Evaluation and Comparison of Three Classes of Central Composite Designs

Fidelia Chinenye Kiwu-Lawrence\(^1\), Lawrence Chizoba Kiwu\(^2\), Desmond Chekwube Bartholomew\(^2\), Chukwudi Paul Obite\(^2\) and Akanno Felix Chikereuba\(^2\)

\(^1\)Department of Statistics, Abia State University Uturu, Abia State, Nigeria.  
\(^2\)Department of Statistics, Federal University of Technology, Owerri, Nigeria.

Authors’ contributions  
This work was carried out in collaboration among all authors. Author FCK-L designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors LCK and DCB managed the analyses of the study. Authors CPO and AFC managed the literature searches. All authors read and approved the final manuscript.

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Abstract  
Three classes of Central Composite Design: Central Composite Circumscribed Design (CCCD), Central Composite Inscribed Design (CCID) and Central Composite Face-Centered Design (CCFD) in Response Surface Methodology (RSM) were evaluated and compared using the A-, D-, and G-efficiencies for factors, \(k\), ranging from 3 to 10, with 0-5 centre points, in order to determine the performances of the designs under consideration. The results show that the CCDs (CCCD, CCFD and CCID) are at their best when the G-efficiency is employed for all the factors considered while the CCID especially behaves poorly when using the A- and D-efficiencies.

Keywords: Response surface methodology; central composite circumscribed design; central composite inscribed design; central composite face-centered design; A-; D-; and G-Efficiencies.

*Corresponding author: Email: Lawrence.kiwu@futo.edu.ng;
1 Introduction

Experiments are performed by researchers in every field of inquiry so as to study and model the effects of several design variables on the responses of interest. The foundation for response surface methodology (RSM) was laid by Box and Wilson [1]. Response surface methodology consists of statistical and mathematical techniques for empirical model building and model exploitation. It seeks to relate a response or output variable to the levels of a number of predictors or input variables that affect it. The form of such relationship is usually unknown, but can be approximated by a low-order polynomial such as the second-order response surface model,

\[ y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{j=i+1}^{k} \sum_{l=i+1}^{k} \beta_{ij} x_i x_j + \varepsilon_{ij} \]  

(1)

where, \( y \) is the measured response, \( \beta_0, \beta_1, \ldots, \beta_k \) are model parameters, \( x_1, x_2, \ldots, x_k \) are the input variables and \( \varepsilon_{ij} \) is an error term.

Most second-order designs especially the central composite designs utilize this stated model. A second-order response surface design is often chosen based on consideration of several criteria, such as those identified by Myers and Montgomery [2]. Among the most important of these is the stability of the prediction variance over the region of interest. However, the Central Composite Design, being the most popular of the many classes of response surface designs, has been studied and used by many researchers. Box and Draper [3] suggested several criteria which can be used in the selection of design. Several second-order model designs exist in the literature and they include CCD, Box-Behnken Designs (BBD), Hoke Designs, Small Composite Designs (SCD), Hybrid Designs, etc.: see, for example, Box and Wilson [1], Myers and Montgomery [2], pp.541-546 and Zahran et al [4].

It is also worthy to note that a design superior to other designs by a given optimality criterion may not be superior when evaluated by another optimality criterion. Therefore, the choice of a design may be dependent on the choice of an evaluation criterion. Four common design evaluation criteria, which exist in the literature, are the alphabetic G-, D-, A- and E-optimality criteria. By condensing a design’s properties to a single value, however, much information is lost regarding a design’s potential performance. For an overview of optimality criteria, see Atkinson and Donev [5]. The Central Composite Design (CCD) is one of the most popular response surface designs. Since it was introduced by Box and Wilson [1], the CCD has been studied and used by many researchers in fitting the second-order model. Box and Draper [3] suggested several criteria which can be used in the selection of the design. These criteria include, the design should: (a) allow a check on the representational adequacy of the polynomial; (b) should not contain an excessively large number of experimental runs; (c) should lend itself to blocking; etc. Dykstra [6] studied the partial duplication of the factorial portion, as well as the partial duplication of the star portion of the CCDs (Rotatable and Orthogonal Central Composite Designs) for factors \( k = 2, 3, \ldots, 8 \). The results showed that the designs with the star portion duplicated seem to have more potential than the designs with their factorial portions duplicated or partially duplicated. Lucas [7] evaluated four types of optimum composite design in different regions of interest. The optimum designs evaluated are: a symmetric composite design (a composite design with star point distance equal to \( \pm \alpha \), a symmetric smallest composite design (a saturated composite design), an asymmetric composite design with star point distance equal to \( \pm \alpha \) and an asymmetric smallest composite design. The result shows that symmetric composite designs are nearly optimum for experiments in a hypercube. Myers [8] suggested optimal CCDs under several design criteria (orthogonality and rotatability). Xianfeng and Zhang [9] evaluated and compared three CCDs – (CCCD, CCID and CCFD), from the view of region of interest, through simulation of a motor assembly. The results show that effective experiment of design cannot be gained without correct selection of these designs. Oyejola and Nwanya [10] used the D-, A-, G- and I-optimality criteria and the Fraction of Design Space graph to evaluate five varieties of CCD, which include: Spherical Central Composite Design (SCCD), Rotatable Central Composite Design (RCCD), Orthogonal Central Composite Design (OCCD), Slope Rotatable Central Composite Design (Slope-R) and Face Centre Cube (FCC), for 3-6 factors, with replicated star portions and increased centre points. Their results show that replicating the star points tend to reduce the D- and G-optimality criteria of the CCDs in all the factors considered, while it is not so for the A-optimality criterion. In I-optimality,
the CCDs are relatively the same, both when the centre points and axial points are increased. The FDS plots indicate that the CCDs maintain relatively low and stable Scaled Prediction Variance (SPV) when the star points are replicated with increased centre points.

