Hamiltonian structure and quantization of 2+1 dimensional gravity coupled to particles

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Abstract

It is shown that the reduced particle dynamics of 2+1 dimensional gravity in the maximally slicing gauge has hamiltonian form. This is proved directly for the two body problem and for the three body problem by using the Garnier equations for isomonodromic transformations. For a number of particles greater than three the existence of the hamiltonian is shown to be a consequence of a conjecture by Polyakov which connects the auxiliary parameters of the fuchsian differential equation which solves the SU(1,1) Riemann-Hilbert problem, to the Liouville action of the conformal factor which describes the space-metric.

We give the exact diffeomorphism which transforms the expression of the spinning cone geometry in the Deser, Jackiw, ’t Hooft gauge to the maximally slicing gauge. It is explicitly shown that the boundary term in the action, written in hamiltonian form gives the hamiltonian for the reduced particle dynamics.

The quantum mechanical translation of the two particle hamiltonian gives rise to the logarithm of the Laplace-Beltrami operator on a cone whose angular deficit is given by the total energy of the system irrespective of the masses of the particles thus proving at the quantum level a conjecture by ’t Hooft on the two particle dynamics. The quantum mechanical Green’s function for the two body problem is given.

1 Introduction

Gravity in 2+1 dimensions has been the object of vast interest both at the classical and quantum level. Several approaches have been pursued. In the maximally slicing gauge, or instantaneous York gauge, was introduced. The application of such a gauge is restricted to universes with spacial topology of genus \( g < 1 \); moreover for the sphere topology it can be applied only to the static problem. Thus the range of applicability of such a gauge is practically restricted to open universes with the topology of the plane; here however it will prove a very powerful tool.

The approach developed in is first order. In the same gauge was exploited in the second order ADM approach; this approach turns out to be more straightforward than
the previous one and being strictly canonical lends itself to be translated at the quantum level. Quantization schemes have been proposed in the absence of particles in [3, 5, 10, 11] and in the presence of particles in [3, 12].

The present paper is the continuation of two previous papers [7, 8] and goes a lot deeper into the problem.

In sect.2 we give a concise summary of the results of the previous papers [7, 8]; in sect.3 we derive generalized conservation laws starting from the time evolution of the analytic component of the energy momentum tensor of the Liouville theory which underlies the conformal factor describing the space metric.

In sect.4 we prove explicitly the hamiltonian nature of the reduced particle dynamics i.e. the fact that one can give a hamiltonian description of the time development of the system in terms of the position and momenta of the particles. Thus this is the counterpart of the hamiltonian description in the absence of particles for closed universes given by Moncrief [13] and Hosoya and Nakao [14].

While for the two particle case the result is elementary, for three particles it involves the exploitation of the Garnier equations, related to the isomonodromic transformations of a fuchsian problem. We recall that in [7, 8] it was proved that such Garnier equations are an outcome of the ADM dynamical equations of 2+1 dimensional gravity. For more than three particles the proof of the hamiltonian nature of the reduced equations of motion and the derivation of the hamiltonian, relies on a conjecture by Polyakov [15] on the relation between the regularized Liouville action and the accessory parameters of the $SU(1,1)$ Riemann-Hilbert problem. Such a conjecture has been proved by Zograf and Takhtajan [15] for the special cases of parabolic singularities and elliptic singularities of finite order, but up to now a proof for general elliptic singularities is absent.

In sect. 5 we give the exact diffeomorphism which relates the conical metric of Deser, Jackiw and 't Hooft (DJH) in the presence of angular momentum to its description in the maximally slicing gauge; as a by-product it gives the exact relation between the asymptotic metrics in the DJH and in the maximally slicing gauge. These results will be useful in the following to understand the boundary terms in the action. We write also the exact expression
of the Killing vectors in the maximally slicing gauge.

In sect. 6 we connect the results of sect. 4 with the boundary terms of the gravitational action; 2+1 dimensional gravity coupled to particles is an example in which one can compute the hamiltonian explicitly as a boundary term. The dynamics is described completely by such boundary terms of the action.

Finally in sect. 7 we treat the quantization of the two particle problem starting form the classical two particle hamiltonian. The quantum hamiltonian turns out to be the logarithm of the Laplace-Beltrami operator on a cone whose aperture is given by the total energy of the system and is independent of the masses of the two particles. This provides a complete proof of the conjecture of 't Hooft [16] about the two particle dynamics, i.e. the equivalence of the relative motion of two particles with that of a test particle on a cone of aperture equal to the total energy. Obviously, the ordering problem is always present but the Laplace-Beltrami operator appears to be the most natural choice. A very similar structure was found and thoroughly examined by Deser and Jackiw [17], when treating the quantum problem of a test particle moving on a cone; the main difference is that in the present treatment its logarithm rather than the Laplace-Beltrami operator appears.

Given the hamiltonian one can easily compute the Green function; it can be written in terms of hypergeometric functions.

The quantum mechanical problem with more than two particles requires a more explicit knowledge of the hamiltonian which is related to the auxiliary parameters \( \beta_B \). The existence of those parameters is assured by the solvability of the Riemann-Hilbert problem and one can try to produce a perturbative expansion of them at least in some limit situations. Here however the ordering problem is likely to be more acute.
2 Hamiltonian approach

To make the paper relatively self-contained we shall summarize in this section some results of the papers [7, 8]. With the usual ADM notation for the metric [18]

\[ ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \]  

(1)

the gravitational action expressed in terms of the canonical variables is [19, 20, 21]

\[
S_{\text{Grav}} = \int dt \int_{\Sigma_t} d^Dx \left[ \pi^{ij} \dot{g}_{ij} - N^i H_i - NH \right] + \nonumber \\
+ 2 \int dt \int_{B_t} d^{(D-1)}x \sqrt{\sigma_{Bt}} N \left( K_{B_t} + \frac{\eta}{\cosh \eta} D_{\alpha} u^\alpha \right) - 2 \int dt \int_{B_t} d^{(D-1)}x r_\alpha \pi^{\alpha\beta}_{(B_t)} N_\beta. 
\]  

(2)

where \( \sinh \eta = n_\mu u^\mu \) with \( n^\mu \) the future pointing unit normal to the time slices \( \Sigma_t \) and \( u^\mu \) the outward pointing unit normal to space-like boundary \( B_t \); \( B_t = \Sigma_t \cap B \), \( \sqrt{\sigma_{Bt}} \) stands for the volume form induced by the space metric on \( B_t \), \( K_{B_t} \) is the extrinsic curvature of \( B_t \) as a surface embedded in \( \Sigma_t \), \( v^\alpha \equiv \frac{1}{\cosh \eta} (n^\alpha - \sinh \eta u^\alpha) \) and \( r_\alpha \) is the versor normal to \( B_t \) in \( \Sigma_t \). The subscript \( \sigma_{Bt} \) in \( \pi^{\alpha\beta}_{(B_t)} \) is a reminder that it has to be considered a tensor density with respect to the measure \( \sqrt{\sigma_{Bt}} \). The explicit form of \( H \) and \( H_i \) can be found in [7].

The matter action can be rewritten as

\[
S_m = \int dt \sum_n \left( P_n \dot{q}_n^i + g^{ij}(q_n) P_{ni} - N(q_n) \sqrt{P_n P_{nj} g^{ij}(q_n) + m^2_n} \right). 
\]  

(3)

In the \( K = 0 \) gauge and using the complex coordinates \( z = x + iy \) the diffeomorphism constraint is simply solved by

\[
\pi_{z} = -\frac{1}{2\pi} \sum_n \frac{P_n}{z - z_n} 
\]  

subject to the restriction \( \sum_n P_n = 0 \) [8]. Always for \( K = 0 \) and using the conformal gauge for the space metric i.e.

\[
ds^2 = -N^2 dt^2 + e^{2\sigma}(dz + N^z dt)(d\bar{z} + N^\bar{z} dt)
\]  

(5)
the Hamiltonian constraint takes the form of the following inhomogeneous Liouville equation

$$2\Delta \tilde{\sigma} = -e^{-2\tilde{\sigma}} - 4\pi \sum_n \delta^2(z - z_n)(\mu_n - 1) - 4\pi \sum_A \delta^2(z - z_A).$$

In eq. (6) $\tilde{\sigma}$ is defined by

$$e^{2\sigma} = 2\pi^{\frac{1}{z}} z^{\pi} e^{2\tilde{\sigma}},$$

$\mu_n$ are the particle masses divided by $4\pi$, $z_n$ the particle positions and $z_A$ the positions of the $(N - 2)$ apparent singularities i.e. of the zeros of eq. (4). The Lagrange multipliers $N$ and $N^z$ where expressed in terms of $\tilde{\sigma}$ through

$$N = \frac{\partial (-2\tilde{\sigma})}{\partial M}$$

and

$$N^z = -2\pi^{\frac{1}{z}} \frac{\partial}{\partial z} N + g(z),$$

with

$$g(z) = \sum_B \frac{\partial \beta_B}{\partial M} \frac{1}{z - z_B} \frac{\mathcal{P}(z_B)}{\prod_{C \neq B}(z_B - z_C)} + p_1(z)$$

and $\mathcal{P}$ is defined by

$$-\frac{\pi^{\frac{1}{z}}(z)}{2} = \frac{1}{4\pi} \sum_n \frac{P_n z}{z - z_n} \equiv \prod_B \frac{(z - z_B)}{\mathcal{P}(z)}.$$

$p_1(z) = c_0(t) + c_1(t) z$ is a first order polynomial. The role of the first term in $g(z)$ is to cancel the poles arising in the first term of eq. (4) due to the zeros of $\pi^{\frac{1}{z}}$ and $\beta_B$ are the accessory parameters of the fuchsian differential equation [25] which underlies the solution of the Liouville equation (3). The equations for the particle motion are given by

$$\dot{z}_n = -N^z(z_n) = -g(z_n) = -\sum_B \frac{\partial \beta_B}{\partial M} \frac{1}{z_n - z_B} \frac{\mathcal{P}(z_B)}{\prod_{C \neq B}(z_B - z_C)} - p_1(z_n),$$

$$\dot{P}_{nz} = 4\pi \frac{\partial p_n}{\partial M} + P_{nz} g'(z_n) =$$
= 4\pi \frac{\partial \beta_n}{\partial M} - P_nz \sum_B \frac{\partial \beta_B}{\partial M} \frac{\mathcal{P}(z_B)}{(z_n - z_B)^2 \prod_{C \neq B}(z_B - z_C)} + P_nz_p'(z_n). \quad (13)

If we want a reference frame which does not rotate at infinity the linear term in \( p_1(z) \) must be chosen so as to cancel in \( Nz \) the term increasing linearly at infinity; such a choice is unique and given by \(-z/(\sum_n P_nz_n)\).

