EIGENVALUES AND HOLOMONY

WERNER BALLMANN, JOCHEN BRÜNING, AND GILLES CARRON

ABSTRACT. We estimate the eigenvalues of connection Laplacians in terms of the non-triviality of the holonomy.

INTRODUCTION

Let \( S_L = \mathbb{R}/L\mathbb{Z} \) be a circle of length \( L \) and \( X \) be the oriented unit vector field on \( S = S_L \). Up to equivalence, there is exactly one Hermitian line bundle, \( E \), over \( S \). For a given complex number \( z \) of modulus 1, there is, again up to equivalence, exactly one Hermitian connection, \( \nabla^E \), on \( E \) with holonomy \( z \) around \( S \).

The Laplace operator \( \Delta^E = (\nabla^E)^* \nabla^E \) is essentially self-adjoint as an operator in \( L^2(E) \) with domain \( C^2(E) \). The spectrum of its closure is discrete and consists of the eigenvalues

\[
\frac{4\pi^2}{L^2}(\rho + k)^2, \quad k \in \mathbb{Z},
\]

where we write \( z = \exp(2\pi i \rho) \). The corresponding eigenspaces are spanned by the functions \( \exp(2\pi i (\rho + k)x/L) \). We see that, for \( z \neq 1 \), the spectrum does not contain 0, and that we can estimate the smallest eigenvalue in terms of \( L \) and \( z \).

The aim of this paper is a corresponding estimate for Hermitian vector bundles over closed Riemannian manifolds in higher dimensions. The results of this paper are of importance in [BBC], but seem to be also of independent interest.

Let \( M \) be a closed Riemannian manifold of dimension \( n \geq 2 \). Let \( -(n-1)\kappa \leq 0 \) be a lower bound for the Ricci curvature of \( M \), i.e. \( \text{Ric}_M \geq -(n-1)\kappa \), and let \( D \) be an upper bound for the diameter of \( M \), \( \text{diam} \, M \leq D \). Let \( E \to M \) be a Hermitian vector bundle over \( M \) and \( \nabla^E \) be a Hermitian connection on \( E \). The kernel of the associated connection Laplacian \( \Delta^E = (\nabla^E)^* \nabla^E \) consists of globally parallel sections of \( E \). The estimates we obtain are in terms of quantitative measures for the non-existence of parallel sections, that is, in terms of the holonomy of \( E \).

Assume first that \( \nabla^E \) is flat and that the holonomy of \( \nabla \) is irreducible (and nontrivial). Recall that for each point \( x \in M \), the fundamental group \( \pi_1(M, x) \) of \( M \) at \( x \) admits a short basis, that is, a generating set represented by loops of length at most \( 2 \text{diam} \, M \), see [Gr]. Hence for each point \( x \in M \), there is a...
constant \( \alpha(x) > 0 \) such that for all \( v \in E_x \) there is a smooth unit speed loop \( c : [0, l] \to M \) at \( x \) of length \( l \leq 2 \text{diam} \ M \) with holonomy \( H_c \) satisfying

\[
|H_c(v) - v| \geq \alpha(x)|v|.
\]

There is also a constant \( \varepsilon(x) > 0 \) such that a loop at \( x \) has length \( > 2 \text{diam} \ M + \varepsilon(x) \) unless it is homotopic to a loop at \( x \) of length \( \leq 2 \text{diam} \ M \). It follows that for any point \( y \in M \) of distance \( < \varepsilon/4 \) to \( x \), the homotopy classes of loops of length \( \leq 2 \text{diam} \ M \) at \( y \) are represented by concatenated curves of the form \( c_{xy}^{-1} \ast c \ast c_{xy} \), where \( c_{xy} \) denotes a fixed minimal geodesic from \( x \) to \( y \) and \( c \) is a loop at \( x \) of length \( \leq 2 \text{diam} \ M \). Since \( \nabla^E \) is flat, parallel translation does not depend on homotopy classes. It follows that for each point \( y \) sufficiently close to \( x \), there is a loop \( c \) of length \( \leq 2 \text{diam} \ M \) at \( y \) which has the same non-trivial holonomy as the loop \( c_{xy} \ast c \ast c_{xy}^{-1} \) at \( x \). In particular, we can choose the constants \( \alpha(x) \) such that they have uniform positive lower bounds locally. By the compactness of \( M \), there is a uniform constant \( \alpha > 0 \) such that, for all \( x \in M \) and \( v \in E_x \), there is a smooth unit speed loop \( c : [0, l] \to M \) at \( x \) of length \( l \leq 2 \text{diam} \ M \) with holonomy \( H_c \) satisfying