Chigbu and Ohaegbulem [11] used the D-optimality criterion to compare partially replicated cube and star portions of the rotatable and orthogonal CCD. Their results indicate that replicating the cube portion enhances the D-optimal performance of the CCD more than replicating the star portion. Lucas [12] compared the performances of several types of quadratic response surface designs in symmetric region. In the study, the CCD, BBD, Hoke designs, Pesotchinsky designs, were compared using the D- and G-efficiencies, and the result showed that the CCD performs better than the other designs, though all of the designs compared have high D- and G-efficiencies.

In this study, three classes of central composite design, namely: central composite circumscribed design (CCCD), central composite inscribed design (CCID), and central composite face-centered (CCFD), will be evaluated and compared using the A-, D- and G-efficiencies for factors, k = 3,4,5,6,7,8,9 and 10. To reduce the number of design runs, which increase rapidly as the number of factors increases, especially from k = 5, the full factorial portions of the CCDs are employed for factors k = 3 and 4, while fraction of the factorial portions of the CCDs are employed for factors k = 5, 6,7,8,9 and 10. The performance of these designs will be considered when the centre point, \( n_0 \), ranges from 0 to 5.

2 Methodology

We present the three Central Composite Designs as well as the optimality criteria, which will be used to assess the designs under consideration.

2.1 Designs for Comparison

The three CCDs that will be examined and compared based on the A-, D- and G-efficiencies, are the Central Composite Circumscribed Design (CCCD), Central Composite Inscribed Design (CCID) and the Central Composite Face-centered Design (CCFD).

2.2 Central Composite Design

The Central Composite Design (CCD) was introduced by Box and Wilson [1], and it is perhaps the most popular class of second-order designs. Assuming \( k \geq 2 \) design variables, the CCD consists of the following:

1. An \( f = 2^{k-p} \) full ( \( p = 0 \) ) or fractional ( \( p > 0 \) ) factorial design of at least Resolution V; each point is of the form \( (x_1, \ldots, x_k) = (\pm 1, \pm 1, \ldots, \pm 1) \);
2. \( 2k \) star points, of the form \( (x_1, \ldots, x_i, \ldots, x_k) = (0, \ldots, 0, \pm \alpha, 0, \ldots, 0) \), for \( 1 \leq i \leq k \);
3. \( n_0 \) replication of the centre points \( (x_1, \ldots, x_k) = (0, 0, \ldots, 0) \): see, for example, Onukogu and Chigbu [13], pp 72-73.

For this work, to reduce the number of design runs, which increases rapidly as the number of factors increases, especially from \( k = 5 \). The full factorial portions of the CCDs are employed for factors \( k = 3 \) and 4, while fraction of the factorial portions of the CCDs are employed for factors \( k = 5, 6,7,8,9 \) and 10. The performance of these designs will be considered when the centre point, \( n_0 \), ranges from 0 to 5.

Let \( N \) denote the total number of experimental runs in the CCD, \( N = f + 2k + n_0 \). where \( f \) is the number of factorial points, \( 2k \) is the number of axial points and \( n_0 \), the number of centre points. The choice of axial distance, \( \alpha \), is based on the region of interest. Choosing the appropriate values of \( \alpha \) specifies the type of CCD.
The $\alpha$ considered in this work is the rotatable $\alpha$ for CCCD and CCID. For the CCFD, the star points are at the centre of each face of the factorial space, so $\alpha = \pm 1$.

