\]

(65)

which is defined on the three dimensional domain \( \bar{D} \). Then,

\[
\Phi_x = \varphi_x, \quad \Phi_y = \varphi_y, \quad \Phi_z = \varphi_z \quad \text{in} \ D_0,
\]

(66)

and that \( \Phi \in W^{1,\infty}_{\text{loc}}(D_0) \cap C^{2,\mu}(D_0) \). Moreover, it follows from (63), (66), and (37) that for any \( \varepsilon > 0 \), \( \Phi \) solves the equation

\[
\text{div}(\tilde{g}(|\nabla \Phi|^2) \nabla \Phi) = 0
\]

(67)

in the three dimensional domain \( D_\varepsilon = \{(x, y, z) | -\infty < x < \infty, \varepsilon < \sqrt{y^2 + z^2} < f(x)\} \).

Therefore, any \( \eta \in C_0^\infty(D) \), one has

\[
\iint_D \tilde{g}(|\nabla \Phi|^2) \nabla \Phi \cdot \nabla \eta dxdydz = 0
\]

(68)

as \( \varepsilon \to 0 \) since \( \nabla \Phi \) is bounded. Therefore,

\[
\iint_D \tilde{g}(|\nabla \Phi|^2) \nabla \Phi \cdot \nabla \eta dxdydz = 0 \quad \forall \eta \in C_0^\infty(D).
\]

Thus, \( \Phi \) is a weak solution of equation (67) in \( D \). Since (67) is elliptic due to (58), thus the standard elliptic regularity theory, [10], shows

\[
\Phi \in C^\infty(D).
\]

Moreover, for \( k = 1, 2, 3 \), \( \partial_k \Phi \) satisfies the equation

\[
\partial_i \left( (\tilde{g}(|\nabla \Phi|^2)) \delta_{ij} + 2\tilde{g}'(|\nabla \Phi|^2) \partial_i \Phi \partial_j \Phi \partial_j (\partial_k \Phi) \right) = 0,
\]

(69)
which is uniformly elliptic due to (38). Thus, by Nash-Moser iteration, there exists a 
\( \beta_1 \in (0, 1) \) such that for suitably small positive constant \( h \),
\[
[\partial_k \Phi]_{C^{\beta_1}(B_h((x,0,0)))} \leq C\|\nabla \Phi\|_{L^\infty(B_{2h}((x,0,0)))}. \tag{70}
\]
Therefore,
\[
|\partial_y \Phi(x, y, z) - \partial_y \Phi(x, 0, 0)| \leq C(y^2 + z^2)^{\beta_1/2} \text{ for } r = (y^2 + z^2)^{1/2} \leq h.
\]
It follows from (66) that
\[
|\varphi_r(x, r) \cos \theta - \Phi_y(x, 0, 0)| \leq Cr^{\beta_1}, \text{ for all } \theta \in [0, 2\pi).
\]
Thus,
\[
\Phi_y(x, 0, 0) = 0.
\]
Similarly, \( \Phi_z(x, 0, 0) = 0 \). Therefore
\[
|\varphi_r(x, r)| \leq Cr^{\beta_1}.
\]
Thus,
\[
\lim_{(x, r) \to (x_0, 0)} \varphi_r(x, r) = 0. \tag{71}
\]
Furthermore, (70) yields
\[
[\nabla \varphi]_{C^{\beta_1}((l_1, l_2) \times (0, h))} \leq C(|l_2 - l_1|). \tag{72}
\]
So the desired estimates (60) and (61) follow.

This finishes the proof of the Lemma. \( \square \)

Step 9. Removal of cutoff. Combining (57), (58) and (61) yields
\[
\left| \frac{\nabla \psi}{r} \right| \leq Cm, \text{ in } \Omega.
\]
If \( m \) is sufficiently small, then \( Cm < \tilde{m} \), therefore
\[
\left| \frac{\nabla \psi}{r} \right| \leq \tilde{m}.
\]
Consequently, $\psi$ solves the problem (34), and moreover, which is uniformly subsonic.

Step 10. Existence of 3-D subsonic flow. It follows from the proof of Lemma 4 and step 1-9 that there exists a three dimensional subsonic solution to problem (11) which satisfies (13) and (14).

4 Existence of The Critical Incoming Mass Flux

In this section, it will be shown that there exists a critical value $\hat{m}$ such that the flow is always subsonic when the three dimensional mass flux $m_0$ is less than $\hat{m}$. To achieve this goal, we first show that the flow approximates to uniform flows at far fields.

Let

$$\hat{H}(s) = \begin{cases} H(s) & \text{if } s < s_0^2, \\ H((s_0^2)^{\frac{1}{2}}) & \text{if } s > (s_0^2)^{\frac{1}{2}} \end{cases}$$

be a smooth decreasing function, where $s_0 \in (0, 1)$. It follows from the proof of Theorem 1 that there exists a solution $\psi$ to the problem

$$\begin{align*}
\text{div} \left( \left( \hat{H} \left( \frac{\nabla \psi}{r} \right) \right)^{-1} \frac{\nabla \psi}{r} \right) = 0, & \quad \text{in } \Omega,
\psi = 0, & \quad \text{on } T_1,
\psi = m, & \quad \text{on } T_2,
\end{align*}$$

for any $m > 0$. Moreover, $\psi$ satisfies

$$\psi \leq Cr^2. \quad (75)$$

If the wall of the nozzle tends to be flat at far fields, i.e., $f$ satisfies (15), then solutions to (74) approximate to uniform flows at far fields, as is described in the following lemma.

