A novel geometric modeling approach for cubic developable C-Bézier surfaces

Gang HU*, Huinan LI* and Xianzhi HU*
*Xi’an University of Technology
5 South Jinhua Road, Xincheng District, Xi’an 710054, China
E-mail: hg_xaut@xaut.edu.cn

Received: 27 September 2019; Revised: 20 January 2020; Accepted: 24 March 2020

Abstract

In this paper, a novel geometric modeling method is proposed to construct generalized cubic developable C-Bézier (GCDC-Bézier, for short) surfaces with shape parameters. By using the control plane with generalized cubic C-Bézier basis function, the GCDC-Bézier surfaces are designed, and the shape of the surfaces are adjusted by changing its shape parameters. In addition, the necessary and sufficient conditions of $G^1$ continuity and $G^2$ Beta smooth continuity between two adjacent GCDC-Bézier surfaces are derived. Finally, we also discuss some properties of the GCDC-Bézier surfaces. The approach proposed in this paper provides a valuable alternative to the existing geometric modeling methods of developable surfaces.

Keywords: Geometric modeling, Developable C-Bézier surfaces, C-Bézier basis functions, Shape parameter, Continuity condition

1. Introduction

A developable surface, as a special ruled surface, can be expanded into a complete plane without stretching or tearing. Because of its good geometric properties, developable surfaces play a crucial part in numerous engineering fields, especially in mechanical engineering design (Kwon et al.,2005; Hu et al.,2017a, 2017b). For instance, the appearance design of automobile bodies, aircraft skins, ship hulls and clothing can be achieved by using the smooth continuity between a number of developable patches. The computer representation of developable surfaces extends its application in engineering and its constructive methods can be classified into two categories: (a) point geometric representation (PGR) (Chu et al.,2008; Gunter et al.,1988; Maekawa et al.,1998); (b) line and plane geometric representation (LPGR) (Bodduluri et al.,1993,1994; Pottmann et al.,1995; Yang et al.,2007). In the former, a developable surface is treated as a ruled surface in Euclidean space and then, in order to achieve developability, constraint conditions that have to be additionally imposed on the ruled surface need to be derived. In the category (b), a developable surface is treated as a curve in three-dimensional projective space, which is used for geometric representation of lines and planes, and constructed by means of duality between points and planes. However, the shape of the developable surfaces generated by LPGR method is determined only by its control planes. In order to overcome the shortcomings of LPGR method, a developable C-Bézier surface is constructed by (Zhou et al., 2013) that using the cubic C-Bézier basis presented in (Zhang, 1996). In the case of keeping the position of the control plane unchanged, the shape of the surface can be changed. Nevertheless, the developable C-Bézier surface has only one shape parameter, which leads to the lack of freedom and limits shape adjustment in the construction of complex developable surfaces. Therefore, in this paper, a new kind of developable surfaces named GCDC-Bézier surfaces are constructed by a class of generalized cubic C-Bézier basis functions associated with three shape parameters. When the control plane is given, the shape of the GCDC-Bézier surfaces can be adjusted locally or globally by modifying the shape parameters.

The remainder of the paper is organized as follows. In Section 2, we propose a new kind of generalized cubic C-Bézier curves. Section 3 describes the constructed generalized cubic developable C-Bézier surfaces, and studies some properties of the surfaces. The continuity conditions of the GCDC-Bézier surfaces are derived in Section 4. In Section 5, we provide some practical modeling examples. Finally, a brief conclusion is provided in Section 6.
2. Generalized cubic C-Bézier curves with shape parameters

2.1 Definition of generalized cubic C-Bézier basis functions

Let the initial functions (Chen et al., 2003)

\[
\begin{align*}
    u_{0,1}(t; \alpha) &= \sin(\alpha - t) / \sin \alpha, \\
    u_{1,1}(t; \alpha) &= \sin(t) / \sin \alpha,
\end{align*}
\]

where \( \alpha \in (0, \pi] \), \( t \in [0, \alpha] \).

According to (1), the quadratic C-Bézier basis are defined as follows (Chen et al., 2003):

\[
\begin{align*}
    u_{0,2}(t; \alpha) &= 1 - \int_0^t \phi_{0,1}u_{0,1}(s; \alpha) ds = \frac{1 - \cos(\alpha - t)}{1 - \cos \alpha}, \\
    u_{1,2}(t; \alpha) &= \int_0^t [\phi_{0,1}u_{0,1}(s; \alpha) - \phi_{1,1}u_{1,1}(s; \alpha)] ds = \frac{1 - \cos t + \cos \alpha - \cos(\alpha - t)}{1 - \cos \alpha}, \\
    u_{2,2}(t; \alpha) &= \int_0^t \phi_{1,1}u_{1,1}(s; \alpha) ds = \frac{1 - \cos t}{1 - \cos \alpha},
\end{align*}
\]

where \( \phi_{1,1} = \left( \int_0^\alpha u_{1,1}(t; \alpha) dt \right)^{-1} \) (\( i = 0, 1 \)).

Analogously, for \( \alpha \in (0, \pi] \) and \( t \in [0, \alpha] \), the traditional cubic C-Bézier basis (TCCB) functions \( u_{i,3}(t; \alpha)(i = 0, 1, 2, 3) \) are defined recursively by (Zhang, 1996; Chen et al., 2003)

\[
\begin{align*}
    u_{0,3}(t; \alpha) &= 1 - \int_0^t \phi_{0,2}u_{0,2}(s; \alpha) ds = 1 - f_1(t; \alpha), \\
    u_{1,3}(t; \alpha) &= \int_0^t [\phi_{0,2}u_{0,2}(s; \alpha) - \phi_{1,2}u_{1,2}(s; \alpha)] ds = f_1(t; \alpha) - f_2(t; \alpha), \\
    u_{2,3}(t; \alpha) &= \int_0^t [\phi_{1,2}u_{1,2}(s; \alpha) - \phi_{2,2}u_{2,2}(s; \alpha)] ds = f_2(t; \alpha) - f_3(t; \alpha), \\
    u_{3,3}(t; \alpha) &= \int_0^t \phi_{2,2}u_{2,2}(s; \alpha) ds = f_3(t; \alpha),
\end{align*}
\]

where \( \phi_{1,2} = \left( \int_0^\alpha u_{1,2}(t; \alpha) dt \right)^{-1} \) (\( i = 0, 1, 2 \)),

\[
\begin{align*}
    f_1(t; \alpha) &= \frac{t - \sin \alpha + \sin(\alpha - t)}{\alpha - \sin \alpha}, \\
    f_2(t; \alpha) &= \frac{t \cos \alpha + \sin(\alpha - t) - \sin \alpha - \sin t + t}{\alpha \cos \alpha - 2 \sin \alpha + \alpha}, \\
    f_3(t; \alpha) &= \frac{t - \sin t}{\alpha - \sin \alpha}.
\end{align*}
\]

