Dynamics of Resonances for 0th Order Pseudodifferential Operators

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Abstract: We study the dynamics of resonances of analytic perturbations of 0th order pseudodifferential operators $P(s)$. In particular, we prove a Fermi golden rule for resonances of $P(s)$ at embedded eigenvalues of $P = P(0)$. We answer the question on the generic absence of eigenvalues asked by Colin de Verdière (Anal PDE 13:1521–1537, 2020). We also study the dynamics of eigenvalues of $P + it\Delta$ as the eigenvalues converge to simple eigenvalues of $P$. The 0th order pseudodifferential operators we consider satisfy natural dynamical assumptions and are used as microlocal models of internal waves.

1. Introduction

In this paper, we are interested in the dynamics of the resonances of 0th order pseudodifferential operators. 0th order pseudodifferential operators arise naturally in the study of fluid, in particular, the study of internal waves. We refer to [Ra73] for the early work.

Colin de Verdière and Saint-Raymond [CS20] used 0th order pseudodifferential operators with dynamical assumptions on 2 dimensional torus as microlocal model for internal waves. Colin de Verdière [CdV19] generalized the results to manifolds of higher dimensions. Dyatlov and Zworski [DyZw19b] provided alternative proofs of the results in [CS20] using tools from scattering theory. Wang [Wan19] studied the scattering matrix for these operators.

In the first part of this paper, we consider the perturbation theory for a 0th order pseudodifferential operator $P$. We consider the case when $P$ has embedded eigenvalues $\lambda$. If $P(s)$ is a family of 0th order operators with $P = P(0)$, under certain conditions, the resonances of $P(s)$ near $\lambda$ converge to $\lambda$. If $\lambda$ has multiplicity $m > 1$, we show the resonances of $P(s)$ allow expansions as Puiseux series. In the case when $\lambda$ is a simple eigenvalue of $P$, we propose and prove a Fermi golden rule – for references of Fermi golden rules, we refer to Simon [Si73] for $n$-body quantum systems; Colin de Verdière [CdV83] for the generic absence of embedded eigenvalues for surfaces with
variable curvature and cusps; Phillips and Sarnak \cite{PhSa92} for the Laplacian operator on automorphic functions; Lee and Zworski \cite{LeZw16} for quantum graphs; \cite[Theorem 4.22]{DyZw19a} for a textbook style presentation of Fermi golden rule for black box scattering.

We are also interested in embedded eigenvalues \( \lambda \) as limits of eigenvalues \( \lambda(t) \) of \( P + it \Delta \). Galkowski and Zworski \cite{GaZw19} defined the set of resonances of \( P \) and showed the resonances can be approximated by the eigenvalues of \( P + it \Delta \). In the case when \( \lambda \) is simple, we compute the first derivative of \( \lambda(t) \) at 0.

1.1. Main results. Let \( \mathbb{T}^n = \mathbb{R}^n / (2\pi \mathbb{Z})^n \) be the torus and \( P(s) \in \Psi^0(\mathbb{T}^n) \) be a family of 0th order self-adjoint pseudodifferential operator on \( \mathbb{T}^n \) defined by

\[
P(s)u(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-x',\xi)} P(s, x, \xi) u(x') dx' d\xi.
\]  

Here \( s \in (-s_0, s_0) \) for some \( s_0 > 0 \) and \( P(s) \in S^0(T^*\mathbb{T}^n) \) is called the full symbol of \( P(s) \). In the integral (1.1) we view \( P(s) \) as a 2\( \pi \) periodic function in \( x \) over \( \mathbb{R}^n \) and the integral is considered in the sense of oscillatory integrals (see \cite[\S 5.3]{Zw12}). We assume

\[
P(s, x, \xi) \text{ has a holomorphic continuation from } (-s_0, s_0) \times T^*\mathbb{T}^n \text{ such that } |P(s, x, \xi)| \leq M \text{ for } |\text{Im}s| < a_0, \quad |\text{Im}x| \leq a_1, \quad |\text{Im}\xi| \leq a_2(\text{Re}\xi),
\]

with \( a_0, a_1, a_2 > 0 \) and some \( M > 0 \). We assume that

\[
P(s) = p(s) + P_{-1}(s) \text{ for } |\xi| > C \text{ where } P_{-1} \in S^{-1}(T^*\mathbb{T}^n) \text{ and } p(s) \in C^\infty(T^*\mathbb{T}^n \setminus 0; \mathbb{R}) \text{ is homogeneous of order 0}.
\]

We call \( p(s) \) the principal symbol of \( P(s) \).

We assume that

\[
\text{there exists } F_0 \in S^1(T^*\mathbb{T}^n; \mathbb{R}) \text{ that is homogeneous of order 1 and } H_{p(s)} F_0(x, \xi) > 0, \text{ for } (x, \xi) \in \{ p(s, x, \xi) = 0 \}, \quad |s| < s_0.
\]

Here \( H_{p(s)} \) is the Hamiltonian vector field of \( p(s) \) defined by

\[
H_{p(s)} := \partial_\xi p(s) \cdot \partial_x - \partial_x p(s) \cdot \partial_\xi.
\]

In this paper, we consider \( P(s) \) as perturbations of \( P = P(0) \). The eigenvalues of \( P \) can be approximated by the resonances of \( P(s) \) as \( s \) goes to 0—see Sect. 2.2 for the definition of resonances. More precisely, we have the following

**Theorem 1.** Assume \( P(s) \in \Psi^0(\mathbb{T}^n) \) are self-adjoint operators satisfying (1.1)–(1.4) and \( \zeta_0, \epsilon, \Omega_{\zeta_0, \epsilon} \) are as in Sect. 2.2. Suppose \( \lambda \in (-\zeta_0, \zeta_0) \) and \( \text{Ker}_{L^2}(P(0) - \lambda) \) has an \( L^2 \) orthonormal basis \( \{u_\ell\}_{1 \leq \ell \leq m} \), \( m \geq 1 \). Then there exists

\[
s_1 \in (0, s_0) \text{ and } \lambda_\ell(\bullet) \in C^\infty((-s_1, s_1) \setminus \{0\}; \Omega_{\zeta_0, \epsilon}), \quad \lim_{s \to 0} \lambda_\ell(s) = \lambda, \quad 1 \leq \ell \leq m,
\]

(1.5)
such that \( \lambda_\ell(s) \) are the only resonances of \( P(s) \) near \( \lambda \). Moreover, \( \lambda_\ell(\bullet) \) are analytic and have Puiseux series expansions (see Sect. 2.4). If

\[
\lambda_\ell(s) = \sum_{k=0}^{+\infty} c_k (e^{2i\pi i s})^{k/q}, \quad q \geq 1, \quad 1 \leq \ell \leq q
\]  

(1.6)
is a Puiseux cycle, then either \( q = 1 \) and \( c_k \in \mathbb{R} \) for all \( k \), or there exists \( J \in \mathbb{N} \) such that when \( 0 \leq k \leq 2Jq \), \( c_k \in \mathbb{R} \); when \( 1 \leq k \leq 2Jq - 1 \), \( k/q \not\in \mathbb{Z} \), \( c_k = 0 \); \( \text{Im} c_{2Jq} < 0 \). If we denote matrices

\[
\mathcal{M} := \left( \langle \dot{P}u_i, u_j \rangle \right)_{1 \leq i,j \leq m}, \quad \mathcal{N} := \left( \frac{1}{2} \langle \ddot{P}u_i, u_j \rangle - \langle \Pi_\perp R(\lambda) \Pi_\perp \dot{P}u_i, \dot{P}u_j \rangle \right)_{1 \leq i,j \leq m},
\]

(1.7)

where \( \Pi_\perp R(\lambda) \Pi_\perp \) is defined in Sect. 2.2, then there exists \( \rho(s) = o(s) \), such that

\[
det(c_q I_m - \mathcal{M} + s(c_{2q} I_m - \mathcal{N}) + \rho(s)) = 0, \quad s \in (-s_1, s_1).
\]

(1.8)

Remarks. 1. In Sect. 3, we show that \( u_\ell \) are analytic. By (1.1) and (1.2), \( \dot{P}u_\ell \) are also analytic. Since \( \Pi_\perp R(\lambda) \Pi_\perp \dot{P}u_\ell \in H_\delta \subset \mathcal{I}_{-\delta} \) (see Sect. 2.1) for sufficiently small \( \delta > 0 \), we know (1.7) is well-defined. 2. Let \( s \to 0 \) in (1.8), and we find \( c_q \) is an eigenvalue of \( \mathcal{M} \). Let \( m_q \in \mathbb{N} \) and \( U \) be a unitary matrix such that

\[
U^* \mathcal{M} U = \begin{pmatrix} c_q I_{m_q} & 0 \\ 0 & D_{m_q} \end{pmatrix}, \quad D_{m_q} \text{ is diagonal and the diagonal entries are not } c_q.
\]

where \( U^* = \overline{U}^T \). Then by (1.8) we have

\[
\lim_{s \to 0} s^{-m_q} \det(c_q I_m - U^* \mathcal{M} U + s(c_{2q} I_m - U^* \mathcal{N} U) + r(s)) = 0,
\]

(1.9)
hence \( c_{2q} \) is an eigenvalue of \((U^* \mathcal{N} U)_{i,j}\) where \( (U^* \mathcal{N} U)_{i,j} \) is the \((i, j)\)-th entry of \( U^* \mathcal{N} U \).

3. If \( m = 1 \), that is, \( \lambda \) is a simple eigenvalue of \( P \), we put \( u = u_1, \lambda(s) = \lambda_1(s) \). Then by (1.8), in (1.6) we have \( q = 1 \) and

\[
c_1 = \langle \dot{P}u, u \rangle, \quad c_2 = \frac{1}{2} \langle \ddot{P}u, u \rangle - \langle \Pi_\perp R(\lambda) \Pi_\perp \dot{P}u, \dot{P}u \rangle.
\]

(1.10)

Therefore

\[
\text{Im} \ddot{\lambda} = -2\text{Im} \langle \Pi_\perp R(\lambda) \Pi_\perp \dot{P}u, \dot{P}u \rangle.
\]

(1.11)

This is called the Fermi golden rule for \( P(s) \). In this case, we can merely assume \( \overline{P(s)} \) is smooth in \( s \) and then Theorem 1, in particular (1.11), holds, except that \( \lambda(\bullet) \) is only smooth in \( s \). An independent proof for the smooth case is provided in the proof of Theorem 1.

In [CdV19], Colin de Verdière asked the question on the generic absence of embedded eigenvalues. We answer this question in the setting of perturbation theory.

For simplicity, we only consider operators on \( \mathbb{T}^2 \) in this part. Let \( \mathcal{O} \subset \mathbb{R} \) be a bounded open interval. Assume \( P \in \Psi^0(\mathbb{T}^2) \) satisfies (1.1)–(1.3) with no dependence on \( s \). For any \( \lambda \in \mathcal{O} \), we denote \( Z_\lambda := p^{-1}(\lambda) \subset T^*\mathbb{T}^2 \setminus 0 \) and assume

\[
dp(x, \xi) \neq 0, \quad (x, \xi) \in Z_\lambda.
\]

(1.12)
(1.12) implies that $Z_\lambda$ is a smooth conic hypersurface of $T^*\mathbb{T}^2 \setminus 0$. The boundary of $Z_\lambda$ is defined by $\Sigma_\lambda := Z_\lambda/\mathbb{R}_+$, where the action of $\mathbb{R}_+$ on $T^*\mathbb{T}^2 \setminus 0$ is given by

$$\mathbb{R}_+ \times (T^*\mathbb{T}^2 \setminus 0) \ni (r, x, \xi) \mapsto (x, r\xi) \in T^*\mathbb{T}^2 \setminus 0.$$  \hspace{1cm} (1.13)

Let $\pi : T^*\mathbb{T}^2 \setminus 0 \to (T^*\mathbb{T}^2 \setminus 0)/\mathbb{R}_+$ be the natural quotient map. Then $V := \pi_*(|\xi|H_0)$ defines a smooth vector field on $\Sigma_\lambda$. We say

$$\exp tV$$

is a Morse-Smale flow on $\Sigma_\lambda$ with no fixed points, \hspace{1cm} (1.14)

if the following two conditions hold (see for instance [NZ99, Definition 5.1.1])

(i) $\exp tV$ has a finite number of hyperbolic limit cycles on $\Sigma_\lambda$;
(ii) every trajectory of $\exp tV$ on $\Sigma_\lambda$ that is not in (i), has unique limit cycles in (i) as its $\alpha, \omega$-limit sets.

Let $\mathcal{A}_O \subset \Psi^0(\mathbb{T}^2)$ such that $\mathcal{A}_O$ consists of self-adjoint operators satisfying (1.1)–(1.3) and (1.12), (1.14) for all $\lambda \in O$.

**Remark.** Notice that under the condition (1.12), [CdV19, Theorem 6.2] shows that (1.14) implies (1.4) (with no $s$ dependence). However, condition (1.4) does not imply (1.14) as we can not rule out fixed points of the flow by just assuming (1.14). Assumption (1.14) is required in our proof of Theorem 2, since we used [Wan19, Proposition 6.5] in the proof of Lemma 5.1. It is not clear if we can relax (1.14) to (1.4) in Theorem 2.