In the simple two particle case one obtains the equations of motion in the relative coordinates \( z'_2 = z_2 - z_1 \), \( P'_2 = P_2 = -P_1 \)

\[
\dot{z}'_2 = \frac{1}{P'_2}; \quad \dot{P}'_2 = -\frac{\mu}{z'_2}. \quad (14)
\]

It is interesting that in order to reach eq. (14) there is no need to solve the Liouville equation; the local properties of the fuchsian differential equation underlying eq.(6) are sufficient. The solution of eq.(14)

\[
z'_2 = \text{const } [(1 - \mu)(t - t_0) - iL/2]^\frac{1}{1-\mu} \quad (15)
\]

agrees with the solution found in [5]. A still simpler derivation of eq.(15) as a ratio of two conservation laws, will be given in the next section.

### 3 Virasoro generators and conservation laws

In ref.[7] the following generalized conservation law (and its complex conjugate) for the \( N \) particle problem was obtained

\[
\sum_n P_nz_n = (1 - \mu)(t - t_0) - iL \quad (16)
\]

by using the particle equations of motion eq.(12, 13), where \( 4\pi\mu \) is the total energy of the system and \( L \) the angular momentum.

In the two particle case eq.(13) is simply \( P'z'_2 = (1 - \mu)(t - t_0) - iL \). As can be easily checked the hamiltonian for eqs.(14) and their complex conjugates is given by the sum of two conserved hamiltonians i.e. \( H = \bar{h} + \bar{h} \) with \( h = \ln(P'z'_2) \). Taking the ratio of
\[ P'z^m = \exp(h) = \text{const.} \]

with the previous equation we obtain the solution eq.(15) without the need to solve the system (14).

In this section we want to give a treatment of these and analogous conservation laws from a more general viewpoint.

In ref.[7] the following equation was derived from the ADM formalism, with regard to the time evolution of the function \( Q(z) \) appearing in the fuchsian differential equation

\[ \dot{Q}(z) = \frac{1}{2}g'''(z) + 2g'(z)Q(z) + g(z)Q'(z). \]  

(17)

\( Q(z) \) can be understood as the analytic component of the energy momentum tensor of the Liouville theory governing the conformal factor \( \tilde{\sigma} \) and the above equation represents the change of this anomalous energy momentum tensor under a conformal transformation generated by \( g(z) \). It was also shown in ref.[7] that eq.(17) contains all the dynamics of the system, i.e. the motion of the particle singularities and auxiliary singularities and the change in time of the residues at such singularities; it provides also an interpretation of 2+1 dimensional gravity à la Einstein-Infeld-Hoffmann [24]. Following a well trodden path, we want now to convert eq.(17) into equations for the Laurent series coefficients of \( Q(z) \). With

\[ \frac{1}{2\pi i} \oint z^{n+1}Q(z)dz = L_n \]  

(18)

we obtain

\[ L_{-1} = \frac{1}{2} \left( \sum_n \beta_n + \sum_B \beta_B \right) \]  

(19)

\[ L_0 = \frac{1}{2} \left( \sum_n \beta_n z_n + \sum_B \beta_B z_B \right) + \frac{1}{4} \left[ \sum_n (1 - \mu_n^2) - 3(\mathcal{N} - 2) \right] \]  

(20)

\[ L_1 = \frac{1}{2} \left( \sum_n \beta_n z_n^2 + \sum_B \beta_B z_B^2 \right) + \frac{1}{2} \left( \sum_n (1 - \mu_n^2) z_n - 3 \sum_B z_B \right) \]  

(21)

and the following equation of motion

\[ \dot{L}_{-1} = \frac{c_1}{2} \left( \sum_n \beta_n + \sum_B \beta_B \right) \]  

(22)
\[ \dot{L}_0 = -\frac{c_0}{2} (\sum_n \beta_n + \sum_B \beta_B) \] (23)

\[ \dot{L}_1 = -c_0 L_0 - c_1 L_1. \] (24)

In this paper we shall restrict ourselves to \( L_{-1}, L_0, L_1 \). We recall that \( c_0 \) and \( c_1 \) are functions of time which specify the translations and roto-dilatations of the reference frame. Eq. (22) is simply a consistency requirement on the first Fuchs relation \( \sum_n \beta_n + \sum_B \beta_B = 0 \). Eq. (23) tells us through the first Fuchs relation that \( L_0 \) is constant. The value of the constant is actually provided by the second Fuchs relation

\[ 4L_0 = 2 \sum_n \beta_n z_n + 2 \sum_B \beta_B z_B + \sum_n (1 - \mu_n^2) - 3(N - 2) = 1 - \mu_\infty^2 = 1 - (1 - \mu)^2. \] (25)

This implicitly shows that the total mass \( \mu \) is constant in time and more importantly, by taking the derivative with respect to \( \mu \), we have

\[ 1 - \mu = \sum_n \frac{\partial \beta_n}{\partial \mu} z_n + \sum_B \frac{\partial \beta_B}{\partial \mu} z_B \] (26)

which combined with the equations of motion provides the generalized conservation law, obviously related to the dilatations

\[ \frac{d}{dt} (\sum_n P_n z_n) = 1 - \mu. \] (27)

We notice that due to \( \sum_n P_n = 0 \), \( \sum_n P_n z_n \) is invariant under translations, in addition to rotations and dilatations. Eq. (24) for the time evolution of \( L_1 \), keeping in mind that \( L_0 \) is a constant is easily solved in the form

\[ L_1(t) = -L_0 \int_0^t c_0(t') dt' \exp(\int_0^{t'} c_1(t'') dt'') \exp(- \int_0^t c_1(t') dt') + \kappa \exp(- \int_0^t c_1(t') dt'). \] (28)

We want now to translate such information on the physical variables \( z_n, P_n \). By using the equation of motion we obtain

\[ \frac{d}{dt} (\sum_n P_n z_n^2) = -2c_0 \sum_n P_n z_n + c_1 \sum_n P_n z_n^2 + \sum_n \frac{\partial \beta_n}{\partial \mu} z_n^2 + \sum_B \frac{\partial \beta_B}{\partial \mu} z_B^2. \] (29)
The time development of

\[ \sum_n \frac{\partial \beta}{\partial \mu} z_n^2 + \sum_B \frac{\partial \beta_B}{\partial \mu} z_B^2 \]  

in eq.(29) depends on the functions \( c_0(t), c_1(t) \). Let us consider first the (rotating) frame defined by \( c_0 = c_1 = 0 \). Then from eq.(24) we have \( \dot{L}_1 = 0 \). Taking the derivative of that equation with respect to \( \mu \) we have

\[ \frac{d}{dt} \left( \sum_n \frac{\partial \beta_n}{\partial \mu} z_n^2 + \sum_B \frac{\partial \beta_B}{\partial \mu} z_B^2 \right) = 0 \]  

and thus in such a reference frame

\[ \sum_n P_n z_n^2 = At + B. \]  

Let us now consider the non rotating frame given by \( c_1(t) = -(1 - \mu) t - \imath b \) and let us consider the case \( c_0(t) = 0 \). We have

\[ \frac{\partial L_1}{\partial \mu} = \frac{1}{2} \left( \sum_n \frac{\partial \beta_n}{\partial \mu} z_n^2 + \sum_B \frac{\partial \beta_B}{\partial \mu} z_B^2 \right) = k_1 [(1 - \mu) t - \imath b]^{1/\mu}. \]  

This result can be substituted in eq.(29) which solved gives

\[ \sum_n P_n z_n^2 = (k_2 t + k_3) [(1 - \mu) t - \imath b]^{1/\mu} \]  

where \( k_2, k_3 \) are constants. We notice that the non rotating frame with \( c_0(t) = 0 \) corresponds to an asymptotic behavior for \( N^z \) given by

\[ N^z = z(\bar{z})^{\mu-1} \ln(\bar{z}) \left( 1 + O \left( \frac{1}{|z|} \right) \right). \]  

Since the l.h.s. of eq.(34) is not translation invariant, it provides, once the relative motion of the particles has been solved, information on the overall motion of the system e.g. on \( z_1 \).
4 Hamiltonian nature of the reduced dynamics

In [7] starting form the ADM action in the presence of particles we have reached the particle equations of motion in the maximally slicing gauge $K = 0$ eqs.[12,13]. As we followed a canonical procedure, we expect equations (12,13) to be canonical i.e. derivable from a hamiltonian. The present section is devoted to the direct proof that such equations are indeed canonical i.e. are generated by a hamiltonian and to the construction of such hamiltonian.