(1) \[
|H_c(v) - v| \geq \alpha|v|.
\]

Our first estimate is as follows.

**Theorem 1.** Suppose that \( \nabla^E \) is flat and that the holonomy of \( \nabla^E \) satisfies (1). Then, for each eigenvalue \( \lambda \) of \( \Delta^E \),

\[
\sqrt{\lambda} \exp \left( c_0 \sqrt{\lambda} + (n - 1) \kappa \text{diam} \ M \right) \geq \frac{\alpha}{2 \text{diam} \ M}
\]

with a constant \( c_0 = c_0(n, \sqrt{\kappa} D) \). In particular,

\[
\sqrt{\lambda} \geq \min \left\{ \frac{1}{c_0 \text{diam} \ M}, \frac{\alpha}{2 \text{diam} \ M} \exp \left( -c_0 \sqrt{\kappa} \text{diam} \ M - 1 \right) \right\}.
\]

For each point \( x \in M \) and unit vector \( v \in E_x \), let \( \beta(v) \) be the supremum of the ratios \( |H_c(v) - v|/L(c) \), where the supremum is taken over all non-constant loops \( c \) starting at \( x \), \( L(c) \) denotes the length of \( c \), and \( H_c \) the holonomy along \( c \). Set

(2) \[
\beta := \inf \{ \beta(v) \mid v \in E, |v| = 1 \}.
\]

Note that by the definition of the constant \( \alpha \) in (1), we have \( \beta \geq \alpha/2 \text{diam} \ M \). In the general case, i.e. if \( \nabla^E \) is not necessarily flat, we have the following estimate.

**Theorem 2.** There are positive constants \( a = a(n) \) and \( c_1 = c_1(n, \sqrt{\kappa} D) \) such that, for each eigenvalue \( \lambda \) of \( \Delta^E \),

\[
\sqrt{\lambda} \exp \left( c_1 \sqrt{\lambda} + (n - 1) \kappa + n^2 r + n^2 r^2/\beta^2 \text{diam} \ M \right) \geq \frac{\beta}{a},
\]
where $r$ is a uniform bound for the pointwise operator norm of $R^E$. In particular,

$$\sqrt{\lambda} \geq \min \left\{ \frac{1}{c_1 \text{diam } M} \beta a \exp \left(-c_1 \sqrt{(n-1)\kappa + n^2r + n^2r^2/\beta^2 \text{diam } M - 1}\right) \right\}.$$  

The constants $a$, $c_0$ and $c_1$ in Theorems 1 and 2 can be determined explicitly. Except for the factor $1/a$, Theorem 2 implies Theorem 1. On the other hand, the proof of Theorem 1 is more elementary than the one of Theorem 2 and exposes the main ideas more clearly. Moreover, the constant $c_0$ is better than the constant $c_1$, that is, $c_0 \leq c_1$. Both proofs rely on a Sobolev inequality of Gallot [Ga] and Moser iteration. In the proof of Theorem 2 we actually need an extension of Moser’s iteration technique.

If part of the holonomy is trivial, then the corresponding space of parallel sections determines a subbundle $E'$ of $E$. The above results then apply to the orthogonal complement $E''$ of $E'$ in $E$. On the other hand, for a section $\sigma = \sum \phi_i \sigma_i$ in $E'$, where the sections $\sigma_i$ are parallel, we have $\Delta E \sigma = \sum (\Delta \phi_i) \sigma_i$, and hence the usual eigenvalue estimates for the Laplace operator on functions as for example in [LY] or [Zh] apply.

