THE ASYMPTOTIC TIAN-YAU-ZELDITCH EXPANSION ON RIEMANN SURFACES WITH CONSTANT CURVATURE

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Abstract. Let $M$ be a regular Riemann surface with a metric which has constant scalar curvature $\rho$. We give the asymptotic expansion of the sum of the square norm of the sections of the pluricanonical bundles $K_M^m$. That is,

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|^2_{h_m} \sim m(1 + \frac{\rho}{2m}) + O\left(e^{-\frac{\|x_0\|^2}{8}}\right),$$

where \{\{S_0, \cdots, S_{d_m-1}\}\} is an orthonormal basis for $H^0(M, K_M^m)$ for sufficiently large $m$.

1. Introduction

Let $M$ be an $n$-dimensional compact complex Kähler manifold with an ample line bundle $L$ over $M$. Let $g$ be the Kähler metric on $M$ corresponding to the Kähler form $\omega_g = \text{Ric}(h)$ for some positive Hermitian $h$ metric on $L$. Such a Kähler metric $g$ is called a polarized Kähler metric. The metric $h$ induces a Hermitian metric $h_m$ on $L^m$ for all positive integers $m$. Let \{\{S_0, \cdots, S_{d_m-1}\}\} be an orthonormal basis of the space $H^0(M, L^m)$ with respect to the inner product

$$\langle S, T \rangle = \int_M \langle S(x), T(x) \rangle_{h_m} dV_g,$$

where $d_m = \dim H^0(M, L^m)$ and $dV_g = \frac{\omega^n}{n!}$ is the volume form of $g$. The quantity

$$\sum_{i=0}^{d_m-1} \|S_i(x)\|^2_{h_m}$$

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is related to the existence of Kähler-Einstein metrics and stability of complex manifolds. A lot of work has been done for (1.2) on compact complex Kähler manifolds. Tian [6] applied Hömander’s $L^2$-estimate to produce peak sections and proved the $C^2$ convergence of the Bergman metrics. Later, Ruan [5] proved the $C^\infty$ convergence. About the same time, Zelditch [7] and Catlin [4] separately generalized the theorem of Tian by showing there is an asymptotic expansion

$$(1.3) \sum_{i=0}^{d_m-1} \|S_i(x)\|_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \ldots$$

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$. In [10], Lu proved that each coefficient $a_j(x)$ is a polynomial of the curvature and its covariant derivatives. In particular, $a_1 = \frac{\rho}{2}$, where $\rho$ is the scalar curvature of $M$. These polynomials can be found by finitely many steps of algebraic operations. Recently, Song [3] generalized Zelditch’s theorem on orbifolds of finite isolated singularities. The information on the singularities can be found in the expansion.

On the Riemann surfaces with bounded curvature, Lu [9] proved that there is a lower bound for (1.2). Later, the result of Lu and Tian [8] implies that on the Riemann surfaces with constant scalar curvature $\rho$, the asymptotic expansion (1.3) is given by

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|_{h_m}^2 \sim m(1 + \frac{\rho}{2m}) + O\left(\frac{1}{m^p}\right)$$

for any $p > 0$. In the current paper, we obtain a more precise result for (1.3).

**Theorem 1.1.** Let $M$ be a regular compact Riemann surface and $K_M$ be the canonical line bundle endowed with a Hermitian metric $h$ such that the curvature $\text{Ric}(h)$ of $h$ defines a Kähler metric $g$ on $M$. Suppose that this metric $g$ has constant scalar curvature $\rho$. Then there is a complete asymptotic expansion:

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|_{h_m}^2 \sim m(1 + \frac{\rho}{2m}) + O\left(e^{-\frac{(\log m)^2}{8}}\right),$$

where $\{S_0, \ldots, S_{d_m-1}\}$ is an orthonormal basis for $H^0(M, K_M^m)$ for some $m > \max\{e^{20\sqrt{5}} + 2|\rho|, |\rho|^{4/3}, \frac{1}{8}\sqrt{\frac{2}{|\rho|}}\}$, where $\delta$ is the injective radius at $x_0$.