Since the TCCB functions in (3) have only one global shape parameter, a natural idea arises to define a new kind of C-Bézier basis with multiple local shape parameters. By improving formula (3), we can construct a class of generalized cubic C-Bézier basis functions with three local shape parameters as follows.

**Definition 1.** Let \( \alpha_1, \alpha_2, \alpha_3 \in (0, 2\pi] \), then for any value of \( t \in [0, 1] \), the generalized cubic C-Bézier basis (GCCB, for short) functions of \( t \) can be defined as follows

\[
\begin{align*}
    w_{0,3}(t; \alpha_1) &= 1 - \tilde{f}_1(t; \alpha_1), \\
    w_{1,3}(t; \alpha_1, \alpha_2) &= \tilde{f}_1(t; \alpha_1) - \tilde{f}_2(t; \alpha_2), \\
    w_{2,3}(t; \alpha_2, \alpha_3) &= \tilde{f}_2(t; \alpha_2) - \tilde{f}_3(t; \alpha_3), \\
    w_{3,3}(t; \alpha_3) &= \tilde{f}_3(t; \alpha_3),
\end{align*}
\]
where \( \tilde{f}_1(t;\alpha_i) = \frac{\alpha_1 t - \sin \alpha_1 + \sin(\alpha_1 - \alpha_i t)}{\alpha_1 - \sin \alpha_1} \),

\( \tilde{f}_2(t;\alpha_2) = \frac{\alpha_2 t \cos \alpha_2 + \sin(\alpha_2 - \alpha_i t) - \sin \alpha_2 - \sin(\alpha_2 t) + \alpha_2 t}{\alpha_2 \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2} \),

\( \tilde{f}_3(t;\alpha_3) = \frac{\alpha_3 t - \sin(\alpha_3 t)}{\alpha_3 - \sin \alpha_3} \).

**Remark 1.** When the three shape parameters are equal to each other, that is \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha \), the GCCB functions degenerate to TCCB functions in (Zhang, 1996; Chen et al., 2003; Zhou et al., 2013).

**Theorem 1.** The properties of GCCB functions associated with shape parameters \( \alpha_i (i=1,2,3) \) are listed as follows:

(a) Non-negativity. For any value of \( \alpha_i (i=1,2,3) \) as well as \( t \in [0,1] \), \( w_{1,3}(t) \geq 0 (i=0,1,2,3) \).

(b) Partition of unity. In other words, it's \( \sum_{i=0}^{3} w_{i,3}(t) = 1 \).

(c) Terminal properties. The GCCB functions \( w_{i,3}(t) (i=0,1,2,3) \) defined by (4) satisfy the following properties:

\[
\begin{align*}
    w_{0,3}(0) &= w_{3,3}(1) = 1, \\
    w_{i,3}(0) &= w_{j,3}(1) = 0, i=1,2,3; j=0,1,2.
\end{align*}
\] (5)

And the first derivatives of the GCCB functions at end-points satisfy

\[
\begin{align*}
    w'_{0,3}(0) &= \alpha_1 \cos \alpha_1 - \alpha_1, \\
    w'_{1,3}(0) &= w'_{0,3}(0), \\
    w'_{2,3}(0) &= w'_{3,3}(0) = w'_{0,3}(1) = w'_{1,3}(1) = 0, \\
    w'_{2,3}(1) &= \alpha_3 \cos \alpha_3 - \alpha_3, \\
    w'_{3,3}(1) &= w'_{2,3}(1). \\
\end{align*}
\] (6)

Furthermore, the second derivatives of the GCCB functions are

\[
\begin{align*}
    w''_{0,3}(0) &= \frac{\alpha_1^2 \sin \alpha_1}{\alpha_1 - \sin \alpha_1}, \\
    w''_{1,3}(0) &= 0, \\
    w''_{2,3}(0) &= w''_{1,3}(1) = - \frac{\alpha_2^2 \sin \alpha_2}{\alpha_2 \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2}, \\
    w''_{3,3}(1) &= 0, \\
    w''_{2,3}(1) &= w''_{3,3}(1) = \frac{\alpha_3^2 \sin \alpha_3}{\alpha_3 - \sin \alpha_3}. \\
\end{align*}
\] (7)

(d) Linear independence. The GCCB functions defined by (4) are linearly independent.

(e) When \( \alpha_1 = \alpha_3 \), the GCCB functions \( w_{1,3}(t) (i=0,1,2,3) \) are symmetric, that is \( w_{0,3}(t) = w_{3,3}(1-t) \) and \( w_{1,3}(t) = w_{2,3}(1-t) \).

