Spatiotemporal localizations of light in quadratically nonlinear media with transversal inhomogeneity

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Abstract. In the current study with the help of mathematical modeling we investigate the generation of second optical harmonic provided various group velocity dispersion (GVD) and third order dispersion effects taken into account. Varying values and signs of the third-order dispersion at quasi-zero values of GVD we reveal the conditions of light bullet formation and stable propagation. To this end we use a generalization of the well-known system of the second harmonic generation in quasi-optical approach. Particular attention is paid to the case when GVD at the second harmonic is close to zero. In addition, we investigate these cases using both focusing and defocusing waveguide which means the presence of transversal inhomogeneity.

1. Introduction
Multicomponent multidimensional solitons at quadratic nonlinearity have been intensively studied for several decades [1, 2, 3]. Nowadays light bullets are promising in various applications, for instance, in ultrafast optics and highly precise interferometry. Recently with the help of the averaged Lagrangian method we have developed a detailed theory of “breathing” two-component light bullets propagating in a microdispersive medium with anomalous GVD [4] and in the waveguide with dispersion of arbitrary sign [5]. These analytical results were obtained under the assumption of group- and phase velocity synchronism and a certain ratio between GVD coefficients of two harmonics. If the pulse frequency is close to zero GVD frequency, a part of pulse spectrum lies in the region of positive dispersion and the rest is in negative one. Recently a possibility of the forming of a stable spatiotemporal soliton was demonstrated near zero GVD at the second harmonic [6]. In the present work using mathematical modeling we study second harmonic generation (SHG) at various GVD and third order dispersion effects. We demonstrate light bullet formation and stable propagation at certain combinations of GVD and third-order dispersion. We show also that a localization of the pulse may occur when special conditions connecting characteristic propagation lengths are satisfied.

2. Mathematical model and numerical method
In “slowly varying envelope approximation” we write down dimensionless equation system for SHG in a focusing waveguide in planar case at group velocity and phase matching.

\[
\frac{i}{2} \frac{\partial A_1}{\partial z} = -D_{q1} x^2 A_1 - D_{r21} \frac{\partial^2 A_1}{\partial \tau^2} + i D_{r31} \frac{\partial^3 A_1}{\partial \tau^3} + A_1^* A_2 + D_x \frac{\partial^2 A_1}{\partial x^2},
\]

(1)
the illustration of “breathing” soliton taken from [4] is given in the upper row of Figure 1. Lower row of Figure 1 illustrates this case. For comparison reduce the computational domain in scheme realized with FFT algorithm for the problem (1)-(2). To accelerate calculations we especially important when modeling a soliton formation and propagation. In [1, 9, 10] the problems of nonlinear optics. In our study we construct an original pseudo-spectral difference conservativeness and effectiveness of Fast Fourier Transform (FFT) methods are proved for the do not preserve motion integrals, in particular, such integrals as (4). But conservativeness is thus, numerical methods used for mathematical modeling must be conservative, namely, to holding difference analogue of (3). Splitting technique is widely applied to such problems [7] 8. Its advantage is in the reduction of computational time. At the same time splitting methods do not preserve motion integrals, in particular, such integrals as [4]. But conservativeness is especially important when modeling a soliton formation and propagation. In [1] [9] [10] the conservativeness and effectiveness of Fast Fourier Transform (FFT) methods are proved for the problems of nonlinear optics. In our study we construct an original pseudo-spectral difference scheme realized with FFT algorithm for the problem (1)-(2). To accelerate calculations we reduce the computational domain in x and τ directions and apply absorbing boundary conditions embedding an artificial absorption in (1)-(2) [11]. In computations we launch the initial pulse at both frequencies \( A_1(z = 0) = \exp(-x^2 - \tau^2) \), \( A_2(z = 0) = 0.5 \exp(-x^2 - \tau^2) \).

3. Numerical results
For a definite ratio between GVD coefficients of the fundamental and second harmonics it was demonstrated earlier that such propagation has a “breathing” character and is accompanied by regular in-phase oscillations of the peak intensities at both harmonics [4] [5] [6]. Oscillations of spatial and temporal pulse widths also have in-phase behavior, which is in antiphase for intensity oscillations. Regime of classic parametric solitons means energy exchange absence between the fundamental and second harmonics. This regime is known as a reactive one, and an optimal relation between phases of interacting waves \( \Phi = 2\phi_1 - \phi_2 \) is its characteristic sign [4].

In the present study we use the above mentioned features for spatiotemporal localizations detection. Our major task is to reveal regions of spatiotemporal soliton and quasi-soliton robust propagation for arbitrary dispersion. Our numerical simulation is divided into two parts. Firstly, we deal with a microdispersive medium and then we proceed with an inhomogenous medium.

