Quantum Spin Hall, triplet Superconductor, and topological liquids on the honeycomb lattice

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We classify the order parameters on the honeycomb lattice using the SO(4) symmetry of the Hubbard model. We will focus on the topologically nontrivial quantum spin Hall order and spin triplet superconductor, which together belong to the (3, 3) representation of the SO(4). Depending on the microscopic parameters, this (3, 3) order parameter has two types of ground states with different symmetries: type A, the ground state manifold is \([S^2 \otimes S^2]/Z_2\); and type B, with ground state manifold SO(3) \(\otimes Z_2\). We demonstrate that phase A is adjacent to a \(Z_2 \otimes Z_2\) topological phase with mutual semi-on statistics between spin and charge excitations, while phase B is adjacent to a nonabelian phase described by SU(2) Chern-Simons theory. Connections of our study to the recent quantum Monte Carlo simulation on the Hubbard model on the honeycomb lattice will also be discussed.

I. INTRODUCTION AND SYMMETRY

We consider a class of (extended) Hubbard models on a bipartite lattice at half filling, with the following form:

\[
H = \sum_{\langle i, j \rangle, \sigma} -t c_{i,\sigma} c_{j,\sigma} + H.c. + U n_{i,\uparrow} n_{i,\downarrow} + H'
\]

\[
H' = \sum_{i \in sA, j \in sB} t_{ij} c_{i,\sigma} c_{j,\sigma} + H.c. + \sum_{i,j} J_{ij} \vec{S}_i \cdot \vec{S}_j + V_{ij} \vec{T}_i \cdot \vec{T}_j + \ldots
\]

\[
\vec{T}_i = (i (-1)^i \text{Re}[\Delta_i], (-1)^i \text{Im}[\Delta_i], n_i - 1), \quad (1)
\]

\[
\vec{S}_i = \frac{1}{2} c_{i,\sigma} \sigma c_{i,\sigma}
\]

the Hamiltonian is almost unchanged except that \(U\) changes sign, and \(J_{ij}\) switches with \(V_{ij}\). This implies that in addition to the apparent SU(2)\(_\text{spin}\) symmetry, this model also has an SU(2)\(_\text{charge}\) symmetry that mixes \((c_{i,\uparrow}, -1)c_{i,\downarrow}\). Therefore the full symmetry of this extended Hubbard model is \(1^2\).

\[
\text{SO}(4) \sim [\text{SU}(2)\(_\text{spin}\) \otimes \text{SU}(2)\(_\text{charge}\)]/Z_2, \quad (3)
\]

for arbitrary parameters in Eq. 1. For instance, this SO(4) symmetry holds for the simplest Hubbard model with only on-site Hubbard interaction and nearest neighbor electron hopping.

Since the SO(4) symmetry is the full symmetry of the extended Hubbard model Eq. 1 on any bipartite lattice, all the order parameters should be classified in terms of the representations of the SO(4) Lie algebra. In this work we will take the honeycomb lattice as an example. Using the notation introduced in Ref. 3, we expand the electron at two Dirac valleys by \(d_{i,1,2} = e^{i\vec{Q} \cdot \vec{x}} e^{i\pi d_{i,1,2}}\) (where \(\vec{Q}_{1,2} = \pm (\frac{\pi}{\sqrt{3}}, 0)\) are the wavevectors of the valleys), and introduce Pauli matrices \(\tau^\alpha\) and \(\mu^\alpha\) which act on the sublattice and valley spaces respectively. Then, after introducing real Majorana fermions \(\zeta^\alpha\) as the real and imaginary parts of \(e^{i\pi \vec{r}^\alpha e^{i\pi d_{i,1,2}}(d_{i,1,2})^\dagger}\), we obtain the continuum Lagrangian for the semimetal phase

\[
L_0 = \sum_{\alpha=1}^8 \zeta_\alpha \mu^\alpha \partial_\mu \zeta_\alpha. \quad (4)
\]

Here \(\mu\) is a 2+1 dimensional spacetime index, and the Dirac \(\gamma\) matrices are \((\gamma_0, \gamma_1, \gamma_2) = (\tau^x, \tau^y, \tau^z)\), \(\bar{\zeta} = \zeta^\dagger \gamma^0\). Using this notation, in the low energy field theory, the SU(2)\(_\text{spin}\) and SU(2)\(_\text{charge}\) symmetries are generated by the following matrices:

\[
S^\alpha = \sigma^\alpha \rho^\alpha, \quad S^\gamma = \sigma^x \rho^y, \quad S^z = \sigma^z \rho^\gamma,
\]

\[
T^\alpha = \sigma^\alpha \rho^x, \quad T^y = \sigma^y \rho^\alpha, \quad T^z = \rho^\gamma. \quad (5)
\]

\(\sigma^\alpha\) are spin Pauli matrices, while \(\rho^\alpha\) are Pauli matrices that mix the real and imaginary parts of electron. Notice that SU(2) \(\otimes\) SU(2) is a double covering of SO(4), which leads to the \(Z_2\) in Eq. 3.

Based on the symmetry Eq. 3 and Lie Algebra Eq. 5 the spin and charge are dual to each other for a large class of extended Hubbard model. This spin-charge duality will lead to many interesting results in our paper. The structure of this paper is as following: in section II we show that the two types of topological orders, the quantum spin Hall (QSH) order and triplet superconductor (T-SC) are unified as one representation of the SO(4) group, and the Ginzburg-Landau theory gives two types of ground states with different symmetry breakings. Section III and IV study the phase diagrams driven by proliferating the topological defects in the two types of
orders described in section II respectively. In both section III and IV, we will first give an argument of the phase diagram based on the quantum numbers of the topological defects, then a more solid description based on the Majorana liquid formalism developed in Ref. will be presented, and the results from these two approaches match perfectly with each other. Section V discusses the situation with $SU(2)_{\text{charge}}$ broken down to $U(1)_{\text{charge}}$ symmetry.