**Proof.** We shall prove (d) and (e). The rest of the properties can be proved with simple computation.

(d) For any \( \alpha_1, \alpha_2, \alpha_3 > 0 \), and \( c_i \in \mathbb{R} (i=0,1,2,3) \), we consider a linear combination as follows:
\[ \sum_{i=0}^{3} c_i w_{i,3}(t) = 0. \]  \hspace{1cm} (8)

By taking the \( k \)-order derivatives of the formula (8) about \( t \) on both sides, we get

\[ \sum_{i=0}^{3} c_i w_{i,3}^{(k)}(t) = 0, \quad k = 0, 1, 2, 3. \]  \hspace{1cm} (9)

From (8) and (9), we get the following system of linear equations at the point \( t = 0 \):

\[
\begin{aligned}
c_0 w_{0,3}^{(0)}(0) &= 0, \\
c_0 w_{0,3}^{(1)}(0) + c_1 w_{1,3}^{(1)}(0) &= 0, \\
c_0 w_{0,3}^{(2)}(0) + c_1 w_{1,3}^{(2)}(0) + c_2 w_{2,3}^{(2)}(0) &= 0, \\
c_0 w_{0,3}^{(3)}(0) + c_1 w_{1,3}^{(3)}(0) + c_2 w_{2,3}^{(3)}(0) + c_3 w_{3,3}^{(3)}(0) &= 0,
\end{aligned}
\]  \hspace{1cm} (10)

where \( w_{0,3}^{(0)}(0) = w_{0,3}(0) = 1, \quad w_{i,3}^{(j)}(0) \neq 0 (i = 0, 1, \cdots, j; j = 1, 2, 3). \)

It is obvious that \( c_i = 0 (i = 0, 1, 2, 3) \), meaning that \( w_{i,3}(t) (i = 0, 1, 2, 3) \) are linearly independent.

(e) Let \( \alpha_1 = \alpha_3 \), we have

\[ w_{0,3}(t) = \frac{\alpha_1 (1-t) - \sin(\alpha_1 - \alpha_3 t)}{\alpha_1 - \sin \alpha_1} = \frac{\alpha_3 (1-t) - \sin[\alpha_3 (1-t)]}{\alpha_3 - \sin \alpha_3} = w_{3,3}(1-t), \]  \hspace{1cm} (11)

and

\[
\begin{aligned}
w_{2,3}(1-t) &= \frac{\alpha_2 (1-t) \cos \alpha_2 + \sin[\alpha_2 - \alpha_3 (1-t)] - \sin \alpha_2 - \sin[\alpha_2 (1-t)] + \alpha_2 (1-t)}{\alpha_2 \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2} \\
&- \frac{\alpha_3 (1-t) - \sin[\alpha_3 (1-t)]}{\alpha_3 - \sin \alpha_3} \\
&= \frac{\alpha_2 \cos \alpha_2 - \alpha_2 \cos \alpha_2 + \sin(\alpha_2 t) - \sin \alpha_2 - \sin(\alpha_2 - \alpha_3 t) + \alpha_2}{\alpha_2 \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2} \\
&- \frac{\alpha_3 - \alpha_3 t - \sin(\alpha_3 - \alpha_3 t)}{\alpha_3 - \sin \alpha_3} \\
&= \frac{1 - \alpha_2 t \cos \alpha_2 + \sin(\alpha_2 - \alpha_2 t) - \sin \alpha_2 - \sin(\alpha_2 t) + \alpha_2}{\alpha_2 \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2} \\
&- \frac{\alpha_3 - \alpha_3 t - \sin(\alpha_3 - \alpha_3 t)}{\alpha_3 - \sin \alpha_3} \\
&= \frac{\alpha_1 t - \sin(\alpha_1 - \alpha_1 t) - \alpha_2 t \cos \alpha_2 + \sin(\alpha_2 - \alpha_2 t) - \sin \alpha_2 - \sin(\alpha_2 t) + \alpha_2}{\alpha_1 - \sin \alpha_1} \\
&- \frac{\alpha_2 t \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2}{\alpha_2 \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2} \\
&= w_{1,3}(t).
\]  \hspace{1cm} (12)

Obviously, (11) and (12) indicate that \( w_{i,3}(t) (i = 0, 1, 2, 3) \) are symmetric under condition that the shape parameter \( \alpha_i \) is equal to \( \alpha_3 \).

Fig.1 shows the GCCB functions with different parameters. As can be seen from Fig. 1, the GCCB functions with three parameters are more flexible than the TCCB ones in shape control.

**2.2 Construction of generalized cubic C-Bézier curves**

**Definition 2.** Given control points \( P_i \in R^d \ (i = 2, 3; \ i = 0, 1, 2, 3) \), the generalized cubic C-Bézier (GCC-Bézier, for short) curves associated with shape parameters are defined below
\[
\mathbf{r}(t; \alpha_1, \alpha_2, \alpha_3) = \sum_{i=0}^{3} P_i w_{i,3}(t) \quad t \in [0,1],
\]

where \( \alpha_i (i = 1, 2, 3) \) are shape parameters, and \( w_{i,3}(t) (i = 0, 1, 2, 3) \) are GCCB functions defined by (4).

\[
\begin{align*}
\alpha_1 = \theta_1 & = \pi / 2 \\
\alpha_2 = \theta_2 & = \pi / 10 \\
\alpha_3 & = \pi
\end{align*}
\]

Fig. 1. The GCCB functions with different shape parameters

On the basis of properties of the GCCB functions defined by (4), it can be easily obtained that GCC-Bézier curves possess the following properties.

**Theorem 2.** The properties of GCC-Bézier curves associated with shape parameters are listed as follows:

(a) **Terminal properties.** For any value of \( t \in [0,1] \) as well as \( \alpha_i (i = 1, 2, 3) \), we have

\[
\begin{align*}
\mathbf{r}(0; \alpha_1, \alpha_2, \alpha_3) &= \mathbf{P}_0, \\
\mathbf{r}(1; \alpha_1, \alpha_2, \alpha_3) &= \mathbf{P}_3, \\
\mathbf{r}'(0; \alpha_1, \alpha_2, \alpha_3) &= \frac{\alpha_1(1 - \cos \alpha_1)}{\alpha_1 - \sin \alpha_1}(\mathbf{P}_1 - \mathbf{P}_0), \\
\mathbf{r}'(1; \alpha_1, \alpha_2, \alpha_3) &= \frac{\alpha_3(1 - \cos \alpha_3)}{\alpha_3 - \sin \alpha_3}(\mathbf{P}_3 - \mathbf{P}_2),
\end{align*}
\]

(b) **Convex hull property.** The whole GCC-Bézier curves must be located within its control polygon spanned by \( \mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \).

