EXISTENCE AND UNIQUENESS OF VERY WEAK SOLUTION OF THE MHD TYPE SYSTEM

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Abstract. This paper studies the very weak solution to the steady MHD type system in a bounded domain. We prove the existence of very weak solutions to the MHD type system for arbitrary large external forces \((f, g)\) in \(L^r(\Omega) \times [X_{\theta', q'}(\Omega)]'\) and suitable boundary data \((B_0, U_0)\) in \(W^{-1/p, p}(\partial\Omega) \times W^{-1/q, q}(\partial\Omega)\), under certain assumptions on \(p, q, r, \theta\). The uniqueness of very weak solution for small data \((f, g, B_0, U_0)\) is also studied.

1. Introduction. In this paper, we consider a system of Magnetohydrodynamic type (MHD type) which has been used in the study of magnetic properties of electrically conducting fluids with applications in the study of Geophysics and Astrophysics. In stationary state and where there is free motion of heavy ions, this model can be reduce to the form (see for instance [23, 32])

\[
\begin{cases}
\text{curl} \left( \frac{1}{\mu \sigma} \text{curl} \mathbf{B} + \mathbf{B} \times \mathbf{u} + \mathbf{f} \right) + \nabla w = 0 & \text{in } \Omega, \\
-\frac{\eta}{\rho} \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \pi - \frac{\mu}{\rho} (\mathbf{B} \cdot \nabla) \mathbf{B} = \mathbf{g} & \text{in } \Omega, \\
\text{div} \mathbf{B} = 0, \quad \text{div} \mathbf{u} = 0 & \text{in } \Omega, \\
\mathbf{B} = \mathbf{B}_0, \quad \mathbf{u} = \mathbf{U}_0 & \text{on } \partial \Omega,
\end{cases}
\]

(1)

where \(\rho, \sigma, \eta, \mu > 0\) are constants and \(\mathbf{B}\) is the magnetic field, \(\mathbf{u}\) is the fluid velocity, \(p\) is the hydrostatic pressure and \(w\) is a function related to the motion of heavy ions. The given vector fields \(\text{curl} \mathbf{f}\) and \(\mathbf{g}\) are external forces on the magnetically charged fluid flows.

There are many studies on the time-dependent system of the above system (1). For example, existence and uniqueness of strong solution for small data in bounded or unbounded domains are studied in [29, 35, 27]. The temporal and spatial decay of the solution are studied in [26, 27]. For more studies on the time-dependent system of (1), we refer to [2, 25] and references therein.

When there is no potential term \(\nabla w\) in the first equation, this model reduce to the usual MHD system. The time-dependent MHD system has been extensively studied by many mathematicians. For the existence of weak and strong solution with smooth enough data, we refer to [13, 15, 31] and references therein. For

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the regularity criteria see for example [11, 22] and references therein. The time-independent MHD system is also well studied. For instance, existence of weak solutions to the steady MHD system has been studied in [1, 21] by applying the Lax-Milgram theorem to some continuous forms on the Sobolev spaces. Regularity of the weak solutions follows from a standard bootstrap argument.

In this paper, we study the existence and uniqueness of very weak solution to the MHD type system (1) with singular datum. This is motivated by the works on the existence and uniqueness of very weak solutions to the stationary Navier-Stokes system, especially by the seminal papers of Kim [24] and Amrouche et al. [8]. For more studies of very weak solution to Navier-Stokes system, see for instance [4, 5, 6, 7, 17, 20, 28, 33] and references therein. See [10] for the study of non-linear Stokes equations with singular forcing.

We mention here that Villamizar-Roa et al. [32] studied this model in two dimensional space. They considered the Dirichlet boundary condition \((u, B) = (u_0, B_0)\) on \(\partial \Omega\) with \(u_0, B_0 \in L^2(\partial \Omega)\) and proved the existence of very weak solution for small \(\mu \sigma\) and the uniqueness for large \(\eta\) of small \(\rho\). Different from their result, we consider three dimensional space and more general boundary datum \(B^0 \in W^{-1/p,p}(\partial \Omega)\) and \(U^0 \in W^{-1/q,q}(\partial \Omega)\) for some \(3 \leq p \leq q < \infty\). Moreover, we shall show the existence of very weak solution for arbitrary external forces and suitable boundary datum.

To formulate our result, we introduce some notations and functional spaces for vector fields. For any functional space \(X\), we denote \(X'\) by its dual space. The notation \(X(\text{div}0, \Omega)\) denotes the space of divergence free vector fields in \(X(\Omega)\). For any constant \(p \in (1, \infty)\), we set \(p' = \frac{p}{p-1}\). Mainly, we follow the notations of Amrouche et al. [8]. For any \(1 < r, p < +\infty\), we define

\[
\mathcal{D}(\Omega) = C_0^\infty(\Omega, \mathbb{R}^3), \quad \mathcal{D}(\Omega) = C_0^\infty(\text{div}0, \Omega), \\
X_{r',p'}(\Omega) = \{u \in W^{1,r'}_0(\Omega, \mathbb{R}^3), \text{div } u \in W^{1,p'}_0(\Omega)\}, \\
T_{p,r}(\Omega) = \{u \in L^p(\Omega, \mathbb{R}^3), \Delta u \in [X_{r',p'}(\Omega)]', \text{div } u = 0\}, \\
T_{p,r,\sigma}(\Omega) = \{u \in T_{p,r}(\Omega), \text{div } u = 0\}, \\
Y_{p'}(\Omega) = \{u \in W^{2,p'}(\Omega, \mathbb{R}^3), u = 0\text{ and } \frac{\partial u}{\partial \nu} = 0\text{ on } \partial \Omega\}.
\]

Note that \(Y_{p'}(\Omega)\) can also be characterized [3] by

\[
Y_{p'}(\Omega) = \left\{ u \in W^{2,p'}(\Omega, \mathbb{R}^3), u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \right\},
\]

where \(\nu\) denotes the unit outer normal vector. Moreover, \(Y_{p'}(\Omega) \subset X_{r',p'}(\Omega)\) if

\[
\frac{1}{r} \leq \frac{1}{p} + \frac{1}{3}
\]

Amrouche et al. [8, Lemma 12] showed that the map \(\gamma_r : u \mapsto u|_{\partial \Omega} = u - (u \cdot \nu)\nu\) can be extended to a linear continuous map from \(T_{p,r}(\Omega)\) to \(W^{-1/p,p}(\partial \Omega)\), and we have the Green formula:

\[
\langle \Delta v, \phi \rangle_{[X_{r',p'}(\Omega)]' \times X_{r',p'}(\Omega)} = \int_{\Omega} v \cdot \Delta \phi \, dx - \left\langle v_{T}, \frac{\partial \phi}{\partial \nu} \right\rangle_{W^{-1/p,p}(\partial \Omega) \times W^{1/p,p'}(\partial \Omega)},
\]

for all \(\phi \in Y_{p'}(\Omega)\). With the help of the above Green formula, we can now give the definition of very weak solutions to the MHD type system (1) with datum

\[
f \in L^r(\Omega, \mathbb{R}^3), \quad g \in [X_{q',q'}(\Omega)]', \\
B^0 \in W^{-1/p,p}(\partial \Omega, \mathbb{R}^3), \quad U^0 \in W^{-1/q,q}(\partial \Omega, \mathbb{R}^3),
\]

(3)
satisfying the compatible condition
\[ \langle B^0, \nu, 1 \rangle_{W^{-1/p, p} (\partial \Omega) \times W^{-1/q, q} (\partial \Omega)} = 0, \quad \langle U^0, \nu, 1 \rangle_{W^{-1/p, q} (\partial \Omega) \times W^{-1/q, q} (\partial \Omega)} = 0. \] \tag{4}

For simplicity, throughout this paper, we assume that \( \rho = \mu = \sigma = \eta = 1. \)

**Definition 1.1.** Assume that \( f, g, B^0, U^0 \) satisfy (3) and the compatible condition (4) with \( 3 \leq p < \infty. \) We say
\[ (B, u, w, \pi) \in L^p(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathbb{R}^3) \times W^{-1, p}(\Omega) \times W^{-1, q}(\Omega) \]
is a very weak solution to (1) provided that
\[
\int_{\Omega} B \cdot (-\Delta \phi) \, dx + \int_{\Omega} B \times u \cdot \text{curl} \phi \, dx + \int_{\Omega} f \cdot \text{curl} \phi \, dx
\]
and
\[
- \int_{\Omega} [u \cdot \nabla \phi] + (u \cdot \nabla) \phi \cdot u - (B \cdot \nabla) \phi \cdot B \, dx - \langle \pi, \text{div} \, \psi \rangle_{W^{-1, p}(\Omega) \times W^{1, q}(\Omega)} = 0,
\]
for all \( \phi \in Y_{p'}(\Omega), \psi \in Y_{q'}(\Omega) \) and \( \zeta \in W^{1, p'}(\Omega), \xi \in W^{1, q'}(\Omega), \) where the dualities on \( \Omega \) and \( \partial \Omega \) are defined as follows:
\[
\langle \cdot, \cdot \rangle_{\Omega ; p, r} = \langle \cdot, [X_{p, r}^{*}(\Omega)]^{*} \rangle_{X_{p, r}^{*}(\Omega)}, \quad \langle \cdot, \cdot \rangle_{\partial \Omega ; p} = \langle \cdot, [W^{-1/p, r}(\partial \Omega) \times W^{1/q, r}(\partial \Omega)]^{*} \rangle_{W^{-1/p, r}(\partial \Omega) \times W^{1/q, r}(\partial \Omega)}.
\]

Note that the Sobolev embedding gives \( W^{1, p'}(\Omega) \hookrightarrow L^p(\Omega) \) and \( W^{1, q'}(\Omega) \hookrightarrow L^q(\Omega) \) with \( \frac{1}{p} = \frac{2}{3} - \frac{1}{p}, \) and \( \frac{1}{q} = \frac{2}{3} - \frac{1}{q}. \) On the other hand, we have
\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{p} = \frac{2}{3} + \frac{1}{q} \leq 1.
\]

By the Hölder’s inequality we see that \( \int_{\Omega} B \times u \cdot \text{curl} \phi \, dx \) is well-defined. Similarly, the other integrations in the above definition are also well-defined.

**Remark 1.** Since \( \phi = 0 \) on \( \partial \Omega, \) for all \( B, u \in C^\infty(\text{div} \, 0, \Omega), \) we have
\[
\int_{\Omega} B \times u \cdot \text{curl} \phi \, dx = \int_{\Omega} \text{curl}(B \times u) \cdot \phi \, dx
\]
\[
= \int_{\Omega} (u \cdot \nabla) B \cdot \phi - (B \cdot \nabla) u \cdot \phi \, dx
\]
\[
= \int_{\Omega} [\text{div} ((B \cdot \phi) u) - (u \cdot \nabla) \phi \cdot B - \text{div} ((u \cdot \phi) B) + (B \cdot \nabla) \phi \cdot u] \, dx
\]
\[
= \int_{\Omega} [(B \cdot \nabla) \phi \cdot u - (u \cdot \nabla) \phi \cdot B \, dx + \int_{\partial \Omega} (B \cdot \phi)(u \cdot \nu) - (u \cdot \phi)(B \cdot \nu) \, d\sigma
\]
\[
= \int_{\Omega} [(B \cdot \nabla) \phi \cdot u - (u \cdot \nabla) \phi \cdot B] \, dx.
\]
By the density of $C^\infty(\text{div}0,\Omega)$ in $L^p(\text{div}0,\Omega)$ (see Lemma 7 in [8]), we see that

$$\int_\Omega \mathbf{B} \times \mathbf{u} \cdot \text{curl} \phi \, dx = \int_\Omega [(\mathbf{B} \cdot \nabla)\phi \cdot \mathbf{u} - (\mathbf{u} \cdot \nabla)\phi \cdot \mathbf{B}] \, dx$$  \hspace{1cm} (5)

holds for all $(\mathbf{B}, \mathbf{u})$ in $L^p(\text{div}0,\Omega) \times L^q(\text{div}0,\Omega)$. This identity will be used in proving Proposition 2.