Lemma 5 Suppose that $f$ satisfies (12) and (15). Let $\psi$ be a solution to (74) and satisfy (75). Then for any $\epsilon > 0$, there exists a constant $L > 0$ such that

$$\left| \frac{\nabla \psi}{r}(x, r) - (0, 2m) \right| < \epsilon, \quad \text{if } x < -L,$$

and

$$\left| \frac{\nabla \psi}{r}(x, r) - (0, \frac{2m}{a^2}) \right| < \epsilon, \quad \text{if } x > L.$$

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Proof: We start with a special case. Assume that \( f(x) = a \) if \( x > L_0 \). Set \( \psi_k(x, r) = \psi(x + k, r) \chi_{\{(x, r)|x > -k + L_0 + 1, 0 < r < a\}} \). It follows from the standard H"older gradient estimates and Lemma \ref{lem:holder} that there exists \( \alpha_2 \in (0, 1) \) such that for any compact set \( K \subset (-\infty, \infty) \times [0, a] \), \( \|\psi_k\|_{C^{1,\alpha_2}(K)} \leq C(K) \) for \( k \) sufficiently large, where \( C(K) \) does not depend on \( k \). Therefore, by Arzela-Ascoli lemma, there exists a subsequence \( \psi_{k_l} \to \psi_0 \) in \( C^{1,\alpha_3}(K) \) with \( \alpha_3 < \alpha_2 \).

Moreover, \( \psi_0 \) solves the following boundary value problem
\[
\begin{cases}
\mathcal{L}\psi_0 = \text{div} \left( \hat{H}(|\nabla \psi_0|^2)^{-1} \nabla \psi_0 \right) = 0, & \text{in } E_0 = \{(x, r)| -\infty < x < \infty, 0 < r < a\}, \\
\psi_0(x, a) = m, & \text{if } -\infty < x < \infty, \\
\psi_0(x, 0) = 0, & \text{if } -\infty < x < \infty.
\end{cases}
\]

(76)

Furthermore, thanks to (75), \( \psi_0 \) satisfies the estimate
\[
\psi_0 \leq Cr^2 \quad \text{in } E_0.
\]

(77)

In fact, problem (76) and (77) has a unique solution
\[
\psi_0 = \frac{mr^2}{a^2},
\]

(78)

which follows from a simple comparison argument in \cite{1}.

Since the solution to problem (76) and (77) is unique, therefore, for \( K = [-2, 2] \times [h, a] \), then
\[
\left\| \frac{\nabla \psi_k}{r} - \frac{\nabla \psi_0}{r} \right\|_{C^{\mu([-2,2] \times [h,a])}} \to 0 \quad \text{for } \forall h > 0, \text{ for } \mu < \alpha_3.
\]

By the definition of \( \psi_k \) and (78), this is equivalent to
\[
\left\| \frac{\nabla \psi}{r} - (0, 2m/a^2) \right\|_{C^{\mu([-2,k+2] \times [h,a])}} \to 0 \quad \text{as } k \to \infty \quad \text{for } \forall h > 0 \text{ for } \mu < \alpha_3.
\]

(79)

In the general case that the wall of the nozzle is not flat at far fields, one can set \( \psi_k(x, r) = \psi(x + k, r) \chi_{\{(x, r)|x > -k + 1, 0 < r < f(x+k)\}} \). Then it follows from a similar analysis that \( \psi_{k_l} \to \psi_0 \) in \( C^{2,\alpha_3}(K) \) for any compact set \( K \subset (-\infty, \infty) \times (0, a) \), here \( K \) may not touch the boundary \( r = a \), but \( \psi_0 \) still satisfies the same boundary value problem (76) and estimate (77). Therefore,
\[
\left\| \frac{\nabla \psi}{r} - (0, 2m/a^2) \right\|_{C^{\mu([-2,k+2] \times [h,a-h])}} \to 0 \quad \text{as } k \to \infty \quad \text{for } \forall h > 0 \text{ for } \mu < \alpha_3.
\]

(79)
However, away from the symmetry axis, $\psi$ possesses Hölder gradient estimates, consequently, there exists $\alpha_4 > 0$ such that

$$\left[ \frac{\nabla \psi}{r} \right]_{C^{\alpha_4}(\{(x,r) | k-2 < x < k+2, r > a-h \})} \leq C. \quad (80)$$

Following the same argument in Section 3, one can show that there exists $\beta_2 \in (0, 1/2)$ such that

$$\left[ \frac{\nabla \psi}{r} \right]_{C^{\beta_2}((k-2,k+2) \times [0,h])} \leq C. \quad (81)$$

It now follows from estimates (79), (80) and (81) that the flow approximates to uniform flows at far fields. Indeed, by (80) and (81),

$$\left| \frac{\nabla \psi}{r}(x,r_1) - \frac{\nabla \psi}{r}(x,r_2) \right| \leq C(h_0^{\beta_2} + h_0^{\alpha_4}) \text{ if } r_1, r_2 > a - h_0 \text{ or } 0 < r_1, r_2 < h_0. \quad (82)$$