Let $\mathcal{W}$ be a subset of $S^0(T^*\mathbb{T}^2)$ that consists of symbols $w$ that satisfy (1.2) with no dependence on $s$ and $w(x, D) \in \Psi^0(\mathbb{T}^2)$ are self-adjoint operators. Here the quantization $w(x, D)$ is given by (1.1) with $P(s)$ replaced by $w$. We equip $\mathcal{W}$ with the following metric:

$$|w|_\mathcal{W} := \sup_{|\text{Im}x| \leq a_1, |\text{Im}\xi| \leq a_2} |w(x, \xi)|.$$  \hspace{1cm} (1.15)

By [GaZw19, Proposition 6.2], [DyZw19a, Proposition E.19] and Cauchy estimates for analytic functions, we know that there exists $C > 0$ such that

$$\|w(x, D)\|_{H_\Lambda \to H_\Lambda} \leq C|w|_\mathcal{W}.$$  \hspace{1cm} (1.16)

One can check that $\mathcal{W}$ is a complete metric space, hence the intersection of countably many open dense subsets of $\mathcal{W}$ is still a dense set.

We now state the theorem for absence of embedded eigenvalues for generic perturbations.

**Theorem 2.** If $P \in \mathcal{A}_O$, then there exist countably many open dense subsets $\mathcal{U}_\alpha, \alpha \in J$, of $\mathcal{W}$ such that for any $w \in \bigcap_{\alpha \in J} \mathcal{U}_\alpha$, any compact subset $K \subset O$, there exists $\delta = \delta(w, K)$, such that

$$\text{Spec}_{pp}(P + sw(x, D)) \cap K = \emptyset, \ s \in (-\delta, \delta) \setminus \{0\}.$$  \hspace{1cm} (1.17)

Finally we consider the viscosity limits. We are interested in the first derivatives of the viscosity limits and we show the following
Theorem 3. Suppose $P \in \Psi^0(\mathbb{T}^n)$ is self-adjoint and satisfies (1.1)–(1.4) with no dependence on $s$. Let $\zeta_0, \epsilon, \Omega_{\zeta_0, \epsilon}$ be as in Sect. 2.2. Assume $\lambda \in (-\zeta_0, \zeta_0)$ is a simple eigenvalue of $P$ with $L^2$ normalized eigenfunction $u \in L^2(\mathbb{T}^n)$. Then there exists

$$t_0 > 0, \quad \lambda(\bullet) \in C^\infty((0, t_0); \Omega_{\zeta_0, \epsilon}), \quad \lim_{t \to 0^+} \lambda(t) = \lambda$$

(1.18)

such that $\lambda(t) \in \text{Spec}_{pp, L^2}(P + it\Delta_{\mathbb{T}^n})$ and

$$\dot{\lambda} = -i \|\nabla u\|_{L^2(\mathbb{T}^n)}^2$$

(1.19)

where $\dot{\lambda} = \dot{\lambda}(0)$.

1.2. Examples. Similar to [Ta19, Example 1], [GaZw19, (B.4)], we consider operators of the following form on $\mathbb{T}^2$:

$$P = \langle D \rangle^{-1} D_{x_2} + \sin(x_1) (I - V_m(D)) + (I - V_m(D)) \sin(x_1) + V_a(D)$$

(1.20)

with $V_m, V_a$ analytic and for some $b, c > 0$,

$$\max \{ |V_m(\xi)|, |V_a(\xi)| \} \leq Ce^{-c|\Re \xi|^2}, \quad |\Im \xi| < b(|\Re \xi|).$$

(1.21)

The principal symbol of $P$ and its corresponding Hamiltonian vector field are

$$p(x, \xi) = \frac{\xi_2}{|\xi|} + 2 \sin(x_1), \quad H_p = -\frac{\xi_1 \xi_2}{|\xi|^2} \partial_{x_1} + \frac{\xi_2^2}{|\xi|^2} \partial_{x_2} - 2 \cos(x_1) \partial_{\xi_1}.$$

Now we put

$$F_0(x, \xi) := -\xi_1 \cos(x_1) \in T^1(T^*\mathbb{T}^2),$$

then $F_0$ is homogeneous of order 1 and

$$H_p F_0(x, \xi) = 2 - \frac{\xi_2^4}{2|\xi|^4} > 0, \quad \text{for } (x, \xi) \in \{ p = 0 \}.$$

Moreover, let $\mathcal{O} = (-1, 1)$. For all $\lambda \in \mathcal{O}$, (1.12), (1.14) are satisfied, and $\Sigma_\lambda$ is a disjoint union of two tori that do not cover $\mathbb{T}^2$. The endpoints $\{ \pm 1 \}$ of $\mathcal{O}$ are not regular points of $p$:

$$\text{for } \xi_2 < 0, \quad (\frac{\pi}{2}, x_2, 0, \xi_2) \in p^{-1}(1), \quad dp(\frac{\pi}{2}, x_2, 0, \xi_2) = 0;$$

$$\text{for } \xi_2 > 0, \quad (\frac{3\pi}{2}, x_2, 0, \xi_2) \in p^{-1}(-1), \quad dp(\frac{3\pi}{2}, x_2, 0, \xi_2) = 0.$$  

(1.22)

Lemma 1.1. Let $P$ be as in (1.20) with $V_\bullet$ satisfy (1.21) and $V_\bullet(k) = 0$ when $k \in \mathbb{Z}^2$, $k_2 \neq 0$, where $\bullet = m, a$. Suppose there exist $N_+ \geq 1, N_- \leq -1$ such that

$$\max \left\{ \left| \frac{2V_m(k_1, 0)}{2 - V_m(k_1, 0) - V_m(k_2 + 1, 0)} \right|, 1 - \left| \frac{2V_m(k_1, 0) - V_m(k_1 + 1, 0)}{2V_m(k_1, 0) - V_m(k_1, 0) - V_m(k_1 + 1, 0)} \right| \right\} \leq \frac{1}{|k_1|}, \quad k_1 \geq N_+ + 1,$$

$$\max \left\{ \left| \frac{2V_m(k_1, 0)}{2 - V_m(k_1 - 1, 0) - V_m(k_2, 0)} \right|, 1 - \left| \frac{2V_m(k_1, 0) - V_m(k_1 - 1, 0)}{2V_m(k_1, 0) - V_m(k_1 - 1, 0) - V_m(k_2, 0)} \right| \right\} \leq \frac{1}{|k_1|}, \quad k_1 \leq N_- - 1.$$  

(1.23)

Assume $u(x) = \sum_{k \in \mathbb{Z}^2} u_k e^{ik \cdot x}$ is analytic and $P u(x) = 0$, then $u_k = 0$ when $k \notin \{ N_- < k_1 < N_+, k_2 = 0 \}$. 
Proof. We first note that
\[ P(e^{ik \cdot x}) = \left( (k)^{-1} k_2 + V_a(k) \right) e^{ik \cdot x} + \sum_{\pm} \frac{\pm V_m(k \pm e_1)}{2i} e^{i(k \pm e_1) \cdot x} \]  \hspace{1cm} (1.24)
with \( e_1 = (1, 0) \in \mathbb{Z}^2 \). Hence \( Pu(x) = 0 \) implies that for any \( k \in \mathbb{Z}^2 \),
\[ \left( (k)^{-1} k_2 + V_a(k) \right) u_k + \sum_{\pm} \frac{\pm V_m(k \mp e_1)}{2i} u_{k \mp e_1} = 0. \]  \hspace{1cm} (1.25)

When \( k_2 \neq 0 \), we have
\[ u_{k-e_1} + i (k)^{-1} k_2 u_k - u_{k+e_1} = 0. \]  \hspace{1cm} (1.26)
This implies that
\[ |u_{k+e_1}| + |u_k| \geq \left( 1 - (k)^{-1} |k_2| \right) \left( |u_k| + |u_{k-e_1}| \right), \quad k_1 \in \mathbb{Z}, \quad k_2 \neq 0. \]  \hspace{1cm} (1.27)

Now use the assumption that \( u \) is analytic and we find \( u_k = 0 \) when \( k_2 \neq 0 \).

When \( k_2 = 0 \), by (1.23), we have
\[ |u_{k+e_1}| + |u_k| \geq \left( 1 - \frac{1}{|k_1|} \right) \left( |u_k| + |u_{k-e_1}| \right), \quad k_1 \geq N_+ + 1, \]
\[ |u_{k-e_1}| + |u_k| \geq \left( 1 - \frac{1}{|k_1|} \right) \left( |u_k| + |u_{k+e_1}| \right), \quad k_1 \leq N_- - 1. \]  \hspace{1cm} (1.28)

Use \( u \) is analytic again and we find \( u_{k_1,0} = 0 \) when \( k_1 \leq N_- \) or \( k_1 \geq N_+ \). \( \Box \)

Lemma 1.2. Let \( P, N_\pm \) be as in Lemma 1.1 and \( w(x) = \sum_{\ell \in \mathbb{Z}} w(0) e^{i \ell x_1} \perp \text{Ker}_{L^2}(P) \). Suppose there exists \( K \in \mathbb{N} \) such that \( w(0) = 0 \) when \( |\ell| > K \). For \( 0 < \epsilon \ll 1, N \geq \max\{K, |N_\pm|\} \), assume \( v_{\epsilon,N}(x), \{c_\ell(\epsilon, N)\} \) satisfy
\[ v_{\epsilon,N}(x) = \sum_{\ell \in \mathbb{Z}} c_\ell(\epsilon, N) e^{i \ell x_1}; \quad c_\ell = \left( \frac{\epsilon + \sqrt{\epsilon^2 + 4}}{2} \right) \ell \pm N \quad c_{\pm N}, \quad \pm \ell \geq N, \]
\[ (V_a(\ell, 0) - i \epsilon) c_\ell + \sum_{\pm} \frac{\pm V_m(\ell \pm 1,0) - V_m(\ell,0)}{2i} c_{\ell \mp 1} = w(0), \quad |\ell| \leq N. \]  \hspace{1cm} (1.29)
Then for any \( \alpha > 0 \), we have
\[ \lim_{\beta \to (0,0)} \| v_{\epsilon,N} - \Pi_{\perp} R(0) \Pi_{\perp} w \|_{H^{-\beta}} = 0. \]  \hspace{1cm} (1.30)

Remark. The limit (1.30) implies that if \( \Pi_{\perp} R(0) \Pi_{\perp} w = \sum_{\ell \in \mathbb{Z}} a_\ell e^{i \ell x_1} \), then for any fixed \( \ell \), \( \lim_{\beta \to (0,0)} c_\ell(\epsilon, N) = a_\ell \).

Proof. By (1.24) and the definition of \( c_\ell \), we know
\[ (P - i \epsilon)v_{\epsilon,N}(x) = w(x) + r_{\epsilon,N}(x), \]
\[ r_{\epsilon,N}(x) := \sum_{|\ell| \geq N+1} \left( V_a(\ell, 0) c_\ell + \sum_{\pm} \frac{\pm V_m(\ell \pm 1,0) - V_m(\ell,0)}{2i} c_{\ell \pm 1} \right) e^{i \ell x_1}. \]  \hspace{1cm} (1.31)
By Lemma 1.1, we know \( r_{\epsilon,N} \perp \text{Ker}_{L^2}(P) \), which further implies \( v_{\epsilon,N} \perp \text{Ker}_{L^2}(P) \).
We also remark that both \( r_{\epsilon,N} \) and \( v_{\epsilon,N} \) are analytic.
We first notice that by (1.21) and the second identity in (1.31), for any \( \alpha > 0 \), there exists \( C_\alpha \) such that for \( N \gg 1, \)
\[
\| r_{\epsilon,N} \|_{H^{\frac{1}{2}+\alpha}} \leq C_{\alpha} e^{-\frac{\epsilon^2}{2}N^2} \| v_{\epsilon,N} \|_{L^{\frac{1}{2}-\alpha}}. \tag{1.32}
\]

We now show that for any \( \alpha > 0 \), \( \| v_{\epsilon,N} \|_{H^{\frac{1}{2}-\alpha}} \) is bounded uniformly in \( \epsilon, N \). Assume this is not true, then there exists \( \alpha > 0 \) such that we can find \( \epsilon_j \to 0^+, N_j \to \infty \), such that \( \| v_{\epsilon_j,N_j} \|_{H^{\frac{1}{2}-\alpha}} \to \infty \). We denote
\[
L_j := \| v_{\epsilon_j,N_j} \|_{H^{\frac{1}{2}-\alpha}}, \quad \tilde{v}_j := v_{\epsilon_j,N_j}/L_j, \quad \tilde{w}_j := w/L_j, \quad \tilde{r}_j := r_{\epsilon_j,N_j}/L_j. \tag{1.33}
\]
Then we have
\[
\| \tilde{v}_j \|_{H^{\frac{1}{2}-\alpha}} = 1, \quad \tilde{w}_j \xrightarrow{C^0} 0, \quad \| \tilde{r}_j \|_{H^{\frac{1}{2}+\alpha}} \leq C_{\alpha} e^{-\frac{\epsilon_j^2}{2}N_j^2} \| \tilde{v}_j \|_{H^{\frac{1}{2}-\alpha}} \to 0. \tag{1.34}
\]
Apply [DyZw19b, (3.5)] (with \( (\beta, N) \) there being replaced by \( (\frac{\alpha}{2}, \frac{1}{2}+\alpha) \)) to the equation
\[
(P - i\epsilon_j)\tilde{v}_j = \tilde{w}_j + \tilde{r}_j \tag{1.35}
\]
and we find
\[
\| \tilde{v}_j \|_{H^{-\frac{1}{2}+\frac{\alpha}{2}}} \leq C \| \tilde{w}_j + \tilde{r}_j \|_{H^{\frac{1}{2}+\frac{\alpha}{2}}} + C \| \tilde{v}_j \|_{H^{\frac{1}{2}-\alpha}} \tag{1.36}
\]
where \( C \) only depends on \( \alpha \). This shows that \( \tilde{v}_j \) is bounded in \( H^{-\frac{1}{2}+\frac{\alpha}{2}} \). Since the embedding \( H^{-\frac{1}{2}+\frac{\alpha}{2}} \hookrightarrow H^{-\frac{1}{2}-\alpha} \) is compact, by passing to a subsequence, we can assume that \( \tilde{v}_j \to \tilde{v} \) in \( H^{-\frac{1}{2}-\alpha} \). Send \( j \to \infty \) in (1.35) and we find \( P\tilde{v} = 0 \). Now apply [DyZw19b, (3.6)] to (1.35) and we see that \( \text{WF}(\tilde{v}) \subset \Gamma^+_0 \) (see Sect. 2.2 for the definition of \( \Gamma^+_0 \)). Use [DyZw19b, Lemma 3.1] and we conclude that \( \tilde{v} \in C^\infty \). In particular, this means \( \tilde{v} \in \text{Ker}_{L^2}(P) \). On the other hand, since \( \tilde{v}_j \perp \text{Ker}_{L^2}(P) \), we also have \( \tilde{v} \perp \text{Ker}_{L^2}(P) \). Therefore we must have \( \tilde{v} = 0 \). This contradicts \( \| \tilde{v}_j \|_{H^{-\frac{1}{2}-\alpha}} = 1 \).