The structure of the CCD matrix, $X$, for any two design variables, $x_i$ and $x_j$, with one centre point is given as:

$$
X = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 0 & \alpha & 0 & \alpha \alpha & 0 \\
1 & 0 & \alpha & 0 & \alpha & 0 \alpha \\
1 & 0 & -\alpha & 0 & \alpha & 0 \alpha \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

The three classes of CCD evaluated in this study are: the central composite circumscribed design, the central composite inscribed design and the central composite face-centered design.

### 2.3 Central Composite Circumscribed Design

The central composite circumscribed design (CCCD) is the original form of the CCD, with the star points located at some distance, $\alpha$, from the centre. The star points establish extremes for the low and high settings for all factors. These designs require five levels for each factor. Augmenting a two-level factorial or fraction (resolution V) with a $2k$ axial or star points and centre points can produce this design. The matrix structure of the central composite circumscribed design for $k = 3$, with $n_0 = 1$ and $\alpha = (f)^{1/2}$ is given as:

$$
X = \begin{pmatrix}
1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 \\
1 & -1.6818 & 0 & 0 \\
1 & 1.6818 & 0 & 0 \\
1 & 0 & -1.6818 & 0 \\
1 & 0 & 1.6818 & 0 \\
1 & 0 & 0 & -1.6818 \\
1 & 0 & 0 & 1.6818 \\
1 & 0 & 0 & 0
\end{pmatrix}
$$
2.4 Central composite inscribed design

The CCID is a scaled down CCD, with each factor level of the CCD divided by, \( \alpha \), to generate the CCID. This design, being a scaled down of the CCD also requires five levels of each factor, because the star points lie within the space of the factorial design. The matrix structure of the central composite inscribed design for \( k = 3 \) with \( n_0 = 1 \) is given as:

\[
\begin{bmatrix}
  x_0 & x_1 & x_2 & x_3 \\
  1 & -0.5946 & -0.5946 & -0.5946 \\
  1 & -0.5946 & -0.5946 & 0.5946 \\
  1 & -0.5946 & 0.5946 & -0.5946 \\
  1 & -0.5946 & 0.5946 & 0.5946 \\
  1 & 0.5946 & -0.5946 & -0.5946 \\
  1 & 0.5946 & -0.5946 & 0.5946 \\
  1 & 0.5946 & 0.5946 & -0.5946 \\
  1 & 0.5946 & 0.5946 & 0.5946 \\
  1 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

2.5 Central composite face-centered design

The CCFD is a special case of a CCD, in which \( \alpha = 1 \). As a result, the CCFD becomes a three-level design, because the star point is located at the centre of the face of the cube, requiring three levels for each face. The axial and the factorial points of face-centered CCD fall onto the surface of the cube. The matrix structure of the central composite face-centered design for \( k = 3 \) with \( n_0 = 1 \) is given as:

\[
\begin{bmatrix}
  x_0 & x_1 & x_2 & x_3 \\
  1 & -1 & -1 & -1 \\
  1 & -1 & -1 & 1 \\
  1 & -1 & 1 & -1 \\
  1 & -1 & 1 & 1 \\
  1 & 1 & -1 & -1 \\
  1 & 1 & -1 & 1 \\
  1 & 1 & 1 & -1 \\
  1 & 1 & 1 & 1 \\
  1 & -1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  1 & 0 & 0 & -1 \\
  1 & 0 & 0 & 1 \\
  1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Two of these designs, CCCD and CCID, have a common characteristic; they are rotatable. A design is rotatable if the estimated response, $y$, has a constant variance at all points which are the same distance from the centre of the design. See, for example, Box and Hunter [14]. For a CCD to be rotatable, $\alpha = (f)^{1/4}$, where $f$ is the number of runs in the factorial portion of the CCD.

### 2.6 Optimality criteria

An optimal design is an experimental design that is based on a particular optimality criterion. Kiefer [15] detailed the theory behind optimum designs, which states that, if $\chi$ is a compact space on which the real function, $f$, are continuous and linearly independent, the probability measure, $\xi$, is D-optimum for an unknown m-vector, $\theta_1, \ldots, \theta_m$ if and only if it is G-optimal. Design optimality criterion could be alphabetic because they are represented by the first letters of the names of the criteria. There commonly used design optimality criteria are the A-, D- and G-optimality criteria. Based on these optimality criteria, the design efficiencies can be calculated and can be used to compare designs. The A-, D- and G-efficiencies are used in this work.

#### 2.7 A-optimality criterion

This criterion, introduced by Chernoff [16], seeks to minimize the trace of the inverse of the information matrix, $(X'X)$. This criterion also results in minimizing the average variance of the estimates of the regression coefficients, and it is given by:

\[
\text{A-criterion} = \min \text{trace}\left( (X'X)^{-1} \right), \text{ and}
\]

The A-efficiency = \(100 \frac{p}{\text{trace}[N(X'X)^{-1}]}\),

where, $N$ is the design size, and $p$ is the number of model parameters.