II. SO(4) CLASSIFICATION AND GINZBURG-LANDAU FORMALISM

Using the SO(4) algebra in Eq. we will classify the order parameters which immediately open up a mass gap for the Dirac fermion in the semi-metal phase. Some simple Dirac mass gap order parameters can be classified with these symmetries straightforwardly. For instance, the quantum Hall order parameter $\zeta$ is a $(1, 1)$ representation of SO(4) i.e. it is a singlet of both SU(2) symmetries. The two sublattice Néel order $N^a = \zeta S^a \mu^b \zeta$ is a $(3, 1)$ representation. The fermion bilinear $M^a = \zeta T^a \mu^b \zeta$ which belongs to the $(1, 3)$ representation is a two sublattice charge-density wave (CDW) and $s$-wave superconductor: $M^z \sim (-1)^j (n_i - 1)$, $M^x + iM^y \sim \epsilon^{ij} \sigma^j \epsilon^c$. $M^a$ can be viewed as the spin-charge dual version of $N^a$. In the simplest Hubbard model, $N^a$ and $M^a$ orders can be realized in the two limits $U \gg |t|$ and $-U \gg |t|$ respectively.

Now we discuss the following order parameters which belong to the $(3, 3)$ representation of SO(4):

$$Q_{ab} = \zeta A_{ab} \zeta;$$

$$A_{ab} = T^a S^b = \begin{pmatrix}
-\sigma^z \rho^z & \rho^z & \sigma^{xy} \\
\sigma^z \rho^z & \rho^z & -\sigma^{xy} \\
\sigma^z & \sigma^{xy} & \sigma^z
\end{pmatrix}$$

This $3 \times 3$ matrix $Q_{ab}$ has SO(3)$_{\text{left}}$ and SO(3)$_{\text{right}}$ transformations, which correspond to SU(2)$_{\text{charge}}$ and SU(2)$_{\text{spin}}$ symmetry respectively, $Q_{3b}$ corresponds to the QSH vector $\Delta^2$, while $Q_{2b} + iQ_{1b}$ is the spin triplet pairing between next nearest neighbor sites:

$$Q_{2b} + iQ_{1b} \sim \sum_{j \in sA, \ a=1,2,3} ic_j^a \sigma^b c_{j+a} + H.c.$$  

$$Q_{2b} + iQ_{1b} \sim \sum_{j \in sA, \ a=1,2,3} ic_j^a \sigma^b c_{j+a}$$

$$(sA \rightarrow sB).$$

FIG. 1: Honeycomb lattice and the vectors $e_a$. 

e_1 = \sqrt{3}x, e_2, e_3 = -\frac{\sqrt{3}}{2}x \pm \frac{3}{2}y$ are three vectors on the honeycomb lattice that connect nearest neighbor sites. Therefore the two types of topological order parameters, QSH and T-SC, are unified through the SO(4) symmetry. Under time reversal symmetry $T$, $Q_{2b}$ and $Q_{2b}$ are even, while $Q_{1b}$ is odd. Under reflection symmetry $P_x : y \rightarrow -y$, $Q_{1b}$ and $Q_{2b}$ are even, while $Q_{2b}$ is odd; under $P_y : x \rightarrow -x$, all components of $Q_{ab}$ are odd.

The low energy dynamics of $Q_{ab}$ can be described by the following Ginzburg-Landau field theory:

$$L_Q = \text{tr}[\partial_t Q^T \partial_t Q] + r(\text{tr}[Q^T Q]) + g(\text{tr}[Q^T Q])^2$$

$$+ u(\text{tr}[Q^T QQ^T Q]) + \cdots$$

The first three terms have an enlarged SO(9) symmetry which corresponds to the rotation between the nine order parameters in matrix $Q$; while the last term $\sim u$ breaks this SO(9) symmetry down to SO(4) symmetry. Another term $\text{Det}[Q]$ is also invariant under SO(4) transformation, but $\text{Det}[Q]$ breaks the time-reversal and reflection symmetry of the honeycomb lattice, therefore $\text{Det}[Q]$ is forbidden in the Lagrangian Eq. However, if the system already breaks the time reversal and reflection symmetry (for instance $(\zeta_\zeta) \neq 0$), $\text{Det}[Q]$ would be allowed. In Eq. when $r < 0$, $Q$ is ordered, and the SO(4) symmetry is broken down to its subgroups. Depending on the sign of $u$, there are two types of ground states:

Type $A$, $u < 0$, one example state of this phase is $(\langle Q_{33} \rangle \neq 0$, and all the other components $(Q_{ab}) = 0$. In this phase the SO(4) symmetry is broken down to its following subgroup:

$$\langle U(1)_{\text{spin}} \otimes U(1)_{\text{charge}} \otimes Z_2 \rangle / Z_2.$$  

The $U(1)_{\text{spin}}$ and $U(1)_{\text{charge}}$ symmetry are generated by matrices $S^z$ and $T^z$ in Eq. The $Z_2$ in the numerator corresponds to reversing the direction of $S^z$ and $T^z$ simultaneously, while keeping $Q_{33}$ invariant. The $Z_2$ in the denominator is the same $Z_2$ as in Eq. which corresponds to changing the sign of electron operator. The ground state manifold (GSM) of this phase is

$$\text{GSM} \sim [S_{\text{spin}}^2 \otimes S_{\text{charge}}^2] / Z_2.$$  

The ground state can be described by two independent unit vectors $\hat{N}_s$ and $\hat{N}_c$ which belong to the $(3, 1)$ and $(1, 3)$ representation of SO(4) respectively, and $Q_{ab} =$
This phase has four independent Goldstone modes. The $Z_2$ in Eq. (10) is due to the fact that $\vec{N}_s$ and $\vec{N}_c$ can reverse direction simultaneously, and the ground state remains invariant.

Type $B$, $u > 0$, one example state of this phase is $\langle Q_{11} \rangle = \langle Q_{22} \rangle = (Q_{33}) \neq 0$. The $SO(4)$ group element $G_{so(4)}$ can be written as $G_{su(2)} = SO(2)_c$, and the type $B$ phase breaks the $SO(4)$ down to its subgroup with $G_{su(2)} = \pm G_{su(2)}$. This implies that the residual symmetry group elements can be parametrized as $\pm R(\theta^a, \vec{n}^a)$ with $\theta^a \in (0, 2\pi)$, which is equivalent to the diagonal subgroup $SU(2)_c$ generated by operators $G^a = S^a + T^a$. Here $R(\theta^a, \vec{n}^a)$ represents spin rotation by angle $\theta^a$ about axis $\vec{n}^a$. The GSM of phase $B$ is

\[
GSM = (SO(4) \otimes T_+) /SU(2)_c = SO(3) \otimes Z_2, \tag{11}
\]

with three Goldstone modes. $T$ denotes the time reversal symmetry, and type $B$ phase spontaneously breaks $T$. Therefore the GSM of phase $B$ contains two disconnected sub-manifolds, with positive and negative $\text{Det}([Q])$ respectively.