The uniform boundedness in \( \epsilon, N \) of \( \| v_{\epsilon,N} \|_{H^{\frac{1}{2}-\alpha}} \) for all \( \alpha > 0 \) has two consequences: (i) By (1.32), we know \( \| r_{\epsilon,N} \|_{H^{\frac{1}{2}+\alpha}} \to 0 \), as \( (\epsilon, 1/N) \to (0, 0) \); (ii) Use the compact embedding \( H^{-\frac{1}{2}+\frac{\alpha}{2}} \hookrightarrow H^{-\frac{1}{2}-\alpha} \) again and we see that for all \( \alpha > 0 \), \( v_{\epsilon,N} \) is precompact in \( H^{-\frac{1}{2}-\alpha} \) as \( (\epsilon, 1/N) \to (0, 0) \). By the first equation in (1.31) and (i) we know every limit point of \( v_{\epsilon,N} \) in \( H^{-\frac{1}{2}-\alpha} \) has to be the solution to
\[
P v = w, \quad \text{WF}(v) \subset \Gamma^+_0, \quad v \perp \text{Ker}_{L^2}(P). \tag{1.37}
\]
By [DyZw19b, Lemma 3.1] and Lemma 2.1, this equation has a unique solution \( v = \Pi_{\perp} R(0) \Pi_{\perp} w \). Hence we know that for any \( \alpha > 0 \),
\[
\lim_{(\epsilon, 1/N) \to (0, 0)} \| v_{\epsilon,N} - \Pi_{\perp} R(0) \Pi_{\perp} w \|_{H^{-\frac{1}{2}-\alpha}} = 0. \tag{1.38}
\]
This concludes the proof. \( \square \)
Fig. 1. Resonances of $P(s)$ where $P(s)$ is of the form (1.20) with $V_m, V_a$ given by (1.39), (1.40). In this case, $0$ is a simple eigenvalue of $P$ with eigenfunction $e^{ix_1}/2\pi$. $a\lambda(\pm)$ denotes the resonances of $P(s)$ with $\pm s > 0$, $-4.5 < s < 5.5$. $b\text{Im}\lambda(s)$ with $s$ small. The dashed line is the theoretic approximation of $\text{Im}\lambda(s)$, see (1.43)

Example 1. (Simple eigenvalues). We first put

$$V_{m,1}(\xi_1) = 2\sin^2\left(\frac{\pi \xi_1}{2}\right)e^{-(\xi_1 - 1)^2}, \quad V_{a,1}(\xi_1) = 5(\xi_1 - 1)e^{-\xi_1^2},$$

$$(1.39)$$

$$V_\bullet,2(\xi_2) = \frac{\sin\pi \xi_2}{\pi \xi_2}e^{-\xi_2^2}, \quad V_\bullet(\xi) = V_\bullet,1(\xi_1)V_\bullet,2(\xi_2), \quad \bullet = m, a.$$ 

$V_\bullet$ are analytic and (1.21) is satisfied. Moreover, $V_\bullet$ satisfies conditions in Lemma 1.1 with $N_+ = 2, N_- = -1$. By Proposition 3.1 and Lemma 1.1, $0$ is an simple eigenvalue of $P$ with eigenfunction $u(x) = e^{ix_1}/2\pi$.

We now consider the following perturbation of $V_m$:

$$V_{m,1}(s, \xi_1) := V_{m,1}(\xi_1) + se^{-(\xi_1 - 1)^2}, \quad V_{m}(s, \xi) := V_{m,1}(s, \xi_1)V_{m,2}(\xi_2).$$

$$(1.40)$$

Figure 1 shows a numerical illustration of the resonances of $P(s)$ near the eigenvalue $0$. The numerical computation is based on the method of complex scaling, see [GaZw19, Appendix B].

To compute $\tilde{\lambda}$ in Theorem 1, we notice that

$$\dot{P}(u) = \frac{1+e^{-\frac{1}{4\pi\epsilon}}}{4\pi\epsilon} \left(1 - e^{2ix_1}\right).$$

$$(1.41)$$

Assume $\Pi_\perp R(0)\Pi_\perp \dot{P} u = \sum_{\ell \in \mathbb{Z}} a_\ell e^{i\ell x_1}, a_\ell \in \mathbb{C}$. We can use Lemma 1.2 to find approximate values of $a_\ell$: let $c_\ell(\epsilon, N)$ be as in Lemma 1.2 with $w = \dot{P} u, N = 3, \epsilon \to 0+$, then we find

$$a_0 \approx c_0(0+, 3) = 0.0003 + 0.0230i, \quad a_2 \approx c_2(0+, 3) = -0.1100 + 0.0101i.$$ 

$$(1.42)$$

As a result, we find

$$\text{Im}\tilde{\lambda} = -2\text{Im}\langle \Pi_\perp R(0)\Pi_\perp \dot{P} u, \dot{P} u \rangle_{L^2} = 2\pi(1 + e^{-1})\Re((a_2 - a_0) \approx -0.9479).$$
**Dynamics of Resonances**

(a) Resonances $\lambda(s)$ of $P(s)$ near 0

(b) $\text{Im}\lambda(s)$

Fig. 2. Resonances of $P(s)$ near $\lambda = 0$, where $P(s)$ is of the form (1.20) with $V_m, V_a$ given by (1.44), (1.45). In this example, 0 is an eigenvalue of $P$ with multiplicity 2 and eigenfunctions $e^{\pm i x_1/2 \pi}$. \( \lambda_{1,2}(\pm) \) denotes resonances of $P(s)$ with $\pm s > 0, -5 < s < 9$. b $\text{Im}\lambda_{1,2}(s)$ with small $s$. Dashed lines are the theoretic approximation of $\text{Im}\lambda_{1,2}(s)$, see (1.50)

Therefore

$$\text{Im}\lambda = \frac{1}{2} (\text{Im}\lambda') s^2 + o(s^2) \approx -0.4739 s^2 + o(s^2). \quad (1.43)$$

The results are shown in Fig. 1.

**Example 2. (Eigenvalues with multiplicities).** Now we consider

$$V_{m,1}(\xi_1) = 2 \sin^2\left(\frac{\pi \xi_1}{2}\right)e^{-((\xi_1)^2-1)^2}, \quad V_{a,1}(\xi_1) = 6(1-\xi_1^2)e^{-(\xi_1-1)^2},$$

$$V_{*,2}(s) \text{, as in (1.39), \quad V}_*(\xi) = V_{*,1}(\xi_1)V_{*,2}(\xi_2), \quad \bullet = m, a. \quad (1.44)$$

Similar to Example 1, one can use Proposition 3.1 and Lemma 1.1 to check that $\text{Ker}_{L_2} (P)$ has an orthonormal basis \( \{u_1(x) = e^{-ix_1/2\pi}, u_2(x) = e^{ix_1/2\pi}\}. \) Let $V_{m}(s, \bullet)$ be a perturbation of $V_{m}$ as follows:

$$V_{m,1}(s, \xi_1) = V_{m,1}(\xi_1) + s \left(e^{-(\xi_1+4/5)^2} + \frac{4}{5}e^{-(\xi_1-4/5)^2}\right),$$

$$V_{m}(s, \xi) := V_{m,1}(s, \xi_1)V_{m,2}(\xi_2). \quad (1.45)$$

Figure 2 shows a numerical result of the resonances of $P(s)$ near 0. Now we compute the first two coefficients of $\lambda_{1,2}(s)$:

$$\lambda_1(s) = c_{11}s + c_{21}s^2 + o(s^2), \quad \lambda_2(s) = c_{12}s + c_{22}s^2 + o(s^2). \quad (1.46)$$

In fact, in this example,

$$M = 0, \quad N = -\left(\langle \Pi_{\perp} R(0)\Pi_{\perp} \hat{P}u_1, \hat{P}u_1 \rangle \langle \Pi_{\perp} R(0)\Pi_{\perp} \hat{P}u_2, \hat{P}u_2 \rangle - \langle \Pi_{\perp} R(0)\Pi_{\perp} \hat{P}u_2, \hat{P}u_1 \rangle \langle \Pi_{\perp} R(0)\Pi_{\perp} \hat{P}u_1, \hat{P}u_2 \rangle \right). \quad (1.47)$$

Therefore $c_{11} = c_{12} = 0$, and $c_{21}, c_{22}$ are two eigenvalues of $N$. 
Now we find an approximation of $\mathcal{N}$. Assume
\[
\Pi_\perp R(0)\Pi_\perp \hat{P}u_1 = \sum a_\ell e^{i\ell x_1}, \quad \Pi_\perp R(0)\Pi_\perp \hat{P}u_2 = \sum b_\ell e^{i\ell x_1}, \quad a_\ell, b_\ell \in \mathbb{C}.
\] (1.48)

Apply Lemma 1.2 to $w = \hat{P}u_1$, $w = \hat{P}u_2$ with $N = 2$, $\epsilon \to 0^+$ and approximate $a_\ell, b_\ell$ using corresponding $c_\ell(\epsilon, N)$. We then have
\[
a_{-2} \approx 0.6147 + 0.0014i, \quad a_0 \approx 0.4397i, \quad b_0 \approx 0.3980i, \quad b_2 \approx -0.0111 - 0.0737i,
\]
\[
\mathcal{N} \approx \begin{pmatrix} -0.4260 - 0.3778i & 0.3863 \\ 0.3863 & -0.3129 - 0.0055i \end{pmatrix}.
\] (1.49)

Therefore
\[
\text{Im}\lambda_1 \approx -0.1611s^2 + o(s^2), \quad \text{Im}\lambda_2 \approx -0.2222s^2 + o(s^2).
\] (1.50)

The results are shown in Fig. 2.

Example 3. (Simple eigenvalues and operators with viscosity). Let $P$ be as in Example 1. If we add the viscosity to $P$ and consider $P_{0,t} := P + it\Delta_{\mathbb{R}^2}$, then
\[
P_{0,t}(e^{ix_1}) = it\Delta_{\mathbb{R}^2}(e^{ix_1}) = -ite^{ix_1}.
\] (1.51)

Therefore $\lambda(t) = -it$ is the eigenvalue of $P_{0,t}$ near 0. Hence $\dot{\lambda} = -i$. This justifies the formula in Theorem 3. As a less trivial example, we consider
\[
V_{m,1}(\xi_1) = (\xi_1 + 3)(\xi_1 - 2) \left( \frac{1}{2}\xi_1^3 + \frac{7}{12}\xi_1^2 - \frac{11}{12}\xi_1 - \frac{2}{3} \right) e^{-\xi_1(\xi_1+3)(\xi_1^2-1)(\xi_1^2-4)},
\]
\[
V_{a,1}(\xi_1) = \left( -\frac{2}{3}\xi_1^3 - \xi_1^2 + \frac{5}{3}\xi_1 + 3 \right) e^{-\xi_1(\xi_1+3)(\xi_1+2)},
\] (1.52)

and $V_{\bullet,2}, V_{\bullet, \bullet} = m, a$ are as in (1.39). Then $P$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$ with eigenfunctions $u_1(x) = \frac{1}{2\sqrt{2\pi}} \left( ie^{-ix_1} + e^{-2ix_1} \right)$, $u_2(x) = \frac{1}{2\sqrt{2\pi}} \left( i + e^{ix_1} \right)$. Thus
\[
\dot{\lambda}_1 = -i\|\nabla u_1\|^2_{L^2(\mathbb{R}^2)} = -\frac{5i}{2}, \quad \dot{\lambda}_2 = -i\|\nabla u_2\|^2_{L^2(\mathbb{R}^2)} = -\frac{i}{2}.
\] (1.53)

Figure 3a shows the numerical results of the eigenvalues of $P(t)$ near 0. Figure 3b justifies (1.53).

