ABSTRACT A q-rung orthopair fuzzy set is one of the effective generalizations of fuzzy set for dealing with uncertainties in information. Under this environment, in this study, we define a new type of extensions of fuzzy sets called n,m-rung orthopair fuzzy sets and investigate their relationship with Fermatean fuzzy sets, Pythagorean fuzzy sets and intuitionistic fuzzy sets. The n,m-rung orthopair fuzzy sets can supply with more doubtful circumstances than Fermatean fuzzy sets, Pythagorean fuzzy sets and intuitionistic fuzzy sets because of their larger range of depicting the membership grades. There is a symmetry between the values of this membership function and non-membership function. Here, any power function scales are utilized to widen the scope of the decision-making problems. In addition, the novel notion of an n,m-rung orthopair fuzzy set through double universes is more flexible when debating the symmetry between two or more objects that are better than the diffusing concept of an n-rung orthopair fuzzy set, as well as m-rung orthopair fuzzy set. The main advantage of n,m-rung orthopair fuzzy sets is that it can describe more uncertainties than Fermatean fuzzy sets, which can be applied in many decision-making problems. Then, we discover the essential set of operations for the n,m-rung orthopair fuzzy sets along with their several properties. Finally, we introduce a new operator, namely, n,m-rung orthopair fuzzy weighted power average (n,m-ROFWPA) over n,m-rung orthopair fuzzy sets and apply this operator to the MADM problems for evaluation of alternatives with n,m-rung orthopair fuzzy information.

INDEX TERMS n,m-rung orthopair fuzzy sets, operations, score function, aggregation operator.

I. INTRODUCTION

Zadeh [27] introduced the concept of fuzzy sets to handle the imprecise information, and after that several researches were conducted on the generalizations of the notion of fuzzy set. It is seen to have many applications related to the fuzzy set theory in both theoretical and practical studies from health sciences to computer science, from physical sciences to arts, and from engineering and humanities to life sciences. For theoretical study examples, refer to the [6], [15], [19], [28], [29], and [30]. The notion of rough set theory originally introduced by Pawlak [20]. The concept of soft sets was first defined by Molodtsov [16] as a general mathematical tool for dealing with uncertain objects. The merging between fuzzy sets and some doubtfulness approaches like rough sets and soft sets have been discussed in [1], [3], and [5]. Intuitionistic fuzzy sets defined by Atanassov [2] and its one of the interesting generalizations of fuzzy sets with excellent applicability. In various fields the applications of intuitionistic fuzzy sets appear, including optimization problems decision-making, medical diagnosis [8], [9], [10]. But, there are many cases where the decision maker may give the degree of membership and non-membership of a specific characteristic in such a way that their sum is greater than one. Hence, Yager [24] presented the concept of Pythagorean fuzzy set which is the generalization of intuitionistic fuzzy sets and it is a more effective tool to solve uncertain problems. The construct of Pythagorean fuzzy sets can be used to describe doubtful
information more adequately and precisely than intuitionistic fuzzy set. Ibrahim et al. [11] defined another type of generalized Pythagorean fuzzy set is called (3, 2)-Fuzzy sets. Fermatean fuzzy sets studied by Senapati et al. [17], and they also introduced fundamental operations on the Fermatean fuzzy sets. Murad et al. [12] presented new kind of fuzzy sets is called (3, 4)-fuzzy sets which is deal with more uncertain situations than (3, 2)-Fuzzy. Yang et al. [26] investigated the construction of shadowed sets from Atanassov intuitionistic fuzzy sets in light of three-way decision.

Multiple Attribute Decision-Making (MADM) is a method that particularly considers the best potential alternatives. A multiplicity of valuable mathematical methods, such as soft sets, and fuzzy sets, were improved to tackle the complexities and complexity of MADM problems. Xu [21] proposed the idea of intuitionistic fuzzy weighted averaging operators. Xu et al. [22] under the environment of intuitionistic fuzzy sets introduced some geometric weighted and geometric hybrid operators. Yager et al. [23], [25], presented the concepts of weighted averaging, and ordered weighted averaging operators over Pythagorean fuzzy sets. Senapati et al. [18] defined Fermatean fuzzy weighted power average (FFWPA) operator over Fermatean fuzzy sets and they discussed their properties in details. Verma [14] defined four new order-α divergence measures between two intuitionistic fuzzy sets MABAC method to solve MAGDM problems in which the information about the attribute weights is completely unknown or partially known. Three different MCDM methods under Pythagorean fuzzy environment, IVPF-AHP for determining weights, PF-TOPSIS and PF-VIKOR for ranking and prioritizing process, were used for prioritization of the risks associated with self-driving vehicles by Bakioglu et al. [4]. Verma [13] defined two new order-α divergence measures between 4RosPFSs based on logarithmic and exponential functions. Deng et al. [7] defined two novel distance measure methods for Fermatean fuzzy sets which are called the FFSH distance and FFSTD distance.

The purpose of writing this paper are 1) to introduce a novel extension of intuitionistic fuzzy set called n,m-rung orthopair fuzzy sets which is not obtained from q-rung orthopair fuzzy sets; 2) to present the set of operations for the n,m-rung orthopair fuzzy sets and discuss their main properties; 3) to investigate a rough topology on n,m-rung orthopair fuzzy sets; and 4) to reinforce a technique for multi-criteria decision making (MCDM) in light of new operator under n.m-rung orthopair fuzzy condition.

In this paper, we present the concept of n,m-rung orthopair fuzzy sets and compare it with the other types of fuzzy sets. Then, we propose the set of operations for the n,m-rung orthopair fuzzy sets and explore their main features. Also, the concept of weighted power average operator for n,m-rung orthopair fuzzy sets is investigated. Thereafter, we describe MADM problems under this operator. Finally, we outline the main achievements of the paper and propose some upcoming works in Section 6. The motivation and objectives of this extended work are given step by step in this paper.

II. n,m-RUNG ORTHOPAIR FUZZY SETS

In this section, we study the notion of n,m-rung orthopair fuzzy sets in details. For computations, we use only four decimal places in whole paper.

A. n,m-RUNG ORTHOPAIR FUZZY SETS

Definition 1: Assume that N is a set of all natural numbers and Z is a universal set. Then, the n,m-rung orthopair fuzzy set (n,m-ROFS) \( \mathcal{A} \), which is a set of ordered pairs over \( Z \), is defined by the following:

\[
\mathcal{A} = \{ (d, \theta_\mathcal{A}(d), \psi_\mathcal{A}(d)) : d \in Z \},
\]

where \( \theta_\mathcal{A}(d) : Z \to [0, 1] \) and \( \psi_\mathcal{A}(d) : Z \to [0, 1] \) denote the degree of membership and the degree of non-membership, respectively, of the element \( d \in Z \) to \( \mathcal{A} \), and including the condition

\[
0 \leq (\theta_\mathcal{A}(d))^n + (\psi_\mathcal{A}(d))^m \leq 1,
\]

for all \( d \in Z \) and \( n, m \in N \) such that \( n \neq m \). Then, there is a degree of indeterminacy of \( d \in Z \) to \( \mathcal{A} \) defined by

\[
\pi_\mathcal{A}(d) = \frac{n^{m/n} \sqrt{1 - [(\theta_\mathcal{A}(d))^n + (\psi_\mathcal{A}(d))^m]}}{n^{m/n}} \text{ and } \pi_\mathcal{A}(d) \in [0, 1].
\]

For the sake of simplicity, we shall mention the symbol \( \mathcal{A} = (\theta_\mathcal{A}, \psi_\mathcal{A}) \) for the n,m-ROFS \( \mathcal{A} = \{ (d, \theta_\mathcal{A}(d), \psi_\mathcal{A}(d)) : d \in Z \} \).