AN APRIORI ESTIMATE

For the convenience of the reader and since we will need a modification further on, we give a short account of Moser iteration as applied in [Li], see also [Ga, Au]. It will give rise to the following infinite product,

$$A(x, y, z) := \prod_{i=0}^{\infty} \left( x + \frac{yz^i}{\sqrt{2z^i - 1}} \right)^{1/z^i},$$

where $x, y > 0$ and $z > 1$. Note that

$$A(tx, ty, z) = t^{z/(z-1)}A(x, y, z) \quad \text{for } t > 0.$$  

We have $\sqrt{2z^i - 1} > z^{i/2}$ and $\ln(1 + yz^{i/2}) < yz^{i/2}$, by the assumptions on $y$ and $z$, hence

$$A(1, y, z) \leq \exp \left( \frac{y}{1 - 1/\sqrt{z}} \right).$$

This estimate will be good enough for the present purposes. For more precise estimates, see [Li, p.467] and [Au, p.7].

Let $M$ be a closed Riemannian manifold of dimension $n$ and volume $V$. Let $\nabla$ and $\Delta$ be the Levi-Civita connection and the Laplace operator on functions of $M$, respectively. Denote by $\| \cdot \|_p$ the $L^p$-norm with respect to the normalized Riemannian measure of $M$. Let $q > 1$ be in the Sobolev range, i.e. there are positive constants $B$ and $C$ such that $M$ satisfies the Sobolev inequality

$$\|f\|_{2q} \leq B\|f\|_2 + CV^{1/n}\|df\|_2,$$  

where $f$ is a uniform bound for the pointwise operator norm of $R^E$. In particular,
for all smooth functions \( f \) on \( M \). Let \( F \to M \) be a Hermitian vector bundle with a Hermitian connection \( \nabla^F \). Let \( \Delta^F \) be the associated connection Laplacian.

**Lemma 3.** Let \( \sigma \in L^2(M,F) \) be a smooth section. Assume that (pointwise)
\[
V^{2/n} \langle \Delta^F \sigma, \sigma \rangle \leq \Lambda^2 |\sigma|^2
\]
for some constant \( \Lambda \geq 0 \). Then
\[
\|\sigma\|_\infty \leq A(B, C\Lambda, q) \|\sigma\|_2.
\]

**Proof.** We may assume that \( V = 1 \). The function \( f_\varepsilon = \sqrt{|\sigma|^2 + \varepsilon^2} \) is smooth, and, by Kato’s inequality and our assumption, respectively, we have the pointwise estimate
\[
f_\varepsilon \Delta f_\varepsilon \leq \text{Re} \langle \sigma, \Delta^F \sigma \rangle \leq \Lambda^2 |\sigma|^2 \leq \Lambda^2 f_\varepsilon^2.
\]
Let \( k > 1/2 \). Since \( df_\varepsilon^k = k f_\varepsilon^{k-1} df_\varepsilon \),
\[
\|df_\varepsilon^k\|_2^2 = \frac{k^2}{2k-1} \langle df_\varepsilon, df_\varepsilon^{2k-1} \rangle_2 = \frac{k^2}{2k-1} \langle df_\varepsilon, df_\varepsilon^{2k-1} \rangle_2
\]
\[
= \frac{k^2}{2k-1} \langle df_\varepsilon, df_\varepsilon^{2k-1} \rangle_2 \leq \frac{\Lambda^2 k^2}{2k-1} \int f_\varepsilon^{2k} = \frac{\Lambda^2 k^2}{2k-1} \|f_\varepsilon\|_{2k}^2.
\]
Using (8), we get
\[
\|f_\varepsilon\|_{2k} \leq B \|f_\varepsilon\|_{2k} + \frac{C\Lambda k}{\sqrt{2k-1}} \|f_\varepsilon\|_{2k}.
\]
By letting \( \varepsilon \to 0 \) we conclude
\[
\|\sigma\|_{2k} \leq \left( B + \frac{C\Lambda k}{\sqrt{2k-1}} \right)^{1/k} \|\sigma\|_{2k}.
\]
Iterating this inequality with \( k = q^j, j = 0, 1, \ldots \), we get
\[
\|\sigma\|_{2q^{j+1}} \leq \left( B + \frac{C\Lambda q^j}{\sqrt{2q^j-1}} \right)^{1/q^j} \|\sigma\|_{2q^j}
\]
\[
\|\sigma\|_{2q^j} \leq \left( B + \frac{C\Lambda q^j}{\sqrt{2q^j-1}} \right)^{1/q^j} \|\sigma\|_{2q^j}
\]
Now \( \|\sigma\|_{2q^{j+1}} \to \|\sigma\|_{\infty} \) as \( j \) tends to \( \infty \). Hence the lemma.