Thus, $\forall \varepsilon > 0$, it follows from (82) that there exists $\bar{h} > 0$ such that

$$\left| \frac{\nabla \psi}{r}(x,r_1) - \frac{\nabla \psi}{r}(x,r_2) \right| \leq \varepsilon, \quad \forall 0 \leq r_1, r_2 < \bar{h}. \quad (83)$$

and

$$\left| \frac{\nabla \psi}{r}(x,r_1) - \frac{\nabla \psi}{r}(x,r_2) \right| \leq \varepsilon, \quad \forall r_1, r_2 > a - \bar{h}. \quad (84)$$

On the other hand, there exists $L > 0$ such that

$$\left| \frac{\nabla \psi}{r}(x,r) - (0,2m/a^2) \right| \leq \varepsilon, \quad \frac{\bar{h}}{2} < r < a - \frac{\bar{h}}{2}, x > L. \quad (85)$$

Thus, combining (83), (84) and (85) yields

$$\left| \frac{\nabla \psi}{r}(x,r) - (0,2m/a^2) \right| \leq \varepsilon, \quad \forall x > L.$$

Similarly, we have

$$\left| \frac{\nabla \psi}{r}(x,r) - (0,2m) \right| \leq \varepsilon, \quad \forall x < -L.$$

This implies that the flow approximates to uniform flows at far fields. \hfill \square

With the help of Lemma 5, one can show the uniqueness of uniformly subsonic flows.
Lemma 6 Suppose that \( f \) satisfies (12) and (15), then uniformly subsonic flows to problem (34) are unique.

Proof: The proof is quite similar to the proof in [13]. Suppose there are two uniformly subsonic flows \( \psi_1 \) and \( \psi_2 \) which satisfy

\[
\left| \frac{\nabla \psi_1}{r} \right|, \left| \frac{\nabla \psi_2}{r} \right| \leq s_0 < 1.
\]

Since \( \psi_i \ (i = 1, 2) \) satisfies the equation

\[
\text{div} \left( \left( \hat{H} \left( \left| \frac{\nabla \psi_i}{r} \right|^2 \right) \right)^{-1} \frac{\nabla \psi_i}{r} \right) = 0
\]

where \( \hat{H} \) is defined in (73). It is easy to check that \( \bar{\psi} = \psi_1 - \psi_2 \) satisfies an equation of the form

\[
\hat{L} \bar{\psi} = a_{ij}(x, r) \partial_{ij} \bar{\psi} + b_i(x, r) \partial_i \bar{\psi} = 0.
\]

By Lemma 5, the flows corresponding to \( \psi_1 \) and \( \psi_2 \) approximate to same uniform flows at the far fields, therefore, for any \( \varepsilon > 0 \), there exists a \( L > 0 \) such that \( |\bar{\psi}(x, r)| < \varepsilon \) if \( |x| > L \). Thus by maximum principle, \( |\bar{\psi}| < \varepsilon, \forall \varepsilon > 0 \), since \( \bar{\psi} = 0 \) on \( T_1 \) and \( T_2 \). Since \( \varepsilon \) is arbitrary, so \( \bar{\psi} = 0 \).

This finishes the proof of the Lemma. \( \square \)

With the help of Lemma 5, Lemma 6, and going back to the original three dimensional flows, we can show in the same way as in [13] that there exists \( \hat{m} \) such that as \( m_0 \to \hat{m} \), \( M(m_0) \to 1 \). Furthermore, it follows from the comparison principle by Gilbarg[7] that as \( m_0 \uparrow \hat{m} \), \( M(m_0) \uparrow 1 \).

5 Properties of Subsonic Flows

In this section, as same as the case for plane flows, we will obtain some properties of subsonic axially symmetric flows, which are useful to show the existence of subsonic-sonic flows.
It follows from the proof of Lemma 4 that for axially symmetric subsonic flows, problem (11) and (34) are equivalent. Thus, in this section, we will use two descriptions simultaneously.

First of all, as for the plane flows in [13], the axial velocity is always positive.

Lemma 7 Suppose that $f$ satisfies (12) and (15). Let $\psi$ be a uniformly subsonic solution to problem (34), then

$$u > 0 \quad \text{in } \bar{D}.$$  

Proof: Since the flow is uniformly subsonic, one can assume that

$$\sup_{\Omega} \frac{\nabla \psi}{r} \leq s_0 < 1.$$  

Define $\hat{g}$ as in (37) with the help of $\hat{H}$ in (73), and $\Phi$ as in (65). Then $\Phi$ satisfies

$$\text{div}(\hat{g}(|\nabla \Phi|^2)\nabla \Phi) = a_{ij}(\nabla \Phi)\partial_{ij} \Phi = 0, \quad \text{in } D.$$  

Set $u = \Phi_x$. Then

$$a_{ij}(\nabla \Phi)\partial_{ij} u + D_{pk} a_{ij}(\nabla \Phi)\partial_{ij} \Phi \partial_k u = 0,$$

which is a uniform elliptic equation of $u$. Since $\psi = m$ on the solid boundary $r = f(x)$, therefore, $\psi_x + \psi_r f'(x) = 0$. On the other hand, $\psi$ attains its maximum on the solid boundary, therefore, by Hopf lemma, $\frac{\partial \psi}{\partial N} > 0$, where $\vec{N}$ is unit outward normal to the 2-D domain $\Omega$. Since

$$\frac{\partial \psi}{\partial N} = \psi_x (-f'(x)/\sqrt{1 + (f'(x))^2}) + \psi_r / \sqrt{1 + (f'(x))^2} = \psi_r \sqrt{1 + (f'(x))^2},$$