Considering a microdispersive medium [4], we firstly remove GVD and third-order dispersion from the second harmonic. Third-order dispersion at the fundamental frequency is also assumed to be zero. The ratio between dispersion and diffraction is varied in a range from \( l_D = 0.24l_{dis} \) to \( l_D = 0.55l_{dis} \). \( l_D \) and \( l_{dis} \) are characteristic lengths at which initial spatial and temporal pulse widths increase in \( \sqrt{2} \) times. The best results correspond to \( l_D = 0.24l_{dis} \) that was predicted analytically in [3]. Lower row of Figure 1 illustrates this case. For comparison the illustration of “breathing” soliton taken from [4] is given in the upper row of Figure 1. Dependencies of the peak intensities of both harmonics and generalized phase on z demonstrate

\[
\frac{i}{D} \frac{\partial A_2}{\partial z} = -D_{q2} x^2 A_2 - D_{c2} \frac{\partial^2 A_2}{\partial \tau^2} + i D_{c3} \frac{\partial^3 A_2}{\partial \tau^3} + \eta A_1^2 + D_{x2} \frac{\partial^2 A_2}{\partial x^2}.
\]
that the regime of robust light bullet propagation is established with the stabilization of energy exchange just for a few nonlinear lengths. This propagation is accompanied by the regular in-phase oscillations of peak intensities and generalized phase at both harmonics. Quite close to the optimal ratio \( l_D = 0.24l_{dis} \) one should observe approximately the same pattern. At the same time, a significant violation of this ratio leads to the absence of light bullet or quasi-light bullet regime. Reliability of our computations is confirmed by the preserving of the motion integrals (3)-(4). Illustration corresponding to the lower row of Figure 1 is given in Figure 2.

Further in a microdispersive medium we vary the ratio between the values of GVD of two harmonics. We return to the initial position with zero second harmonic GVD and third-order dispersion at both frequencies. Then we add a relatively small GVD coefficient to the second harmonic and study the cases of anomalous (negative) and normal (positive) GVD. Provided relatively small second harmonic GVD coefficient of arbitrary sign and the ratio \( l_D = 0.24l_{dis} \) one should observe quasi-light bullet regime. We gradually increase second harmonic GVD coefficient and conclude that the robust light bullet or quasi-light bullet regime cannot be observed anymore at \( l_D = 0.24l_{dis} \). To reach the desired regime we vary the ratio between diffraction and dispersion. When GVD coefficient of the second harmonic is equal to the doubled GVD coefficient of the fundamental wave new optimal ratio between diffraction and dispersion is approximately the same as it was predicted theoretically in [4]: \( l_D = 0.55l_{dis} \). Upper row of Figure 1 illustrates this case. Dependencies of the peak intensities of both harmonics and generalized phase on the propagation coordinate \( z \) prove that the regime of robust light bullet propagation is established. Further we gradually reduce the GVD coefficient of the fundamental harmonic. We observe that small reduction actually does not influence bullet robust propagation. At the same time, a significant reduction leads to the complete termination of bullet or quasi-bullet regime. Therefore, one may conclude that dispersion at the fundamental frequency plays a dominant role in a two-component light bullet formation.

![Figure 1. Two-color light bullet with anomalous group velocity dispersion in a microdispersive medium. Peak intensities of the fundamental (black lines) and second (red lines) harmonics along the propagation coordinate \( z \) (left), generalized phase \( \Phi = 2\phi_1 - \phi_2 \) along \( z \) (right). Upper row: \( 2D_{\tau 21} = D_{\tau 22}, l_D = 0.55l_{dis} \) [4], lower row: \( D_{\tau 22} = 0, l_D = 0.24l_{dis} \) [6].](image)

When considering ultrashort pulses, it is necessary to take into account the influence of third-order dispersion. Let us add small third-order dispersion to the fundamental and second harmonics \( D_{\tau 31}, D_{\tau 32} \neq 0 \) for the case illustrated in Figure 1 (lower row) and consider various
values of the ratio between $D_{\tau 21}$ and $D_{\tau 32}$. The amplitude profiles show that with an increase in $D_{\tau 32}$, a positive drift occurs and the group velocity changes. In this case, a soliton solution is still observed, but only up to $z_{nl} = 500$ for $D_{\tau 32} = 0.01$ ($z_{nl} = 200$ for $D_{\tau 32} = 0.05$ and $z_{nl} = 150$ for $D_{\tau 32} = 0.1$). $z_{nl}$ is the characteristic length at which nonlinear effects manifest themselves.

We widen the range of GVD frequencies, considering a focusing waveguide at normal second-order dispersion. We repeat all the steps made in the previous cases in the range of soliton stability [5], but now we also vary signs of the fundamental harmonic GVD coefficient. Our modeling proves that a balance between normal dispersion and focusing waveguide forms a stable two-component bullet.

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