**Proof of Theorem 4**

From now on, we assume that \( M \) satisfies \( \text{Ric}_M \geq -(n-1)\kappa \) and \( \text{diam} M \leq D \). We will use the following Sobolev inequality.
Lemma 4 (Gallot [Ga]). There is a positive constant $c' = c'(n, \sqrt{\kappa D})$ such that, for all $p \in [1, \frac{2}{n-1}]$ and all smooth functions $f$ on $M$,

$$\|f\|_{2^{-p}} \leq \|f\|_2 + \frac{2c'}{2-p} \text{diam } M \|df\|_2.$$

In other words, if the assumptions of Lemma 4 are satisfied, then $M$ satisfies a Sobolev inequality of the type (6) with $q = p^2 - p, B = 1, C = \frac{2c'}{2-p} \text{diam } M V^{1-1/n}$.

Note also that the function $c'$ can be chosen to be equal to

$$c'(n, d) = \left\{ \frac{1}{d} \int_0^d \left( \frac{1}{2} e^{(n-1)d} \cosh t + \frac{1}{nd} \sinh t \right)^{n-1} dt \right\}^{1/n}$$

with $d = \sqrt{\kappa D}$, compare [Ga].

Theorem 5. Suppose that $\nabla^E R^E = 0$. Then, for each eigenvalue $\lambda$ of $\Delta^E$,

$$\sqrt{\lambda} \exp \left( c_0 \sqrt{\lambda + (n-1)\kappa + n^2 r \text{diam } M} \right) \geq \beta$$

with $r$ and $\beta$ as in Theorem 2 and $c_0 = c_0(n, \kappa \sqrt{D})$.

Recall that $\beta \geq \alpha/2 \text{diam } M$ and $r = 0$ under the assumptions of Theorem 1. Hence Theorem 5 implies Theorem 1.

Proof of Theorem 3. Let $\sigma$ be a nonzero section of $E$ with $\Delta^E \sigma = \lambda \sigma$. Let $x \in M$ and choose $\beta' < \beta$. Then there is a unit speed loop $c : [0, l] \to M$ at $x$, of length $l$, with holonomy $H_c : E_x \to E_x$ satisfying

$$|H_c(\sigma(x)) - \sigma(x)| \geq \beta' l |\sigma(x)|.$$

Let $F_1, \ldots, F_k : [0, l] \to E$ be a parallel orthonormal frame along $c$. Express $\sigma \circ c$ as a linear combination of this frame, $\sigma \circ c = \sum \phi^i F_i$. By the assumption on the holonomy, we have

$$\beta' l |\sigma(x)| = \beta' l |\phi(0)| \leq |\phi(l) - \phi(0)| \leq \int_0^l |\phi'| dt$$

$$\leq \int_0^l |(\nabla^E \sigma) \circ c| dt \leq l \|\nabla^E \sigma\|_\infty.$$

Since we use the normalized volume element for our norms, this gives

$$\beta \|\sigma\|_2 \leq \beta \|\sigma\|_\infty \leq \|\nabla^E \sigma\|_\infty.$$