thus, $\psi_r > 0$. By the definition of $\varphi$ and $\Phi$, $\psi_r > 0$ is equivalent to $\Phi_x > 0$ on the three dimensional solid boundary $\{(x, r, z) | f(x) = \sqrt{y^2 + z^2}\}$. Since the flow approximates to uniform flows at far fields, moreover, $\Phi_x \rightarrow \{G^{-1}(4m^2/\alpha^2)\}^{1/2}$ as $x \rightarrow \infty$, and $\Phi_x \rightarrow \{G^{-1}(4m^2)\}^{1/2}$ as $x \rightarrow -\infty$, therefore, by the maximum principle

$$u > 0 \quad \text{in } \bar{D}.$$
Therefore, the proof of the Lemma is complete. □

Since the axial velocity is positive, then we can define the flow angle by

\[ \omega = \arctan \frac{V}{U}. \]  \hspace{1cm} (86)

Moreover, we have the following estimate on the flow angles.

**Lemma 8** Suppose that \( f \) satisfies (12) and (15). Let \( \psi \) be a uniformly subsonic solution to the problem (34), then the angle \( \omega \) defined by (86) satisfies

\[ \underline{\omega} \leq \omega \leq \bar{\omega}, \] \hspace{1cm} (87)

where

\[ \underline{\omega} = \min \{0, \inf_{x} \arctan f'(x)\}, \quad \bar{\omega} = \max \{0, \sup_{x} \arctan f'(x)\}. \]

**Proof:** The basic idea for the proof of the lemma is the same as that in [13], i.e., using hodograph transformation to obtain an elliptic equation for the angle, then the estimate (87) will be obtained by a comparison principle for elliptic equations. However, for axially symmetric flow, this procedure is more involved.

Let us first go back to the equations for axially symmetric flows (see [24] and [25]), which reads

\[
\begin{cases}
(rgU)_x + (rgV)_r = 0, \\
U_r - V_x = 0,
\end{cases}
\]

where \( U = \frac{\psi}{rg}, \quad V = -\frac{\varphi}{rg}, \quad g = g(q^2), \quad q = \sqrt{U^2 + V^2}, \) and \( g \) is defined by (5). Since \( \psi \) satisfies (34), so \( \varphi \) as in (62) is well-defined, moreover,

\[ J = \frac{\partial(\varphi, \psi)}{\partial(x, r)} = rg(q^2)q^2 \geq 0, \]

and which is strictly positive for \( r > 0 \). For \( r > 0 \), the mapping \( (x, r) \leftrightarrow (\varphi, \psi) \) is a local differmorphism. In fact, the mapping is globally invertible. Indeed, suppose that there are two points \( (x_1, r_1) \) and \( (x_2, r_2) \) such that \( \varphi(x_1, r_1) = \varphi(x_2, r_2) \) and \( \psi(x_1, r_1) = \psi(x_2, r_2) \). If \( \psi(x_1, r_1) = \psi(x_2, r_2) = 0 \), then it is obvious that \( x_1 = x_2 \) and \( r_1 = r_2 = 0 \) due to maximum
principle and Lemma 7. Let \( \psi(x_1, r_1) = \psi(x_2, r_2) = d > 0 \), then \((x_1, r_1)\) and \((x_2, r_2)\) are both on the streamline defined as follows

\[
\begin{align*}
\frac{dx}{ds} &= U(x, r), \\
\frac{dy}{ds} &= V(x, r), \\
x(0) &= x_1, r(0) = r_1.
\end{align*}
\]

Moreover, this streamline is uniformly away from the symmetry axis, then it follows from the argument in [13] that \(x_1 = x_2\) and \(r_1 = r_2\).

Now direct calculations show that

\[
r((g(q^2)U)_x + (g(q^2)V)_r) + g(q^2)V = r \left( qg(q^2)(1 - \frac{q^2}{c^2})q_\varphi + rg(q^2)q_\varphi \psi \right) + g(q^2)q \sin \omega,
\]

\[
U_r - V_x = -q^2 \omega \varphi + rg(q^2)q \psi,
\]

where \(c\) is the sound speed. Therefore

\[
\left( \frac{q}{rg(q^2)} \omega_\varphi \right)_\varphi + \left( \frac{rg(q^2)q}{1 - \frac{q^2}{c^2}} \omega_\psi \right)_\psi + \left( \frac{1}{r(1 - \frac{q^2}{c^2})} \sin \omega \right)_\psi = 0, \tag{88}
\]

which can be rewritten as

\[
\begin{align*}
\left( \frac{q}{rg(q^2)} \omega_\varphi \right)_\varphi + \left( \frac{rg(q^2)q}{1 - \frac{q^2}{c^2}} \omega_\psi \right)_\psi &+ \frac{1}{r} \frac{d}{dq}(1 - q^2/c^2) \frac{\sin \omega}{rg(q^2)} \omega_\varphi \\
+ \cos \omega &\frac{\omega_\psi}{r(1 - \frac{q^2}{c^2})} - \frac{\sin \omega}{r^2(1 - \frac{q^2}{c^2})} \psi = 0. \tag{89}
\end{align*}
\]

It follows from definitions of \(\varphi\) and \(\psi\) that

\[
r_\psi = \frac{\varphi_x}{\varphi_x \psi_r - \psi_x \varphi_r} = \frac{U}{rg(q^2)q^2}. \tag{90}
\]