Our result indicates that the asymptotic expansion (1.3) is in exponential decay. Englis [2] has an asymptotically expansion for the Berezin transformation on any planar domain of hyperbolic type. He also showed that Berezin kernel [1] has

$$\tilde{B}(\eta, \eta) = m \left(1 + O(1)\rho_0(0)^{\frac{m-1}{2}}\right),$$

where $\rho_0(0)$ is a positive constant.
2. General Set Up

Let \( M \) be an \( n \)-dimensional compact complex Kähler manifold with a polarized line bundle \((L, h) \to M\). Choose the \( K \)-coordinates \((z_1, \cdots, z_n)\) on an open neighborhood \( U \) of a fixed point \( x_0 \in M \). Then the Kähler form
\[
\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta=1}^n g_{\alpha\bar{\beta}} \, dz_\alpha \wedge d\bar{z}_\beta
\]
satisfies
\[
(2.1) \quad g_{\alpha\bar{\beta}}(x_0) = \delta_{\alpha\bar{\beta}}, \quad \frac{\partial^{p_1+\cdots+p_n}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}} (x_0) = 0,
\]
for \( \alpha, \beta = 1, \cdots, n \) and any nonnegative integers \( p_1, \cdots, p_n \) with \( p_1 + \cdots + p_n \neq 0 \).

We also choose a local holomorphic frame \( e_L \) of the line bundle \( L \) at \( x_0 \) such that \( a \) is the local representation function of the Hermitian metric \( h \). That is,
\[
\text{Ric}(h) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log a.
\]
Under the \( K \)-coordinate, the function \( a \) has the properties
\[
(2.2) \quad a(x_0) = 1, \quad \frac{\partial^{p_1+\cdots+p_n}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}} (a)(x_0) = 0
\]
for any nonnegative integers \( p_1, \cdots, p_n \) with \( p_1 + \cdots + p_n \neq 0 \).

Let \( \{S_0, \cdots, S_{d_m-1}\} \) be a basis of \( H^0(M, L^m) \). Assume that at the point \( x_0 \in M \),
\[
S_0(x_0) \neq 0, \quad S_i(x_0) = 0, \quad i = 1, \cdots, d_m - 1.
\]
If the set \( \{S_0, \cdots, S_{d_m-1}\} \) is not an orthonormal basis, we may do the following: Let the metric matrix
\[
F_{ij} = (S_i, S_j), \quad i, j = 0, \cdots, d_m - 1
\]
with respect to the inner product (1.1). By definition, \( (F_{ij}) \) is a positive definite Hermitian matrix. We can find a \( d_m \times d_m \) matrix \( G_{ij} \) such that
\[
F_{ij} = \sum_{k=0}^{d_m-1} G_{ik} G_{jk}.
\]
Let \( (H_{ij}) \) be the inverse of \( (G_{ij}) \). Then \( \{\sum_{j=0}^{d_m-1} H_{ij} S_j\} \) forms an orthonormal basis of \( H^0(M, L^m) \). The left hand side of (1.2) is equal to
\[
(2.3) \quad \sum_{i=0}^{d_m-1} \left\| \sum_{j=0}^{d_m-1} H_{ij} S_j(x_0) \right\|^2_{h_m} = \sum_{i=0}^{d_m-1} |H_{i0}|^2 \|S_0(x_0)\|^2_{h_m}.
\]
Let \((I_{ij})\) be the inverse matrix of \((F_{ij})\). Denote that
\[
(2.4) \quad \sum_{i=0}^{d_m-1} |H_{i0}|^2 = I_{00}.
\]
In order to compute (2.4), we need a suitable choice of the basis \(\{S_0, \ldots, S_{d_m-1}\}\).
We select some of Tian’s peak sections in our basis. The following lemma is an improved version of Tian’s result [6, Lemma 1.2], which is done by Lu and Tian.

Let \(Z^m_+\) be the set of \(n\)-tuple integers \(P = (p_1, \ldots, p_n)\) such that each \(p_i\) is a nonnegative integer for \(i = 1, \ldots, n\). For \(P \in Z^m_+\), we denote that \(z^P = z_{p_1} \cdots z_{p_n}\) and \(|P| = p_1 + \cdots + p_n\).

