Collective Excitations of a Two-Component Bose Condensate at Finite Temperature

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We compare the collective modes for Bose-condensed systems with two degenerate components with and without spontaneous intercomponent coherence at finite temperature using the time-dependent Hartree-Fock approximation. We show that the interaction between the condensate and non-condensate in these two cases results in qualitatively different collective excitation spectra. We show that at zero temperature the single-particle excitations of the incoherent Bose condensate can be probed by intercomponent excitations.

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Introduction. For a Bose-condensed system with two internal components, one of the fundamental questions is whether one can distinguish a system with spontaneous intercomponent coherence from one without it. At zero temperature, the ground state energies calculated with the static Gross-Pitaevskii energy functional are the same for both cases. The collective excitations calculated with the linearized time-dependent Gross-Pitaevskii (TDGP) equation are also identical for both cases. For example, the collective modes for a homogeneous two-component system are

\[
\omega_{\pm}^2 = \omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 16g_{12}^2 \rho_1 \rho_2 (\epsilon_0^2)^2} / 2
\]

for both cases, as obtained from the linearized TDGP or the Bogoliubov theory, where \(\epsilon_0^2 = \hbar^2 q^2 / 2m\), \(\omega_i^2 = \epsilon_0^2 (\epsilon_0^2 + 2g_{ii} \rho_i)\) is the Bogoliubov mode for a single component system, \(g_{ii}\) is the \(i\)-th intracomponent contact interaction, \(g_{12}\) the intercomponent contact interaction, and \(\rho_i\) is the density of \(i\)-th component with \(i = 1, 2\). The excitations in Eq. may be interpreted differently for the incoherent and coherent cases are different. Without intercomponent coherence, there are two macroscopically occupied states, a situation that has come to be known as a fragmented condensate. In this case \(\omega_{\pm}\) may be interpreted as a density wave in which the two condensates move in phase, while \(\omega_{-}\) is a collective excitation associated with out of phase motion between the two condensates. With intercomponent coherence, there is only a single condensate. In this case \(\omega_{+}\) is interpreted as the superfluid mode, and \(\omega_{-}\) is a Goldstone mode corresponding to a spontaneously broken U(1) symmetry in the relative phase between the two components. As pointed out by Leggett, in the absence of particle exchange between the two components, no physical quantities can depend on the relative phase between the them. However, at finite temperatures, we will show that the coupling between the condensate and non-condensate particles provides a way of probing the nature of the condensates.

In this work, we compare the collective excitations for a two-component Bose-condensed system with and without intercomponent coherence at finite temperature using the time-dependent Hartree-Fock (TDHF) approximation, for the special case \(g_{11} = g_{22} > g_{12}\). Our principal findings are as follows. Without intercomponent coherence: (i) The motions of the two components and uncondensed particles give rise to gapless collective excitations. We find one in-phase (superfluid) mode. However, there can be two out-of-phase modes within some range of intercomponent interaction strengths at appropriate temperatures. This is illustrated in Fig. (ii) The perturbative response to a field that allows intercomponent conversion has a resonance at frequencies given by the single-particle excitation energies. With intercomponent coherence, we find a gapless superfluid mode, and two "pseudo-spin waves", one gapless and the other gapped, as illustrated in Fig. The gapped pseudo-spin wave has vanishing weight as temperature goes to zero. The differences in these spectra allow one to distinguish a single condensate from a fragmented one.

Model. We consider the collective excitations in a two-component homogenous Bose-condensed atomic gas at finite temperatures \(k_B T = 1/\beta\). To simplify the physical picture, we assume the two internal states are degenerate and take the two-body interactions as contact forms with strength parameters \(g_{11} = g_{22} = g_s = 4\pi \hbar^2 / m a_s\) and...
of energy is written as spinors with components ψ_{\lambda(\mathbf{r})} = V^{-\frac{1}{4}} e^{i \mathbf{k} \mathbf{r}} \eta_{\lambda}, with V the volume of the system and \eta_{\lambda} the spin state functions. In this work, the spinor index is denoted by Greek letters. For incoherent condensates, we write \lambda = 1, 2 with \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} and \eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} the pseudospin states of the two condensates.

