A NOTE ON THE CONVEX BODY ISOPERIMETRIC CONJECTURE IN THE PLANE

BO-HSHIUNG WANG AND YE-KAI WANG

ABSTRACT. The convex body isoperimetric conjecture in the plane asserts that the least perimeter to enclose given area inside a unit disk is greater than inside any other convex set of area $\pi$. In this note we confirm two cases of the conjecture: domains symmetric to both coordinate axes and perturbations of unit disk.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. For $0 < A < \text{Area}(\Omega)$, consider the variational problem

$$I_\Omega(A) = \min \{ \text{Length}(\gamma) : \gamma \text{ encloses a region of area } A \text{ inside } \Omega \}.$$ (1)

The function $I_\Omega : (0, \text{Area}(\Omega)) \to (0, \infty)$ is called the isoperimetric profile of $\Omega$. Note that

$$I_\Omega(A) = I_\Omega(\text{Area}(\Omega) - A).$$ (2)

The convex body isoperimetric conjecture in the plane asserts that if $\Omega$ is a convex domain of area $\pi$, then

$$I_\Omega(A) \leq I_{B_1}(A)$$ (3)

for $0 < A < \pi$.

We first learned this conjecture in M. Hutching’s webpage [7]. He attributes it to Wicharamala. F. Morgan’s blog contains an extensive discussion of the conjecture [8], mostly focusing on its higher dimensional version.

The conjecture is completely solved for $A = \pi/2$ by Esposito et. al.

Theorem 1. [5 Theorem 1] If $K$ is an open convex set of $\mathbb{R}^2$, we have:

$$\inf_{G \subset K, |G| = |K|/2} \text{Per}(G; K)^2 \leq \frac{4}{\pi} |K|.$$

Moreover, equality holds if and only if $K$ is a disk.

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In addition to being interesting on its own, the theorem leads to several relative isoperimetric inequalities. The conjecture also holds true for regular polygons \([3, \text{Theorem } 4.1]\). Besides these two cases, little is known.

In this note, we confirm the conjecture for two special cases. The first is

**Theorem 2 (Theorem [13])**. Let \(\mathcal{A}\) be the class of domains \(\Omega\) bounded by smooth convex curves that are symmetric in both coordinate axes and have exactly four vertices. Suppose that \(\Omega \in \mathcal{A}\) has area \(\pi\) and is not a unit disk. Then

\[
I_{\Omega}(A) < I_{B_1}(A)
\]

for \(0 < A < \pi\).

The class \(\mathcal{A}\) has been studied by [1]. It serves as a model for a comparison theorem of isoperimetric profile which leads to a new proof of Grayson-Gage-Hamilton Theorem for curve-shortening flow. The crucial fact about \(\mathcal{A}\) is that the minimizers of (1) for \(\Omega \in \mathcal{A}\) admits a simple characterization.

The second result is obtained by analyzing the first and second variation of length under area-preserving perturbation.

**Theorem 3 (Theorem [14])**. The conjecture holds for perturbations of the unit disk.

The paper is organized as follows. In section 2, we set up the notations and present background materials. Theorem 2 and Theorem 3 are proved in Section 3 and 4 respectively. We include two appendices describing relevant results to our main theorems.

## 2. Preliminaries

All domains considered in this note are assumed to be bounded, connected and have smooth boundary \(\partial \Omega\) whose (signed) curvature is denoted by \(\kappa\).

We start with the following well-known description of the minimizers of the variational problem (1).

**Proposition 4**. A least-perimeter curve enclosing a given area within a region consists of circular arcs or straight line segments meeting the boundary orthogonally. Moreover, if the region is convex, then a least-perimeter curve is connected.

**Proof**. The first assertion is a consequence of the first variation of length. See Lemma 3.2 of [1] for example. The second assertion follows from an observation of Kuwert that \(I_{\Omega}^2\) is concave for convex domains. See Section 2 of [3]. \(\square\)
Definition 5. A perfect arc in $\Omega$ is a circular arc or straight line segment (not necessary a minimizer of (1)) inside $\Omega$ that meets $\partial \Omega$ orthogonally.

We have another consequence of the first variation formula.

Proposition 6. Suppose $\gamma(t)$ is a family of perfect arcs with constant curvature $k(t)$. Let $L(t)$ and $A(t)$ denote the length and the enclosed area of $\gamma(t)$. Then
\[
\frac{dL}{dt} = k(t) \frac{dA}{dt}.
\]

Next, we recall the fundamental Pestov-Ionin inequallity.