1.3. Organization of the paper. In Sect. 2, we review some preliminaries including the construction of the space of hyperfunctions, properties of the resolvent, the definition of resonances, Grushin problems, and Puiseux series. In Sect. 3, we show the analyticity of the eigenfunctions of $P$. In Sect. 4, we give a proof to Theorem 1. In Sect. 5, we prove Theorem 2. In Sect. 6, we prove Theorem 3.
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(a) Eigenvalues $\lambda(t)$ of $P(t)$ near 0

Fig. 3. Eigenvalues of $P(t) = P + it\Delta$, where $P$ is of the form (1.20) with $V_m$, $V_a$ given by (1.52). In this example, $P$ has eigenvalues $\lambda_1 = 0, \lambda_2 = 1$ with eigenfunctions $u_1 = \frac{1}{2\sqrt{2\pi}}(e^{ix_1} + e^{-2ix_1}), u_2 = \frac{1}{2\sqrt{2\pi}}(i + e^{ix_1})$. As computed in (1.53), $\dot{\lambda}_1 = -\frac{5}{2}i, \dot{\lambda}_2 = -\frac{i}{2}\

2. Preliminaries

2.1. Space of hyperfunctions. We first review the function spaces $H_\Lambda$ constructed in [GaZw19, (4.7)]. Such spaces are motivated by [HeSj86] and [Sj96], and are used to study the resolvent of 0th order operators in [GaZw19].

Let $\tilde{T}^\ast T_n = \{(y, \eta) : y \in \mathbb{C}^n/2\pi\mathbb{Z}^n, \eta \in \mathbb{C}^n\}$ be the complexification of $T^\ast T^n$. Let $\sigma = \eta dy$ be the complex symplectic form over $\tilde{T}^\ast T_n$, $F_0$ be as in (1.4) and $0 < \epsilon \ll 1$. We consider the complex deformation of $T^\ast T_n$ given by

$$\Lambda(\epsilon) := \{(x + i\epsilon \partial_x F_0, \xi - i\epsilon \partial_\xi F_0) : (x, \xi) \in T^\ast T^n\} \subset \tilde{T}^\ast T^n. \quad (2.1)$$

To simplify the notation, we suppress the dependence of $\Lambda(\epsilon)$ on $\epsilon$. We have

$$\text{Im}(\sigma|_\Lambda) \equiv 0, \quad \Re(\sigma|_\Lambda) \text{ is non-degenerate.} \quad (2.2)$$

If we use $\alpha = (y, \eta) = (x + i\epsilon \partial_\xi F_0, \xi - i\epsilon \partial_x F_0)$ as coordinates of $\Lambda$, then $d\alpha = (\sigma|_\Lambda)^n/n!$ is the natural volume form on $\Lambda$.

Let $\mathcal{F}_\delta$ be the function space defined by

$$\mathcal{F}_\delta := \{u \in L^2(T^n) \left| \|u\|_{\mathcal{F}_\delta} < \infty\right.\}, \quad (2.3)$$

where

$$\|u\|_{\mathcal{F}_\delta}^2 := \sum_{\ell \in \mathbb{Z}^n} |u_\ell|^2 e^{4\delta |\ell|}, \quad u_\ell = \frac{1}{(2\pi)^n} \int_{T^n} e^{-i\ell \cdot x} u(x) dx. \quad (2.4)$$

Notice that $\cup_{\delta > 0} \mathcal{F}_\delta$ is the space of analytic functions on $T^n$, and functions in $\mathcal{F}_\delta, \delta > 0$, are called hyperfunctions. Let $\tilde{T}_\Lambda$ be the complex deformation of the FBI transform (see [GaZw19, §4]). By [GaZw19, Lemma 4.1, Lemma 4.2], there exists $\delta_0 > 0, C > 0$, such that for $0 < \delta < \delta_0, 0 < \epsilon \ll 1$, $\tilde{T}_\Lambda$ has the mapping property

$$\tilde{T}_\Lambda : \mathcal{F}_\delta \rightarrow e^{-\delta C(\xi)} L^2_\Lambda.$$
The space $H_\Lambda$ is then defined as the closure of $\mathcal{I}_\delta$ with respect to the norm $\| \cdot \|_{H_\Lambda}$ given by the formula

$$\|u\|_{H_\Lambda} := \int_{\Lambda} |\tilde{T}_\Lambda u(\alpha)|^2 d\alpha.$$ (2.5)

By the remark in [GaZw19, §4], there exists $\delta > 0$ such that

$$\mathcal{I}_\delta \subset H_\Lambda \subset \mathcal{I}_{-\delta}.$$ (2.6)

2.2. Resolvents and resonances. In this paper, we always assume $\varepsilon, \Lambda, H_\Lambda$ are as in Sect. 2.1. Then by [GaZw19, Lemma 7.4], there exists $\zeta_0 > 0$ such that if $\varepsilon > 0$ is sufficiently small, $\zeta \in (-\zeta_0, \zeta_0) + i(-\zeta_0 \varepsilon, \infty)$, then (after picking smaller $s_0$ if necessary)

$$P(s) - \zeta : H_\Lambda \rightarrow H_\Lambda$$ (2.7)

is a Fredholm operator. For such $\zeta_0, \varepsilon$, we denote

$$\Omega_{\zeta_0, \varepsilon} := (-\zeta_0, \zeta_0) + (-\zeta_0 \varepsilon, \infty).$$ (2.8)

Moreover, the resolvent of $P(s)$

$$R(s, \zeta) := (P(s) - \zeta)^{-1} : H_\Lambda \rightarrow H_\Lambda$$ (2.9)

is a meromorphic family of operators in $\zeta$ for $\zeta \in \Omega_{\zeta_0, \varepsilon}$. The poles of $R(s, \zeta)$ are then called the resonances of $P(s)$ in $\Omega_{\zeta_0, \varepsilon}$.

Remark. In this paper, we always assume $\zeta_0, \varepsilon$ satisfy conditions above. In Theorem 3 and Sect. 6, we further assume that $\zeta_0, t_0, \varepsilon$ satisfy conditions in [GaZw19, Lemma 7.6] (notice that there they are called $\omega_0, \nu_0, \theta_0$ respectively).

We denote

$$R(\zeta) := R(0, \zeta) : H_\Lambda \rightarrow H_\Lambda.$$ If $\lambda \in (-\zeta_0, \zeta_0)$ is an eigenvalue of $P(0)$, then by [GaZw19, Lemma 7.9], there exists $A(\zeta) : H_\Lambda \rightarrow H_\Lambda$ such that $A(\zeta)$ is holomorphic near $\lambda$ and

$$R(\zeta) = A(\zeta) + \frac{\Pi}{\lambda - \zeta},$$ (2.10)

where $\Pi$ is the orthogonal projection onto the $L^2$ eigenspace of $P(0)$ at $\lambda$. Let $\Pi_\perp = I - \Pi : L^2(\mathbb{T}^n) \rightarrow L^2(\mathbb{T}^n)$. In Sect. 3, we show that the eigenfunctions are analytic. As a result, $\Pi_\perp - I \in \Psi^{-\infty}(\mathbb{T}^n)$ and $\Pi_\perp$ can be extended to

$$\Pi_\perp : \mathcal{D}'(\mathbb{T}^n) \rightarrow \mathcal{D}'(\mathbb{T}^n), \quad \Pi_\perp : H_\Lambda \rightarrow H_\Lambda.$$ (2.11)

For $\zeta \neq \lambda$, we have

$$\Pi_\perp R(\zeta) \Pi_\perp = A(\zeta) : H_\Lambda \rightarrow H_\Lambda.$$ Thus $\Pi_\perp R(\zeta) \Pi_\perp$ has a limit $A(\lambda)$ as $\zeta \rightarrow \lambda$. We also use the notation

$$\Pi_\perp R(\lambda) \Pi_\perp := A(\lambda) : H_\Lambda \rightarrow H_\Lambda.$$ (2.12)

We record the limiting absorption principle for $P$ at embedded eigenvalues:
Lemma 2.1. Suppose \( P \in \Psi^0(\mathbb{T}^2) \) is self-adjoint with an embedded eigenvalue \( \lambda \in \mathbb{R} \). Assume \( P, \lambda \) satisfies (1.12), (1.14). Then for any \( f \in C^\infty(\mathbb{T}^2) \), the limit

\[
\lim_{\epsilon \to 0^+} \Pi_\perp (P - \lambda - i\epsilon)^{-1} \Pi_\perp f
\]

exists in \( H^{-1/2}(-\mathbb{T}^2) \). Moreover, let \( \Pi_\perp (P - \lambda - i0)^{-1} \Pi_\perp f \) be the limit, then

\[
\Pi_\perp (P - \lambda - i0)^{-1} \Pi_\perp f \in I^0(\Gamma^+_\lambda).
\]

Remark. Here \( I^0(\Gamma^+_\lambda) \) is the set of 0th order Lagrangian distributions (see [Wan19, §2.2]) associated to \( \Gamma^+_\lambda \), and \( \Gamma^+_\lambda \) is the attractive Lagrangian defined as follows: assume \( \gamma^+_\lambda \subset \Sigma_\lambda \) are the attractive cycles of \( \exp i V \) on \( \Sigma_\lambda \), then \( \Gamma^+_\lambda := \pi^{-1}(\gamma^+_\lambda) \), where \( \pi : T^*\mathbb{T}^2 \setminus 0 \to (T^*\mathbb{T}^2 \setminus 0)/\mathbb{R}^+ \) is the quotient map in Sect. 1.1. We remark that if \( \gamma^-_\lambda \subset \Sigma_\lambda \) are the repulsive cycles of \( \exp i V \), then \( \Gamma^-_\lambda := \pi^{-1}(\gamma^-_\lambda) \) is called the repulsive Lagrangian. Recall that [DyZw19b, Lemma 2.1] states that \( \gamma^+_\lambda \) are the radial sink (+) and radial source (−) for the Hamiltonian flow of \( |\xi|/(p - \lambda) \) (see also [DyZw19a, Definition E.50]).

Proof of Lemma 2.1. Notice that for \( f \in C^\infty(\mathbb{T}^2), \epsilon > 0, \)

\[
\Pi_\perp (P - \lambda - i\epsilon)^{-1} \Pi_\perp f = (P + \Pi - \lambda - i\epsilon)^{-1} \Pi_\perp f.
\]

Since \( \Pi \in \Psi^{-\infty}(\mathbb{T}^m) \), we know \( P + \Pi \in \Psi^0(\mathbb{T}^m) \) is a self-adjoint operator and satisfies (1.12), (1.14). One can check that \( \lambda \) is not an eigenvalue of \( P + \Pi \), hence [DyZw19b, Lemma 3.3, Lemma 4.1] applies with \( (P, \omega) \) being replaced by \( (P + \Pi, \lambda) \). This concludes the proof. \( \square \)

Remark. If we use the notations in (2.12) and Lemma 2.1, then

\[
\Pi_\perp R(\lambda) \Pi_\perp = \Pi_\perp (P - \lambda - i0)^{-1} \Pi_\perp : C^\infty(\mathbb{T}^2) \to I^0(\Gamma^+_\lambda).
\]

In fact, by (2.10) and [GaZw19, Lemma 7.9], for \( f \in C^\infty(\mathbb{T}^2), \epsilon > 0 \), we have

\[
A(\lambda + i\epsilon)f = \Pi_\perp R(\lambda + i\epsilon) \Pi_\perp f = \Pi_\perp R_L^2(\lambda + i\epsilon) \Pi_\perp f = \Pi_\perp (P - \lambda - i\epsilon)^{-1} \Pi_\perp f,
\]

where \( R_L^2 \) is the \( L^2 \) resolvent of \( P \). Let \( \epsilon \to 0^+ \) and we get what we want.

2.3. Grushin problems. We briefly review Grushin problems. For a complete introduction, see for instance [DyZw19a, §C.1]. See also [SjZw07] for applications of Grushin problems.

Suppose \( P : X_1 \to X_2, R_- : X_- \to X_2, R_+ : X_1 \to X_+ \) are bounded operators on Banach spaces \( X_1, X_2, X_-, X_+ \). We call the equation

\[
\begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}
\]

(2.15)

a Grushin problem. We call the Grushin problem (2.15) well-posed if it is invertible, and in this case we write the inverse as

\[
\begin{pmatrix} u \\ v_- \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_{--} \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix},
\]

(2.16)
with operators \( E : X_2 \to X_1, E_+ : X_+ \to X_1, E_- : X_2 \to X_-, X_+ \to X_- \). We also know that \( P \) is invertible if and only if \( E_-+ \) is invertible. Moreover, when \( P \) and \( E_-+ \) are invertible, we have
\[
P^{-1} = E - E_+E_-^{-1}E_-.
\] (2.17)

We also record the following formula for the perturbed Grushin problem

**Lemma 2.2** ([DyZw19a, Lemma C.3]). Suppose the Grushin problem (2.15) is well-posed and (2.16) is the inverse. If \( B : X_1 \to X_2 \) is a bounded operator such that
\[
\|EB\|_{X_1 \to X_1} < 1, \quad \|BE\|_{X_2 \to X_2} < 1,
\] (2.18)

then the Grushin problem
\[
\left(\begin{array}{cc}
P + B & R_-
\end{array}\right): X_1 \times X_- \to X_2 \times X_+
\] (2.19)
is well-posed with inverse
\[
\left(\begin{array}{cc}
G & G_+
\end{array}\right)
\] (2.20)
such that
\[
G_{-+} = E_{-+} + \sum_{\ell=1}^{\infty} (-1)^\ell E_- B(EB)^{\ell-1} E_+.
\] (2.21)

### 2.4. Puiseux series

We briefly introduce Puiseux series in this section. For applications of Puiseux series to spectral problems, one can see for instance [Ka80, Chapter 2], [Ho74].