Throughout this paper, $\Omega \subset \mathbb{R}^3$ always denotes a bounded connected $C^2$ domain, which is needed when applying the $L^q$ theory for the Stokes equations (see [12] or [9, Theorem IV.6.6, p.303]). For convenience, we set

$$\|f\|_{L^q(\Omega)} + \|g\|_{W^{1,p}(\partial \Omega)} = \|\mathbf{B}^0\|_{W^{-1/p,p}(\partial \Omega)} + \|\mathcal{U}^0\|_{W^{-1/q,q}(\partial \Omega)}.$$  \hspace{1cm} (6)

2. Main results.

**Proposition 1.** Let $f, g, \mathcal{B}^0, \mathcal{U}^0$ satisfy (3) and the compatible condition (4). Assume that

$$3 \leq q \leq p < +\infty, \quad r \leq p;$$

$$\frac{1}{p} + \frac{1}{q} \leq 1 < \frac{1}{r} + \frac{1}{3}, \quad \frac{1}{q} \leq \frac{1}{r} \leq \frac{1}{q} + \frac{1}{3}. \hspace{1cm} (6)$$

There exists a constant $\delta_0 = \delta_0(p, q, r, \theta, \Omega) > 0$ such that if

$$\|(f, g, \mathcal{B}^0, \mathcal{U}^0)\| \leq \delta_0,$$  \hspace{1cm} (7)

then (1) admits a unique very weak solution

$$(\mathbf{B}, \mathbf{u}, w, \pi) \in L^p(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathbb{R}^3) \times W^{-1,p}(\Omega) \times W^{-1,q}(\Omega)$$

such that

$$\|\mathbf{B}\|_{L^p(\Omega)} + \|\mathbf{u}\|_{L^q(\Omega)} + \|w\|_{W^{-1,p}(\Omega)} + \|\pi\|_{W^{-1,q}(\Omega)} \leq C\|(f, g, \mathcal{B}^0, \mathcal{U}^0)\|,$$  \hspace{1cm} (8)

where $C$ depends only on the constant $C_1$ appears in Lemma 3.2.

Note that the last condition in (6) implies that $\theta \leq \frac{q}{2}$. This proposition shows only the uniqueness of “small solution”. Generally, if we have another very weak solution, we do not know whether it coincides with the small solution or not. However, under some additional assumptions, we can show that any weak solution coincides with the small solution.

**Proposition 2.** Let $f, g, \mathcal{B}^0, \mathcal{U}^0$ satisfy (3) and the compatible condition (4). Assume that $3 \leq q = p \leq 6$ or $3 < q < p \leq 6$ and $r, \theta$ satisfy (6). There exists a positive constant $\delta_1 = \delta_1(p, q, r, \theta, \Omega)$ such that if

$$\|(f, g, \mathcal{B}^0, \mathcal{U}^0)\| \leq \delta_1,$$  \hspace{1cm} (9)

problem (1) admits a unique very weak solution

$$(\mathbf{B}, \mathbf{u}, w, \pi) \in L^p(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathbb{R}^3) \times W^{-1,p}(\Omega) \times W^{-1,q}(\Omega).$$

For arbitrary data $(f, g)$ and suitable $(\mathcal{B}^0, \mathcal{U}^0)$, we have the following existence result.

**Proposition 3.** Let $f, g, \mathcal{B}^0, \mathcal{U}^0$ satisfy (3) and the compatible condition (4). Suppose that $p, q, r, \theta$ satisfy (6). Assume either $3 < q \leq p < +\infty$ or $3 = q \leq p \leq 6$. Then the following results hold:

(i) If $\Omega$ is simply-connected, then there exists at least one very weak solution

$$(\mathbf{B}, \mathbf{u}, w, \pi) \in L^p(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathbb{R}^3) \times W^{-1,p}(\Omega) \times W^{-1,q}(\Omega)$$

to (1);
(ii) If $\Omega$ is multi-connected, then there exists a positive constant $\delta_2$ which depends only on $\Omega$ such that (1) admits at least one very weak solution if

$$
\sum_{i=1}^{m} (|\langle B^i \cdot \nu, 1 \rangle_{\Gamma_i, p}| + |\langle U_0 \cdot \nu, 1 \rangle_{\Gamma_i, q}|) \leq \delta_2,
$$

where $\Gamma_i$ denotes the connected components of the boundary $\partial \Omega$ of the open set $\Omega$, $m$ denotes the number of the connected components of the boundary $\partial \Omega$.

**Remark 2.** Our result generalizes the work of Amrouche et al. [8, Theorem 4, Theorem 5] on Navier-Stokes equations. In [8], the authors proved the existence of very weak solution $(u, \pi) \in L^3(\Omega) \times W^{-1,3}(\Omega)$ under the assumptions that large external force $g \in [X_{3,3/2}(\Omega)]'$ and suitable boundary data $U_0 \in W^{-1/3,3}(\partial \Omega)$. Also they proved such solution belongs to $L^p(\Omega) \times W^{-1,q}(\Omega)$ if $g \in [X_{\theta,\eta}(\Omega)]'$ and $U_0 \in W^{-1/\eta,\eta}(\partial \Omega)$ with $\frac{1}{\eta} \leq \frac{1}{p} + \frac{1}{q}$ and $\max(\theta, 3) \leq q$. When considering (1), condition (6) is a natural generalization of such conditions.

3. Preliminaries. To prove our main results, we need some auxiliary lemmas. The first lemma is a variant of the Gerhardt inequality [18]. See also [24, Lemma 1].

**Lemma 3.1.** Suppose that $u \in L^3(\Omega)$, $v \in W^{1,s}(\Omega)$ and $1 \leq s < 3$.

(i) We have $uv \in L^s(\Omega)$ and

$$
\|uv\|_{L^s(\Omega)} \leq C \|u\|_{L^3(\Omega)} \|v\|_{W^{1,s}(\Omega)}
$$

for some positive constant $C = C(s, t, \Omega)$.

(ii) For any constant $\varepsilon > 0$, there exists a constant $C_\varepsilon = C(\varepsilon, s, \Omega, \|u\|_{L^3(\Omega)}) > 0$ such that

$$
\|uv\|_{L^s(\Omega)} \leq \varepsilon \|v\|_{W^{1,s}(\Omega)} + C_\varepsilon \|u\|_{L^s(\Omega)}.
$$

The second lemma is a result of Amrouche [8] which concerns the Stokes equation, we quote it here for convenience of the reader.

**Lemma 3.2.** [8, Theorem 11] Let $h \in [X_{r',\eta}(\Omega)]'$ and $u^0 \in W^{-1/p, p}(\partial \Omega)$. Assume that $1/r \leq 1/p + 1/3$, $r \leq p$ and $\langle u^0 \cdot \nu, 1 \rangle_{\partial \Omega, p} = 0$. Then the following problem

$$
\begin{cases}
-\Delta u + \nabla P = h, & \text{div } u = 0 \quad \text{in } \Omega, \\
u = u^0 & \text{on } \partial \Omega,
\end{cases}
$$

admits a unique very weak solution $u \in L^p(\text{div } 0, \Omega)$ and $P \in W^{-1/p}(\Omega)/\mathbb{R}$ such that

$$
\|u\|_{L^p(\Omega)} + \|P\|_{W^{-1/p}(\Omega)/\mathbb{R}} \leq C_1(\|h\|_{X_{r',\eta}(\Omega)}') + \|u^0\|_{W^{-1/p, p}(\partial \Omega)}.
$$

The following lemma is needed in proving Proposition 2.

**Lemma 3.3.** Assume that $3 \leq q = p \leq 6$ or $3 < q < p \leq 6$. Given $B_1, B_2 \in T_{p,r,\sigma}(\Omega)$ and $u_1, u_2 \in T_{q,\theta,\sigma}(\Omega)$. There exists a constant $\delta_2 = \delta_2(\Omega, p, q) > 0$ such that if $(B_1, u_1, u_2)$ satisfies

$$
\|B_1\|_{L^p(\Omega)} + \|u_1\|_{L^q(\Omega)} + \|u_2 \cdot \nu\|_{W^{-1/q,\sigma}(\partial \Omega)} \leq \delta_2,
$$

where $\sigma$ denotes the connected components of the boundary $\partial \Omega$. Also they proved such solution belongs to $L^p(\Omega) \times W^{-1,q}(\Omega)$ if $g \in [X_{\theta,\eta}(\Omega)]'$ and $U_0 \in W^{-1/\eta,\eta}(\partial \Omega)$ with $\frac{1}{\eta} \leq \frac{1}{p} + \frac{1}{q}$ and $\max(\theta, 3) \leq q$. When considering (1), condition (6) is a natural generalization of such conditions.
then for any \((w, \eta) \in L^{p'}(\Omega, \mathbb{R}^3) \times L^{q'}(\Omega, \mathbb{R}^3),\) the problem

\[
\begin{align*}
-\Delta H - (u_2 \cdot \nabla)H + (\nabla H)u_1 + (\nabla \xi)B_1 + (B_2 \cdot \nabla)\xi + \nabla \Pi_1 = w & \quad \text{in } \Omega, \\
-\Delta \xi - (\nabla \xi)u_1 - (u_2 \cdot \nabla)\xi - (\nabla H)B_1 + (B_2 \cdot \nabla)H + \nabla \Pi_2 = \eta & \quad \text{in } \Omega, \\
\text{div } H = \text{div } \xi = 0 & \quad \text{in } \Omega, \\
H = 0, \quad \xi = 0 & \quad \text{on } \partial \Omega, \\
\end{align*}
\]

admits a unique solution

\((H, \xi, \Pi_1, \Pi_2) \in W^{2,p'}(\text{div } 0, \Omega) \times W^{2,q'}(\text{div } 0, \Omega) \times W^{1,p'}(\Omega)/\mathbb{R} \times W^{1,q'}(\Omega)/\mathbb{R}.

Here \((\nabla H)u_1\) is understood as \((u_1 \cdot \partial_1 H, u_1 \cdot \partial_2 H, u_1 \cdot \partial_3 H)\) for vector fields \(H\) and \(u_1\) in \(\mathbb{R}^3\), the other terms are understand in the same way.

**Proof.** We shall first make use of the Leray-Schauder fixed point theorem to show the existence of unique solution

\[(H, \xi, \Pi_1, \Pi_2) \in W^{2,p'}(\text{div } 0, \Omega) \times W^{2,q'}(\text{div } 0, \Omega) \times W^{1,p'}(\Omega)/\mathbb{R} \times W^{1,q'}(\Omega)/\mathbb{R}
\]

to the problem (14), and then prove that \((\xi, \Pi_2) \in W^{2,q'}(\text{div } 0, \Omega) \times W^{1,q'}(\Omega)/\mathbb{R}\) by using the \(L^q\) theory for the Stokes equations.