Theorem 7. If $\gamma$ is a simple closed smooth curve, then
\[
\kappa_{\max} \geq \sqrt{\frac{\pi}{A}},
\]
where $\kappa_{\max}$ is the maximum curvature and $A$ is the enclosed area. The equality holds if and only if $\gamma$ is a circle.

The original proof [9] is not easily accessible. See the recent lecture note [10] for an elementary account or [11] for a proof using curve shortening flow. We will only use a simple implication of the inequality: If $\Omega$ is a domain with area $\pi$ and is not a unit disk, then the maximum curvature of $\partial \Omega$ is greater than 1 and, as a result of $\int_{\partial \Omega} \kappa ds = 2\pi$ and the isoperimetric inequality, the minimum curvature is less than 1.

An immediate consequence is that the conjecture is true for sufficiently small $A$ without the convexity assumption.

Theorem 8. Let $\Omega$ be a domain with area $\pi$. Then there exists $\delta > 0$ such that
\[
I_\Omega(A) < I_{B_1}(A)
\]
for $0 < A < \delta$.

The assertion follows from Proposition 2.1 of [1]:
\[
\lim_{a \to 0} I_\Omega(a) - \frac{\sqrt{2\pi a}}{a} = -4 \max_{\partial \Omega} \kappa.
\]

See also Section 5 of [3] for a proof that applies to convex regions based on the analysis of regular polygons.

3. Symmetric Domains

We fix a domain $\Omega \in \mathcal{A}$. We assume that the major axis of $\Omega$ lies on the $x$-axis and the minor axis lies on the $y$-axis. Write $C$ for $\partial \Omega$ and denote the unit tangent and unit outward normal of $C$ by $T$ and $N$. Since $C$ is convex, we can parametrize it by the normal vector. Namely, $C = C(\theta)$ with $N = (\cos \theta, \sin \theta)$. 

The following three lemmas were obtained in Section 4 of [1]. We present an elementary proof of them.

**Lemma 9.** We have $C \cdot T < 0$ when $0 < \theta < \pi/2$. In particular, the maximum radius of $\Omega$ are attained at the vertices on the $x$-axis and the minimum radius are attained at the vertices on the $y$-axis.

**Proof.** Denote $\frac{df}{d\theta} = f'$ in the proof. Direct computation yields $(C \cdot N)' = C \cdot T$ and $(C \cdot T)' = \frac{1}{\kappa} - C \cdot N$. Write $\Psi = -C \cdot N$ and we get $\Psi' + \Psi'' = \frac{\kappa'}{\kappa} < 0$ on $(0, \pi/2)$.

Let

$$
P = \Psi' \cos \theta - \Psi'' \sin \theta$$

$$Q = \Psi' \sin \theta + \Psi'' \cos \theta.$$

Then $P(0) = Q(\pi/2) = 0$. Moreover, we have $P' = -\sin \theta (\Psi' + \Psi'') > 0$ and $Q' = \cos \theta (\Psi' + \Psi'') < 0$ on $(0, \pi/2)$. It follows that $P, Q,$ and $\Psi' = -C \cdot T = P \cos \theta + Q \sin \theta$ are all positive on $(0, \pi/2)$. □

**Lemma 10.** The perfect arc that is symmetric with respect to the $x$-axis is contained inside $\Omega$.

**Proof.** Suppose the end points of the perfect arc are $C_1 = C(\theta)$ and $C_2$. Then the normal lines at $C_1$ and $C_2$ intersect at $p = (-\frac{C(\theta) \cdot T(\theta)}{\sin \theta}, 0)$. Since $p \in \Omega$ and the perfect arc is contained in the triangle $pC_1C_2$, the perfect arc is contained in $\Omega$ by the convexity. □

We parametrize the family of perfect arcs that are symmetric with respect to the $x$-axis by $\theta$ and denote their length by $L(\theta)$ and enclosed area by $A(\theta)$.

**Lemma 11.** Both $L(\theta)$ and $A(\theta)$ are strictly increasing functions in $\theta$.

This is not obvious as the perfect arcs may cross each other.