Substituting (90) into (88) yields

\[
\begin{align*}
\left( \frac{q}{rg(q^2)} \omega_\varphi \right)_\varphi + \left( \frac{rg(q^2)q}{1 - \frac{q^2}{c^2}} \omega_\psi \right)_\psi &+ \frac{1}{r} \frac{d}{dq}(1 - q^2/c^2) \frac{\sin \omega}{rg(q^2)} \omega_\varphi \\
+ \cos \omega &\frac{\omega_\psi}{r(1 - \frac{q^2}{c^2})} = \frac{U}{r^3g(q^2)q^2(1 - \frac{q^2}{c^2})} \sin \omega = 0. \tag{91}
\end{align*}
\]

Since the flow is subsonic, \(1 - \frac{q^2}{c^2} > 0\), therefore, equation (91) is an elliptic equation. Note that, by Lemma 7, \(\omega \in (-\frac{\pi}{2}, \frac{\pi}{2})\). Moreover, in the domain \(\Omega^+ = \{\omega > 0\} \cap \Omega\), it follows
from Lemma 7 that $\omega$ satisfies that

$$Q\omega = \left(\frac{q}{rg(q^2)} \omega_\phi\right) + \left(\frac{rg(q^2)q}{1 - \frac{q^2}{c^2}} \omega_\psi\right) + \frac{1}{r dq} (1 - q^2/c^2) \sin \frac{\omega}{r} \frac{dg(q^2)}{dq} \omega_\phi + \frac{\cos \frac{\omega}{r}}{r(1 - \frac{q^2}{c^2})} \omega_\psi \geq 0.$$ 

By the maximum principle, Theorem 3.1 in [10],

$$\sup_{\Omega^+} \omega \leq \sup_{\omega \in \partial \Omega^+} \omega = \max\{\sup_{\partial \Omega} \omega, 0\}. \tag{92}$$

Similarly, in the domain $\Omega^- = \{\omega < 0\} \cap \Omega$, $\omega$ satisfies

$$Q\omega = \left(\frac{q}{rg(q^2)} \omega_\phi\right) + \left(\frac{rg(q^2)q}{1 - \frac{q^2}{c^2}} \omega_\psi\right) + \frac{1}{r dq} (1 - q^2/c^2) \sin \frac{\omega}{r} \frac{dg(q^2)}{dq} \omega_\phi + \frac{\cos \frac{\omega}{r}}{r(1 - \frac{q^2}{c^2})} \omega_\psi \leq 0,$$

by the maximum principle,

$$\inf_{\Omega^-} \omega \geq \inf_{\omega \in \partial \Omega^-} \omega = \min\{\inf_{\partial \Omega} \omega, 0\}. \tag{93}$$

Combining (92) and (93) together, we have

$$\min(\inf_{\partial \Omega} \omega, 0) \leq \omega \leq \max(\sup_{\partial \Omega} \omega, 0).$$

This finishes the proof of the Lemma. \hfill \Box

At the end of this section, we would like to study the relationship between flow speed and incoming mass flux.

**Lemma 9** (a) Let $0 < m_1 \leq m_2 < \hat{m}$. Suppose that $\psi_i$ are uniform subsonic solutions to (34) associated with the incoming mass flux $m_i (i = 1, 2)$. Then

$$|\nabla \psi_1(x, f(x))| \leq |\nabla \psi_2(x, f(x))|, \quad \forall x \in \mathbb{R}$$

\tag{94}

(b) Both the supremum of flow speed and infimum of horizontal velocity for uniformly subsonic solutions to (34) are monotone increasing with respect to the incoming mass flux. Moreover, for any given $\underline{m} \in (0, \hat{m})$, there exist a positive constant $\delta = \delta(\underline{m}) > 0$, such that if $m \in [\underline{m}, \hat{m})$, then

$$q(m) = \inf_{\Omega} \frac{|\nabla \psi|}{r} \geq \delta.$$
Proof:

(a) It is essentially proved in [7], however, regularity of solutions in our case near solid boundary is only $C^{1, \alpha}$, we can not use Hopf Lemma for derivative of solutions.

Set $\psi = \psi_2 - \psi_1$, then $\psi$ satisfies

$$A_{ij}(x,r)\partial_{ij}\psi + B_i(x,r)\partial_i\psi = 0,$$

where $A_{ij}, B_i \in L^\infty_{loc}(\Omega)$. Therefore $\psi$ achieves its maximum either on the solid boundary or at far fields. Note that $\psi$ tends to uniform flow at far fields, thus $\psi$ achieves its maximum on the nozzle wall. Therefore,

$$\frac{\partial \psi}{\partial n} \geq 0,$$

where $n$ is the unit outer normal of the nozzle wall. On the other hand, $\psi_i$ achieve their maximum on the whole nozzle wall where they are constants, so

$$\frac{\partial \psi_i}{\partial n} > 0,$$

and $\frac{\partial \psi_i}{\partial \tau} = 0$, on $T_2$, for $i = 1, 2$,

where $\tau$ is the tangential direction of $T_2$. Thus

$$|\nabla_{\tau} \psi_2|^2 \geq |\nabla_{\tau} \psi_1|^2.$$

(b) It follows from the proof of Lemma [7] for any $m \in (0, \hat{m})$, there exists $\delta = \delta(m)$ such that the corresponding solution $\psi$ with $m$ satisfies

$$\inf_{\Omega} \left| \frac{\partial \psi}{\partial r} \right| \geq \delta.$$

Furthermore, $\frac{\partial \psi}{\partial r}$ achieves its infimum either on the nozzle wall or at far fields.