Define

\[
\mathcal{X} = W^{2,p'}(\text{div } 0, \Omega) \times W^{2,q'}(\text{div } 0, \Omega), \quad \mathcal{Y} = W^{1,p'}(\Omega)/\mathbb{R} \times W^{1,q'}(\Omega)/\mathbb{R}.
\]

Fix \((H, \xi) \in \mathcal{X}, \) since \(p' \leq q' \leq \frac{3}{2} < 3 \leq q \leq p\), by using Lemma 3.1, it is easily checked that

\[
- (u_2 \cdot \nabla)H + (\nabla H)u_1 + (\nabla \xi)B_1 + (B_2 \cdot \nabla)\xi \in L^{p'}(\Omega, \mathbb{R}^3),
\]

\[
- (\nabla \xi)u_1 - (u_2 \cdot \nabla)\xi - (\nabla H)B_1 + (B_2 \cdot \nabla)H \in L^{p'}(\Omega, \mathbb{R}^3).
\]

Therefore, the \(L^q\) theory for the Stokes system then ensures a unique \((\tilde{H}, \tilde{\xi}) \in \mathcal{X}\) and \((\tilde{P}_1, \tilde{P}_2) \in \mathcal{Y}\) such that

\[
\begin{align*}
-\Delta \tilde{H} + \nabla \tilde{P}_1 = w + (u_2 \cdot \nabla)H - (\nabla H)u_1 - (\nabla \xi)B_1 - (B_2 \cdot \nabla)\xi & \quad \text{in } \Omega, \\
-\Delta \tilde{\xi} + \nabla \tilde{P}_2 = \eta + (\nabla \xi)u_1 + (u_2 \cdot \nabla)\xi + (\nabla H)B_1 - (B_2 \cdot \nabla)H & \quad \text{in } \Omega, \\
\text{div } \tilde{H} = \text{div } \tilde{\xi} = 0 & \quad \text{in } \Omega, \\
\tilde{H} = 0, \quad \tilde{\xi} = 0 & \quad \text{on } \partial \Omega.
\end{align*}
\]

Denotes \((\tilde{H}, \tilde{\xi}) = \mathfrak{F}(H, \xi),\) the above shows that \(\mathfrak{F}\) maps \(\mathcal{X}\) into itself.

**Step 1.** We show that \(\mathfrak{F}\) is compact. Let \((H_k, \xi_k) \to (H, \xi)\) weakly in \(\mathcal{X},\) and set \((\tilde{H}_k, \tilde{\xi}_k) = \mathfrak{F}(H_k, \xi_k).\) We have

\[
\begin{align*}
||\tilde{H}_k - H||_{W^{2,p'}(\Omega)} & \leq C \|(u_2 \cdot \nabla)(H_k - H) - [\nabla (H_k - H)]u_1 \\
& \quad - (\nabla (\xi_k - \xi)) B_1 - (B_2 \cdot \nabla)(\xi_k - \xi)||_{L^{p'}(\Omega)}. \quad (15)
\end{align*}
\]

Using Lemma 3.1 again, we get

\[
||\tilde{H}_k - H||_{W^{2,p'}(\Omega)} \leq \varepsilon(||H_k - H||_{W^{2,p'}(\Omega)} + ||\xi_k - \xi||_{W^{2,p'}(\Omega)}) \quad (16)
\]

and similarly

\[
||\tilde{\xi}_k - \xi||_{W^{2,q'}(\Omega)} \leq \varepsilon(||H_k - H||_{W^{2,q'}(\Omega)} + ||\xi_k - \xi||_{W^{2,q'}(\Omega)}) \quad (17)
\]

\[
||\tilde{\xi}_k - \xi||_{W^{1,p'}(\Omega)} \leq \varepsilon(||H_k - H||_{W^{1,p'}(\Omega)} + ||\xi_k - \xi||_{W^{1,p'}(\Omega)}) \quad (18)
\]

\[
||\tilde{\xi}_k - \xi||_{W^{1,q'}(\Omega)} \leq \varepsilon(||H_k - H||_{W^{1,q'}(\Omega)} + ||\xi_k - \xi||_{W^{1,q'}(\Omega)})
\]

\[
||\tilde{\xi}_k - \xi||_{W^{1,q'}(\Omega)} \leq \varepsilon(||H_k - H||_{W^{2,q'}(\Omega)} + ||\xi_k - \xi||_{W^{2,q'}(\Omega)})
\]

\[
||\tilde{\xi}_k - \xi||_{W^{1,q'}(\Omega)} \leq \varepsilon(||H_k - H||_{W^{1,q'}(\Omega)} + ||\xi_k - \xi||_{W^{1,q'}(\Omega)})
\]
for any $\varepsilon > 0$, where $C_\varepsilon = C(\varepsilon, p, q, \Omega, \|B_i\|_{L^p(\Omega)}), \|u_i\|_{L^q(\Omega)})$, $i = 1, 2$, which is independent of $k$. Since $(H_k, \xi_k) \to (H, \xi)$ weakly in $X$, we have that $\{(H_k, \xi_k)\}_{k=1}^\infty$ is bounded in $X$ and $(H_k, \xi_k) \to (H, \xi)$ strongly in $W^{1,p'}(\Omega) \times W^{1,p'}(\Omega)$. Letting $k \to +\infty$ and then $\varepsilon \to 0$ in (17) and (18), we deduce that $(H_k, \xi_k) \to (H, \xi)$ strongly in $X$, which shows that $\mathfrak{F}$ is compact.

**Step 2.** We show that

$$(H, \xi) = \lambda \mathfrak{F}(H, \xi), \quad (H, \xi) \in X, \quad \lambda \in [0, 1] \Rightarrow \|(H, \xi)\|_X \leq C\|(w, \eta)\|_{L^p(\Omega) \times L^{p'}(\Omega)}$$

for some constant $C = C(p, q, \|B_i\|_{L^p(\Omega)}, \|u_i\|_{L^q(\Omega)}, \Omega)$, $i = 1, 2$. Suppose that $(H, \xi) = \lambda \mathfrak{F}(H, \xi), \quad (H, \xi) \in X, \quad \lambda \in [0, 1]$. Then $(H, \xi) = \mathfrak{F}(H, \xi)$ satisfies

\[
\begin{cases}
-\Delta H + \nabla P_1 = w + \lambda[(u_2 \cdot \nabla)H - (\nabla H)u_1] - (\nabla \xi)B_1 - (B_2 \cdot \nabla)\xi & \text{in } \Omega, \\
-\Delta \xi + \nabla P_2 = \eta + \lambda[(\nabla \xi)u_1 + (u_2 \cdot \nabla)\xi - (\nabla H)B_1 - (B_2 \cdot \nabla)H] & \text{in } \Omega, \\
\text{div } H = \text{div } \xi = 0 & \text{in } \Omega, \\
H = 0, \quad \xi = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Using Lemma 3.1 again, we find that

\[
\|H\|_{W^{2,p'}(\Omega)} \leq C\|w + \lambda[(u_2 \cdot \nabla)H - (\nabla H)u_1] - (\nabla \xi)B_1 - (B_2 \cdot \nabla)\xi\|_{L^{p'}(\Omega)}
\]

\[
\leq \frac{1}{4}(\|H\|_{W^{2,p'}(\Omega)} + \|\xi\|_{W^{2,p'}(\Omega)}) + C(\|w\|_{L^p(\Omega)} + \|H\|_{W^{1,p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)}),
\]

and

\[
\|\xi\|_{W^{2,p'}(\Omega)} \leq C\|\eta + \lambda[(\nabla \xi)u_1 + (u_2 \cdot \nabla)\xi - (\nabla H)B_1 - (B_2 \cdot \nabla)H]\|_{L^{p'}(\Omega)}
\]

\[
\leq \frac{1}{4}(\|H\|_{W^{2,p'}(\Omega)} + \|\xi\|_{W^{2,p'}(\Omega)}) + C(\|\eta\|_{L^p(\Omega)} + \|H\|_{W^{1,p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)}),
\]

for some constant $C = C(p, q, \|B_i\|_{L^p(\Omega)}, \|u_i\|_{L^q(\Omega)}, \Omega)$, $i = 1, 2$. Thus, we get

\[
\|H\|_{W^{2,p'}(\Omega)} + \|\xi\|_{W^{2,p'}(\Omega)} \leq C(\|w\|_{L^p(\Omega)} + \|\eta\|_{L^p(\Omega)} + \|H\|_{W^{1,p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)}).
\]

Recall the interpolation inequality (see for instance [19, Theorem 7.28, p. 173])

\[
\|H\|_{W^{1,p'}(\Omega)} + \|\xi\|_{W^{1,p'}(\Omega)} \leq \frac{1}{4}(\|H\|_{W^{2,p'}(\Omega)} + \|\xi\|_{W^{2,p'}(\Omega)}) + C(\Omega)(\|H\|_{L^p(\Omega)} + \|\xi\|_{L^{p'}(\Omega)}),
\]

we then have

\[
\|H\|_{W^{2,p'}(\Omega)} + \|\xi\|_{W^{2,p'}(\Omega)} \leq C(\|w\|_{L^p(\Omega)} + \|\eta\|_{L^p(\Omega)} + \|H\|_{L^p(\Omega)} + \|\xi\|_{L^{p'}(\Omega)}). \tag{20}
\]

We claim that

\[
\|H\|_{L^p(\Omega)} + \|\xi\|_{L^{p'}(\Omega)} \leq C(\Omega, p, q)(\|w\|_{L^p(\Omega)} + \|\eta\|_{L^p(\Omega)}) + \frac{1}{2} \left(\|H\|_{W^{2,p'}(\Omega)} + \|\xi\|_{W^{2,p'}(\Omega)}\right) \tag{21}
\]

when $\delta_0$ small enough. In fact, since $3 \leq p \leq 6$, we have $6/5 \leq p' \leq 3/2$, thus we have the following embedding relations

\[
W^{2,p'} \hookrightarrow L^{\frac{6}{5}}, \quad W^{2,p'} \hookrightarrow W^{1,\frac{3}{2}}, \tag{22}
\]
Thus, we have 
\[
\nabla \bar{u} \cdot \bar{H} - \lambda(\nabla \bar{H})u_1 \cdot \bar{H} - \lambda(\nabla \bar{H})B_1 \cdot \bar{H} dx
\]
(24)

Similarly, we have 
\[
\int_\Omega |\nabla \xi|^2 dx = \int_\Omega \{\eta + \lambda(\nabla \xi)u_1 + (u_2 \cdot \nabla)\xi + (\nabla \xi)B_1 - (B_2 \cdot \nabla)\bar{H}\} \cdot \xi dx
\]
(25)

On the other hand, we observe that \((\nabla \bar{H})\bar{H} \in L^{q'}(\Omega)\). In fact, if \(p = 3\), then \((\nabla \bar{H})\bar{H} \in L^{p^2}(\Omega)\) for all \(p^2 < 3\), particularly, \((\nabla \bar{H})\bar{H} \in L^{q'}(\Omega)\). If \(3 < p \leq 6\), by the previous embedding relation (22), we have \((\nabla \bar{H})\bar{H} \in L^{p^2}(\Omega)\) with
\[
\frac{1}{p^2} = \frac{p-3}{3p} + \frac{2p-3}{3p} = 1 - \frac{2}{p} \leq 1 - \frac{1}{q} = \frac{1}{q'}.
\]

Thus, we have \((\nabla \bar{H})\bar{H} \in L^{q'}(\Omega)\). Therefore, the Sobolev embedding theorem and Hölder inequality imply
\[
\|\bar{H}\|_{W^{1,q'}(\Omega)} \leq C(\Omega, p, q) \|\bar{H}\|_{W^{2,p'}(\Omega)}.
\]
(26)

Thus, we have 
\[
\int_\Omega (u_2 \cdot \nabla)\bar{H} \cdot \bar{H} dx = \frac{1}{2} \int_\Omega u_2 \cdot \nabla(|\bar{H}|^2) dx = \frac{1}{2} \langle u_2 \cdot \nu, |\bar{H}|^2 \rangle_{W^{-1,q,\nu}(\partial\Omega) \times W^{1,q'}(\partial\Omega)},
\]
hence,
\[
\int_\Omega (u_2 \cdot \nabla)\bar{H} \cdot \bar{H} dx \leq \|u_2 \cdot \nu\|_{W^{-1,q,\nu}(\partial\Omega)} \|\bar{H}\|_{W^{1,q'}(\Omega)}.
\]
(27)