The order of $Q$ can be obtained through the following $SO(4)$ invariant interacting Lagrangian for Dirac fermions on the honeycomb lattice:

\[
\mathcal{L} = \sum_{a=1}^{8} \bar{\zeta}_a \gamma_{\mu} \partial_{\mu} \zeta_a - g \text{tr}[Q^\dagger Q], \tag{12}
\]

The interaction $-g \text{tr}[Q^\dagger Q]$ can be generated with $SO(4)$ invariant interaction on the lattice, for instance $\sum_{i,j} (\vec{S}_i \cdot \vec{S}_j) \sim -\text{tr}[Q^\dagger Q]/8 + \cdots$. A simple mean field calculation after the standard Hubbard-Stratonovich Transformation of Eq. (12) shows that the type $A$ phase has more favorable ground state energy compared with the type $B$ phase on the honeycomb lattice. In the following we will mainly focus on the analysis on the type $A$ phase.

### III. PHASE DIAGRAM AROUND TYPE A PHASE

Now we hope to understand the topological defects, and the phase transitions driven by topological defects in phase $A$. We will first give an argument about the phase diagram and phase transitions using the quantum numbers carried by the topological defects, and then a systematic description based on the Majorana liquid formalism developed in Ref. [2] will be presented. We will demonstrate that these two approaches match very well.

#### A. Topological defects and phase transitions

In phase $A$, since the GSM is $[S^2 \otimes S^2]/Z_2$, both spin and charge sectors can have Skyrmion like defects characterized by homotopy group $\pi_2[S^2]$. Again, let us assume that $Q_{33}$ is the only component that acquires a nonzero expectation value, then $\langle Q_{33} \rangle$ breaks the $SO(4)$ symmetry down to residual symmetries generated by $S^2$ and $T^2$ in Eq. (5). According to Ref. [3], under our current assumption that $\vec{N}_c \parallel \vec{z}$ ($Q_{ab}$ is the QSH vector), a Skyrmion of the spin sector manifold $S^2_{\text{spin}}$ carries charge $2e$; and a Skyrmion current is identified as the charge current:

\[
J^\mu = \frac{2e}{8\pi} \epsilon_{\mu\nu\rho} \int d^2 x e_{abc} \vec{N}_a^b \partial_\nu \vec{N}_c^b \phi^c. \tag{13}
\]

For the same reason, a Skyrmion of the charge sector manifold will carry spin-1: $S^2 = 1$. For a general state with $\langle Q_{ab} \rangle \neq 0$, the spin-Skyrmion carries the quantum number of the $U(1)_c$ residual symmetry, while the charge-Skyrmion carries $U(1)_c$ quantum number $i.e.$ spin and charge view each other as topological defects.

The condensation of Skyrmions with nontrivial quantum numbers can lead to unconventional quantum phase transitions. For instance the proposal of deconfined quantum criticality is based on the observation that the Skyrmion of the Néel order carries lattice momentum $2k_\text{F}$, hence the condensate of the Skyrmion is equivalent to the valence bond solid state. In our current case, since a charge-Skyrmion carries spin-1, if the charge-Skyrmion is condensed, then the $SU(2)_c$ symmetry is fully restored, which implies that the condensate of the charge Skyrmion is a Mott insulator. Meanwhile, the residual spin symmetry is further spontaneously broken down to $Z_2 \otimes Z_2$. One of these $Z_2$ corresponds to changing the sign of electron, the other one corresponds to reversing the direction of $\vec{N}_s$. The GSM of the charge Skyrmion condensate is

\[
GSM = SU(2)_c / [Z_2 \otimes Z_2] = SO(3)/Z_2. \tag{14}
\]

We mod $Z_2$ from $SO(3)$, because after the proliferation of charge Skyrmion, $\vec{N}_s$ is completely disordered, and $\vec{N}_s$ becomes a headless vector, due to the $Z_2$ in Eq. (10).

How do we determine the order of the charge-Skyrmion condensate unambiguously? As was pointed out in Ref. [2], the phase of the $O(3)$ Skyrmion condensate can be identified as order parameters that share an $O(3)$ Wess-Zumino-Witten (WZW) term with the $O(3)$ order parameter. Therefore, to unambiguously identify the order of spin-Skyrmion condensate, we need to seek for order parameters that have an $O(5)$ WZW term with vector $\varphi^a = Q_{ab}$. It turns out that the Néel order parameter $\vec{N}$ is the only candidate of the charge-Skyrmion condensate. For arbitrary $b$, we obtain the following WZW term between $\varphi^a = Q_{ab}$ and $\vec{N} \sim \chi^{\mu\nu} \vec{S}_{\chi}$:

\[
\mathcal{L} = \sum_{a=1}^{5} \frac{1}{9} (\partial_{\mu} \varphi^a)^2 - \frac{3i}{4\pi} \int d^2 x \epsilon_{abcde} \varphi^a \partial_{\mu} \varphi^b \partial_{\nu} \varphi^c \partial_{\nu} \partial_{\mu} \varphi^d \varphi^e,
\]
\[ \phi^a = \varphi^a = \bar{\chi} T^a S_b \chi, \quad a = 1, 2, 3, \]
\[ \phi^4 = N^c \sim \bar{\chi} \mu^y S^c \chi, \]
\[ \phi^5 = N^d \sim \bar{\chi} \mu^y S^4 \chi, \quad c, d \neq b. \] (15)

Therefore the charge-Skyrmion condensate contains both headless vector \( \vec{N}_e \) and Néel order \( \vec{N} \), and \( \vec{N}_s \perp \vec{N} \). Physically the headless vector \( \vec{N}_e \) corresponds to spin nematic order \( S_{ab} = 3 N_a N_b - \delta_{ab} (N_c)^2 \), which is invariant under reversing the direction of \( \vec{N}_e \). All these results will be confirmed later with the Majorana liquid formalism.