**Lemma 2.1.** ([Tian]). Suppose \(\text{Ric}(g) \geq -K \omega g\), where \(K > 0\) is a constant.

For \(P \in Z^m_+\) and an integer \(p' > |P|\), let \(m\) be an integer such that
\[
m > \max \{e^{20\sqrt{n+2p'}} + 2K, e^{8(p'-1+n)}\}.
\]
Then there is a holomorphic section \(S_{P,m} \in H^0(M, L^m)\), satisfying
\[
\left| \int_M \|S_{P,m}\|^2 h_m dV_g - 1 \right| \leq Ce^{-\frac{1}{8}(\log m)^2}.
\]
Moreover, \(S_{P,m}\) can be decomposed as
\[
S_{P,m} = \tilde{S}_{P,m} - u_{P,m}
\]
such that
\[
\tilde{S}_{P,m}(x) = \lambda_P \eta \left( \frac{m|z|^2}{(\log m)^2} \right) z^P e^{m} = \begin{cases} \lambda_P z^P e^{m} & x \in \{|z| \leq \frac{\log m}{\sqrt{2m}}\}, \\ 0 & x \in M \setminus \{|z| \leq \frac{\log m}{\sqrt{m}}\}, \end{cases}
\]
and
\[
\int_M \|u_{P,m}\|^2 h_m dV_g \leq Ce^{-\frac{1}{4}(\log m)^2},
\]
where \(\eta\) is a smoothly cut-off function
\[
\eta(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{for } t \geq 1. \end{cases}
\]
satisfying \(0 \leq -\eta'(t) \leq 4\) and \(|\eta''(t)| \leq 8\) and
\[
(2.8) \quad \lambda^2_P = \int_{|z| \leq \frac{\log m}{\sqrt{2m}}} |z^P|^2 \omega \eta^2 dV_g.
\]
Proof. Define the weight function
\[ \Psi(z) = (n + 2p') \eta \left( \frac{m|z|^2}{(\log m)^2} \right) \log \left( \frac{m|z|^2}{(\log m)^2} \right). \]
A straightforward computation gives
\[ (2.9) \quad \sqrt{-1} \partial \bar{\partial} \Psi \geq -100m(n + 2p') \frac{(\log m)^2}{(\log m)^2} \omega_g. \]
By using (2.9), we can verify that
\[ \langle \partial \bar{\partial} \Psi + \frac{2\pi}{\sqrt{-1}} (Ric(h^m) + Ric(g)), v \wedge \bar{v} \rangle_g \geq \frac{1}{4} m \|v\|^2_g. \]
For \( P \in \mathbb{Z}^n_+ \), consider the 1-form
\[ w_P = \bar{\partial} \eta \left( \frac{m|z|^2}{(\log m)^2} \right) z^P e^m. \]
Since \( w_P \equiv 0 \) in a neighborhood of \( x_0 \), we have
\[ \int_M \|w_P\|^2 h^m e^{-\Psi} dV_g < +\infty. \]
By [6, Prop. 2.1], there exists a smooth \( L^m \)-valued section \( u_P \) such that \( \bar{\partial}u_P = w_P \) and
\[ (2.10) \quad \int_M \|u_P\|^2 h^m e^{-\Psi} dV_g \leq \frac{4}{m} \int_M \|w_P\|^2 h^m e^{-\Psi} dV_g < \infty. \]
By direct computation, we get
\[ \int_M \|u_P\|^2 h^m e^{-\Psi} dV_g \leq C_1 (\log m)^2 (p-1) \int_{\log m \leq |z| \leq \log m} a^m dV_0. \]
Under the \( K \)-coordinate, we have
\[ a^m = e^{m \log a} = e^{m(-|z|^2 + O(|z|^4))}. \]
Hence we get
\[ \int_M \|u_P\|^2 h^m e^{-\Psi} dV_g \leq C_1 (\log m)^2 (p-1+n) \frac{e^{-\frac{1}{2}(\log m)^2}}{m^{p+n}} \]
for some constant \( C_1 \). Let \( \tilde{S}_{P,m} = \lambda_P \eta \left( \frac{m|z|^2}{(\log m)^2} \right) z^P e^m \) and \( u_{P,m} = \lambda_P u_P \). Use a result in [10]
\[ \lambda_P^2 \leq C_2 m^{n+|P|} \]
for some constant $C_2$. Then we have
\[ \int_M \|u_{P,m}\|_{h_m}^2 dV_g \leq C (\log m)^2 (P|\rho|^{-1+n}) e^{-\frac{1}{2} (\log m)^2}. \]
Choosing $m > e^{8(p'-1+n)}$, we obtain
\[ \int_M \|u_{P,m}\|_{h_m}^2 dV_g \leq C e^{-\frac{1}{4} (\log m)^2}. \]