On the other hand, $\nabla^E \sigma$ is a one-form with values in $E$, that is, a section of the bundle $F = \Lambda^1 (T^*M) \otimes E$. This bundle inherits a connection, $\nabla^F$, from the Levi–Civita connection $\nabla$ of $M$ and the connection $\nabla^E$ of $E$. In terms of a
local orthonormal frame $X_1, \ldots, X_n$ of $M$ and a further local vector field $Z$, the corresponding Bochner formula is

$$(15) \quad (\Delta^F \nabla^E \sigma)(Z) = \nabla^E \nabla^E \sigma - \nabla^E_{\text{Ric}} Z \sigma - 2 \sum R^E(X_i, Z) \nabla^E_{X_i} \sigma - \sum (\nabla^E_{X_i} R^E)(X_i, Z) \sigma,$$

see e.g. Lemma 3.3.1 of [LR]. In particular, since $\Delta^E \sigma = \lambda \sigma$ and $\nabla^E R^E = 0$,

$$(16) \quad \langle \Delta^F \nabla^E \sigma, \nabla^E \sigma \rangle \leq (\lambda + (n - 1)\kappa + 2n^2 r)|\nabla^E \sigma|^2,$$

where we are somewhat generous in the estimate of the curvature term. From (16) and Lemmas 3 and 4, where we choose $p = (n + 2)/(n + 1)$ and $q = (n + 2)/n$ and $C$ as in (12), we conclude that

$$\|\nabla^E \sigma\|_{\infty} \leq A (1, \frac{(2n + 2)c'}{n} \sqrt{\lambda + (n - 1)\kappa + 2n^2 r} \text{diam } M, \frac{n + 2}{n}) \|\nabla^E \sigma\|_2.$$ 

Now $\|\nabla^E \sigma\|_2 = \sqrt{\lambda} \|\sigma\|_2$ since $\Delta^E \sigma = \lambda \sigma$. In combination with (5) and (14), this proves the asserted inequality.

**Proof of Theorem 2**

We cannot apply the previous argument directly to prove Theorem 2. The reason is that, in general, the Bochner formula (15) only gives the estimate

$$(17) \quad \langle \nabla^E \sigma, \Delta^F \nabla^E \sigma \rangle \leq (\lambda + (n - 1)\kappa + n^2 r)|\nabla^E \sigma|^2 - \sum_{i,j} \langle (\nabla^E_{X_i} R^E)(X_i, X_j) \sigma + R^E(X_i, X_j) \nabla^E_{X_i} \sigma, \nabla^E_{X_j} \sigma \rangle.$$ 

Note that we distributed the terms arising from $2 \sum R^E(X_i, Z) \nabla^E_{X_i} \sigma$ in (15) to both terms on the right hand side in (17). Now (17) involves $\sigma$ on the right hand side. To overcome this problem, we have to modify the argument in the proof of Lemma 3. We replace $\sigma$ there by the section $\nabla^E \sigma$ under discussion here and set $f_\varepsilon := \sqrt{|\nabla^E \sigma|^2 + \varepsilon^2}$. Instead of (7), we now have the pointwise estimate

$$f_\varepsilon \Delta f_\varepsilon \leq \text{Re} \langle \nabla^E \sigma, \Delta^F \nabla^E \sigma \rangle \leq (\lambda + (n - 1)\kappa + n^2 r)f_\varepsilon^2 - \sum_{i,j} \langle (\nabla^E_{X_i} R^E)(X_i, X_j) \sigma + R^E(X_i, X_j) \nabla^E_{X_i} \sigma, \nabla^E_{X_j} \sigma \rangle.$$ 

Let $k \geq 1$. Then

$$\int_M |d f_\varepsilon^k|^2 \leq \frac{k^2}{2k - 1} (\lambda + (n - 1)\kappa + n^2 r) \int_M f_\varepsilon^{2k} - \frac{k^2}{2k - 1} \int_M \sum_{i,j} \langle \nabla^E_{X_i} (R^E(X_i, X_j) \sigma), \nabla^E_{X_j} \sigma \rangle f_\varepsilon^{2k-2},$$
where it is understood that we choose, for each point \( x \in M \), an orthonormal frame \( X_1, \ldots, X_n \) with \( (\nabla_{X_i} X_j)(x) = 0 \). As in \([LR]\), the divergence theorem gives

\[
- \int_M \sum_{i,j} \langle \nabla^E_{X_i} (R^E(X_i, X_j) \sigma), \nabla^E_{X_j} \sigma \rangle f_{\varepsilon}^{2k-2}
\]

\[
= \int_M \sum_{i,j} \langle R^E(X_i, X_j) \sigma, \nabla^E_{X_i} \nabla^E_{X_j} \sigma \rangle f_{\varepsilon}^{2k-2}
\]

\[
+ 2(k - 1) \int_M f_{\varepsilon}^{2k-3} \sum_{i,j} df_{\varepsilon}(X_i) \langle R^E(X_i, X_j) \sigma, \nabla^E_{X_j} \sigma \rangle.
\]