To start, by means of the transformation of generator

$$G(z, \tilde{P}) = \sum_n (a_1(t)z_n + a_0(t))\tilde{P}_n$$  \hspace{1cm} (36)

with

$$c_1(t) = -\frac{\dot{a}_1(t)}{a_1(t)}; \hspace{0.5cm} c_0(t) = -\dot{a}_0(t) + \frac{a_0(t)}{a_1(t)}\dot{a}_1(t)$$ \hspace{1cm} (37)

one can get rid of $p_1$ and $p'_1$ in eqs.(12,13). This is due to the covariance of eqs.(12,13) under the transformation

$$z_n(t) \rightarrow a_1(t)z_n(t) + a_0(t), \hspace{0.5cm} P_n(t) \rightarrow \frac{P_n(t)}{a_1(t)}.$$ \hspace{1cm} (38)

As we are working in the gauge $\sum_n P_n = 0$ it is useful to perform the canonical transformation generated by

$$G(z, P') = (z_1 + \cdots + z_N)P'_1 + (z_2 - z_1)P'_2 + \cdots + (z_N - z_1)P'_n$$ \hspace{1cm} (39)

i.e. the change of variables

$$z'_1 = z_1 + \cdots + z_N$$ \hspace{1cm} (40)

$$z'_2 = z_2 - z_1$$ \hspace{1cm} (41)

$$z'_N = z_N - z_1$$ \hspace{1cm} (42)
\[ P_1' = \frac{P_1 + \ldots + P_N}{N} \]

\[ P_n' = P_n - \frac{P_1 + \ldots + P_N}{N}, \quad n > 1. \]

The reduced hamiltonian will be translational invariant, i.e. independent of \( z_1' \) to be consistent with \( \sum P_n = 0 \) and our canonical variables will be \( z_2' \ldots z_N' \) and \( P_2' \ldots P_N' \).

Using the definition of \( \mathcal{P}(z) \)

\[ \frac{1}{4\pi} \sum_n \frac{P_{nz}}{z - z_n} = \frac{\prod_B (z - z_B)}{\mathcal{P}(z)} \]

and the properties of the locations \( z_B \) of the apparent singularities

\[ \sum_n \frac{P_{nz}}{z_B - z_n} = 0, \]

one easily derives

\[ 4\pi \frac{\partial z_B'}{\partial P_n'} = \left( \frac{1}{z_B' - z_n'} - \frac{1}{z_B'} \right) \frac{\mathcal{P}(z_B' + z_1)}{\prod_{C \neq B} (z_B' - z_C')} \]

\[ 4\pi \frac{\partial z_n'}{\partial z_n'} = -\frac{P_n'}{(z_B' - z_n')^2} \frac{\mathcal{P}(z_B' + z_1)}{\prod_{C \neq B} (z_B' - z_C')} \]

from which

\[ z_n' = -\sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z_B'}{\partial P_n'} - c_1(t) z_n' \quad n = 2, \ldots N \]

and

\[ P_n' = \frac{\partial \beta_n}{\partial \mu} + \sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z_B'}{\partial z_n'} + c_1(t) P_n' \quad n = 2, \ldots N. \]

Again by means of a canonical transformation of generator

\[ G(z', P'') = \sum_n a_1(t) z_n' P_n'' \]
one can get rid of $c_1(t)$ in eq. (49). This holds when $c_1$ is simply a function of $t$. If one wants to write eq. (49) in the frame which does not rotate at infinity, $c_1$ has to be chosen

$$c_1 = -\frac{1}{\sum_n P_n z_n}$$

which is not simply a function of $t$. At any rate it is immediately seen that if $H$ generates eqs. (49) with $c_1 = 0$ the hamiltonian $H + \ln(\sum_n P_n z_n \sum_n \tilde{P}_n \tilde{z}_n)$ generates eqs. (49) with $c_1$ given by eq. (52). Thus we shall here examine the case $c_1 = 0$.

It is instructive to treat first the three body case: Since there is only one apparent singularity, the equations of motion (49) become

$$\dot{z}_n = -\frac{\partial \beta}{\partial \mu} \frac{\partial \beta}{\partial P_n}$$

and

$$\dot{P}_n = \frac{\partial \beta}{\partial \mu} + \frac{\partial \beta}{\partial z_n} \frac{\partial \beta}{\partial z_n}.$$  

From eq. (53) we see that the hamiltonian must be of the form

$$H(z_2', z_2, P_2', P_3) = -\int_{z_0}^{z_2'} \frac{\partial \beta}{\partial \mu} (z_2', z_3', z_4') \, dz_A + f(z_2', z_4')$$

where $z_A'$ is a function of $z_n'$ and $P_n'$ through the relation eq. (46). With regard to the integral in the above equation, it can be related to the $\beta_{Ar}$ appearing in the reduced SL(2C) canonical equation, i.e. with $z_1 \equiv 0$, $z_2 \equiv 1$ recalling that

$$\beta_A(z_2', z_3, z_A', \mu) = \frac{1}{z_2'} \beta_A(1, \frac{z_3'}{z_2'}, \frac{z_A'}{z_2'}, \mu) \equiv \frac{1}{z_2'} \beta_{Ar}(u, v, \mu)$$

and thus

$$-\int_{z_0}^{z_2'} \frac{\partial \beta}{\partial \mu} (z_2', z_3', z_4', \mu) \, dz_A = -\int_{z_0/z_2'}^{v} \frac{\partial \beta_{Ar}}{\partial \mu} (u', v, \mu) \, dv'$$

with $u = z_3'/z_2'$ and $v = z_A'/z_2'$.

We must now check that with a proper choice of $f(z_2', z_4')$ the hamiltonian, which already generates eqs. (53), generates also eqs. (54). We have

$$-\frac{\partial H}{\partial z_n'} = \int_{z_0}^{z_2'} \frac{\partial \beta}{\partial z_n'} \frac{\partial \beta}{\partial z_n'} \, dz_A + \frac{\partial \beta}{\partial z_n'} \frac{\partial \beta}{\partial z_n'} - \frac{\partial f(z_2', z_4')}{\partial z_n'}.$$
In appendix 1 it is proved that
\[ \frac{\partial^2 \beta_A}{\partial \mu \partial z''_n} = \frac{\partial^2 \beta_n}{\partial \mu \partial z''_A}, \] (59)
from which
\[ -\frac{\partial H}{\partial z''_n} = \frac{\partial \beta_n}{\partial \mu} (z'_2, z'_3, z''_A, \mu) - \frac{\partial \beta_n}{\partial \mu} (z'_2, z'_3, z_0, \mu) + \frac{\partial \beta_A}{\partial \mu} \frac{\partial z'_A}{\partial z''_n} (z'_2, z'_3, z''_A, \mu) - \frac{\partial f(z'_2, z'_3)}{\partial z''_n} \] (60)
and thus \( f(z'_2, z'_3) \) has to satisfy
\[ \frac{\partial f}{\partial z''_n} = -\frac{\partial \beta_n}{\partial \mu}. \] (61)
The integrability in \( f \) of such a relation is provided by
\[ \frac{\partial \beta_n}{\partial z''_m} = \frac{\partial \beta_m}{\partial z''_n}, \] (62)
which is also proved in appendix 1.

We come now to the \( \mathcal{N} \) particle case. The natural extension of the three particle hamiltonian (55) is
\[ H(z'_2, \ldots, z'_N, P'_2, \ldots, P'_N) = -\int_{(z_0)}^{(z_B)} \sum_B \frac{\partial \beta_B}{\partial \mu} (z'_2, \ldots, z'_N, z''_A, \ldots, \mu) \, dz''_B + f(z'_2, \ldots, z'_N). \] (63)
In order eq. (63) to make sense we need that the integral be independent of the path in the \( \mathcal{N} - 2 \) dimensional space of the \( z_B \), namely the form \( \omega = \sum_A \frac{\partial \beta_A}{\partial \mu} d z_A \) is exact. Such a property is a consequence of a conjecture due to Polyakov [15] to which now we turn. Such a property states that the accessory parameters in the fuchsian differential equation which solves the Liouville equation are obtained as derivatives of the regularized Liouville action [26]
\[ S_\epsilon[\phi] = \frac{i}{2} \int_{X_t} (\partial_x \phi \partial_x \phi + \frac{e^\phi}{2}) d z \wedge d \bar{z} - \frac{i}{2} \sum_n (1 - \mu_n) \int_n \phi \left( \frac{d \bar{z}}{\bar{z} - z_n} - \frac{d z}{z - z_n} \right) \]
\[ + \frac{i}{2} \sum_B \int_B \phi \left( \frac{d \bar{z}}{\bar{z} - z_B} - \frac{d z}{z - z_B} \right) - \frac{i}{2} (\mu - 2) \int_\infty \phi \left( \frac{d \bar{z}}{\bar{z} - z} - \frac{d z}{z} \right) \]
computed on the solution of the Liouville equation. In \((64)\) \(idz \wedge d\bar{z}/2 = dxdy\) and \(X_\epsilon\) is a large disk of radius \(1/\epsilon\) from which small disks of radius \(\epsilon\) around the particles and apparent singularities have been removed. The line integrals are all taken counterclockwise and they impose the correct behavior on \(\phi\) around the singularities and at infinity. Polyakov conjecture states that

\[
-\frac{1}{2\pi} dS_\epsilon = \sum_n \beta_n dz_n + \sum_B \beta_B dz_B. \tag{65}
\]

In other words, the accessory parameters \(\beta_n\) and \(\beta_B\) which provide \(SU(1, 1)\) monodromies i.e. a monodromic conformal factor, define an exact 1-form. Such a conjecture has been proved by Zograf and Tahktajan [15] for fuchsian differential equations with parabolic singularities; in addition they remark that the proof can be extended in a straightforward way to the case of elliptic singularities of finite order. We are obviously interested in the generic elliptic case including non algebraic singularities (any real \(\mu_l\) with \(0 < \mu_l < 1\) not necessarily of the form \(1/n\)). The extension of the proof to this case seems not as straightforward since the main tool of the proof, i.e. the mapping of the upper complex half plane into the punctured Riemann surface through a properly discontinuous group, is not available. Nevertheless from what follows it appears that such an extension is of great relevance for the hamiltonian structure of 2 + 1 gravity.