Notice that manifold SO(3) is equivalent to the projected manifold \( S^3/Z_2 \), which gives us a convenient way of parameterizing SO(3). Let us introduce SU(2) spinon \( z_\alpha \) with constraint \( |z_1|^2 + |z_2|^2 = 1 \). This constraint implies that the SU(2) spinon \( z_\alpha \) parametrizes \( S^3 \). Then by coupling \( z_\alpha \) to a Z_2 gauge field, the gauge invariant GSM of the condensate of \( \vec{z}_\alpha \) automatically becomes SO(3). SO(3) manifold can also be viewed as the manifold of all the configurations of three perpendicular unit vectors \( \vec{q}_1, \vec{q}_2, \vec{q}_3 \). These three vectors can be parametrized as
\[ \vec{q}_1 = z^1 i \sigma^y z, \quad \vec{q}_2 = i \vec{q}_3 = z^2 i \sigma^y \sigma^z, \]
which automatically guarantees the perpendicularity of these vectors. In our situation, the three perpendicular vectors that characterize the GSM are \( \vec{N}_s, \vec{N} \) and \( \vec{N}_s \times \vec{N} \). Since \( \vec{N}_s \) is headless, the GSM is in fact SO(3)/Z_2. And in the next subsection we will demonstrate that it is most convenient to describe this GSM by introducing a \( Z_2 \times Z_2 \) or \( Z_4 \) gauge field.

Manifold SO(3) has homotopy group \( \pi_1 [SO(3)] = Z_2 \), therefore phase \( B \) has topologically stable half vortex. Using the CP(1) spinon description introduced in Eq. (10), this half-vortex can also be viewed as the vison (a dynamical \( \pi \)-flux) of the Z_2 gauge field coupled to \( z_\alpha \). Pictorially, a vison can be viewed as a configuration with (for instance) \( \vec{q}_1 \) being uniform in space, while \( \vec{q}_2 \) and \( \vec{q}_3 \) have a vortex. Now since \( \vec{N}_s \) and \( \vec{N}_s \times \vec{N} \) are both headless vectors, this state also supports “half vison”, where \( \vec{N} \) is uniform, while \( \vec{N}_s \) has a half vortex in space. In fact, this half vison has a counterpart in phase A. Since phase A has GSM \( [S^2 \otimes S^2]/Z_2 \), there exists “double half vortex”, and both \( \vec{N}_s \) and \( \vec{N}_c \) reverse direction after encircling this double half vortex. After phase A is destroyed by proliferating the charge-Skyrmion, this double half vortex becomes the half vison of SO(3)/Z_2.

In the spin-charge dual side of the theory, all the conclusions can be obtained by straightforward generalization. Once the spin-Skyrmion is condensed, the system will also enter a phase with GSM SO(3)/Z_2, and the SU(2)spin symmetry is fully restored, which implies that the condensate of the spin-Skyrmion is spin singlet. In Ref. 8, the authors proposed that after the proliferation of Skyrmions of the QSH vector, the system enters a spin singlet s-wave superconductor. In our situation, since there is a generic SU(2)_charge symmetry, the s-wave superconductor is promoted to a phase with GSM SO(3)/Z_2. If \( \vec{N}_c \parallel \vec{z} \) (QSH vector), the SO(3)/Z_2 manifold is characterized with headless vector \( \vec{N}_c \) and s-wave superconductor. The order of headless vector \( \vec{N}_c \) implies that the degeneracy between CDW and s-wave superconductor is spontaneously lifted. In general, the order after spin-Skyrmion proliferation can also be determined with the same WZW term analysis in Eq. (15). A full list of order parameters with WZW terms can be found in Ref. 10.

This Skyrmion condensation transition is described by the same CP(1) field theory as the deconfined quantum criticality[10,11]. How do we see the CP(1) transition directly? The CP(1) model \( \mathcal{L} = \frac{1}{8}(\bar{\theta}_a - i A_\mu) \bar{\zeta} \zeta \) describes a transition between a condensate of spinon \( z_\alpha \) and a photon phase. The spin condensate has GSM \( S^2 \), while the photon phase has GSM \( S^1 \), as it is a condensate of the U(1) gauge flux. In our case the Skyrmion condensation is a transition between \( [S^2 \otimes S^2]/Z_2 \) and SO(3)/Z_2, while SO(3) can be roughly viewed as \( S^2 \otimes S^1 \), therefore effectively the Skyrmion condensation is more or less also a transition between \( S^2 \) and \( S^1 \), so it is equivalent to the CP(1) transition. This hand-waving argument will be made precise in the next subsection by the Majorana liquid formalism.

Since a spin-Skyrmion (charge-Skyrmion) carries charge-2e (spin-1), then the corresponding half-Skyrmion (vortex) will carry charge-e and spin-1/2 respectively. If a charge-e excitation encircles around a spin-1/2 excitation bound with a charge-vortex, the charge-e excitation will acquire a \( \pi \) phase shift; on the other hand, if a spin-1/2 excitation encircles a charge-e excitation bound with a spin-vortex, the spin-1/2 excitation will also gain a \( \pi \) Berry phase. This implies that in phase A charge-e and spin-1/2 excitations have mutual semion statistics.

In phase A, a vortex is not a local excitation, and the gapless Goldstone mode of phase A makes the adiabatic braiding between two excitations impossible, therefore the semion statistics in phase A is not well defined. However, later we will see that phase A is adjacent to a liquid phase where the spin-charge mutual semion statistics persists, and it becomes a well defined property.

### B. Phase diagram with Majorana liquid formalism

From now on we hope to understand the phase diagrams discussed above with a more solid formalism. In Ref. 8, we discussed a fractionalized phase of electrons by decomposing \( \zeta \) as following:
\[ \zeta = Z_s Z_c \chi, \]
\[ Z_s = \phi_0^s + i \phi_1^s S^x + i \phi_2^s S^y + i \phi_3^s S^z, \]
\[ Z_c = \phi_0^c + i \phi_1^c T^x + i \phi_2^c T^y + i \phi_3^c T^z. \] (17)
The electron $\zeta$ decomposes into the bosonic fields $Z_s$ and $Z_c$ carrying its spin and charge respectively, and into the Majorana fermion $\chi$ carrying the Fermi statistics. The resulting theory has a $SO(4)_{g} \cong SU(2)_{s,g}$ gauge invariance: $Z_s$ and $\chi$ carry $SU(2)_{s,g}$ charges, and $Z_c$ and $\chi$ carry $SU(2)_{c,g}$ charges.

