Vacuum energy of the supersymmetric $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ in the $1/N$ expansion

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By employing the $1/N$ expansion, we compute the vacuum energy $E(\delta \epsilon)$ of the two-dimensional supersymmetric (SUSY) $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with $\mathbb{Z}_N$ twisted boundary conditions to the second order in a SUSY breaking parameter $\delta \epsilon$. This quantity was vigorously studied recently by Fujimori et al. by using a semi-classical approximation based on the bion, motivated by a possible semi-classical picture on the infrared renormalon. In our calculation, we find that the parameter $\delta \epsilon$ receives renormalization and, after this renormalization, the vacuum energy becomes ultraviolet finite. To the next-to-leading order of the $1/N$ expansion, we find that the vacuum energy normalized by the radius of the $S^1$, $R$, $RE(\delta \epsilon)$ behaves as $(\Lambda R)^{-3}$ for $\Lambda R$ small, where $\Lambda$ is the dynamical scale. Since $\Lambda$ is related to the renormalized 't Hooft coupling $\lambda_R$ as $\Lambda \sim e^{-2\pi/\lambda_R}$, to the order of the $1/N$ expansion we work out, the vacuum energy is a purely non-perturbative quantity and has no well-defined weak coupling expansion in $\lambda_R$. 

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1. Introduction

In this paper, by employing the $1/N$ expansion (for a classical exposition, see Ref. [1]), we compute the vacuum energy $E(\delta \epsilon)$ of the two-dimensional (2D) supersymmetric (SUSY) $\mathbb{C}P^{N-1}$ model [2,3] on $\mathbb{R} \times S^1$ with $Z_N$ twisted boundary conditions to the second order in a SUSY breaking parameter $\delta \epsilon$. This quantity was vigorously studied recently by Fujimori et al. [4] (see also Refs. [5–8]) by using a semi-classical approximation based on the bion [9–14]. One of motivations of their study was a possible semi-classical picture on the infrared (IR) renormalon [15, 16] advocated in Refs. [17–20]. In these works, in the context of the resurgence program (for a review, see Ref. [21] and the references cited therein), it is proposed that the ambiguity caused by the IR renormalon through the Borel resummation (for a review, see Ref. [22]) be cancelled by the ambiguity associated with the integration of quasi-collective coordinates of the bion; this scenario is quite analogous to the Bogomolny–Zinn-Justin mechanism for the instanton–anti-instanton pair [23, 24].

In Ref. [5], by using the Lefschetz thimble method [25–27], the integration over quasi-collective coordinates of the bion is explicitly carried out and it was found that the vacuum energy $E(\delta \epsilon)$ possesses the imaginary ambiguity which is of the same order as that caused by the so-called $u = 1$ IR renormalon. On the other hand, for the four-dimensional $SU(N)$ gauge theory with the adjoint fermion (4D QCD(adj.)), for $N = 2$ and 3, it has been found that [28] when the spacetime is compactified as $\mathbb{R}^3 \times S^1$, the logarithmic behavior of the vacuum polarization of the gauge boson associated with the Cartan subalgebra (“photon”) disappears. Since the IR renormalon is attributed to such a logarithmic behavior, in Ref. [28], it is concluded that the circle compactification generally eliminates the IR renormalon. This appears inconsistent with the renormalon interpretation of the result in Ref. [5].

The original motivation in a series of works [29–31] by a group including the present authors was to investigate the fate of the IR renormalon under the circle compactification to understand the above inconsistency. For this, we employed the $1/N$ expansion (i.e., the large-$N$ limit), in which

$$\Lambda R = \text{const. as } N \to \infty, \quad (1.1)$$

where $\Lambda$ is a dynamical scale and $R$ is the $S^1$ radius. We expected that in this way the IR renormalon and the bion can be highlighted, because the beta function of the ’t Hooft coupling and the bion action remain non-trivial in the large-$N$ limit [1.1], whereas other sources to the Borel singularity such as the instanton–anti-instanton pair is suppressed. This intention was not so successful, because the calculations in Refs. [29–31] show that the behavior of the IR renormalon rather depends on the system; in the 2D SUSY $\mathbb{C}P^{N-1}$ model, the compactification from $\mathbb{R}^2$ to $\mathbb{R} \times S^1$ shifts the location of the Borel singularity associated with the IR renormalon [29, 31]. In the 4D QCD(adj.), because of the twisted momentum of the gauge boson associated with the root vectors (“W boson”), $\mathbb{R}^3 \times S^1$ is effectively decompactified in the large-$N$ limit [35-37] and the IR renormalon gives rise to the same Borel singularity as the uncompactified $\mathbb{R}^4$ [30]. It appears that a unified picture on the semi-classical understanding of the IR renormalon is still missing.

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1. Recent related works are Refs. [32–34].

2. In this analysis, we relied on the so-called large-$\beta_0$ approximation [38–40].
In the present paper, as announced in Ref. [29], we compute the vacuum energy $E(\delta E)$ of the 2D SUSY $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with $\mathbb{Z}_N$ twisted boundary conditions to the second order in a SUSY breaking parameter $\delta \epsilon$ in the $1/N$ expansion with Eq. (1.1); this is the quantity computed in Ref. [3] by the bion calculus. First, we find that the parameter $\delta \epsilon$ receives renormalization and, after this renormalization, the vacuum energy becomes ultraviolet (UV) finite. To the next-to-leading order of the $1/N$ expansion, we find that the vacuum energy is IR finite as should be the case for a physical quantity. Finally, we find that the vacuum energy normalized by the radius of the $S^1$, $RE(\delta \epsilon)$ behaves as $(\Lambda R)^{-3}$ for $\Lambda R$ small as shown in Eqs. (3.52)–(3.57) and Figs. 2 and 3. Since $\Lambda$ is related to the renormalized ’t Hooft coupling $\lambda_R$ as $\Lambda \sim e^{-2\pi/\lambda_R}$, to the order of the $1/N$ expansion we work out, the vacuum energy is a purely non-perturbative quantity and has no well-defined weak coupling expansion in $\lambda_R$. This implies that one cannot even define the perturbative expansion for this quantity computed in the $1/N$ expansion and cannot even discuss the renormalon problem. Therefore, although our $1/N$ calculation is robust, it does not give any clue to the issue. We are not well understanding yet why the semi-classical calculation on the basis of the bion cannot be observed in the $1/N$ expansion. Nevertheless, we believe that it is worthwhile to report our $1/N$ calculation for future considerations because our calculation itself is rather nontrivial.

2. 2D SUSY $\mathbb{C}P^{N-1}$ model

2.1. Action and boundary conditions

Our spacetime is $\mathbb{R} \times S^1$ and $-\infty < x < \infty$ denotes the coordinate of $\mathbb{R}$ and $0 \leq y < 2\pi R$ the coordinate of $S^1$. The Euclidean action of the 2D SUSY $\mathbb{C}P^{N-1}$ model in terms of the homogeneous coordinate variables [2–4] is, in the notation of Eq. (2.24) of Ref. [29],

$$S = \int d^2x N_\chi \left[ -f + \delta \sigma + \bar{z}^A (D_\mu D_\mu + f)z^A 
+ \chi^A (\bar{\psi} + \sigma P_+ + \sigma P_-)\chi^A + 2\bar{z}^A z^A \eta + 2\bar{\eta} z^A \chi^A \right]
- \int d^2x \frac{i\theta}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu.$$  (2.1)

Here and in what follows, it is understood that repeated indices are summed over; the lower Greek indices, $\mu, \nu, \ldots$, take the value $x$ or $y$ and the uppercase Roman indices, $A, B, \ldots$, run from 1 to $N$. $\lambda$ is the bare ‘t Hooft coupling and $\theta$ is the theta parameter. Also,

$$D_\mu z^A \equiv (\partial_\mu + iA_\mu)z^A, \quad D^A \chi^A \equiv \gamma_\mu (\partial_\mu + iA_\mu)\chi^A,$$

$$P_\pm \equiv \frac{1 \pm \gamma_5}{2}, \quad \gamma_5 \equiv -i\gamma_x \gamma_y, \quad \gamma_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$  (2.2)

and $\epsilon_{xy} = -\epsilon_{yx} = +1$.

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3 The theta parameter $\theta$ may be eliminated by the anomalous chiral rotation, $\chi^A \rightarrow e^{i\alpha\gamma_5} \chi^A$, $\bar{\chi}^A \rightarrow \bar{\chi}^A e^{i\alpha\gamma_5}$, $\eta \rightarrow e^{-i\alpha\gamma_5} \eta$, $\bar{\eta} \rightarrow \bar{\eta} e^{-i\alpha\gamma_5}$, and $\sigma \rightarrow e^{2i\alpha} \sigma$.  