#### 2.8 D-optimality criterion

D-optimality criterion, developed by Wald [17], was the first alphabetical optimality criterion developed. It is the most widely used criterion because of its computational ease. The D-optimality criterion focuses on the estimation of model parameters through the good attributes of the moment matrix, $M$, which is defined as,

\[
M = \begin{bmatrix} X'X \\ N \end{bmatrix}
\]

where, $X'X$ is the information matrix, and $N$, is the total number of runs. The D-optimality criterion seeks to maximize the determinant of the information matrix, $X'X$, or equivalently seeks to minimize the inverse of the information matrix. That is, Max $|X'X|$ or Min $(X'X)^{-1}$, respectively. That is, the D-criterion = minimize $|\left(X'X\right)^{-1}|$ or equivalently, maximize $|X'X|$, and

the D-efficiency = \(100 \frac{|X'X|^{1/p}}{N}\),

where $p$ and $N$, are as defined above.
2.9 G-optimality criterion

This criterion is concerned with the prediction variance. It may be that the aim of the practitioner is to have good prediction at a particular location in the design space, or throughout the design region. To attain this, Box and Hunter [14] defined a variance function, the Scaled Prediction Variance (SPV), as:

\[
\frac{\text{NVar}[y(x)]}{\sigma^2} = Nf'(x)(X'X)^{-1}f(x)
\]

(4)

where, \( f(x) \) is the vector coordinates of points in the region of interest, that is, \( f'(x) = [1, x_1, \ldots, x_k, x_1^2, \ldots, x_k^2, x_1x_2, \ldots, x_{k-1}x_k] \), \( N \) is the total sample size, \( X \) is the design matrix and \( \sigma^2 \) is the process variance of the design. A G-optimal design is one that minimizes the maximum SPV over the experimental design region. Symbolically, it is written as

\[
\min \{ \max N \text{ var } y(x) \} = \min \{ N \max f'(x)(X'X)^{-1}f(x) \}
\]

(5)

The G-efficiency = \( 100 \frac{N}{\sigma_{\text{max}}^2} \).

where, \( p \) and \( N \) are as defined before, and \( \sigma_{\text{max}}^2 \) is the maximum of \( f'(x)(X'X)^{-1}f(x) \) : see Borkowski and Valerozo [18].

The A-, D- and G-efficiencies are here computed using the MATLAB program for each of the classes of designs compared and the results are shown in Tables 1 to 8 of section four.

3 Comparison of the Designs

In this section, the three classes of CCD (CCCD, CCID and CCFD) for factors \( 3 \leq k \leq 10 \) are compared using the optimality criteria.

3.1 Design comparison using optimality criteria

In this section, the A-, D- and G-efficiencies of the three designs considered will be compared, the result will also be shown graphically. Let \( N_0 \) indicate the number of centre points and \( N \) the number of design runs.

The expanded design matrix for CCCD for \( k = 3 \) with \( N_0 = 0 \) is given by:

\[
X = \begin{bmatrix}
1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1.68 & 0 & 0 & 2.83 & 0 & 0 & 0 & 0 & 0 \\
1 & 1.68 & 0 & 0 & 2.83 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1.68 & 0 & 0 & 2.83 & 0 & 0 & 0 & 0 \\
1 & 0 & 1.68 & 0 & 0 & 2.83 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1.68 & 0 & 0 & 2.83 & 0 & 0 & 0 \\
1 & 0 & 0 & 1.68 & 0 & 0 & 2.83 & 0 & 0 & 0 \\
\end{bmatrix}
\]
The D-efficiency is obtained as:

$$\text{Det}(X'X) = 1.5725e + 008.$$  

$$D-\text{efficiency} = 100 \left[ \frac{(1.5725e + 008)^{1/10}}{14} \right] = 47.1554.$$  

$$A\text{-efficiency is obtained as:} \quad 100 \left[ \frac{p}{\text{trace}[N(X'X)^{-1}]} \right]$$

$$N[(X'X)^{-1}] =$$

$$1.1890 \quad 0 \quad 0 \quad 0 \quad -0.4059 \quad -0.4059 \quad -0.4059 \quad 0 \quad 0 \quad 0$$

$$0.00010 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

$$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0$$

$$-0.4059 \quad 0 \quad 0 \quad 0 \quad 0.1393 \quad 0.1384 \quad 0.1384 \quad 0 \quad 0 \quad 0$$

$$-0.4059 \quad 0 \quad 0 \quad 0 \quad 0.1384 \quad 0.1393 \quad 0.1384 \quad 0 \quad 0 \quad 0$$