3. PROOF OF THEOREM 1.1

**Proof.** Let $M$ be a smooth compact Riemann surface with a metric $g$ that has constant scalar curvature. Let $x_0$ be a fixed point. Let
\[ U = \{ x : \text{dist}(x, x_0) < \delta \}, \]
where $\delta$ is the injective radius at $x_0$. It is well known that on a Riemann surface there is an isothermal coordinate at each point on $U$. We may assume that there is a holomorphic function $z$ on $U$ and it defines the conformal structure on $U$. That is,
\[ ds^2 = g dz d\bar{z} \]
and $g > 0$. The metric $g$ satisfies
\begin{equation}
\triangle \log g = -\rho, \quad g(x_0) = 1, \quad \text{and} \quad \frac{\partial g}{\partial z}(x_0) = 0,
\end{equation}
where
\[ \triangle = g^{-1} \frac{\partial^2}{\partial z \partial \bar{z}} \]
is the complex Laplace of $M$. Since the metric $g$ has conformal structure, it is rotationally symmetric. We can write (3.1) in polar coordinates $(r, \theta)$:
\begin{equation}
\frac{\partial^2 g}{\partial r^2} + \frac{1}{r} \frac{\partial g}{\partial r} - \frac{1}{g} \left( \frac{\partial g}{\partial r} \right)^2 = -4 \rho g^2, \quad g(0, \theta) = 1, \quad \frac{\partial g}{\partial r}(0, \theta) = 0,
\end{equation}
where $z = re^{i\theta}$, and $|z|^2 = r^2$. There exists a solution
\begin{equation}
g = \frac{1}{(1 + \frac{\rho}{2} |z|^2)^2}
\end{equation}
to (3.2) for $|z| < \sqrt{-\frac{2}{\rho}}$ if $\rho < 0$. Suppose that there exists another solution $g_1$ to (3.2). We have
\[ \triangle \log (g_1/g) = 0 \quad \text{and} \quad g_1(x_0) = 1. \]
For $\rho < 0$, let $r_0 < \sqrt{-2/\rho}$. Since $g$ and $g_1$ are rotationally symmetric, they remain constant on $|z| = r_0$. The harmonic function $\log(g/g_1)$ is a constant on $|z| \leq r_0$ by Maximum Principle. By definition, we have $g(x_0) = g_1(x_0) = 1$. Therefore, the solution in (3.3) is unique around $x_0$. By the same reason, $g = g_1$ on $\{\text{dist}(x, x_0) \leq \delta_1\}$ for some $\delta_1 < \delta$ for $\rho \leq 0$.

Let $a$ be the local representation of the metric $h$ on $K_M$ such that

$$-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log a = \omega_g.$$  

If we normalize $a$ and $a$ satisfies

$$\triangle \log a = -1, \quad a(x_0) = 1, \quad \frac{\partial a}{\partial z}(x_0) = 0.$$  

Since

$$-\frac{\partial^2}{\partial z \partial \bar{z}} \log a = g,$$

$\log a$ is also rotationally symmetric. Since

$$a = \begin{cases} \left(1 + \frac{\rho}{2} |z|^2\right)^{-\frac{2}{\rho}}, & \text{if } \rho \neq 0; \\
 e^{-|z|^2}, & \text{if } \rho = 0. \end{cases}$$

satisfies (3.4), the local uniqueness is due to the same reason.

We need to choose sufficient large $m$ such that $\frac{\log m}{\sqrt{m}} < \min\{\delta, \sqrt{\frac{2}{|\rho|}}\}$. With these particular solutions of $g$ and $a$, we further compute

$$\lambda_0^{-2} = \int_{|z| \leq \sqrt{\frac{m}{m}}} a^m g \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}$$

$$= 2 \int_0^{\sqrt{\frac{m}{m}}} (1 + \frac{\rho}{2} |z|^2)^{-\frac{2m}{\rho}} \sqrt{\frac{m}{m}} r dr$$

$$= \frac{1}{m + \frac{\rho}{2}} \left(1 - \left(1 + \frac{\rho \left(\log m\right)^2}{m}\right)^{-1 - \frac{2m}{\rho}}\right) \quad \text{for } \rho \neq 0.$$  

For $m > \max\{|\rho|^{1/3}, 10\}$, we have $\left|\frac{\rho \left(\log m\right)^2}{m}\right| < 1/2$. For $\rho \neq 0$, this gives

$$\left(1 + \frac{\rho \left(\log m\right)^2}{m}\right)^{-1 - \frac{2m}{\rho}} \leq 2e^{-2m/\rho} \left(\frac{\left(\log m\right)^2}{m} \cdot 1 + \frac{\rho \left(\log m\right)^2}{m} \cdot 2 + \cdots\right) \leq Ce^{-\left(\log m\right)^2}.$$  