For \( \alpha > 0 \), we define the function
\[
\exp^{-1}(\mathbb{C} \setminus \{0\}) \ni z \mapsto z^\alpha := e^{\alpha \log z} \in \mathbb{C}, \quad \log(0, +\infty) \subset \mathbb{R},
\]
where \( \exp^{-1}(\mathbb{C} \setminus \{0\}) \) is the logarithmic plane. We define the field of Puiseux series as follows:

**Definition 2.3.** For \( q \in \mathbb{N}, q \geq 1 \), let
\[
K((z^{1/q})) := \left\{ \sum_{\ell=0}^{+\infty} c_{\ell} z^{\ell/q} \left| \ell_0 \in \mathbb{Z}, c_{\ell} \in \mathbb{C}, \ell \geq \ell_0 \right. \right\},
\] (2.22)
\[
K((z)) := \bigcup_{q=1}^{\infty} K((z^{1/q})).
\]

\( K((z)) \) is called the field of Puiseux series.

We remark the following result from algebra:

**Lemma 2.4** (Newton-Puiseux Theorem). \( K((z)) \) is an algebraically closed field.
For a full discussion of the theorem, see for instance [Wal78, Theorem 3.1]. Recall that Lemma 2.4 means that roots of any polynomial with coefficients in $K((z))$ are still in $K((z))$. In particular, when the coefficients are analytic functions, the Puiseux series of the roots do not admit negative exponents and we further have the following:

**Lemma 2.5.** Let $m \in \mathbb{N}$, $Q(z, w) := \sum_{\ell=0}^{m} c_\ell(z)w^\ell$, where $c_\ell(\bullet)$ are analytic functions on $\mathbb{C}$. If $w(z) \in K((z^{1/q}))$, $q \geq 1$ is a root of $Q$, then for $k \in \mathbb{Z}$, $0 \leq k \leq q - 1$, $w(2^{2k}i z) \in K((z^{1/q}))$ are also roots of $Q$.

**Remark.** For $w \in K((z^{1/q}))$, we say $\{w(2^{2k}i z)\}_{0 \leq k \leq q-1}$ is a Puiseux cycle. Lemma 2.5 shows that roots of $Q$ can be grouped into finitely many Puiseux cycles.

**Proof.** Since $w(z)$ is a root, we know $\sum_{\ell=0}^{m} c_\ell(z)w(z)^\ell = 0$. Replace $z$ by $2^{2k}i z$ and use the fact that $c_\ell(\bullet)$ are analytic on $\mathbb{C}$, we find $\sum_{\ell=0}^{m} c_\ell(z)w(2^{2k}i z)^\ell = 0$, which justifies the lemma. \qed

### 3. Analyticity of Eigenfunctions

In this section we prove the analyticity of the eigenfunctions of $P$. The analyticity of the eigenfunctions allows us to define the $L^2$ pairing of eigenfunctions with hyperfunctions in $H_\Lambda$. These pairings are used throughout our proofs of Theorem 1–3.

We first introduce the standard FBI transform

$$ T v(x, \xi; h) := h^{-\frac{3n}{4}} \int_{\mathbb{R}^n} e^{i\frac{1}{h}(x-y, \xi) + \frac{i}{2}(x-y)^2} v(y) dy, \quad v \in \mathcal{S}'(\mathbb{R}^n), \tag{3.1} $$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. The analytic wavefront set $\text{WF}_a(v)$ of a distribution $v$ is define as follows: $(x_0, \xi_0) \notin \text{WF}_a(v)$ if and only if there exists $h_0 > 0$, $\delta > 0$ and a neighborhood $U$ of $(x_0, \xi_0)$, such that

$$ |Tv(x, \xi, h)| \leq C e^{-\delta/h}, \quad (x, \xi) \in U, \quad 0 < h < h_0. \tag{3.2} $$

We refer to [Ma02, Chapter 3] and [Hö83a, §9.6] for details of the definitions. For a distribution $v \in \mathcal{S}'(\mathbb{T}^n)$, we can consider $v$ as a distribution in $\mathcal{S}'(\mathbb{R}^n)$ by extending it $2\pi$ periodically. We now state the following

**Proposition 3.1.** Assume $P \in \mathcal{S}^{0}(\mathbb{T}^n)$ is self-adjoint satisfying (1.1)–(1.4), and $\zeta_0$ is as in Sect. 2.2. If $u \in \text{Ker}_{L^2}(P - \lambda)$ with $\lambda \in (-\zeta_0, \zeta_0)$, then there exists $\delta > 0$ such that $u \in \mathcal{S}_\delta$.

To prove Proposition 3.1, we use a result from [GaZw20]. Before stating the lemma, we introduce the deformed FBI transform ([GaZw20, §2.1]): for $h > 0, u \in C_{c}^{\infty}(\mathbb{R}^n)$, we define

$$ \tilde{T} u(x, \xi) := h^{-\frac{3n}{4}} \int_{\mathbb{R}^n} e^{i\frac{1}{h}(x-y, \xi)} e^{\frac{i}{2}(\xi(y)^2)} \langle \xi \rangle^{\frac{3}{4}} u(y) dy. \tag{3.3} $$

Then we have

**Lemma 3.2 ([GaZw20, Proposition 2.3]).** Let $\tilde{T}$ be defined by (3.3). Suppose $U \subset \mathbb{R}^n$ is open, $\Gamma \subset \mathbb{R}^n \setminus 0$ is open and conic. If $\phi \in S^{1}(T^*\mathbb{R}^n)$ satisfies $\phi \geq c|\xi|$ over $U \times \Gamma$, $c > 0$. Then for any $u \in H^{-N}(\mathbb{R}^n)$,

$$ e^{\phi(\xi)^{-N}} \tilde{T} u \in L^2(T^*\mathbb{R}^n) \text{ implies } \text{WF}_a(u) \cap (U \times \Gamma) = \emptyset. $$
The following lemma from [GaZw20] shows that we can apply \( \widetilde{T} \) to functions in \( H_\Lambda \).

**Lemma 3.3** ([GaZw20, Proposition 2.5]). Let \( \epsilon, F_0, \Lambda \) be as in Sect. 2.1. Then there exists \( \phi \in S^1(T^*\mathbb{R}^n) \) such that

\[
\phi(x, \xi) = \epsilon F_0(x, \xi) + O(\epsilon^2) \quad S^1(T^*\mathbb{R}^n)
\]

and \( \widetilde{T} : \mathcal{H}_\delta \rightarrow L^2(T^*\mathbb{R}^n, e^{\delta(\xi)/Ch} \, dx \, d\xi) \) can be extended to a bounded operator

\[
\widetilde{T} : H_\Lambda \rightarrow L^2(T^*\mathbb{R}^n, e^{2\phi/h} \, dx \, d\xi).
\]

We now prove the analyticity of the eigenfunctions.

**Proof of Proposition 3.1.** Let \( \epsilon, F_0, \Lambda \) be as in Sect. 2.1. Suppose \( u \in L^2(T^*\mathbb{R}^n) \) is an eigenfunction of \( P \) with eigenvalue \( \lambda \). By Lemma [GaZw19, Lemma 7.9], \( u \in H_\Lambda \). By Lemma 3.3, there exists \( \phi \in S^1(T^*\mathbb{R}^n) \) and a constant \( C > 0 \) such that

\[
\phi(x, \xi) = \epsilon F_0(x, \xi) + r(x, \xi), \quad |r(x, \xi)| \leq C \epsilon^2 |\xi|,
\]

\[
e^{-2\phi/h} \widetilde{T} u \in L^2(T^*\mathbb{R}^n).
\]

Now we define sets \( \mathcal{E}(\pm, \epsilon) \subset T^*\mathbb{R}^n \setminus 0 \) as the following:

\[
\mathcal{E}(\pm, \epsilon) := \{(x, \xi) \in T^*\mathbb{R}^n : \pm F_0(x, \xi) \geq (C + 1)\epsilon|\xi|\}.
\]

For \( (x, \xi) \in \mathcal{E}(+, \epsilon) \), we have

\[
\phi(x, \xi) \geq \epsilon F_0(x, \xi) - C \epsilon^2 |\xi| \geq \epsilon^2 |\xi|,
\]

hence by Lemma 3.2,

\[
\WF_a(u) \cap \mathcal{E}(+, \epsilon) = \emptyset. \tag{3.4}
\]

Notice that if we replace \( P \) by \( -P \), \( F_0 \) by \( -F_0 \), the previous argument still applies as \( u \) is an eigenfunction of \( -P \) with eigenvalue \(-\lambda\). For \( (x, \xi) \in \mathcal{E}(-, \epsilon) \), we have

\[
\phi(x, \xi) \geq -\epsilon F_0(x, \xi) - C \epsilon^2 |\xi| \geq \epsilon^2 |\xi|.
\]

By Lemma 3.2, we find

\[
\WF_a(u) \cap \mathcal{E}(-, \epsilon) = \emptyset. \tag{3.5}
\]

Since (3.4), (3.5) holds for all small \( \epsilon > 0 \), we have

\[
\WF_a(u) \cap (\cup_{\epsilon > 0} \cup_{\pm} \mathcal{E}(\pm, \epsilon)) = \emptyset.
\]

Notice that \( \mathcal{E}(F_0) := \cup_{\epsilon > 0} \cup_{\pm} \mathcal{E}(\pm, \epsilon) \) is the elliptic set of \( F_0 \), and we now have

\[
\WF_a(u) \cap \mathcal{E}(F_0) = \emptyset.
\]

Now for \( \rho > 0 \), we define \( F_\rho(x, \xi) = F_0(x, \xi) + \rho|\xi| \in S^1(T^*\mathbb{T}^n) \). When \( \rho \) is sufficiently small, \( F_\rho \) still satisfies the conditions in (1.4). Therefore we can repeat the previous argument and conclude that

\[
\WF_a(u) \cap \mathcal{E}(F_\rho) = \emptyset.
\]

Notice that \( \mathcal{E}(F) \cup \mathcal{E}(F_\rho) = T^*\mathbb{R}^n \), we conclude that \( \WF_a(u) = \emptyset \), i.e., \( u \) is analytic. \( \square \)
4. Puiseux Series Expansions and the Fermi Golden Rule

We now give a proof to Theorem 1.

Proof of Theorem 1. Part 1. We first notice that by [GaZw19, Lemma 7.9], \( u_\ell \in H_\Lambda \), \( 1 \leq \ell \leq m \). We consider the Grushin problem

\[
\mathcal{P}(s, \zeta) = \left( \begin{array}{c}
P(s) - \zeta R_+ \\
R_+
\end{array} \right) : H_\Lambda \times \mathbb{C}^m \to H_\Lambda \times \mathbb{C}^m, \quad (s, \zeta) \in (-s_0, s_0) \times \Omega_{\zeta_0, \varepsilon},
\]

(4.1)

where \( R_- : \mathbb{C}^m \to H_\Lambda \) and \( R_+ : H_\Lambda \to \mathbb{C}^m \) are defined by

\[
R_-(u_1, \ldots, u_m) = \sum_{\ell=1}^m u_\ell, \quad R_+u = (\langle u_1 \rangle_{L^2(\mathbb{T}^n)}, \ldots, \langle u, u_m \rangle_{L^2(\mathbb{T}^n)}).
\]

(4.2)

By Proposition 3.1, \( u_\ell \in \mathcal{I}_\delta \) for small \( \delta > 0 \). Therefore (see (2.6))

\[
|\langle u, u_\ell \rangle_{L^2(\mathbb{T}^n)}| \leq \|u_\ell\|_{\mathcal{I}_\delta} \|u\|_{\mathcal{I}_\delta} \leq C \|u_\ell\|_{\mathcal{I}_\delta} \|u\|_{H_\Lambda}.
\]

(4.3)

This implies \( R_\pm \) are well-defined and bounded operators.