$$-0.4059 \quad 0 \quad 0 \quad 0 \quad 0.1384 \quad 0.1384 \quad 0.1393 \quad 0 \quad 0 \quad 0$$

$$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.0018 \quad 0 \quad 0$$

$$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.0018 \quad 0$$

$$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.0018$$

$$1.0e + 003$$

$$X = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$$

$$x'(X'X)x = 0.7197;$$

then the scaled prediction variance, \( \text{Var}(x) \), is given by

$$V\text{ar}(x) = Nx'(X'X)^{-1}x = 14(0.7197) = 10.0758;$$

And the G-efficiency is obtained as\(100 \left[ \frac{10}{0.7197} \right] = 99.2477.$$
Using the same procedure, the following results, presented in Tables 1 to 8, are obtained for CCCD, CCID and CCFD for 3 to 10 factors with 0 to 5 centre points for each factor.

Table 1. Summary Statistics for the Three Classes of CCD for k = 3

| Design | n_0 | n | N  | D-efficiency | G-efficiency | A-efficiency |
|--------|-----|---|----|-------------|-------------|-------------|
| CCCD   | 0   | 14| 47.1554 | 99.2477     | 0.6191      |
|        | 1   | 15| 68.7038 | 67.4491     | 32.0718     |
|        | 2   | 16| 68.9923 | 93.2836     | 44.1517     |
|        | 3   | 17| 67.6076 | 87.8094     | 49.3150     |
|        | 4   | 18| 65.7088 | 82.9435     | 51.3877     |
|        | 5   | 19| 63.6514 | 78.5781     | 51.9060     |
| CCID   | 0   | 14| 9.9132  | 99.2477     | 0.2182      |
|        | 1   | 15| 14.4432 | 67.4491     | 7.7747      |
|        | 2   | 16| 14.5038 | 93.2836     | 9.3158      |
|        | 3   | 17| 14.2127 | 87.8094     | 9.6713      |
|        | 4   | 18| 13.8136 | 82.9435     | 9.6318      |
|        | 5   | 19| 13.3810 | 78.5781     | 9.4338      |
| CCFD   | 0   | 14| 46.3045 | 89.2856     | 31.0559     |
|        | 1   | 15| 44.7163 | 83.6260     | 31.2908     |
|        | 2   | 16| 42.9990 | 78.5472     | 30.6813     |
|        | 3   | 17| 41.2965 | 74.0198     | 29.7416     |
|        | 4   | 18| 39.6635 | 69.9692     | 28.6826     |
|        | 5   | 19| 38.1206 | 66.3200     | 27.5998     |

Table 2. Summary Statistics for the Three Classes of CCD for k = 4

| Design | n_0 | n | N  | D-efficiency | G-efficiency | A-efficiency |
|--------|-----|---|----|-------------|-------------|-------------|
| CCCD   | 0   | 24| 0  | 0           | 0           | 0           |
|        | 1   | 25| 76.7266 | 60          | 31.6484     |
|        | 2   | 26| 77.2647 | 98.9068     | 45.3972     |
|        | 3   | 27| 76.4417 | 95.2435     | 55.2876     |
|        | 4   | 28| 75.1389 | 91.8420     | 55.9007     |
|        | 5   | 29| 73.6354 | 88.6750     | 57.7385     |
| CCID   | 0   | 24| 0  | 0           | 0           | 0           |
|        | 1   | 25| 8.3493  | 60          | 4.5000      |
|        | 2   | 26| 8.4079  | 98.9068     | 5.3254      |
|        | 3   | 27| 8.3183  | 95.2435     | 5.5556      |
|        | 4   | 28| 8.1765  | 91.8420     | 5.5901      |
|        | 5   | 29| 8.0129  | 88.6750     | 5.5419      |
| CCFD   | 0   | 24| 45.7448 | 94.7400     | 25.9366     |
|        | 1   | 25| 44.5232 | 90.9918     | 25.4855     |
|        | 2   | 26| 43.3015 | 87.5319     | 24.9074     |
|        | 3   | 27| 42.1055 | 84.3028     | 24.2751     |
|        | 4   | 28| 40.9479 | 81.3167     | 22.0506     |
|        | 5   | 28| 39.8347 | 78.5246     | 22.9789     |

where Infinity implies that the D- and G-efficiencies are at infinity.
Table 3. Summary Statistics for the Three Classes of CCD for k = 5