**Figure 1.** Intersection of perfect arcs. Readers should smooth out the corners.

**Proof.** Suppose $C(\theta) = (x(\theta), y(\theta))$. Then we have $y(\theta) = \int_0^\theta \frac{\cos \omega}{\kappa(\omega)} d\omega$ and $L(\theta) = \frac{\pi - 2\theta}{\cos \theta} \int_0^\theta \frac{\cos \omega}{\kappa(\omega)} d\omega$. Since $\kappa$ is decreasing on $(0, \pi/2)$, the derivative of
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$L$ on $(0, \pi/2)$ satisfies

\[
\frac{dL}{d\theta} = \frac{(\pi - 2\theta) \sin \theta - 2 \cos \theta}{\cos^2 \theta} \int_0^\theta \frac{\cos \omega}{\kappa(\omega)} d\omega + \frac{\pi - 2\theta \cos \theta}{\cos \theta \kappa(\theta)}
\]

\[
\geq \frac{(\pi - 2\theta) \sin \theta - 2 \cos \theta}{\cos^2 \theta} \int_0^\theta \frac{\cos \omega}{\kappa(0)} d\omega + \frac{\pi - 2\theta \kappa(0)}{\kappa(0)}
\]

\[
= \frac{(\pi - 2\theta) - \sin 2\theta}{\kappa(0) \cos^2 \theta} > 0.
\]

By Proposition 6, we also get $\frac{dA}{d\theta} > 0$.

Now we further impose that $\Omega$ has area $\pi$ and is not a unit disk. We denote the length of perfect arcs in the unit disk by $L^*(\theta)$.

**Lemma 12.** If $0 < \theta \leq \theta^* \leq \pi/2$, then $L(\theta) < L^*(\theta^*)$.

**Proof.** Since $L$ is increasing in $\theta$, it suffices to show that $L(\theta^*) < L^*(\theta^*)$. Moreover, it suffices to show that $y(\theta^*) < y^*(\theta^*)$ where $(x^*(\theta), y^*(\theta))$ is the upper endpoint of the perfect arc in the unit disk.

Let $s$ be the arclength parameter. We have $\frac{dy}{ds} = \cos \theta \frac{d\theta}{ds} = \kappa$ and hence the relation $y(\theta) - y^*(\theta) = \int_0^\theta \cos \omega (\frac{1}{\kappa(\omega)} - 1) d\omega$, which implies that $y - y^*$ is decreasing on $(0, \bar{\theta})$ and increasing on $[\bar{\theta}, \pi/2]$, where $\kappa(\bar{\theta}) = 1$.

Suppose $y(\theta^*) \geq y^*(\theta^*)$ for some $\theta^*$. Then $\theta^* \in (\bar{\theta}, \pi/2]$ and we infer that $y(\pi/2) \geq y^*(\pi/2)$. Therefore the minor axis of $\Omega$ is longer than 2 and $\Omega$ contains a unit disk by Lemma 9. This contradicts the assumption that $\Omega$ has area $\pi$.

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**Figure 2.** Illustration for Lemma 12

We are ready to prove the main theorem of this section.

**Theorem 13.** Let $\mathcal{A}$ be the class of domains $\Omega$ bounded by smooth convex curves that are symmetric in both coordinate axes and have exactly four vertices. Suppose that $\Omega \in \mathcal{A}$ has area $\pi$ and is not a unit disk. Then

\[ I_{\Omega}(A) < I_{B_1}(A) \]

for $0 < A < \pi$. 
Proof. Since $A(\theta)$ is strictly increasing on $(0, \frac{\pi}{2})$, we change variable to consider $L$ as a function of $A$. Since $I_{\Omega}(A) \leq L(A)$ and $I_{B_1}(A) = L^*(A)$, it suffices to show that supremum of the function $\frac{L}{L^*}(A)$ on $A \in (0, \pi)$, which is symmetric with respect to $\pi/2$, is less than 1. If the supremum occurs at $A = 0$, then the assertion follows from Theorem 8 so we assume the absolute maximum occurs at $\bar{A} \in (0, \frac{\pi}{2}]$. We have

$$0 = \left( \frac{L}{L^*} \right)'(\bar{A}) = \frac{kL^* - k^*L}{L^{*2}}(\bar{A})$$

Since $L = \frac{\pi - 2\theta}{k}$, we obtain

$$\left( \frac{\pi - 2\theta}{\pi - 2\theta^*} \right)^2 = \left( \frac{L}{L^*} \right)^2$$

at $\bar{A}$. Either $\theta > \theta^*$ or $\theta \leq \theta^*$ leads to $L < L^*$, where Lemma [12] is used in the latter case.  