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For the fields with the index $A$ (we call them $N$-fields), we impose the $\mathbb{Z}_N$ twisted boundary conditions along $S^1$,

$$
\begin{align*}
  z^A(x, y + 2\pi R) &= e^{2\pi im_A R} z^A(x, y), \\
  \chi^A(x, y + 2\pi R) &= e^{2\pi im_A R} \chi^A(x, y), \\
  \bar{\chi}^A(x, y + 2\pi R) &= e^{-2\pi im_A R} \bar{\chi}^A(x, y),
\end{align*}
$$

(2.3)

where the twist angle $m_A$ in these expressions depends on the index $A$ as

$$
m_A \equiv \frac{A}{NR} \quad \text{for } A = 1, \ldots, N - 1, \quad m_N \equiv 0.
$$

(2.4)

These twisted boundary conditions allow the fractional instanton/anti-instanton, the constituent of the bion.

For the auxiliary fields, $f$, $\sigma$, $\bar{\sigma}$, $A_\mu$, $\eta$, and $\bar{\eta}$, on the other hand, we assume the periodic boundary conditions along $S^1$.

For the calculation below, however, it turns out that an alternative form of the action, that is obtained by

$$
f \to f + \bar{\sigma}\sigma,
$$

(2.5)

from Eq. (2.1), that is,

$$
S = \int d^2 x \frac{N}{\lambda} \left[ -f + \bar{z}^A (-D_\mu D^\mu + f + \bar{\sigma}\sigma) z^A \\
+ \bar{\chi}^A (\slashed{D} + \bar{\sigma} P_+ + \sigma P_-) \chi^A + 2 \bar{\chi}^A z^A \eta + 2 \bar{\eta} \bar{z}^A \chi^A \right] \\
- \int d^2 x \frac{i \theta}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu,
$$

(2.6)

is more convenient. This is because renormalization with the action (2.1) requires an infinite shift of the field $f$ in addition to the multiplicative renormalization of the ’t Hooft coupling (Eq. (2.10) below), whereas the action (2.6) does not require such a shift. This difference comes from the fact that $\bar{\sigma}\sigma$ in Eq. (2.5) is a composite operator and UV divergent. In fact, the action (2.6) can be obtained by the dimensional reduction of a manifestly SUSY invariant nonlinear sigma model in four dimensions [41]; we thus expect a simpler UV divergent structure. For this reason, we adopt the action (2.6) in the present paper.

2.2. Saddle point and propagators in the leading order of the $1/N$ expansion

Now, since the action (2.1) (i.e., Eq. (2.24) of Ref. [29]) and the action (2.6) are simply related by the change of variable (2.5), we can borrow the results in Ref. [29] in the leading order of the $1/N$ expansion.

First, setting

$$
A_\mu = A_{\mu 0} + \delta A_\mu, \quad f = f_0 + \delta f, \quad \sigma = \sigma_0 + \delta \sigma,
$$

(2.7)

where the subscript 0 indicates the value at the saddle point in the $1/N$ expansion and $\delta$ denotes the fluctuation, in the leading order of the $1/N$ expansion in Eq. (1.1), we have

$$
A_{\mu 0} = A_\mu 0 \delta_{\mu y}, \quad f_0 = 0, \quad \bar{\sigma}_0 \sigma_0 = \Lambda^2,
$$

(2.8)

where $\Lambda$ is the dynamical scale,

$$
\Lambda = \mu e^{-2\pi/\lambda R},
$$

(2.9)
defined from the renormalized ’t Hooft coupling $\lambda_R$ in the “$\overline{\text{MS}}$ scheme”,

$$\lambda = \left( \frac{e^\gamma \mu^2}{4\pi} \right)^\varepsilon \lambda_R \left( 1 + \frac{\lambda_R}{4\pi \varepsilon} \right)^{-1}. \quad (2.10)$$

Here, we have used the dimensional regularization with the complex dimension $D = 2 - 2\varepsilon$; $\mu$ is the renormalization scale. In Eq. (2.8), the constant $A_{y0}$ is not determined from the saddle point condition in the present supersymmetric theory and, for $\mathbb{Z}_N$ invariant quantities such as the partition function and the vacuum energy considered below, it should be integrated over with the measure $\int_0^1 d(A_{y0}RN)$. \quad (2.11)

Next, we need the propagators among fluctuations of the auxiliary fields. To obtain these, we add the gauge fixing term

$$S_{gf} = \frac{N}{4\pi} \int d^2x d^2x' \frac{1}{2} \partial_\mu A_\mu(x) \partial_\nu A_\nu(x') \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{-ip(x-x')} \mathcal{L}(p), \quad (2.12)$$

and a local counter term

$$S_{\text{local}} = \frac{N}{4\pi} \int d^2x \left( -\frac{1}{2} \right) [\delta\sigma(x) - \delta\tilde{\sigma}(x)]^2, \quad (2.13)$$

to the action $\mathcal{S}_{gf}$ $\mathcal{S}_{\text{local}}$. Then, in the leading order of the $1/N$ expansion, we have

$$\langle \delta A_\mu(x) \delta A_\nu(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip(x-x')} \frac{\mathcal{L}(p)}{D(p)} \left\{ \delta_{\mu\nu} + 4 \left[ \Lambda^2 + \frac{\bar{p}_y^2 K(p)^2}{p^2 \mathcal{L}(p)^2} \right] \frac{p_\mu p_\nu}{(p^2)^2} \right\},$$

$$\langle \delta A_\mu(x) \delta R(x') \rangle = \langle \delta R(x) \delta A_\mu(x') \rangle = 0,$$

$$\langle \delta A_\mu(x) \delta I(x') \rangle = -\langle \delta I(x) \delta A_\mu(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip(x-x')} \frac{\mathcal{L}(p)}{D(p)} \frac{2\Lambda^2 \bar{p}_y}{p^2},$$

$$\langle \delta A_\mu(x) \delta f(x') \rangle = \langle \delta f(x) \delta A_\mu(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip(x-x')} \frac{K(p)}{D(p)} \frac{2\bar{p}_\mu \bar{p}_y}{p^2},$$

$$\langle \delta R(x) \delta R(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip(x-x')} \frac{\mathcal{L}(p)}{D(p)} \Lambda^2,$$

$$\langle \delta R(x) \delta I(x') \rangle = -\langle \delta I(x) \delta R(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip(x-x')} \frac{K(p)}{D(p)} \frac{2\bar{p}_\mu}{p^2},$$

$$\langle \delta R(x) \delta f(x') \rangle = \langle \delta f(x) \delta R(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip(x-x')} \frac{\mathcal{L}(p)}{D(p)} (-2\Lambda^2),$$

$$\langle \delta I(x) \delta I(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip(x-x')} \frac{\mathcal{L}(p)}{D(p)} \Lambda^2,$$

$$\langle \delta I(x) \delta f(x') \rangle = -\langle \delta f(x) \delta I(x') \rangle = 0,$$

$$\langle \delta f(x) \delta f(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip(x-x')} \frac{\mathcal{L}(p)}{D(p)} (-p^2),$$
\[
\langle \eta(x) \bar{\eta}(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \sum_{p_y} \frac{1}{2\pi R} e^{ip(x-x')} \left( (i\not{p} + 2\bar{\sigma}_0 P_+ + 2\sigma_0 P_-) \mathcal{L}(p) + 2i\bar{\rho}_y / p^2 \mathcal{K}(p) \right) \left( -\frac{1}{2} \right), \quad (2.14)
\]

where the Kaluza–Klein (KK) momentum along \( S^1 \), \( p_y \), takes discrete values \( p_y = n/R \) with \( n \in \mathbb{Z} \). We also have introduced the notations,

\[
\bar{\rho}_\mu \equiv e_{\mu\nu} p_\nu, \quad (2.15)
\]

and

\[
\delta R(x) \equiv \frac{1}{2} [\bar{\sigma}_0 \delta \sigma(x) + \sigma_0 \delta \bar{\sigma}(x)], \quad \delta I(x) \equiv \frac{1}{2t} [\bar{\sigma}_0 \delta \sigma(x) - \sigma_0 \delta \bar{\sigma}(x)]. \quad (2.16)
\]