(c) **Geometric invariance.** The shape of a GCC-Bézier curve is independent of the selection of coordinate.

(d) **Shape adjustable property.** In the case of keeping control polygon unchanged, the shape of the GCC-Bézier curves can be adjusted by modifying three shape parameters.

According to the definition of (5), when the shape parameters are different, a family of GCC-Bézier curves will be generated if the control points \( \mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \) are given. Fig. 1 shows the effect of changing shape parameters \( \alpha_i (i = 1, 2, 3) \) on the GCC-Bézier curves when the control points are fixed. Fig. 1 (a) displays curves with two fixed \( \alpha_2 = \alpha_3 = \pi / 2 \), and changing \( \alpha_1 \) to \( \alpha_1 = 0.1\pi \) (black dotted line), \( 0.8\pi \) (blue dashed line), \( 1.2\pi \) (magenta dash-dotted line) and \( 1.5\pi \) (green solid line); the curves gradually move away from the edge \( \mathbf{P}_1 - \mathbf{P}_0 \). Fig. 1 (b) displays curves with two fixed \( \alpha_1 = \alpha_3 = \pi / 12 \), and changing \( \alpha_2 \) to \( \alpha_2 = 0.1\pi \) (black dotted line), \( 1.5\pi \) (blue
dashed line), $1.8\pi$ (magenta dash-dotted line) and $2\pi$ (green solid line); the curves gradually approach edges $P_1 - P_0$ and $P_2 - P_3$ simultaneously. Fig. 1 (c) displays curves with two fixed $\alpha_1 = \alpha_2 = \pi/2$, and changing $\alpha_3$ to $0.1\pi$ (black dotted line), $0.8\pi$ (blue dashed line), $1.25\pi$ (magenta dash-dotted line) and $1.5\pi$ (green solid line); the curves gradually move away from the the edge $P_2 - P_3$. Fig. 1 (d) shows curves with a fixed $\alpha_2$ and increases the other two shape parameters $\alpha_1, \alpha_3$ simultaneously, the curves gradually move away from the control polygon.

![Fig. 2. GCC-Bézier curves with different shape parameters](image)

3. Construction of generalized cubic developable C-Bézier surfaces

3.1 Generalized cubic developable C-Bézier surfaces

On the basis of the theory of differential geometry, the definition of developable surfaces can be given as follows.

**Definition 3.** The developable surface can be described as an envelope of a single-parameter family of planes (Do Carmo, 1976; Mei et al., 2003; Struik, 1998).

According to the duality between points and planes in 3D projective space, the single-parameter family of planes can be obtained when the control points of the GCC-Bézier curves is regarded as the control planes (Bodduluri et al., 1993; Zhou et al., 2013; Hu et al., 2017a, 2017b). By means of (13), the equation of determining the single-parameter family can be expressed as follows:

$$\{\tilde{\Pi}_i\} : S(t; \alpha_1, \alpha_2, \alpha_3) = \sum_{i=0}^{3} w_{i,1}(t) X_i, \quad (15)$$
where \( X_i = (X_{i,1}, X_{i,2}, X_{i,3}, X_{i,4}), X_{i,j} \in \mathbb{R}, (i, j = 0,1,2,3) \) are control planes, and \( \alpha_i (i = 1,2,3) \) are shape control parameters.

Obviously, on the basis of (15) and combining duality principle, the equation of the single-parameter family of planes \( \{\Pi_i\} \) can be represented as

\[
S(t;\alpha_1,\alpha_2,\alpha_3) = \sum_{i=0}^{3} w_{i,3}(t) (X_{i,0}, X_{i,1}, X_{i,2}, X_{i,3}) = \{u_0(t), u_1(t), u_2(t), u_3(t)\},
\]

where \( u_j(t) = \sum_{i=0}^{3} w_{i,3}(t) X_{i,j} (j = 0,1,2,3). \)

According to Definition 3, the enveloping surface of the single-parameter family of planes is developable surface. By using the method based on line and plane in (Bodduluri et al.,1993; Zhou et al., 2013; Hu et al.,2017a, 2017b), the generalized cubic enveloping developable C-Bézier surface of the single-parameter family of planes \( \{\Pi_i\} \) can be constructed as follows:

\[
K(t;\alpha_1,\alpha_2,\alpha_3) = k m(t) + V(t), \quad (k \in (-\infty, +\infty)),
\]

where

\[
\begin{align*}
&m(t) = u(t) \times u'(t) = \{u_1(t)u_2'(t) - u_2(t)u_1'(t), u_2(t)u_3'(t) - u_3(t)u_2'(t), u_3(t)u_0'(t) - u_0(t)u_3'(t), u_0(t)u_1'(t) - u_1(t)u_0'(t)\}, \\
&V(t) = m \times \sigma / m \cdot m, t \in [0,1].
\end{align*}
\]

Let

\[
q(t) = \frac{u_3'(t)(u(t) \times u'(t)) + u_1(t)(u'(t) \times u''(t)) + u_2'(t)(u''(t) \times u(t))}{u(t) \cdot [u'(t) \times u''(t)]}.
\]

A spatial parametric curve \( q(t) \) will be formed when \( t \) varies on \([0, 1]\). If taking \( q(t) \) as a spine curve and considering the developable surface as the tangent surface of the spine curve, the following generalized cubic spine curve developable H-Bézier surface can be constructed (Bodduluri et al.,1993; Zhou et al., 2013; Hu et al.,2017a, 2017b):

\[
Y(t, v : \alpha_1, \alpha_2, \alpha_3) = q(t) + v q'(t),
\]

where \( t \in [0,1], v \in (-\infty, +\infty), \alpha_i (i = 1,2,3) \) are shape parameters.

**Definition 4.** The surface formed by straight line traces is ruled surface, and its parametric equation is (Do Carmo, 1976; Mei et al., 2003; Struik, 1998)

\[
L(t, v) = a(t) + v b(t) \quad (t_0 \leq t \leq t_1, v \in \mathbb{R}),
\]

where \( a(t) \) and \( b(t) \) are all differentiable continuous functions, and \( a(t) \) is directrix of the ruled surface.