Combining (24), (25), (26), (28) and the fact that 
\[
\|\bar{H}\|_{H^1(\Omega)}^2 + \|\xi\|_{H^1(\Omega)}^2 \leq C(\Omega)(\|\nabla \bar{H}\|_{L^2(\Omega)}^2 + \|\nabla \xi\|_{L^2(\Omega)}^2),
\]
we find that
\[
\|\bar{H}\|_{H^1(\Omega)}^2 + \|\xi\|_{H^1(\Omega)}^2 \leq C(\Omega, p, q) \left\{ \|w\|_{L^{p'}(\Omega)} \|\bar{H}\|_{L^p(\Omega)} + \|u_1\|_{L^p(\Omega)} \|\nabla \bar{H}\|_{L^2(\Omega)} \|\bar{H}\|_{L^\infty(\Omega)}
\right.
\]
\[
+ \|B\|_{L^p(\Omega)} \|\nabla \xi\|_{L^2(\Omega)} \|\bar{H}\|_{L^p(\Omega)} + \|u_2 \cdot \nu\|_{W^{-1,q,\nu}(\partial\Omega)} \|\bar{H}\|_{W^{1,q'}(\Omega)}
\right.
\]
\[
+ \|\eta\|_{L^{p'}(\Omega)} \|\xi\|_{L^p(\Omega)} + \|u_1\|_{L^p(\Omega)} \|\nabla \xi\|_{L^2(\Omega)} \|\xi\|_{L^\infty(\Omega)}
\right\}
\]
Thus, we estimate (29).

Concluding the above two steps, the Leray-Schauder fixed point theorem then gives the existence of \((\tilde{\mathbf{H}}, \tilde{\xi}) \in \mathbb{X}\) to (14). The existence of \(\Pi_1\) and \(\Pi_2\) then follows from the de Rham theorem. The uniqueness follows immediately from the estimate (29).
Step 3. We show that \((\xi, \Pi_2) \in W^{2,q'}(\text{div } 0, \Omega) \times W^{1,q'}(\Omega)/\mathbb{R}\). If \(p = q\), we are done. We now consider the case of \(3 < q < p \leq 6\). Recall the embedding relation \((22)\), we see that \(\nabla \xi, \nabla H \in L^p(\Omega)\). Therefore, 
\[
(\nabla H)B_1, (B_2 \cdot \nabla)H \in L^{3/2}(\Omega) \hookrightarrow L^{q'}(\Omega).
\]
On the other hand, we find that \((\nabla \xi)u_1, (u_2 \cdot \nabla)\xi \in L^s(\Omega)\), where \(s\) satisfies 
\[
\frac{1}{s} = \frac{1}{q} + \frac{1}{p} = \frac{1}{q} + \frac{2}{3} - \frac{1}{p}.
\]
Therefore, the \(L^q\) theory of the Stokes equations gives \(\xi \in W^{2,s}(\text{div } 0, \Omega)\). If \(s \geq q'\), namely, \(2/q - 1/p \leq 1/3\), we are done. Otherwise, we repeat the previous argument. From the embedding relation 
\[
W^{2,s} \hookrightarrow W^{1,t_1}, \quad \frac{1}{t_1} = \frac{1}{s} - \frac{1}{3} = \frac{1}{q} - \frac{1}{p} + \frac{1}{3},
\]
we have \((\nabla \xi)u_1, (u_2 \cdot \nabla)\xi \in L^{s_1}(\Omega)\), where 
\[
\frac{1}{s_1} = \frac{1}{q} + \frac{1}{t_1} = \frac{2}{q} - \frac{1}{p} + \frac{1}{3},
\]
thus \(\xi \in W^{2,s_1}(\text{div } 0, \Omega)\). If \(s_1 \geq q'\), namely, \(3/q - 1/p \leq 2/3\), we are done. Otherwise, repeat the previous argument again. It is easily checked that after \(k\) times repeat, we formally have \(\xi \in W^{2,s_k}(\text{div } 0, \Omega)\), where \(\frac{1}{s_k} = \frac{k+1}{q} - \frac{1}{p} + \frac{2}{3}k\). Therefore, at most repeat 
\[
k = \left\lfloor \frac{1}{q} - \frac{1}{p} \right\rfloor \frac{3}{2} - \frac{1}{q}
\]
times, we will see that 
\[
\frac{1}{s_k} - \frac{1}{q'} = (k + 1) \left(\frac{1}{q} - \frac{1}{3}\right) + \frac{1}{q} - \frac{1}{p} \leq 0,
\]
that is, \(s_k \geq q'\). Therefore, \(\xi \in W^{2,q'}(\text{div } 0, \Omega)\). In the same way we find that \(\Pi_2 \in W^{1,q'}(\Omega)/\mathbb{R}\). We are done. \(\square\)

4. Proof of Proposition 1. Now we are ready to prove Proposition 1.

Proof of Proposition 1. We shall use a fixed point argument.

Step 1. Define \(\mathfrak{M} = \{(B, u) \in L^p(\text{div } 0, \Omega) \times L^q(\text{div } 0, \Omega) : ||B||_{L^p(\Omega)} + ||u||_{L^q(\Omega)} \leq \delta_3\}\). For \((B, u) \in \mathfrak{M}\), denotes \((D, v) = \mathfrak{T}(B, u)\) by the unique very weak solution to the following problem: 
\[
\begin{cases}
\text{curl (curl } D + B \times u + f) + \nabla w_0 = 0 & \text{in } \Omega, \\
\Delta v + \nabla \pi_0 = g + (B \cdot \nabla)B - (u \cdot \nabla)u & \text{in } \Omega, \\
\text{div } D = \text{div } v = 0 & \text{in } \Omega, \\
D = B^0, & v = U^0 & \text{on } \partial \Omega.
\end{cases}
\]
(30)

We shall show that when \(||(f, g, B^0, U^0)|| \leq \delta_0\), and \(\delta_0\) and \(\delta_3\) both small enough, the map \(\mathfrak{T}\) defined above then maps \(\mathfrak{M}\) into itself.

We first show that \(\mathfrak{T}\) is well defined. Let us define \(\tilde{f} = -\text{curl } (B \times u + f)\). Recall that \(B \in L^p(\text{div } 0, \Omega), u \in L^q(\text{div } 0, \Omega)\) and \(f \in L^r(\Omega, \mathbb{R}^3)\). On the other hand, for
any \( \phi \in X_{r',p'}(\Omega) \), we have \( \text{curl} \phi \in L^{r'}(\Omega, \mathbb{R}^3) \). Thus, we have \((B \times u + f) \cdot \text{curl} \phi \in L^1(\Omega)\) by noting that
\[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r'} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1.
\]
Therefore, for the given \( \hat{f} \) and any \( \phi \in X_{r',p'}(\Omega) \),
\[
\langle \hat{f}, \phi \rangle = \int_{\Omega} (B \times u + f) \cdot \text{curl} \phi \, dx
\]
defines a bounded linear map on \( X_{r',p'}(\Omega) \), which implies that \( \hat{f} = -\text{curl}(B \times u + f) \in [X_{r',p'}(\Omega)]' \). Moreover,
\[
\| \hat{f} \|_{X_{r',p'}(\Omega)'} = \sup_{\phi \in X_{r',p'}(\Omega)} \frac{\| \langle \hat{f}, \phi \rangle \|}{\| \phi \|_{X_{r',p'}(\Omega)}} \leq \frac{\| B \times u + f \|_{L^r(\Omega)} \| \text{curl} \phi \|_{L^{r'}(\Omega)}}{\| \phi \|_{W^{1,r'}(\Omega)}}
\leq \| B \times u + f \|_{L^r(\Omega)} \| f \|_{L^r(\Omega)} + \| B \|_{L^p(\Omega)} \| u \|_{L^q(\Omega)} |\Omega|^{\frac{1}{p} - \frac{1}{p} - \frac{1}{q}}.
\]
Using Lemma 3.2, there exists a unique \((D, w_0)\) which solves the equation of \( D \) in (30) such that
\[
\| D \|_{L^p(\Omega)} + \| w_0 \|_{W^{-1,p}(\Omega)} \leq C_1(\| \hat{f} \|_{X_{r',p'}(\Omega)'} + \| B \|_{W^{-1,p}(\Omega)})
\leq C_1(\| f \|_{L^r(\Omega)} + \| B \|_{L^p(\Omega)} \| u \|_{L^q(\Omega)} |\Omega|^{\frac{1}{p} - \frac{1}{p} - \frac{1}{q}} (31)
+ \| B \|_{W^{-1,p}(\Omega)})
\]
Similarly, we have
\[
\hat{g} = g + (B \cdot \nabla)B - (u \cdot \nabla)u \in [X_{r',q'}(\Omega)]'
\]
with
\[
\| \hat{g} \|_{X_{r',q'}(\Omega)'} \leq \| g \|_{X_{r',q'}(\Omega)'} + |\Omega|^{\frac{1}{q} - \frac{1}{q}} \| B \|_{L^p(\Omega)}^2 + |\Omega|^{\frac{1}{q} - \frac{1}{q}} \| v \|_{L^{q'}(\Omega)}^2.
\]
Thus, by Lemma 3.2, there exists a unique \((v, \pi_0)\) which solves the equation of \( v \) in (30) such that
\[
\| v \|_{L^q(\Omega)} + \| \pi_0 \|_{W^{-1,q}(\Omega)} \leq C_1(\| \hat{g} \|_{X_{r',q'}(\Omega)'} + \| \Omega \|_{W^{-1,q}(\Omega)}
\leq C_1(\| g \|_{X_{r',q'}(\Omega)'} + |\Omega|^{\frac{1}{q} - \frac{1}{q}} \| B \|_{L^p(\Omega)}^2 + |\Omega|^{\frac{1}{q} - \frac{1}{q}} \| v \|_{L^{q'}(\Omega)}^2 (32)
\]
This shows that \( \mathcal{F} \) is well defined. Combining (31) and (32), we see that
\[
\| \mathcal{F}(B, u) \|_{L^p(\Omega) \times L^{q'}(\Omega)} + \| w_0 \|_{W^{-1,p}(\Omega)} + \| \pi_0 \|_{W^{-1,q}(\Omega)}
= \| D \|_{L^p(\Omega)} + \| v \|_{L^{q'}(\Omega)} + \| \pi_0 \|_{W^{-1,q}(\Omega)} + \| w_0 \|_{W^{-1,p}(\Omega)}
\leq C_1(\| f \|_{L^r(\Omega)} + \| g \|_{X_{r',p'}(\Omega)'} + \| B \|_{W^{-1,p}(\Omega)} + |\Omega|^{\frac{1}{p} - \frac{1}{p}} \| B \|_{L^p(\Omega)}^2 + |\Omega|^{\frac{1}{q} - \frac{1}{q}} \| v \|_{L^{q'}(\Omega)}^2 (33)
+ \| B \|_{L^p(\Omega)} \| u \|_{L^q(\Omega)} |\Omega|^{\frac{1}{p} - \frac{1}{p} - \frac{1}{q}} + |\Omega|^{\frac{1}{q} - \frac{1}{q} - \frac{1}{q}} \| B \|_{L^p(\Omega)}^2 + |\Omega|^{\frac{1}{q} - \frac{1}{q} - \frac{1}{q}} \| v \|_{L^{q'}(\Omega)}^2)
\leq C_1(\delta_0 + 3C_3 \delta_3^2),
\]
where \( C_3 = \max(\| \Omega \|_{\frac{1}{p} - \frac{1}{p}}, |\Omega|_{\frac{1}{q} - \frac{1}{q}}, |\Omega|_{\frac{1}{q} - \frac{1}{q}}) \). Thus, once we choosing
\[
\delta_0 = \frac{\delta_3}{2C_1}, \quad \delta_3 \leq \frac{1}{6C_1C_3},
\]
we immediately get
\[ \| \mathcal{F}(B, u) \|_{L^p(\Omega) \times L^q(\Omega)} + \| w_0 \|_{W^{-1,p}(\Omega) / \mathbb{R}} + \| \pi_0 \|_{W^{-1,q}(\Omega) / \mathbb{R}} \leq \delta_3. \] (35)

Thus, \( \mathcal{F} \) maps \( \mathfrak{M} \) into itself provided that (34) holds.