Note that on the nozzle boundary

$$\frac{\partial \psi}{\partial n} = \frac{\partial \psi}{\partial r} \sqrt{1 + (f'(x))^2},$$

therefore, for any $m \in (m, \hat{m})$,

$$|\nabla \psi| \geq \frac{\partial \psi}{\partial r} \geq \inf \frac{\partial \psi}{\partial r} \geq \inf \frac{\partial \psi}{\partial r} \geq \delta.$$
Collecting all these lemmas together, we finish the proof of Theorem 2.

6 Subsonic-Sonic Flows

To take limit for $m_0 \to \hat{m}$, let us recall the compensated compactness framework in [13].

Theorem 10 Let $w^\varepsilon(x,r) = (q^\varepsilon, \omega^\varepsilon)(x,r)$ be a sequence of functions satisfying the Conditions (C):

(C.1) $0 < \delta \leq q^\varepsilon(x,r) \leq 1$ a.e. in $\Omega$ for some positive constant $\delta$.
(C.2) $|\omega^\varepsilon(x,r)| \leq \hat{\omega} < \frac{\pi}{2}$, for some constant $\hat{\omega}$ independent of $\varepsilon$.
(C.3) $\partial_x \eta_\pm(w^\varepsilon) + \partial_r \Lambda_\pm(w^\varepsilon)$ are confined in a compact set in $H^{-1}_{loc}(\Omega)$ for the momentum entropy-entropy flux pair

$$(\eta_+, \Lambda_+) = (\rho q^2 \cos^2 \omega + p, \rho q^2 \sin \omega \cos \omega), \quad (\eta_-, \Lambda_-) = (\rho q^2 \sin \omega \cos \omega, \rho q^2 \sin^2 \omega + p),$$

where $p = p(\rho)$, and $\rho = g(q^2)$ is determined by (9) through Bernoulli’s law.

Then there exists a subsequence $\{w^{\varepsilon_k}\}$ of $\{w^\varepsilon\}$ and $w(x,r) = (q, \omega)(x,r)$ such that

$$(q^{\varepsilon_k}, \omega^{\varepsilon_k}) \to (q, \omega), \quad (95)$$

$$q^{\varepsilon_k} \cos \omega^{\varepsilon_k} \rightarrow q \cos \omega, \quad q^{\varepsilon_k} \sin \omega^{\varepsilon_k} \rightarrow q \sin \omega, \quad (96)$$

$$g((q^{\varepsilon_k})^2)q^{\varepsilon_k} \cos \omega^{\varepsilon_k} \rightarrow g(q^2)q \cos \omega, \quad g((q^{\varepsilon_k})^2)q^{\varepsilon_k} \sin \omega^{\varepsilon_k} \rightarrow g(q^2)q \sin \omega, \quad (97)$$

where all the convergence in (95)-(97) are almost everywhere convergence, and $w = (q, \omega)$ satisfies

$$0 < \delta \leq q(x,r) \leq 1,$$

$$|\omega(x,r)| \leq \hat{\omega}.$$

Remark 6 The strong convergence of velocity field $(U,V) = (q \cos \omega, q \sin \omega)$ instead of $(q, \omega)$ was first proved in [4]. Since we have good control on flow speed, we can also get
strong convergence on flow angles. The difference between assumptions on Theorem 10 and Theorem 1 in [4] is that they use one more entropy-entropy flux pair instead of the condition on the lower bound on flow speed.

Let $\psi$ satisfies (34). Set, as before,

$$U = \frac{\psi_r}{r}, \quad V = -\frac{\psi_x}{r}, \quad q^2 = U^2 + V^2, \quad \rho = g(q^2).$$

By direct calculations, it is easy to see that $(\rho, U, V)$ satisfies

$$\begin{cases}
\left(\rho U^2 + p(\rho)\right)_x + (\rho UV)_r = -\frac{\rho UV}{r}, \\
(\rho UV)_x + (\rho V^2 + p(\rho))_r = -\frac{\rho V^2}{r}.
\end{cases}$$

Let $m^\varepsilon \to \hat{m}$, and $(q^\varepsilon, \omega^\varepsilon)$ be the solutions to (34) corresponding to $m^0_0$, then away from the symmetry axis, on any compact subset in $\Omega$, $(-\frac{g((q^\varepsilon)^2)U^\varepsilon V^\varepsilon}{r}, -\frac{g((q^\varepsilon)^2)(V^\varepsilon)^2}{r})$ are uniformly bounded, therefore, precompact in $H^{-1}_{loc}$. Furthermore, due to Lemma 8, the associated flow angles satisfy the estimates

$$-\frac{\pi}{2} < \min\left(\inf_{x} \arctan f'(x), 0\right) \leq \omega^\varepsilon \leq \max\left(\sup_{x} \arctan f'(x), 0\right) < \frac{\pi}{2}.$$

Therefore, conditions (C.1), (C.2) and (C.3) in Theorem 10 are all satisfied. Hence it follows from Theorem 10 that $(U^\varepsilon, V^\varepsilon)$ has weak-* limit $(U, V)$ such that

$$\begin{cases}
(rg(q^2)U)_x + (rg(q^2)V)_r = 0, \\
U_r - V_x = 0
\end{cases} \quad (98)$$

holds in $\Omega$ in the sense of distribution, where $q^2 = U^2 + V^2$, and $g$ is defined in (5).