\begin{equation}
\left( \omega + \varepsilon_{\mathbf{k}+\mathbf{q}} - \varepsilon_{-\mathbf{k}\lambda} \right) \chi_{\lambda\mu,\alpha\beta}(q^2, \omega) = (n_{\mathbf{k}+\mathbf{q}} - n_{-\mathbf{k}\lambda}) \delta_{\mu\lambda} \delta_{\beta\alpha} + (n_{\mathbf{k}+\mathbf{q}} - n_{-\mathbf{k}\lambda}) \sum_{ss',\lambda_1\mu_1,\mathbf{k}_1} K_{ss',\lambda_1\mu_1,\lambda\mu}(q^2, \omega) \chi_{ss',\lambda_1\mu_1,\mathbf{k}_1} \tag{2}
\end{equation}

with a vertex

\begin{equation}
K_{\lambda\mu_1,\lambda_1\mu}(q^2) = g_{ss',\lambda_1\mu_1,\mu}(1 - \delta_{\lambda\lambda_1} \epsilon_{\mathbf{k},0} - \mu_{\lambda_1} c \delta_{\mathbf{k},-\mathbf{q}}) + \epsilon_{g_{\lambda\mu_1,\mu_1}} \left[ 1 - (\delta_{\lambda\lambda_1} + \mu_{\lambda_1} \epsilon_{\mathbf{k},-\mathbf{q}}) \right]. \tag{3}
\end{equation}

Here \( s \) and \( s' \) denote the components of each spinor, and \( g_{ss',\lambda_1\mu_1,\mu} = g \delta_{s,s'} \delta_{\lambda_1,\lambda} \delta_{\mu_1,\mu} \). The Kronecker delta terms \( \delta_{\lambda\lambda_1} \) are 1 when \( \lambda \) is a condensate state, and 0 otherwise. Poles of the response functions correspond to collective excitations of the system.

**Results.** We first consider incoherent condensates. In this case, the density-density response function matrix has only six non-zero matrix elements: \( \chi_{11,11}(q^2, \omega) = \chi_{22,22}(q^2, \omega), \chi_{11,22}(q^2, \omega) = \chi_{22,11}(q^2, \omega), \) and \( \chi_{12,21}(q^2, \omega) = \chi_{21,12}(-q^2, -\omega), \) where \( \chi_{\lambda\mu,\alpha\beta}(q^2, \omega) = \frac{1}{V} \sum_{\mathbf{q}_1\mathbf{q}_2} \chi_{\lambda_1\mu_1,\alpha\beta}(q^2, \omega). \) The equations for the first four functions are decoupled from those of the latter two.

The functions \( \chi_{11,11}(q^2, \omega) \) and \( \chi_{22,11}(q^2, \omega) \) describe the density response of the two components. From Eqs. 2 and 3, one obtains

\begin{equation}
\begin{pmatrix}
1 - 2g_{s} P & -g_{s} P \\
-g_{s} P & 1 - 2g_{s} P
\end{pmatrix}
\begin{pmatrix}
\chi_{11,11} \\
\chi_{22,11}
\end{pmatrix}
= \begin{pmatrix}
P \\
0
\end{pmatrix}, \tag{4}
\end{equation}

where \( P(q^2, \omega) = P_c(q^2, \omega) + P_n(q^2, \omega) \) with \( P_c(q^2, \omega) = \frac{2m \omega x^2}{\omega^2 - (\varepsilon_{q^2} + 2g_{s} P_n)^2} \) and \( P_n(q^2, \omega) = \frac{n_{\varepsilon_1}}{n_{\omega} + \varepsilon_{q^2}} \), with \( n_{\varepsilon_1} = n_{\varepsilon_2}. \) The poles of these \( \chi \)’s occur when the determinant of the matrix in Eq. 4 vanishes,

\begin{equation}
1 - 2g_{s} \pm g_{s} \left[ P_c(q^2, \omega) + P_n(q^2, \omega) \right] = 0. \tag{5}
\end{equation}

Equation 5 shows the effect of intercomponent interaction and the interaction between the condensate and non-condensate on the collective excitations. The plus