For \( \zeta \in \Omega_{\zeta_0, \varepsilon} \), let \( A(\zeta) \) be as in (2.10). We define

\[
\mathcal{E}(\zeta) := \left( \begin{array}{cc}
A(\zeta) & E_+ \\
E_- & E_{-+}
\end{array} \right) : H_\Lambda \times \mathbb{C}^m \to H_\Lambda \times \mathbb{C}^m,
\]

(4.4)

where \( E_+ : \mathbb{C}^m \to H_\Lambda \), \( E_- : H_\Lambda \to \mathbb{C}^m \), \( E_{-+} : \mathbb{C}^m \to \mathbb{C}^m \) are given by

\[
E_+(v_1, \ldots, v_m) = \sum_{\ell=1}^m v_\ell, \quad E_-v = (\langle v, u_1 \rangle_{L^2(\mathbb{T}^n)}, \ldots, \langle v, u_m \rangle_{L^2(\mathbb{T}^n)}),
\]

\[
E_{-+} = (\zeta - \lambda) I_m.
\]

One can check that \( \mathcal{E}(\zeta) \) is the inverse of \( \mathcal{P}(0, \zeta) \). By (1.2), there exists \( 0 < s_1 < s_0 \) such that for \( s \in (-s_1, s_1) \),

\[
\max \left( \|P(s) - P(\zeta)\|_{H_\Lambda \to H_\Lambda}, \|A(\zeta)(P(s) - P)\|_{H_\Lambda \to H_\Lambda} \right) < 1.
\]

(4.5)

Hence by [DyZw19a, Lemma C.3], for \( (s, \zeta) \in (-s_1, s_1) \times \Omega_{\zeta_0, \varepsilon}, \mathcal{P}(s, \zeta) \) has inverse

\[
\mathcal{E}(s, \zeta) = \left( \begin{array}{cc}
E(s, \zeta) & E_+(s, \zeta) \\
E_-(s, \zeta) & E_{-+}(s, \zeta)
\end{array} \right)
\]

(4.6)

such that \( \mathcal{E}(s, \zeta) \) is analytic in \( s, \zeta \) and \( \mathcal{E}(0, \zeta) = \mathcal{E}(\zeta) \). Moreover, \( P(s) - \zeta \) is invertible if and only if \( E_{-+}(s, \zeta) \) is invertible. We now see that the resonances \( \lambda_\ell(s) \), \( 1 \leq \ell \leq m \), of \( P(s) \) in \( \Omega_{\zeta_0, \varepsilon} \) must satisfy

\[
\det(E_{-+}(s, \lambda_\ell(s))) = 0, \quad \lambda_\ell(0) = \lambda.
\]

(4.7)

Let \( L(s, \zeta) := \det(E_{-+}(s, \zeta)) \), \( (s, \zeta) \in (-s_1, s_1) \times \Omega_{\zeta_0, \varepsilon} \). Then \( L(s, \zeta) \) is analytic and

\[
L(0, \zeta) = \det(E_{-+}(0, \zeta)) = (\zeta - \lambda)^m.
\]

(4.8)
By the Weierstrass preparation theorem (see for instance [Hö90, Theorem 6.1.1]), there exist analytic functions \( g_\ell, g_\ell(\lambda) = 0, 0 \leq \ell \leq m - 1 \), and an analytic function \( N(s, \zeta) \), \( N(s, \zeta) \neq 0 \) in \((-s_1, s_1) \times \Omega_{g_0, \epsilon} \), such that
\[
L(s, \zeta) = \left( (\zeta - \lambda)^m + g_{m-1}(s)(\zeta - \lambda)^{m-1} + \cdots + g_0(s) \right) N(s, \zeta). \tag{4.9}
\]

Since \( \lambda_\ell \) are roots of \( L(s, \zeta)/N(s, \zeta) \), which is a polynomial with analytic coefficients, Lemma 2.5 (or a direct argument as in [Ka80, Chapter 2, §1.2]) shows that \( \lambda_\ell \) have Puiseux series expansions and can be grouped into Puiseux cycles. For a Puiseux cycle,
\[
\lambda_\ell(s) = \sum_{k=0}^{+\infty} c_\ell(e^{2\ell \pi i} s)^{k/q}, \quad q \geq 1, \quad 1 \leq \ell \leq q, \tag{4.10}
\]
if \( \lambda_\ell(s) \in \mathbb{R} \) for all \( s \in (-s_1, s_1) \), then \( q = 1 \) and \( c_\ell \in \mathbb{R} \) for all \( k \); if \( \lambda_\ell \notin \mathbb{R} \), using the fact that \( \text{Im} \lambda_\ell(s) \leq 0 \) for all \( s \in (-s_1, s_1) \) (see [GaZw19, Lemma 7.9]), we know the alternative results for the coefficients in Theorem 1 holds.

**Part 2.** Now we prove (1.8). By Lemma 2.2, for small \( s \), we have
\[
E_- (s, \zeta) = E_+ + \sum_{k=1}^{\infty} (-1)^k E_- (0, \zeta)(P(s) - P)(A(\zeta)(P(s) - P))^{k-1} E_+
\]
\[
= (\zeta - \lambda) I_m + \sum_{k=1}^{\infty} (-1)^k \left( \left[ (P(s) - P)(A(\zeta)(P(s) - P))^{k-1} u_i, u_j \right] \right) \tag{4.11}
\]
Notice that by Part 1,
\[
\lambda_\ell(s) = \lambda + c_\ell s + c_{2\ell} s^2 + o(s^2), \quad P(s) - P = \dot{P}s + \frac{1}{2} \ddot{P}s^2 + o(s^2),
\]
\[
A(\lambda_\ell(s)) = A(\lambda) + c_\ell \dot{A}(\lambda)s + \left( c_{2\ell} \ddot{A}(\lambda) + \frac{1}{2} c_{2\ell} \dot{A}(\lambda) \right) s^2 + o(s^2). \tag{4.12}
\]
Inserting (4.12) into (4.11) and we find
\[
E_- (s, \lambda_\ell(s)) = s(c_\ell I_m - M) + s^2(c_{2\ell} I_m - N) + o(s^2). \tag{4.13}
\]
(1.8) then follows from \( \text{det}(E_- (s, \lambda_\ell(s))) = 0, \quad s \in (-s_1, s_1) \).

**Part 3.** Now we consider the case when \( \lambda \) is a simple eigenvalue and prove the Fermi golden rule under the assumption that \( P(s) \) depends only smoothly on \( s \).

The proof above still applies to this case except for the fact that \( L(s, \zeta) \) depends only smoothly on \( s \), hence \( \lambda(s) \) does not have the Puiseux series expansion (4.10). We still have (4.12) though. We can again use (1.8), which is a consequence of (4.12), and get
\[
c_1 - M + s(c_2 - N) + \rho(s) = 0, \quad \rho(s) = o(s). \tag{4.14}
\]
Therefore we have (1.10) and (1.11). \( \square \)
5. Absence of Embedded Eigenvalues for Generic Perturbations

In this section, we prove Theorem 2. To begin with, we introduce two linear operators constructed in [Wan19] and use them to prove an alternative formula for Im$\hat{\lambda}$ in the case where $\lambda$ is a simple eigenvalue of $P$.

Assume $P \in \Psi^0(M)$ is self-adjoint and $P$, $\lambda$ satisfy (1.12), (1.14). Then the conditions in [Wan19, Theorem 1] is satisfied and we denote the invertible linear operator $H^+_{\lambda, 0}$ there by $H^+_{\lambda}$. Let $G^+_{\lambda}$ be the inverse of $H^+_{\lambda}$, then

$$H^+_{\lambda} : C^\infty(S^1; \mathbb{C}^d) \rightarrow D^+(P, \lambda), \quad G^+_{\lambda} : D^+(P, \lambda) \rightarrow C^\infty(S^1; \mathbb{C}^d),$$

$$D^+(P, \lambda) := \Pi_\perp \left\{ u \in \Pi_0(\Gamma^+_{\lambda}) \mid (P - \lambda)u \in C^\infty(\mathbb{T}^2) \right\} / \Pi_\perp C^\infty(\mathbb{T}^2).$$

(5.1)

Here $d$ is the number of connected components of $\Gamma^+_{\lambda}$ (see Lemma 2.1). Roughly speaking, $D^+(P, \lambda)$ is the set of microlocal solutions in $\Pi_0(\Gamma^+_{\lambda})$ to $(P - \lambda)u = 0$. $G^+_{\lambda}$ maps distributions in $D^+(P, \lambda)$ to its “initial data”, and $H^+_{\lambda}$ recovers a distribution in $D^+(P, \lambda)$ by its “initial data”.

We remark that in the case where $\lambda$ is an embedded eigenvalue of $P$, the construction of $H^+_{\lambda}$ is explained in [Wan19, §11].

We now state the formula for Im$\hat{\lambda}$ by using $G^+_{\lambda}$.

**Lemma 5.1.** Let $P(s) \in \Psi^0(\mathbb{T}^2)$ be as in Theorem 1, $\zeta_0$, $\epsilon$, $\Omega_{\zeta_0, \epsilon}$ be as in Sect. 2.2. Assume $\lambda \in (-\zeta_0, \zeta_0)$ is a simple embedded eigenvalue of $P = P(0)$ with an $L^2$ normalized eigenfunction $u$. Assume (1.12), (1.14) hold for $P$, $\lambda$, and $G^+_{\lambda}$ is as in (5.1), then for resonances $\lambda(\bullet)$ in Theorem 1, we have

$$\text{Im} \hat{\lambda} = -4\pi^2 \int_{S^1} |G^+_{\lambda} \Pi_\perp R(\lambda) \Pi_\perp \hat{u} |^2 dS.$$  

(5.2)

**Proof of Lemma 5.1.** By Lemma 2.1 and the boundary pairing formula [Wan19, Proposition 6.5], for any $v_1, v_2 \in C^\infty(\mathbb{T}^2)$, we have

$$\langle \Pi_\perp R(\lambda) \Pi_\perp v_1, \Pi_\perp v_2 \rangle - \langle \Pi_\perp v_1, \Pi_\perp R(\lambda) \Pi_\perp v_2 \rangle = \langle \Pi_\perp R(\lambda) \Pi_\perp v_1, (P - \lambda) \Pi_\perp R(\lambda) \Pi_\perp v_2 \rangle - \langle (P - \lambda) \Pi_\perp R(\lambda) \Pi_\perp v_1, \Pi_\perp R(\lambda) \Pi_\perp v_2 \rangle$$

$$= 4\pi^2 i \int_{S^1} G^+_{\lambda} \Pi_\perp R(\lambda) \Pi_\perp v_1 \cdot G^+_{\lambda} \Pi_\perp R(\lambda) \Pi_\perp v_2 dS.$$  

Here $\cdot$ is the Hermitian product on $\mathbb{C}^d$.

Now by Theorem 1, we have

$$\text{Im} \hat{\lambda} = -2\text{Im} \langle \Pi_\perp R(\lambda) \Pi_\perp \hat{u}, \hat{u} \rangle = -4\pi^2 \int_{S^1} |G^+_{\lambda} \Pi_\perp R(\lambda) \Pi_\perp \hat{u} |^2 dS.$$  

This concludes the proof. \hfill $\square$

We need Im$\hat{\lambda}$ to be nonzero for certain perturbation in the proof of Theorem 2. We show the following

**Lemma 5.2.** Let $P, \lambda$ be as in Lemma 5.1. Then there exists $u_0 \in C^\infty(\mathbb{T}^2)$ such that $u_0$ is analytic and $G^+_{\lambda} \Pi_\perp R(\lambda) \Pi_\perp u_0 \neq 0$. 
Proof of Lemma 5.2. \textbf{Step 1.} We first show that there exists \( v \in C^\infty(\mathbb{T}^2) \) such that 
\[ G_\lambda^+ \Pi_\perp R(\lambda) \Pi_\perp v \neq 0. \]

In fact, let \( H_\lambda^+ \) be as in (5.1). For \( f \in C^\infty(\mathbb{S}^1; \mathbb{C}^d) \), let \( H_\lambda^+(f) \) be any representative in the equivalent class. If we put 
\[ v := (P - \lambda) H_\lambda^+(f) \in C^\infty(\mathbb{T}^2), \]
then 
\[ G_\lambda^+ \Pi_\perp R(\lambda) \Pi_\perp v = f. \]

To see this, we define 
\[ \psi := \Pi_\perp H_\lambda^+(f) - \Pi_\perp R(\lambda) \Pi_\perp v. \]

One can see that 
\[ (P - \lambda) \psi = (P - \lambda) \Pi_\perp H_\lambda^+(f) - \Pi_\perp v = \Pi_\perp (P - \lambda) H_\lambda^+(f) - \Pi_\perp v = 0 \]
and since \( H_\lambda^+(f) \in L^0(\Gamma_\lambda^+) \), \( \Pi_\perp R(\lambda) \Pi_\perp v \in L^0(\Gamma_\lambda^+) \), we find 
\[ \text{WF}(\psi) \subset \Gamma_\lambda^+. \]

By [DyZw19b, Lemma 3.1], \( \psi \in C^\infty(\mathbb{T}^2) \), which implies \( G_\lambda^+(\psi) = 0 \). Hence 
\[ G_\lambda^+ \Pi_\perp R(\lambda) \Pi_\perp v = G_\lambda^+ \Pi_\perp H_\lambda^+(f) = G_\lambda^+ H_\lambda^+(f) = f. \]

As long as \( f \neq 0 \), we have \( G_\lambda^+ \Pi_\perp R(\lambda) \Pi_\perp v \neq 0 \).

\textbf{Step 2.} We now show that there exists an analytic \( u_0 \) such that \( G_\lambda^+ \Pi_\perp R(\lambda) \Pi_\perp u_0 \neq 0 \).

In fact, assume this is not true, then for any analytic \( u \), we have \( \Pi_\perp R(\lambda) \Pi_\perp u \in C^\infty(\mathbb{T}^2) \). Since analytic functions are dense in \( C^\infty(\mathbb{T}^2) \), for any \( v \in C^\infty(\mathbb{T}^2) \), there exist analytic functions \( u_\ell, \ell \geq 1 \), such that \( u_\ell \to v \) in \( C^\infty(\mathbb{T}^2) \). Put 
\[ w_\ell := \Pi_\perp R(\lambda) \Pi_\perp u_\ell \in C^\infty(\mathbb{T}^2), \quad w := \Pi_\perp R(\lambda) \Pi_\perp v \in L^0(\Gamma_\lambda^+) \subset H^{-\frac{1}{2}} - (\mathbb{T}^2). \]

We would like to show that \( w \in C^\infty(\mathbb{T}^2) \).