After the operator decomposition, when both $Z_s$ and $Z_c$ are gapped out, one obtains the parent state, i.e. the Algebraic Majorana Liquid (AML) state with Lagrangian

$$\mathcal{L}_{\text{AML}} = \bar{\chi} \gamma_{\mu} \left( \partial_{\mu} - i A_{s,\mu}^{a} S^{a} - i A_{c,\mu}^{a} T^{a} \right) \chi. \tag{18}$$

The fractionalized Majorana fermion $\chi$ fills the same mean field band structure as the physical Majorana fermion $\zeta$. The gauge field $A_{s,\mu}^{a}$ and $A_{c,\mu}^{a}$ also couple to the spin and charge $SU(2)$ rotors $Z_s$ and $Z_c$ as well. Since $\chi$ no longer carries physical spin and charge quantum numbers, the fermion bilinears of $\chi$ can only break the gauge symmetry, but not physical symmetry. If $Z_s$ or $Z_c$ condense, the formalism reduces to the two standard slave particle formalisms, with fermionic or bosonic spinons respectively.

Now let us assume $\chi$ enters a type A phase i.e. the matrix field $\tilde{Q}_{ab} = \bar{\chi} A_{ab} \chi$ condenses. For instance let us take

$$\langle \tilde{Q}_{33} \rangle = \langle \bar{\chi} \sigma^z \chi \rangle \neq 0. \tag{19}$$

Although the fractionalized Majorana fermion $\chi$ fills the same mean field band structure as $\zeta$, unlike the physical QSH vector, nonzero $\langle \tilde{Q}_{33} \rangle$ breaks no discrete symmetries (time-reversal, refection) when rotor fields $Z_s$ and $Z_c$ are gapped. This is because the gauge symmetry released from gapping out the rotor fields can always reverse the sign of $\langle \tilde{Q}_{33} \rangle$. This condensate of $\tilde{Q}_{33}$ breaks the $SU(2)_{s,g} \otimes SU(2)_{c,g}$ gauge invariance down to $U(1)_{s,g} \otimes U(1)_{c,g}$ gauge symmetry generated by $S^{z}$ and $T^{z}$. Sometimes it will be convenient to use the following spin and charge CP(1) field

$$z^{s} = (z^{s}_{1}, \ z^{s}_{2})^{\dagger} = (\phi_{0}^{s*} + i \phi_{3}^{s*}, \ -\phi_{2}^{s*} + i \phi_{1}^{s*})^{\dagger};$$

$$z^{c} = (z^{c}_{1}, \ z^{c}_{2})^{\dagger} = (\phi_{0}^{c*} + i \phi_{3}^{c*}, \ -\phi_{2}^{c*} + i \phi_{1}^{c*})^{\dagger}. \tag{20}$$

It was discussed in our previous work that, after integrating out $\chi$, we obtain the low energy theory when $\langle \tilde{Q}_{33} \rangle \neq 0$, which is a mutual Chern-Simons theory:

$$\mathcal{L}_{\text{CS}} = \frac{2i}{2\pi} \epsilon_{\mu\nu\rho} A_{c,\mu}^{a} \partial_{\nu} A_{c,\rho}^{a} + |(\partial_{\mu} - i \bar{A}_{c,\mu}^{a}) z_{a}^{c}|^{2} + r_{s}|z_{a}^{s}|^{2} + \cdots. \tag{21}$$

The CP(1) fields $z^{s}_{a}$ and $z^{c}_{a}$ carry spin and charge respectively, and we have chosen the notation to make both spin and charge $SU(2)$ physical global symmetries manifest. Eq. (21) implies that there is a mutual semion statistics between the charge and spin CP(1) fields $z^{c}_{a}$ and $z^{s}_{a}$, which verifies the observation in section IIIA.

This field theory is similar to the one on the triangular lattice, with mutual semion statistics between charge and vison, except there the $SU(2)$ symmetry of the vison is broken down to discrete symmetry by higher order terms, while here the $SU(2)$ charge symmetry is exact. Based on this analogy, we can propose a similar global phase diagram with tuning parameters $r_{s}$ and $r_{c}$ (Fig. 2), with a different interpretation of the phases:

1. **Phase A**

Phase A is the phase with both $z^{s}_{a}$ and $z^{c}_{a}$ condensed, and the $SU(2)_{s,g}$ and $SU(2)_{c,g}$ gauge fields are both Higgsed and gapped out from the spectrum. This phase is characterized by the $SU(2)$ vectors:

$$N_{s}^{a} = z^{s}_{a} = |z_{a}^{s}|^{2} \sim \text{tr}[Z_{s}^{a} S^{a} Z_{s}^{a}];$$

$$N_{c}^{a} = z^{c}_{a} = |z_{a}^{c}|^{2} \sim \text{tr}[Z_{c}^{a} T^{a} Z_{c}^{a}], \tag{22}$$

and the gauge invariant physical order parameter is

$$Q_{ab} = z_{ab} = \bar{z}_{ab} = \text{tr} \left[ Z_{a}^{s} S^{a} Z_{b}^{s} \right] \sim \langle \tilde{Q}_{33} \rangle N_{s}^{a} N_{b}^{a}, \tag{23}$$