**Step 2.** We show that \( \mathcal{F} \) is a contraction map on \( \mathfrak{M} \) provided that \( \delta_3 \) is small enough.

Let \((B_1, u_1), (B_2, u_2) \in \mathfrak{M}, (D_1, v_1) = \mathcal{F}(B_1, u_1), (D_2, v_2) = \mathcal{F}(B_2, u_2)\) and \((w_1, \pi_1), (w_2, \pi_2)\) be the associated pressure terms of \((D_1, v_1), (D_2, v_2)\) given by Lemma 3.2, respectively.

Denote \(B = B_1 - B_2, u = u_1 - u_2, D = D_1 - D_2, v = v_1 - v_2, w_3 = w_1 - w_2\) and \(\pi_3 = \pi_1 - \pi_2\). It is easily checked that \((D, v)\) satisfies
\[
\left\{ \begin{array}{l}
\text{curl} (\text{curl} D + B \times u_1 + B_2 \times u) + \nabla w_3 = 0 \quad \text{in } \Omega, \\
-\Delta v + \nabla \pi_3 = (B \cdot \nabla) B_1 + (B_2 \cdot \nabla) B - (u \cdot \nabla) u_1 - (u_2 \cdot \nabla) u \quad \text{in } \Omega, \\
\text{div} D = \text{div} v = 0 \quad \text{on } \partial \Omega.
\end{array} \right.
\] (36)

Thus, we have
\[
\| D \|_{L^p(\Omega)} + \| v \|_{L^q(\Omega)} 
\leq C(\Omega) \left( \| B \|_{L^p(\Omega)} \| u_1 \|_{L^q(\Omega)} + \| B_2 \|_{L^p(\Omega)} \| u \|_{L^q(\Omega)} + \| B \|_{L^p(\Omega)} \| B_1 \|_{L^p(\Omega)} + \| B_2 \|_{L^p(\Omega)} \| \right. \\
+ \| u \|_{L^q(\Omega)} \| u_1 \|_{L^q(\Omega)} + \| u_2 \|_{L^q(\Omega)} \| u \|_{L^q(\Omega)} + \| u \|_{L^q(\Omega)} \| u_2 \|_{L^q(\Omega)} \| \\
= C(\Omega) \left[ \| B \|_{L^p(\Omega)} \| u_1 \|_{L^q(\Omega)} + \| B_2 \|_{L^p(\Omega)} \| u \|_{L^q(\Omega)} + \| B_2 \|_{L^p(\Omega)} \right. \\
+ \| u \|_{L^q(\Omega)} \| B_2 \|_{L^p(\Omega)} + \| u_2 \|_{L^q(\Omega)} \| u \|_{L^q(\Omega)} + \| u \|_{L^q(\Omega)} \| \\
\leq 2\delta_3 C(\Omega) (\| B \|_{L^p(\Omega)} + \| u \|_{L^q(\Omega)})
\]
where the last inequality used that fact that
\[
\| B_1 \|_{L^p(\Omega)} + \| u_1 \|_{L^q(\Omega)} \leq \delta_3, \quad \| B_2 \|_{L^p(\Omega)} + \| u_2 \|_{L^q(\Omega)} \leq \delta_3.
\]

Therefore, once we choosing
\[
\delta_3 \leq \frac{1}{4C(\Omega)}, \tag{37}
\]
we then have
\[
\| \mathcal{F}(B_1, u_1) - \mathcal{F}(B_2, u_2) \|_{L^p(\Omega) \times L^q(\Omega)} \leq \frac{1}{2} \| (B_1, u_1) - (B_2, u_2) \|_{L^p(\Omega) \times L^q(\Omega)}. \tag{38}
\]

Hence, \( \mathcal{F} \) is a contraction map on \( \mathfrak{M} \) provided that
\[
\delta_0 = \frac{\delta_3}{2C_1}, \quad \delta_3 \leq \max \left( \frac{1}{6C_1 C_3}, \frac{1}{4C(\Omega)} \right).
\]

The Banach fixed point theorem then give a unique \((B, u) \in \mathfrak{M}\) such that \( \mathcal{F}(B, u) = (B, u) \). The existence of \( \omega, \pi \) then follows from the de Rham theorem.

**Step 3.** Now we show that \((B, u, w, \pi)\) satisfies (8). To this end, recall the first inequality of (33), we conclude that
\[
\| B \|_{L^p(\Omega)} + \| u \|_{L^q(\Omega)} + \| w \|_{W^{-1,p}(\Omega) / \mathbb{R}} + \| \pi \|_{W^{-1,q}(\Omega) / \mathbb{R}} 
\leq C_1 \| (f, g, B^0, U^0) \| + C_1 C_3 \delta_3 \| B \|_{L^p(\Omega)} + \| u \|_{L^q(\Omega)},
\]
by the definition of \( \delta_3 \), we immediately get
\[
\| B \|_{L^p(\Omega)} + \| u \|_{L^q(\Omega)} + \| w \|_{W^{-1,p}(\Omega) / \mathbb{R}} + \| \pi \|_{W^{-1,q}(\Omega) / \mathbb{R}} \leq \frac{3}{2} C_1 \| (f, g, B^0, U^0) \|.
\]
5. Proof of Proposition 2.

Proof of Proposition 2. Let \( \delta_1 \leq \min(\delta_0, \delta_2)/2C \), where \( \delta_0 \) is determined in Proposition 1, \( \delta_2 \) is determined in Lemma 3.3 and \( C \) is the constant appears in (8). We may assume that \( C > 1 \). Denotes \((B_1, u_1, w_1, \pi_1)\) by the very weak solution obtained in Proposition 1 and let \((B_2, u_2, w_2, \pi_2)\) in Proposition 1 and let \((B_2, u_2, w_2, \pi_2) \in L^p(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathbb{R}^3) \times W^{-1,p}(\Omega) \times W^{-1,q}(\Omega) \) be another very weak solution of \((\text{1})\) corresponding to the same data. It is easily seen that

\[
\|B_1\|_{L^p(\Omega)} + \|u_1\|_{L^q(\Omega)} + \|u_2 \cdot \nu\|_{W^{-1,q}(\partial \Omega)} \leq \frac{1}{2} \min(\delta_0, \delta_2) + \frac{1}{2C} \min(\delta_0, \delta_2) \leq \min(\delta_0, \delta_2).
\]

Set \( B = B_1 - B_2, u = u_1 - u_2, w = w_1 - w_2 \) and \( \pi = \pi_1 - \pi_2 \). Then from Definition 1.1 and Remark 1, \((B, u, w, \pi) \in L^p(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathbb{R}^3) \times W^{-1,p}(\Omega) \times W^{-1,q}(\Omega)\) satisfies

\[
\begin{aligned}
&\int_{\Omega} (B \cdot (-\Delta \phi) - (u \cdot \nabla) \phi \cdot B_1 - (u_2 \cdot \nabla) \phi \cdot B + (B \cdot \nabla) \phi \cdot u_1 + (B_2 \cdot \nabla) \phi \cdot u) \, dx \\
&- \langle w, \text{div} \phi \rangle_{W^{-1,p}(\Omega) \times W^{1,p'}(\Omega)} + \int_{\Omega} B \cdot \nabla \zeta \, dx = 0, \\
&\int_{\Omega} (u \cdot (-\Delta \psi) - (u \cdot \nabla) \psi \cdot u_1 - (u_2 \cdot \nabla) \psi \cdot u + (B \cdot \nabla) \psi \cdot B_1 + (B_2 \cdot \nabla) \psi \cdot B) \, dx \\
&- \langle \pi, \text{div} \psi \rangle_{W^{-1,q}(\Omega) \times W^{1,q'}(\Omega)} + \int_{\Omega} u \cdot \nabla \xi \, dx = 0
\end{aligned}
\]

for all \( \phi \in Y_p'(\Omega), \psi \in Y_q'(\Omega) \) and \( \zeta \in W^{1,p'}(\Omega), \xi \in W^{1,q'}(\Omega) \). Note that

\[
(B \cdot \nabla) \phi \cdot u_1 = \sum_{i,j=1}^3 B_i \partial_i \phi_j u_{i,j} = \sum_{i=1}^3 B_i (u_i \cdot \partial_i \phi) = B \cdot ((\nabla \phi) u_1).
\]

Similarly, we have

\[
(u \cdot \nabla) \phi \cdot B_1 = u \cdot ((\nabla \phi) B_1), \quad (u \cdot \nabla) \psi \cdot u_1 = u \cdot ((\nabla \psi) u_1), \quad (B \cdot \nabla) \psi \cdot B_1 = B \cdot ((\nabla \psi) B_1).
\]

Therefore, we can write (39) as follows

\[
\begin{aligned}
&\int_{\Omega} B \cdot [-\Delta \phi - (u_2 \cdot \nabla) \phi + (\nabla \phi) u_1 + (\nabla \psi) B_1 + (B_2 \cdot \nabla) \psi] \, dx \\
&+ \int_{\Omega} u \cdot [-\Delta \psi - (\nabla \psi) u_1 - (u_2 \cdot \nabla) \psi - (\nabla \phi) B_1 + (B_2 \cdot \nabla) \phi] \, dx = 0
\end{aligned}
\]

for all \( \phi \in Y_p'(\Omega) \) and \( \psi \in Y_q'(\Omega) \) with \( \text{div} \phi = \text{div} \psi = 0 \). Take any \((w, \eta) \in L^p(\Omega, \mathbb{R}^3) \times L^q(\Omega, \mathbb{R}^3)\), by Lemma 3.3, there exists a unique

\[
(H, \xi, \Pi_1, \Pi_2) \in W^{2,p'}(\text{div}0,0,\Omega) \times W^{2,q'}(\text{div}0,0,\Omega) \times W^{1,p'}(\Omega)/\mathbb{R} \times W^{1,q'}(\Omega)/\mathbb{R}
\]

such that (14) holds. Clearly, \((H, \xi) \in Y_p'(\Omega) \times Y_q'(\Omega)\). Therefore, by taking \((\phi, \psi) = (H, \xi) \) in (40), we see that

\[
\int_{\Omega} (B \cdot w + u \cdot \eta) \, dx = 0.
\]

Since \((w, \eta)\) is arbitrary, we conclude that \((B, u) = (0, 0)\). This completes the proof of Proposition 2. \(\square\)
6. Proof of Proposition 3. We need a density result which is an analogue of Lemma 15 in [8], we give the proof of this lemma in the Appendix for completeness.