The space of some types of n,m-rung orthopair fuzzy membership grades can be shown as in the Figures 1, 2, 3 and 4.

Definition 2: Let \( Z \) be a universal set. The Fermatean fuzzy set (FFS) [17] (resp. Pythagorean fuzzy set (FPS) [24] and intuitionistic fuzzy set (IFS) [2]) is presented as the following:

\[
\mathcal{A} = \{ (d, \theta_\mathcal{A}(d), \psi_\mathcal{A}(d)) : d \in Z \},
\]

including the condition \( 0 \leq (\theta_\mathcal{A}(d))^3 + (\psi_\mathcal{A}(d))^3 \leq 1 \) (resp. \( 0 \leq (\theta_\mathcal{A}(d))^2 + (\psi_\mathcal{A}(d))^2 \leq 1 \) and \( 0 \leq \theta_\mathcal{A}(d) + \psi_\mathcal{A}(d) \leq 1 \)), where \( \theta_\mathcal{A}(d) : Z \to [0, 1] \) is the degree of membership and \( \psi_\mathcal{A}(d) : Z \to [0, 1] \) is the degree of non-membership of every \( d \in Z \) to \( \mathcal{A} \).

Remark 3: From Figure 1, the set of

1) intuitionistic membership grades is smaller than the set of 1,m-rung orthopair fuzzy membership grades for \( m > 1 \).

2) 1,2-rung orthopair fuzzy membership grades is smaller than the set of Pythagorean membership grades.

3) Fermatean membership grades is larger than the set of 1,3-rung orthopair fuzzy and 2,3-rung orthopair fuzzy membership grades.

Remark 4: From Figure 2, the set of

1) 2,m-rung orthopair fuzzy membership grades is larger than the set of Pythagorean membership grades for \( m > 2 \).

2) 2,1-rung orthopair fuzzy membership grades is smaller than the set of Pythagorean membership grades.
Remark 5: From Figure 3, the set of Fermatean membership grades is

1) smaller than the set of 3,m-rung orthopair fuzzy membership grades for \( m > 3 \).

2) larger than the set of 3,1-rung orthopair fuzzy and 3,2-rung orthopair fuzzy membership grades.

Example 7: Suppose that \( \vartheta(0.9) = 0.9 \) and \( \psi(0.7) = 0.7 \) for \( Z = \{d\} \). Hence, \( 0.9 + 0.7 = 1.6 > 1 \), \((0.9)^2 + (0.7)^2 = 1.3 > 1 \) and \((0.9)^3 + (0.7)^3 = 1.072 > 1 \) but \((0.9)^4 + (0.7)^4 = 0.9991 \leq 1 \) and \((0.9)^3 + (0.7)^4 = 0.9691 \leq 1 \). Thus, \( \mathcal{R} = (0.9, 0.7) \) is both 3,4-ROFS and 4,3-ROFS, but \( \mathcal{R} \) is neither IFS nor PFS nor FFS.

Definition 8: Let \( \mathcal{R} = (\vartheta_{\overline{N}}, \psi_{\overline{N}}) \), \( \mathcal{R}_1 = (\vartheta_{\overline{N}_1}, \psi_{\overline{N}_1}) \) and \( \mathcal{R}_2 = (\vartheta_{\overline{N}_2}, \psi_{\overline{N}_2}) \) be three n,m-ROFSs and \( \eta \) be a positive real number (\( \eta > 0 \)), then their operations are defined as follows:

1) \( \mathcal{R}_1 \cap \mathcal{R}_2 = (\min\{\vartheta_{\overline{N}_1}, \vartheta_{\overline{N}_2}\}, \max\{\psi_{\overline{N}_1}, \psi_{\overline{N}_2}\}) \).
2) \( \mathcal{R}_1 \cup \mathcal{R}_2 = (\max\{\vartheta_{\overline{N}_1}, \vartheta_{\overline{N}_2}\}, \min\{\psi_{\overline{N}_1}, \psi_{\overline{N}_2}\}) \).
3) \( \mathcal{R}^\eta = (\sqrt[\eta]{\vartheta_{\overline{N}}}, \sqrt[\eta]{\psi_{\overline{N}}}) \), for \( n, \vartheta \in \mathbb{N} \setminus \{1\} \).
4) \( \mathcal{R}_1 \oplus \mathcal{R}_2 = \left(\frac{\vartheta_{\overline{N}_1} + \vartheta_{\overline{N}_2}}{\vartheta_{\overline{N}_1} \vartheta_{\overline{N}_2}}, \frac{\psi_{\overline{N}_1} + \psi_{\overline{N}_2}}{\psi_{\overline{N}_1} \psi_{\overline{N}_2}}\right) \), for \( n, \vartheta \in \mathbb{N} \setminus \{1\} \).
5) \( \mathcal{R}_1 \otimes \mathcal{R}_2 = \left(\frac{\vartheta_{\overline{N}_1} \vartheta_{\overline{N}_2}}{\vartheta_{\overline{N}_1} + \vartheta_{\overline{N}_2}}, \frac{\psi_{\overline{N}_1} \psi_{\overline{N}_2}}{\psi_{\overline{N}_1} + \psi_{\overline{N}_2}}\right) \), for \( n, \vartheta \in \mathbb{N} \setminus \{1\} \).
6) \( \eta\mathcal{R} = (\sqrt[\eta]{\vartheta_{\overline{N}}}, \sqrt[\eta]{\psi_{\overline{N}}}) \), for \( n, \vartheta \in \mathbb{N} \setminus \{1\} \).
7) \( \mathcal{R}^\eta = (\sqrt[\eta]{\vartheta_{\overline{N}}}, \sqrt[\eta]{\vartheta_{\overline{N}}}) \), for \( n, \vartheta \in \mathbb{N} \setminus \{1\} \).

Example 9: Assume that \( \mathcal{R}_1 = (0.92, 0.83) \) and \( \mathcal{R}_2 = (0.93, 0.82) \) are both 5,6-ROFSs for \( Z = \{d\} \). Then:

1) \( \mathcal{R}_1 \cap \mathcal{R}_2 = (\min\{0.92, 0.93\}, \max\{0.83, 0.82\}) = (0.92, 0.83) \).
2) $\mathfrak{N}_1 \cup \mathfrak{N}_2 = (\max\{\vartheta_{\mathfrak{N}_1}, \vartheta_{\mathfrak{N}_2}\}, \min\{\varphi_{\mathfrak{N}_1}, \varphi_{\mathfrak{N}_2}\}) = (\max(0.92, 0.93), \min(0.83, 0.82)) = (0.93, 0.82).

3) $\mathfrak{N}_1 \approx (0.7996, 0.9329).

4) $\mathfrak{N}_1 \oplus \mathfrak{N}_2 = \left(\sqrt{\vartheta_{\mathfrak{N}_1}^n + \vartheta_{\mathfrak{N}_2}^n - \vartheta_{\mathfrak{N}_1}^m \vartheta_{\mathfrak{N}_2}^m}, \varphi_{\mathfrak{N}_1} \varphi_{\mathfrak{N}_2}\right) = \left(\sqrt{0.92^3 - 0.92^{2.93} - 0.92^{3.82}}, \sqrt{0.83}\right) \approx (0.9783, 0.6806).

5) $\mathfrak{N}_1 \otimes \mathfrak{N}_2 = \left(\vartheta_{\mathfrak{N}_1}, m \varphi_{\mathfrak{N}_1} + m \varphi_{\mathfrak{N}_2}, \varphi_{\mathfrak{N}_1} \varphi_{\mathfrak{N}_2}\right) = \left(0.92, 0.93\right) \geq \left(0.83^{0.82}, 0.83\right) \approx (0.8556, 0.9000).