| Design | \( n_o \) | N   | D-efficiency | G-efficiency | A-efficiency |
|--------|---------|-----|--------------|--------------|--------------|
| CCCD   | 0       | 26  | 70.0500      | 90.1644      | 15.3238      |
|        | 1       | 27  | 72.4642      | 88.1934      | 40.4332      |
|        | 2       | 28  | 71.8171      | 85.2079      | 49.8270      |
|        | 3       | 29  | 70.5492      | 82.3352      | 53.9832      |
|        | 4       | 30  | 69.0660      | 79.6178      | 55.8140      |
|        | 5       | 31  | 67.5089      | 77.0758      | 56.4569      |
| CCID   | 0       | 26  | 6.9498       | 90.1644      | 2.7535       |
|        | 1       | 27  | 7.1893       | 88.1934      | 4.6823       |
|        | 2       | 28  | 7.1252       | 85.2079      | 5.0292       |
|        | 3       | 29  | 6.9994       | 82.3352      | 5.0292       |
|        | 4       | 30  | 6.8522       | 79.6178      | 4.9763       |
|        | 5       | 31  | 6.6977       | 77.0758      | 4.8875       |
| CCFD   | 0       | 26  | 44.0194      | 99.6654      | 1.5714       |
|        | 1       | 27  | 42.6900      | 80.5904      | 25.2012      |
|        | 2       | 28  | 41.4195      | 77.7283      | 24.4658      |
|        | 3       | 29  | 40.2099      | 75.0558      | 23.7483      |
|        | 4       | 30  | 39.0602      | 72.5614      | 23.0558      |
|        | 5       | 31  | 37.9682      | 70.2280      | 22.3915      |

Table 4. Summary Statistics for the Three Classes of CCD for k = 6

| Design | \( n_o \) | N   | D-efficiency | G-efficiency | A-efficiency |
|--------|---------|-----|--------------|--------------|--------------|
| CCCD   | 0       | 44  | 73.6955      | 99.6654      | 1.5714       |
|        | 1       | 45  | 81.8114      | 64.0542      | 34.0467      |
|        | 2       | 46  | 81.9967      | 96.4958      | 48.0542      |
|        | 3       | 47  | 81.4085      | 94.4427      | 55.3616      |
|        | 4       | 48  | 80.5287      | 92.4752      | 59.5174      |
|        | 5       | 49  | 79.5123      | 90.5879      | 61.9579      |
| CCID   | 0       | 44  | 3.7784       | 99.6654      | 0.2662       |
|        | 1       | 45  | 4.1946       | 64.0542      | 2.5171       |
|        | 2       | 46  | 4.2040       | 96.4958      | 2.8168       |
|        | 3       | 47  | 4.1739       | 94.4427      | 2.8986       |
|        | 4       | 48  | 4.1288       | 92.4752      | 2.9136       |
|        | 5       | 49  | 4.0767       | 90.5879      | 2.9006       |
| CCFD   | 0       | 44  | 45.6289      | 94.0671      | 19.3190      |
|        | 1       | 45  | 44.7976      | 91.9903      | 18.9792      |
|        | 2       | 46  | 43.9848      | 89.9905      | 18.6377      |
|        | 3       | 47  | 43.1923      | 88.0758      | 18.2987      |
|        | 4       | 48  | 42.4213      | 86.2409      | 17.9649      |
|        | 5       | 49  | 41.6723      | 84.4934      | 17.6379      |

Table 5. Summary Statistics for the Three Classes of CCD for k = 7

| Design | \( n_o \) | N   | D-efficiency | G-efficiency | A-efficiency |
|--------|---------|-----|--------------|--------------|--------------|
| CCCD   | 0       | 78  | 87.3555      | 76.9231      | 8.2834       |
|        | 1       | 79  | 90.4324      | 55.6950      | 32.7157      |
|        | 2       | 80  | 90.7973      | 81.8182      | 46.1538      |
|        | 3       | 81  | 90.6068      | 81.0586      | 54.4337      |
|        | 4       | 82  | 90.1763      | 80.2018      | 59.8893      |
|        | 5       | 83  | 89.6175      | 79.3224      | 63.6365      |
| CCID   | 0       | 78  | 2.2955       | 76.9231      | 0.6767       |
|        | 1       | 79  | 2.3764       | 55.6950      | 1.3936       |
|        | 2       | 80  | 2.3860       | 81.8182      | 1.5437       |
|        | 3       | 81  | 2.3810       | 81.0586      | 1.5985       |
Table 6. Summary Statistics for the Three Classes of CCD for \( k = 8 \)