Lemma 6.1. Assume that $p, q, r, \theta \in (1, +\infty)$. There exists a sequence $(f_\varepsilon, g_\varepsilon, B_\varepsilon^0, U_\varepsilon^0) \in [C^\infty_0(\Omega, \mathbb{R}^3)]^2 \times [C^\infty(\partial\Omega, \mathbb{R}^3)]^2$

such that

$$
\begin{align*}
(f_\varepsilon, g_\varepsilon, B_\varepsilon^0, U_\varepsilon^0) &\rightarrow f, g, B^0, U^0 &\text{in } L^r(\Omega, \mathbb{R}^3), \\
B_\varepsilon^0 &\rightarrow B^0 &\text{in } W^{-1/p,p}(\partial\Omega), \\
U_\varepsilon^0 &\rightarrow U^0 &\text{in } W^{-1/q,q}(\partial\Omega),
\end{align*}
$$

as $\varepsilon \rightarrow 0$. Moreover, it holds that

$$
\int_{\partial\Omega} B_\varepsilon^0 \cdot \nu \, d\sigma = 0, \quad \int_{\partial\Omega} U_\varepsilon^0 \cdot \nu \, d\sigma = 0.
$$

With the help of the above density result, we can now prove Proposition 3. The general idea is that divide the MHD system (1) into two systems: one is an MHD system with small datum which is solved by Proposition 1, the other is a system with more regular datum that can be solved by using the Galerkin approximation method.

Proof of Proposition 3. Step 1. We divide the MHD system (1) into two problems. Thanks to Lemma 6.1, we can choose smooth $f_\varepsilon, g_\varepsilon, B_\varepsilon^0, U_\varepsilon^0$ such that

$$
|| (f - f_\varepsilon, g - g_\varepsilon, B^0 - B_\varepsilon^0, U^0 - U_\varepsilon^0) || \leq \delta,
$$

where $\delta < \delta_0$ to be determined. Then we can divide the MHD system (1) into two problems:

$$
\begin{align*}
\text{curl} \left( \text{curl} B_1 + B_1 \times u_1 + f - f_\varepsilon \right) + \nabla w_1 = 0 &\quad \text{in } \Omega, \\
-\Delta u_1 + (u_1 \cdot \nabla) u_1 + \nabla \pi_1 - (B_1 \cdot \nabla) B_1 = g - g_\varepsilon &\quad \text{in } \Omega, \\
\text{div } B_1 = \text{div } u_1 = 0 &\quad \text{in } \Omega, \\
B_1 = B^0 - B_\varepsilon^0, &\quad u_1 = U^0 - U_\varepsilon^0 &\quad \text{on } \partial\Omega,\tag{41}
\end{align*}
$$

and

$$
\begin{align*}
\text{curl} \left( \text{curl} B_2 + B_1 \times u_2 + B_2 \times u_1 + B_2 \times u_2 + f_\varepsilon \right) + \nabla w_2 = 0 &\quad \text{in } \Omega, \\
-\Delta u_2 + (u_1 \cdot \nabla) u_2 + (u_2 \cdot \nabla) u_1 + (u_2 \cdot \nabla) u_2 + \nabla \pi_2 - (B_1 \cdot \nabla) B_1 - (B_2 \cdot \nabla) B_2 = g_\varepsilon &\quad \text{in } \Omega, \\
\text{div } B_2 = \text{div } u_2 = 0 &\quad \text{in } \Omega, \\
B_2 = B_\varepsilon^0, &\quad u_2 = U_\varepsilon^0 &\quad \text{on } \partial\Omega.\tag{42}
\end{align*}
$$

Since $B_\varepsilon^0$ and $U_\varepsilon^0$ are smooth, and

$$
\int_{\partial\Omega} B_\varepsilon^0 \cdot \nu \, d\sigma = \int_{\partial\Omega} U_\varepsilon^0 \cdot \nu \, d\sigma = 0, \tag{43}
$$

$B_\varepsilon^0$ and $U_\varepsilon^0$ satisfy the assumptions of Lemma IX.4.2 in [16, p.610], analog the proof of Lemma IX.4.2, for any $\gamma > 0$, and $\eta_1 > 0$, $\eta_2 > 0$, there exists $\mathbf{B} = \mathbf{B}(\eta_1) \in H^1(\text{div } 0, \Omega)$, $\mathbf{u} = \mathbf{u}(\eta_2) \in H^1(\text{div } 0, \Omega)$ such that $\mathbf{B} = B_\varepsilon^0$, $\mathbf{u} = U_\varepsilon^0$ in the sense of $H^{1/2}(\partial\Omega)$. Moreover,

$$
\| \mathbf{w} \mathbf{B} \|_{L^2(\Omega)} \leq C(\Omega)(\eta_1\|B_\varepsilon^0\|_{H^{1/2}(\partial\Omega)} + \sum_{i=1}^m (|B_\varepsilon^0 \cdot \nu, \mathbf{1}_{\Gamma_i, p}|)\|\nabla \mathbf{w}\|_{L^2(\Omega)},
$$
It is easily seen that for all $w \in H^1_0(\Omega, \mathbb{R}^3)$

\[ \vert \vert w \vert \vert_{L^2(\Omega)} \leq C(\Omega)(\eta_2 \vert \vert \mathcal{U}_c^0 \vert \vert_{H^{1/2}(\partial \Omega)} + \sum_{i=1}^m \vert \langle \mathcal{U}_c^0 \cdot \nu, 1 \rangle_{\Gamma_{1,i}} \vert \vert \vert \vert \nabla w \vert \vert_{L^2(\Omega)}, \]

for all $w \in H^1_0(\Omega, \mathbb{R}^3)$.

If $\Omega$ is simply-connected, then $m = 1$ and $\Gamma_1 = \partial \Omega$, hence

\[ \langle \mathcal{B}_c^0 \cdot \nu, 1 \rangle_{\Gamma_1} = \langle \mathcal{U}_c^0 \cdot \nu, 1 \rangle_{\Gamma_1} = 0 \]

from the compatible condition (43), thus for any fixed $\gamma$, we can choose

\[ \eta_1 \leq \frac{\gamma}{C(\Omega)\vert \vert \mathcal{B}_c^0 \vert \vert_{H^{1/2}(\partial \Omega)}}, \quad \eta_2 \leq \frac{\gamma}{C(\Omega)\vert \vert \mathcal{U}_c^0 \vert \vert_{H^{1/2}(\partial \Omega)}} \]

to obtain

\[ \vert \vert w \vert \vert_{L^2(\Omega)} \leq \gamma \vert \vert \nabla w \vert \vert_{L^2(\Omega)}, \quad \vert \vert w \vert \vert_{L^2(\Omega)} \leq \gamma \vert \vert \nabla w \vert \vert_{L^2(\Omega)} \] (44)

for all $w \in H^1_0(\Omega, \mathbb{R}^3)$.

If $\Omega$ is multi-connected, since

\[ \mathcal{B}_c^0 \rightarrow \mathcal{B}_c^0 \text{ in } W^{-1/p,p}(\partial \Omega), \quad \mathcal{U}_c^0 \rightarrow \mathcal{U}_c^0 \text{ in } W^{-1/q,q}(\partial \Omega), \]

it is easily seen that

\[ \sum_{i=1}^m \vert \langle \mathcal{B}_c^0 \cdot \nu, 1 \rangle_{\Gamma_{1,i}} \vert \leq 2 \sum_{i=1}^m \vert \langle \mathcal{U}_c^0 \cdot \nu, 1 \rangle_{\Gamma_{1,i}} \vert \leq 2 \delta_2, \]

and

\[ \sum_{i=1}^m \vert \langle \mathcal{U}_c^0 \cdot \nu, 1 \rangle_{\Gamma_{1,i}} \vert \leq 2 \sum_{i=1}^m \vert \langle \mathcal{U}_c^0 \cdot \nu, 1 \rangle_{\Gamma_{1,i}} \vert \leq 2 \delta_2. \]

Therefore, by choosing

\[ \eta_1 \leq \frac{\gamma}{2C(\Omega)\vert \vert \mathcal{B}_c^0 \vert \vert_{H^{1/2}(\partial \Omega)}}, \quad \eta_2 \leq \frac{\gamma}{2C(\Omega)\vert \vert \mathcal{U}_c^0 \vert \vert_{H^{1/2}(\partial \Omega)}}, \quad \delta_2 = \frac{\gamma}{4C(\Omega)}, \] (45)

we can also get (44).

By setting $\hat{B} = \hat{D} + \mathcal{B}$, $u_2 = v + \hat{u}$, we rewrite (42) in the following form

\[
\begin{cases}
\text{curl} (\text{curl} D + D \times v + \hat{B} \times v + D \times (u_1 + \hat{u}) + \hat{f}) + \nabla w = 0 & \text{in } \Omega, \\
-\Delta v + (v \cdot \nabla)(u_1 + \hat{u}) + ((v + u_1 + \hat{u}) \cdot \nabla)v + \nabla \pi_2 \\
- (\hat{B} \cdot \nabla)\hat{B} - (D \cdot \nabla)D = \hat{g} & \text{in } \Omega, \\
\text{div} D = \text{div} v = 0 & \text{in } \Omega, \\
D = 0, \quad v = 0 & \text{on } \partial \Omega,
\end{cases}
\] (46)

where

\[
\begin{align*}
\hat{B} &= B_1 + \hat{B}, \quad \hat{f} = B_1 \times \hat{u} + \hat{B} \times u_1 + \hat{B} \times \hat{u} + f_c, \\
\hat{g} &= g_c + \Delta \hat{u} - (u_1 \cdot \nabla)\hat{u} - (\hat{u} \cdot \nabla)u_1 - (\hat{u} \cdot \nabla)\hat{u} \\
&\quad + (B_1 \cdot \nabla)\hat{B} + (\hat{B} \cdot \nabla)B_1 + (\hat{B} \cdot \nabla)\hat{B}.
\end{align*}
\]

The existence of

\[(B_1, u_1, w_1, \pi_1) \in L^p(\text{div } 0, \Omega) \times L^q(\text{div } 0, \Omega) \times W^{-1,q}(\Omega)/\mathbb{R} \times W^{-1,q}(\Omega)/\mathbb{R}\]

such that

\[ \|B_1\|_{L^p(\Omega)} + \|u_1\|_{L^q(\Omega)} + \|w_1\|_{W^{-1,p}(\Omega)/\mathbb{R}} + \|\pi_1\|_{W^{-1,q}(\Omega)/\mathbb{R}} \leq C\delta \] (47)
is guaranteed by Proposition 1. Note the fact that \( \mathbf{B}_1 \in L^p(\text{div} \, 0, \Omega) \) \( (p > 3) \), \( \mathbf{u} \in H^1(\text{div} \, 0, \Omega) \) and \( \mathbf{\hat{B}} \in H^1(\text{div} \, 0, \Omega) \), \( \mathbf{f}_i \in C^\infty(\Omega, \mathbb{R}^3) \), it is easily checked that \( \mathbf{\hat{f}} \in L^2(\Omega, \mathbb{R}^3) \). To show \( \mathbf{g} \in H^{-1}(\Omega) \), it is enough to show \( -(\mathbf{\hat{u}} \cdot \nabla)\mathbf{u}_1 \in H^{-1}(\Omega) \).

Indeed, define
\[
\langle -(\mathbf{\hat{u}} \cdot \nabla)\mathbf{u}_1, \phi \rangle = \int_{\Omega} (\mathbf{\hat{u}} \cdot \nabla)\phi \cdot \mathbf{u}_1 \, dx, \quad \text{for all } \phi \in H^1_0(\Omega, \mathbb{R}^3),
\]

(48)
since \( \mathbf{\hat{u}} \in H^1(\text{div} \, 0, \Omega) \), \( \mathbf{u}_1 \in L^q(\text{div} \, 0, \Omega), q \geq 3 \), we see that
\[
\left| \int_{\Omega} (\mathbf{\hat{u}} \cdot \nabla)\phi \cdot \mathbf{u}_1 \, dx \right| \leq C\|\mathbf{\hat{u}}\|_{L^q(\Omega)}\|\phi\|_{H^1(\Omega)}\|\mathbf{u}_1\|_{L^q(\Omega)}\|\mathbf{u}_1\|_{L^p(\Omega)},
\]

where \( \frac{1}{q} = \frac{1}{3} - \frac{1}{q} \). Thus (48) defines a bounded linear functional on \( H^1_0(\Omega, \mathbb{R}^3) \), hence \( -(\mathbf{\hat{u}} \cdot \nabla)\mathbf{u}_1 \in H^{-1}(\Omega) \). Similarly, we can show that \( \mathbf{g} \in H^{-1}(\Omega) \).