We now verify that $(\rho, U, V)$ gives rise to a global subsonic-sonic weak solution to (11) on $D$. Thus, define

$$\rho(x, y, z) = g(q^2)(x, r), \quad u(x, y, z) = U(x, r), \quad v(x, y, z) = V(x, r) \frac{y}{r}, \quad w(x, y, z) = V(x, r) \frac{z}{r}.$$
where \( r = \sqrt{y^2 + z^2} \). First, note that for \( \eta \in C^\infty_0(D_0) \),

\[
\begin{align*}
&\iint_D u\eta_y - v\eta_x \, dx \, dy \, dz \\
&= \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^\infty f(x) \left( U(x, r)(\eta_y(x, r, \theta) \cos \theta - \eta_x(x, r, \theta) \frac{\sin \theta}{r}) \right) r \, dr \, dx \, d\theta \\
&\quad - \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^\infty f(x) V(x, r) \cos \theta \eta_x(x, r, \theta) r \, dr \, dx \, d\theta \\
&= \int_{-\infty}^{\infty} \int_0^\infty f(x) \left( r U(x, r) \eta_y \cos \theta - V \eta_x \cos \theta \right) r \, dr \, dx \, d\theta \\
&\quad - \int_{-\infty}^{\infty} \int_0^\infty \left( r U(x, r) \eta_y \cos \theta - V \eta_x \cos \theta \right) r \, dr \, dx \, d\theta \\
&= 0,
\end{align*}
\]

where we have used the facts that

\[
\int_0^{2\pi} r \eta(x, r, \theta) \cos \theta \, d\theta \in C^\infty_0(\Omega)
\]

and (98) holds in the sense of distribution in \( \Omega \).

Define \( \zeta = \zeta(s) \) such that

\[
\zeta(s) = 1 \text{ if } s > 1, \quad \zeta(s) = 0, \text{ if } s < \frac{1}{4}, \quad |\zeta'\zeta| < 2, \text{ and } \zeta \in C^\infty(\mathbb{R}),
\]

and set \( \zeta_\delta(s) = \zeta(\frac{s}{\delta}) \), where \( 0 < \delta < b \).

Then for any \( \eta \in C^\infty_0(D) \), \( \zeta_\delta(r)\eta \in C^\infty_0(D_0) \), thus

\[
\begin{align*}
&\iint_D u\eta_y - v\eta_x \, dx \, dy \, dz \\
&= \int_D u(\zeta_\delta(r)\eta)_y - v(\zeta_\delta(r)\eta)_x \, dx \, dy \, dz \\
&\quad + \int_D u((1 - \zeta_\delta(r))\eta)_y - v((1 - \zeta_\delta(r))\eta)_x \, dx \, dy \, dz \\
&= \int_D u((1 - \zeta_\delta(r))\eta)_y - v((1 - \zeta_\delta(r))\eta)_x \, dx \, dy \, dz \\
&= \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^\delta \left( U(1 - \zeta_\delta(r))\eta + U(1 - \zeta_\delta(r))\eta - V \cos \theta(1 - \zeta_\delta(r))\eta_x \right) r \, dr \, dx \, d\theta \\
&\to 0
\end{align*}
\]

as \( \delta \to 0 \). Thus for any \( \eta \in C^\infty_0(D) \)

\[
\iint_D u\eta_y - v\eta_x \, dx \, dy \, dz = 0.
\]
Similarly, one can show that
\[ \iiint_D u \eta_z - w \eta_x dxdydz = 0, \]
and
\[ \iiint_D v \eta_z - w \eta_y dxdydz = 0, \]
for any \( \eta \in C_0^\infty(D) \).