| Design | \( n_b \) | N  | D-efficiency | G-efficiency | A-efficiency |
|--------|----------|----|--------------|--------------|--------------|
| CCFD   | 0        | 80 | 41.7017      | 93.7500      | 0            |
|        | 1        | 81 | 87.8689      | 55.5556      | 32.3232      |
|        | 2        | 82 | 88.1446      | 99.7783      | 47.4621      |
|        | 3        | 83 | 87.8708      | 98.5761      | 55.9658      |
|        | 4        | 84 | 87.3816      | 97.4026      | 61.2245      |
|        | 5        | 85 | 86.7829      | 96.2567      | 64.6610      |
| CCID   | 0        | 80 | 1.0343       | 93.7500      | 0            |
|        | 1        | 81 | 2.1794       | 55.5556      | 1.3419       |
|        | 2        | 82 | 2.1863       | 99.7783      | 1.4872       |
|        | 3        | 83 | 2.1795       | 98.5761      | 1.5315       |
|        | 4        | 84 | 2.1673       | 97.4026      | 1.5461       |
|        | 5        | 85 | 2.1525       | 96.2567      | 1.5480       |
| CCFD   | 0        | 80 | 47.3635      | 97.9453      | 13.5431      |
|        | 1        | 81 | 46.8607      | 96.7361      | 13.3985      |
|        | 2        | 82 | 46.3644      | 95.5564      | 13.2544      |
|        | 3        | 83 | 45.8751      | 94.4051      | 13.1113      |
|        | 4        | 84 | 45.3932      | 93.2813      | 12.9697      |
|        | 5        | 85 | 44.9189      | 92.1838      | 12.8297      |

Table 7. Summary Statistics for the Three Classes of CCD for \( k = 9 \)

| Design | \( n_b \) | N  | D-efficiency | G-efficiency | A-efficiency |
|--------|----------|----|--------------|--------------|--------------|
| CCFD   | 0        | 146| 95.1961      | 65.5038      | 24.6728      |
|        | 1        | 147| 95.7713      | 67.7685      | 40.1483      |
|        | 2        | 148| 95.8360      | 68.2501      | 49.9540      |
|        | 3        | 149| 95.6959      | 68.2558      | 56.7398      |
|        | 4        | 150| 95.4467      | 68.0777      | 61.6512      |
|        | 5        | 151| 95.1310      | 67.8158      | 65.3207      |
| CCID   | 0        | 146| 1.2082       | 65.5038      | 0.6807       |
|        | 1        | 147| 1.2155       | 67.7685      | 0.7802       |
|        | 2        | 148| 1.2163       | 68.2501      | 0.8161       |
|        | 3        | 149| 1.2145       | 68.2558      | 0.8326       |
|        | 4        | 150| 1.2114       | 68.0777      | 0.8406       |
|        | 5        | 151| 1.2073       | 67.8158      | 0.8443       |
| CCFD   | 0        | 146| 48.0001      | 74.1998      | 8.5040       |
|        | 1        | 147| 47.7326      | 74.2360      | 8.4560       |
|        | 2        | 148| 47.4650      | 74.2056      | 8.4075       |
|        | 3        | 149| 47.1978      | 74.1220      | 8.3586       |
|        | 4        | 150| 46.9314      | 73.9993      | 8.3096       |
|        | 5        | 151| 46.6660      | 73.8521      | 8.2605       |
Table 8. Summary statistics for the Three Classes of CCD for k = 10

| Design | n_0 | N   | D-efficiency | G-efficiency | A-efficiency |
|--------|-----|-----|--------------|--------------|--------------|
| CCCD   | 0   | 148 | 92.2430      | 77.5423      | 10.4567      |
|        | 1   | 149 | 93.7285      | 57.0375      | 33.9412      |
|        | 2   | 150 | 93.9179      | 81.6781      | 47.1633      |
|        | 3   | 151 | 93.8100      | 81.3789      | 55.5256      |
|        | 4   | 152 | 93.5685      | 80.9792      | 61.2067      |
|        | 5   | 153 | 93.2527      | 80.5250      | 65.2541      |
| CCID   | 0   | 148 | 1.1202       | 77.5423      | 0.4616       |
|        | 1   | 149 | 1.1382       | 57.0375      | 0.7301       |
|        | 2   | 150 | 1.1405       | 81.6781      | 0.7836       |
|        | 3   | 151 | 1.1392       | 81.3789      | 0.8037       |
|        | 4   | 152 | 1.1363       | 80.9792      | 0.8125       |
|        | 5   | 153 | 1.1324       | 80.5250      | 0.8162       |
| CCFD   | 0   | 148 | 49.3425      | 87.8365      | 8.9235       |
|        | 1   | 149 | 49.0560      | 87.2470      | 8.8706       |
|        | 2   | 150 | 48.7707      | 87.6145      | 8.9369       |
|        | 3   | 151 | 48.4871      | 87.4347      | 8.7649       |
|        | 4   | 152 | 48.2052      | 87.2259      | 8.7122       |
|        | 5   | 153 | 47.9252      | 86.9878      | 8.6598       |

3.2 Graphical presentation of results and discussion

Graphical presentation of results in Fig. 1 – 8, for factors k = 3 – 10 and with centre points 0 -5

Three-Factor Design: Fig. 1 shows that with n_0 = 0, the D-efficiency values for the CCCD and CCID are low; an increase in n_0 to 1 and 2, increases the D-efficiency values, and decreases as n_0 increases. For CCFD, the D-, G- and A-efficiency values are high with n_0 = 0, but reduce as n_0 increases.