therefore the physical GSM is $[S^{z} \otimes S^{z}]/Z_{2}$. This phase is precisely the phase A obtained in the GL formalism in section II. The Skyrmion of vector $\vec{N}_{s}$ is equivalent to the flux of gauge field $A_{s,\mu}^{z}$, and due to the mutual CS interaction, the gauge flux of $A_{s,\mu}^{z}$ (Skyrmion of $\vec{N}_{s}$) carries charge excitation $z_{a}^{s}$, which confirms our analysis in section IIIA.
Phase A2 has $r_s < 0$, $r_c > 0$, hence it is a phase with $z^c_s$ condensed while $z^a_s$ gapped. In section IIIA we concluded that the GSM of this phase is SO(3)/$Z_2$, and it has both nematic order and Néel order by directly calculating the WZW term. How do we understand the physical orders using the Majorana liquid formalism? Since $z^c_s$ is gapped, in phase A2 there is no charge degrees of freedom, therefore the “QSH” vector $\tilde{Q}_V$ should correspond to a pure spin operator. In fact, since $z^c_s$ is gapped, $\langle Q_{ab} \rangle$ in Eq. 23 vanishes, hence the only gauge invariant operator which acquires a nonzero expectation value is

$$3\tilde{Q}_{cd}^I\tilde{Q}_{de} \text{tr}[Z^a_s S^a Z^c_s S^c]\text{tr}[Z^I_s S^b Z^c_s S^c]$$

$$\sim \langle \tilde{Q}_{33} \rangle^2 N^a_s N^b_s,$$

(24)

therefore we can define the physical order parameter as

$$S_{ab} \sim 3N^a_s N^b_s - \delta^{ab}(\tilde{N}_s)^2 \sim 3S^a_i S^b_j - \delta^{ab}\tilde{S}_i \cdot \tilde{S}_j.$$  

(25)

Hence $S_{ab}$ is the spin-2 nematic order parameter which breaks the spin rotation symmetry down to U(1) $\otimes$ $Z_2$, but preserves the discrete symmetries. Notice that the physical order parameter is always a bilinear of $\tilde{N}_s$.

When $z^a_s$ is condensed, the SU(2)$_{s,g}$ gauge field is Higgsed, then the SU(2)$_{s,g}$ gauge charge of $\chi$ becomes equivalent to the physical spin quantum number of $\xi$ after a SU(2) gauge transformation. If we take $\langle \tilde{Q}_{33} \rangle \neq 0$, the low energy field theory for fermions in phase A2 reads:

$$\mathcal{L} = \bar{\psi}i\gamma_\mu(\partial_\mu - iA^a_{c,\mu})\psi + m(\tilde{Q}_{33}) \cdot \bar{\psi}\sigma^z\psi,$$

(26)

where $\psi = \chi_1 + i\chi_2$. Based on the QSH physics, the flux of gauge field $A^a_{c,\mu}$ carries spin:

$$\nabla \cdot \tilde{A}^a_s \sim \psi^\dagger \sigma^z \psi \text{tr}[Z^a_s S^a Z^c_s S^c].$$

(27)

This equivalence between the spin and flux is the phase of gauge field $A^a_{c,\mu}$, which is the condensate of the flux is a spin XY order. This effect was studied in Ref. 17, 18 with projected wave-function calculation, and the photon phase of the U(1) gauge field is precisely the Néel order:

$$N^a \sim \bar{\chi}\mu^\nu S^\nu\chi \text{tr}[Z^a_s S^a Z^c_s S^c].$$

(28)

Based on these analysis, we conclude that phase A2 is a phase with both nematic vector $\tilde{N}_s$ in Eq. 22 and AF Néel order $\bar{N}$ in Eq. 28, Eq. 22 and Eq. 28 guarantee that $\tilde{N}_s \perp \bar{N}$:

$$\tilde{N}_s \cdot \bar{N} \sim \sum_a \text{tr}[Z^a_s S^a Z^a_s S^c] \text{tr}[Z^a_s S^a Z^c_s S^c] = 0.$$  

(29)

As we mentioned before, since the nematic vector is headless, the GSM should be SO(3)/$Z_2$. This analysis again confirms our prediction in section IIIA with the WZW term.

Phase A4 is the spin-charge dual phase of phase A2, the GSM is also SO(3)/$Z_2$ with three branches of Goldstone modes. The spin-charge dual of the inplane Néel order is precisely a $s$-wave superconductor. The spin-charge dual of the nematic order $S_{ab}$ will break the SU(2)$_{charge}$, for instance:

$$S_{zz} \sim 2T^z_i T^z_j - T^x_i T^x_j - T^y_i T^y_j,$$

(30)

with $\hat{T}_i$ given by Eq. 4. Therefore, if we turn on an extra density repulsion between next nearest neighbor sites in Eq. 1, it corresponds to turning on $S_{zz}$, and breaks the SU(2)$_{charge}$ down to U(1) $\otimes$ $Z_2$. This U(1) corresponds to the ordinary electron charge conservation, and $Z_2$ corresponds to the discrete particle-hole symmetry.

Phase A3 is a liquid state with $r_s > 0$, $r_c > 0$, both $z^a_s$ and $z^c_s$ are gapped. When $z^c_s$ is gapped, $A^a_{c,\mu}$ in the photon phase. Since the photon phase of 2+1d U(1) gauge field is also the condensate of gauge flux based on the standard QED-superfluid duality, the mutual CS coupling in Eq. 21 implies that the photon phase of $A^a_{c,\mu}$ breaks the U(1)$_{s,g}$ down to $Z_2$ gauge symmetry. For the same reason, U(1)$_{c,g}$ is also broken down to $Z_2$. The mutual CS theory in Eq. 21 has the same topological degeneracy as the standard $Z_2$ gauge field on the torus, 15, 19, and the mutual statistics between charge and spin is an analogue of the mutual statistics between charge and vison of the well-known toric code model. 20

In addition to the $Z_2$ gauge field coming from the mutual CS coupling, there is one extra residual $Z_2$ gauge symmetry which corresponds to reversing the sign of gauge symmetry generators $S^x$ and $T^x$ simultaneously, while leaving $\tilde{Q}_{33}$ invariant. This extra discrete $Z_2$ gauge symmetry contains group elements

$$G^{(z_2)} = I_{4 \times 4}, \text{ or } S^x T^x.$$  

(31)

This $Z_2$ gauge field couples to both spin and charge SU(2) rotors $Z_s$ and $Z_c$, but it was not explicit in our continuum limit field theory. Therefore phase A3 is characterized by $Z_2 \otimes Z_2$ gauge fields. Under $Z_2$ gauge symmetry $G^{(z_2)}$, the fractionalized particles transform as

$$Z_{s,j} \rightarrow Z_{s,j} \frac{1}{2} ((1 + \mu_j)I_{4 \times 4} + (1 - \mu_j)iS^x),$$

$$Z_{c,j} \rightarrow Z_{c,j} \frac{1}{2} ((1 + \mu_j)I_{4 \times 4} + (1 - \mu_j)iT^x),$$

$$\chi_j \rightarrow \frac{1}{2} ((1 + \mu_j)I_{4 \times 4} + (1 - \mu_j)S^x T^x) \chi_j,$$

$$\mu_j = \pm 1.$$  

(32)

If the matter fields are ignored, these two $Z_2$ gauge fields are equivalent to a $Z_4$ gauge field with group elements.
$G^{(z_4)} = \exp[i\theta S^z T^x]$, $\theta = 0, \pi/2, \pi, 3\pi/2$.