Now \( R(X_i, X_j) = -R(X_j, X_i) \); therefore, with the above choice of frames,

\[
\sum_{i,j} \langle R^E(X_i, X_j) \sigma, \nabla^E_{X_i} \nabla^E_{X_j} \sigma \rangle = \frac{1}{2} \sum_{i,j} |R^E(X_i, X_j) \sigma|^2.
\]

Hence

\[
- \int_M \sum_{i,j} \langle \nabla^E_{X_i} (R^E(X_i, X_j) \sigma), \nabla^E_{X_j} \sigma \rangle f_{\varepsilon}^{2k-2}
\]

\[
\leq \frac{n^2 r^2}{2} \int_M |\sigma|^2 f_{\varepsilon}^{2k-2} + 2(k - 1)nr \int_M |\sigma| f_{\varepsilon}^{2k-2} |df_{\varepsilon}|
\]

\[
\leq \frac{n^2 r^2}{2} \|\sigma\|_\infty^2 \int_M f_{\varepsilon}^{2k-2} + 2 \frac{k - 1}{k} nr \|\sigma\|_\infty \int_M f_{\varepsilon}^{k-1} |df_{\varepsilon}^k|
\]

\[
\leq \frac{n^2 r^2}{2} \|\sigma\|_\infty^2 \int_M f_{\varepsilon}^{2k-2} + 2nr \|\sigma\|_\infty \int_M f_{\varepsilon}^{k-1} |df_{\varepsilon}^k|.
\]

But

\[
\frac{2k(k - 1)}{2k - 1} nr \|\sigma\|_\infty \int_M f_{\varepsilon}^{k-1} |df_{\varepsilon}^k| \leq \frac{1}{2} \int_M |df_{\varepsilon}^k|^2 + \left( \frac{k(k - 1)}{2k - 1} \right)^2 2n^2 r^2 \|\sigma\|_\infty^2 \int_M f_{\varepsilon}^{2k-2}
\]

and \( \|\sigma\|_\infty \leq \|\nabla^E \sigma\|_\infty / \beta \leq \|f_{\varepsilon}\|_\infty / \beta \), hence

\[
\|df_{\varepsilon}^k\|_2^2 \leq \frac{2k^2}{2k - 1} \left( \lambda + (n - 1) \kappa + n^2 r + \left( \frac{1}{2} + \frac{2(k - 1)^2}{2k - 1} \right) \frac{n^2 r^2}{\beta^2} \right) \|f_{\varepsilon}\|_\infty^2 \|f_{\varepsilon}^{k-1}\|_2^2
\]

\[
\leq 2k^2 \left( \lambda + (n - 1) \kappa + n^2 r + \left( \frac{1}{2} + \frac{2(k - 1)^2}{2k - 1} \right) \frac{n^2 r^2}{\beta^2} \right) \|f_{\varepsilon}\|_\infty^2 \|f_{\varepsilon}^{k-1}\|_2^2.
\]