Thus the hamiltonian

\[
H = \frac{1}{2\pi} \frac{\partial S_\epsilon}{\partial \mu}. \tag{66}
\]

already provides the correct expression for \(z'_n\). It is now straightforward to prove that with \((66)\) also the equations for \(\dot{P}_n\) are satisfied. In fact we have

\[
-\frac{\partial H}{\partial z'_n} = -\frac{1}{2\pi} \frac{\partial^2 S_\epsilon}{\partial \mu \partial z'_n} - \frac{1}{2\pi} \sum_B \frac{\partial^2 S_\epsilon}{\partial \mu \partial z'_B} \frac{\partial z'_B}{\partial z'_n} = \frac{\partial \beta_n}{\partial \mu} + \sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z'_B}{\partial z'_n}. \tag{67}
\]

We recall that in the non rotating frame the hamiltonian contains an additional contribution, as already observed at the beginning of this section. Its complete form in that frame is indeed
given by

\[ H = \ln \left( \sum_n P_n \bar{z}_n \right) + \frac{1}{2\pi} \frac{\partial S}{\partial \mu}. \]  

(68)

Note that this hamiltonian, being time-independent, provides a further conservation law in the \( N \)-particle problem.

On a more formal grounds, it is interesting to notice that the closedness of the 1-form (65) is implied by the weaker relation

\[ \frac{\partial \beta_A}{\partial z_B'} = \frac{\partial \beta_B}{\partial z_A'}. \]  

(69)

for the auxiliary parameters, as from these it follows through the Garnier equations

\[ \frac{\partial \beta_n}{\partial z_A'} = \frac{\partial \beta_A}{\partial z_n'}. \]  

(70)

In fact from the Garnier equations we have

\[ \frac{\partial \beta_A}{\partial z_B'} = -2 \frac{\partial H_n}{\partial z_B'} - 2 \sum_B \frac{\partial \beta_A}{\partial z_B'} \frac{\partial z_B'}{\partial z_A'} = -2 \frac{\partial H_n}{\partial z_A'} - 2 \sum_B \frac{\partial \beta_A}{\partial z_B'} \frac{\partial H_n}{\partial z_B'} \]  

(71)

while

\[ \frac{\partial \beta_n}{\partial z_A'} = -2 \frac{\partial H_n}{\partial z_A'} - 2 \sum_B \frac{\partial H_n}{\partial \beta_B} \frac{\partial \beta_B}{\partial z_A'}. \]  

(72)

and thus from eq.(69) we obtain eq.(70).

Similarly we have

\[ \frac{\partial \beta_n}{\partial z_m'} = -2 \frac{\partial H_n}{\partial z_m'} - 2 \sum_B \frac{\partial H_n}{\partial \beta_B} \frac{\partial \beta_B}{\partial z_m'} = -2 \frac{\partial H_n}{\partial z_m'} + 4 \sum_B \frac{\partial H_m}{\partial \beta_B} \frac{\partial H_n}{\partial \beta_B} - \]  

\[ -4 \sum_B \frac{\partial H_n}{\partial \beta_B} \sum_C \frac{\partial H_m}{\partial \beta_C} \frac{\partial \beta_B}{\partial z_C}. \]  

(73)

i.e. if eq.(69) holds we have

\[ \frac{\partial \beta_n}{\partial z_m'} \frac{\partial \beta_m}{\partial z_n'} = 2 \left[ \frac{\partial H_m}{\partial z_m'} \frac{\partial H_n}{\partial z_n'} + 2 \sum_B \left( \frac{\partial H_m}{\partial z_B'} \frac{\partial H_n}{\partial \beta_B} \right) \right] \]  

(74)
which due to a general identity vanishes. Thus
\[
\frac{\partial \beta_n}{\partial z'_m} = \frac{\partial \beta_m}{\partial z'_n}.
\] (75)

Finally we notice that eq.(66) assures that the hamiltonian \( H \) is globally defined, while the integrability condition we proved in the three particle case assures only the local existence of the hamiltonian.

5 The asymptotic metric

In the previous section we constructed the reduced particle hamiltonian from the equations of motion. On the other hand one could follow a different path, i.e. to recover the hamiltonian as a boundary term in the gravitational action. In order to do so we shall first investigate the diffeomorphism which connects the metric for a spinning particle in the DJH gauge and the same geometry in the \( K = 0 \) gauge. It turns out that such a diffeomorphism can be computed exactly and that will also allow us to compute the expression of the Killing vectors of the spinning cone geometry in our coordinates.

The DJH metric is given by
\[
ds^2 = -(dT + Jd\phi)^2 + dR^2 + \alpha^2 R^2 d\phi^2.
\] (76)

With the transformation \( R = r_0 \zeta^\alpha \) it can be put into conformal form
\[
ds^2 = -(dT + Jd\phi)^2 + \alpha^2 r_0^2 (\zeta^\alpha - 1)^2 (d\zeta^2 + \zeta^2 d\phi^2).
\] (77)

It is a solution of the sourceless 2+1 Einstein’s equations with a single source located at \( r = 0, \forall t \). It possesses two Killing vector fields, \( \frac{\partial}{\partial T} \) and \( \frac{\partial}{\partial \phi} \).

For the metric in the maximally slicing gauge we shall use the ADM form
\[
ds^2 = -N^2 dt^2 + e^{2\sigma} (dz + N^z dt)(d\bar{z} + N^\bar{z} dt).
\] (78)
We shall set with \( r = |z| \)
\[
e^{2\sigma} = f^2(r,t); \quad N^z = zn(r,t); \quad N^\bar{z} = \bar{z}\bar{n}(r,t).
\]
(79)

Such a metric possesses the Killing vector field \( \frac{\partial}{\partial \theta} \). Moreover
\[
g_{tt} = -N^2 + r^2n\bar{n}f^2; \quad g_{t\theta} = r^2f^2\frac{n - \bar{n}}{2i}; \quad g_{tr} = rf^2\frac{n + \bar{n}}{2};
\]
\[
g_{rr} = f^2; \quad g_{\theta\theta} = r^2f^2; \quad g_{r\theta} = 0.
\]
(80)

A solution of the Einstein equations which comply to the York instantaneous gauge is provided by
\[
e^{2\sigma} = 2\pi \pi^z \pi_{z} e^{2\tilde{\sigma}}
\]
(81)

with
\[
e^{-2\tilde{\sigma}} = \frac{8\alpha^2}{\Lambda^2} \frac{(\frac{z^z}{\Lambda^2})^{-\alpha - 1}}{(1 - (\frac{z^z}{\Lambda^2})^{-\alpha})^2}
\]
(82)

where \( \alpha = 1 - \mu = 1 - \frac{M}{4\pi} \). \( \tilde{\sigma} \) solves
\[
\Delta(2\tilde{\sigma}) = -e^{-2\tilde{\sigma}}
\]
(83)

for \( z \neq 0 \) and \( \pi^z \) is given by
\[
\pi^z_z = -\frac{1}{2\pi z^2} \sum_n P_n z_n \equiv \frac{p(t)}{z^2}.
\]
(84)

Eq.(84) is the asymptotic form of the expression
\[
-\frac{1}{2\pi} \sum_n \frac{P_n}{z - z_n}
\]
(85)

subject to the constraint \( \sum_n P_n = 0 \).

\( N \) and \( N^z \) are given by eq.(81). We know that \( p(t) \) evolves according to
\[
p(t) = -\frac{1}{2\pi}[\alpha t - ib]
\]
(86)
and $\Lambda$ is given by

$$\Lambda^2(t) = c_\Lambda [p(t)p(t)]^{\frac{1}{4}}. \quad (87)$$

In fact if the conformal factor $e^{2\sigma}$ has to provide a solution of Einstein’s equations the coefficient $s^2$ which appears in its asymptotic expansion

$$e^{2\sigma} \approx s^2 (z\bar{z})^{-\mu} \quad (88)$$

has to be time independent. We shall see in the following section that ln $s^2$ coincides with the hamiltonian, which is obviously conserved. One checks that the metric (78) with positions (8,9,82,84,87) satisfy Einstein’s equation in all space with a source confined to $z = 0$.