**Lemma 1.** The following conditions are all necessary and sufficient conditions for the ruled surface \( L(t, v) \) to be a developable surface (Do Carmo, 1976; Mei et al., 2003; Struik, 1998)

(1) \( (a'(t), b(t), b'(t)) = 0. \)

(2) The surface \( L(t, v) \) is an envelope of a single-parameter family of planes.

(3) The surface \( L(t, v) \) is a tangent surface of a spine curve.

(4) Its Gaussian curvature is equal to zero.

**Theorem 3.** The surfaces constructed by (17) and (19) are developable surfaces, and their Gaussian curvatures are equal to zero.
Proof. Conclusions in Theorem 3 can be directly proved by Lemma 1.

**Remark 2.** For convenience, the surfaces in (17) and (19) are called generalized cubic developable C-Bézier (GCDC-Bézier, for short) surfaces. Obviously, for \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha \), the GDCB-Bézier surfaces degenerate into the developable C-Bézier surface in (Zhou et al., 2013).

### 3.2 Analysis for the properties of the GCDC-Bézier surfaces

In terms of the expression of the GCDC-Bézier surfaces, there is a similar relationship between the control planes and the generated developable surfaces. Based on the definition of the single-parameter family of planes as well as the first and the second order derivatives of Equation (4), the following equations can be obtained when \( t = 0 \) and \( t = 1 \):

\[
\begin{align*}
S(0; \alpha_1, \alpha_2, \alpha_3) &= X_0, S'(0; \alpha_1, \alpha_2, \alpha_3) = \frac{\alpha_1(1 - \cos \alpha_1)}{\alpha_1 - \sin \alpha_1} (X_1 - X_0), \\
S^*(0; \alpha_1, \alpha_2, \alpha_3) &= \frac{\alpha_2^2 \sin \alpha_1}{\alpha_1 - \sin \alpha_1} (X_0 - X_1) + \frac{\alpha_2 \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2}{\alpha_1 - \sin \alpha_1} (X_1 - X_2),
\end{align*}
\]

and

\[
\begin{align*}
S(1; \alpha_1, \alpha_2, \alpha_3) &= X_3, S'(1; \alpha_1, \alpha_2, \alpha_3) = \frac{\alpha_3(1 - \cos \alpha_3)}{\alpha_3 - \sin \alpha_3} (X_3 - X_2), \\
S^*(1; \alpha_1, \alpha_2, \alpha_3) &= \frac{\alpha_3^2 \sin \alpha_2}{\alpha_2 \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2} (X_2 - X_1) + \frac{\alpha_3 \sin \alpha_3}{\alpha_3 - \sin \alpha_3} (X_3 - X_2). 
\end{align*}
\]

The first equations of (20) and (21) indicate that the first and the last planes in \( \{\vec{P}_i\} \) are the first and the last control planes given by designers. And the two planes are tangent to the GCDC-Bézier surfaces along its generators at \( t = 0 \) and \( t = 1 \), respectively. Furthermore, the generator of the developable surface at \( t = 0 \) is the intersection of the two planes defined by the first two equations of (20), called the starting generator. In other words, the starting generator \( K(0; \alpha_1, \alpha_2, \alpha_3) \) can be represented in the following vector form

\[
q_0 \cdot Z = X_{0,3},
\]

\[
\frac{\alpha_i(1 - \cos \alpha_1)}{\alpha_i - \sin \alpha_1} \cdot (q_1 - q_0) \cdot Z = \frac{\alpha_1(1 - \cos \alpha_1)}{\alpha_1 - \sin \alpha_1} (X_{1,3} - X_{0,3}),
\]

where \( q_i = (X_{i,0}, X_{i,1}, X_{i,2}) \), \( i = 0, 1 \), and \( Z = (x, y, z) \).

It is obvious that the control plane \( X_1 \) can be yielded according to the linear combination of the two equations in (22), therefore the generator \( K(0) \) is the intersection of the first two control planes \( X_0 \) and \( X_1 \). Meanwhile, the starting generator \( K(0) \) is dual to the line connecting \( X_0 \) and \( X_1 \). (Here, the control planes in 3D Euclidean space are regarded as points in 4D homogeneous space), which is tangent to the GCC-Bézier curves determined by \( X_i (i = 0, 1, 2, 3) \) at \( t = 0 \). Similarly, the generator \( K(1) \) is the intersection of the last two control planes \( X_2 \) and \( X_3 \) according to the first two equations in (21), called the ending generator. Furthermore, (20) can be represented as following form:

\[
\begin{align*}
X_0 &= S(0), X_1 &= S(0) + \frac{\alpha_1 - \sin \alpha_1}{\alpha_1(1 - \cos \alpha_1)} S'(0), \\
X_2 &= S(0) + \left( \frac{\alpha_1 - \sin \alpha_1}{\alpha_1(1 - \cos \alpha_1)} \cdot \frac{\alpha_1 \sin \alpha_1 Q}{\alpha_2^2 \sin \alpha_2 (1 - \cos \alpha_1)} \right) S'(0) - \frac{Q}{\alpha_2^2 \sin \alpha_2} S^*(0),
\end{align*}
\]

where \( Q = \alpha_2 \cos \alpha_2 - 2 \sin \alpha_2 + \alpha_2 \).

Equation (23) implies that the coordinates of each control plane \( X_0, X_1, X_2 \) can be derived by a linear combination of \( S(0), S'(0), S^*(0) \). On account of the intersection of the planes \( S(0), S'(0), S^*(0) \) is the...
characteristic point of developable surface, the intersection of the control planes \( X_0, X_1, X_2 \) is the characteristic point of spine curve developable surface on the starting generator \( Y(0) \). Analogously, the characteristic point on the ending generator \( Y(1) \) is determined by the control planes \( X_1, X_2, X_3 \).