**Method.** We briefly outline our method here; details will be presented elsewhere. It is well known that to treat the dynamics of a condensate and thermally excited (non-condensate) particles in a fully consistent manner is theoretically challenging [8]. Recently we have shown this can be done within a TDHF approximation [9] by using a constrained grand canonical ensemble [10] in which the condensate particle number is fixed. We first determine the static Hartree-Fock (HF) ground state properties, including the chemical potential \( \mu \), the single-particle energies \( \varepsilon_{\mathbf{k}a} \), the occupation numbers \( n_{\mathbf{k}a} \), and the critical temperature \( T_c \). This involves defining a self energy \( \Sigma \) which is a functional of the densities \( \rho_{\mathbf{k}1a_1,\mathbf{k}2a_2} = a_{\mathbf{k}1a_1}^{\dagger} a_{\mathbf{k}2a_2} \), with \( a_{\mathbf{k}a} \) the boson destruction operator, and the condensate wavefunction(s) \( \psi_{\lambda_0} \). We then introduce a small perturbation \( \delta U \), and assume \( \Sigma \) keeps its functional form with respect to the \( \rho \)'s and \( \psi_{\lambda_0} \), which are now time-dependent. This results in self-consistent equations which can be solved in the linear response regime. This allows us to obtain equations for the response functions \( \chi_{\lambda_0,\mu_0,\alpha,\beta}(q, \omega) = \int_{0}^{\infty} e^{-i\Omega t} \langle a_{\mathbf{k}1a_1}^{\dagger}(t) a_{\mathbf{k}2a_2}(0) a_{\mathbf{k}3a_3}(0) a_{\mathbf{k}4a_4}(0) \rangle \).

**FIG. 2:** (Color online) The two pseudo-spin modes in the system with intercomponent coherence.
and minus signs in Eq. (6) can be interpreted as resulting from the in-phase and out-of-phase motion of the two components, respectively. We find only one in-phase (gapless) mode as a solution, which reduces to $\omega_+$ in Eq. (1) in the zero temperature limit. Thus the interaction between the condensate and non-condensate has little effect on the in-phase motion. However, this interaction has a striking effect on the out-of-phase motion. Within a range $g_x < g_{x,1} < g_{x,2}$, the values $g_x,1$ and $g_x,2$ depending on the gas parameter $\rho a^3$ and temperature $T$, the system can support a new out-of-phase mode for small $q$.

This is illustrated Fig. 3 for $\rho a^3 = 10^{-4}$ and $\beta/\beta_c = 3$, where it can be seen that $\frac{1}{\omega^2} [\chi_{1111}(\vec{q};\omega) - \chi_{2211}(\vec{q};\omega)]$ carries an extra resonance. In order to see why this occurs, we show the polarization functions $P_c$ and $P_n$ in Fig. 3. For $\varepsilon_0^0 < 2g_x\rho_0$, $P_c(\vec{q};\omega)$ is a monotonic function of $\omega$ and approaches $1/g_x$ with zero slope as $\omega \to 0$, while $P_n(\vec{q};\omega)$ is non-monotonic at small $\omega$. The combination $g_x(2 - \gamma)P(\vec{q};\omega)$ is then non-monotonic at small $\omega$, so for certain ranges of $\gamma$, Eq. (3) is satisfied twice. This is shown in the inset in Fig. 3.

The function $\chi_{21,12}(\vec{q};\omega)$ supports a collective mode [11, 12, 13], corresponding to an excitation in which the “flavor” of a particle changes. From Eq. (2) one finds

$$\chi_{21,12}(\vec{q};\omega) = \frac{P_{12}^c(\vec{q};\omega) + P_n^c(\vec{q};\omega)}{1 - g_x [P_{12}^c(\vec{q};\omega) + P_n^c(\vec{q};\omega)]},$$

(6)

where

$$P_{12}^c(\vec{q};\omega) = \frac{2\rho_0\left(\varepsilon_0^0 + g_x\rho_0\right)}{\omega^2 - \left(\varepsilon_0^0 + g_x\rho_0\right)^2}.$$