From the above results, we also have

\[
\langle \delta \sigma(x) \delta \bar{\sigma}(x') \rangle = \frac{4\pi}{N} \int \frac{dp_x}{2\pi} \sum_{p_y} \frac{1}{2\pi R} e^{ip(x-x')} \left( 2\mathcal{L}(p) + 4i\bar{\rho}_y / p^2 \mathcal{K}(p) \right). \quad (2.17)
\]

Various functions used in the above expressions are defined by

\[
\mathcal{L}(p) \equiv \mathcal{L}_\infty(p) + \hat{\mathcal{L}}(p),
\]

\[
\mathcal{L}_\infty(p) \equiv \frac{2}{\sqrt{p^2(2 + 4\Lambda^2)}} \ln \left( \frac{\sqrt{p^2 + 4\Lambda^2} + \sqrt{p^2}}{\sqrt{p^2 + 4\Lambda^2} - \sqrt{p^2}} \right),
\]

\[
\hat{\mathcal{L}}(p) \equiv \int_0^1 dx \sum_{m \neq 0} e^{-iA_0 2\pi R n m} e^{i p_y 2\pi R n m} \frac{2\pi R |m|}{\sqrt{\Lambda^2 + x(1-x)p^2}} K_1(\sqrt{\Lambda^2 + x(1-x)p^2} 2\pi R |m|),
\]

\[
\mathcal{K}(p) \equiv i \int_0^1 dx \sum_{m \neq 0} e^{-iA_0 2\pi R n m} e^{i p_y 2\pi R n m} 2\pi R n m K_0(\sqrt{\Lambda^2 + x(1-x)p^2} 2\pi R |m|),
\]

\[
\mathcal{D}(p) \equiv (p^2 + 4\Lambda^2) \mathcal{L}(p)^2 + 4\bar{\rho}_y^2 / p^2 \mathcal{K}(p)^2, \quad (2.18)
\]

where \( K_\nu(z) \) denotes the modified Bessel function in the second kind. For later calculations, it is important to note the properties,

\[
\mathcal{L}(p) = \mathcal{L}(-p), \quad \mathcal{K}(p) = \mathcal{K}(-p). \quad (2.19)
\]

These can be shown by the change of the Feynman parameter, \( x \to 1-x \), noting \( p_y \in \mathbb{Z}/R \).

Going back to the action \( S \) (2.4), with the saddle point values (2.8), the propagators of the \( N \)-fields in the leading order of the \( 1/N \) expansion are given by

\[
\langle \zeta^A(x) \bar{\zeta}^B(x') \rangle = \delta^{AB} \frac{\lambda}{N} \int \frac{dp_x}{2\pi} \sum_{p_y} \frac{1}{2\pi R} e^{ip_x(x-x')} e^{ip_y + m_{A}}(y-y')
\]

\[
\times \left[ p_x^2 + (p_y + A_y 0 + m_{A})^2 + \Lambda^2 \right]^{-1},
\]

\[
\langle \chi^A(x) \bar{\chi}^B(x') \rangle = \delta^{AB} \frac{\lambda}{N} \int \frac{dp_x}{2\pi} \sum_{p_y} \frac{1}{2\pi R} e^{ip_x(x-x')} e^{ip_y + m_{A}}(y-y')
\]

\[
\times \left[ i\gamma_x p_x + i\gamma_y (p_y + A_y 0 + m_{A}) + \bar{\sigma}_0 P_+ + \sigma_0 P_- \right]^{-1}. \quad (2.20)
\]

To obtain these, we noted the twisted boundary conditions (2.3).
3. Computation of the vacuum energy

3.1. General strategy

Our objective in this paper is to compute the vacuum energy of the present system as a power series of the coefficient $\delta \epsilon$ of a supersymmetry breaking term—the quantity computed in Ref. [5]:

$$E(\delta \epsilon) = E(0) + E^{(1)} \delta \epsilon + E^{(2)} \delta \epsilon^2 + \cdots.$$  

(3.1)

Here, the supersymmetry breaking term introduced in Ref. [5] is,

$$\delta S \equiv \int d^2x \frac{\delta \epsilon}{\pi R} \sum_{A=1}^N m_A \left( \bar{z}^A z^A - \frac{1}{N} \right).$$  

(3.2)

Note that this depends on the twist angles in Eq. (2.4). A quick way to incorporate the effect of Eq. (3.2) systematically is to regard $\delta S$ as a mass term of the $z^A$-field, as

$$S + \delta S = \int d^2x \frac{N}{\lambda} z^A (-\partial_\mu \partial_\mu + \Lambda^2 + \delta_A) z^A + \cdots,$$  

(3.3)

where

$$\delta_A \equiv \frac{\lambda \delta \epsilon}{\pi R N} m_A.$$  

(3.4)

With this modification, the vacuum energy is given by

$$- \int dx E(\delta \epsilon) = \int d^2x \frac{1}{\lambda} \sum_A \delta_A - \sum_A \ln \det(-\partial_\mu \partial_\mu + \Lambda^2 + \delta_A)$$

$$+ \text{(connected vacuum bubble diagrams)}.$$  

(3.5)

Here, the vacuum bubble diagrams, that start from two-loop order, are computed by using the modified $z^A$-propagator,

$$\langle z^A(x)z^B(x') \rangle = \delta^{AB} \frac{\lambda}{N} \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} e^{ip_x(x-x')} e^{i(p_y + m_A)(y-y')} \left[p_x^2 + (p_y + A_y0 + m_A)^2 + \Lambda^2 + \delta_A\right]^{-1},$$  

(3.6)

instead of the one in Eq. (2.20). Then, by expanding Eq. (3.5) with respect to $\delta_A$, we have the series expansion (3.1). In the following calculations, we set $E^{(0)} = 0$ assuming that the bare vacuum energy at $\delta \epsilon = 0$ is chosen so that the system is supersymmetric for $\delta \epsilon = 0$. This amounts to compute the difference $E(\delta \epsilon) - E(\delta \epsilon = 0)$.

If all the $N$-fields obey the same boundary conditions along $S^1$, all $z^A$ (or $\chi^A$ and $\bar{\chi}^A$) equally contribute and the order of the loop expansion with the use of the auxiliary fields and the order of the $1/N$ expansion would coincide [1]. With the twisted boundary conditions [2.3], however, not all $N$-fields contribute equally. The SUSY breaking term (3.2) also treats each of $N$-fields differently. For these reasons, in the present system, the order of the loop expansion and that of the $1/N$ expansion do not necessarily coincide; we have to distinguish both expansions. For instance, although the one-loop Gaussian determinant in Eq. (3.5) gives rise to the contribution of $O(1/N)$, it contains also terms of subleading orders, $O(1/N^2)$ and $O(1/N^3)$, as well (see Eq. 3.49, for instance).
3.2. One-loop Gaussian determinant

Let us start with the one-loop Gaussian determinant in Eq. (3.5). We first note

\[
- \sum_A \ln \det \left( -\partial_\mu \partial_\mu + \Lambda^2 + \delta A \right)
\]

\[
= - \sum_A \int d^2x \int \frac{dp_x}{2\pi} \frac{1}{2\pi R} \sum_{p_y} \ln \left[ p_x^2 + (p_y + m_A + A_{y0})^2 + \Lambda^2 + \delta A \right]
\]

\[
= - \int d^2x \sum_A \sum_{n=-\infty}^\infty \int \frac{d^2p}{(2\pi)^2} e^{i(p_y - m_{A_y} - A_{y0})2\pi R n} \ln(p^2 + \Lambda^2 + \delta A),
\]

where we have used the identity

\[
\frac{1}{2\pi R} \sum_{n=-\infty}^\infty F(n/R) = \sum_{n=-\infty}^\infty \int \frac{dp_y}{2\pi} e^{ip_y 2\pi R n} F(p_y),
\]

Hence, subtracting the logarithm of the Gaussian determinant at \(\delta \epsilon = 0\), we have

\[
- \sum_A \ln \det \left( -\partial_\mu \partial_\mu + \Lambda^2 + \delta A \right)
\]

\[
= - \int d^2x \sum_A \sum_{n=-\infty}^\infty \int \frac{d^2p}{(2\pi)^2} e^{i(p_y - m_{A_y} - A_{y0})2\pi R n} \ln \left( \frac{p^2 + \Lambda^2 + \delta A}{p^2 + \Lambda^2} \right).
\]