These two $Z_2$ gauge field together again implies that the GSM of $A_2$ (the condensate of $z_{\alpha}^c$) is $SO(3)/Z_2$, as we already concluded. If we approach phase $A_2$ from phase $A_3$, we can interpret phase $A_2$ as the condensate of SU(2) spin rotor $Z_3$ which couples to the two $Z_2$ gauge groups discussed above. With the condensate of $Z_3$, we can again define three perpendicular vectors:

\[
\begin{align*}
\vec{N}_s &= \text{tr}[Z_i^c S_i S^c] \sim z^s \sigma^z, \\
\vec{N}_1 &= \text{tr}[Z_i^c S_i S^c] \sim \text{Re}[(z^s)^* i\sigma^y \sigma^z], \\
\vec{N}_2 &= \text{tr}[Z_i^c S_i S^c] \sim \text{Im}[(z^s)^* i\sigma^y \sigma^z].
\end{align*}
\] (33)

$\vec{N}_s$ and $\vec{N}_2$ change sign under gauge transformation $Z_{c} \rightarrow Z_{c} S^{s}$ (i.e. $\mu_j = -1$), while $\vec{N}_1$ never changes sign, therefore $\vec{N}_s$ and $\vec{N}_2$ become headless nematic vectors by coupling to the $Z_2 \otimes Z_2$ gauge group. Hence the manifold formed with $\vec{N}_s, \vec{N}_1$ and $\vec{N}_2$ is $SO(3)/Z_2$. This is completely consistent with our description of phase $A_2$ in section [III A] and [III B].

In addition to the mutual statistics between $Z_s$ and $Z_c$, there is one more topological defect in phase $A_3$ with

\[
\prod_{\mathcal{C}} G^{(z_2)} = S^z T^x,
\] (34)

$\mathcal{C}$ is a closed loop on the lattice. This defect carries a gauge flux $S^z T^x$. After encircling this defect, $Z_s \rightarrow Z_s S^z$, $Z_c \rightarrow Z_c T^x$. The vectors $\vec{N}_s$ and $\vec{N}_2$ always acquire a minus sign after encircling this defect. Therefore this defect is a counterpart of the “double half vortex” and “half vison” discussed in section [III A].

4. Discussion

The universality class of the phase transitions in Fig. 2 can also be analyzed with field theory Eq. (21) in the same way as Ref.[15]. Quoting the results in Ref.[15], the transition between phases $A_3, A_2$, and the transition between phases $A_3, A_4$ are 3d O(4) transitions, because spinon $z_{\alpha}^c$ and $z_{\alpha}^s$ are O(4) vectors, and the fully gapped discrete gauge fields coupled to the O(4) vector do not change the O(4) universality class[24]. The transition $(A, A_3)$, and the transition $(A, A_4)$ are CP(1) transitions, which become manifest with the CP(1) fields $z_{\alpha}^c$ and $z_{\alpha}^s$, and U(1) gauge fields $A_{\mu}^{c,\mu}, A_{\mu}^{c,\mu}$.

A recent quantum Monte carlo simulation of the Hubbard model on the honeycomb lattice suggests that there is a fully gapped spin liquid phase [25] sandwiched between the ordinary Néel order and semi-metal phase, which has motivated spin liquid analysis on the honeycomb lattice using either slave boson or slave fermion techniques[22, 23]. In our formalism, phase $A_3$ in phase diagram Fig. 2 is a candidate of this gapped spin liquid. However, based on our analysis, phase $A_3$ is not directly adjacent to a pure Néel order, instead phase $A_3$ is adjacent to phase $A_2$ with both Néel order and nematic order. Starting with phase $A_2$, we need to go through one more transition which suppresses the nematic order, and enters the final Néel order in the large Hubbard $U$ limit.

The spin-2 nematic order is a natural candidate of the ground state of spin-1 systems, with bi-quadratic interactions [24]. For spin-1/2 model, nematic order can exist when there is a considerable ring exchange or multi-spin interaction, which can be generated in the weak Mott insulator phase of the simplest Hubbard model with high order perturbation of $t/U$. Our prediction of a phase with coexistence of nematic and Néel order can be checked numerically in future. We will present a general classification about nematic orders and their adjacent spin liquid phases in future [25].

A similar analysis can be applied to the $Z_2$ liquid phase obtained from the standard Schewinger boson formalism. Since the spinon $z_{\alpha}$ always couples to a $Z_2$ gauge field, the condensate of $z_{\alpha}^c$ is not an ordinary Néel order, because there always exists three perpendicular gauge invariant vectors like Eq. (10). In Ref.[24], the authors proposed that there is an intermediate chiral antiferromagnetic order between a fully gapped $Z_2$ liquid phase and a Néel order. This chiral AF state has GSM SO(3), which is different from the phase predicted in our paper with both nematic and Néel order.

IV. PHASE DIAGRAM AROUND TYPE B PHASE

Now let us move on the phase $B$ with GSM $SO(3) \otimes Z_2$. All the phases discussed in this section breaks time-reversal symmetry $T$, therefore we only focus on one of the two disconnected sub-manifolds $SO(3)$. Phases with GSM $SO(3)$ have been studied extensively with non-collinear spin density vector [26]. After disordering the state, both $SU(2)_{\text{spin}}$ and $SU(2)_{\text{charge}}$ are restored, while the vison of the SO(3) manifold is still locally conserved, and the system most naturally enters a $Z_2$ liquid phase.