Finally, we check the continuity equation. For any \( \eta \in C^\infty_c(D) \),
\[
\begin{align*}
\iiint_D \rho u \eta_x + \rho v \eta_y + \rho w \eta_z dxdydz \\
= \iiint_D \rho u (\zeta_\delta(r) \eta)_x + \rho v (\zeta_\delta(r) \eta)_y + \rho w (\zeta_\delta(r) \eta)_z dxdydz \\
+ \iiint_D \rho u (1 - \zeta_\delta(r)) \eta_x + \rho v ((1 - \zeta_\delta(r)) \eta)_y + \rho w ((1 - \zeta_\delta(r)) \eta)_z dxdydz
\end{align*}
\]
Note that (34) shows that
\[
\begin{align*}
\iiint_D \rho^\varepsilon u^\varepsilon (\zeta_\delta \eta)_x + \rho^\varepsilon v^\varepsilon (\zeta_\delta \eta)_y + \rho^\varepsilon w^\varepsilon (\zeta_\delta \eta)_z dxdydz = 0,
\end{align*}
\]
while (97) yields
\[
\begin{align*}
\lim_{\varepsilon \to 0} \iiint_D \zeta_\delta(f(x) - r) \left( \rho^\varepsilon u^\varepsilon (\zeta_\delta \eta)_x + \rho^\varepsilon v^\varepsilon (\zeta_\delta \eta)_y + \rho^\varepsilon w^\varepsilon (\zeta_\delta \eta)_z \right) dxdydz \\
= \iiint_D \zeta_\delta(f(x) - r) \left( \rho u (\zeta_\delta(r) \eta)_x + \rho v (\zeta_\delta(r) \eta)_y + \rho w (\zeta_\delta(r) \eta)_z \right) dxdydz,
\end{align*}
\]
where \( (\rho^\varepsilon, u^\varepsilon, v^\varepsilon, w^\varepsilon) \) denotes the three dimensional flow associated with the incoming mass flux \( m^\varepsilon \). Therefore,
\[
\begin{align*}
\iiint_D \rho u (\zeta_\delta(r) \eta)_x + \rho v (\zeta_\delta(r) \eta)_y + \rho w (\zeta_\delta(r) \eta)_z dxdydz \\
= \iiint_D (1 - \zeta_\delta(f(x) - r)) \left( \rho u (\zeta_\delta(r) \eta)_x + \rho v (\zeta_\delta(r) \eta)_y + \rho w (\zeta_\delta(r) \eta)_z \right) dxdydz \\
+ \iiint_D \zeta_\delta(f(x) - r) \left( \rho u (\zeta_\delta(r) \eta)_x + \rho v (\zeta_\delta(r) \eta)_y + \rho w (\zeta_\delta(r) \eta)_z \right) dxdydz \\
= \iiint_D (1 - \zeta_\delta(f(x) - r)) \left( \rho u (\zeta_\delta(r) \eta)_x + \rho v (\zeta_\delta(r) \eta)_y + \rho w (\zeta_\delta(r) \eta)_z \right) dxdydz \\
+ \lim_{\varepsilon \to 0} \iiint_D \zeta_\delta(f(x) - r) \left( \rho^\varepsilon u^\varepsilon (\zeta_\delta \eta)_x + \rho^\varepsilon v^\varepsilon (\zeta_\delta \eta)_y + \rho^\varepsilon w^\varepsilon (\zeta_\delta \eta)_z \right) dxdydz
\end{align*}
\]
\[
\begin{align*}
  &\int\int\int_D (1 - \zeta_\delta(f(x) - r))(\rho u(\zeta_\delta(r))\eta_x + \rho v(\zeta_\delta(r))\eta_y + \rho w(\zeta_\delta(r))\eta_z) \, dxdydz \\
  &+ \lim_{\varepsilon \to 0} \int\int\int_D (\rho^\varepsilon u^\varepsilon(\zeta_\delta\eta_x) + \rho^\varepsilon v^\varepsilon(\zeta_\delta\eta_y) + \rho^\varepsilon w^\varepsilon(\zeta_\delta\eta_z)) \, dxdydz \\
  &+ \lim_{\varepsilon \to 0} \int\int\int_D (\zeta_\delta(f(x) - r) - 1)(\rho^\varepsilon u^\varepsilon(\zeta_\delta\eta_x) + \rho^\varepsilon v^\varepsilon(\zeta_\delta\eta_y) + \rho^\varepsilon w^\varepsilon(\zeta_\delta\eta_z)) \, dxdydz \\
  = &\int\int\int_D (1 - \zeta_\delta(f(x) - r))(\rho u(\zeta_\delta(r))\eta_x + \rho v(\zeta_\delta(r))\eta_y + \rho w(\zeta_\delta(r))\eta_z) \, dxdydz \\
  &+ \lim_{\varepsilon \to 0} \int\int\int_D (\zeta_\delta(f(x) - r) - 1)(\rho^\varepsilon u^\varepsilon(\zeta_\delta\eta_x) + \rho^\varepsilon v^\varepsilon(\zeta_\delta\eta_y) + \rho^\varepsilon w^\varepsilon(\zeta_\delta\eta_z)) \, dxdydz.
\end{align*}
\]

Thus
\[
\int\int\int_D \rho u\eta_x + \rho v\eta_y + \rho w\eta_z \, dxdydz
\]
\[
= \int\int\int_D \rho u((1 - \zeta_\delta(r))\eta_x + \rho v((1 - \zeta_\delta(r))\eta_y + \rho w((1 - \zeta_\delta(r))\eta_z) \, dxdydz \\
  + \int\int\int_D (1 - \zeta_\delta(f(x) - r))(\rho u(\zeta_\delta(r))\eta_x + \rho v(\zeta_\delta(r))\eta_y + \rho w(\zeta_\delta(r))\eta_z) \, dxdydz \\
  + \lim_{\varepsilon \to 0} \int\int\int_D (\zeta_\delta(f(x) - r) - 1)(\rho^\varepsilon u^\varepsilon(\zeta_\delta\eta_x) + \rho^\varepsilon v^\varepsilon(\zeta_\delta\eta_y) + \rho^\varepsilon w^\varepsilon(\zeta_\delta\eta_z)) \, dxdydz \\
  \to 0
\]

as \(\delta \to 0\). Therefore, for any test function \(\eta \in C_c^\infty(\bar{D})\),
\[
\int\int\int_D \rho u\eta_x + \rho v\eta_y + \rho w\eta_z \, dxdydz = 0. \tag{99}
\]

Moreover, the equation \(99\) also implies that \((u, v, w)\) satisfies the boundary condition \(9\) actually as the normal trace of divergence measure field \((\rho u, \rho v, \rho w)\) on the boundary in the sense of Anzellotti\cite{2}. So we finish the proof of Theorem 3.

Further characterizations of the subsonic-sonic flow we obtained are left for future.

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