In this expression, since \(n \neq 0\) terms are Fourier transforms, only the \(n = 0\) term is UV divergent. Under the dimensional regularization with \(D = 2 - 2\epsilon\), the momentum integration yields

\[
- \sum_A \ln \det \left( -\partial_\mu \partial_\mu + \Lambda^2 + \delta A \right)
\]

\[
= \int d^2x \frac{1}{4\pi} \left[ \frac{1}{\epsilon} - \ln \left( \frac{e^{\gamma E}}{4\pi} \right) \right] \sum_A \delta A
\]

\[- \int d^2x \sum_A \frac{1}{4\pi} \left[ \delta A - (\Lambda^2 + \delta A) \ln \left( 1 + \frac{\delta A}{\Lambda^2} \right) \right]
\]

\[- \int d^2x \sum_A \sum_{n \neq 0} e^{-i(m_{A_y} + A_{y0})2\pi R n} \frac{1}{4\pi} \left[ (-4) \frac{1}{2\pi R |n|} \left[ \sqrt{\Lambda^2 + \delta A K_1(\sqrt{\Lambda^2 + \delta A 2\pi R |n|})} - \Lambda K_1(\Lambda 2\pi R |n|) \right] \right].
\]

Since Eqs. (2.9) and (2.10) imply

\[
\frac{1}{4\pi} \left[ \frac{1}{\epsilon} - \ln \left( \frac{e^{\gamma E}}{4\pi} \right) \right] = \frac{1}{\lambda},
\]

we see that the first term in the right-hand side of Eq. (3.10) is precisely canceled by the first term in the right-hand side of Eq. (3.5).
Fig. 1: Two-loop vacuum bubble diagrams that contribute to $E(\delta \epsilon)|_{2\text{-loop}}$ (3.13). The solid line denotes the $z^A$-propagator (3.6). The wavy line denotes the $\delta A^\mu$-propagator, the dotted line the $\delta f^\mu$-propagator, the broken line the $\delta \sigma$-propagator, the arrowed solid line the $\eta$-propagator in Eqs. (2.17) and (2.14), respectively.

In this way, from Eq. (3.5), we have

\[
E(\delta \epsilon)|_{1\text{-loop}} = 2\pi R \sum_A \frac{1}{4\pi} \left[ \delta_A - (\Lambda^2 + \delta A) \ln \left( 1 + \frac{\delta A}{\Lambda^2} \right) \right] 
+ 2\pi R \sum_A \sum_{n \neq 0} e^{-i(m_A + A_{\alpha})2\pi R n} 
\times \frac{1}{4\pi} \left( -4 \right) \frac{1}{2\pi R |n|} \left[ \sqrt{\Lambda^2 + \delta A} K_1(\sqrt{\Lambda^2 + \delta A} 2\pi R |n|) - \Lambda K_1(\Lambda 2\pi R |n|) \right]. \tag{3.12}
\]

3.3. Two-loop vacuum bubble diagrams

Next, we work out the vacuum bubble diagrams in the two-loop level; they are depicted in Fig. 1. By using the propagators in Eqs. (2.14), (2.17), (2.20), and (3.6), and interaction vertices in Eq. (2.6), from Eq. (3.5), we have

\[
E(\delta \epsilon)|_{2\text{-loop}} = -2\pi R^4 \frac{4\pi}{N} \sum_A \sum_{n=-\infty}^{\infty} \int \frac{d^2 p}{(2\pi)^2} e^{i(p_y - A_{\alpha 0} - m_A)2\pi R n} \frac{1}{p^2 + \Lambda^2 + \delta A} 
\times \int \frac{d\ell \times 1}{2\pi} \frac{1}{2\pi R} \sum_{\ell y} \frac{1}{(p - \ell)^2 + \Lambda^2 + \delta A} 
\times \left( \frac{1}{2} (2p_\mu - \ell_\mu)(2p_\nu - \ell_\nu) \frac{L(\ell)}{D(\ell)} \right) \left[ \delta_{\mu\nu} + 4 \left( \Lambda^2 + \frac{\ell_y^2}{\ell^2} K(\ell)^2 \right) \right] \frac{\ell_\mu \ell_\nu}{(\ell^2)^2} \right) \tag{Fig. 1a}
\]

\[
- \frac{1}{2} \frac{L(\ell)}{D(\ell)} \ell^2 \tag{Fig. 1b}
\]

\[
- 4p_\mu \frac{K(\ell)}{D(\ell)} \ell_\mu \ell_y \tag{Fig. 1c}
\]
To examine the renormalizability of this expression, we first note that this can be written as,

\[
E(\delta \epsilon) \big|_{2\text{-loop}} = \left. \frac{4 \pi}{N} \sum_{n=0}^{\infty} \int \frac{d^2 p}{(2\pi)^2} e^{i(p_y - A_0 - m_A)2\pi R_n} \frac{1}{p^2 + \Lambda^2 + \delta_A} \right.
\]

\[
\times \left\{ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \left[ 2p^2 - 2p \cdot \ell - 8\Lambda^2 \frac{p \cdot \ell}{\ell^2} + 8\Lambda^2 \left( \frac{p \cdot \ell}{\ell^2} \right)^2 \right] + \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)} \left( \ell^2 \right) \left[ 2 - 8 \frac{p \cdot \ell}{\ell^2} + 8 \left( \frac{p \cdot \ell}{\ell^2} \right)^2 \right] + \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)} \left( -4 \right) \frac{p \cdot \ell \bar{\ell} y}{\ell^2} \right\}
\]

\[
+ \left. \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \frac{1}{(p - \ell)^2 + \Lambda^2} \right.
\]

\[
\times \left\{ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \left[ -4p^2 + 4p \cdot \ell + 8\Lambda^2 \frac{p \cdot \ell}{\ell^2} - 4\Lambda^2 \frac{p^2}{\ell^2} - 4\Lambda^4 \frac{1}{\ell^2} \right] + \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)} \left( \ell^2 \right) \left[ -4 + 8 \frac{p \cdot \ell}{\ell^2} - 4 \frac{p^2}{\ell^2} - 4\Lambda^2 \frac{1}{\ell^2} \right] + \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)} \left( 8 \right) \frac{p \cdot \ell \bar{\ell} y}{\ell^2} \right\}
\]

\[- \text{ (terms with } \delta \epsilon = 0)\right. \right.
\]

(3.13)

where the contributions of each diagrams in Fig. 1 are separately indicated by the equation numbers. The total sum is

\[
E(\delta \epsilon) \big|_{2\text{-loop}} = -2\pi R \frac{4\pi}{N} \sum_{\delta \epsilon} \left\{ \left( e^{\delta \epsilon} e^{\delta \epsilon} - 1 \right) I(\xi, \eta) + \left( e^{\delta \epsilon} e^{\delta \epsilon} - 1 \right) [-2I(\xi, 0) + J(\xi)] \right\}_{|\xi = \eta = 0}, \quad (3.15)
\]
where

\[
I(\xi, \eta) \equiv \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \sum_{n=-\infty}^{\infty} \int \frac{d^2 p}{(2\pi)^2} e^{i(p_y - A_{yo} - m_{A})2\pi R n} \frac{1}{p^2 + \Lambda^2 + \xi (p - \ell)^2 + \Lambda^2 + \eta} \\
\times \left\{ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \left[ 2p^2 - 2p \cdot \ell - 8\Lambda^2 \frac{p \cdot \ell}{\ell^2} + 8\Lambda^2 \frac{(p \cdot \ell)^2}{(\ell^2)^2} \right] + \frac{K(\ell)}{\mathcal{D}(\ell)\mathcal{L}(\ell)} \left( -4 \frac{\ell^2}{\ell^2} + 8 \frac{(p \cdot \ell)^2}{(\ell^2)^2} \right) \right\},
\]

(3.16)

and

\[
J(\xi) \equiv \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \sum_{n=-\infty}^{\infty} \int \frac{d^2 p}{(2\pi)^2} e^{i(p_y - A_{yo} - m_{A})2\pi R n} \frac{1}{p^2 + \Lambda^2 + \xi (p - \ell)^2 + \Lambda^2} \\
\times \left[ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \Lambda^2 + \frac{K(\ell)}{\mathcal{D}(\ell)\mathcal{L}(\ell)} \frac{\ell^2}{\ell^2} \right] \left[ -8\frac{p \cdot \ell}{\ell^2} - 4 \frac{p^2}{\ell^2} - 4 \Lambda^2 \frac{1}{\ell^2} + 16 \frac{(p \cdot \ell)^2}{(\ell^2)^2} \right].
\]

(3.17)