We then show the existence of weak solution
\[
(\mathbf{D}, \mathbf{v}, w_2, \pi_2) \in H^1(\text{div} \, 0, \Omega) \times H^1(\text{div} \, 0, \Omega) \times L^2(\Omega) \times L^2(\Omega)
\]
to (46).

Step 2. We shall use the Galerkin Approximation method to show the existence of weak solution to (46). This is similar to the proof of existence of weak solutions to Navier-Stokes equations (see for instance [9, Theorem V.3.1, p.392]) and the steady Hall-MHD system (see [34]), with a delicately different while dealing with the coupled terms, especially the terms including \( \mathbf{\hat{B}} \) and \( \mathbf{\hat{u}} \). We give the proof here for completion. Let \( (\eta_i, \mu_i) \) be the eigenpairs of the Stokes operator, where \( \eta_i \in H^1_0(\text{div} \, 0, \Omega) \). Namely,
\[
\begin{aligned}
\{ -\Delta \eta_i + \nabla \mu_i & = \mu_i \eta_i, \quad \text{div} \eta_i = 0 \quad \text{in } \Omega, \\
\eta_i & = 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

For each integer \( N \), define finite dimensional space
\[
H_N = \text{span}\{\eta_i\}_{i=1}^N
\]
and consider the following approximation problem: Find \( (\mathbf{D}_N, \mathbf{v}_N) \in H^1_N \times H_N \) such that
\[
\left\{ \begin{array}{l}
\int_{\Omega}[\text{curl} \, \mathbf{D}_N \cdot \text{curl} \, \Psi + \mathbf{D}_N \times \mathbf{v}_N \cdot \text{curl} \, \Psi + \mathbf{\hat{B}} \times \mathbf{v}_N \cdot \text{curl} \, \Psi \\
+ \mathbf{D}_N \times (\mathbf{u}_1 + \mathbf{\hat{u}}) \cdot \text{curl} \, \Psi + \mathbf{\hat{f}} \cdot \text{curl} \, \Psi] \, dx = 0,
\end{array} \right.
\]
\[
\int_{\Omega}[\nabla \mathbf{v}_N : \nabla \Phi - (\mathbf{v}_N \cdot \nabla)\Phi \cdot (\mathbf{u}_1 + \mathbf{\hat{u}}) + ((\mathbf{v}_N + \mathbf{u}_1 + \mathbf{\hat{u}}) \cdot \nabla)\mathbf{v}_N \cdot \Phi \\
- (\mathbf{\hat{B}} \cdot \nabla)\mathbf{D}_N \cdot \Phi + (\mathbf{D}_N \cdot \nabla)\Phi \cdot \mathbf{\hat{B}} - (\mathbf{D}_N \cdot \nabla)\mathbf{D}_N \cdot \Phi] \, dx = \langle \mathbf{g}, \Phi \rangle_{H^{-1} \times H^1},
\]

(49)
for all \( \Psi \) and \( \Phi \) in \( H_{2,N} \).

To solve (49), we shall use the Brouwer’s fixed point theorem [9, Proposition II.3.11, p.83]. Since \( (\mathbf{D}_N, \mathbf{v}_N) \in H^1_N \times H_N \), we write
\[
\mathbf{D}_N = \sum_{i=1}^N \alpha_i^1 \eta_i, \quad \mathbf{v}_N = \sum_{i=1}^N \alpha_i^2 \eta_i,
\]
where \( \alpha = (\alpha^1, \alpha^2) \in \mathbb{R}^N \times \mathbb{R}^N \). Now we define a map \( P \) from \( \mathbb{R}^N \times \mathbb{R}^N \) to itself such that \( P(\alpha) = \beta = (\beta^1, \beta^2) \), with components

\[
\beta^1_i = \int_\Omega [\text{curl} D_N \cdot \text{curl} \omega_i + D_N \times v_N \cdot \text{curl} \omega_i + \hat{B} \times v_N \cdot \text{curl} \omega_i \\
+ D_N \times (u_1 + \hat{u}) \cdot \text{curl} \omega_i + \hat{f} \cdot \text{curl} \omega_i] \, dx,
\]
\[
\beta^2_i = \int_\Omega [\nabla v_N : \nabla \eta_i - (v_N \cdot \nabla) \eta_i \cdot (u_1 + \hat{u})] + ((v_N + u_1 + \hat{u}) \cdot \nabla) v_N \cdot \eta_i \\
- (\hat{B} \cdot \nabla) D_N \cdot \eta_i + (D_N \cdot \nabla) \eta_i \cdot \hat{B} - (D_N \cdot \nabla) D_N \cdot \eta_i \, dx \\
- \langle \tilde{g}, \eta_i \rangle_{H^{-1} \times H^1}.
\]

By the Brouwer’s fixed point theorem, it suffices to check that \( \alpha \cdot P(\alpha) \geq 0 \) when \( |\alpha| \) large enough. Direct computation gives

\[
\alpha \cdot P(\alpha) = \int_\Omega \left(|\text{curl} D_N|^2 + |\nabla v_N|^2 + \hat{f} \cdot \text{curl} D_N\right) \, dx - \langle \tilde{g}, v_N \rangle_{H^{-1} \times H^1} \\
+ \int_\Omega \left\{D_N \times v_N \cdot \text{curl} D_N + \hat{B} \times v_N \cdot \text{curl} D_N + D_N \times u_1 \cdot \text{curl} D_N \\
+ D_N \times \hat{u} \cdot \text{curl} D_N \right\} \, dx - \langle \tilde{g}, v_N \rangle_{H^{-1} \times H^1} \\
- \int_\Omega \{D_N \times \hat{u} \cdot \text{curl} D_N \right\} \, dx
\]

\[
\quad : = \int_\Omega \left(|\text{curl} D_N|^2 + |\nabla v_N|^2 + \hat{f} \cdot \text{curl} D_N - \langle \tilde{g}, v_N \rangle_{H^{-1} \times H^1} + \sum_{i=1}^9 T_i \right). \tag{50}
\]

We now estimate \( T_i, i = 1, \cdots, 9 \) term by term. Thanks to (5), we see that

\[
T_1 + T_9 = \int_\Omega \left|\{D_N \cdot \nabla\} D_N \cdot v_N - (v_N \cdot \nabla) D_N \cdot D_N \right| \, dx - \int_\Omega (D_N \cdot \nabla) D_N \cdot v_N \, dx \\
= - \int_\Omega (v_N \cdot \nabla) D_N \cdot D_N \, dx = - \frac{1}{2} \int_{\partial \Omega} |D_N|^2 (v_N \cdot \nu) \, d\sigma = 0.
\]

Recall that \( 3 \leq q \leq p < +\infty \), by choosing \( \gamma = \delta \) and using (44), we find that

\[
|T_2| = \int_\Omega B_1 \times v_N \cdot \text{curl} D_N \, dx + \int_\Omega \hat{B} \times v_N \cdot \text{curl} D_N \, dx \\
\quad \leq C(\Omega) \|B_1\|_{L^p(\Omega)} \|v_N\|_{L^q(\Omega)} \|\text{curl} D_N\|_{L^2(\Omega)} \\
\quad + ||\hat{B}\|_{L^2(\Omega)} \|\text{curl} D_N\|_{L^2(\Omega)} \\
\quad \leq C(\Omega) \delta \|v_N\|_{L^2(\Omega)} \|\text{curl} D_N\|_{L^2(\Omega)},
\]
\[
|T_3| + |T_4| = \int_\Omega D_N \times u_1 \cdot \text{curl} D_N \, dx + \int_\Omega D_N \times \hat{u} \cdot \text{curl} D_N \, dx \\
\quad \leq C(\Omega) \|u_1\|_{L^q(\Omega)} \|D_N\|_{L^q(\Omega)} \|\text{curl} D_N\|_{L^2(\Omega)} \\
\quad + ||\hat{u}\|_{L^2(\Omega)} \|\text{curl} D_N\|_{L^2(\Omega)} \\
\quad \leq C(\Omega) \delta \|\text{curl} D_N\|_{L^2(\Omega)}^2,
\]
\[
|T_5| + |T_6| = \int_\Omega (v_N \cdot \nabla) v_N \cdot u_1 \, dx + \int_\Omega (v_N \cdot \nabla) v_N \cdot \hat{u} \, dx \\
\quad \leq C(\Omega) \|u_1\|_{L^q(\Omega)} \|v_N\|_{L^q(\Omega)} \|\nabla v_N\|_{L^2(\Omega)} + ||\hat{u}\|_{L^2(\Omega)} \|\nabla v_N\|_{L^2(\Omega)} \\
\quad \leq C(\Omega) \delta \|\nabla v_N\|_{L^2(\Omega)}^2,
\]

\[
|T_7| + |T_8| = \int_\Omega (v_N \cdot \nabla) v_N \cdot \hat{u} \, dx + \int_\Omega (v_N \cdot \nabla) u_1 \cdot \hat{u} \, dx \\
\quad \leq C(\Omega) \|u_1\|_{L^q(\Omega)} \|v_N\|_{L^q(\Omega)} \|\nabla v_N\|_{L^2(\Omega)} + ||\hat{u}\|_{L^2(\Omega)} \|\nabla u_1\|_{L^2(\Omega)} \\
\quad \leq C(\Omega) \delta \|\nabla v_N\|_{L^2(\Omega)}^2,
\]

\[
|T_{10}| = \int_\Omega (v_N \cdot \nabla) v_N \cdot \hat{f} \, dx + \int_\Omega (v_N \cdot \nabla) v_N \cdot \hat{f} \, dx \\
\quad \leq C(\Omega) \|v_N\|_{L^q(\Omega)} \|\hat{f}\|_{L^p(\Omega)} \|\nabla v_N\|_{L^2(\Omega)} + ||\hat{f}\|_{L^2(\Omega)} \|\nabla v_N\|_{L^2(\Omega)} \\
\quad \leq C(\Omega) \delta \|\nabla v_N\|_{L^2(\Omega)}^2.
\]
Substitute these estimates into (50), we find that
\[ |T_7| + |T_8| = \int_{\Omega} \left[ (B_1 \cdot \nabla)D_N \cdot v_N + (\hat{B} \cdot \nabla)D_N \cdot v_N \right] dx \]
\[ \leq C(\Omega) \left\| B_1 \left\|_{L^p(\Omega)} \right\| D_N \right\|_{L^q(\Omega)} \| \nabla v_N \|_{L^2(\Omega)} + \| \hat{B} \|_{L^q(\Omega)} \| v_N \|_{L^2(\Omega)} \| \nabla D_N \|_{L^2(\Omega)} \]
\[ + \| \hat{B} \|_{L^q(\Omega)} \| \nabla v_N \|_{L^2(\Omega)} + \| \nabla v_N \|_{L^2(\Omega)} + \| \nabla v_N \|_{L^2(\Omega)} \| \nabla D_N \|_{L^2(\Omega)} \]
\[ \leq C(\Omega) \delta \| \nabla v_N \|_{L^2(\Omega)} \| \nabla v_N \|_{L^2(\Omega)} \| \nabla v_N \|_{L^2(\Omega)} \| \nabla D_N \|_{L^2(\Omega)} \]