Set

\[
L^2 := 2(\lambda + (n - 1) \kappa + n^2 r + n^2 r^2 / \beta^2).
\]
Since \( k \geq 1 \), we now have, instead of (8),
\[
\| d f^k \|_2^2 \leq L^2 k^2 \| f \|_\infty^2 \| f^k \|_\infty^{2k-2} \leq L^2 k^2 \| f \|_\infty^2 \| f^k \|_\infty^{2k-2}.
\]
Using Lemma 4 with \( q = (n+2)/n \) and \( C \) as before, we replace (4) by
\[
\| f \|_{2kq}^k = \| f \|_{2q}^k \leq \| f \|_{2k}^k + CLk \| f \|_\infty \| f^k \|_{2k}^{k-1}
\leq (1 + CLk) \| f \|_\infty \| f^k \|_{2k}^{k-1}.
\]
Instead of (8), we conclude now, by letting \( \varepsilon \to 0 \), that
\[
\| \nabla^E \sigma \|_{2kq} \leq (1 + CLq)^{1/k} \| \nabla^E \sigma \|_{\infty}^{1/k} \| \nabla^E \sigma \|_{2kq}^{1-1/k}.
\]
As in (19), we iterate this inequality with \( k = q^j \), but now only for \( j = 1, 2 \ldots \) since (19) is useless in the case \( k = 1 \). Setting \( p_i := 1 - 1/q^i \), we get
\[
\| \nabla^E \sigma \|_{2q^{j+1}} \leq (1 + CLq^j)^{1/q^j} \| \nabla^E \sigma \|_{\infty}^{1-p_j} \| \nabla^E \sigma \|_{2q^{j}}^{p_j}
\leq \prod_{i=1}^j (1 + CLq^i)^{p_i+1 \ldots p_j/q^i} \| \nabla^E \sigma \|_{\infty}^{1-p_1 \ldots p_j} \| \nabla^E \sigma \|_{2q^{j}}^{p_1 \ldots p_j}
\leq \prod_{i=1}^j (1 + CLq^i)^{1/q^i} \| \nabla^E \sigma \|_{\infty}^{1-p_1 \ldots p_j} \| \nabla^E \sigma \|_{2q^{j}}^{p_1 \ldots p_j},
\]
where we use, for the latter inequality, that \( 0 < p_i < 1 \) and that \( x^p \leq x \) if \( x \geq 1 \) and \( 0 < p < 1 \). The limit
\[
\varepsilon = \varepsilon(n) := \prod_{i=1}^\infty p_i
\]
exists and satisfies \( 0 < \varepsilon < 1 \). Moreover, using the inequality
\[
1 + CLq^i \leq (1 + CL)q^i
\]
we obtain
\[
\prod_{i=1}^\infty (1 + CLq^i)^{1/q^i} \leq (1 + CL)\sum_{i=1}^\infty 1/q^i \cdot q\sum_{i=1}^\infty i/q^i \leq a_1(n)e^{b(n)CL}
\]
with \( a_1(n) = q\sum_{i=1}^\infty i/q^i \) and \( b(n) = \sum_{i=1}^\infty 1/q^i \). We conclude that
\[
\| \nabla^E \sigma \|_{\infty} \leq a_2(n) \exp (b(n)CL/\varepsilon(n)) \| \nabla^E \sigma \|_{2q}
\]
with \( a_2(n) = a_1(n)^{1/\varepsilon(n)} \). We also have
\[
\| \nabla^E \sigma \|_{2q} \leq \| \nabla^E \sigma \|_{2}^{1/q} \cdot \| \nabla^E \sigma \|_{\infty}^{(q-1)/q}
\leq \| \nabla^E \sigma \|_{2}^{n/(n+2)} \cdot \| \nabla^E \sigma \|_{\infty}^{2/(n+2)},
\]
where we recall that \( q = (n+2)/n \). Hence finally
\[
\| \nabla^E \sigma \|_{\infty} \leq a(n) \exp (n+2)b(n)CL/(n\varepsilon(n)) \| \nabla^E \sigma \|_{2}
\]
with \( a(n) = a_2(n)^{(n+2)/n} \). The rest of the argument is as before.
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Mathematisches Institut, Universität Bonn, Beringstrasse 1, D-53115 Bonn,
E-mail address: ballmann@math.uni-bonn.de

Institut für Mathematik, Humboldt–Universität, Rudower Chaussee 5, 12489 Berlin, Germany,
E-mail address: bruening@mathematik.hu-berlin.de

Département de Mathématiques, Université de Nantes, 2 rue de la Houssinière,
BP 92208, 44322 Nantes Cedex 03, France,
E-mail address: Gilles.Carron@math.univ-nantes.fr