From Eq. (3.18), we see that, for \(|\ell| \to \infty\), \(\mathcal{L}(p)\) and \(K(p)\) are exponentially small because of the Bessel functions, and thus

\[
\mathcal{L}(\ell) \xrightarrow{|\ell| \to \infty} \frac{2}{\ell^2} \ln(\ell^2/\Lambda^2), \quad \mathcal{D}(\ell) \xrightarrow{|\ell| \to \infty} \ell^2 \mathcal{L}(\ell)^2.
\]

(3.18)

From these, we see that, in \(I(\xi, \eta)\) (3.16), the integration over \(\ell\) as well as the integration over \(p\) are logarithmically UV divergent. In \(J(\xi)\) (3.17), the integration over \(p\) is logarithmically UV divergent but the integration over \(\ell\) is UV convergent. Assuming (say) the dimensional regularization, the change of integration variables \((p, \ell) \to (p - \ell, -\ell)\) in \(I(\xi, \eta)\) (3.16) shows that

\[
I(\xi, \eta) = I(\eta, \xi).
\]

(3.19)

Now, in Eq. (3.15), using the identity

\[
e^{\delta_A \partial_\ell} e^{\delta_A \partial_n} - 1 = \left( e^{\delta_A \partial_\ell} - 1 \right) \left( e^{\delta_A \partial_n} - 1 \right) + e^{\delta_A \partial_\ell} + e^{\delta_A \partial_n} - 2,
\]

(3.20)

and noting the property (3.19), we have the following very convenient representation:

\[
E(\delta \epsilon)_{|2\text{-loop}} = -2\pi R \frac{4\pi}{N} \sum_A \left[ \left( e^{\delta_A \partial_\ell} - 1 \right) \left( e^{\delta_A \partial_n} - 1 \right) I(\xi, \eta) + \left( e^{\delta_A \partial_\ell} - 1 \right) J(\xi) \right] \bigg|_{\xi=\eta=0}.
\]

(3.21)

This shows that \(E(\delta \epsilon)_{|2\text{-loop}}\) is UV finite provided that the parameter \(\delta_A\) is UV finite. That is, the operator \(e^{\delta_A \partial_\ell} - 1\) acting on \(J(\xi)\) increases the power of \(p^2 + \Lambda^2\) in the denominator in Eq. (3.17) and makes the \(p\) integration UV finite. Similarly, the operator \((e^{\delta_A \partial_\ell} - 1)(e^{\delta_A \partial_n} - 1)\) acting on \(I(\xi, \eta)\) increases the power of \((p^2 + \Lambda^2)[(p - \ell)^2 + \Lambda^2]\) in the denominator of Eq. (3.16) and makes the integrations over \(p\) and \(\ell\) UV convergent.
3.4. Renormalizability to the two-loop order

So far, we have observed that, from Eq. (3.12),

\[
E(\delta \epsilon)_{1\text{-loop}} = 2\pi R \sum_A \frac{1}{4\pi} \left[ \delta_A - (\Lambda^2 + \delta A) \ln \left( 1 + \frac{\delta A}{\Lambda^2} \right) \right]
\]

\[+ 2\pi R \sum_A \sum_{n \neq 0} e^{-i(m_A + A_\omega)2\pi Rn} \times \frac{1}{4\pi} (-4) \left[ \sqrt{\Lambda^2 + \delta A} K_1(\sqrt{\Lambda^2 + \delta A}2\pi R|n|) - \Lambda K_1(\Lambda 2\pi R|n|) \right], \tag{3.22}
\]

and, from Eq. (3.21),

\[
E(\delta \epsilon)_{2\text{-loop}} = -2\pi R \frac{4\pi}{N} \sum_A \left[ (e^{\delta A \partial_\xi} - 1) (e^{\delta A \partial_\eta} - 1) I(\xi, \eta) + (e^{\delta A \partial_\xi} - 1) J(\xi) \right] \bigg|_{\xi = \eta = 0}. \tag{3.23}
\]

These representations show that the vacuum energy to the two-loop order is UV finite, if the parameter \( \delta A \) defined in Eq. (3.4) is UV finite. This implies that the parameter \( \delta \epsilon \) must receive a non-trivial renormalization, as

\[
\delta A = \frac{\lambda \delta \epsilon}{\pi N} m_A \text{ is UV finite } \Rightarrow \delta \epsilon = \left( \frac{e^{\gamma_E} \mu^2}{4\pi} \right)^{-\varepsilon} \left( 1 + \frac{\lambda R}{4\pi \varepsilon} \right) \delta \epsilon_R, \tag{3.24}
\]

so that \( \lambda \delta \epsilon = \lambda R \delta \epsilon_R \) is UV finite; here we have used Eq. (2.10).

In terms of the renormalized parameters, the expansion of Eq. (3.22) with respect to \( \delta \epsilon \) yields

\[
E^{(1)}(\delta \epsilon)_{1\text{-loop}} = N\Lambda \frac{1}{R} \frac{\lambda R \delta \epsilon_R}{\pi N} \frac{R}{N} \sum_A m_A \sum_{n \neq 0} e^{-i(m_A + A_\omega)2\pi Rn} K_0(2\pi R|n|) \cdot \sum_{n \neq 0} e^{-i(m_A + A_\omega)2\pi Rn} 2\pi \Lambda R|n| K_1(2\pi \Lambda R|n|). \tag{3.25}
\]

For Eq. (3.23), we need to carry out momentum integrations in Eqs. (3.16) and (3.17). This is the subject of the next subsection.

3.5. \( p \)-integration in \( E^{(1)}(\delta \epsilon)_{2\text{-loop}} \) and \( E^{(2)}(\delta \epsilon^2)_{2\text{-loop}} \)

Let us next consider \( E^{(1)}(\delta \epsilon)_{2\text{-loop}} \) that is given by the \( O(\delta A) \) term of Eq. (3.23). By using the formulas in Appendix A, \( p \)-integration in Eq. (3.17) yields

\[
E^{(1)}(\delta \epsilon)_{2\text{-loop}} = 2\pi R \frac{1}{N} \sum_A \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \left[ \frac{L(\ell)}{D(\ell)} A^2 + \frac{K(\ell)^2 \ell_y^2}{D(\ell) L(\ell) \ell^2} \right] \times \int_0^1 dx \frac{1}{2} \sum_{n \neq 0} e^{-i(m_A + A_\omega)2\pi Rn} e^{i\ell_y 2\pi Rn}
\]
\[
\times \left\{ (2\pi Rn)^2 [K_0(z) - K_2(z)] \frac{2}{\ell^2} + (2\pi Rn)^2 K_0(z)(-8) \frac{\ell^2_y}{(\ell^2)^2} \right. \\
+ \frac{2\pi R|n|}{\sqrt{x(1-x)}\ell^2 + \Lambda^2} K_1(z) \left[ \frac{4}{\ell^2} + i2\pi Rn \frac{\ell^2_y}{\ell^2}(-4)(1-2x) \right]\right\},
\]

where
\[
z \equiv \sqrt{x(1-x)}\ell^2 + \Lambda^22\pi R|n|.
\]

Actually, the form of the integrand in the above expression depends on the choice of the Feynman parameter \(x\). It can be changed by the change of variables, \(x \rightarrow 1 - x\) and \(\ell_y \rightarrow -\ell_y\), which keeps the integration region and the factor \(e^{ix\ell_y2\pi Rn}\) intact.\(^4\) It is convenient to fix the form of the integrand \(I(x, \ell_y)\) by
\[
\int_0^1 dx \sum_{\ell_y} I(x, \ell_y) \rightarrow \int_0^1 dx \sum_{\ell_y} \frac{1}{2} [I(x, \ell_y) + I(1-x, -\ell_y)],
\]
so that the form of the integrand is invariant under the above change of variables. The particular expression in Eq. (3.26) has been obtained in this way.