6) $\eta \mathfrak{N}_1 = \left(\sqrt{1 - (1 - \vartheta_{\mathfrak{N}_1}^m)^n}, \varphi_{\mathfrak{N}_1}\right) = \left(\sqrt[3]{1 - (1 - 0.92^5)^3}, 0.83\right) \approx (0.9920, 0.5718), \text{ for } \eta = 3.

7) $\eta \mathfrak{N}_2 = \left(\vartheta_{\mathfrak{N}_2}, m \varphi_{\mathfrak{N}_2} - \vartheta_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_2}, \varphi_{\mathfrak{N}_1} \varphi_{\mathfrak{N}_2}\right) = \left(0.92, 0.93\right) \geq \left(0.83^{0.82}, 0.83\right) \approx (0.7787, 0.9412), \text{ for } \eta = 3.

Theorem 10: If $\mathfrak{N} = (\vartheta_{\mathfrak{N}}, \varphi_{\mathfrak{N}})$ is a n,m-ROFS, then $\mathfrak{N}^c$ is also a n,m-ROFS and $\mathfrak{N}^c = \mathfrak{N}$.

Proof: Since $0 \leq \vartheta_{\mathfrak{N}}^n \leq 1$, $0 \leq \varphi_{\mathfrak{N}}^m \leq 1$, and $0 \leq \vartheta_{\mathfrak{N}}^m \leq 1$, then $0 \leq \sqrt[3]{\varphi_{\mathfrak{N}}^m} + m \varphi_{\mathfrak{N}}^m + m \varphi_{\mathfrak{N}}^m \varphi_{\mathfrak{N}}^m \leq 1$. Hence, $\mathfrak{N}^c$ is a n,m-ROFS and it is obvious that $\left(\mathfrak{N}^c\right)^c = \left(\sqrt[3]{\varphi_{\mathfrak{N}}^m} + m \varphi_{\mathfrak{N}}^m + m \varphi_{\mathfrak{N}}^m \varphi_{\mathfrak{N}}^m\right)^c = \left(\vartheta_{\mathfrak{N}}, \varphi_{\mathfrak{N}}\right) = \mathfrak{N}$.

Theorem 11: If $\mathfrak{N}_1 = (\vartheta_{\mathfrak{N}_1}, \varphi_{\mathfrak{N}_1})$ and $\mathfrak{N}_2 = (\vartheta_{\mathfrak{N}_2}, \varphi_{\mathfrak{N}_2})$ are two n,m-ROFSs, then $\mathfrak{N}_1 \oplus \mathfrak{N}_2$ and $\mathfrak{N}_1 \otimes \mathfrak{N}_2$ are also n,m-ROFSs.

Proof: For n,m-ROFSs $\mathfrak{N}_1 = (\vartheta_{\mathfrak{N}_1}, \varphi_{\mathfrak{N}_1})$ and $\mathfrak{N}_2 = (\vartheta_{\mathfrak{N}_2}, \varphi_{\mathfrak{N}_2})$ the following relations are evident:

$$0 \leq \vartheta_{\mathfrak{N}_1}^n \leq 1, 0 \leq \varphi_{\mathfrak{N}_1}^m \leq 1, 0 \leq \vartheta_{\mathfrak{N}_2}^n \leq 1, 0 \leq \varphi_{\mathfrak{N}_2}^m \leq 1,$$

and

$$0 \leq \vartheta_{\mathfrak{N}_1}^m \leq 1, 0 \leq \varphi_{\mathfrak{N}_1}^m \leq 1, 0 \leq \vartheta_{\mathfrak{N}_2}^m \leq 1, 0 \leq \varphi_{\mathfrak{N}_2}^m \leq 1.$$

Then, we have

$$\vartheta_{\mathfrak{N}_1}^n \geq \vartheta_{\mathfrak{N}_1}^n \varphi_{\mathfrak{N}_2}^m \varphi_{\mathfrak{N}_2}^m, \vartheta_{\mathfrak{N}_2}^n \geq \vartheta_{\mathfrak{N}_2}^n \varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_1}^m, 1 \geq \vartheta_{\mathfrak{N}_1}^n \varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_1}^m \geq 0$$

and

$$\varphi_{\mathfrak{N}_1}^m \geq \varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_2}^m \varphi_{\mathfrak{N}_2}^m, \varphi_{\mathfrak{N}_2}^m \geq \varphi_{\mathfrak{N}_2}^m \varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_1}^m, 1 \geq \varphi_{\mathfrak{N}_2}^m \varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_1}^m \geq 0$$

which indicates that $\vartheta_{\mathfrak{N}_1}^n + \vartheta_{\mathfrak{N}_2}^n - \vartheta_{\mathfrak{N}_1}^m \vartheta_{\mathfrak{N}_2}^m \geq 0$, implies $\sqrt[3]{\varphi_{\mathfrak{N}_1}^n + \varphi_{\mathfrak{N}_2}^n} \geq \sqrt[3]{\varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_2}^m}$, and $\varphi_{\mathfrak{N}_1}^m + \varphi_{\mathfrak{N}_2}^m - \varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_1}^m \geq 0$, implies $\sqrt[3]{\varphi_{\mathfrak{N}_2}^m \varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_1}^m} \geq 0$.

Since $\vartheta_{\mathfrak{N}_1}^n \leq 1$ and $0 \leq \varphi_{\mathfrak{N}_1}^m \leq 1$, then $\vartheta_{\mathfrak{N}_1}^n (1 - \varphi_{\mathfrak{N}_1}^m) \leq (1 - \vartheta_{\mathfrak{N}_2}^m)$ and we get $\vartheta_{\mathfrak{N}_1}^n + \vartheta_{\mathfrak{N}_2}^n - \vartheta_{\mathfrak{N}_1}^m \vartheta_{\mathfrak{N}_2}^m \leq 1$ and hence $\sqrt[3]{\varphi_{\mathfrak{N}_1}^n + \varphi_{\mathfrak{N}_2}^n} \leq 1$.

Similarly, we can get

$$\sqrt[3]{\varphi_{\mathfrak{N}_1}^n + \varphi_{\mathfrak{N}_2}^n} \left(1 - \varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_1}^m \varphi_{\mathfrak{N}_1}^m\right) \leq 1.$$

It is obvious that

$$0 \leq \varphi_{\mathfrak{N}_2}^n \leq 1 - \varphi_{\mathfrak{N}_1}^n \leq 0 \leq \varphi_{\mathfrak{N}_2}^m \leq 1 \geq \varphi_{\mathfrak{N}_1}^m.$$
\[
\min(\max\{\psi_{R_1}, \psi_{R_2}\}, \psi_{R_3}) = (\min(\psi_{R_2}, \psi_{R_3}), \psi_{R_3})
\]
\[
(\min(\psi_{R_2}, \psi_{R_3}), \psi_{R_3})
\]
2) We can prove in a similar fashion to (1).

**Theorem 15:** Let \( R = (\varnothing_R, \psi_R), R_1 = (\varnothing_R, \psi_{R_1}) \) and \( R_2 = (\varnothing_R, \psi_{R_2}) \) be three n,m-ROFSs, then

1) \( \eta(\varnothing_R \otimes \varnothing_R) = \eta \varnothing_R \otimes \eta \varnothing_R \), for \( \eta > 0 \).
2) \( \eta(\varnothing_R \otimes \varnothing_R) = \eta \varnothing_R \otimes \varnothing_R \), for \( \eta > 0 \).
3) \( \eta(\varnothing_R \otimes \varnothing_R) = \eta \varnothing_R \otimes \varnothing_R \), for \( \eta > 0 \).
4) \( \varnothing_R^1 \otimes \varnothing_R^2 = (\min(\varnothing_R^1, \varnothing_R^2), \max(\psi_{R_1}, \psi_{R_2})) \), for \( \eta > 0 \).