If \( \| w_\ell \|_{L^2} \geq 1 \) is unbounded, by passing to a subsequence, we assume that \( \| w_\ell \|_{L^2} \to \infty, \ell \to \infty \). If we put 
\[ \tilde{w}_\ell := w_\ell/\| w_\ell \|_{L^2}, \quad \tilde{u}_\ell := u_\ell/\| w_\ell \|_{L^2}, \]
then 
\[ \tilde{w}_\ell \in \Pi_\perp L^2, \quad \| \tilde{w}_\ell \|_{L^2} = 1, \quad \tilde{u}_\ell \to 0 \text{ in } C^\infty, \quad (P - \lambda) \tilde{w}_\ell = \Pi_\perp \tilde{u}_\ell. \quad (5.3) \]

Now for \( \ell, k \in \mathbb{N} \), we consider 
\[ (P - \lambda)(\tilde{w}_\ell - \tilde{w}_k) = \Pi_\perp (\tilde{u}_\ell - \tilde{u}_k). \]

Since \( \tilde{w}_\ell, \tilde{w}_k \) are smooth, we can apply the high regularity radial estimates [DyZw19a, Theorem E.52] for \( \langle D \rangle^{1/2} (P - \lambda) \langle D \rangle^{1/2} \) to \( \langle D \rangle^{-1/2}(\tilde{w}_\ell - \tilde{w}_k) \) at the radial source of the Hamiltonian flow of \( |\xi|(p - \lambda) \) and at the radial source \( -|\xi|(p - \lambda) \) (which is the radial sink of \( |\xi|(p - \lambda) \)). For the radial set structure of the Hamiltonian flow of \( \pm |\xi|(p - \lambda) \), we refer to the remark after Lemma 2.1. Combining these radial estimates
and the propagation estimates \[\text{[DyZw19a, Theorem E.47]}\] for \((D)^{1/2}(P - \lambda)(D)^{1/2}\), we find that there exists \(s_0 \in \mathbb{R}\) such that for any \(s > s_0, N \in \mathbb{R}\),

\[
\|\tilde{w}_\ell - \tilde{w}_k\|_{H^s} \leq C\|\tilde{u}_\ell - \tilde{u}_k\|_{H^{s+1}} + C\|\bar{w}_\ell - \bar{w}_k\|_{H^{-N}},
\]

(5.4)

where \(C\) does not depend on \(\ell, k\). Let \(N = 1\) and we find that for any \(s \in \mathbb{R}\),

\[
\|\tilde{w}_\ell - \tilde{w}_k\|_{H^s} \leq C\|\tilde{u}_\ell - \tilde{u}_k\|_{H^{\max\{s+1,2\}}} + C\|\bar{w}_\ell - \bar{w}_k\|_{H^{-1}}.
\]

(5.5)

Since \(\|\tilde{w}_\ell\|_{L^2}\) are bounded, by passing to a subsequence, we can assume that there exists \(\tilde{w}_0 \in L^2\) such that \(\tilde{w}_\ell \rightharpoonup \tilde{w}_0\) in \(L^2\) and hence \(\tilde{w}_\ell \to \tilde{w}_0\) in \(H^{-1}\). Now from (5.5) we know that \(\tilde{w}_\ell\) converges in \(H^s\) for any \(s \in \mathbb{R}\). Hence we can write \(\tilde{w}_\ell \to \tilde{w}_0\) in \(C^\infty\). Let \(\ell \to \infty\) in (5.3) and we find

\[
\tilde{w}_0 \in \Pi_{\perp} L^2, \quad \|\tilde{w}_0\|_{L^2} = 1, \quad (P - \lambda)\tilde{w}_0 = 0.
\]

Such \(\tilde{w}_0\) does not exist, as \((P - \lambda)\tilde{w}_0 = 0\) and \(\tilde{w}_0 \in \Pi_{\perp} L^2\) implies \(\tilde{w}_0 = 0\), which contradicts \(\|\tilde{w}_0\|_{L^2} = 1\).

From the argument above, we know \(\{\|w_\ell\|_{L^2}\}_{\ell \geq 0}\) is bounded. Without loss of generality, we assume \(\|w_\ell\|_{L^2(\mathbb{T}^d)} = 1\). Similar to the argument about \(\tilde{w}_\ell\) above, we know that there exists \(w_0 \in C^\infty\) such that \(w_\ell \to w_0\) as \(\ell \to \infty\). Notice that

\[
(P - \lambda)w_\ell = \Pi_{\perp} u_\ell, \quad (P - \lambda)w = \Pi_{\perp} v.
\]

Let \(\ell \to \infty\) and we find

\[
(P - \lambda)(w_0 - w) = 0.
\]

We also have

\[
\WF(w) \subset \Gamma^\pm_\lambda, \quad w_0 \in C^\infty \Rightarrow \WF(w_0 - w) \subset \Gamma^\pm_\lambda.
\]

Now [\text{DyZw19b, Lemma 3.1}] implies that \(w_0 - w \in C^\infty\), hence \(w_0 - w\) is an eigenfunction of \(P\) with eigenvalue \(\lambda\). On the other hand, \(w_0 - w \in \Pi_{\perp} L^2\). Thus \(w = w_0 \in C^\infty(\mathbb{T}^2)\).

As a conclusion, we showed that for any \(v \in C^\infty(\mathbb{T}^2)\), one always has \(\Pi_{\perp} R(\lambda)\Pi_{\perp} v \in C^\infty(\mathbb{T}^2)\). This contradicts Step 1.

\(\square\)

**Proof of Theorem 2.** Let \(K \subset O\) be a closed interval. Suppose \(w \in \mathbb{H}, W := w(x, D)\) and there exists \(s_k \to 0, \lambda_k \in K\) such that \(\lambda_k \in \Spec_{pp}(P + s_k W)\). Suppose \(\lambda \in K\) is a limit point of \(\lambda_k\). Without loss of generality, we assume \(\lambda_k \to \lambda, k \to \infty\). If \(\lambda\) is not an eigenvalue of \(P\), then \(R(\xi) = (P - \xi)^{-1} : H_\Lambda \to H_\Lambda\) is holomorphic near \(\lambda\) (see [\text{GaZw19, Lemma 7.9}]). Note that

\[
(P + sW - \xi)^{-1} = (I + sR(\xi)W)^{-1}R(\xi).
\]

(5.6)

When \(s < 1/(C\|R(\lambda)\|\|W\|)\) for large \(C\), we know \((P + sW - \xi)^{-1}\) is holomorphic near \(\lambda\). Therefore \(P + sW\) does not have eigenvalues near \(\lambda\) for small \(s\). This contradiction shows that

\[
\Spec_{pp}(P + sW) \cap K \to \Spec_{pp}(P) \cap K, \quad s \to 0.
\]

(5.7)

Recall that by [\text{CS20, Theorem 5.1}] or [\text{DyZw19b, Lemma 3.2}], we know

\[
\left|\Spec_{pp}(P) \cap K\right| < \infty.
\]

(5.8)
For any $\lambda \in \text{Spec}_{pp}(P) \cap \mathcal{K}$, let $\{u_1, \ldots, u_m\}$ be an orthonormal basis of $\text{Ker}_{L^2}(P - \lambda)$. Let $\mathcal{M}(\lambda, W), \mathcal{N}(\lambda, W)$ be the matrices $\mathcal{M}, \mathcal{N}$ defined in Theorem 1 with $P(s) = P + sW$ and eigenvalue $\lambda$. We define $\mathcal{U}_\lambda \subset \mathcal{H}$ to be the set that consists of $w \in \mathcal{H}$ that satisfies the following condition:

$$W = w(x, D); \quad \mathcal{M}(\lambda, W) \text{ has simple eigenvalues, and } \exists \text{ a unitary matrix } U \text{ s.t.}$$

$$U^*\mathcal{M}(\lambda, W)U \text{ is diagonal and } \text{Im}(U^*\mathcal{N}(\lambda, W)U)_{\ell, \ell} < 0, 1 \leq \ell \leq m,$$

where $(U^*\mathcal{N}U)_{\ell, \ell}$ is the $\ell$-th diagonal entry of $U^*\mathcal{N}U$. We show that $\mathcal{U}_\lambda$ is open dense in $\mathcal{H}$.

$\mathcal{U}_\lambda$ is open. Suppose $W \in \mathcal{U}_\lambda$. Let $c_1, \ldots, c_m$ be the eigenvalues of $\mathcal{M} := \mathcal{M}(\lambda, W)$ with unit eigenvectors $v_1, \ldots, v_m \in \mathbb{C}^m$. For $1 \leq \ell \leq m$, let $\gamma_\ell(\epsilon)$ be a circle centered at $c_\ell$ with radius $\epsilon > 0$. Since all the eigenvalues of $\mathcal{M}$ are simple, there exists $C(\epsilon)$ such that if $|\tilde{w} - w|_{\mathcal{H}} < C(\epsilon)$, $\tilde{W} := \tilde{w}(x, D), \tilde{\mathcal{M}} := \mathcal{M}(\lambda, \tilde{W})$ has only simple eigenvalues $\tilde{c}_\ell$ in each disk enclosed by $\gamma_\ell(\epsilon/2)$, with unit eigenvectors $\tilde{v}_\ell$. Let

$$\Pi_\ell := \frac{1}{2\pi i} \oint_{\gamma_\ell(\epsilon)} (\xi I - \tilde{\mathcal{M}})^{-1} d\xi, \quad \tilde{\Pi}_\ell := \frac{1}{2\pi i} \oint_{\gamma_\ell(\epsilon)} (\xi I - \mathcal{M})^{-1} d\xi. \quad (5.9)$$

Then when $|\tilde{w} - w|_{\mathcal{H}} \leq \min\{C(\epsilon), \epsilon^2/C\}$ such that $\|\tilde{\mathcal{M}} - \mathcal{M}\| \leq \epsilon^2/2$, we have

$$\|\tilde{\Pi}_\ell - \Pi_\ell\| \leq \frac{1}{2\pi} \oint_{\gamma_\ell(\epsilon)} \|(\xi I - \tilde{\mathcal{M}})^{-1}\|(\xi I - \mathcal{M})^{-1}\|d\xi \leq \frac{2}{\epsilon} \|\tilde{\mathcal{M}} - \mathcal{M}\| \leq \epsilon.$$

Note that $\Pi_\ell = v_\ell \otimes v_\ell, \tilde{\Pi}_\ell = \tilde{v}_\ell \otimes \tilde{v}_\ell$, hence if we choose $\theta_0$ such that $e^{i\theta_0}(\tilde{v}_\ell, v_\ell) = |\langle \tilde{v}_\ell, v_\ell \rangle|$, then

$$\|v_\ell - e^{i\theta_0}\tilde{v}_\ell\| \leq \sqrt{2}\|\tilde{\Pi}_\ell - \Pi_\ell\|(\tilde{v}_\ell) \leq \sqrt{2}\epsilon. \quad (5.10)$$

This shows that we can find a unitary matrix $\tilde{U}$ such that

$$\tilde{U}^*\tilde{\mathcal{M}}\tilde{U} \text{ is diagonal and } \|\tilde{U} - U\| \leq C\epsilon. \quad (5.11)$$

By (1.7), we know $\mathcal{N}(\lambda, W)$ depends continuously on $W$. Hence for $C(\epsilon)$ small and $|\tilde{w} - w|_{\mathcal{H}} < C(\epsilon)$, we have $\|\mathcal{N} - \tilde{\mathcal{N}}\| \leq \epsilon$. Since

$$\tilde{U}^*\tilde{\mathcal{N}}\tilde{U} - U^*\mathcal{N}U = (\tilde{U} - U)^*\tilde{\mathcal{N}}\tilde{U} + U^*\mathcal{N}(\tilde{U} - U)$$

and each term on the right hand side can be bounded by $O(\epsilon)$, we find

$$\|\tilde{U}^*\tilde{\mathcal{N}}\tilde{U} - U^*\mathcal{N}U\| \leq C\epsilon.$$

For $1 \leq \ell \leq m$, since $\text{Im}(U^*\mathcal{N}U)_{\ell, \ell} < 0$, we know $\text{Im}(\tilde{U}^*\tilde{\mathcal{N}}\tilde{U})_{\ell, \ell} < 0$. This implies $\mathcal{U}_\lambda$ is open.

$\mathcal{U}_\lambda$ is dense. Suppose $w \in \mathcal{H}, W = w(x, D)$. We can find orthonormal eigenfunctions $u_1, \ldots, u_m$ of $P$ such that $\mathcal{M}(\lambda, W)$ is diagonal. Let $u_0$ be an analytic function as in Lemma 5.2, $\|u_0\|_{L^2} = 1$. We can further assume that $u_0 \perp \text{Ker}_{L^2}(P - \lambda)$. Now we consider

$$W_1 = \sum_{1 \leq \ell \leq m} \alpha_\ell u_\ell \otimes u_\ell + \frac{1}{2} \sum_{1 \leq \ell \leq m} \beta_\ell (u_0 \otimes u_\ell + u_\ell \otimes u_0), \quad \alpha_\ell, \beta_\ell \in \mathbb{R}. \quad (5.12)$$
Since $u_\ell, 0 \leq \ell \leq m$, are analytic, we know the symbol $\sigma(W_1) \in \mathcal{W}$ and
\[ |\sigma(W_1)|_{\mathcal{W}} \leq C \max_{1 \leq k \leq m} \{ |\alpha_k|, |\beta_k| \}. \tag{5.13} \]

Since $\mathcal{M}(\lambda, W)$ is diagonal, we have
\[ \mathcal{M}(\lambda, W + W_1) = \mathcal{M}(\lambda, W) + \text{diag}\{\alpha_1, \cdots, \alpha_m\} \tag{5.14} \]
is also diagonal and for any $\epsilon > 0$, we can find
\[ \max\{ |\alpha_1|, \cdots, |\alpha_m| \} \leq \epsilon \tag{5.15} \]
such that $\mathcal{M}(\lambda, W + W_1)$ has only simple eigenvalues.