In order to find the diffeomorphism which connects the two metrics (76) and (78) it is useful to introduce the intermediate variable $\rho = \left(\frac{r}{\Lambda}\right)^{\frac{1}{\alpha}}$. We have

$$-2\tilde{\sigma} = \ln\left(\frac{8}{\Lambda^2}\right) + 2 \ln \alpha - \frac{1 - \alpha}{\alpha} \ln \rho^2 - 2 \ln(1 - \rho^{-2}) \quad (89)$$

and

$$N = \frac{1}{2\pi \alpha} \left[ \frac{\ln(\rho^2 k^2)}{2} - 1 + \frac{1}{\rho^2 - 1} \ln(\rho^2 k^2) \right] \quad (90)$$

where

$$\ln k^2 = 2\alpha^2 \frac{\partial \ln \Lambda^2}{\partial \mu}. \quad (91)$$

$k^2$ in general will be a function of time and given a solution of the $N$- particle problem it is a well defined function; e.g. it can be explicitly computed in the two body case. On the other hand one can verify that the asymptotic metric provided by eq.(8,9,82,84) is a solution of Einstein’s equations for any $k^2(t)$ (see appendix 2). It is important to note that the solution of

$$\Delta N = e^{-2\sigma} N \quad (92)$$
obtained by taking the derivative of eq.(83) with respect to \( \mu \) is performed at fixed time and the \( \Lambda \) appearing in eq.(82) is a function of \( \mu \). From eq.(9) \( N_z \) is given by

\[
N_z = \frac{z}{\pi p(t)} \left[ \frac{\rho^2 \ln(\rho^2 k^2)}{(\rho^2 - 1)^2} - \frac{1}{(\rho^2 - 1)} \right] \equiv z_n. \tag{93}
\]

The most general transformation which transforms \( \frac{\partial}{\partial \theta} \) into \( \frac{\partial}{\partial \phi} \) is

\[
R = R(\rho, t); \quad T = T(\rho, t); \quad \phi = \theta + \omega(\rho, t). \tag{94}
\]

Equating the coefficient of \( d\theta^2 \) one obtains with \( f^2 = e^{2\sigma} \)

\[
g_{\theta\theta} = r^2 f^2 = \alpha^2 R^2 - J^2 \tag{95}
\]

and we have for the metric in the variables \( t, \theta, \rho \)

\[
g_{\rho\rho} = \frac{|p(t)|^2}{4\alpha^4} (1 - \rho^{-2})^2 \tag{96}
\]

and

\[
g_{\theta\theta} = \frac{|p(t)|^2}{4\alpha^2} (\rho - \rho^{-1})^2. \tag{97}
\]

From eqs.(95,97) we obtain

\[
R^2 = \frac{1}{\alpha^2} (J^2 + \frac{|p(t)|^2}{4\alpha^2} (\rho - \rho^{-1})^2). \tag{98}
\]

The matching of the \( \rho \theta \) and \( \rho \rho \) components of the metric gives

\[
0 = g_{\theta\rho} = -J(\partial_\rho T + J\partial_{\rho}\omega) + \alpha^2 R^2 \partial_{\rho}\omega \tag{99}
\]

\[
g_{\rho\rho} = -(\partial_\rho T + J\partial_{\rho}\omega)^2 + (\partial R)^2 + \alpha^2 R^2(\partial_{\rho}\omega)^2 \tag{100}
\]

from which we deduce

\[
(\partial_{\rho}\omega)^2 = \frac{J^2[(\partial_\rho R)^2 - g_{\rho\rho}]}{\alpha^2 R^2 g_{\theta\theta}} \tag{101}
\]
and substituting eqs. (95, 98, 100) into (101) we have

\[ \partial_\rho \omega = \frac{4J \rho V(t)}{|p(t)|^2(p^2 - 1) + 4\alpha^2 J^2 \rho^2} = \frac{2\rho B}{\alpha[(\rho^2 - A)^2 + B^2]} \]  

(102)

with \( V(t) = \sqrt{|p(t)|^2 - \alpha^2 J^2} \), \( A(t) = 1 - \frac{2\alpha^2 J^2}{|p(t)|^2} \), \( B(t) = \frac{2\alpha J V(t)}{|p(t)|^2} \), which integrated gives

\[ \omega = \frac{1}{\alpha} \left[ \arctan \left( \frac{\rho^2 - A}{B} \right) - \frac{\pi}{2} \right] + f(t) \equiv \bar{\omega}(t, \rho) + f(t). \]  

(103)

\( \bar{\omega} \) for \( \rho \to \infty \) goes to zero.

Similarly from eqs. (99, 100) we find

\[ \partial_\rho T = \frac{V(t)}{\alpha^2 \rho} - J \partial_\rho \omega \]  

(104)

from which

\[ T = \frac{V(t)}{2\alpha^2} \ln \rho^2 - J \bar{\omega}(t, \rho) + h(t) \equiv \bar{T}(t, \rho) + h(t). \]  

(105)

The matching of the \( t\theta \) component of the metric gives

\[ g_{t\theta} = -J \partial_\rho T + (\alpha^2 R^2 - J^2) \partial_\rho \omega \]  

(106)

i.e.

\[ -\frac{b}{8\pi^2 \alpha^2} \left[ \ln(k^2 \rho^2) - 1 + \rho^{-2} \right] = \]

\[ -J \dot{h}(t) - \frac{JV}{2\alpha^2} \ln \rho^2 + \frac{|p(t)|^2}{4\alpha^3} \left[ -\dot{B} + (\dot{A}\dot{B} - A\dot{B}) \rho^{-2} \right] + \alpha^2 R^2 \dot{f}(t). \]  

(107)

As \( R^2(\rho) \) behaves like \( \rho^2 \) for large \( \rho \), we have \( \dot{f}(t) = 0 \) which means that the two frames asymptotically do not rotate one with respect to the other. The \( \ln \rho^2 \) terms fixes \( b = 2\pi \alpha J \) thus giving \( V(t) = \frac{\alpha^2}{2\pi} \), while the matching of the constant terms gives

\[ \dot{h}(t) = \frac{1}{4\pi \alpha} \left[ \ln k^2(t) - \frac{2\alpha^2 J^2}{|p(t)|^2} \right], \]  

(108)
which defines $h(t)$ up to a constant; this is due to the fact that in the DJH gauge the time like Killing vector is simply $\frac{\partial}{\partial t}$. Now the diffeomorphism is completely fixed and one can check that the remaining equations for $g_{tt}$ and $g_{t\rho}$ are satisfied.

Summarizing the diffeomorphism is given by

$$R^2 = \frac{1}{\alpha^2} \left[ J^2 + \frac{r^{2\alpha}}{4c_\Lambda \alpha^2} (1 - c_\Lambda |p(t)|^2 r^{-2\alpha})^2 \right]$$  \hspace{1cm} (109)

$$\phi = \theta + \omega \equiv \theta + \frac{1}{\alpha} \arctan \left[ 2\pi c_\Lambda^{-1} r^{2\alpha} - |p(t)|^2 + 2\alpha^2 J^2 \right] \frac{2\alpha^2 Jt}{2\alpha^2 Jt}$$ \hspace{1cm} (110)

$$T = \frac{t}{4\pi} \left[ \ln \frac{r^2}{c_\Lambda} - \frac{1}{\alpha} \ln |p(t)|^2 \right] - J\omega + h(t)$$ \hspace{1cm} (111)

with $h(t)$ obeying eq. (108). This shows that two asymptotic solutions with different $k^2(t)$ are diffeomorphic to the same DJH metric and thus are diffeomorphic to each other. This explains why for any choice of $k^2(t)$ eq.s (8, 9, 82) are solutions of Einstein’s equations. For large $r$ eq.s (109, 110, 111) become

$$R^2 \approx \frac{r^{2\alpha}}{4c_\Lambda \alpha^4}$$ \hspace{1cm} (112)

$$\phi \approx \theta + \frac{\pi}{2\alpha}$$ \hspace{1cm} (113)

$$T \approx \frac{t}{4\pi} \ln(\frac{r^2}{c_\Lambda |p(t)|^2}) - \frac{\pi J}{2\alpha} + h(t).$$ \hspace{1cm} (114)

In the DJH gauge a finite transformation along the Killing vector $\frac{\partial}{\partial T}$ is simply given by $T \rightarrow T + c$ while in the York instantaneous gauge it is more complicated. The time-like Killing vector in the instantaneous York gauge is simply computed and given by

$$\frac{(2\pi)^3 t (p^2 + 1) |p(t)|^2}{\mathcal{D}} \frac{\partial}{\partial t} + \frac{8\pi^2 J \alpha^2}{\mathcal{D}} \frac{\partial}{\partial \theta} + \frac{4\pi \alpha^2 r t}{\mathcal{D}} \frac{\partial}{\partial r}$$ \hspace{1cm} (115)
\[ D = 4\pi^2 |p(t)|^2 (\rho^2 + 1) \left[ \ln \rho + 2\pi \alpha \hat{h}(t) \right] + \alpha^2 t^2 (1 - \rho^2) + 8\pi^2 J^2 \alpha^2. \]  

(116)

For large \( r \) the vector (115) reduces to

\[ \frac{4\pi}{\ln \left( \frac{r^2}{c_\Lambda |p(t)|^2} \right)} \left( \frac{\partial}{\partial t} + \frac{\alpha J c^\alpha}{\pi r^{2\alpha}} \frac{\partial}{\partial \theta} + \frac{\alpha t c^\alpha}{2\pi r^{2\alpha}} r \frac{\partial}{\partial r} \right). \]  

(117)

6 The hamiltonian as a boundary term

We have solved the hamiltonian and diffeomorphism constraints and moreover in the \( K = 0 \) conformal gauge we have \( \pi^{ij} \dot{g}_{ij} \equiv 0 \). Thus the action of the particles plus gravity reduces to

\[ S = \int dt \left( \sum_n P_n q_n - H_B \right) \]  

(118)

with

\[ H_B = -2 \int dt \int_{B_t} d^{(D-1)} x \sqrt{\sigma_{Bt}} N \left( K_{Bt} + \frac{\eta}{\cosh \eta} \mathcal{D}_\alpha v^\alpha \right) + 2 \int dt \int_{B_t} d^{(D-1)} x r_\alpha \pi^\alpha_{\beta (B_t)} N_\beta. \]  

(119)

We want now to extract from \( H_B \) the reduced particle hamiltonian and compare it to the hamiltonian \( H \) derived directly from the particle equations of motion.