**Proof:** For the three n,m-ROFSs \( R_R, R_1 \) and \( R_2 \), and \( \eta > 0 \), according to Definition 8, we can obtain:

1) \( \eta\varnothing_R \otimes \varnothing_R = \eta \left( (1 - \varnothing_R^{+m, n+2}) \psi_{R_1} \psi_{R_2} \right) = (1 - (1 - \varnothing_R^{+m, n+2})^{\eta}) \psi_{R_1} \psi_{R_2} \)
2) \( \varnothing_R \otimes \varnothing_R = (1 - (1 - \varnothing_R^{+m, n+2})^{\eta}) \psi_{R_1} \psi_{R_2} \)
3) \( \varnothing_R \otimes \varnothing_R = (1 - (1 - \varnothing_R^{+m, n+2})^{\eta}) \psi_{R_1} \psi_{R_2} \)
4) \( \varnothing_R \otimes \varnothing_R = (1 - (1 - \varnothing_R^{+m, n+2})^{\eta}) \psi_{R_1} \psi_{R_2} \)

**Theorem 16:** Let \( R_1 = (\varnothing_R, \psi_{R_1}), R_2 = (\varnothing_R, \psi_{R_2}) \) and \( R_3 = (\varnothing_R, \psi_{R_3}) \) be three n,m-ROFSs and \( \eta > 0 \), then

1) \( R_1 \cap (R_2 \cap R_3) = (R_1 \cap R_2) \cap R_3 \).
2) \( R_1 \cup (R_2 \cup R_3) = (R_1 \cup R_2) \cup R_3 \).
3) \( \eta(R_1 \cup R_2) = \eta R_1 \cup \eta R_2 \).
4) \( \eta(R_1 \cap R_2) = \eta R_1 \cap \eta R_2 \).

**Proof:** For the three n,m-ROFSs \( R_1, R_2, \) and \( R_3 \), and \( \eta > 0 \), according to Definition 8, we can obtain:

1) \( R_1 \cap (R_2 \cap R_3) = (\varnothing_R, \psi_{R_1}) \cap (\min(\varnothing_R^1, \varnothing_R^2), \max(\psi_{R_1}, \psi_{R_2})) \)
2) \( R_1 \cup (R_2 \cup R_3) = (\varnothing_R, \psi_{R_1}) \cup (\max(\varnothing_R^1, \varnothing_R^2), \max(\psi_{R_1}, \psi_{R_2})) \)
3) \( \eta(R_1 \cup R_2) = \eta \varnothing_R \otimes \eta \varnothing_R \).
4) \( \eta(R_1 \cap R_2) = \eta \varnothing_R \otimes \eta \varnothing_R \).

**Theorem 17:** Let \( R_1 = (\varnothing_R, \psi_{R_1}) \) and \( R_2 = (\varnothing_R, \psi_{R_2}) \) be two n,m-ROFSs, then

1) \( (R_1 \cap R_2)^{c} = \varnothing_R^c \cup \varnothing_R^c \).
2) \( R_1 \cup R_2)^{c} = \varnothing_R^c \cap \varnothing_R^c \).

**Proof:** For the two n,m-ROFSs \( R_1 \) and \( R_2 \), according to Definition 8, we can obtain:

1) \( (R_1 \cap R_2)^{c} = \min(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \)
2) \( (R_1 \cup R_2)^{c} = \max(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \)
3) \( \eta(R_1)^{c} = \eta \varnothing_R^{c} \).
4) \( \eta(R_1)^{c} = \eta \varnothing_R^{c} \).

**Theorem 18:** Let \( R = (\varnothing_R, \psi_R), R_1 = (\varnothing_R, \psi_{R_1}) \) and \( R_2 = (\varnothing_R, \psi_{R_2}) \) be three n,m-ROFSs, and \( \eta > 0 \), then

1) \( (R_1 \cap R_2)^{c} = \min(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \)
2) \( (R_1 \cup R_2)^{c} = \max(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \)
3) \( \eta(R_1)^{c} = \min(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \)
4) \( \eta(R_1)^{c} = \min(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \).

**Proof:** For the three n,m-ROFSs \( R_1, R_2, \) and \( R_3, \eta > 0 \), according to Definitions 8, we can obtain:

1) \( (R_1 \cap R_2)^{c} = \max(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \)
2) \( (R_1 \cup R_2)^{c} = \min(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \)
3) \( \eta(R_1)^{c} = \min(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \)
4) \( \eta(R_1)^{c} = \min(\varnothing_R^{+m, n+2}, \varnothing_R^{+m, n+2}) \).

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\[
= (\theta_1^n, \psi_1^m) (1 - (1 - \psi_2^m))C
= (\theta_2^n, \psi_2^m)C.
\]

**Theorem 19:** Let \(\mathcal{A}_1 = (\theta_{\mathcal{A}_1}, \psi_{\mathcal{A}_1})\), \(\mathcal{A}_2 = (\theta_{\mathcal{A}_2}, \psi_{\mathcal{A}_2})\) and \(\mathcal{A}_3 = (\theta_{\mathcal{A}_3}, \psi_{\mathcal{A}_3})\) be three n,m-ROFSs, then

1) \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3 = (\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3\)
2) \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3 = (\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3\)
3) \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3 = (\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3\)
4) \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3 = (\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3\)

**Proof:** By Definition 8, we can obtain:

1) \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3 = (\min(\theta_{\mathcal{A}_1}, \theta_{\mathcal{A}_2}), \max(\psi_{\mathcal{A}_1}, \psi_{\mathcal{A}_2})) \triangle (\theta_{\mathcal{A}_3}, \psi_{\mathcal{A}_3})
= \left(\frac{\min(\theta_{\mathcal{A}_1}, \theta_{\mathcal{A}_2}) + \theta_{\mathcal{A}_3} - \min(\theta_{\mathcal{A}_1}, \theta_{\mathcal{A}_2})}{\theta_{\mathcal{A}_3}}, \max(\psi_{\mathcal{A}_1}, \psi_{\mathcal{A}_2})\right)
\]

2) We can prove in a similar fashion to (1).
3) \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3 = (\min(\theta_{\mathcal{A}_1}, \theta_{\mathcal{A}_2}), \max(\psi_{\mathcal{A}_1}, \psi_{\mathcal{A}_2})) \triangle \mathcal{A}_3
= (\min(\theta_{\mathcal{A}_1}, \theta_{\mathcal{A}_2}), \theta_{\mathcal{A}_3}, \max(\psi_{\mathcal{A}_1}, \psi_{\mathcal{A}_2}))
\]

Thus, \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3 = (\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3\)

4) We can prove in a similar fashion to (3).