4. PERTURBATION OF THE UNIT DISK

4.1. Prefect arcs in the unit disk. We begin this section by describing the perfect arcs in the unit disk, following Section 2 of [1]. The isoperimetric profile of $B_1$ is given implicitly by

$$I_{B_1}(a) = (\pi - 2\theta) \tan \theta, \quad a = \theta - \tan \theta + \left( \frac{\pi}{2} - \theta \right) \tan^2 \theta.$$

The perfect arc $\tilde{\sigma}$ is given by

$$\tilde{\sigma}(x) = (\sec \theta, 0) + \tan \theta \left( \cos \left( (\pi - 2\theta)x + \frac{\pi}{2} + \theta \right), \sin \left( (\pi - 2\theta)x + \frac{\pi}{2} + \theta \right) \right), \quad 0 \leq x \leq 1.$$

The other perfect arcs $\tilde{\sigma}(x; u)$ are obtained by rotating $\tilde{\sigma}$ counterclockwise by an angle of $u$.  

Figure 3. Perfect arcs of the unit disk.
4.2. Setup of the perturbative analysis. We follow closely the convention in [1]. See their Figure 2 for an instructive summary. Let \( X(u, s) : [0, 2\pi] \times [0, \delta) \to \mathbb{R}^2 \) be a perturbation of the unit circle that preserves area. Namely, denoting the domain enclosed by \( X(\cdot, s) \) by \( \Omega_s \), \( X(u, s) \) satisfies \( X(u, 0) = (\cos u, \sin u) \) and \( \text{Area}(\Omega_s) = \pi \).

We consider a family of curves \( \sigma(x, s; u) : [0, 1] \times [0, \delta) \times [0, 2\pi] \) inside \( \Omega_s \) satisfying

1. \( \sigma(x, 0; u) = \tilde{\sigma}(x; u) \) is the perfect arc of \( B_1 \).
2. The endpoints of \( \sigma(\cdot, s; u) \) lie on \( \partial \Omega_s \):
   \[ \sigma(0, s; u) = X(u_+ + s, s), \quad \sigma(1, s; u) = X(u_- + s, s) \]
3. \( \sigma(\cdot, s) \) encloses area \( a \) together with \( \partial \Omega_s \).

Let

\[
\frac{\partial X}{\partial s} = fN + gT
\]

be the variational field of \( X(u, s) \) and

\[
\frac{\partial \sigma}{\partial s} = \eta n + \xi t
\]

be the variational field of the arcs. To simplify notation, we write

\[
\eta_0(x) = \eta(x, 0), \quad f_0(u) = f(u, 0)
\]

and

\[
\xi_0(x) = \xi(x, 0), \quad g_0(u) = g(u, 0).
\]

4.3. First variations. Since \( t(0) = -N(u_+), t(1) = N(u_-), n(0) = T(u_+), n(1) = T(u_-) \), differentiating (4) yields

\[
\eta_0(0) = \dot{u}_+ + g_0, \quad \xi_0(0) = -f_0(u_+)
\]

and similarly

\[
\eta_0(1) = -\dot{u}_- - g_0, \quad \xi_0(1) = f_0(u_-)
\]

where \( \dot{u}_\pm = \left. \frac{\partial}{\partial s} \right|_{s=0} u_\pm \).

By assumption, the first variation of the area enclosed by the arc is zero

\[
\frac{\partial A}{\partial s} \bigg|_{s=0} = \int_{u_-}^{u_+} f_0 \left| \frac{\partial X}{\partial u} \right| du + \int_{0}^{1} \eta_0 \left| \frac{\partial \sigma}{\partial x} \right| dx = 0.
\]

Together with the evolution of arc length (see (3.3) of [1])

\[
\frac{\partial}{\partial s} \left| \frac{\partial \sigma}{\partial x} \right| = \eta k \left| \frac{\partial \sigma}{\partial x} \right| + \frac{\partial \xi}{\partial x}.
\]
we obtain the first variation of the length of the arc

\[
\frac{\partial L}{\partial s} \bigg|_{s=0} = \int_0^1 \eta_0 k \left| \frac{\partial \sigma}{\partial x} \right| \, dx + \xi_0(1) - \xi_0(0)
\]

(8)

\[
= -k \int_{u_-}^{u_+} f_0 \, du + f_0(u_-) + f_0(u_+) =: l(u)
\]

(9)

noting that it does not depend on how the arc moves.