Next, in Eq. (3.26), we use the identity,
\[
K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z}K_{\nu}(z),
\]
with \(\nu = 1\). Then, by further using
\[
K'_0(z) = -K_1(z),
\]
and
\[
\frac{\partial z}{\partial x} = \frac{2\pi R|n|}{\sqrt{x(1-x)}\ell^2 + \Lambda^2(1-2x)\ell^2/2},
\]
which follows from Eq. (3.27), we have
\[
E^{(1)}\delta\ell \bigg|_{\text{2-loop}} = 2\pi R \frac{1}{N} \sum_A \delta A \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \left[ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \Lambda^2 + \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)\mathcal{L}(\ell)\ell^2} \ell^2_y \right] \\
\times \int_0^1 dx \frac{1}{2} \sum_{n \neq 0} e^{-i(m_A + A_y)2\pi Rn} e^{ix\ell_y2\pi Rn} \\
\times \left[ 2\pi Rn\ell_y K_0(z) - i \frac{\partial}{\partial x} K_0(z) \right] 2\pi Rn(-8) \frac{\ell^2_y}{(\ell^2)^2}. \tag{3.32}
\]

Finally, the integration by parts with respect to \(x\) yields
\[
E^{(1)}\delta\ell \bigg|_{\text{2-loop}} = 0. \tag{3.33}
\]

\(^4\)Recall that \(\ell_y \in \mathbb{Z}/R\).
Next, let us consider $E^{(2)}\delta e^2|_{2\text{-loop}}$ which is given by the $O(\delta_A^2)$ terms in Eq. (3.23). First, the $p$-integration in the function $J$ in Eq. (3.17) gives

$$E^{(2)}\delta e^2|_{2\text{-loop}}^{(J)} = -2\pi R \frac{1}{N} \sum_A \delta_A \frac{1}{2\pi R} \sum_{\ell_y} \frac{1}{\ell_y} \int_0^1 dx \left[ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \Lambda^2 + \frac{\mathcal{K}(\ell)^2}{\mathcal{D}(\ell)\mathcal{P}(\ell)^2} \right]$$

$$\times \left( \frac{1}{[x(1-x)\ell^2 + \Lambda^2]^3} \left[ -x(1-x)(3 - 10x + 10x^2) - (1 - 2x + 2x^2) \frac{\Lambda^2}{\ell^2} \right] \right.$$

$$\times \left( 2(1 - 2x + 2x^2) \frac{1}{\ell^2} + i2\pi Rn \frac{\ell_y}{\ell^2} (-2)(1 - 2x)(1 - 3x + 3x^2) \right).$$

On the other hand, from the function $I$ (3.16),

$$E^{(2)}\delta e^2|_{2\text{-loop}}^{(I)} = -2\pi R \frac{1}{N} \sum_A \delta_A \frac{1}{2\pi R} \sum_{\ell_y} \frac{1}{\ell_y} \int_0^1 dx \left[ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \right.$$

$$\times \left( \frac{1}{[x(1-x)\ell^2 + \Lambda^2]^3} \left[ -x(1-x)(1 - 3x + 3x^2) - (1 - 2x + 2x^2) \frac{\Lambda^2}{\ell^2} \right] \right.$$

$$\times \left( 1 - 2x + 2x^2 \right) \left( 1 + 4 \frac{\Lambda^2}{\ell^2} \right) \right).$$
\[
\frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell^2 + \Lambda^2}} K_1(z)(-2)x(1-x) \left[ \frac{1}{\ell^2} + 4\Lambda^2 \frac{\ell_y^2}{(\ell^2)^2} \right]
\]
\[
\frac{\mathcal{K}(\ell)^2}{D(\ell)\mathcal{L}(\ell)\ell^2} \left( \frac{1}{x(1-x)\ell^2 + \Lambda^2} \right)^3 K_3(z)2x(1-x)(1-2x)^2
\]
\[
+ \frac{(2\pi Rn)^2}{x(1-x)\ell^2 + \Lambda^2} K_2(z)8x(1-x) \left[ \frac{1}{\ell^2} - i2\pi Rn \frac{\ell_y}{\ell^2}(1-2x) \right]
\]
\[
+ \frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell^2 + \Lambda^2}} K_1(z)(-8)x(1-x) \left[ \frac{\ell_y^2}{(\ell^2)^2} \right]
\]
\[
+ \frac{\mathcal{K}(\ell)^2}{D(\ell)\ell^2} \frac{1}{4} \sum_{n \neq 0} e^{-i(m_A + A_{y0})2\pi Rn} e^{ix\ell_y} 2\pi Rn
\]
\[
\times \left( \frac{(2\pi Rn)^2}{x(1-x)\ell^2 + \Lambda^2} K_2(z)2i2\pi Rn(4)x(1-x) \right).
\]  
(3.35)

To obtain the above expressions (3.34) and (3.35), we applied the procedure (3.28).

To further simplify the above expressions, we first note that all the terms linear in \(\ell_y\) are proportional to \(1 - 2x\) and thus to \(\partial z/\partial x\) (3.31). Using this fact and the identity,

\[
K_2(z) = -z \left[ \frac{1}{x} K_1(z) \right]'.
\]  
(3.36)

we can carry out the integration by parts with respect to \(x\) in those terms linear in \(\ell_y\). We then use the identity (3.29) with \(\nu = 2\) to express \(K_3(z)\) in terms of \(K_1(z)\) and \(K_2(z)\). The resulting expression contains the term \(K_1(z)x(1-x)(1-2x)^2\) to which we use Eq. (3.31). We repeat the integration by parts as long as the factor \(1 - 2x\) remains. In an intermediate step, we use

\[
K_0(z) = -z \left[ xK_1(z) \right]'.
\]  
(3.37)

Finally, we can carry out the \(x\)-integration in terms that do not contain the Bessel function. In this way, we have the following rather simple expression,

\[
E^{(2)} \delta e^2 |_{\text{2-loop}} = -2\pi R \frac{1}{N} \sum_A \delta_A^2 \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \sum
\]

\[\text{We note}
\tanh^{-1} \left( \sqrt{\frac{\ell_y^2}{\ell^2 + 4\Lambda^2}} \right) = \frac{1}{4} \sqrt{\ell^2 + 4\Lambda^2} \mathcal{L}_\infty(\ell).
\]  
(3.38)
This completes the $p$-integration in $E^{(2)}\delta\epsilon^2_{\text{2-loop}}$.

Let us examine whether the expression (3.39) is IR finite or not. From the expressions in Eq. (2.18) and

$$L_\infty(\ell) = \frac{1}{\Lambda^2} - \frac{1}{6\Lambda^2} + O((\ell^2)^2),$$

we see that the above $\ell_x$-integral for $E^{(2)}\delta\epsilon^2_{\text{2-loop}}$ is IR finite, as should be the case for any physical quantity.

In what follows, we carry out the summation over the index $A$ in Eqs. (3.25) and (3.39) and integrate the resulting expressions over the “vacuum moduli” $A_{\gamma\bar{0}}$ as Eq. (2.11). Then, we organize them according to the powers of $1/N$. Before doing these, however, it is helpful to further simplify Eq. (3.39) by noting that $\tilde{L}(p)$ and $K(p)$ in Eqs. (2.18) are exponentially suppressed for $N \to \infty$ as $\lesssim e^{-ARN}$ because of the asymptotic behavior of the Bessel function, $K_{\nu}(z) \sim \sqrt{\pi/(2z)}e^{-z}$. Therefore, these functions can be neglected in the power series expansion in $1/N$ and we can set $L(\ell) \to L_\infty(\ell)$, $K(\ell) \to 0$, and $D(\ell) \to (p^2 + 4\Lambda^2)L_\infty(\ell)^2$ in Eq. (3.39) to yield

$$E^{(2)}\delta\epsilon^2_{\text{2-loop}} = -2\pi R \frac{1}{N} \sum_A \delta_A^2 \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \left[ 4 - 2(\ell^2 + 2\Lambda^2)L_\infty(\ell) \right.

\left. \right] \frac{1}{\ell^2(\ell^2 + 4\Lambda^2)^2L_\infty(\ell)}

+ \int_0^1 dx \sum_{n \neq 0} e^{-i(m_A + A_{\gamma\bar{0}})2\pi R_n} e^{ix\ell_y2\pi R_n} \frac{1}{(\ell^2 + 4\Lambda^2)L_\infty(\ell)} \left. \right] \times \left\{ \frac{(2\pi R|n|)^3}{\sqrt{x(1-x)^2 + \Lambda^2}} K_1(z)x(1-x) - \frac{(2\pi Rn)^2}{x(1-x)^2 + \Lambda^2} K_2(z)x(1-x) \right\}.$$
\[ + \frac{\ell_y^2}{\pi^2} \left[ \frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell_y^2 + \Lambda^2}} K_1(z)x(1-x) + (2\pi Rn)^2 K_0(z) \right] \] \] \quad (3.41)

up to exponentially small terms.