**Theorem 20:** Let \(\mathcal{A}_1 = (\theta_{\mathcal{A}_1}, \psi_{\mathcal{A}_1})\), \(\mathcal{A}_2 = (\theta_{\mathcal{A}_2}, \psi_{\mathcal{A}_2})\) and \(\mathcal{A}_3 = (\theta_{\mathcal{A}_3}, \psi_{\mathcal{A}_3})\) be three n,m-ROFSs, then

1) \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3 = (\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3\)
2) \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3 = (\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3\)

**Proof:**
1) \((\mathcal{A}_1 \triangle \mathcal{A}_2) \triangle \mathcal{A}_3
= (\theta_{\mathcal{A}_1}, \psi_{\mathcal{A}_1}) \triangle (\theta_{\mathcal{A}_2}, \psi_{\mathcal{A}_2}) \triangle (\theta_{\mathcal{A}_3}, \psi_{\mathcal{A}_3})
= \left(\frac{\max(\theta_{\mathcal{A}_1}, \theta_{\mathcal{A}_2}) + \theta_{\mathcal{A}_3} - \max(\theta_{\mathcal{A}_1}, \theta_{\mathcal{A}_2})}{\theta_{\mathcal{A}_3}}, \max(\psi_{\mathcal{A}_1}, \psi_{\mathcal{A}_2})\right) \triangle (\theta_{\mathcal{A}_3}, \psi_{\mathcal{A}_3})
= (\frac{\max(\theta_{\mathcal{A}_1}, \theta_{\mathcal{A}_2}) + \theta_{\mathcal{A}_3} - \max(\theta_{\mathcal{A}_1}, \theta_{\mathcal{A}_2})}{\theta_{\mathcal{A}_3}}, \max(\psi_{\mathcal{A}_1}, \psi_{\mathcal{A}_2}))\)

2) We can prove in a similar fashion to (1).

**Definition 21:** Let \(\mathcal{A}_1 = (\theta_{\mathcal{A}_1}, \psi_{\mathcal{A}_1})\) and \(\mathcal{A}_2 = (\theta_{\mathcal{A}_2}, \psi_{\mathcal{A}_2})\) be two n,m-ROFSs, then

1) \(\mathcal{A}_1 = \mathcal{A}_2\) if and only if \(\theta_{\mathcal{A}_1} = \theta_{\mathcal{A}_2}\) and \(\psi_{\mathcal{A}_1} = \psi_{\mathcal{A}_2}\).
2) \(\mathcal{A}_1 \geq \mathcal{A}_2\) if and only if \(\theta_{\mathcal{A}_1} \geq \theta_{\mathcal{A}_2}\) and \(\psi_{\mathcal{A}_1} \leq \psi_{\mathcal{A}_2}\).
3) \(\mathcal{A}_2 \subseteq \mathcal{A}_1\) or \(\mathcal{A}_1 \supseteq \mathcal{A}_2\) if \(\mathcal{A}_1 \geq \mathcal{A}_2\).

**Example 22:** Consider \(\mathcal{A}_1 = (0.95, 0.91)\) and \(\mathcal{A}_2 = (0.95, 0.91)\) are two 7,13-ROFSs for \(Z = \{d\}\), then \(\mathcal{A}_1 = \mathcal{A}_2\).

2) Consider \(\mathcal{A}_1 = (0.95, 0.91)\) and \(\mathcal{A}_2 = (0.94, 0.92)\) are two 7,13-ROFSs for \(Z = \{d\}\), then \(\mathcal{A}_2 \leq \mathcal{A}_1\) and hence \(\mathcal{A}_2 \subset \mathcal{A}_1\).

**III. n,m-RUNG ORTHOPAIR FUZZY WEIGHTED POWER AVERAGE**

In this section, we propose a new operation on n,m-rung orthopair fuzzy sets, and some interesting properties are indicated in details.

**Definition 23:** Let \(\mathcal{A}_i = (\theta_{\mathcal{A}_i}, \psi_{\mathcal{A}_i})\) \((i = 1, 2, \ldots, k)\) be a value of n,m-ROFSs and \(\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)^T\) be weight vector of \(\mathcal{A}_i\) with \(\epsilon_i > 0\), \(\sum_{i=1}^{k} \epsilon_i = 1\) and \(n, m > 1\). Then, an n,m-rung orthopair fuzzy weighted power average (n,m-ROFWPA) operator is a function \(n, m\)-ROFWPA : \(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k \rightarrow \mathcal{A}\), where

\[
\mathcal{A} = (\sum_{i=1}^{k} \epsilon_i \theta_{\mathcal{A}_i}^{m_i} \psi_{\mathcal{A}_i}^{1-n_i})^{ \frac{1}{n_i}}, \quad \sum_{i=1}^{k} \epsilon_i \psi_{\mathcal{A}_i}^{m_i} = 1.
\]

**Example 24:** Consider \(\mathcal{A}_1 = (0.8, 0.4), \mathcal{A}_2 = (0.3, 0.9), \mathcal{A}_3 = (0.6, 0.7), \mathcal{A}_4 = (0.8, 0.7), \) and \(\mathcal{A}_5 = (0.9, 0.5)\) are five n,m-rung orthopair fuzzy sets, and let \(\epsilon = (0.18, 0.27, 0.25, 0.16, 0.14)^T\) be a weight vector of \(\mathcal{A}_i\) \((i = 1, 2, \ldots, 5)\). Then, \(n, m\)-ROFWPA(\(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_5\)) = \(((0.8^m \times 0.18 + 0.3^m \times 0.27 + 0.6^m \times 0.25 + 0.8^m \times 0.16 + 0.9^m \times 0.14)^{ \frac{1}{n}}, (0.4^m \times 0.18 + 0.9^m \times 0.27 + 0.7^m \times 0.25 + \ldots)^{ \frac{1}{n}}\).

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Then, \( n, m\text{-ROFWPA}(\mathfrak{R}_1 \oplus \mathfrak{R}, \mathfrak{R}_2 \oplus \mathfrak{R}, \ldots, \mathfrak{R}_k \oplus \mathfrak{R}) \geq n, m\text{-ROFWPA}(\mathfrak{R}_1 \oplus \mathfrak{R}, \mathfrak{R}_2 \oplus \mathfrak{R}, \ldots, \mathfrak{R}_k \oplus \mathfrak{R}) \).

**Proof:** For any \( \mathfrak{R}_i = (\vartheta_{\mathfrak{R}_i}, \psi_{\mathfrak{R}_i}) \) and \( \mathfrak{R} = (\vartheta_{\mathfrak{R}}, \psi_{\mathfrak{R}}) \), we have
\[
\sum_{i=1}^{k} \sum_{j=1}^{m} \vartheta_{\mathfrak{R}_i}(\vartheta_{\mathfrak{R}_j}^{\mathfrak{R}} - \vartheta_{\mathfrak{R}_j}^{\mathfrak{R}})^{1/2} \geq \sum_{i=1}^{k} \sum_{j=1}^{m} \vartheta_{\mathfrak{R}_j}(\vartheta_{\mathfrak{R}_j}^{\mathfrak{R}} - \vartheta_{\mathfrak{R}_j}^{\mathfrak{R}})^{1/2},
\]
that is,
\[
\sum_{i=1}^{k} \sum_{j=1}^{m} \vartheta_{\mathfrak{R}_i}(\vartheta_{\mathfrak{R}_j}^{\mathfrak{R}} - \vartheta_{\mathfrak{R}_j}^{\mathfrak{R}})^{1/2} \geq \sum_{i=1}^{k} \sum_{j=1}^{m} \vartheta_{\mathfrak{R}_j}(\vartheta_{\mathfrak{R}_j}^{\mathfrak{R}} - \vartheta_{\mathfrak{R}_j}^{\mathfrak{R}})^{1/2}.
\]

**Theorem 25:** Let \( \mathfrak{R}_i = (\vartheta_{\mathfrak{R}_i}, \psi_{\mathfrak{R}_i}) \) be a value of \( n,m\text{-ROFSs} \) and \( \epsilon = (e_1, e_2, \ldots, e_k)^T \) be weight vector of \( \mathfrak{R}_i \) with \( e_i > 0 \) and \( \sum_{i=1}^{k} e_i = 1 \). Then, \( n, m\text{-ROFWPA}(\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_k) \) is an \( n,m\text{-ROFS}. \)