Again, we hope to understand the phase diagram with the Majorana liquid formalism. Let us assume $\chi$ in the parent state Eq. (18) enters the type $B$ phase, for instance $\langle Q_{11} \rangle = \langle Q_{22} \rangle = \langle Q_{33} \rangle \neq 0$. It would be convenient to introduce the following CP(1) fields

\[
\begin{align*}
z^s &= (z_1^s, z_2^s)^t = (\phi_0 - i\phi_5, \phi_2 - i\phi_1)^t, \\
z^c &= (z_1^c, z_2^c)^t = (\phi_0 - i\phi_5, \phi_2 - i\phi_1)^t. 
\end{align*}
\] (35)

The phase diagram around type $B$ order is depicted in Fig. 3.

Phase $B$ in Fig. 3 with both $z_{\alpha}^s$ and $z_{\alpha}^c$ condensed is precisely the phase $B$ in the Ginzburg-Landau description in section [II] with GSM $SO(3) \otimes Z_2$. In phase $B_2$, the $SU(2)_{s,g}$ gauge symmetry is Higgsed by the condensation of $z_{\alpha}^s$, while the $SU(2)_{c,g}$ is broken down to $Z_2$.
There is one extra in the Lie Algebra of the gauge group, but it implies that either one of \( z \) becomes the SU(2) CS theory at level 2 in Eq. 36. Now \( \chi \) becomes the SU(2) CS theory at level 2 in Eq. 36. Now \( \chi \) requires \( A^a_{\mu} \) is Higgsed, but the system still has a \( Z_2 \) gauge symmetry, which characterizes the \( Z_2 \) liquid phase B2 and B4.

The transition (B2, B3) and transition (B3, B4) is a Higgs transition, described by spinon \( z^a_\alpha \) or \( z^c_\alpha \) coupled with SU(2) CS theory in Eq. 39. The universality class of these transitions is not understood yet.

V. SITUATION WITH SU(2)_{charge} BROKEN TO U(1)_{charge} \otimes Z_2

When the SU(2)_{charge} symmetry is broken down to U(1)_{charge} \otimes Z_2 symmetry which corresponds to charge conservation and particle-hole transformation, the degeneracy between \( Q_{3b} \) and \( Q_{1b}, Q_{2b} \) is lifted. For instance, if an extra repulsive next nearest neighbor density interaction (linear with \( \mathcal{S}_{zz} \) in Eq. 30) is turned on, the system favors to develop \( Q_{3b} \), i.e. the system only has QSH order with GSM \( S^2 \). Then according to Ref. 39, the Skyrmion of the QSH vector carries charge-2, and Skyrmion condensate is a s-wave superconductor.

If the system favors to have T-SC \( Q_{1b}, Q_{2b} \) rather than \( Q_{3b} \), then depending on the microscopic parameters the T-SC can have orders with either \( Q_{1b}, Q_{2b} \) (type A) or \( Q_{1b} \perp Q_{2b} \) (type B). The type A phase has fully gapped fermion spectrum, with GSM~ \( [S^1 \otimes S^1]/Z_2 \). Here \( S^2 \) corresponds to the spin direction of the triplet Cooper pair, while \( S^1 \) corresponds to the pairing phase angle. Again, there are spin and charge topological defects. For instance, the charge vortex (defect of the \( S^1 \) part of the GSM) carries spin-1/2 quantum number (quantum number of the residual U(1)_{spin} symmetry). The proliferation of the charge vortex leads to the phase A2 in phase diagram Fig. 2, and the transition is a CP(1) theory with easy plane anisotropy on charge CP(1) field \( z^c_\alpha \) introduced in Eq. 20, which is consistent with the conclusion in Ref. 39.

The proliferation of the spin-Skyrmion restores the SU(2)_{spin} symmetry, but the U(1)_{charge} symmetry is still broken. However, since the spin and charge sectors can change sign simultaneously without modifying the ground state, after the proliferation of the spin-Skyrmion there is still a \( Z_2 \) gauge symmetry for the charge manifold. Therefore the GSM of this phase is \( S^1 \otimes Z_2 \), which also has the GSM \( [S^2 \otimes S^1]/Z_2 \).

Unlike type A order, the type B phase with \( Q_{1b} \perp Q_{2b} \) does not have a fully gapped fermion spectrum. For instance, with \( \langle Q_{11} \rangle = \langle Q_{22} \rangle \neq 0 \), only spin-up is paired and gapped out, while spin-down is not gapped. Since the fermion spectrum is gapless, the quantum number of defects is no longer topologically stable.

FIG. 3: Phase diagram around type B phase with \( \langle Q_{11} \rangle = \langle Q_{22} \rangle = \langle Q_{33} \rangle \neq 0 \).
VI. SUMMARY AND DISCUSSION

In this work we have classified the QSH and T-SC states on the honeycomb lattice using the SO(4) symmetry for a large class of extended Hubbard models. By analyzing the quantum numbers of topological defects, we obtained two different phase diagrams, which were also confirmed by the Majorana liquid formalism. The results of this paper can be straightforwardly generalized to other bipartite lattice. Our formalism also predicts a phase with both spin nematic and Néel order, sandwiched between a fully gapped $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ liquid phase and the ordinary Néel order, which can be checked in future using the similar method as Ref. 13.

The results we obtained in this work explicitly demonstrates the spin-charge duality of the Hubbard model. For instance, in phase diagram Fig. 2, spin and charge view each other as topological defects. A similar spin-charge duality was applied to the cuprates high temperature superconductor wave superconductor. In our current work we showed that the generic symmetry of the Hubbard model and the condensate of matrix order parameter $Q$ in Eq. 4 give us a complete and explicit duality between spin and charge in interacting electrons.

In both Fig. 2 and Fig. 3 there is a multi-critical point with $r_s = r_c = 0$. The multi-critical point in Fig. 2 was analyzed in Ref. 15, and for large enough spinon number this multi-critical point is stable. Also, it has been proposed that a similar multi-critical point is responsible for the spin liquid behavior in material $\kappa - (ET)_2Cu_2(CN)_3$ on the triangular lattice. The multi-critical point in Fig. 3 is more complicated, we will leave this multi-critical point to future study.

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