(7)

At zero temperature the collective mode requires exciting a particle from one of the condensates to an excited state of the other component, so $\chi_{21,12}$ has a pole at the single particle excitation spectrum, $\omega = \varepsilon_q - \mu = \varepsilon_0^0 + g_x\rho_0$. Thus, this provides a way to probe the single-particle spectrum. We note that, since $\chi_{21,12}$ is the response to a “flavor-changing” perturbation, this collective mode is not found in the TDGP approximation when species exchange is absent from the unperturbed Hamiltonian. It is apparent that our TDHF approach is able to capture a broader variety of excitations for this system.

We now turn to the coherent case. The system can condense into any spinor given by $\eta(\theta) = \frac{1}{2}\left(\eta_1 + e^{i\theta}\eta_2\right)$, resulting in a single condensate. We choose the system condensing into $\eta_+ = \eta(\theta = 0)$, which spontaneously breaks the U(1) symmetry. Therefore, we would expect at least two gapless modes: a superfluid density mode and a Goldstone mode (a pseudo-spin wave) for the spontaneous U(1) symmetry breaking.

The density wave may be found from the poles of $\chi_+(\vec{q};\omega) = \sum_{k,\gamma\alpha\beta} \chi_+(\vec{q};\omega)$ and $\chi_-(\vec{q};\omega) = \sum_{k,\gamma\alpha\beta} \chi_-(\vec{q};\omega)$, which follow a matrix equation obtained from Eq. (2).

$$\begin{pmatrix}
1 - g_x P_{++} & -g_x P_{++} \\
-g_x P_{--} & 1 - g_x P_{--}
\end{pmatrix}
\begin{pmatrix}
\chi_+ \\
\chi_-
\end{pmatrix}
= \begin{pmatrix}
P_{++} \\
P_{--}
\end{pmatrix},$$

(8)

where $g_P = g_x + g_{x,1}$, $P_{++}(\vec{q};\omega) = P_{++}^c(\vec{q};\omega) + P_{++}^n(\vec{q};\omega)$, $P_{--}(\vec{q};\omega) = P_{--}^c(\vec{q};\omega) + P_{--}^n(\vec{q};\omega)$ and $P_{\alpha\beta}^c(\vec{q};\omega) = \frac{1}{n} \sum_k \frac{\eta_+ e_k \eta_- e_k}{\omega^2 - (\varepsilon_0^0 + g_x\rho_0)^2}$, where the prime excludes the momentum of the condensate modes. As in the single component case, the non-condensate is too dilute to sustain a propagating second sound mode within this approximation. The resulting response and superfluid mode are thus qualitatively similar to the single component case.

By contrast, the non-condensate component has a dramatic effect on the pseudo-spin waves. In order to see this, we define $\chi_{21,12}(\vec{q};\omega) = \sum_{k,\gamma\alpha\beta} \chi_{21,12}(\vec{q};\omega)$ and split these into condensate and non-condensate parts: $\chi_+ - (\vec{q};\omega) = \sum_{k,\gamma\alpha\beta} \chi_{21,12}^c(\vec{q};\omega)$, $\chi_- - (\vec{q};\omega) = \sum_{k,\gamma\alpha\beta} \chi_{21,12}^n(\vec{q};\omega)$, and $\chi_+ - (\vec{q};\omega) = \sum_{k,\gamma\alpha\beta} \chi_{21,12}'(\vec{q};\omega)$, where the prime on the sum excludes the momentum of the condensate mode. In terms of these quantities, Eq. (2) yields

$$\begin{align*}
\sum_{k,\gamma\alpha\beta} \chi_{21,12}(\vec{q};\omega) & = \frac{2\rho_0\left(\varepsilon_0^0 + g_x\rho_0\right)}{\omega^2 - \left(\varepsilon_0^0 + g_x\rho_0\right)^2}, \\
\sum_{k,\gamma\alpha\beta} \chi_{21,12}^c(\vec{q};\omega) & = \frac{2\rho_0\left(\varepsilon_0^0 + g_x\rho_0\right)}{\omega^2 - \left(\varepsilon_0^0 + g_x\rho_0\right)^2} \sum_{k,\gamma\alpha\beta} \chi_{21,12}'(\vec{q};\omega), \\
\sum_{k,\gamma\alpha\beta} \chi_{21,12}^n(\vec{q};\omega) & = \frac{2\rho_0\left(\varepsilon_0^0 + g_x\rho_0\right)}{\omega^2 - \left(\varepsilon_0^0 + g_x\rho_0\right)^2} \sum_{k,\gamma\alpha\beta} \chi_{21,12}'(\vec{q};\omega).
\end{align*}$$