**Proof:** For any \( n,m\text{-ROFS} \) \( \mathfrak{R}_i = (\vartheta_{\mathfrak{R}_i}, \psi_{\mathfrak{R}_i}) \), we have
\[
0 \leq \vartheta_{\mathfrak{R}_i}^{\mathfrak{R}} \leq 1 \quad \text{and} \quad 0 \leq \psi_{\mathfrak{R}_i}^{\mathfrak{R}} \leq 1
\]
and
\[
0 \leq \epsilon \vartheta_{\mathfrak{R}_i}^{\mathfrak{R}} + \epsilon \psi_{\mathfrak{R}_i}^{\mathfrak{R}} \leq \epsilon
\]
and so 0 \( \leq (\epsilon \vartheta_{\mathfrak{R}_i}^{\mathfrak{R}} + \epsilon \psi_{\mathfrak{R}_i}^{\mathfrak{R}}) + (\epsilon \vartheta_{\mathfrak{R}_j}^{\mathfrak{R}} + \epsilon \psi_{\mathfrak{R}_j}^{\mathfrak{R}}) + \ldots + (\epsilon \vartheta_{\mathfrak{R}_k}^{\mathfrak{R}} + \epsilon \psi_{\mathfrak{R}_k}^{\mathfrak{R}}) \) \( \leq \epsilon + \epsilon + \ldots + \epsilon \), implies that
\[
0 \leq \sum_{i=1}^{k} \epsilon \vartheta_{\mathfrak{R}_i}^{\mathfrak{R}} + \sum_{i=1}^{k} \epsilon \psi_{\mathfrak{R}_i}^{\mathfrak{R}} \leq \sum_{i=1}^{k} \epsilon = 1.
\]
Therefore,
\[
0 \leq \left( \sum_{i=1}^{k} \epsilon \vartheta_{\mathfrak{R}_i}^{\mathfrak{R}} \right)^{1/2} + \left( \sum_{i=1}^{k} \epsilon \psi_{\mathfrak{R}_i}^{\mathfrak{R}} \right)^{1/2} \leq 1
\]
and
\[
0 \leq \left( \sum_{i=1}^{k} \epsilon \vartheta_{\mathfrak{R}_i}^{\mathfrak{R}} \right)^{1/2} \leq 1
\]
and
\[
0 \leq \left( \sum_{i=1}^{k} \epsilon \psi_{\mathfrak{R}_i}^{\mathfrak{R}} \right)^{1/2} \leq 1
\]
This indicate that \( n, m\text{-ROFWPA}(\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_k) \) is an \( n,m\text{-ROFS}. \)

**Theorem 26:** Let \( \mathfrak{R}_i = (\vartheta_{\mathfrak{R}_i}, \psi_{\mathfrak{R}_i}) \) be a value of \( n,m\text{-ROFSs} \), \( \mathfrak{R} = (\vartheta_{\mathfrak{R}}, \psi_{\mathfrak{R}}) \) be \( n,m\text{-ROFS} \) and \( \epsilon = (e_1, e_2, \ldots, e_k)^T \) be weight vector of \( \mathfrak{R}_i \) with \( \sum_{i=1}^{k} e_i = 1 \).
and

\[ n, m, \text{ROFWPA}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k) \otimes \mathbb{R} \]
\[ = (\sum_{i=1}^{k} e_i \theta_i^{n})^{\frac{1}{n}} \left( \sum_{i=1}^{k} e_i \psi_i^{m} \right)^{\frac{1}{m}} \otimes (\theta_{\mathcal{N}_i}, \psi_{\mathcal{N}_i}) \]
\[ = (\sum_{i=1}^{k} e_i \theta_i^{n})^{\frac{1}{n}} \theta_{\mathcal{N}_i} + (\sum_{i=1}^{k} e_i \psi_i^{m})^{\frac{1}{m}} \psi_{\mathcal{N}_i} \]
\[ = \left( \sum_{i=1}^{k} e_i \theta_i^{n} \right)^{\frac{1}{n}} \theta_{\mathcal{N}_i} + \left( \sum_{i=1}^{k} e_i \psi_i^{m} \right)^{\frac{1}{m}} \psi_{\mathcal{N}_i} \].

Then, from (1) and (2) we get \( n, m, \text{ROFWPA}(\mathcal{N}_1 \oplus \mathcal{N}_2, \mathcal{N}_3 \oplus \mathcal{N}_4, \mathcal{N}_5 \oplus \mathcal{N}_6, \mathcal{N}_7 \oplus \mathcal{N}_8, \mathcal{N}_9 \oplus \mathcal{N}_10) \geq n, m, \text{ROFWPA}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k) \otimes \mathbb{R} \).

**Theorem 28:** Let \( \mathcal{N}_i = (\theta_{\mathcal{N}_i}, \psi_{\mathcal{N}_i}) \) and \( L_i = (\theta_{L_i}, \psi_{L_i}) \) \((i = 1, 2, \ldots, k)\) be two values of \( n, m, \text{ROFPS} \)s, and \( \epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k)^T \) be a weight vector of them with \( \sum_{i=1}^{k} \epsilon_i = 1 \). Then

1. \( n, m, \text{ROFWPA}(\mathcal{N}_1 \oplus L_1, \mathcal{N}_2 \oplus L_2, \ldots, \mathcal{N}_k \oplus L_k) \geq n, m, \text{ROFWPA}(\mathcal{N}_1 \otimes \mathcal{N}_2, \ldots, \mathcal{N}_k \otimes L_k) \).

2. \( n, m, \text{ROFWPA}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k) \otimes \mathbb{R} \)
\[ n, m, \text{ROFWPA}(L_1, L_2, \ldots, L_k) \geq n, m, \text{ROFWPA}(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k) \otimes \mathbb{R} \).

**Proof:** For any \( \mathcal{N}_i = (\theta_{\mathcal{N}_i}, \psi_{\mathcal{N}_i}) \) and \( L_i = (\theta_{L_i}, \psi_{L_i}) \) \((i = 1, 2, \ldots, k)\), we have

1. \( \theta_{\mathcal{N}_i}^{n} + \theta_{L_i}^{n} - \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \leq 2 \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} - \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} = \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \)
\( \psi_{\mathcal{N}_i}^{m} + \psi_{L_i}^{m} - \psi_{\mathcal{N}_i}^{m} \psi_{L_i}^{m} \leq 2 \psi_{\mathcal{N}_i}^{m} \psi_{L_i}^{m} - \psi_{\mathcal{N}_i}^{m} \psi_{L_i}^{m} = \psi_{\mathcal{N}_i}^{m} \psi_{L_i}^{m} \),

that is,
\[ \sum_{i=1}^{k} e_i (\theta_{\mathcal{N}_i}^{n} + \psi_{\mathcal{N}_i}^{m}) \geq \sum_{i=1}^{k} e_i \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \]
\[ \Rightarrow \left( \sum_{i=1}^{k} e_i (\theta_{\mathcal{N}_i}^{n} + \psi_{\mathcal{N}_i}^{m}) \right)^\frac{1}{n} \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \]
\[ \geq \left( \sum_{i=1}^{k} e_i \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \right)^\frac{1}{n} \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \] (***)

and
\[ \sum_{i=1}^{k} e_i (\theta_{\mathcal{N}_i}^{n} + \psi_{\mathcal{N}_i}^{m}) \geq \sum_{i=1}^{k} e_i \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \]
\[ \Rightarrow \left( \sum_{i=1}^{k} e_i (\theta_{\mathcal{N}_i}^{n} + \psi_{\mathcal{N}_i}^{m}) \right)^\frac{1}{n} \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \]
\[ \geq \left( \sum_{i=1}^{k} e_i \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \right)^\frac{1}{n} \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \] (****)