The last term in the above equation can be computed as follows: on the boundary \( x^2 + y^2 = r_0^2 = \text{const} \) we have

\[ r_\alpha \pi^\alpha_{\beta (B_t)} N_\beta = -2(\bar{z} \partial \bar{z} N + z \partial z N) + \bar{z}g(\bar{z})\pi^z_{\bar{z}} + zg(z)\pi^z_z \]  

(120)

whose integral in \( d\theta \) between 0 and \( 2\pi \) is given by

\[ -2 \oint (\bar{z} \partial \bar{z} + z \partial z) N d\theta + i \oint d\bar{z}g(\bar{z})\pi^z_{\bar{z}} - i \oint dzg(z)\pi^z_z. \]  

(121)

As for large \( |z| \), \( N \) behaves like \( \ln (z\bar{z})/4\pi \) we see that the first term in the above expression goes over to the constant \( -2 \). In the computation of the remaining terms as already noticed
in the only contribution in \( g(z) \) which survives in the sum is the one arising from the linear term in the first order polynomial \( p_1(z) \) which in the frame non rotating at infinity is given by

\[
p_1(z) = c_0 - \frac{1}{\sum_n P_n z_n} z.
\]  

(122)

Using this result we find zero for eq.(121) i.e. for the last term in (119). Similarly one proves that the contribution of the term \( D_\alpha v^\alpha \) goes to zero like \( (r_0^2)^{\mu - 1} \ln r_0^2 \) for \( r_0 \to \infty \). Thus we are left with the boundary term

\[
H_B = -2 \int_{B_t} d^{(D-1)}x \sqrt{\sigma_{Bt}} N K_{Bt}.
\]  

(123)

By inserting the metric eqs.(8,9,82) into the expression for \( K_{Bt} \) and \( \sigma_{Bt} \) we obtain for the integral

\[
H_B = -4\pi N r_0 \partial_r [\ln(r e^\sigma)]
\]  

(124)

and thus for large \( r_0 \) the boundary term becomes

\[
H_B = -r_0 \ln r_0^2 (\frac{1}{r_0} + \partial_r \sigma) = (\mu - 1) \ln r_0^2.
\]  

(125)

We recall now that the equations of motion are obtained from the action by keeping the values of the fields fixed at the boundary, or equivalently [21] by keeping fixed the intrinsic metric of the boundary. In our case the variations should be performed keeping fixed the fields \( N, N^\alpha \), and \( \sigma \) at the boundary. We shall perform the computation for the boundary given by a circle of radius \( r_0 \) for a very large value of \( r_0 \). If we change the positions of particle positions and momenta, \( \Lambda \) varies and in order to keep the value of \( \sigma \) fixed at the boundary we must vary \( \mu \) as to satisfy the following equality

\[
\ln\{(\sum_n P_n z_n)(\sum_n \bar{P}_n \bar{z}_n)\} - \mu \ln r_0^2 + (\mu - 1) \ln \Lambda^2 - \ln 16\pi^2 \equiv -\mu \ln r_0^2 + \ln s^2 = \text{const}.
\]  

(126)
Thus

\[ 0 = -\delta \mu \ln r_0^2 + \sum_n (\delta z_n \frac{\partial \ln s^2}{\partial z_n} + \delta P_n \frac{\partial \ln s^2}{\partial P_n} + \text{c.c.}) + \delta \mu \frac{\partial \ln s^2}{\partial \mu} \]  

(127)

i.e. for large \( r_0 \)

\[ \delta \mu \approx \frac{1}{\ln r_0^2} \sum_n (\delta z_n \frac{\partial \ln s^2}{\partial z_n} + \delta P_n \frac{\partial \ln s^2}{\partial P_n} + \text{c.c.}). \]  

(128)

Substituting into eq.(125) we have

\[ \delta H_B = \sum_n (\delta z_n \frac{\partial \ln s^2}{\partial z_n} + \delta P_n \frac{\partial \ln s^2}{\partial P_n} + \text{c.c.)} \]  

(129)

i.e. apart for a constant \( H_B \) equals \( \ln s^2 \)

\[ H_B = \ln s^2 + \text{const.} = \ln \left[ \left( \sum_n P_n z_n \right) \left( \sum_n \bar{P}_n \bar{z}_n \right) \right] + (\mu - 1) \ln \Lambda^2 + \text{const.} \]  

(130)

In the two particle case one can check that eq.(130) coincides with the hamiltonian derived directly from the equations of motion. In fact explicit computation by using the expression of \( \Lambda \) in terms of hypergeometric functions gives

\[ \Lambda^2 = |z_2'|^2 \left[ \frac{1}{8(1 - \mu)^2 G(\mu)} \right]^\frac{1}{\mu} \]  

(131)

where

\[ G(\mu) = \pi^{-2} \Gamma^4(1 - \mu) \sin^2(\pi \mu) \times \]

\[ \Delta((\mu + \mu_1 + \mu_2)/2) \Delta((\mu - \mu_1 + \mu_2)/2) \Delta((\mu + \mu_1 - \mu_2)/2) \Delta((\mu - \mu_1 - \mu_2)/2). \]  

(132)

The boundary term eq.(123) depends on the fields on the boundary and also on the derivative of the fields directed towards the interior i.e. derivative with respect to \( r \). By keeping the values of the fields fixed on the boundary it provides the hamiltonian, i.e. a function of \( z_n \) and \( P_n \) which through Hamilton’s equations give rise to the equations of motion. It is not however the energy as usually defined i.e. the value of the boundary term when \((N, N^i)\)
take the values of the asymptotic time-like Killing vector. In our case due to the choice of the \( K = 0 \) gauge which vastly simplifies the dynamics, the \( (N, N^i) \) differ from the timelike asymptotic Killing vector. The energy of a solution is easily obtained in the DJH gauge, where one checks from the metric eq.(76) that with \( (N, N^i) = (1, 0, 0) \) i.e the normalized Killing vector, one obtains for \( H_B \) the value \( 4\pi(\mu - 1) \) as expected.

It is of interest to examine how \( \ln \Lambda^2 \) behaves under a complex scaling \( z' = \alpha z \). It is easily seen from the Liouville equation that if \( 2\tilde{\sigma}(z) \) is a solution with singularities in \( z_n \) and \( z_B(z_n, P_n) \) the solution with singularities in \( \alpha z_n \) and \( z_B(\alpha z_n, P_n/\alpha) = \alpha z_B(z_n, P_n) \) is given by

\[
2\tilde{\sigma}(z) = 2\tilde{\sigma}(\frac{z}{\alpha}) + \ln(\alpha \bar{\alpha}).
\]

(133)

It implies the following transformation law on \( \ln \Lambda \)

\[
\ln \Lambda^2(\alpha z_n, \frac{P_n}{\alpha}) = \ln \Lambda^2(z_n, P_n) + \ln(\alpha \bar{\alpha})
\]

(134)

which provides the following Poisson bracket

\[
[H, \sum_n P_n z_n] = [\sum_n P_n z_n, (\mu - 1) \ln \Lambda^2] = \mu - 1
\]

(135)

and thus we have reached a hamiltonian derivation of the generalized conservation law

\[
\sum_n P_n z_n = (1 - \mu)(t - t_0) - ib.
\]

(136)

We want now to relate the result eq.(130) to the results of sect.4. Let us now consider the value of the action \( S_\epsilon \) on the solution of the Liouville equation and let us compute its derivative with respect to \( \mu \). As we are varying around a stationary point the only contribution is provided by the terms in eq.(64) which depend explicitly on \( \mu \) i.e.

\[
\frac{\partial S_\epsilon}{\partial \mu} = -i \frac{1}{2} \int_{\infty} \phi \left( \frac{d\bar{z}}{z} - \frac{dz}{\bar{z}} \right) - 2\pi(\mu - 2) \ln \epsilon^2
\]

(137)

and as \( \phi \equiv -2\tilde{\sigma} \) at infinity behaves like

\[
\phi \approx \ln 8(1 - \mu)^2 + (\mu - 2) \ln z\bar{z} - (\mu - 1) \ln \Lambda^2
\]

(138)
we have
\[ \frac{\partial S_\epsilon}{\partial \mu} = -2\pi \ln 8(1 - \mu)^2 + 2\pi(\mu - 1) \ln \Lambda^2. \] (139)
Thus we can rewrite eq. (140) as
\[ H_r = \ln \left[ \left( \sum_n P_n z_n \right) \left( \sum_n \bar{P}_n \bar{z}_n \right) \right] + \frac{1}{2\pi} \frac{\partial S_\epsilon}{\partial \mu} + \text{const} \] (140)
in agreement with the result of sect. 4 obtained through Polyakov’s conjecture.