3.3 Comparisons with the previous works

The LPGR methods first proposed by (Bodduluri et al., 1993 and 1994) are also called the dual representation methods, which use cubic Bernstein and B-spline basis functions combined with the duality theory to construct cubic developable Bézier (CD-Bézier, for short) and cubic developable B-spline surfaces respectively and avoid the shortcoming of the above PGR methods. Analogously, (Yang et al., 2007) used quintic Bernstein basis functions to construct quintic developable Bézier surface. Nevertheless, once the control planes are given, the shapes of the developable surfaces generated by the above LPGR methods are formed uniquely. Modifying the shapes of these developable surfaces inevitably requires the adjustments of their control planes, which is very inconvenient. Furthermore, the above LPGR methods exist a distinct inadequacy, i.e. weeny changes of the position of control planes will lead to a prodigious impact on the shape of developable surface. This means that the shapes of the developable surfaces generated by this kind of methods are difficult to control, modify and optimize. As a consequence, the LPGR methods in (Bodduluri et al., 1993, 1994; Yang et al., 2007) usually cannot meet different practical needs.

To conquer the shortcomings of the above LPGR methods, (Hu et al., 2017a) constructed developable surface with one shape parameter by a kind of \( \lambda \)-Bézier basis functions. The shape of the developable \( \lambda \)-Bézier (D-\( \lambda \)-Bézier, for short) surfaces in (Hu et al., 2017a) can be modified without altering their control planes, whereas the D-\( \lambda \)-Bézier surfaces have only one global shape parameter, which limits the adjustment of its own shape. That is, the surfaces have no local shape adjustability. Therefore, multiple shape parameters are applied to generate the developable Bézier-like (D-Bézier-like, for short) surfaces whose shapes can be adjusted locally (Hu et al., 2017b). However it is noteworthy that the developable surfaces generated by (Bodduluri et al., 1993, 1994; Yang et al., 2007; Hu et al., 2017a, 2017b) were constructed in polynomial function space.

Table 1: Performance comparisons of five different developable surfaces.

| Property of basis functions | CD-Bézier surfaces | D-\( \lambda \)-Bézier surfaces | D-Bézier-like surfaces | CDC-Bézier surfaces | GCDC-Bézier surfaces |
|-----------------------------|-------------------|-------------------------------|-----------------------|-------------------|---------------------|
| Property of developable surfaces | Polynomial | Polynomial | Polynomial | Trigonometric polynomial | Trigonometric polynomial |
| Degeneracy                  | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| Local shape adjustable property | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| Global shape adjustable property | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| Extra degree of freedom     | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| Affine invariability        | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
| Symmetry                    | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
| Boundary property           | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
In surface modeling, how to construct a developable surface in trigonometric polynomial function space (TPFS, for short) is an interesting problem. During the past 20 years, C-Bézier model defined in TPFS is adopted to design free-form curves and surfaces in CAD/CAM widely (Hoffmann et al., 2006; Shen and Wang, 2015; Yang and Wang, 2004; Zhou et al., 2017). The C-Bézier model retains the excellent properties of the Bézier one, and can be utilized to accurately represent circles and ellipses (Zhang, 1996; Chen et al., 2003). Undoubtedly, the construction about C-Bézier-based developable surfaces is a significant issue in practical applications. Inspired by the LPGR methods, (Zhou et al., 2013) first constructed cubic developable C-Bézier (CDC-Bézier, for short) surfaces in TPFS. Nevertheless, the CDC-Bézier surfaces have only one global shape parameter. In many cases, when adjusting the appearance of developable surfaces, effective methods are often desired to adjust the local and global shapes of developable surfaces. For this purpose, in this paper, the local controlled GCDC-Bézier surfaces with three shape parameters are constructed in TPFS, which are an extension of CDC-Bézier surfaces. To sum up, with the extra degree of freedom provided by the shape parameters, the GCDC-Bézier surfaces can be freely adjusted and controlled by changing the value of $\alpha_i (i = 1, 2, 3)$ instead of changing the control planes. Performances of CD-Bézier surfaces, D-$\lambda$-Bézier surfaces, D-Bézier-like surfaces, CDC-Bézier surfaces and GCDC-Bézier surfaces are compared in detail in Table 1.

4. Continuity conditions of the GCDC-Bézier surfaces

Firstly, the GCC-Bézier curves with weight coefficients in 4D homogeneous space are considered here. The expression is defined as follows:

$$\tilde{R}(t;\alpha_1,\alpha_2,\alpha_3) = \sum_{i=0}^{3} \tilde{P}_i w_i(t), t \in [0,1],$$

where $\tilde{P}_i = (\alpha_i P_i, \alpha_i) (i = 0,1,2,3)$ denote the weighted control points of $P_i (i = 0,1,2,3)$, and $\alpha_i (i = 01,2,3)$ are scalars, called weight coefficients.

When the coordinates $\tilde{P}_i = (\alpha_i P_i, \alpha_i) (i = 0,1,2,3)$ in 4D homogeneous space are understood as control planes $X_i (i = 0,1,2,3)$ in 3D Euclidean space, the single-parameter family of planes $\{\tilde{P}_i\}$ is dual to the weighted GCC-Bézier curves $\tilde{R}(t;\alpha_1,\alpha_2,\alpha_3)$. Therefore, some geometric properties (such as continuity conditions, etc.) of GCDC-Bézier surfaces are similar to the weighted GCC-Bézier curves.

For the sake of convenience, it is assumed that there are two cubic generalized developable C-Bézier surfaces need to achieve smooth continuity, whose single-parameter families of planes are

$$\begin{align*}
\{\tilde{P}_{i,1}\} & : S_1(t;\alpha_1,\alpha_2,\alpha_3) = w_{0,3}(t)X_{0,1} + w_{1,3}(t)X_{1,1} + w_{2,3}(t)X_{2,1} + w_{3,3}(t)X_{3,1}, \\
\{\tilde{P}_{i,2}\} & : S_2(t;\beta_1,\beta_2,\beta_3) = w_{0,3}(t)X_{0,2} + w_{1,3}(t)X_{1,2} + w_{2,3}(t)X_{2,2} + w_{3,3}(t)X_{3,2},
\end{align*}$$

where $\alpha_i, \beta_j (i, j = 1,2,3)$ are shape parameters; $X_{i,1} (i = 0,1,2,3)$ and $X_{j,2} (j = 0,1,2,3)$ are control planes of planes $\{\tilde{P}_{i,1}\}$ and $\{\tilde{P}_{i,2}\}$, respectively.