### 3.6 Summation over \(A\) and the integration over \(A_{y0}\)

We thus consider the sum over the index \(A\) and the integration over the vacuum moduli \(A_{y0}\) in Eq. (2.11). The summation over \(A\) can be carried out as

\[ \sum_A e^{-im_A 2\pi Rn} = \sum_{j=0}^{N-1} (e^{-2\pi j/N})^j = N \begin{cases} 1, & \text{for } n = 0 \mod N, \\ 0, & \text{for } n \neq 0 \mod N, \end{cases} \] \quad (3.42)

and

\[ \sum_A m_A e^{-im_A 2\pi Rn} = \frac{N}{2R} \begin{cases} 1 - \frac{1}{N}, & \text{for } n = 0 \mod N, \\ 1 - \frac{2}{N e^{-2\pi j/N} - 1}, & \text{for } n \neq 0 \mod N, \end{cases} \] \quad (3.43)

On the other hand, the integration over \(A_{y0}\) with the measure (2.11) results in

\[ \int_0^1 d(A_{y0} R N) e^{-i A_{y0} 2\pi R n} = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \neq 0, \text{ } n = 0 \mod N, \end{cases} \] \quad (3.45)

The combination of the above two operations yield, therefore,

\[ \int_0^1 d(A_{y0} R N) \sum_A m_A e^{-i(m_A + A_{y0}) 2\pi R n} = \frac{N}{2R} \begin{cases} 1 - \frac{1}{N}, & \text{for } n = 0, \\ 0, & \text{for } n \neq 0, \text{ } n = 0 \mod N, \end{cases} \] \quad (3.46)

and

\[ \int_0^1 d(A_{y0} R N) \sum_A m_A^2 e^{-i(m_A + A_{y0}) 2\pi R n} = \frac{N}{3R^2} \begin{cases} 1 - \frac{3}{2N} + \frac{1}{2N^2}, & \text{for } n = 0, \\ 0, & \text{for } n \neq 0, \text{ } n = 0 \mod N, \end{cases} \] \quad (3.47)

Using Eqs. (3.46) and (3.47) for Eq. (3.25), under the integration over \(A_{y0}\),

\[ E^{(1)} \delta \epsilon \bigg|_{1\text{-loop}} = N \Lambda \frac{1}{AR} \frac{\lambda_R \delta \epsilon_R}{\pi N} \frac{1}{2} \sum_{n \neq 0 \mod N} \frac{i}{\pi n} K_0(2\pi AR|n|) = 0, \] \quad (3.48)
and
\[
E^{(2)} \delta \epsilon^2 \Big|_{\text{1-loop}} = N \Lambda \frac{1}{(AR)^3} \left( \frac{\lambda_R \delta \epsilon_R}{\pi N} \right)^2 \left( -\frac{1}{12} \right) \left[ 1 - \frac{3}{2N} + \frac{1}{2N^2} + \frac{6}{N} \sum_{n>0, n \neq 0 \mod N} \frac{\Lambda R K_1(2\pi AR_n)}{\tan(\pi n/N)} \right],
\]
(3.49)

For the two-loop corrections, from Eq. (3.33),
\[
E^{(1)} \delta \epsilon \Big|_{\text{2-loop}} = 0,
\]
(3.50)

and, for Eq. (3.40), we have
\[
E^{(2)} \delta \epsilon^2 \Big|_{\text{2-loop}} = -\frac{2\pi}{3} \left( \frac{\lambda_R \delta \epsilon_R}{\pi RN} \right)^2 \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\epsilon_y} \left[ \frac{4 - 2(\ell^2 + 2\Lambda^2) L_\infty(\ell)}{\ell^2(\ell^2 + 4\Lambda^2) L_\infty(\ell)} \right]
\times \left[ \frac{1}{N \tan(\pi n/N)} - 6 \left( 1 - \frac{1}{N} \right) \sin(x\ell_y 2\pi Rn) \right] \frac{1}{(\ell^2 + 4\Lambda^2) L_\infty(\ell)} \times \left\{ -\frac{(2\pi Rn)^2}{\sqrt{x(1-x)\ell^2 + \Lambda^2}} K_1(x(1-x)) - \frac{2\pi Rn}{x(1-x)\ell^2 + \Lambda^2} K_2(x(1-x)) + \frac{\ell_x^2}{\ell^2} \left[ \frac{(2\pi Rn)^2}{\sqrt{x(1-x)\ell^2 + \Lambda^2}} K_1(x(1-x)) + 2\pi Rn K_0(x(1-x)) \right] \right\}
\]
(3.51)

up to exponentially small terms.

3.7 Final results
Finally, we arrange the above results in the powers of $1/N$. From Eqs. (3.48) and (3.50), we have
\[
E^{(1)} \delta \epsilon = 0 \cdot N^0 + 0 \cdot N^{-1} + O(N^{-2}).
\]
(3.52)

Thus $E^{(1)} \delta \epsilon$ vanishes to the order we worked out.

For $E^{(2)} \delta \epsilon^2$, setting
\[
E^{(2)} \delta \epsilon^2 = E^{(2)} \delta \epsilon^2 \Big|_{\text{O}(N^{-1})} + E^{(2)} \delta \epsilon^2 \Big|_{\text{O}(N^{-2})} + O(N^{-3}),
\]
(3.53)
from Eq. (3.49),
\[
RE^{(2)} \delta \epsilon^2 \Big|_{\text{O}(N^{-1})} = N^{-1}(\lambda_R \delta \epsilon_R)^2 (\Lambda R)^{-3} F(\Lambda R),
\]
(3.54)
where
\[
F(\xi) \equiv -\frac{1}{12\pi^2} \left[ \xi + c(\xi) \right], \quad c(\xi) \equiv \lim_{N \to \infty} \frac{6}{N} \sum_{n > 0, n \neq 0 \mod N} \frac{\xi^2 K_1(2\pi \xi n)}{\tan(\pi n / N)}.
\]

From Eqs. (3.49) and (3.51), on the other hand,
\[
\left. RE^{(2)}(\delta \epsilon) \right|_{O(N^{-2})} = N^{-2}(\lambda R \delta \epsilon R)^2 (\Lambda R)^{-3} G(\Lambda R),
\]
where
\[
G(\xi) \equiv -\frac{1}{12\pi^2} \left\{ -\frac{3}{2} \xi + \lim_{N \to \infty} \left[ \frac{6}{N} \sum_{n > 0, n \neq 0 \mod N} \frac{\xi^2 K_1(2\pi \xi n)}{\tan(\pi n / N)} - N c(\xi) \right] \right\}
\]
\[
- \frac{1}{6\pi^3 \xi^3} \int_{-\infty}^{\infty} d\tilde{\ell}_x \sum_{\tilde{\ell}_y \in \mathbb{Z}} \left( \frac{4 - 2(\tilde{\ell}_x^2 + 2\xi^2)}{\tilde{\ell}_x^2 (\tilde{\ell}_x^2 + 4\xi^2)^2} \tilde{\ell}_x^2 \right) G(\tilde{\ell}_x, \tilde{\ell}_y)
\]
\[
+ \lim_{N \to \infty} \int_0^1 dx \sum_{n > 0, n \neq 0 \mod N} \left[ \frac{6 \cos(x \tilde{\ell}_y 2\pi n)}{N \tan(\pi n / N)} - 6 \sin(x \tilde{\ell}_y 2\pi n) \right] \frac{1}{(\tilde{\ell}_x^2 + 4\xi^2) \tilde{\ell}_x^2}
\]
\[
\times \left\{ -\frac{(2\pi n)^2}{\sqrt{x(1-x)\tilde{\ell}_x^2 + \xi^2}} \left[ -\frac{2\pi n}{\sqrt{x(1-x)\tilde{\ell}_x^2 + \xi^2}} K_1(z) x(1-x) \right.ight.
\]
\[
- \left. \left. \frac{2\pi n}{\tilde{\ell}_x^2} \left[ \frac{(2\pi n)^2}{\sqrt{x(1-x)\tilde{\ell}_x^2 + \xi^2}} K_1(z) x(1-x) + \frac{2\pi n K_0(z)}{\tilde{\ell}_x^2} \right] \right\} \right\}.
\]

In this expression, we have defined
\[
\tilde{\ell}_x(\tilde{\ell}_x, \tilde{\ell}_y) \equiv \frac{2}{\sqrt{\tilde{\ell}_x^2 + 4\xi^2}} \ln \left( \frac{\sqrt{\tilde{\ell}_x^2 + 4\xi^2} + \sqrt{\tilde{\ell}_x^2}}{\sqrt{\tilde{\ell}_x^2 + 4\xi^2} - \sqrt{\tilde{\ell}_x^2}} \right),
\]
and
\[
z \equiv \sqrt{x(1-x)\tilde{\ell}_x^2 + \xi^2} 2\pi |n|. \tag{3.59}
\]

We plot the function $F(\Lambda R)$ in Eq. (3.54) in Fig. 2 and the function $G(\Lambda R)$ in Eq. (3.56) in Fig. 3. These plots clearly show that, to the order of the $1/N$ expansion we worked out, the vacuum energy is a well-defined finite quantity under the parameter renormalization in Eqs. (2.10) and (3.24). Eqs. (3.52)–(3.57) and Figs. 2 and 3 are our main results in this paper. Since Figs. 2 and 3 show that the functions $F(\Lambda R)$ and $G(\Lambda R)$ remain finite as $\Lambda R \to 0$, Eqs. (3.54) and (3.56) (and Eq. (3.52)) show that the vacuum energy normalized by the radius of the $S^1$, $RE(\delta \epsilon)$ behaves as $(\Lambda R)^{-3}$ for $\Lambda R$ small. Since $\Lambda$ is given by Eq. (2.9),

this result implies that to the order of the $1/N$ expansion we worked out, the vacuum energy is a purely non-perturbative quantity and it has no well-defined weak coupling expansion in $\lambda_R$.