At \( s = 0 \), we have \( u_+ = u_- + 2b, \ k = \cot b \) and hence

\[
l(u) = -\cot b \int_u^{u+2b} f_0(\tilde{u}) \, d\tilde{u} + f_0(u) + f_0(u+2b).
\]

Since \( X \) preserves enclosed area, we have \( \int_0^{2\pi} f \, du = 0 \) and hence \( \int_0^{2\pi} l \, du = 0 \).

If \( l(u) \) is not identically zero, then there exists a perfect arc with \( \frac{\partial L}{\partial s} \bigg|_{s=0} < 0 \) and the isoperimetric profile decreases. However, it is shown in Appendix A that there exists nontrivial variation such that \( l(u) \equiv 0 \). Therefore, we resort to the second variations.

4.4. Second variations. Firstly, since \( \sigma(x, s) \) always encloses area \( a \), we further differentiate (6) to get

\[
0 = \frac{\partial}{\partial s} \bigg|_{s=0} \left( \int_0^1 \eta_0 k \left| \frac{\partial \sigma}{\partial x} \right| \, dx + \int_{u_-}^{u_+} f \left| \frac{\partial X}{\partial u} \right| \, du \right)
\]

\[
= \int_0^1 \frac{\partial \eta}{\partial s} \bigg|_{s=0} \left| \frac{\partial \sigma}{\partial x} \right| + \eta_0 \left( \eta_0 k \left| \frac{\partial \sigma}{\partial x} \right| + \frac{\partial \xi_0}{\partial x} \right) \, dx + f_0(u_+ \dot{u}_+ - f_0(u_-) \dot{u}_- + \int_{u_-}^{u_+} \frac{\partial f}{\partial s} \bigg|_{s=0} + f(\dot{f} + \frac{\partial g_0}{\partial u}) \, du.
\]

(10)

By the evolution of unit tangent and normal vector of \( \sigma \) (see page 514 of [1])

\[
\frac{\partial t}{\partial s} \bigg|_{s=0} = \tilde{\varphi} n,
\]

\[
\frac{\partial n}{\partial s} \bigg|_{s=0} = -\tilde{\varphi} t
\]

with

\[
\tilde{\varphi} = \frac{\partial \eta_0}{\partial x} \bigg|_{\sigma} - k \xi_0,
\]
we get the second derivative of arc length
\[
\frac{\partial^2}{\partial s^2} \bigg|_{s=0} \left[ \frac{\partial \sigma}{\partial x} \right] = \frac{\partial n}{\partial s} k \left[ \frac{\partial \sigma}{\partial x} \right] - \frac{\partial}{\partial x} \left( \eta_0 \tilde{\varphi} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \xi}{\partial s} \bigg|_{s=0} \right) + \frac{\varphi}{\partial x}. \]

Combining it with (10) we obtain the second variation of the length of the arc
\[
\frac{\partial^2 L}{\partial s^2} \bigg|_{s=0} = \int_0^1 \left( -k^2 \eta_0^2 \frac{\partial \sigma}{\partial x} + \frac{\partial \eta_0}{\partial x} \right) dx + k \left[ -\eta_0 \xi_0 \bigg|_{x=1} - f_0(u_+) \ddot{u}_+ + f_0(u_-) \ddot{u}_- - \int_{u_-}^{u_+} \frac{\partial f}{\partial s} \bigg|_{s=0} + f \left( f + \frac{\partial g_0}{\partial u} \right) \right] + k \left[ \frac{\partial \xi}{\partial s} \bigg|_{s=0} - \eta_0 \tilde{\varphi} \right].
\]

To simplify the last line, we differentiate (4) in \( s \) twice
\[
\frac{\partial n}{\partial s} \mathbf{n} - \eta_0 \tilde{\varphi} \mathbf{t} + \frac{\partial \xi}{\partial s} \mathbf{t} + \xi_0 \tilde{\varphi} \mathbf{n}
\]
\[
= \frac{\partial^2 X}{\partial s \partial u} \dddot{u}_+ + \frac{\partial X}{\partial u} \ddot{u}_+ + \frac{\partial f}{\partial s} \mathbf{N} - f \left( \frac{\partial f}{\partial u} - g_0 \right) \mathbf{T} + \frac{\partial g}{\partial s} \mathbf{T} + g \left( \frac{\partial f}{\partial u} - g_0 \right) \mathbf{N}.
\]