4.1 The $G^1$ continuity conditions of GCDC-Bézier surfaces

Based on the conditions of $G^1$ smooth continuity between two kinds of parametric curves in (Hu et al., 2017; Zhou et al., 2013), the $G^1$ continuity conditions for GCDC-Bézier surfaces in (25) can be obtained as follows:

$$\begin{align*}
S_1 (0;\beta_1,\beta_2,\beta_3) &= S_1 (1;\alpha_1,\alpha_2,\alpha_3), \\
S_2' (0;\beta_1,\beta_2,\beta_3) &= \lambda S_1'(1;\alpha_1,\alpha_2,\alpha_3),
\end{align*}$$

where $\lambda > 0$ is a constant. Then in terms of (20), (21) as well as (26), the conclusions of Theorems 5-6 can be easily obtained.

Theorem 4. For two adjacent GCDC-Bézier surfaces in (25), the necessary and sufficient conditions for $G^1$ smooth continuity at the joint are
\[
\begin{align*}
X_{0,2} &= X_{3,1}, \\
X_{1,2} &= \lambda \frac{\alpha_3(1 - \cos \alpha_3)}{\alpha_3 - \sin \alpha_3} \frac{\beta_1 - \sin \beta_1}{\beta_1(1 - \cos \beta_1)} (X_{3,1} - X_{2,1}) + X_{3,1},
\end{align*}
\]

(27)

where \( \lambda > 0 \) is a constant, which called scale factor.

4.2 \( G^2 \) Beta continuity conditions of GCDC-Bézier surfaces

Analogously, if two adjacent GCDC-Bézier surfaces in (25) need to reach \( G^2 \) Beta smooth continuity, they are required to satisfy the conditions of \( G^1 \) continuity as well as formula (28) (Hu et al., 2017; Zhou et al., 2013):

\[
S_2^*(0; \beta_1, \beta_2, \beta_3) = \lambda^2 S_1^*(1; \alpha_1, \alpha_2, \alpha_3) + \mu S_1^*(1; \alpha_1, \alpha_2, \alpha_3),
\]

(28)

where \( \mu \) is an arbitrary constant. Therefore, Theorem 6 can be easily obtained.

Theorem 5. For two adjacent GCDC-Bézier surfaces in (25), the necessary and sufficient conditions for \( G^2 \) Beta smooth continuity at the joint are

\[
\begin{align*}
X_{0,2} &= X_{3,1}, \\
X_{1,2} &= \lambda \frac{\alpha_3(1 - \cos \alpha_3)}{\alpha_3 - \sin \alpha_3} \frac{\beta_1 - \sin \beta_1}{\beta_1(1 - \cos \beta_1)} (X_{3,1} - X_{2,1}) + X_{3,1},
\end{align*}
\]

(29)

where

\[
\begin{align*}
f_{1,1} &= \frac{\lambda^2 \alpha_3^2 \sin \alpha_2}{\alpha_3 \cos \alpha_3 - 2 \sin \alpha_2 + \alpha_2} \frac{\beta_1 \cos \beta_1 - 2 \sin \beta_1 + \beta_2}{\beta_1 \sin \beta_1}, \\
f_{2,1} &= \frac{\lambda \alpha_3 (1 - \cos \alpha_3)}{\alpha_3 - \sin \alpha_3} \frac{\beta_1 - \sin \beta_1}{\beta_1(1 - \cos \beta_1)} \left[ \frac{\lambda \alpha_3^2 \sin \alpha_2}{\alpha_3 \cos \alpha_3 - 2 \sin \alpha_2 + \alpha_2} \frac{\lambda \alpha_3^2 \sin \alpha_2}{\alpha_3 - \sin \alpha_3} \right] - \frac{\lambda \alpha_3 (1 - \cos \alpha_3)}{\alpha_3 - \sin \alpha_3} \frac{\beta_1 - \sin \beta_1}{\beta_1(1 - \cos \beta_1)} \left[ \frac{\lambda \alpha_3^2 \sin \alpha_2}{\alpha_3 \cos \alpha_3 - 2 \sin \alpha_2 + \alpha_2} \frac{\lambda \alpha_3^2 \sin \alpha_2}{\alpha_3 - \sin \alpha_3} \right] \\
&+ \frac{\lambda \alpha_3 (1 - \cos \alpha_3)}{\alpha_3 - \sin \alpha_3} \frac{\beta_1 - \sin \beta_1}{\beta_1(1 - \cos \beta_1)} \left[ \frac{\lambda \alpha_3^2 \sin \alpha_2}{\alpha_3 \cos \alpha_3 - 2 \sin \alpha_2 + \alpha_2} \frac{\lambda \alpha_3^2 \sin \alpha_2}{\alpha_3 - \sin \alpha_3} \right] + 1.
\end{align*}
\]

5. Design examples of GCDC-Bézier surfaces

In this section, an example of constructing a cubic generalized developable C-Bézier surface is given. The coordinates \((a_i, b_i, c_i, d_i)\) of each control plane \(X_i (i = 0, 1, 2, 3)\) satisfy the condition of \(d_i = a_i^2 + b_i^2 + c_i^2\).

Example 1. First of all, we give an example to illustrate the construction of enveloping GCDC-Bézier surfaces, which is the envelope of the single-parameter family of planes (15) (see Fig. 3). In the case where the shape parameters take different values, different enveloping GCDC-Bézier surfaces can be constructed under the condition of given control plane.

Example 2. In this example, we present an example to show the construction of spine curve GCDC-Bézier surfaces which is consisted of the tangents of the spine curves (see Fig. 4). The influence of shape parameters on the constructed surfaces is shown in Fig. 4 clearly. And we can modify the shapes of the constructed developable surfaces flexibly according to actual requirement.