4. Conclusion and discussion
In this paper, by employing the $1/N$ expansion, we computed the vacuum energy $E(\delta \epsilon)$ of the 2D SUSY $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with $\mathbb{Z}_N$ twisted boundary conditions to the second order in the SUSY breaking parameter $\delta \epsilon$ in Eq. (3.2). We found that the vacuum energy is purely non-perturbative and, although it is a perfectly well-defined physical quantity in the $1/N$ expansion, it has no sensible weak coupling expansion in $\lambda_R$.

Our original intention was to compare our result in the $1/N$ expansion with the result by the bion calculus in Ref. [5], because it appears that, at least seemingly, the calculation in Ref. [5] holds even under the limit (1.1).
According to Ref. [5], the contribution of a single bion to the vacuum energy (3.1) is given by \((E^{(0)}\) is set to be zero),

\[
RE^{(1)} \delta \epsilon = -R \sum_{b=1}^{N-1} 2m_b A_b (\Lambda R)^{2Rm_b N} \delta \epsilon, \tag{4.1}
\]

and

\[
RE^{(2)} \delta \epsilon^2 = -R \sum_{b=1}^{N-1} 2m_b A_b (\Lambda R)^{2Rm_b N} \left[ -2\gamma_E - 2 \ln \left( \frac{4\pi Rm_b N}{\lambda R} \right) \mp \pi i \right] \delta \epsilon^2, \tag{4.2}
\]

where the last \(\mp \pi i\) term is the imaginary ambiguity caused by the integration over quasi-collective coordinates of the bion. In these expressions, the index \(b\) corresponds to the "species" of the bion and the coefficient \(A_b\) is given by using the twist angle \(m_A\) in Eq. (2.4) as

\[
A_b = \frac{\left[ \Gamma(1 - m_b R) / \Gamma(1 + m_b R) \right]^2 \prod_{a=1, a \neq b}^{N-1} \frac{m_a}{m_a - m_b} \frac{\Gamma(1 + (m_a - m_b) R)}{\Gamma(1 - (m_a - m_b) R)} \frac{\Gamma(1 - m_a R)}{\Gamma(1 + m_a R)} }{(-1)^{b+1} \frac{N^{2b}}{(b!)^2}}. \tag{4.3}
\]

Using this, the coefficient of the imaginary ambiguity in Eq. (4.2) is given by

\[
-R \sum_{b=1}^{N-1} 2m_b A_b (\Lambda R)^{2Rm_b N} = \frac{2}{N} \sum_{b=1}^{N-1} (-1)^b \frac{b}{(b!)^2} (\Lambda R N)^{2b}. \tag{4.4}
\]

When \(N\) is fixed, in the weak coupling limit \(\Lambda R \ll 1\) for which the semi-classical approximation should be valid, the \(b = 1\) term \(-2\Lambda^2 R^2 N\) dominates the sum (4.4). \(\Lambda^2 = \mu^2 e^{-4\pi/\lambda} R\) is the exponential of the action of the constituent of the minimum bion (the minimal fractional instanton–anti-instanton pair) and, at the same time, is consistent with the order of the \(u = 1\) IR renormalon ambiguity. On the other hand, in the large-\(N\) limit in Eq. (1.1), whether Eq. (4.4) possesses a sensible \(1/N\) expansion or not is not clear, because each term behaves as \(O(N)\), \(O(N^3)\), \(O(N^5)\), \ldots; we could not estimate the sum as the whole in the large \(N\) limit.

Thus, we cannot compare our result in the \(1/N\) expansion with the result in Ref. [5] by the bion calculus. We have no clear idea yet why this comparison is impossible. One phenomenological observation from Eq. (4.4) is that it is a series in the combination \(\Lambda R N\) and thus the result in Ref. [5] seems meaningful for \(\Lambda R N \ll 1\) instead of our large \(N\) limit in Eq. (1.1) with which \(\Lambda R N \gg 1\). More thought seems to be necessary to clearly understand the relation between bions, the IR renormalon, and the \(1/N\) expansion.

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A. Integration formulas

In Sect. 3.3 we have used the following integration formulas (in practice, we are interested in the cases, \((\alpha, \beta) = (1, 2), (1, 3), \text{and } (2, 2)\):

\[
\int \frac{d^2 p}{(2\pi)^2} e^{ip\mu x} \frac{1}{[(p - \ell)^2 + \Lambda^2]^{\alpha/2}} \frac{1}{(p^2 + \Lambda^2)^\beta} \left\{ \begin{array}{ll} 1 & \text{if } p\mu \\ p\mu p_\nu & \text{if } p\nu \end{array} \right. 
\]

\[
n=0 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \, x^{\alpha-1}(1-x)^{\beta-1} \times \frac{1}{4\pi} \begin{cases} \Gamma(\alpha + \beta - 1) \left[ x(1-x)\ell^2 + \Lambda^2 \right]^{1-\alpha-\beta}, \\ \Gamma(\alpha + \beta - 1) \left[ x(1-x)\ell^2 + \Lambda^2 \right]^{1-\alpha-\beta} x\ell_\mu, \\ \Gamma(\alpha + \beta - 1) \left[ x(1-x)\ell^2 + \Lambda^2 \right]^{1-\alpha-\beta} x^2\ell_\mu\ell_\nu \\ \quad + \frac{1}{2} \Gamma(\alpha + \beta - 2) \left[ x(1-x)\ell^2 + \Lambda^2 \right]^{2-\alpha-\beta} \delta_{\mu\nu}, \end{cases}
\]

\[
n\neq 0 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \, x^{\alpha-1}(1-x)^{\beta-1} \times \frac{1}{4\pi} 2^{2-\alpha-\beta} e^{ix\ell_\mu 2\pi R n} \left( \begin{array}{c} 2\pi R |n| \sqrt{\frac{x(1-x)\ell^2 + \Lambda^2}{\ell^2 + \Lambda^2}} \quad K_{\alpha+\beta-1}(z), \\ 2\pi R |n| \sqrt{\frac{x(1-x)\ell^2 + \Lambda^2}{\ell^2 + \Lambda^2}} \quad K_{\alpha+\beta-1}(z) x\ell_\mu \\ + \left( \frac{2\pi R |n|}{\sqrt{x(1-x)\ell^2 + \Lambda^2}} \right)^{\alpha+\beta-2} K_{\alpha+\beta-2}(z) i2\pi R n \delta_{\mu\nu} \end{array} \right) \]

\[
\times \left( \begin{array}{c} 2\pi R |n| \sqrt{\frac{x(1-x)\ell^2 + \Lambda^2}{\ell^2 + \Lambda^2}} \quad K_{\alpha+\beta-1}(z) x^2\ell_\mu\ell_\nu \\ + \left( \frac{2\pi R |n|}{\sqrt{x(1-x)\ell^2 + \Lambda^2}} \right)^{\alpha+\beta-2} K_{\alpha+\beta-2}(z) \\ \times (\delta_{\mu\nu} + ix\ell_\mu 2\pi R n \delta_{\nu\gamma} + i2\pi R n \delta_{\mu\gamma} x\ell_\nu) \\ - \left( \frac{2\pi R |n|}{\sqrt{x(1-x)\ell^2 + \Lambda^2}} \right)^{\alpha+\beta-3} K_{\alpha+\beta-3}(z) \end{array} \right) \]

\[
\times (2\pi R)^2 \delta_{\mu\mu} \delta_{\nu\nu},
\]

where

\[
z \equiv \sqrt{x(1-x)\ell^2 + \Lambda^2} 2\pi R |n|.
\]

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