Therefore, we have

\[ n, m, \text{ROFWPA}(\mathcal{N}_1 \oplus L_1, \mathcal{N}_2 \oplus L_2, \ldots, \mathcal{N}_k \oplus L_k) \]
\[ = \left( \sum_{i=1}^{k} e_i (\theta_{\mathcal{N}_i}^{n} + \psi_{\mathcal{N}_i}^{m}) \right)^\frac{1}{n} \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \]
\[ \oplus \mathbb{R} \]
\[ = \left( \sum_{i=1}^{k} e_i (\theta_{\mathcal{N}_i}^{n} + \psi_{\mathcal{N}_i}^{m}) \right)^\frac{1}{n} \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \]
\[ = \left( \sum_{i=1}^{k} e_i (\theta_{\mathcal{N}_i}^{n} + \psi_{\mathcal{N}_i}^{m}) \right)^\frac{1}{n} \theta_{\mathcal{N}_i}^{n} \theta_{L_i}^{n} \]
\[ \oplus \mathbb{R} \]

and, as shown in the equation at the bottom of the next page. Thus, from (*** and **** we get
\[ n, m, \text{ROFWPA}(\mathcal{N}_1 \oplus L_1, \mathcal{N}_2 \oplus L_2, \ldots, \mathcal{N}_k \oplus L_k) \]
\[ \geq n, m, \text{ROFWPA}(\mathcal{N}_1 \oplus L_1, \mathcal{N}_2 \oplus L_2, \ldots, \mathcal{N}_k \oplus L_k, \mathcal{N}_1 \oplus L_1, \mathcal{N}_2 \oplus L_2, \ldots, \mathcal{N}_k \oplus L_k) \]
\[ \oplus \mathbb{R} \]
In order to rank n,m-ROFSs, we present the score function and accuracy function of the n,m-ROFS:

**Definition 29:** 1) The score function of an n,m-ROFS \( \mathfrak{R} = (\vartheta_{\mathfrak{R}}, \psi_{\mathfrak{R}}) \) can be represented as \( \bar{s}(\mathfrak{R}) = \vartheta_{\mathfrak{R}} + \psi_{\mathfrak{R}}^m \).
2) The accuracy function of an n,m-ROFS \( \mathfrak{R} = (\vartheta_{\mathfrak{R}}, \psi_{\mathfrak{R}}) \) can be represented as \( \bar{a}(\mathfrak{R}) = \vartheta_{\mathfrak{R}}^n + \psi_{\mathfrak{R}}^m \).

**Example 30:** Consider \( \mathfrak{R} = (0.91, 0.82) \) is n,m-ROFS, then

\[
\bar{a}(\mathfrak{R}) \approx \begin{cases} 0.9949 & \text{if } n = 12 \text{ and } m = 2, \\ 0.8875 & \text{if } n = 7 \text{ and } m = 5, \\ 0.9386 & \text{if } n = 6 \text{ and } m = 5, \\ 0.9898 & \text{if } n = 4 \text{ and } m = 6, 
\end{cases}
\]
and

\[
\bar{s}(\mathfrak{R}) \approx \begin{cases} -0.3499 & \text{if } n = 12 \text{ and } m = 2, \\ 0.1460 & \text{if } n = 7 \text{ and } m = 5, \\ 0.1971 & \text{if } n = 6 \text{ and } m = 5, \\ 0.3817 & \text{if } n = 4 \text{ and } m = 6. 
\end{cases}
\]

**Theorem 31:** Let \( \mathfrak{R} = (\vartheta_{\mathfrak{R}}, \psi_{\mathfrak{R}}) \) be any n,m-ROFS, then the suggested score function \( \bar{s}(\mathfrak{R}) \in [1, 1] \).

**Proof:** Since for any n,m-ROFS \( \mathfrak{R} \), we have \( \vartheta_{\mathfrak{R}}^n + \psi_{\mathfrak{R}}^m \leq 1 \). Hence, \( \vartheta_{\mathfrak{R}}^n - \psi_{\mathfrak{R}}^m \leq \vartheta_{\mathfrak{R}}^n \leq 1 \) and \( \vartheta_{\mathfrak{R}}^n - \psi_{\mathfrak{R}}^m \geq -1 \). Thus, \( -1 \leq \vartheta_{\mathfrak{R}}^n - \psi_{\mathfrak{R}}^m \leq 1 \), namely \( \bar{s}(\mathfrak{R}) \in [1, 1] \).

In particular, if \( \mathfrak{R} = (0, 1) \), then \( \bar{s}(\mathfrak{R}) = -1 \) and if \( \mathfrak{R} = (1, 0) \), then \( \bar{s}(\mathfrak{R}) = 1 \).

**Remark 32:** For any n,m-ROFS \( \mathfrak{R} = (\vartheta_{\mathfrak{R}}, \psi_{\mathfrak{R}}) \), the suggested accuracy function \( \bar{a}(\mathfrak{R}) \in [0, 1] \).

**Definition 33:** For any n,m-ROFSs \( \mathfrak{R}_i = (\vartheta_{\mathfrak{R}_i}, \psi_{\mathfrak{R}_i}) \) the comparison technique supposed as,

1. if \( \bar{s}(\mathfrak{R}_1) < \bar{s}(\mathfrak{R}_2) \), then \( \mathfrak{R}_1 < \mathfrak{R}_2 \),
2. if \( \bar{s}(\mathfrak{R}_1) > \bar{s}(\mathfrak{R}_2) \), then \( \mathfrak{R}_1 > \mathfrak{R}_2 \),
3. if \( \bar{s}(\mathfrak{R}_1) = \bar{s}(\mathfrak{R}_2) \), then
   a) if \( \bar{a}(\mathfrak{R}_1) < \bar{a}(\mathfrak{R}_2) \), then \( \mathfrak{R}_1 < \mathfrak{R}_2 \),
   b) if \( \bar{a}(\mathfrak{R}_1) > \bar{a}(\mathfrak{R}_2) \), then \( \mathfrak{R}_1 > \mathfrak{R}_2 \),
   c) if \( \bar{a}(\mathfrak{R}_1) = \bar{a}(\mathfrak{R}_2) \), then \( \mathfrak{R}_1 \approx \mathfrak{R}_2 \).

**IV. APPLICATION of n,m-ROFSs to SELECT THE BEST JOURNAL**

In this section, we use the n,m-ROFWPA operator to select the best journal between different journals. In this approach, the decision makers provide the data in the form of n,m-rung orthopair fuzzy sets. In light of this, we first, show the steps used in the suggested methodology for MCDM:

1. **Step 1:** Represent a MCDM problem under study using n,m-rung orthopair fuzzy decision matrix.
2. **Step 2:** Convert n,m-rung orthopair fuzzy decision matrix into the normalized n,m-rung orthopair fuzzy decision matrix.
3. **Step 3:** Using proposed n,m-ROFWPA operator to compute the alternative preference values with associated weights.
4. **Step 4:** Find the score value for each alternative.
5. **Step 5:** Select the maximum score value to choose the best alternative.

**A. APPLICATION EXAMPLE**

Suppose that \( J = \{J_1, J_2, J_3, J_4, J_5\} \) is a set of alternatives (Journals), and \( D = \{D_1, D_2, D_3, D_4, D_5\} \) is a set of five attributes for the selection of journals, where \( D_1 = \{\text{represents Indexing}\} \), \( D_2 = \{\text{represents Editorial Board}\} \), \( D_3 = \{\text{represents Journal Rank}\} \), \( D_4 = \{\text{represents Reviewer Report}\} \), and \( D_5 = \{\text{represents Impact Factor}\} \). Construct the n,m-rung orthopair fuzzy set decision-making matrices are shown in Table 1, where \( \vartheta_{d_i} \) is the positive membership degree for which alternative complies the given attribute and \( \psi_{d_i} \) is the membership degree for which alternative does not complies the given attribute such that \( 0 \leq (\vartheta_{d_i})^n + (\psi_{d_i})^m \leq 1 \) and \( \vartheta_{d_i}, \psi_{d_i} \in [0, 1] \).