7 Quantization: the two particle case

We recall that the classical two particle hamiltonian in the reference system which does not rotate at infinity is given by
\[ H = \ln(P z \bar{P} \bar{z}) + (\mu - 1) \ln(z \bar{z}) = \ln(P z^\mu) + \ln(\bar{P} \bar{z}^\mu) = h + \bar{h} \] (141)
with \( P = P'_z \) and \( z = z'_z \). \( h \) and \( \bar{h} \) are separately constant of motion and if we combine them with the generalized conservation law \( P z = (1 - \mu)(t - t_0) - ib \) (see eq. (27)) we obtain the solution for the motion
\[ z = \text{const} [(1 - \mu)(t - t_0) - ib]^{1/\mu}. \] (142)
\( H \) can be rewritten as
\[ H = \ln((x^2 + y^2)^\mu ((P_x)^2 + (P_y)^2)). \] (143)
Keeping in mind that with our definitions \( P \) is the momentum multiplied by \( 16\pi G_N/c^3 \), applying the correspondence principle we have
\[ [x, P_x] = [y, P_y] = il_P \] (144)
where \( l_P = 16\pi G_N \hbar/c^3 \), all the other commutators equal to zero. \( H \) is converted into the operator
\[ \ln[-(x^2 + y^2)^\mu \Delta] + \text{constant}. \] (145)
The argument of the logarithm is the Laplace-Beltrami $\Delta_{LB}$ operator on the metric $ds^2 = (x^2 + y^2)^{-\mu}(dx^2 + dy^2)$. Following an argument similar to the one presented in [27] one easily proves that if we start from the domain of $\Delta_{LB}$ given by the infinite differentiable functions of compact support $C^\infty_0$ which can also include the origin, then $\Delta_{LB}$ has a unique self-adjoint extension in the Hilbert space of functions square integrable on the metric $ds^2 = (x^2 + y^2)^{-\mu}(dx^2 + dy^2)$ and as a result since $\Delta_{LB}$ is a positive operator, $\ln(\Delta_{LB})$ is also self-adjoint. In fact expanding in circular harmonics

$$\psi(x, y) = \sum_m e^{im\theta} \phi_m(r)$$

(146)

the indicial equation furnishes the behaviors $r^{\pm m}$ at the origin for $m \neq 0$ and $r^0$, $\ln(r)$ for $m = 0$. Then for $m \neq 0$ only the behavior $r^{|m|}$ gives rise to a square integrable function (we recall that $\mu < 1$). For $m = 0$ if $\Delta_{LB}$ is defined already on the $C^\infty_0$ functions with support which can include the origin, one sees that the equation $(\Delta_{LB}^* \pm i)\phi = 0$ has no solution for $\phi \in D(\Delta_{LB}^*)$. In fact if $D(\Delta_{LB})$ includes the $C^\infty_0$ functions whose support can include the origin, then $D(\Delta_{LB}^*)$ cannot contain functions which diverge logarithmically at the origin. But if the $\phi$ which solves $(\Delta_{LB}^* \pm i)\phi = 0$ has no logarithmically divergent part one proves easily that $(\phi, \Delta_{LB}^* \phi) = \text{real}$, which is a contradiction. Obviously in rewriting $H$ in the from eq.(145) a well defined ordering has been chosen; one that appears rather appealing due the simplicity and covariant nature of the result.

Deser and Jackiw [17] considered the quantum scattering of a test particle on a cone both at the relativistic and non relativistic level. Most of the techniques developed there can be transferred here. The main difference is the following; instead of the hamiltonian $(x^2 + y^2)^\mu(p_x^2 + p_y^2)$ which appears in their non relativistic treatment, we now have the hamiltonian $\ln[(x^2 + y^2)^\mu(p_x^2 + p_y^2)]$. The partial wave eigenvalue differential equation

$$(r^2)^\mu \left[- \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{n^2}{r^2}\right] \phi_n(r) = k^2 \phi_n(r)$$

(147)

with $\mu = 1 - \alpha$ is solved by

$$\phi_n(r) = J_{|n|}^{\alpha}(\frac{k}{\alpha} r^\alpha)$$

(148)
and we have the completeness relation

\[ \delta^2(z - z') = \alpha \sum_n \frac{e^{in(\phi - \phi')}}{2\pi} \int_0^{\infty} \frac{r^{(\alpha-1)}}{\alpha} J_{|n|} \left( \frac{k}{\alpha} r'^{\alpha} \right) kdk \frac{\alpha}{\alpha} J_{|n|} \left( \frac{k}{\alpha} r^{\alpha} \right) \]

from which the logarithm of the operator \(-\Delta_{LB}\), which is our hamiltonian, becomes

\[ \alpha \sum_n \frac{e^{in(\phi - \phi')}}{2\pi} \int_0^{\infty} \frac{r^{(\alpha-1)}}{\alpha} J_{|n|} \left( \frac{k}{\alpha} r'^{\alpha} \right) \ln \left( \frac{k}{\alpha} r'^{\alpha} \right) \frac{\alpha}{\alpha} J_{|n|} \left( \frac{k}{\alpha} r^{\alpha} \right). \]  

(150)

It is a self-adjoint operator with domain [28] given by those \(f(z)\) such that

\[ \alpha \sum_n \int (\ln(k^2))^2dk | \int_0^{\infty} J_{|n|} \left( \frac{k}{\alpha} r'^{\alpha} \right) f_n(r)rdr|^2 < \infty. \]

(151)

The Green function is given by

\[ G(z, z', t) = \alpha \sum_n \frac{e^{in(\phi - \phi')}}{2\pi} \int_0^{\infty} \frac{r^{(\alpha-1)}}{\alpha} J_{|n|} \left( \frac{k}{\alpha} r'^{\alpha} \right) \ln(k^2) kdk \frac{\alpha}{\alpha} J_{|n|} \left( \frac{k}{\alpha} r^{\alpha} \right). \]

(152)

The integral in \(k\) can be performed in terms of hypergeometric functions, to obtain

\[ G(z, z', t) = \frac{2}{\alpha \Gamma(\frac{ict}{l_P}) r'^{\alpha}} \left( \frac{r^{\alpha} + r'^{\alpha}}{2\alpha} \right)^{\frac{2ict}{l_P}} \]

\[ \sum_n \frac{e^{in(\phi - \phi')}}{2\pi} \frac{\Gamma\left( \frac{|n|}{\alpha} + 1 \right) - i\frac{ict}{l_P}}{\Gamma\left( \frac{|n|}{\alpha} + 1 \right)} \rho^{\alpha+1} \frac{1}{2} F_1 \left( \frac{|n|}{\alpha} + 1 - \frac{ict}{l_P}; \frac{|n|}{\alpha} + 1; \frac{1}{2}, \frac{2|n|}{\alpha} + 1; 4\rho \right) \]

where

\[ \rho = \frac{r^{\alpha} r'^{\alpha}}{r^{\alpha} + r'^{\alpha}}. \]

(153)

(154)

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Appendix 1: Properties of the residues $\beta_n$, $\beta_A$ in the three body problem

In this appendix we outline the derivation of relations (59, 62) which were used in sect. 4 to prove the hamiltonian nature of the equations for $\dot{z}_n'$ and $\dot{P}_n'$ of the three body problem.

To this end we shall exploit the Garnier equations which express the isomonodromic evolution of the apparent singularity [22]. We recall that the Garnier equations are a direct outcome of the ADM treatment of the particle dynamics [7]. The Garnier hamiltonian for the standard parameters $z_A$ and $b_A$ ($z_1 \equiv 0$, $z_2 \equiv 1$ see [22, 23]) is given by

$$H_G = \frac{z_A(z_A - 1)(z_A - z_3)}{z_3(z_3 - 1)} \left\{ b_A^2 - \frac{\mu_1}{z_A} + \frac{\mu_2}{z_A - 1} + \frac{\mu_3 - 1}{z_A - z_3} b_A + \frac{\kappa}{z_A(z_A - 1)} \right\}$$  \hspace{1cm} (155)

and we have

$$\frac{\partial z_A}{\partial z_3} = \frac{\partial H_G}{\partial b_A}; \quad \frac{\partial b_A}{\partial z_3} = -\frac{\partial H_G}{\partial z_A}. \hspace{1cm} (156)$$

$b_A$ is related to the $\beta_A$ appearing in the $SL(2C)$ canonical form (again $z_1 \equiv 0$, $z_2 \equiv 1$) by

$$b_A = \frac{\beta_A}{2} - \frac{1}{2} \left( \frac{1 - \mu_1}{z_A} + \frac{1 - \mu_2}{z_A - 1} + \frac{1 - \mu_3}{z_A - z_3} \right). \hspace{1cm} (157)$$

Starting from the previous equations it is not difficult to verify directly (see also [23]) that the two Garnier hamiltonians $H_2$ and $H_3$ which supervise the evolution of the auxiliary parameters $z_A'$ and $\beta_A \equiv 2\beta_A$ appearing in the $SL(2C)$ canonical differential equation

$$y''(z) + Q(z)y(z) = 0 \hspace{1cm} (158)$$

with

$$Q(z) = \frac{1 - \mu_1^2}{4z^2} + \frac{1 - \mu_2^2}{4(z - z_2')^2} + \frac{1 - \mu_3^2}{4(z - z_3')^2} - \frac{3}{4(z - z_A')^2} \hspace{1cm} (159)$$

are given by $H_2 = -\beta_2/2$ and $H_3 = -\beta_3/2$, i.e. by the simple residues at $z_2'$ and $z_3'$ of the function $Q(z)$. This is obtained by expressing $\beta_2$ and $\beta_3$ in terms of $z_2'$, $z_3'$, $z_A'$, $\beta_A$, $\mu$ using the
Fuchs and no-logarithm conditions at $z'_A$. Under isomonodromic transformations generated by a change in the value of $z'_n$ we have

$$\frac{dz'_A}{dz'_n} = \frac{\partial H_n}{\partial \beta_A}; \quad \frac{d\beta_A}{dz'_n} = -\frac{\partial H_n}{\partial z'_A}. \quad (160)$$

The monodromy condition on $\tilde{\sigma}$ or equivalently the imposition of the $SU(1,1)$ nature of the monodromies described by the fuchsian differential equation fixes $\beta_A$ and consequently, through Fuchs and no logarithm conditions, $\beta_1, \beta_2, \beta_3$ as functions of $z'_1, z'_3, z'_A, \mu$, i.e.

$$H_n(z'_2, z'_3, z'_A, \mu, \beta_A(z'_2, z'_3, z'_A, \mu)) = -\frac{1}{2} \beta_n(z'_2, z'_3, z'_A, \mu). \quad (161)$$

Consider now that due to the Garnier equations we have

$$\frac{\partial \beta_A}{\partial z'_n} = -2 \frac{\partial H_n}{\partial z'_A} - 2 \frac{\partial \beta_A}{\partial z'_A} \frac{\partial H_n}{\partial \beta_A}, \quad (162)$$

and also

$$\frac{\partial \beta_n}{\partial z'_A} = -2 \frac{\partial H_n}{\partial z'_A} - 2 \frac{\partial H_n}{\partial \beta_A} \frac{\partial \beta_A}{\partial z'_A} \frac{\partial \beta_A}{\partial z'_n} \quad (163)$$

which proves eq.(59) of the text.