Since \( \mathbf{t}(0) = -\mathbf{N}(u_+), \mathbf{t}(1) = \mathbf{N}(u_-), \mathbf{n}(0) = \mathbf{T}(u_+), \mathbf{n}(1) = \mathbf{T}(u_-), \) we get
\[
\left. \frac{\partial \xi}{\partial s} \right|_{s=0} - \eta_0 \tilde{\varphi} = -\eta_0 \left( \frac{\partial f_0}{\partial u} - g_0 \right) (u_+) - \left. \frac{\partial f}{\partial s} \right|_{s=0} (u_+)
\]
at \( x = 0 \) and
\[
\left. \frac{\partial \xi}{\partial s} \right|_{s=0} - \eta_0 \tilde{\varphi} = -\eta_0 \left( \frac{\partial f_0}{\partial u} - g_0 \right) (u_-) + \left. \frac{\partial f}{\partial s} \right|_{s=0} (u_-)
\]
at \( x = 1 \). On the other hand, since \( \Omega_s \) has area \( \pi \), we have
\[
(11) \quad 0 = \left. \frac{\partial}{\partial s} \right|_{s=0} \int_0^{2\pi} f \left. \frac{\partial X}{\partial u} \right| du = \int_0^{2\pi} \left. \frac{\partial f}{\partial s} \right|_{s=0} + f_0 \left( f_0 + \frac{\partial g_0}{\partial u} \right) du.
\]

Next, to kill the effect of rigid motions in \( \mathbb{R}^2 \), we require that \( X(u, s) \) is a normal variation in the first approximation:
\[
g_0(u) \equiv 0.
\]

Moreover, we move the arcs tangentially in the first approximation:
\[
\eta_0(x) \equiv 0.
\]

Putting these together, we obtain
\[
\frac{\partial^2 L}{\partial s^2} \bigg|_{s=0} = -k \int_{u_-}^{u_+} \left( \left. \frac{\partial f}{\partial s} \right|_{s=0} + f^2 \right) du + \left. \frac{\partial f}{\partial s} \right|_{s=0} (u_-) + \left. \frac{\partial f}{\partial s} \right|_{s=0} (u_+).
\]
Recall that we fix one arc in the above calculations. Taking all arcs into account, we obtain, by (11),
\[
\int_0^{2\pi} \frac{\partial^2 L}{\partial s^2} \bigg|_{s=0}^0 (u) \, du = -2 \int_0^{2\pi} f_0^2(u) \, du < 0
\]
for nontrivial variations. Hence, at least one arc becomes shorter while enclosing the same amount of area in the variation. In summary, we prove

**Theorem 14.** Let \( \Omega_s, 0 \leq s < \epsilon \) be a family of domains with \( \Omega_0 = B_1 \) and area(\( \Omega_s \)) = \( \pi \). We assume that \( \Omega_s \) does not arise from rigid motion. Then for any \( 0 < A < \pi \) the following dichotomy on the perfect arcs that minimizes \( I_{B_1}(A) \) holds.

1. There exists a perfect arc whose length satisfies \( \frac{d}{ds} \bigg|_{s=0} L < 0 \).
2. We have \( \frac{d^2}{ds^2} \bigg|_{s=0} L = 0 \) for all perfect arcs and there exists a perfect arc whose length satisfies \( \frac{d^2}{ds^2} \bigg|_{s=0} L < 0 \) under tangential variation.

Consequently, the isoperimetric profile must decrease. This completes the proof of Theorem 3.

**APPENDIX A. LOCAL EXISTENCE OF PERFECT ARCS**

Given two points \( C_1 \) and \( C_2 \) on a curve \( C \subset \mathbb{R}^2 \), denote the unit tangent and unit normal vector at \( C_1 \) and \( C_2 \) by \( T_1, T_2 \) and \( N_1, N_2 \). We observe an elementary criterion for two points to be joined by a perfect arc.

**Proposition 15.** For two points \( C_1 = C(s_1), C_2 = C(s_2) \) on a curve \( C \), a necessary and sufficient condition that there is a perfect arc passing through \( C_1 \) and \( C_2 \) is either the function \( f : C \times C \rightarrow \mathbb{R} \) satisfies
\[
f(s_1, s_2) := (C_1 - C_2) \cdot (N_1 + N_2) = 0 \text{ with } N_1 + N_2 \neq 0
\]
or there is a straight line passing through \( C_1 \) and \( C_2 \) with direction \( N_1 = -N_2 \).

**Proof.** The assertion follows by observing that the angle between \( C_1 - C_2 \) and \( T_1 \) is equal to the angle between \( C_1 - C_2 \) and \( T_2 \).