By choosing \( \delta < \frac{1}{8C(\Omega)} \) and applying the Hölder inequality, we immediately get
\[ \alpha \cdot P(\alpha) \geq \frac{1}{2} \left( \| \nabla v_N \|_{L^2(\Omega)}^2 + \| \nabla v_N \|_{L^2(\Omega)}^2 \right) - C(\Omega)(\| \hat{f} \|_{L^2(\Omega)}^2 + \| \hat{g} \|_{H^{-1}(\Omega)}^2). \]

Note that
\[ \| \nabla v_N \|_{L^2(\Omega)}^2 + \| \nabla v_N \|_{L^2(\Omega)}^2 = \sum_{i=1}^{N} (\lambda_i(\alpha_1^2) + \lambda_i(\alpha_2^2))^2 \geq \lambda_1 |\alpha|^2, \]
we have
\[ \alpha \cdot P(\alpha) \geq 0 \]
for all \( \alpha \in \mathbb{R}^N \times \mathbb{R}^N \) such that
\[ |\alpha| \geq \sqrt{\lambda_1^{-1} C(\Omega)(\| \hat{f} \|_{L^2(\Omega)}^2 + \| \hat{g} \|_{H^{-1}(\Omega)}^2)}. \]

Thus, for each \( N > 0 \), problem (49) admits a solution \((D_N, v_N) \in H_{1,N} \times H_{2,N}\) such that
\[ \| \nabla v_N \|_{L^2(\Omega)} + \| \nabla v_N \|_{L^2(\Omega)} \leq C(\Omega)(\| \hat{f} \|_{L^2(\Omega)} + \| \hat{g} \|_{H^{-1}(\Omega)}), \]
which further implies that (see for example [14, Theorem 3, p.209])
\[ \| D_N \|_{H^1(\Omega)} + \| v_N \|_{H^1(\Omega)} \leq C(\Omega)(\| \hat{f} \|_{L^2(\Omega)} + \| \hat{g} \|_{H^{-1}(\Omega)}). \]

Therefore, up to a subsequence, there exists a \((D, v) \in H^1(\text{div} 0, \Omega) \times H^1(\text{div} 0, \Omega)\) such that
\[ v_N \to v \quad \text{weakly in } H^1(\text{div} 0, \Omega) \quad \text{and strongly in } L^r(\Omega, \mathbb{R}^3) \quad \text{for all } r \in [1, 6]; \]
\[ D_N \to D \quad \text{weakly in } H^1(\text{div} 0, \Omega) \quad \text{and strongly in } L^r(\Omega, \mathbb{R}^3) \quad \text{for all } r \in [1, 6]; \]
and
\[ (v_N \cdot \nabla)v_N \to (v \cdot \nabla)v \quad \text{weakly in } L^r(\Omega, \mathbb{R}^3) \quad \text{for all } r \in [1, 3/2]; \]
\[ (D_N \cdot \nabla)D_N \to (D \cdot \nabla)D \quad \text{weakly in } L^r(\Omega, \mathbb{R}^3) \quad \text{for all } r \in [1, 3/2]; \]
\[ D_N \times v_N \to D \times v \quad \text{strongly in } L^r(\Omega, \mathbb{R}^3) \quad \text{for all } r \in [1, 3]. \]
Passing to the limit $N \to +\infty$ in (49), we find that
\begin{equation}
\begin{cases}
\int_\Omega [\text{curl } D \cdot \text{curl } \Psi + D \times v \cdot \text{curl } \Psi + \tilde{B} \times v \cdot \text{curl } \Psi \\
+ D \times (u_1 + \hat{u}) \cdot \text{curl } \Psi + \hat{f} \cdot \text{curl } \Psi] \, dx = 0,
\end{cases}
\end{equation}
holds for all $\Psi$ and $\Phi$ in $H_N$. By the density of $\bigcup_{N=1}^\infty H_N$ in $H^1_0(\text{div } 0, \Omega)$, we immediately deduce that $(D, v)$ with the associated pressure $w_2, \pi_2$ is a weak solution to (46).

**Step 3.** We show that $(B, u, w, \pi) = (B_1 + \tilde{B} + D, u_1 + \hat{u} + v, w_1 + w_2, \pi_1 + \pi_2)$ is a very weak solution to (1). We first note that we have in fact that $\hat{\pi} = 0$ since $\tilde{L}^2$ and $L^2_0$ are smooth. Particularly, we have $\hat{B} \in L^p(\text{div } 0, \Omega)$, $\hat{u} \in L^q(\text{div } 0, \Omega)$. It then suffices to show that we have $D \in L^p(\text{div } 0, \Omega)$, $v \in L^q(\text{div } 0, \Omega)$, $w_2 \in W^{-1, p}(\Omega)$ and $\pi_2 \in W^{-1, q}(\Omega)$. There are three cases to deal with.

**Case 1.** $3 \leq q \leq p \leq 6$, this follows immediately from the Sobolev embedding theorem.

**Case 2.** $6 \leq q \leq p < +\infty$. In this case, we have particularly $D, v, \hat{B}, u_1, \hat{u} \in L^6(\Omega, \mathbb{R}^3)$. Therefore,
\[
\hat{f} := D \times v + \hat{B} \times v + D \times (u_1 + \hat{u}) + \hat{f} \in L^3(\Omega, \mathbb{R}^3),
\]
applying the $L^p$ estimates for Stokes equations to the first equation of (46), we get $D \in W^{1, 3}(\text{div } 0, \Omega)$ and $w_2 \in L^3(\Omega) \hookrightarrow W^{-1, \infty}(\Omega)$, thus $D \in L^p(\text{div } 0, \Omega)$ and $w_2 \in W^{-1, p}(\Omega)$ for any $1 \leq p < \infty$. Similarly, we have $u \in W^{1, 3}(\text{div } 0, \Omega), \pi_2 \in L^3(\Omega)$ and thus $v \in L^3(\text{div } 0, \Omega), \pi_2 \in W^{-1, q}(\Omega)$ for any $1 \leq q < \infty$. Therefore, $(B, u, w, \pi) \in L^p(\text{div } 0, \Omega) \times L^3(\text{div } 0, \Omega) \times W^{-1, p}(\Omega) \times W^{-1, q}(\Omega)$ is a very weak solution to (1).

**Case 3.** $3 < q < 6 < p$. We need only to show that $D \in L^p(\text{div } 0, \Omega)$. To this end, we use a bootstrap argument. Since $D \in H^1(\text{div } 0, \Omega), v \in L^q(\text{div } 0, \Omega)$, we see that $\hat{f} \in L^{q/6}(\Omega, \mathbb{R}^3)$. Thus, applying the $L^q$ estimates for Stokes equation to (46), we find that
\[
D \in W^{1, \frac{q}{1+q/6}}(\text{div } 0, \Omega) \hookrightarrow L^{r_1}(\text{div } 0, \Omega) \quad \text{with } 1/r_1 = 1/q - 1/6
\]
and
\[
w_2 \in L^{\frac{q}{1+q/6}}(\Omega) \hookrightarrow W^{-1, \tau_1}(\Omega) \quad \text{for } \tau_1 < (1/q - 1/6)^{-1} = r_1.
\]
If $1/r_1 \leq 1/p$, namely, $1/q - 1/6 \leq 1/p$, we are done. If $1/q - 1/6 > 1/p$, we repeat the previous argument. Since $\hat{f} \in L^{q/6}(\Omega, \mathbb{R}^3)$, we find that
\[
D \in W^{1, \frac{r_1 q}{1+q/6}}(\text{div } 0, \Omega) \hookrightarrow L^{r_2}(\text{div } 0, \Omega)
\]
with $1/r_2 = 2/q - 1/2$ if $3 < q < 4$ and arbitrary $1 \leq r_2 < +\infty$ if $4 \leq q < 6$, $w_2 \in L^{\frac{r_1 q}{1+q/6}}(\Omega) \hookrightarrow W^{-1, \tau_2}(\Omega)$ for $\tau_2 < (2/q - 1/2)^{-1} = r_2$ if $3 < q < 4$ and $\tau_2 \in [1, +\infty)$ arbitrary if $4 \leq q < 6$. Thus, if $4 \leq q < 6$ or $3 < q < 4$ but $1/r_2 < 1/p$, we are done. Else, we repeat the above argument again.
It is easily checked that after $k$ times iteration, we will formally deduce that
\[ D \in L^{r_k}(\text{div } 0, \Omega) \quad \text{with } 1/r_k = k/q - (2k - 1)/6 \]
and
\[ w_2 \in L^{r_k}(\text{div } 0, \Omega) \quad \text{with } r_k = (k/q - (2k - 1)/6)^{-1} = r_k. \]
Note that if $k = \left[\frac{q}{2q-6}\right] + 1$, then we have $k/q - (2k - 1)/6 < 0$, which means that we have in fact that $D \in L^{r_k}(\text{div } 0, \Omega)$ and $w_2 \in W^{-1,r_k}(\Omega)$ for any $1 \leq r_k, r_k < +\infty$, particularly, $D \in L^p(\text{div } 0, \Omega)$ and $w_2 \in W^{-1,p}(\Omega)$. On the other hand, if $k = \left[\frac{q(p-6)}{2p(q-3)}\right] + 1$, we then have
\[ \frac{1}{r_k} - \frac{1}{p} = \frac{k - 2k - 1}{q} - \frac{1}{p} = k \frac{3 - q}{3q} + \frac{p - 6}{6p} < \frac{q(p-6)}{2p(q-3)} \frac{3 - q}{3q} + \frac{p - 6}{6p} = 0, \]
thus, $D \in L^p(\text{div } 0, \Omega)$ and $w_2 \in W^{-1,p}(\Omega)$. Therefore, at most repeat
\[ k = \min\left( \left[\frac{q}{2q-6}\right] + 1, \left[\frac{q(p-6)}{2p(q-3)}\right] + 1 \right) \]
times iteration, we will get $D \in L^p(\text{div } 0, \Omega)$ and $w_2 \in W^{-1,p}(\Omega)$. We are done. \(\square\)

**Remark 3.** From (45) and the choosing of $\gamma$ and $\delta$ in Step 2, we see that $\delta_2$ is a constant depends only on $\Omega$, $p$ and $q$.

**Appendix.**

**Proof of Lemma 6.1.** The existence of such $f_\varepsilon$ is obvious. The existence of $g_\varepsilon$ is ensured by Lemma 14 in [8]. To show the existence of $E_\varepsilon^0$, we consider the following Stokes problem:
\[
\begin{cases}
-\Delta v + \nabla \pi = 0, & \text{in } \Omega, \\
v = E_\varepsilon^0 & \text{on } \partial \Omega. 
\end{cases}
\]
By Lemma 3.2, there exists a unique very weak solution $(v, \pi) \in L^p(\Omega, \text{div } 0) \times W^{-1,p}(\Omega)/\mathbb{R}$ to (52). Clearly, $-\Delta v \in [X', p', \Omega)'$ and thus $v \in T_{p,p,\sigma}(\Omega)$. Using Lemma 11 in [8], there exists $v_\varepsilon \in D_\sigma(\Omega)$ such that
\[ v_\varepsilon \rightarrow v \quad \text{in } T_{q,q,\sigma}(\Omega). \]
Let $E_\varepsilon^0 = v_\varepsilon|_{\partial \Omega}$, we see that
\[ E_\varepsilon^0 \rightarrow E_0^0 \quad \text{in } W^{-1/p,p}(\partial \Omega), \quad \int_{\partial \Omega} E_\varepsilon^0 \cdot \nu \, d\sigma = 0. \]
The existence of $U_0^0$ follows in the same way. \(\square\)

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