Similarly we have

$$\frac{\partial \beta_n}{\partial z'_m} = -2 \frac{\partial H_n}{\partial z'_m} - 2 \frac{\partial H_n}{\partial \beta_A} \frac{\partial \beta_A}{\partial z'_m} = -2 \frac{\partial H_n}{\partial z'_m} + 4 \frac{\partial H_m}{\partial \beta_A} \frac{\partial \beta_A}{\partial z'_m} \frac{\partial \beta_A}{\partial z'_n} \quad (164)$$

and thus

$$\frac{\partial \beta_n}{\partial z'_m} - \frac{\partial \beta_m}{\partial z'_n} = 2[ \frac{\partial H_m}{\partial z'_m} - \frac{\partial H_n}{\partial z'_n} + 2( \frac{\partial H_m}{\partial \beta_A} \frac{\partial \beta_A}{\partial z'_m} - \frac{\partial H_n}{\partial \beta_A} \frac{\partial \beta_A}{\partial z'_n})]. \quad (165)$$

Using the expression of $H_2$ and $H_3$ in terms of $z'_n, z'_A, \beta_A$ and $\mu$, obtained from the Fuchs and no logarithm relations, one can check directly that the r.h.s. of eq.(165) is zero thus proving eq.(62) of the text.
Appendix 2: General solution for the N with spherical symmetry

In the following we shall investigate the structure of the most general spherically symmetric solution of the equation

$$\triangle N = N \exp(-2\tilde{\sigma}),$$  \hfill (166)

where the conformal factor is given by (82). In polar coordinates, where the requirement of spherical symmetry becomes $\frac{\partial N}{\partial \theta} = 0$, the above equation takes the form

$$r \frac{\partial}{\partial r} \left( r \frac{\partial N}{\partial r} \right) = r^2 \exp(-2\tilde{\sigma}(t,r))N. \hfill (167)$$

If we set $r = \exp(w - w_0)$, it reduces to the following ordinary linear differential equation of the second order

$$\frac{\partial^2 N}{\partial w^2} = \exp(-2\tilde{\sigma}(t,w) + 2(w - w_0))N. \hfill (168)$$

Given a particular solution $\bar{N}$ of eq.(168), a second one can be sought in the factorized form $\phi(w,t)\bar{N}(w,t)$. A short calculation shows that $\phi(w,t)$ must satisfy

$$\frac{\partial}{\partial w} \left( \log \left( \frac{\partial \phi}{\partial w} \right) \right) = -\frac{\partial}{\partial w} \log \bar{N}^2, \hfill (169)$$

from which

$$\phi(t,w) = -\int^w dw' \frac{dw'}{\bar{N}^2(w',t)}. \hfill (170)$$

Thus the most general solution of eq.(168) has the form

$$N_{gen.}(w,t) = a(t)\bar{N}(w,t) + b(t)\bar{N}(w,t) \int^w dw' \frac{dw'}{\bar{N}^2(w',t)}, \hfill (171)$$

where $a(t)$ and $b(t)$ are two arbitrary function depending on time, but not on $w$. 
As we have seen, a particular solution is provided by
\[
N = \frac{\partial(-2\bar{\sigma})}{\partial M} = \frac{1}{2\pi} \log \left( \frac{r}{\Lambda} \right) - \frac{1}{2\pi(1-\mu)} + \frac{1}{2\pi} \frac{2}{(\frac{r}{\Lambda})^{2(1-\mu)} - 1} \log \left( \frac{r}{\Lambda} \right) =
\]
\[
= \frac{w}{2\pi} - \frac{1}{2\pi(1-\mu)} + \frac{1}{2\pi} \frac{2w}{e^{2(1-\mu)w} - 1},
\]
which substituted in eq.(171) gives
\[
N_{\text{gen.}}(t,w) = a(t) \left[ \frac{w}{2\pi} - \frac{1}{2\pi(1-\mu)} \frac{b(t)/a(t)}{2\pi(1-\mu)} + \frac{2}{e^{2(1-\mu)w} - 1} \left( w - \frac{b(t)/a(t)}{2\pi(1-\mu)} \right) \right].
\]
Defining \( k(t) \equiv a(t)/b(t)(1-\mu) \) and restoring the variable \( r \), the above general solution takes the known form
\[
N_{\text{gen.}}(r,t) = a(t) \left[ \frac{1}{2\pi} \log \left( \frac{r}{\Lambda \kappa(t)} \right) - \frac{1}{2\pi(1-\mu)} + \frac{1}{2\pi} \frac{2}{(\frac{r}{\Lambda})^{2(1-\mu)} - 1} \log \left( \frac{r}{\Lambda \kappa(t)} \right) \right].
\]
Requiring that \( N \to \frac{1}{2\pi} \log \left( \frac{r}{\Lambda} \right) \) when \( r \) approaches infinity fixes \( a(t) \) to be 1 and thus we are left with one arbitrary function given by \( k(t) \).

References

[1] A. Staruszkiewicz, Acta Phys. Polonica 24 (1963) 734; S. Deser, R. Jackiw and G. ’t Hooft, Ann. Phys. (NY) 152 (1984) 220; S. Deser and R. Jackiw, Ann. Phys. 153 (1984) 405

[2] G. ’t Hooft, Class. Quantum Grav. 9 (1992) 1335; Class. Quantum Grav. 10 (1993) 1023

[3] G. ’t Hooft, Class. Quantum Grav. 10 (1993) 1653; Class. Quantum Grav. 13 (1996) 1023

[4] H. Waelbroeck, Class. Quantum Grav. 7 (1990) 751; Phys. Rev. D50 (1994) 4982

[5] A. Bellini, M. Ciafaloni, P. Valtancoli, Physics Lett. B 357 (1995) 532; Nucl. Phys. B 462 (1996) 453

[6] M. Welling, Class. Quantum Grav. 13 (1996) 653; Nucl. Phys. B 515 (1998) 436
[7] P. Menotti, D. Seminara, Ann. Phys. 279 (2000) 282

[8] P. Menotti, D. Seminara, Nucl. Phys. (Proc. Suppl.) 88 (2000) 132.

[9] A. Hosoya and K. Nakao, Progr. Theor. Phys. 84 (1990) 739

[10] J. Nelson, T. Regge, Phys. Lett. B273 (1991) 213

[11] S. Carlip, Phys. Rev. D42 (1990) 2647; Phys. Rev. D45 (1992) 3584

[12] M. Welling, Class. Quant. Grav. 14 (1997) 3313; H-J. Matschull and M. Welling, Class. Quantum Grav. 15 (1998) 2981

[13] V. Moncrief, J. Math. Phys. 30 (1989) 2907

[14] A. Hosoya and K. Nakao, Class. Quantum Grav. 7 (1990) 163

[15] P. G. Zograf, L. A. Tahktajan, Math. USSR Sbornik 60 (1988) 143

[16] G. ’t Hooft, Comm. Math. Phys. 117 (1988) 685

[17] S. Deser, R. Jackiw, Comm. Math. Phys. 118 (1988) 495

[18] R. Arnowitt, S. Deser and C.W. Misner, in “Gravitation: an introduction to current research” Edited by L.Witten, John Wiley & Sons New York, London 1962

[19] S.W. Hawking and C. J. Hunter, Class. Quantum Grav. 13 (1996) 2735; G. Hayward, Phys. Rev. D 47 (1993) 3275

[20] R. M. Wald, “General Relativity”, The University of Chicago Press, Chicago and London (1984)

[21] C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation”, W. H. Freeman and Co., New York (1973)

[22] M. Yoshida, “Fuchsian differential equations”, Fried. Vieweg & Sohn, Braunschweig (1987)
[23] K. Okamoto, J. Fac. Sci. Tokio Univ. 33 (1986)

[24] A. Einstein, L. Infeld and B. Hoffmann, Ann. Math. 39 (1938) 65; J. N. Goldberg, in “Gravitation: an introduction to current research” Edited by L. Witten, John Wiley & Sons New York, London 1962

[25] J. Liouville J. Math. Pures Appl. 18 (1853) 71; H. Poincaré, J. Math. Pures Appl. 5 ser.2 (1898) 157; P. Ginsparg and G. Moore, hep-th/9304011; L. Takhtajan, “Topics in quantum geometry of Riemann surfaces: two dimensional quantum gravity”, Como Quantum Groups (1994) 541, hep-th/9409088; I. Kra, “Automorphic forms and Kleinian groups” Benjamin, Reading Mass., 1972.

[26] L. A. Takhtajan, Mod. Phys. Lett. A11 (1996) 93

[27] M. Bourdeau, S. D. Sorkin, Phys. Rev. D 45 (1992) 687

[28] F. Riesz and B. Sz. Nagy, “Functional Analysis” Dover Publications, Inc. New York 1990.