For $\rho = 0$, we have

$$\lambda_0^{-2} = \int_{|z| \leq \sqrt{\frac{m}{m}}} e^{-m |z|^2} \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} = \frac{1}{m} (1 + O(e^{-\left(\log m\right)^2})).$$
Hence we obtain

\[ \lambda_0^{-2} = \frac{1}{m + \frac{2}{5}} \left( 1 + O\left( e^{-\left(\log m\right)^2} \right) \right). \]  

From the properties of \( g \) and \( a \), the isothermal coordinate \((U, z)\) is a \( K \)-coordinate. According to Lemma 2.1, we may choose two peak sections

\[ S_{0,m} = \lambda_0(\eta(\frac{m|z|^2}{\log m^2}))(dz)^m - u_0 \]
\[ S_{1,m} = \lambda_1(\eta(\frac{m|z|^2}{\log m^2}))(dz)^m - u_1 \]

in \( H^0(M, K_M^{m}) \) for some \( m > e^{20\sqrt{5} + 2|\rho|} \). Obviously, we have \( S_{0,m}(x_0) \neq 0 \) and \( S_{1,m}(x_0) = 0 \). Let the subspace

\[ V = \{ S \in H^0(M, K_M^{m}) | S(x_0) = 0, DS(x_0) = 0 \}, \]

where \( D \) is a covariant derivative on \( K_M^{m} \). Let \( T_1, \cdots, T_{d_m-2} \) be an orthonormal basis of \( V \). Let

\[ S_i = \begin{cases} S_{i,m} & \text{if } i = 0, 1 \\ T_{i-1} & \text{if } 2 \leq i \leq d_m - 1 \end{cases} \]  

Then \( \{S_i\}_{i=0}^{d_m-1} \) forms a basis for \( H^0(M, K_M^{m}) \). Locally, each \( T_i \) has the form \( f_i(dz)^m \) for some holomorphic function \( f_i \) defined in \( U \). The holomorphic function \( f_i \) has Taylor expansion as \( f_i = \sum_{\alpha=2}^{\infty} b_{i\alpha} z^\alpha \) in \( U \), since \( T_i(x_0) = 0 \) and \( DT_i(x_0) = 0 \) for \( 1 \leq i \leq d_m - 2 \).

**Lemma 3.2.** Let \( \{S_i\}_{i=0}^{d_m-1} \) be the basis of \( H^0(M, K_M^{m}) \), defined in (3.8). For \( m > e^{20\sqrt{5} + 2|\rho|} \), the Hermitian matrix

\[ (S_i, S_j) = \int_M \langle S_i(x), S_j(x) \rangle_{h^m} dV_g \]

is given by

\begin{align*}
(S_0, S_0) &= 1 + O\left( e^{-\left(\log m\right)^2} \right), \\
(S_0, S_1) &= O\left( e^{-\left(\log m\right)^2} \right), \\
(S_1, S_0) &= 1 + O\left( e^{-\left(\log m\right)^2} \right), \\
(S_0, S_1) &= O\left( e^{-\left(\log m\right)^2} \right).
\end{align*}
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\[(S_1, S_i) = O \left( e^{-\left(\frac{\log m}{8}\right)^2} \right), \]
\[(S_i, S_j) = \delta_{ij} \] for \(i, j = 2, \ldots, d_m - 1.\)

**Proof.** By definition, we have \((S_i, S_j) = \delta_{ij}\) for \(2 \leq i, j \leq d_m - 1.\) The inner product of \((S_i, S_i)\) for \(0 \leq i \leq 1\) is directly from Lemma 2.1. Since \(a^m g\) is rotationally symmetric, we have
\[
\int_{|z| \leq \frac{\log m}{\sqrt{m}}} z^\alpha a^m gdV_0 = 0 \quad \text{for arbitrary positive integer } \alpha.
\]

Then we get
\[
(S_0, S_1) = (\tilde{S}_0, \tilde{S}_1) + (\lambda_0 u_0, \tilde{S}_1) + (\tilde{S}_0, \lambda_1 u_1) + (u_0, u_1)
= O \left( e^{-\left(\frac{\log m}{8}\right)^2} \right).
\]