Example 3. Fig. 5 displays examples of \( G^2 \) Beta continuity between two adjacent enveloping GCDC-Bézier surfaces. In Fig. 5, the yellow surfaces are the original GCDC-Bézier surfaces \(S_j (t; \alpha_1, \alpha_2, \alpha_3)\), whose shape parameters are...
defined as $\alpha_i = 0.1\pi$ ($i = 1, 2, 3$). The green surfaces are the spliced developable surfaces $S_2(t; \beta_1, \beta_2, \beta_3)$, which is constructed by the $G^2$ Beta smooth continuity method. Compared with Fig. 5(a), Fig. 5(b) displays the composite developable surface after modifying the scale factors $\lambda$ and $\mu$; Fig. 5(c) shows the composite surface with the shape parameters $\beta_j (j = 1, 2, 3)$ changed while the rest of shape parameters remain unchanged. Furthermore, some examples of smooth continuity between two adjacent spine curve GCDC-Bézier surfaces can be given similarly.

![Fig. 3. The enveloping GCDC-Bézier surfaces with different shape parameters](image1)

![Fig. 4. The spine curve GCDC-Bézier surfaces with different shape parameters](image2)
6. Conclusions

In this paper, a class of new generalized enveloping and spine curve developable C-Bézier surfaces associated with multiple shape parameters are constructed, and some properties of the proposed surfaces are analyzed as well. In addition, in order to solve the problem of constructing complex developable surfaces, we research the geometric conditions of $G^1$ continuity and $G^2$Beta continuity between two adjacent GCDC-Bézier surfaces. Some representative and convicitive examples show the effectiveness of the proposed surfaces: the GCDC-Bézier surfaces are more advantageous than the existing CDC-Bézier surfaces described in (Zhou et al., 2013). The advantages of the GCDC-Bézier surfaces can be summarized as follows:

(a) Our proposed GCDC-Bézier surfaces extend the conclusions of developable C-Bézier surfaces presented in (Zhou et al., 2013).

(b) The GCDC-Bézier surfaces inherit entire properties of the traditional CD-Bézier surfaces; the GCDC-Bézier surfaces have more powerful shape adjustability than the classical CD-Bézier surfaces.

(c) For a composite GCDC-Bézier surface with geometric continuity condition, designers can adjust the global and local shape of the surface by changing the shape parameters without having to redetermine the control planes.
Another interesting direction for future research would be to realize shape optimization for GCDC-Bézier surfaces by finding optimal shape parameters.

Acknowledgments

The authors are very grateful to the referees for their helpful suggestions and comments which have improved the paper. This work is supported by the National Natural Science Foundation of China (No.51875454, No.11971379, and No.11702214).

References

Bodduluri, R. and Ravani, B., Design of developable surfaces using duality between plane and point geometries, Computer-Aided Design, Vol.25, No.10 (1993), pp.621–632.
Bodduluri, R. and Ravani, B., Geometric design and fabrication of developable Bézier and B-spline surfaces, ASME Transactions Journal of Mechanical Design, Vol.116, No.4 (1994), pp.1042–1048.
Chu, C.H., Wang, C. and Tsai, C.R., Computer aided geometric design of strip using developable Bézier patches, Computers in Industry, Vol.59, No.6 (2008), pp.601–611.
Chen, Q. and Wang, G., A class of Bézier-like curves, Computer Aided Geometric Design, Vol.20, No.1 (2003), pp.29–39.
Do Carmo, M., Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, 1976.
Gunter, W. and Peter, P., Computer-aided treatment of developable surfaces, Computers & Graphics, Vol.12, No.1 (1988), pp.39–51.
Hu, G., Cao, H.X., Qin, X.Q. and Wang X., Geometric design and continuity conditions of developable λ-Bézier surfaces, Adv. Eng. Soft. Vol.114, No.1 (2017a), 235–245.
Hu, G., Cao, H.X., Zhang, S.X., et al., Developable Bézier-like surfaces with multiple shape parameters and its continuity conditions, Applied Mathematics Modelling, Vol.45, No.5 (2017b), pp.728–747.
Hoffmann, M., Li, Y.J. and Wang, G.Z., Paths of C-Bézier and C-B-spline curves, Computer Aided Geometric Design, Vol.23, No.5 (2006), pp.463–475.
Kwon, D.Y., Park, F.C. and Chi, D.P., Inextensible flows of curves and developable surfaces, Applied Mathematics Letters, Vol.18, No.10 (2005), pp.1156–1162.
Maekawa, T. and Chalfant, J.S., Design and tessellation of B-spline developable surfaces, ASME Transactions Journal of Mechanical Design, Vol.120, No.3 (1998), pp.453–461.
Mei, X.M., Huang, J.Z., Differential geometry, Higher Education Press , Bei Jing, 2003. (in Chinese)
Pottmann, H. and Farin, G., Developable rational Bézier and B-spline surfaces, Computer Aided Geometric Design, Vol.12, No.5 (1995), pp.513–531.
Struik, D.J., Lectures on Classical Differential Geometry, Dover, USA, 1998.
Shen, W.Q. and Wang, G.Z., Geometric shapes of C-Bézier curves, Computer-Aided Design, Vol.58, No.1 (2015), pp.242–247.
Yang, J.Q., Zhou, M., Ye, Z.L., et al., Geometric design of adjustably developable surfaces, Chinese Journal of Mechanical Engineering, Vol.18, No.12 (2007), pp.1425–1429. (in Chinese)
Yang, Q.M. and Wang, G.Z., Inflection points and singularities on C-curves, Computer Aided Geometric Design, Vol.21, No.2 (2004), pp.207–213.
Zhang, J.W., C-curves: an extension of cubic curves, Computer Aided Geometric Design, Vol. 13, No.11 (1996), pp.199–217.
Zhou, L., Lin, X.H., Zhao, H.Y. and Chen, J., Optimal multi-degree reduction of C-Bézier surfaces with constraints, Frontiers of Information Technology and Electronic Engineering, Vol.18, No.12 (2017), pp.2009–2016.
Zhou, M., Yang, J.Q., Zheng, H.C. and Song, W.J., Design and shape adjustment of developable surfaces, Applied Mathematics Modelling, Vol.37, No.6 (2013), pp.3789–3801.