**Proposition 16.** Let \( \gamma \) be a perfect arc of \( C \) with nonzero curvature and \( C_1 \) and \( C_2 \) be its endpoints. Denote the curvature of \( C \) at \( C_1 \) and \( C_2 \) by \( k_1 \) and \( k_2 \). Then there exists a nontrivial family of perfect arcs \( \gamma(t), -\delta < t < \delta \) such that \( \gamma(0) = \gamma \) unless
\[
k_1(C_1 - C_2) = N_2 - N_1 = k_2(C_1 - C_2).
\]
Proof. Consider the two-point function $f: C \times C \to \mathbb{R}$ defined in the previous proposition. The partial derivatives of $f$ at $(C_1, C_2)$ are given by

$$\frac{\partial f}{\partial s_1} = T_1 \cdot N_2 - (C_1 - C_2) \cdot k_1 T_1$$
$$\frac{\partial f}{\partial s_2} = -T_2 \cdot N_1 - (C_1 - C_2) \cdot k_2 T_2$$

We claim that $\frac{\partial f}{\partial s_1}(C_1, C_2)$ and $\frac{\partial f}{\partial s_2}(C_1, C_2)$ do not vanish at the same time. Indeed, if $\frac{\partial f}{\partial s_1}(C_1, C_2) = \frac{\partial f}{\partial s_2}(C_1, C_2) = 0$, then

$$N_2 - k_1(C_1 - C_2) = \alpha N_1, \quad N_1 + k_2(C_1 - C_2) = \beta N_2$$

for some constants $\alpha, \beta$. Since $\gamma$ has nonzero curvature, $N_1 + N_2 \neq 0$. Taking inner product with $N_1 + N_2$, we get $\alpha = \beta = 1$ and hence

$$k_1(C_1 - C_2) = N_2 - N_1 = k_2(C_1 - C_2).$$

Without loss of generality, we assume $\frac{\partial f}{\partial s_2}(C_1, C_2) \neq 0$. By the implicit function theorem, there is a function $g(s_1)$ defined on some open interval such that $f(s_1, g(s_1)) = 0$. By Proposition 15, we obtain a family of perfect arcs. □

We now present a local existence result of perfect arcs around a vertex. It is reminiscent of the existence of constant mean curvature foliation in a neighborhood of a point [12].

**Proposition 17.** Suppose there is a family of perfect arcs shrinking to $p$. Then $p$ must be a vertex of $C$. Conversely, if $p \in C$ is a non-degenerated vertex ($k' = 0$ but $k'' \neq 0$), then there is a family of perfect arcs shrinking to $p$.

**Proof.** We parametrize $C$ by arclength $s$. Suppose $p = C(0) = (0, 0)$. The local canonical form of plane curves [4 Section 1.6] says

$$C(s) = \left(s - \frac{k^2 s^3}{3!}\right) t + \left(\frac{s^2 k}{2} + \frac{s^3 k'}{3!}\right) n + O(s^4),$$
$$t(s) = \left(1 - \frac{k^2 s^2}{2}\right) t + \left(k s + \frac{k' s^2}{2}\right) n + O(s^3),$$
$$n(s) = \left(1 - \frac{k^2 s^2}{2}\right) n - \left(k s + \frac{k' s^2}{2}\right) t + O(s^3)$$

where all terms on the right-hand side of the following equation are evaluated at $s = 0$.

Suppose $\gamma_t, 0 < t < \epsilon$ is a family of perfect arcs shrinking to $p$ as $t \to 0$. Let $C_1 = C(s_1), C_2 = C(s_2)$ be the endpoints of $\gamma$ where $s_1$ and $s_2$ depend on $t$ smoothly. Without loss of generality, we assume $s_1(t) = t$ and use
s_1 as the parameter of the family \( \gamma_t \); moreover, we assume \( s_2 = O(s_1) \) as \( s_1 \to 0 \). Recall that \( f(s_1, s_2) = (C_1 - C_2) \cdot (N_1 + N_2) = 0 \). We compute the expansion of \( f(s_1, s_2) \) with respect to \( s_1 \) to get
\[
0 = (C(s_1) - C(s_2))(n(s_1) - n(s_2)) = -\frac{k'}{6}(s_1 - s_2)^2 + O(s_1^4).
\]

By the assumption of \( s_2 \), \( k' = 0 \) and hence \( p \) is a vertex of \( C \).