Consider
\[
(S_0, S_i) = \int_M \langle \lambda_0 (\eta \left( \frac{m|z|^2}{(\log m)^2} \right)(dz)^m - u_0), f_{i-1}(dz)^m \rangle dV_g
\leq \lambda_0 \int_{|z| \leq \frac{\log m}{\sqrt{m}}} \sum_{\alpha = 2}^\infty (i-1)\bar{z}^\alpha a^m gdV_0 + \lambda_0 \|u_0\| \cdot \|S_i\|.
\]

Thus we have
\[
(S_0, S_i) = O \left( e^{-\left(\frac{\log m}{8}\right)^2} \right) \quad \text{for } 2 \leq i \leq d_m - 1.
\]

Similarly, consider
\[
(S_1, S_j) \leq \lambda_0 \int_{|z| \leq \frac{\log m}{\sqrt{m}}} \sum_{\alpha = 2}^\infty b_{(i-1)\alpha} \bar{z}^\alpha a^m gdV_0 + \lambda_1 \|u_1\| \cdot \|S_i\| \quad \text{for } 2 \leq i \leq d_m - 1.
\]

Since \(a^m g\) is rotationally symmetric, \(\int_{|z| \leq \frac{\log m}{\sqrt{m}}} z \bar{z}^\alpha a^m gdV_0 = 0\) for \(\alpha \geq 2.\) Hence we obtain
\[
(S_1, S_i) = O \left( e^{-\left(\frac{\log m}{8}\right)^2} \right). \quad \blacksquare
\]

According to [10, Definition 3.1], the metric matrix \((F_{ij})\) can be represented by the block matrices.
\[(F_{ij}) = \begin{pmatrix} (S_0, S_0) & (S_0, S_1) & M_{13} \\ (S_1, S_0) & (S_1, S_1) & M_{23} \\ M_{31} & M_{32} & E \end{pmatrix},\]

where \(M_{13} = ((S_0, S_2), \ldots, (S_0, S_{d_m - 1}))\), \(M_{23} = ((S_1, S_2), \ldots, (S_1, S_{d_m - 1}))\), \(M_{31} = M_{13}^T\), \(M_{32} = M_{23}^T\), and \(E\) is a \((d_m - 2) \times (d_m - 2)\) identity matrix. By using [10, Lemma 3.1], we obtain

\[
(3.10) \quad I_{00} = \frac{1}{(S_0, S_0)} + \frac{1}{(S_0, S_0)^2} \left( (S_0, S_1) M_{13} \right) \tilde{M}^{-1} \left( \begin{pmatrix} (S_1, S_0) \\ M_{31} \end{pmatrix} \right),
\]

where

\[
\tilde{M} = \begin{pmatrix} (S_1, S_1) & M_{23} \\ M_{32} & E \end{pmatrix} - \frac{1}{(S_0, S_0)} \begin{pmatrix} (S_1, S_0) \\ M_{31} \end{pmatrix} \begin{pmatrix} (S_0, S_1) & M_{13} \end{pmatrix}.
\]

Applying Lemma 3.2 in (3.10), we get

\[
(3.11) \quad I_{00} = 1 + O \left( e^{-\left(\log m\right)^2 / 8} \right).
\]

In order to evaluate the expansion of (2.3), we are left to find \(\|S_0(x_0)\|_{h_m}^2 = \lambda_0^2\).

From (3.7), we have

\[
\lambda_0^2 = m \left( 1 + \frac{\rho}{2m} \right) \left( 1 + O \left( e^{-\left(\log m\right)^2 / 8} \right) \right).
\]

Therefore, the Tian-Yau-Zelditch expansion according to (2.3) on a Riemann surface with constant scalar curvature \(\rho\) is

\[
I_{00} \lambda_0^2 = (1 + O \left( e^{-\left(\log m\right)^2 / 8} \right)) m \left( 1 + \frac{\rho}{2m} \right) \left( 1 + O \left( e^{-\left(\log m\right)^2 / 8} \right) \right)
\]

\[
= m \left( 1 + \frac{\rho}{2m} \right) + O \left( e^{-\left(\log m\right)^2 / 8} \right)
\]

for \(m > \max\{e^{20\sqrt{5} / 3} + 2|\rho|, |\rho|^{4/3}, 1 \sqrt{\frac{2}{3}} \sqrt{\frac{7}{|\rho|}}\}\).

\[\blacksquare\]

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