For the G-efficiency, the CCCD and CCID tend to have high values with n_0 = 0, reduce with n_0 = 1, increase with n_0 = 2, and thereafter, an increase of the n_0 reduces the G-efficiency value.
Four-Factor Design: Fig. 2. shows that the D- and G-efficiency values for the CCCD and CCID with \( n_0 = 0 \), are at infinity. Increasing \( n_0 \) tends to fluctuate the D- and G-efficiency values for the CCCD and CCID; an increase of \( n_0 \) increases the A-efficiency values for the CCID. The D- and A-efficiency values for the CCID from \( n_0 = 1 \) to \( 5 \) are relatively the same. For the CCFD, the D-, G- and A-efficiency values tend to reduce with an increase in \( n_0 \).

Five-Factor Design: Fig. 3. shows that increasing \( n_0 \) reduces the D-, G- and A-efficiency values of the CCFD. At \( n_0 = 0 \), the G-efficiency values of the CCCD and CCID tend to be high, which reduces as \( n_0 \) increases. The D- and A-efficiency values of the CCCD fluctuate as \( n_0 \) increases.

Six-Factor Design: Fig. 4. shows that increasing \( n_0 \) increases the D- and A-efficiency values for the CCCD and CCID, but for the CCCD, the D-efficiency values started to decline at \( n_0 = 3 \). Also at \( n_0 = 0 \), the G-efficiency values for the CCCD and CCID are high; thereafter, it begins to fluctuate. For the CCFD, the D-, G- and A-efficiency values reduce as \( n_0 \) increases.
Seven-Factor Design: Fig 5 shows the same conclusion as in the case of the six-factor design

Eight-Factor Design: Fig. 6. Shows the same conclusion as in the case of six-factor design, only that the A-efficiency values for the CCCD and CCID at $n_0 = 0$ is 0

Nine-Factor Design: Fig. 7 shows an almost equal D- and G-efficiency values for the CCCD and CCID while the A-efficiency values for both CCCD and CCID increases as $n_0$ increases. For the CCFD, the D-, G- and A-efficiency values tend to reduce slightly with an increase in $n_0$.

Ten-Factor Design: Fig. 8. Shows slightly equal D-efficiency values for the CCCD and CCID. Also, the A-efficiency values for the CCCD and CCID tend to increase as $n_0$ increases. However, their G-efficiency values fluctuate with increase in $n_0$. For the CCFD, Increasing $n_0$ tend to reduce the D-, G- and A-efficiency values.
4 Findings and Conclusions

4.1 Findings

Three classes of central composite design, namely: Central Composite Circumscribed Design, Central Composite Inscribed Design and Central Composite Face-Centered Design are compared for factors, \( k \), ranging from 3 to 10 with 0 – 5 centre points, respectively, using the \( D \)-, \( G \)- and \( A \)-efficiencies. The results show that the CCDs perform better when the \( G \)-efficiency is employed for all the factors considered. Also increasing the centre points tend to reduce the \( D \)-, \( G \)- and \( A \)-efficiency values of the CCFD. The CCCD and CCID behave alike in terms of the \( G \)-efficiency criterion; the CCCD performs better than the CCID and CCFD when the \( D \)- and \( A \)-efficiency criteria are employed, but with centre points greater than zero.

4.2 Conclusion

From the foregoing, it can be seen that for factors \( k = 3, 4, 5, 6 \) and 8, the \( G \)-efficiency performs better than the \( D \)- and \( A \)-efficiencies for the number of parameter, \( N \), and the number of centre points \( n_0 \) considered. For factor \( k = 7, 9 \) and 10 the \( D \)-efficiency performs better than the \( G \)- and \( A \)-efficiencies for CCCD, while the \( G \)-efficiency performs better than the \( D \)- and \( A \)-efficiencies for CCID and CCFD, for the number of parameter, \( N \), and the number of centre points \( n_0 \) considered.

In general the CCDs give high efficiency values when the \( G \)-efficiency is employed and it can also be seen that the CCID and CCFD have low efficiencies values under the \( D \)- and \( A \)-eficiencies respectively for the number of parameter, \( N \), and the number of centre points \( n_0 \) considered. Finally, the CCCD performs better than the CCID and CCFD when the \( D \)- and \( A \)-efficiency criteria are employed, but with centre points greater than zero which implies that the CCCD is a better CCD, when compared but the inclusion of centre points is recommended.

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Competing Interests

Authors have declared that no competing interests exist.

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