For the converse, suppose \( p \) is a non-degenerated vertex. We expand \( \alpha(s) \) to higher order:

\[
C(s) = \left( s - \frac{k^2}{6}s^3 \right) t + \left( \frac{k^4}{24} + \frac{k''}{120}s^5 \right) t + O(s^6)
\]

and it follows that
\[
n(s) = \left( 1 - \frac{k^2}{2}s^2 + \frac{k^4}{24}s^4 \right) n - \left( ks + \frac{k''}{6}s^3 + \frac{k''}{24}s^4 \right) t + O(s^5).
\]

By direct computation, \( 0 = f(s_1, s_2) = -\frac{k''}{12}(s_1^2 - s_2^2)(s_1 - s_2)^2 + O(s_1^4) \).

Since \( k'' \neq 0 \), we obtain \( a_1 = -1 \). We have in the next order
\[
0 = f(s_1, s_2) = (s_1 - s_2) \left[ (s_1 + s_2)A - \frac{k''}{24}(s_1^4 + s_2^4) + \frac{k''}{60}(s_1^4 + s_1^3s_2 + s_1s_2^3 + s_2^4) \right] + O(s_1^6)
\]

where
\[
A = -\frac{k''}{6}(s_1^2 - s_1s_2 + s_2^2) + \frac{k^3}{6}(s_1^2 + s_1s_2 + s_2^2) + \left( \frac{k''}{12} - \frac{k^3}{3} \right)(s_1^2 + s_2^2).
\]

Plugging in \( s_2 = -s_1 + a_2s_2 + O(s_1^4) \), the term in the bracket simplifies to
\[
s_1^4 \left( \frac{-k''}{3}a_2 - \frac{1}{15}k'' \right) + O(s_1^5).
\]

Since \( k'' \neq 0 \), for \( s_1 \) sufficiently small we can find two points \( q^+, q^- \) on \( \partial \Omega \) such that \( f(C_1, q^+) > 0 \) and \( f(C_1, q^-) < 0 \). For example, take \( s_2 = -s_1 - \frac{k''}{6}k'' s_1^2 \). By the intermediate value theorem, there exists a point \( q \) between \( q^+ \) and \( q^- \) satisfying \( f(C_1, q) = 0 \). \( C_1 \) and \( q \) would give us a perfect arc.

We close this section by commenting the hypothesis on the number of vertices in Theorem 13. The above proposition shows that additional vertex may lead to additional minimizers of the variational problem (1). Nonuniqueness of minimizers then cause isoperimetric profile non-differentiable. In
general the isoperimetric profile of a compact Riemannian manifold is at best piecewise differentiable, see [2, 6] for example.

**APPENDIX B. PERTURBATIONS THAT PRESERVE THE ISOPERIMETRIC PROFILE OF $B_1$ IN THE FIRST VARIATION**

In Section 4.1 we show that a periodic function $f : [0, 2\pi] \rightarrow \mathbb{R}$ satisfying

$$l(u) := -\cot b \int_u^{u+2b} f(\tilde{u}) \, d\tilde{u} + f(u) + f(u + 2b) \equiv 0$$

gives rise to a perturbation that causes isoperimetric profile of $B_1$ to have zero first variation. It is elementary to show that the relation holds for $\cos u$ and $\sin u$, which corresponds to translation of unit circle. For the perturbation to preserve enclosed area $\pi$, we also require $\int_0^{2\pi} f \, du = 0$. To construct nontrivial perturbations, we consider Fourier series $f(\theta) = \sum'_{n \in \mathbb{R}} c_n e^{in\theta}$ with $c_n \in \mathbb{R}$. Here $\sum'$ means summation from $-\infty$ to $+\infty$ except $n = 0$. Direct computation yields

$$l(\theta) = \sum' \frac{c_n e^{in\theta}}{in} \left( -\cos b(e^{inb} - 1) + i n \sin b(1 + e^{inb}) \right)$$

and

$$-\cos b(e^{inb} - 1) + i n \sin b(1 + e^{inb})$$

$$= 2 \sin nb(\cos b \sin nb - n \sin b \cos nb) + i \cdot 2 \cos nb(\cos b \sin nb + n \sin b \cos nb)$$

The figure below shows the implicit equation $\cos y \sin xy - x \sin y \cos xy = 0$. Except the vertical lines $x = \pm 1, 0$ (they correspond to translations), every intersection of the curve with $y = n, n \in \mathbb{Z}$ gives rise to a nontrivial $f$ with $l(u) \equiv 0$. 

![Implicit Equation Graph](image-url)
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