Now, applying the n,m-ROFWPA operator with weight vectors \( \epsilon = (0.21, 0.15, 0.22, 0.13, 0.29)^T \), and putting \( n = 2, 3, 4, 5, 10, 100 \) and \( m = 3, 2, 5, 4, 100, 10 \) as follows in Table 2.

Now, we find the score value of each alternative and their ranking as demonstrated in Table 3.

To explain the effect of the parameters \( n \) and \( m \) on MADM end results, we have utilized different values of \( n \) and \( m \) to rank the alternatives. The results of ranking order of the alternatives based on n,m-ROFWPA operator are presented in Table 3. When \( n, m = 2, 3, n, m = 3, 2, n, m = 4, 5 \) and \( n, m = 10, 100 \), we obtained a rank of alternatives as \( J_2 > J_1 > J_3 > J_5 \), here, \( J_4 \) is the best choice and \( J_2 \) is the best second choice, but, when \( n, m = 5, 4 \) and...
TABLE 1. n,m-rung orthopair fuzzy values.

| Journals | D_1           | D_2           | D_3           | D_4           | D_5           |
|----------|---------------|---------------|---------------|---------------|---------------|
| J_1      | (0.61, 0.80)  | (0.69, 0.73)  | (0.48, 0.79)  | (0.59, 0.58)  | (0.81, 0.68)  |
| J_2      | (0.62, 0.81)  | (0.81, 0.63)  | (0.66, 0.77)  | (0.65, 0.76)  | (0.75, 0.73)  |
| J_3      | (0.65, 0.78)  | (0.71, 0.79)  | (0.75, 0.75)  | (0.64, 0.77)  | (0.49, 0.91)  |
| J_4      | (0.81, 0.61)  | (0.83, 0.59)  | (0.75, 0.74)  | (0.85, 0.57)  | (0.83, 0.59)  |
| J_5      | (0.59, 0.86)  | (0.81, 0.68)  | (0.61, 0.80)  | (0.48, 0.79)  | (0.47, 0.89)  |

TABLE 2. Aggregated n,m-rung orthopair fuzzy information matrix.

| Operator           | J_1 | J_2 | J_3 | J_4 | J_5 |
|--------------------|-----|-----|-----|-----|-----|
| 2.3-ROFWPA         | 0.6601 | 0.7312 | 0.7022 | 0.7485 | 0.6411 | 0.8165 | 0.8115 | 0.6310 | 0.5894 | 0.8252 |
| 3.2-ROFWPA         | 0.6710 | 0.7276 | 0.7054 | 0.7466 | 0.6482 | 0.8139 | 0.8122 | 0.6277 | 0.6012 | 0.8224 |
| 4.5-ROFWPA         | 0.6812 | 0.7376 | 0.7087 | 0.7522 | 0.6547 | 0.8219 | 0.8129 | 0.6382 | 0.6134 | 0.8303 |
| 5.4-ROFWPA         | 0.6907 | 0.7345 | 0.7120 | 0.7504 | 0.6606 | 0.8192 | 0.8136 | 0.6345 | 0.6256 | 0.8278 |
| 10,100-ROFWPA      | 0.7270 | 0.7897 | 0.7275 | 0.7975 | 0.6822 | 0.8988 | 0.8166 | 0.7289 | 0.6799 | 0.8793 |
| 100,10-ROFWPA      | 0.8000 | 0.7503 | 0.7948 | 0.7956 | 0.7388 | 0.8359 | 0.8352 | 0.6573 | 0.7948 | 0.8405 |

TABLE 3. Ranking using score value.

| n, m     | \( \bar{\xi}(J_1) \) | \( \bar{\xi}(J_2) \) | \( \bar{\xi}(J_3) \) | \( \bar{\xi}(J_4) \) | \( \bar{\xi}(J_5) \) | Ranking |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|---------|
| 2, 3     | 0.0448          | 0.0737          | -0.1333         | 0.4073          | -0.2145         | J_4 ≻ J_2 ≻ J_1 ≻ J_3 ≻ J_5 |
| 3, 2     | -0.2273         | -0.2064         | -0.3901         | 0.1418          | -0.4590         | J_4 ≻ J_2 ≻ J_1 ≻ J_3 ≻ J_5 |
| 4, 5     | -0.0030         | 0.0115          | -0.1913         | 0.3308          | -0.2530         | J_4 ≻ J_2 ≻ J_1 ≻ J_3 ≻ J_5 |
| 5, 4     | -0.1339         | -0.1341         | -0.3246         | 0.1944          | -0.3737         | J_4 ≻ J_2 ≻ J_1 ≻ J_3 ≻ J_5 |
| 10, 100  | 0.0412          | 0.0415          | 0.0218          | 0.1319          | 0.0211          | J_4 ≻ J_2 ≻ J_1 ≻ J_3 ≻ J_5 |
| 100, 10  | -0.0064         | -0.0060         | -0.1665         | -0.0151         | -0.1759         | J_4 ≻ J_2 ≻ J_1 ≻ J_3 ≻ J_5 |

n, m = 100, 10, we obtained a rank of alternatives as \( J_4 ≻ J_1 ≻ J_2 ≻ J_3 ≻ J_5 \), here, \( J_4 \) is the best choice, and \( J_1 \) is the best second choice. Thus, the overall best rank is \( J_4 \). To these MADM problems based on n,m-ROFWPA operator, we notice that the corresponding ranking orders of the alternatives can change for the different values of parameters \( n \) and \( m \).

V. COMPARISON ANALYSIS

This section gives the comparison analysis of the proposed n,m-ROFWPA operator under n,m-rung orthopair fuzzy numbers with other well known operator. We compared the results of n,m-ROFWPA operator with FFWPA operator [18]. The results are summed up as follows in Table 4.

It is clear that our results of 2.3-ROFWPA, 3.2-ROFWPA, 4.5-ROFWPA and 10,100-ROFWPA are similar to Fermatean fuzzy weighted power average (FFWPA) operator, but results of 4.5-ROFWPA and 10,100-ROFWPA gives \( J_1 \) is the best second choice while FFWPA give \( J_2 \) is the best second choice. Therefore, our proposed method is more flexible than other existing methods.

VI. CONCLUSION

For express fuzzy information, n,m-rung orthopair fuzzy is a good tool. It has a parameter \( n \) and \( m \), so it holds a wider range of fuzzy information than IFS, PFS, FFS and q-rung orthopair fuzzy sets for \( q > 3 \). In this study, we have proposed a new generalized intuitionistic fuzzy set is called n,m-rung orthopair fuzzy sets and compared their relationship with other kinds of the generalize of fuzzy sets. Moreover, some operators on n,m-rung orthopair fuzzy sets are introduced and their relationship have been proved. Also, we have presented new weighted aggregated operator over n,m-rung orthopair fuzzy sets and discussed their properties in details. Finally, we have showed this procedure with one practical fully developed example.

In future works, the weighted average operator, weighted geometric operator and weighted power geometric operator over n,m-rung orthopair fuzzy sets may be investigated and study a MCDM methods depending on these operators. In addition, we will try to define the topology on the collection of n,m-rung orthopair fuzzy sets and introduce the ideas of connectedness in n,m-rung orthopair fuzzy topology.

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