Let $c_{2,\ell}, \tilde{c}_{2,\ell}, 1 \leq \ell \leq m$, be the diagonal entries of $\mathcal{N}(\lambda, W)$ and $\mathcal{N}(\lambda, W + W_1)$. Then we have
\[
\begin{align*}
\text{Im} \tilde{c}_{2,\ell} &= \text{Im} c_{2,\ell} + b_\ell \beta_\ell + a_\ell \beta_\ell^2, \\
\text{Im} c_{2,\ell} &= -\text{Im} \langle \Pi_\perp R(\lambda) \Pi_\perp W u_\ell, Wu_\ell \rangle, \\
a_\ell &= -\frac{1}{4} \text{Im} \langle \Pi_\perp R(\lambda) \Pi_\perp u_0, u_0 \rangle, \\
b_\ell &= -\frac{1}{2} \text{Im} \langle \Pi_\perp (R(\lambda) - R(\lambda)^*) \Pi_\perp Wu_\ell, u_0 \rangle. \tag{5.16}
\end{align*}
\]

By the proof of Lemma 5.1 and the choice of $u_0$, we know
\[ \text{Im} \tilde{c}_{2,\ell} \leq 0, \ a_\ell = -\frac{\pi^2}{2} |\beta_\ell|^2 \int_{\mathbb{S}^1} |G^+_\lambda \Pi_\perp R(\lambda) \Pi_\perp u_0|^2 dS < 0, \ 1 \leq \ell \leq m. \tag{5.17} \]

Therefore, we can always choose $\beta_\ell$ such that
\[ \max\{ |\beta_1|, \cdots, |\beta_m| \} \leq \epsilon, \text{ and } \text{Im} \tilde{c}_{2,\ell} < 0, \ 1 \leq \ell \leq m. \tag{5.18} \]

Notice that (5.13), (5.15), (5.18) guarantee that
\[ |\sigma(W_1)|_{\mathcal{W}} \leq C \epsilon. \tag{5.19} \]

This shows that $\mathcal{U}_\lambda$ is dense in $\mathcal{W}$.

Now we put
\[ \mathcal{U}_K := \cap_{\lambda \in \text{Spec}_{pp}(P) \cap K} \mathcal{U}_\lambda. \tag{5.20} \]

By (5.7), the definition of $\mathcal{U}_\lambda$ and Remark 2 after Theorem 1, $P + sW$ has only simple non-real resonances in an open neighborhood of $K \subset \mathbb{C}$ for $s \neq 0$. This implies that there exists $\delta = \delta (w, K) > 0$ such that
\[ \text{Spec}_{pp}(P + sW) \cap K = \emptyset, \ s \in (-\delta, \delta) \setminus \{0\}. \tag{5.21} \]

Let $\{K_j\}_{j \geq 0}$ be a sequence of closed intervals such that $\cup_{j \geq 0} K_j = \emptyset$. Then $\{\mathcal{U}_{K_j}\}_{j \geq 0}$ satisfies conditions in Theorem 2. \qed
6. Viscosity Limits

In this section, we prove Theorem 3 by proposing a Grushin problem. We first introduce some notations. Let $\Lambda$ be as in Sect. 2.1. Let $\overline{\mathcal{T}}_\Lambda$ be the complex deformation of $\mathcal{T}$ associated to $\Lambda$. Let $\overline{S}_\Lambda$ be the inverse of $\overline{\mathcal{T}}_\Lambda$, $\Pi_\Lambda : L^2_\Lambda \to H_\Lambda$ be the orthogonal projection. For the construction and the properties of $\overline{\mathcal{T}}_\Lambda$, $\overline{S}_\Lambda$, $\Pi_\Lambda$, see [GaZw19, §4,5].

**Lemma 6.1.** Assume $P \in \Psi^0(T^n)$ satisfies conditions in Theorem 3, and $\zeta_0, t_0, \epsilon$ are as in Sect. 2.2. Let $q = q(\frac{1}{2}, \epsilon) \in C_c^\infty(\Lambda; [0, \infty))$ be as in [GaZw19, Lemma 7.6] (with $(\epsilon, \theta)$ there being replaced by $(\frac{1}{2}, \epsilon)$), $Q := \overline{S}_\Lambda \Pi_\Lambda q \Pi_\Lambda \overline{\mathcal{T}}_\Lambda$. Then for $\zeta \in \Omega_{\zeta_0, \epsilon}$,

$$\frac{\partial}{\partial t}|_{t=0}(R_{q,t}(\zeta)Q) = -iR_{q,0}(\zeta)\Delta R_{q,0}(\zeta)Q : H_\Lambda \to H_\Lambda$$

(6.1)

is bounded, where $R_{q,t}(\zeta)$ is the resolvent of $P + it\Delta - iQ$.

**Proof.** The existence and boundedness of $R_{q,t}(\zeta) : H_\Lambda \to H_\Lambda$ is proved in [GaZw19, Lemma 7.6]. For $t > 0$, we have

$$t^{-1}(R_{q,t}(\zeta) - R_{q,0}(\zeta))Q + iR_{q,0}(\zeta)\Delta R_{q,0}(\zeta)Q = -tR_{q,t}(\zeta)\Delta R_{q,0}(\zeta)\Delta R_{q,0}(\zeta)Q$$

(6.2)

Let $H^s_\Lambda$ be closure of $\mathcal{S}_\delta, 0 < \delta \ll 1$, with the norm defined by (2.5) and $d\alpha$ replaced by $(\Re \alpha)^{2s}d\alpha$ – see [GaZw19, (4.7)]. Then by [GaZw19, Lemma 7.6], for any $N > 0$,

$$Q : H^s_\Lambda \to H^{s+N}_\Lambda, R_{q,t}(\zeta) : H^s_\Lambda \to H^{s-\frac{1}{2}}_\Lambda$$

(6.3)

are bounded uniformly in $(t, \zeta) \in [0, t_0] \times \Omega_{\zeta_0, \epsilon}$. Thus there exists $0 < C < \infty$ that does not depend on $t, \zeta$ and

$$\|R_{q,t}(\zeta)\Delta R_{q,0}(\zeta)\Delta R_{q,0}(\zeta)\|_{H_\Lambda \to H_\Lambda} \leq C, \ (t, \zeta) \in [0, t_0] \times \Omega_{\zeta_0, \epsilon}.$$  

This and (6.2) implies (6.1). □

**Proof of Theorem 3.** Let $q$, $Q$, $R_{q,t}(\zeta)$ be as in Lemma 6.1. We denote

$$P_{q,t} := P + it\Delta - iQ,$$

(6.4)

Note that

$$R_{0,t}(\zeta) = (I + iR_{q,t}(\zeta)Q)^{-1}R_{q,t}(\zeta).$$

(6.5)

Since $R_{q,t}(\zeta) : H_\Lambda \to H_\Lambda$ exists for $(t, \zeta) \in [0, t_0] \times \Omega_{\zeta_0, \epsilon}$ by [GaZw19, Lemma 7.6], we know the eigenvalues of $P_{0,t}$ in $\Omega_{\zeta_0, \epsilon}$ are values of $\zeta$ such that $I + iR_{q,t}(\zeta)Q : H_\Lambda \to H_\Lambda$ is not invertible. Since by (2.10),

$$R_{0,0}(\zeta) = A(\zeta) + \frac{u \otimes u}{\lambda - \zeta},$$

(6.6)

we have

$$(I + iR_{q,0}(\zeta)Q)^{-1} = A(\zeta)(P_{q,0} - \zeta) + \frac{u \otimes u}{\lambda - \zeta}(P_{q,0} - \zeta)$$

(6.7)
We now consider the Grushin problem
\[
P_{q,t}(\xi) := \left( I + i R_{q,t}(\xi) Q \frac{R_+}{R_-} \right) : H_\Lambda \times \mathbb{C} \to H_\Lambda \times \mathbb{C}, \quad (t, \xi) \in [0, t_0] \times \Omega_{\xi_0, \epsilon},
\]
(6.8)
where $R_+ : H_\Lambda \to \mathbb{C}$, $R_- : \mathbb{C} \to H_\Lambda$ are defined by
\[
R_+ w = \langle w, u \rangle_{L^2(T^n)}, \quad R_- w_\perp = w_\perp R_{q,0}(\xi) u.
\]
As in the proof of Theorem 1, this Grushin problem is well-posed and has an inverse
\[
E_{q,t}(\xi) := \left( E_{q,t}(\xi) E_{+,q,t}(\xi) \right) : H_\Lambda \times \mathbb{C} \to H_\Lambda \times \mathbb{C}
\]
(6.9)
with
\[
E_{q,0}(\xi) = A(\xi)(P_{q,0} - \xi), \quad E_{+,q,0}(\xi) = \xi - \lambda,
\]
\[
E_{-,q,0}(\xi)v = \langle (P_{q,0} - \xi)v, u \rangle_{L^2(T^n)}, \quad E_{+,q,0}(\xi)v_+ = v_+ u.
\]
Since $I + i R_{q,t}(\xi) Q$ is invertible if and only if $E_{+,q,t}(\xi)$ is invertible, we know $\lambda(t)$ must satisfy
\[
E_{+,q,t}(\lambda(t)) = 0.
\]
Differentiate with respect to $t$ and we find
\[
\dot{\lambda} = -\frac{\partial_t E_{+,q,0}(\lambda)}{\partial \xi E_{+,q,0}(\lambda)} = -\partial_t E_{+,q,0}(\lambda).
\]
Now differentiate $P_{q,t}(\xi) E_{q,t}(\xi) = I$ with respect to $t$ and we find
\[
\dot{E}_{q,0}(\lambda) = -E_{q,0}(\lambda) P_{q,0}(\xi) E_{q,0}(\lambda).
\]
This shows that
\[
\partial_t E_{+,q,0}(\lambda) = -E_{-,q,0}(\lambda) \frac{\partial}{\partial t} \big|_{t=0} (I + i R_{q,t}(\xi) Q) E_{+,q,0}(\lambda).
\]
(6.10)
Use Lemma 6.1 and we find
\[
\dot{\lambda} = E_{-,q,0}(\lambda) R_{q,0}(\lambda) \Delta R_{q,0}(\lambda) Q E_{+,q,0}(\lambda) = -i \| \nabla u \|^2_{L^2(T^n)}.
\]
(6.11)
Here we used the fact that $R_{q,0}(\lambda) u = i u$. This completes the proof.

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References

[CdV83] Colin de Verdière, Y.: Pseudo-Laplacian. II. Ann. Inst. Fourier 33, 87–113 (1983)
[CdV19] Colin de Verdière, Y.: Spectral theory of pseudo-differential operators of degree 0 and applications to forced waves. Anal. PDE 13(5), 1521–1537 (2020)
[CS20] Colin de Verdière, Y., Saint-Raymond, L.: Attractors for two dimensional waves with homogeneous Hamiltonian of degree 0. Commun. Pure Appl. Math. 73, 421–462 (2020)
[DyZw19a] Dyatlov, S., Zworski, M.: Mathematical theory of scattering resonances. Graduate studies in mathematics, vol. 200. AMS, Providence (2019)
[DyZw19b] Dyatlov, S., Zworski, M.: Microlocal analysis of forced waves. Pure Appl. Anal. 1, 359–394 (2019)
[GaZw19] Galkowski, J., Zworski, M.: Viscosity limits for 0th order pseudodifferential operators. arXiv:1912.09840, to appear in Communications on Pure and Applied Mathematics
[GaZw20] Galkowski, J., Zworski, M.: Analytic hypoellipticity of Keldysh operators. arXiv:2003.08106, to appear in Proceedings of London Mathematical Society
[HeSj86] Helffer, Bernard, Sjöstrand, Johannes: Resonances en limite semiclassique. Bull. Soc. Math. France 114, 24–25 (1986)
[Ho74] Howland, James: Puiseux series for resonances at an embedded eigenvalue. Pacific J. Math. 55, 157–176 (1974)
[Hö83a] Hörmander, Lars: The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis. Springer, Berlin (1983)
[Hö90] Hörmander, L.: An Introduction to Complex Analysis in Several Variables, 3rd edn. Elsevier, Amsterdam (1990)
[Ka80] Kato, Tosio: Perturbation Theory for Linear Operators. Springer, Berlin, Heidelberg (1980)
[LeZw16] Lee, Minjae, Zworski, Maciej: A Fermi golden rule for quantum graphs. J. Math. Phys. 57, 092101 (2016)
[Ma02] Martinez, A.: An Introduction to Semiclassical and Microlocal Analysis. Springer, New York (2002)
[NZ99] Nikolaev, I., Zhuzhoma, E.: Flows on 2-Dimensional Manifolds. An Overview. Springer, Berlin (1999)
[PhSa92] Phillips, R., Sarnak, P.: Perturbation theory for the Laplacian on automorphic functions. J. Am. Math. Soc. 5, 1–32 (1992)
[Ra73] Ralston, James: On stationary modes in inviscid rotating fluid. J. Math. Anal. Appl. 44, 366–383 (1973)
[Si73] Simon, Barry: Resonances in n-body quantum systems with dilation analytic potentials and the foundations of time-dependent perturbation theory. Ann. Math. 97, 247–274 (1973)
[Sj96] Sjöstrand, Johannes: Density of resonances for strictly convex analytic obstacles. Can. J. Math. 48, 397–447 (1996)
[SjZw07] Sjöstrand, Johannes, Zworski, Maciej: Elementary linear algebra for advanced spectral problems. Ann. de l’Institut Fourier 57, 2095–2141 (2007)
[Ta19] Tao, Z.: 0-th order pseudodifferential operators on the circle. arXiv:1909.06316, to appear in Proceedings of AMS
[Wal78] Walker, J.R.: Algebraic Curves. Springer-Verlag, Berlin (1978)
[Wan19] Wang, J.: The scattering matrix for 0th order pseudodifferential operators. arXiv:1909.06484
[Zw12] Zworski, M.: Semiclassical Analysis. Graduate Studies in Mathematics, vol. 138. AMS, Providence (2012)

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