FIG. 3: (Color online) The condensate and non-condensate polarization functions at which the new out-of-phase mode appears. The inset shows $g_x(2 - \gamma)P(\vec{q};\omega)$. The arrows show the locations of the two out-of-phase modes.

\begin{align*}
\sum_{k,\gamma\alpha\beta} \chi_{21,12}(\vec{q};\omega) & = \frac{2\rho_0\left(\varepsilon_0^0 + g_x\rho_0\right)}{\omega^2 - \left(\varepsilon_0^0 + g_x\rho_0\right)^2}, \\
\sum_{k,\gamma\alpha\beta} \chi_{21,12}^c(\vec{q};\omega) & = \frac{2\rho_0\left(\varepsilon_0^0 + g_x\rho_0\right)}{\omega^2 - \left(\varepsilon_0^0 + g_x\rho_0\right)^2} \sum_{k,\gamma\alpha\beta} \chi_{21,12}'(\vec{q};\omega), \\
\sum_{k,\gamma\alpha\beta} \chi_{21,12}^n(\vec{q};\omega) & = \frac{2\rho_0\left(\varepsilon_0^0 + g_x\rho_0\right)}{\omega^2 - \left(\varepsilon_0^0 + g_x\rho_0\right)^2} \sum_{k,\gamma\alpha\beta} \chi_{21,12}'(\vec{q};\omega).
\end{align*}
where $g_m = g_s - g_x$. Eq. 9 yields two spin waves, as shown in Fig. 2, one of which is gapless and originates from the U(1) symmetry breaking of the relative phase of the two internal states. At $T = 0$, this gapless mode is reduced to $\omega_-$ given by Eq. 11. The other spin wave has a gap as $q \to 0$ given by

$$\Delta = \sqrt{g_x^2 \left( \rho_0 + (\rho_+ - \rho_-)^2 \right) + g_x g_p \rho_0 (\rho_+ - \rho_-).} \quad (10)$$

Here $\rho_+$ and $\rho_-$ are the non-condensate density in spinor $\eta_+$ and $\eta_-$, respectively. The gapped spin wave, unlike the gapped flavor-changing mode obtained from Eq. 10 in the incoherent case, disappears as $T \to 0$ (i.e., its weight in the response functions vanishes), which can be seen directly from Eq. 9 since both $P_{n-}^0$ and $P_{n+}^0$ are zero in this limit.

In the above calculations we have discussed the special case of interaction strengths $g_{11} = g_{22}$. For $g_{11} \neq g_{22}$ but $g_{12}^2 < g_{11} g_{22}$ so that the homogeneous state is still stable against phase separation, we find for the incoherent case results which are qualitatively the same. The coherent system, by contrast, is more complicated. In particular one finds that the spinors of the non-condensate modes are neither parallel nor anti-parallel to the condensate mode spinor. This leads to the response functions in Eq. 2 all being coupled together, so a classification of the modes as flavor-changing and flavor-preserving is no longer possible. Nevertheless, one still finds two gapless linear modes and one gapped mode as we found in the $g_{11} = g_{22}$ case. Details of this more complicated situation will be presented elsewhere.

In conclusion, we have compared the collective excitations of two-component Bose-condensed systems with and without coherence at finite temperature. The coupling between the condensate and non-condensate depends on whether there is intercomponent coherence, and the two identical dispersions given by Eq. 11 at zero temperature evolve to quite different structures. We also have shown that, at zero temperature the single-particle excitations in the incoherent case can be probed by flavor-changing excitations, which can not be done in the coherent case.

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